Classical ergodicity and quantum eigenstate thermalization: analysis in fully-connected Ising ferromagnets

Takashi Mori
Department of Physics, Graduate School of Science, University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan

We investigate the relation between the classical ergodicity and the quantum eigenstate thermalization in the fully-connected Ising ferromagnets. In the case of spin-1/2, an expectation value of an observable in a single energy eigenstate coincides with the long-time average in the underlying classical dynamics, which is the trivial result of the WKB approximation. In the case of spin-1, the underlying classical dynamics is not necessarily ergodic. In that case, it turns out that, in the thermodynamic limit, the statistics of the expectation values of an observable in the energy eigenstates coincides with the statistics of the long-time averages in the underlying classical dynamics starting from random initial states sampled uniformly from the classical phase space. This feature seems to be a general property in semiclassical systems such as several fully-connected many-body systems, and the result presented here clarifies a close relationship between the long-time behavior of classical dynamics and the property of individual energy eigenstates.

The study of out-of-equilibrium dynamics in isolated quantum many-body systems has been paid much attention triggered by experimental advance in ultra-cold atomic systems [1-3]. One of the key problems is to understand the property of the steady state reached after sufficiently long times starting from a certain out-of-equilibrium initial state [4-21]. Theoretical studies on this fundamental problem has been progressed in recent years owing to theoretical development and rearrangement of some fundamental old ideas [4-7]. Eigenstate thermalization hypothesis (ETH) is one of the important notions [8-11]. It insists that $\langle \phi_n | O | \phi_n \rangle \approx \text{Tr} \rho_{mc}$ for any energy eigenstate $\phi_n$ and any local observable $O$, where $\rho_{mc}$ is the microcanonical density matrix with the energy identical to the energy eigenvalue $E_n$ of $\phi_n$.

ETH is equivalent to the following statement:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \psi(t) | O | \psi(t) \rangle \approx \text{Tr} \rho_{mc}$$

(1)

for any initial state $\psi(0)$ picked up from the microcanonical energy shell, $| \psi(0) \rangle = \sum_n c_n | \phi_n \rangle$ with $c_n$ nonzero only for $n$ such that $E_n \in [E, E+\Delta E)$, where $\Delta E$ is the energy width in the microcanonical ensemble. The fluctuation in time of $\langle \psi(t) | O | \psi(t) \rangle$ is typically very small for macroscopic systems [8, 11, 22], so that eq. (1) implies $\langle \psi(t) | O | \psi(t) \rangle \approx \text{Tr} \rho_{mc}$ for a sufficiently large typical time $t$, which implies thermalization of the system.

From eq. (1), ETH is regarded as a quantum counterpart of ergodicity in classical systems. Naturally, it is expected that there is a close relation between classical ergodicity and quantum ETH. Indeed, in his pioneering work, Berry conjectured that the classical ergodicity implies the quantum ETH in the semiclassical regime [5]. A recent numerical study [23] demonstrated this relation in a periodically-driven system. On the other hand, the relation between the classical dynamics and the property of quantum energy eigenstates is poorly understood when the classical dynamics is not ergodic.

In this paper, we study the relation between the long-time behavior of the underlying classical dynamics and the property of energy eigenstates in fully-connected Ising ferromagnets, including the case in which the classical dynamics is not ergodic. The Hamiltonian is given by

$$H = -\frac{J}{2N} \sum_{i \neq j} S_i^z S_j^z - h_x \sum_{i=1}^N S_i^x - h_z \sum_{i=1}^N S_i^z + D \sum_{i=1}^N (S_i^z)^2,$$

(2)

where $S_i$ is the spin-1/2 or spin-1 operator of $i$th spin, $J$ is the exchange interaction, $h_x$ and $h_z$ are the magnetic field along $x$ and $z$ directions, respectively, and $D \geq 0$ is the anisotropic term, which plays the role only for the spin-1 case. We consider the ferromagnetic coupling and put $J = 1$, but essentially the same result also holds for anti-ferromagnetic coupling $J = -1$. We set $\hbar = 1$ throughout the paper.

In fully-connected spin systems, there is the permutation symmetry of any $i$th and $j$th spins. Therefore, if the initial state is totally symmetric, i.e., symmetric for any permutation of spins, the state remains in the totally symmetric subspace during the quantum dynamics. We assume it, and always consider in the totally symmetric subspace.

