BRST COHOMOLOGY IN BELTRAMI PARAMETRIZATION

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1 April 1995

Abstract

We study the BRST cohomology within a local conformal Lagrangian field theory model built on a two dimensional Riemann surface with no boundary. We deal with the case of the complex structure parametrized by Beltrami differential and the scalar matter fields. The computation of all elements of the BRST cohomology is given.
1 Introduction

The Beltrami differentials \[1\] have turned out to be basic variables for parametrizing complex structures for a bidimensional theory, in a conformally invariant way \[2, 3, 4\]. The advantage of the Beltrami differentials is the fact that they changes only under reparametrization transformations and diffeomorphisms while the Weyl rescaling is factorized out. Thus the Weyl degree of freedom is eliminated from the very beginning and some advantages in the quantization are reached.

In this paper we shall describe models for which a conformal matter field of weight \((j, \bar{j})\) coupled to a 2D gravity, is characterized by the Beltrami differentials, in a reparametrization invariant way \[2, 8, 5, 6, 7\]. To accomplish this aim we shall introduce a BRST symmetry \[3, 9, 2, 10\] carried out through a nilpotent operator \(s\) and we shall calculate its cohomology group \(H(s)\) within a local Lagrangian field theory formulation. We are not going to characterize these models by a specific conformally invariant classical action but rather we shall specify the field content and the gauge invariances of the classical theory. In this framework the search for the invariant Lagrangians, the anomalies and the Schwinger terms can be done in a purely algebraic way, along the line of algebraic topology \[9, 11, 12\]. In fact, the main purpose of our search is to find out all nontrivial solutions of the equation

\[
sA = 0
\]

with \(s\) the nilpotent BRST differential and \(A\) is an integrated local functional \(A = \int d^2xf\). The condition (1.1) can be translated into the local descent equations \[3, 13, 8\]

\[
s\omega_2 + d\omega_1 = 0 , \quad s\omega_1 + d\omega_0 = 0
\]

\[
s\omega_0 = 0 .
\]

(1.2)

where \(\omega_2\) is a 2-form with \(A = \int \omega_2\) and \(\omega_1, \omega_0\) are local 1- and 0-forms, respectively. It is well known \[14\] that the descent equations (1.2) end, for the Beltrami parametrization, always with
a nontrivial 0-form $\omega_0$ and that their ”integration” is trivial

$$\omega_1 = \delta \omega_0, \quad \omega_2 = \frac{1}{2} \delta^2 \omega_0,$$

(1.3)

where the operator $\delta$ was introduced by Sorella \[3, 8\] and it allows to express the exterior derivative $d$ as a BRST commutator:

$$d = -[s, \delta].$$

(1.4)

Thus it is sufficient to find out the general solution of the equation

$$s \omega_0 = 0,$$

(1.5)

in the space of local functions of the fields and their derivatives i.e. to calculate the BRST cohomology group $H(s)$.

In this paper we shall calculate all elements of $H(s)$ for the string theory in the Beltrami parametrization in the presence of one scalar matter field of weight $(0, 0)$. The basic ingredients of our calculations are the choice of an appropriate new basis and the existence in this basis a contracting homotopy, which reduces considerably the number of the elements from the basis for the solutions of (1.3). In this way we shall obtain a very limited possible solutions of (1.5) which can be listed and studied. We want to stress that the basis used in this paper is very closed to the one proposed by Brandt, Troost and Van Proyen, in a very interesting paper \[11\] but the BRST transformations of this basis and the contracting homotopy differ and our method can be easily generalized for other models as superstring model in the super-Beltrami parametrization \[15\] and $W_3$-gravity \[16\].

The paper is organized as follows. In Sect.2 we briefly recall the Beltrami parametrization and its BRST symmetry. In Sect. 3 we define the differential algebra of all fields and their derivatives $\mathcal{A}$ and we show that it can be split in a contractive part $\mathcal{C}$ and a minimal one $\mathcal{M}$. Only the minimal part does contribute to the BRST cohomology. In Sect. 4 we introduce a new basis and show that, in the presence of the nonlocal fields $\ln \lambda$ and $\ln \bar{\lambda}$ the equation (1.5) has nontrivial solutions in a very small subalgebra, which we are going to describe. In this subalgebra we find all
nontrivial elements of $H(s)$. Thus we can find out all solutions of the descent equations (1.2). In Sec. 5 the cohomology group $H(s)$ and the solutions of the decent equation (1.2) are constructed for theories without the fields $\ln \lambda$ and $\ln \bar{\ell}$.

2 The diffeomorphism BRST cohomology

Let we start by introducing the setup for the string theory in the Beltrami parametrization. We will work on a Riemann surface $M$ equipped with a complex structure or, equivalently, with a conformal class of metrics [1]. Using the complex notations $dz = dx + idy, d\bar{z} = dx - idy$ the line element associated to the metric can be written as:

$$ds^2 = |\rho|^2 |dz + \mu d\bar{z}|^2$$  \hspace{1cm} (2.1)

where $\rho$ and $\mu$ are smooth complex-valued functions of $z, \bar{z}$ and the positive-definiteness of the metric is expressed by the condition $|\mu| < 1$. The function $\rho$ is usually called the conformal factor and $\mu$ the Beltrami differential (or parameter) [1] and (2.1) is often called the Beltrami parametrization of the metric. [4]. The line element $ds^2$ can be written in terms of isothermal coordinated $(Z, \bar{Z})$ such that $ds^2 \sim |dZ|^2$. These isothermal coordinates are defined by

$$dZ = \lambda(z, \bar{z}) [dz + \mu d\bar{z}]$$  \hspace{1cm} (2.2)

with $\lambda$ a smooth complex-valued function, called the integrating factor. The condition $d^2 = 0$ yields

$$(\bar{\partial} - \mu \partial)(\ln \lambda) = \partial \mu.$$  \hspace{1cm} (2.3)

The line element $ds^2$ has a very simple form in the isothermal coordinates

$$ds^2 = |dZ|^2.$$  \hspace{1cm} (2.4)

Despite of the tact that the conformal factor $\rho$ and the integrating factor $\lambda$ look very similar, they have different transformations laws and they are very different in many respects.
The matter fields, in our models, are realized by local tensor fields

\[ \Phi_{j,\bar{j}}(Z, \bar{Z})dZ^jd\bar{Z}^\bar{j} \]  

(2.5)
of the weight \((j, \bar{j})\) invariant under change of holomorphic charts. The matter fields \(\varphi_{j,\bar{j}}\) are defined by (2.5) written in terms of the local coordinates \((z, \bar{z})\):

\[ \Phi_{j,\bar{j}}(Z, \bar{Z})dZ^jd\bar{Z}^\bar{j} = \varphi_{j,\bar{j}}(z, \bar{z})(dz + \mu d\bar{z})^j(d\bar{z} + \bar{\mu}dz)^\bar{j}, \]  

(2.6)
with

\[ \Phi_{j,\bar{j}} = \frac{\varphi_{j,\bar{j}}(z, \bar{z})}{\lambda^j(z, \bar{z})\lambda^\bar{j}(z, \bar{z})}. \]  

