KAM FOR THE NON-LINEAR BEAM EQUATION 2: A NORMAL FORM THEOREM

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Abstract. We prove an abstract KAM theorem adapted to space-multidimensional
hamiltonian PDEs with regularizing nonlinearities. It applies in particular to
the singular perturbation problem studied in the first part of this work.

8/2/ 2015

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1991 Mathematics Subject Classification.
Key words and phrases. KAM theory, Hamiltonian systems, multidimensional PDEs.
1. Introduction

1.0.1. The phase space. Let \( A \) and \( \mathcal{F} \) be two finite sets in \( \mathbb{Z}^d \) and let \( L_\infty \) be an infinite subset of \( \mathbb{Z}^d \). Let \( \mathcal{L} \) be the disjoint union \( A \sqcup \mathcal{F} \sqcup L_\infty \) and consider \((\mathbb{C}^2)^\mathcal{L}\).

For any subset \( X \) of \( \mathcal{L} \), consider the projection
\[
\pi_X : (\mathbb{C}^2)^\mathcal{L} \to (\mathbb{C}^2)^X = \{ \zeta \in (\mathbb{C}^2)^\mathcal{L} : \zeta_a = 0 \ \forall a \notin X \}.
\]

We can thus write \((\mathbb{C}^2)^\mathcal{L} = (\mathbb{C}^2)^X \times (\mathbb{C}^2)^{\mathcal{L}\setminus X} \), \( \zeta = (\zeta_X, \zeta_{\mathcal{L}\setminus X}) \), and when \( X \) is finite this gives an injection
\[
\iota_X : (\mathbb{C}^2)^\#X \hookrightarrow (\mathbb{C}^2)^\mathcal{L}
\]
whose image is \((\mathbb{C}^2)^X\).

Let \( \gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2 \) and let \( Y_\gamma \) be the space of sequences \( \zeta \in (\mathbb{C}^2)^\mathcal{L} \) such that
\[
||\zeta||_\gamma = \sqrt{\sum_{a \in \mathcal{L}} |\zeta_a|^2 e^{2\gamma_1 |a|} (\langle a \rangle^{2\gamma_2} < \infty
\]
- here \( \langle a \rangle = \max(|a|, 1) \) and \( |\cdot| \) is the standard Hermitian norm on \( \mathbb{C}^n \) associated with the standard scalar product \( \langle \cdot, \cdot \rangle_{\mathbb{C}^n} \).

Write \( \zeta_a = (p_a, q_a) \) and let
\[
\Omega(\zeta, \zeta') = \sum_{a \in \mathcal{L}} p_a q_a' - q_a p_a'.
\]
\( \Omega \) is an anti-symmetric bi-linear form which is continuous on
\[
Y_\gamma \times Y_{-\gamma} \cup Y_{-\gamma} \times Y_\gamma \to \mathbb{C}
\]
with norm \( ||\Omega|| = 1 \). The subspaces \((\mathbb{C}^2)^{\{a\}}\) are symplectic subspaces of two (complex) dimensions carrying the canonical symplectic structure.

\( \Omega \) defines as usual (by contraction on the first factor) a bounded bijective operator
\[
Y_\gamma \ni \zeta \mapsto \Omega(\zeta, \cdot) \in Y_{-\gamma}^\ast
\]
We shall denote its inverse by
\[
J : Y_{-\gamma}^\ast \to Y_\gamma.
\]

NB. There is another common way to identify \( Y_{-\gamma}^\ast \) with \( Y_\gamma \), the \( L^2 \)-pairing. This pairing defines an isomorphism \( \nabla : Y_{-\gamma}^\ast \to Y_\gamma \) such that
\[
J \circ \nabla^{-1} \zeta = \left\{ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right\} \zeta_a : a \in \mathcal{L} \right\}
\]
The operator \( J \circ \nabla^{-1} \) is a complex structure compatible with \( \Omega \) which is customarily denoted by \( J \), and we shall follow this tradition. This abuse of notation will cause no confusion since the two \( J \)'s act on different objects: one acts on one-forms and the other on vectors, and which is the case will be clear from the context.

A bounded map \( A : Y_\gamma \to Y_{\gamma}, \gamma \geq (0, 0) \) is symplectic if, and only if, it extends to a bounded map \( A : Y_{-\gamma} \to Y_{-\gamma} \) and verifies
\[
\Omega(A\zeta, A\zeta') = \Omega(\zeta, \zeta'), \quad \zeta \in Y_\gamma, \zeta' \in Y_{-\gamma},
\]
or, equivalently, \( A^\ast \circ J^{-1} \circ A = J^{-1} \) on \( Y_\gamma \) and on \( Y_{-\gamma} \). If \( A \) is bijective, then it is symplectic if, and only if, \( A^\ast \circ J^{-1} \circ A = J^{-1} \) on \( Y_\gamma \) (see [12]).

\(^1 \) \( Y_\gamma^\ast \) denote the Banach space dual of \( Y_\gamma \)

\(^2 \) \((\gamma_1', \gamma_2') \leq (\gamma_1, \gamma_2) \) if, and only if \( \gamma_1' \leq \gamma_1 \) and \( \gamma_2' \leq \gamma_2 \)
Let
\[ \mathbb{A}^A = \mathbb{C}^A \times (\mathbb{C} / 2\pi\mathbb{Z})^A \]
and consider the Banach manifold \( \mathbb{A}^A \times \pi_{\mathcal{L}\setminus A}Y_\gamma \) whose elements are denoted \( x = (r, \theta = [z], w) \).

We provide this manifold with the metric
\[ \| x - x' \|_\gamma = \inf_{p \in \mathbb{Z}^d} \| (r, z + 2\pi p, w) - (r', z', w') \|_\gamma. \]

We provide \( \mathbb{A}^A \times \pi_{\mathcal{L}\setminus A}Y_\gamma \) with the symplectic structure \( \Omega \). To any \( C^1 \)-function \( f(r, \theta, w) \) on (some open set in) \( \mathbb{A}^A \times \pi_{\mathcal{L}\setminus A}Y_\gamma \) it associates a vector field \( X_f = -J(\partial f / \partial \theta) \) – the Hamiltonian vector field of \( f \) – which in the coordinates \( (r, \theta, w) \) takes the form
\[
\begin{pmatrix}
\dot{r}_a \\
\dot{\theta}_a
\end{pmatrix} = \frac{\partial}{\partial \theta} f(r, \theta, w) \quad \text{and} \quad
\begin{pmatrix}
\dot{p}_a \\
\dot{q}_a
\end{pmatrix} = \frac{\partial}{\partial q} f(r, \theta, w).
\]

1.0.2. An integrable Hamiltonian system in \( \infty \) many dimensions.

In this paper we are considering an infinite dimensional Hamiltonian system given by a function \( h(r, w, \rho) \) of the form
\[
\langle r, \omega(\rho) \rangle + \frac{1}{2} \langle w, A(\rho)w \rangle = \langle r, \omega(\rho) \rangle + \frac{1}{2} \langle w, H(\rho)w \rangle + \frac{1}{2} \sum_{a \in \mathcal{L}_\infty} \lambda_a (p^2_a + q^2_a),
\]
where \( w_a = (p_a, q_a) \) and
\[
\begin{align*}
\omega : \mathcal{D} &\to \mathbb{R}^A \\
\lambda_a : \mathcal{D} &\to \mathbb{R}, \quad a \in \mathcal{L}_\infty \\
H : \mathcal{D} &\to gl(\mathbb{R}^q \times \mathbb{R}^q), \quad \hbar H = H
\end{align*}
\]
are \( C^{s_*} \), \( s_* \geq 1 \), functions of \( \rho \in \mathcal{D} \), the unit ball in \( \mathbb{R}^P \), parametrized by some finite subset \( \mathcal{P} \) of \( \mathbb{Z}^d \).

The Hamiltonian vector field of \( h \) is not \( C^1 \) on \( \mathbb{A}^A \times \pi_{\mathcal{L}\setminus A}Y_\gamma \), but its Hamiltonian system still has a well defined flow with a finite-dimensional invariant torus
\[
\{ 0 \} \times \mathbb{T}^A \times \{ 0, 0 \}
\]
which is reducible, i.e. the linearized equation on this torus (is conjugated to a system that) does not depend on the angles \( \theta \). This linearized equation has infinitely many elliptic directions with purely imaginary eigenvalues
\[
\{ i \lambda_a(\rho) : a \in \mathcal{L}_\infty \}
\]
and finitely many other directions given by the system
\[
\dot{\zeta}_x = JH(\rho)\zeta_x.
\]

\[ ^3 \] \([z]\) being the class of \( z \in \mathbb{C}^A \)

\[ ^4 \] there is no agreement as to the sign of the Hamiltonian vectorfield - we’ve used the choice of Arnold \([1]\)
A perturbation problem. The question here is if this invariant torus for \( h \) persists under perturbations \( h + f \), and, if so, if the persisted torus is reducible.

In finite dimension the answer is yes under very general conditions – for the first proof in the purely elliptic case see [3], and for a more general case see [8]. These statements say that, under general conditions, the invariant torus persists and remains reducible under sufficiently small perturbations for a subset of parameters \( \rho \) of large Lebesgue measure. Since the unperturbed problem is linear, parameter selection can not be avoided here.

In infinite dimension the situation is more delicate, and results can only be proven under quite severe restrictions on the normal frequencies (i.e. the eigenvalues \( \lambda_a \)). Such restrictions are fulfilled for many PDE’s in one space dimension – the first such result was obtained in [11].

For PDE’s in higher space dimension the behavior of the normal frequencies is much more complicated and the results are more sparse. A result for the Beam equation (which is simpler model than the Schrödinger equation and the Wave equation, and frequently considered in works on nonlinear PDE’s) was first obtained in [9] and [10]. For other results on PDE’s in higher space dimension see the discussion in the first part of this work [5].

Conditions on the unperturbed Hamiltonian. The function \( h \) we shall consider will verify several assumptions.

A1 – spectral asymptotics. There exist constants \( 0 < c', c \leq 1 \) and exponents \( \beta_1 = 2, \beta_2 \geq 0, \beta_3 > 0 \) such that for all \( \rho \in D \):

\[
|\lambda_a(\rho) - |a|^{\beta_1}| \leq c \frac{1}{\langle a \rangle^{\beta_2}} \quad a \in L_\infty;
\]

\[
|\lambda_a(\rho) - \lambda_b(\rho)| - (|a|^{\beta_1} - |b|^{\beta_1})| \leq c' \max(\frac{1}{\langle a \rangle^{\beta_3}}, \frac{1}{\langle b \rangle^{\beta_3}}), \quad a, b \in L_\infty;
\]

\[
\left\{ \begin{array}{l}
\lambda_a(\rho) \geq c' \quad a \in L_\infty \\
\|(JH(\rho))^{-1}\| \leq \frac{1}{\rho};
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
|\lambda_a(\rho) - \lambda_b(\rho)| \geq c' \quad a, b \in L_\infty, \ |a| \neq |b| \\
\|(\lambda_a(\rho)I - iJH(\rho))^{-1}\| \leq \frac{1}{\rho} \quad a \in L_\infty.
\end{array} \right.
\]

A2 – transversality. We refer to section [32] for the precise formulation of this condition which describes how the eigenvalues vary with the parameter \( \rho \).

Conditions on the perturbation. For \( \sigma > 0 \) we let \( \mathcal{O}_\gamma(\sigma, \mu) \) be the set

\[
\{ x = (r, \theta, w) \in \mathbb{R}^3 \times \pi \gamma : \left\| \begin{pmatrix} r \\ \frac{3}{\mu} \theta \\ \frac{w}{\mu} \end{pmatrix} - 0 \right\|_\gamma < 1 \}.
\]

It is often useful to scale the action variables by \( \mu^2 \) and not by \( \mu \), but in our case \( \mu \) will be \( \approx 1 \), and then there is no difference.

We shall consider perturbations

\[
f : \mathcal{O}_\gamma(\sigma, \mu) \to \mathbb{C}, \quad \gamma_* = (0, m_*) \geq (0, 0),
\]
that are real holomorphic up to the boundary (rhb). This means that it gives real values to real arguments and extends holomorphically to a neighborhood of the closure of $O_{\gamma_*}(\sigma, \mu)$. $f$ is clearly also rhb on $O_{\gamma'}(\sigma, \mu)$ for any $\gamma' \geq \gamma_*$, and

$$Jdf : O_{\gamma'}(\sigma, \mu) \to Y_{-\gamma'}$$

is rhb. But we shall require more:

R1 – first differential

$$Jdf : O_{\gamma'}(\sigma, \mu) \to Y_{\gamma'}$$

is rhb for any $\gamma_* \leq \gamma' \leq \gamma$.

This is a natural smoothness condition on the space of holomorphic functions on $O_{\gamma_*}(\sigma, \mu)$, and it implies, in particular, that $Jd^2 f(x) \in B(Y_{\gamma}; Y_{\gamma})$ for any $x \in O_{\gamma}(\sigma, \mu)$.

That $Jd^2 f(x) \in B(Y_{\gamma}; Y_{\gamma})$ implies in turn that

$$|Jd^2 f(x)[e_a, e_b]| \leq C t e^{-\gamma_1|a|-\beta} \min\left(\frac{(a)}{(b)}, \frac{(b)}{(a)}\right) \gamma_2$$

for any two unit vectors $e_a \in (\mathbb{C}^2)^{(a)}$ and $e_b \in (\mathbb{C}^2)^{(b)}$. But many Hamiltonian PDE’s verify other, and stronger, decay conditions in terms of

$$\min(|a - b|, |a + b|).$$

Such decay conditions do not seem to be naturally related to any smoothness condition of $f$, but they may be instrumental in the KAM-theory for multidimensional PDE’s: see for example [7] where such conditions were used to build a KAM-theory for some multidimensional non-linear Schrödinger equations.

The decay condition needed in this work depends on a parameter $0 \leq \kappa \leq m_*$ and defines a Banach sub-algebra $M^b_{\gamma, \kappa}$ of $B(Y_{\gamma}; Y_{\gamma})$, with norm $\|\cdot\|_{\gamma, \kappa}$ – its precise definition will be given in section 2.1. We shall require:

R2 – second differential

$$Jd^2 f : O_{\gamma'}(\sigma, \mu) \to M^b_{\gamma', \kappa}$$

is rhb for any $\gamma_* \leq \gamma' \leq \gamma$.

Denote by $T_{\gamma, \kappa}(\sigma, \mu)$ the space of functions

$$f : O_{\gamma_*}(\sigma, \mu) \to \mathbb{C},$$

real holomorphic up to the boundary, verifying R1 and R2. We provide $T_{\gamma, \kappa}(\sigma, \mu)$ with the norm

$$(1.7) \quad |f|_{\gamma, \kappa, \mu} = \max\left\{ \sup_{x \in O_{\gamma_*}(\sigma, \mu)} |f(x)|, \sup_{\gamma_* \leq \gamma' \leq \gamma} \sup_{x \in O_{\gamma'}(\sigma, \mu)} \|Jdf(x)\|_{\gamma'}, \sup_{\gamma_* \leq \gamma' \leq \gamma} \sup_{x \in O_{\gamma'}(\sigma, \mu)} \|Jd^2 f(x)\|_{\gamma', \kappa} \right\}$$

making it into a Banach space. Notice that the first two “components” of this norm are related to the smoothness of $f$, while the third “component” imposes a further decay condition on $Jd^2 f(x)$. 
1.0.6. The normal form theorem. For any $a \in L_\infty$, let
\[ [a] = \{ b \in L_\infty : |b| = |a| \}. \]

**Theorem.** Let $h$ be a Hamiltonian defined by (1.1) and verifying Assumptions A1-2.

Let $f : \mathcal{O}_{\gamma_*}(\sigma, \mu) \to \mathbb{C}$ be real holomorphic and verifying Assumptions R1-2 with
\[ \gamma = (\gamma_1, m_*) > \gamma_* = (0, m_*) \quad \text{and} \quad 0 < \kappa \leq m_* . \]

If $\varepsilon = \| f \|_{\gamma_*}^\sigma, \mu$ is sufficiently small, then there is a set $D' \subset D$ with
\[ \text{Leb}(D \setminus D') \to 0, \quad \varepsilon \to 0 \]
and a $C^{s_*}$ mapping
\[ \Phi : \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2) \times D \to \mathcal{O}_{\gamma_*}(\sigma, \mu) , \]
real holomorphic and symplectic for each parameter $\rho \in D$, such that
\[ (h + f) \circ \Phi = h' + f' \in T_{\gamma_*, \kappa} , \]
and
(i) for $\rho \in D'$ and $\zeta = r = 0$
\[ d_r f' = d_\theta f' = d_\zeta f' = d_\tau^2 f' = 0; \]
(ii) $h' : \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2) \times D \to \mathbb{C}$ is a $C^{s_*}$-function, real holomorphic for each parameter $\rho \in D$, and
\[ \| \partial_{\rho j} (h'(\cdot, \rho) - h(\cdot, \rho)) \|_{\gamma_*}^\sigma, \mu < C\varepsilon , \quad |j| \leq s_* - 1; \]
(iii) $h'$ has the form
\[ \langle r, \omega'(\rho) \rangle + \frac{1}{2} \langle \zeta_\tau, H'(\rho) \zeta_\tau \rangle + \]
\[ + \frac{1}{2} \sum_{[a]} \left( \langle p_{[a]}, A'_{[a]}(\rho)p_{[a]} \rangle + \langle q_{[a]}, A'_{[a]}(\rho)q_{[a]} \rangle \right) \]
where the matrix $A'_{[a]}$ is symmetric;
(iv) for any $x \in \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2)$
\[ \| \partial_{\rho j} (\Phi(x, \rho) - x) \|_{\gamma_*} \leq C\varepsilon , \quad |j| \leq s_* - 1. \]

The constant $C$ depends on $\# A, \# F, \# P, d_*, s_*, m_*, \kappa$ and $h$, but not on $\varepsilon$.

We shall give a more precise formulation of this result in Theorem 3.6 and its Corollary 3.7.

1.0.7. A singular perturbation problem. We want to apply this theorem to construct small-amplitude solutions of the multi-dimensional beam equation on the torus:
\[ u_{tt} + \Delta^2 u + nu = -g(x, u), \quad u = u(t, x), \quad x \in T^{d_*}. \]

Here $g$ is a real analytic function satisfying
\[ g(x, u) = 4u^3 + O(u^4). \]

Writing it in Fourier components, and introducing action-angle variables for the modes in (an arbitrary finite subset) $\mathcal{A}$, the linear part becomes a Hamiltonian system with a Hamiltonian $h$ of the form (1.1), with $F$ void. $h$ satisfies (for all $m > 0$) condition A1, but not condition A2.
The way to improve on \( h \) is to use a (partial) Birkhoff normal form around \( u = 0 \) in order to extract a piece from the non-linear part which improves on \( h \). This leads to a situation where the assumptions A1 and A2 and the size of the perturbation are linked – a singular perturbation problem.

In order to apply the theorem to such a singular situation one needs a careful and precise description of how the smallness requirement depends on the (parameters determining) assumptions A1 and A2. This is quite a serious complication which is carried out in this paper and the precise description of the smallness requirement is given in Theorem 3.6.

This normal form theorem improves on the result in [9] and [10] in two respects.

- We have imposed no “conservation of momentum” on the perturbation – this has the effect that our normal form is not diagonal in the purely elliptic directions. In this respect it resembles the normal form obtained for the non-linear Schrödinger equation obtained in [7] and the block diagonal form is the same.
- We have a finite-dimensional, possibly hyperbolic, component, whose treatment requires higher smoothness in the parameters.

The proof has no real surprises. It is a classical KAM-theorem carried out in a complex situation. The main part is, as usual, the solution of the homological equation with reasonable estimates. The fact that the block structure is not diagonal complicates, but this was also studied in for example [7]. The iteration combines a finite linear iteration with a “super-quadratic” infinite iteration. This has become quite common in KAM and was also used for example in [7].

1.0.8. Notation and agreements. \( \langle a \rangle = \max(|a|, 1) \). \( i \) will denote the complex imaginary unit. \( \overline{z} \) is the complex-conjugate of \( z \in \mathbb{C} \). By \( \langle \zeta, \zeta' \rangle_{\mathbb{C}^n} \) we denote the standard Hermitian scalar product in \( \mathbb{C}^n \), conjugate-linear in the first variable and linear in the second variable.

All Euclidean spaces, unless otherwise stated, are provided with the Euclidean norm denoted by \( |\cdot| \). For two subsets \( X \) and \( Y \) of a Euclidean space we denote

\[
\text{dist}(X, Y) = \inf_{x \in X, \ y \in Y} |x - y|
\]

and

\[
\text{diam}(X) = \sup_{x, y \in X} |x - y|.
\]

The space of bounded linear operators between two Banach spaces \( X \) and \( Y \) is denoted \( B(X; Y) \). Its operator norm will usually be denoted \( \|\cdot\| \) without specification of the spaces. Our complex Banach spaces will be complexifications of some “natural” real Banach spaces which in general are implicit. An analytic function between domains of two complex Banach spaces is called real holomorphic if it gives real values to real arguments.

The sets \( \mathcal{A}, \mathcal{F}, \mathcal{L}_\infty, \mathcal{P} \), as well as the “starred” constants \( d_*, s_*, m_* \) will be fixed in this paper – the dependence of them is usually not indicated.

Constants depending only on the dimensions \#\( \mathcal{A}, \#\mathcal{F}, \#\mathcal{P} \), on \( d_*, s_*, m_* \) and on the choice of finite-dimensional norms are regarded as absolute constants. An absolute constant only depending on \( \beta \) is thus a constant that only depends on \( \beta \), besides these factor. Arbitrary constants will often be denoted by \( C_t, c_t \) and, when they occur as an exponent, exp. Their values may change from line to line. For example we allow ourselves to write \( 2C_t \leq C_t \).
1.0.9. **Acknowledgement.** The authors acknowledge the support from the project ANR-10-BLAN 0102 of the Agence Nationale de la Recherche.

2. **Preliminaries.**

2.1. **A matrix algebra.** The mapping

\[(a, b) \mapsto |a - b| = \min(|a - b|, |a + b|)\]

is a pseudo-metric on \(\mathbb{Z}^d\), i.e. verifying all the relations of a metric with the only exception that \([a - b] = 0\) for some \(a \neq b\). This is most easily seen by observing that \([a - b] = d_{\text{Hausdorff}}((\{\pm a\}, \{\pm b\})\). We have \([a - 0] = |a|\).

Define, for any \(\gamma = (\gamma_1, \gamma_2) \geq (0, 0)\) and \(\kappa \geq 0\),

\[e_{\gamma, \kappa}(a, b) = Ce^{\gamma_1|a - b|} \max(|a - b|, 1)^{\gamma_2} \min(|a|, |b|)^{\kappa}.\]

**Lemma 2.1.**

(i) If \(\gamma_1, \gamma_2 - \kappa \geq 0\), then

\[e_{\gamma, \kappa}(a, b) \leq e_{\gamma, \kappa}(a, c)e_{\gamma, \kappa}(c, b), \quad \forall a, b, c,\]

if \(C\) is sufficiently large (bounded with \(\gamma_2, \kappa\)).

(ii) If \(\gamma - \gamma \leq \gamma \leq \gamma\), then

\[e_{\gamma, \kappa}(a, 0) \leq e_{\gamma, \kappa}(a, b)e_{\gamma, \kappa}(b, 0), \quad \forall a, b\]

if \(C\) is sufficiently large (bounded with \(\gamma_2, \kappa\)).

