Full groups of Cuntz–Krieger algebras and Higman–Thompson groups

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Abstract

In this paper, we will study presentations of the continuous full group $\Gamma_A$ of a one-sided topological Markov shift $(X_A, \sigma_A)$ for an irreducible matrix $A$ with entries in $\{0,1\}$ as a generalization of Higman–Thompson groups $V_N, 1 < N \in \mathbb{N}$. We will show that the group $\Gamma_A$ can be represented as a group $\Gamma_{A_{\mathbb{Z}}}^a$ of matrices, called $A$-adic tables, with entries in admissible words of the shift space $X_A$, and a group $\Gamma_{A_{\mathbb{Z}}}^{PL}$ of right continuous piecewise linear functions, called $A$-adic PL functions, on $[0,1]$ with finite singularities.

1 Introduction

In 1960’s, R. J. Thompson has initiated a study of finitely presented simple infinite groups. He has discovered first two such groups in [23]. They are now known as the groups $V_2$ and $T_2$. G. Higman has generalized the group $V_2$ to infinite family of finitely presented infinite groups. One of such family is the groups written $V_N, 1 < N \in \mathbb{N}$ which are called the Higman–Thompson groups. They are finitely presented and their commutator subgroups are simple. Their abelianizations are trivial if $N$ is even, and $\mathbb{Z}_2$ if $N$ is odd. K. S. Brown has extended the groups $V_N$ to triplets of infinite families $F_N \subset T_N \subset V_N, 1 < N \in \mathbb{N}$, and proved that each of the groups is finitely presented ([1]). The Higman–Thompson group $V_N$ is known to be represented as the group of right continuous piecewise linear functions $f : [0,1) \to [0,1)$ having finitely many singularities such that all singularities of $f$ are in $\mathbb{Z}[\frac{1}{N}]$, the derivative of $f$ at any non-singular point is $N^k$ for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[\frac{1}{N}] \cap [0,1)) = \mathbb{Z}[\frac{1}{N}] \cap [0,1)$ ([23]). See [2] for general reference on these groups.

V. Nekrashevych [19] has shown that the Higman–Thompson group $V_N$ appears as a certain subgroup of the unitary group of the Cuntz algebra $O_N$. The second named author has observed in [18, Remark 6.3] that the subgroup is nothing but the continuous full group $\Gamma_N$ of $O_N$, which is also realized as the topological full group of the associated groupoid. Such full groups have arisen from a study of orbit equivalence of symbolic dynamics.

Recently the authors have studied full groups of the Cuntz–Krieger algebras and full groups of the groupoids coming from shifts of finite type. The first named author has studied the normalizer groups of the canonical maximal abelian $C^*$-subalgebras in the Cuntz–Krieger algebras which are called the continuous full groups from the view point of
orbit equivalences of topological Markov shifts and classification of $C^*$-algebras ([9], [10], etc.), and showed that the continuous full groups are complete invariants for the continuous orbit equivalence classes of the underlying topological Markov shifts ([12], more generally [18]). The second named author has studied the continuous full groups of more general étale groupoids ([16], [17], [18], etc.), and called them the topological full groups of étale groupoids. He has proved that if an étale groupoid is minimal, the topological full group of the groupoid is a complete invariant for the isomorphism class of the groupoid. He has also shown that if a groupoid comes from a shift of finite type, the topological full group is of type $F_\infty$ and in particular finitely presented. He has furthermore obtained that the topological full groups for shifts of finite type are simple if and only if its homology group $H_0(G_A)$ of the groupoid $G_A$ is 2-divisible, and that its commutator subgroups are always simple. We have obtained the following results on the group $\Gamma_A$ for the topological Markov shift $(X_A, \sigma_A)$ defined by an irreducible square matrix with entries in $\{0, 1\}$.

Theorem 1.1 ([12], [14], [18]). Let $A$ and $B$ be irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I) in [5]. The following conditions are equivalent:

1. The one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent.
2. The étale groupoids $G_A$ and $G_B$ are isomorphic.
3. The groups $\Gamma_A$ and $\Gamma_B$ are isomorphic.
4. The Cuntz–Krieger algebras $O_A$ and $O_B$ are isomorphic and $\det(id - A) = \det(id - B)$.

Suppose that $A$ is an $N \times N$ matrix and $B$ is an $M \times M$ matrix. It is well-known that the Cuntz–Krieger algebras $O_A$ and $O_B$ are isomorphic if and only if there exists an isomorphism $\Phi$ of groups from $\mathbb{Z}^N/(id - A)^t\mathbb{Z}^N$ to $\mathbb{Z}^M/(id - B)^t\mathbb{Z}^M$ such that $\Phi(u_A) = u_B$ where $u_A$ and $u_B$ are the classes of the vectors $[1, \ldots, 1]$ ([22]). Hence the isomorphism classes of the groups $\Gamma_A$ are completely classified in terms of the underlying matrices $A$, so that there exist an infinite family of finitely presented infinite simple groups of the form $\Gamma_A$.

In this paper, we will study presentations of the group $\Gamma_A$ for an irreducible matrix $A$ with entries in $\{0, 1\}$ as a generalization of the Higman–Thompson groups $V_N, 1 < N \in \mathbb{N}$. The group $\Gamma_A$ is defined as a group of homeomorphisms $\tau$ on the shift space $X_A$ of a topological Markov shift $(X_A, \sigma)$ such that

$$\sigma_A^{k_\tau(x)}(\tau(x)) = \sigma_A^{l_\tau(x)}(x), \quad x \in X_A$$

(1.1)

for some continuous functions $k_\tau, l_\tau : X_A \rightarrow \mathbb{Z}_+$. If the matrix $A$ is the $N \times N$-matrix whose entries are all 1’s, the group $\Gamma_A$ coincides with the Higman–Thompson group $V_N$ of order $N$.

We will introduce a notion of $A$-adic PL (piecewise linear) function which is a right continuous piecewise linear function on the interval $[0, 1)$ associated with the matrix $A$ to represent an element of the group $\Gamma_A$. The set of $A$-adic PL functions forms a group
under composition of functions. We also introduce a notion of $A$-adic table in order to represent elements of $\Gamma_A$ which is a matrix
\[
\begin{bmatrix}
\mu(1) & \mu(2) & \cdots & \mu(m) \\
\nu(1) & \nu(2) & \cdots & \nu(m)
\end{bmatrix}
\]
with entries in admissible words $\nu(i), \mu(i), i = 1, \ldots, m$ of the one-sided topological Markov shift $(X_A, \sigma_A)$ satisfying certain properties. We may define an equivalence relation of the $A$-adic tables, and a product structure in the set $\Gamma_{\text{tab}}^A$ of the equivalence classes of $A$-adic tables which makes it a group. We will show the following theorem which is a generalization of a well-known result for the Higman–Thompson groups. Assume that $A$ is an irreducible square matrix with entries in $\{0, 1\}$ satisfying condition (I) in [5].

**Theorem 1.2** (Theorem 6.3). There exist canonical isomorphisms of discrete groups among the continuous full group $\Gamma_A$, the group $\Gamma_{\text{tab}}^A$ of the equivalence classes of $A$-adic tables, and the group $\Gamma_{\text{PL}}^A$ of the $A$-adic PL functions on $[0, 1)$, that is
\[
\Gamma_A \cong \Gamma_{\text{tab}}^A \cong \Gamma_{\text{PL}}^A.
\]

Let $1 < \beta \in \mathbb{R}$ be the Perron eigenvalue of $A$. For $\tau \in \Gamma_A$, we put $d_\tau(x) = l_\tau(x) - k_\tau(x)$ for the continuous functions $k_\tau, l_\tau$ satisfying (1.1). We define the derivative $D_\tau$ of $\tau$ as a real valued continuous function on $X_A$:
\[
D_\tau(x) = \beta^{d_\tau(x)}, \quad x \in X_A.
\]
We know that $D_\tau$ satisfies the following law of derivative:
\[
D_{\tau_2 \circ \tau_1} = D_{\tau_1} \cdot (D_{\tau_2} \circ \tau_1), \quad D_{\tau^{-1}} = (D_\tau \circ \tau^{-1})^{-1}
\]
for $\tau, \tau_1, \tau_2 \in \Gamma_A$ (Proposition 7.11).

The continuous full group $\Gamma_A$ is isomorphic to the group $\Gamma_{\text{PL}}^A$ of all $A$-adic PL functions on $[0, 1)$ by the above theorem. We will show that $\tau \in \Gamma_A$ is realized as an $A$-adic PL function on $[0, 1)$ in the following way, where $X_A$ is endowed with lexicographic order.

**Theorem 1.3** (Theorem 7.10). There exists an order preserving continuous surjection $\rho_A : X_A \rightarrow [0, 1]$ from the shift space $X_A$ of a one-sided topological Markov shift $(X_A, \sigma_A)$ to the closed interval $[0, 1]$ such that for any element $\tau \in \Gamma_A$, there exists an $A$-adic PL function $f_\tau$ and a finite set $S_\tau \subset X_A$ satisfying the following properties:

(i) $f_\tau(\rho_A(x)) = \rho_A(\tau(x))$ for $x \in X_A \setminus S_\tau$,

(ii) $\frac{df_\tau}{dx}(\rho_A(x)) = D_\tau(x)$ for $x \in X_A \setminus S_\tau$.

K. S. Brown has extended the groups $V_N, 1 < N \in \mathbb{N}$ to triplets $F_N \subset T_N \subset V_N$ of infinite discrete groups. In the final section, we will generalize the triplet to the triplet $F_A \subset T_A \subset \Gamma_A$ of infinite discrete groups.

Throughout the paper, we denote by $\mathbb{N}$ and by $\mathbb{Z}_+$ the set of positive integers and the set of nonnegative integers respectively.
2 Preliminaries

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in \{0,1\}, where $1 < N \in \mathbb{N}$. In what follows, we assume that $A$ is irreducible and has no rows or columns identically equal to zero. We denote by $X_A$ the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

of the right one-sided topological Markov shift for $A$. It is a compact Hausdorff space in natural product topology. The shift transformation $\sigma$ on $X_A$ defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ is a continuous surjection on $X_A$. The topological dynamical system $(X_A, \sigma)$ is called the (right one-sided) topological Markov shift for $A$. We henceforth assume that $A$ satisfies condition (I) in the sense of Cuntz–Krieger [5]. The condition (I) for $A$ is equivalent to the condition that $X_A$ is homeomorphic to a Cantor discontinuum.

A word $\mu = (\mu_1, \ldots, \mu_m)$ for $\mu_i \in \{1, \ldots, N\}$ is said to be admissible for $X_A$ if $\mu$ appears in somewhere in some element $x$ in $X_A$. The length of $\mu$ is $m$ and denoted by $|\mu|$. We denote by $B_m(X_A)$ the set of all admissible words of length $m \in \mathbb{N}$. For $m=0$ we denote by $B_0(X_A)$ the empty word $\emptyset$. We put $B_*(X_A) = \cup_{m=0}^\infty B_m(X_A)$ the set of admissible words of $X_A$. For two words $\mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A)$, $\nu = (\nu_1, \ldots, \nu_n) \in B_n(X_A)$, we denote by $\mu \nu$ the word $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$. For a word $\mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A)$, the cylinder set $U_\mu \subset X_A$ is defined by

$$U_\mu = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \ldots, x_m = \mu_m\}.$$

We put

$$\Gamma_k^+(\mu) = \{(\eta_1, \ldots, \eta_k) \in B_k(X_A) \mid (\mu_1, \ldots, \mu_m, \eta_1, \ldots, \eta_k) \in B_{m+k}(X_A)\}, \quad k \in \mathbb{Z}_+$$

and $\Gamma_*^+(\mu) = \cup_{k=1}^\infty \Gamma_k^+(\mu)$ which is called the follower set of $\mu$. For two words $\mu, \nu \in B_*(X_A)$, we see that $\Gamma_*^+(\mu) = \Gamma_*^+(\nu)$ if and only if $\Gamma_k^+(\mu) = \Gamma_k^+(\nu)$.

