SYMMETRY RESULTS FOR FINITE-TEMPERATURE, RELATIVISTIC THOMAS-FERMI EQUATIONS*

Michael K.-H. Kiessling
Department of Mathematics, Rutgers University
110 Frelinghuysen Rd., Piscataway, N.J. 08854

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Abstract

In the semi-classical limit the relativistic quantum mechanics of a stationary beam
of counter-streaming (negatively charged) electrons and one species of positively
charged ions is described by a nonlinear system of finite-temperature Thomas–Fermi
equations. In the high temperature / low density limit these Thomas–Fermi equa-
tions reduce to the (semi-)conformal system of Bennett equations discussed earlier
by Lebowitz and the author. With the help of a sharp isoperimetric inequality it is
shown that any hypothetical particle density function which is not radially symmet-
ric about and decreasing away from the beam’s axis would violate the virial theorem.
Hence, all beams have the symmetry of the circular cylinder.

*In celebration of the 70th birthday of Joel L. Lebowitz.
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poses is permitted.
1 INTRODUCTION

Modern books on charged-particle beams, e.g. [21], usually contain a chapter about the Bennett model [1], but back in the early 50’s when regular research on charged-particle beams came into sharper focus, W.H. Bennett’s pioneering pre-WWII paper [1] on the statistical mechanics of a relativistic, stationary particle beam had been forgotten, apparently, and so in 1953 Bennett sent out a reminder note [5]. For some reason or other, Bennett’s note did not appear until 1955 [5], the very year when Joel L. Lebowitz was launching his stellar career [30] with center of gravity in stationary non-equilibrium statistical mechanics [31, 32, 33]. At that time, a single issue of *The Physical Review* was still of a decent size and could be consumed from first to last page by an individual with huge scientific appetite such as Joel, and Bennett’s note [5] did not pass unnoticed before Joel’s hungry eyes. All this happened a few years before I was born, but when I came to spend some postdoctoral time with Joel nearly 40 years later, several interesting questions raised by Bennett’s work were still unanswered, and so we began to answer some of these [29].

One of the problems we had to leave open was that of the symmetry of a beam. Following Bennett we only inquired into circular-cylindrically symmetric solutions. While it is a natural conjecture that in the absence of external fields an unbounded straight particle beam with finite electrical current through its cross-section necessarily possesses the symmetry of the circular cylinder, how to prove it is not quite so obvious. It is with great pleasure when in this paper I present a rigorous proof to Joel.

Fitting for the occasion, the proof of the cylindrical symmetry of the beam involves statistical mechanics in an essential way. Namely, it is shown that any hypothetical stationary beam with finite electrical current whose particle density functions are not radially symmetric about and decreasing away from the beam’s axis would violate the virial theorem for this many-particle system. This symmetry proof covers Bennett’s strictly classical model as well as its semi-classical upgrade, i.e. a system of relativistic, finite-temperature Thomas–Fermi equations which in the high-temperature / low-density limit reduce to the (semi-)conformal Bennett equations. The proof is, however, restricted to a system of merely two equations because the coefficient matrix for the beam equations has rank 2. Our symmetry theorem therefore does not apply to beams that consist of the negatively charged electrons and more than one, differently positively charged ion species; but then again, our method of proof not only yields the cylindrical symmetry of the beam, it also yields monotonic radial decrease of the particle density functions. Hence, it is conceivable that monotonicity of the density functions may be violated in an electron / multi-ion species beam while cylindrical symmetry might still hold — yet to prove that would seem to require an entirely new argument.
Incidentally, our result also sheds some new light on the theory of white dwarfs [8]. These Earth-sized, expired stellar objects shine in bright white light because they are still incredibly hot compared to our Sun, yet they are relatively cold compared to their Fermi temperature and therefore essentially in their quantum ground state. This justifies using zero-temperature Thomas–Fermi theory for the description of their overall structure [8] — a fortunate happening, for finite-temperature Thomas–Fermi theory could not be used in three dimensions since it does not have solutions with finite mass. Interestingly, the finite-temperature Thomas–Fermi equations of the two-dimensional caricature of such a white dwarf star should have solutions with finite mass, because the gravitational potential in two dimensions is sufficiently strongly confining for this purpose. In any event, relatively little is known rigorously[1] for such a gravitating plasma of negative electrons and positive nuclei (all species treated as fermions) in either two or three dimensions; see the discussion of this model by W.E. Thirring in the preface to the E.H. Lieb jubilee volume [41], where Thirring gives an amusing account of the pitfalls associated with the fact that the Thomas–Fermi equations are the Euler–Lagrange equations for the saddle points of a variational functional. When dealing with saddle points, existence and symmetry of solutions via minimization by radial decreasing rearrangement [1, 6, 7, 27] is not an option, and neither is symmetry via uniqueness by convexity [27] of the functional.

Now recall that by the Biot-Savart law the magnetic interactions of straight, parallel electrical current filaments are attractive, with a distance law that is identical to the Newtonian gravity law in two dimensions. From this it follows that the finite-temperature Thomas–Fermi beam equations are identical to the finite-temperature Thomas–Fermi equations of the two-dimensional caricature of a white dwarf model, with the magnetic flux function re-interpreted as the gravitational Newton potential in two dimensions, and the mean electric current of each species (positive after at most a joint space rotation) re-interpreted as the mass of that species. Our symmetry result can be rephrased thus: two-dimensional finite-temperature white dwarfs are radially symmetric.

Our proof of symmetry, which is based on the Rellich [39]–Pokhozaev [38] identity (which expresses the virial theorem) and the classical isoperimetric inequality [2, 6], does involve radial rearrangements in a strategy that goes back at least as far as [2], where it

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More is known rigorously [8, 35] for the locally neutral approximation of the model, where the positive and negative charges are distributed identically and Coulomb’s law is discarded. In particular, radial symmetry of solutions for this locally neutral model has been proven by energy minimization through radial rearrangement [35]. We remark that due to the enormous ratio of the electrical and gravitational coupling constants the locally neutral approximation is expected to be an excellent approximation for a white dwarf; however, this is not generally the case for a particle beam, where the ratio of electric and magnetic coupling constants may be arbitrarily close to 1.
is applied to Liouville’s equation in a disk $\subset \mathbb{R}^2$. In [11] this strategy was generalized to systems of PDEs of Liouville type in all $\mathbb{R}^2$ which are unrestricted in size but which have a symmetric, fully stochastic coefficient matrix of full rank. The Bennett equations also constitute a Liouville system, but are not covered by the theorem of [11] because their coefficient matrix is generally not symmetric, has some negative elements, and is always rank 2. The present paper develops the necessary generalizations of [11] to overcome the first two peculiarities of the Bennett equations, but the rank 2 restricts the proof to a system of two equations. By adapting the treatment of single PDEs with more general nonlinearities developed in [36] (cf. also [28]) and [11, 12] to the system case we are able to extend our proof of symmetry for the Bennett equations to the relativistic, finite-temperature Thomas–Fermi beam equations.

Our proof simplifies considerably when the systems of Thomas–Fermi and Bennett equations are restricted to a disk with 0-Dirichlet boundary conditions for the electric and magnetic potentials. In this compact case, an alternate proof of the radial symmetry and decrease of the solutions to systems of PDE which includes the finite-temperature Thomas–Fermi and Bennett equations, was given by Troy [12], who exploited Alexandroff’s method of moving planes. For more on the moving-planes method, see [10, 12, 34, 41, 6, 13]. Troy’s proof has been extended to Liouville systems in unbounded domains, the Bennett equations not included though, in [13]. Presumably, the moving planes method can be made to work also for the system of Thomas–Fermi equations studied here; however, this is not done in this paper.

