ON FIXED POINTS OF A GENERALIZED MULTIDIMENSIONAL AFFINE RECURSION

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Abstract. Let $G$ be a multiplicative subsemigroup of the general linear group $\text{GL}(\mathbb{R}^d)$ which consists of matrices with positive entries such that every column and every row contains a strictly positive element. Given a $G$–valued random matrix $A$, we consider the following generalized multidimensional affine equation

$$R = \sum_{i=1}^{N} A_i R_i + B,$$

where $N \geq 2$ is a fixed natural number, $A_1, \ldots, A_N$ are independent copies of $A$, $B \in \mathbb{R}^d$ is a random vector with positive entries, and $R_1, \ldots, R_N$ are independent copies of $R \in \mathbb{R}^d$, which have also positive entries. Moreover, all of them are mutually independent and $\Xi$ stands for the equality in distribution. We will show with the aid of spectral theory developed by Guivarc’h and Le Page [6, 16], that under appropriate conditions, there exists $\chi > 0$ such that $\mathbb{P}(\{(R, u) > t\}) \sim t^{-\chi}$, as $t \to \infty$, for every unit vector $u \in S^{d-1}$ with positive entries.

1. Introduction and statement of the results

We consider the Euclidean space $\mathbb{R}^d$ endowed with the scalar product $\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i$, the norm $|x| = \sqrt{x, x}$, and its Borel $\sigma$–field $\text{Bor}(\mathbb{R}^d)$. We say that $\mathbb{R}^d \ni x = (x_1, \ldots, x_d) \geq 0$ is positive (resp. $\mathbb{R}^d \ni x = (x_1, \ldots, x_d) > 0$ is strictly positive), when $x_n \geq 0$, (resp. $x_n > 0$) for every $1 \leq n \leq d$. By $\mathbb{R}_+^d$ we denote the set of all positive vectors, and we define the set $S^+ = \mathbb{R}_+^d \cap S^{d-1}$ of all positive vectors on the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ with the distance being the restriction of the Euclidean norm to $S^+$. Given $x \in \mathbb{R}^d$ we denote its projection on $S^{d-1}$ by $\mathfrak{p} = \frac{x}{|x|}$.

Let $\text{GL}(\mathbb{R}^d)$ be the group of $d \times d$ invertible matrices on $\mathbb{R}^d$, with the operator norm $\| \cdot \|$ associated with the Euclidean norm $| \cdot |$ on $\mathbb{R}^d$, i.e. $\|a\| = \sup_{x \in S^{d-1}} |ax|$ for every $a \in \text{GL}(\mathbb{R}^d)$.

Suppose that $G$ is a multiplicative subsemigroup of $\text{GL}(\mathbb{R}^d)$ which consists of matrices with positive entries such that every column and every row contains a strictly positive element. By $G^+$ we denote the multiplicative subsemigroup of $G$ composed of matrices with strictly positive entries. It is easy to see that $G$ provides a projective action on $S^+$ which is given by

$$G \times S^+ \ni (a, x) \mapsto a \cdot x = \frac{ax}{|ax|} \in S^+.$$

Let $A$ be a $G$–valued random matrix distributed according to a probability measure $\mu$ on $G$, and $B$ be a random vector independent of $A$, taking its values in $\mathbb{R}_+^d$.

Let $A_1, \ldots, A_N$ and $B_0$ be independent random variables, where $N \geq 2$ is a fixed natural number, $A_1, \ldots, A_N$ are independent copies of $A$, and $B_0$ is an independent copy of $B$.

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The aim of this paper is to find a random vector \( R \in \mathbb{R}_+^d \), independent of \( A \) and \( B \), which solves (in law \( \mathbb{D} \)) a generalized multidimensional affine equation i.e.

\[
R \overset{\mathbb{D}}{=} \sum_{i=1}^{N} A_i R_i + B_0,
\]

where \( R_1, \ldots, R_N \) are independent copies of \( R \in \mathbb{R}_+^d \) and independent of \( A, A_1, \ldots, A_N, B, B_0 \), (see Theorem 1.7 stated below).

Furthermore, we would like to find possibly mild conditions, which allows us to establish an asymptotic tail formula for \( R \). More precisely, we are interested in the existence \( \chi > 0 \), such that

\[
\mathbb{P}(\{\langle R, u \rangle > t\}) \asymp t^{-\chi}, \quad \text{as} \quad t \to \infty,
\]

for every \( u \in \mathbb{S}^+ \), (see Theorem 1.8 stated below).

The one dimensional version of the equation (1.1) has been considered recently by Jelenković and Olera-Cravioto [12, 13] and [14] in the context of Google’s PageRank algorithm. The authors solved equation (1.1) and justified formula (1.2) using the renewal theorem. It is worth to emphasize that the one dimensional version of equation (1.1) with \( B = 0 \), was studied by Liu in a series of articles (see for instance [17] and the references given there).

We are also motivated by the recent results of Buraczewski, Damek and Guivarc’h [3], where the authors considered the multidimensional version of equation (1.1) with \( B = 0 \), and established formula (1.2) with the help of Kesten’s renewal theorem [16] and the spectral method developed by Guivarc’h and Le Page [5] and [6]. Their approach sheds some new light on multidimensional problems and fits perfectly to our situation.

In order to avoid repetitions in the sequel, and shorten article we have decided to state all necessary definitions and notations in the introduction, and formulate our main results as general as it is possible.

Let \( M^1(G) \) denote the set of all probability measures on \( G \) endowed with the weak topology. We denote by supp\( \mu \) the support of the measure \( \mu \in M^1(G) \). If \( E \subseteq G \), let \( \langle E \rangle \) be the subsemigroup of \( G \) generated by the set \( E \). For \( n \in \mathbb{N} \) let \( S_n = A_n \cdot \ldots \cdot A_1 \in G \), where \( A_1, A_2, \ldots \in G \) is a sequence of independent copies of \( G \)-valued random matrix \( A \) distributed according to \( \mu \).

A subsemigroup \( \text{supp} \mu \) of \( G \) is called \textbf{contractive} if \( \text{supp} \mu \cap G^0 \neq \emptyset \). In other words,

\[
\mathbb{P} \left( \bigcup_{n \in \mathbb{N}} \{S_n \in G^0\} \right) > 0.
\]

The condition (1.3) was considered by Hennion [9], Hennion and Hervé [10] in the context of limit theorems for the products of positive random matrices.

An element \( a \in \text{Gl}(\mathbb{R}^d) \) is \textbf{proximal} if there exists a unique eigenvalue \( \lambda_a \) (the dominant eigenvalue) of \( a \), such that \( r(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = |\lambda_a| \).

According to the Perron–Frobenius theorem [11] every \( a \in G^0 \) is proximal. Moreover, for every \( a \in G^0 \) and its adjoint \( a^* \in G^0 \) it is possible to choose \( v_a, w_a \in \mathbb{R}^d_+ \) such that \( v_a > 0, w_a > 0 \) and

\[
aw_a = \lambda_a v_a, \quad a^*w_a = \lambda_a w_a, \quad \langle v_a, w_a \rangle = 1, \quad |w_a| = 1.
\]

The eigenvector \( v_a \) determined by these relations will be called the dominant eigenvector of \( a \in G^0 \). This means that we can write \( \mathbb{R}^d = \mathbb{R} \cdot v_a \oplus v_a^\perp \), and the spectral radius of \( a \), restricted to \( v_a^\perp = \{x \in \mathbb{R}^d : \langle x, v_a \rangle = 0\} \) is strictly less than \( |\lambda_a| \). Furthermore, by the preceding relations we have

\[
\lim_{n \to \infty} \frac{a^n}{r(a)^n} = v_a \oplus w_a,
\]
contains a detailed proof of Theorem 1.7.

A subsemigroup \( \Gamma \subseteq \text{Gl}(\mathbb{R}^d) \) is **strongly irreducible** if there does not exist a finite number \((k \in \mathbb{N}) \) of proper linear subspaces \( V_1, \ldots, V_k \) of \( \mathbb{R}^d \) such that

\[
\Gamma \left( \bigcup_{i=1}^k V_i \right) \subseteq \bigcup_{i=1}^k V_i.
\]

If \( E \subseteq \text{Gl}(\mathbb{R}^d) \) we denote by \( E^{\text{prox}} \) the set of all proximal elements of \( E \). A subsemigroup \( \Gamma \subseteq \text{Gl}(\mathbb{R}^d) \) is said to satisfy condition \((i - p)\) if \( \Gamma \) is strongly irreducible and \( \Gamma^{\text{prox}} \neq \emptyset \). This condition was widely investigated by Guivarc’h and Le Page [5] and [6], see also [7, 8, 3] and the references given there.

A subsemigroup \([\text{supp} \mu] \subseteq G\), where \( \mu \in M^1(G) \), is said to satisfy **condition \((C)\)** if \([\text{supp} \mu]\) is contractive and strongly irreducible. Clearly, condition \((C)\) implies condition \((i - p)\) with \( \Gamma = [\text{supp} \mu]\).

For \( s \geq 0 \) we write

\[
\kappa(s) = \kappa_\mu(s) = \lim_{n \to \infty} \left( \int_G \|a\|^s \mu^{*n}(da) \right)^{\frac{1}{n}},
\]

where \( \mu^{*n} \) is the \( n \)-th convolution power of \( \mu \in M^1(G) \). The limit above exists and it is equal to \( \inf_{n \in \mathbb{N}} \left( \int_G \|a\|^s \mu^{*n}(da) \right)^{\frac{1}{n}} \), because \( u_n(s) = \int_G \|a\|^s \mu^{*n}(da) \) is submultiplicative, i.e. \( u_{m+n}(s) \leq u_m(s) u_n(s) \) for every \( m, n \in \mathbb{N} \). Moreover,

\[
I_\mu = \{ s \in [0, \infty) : \kappa_\mu(s) < \infty \} = \left\{ s \in [0, \infty) : \int_G \|a\|^s \mu(da) < \infty \right\}.
\]

Let \( s_\infty = \sup \{ s \geq 0 : \kappa_\mu(s) < \infty \} \in \mathbb{R}_+ \cup \{ \infty \} \), then by the Hölder inequality \( I_\mu = [0, s_\infty) \) or \( I_\mu = [0, s_\infty] \). For technical reasons we have to assume that there is \( s_1 < \frac{1}{\xi} \) such that \( \mathbb{E}(\| A \|^s_1) \leq \frac{1}{\xi} \).

Our “existence” result is the following

**Theorem 1.7.** Assume that \( A \) is a \( G \)-valued random matrix distributed according to a probability measure \( \mu \) on \( G \), and \( B \) is a random vector independent of \( A \), taking its values in \( \mathbb{R}_+^d \), such that \( \mathbb{P}(\{ B > 0 \}) > 0 \). Let \( A_1, \ldots, A_N \) and \( B_0 \) be independent random variables as in \((1.1)\), where \( N \geq 2 \) is a fixed natural number, \( A_1, \ldots, A_N \) are independent copies of \( A \), and \( B_0 \) is an independent copy of \( B \). Suppose further that \([\text{supp} \mu] \subseteq G\) satisfies condition \((C)\) and there exist \( s_1 \in (0, 1/2] \), and \( s_2 > s_1 \) such that \( \mathbb{E}(\| A \|^{s_1}) \leq \frac{1}{\xi} \), \( \mathbb{E}(\| A \|^{s_2}) \leq \frac{1}{\xi} \), and \( \mathbb{E}(\| B \|^{s_2}) < \infty \). Then there exists a unique vector \( R \in \mathbb{R}_+^d \) and its independent copies \( R_1, \ldots, R_N \) independent of \( A, A_1, \ldots, A_N, B, B_0 \) which solve \((1.1)\) in law. Moreover, \( \mathbb{E}(\| R \|^s) < \infty \) for every \( s < s_2 \).

Section 3 contains a detailed proof of Theorem 1.7, which is similar in spirit to that of [12]. However, the multidimensional framework, we consider, provides some difficulties which do not appear in the one dimensional case. Namely, the method developed in [12], which gives finiteness of appropriate moments for the solution of \((1.1)\), breaks down in higher dimensions. This problem will be dealt with the help of condition \((C)\).

Let \( \lambda_d \) be the Lebesgue measure on \( \mathbb{R}^d \). If \( \nu \) is a probability measure on \( \mathbb{R}^d \), then by \( \nu = \nu_a + \nu_s \) we denote its Lebesgue decomposition with respect to \( \lambda_d \), where \( \nu_a \) is the absolutely continuous part with respect to \( \lambda_d \), i.e. \( \nu_a \ll \lambda_d \), and \( \nu_s \) is the singular part with respect to \( \lambda_d \), i.e. \( \nu_s \perp \lambda_d \). We have also \( \nu_s \perp \nu_s \). Since \( \nu \) is positive then its total variation \( \| \nu \| = \nu(\mathbb{R}^d) = 1 \). We say that the measure \( \nu \) is singular if \( \| \nu_s \| = 1 \), otherwise \( \nu \) is nonsingular, i.e. \( \| \nu_s \| < 1 \).
Now we can state our main “tail” result.

**Theorem 1.8.** Fix a natural number $N \geq 2$, a $G$-valued random matrix $A$ distributed according to $\mu$, and a random vector $B$ with law $\eta$, independent of $A$, taking its values in $\mathbb{R}^d$, such that $\mathbb{P}(\{B > 0\}) > 0$.

- Assume that $[\text{supp}\mu] \subseteq G$ satisfies condition (C), and there is $s_1 \in (0, 1/2]$, such that $\mathbb{E}(\|A\|^{s_1}) \leq \frac{1}{N}$. Moreover, we assume $s_\infty > s_1$ and $\lim_{s \to s_\infty} \kappa(s) > \frac{1}{N}$. Then there exists $\chi > s_1$ such that $N\kappa(\chi) = 1$.

- Furthermore, if $\mathbb{E}(\|A\|^\log^+\|A\|) < \infty$, $E(|B|^\chi^\varepsilon) < \infty$ for some $\varepsilon > 0$, and either
  1. $\eta$ is nonsingular, i.e. $\|\eta_x\| < 1$, or
  2. $\eta$ is singular, i.e. $\|\eta_x\| = 1$, and $\mathbb{P}(\{(B, u) = r\}) = 0$ for every $(u, r) \in S^+ \times \mathbb{R}_+$.

Then there exists a positive function $e^\chi : S^+ \mapsto (0, \infty)$ and a constant $C_\chi \geq 0$ such that

$$\lim_{t \to \infty} t^\chi \mathbb{P}(\{(R, u) > t\}) = C_\chi e^\chi(u) \geq 0,$$

for every $u \in S^+$, where $R \in \mathbb{R}^d$ is the stationary solution of the equation (1.1) as in Theorem 1.7. Moreover, if $\chi \geq 1$ then $C_\chi > 0$, and the limit in (1.9) is strictly positive.