The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix $A$ has been defined in [5] as the universal $C^*$-algebra generated by $N$ partial isometries $S_1, \ldots, S_N$ subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1, \ldots, N. \quad (2.1)$$

The algebra $\mathcal{O}_A$ is the unique $C^*$-algebra subject to the above relations under the condition (I) for $A$. By the universality for the relation (2.1), the correspondence $S_i \rightarrow e^{2\pi \sqrt{-1} t} S_i$, $i = 1, \ldots, N$, $e^{2\pi \sqrt{-1} t} \in \mathbb{T} = \{e^{2\pi \sqrt{-1} t} \mid t \in [0, 1]\}$ yields an action $\rho : \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_A)$ called the gauge action. For a word $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_i \in \{1, \ldots, N\}$, we denote the product $S_{\mu_1} \cdots S_{\mu_k}$ by $S_\mu$. Then $S_\mu \neq 0$ if and only if $\mu \in B_*(X_A)$. Let $C^*(S_\mu S_\mu^*; \mu \in B_*(X_A))$ be the $C^*$-subalgebra of $\mathcal{O}_A$ generated by the projections of the form $S_\mu S_\mu^*; \mu \in B_*(X_A)$, which we denote by $\mathcal{D}_A$. It is isomorphic to the commutative $C^*$-algebra $C(X_A)$ of all complex valued continuous functions on $X_A$ through the correspondence $S_\mu S_\mu^* \in \mathcal{D}_A \leftrightarrow \chi_\mu \in C(X_A)$ where $\chi_\mu$ denotes the characteristic function on $X_A$ for the cylinder set $U_\mu$ for $\mu \in B_*(X_A)$. We will identify $C(X_A)$ with the subalgebra $\mathcal{D}_A$ of $\mathcal{O}_A$. It is
well-known that the algebra \( \mathcal{D}_A \) is maximal abelian in \( \mathcal{O}_A \) ([5] Remark 2.18). We denote by \( U(\mathcal{O}_A) \) and \( U(\mathcal{D}_A) \) the group of unitaries in \( \mathcal{O}_A \) and the group of unitaries in \( \mathcal{D}_A \) respectively. The normalizer \( N(\mathcal{O}_A, \mathcal{D}_A) \) of \( \mathcal{D}_A \) in \( \mathcal{O}_A \) is defined by

\[
N(\mathcal{O}_A, \mathcal{D}_A) = \{ u \in U(\mathcal{O}_A) \mid u\mathcal{D}_Au^* = \mathcal{D}_A \}.
\]

For \( x = (x_n)_{n \in \mathbb{N}} \in X_A \), the orbit \( \text{orb}_{\sigma_A}(x) \) of \( x \) under \( \sigma_A \) is defined by

\[
\text{orb}_{\sigma_A}(x) = \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_{A}^{-k}(\sigma_{A}^{l}(x)) \subset X_A.
\]

Hence \( y = (y_n)_{n \in \mathbb{N}} \in X_A \) belongs to \( \text{orb}_{\sigma_A}(x) \) if and only if there exist \( k, l \in \mathbb{Z}_+ \) and an admissible word \( (\mu_1, \ldots, \mu_k) \in B_k(X_A) \) such that

\[
y = (\mu_1, \ldots, \mu_k, x_{l+1}, x_{l+2}, \ldots).
\]

We denote by \( \text{Homeo}(X_A) \) the group of all homeomorphisms on \( X_A \). We have defined in [9] the continuous full group \( \Gamma_A \) for \( (X_A, \sigma_A) \) as in the following way.

**Definition 2.1 ([9])**. Let \( \tau \) be a homeomorphism on \( X_A \) such that \( \tau(x) \in \text{orb}_{\sigma_A}(x) \) for all \( x \in X_A \), so that there exist functions \( k_{\tau}, l_{\tau} : X_A \to \mathbb{Z}_+ \) satisfying

\[
\sigma_{A}^{k_{\tau}(x)}(\tau(x)) = \sigma_{A}^{l_{\tau}(x)}(x) \quad \text{for all } x \in X_A.
\]

Let \( \Gamma_A \) be the set of all homeomorphisms \( \tau \) such that there exist continuous functions \( k_{\tau}, l_{\tau} : X_A \to \mathbb{Z}_+ \) satisfying (2.2). The set \( \Gamma_A \) is a subgroup of \( \text{Homeo}(X_A) \) and is called the **continuous full group** of \( (X_A, \sigma_A) \).

The functions \( k_{\tau}, l_{\tau} \) above are called the orbit cocycles for \( \tau \). They are not necessarily uniquely determined by \( \tau \).

The étale groupoid \( G_A \) for the topological Markov shift \( (X_A, \sigma_A) \) is given by

\[
G_A = \{(x, n, y) \in X_A \times \mathbb{Z}_+ \times X_A \mid \text{there exist } k, l \in \mathbb{Z}_+; n = k - l, \sigma_{A}^{k}(x) = \sigma_{A}^{l}(y) \}.
\]

The topology of \( G_A \) is generated by the sets

\[
\{(x, k - l, y) \in G_A \mid x \in V, y \in W, \sigma_{A}^{k}(x) = \sigma_{A}^{l}(y) \}
\]

for open sets \( V, W \subset X_A \) and \( k, l \in \mathbb{Z}_+ \). Two elements \( (x, n, y), (x', n', y') \in G_A \) are composable if and only if \( y = x' \) and the product and the inverse are given by

\[
(x, n, y) \cdot (x', n', y') = (x, n + n', y'), \quad (x, n, y)^{-1} = (y, -n, x).
\]

The unit space \( G_A^{(0)} \) is defined by \( \{(x, 0, x) \mid x \in X_A \} \), which is identified with \( X_A \). The range map, source map \( r, s : G_A \to G_A^{(0)} \) are defined by \( r(x, n, y) = x, s(x, n, y) = y \) respectively. A subset \( U \subset G_A \) is called a \( G_A \)-set if \( r|_U, s|_U \) are injective. For an open \( G_A \)-set \( U \), denote by \( \pi_U \) the homeomorphism \( r \circ (s|_U)^{-1} \) from \( s(U) \) to \( r(U) \). The topological full group \([G_A]\) of \( G_A \) is defined by the group of all homeomorphisms \( \pi_U \) for some compact open \( G_A \)-set \( U \) such that \( s(U) = r(U) = G_A^{(0)} \) (see [18]). The topological full group \([G_A]\) of the étale groupoid \( G_A \) for the topological Markov shift \( (X_A, \sigma_A) \) is naturally identified with the continuous full group \( \Gamma_A \) ([18]).
Proof. Since a homeomorphism \( \tau \) on \( X_A \) is said to be a cylinder map if there exist two families
\[
\mu(i) = (\mu_1(i), \mu_2(i), \ldots, \mu_k(i)) \in B_k(X_A), \quad i = 1, \ldots, m,
\]
\[
\nu(i) = (\nu_1(i), \nu_2(i), \ldots, \nu_l(i)) \in B_l(X_A), \quad i = 1, \ldots, m
\]
of words such that
\[
U_{\nu(i)} \cap U_{\nu(j)} = U_{\mu(i)} \cap U_{\mu(j)} = \emptyset, \quad \text{for } i \neq j, \quad (2.3)
\]
\[
\bigcup_{i=1}^{m} U_{\nu(i)} = \bigcup_{i=1}^{m} U_{\mu(i)} = X_A, \quad (2.4)
\]
\[
\Gamma_{\ast}^{+}(\nu(i)) = \Gamma_{\ast}^{+}(\mu(i)) \quad \text{for } i = 1, \ldots, m, \quad (2.5)
\]
and
\[
\tau((\nu_1(i), \nu_2(i), \ldots, \nu_l(i), x_{l+1}, x_{l+2}, \ldots)) = (\mu_1(i), \mu_2(i), \ldots, \mu_k(i), x_{l+1}, x_{l+2}, \ldots) \quad (2.6)
\]
for \( (x_{l+1}, x_{l+2}, \ldots) \in \Gamma_{\ast}^{+}((\nu(i)) \) and \( i = 1, \ldots, m \). Since the words \( \nu(i), i = 1, \ldots, m \) satisfy \( U_{\nu(i)} \cap U_{\nu(j)} = \emptyset \), we may reorder them such as \( \nu(1) < \nu(2) < \cdots < \nu(m) \). We see that a homeomorphism \( \tau \) of \( X_A \) belong to \( \Gamma_A \) if and only if \( \tau \) is a cylinder map (\[12\]).

Lemma 2.2. For \( \tau \in \Gamma_A \), there exist \( u_{\tau} \in N(O_A, D_A) \) and \( \mu(i), \nu(i) \in B_{\ast}(X_A), i = 1, \ldots, m \) such that

1. \( u_{\tau} = \sum_{i=1}^{m} S_{\mu(i)} S_{\nu(i)}^* \) and
   
   (a) \( S_{\nu(i)} S_{\nu(i)}^* = S_{\mu(i)}^* S_{\mu(i)}, \quad i = 1, \ldots, m \),
   (b) \( \sum_{i=1}^{m} S_{\nu(i)} S_{\nu(i)}^* = \sum_{i=1}^{m} S_{\mu(i)} S_{\mu(i)}^* = 1 \).

2. \( f \circ \tau^{-1} = u_{\tau} f u_{\tau}^* \) for \( f \in D_A \).

Proof. Since \( \tau \) is a cylinder map, there exist two families of words \( \mu(1), \ldots, \mu(m) \) and \( \nu(1), \ldots, \nu(m) \) satisfying (2.3), (2.4), (2.5) and (2.6). Hence we have
\[
\sum_{i=1}^{m} S_{\nu(i)} S_{\nu(i)}^* = \sum_{i=1}^{m} S_{\mu(i)} S_{\mu(i)}^* = 1, \quad S_{\nu(i)} S_{\nu(i)}^* = S_{\mu(i)}^* S_{\mu(i)}, \quad i = 1, \ldots, m.
\]
By putting \( u_{\tau} = \sum_{i=1}^{m} S_{\mu(i)} S_{\nu(i)}^* \) we see that \( u_{\tau} \) belongs to \( N(O_A, D_A) \) and satisfies \( \chi_{U_\eta} \circ \tau^{-1} = u_{\tau} \chi_{U_\eta} u_{\tau}^* \) for all \( \eta \in B_{\ast}(X_A) \) where \( \chi_{U_\eta} = S_{\eta} S_{\eta}^* \) so that \( f \circ \tau^{-1} = u_{\tau} f u_{\tau}^* \) for all \( f \in D_A \).

As in [9, Theorem 1.2], [10, Proposition 5.6], there exists a short exact sequence
\[
1 \rightarrow U(D_A) \rightarrow N(O_A, D_A) \rightarrow \Gamma_A \rightarrow 1
\]
that splits.

It has been proved by the second named author [13] that the homology group \( H_0(G_A) \) of the groupoid \( G_A \) is isomorphic to the \( K_0 \)-group \( K_0(O_A) = \mathbb{Z}^N/(I - A') \mathbb{Z}^N \) of the \( C^* \)-algebra \( O_A \). He has proved that the group \( \Gamma_A \) is simple if and only if \( H_0(G_A) \) is 2-divisible. He has also proved that \( \Gamma_A \) is finitely presented and its commutator subgroup \( D(\Gamma_A) \) is always simple. As the group \( \Gamma_A \) is non-amenable (\[11\], \[13\]), we have
Theorem 2.3 \cite{[19]}: The group $\Gamma_A$ is a countably infinite, non-amenable, finitely presented discrete group. It is simple if and only if the group $\mathbb{Z}^N/(I-A^i)\mathbb{Z}^N$ is 2-divisible.

