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The Bochner Technique and Weighted Curvatures

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Abstract. In this note we study the Bochner formula on smooth metric measure spaces. We introduce weighted curvature conditions that imply vanishing of all Betti numbers.

Key words: Bochner technique; smooth metric measure spaces; Hodge theory

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1 Introduction

Let $(M, g)$ be an oriented Riemannian manifold, let $\text{vol}_g$ denote its volume form and let $f$ be a smooth function on $M$. The triple $(M, g, e^{-f} \text{vol}_g)$ is called a smooth metric measure space. Based on considerations from diffusion processes, Bakry–Émery [1] introduced the tensor $\text{Ric}_f = \text{Ric} + \text{Hess} f$ as a weighted Ricci curvature for a geometric measure space. In fact, this tensor appeared earlier in work of Lichnerowicz [3]. Volume comparison theorems for smooth metric measure spaces with $\text{Ric}_f$ bounded from below have been established by Qian [7], Lott [4], Bakry–Qian [2] and Wei–Wylie [8].

In this note we study the Bochner technique on smooth metric measure spaces. The distortion of the volume element introduces a diffusion term to the Bochner formula

$$\Delta_f \omega = (dd^*_f+d^*_f d)\omega = \nabla^*_f \nabla \omega + \text{Ric}(\omega) - (\text{Hess} f)\omega,$$

where $\text{Ric}$ is the Bochner operator on $p$-forms. Lott [4] proved that if $\text{Ric}_f \geq 0$, then all $\Delta_f$-harmonic 1-forms are parallel and, for compact manifolds, $H^1(M; \mathbb{R})$ is isomorphic to the space of all parallel 1-forms $\omega$ which satisfy $\langle \nabla e^{-f}, \omega \rangle = 0$. Moreover, if $\text{Ric}_f > 0$, then all $\Delta_f$-harmonic 1-forms vanish.

We introduce new weighted curvature conditions that imply rigidity and vanishing results for $\Delta_f$-harmonic $p$-forms for $p \geq 1$. We can restrict to $p$-forms $\omega$ for $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ since $\omega$ is parallel if and only if $\ast \omega$ is parallel, where $\ast$ denotes the Hodge star.

By convention, we will refer to the eigenvalues of the curvature operator simply as the eigenvalues of the associated curvature tensor.

Theorem. Let $(M^n, g, e^{-f} \text{vol}_g)$ be a smooth metric measure space. For $1 \leq p < \frac{n}{2}$ set

$$h = \frac{1}{n-2p} \text{Hess} f - \frac{\Delta_f}{2(n-p)(n-2p)} g.$$

Let $\omega$ be a $\Delta_f$-harmonic $p$-form with $|\omega| \in L^2(M, e^{-f} \text{vol}_g)$ for $1 \leq p < \frac{n}{2}$. Let $\lambda_1 \leq \cdots \leq \lambda_{\lfloor \frac{n}{2} \rfloor}$ denote the eigenvalues of the weighted curvature tensor $Rm + h \otimes g$.

This paper is a contribution to the Special Issue on Scalar and Ricci Curvature in honor of Misha Gromov on his 75th Birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Gromov.html.
If \( \lambda_1 + \cdots + \lambda_{n-p} \geq 0 \), then \( \omega \) is parallel. If in addition \( M \) is compact, then \( H^p(M) = \{ \omega \in \Omega^p(M) \mid \nabla \omega = 0 \text{ and } i_{\vec{\nabla} f} \omega = 0 \} \).

If \( \lambda_1 + \cdots + \lambda_{n-p} > 0 \), then \( \omega \) vanishes. If in addition \( M \) is compact, then the Betti numbers \( b_p(M) \) and \( b_{n-p}(M) \) vanish for \( 1 \leq p < \frac{n}{2} \).

For \( p = 1 \) the Ricci curvature of the modified curvature tensor is the Bakry–Émery Ricci tensor, and the assumption in the Theorem implies that it is nonnegative. In this sense the Theorem is a generalization of Lott’s [4] results for 1-forms.

A stronger curvature assumption also allows control in the middle dimension \( p = \frac{n}{2} \). Recall that a curvature tensor is \( l \)-nonnegative (positive) if the sum of its lowest \( l \) eigenvalues is nonnegative (positive).

**Proposition.** Let \( (M^n, g, e^{-f} \text{vol}_g) \) be a smooth metric measure space. Let \( \mu_1 \leq \cdots \leq \mu_n \) denote the eigenvalues of \( \text{Hess} f \) and let \( 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \).