**Proof.** (i). Since \(|a - b| \leq |a - c| + |c - b|\) it is sufficient to prove this for \(\gamma_1 = 0\). If \(\gamma_2 = 0\) then the statement holds for any \(C \geq 1\), so it is sufficient to consider \(\gamma_2 > 0\) and, hence \(\gamma_2 = 1\). Then we want to prove

\[
\max(|a - b|, 1) \min(|a|, |b|)^{\kappa} \leq C \max(|a - c|, 1) \max(|c - b|, 1) \min(|c|, |b|)^{\kappa}.
\]

Now \(\max(|a - b|, 1) \leq \max(|a - c|, 1) + \max(|c - b|, 1),\)

\[
\max(|c - b|, 1) \min(|c|, |b|)^{\kappa} \geq |b|^{\kappa},
\]

and

\[
\max(|a - c|, 1) \min(|c|, |b|)^{\kappa} \geq \min(|a|, |b|)^{\kappa}.
\]

This gives the estimate.

(ii) Again it suffices to prove this for \(\gamma_1 = 0\) and \(\gamma_2 = 1\). Then we want to prove

\[
\max(|a|, 1)^{\gamma_2} \leq C \max(|a - b|, 1) \min(|a|, |b|)^{\kappa} \max(|b|, 1)^{\gamma_2}.
\]

The inequality is fulfilled with \(C \geq 1\) if \(a\) or \(b\) equal 0. Hence we need to prove

\[
|a|^{\gamma_2} \leq C \max(|a - b|, 1) \min(|a|, |b|)^{\kappa} |b|^{\gamma_2}.
\]

Suppose \(\gamma_2 \geq 0\). If \(|a| \leq 2|b|\) then this holds for any \(C \geq 2\). If \(|a| \geq 2|b|\) then \(|a - b| \geq \frac{1}{2}|a|\) and the statement holds again for any \(C \geq 2\).

If instead \(\gamma_2 < 0\), then we get the same result with \(a\) and \(b\) interchanged. \(\square\)
Proposition 2.3. Let \( \gamma \geq \kappa \). We define (formally) the matrix product
\[
(AB)_a^b = \sum_c A^c_a B^b_c.
\]
Notice that complex conjugation, transposition and taking the adjoint behave in the usual way under this formal matrix product.

Proposition 2.2. Let \( \gamma_2 \geq \kappa \). If \( A \in \mathcal{M}_{\gamma,0} \) and \( B \in \mathcal{M}_{\gamma,\gamma} \), then \( AB \) and \( BA \in \mathcal{M}_{\gamma,\gamma} \) and
\[
|AB|_{\gamma,\gamma} \quad \text{and} \quad |BA|_{\gamma,\gamma} \leq |A|_{\gamma,0} |B|_{\gamma,\gamma}.
\]
Proof. (i) We have, by Lemma 2.1(i),
\[
\sum_b (AB)_a^b \gamma,\kappa(a,b) \leq \sum_b |A^b_a| |B^b_a| \gamma,\kappa(a,b) \leq \sum_{b,c} |A^c_a| |B^b_c| \gamma,\kappa(a,c) \gamma,\kappa(c,b)
\]
which is \( \leq |A|_{\gamma,0} |B|_{\gamma,\gamma} \). This implies in particular the existence of \( (AB)_a^b \).
The sum over \( a \) is shown to be \( \leq |A|_{\gamma,0} |B|_{\gamma,\gamma} \) in a similar way. The estimate of \( BA \) is the same. \( \square \)

Hence \( \mathcal{M}_{\gamma,0} \) is a Banach algebra, and \( \mathcal{M}_{\gamma,\gamma} \) is a closed ideal in \( \mathcal{M}_{\gamma,0} \).

2.1.3. The space \( \mathcal{M}_{\gamma,\gamma} \). We define (formally) on \( Y_\gamma \) (see section 1.0.1)
\[
(A\zeta)_a = \sum_b A^b_a \zeta_b.
\]

Proposition 2.3. Let \( \gamma \geq \kappa \). If \( A \in \mathcal{M}_{\gamma,\gamma} \) and \( \zeta \in Y_\gamma \), then \( A\zeta \in Y_\gamma \) and
\[
\|A\zeta\|_{\gamma} \leq |A|_{\gamma,\gamma} \|\zeta\|_{\gamma}.
\]
Proof. Let \( \zeta' = A\zeta \). We have
\[
\sum_a |\zeta'_a|^2 = \sum_a |A^b_a| |\zeta_b| e_{\gamma,0}(a,0)^2 \leq \sum_a \left( \sum_b |A^b_a| |\zeta_b| e_{\gamma,0}(a,0) \right)^2.
\]
Write
\[
|A^b_a| |\zeta_b| e_{\gamma,0}(a,0) = I \times (I |\zeta_b| e_{\gamma,0}(b,0)) \times J
\]
where
\[
I_{a,b} = \sqrt{|A^b_a| e_{\gamma,\kappa}(a,b)}.
\]
Since, by Lemma 2.1(ii),
\begin{align*}
J &= \frac{e_{\gamma,0}(a,0)}{e_{\gamma,\kappa}(a,b) e_{\gamma,0}(b,0)} \\
&\leq 1,
\end{align*}
we get, by Hölder,
\begin{align*}
\sum_{a} |\zeta_{a}^{i}| e_{\gamma,0}(a,0)^2 &\leq \sum_{a} \left( \sum_{b} I_{a,b}^{2} |\zeta_{b}| \right) \left( \sum_{b} I_{a,b}^{2} e_{\gamma,0}(b,0)^2 \right) \\
&\leq |A|_{\gamma,\kappa} \sum_{a,b} I_{a,b}^{2} |\zeta_{b}| e_{\gamma,0}(b,0)^2 \leq |A|_{\gamma,\kappa} \sum_{b} |\zeta_{b}| e_{\gamma,0}(b,0)^2 \sum_{a} I_{a,b}^{2} \\
&\leq |A|_{\gamma,\kappa} \|\zeta\|^{2}_{\bar{\gamma}}.
\end{align*}
This shows that \(y_{a}\) exists for all \(a\), and it also proves the estimate. \(\square\)

We have thus, for any \(-\gamma \leq \bar{\gamma} \leq \gamma\), a continuous embedding of \(\mathcal{M}_{\gamma,\kappa}\),
\[\mathcal{M}_{\gamma,\kappa} \hookrightarrow \mathcal{M}_{\gamma,0} \to \mathcal{B}(Y_{\bar{\gamma}};Y_{\bar{\gamma}}),\]
into the space of bounded linear operators on \(Y_{\bar{\gamma}}\). Matrix multiplication in \(\mathcal{M}_{\gamma,\kappa}\) corresponds to composition of operators.

For our applications we must consider a larger sub algebra with somewhat weaker decay properties. For \(\gamma = (\gamma_{1}, \gamma_{2}) \geq \gamma_{*} = (0, m_{*})\), let
\[\mathcal{M}_{\gamma,\kappa}^{b} = \mathcal{B}(Y_{\bar{\gamma}};Y_{\bar{\gamma}}) \cap \mathcal{M}_{(\gamma_{1}, \gamma_{2}+\kappa-m_{*}),\kappa}\]
which we provide with the norm
\[\|A\|_{\gamma,\kappa} = \|A\|_{\mathcal{B}(Y_{\bar{\gamma}};Y_{\bar{\gamma}})} + |A|_{(\gamma_{1}, \gamma_{2}+\kappa-m_{*}),\kappa} .\]
This norm makes \(\mathcal{M}_{\gamma,0}^{b}\) into a Banach sub-algebra of \(\mathcal{B}(Y_{\bar{\gamma}};Y_{\bar{\gamma}})\) and \(\mathcal{M}_{\gamma,\kappa}^{b}\) becomes a closed ideal in \(\mathcal{M}_{\gamma,0}^{b}\).

2.2. Functions. Let
\[0 \leq \kappa \leq m_{*}\]
and let
\[\gamma = (\gamma_{1}, m_{*}) \geq \gamma_{*} = (0, m_{*}) .\]

2.2.1. The function space \(\mathcal{T}_{\gamma,\kappa}\). Consider the space of functions \(f : \mathcal{O}_{\gamma}(\sigma, \mu) \to \mathbb{C}\) which are real holomorphic up to the boundary (rhh) of \(\mathcal{O}_{\gamma}(\sigma, \mu)\). This implies that for any \(\gamma \geq \gamma_{*}\)
\[f : \mathcal{O}_{\gamma}(\sigma, \mu) \to \mathbb{C}\]
and
\[Jd f : \mathcal{O}_{\gamma}(\sigma, \mu) \to Y_{-\gamma}\]
are also rhh. We define \(\mathcal{T}_{\gamma,\kappa}(\sigma, \mu)\) to be the space of such functions such that for any \(\gamma_{*} \leq \gamma' \leq \gamma\),
\[R_{1} - Jd f : \mathcal{O}_{\gamma'}(\sigma, \mu) \to Y_{\gamma'}\]
\[R_{2} - Jd^{2} f : \mathcal{O}_{\gamma}(\sigma, \mu) \to \mathcal{M}_{\gamma',\kappa}^{b}\]
are rhh.

We provide \(\mathcal{T}_{\gamma,\kappa}(\sigma, \mu)\) with the norm (1.7). The higher differentials \(d^{k+2} f\) can be estimated by Cauchy estimates on some smaller domain in terms of this norm. This norm makes \(\mathcal{T}_{\gamma,\kappa}(\sigma, \mu)\) into a Banach space and a Banach algebra with the constant function \(f = 1\) as unit.
Remark. The differential forms $d^{k+2} f(x)$, $x \in \mathcal{O}_f(\sigma, \mu)$, are canonically identified with three bounded linear maps

$$
R_0 = \bigotimes_{k+2} Y_{\gamma} = Y_{\gamma} \otimes \cdots \otimes Y_{\gamma} \to \mathbb{C}
$$

$$
R_1 = \bigotimes_{k+1} J_{\gamma} Y_{\gamma}^* \to Y_{\gamma}^* \to Y_{\gamma}
$$

$$
R_2 = \bigotimes_k J^*_{\gamma} M_{\gamma, \gamma} \to M_{\gamma, \gamma}
$$

where $\otimes$ is the symmetric tensor product (over $\mathbb{C}$).

2.2.2. The function space $T_{\gamma, \mu}$. Let $\mathcal{D}$ be the unit ball in $\mathbb{R}^P$. We shall consider functions

$$
f : \mathcal{O}_f(\sigma, \mu) \times \mathcal{D} \to \mathbb{C}
$$

which are of class $C^s$. We say that $f \in T_{\gamma, \mu}(\sigma, \mu)$ if, and only if,

$$
\frac{\partial^j f}{\partial \rho^j}(\cdot, \rho) \in T_{\gamma, \mu}(\sigma, \mu)
$$

for any $\rho \in \mathcal{D}$ and any $|j| \leq s$. We provide this space by the norm

$$
|f|_{\sigma, \mu} = \max_{|j| \leq s, \rho \in \mathcal{D}} \frac{\partial^j f}{\partial \rho^j}(\cdot, \rho)
$$

This norm makes $T_{\gamma, \mu}(\sigma, \mu)$ into a Banach space and a Banach algebra.

2.2.3. Jets of functions. For any function $f \in T_{\gamma, \mu}(\sigma, \mu)$ we define its jet $f^T(x)$, $x = (r, \theta, w)$, as the following Taylor polynomial of $f$ at $r = 0$ and $w = 0$

$$
(2.2) \quad f(0, \theta, 0) + d_r f(0, \theta, 0)[r] + d_w f(0, \theta, 0)[w] + \frac{1}{2} d_w^2 f(0, \theta, 0)[w, w]
$$

Functions of the form $f^T$ will be called jet-functions.

Proposition 2.4. Let $f \in T_{\gamma, \mu}(\sigma, \mu)$. Then $f^T \in T_{\gamma, \mu}(\sigma, \mu)$ and

$$
|f^T|_{\sigma, \mu} \leq C|f|_{\sigma, \mu}
$$

$C$ is an absolute constant.

Proof. The first part follows by general arguments. Look for example on

$$
g(x) = d^2_w f \circ p(x)[w, w], \quad x = (r, \theta, w),
$$

where $p(x)$ is the projection onto $(0, \theta, 0)$. This function $g$ is real holomorphic up to the boundary (rbh) on $\mathcal{O}_f(\sigma, \mu)$, being a composition of such functions. Its sup-norm is obtained by a Cauchy estimate of $f$:

$$
\|d^2_w f(p(x))\|_{\mathcal{B}(\mathcal{O}_f(\sigma, \mu), \mathbb{C})} \leq \frac{1}{\mu^2} \sup_{\mathcal{O}_f(\sigma, \mu)} |f(y)| \|w\|_{\gamma}^2 \leq C \sup_{y \in \mathcal{O}_f(\sigma, \mu)} |f(y)|.
$$

Since $Jd_g(x)[\cdot]$ equals

$$
(Jd^2_w f \circ p(x)[w, w])[dp[\cdot]] + 2(Jd^2_w f \circ p(x)[w])[\cdot],
$$

and

$$
Jd^2_w f : \mathcal{O}_f(\sigma, \mu) \to \mathcal{B}(\mathcal{O}_f(\sigma, \mu), \mathbb{C})
$$

and

$$
Jd^2_w f = Jd^2_w df : \mathcal{O}_f(\sigma, \mu) \to \mathcal{B}(\mathcal{O}_f(\sigma, \mu), \mathbb{C})
$$
are rhb, it follows that \( dg \) verifies R1 and is rhb. The norm \( \|Jd g(x)\|_{\gamma'} \) is less than
\[
\|Jd^2_w f(p(x))\|_{\mathcal{B}(Y,\gamma')} \|w\|_{\gamma'}^2 + 2 \|Jd_w f(p(x))\|_{\mathcal{B}(Y,\gamma')} \|w\|_{\gamma'}^2,
\]
which is \( \leq \text{Ct. sup}_{y \in \mathcal{O}_{\gamma'}(\sigma,\mu)} \|Jdf(y)\|_{\gamma'} \) — this follows by Cauchy estimates of derivatives of \( Jdf \).

Since \( Jd^2 g(x) \) satisfies R1 and is rhb, the norm \( \|Jd^2 g\|_{\gamma',\kappa} \) is less than
\[
\|Jd^2_w d^2_w f(p(x))\|_{\mathcal{B}(Y,\gamma')} \|w\|_{\gamma'}^2 + 2 \|Jd_w d^2_w f(p(x))\|_{\mathcal{B}(Y,\gamma')} \|w\|_{\gamma'}^2 + 2 \|Jd^2 f(x)\|_{\gamma',\kappa},
\]
which is \( \leq \text{Ct. sup}_{y \in \mathcal{O}_{\gamma'}(\sigma,\mu)} \|Jd^2 f(y)\|_{\gamma',\kappa} \) — this follows by a Cauchy estimate of \( Jd^2 f \).

The derivatives with respect to \( \rho \) are treated alike. \( \square \)

2.3. Flows.

2.3.1. Poisson brackets. The Poisson bracket \( \{f, g\} \) of two \( C^1 \)-functions \( f \) and \( g \) is (formally) defined by
\[
\Omega(Jdf, Jdg) = df[Jdg] = -dg[Jdf]
\]
If one of the two functions verify condition R1, this product is well-defined. Moreover, if both \( f \) and \( g \) are jet-function, then \( \{f, g\} \) is also a jet-function.

**Proposition 2.5.** Let \( f, g \in \mathcal{T}_{\gamma,\kappa,\mathcal{D}}(\sigma,\mu) \), and let \( \sigma' < \sigma \) and \( \mu' < \mu \leq 1 \). Then

(i) \( \{g, f\} \in \mathcal{T}_{\gamma,\kappa,\mathcal{D}}(\sigma,\mu) \) and
\[
[\{g, f\}|_{\gamma',\mu'} \leq C_{\mu - \mu'}^{\mu - \mu'} \{g|_{\gamma,\kappa,\mathcal{D}} \{f|_{\gamma,\kappa,\mathcal{D}}
\]
for
\[
C_{\mu - \mu'}^{\mu - \mu'} = C\left(\frac{1}{\sigma - \sigma'} + \frac{1}{\mu - \mu'}\right).
\]

(ii) the n-fold Poisson bracket \( \{P^n f, g\} \in \mathcal{T}_{\gamma,\kappa,\mathcal{D}}(\sigma,\mu) \) and
\[
[\{P^n f, g\}|_{\gamma',\mu'} \leq \left(C_{\mu - \mu'}^{\mu - \mu'} \{g|_{\gamma,\kappa,\mathcal{D}} \right)^n \{f|_{\gamma,\kappa,\mathcal{D}}
\]
where \( P^n f = \{g, f\} \).

\( C \) is an absolute constant.
Proof: (i) We must first consider the function $h = \Omega(Jdg, Jdf)$ on $\mathcal{O}_{\gamma,\sigma}(\sigma, \mu)$. Since $Jdg$, $Jdf : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to Y_{\gamma'}$ are real holomorphic up to the boundary (rhh), it follows that $h : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to \mathbb{C}$ is rhb, and

$$|h(x)| \leq \|Jdg(x)\|_{\gamma,\sigma} \|Jdf(x)\|_{\gamma,\sigma}.$$ 

The vector $Jdh(x)$ is a sum of

$$J\Omega(Jd^2g(x), Jdf(x)) = Jd^2g(x)[Jdf(x)]$$

and another term with $g$ and $f$ interchanged. Since $Jd^2g : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to \mathcal{B}(Y_{\gamma'}, Y_{\gamma'})$ and $Jdg$, $Jdf : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to Y_{\gamma'}$ are rhb, it follows that $Jdh$ verifies R1 and is rhb. Moreover

$$\|Jd^2g(x)[Jdf(x)]\|_{\gamma,\sigma} \leq \|Jd^2g(x)\|_{\mathcal{B}(Y_{\gamma'}, Y_{\gamma'})} \|Jdf(x)\|_{\gamma,\sigma}$$

and, by definition of $\mathcal{M}^b_{\gamma,\sigma}$,

$$\|Jd^2g(x)\|_{\mathcal{B}(Y_{\gamma'}, Y_{\gamma'})} \leq \|Jd^2g(x)\|_{\gamma',0}.$$

The operator $Jd^2h(x) = d(Jdh)(x)$ is a sum of

$$Jd^3g(x)[Jdf(x)]$$

and

$$Jd^2g(x)[Jdf^2(x)]$$

and two other terms with $g$ and $f$ interchanged.

Since $Jd^3g : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{M}^b_{\gamma',\sigma})$ and $Jdf : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to Y_{\gamma'}$ are rhb, it follows that the first function $\mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{M}^b_{\gamma',\sigma})$ is rhb. It can be estimated on a smaller domain using a Cauchy estimate for $Jd^3g(x)$.

The second term is treated differently. Since $Jd^2f$, $Jd^2g : \mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to \mathcal{M}^b_{\gamma,\sigma}$ are rhb, and since, by Proposition 2.2 taking products is a bounded bi-linear maps with norm $\leq 1$, it follows that the second function $\mathcal{O}_{\gamma,\sigma}(\sigma, \mu) \to \mathcal{M}^b_{\gamma,\sigma}$ is rhb and

$$\|Jd^2g(x)[Jd^2f(x)]\|_{\gamma',\sigma} \leq \|Jd^2g(x)\|_{\gamma',\sigma} \|Jd^2f(x)\|_{\gamma',\sigma}.$$ 

The derivatives with respect to $\rho$ are treated alike.

(ii) That $F^n g \in \mathcal{T}_{\gamma,\sigma,D}(\sigma, \mu)$ follows from (ii), but the estimate does not follow from the estimate in (ii). The estimate follows instead from Cauchy estimates of $n$-fold product $F^n g$.

Remark 2.6. The proof shows that the assumptions can be relaxed when $g$ is a jet function: it suffices then to assume that $g \in \mathcal{T}_{\gamma,0,D}(\sigma, \mu)$ and $g = \hat{g}(\cdot, 0, \cdot) \in \mathcal{T}_{\gamma,\sigma,D}(\sigma, \mu)$. Then $\{g, f\}$ will still be in $\mathcal{T}_{\gamma,\sigma,D}(\sigma, \mu)$ but with the bound

$$|\{g, f\}|_{\gamma',\sigma} \leq C^\mu_{\sigma,\sigma} \|g\|_{\gamma,0,D} \|g = \hat{g}(\cdot, 0, \cdot)\|_{\gamma,\sigma,D} \|f\|_{\gamma,\sigma,D}.$$ 

To see this it is enough to consider a jet-function $g$ which does not depend on $\theta$. The only difference with respect to case (i) is for the second differential. The second term is fine since, by Proposition 2.2, $\mathcal{M}^b_{\gamma',\sigma}$ is a two-sided ideal in $\mathcal{M}^b_{\gamma',0}$ and

$$\|Jd^2g(x)[Jd^2f(x)]\|_{\gamma',\sigma} \leq \|Jd^2g(x)\|_{\gamma',0} \|Jd^2f(x)\|_{\gamma',\sigma}.$$ 

5 $\hat{g}(\cdot, 0, \cdot)$ is the 0:th Fourier coefficient of the function $\theta \mapsto g(\cdot, \theta, \cdot)$.
For the first term we must consider \( J d^2 g (x) [J d f (x)] \) which, a priori, takes its values in \( \mathcal{M}_{\nu, 0}^b \) and not in \( \mathcal{M}_{\nu, \kappa}^b \). But since \( g \) is a jet-function independent of \( \theta \) this term is \( = 0 \).

2.3.2. Hamiltonian flows. The Hamiltonian vector field of a \( \mathcal{C}^1 \)-function \( g \) on (some open set in) \( Y_\gamma \) is \( -J dg \). Without further assumptions it is an element in \( Y_{-\gamma} \), but if \( g \in \mathcal{T}_{\gamma, \kappa} \) then it is an element in \( Y_\gamma \) and has a well-defined local flow \( \Phi_g \).

**Proposition 2.7.** Let \( g \in \mathcal{T}_{\gamma, \kappa, D}(\sigma, \mu) \), and let \( \sigma^* < \sigma \) and \( \mu^* < \mu \leq 1 \). If

\[
|g|_{\gamma, \kappa, D}^{\sigma, \mu} \leq \frac{1}{C} \min(\sigma - \sigma^*, \mu - \mu^*),
\]

then

(i) the Hamiltonian flow map \( \Phi^t = \Phi_g \) is, for all \( |t| \leq 1 \) and all \( \gamma^* \leq \gamma \leq \gamma', \) a \( \mathcal{C}^\ast \)-map

\[
\mathcal{O}_{\gamma'}(\sigma', \mu') \times D \to \mathcal{O}_{\gamma'}(\sigma, \mu)
\]

which is real holomorphic and symplectic for any fixed \( \rho \in D \).

Moreover,

\[
\left\| \partial_{\sigma}^j (\Phi^t (x, \rho) - x) \right\|_{\gamma, \kappa} \leq C |g|_{\gamma, \kappa, D}^{\sigma, \mu},
\]

and

\[
\left\| \partial_{\sigma}^j (d\Phi^t (x) - I) \right\|_{\gamma', \kappa} \leq C |g|_{\gamma, \kappa, D}^{\sigma, \mu},
\]

for any \( x \in \mathcal{O}_{\gamma'}(\sigma', \mu') \), \( \gamma^* \leq \gamma' \leq \gamma \), and \( 0 \leq |j| \leq s^* \).

(ii) \( f \circ \Phi^t \in \mathcal{T}_{\gamma, \kappa}(\sigma, \mu, D) \) for \( |t| \leq 1 \) and

\[
|f \circ \Phi^t|_{\gamma, \kappa, D}^{\sigma, \mu} \leq C |f|_{\gamma, \kappa, D}^{\sigma, \mu}.
\]

\( C \) is an absolute constant.