Now we give an example of singular measure $\eta$, i.e. $\|\eta_x\| = 1$, on the plane $(d = 2)$, such that $\eta(\{x \in \mathbb{R}^2 : \langle x, u \rangle = r\}) = 0$ for every $(u, r) \in S^+ \times \mathbb{R}_+$, and $\eta(\{x \in \mathbb{R}^2 : x > 0\}) > 0$. Define $S = \{(\cos \alpha, \sin \alpha) : 0 < \alpha < \pi/2\} \subseteq S^+$ and let $\eta$ be the normalized one dimensional Lebesgue measure on $S$, i.e. $\text{supp}\eta = S^+$ and $\eta(S) = 1$. It is not hard to see that $\eta$ is singular with respect to two dimensional Lebesgue measure $\lambda_2$. Obviously $\eta(\{x \in \mathbb{R}^2 : x \neq 0\}) = \eta(S) = 1$, and notice that $\{x \in \mathbb{R}^2 : \langle x, u \rangle = r\}$ intersects $S$ at most two points, hence finally $\eta(\{x \in \mathbb{R}^2 : \langle x, u \rangle = r\}) = 0$.

As we mentioned before, the proof is based on concepts of [3] with considerable complications determined by the structure of equation (1.1). The most important tool which allows us to establish relation (1.9) is Kesten’s renewal theorem [16]. We need to check that its assumptions are satisfied (see Section 4). This is the most difficult part of the paper and requires the spectral theory of transfer operators developed by Guivarc’h and Le Page ([5], [6] and [3]), which is summarized in Section 2. But we touch only a few aspects of their theory and restrict our attention to the results which will be used in Sections 3 and 4. Guivarc’h and Le Page approach significantly simplifies and clarifies proofs developed by Kesten in [15], and what is most important for us, it is applicable to our situation.

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2. **Transfer operators**

Let $C(S^+)$ be the space of continuous functions on $S^+$ with the supremum norm $|\cdot|_\infty$.

$$H_\varepsilon = \{\phi \in C(S^+) : \|\phi\|_\varepsilon = |\phi|_\infty + |\phi|_\varepsilon < \infty, \varepsilon \in (0, 1]\}$$

is the space of all $\varepsilon$-Hölder functions on $S^+$ with

$$|\phi|_\varepsilon = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\varepsilon}.$$  

Given a closed subset $V$ of $S^+$, $M^1(V)$ denotes the set of all probability measures on $V$, endowed with the weak topology. We say that $U \subseteq S^+$ is a subspace of $S^+$, if $U = V \cap S^+$ for some subspace $V \subseteq \mathbb{R}^d$. A measure $\nu \in M^1(S^+)$ is said to be proper if $\nu(U) = 0$ for every subspace $U \not\subseteq S^+$. Here
and subsequently, \( \Lambda(\Gamma) = \{ \pi_a \in S^+ : v_a \) is the dominant eigenvector of \( a \in \Gamma^{prox} \} \), where \( \Gamma \) is a subsemigroup of \( G \) such that \( \Gamma^{prox} \neq \emptyset \).

The following Proposition 2.1 due to Guivarc'h and Raugi [7] (see also [8]) contains the relevant properties of \( (i-p) \) semigroups which will be used in the sequel.

**Proposition 2.1.** Let \( \mu \in M^1(G) \) and \( \Gamma = [\text{supp} \mu] \) satisfies condition \((i-p)\). Then there exists a unique proper \( \mu \)-stationary measure \( \nu \in M^1(S^+) \) such that \( \text{supp} \mu = \Lambda(\Gamma) \). Furthermore, \( \Lambda(\Gamma) \) is the unique \( \Gamma \)-minimal subset of \( S^+ \) (i.e. if \( Z \subseteq S^+ \) is closed and \( \Gamma \cdot Z \subseteq Z \), then \( \Lambda(\Gamma) \subseteq Z \)), and the subgroup of \( R^*_+ \) generated by the set \( \{ |\lambda_a| : a \in \Gamma^{prox} \} \) is dense in \( R^*_+ \).

Let \( \mu \in M^1(G) \). For \( s \in I_\mu \), \( x \in S^+ \) and a measurable function \( \phi \) on \( S^+ \) we consider the following transfer operators

\[
P^s\phi(x) = \int_G |ax|^s \phi(a \cdot x) \mu(da),
\]

\[
P^*_s\phi(x) = \int_G |a^s x|^s \phi(a^s \cdot x) \mu(da) = \int_G |ax|^s \phi(a \cdot x) \mu_s(da),
\]

where \( \mu_s \in M^1(G) \) and \( \mu_s(U) = \mu(\{ a \in G : a^s \in U \}) \) for every \( U \in \text{Bor}(G) \).

The main purpose of this section is to summarize a number of properties of operators \( P^s, P^*_s \), see Theorem 2.3 below.

**Theorem 2.3.** Assume that \( \mu \in M^1(G) \), \( s \in I_\mu \) and \( \Gamma = [\text{supp} \mu] \) satisfies condition \((i-p)\). Then

- there exists a unique probability measure \( \nu^s \in M^1(S^+) \), \( (\nu^s \in M^1(S^+)) \) such that
  - (i) \( P^s \nu = \kappa(s) \nu^s \) \( (P^*_s \nu^s = \kappa(s) \nu^s) \).
- \( \text{supp} \nu^s = \Lambda([\text{supp} \mu]) \), \( (\text{supp} \nu^s = \Lambda([\text{supp} \mu])) \) and it is not contained in any proper subspace of \( S^+ \).
- \( I_\mu \supseteq s \mapsto \nu^s \in M^1(S^+) \), \( (I_\mu \supseteq s \mapsto \nu^s \in M^1(S^+)) \) is continuous in the weak topology.
- \( I_\mu \supseteq s \mapsto \kappa(s) \) is strictly \( \text{log-convex} \) function.
- there exists a unique \( s \)-\( \text{Hölder continuous} \) function \( e^s : S^+ \mapsto (0, \infty) \) with \( s = \min\{s, 1\} \) such that
  - (i) \( P^s e^s = \kappa(s) e^s \).
  - (ii) \( e^s \) is given by the formula
    \[
e^s(x) = \int_{S^+} \langle x, y \rangle^s \nu^s(dy), \quad \text{for} \ x \in S^+.
\]
- Moreover, there exists a unique stationary measure \( \pi^s \in M^1(S^+) \), \( (\pi^s \in M^1(S^+)) \) for operator \( Q^s f = \frac{P^s(e^s f)}{\kappa(s) e^s} \), \( (Q^s f = \frac{P^s(e^s f)}{\kappa(s) e^s}) \) where \( f \in C(S^+) \), such that
  - (i) \( \pi^s = \frac{\kappa(s)}{\nu^s(e)} \), \( (\pi^s = \frac{\kappa(s)}{\nu^s(e)}) \).
  - (ii) \( (Q^s)^n f \) \( ((Q^s)^n f) \) converges uniformly to \( \pi^s(f) \), \( (\pi^s(f)) \) for any \( f \in C(S^+) \).
  - (iii) \( \text{supp} \pi^s = \Lambda([\text{supp} \mu]) \), \( (\text{supp} \pi^s = \Lambda([\text{supp} \mu])) \).

This result was proved by Guivarc'h and Le Page and its, quite long and far from being obvious, proof can be found in [5] and [6]. Notice that in view of the cocycle property \( \sigma^s(x, a_2 a_1) = \sigma^s(x, a_1) \sigma^s(a_1 \cdot x, a_2) \), \( a_1, a_2 \in G, x \in S^+ \) of

\[
(2.4) \quad \sigma^s(x, a) = |ax|^s \frac{e^s(a \cdot x)}{e^s(x)},
\]
the Markov operators $Q^*$ and $Q_n^*$ defined in Theorem 2.3 can be rewritten in the following form

\begin{align}
(2.5) & \quad (Q^*)^n \phi(x) = \int_G \phi(a \cdot x) q_n^*(x, a) \mu^n(da), \\
(2.6) & \quad (Q_n^*)^n \phi(x) = \int_G \phi(a \cdot x) q_n^*(x, a) \mu_n^n(da),
\end{align}

where

\begin{equation}
q_n^*(x, a) = \frac{1}{\kappa_n(s)} \frac{\sigma^*(x, a)}{e^*(x)} |ax|^s = \frac{\sigma^*(x, a)}{\kappa_n(s)},
\end{equation}

$n \in \mathbb{N}$, $x \in \mathbb{S}^+$, $a \in G$ and $\phi$ is an arbitrary measurable function on $\mathbb{S}^+$.

3. Construction of the solution

Recall that $A$ stands for a $G$-valued random matrix distributed according to the measure $\mu \in M^1(G)$, and $B$ for a random vector taking its values in $\mathbb{R}_+^d$, independent of $A$. In this section we construct a solution of the equation (1.1). The idea of the construction goes back to [12]. It is not difficult to imagine that we have to study a sequence of random variables that are obtained by iterating (1.1). Let $N \geq 2$ be a fixed natural number and $R_{0,1}, \ldots, R_{0,N}$ be independent and identically distributed (i.i.d.) copies of the initial random variable $R_0 \in \mathbb{R}_+^d$. We consider the sequence $(R_n^*)_{n \geq 0}$ such that

\begin{equation}
R_{n+1}^* = \sum_{k=1}^N A_{n+1,k} R_{n,k}^* + B_{n+1}, \quad \text{for every } n \geq 0,
\end{equation}

where $A_{n+1,1}, \ldots, A_{n+1,N}$, $B_{n+1}$ and $R_{n,1}, \ldots, R_{n,N}$, $n \geq 0$ are independent. Moreover, for $n \geq 1$ $R_{n,1}^*, \ldots, R_{n,N}^*$ are i.i.d. copies of $R_0^*$ from the previous iteration. For $n \geq 0$ $A_{n+1,1}, \ldots, A_{n+1,N}$ are i.i.d. copies of $A$ and $B_{n+1}$ is an independent copy of $B$.

We will look more closely at the sequence $(R_n^*)_{n \geq 0}$. Let $\mathcal{A} = \{A_{i_1,\ldots,i_n} : (i_1, \ldots, i_n) \in \{1, \ldots, N\}^n, n \in \mathbb{N}\}$ be the set consisting of i.i.d. copies of $A$, and $\mathcal{B} = \{B_{i_1,\ldots,i_n} : (i_1, \ldots, i_n) \in \{1, \ldots, N\}^n, n \in \mathbb{N}\} \cup \{B_0\}$ the set consisting of i.i.d. copies of $B$ independent of $A$. Additionally we assume that $A_0 = \text{Id}$ a.s. and the initial random variable $R_0^*$ is always independent of $A$, $B$, $\mathcal{A}$ and $\mathcal{B}$.

Now let $W_0 = A_0 B_0 = B_0$ a.s.,

\begin{equation}
W_n = \sum_{(i_1,\ldots,i_n) \in \{1, \ldots, N\}^n} A_{i_1,\ldots,i_n} A_{i_1,i_2} \cdots A_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}, \quad n \geq 1,
\end{equation}

and for $n \geq 0$

\begin{equation}
R^{(n)} = \sum_{i=0}^n W_i
\end{equation}

be the partial sum of the sequence $(W_n)_{n \geq 0}$. Then

\begin{equation}
R = \lim_{n \to \infty} R^{(n)} = \sum_{i=0}^\infty W_i \text{ a.s.}
\end{equation}
is a candidate for a solution of (1.1). It is not hard to see that $W_n$ satisfies

$$W_n = \sum_{(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n} A_{i_1} A_{i_1, i_2} \cdots A_{i_1, \ldots, i_n} B_{i_1, \ldots, i_n}$$

$$= \sum_{k=1}^{N} A_k \left( \sum_{(k, i_2, \ldots, i_n) \in \{1, \ldots, N\}^n} A_{k, i_2} \cdots A_{k, i_2, \ldots, i_n} B_{k, i_2, \ldots, i_n} \right) = \sum_{k=1}^{N} A_k W_{n-1, k},$$

where $A_k$ and $W_{n-1, k}$ are independent of each other and $W_{n-1, 1}, \ldots, W_{n-1, N}$ have the same distribution as $W_n$. In view of the above calculations, $R^{(n)}$ satisfies the recursion

$$R^{(n)} = \sum_{k=1}^{N} A_k R^{(n-1)} + B_0,$$

for every $n \in \mathbb{N}$, where $R_1^{(n-1)}, \ldots, R_N^{(n-1)}$ are independent copies of $R^{(n-1)}$.

To obtain a solution with an initial condition, let $R_0^{*} \in \mathbb{R}_+^d_n$, $n \in \mathbb{N}$, be i.i.d. copies of the initial random variable $R_0^*$. For $n \geq 1$, we define similarly as in (3.2)

$$W_n(R_0^*) = \sum_{(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n} A_{i_1} A_{i_1, i_2} \cdots A_{i_1, \cdots, i_n} R_0^*(i_1, \ldots, i_n).$$

Moreover, as in (3.5), we obtain $W_n(R_0^*) \overset{D}{=} \sum_{k=1}^{N} A_k W_{n-1, k}(R_0^*)$, where $A_k$ and $W_{n-1, k}(R_0^*)$ are independent of each other and $W_{n-1, 1}(R_0^*), \ldots, W_{n-1, N}(R_0^*)$ have the same distribution as $W_{n-1}(R_0^*)$. Now we have following

**Lemma 3.8.** Assume now that $(R_n^*)_{n \geq 0}$ and $(R^{(n)})_{n \geq 0}$ are the sequences defined in (3.1) and (3.3) respectively, then for every $n \in \mathbb{N}$ we have

$$R_n^* \overset{D}{=} R^{(n-1)} + W_n(R_0^*).$$

**Proof.** Observe that for $n = 1$, (3.9) follows from definition. For more details we refer to [12]. □

Now we have simple, but very useful

**Lemma 3.10.** Under the assumptions of Theorem 2.3 there exists $c_s > 0$ such that for every $n \in \mathbb{N}$ we have

$$c_s \int_G \|a\|^s \mu^n(da) \leq \kappa^n(s) \leq \int_G \|a\|^s \mu^n(da).$$

**Proof.** We refer to [5]. □

To take the limit in (3.4) we need an estimate for $\mathbb{E}(\|W_n\|^s)$. Suppose for a moment that $s \leq 1$. Then, in view of inequality (3.11), we have

$$\mathbb{E}(\|W_n\|^s) \leq \mathbb{E} \left( \sum_{(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n} \|A_{i_1} A_{i_1, i_2} \cdots A_{i_1, \ldots, i_n}\|^s |B_{i_1, \ldots, i_n}| \right)$$

$$\leq \sum_{(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n} \mathbb{E} \left( \|A_{i_1} A_{i_1, i_2} \cdots A_{i_1, \ldots, i_n}\|^s \right) \mathbb{E}(\|B\|^s)$$

$$= N^n \int_G \|a\|^s \mu^n(da) \mathbb{E}(\|B\|^s) \leq \frac{1}{c_s} \mathbb{E}(\|B\|^s) N^n \kappa^n(s).$$
We would like to show that for an appropriate \( s > 0 \), not necessarily less or equal 1, \( E(\|W_n\|^s) \) decays exponentially. This is contained in Lemma 3.12. For the sake of computations we have to assume that there exist \( s_1 \in (0, 1/2] \) such that \( E(\|A\|^s_1) \leq \frac{1}{16} \).

