On the spectrum of DW Hamiltonian of quantum SU(2) gauge field

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Abstract

The spectrum of masses of the colorless states of the DW (De Donder-Weyl) Hamiltonian operator of quantum SU(2) Yang-Mills field theory on $\mathbb{R}^D$ obtained via the precanonical quantization is shown to be purely discrete and bounded from below. The scale of the mass gap is estimated to be of the order of magnitude of the scale of the ultra-violet parameter $\kappa$ introduced by precanonical quantization on dimensional grounds.

Keywords: Quantum Yang-Mills theory; De Donder–Weyl formalism; precanonical quantization; Clifford analysis; Schrödinger operators; Airy function; discrete spectrum; eigenvalues; mass gap.

1 Introduction

In this paper, we study the spectrum of the DW (De Donder-Weyl [1, 2]) Hamiltonian operator which was obtained within the precanonical quantization of pure YM theory [3, 4]:

$$\hat{H} = \frac{1}{2} \hbar^2 \kappa^2 \frac{\partial^2}{\partial A_\mu^a \partial A^a_\mu} - \frac{1}{2} i g \hbar \kappa C^a_{bc} A^b_\mu A^c_\nu \gamma^\nu \frac{\partial}{\partial A^a_\mu}. \tag{1.1}$$

Here and in what follows, $A_\mu^a$ are YM gauge potentials, $\gamma^\mu$ are Dirac matrices, $C^a_{bc}$ are the structure constants of the gauge group, $g$ is the gauge coupling constant, and $\kappa$ is an ultraviolet parameter of dimension $\text{length}^{-(n-1)}$ in $n$ space-time dimensions, which is introduced by precanonical quantization (when the differential forms corresponding to dynamical variables are represented by Clifford-algebra-valued operators [5]). In what follows, we also denote $\frac{\partial}{\partial A_\mu^a}$ as $\partial_{A_\mu^a}$ and $\frac{\partial}{\partial x_\mu}$ as $\partial_\mu$.

Let us recall that the importance of the DW Hamiltonian operator within the precanonical quantization of fields is that the evolution of Clifford-algebra-valued
precanonical wave functions $\Psi(A, x)$ on the bundle of field variables $A$ over spacetime (whose coordinates are $x^\mu$) is controlled by the Dirac-like equation [5]

$$i\hbar \gamma^\mu \partial_\mu \Psi = \hat{H} \Psi. \tag{1.2}$$

This equation appears as a generalization of the Schrödinger equation to field theory when all space-time variables are being treated on an equal footing. In this formulation, the only entities which propagate are different modes of precanonical wave function on the total space of field variables and space-time variables. Correspondingly, the masses of propagating modes of (the precanonical wave function of) quantum fields are given by the spectrum of $\frac{1}{\kappa} \hat{H}$. Note that in the classical limit (1.2) is consistent with the classical field equations: in [4], we have shown that the classical YM field equations arise from precanonical formulation as the equations satisfied by the expectation values of the corresponding field operators defined by precanonical quantization. The expectation values are calculated using the scalar product

$$\langle \Psi | \Psi \rangle = \text{Tr} \int [dA] \overline{\Psi} \Psi, \tag{1.3}$$

where $[dA] := \prod_{\mu,a} dA^a_\mu$ and $\overline{\Psi} := \gamma^0 \Psi^\dagger \gamma^0$. The derivation of this generalization of the Ehrenfest theorem requires that $\hat{H}$ is pseudo-Hermitian with respect to (1.3):

$$\overline{\hat{H}} = \hat{H} \quad (\text{cf.} \quad [6]).$$

A relation between the precanonical field quantization and the canonical quantization in the functional Schrödinger representation [7] was established in [8] and extended to quantum YM fields in [9]. In [8], it was argued that the standard QFT in the functional Schrödinger representation can be derived from precanonical formulation in the limiting case of an infinitely large ultra-violet scale $\kappa$. In this limiting case, the Schrödinger wave functional of a field configuration $y = y(x)$ at a moment of time $t$ is expressed as a product integral [10] of precanonical wave functions restricted to this field configuration. The canonical functional derivative Schrödinger equation can be derived from (1.2) by means of the space+time splitting $x^\mu \to (t, x)$ and restriction to the surface representing the abovementioned field configuration.

