A.Cormack’s last inversion formula and a FBP reconstruction

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Abstract

A reconstruction of a function from integrals over the family of confocal paraboloids in \( \mathbb{R}^n \) is given by a FBP formula.

1 Introduction

Cormack proposed a clever method for determination of distribution of a scattered material in a space from data of scattered light filtered on travel times [5]. He wrote in the introduction to [5]: "Suppose that waves travel in this space with a speed \( v \), and that the space contains a distribution of scattering material with a density \( f \) which is assumed to be smooth and either rapidly decreasing or of compact support... An impulsive plane wave front with a normal \( \omega \) reaches 0 at \( t = 0 \). As result of the well-known focusing property of paraboloids the scattering waves arriving at 0 at time \( t = 2p/v \) will all have originated on the paraboloid defined by \( r(1 + \langle \xi, \omega \rangle) = 2p..." The mathematical problem is to reconstruct a function in a plane from data of integrals \( Rf \) over a family of confocal parabolas or confocal rotation paraboloids. "The problem of determining \( f \) from \( Rf \) for a family of curves in \( \mathbb{R}^2 \) was discussed in [1], [2], [3] and in [3] it was shown that the parabolic case could be reduced to the ordinary Radon transform by a coordinate transformation which does not have an analog in \( \mathbb{R}^3 \). The case of \( \mathbb{R}^n \) was considered in [4], in which \( f \) was considered to be expanded in spherical harmonics; some properties of the harmonic components were given, and a solution to the resulting integral equation was given. This solution, however, had a very awkward form and as a result it was not investigated in detail. Given in [5] is a simpler solution for the case of \( \mathbb{R}^3..." 

This solution is

\[
\frac{1}{e^{r/2}} f(x) = \frac{1}{(4\pi)^\frac{n}{2}} \int_{S^2} \left( \frac{\partial}{\partial r} r \right)^2 \frac{g(p, \omega)}{p^{n/2}} \left. \frac{\partial}{\partial p} \frac{\partial}{\partial \Omega} \right|_{p=r(1+\langle \xi, \omega \rangle)/2} \Omega, \tag{1}
\]

where \( r = |x|, \ \xi = x/|x|, \ \Omega \) is the area form in \( S^2 \) and \( g \) is the surface integral of \( f \).

Our goal is a reconstruction by means of an integral transform of FBP type to avoid divergent integrals.

2 Confocal paraboloids and integrals

Confocal parabolas
The family of paraboloids $Z(p, \omega)$ in an Euclidean space $E^n$ with the focus at the origin can be given by the equation $r - \langle x, \omega \rangle = 2p$ where $p > 0$ and $\omega \in S^{n-1}$ are parameters. For a bounded function $f$ in $E^n$ with compact support we consider the integrals over paraboloids

$$Rf(p, \omega) = \int_{Z(p, \omega)} f \, dS,$$

$$Mf(p, \omega) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{p \leq r - \langle x, \omega \rangle \leq p + \varepsilon} f \, dx = \int_{Z(p, \omega)} \frac{f \, dS}{|\nabla \theta(x, \omega)|},$$

where $dS$ is the Euclidean surface density and $Mf$ is equal to the integral of $f \, dx$ between to close paraboloids. The function can be reconstructed from $Rf$ and from $Mf$ according to the following
Theorem 1. For an arbitrary function \( f \in C^2(E^2) \) with compact support, a reconstruction is given by

\[
f(x) = -\frac{1}{4\pi^2 r^{1/2}} \int_{S^1} \int_{R} \frac{\partial}{\partial p} \left( p^{1/2} R f(p, \omega) \right) \frac{dp}{\theta(x, \omega) - p} \Omega \quad (3)
\]

\[
f(x) = -\frac{1}{4\pi^2 r^{1/2}} \int_{S^1} \int_{R} \frac{\partial^2}{\partial p^2} \left( p^{1/2} R f(p, \omega) \right) \frac{dp}{\theta(x, \omega) - p} \Omega, \quad x \in E^2 \setminus 0.
\]

For \( f \in C^2(E^3) \), we have

\[
f(x) = -\frac{1}{8\pi^2 r^{1/2}} \int_{S^2} \frac{\partial^2}{\partial p^2} \left( p^{1/2} R f(p, \omega) \right) \frac{dp}{\theta(x, \omega) - p} \Omega
\]

\[
f(x) = -\frac{1}{8\pi^2 r^{1/2}} \int_{S^2} \frac{\partial^2}{\partial p^2} \left( p M f(p, \omega) \right) \frac{dp}{\theta(x, \omega) - p} \Omega, \quad x \in E^3 \setminus 0.
\]

Remark. The first formula (4) reminds Cormack’s reconstruction (8) where apparently \( g = R f \). However the integral \( M f \) is a correct model for the “scattered light filtrated on travel times”. It is different from the Euclidean surface integral \( R f \) since the dominator in (2) is not a constant.