(2.7)
The BRST symmetry can be obtained by considering an infinitesimal change of the coordinate \((z, \bar{z})\) generated by a vector field:

\[ \xi.\partial = \xi \partial_z + \bar{\xi} \partial_{\bar{z}} = \xi \partial + \bar{\xi} \partial, \]  

(2.8)
and then replacing the parameters \((\xi, \bar{\xi})\) by the ghosts \((c, \bar{c})\). Thus the BRST differential \(s\) acts on \(Z\) and \(\varphi_{j,\bar{j}}\) as the Lie derivatives

\[ sZ = L_c\partial Z = \lambda(c + \mu \bar{c}) \]  

(2.9)
\[ s\varphi_{j,\bar{j}} = L_c\partial \varphi_{j,\bar{j}} = (c \cdot \partial)\varphi_{j,\bar{j}} + \]  

(2.10)
\[ + [j(\partial c + \mu \partial \bar{c}) + \bar{j}(\partial \bar{c} + \bar{\mu} \partial c)]\varphi_{j,\bar{j}}. \]  

(2.11)
The operator \(s\) acts as an antiderivation from the left and the graduation is given by adding the form degree to the ghost number.

The corresponding transformation laws of \(\mu\) and \(\lambda\) follow by evaluating the variation of \(dZ\) in two different ways

\[ s(dZ) = -d(sZ) = -d[\lambda(c + \mu \bar{c})] \]

and

\[ s(dZ) = s[\lambda(dz + \mu d\bar{z})] \]
By comparing the different coefficients of $dz$ and $d\bar{z}$ one finds
\begin{align}
s\mu &= (c \cdot \partial)\mu - \mu(\partial c + \mu \partial \bar{c}) + \bar{\partial}c + \mu \bar{\partial} \bar{c} \tag{2.12} \\
s\lambda &= \partial[\lambda(c + \mu \bar{c})]. \tag{2.13}
\end{align}

The nilpotency of $s$ requires
\[0 = s^2 Z = [sc - c\partial c]\lambda\]
and thereby
\[sc = c\partial c. \tag{2.14}\]

It is very convenient to replace the ghosts $(c, \bar{c})$ with the Becchi’s reparametrization \[2\]
\[C = c + \mu \bar{c}, \quad \bar{C} = \bar{c} + \bar{\mu}c. \tag{2.15}\]

This reparametrization ensures the holomorphic factorization of the BRST variations of $\mu$ and $\lambda$. \[10, 4\]. Eqs. (2.9)-(2.14) can be rewritten as
\begin{align}
sZ &= \lambda C \tag{2.16} \\
s\varphi_{jj} &= CD\varphi_{jj} + \bar{C}\bar{D}\varphi_{jj} + \\
&\quad + [j(\partial C) + \bar{j}(\bar{\partial} \bar{C})]\varphi_{jj} \tag{2.17} \\
s(ln \lambda) &= \partial C \tag{2.18} \\
s\mu &= \bar{\partial}C + C\partial \mu - \mu \partial C \tag{2.19} \\
sC &= C\partial C(\ln \lambda)C \tag{2.20}
\end{align}

and their complex conjugate expressions, with
\[
D = \lambda \frac{\partial}{\partial Z} = \frac{1}{1 - \mu \bar{\mu}}(\partial - \bar{\mu} \bar{\partial})
\]
\[
D = \bar{\lambda} \frac{\partial}{\partial \bar{Z}} = \frac{1}{1 - \mu \bar{\mu}}(\bar{\partial} - \mu \partial)
\]
are the “covariant derivatives” \[18, 13, 19, 20\]. Now we introduce the ghost number (or Faddeev-Popov charge) gh=g, which is one for the ghost fields $c$ and $\bar{c}$ (or equivalently $C$ and $\bar{C}$) and the BRST differential $s$ and zero for the other fields $\{Z, \bar{Z}, \varphi_{jj}, \lambda, \bar{\lambda}, \mu, \bar{\mu}\}$ and the differential $d$. 

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It is easy to see that the nilpotency of $s$ is equivalent to the differential equation (2.23) and the commutation relations
\[ [s, \partial] = [s, \bar{\partial}] = 0 \] (2.22)
or equivalently
\[ \{s, d\} = 0 \] (2.23)
with
\[ d = dz\partial + d\bar{z}\bar{\partial} \]
the exterior derivative.

The main purpose of the present paper is to give the most general solution of the equation
\[ (2.24) \]
\[ sA^p = 0 \quad \text{with} \quad A^p = \int \Delta^p_r(z, \bar{z}) \]
where $A^p$ has the ghost number two and $\Delta^p_r(z, \bar{z})$ is a $r$-form with the ghost number $p$. In eq. (2.24) $r$ can take two values $r = 1, 2$ and in these cases we have the 1-form descent equations (2.24) and two-form descent equations (19, 13).

In terms of local quantum eq. (2.24) is expressed by the $s$-cohomology modulo $d$:
\[ s\Delta^p_1(z, \bar{z}) + d\Delta^p_0(z, \bar{z}) = 0 \]
\[ s\Delta^p_0(z, \bar{z}) = 0 \] (2.25)
or
\[ s\Delta^p_0(z, \bar{z}) + d\Delta^p_1(z, \bar{z}) = 0 \]
\[ s\Delta^p_1(z, \bar{z}) + d\Delta^p_0(z, \bar{z}) = 0 \]
\[ s\Delta^p_0(z, \bar{z}) = 0 \] (2.26)

The ladder (2.25) or (2.26) could be solved thanks to an operator $\delta$ introduced by Sorella for the Yang-Mills BRST cohomology [8, 9], bosonic string [13] and superbosonic string [22] (see
also \[11\] for \(W_3\)-gravity). The operator \(\delta\) allows us to express the exterior derivative \(d\) as a BRST commutator, i.e.:

\[
d = -[s, \delta]. \tag{2.27}
\]

Now it is easy to see that, once the decomposition (2.27) has been found, repeated application of the operator \(\delta\) on the local functions \(\{\Delta_0^{p+2}(z, \bar{z}), \Delta_0^{p+1}(z, \bar{z})\}\) that solve the last equation of (2.25) or (2.26) given an explicit and nontrivial solution for the other cocycles \(\Delta_n^{p+n}(z, \bar{z})\). In other words, with the operator \(\delta\) we can go from the cohomology \(H(s)\) to the relative cohomology \(H(s \mod d)\).