It has been shown that for two irreducible square matrices $A$ and $B$, the groups $\Gamma_A$ and $\Gamma_B$ are isomorphic if and only if the $C^*$-algebras $O_A$ and $O_B$ are isomorphic and $\det(1-A) = \det(1-B)$ \cite{[19]}. Hence there are many mutually non-isomorphic continuous full groups $\Gamma_A$.

3 Realization of $O_A$ on $L^2([0,1])$

The Higman–Thompson group $V_N$, $1 < N \in \mathbb{N}$ is represented as the group of right continuous piecewise linear bijective functions $f : [0,1) \rightarrow [0,1)$ having finitely many singularities such that all singularities of $f$ are in $\mathbb{Z}[\frac{1}{N}]$, the derivative of $f$ at any non-singular point is $N^k$ for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[\frac{1}{N}] \cap [0,1)) = \mathbb{Z}[\frac{1}{N}] \cap [0,1)$. In order to represent our group $\Gamma_A$ as a group of piecewise linear functions on $[0,1)$, we will represent the algebra $O_A$ on the Hilbert space $H$ of the square integrable functions $L^2([0,1])$ on $[0,1]$ with respect to the Lebesgue measure in the following way. We note that the essentially bounded measurable functions $L^\infty([0,1])$ act on $H$ by left multiplication.

Since $A$ is irreducible and satisfies condition (I), its Perron eigenvalue $\beta$ is greater than one. Let $\varphi$ be the unique KMS state on $O_A$ for the gauge action $\rho$ of $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ (see \cite{[6]}). It satisfies

$$\varphi(S_\mu a S_\mu^*) = \frac{1}{\beta|\mu|} \varphi(a S_\mu S_\mu^*), \quad a \in D_A, \ \mu \in B_+(X_A).$$

In particular,

$$\varphi(S_i S_i^*) = \frac{1}{\beta} \varphi(S_i^* S_i), \quad i = 1, \ldots, N.$$

Since $\varphi(S_i^* S_i) = \sum_{j=1}^N A(i,j) \varphi(S_j S_j^*)$, the equality

$$\sum_{j=1}^N A(i,j) \varphi(S_j S_j^*) = \beta \varphi(S_i S_i^*), \quad i = 1, \ldots, N \quad (3.1)$$

holds. Put $p_j = \varphi(S_j S_j^*)$, $j = 1, \ldots, N$. The equality (3.1) implies that the vector

$$\begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix}$$

is a unique normalized positive eigenvector for the Perron eigenvalue $\beta$ of the matrix $A$. For $i, j = 1, 2, \ldots, N$, put $p_{ij} = \varphi(S_i S_j S_j^* S_i^*)$ so that

$$p_{ij} = \frac{1}{\beta^2} \varphi(S_j S_j^* S_i S_i^*) = \frac{1}{\beta^2} A(i,j) \varphi(S_j S_j^*) = \frac{1}{\beta^2} A(i,j) p_j.$$

We set for $i, j = 1, 2, \ldots, N$,

$$p(0) = 0, \quad p(i) = \sum_{k=1}^i p_k, \quad q(0,0) = q(i,0) = 0, \quad q(i,j) = \sum_{k=1}^j p_{ik}$$
and define the intervals $I_i$, $I_{ij}$ in $[0,1)$ by
\begin{align}
I_i &= [p(i-1), p(i)), \quad (3.2) \\
I_{ij} &= [p(i-1) + q(i, j-1), p(i-1) + q(i, j)). \quad (3.3)
\end{align}

The latter interval $I_{ij}$ is empty if $A(i,j) = 0$. We set
\begin{align}
l(I_i) &= p(i-1), \quad r(I_i) = p(i), \quad (3.2) \\
l(I_{ij}) &= p(i-1) + q(i, j-1), \quad r(I_{ij}) = p(i-1) + q(i, j) \quad (3.3)
\end{align}
so that $I_i = [l(I_i), r(I_i)), \quad I_{ij} = [l(I_{ij}), r(I_{ij})]$.}

**Lemma 3.1.** Keep the above notations.

(i) $[0,1) = \bigsqcup_{i=1}^N I_i$ : disjoint union.

(ii) $I_i = \bigsqcup_{j=1}^N I_{ij}$ : disjoint union.

**Proof.** (i) is clear. (ii) Let $N_i = \text{Max}\{j = 1, \ldots, N \mid A(i,j) = 1\}.$ As we have
\[
q(i, N_i) = \sum_{k=1}^{N_i} p_{ik} = \frac{1}{\beta} \sum_{k=1}^{N_i} A(i,k)p_k = p_i,
\]
the equality $p(i-1) + q(i, N_i) = p(i)$ holds so that $r(I_{i,N_i}) = r(I_i)$. As the intervals $I_{ij}, I_{ij'}$ are disjoint for $j \neq j'$, one easily sees that $I_i = \bigsqcup_{j=1}^{N_i} I_{ij} = \bigsqcup_{j=1}^{N_i} I_{ij}$. \(\square\)

We define right continuous functions $f_A, g_1, \ldots, g_N$ in the following way. The function $f_A : [0,1) \rightarrow [0,1)$ is defined by
\[
f_A(x) = \beta(x - l(I_{ij})) + l(I_j) \quad \text{for} \ x \in I_{ij}
\]
so that $f_A$ is linear on $I_{ij}$ with slope $\beta$ and $f_A(I_{ij}) = I_j$. We set
\[
J_i = \bigcup_{j=1}^{N_i} I_{ij}.
\]
The function $g_i : J_i \rightarrow I_i$ for each $i = 1, \ldots, N$ is defined by
\[
g_i(x) = \frac{1}{\beta} (x - l(I_j)) + l(I_{ij}) \quad \text{for} \ x \in I_j \text{ with } A(i,j) = 1
\]
so that $g_i$ is linear on $I_j$ for $A(i,j) = 1$ with slope $\frac{1}{\beta}$ and $g_i(I_j) = I_{ij}, g_i(J_i) = I_i$. The following lemma is direct.

**Lemma 3.2.** For $i = 1, \ldots, N$, we have

(i) $f_A(g_i(x)) = x$ for $x \in J_i$.

(ii) $g_i(f_A(x)) = x$ for $x \in I_i$. 

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For a measurable subset $E$ of $[0, 1)$, denote by $\chi_E$ the multiplication operator on $H$ of the characteristic function of $E$. Define the bounded linear operators $T_{fA}, T_{g_i}, i = 1, \ldots, N$ on $H$ by

$$(T_{fA}\xi)(x) = \xi(f_A(x)), \quad (T_{g_i}\xi)(x) = \chi_{J_i}(x)\xi(g_i(x)) \quad \text{for } \xi \in H, x \in [0, 1).$$

The following lemma is straightforward:

**Lemma 3.3.** Keep the above notations. We have

(i) $T_{fA}^* = \frac{1}{\beta} \sum_{i=1}^N T_{g_i}$.

(ii) $T_{fA}^* T_{fA} = \frac{1}{\beta} \sum_{i=1}^N \chi_{J_i}$.

(iii) $T_{g_i}^* T_{g_i} = \beta \chi_{I_i}$ for $i = 1, \ldots, N$ and hence $\sum_{i=1}^N T_{g_i}^* T_{g_i} = \beta 1$.

(iv) $T_{g_i}^* T_{g_i} = \beta \chi_{J_i}$ for $i = 1, \ldots, N$.

We define the operators $s_i, i = 1, \ldots, N$ on $H$ by setting

$$s_i = \frac{1}{\sqrt{\beta}} T_{g_i}^*, \quad i = 1, \ldots, N.$$

By the above lemma, we have

**Proposition 3.4.** The operators $s_i, i = 1, \ldots, N$ are partial isometries such that

$$s_i^* s_i = \chi_{I_i}, \quad s_i^* s_i = \chi_{J_i}, \quad i = 1, \ldots, N.$$

Hence they satisfy the relations

$$\sum_{j=1}^N s_j^* s_j = 1, \quad s_i^* s_i = \sum_{j=1}^N A(i, j) s_j^* s_j, \quad i = 1, \ldots, N.$$

Therefore the correspondence $s_i \rightarrow S_i, i = 1, \ldots, N$ gives rise to an isomorphism from the $C^*$-algebra $C^*(s_1, \ldots, s_N)$ on $H$ onto the Cuntz–Krieger algebra $\mathcal{O}_A$.

## 4 $\beta$-adic PL functions

By Proposition 3.4, we may represent $\mathcal{O}_A$ on $H$ by identifying $S_i$ with $s_i$ for $i = 1, \ldots, N$. In this section, we will define piecewise linear (PL) functions on $[0, 1)$ associated to the topological Markov shift $(X_A, \sigma_A)$. For $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_A)$, define

$$l(\mu) = \sum_{\nu \in B_n(X_A)} \varphi(S_\nu S_\nu^*), \quad r(\mu) = l(\mu) + \varphi(S_\mu S_\mu^*).$$

Put the interval

$$I_\mu = [l(\mu), r(\mu)).$$

The following lemma is clear.
Lemma 4.1. For each $n \in \mathbb{N}$ we have

(i) $I_{\mu} \cap I_{\nu} = \emptyset$ for $\mu, \nu \in B_n(X_A)$ with $\mu \neq \nu$.

(ii) $\cup_{\mu \in B_n(X_A)} I_{\mu} = [0,1)$.

For $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_A)$, we note that the following equalities hold

$$
\varphi(S_{\mu}S_{\mu}^*) = \frac{1}{\beta^n}\varphi(S_{\mu}^*S_{\mu}) = \frac{1}{\beta^n}\varphi(S_{\mu_n}^*S_{\mu_n}) = \frac{1}{\beta^n}\sum_{j=1}^{N} A(\mu_n, j)p_j. \quad (4.1)
$$

For $i, j = 1, \ldots, N$ with $A(i,j) = 1$, we apply (4.1) for $\mu = i, (i,j)$ so that

$$
l(i) = \sum_{j<i} \varphi(S_jS_i^*) = \sum_{j=1}^{i-1} p_j = p(i-1),
$$

$$
r(i) = l(i) + \varphi(S_iS_i^*) = p(i-1) + p_i = p(i)
$$

and

$$
l(i,j) = \sum_{(\mu_1, \mu_2) \prec (i,j)} \varphi(S_{\mu_1}S_{\mu_2}S_{\mu_2}^*S_{\mu_1}^*) = \sum_{(\mu_1, \mu_2) \prec (i,j)} p_{\mu_1\mu_2}
$$

$$
= \sum_{\mu_1 = 1}^{i-1} \sum_{\mu_2 = 1}^{j-1} p_{\mu_1\mu_2} + \sum_{\mu_2 = 1}^{j-1} p_{i\mu_2}
$$

$$
= \sum_{\mu_1 = 1}^{i-1} \sum_{\mu_2 = 1}^{j-1} A(\mu_1, \mu_2)\frac{1}{\beta}\sum_{j=1}^{N} p_{\mu_2} + q(i, j-1) = p(i-1) + q(i, j-1),
$$

$$
r(i,j) = l(i,j) + \varphi(S_iS_jS_j^*S_i^*) = p(i-1) + q(i, j-1) + p_{ij} = p(i-1) + q(i, j).
$$

Hence we see that

$$
[l(i), r(i)] = [p(i-1), p(i)) = I_i : \text{the interval defined in (3.2)},
$$

$$
[l(i,j), r(i,j)] = [p(i-1) + q(i, j-1), p(i-1) + q(i, j)) = I_{ij} : \text{the interval defined in (3.3)}.
$$