This result is consistent with the fact that the standard formulation of QFT leads to the meaningful results only upon regularization and renormalization. Precanonical quantization with its inbuilt ultraviolet scale $\kappa$ appears as an “already regularized” QFT because the divergent integrals in the usual formulations are replaced by expressions containing $\kappa$ in the precanonical formulation [11]. However, unlike the nonlocal theories or theories on noncommutative, discrete or fractal space-times, or theories which attempt to take into account the effects of quantum gravity, LQG or string theories, in precanonical quantization, the finite expressions are obtained without manually modifying the relativistic space-time at small distances in order to regularize the infinities. The quantum-gravitational geometry of space-time has been discussed within the precanonical quantization of Einstein gravity in [12, 13].
Our interest in the study of the spectrum of (1.1) originates from the observation that the spectrum of the DW Hamiltonian of a free scalar field is that of the harmonic oscillator in the field space: $\kappa m(N + 1/2)$, where $N \in \mathbb{Z}_+$ is a non-negative integer. When perturbed, this system radiates excitations corresponding to the only allowed transitions between the nearby levels with $\Delta N = \pm 1$, that corresponds to the excitations of mass $m$, which we usually interpret as the free massive quantum scalar particles. In the case of free massless fields, the spectrum of the DW Hamiltonian is continuous and the corresponding propagating excitations are massless. Hence, the discreteness or continuity of the spectrum of (1.1) will tell us whether the propagating excitations of quantum pure YM field are massive or massless.

Let us recall that precanonical quantization of fields [5] is the result of quantization of the Poisson-Gerstenhaber (PG) brackets found in [14] within the DW Hamiltonian theory [1, 2]. Those brackets are defined on differential forms which represent the dynamical variables or observables of the theory. Their generalization to singular DW Hamiltonian theories is discussed in [15]. The Clifford-Dirac algebra appears as the result of quantization of bi-graded PG brackets of forms [5, 16]. The Dirac-like precanonical analogue of the Schrödinger equation in (1.2) is a quantum counterpart of the classical expression of the DW Hamiltonian equations in terms of the PG brackets, where the bracket with the DW Hamiltonian function $H$ is related to the operation of the exterior differential of forms [14], whose quantum version is the Dirac operator [5, 16].

### 2 Elements of precanonical quantum YM on $\mathbb{R}^D$

In this paper, we consider a Euclidean theory over $\mathbb{R}^D$. The DW Hamiltonian operator is given by

$$\frac{1}{\kappa} \hat{H} = \frac{1}{2} \hbar^2 \kappa \partial^2_{A^p} - \frac{1}{2} i g \hbar C_{bc} A^b_i A^c_j \gamma^i \partial A^p_i, \quad (2.1)$$

where $\gamma^i, i = 1, ..., D$, are anti-Hermitian and $\gamma^i \gamma^j + \gamma^j \gamma^i = -\delta^{ij}$. This operator acts on the space of Clifford-valued wave functions

$$\Psi(A) = \psi(A) + \psi_1(A) \gamma^i + \psi_{ii_2}(A) \gamma^{i_1 i_2} + ... + \psi_{ii_2...i_D}(A) \gamma^{i_1 i_2...i_D}. \quad (2.2)$$

It is Hermitian on the space of those functions equipped with the scalar product

$$\langle \Phi, \Psi \rangle = \int [dA] [\Phi^* r \Psi]_0, \quad (2.3)$$

where $* \text{ is the complex conjugation of the components of } \Phi, \ r$ is the reversal anti-automorphism on the Clifford algebra such that, e.g., $(\gamma^i \gamma^j)^r = \gamma^j \gamma^i$, $[...]_0$ denotes the scalar part of the Clifford number under the square brackets, and $[dA] := \prod_a dA_a$. This product is not positive definite, however, because

$$\Psi^* r = \psi^*(A) + \psi^*_1 \gamma^i - \psi^*_{ii_2}(A) \gamma^{i_1 i_2} + ... + (-1)^{D(D-1)/2} \psi^*_{ii_2...i_D}(A) \gamma^{i_1 i_2...i_D}. \quad (2.4)$$

\footnote{For the relevant mathematics consult e.g. [17, 18, 19].}
The positive definite scalar product is given by
\[
\langle \Phi, \Psi \rangle = \int \[ dA \] \[ \Phi^* a \Psi \]_0 ,
\] (2.5)
where \( a \) is the main Clifford automorphism: \( \gamma^i a = -\gamma^i \). \( \hat{H} \) is pseudo-Hermitian with respect to (2.5) in the sense that \( \langle \Phi, \hat{H} \Psi \rangle = \langle \hat{H} \Phi, \Psi \rangle \). The Hermitian conjugate matrix \( \Psi^\dagger = \Psi^{* a} \), so that the integrand in (2.5) is just the Frobenius inner product of matrices. Note that, in pseudo-Euclidean space-times, the integrand in (2.3) coincides with \( [\beta \Phi^\dagger / \beta \Psi]_0 =: [\Psi_0 \Phi]_0 \) used in our previous papers.

