Abstract. We determine sufficient conditions for the occurrence of a pointwise gradient estimate for the evolution operators associated to nonautonomous second order parabolic operators with (possibly) unbounded coefficients. Moreover we exhibit a class of operators which satisfy our conditions.

1. Introduction

Let $I$ be an open right halfline and let $\{A(t)\}_{t \in I}$ be a family of second order differential operators defined on smooth functions $\zeta$ by

$$
(A(t)\zeta)(x) = \text{Tr}(Q(t,x)D^2\zeta(x)) + \langle b(t,x), \nabla \zeta(x) \rangle,
$$

where the (possibly) unbounded coefficients $Q = \{q_{ij}\}_{i,j=1,\ldots,d}$ and $b = (b_1,\ldots,b_d)$ are defined in $I \times \mathbb{R}^d$. Let us consider the nonautonomous Cauchy problem

$$
\begin{aligned}
D_t u(t,x) &= A(t)u(t,x), & (t,x) &\in (s, +\infty) \times \mathbb{R}^d, \\
u(s,x) &= f(x), & x &\in \mathbb{R}^d,
\end{aligned}
$$

with $s \in I$ and $f \in C_b(\mathbb{R}^d)$. In the pioneering paper [6], under suitable assumptions on the coefficients $q_{ij}$ and $b_i$, the authors prove the wellposedness of the problem (1.2) in the space of continuous and bounded functions defined in $\mathbb{R}^d$. The unique bounded solution of (1.2) can be written in terms of an evolution operator $G(t,s)$ associated to $A(t)$, i.e.,

$$u(t,x) = (G(t,s)f)(x), \quad t > s, \ x \in \mathbb{R}^d.$$

Many properties of the solution of problem (1.2) are investigated in [6]; in particular, in [6, Sect. 4] some sufficient conditions on the coefficients are provided in order that the pointwise gradient estimates

$$
|\langle \nabla_x G(t,s)f(x) \rangle|^p \leq e^{c_p(t-s)}(G(t,s)|\nabla f|^p)(x), \quad t > s, \ x \in \mathbb{R}^d,
$$

hold for every $p > 1$, $f \in C^1_b(\mathbb{R}^d)$ and some $c_p \in \mathbb{R}$.

The interest in this kind of estimates is due to the fact that they play a crucial role in the analysis of many qualitative properties of $G(t,s)$. Already in the autonomous case, they have been used to study the asymptotic behavior of the semigroup $T(t)$ generated by the operator in (1.1) (when $Q(t,x) = Q(x)$ and $b(t,x) = b(x)$) in $L^p(\mathbb{R}^d,\mu)$, where $\mu$ is an invariant measure of $T(t)$, i.e., a Borel probability measure such that $\int_{\mathbb{R}^d} T(t) f \, d\mu = \int_{\mathbb{R}^d} f \, d\mu$, for every $f \in C_b(\mathbb{R}^d)$ and any $t > 0$. In fact, this is the case also in the nonautonomous setting, where $T(t)$ is replaced by $G(t,s)$ and...
the single invariant measure $\mu$ is replaced by a family of Borel probability measures $\{\mu_t : t \in I\}$ called evolution system of measures, satisfying
\[
\int_{\mathbb{R}^d} (G(t,s)f)(x) d\mu_t(x) = \int_{\mathbb{R}^d} f(x) d\mu_s(x), \quad t > s \in I, \ f \in C_b(\mathbb{R}^d).
\]
In the case of $T$-time periodic (unbounded) coefficients, it has been proved in \[8\] that, if the coefficients are smooth enough and a weak dissipativity condition on the drift $b$ is assumed, then
\[
\lim_{t \to +\infty} \|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_t)} = 0, \quad s \in \mathbb{R}, \ f \in L^p(\mathbb{R}^d, \mu_s),
\]
for every $p \in [1, +\infty)$, where
\[
m_s(f) = \int_{\mathbb{R}^d} f(y) d\mu_s(y)
\]
and $\{\mu_t : t \in \mathbb{R}\}$ is the $T$-periodic evolution system of measures.

The asymptotic behavior stated in \[13\] still holds also in a non-periodic setting provided that estimate \[13\] holds for $p = 1$ and some $c_1 < 0$ (see \[1\]). Hence the problem is reduced to find conditions that imply
\[
\|\nabla G(t,s)f(x)\| \leq e^{c_1(t-s)} \|G(t,s)\nabla f\| \|f\|_{L^p(x)}, \quad t > s \in I, \ x \in \mathbb{R}^d,
\]
for functions $f \in C^1_b(\mathbb{R}^d)$. This is the case (see \[6\] Thm. 4.5) if the coefficients $q_{ij}$ $(i,j = 1, \ldots, d)$ do not depend on $x$ and
\[
\langle \nabla x b(t,x)\xi, \xi \rangle \leq r_0|\xi|^2, \quad (t,x) \in I \times \mathbb{R}^d,
\]
for some $r_0 \in \mathbb{R}$. In this case, estimate \[15\] is satisfied with $c_1 = r_0$. Actually, the gradient estimate \[15\] gives sharper information than formula \[14\]. When it is satisfied (as it has been proved in \[1\] Cor. 5.4]), the exponential decay estimate
\[
\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_t)} \leq C_p e^{c_1(t-s)} \|f\|_{L^p(\mathbb{R}^d, \mu_s)}, \quad t > s \in I,
\]
holds for every $p > 1$, $f \in L^p(\mathbb{R}^d, \mu_s)$ and some $C_p > 0$.

The fact that estimate \[15\] has been proved only when the diffusion coefficients do not depend on $x$ is not surprising since, already in the autonomous case, Wang \[2\] proved that the gradient estimate $|\nabla T(t)f| \leq e^{ct} T(t)|\nabla f|$ cannot hold if the coefficients $q_{ij}$ do not satisfy the algebraic condition:
\[
D_i q_{ij}(x) + D_j q_{ji}(x) + D_k q_{ik}(x) = 0, \quad 1 \leq i,j,k \leq d, \ x \in \mathbb{R}^d.
\]

Estimate \[15\] has been also the key formula to establish many other results on the summability improving properties of $G(t,s)$ in the $L^p$-spaces related to the unique tight evolution system of measures $\{\mu_t : t \in I\}$. In \[1\], we use \[15\] in order to prove a Logarithmic-Sobolev inequality with respect to the tight system $\{\mu_t : t \in I\}$. Moreover, we establish a connection between the Logarithmic-Sobolev inequality and the hypercontractivity of the evolution operator $G(t,s)$ in the $L^p$-spaces related to the evolution system of measures $\{\mu_t : t \in I\}$.