Lemma 4.2. For $\mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A)$, we have

$$
f_A(I_{\mu}) = I_{\mu_2 \cdots \mu_m} \quad \text{and hence} \quad f_A^{m-1}(I_{\mu}) = I_{\mu_m} (= [l(\mu_m), r(\mu_m))).
$$

Proof. The algebra $O_A$ is represented on $H$ by identifying $S_i$ with $s_i$ for $i = 1, \ldots, N$. We then see

$$
S_{\mu}S_{\mu}^* = \chi_{I_{\mu}} \quad \text{and} \quad \lambda_A(S_{\mu}S_{\mu}^*) = \chi_{f_A(I_{\mu})}.
$$

Since $S_{\mu_1}S_{\mu_1} \geq S_{\mu_2}S_{\mu_2}^*$, we have

$$
\lambda_A(S_{\mu_1}S_{\mu_1}^*) = S_{\mu_2}^*S_{\mu_2} \cdots S_{\mu_m}^*S_{\mu_m} \cdots S_{\mu_2}^*S_{\mu_2}^*S_{\mu_1}S_{\mu_1},
$$

$$
= S_{\mu_2} \cdots S_{\mu_m}S_{\mu_m}^* \cdots S_{\mu_2}^*S_{\mu_2},
$$

so that $\chi_{I_{\mu_2 \cdots \mu_m}} = \chi_{f_A(I_{\mu})}$. \hfill \square
Lemma 4.3. For $\mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A)$, $\nu = (\nu_1, \ldots, \nu_n) \in B_n(X_A)$, the condition $S_{\mu}^*S_{\nu} = S_{\nu}^*S_{\mu}$ implies
\[
\frac{r(\mu) - l(\mu)}{r(\nu) - l(\nu)} = \beta^{n-m}. \tag{4.2}
\]

Proof. Since $r(\mu) - l(\mu) = \varphi(S_{\mu}^*S_{\mu}) = \frac{1}{\beta^m}\varphi(S_{\mu}^*S_{\mu})$ and similarly $r(\nu) - l(\nu) = \frac{1}{\beta^n}\varphi(S_{\nu}^*S_{\nu})$, the condition $S_{\mu}^*S_{\mu} = S_{\nu}^*S_{\nu}$ implies (4.2). \qed

Lemma 4.4. For $\mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A)$, $\nu = (\nu_1, \ldots, \nu_n) \in B_n(X_A)$, the following five conditions are equivalent:

1. $\Gamma_+^+(\mu) = \Gamma_+^+(\nu)$.
2. $S_{\mu}^*S_{\nu} = S_{\nu}^*S_{\mu}$.
3. $S_{\mu}^*S_{\mu} = S_{\nu}^*S_{\nu}$.
4. $f^m_A(I\mu) = f^n_A(I\nu)$.
5. $f_A(I\mu) = f_A(I\nu)$.

Proof. For $\mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A)$, the following identities
\[
Xf^m_A(I\mu) = Xf_A(I\mu) = \lambda_A(S_{\mu_1}^*S_{\mu_2}^*) = S_{\mu_1}^*S_{\mu_2} = S_{\mu_2}^*S_{\mu_1}
\]
hold. They imply the desired assertion. \qed

Definition 4.5. (i) For a word $\nu \in B_n(X_A)$, an interval $[x_1, x_2]$ in $[0, 1)$ is said to be an $A$-adic interval for $\nu$ if $x_1 = l(\nu)$ and $x_2 = r(\nu)$.

(ii) A rectangle $I \times J$ in $[0, 1) \times [0, 1)$ is said to be an $A$-adic rectangle if both the intervals $I, J$ are $A$-adic intervals for words $\nu \in B_n(X_A), \mu \in B_m(X_A)$ respectively such that
\[
I = [l(\nu), r(\nu)), \quad J = [l(\nu), r(\nu)) \quad \text{and} \quad f^m_A(I) = f^n_A(J).
\]

(iii) For two partitions
\[
0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1, \quad 0 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 1
\]
of $[0, 1)$, put
\[
I_p = [x_{p-1}, x_p), \quad J_p = [y_{p-1}, y_p) \quad \text{for} \quad p = 1, 2, \ldots, m.
\]
The partition $I_p \times J_q, p, q = 1, \ldots, m$ of $[0, 1) \times [0, 1)$ is said to be an $A$-adic pattern of rectangles if there exists a permutation $\sigma$ on $\{1, 2, \ldots, m\}$ such that the rectangles $I_p \times J_{\sigma(p)}$ are $A$-adic rectangles for all $p = 1, 2, \ldots, m$.

For an $A$-adic pattern of rectangles above, the slopes of diagonals
\[
s_p = \frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}}, \quad p = 1, 2, \ldots, m
\]
are said to be rectangle slopes.
A piecewise linear function \( f \) on \([0, 1)\) is called an \( A \)-adic \( PL \) function if \( f \) is a right continuous bijection on \([0, 1)\) such that there exists an \( A \)-adic pattern of rectangles \( I_p \times J_p, p = 1, 2, \ldots, m \) where \( I_p = [x_{p-1}, x_p), J_p = [y_{p-1}, y_p), p = 1, \ldots, m \) with a permutation \( \sigma \) on \( \{1, 2, \ldots, m\} \) such that

\[
    f(x_{p-1}) = y_{\sigma(p)-1}, \quad f(x_p) = y_{\sigma(p)+1}, \quad p = 1, 2, \ldots, m
\]

where \( f_-(x_p) = \lim_{h \to 0^+} f(x_p - h) \), and \( f \) is linear on \([x_{p-1}, x_p)\) with slope \( \frac{y_{\sigma(p)} - y_{\sigma(p)+1}}{x_p - x_{p-1}} \) for \( p = 1, 2, \ldots, m \).

**Lemma 4.7.** The composition of two \( A \)-adic \( PL \) functions and the inverse function of an \( A \)-adic \( PL \) function are also \( A \)-adic \( PL \) functions.

By the above lemma, the set of \( A \)-adic \( PL \) functions forms a group under compositions of functions.

**Definition 4.8.** We denote by \( \Gamma_A^{PL} \) the group of \( A \)-adic \( PL \) functions.

The following proposition is immediate by definition of \( A \)-adic \( PL \) functions.

**Proposition 4.9.** An \( A \)-adic \( PL \) function naturally gives rise to an \( A \)-adic pattern of rectangles, whose rectangle slopes are the slope of the \( A \)-adic \( PL \) function. Conversely, an \( A \)-adic pattern of rectangles gives rise to an \( A \)-adic \( PL \) function by taking the diagonal lines of the corresponding rectangles.

## 5 \( A \)-adic Tables

Recall that for two words \( \mu = (\mu_1, \ldots, \mu_m) \in B_m(X_A), \nu = (\nu_1, \ldots, \nu_n) \in B_n(X_A) \) with \( U_\mu \cap U_\nu = \emptyset \), we write \( \mu \prec \nu \) if \( \mu_1 = \nu_1, \ldots, \mu_k = \nu_k \) and \( \mu_{k+1} < \nu_{k+1} \) for some \( k \).

Nekrashevych in [19] has introduced a notion of table to represent the Higman–Thompson group \( V_N \). We will generalize the Nekrashevych’s notion to a notion of \( A \)-adic table in order to represent elements of \( \Gamma_A \).

**Definition 5.1.** An \( A \)-adic table is a matrix \( T \)

\[
    T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}
\]

for \( \mu(i), \nu(i) \in B_*(X_A), i = 1, \ldots, m \) such that

1. \( \Gamma_+^\mu(\nu(i)) = \Gamma_+^\nu(\mu(i)), i = 1, \ldots, m \),

2. \( X_A = \bigsqcup_{i=1}^m U_\nu(i) = \bigsqcup_{i=1}^m U_\mu(i) \) disjoint unions.

Since the words \( \nu(i), i = 1, \ldots, m \) satisfy \( U_{\nu(i)} \cap U_{\nu(j)} = \emptyset \) for \( i \neq j \), we may reorder them such as \( \nu(1) < \nu(2) < \cdots < \nu(m) \). As the above two conditions (a), (b) are equivalent to the conditions (a), (b) in Lemma 2.2 (1) respectively, we have
Lemma 5.2. For an element $\tau \in \Gamma_A$, let words $\mu(i), \nu(i), i = 1, \ldots, m$ and the unitary $u_\tau = \sum_{i=1}^{m} S_{\mu(i)} S^*_{\nu(i)}$ satisfy the conditions (1) and (2) in Lemma 2.2. Then the matrix

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

is an $A$-adic table.

The $A$-adic table $T$ above is called a presentation of $\tau$. It is also called that $T$ presents $\tau$.

For an $A$-adic table $T = [\mu(1) \mu(2) \cdots \mu(m)]$ and $i = 1, 2, \ldots, m$, let $\eta(i, j) \in B_*(X_A), j = 1, \ldots, n_i$ be a family of (possibly empty) words satisfying the following three conditions:

(i) $\eta(i, 1) < \eta(i, 2) < \cdots < \eta(i, n_i)$,

(ii) $\eta(i, j) \in \Gamma^+_*(\nu(i))$ for $j = 1, \ldots, n_i$,

(iii) $U_{\nu(i)} = \bigcup_{j=1}^{n_i} U_{\nu(i)} \eta(i, j)$.

Since $\Gamma^+_*(\nu(i)) = \Gamma^+_*(\mu(i))$, one has $\eta(i, j) \in \Gamma^+_*(\mu(i))$ and $U_{\mu(i)} = \bigcup_{j=1}^{n_i} U_{\mu(i)} \eta(i, j)$. Put

$$\nu(i, j) = \nu(i) \eta(i, j), \quad \mu(i, j) = \mu(i) \eta(i, j), \quad j = 1, \ldots, n_i, \quad i = 1, \ldots, m. \quad (5.1)$$

Then the matrix

$$\begin{bmatrix} \mu(1, 1) & \cdots & \mu(1, n_1) & \mu(2, 1) & \cdots & \mu(2, n_2) & \cdots & \mu(m, 1) & \cdots & \mu(m, n_m) \\ \nu(1, 1) & \cdots & \nu(1, n_1) & \nu(2, 1) & \cdots & \nu(2, n_2) & \cdots & \nu(m, 1) & \cdots & \nu(m, n_m) \end{bmatrix}$$

is an $A$-adic table, which is called an expansion of $T$. Let us denote by $\approx$ the equivalence relation in the $A$-adic tables generated by the expansions. This means that two $A$-adic tables

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}, \quad T' = \begin{bmatrix} \mu'(1) & \mu'(2) & \cdots & \mu'(m) \\ \nu'(1) & \nu'(2) & \cdots & \nu'(m) \end{bmatrix},$$

are equivalent and written $T \approx T'$ if there exists a finite sequence $T_1, T_2, \ldots, T_k$ of $A$-adic tables such that $T = T_1, T' = T_k$ and $T_i$ is an expansion of $T_{i+1}$, or $T_{i+1}$ is an expansion of $T_i$.

Lemma 5.3. For $\tau, \tau' \in \Gamma_A$, let $T, T'$ be $A$-adic tables presenting $\tau, \tau'$ respectively. Then $\tau = \tau'$ if and only if $T \approx T'$.