**Lemma 3.12.** Assume that \( \{\text{supp}\mu\} \subseteq G \) satisfies condition (C), and there exist \( s_1 \in (0, 1/2] \), and \( s_2 > 1 \) such that \( E(\|A\|^s_1) \leq \frac{1}{16} \), \( E(\|A\|^s_2) \leq \frac{1}{16} \), and \( E(\|B\|^s_2) \leq 1 \). Then for every \( s \in (s_1, s_2) \), there exist finite constants \( K_s > 0 \) and \( \eta < 1 \) such that for every \( n \in \mathbb{N} \)

\[
E(\|W_n\|^s) \leq K_s \eta^n.
\]

**Proof.** By Theorem 2.3 \( \kappa(s) \) is strictly log-convex so \( N\kappa(s) < 1 \), for every \( s \in (s_1, s_2) \) and for \( s \leq 1 \), (3.13) follows from the calculation above. From now we assume that \( s \in (1, s_2) \) and it is fixed. Let \( S_{i_1,\ldots,i_n} = A_{i_1} A_{i_1,i_2} \cdots A_{i_1,\ldots,i_n} \) for \( (i_1, \ldots, i_n) \in \{1, \ldots, N\}^n \) and \( n \in \mathbb{N} \). We order the set of indices writing \( \{1, \ldots, N\}^n = \{i_1, \ldots, i_{N^n}\} \) and we choose \( p \in \mathbb{N} \) and \( p \geq 2 \), such that \( p - 1 < s/p \leq 1 \), and

\[
\mathbb{E}(\|W_n\|^s) \leq \mathbb{E}\left( \left( \sum_{(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n} |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/p} \right)^p \right)
\]

\[
= \mathbb{E}\left( \sum_{j_1 + \cdots + j_{N^n} = p} \left( \sum_{j_1, \ldots, j_{N^n}} \mathbb{E}\left( \left( |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/j_{i_1}} \right)^{s/j_{i_1}} \right) \cdots \mathbb{E}\left( \left( |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/j_{i_{N^n}}} \right)^{s/j_{i_{N^n}}} \right) \right) \right)
\]

Notice that \( E(\|B_{i_1}^{s/j_{i_1}}/p\cdots\cdot E(\|B_{N^n}^{s/j_{N^n}}/p\) = \( \|B\|^{s/j_{i_1}}/p \cdots \cdot \|\|B_{N^n}^{s/j_{N^n}}/p \leq \|B\|^s \), since \( \|B\|^r = E(|B|^r) \) is increasing and \( \|B\|_0 = 1 \). This implies that

\[
\sum_{j_1 + \cdots + j_{N^n} = p} \left( \sum_{j_1, \ldots, j_{N^n}} \mathbb{E}\left( \left( |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/j_{i_1}} \right)^{s/j_{i_1}} \right) \cdots \mathbb{E}\left( \left( |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/j_{i_{N^n}}} \right)^{s/j_{i_{N^n}}} \right) \right)
\]

\[
\leq \mathbb{E}(\|B\|^s) \sum_{j_1 + \cdots + j_{N^n} = p} \left( \sum_{j_1, \ldots, j_{N^n}} \mathbb{E}\left( \left( |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/j_{i_1}} \right)^{s/j_{i_1}} \right) \cdots \mathbb{E}\left( \left( |S_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n}|^{s/j_{i_{N^n}}} \right)^{s/j_{i_{N^n}}} \right) \right)
\]

\[
= \mathbb{E}(\|B\|^s) \sum_{j_1 + \cdots + j_{N^n} = p} \left( \sum_{j_1, \ldots, j_{N^n}} \right) \int_G \|a\|^{s/j_{i_1}/p} \mu^s(da) \cdots \int_G \|a\|^{s/j_{i_{N^n}}/p} \mu^s(da).
\]

Observe that by the inequality (3.11), there exist constants \( c_{s/j_{i_1}/p}, c_{s/j_{i_2}/p}, \ldots, c_{s/j_{i_{N^n}}}/p \in (0, 1], \) such that for all \( n \in \mathbb{N} \)

\[
\int_G \|a\|^{s/j_{i_1}/p} \mu^s(da) \leq c_{s/j_{i_1}/p}^{-1}(s/j_{i_1}/p),
\]

\[
\int_G \|a\|^{s/j_{i_2}/p} \mu^s(da) \leq c_{s/j_{i_2}/p}^{-1}(s/j_{i_2}/p),
\]

\[
\vdots
\]

\[
\int_G \|a\|^{s/j_{i_{N^n}}/p} \mu^s(da) \leq c_{s/j_{i_{N^n}}/p}^{-1}(s/j_{i_{N^n}}/p).
\]

Since \( j_{i_1}, j_{i_2}, \ldots, j_{i_{N^n}} \in \{0, 1, \ldots, p\} \), the constants above do not depend on \( n \in \mathbb{N} \) and we may define \( c_{p,n} = \max\{c_0, \cdots, c_{(p-1)/s/p}, c_{s}^{-1}\} \) that dominates all of them.
When \( N^n \leq p \), we have

\[
(3.14) \quad \int_G \|a\|^{s j_1/p} \mu^{s^n}(da) \cdots \int_G \|a\|^{s j_N/p} \mu^{s^n}(da) \leq c_{p,s}^{p} \kappa^n(s j_1/p) \cdots \kappa^n(s j_N/p).
\]

Therefore,

\[
\sum_{j_1 + \ldots + j_N = p} \left( \int_G \|a\|^{s j_1/p} \mu^{s^n}(da) \cdots \int_G \|a\|^{s j_N/p} \mu^{s^n}(da) \right) \leq c_{p,s}^{p} \sum_{j_1 + \ldots + j_N = p} \left( \int_G \|a\|^{s j_1/p} \mu^{s^n}(da) \right) \cdots \kappa^n(s j_N/p) \leq c_{p,s}^{p} N^{n^{p}} \cdot \max\{\kappa(s/p), \kappa(2s/p), \ldots, \kappa((p-1)s/p), \kappa(s)\} \cdot N^{n^{p-1}} \cdot \kappa(s/p) \cdots \kappa((p-1)s/p), \kappa(s)\},
\]

since \( ks/p \in (s_1, s_2) \) for every \( k \in \{1, \ldots, p\} \). This yields (3.13) with \( K_s = c_{p,s}^{p} p^{p-1} \|B\| < \infty \) and \( \eta = N \cdot \max\{\kappa(s/p), \ldots, \kappa(s)\} < 1 \). Here the assumption \( s_1 \leq 1/2 \) is indispensable, because it guarantees that \( N \cdot \kappa(s/p) < 1 \).

When \( N^n > p \), (3.14) also holds with the universal constant \( c_{p,s}^{p} \) which does not depend on \( n \in \mathbb{N} \), but we have to estimate

\[
\sum_{j_1 + \ldots + j_N = p} \left( \int_G \|a\|^{s j_1/p} \mu^{s^n}(da) \cdots \int_G \|a\|^{s j_N/p} \mu^{s^n}(da) \right) \kappa^n(s j_1/p) \cdots \kappa^n(s j_N/p),
\]

in a more subtle way. Before we do that we need to introduce a portion of necessary definitions.

For every \( r \leq k \), and \( j_1 \leq \ldots \leq j_k \), let

\[
L(j_1, \ldots, j_k) = \binom{k}{l_1, l_2, \ldots, l_r},
\]

when \( j_1 = \ldots = j_l < j_{l+1} = \ldots = j_{l+t_l} < j_{l+t_l+1} = \ldots = j_{l+t_l+t_{l+1}} < \ldots < j_{l+\ldots+t_l+t_{l+1}} = \ldots = j_{r+t_1+t_2+\ldots+t_r} \) and \( l_1 + l_2 + \ldots + l_r = k \). Then it is not difficult to see that for every \( k \leq p \)

\[
L(j_1, \ldots, j_k) \leq k!,
\]

\[
\binom{p}{j_1, \ldots, j_k} \leq p!,
\]

\[
\binom{N^n}{k} L(j_1, \ldots, j_k) \leq \frac{N^n!}{(N^n-k)!} \leq N^{kn}.
\]

Let now \( \eta = \max\{\eta_1, \eta_2, \ldots, \eta_p\} \leq 1 \), where

\[
\eta_k = \max\{(N\kappa(s j_1/p)) \cdots (N\kappa(s j_k/p)) : j_1 + \ldots + j_k = p, \text{ and } j_1 \leq \ldots \leq j_k \} < 1.
\]
This implies that
\[
\sum_{j_1 + \ldots + j_N = p} \left(\sum_{j_1, \ldots, j_N} \kappa^n(s_{j_1}/p) \ldots \kappa^n(s_{j_N}/p) \right) = N^n \kappa^n(s)
\]
\[
\left(\frac{N^n}{2}\right) \sum_{j_1 + j_2 = p, j_1 \leq j_2} \left(\sum_{j_1, j_2} \kappa^n(s_{j_1}/p) \kappa^n(s_{j_2}/p) \right)
\]
\[
\left(\frac{N^n}{3}\right) \sum_{j_1 + j_2 + j_3 = p, j_1 \leq j_2 \leq j_3} \left(\sum_{j_1, j_2, j_3} \kappa^n(s_{j_1}/p) \kappa^n(s_{j_2}/p) \kappa^n(s_{j_3}/p) \right)
\]
\]
\[
\vdots
\]
\[
\left(\frac{N^n}{p}\right) \sum_{j_1 + \ldots + j_p = p, j_1 \leq \ldots \leq j_p} \left(\sum_{j_1, \ldots, j_p} \kappa^n(s_{j_1}/p) \ldots \kappa^n(s_{j_p}/p) \right)
\leq N^n \kappa^n(s) + \sum_{j_1 + j_2 = p, j_1 \leq j_2} p! N^{2n} \kappa^n(s_{j_1}/p) \kappa^n(s_{j_2}/p)
\]
\[
+ \ldots + \sum_{j_1 + \ldots + j_p = p, j_1 \leq \ldots \leq j_p} p! N^{pn} \kappa^n(s_{j_1}/p) \ldots \kappa^n(s_{j_p}/p)
\]
\[
\leq \eta^n \cdot \sum_{k=1}^{p} \sum_{j_1 + \ldots + j_k = p, j_1 \leq \ldots \leq j_k} 1
\]
\[
\leq \eta^n \cdot p! \sum_{k=1}^{p} \binom{p - 1}{k - 1} \leq 2^{p-1} p! \cdot \eta^n.
\]

Hence in this case (3.13) follows with \( K_s = 2^{p-1} p! \eta^n \mathbb{E}(\|B\|^s) < \infty \) and \( \eta < 1 \). \( \square \)

**Proof of Theorem 1.7.** First of all we show that \( \mathbb{E}(\|R\|^s) < \infty \) for every \( s < s_2 \). By Lemma 3.12 there exist \( \eta < 1 \) and \( K_s < \infty \) such that for every \( n \in \mathbb{N} \) we have \( \mathbb{E}(\|W_n\|^s) \leq K_s \eta^n \). Observe now

\[
\mathbb{E}(\|R\|^s) = \mathbb{E}(\liminf_{n \to \infty} \|R^{(n)}\|^s) \leq \liminf_{n \to \infty} \mathbb{E}(\|R^{(n)}\|^s) \leq \liminf_{n \to \infty} \mathbb{E} \left( \sum_{k=0}^{n} |W_k| \right)^s.
\]

When \( 0 < s \leq 1 \), we have

\[
\liminf_{n \to \infty} \mathbb{E} \left( \sum_{k=0}^{n} |W_k| \right)^s \leq \liminf_{n \to \infty} \mathbb{E} \left( \sum_{k=0}^{n} |W_k| \right) \leq \liminf_{n \to \infty} \mathbb{E} \left( \sum_{k=0}^{n} |W_k|^s \right) \leq \liminf_{n \to \infty} K_s \sum_{k=0}^{n} \eta^k = \frac{K_s}{1 - \eta} < \infty.
\]
When \( s > 1 \), we have
\[
\liminf_{n \to \infty} \mathbb{E} \left( \sum_{k=0}^{n} |W_k| \right)^s \leq \liminf_{n \to \infty} \left( \sum_{k=0}^{n} \mathbb{E} (|W_k|^s)^{1/s} \right)^s \leq \lim_{n \to \infty} K_s \left( \sum_{k=0}^{n} \eta^{k/s} \right)^s = \frac{K_s}{(1 - \eta^{1/s})^s} < \infty.
\]
It immediately implies that \( \mathbb{E}(|R|^s) < \infty \), which in turn gives \( |R| < \infty \) a.s.

Now we want to show that \( R \) is the unique solution of (1.1). It is enough to show that \( R_n^* \), with arbitrary initial random variable \( R_0^* \in \mathbb{R}_+^d \), such that \( \mathbb{E}(|R_0^*|^s) < \infty \) where \( r > 0 \), converge weakly to \( R \) as \( n \to \infty \). We show that \( \mathbb{E}(f(R_n^*)) \to \mathbb{E}(f(R)) \) for an arbitrarily uniform continuous function \( f \) defined on \( \mathbb{R}^d \). Fix \( \varepsilon > 0 \), and choose \( \delta > 0 \) such that
\[
|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.
\]
By (3.9) we know that \( R_n^* \overset{D}{=} R^{(n-1)} + W_n(R_0^*) \) for every \( n \in \mathbb{N} \), hence
\[
\mathbb{E}(f(R_n^*) - f(R)) \leq \mathbb{E}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)})) + \mathbb{E}(f(R^{(n-1)}) - f(R)).
\]
It is enough to show that \( \mathbb{E}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)})) \to 0 \) as \( n \to \infty \). Fix \( s < \min\{r, 1\} \) and observe that
\[
\mathbb{E}(f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)})) \leq \mathbb{E}(|1_{|W_n(R_0^*)| \leq \delta} (f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)})))
+ \mathbb{E}(|1_{|W_n(R_0^*)| > \delta} (f(R^{(n-1)} + W_n(R_0^*)) - f(R^{(n-1)})))
\leq \varepsilon + \frac{2M}{\delta} \mathbb{P}(|W_n(R_0^*)| > \delta) \leq \varepsilon + \frac{2M}{\delta} K_s \mathbb{E}(|W_n(R_0^*)|^s)
\leq \varepsilon + \frac{2M}{\delta} K_s \mathbb{E}(|W_n(R_0^*)|^s) (N \kappa(s)) \to \varepsilon,
\]
since \( \varepsilon > 0 \) is arbitrary we have shown \( \mathbb{E}(f(R_n^*)) \to \mathbb{E}(f(R)) \), and Theorem 1.7 follows.

4. Application of Kesten’s renewal theorem

In order to prove Theorem 1.8, as mentioned in the introduction, we will use Kesten’s renewal theorem [16] which allows us to describe the desired tail asymptotic (1.9). Before we state Kesten’s theorem we have to introduce necessary definitions and to prove a number of auxiliary results. They are contained in the three lemmas of Section 4.1 and they will be used later on to check that the assumptions of Kesten’s renewal theorem are satisfied in our settings. The material presented in this section is adapted from [3], [5], [6] and [15].