Let \( \omega \) be a \( \Delta_f \)-harmonic \( p \)-form with \( |\omega| \in L^2(M, e^{-f} \text{vol}_g) \). If the weighted curvature tensor

\[
Rm + \sum_{i=1}^{p} \frac{\mu_i}{2p(n-p)} g \otimes g
\]

is \( (n-p) \)-nonnegative, then \( \omega \) is parallel. If it is \( (n-p) \)-positive, then \( \omega \) vanishes.

In particular, if \( M \) is compact, then \( H^p(M) = \{ \omega \in \Omega^p(M) \mid \nabla \omega = 0 \text{ and } i_{\vec{\nabla} f} \omega = 0 \} \) and in case the weighted curvature tensor is \( (n-p) \)-positive, the Betti numbers \( b_p(M) \) and \( b_{n-p}(M) \) vanish.

The notation in this paper builds up on the presentation in [5, Chapter 9] and [6].

## 2 Preliminaries

### 2.1 Algebraic curvature tensors

For an \( n \)-dimensional Euclidean vector space \((V, g)\) let \( \mathcal{T}^{(0,k)}(V) \) denote the vector space of \( (0,k) \)-tensors and \( \text{Sym}^2(V) \) the vector space of symmetric \((0,2)\)-tensors on \( V \).

Let \( \mathcal{C}(V) \) denote the vector space of \((0,4)\)-tensors with \( T(X,Y,Z,W) = -T(Y,X,Z,W) = T(Z,W,X,Y) \). If \( T \) also satisfies the algebraic Bianchi identity, then \( T \) is called algebraic curvature tensor, \( T \in \mathcal{C}_B(V) \).

The Kulkarni–Nomizu product of \( S_1, S_2 \in \text{Sym}^2(V) \) is given by

\[
(S_1 \otimes S_2)(X,Y,Z,W) = S_1(X,Z)S_2(Y,W) - S_1(X,W)S_2(Y,Z) \\
+ S_1(Y,W)S_2(X,Z) - S_1(Y,Z)S_2(X,W).
\]

With this convention the algebraic curvature tensor \( I = \frac{1}{2} g \otimes g \) corresponds to the curvature tensor of the unit sphere.

Recall that the decomposition of \( \mathcal{C}(V) \) into \( O(n) \)-irreducible components is given by

\[
\mathcal{C}(V) = \langle I \rangle \oplus \langle \hat{\text{Ric}} \rangle \oplus \langle W \rangle \oplus \Lambda^4 V,
\]

where \( \langle \hat{\text{Ric}} \rangle = S^2_0(V) \oplus g \) is the subspace of algebraic curvature tensors of trace-free Ricci type, \( S^2_0(V) = \{ h \in \text{Sym}^2(V) \mid \text{tr}(h) = 0 \} \), and \( \langle W \rangle \) denotes the subspace of Weyl tensors.

Explicitly, every algebraic curvature tensor decomposes as

\[
Rm = \frac{\text{scal}}{2(n-1)n} g \otimes g + \frac{1}{n-2} \hat{\text{Ric}} \otimes g + W.
\]
2.2 Lichnerowicz Laplacians on smooth metric measure spaces

Let \((M, g, f)\) be a smooth metric measure space. The formal adjoints of the exterior and covariant derivative with respect to the measure \(e^{-f} \text{vol}_g\) are given by

\[
d^*_f = d^* + i \nabla f \quad \text{and} \quad \nabla^*_f = \nabla^* + i \nabla f.
\]

More generally, for a vector field \(U\) on \(M\), we will consider

\[
d^*_U = d^* + i_U \quad \text{and} \quad \nabla^*_U = \nabla^* + i_U.
\]

The associated generalized Lichnerowicz Laplacian on \((0,k)\)-tensors is given by

\[
\Delta_U T = \nabla^*_U \nabla T + \text{Ric}(T) - (\nabla U) T,
\]

where the curvature term is given by

\[
\text{Ric}(T)(X_1, \ldots, X_k) = \sum_{i=1}^k \sum_{j=1}^n (R(X_i, e_j)T)(X_1, \ldots, e_j, \ldots, X_k).
\]

A tensor \(T\) is called \(U\)-harmonic if \(\Delta_U T = 0\).

