THE $L^1$ NORMS OF DE LA VALLÉE POUSSEN KERNELS

HARSH MEHTA

Abstract. Charles de la Vallée Poussin defined two different kernels that bear his name. This paper considers the ones are a linear combinations of two Fejér kernels, which are known as the delayed means. We show that the $L^1$ norms are constant in families of delayed means, and determine the exact value of the $L^1$ norm for some of them.

1. Introduction

This paper studies properties of certain summability kernels for Fourier series, the de la Vallée Poussin kernels defined below. Let $f(x)$ be a periodic function on $\mathbb{R}$ of period 1, of finite $L^1$-norm in $[0,1]$, with Fourier series denoted

$$ f(x) \sim \sum_{k \in \mathbb{Z}} \hat{f}(k)e(kx), $$

where $e(x) := e^{2\pi ix}$. Define the partial sum of its Fourier series

$$ S_n(f,x) := \sum_{k=-n}^{n} \hat{f}(k)e(kx) \quad (1.1) $$

The Fejér mean (of period 1) with parameter $n$ is

$$ \sigma_n(f,x) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f,x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k)e(kx). \quad (1.2) $$

This mean is given as singular integral of convolution type

$$ \sigma_n(f,x) = \int_0^1 f(x-u)K_n(u)du, $$

where the Fejér kernel $K_n(x)$, rescaled for functions of period 1, is

$$ K_n(x) := \Delta_{n+1}(x) = \frac{1}{n+1} \left(\frac{\sin \pi(n+1)x}{\sin \pi x}\right)^2. \quad (1.3) $$

In 1918 de la Vallée Poussin [17] introduced the delayed means (also called de la Vallée Poussin sums [5, 10]), as

$$ \sigma_{n,p}(f,x) := \frac{1}{p} \sum_{k=-n}^{n+p-1} S_n(f,x) = \frac{n+p}{p} \sigma_{n+p-1}(f,x) - \frac{n}{p} \sigma_{n-1}(f,x). \quad (1.4) $$

Here we follow the notation for these means used in Zygmund [21, Chap. III.1, (1.30)], after rescaling the periodicity interval from $[0,2\pi]$ to $[0,1]$. The means above are given

Date: November 2, 2013.

2010 Mathematics Subject Classification. Primary: 42A05.

Key words and phrases. de la Vallée Poussin kernel.
by the convolution integral

\[ \sigma_{n,p}(f, x) = \int_0^1 f(x - u)K_{n,p}(u)du, \]  

(1.5)
in which \( K_{n,p}(x) \) is the de la Vallée Poussin kernel function \[17] with parameters \((n, p)\). This is found to be

\[ K_{n,p}(x) = \frac{n + p}{p} \Delta_{n+p}(t) - \frac{n}{p} \Delta_n(t) \quad (1.6) \]

\[ = \frac{1}{p} \left( \frac{(\sin \pi(n + p)x)^2 - (\sin \pi nx)^2}{(\sin \pi x)^2} \right). \quad (1.7) \]

Taking \( n = 0 \) and \( p = n \) we obtain \( K_{0,n}(x) = \Delta_n(x) = K_{n-1}(x) \), the Fejér kernel with a shifted parameter. All of these kernels have

\[ \int_0^1 K_{n,p}(x)dx = 1. \quad (1.8) \]

For each parameter set \((n, p)\) there is a family \( F_{n,p} := \{K_{nN,pN}(x) : N \geq 1\} \) of kernel functions indexed by the positive integer parameter \( N \geq 1 \). This family forms a summation kernel in the sense of Walker \[20\] (8.1), (8.23)], or a finite \( \theta \)-factor in the sense of Butzer and Nessel \[2\], Sec. 1.2.5]. These authors define a general finite \( \theta \)-factor to be an infinite family of data \( \{\theta_N(j) : N \geq 1, j \in \mathbb{Z}\} \) with

\[ S_{\theta}(f, x) = \sum_{j=-m(N)}^{m(N)} \theta_N(j)\hat{f}(j)e(jx) \]

and with a function \( m(N) \to \infty \) as \( N \to \infty \). For the de la Vallée Poussin kernel with parameters \((n, p)\) the associated \( \theta \)-factor takes \( m(N) = N \) and sets

\[ \theta_N(j) := v_{n,p}(\frac{j}{N}), \]

where \( v_{n,p}(x) \) being a compactly supported piecewise linear function on \( \mathbb{R} \) given by

\[ v_{n,p}(u) := \begin{cases} 
1 & \text{if } |u| \leq n, \\
\frac{n+p-|u|}{p} & \text{if } n \leq |u| \leq n + p, \\
0 & \text{if } |u| \geq n + p
\end{cases} \]

Special cases of function are pictured in Figure 1 and Figure 2.