While the present paper addresses only the question whether invariance of the PDEs under rotations implies radial symmetry of their solutions, these PDEs feature other symmetries which deserve mentioning. The system of Thomas–Fermi equations is invariant under the isometries of Euclidean space, simple gauges, and Lorentz boosts along the beam. The Bennett equations are in addition to that invariant under isotropic scaling in $\mathbb{R}^2$, and for a special family of parameter values also under Kelvin transformations, in which case they are invariant under the Euclidean conformal group of $\mathbb{R}^2$. In this fully conformal case the conformal orbit of the finite current solutions is connected and itself invariant [29]. Invariance under the Euclidean conformal group holds also for the Liouville systems studied in [10], but their conformal orbit of finite mass solutions is generally not connected, and each component not invariant under inversions [14]. Toda systems in $\mathbb{R}^2$, which are Liouville systems with symmetric coefficient matrix given by the $SU(N)$ Cartan matrix, are studied in [22, 23]. The distribution of negative and positive signs in the $SU(2)$ Cartan

\footnote{The elliptic Liouville equation, known from two-dimensional differential geometry, is meant and not the evolution equation on phase space known from statistical mechanics.}
matrix is opposite to that in our Bennett equations, and sure enough, our radial symmetry proof fails in this case. Interestingly, in this case one can show that radial symmetry is in fact broken by some solutions, see the bifurcation argument with $n = 2$ in (1.7) of [10], and see [23] for the construction of the complete solution family with finite masses. Another interesting topic not discussed further here is whether the translation invariance along the beam can be broken, as is suggested by various dynamical beam instabilities [14].

The remainder of this paper is structured as follows. In the next section we formulate the basic equations of the semi-classical beam model and its classical limit. Existence of solutions is briefly touched upon. In section 3 we state our two main theorems, and in section 4 we present their proofs.

## 2 RELATIVISTIC BEAM EQUATIONS

We let $a \in S^2$ denote the fixed axis of the beam, $x \in \mathbb{R}^2$ a point in the cross-section of the beam containing the coordinate origin, and $p \in \mathbb{R}^3$ the kinematical particle momentum. The self-consistent electric field of the beam is given by $E(x) = -\nabla \phi(x)$, where $\phi$ is the electric potential, and the magnetic field by $B(x) = \nabla \psi(x) \wedge a$, where $\psi$ is the magnetic flux function. The beam consists of spin $1/2$ electrons (negatively charged, thus indexed by $s = -$) and one species of positively charged spin $1/2$ fermions (indexed by $s = +$), characterized by the following parameters: the particle charges $q_s$ and rest masses $m_s$; the rest frame temperatures $T_s$; the external chemical potentials $\mu_s$; and lab frame drift speeds $c \nu_s$, where $c$ is the speed of light and $\nu_s \in (-1, 1)$. We demand $\nu_+ \neq \nu_-$, as appropriate for counter-streaming particle species. The temperatures and drift speeds combine into the thermal lab frame parameters $\beta_s^{-1} = k_B T_s \sqrt{1 - \nu_s^2}$.

### 2.1 The semi-classical model (Thomas–Fermi theory)

The finite-temperature Thomas–Fermi model of a straight, relativistic beam is set up as follows. In the lab frame the density of $s$-particles at $x$ is given by $\rho_s(x) = G_{s}^{TF}(\phi, \psi)(x)$, where

$$G_{s}^{TF}(\phi, \psi) = \frac{2}{h^3} \int_{\mathbb{R}^3} \frac{dp}{1 + e^{-\beta_s (\mu_s - c \sqrt{m_s^2 c^2 + |p|^2 + w_s c \nu_s - q_s (\phi - \nu_s \psi)}}}$$

is the finite-temperature Thomas–Fermi density function for the relativistic $s$-species, which is subjected to the integrability condition

$$\int_{\mathbb{R}^2} G_{s}^{TF}(\phi, \psi)(x) dx = N_s,$$
where \( N_s \) is the number of \( s \)-particles per unit length of beam. The phase-space density function under the integral in (1) is the drifting Fermi–Dirac–Jüttner function \([26]\) with local chemical self-potential 

\[-q_s(\phi(x) - \nu_s \psi(x)).\]

The electric charge and current densities in the Poisson equations for the electric potential \( \phi \) and the magnetic flux function \( \psi \) are computed with the density functions (1), which leads to the system of nonlinear PDEs

\[
-\Delta \phi = 4\pi \sum_s q_s G_{s}^{TF}(\phi, \psi) \tag{3}
\]

\[
-\Delta \psi = 4\pi \sum_s \nu_s q_s G_{s}^{TF}(\phi, \psi). \tag{4}
\]

Here and in the following, \( \sum_s \) or \( \sum_t \) always stands for summation over the particle species, i.e. \( s = \mp \) and \( t = \mp \).

The Thomas–Fermi equations (3), (4), are invariant under the isometries of three-dimensional Euclidean space, Lorentz boosts along the beam’s axis \( a \), and the gauge transformation

\[
\phi(x) \to \phi(x) + \phi_0; \quad \psi(x) \to \psi(x) + \psi_0; \quad \mu_s \to \mu_s + q_s(\phi_0 - \nu_s \psi_0), \tag{5}
\]

where \( \phi_0 \) and \( \psi_0 \) are arbitrary constants.

Since we are interested in the beam’s natural symmetries, we will not allow “sources at infinity” which would deform the beam; hence, we supplement (3) and (4) with the asymptotic conditions that, uniformly as \( |x| \to \infty \),

\[
\lim_{|x| \to \infty} \frac{\phi(x)}{Q \ln \frac{1}{|x|}} = 2 = \lim_{|x| \to \infty} \frac{c \psi(x)}{I \ln \frac{1}{|x|}}, \tag{6}
\]

with \( I \neq 0 \) and \( Q \neq 0 \), where \( I = \sum s N_s q_s \nu_s c \) is the total electrical current through the beam’s cross-section and \( Q = \sum s N_s q_s \) the total charge per unit length of beam in the lab system; if \( Q = 0 \), the left equation in (3) is to be replaced by the condition that \( \phi(x) \to \text{const} \) uniformly as \( |x| \to \infty \). The situation \( I = 0 \) is not considered here, for then of course there is no stationary beam.

Remark: There are good reasons to conjecture that the asymptotic conditions (6) are in fact implied by (1)-(4). Analogous results have been proven for Liouville’s equation \([14]\) and for some Liouville systems \([10, 15]\). No attempt will be made here to generalize these results to (1)-(4). However, we note that such a generalization would have the interesting physical implication (within the limits of applicability of the model) that one cannot maintain a stationary straight beam of finite current, whatever the geometry of its cross section, when there are magnetic or electric multipole sources “at infinity.”