To emphasize that the curvature term is calculated with respect to the curvature tensor \(R_m\), we will also write \(\text{Ric}_{R_m}(T)\) for \(\text{Ric}(T)\).

Recall that for an endomorphism \(L\) of \(V\) and a \((0,k)\)-tensor \(T\) we have

\[
(LT)(X_1, \ldots, X_k) = -\sum_{i=1}^k T(X_1, \ldots, L(X_i), \ldots, X_k).
\]

In particular, the Ricci identity implies that the definition of the curvature term in the Lichnerowicz Laplacian naturally carries over to algebraic curvature tensors.

**Proposition 2.1.** Let \((M, g)\) be a Riemannian manifold and \(U\) a vector field on \(M\). For a \((0,k)\)-tensor \(T\) on \(M\) set \(\text{Ric}_U(T) = \text{Ric}(T) - (\nabla U) T\).

(a) Every \(p\)-form satisfies

\[
(dd^*_U + d^*_U d) \omega = \nabla^*_U \nabla \omega + \text{Ric}_U(\omega).
\]

(b) Every symmetric \((0,2)\)-tensor satisfies

\[
(\nabla_X \nabla^*_U T)(X) + (\nabla^*_U d^T)(X, X) = (\nabla^*_U \nabla T)(X, X) + \frac{1}{2}(\text{Ric}_U T)(X, X),
\]

where \(d^T(Z, X, Y) = (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z)\).

**Proof.** (a) The case \(U = 0\) recovers the well-known Bochner formula. The generalized Hodge Laplacian satisfies

\[
dd^*_U + d^*_U d = dd^* + d^* d + di_U + i_U d = \Delta + L_U.
\]

In addition to the classical Lichnerowicz Laplacian we have on the right hand side

\[
\nabla_U - (\nabla U) = L_U
\]

and thus all diffusion terms balance out.
(b) As in (a), it suffices to consider all terms that depend on $U$ and show that

$$(\nabla_X i_U h)(X) + (i_U d\nabla h)(X, X) = (\nabla_U h)(X, X) - \frac{1}{2}((\nabla U)h)(X, X).$$

This is a straightforward calculation

$$(\nabla_X i_U h)(X) + (i_U d\nabla h)(X, X) = (\nabla_X h)(U, X) + h(\nabla_X U) + (\nabla_U h)(X, X) - (\nabla_X h)(U, X)$$

$$(\nabla_U h)(X, X) - \frac{1}{2}((\nabla U)h)(X, X).$$

\[\blacksquare\]

Remark 2.2. The curvature tensor $Rm$ of a Riemannian manifold satisfies

$$\nabla^*_U \nabla Rm + \frac{1}{2} \text{Ric}_U(Rm) = \frac{1}{2}\nabla_X \nabla^*_U Rm(Y, Z, W) - \frac{1}{2}\nabla_Y \nabla^*_U Rm(X, Z, W)$$

$$+ \frac{1}{2}\nabla_Z \nabla^*_U Rm(W, X, Y) - \frac{1}{2}\nabla_W \nabla^*_U Rm(Z, X, Y).$$

A straightforward computation based on the second Bianchi identity shows that all terms that involve $U$ cancel.

The Bochner technique with diffusion relies on the following basic observations. Firstly, the maximum principle implies:

Lemma 2.3. Let $(M, g)$ be a Riemannian manifold, $U$ a vector field on $M$. Let $T$ be a tensor such that

$$g(\nabla^*_U \nabla T, T) \leq 0.$$

If $|T|$ has a maximum, then $T$ is parallel.

Remark 2.4. Note that a $p$-form $\omega$ satisfies $(dd^*_U + d^*_U d)\omega = 0$ if and only if $d\omega = 0$ and $d^*_U \omega = 0$.

As in [4], if $M$ is compact and oriented, standard elliptic theory implies that

$$H^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0 \text{ and } d^*_U \omega = 0\}.$$

Suppose that $\text{Ric}_U \geq 0$ on $p$-forms. It follows that a $p$-form $\omega$ is $U$-harmonic if and only if $\omega$ is parallel and $i_U \omega = 0$. Thus,

$$H^p(M) = \{\omega \in \Omega^p(M) \mid \nabla \omega = 0 \text{ and } i_U \omega = 0\}.$$

If $U = \nabla f$, then we can use integration to conclude:

Lemma 2.5. Let $(M, g, f)$ be a smooth metric measure space with $\int_M e^{-f} \text{vol}_g < \infty$. If $T$ is a $(0, k)$-tensor with $|T| \in L^2(M, e^{-f} \text{vol}_g)$ and

$$g(\nabla^*_f \nabla T, T) \leq 0,$$

then $T$ is parallel.
3 Weighted Lichnerowicz Laplacians

The idea of this section is to define a weighted curvature tensor \( \widetilde{Rm} \) so that for a given symmetric tensor \( S \) the curvature term of the Lichnerowicz Laplacian satisfies

\[
g(\text{Ric}_{\widetilde{Rm}}(T) - (S)T, T) = g(\text{ Ric}_{\widetilde{Rm}}(T), T).
\]

This will be achieved by adding a weight to the Ricci tensor of \( Rm \), leaving the Weyl curvature unchanged. The specific weight will depend on the irreducible components of the tensors of type \( T \), e.g., it is different for forms and symmetric tensors.

Let \( T \) be a \((0, k)\)-tensor. For \( \tau_{ij} \in S_k \) let \( T \circ \tau_{ij} \) denote the transposition of the \( i \)-th and \( j \)-th entries of \( T \) and for \( h \in \text{Sym}^2(V) \) let \( c_{ij}(h \otimes T) \) denote the contraction of \( h \) with the \( i \)-th and \( j \)-th entries of \( T \).

**Proposition 3.1.** For \( h \in \text{Sym}^2(V) \) let \( H : V \to V \) denote the associated symmetric operator. If \( T \in \mathcal{T}^{(0, k)}(V) \), then

\[
\text{Ric}_{h \otimes g}(T)(X_1, \ldots, X_k) = 2 \sum_{i \neq j} (T \circ \tau_{ij})(X_1, \ldots, H(X_i), \ldots, X_k) \nonumber \] 

\[
- \sum_{i \neq j} g(X_i, X_j)c_{ij}(h \otimes T)(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \nonumber \] 

\[
- \sum_{i \neq j} h(X_i, X_j)c_{ij}(g \otimes T)(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \nonumber \] 

\[
- (n - 2)(HT)(X_1, \ldots, X_k) + k \cdot \text{tr}(h)T(X_1, \ldots, X_k). \nonumber \]

**Proof.** The algebraic curvature tensor \( R = h \otimes g \) satisfies

\[
R(X, Y, Z, W) = g(H(X), Z)g(Y, W) - g(Y, Z)g(H(X), W) \nonumber \] 

\[
+ g(X, Z)g(H(Y), W) - g(H(Y), Z)g(X, W) \nonumber \]

and hence

\[
R(X, Y)Z = (H(X) \wedge Y + X \wedge H(Y))Z \nonumber \]

is the corresponding \((1, 3)\)-tensor. It follows that

\[
\text{Ric}_{h \otimes g}(T)(X_1, \ldots, X_k) = \sum_{i=1}^{k} \sum_{a=1}^{n} (R(X_i, e_a)T)(X_1, \ldots, e_a, \ldots, X_k) \nonumber \]

\[
= \sum_{i=1}^{k} \sum_{a=1}^{n} ((H(X_i) \wedge e_a)T)(X_1, \ldots, e_a, \ldots, X_k) \nonumber \]

\[
+ \sum_{i=1}^{k} \sum_{a=1}^{n} ((X_i \wedge H(e_a))T)(X_1, \ldots, e_a, \ldots, X_k). \nonumber \]

It is straightforward to calculate

\[
\sum_{i=1}^{k} \sum_{a=1}^{n} ((X_i \wedge H(e_a))T)(X_1, \ldots, e_a, \ldots, X_k) \nonumber \]

\[
= \sum_{i \neq j} \sum_{a=1}^{n} T(X_1, \ldots, (H(e_a) \wedge X_i)X_j, \ldots, e_a, \ldots, X_k) \nonumber \]
\[ + \sum_{i=1}^{k} \sum_{a=1}^{n} T(X_1, \ldots, (H(e_a) \wedge X_i)e_a, \ldots, X_k) \]

\[ = \sum_{i \neq j}^{n} \sum_{a=1}^{n} T(X_1, \ldots, g(H(e_a), X_j)X_i - g(X_i, X_j)H(e_a), \ldots, e_a, \ldots, X_k) \]

\[ + \sum_{i=1}^{k} \sum_{a=1}^{n} T(X_1, \ldots, g(H(e_a), e_a)X_i - g(e_a, X_i)H(e_a), \ldots, X_k) \]