For later convenience we make a linear change of variables in the parameters, setting \( N = \gcd(n, n + p) \) and writing \( Nr = n \) and \( Ns = n + p \), so that \( 0 \leq r < s \) and \( r \) and \( s \) are relatively prime. (Thus \( n = Nr \) and \( p = N(s - r) \).) In the new parameters the de la Vallée Poussin kernels \( (1.6) \) become

\[ V_{rN,sN}(x) := K_{rN,(s-r)N}(x) = \frac{s\Delta_sN(x) - r\Delta_rN(s)}{s-r}. \quad (1.9) \]

In consequence

\[ V_{rN,sN}(x) = \frac{(\sin sN \pi x)^2 - (\sin rN \pi x)^2}{(s-r)N(\sin \pi x)^2} \]

\[ = \sum_{n=-sN+1}^{sN-1} v_{r,s-r(n/N)}e(nx) \quad (1.11) \]
Figure 1. \( v_{n,p}(x) \) with \((n, p) = (1, 1)\), so \( N = 1 \) and \((r, s) = (1, 2)\).

Figure 2. \( v_{n,p}(x) \) with \((n, p) = (2, 2)\), so \( N = 2 \) and \((r, s) = (1, 2)\).

where for real \( u \)

\[
v_{r,s-r}(u) = \begin{cases} 
1 & (|u| \leq r), \\
\frac{s-|u|}{s-r} & (r \leq |u| \leq s), \\
0 & (|u| > s).
\end{cases} \tag{1.12}
\]
A particularly well known case of his summability kernel [18, Sect. 29] occurs for \( r = 1, s = 2 \), and is

\[
K_{N,N}(x) = V_{N,2N}(x) = 2\Delta_{2N}(x) - \Delta_N(x)
\]

\[
= \frac{(\sin 2N\pi x)^2 - (\sin N\pi x)^2}{N(\sin \pi x)^2}
\]

\[
= \sum_{|n|\leq N} e(nx) + \sum_{N<|n|<2N} (2 - |n|/N)e(nx).
\]

The shown for \( N = 1 \) in Figure 3 and for \( N = 2 \) in Figure 4.

Figure 3. de la Vallée Poussin kernel \( K_{1,1}(x) = V_{1,2}(x) \) on \([0, 1]\).

This paper studies the \( L^1 \)-norm of \( V_{r,s,N}(x) \). From (1.8) and (1.9) we have the easy bounds

\[
1 \leq \|V_{r,s,N}(x)\|_{L^1(T)} = \int_0^1 |V_{r,s,N}(x)|dx \leq \frac{s+r}{s-r}.
\]

Our main result is the observation that the \( L^1 \)-norms of the kernels in these families are independent of the kernel family parameter \( N \).

**Theorem 1.1.** Let \( r \) and \( s \) be fixed integers with \( 0 \leq r < s \) and \((r,s) = 1\). Then all members of the kernel family \( \mathcal{G}_{r,s} = \{V_{r,s,N} : N \geq 1\} \) have the same \( L^1 \)-norm. That is,

\[
\|V_{r,s,N}\|_{L^1(T)} = \|V_{r,s}\|_{L^1(T)} = \int_0^1 |V_{r,s}(x)|dx.
\]

This result is surprising because of the oscillatory nature of these functions, which have increasing numbers of sign changes as \( N \) increases, visible in Figure 3 and Figure 4; nevertheless both functions have the same \( L^1 \)-norm on \([0, 1]\) by Theorem 1.1.

For individual values of \( r \) and \( s \), on taking \( N = 1 \), the value can in principle be explicitly determined. For the special case \((r, s) = (n, n+1)\) we observe that the kernel
Figure 4. de la Vallée Poussin kernel $K_{2,2}(x) = V_{2,4}(x)$ on $[0, 1]$.