3 Spectrum of DW Hamiltonian of SU(2) YM

We are interested in the eigenvalue problem for the DW Hamiltonian operator divided by \( \kappa \):
\[
\frac{1}{\kappa} \hat{H} \Phi = \mu \Phi .
\] (3.1)
In the case of SU(2) gauge group, the structure constants \( C_{abc} \) are the Levi-Civita symbol \( \epsilon_{abc} \) and \( A^i_a \) is a triplet of vector fields \( A, B, C \) with the components \( A_i, B_i, C_i, i = 1, ..., D \). In this notation, the operator in Eq. (2.1) takes the form
\[
\frac{1}{\kappa} \hat{H} = -\frac{\hbar^2 \kappa}{2} (\partial AA + \partial BB + \partial CC) + \frac{g \hbar}{2} (\hat{A} \hat{L}_{BC} + \hat{B} \hat{L}_{CA} + \hat{C} \hat{L}_{AB}) ,
\] (3.2)
where \( \partial AA := \sum_i \partial A_i A_i, \hat{A} := A_i \gamma^i \) and \( \hat{L}_{AB} := i(A_i \partial B_i - B_i \partial A_i) \). Let us notice the permutation symmetry between \( A, B \) and \( C \) which will allow us to simplify the problem.

In order to analyze the spectrum of (3.2), let us rewrite it in the form
\[
\frac{1}{\kappa} \hat{H} = -\frac{\hbar^2 \kappa}{2} (\partial AA + \partial BB + \partial CC) + \frac{g \hbar}{2} (\hat{A} \hat{L}_{BC} + \hat{B} \hat{L}_{CA} + \hat{C} \hat{L}_{AB}) - \frac{\hbar^2 \kappa}{2} (\partial BB + \partial CC) =: \hat{G}_A - \frac{\hbar^2 \kappa}{2} (\partial BB + \partial CC) .
\] (3.3)
Here, the permutation symmetry between \( A, B \) and \( C \) is not manifest anymore. In fact, the arguments below can be presented in a manifestly permutation symmetric way by considering \( \frac{1}{\kappa} \hat{H} \) in the form
\[
\frac{1}{3} \left( \hat{G}_A + \hat{G}_B + \hat{G}_C - \hbar^2 \kappa (\partial AA + \partial BB + \partial CC) \right) ,
\] (3.4)
where \( \hat{G}_B \) and \( \hat{G}_C \) are obtained from \( \hat{G}_A \) by cyclic permutation of \( A, B, C \). However, it would make the formulae more cumbersome and the presentation lengthier without adding to its essence.

Thus, we proceed by concentrating on the operator \( \hat{G}_A \) which has absorbed all the terms with \( A \) and \( \partial A \) in (3.3). At first we note that \( \hat{L}_{BC} = \sum_i i(B_i \partial C_i - C_i \partial B_i) \).
is similar to the angular momentum operator $\hat{L}_z$ in quantum mechanics. Therefore, the eigenvalues of $\hat{L}_{BC}$ are integers $m := \sum_i m_i = m_1 + m_2 + \ldots + m_D \in \mathbb{Z}$ with the admissible values of each $m_i$ being limited by the quantum numbers $l_i$ of the operators $\hat{L}_{A,B_i}^2 + \hat{L}_{A,C_i}^2 + \hat{L}_{B,C_i}^2$ for the corresponding value of $i$ (no summation over $i$ here): $|m_i| \leq l_i$. Similarly, the eigenvalues of $\hat{L}_{ij} := i(B_i\partial_{C_j} - C_j\partial_{B_i})$ are integers $m_{ij} \in \mathbb{Z}$ whose absolute values are restricted by the quantum numbers $l_{ij}$ of the operators $\hat{L}_{ij}$. Because $[\hat{L}_{BC}, \hat{L}_{ij}] = 0$, we can chose the basis $|m, m_{ij}\rangle$ in which both operators are diagonal. Then the operator in the first line of (3.3) takes the form

$$\hat{G}_A := -\frac{\hbar^2 \gamma}{2} \partial_{AA} + \frac{i\hbar}{2} (BC - CB) \cdot \partial_A + \frac{g\hbar}{2} (m A + m_{ij} A_i \gamma^j). \quad (3.5)$$