Proof. We assume at the beginning that \( n \geq 2 \) is arbitrary. The family of rotation paraboloids is generated by the function \( \Phi(x, p, \omega) = \theta(x, \omega) - p, \theta(x, \omega) = (r - (x, \omega))/2 \) defined in the manifold \( X \times \Sigma, X = E^n \setminus 0, \Sigma = \mathbb{R}_+ \times S^{n-1} \). The set \( X \) is supplied with the Euclidean volume form \( d\xi \) and the manifold \( \Sigma \) with the volume form \( dp \wedge \Omega \). The map \( \pi_X : Z \to X \) is proper where \( Z = \Phi^{-1}(0) \) and the function \( \Phi \) but satisfies the conditions: \( \pi_X \) has rank \( n \) and the map \( \pi^* \) is a bijection see §3. We have for any \( n \geq 2 \)

\[
\nabla_x \theta = \frac{\xi - \omega}{2}, \quad \text{where } \xi = \frac{x}{r}, \quad \text{and } |\nabla_x \theta| = \theta^{1/2} r^{-1/2},
\]

\[
2^n (n - 1)! J(x, \omega) = \nabla \theta \wedge (d\omega \nabla \theta)^{\wedge n-1} = \langle \xi - \omega, \omega \rangle \quad \omega \wedge (d\omega)^{\wedge n-1} = \langle \xi - \omega, \omega \rangle \Omega.
\]

It follows that \( \det J(x, \omega) \neq 0 \) except for the submanifold \( V \setminus \{ \omega = \xi \} \in X \times \Sigma \). However \( \pi^* \) is a bijection since \( \langle \xi - \omega, \omega \rangle \geq 0 \). Choose a weight function \( b(x, \omega) = |\nabla_x \theta(x, \omega)|^2 \) and consider the weighted Funk transform

\[
M_b f(p, \omega) = \int_{\theta(x, \omega) = p} f(x) |\nabla_x \theta(x, \omega)| dS = p^{1/2} R \left( r^{-1/2} f \right) = p M \left( r^{-1} f \right), \quad (5)
\]

where \( R \) is the Euclidean integral transform as above. We want to invert \( M_b \) by means of a filtered back projection. Take a number \( \alpha > 0 \) and consider the \( \varepsilon \)-neighborhood \( V(\varepsilon) \). The generating function \( \Phi \) satisfies (i) for the map \( \pi^* \) restricted to the set \( Z \setminus V(\varepsilon) \times \mathbb{R}_+ \). Let \( b(\varepsilon) \) be a function in \( X \times \Sigma \) that vanishes in \( V(\varepsilon) \) and coincides with \( b \) otherwise. Take \( \beta = 1 \) as the second weight functions and apply Theorem 5§3 to \( \Phi \) and \( Z \setminus V(\varepsilon) \). Note that equation \( d\omega \theta(x, \omega) = -x/2 \) implies (ii). This yields

\[
D_b(\varepsilon) f = N M_b f + \Theta_b(\varepsilon) f,
\]

where \( N = N_\beta \) and \( \Theta_b(\varepsilon) \) is the operator with the kernel

\[
\Theta_b(x, y) = \text{Re} \int_{S^2} \frac{b(x, \omega) \Omega}{\theta(x, \omega) - \theta(y, \omega) - i0}^n.
\]
Lemma 2 We have $D_{b(e)}(x) \to D_b(x)$ for $x \neq 0$ and $\Theta_{b(e)}(x,y) \to 0$ for arbitrary $x \neq y \in \mathbb{E}^n \setminus 0$ as $\varepsilon \to 0$ where

$$D_b(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \frac{b(\varepsilon) \Omega}{|\nabla \theta|^n}.$$  \hspace{1cm} (6)

for any $n \geq 2$.