$V_{n,n+1}(x)$ coincides with the Dirichlet kernel $D_n(x)$ and therefore obtain the following well known answer.

**Theorem 1.2.** For $n \geq 1$ we have

$$V_{n,n+1}(x) = D_n(x) = \frac{\sin(2n + 1)x}{\sin \pi x}.$$ 

Here $D_n(x)$ is the Dirichlet kernel for period 1 functions. In particular,

$$\|V_{n,n+1}\|_{L^1(T)} = \|D_n\|_{L^1(T)} = L_n,$$

where $L_n$ is the $n$-th Lebesgue constant.

The Lebesgue constants $L_n$ solve the extremal problem of giving the supremum of the $n$-th partial sum $|S_n(f, x)|$ of the Fourier series of $f(x)$ where $f(x)$ is any periodic continuous function of period 1 having $|f(x)| \leq 1$ everywhere (Lebesgue [8], see also Timan [15 Sects. 4.5, 8.2], Zygmund [21 Chap. II, Sect. 12].) They are given by

$$L_n = \frac{2}{\pi} \int_0^{2\pi} \left| \frac{\sin((n+1)/2)\theta}{2\sin(1/2)\theta} \right| d\theta.$$ 

(1.16)

It is well known that

$$L_1 = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} = 1.435991 \ldots$$

and

$$L_2 = \frac{1}{5} + \frac{\sqrt{10 - 2\sqrt{5}}}{4\pi} \left(1 + 3\sqrt{5}\right) = 1.642188 \ldots$$

Combining these two results we obtain for the original kernel of de la Vallée Poussin [17 p. 801 top] and [18 Sect. 26], the following answer.
Corollary 1.3. Let $V_{N,2N}(x) = 2\Delta_{2N}(x) - \Delta_N(x)$ be the de la Vallée Poussin kernel. Then
\begin{equation}
\int_0^1 |V_{N,2N}(x)| \, dx = L_1 = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} = 1.43599112 \ldots \tag{1.17}
\end{equation}
for all $N$.

We remark that de la Vallée Poussin introduced the delayed to study pointwise approximation to Fourier series in his 1919 book on approximation \cite{15}. Further work includes Efimov \cite{5}, \cite{6}, Teljakovski \cite{16}, Dahmen \cite{4}, Stechkin \cite{13}, and Serdyuk et al. \cite{11}, \cite{10}, \cite{12}.

2. Proof of Theorem 1.1

Let $r$ and $s$ be integers with $0 \leq r < s$ and $(r, s) = 1$. The parameters $r, s$ will remain fixed throughout this section, we simplify the notation by omitting mention of $r$ and $s$ when naming functions. In particular, $V_N$ means $V_{r,s,N}$.

From (1.10) one obtains using trigonometric addition formulas\footnote{The addition formulas yield $(\sin(a+b)x)(\sin(a-b)x) = (\sin ax)^2(\cos bx)^2 - (\sin bx)^2(\cos ax)^2$. The right side then equals $(\sin ax)^2 - (\sin bx)^2$, after adding $(\sin ax)^2(\sin bx)^2$ to both the opposing terms.} that
\begin{equation}
V_N(x) = \frac{(\sin \pi(s+r)N)(\sin \pi(s-r)N)x}{(s-r)N(\sin \pi x)^2}. \tag{2.1}
\end{equation}
Hence $V_N$ has zeros at points of the form \{$(a/((s+r)N)) : 0 \leq a \leq (s+r)N-1$\} and also at points of the form \{$(b/((s-r)N)) : 0 \leq b \leq (s-r)N-1$\}, excluding points where $(s+r)N|a$ or $(s-r)N|b$. Thus there are $(s+r)N-1$ zeros of the first kind and $(s-r)N-1$ zeros of the second kind, making a total of $2sN-2$ zeros, counting multiplicity.

Since $V_N$ is a trigonometric polynomial of degree $sN-1$, we know that it could have at most $2sN-2$ zeros. By taking $a = (s+r)n$ and $b = (s-r)n$ We can see that $V_N$ has a double zero at $n/N$, for $0 < n < N$, taking $a = (s+r)n$ and $b = (s-r)n$ above.