In \[2\], assuming \[15\], we prove some Harnack type estimates and stronger results than hypercontractivity for the evolution operator $G(t,s)$.

These results have been proved assuming that the diffusion coefficients do not depend on $x$ and formula \[16\] is satisfied, so that \[15\] holds.

Because of the great importance of formula \[15\], in this paper we provide two sufficient conditions on the coefficients $q_{ij}$ and $b_j$ in order that \[15\] is satisfied in the general case, and we show that one of them is also necessary. More precisely we prove that, if the algebraic pointwise condition
\[
D_k q_{ij}(t,x) + D_j q_{ik}(t,x) + D_i q_{kj}(t,x) = 0, \quad (t,x) \in I \times \mathbb{R}^d,
\]
is satisfied for every $i,j,k \in \{1,\ldots,d\}$ and if the dissipativity condition (which includes also the spatial derivatives of the diffusion coefficients $q_{ij}$)

$$\frac{1}{2\eta(t,x)} \sum_{i,j=1}^{d} (\nabla_x q_{ij}(t,x),\xi)^2 + (\nabla_x b(t,x)\xi,\xi) \leq c_0 |\xi|^2,$$

holds for every $\xi \in \mathbb{R}^d$, $(t,x) \in I \times \mathbb{R}^d$ and some $c_0 \in \mathbb{R}$, (see (2.1) for the definition of $\eta$), then the gradient estimate (1.5) is satisfied. Moreover, as in the autonomous case, condition (1.8) is necessary for estimate (1.5).

The proof of these facts relies on the connection between the gradient estimate (1.5) and the uniform Bakry type estimate

$$\langle \nabla f, \nabla_x (A(s)f) \rangle \leq |\nabla f|(A(s)|\nabla f|) + c|\nabla f|^2, \quad f \in C^\infty(\mathbb{R}^d), \ s \in I.$$  

Unfortunately, differently from the autonomous case (where they are equivalent, see [3]), we are able to prove only that estimate (1.9) is a necessary condition for the gradient estimate (1.5) hold, hence we prove the main result of the paper following [3], we are able to prove only that estimate (1.9) is a necessary condition for the gradient estimate (1.5) hold, hence we prove the main result of the paper following [9].

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Notations. Let $k \in [0, +\infty)$, we denote by $C^k_b(\mathbb{R}^d)$ the set of functions in $C^k(\mathbb{R}^d)$ which are bounded together with all their derivatives up to the $[k]$-th order and such that the $[k]$-th order derivatives are $k$-Hölder continuous in $\mathbb{R}^d$. We use the subscript “$c$” instead of “$b$” for the subsets of the above spaces consisting of functions with compact support.

If $J \subset \mathbb{R}$ is an interval and $\alpha \in (0, 1)$, $C^{k+\alpha/2,2k+\alpha}_{\text{loc}}(J \times \mathbb{R}^d)$ $(k = 0, 1)$ denotes the set of functions $f : J \times \mathbb{R}^d \to \mathbb{R}$ such that the time derivatives up to the $k$-th order and the spatial derivatives up to the $2k$-th order are Hölder continuous with exponent $\alpha$, with respect to the parabolic distance, in any compact set of $J \times \mathbb{R}^d$.

Analogously we define the space of functions $C^{k+\alpha/2,2k+\alpha}_{\text{loc}}(J \times \mathbb{R}^d)$.

About partial derivatives, the notations $D_1 f := \frac{\partial f}{\partial t}$, $D_i f := \frac{\partial f}{\partial x_i}$, $D_{ij} f := \frac{\partial^2 f}{\partial x_i \partial x_j}$ are extensively used.

About matrices and vectors, we denote by $\text{Tr}(Q)$, $\langle x, y \rangle$ and $|x|$ the trace of the square matrix $Q$, the inner product of the vectors $x, y \in \mathbb{R}^d$ and the Euclidean norm of $x$, respectively.

The ball in $\mathbb{R}^d$ centered at $x_0$ with radius $r > 0$ is denoted by $B(x_0, r)$. When $x_0 = 0$, we simply write $B_r$ instead of $B(x_0, r)$.

2. Assumptions, definitions and a review of some properties of $G(t,s)$

First we state our standing assumptions and we collect some known results.

Let $I$ be an open right halfline. For every $t \in I$, we consider the linear second order differential operator $A(t)$ defined on smooth functions $\zeta$ by

$$(A(t)\zeta)(x) = \sum_{i,j=1}^{d} q_{ij}(t,x)D_{ij}\zeta(x) + \sum_{i=1}^{d} b_i(t,x)D_i\zeta(x)$$

$$= \text{Tr}(Q(t,x)D^2\zeta(x)) + \langle b(t,x), \nabla\zeta(x) \rangle, \quad x \in \mathbb{R}^d.$$  

The standing hypotheses on the data $Q = [q_{ij}]_{i,j=1,\ldots,d}$ and $b = (b_1, \ldots, b_d)$ are the following:
Hypotheses 2.1. (i) The coefficients $a_{ij}$ and $b_i (i, j = 1, \ldots, d)$ and their first order spatial derivatives belong to $C^{\alpha/2,\alpha}_{\text{loc}} (I \times \mathbb{R}^d)$ for some $\alpha \in (0, 1)$; (ii) the symmetric matrix $Q(t, x) = [q_{ij}(t, x)]_{i, j = 1, \ldots, d}$ is uniformly elliptic, i.e., there exists a function $\eta : I \times \mathbb{R}^d \to \mathbb{R}$ such that $0 < \eta_0 = \inf_{I \times \mathbb{R}^d} \eta$ and
\[ (Q(t, x)\xi, \xi) \geq \eta(t, x)|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad (t, x) \in I \times \mathbb{R}^d; \] (2.1) (iii) for every bounded interval $J \subset I$ there exist a function $\varphi = \varphi_J \in C^2(\mathbb{R}^d)$ with positive values, such that $\lim_{|x| \to +\infty} \varphi(x) = +\infty$, and a positive number $\gamma = \gamma_J$ such that
\[ (A(t)\varphi)(x) \leq \gamma \varphi(x), \quad (t, x) \in J \times \mathbb{R}^d. \] (2.2)