**Proof.** It follows by general arguments that \( \Phi = \Phi_g : U \to \mathcal{O}_{\gamma}(\sigma, \mu) \) is real holomorphic in \( (t, \zeta) \in U \subset \mathbb{C} \times \mathcal{O}_x(\sigma, \mu) \) and depends smoothly on any smooth parameter in the vector field. Clearly, for \( |t| \leq 1 \) and \( x \in \mathcal{O}_{\gamma'}(\sigma', \mu') \)

\[
\left\| \Phi^t (x, \rho) - x \right\|_\gamma \leq \sup_{x \in \mathcal{O}_{\gamma}(\sigma, \mu)} \left\| J d g (x) \right\|_\gamma \leq |g|_{\gamma, \kappa, D}^{\sigma, \mu}
\]

as long as \( \Phi^t (x) \) stays in the domain \( \mathcal{O}_{\gamma}(\sigma, \mu) \). It follows by classical arguments that this is the case if

\[
|g|_{\gamma, \kappa, D}^{\sigma, \mu} \leq c t \min(\sigma - \sigma^*, \mu - \mu^*).
\]

The differential. We have

\[
\frac{d}{dt} d\Phi^t (x) = -J d^2 g (\Phi^t (x)) d\Phi^t (x) = B(t) d\Phi^t (x),
\]

where \( B(t) \in \mathcal{M}_{\nu, \kappa}^b \). By re-writing this equation in the integral form \( d\Phi^t (x) = Id + \int_0^t B(s) d\Phi^t (x) ds \) and iterating this relation, we get that \( d\Phi^t (x) - Id = B^\infty (t) \) with

\[
B^\infty (t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{j=1}^k B(t_j) dt_k \cdots dt_2 dt_1.
\]
We get, by Proposition 2.2, that
\[ d \Phi t(x) - \text{Id} \in M_{\gamma,\kappa} \]
and, for \(|t| \leq 1\),
\[ \|d \Phi t(x) - \text{Id}\|_{\gamma,\kappa} \leq \sum_{k \geq 1} \|Jd^2 g(\Phi t(x))\|_{\gamma,\kappa} \frac{t^k}{k!} \leq \|Jd^2 g(\Phi t(x))\|_{\gamma,\kappa}. \]

In particular, \( A = d \Phi t(x) \) is a bounded bijective operator on \( Y_{\gamma} \). Since \( Jd^2 g \) is a Hamiltonian vector field we clearly have that
\[ \Omega(A\zeta, A\zeta') = \Omega(\zeta, \zeta'), \quad \forall \zeta, \zeta' \in Y_{\gamma}, \]
so \( A \) is symplectic.

Parameter dependence. For \(|j| = 1\), we have
\[ \frac{d}{dt} Z(t) = \frac{d}{dt} \partial_j \Phi t(x, \rho) = B(t, \rho) Z(t) - \partial_j Jd g(x, \rho) = B(t) Z(t) + A(t). \]
Since
\[ \|A(t)\|_{\gamma} + \|B(t)\|_{\gamma,\kappa} \leq C t |g|_{\gamma,\kappa, D}, \]
it follows by classical arguments, using Gronwall, that
\[ \|Z(t)\|_{\gamma,0} \leq C t |g|_{\gamma,\kappa, D} |t|. \]

The higher order derivatives (with respect to \( \rho \)) of \( \Phi t(x, \rho) \), and the derivatives of \( d \Phi t(x, \rho) \) are treated in the same way.

The same argument applies to any \( \gamma_s \leq \gamma' \leq \gamma \).
Since
\[ f \circ \Phi g = \sum_{n \geq 0} \frac{1}{n!} p^n f, \]
(ii) is a consequence of Proposition 2.5(ii). \( \square \)

3. Normal Form Hamiltonians and the KAM theorem

3.1. Block decomposition, normal form matrices. In this subsection we recall two notions introduced in \([7]\) for the nonlinear Schrödinger equation. They are essential to overcome the problems of small divisors in a multidimensional context. Since the structure of the spectrum for the beam equation, \( \{ \sqrt{|a|}^4 + m, \ a \in \mathbb{Z}^{d^*} \} \), is similar to that for the NLS equation, \( \{ |a|^2 + \hat{V}_a, \ a \in \mathbb{Z}^{d^*} \} \), then to study the beam equation we will use tools, similar to those used to study the NLS equation.

**Block decomposition:** For any \( \Delta \in \mathbb{N} \cup \{\infty\} \) we define an equivalence relation on \( \mathbb{Z}^{d^*} \), generated by the pre-equivalence relation
\[ a \sim b \iff \begin{cases} |a| = |b| \\ |a - b| \leq \Delta. \end{cases} \]
(see (2.1)). Let \([a]_{\Delta}\) denote the equivalence class of \( a \) – the block of \( a \). For further references we note that
\[ |a| = |b| \text{ and } [a]_{\Delta} \neq [b]_{\Delta} \Rightarrow |a - b| \geq \Delta \]
The crucial fact is that the blocks have a finite maximal “diameter”
\[ d_{\Delta} = \max_{|a| = |b|} |a - b| \]
which do not depend on \( a \) but only on \( \Delta \).
Proposition 3.1.

\[ d_\Delta \leq C\Delta^{\frac{(d_\ast +1)}{2}}. \]

The constant \( C \) only depends on \( d_\ast \).

Proof. In \([7]\) it was considered the equivalence relation on \( \mathbb{Z}^{d_\ast} \), generated by the pre-equivalence

\[ a \approx b \text{ if } |a| = |b| \text{ and } |a - b| \leq \Delta. \]

Denote by \([a]_\Delta^0\) and \(d_\Delta^0\) the corresponding equivalence class and its diameter (with respect to the usual distance). Since \( a \sim b \) if and only if \( a \approx b \) or \( a \approx -b \), then

\[ [a]_\Delta = [a]_\Delta^0 \cup -[a]_\Delta^0, \]

provided that the union in the r.h.s. is disjoint. It is proved in \([7]\) that \( d_\Delta^0 \leq D_\Delta =: C\Delta^{\frac{(d_\ast +1)}{2}} \). Accordingly, if \( |a| \geq D_\Delta \), then the union above is disjoint, (3.3) holds and diameter of \([a]_\Delta\) satisfies (3.2). If \( |a| < D_\Delta \), then \([a]_\Delta\) is contained in a sphere of radius \( < D_\Delta \). So the block’s diameter is at most \( 2D_\Delta \). This proves (3.2) if we replace there \( C\Delta^{\frac{(d_\ast +1)}{2}} \) by \( 2C\Delta^{\frac{(d_\ast +1)}{2}} \). \( \square \)

If \( \Delta = \infty \) then the block of \( a \) is the sphere \( \{ b : |b| = |a| \} \). Each block decomposition is a sub-decomposition of the trivial decomposition formed by the spheres \( \{|a| = \text{const}\}\).

Normal form matrices. Let \( E_\Delta \) be the decomposition of \( \mathcal{L} = \mathcal{F} \sqcup \mathcal{L}_\infty \) into the subsets

\[ [a]_\Delta = \begin{cases} [a]_\Delta^0 \cap \mathcal{L}_\infty & a \in \mathcal{L}_\infty \subset \mathbb{Z}^{d_\ast} \\ a \in \mathcal{F}. \end{cases} \]

Remark 3.2. Now the diameter of each block \([a]_\Delta\) is bounded by

\[ d_\Delta \leq C\Delta^{\frac{(d_\ast +1)}{2}} \]

if moreover we let \( C \geq \#\mathcal{F} \).

On the space of \( 2 \times 2 \) complex matrices we introduce a projection

\[ \Pi : \text{Mat}(2 \times 2, \mathbb{C}) \rightarrow \mathbb{C}I + \mathbb{C}J, \]

orthogonal with respect to the Hilbert-Schmidt scalar product. Note that \( \mathbb{C}I + \mathbb{C}J \) is the space of matrices, commuting with the symplectic matrix \( J \).

Definition 3.3. We say that a matrix \( A : \mathcal{L} \times \mathcal{L} \rightarrow \text{Mat}(2 \times 2, \mathbb{C}) \) is on normal form with respect to \( \Delta \), \( \Delta \in \mathbb{N} \cup \{\infty\} \), and write \( A \in \mathcal{NF}_\Delta \), if

(i) \( A \) is real valued,
(ii) \( A \) is symmetric, i.e. \( A_0^\ast = \frac{1}{2}A_0 \),
(iii) \( A \) is block diagonal over \( E_\Delta \), i.e. \( A_0^\ast = 0 \) if \([a]_\Delta \neq [b]_\Delta \),
(iv) \( A \) satisfies \( \Pi A_0^\ast \equiv A_0^\ast \) for all \( a, b \in \mathcal{L}_\infty \).

Any quadratic form \( q(w) = \frac{1}{2}(w, Aw), w = (p, q) \), can be written as

\[ \frac{1}{2}(p, A_{11}p) + (p, A_{12}q) + \frac{1}{2}(q, A_{22}q) + \frac{1}{2} \langle w_\mathcal{F}, H(p)w_\mathcal{F} \rangle \]

where \( A_{11}, A_{22} \) and \( H \) are real symmetric matrices and \( A_{12} \) is a real matrix.
We now pass from the real variables \( w_a = (p_a, q_a) \) to the complex variables \( z_a = (\xi_a, \eta_a) \) by \( w = U z \) defined through

\[
(3.4) \quad \xi_a = \frac{1}{\sqrt{2}} (p_a + i q_a), \quad \eta_a = \frac{1}{\sqrt{2}} (p_a - i q_a),
\]

for \( a \in L_\infty \), and acting like the identity on \( (C^2)^F \). Then we have

\[
q(U z) = \frac{1}{2} \langle \xi, P \xi \rangle + \frac{1}{2} \langle \eta, P \eta \rangle + \langle \xi, Q \eta \rangle + \frac{1}{2} \langle z_F, H(\rho) z_F \rangle,
\]

where

\[
P = \frac{1}{2} \left( (A_{11} - A_{22}) - i (A_{12} + t A_{12}) \right)
\]

and

\[
Q = \frac{1}{2} \left( (A_{11} + A_{22}) + i (A_{12} - t A_{12}) \right).
\]

Hence \( P \) is a complex symmetric matrix and \( Q \) is a Hermitian matrix. If \( A \) is on normal form, then \( P = 0 \).

Notice that this change of variables is not symplectic but

\[
U \left( \begin{array}{cc} J_\infty & 0 \\ 0 & J_F \end{array} \right) U = \left( \begin{array}{cc} i J_\infty & 0 \\ 0 & J_F \end{array} \right).
\]

3.2. The unperturbed Hamiltonian. Let \( h \) be a function as in (1.1), i.e.

\[
(3.5) \quad h(r, w, \rho) = \langle r, \omega(\rho) \rangle + \frac{1}{2} \langle w, A(\rho) w \rangle,
\]

where

\[
\langle w, A(\rho) w \rangle = \langle w_F, H(\rho) w_F \rangle + \frac{1}{2} \left( \langle p_\infty, Q(\rho) p_\infty \rangle + \langle q_\infty, Q(\rho) q_\infty \rangle \right)
\]

and

\[
Q(\rho) = \text{diag}\{\lambda_a(\rho) : a \in L_\infty\}.
\]

Assume \( \omega, \lambda_a, H \) verify (1.2). We shall denote by

\[
\chi = |\nabla \omega|_{C^{\beta-1}(\Omega)} + \sup_{a \in L} |\nabla \lambda_a|_{C^{\beta-1}(\Omega)} + ||\nabla H||_{C^{\beta-1}(\Omega)}
\]

since this quantity will play an important role in our analysis.

3.2.1. Assumption A1 - spectral asymptotic.

There exist constants \( 0 < c, c' \leq 1 \) and exponents \( \beta_1 = 2, \beta_2 \geq 0, \beta_3 > 0 \) such that for all \( \rho \in \mathcal{D} \) the relations (1.5), (1.4), (1.6) and (1.3) hold.

3.2.2. Assumption A2 - transversality. Let \( [a] = [a]_\infty \) so that \( [a] \) equals \( \{ b : |b| = |a| \} \) when \( a \in L_\infty \) and equals \( F \) when \( a \in F \). Denote by \( Q_{[a]} \) the restriction of the matrix \( Q \) to \( [a] \times [a] \) and let \( Q_{[0]} = 0 \). Let also \( JH(\rho)_{[0]} = 0 \) and \( m = 2#F \).

There exists a \( 1 \geq \delta_0 > 0 \) such that for all \( C^{\beta'} \)-functions

\[
\omega' : \mathcal{D} \to \mathbb{R}^n, \quad |\omega' - \omega|_{C^{\beta'}(\Omega)} < \delta_0,
\]

the following hold for each \( k \in \mathbb{Z}^n \setminus \{0\} \):
(i) for any \( a, b \in \mathcal{L}_\infty \cup \{ \emptyset \} \) let
\[
L(\rho) : X \mapsto \langle k, \omega'(\rho) \rangle X + Q_{[a]}(\rho)X \pm XQ_{[b]};
\]
then either \( L(\rho) \) is \( \delta_0 \)-invertible for all \( \rho \in \mathcal{D} \), i.e.
\[
\| L(\rho)^{-1} \| \leq \frac{1}{\delta_0}, \quad \forall \rho \in \mathcal{D},
\]
or there exists a unit vector \( z \) such that
\[
|\langle v, \partial_n L(\rho)v \rangle| \geq \delta_0, \quad \forall \rho \in \mathcal{T}^\rho
\]
and for any unit-vector \( v \) in the domain of \( L(\rho) \);
(ii) let
\[
L(\rho, \lambda) : X \mapsto \langle k, \omega'(\rho) \rangle X + \lambda X + iXJH(\rho)
\]
and
\[
P(\rho, \lambda) = \det L(\rho, \lambda) ;
\]
then either \( L(\rho, \lambda_a(\rho)) \) is \( \delta_0 \)-invertible for all \( \rho \in \mathcal{D} \) and \( a \in [a]_\infty \), or there exists a unit vector \( z \) such that
\[
|\partial_j P(\rho, \lambda_a(\rho)) + \partial_\lambda P(\rho, \lambda_a(\rho)) \langle v, \partial_j Q(\rho)v \rangle| \geq \delta_0 \| L(\cdot, \lambda_a(\cdot)) \|^{m-1}_{C^1(\mathcal{D})}
\]
for all \( \rho \in \mathcal{D} \), \( a \in [a]_\infty \) and for any unit-vector \( v \in (\mathbb{C}^2)^{|a|} \);
(iii) for any \( a, b \in \mathcal{F} \cup \{ \emptyset \} \) let
\[
L(\rho) : X \mapsto \langle k, \omega'(\rho) \rangle X - iJH(\rho)_{[a]}X + iXJH(\rho)_{[b]};
\]
then either \( L(\rho) \) is \( \delta_0 \)-invertible for all \( \rho \in \mathcal{D} \), or there exists a unit vector \( \zeta \) and an integer \( 1 \leq j \leq s_a \) such that
\[
|\partial_j \det L(\rho)| \geq \delta_0 \| L(\rho) \|^{m-1}_{C^1(\mathcal{D})}, \quad \forall \rho \in \mathcal{D}.
\]

**Remark 3.4.** The dichotomy in A2 is imposed not only on \( \omega \) but also on \( C^1 \)-perturbations of \( \omega' \), because, in general, the dichotomy for \( \omega' \) does not imply that for perturbations.

If, however, any \( C^{s_\ast} \) perturbation \( \omega' \) of \( \omega \) can be written as \( \omega' = \omega \circ f \) for some diffeomorphism \( f = id + O(\delta_0) \) – this is for example the case when \( \omega(\rho) = \rho, \) – then the dichotomy on \( \omega \) implies a dichotomy on \( C^{s_\ast} \)-perturbations.

**3.3. Normal form Hamiltonians.** Consider now the function \( (3.5) \) defined on the set \( \mathcal{D} \). This function will be fixed throughout this paper and we shall denote it and its “components” by
\[
h_{up}, \omega_{up}, A_{up}, Q_{up}, H_{up}.
\]

**Remark 3.5.** The essential properties of \( h_{up} \) are given by the constants
\[
\chi, c', c, \beta = (\beta_1, \beta_2, \beta_3), \delta_0.
\]
These will be fixed now, once and for all. All estimates will depend on \( h_{up} \) only through these constants. Since it will be important for our analysis of the Beam equation we shall track this dependence with respect to \( c', \delta_0, \chi \). In order to simplify the estimates a little we shall assume that
\[
0 < c' \leq \delta_0 \leq \chi \leq c. \tag{3.6}
\]
\(^6\) \( \partial_\lambda \) denotes here the directional derivative in the direction \( \zeta \in \mathbb{R}^p \)
We shall consider functions of the form
\[ h(r, w, \rho) = \langle \omega(\rho), r \rangle + \frac{1}{2} \langle w, A(\rho)w \rangle \]
which satisfies
\[ \text{Hypothesis } \omega: \text{ } \omega \text{ is of class } \mathcal{C}^* \text{ on } D \text{ and } \]
\[ |\omega - \omega_{up}|_{\mathcal{C}^*(D)} \leq \delta. \]
\[ \text{Hypothesis B: } A - A_{up} : D \to M^h_{0, \kappa} \text{ is of class } \mathcal{C}^*, A(\rho) \text{ is on normal form } \in \mathcal{NF}_\Delta \text{ for all } \rho \in D \text{ and } \]
\[ ||\partial^j_\rho (A(\rho) - A_{up}(\rho))_a|| \leq \frac{1}{(a)^s} \]
for \( |j| \leq s, a \in \mathcal{L} \text{ and } \rho \in D. \]
We also require that
\[ \kappa > 0. \]
A function verifying these assumptions is said to be on \textit{normal form.} and we shall denote this by
\[ h \in \mathcal{NF}_{\kappa, h_{up}}(\Delta, \delta). \]
Since the unperturbed Hamiltonian \( h_{up} \) will be fixed in this paper we shall often suppress it in this notation writing simply \( h \in \mathcal{NF}_{\kappa}(\Delta, \delta). \)

3.4. The normal form theorem. In this section we state an abstract KAM result for perturbations of normal form Hamiltonians by a function in \( T_{\gamma, \kappa, D}(\sigma, \mu), 0 < \sigma, \mu \leq 1. \) Let
\[ \gamma = (\gamma_1, m_*), \gamma_s = (0, m_*) \text{ and } \kappa > 0. \]

**Theorem 3.6.** There exist positive constants \( C, \alpha, \exp \) and \( \exp_3 \) such that, for any \( h \in \mathcal{NF}_{\kappa, h_{up}}(\Delta, \delta) \) and for any \( f \in T_{\gamma, \kappa, D}(\sigma, \mu), \)
\[ \varepsilon = |f^T|_{\gamma, \kappa, D} \text{ and } \xi = |f|_{\gamma, \kappa, D}, \]
if
\[ \delta \leq \frac{1}{2C} (\varepsilon \exp_3 \frac{c'}{\varepsilon} \exp) \]
and
\[ \varepsilon (\log \frac{1}{\varepsilon}) \exp \leq C (\frac{\max(\gamma_1^{-1}, d_\Delta)}{\sigma \mu}) - \exp \left( \frac{c'}{\chi + \xi} \right) \exp_3 \exp_3 \frac{c'}{\varepsilon} \]
then there exist a set \( D' = D'(h, f) \subset D, \)
\[ \text{Leb}(D \setminus D') \leq C (\log \frac{1}{\varepsilon} \max(\gamma_1^{-1}, d_\Delta) \exp (\chi + \xi))^{1+\alpha} \frac{\varepsilon}{\delta_0} \]
and a \( \mathcal{C}^* \) mapping
\[ \Phi : O_{\gamma_s}(\sigma/2, \mu/2) \times D \to O_{\gamma_s}(\sigma, \mu), \]
real holomorphic and symplectic for each parameter \( \rho \in D, \) such that
\[ (h + f) \circ \Phi = h' + f' \]
and
\[ ^7 \text{here it is important that } || \text{ is the matrix operator norm} \]
\[ ^8 \text{remember the convention (3.6)} \]
Proof. Let us denote

\[ \|s\| \text{ depends on } x \]

and equal to the identity when \( \epsilon = 0 \).

Remark 3.8. The assumption (3.11) on \( \epsilon \) involves many constants and parameters. Then (3.11) takes the form

\[ \epsilon (\log \frac{1}{\epsilon})^{\exp} \leq C' \left( \frac{c'}{\chi + \xi} \right)^{\exp} \epsilon \]

The constant \( C \) is an absolute constant that only depends on \( \beta, \chi, c \) and \( \sup_{\mathcal{D}} |\omega| \). The exponent \( \exp' \) is an absolute constant that only depends on \( \beta \) and \( \chi \). The exponent \( \exp_3 \) only depends on \( s_* \). The exponent \( \alpha' \) is a positive constant only depending on \( s_*, \frac{d_1}{\chi}, \frac{d_2}{p/3} \).

The condition on \( \Phi \) and \( h' - h \) may look bad but it is not.

Corollary 3.7. Under the assumption of the theorem, let \( s_* \) be the largest positive number such that (3.11) holds. Then, for any \( \rho \in \mathcal{D} \) and \( |j| \leq s_* - 1 \),

(i) \( \|\partial_\rho^j(h' - h)\|_{\gamma, \om, \mathcal{D}} \leq \frac{C}{\epsilon} |f|^\rho_{\sigma, \mu/2}^{\gamma, \om, \mathcal{D}} \); 

(ii) for any \( x \in \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2) \) and \( \|\partial_\rho^j(d\Phi(x, \cdot) - I)\|_{\gamma, \om, \mathcal{D}} \leq C \); 

(iii) \( \|\partial_\rho^j(d\Phi(x, \cdot) - x)\|_{\gamma, \om, \mathcal{D}} \leq C \).

The argument for \( h' - h \) is the same.

We can take \( \delta = 0 \) here, in which case \( h \) equals the unperturbed Hamiltonian \( h_{up} \) - this is the case described in theorem of the Introduction.

Remark 3.8. The assumption (3.11) on \( \epsilon \) involves many constants and parameters. Then (3.11) takes the form

\[ \epsilon (\log \frac{1}{\epsilon})^{\exp} \leq C' \left( \frac{c'}{\chi + \xi} \right)^{\exp} \epsilon \]
where $C'$ depends on $C_\gamma, \sigma, \mu, \Delta$. If we assume that

$$\xi, \chi = O(\delta_0^{-N}) \quad \text{and} \quad \epsilon' = O(\delta_0^{1+N})$$

for some $N > 0$, then assumption (3.11) reduces to

$$\epsilon (\log \frac{1}{\epsilon}) \exp \lesssim C' \delta_0^{1+N+2N \exp_3},$$

which implies

$$\epsilon_0^{\ast} \gtrsim C' \delta_0^{1+2N+2N \exp_3},$$

when $\delta_0$ is sufficiently small.

Actually in paper [5] Theorem 3.6 this is used in this context.

4. SMALL DIVISORS

For a mapping $L: D \rightarrow \text{gl}(\dim, \mathbb{R})$ define, for any $\kappa > 0$,

$$\Sigma(L, \kappa) = \{ \rho \in D : ||L^{-1}|| > \frac{1}{\kappa} \}.$$ 

Let

$$h(r, w, \rho) = \langle r, \omega(\rho) \rangle + \frac{1}{2} \langle w, A(\rho) w \rangle$$

be a normal form Hamiltonian in $\mathcal{NF}_\kappa(\Delta, \delta)$. Recall the convention in Remark 3.5 and assume $\kappa > 0$ and

$$\delta \leq \frac{1}{C' \kappa},$$

where $C$ is to be determined.

**Lemma 4.1.** Let

$$L_k = \langle k, \omega(\rho) \rangle.$$ 

There exists a constant $C$ such that if (4.1) holds, then

$$\text{Leb}\left( \bigcup_{0 < |k| \leq N} \Sigma(L_k, \kappa) \right) \leq C N^{\exp \chi + \frac{\delta}{\delta_0} \kappa}$$

and

$$\text{dist}(D \setminus \Sigma(L_k, \kappa), \Sigma(L_k, \kappa/2)) > \frac{1}{C} \frac{\kappa}{N(\chi + \delta)}$$

for any $\kappa > 0$.

The exponent $\exp$ only depends on $\# A$. $C$ is an absolute constant.