To the best of the author’s knowledge, the existence of beam solutions in the Thomas–Fermi model (1)-(4) with asymptotics (6) has not yet been studied rigorously. However, this semi-classical model is surely more regular than the classical one, addressed next.
2.2 The classical limit (Bennett theory)

In the high-temperature / low-density limit, i.e. formally \(0 < \beta_s \ll 1\) and \(\beta_s \mu_s \ll -1\), the Fermi–Dirac–Jüttner functions \([24]\) reduce to the Maxwell–Boltzmann–Jüttner functions \([24]\) (see also \([17]\), p.46, eq. (24)), so that the Thomas–Fermi densities \((1)\) simplify to Boltzmann densities,

\[
G^B_s(\phi, \psi) = \frac{2}{\hbar^3} \int_{\mathbb{R}^3} e^{-\beta_s \left(c\sqrt{m^2 c^2 + |p|^2} - \nu_s c^2 p \cdot a \right)} dp \ e^{\beta_s (\mu_s - \nu_s [\phi - \nu_s \psi])},
\]

(7)

and \((3)\) becomes

\[
\int_{\mathbb{R}^2} G^B_s(\phi, \psi)(x) dx = N_s.
\]

(8)

The system of equations \((3)\) and \((4)\) then reduces to the Bennett equations

\[
-\Delta \phi = 4\pi \sum_s N_s q_s \frac{e^{-\beta_s q_s (\phi - \nu_s \psi)}}{\int_{\mathbb{R}^2} e^{-\beta_s q_s (\phi - \nu_s \psi)} dx},
\]

(9)

\[
-\Delta \psi = 4\pi \sum_s N_s q_s \nu_s \frac{e^{-\beta_s q_s (\phi - \nu_s \psi)}}{\int_{\mathbb{R}^2} e^{-\beta_s q_s (\phi - \nu_s \psi)} dx},
\]

(10)

see \([4]\) eq.’s(8),(9), and \([3]\) eq.(7), where we have eliminated the external chemical potentials \(\mu_s\) via \((8)\).

The Bennett system is invariant under the isometries of three-dimensional Euclidean space and under Lorentz boosts along the beam’s axis, \(a\). Restricted to the beam’s cross-section, it is also invariant under isotropic scaling, and in the special case when the parameters satisfy

\[
\beta_s q_s (\nu_s c^{-1} I - Q) = 2, \quad s = \mp,
\]

(11)

also invariant under translated inversions. Thus, \((11)\) implies invariance of the Bennett system under the conformal group of two-dimensional Euclidean space, acting in the beam’s cross-section. In addition, the Bennett equations are invariant under a gauge transformation \(\phi(x) \to \phi(x) + \phi_0, \ \psi(x) \to \psi(x) + \psi_0\). Recall that we already eliminated the external chemical potentials via the constraint equations \((4)\) in the Bennett limit.

\[3\] In his papers \([4, 5]\), Bennett employed a classical, semi-relativistic setup, assuming drifting Maxwell-Boltzmann distributions with relativistic drift speeds, yet with non-relativistic velocity dispersion in the cross-section of the beam; the relativistic model with drifting Jüttner functions was used in \([3]\). It should be noticed, though, that after integration over momentum space the very system of equations \((4)\), \((10)\) results in either case, and it does so also in the strictly non-relativistic limit \([29]\) — except for minor re-interpretations of the parameters in each case.
In the conformally invariant case (11), Bennett’s Ansatz
\[ I^{-1}c\psi(x) = v(x) = Q^{-1}\phi(x) \] (12)
maps (9) and (10) separately into Liouville’s equation [37]
\[ -\Delta v = 4\pi e^{2v} \int_{\mathbb{R}^2} e^{2v} \, dx. \] (13)

As remarked above, it has been proven in [14] that any regular solution of (13), with the understanding that \( \int \exp(2v)dx < \infty \), satisfies
\[ \lim_{|x| \to \infty} \frac{v(x)}{\ln \frac{1}{|x|}} = 2, \] (14)
uniformly as \(|x| \to \infty\), which implies that the asymptotic conditions (6) are automatically satisfied if \( \phi \) and \( \psi \) are given by (12). It has also been proven in [14], and subsequently in [10, 16] by using alternate techniques, that (13) has only one regular family of solutions, given by
\[ v(x|x_0;k) = v_0 + \ln \frac{1}{1 + k^2|x - x_0|^2}, \] (15)
where \( k^{-1} > 0 \) is an arbitrary scale length, \( x_0 \) the arbitrary center of rotational symmetry of the solution, and \( v_0 \) an arbitrary gauge constant. The corresponding current density \( j(x) \) and charge density \( q(x) \) are given by
\[ I^{-1}j(x) = \frac{1}{\pi} \frac{k^2}{(1 + k^2|x - x_0|^2)^2} = Q^{-1}q(x). \] (16)

The density profile (16) is the celebrated Bennett beam profile.

Bennett speculated about the existence of other solutions to (9) and (10) with asymptotics (6); see [4] p.893, and [5] p.1587. (In the punctured plane additional solutions are readily found, see e.g. [3]; however, they all lack regularity, due to a point source, at the origin.) In [29] we proved that in the conformal case (11), Bennett’s system of equations (9) and (10), supplemented by the asymptotic conditions (6), are in fact equivalent to (13) (with asymptotic condition (14) automatically satisfied, see above) so that (15) then

4Bennett actually made the Ansatz that \( \rho_+(x)/\rho_-(x) = \text{const} \), which up to gauge freedom for the potentials is equivalent to (12).
exhausts all possibilities. Moreover, for the semi-conformal case where (11) does not hold we proved the existence of a continuous parameter family of smooth radial solutions to (9) and (10) with asymptotics (3) which are not invariant under inversions.

All the solutions of our beam equations are automatically also stationary solutions of the equations of Vlasov’s relativistic kinetic theory [43]. In [29] we showed that the Bennett equations can also be realized as transversal part of stationary dissipative kinetic equations in which the dissipation, modeled by a thermostat, compensates the action of an applied longitudinal electromotive force that drives the current.

In [29] we also gave a rigorous proof that all radial solutions of (9) and (10) satisfying (6) also satisfy the Bennett identity

\[ c^{-2}I^2 - Q^2 = 2 \sum_s N_s k_B T_s \sqrt{1 - \nu_s^2}. \]  

The identity (17) was originally obtained by Bennett [5] in a formal (and not entirely compelling) manner by studying the radial time-dependent virial. In this paper we will show that the Bennett identity (17), respectively its counterpart for the Thomas–Fermi model, holds \textit{a priori} without assuming symmetry, and this fact will be one major ingredient in our proof of the cylindrical symmetry of the beams.

### 3 MAIN RESULTS

To state our virial theorem, we introduce the thermodynamic potentials (per unit length of beam), given by

\[ J_{TF} = \sum_s \beta_s^{-1} \frac{2}{h^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln \left[ 1 + e^{\beta_s (\mu_s - c \sqrt{m_s^2 c^2 + |p|^2 + \nu_s c_p a - q_s [\phi - \nu_s \psi]})} \right] dp \, dx \]  

for the semi-classical model, respectively by

\[ J_B = \sum_s \beta_s^{-1} \frac{2}{h^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} e^{\beta_s (\mu_s - c \sqrt{m_s^2 c^2 + |p|^2 + \nu_s c_p a - q_s [\phi - \nu_s \psi]})} dp \, dx \]  

for the classical model.

**Theorem 3.1:** (Virial identity.) Let \( \phi \in C^{2,\alpha}(\mathbb{R}^2) \) and \( \psi \in C^{2,\alpha}(\mathbb{R}^2) \) solve (3) and (4) under the constraints (2), respectively solve (9) and (10) under the constraints (8), \( s = \mp \), in either case subjected to the asymptotic conditions (3). Then

\[ c^{-2}I^2 - Q^2 = 2J, \]  

where \( J \) stands for either \( J_{TF} \) or \( J_B \).
We also show that deviations from cylindrical symmetry violate (20), which is expressed in the next theorem.