Proof. Let $T$ and $T'$ be the matrices

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}, \quad T' = \begin{bmatrix} \mu'(1) & \mu'(2) & \cdots & \mu'(m) \\ \nu'(1) & \nu'(2) & \cdots & \nu'(m) \end{bmatrix}.$$ 

Suppose that $T'$ is an expansion of $T$. We write $T'$ as

$$\begin{bmatrix} \mu(1, 1) & \cdots & \mu(1, n_1) & \mu(2, 1) & \cdots & \mu(2, n_2) & \cdots & \mu(m, 1) & \cdots & \mu(m, n_m) \\ \nu(1, 1) & \cdots & \nu(1, n_1) & \nu(2, 1) & \cdots & \nu(2, n_2) & \cdots & \nu(m, 1) & \cdots & \nu(m, n_m) \end{bmatrix}.$$
where \(\mu(i,j)\) and \(\nu(i,j)\) are words for \(\eta(i,j)\) as in (5.1). The homeomorphisms \(\tau\) and \(\tau'\) are induced by the unitaries \(u_T\) and \(u_{T'}\) defined by

\[
u_T = \sum_{i=1}^{m} S_{\mu(i)} S_{\nu(i)}^* \quad \text{and} \quad u_{T'} = \sum_{i=1}^{m} S_{\mu'(i)} S_{\nu'(i)}^*
\]
such as \(f \circ \tau^{-1} = \text{Ad}(u_T)(f)\) and \(f \circ \tau'^{-1} = \text{Ad}(u_{T'})(f)\) for \(f \in C(X_A) = \mathcal{D}_A\). As

\[
S_{\mu(i)} S_{\nu(i)}^* = \sum_{j=1}^{n} S_{\mu(i)} S_{\eta(i,j)} S_{\eta(i,j)}^* S_{\nu(i)}^* = \sum_{j=1}^{n} S_{\mu(i,j)} S_{\nu(i,j)}^*
\]
we have

\[
u_T = \sum_{i=1}^{m} S_{\mu(i)} S_{\nu(i)}^* = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{\mu(i)} S_{\eta(i,j)} S_{\eta(i,j)}^* S_{\nu(i)}^* = \sum_{j=1}^{n} S_{\mu(i,j)} S_{\nu(i,j)}^* = u_{T'}
\]
so that \(\tau = \tau'\).

Conversely, suppose that \(\tau = \tau'\). Let

\[
K' = \text{Max}\{|\nu'(i)| \mid 1 \leq i \leq m'\}, \quad L' = \text{Max}\{|\mu'(i)| \mid 1 \leq i \leq m'\}.
\]
There exist admissible words \(\eta(i,j) \in B_s(X_A), j = 1, \ldots, n_i, i = 1, \ldots, m\) such that

(a) \(\eta(i,1) \prec \eta(i,2) \prec \cdots \prec \eta(i,n_i)\),

(b) \(\eta(i,j) \in \Gamma^*_+ (\nu(i))\),

(c) \(|\nu(i)\eta(i,j)| \geq K', |\mu(i)\eta(i,j)| \geq L'\),

(d) \(U_{\nu(i)} = \bigsqcup_{j=1}^{n_i} U_{\nu(i)\eta(i,j)}, U_{\mu(i)} = \bigsqcup_{j=1}^{n_i} U_{\mu(i)\eta(i,j)}\).

Put

\[
\nu(i,j) = \nu(i)\eta(i,j), \quad \mu(i,j) = \mu(i)\eta(i,j), \quad j = 1, \ldots, n_i, i = 1, \ldots, m
\]
and

\[
T^n = \begin{bmatrix}
\mu(1,1) & \cdots & \mu(1,n_1) & \mu(2,1) & \cdots & \mu(2,n_2) & \cdots & \mu(m,1) & \cdots & \mu(m,n_m)
\end{bmatrix}
\begin{bmatrix}
\nu(1,1) & \cdots & \nu(1,n_1) & \nu(2,1) & \cdots & \nu(2,n_2) & \cdots & \nu(m,1) & \cdots & \nu(m,n_m)
\end{bmatrix}.
\]
Hence \(T^n\) is an expansion of \(T\). We will compare \(T^n\) and \(T'\). Put

\[
F_k = \{(i,j) \mid \nu(i,j) \prec \nu'(k)\}, \quad k = 1, \ldots, m'.
\]
Since \(|\nu(i,j)| \geq K', |\mu(i,j)| \geq L'\), one has

\[
\nu'(k) = \bigsqcup_{(i,j) \in F_k} \nu(i,j).
\]
Since \(|\nu(i,j)| \geq |\nu'(k)|\), there exist \(\eta'(k,i,j) \in B_s(X_A)\) such that

\[
\nu(i,j) = \nu'(k)\eta'(k,i,j) \quad \text{for} \quad (i,j) \in F_k.
\]
As \(\tau = \tau'\), we have

\[
\tau(\chi U_{\nu(i,j)}) = \chi U_{\mu(i)\eta(i,j)} = \chi U_{\nu(i,j)} = \tau'(\chi U_{\nu(i,j)}) = \chi U_{\mu'(i)\eta'(k,i,j)}
\]
so that

\[
\mu(i,j) = \mu'(k)\eta'(k,i,j) \quad \text{for} \quad (i,j) \in F_k.
\]
This implies that \(T^n\) is an expansion of \(T'\) to prove that \(T\) is equivalent to \(T'\).
We denote by \([T]\) the equivalence class of an \(A\)-adic table \(T\). For \(\tau \in \Gamma_A\), denote by \(T_\tau\) an \(A\)-adic table presenting \(\tau\). The preceding lemma says that its equivalence class \([T_\tau]\) does not depend on the choice of \(T_\tau\) presenting \(\tau\).

An \(A\)-adic table \([\mu(1) \mu(2) \ldots \mu(m)]\) presenting \(\tau \in \Gamma_A\) is said to be reduced if it has a minimal length \(m\) in the set of \(A\)-adic tables presenting \(\tau\). Recall that for a word \(\mu = (\mu_1, \ldots, \mu_m) \in B_*(X_A)\), we write \(\Gamma_1^+(\mu) = \{j \in \{1, \ldots, N\} \mid A(\mu_m, j) = 1\}\). The following lemma is obvious.

**Lemma 5.4.** For an \(A\)-adic table \(T = [\mu(1) \mu(2) \ldots \mu(m)]\) and \(i = 1, \ldots, m\), let \(\Gamma_1^+(\mu(i)) = \{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\}\) such that \(\alpha_{i1} < \alpha_{i2} < \ldots < \alpha_{in}\). Put the words

\[
\begin{align*}
\mu(i, 1) &= \mu(i)\alpha_{i1}, \\
\mu(i, 2) &= \mu(i)\alpha_{i2}, \\
&\quad \ldots, \\
\mu(i, n) &= \mu(i)\alpha_{in}, \\
\nu(i, 1) &= \nu(i)\alpha_{i1}, \\
\nu(i, 2) &= \nu(i)\alpha_{i2}, \\
&\quad \ldots, \\
\nu(i, n) &= \nu(i)\alpha_{in}.
\end{align*}
\]

Then the \(A\)-adic table \(T_i'\) defined by

\[
T_i' = \left[\begin{array}{cccccc}
\mu(1) & \ldots & \mu(i-1) & \mu(i, 1) & \mu(i, n) & \mu(i+1) & \ldots & \mu(m) \\
\nu(1) & \ldots & \nu(i-1) & \nu(i, 1) & \nu(i, n) & \nu(i+1) & \ldots & \nu(m)
\end{array}\right]
\]

is equivalent to \(T\).

For an \(A\)-adic table \(T = [\mu(1) \mu(2) \ldots \mu(m)]\), define the range depth \(R(T)\) and the domain depth \(D(T)\) by

\[
R(T) = \max\{|\mu(i)| \mid 1 \leq i \leq m\}, \quad D(T) = \max\{|\nu(i)| \mid 1 \leq i \leq m\}.
\]

By using the above lemma recursively, we know the following lemma.

**Lemma 5.5.** Let \(T = [\mu(1) \mu(2) \ldots \mu(m)]\) be an \(A\)-adic table.

(i) For an integer \(p \geq D(T)\), there exists an \(A\)-adic table \(T' = [\mu'(1) \mu'(2) \ldots \mu'(m')]\) such that \(T' \approx T\) and \(\{\nu'(i) \mid i = 1, \ldots, m'\} = B_p(X_A)\).

(ii) For an integer \(q \geq R(T)\), there exists an \(A\)-adic table \(T'' = [\mu''(1) \mu''(2) \ldots \mu''(m'')]\) such that \(T'' \approx T\) and \(\{\mu''(i) \mid i = 1, \ldots, m''\} = B_q(X_A)\).

Let \(T_1, T_2\) be two \(A\)-adic tables. Take \(M\) such that \(M \geq D(T_1), R(T_2)\). By the preceding lemma, there exist \(A\)-adic tables

\[
T_1' = \left[\begin{array}{cccc}
\mu'_1(1) & \mu'_1(2) & \ldots & \mu'_1(p) \\
\nu'_1(1) & \nu'_1(2) & \ldots & \nu'_1(p)
\end{array}\right], \quad T_2' = \left[\begin{array}{cccc}
\mu'_2(1) & \mu'_2(2) & \ldots & \mu'_2(q) \\
\nu'_2(1) & \nu'_2(2) & \ldots & \nu'_2(p)
\end{array}\right]
\]

such that \(T_1' \approx T_1\) and \(T_2' \approx T_2\) and

\[
|\nu'_1(1)| = \cdots = |\nu'_1(p)| = |\mu'_2(1)| = \cdots = |\mu'_2(q)| = M.
\]

Hence we have \(p = q = |B_M(X_A)|\). One may take

\[
\nu'_1(1) < \nu'_1(2) < \cdots < \nu'_1(p), \quad \mu'_2(1) < \mu'_2(2) < \cdots < \mu'_2(q)
\]
so that
\[ \nu'_1(1) = \mu'_2(1), \quad \nu'_1(2) = \mu'_2(2), \quad \ldots, \quad \nu'_1(p) = \mu'_2(q). \]

Define the product \( T'_1 \circ T'_2 \) by the \( A \)-adic table
\[
T'_1 \circ T'_2 = \begin{bmatrix}
\mu'_1(1) & \mu'_1(2) & \cdots & \mu'_1(p) \\
\nu'_2(1) & \nu'_2(2) & \cdots & \nu'_2(p)
\end{bmatrix}.
\]

It is easy to see that \( T'_1 \circ T'_2 \) is an \( A \)-adic table. It is straightforward to see that the equivalence class \([ T'_1 \circ T'_2 ]\) does not depend on the choice of representatives \( T'_1 \) of \([ T'_1 ]\) and \( T'_2 \) of \([ T'_2 ]\). Hence one may define the product \([ T_1 \circ T_2 ]\) by the equivalence class \([ T'_1 \circ T'_2 ]\) of the product \( T'_1 \circ T'_2 \).

For an \( A \)-adic table \( T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix} \), define an \( A \)-adic table
\[
T^{-1} = \begin{bmatrix} \nu(1) & \nu(2) & \cdots & \nu(m) \\ \mu(1) & \mu(2) & \cdots & \mu(m) \end{bmatrix}.
\]

The identity table denoted by \( I \) is defined by
\[
I = \begin{bmatrix} 1 & 2 & \cdots & N \\ 1 & 2 & \cdots & N \end{bmatrix}
\]
where the two rows of \( I \) denote the list of the ordered symbols \( \{1, 2, \ldots, N\} = B_1(X_A) \).

**Lemma 5.6.** Keep the above notations.

(i) The equivalence class \([ I ]\) of \( I \) is the unit of the product operations in the equivalence classes of the \( A \)-adic tables.

(ii) If \( T \approx T' \), then \( T^{-1} \approx T'^{-1} \).

Since \( T^{-1} \circ T \approx I \) and \( T \circ T^{-1} \approx I \), the class \([ T^{-1} ]\) of \( T^{-1} \) is the inverse of \([ T ]\) in the equivalence classes of the \( A \)-adic tables.