**Proof.** Since $\delta \leq \delta_0$, using Assumption A2(i), with $a = b = \emptyset$, we have, for each $k \neq 0$, either that

$$|\langle \omega(\rho), k \rangle| \geq \delta_0 \geq \kappa \quad \forall \rho \in D$$

or that

$$\partial_k \langle \omega(\rho), k \rangle \geq \delta_0 \quad \forall \rho \in D$$

(for some suitable choice of a unit vector $f$). The first case implies $\Sigma(L_k, \kappa) = \emptyset$. The second case implies that $\Sigma(L_k, \kappa)$ has Lebesgue measure

$$\lesssim \frac{N(\chi + \delta)}{\delta_0} \frac{\kappa}{\delta_0}.$$ 

---

9 this is assumed to be fulfilled if $\Sigma_L(\frac{\kappa}{2}) = \emptyset$
Summing up over all $0 < |k| \leq N$ gives the first statement. The second statement follows from the mean value theorem and the bound

$$|\nabla \rho L_k(\rho)| \leq N(\chi + \delta).$$

□

Lemma 4.2. Let

$$L_{k,[a]} = (\langle k, \omega \rangle I - JA)_{[a]}.$$

There exists a constant $C$ such that if (4.1) holds, then,

$$\text{Leb} \left( \sum_{0 < |k| \leq N} \frac{(\chi + \delta)}{\delta_0} \sum_{\kappa} \frac{\kappa}{2} \right) \leq C N \exp \left( \frac{\kappa}{\delta_0} \frac{1}{\gamma^*} \right)$$

and

$$\text{dist} \left( D \setminus \Sigma(L_{k,[a]}, \kappa), \Sigma(L_{k,[a]}, \frac{\kappa}{2}) \right) > \frac{1}{C N(\chi + \delta)}$$

for any $\kappa > 0$.

$\exp$ only depends on $d_{\gamma^*}$ and $\#A$. $C$ is an absolut constant that depends on $c$ and $\sup_D |\omega|$.

Proof. Consider first $a \in L_\infty$. Then $L_{k,[a]}$ decouples into to a sum of two Hermitian operators of the form

$$\langle k, \omega \rangle I + Q_{[a]}$$

– denoted $L = L_{k,[a]}$ – where $Q_{[a]}$ is the restriction of $Q$ to $[a] \times [a]$. Since $L$ is Hermitian:

- $\|L(\rho)^{-1}\| \leq \max \|\langle k, \omega(\rho) \rangle + \lambda(\rho)\|^{-1}$, where the maximum is taken over all eigenvalues $\lambda(\rho)$ of $Q(\rho)$;
- for any $\rho_0 \in D$,

$$\partial_3 (v(\rho)L(\rho), v(\rho))_{\rho = \rho_0} = \partial_3 (v(\rho_0)L(\rho), v(\rho_0))_{\rho = \rho_0}$$

for any eigenvector $v(\rho)$ of $L(\rho)$ (associated to an eigenvalue which is $C^1$ in the direction $\partial_3$).

If we let

$$L_{\text{up}} = \langle k, \omega \rangle I + (Q_{\text{up}})_{[a]},$$

where $Q_{\text{up}}$ comes from the unperturbed Hamiltonian, then it follows, from (3.9) and (4.1), that

$$\|L - L_{\text{up}}\|_{C^*}< \delta \leq \frac{\delta_0}{2}$$

and, hence,

$$\text{d}_{\text{Hausdorff}}(\sigma(L), \sigma(L_{\text{up}})) < \delta.$$

If now $L_{\text{up}}$ is $\delta_0$-invertible, then this implies that $L$ is $\frac{\delta_0}{2}$-invertible. Otherwise

$$\partial_3 (\langle k, \omega(\rho) + \lambda(\rho)\rangle_{\rho = \rho_0} = \langle v(\rho), \partial_3 L(\rho)v(\rho)\rangle_{\rho = \rho_0} =$$

$$= \langle v(\rho), \partial_3 L_{\text{up}}(\rho)v(\rho)\rangle_{\rho = \rho_0} + \mathcal{O}(\delta),$$

where the maximum is taken over all eigenvalues $\lambda(\rho)$ of $Q(\rho)$.
where \( v(\rho) \) is a unit eigenvector of \( L(\rho) \) associated with the eigenvalue \( \lambda(\rho) \). If \( L_{up} \) is not \( \delta_0 \)-invertible, then, by assumption A2(i) there exists a unit vector \( \zeta \) such that

\[
|\partial_j \left( \langle k, \omega(\rho) \rangle + \lambda(\rho) \right) / \rho = \rho_0 | \geq \delta_0 - \delta \geq \frac{\delta_0}{2}.
\]

Hence, the Lebesgue measure of \( \Sigma(L_{k,[a]}, \kappa) \) is \( \leq \delta_{\Delta} \frac{\kappa}{\delta_0} \) — recall that, by Remark C.2, the operator is of dimension \( \leq d_{\Delta}^2 \). (This argument is valid if \( \lambda(\rho) \) is \( C^1 \) in the direction \( \zeta \) which can always be assumed when \( Q \) is analytic in \( \rho \). The non-analytic case follows by analytical approximation.)

Since \( |\langle k, \omega(\rho) \rangle| \leq k \leq N \), it follows, by (1.3), that

\[
|\langle k, \omega(\rho) \rangle + \lambda(\rho)| \geq |\lambda_\rho(\rho)| - \delta - \text{Ct.} |k| \geq |a|^{\beta_1} - c(a)^{-\beta_2} - \delta - \text{Ct.} |k|
\]

for some appropriate \( a \in [a] \). Hence, \( \Sigma(L_{k,[a]}, \kappa) = \emptyset \) for \( |a| \geq N^{\frac{1}{\beta_2}} \).

Summing up over all \( 0 < |k| \leq N \) and all \( |a| \geq N^{\frac{1}{\beta_2}} \) gives the first estimate.

Consider now \( a \in \mathcal{F} \). Then \( L(\rho) = (\langle k, \omega I - iJH \rangle) \) and it follows, by (3.10) and (4.1), that

\[
\| L - L_{up} \|_{C^s} \leq \delta \leq \frac{1}{2} \delta_0,
\]

where \( L_{up}(\rho) = (\langle k, \omega I - iJH_{up} \rangle) \). If now \( L_{up} \) is \( \delta_0 \)-invertible, then \( L \) will be \( \frac{\delta_0}{2} \)-invertible.

Otherwise,

\[
|\text{det} L - \text{det} L_{up}\|_{C^j} \leq \text{Ct.} \delta \left( \| L_{up} \|_{C^j} + \delta \right)^{m-1}
\]

and, by assumption A2(iii), there exists a unit vector \( \zeta \) and an integer \( 1 \leq j \leq s \) such that

\[
|\partial_j^j \text{det} L_{up}(\rho)| \geq \delta_0 \| L_{up} \|_{C^j(D)}^{m-1}, \quad \forall \rho \in \mathcal{D}.
\]

This implies that \( |L_{up}\|_{C^j} \geq \text{Ct.} \delta_0 \) and, hence,

\[
|\text{det} L - \text{det} L_{up}\|_{C^j} \leq \text{Ct.} \delta \| L_{up} \|_{C^j}^{m-1}.
\]

Thus

\[
|\partial_j^j \text{det} L(\rho)| \geq (\delta_0 - \text{Ct.} \delta) \| L \|_{C^j(D)}^{m-1}, \quad \forall \rho \in \mathcal{D},
\]

and \( \delta_0 - \text{Ct.} \delta \geq \frac{\delta_0}{2} \).

Then, by Lemma A.4.1

\[
\frac{\text{det} L(\rho)}{\| L \|_{C^j(D)}^{m-1}} \geq \frac{\kappa}{\delta_0},
\]

outside a set \( \Sigma' \) of Lebesgue measure

\[
\leq \text{Ct.} \frac{\| \nabla_\rho L_{k,\kappa-1}(\mathcal{D}) \|_{C^j(D)}}{\delta_0} \left( \frac{\kappa}{\delta_0} \right)^{\frac{1}{2}}.
\]

Hence, by Cramer’s rule,

\[
\text{Leb} \Sigma(L, \varepsilon) \leq \text{Ct.} \frac{\| \nabla_\rho L_{k,\kappa-1}(\mathcal{D}) \|_{C^j(D)}}{\delta_0} \left( \frac{\kappa}{\delta_0} \right)^{\frac{1}{2}} \leq \text{Ct.} \frac{N(\chi + \delta)}{\delta_0} \left( \frac{\kappa}{\delta_0} \right)^{\frac{1}{2}}.
\]

Summing up over all \( |k| \leq N \) gives the first estimate.

The second estimate follows from the mean value theorem and the bound

\[
|\nabla_\rho L_{k,[a]}(\rho)| \leq N(\chi + \delta).
\]

\[\square\]
Lemma 4.3. Let

\[ L_{k,[a],[b]} = (\langle k, \omega \rangle I - i \text{ad}_A)_{[a][b]} \]

There exists a constant \( C \) such that if \( \text{(3.3)} \) holds, then,

\[ \mathcal{U}(L_{k,[a],[b]}, \kappa) \leq C \eta \epsilon (\frac{\chi + \delta}{\delta_0})^{\alpha(\kappa)} \]

and

\[ \text{dist}(\mathcal{D} \setminus \Sigma(L_{k,[a],[b]}, \kappa), \Sigma(L_{k,[a],[b]}, \kappa/2)) > \frac{\kappa}{CN(\chi + \delta)} \]

for any \( \kappa > 0 \).

The exponent \( \alpha \) only depends on \( \frac{\delta_0}{C} \) and \( \#A \). The exponent \( \mu \) is a positive constant only depending on \( s_\ast, \frac{\delta_0}{C} \). \( C \) is an absolute constant that depends on \( c \) and \( \sup_{\mathcal{D}} |\omega| \).

Proof. Consider first \( a, b \in \mathcal{F} \). This case is treated as the operator \( L(\rho) = (\langle k, \omega \rangle I - iJH) \) in the previous lemma.

Consider then \( a \in \mathcal{L}_\infty \) and \( b \in \mathcal{F} \), so that

\[ L_{k,[a]}(\rho) : X \mapsto \langle k, \omega(\rho) \rangle X + Q_{[a]}(\rho)X + XJH(\rho). \]

Let

\[ L(\rho, \lambda) : X \mapsto \langle k, \omega(\rho) \rangle X + \lambda X + iXJH(\rho) \]

and

\[ P(\rho, \lambda) = \det L(\rho, \lambda). \]

Since \( L_{k,[a]}(\kappa) \) is “partially” Hermitian,

\[ ||L_{k,[a]}(\rho)^{-1}|| \leq \max ||L(\rho, \lambda(\rho))^{-1}||, \]

where the maximum is taken over all eigenvalues \( \lambda(\rho) \) of \( Q(\rho) \).

If we let

\[ L_{up}(\rho, \lambda) : X \mapsto \langle k, \omega(\rho) \rangle X + \lambda X + XJH_{up}(\rho), \]

then it follows, from \( \text{(3.9)} \) and \( \text{(4.1)} \), that

\[ ||L - L_{up}||_{C^{\ast}(\mathcal{D})} \leq \delta \leq \frac{\delta_0}{2}. \]

If \( L_{up} \) is \( \delta_0 \)-invertible, then this implies that \( L \) is \( \frac{\delta_0}{2} \)-invertible.

Otherwise

\[ \frac{d}{d_3}P(\rho, \lambda(\rho)) = \partial_3 P(\rho, \lambda(\rho)) + \partial_3 P(\rho, \lambda(\rho))(v(\rho), \partial_3 Q(\rho)v(\rho)) = \]

\[ = \partial_3 P_{up}(\rho, \lambda_u(\rho)) + \partial_3 P_{up}(\rho, \lambda_u(\rho))(v(\rho), \partial_3 Q_{up}(\rho)v(\rho)) + O(\delta||L||_{C^1(\mathcal{D})}^{m-1}). \]

By Assumption A2(ii) there exists a unit vector \( \mathbf{z} \) such that

\[ \left| \frac{d}{d_3}P(\rho, \lambda(\rho)) \right| \geq \frac{\delta_0}{2} ||L||_{C^1(\mathcal{D})}^{m-1}. \]

Hence, the Lebesgue measure of \( \Sigma(L_{k,[a]}, \kappa) \) is \( \lesssim d_\Delta \frac{\delta_0}{C} \) — recall that, by Remark \( \text{(3.2)} \), the operator is of dimension \( \lesssim d_\Delta \). (This argument is valid if \( \lambda(\rho) \) is \( C^1 \) in the direction \( \mathbf{z} \) which can always be assumed when \( Q \) is analytic in \( \rho \). The non-analytic case follows by analytical approximation.)
Let $\lambda(\rho)$ be an eigenvalue of $Q_{[a]}$. Since $|\langle k, \omega(\rho) \rangle| \lesssim |k| \lesssim N$, it follows, by (1.3), that

$$|\langle k, \omega(\rho) \rangle + \lambda(\rho)| \geq |\lambda(\rho)| - \delta - \text{Ct.}|k| \geq |a|^{\beta_1} - c|a|^{-\beta_2} - \delta - \text{Ct.}|k|$$

for some appropriate $a \in [a]$. Hence, $\Sigma(L_{k,[a]}, \kappa) = \emptyset$ for $|a| \gtrsim (N)^{\frac{\alpha}{\beta_1}}$.

Summing up over all $0 < |k| \leq N$ and all $|a| \lesssim (N)^{\frac{\alpha}{\beta_1}}$ gives the first estimate.

Consider finally $a, b \in L_\infty$. Then $L_{k,[a],[b]}$ decouples into a sum of four Hermitian operators of the forms

$$L_{k,[a],[b]}(\rho) : X \mapsto \langle k, \omega \rangle X + Q_{[a]}X + X^\dagger Q_{[b]}$$

and

$$L_{k,[a],[b]}(\rho) : X \mapsto \langle k, \omega \rangle X + Q_{[a]}X - XQ_{[b]}.$$  

The first one is treated exactly as the operator $X \mapsto \langle k, \omega \rangle X + Q_{[a]}X$ in the previous lemma, so let us concentrate on the second one. It follows as in the previous lemma that the Lebesgue measure of $\Sigma(L_{k,[a],[b]}, \kappa)$ is $\lesssim \ell_\Delta^{2d+\frac{\alpha}{\beta_1}}$ — recall that, by Remark 3.2, the operator is of dimension $\lesssim \ell_\Delta^{2d+\alpha}$.

The problem now is the measure estimate of $\bigcup \Sigma(L_{k,[a],[b]}, \kappa)$ since, a priori, there may be infinitely many $\Sigma(L_{k,[a],[b]}, \kappa)$ that are non-void. We can assume without restriction that $|a| \leq |b|$. Since $|\langle k, \omega(\rho) \rangle| \leq \text{Ct.}|k| \leq \text{Ct.}N$, it is enough to consider $|b| - |a| \leq \text{Ct.}N$ (because $\beta_1 \geq 1$).

Since $\beta_1 = 2$, $|a|^{\beta_1} - |b|^{\beta_1}$ is an integer, and outside a set $\Sigma(2\kappa')$ of Lebesgue measure

$$\lesssim \text{Ct.}N^{\kappa'}$$

we have

$$|\langle k, \omega(\rho) \rangle + |a|^{\beta_1} - |b|^{\beta_1}| \geq 2\kappa'.$$

Then, by (1.4),

$$|\langle k, \omega(\rho) \rangle + \alpha(\rho) - \beta(\rho)| \geq 2\kappa' - 2\frac{\delta}{(a)^{\kappa'}} - 2\frac{c'c}{(a)^{\beta_3}}$$

for any $\alpha(\rho)$ and $\beta(\rho)$, eigenvalues of $Q_{[a]}(\rho)$ and $Q_{[b]}(\rho)$, respectively. Now this is $\geq \kappa'$ unless

$$|a| \leq \text{Ct.} \min \left( \left( \frac{\delta}{\kappa'} \right)^{\frac{1}{\beta_1}}, \left( \frac{c'c}{\kappa'} \right)^{\frac{1}{\beta_3}} \right) =: M.$$  

Hence, if $\kappa' \geq \kappa$, then

$$\bigcup_{[a],[b],k} \Sigma(L_{k,[a],[b]}, \kappa) \subset \Sigma(2\kappa') \cup \bigcup_{|a|,|b| \leq M + \text{Ct.}N,k} \Sigma(L_{k,[a],[b]}, \kappa).$$

This set has measure

$$\lesssim N^{\kappa'} \frac{\delta_0}{\delta_0} + \left( N^{\frac{\delta_0}{\kappa'}} \right)^{2d} \max \left( \frac{\delta_0}{\beta_3}, \frac{\delta_0}{\alpha} \right)^{\frac{\kappa'}{\delta_0}}.$$

By an appropriate choice of $\kappa' \in [\kappa, \delta_0]$, this becomes

$$\leq \text{Ct.}N^{\exp \left( \frac{\kappa}{\delta_0} \right)^{\alpha}}$$

for some $\alpha > 0$ depending on $\frac{\delta_0}{\beta_3}, \frac{\delta_0}{\alpha}$.

\qed
5. Homological equation

Let \( h \) be a normal form Hamiltonian (3.7),
\[
h(r, w, \rho) = \langle \omega(\rho), r \rangle + \frac{1}{2} \langle w, A(\rho)w \rangle \in N\mathcal{F}_\kappa(\Delta, \delta)
\]
and assume \( \kappa > 0 \) and
\[
\delta \leq \frac{1}{C} c',
\]
where \( C \) is to be determined. Let
\[
\gamma = (\gamma, m_\star) \geq \gamma_\star = (0, m_\star).
\]

Remark 5.1. Notice the abuse of notations here. It will be clear from the context when \( \gamma \) is a two-vector, like in \( \| \cdot \|_\gamma, \kappa \), and when it is a scalar, like in \( e^{\gamma d} \).

Let \( f \in T_{\gamma, \kappa, D}(\sigma, \mu) \). In this section we shall construct a jet-function \( S \) that solves the non-linear homological equation
\[
\{ h, S \} + \{ f - f^T, S \}^T + f^T = 0
\]
as good as possible — the reason for this will be explained in the beginning of the next section. In order to do this we shall start by analyzing the homological equation
\[
\{ h, S \} + f^T = 0.
\]
We shall solve this equation modulo some “cokernel” and modulo an “error”.

5.1. Three components of the homological equation. Let us write
\[
f^T(\theta, r, w) = f_r(r, \theta) + \langle f_w(\theta), w \rangle + \frac{1}{2} \langle f_{ww}(\theta)w, w \rangle
\]
and recall that, by Proposition 2.4, \( f^T \in T_{\gamma, \kappa, D}(\sigma, \mu) \). Let
\[
S(\theta, r, w) = S_r(r, \theta) + \langle S_w(\theta), w \rangle + \frac{1}{2} \langle S_{ww}(\theta)w, w \rangle,
\]
where \( f_r \) and \( S_r \) are affine functions in \( r \) — here we have not indicated the dependence on \( \rho \).

Then the Poisson bracket \( \{ h, S \} \) equals
\[
- (\partial_\omega S_r(r, \theta) + \langle \partial_\omega S_w(\theta), w \rangle + \frac{1}{2} \langle \partial_\omega S_{ww}(\theta), w \rangle + \\
+ \langle AJS_w(\theta), w \rangle + \frac{1}{2} \langle AJS_{ww}(\theta)w, w \rangle - \frac{1}{2} \langle S_{ww}(\theta)JA w, w \rangle
\]
where \( \partial_\omega \) denotes the derivative of the angles \( \theta \) in direction \( \omega \). Accordingly the homological equation (5.3) decomposes into three linear equations:
\[
\begin{align*}
\partial_\omega S_r(r, \theta) &= f_r(r, \theta), \\
\partial_\omega S_w(\theta) - AJS_w(\theta) &= f_w(\theta), \\
\partial_\omega S_{ww}(\theta) - AJS_{ww}(\theta) + S_{ww}(\theta)JA &= f_{ww}(\theta).
\end{align*}
\]
5.2. The first equation.

**Lemma 5.2.** There exists constant \( C \) such that if \((6.3)\) holds, then, for any \( N \geq 1 \) and \( \kappa > 0 \), there exists a closed set \( D_1 = D_1(h, \kappa, N) \subset D \), satisfying

\[
\text{Leb}(D \setminus D_1) \leq CN^{\exp \frac{\chi + \frac{\delta}{\kappa}}{\delta_0}}
\]

and there exist \( C^* \) functions \( S_r \) and \( R_r \) on \( C^A \times T^A \times D \rightarrow \mathbb{C} \), real holomorphic in \( r, \theta \), such that for all \( \rho \in D_1 \)

\[
(5.4) \quad \partial_\omega S_r(r, \theta, \rho) = f_r(r, \theta, \rho) - \hat{f}_r(r, 0, \rho) - R_r(\theta, \rho)
\]

and for all \((r, \theta, \rho) \in C^A \times T^A \times D \), \(|r| < \mu, \sigma' < \sigma\), and \(|j| \leq s_\ast\)

\[
(5.5) \quad |\partial_\rho^j S_r(r, \theta, \rho)| \leq C \left( \frac{1}{\kappa} \frac{|\sigma - \sigma'|^n}{|\kappa|} \right) |f^T|_{\gamma, \kappa, \nu, D},
\]

\[
(5.6) \quad |\partial_\rho^j R_r(r, \theta, \rho)| \leq C \exp |\sigma - \sigma'|^n |f^T|_{\gamma, \kappa, \nu, D}.
\]

The exponent \( \exp \) only depends on \( n = |A| \), and \( C \) is an absolute constant.

**Proof.** Written in Fourier components the equation \((5.4)\) then becomes, for \( k \in \mathbb{Z}^A \),

\[
L_k(\rho) \hat{S}(k) = (k, \omega(\rho)) \hat{S}(k) = -i(\hat{F}(k) - \hat{R}(k))
\]

where we have written \( S, F \) and \( R \) for \( S_r, (f_r - \hat{f}_r) \) and \( R_r \) respectively. Therefore \((5.4)\) has the (formal) solution

\[
S(r, \theta, \rho) = \sum \hat{S}(r, k, \rho) e^{i(k, \theta)} \quad \text{and} \quad R(r, \theta, \rho) = \sum \hat{F}(r, k, \rho) e^{i(k, \theta)}
\]

with

\[
\hat{S}(r, k, \rho) = \begin{cases} -L_k(\rho)^{-1} i \hat{F}(r, k, \rho) & \text{if } 0 < |k| \leq N \\ 0 & \text{if not} \end{cases}
\]

and

\[
\hat{R}(r, k, \rho) = \begin{cases} \hat{F}_0(r, k, \rho) & \text{if } |k| > N \\ 0 & \text{if not} \end{cases}
\]

By Lemma \(4.1\)

\[
\|((L_k(\rho))^{-1})\| \leq 1/\kappa
\]

for all \( \rho \) outside some set \( \Sigma(L_k, \kappa) \) such that

\[
\text{dist}(D \setminus \Sigma(L_k, \kappa), \Sigma(L_k, \frac{\kappa}{2})) \geq c, \frac{\kappa}{N(\chi + \delta)}
\]

and

\[
D_1 = D \setminus \bigcup_{0 < |k| \leq N} \Sigma(L_k, \kappa)
\]

fulfills the estimate of the lemma. For \( \rho \notin \Sigma(L_k, \frac{\kappa}{2}) \) we get

\[
|\hat{S}(r, k, \rho)| \leq C, \frac{1}{\kappa} |\hat{F}(r, k, \rho)|.
\]

Differentiating the formula for \( \hat{S}(r, k, \rho) \) once we obtain

\[
\partial_\rho^2 \hat{S}(r, k, \rho) = \left( -\frac{i}{(\omega, k)} \partial_\rho^1 \hat{F}(r, k, \rho) + \frac{i}{(\omega, k)^2} (\partial_\rho^0 \omega, k) \hat{F}(r, k, \rho) \right)
\]

10 \( f_r(r, 0, \rho) \) is the 0:th Fourier coefficient, or the mean value, of the function \( \theta \mapsto f_r(r, \theta, \rho) \)
which gives, for \( \rho \notin \Sigma(L_k, \frac{\kappa}{2}) \),
\[
|\partial^j_\rho \hat{S}(r, k, \rho)| \leq C t \left(\frac{N \chi + \delta}{\kappa}\right) \max_{0 \leq i \leq j} |\partial_i \hat{F}(r, k, \rho)|.
\]
(Here we used that \( |\partial_\rho \omega(\rho)| \leq \chi + \delta \).) The higher order derivatives are estimated in the same way and this gives
\[
|\partial^j_\rho \hat{S}(r, k, \rho)| \leq C t \left(\frac{N \chi + \delta}{\kappa}\right)^{|j|} \max_{0 \leq i \leq j} |\partial_i \hat{F}(r, k, \rho)|
\]
for any \(|j| \leq s_*\), where \(Ct\) is an absolute constant.