In our theory the operator \(\delta\) from the decomposition (1.4) can be defined by

\[
\delta C = dz + \mu d\bar{z} \quad \delta \bar{C} = d\bar{z} + \bar{\mu} dz
\]

\[
\delta \Phi = 0 \quad \text{for} \quad \Phi = \{\mu, \bar{\mu}, \lambda, \bar{\lambda}, \varphi_{j\bar{j}}\}. \tag{2.28}
\]

Now it is easy to verify that \(\delta\) is of degree 0 and obeys the following algebraic relations:

\[
d = -[s, d], \quad [d, \delta] = 0. \tag{2.29}
\]

To solve the towers (2.25) or (2.26) we shall make use of the following identity

\[
e^\delta s = (s + d)e^\delta \tag{2.30}
\]

that is a direct consequence of (1.4) and (2.29) (see \[8\]). Therefore, once a non-trivial solution. In this way we get

\[
(s + d)\left[e^\delta \Delta_0^{p+n}\right] = 0 \quad (n = 1, 2). \tag{2.31}
\]

But, as one can see from (2.28), the operator \(\delta\) acts as a translation on the ghosts \((C, \bar{C})\)

\[
C \rightarrow C + dz + \mu d\bar{z} \quad \bar{C} \rightarrow \bar{C} + d\bar{z} + \bar{\mu} dz
\]

and eq. (2.31) can be rewritten as

\[
(s + d)\Delta_0^{p+n}(C + dz + \mu d\bar{z}, \bar{C} + d\bar{z} + \bar{\mu} dz, \mu, \bar{\mu}, \lambda, \bar{\lambda}, \varphi_{j\bar{j}}) = 0. \tag{2.32}
\]

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Thus the expansion of the zero form cocycle $\Delta_{0}^{p+n}$ in power of the one-forms $(dz + \mu d\bar{z}, d\bar{z} + \bar{\mu} dz)$ yields all the cocycles $\Delta_{p}^{n-r}$. The operator $\delta$, defined in (2.28) is closed connected with the operators $(\mathcal{W}, \bar{\mathcal{W}})$ introduced in [13] and defined as:

$$\mathcal{W} = \int dzd\bar{z} \left(\bar{\mu} \frac{\delta}{\delta C} + \frac{\delta}{\delta C}\right)$$

(2.33)

$$\bar{\mathcal{W}} = \int dzd\bar{z} \left(\mu \frac{\delta}{\delta C} + \frac{\delta}{\delta C}\right)$$

(2.34)

Our $\delta$ is related to $(\mathcal{W}, \bar{\mathcal{W}})$ by the relation

$$\delta = dz\mathcal{W} + d\bar{z}\bar{\mathcal{W}}$$

(2.35)

and the relations (2.29) imply

$$\{s, \mathcal{W}\} = \partial \quad \{s, \bar{\mathcal{W}}\} = \bar{\partial}$$

$$\{\mathcal{W}, \mathcal{W}\} = \{\mathcal{W}, \bar{\mathcal{W}}\} = \{\bar{\mathcal{W}}, \bar{\mathcal{W}}\} = 0.$$  

(2.36)

3 BRST symmetry in a new basis

In his section we are going to solve the equation

$$s \omega_0 = 0$$

(3.1)

in the algebra of local analytic function of all fields and their derivatives $\mathcal{A}$. A basis of this algebra can be chosen to be

$$\{\partial^p \bar{\partial}^q \Psi, \partial^p \bar{\partial}^q \bar{\Psi}, \partial^p \bar{\partial}^q \bar{C}\}$$

(3.2)

where $\Psi = \{\mu, \bar{\mu}, \lambda, \bar{\lambda}, \phi_{i,j}\}$ and $p, q = 0, 1, 2, \cdots$. However, the BRST transformations of the elements of this basis are quite complicated and there are many terms which can be eliminated in $H(s)$. In fact the algebra $\mathcal{A}$ with the BRST differential $s$ form a free differential algebra, which can be decomposed, by using a theorem due to Sullivan [25], as a tensor product of a contractible algebra $\mathcal{C}$ and a minimal one $\mathcal{M}$. A contractible differential algebra $\mathcal{C}$ is an algebra isomorphic to
a tensor product of algebras of the form $\wedge(x, sx)$ and a minimal one $\mathcal{M}$ is an algebra for which
$s\mathcal{M} \subseteq \mathcal{M}^+ \cdot \mathcal{M}^+$. with $\mathcal{M}^+$ the part of $\mathcal{M}$ in positive degree. The remarkable point in Sullivan decomposition is the fact that the contractible part of the algebra $\mathcal{A}$ does not contribute to the cohomology group $H(s)$. Thus, to calculate $H(s)$ it is enough to separate from the differential algebra $\mathcal{A}$ its minimal part $\mathcal{C}$ and to calculate the cohomology group of $\mathcal{M}$. Indeed, according to the K"uneth theorem

$$H(\mathcal{M} \otimes \mathcal{C}) = H(\mathcal{M}) \otimes H(\mathcal{C})$$

since $\mathcal{C}$ is contractible and its cohomology group is zero.

The separation of the algebra $\mathcal{A}$ in two parts is easier to be accomplished if we introduce a new basis of variables substituting the fields and their derivatives (3.2). The new basis consist of

1. the variables $\phi^{p,q}_{j\bar{j}}$, substituting one-by-one the partial derivatives $\partial^p \bar{} \partial^q \phi^{j\bar{j}}$ of the matter fields

$$\phi^{p,q}_{j\bar{j}} = \Delta^p \bar{\Delta}^q \phi^{j\bar{j}}$$

where the even differentials $\{\Delta, \bar{\Delta}\}$ are defined by

$$\Delta = \{s, \frac{\partial}{\partial C}\} \quad \bar{\Delta} = \{s, \frac{\partial}{\partial \bar{C}}\};$$

(3.4)

2. the ghost variables

$$C^m = \frac{1}{(n + 1)!} \Delta^{n+1} C \quad \bar{C}^m = \frac{1}{(n + 1)!} \bar{\Delta}^{n+1} \bar{C}.$$  (3.5)

All the variables (3.3) and (3.5) have two remarkable properties. First, they have very simple BRST transformation properties, being the basis for the minimal part of $\mathcal{A}$ and second, they have a special total weight which allow us to select only very few possibilities for the solutions of equation (3.1). The transformation properties of these variables can be obtained from their definitions, the commutation relations

$$\Delta s = s\Delta, \quad \bar{\Delta} s = s\bar{\Delta}$$

(3.6)
and the fact that \( \{ \Delta, \bar{\Delta} \} \) are even differentials i.e. they satisfy the Leibniz rule

\[
\Delta(ab) = (\Delta a)b = a(\Delta b)
\]  

(3.7)

and its generalized form

\[
\Delta^n(ab) = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k a)(\Delta^{n-k} b).
\]  

(3.8)

The BRST transformation of the basis (3.5) can be written as

\[
sC = \frac{1}{(n+1)!} \Delta^{n+1}(C\partial C) = \sum_{k=-1}^{n-1} (n-k)C^k C^{n-k} = \frac{1}{2}f_{pq} \ n^q C^n C^q,
\]  

(3.9)

where

\[f_{pq} = (p-q)\delta_{p+q}^n\]  

(3.10)

since the BRST transformation of \( C \) is given by (2.20), and of course the complex conjugate expressions.