It can be shown that $1/N$ plays the role of the Planck constant in the totally symmetric subspace, and the thermodynamic limit $N \to \infty$ corresponds to the classical limit [24]. When $N$ is large but finite, therefore, the system described by eq. (2) is regarded as a semiclassical system, and thus these models are good starting points to investigate the relation between the long-time classical dynamics and the property of the quantum eigenstates. Moreover, fully-connected Ising (anti-)ferromagnets can be realized in ion-trap experiment [25-28], and we can compare theory to experiment.

Let us start from the simple spin-1/2 case. The totally symmetric subspace corresponds to the subspace with
the maximum total spin, \( \left( \sum_{i=1}^{N} S_i \right)^2 = (N/2)(N/2+1) \). The energy eigenstates are labeled by a single variable \( z \), where

\[
\frac{1}{N} \sum_{i=1}^{N} S_i^z |z\rangle = z |z\rangle.
\]

(3)

The wave function is defined as \( \psi_t(z) := \langle z | \psi(t) \rangle \) for a state \( |\psi(t)\rangle \), and then, it is shown that the Schrödinger equation \( i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \) reduces to

\[
\frac{i}{N} \frac{\partial}{\partial t} \psi_t(z) = \left( -\frac{1}{2} z^2 - h_z z - h_x \sqrt{1 - z^2 \cos p_z} \right) \psi_t(z) =: \hat{H} \psi_t(z)
\]

(4)

in the leading order in \( 1/N \) [24]. Here, \( 1/N \) plays the role of the Planck constant, and the canonical momentum conjugate to \( z \) is defined as \( p_z := (-i/N) \partial / \partial z \).

When \( N \gg 1 \), an energy eigenstate \( \phi_n(z) = \langle z | \phi_n \rangle \) with the energy eigenvalue \( E_n = N \varepsilon \) is given by the Wentzel-Kramers-Brillouin (WKB) approximation. The immediate consequence of the WKB approximation is that the eigenstate expectation value of an observable \( f(z,p_z) \), where \( f \) is a function independent of \( N \), asymptotically equals the long-time average of the same quantity in the underlying classical dynamics for large \( N \) [24],

\[
\langle \phi_n | f(z,p_z) | \phi_n \rangle \approx \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt f(z(t),p_z(t)) =: f(z(t),p_z(t)),
\]

(5)

where \( z(t) \) and \( p_z(t) \) are the solutions of the classical equations of motion, \( dz(t)/dt = \partial \hat{H} / \partial p_z \) and \( dp_z(t)/dt = -\partial \hat{H} / \partial z \). Because the equal energy surface \( \hat{H}(z,p_z) = \varepsilon \) is one dimensional in the phase space, the ergodicity of the classical dynamics is trivial as long as the equal energy surface is simply connected. As a result,

\[
\langle \phi_n | f(z,p_z) | \phi_n \rangle \approx f(z(t),p_z(t)) \approx f_{eq},
\]

(6)

where \( f_{eq} \) is the equilibrium value of \( f \) calculated in the microcanonical ensemble of \( H \) with the energy \( N \varepsilon \) within the totally symmetric subspace [29]. This implies that quantum energy eigenstates \( \phi_n \) satisfy ETH within the totally symmetric subspace [30].

In this model, in some choice of parameters \{\( h_x, h_z, \varepsilon \)\}, the equal-energy surface of \( \hat{H} \) is separated into the two disconnected parts (ergodic regions), see [31]. In that case, the energy eigenstates are also divided into the two groups, corresponding to the two separated ergodic regions of the underlying classical dynamics. In the spin-1/2 case, this correspondence between the long-time average in the classical dynamics and the quantum mechanical average in a single energy eigenstate is a consequence of the WKB approximation.

In the spin-1 case, the situation becomes complicated. Because of the presence of the anisotropy \( D \sum_{i=1}^{N} (S_i^z)^2 \), the total spin is not conserved. The energy eigenstates in the totally symmetric subspace are characterized by the two variables \( x \) and \( y \), where

\[
\begin{align*}
\frac{1}{N} \sum_{i=1}^{N} S_i^z(S_i^z + 1)|x,y\rangle &= x|x,y\rangle, \\
\frac{1}{N} \sum_{i=1}^{N} S_i^z(S_i^z - 1)|x,y\rangle &= y|x,y\rangle.
\end{align*}
\]

(7)