By Lemma A.2 there exists a \( C^\infty \)-function \( g_k : D \to \mathbb{R} \), being \( 1 \) outside \( \Sigma(L_k, \kappa) \) and \( 0 \) on \( \Sigma(L_k, \frac{\kappa}{2}) \) and such that for all \( j \geq 0 \)
\[
|g_k|_{C^j(D)} \leq (Ct \left(\frac{N \chi + \delta}{\kappa}\right))^j.
\]
Multiplying \( \hat{S}(r, k, \rho) \) with \( g_k(\rho) \) gives a \( C^* \)-extension of \( \hat{S}(r, k, \rho) \) from \( D \setminus \Sigma(L_k, \kappa) \) to \( D \) satisfying the same bound as \( \hat{S}(r, k, \rho) \).

It now follows, by a classical argument, that the formal solution converges and that \( |\partial_\rho S(r, \theta, \rho)| \) and \( |\partial_\rho R(r, \theta, \rho)| \) fulfills the estimates of the lemma. When summing up the series for \( |\partial_\rho R(r, \theta, \rho)| \) we get a term \( e^{-t \hat{\omega}(\rho - \sigma)} \), but the factor \( \frac{1}{t} \) disappears by replacing \( N \) by \( CN \).

By construction \( S \) and \( R \) solve equation (5.4) for any \( \rho \in D_1 \). \( \square \)

5.3. The second equation. Concerning the second component of the homological equation we have

**Lemma 5.3.** There exists an absolut constant \( C \) such that if (5.4) holds, then, for any \( N \geq 1 \) and
\[
0 < \kappa \leq \kappa',
\]
there exists a closed set \( D_2 = D_2(h, \kappa, N) \subset D \), satisfying
\[
\text{Leb}(D \setminus D_2) \leq CN^{\exp\left(\frac{\chi + \delta}{\delta_0}\right)}(\kappa, \frac{\kappa}{\delta_0}),
\]
and there exist \( C^* \)-functions \( S_w \) and \( R_w \) on \( T^\mathcal{A} \times D \to Y_\gamma \), real holomorphic in \( \theta \), such that for \( \rho \in D_2 \)
\[
\partial_{\phi(\rho)} S_w(\theta, \rho) - A(\rho) J S_w(\theta, \rho) = f_w(\theta, \rho) - R_w(\theta, \rho)
\]
and for all \( (\theta, \rho) \in T^\mathcal{A} \times D \), \( \sigma' < \sigma \), and \( |j| \leq s_* \)
\[
||\partial_\rho^j S_w(\theta, \rho)||_{\gamma} \leq C \frac{1}{\kappa(\sigma - \sigma')} \left(\frac{N \chi + \delta}{\kappa}\right)^{|j|} |f_T|_{\gamma, \kappa, D}.
\]
\[
||\partial_\rho^j R_w(\theta, \rho)||_{\gamma} \leq C e^{-t \hat{\omega}(\rho - \sigma')} \left(\frac{N \chi + \delta}{\kappa}\right)^{|j|} |f_T|_{\gamma, \kappa, D}.
\]
The exponent \( \exp \) only depends on \( \frac{\kappa}{\delta_1} \) and \( n = \#A \), and \( C \) is an absolut constant that depends on \( c \) and \( \sup |\omega| \).

**Proof.** Let us re-write (5.7) in the complex variables \( \xi \) and \( \eta \) described in section 3.2. The quadratic form \( (1/2)\langle w, A(\rho)w \rangle \) gets transformed, by \( w = Uz \), to
\[
\langle \xi, Q(\rho)\eta \rangle + \frac{1}{2} \langle z, H'(\rho)z \rangle,
\]
where \( Q' \) is a Hermitian matrix and \( H' \) is a real symmetric matrix. Then we make in 5.7 the substitution \( S = US^\omega, \quad R = U^\omega R^w \) and \( F = tU_f^w, \) where \( S = t(S^\omega, S^\gamma, S^\xi), \) etc. In this notation eq. 5.7 decouples into the equations
\[
\begin{align*}
\partial_\omega S^\xi + i Q S^\xi &= F^\xi - R^\xi, \\
\partial_\omega S^\gamma - i Q S^\gamma &= F^\gamma - R^\gamma, \\
\partial_\omega S^\xi - HJS^\xi &= F^\xi - R^\xi.
\end{align*}
\]

Let us consider the first equation. Written in the Fourier components it becomes
\[
(k, \omega(\rho)) I + Q) \hat{S}^\xi(k) = -i(\hat{F}^\xi(k) - \hat{R}^\xi(k)).
\]
This equation decomposes into its “components” over the blocks \( [a] = [a]_\Delta \) and takes the form
\[
L_{k, [a]}(\rho) \hat{S}_{[a]}(k) =: ((k, \omega(\rho)) + Q_{[a]}) \hat{S}_{[a]}(k) = -i(\hat{F}_{[a]}(k) - \hat{R}_{[a]}(k))
\]
- the matrix \( Q_{[a]} \) being the restriction of \( Q^\xi \) to \( [a] \times [a], \) the vector \( F_{[a]} \) being is the restriction of \( F^\xi \) to \( [a] \) etc.

Equation (5.10) has the (formal) solution
\[
\hat{S}_{[a]}(k, \rho) = \begin{cases} \frac{-i(L_{k,[a]}(\rho))^{-1} \hat{F}_{[a]}(k, \rho)}{0} & \text{if } |k| \leq N \\ 0 & \text{if not.} \end{cases}
\]
and
\[
\hat{R}_{[a]}(k, \rho) = \begin{cases} \hat{F}_{[a]}(k, \rho) & \text{if } |k| > N \\ 0 & \text{if not.} \end{cases}
\]
For \( k \neq 0, \) by Lemma 4.2
\[
\|(L_{k,[a]}(\rho))^{-1}\| \leq \frac{1}{\kappa}
\]
for all \( \rho \) outside some set \( \Sigma(L_{k,[a]}, \kappa) \) such that
\[
\text{dist}(\mathcal{D} \setminus \Sigma(L_{k,[a]}, \kappa), \Sigma(L_{k,[a]}, \frac{\kappa}{2})) \geq c \frac{\kappa}{N(\chi + \delta)}
\]
and
\[
\mathcal{D}_2 = \mathcal{D} \setminus \bigcup_{0 < |k| \leq N} \Sigma_{k,[a]}(\kappa),
\]
fulfills the required estimate. For \( k = 0, \) it follows by 6.4 and 1.5 that
\[
\|(L_{k,[a]}(\rho))^{-1}\| \leq \frac{1}{c} \leq \frac{1}{\kappa}.
\]

We then get, as in the proof of Lemma 5.2, that \( \hat{S}_{[a]}(k, \cdot) \) and \( \hat{R}_{[a]}(k, \cdot) \) have \( C^\ast \)-extension to \( \mathcal{D} \) satisfying
\[
\|\partial_\rho^j \hat{S}_{[a]}(k, \rho)\| \leq C t \left( \frac{N \chi + \delta}{\kappa} \right)^{|j|} \max_{0 \leq i \leq j} \|\partial_\rho^i \hat{F}_{[a]}(k, \rho)\|
\]
and
\[
\|\partial_\rho^j \hat{R}_{[a]}(k, \rho)\| \leq C t \|\partial_\rho^j \hat{F}_{[a]}(k, \rho)\|,
\]
and satisfying (5.11) for \( \rho \in \mathcal{D}_2. \)

These estimates imply that
\[
\|\partial_\rho^j \hat{S}^\xi(k, \rho)\|_\gamma \leq C t \left( \frac{N \chi + \delta}{\kappa} \right)^{|j|} \max_{0 \leq i \leq j} \|\partial_\rho^i \hat{F}^\xi(k, \rho)\|_\gamma
\]
and
\[ \| \partial_\rho^j R^\xi (k, \rho) \| \leq C_t \| \partial_\rho^j F^\xi (k, \rho) \|_\gamma. \]

Summing up the Fourier series, as in Lemma 5.2, we get
\[ \| \partial_\rho^j S^\xi (\theta, \rho) \| \leq C_t \frac{1}{\kappa (\sigma - \sigma')^n} \left( N \frac{\chi + \delta}{\kappa} \right) |j| \max_{0 \leq |\theta| < \sigma} \| \partial_\rho^j F^\xi (\cdot, \rho) \|_\gamma \]
and
\[ \| \partial_\rho^j R^\xi (\theta, \rho) \|_\gamma \leq C_t \frac{e^{- \frac{1}{C_0} (\sigma - \sigma')^n}}{(\sigma - \sigma')^n} \sup_{|\theta| < \sigma} \| \partial_\rho^j F^\xi (\cdot, \rho) \|_\gamma \]
for \((\theta, \rho) \in T^A_0 \times D, 0 < \sigma' < \sigma, \) and \(|j| \leq s_*\). This implies the estimates (5.8) and (5.9) — the factor \(\frac{1}{C_0}\) disappears by replacing \(N\) by \(C_t N\).

The other two equations are treated in exactly the same way.

5.4. The third equation. Concerning the third component of the homological equation, (5.3), we have the following result, where for a solution \(S_{ww}(\theta, \rho)\) we estimate separately its mean-value \(\bar{S}_{ww}(0, \rho)\) and the deviation from the mean-value \(S_{ww}(\theta, \rho) - \bar{S}_{ww}(0, \rho)\).

**Lemma 5.4.** There exists an absolute constant \(C\) such that if (5.24) holds, then, for any \(N \geq 1, \Delta' \geq \Delta \geq 1, \)
\[ \kappa \leq \frac{1}{C'}, \]
there exist subsets \(D_3 = D_3(h, \kappa, N, \Delta') \subset D,\) satisfying
\[ \text{Leb}(D \setminus D_3) \leq CN^\exp \left( \frac{\chi + \delta}{\delta_0} \right) \left( \frac{\kappa}{\delta_0} \right)^n \]
and there exist real \(C^*\)-functions \(B_{ww} : D \to M_{\gamma, \kappa} \cap N^{F \Delta'}\) and \(S_{ww}, R_{ww} : T^A \times D \to M_{\gamma, \kappa},\) real holomorphic in \(\theta,\) such that for all \(\rho \in D_3\)
\[ (5.12) \quad \partial_\nu_S S_{ww}(\theta, \rho) = A(\rho) JS_{ww}(\theta, \rho) + S_{ww}(\theta, \rho) J A(\rho) = f_{ww}(\theta, \rho) - B_{ww}(\theta, \rho) - R_{ww}(\theta, \rho) \]
and for all \((\theta, \rho) \in T^A_0 \times D, \sigma' < \sigma, \) and \(|j| \leq s_*\)
\[ (5.13) \quad \| \partial_\rho^j S_{ww}(\theta, \rho) \|_{\gamma', \kappa} \leq C \Delta \Delta^{\exp 2} e^{2 \gamma d_\Delta} \frac{(N \frac{\chi + \delta}{\kappa}) |j|}{f_T_{\gamma', \kappa, D}}, \]
\[ (5.14) \quad \| \partial_\rho^j R_{ww}(\theta, \rho) \|_{\gamma', \kappa} \leq C \Delta' \Delta^{\exp 2} \left( \frac{e^{-(\sigma - \sigma')} N + e^{-(\gamma - \gamma') \Delta'}}{(\sigma - \sigma')^n} \right) |f_T_{\gamma', \kappa, D'}|, \]
\[ (5.15) \quad \| \partial_\rho^j B_{ww}(\rho) \|_{\gamma', \kappa} \leq C \Delta' \Delta^{\exp 2} |f_T|_{\gamma', \kappa, D'} \]
for any \(\gamma_* \leq \gamma' \leq \gamma.\)

The exponent \(\exp\) only depends on \(d_\kappa\) and \(n = \# A.\) The exponent \(\exp\) only depends on \(d_*, m_*\). The exponent \(\alpha\) is a positive constant only depending on \(s_*\) and \(\frac{d_\kappa}{\gamma},\) \(\frac{d_\kappa}{\gamma'}\) and \(C\) is an absolute constant that depends on \(c\) and \(\sup_D |\omega|\).
\textbf{Proof.} It is also enough to find complex solutions $S_{ww}$, $R_{ww}$ and $B_{ww}$ verifying the estimates, because then their real parts will do the job.

As in the previous section, and using the same notation, we re-write (5.12) in complex variables. So we introduce $S = \mathcal{U}S_{\zeta}U$, $R = \mathcal{U}R_{\zeta}U$, $B = \mathcal{U}B_{\zeta}U$ and $F = \mathcal{U}Jf_{\zeta}U$. In appropriate notations (5.12) decouples into the equations

$$
\partial_\omega S^{\xi\xi} + iQS^{\xi\xi} + iS^{\xi\xi}tQ = F^{\xi\xi} - B^{\xi\xi} - R^{\xi\xi},
$$

$$
\partial_\omega S^{\xi\eta} + iQS^{\xi\eta} - iS^{\xi\eta}Q = F^{\xi\eta} - B^{\xi\eta} - R^{\xi\eta},
$$

$$
\partial_\omega S^{\xi\zeta} + iQS^{\xi\zeta} + S^{\xi\zeta}JH = F^{\xi\zeta} - B^{\xi\zeta} - R^{\xi\zeta},
$$

$$
\partial_\omega S^{\xi\zeta\zeta} + HJS^{\xi\zeta\zeta} - S^{\xi\zeta\zeta}JH = F^{\xi\zeta\zeta} - B^{\xi\zeta\zeta} - R^{\xi\zeta\zeta},
$$

and equations for $S^{\eta\eta}, S^{\eta\zeta}, S^{\zeta\zeta}, S^{\zeta\eta\eta}, S^{\zeta\zeta\zeta}$. Since those latter equations are of the same type as the first four, we shall concentrate on these first.

\textit{First equation.} Written in the Fourier components it becomes

$$
\langle k, \omega(\rho) \rangle I + Q)\hat{S}^{\xi\xi}(k) + \hat{S}^{\xi\xi}(k)^4Q = -i(\hat{F}^{\xi\xi}(k) - \delta_{k,0}B - \check{R}^{\xi\xi}(k)).
$$

This equation decomposes into its “components” over the blocks $[a] \times [b]$, $[a] = [a]_\Delta$, and takes the form

$$
L(k, [a], [b], \rho)\hat{S}^{[b]}_{[a]}(k) = \langle k, \omega(\rho) \rangle \hat{S}^{[b]}_{[a]}(k) + Q_{[a](\rho)}\hat{S}^{[b]}_{[a]}(k) + \hat{S}^{[b]}_{[a]}(k)^4Q_{[a]}(\rho) = -i(\hat{F}^{[b]}_{[a]}(k, \rho) - \check{R}^{[b]}_{[a]}(k) - \delta_{k,0}B^{[b]}_{[a]}),
$$

the matrix $Q_{[a]}$ being the restriction of $Q^{\xi\xi}$ to $[a] \times [a]$, the vector $F^{[b]}_{[a]}$ being the restriction of $F^{\xi\xi}$ to $[a] \times [b]$ etc.

Equation (5.10) has the (formal) solution

$$
\hat{S}^{[b]}_{[a]}(k, \rho) = \begin{cases} 
-L(k, [a], [b], \rho)^{-1}i\hat{F}^{[b]}_{[a]}(k, \rho) & \text{if } \text{dist}([a], [b]) \leq \Delta' \text{ and } |k| \leq N \\
0 & \text{if not,}
\end{cases}
$$

$$
\check{R}^{[b]}_{[a]}(k, \rho) = \begin{cases} 
\hat{F}^{[b]}_{[a]}(k, \rho) & \text{if } \text{dist}([a], [b]) \geq \Delta' \text{ or } |k| > N \\
0 & \text{if not.}
\end{cases}
$$

For $k \neq 0$, by Lemma 1.3,

$$
||L_{[a], [b]}(\rho)||^{-1} \leq \frac{1}{\kappa}
$$

for all $\rho$ outside some set $\Sigma_{k,[a],[b]}(\kappa)$ such that

$$
\text{dist}(D \setminus \Sigma_{k,[a],[b]}(\kappa), \Sigma_{k,[a],[b]}(\frac{K}{2})) \geq c_{t} \frac{K}{N(\chi + \delta)},
$$

and

$$
D_3 = D \setminus \bigcup_{0 < |k| \leq N} \Sigma_{k,[a],[b]}(\kappa)
$$

fulfills the required estimate. For $k = 0$, it follows by 6.4 and 1.5 that

$$
||L_{[a], [b]}(\rho)||^{-1} \leq \frac{1}{c_t} \leq \frac{1}{\kappa}.
$$
We then get, as in the proof of Lemma 5.2 that $S^{[b]}_{[a]}(k, \cdot)$ and $R^{[b]}_{[a]}(k, \cdot)$ have $C^\infty$-extension to $D$ satisfying
\[
\| {\partial^j\hat{S}^{[b]}_{[a]} (k, \rho) } \| \leq C_t \left( N \frac{\chi + \delta}{\kappa} \right)^{|j|} \max_{0 \leq l \leq j} \| {\partial^l \hat{F}^b_{[a]} (k, \rho) } \|
\]
and
\[
\| {\partial^j \hat{R}^b_{[a]} (k, \rho) } \| \leq C_t \| {\partial^j \hat{F}^b_{[a]} (k, \rho) } \|
\]
and satisfying (5.17) for $\rho \in D_3$.

These estimates imply that, for any $\gamma_* \leq \gamma' \leq \gamma$,
\[
\| {\partial^j \hat{S}^\xi \xi \xi_{[a]} (k, \rho) } \|_{B(Y'_r, Y'_r)} \leq C_t \Delta'_\exp \left( N \frac{\chi + \delta}{\kappa} \right)^{|j|} \max_{0 \leq l \leq j} \| {\partial^l \hat{F}^\xi \xi \xi_{[a]} (k, \rho) } \|_{B(Y'_r, Y'_r)}
\]
and
\[
\| {\partial^j \hat{R}^\xi \xi \xi_{[a]} (k, \rho) } \|_{B(Y'_r, Y'_r)} \leq C_t \Delta'\Delta'_\exp \| {\partial^j \hat{F}^\xi \xi \xi_{[a]} (k, \rho) } \|_{B(Y'_r, Y'_r)}.
\]
The factor $\Delta'_\exp e^{2\nu d_\Delta}$ occurs because the diameter of the blocks $\leq d_\Delta$ interferes with the exponential decay and influences the equivalence between the $l_1$-norm and the operator-norm. The factor $\Delta'\Delta'_\exp$ occurs because the truncation $\lesssim \Delta' + d_\Delta$ of diagonal influences the equivalence between the sup-norm and the operator-norm.

These estimates gives estimates for the matrix norms and, for any $\gamma_* \leq \gamma' \leq \gamma$,
\[
\| {\partial^j \hat{S}^\xi \xi \xi_{[a]} (k, \rho) } \|_{\gamma', \kappa} \leq C_t \left( N \frac{\chi + \delta}{\kappa} \right)^{|j|} \max_{0 \leq l \leq j} \| {\partial^l \hat{F}^\xi \xi \xi_{[a]} (k, \rho) } \|_{\gamma', \kappa}
\]
and
\[
\| {\partial^j \hat{R}^\xi \xi \xi_{[a]} (k, \rho) } \|_{\gamma', \kappa} \leq C_t \| {\partial^j \hat{F}^\xi \xi \xi_{[a]} (k, \rho) } \|_{\gamma', \kappa}.
\]

Summing up the Fourier series, as in Lemma 5.3 we get that $S^\xi \xi \xi (\theta, \rho)$ satisfies the estimate (5.17) and that $R^\xi \xi \xi (\theta, \rho)$ satisfies the estimate (5.14).

The third equation. We write the equation in Fourier components and decompose it into its “components” on each product block $[a] \times [b]$, $[b] = F$:
\[
L(k, [a], [b], \rho) \hat{S}^{[b]}_{[a]}(k, \omega) = \langle k, \omega(\rho) \rangle \hat{S}^{[b]}_{[a]}(k) + Q_{[a]}(\rho) \hat{S}^{[b]}_{[a]}(k) - \hat{1} \Re \hat{S}^{[b]}_{[a]}(k)JH(\rho) = -\hat{1} \Re \hat{F}^{[b]}_{[a]}(k, \rho) - \delta_{k, 0} B^{[b]}_{[a]} - \hat{R}^{[b]}_{[a]}(k))
\]
– here we have suppressed the upper index $\xi_z F$.

The formal solution is the same as in the previous case and it converges to functions verifying (5.13), (5.14) and (5.15), by Lemma 4.3 and by (1.6).

The fourth equation. We write the equation in Fourier components:
\[
L(k, [a], [b], \rho) \hat{S}^{[b]}_{[a]}(k, \omega) = \langle k, \omega(\rho) \rangle \hat{S}^{[b]}_{[a]}(k) - \Re \hat{1} HJ(\rho) \hat{S}^{[b]}_{[a]}(k) + \hat{1} \Re \hat{S}^{[b]}_{[a]}(k)JH(\rho) = -\hat{1} \Re \hat{F}^{[b]}_{[a]}(k, \rho) - \delta_{k, 0} B^{[b]}_{[a]} - \hat{R}^{[b]}_{[a]}(k)),
\]
where $[a] = [b] = F$ – here we have suppressed the upper index $\xi_z F$.

The equation is solved (formally) by
\[
\hat{S}^{[b]}_{[a]}(k, \rho) = \begin{cases} -L(k, [a], [b], \rho)^{-1} \Re \hat{F}^{[b]}_{[a]}(k, \rho) & \text{if } 0 < |k| \leq N \\ 0 & \text{if not}, \end{cases}
\]
\[
\hat{R}^{[b]}_{[a]}(k, \rho) = \begin{cases} \hat{F}^{[b]}_{[a]}(k, \rho) & \text{if } |k| > N \\ 0 & \text{if not}; \end{cases}
\]
and

\[ B_{[a]}^{[b]}(\rho) = \tilde{F}_{[a]}^{[b]}(0, \rho). \]

The formal solution now converges a solution verifying (6.13, 6.14) and (6.15) by Lemma 4.3.

The second equation. We write the equation in Fourier components and decom- 
pose it into its “components” on each product block \([a] \times [b]::

\[
L(k, [a], [b], \rho) \tilde{S}_{[a]}^{[b]}(k) =: \langle k, \omega(\rho) \rangle \tilde{S}_{[a]}^{[b]}(k) + Q_{[a]}(\rho)\tilde{S}_{[a]}^{[b]}(k) -
\tilde{S}_{[a]}^{[b]}(k)Q_{[b]}(\rho) = -i(\tilde{F}_{[a]}^{[b]}(k, \rho) - \tilde{R}_{[a]}^{[b]}(k) - \delta_{k,0}B_{[a]}^{[b]})
\]

– here we have suppressed the upper index \(\xi\eta\). This equation is now solved (formally) by

\[
\tilde{S}_{[a]}^{[b]}(\theta, \rho) = \sum \tilde{S}_{[a]}^{[b]}(k, \rho)e^{ik\theta} \quad \text{and} \quad \tilde{R}_{[a]}^{[b]}(\theta, \rho) = \sum \tilde{R}_{[a]}^{[b]}(k, \rho)e^{ik\theta},
\]

with

\[
\tilde{S}_{[a]}^{[b]}(k, \rho) = \begin{cases} 
L(k, [a], [b], \rho)^{-1}i\tilde{F}_{[a]}^{[b]}(k, \rho) & \text{if } \text{dist}([a], [b]) \leq \Delta' \text{ and } 0 < |k| \leq N \\
0 & \text{if not},
\end{cases}
\]

\[
\tilde{R}_{[a]}^{[b]}(k, \rho) = \begin{cases} 
\tilde{F}_{[a]}^{[b]}(k, \rho) & \text{if } \text{dist}([a], [b]) \geq \Delta' \text{ or } |k| > N \\
0 & \text{if not}.
\end{cases}
\]

and

\[
B_{[a]}^{[b]}(\rho) = \begin{cases} 
\tilde{F}_{[a]}^{[b]}(0, \rho) & \text{if } \text{dist}([a], [b]) \leq \Delta' \text{ and } k = 0 \\
0 & \text{if not}.
\end{cases}
\]

We have to distinguish two cases, depending on when \(k = 0\) or not.