For the elements of the basis (3.3) the BRST transformation can be obtained from their definition and the transformation of the matter fields \( \phi_{j,\bar{j}} \) (2.17). Therefore eqs. (3.7) and (3.8) yield

\[
s\phi_{j,\bar{j}}^{p,q} = \Delta^p \bar{\Delta}^q(s\phi_{j,\bar{j}}) = \sum_{k=-1}^{p-1} A^k_p C^k \phi_{j,\bar{j}}^{p-k,q} + \sum_{k=-1}^{q-1} A^k_q \bar{C}^k \phi_{j,\bar{j}}^{p,q-k} = \sum_{k=-1}^{\infty} (C^k L_k + \bar{C}^k \bar{L}_k)\phi_{j,\bar{j}}^{p,q}
\]  

(3.11)

where

\[
L_k \phi_{j,\bar{j}}^{p,q} = A^k_p(j) \phi_{j,\bar{j}}^{p-k,q}
\]

\[
\bar{L}_k \bar{\phi}_{j,\bar{j}}^{p,q} = A^k_p(\bar{j}) \bar{\phi}_{j,\bar{j}}^{p-k,q}
\]  

(3.12)

and

\[
A^k_p(j) = \frac{p!}{(p-k)!}[j(k+1) + p - k].
\]  

(3.13)

On the basis \( \{ \phi_{j,\bar{j}}^{p,q} \} \) the operators \( \{ L_k, \bar{L}_k \} \) represent two copies of the Virasoro algebra, fact that can be seen from their definitions (3.12). This fact was pointed out by Brandt, Troost and
Van Proeyen [11] for 2D conformal gravity in a basis formed by $\phi_{0,0}$ matter fields. In fact the definitions (3.12) yield

$$[L_m, L_n] = f^k_{mn} L_k, \quad [\bar{L}_m, \bar{L}_n] = f^k_{mn} \bar{L}_k, \quad [L_m, \bar{L}_n] = 0$$

(3.14)

where $f^k_{mn}$ are the structure constants of the Virasoro algebra given by (3.10).

On the other hand, the relation (3.11) shows that on the basis $\{\phi^p_{j,\bar{j}}\}$ the generators of the Virasoro algebra $\{L_k, \bar{L}_k\}$ have the following expressions:

$$L_k = \left\{s, \frac{\partial}{\partial C^k}\right\}, \quad \bar{L}_k = \left\{s, \frac{\partial}{\partial \bar{C}^k}\right\}, \quad k > -2.$$  

(3.15)

Hitherto we have not said anything about the other members of the basis for the algebra $\mathcal{A}$ (3.2), i.e. about $\{\mu, \bar{\mu}, \lambda, \bar{\lambda}\}$ and their derivatives. The structure of the free differential algebra $\mathcal{A}$ strongly depends on the fact that we allow the variables $\{\ln \lambda, \ln \bar{\lambda}\}$ in it.

4 BRST cohomology with the variable $\ln \lambda$

Now if we consider the $\ln \lambda$ and $\ln \bar{\lambda}$ as variables in our differential algebra $\mathcal{A}$ then the BRST cohomology has a very simple form. Indeed, in this case the BRST transformations

$$s(\ln \lambda) = \partial C + \partial (\ln \lambda) C, \quad s(\ln \bar{\lambda}) = \partial \bar{C} + \partial (\ln \bar{\lambda}) \bar{C},$$

$$s\mu = \bar{\partial} C + C \partial \mu - \mu \partial C, \quad s\bar{\mu} = \partial \bar{C} + \bar{C} \partial \bar{\mu} - \bar{\mu} \partial \bar{C}$$

(4.1)

show that the subalgebra $\mathcal{C}$ could be generated by the elements

$$\mathcal{C} = \{\partial^p \bar{\partial}^q \mu, \partial^p \bar{\partial}^q \bar{\mu}, \partial^p \bar{\partial}^q \lambda, \partial^p \bar{\partial}^q \bar{\lambda}, s(\partial^p \bar{\partial}^q \mu), s(\partial^p \bar{\partial}^q \bar{\mu}), s(\partial^p \bar{\partial}^q \lambda), s(\partial^p \bar{\partial}^q \bar{\lambda})\}.$$  

(4.2)

A possible candidate for the minimal subalgebra $\mathcal{M}$ might be generated by the elements

$$\mathcal{M} = \{C, \bar{C}, \phi^p_{j,\bar{j}}\}.$$  

(4.3)

For example all the derivatives of $C$ and $\bar{C}$ can be expressed as polynomials of the elements of the basis (4.2) and (4.3).
However, as it can be seen by a simple inspection of the BRST transformations of $\phi^{p,q}_{j\bar{j}}$, the algebra $\mathcal{M}'$ does not satisfy the condition for a minimal algebra

$$s\mathcal{M}' \subseteq \mathcal{M}^+ \cdot \mathcal{M}^+.$$ 

But we can slightly modify the subalgebra $\mathcal{M}'$ to obtain a minimal one. Instead of the matter fields $\phi^{p,q}_{j\bar{j}}$ we shall use the matter fields $\Phi^{j\bar{j}}$ in the $(Z, \bar{Z})$ complex analytic coordinates defined by

$$\Phi^{j\bar{j}}(Z, \bar{Z}) = \frac{\phi^{j\bar{j}}(z, \bar{z})(dz + \mu d\bar{z})^{j}(d\bar{z} + \bar{\mu}dz)^{\bar{j}},}{\lambda^{j}(z, \bar{z})\lambda^{\bar{j}}(z, \bar{z})} \quad (4.4)$$

or

$$\Phi^{j\bar{j}}(Z, \bar{Z}) = \frac{\phi^{j\bar{j}}(z, \bar{z})}{\lambda^{j}(z, \bar{z})\lambda^{\bar{j}}(z, \bar{z})} \quad (4.5)$$

Here it is crucial to point out that the fields $\Phi^{j\bar{j}}$ behaves like scalar quantities in the $(Z, \bar{Z})$ coordinates whereas the old matter fields $\phi^{j\bar{j}}$ have a tensorial nature in the $(z, \bar{z})$ coordinates since the diffeomorphism action is performed in the background coordinates $(z, \bar{z})$.

Since the diffeomorphisms only move the coordinates $(Z, \bar{Z})$, the BRST transformations for the new fields have the form:

$$sZ = \gamma = \lambda(c + \mu\bar{c}) = \lambda C \quad , \quad s\bar{Z} = \bar{\gamma} = \bar{\lambda}(\bar{c} + \bar{\mu}c) = \bar{\lambda}\bar{C},$$

$$s\Phi^{j\bar{j}} = (\gamma \partial_{Z} + \bar{\gamma} \partial_{\bar{Z}})\Phi^{j\bar{j}},$$

$$s\gamma = s\bar{\gamma} = 0 \quad (4.6)$$

where the Cauchy-Riemann operators $\partial_{Z}$ and $\partial_{\bar{Z}}$ read in term of the $(z, \bar{z})$ coordinates

$$\partial_{Z} = \frac{\partial - \bar{\mu}\bar{\partial}}{\lambda(1 - \mu\bar{\mu})}, \quad \partial_{\bar{Z}} = \frac{\bar{\partial} - \mu\partial}{\bar{\lambda}(1 - \mu\bar{\mu})}. \quad (4.7)$$