Proof of Lemma 2 We have $|\nabla \theta|^2 = r^{-1} \theta$, hence $|\nabla \theta(x,\omega)|^{-1} \approx \sin \gamma/2$ where $\gamma$ is the angle between $x$ and $\omega$. Therefore integral (6) converges to (7). This proves the first statement. To check the second statement we note that

$$\theta(x,\omega) - \theta(y,\omega) = \frac{1}{2}(|x| - |y| - \langle x - y, \omega \rangle)$$

is a linear function of $\omega$ for arbitrary $x \neq y$. This function has a zero $\omega \in S^{n-1}$ since $||x| - |y|| \leq |x - y|$. By (6) it follows that $\Theta_b(x,y) = 0$ since $\theta(x,\omega) - \theta(y,\omega)$ is a linear function of $\omega$ and $\deg b + n - 1 = n$. Now it is sufficient to check that $\Theta_{b(e)} \to \Theta_b$ as $\varepsilon \to 0$. For this we show that the dominator $\theta(x,\omega) - \theta(y,\omega)$ does not vanish at the point $\omega = \xi / |x|$. We have

$$\theta(x,\xi) - \theta(y,\xi) = -\frac{|x| |y| + \langle y, x \rangle}{2 |x|} < 0$$

since $y \neq x$ and $x \neq 0, y \neq 0$. This completes the proof.  

By Lemma 2 we can take the limit in (10) and obtain the equation $D_b f = N b f$. Due to (5) we can replace the integrals $M_b f$ by data of $M(r^{-1} f)$ and or by data of $R(r^{-1/2} f)$. By substitution $f = r f$ or $f = r^{1/2} \tilde{f}$ we finally come up to the equations (3) and (4). This completes the proof of Theorem 1.

Proposition 3 Integrals (3) and (4) have the classical sense for sufficiently smooth functions $f$.

Proof. For $n = 2$, we have

$$dZ = (dr^2 + r^2 d\gamma)^{1/2} = \frac{dr}{\cos \gamma/2} = \frac{dr}{(1 - p/r)^{1/2}}$$

and

$$p^{1/2} R(r^{1/2} f)(p,\omega) = p^{1/2} \int_{\sin^2(\gamma/2) = p/r} r^{1/2} f(r,\gamma) \, dZ = 2 \int_{p}^{\infty} f(r,\gamma) \, \frac{r dr}{(r/p - 1)^{1/2}} = 2 \int_{1}^{\infty} f(s,\gamma) \, \frac{sd s}{(s - 1)^{1/2}}.$$ 

The right hand side has continuous second derivative with respect to $p$ and $\omega$ if $f \in C^2(E^2)$.

Lemma 4 For any $f \in C^4(E^3)$, we have

$$M f(p,\omega) = p^{-1/2} R(r^{1/2} f)(p,\omega) = g_0(\omega) + pg_1(p,\omega)$$

as $p \to 0$ where $g_0$ and $g_1$ are $C^2$-continuous functions.
Theorem 1 will follow since \( Mf = p^{-1/2}R \left( r^{1/2}f \right) \) and
\[
\frac{\partial^2}{\partial p^2} \left( pMf \left( p, \omega \right) \right) = \frac{\partial^2}{\partial p^2} \left( p^{1/2}R \left( r^{1/2}f \right) \right) \left( p, \omega \right) = 2g_1 \left( p, \omega \right) + o \left( 1 \right).
\]