The case \(k \neq 0\).

We have, by Lemma 4.3,

\[
||(L_{k,[a],[b]}(\rho))^{-1}|| \leq \frac{1}{\kappa}
\]

for all \(\rho\) outside some set \(\Sigma_{k,[a],[b]}(\kappa)\) such that

\[
\text{dist}(D \setminus \Sigma_{k,[a],[b]}(\kappa), \Sigma_{k,[a],[b]}(\frac{\kappa}{2})) \geq \text{ct.} \frac{\kappa}{N(\chi + \delta)}.
\]

and

\[
\mathcal{D}_3 = D \setminus \bigcup_{0 < |k| \leq N} \Sigma_{k,[a],[b]}(\kappa)
\]

fulfills the required estimate.

The case \(k = 0\). In this case we consider the block decomposition \(\mathcal{E}_{\Delta'},\) and we 
distinguish whether \(|a| = |b|\) or not.

If \(|a| > |b|\), we use (6.4) and (1.6) to get

\[
|\alpha(\rho) - \beta(\rho)| \geq c' - \frac{\delta}{(a)^{\alpha}} - \frac{\delta}{(b)^{\alpha}} \geq c' - \frac{\delta}{2} \geq \kappa.
\]

This estimate allows us to solve the equation by choosing

\[
B_{[a]}^{[b]} = \tilde{R}_{[a]}^{[b]}(0) = 0
\]
and
\[ S_{[a]}^{[b]}(0, \rho) = L(0, [a], [b], \rho)^{-1} \hat{F}_{[a]}^{[b]}(0, \rho) \]
with
\[ \| \partial^j \rho S_{[a]}^{[b]}(0, \rho) \| \leq C t \left( N \chi + \delta \right)^{j|} \max_{0 \leq i \leq j} \| \partial^i \rho \hat{F}_{[a]}^{[b]}(0, \rho) \|, \]
which implies (5.13).

If \(|a| = |b|\), we cannot control \(|\alpha(\rho) - \beta(\rho)|\) from below, so then we define
\[ S_{[a]}^{[b]}(0) = 0 \]
and
\[ B_a^b(\rho) = \hat{F}_a^b(0, \rho), \quad \hat{R}_a^b(0) = 0 \quad \text{for } [a]_{\Delta'} = [b]_{\Delta'}. \]
Clearly \(R\) and \(B\) verify the estimates (5.14) and (5.15).

Hence, the formal solution converges to functions verifying (5.13) and (5.14), Moreover, for \(\rho \in \mathcal{D}'\), these functions are a solution of the fourth equation.

\[ \square \]

5.5. The homological equation.

**Lemma 5.5.** There exists a constant \(C\) such that if (6.4) holds, then, for any \(N \geq 1, \Delta' \geq \Delta \geq 1\) and
\[ \kappa \leq \frac{1}{C \varepsilon}, \]
there exist subsets \(\mathcal{D}' = \mathcal{D}(h, \kappa, N) \subset \mathcal{D}\), satisfying
\[ \text{Leb}(\mathcal{D} \setminus \mathcal{D}') \leq C N^{\exp \left( \frac{\chi + \delta}{\delta_0} \right) \left( \frac{\kappa}{\delta_0} \right)^N} \]
and there exist real jet-functions \(S, R \in \mathcal{T}_{\gamma, \kappa, \mathcal{D}}(\sigma, \mu)\) and \(h_+\) verifying, for \(\rho \in \mathcal{D}'\),
\begin{align*}
\{h, S\} + f^T &= h_+ + R, \\
\text{and such that} \\
&h + h_+ \in \mathcal{N}\mathcal{F}_{\gamma'}(\Delta', \delta_+) \end{align*}
and, for all \(0 < \sigma' < \sigma\),
\begin{align*}
|h_+|_{\sigma, \mu} &\leq X \left| f^T \right|_{\sigma, \mu} \\
|S|_{\sigma, \mu} &\leq \frac{1}{\kappa} X \left( N \chi + \frac{\delta}{\kappa} \right)^{\kappa^*} \left| f^T \right|_{\sigma, \mu} \\
|R|_{\sigma, \mu} &\leq X \left( e^{-(\sigma - \sigma')N} + e^{-(\gamma - \gamma') \Delta'} \right) \left| f^T \right|_{\gamma, \mu}, \end{align*}
for \(\gamma_* \leq \gamma' \leq \gamma\), where
\[ X = C \Delta' \left( \frac{\Delta}{\sigma - \sigma'} \right)^{\exp_2 e^{2\gamma \Delta}} \max(1, \mu^2). \]
The exponent $\exp_1$ only depends on $\frac{d_s}{m}$ and $\#A$. The exponent $\exp_2$ only depends on $d_s, m_s$ and $\#A$. The exponent $\alpha$ is a positive constant only depending on $s_\ast, \frac{d_s}{\beta_s}$, $d_s$. $C$ is an absolute constant that depends on $c$ and $\sup_D |\omega|$.

Remark 5.6. The estimates (5.19) provides an estimate of $\delta_\ast$. Indeed, for any $a, b \in [a]_{\Delta'}$

$$|\partial^j B_\alpha^b| \leq \frac{1}{C} |\partial^j B|_{\gamma, \infty}(a, b)^{-1} \leq \text{Ct.} (\Delta')^\sigma |f^T|_{\gamma, \infty, D} \frac{1}{\langle \sigma \rangle^\delta}.$$  

Since $\#[a]_{\Delta'} \leq (\Delta')^{\exp}$ we get

$$||| \partial^j B(a) ||| \leq \text{Ct.} (\Delta')^{\exp} |f^T|_{\gamma, \infty, D} \frac{1}{\langle \sigma \rangle^\delta}.$$ 

This gives the estimate of $\delta_\ast - \delta$.

**Proof.** The set $\mathcal{D}'$ will now be given by the intersection of the sets in the three previous lemma of this section. We set

$$h_+(r, w) = \hat{f}_c(r, 0) + \frac{1}{2} \langle w, Bw \rangle$$

$$S(r, \theta, w) = S_r(\theta, r) + \langle S_w(\theta)w \rangle + \frac{1}{2} \langle S_{ww}(\theta)w, w \rangle$$

and

$$R(r, \theta, w) = R_r(\theta, r) + \langle R_w(\theta), w \rangle + \frac{1}{2} \langle R_{ww}(\theta)w, w \rangle.$$ 

These functions also depend on $\rho \in \mathcal{D}$ and they verify equation (5.18) for $\rho \in \mathcal{D}'$.

If $x = (r, \theta, w) \in \mathcal{O}_{\gamma_s}(\sigma, \mu)$, then

$$|h_+(x)| \leq |f^T|_{\gamma, \infty, D} \frac{1}{2} ||Bw||_{\gamma_s} ||w||_{\gamma_s}.$$ 

Since

$$||B||_{\gamma, \infty} \geq ||B||_{\gamma_s, \infty} \geq ||B||_{\mathcal{B}(Y_s, Y_\ast)}$$

it follows that

$$|h_+(x)| \leq \text{Ct.} ||f^T||_{\gamma, \infty, D} \max(1, \mu^2).$$

We also have for any $x = (r, \theta, w) \in \mathcal{O}_{\gamma_s}(\sigma, \mu)$, $\gamma_s \leq \gamma' \leq \gamma$,

$$||Jdh_+(x)||_{\gamma'} \leq \text{Ct.} ||f^T||_{\gamma, \infty, D} ||Bw||_{\gamma_s}.$$ 

Since

$$||B||_{\gamma, \infty} \geq ||B||_{\gamma', \infty} \geq ||B||_{\mathcal{B}(Y_\gamma, Y_{\gamma'})}$$

it follows that

$$||Jdh_+(x)||_{\gamma'} \leq \text{Ct.} ||f^T||_{\gamma, \infty, D} \max(1, \mu).$$

Finally $J\dot{d}^2 h_+(x)$ equals $JB$ which satisfies the required bound.

The estimates of the derivatives with respect to $\rho$ are the same and obtained in the same way.

The functions $S(\theta, r, \zeta)$ and $R(\theta, r, \zeta)$ are estimated in the same way. 

$\square$
5.6. The non-linear homological equation. The equation (5.2) can now be solved easily.

**Proposition 5.7.** There exists a constant $C$ such that for any

$$h \in \mathcal{N} \mathcal{F}_\kappa(\Delta, \delta), \quad \delta \leq \frac{1}{C} \varepsilon',$$

and for any

$$N \geq 1, \quad \Delta' \geq \Delta \geq 1, \quad \kappa \leq \frac{1}{C} \varepsilon',$$

there exists a subset $\mathcal{D}' = \mathcal{D}(h, \kappa, N) \subset \mathcal{D}$, satisfying

$$\text{Leb}(\mathcal{D} \setminus \mathcal{D}') \leq CN^{\exp_1\left(\frac{\chi + \delta}{\delta_0}\right)}\left(\frac{\kappa}{\delta_0}\right)^\alpha,$$

and, for any $f \in T_{\gamma, \kappa}(\sigma, \mu, \mathcal{D})$, $\mu \leq 1$,

$$\varepsilon = |f^T|_{\sigma, \mu, \gamma, \kappa, \mathcal{D}} \quad \text{and} \quad \xi = |f|_{\sigma, \mu, \gamma, \kappa, \mathcal{D}},$$

there exist real jet-functions $S, R \in T_{\gamma, \kappa}(\sigma, \mu)$ and $h_+$ verifying, for $\rho \in \mathcal{D}'$,

$$(5.22) \quad \{h, S\} + \{f - f^T, S\}^T + f^T = h_+ + R$$

and such that

$$h + h_+ \in \mathcal{N} \mathcal{F}_\kappa(\Delta', \delta_+)$$

and, for all $\sigma' < \sigma$ and $\mu' < \mu$,

$$(5.23) \quad |h_+|_{\sigma', \mu, \gamma, \kappa, \mathcal{D}} \leq CXY\varepsilon$$

$$(5.24) \quad |S|_{\sigma', \mu, \gamma, \kappa, \mathcal{D}} \leq C X Y \xi \varepsilon$$

$$(5.25) \quad |R|_{\sigma', \mu, \gamma, \kappa, \mathcal{D}} \leq C \left(e^{-(\sigma - \sigma')N} + e^{-(\gamma - \gamma')\Delta'}\right) XY \varepsilon,$$

for $\gamma' \leq \gamma$, where

$$X = \left(\frac{N \Delta' e^{\gamma \Delta}}{\sigma - \sigma'}(\mu - \mu')\right)^{\exp_2}$$

and

$$Y = \left(\frac{\chi + \delta + \xi}{\kappa}\right)^{4s_* + 3}.$$

The exponent $\exp_1$ only depends on $d_1, m_*, s_*$ and $A$. The exponent $\exp_2$ only depends on $d_1, d_2$, $s_*$ and $#A$. The exponent $\alpha$ is a positive constant only depending on $s_*, \frac{d_1}{\beta_1}, \frac{d_2}{\beta_2}$. $C$ is an absolute constant that depends on $c$ and $\sup_{\mathcal{D}} |\omega|$.

**Remark 5.8.** Notice that the “loss” of $S$ with respect to $\kappa$ is of “order” $4s_* + 3$. However, if $\chi$, $\delta$ and $\xi = |f|_{\sigma, \mu, \gamma, \kappa, \mathcal{D}}$ are of size $\lesssim \kappa$, then the loss is only of “order” 1.
Proof. Let \( S = S_0 + S_1 + S_2 \) be a jet-function such that \( S_1 \) starts with terms of degree 1 in \( r, w \) and \( S_2 \) starts with terms of degree 2 in \( r, w \). Let \( f \) be a jet function and we give (as is usual) \( w \) degree 1 and \( r \) degree 2.

Let now \( \sigma' = \sigma_5 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 = \sigma \) be a (finite) arithmetic progression, i.e. \( \sigma_j - \sigma_{j+1} \) do not depend on \( j \), and let and \( \mu' = \mu_5 < \mu_4 < \mu_3 < \mu_2 < \mu_1 < \mu_0 = \mu \) be another arithmetic progressions.

Then \( \{ h', S \} + \{ f - f^T, S \}^T + f^T = h_+ + R \) decomposes into three homological equations

\[
\{ h', S_0 \} + f^T = (h_+) + R_0, \\
\{ h', S_1 \} + f^T = (h_+) + R_1, \\
\{ h', S_2 \} + f^T = (h_+) + R_2.
\]

By Lemma 5.5 we have for the first equation

\[
|h_+|_{0} \leq X_\varepsilon, \\
|R_0|_{1} \leq XZ_\varepsilon,
\]

where

\[
X = C\Delta'(\frac{5\Delta}{\sigma - \sigma'})\exp\varepsilon^{\gamma_1d_\Delta}.
\]

and where \( Y, Z \) are defined by the right hand sides in the estimates (5.20) and (5.21).

By Proposition 2.3 we have

\[
\xi_1 = |f_1|_{1} \leq \frac{1}{\kappa} X_\varepsilon W \varepsilon
\]

where

\[
W = C(\frac{5}{\sigma - \sigma'}) + \frac{5}{\mu - \mu'}.
\]

By Proposition 2.4 \( \xi_1 = |f_1^T|_{1} \) satisfies the same bound as \( \xi_1 \).

By Lemma 5.5 we have for the second equation

\[
|h_+|_{1} \leq X_\varepsilon, \\
|R_1|_{1} \leq XZ_\varepsilon,
\]

where

\[
\varepsilon = \frac{1}{\kappa} X_\varepsilon \varepsilon_1.
\]

By Proposition 2.4 we have

\[
\xi_2 = |f_2|_{1} \leq \frac{1}{\kappa} X_\varepsilon \xi_1,
\]

and \( \xi_2 = |f_2^T|_{1} \) satisfies the same bound.

By Lemma 5.5 we have for the third equation

\[
|h_+|_{2} \leq X_\varepsilon, \\
|R_2|_{2} \leq XZ_\varepsilon,
\]

where

\[
\varepsilon = \frac{1}{\kappa} X_\varepsilon \varepsilon_2.
\]

Putting this together we find that

\[
\varepsilon + \varepsilon_1 + \varepsilon_2 \leq (1 + \frac{1}{\kappa} X_\varepsilon \varepsilon)^3 \varepsilon = T\varepsilon
\]
and
\[ |h_{\gamma', \mu'}| \leq X T \varepsilon, \quad |R_{\gamma', \mu'}| \leq X Z T \varepsilon, \]
\[ |S_{\gamma', \mu'}| \leq \frac{1}{\kappa} X Y T \varepsilon. \]

Renaming X and Y gives now the estimates. \( \square \)

6. Proof of the KAM Theorem

Theorem 3.6 is proved by an infinite sequence of change of variables typical for KAM-theory. The change of variables will be done by the classical Lie transform method which is based on a well-known relation between composition of a function with a Hamiltonian flow \( \Phi^t_S \) and Poisson brackets:
\[
\frac{d}{dt} f \circ \Phi^t_S = \{ f, S \} \circ \Phi^t_S
\]
from which we derive
\[
f \circ \Phi^1_S = f + \{ f, S \} + \int_0^1 (1 - t) \{ \{ f, S \}, S \} \circ \Phi^t_S \, dt.
\]
Given now three functions \( h, k \) and \( f \), then
\[
(h + k + f) \circ \Phi^1_S = h + k + f + \{ h + k + f, S \} + \int_0^1 (1 - t) \{ \{ h + k + f, S \}, S \} \circ \Phi^t_S \, dt.
\]
If now \( S \) is a solution of the equation
\[
\{ h, S \} + \{ f - f^T, S \}^T + f^T = h_+ + R,
\]
then
\[
(h + k + f) \circ \Phi^1_S = h + k + h_+ + f_+
\]
with
\[
f_+ = R + (f - f^T) + \{ k + f^T, S \} + \{ f - f^T, S \} - \{ f - f^T, S \}^T + \int_0^1 (1 - t) \{ \{ h + k + f, S \}, S \} \circ \Phi^t_S \, dt
\]
and
\[
f_+^T = R + \{ k + f^T, S \}^T + \int_0^1 (1 - t) \{ \{ h + k + f, S \}, S \} \circ \Phi^t_S \, dt)^T.
\]

If we assume that \( S \) is “small as” \( f^T \), then \( f_+^T \) is is “small as” \( k f^T \) — this is the basis of a linear iteration scheme with (formally) linear convergence. But if also \( k \) is of the size \( f^T \), then \( f_+^T \) is is “small as” the square of \( f^T \) — this is the basis of a quadratic iteration scheme with (formally) quadratic convergence. We shall combine both of them.

First we shall give a rigorous version of the change of variables described above.

---

11 it was first used by Poincaré, credited by him to the astronomer Delauney, and it has been used many times since then in different contexts.
6.1. The basic step. Let \( h \in \mathcal{N}_F(\Delta, \delta) \) and assume \( \kappa > 0 \) and
\[
\delta \leq \frac{1}{C} e',
\]
where \( C \) is to be determined.

Let \( \gamma = (\gamma, m_\ast) \geq \gamma_\ast = (0, m_\ast) \) and recall Remark 5.1. Let \( N \geq 1, \Delta' \geq \Delta \geq 1 \) and \( \kappa \leq \frac{1}{C} e' \).

Proposition 5.7 then gives, for any \( f \in T_{\gamma, \kappa, D}(\sigma, \mu) \), \( \mu \leq 1 \),
\[
\varepsilon = \left| f^T \right|_{\sigma, \mu} \text{ and } \xi = \left| f \right|_{\sigma, \mu},
\]
a set \( D' = D'(h, \kappa, N) \subset D \) and functions \( S, h, R \)
satisfying (5.23)+(5.24)+(5.25) and solving the equation (6.1),
\[
\{ h, S \} + \{ f - f^T, S \} + f^T = h + R.
\]
for any \( \rho \in D' \). Let now \( 0 < \sigma' = \sigma_1 < \sigma_2 < \sigma_1 < \sigma_0 = \sigma \) and \( 0 < \mu' = \mu_4 < \mu_3 < \mu_2 < \mu_1 < \mu_0 = \mu \) be (finite) arithmetic progressions.

The flow \( \Phi^t_S \).

We have, by (5.24),
\[
|S|_{\sigma_1, \mu_1} \leq C t \frac{X Y \varepsilon}{\kappa}
\]
where \( X, Y \) and \( C t \) are given in Proposition 5.7, i.e.
\[
X = \left( \frac{\Delta' e^{\gamma d N}}{(\sigma_0 - \sigma_1)(\mu_0 - \mu_1)} \right)^{exp_2} \leq \left( \frac{4^2 \Delta' e^{\gamma d N}}{(\sigma - \sigma')(\mu - \mu')} \right)^{exp_2},
\]
\[
Y = \left( \frac{\chi + \delta + \xi}{\kappa} \right)^{4s_\ast + 3}
\]
– we can assume without restriction that \( exp_2 \geq 1 \).

If
\[
\varepsilon \leq \frac{1}{C} \frac{\kappa}{X^2 Y}.
\]
and \( C \) is sufficiently large, then we can apply Proposition 2.7(i). By this proposition it follows that for any \( 0 \leq t \leq 1 \) the Hamiltonian flow map \( \Phi^t_S \) is a \( C^\infty \)-map
\[
\mathcal{O}_{\gamma}(\sigma_{i+1}, \mu_{i+1}) \times D \to \mathcal{O}_{\gamma'}(\sigma_i, \mu_i), \quad \forall \gamma_\ast \leq \gamma' \leq \gamma, \quad i = 1, 2, 3,
\]
real holomorphic and symplectic for any fixed \( \rho \in D \). Moreover,
\[
|| \partial^i_{\rho}(\Phi_S^t(x, \cdot) - x) ||_{\gamma'} \leq C t \frac{X Y \varepsilon}{\kappa}
\]
and
\[
|| \partial^i_{\rho}(d\Phi_S^t(x, \cdot) - I) ||_{\gamma', \infty} \leq C t \frac{X Y \varepsilon}{\kappa}
\]
for any \( x \in \mathcal{O}_{\gamma'}(\sigma_2, \mu_2), \gamma_\ast \leq \gamma' \leq \gamma, \) and \( 0 \leq |j| \leq s_\ast \).

A transformation.

Let now \( k \in T_{\gamma, \kappa, D}(\sigma, \mu) \) and set
\[
\eta = |k|_{\sigma, \mu}.\]
Then we have

\[(h + k + f) \circ \Phi_S^t = h + k + h_+ + f_+\]

where \(f_+\) is defined by (6.2), i.e.

\[f_+ = R + (f - f^T) + \{k + f^T, S\} + \{f - f^T, S\} - \{f - f^T, S\}^T + \int_0^1 (1 - t)\{\{h + k + f, S\}, S\} \circ \Phi_S^t \, dt\]

The integral term is the sum

\[\int_0^1 (1 - t)\{h + R - f^T, S\} \circ \Phi_S^t \, dt + \int_0^1 (1 - t)\{k + f, S\} - \{f - f^T, S\}^T, S\} \circ \Phi_S^t \, dt\]

\[\text{The estimates of } \{k + f^T, S\} \text{ and } \{f - f^T, S\}.
\]

By Proposition 2.5(i)

\[|\{k + f^T, S\}|_{\gamma, \alpha, D} \leq C_E X |S|_{\gamma, \alpha, D} |k + f^T|_{\gamma, \alpha, D}.
\]

Hence

\[(6.6) \quad |\{k + f^T, S\}|_{\gamma, \alpha, D} \leq C_E X |S|_{\gamma, \alpha, D} |k + f^T|_{\gamma, \alpha, D}.
\]

Similarly,

\[(6.7) \quad |\{f - f^T, S\}|_{\gamma, \alpha, D} \leq C_E X |S|_{\gamma, \alpha, D} |f - f^T|_{\gamma, \alpha, D}.
\]

\[\text{The estimate of } h_+ \text{ and } f_+ \circ \Phi_S^t.
\]

\[|h_+|_{\gamma, \alpha, D} \leq C_E X Y \varepsilon.
\]

This gives, again by Proposition 2.5(ii),

\[|\{h_+ - f^T, S\}|_{\gamma, \alpha, D} \leq C_E X^3 Y^2 \varepsilon^2.
\]

Let now \(F = \{h_+ - f^T, S\}\). If \(\varepsilon\) verifies (6.5) for a sufficiently large constant \(C\), then we can apply Proposition 2.7(ii). By this proposition, for \(|t| \leq 1\), the function \(F \circ \Phi_S^t \in \mathcal{T}_{\gamma, \alpha, D}(\sigma_3, \mu_3)\) and

\[(6.8) \quad |\{h_+ - f^T, S\} \circ \Phi_S^t|_{\gamma, \alpha, D} \leq C_E X^3 Y^2 \varepsilon^2.
\]

\[\text{The estimate of } R, S \circ \Phi_S^t.
\]

\[|R|_{\gamma, \alpha, D} \leq C_E X Y Z_{\gamma} \varepsilon,
\]

where

\[Z_{\gamma} = \left(e^{-(\sigma - \sigma_2)N} + e^{-(\gamma - \gamma')\Delta'}\right).
\]

Then, as in the previous case,

\[(6.9) \quad |\{R, S\} \circ \Phi_S^t|_{\gamma, \alpha, D} \leq C_E X^3 Y^2 Z_{\gamma} \varepsilon^2.
\]

\[\text{The estimate of } \{k + f, S\} - \{f - f^T, S\}^T, S\} \circ \Phi_S^t.
\]
This function is estimated as above. If $F = \{(k + f, S) - \{f - f^T, S\}^T, S\}$, then, by Proposition 2.4 and Proposition 2.5(i),
\[ |F|_{\gamma, \alpha, D} \leq C_{t} \left( \frac{1}{k} X^2 Y \right)^2 (\eta + \xi) \epsilon^2 \]
and by Proposition 2.7(ii)
\[ (6.10) \quad |\{(k + f, S) - \{f - f^T, S\}^T, S\} \circ \Phi|_{\gamma, \alpha, D} \leq C_{t} \left( \frac{1}{k} X^2 Y \right)^2 (\eta + \xi) \epsilon^2. \]

Renaming now $X$ and $Y$ and replacing $N$ by $2N$ now gives the following lemma.