If $r$ and $s$ are of opposite parity, then gcd$(s+r, s-r) = 1$ implies there are no other double zeros. If however $r$ and $s$ are both odd, then gcd$(s+r, s-r) = 2$, and by taking $a = n(s+r)/2$, $b = n(s-r)/2$ we find that $V_N$ has double zeros at $n/(2N)$ for $0 < n < 2N$.
As an example, in Figure 4, we have \( r + s = 3 \), and \( V_{2,4}(x) \) has 5 zeros of the first kind and 1 zero of the second kind; the five zeros of first kind are located at \( x = \frac{j}{6} \), \( 1 \leq j \leq 5 \), and the zero of second kind at \( x = \frac{1}{2} \), so there is a double zero at \( x = \frac{1}{2} \).

To compute \( \int_0^1 |V_N(x)| \, dx \), we break the interval \([0, 1]\) into subintervals running from one simple zero of \( V_N \) to the next and then summing the area enclosed between each interval. In order to know what sign to attach to each interval (to obtain \( |V_N(x)| \)) we need to consider whether \( V_N \) is increasing or decreasing at a simple zero.

**Proof of Theorem 1.** We define the function \( F_N(x) \) by the equality \( V_N(x) = F_N(x) \sin(\pi(s + r)N x) \), so

\[
F_N(x) = \frac{\sin(\pi(s - r)N x)}{(s - r)N (\sin \pi x)^2}
\]

using (2.1). Then for integer \( a \) with \( \frac{a}{s + r}N \notin \mathbb{Z} \), we have

\[
V_N\left( \frac{a}{(s + r)N} \right) = (-1)^a F_N\left( \frac{a}{(s + r)N} \right) \pi(s + r) N.
\]

Hence

\[
\text{sgn} \, V_N\left( \frac{a}{(s + r)N} \right) = (-1)^a \text{sgn} \left( \sin \frac{\pi(s - r)a}{s + r} \right) \quad (2.2)
\]

\[
= (-1)^a \text{sgn} \left( \sin \left( \pi a - \frac{2\pi ar}{s + r} \right) \right) = -\text{sgn} \left( \sin \frac{2\pi ar}{s + r} \right).
\]

Accordingly, we set

\[
\varepsilon(a) := \text{sgn} \left( \sin \frac{2\pi ar}{s + r} \right). \quad (2.3)
\]

We note that

\[
\varepsilon(a + s + r) = \text{sgn} \left( \sin \frac{2\pi(a + s + r)r}{s + r} \right) = \text{sgn} \left( \sin \frac{2\pi ar}{s + r} \right) = \varepsilon(a).
\]

Hence the values \( \varepsilon(a) \) are periodic with period \( s + r \).

We define the function \( G_N(x) \) so that \( V_N(x) = G_N(x) \sin(\pi(s - r)N x) \), so

\[
G_N(x) = \frac{\sin(\pi(s + r)N x))}{(s - r)N (\sin \pi x)^2},
\]

using (2.1). Then for integer \( b \) with \( \frac{b}{(s - r)N} \notin \mathbb{Z} \), one has

\[
V_N\left( \frac{b}{(s - r)N} \right) = (-1)^b G_N\left( \frac{b}{(s - r)N} \right) \pi(s - r) N.
\]

Hence

\[
\text{sgn} \, V_N\left( \frac{b}{(s - r)N} \right) = (-1)^b \text{sgn} \left( \sin \frac{\pi(s + r)b}{s - r} \right) \quad (2.4)
\]

\[
= (-1)^b \text{sgn} \left( \sin \left( \pi b + \frac{2\pi br}{s - r} \right) \right) = \text{sgn} \left( \sin \frac{2\pi br}{s - r} \right).
\]

Accordingly, we set

\[
\delta(b) := -\text{sgn} \left( \sin \frac{2\pi br}{s - r} \right). \quad (2.5)
\]

We note that

\[
\delta(b + s - r) = -\text{sgn} \left( \sin \frac{2\pi(b + s - r)r}{s - r} \right) = -\text{sgn} \left( \sin \frac{2\pi br}{s - r} \right) = \delta(b).
\]