\[ = \sum_{i \neq j}^{n} \sum_{a=1}^{n} T(X_1, \ldots, g(e_a, H(X_j))X_i, \ldots, e_a, \ldots, X_k) \]

\[ - \sum_{i \neq j}^{n} \sum_{a=1}^{n} g(X_i, X_j)T(X_1, \ldots, H(e_a), \ldots, e_a, \ldots, X_k) \]

\[ + \sum_{i=1}^{k} \sum_{a=1}^{n} h(e_a, e_a)T(X_1, \ldots, X_k) - \sum_{i=1}^{k} \sum_{a=1}^{n} T(X_1, \ldots, H(g(e_a, X_i)e_a), \ldots, X_k) \]

\[ = \sum_{i \neq j}^{n} T(X_1, \ldots, X_i, \ldots, H(X_j), \ldots, X_k) \text{ [here } X_i \text{ is in the } j\text{-th position]} \]

\[ - \sum_{i \neq j}^{n} \sum_{a=1}^{n} g(X_i, X_j)h(e_a, e_b)T(X_1, \ldots, e_b, \ldots, e_a, \ldots, X_k) + k \cdot \text{tr}(h)T(X_1, \ldots, X_k) \]

\[ - \sum_{i=1}^{k} T(X_1, \ldots, H(X_i), \ldots, X_k) \]

\[ = \sum_{i \neq j}^{n} (T \circ \tau_{ij})(X_1, \ldots, H(X_j), \ldots, X_i, \ldots, X_k) \text{ [here } H(X_j) \text{ is in the } j\text{-th position]} \]

\[ - \sum_{i \neq j}^{n} g(X_i, X_j)c_{ij}(h \otimes T)(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \]

\[ + k \cdot \text{tr}(h)T(X_1, \ldots, X_k) + (HT)(X_1, \ldots, X_k). \]

Similarly one computes

\[ \sum_{i=1}^{k} \sum_{a=1}^{n} ((H(X_i) \wedge e_a)T)(X_1, \ldots, e_a, \ldots, X_k) \]

\[ = \sum_{i \neq j}^{n} (T \circ \tau_{ij})(X_1, \ldots, X_j, \ldots, H(X_i), \ldots, X_k) \text{ [here } X_j \text{ is in the } j\text{-th position]} \]

\[ - \sum_{i \neq j}^{n} h(X_i, X_j)c_{ij}(g \otimes T)(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) - (n - 1)(HT)(X_1, \ldots, X_k). \]

Adding up both terms yields \( \text{Ric}_{h \otimes g}(T) \) as claimed. \( \square \)

**Proposition 3.2.** Let \((V, g)\) be an \(n\)-dimensional Euclidean vector space and \(h \in \text{Sym}^2(V)\). The following hold:

1. Every \(T \in \text{Sym}^2(V)\) satisfies

\[ \text{Ric}_{h \otimes g}(T) = -nHT - 2\langle T, h \rangle g - 2 \text{tr}(T)h + 2 \text{tr}(h)T, \]

\[ g(\text{Ric}_{h \otimes g}(T), T) = -ng(HT, T) - 4 \text{tr}(T \langle T, h \rangle) + 2 \text{tr}(h|T|^2). \]
2. Every $p$-form $\omega$ satisfies
\[
\text{Ric}_{h \otimes g}(\omega) = -(n - 2p)H\omega + p\text{tr}(h)\omega,
\]
\[
g(\text{Ric}_{h \otimes g}(\omega), \omega) = -(n - 2p)g(H\omega, \omega) + p\text{tr}(h)|\omega|^2.
\]

3. Every algebraic $(0,4)$-curvature tensor $R_m$ satisfies
\[
\text{Ric}_{h \otimes g}(R_m) = -2(h \otimes \text{Ric}) - 2g \otimes (c_{24}(h \otimes R_m)) - (n - 2)H R_m + 4 \text{tr}(h) R_m.
\]

Proof. (a) Due to the symmetry of $T$ it follows that
\[
\text{Ric}_{h \otimes g}(T)(X_1, X_2) = 2\{T(H(X_1), X_2) + T(X_1, H(X_2)) \}
\]
\[
- 2\{g(X_1, X_2)\langle h, T\rangle + h(X_1, X_2)\text{tr}(T)\}
\]
\[
- (n - 2)(HT)(X_1, X_2) + 2\text{tr}(h)T(X_1, X_2).
\]