Let us estimate the spectrum of $\hat{G}_A$ when $B$ and $C$ are treated as external parameters. By denoting $\tilde{m}_{ij} := m \delta_{ij} + m_{ij}$ and $D^2 := B^2C^2 - (B \cdot C)^2$ we rewrite (3.5) in the form

$$\hat{G}_A = \frac{\hbar^2 \gamma}{2} \left(i \partial_{A_i} + \frac{g}{2\kappa} \gamma^j \left(B_j C_i - C_j B_i\right)\right)^2 + \frac{g^2}{4\kappa} D^2 + \frac{g\hbar}{2} \gamma^j \tilde{m}_{ij} A_j, \quad (3.6)$$

which resembles the magnetic Schrödinger operator in $A$-space with the Clifford-algebra-valued analogs of electric and magnetic potentials in that space. The “magnetic potential” term $A_i := \frac{g}{2\kappa} \gamma^j (B_j C_i - C_j B_i)$ is constant in $A$-space: $\partial_A A = 0$, hence it can be expected to contribute only a phase factor to the solution and does not contribute to the eigenvalues of $\hat{G}_A$. The “electric potential” term contains the linear term in $A_i$ and the constant term $\frac{g^2}{4\kappa} D^2$ (in the sense that $\partial_A D^2 = 0$) which just shifts the eigenvalues.

At first, let us consider the case when all quantum numbers $\tilde{m}_{ij} = 0$. In order to find the magnetic phase factor $U$, let us set $\Psi(A) = U(A; B, C)\Phi(A)$, so that all the parametric dependence of the eigenstates of $\hat{G}_A$ of $B$ and $C$ is absorbed in the phase factor $U$. Obviously, the eigenvalue problem

$$\left(-\frac{\hbar^2 \gamma}{2} \partial_{AA} + \frac{i\hbar}{2} (BC - CB) \cdot \partial_A\right) \Psi = \chi \Psi \quad (3.7)$$

leads to the equations

$$-\hbar^2 \gamma \partial_A U + \frac{i\hbar}{2} (BC - CB) U = 0 \quad (3.8)$$

$$-\hbar^2 \gamma \partial_{AA} U + \frac{i\hbar}{2} (BC - CB) \cdot \partial_A U = \xi U \quad (3.9)$$

$$-\hbar^2 \gamma \partial_{AA} \Phi = (\chi - \xi) \Phi. \quad (3.10)$$

From (3.8) it follows

$$-\hbar^2 \gamma \partial_{AA} U + \frac{g^2}{4\kappa} D^2 U = 0. \quad (3.11)$$
By substituting (3.8) in (3.9), we obtain
\[- \frac{\hbar^2 \kappa}{2} \partial_{AA} U + \frac{g^2}{4 \kappa} D^2 U = \xi U. \tag{3.12}\]

Therefore, \(\xi = \frac{g^2}{4 \kappa} D^2\) and eq. (3.7) for \(\Psi\) is equivalent to the following equations for \(\Phi(A)\)
\[- \frac{\hbar^2 \kappa}{2} \partial_{AA} \Phi + \frac{g^2}{4 \kappa} D^2 \Phi = \chi \Phi \tag{3.13}\]
and the phase factor \(U\)
\[\gamma^i \partial_{A_i} U = - \frac{ig}{2\hbar \kappa} \gamma^{ij} (B_i C_j - B_j C_i) U. \tag{3.14}\]

The latter is obtained by contracting (3.8), which is not an integrable equation, with \(\gamma^i\). The solutions of (3.14) which satisfy the condition \(U(A = 0) = 1\) can be viewed as a hypercomplex generalizations of the exponential function (different from those considered in Clifford analysis (c.f. [19, 20, 21])).

From (3.13) we conclude that \(\chi \geq \frac{g^2}{4 \kappa} D^2\). Hence, for the states of (3.6) corresponding to the lowest quantum numbers \(\tilde{m}_{ij} = 0\), we can estimate the lower bound of the eigenvalues of \(\widehat{G}_A\):
\[\widehat{G}_A \geq \frac{g^2}{4 \kappa} D^2. \tag{3.15}\]

This estimation is suitable for the eigenvalues of \(\widehat{H}\) corresponding to the “colorless” states (i.e. those invariant with respect to the internal SU(2) rotations).

The states with non-vanishing quantum numbers \(m\) and/or \(m_{ij}\) correspond to the “colored” states. Let us consider the influence of the last term in (3.6) on the estimation (3.15). To this end, we need to study the eigenvalue problem of \(\widehat{G}_A\) in more detail.

We use the fact that the second term in (3.6) leads to the phase factor \(U\) in the wave function, which transforms (3.7) to (3.13). Let us omit in (3.6) the unnecessary coefficients and the additive term \(D^2\), and consider the eigenvalue problem
\[- \partial_{AA} + g \gamma^i \tilde{m}_{ij} A_j \right) \Phi(A) = \lambda' \Phi(A) \tag{3.16}\]
for arbitrary fixed values of \(\tilde{m}_{ij} = m \delta_{ij} + m_{ij}\). The estimation (3.15) will hold also for the “colored” states if \(\lambda' \geq 0\) for any \(\tilde{m}_{ij}\).