4.1. Some general results. At first we define the probability space \( \Omega = G^\mathbb{N} \). \( \text{Bor}(X) \) stands for the Borel \( \sigma \)-field of the space \( X \). For any sequence \( \omega = (a_1, a_2, \ldots) \in \Omega \) we write
\[
S_n(\omega) = a_n \cdot \cdots \cdot a_1 \in G, \quad \text{for } n \in \mathbb{N} \text{ and } S_0(\omega) = \text{Id} \in G.
\]
Let \( \theta : \Omega \to \Omega \) be the shift on \( \Omega \), i.e.
\[
\theta((a_1, a_2, \ldots)) = (a_2, a_3, \ldots), \quad \text{for every } \omega = (a_1, a_2, \ldots) \in \Omega.
\]
As in Section 2 (see (2.4) and (2.7)), for every \( n \in \mathbb{N} \), we define the kernel
\[
q_n^\omega(x, \omega) = \prod_{k=1}^{n} q_1^\omega(S_{k-1}(\omega) \cdot x, a_k), \quad \text{for every } x \in S^+ \text{ and } \omega = (a_1, a_2, \ldots) \in \Omega.
\]
The cocycle property gives a very useful relation, i.e. for every \( m, n \in \mathbb{N}, x \in \mathbb{S}^+ \) and \( \omega \in \Omega \) we have
\[
q_{m+n}(x, \omega) = q_m(x, S_n(\omega))q_n(S_n(\omega) \cdot x, S_m(\theta^\omega(\omega))).
\]

The Kolmogorov’s consistency theorem guarantees the existence of the probability measure \( Q^\pi_\ast \) on \( \Omega \) being the unique extension of measures \( q_k(x, g)\mu^k(dg) \). Next we define the probability measure
\[
Q^\pi_\ast = \int_{\mathbb{S}^+} Q^\ast_\pi^\ast(dx), \quad \Omega,
\]
where \( \pi^\ast \) is the unique \( Q^\ast \) stationary measure on \( \mathbb{S}^+ \) (see Theorem 2.3). By \( E^\ast_\pi \) we denote the expectation corresponding to \( Q^\ast_\pi \). We extend the probability space \( \Omega \) to \( ^\ast\Omega = \mathbb{S}^+ \times \Omega \). Let \( ^\ast\theta : ^\ast\Omega \to ^\ast\Omega \) be the shift defined by
\[
^\ast\theta(x, \omega) = (a_1 \cdot x, \theta(\omega)), \quad \text{for every} \ x \in \mathbb{S}^+ \text{and} \ \omega = (a_1, a_2, \ldots) \in \Omega.
\]
We now define the probability measure \( ^\astQ^\ast \) on \( ^\ast\Omega \) as follows
\[
^\astQ^\ast = \int_{\mathbb{S}^+} \delta_x \otimes Q^\ast_\pi^\ast(dx).
\]

In the same way, starting with \( \mu_\ast \) instead of \( \mu \), we define the measure \( Q^\ast_\pi^\ast \) and \( E^\ast_\pi \) denotes its expectation. Moreover, the probabilities \( Q^\ast_\pi^\ast \) and \( ^\astQ^\ast_\pi^\ast \) are defined similarly, i.e.
\[
Q^\ast_\pi^\ast = \int_{\mathbb{S}^+} Q^\ast_\pi^\ast(dx), \quad \text{and} \quad ^\astQ^\ast_\pi^\ast = \int_{\mathbb{S}^+} \delta_x \otimes Q^\ast_\pi^\ast(dx),
\]
where \( \pi^\ast \) is the unique \( Q^\ast \) stationary measure on \( \mathbb{S}^+ \) (see Theorem 2.3). Let \( \omega^\ast = (a_1^\ast, a_2^\ast, \ldots) \in \Omega \). Then \( S_n(\omega^\ast) = a_1^\ast \cdot \ldots \cdot a_n^\ast \in G \).

**Remark 4.2.** The properties of the stationary measures \( \pi^\ast \) and \( \pi^\ast_\pi \) developed in Section 2 imply that \((\Omega, \text{Bor}(\Omega), Q^\ast, \theta), (\Omega, \text{Bor}(\Omega), Q^\ast_\pi, \theta), (^\ast\Omega, \text{Bor}(^\ast\Omega), ^\astQ^\ast, ^\ast\theta) \) and \((\Omega, \text{Bor}(\Omega), ^\astQ^\ast_\pi, ^\ast\theta) \) are ergodic.

From now we will work with the measures \( Q^\ast_\pi^\ast, \pi^\ast_\pi, Q^\ast_\pi^\ast \) and \( ^\astQ^\ast_\pi^\ast \). Clearly, all the results stated below remain valid for the measures \( Q^\ast_\pi^\ast, \pi^\ast_\pi, Q^\ast_\pi^\ast \) and \( ^\astQ^\ast_\pi^\ast \).

We begin with following

**Lemma 4.3.** Assume that \( \mu \in M^1(G), s \in I_\mu \) and \( \Gamma = \{ \text{supp}\mu \} \) satisfies condition \((i-p)\). Then there exists \( c > 0 \) such that \( Q^\ast_\pi^\ast(\{ \omega \in \Omega : \exists C > 0 \ \forall n \in \mathbb{N} \ | S_n(\omega)x| \geq C\| S_n(\omega) \| \}) = 1 \). Moreover the constant \( c \) does not depend on \( x \in \mathbb{S}^+ \).

**Proof.** We can repeat the argument from Section 3 in [3]. \( \square \)

**Lemma 4.4.** Assume that \( \mu \in M^1(G), s \in I_\mu \) and \( \Gamma = \{ \text{supp}\mu \} \) satisfies condition \((i-p)\). Then for every \( x \in \mathbb{S}^+ \) we have
\[
Q^\ast_\pi^\ast(\{ \omega \in \Omega : \exists C > 0 \ \forall n \in \mathbb{N} \ | S_n(\omega)x| \geq C\| S_n(\omega) \| \}) = 1, \quad \text{and}
\]
\[
Q^\ast_\pi^\ast(\{ \omega \in \Omega : \exists C > 0 \ \forall n \in \mathbb{N} \ | S_n(\omega)x| \geq C\| S_n(\omega) \| \}) = 1.
\]

**Proof.** Observe that (4.6) implies (4.5). Indeed, let
\[
Z_\ast = \{ \omega \in \Omega : \exists C > 0 \ \forall n \in \mathbb{N} \ | S_n(\omega)x| \geq C\| S_n(\omega) \| \},
\]
and let \( Z_\ast^c \) be the complement of \( Z_\ast \). Then by Lemma 4.3
\[
Q^\ast_\pi^\ast(Z_\ast^c) \leq cQ^\ast_\pi^\ast(Z_\ast^c) = 0.
\]
The proof of (4.6) is adapted from [15]. Conditions (1.3) yields the existence of \( n_0 \in \mathbb{N} \) and \( 0 < \tau < 1 \) such that

\[
(4.7) \quad p = \mathbb{P}^* \left( \{ \omega \in \Omega : S_{n_0}(\omega)(i, j) > \tau, \text{ for all } 1 \leq i, j \leq d \} \right) > 0,
\]

where \( \mathbb{P}^* = \mu_{\omega}^{\otimes \mathbb{N}} \). First of all we need to show that

\[
(4.8) \quad Q_{x}^{*,*}(\{ \omega \in \Omega : T(\omega) < \infty \}) = 1, \quad \text{for every } x \in \mathbb{S}^+ \quad \text{and}
\]

\[
(4.9) \quad Q_{x}^{*,*}(\{ \omega \in \Omega : T(\omega) < \infty \}) = 1, \quad \text{where } T(\omega) = \min\{n \geq n_0 : S_{n_0}^{\omega}(\theta^{n-n_0}(\omega)) \in G^0\}.
\]

Notice that (4.8) immediately gives (4.9), since the event \( \{ T < \infty \} \) does not depend on \( x \in \mathbb{S}^+ \) and \( Q_{x}^{*,*} = \int_{\mathbb{S}^+} Q_{x}^{*,*} \pi_\omega^s(dx) \).

Assume for a moment that (4.9) holds and prove (4.6). If \( x \in \mathbb{S}^+ \) such that \( x > 0 \) then for any \( a \in G \) we have

\[
|ax| \geq d^{-1/2} \sum_{i=1}^{d} (ax)_i \geq d^{-1/2} \min_{1 \leq i \leq d} x_i \sum_{i, j=1}^{d} a(i, j) \geq d^{-1/2} \min_{1 \leq i \leq d} x_i \|a\|,
\]

hence (4.6) holds with \( C = d^{-1/2} \min_{1 \leq i \leq d} x_i > 0 \). Now fix an arbitrary \( x \in \mathbb{S}^+ \) and let \( \Omega_1 = \{ T < \infty \} \subseteq \Omega \). By assumption, \( Q_{x}^{*,*}(\Omega_1) = 1 \). It is easy to see that \( S_{T}(\omega^*)x > 0 \) for \( \omega^* \in \Omega_1 \). Thus for any \( n \geq T \) and \( \omega^* \in \Omega_1 \) we have

\[
|S_{n}(\omega^*)x| = |S_{n-T}(\theta^T(\omega^*))S_{T}(\omega^*)x| \geq d^{-1/2} \min_{1 \leq i \leq d} (S_{T}(\omega^*)x)_i \|S_{n-T}(\theta^T(\omega^*))\|
\]

\[
\geq d^{-1/2} \min_{1 \leq i \leq d} (S_{T}(\omega^*)x)_i \|S_{n}(\omega^*)\|.
\]

It implies that \( |S_{n}(\omega^*)x| \geq C_{T,x}(\omega^*)\|S_{n}(\omega^*)\| \) holds with the constant \( C_{T,x}(\omega^*) > 0 \) independent of \( n \geq T \), for every \( \omega^* \in \Omega_1 \). Recall that \( G \) is the multiplicative semigroup of \( d \times d \) invertible matrices with positive entries such that every row and every column contains a strictly positive element. Now take \( n \leq T \) and notice that \( C_{n,x}(\omega^*) = \frac{\|S_{n}(\omega^*)\|}{\|S_{n}(\omega^*)\|} > 0 \), for every \( \omega^* \in \Omega_1 \) by the definition of \( G \) and \( x \in \mathbb{S}^+ \). Therefore, we take \( C(\omega^*) = \min\{C_{1,x}(\omega^*), \ldots, C_{T,x}(\omega^*)\} > 0 \), and (4.6) follows.

We need only to prove (4.8). In this purpose we define the events

\[
E_k = \{ \omega \in \Omega : S_{n_0}(\theta^k(\omega))(i, j) \geq \tau, \text{ for all } 1 \leq i, j \leq d \}, \quad k \in \mathbb{N}.
\]

We show that there exists \( \gamma \in [0, 1) \) such that for all \( l \in \mathbb{N} \)

\[
(4.10) \quad Q_{x}^{*,*}(\{ T > ln_0 \}) \leq \gamma^l Q_{x}^{*,*}(\{ E_{j,n_0} \text{ does not occur for any } 0 \leq j < l \}) \leq \gamma^l.
\]

Then (4.10) with Borel–Cantelli lemma yield \( Q_{x}^{*,*}(\{ T < \infty \}) = 1 \). In fact it is enough to show that

\[
(4.11) \quad Q_{x}^{*,*}(E_0^c \cap \ldots \cap E_{(i-1)n_0}^c) \leq \gamma^2 Q_{x}^{*,*}(E_0^c \cap \ldots \cap E_{(i-3)n_0}^c) \leq \gamma^l \quad \text{and inductively } \ldots \leq \gamma^l.
\]
Let \( r_s = \inf_{x \in F_n} e^s(x) \sup_{x \in E_n} e^{-s}(x) \). Then

\[
(4.12) \quad Q^{s,*}_x(E_0^c \cap \ldots \cap E_{(l-2)n_0}^c \cap E_{(l-1)n_0}^c) \]

\[
= \int \Omega 1_{E_0^c \cap \ldots \cap E_{(l-2)n_0}^c \cap E_{(l-1)n_0}^c} (\omega^s) q_{l_0}^*(x, S_{l_0}(\omega^s)) \mu^{s,l_0} (d\omega) \\
\geq \frac{r_s \tau^s}{d^{s/2} K_0(s)} \int \Omega 1_{E_0^c \cap \ldots \cap E_{(l-2)n_0}^c} (\omega^*) q_{l_0}^*(x, S_{l_0}(\omega^*)) \mu^{s,l_0} (d\omega) \\
= \frac{r_s \tau^s}{d^{s/2} K_0(s)} E_{(l-1)n_0}^c (E_0^c \cap \ldots \cap E_{(l-2)n_0}^c) = \frac{pr_s \tau^s}{d^{s/2} K_0(s)} Q^{s,*}_x (E_0^c \cap \ldots \cap E_{(l-2)n_0}^c),
\]

since by (4.1) we have the following lower bound

\[
1_{E_{(l-1)n_0}^c} (\omega^s) q_{l_0}^*(x, S_{l_0}(\omega^s)) \\
= 1_{E_{(l-1)n_0}^c} (\omega^s) q_{l_0}^*(x, S_{l_0}(\omega^s)) q_{l_0}^*(S_{l_0}(\omega^s) \cdot x, S_{l_0}(\theta^{(l-1)n_0}(\omega^s))) \\
\geq \frac{r_s}{K_0(s)} 1_{E_{(l-1)n_0}^c} (\omega^*) q_{l_0}^*(x, S_{l_0}(\omega^*)) |S_{l_0}(\theta^{(l-1)n_0}(\omega^*))| |S_{l_0}(\omega^s) \cdot x| \\
\geq \frac{r_s}{d^{s/2} K_0(s)} 1_{E_{(l-1)n_0}^c} (\omega^*) \left( \sum_{i=1}^d S_{l_0}(\theta^{(l-1)n_0}(\omega^s))(S_{l_0}(\omega^s) \cdot x)_i \right)^s \\
\geq \frac{r_s \tau^s}{d^{s/2} K_0(s)} 1_{E_{(l-1)n_0}^c} (\omega^*) q_{l_0}^*(x, S_{l_0}(\omega^s)).
\]

Let \( 0 < \gamma_s = \min \{ 1, \frac{pr_s \tau^s}{d^{s/2} K_0(s)} \} \). For \( \gamma = 1 - \gamma_s \in [0, 1) \), by (4.12), we obtain that

\[
Q^{s,*}_x (E_0^c \cap \ldots \cap E_{(l-2)n_0}^c \cap E_{(l-1)n_0}^c) \leq \gamma Q^{s,*}_x (E_0^c \cap \ldots \cap E_{(l-2)n_0}^c \cap G_{(l-1)n_0}^c) \\
= \gamma Q^{s,*}_x (E_0^c \cap \ldots \cap E_{(l-2)n_0}^c).
\]