(b) Since $\omega \circ \tau_{ij} = -\omega$ for every transposition $\tau_{ij}$ it follows that
\[
\sum_{i \neq j}(\omega \circ \tau_{ij})(X_1, \ldots, H(X_i), \ldots, X_p) = -\sum_{i \neq j}\omega(X_1, \ldots, H(X_i), \ldots, X_p)
\]
\[
= -(p - 1)\sum_{i=1}^{p}\omega(X_1, \ldots, H(X_i), \ldots, X_p)
\]
\[
= (p - 1)(H\omega)(X_1, \ldots, X_p)
\]
and furthermore $c_{ij}(g \otimes \omega) = c_{ij}(h \otimes \omega) = 0$ for all $i \neq j$. This implies the claim.

(c) The symmetries of the curvature tensor imply that
\[
\sum_{i \neq j}((R_m \circ \tau_{ij})(X_1, \ldots, H(X_i), \ldots, X_4)
\]
\[
= (H R_m)(X_1, X_2, X_3, X_4) + (H R_m)(X_2, X_3, X_1, X_4) + (H R_m)(X_3, X_1, X_2, X_4) = 0
\]
due to the first Bianchi identity.

Computing with respect to an orthonormal eigenbasis of $H$ it follows that
\[
(g(\cdot, \cdot)c_{12}(h \otimes R_m))(X, Y, Z, W) = 0,
\]
\[
(g(\cdot, \cdot)c_{13}(h \otimes R_m))(X, Y, Z, W) = \sum_{a, b=1}^{n} g(X, Z) R_m(g(H(e_a), e_b)e_b, Y, e_a, W)
\]
\[
= \sum_{a=1}^{n} g(X, Z) R_m(H(e_a), Y, e_a, W)
\]
\[
= \sum_{a=1}^{n} g(Z, X) R_m(e_a, Y, H(e_a), W)
\]
\[
= (g(\cdot, \cdot)c_{31}(h \otimes R_m))(X, Y, Z, W).
\]

This implies
\[
\sum_{i \neq j}(g(\cdot, \cdot)c_{ij}(h \otimes R_m))(X, Y, Z, W)
\]
\[
= 2 \sum_{i=1}^{n} g(X, Z) R_m(H(e_i), Y, e_i, W) + g(X, W) R_m(H(e_i), Y, Z, e_i)
\]
It is worth noting that there are trace-free symmetric $(0,2)$-tensors $h_1, h_2$ such that the curvature tensor $h_1 \otimes h_2$ is Weyl.

The main Theorem follows as in Proposition 3.4 below by using Lemma 2.5 instead of Lemma 2.3. The description of the de Rham cohomology groups follows from Remark 2.4.

**Proposition 3.4.** Let $(M, g)$ be a Riemannian manifold and let $U$ be a vector field on $M$. Set $S = \nabla U$ and for $1 \leq p < \frac{n}{2}$ set

$$H = \frac{1}{n - 2p} S - \frac{1}{2(n - p)(n - 2p)} \text{tr}(S) I,$$

where $I : TM \to TM$ denotes the identity operator.

Suppose that the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_{\binom{n}{2}}$ of the weighted curvature tensor $\text{Rm} + h \otimes g$ satisfy

$$\lambda_1 + \cdots + \lambda_{n - p} \geq 0$$

and let $\omega$ be a $U$-harmonic $p$-form for $1 \leq p < \frac{n}{2}$.

If $|\omega|$ achieves a maximum, then $\omega$ is parallel. If in addition the inequality is strict, then $\omega$ vanishes.

**Proof.** Proposition 3.2 (b) and $-I \omega = p \omega$ imply that

$$g(\text{Ric}_{h \otimes g} \omega, \omega) = -(n - 2p) g(H \omega, \omega) + p \text{tr}(h)|\omega|^2 = -g(((n - 2p) H + \text{tr}(h) I) \omega, \omega)$$

$$= -g\left(\left( S - \frac{\text{tr}(S)}{2(n - p)} I + \frac{\text{tr}(S)}{2(n - p)} I \right) \omega, \omega \right) = -g(S \omega, \omega).$$

Thus the Bochner formula takes the form

$$\Delta_U \omega = \nabla_U^* \nabla \omega + \text{Ric}(\omega) - (\nabla U) \omega = \nabla_U^* \nabla \omega + \text{Ric}_{\text{Rm} + h \otimes g}(\omega).$$
The argument in [6, proof of Theorem A] shows that $\text{Ric}_{\text{Rm} + h \otimes g}(\omega) \geq 0$. Lemma 2.3 implies the claim.