Under these assumptions, for every $s \in I$ and $f \in C^0_b(\mathbb{R}^d)$, the problem
\[ \begin{aligned}
D_t u(t, x) &= A(t)u(t, x), \quad (t, x) \in (s, +\infty) \times \mathbb{R}^d, \\
u(s, x) &= f(x), \quad x \in \mathbb{R}^d,
\end{aligned} \] (2.3)

admits a unique bounded classical solution, i.e., there exists a unique function $u \in C^0_b([s, +\infty) \times \mathbb{R}^d) \cap C^{1,2}((s, +\infty) \times \mathbb{R}^d)$ that satisfies (2.3). Moreover,
\[ \|u(t, \cdot)\| \leq \|f\|, \quad t \geq s. \] (2.4)

We point out that condition (i) is not minimal for the well-posedness of problem (2.3). In order to get existence and uniqueness of a solution to the problem (2.3), besides Hypotheses 2.1 (ii)-(iii), it suffices to require only that the coefficients $a_{ij}$ and $b_i$ belong to $C^{\alpha/2,\alpha}_{\text{loc}} (I \times \mathbb{R}^d)$. The additional hypothesis on the regularity of the first-order spatial derivatives of the coefficients is used to prove that the solution is smoother.

The unique bounded solution $u$ to the problem (2.3) can be represented by means of a positive evolution operator $G(t, s)$ associated to $A(t)$, by setting $G(t, t) := \text{id}_{C^0_b(\mathbb{R}^d)}$ for every $t \in I$ and
\[ (G(t, s)f)(x) := u(t, x), \quad (t, x) \in (s, +\infty) \times \mathbb{R}^d. \]

As already noticed, uniqueness of the solution of (2.3) is immediate consequence of Hypothesis 2.1 (iii) and is proved by means of the following maximum principle.

Proposition 2.2. Let $s \in I$ and $T > s$. If $u \in C^0_b([s, T] \times \mathbb{R}^d) \cap C^{1,2}((s, T) \times \mathbb{R}^d)$ satisfies
\[ \begin{aligned}
D_t u(t, x) - A(t)u(t, x) &\leq 0, \quad (t, x) \in (s, T] \times \mathbb{R}^d, \\
u(s, x) &\leq 0, \quad x \in \mathbb{R}^d,
\end{aligned} \]
then $u(t, x) \leq 0$ for every $(t, x) \in [s, T] \times \mathbb{R}^d$.

Proof. See [3] Thm. 2.1 and the reference therein. 

The next lemma provides a regularity result when the initial datum $f$ is smooth enough.

Lemma 2.3. If $f \in C^{3+\alpha}_{\text{c}}(\mathbb{R}^d)$, then the solution $u$ to the problem (2.3) belongs to $C^{1+\alpha/2,3+\alpha}_{\text{loc}}([s, +\infty) \times \mathbb{R}^d)$.

Proof. Assume that $f$ belongs to $C^{3+\alpha}_{\text{c}}(\mathbb{R}^d)$. Let $m$ be the smallest integer such that $\text{supp} f \subset B_m$. For every $n > m$, we consider the Cauchy-Dirichlet problem
\[ \begin{aligned}
D_t u(t, x) &= A(t)u(t, x), \quad (t, x) \in (s, +\infty) \times B_n, \\
u(s, x) &= f(x), \quad x \in B_n, \\
u(t, x) &= 0, \quad (t, x) \in (s, +\infty) \times \partial B_n.
\end{aligned} \] (2.5)
From Hypothesis (2.1(i)) and the classical results in [3] Thms. 3.3.7–3.5.12, problem (2.5) admits a unique solution $u_n \in C^{1+\alpha/2,3+\alpha}([s, +\infty) \times \mathbb{B}_n)$ such that
$$\|u_n(t, \cdot)\| \leq \|f\|, \quad t \geq s. \quad (2.6)$$

The local Schauder estimates (see [7] Thm. IV.10.1) and estimate (2.6) yield that, for every $k < n$, there exists a positive constant $c_k$, independent on $n$, such that
$$\|u_n\|_{C^{1+\alpha/2,3+\alpha}([s, +\infty) \times \mathbb{B}_k)} \leq c_k \|f\|_{C^{3+\alpha}([s, +\infty) \times \mathbb{R}^d)}. \quad (2.7)$$

By the Arzelà-Ascoli theorem we deduce that there exists a subsequence $(u_{n_k})$ of $(u_n)$ which converges in $C^{1,3}([s, s+k] \times \mathbb{B}_k)$ to a function $u^k \in C^{1+\alpha/2,3+\alpha}([s, s+k] \times \mathbb{B}_k)$, which satisfies the equation $u^k = A(t)u^k$ in $(s, s+k) \times \mathbb{B}_k$. Moreover, $u^k(s, \cdot) = f$ in $\mathbb{B}_k$. Since, without loss of generality, we can assume that $(u_{n_k}^k)$ is a subsequence of $(u_n^k)$ and hence $u^{k+1} = u^k$ in $(s, s+k) \times \mathbb{B}_k$, we can define the function $u : [s, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ by setting $u(t, x) = u^k(t, x)$ for every $(t, x) \in (s, s+k) \times \mathbb{B}_k$ and every $k \in \mathbb{N}$. The function $u$ belongs to $C^{1+\alpha/2,3+\alpha}_{loc}([s, +\infty) \times \mathbb{R}^d)$, satisfies (2.6) and it is the unique solution of problem (2.3), due to Proposition 2.2.

In the next proposition, following the ideas in [3], we establish a connection between the gradient estimate satisfied by $G(t, s)$ and the Bakry type estimate (1.9) (introduced in the autonomous setting in [3]) satisfied by the operator $A(t)$. More precisely, we prove that the Bakry type estimate is a necessary condition for the gradient estimate (1.9) hold.