The construction presented in the previous section can be accommodated for these new variables. Indeed one can construct a suitable basis by introducing the differential operators $(\tilde{\Delta}, \tilde{\bar{\Delta}})$ defined by

$$\tilde{\Delta} = \left\{ s, \frac{\partial}{\partial\gamma} \right\}, \quad \tilde{\bar{\Delta}} = \left\{ s, \frac{\partial}{\partial\bar{\gamma}} \right\} \quad (4.8)$$
and defining
\[ \Phi_{j,\bar{j}}^{p,q} = \tilde{\Delta}^p \bar{\Delta}^q \Phi_{j,\bar{j}} \] (4.9)

By using (4.6) one can see that the BRST transformations of \( \Phi_{j,\bar{j}}^{p,q} \) have the form
\[ s\Phi_{j,\bar{j}}^{p,q} = \gamma \Phi_{j,\bar{j}}^{p+1,q} + \bar{\gamma} \Phi_{j,\bar{j}}^{p,q+1} = (\gamma \tilde{\Delta} + \bar{\gamma} \bar{\Delta}) \Phi_{j,\bar{j}}^{p,q}. \] (4.10)
The fields \( \Phi_{j,\bar{j}}^{p,q} \) have, in fact, a very simple form in the \((Z, \bar{Z})\) coordinates. Indeed for an arbitrary function of \((Z, \bar{Z})\) one can write
\[ sF(Z, \bar{Z}) = (\gamma \frac{\partial}{\partial Z} + \bar{\gamma} \frac{\partial}{\partial \bar{Z}}) F(Z, \bar{Z}) \] (4.11)
which allows us to rewrite \( \Phi_{j,\bar{j}}^{p,q} \) in a simpler form. The BRST transformations of \( \gamma, \bar{\gamma} \) and \( \Phi_{j,\bar{j}}^{p,q} \) (4.6) and the identity (4.11) yield
\[ \tilde{\Delta}^2 \Phi_{j,\bar{j}} = \frac{\partial}{\partial \gamma} \left[ s(\partial_Z \Phi_{j,\bar{j}}) \right] = \partial_Z^2 \Phi_{j,\bar{j}}. \] (4.12)
The relation (4.12) can be easily generalized and we eventually get
\[ \Phi_{j,\bar{j}}^{p,q} = \partial_Z \partial_{\bar{Z}} \Phi_{j,\bar{j}}. \] (4.13)
Therefore the basis
\[ \mathcal{M} = \{ \Phi_{j,\bar{j}}^{p,q}, \gamma, \bar{\gamma} \} \quad (p, q = 0, 1, \cdots) \] (4.14)
represents a basis for the minimal subalgebra \( \mathcal{M} \) of the algebra \( \mathcal{A} \).

By using the BRST transformations of \( \gamma \) and \( \Phi_{j,\bar{j}}^{p,q} \) it is easy to see that the weights of both \( \gamma \) and \( \Phi_{j,\bar{j}}^{p,q} \), defined in the previous section, are zero i.e.
\[ L_0 \gamma = L_0 \bar{\gamma} = L_0 \Phi_{j,\bar{j}}^{p,q} = 0 \quad , \quad \bar{L}_0 \gamma = \bar{L}_0 \bar{\gamma} = \bar{L}_0 \Phi_{j,\bar{j}}^{p,q} = 0. \] (4.15)
The BRST cohomology group here have to be calculate only in the basis (4.14) and in this new basis, due to the nilpotency of \( \gamma \) and \( \bar{\gamma} \) we have only several of candidates for the solutions of equation (3.1). In fact we have only two possibilities:
\[ \omega^{(1)}_0 = c_1 \gamma \Phi_{j_{1},\bar{j}_{1}}^{p_{1},q_{1}} \cdots \Phi_{j_{n},\bar{j}_{n}}^{p_{n},q_{n}} + c_2 \bar{\gamma} \Phi_{k_{1},\bar{k}_{1}}^{r_{1},s_{1}} \cdots \Phi_{k_{m},\bar{k}_{m}}^{r_{m},s_{m}} = \gamma \Pi_1 + \bar{\gamma} \Pi_2 \] (4.16)
\[ \omega^{(2)}_0 = c_3 \gamma \bar{\gamma} \Phi_{j_{1},\bar{j}_{1}}^{p_{1},q_{1}} \cdots \Phi_{j_{n},\bar{j}_{n}}^{p_{n},q_{n}} = \gamma \bar{\gamma} \Pi_3. \] (4.17)
The possibility $\omega_0^{(1)}$ can be a solution of (3.1) only for some particular values of $\Pi_1$ and $\Pi_2$. Indeed if one use (4.6) and (4.10) then we can write

$$s\omega_0^{(1)} = -\gamma(\gamma\partial_Z + \bar{\gamma}\partial_{\bar{Z}})\Pi_1 - \bar{\gamma}(\gamma\partial_Z + \bar{\gamma}\partial_{\bar{Z}})\Pi_2 = -\gamma\bar{\gamma}[\partial_Z\Pi_1 - \partial_{\bar{Z}}\Pi_2] = 0.$$

Thus a solution of (3.1) in the basis considered in this section has the form (4.16) with

$$\partial_{\bar{Z}}\Pi_1 = \partial_Z\Pi_2.$$  

The solution of (4.18) has the form

$$\Pi_1 = \partial\Pi \quad \Pi_2 = \partial_Z\Pi$$

and $\omega_0^{(1)}$ is s-exact

$$\omega_0^{(1)} = (\gamma\partial_Z + \bar{\gamma}\partial_{\bar{Z}})\Pi = s\Pi.$$  

The candidate $\omega_0^{(2)}$ is a solution of (3.1) fact that can be seen easily

$$s\omega_0^{(3)} = \gamma\bar{\gamma}\left[(\gamma\tilde{\Delta} + \bar{\gamma}\tilde{\bar{\Delta}})\Pi = 0.$$

We can resume all these discussions by saying that in the differential algebra $A$, which includes the fields $\ln \lambda$, $\ln \bar{\lambda}$, the general solution of the equation (3.1) is given by

$$\omega_p^r(z, \bar{z}) = C(z, \bar{z})\bar{C}(z, \bar{z})\Pi_p^{r-2}(z, \bar{z})\delta_0^2 + s\omega_p^{r-1}$$

where the redefined ghost fields $C$ and $\bar{C}$ (see (2.13)) occur and $\omega_p^r$ is a r-form with the ghost p.

These results represent in fact the main results obtained by Bandelloni and Lazzarini in [19, 20] by using the spectral sequence method to calculate the local BRST cohomology modulo $d$. In the present paper we have obtained these crucial results just by using a very convenient basis and Sullivan’s theorem which has allowed us to work only within the minimal subalgebra $M$ (4.3). Starting from this result one can obtain the BRST cohomology with or without the fields $\ln \lambda$ and $\ln \bar{\lambda}$. The local anomalies as well as the vertex operators, which are used to build up the classical action cannot depend on the ”nonlocal” fields $\lambda$ or $\bar{\lambda}$. Therefore the anomalies and
the vertex operators are elements of the local BRST cohomology on the field \{ϕ, \bar{ϕ}, µ, \bar{µ}, C, \bar{C}\}. In the next section we shall give a proper account of this problem.