This finishes the proof of (4.11) and completes the proof of the lemma. \( \square \)

**Lemma 4.13.** Assume that \( \mu \in M^1(G), s \in I_\mu \) and \( \Gamma = \text{supp} \mu \) satisfies condition \( (i-p) \). Assume additionally that \( \int_G \| a \|^p \log^+ \| a \| d\mu < \infty \). Then for any \( x \in S^+ \)

\[
(4.14) \quad \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)| = \lim_{n \to \infty} \frac{1}{n} \log \| S_n(\omega) \| = \alpha(s), \quad Q^{s,*}_x \text{ and } Q^{s,*}_x \text{ a.s.,}
\]

where

\[
(4.15) \quad \alpha(s) = \int_{S^+} \int_G \log |ax| q^*_x(a \cdot x) \mu_x(dax) \pi^*_s(dx).
\]

**Proof.** We show that \( f(x, \omega) = \log |S_1(\omega)| \) is \( |x|^{-p} \) integrable. Observe that there exists \( 0 < \delta < 1 \) such that

\[
0 < |ax| < \delta \implies |ax|^s \log |ax|^{-1} \leq 1.
\]
Then

\[ aQ^{s,*}(f) = \int_{S^+} \int_{\Omega} |\log |S_1(\omega)|\|\delta_x(dy)\Omega_x^{s,*}(\omega)| \pi^x_\alpha(dx) \]

\[ = \int_{S^+} \int_{G} |ax|\log |ax|\|\frac{C(a \cdot x)}{\kappa(s)}e(s)\mu_\alpha(dx)\pi^x_\alpha(dx) \]

\[ \leq C_s \int_{S^+} \int_{G} |ax|\log |ax|\|\mu_\alpha(dx)\pi^x_\alpha(dx) \]

\[ \leq C_s \int_{G} \|a\|\log |a|\|\mu(dx)\pi^x_\alpha(dx) \]

\[ \leq C_s \left( \|a\|\log |a|\|\mu(dx)\pi_\alpha^x\right) + C_s \left( \|a\|\log |a|\|\mu_\alpha(dx)\pi^x_\alpha\right) \]

\[ \leq C_s \left( \|a\|\log |a|\|\mu(dx)\pi_\alpha\right) + C_s \left( \|a\|\log |a|\|\mu_\alpha(dx)\pi^x_\alpha\right) \]

\[ \leq C_s \left( \|a\|\log |a|\|\mu(dx)\pi_\alpha\right) + C_s \left( \|a\|\log |a|\|\mu_\alpha(dx)\pi^x_\alpha\right) \]

\[ \leq C_s \left( \|a\|\log |a|\|\mu(dx)\pi_\alpha\right) + C_s \left( \|a\|\log |a|\|\mu_\alpha(dx)\pi^x_\alpha\right) \]

Hence in view of Remark 4.2, on the one hand, by the Birkhoff ergodic theorem (applied to \(aQ^{s,*}\)

\[ aQ^{s,*}\left( \{ (x, \omega) \in a\Omega : \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)x| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ a^k(x, \omega) = aQ^{s,*} f = a(\alpha) \} \right) = 1. \]

On the other hand by the Kingman subadditive ergodic theorem (applied to \(Q^{s,*}\) and \(\theta\)) we have

\[ Q^{s,*}\left( \{ \omega \in \Omega : \lim_{n \to \infty} \frac{1}{n} \log \|S_n(\omega)\| = \alpha_s \} \right) = 1. \]

Define \(\Omega' = \{ \omega \in \Omega : \exists C > 0 \forall n \in \mathbb{N} \|S_n(\omega)x\| \geq C\|S_n(\omega)\|\text{ and } \lim_{n \to \infty} \frac{1}{n} \log \|S_n(\omega)\| = \alpha_s \}, \)

for every \(x \in S^+\). By Lemma 4.4 and calculations stated above we know that \(Q^{s,*}(\Omega') = 1.\)

Fix arbitrary \(x \in S^+\), take any \(\omega^* \in \Omega'\) and notice that

\[ 0 < C_x(\omega^*) \leq \frac{|S_n(\omega^*)x|}{\|S_n(\omega^*)\|} \leq 1, \]

imply

\[ \frac{1}{n} \log C_x(\omega^*) + \frac{1}{n} \log \|S_n(\omega^*)\| \leq \frac{1}{n} \log \frac{|S_n(\omega^*)x|}{\|S_n(\omega^*)\|} + \frac{1}{n} \log \|S_n(\omega^*)\| \leq \frac{1}{n} \log \|S_n(\omega^*)\|. \]

Since \(\lim_{n \to \infty} \frac{1}{n} \log C_x(\omega^*) = 0 \) we have

\[ Q^{s,*}\left( \{ \omega \in \Omega : \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)x| = \alpha_s \} \right) = 1. \]

And so, in view of Lemma 4.3,

\[ Q^{s,*}_x\left( \{ \omega \in \Omega : \lim_{n \to \infty} \frac{1}{n} \log |S_n(\omega)x| = \alpha_s \} \right) = 1. \]

for all \(x \in S^+\) (by considering complements). Since \(aQ^{s,*} = \int_{S^+} \delta_x \otimes Q^{s,*}_x(\pi^x_\alpha(dx))\) we get \(\alpha(s) = \alpha_s\)

and Lemma 4.13 follows. \(\square\)
4.2. Kesten's renewal theorem. For $x \in \mathbb{S}^+$ and $\omega \in \Omega$ define $X_0(\omega) = x$, and for $n \in \mathbb{N}$

$$X_n(\omega) = g_n(\omega) \cdot X_{n-1}(\omega) = S_n(\omega) \cdot x,$$

and

$$V_n(\omega) = \log |S_n(\omega)x| = \sum_{i=1}^{n} U_i(\omega), \quad \text{where} \quad U_i(\omega) = \log |g_i(\omega)X_{i-1}(\omega)|.$$

Let $F(dt|x, y)$ be the conditional law of $U_1$, given $X_0 = x$, $X_1 = y$, i.e.

$$Q_{x}^*(X_1 \in A, U_1 \in B) = \int_A \int_B F(dt|x, y)Q^*_x(x, dy).$$

A function $g : \mathbb{S}^+ \times \mathbb{R} \to \mathbb{R}$ is called direct Riemann integrable (dRi), if it is $\text{Bor}(\mathbb{S}^+) \times \text{Bor}(\mathbb{R})$ measurable and for every fixed $x \in \mathbb{S}^+$ and $0 < L < \infty$ the function $t \mapsto g(x, t)$ is Riemann integrable on $[-L, L]$, and satisfies

$$\sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \sup \{|g(x, t)| : x \in C_{k+1} \setminus C_k, \text{ and } t \in [l, l+1]\} < \infty,$$

where

$$C_k = \left\{ x \in \mathbb{S}^+ : Q_{x}^*(\left\{ \left\{ \frac{V_m}{m} \geq \frac{1}{k}, \text{ for all } m \geq k \right\} \right\) \geq \frac{1}{2} \right\}, \quad \text{for all } k \in \mathbb{N}.$$

For the reader's convenience we formulate Kesten's renewal theorem [16].

**Theorem 4.18.** Assume the following conditions are satisfied:

- **Condition I.1** There exists $\pi^*_x \in M^1(\mathbb{S}^+)$ such that $\pi^*_x Q^*_x = \pi^*_x$ and for every open set $U \subseteq \mathbb{S}^+$ with $\pi^*_x(U) > 0$, $Q_{x}^*(X_n \in U \text{ for some } n \in \mathbb{N}) = 1$ for every $x \in \mathbb{S}^+$.

- **Condition I.2**

$$\int_{\mathbb{S}^+} \int_{\mathbb{S}^+} \int_{\mathbb{R}} |t| F(dt|x, y)Q^*_x(x, dy)\pi^*_x(dx) < \infty,$$

and for all $x \in \mathbb{S}^+$,

$$\lim_{n \to \infty} \frac{V_n}{n} = \alpha(s) = \int t F(dt|x, y)Q^*_x(x, dy)\pi^*_x(dx) > 0 \quad Q_{x}^* - \text{a.e.}$$

- **Condition I.3** There exists a sequence $\{\zeta_i\} \subset \mathbb{R}$ such that the group generated by $\zeta_i$ is dense in $\mathbb{R}$ and such that for each $\zeta_i$ and $\lambda > 0$ there exists $y = y(\zeta_i, \lambda) \in \mathbb{S}^+$ with the following property: for each $\varepsilon > 0$, there exists an $A \in \text{Bor}(\mathbb{S}^+)$ with $\pi^*_x(A) > 0$ and $m_1, m_2 \in \mathbb{N}$, $\tau \in \mathbb{R}$ such that for any $x \in A$

$$Q_{x}^*\{|X_{m_1} - y| < \varepsilon, |V_{m_1} - \tau| \leq \lambda\} > 0,$$

$$Q_{x}^*\{|X_{m_2} - y| < \varepsilon, |V_{m_2} - \tau - \zeta_i| \leq \lambda\} > 0.$$

- **Condition I.4** For each fixed $x \in \mathbb{S}^+$, $\varepsilon > 0$ there exists $r_0 = r_0(x, \varepsilon) > 0$ such that for all real valued functions $f$ measurable with respect to $\text{Bor}((\mathbb{S}^+ \times \mathbb{R})^\mathbb{N})$ and for all $y \in \mathbb{S}^+$ with $|x - y| < r_0$ one has:

$$\mathbb{E}_{x}^* f(x_0, V_0, X_1, V_1, \ldots) \leq \mathbb{E}_{y}^* f^*(x_0, V_0, X_1, V_1, \ldots) + \varepsilon |f|_{\infty},$$

$$\mathbb{E}_{y}^* f(x_0, V_0, X_1, V_1, \ldots) \leq \mathbb{E}_{x}^* f^*(x_0, V_0, X_1, V_1, \ldots) + \varepsilon |f|_{\infty},$$

where $f^*(x_0, v_0, x_1, v_1, \ldots) = \sup \{f(y_0, u_0, y_1, u_1, \ldots) : \forall i \in \mathbb{N} |x_i - y_i| + |v_i - u_i| < \varepsilon\}.$
If a function $g : R^+ \times R \mapsto R$ is jointly continuous and $(dRt)$, then for every $x \in S^+$

$$\lim_{t \to \infty} E^{x,t}_{\nu} \left( \sum_{n=0}^{\infty} g(X_n, t - V_n) \right) = \frac{1}{\alpha(s)} \int_{S^+} \left( \int_R g(y, x) dx \right) \nu(dy),$$

for $\alpha(s)$ defined in (4.19).

In the next four subsections we indicate how the material developed in Section 2 and 4.1, under the hypotheses of Theorem 1.8, may be used to check the assumptions of Theorem 4.18. From now we will work with the measures $Q^x_{\mu^s}$ for $x \in S^+$, where $\chi > 0$ solves equation $\kappa(\chi) = \frac{1}{\chi}$. Such $\chi > 0$ exists since $\kappa(\mu) = \frac{1}{\mu}$, (see Theorem 1.8 and Theorem 2.3). We are going to prove that Conditions I.1–I.4 are satisfied for $s = \chi$.

4.3. Condition I.1.

Proof of Condition I.1. Theorem 2.3 with Breiman’s strong law of large numbers [2] allow us to repeat the argument contained in Section 5 in [3].

4.4. Condition I.2.

Proof of Condition I.2. We know that $\int_G ||a||^\chi \log^+ ||a|| \mu(da) < \infty$, hence

$$\int_{S^+} \int_{S^+} \int_R |t| F(dt, x, y) Q^x_{\mu^s} (x, dy) = \int_{S^+} \int_{\Omega} |t| \log ||a|| Q^x_{\mu^s} (a, x) \mu^s(da) \nu(dx) < \infty,$$

by the arguments of Lemma 4.13 applied to $s = \chi$. The only point remaining concerns the positivity of $\alpha(\chi)$ defined in Lemma 4.13 (see also (4.19)).

Notice that if $\varepsilon > 0$ is sufficiently small, then for every $t \in (\chi - \varepsilon, \chi)$, we have $\kappa(t) < \kappa(\chi)$, since $\kappa(s)$ is strictly log-convex and $\lim_{s \to \infty} \kappa(s) > \frac{1}{\mu}$, (see Theorem 1.8 and Theorem 2.3). Fix $t \in (\chi - \varepsilon, \chi)$ such that $\chi/t \leq 4/3$ and take $\gamma > 0$ such that $\kappa(t)e^{\gamma} < \kappa(\chi)$. In view of inequality (3.11), there is $C > 0$ such that

$$\int_G ||a||^\delta \mu^s(a) \leq C \kappa^n(t)e^{\gamma n/3}, \quad \text{for every } n \in \mathbb{N},$$

since $1 \leq e^{\gamma/3}$. Fix $x \in S^+$. Then for $\delta = \gamma/3$ we have

$$\mu^s \left( \left\{ a \in G : ||a||^\delta > e^{-\delta n} \right\} \right) \leq \delta n \int_G ||a||^\delta \mu^s(a), \leq C \kappa^n(t)e^{2 \gamma n/3},$$

Now let $\rho = \gamma/6$. Then

$$Q^x_{\mu}(\omega \in G : ||S_n(\omega)|^t < e^{\rho n}) \leq \max \left\{ 1, 11 \right\} \int_G \frac{1}{\mu^s(a)} \left| a \right| \mu^s(a) \leq \frac{C}{\kappa^n(\chi)} \int_G \frac{1}{\mu^s(a)} \left| a \right| \mu^s(a) \leq \frac{C \kappa^n(t)}{\kappa^n(\chi)} \int_G \frac{1}{\mu^s(a)} \left| a \right| \mu^s(a) \leq Ce^{-\gamma n} e^{\gamma n/3} e^{\rho n} t \leq Ce^{-\gamma n} e^{\gamma n/3} e^{\delta n} = Ce^{-\delta n},$$
for some $\beta > 0$. Thus
\[
\sum_{n \in \mathbb{N}} \mathbb{Q}_x^\omega \left( \left\{ \omega \in \Omega : \frac{\log |S_n(\omega)x|}{n} < \frac{\rho t}{l} \right\} \right) < \infty.
\]
Therefore, by the Borel–Cantelli lemma we obtain that for every $x \in \mathbb{S}^+$
\[
\mathbb{Q}_x^\omega \left( \left\{ \omega \in \Omega : \liminf_{n \to \infty} \frac{\log |S_n(\omega)x|}{n} \geq \frac{\rho}{l} > 0 \right\} \right) = 1.
\]
This shows that $\alpha(\chi) > 0$ $\mathbb{Q}_x^\omega$ a.s. for every $x \in \mathbb{S}^+$ and finishes the proof of Condition I.2. □

4.5. Condition I.3.

Proof of Condition I.3. Proposition 2.1 and Theorem 2.3 allow us to use arguments from Section 5 in [3]. □

4.6. Condition I.4.

Proof of Condition I.4. The proof is a consequence of Lemma 4.4 and the argument given by Kesten [15]. □

4.7. Direct Riemann integrability. Now we derive an interesting criterium which significantly simplifies condition (4.16).