If the inequality is strict, then the same argument shows that $\text{Ric}_{\text{Rm} + h \otimes g}(\omega) > 0$ unless $\omega = 0$. ■

The above approach only works for $p = \frac{n}{2}$ if $S$ is a multiple of the identity. However, we have

**Proposition 3.5.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $U$ be a vector field on $M$. Set $S = \nabla U$ and fix $1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor$. Let $\mu_1 \leq \cdots \leq \mu_n$ denote the eigenvalues of $S$. Suppose that the weighted curvature tensor

$$\text{Rm} + \frac{\sum_{i=1}^{p} \mu_i}{2p(n-p)}g \otimes g$$

is $(n-p)$-nonnegative. If $\omega$ is a $U$-harmonic $p$-form $\omega$ such that $|\omega|$ has a maximum, then $\omega$ is parallel. If in addition the weighted curvature tensor is $(n-p)$-positive, then $\omega$ vanishes.

**Proof.** Calculating with respect to an orthonormal eigenbasis for $S$ it follows that

$$-g((S\omega),\omega) = -\sum_{i_1 \cdots < i_p} (S\omega)_{i_1 \cdots i_p}\omega_{i_1 \cdots i_p} = \sum_{i_1 \cdots < i_p} \left(\sum_{j=1}^{p} \mu_{i_j}\right)^2 (\omega_{i_1 \cdots i_p})^2 \geq \left(\sum_{i=1}^{p} \mu_i\right) |\omega|^2.$$

Let $\{\lambda_\alpha\}$ denote the eigenvalues of (the curvature operator associated to) $\text{Rm}$ and let $\{\Xi_\alpha\}$ be an orthonormal eigenbasis. It follows from [6, Proposition 1.6] that

$$g(\text{Ric}_{\text{Rm}}(\omega),\omega) - g(S\omega,\omega) \geq \sum_{\alpha} \lambda_\alpha |\Xi_\alpha \omega|^2 + \left(\sum_{i=1}^{p} \mu_i\right) |\omega|^2 = \sum_{\alpha} \left(\lambda_\alpha + \frac{\sum_{i=1}^{p} \mu_i}{p(n-p)}\right) |\Xi_\alpha \omega|^2.$$

The proof can now be completed as in Proposition 3.4. ■

This principle can also be applied to $(0,2)$-tensors.

**Proposition 3.6.** Let $T \in \text{Sym}^2(V)$ with $\text{tr}(T) = 0$, let $S = \nabla U$ and set

$$H = \frac{S}{n} - \frac{\text{tr}(S)}{2n^2} I.$$

Let $\lambda_1 \leq \cdots \leq \lambda_{\frac{n}{2}}$ denote the eigenvalues of the weighted curvature tensor $\text{Rm} + h \otimes g$ and suppose that

$$\lambda_1 + \cdots + \lambda_{\frac{n}{2}} \geq 0.$$

If $T$ is $U$-harmonic and $|T|$ has a maximum, then $T$ is parallel. If in addition the inequality is strict, then $T$ vanishes.

**Proof.** Proposition 3.2(a) implies that

$$g(\text{Ric}_{h \otimes g}(T),T) = -ng\left(\left(\frac{H}{n} + \frac{\text{tr}(h)}{n} I\right)T,T\right) = -ng\left(\left(\frac{S}{n} - \frac{\text{tr}(S)}{2n^2} I + \frac{\text{tr}(S)}{2n^2} I\right)T,T\right) = -g(ST,T).$$
It follows from Proposition 2.1(b) that
\[
(\nabla_X \nabla^*_U T) (X, X) + (\nabla^*_U d^T) (X, X) = (\nabla^*_U \nabla T) (X, X) + \frac{1}{2} (\text{Ric}_{\text{Rm}} + h g_T) (X, X).
\]

As in [6, Lemma 2.1 and Proposition 2.9] we conclude that \( \text{Ric}_{\text{Rm}} + h g_T (T) \geq 0 \). When the inequality is strict, the argument shows moreover \( \text{Ric}_{\text{Rm}} + h g_T (T) > 0 \) unless \( T = 0 \). This uses again that \( T \) is trace-less.

An application of Lemma 2.5 as before implies the claim. ■

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