In the new fields \{γ, \bar{γ}\} the operator δ has a very simple action

\[ \delta γ = dZ, \quad \delta \bar{γ} = d\bar{Z} \] (4.21)

and we can obtain all possible vertex operators starting from (4.20) and applying \( \frac{1}{2} \delta^2 \). In this way we can obtain the tachyon, graviton and dilaton vertex operators see [26, 27].

The tachyon vertex operator is generated by

\[ γ\bar{γ}\left[ f(\Phi_{0,0}(Z, \bar{Z})) \right]_{Z, \bar{Z}} \] (4.22)

where \([f]_{Z, \bar{Z}}\) is the \((Z, \bar{Z})\)-componant in the ”big” indices of the function in the scalar field. This component \([f]_{z, \bar{z}}\) is related to the corresponding \((z, \bar{z})\)-component \([f]_{z, \bar{z}}\) in the ”little” indices by the relation

\[ [f]_{Z, \bar{Z}} = \frac{1}{\lambda \bar{λ}} [f]_{z, \bar{z}}. \] (4.23)

The tachyon vertex operator obtained from (4.22) has the usual form [26, 27]:

\[ (V_{z\bar{z}}(z, \bar{z}))_{\text{tachyon}} = \left[ f(Φ(Z, \bar{Z})) \right]_{Z, \bar{Z}} dZ \wedge d\bar{Z} = (1 - \mu \bar{µ}) \left[ f(\phi_{0,0}) \right]_{z, \bar{z}} dz \wedge d\bar{z}. \] (4.24)

The scalar function \( f \) can be determined from the conformal Ward identities governing the vertex inseration [26, 27].

The graviton and dilaton vertex operators can be obtained in the same way from the general form of the solution for the equation (3.1) given by (4.20) for different choices possible. Therefore we can chose the following solutions

\[ (ω₀)_{\text{grav}} = γ\bar{γ}\partial_Z Φ_{00}(Z, \bar{Z})\partial_{\bar{Z}} Φ_{00}(Z, \bar{Z}) g(Φ_{00}(Z, \bar{Z})) \]
\[ (ω₀)_{\text{dilaton}} = γ\bar{γ}\left( (\partial_{Z} Φ_{00}(Z, \bar{Z}))^2 h(Φ_{00}(Z, \bar{Z})) + c.c. \right) \] (4.25)

where the functions \( g \) and \( h \) could be fixed by the conformal Ward identities.
The corresponding vertex operators are

\[(V)_{grav} = (1 - \mu \bar{\mu})(D\phi_{00})(\bar{D}\phi_{00})g(\phi_{00})dz \wedge d\bar{z} \quad (4.26)\]

\[(V)_{dilaton} = (1 - \mu \bar{\mu})\left[\frac{\bar{\lambda}}{\lambda}(D\phi_{00})^2h(\phi_{00}) + c.c.\right]dz \wedge d\bar{z}. \quad (4.27)\]

The graviton vertex operator does not depend on \(\lambda\) and \(\bar{\lambda}\). This should lead to an independence of this vertex operator from the \(Z\) and \(\bar{Z}\) indices. On the other hand the dilaton vertex does depend on \(\lambda\) and \(\bar{\lambda}\) fact that implies the nonlocality of this vertex. In fact a necessary but not sufficient condition for the locality in \(\mu\) and \(\bar{\mu}\) is the disappearance of \(\lambda\) and \(\bar{\lambda}\) from the vertex operators \(V_{z,\bar{z}}(z, \bar{z})\) since \(\lambda\) (and \(\bar{\lambda}\)) is a nonlocal holomorphic function in \(\mu\) and \(\bar{\mu}\) (see [26, 27, 19]).

### 5 BRST cohomology without the variable \(\ln \lambda\)

In this section we shall calculate the BRST cohomology group for a BRST differential algebra \(\mathcal{A}\) without the variables \(\ln \lambda\) and \(\ln \bar{\lambda}\). The members of this cohomology are the ones which represent the physical quantities since all operators in a local quantum field theory must be local field i.e. a monomial in the basic fields and their derivatives. Besides they must be unintegrated function in the fields which means they should be differential forms with coefficients analytic functions on the local fields.

There are two ways to calculate this cohomology:

- To use the result of the previous section and to calculate all possible solutions of eq. (3.1), which can be even s-exact, and to select the ones without and dependence of \(\lambda\) and \(\bar{\lambda}\). This is the procedure used by Bandelloni and Lazzarini in [19, 20].

- To calculate the BRST cobomology in a reduced algebra \(\mathcal{A}'\), which contains all field but \(\{\lambda, \bar{\lambda}\}\).

We shall adopt the second point of view since in a particular case we can reduce the present problem to the one corresponding to the BRST cohomology of the 2D conformal gravity. The last
problem have been solved by Brandt, Troot and Van Proeyen [11] (see also [28] for a complete solution).

In the first part of this section we will accomodate the results from [11, 28] for the Beltrami parametrization with only one matter field $\phi_{0,0}$ and in the second part we give some examples for the members of the BRST cohomology in the presence of some different matter fields $\phi_{j,j}$.

In the case of only one matter field $\phi_{0,0}$ with the conformal weight $(0,0)$ the minimal subalgebra $\mathcal{M}$ is generated by the basis $\{\phi_{p,q}^{0,0}, C^n, \bar{C}^n\}$ and we have to calculate the solution of the eq. (3.1) in this basis. All nontrivial solutions must have the total weight $(0,0)$. Therefore we have to eliminate many possible solutions and we are left with only a few possibilities.

A basis which is more convenient for our purposes is one which contains only monomials with the total weight $(0,0)$. Due to the fact that all ghosts anticommute and only the ghosts $C = C^{-1}$ and $\bar{C}^{-1}$ have negative weights (-1,0) respectively (0,-1) we can calculate the members of BRST-cohomologies built up from a reduced basis with only eight elements:

$$
\begin{align*}
\psi_0^1 &= \phi_{0,0}^{0,0} \\
\psi_2^1 &= C^0, \quad \psi_3^1 = \bar{C}^0, \quad \psi_4^1 = C\phi_{0,0}^{1,0}, \quad \psi_5^1 = \bar{C}\phi_{0,0}^{0,1} \\
\psi_6^2 &= CC^1, \quad \psi_7^2 = \bar{C}\bar{C}^1, \quad \psi_8^2 = C\bar{C}\phi_{0,0}^{0,0}.
\end{align*}
$$

All these elements have the total weight $(0,0)$ and generate a minimal algebra since

$$
\begin{align*}
 s\psi_0^1 &= \psi_4^1 + s\psi_5^1 \\
 s\psi_2^1 &= \psi_6^2, \quad s\psi_3^1 = \psi_7^2, \quad s\psi_4^1 = \psi_8^2, \quad s\psi_5^1 = -\psi_8^2 \\
 s\psi_6^2 &= 0, \quad s\psi_7^2 = 0, \quad s\psi_8^2 = 0.
\end{align*}
$$

Now by writing down all possible monomial constructed from this basis and just by simple inspection we have found out all solutions of eq. (3.1) for different values of the ghost number.