**Lemma 6.1.** There exists an absolute constant $C$ such that, for any
\[ h \in \mathcal{N} \mathcal{F}_{\nu}(\Delta, \delta), \quad \kappa > 0, \quad \delta \leq \frac{1}{C} \epsilon', \]
and for any
\[ N \geq 1, \quad \Delta' \geq \Delta \geq 1, \quad \kappa \leq \frac{1}{C} \epsilon', \]
there exists a subset $D' \in D(h, \kappa, N) \subset D$, satisfying
\[ \text{Leb}(D \setminus D') \leq CN \exp \left( \frac{X + \delta}{\delta_0} \left( \frac{K}{\delta_0} \right) \right), \]
and, for any $f \in T_{\gamma, \kappa, D}(\sigma, \mu), \mu \leq 1,$
\[ \epsilon = |f|_{\gamma, \kappa, D} \quad \text{and} \quad \xi = |f|_{\gamma, \kappa, D}, \]
satisfying
\[ \epsilon \leq \frac{1}{C} \frac{\kappa}{XY}, \quad \begin{cases} X = \left( \frac{N \Delta \epsilon^{\delta} \Delta}{(\sigma - \sigma') (\mu - \mu')^{\epsilon}} \right) \exp \epsilon, \quad \sigma' < \sigma, \quad \mu' < \mu \\ Y = \left( \frac{X + \delta}{\delta_0} \right) \exp \epsilon. \end{cases} \]
and for any $k \in T_{\gamma, \kappa, D}(\sigma, \mu)$, there exists a $C^{\ast\ast}$ mapping
\[ \Phi : \mathcal{O}_{\gamma}(\sigma', \mu') \times D \to \mathcal{O}_{\gamma}(\sigma - \frac{\sigma - \sigma'}{2}, \mu - \frac{\mu - \mu'}{2}), \quad \forall \gamma_k \leq \gamma' \leq \gamma, \]
real holomorphic and symplectic for each fixed parameter $\rho \in D$, and functions $f_+, R_+ \in T_{\gamma, \kappa, D}(\sigma', \mu')$ and
\[ h + h_+ \in \mathcal{N} \mathcal{F}_{\nu}(\Delta', \delta_+), \]
such that
\[ (h + k + f) \circ \Phi = h + k + h_+ + f_+ + R_+, \quad \forall \rho \in D', \]
and
\[ |h_+|_{\gamma, \kappa, D} + |f_+ - f|_{\gamma, \kappa, D} \leq CXY \epsilon, \]
\[ |f_+|_{\sigma', \mu'} \leq C \left( \frac{1}{k} \right) XY (\epsilon + \epsilon) \epsilon \]
and
\[ |R_+|_{\gamma', \kappa, D} \leq CXY e^{-(\gamma - \gamma') \Delta \epsilon} \]
for any $\gamma \leq \gamma' \leq \gamma$.
Moreover,
\[ ||\partial^j_{\Phi}(\Phi(x, \rho) - x)||_{\gamma'} + ||\partial^j_{\rho}(d\Phi(x, \rho) - I)||_{\gamma', \kappa} \leq C \left( \frac{1}{k} \right) XY \epsilon \]
for any $x \in \mathcal{O}_{\gamma}(\sigma', \mu'), \gamma \leq \gamma' \leq \gamma$ and $|j| \leq s_*$, and for any $\rho \in D$. 

Finally, if $\tilde{\rho} = (0, \rho_2, \ldots, \rho_p)$ and $f^T(\cdot, \tilde{\rho}) = 0$ for all $\tilde{\rho}$, then $f_+ - f = R_+ = h_+ = 0$ and $\Phi(x, \cdot) = x$ for all $\tilde{\rho}$.

The exponent $\exp_1$ only depends on $\frac{h}{\delta_1}$ and $\#A$. The exponent $\exp_2$ only depends on $d_* m_*, s_*$ and $\#A$. The exponent $\exp_3$ only depends on $s_*$. The exponent $\alpha$ is a positive constant only depending on $d_* \frac{d_+}{s_*}, \frac{d_*}{s_*}$. $C$ is an absolute constant that depends on $c$ and $\sup D |\omega|$.

6.2. A finite induction. We shall first make a finite iteration without changing the normal form in order to decrease strongly the size of the perturbation. We shall restrict ourselves to the case when $N = \Delta'$.

Lemma 6.2. There exists a constant $C$ such that, for any

$$h \in \mathcal{NF}_\kappa(\Delta, \delta), \quad \kappa > 0, \quad \delta \leq \frac{1}{C'}$$

and for any

$$\Delta' \geq \Delta \geq 1, \quad \kappa \leq \frac{1}{C'}$$

there exists a subset $D' = D(h, \kappa, \Delta') \subset D$, satisfying

$$\text{Leb}(D \setminus D') \leq C(\Delta')^{\exp_1}(\frac{\kappa}{\delta_0})^{(\frac{\kappa}{\delta_0})^2},$$

and, for any $f \in T_{\gamma, \kappa, D}(\sigma, \mu), \quad \mu \leq 1,$

$$\varepsilon = |f^T|_{\gamma, \kappa, D} \quad \text{and} \quad \xi = |f|_{\gamma, \kappa, D},$$

satisfying

$$\varepsilon \leq \frac{1}{C} \frac{\kappa}{XY}, \quad \left\{ \begin{array}{l}
X = (\Delta'^{\alpha_0} \log \frac{1}{\varepsilon})^{\exp_2}, \quad \alpha' < \sigma, \ \mu' < \mu
Y = (\Delta'^{\alpha_0})^{\exp_3},
\end{array} \right.$$

there exists a $C^{\alpha'}$-mapping

$$\Phi: \mathcal{O}_{\gamma'}(\sigma', \mu') \times D \to \mathcal{O}_{\gamma}(\sigma - \frac{\sigma - \sigma'}{2}, \mu - \frac{\mu - \mu'}{2}), \quad \forall \gamma_* \leq \gamma', \gamma,$$

real holomorphic and symplectic for each fixed parameter $\rho \in D$, and functions $f' \in T_{\gamma, \kappa, D}(\sigma', \mu')$ and

$$h' \in \mathcal{NF}_\kappa(\Delta', \delta'),$$

such that

$$(h + f) \circ \Phi = h' + f', \quad \forall \rho \in D',$$

and

$$|h' - h|_{\gamma', \kappa, D} \leq CXY\varepsilon,$$

$$\xi' = |f'|_{\gamma', \kappa, D} \leq \xi + CXY\varepsilon$$

and

$$\varepsilon' = |(f')^T|_{\gamma', \kappa, D} \leq CXY(e^{-\frac{1}{2}(\sigma - \sigma')\Delta'} + e^{-\frac{1}{2}(\gamma - \gamma')\Delta'})\varepsilon,$$

for any $\gamma_* \leq \gamma' \leq \gamma$.

Moreover,

$$||\partial^j\rho(\Phi(x, \rho) - x)||_{\gamma'} + ||\partial^j\rho(d\Phi(x, \rho) - I)||_{\gamma', \kappa} \leq \frac{C}{\kappa} XY\varepsilon$$

for any $x \in \mathcal{O}_{\gamma'}(\sigma', \mu'), \gamma_* \leq \gamma' \leq \gamma$ and $|j| \leq s_*$, and for any $\rho \in D$. 

Finally, if $\tilde{\rho} = (0, \rho_2, \ldots, \rho_p)$ and $f^T(\cdot, \tilde{\rho}) = 0$ for all $\tilde{\rho}$, then $f' - f = h' = 0$ and $\Phi(x, \cdot) = x$ for all $\tilde{\rho}$.

The exponent $\exp_1$ only depends on $\frac{d}{\gamma_1}$ and $\#A$. The exponent $\exp_2$ only depends on $d_*, m_*, s_*$ and $\#A$. The exponent $\exp_3$ only depends on $s_*$. The exponent $\alpha$ is a positive constant only depending on $s_*, \frac{ds}{\alpha}, \frac{ds}{\beta}$. $C$ is an absolute constant that depends on $c$ and $\sup_d |\omega|$.

Proof. Let $N = \Delta'$. Let $\sigma_1 = \sigma - \frac{\sigma - \sigma'}{2}, \mu_1 = \mu - \frac{\mu - \mu'}{2}$ and $\sigma_{K+1} = \sigma', \mu_{K+1} = \mu'$, and let $\{\sigma_j\}^{K+1}_{j=1}$ and $\{\mu_j\}^{K+1}_{j=1}$ be arithmetical progressions. We take $K$ such that
\[ K \leq (\sigma - \sigma')\Delta'(\log \frac{\Delta'}{\gamma})^{-1}. \]
i.e. $K \leq (\sigma - \sigma')\Delta'(\log \frac{\Delta'}{\gamma})^{-1}$.

We let $f_1 = f$ and $k_1 = 0$, and we let $\epsilon_1 = [f^T_1]_{\gamma, \alpha, D} = \epsilon$, $\xi_1 = [f^T_1]_{\gamma, \alpha, D} = \xi$, $\delta_1 = \delta$ and $\eta_1 = [k_1]_{\sigma, \mu} = 0$.

Define now
\[ \epsilon_{j+1} = C\frac{1}{K}X_j Y_j (\eta_j + \epsilon_1)\epsilon_j, \]
\[ \xi_{j+1} = \xi_j + CX_j Y_j \epsilon_j, \quad \eta_{j+1} = \eta_j + CX_j Y_j \epsilon_j, \]
with
\[ X_j \left( \frac{N \Delta' e^{\gamma \Delta}}{(\sigma_j - \sigma_{j+1}) (\mu_j - \mu_{j+1})} \right)^{\exp_2}, \quad Y_j = \left( \frac{\epsilon + \delta + \xi_j}{\kappa} \right)^{\exp_3}, \]
where $C, \exp_2, \exp_3$ are given in Lemma 6.1. One verifies by an immediate induction that

Sublemma. There exists an absolute constant $C'$ such that if
\[ \epsilon_1 \leq \frac{1}{C' X_1^2 Y_1^2} \]
then, for all $j \geq 1$,
\[ \epsilon_j \leq \frac{1}{C} \frac{K}{X_j^2 Y_j^2} \quad \text{and} \quad \epsilon_j \leq (\text{Ct.} \frac{X_j^2 Y_j^2}{\kappa} \epsilon_1)^{-1} \epsilon_1, \]
\[ (\xi_j - \xi_1) + (\eta_j - \eta_1) \leq C' X_1 Y_1 \epsilon_1. \]
The constant Ct. only depends on $C$ and $\exp_3$.

We can then apply Lemma 6.1 $K$ times to get
\[ \Phi_j : \mathcal{O}_{\gamma'}(\sigma_{j+1}, \mu_{j+1}) \times D' \to \mathcal{O}_{\gamma'}(\sigma_j - \frac{\sigma_j - \sigma_{j+1}}{2}, \mu_j - \frac{\mu_j - \mu_{j+1}}{2}), \gamma_* \leq \gamma' \leq \gamma_j \]
and $f_{j+1}$ and $R_{j+1}$ such that, for $\rho \in D'$,
\[ (h + k_j + f_j + S_j) \circ \Phi_j = h + k_j + h_{j+1} + f_{j+1} + R_{j+1} + S_j \circ \Phi_j \]
with $k_{j+1} = k_j + k_{j+1}, k_1 = 0, S_{j+1} = R_{j+1} + S_j \circ \Phi_j, S_1 = 0$.

We then take $\Phi = \Phi_1 \circ \cdots \circ \Phi_K, h' = h + k_{K+1}$ and $f' = f_{K+1} + S_{K+1}$.

Then
\[ |h' - h|_{\sigma', \mu', \gamma, \alpha, D} \leq C' X_1 Y_1 \epsilon, \quad \delta' \leq C (\Delta')^{\exp_2} X_1 Y_1 \epsilon \]
and
\[ \delta' = |f'|_{\sigma', \mu', \gamma, \alpha, D} \leq \xi + C' X_1 Y_1 \epsilon \]
and, for \( \rho \in \mathcal{D}' \),
\[
\varepsilon' = |(f')^T_{\gamma',\mu'}| \leq (C, \frac{X^2 Y^2}{K} \varepsilon_1)^{K} \varepsilon + CX_1 Y_1 e^{-\gamma (\gamma')} \Delta' \varepsilon.
\]

For the estimates of \( \Phi \), write \( \Psi_j = \Phi_j \circ \cdots \circ \Phi_K \) and \( \Psi_{K+1} = \text{id} \). For \( (x, \rho) \in \mathcal{O}_{\gamma'}(\sigma', \mu') \times \mathcal{D} \) we then have
\[
||\Phi(x, \rho) - x||_{\gamma'} \leq \sum_{j=1}^{K} ||\Psi_j(x, \rho) - \Psi_{j+1}(x, \rho)||_{\gamma'}.
\]

Then
\[
||\Psi_j(x, \rho) - \Psi_{j+1}(x, \rho)||_{\gamma'} = ||\Phi_j(\Psi_{j+1}(x, \rho), \rho) - \Psi_{j+1}(x, \rho)||_{\gamma'}
\]
is
\[
\leq \text{Ct,} \frac{1}{K} X_1 Y_1 \varepsilon_j \max\left( \frac{1}{|\sigma - \sigma'|}, \frac{1}{|\mu - \mu'|} \right),
\]
by a Cauchy estimate. Hence
\[
||\Phi(x, \cdot) - x||_{\gamma'} \leq \text{Ct,} \frac{1}{K} X_1 Y_1 \max\left( \frac{1}{|\sigma - \sigma'|}, \frac{1}{|\mu - \mu'|} \right) \varepsilon.
\]

The derivatives with respect to \( \rho \) are obtained in the same way, as is also the estimates of \( d\Phi \).

The result now follows if we take \( C' \) sufficiently large and increases the exponent \( \exp_2 \).

\[ \square \]

**Proof of sublemma.** The estimates are true for \( j = 1 \) so we proceed by induction on \( j \). Let us assume the estimates hold up to \( j \). Then, for \( k \leq j \),
\[
Y_k \leq \left( \frac{\chi + \delta + \xi_1 + 2C' X_1 Y_1 \varepsilon_1}{K} \right)^{\exp_3} = 2^{\exp_3} Y_1
\]
and
\[
\varepsilon_{j+1} \leq 2^{\exp_3} \frac{X_1 Y_1}{K} [2 C X_1 Y_1 \varepsilon_1 + \varepsilon_1 + \varepsilon_j] \varepsilon_j \leq \text{Ct,} \frac{X^2 Y^2}{K} \varepsilon_1 \varepsilon_j.
\]
Then
\[
\xi_{j+1} \leq \xi_1 + \text{Ct,} X_1 Y_1 (\varepsilon_1 + \cdots + \varepsilon_{j+1}) \leq \xi_1 + 2 \text{Ct,} X_1 Y_1 \varepsilon_1
\]
and similarly for \( \eta_{j+1} \).

6.3. **The infinite induction.** We are now in position to prove our main result, Theorem 3.6.

Let \( h \) be a normal form Hamiltonian in \( \mathcal{N}\mathcal{F_{\gamma'}}(\Delta, \delta) \) and let \( f \in \mathcal{T}_{\gamma',\sigma, \mu}^1(\sigma, \mu) \) be a perturbation such that
\[
0 < \varepsilon = |f^T_{\gamma',\sigma, \mu}|, \quad \xi = |f|_{\gamma',\sigma, \mu}.
\]

We construct the transformation \( \Phi \) as the composition of infinitely many transformations \( \Phi \) as in Lemma 6.2. We first specify the choice of all the parameters for \( j \geq 1 \).

Let \( C, \exp_1, \exp_2, \exp_3 \) and \( \alpha \) be the constants given in Lemma 6.2.
6.3.1. Choice of parameters. We assume (to simplify) \(\gamma, \sigma, \mu \leq 1\) and \(\Delta \geq 1\). By decreasing \(\gamma\) or increasing \(\Delta\) we can also assume \(\gamma = (d\Delta)^{-1}\).

We choose for \(j \geq 1\)

\[
\mu_j = \left(\frac{1}{2} + \frac{1}{2j}\right) \mu \quad \text{and} \quad \sigma_j = \left(\frac{1}{2} + \frac{1}{2j}\right) \sigma.
\]

We define inductively the sequences \(\varepsilon_j, \Delta_j, \delta_j\) and \(\xi_j\) by

\[
\begin{aligned}
&\varepsilon_{j+1} = \varepsilon^{K_j} \left(\frac{1}{2} + \frac{1}{2j}\right) \log \frac{1}{\varepsilon} \exp_{\varepsilon} 2 \varepsilon_{j+1} = 4K_j \max\left(\frac{1}{\sigma_j - \sigma_{j+1}}, d\Delta_j\right) \log \frac{1}{\varepsilon} \Delta_{j+1} = \Delta \\
&\gamma_{j+1} = (d\Delta_j)^{-1} \gamma \quad \varepsilon_1 = \varepsilon \\
&\delta_{j+1} = \delta_j + C\Delta_j \varepsilon_j \quad \delta_1 = \delta \geq 0 \\
&\xi_{j+1} = \xi_j + C\Delta_j \varepsilon_j \quad \xi_1 = \xi \geq \varepsilon,
\end{aligned}
\]

where

\[
\begin{aligned}
&X_j = \left(\frac{4K_j \max\left(\frac{1}{\sigma_j - \sigma_{j+1}}, d\Delta_j\right) \log \frac{1}{\varepsilon} \Delta_{j+1} = \Delta}{\sigma_{j+1} \delta \kappa_j} \right)^{\exp_{\varepsilon} 3} \\
&Y_j = \left(\frac{4K_j \max\left(\frac{1}{\sigma_j - \sigma_{j+1}}, d\Delta_j\right) \log \frac{1}{\varepsilon} \Delta_{j+1} = \Delta}{\delta \kappa_j} \right)^{\exp_{\varepsilon} 3}
\end{aligned}
\]

The \(\kappa_j\) is defined implicitly by

\[
\varepsilon_j = \frac{1}{C} \frac{\kappa_j}{X_j Y_j}.
\]

These sequences depend on the choice of \(K_j\). We shall let \(K_j\) increase like

\[
K_j = K^j
\]

for some \(K\) sufficiently large.

**Lemma 6.3.** There exist constants \(C'\) and \(\exp'\) such that, if

\[
K \geq C'
\]

and

\[
\varepsilon \leq \frac{1}{C'} \left(\frac{\sigma \mu}{K \Delta \log \frac{1}{\varepsilon}}\right)^{\exp' \left(\frac{\varepsilon}{\chi + \delta + \xi}\right)^{\exp_{\varepsilon} 3} c'},
\]

then

(i)

\[
\sum_{k=1}^{\infty} C X_k Y_k \varepsilon_k \leq 2C X_1 Y_1 \varepsilon \leq \frac{1}{2} C' c'.
\]

(ii)

\[
\sum_{k=1}^{\infty} C \max\left(\frac{1}{\sigma_k - \sigma_{k+1}}, \frac{1}{\mu_k - \mu_{k+1}}\right) X_k Y_k \varepsilon_k \leq 5C \max\left(\frac{1}{\sigma}, \frac{1}{\mu}\right) X_1 Y_1 \varepsilon \leq \frac{1}{2} C' c'.
\]

(iii)

\[
\sum_{j=1}^{\infty} C \Delta_j^{\exp_{\varepsilon} 3} \left(\frac{\chi + \delta_j}{\delta_0}\right)^{\left(\frac{K_j}{\delta_0}\right)^{\alpha}} \leq C' \left(\frac{K \Delta \log \frac{1}{\varepsilon}}{\sigma \mu}\right)^{\exp_{\varepsilon} 3} \left(\frac{\chi + \delta + \xi}{\delta_0}\right)^{\beta} \left(\frac{\varepsilon}{\delta_0}\right)^{\alpha'}.
\]

\(C'\) is an absolute constant that only depends on \(\beta, \chi, c\) and \(\sup_{D} |\omega|\). The exponent \(\exp'\) is an absolute constant that only depends on \(\beta\) and \(\chi\). The exponents \(\alpha'\) and \(\beta\) are positive constants only depending on \(s, d, \frac{d}{\chi}, \frac{d}{\beta}\).
Notice that (i) implies that
\[ CX_j Y_j (e^{-\frac{1}{2}(\sigma_j - \gamma_j - 1) \Delta_{j+1}} + e^{-\frac{1}{2}(\gamma_j + 1) \Delta_{j+1}}) \epsilon_j \leq \epsilon_{j+1}. \]

Proof. \( \Delta_{j+1} \) is equal to
\[ 4K_j \max \left( \frac{1}{\sigma_j - \gamma_j}, d_{\Delta_j} \right) \log \frac{1}{\epsilon} \leq \left( Ct. \frac{K \log \frac{1}{\epsilon}}{\sigma \mu} \right) \exp' (\chi + \delta + \xi) \exp_3 \epsilon, \]
which, by an induction, is seen to be, by assumption on \( \epsilon \),
\[ \leq (A(2K)^a \Delta)^{a_j} \leq \left( \frac{1}{\epsilon} \right)^{a_j} \]
if \( a \) is, say, at least 6. In the same way one sees that
\[ X_j \leq \left( \frac{1}{\epsilon} \right)^2 \exp_2 a_j. \]

(i). For \( j = 1 \), (i) holds by assumption. Indeed, by definition
\[ (CX_1Y_1 \epsilon_1)^{1+\exp_3} = \kappa_1^{1+\exp_3} \leq CX_1Y_1 \kappa_1^{\exp_3} \epsilon_1 \leq Ct. \left( \frac{K \log \frac{1}{\epsilon}}{\sigma \mu} \right) \exp' (\chi + \delta + \xi) \exp_3 \epsilon, \]
which is
\[ \leq \left( \frac{1}{4C^a} \right) \exp_3 + 1 \]
by assumption on \( \epsilon \).
Assume now (i) holds up to \( j - 1 \geq 1 \). Then \( \delta_j \leq \delta + 2CX_1Y_1 \epsilon \) and \( \xi_j \leq \xi + 2CX_1Y_1 \epsilon \), and hence
\[ Y_j \leq \left( \frac{\chi + \delta + \xi + 4CX_1Y_1 \epsilon}{\kappa_j} \right) \exp_3 \leq Ct. Y_1 \left( \frac{\kappa_1}{\kappa_j} \right)^{\exp_3}, \]
and, by the definition of \( \kappa_j \),
\[ \kappa_j^{1+\exp_3} \leq CX_jY_j \epsilon_j \kappa_j^{\exp_3} \leq Ct. Y_1 \kappa_1^{\exp_3} X_j \epsilon_j \leq X_j \epsilon^{K_j} \]
by assumption on \( \epsilon \). Hence
\[ CX_j Y_j \epsilon_j = \kappa_j \leq X_j \epsilon^{2bK_j} \leq \epsilon^{2bK_j-2 \exp_2 a_j} \leq \epsilon^{bK_j}, \quad b = \frac{1}{2(\exp_3 + 1)}, \]
if \( K \) is large enough – notice that \( j \geq 2 \). This implies that
\[ \sum_{k=2}^{j} CX_k Y_k \epsilon_k \leq 2\epsilon^{bK_2} \leq \epsilon \leq CX_1 Y_1 \epsilon_1 \]
if \( K \) is large enough.