Lemma 4.22. Assume that the hypotheses of Theorem 1.8 are satisfied. If $h$ is any bounded and continuous function on $\mathbb{S}^+ \times \mathbb{R}$ which satisfies
\[
\sum_{l=-\infty}^{\infty} \sup_{i \in [l, l+1]} |h(x, t)| : x \in \mathbb{S}^+, \text{ and } t \in [l, l+1] < \infty,
\]
then $h$ is direct Riemann integrable i.e. it satisfies condition (4.16).

Proof. We give only a sketch of the proof, for more details we refer to [3]. First of all we prove that $C_k = \mathbb{S}^+$, for some sufficiently large $k \in \mathbb{N}$, ($C_k$ was defined in (4.17)). Then obviously (4.23) implies (4.16). There is a finite number $N_1$ of points such that $\mathbb{S}^+ \subseteq \bigcup_{i=1}^{N_1} B(x_i, 2)$, since $\mathbb{S}^+$ is compact. Let
\[
\Omega' = \left\{ \lim_{n \to \infty} \frac{\log |S_n x_i|}{n} = \alpha(\chi) > 0, \text{ and } \exists C > 0 \forall n \in \mathbb{N} \ |S_n x_i| \geq C\|S_n\|, \text{ for all } 1 \leq i \leq N_1 \right\}.
\]
Then $\mathbb{Q}_x^\omega(\Omega') = 1$, by Lemma 4.4 and 4.13. Take any $y \in \mathbb{S}^+$, then there exists $1 \leq i \leq N_1$ such that $y \in B(x_i, 2)$. This implies the existence of $m_0 \in \mathbb{N}$ such that
\[
\mathbb{Q}_x^\omega \left( \left\{ \omega \in \Omega : \frac{\log |S_n(\omega)y|}{n} > \frac{\alpha(\chi)}{2}, \text{ for all } n \geq m_0 \right\} \right) \geq 1 - \frac{1}{2e},
\]
with the constant $c > 0$ defined in Lemma 4.3. Taking any $1/k \leq \min\{\alpha(s)/2, 1/m_0\}$ Lemma 4.22 follows. □
5. Proof of the main Theorem

In this section we give a detailed proof of Theorem 1.8. For that we consider the following smooth version of $\mathbb{P}((R, u) > t)$

\begin{equation}
G(u, t) = \frac{1}{e^{t}e^{\chi}(u)} \int_{0}^{e^{t}} r \mathbb{P}((R, u) > r) dr, \quad \text{where} \ (u, t) \in \mathbb{S}^{+} \times \mathbb{R}, \tag{5.1}
\end{equation}

where $R \in \mathbb{R}^{d}_{+}$ solves equation (1.1). Let $\mathcal{B}(\mathbb{S}^{+} \times \mathbb{R})$ be the space of all bounded measurable functions on $\mathbb{S}^{+} \times \mathbb{R}$. Define a linear operator $\Theta : \mathcal{B}(\mathbb{S}^{+} \times \mathbb{R}) \mapsto \mathcal{B}(\mathbb{S}^{+} \times \mathbb{R})$ given by the formula

\begin{equation}
\Theta f(u, t) = \mathbb{E}^{\chi}_{0} (f(X_{1}, t - V_{1}))
= \frac{1}{\kappa(\chi)} \int_{\Omega} f(S_{1}(\omega^{*}) \cdot u, t - \log |S_{1}(\omega^{*})u|) \frac{e^{\chi}(S_{1}(\omega^{*})u)}{e^{\chi}(u)} |S_{1}(\omega^{*})u| \mathbb{P}(d\omega).
\end{equation}

Observe that for every $n \in \mathbb{N}$

\begin{equation}
\Theta^{n} f(u, t) = \mathbb{E}^{\chi}_{0} (f(X_{n}, t - V_{n})).
\end{equation}

First we express $G(u, t)$ as a potential of a function $g(u, t)$ that turns out later on to be direct Riemann integrable.

**Lemma 5.2.** Assume that the hypotheses of Theorem 1.8 are satisfied. Let $G(u, t)$ be the function defined in (5.1), then

\begin{equation}
G_{0}(u, t) = \frac{N}{e^{t}e^{\chi}(u)} \int_{0}^{e^{t}} r \mathbb{P}((AR, u) > r) dr = \Theta G(u, t), \quad \text{and} \tag{5.3}
\end{equation}

\begin{equation}
\lim_{n \to \infty} \Theta^{n} G(u, t) = \lim_{n \to \infty} \mathbb{E}^{\chi}_{0} (G(X_{n}, t - V_{n})) = 0. \tag{5.4}
\end{equation}

Moreover,

\begin{equation}
G(u, t) = \sum_{n=0}^{\infty} \Theta^{n} g(u, t), \quad \text{where} \tag{5.5}
\end{equation}

\begin{equation}
g(u, t) = \frac{1}{e^{t}e^{\chi}(u)} \int_{0}^{e^{t}} r \mathbb{P}(\mathbb{P}((R, u) > \cdot)|A^{*}u > \cdot) - N \mathbb{P}(\mathbb{P}(\cdot) > \cdot) dr. \tag{5.6}
\end{equation}

**Proof.** First of all we show $G_{0}(u, t) = \Theta G(u, t)$. Indeed,

\begin{align*}
G_{0}(u, t) &= \frac{N}{e^{t}e^{\chi}(u)} \int_{0}^{e^{t}} r \mathbb{P}((R, A^{*} \cdot u) | A^{*}u > \cdot) dr \\
&= \mathbb{E} \left( \frac{N}{e^{t}e^{\chi}(u)} \int_{0}^{e^{t}} r \mathcal{1}_{(r, \infty)}((R, A^{*} \cdot u)) dr \right) \\
&= \mathbb{E} \left( \frac{N}{e^{t}e^{\chi}(u)} \int_{0}^{e^{t}} r\mathcal{1}_{(r, \infty)}((R, A^{*} \cdot u)) | A^{*}u > \cdot dr \right) \\
&= \mathbb{E} \left( \frac{1}{e^{\chi}(A^{*} \cdot u)} \int_{0}^{e^{t}} r \mathcal{1}_{(r, \infty)}((R, A^{*} \cdot u)) dr \frac{1}{\kappa(\chi)} \frac{e^{\chi}(A^{*} \cdot u)}{e^{\chi}(u)} | A^{*}u | \mathcal{1}_{(r, \infty)}((R, A^{*} \cdot u)) dr \right) \\
&= \Theta G(u, t).
\end{align*}
Now we have

\[ \Theta^n G(u, t) = E^*_u \left( G(X_n, t - V_n) \right) = E^* \left( G(S_n \cdot u, t - \log |S_n u|) \frac{1}{\kappa^n(\chi)} e^{\chi(S_n \cdot u)} |S_n u|^\chi \right) \]

\[ = N^n E^* \left( \frac{|S_n u|^\chi}{e^{\chi(S_n \cdot u)}} \int_0^{\frac{\epsilon}{1 - e^{\lambda}}(\epsilon - (\log |S_n u|))} r^\chi 1_{(r, \infty)}((\langle R, S_n \cdot u \rangle) e^{\chi(S_n \cdot u)} |S_n u|^\chi dr \right) \]

\[ = N^n E^* \left( \frac{|S_n u|^\chi+1}{e^{\chi(S_n \cdot u)}} \int_0^{\frac{\epsilon}{1 - e^{\lambda}}} r^\chi 1_{(r, \infty)}((\langle R, S_n \cdot u \rangle) dr \right) \]

\[ = N^n E^* \left( \frac{|S_n u|^\chi+1}{e^{\chi(S_n \cdot u)}} \int_0^{\frac{\epsilon}{1 - e^{\lambda}}} r^\chi 1_{\langle n \cdot r, \infty \rangle}((\langle S_n^* R, u \rangle) dr \right) \]

\[ = \frac{N^n}{e^{\chi(S_n \cdot u)}} \int_0^{\epsilon} r^\chi E^* (1_{(r, \infty)}((\langle A_1, \ldots, A_n \rangle R, u)) dr \right) \]

where \( S_n = A_n \cdots A_1 \). By the continuity of \( I_\nu \equiv \eta(s) \) (see Theorem 2.3) we can find \( p < \chi \), such that \( \kappa(p) = \frac{1}{N^n} \), for some \( \epsilon > 0 \), then

\[ E (1_{(r, \infty)}((\langle A_1, \ldots, A_n \rangle R, u))) \leq \frac{E(||A_1, \ldots, A_n||^p) E(|R|^p)}{r^p} \leq \frac{C \kappa^n(p) E(|R|^p)}{r^p}. \]

This implies that

\[ \Theta^n G(u, t) = \frac{N^n}{e^{\chi(S_n \cdot u)}} \int_0^{\epsilon} r^\chi E^* (1_{(r, \infty)}((\langle A_1, \ldots, A_n \rangle R, u)) dr \right) \]

\[ \leq \frac{C N^n}{e^{\chi(S_n \cdot u)}} \int_0^{\epsilon} r^{\chi-\gamma} \kappa^n(p) E(|R|^p) dr \]

\[ \leq \frac{C N^n}{e^{\chi(S_n \cdot u)}} E(||R|^p) \left( \frac{1 - \epsilon}{N^n} \right) \int_0^{\epsilon} r^{\chi-\gamma} dr \]

\[ \leq \frac{CE(||R|^p)}{e^{\chi(S_n \cdot u)}} \left( \frac{1 - \epsilon}{N^n} \right)^n \rightarrow 0. \]

Now it is easy to see that for any \( n \in \mathbb{N} \) we have

\[ G(u, t) = g(u, t) + \Theta g(u, t) + \Theta^2 g(u, t) + \ldots + \Theta^{n-1} g(u, t) + \Theta^n G(u, t), \]

and (5.5) follows. This completes the proof of Lemma 5.4. \( \square \)

Lemmas 5.8 and 5.16 below imply that \( g(u, t) \) is direct Riemann integrable. Lemmas 5.7, 5.12 and 5.14 contain some necessary technicalities.

**Lemma 5.7.** Assume that the hypotheses of Theorem 1.8 are satisfied. Then \( \mathbb{P}(\{\langle R, u \rangle = r \}) = 0 \), for every \( (u, r) \in S^+ \times \mathbb{R}^+ \cup \{0\} \). Moreover, for every \( r \geq 0 \) the functions

\[ S^{d-1} \ni u \mapsto \mathbb{P}(\{\langle R, u \rangle > r \}), \quad \text{and} \quad S^{d-1} \ni u \mapsto \mathbb{P}(\{|AR, u \rangle > r \}), \]

are continuous.

**Proof.** At the beginning, we assume that the law \( \eta \) of \( B \) is nonsingular, i.e. \( \|\eta_t\| < 1 \). Let \( \nu \) be the law of \( R \) and \( \mu \) be the law of \( A \in G \). Let \( * \) be the classical convolution on \( \mathbb{R}^d \). Moreover, we define \( \xi = \mu * G \nu \), where \( \mu * G \nu(D) = \int_G \int_{\mathbb{R}^d} 1_D(ax)\nu(dx)\mu(da) \) and \( D \in \text{Bor}(\mathbb{R}^d) \). Obviously \( \xi \) defines a
probability measure on $\mathbb{R}^d$ which coincide with the distribution of $AR$. Notice that $\nu = \xi^* N \ast \eta$, since $R = \sum_{i=1}^{N} A_i R_i + B$, and observe that by the Lebesgue decomposition we obtain

$$\nu_a + \nu_s = \nu = (\xi_a + \xi_s)^* N \ast (\eta_a + \eta_s) = \sum_{n=0}^{N} \binom{N}{n} \xi_a^N \ast \xi_s^{(N-n)} \ast \eta_a + \sum_{n=1}^{N} \binom{N}{n} \xi_a^N \ast \xi_s^{(N-n)} \ast \eta_s$$

and by its uniqueness $\nu_s = (\xi_s^* N \ast \eta_s)_s$. This gives $\|\nu_s\| \leq \|\xi_s\| N \|\eta_s\|$. Again by the Lebesgue decomposition and its uniqueness we have $\xi = \mu \ast G \nu = \mu \ast G \nu_a + \mu \ast G \nu_s$, hence $\|\xi\| = \|(\mu \ast G \nu)\| \leq \|\mu \ast G \nu_s\| \leq \|\nu_s\|$. Now combining $\|\nu_s\| \leq \|\xi_s\| N \|\eta_s\|$ and $\|\xi_s\| \leq \|\nu_s\|$ we get $\|\nu_s\| \leq \|\nu_s\| N \|\eta_s\|$, if $\|\nu_s\| > 0$, then $1 \leq \|\nu_s\| N^{-1} \|\eta_s\| \leq \|\eta_s\| < 1$. This contradiction shows that $\|\nu_s\| = 0$ hence $\nu$ is absolutely continuous with respect to the Lebesgue measure, which in turn implies that $P(\{(R, u) = r\}) = 0$, for every $(u, r) \in \mathbb{S}^+ \times \mathbb{R}^+ \cup \{0\}$.

If the law $\eta$ of $B$ is singular, i.e. $\|\eta_s\| = 1$, then for fixed $(u, r) \in \mathbb{S}^+ \times \mathbb{R}^+ \cup \{0\}$, we have $P(\{(R, u) = r\}) = 0$, since $P(\{(B, u) = r\}) = 0$.

Now we prove that $\mathbb{S}^{d-1} \ni u \mapsto P(\{(R, u) > r\})$ is continuous. Take any $(u_n)_{n \in \mathbb{N}} \subseteq \mathbb{S}^+$ such that $\lim_{n \to \infty} u_n = u \in \mathbb{S}^+$ and consider $|P(\{(R, u_n) > r\}) - P(\{(R, u) > r\})| \leq P(\{|\langle R, u_n \rangle > r, \text{ and } \langle R, u \rangle \leq r\})$

$$+ P(\{|\langle R, u \rangle \leq r, \text{ and } \langle R, u_n \rangle > r\}),$$

then

$$P(\{|R, u\} \leq r \leq \langle R, u_n \rangle \rangle) = P(\{0 \leq r - \langle R, u \rangle \leq \langle R, u_n \rangle - \langle R, u \rangle\}) \leq P(\{0 \leq r - \langle R, u \rangle \leq |R| u_n - u\}), \text{ and}$$

$$P(\{|R, u_n| \leq r \leq \langle R, u \rangle \rangle) = P(\{\langle R, u_n \rangle - \langle R, u \rangle \leq r - \langle R, u \rangle < 0\}) \leq P(\{-|R| u_n - u \leq r - \langle R, u \rangle < 0\}).$$

If $|u_n - u| < 1/m$, then $|P(\{(R, u_n) > r\}) - P(\{(R, u) > r\})| \leq P(\{|\langle R, u \rangle - r| \leq |R| u_n - u\}) \leq P(\{|\langle R, u \rangle - r| \leq |R|/m\}).$

We also know that $\lim_{m \to \infty} P(\{|\langle R, u \rangle - r| \leq |R|/m\}) = P(\{|\langle R, u \rangle = r\}) = 0$, hence

$$\lim_{n \to \infty} |P(\{|\langle R, u_n \rangle > r\}) - P(\{|\langle R, u \rangle > r\})| = 0.$$

The same arguments work for $u \mapsto P(\{|AR, u\} > r\}$, since $A \in G$ is independent of $R$. \qed

Lemma 5.8. Under the assumptions of Theorem 1.8, there exists $0 < \beta_1 < 1$ such that for every $\beta \in [0, \beta_1)$, there is a finite constant $C_\beta > 0$, such that for every $(u, t) \in \mathbb{S}^+ \times \mathbb{R}$ we have

$$g_1(t, u) = \frac{1}{e^{t u} / (u)} \int_{0}^{t} e^{v} \left( \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r \right) dr \leq C_\beta e^{-\beta |t|},$$

and

$$\int_{0}^{\infty} \left( N P(\{|AR, u\} > r\}) - P\left(\left\{ \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r \right\}\right) r^{x + \beta - 1} dr$$

$$= \frac{1}{\chi + \beta} \left( \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{x+\beta} - \left( \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle \right)^{x+\beta} \right).$$
Moreover, \( S^+ \times \mathbb{R} \ni (u, t) \mapsto g_1(u, t) \) is continuous.