**Proof.** Variables \( r, \gamma \) and \( \phi \) are spherical coordinates in \( \mathbb{R}^3 \) where \( \gamma, 0 \leq \gamma < \pi \) is the spherical distance between points \( \xi, \omega \in S^2 \) and \( \phi \) is rotation angle about \( \omega \) and \( Z \left( p, \omega \right) \). Equation \( p = \left( x - \left\langle x, \omega \right\rangle \right)/2 = r \sin^2 \left( \gamma/2 \right) \) defines the surface \( Z \left( p, \omega \right) \) and we have
\[
dZ = \left( dr^2 + r^2 d\gamma^2 \right)^{1/2} r \sin \gamma d\phi
\]
for the Euclidean area \( dZ \) of \( Z \left( p, \omega \right) \). We have
\[
\left( dr^2 + r^2 d\gamma^2 \right)^{1/2} = \frac{dr}{\cos \gamma/2}, \quad dZ = 2 \left( pr \right)^{1/2} dr d\phi.
\]
This yields
\[
p^{-1/2}R \left( r^{1/2}f \right) \left( p, \omega \right) = p^{-1/2} \int_{\sin^2 \left( \gamma/2 \right) = p/r} r^{1/2} f \left( r, \gamma, \phi \right) dZ = 2 \int_p^\infty \int_0^{2\pi} r f \left( r, \gamma, \varphi \right) d\varphi dr.
\]
For a plane \( P \) and a function \( \phi \in C^4 \left( P \right) \), the circle integral can be represented in the form
\[
\int_{|y|=s} \phi \left( y \right) d\varphi = 2\pi \phi \left( 0 \right) + \tau \left( s \right) s^2
\]
with a remainder \( \tau \in C^2 \). Applying this equation to \( f \) in a plane \( P \) orthogonal to \( \omega \) for \( s = r \sin \gamma \) we obtain
\[
\int_0^{2\pi} f \left( r, \gamma, \varphi \right) d\varphi = 2\pi f \left( r, 0, \varphi \right) + \left( r \sin \gamma \right)^2 h \left( r, \gamma, \varphi \right), \quad h \in C^2.
\]
We have \( \left( r \sin \gamma \right)^2 = 4p \left( r - p \right) \) and \( f \left( r, 0, \varphi \right) = f \left( r\omega \right) \) which implies
\[
p^{-1/2}R \left( r^{1/2}f \right) \left( p, \omega \right) = 4\pi \int_p^\infty r f \left( r\omega \right) dr + 8p \int_p^\infty \tau \left( r, \gamma, \omega \right) \left( r - p \right) r dr.
\]
Now Lemma 1 follows for the functions
\[
g_0 \left( \omega \right) = 4\pi \int_p^\infty r f \left( r\omega \right) dr, \quad g_1 \left( p, \omega \right) = 8 \int_p^\infty \tau \left( r, \gamma, \omega \right) \left( r - p \right) r dr
\]
which belong to \( C^2 \). ▶

## 3 Weighted integral transform and reconstruction

Let \( X, \Sigma \) be smooth manifolds of dimension \( n > 1 \) and \( Z \subset X \times \Sigma \) be a hypersurface. Let \( \pi_X : Z \to X, \pi_{\Sigma} : Z \to \Sigma \) be the natural projections. A family of hypersurfaces \( Z \left( \sigma \right) = \pi_{\Sigma}^{-1} \left( \sigma \right) \) in \( X \) is defined as well as the family of hypersurfaces \( Z \left( x \right) = \pi_X^{-1} \left( x \right) \) in \( \Sigma \). Suppose that there exists a real function \( \Phi \in C^2 \left( X \times \Sigma \right) \) (called generating function) such that \( Z = \Phi^{-1} \left( 0 \right) \) and \( d\Phi \neq 0 \) in \( Z \). Let \( T^* \left( X \right) \) be the cotangent bundle of \( X \) and \( T^* \left( X \right) /\mathbb{R}_+ \) be the bundle of rays in \( T^* \left( X \right) \). We suppose that
(i) The map $\pi^*: \mathbb{R}^n \to T^*(X)$ is a diffeomorphism, where $\pi^*(x, \sigma; t) = (x, t d_x \Phi (x, \sigma))$, $t \in \mathbb{R}_+$. This condition implies that $\pi_X$ has rank $n$ and $Z(\sigma)$ is a smooth hypersurface in $X$ for any $\sigma \in \Sigma$. The function $\det J (\Phi)$ vanishes if where $\Pi^*$ is not a local diffeomorphism and vice versa.

**Definition.** We say that points $x \neq y \in X$ are conjugate with respect to $\Phi$, if $\Phi (x, \sigma) = \Phi (y, \sigma)$ and $d_x \Phi (x, \sigma) = d_y \Phi (y, \sigma)$ for some $\sigma \in \Sigma$. We call a generating function $\Phi$ regular, if the projection $\pi : Z \to X$ is proper, $\Phi$ satisfies the conditions (i) and (ii): there are no conjugate points.