**Definition 5.7.** Denote by \( \Gamma_{A}^{\text{tab}} \) the group of the equivalence classes of \( A \)-adic tables.

Therefore we have

**Proposition 5.8.** The correspondence \( \tau \in \Gamma_A \longrightarrow [ T_{\tau} ] \in \Gamma_A^{\text{tab}} \) gives rise to an isomorphism of groups.

**Proof.** Let \( \tau, \tau' \in \Gamma_A \). By Lemma 5.3, \( \tau = \tau' \) if and only if \([ T_{\tau} ] = [ T_{\tau'} ]\). It is direct to see that for \( \tau_1, \tau_2 \in \Gamma_A \), the equivalence class \([ T_{\tau_1 \circ \tau_2} ]\) of an \( A \)-adic table \( T_{\tau_1 \circ \tau_2} \) presenting the composition \( \tau_1 \circ \tau_2 \) is the class of the product \([ T_{\tau_1} ] \circ [ T_{\tau_2} ]\). Hence the correspondence \( \tau \in \Gamma_A \longrightarrow [ T_{\tau} ] \in \Gamma_A^{\text{tab}} \) gives rise to an isomorphism of groups. \( \square \)
6 Isomorphisms among $\Gamma_A$, $\Gamma^\text{tab}_A$ and $\Gamma^\text{PL}_A$

In the preceding section, we have shown that the two groups $\Gamma_A$, $\Gamma^\text{tab}_A$ are isomorphic. In this section, we will show that these two groups are isomorphic to the group $\Gamma^\text{PL}_A$ of $A$-adic PL functions.

Lemma 6.1. For an $A$-adic table $T = [\mu(1) \mu(2) \ldots \mu(m)]$, there exist an $A$-adic pattern of rectangles whose rectangle slopes are

$$\beta^{\nu(1)} - |\mu(1)|, \beta^{\nu(2)} - |\mu(2)|, \ldots, \beta^{\nu(m)} - |\mu(m)|,$$

and an $A$-adic PL function $f_T$ having these rectangle slopes such that

$$f_T(I_{\nu(i)}) = I_{\mu(i)}, \quad i = 1, 2, \ldots, m.$$  \hspace{1cm} (6.1)

Conversely, for an $A$-adic PL function $f$ with the $A$-adic pattern of rectangles $I_p \times J_\sigma(p), p = 1, 2, \ldots, m$ and a permutation $\sigma$ on $\{1, \ldots, m\}$, there exists an $A$-adic table $T_f = [\mu(1) \mu(2) \ldots \mu(m)]$ such that

$$I_p = I_{\nu(p)}, \quad J_{\sigma(p)} = I_{\mu(p)}, \quad p = 1, 2, \ldots, m.$$  \hspace{1cm} Proof. We are assuming the ordering such as $\nu(1) < \cdots < \nu(m)$. Since $X_A$ is a disjoint union $X_A = \bigcup_{j=1}^m \{U_\mu(j)\}$, there exists a permutation $\sigma_0$ on $\{1, 2, \ldots, m\}$ such that $\mu(\sigma_0(1)) < \mu(\sigma_0(2)) < \cdots < \mu(\sigma_0(m))$. Put

$$x_i = l(\nu(i + 1)), \quad y_i = l(\mu(\sigma_0(i + 1))), \quad i = 0, 1, \ldots, m - 1$$

so that $x_0 = y_0 = 0$ and

$$I_p = [x_{p-1}, x_p], \quad J_p = [y_{p-1}, y_p], \quad p = 1, 2, \ldots, m$$

where $x_p = y_p = 1$. Define the permutation $\sigma := \sigma_0^{-1}$ on $\{1, 2, \ldots, m\}$. We note that $r(\nu(i)) = l(\nu(i + 1)), r(\mu(\sigma_0(i))) = l(\mu(\sigma_0(i + 1)))$ for $i = 1, \ldots, m - 1$. Then the rectangles $I_p \times J_{\sigma(p)}$, $p = 1, 2, \ldots, m$ are $A$-adic rectangles by Lemma 4.4 such that

$$\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}} = \frac{r(\mu(p)) - l(\mu(p))}{r(\nu(p)) - l(\nu(p))}$$

We then have

$$r(\nu(p)) - l(\nu(p)) = \varphi(S_{\nu(p)} S_{\nu(p)}^*) = \frac{1}{\beta^{\nu(p)}} \varphi(S_{\nu(p)}^* S_{\nu(p)})$$

and similarly $r(\mu(p)) - l(\mu(p)) = \frac{1}{\beta^{\mu(p)}} \varphi(S_{\mu(p)}^* S_{\mu(p)})$. As the condition $\Gamma^+_s(\nu(p)) = \Gamma^+_s(\mu(p))$ implies $S^*_{\nu(p)} S_{\nu(p)} = S^*_{\mu(p)} S_{\mu(p)}$, we have

$$\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}} = \beta^{\nu(p) - |\mu(p)|}, \quad p = 1, 2, \ldots, m.$$  \hspace{1cm} (6.1)

By Proposition 4.9, one immediately knows that the associated $A$-adic PL function denoted by $f_T$ with the above $A$-adic pattern of rectangles satisfies the condition (6.1). The converse implication is straightforward by Lemma 4.4. \hfill \Box
We may directly construct an $A$-adic PL function $f_T$ from an $A$-adic table $T = [\mu(1) \mu(2) \cdots \mu(m)]$ as follows. Put $x_i = l(\nu(i + 1)), \hat{y}_i = l(\mu(i + 1))$ and $f_T(x_i) = \hat{y}_i, i = 0, 1, \ldots, m - 1$ Define $f_T(x)$ on $[x_{i-1}, x_i)$ as a linear function with slope $\beta^{l(\mu(i)) - l(\nu(i))}(= \frac{\hat{y}_i - \hat{y}_{i-1}}{x_i - x_{i-1}})$ for $i = 1, 2, \ldots, m$. It is easy to see that the function $f_T$ is an $A$-adic PL function. Let us denote by $\iota$ the $A$-adic PL function defined by $\iota(x) = x, x \in [0, 1)$. The following lemma is direct.

**Lemma 6.2.** For two $A$-adic tables $T_1, T_2$, we have

1. $T_1$ is equivalent to $T_2$ if and only if $f_{T_1} = f_{T_2}$ as functions. Hence we may write $f_T$ as $f_{[T]}$.

2. $f_{[T_1] \circ [T_2]} = f_{[T_1]} \circ f_{[T_2]}$.

3. $\iota = f_{[1]}$.

We reach the main result of the paper.

**Theorem 6.3.** There exist canonical isomorphisms of discrete groups among the continuous full group $\Gamma_A$, the group $\Gamma_{\text{tab}}^{\text{PL}}$ of the equivalence classes of $A$-adic tables, and the group $\Gamma_{\text{PL}}^A$ of $A$-adic PL functions on $[0, 1)$, that is

$$\Gamma_A \cong \Gamma_{\text{tab}}^{\text{PL}} \cong \Gamma_{\text{PL}}^A.$$ 

In particular, the continuous full group $\Gamma_A$ for a topological Markov shift $(X_A, \sigma)$ is realized as the group of all $A$-adic PL functions on $[0, 1)$.

**Proof.** By Proposition 5.8, we have an isomorphism from the continuous full group $\Gamma_A$ to the group $\Gamma_{\text{tab}}^{\text{PL}}$ of the equivalence classes of $A$-adic tables. By Lemma 6.1 and Lemma 6.2 the correspondence $[T] \in \Gamma_{\text{tab}}^{\text{PL}} \rightarrow f_T \in \Gamma_{\text{PL}}^A$ yields an isomorphism. 

### 7 A realization of $\Gamma_A$ as $A$-adic PL functions

In this section, we will construct a continuous surjection of the shift space $X_A$ onto the interval $[0, 1]$ which yields a presentation of elements of the continuous full group $\Gamma_A$ to the group $\Gamma_{\text{PL}}^A$ of $A$-adic PL functions. For $x = (x_i)_{i \in \mathbb{N}} \in X_A$ and $n \in \mathbb{Z}_+$, consider the word $(x_1, \ldots, x_n) \in B_n(X_A)$. We set

$$l_n(x) = l((x_1, \ldots, x_n)), \quad r_n(x) = r((x_1, \ldots, x_n)).$$

**Lemma 7.1.** For $x = (x_i)_{i \in \mathbb{N}} \in X_A$ and $n \in \mathbb{Z}_+$, we have

1. $l_n(x) \leq l_{n+1}(x) \leq r_{n+1}(x) \leq r_n(x)$.

2. $|r_n(x) - l_n(x)| \leq \frac{1}{n}$. 

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Proof. (i) For $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_A)$, the condition $\mu \prec (x_1, \ldots, x_n)$ implies $\mu_j \prec (x_1, \ldots, x_n, x_{n+1})$ for all $j$ with $A(\mu_j, j) = 1$ so that

$$l_n(x) = \sum_{\mu \in B_n(X_A), \mu \prec (x_1, \ldots, x_n)} \varphi(S_\mu S_\mu^*) = \sum_{j=1}^N A(\mu_n, j) \sum_{\mu \in B_n(X_A), \mu \prec (x_1, \ldots, x_n)} \varphi(S_\mu S_\mu^*)$$

$$\leq \sum_{\nu \in B_{n+1}(X_A), \nu \prec (x_1, \ldots, x_n, x_{n+1})} \varphi(S_\nu S_\nu^*) = l_{n+1}(x).$$

We note that

$$l_{n+1}(x) = l_n(x) + \sum_{j < x_{n+1}} \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*)$$

so that

$$r_{n+1}(x) = l_{n+1}(x) + \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*) = l_n(x) + \sum_{j < x_{n+1}} \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*)$$

$$\leq l_n(x) + \sum_{j=1}^N \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*) = l_n(x) + \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*) = r_n(x).$$

(ii) By the equality $r_n(x) = l_n(x) + \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*)$ with

$$\varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*) = \frac{1}{\beta n} \sum_{j=1}^N A(x_n, j)p_j,$$

we have $|r_n(x) - l_n(x)| \leq \frac{1}{\beta n}$.

\[ \square \]

Lemma 7.2. For $x = (x_i)_{i \in \mathbb{N}} \in X_A$ and $n \in \mathbb{Z}_+$, we have

(i) $l_n(x) = l_{n+1}(x)$ if and only if $x_{n+1} = \min\{j = 1, \ldots, N \mid A(x_n, j) = 1\}$.

(ii) $r_n(x) = r_{n+1}(x)$ if and only if $x_{n+1} = \max\{j = 1, \ldots, N \mid A(x_n, j) = 1\}$.

Proof. (i) By (7.1), one sees that $l_{n+1}(x) = l_n(x)$ if and only if $\sum_{j < x_{n+1}} \varphi(S_{x_1 \cdots x_{n+1}} S_{x_1 \cdots x_{n+1}}^*) = 0$. Since the state $\varphi$ on $D_A$ is faithful, the latter condition is equivalent to the condition that there does not exist $j = 1, \ldots, N$ such that $j < x_{n+1}$ and $A(x_n, j) = 1$. Hence we have the desired assertion.