In the proof we extend the approach developed in [12].

Proof. Let \( \beta_1 \in (0, \min\{1, \chi/2\}) \) and take any \( 0 < \beta < \beta_1 \). Then for every \( t > 0 \)

\[
I_1 = e^{-\beta t} e^{-(1-\beta) t} \int_0^t r^\chi \left| \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r \right\} \right) - \mathbb{P}(\{\langle AR, u \rangle > r\}) \right| dr
\leq e^{-\beta t} \int_0^t r^{\chi+\beta-1} \left| \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r \right\} \right) - \mathbb{P}(\{\langle AR, u \rangle > r\}) \right| dr.
\]

Now observe that \( \mathbb{P}(\{\langle AR, u \rangle > r\}) \geq \mathbb{P}(\{\max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r\}) \), then

\[
\int_0^1 \left( \mathbb{P}(\{\langle AR, u \rangle > r\}) - \mathbb{P}(\{\max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r\}) \right) r^{\chi+\beta-1}dr \leq N \int_0^1 r^{\chi+\beta-1}dr < \infty.
\]

Let us define \( \mathcal{F}(y) = \mathbb{P}(\{\langle AR, u \rangle > y\}) \), and \( \gamma = \chi + \beta - \beta_1 \), and notice \( \mathbb{P}(\{\langle AR, u \rangle > r\}) - \mathbb{P}(\{\max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r\}) = (1 - \mathcal{F}(r))^N - 1 + N\mathcal{F}(r) \leq e^{-\gamma \mathcal{F}(r)} - 1 + N\mathcal{F}(r) \), and for some \( c > 0 \)

\[
\mathcal{F}(r) = \mathbb{P}(\{\langle AR, u \rangle > r\}) \leq r^{-\gamma}(\langle AR, u \rangle ^\gamma) \leq cr^{-\gamma}.
\]

Clearly, \( 1 < \frac{\chi+\beta}{\gamma} \), and \( \beta_1 < \chi/2 \) implies \( \gamma = \chi + \beta - \beta_1 \geq \chi/2 + \beta/2 \), hence \( \frac{\chi+\beta}{\gamma} < 2 \). Then

\[
\int_1^\infty \left( \mathbb{P}(\{\langle AR, u \rangle > r\}) - \mathbb{P}(\{\max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r\}) \right) r^{\chi+\beta-1}dr \leq \int_1^\infty (e^{-cN r^{-\gamma}} - 1 + cNr^{-\gamma}) r^{\chi+\beta-1}dr
\]

\[
= \int_1^\infty (e^{-cN r^{-\gamma}} - 1 + cNr^{-\gamma}) \left( \frac{cNr^{-\gamma}}{cN} \right) \frac{\gamma}{\gamma} \frac{dr}{r}
\]

\[
= \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma} \int_0^cN (e^{-r} - 1 + r)r^{-\frac{\chi+\beta}{\gamma}}dr \leq \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma} \int_0^\infty (e^{-r} - 1 + r)r^{-\frac{\chi+\beta}{\gamma}}dr
\]

\[
\leq \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma} \left( \frac{1}{2} \int_0^1 r^{-\frac{\chi+\beta}{\gamma}}dr + \int_1^\infty r^{-\frac{\chi+\beta}{\gamma}}dr \right) = \frac{(cN)^{\frac{\chi+\beta}{\gamma}}}{\gamma} \left( \frac{1}{2 \left( \frac{\chi+\beta}{\gamma} \right)} + \frac{1}{\frac{\chi+\beta}{\gamma} - 1} \right) < \infty.
\]

We have shown that \( I_1 \leq C_\beta e^{-\beta t} \), for every \( \beta \in [0, \beta_1] \) with the constant \( C_\beta > 0 \) which does not depend on \( u \in S^+ \). A straightforward applications of Fubini theorem yields
\[
\int_0^\infty \left( N\mathbb{P}(\{AR, u > r\}) - \mathbb{P}\left( \left\{ \max_{1 \leq i \leq N} (A_i R_i, u) > r \right\} \right) \right) r^{\chi+\beta-1} dr \\
= \int_0^\infty \left( \mathbb{E} \left( \sum_{i=1}^N 1_{\{AR, u > r\}} \right) - \mathbb{E} \left( 1_{\{\max_{1 \leq i \leq N} (A_i R_i, u) > r\}} \right) \right) r^{\chi+\beta-1} dr \\
= \mathbb{E} \left( \int_0^\infty \left( \sum_{i=1}^N 1_{\{AR, u > r\}} - 1_{\{\max_{1 \leq i \leq N} (A_i R_i, u) > r\}} \right) r^{\chi+\beta-1} dr \right) \\
= \mathbb{E} \left( \sum_{i=1}^N \int_0^\infty (A_i R_i, u)^{\chi+\beta} - (\max_{1 \leq i \leq N} (A_i R_i, u))^{\chi+\beta} \right) \\
= \frac{1}{\chi+\beta} \mathbb{E} \left( \sum_{i=1}^N (A_i R_i, u)^{\chi+\beta} - \left( \max_{1 \leq i \leq N} (A_i R_i, u) \right)^{\chi+\beta} \right).
\]

In order to show the continuity of \( S^+ \times \mathbb{R} \ni (u, t) \mapsto g_1(u, t) \) it is enough to prove the continuity of

\[
(5.11) \quad u \mapsto \frac{1}{e^t} \int_0^{e^t} r^{\chi+\beta} \left( N\mathbb{P}(\{AR, u > r\}) - \mathbb{P}\left( \left\{ \max_{1 \leq i \leq N} (A_i R_i, u) > r \right\} \right) \right) dr.
\]

In this purpose observe that \( \mathbb{P}(\{\max_{1 \leq i \leq N} (A_i R_i, u) > r\}) = 1 - (1 - \mathbb{P}(\{(AR, u) > r\}))^N \), hence Lemma 5.7 guarantees that

\[
u \mapsto N\mathbb{P}(\{AR, u > r\}) - \mathbb{P}\left( \left\{ \max_{1 \leq i \leq N} (A_i R_i, u) > r \right\} \right),
\]
is continuous. Observe that

\[
N\mathbb{P}(\{AR, u > r\}) - \mathbb{P}\left( \left\{ \max_{1 \leq i \leq N} (A_i R_i, u) > r \right\} \right) \leq \begin{cases} N, & \text{if } r \leq 1, \\ N e^{-\frac{r}{\mathbb{E}(Y^p) - 1} + N \mathbb{E}(Y^p)} & \text{if } r > 1,
\end{cases}
\]
then arguing in a similar way as above with \( \beta = 0 \), and using Lebesgue dominated convergence theorem we obtain the continuity of (5.11) and the lemma follows. \( \square \)

Now we are going to prove inequality (5.13) and (5.15), that will provide necessary estimates for Lemma 5.16. The first one was proved in [12] and was sufficient in the one dimensional case discussed there. The second one is more subtle and allows us to deal with our situation.

Lemma 5.12. Let \( \alpha > 1 \) and \( p = [\alpha] \geq 2 \). For any sequence of nonnegative i.i.d. random variables \( Y, Y_1, Y_2, \ldots \) such that \( \mathbb{E}(Y^{p-1}) < \infty \), and any \( k \in \mathbb{N} \) we have

\[
(5.13) \quad \mathbb{E} \left( \left( \sum_{i=1}^k Y_i \right)^\alpha - \sum_{i=1}^k Y_i^\alpha \right) \leq k^\alpha \mathbb{E} \left( Y^{p-1} \right)^{\frac{\alpha}{\alpha - 1}}.
\]

Proof. As mentioned before the proof is contained in [12]. \( \square \)

Lemma 5.14. Let \( p \in \mathbb{N} \) and \( \beta \in (0, 1) \). Then for any \( \delta \in \left( 0, \frac{p(1-\beta)}{p+\beta} \right) \), for any sequence of nonnegative i.i.d. random variables \( Y, Y_1, Y_2, \ldots \) such that \( \mathbb{E}(Y^{p-\delta}) < \infty \), and any \( k \in \mathbb{N} \) we have

\[
(5.15) \quad \mathbb{E} \left( \left( \sum_{i=1}^k Y_i \right)^{p+\beta} - \sum_{i=1}^k Y_i^{p+\beta} \right) \leq k^{p+1} \mathbb{E} \left( Y^{p-\delta} \right)^{\frac{p+\beta}{p-\delta}}.
\]
Now observe that \( \beta + \delta < \beta + \frac{p(p-\delta)}{p+1} < 1 \). By the above inequality
\[
\left( \sum_{i=1}^{k} Y_i \right)^{p+\beta} = \left( \sum_{i=1}^{k} Y_i \right)^{p-\delta} \left( \sum_{i=1}^{k} Y_i \right)^{\beta+\delta} = \left( \sum_{i=1}^{k} Y_i \right)^{p-\delta} - \sum_{i=1}^{k} Y_i^{p-\delta} \left( \sum_{i=1}^{k} Y_i \right)^{\beta+\delta} + \sum_{i=1}^{k} Y_i^{p-\delta} \left( \sum_{i=1}^{k} Y_i^{\beta+\delta} \right).
\]
It follows that
\[
\left( \sum_{i=1}^{k} Y_i \right)^{p+\beta} - \sum_{i=1}^{k} Y_i^{p+\beta} \leq \sum_{i=1}^{k} \sum_{i(j_1, \ldots, j_k) \in A_\beta(k)} \left( \sum_{j=1}^{p} \left( \sum_{i=1}^{k} Y_i^{j_1} \ldots Y_j^{j_k} \right)^{p-\delta} \right) Y_i^{\beta+\delta} + \sum_{i \neq j} Y_i^{p-\delta} Y_j^{\beta+\delta}.
\]
But \( j_i \leq p - 1 \). Hence \( \frac{\beta(p-\delta)}{p} + \beta + \delta \leq \frac{1}{p}(p-1)(p-\delta) + \beta + \delta \leq p + \beta - 1 + \frac{\delta}{p} < p - \delta \), since
\[
0 < \delta < \frac{p(1-\beta)}{p+1} \implies \delta \left( 1 + \frac{1}{p} \right) < 1 - \beta \implies \beta - 1 + \frac{\delta}{p} < -\delta \implies p + \beta - 1 + \frac{\delta}{p} < p - \delta.
\]
Now we have
\[
E \left( Y_1^{j_1(p-\delta)} \ldots Y_i^{j_i(p-\delta)} \ldots Y_k^{j_k(p-\delta)} \right) \leq \|Y\|_{p-\delta}^{j_1(p-\delta)} \ldots \|Y\|_{p-\delta}^{j_i(p-\delta)} \ldots \|Y\|_{p-\delta}^{j_k(p-\delta)} = \|Y\|_{p-\delta}^{p+\beta},
\]
because \( j_1 + \ldots + j_k = p \). Observe that
\[
\delta < \frac{p(1-\beta)}{p+1} \implies \delta < \frac{p-\beta}{2} \implies \beta + \delta < p - \delta,
\]
hence $E\left(Y_{p-\delta}^{y^{p-\delta}}Y_{p+\delta}^{\gamma}\right) = \|Y\|_{p-\delta}^{\beta}\|Y\|_{\beta+\delta}^{\beta+\delta} \leq \|Y\|_{p-\delta}^{\beta}$, and so

$$E\left(\sum_{i=1}^{k} Y_{i}^{p+\beta} - \sum_{i=1}^{k} Y_{p+\beta}^{i}\right) \leq k(k^{P} - k)E\left(Y_{p-\delta}^{\gamma}\right) + k^{2}E\left(Y_{p-\delta}^{\gamma}\right) = k^{p+1}E\left(Y_{p-\delta}^{\gamma}\right).$$

\[\square\]

**Lemma 5.16.** Under the assumptions of Theorem 1.8, there exists $0 < \beta_{2} < 1$ such that for every $\beta \in [0, \beta_{2})$, there is a finite constant $C_{\beta} > 0$, such that for every $(u, t) \in \mathbb{S}^{+} \times \mathbb{R}$ we have

$$g_{2}(u, t) = \frac{1}{e^{t}e^{\alpha}(u)} \int_{0}^{\tau} r^{x} \left|\mathbb{P}\{(R, u) > r\} - \mathbb{P}\left\{\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle > r\right\}\right| dr \leq C_{\beta} e^{-\beta|t|},$$

and

$$\int_{0}^{\infty} r^{x+\beta-1} \left(\mathbb{P}\{(R, u) > r\} - \mathbb{P}\left\{\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle > r\right\}\right) dr = \frac{1}{\chi + \beta} E\left(\langle R, u \rangle^{\chi+\beta} - \left(\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle\right)^{\chi+\beta}\right).$$

Moreover, $\mathbb{S}^{+} \times \mathbb{R} \ni (u, t) \mapsto g_{2}(u, t)$ is continuous.