The number of the spins in \( S_i^z = \pm 1 \) equals \( Nx(Ny) \). Similarly to the spin-1/2 case, by deriving the Schrödinger equation for the wave function \( \psi_t(x,y) := \langle x,y | \psi(t) \rangle \) up to the leading order in \( 1/N \), we obtain

\[
\frac{i}{N} \frac{\partial}{\partial t} \psi_t(x,y) = \hat{H} \psi_t(x,y),
\]

(8)

with

\[
\hat{H}(x,y,p_x,p_y) = -\frac{1}{2} (x - y)^2 + D(x + y) - h_z(x - y) - h_x \left( \sqrt{2x(1 - x - y)} \cos p_x + \sqrt{2y(1 - x - y)} \cos p_y \right).
\]

(9)

Again the effective Planck constant is given by \( 1/N \) and the canonical momenta are defined as \( p_x = (-i/N) \partial / \partial x \) and \( p_y = (-i/N) \partial / \partial y \). The underlying classical dynamics \( \{x(t),y(t),p_x(t),p_y(t)\} \) is given by the Hamilton equation in terms of the classical Hamiltonian \( \hat{H} \), but now there are two degrees of freedoms \( (x,y) \), and hence the classical dynamics can be regular or chaotic. The equal energy surface \( \hat{H} = \varepsilon \) is three dimensional, and the classical ergodicity is not trivial.

In the case of spin-1/2, there was a close relation (5) between the long-time average in the classical dynamics and the quantum energy eigenstate expectation value. Also in the case of spin-1, it is expected that there is a close relation between them, although we cannot apply the WKB approximation.

Here, we shall compare the distribution of the long-time averages of some quantities in the classical dynamics with the initial states sampled randomly from the phase space to the distribution of the energy eigenstate expectation values of the same quantities. In FIG. 1, the results for (a) \( m^x = (1/N) \sum_{i=1}^{N} S_i^x = x - y \), (b) \( m^z = (1/N) \sum_{i=1}^{N} S_i^z \approx \sqrt{2x(1 - x - y)} \cos p_x + \sqrt{2y(1 - x - y)} \cos p_y \), and (c) \( m^0 = (1/N) \sum_{i=1}^{N} [1 - (S_i^z)^2] \approx 1 - x - y \) are shown. The transverse axis is the energy density \( \varepsilon \). The red (blue) points are the long-time averages in the classical dynamics \( \langle O(t) \rangle \) (the energy eigenstate expectation values \( \langle \phi_n | O | \phi_n \rangle \) for \( O = m^x, m^z \), and \( m^0 \). They agree very well including the strongly non-ergodic energy region. The distribution functions also agree very well. In FIG. 2, \( P_N(m) \) and \( P_{eq}(m) \) are shown.
for $N = 240$, where $P_N(m)$ is the probability distribution function of $m = \langle \phi_n|m^z|\phi_n \rangle$ obtained by all the energy eigenstates in a finite $N$ quantum system, and $P_{ct}(m)$ is the probability distribution function of the long-time average of $m^z(t)$ in the underlying classical dynamics $m = m^z(t)$ starting from randomly sampled initial states. They agree very well. We can also consider the probability distribution function of $m^z$ or $m^0$, but the result does not change.

Next, we consider the finite-size scaling of the deviation of the two distributions defined as

$$\Delta_N := \int dm |P_N(m) - P_{ct}(m)|.\quad (10)$$

In FIG. 3, we see that $\Delta_N$ decreases towards zero as $\Delta_N \propto N^{-1/2}$ at least up to $N = 240$. This finite-size analysis strongly indicates that the two distributions coincide in the limit of $N \rightarrow \infty$.

Although we only show the result for the fully-connected Ising ferromagnets, the similar result will hold for other fully-connected systems. Indeed, in the Dicke model, which is regarded as another semiclassical model, the same relation between long-time behavior of the classical dynamics and the expectation values in the energy eigenstates is confirmed, see [31]. From these observations in the fully-connected models, it is conjectured that this coincidence of the distribution of the quantum eigenstate expectation values and that of the long-time averages in the underlying classical dynamics is a general feature in semiclassical systems.

This conjecture implies that the quantum ETH is satisfied within the totally symmetric subspace if the underlying classical dynamics is ergodic. On the other hand, if the classical dynamics is not ergodic, the quantum ETH is violated. In that case, according to our conjecture, the statistics of $\langle \phi_n|O|\phi_n \rangle$ for many different $n$ such that $E_n/N \in [\varepsilon, \varepsilon']$ for arbitrary $\varepsilon < \varepsilon'$ coincides with the statistics of $O(t)$ for random initial states sampled uniformly from the phase space with the energy between $\varepsilon$ and $\varepsilon'$.