**Theorem 3.2:** (Cylindrical symmetry.) Let \( \phi \in C^{2,\alpha}(\mathbb{R}^2) \) and \( \psi \in C^{2,\alpha}(\mathbb{R}^2) \) solve (3) and (4) under the constraints (3), respectively solve (9) and (10) under the constraints (8), \( s = \mp \), subjected to the asymptotic conditions (6). Then there exists a point \( x_0 \in \mathbb{R}^2 \) such that both \( \phi \) and \( \psi \) are radially symmetric about \( x_0 \), and the density functions \( G_s(\phi, \psi)(x) \) are decreasing away from \( x_0 \), where \( G_s \) here stands for either the Thomas–Fermi or the Boltzmann density function.

4 PROOFS

We rewrite the Thomas–Fermi, respectively Bennett system in two equivalent versions, which may be called the “density potential representation” and the “chemical self-potential representation.” We will switch between these representations at our convenience to obtain the asymptotic estimates, as \( |x| \to \infty \), and the isoperimetric estimates needed for our proofs of Theorems 3.1 and 3.2.

4.1 The alternate PDE representations

The chemical self-potentials \( U_s(x), \ x \in \mathbb{R}^2 \), are given by

\[
U_s = -q_s(\phi - \nu_s \psi). \tag{21}
\]

We also introduce density potentials \( u_s(x), \ x \in \mathbb{R}^2 \), defined by the invertible linear system

\[
\phi = \sum_s q_s u_s, \tag{22}
\psi = \sum_s \nu_s q_s u_s. \tag{23}
\]

Clearly,

\[
U_s = \sum_t \gamma_{s,t} u_t, \tag{24}
\]

where

\[
\gamma_{s,t} = -q_s q_t (1 - \nu_s \nu_t) \tag{25}
\]

denotes the entries of the matrix of coupling constants. Notice that

\[
\det(\gamma) = -(q_+ q_-)^2(\nu_+ - \nu_-)^2, \tag{26}
\]

so that for \( \nu_+ \neq \nu_- \), we have

\[
\text{rank} (\gamma) = 2, \tag{27}
\]
hence

\[ u_s = \sum_t \gamma_{s,t}^{-1} U_t \quad (28) \]

where \( \gamma_{s,t}^{-1} \) denotes the entries of the inverse matrix \( \gamma^{-1} \) to \( \gamma \).

Now let \( G_s \) stand for either \( G_{s}^{TF} \) or \( G_{s}^{B} \). We note that \( G_s(\phi, \psi) \) depends on \( \phi \) and \( \psi \) only through the combination \(-q_s(\phi - \nu_s \psi) = U_s\); thus we can write \( G_s(\phi, \psi) = G_s(U_s) = G_s(\sum_t \gamma_{s,t} u_t) \), where of course \( G_s \) stands for either \( G_{s}^{TF} \) or \( G_{s}^{B} \). In either case, the map \( w \mapsto G_s(w) \) is monotonic increasing.

It then follows at once that the chemical self-potentials \( U_s \) solve the system of nonlinear PDEs

\[ -\Delta U_s = 4\pi \sum_t \gamma_{s,t} G_t(U_t), \quad (29) \]

supplemented by the integrability conditions

\[ \int_{\mathbb{R}^2} G_s(U_s) \, dx = N_s \quad (30) \]

and by the asymptotic conditions that, uniformly as \( |x| \to \infty \),

\[ \lim_{|x|\to\infty} \frac{U_s(x)}{\ln \frac{1}{|x|}} = 2 \sum_t \gamma_{s,t} N_t. \quad (31) \]

Alternately, in terms of the \( u_s \) we get the following representation for our Thomas–Fermi / Bennett models,

\[ -\Delta u_s = 4\pi G_s(\sum_t \gamma_{s,t} u_t), \quad (32) \]

supplemented by the integrability conditions

\[ \int_{\mathbb{R}^2} G_s(\sum_t \gamma_{s,t} u_t) \, dx = N_s \quad (33) \]

and by the asymptotic conditions that, uniformly as \( |x| \to \infty \),

\[ \lim_{|x|\to\infty} \frac{u_s(x)}{\ln \frac{1}{|x|}} = 2 N_s, \quad (34) \]

for \( s = \mp \). This constitutes the density potential representation of our Thomas–Fermi / Bennett models.

**Remark:** For the sake of completeness, we also state the PDEs of the Bennett model explicitly as a Liouville system. We readily eliminate the \( \mu_s \) in terms of the \( N_s \), using (33).
Setting now \( u_s = N_s v_s \) and \( \beta_s \gamma_{s,t} N_t = \kappa_{s,t} \), and furthermore \( \sum_t \kappa_{s,t} v_t = 2V_s \) (equivalently, \( \beta_s U_s = 2V_s \)), with \( s \) and \( t \) taking the “values” \( \pm \), we rewrite (32) into the form

\[
- \Delta v_s = 4\pi \frac{\exp \left( \sum_t \kappa_{s,t} v_t \right)}{\int_{\mathbb{R}^2} \exp \left( \sum_t \kappa_{s,t} v_t \right) dx},
\]

(35)

and (29) into

\[
- \Delta V_s = 2\pi \sum_t \kappa_{s,t} \frac{\exp \left( 2V_t \right)}{\int_{\mathbb{R}^2} \exp \left( 2V_t \right) dx},
\]

(36)

Equations (35) and (36) are explicit alternate representations of the Liouville system associated to the Bennett model. The coefficient matrix \( \kappa \) is manifestly non-symmetric in general, having negative diagonal and positive off-diagonal elements. Note that in the conformal case (11), viz. \( \sum_t \kappa_{s,t} = 2 \) for \( s = \pm \), the Ansatz \( v_+ = v_− = v \) in (35), respectively \( V_+ = V_− = v \) in (36), reduces both (35) and (36) to Liouville’s equation (13).

### 4.2 Isoperimetric estimates

Let \( G_s \) continue to stand for either \( G_s^{TF} \) or \( G_s^{B} \). We introduce \( g_s \), the primitive of \( G_s \), i.e., \( g'_s(w) = G_s(w) \) for \( w \in \mathbb{R} \), such that the integrals

\[
\int_{\mathbb{R}^2} g_s(U_s) \, dx = M_s
\]

exist (notice that \( M_s \) is defined by (37)). In each case this primitive \( g_s \) is unique and given by

for the semi-classical model, and by

\[
g_s^{B}(U_s) = \beta_s^{-1} \frac{2}{\hbar^3} \int_{\mathbb{R}^3} e^{\beta_s \left( \mu_s - c \sqrt{m^2 c^2 + |p|^2 + \nu_s cp \cdot a + U_s} \right)} \, dp
\]

(39)

for the classical model. Notice that in the classical model we have \( M_s = \beta_s^{-1} N_s \), while in the semi-classical model we have \( M_s > \beta_s^{-1} N_s \) by the simple convexity inequality \( \ln x \leq -1 + x \), with “=” only for \( x = 1 \). Notice furthermore that, in either case, the map \( w \mapsto g_s(w) \) is monotonic increasing.