Hence the values \( \delta(b) \) are periodic with period \( s - r \).
Next set
\[ W_N(x) := x + \sum_{n \neq 0} \frac{v_r s-r(n/N)}{2\pi in} e(nx), \]
and note that \( W'_N(x) = V_N(x) \) by (1.11). Suppose that \( x_{k-1}, x_k, x_{k+1} \) are three consecutive simple zeros of \( V_N \) in \((0, 1)\), and suppose that \( x_k = a/((s+r)N) \). If \( V'_N(x_k) < 0 \), then \( V_N(x) > 0 \) for \( x_{k-1} < x < x_k \) and \( V_N(x) < 0 \) for \( x_k < x < x_{k+1} \). These intervals contribute to the integral an amount
\[ (W_N(x_k) - W_N(x_{k-1})) - (W_N(x_{k+1}) - W_N(x_k)) = -W_N(x_{k-1}) + 2W_N(x_k) - W_N(x_{k+1}). \]
In this situation \( \varepsilon(a) = 1 \), so the point \( x_k \) contributes \( 2\varepsilon(a)W_N(x_k) \). If \( V'_N(x_k) > 0 \), then all signs are reversed, and the contribution of \( x_k \) is still \( 2\varepsilon(a)W_N(x_k) \). Now suppose that \( x_k \) is a double zero, and that \( V_N(x) > 0 \) in the two intervals. Then these intervals contribute
\[ (W_N(x_k) - W_N(x_{k-1})) + (W_N(x_{k+1}) - W_N(x_k)) = W_N(x_{k+1}) - W_N(x_{k-1}). \]
In this case, \( x_k \) makes no contribution, but \( \varepsilon(a) = 0 \), so the contribution is still \( 2\varepsilon(a)W_N(x_k) \).

The interval \([0, 1/((s+r)N)]\) contributes \( W_N(1/((s+r)N)) - W_N(0) \). The first term here is half of the contribution made by the point \( 1/((s+r)N) \), since \( \varepsilon(1) = 1 \). The contribution made by the interval \([1 - 1/((s+r)N), 1]\) is \( W_N(1) - W_N(1-1/((s+r)N)) \). The latter term is half the contribution made by the point \( 1 - 1/((s+r)N) \), since \( \varepsilon((s+r)N - 1) = -1 \). We note that \( W_N(1) - W_N(0) = 1 \). Hence we conclude that
\[ \int_0^1 |V_N(x)| \, dx = 1 + 2 \sum_{a=1}^{(s+r)N} \varepsilon(a)W_N \left( \frac{a}{(s+r)N} \right) \]
\[ + 2 \sum_{b=1}^{(s-r)N} \delta(b)W_N \left( \frac{b}{(s-r)N} \right). \]
Here the terms \( a = (r+s)N \) and \( b = (s-r)N \) ought not to be included in the above, since \( V_N(1) \neq 0 \). However, \( \varepsilon((s+r)N) = 0 \) and \( \delta((s-r)N) = 0 \), so no harm is done.

We write \( W_N(x) = x + X_N(x) \) and evaluate the contributions of the two terms separately to the right side of (2.7). The contribution of the linear term \( x \) to the sum (2.7) is
\[ 2 \sum_{a=1}^{(s+r)N} \varepsilon(a) \frac{a}{(s+r)N} + 2 \sum_{b=1}^{(s-r)N} \delta(b) \frac{b}{(s-r)N}. \]
Since the \( \varepsilon(a) \) are periodic with period \( s+r \), and the \( \delta(b) \) are periodic with period \( s-r \), the above is
\[ = 2 \sum_{n=0}^{N-1} \left( \sum_{a=1}^{s+r} \varepsilon(a) \left( \frac{a}{(s+r)N} + \frac{n}{N} \right) + \sum_{b=1}^{s-r} \delta(b) \left( \frac{b}{(s-r)N} + \frac{n}{N} \right) \right). \]
On reversing the order of the double sums, we see that this is
\[ = 2 \sum_{a=1}^{s+r} \varepsilon(a) \left( \frac{a}{s+r} + N - 1 \right) + 2 \sum_{b=1}^{s-r} \delta(b) \left( \frac{b}{s-r} + N - 1 \right). \]
We write \( s + r > 0 \), in the interval \([0, 1]\) we pass from positive values to negative the same number of times that we pass from negative values to positive. That is, 
\[
0 = \sum_{a=1}^{(s+r)N} \varepsilon(a) + \sum_{b=1}^{(s-r)N} \delta(b) = N \sum_{a=1}^{s+r} \varepsilon(a) + N \sum_{b=1}^{s-r} \delta(b).
\]