(ii) is similar to (i).

\[ \square \]

For a word $\omega = (\omega_1, \ldots, \omega_n) \in B_n(X_A)$, let us denote by $\omega_{\min} = (\omega_i)_{i \in \mathbb{N}} \in X_A$ (resp. $\omega_{\max} = (\omega_i)_{i \in \mathbb{N}} \in X_A$) its minimal (resp. maximal) extension to a right infinite sequence in $X_A$, which is defined by setting

$$\omega_i = \omega_i \quad (\text{resp. } \overline{\omega}_i = \omega_i) \quad \text{for } i = 1, \ldots, n,$$

$$\omega_{n+k} = \min\{j = 1, 2, \ldots, N \mid A(\omega_{n+k-1}, j) = 1\} \quad (\text{resp. } \overline{\omega}_{n+k} = \max\{j = 1, 2, \ldots, N \mid A(\overline{\omega}_{n+k-1}, j) = 1\}) \quad \text{for } k = 1, 2, \ldots.$$
By Lemma 7.2, one has $l(\omega) = l_{n+k}(\omega_{\min})$ and $r(\omega) = r_{n+k}(\omega_{\max})$ for all $k \in \mathbb{N}$. For the two symbols $1, N \in B_1(X_A)$, we may consider the elements $1_{\min}, N_{\max}$ in $X_A$ so that we see

**Lemma 7.3.** $l_n(1_{\min}) = 0, \ r_n(N_{\max}) = 1$ for all $n \in \mathbb{N}$.

For two sequences $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in X_A$, we write $x < y$ if $x_1 = y_1, \ldots, x_n = y_n, x_{n+1} < y_{n+1}$ for some $n \in \mathbb{Z}_+$. Hence $X_A$ becomes an ordered space such that $1_{\min}$ (resp. $N_{\max}$) is minimum (resp. maximum). Recall that for a word $\mu \in B_s(X_A)$, denote by $I_\mu$ the interval $[l(\mu), r(\mu))$, so that $I_\mu = [l(\mu), r(\mu)]$.

**Proposition 7.4.** There exists an order preserving surjective continuous map $\rho_A : X_A \rightarrow [0,1]$ such that

$$\rho_A(1_{\min}) = 0, \quad \rho_A(N_{\max}) = 1 \quad \text{and} \quad \rho_A(U_\mu) = I_\mu \quad \text{for} \quad \mu \in B_n(X_A).$$

**Proof.** For $x = (x_i)_{i \in \mathbb{N}} \in X_A$, there exists an element $\lim_{n \to \infty} l_n(x) = \lim_{n \to \infty} r_n(x)$ in $[0,1]$ which we denote by $\rho_A(x)$. It satisfies the inequalities $l_n(x) \leq \rho_A(x) \leq r_n(x)$ for all $n \in \mathbb{N}$. By the above lemma, we have

$$\rho_A(1_{\min}) = \lim_{n \to \infty} l_n(1_{\min}) = 0, \quad \rho_A(N_{\max}) = \lim_{n \to \infty} r_n(N_{\max}) = 1.$$ 

We will next show that $\rho_A : X_A \rightarrow [0,1]$ is surjective. For $t \in [0,1]$, we may assume that $t < 1$ because $\rho_A(N_{\max}) = 1$. For $n \in \mathbb{N}$, by Lemma 4.4 (ii), one may find a word $\mu^{(n)} \in B_n(X_A)$ such that $t \in I_{\mu^{(n)}}$. The first $n$-symbols of $\mu^{(n+1)}$ coincide with $\mu^{(n)}$ so that the sequence $\{\mu^{(n)}\}_{n \in \mathbb{N}}$ of words defines a right infinite sequence $x_t = (x_n)_{n \in \mathbb{N}}$ of $X_A$ such that $(x_1, \ldots, x_n) = \mu^{(n)}$. Since $l(\mu^{(n)}) \leq t \leq r(\mu^{(n)})$ and $|r(\mu^{(n)}) - l(\mu^{(n)})| < \frac{1}{2^n}$, one sees that $\rho_A(x_t) = \lim_{n \to \infty} l(\mu^{(n)}) = t$ so that $\rho_A : X_A \rightarrow [0,1]$ is surjective.

For $\mu \in B_n(X_A)$ and $x \in U_\mu$, one sees that $l(\mu) = l_n(x) \leq \rho_A(x) \leq r_n(x) = r(\mu)$ so that $\rho_A(x) \in [l(\mu), r(\mu)]$. Hence we have $\rho_A(U_\mu) \subset I_\mu$. As $\rho_A(X_A) = [0,1]$ and $[0,1] = \bigcup_{\mu \in B_n(X_A)} I_\mu$ is a disjoint union for a fixed $n \in \mathbb{N}$, one has $I_\mu \subset \rho_A(U_\mu)$ so that $\rho_A(U_\mu) = I_\mu$. This also shows that $\rho_A$ is order preserving. \[\square\]

We will represent $A$-adic PL functions on $[0,1]$ by using the surjection $\rho_A : X_A \rightarrow [0,1]$. For $\tau \in \Gamma_A$, let $T_\tau = [\mu^{(1)}, \mu^{(2)} \ldots, \mu^{(m)}]$ be its reduced presentation. Let $C_\tau$ be the finite subset of $[0,1]$ defined by

$$C_\tau = \{l(\nu(i)) \mid i = 2, 3, \ldots, m\}(= \{r(\nu(i)) \mid i = 1, 2, \ldots, m - 1\}).$$

Then the $A$-adic PL function $f_\tau$ associated with the $A$-adic table $T_\tau$ is continuous and linear on $[0,1]$ except $C_\tau$. We define a finite subset $S_\tau$ of $X_A$ by

$$S_\tau = \{\nu(i)_{\min} \in X_A \mid i = 1, 2, \ldots, m\}$$

so that $\rho_A(S_\tau) = C_\tau$.

**Proposition 7.5.** For $\tau \in \Gamma$, we have $f_\tau(\rho_A(x)) = \rho_A(\tau(x))$ for all $x \in X_A \setminus S_\tau$. 

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Proof. Since $X_A$ is a disjoint union $\sqcup_{i=1}^M U_{\nu(i)}$, for $x \in X_A \setminus S_\tau$ we may take $\nu(i) = (\nu(i)_1, \ldots, \nu(i)_l_i)$ such that $x \in U_{\nu(i)}$. We write $x = (\nu(i)_1, \ldots, \nu(i)_l_i, x_{l_i+1}, x_{l_i+2}, \ldots)$. As $x \notin S_\tau$, the function $f_\tau$ is continuous at $x$. It then follows that

$$f_\tau(\rho_A(x)) = f_\tau(\lim_{n \to \infty} r(\nu(i)_1, \ldots, \nu(i)_l_i, x_{l_i+1}, \ldots, x_{l_i+n}))$$

$$= \lim_{n \to \infty} f_\tau(r(\nu(i)_1, \ldots, \nu(i)_l_i, x_{l_i+1}, \ldots, x_{l_i+n}))$$

$$= \lim_{n \to \infty} r(\mu(i)_1, \ldots, \mu(i)_k, x_{l_i+1}, \ldots, x_{l_i+n}))$$

$$= \rho_A(\tau(x)).$$

We will next define the derivative of $\tau \in \Gamma_A$. For $\tau \in \Gamma_A$, let $l_\tau, k_\tau$ be $\mathbb{Z}_+$-valued continuous functions on $X_A$ satisfying (2.2).

**Lemma 7.6.** For $\tau \in \Gamma_A$, define $d_\tau : X_A \to \mathbb{Z}$ by setting

$$d_\tau(x) = l_\tau(x) - k_\tau(x), \quad x \in X_A.$$ 

Then $d_\tau$ does not depend on the choice of the functions $l_\tau, k_\tau$ satisfying (2.2).

**Proof.** Let $l'_\tau, k'_\tau$ be another continuous functions on $X_A$ such that

$$\sigma_A^{k'_\tau(x)}(\tau(x)) = \sigma_A^{l'_\tau(x)}(x), \quad x \in X_A.$$

For $x = (x_i)_{i \in \mathbb{N}} \in X_A$, the identities (2.2) and (7.2) ensure us that there exist words $(\mu_1(x), \ldots, \mu_{k_\tau(x)}(x)) \in B_{k_\tau(x)}(X_A)$ and $(\mu'_1(x), \ldots, \mu'_{k'_\tau(x)}(x)) \in B_{k'_\tau(x)}(X_A)$ such that

$$\tau(x) = (\mu_1(x), \ldots, \mu_{k_\tau(x)}(x), x_{l_\tau(x)+1}, x_{l_\tau(x)+2}, \ldots)$$

$$= (\mu'_1(x), \ldots, \mu'_{k'_\tau(x)}(x), x'_{l'_\tau(x)+1}, x'_{l'_\tau(x)+2}, \ldots).$$

For any $n > k_\tau(x), k'_\tau(x)$, by taking the $n$th coordinates of the above sequences, we see that

$$x_{n-k_\tau(x)+l_\tau(x)} = x_{n-k'_\tau(x)+l'_\tau(x)}.$$ 

Put $d'_\tau(x) = l'_\tau(x) - k'_\tau(x)$ and $K(x) = \max\{k_\tau(x), k'_\tau(x)\}$, so that

$$\sigma_A^{K(x)+d_\tau(x)}(x) = \sigma_A^{K(x)+d'_\tau(x)}(x).$$

Suppose that $d_\tau(x) \neq d'_\tau(x)$ for some $x \in X_A$. The above equality implies that $x$ is an eventually periodic point. As the functions $K, d_\tau, d'_\tau$ are all continuous, all elements of some neighborhood of $x$ are eventually periodic. Since the set of non-eventually periodic points is dense in $X_A$, we have a contradiction and hence $d_\tau = d'_\tau$.

**Lemma 7.7.** For $\tau, \tau_1, \tau_2 \in \Gamma_A$, we have

(i) $d_{\tau_2 \tau_1} = d_{\tau_1} + d_{\tau_2} \circ \tau_1$.

(ii) $d_{\tau^{-1}} = -d_\tau \circ \tau^{-1}$. 

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Proof. (i) For \( \tau_1 \in \Gamma_A \), take continuous functions \( k_{\tau_1}, l_{\tau_1} : X_A \rightarrow \mathbb{Z}_+ \) such that

\[
\sigma_A^{k_{\tau_1}(x)}(\tau_1(x)) = \sigma_A^{l_{\tau_1}(x)}(x), \quad i = 1, 2, \ x \in X_A
\]

so that

\[
\sigma_A^{k_{\tau_2}(\tau_1(x))}(\tau_2(\tau_1(x))) = \sigma_A^{l_{\tau_2}(\tau_1(x))}(\tau_1(x)), \quad x \in X_A.
\]

It then follows that

\[
\sigma_A^{k_{\tau_1}(x)}(\sigma_A^{k_{\tau_2}(\tau_1(x))}(\tau_2(\tau_1(x)))) = \sigma_A^{l_{\tau_1}(x)}(\sigma_A^{k_{\tau_2}(\tau_1(x))}(\tau_1(x))) = \sigma_A^{l_{\tau_2}(\tau_1(x))}(\sigma_A^{l_{\tau_1}(x)}(x))
\]

so that

\[
\sigma_A^{k_{\tau_1}(x)+k_{\tau_2}(\tau_1(x))}(\tau_2 \circ \tau_1(x)) = \sigma_A^{l_{\tau_1}(x)+l_{\tau_2}(\tau_1(x))}(x).
\]

Hence we have

\[
d_{\tau_2 \circ \tau_1}(x) = \{l_{\tau_1}(x) + l_{\tau_2}(\tau_1(x))\} - \{k_{\tau_1}(x) + k_{\tau_2}(\tau_1(x))\} = d_{\tau_1}(x) + d_{\tau_2}(\tau_1(x)).
\]

(ii) By (2.2), we have

\[
\sigma_A^{k_\tau(x)(\tau^{-1}(x))}(x) = \sigma_A^{l_\tau(x)(\tau^{-1}(x))}(x), \quad x \in X_A
\]

so that

\[
d_{\tau^{-1}}(x) = k_\tau(x) - l_\tau(x) = -d_\tau(x).
\]

\[\square\]

Definition 7.8. For an element \( \tau \in \Gamma_A \), the derivative \( D_\tau \) of \( \tau \) is defined by a real valued continuous function \( D_\tau \) on \( X_A \):

\[
D_\tau(x) = \beta^{d_\tau(x)}, \quad x \in X_A, \quad (7.3)
\]

where \( \beta \) is the Perron eigenvalue of the matrix \( A \).