ETH implies that the long-time average of local observables coincides with the equilibrium average. If the initial state is given by a superposition of a large number of energy eigenstates, the time fluctuation of $\langle \psi(t)|O|\psi(t) \rangle$ is very small. It is shown that for the effective dimension $d_{eff}$ of the initial state defined as...
Hamiltonian employed in FIG. 4.

Other parameters are fixed as $h$ which

$\langle \psi(t) | \psi(t) \rangle$ approaches the stationary value almost identical to the

dynamics for spin-1 case, we numerically calculate the dy-

namic, this stationary value will be equal to the equilib-

ics is ergodic, see FIG. 1. In this case, as is clearly observed in the bottom
of FIG. 4, $\langle m^x(t) \rangle$ approaches a stationary value but it

which the classical dynamics is strongly nonergodic, see

FIG. 1. In this case, as is clearly observed in the bottom
of FIG. 4, $\langle m^x(t) \rangle$ approaches a stationary value but it

is different from the equilibrium value. The absence of

thermalization is the consequence of the lack of ETH in

the classically nonergodic region.

The timescale in which observables reach their station-

ary values is numerically found to be proportional to $N^{1/2}$. In the limit of $N \to \infty$, the classical dynamics

is exact forever, and the relaxation to the steady state
does not occur. This size dependence of the relaxation
time is consistent with the result on the exactly solv-

able Emch-Radin model (no $h_x$ and $D$ terms) obtained

in Refs. [32, 33].

In summary, the relation between the property of individual

energy eigenstates and the long-time behavior of

the underlying classical dynamics has been investigated

in fully-connected Ising ferromagnets, and it has been

shown that the distribution of the expectation values in

the quantum energy eigenstates converges to the distri-

bution of the long-time averages in the classical dynam-

ics starting from random initial states sampled uniformly

from the classical phase space. This feature is also ob-

served in another fully-connected model, i.e. the Dicke

model [31], and based on those observations, it is conjec-
tured that this is a general feature in semiclassical sys-

tems. Obviously, we need the numerical analysis in larger

system sizes and should study other semiclassical models

comprehensively in order to firmly verify the correctness

of the conjecture.

In experiment, long-range Ising models with the pair

interactions decaying as $1/r^\alpha$ with $0 \leq \alpha < d$, where $r$ is the distance and $d$ is the spatial dimension, have been

realized in trapped ions [25–28]. Our result is valid only

for the case of $\alpha = 0$, but it is expected that the system

with $0 < \alpha < d$ would have some common feature with

the system with $\alpha = 0$ like in equilibrium [34, 35], and

studying the case of $0 < \alpha < d$ is an experimentally

relevant interesting problem. The study of this problem

will be reported elsewhere [36].

This work was financially supported by JSPS KAK-

ENHI Grant No. 15K17718.

\[ d_{\text{eff}} := (\sum_{n} |c_n|^4)^{-1} \text{ with } |\psi(0)\rangle = \sum_{n} c_n |\phi_n\rangle, \text{ the fluctu-
tation in time } \left( \langle \psi(t) | \mathcal{O} | \psi(t) \rangle^2 - \langle \psi(t) | \mathcal{O} | \psi(t) \rangle^2 \right)^{1/2} \]

is smaller than a quantity of $O(d_{\text{eff}}^{-1/2})$ under the non-

resonance condition [22]. In the case of fully-connected

Ising ferromagnets, the number of energy eigenstates in

the totally symmetric subspace with the energy density

between $\varepsilon$ and $\varepsilon + \delta\varepsilon$ is proportional to

$N\delta\varepsilon$ in the case of spin-1/2 and $N^2\delta\varepsilon$ in the case of spin-1. Usually,$\delta\varepsilon \sim N^{-1/2}$, and hence $d_{\text{eff}}$ is typically very large (scaled as $d_{\text{eff}} \sim N^{1/2}$ for spin-1/2 and $N^{3/2}$ for spin-1). It is therefore expected that $\langle \psi(t) | \mathcal{O} | \psi(t) \rangle$ will reach an al-

most stationary value without any large fluctuation after

a sufficiently long time. If the classical dynamics is er-

godic, this stationary value will be equal to the equilib-

rium value.