- For ghost number $g=0$ and $g=1$ we have not found any solution;

- for $g=2$ we have found only one independent solution

$$
\omega_2^1 = \psi_4^1 \psi_5^1 F(\psi_1^0). 
$$

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where $F = F(\psi^0_1)$ is an arbitrary smooth function of the matter field.

- For $g=3$ there are four independent solutions
  \[
  \omega_2^3 = \psi_2^1 \psi_3^1 \psi_4^1 F(\psi^0_1), \quad \omega_3^3 = \psi_2^1 \psi_3^1 \psi_5^1 F(\psi^0_1) \\
  \omega_4^3 = \psi_2^1 \psi_6^2, \quad \omega_6^3 = \psi_3^1 \psi_7^2. \tag{5.4}
  \]
  The last two solutions coincide with the Guelfand and Fuks cocycles\[29\]

- For $g=4$ there are three independent solutions
  \[
  \omega_6^4 = \psi_2^1 \psi_3^1 \psi_7^2 F(\psi^0_1), \quad \omega_7^4 = \psi_3^1 \psi_5^2 \psi_6^1 F(\psi^0_1). \tag{5.5}
  \]

- For $g=5$ there are two independent solutions
  \[
  \omega_8^5 = \psi_2^1 \psi_3^1 \psi_6^1 \psi_7^2 F(\psi^0_1), \quad \omega_9^5 = \psi_2^1 \psi_3^1 \psi_4^1 \psi_7^2 F(\psi^0_1). \tag{5.6}
  \]

- For $g=6$ there is only one solution
  \[
  \omega_{10}^6 = \psi_2^1 \psi_3^1 \psi_6^2 \pi_7^2 F(\psi^0_1). \tag{5.7}
  \]

Now the members of the functional cohomology are, in fact, the solutions of the descent equations\[1.2\] and they can be obtained using the operator $\delta$ introduced in\[2.28\] by using\[2.32\]. The action of $\delta$ is simpler if we use the diffeomorphism ghosts $c$ and $\bar{c}$ related to $C$ and $\bar{C}$ by the relations
\[
C = c + \mu \bar{c}, \quad \bar{C} = \bar{c} + \bar{\mu} c. \tag{5.8}
\]
and the equation\[2.32\] can be rewritten as
\[
(s + d)\omega_0(c + dz, \bar{c} + d\bar{z}, \Psi) = 0 \tag{5.9}
\]
where $\Psi$ represent all the fields except $c$ and $\bar{c}$. Now if one use the solutions of\[3.1\] just given the equation\[5.3\] yields the results which are presented in the Table. In this Table we made
use of the following notations

\[ c^0 = \partial c + \bar{\mu} \partial \bar{c} \]
\[ c^0 = \bar{\partial} \bar{c} + \mu \partial c \]
\[ c^1 = \partial^2 c + 2 \partial \bar{\mu} \partial \bar{c} + \bar{\mu} \partial^2 \bar{c} \]
\[ c^1 = \bar{\partial}^2 \bar{c} + 2 \partial \mu \partial c + \mu \bar{\partial}^2 c \]

\[ y = 1 - \mu \bar{\mu} \]

(5.10)

\[ \bar{c}^1 = \bar{\partial}^2 \bar{c} + 2 \bar{\partial} \mu \partial c + \mu \bar{\partial}^2 c \]

(5.11)

| Ghost | Monomial | \[\delta^2(\text{Monomial})/dz \wedge d\bar{z}\] |
|-------|----------|---------------------------------|
| 0     | -        | -                               |
| 1     | -        | -                               |
| 2     | \(C \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \(2(1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
| 3     | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
|      | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
|      | \(CC \phi_{0,0}^{1,0} C\) | \(\partial C \bar{\partial}^2 \bar{\mu} - \bar{\partial}^2 C \partial \bar{\mu}\) |
|      | \( \bar{C} \bar{C} \phi_{0,0}^{1,0} \bar{C}\) | \(\partial \bar{C} \bar{\partial}^2 \mu - \bar{\partial}^2 \bar{C} \partial \mu\) |
| 4     | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
|      | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
|      | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
| 5     | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
|      | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |
| 6     | \(C \bar{C} \bar{C} \phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) | \((1 - y)\phi_{0,0}^{1,0} \phi_{0,0}^{1,1} F\) |

**TABLE**

From this table we can see that for a theory with only one scalar matter field there are only ten independent solutions of the descent equations (1.2) and for ghost number bigger then four we have no solution.
For \( g=0 \) we get only one solution of the form

\[
(1 - \mu \bar{\mu}) \phi_0^0 \phi_0^0 F(\phi_0,0) dz \wedge d\bar{z} = (1 - \mu \bar{\mu}) D\phi_0,0 \bar{D}\phi_0,0 F(\phi_0,0) dz \wedge d\bar{z}
\] (5.12)
with

\[
D = \frac{1}{1 - \mu \bar{\mu}} (\partial - \mu \bar{\partial}) \quad , \quad \bar{D} = \frac{1}{1 - \mu \bar{\mu}} (\bar{\partial} - \bar{\mu} \partial)
\]

and \( F(\phi_0,0) \) an analytic function of \( \phi_0,0 \). For \( F = 1 \) we obtain the classical action for the bosonic string in the Beltrami parametrization [2, 18, 29, 27]:

\[
S_{cl} = \int dzd\bar{z} \frac{1}{1 - \mu \bar{\mu}} [(1 + \mu \bar{\mu}) \partial X \cdot \bar{\partial} X - \mu \partial X \cdot \partial X - \bar{\mu} \partial X \cdot \bar{\partial} X]
\] (5.13)

with \( X = \phi_0,0 \).

For \( gh=1 \) we get four solutions, which can be rewritten as

\[
\mathcal{A}_2^1 = \int [a_1 \mu \partial^2 C + a_2 \bar{\mu} \bar{\partial}^2 \bar{C} + \\
+ \frac{1}{1 - \mu \bar{\mu}} (\partial \bar{c} + \bar{\mu} \partial \bar{c}) \nabla X \cdot \bar{\nabla} X F_2(X) + \\
+ \frac{1}{1 - \mu \bar{\mu}} (\bar{\partial} \bar{c} + \mu \partial \bar{c}) \nabla X \cdot \bar{\nabla} X F_3(X)] dz \wedge d\bar{z}
\] (5.14)

where

\[
\nabla = \partial - \mu \bar{\partial} \quad , \quad \bar{\nabla} = \bar{\partial} - \bar{\mu} \partial
\] (5.15)

and \( a_j \), \( j = 1,2 \), are constants and \( F_2(X) \), \( F_3(X) \) are arbitrary functions of \( X \). These solutions play a special role being the possible candidates for anomalies. Actually the matter dependent part of the anomaly (5.14) cannot be associated to a true diffeomorphism anomaly if one use as a classical action (5.13) since this action does not contain a self-interaction term in the matter fields. It follows that in the framework of the perturbation theory the numerical coefficients of the corresponding Feynman diagrams automatically vanishes i.e. in this case \( a_3 = a_4 = 0 \) and the unique breaking of the diffeomorphism invariance at the quantum level has the form