Hence the expression (2.8) is
\[
= \frac{2}{s + r} \sum_{a=1}^{s+r} \varepsilon(a)a + \frac{2}{s - r} \sum_{b=1}^{s-r} \delta(b)b.
\]  
(2.9)

The contribution of \( X_N(x) \) to the right side of (2.7) is
\[
2 \sum_{a=1}^{(s+r)N} \varepsilon(a)X_N \left( \frac{a}{s + r} \right) + 2 \sum_{b=1}^{(s-r)N} \delta(b)X_N \left( \frac{b}{s - r} \right).
\]  
(2.10)

Here the sum over \( a \) is
\[
\sum_{n \neq 0} v_{r, s-r}(n/N) \frac{(s+r)N}{\pi in} \sum_{a=1}^{s+r} \varepsilon(a)e \left( \frac{an}{s + r} \right).
\]  
(2.11)

where \( v_{r, s-r}(n/N) \) are given in (1.12). Since the \( \varepsilon(a) \) have period \( s + r \), we know by the theory of the Discrete Fourier Transform that there exist numbers \( \hat{\varepsilon}(k) \) such that
\[
\varepsilon(a) = \sum_{k=1}^{s+r} \hat{\varepsilon}(k)e \left( \frac{ak}{s + r} \right)
\]  
(2.12)

holds for all integer \( a \). Hence the expression (2.11) is
\[
= \sum_{n \neq 0} v_{r, s-r}(n/N) \frac{(s+r)N}{\pi in} \sum_{k=1}^{s+r} \hat{\varepsilon}(k) \sum_{a=1}^{s+r} e \left( \frac{a(n + kN)}{(s + r)N} \right).
\]

Here the innermost sum is \((s + r)N\) if \( n \equiv -kN \pmod{(s + r)N}\), and is 0 otherwise. We write \( n = -kN + m(r + s)N \). Then the above is
\[
= (s + r) \sum_{k=1}^{s+r} \hat{\varepsilon}(k) \sum_{m \in \mathbb{Z}} \frac{v_{r, s-r}(-k + m(s + r))}{\pi i(-k + m(s + r))}.
\]  
(2.13)

We note that if \( 1 \leq k \leq s + r \), then \( v(-k + m(s + r)) = 0 \) if \( m > 1 \) or if \( m < 0 \). Thus the sum over \( m \) can be restricted to just \( m = 0, 1 \). However, the main point of the above is that it is independent of \( N \).

The sum over \( b \) in (2.10) is
\[
\sum_{n \neq 0} v_{r, s-r}(n/N) \frac{(s-r)N}{\pi in} \sum_{b=1}^{s-r} \delta(b)e \left( \frac{bn}{s - r} \right).
\]  
(2.14)

The \( \delta(b) \) have period \( s - r \), so let numbers \( \hat{\delta}(n) \) be determined so that
\[
\delta(b) = \sum_{k=1}^{s-r} \hat{\delta}(k)e \left( \frac{kb}{s - r} \right)
\]  
(2.15)
for all $b$. Hence the expression (2.14) is
\[
= \sum_{n \neq 0} \frac{v_{r,s-r}(n/N)}{\pi in} \sum_{k=1}^{s-r} \hat{\delta}(k) \sum_{b=1}^{(s-r)N} e\left( \frac{b(n + kN)}{(s-r)N} \right).
\]
The innermost sum is $(s-r)N$ if $n \equiv -kN \pmod{(s-r)N}$, and is 0 otherwise. We write $n = -kN + m(s-r)N$. Then the above is
\[
(s-r) \sum_{k=1}^{s-r} \hat{\delta}(k) \sum_{m \in \mathbb{Z}} \frac{v_{r,s-r}(-k + m(s-r))}{\pi i(-k + m(s-r))}.
\]
(2.16)

This formula and (2.13) serve to evaluate the expression (2.10). On combining this evaluation with (2.9) in (2.7), we conclude that
\[
\int_0^1 |V_N(x)| \, dx = 1 + 2 \sum_{a=1}^{s+r} \varepsilon(a) a + 2 \sum_{b=1}^{s-r} \delta(b) b
\]
\[
+ (s-r) \sum_{k=1}^{s+r} \hat{\varepsilon}(k) \sum_{m \in \mathbb{Z}} \frac{v_{r,s-r}(-k + m(s+r))}{\pi i(-k + m(s+r))}
\]
\[
+ (s-r) \sum_{k=1}^{s-r} \hat{\delta}(k) \sum_{m \in \mathbb{Z}} \frac{v_{r,s-r}(-k + m(s-r))}{\pi i(-k + m(s-r))}.
\]
\]
(2.17)

Since this value is independent of $N$, the proof is complete.