In order to check the above scenario on quantum dy-

namics for spin-1 case, we numerically calculate the dyna-

mics of $\langle m^x(t) \rangle := \langle \psi(t) | m^x | \psi(t) \rangle$ after the quench, in

which $h_x$ is suddenly quenched from $h_x^{(i)}$ to $h_x^{(f)} = 0.2$. Other parameters are fixed as $h_z = 0.01$ and $D = 0.4$, and the post-quench Hamiltonian is the same as the

Hamiltonian employed in FIG. 1. The initial state is
given as the ground state of the pre-quench Hamiltonian.

In the top of FIG. 4, the time evolution in the case

of $h_x^{(i)} = -0.3$ is shown. In this case, the energy density

after the quench is 0.07, in which the classical dynamics

is ergodic, see FIG. 1. In the top of FIG. 4, $\langle m^x(t) \rangle$ approaches the stationary value almost identical to the equilibrium value (the red dashed line). In the bottom

of FIG. 4, the dynamics in the quench of $h_x^{(i)} = -0.39$ is shown. The energy density after the quench is 0.21, in

which the classical dynamics is strongly nonergodic, see

FIG. 1. In this case, as is clearly observed in the bottom
of FIG. 4, $\langle m^x(t) \rangle$ approaches a stationary value but it

is different from the equilibrium value. The absence of

thermalization is the consequence of the lack of ETH in

the classically nonergodic region.

The timescale in which observables reach their station-

ary values is numerically found to be proportional to $N^{1/2}$. In the limit of $N \to \infty$, the classical dynamics

is exact forever, and the relaxation to the steady state
does not occur. This size dependence of the relaxation
time is consistent with the result on the exactly solv-

able Emch-Radin model (no $h_x$ and $D$ terms) obtained

in Refs. [32, 33].

In summary, the relation between the property of individual

energy eigenstates and the long-time behavior of

the underlying classical dynamics has been investigated

in fully-connected Ising ferromagnets, and it has been

shown that the distribution of the expectation values in

the quantum energy eigenstates converges to the distri-

bution of the long-time averages in the classical dynam-

ics starting from random initial states sampled uniformly

from the classical phase space. This feature is also ob-

served in another fully-connected model, i.e. the Dicke

model [31], and based on those observations, it is conjec-
tured that this is a general feature in semiclassical sys-

tems. Obviously, we need the numerical analysis in larger

system sizes and should study other semiclassical models

comprehensively in order to firmly verify the correctness

of the conjecture.

In experiment, long-range Ising models with the pair

interactions decaying as $1/r^\alpha$ with $0 \leq \alpha < d$, where $r$ is the distance and $d$ is the spatial dimension, have been

realized in trapped ions [25–28]. Our result is valid only

for the case of $\alpha = 0$, but it is expected that the system

with $0 < \alpha < d$ would have some common feature with

the system with $\alpha = 0$ like in equilibrium [34, 35], and

studying the case of $0 < \alpha < d$ is an experimentally

relevant interesting problem. The study of this problem

will be reported elsewhere [36].

This work was financially supported by JSPS KAK-

ENHI Grant No. 15K17718.

1. T. Kinoshita, T. Wenger, and D. S. Weiss, Nature 440, 900 (2006).
2. S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature 449, 324 (2007).
3. S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Nature Phys. 8, 325 (2012).
4. J. v. Neumann, Z. Phys. 57, 30 (1929).
5. M. V. Berry, J. Phys. A 10, 2083 (1977).
6. M. Srednicki, Phys. Rev. E 50, 888 (1994).
7. J. M. Deutsch, Phys. Rev. A 43, 2046 (1991).
M. Rigol, V. Dunjko, and M. Olshanii, *Nature* **452**, 854 (2008).

G. Biroli, C. Kollath, and A. M. Läuchli, *Phys. Rev. Lett.* **105**, 250401 (2010).

A. C. Cassidy, C. W. Clark, and M. Rigol, *Phys. Rev. Lett.* **106**, 140405 (2011).

H. Tasaki, *J. Stat. Phys.* **163**, 937 (2016).

H. Tasaki, *Phys. Rev. Lett.* **80**, 1373 (1998).

J. Berges, S. Borsányi, and C. Wetterich, *Phys. Rev. Lett.* **93**, 142002 (2004).

P. Reimann, *Phys. Rev. Lett.* **101**, 190403 (2008).

S. Goldstein, J. L. Lebowitz, C. Mastrodonato, R. Tumulka, and N. Zanghi, *Phys. Rev. E* **81**, 011109 (2010).

P. Calabrese, F. H. L. Essler, and M. Fagotti, *Phys. Rev. Lett.* **106**, 227203 (2011).

J. Sato, R. Kanamoto, E. Kaminishi, and T. Deguchi, *Phys. Rev. Lett.* **108**, 110401 (2012).

J.-S. Caux and F. H. L. Essler, *Phys. Rev. Lett.* **110**, 257203 (2013).