**Proposition 2.4.** Assume that there exists $c \in \mathbb{R}$ such that, for every $f \in C^1_0(\mathbb{R}^d)$ and $I \ni s \leq t$,
$$|\nabla_x G(t, s, f)| \leq e^{c(t-s)}G(t, s)|\nabla f|. \quad (2.7)$$

Then, the estimate
$$|\nabla f, \nabla_x (A(s)f)| \leq |\nabla f|A(s)|\nabla f| + c|\nabla f|^2 \quad (2.8)$$
holds for every $f \in C^2(\mathbb{R}^d)$ and $s \in I$.

**Proof.** It suffices to prove (2.8) at any $x_0 \in \mathbb{R}^d$ such that $|\nabla f(x_0)| > 0$. Formula (2.7) yields
$$\frac{1}{t-s} \left| (\nabla_x G(t, s, f) - \nabla_x G(s, s, f) - \nabla_x G(t, s, f) - \nabla_x G(s, s, f)) \right|^2 \leq \frac{1}{t-s} \left( e^{2c(t-s)}G(t, s)|\nabla f|^2 - |\nabla f|^2 \right). \quad (2.9)$$

for any $t > s \in I$. We notice that the left and the right hand sides of (2.8) represent, respectively, the incremental ratio at $t = s$ of the functions $t \mapsto |\nabla_x G(t, s, f)|^2 =: h_1(t)$ and $t \mapsto e^{2c(t-s)}G(t, s)|\nabla f|^2 =: h_2(t)$.

We prove first (2.8) for $f \in C^\infty_c(\mathbb{R}^d)$. The smoothness of the coefficients $q_{ij}$ and $b_i$ and Lemma 2.3 yield that the first-order spatial derivatives of $G(t, s)f$ belong to $C^{1+\alpha/2,2+\alpha}_{loc}((s, +\infty) \times \mathbb{R}^d)$, hence
$$D_i(\nabla_x G(t, s, f)) = \nabla_x (D_i(G(t, s, f))) = \nabla_x (A(t)G(t, s, f))$$
and consequently
$$h_1(t) = 2\nabla_x (A(t)G(t, s, f)), \quad t > s.$$ 

Moreover, again the smoothness of $f$ and of the coefficients of $A(t)$, together with Lemma 2.3, imply that the functions $\nabla_x G(\cdot, s, f)$ and $\nabla_x (A(\cdot)G(\cdot, s, f))$ are continuous in $(s, +\infty) \times \mathbb{R}^d$. Hence $h_1$ is differentiable also in $t = s$ and $h_1'(s) = 2\nabla_x (A(s,f), \nabla f)$. Let us observe that the derivative of the function $h_2$ is given by
$$h_2'(t) = 2ch_2(t) + 2e^{2c(t-s)}G(t, s)|\nabla f|G(t, s)|\nabla f|, \quad t > s.$$


Since the function $t \mapsto A(t)G(t,s)|\nabla f|$ is not (necessarily) continuous up to $s$, we consider a function

$$g \in C^{\infty}_c(\mathbb{R}^d), \quad g = |\nabla f| \text{ in a neighborhood of } x_0 \quad \text{and} \quad g \geq |\nabla f| \text{ in } \mathbb{R}^d. \quad (2.10)$$

In this case, $G(t,s)g \in C^{1+\alpha/2,2+\alpha}_{loc}([s, +\infty) \times \mathbb{R}^d)$ and $(A(s)g)(x_0) = (A(s)|\nabla f|)(x_0)$.

From (2.7), (2.10) and the positivity of $G(t,s)$ we deduce that

$$|\nabla_x G(t,s)f|^2 \leq e^{2c(t-s)}(G(t,s)g)^2, \quad (2.11)$$

with equality at $t = s$. Taking the derivatives with respect to $t$ at $t = s$ of both sides in (2.11), we get

$$2\langle \nabla_x (A(s)f), \nabla f \rangle \leq 2(eg^2 + g(A(s)g)),$$

hence,

$$\langle \nabla_x (A(s)f)(x_0), \nabla f(x_0) \rangle \leq c|\nabla f(x_0)|^2 + |\nabla f(x_0)|(A(s)|\nabla f|)(x_0), \quad s \in I. \quad (2.12)$$

To conclude the proof in this case, we determine a function $g$ which satisfies (2.10).

Let $r > 0$ be such that $|\nabla f(y)| > 0$ for $|y - x_0| \leq r$. Let us consider two functions $\theta, \psi \in C^\infty_c(\mathbb{R}^d)$ such that $\theta = 1$ in $B(x_0, r/2)$, $\theta = 0$ in $\mathbb{R}^d \setminus B(x_0, r)$ and $\psi = 1$ in the support of $f$. Then, the function $g(y) := \psi(y)|\theta(y)\nabla f(y)| + (1 - \theta(y))|\nabla f||_\infty$, $y \in \mathbb{R}^d$, satisfies all the properties claimed in (2.10). By the arbitrariness of $x_0 \in \mathbb{R}^d$ we get (2.3) for any function $f \in C^\infty_c(\mathbb{R}^d)$.

Finally, if $f \in C^1(\mathbb{R}^d)$ we can consider a sequence of functions $f_n \in C^\infty(\mathbb{R}^d)$, which converges locally uniformly to $f$, and the sequence of functions $\tilde{f}_n := \theta_n f_n$, where $\theta_n$ is defined as follows

$$\theta_n(x) = \psi \left( \frac{|x|}{n} \right), \quad x \in \mathbb{R}^d, \quad n \in \mathbb{N}, \quad (2.12)$$

and $\psi \in C^\infty(\mathbb{R})$ satisfies $\chi_{(-\infty,1]} \leq \psi \leq \chi_{(-\infty,2]}$. Then, $\tilde{f}_n \in C^\infty_c(\mathbb{R}^d)$ for every $n \in \mathbb{N}$ and $(D^{(\alpha)}f_n)(x)$ converges to $(D^{(\alpha)}f)(x)$ as $n \to +\infty$ for every $x \in \mathbb{R}^d$ and $0 \leq |\alpha| \leq 3$. Hence, writing (2.3) for $\tilde{f}_n$ and letting $n \to +\infty$ we get the claim. \hfill \Box

3. Main theorem

This section is devoted to prove the main result of the paper. In the following theorem some sufficient conditions in order that the pointwise gradient estimate (1.5) hold are given.