The derivative \( D_\tau \) of \( \tau \) is regarded as an element of \( D_A \). Recall that \( \varphi \) stands for the unique KMS-state on \( O_A \) for gauge action. The following proposition shows that \( D_\tau \) satisfies the law of derivatives.

Proposition 7.9. For \( \tau, \tau_1, \tau_2 \in \Gamma_A \), we have

(i) \( \varphi(D_\tau) = 1 \).

(ii) \( D_{\tau_2 \circ \tau_1} = D_{\tau_1} \cdot (D_{\tau_2} \circ \tau_1) \).

(iii) \( D_{\tau^{-1}} = (D_\tau \circ \tau^{-1})^{-1} \).

Proof. Suppose that \( \tau \) is given by an \( A \)-adic table \( T = [\mu^{(1)} \mu^{(2)} \ldots \mu^{(m)}] \) so that \( u_\tau = \sum_{i=1}^m S_\mu(i) S_\mu^* \), \( S_\nu(i) S_\mu(i) = S_\nu(i) S_\mu(i) \) and \( \sum_{i=1}^m S_\mu(i) S_\mu^* = \sum_{i=1}^m S_\nu(i) S_\nu^* = 1 \). Put \( \lambda_A(f) = \sum_{i=1}^N S_\tau f S_i \) for \( f \in D_A \). It then follows that

\[
\lambda_A^{[\mu(i)]}(u_\tau S_\nu(i) S_\mu^*) = \lambda_A^{[\mu(i)]}(S_\mu(i) S_\nu^*) = S_\mu^* S_\mu(i) = S_\nu^* S_\nu(i)
\]
so that
\[ \lambda_A^{\mu(i)}(u_{\tau}S_{\nu(i)}S^*_{\nu(i)}u^*_\tau) = \lambda_A^{\nu(i)}(S_{\nu(i)}S^*_{\nu(i)}), \quad i = 1, \ldots, m. \]

As \( \varphi \circ \lambda_A = \beta \varphi \) on \( D_A \), we have
\[ \varphi(u_{\tau}S_{\nu(i)}S^*_{\nu(i)}u^*_\tau) = \beta^{\nu(i)-\mu(i)}\varphi(S_{\nu(i)}S^*_{\nu(i)}), \quad i = 1, \ldots, m. \] (7.4)

Since \( d_{\tau}(x) = l_{\tau}(x) - k_{\tau}(x) = |\nu(i)| - |\mu(i)| \) for \( x \in U_{\nu(i)} \), the derivative \( D_{\tau} \) is expressed as
\[ D_{\tau} = \sum_{i=1}^{m} \beta^{\nu(i)-\mu(i)}S_{\nu(i)}S^*_{\nu(i)} \]
so that by the equality (7.4) one obtains that
\[ \varphi(D_{\tau}) = \sum_{i=1}^{m} \beta^{\nu(i)-\mu(i)}\varphi(S_{\nu(i)}S^*_{\nu(i)}) = \sum_{i=1}^{m} \varphi(u_{\tau}S_{\nu(i)}S^*_{\nu(i)}u^*_\tau) = \varphi(1) = 1. \]

(ii), (iii) By the previous lemma, we have
\[ D_{\tau_2 \circ \tau_1} = \beta^{d_{\tau_2} \circ \tau_1} = \beta^{d_{\tau_1}} \cdot \beta^{d_{\tau_2} \circ \tau_1} = D_{\tau_1} \cdot D_{\tau_2} \circ \tau_1, \]
\[ D_{\tau^{-1}} = \beta^{-d_{\tau} \circ \tau^{-1}} = [D_{\tau} \circ \tau^{-1}]^{-1}. \]

□

As the function \( f_{\tau} \) is linear on the interval \( I_{\nu(i)} = [l(\nu(i)), r(\nu(i))] \) with slope \( \beta^{\nu(i)-\mu(i)} \), we may summarize the above discussions in the following theorem.

**Theorem 7.10.** There exists an order preserving continuous surjection \( \rho_A : X_A \rightarrow [0, 1] \) from the shift space \( X_A \) of a one-sided topological Markov shift \((X_A, \sigma_A)\) to the closed interval \([0, 1]\) such that for any element \( \tau \in \Gamma_A \), there exists a finite set \( S_{\tau} \subset X_A \) such that the corresponding \( A \)-adic PL function \( f_{\tau} \) for \( \tau \) satisfies the following properties:

(i) \( f_{\tau}(\rho_A(x)) = \tau(\rho_A(x)) \) for \( x \in X_A \backslash S_{\tau}, \)

(ii) \( \frac{df_{\tau}}{dx}(\rho_A(x)) = D_{\tau}(x) = \beta^{d_{\tau}(x)} \) for \( x \in X_A \backslash S_{\tau}, \)

where \( d_{\tau}(x) = l_{\tau}(x) - k_{\tau}(x) \) for the continuous functions \( k_{\tau}, l_{\tau} : X_A \rightarrow \mathbb{Z}_+ \) satisfying \( \sigma^{k_{\tau}}(\tau(x)) = \sigma^{l_{\tau}}(x), x \in X_A \) and \( \beta \) is the Perron eigenvalue of \( A \).

8 Generalizations of other Thompson groups

R. J. Thompson has defined finitely presented infinite subgroups \( F_2, T_2 \) of \( V_2 \) which satisfy \( F_2 \subset T_2 \subset V_2 \). K. S. Brown [1] has extended the subgroups \( F_2, T_2 \) of \( V_2 \) to the family \( F_N \subset T_N \subset V_N \) of finitely presented subgroups \( F_N, T_N \) of \( V_N \) such that \( T_N \) is a group of piecewise linear homeomorphisms \( f : [0, 1] \rightarrow [0, 1] \) on the unit circle having finitely many singularities such that all singularities of \( f \) are in \( \mathbb{Z}[\frac{1}{N}] \), the derivative of \( f \) at any nonsingular point is \( N^k \) for some \( k \in \mathbb{Z} \), and \( F_N \) is a subgroup of \( T_N \) consisting of piecewise linear homeomorphisms \( f : [0, 1] \rightarrow [0, 1] \) on the unit interval.
In this section, we generalize the groups $F_N, T_N$ for $1 < N \in \mathbb{N}$ to $F_A, T_A$ for irreducible square matrices $A$ with entries in $\{0,1\}$ by using the techniques of the preceding sections.

Recall that an element $\tau \in \Gamma_A$ is represented as a cylinder map given by two families $\mu(i), \nu(i), i = 1, \ldots, m$ of words satisfying (2.3), (2.4), (2.5) and (2.6). We may assume that the words $\nu(i), i = 1, \ldots, m$ are ordered such as $\nu(1) < \nu(2) < \cdots < \nu(m)$. We define further properties for $\tau \in \Gamma_A$ as follows. $\tau \in \Gamma_A$ is said to be

(i) **order preserving** if one may take the words $\mu(i), i = 1, \ldots, m$ such as

$$
\mu(1) < \mu(2) < \cdots < \mu(m),
$$

(ii) **cyclic order preserving** if one may take the words $\mu(i), i = 1, \ldots, m$ such as

$$
\mu(k) < \mu(k+1) < \cdots < \mu(m) < \mu(1) < \mu(2) < \cdots < \mu(k-1)
$$

for some $k \in \{1,2,\ldots,m\}$. If $\tau$ is order preserving, it is cyclic order preserving. It is easy to see that the set of order preserving cylinder maps forms a subgroup of $\Gamma_A$, and the set of cyclic order preserving cylinder maps forms a subgroup of $\Gamma_A$. We denote them by $F_A$ and by $T_A$ and call them the order preserving continuous full group and the cyclic order preserving continuous full group, respectively.

In Definition 4.6, if one may take $\sigma$ such as (8.1) for some $k \in \{1,\ldots,m\}$, an $A$-adic PL function $f$ is called a cyclic order preserving $A$-adic PL function. If in particular, one may take $\sigma = \text{id}$, $f$ is called an order preserving $A$-adic PL function.

It is easy to see that the set $F_A^{\text{PL}}$ of order preserving $A$-adic PL functions and the set $T_A^{\text{PL}}$ of cyclic order preserving $A$-adic PL functions form subgroups of the group of the $A$-adic PL functions. Hence we have subgroups of inclusion relations:

$$
F_A^{\text{PL}} \subset T_A^{\text{PL}} \subset \Gamma_A^{\text{PL}}.
$$

The following proposition is immediate by definition of order preserving (resp. cyclic order preserving) $A$-adic PL functions.

**Proposition 8.1.** An $A$-adic order preserving (resp. cyclic order preserving) PL function naturally gives rise to an $A$-adic order preserving (resp. cyclic order preserving) pattern of rectangles, whose rectangle slopes are the slope of the $A$-adic PL function. Conversely, an $A$-adic order preserving (resp. cyclic order preserving) pattern of rectangles gives rise to an $A$-adic order preserving (resp. cyclic order preserving) PL function by taking the diagonal lines of the corresponding rectangles.

In Definition 5.1 let $T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$ be an $A$-adic table such that $\nu(1) < \nu(2) < \cdots < \nu(m)$. Then $T$ is said to be
(i) **order preserving** if $\mu(1) \prec \mu(2) \prec \cdots \prec \mu(m)$,

(ii) **cyclic order preserving** if

$$\mu(k) \prec \mu(k+1) \prec \cdots \prec \mu(m) \prec \mu(1) \prec \mu(2) \prec \cdots \prec \mu(k-1)$$

for some $k \in \{1, 2, \ldots, m\}$.

If $T$ is order preserving, it is cyclic order preserving. These two properties of $A$-adic tables are closed under taking expansions of $A$-adic tables respectively. We see that the set $F_A^{\text{tab}}$ of the equivalence classes of order preserving $A$-adic tables and the set $T_A^{\text{tab}}$ of the equivalence classes of cyclic order preserving $A$-adic tables form subgroups of $\Gamma_A^{\text{tab}}$, respectively. Hence we have subgroups of inclusion relations:

$$F_A^{\text{tab}} \subset T_A^{\text{tab}} \subset \Gamma_A^{\text{tab}}.$$

We further see the following:

**Lemma 8.2.** For a table $T$, let $f_T$ be the associated $A$-adic PL function. Then $T$ is order preserving (resp. cyclic order preserving) if and only if the function $f[T]$ is order preserving (resp. cyclic order preserving).

We thus have

**Proposition 8.3.** There exist canonical isomorphisms of discrete groups among the order preserving (resp. cyclic order preserving) continuous full group $F_A$ (resp. $T_A$), the group $F_A^{\text{tab}}$ (resp. $T_A^{\text{tab}}$) of the equivalence classes of order preserving (resp. cyclic order preserving) $A$-adic tables and the group $F_A^{\text{PL}}$ (resp. $T_A^{\text{PL}}$) of the order preserving (resp. cyclic order preserving) $A$-adic PL functions on $[0,1)$, that is

$$F_A \cong F_A^{\text{tab}} \cong F_A^{\text{PL}}, \quad T_A \cong T_A^{\text{tab}} \cong T_A^{\text{PL}}.$$

**Proof.** The isomorphisms in Proposition 5.8 and Theorem 6.3 among $\Gamma_A$, $\Gamma_A^{\text{tab}}$ and $\Gamma_A^{\text{PL}}$ preserve the orders of words, so that its restrictions yield desired isomorphisms.

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