To check if this is the case, let us first consider a special case when \(\tilde{m}_{ij}\) has only one independent non-vanishing component \(m_{12}\), which can be either positive or negative integer. Then (3.16) reads
\[\left(- \sum_{k \neq 1,2} \partial_{A_k A_k} + \partial_{A_1 A_1} - \partial_{A_2 A_2} + g m_{12} (\gamma^1 A_2 - \gamma^2 A_1)\right) \Phi = \lambda' \Phi. \tag{3.17}\]
By separating the variables with $k \neq 1, 2$ and $k = 1, 2$, we obtain a two-dimensional eigenvalue problem

$$
(-\partial_{A_1 A_1} - \partial_{A_2 A_2} + g m_{12}(\gamma^1 A_2 - \gamma^2 A_1)) \Phi(A) = \lambda'' \Phi(A),
$$

(3.18)

which can be seen as a two-dimensional Clifford-algebraic generalization of the Airy equation [22, 24] for the Clifford-valued wave function

$$
\Phi(A) = \phi(A) + \phi_1(A) \gamma^i.
$$

(3.19)

Using the notation $\Delta_{12} := \partial_{A_1 A_1} + \partial_{A_2 A_2}$, let us write (3.18) in the component form

$$
-\Delta_{12} \phi - g m_{12}(A_2 \phi_1 - A_1 \phi_2) = \lambda'' \phi
$$

$$
-\Delta_{12} \phi_1 + g m_{12} A_2 \phi = \lambda'' \phi_1
$$

$$
-\Delta_{12} \phi_2 - g m_{12} A_1 \phi = \lambda'' \phi_2
$$

$$
A_1 \phi_1 + A_2 \phi_2 = 0
$$

$$
A_i \phi_k = 0 \text{ for } k \neq 1, 2.
$$

(3.20)

The solutions are

$$
\phi_1 = \pm \frac{A_2}{|A|} \phi, \quad \phi_2 = \pm \frac{A_1}{|A|} \phi \text{ and } \phi_k = 0 \text{ for } k \neq 1, 2,
$$

where $|A| := \sqrt{A_1^2 + A_2^2}$, and the non-vanishing components of $\Phi$ obey

$$
-\Delta_{12} \phi \mp g m_{12}|A|\phi = \lambda'' \phi
$$

$$
-\Delta_{12} \phi_i \pm g m_{12}|A|\phi_i = \lambda'' \phi_i, \quad i = 1, 2.
$$

(3.21)

(3.22)

This system has a discrete positive spectrum both for positive and negative values of $m_{12}$. For example, with the upper choice of the sign, the eigenfunctions $\Phi(u)$ have a vanishing scalar part $\phi$ for $m_{12} > 0$ or vanishing vector components $\phi_{1,2}$ for $m_{12} < 0$. In both cases, the separation of angular and radial variables in the two-dimensional Laplacian $\Delta_{12}$ leads to the radial equation for the lowest (vanishing) quantum number of the orbital angular momentum corresponding to the rotations in $(A_1, A_2)$-plane:

$$
(-\partial_{\rho \rho} - \frac{1}{\rho} \partial_\rho + g|m_{12}| \rho) \phi(\rho) = \lambda'' \phi(\rho),
$$

(3.23)

where $\rho := |A| \geq 0$. The linear growth of the potential term ensures that the spectrum is discrete with the eigenvalues corresponding to the boundary condition $\partial_A \langle \Phi^* \Phi \rangle = 0$ at $|A| = 0$. Hence, for any value of $m_{12}$ the spectrum of (3.18) is positive and discrete. Correspondingly, the spectrum of (3.17) is positive and bounded from below by the lowest eigenvalue of (3.18): $\lambda' = \lambda'' + (p_k)^2$, where the last term $(p_k)^2 > 0$ is the continuous spectrum of $-\sum_{k \neq 1,2} \partial_{A_k A_k}$.

Similarly, for arbitrary $\tilde{m}_{ij}$, we obtain from (3.16)

$$
-\partial_{A A} \phi - g \tilde{m}_{ij} A_j \phi_i = \lambda'' \phi
$$

$$
-\partial_{A A} \phi_i + g \tilde{m}_{ij} A_j \phi = \lambda'' \phi_i
$$

$$
\tilde{m}_{ij} A_j \phi_k = 0.
$$

(3.24)
The last equation is solved by \( \phi_i = \pm \tilde{m}_{ij} A_j |\tilde{m}A| \), where \( |\tilde{m}A| := \sqrt{\tilde{m}_{ij} A_j \tilde{m}_{ik} A_k} \). Then (3.24) yields

\[
-\partial AA \phi \mp g |\tilde{m}A| \phi = \lambda'' \phi, \quad (3.25)
\]

\[
-\partial AA \phi_i \mp g |\tilde{m}A| \phi_i = \lambda'' \phi_i. \quad (3.26)
\]