**Theorem 3.1.** Assume that, for every $i, j, k = 1, \ldots, d$,

$$D_kq_{ij}(t,x) + D_jq_{ik}(t,x) + D_ig_{jk}(t,x) = 0, \quad (t,x) \in I \times \mathbb{R}^d, \quad (3.1)$$

and that there exists $c_0 \in \mathbb{R}$ such that

$$\left( \frac{1}{2\eta(t,x)} \sum_{i,j=1}^d (\nabla_x q_{ij}(t,x), \xi)^2 \right) + \langle \nabla_x b(t,x)\xi, \xi \rangle \leq c_0|\xi|^2, \quad (3.2)$$

for every $\xi \in \mathbb{R}^d$ and $(t,x) \in I \times \mathbb{R}^d$, where $\eta$ is the function defined in (2.1). Then, for every $f \in C^1_b(\mathbb{R}^d)$ and $I \ni s \leq t$,

$$|\nabla_x G(t,s)f(x)| \leq c_{c_0(t-s)}(G(t,s)|\nabla f|)(x), \quad x \in \mathbb{R}^d. \quad (3.3)$$

Conversely, assume that the gradient estimate (3.3) is satisfied for some $c_0 \in \mathbb{R}$. Then (3.1) holds for every $t \in I$, $x \in \mathbb{R}^d$ and $i, j, k = 1, \ldots, d$. \hfill \Box
Proof. We prove the first part of the statement by using a variant of the Bernstein method. Fix $s \in I$ and $\varepsilon > 0$. For every $f \in C^1_b(\mathbb{R}^d)$, set $u(t, x) := (G(t, s)f)(x)$ and define

$$w(t, x) = (|\nabla_x u(t, x)|^2 + \varepsilon)^{1/2}, \quad t \geq s, \quad x \in \mathbb{R}^d.$$ 

By [5, Thm. 3.10] and [6, Thm. 4.1, Cor. 4.4], $w \in C_0([s, T] \times \mathbb{R}^d) \cap C^{1,2}(s, T) \times \mathbb{R}^d)$ for every $T > s$ and a straightforward computation shows that

$$D_tw - A(t)w = F,$$

where

$$F = (|\nabla_x u|^2 + \varepsilon)^{-1/2} \left( \langle \nabla_x b \nabla_x u, \nabla_x u \rangle - \sum_{k=1}^d \langle Q \nabla_x D_k u, \nabla_x D_k u \rangle \right),$$

$$+ (|\nabla_x u|^2 + \varepsilon)^{-1/2} \sum_{k=1}^d D_k u \cdot \text{Tr}(D_k Q \cdot D_k^2 u)$$

$$+ (|\nabla_x u|^2 + \varepsilon)^{-3/2} \langle Q D_x^2 u \nabla_x u, D_x^2 u \nabla_x u \rangle.$$ 

First of all, let us observe that

$$F \leq (|\nabla_x u|^2 + \varepsilon)^{-1/2} \left( \langle \nabla_x b \nabla_x u, \nabla_x u \rangle - \sum_{k=1}^d \langle Q \nabla_x D_k u, \nabla_x D_k u \rangle \right),$$

$$+ \sum_{k=1}^d D_k u \cdot \text{Tr}(D_k Q \cdot D_k^2 u) + \left( \langle Q D_x^2 u \frac{\nabla_x u}{|\nabla_x u|}, D_x^2 u \frac{\nabla_x u}{|\nabla_x u|} \rangle \right).$$

Moreover,

$$\langle Q D_x^2 u \frac{\nabla_x u}{|\nabla_x u|}, D_x^2 u \frac{\nabla_x u}{|\nabla_x u|} \rangle = \sum_{k=1}^d \langle Q \nabla_x D_k u, \nabla_x D_k u \rangle$$

$$= \sum_{i,j=1}^d q_{ij} \left( \langle \nabla_x u, \nabla_x D_i u \rangle \langle \nabla_x u, \nabla_x D_j u \rangle - \langle \nabla_x D_i u, \nabla_x D_j u \rangle \right)$$

$$= - \sum_{i,j=1}^d q_{ij} \langle P(\nabla_x D_i u), P(\nabla_x D_j u) \rangle \leq -n \sum_{i=1}^d |P(\nabla_x D_i u)|^2,$$ (3.4)

where $P$ denotes the projection

$$P(v) = v - \left\langle v, \frac{\nabla_x u}{|\nabla_x u|} \right\rangle \frac{\nabla_x u}{|\nabla_x u|}, \quad v \in \mathbb{R}^d.$$ (3.5)

Hence, we have

$$F \leq \frac{1}{w} \left( \langle \nabla_x b \nabla_x u, \nabla_x u \rangle - \sum_{i,j=1}^d q_{ij} \langle P(\nabla_x D_i u), P(\nabla_x D_j u) \rangle \right)$$

$$+ \sum_{k=1}^d D_k u \cdot \text{Tr}(D_k Q \cdot D_k^2 u) \right) =: \frac{1}{w} I.$$ (3.6)

The crucial point of the first part of the proof consists in proving that

$$I(t, x) \leq c_0 |\nabla_x u(t, x)|^2,$$ (3.7)

for every $t > s$ and $x \in \mathbb{R}^d$, where $c_0$ is the constant in assumption (5.2). Indeed, in this case we obtain $D_t w - A(t)w \leq c_0w$. Since, on the other hand, the function

$$z(t, \cdot) = e^{c_0(t-s)}G(t, s)(|\nabla f|^2 + \varepsilon)^{1/2}, \quad t > s,$$
satisfies $D_t z - A(t) z = c_0 z$, we get
\[
\begin{cases}
D_t (w - z)(t, x) - [(A(t) + c_0)(w - z)](t, x) \leq 0, \\
(w - z)(s, x) = 0,
\end{cases}
\]
for every fixed $(t, x) \in (s, +\infty) \times \mathbb{R}^d$, $x \in \mathbb{R}^d$.