The proof of (ii) is similar. To see (iii) we have for \( j \geq 2 \)
\[ \Delta_{j+1}^{\exp_3} \kappa_j^\alpha = (\Delta_{j+1}^{\exp'_j} \kappa_j^\alpha) \leq \left( X_j^{\exp'_j} \kappa_j^\alpha \right) \leq (\epsilon^{2bK_j-2(\exp'_j + 1) \exp_2 a_j})^\alpha \]
which is
\[ \leq \epsilon^{bK_j \alpha} \]
if \( K \) is large enough.
Therefore
\[ \sum_{j \geq 1} \Delta_{j+1}^{\exp_3} \kappa_j^\alpha = \Delta_{\alpha}^{\exp_3} \kappa_1^\alpha + 2\epsilon^{bK_2 \alpha} \leq 2\Delta_{\alpha}^{\exp_3} \kappa_1^\alpha \]
if \( K \) is large enough.
6.3.2. The iteration.

**Proposition 6.4.** There exist positive constants $C'$, $\alpha'$ and $\exp'$ such that, for any $h \in \mathcal{N}F(x)(\Delta, \delta)$ and for any $f \in T_{\gamma, x, D}(\sigma, \mu)$, $0 < \gamma, \sigma, \mu \leq 1$,

$$
\varepsilon = \left| f^T \right|_{\gamma, x, D}, \quad \xi = \left| f \right|_{\gamma, x, D},
$$

if

$$
\delta \leq \frac{1}{C'} c'
$$

and

$$
\varepsilon(\log \frac{1}{\varepsilon})^{\exp'} \leq \frac{1}{C} \left( \frac{\max(\gamma^{-1}, d_{\Delta})}{\sigma \mu} \right)^{\exp'} \left( \frac{c'}{\chi + \delta + \xi} \right)^{\exp_3} c'
$$

then there exist a set $D' = D'(h, f) \subset D$,

$$
\text{Leb}(D \setminus D') \leq C \left( \log \frac{1}{\varepsilon} \right)^{\exp'} \left( \frac{\chi + \delta + \xi}{\delta_0} \right)^{\exp} \left( \frac{\varepsilon}{\chi \delta_0} \right)^{\exp'},
$$

and a $C^\infty$ mapping

$$
\Phi : O_{\gamma, x}(\sigma/2, \mu/2) \times D \to O_{\gamma, x}(\sigma, \mu),
$$

real holomorphic and symplectic for given parameter $\rho \in D$, and

$$
h' \in \mathcal{N}F(x)(\infty, \delta'), \quad \delta' \leq \frac{c'}{2},
$$

such that

$$(h + f) \circ \Phi = h' + f'$$

verifies

$$
\left| f' - f \right|_{\gamma/2, \mu/2} \leq C'
$$

and, for $\rho \in D'$, $(f')^T = 0$.

Moreover,

$$
\left| h' - h \right|_{\gamma/2, \mu/2} \leq C'
$$

and

$$
\left\| \partial^j_\rho (\Phi(x, \cdot) - x) \right\|_{\gamma, x} + \left\| \partial^j_\rho (d\Phi(x, \cdot) - I) \right\|_{\gamma, x} \leq C'
$$

for any $x \in O_{(0, m)}(\sigma', \mu')$ and $|j| \leq s_*$, and for any $\rho \in D$.

Finally, if $\tilde{\rho} = (0, \rho_2, \ldots, \rho_p)$ and $f^T(\cdot, \tilde{\rho}) = 0$ for all $\tilde{\rho}$, then $h' = h$ and $\Phi(x, \cdot) = x$ for all $\tilde{\rho}$.

$C'$ is an absolute constant that only depends on $\beta, \zeta, c$ and $\sup_D |\omega|$. The exponent $\exp'$ is an absolute constant that only depends on $\beta$ and $\zeta$. The exponent $\exp_3$ only depends on $s_*$. The exponent $\alpha'$ is a positive constant only depending on $s_*, \frac{d_{\Delta}}{\beta}, \frac{d_{\Delta}}{\beta_3}$.

**Proof.** Assume first that $\gamma = d_{\Delta}^{-1}$.

Choose the number $\mu_j, \sigma_j, \varepsilon_j, \Delta_j, \gamma_j, \delta_j, \xi_j, X_j, Y_j, \kappa_j$ as above in Lemma 6.3 with $K = C'$. By the assumption on $\varepsilon$ we can apply Lemma 6.3.

Let $h_1 = h$, $f_1 = f$ and $D_1 = D$. Lemma 6.3 now implies that we can apply Lemma 6.2 iteratively to get for all $j \geq 1$ a set $D_{j+1} \subset D_j$ such that

$$
\text{Leb}(D_j \setminus D_{j+1}) \leq C \Delta_{j+1}^{\exp_2} \left( \frac{\chi + \delta_j}{\delta_0} \right)^{\kappa_j / \delta_0}.
$$
a $C^\infty$ mapping

$$\Phi_{j+1} : \mathcal{O}'(\sigma_{j+1}, \mu_{j+1}) \times D_{j+1} \to \mathcal{O}'(\sigma_j - \frac{\sigma_j - \sigma_{j+1}}{2}, \mu_j - \frac{\mu_j - \mu_{j+1}}{2}), \quad \forall \gamma_* \leq \gamma' \leq \gamma_{j+1},$$

real holomorphic and symplectic for each fixed parameter $\rho$, and functions $f_{j+1} \in T_{\gamma_* \mu}(\sigma_{j+1}, \mu_{j+1})$ and

$$h_{j+1} \in \mathcal{N}\mathcal{F}_\kappa(\Delta_{j+1}, \delta_{j+1})$$

such that

$$(h_j + f_j) \circ \Phi_{j+1} = h_{j+1} + f_{j+1}, \quad \forall \rho \in D_{j+1},$$

with

$$|f_{j+1} T|_{\gamma_{j+1}, \mu_{j+1}, D} \leq \varepsilon_{j+1}$$

and

$$|f_{j+1} |_{\gamma_{j+1}, \mu_{j+1}, D} \leq \xi_{j+1}.$$ 

Moreover,

$$|h_{j+1} - h_j|_{\gamma_{j+1}, \mu_{j+1}, D} \leq CX_j Y_j \varepsilon_j$$

and

$$||\partial_p^l(\Phi_{j+1}(x, \cdot) - x)||_{\gamma_*} + ||\partial_p^l(d(\Phi_{j+1}(x, \cdot) - I))||_{\gamma_*} \leq C \frac{1}{\kappa_j} X_j Y_j \varepsilon_j$$

for any $x \in \mathcal{O}'(\sigma_{j+1}, \mu_{j+1})$, $\gamma_* \leq \gamma' \leq \gamma_{j+1}$ and $|l| \leq s_*$. 

We let $h' = \lim h_j$, $f' = \lim f_j$ and $\Phi = \Phi_2 \circ \cdots \circ \Phi_3 \circ \cdots$. Then $(h + f) \circ \Phi = h' + f'$ and $h'$ and $f'$ verify the statement. The convergence of $\Phi$ and its estimates follows by Cauchy estimates as in the proof of Lemma 6.2.

The last statement is obvious.

If $\gamma > (d\Delta)^{-1}$, then we can just decrease $\gamma$ and we obtain the same result. If $\gamma < (d\Delta)^{-1}$, then we increase $\Delta$ and we obtain the same result. \(\square\)

Theorem 3.6 now follows from this proposition.

7. Examples

7.1. Beam equation with a convolutive potential. Consider the $d_*$ dimensional beam equation on the torus

$$(7.1) \quad u_{tt} + \Delta^2 u + V * u + \varepsilon g(x, u) = 0, \quad x \in \mathbb{T}^{d_*}.$$ 

Here $g$ is a real analytic function on $\mathbb{T}^{d_*} \times I$, where $I$ is a neighborhood of the origin in $\mathbb{R}$, and the convolution potential $V : \mathbb{T}^{d_*} \to \mathbb{R}$ is supposed to be analytic with real Fourier coefficients $\hat{V}(a), \ a \in \mathbb{Z}^{d_*}$.

Let $\mathcal{A}$ be any subset of cardinality $n$ in $\mathbb{Z}^{d_*}$. We set $\mathcal{L} = \mathbb{Z}^{d_*} \setminus \mathcal{A}$, $\rho = (\hat{V}_a)_{a \in \mathcal{A}}$, and treat $\rho$ as a parameter of the equation,

$$\rho = (\rho_a, \ldots, \rho_a) \in D = [\rho_a, \rho_a'] \times \cdots \times [\rho_a', \rho_a''],$$

(all other Fourier coefficients are fixed). We denote $\mu_a = |a|^4 + \hat{V}(a), \ a \in \mathbb{Z}^{d_*}$, and assume that $\mu_a > 0$ for all $a \in \mathcal{A}$, i.e. $|a|^4 + \rho_a' > 0$ if $a \in \mathcal{A}$. We also suppose that

$$\mu_1 \neq 0, \quad \mu_1 \neq \mu_2, \quad \forall l, l_1, l_2 \in \mathcal{L}, \ l_1 \neq l_2.$$ 

Denote

$$\mathcal{F} = \{a \in \mathcal{L} : \mu_a < 0\}, \quad |\mathcal{F}| =: N, \quad \mathcal{L}_\infty = \mathcal{L} \setminus \mathcal{F},$$
consider the operator
\[ \Lambda = |\Delta^2 + V*|^1/2 = \text{diag}\{\lambda_a, a \in \mathbb{Z}^d\}, \quad \lambda_a = \sqrt{|\mu_a|}, \]
and the following operator $\Lambda^\#$, linear over real numbers:
\[ \Lambda^\#(ze^{i(a,x)}) = \begin{cases} z\lambda_a e^{i(a,x)}, & a \in \mathcal{L}_\infty, \\ -z\lambda_a e^{i(a,x)}, & a \in \mathcal{F}, \end{cases} \]
Introducing the complex variable
\[ \psi = \frac{1}{\sqrt{2}}(\Lambda^{1/2} u - i\Lambda^{-1/2} \dot{u}) = (2\pi)^{-d/2} \sum_{a \in \mathbb{Z}^d} \psi_a e^{i(a,x)}, \]
we get for it the equation (cf. \[5,\text{Section 1.2}\])
\[ (7.2) \quad \dot{\psi} = i(\Lambda^\# \psi + \frac{1}{\sqrt{2}} \Lambda^{-1/2} g(x, \Lambda^{-1/2} \left( \frac{\psi + \bar{\psi}}{\sqrt{2}} \right))). \]
Writing $\psi_a = (u_a + iv_a)/\sqrt{2}$ we see that eq. (7.2) is a Hamiltonian system with respect to the symplectic form $\sum dv_x \wedge du_x$ and the Hamiltonian $h = h_{up} + \varepsilon F$, where
\[ P = \int_{\Omega^d_1} G(x, \Lambda^{-1/2} \left( \frac{\psi + \bar{\psi}}{\sqrt{2}} \right)) \, dx, \quad \partial_u G(x, u) = g(x, u), \]
and $h_{up}$ is the quadratic Hamiltonian
\[ h_{up}(u, v) = \sum_{a \in \mathcal{A}} \lambda_a |\psi_a|^2 + \left\langle \begin{pmatrix} u_F \\ v_F \end{pmatrix}, H \begin{pmatrix} u_F \\ v_F \end{pmatrix} \right\rangle + \sum_{a \in \mathcal{L}_\infty} \lambda_a |\psi_a|^2. \]
Here $u_F = \imath(u_a, a \in \mathcal{F})$ and $H$ is a symmetric $2N \times 2N$-matrix. The $2N$ eigenvalues of the Hamiltonian operator with the matrix $H$ are the real numbers $\{\pm \lambda_a, a \in \mathcal{F}\}$. So the linear system (7.2) is stable if and only if $n = 0$.

Let us fix any $n$-vector $I = \{I_a > 0, a \in \mathcal{A}\}$ with positive components. The $n$-dimensional torus
\[ \{ |\psi_a|^2 = I_a, \quad a \in \mathcal{A} \}
\[ \psi_a = 0, \quad a \in \mathcal{L} = \mathbb{Z}^d_\ast \setminus \mathcal{A}, \]
is invariant for the unperturbed linear equation; it is linearly stable if and only if $N = 0$. In the linear space $\text{span}\{\psi_a, a \in \mathcal{A}\}$ we introduce the action-angle variables $(r_a, \theta_a)$ through the relations $\psi_a = \sqrt{(I_a + r_a)} e^{i\theta_a}, a \in \mathcal{A}$. The unperturbed Hamiltonian becomes
\[ h_{up} = \text{const} + \langle r, \omega(\rho) \rangle + \left\langle \begin{pmatrix} u_F \\ v_F \end{pmatrix}, H \begin{pmatrix} u_F \\ v_F \end{pmatrix} \right\rangle + \sum_{a \in \mathcal{L}_\infty} \lambda_a |\psi_a|^2, \]
with $\omega(\rho) = (\omega_a, a \in \mathcal{A})$, and the perturbation becomes
\[ P = \varepsilon \int_{\Omega^d_1} G(x, \dot{u}(r, \theta; \zeta)(x)) \, dx, \quad \dot{u}(r, \theta; \zeta)(x) = \Lambda^{-1/2} \left( \frac{\psi + \bar{\psi}}{\sqrt{2}} \right), \]
i.e.
\[ \dot{u} = \sum_{a \in \mathcal{A}} \sqrt{(I_a + r_a)} (e^{i\theta} \varphi_a + e^{-i\theta} \varphi_{-a}) \sqrt{2\lambda_a} + \sum_{a \in \mathcal{L}} \frac{\psi_a \varphi_a + \bar{\psi}_a \varphi_{-a}}{\sqrt{2\lambda_a}}. \]
In the symplectic coordinates $((u_a, v_a), a \in \mathcal{L})$ the Hamiltonian $h_{up}$ has the form (1.1), and we wish to apply to the Hamiltonian $h = h_{up} + \varepsilon F$ Theorem 3.6 and Corollary 3.7. The assumption A1 with constants $c, c'$ of order one, $\beta_1 = \beta_2 = \beta_3 = \ldots$
2 holds trivially. The assumption A2 also holds since for each case (i)-(iii) the second alternative with \( \omega(\rho) = \rho \) is fulfilled for some \( \delta_0 \sim 1 \). Finally, the assumptions R1 and R2 with \( \kappa = 1 \) and suitable constants \( \gamma_1, \sigma, \mu > 0 \) and \( \gamma_2 = m_\ast \) are valid in view of Lemma 3.2 in [5]. More exactly, the validity of the assumption R1 is a part of the lemma’s assertion. The lemma also states that the second differential \( Jd^2 f \) defines holomorphic mappings

\[
Jd^2 f : \mathcal{O}_\gamma(\sigma, \mu) \to M^D_\gamma, \quad \gamma' \leq \gamma,
\]

where \( M^D_\gamma \) is the space of matrices \( A \), formed by \( 2 \times 2 \)-blocks \( A^b \), such that

\[
\| A \|^b \sup_{a, b} (a, \langle | \rangle b) | A |^b_m \max (| a - b |, 1)^\gamma_2 e^{\gamma_1 | a - b |} < \infty.
\]

It is easy to see that \( M^D_\gamma \subset M^b_\gamma, \kappa \) if \( \kappa = 1 \) and \( m_\ast \), entering the definition of \( M^b_\gamma, \kappa \), is sufficiently big. So R2 also holds.

Let us set \( u_0(\theta, x) = \hat{u}(0, \theta, 0)(x) \). Then for every \( I \in \mathbb{R}_+^n \) and \( \theta_0 \in \mathbb{T}_d^d \), the function \( (t, x) \mapsto u_0(\theta_0 + t \omega, x) \) is a solution of (7.1) with \( \epsilon = 0 \). Application of Theorem 3.6 and Corollary 3.7 gives us the following result:

**Theorem 7.1.** For \( \epsilon \) sufficiently small there is a Borel subset \( D_\epsilon \subset D \), \( \text{meas}(D \setminus D_\epsilon) \leq C\epsilon^n, \alpha > 0 \), such that for \( \rho \in D_\epsilon \) there is a function \( u_1(\theta, x) \), analytic in \( \theta \in \mathbb{T}_b^d \) and \( H^d \)-smooth in \( x \in \mathbb{T}_d^d \), satisfying

\[
\sup_{| \theta_0 | < \frac{\epsilon}{2}} \| u_1(\theta, \cdot) - u_0(\theta, \cdot) \|_{H^b(\mathbb{T}_b^d)} \leq \beta \epsilon,
\]

and there is a mapping \( \omega' : D_\epsilon \to \mathbb{R}_+^n \), \( \| \omega' - \omega \|_{C^1(D_\epsilon)} \leq \beta \epsilon \), such that for \( \rho \in D_\epsilon \) the function \( u(t, x) = u_1(\theta + t \omega'(\rho), x) \) is a solution of the beam equation (7.1). Equation (7.2), linearised around its solution \( \psi(t) \), corresponding to the solution \( u(t, x) \) above, has exactly \( N \) unstable and \( N \) stable directions.

The last assertion of this theorem follows from the item (iii) of Theorem 3.6 which implies that the linearised equation, in the directions, corresponding to \( L \), reduces to a linear equation with a coefficient matrix which can be written as \( B = B_\infty + B_\infty \). The operator \( B_\infty \) is close to the Hamiltonian operator with the matrix \( H \), so it has \( N \) stable and \( N \) unstable directions, while the matrix \( B_\infty \) is skew-symmetric, so it has imaginary spectrum.

**Remark 7.2.** This result was proved by Geng and You [9] for the case when the perturbation \( g \) does not depend on \( x \) and the unperturbed linear equation is stable.

### 7.2. NLS equation with a smoothing nonlinearity

Consider the NLS equation with the Hamiltonian

\[
g(u) = \frac{1}{2} \int |\nabla u|^2 dx + \frac{m}{2} \int |u(x)|^2 dx + \epsilon \int f(t, (-\Delta)^{-\alpha} u(x), x) dx,
\]

where \( m \geq 0, \alpha > 0 \), \( u(x) \) is a complex function on the torus \( \mathbb{T}_d^d \), and \( f \) is a real-analytic function on \( \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}_d^d \) (here we regard \( \mathbb{C} \) as \( \mathbb{R}^2 \)). The corresponding Hamiltonian equation is

\[
\dot{u} = i \left( -\Delta + mu + \epsilon (-\Delta)^{-\alpha} \nabla_2 f(t, (-\Delta)^{-\alpha} u(x), x) \right),
\]

where \( \nabla_2 \) is the gradient with respect to the second variable, \( u \in \mathbb{R}^2 \). We have to introduce in this equation a vector-parameter \( \rho \in \mathbb{R}^n \). To do this we can either assume that \( f \) is time-independent and add a convolution-potential term \( V(x, \rho) \ast u \)
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Let us discuss the second option. In this case the non-autonomous equation (7.3) can be written as an autonomous system on the extended phase-space \( \mathcal{O} \times T^n \times L^2_2 = \{(r, \theta, u(\cdot))\} \), where \( L^2_2 = L^2_2(T^d; \mathbb{R}^2) \) and \( \mathcal{O} \) is a ball in \( \mathbb{R}^n \), with the Hamiltonian

\[
g(r, u, \rho) = h_{uv}(r, u, \rho) + \varepsilon \int F(\theta, (-\Delta)^{-\alpha} u(x), x) \, dx,
\]

\[
h_{uv}(r, u, \rho) = \langle \rho, r \rangle + \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{m}{2} \int |u(x)|^2 \, dx.
\]

Assume that \( m > 0 \) and take for \( A_{uv} \) the operator \( -\Delta + m \) with the eigenvalues \( \lambda_a = |a|^2 + m \). Then the Hamiltonian \( g(r, u, \rho) \) has the form, required by Theorem 3.6 with \( \mathcal{L} = T^d, \quad \mathcal{F} = \emptyset, \quad \kappa = \min(2\alpha, 1), \quad \beta_1 = 2, \quad \beta_2 = 0, \quad \beta_3 = 2 \)

(any \( \beta_3 \) will do here in fact) and suitable \( \sigma, \mu, \gamma_1 > 0 \) and \( \gamma_2 = m_* \). The theorem applies and implies that, for a typical \( \rho \), equation (7.3) has time-quasiperiodic solutions of order \( \varepsilon \). The equation, linearised about these solutions, reduces to constant coefficients and all its Lyapunov exponents are zero.

If \( \alpha = 0 \), equations (7.3) become significantly more complicated. Still the assertions above remain true since they follow from the KAM-theorem in [7]. Cf. [6], where is considered nonautonomous linear Schrödinger equation, which is equation (7.3) with the perturbation \( \varepsilon (-\Delta)^{-\alpha} \nabla_2 f \) replaced by \( \varepsilon V(\rho t, x) u \), and it is proved that this equation reduces to an autonomous equation by means of a time-quasiperiodic linear change of variable \( u \). In [2] equation (7.3) with \( \alpha = 0 \) and \( f = F(\rho t, (-\Delta)^{-\alpha} u(x), x) \) is considered for the case when the constant-potential term \( m u \) is replaced by \( V(x) u \) with arbitrary sufficiently smooth potential \( V(x) \). It is proved that for a typical \( \rho \) the equation has small time-quasiperiodic solutions, but not that the linearised equations are reducible to constant coefficients.

**Appendix A.**

A.0.1. **Transversality.**

Let \( \mathcal{D} \) be the unit ball in \( \mathbb{R}^p \). For any matrix-valued function

\[
f : \mathcal{D} \to gl(\dim, \mathbb{C}),
\]

let

\[
\Sigma(f, \varepsilon) = \{ \rho \in \mathcal{D} : \| f(\rho)^{-1} \| > \frac{1}{\varepsilon} \},
\]

where \( \| \| \) is the operator norm.

**Lemma A.1.** Let \( f : \mathcal{D} \to \mathbb{C} \) be a \( C^{s_*} \)-function which is \((3, j, \varepsilon_0)\)-transverse, \( 1 \leq j \leq s_* \).

Then,

\[
\text{Leb}\{ \rho \in \mathcal{D} : |f(\rho)| < \varepsilon \} \leq C \frac{|\nabla_p f|^{C^{s_*-1}(\mathcal{D})}}{\varepsilon_0^\frac{1}{j}}.
\]

C is a constant that only depends on \( s_* \) and \( p \).

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12 If undesirable, the term \( i m u \) can be removed from eq. (7.3) by means of the substitution \( u(t, x) = u'(t, x)e^{imt} \).
Proof. It is enough to prove this for \( z = (1, 0, \ldots, 0) \), i.e. for a scalar \( \rho \). It is a well-known result, see for example Lemma B.1 in [4], that
\[
\text{Leb}\left( \Sigma(f, \varepsilon) \right) \leq C \frac{|f|_{C^0(D)}}{\delta_0} \left( \frac{\varepsilon}{\delta_0} \right)^j.
\]
This implies the claim. \( \square \)

A.0.2. Extension.

Lemma A.2. Let \( X \subset Y \) be subsets of \( D_0 \) such that
\[
\text{dist}(D_0 \setminus Y, X) \geq \varepsilon,
\]
then there exists a \( C^\infty \)-function \( g : D_0 \to \mathbb{R} \), being 1 on \( X \) and 0 outside \( Y \) and such that for all \( j \geq 0 \)
\[
|g|_{C^j(D_0)} \leq C \left( \frac{C}{\varepsilon} \right)^j.
\]
\( C \) is an absolute constant.

Proof. This is a classical result obtained by convoluting the characteristic function of \( X \) with a \( C^\infty \)-approximation of the Dirac-delta supported in a ball of radius \( \frac{\varepsilon}{2} \). \( \square \)

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