**Lemma 4.1:** Let the pair \( (u_+, u_-) \) solve equations (33) and (34), \( s = \mp \), under the asymptotic conditions (24), with \( \gamma \) given in (23) satisfying (24). Then

\[
\frac{1}{2} \sum_{s,t} \gamma_{s,t} N_s N_t - \sum_s M_s \geq 0,
\]

(40)
and equality in (41) holds if and only if both \( u_+ \) and \( u_- \) are radially symmetric and decreasing about the same point.

**Proof:** We follow the general reasoning of [10, 11, 12].

Since, by hypothesis, the pair \((u_+, u_-)\) solves the equations (32) and (33), \( s = \mp \), under the asymptotic conditions (34), then \((U_+, U_-)\) satisfies (29) and (33), \( s = \mp \), under the asymptotic conditions (31). Therefore, as \(|x| \to \infty\),

\[
G_s(U_s)(x) = \frac{2}{h^3} \int_{\mathbb{R}^3} e^{\beta(s, c) \left( \mu_s - c \sqrt{m_s^2 + |p|^2 + \nu_s c p} \cdot a \right)} \, dp \, |x|^{-2\beta_s \sum t \gamma_s t N_t (1+\theta(x))},
\]

with \( \theta(x) = o(1) \). Also by hypothesis, (30) is satisfied, so that from (41) we conclude that \( \beta_s \sum t \gamma_s t N_t \neq 1 \). Then, by (11) again, and since \( U_s \in C^{2,\alpha} \) (hence, \( U_s \in C^\infty \) by bootstrapping), the level sets \( \Lambda_s^\xi = \{ x | U_s(x) \geq \xi \} \) are compact, hence \( |\Lambda_s^\xi| < \infty \).

Let \( x \mapsto U_*^s(|x|) \) denote the equi-measurable, radially symmetric, non-increasing rearrangement of \( x \mapsto U_s(x) \), centered at the origin, and denote by \( \Lambda_*^\xi = \{ x | U_*^s(x) \geq \xi \} \) the ball of radius \( r_*^s \), centered at the origin. By Sard’s theorem the \( C^\infty \) regularity of the \( U_s \) implies that the outward normal \( \hat{\lambda} \) to \( \partial \Lambda_*^\xi \) exists except at most for \( \xi \)-values in a set of measure zero, so that the ensuing manipulations involving \( \hat{\lambda} \) to \( \partial \Lambda_*^\xi \) a.e.

First, recalling that \( G_s > 0 \), we note that on \( \partial \Lambda_*^\xi \) we have \( \langle \hat{\lambda}, \nabla U_s \rangle = |\nabla U_s| \), by the Hopf lemma. Integration of this identity over \( \partial \Lambda_*^\xi \), a trivial rewriting, and an application of the Cauchy–Schwarz inequality now gives the estimate

\[
-\int_{\partial \Lambda_*^\xi} \langle \hat{\lambda}, \nabla U_s \rangle \, d\sigma = \int_{\partial \Lambda_*^\xi} |\nabla U_s|^2 \frac{1}{|\nabla U_s|} \, d\sigma \geq \left( \int_{\partial \Lambda_*^\xi} \frac{d\sigma}{|\nabla U_s|} \right)^2 \left( \int_{\partial \Lambda_*^\xi} \frac{1}{|\nabla U_s|} \, d\sigma \right)^{-1},
\]

with equality holding if and only if \( |\nabla U_s| \) is constant on \( \partial \Lambda_*^\xi \). Noting that

\[
\int_{\partial \Lambda_*^\xi} \frac{d\sigma}{|\nabla U_s|} = |\Lambda_*^\xi|,
\]

and applying the classical isoperimetric inequality [2], we have

\[
|\Lambda_*^\xi| \geq |\partial \Lambda_*^{\xi^*}|,
\]

with equality holding if and only if, up to translation, \( \partial \Lambda_*^\xi = \partial \Lambda_*^{\xi^*} \). By the co-area formula [15],

\[
\int_{\partial \Lambda_*^\xi} \frac{1}{|\nabla U_s|} \, d\sigma = \int_{\partial \Lambda_*^{\xi^*}} \frac{1}{|\nabla U_*^s|} \, d\sigma.
\]
Pulling these estimates together we have

$$- \int_{\partial \Lambda_s^*} \langle \hat{\lambda}, \nabla U_s \rangle \, d\sigma \geq |\partial \Lambda_s^*| \left( \int_{\partial \Lambda_s^*} \frac{1}{|\nabla U_s|} \, d\sigma \right)^{-1},$$

(46)

with equality holding if and only if, (i), \(|\nabla U_s|\) is constant on \(\partial \Lambda_s^*\), and (ii), \(\partial \Lambda_s^* = \partial \Lambda_s^*\), up to translation. This last remark implies in particular that we can restate (46) as

$$- \int_{\partial \Lambda_s^*} \langle \hat{\lambda}, \nabla U_s \rangle \, d\sigma \geq - \int_{\partial \Lambda_s^*} \partial_t U_s^* \, d\sigma.$$  

(47)

Next, using Green’s theorem and (29), then rearrangement identity for \(s = t\), then rearrangement inequality for \(s \neq t\) (in which case \(t = -s\), noting that \(\gamma_{s,-s} > 0\) and recalling that \(w \mapsto G_s(w)\) is increasing, we have

$$- \int_{\partial \Lambda_s^*} \langle \hat{\lambda}, \nabla U_s \rangle \, d\sigma = - \int_{\Lambda_s^*} \Delta U_s \, dx$$

(48)

$$= 4\pi \sum_t \gamma_{s,t} \int_{\Lambda_t^*} G_t(U_t) \, dx$$

(49)

$$= 4\pi \left( \gamma_{s,s} \int_{\Lambda_s^*} G_s(U_s^*) \, dx + \gamma_{s,-s} \int_{\Lambda_s^*} G_{-s}(U_{-s}) \, dx \right)$$

(50)

$$\leq 4\pi \sum_t \gamma_{s,t} \int_{\Lambda_t^*} G_t(U_t^*) \, dx$$

(51)

where equality in (51) can hold only if \(U_t\) and \(U_s\) share their level lines (up to the labelling) in \(\Lambda_t^*\), for our \(\gamma\) is irreducible.

Combining inequalities (47) and (51), we arrive at the inequality

$$- \int_{\partial \Lambda_s^*} \partial_t U_s^* \, d\sigma \leq 4\pi \sum_t \gamma_{s,t} \int_{\Lambda_t^*} G_t(U_t^*) \, dx,$$

(52)

where equality can hold if and only if each \(\Lambda_s^*\) is a disk, with \(|\nabla U_s|\) constant on \(\partial \Lambda_s^*\), and all the \(U_s\) share their level lines (up to the labelling). Thus, in case of equality in (52), from the first two conditions for equality it follows that the family of disks \(\Lambda_s^+\) and the family of disks \(\Lambda_s^-\) are separately concentric, while from the third condition for equality it then follows that the families of disks must be jointly concentric.
On the other hand, if at least one of the $U_s$ is not radially symmetric decreasing about any point, let $\Xi^s$ be the image under $U_s$ of the (generally non-radial) set $\subset \mathbb{R}^2$ which supports the non-radial parts of $U_s$. Then $\Xi^s$ has finite measure. Since equality in (52) cannot hold for $\xi \in \Xi^s$, for $\xi \in \Xi^s$ we now conclude that we have strict inequality in (52),

$$-2\pi r_s^s U_s''(r_s^s) < 4\pi \sum_t \gamma_{s,t} \int_{\Lambda_{t}^s} G_t(U_s^*) \, dx$$

(53)

for both $s = \mp$.