The positive discrete spectrum is obtained either for purely scalar wave functions \( \Phi = \phi \) or purely vector ones \( \Phi = \phi_i \gamma^i \) for the lower or upper choice of the signs, respectively. The eigensolutions correspond to the boundary conditions \( \partial_{\mathbf{A}} \langle \Phi^\dagger \Phi \rangle = 0 \) at \( |\tilde{m}A| = 0 \). If \( \text{rank}(\tilde{m}_{ij}) < D \), the spectrum has the form \( \lambda'' = \lambda' + p^2 \), where \( \lambda' \) are purely discrete and \( p^2 \) is the continuous norm squared of a vector in the Euclidean \((D - \text{rank}(\tilde{m}))\)-dimensional space.

Therefore, it is shown that

**Lemma 1.** For any non-vanishing values of quantum numbers \( m \) and \( m_{ij} \) the spectrum of (3.16) is discrete and bounded from below.

Note that this Lemma is the necessary condition for the validity of the estimation (3.14) for the “colored” states with \( \tilde{m}_{ij} \neq 0 \). The sufficient condition has to take into account the non-commutativity of the Clifford-algebra-valued potential term \( g\gamma^i \tilde{m}_{ij} A_j U \) with the phase factor \( U \) which is a solution of (3.14). It requires an analysis of a generalization of (3.16) with the potential term \( g\gamma^i \tilde{m}_{ij} A_j U^{-1} \) which will parametrically depend on \( \mathbf{B} \) and \( \mathbf{C} \). As our main interest here is the spectrum of the “colorless” states of \( \hat{H} \) with vanishing \( \tilde{m}_{ij} \), we leave this part of the problem beyond the scope of the paper.

Now, using this result in eq. (3.3), we can write the following inequality for the operator \( \frac{1}{\kappa} \hat{H} \), which is valid at least for the colorless states:

\[
\frac{1}{\kappa} \hat{H} \geq -\frac{\hbar^2}{2} \left( \partial_{\mathbf{B}} \partial_{\mathbf{B}} + \partial_{\mathbf{C}} \partial_{\mathbf{C}} \right) + \frac{g^2}{4\kappa} \left( \mathbf{B}^2 \mathbf{C}^2 - (\mathbf{B} \cdot \mathbf{C})^2 \right). \quad (3.27)
\]

The spectrum of the operator in the r.h.s. of (3.27) is purely discrete and bounded from below, as it was already proven e.g. in sect. 7 of [23] using the Fefferman-Phong theorem. Then the inequality (3.27) proves the main assertion of this paper:

**Theorem 1.** The spectrum of colorless states of the DW Hamiltonian operator of quantum SU(2) Yang-Mills field is purely discrete and bounded from below.

An immediate consequence of the discreteness of the spectrum of colorless states of the DW Hamiltonian operator is the nonvanishing gap between the ground state and the lowest colorless excited state. It means that the first propagating colorless excited mode of quantum YM field is massive.

It is interesting to consider the eigenvalue problem of the operator in the r.h.s. of (3.27) in more detail. In order to estimate its spectrum, let us simplify the problem by approximating the potential term \( \mathbf{B}^2 \mathbf{C}^2 - (\mathbf{B} \cdot \mathbf{C})^2 = \mathbf{B}^2 \mathbf{C}^2 \sin^2 \theta \), where \( \theta \) is the angle between \( \mathbf{B} \) and \( \mathbf{C} \), by its average over \( \theta \in [0, 2\pi] \):

\[
\frac{1}{2\pi} \int_0^{2\pi} d\theta \mathbf{B}^2 \mathbf{C}^2 \sin^2 \theta = \frac{1}{2} \mathbf{B}^2 \mathbf{C}^2.
\]
Then the estimation in (3.27) is replaced by an approximate inequality

\[ \frac{1}{\kappa} \hat{H} \gtrsim -\frac{\hbar^2 \kappa}{2} (\partial_{BB} + \partial_{CC}) + \frac{g^2}{8\kappa} B^2 C^2. \] (3.28)

By employing the argument by B. Simon in sect. 2 of [23] and using the exactly known ground state of isotropic harmonic oscillator in \( D \)-dimensions, we further obtain

\[ \frac{1}{\kappa} \hat{H} \gtrsim -\frac{\hbar^2 \kappa}{2} \partial_{BB} + \frac{D g \hbar}{4} |B|. \] (3.29)