3. Proofs of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Recall from (2.1) that for $(r,s) = 1$,
\[
V_{r,s}(x) = \frac{(\sin \pi(s+r)x)(\sin \pi(s-r)x)}{(s-r)(\sin \pi x)^2}.
\]
(3.1)

In the special case $s-r = 1$, which corresponds to $(r,s) = (n,n+1)$ we obtain the simplification
\[
V_{n,n+1}(x) = \frac{(\sin \pi(2n+1)x)(\sin \pi x)}{\sin \pi x} = \frac{\sin \pi(2n+1)x}{\sin \pi x} = D_n(x)
\]

The right hand side is exactly the Dirichlet kernel $D_n(x)$, rescaled to the interval $[0,1]$. By definition the Lebesgue constant
\[
L_n = \|D_n(x)\|_{L^1(T)} = \int_0^1 |D_n(x)| \, dx,
\]
which on rescaling to the usual interval $[0,2\pi]$ recovers the usual definition (1.16).

We give explicit computations yielding Corollary 1.3.

Proof of Corollary 1.3. The result follows on combining Theorems 1.1 and 1.2. For the explicit value we have
\[
V_{1,2}(x) = 1 + 2 \cos 2\pi x = \frac{\sin 3\pi x}{\sin \pi x}.
\]
by (1.15) and (2.1). Hence
\[
\int_0^1 |V_{1,2}(x)| \, dx = \left[ x + \frac{\sin 2\pi x}{\pi} \right]_{0}^{1/3} - \left[ x + \frac{\sin 2\pi x}{\pi} \right]_{1/3}^{2/3} + \left[ x + \frac{\sin 2\pi x}{\pi} \right]_{2/3}^1
\]
\[
= \frac{1}{3} + \frac{2\sqrt{3}}{\pi}.
\]
Alternatively, one can argue from (2.17). We find that \( \varepsilon(1) = 1, \varepsilon(2) = -1, \varepsilon(3) = 0, \) \( \hat{\varepsilon}(1) = -i/\sqrt{3}, \hat{\varepsilon}(2) = i/\sqrt{3}, \hat{\varepsilon}(3) = 0, \) and \( \delta(1) = \delta(1) = 0. \) The result is the same. \( \square \)

4. Concluding Remarks

We showed that the \( L^1 \)-norm of \( V_{rN,sN}(x) = (\sin \pi (s + r)Nx)(\sin \pi (s - r)Nx) \) \( (s - r)N \sin^2 \pi x \)
is independent of \( N \). If we let \( A^+_{r,s,N} \) resp. \( A^-_{r,s,N} \) denote the positive and negative areas of the graph then we have shown
\[
A^+_{r,s,N} - A^-_{r,s,N} = \|V_{r,s}\|_{L^1(T)}
\]
is independent of \( N \). Since \( A^+_{r,s,N} + A^-_{r,s,N} = 1 \) we have that both these areas are independent of \( N \), with
\[
A^+_{r,s,N} = \frac{1}{2} \left( 1 + \|V_{r,s}\|_{L^1(T)} \right).
\]
The effect of increasing \( N \) is does not change the area, but shifts its location. As \( N \) increases most area (both positive and negative) is concentrated near integer values of \( x \). One can show that
\[
|V_{rN,sN}(x)| \leq \frac{4}{\sqrt{N}} \text{ for } \frac{1}{\sqrt{N}} \leq x \leq 1 - \frac{1}{\sqrt{N}}.
\]

5. Acknowledgments

The author thanks H. L. Montgomery for suggesting the project of improving the bounds for \( L^1 \) norms of de la Vallee Poussin kernels as part of an REU program at the University of Michigan in summer 2012. He thanks H. L. Montgomery and J. C. Lagarias for references and for editorial assistance with exposition. The author thanks P. Nevai for helpful comments and references. P. Nevai observed that Theorem 1.1 can be deduced from results which are contained in his unpublished 1969 manuscript (Nevai [9, Lemma 3]).