We now set

$$\mathcal{N}_s(r) = \int_{B_r(0)} G_s(U_s^*) \, dx,$$

(54)

and

$$\mathcal{M}_s(r) = \int_{B_r(0)} g_s(U_s^*) \, dx.$$  

(55)

We have $\lim_{r \to \infty} \mathcal{N}_s(r) = N_s$ and $\lim_{r \to \infty} \mathcal{M}_s(r) = M_s$, for

$$\int_{\mathbb{R}^2} f(U_s^*) \, dx = \int_{\mathbb{R}^2} f(U_s) \, dx,$$

(56)

where $f$ stands for either $g_s$ or $G_s$. By (53),

$$2\pi r U_s''(r) \geq -4\pi \sum_t \gamma_{s,t} \mathcal{N}_t(r),$$

(57)

from which we conclude that

$$r \mathcal{M}_s''(r) \geq \mathcal{M}_s'(r) - 2\mathcal{N}_s'(r) \sum_t \gamma_{s,t} \mathcal{N}_t'(r),$$

(58)

with “$>$” valid for all $r > 0$ for which $U_s^*(r) \in \Xi^s$, while “$=$” holds for $U_s^*(r) \in \Xi^s$. We now sum (58) w.r.t. $s = \mp$, obtaining

$$\sum_s r \mathcal{M}_s''(r) \geq \sum_s \mathcal{M}_s'(r) - \sum_{s,t} \gamma_{s,t} \left( \mathcal{N}_s(r) \mathcal{N}_t'(r) \right),$$

(59)

where we made use of the fact that $\gamma$ is real symmetric. Next we integrate (53) from $r = 0$ to $r = \infty$, using integration by parts on the left-hand side. Since $g_s(U_s^*) \in L^1(\mathbb{R}^2)$ is radially decreasing, we have

$$\lim_{r \to \infty} r \mathcal{M}_s'(r) = 0,$$

(60)
thus we get the result

\[ \frac{1}{2} \sum_{s,t} \gamma_{s,t} N_s N_t - \sum_s M_s \geq 0. \]  

(61)

Now, if “=” holds in (61), then all the level curves \( \partial \Lambda_s^\xi \) are circles with \( |\nabla U_s| \) constant on \( \partial \Lambda_s^\xi \): hence the circular level curves of each \( U_s \) are concentric, and then \( U_s(x) = U_s^\ast(|x - x_0^s|) \) for some \( x_0^s \). Moreover, since in case of “=” in \( (61) \) \( (51) \) tells us that the two \( U_s \) must share their level curves (with generally different level values, of course), we conclude that \( x_0^+ = x_0^- \), i.e. \( U_+ \) and \( U_- \) are then radially symmetric and decreasing about the same center of symmetry, \( x_0 \).

On the other hand, if at least one of the \( U_s \) is not radially symmetric and decreasing about any point, then the integration picks up all the strict inequalities from \( \xi \in \Xi^s \), and “>” holds in \( (61) \).

Finally, since rank \( (\gamma) = 2 \), it follows that at least one \( u_s \) is not radially symmetric and decreasing about any point if at least one \( U_s \) is not. In the same vein, \( u_+ \) and \( u_- \) are radially symmetric and decreasing about the same center of symmetry, \( x_0 \), whenever both \( U_+ \) and \( U_- \) are.

This proves Lemma 4.1. Q.E.D.

### 4.3 Asymptotic control near infinity

Standard harmonic analysis gives us:

**Proposition 4.2:** Under the hypothesis stated in Lemma 4.1, each solution pair \((u_+, u_-)\) of \((32), (33), (34)\) satisfies the integral representation

\[
\begin{align*}
    u_s(x) - u_s(0) &= \int_{\mathbb{R}^2} \left( \ln \frac{1}{|x-y|^2} - \ln \frac{1}{|y|^2} \right) G_s \left( \sum_t \gamma_{s,t} u_t \right)(y) dy. \\
\end{align*}
\]

(62)

**Corollary 4.3:** By \((62)\) we have

\[
\nabla u_s(x) = -2 \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} G_s \left( \sum_t \gamma_{s,t} u_t \right)(y) dy.
\]

(63)

With the help of \((62)\) and \((63)\) we obtain asymptotic control over the r.h.s. of \((32)\), expressed in terms of the \( U_s \).

**Lemma 4.4:** Under the hypotheses of Lemma 4.1, there exists an \( r_0(U_s) > 0 \), a constant \( C_s > 0 \), and a monotonic decreasing \( h_s(|x|) > 0 \) satisfying

\[
\lim_{|x| \to \infty} |x|^{-h_s(|x|)} = 0,
\]

(64)
such that for $s = \mp$ we have, for $|x| > r_0$,
\[ G_s(U_s)(x) \leq C_s|x|^{-2-h_s(|x|)}. \] (65)

Furthermore, for at least one $s$, we have $h_s(|x|) \geq \epsilon_s > 0$ for $|x| > r_0$.

Proof: The bound (65), with $h_s(|x|) = O(1)$ monotonic decreasing, follows directly from (41) and (30). Furthermore, by (41) and (30) we can find $h$ such that $\int_1^\infty |x|^{-1-h_s(|x|)}d|x| < \infty$, but this is impossible if $|x|^{-h_s(|x|)} \not\rightarrow 0$; hence, (64) follows.

This still allows $h_s(|x|) = o(1)$ for both $s$, but by Lemma 4.1 and the fact that $M_s \geq \beta_s^{-1}N_s$ (see the definition of the $g_s$ above), we find, after multiplying (61) by $-2$ and re-grouping terms, that
\[ \sum_s \beta_s^{-1}N_s(2 - \beta_s\sum_t \gamma_{s,t}N_t) \leq 0. \] (66)

Thus, if for one of the $s$ we have $h_s(|x|) = o(1)$, say for $s = +$, then by (11) we have $\beta_+ \sum_t \gamma_{+,t}N_t = 1$, so that (66) gives us right away
\[ \beta_-\sum_t \gamma_{-,t}N_t \geq 2 + \frac{\beta_- N_+}{\beta_+ N_-}. \] (67)

By symmetry, the analog conclusion holds for $\beta_+ \sum_t \gamma_{+,t}N_t$ if $h_-(|x|) = o(1)$. Hence, $h_s(|x|) = o(1)$ for at most one of the $s$.

This proves Lemma 4.4. Q.E.D.

**Corollary 4.5:** For at least one $s$, we have $\left| \int_{\mathbb{R}^2} \ln |y| G_s(U_s)(y) dy \right| < \infty$, so that for this $s$, we have
\[ \lim_{|x| \rightarrow \infty} \left( u_s(x) + 2N_s \ln |x| \right) = u_s(0) + 2 \int_{\mathbb{R}^2} \ln |y| G_s(U_s)(y) dy. \] (68)

We proceed with gradient estimates.