E. Kaminishi, T. Mori, T. N. Ikeda, and M. Ueda, *Nature Phys.* **11**, 1050 (2015).

P. Reimann, *Phys. Rev. Lett.* **115**, 010403 (2015).

L. Vidmar and M. Rigol, *J. Stat. Mech.* **2016**, P06007 (2016).

A. J. Short, *New J. Phys.* **13**, 053009 (2011).

A. Russomanno, R. Fazio, and G. E. Santoro, *EPL* **110**, 37005 (2015).

B. Sciolla and G. Biroli, *J. Stat. Mech.* **2011**, P11003 (2011).

D. Porras and J. I. Cirac, *Phys. Rev. Lett.* **92**, 207901 (2004).

J. W. Bittman, B. C. Sawyer, A. C. Keith, C.-C. J. Wang, J. K. Freericks, H. Uys, M. J. Biercuk, and J. J. Bollinger, *Nature* **484**, 489 (2012).

R. Islam, C. Senko, W. Campbell, S. Korenblit, J. Smith, A. Lee, E. Edwards, C.-C. Wang, J. Freericks, and C. Monroe, *Science* **340**, 583 (2013).

P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, *Nature* **511**, 198 (2014).

\(f_{eq}\) is not equal to the expectation value in the microcanonical ensemble of the whole Hilbert space. We should consider the equilibrium state within the totally symmetric subspace.

It might be better to call the ETH within the totally symmetric subspace the “generalized eigenstate thermalization” in order to distinguish it from the ETH in the whole Hilbert space, see Ref. [10, 21].

Supplementary Information.

M. Kastner, *Phys. Rev. Lett.* **106**, 130601 (2011).

M. van den Worm, B. C. Sawyer, J. J. Bollinger, and M. Kastner, *New J. Phys.* **15**, 083007 (2013).

T. Mori, *Phys. Rev. E* **84**, 031128 (2011).

T. Mori, *Phys. Rev. E* **86**, 021132 (2012).

T. Mori, in preparation.
In the spin-1/2 fully-connected Ising ferromagnet, the classical Hamiltonian obtained in the limit of $N \to \infty$ is given by

$$\tilde{H}(z,p_z) = -\frac{1}{2}z^2 - h_z z - h_x \sqrt{1 - z^2} \cos p_z. \quad (S1)$$

The classical trajectory is determined by the equal-energy surface $\tilde{H}(z,p_z) = \varepsilon$ because it is one-dimensional in the phase space.

Typical shapes of the equal-energy surfaces are given in FIG. S1. The parameters are chosen as $h_x = 0.2$ and $h_z = 0.001$. The equal energy surface $\tilde{H}(z,p_z) = \varepsilon$ with $\varepsilon = 0$ is depicted by the solid line and that with $\varepsilon = -0.11$ is depicted as the dashed line. For $\varepsilon = 0$, the equal-energy surface is simply connected. If any point on the equal-energy surface is chosen as an initial state, the classical trajectory passes through all the points on the same equal-energy surface, i.e. the classical dynamics is trivially ergodic. On the other hand, for $\varepsilon = -0.11$, the equal-energy surface is divided into the two disconnected regions (it is noted that $p_z = 0$ is equivalent to $p_z = 2\pi$). In that case, the classical orbit is given by either of the two curves depending on the initial state. In that case, the corresponding quantum eigenstates are also divided into the two branches, shown in FIG. S2.

It should be pointed out that $h_z = 0$ is an exception. When $h_z = 0$, two disconnected equal-energy surfaces with $m^z(t) > 0$ and $m^z(t) < 0$ appear in the classical dynamics, but the quantum expectation values of $m^z$ are zero, $\langle \phi_n | m^z | \phi_n \rangle = 0$ for all $n$ because of the inversion symmetry of the $z$-component of the magnetization. It is interpreted that the two equal-energy surfaces are connected through the resonant quantum tunneling. However, as long as $h_z$ is nonzero and there is no symmetry between the two disjoint ergodic regions, such resonant tunneling is suppressed and we can see that $\langle \phi_n | m^z | \phi_n \rangle > 0$ for some $n$ and $\langle \phi_n | m^z | \phi_n \rangle < 0$ for the others when $N$ is large.