Thus, the maximum principle in Proposition 2.2 implies that $w \leq z$. Letting $\varepsilon \to 0^+$ and using the continuity property of $G(t, s)$ that follows from estimate (2.4), we get (3.3).

Now, let us fix $x_0 \in \mathbb{R}^d$ and $t > s$ and prove that $I(t, x_0) \leq c_0 |\nabla_x u(t, x_0)|^2$. We point out that it is not restrictive, from now on, to assume that the coefficients $q_{ij}$ are linear functions. Indeed, if we denote by $\tilde{I}$ the sum in brackets in formula (3.6) where the $q_{ij}$’s are replaced by the $\tilde{q}_{ij}$’s, defined by $\tilde{q}_{ij}(t, x_0) + \langle \nabla_x \tilde{q}_{ij}(t, x_0), x - x_0 \rangle$, $(i, j = 1, \ldots, d)$, we notice that $\tilde{I}(t, x_0) = I(t, x_0)$. Moreover $\tilde{q}_{ij}$ and $b_i$ satisfy the assumptions (3.1) and (3.2) at $(t, x_0)$ with the same constant $c_0$, and this is enough to complete the proof.

We have
\[
\sum_{k=1}^d D_k u \cdot \text{Tr}(D_k Q \cdot D_k^2 u) = \sum_{j=1}^d \langle \nabla_x D_j u, Q^j \rangle,
\]
where, for every $j = 1, \ldots, d$ and $(t, x) \in I \times \mathbb{R}^d$, $Q^j(t, x)$ is the vector with components $Q^j_i(t, x) = \sum_{l=1}^d D_l u(t, x)D_l q_{ij}(t, x)$ for $1 \leq i \leq d$. Taking into account the definition of $P$ in (3.3), we can write
\[
\langle \nabla_x D_j u, Q^j \rangle = \langle P(\nabla_x D_j u), P(Q^j) \rangle + \frac{1}{|\nabla_x u|^2} \langle \nabla_x u, \nabla_x D_j u \rangle \langle \nabla_x u, Q^j \rangle.
\]

Moreover, being
\[
\langle \nabla_x u, \nabla_x D_j u \rangle = \langle \nabla_x u, \nabla_x (D_j u(\nabla_x u, Q^j)) \rangle - D_j u(\nabla_x u, \nabla_x (\nabla_x u, Q^j))
\]
and $\sum_{j=1}^d D_j u(\nabla_x u, Q^j) = 0$ by the assumption (3.3), we get
\[
\sum_{j=1}^d \langle \nabla_x u, \nabla_x D_j u \rangle \langle \nabla_x u, Q^j \rangle = - \sum_{j=1}^d D_j u(\nabla_x u, \nabla_x (\nabla_x u, Q^j))
\]
\[
= - \sum_{i,j,k,l=1}^d D_j u D_k u (D_l u D_l q_{ij} + D_l u D_k u D_l q_{ij})
\]
\[
= - \sum_{i,j,k,l=1}^d D_j u D_k u [D_l u D_k u D_l q_{ij} + D_l u D_l k u (-D_l q_{ij} - D_l q_{ij})]
\]
\[
= \sum_{i,j,k,l=1}^d D_j u D_k u D_l u D_l u q_{ij},
\]
where we have used the linearity of $q_{ij}$ and again assumption (3.1). Then,
\[
\sum_{j=1}^d \langle \nabla_x u, \nabla_x D_j u \rangle \langle \nabla_x u, Q^j \rangle = \sum_{k=1}^d \langle R^k, \nabla_x D_k u \rangle,
\]
where, for every fixed $k = 1, \ldots, d$ and $(t, x) \in I \times \mathbb{R}^d$, $R^k(t, x)$ denotes the vector with components $R^k_i(t, x) = D_k u(t, x) \sum_{j=1}^d D_j u(t, x)D_j q_{ij}(t, x)$ for $1 \leq i \leq d$. Finally, since assumption (3.1) implies $\langle R^k, \nabla_x u \rangle = 0$, we have
\[
\sum_{j=1}^d \langle \nabla_x u, \nabla_x D_j u \rangle \langle \nabla_x u, Q^j \rangle = \sum_{k=1}^d \langle R^k, P(\nabla_x D_k u) \rangle.
\]
Putting together all these results, we deduce
\[ \sum_{k=1}^{d} D_k u \cdot \text{Tr}(D_k Q \cdot D_x^2 u) = \sum_{j=1}^{d} \left( P(\nabla_x D_j u), P(Q^j) + \frac{|R^j|}{|\nabla_x u|^2} \right). \]

The Cauchy-Schwarz and the Young inequalities yield that
\[ \sum_{k=1}^{d} D_k u \cdot \text{Tr}(D_k Q \cdot D_x^2 u) \leq \sum_{j=1}^{d} \left| P(\nabla_x D_j u) \right| \left( |P(Q^j)| + \frac{|R^j|}{|\nabla_x u|^2} \right) \]
\[ \leq \sum_{j=1}^{d} \left| P(\nabla_x D_j u) \right| \left( 2|P(Q^j)|^2 + 2 \frac{|R^j|^2}{|\nabla_x u|^4} \right) \]
\[ \leq \left( \sum_{j=1}^{d} \left| P(\nabla_x D_j u) \right|^2 \right)^{\frac{1}{2}} \left( 2 \sum_{j=1}^{d} \left( |P(Q^j)|^2 + \frac{|R^j|^2}{|\nabla_x u|^4} \right) \right)^{\frac{1}{2}} \]
\[ \leq \varepsilon \sum_{j=1}^{d} \left| P(\nabla_x D_j u) \right|^2 + \frac{1}{2 \varepsilon} \sum_{j=1}^{d} \left( |P(Q^j)|^2 + \frac{|R^j|^2}{|\nabla_x u|^4} \right). \]

Choosing \( \varepsilon = \eta(t, x_0) \) in (3.3) and using (3.4), we get
\[ I(t, x_0) \leq \frac{1}{2 \eta(t, x_0)} \sum_{j=1}^{d} \left( |P(Q^j)|^2 + \frac{|R^j|^2}{|\nabla_x u|^4} \right) + \langle \nabla_x b \nabla_x u, \nabla_x u \rangle. \]