\[
\mathcal{A}_2^1 = \int [a_1 \mu \partial^2 C + a_2 \bar{\mu} \bar{\partial}^2 \bar{C}] dz \wedge d\bar{z}.
\] (5.16)
For $gh=2$ there are three independent solutions of the following form

$$
\mathcal{A}_2^2 = \int [(\partial \bar{c} + \mu \partial \bar{c})(\partial^2 \bar{c} + 2 \partial \bar{\mu} \partial \bar{c} + \mu \partial^2 \bar{c}) \nabla X F_4(X) \\
+ (\partial \bar{c} + \bar{\mu} \partial \bar{c})(\partial^2 \bar{c} + 2 \partial \bar{\mu} \partial \bar{c} + \bar{\mu} \partial^2 \bar{c}) \nabla X F_5(X) \\
+ (\partial \bar{c} + \mu \partial \bar{c})(\partial \bar{c} + \mu \partial \bar{c}) \nabla X \bar{\nabla} X F_6(X)]
$$

(5.17)

where $F_4(X), F_5(X), F_6(X)$ are arbitrary functions of $X$.

For $gh=3$ and $gh=4$ the solutions in number of two, respectively one, can be rewritten as

$$
\mathcal{A}_2^3 = \int [(\partial \bar{c} + \mu \partial \bar{c})(\partial \bar{c} + \mu \partial \bar{c})(\partial^2 \bar{c} + 2 \partial \bar{\mu} \partial \bar{c} + \mu \partial^2 \bar{c}) \nabla X \bar{\nabla} X F_7(X) \\
+ (\partial \bar{c} + \bar{\mu} \partial \bar{c})(\partial \bar{c} + \mu \partial \bar{c})(\partial \bar{c} + \mu \partial \bar{c}) \nabla X \bar{\nabla} X F_8(X)]
$$

(5.18)

and

$$
\mathcal{A}_2^4 = \int [\frac{1}{1 - \mu \bar{\mu}} (\partial \bar{c} + \mu \partial \bar{c})(\partial \bar{c} + \mu \partial \bar{c})(\partial^2 \bar{c} + 2 \partial \bar{\mu} \partial \bar{c} + \mu \partial^2 \bar{c})(\partial^2 \bar{c} + 2 \partial \bar{\mu} \partial \bar{c} + \mu \partial^2 \bar{c}) \nabla X \bar{\nabla} X F_9(X)]
$$

(5.19)

where $F_7(X), F_8(X), F_9(X)$ are arbitrary functions of $X$.

In the situation when there are more than one matter field $\{\phi_{j_1, \bar{j}_1}, \cdots \phi_{j_n, \bar{j}_n}\}$ with the conformal weights $(j_1, \bar{j}_1) \cdots (j_n, \bar{j}_n)$ the simplicity of the previous basis disappears since in this case there are an infinite number of possibilities to construct local functions with the total weight $(0,0)$.

For $gh=2$ we take the solutions of the (3.1) of the form

$$
\omega_0 = C \bar{C} \phi_{p_1, q_1}^{j_1, \bar{j}_1} \cdots \phi_{p_n, q_n}^{j_n, \bar{j}_n} = C \bar{C} \Pi.
$$

(5.20)

Since the total weight of (5.20) must be $(0,0)$ we have to impose the following conditions on the indices

$$
\begin{align*}
  j_1 + \cdots + j_n + p_1 + \cdots p_n &= 1 \\
  \bar{j}_1 + \cdots + \bar{j}_n + q_1 + \cdots q_n &= 1
\end{align*}
$$

(5.21)

The equations (3.1) and (3.11) yield

$$
\begin{align*}
  s \omega_0^2 &= c \bar{c}[c^0 - \bar{c}^0 + \sum_{k=-1}^1 (c^k L_k + \bar{c}^k \bar{L}_k)] \Pi = 0
\end{align*}
$$

(5.22)

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since $L_k$ and $\bar{L}_k$ are even derivatives. Taking into account that

$$L_k \phi_{j,j}^{p,q} = 0 \quad \text{for} \quad k > p \quad \text{or} \quad k = p, j = 0$$

$$\bar{L}_k \phi_{j,j}^{p,q} = 0 \quad \text{for} \quad k > q \quad \text{or} \quad k = q, \bar{j} = 0$$

(5.23)

equation (5.22) yields

$$p_l = 0 \quad \text{or} \quad p_l = 1, \ j_l = 0$$

$$q_l = 0 \quad \text{or} \quad q_l = 1, \ \bar{j}_l = 0$$

(5.24)

With these conditions at hand we can write down the most general form of solution of the equation (3.1)

$$\phi_{j_1,j_1}^{0,0} \cdots \phi_{j_n,j_n}^{0,0} \phi_{0,k_1}^{1,0} \cdots \phi_{0,k_m}^{1,0} \phi_{l_1,0}^{0,1} \cdots \phi_{l_s,0}^{0,1}\theta(\phi_{0,0}^{1,1})$$

(5.25)

with

$$j_1 + \cdots j_n + \bar{j}_1 + \cdots \bar{j}_n + k_1 + 1 + \cdots k_m + 1 + l_1 + 1 + \cdots l_s + 1 + 2q = 1$$

(5.26)

### 6 Conclusions

We have calculated the complete BRST cohomology in the space of the local functions for the local field theories on a Riemann surface which contain the conformal matter field coupled with a complex structure parametrized by a Beltrami differential without any reference to metrics.

For the theory with only one scalar matter field with $\phi_{0,0}$ and without the integrating factor $\lambda$, $\bar{\lambda}$, we have calculated all members of the BRST cohomology. For the theory with several matter fields we have found out only a limited number of $H(s)$, but they include all cases considered by Bandelloni and Lazzarini [13, 21].

The simplest case is the one where we have introduced the integrating factors $\lambda$ and $\bar{\lambda}$. Here the BRST cohomology contains only terms of the form

$$C\bar{C}\phi_{j_1,j_1}^{p_1,q_1} \cdots \phi_{j_n,j_n}^{p_n,q_n}$$

(6.1)
i.e., $H^g(s) = 0$ if the ghost number $g \neq 2$. However, it is worth reminding the reader that the form (6.1) has been obtained only from geometrical point of view. If we want to impose the *locality* assumption for our model then the factors $\lambda, \bar{\lambda}$ should disappear in (6.1) since $\lambda$ is a *nonlocal* holomorphic function. Thus the locality assumption bounds considerably the form of the solutions of (3.1). We could assure the locality from the very beginning if we start to calculate the BRST cohomology without the fields $\lambda, \bar{\lambda}$.

The technique presented in this paper can be used to study the BRST cohomology for other models, as $W_3$-gravity \([16]\) or the superstring in the super-Beltrami parametrization \([15]\). Also it has been used to calculate the BRST cohomology of the Slavnov operator \([11]\) or the BRST-antibracket cohomology for 2D gravity.

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