**Lemma 4.6:** Under the hypothesis stated in Lemma 4.1, each solution pair $(u_+, u_-)$ of (32), (33), (34) satisfies the gradient estimates
\[ \limsup_{|x| \rightarrow \infty} |x||\nabla u_s| \leq 2N_s. \] (69)

Proof: By Corollary 4.3, we have
\[ |\nabla u_s(x)| \leq 2 \int_{\mathbb{R}^2} \frac{G_s(U_s)(y)}{|x-y|} dy. \] (70)
After multiplying (70) by $|x|$, a simple rewriting of the r.h.s. gives

$$|x|\nabla u_s(x) \leq 2 \int_{\mathbb{R}^2} G_s(U_s)(y)\,dy + 2 \int_{\mathbb{R}^2} \left(\frac{|x|}{|x-y|} - 1\right) G_s(U_s)(y)\,dy.$$  \hspace{1cm} (71)

By (30) the first integral on the r.h.s. of (71) equals $N_s$. By the triangle inequality, the second integral on the r.h.s. of (71) is bounded in absolute value by

$$2 \int_{\mathbb{R}^2} \frac{|y|}{|x-y|} G_s(U_s)(y)\,dy.$$  \hspace{1cm} (72)

We now show that

$$\lim_{|x| \to \infty} \int_{\mathbb{R}^2} \frac{|y|}{|x-y|} G_s(U_s)(y)\,dy = 0,$$  \hspace{1cm} (73)

from which the lemma follows.

We split the domain of integration in (72) as follows: $\mathbb{R}^2 = \Omega_1 \cup \Omega_2 \cup \Omega_3$, with $\Omega_1 = \{y \mid |y| < |x|/2\}$, $\Omega_2 = \{y \mid |x|/2 \leq |y| \leq 2|x|\}$, and $\Omega_3 = \{y \mid |y| > 2|x|\}$. If $G_s(U_s)(y) \leq C|y|^{-2-\epsilon}$, with $0 < \epsilon < 1$, then the estimates are precisely the same as in [10], section 2, with exp replaced by $G_s$; this is the case for at least one of the $s$. It remains to provide estimates when $h_s(|x|) = o(1)$ for one of the $s$.

To estimate the contribution from $\Omega_1$ when $G_s(U_s)(y) \leq C|y|^{-2-h_s(|y|)}$ with $h_s(|y|) = o(1)$, we note that

$$\int_{\Omega_1} \frac{|y|}{|x-y|} G_s(U_s)(y)\,dy \leq \frac{C'}{|x|} \int_{\Omega_1} |y| G_s(U_s)(y)\,dy \leq \frac{C''}{|x|} \int_0^{|x|} \zeta^{-h_s(\zeta)} d\zeta.$$  \hspace{1cm} (74)

As for the right hand side of (74), L’Hopital’s Rule gives us

$$\lim_{|x| \to \infty} \frac{C''}{|x|} \int_0^{|x|} \zeta^{-h_s(\zeta)} d\zeta = C'' \lim_{|x| \to \infty} |x|^{-h_s(|x|)} = 0,$$  \hspace{1cm} (75)

the last step by Lemma 4.4. Hence, the l.h.s. (74) vanishes as $|x| \to \infty$.

Similarly, the contribution from $\Omega_2$ is estimated by using again that $G_s(U_s)(y) \leq C|y|^{-2-h_s(|y|)}$, so that for $|x|$ large enough we have the bound

$$\int_{\Omega_2} \frac{|y|}{|x-y|} G_s(U_s)(y)\,dy \leq \frac{C}{|x|^{1+h_s(|y|)}} \int_{|y| < 4|x|} \frac{dy}{|y|} \leq C|x|^{-h_s(|x|)}.$$  \hspace{1cm} (76)

Clearly r.h.s. (76) $\to 0$ as $|x| \to \infty$, by the same reasoning as for $\Omega_1$. 

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Finally, the contribution from $\Omega_3$ is dominated by
\[ \int_{\Omega_3} \frac{|y|}{|x-y|} G_s(U_s)(y) dy \leq C \int_{|y| > 2|x|} G_s(U_s)(y) dy, \] (77)
which vanishes as $|x| \to \infty$, by hypothesis (30).

This concludes the proof of Lemma 4.6. Q.E.D.

**Lemma 4.7:** Under the hypotheses of Lemma 4.1, we have, uniformly in $x$,
\[ \lim_{|x| \to \infty} \langle x, \nabla u_s \rangle = -2N_s. \] (78)

**Proof:** Let $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$, with $|x| = |y|$. Now fix $\hat{x} \in S^1$. By (34), we have
\[ \lim_{\tau \to \infty} u_s(\tau \hat{x}) \ln \tau = -2N_s. \] (79)

Thus, by L’Hospital’s Rule,
\[ \lim_{\tau \to \infty} \tau \frac{d}{d\tau} u_s(\tau \hat{x}) = \lim_{|x| \to \infty} \langle x, \nabla u_s \rangle = -2N_s \] (80)
for $x = |x|\hat{x}$. It remains to establish uniformity of (80). To this extent, we show that there exist $R$ and $\delta$, such that, if $|x| > R$ and $|\hat{x} - \hat{y}| < \delta$, then
\[ |\langle x, \nabla u_s(x) \rangle - \langle y, \nabla u_s(y) \rangle| < \epsilon. \] (81)

We first show that for $|x| > R$ and $|x - y| < |x|/10$, we have,
\[ |x| |\nabla u_s(x) - \nabla u_s(y)| \leq C|\hat{x} - \hat{y}| + C'|x|^{-h_s(|x|)}. \] (82)

By Corollary 4.3,
\[ |\nabla u_s(x) - \nabla u_s(y)| \leq 2 \int_{\mathbb{R}^2} G_s(U_s)(z) \left| \frac{x-z}{|x-z|^2} - \frac{y-z}{|y-z|^2} \right| dz. \] (83)

We break up the domain of integration in the above integral exactly as in the proof of Lemma 4.6. (Notice the integration variable is now $z$.) The integration over $\Omega_1$ is estimated exactly as in section 2 of [11] to get
\[ \int_{\mathbb{R}^2} G_s(U_s)(z) \frac{|x-y|}{|x|^2} dz \leq C|S_0|. \] (84)
The integral over $\Omega_2$ is dominated by
\[ \int_{|z| \sim |x|} G_s(U_s)(z) \left( \frac{1}{|x - z|} + \frac{1}{|y - z|} \right) \, dz \leq C|x|^{-1-h_s(|x|)}. \] (85)

The final estimate above was identical to that made in the proof of Lemma 4.6. Use was made of $G_s(U_s)(z) \leq C|x|^{-2-h_s(|x|)}$ on $\Omega_2$, which holds by Lemma 4.4. The contribution from $\Omega_3$ is estimated once again exactly as in section 2 of [10] to be dominated by
\[ C|x - y| \int_{|z| > 2|x|} G_s(U_s)(z) \, dz \leq C'' \frac{|x - y|}{|x|^2}, \] (86)
where the last step follows by (30). By these estimates,
\[ |\langle \hat{x}, \hat{\omega} \rangle - \langle \hat{y}, \hat{\omega} \rangle| \leq |\hat{x} - \hat{y}| \, |\nabla u_s(x)| + |\hat{y}| \, |\nabla u_s(x) - \nabla u_s(y)| \] (87)
\[ \leq |\hat{x}| \, |\nabla u_s(x)| |\hat{x} - \hat{y}| + |\hat{x} - \hat{y}| + C|x|^{-h_s(|x|)}. \] (88)

By Lemma 4.6, the last expression above is at most $C'd + C|x|^{-h_s(|x|)}$. Thus our claim (81) follows now from Lemma 4.4 for suitably large $R$ and small $\delta$. Since $S^1$ is compact, uniformity of the limit in Lemma 4.7 now follows. Q.E.D.