Now, since
\[ |P(Q^j)|^2 = |Q^j|^2 - \left( \frac{\nabla_x u}{|\nabla_x u|^2} \right)^2 \]
and
\[ \sum_{j=1}^{d} |R^j|^2 = |\nabla_x u|^2 \sum_{j=1}^{d} (Q^j, \nabla_x u)^2, \]
we conclude that
\[ I(t, x_0) \leq \frac{1}{2 \eta(t, x_0)} \sum_{j=1}^{d} |Q^j|^2 + \langle \nabla_x b \nabla_x u, \nabla_x u \rangle. \]

Finally, being
\[ \sum_{j=1}^{d} |Q^j|^2 = \sum_{i,j=1}^{d} (Q^j)^2 = \sum_{i,j=1}^{d} \left( \sum_{k=1}^{d} D_k u D_k q_{ij} \right)^2 = \sum_{i,j=1}^{d} (\nabla_x u, \nabla_x q_{ij})^2, \]
by assumption (3.2), we deduce that \( I(t, x_0) \leq c_0 |\nabla_x u(t, x_0)|^2 \) as claimed.

The second part of the statement can be obtained arguing as in [9, Thm. 1.1(1)] but, for the readers convenience, we give a sketch of the proof.

Let us assume that estimate (3.3) holds for some \( c_0 \in \mathbb{R} \). Then, Proposition 2.4 implies that estimate (2.8) is satisfied too. We show how, throughout a suitable choice of smooth functions \( f \), formula (2.8) implies (3.1) in the three cases, respectively \( i = j = k \), \( i \neq j \) with \( k \in \{i,j\} \) and \( i \neq j \) with \( k \notin \{i,j\} \).

Fix \( t \in I, x \in \mathbb{R}^d \); let us assume that \( i = j = k \) and consider the function \( f \) defined by \( f(y) = \cos(y_i - x_i) \) for any \( y \in \mathbb{R}^d \); from (2.8), for every \( t \in I, y \in \mathbb{R}^d \) and \( \varepsilon > 0 \) small enough, we get
\[ \begin{align*}
D_i q_{ij}(t, y) &\leq (c_0 - D_ib_i(t, y)) \tan(y_i - x_i), & y_i - x_i &\in (0, \varepsilon), \\
D_i q_{ij}(t, y) &\geq (c_0 - D_ib_i(t, y)) \tan(y_i - x_i), & y_i - x_i &\in (-\varepsilon, 0). 
\end{align*} \]
In the second case, if, for instance, \( i \neq j \) and \( k = i \), we have to prove that
\[
2D_i q_{ij}(t, y) + D_j q_{ii}(t, x) = 0.
\]
For every \( \varepsilon > 0 \), let us consider the function \( f \) defined by
\[
f(y) = [\varepsilon (y_j - x_j) + (y_i - x_i)]^2
\]
for any \( y \in \mathbb{R}^d \). From (2.8), taking into account that, by the previous step, \( D_k q_{kk}(t, x) = 0 \) for every \( (t, x) \in I \times \mathbb{R}^d \) and \( k = 1, \ldots, d \), we get that, if \( y_j - x_j > 0 \) and \( y_i - x_i > 0 \), then
\[
2D_i q_{ij}(t, y) + D_j q_{ii}(t, y) \leq -\varepsilon (2D_i q_{ij}(t, y) + D_i q_{ij}(t, y)) + \frac{\varepsilon(y_j - x_j) + (y_i - x_i)}{\varepsilon} \psi(t, y),
\]
where
\[
\psi(t, y) = [c_0(1 + \varepsilon^2) - \varepsilon^2 D_j b_j + \varepsilon(D_i b_i + D_i b_j) + D_i b_i].
\]
Analogously, if \( y_j - x_j < 0 \) and \( y_i - x_i < 0 \), we get the inverse inequality of (3.11).
Therefore, letting first \( y \to x \) and then \( \varepsilon \to 0^+ \) in both of the obtained inequalities, we get
\[
2D_i q_{ij}(t, x) + D_j q_{ii}(t, x) = 0.
\]
In the last case, if \( i \neq j \) and \( k \notin \{ i, j \} \), we consider the function \( f \) defined by
\[
f(y) = [(y_k - x_k) + (y_i - x_i) + (y_j - x_j)]^2
\]
for any \( y \in \mathbb{R}^d \). Using again (2.8), the results obtained in the previous two cases and arguing as before (distinguishing the two cases \( y_j - x_j > 0 \) and \( y_j - x_j < 0 \) \( l \in \{ i, j, k \} \)), we deduce that \( D_k q_{ij}(t, x) + D_i q_{ij}(t, x) + D_j q_{ii}(t, x) = 0 \), and the proof is now complete.

4. Comments and Examples

In [6] Thm. 4.5, estimate (2.7) has been proved when the diffusion coefficients of \( A(t) \) do not depend on \( x \). In this section we provide concrete examples of nonautonomous operators like (1.1) whose diffusion matrices depend also on \( x \) and whose associated evolution operators \( G(t, s) \) satisfy the gradient estimate (2.7).

First, in the following remark we point out that, in some simple case, the algebraic condition (3.1) forces the diffusion matrix to be independent of \( x \), coming back trivially to the case considered in [6].

**Remark 4.1.**

(i) Let us consider the nonautonomous operator (1.1) whose diffusion matrix \( Q(t, x) \) is of the form \( q(x)H(t) \) where \( q \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^d) \) and \( H(t) = [b_ij(t)]_{i,j=1}^{d} \) has entries \( b_ij \in C_{\text{loc}}^{\alpha/2}(I) \) for every \( i,j = 1,\ldots,d \). If (3.1) is satisfied, then \( q(x) = c \) for every \( x \in \mathbb{R}^d \) and some \( c \in \mathbb{R} \). To check this fact, it suffices to write (3.1) for \( i = j = k \in \{1,\ldots,d\} \).

(ii) Assume that the matrix \( Q(t, x) = [q_{ij}(t, x)]_{i,j=1}^{d} \) in (1.1) is such that
\[
q_{ij}(t, x) = a_i(t, x) \delta_{ij}
\]
for every \( i,j = 1,\ldots,d \). If (3.1) is assumed to hold, then \( Q(t, x) = Q(t) \); indeed, if \( i = j \neq k \) formula (3.1) yields \( D_k a_i(t, x) = 0 \) for every \( k \neq i \), moreover, if \( i = j = k \) we also deduce that \( D_k a_i(t, x) = 0 \).