**Proof.** Let $0 < \beta_{2} < \min\{\varepsilon, \beta_{1}\}$ ($\varepsilon > 0$ as in Theorem 1.8 and $\beta_{1} > 0$ as in Lemma 5.8) and take $\beta \in [0, \beta_{2})$. Then for every $t > 0$

$$I_{2} = e^{-\beta t}e^{-(1-\beta)t} \int_{0}^{\tau} r^{x} \left|\mathbb{P}\{(R, u) > r\} - \mathbb{P}\left\{\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle > r\right\}\right| dr \leq e^{-\beta t} \int_{0}^{\infty} r^{x+\beta-1} \left|\mathbb{P}\{(R, u) > r\} - \mathbb{P}\left\{\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle > r\right\}\right| dr.$$

Observe that $(R, u) \geq \max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle$. Then applying Fubini theorem as in Lemma 5.8 we obtain

$$\int_{0}^{\infty} r^{x+\beta-1} \left(\mathbb{P}\{(R, u) > r\} - \mathbb{P}\left\{\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle > r\right\}\right) dr = \frac{1}{\chi + \beta} E\left(\langle R, u \rangle^{\chi+\beta} - \left(\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle\right)^{\chi+\beta}\right).$$

If $0 < \chi < 1$, take any $\beta \in [0, \beta_{2})$ such that $0 < \chi + \beta \leq 1$ and notice

$$E\left(\langle R, u \rangle^{\chi+\beta} - \left(\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle\right)^{\chi+\beta}\right) \leq E\left(\langle B, u \rangle^{\chi+\beta}\right) + E\left(\sum_{i=1}^{N} \langle A_{i}R_{i}, u \rangle^{\chi+\beta} - \left(\max_{1 \leq i \leq N} \langle A_{i}R_{i}, u \rangle\right)^{\chi+\beta}\right) < \infty,$$
since \( \mathbb{E}(|B|^{\chi+\varepsilon}) < \infty \) for some \( \varepsilon > 0 \), and second term is finite by Lemma 5.8.

If \( \chi \geq 1 \) we write

\[
\mathbb{E} \left( (R, u)^{\chi+\beta} - \left( \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle \right)^{\chi+\beta} \right) = \mathbb{E} \left( (R, u)^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta} \right) + \mathbb{E} \left( \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta} - \left( \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle \right)^{\chi+\beta} \right).
\]

We have to estimate only the first term, since the second one is finite by Lemma 5.8. In this purpose we use Lemma 5.12 and 5.14. Notice that

\[
\mathbb{E} \left( (R, u)^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta} \right) = \mathbb{E} \left( \sum_{i=1}^{N} A_i R_i + B, u \right)^{\chi+\beta} - \mathbb{E} \left( \sum_{i=1}^{N} A_i R_i, u \right)^{\chi+\beta}
\]

\[
+ \mathbb{E} \left( \sum_{i=1}^{N} A_i R_i, u \right)^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta}
\]

\[
\leq (\chi + \beta) \mathbb{E} \left( |B| \left( \sum_{i=1}^{N} |A_i R_i| + |B| \right)^{\chi+\beta-1} \right)
\]

\[
+ \mathbb{E} \left( \sum_{i=1}^{N} A_i R_i, u \right)^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta}
\]

\[
\mathbb{E} \left( |B| \left( \sum_{i=1}^{N} |A_i R_i| + |B| \right)^{\chi+\beta-1} \right)
\]

is finite, since \( \mathbb{E}(|A|^{\chi+\beta-1}) < \infty \), \( \mathbb{E}(|B|^{\chi+\varepsilon}) < \infty \) and Theorem 1.7 yields \( \mathbb{E}(|R|^{\chi+\beta-1}) < \infty \).

If \( \chi \notin \mathbb{N} \) we assume additionally that \( \lceil \chi + \beta_2 \rceil = \lceil \chi \rceil \), (which holds for sufficiently small \( \beta_2 > 0 \)). Applying inequality (5.13) with \( p = \lceil \chi \rceil = \lceil \chi + \beta \rceil \) and \( \beta \in [0, \beta_2) \) we obtain

\[
\mathbb{E} \left( \sum_{i=1}^{N} A_i R_i, u \right)^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta} \leq N^{\chi+\beta} \left( \mathbb{E} \left( |AR, u|^{p-1} \right) \right)^{\frac{1}{p-1}} < \infty,
\]

since \( p-1 < \chi \).

If \( \chi \in \mathbb{N} \) and \( \beta \in [0, \beta_2) \) take any \( \delta \in \left( 0, \frac{p(1-\beta)}{p+1} \right) \) as in Lemma 5.14 with \( p = \chi \), then by inequality (5.15) we get

\[
\mathbb{E} \left( \sum_{i=1}^{N} A_i R_i, u \right)^{\chi+\beta} - \sum_{i=1}^{N} \langle A_i R_i, u \rangle^{\chi+\beta} \leq N^{\chi+1} \left( \mathbb{E} \left( |AR, u|^{\chi-\delta} \right) \right)^{\frac{1}{\chi-\delta}} < \infty.
\]

Finally, we have proved \( I_2 \leq C_\beta e^{-\beta|u|} \), for every \( \beta \in [0, \beta_2) \) with \( C_\beta < \infty \) independent of \( u \in \mathbb{S}^+ \).

It remains to prove that \( \mathbb{S}^+ \times \mathbb{R} \ni (u, t) \mapsto g_2(u, t) \) is continuous. In this purpose it suffices to show continuity of

\[
u \mapsto \frac{1}{\varepsilon^{\delta}} \int_{0}^{e^{\delta}} r^{\chi} \left( \mathbb{P}(\{R > r\}) - \mathbb{P} \left( \max_{1 \leq i \leq N} \langle A_i R_i, u \rangle > r \right) \right) dr.
\]
Observe that
\[
\frac{1}{e^t} \int_0^e r^x \left[ \mathbb{P}(\{ (R, u_n) > r \}) - \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R, u_n \rangle > r \right\} \right) \right] \, dr \\
\leq \int_0^\infty r^{x-1} \left| \mathbb{P}(\{ (R, u_n) > r \}) - \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R, u_n \rangle > r \right\} \right) \right| \, dr.
\]

It is enough to show that the last integral converges to 0 as \( \lim_{n \to \infty} u_n = u_0 \). In this purpose we will use an extended version of Lebesgue dominated convergence theorem (see for instance in [1]). Namely,

**Theorem 5.20.** Given a measure space \((X, \mathcal{M}, \mu)\) (where \( \mu \) may takes values in \([0, \infty]\)). Let \((f_n)_{n \in \mathbb{N}}\) and \((h_n)_{n \in \mathbb{N}}\), \(f\) and \(h\) be \( \mathcal{M} \) measurable, real valued functions on \(X\). Suppose

- \(\lim_{n \to \infty} f_n = f\) and \(\lim_{n \to \infty} h_n = h\) a.e. on \(X\),
- \((h_n)_{n \in \mathbb{N}}\) and \(h\) are all \( \mu \) integrable on \(X\) and \(\lim_{n \to \infty} \int_X h_n \, d\mu = \int_X h \, d\mu\),
- \(|f_n| \leq h_n\) a.e. on \(X\) for every \(n \in \mathbb{N}\).

Then \(f\) is \( \mu \) integrable on \(X\) and \(\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu\).

We will apply Theorem 5.20 with
\[
f_n(r) = r^{x-1} \left[ \mathbb{P}(\{ (R, u_n) > r \}) - \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R, u_n \rangle > r \right\} \right) \right],
\]
\[
h_n(r) = r^{x-1} \left[ \mathbb{P}(\{ (R, u_n) > r \}) - \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R, u_n \rangle > r \right\} \right) \right] + \left( \mathbb{P}(\{ (R, u_0) > r \}) - \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R, u_0 \rangle > r \right\} \right) \right),
\]
and
\[
h(r) = 2r^{x-1} \left( \mathbb{P}(\{ (R, u_0) > r \}) - \mathbb{P} \left( \left\{ \max_{1 \leq i \leq N} \langle A_i R, u_0 \rangle > r \right\} \right) \right).
\]

Clearly, \(|f_n| \leq h_n\) for every \(n \in \mathbb{N}\), and by the previous part of the lemma \((h_n)_{n \in \mathbb{N}}\) and \(h\) are all integrable. Lemma 5.7 guarantees that \(\lim_{n \to \infty} f_n(r) = 0\) and \(\lim_{n \to \infty} h_n(r) = h(r)\). In order to show that \(\lim_{n \to \infty} \int_0^\infty h_n(r) \, dr = \int_0^\infty h(r) \, dr\), notice that by (5.18) with \(\beta = 0\) we have to show that
\[
(5.21) \quad \lim_{n \to \infty} \mathbb{E} \left( (R, u_n)^X - \left( \max_{1 \leq i \leq N} \langle A_i R, u_n \rangle \right)^X \right) = \mathbb{E} \left( (R, u_0)^X - \left( \max_{1 \leq i \leq N} \langle A_i R, u_0 \rangle \right)^X \right).
\]
But in view of the first part of the lemma and the estimates given there (5.21) is a simple consequence of a classical Lebesgue dominated convergence theorem. This finishes the proof of Lemma 5.16.

Proof of Theorem 1.8. From Lemma 5.2 we know that

\[ G(u, t) = \sum_{n=0}^{\infty} \Theta^n g(u, t), \]

where

\[ g(u, t) = \frac{1}{e^t e^u} \int_0^{e^t} r^\chi \left( \mathbb{P}(\{(R, u) > r\}) - N\mathbb{P}(\{(AR, u) > r\}) \right) dr. \]

As a consequence of Lemma 5.8 and Lemma 5.16 the function \( S^+ \times \mathbb{R} \ni (u, t) \mapsto g(u, t) \) is jointly continuous. Moreover, it is possible to find \( \beta > 0 \) and a positive constant \( C_\beta < \infty \) such that

\[ |g(u, t)| \leq C_\beta e^{-\beta|t|}, \quad \text{for every} \ (u, t) \in S^+ \times \mathbb{R}, \]

since \( |g(u, t)| \leq g_1(u, t) + g_2(u, t) \), for \( g_1(u, t) \) and \( g_2(u, t) \) defined in Lemma 5.8 and Lemma 5.16 respectively. This shows that \( g(u, t) \) satisfies condition (4.23). By the Kesten’s renewal theorem 4.18 we obtain

\[ \lim_{t \to \infty} G(u, t) = \lim_{t \to \infty} \mathbb{E}_{\chi}^t \left( \sum_{n=0}^{\infty} g(X_n, t - V_n) \right) = \frac{1}{\alpha(\chi)} \int_{S^+} \left( \int \mathbb{E}_{\chi}^t g(y, x) dx \right) \pi^\chi(dy) = C_\chi. \]

In other words we have proved that for every \( u \in S^+ \)

\[ \lim_{t \to \infty} G(u, t) = \lim_{t \to \infty} \frac{1}{e^t e^u} \int_0^{e^t} r^\chi \mathbb{P}(\{(R, u) > r\}) dr = C_\chi \geq 0. \]

Hence in view of Lemma 9.3 of [4], for every \( u \in S^+ \)

\[ \lim_{t \to \infty} t^\chi \mathbb{P}(\{(R, u) > t\}) = C_\chi e^\chi(u). \]

It remains to prove that \( C_\chi > 0 \) for every \( \chi \geq 1 \). In this purpose notice that

\[ C_\chi = \frac{1}{\alpha(\chi)} \int_{S^+} \left( \int g(u, t) dt \right) \pi^\chi(du) \]

\[ = \frac{1}{\alpha(\chi)} \int_{S^+} \int_{\mathbb{R}} \left( \frac{1}{e^t e^u} \int_0^{e^t} r^\chi \left( \mathbb{P}(\{(R, u) > r\}) - N\mathbb{P}(\{(AR, u) > r\}) \right) dr \right) dt \pi^\chi(du) \]

\[ = \frac{1}{\alpha(\chi)} \int_{S^+} \int_{\mathbb{R}} \left( \frac{1}{e^t e^u} \int_{-\infty}^{e^t} e^{s(\chi+1)} \left( \mathbb{P}(\{(R, u) > e^s\}) - N\mathbb{P}(\{(AR, u) > e^s\}) \right) ds \right) dt \pi^\chi(du) \]

\[ = \frac{1}{\alpha(\chi)} \int_{S^+} \int_{\mathbb{R}} \int_0^{\infty} \left( e^{s(\chi+1)} \mathbb{P}(\{(R, u) > e^s\}) - N\mathbb{P}(\{(AR, u) > e^s\}) \right) ds \pi^\chi(du) = \]
\[
\frac{1}{\alpha(x)} \int_{\mathbb{S}^+} \int_{\mathbb{R}^d} e^x \left( \Pr(\{\langle R, u \rangle > e^x\}) - N \Pr(\{\langle AR, u \rangle > e^x\}) \right) ds \pi^x_s(du)
\]
\[
= \frac{1}{\alpha(x)} \int_{\mathbb{S}^+} \int_{\mathbb{R}^d} e^x \left( r^{x-1} \Pr(\{\langle R, u \rangle > r\}) - N \Pr(\{\langle AR, u \rangle > r\}) \right) dr \pi^x_s(du)
\]
\[
= \frac{1}{\alpha(x)} \int_{\mathbb{S}^+} \int_{\mathbb{R}^d} e^x \left( \sum_{i=1}^N 1_{\{\langle A_i R_i + B, u \rangle > r\}} - \sum_{i=1}^N 1_{\{\langle A_i R_i, u \rangle > r\}} \right) \pi^x_s(du)
\]
\[
\geq \frac{1}{\alpha(x)} \int_{\mathbb{S}^+} \frac{1}{e^x} \mathbb{E} \left( \langle B, u \rangle^x \pi^x_s(du) \right)
\]

since we have used
\[
\left( \sum_{i=1}^N \langle A_i R_i, u \rangle^x + \langle B, u \rangle^x \right)^{1/x} \leq \sum_{i=1}^N \langle A_i R_i, u \rangle + \langle B, u \rangle.
\]

We need only to show that
\[
\int_{\mathbb{S}^+} \frac{1}{e^x} \mathbb{E} \left( \langle B, u \rangle^x \pi^x_s(du) \right) > 0.
\]

We will show that there exists \( c_x > 0 \) such that
\[
\int_{\mathbb{S}^+} (x, u)^x \pi^x_s(du) \geq c_x \|x\|^x,
\]
for every \( x \in \mathbb{R}^d_+ \). Observe that \( \mathbb{S}^+ \ni x \mapsto \int_{\mathbb{S}^+} (x, u)^x \pi^x_s(du) \) is continuous and nonzero for every \( x \in \mathbb{S}^+ \), since \( \text{supp} \pi^x_s \) is not contained in any proper subspace of \( \mathbb{S}^+ \) (see Section (2)). This allows us to conclude that \( x \mapsto \int_{\mathbb{S}^+} (x, u)^x \pi^x_s(du) \) attains its minimum \( c_x > 0 \) on \( \mathbb{S}^+ \), and in fact this proves (5.23).

In order to prove (5.22) notice that by (5.23) we obtain
\[
\int_{\mathbb{S}^+} \frac{1}{e^x} \mathbb{E} \left( \langle B, u \rangle^x \pi^x_s(du) \right) \geq \frac{1}{\sup_{u \in \mathbb{S}^+} e^x} \int_{\mathbb{S}^+} \mathbb{E} \left( \langle B, u \rangle^x \pi^x_s(du) \right) \geq \frac{1}{\sup_{u \in \mathbb{S}^+} e^x} \mathbb{E} \left( \int_{\mathbb{S}^+} \langle B, u \rangle^x \pi^x_s(du) \right) \geq \frac{c_x}{\sup_{u \in \mathbb{S}^+} e^x} \mathbb{E} (\|B\|^x) > 0,
\]

since \( \Pr(\{|B| > 0\}) > 0 \). This completes the proof of Theorem 1.8. \( \square \)

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