FIG. S1. The equal energy surfaces for $\varepsilon = 0$ (solid line) and for $\varepsilon = -0.11$ (dashed line). The parameters are chosen as $h_x = 0.2$ and $h_z = 0.001$.

FIG. S2. The eigenstate expectation values $\langle \phi_n | m^z | \phi_n \rangle$. The horizontal axis is the energy eigenvalue divided by $N$. The parameters are chosen as $h_x = 0.2$, $h_z = 0.001$, and $N = 5000$. 
THE RESULT IN THE DICKE MODEL

We consider the Dicke model, whose Hamiltonian is given by

\[ H_D = \omega_p a^\dagger a + \omega_a \sum_{i=1}^N S_i^z - \frac{g}{\sqrt{N}} (a + a^\dagger) \sum_{i=1}^N S_i^x, \]  

(S2)

where \( a \) and \( a^\dagger \) are the annihilation and the creation operators of cavity photons, and \( S_i \) is the spin-1/2 operator of \( i \)th spin. The collective interaction between cavity photons and \( N \) spins is given by \( g \). Here we set \( \omega_p = \omega_a = 1 \) and \( g = 0.6 \).

In the totally symmetric subspace, the basis state \( |x,y\rangle \) is characterized by the two variables \( x \) and \( y \), where

\[ \frac{1}{N} a^\dagger a |x,y\rangle = x|x,y\rangle, \quad \frac{1}{N} \sum_{i=1}^N S_i^z |x,y\rangle = y|x,y\rangle. \]  

(S3)

The Schrödinger equation for the wave function \( \psi_t(x,y) := \langle x,y|\psi(t) \rangle \), where \( |\psi(t)\rangle \) obeys \( i\partial|\psi(t)\rangle/\partial t = H_D|\psi(t)\rangle \), is obtained as, in the leading order in \( N \),

\[ \frac{i}{N} \frac{\partial}{\partial t} \psi_t(x,y) = \left[ \omega_p x + \omega_a y - 2g \sqrt{x} \left( \frac{1}{4} - y^2 \right) \cos p_x \cos p_y \right] \psi_t(x,y) =: \tilde{H}_D \psi_t(x,y), \]  

(S4)

where \( p_x = -(i/N)\partial/\partial x \) and \( p_y = (-i/N)\partial/\partial y \) are the canonical momenta conjugate to \( x \) and \( y \), respectively. It is noted that \( 1/N \) plays the role of the Planck constant \( h \).

In the limit of \( N \to \infty \), the system becomes classical, and the corresponding classical equations of motion for \( \{x(t), y(t), p_x(t), p_y(t)\} \) are given by the Hamilton equations under the classical Hamiltonian \( \tilde{H}_D \). As in the spin-1 fully-connected Ising ferromagnet, we calculate the distribution of quantum eigenstate expectation values of \( n := a^\dagger a/N \) and the interaction energy per spin,

\[ h_{\text{int}} := \frac{1}{N} \left[ -\frac{g}{\sqrt{N}} (a + a^\dagger) \sum_{i=1}^N S_i^x \right], \]  

(S5)

and the distribution of the long-time averages of the same quantities in the classical dynamics with random initial states sampled uniformly from the phase space. In the calculation of quantum eigenstate expectation values, the number of cavity photons is truncated at \( N_{\text{max}} = 6N \) in order to avoid the infinite dimension of the Hilbert space.

The result is presented in FIG. S3, and one can see good agreement of the two distributions for both observables, and the conjecture discussed in the main text also holds in the Dicke model.

![FIG. S3](image-url) Long-time averages of (a) \( n = a^\dagger a/N \) and (b) \( h_{\text{int}} \) in the classical dynamics starting from random initial states sampled uniformly from the phase space (red points) and expectation values of the same quantities in each quantum energy eigenstates for \( N = 40 \) (blue points). The horizontal axis is the energy density \( \varepsilon \). The number of the samples of the random initial states in the classical dynamics is 2000. The classical dynamics is calculated up to \( t = 10000 \) and the long-time averages are calculated within this time interval.