By going to the spherical coordinates in \( B \)-space and neglecting the knowingly positive contribution of the square of the orbital angular momentum operator in \( B \)-space, we obtain a further approximate estimation

\[ \frac{1}{\kappa} \hat{H} \gtrsim -\frac{\hbar^2 \kappa}{2} \left( \partial_{rr} + \frac{D - 1}{r} \partial_r \right) + \frac{D g \hbar}{4} r =: \hat{R}, \] (3.30)

where \( r \) denotes \( |B| \). The eigenvalue problem \( \hat{R} f = \mu f \) with \( r \in \mathbb{R}_+ = [0, \infty) \) leads to the solutions which interpolate between the Bessel function \( J_{D/2 - 1} (\sqrt{\mu} r) / r^{D/2 - 1} \) at \( r \to 0 \) and the Airy function \( \text{Ai}(a(r - b)) \) with \( a = \left( \frac{D g}{2 \hbar} \right)^{1/3} \) and \( b = \frac{4}{D g \hbar} \) at \( r \to \infty \). The spectrum is discrete and positive, and it follows from the condition \( \partial_r f^2 = 0 \) at \( r = 0 \): \( \mu_K = f(K, D) \left( \frac{2^{D/2 - 3/2} \kappa}{D g^2 \hbar} \right)^{1/3} \), where the coefficients \( f(K, D) \) labeled by \( K \in \mathbb{Z}_+ \) are given by the roots of \( f(r) \) and its derivatives (similarly to the one dimensional \( |x| \) problem in quantum mechanics, see [22, 24, 25]).

At \( D = 3 \), the spectral problem for the operator in the r.h.s. of (3.30) is quasi-exactly solvable [26, 27]. Namely, an exact solution can be obtained for the lowest eigenstates corresponding to the vanishing orbital angular momentum quantum number in \( B \)-space. This fact allows us to estimate the lower bound of \( \frac{1}{\kappa} \hat{H} \) in the physically relevant case of \( D = 3 \) which can be related to the YM theory in (3 + 1)-dimensional pseudoeuclidean space-time in the temporal gauge \( A_0 = 0 \) (c.f. [3]). In this case,

\[ \frac{1}{\kappa} \hat{H} \gtrsim \hat{R} = -\frac{\hbar^2 \kappa}{2} \left( \partial_{rr} + \frac{2}{r} \partial_r \right) + \frac{3}{4} g \hbar r. \] (3.31)

The solutions of the eigenvalue problem \( \hat{R} f = \mu f \) (up to a normalization factor \( \mathcal{N} \)) are

\[ f(r) = \mathcal{N} r^{-1} \text{Ai} (\gamma r - \delta), \] (3.32)

where

\[ \gamma = \left( \frac{3g}{2 \hbar \kappa} \right)^{1/3}, \quad \delta = \mu \left( \frac{32}{9 g^2 \hbar^4 \kappa} \right)^{1/3}. \] (3.33)

Note that the wave functions \( f(r) \) are normalized to a delta function on \( r \in [0, +\infty) \) with the integration measure \( \sim r^2 dr \). Again, there are two families of eigenvalues due to the boundary conditions \( f(r) = 0 \) and \( f'(r) = 0 \) at \( r = 0 \), which require
the admissible values of $\delta$ to be roots of the Airy function $Ai$ and its derivative. Hence, the spectrum of $\hat{R}$ is discrete and bounded from below, and the eigenvalues are labeled by the non-negative integers $K \in \mathbb{Z}_+$

$$
\mu_K = \left( \frac{9g^2\hbar^4}{32} \right)^{1/3} \left| a_{K/2+1}' \right| \quad \text{for even } K, \\
\mu_K = \left( \frac{9g^2\hbar^4}{32} \right)^{1/3} \left| a_{(K+1)/2} \right| \quad \text{for odd } K,
$$

(3.34)

where $a_M$ and $a'_M$ denote the $M$-th root of the Airy function $Ai$ and its derivative. Then the approximate estimation $\frac{1}{\kappa} \hat{H} \gtrsim \hat{R}$ proves Theorem 1 independently from the result by B. Simon quoted above (though still using the insights from his paper $[23]$).