Let $dX$ be a volume form in $X$ and $b = b(x, \sigma)$ be a continuous function in $Z$. For any bounded arbitrary function $f$ in $X$ with compact support, the integral

$$
M_b f (\sigma) \doteq \int \delta (\Phi (x, \sigma)) f(x) b(x, \sigma) dX = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|\Phi (., \sigma)| \leq \varepsilon} f(x) b(x, \sigma) dX
$$

converges for any $\sigma \in \Sigma$. We suppose now that $dX$ is the Riemannian volume form of a Riemannian metric $g$ in $X$. We have:

$$
M_b f (\sigma) = \int_{Z(\sigma)} \frac{f(x) b(x, \sigma)}{|d_x \Phi (x, \sigma)|_g} d_S,
$$

where $d_S$ is the Riemannian volume form in a hypersurface in $X$. Here $Z(\sigma) = \{ x \in X, \Phi (x, \sigma) = 0 \}$ and $|.|_g$ means the Riemannian norm of a covector. Let $\beta$ be a locally bounded function in $X \times \Sigma$. The singular integral

$$
\Theta_{b, \beta} (x, y) = (-1)^n \int_{Z(y)} \frac{b(y, \sigma) \beta(x, \sigma)}{(\Phi (x, \sigma) - i0)^n} \frac{d\Sigma}{d_x \Phi (y, \sigma)}
$$

is well defined for any $x, y \in X$, $y \neq x$ since $d_x \Phi (x, \sigma) \neq 0$ in $Z(y) \doteq \{ \sigma \in \Sigma; \Phi (y, \sigma) = 0 \}$ due to (ii).

**Theorem 5** Suppose that a regular generating function $\Phi$ satisfies condition (iii):

$$
\Re i^n \Theta_{b, \beta} (x, y) = 0 \text{ for any } x, y \in X, \ x \neq y.
$$

Then an arbitrary function $f \in L_{2\text{comp}} (X)$ can be reconstructed from data of $M_b f$ by

$$
D_{b, \beta} f = N_{\beta} M_b f + \Theta_{b, \beta} f
$$

where $\Theta_{b, \beta}$ is an operator with the kernel $\Theta_{b, \beta} (x, y)$ and for even $n$,

$$
N_{\beta} g (x) = \frac{(n - 1)!}{n!} \int_{Z(x)} \frac{g(\sigma) \beta(x, \sigma)}{\Phi (x, \sigma)^n} \frac{d\Sigma}{d_x \Phi},
$$

for odd $n$,

$$
N_{\beta} g (x) = \frac{1}{2^{n-1}} \int_{Z(x)} \delta^{(n-1)} (\Phi (x, \sigma)) g(\sigma) \beta(x, \sigma) \frac{d\Sigma}{d_x \Phi}
$$

is a bounded operator $H_{2\text{comp}}^{(n-1)/2} (\Sigma) \to L_{2\text{loc}} (X)$; and

$$
D_{b, \beta} (x) = \frac{1}{|s^{n-1}|} \int_{Z(x)} \frac{b(x, \sigma) \beta(x, \sigma)}{|d_x \Phi (x, \sigma)|_g^n} \frac{d\Sigma}{d_x \Phi}.
$$

A proof is given in [6] for the case $b = \beta = 1$. The general case can be obtained in the same lines.
References

[1] Cormack, A.M.: The Radon transform on a family of curves in the plane I. Proc. Am. Math. Soc. 83, 325-330 (1981)

[2] Cormack, A.M.: The Radon transform on a family of curves in the plane II. Proc. Am. Math. Soc. 86, 293-298 (1982)

[3] Cormack, A.M.: Radon’s problem - old and new. SIAM-AMS Proceedings, Amer. Math. Soc., Providence 14 33-39 (1984)

[4] Cormack, A.M.: Radon’s problem for some surfaces in $\mathbb{R}^n$. Proc. Am. Math. Soc. 99, 305-312 (1987)

[5] Cormack, A.M.: A paraboloidal Radon transform. 75 years of Radon transform (Vienna, 1992), 105–109, Conf. Proc. Lecture Notes Math. Phys., IV, Int. Press, Cambridge, MA (1994)

[6] Palamodov, V. P.: A uniform reconstruction formula in integral geometry. Inverse Probl. 28, 065014 (2012)