References

[1] S. N. Bernstein, *Sur l’ordre de la meilleure approximation des fonctions continues par les polynomes de degré donne*, Académie Royale de Belgique, Cl. Sci. Mém. Coll. in quarto, Ser. 2 4 (1922), No. 1, 104 pages. [This paper submitted in 1912 to Concors Annuel.]
[2] P. L. Butzer and R. Nessel, *Fourier Analysis and Approximation. Volume 1. One-Dimensional Theory*, Academic Press: New York and London 1971.
[3] P.L. Butzer and R. Nessel, *Work of de La Vallée Poussin in Approximation Theory and its Impact*, In: Collected Works= Oeuvres scientifiques, Volume 2, pp. 375-414.
[4] W. Dahmen, *Best approximation and de la Vallée Poussin sums,(Russian) Mat. Zametki* 23 (1978), no. 5, 671–683
[5] A. V. Efimov, *Approximation of periodic functions by de la Vallée Poussin sums, (Russian)* Izv. Akad. Nauk SSSR . Ser Mat. 23 (1959), 737–770.
[6] A. V. Efimov, *Approximation of periodic functions by de la Vallée Poussin sums II, (Russian)* Izv. Akad. Nauk SSSR . Ser Mat. 24 (1960), 431–468.
[7] L. Fejér, *Untersuchungen übe Fouriersche reihen*, Math. Annalen 58 (1904), 501–569.
[8] H. Lebesgue, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, Bull. Soc. Math. France. [Also: *Oeuvres Scientifiques*, vol. 3, pp. 363–389, L’Enseign. Math. 1972.]

[9] P. Nevai, *A Vallée Poussin-Féle Operátorról, valamint a trigonomerikus polinomiális operációkról*, (Hungarian), typed manuscript, Leningrad State University 29 Dec 1969.

[10] A. S. Serdyuk, *Approximation of Poisson integrals by de la Vallée Poussin sums in uniform and integral metric*, Ukrainian Math. J. 62 (2011), no. 12, 1941–1977.

[11] A. S. Serdyuk and I. E. Ovsii, *Approximation on classes of entire functions by de la Vallée Poussin sums*, Zb. Pr. Inst. Mat. NAN Ukr. 5, No.1 (2008), 334–351.

[12] A. S. Serdyuk, I. E. Ovsii, and A. P. Musienko, *Approximation of classes of analytic functions by de la Vallée Poussin sums in uniform metric*, Redicondi di Math., Ser. VII, 32 (2012), 1–15.

[13] S. B. Stechkin, *On the approximation of periodic functions by de la Vallée Poussin sums*, Anal. Math. 4 (1978), no. 1, 61–74.

[14] G. Szegö, *Über die Lebesguesschen Konstanten bei den Fourierschen Reihen*, Math. Zeitschrift 9 (1921), 163–166.

[15] A. F. Timan, *Theory of Approximations of Functions of a Real Variable*, Translated by J. Berry. MacMillan Co, New York 1963.

[16] S. A. Teljakovskii, *Approximations to differentiable functions by linear means of their Fourier series*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 24 (1960), 213–242.

[17] C. de la Vallée Poussin, *Sur la meilleure approximation des fonctions d’une variable réelle par des expressions d’ordre donné*, C.R. Acad. Sci. Paris 166 (1918), 799–802. [Also in: *Collected Works*, Volume 3, pp. 233–236.]

[18] C. de la Vallée Poussin, *Lecons sur l’approximation des fonctions de variable réelle*, Gauthier-Villars: Paris 1919. (Reprint: Chelsea: New York 1970).

[19] C. de la Vallée Poussin, *Collected Works= Oeuvres scientifiques*, Volume 3, *Fourier Analysis and Approximation Theory* Edited by Paul Butzer, Jean Mahwin, Pasquale Vetro. Circolo Matematico di Palermo: Palermo 2000.

[20] J. S. Walker, *Fourier Analysis*, Oxford University Press: Oxford 1988.

[21] A. Zygmund, *Trigonometric Series, Second Edition, Volumes I and II Combined*, Cambridge University Press: Cambridge 1968.

**Department of Mathematics, University of Michigan, 530 Church St., Ann Arbor, MI 48109–1043 (USA)**

*E-mail address: hmehta@umich.edu*