Moreover, from (3.34) we obtain an estimation for the ground state of the DW Hamiltonian operator at $D = 3$:

$$
\left\langle \frac{1}{\kappa} \hat{H} \right\rangle_0 \gtrsim \mu_0 = \left( \frac{9g^2\hbar^4}{32} \right)^{1/3} \left| a'_1 \right| \quad (3.35)
$$

and the gap between the first excited state and the ground state:

$$
\Delta \mu \approx \mu_1 - \mu_0 = \left( |a_1| - |a'_1| \right) \left( \frac{9g^2\hbar^4}{32} \right)^{1/3} \approx 0.86 \left( g^2\hbar^4 \right)^{1/3}. \quad (3.36)
$$

Note that, according to our conventions in (1.1), the gauge coupling constant $g$ in $3 + 1$ dimensions (i.e. the bare constant present in the Lagrangian) is given in the units of $1/\sqrt{\hbar}$, hence it is dimensionless in the units with $\hbar = 1$. Correspondingly, the dimensionality of the r.h.s of (3.36) is that of a mass. Eq. (3.36) tells us that the scale of the ultra-violet parameter $\kappa$ introduced in precanonical quantization is connected with the scale of the first massive excitation of quantum pure YM theory. Interestingly, the ground state of $\hat{H}$ in (1.1) is independent of the ordering of the multiplicative operators $A$ and the differential operators $\partial_A$ because of the specific structure of the interaction term in the DW Hamiltonian operator and the antisymmetry of the structure constants.

Let us also note that a comparison of the coefficient $Dg\hbar/4$ in front of the $|B|$ term in (3.29) with the coefficient $g^2 \tilde{m}$ in front of the linear term in (3.6) indicates that at $D > 2$ it is not excluded that there can be colored mass excitations with nonvanishing quantum numbers $\tilde{m}_{ij}$ which are lying below the estimated mass gap (3.36) for the colorless excitations with $\tilde{m}_{ij} = 0$.

4 Conclusion

The approach of precanonical quantization leads to the description of quantum pure Yang-Mills theory in terms of the Clifford-algebra-valued precanonical wave function on the space of Yang-Mills field variables $A^a_\mu$ and space-time coordinates
\[ x^\mu. \] Hence, the quantum YM field is understood as a section in the Clifford bundle over the bundle of gauge field components over space-time. This contrasts with the usual description of quantum fields in terms of the Schrödinger wave functionals or operator-valued distributions, or operator algebras. The precanonical wave function satisfies the covariant analogue of the Schrödinger equation defined on the aforementioned bundle, eq. (1.2), which is a Dirac-like PDE on this bundle, with the mass term replaced by the DW Hamiltonian operator. Note that, unlike the quantization based on the canonical Hamiltonian formalism, this approach treats all space-time dimensions on an equal footing. Moreover, the construction of precanonical quantum field theory of YM fields is intrinsically nonperturbative. We have demonstrated elsewhere [4] that this formulation is also consistent with a generalized version of the Ehrenfest theorem, i.e. the classical field equations are reproduced as the equations satisfied by the expectation values of precanonical operators [4]. Moreover, the standard quantum YM theory in the functional Schrödinger representation can be derived from our precanonical formulation in the limiting case of an infinite value of the ultraviolet parameter \( \kappa \) (see also [8]). Thus the standard QFT, which requires a UV regularization, appears as a limiting case of the precanonical formulation which has the UV scale \( \kappa \) built in \textit{ab initio}.

The appearance of the DW Hamiltonian operator in place of the mass term in the Dirac-like precanonical Schrödinger equation (1.2) indicates that its spectrum has to do with the spectrum of propagating excitations of the field, which one usually calls particles. This has motivated our interest in the spectrum of the DW Hamiltonian operator of pure YM theory. The discreteness of the spectrum, which we have proven for the colorless states, indicates that the propagating excitations in quantum YM theory are massive. Our consideration has lead to the estimation of the ground state of the DW Hamiltonian operator and the gap between the ground state and the first colorless excited state. Both expressions are \( \sim \frac{g^2}{\kappa} \frac{1}{3} \) in three spatial dimensions. This relates the scale of \( \kappa \), up to a coefficient given by the bare gauge coupling \( g \) in the Lagrangian, to the scale of the mass gap of the quantum nonabelian gauge theory under consideration.

Recently, a rough estimation of the scale of \( \kappa \) was obtained by us by a completely independent consideration based on the precanonical quantization of gravity [12]. There, \( \kappa \) appears in the ordering-dependent dimensionless combination with \( \hbar \), Newton’s \( G \) and the cosmological constant \( \Lambda \). A preliminary consideration has shown that the value of \( \kappa \) consistent with the observable values of the constants of nature is roughly at the subnuclear scale. An order of magnitude coincidence of this estimation with the above estimation of the mass gap in \( SU(2) \) quantum gauge theory indicates, albeit preliminarily, that \( \kappa \) is a fundamental scale rather than a kind of renormalization scale to be removed from the final results. Note that in spite of this indication to the fundamental scale at such a low energy, its existence in our formalism does not contradict the current experimental evidence that the relativistic space-time holds at least till the \( TeV \) energies, because the scale \( \kappa \) has been introduced in precanonical quantization without a manual modification of the relativistic space-time at small distances.
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