Lemmata 4.6 and 4.7 imply

**Corollary 4.8:** Under the hypotheses expressed in Lemma 4.1, we have, uniformly in $x$,
\[ \lim_{|x| \to \infty} |x| |\nabla u_s| = 2N_s. \] (89)

**Proof:** Follows essentially verbatim [10], proof of Corollary 2.2, with $\exp$ replaced by $G_s$. Q.E.D.

Let $\Omega_\xi = \{ x \mid u_s(x) \geq \xi \}$, where $\xi \ll -1$. By (84) it follows that if $x \in \partial \Omega_\xi$, then $|x| \geq R(c)$ with $R(c)$ large. For such $x$, it follows from Corollary 4.8 that $\nabla u_s \neq 0$. Since $u \in C^{2,\alpha}_{\text{loc}}$, we easily see that therefore $\partial \Omega_\xi \in C^{2,\alpha}$. Thus the unit outward normal $\hat{\omega}(x)$ to $\partial \Omega_\xi$ exists at all $x \in \partial \Omega_\xi$ for $\xi$ sufficiently negative.

**Lemma 4.9.** Let $\hat{\omega}(x)$ be the unit outward normal to $\partial \Omega_\xi$ at $x$, and let $\hat{x} = x/|x|$. We have, uniformly in $x$,
\[ \lim_{c \to -\infty} \langle \hat{x}, \hat{\omega} \rangle = 1. \] (90)

**Proof:** Identical to [10], proof of Lemma 2.8. Q.E.D.

**Remark:** Lemma 4.9 implies that asymptotically for large $x$ the $\partial \Omega_\xi$ become concentric circles.

We are now in a position to prove our Theorem 3.1.
4.4 Proof of the Virial Theorem

**Proposition 4.10:** (Rellich–Pokhozaev identity.) Under the hypotheses expressed in Theorem 3.1, any solution pair \((u_+, u_-)\), of \((32), (33), (34)\) satisfies the Rellich–Pokhozaev identity

\[
\frac{1}{2} \sum_{s,t} \gamma_{s,t} N_s N_t - \sum_s M_s = 0. \tag{91}
\]

**Remark:** The Rellich–Pokhozaev identity \((91)\) is identical to the identity expressed in the Virial Theorem 3.1.

**Proof of Proposition 4.10:** For \((u_+, u_-)\) a solution pair of \((32), (33), (34)\), we have the partial differential identity

\[
\text{div}(\langle x, \nabla u_t \rangle \nabla u_s) = \langle \nabla u_s, (1 + \langle x, \nabla \rangle) \nabla u_t \rangle - 4\pi \langle x, \nabla u_t \rangle G_s \left( \sum_t \gamma_{s,t} u_t \right). \tag{92}
\]

We will multiply \((92)\) by \(\gamma_{s,t}\), sum over \(s\) and \(t\), integrate over \(B_R\), use some partial integrations, then take the limit \(R \to \infty\).

We evaluate first the left-hand side of \((92)\). Green’s theorem gives us

\[
\int_{B_R} \text{div}(\langle x, \nabla u_t \rangle \nabla u_s) \, dx = \int_{\partial B_R} |x|^{-1} \langle x, \nabla u_t \rangle \langle x, \nabla u_s \rangle \, d\sigma. \tag{93}
\]

Taking the limit \(R \to \infty\), using \((78)\), we get

\[
\lim_{R \to \infty} \int_{\partial B_R} |x|^{-1} \langle x, \nabla u_t \rangle \langle x, \nabla u_s \rangle \, d\sigma = 8\pi N_s N_t, \tag{94}
\]

hence

\[
\lim_{R \to \infty} \sum_{s,t} \gamma_{s,t} \int_{B_R} \text{div}(\langle x, \nabla u_t \rangle \nabla u_s) \, dx = 8\pi \sum_{s,t} \gamma_{s,t} N_s N_t. \tag{95}
\]

On the other hand, the last term in the right-hand side of \((92)\) gives us

\[
\sum_{s,t} \gamma_{s,t} \int_{B_R} \langle x, \nabla u_t \rangle G_s (\sum_t \gamma_{s,t} u_t) \, dx = \sum_s \int_{B_R} \langle x, \nabla g_s (\sum_t \gamma_{s,t} u_t) \rangle \, dx \\
= \sum_s \int_{\partial B_R} |x| \, g_s (\sum_t \gamma_{s,t} u_t) \, d\sigma - 2 \sum_s \int_{B_R} g_s (\sum_t \gamma_{s,t} u_t) \, dx. \tag{96}
\]
We now take the limit \( R \to \infty \) in (96). As for the surface integrals, we note that by Lemma 4.4 we have \( g_s(U_s)(x) \sim CG_s(U_s)(x) \) as \( |x| \to \infty \), so that once again by Lemma 4.4, we have

\[
\lim_{R \to \infty} \int_{\partial B_R} |x| g_s(\sum_t \gamma_{s,t} u_t) \, d\sigma = 0.
\]

(97)

As for the volume integrals, we get

\[
\lim_{R \to \infty} \int_{B_R} g_s(\sum_t \gamma_{s,t} u_t) \, dx = M_s.
\]

(98)

Turning now to the first term in the right-hand side of (92), we use the symmetry of \( \gamma \), an integration by parts and (32), to get

\[
\sum_{s,t} \gamma_{s,t} \int_{B_R} \langle \nabla u_s, (1 + \langle x, \nabla \rangle) \nabla u_t \rangle \, dx = \frac{1}{2} \sum_{s,t} \gamma_{s,t} \int_{\partial B_R} |x| \langle \nabla u_s, \nabla u_t \rangle \, d\sigma
\]

(99)

\[
= \frac{1}{2} \sum_{s,t} \gamma_{s,t} \int_{\partial B_R} \left( |x|^{-1} \langle x, \nabla u_s \rangle \langle x, \nabla u_t \rangle + |x| \langle \nabla_T u_s, \nabla_T u_t \rangle \right) \, d\sigma
\]

(100)

where \( \nabla_T \) denotes tangential derivative. By Lemma 4.7 and Corollary 4.8, we have

\[
|x|^2 |\nabla u_s|^2 = |x|^2 |\nabla u_s|^2 - \langle x, \nabla u_s \rangle^2 \to 0,
\]

(101)

uniformly as \( |x| \to \infty \). Thus as \( R \to \infty \),

\[
\int_{\partial B_R} |x| \langle \nabla_T u_s, \nabla_T u_t \rangle \, d\sigma \to 0
\]

(102)

and therefore

\[
\lim_{R \to \infty} \sum_{s,t} \gamma_{s,t} \int_{B_R} \langle \nabla u_s, (1 + \langle x, \nabla \rangle) \nabla u_t \rangle \, dx = 4\pi \sum_{s,t} \gamma_{s,t} N_s N_t.
\]

(103)

Pulling all limit results together, we obtain Proposition 4.10. Q.E.D.

Remark. The proof of the virial theorem extends to situations when \( \gamma \) does not have full rank, hence to more-than-two species beams.
4.5 Concluding the proof of the Symmetry Theorem

By Lemma 4.1, and by Proposition 4.10, the solutions $u_s$ of (32), (33), (34), have to be radially symmetric and decreasing about a common center $x_0$. Since the coupling matrix $\gamma$ is invertible, the same conclusion holds for the solutions $U_s$ of (29), (30), (31). The proof is complete. Q.E.D.

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