Now, we exhibit some class of nonautonomous operators whose diffusion coefficients depend on the space variable \( x \) and to which the result in Theorem 3.1 may be applied.

**Example 4.2.** Consider the class of nonautonomous elliptic operators defined on smooth functions \( \zeta \) by
\[
(A(t)\zeta)(x) = \text{Tr}(Q(t, x)D^2\zeta(x)) + \langle b(t, x), \nabla \zeta(x) \rangle, \quad t \in I, \ x \in \mathbb{R}^3.
\]

Here,
\[
Q(t, x_1, x_2, x_3) = \begin{pmatrix}
a_1(t) + \psi(t)x_1^2 & -\psi(t)x_1x_2 & 0 \\
-\psi(t)x_1x_2 & a_2(t) + \psi(t)x_2^2 & 0 \\
0 & 0 & a_3(t)
\end{pmatrix}
\]
(4.1)
and
\[ b(t, x) = -\gamma(t)|x|^{2\beta}, \quad \beta \in [1, +\infty). \]

The positive functions \( a_i, \psi, \gamma \) satisfy the following conditions:

(i) \( a_i, \psi, \gamma \in C^\alpha_{\text{loc}}(I) \) for \( i = 1, 2, 3 \);
(ii) \( \inf_{t \in I} a_i(t) > 0 \) for \( i = 1, 2, 3 \);
(iii) \( \gamma > \max \{ \bar{a}, \psi, 2\psi^2/\bar{a} \} \) where \( \bar{a}(t) := \min_{i=1,2,3} \{ a_i(t) \} \).

As it can be easily seen, \( Q(t, x) \) is a positive definite matrix for any \( (t, x) \in I \times \mathbb{R}^d \) and satisfies the condition (3.1). Moreover, the function
\[ \varphi(x) = 1 + |x|^2, \quad x \in \mathbb{R}^3, \]
satisfies Hypothesis 2.1(iii). Indeed, for \( t \in I \) and \( x \in \mathbb{R}^3 \) we have
\[
( A(t) \varphi)(x) = 2 \left[ \text{Tr}(Q(t, x)) + b(t, x) \right] \\
\leq 2 \left[ a_1(t) + a_2(t) + a_3(t) + \psi(t)|x|^2 - \gamma(t)|x|^{2+2\beta} \right].
\]
Hence,
\[
\left( A(t) \varphi \right)(x) \to -\infty, \quad \text{as } |x| \to +\infty,
\]
uniformly with respect to \( t \in I \). Thus, formula (2.2) is satisfied.

Finally, we prove estimate (3.2). Let us observe that the matrix \( Q \) is the sum of a semi-definite matrix and of a diagonal matrix whose diagonal elements are respectively \( a_1(t), a_2(t) \) and \( a_3(t) \), hence the function \( \eta \) in Hypothesis 2.1(ii) is such that \( \eta(t, x) \geq \bar{a}(t) \) for any \( t \in I \) and \( x \in \mathbb{R}^3 \). Moreover,
\[
\sum_{i,j=1}^{3} \langle \nabla_x q_{ij}(t, x), \xi \rangle^2 = 2\psi^2(t)[2x_2^2\xi_1^2 + x_2^2\xi_2^2 + x_2^2\xi_3^2 + 2x_1x_2\xi_1\xi_2] \\
\leq 2\psi^2(t)[2x_2^2\xi_1^2 + 2x_2^2\xi_2^2 + 2x_2^2\xi_3^2 + 2x_2^2\xi_1^2] \\
\leq 4\psi^2(t)|x|^2|\xi|^2
\]
and
\[
\langle \nabla_x b(t, x)\xi, \xi \rangle = -\gamma(t)|x|^{2\beta}|\xi|^2 - 2\beta\gamma(t)|x|^{2\beta-1} \langle x, \xi \rangle^2,
\]
for any \( t \in I \) and \( x, \xi \in \mathbb{R}^3 \). Therefore, we get
\[
\frac{1}{2\eta(t, x)} \sum_{i,j=1}^{3} \langle \nabla_x q_{ij}(t, x), \xi \rangle^2 + \langle \nabla_x b(t, x)\xi, \xi \rangle \leq \left( \frac{2\psi^2(t)}{\bar{a}(t)}|x|^2 - \gamma(t)|x|^{2\beta} \right)|\xi|^2 \\
=: c(t, x)|\xi|^2.
\]
Since, as \( |x| \to +\infty \), the function \( c \) tends to \( -\infty \) uniformly with respect to \( t \in I \), we can conclude that there exist a constant \( c_0 \in \mathbb{R} \) such that \( c(t, x) \leq c_0 \) for every \( t \in I \) and \( x \in \mathbb{R}^3 \). Hence, (3.2) holds.

**Remark 4.3.** The Example 1.2 can be extended to the \( d \)-dimensional case. Indeed, we can consider a block diagonal matrix of the form
\[
Q(t, x) = \begin{pmatrix}
Q_1(t, x) & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \vdots \\
\vdots & \ddots & Q_i(t, x) & \ddots \\
0 & \cdots & \cdots & 0
\end{pmatrix}, \quad (t, x) \in I \times \mathbb{R}^d,
\]
where each block $Q_i$ is either a three-dimensional matrix of the form of $Q$ in (4.1) or a two-dimensional matrix of the form

$$Q(t,x,y) = \begin{pmatrix} a_1(t) + \psi(t)y^2 & -\psi(t)xy \\ -\psi(t)xy & a_2(t) + \psi(t)x^2 \end{pmatrix},$$

and the functions $a_1, a_2$ and $\psi$ satisfy conditions (i),(ii) and (iii) in Example 4.2.

Moreover, for every $i = 1,\ldots,k$, $Q_i(t,x) = Q_i(t,x_{n_i-1}+1,\ldots,x_{n_i-1}+n_i)$, where $n_0 = 0$ and $n_i \in \{2,3\}$ denotes the dimension of the block $Q_i$.

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