On classification of higher-order integrable nonlinear partial differential equations

Il’in I.A., Noshchenko D.S., Perezhogin A.S.

IKIR FEB RAS, 684034 Kamchatka region, Elizovskiy district, Paratunka, Mirnaya str., 7.
KamGU Vitus Bering, 683032, Kamchatka region, Elizovskiy district, Petropavlovsk-Kamchatsky, Pogranichnaya str., 4.

Abstract

In the paper we investigate existence of soliton solutions for higher-order partial differential equations with polynomial nonlinearities. Using the \(\tau\)-function method we obtain classification for high-order integrable systems.

1 Introduction

In nonlinear physics solitons are defined as solitary waves which interact in elastic way, like particles.

Classical example – the Kortevieg de Vries (KdV) equation

\[
    u_t + 6uu_x + u_{xxx} = 0
\]

admits \(n\)-soliton solution that can be expressed in terms of Hirota’s \(\tau\)-function.

Consider some \(n\)-th order partial differential equation with polynomial nonlinearities in 1+1 variables \(E(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, \ldots)\). Let \(u(x, t)\) its meromorphic solution (only finite number of negative powers contained in its Laurent expansion)

\[
    u(x, t) = \sum_{k=-p}^{\infty} c_k(x, t)k
\]

Integer \(p\) is called a singular order of equation [4, 5]. It can be obtained by substitution \(u(\xi) = \frac{1}{\xi^p}\), where \(\partial_\xi \sim \partial_x\). For the KdV equation [1] we have

\[
    \frac{d^3}{d\xi^3} u(\xi) + 6 \left( \frac{d}{d\xi} u(\xi) \right) u(\xi) = (p^2 + 3p + 2)\xi^{-p-3} + 6\xi^{-2p-1}
\]

To equal poles we take \(p + 3 = 2p + 1\), so \(p = 2\).
We denote $n$-th order partial differential equation of singular order $p$ as $E^n_p(u)$. Hirota’s $\tau$-function [2, 5] for $E^n_p(u)$ equation is defined as

$$u(x,t) = K \frac{\partial^p}{\partial x^p} \log \tau(x,t)$$ (4)

For example, for the KdV equation one-soliton solution (solitary wave) can be expressed as

$$u(x,t) = \frac{\partial^2}{\partial x^2} \log(1 + \exp(px - qt))$$ (5)

and two-soliton solution

$$u(x,t) = \frac{\partial^2}{\partial x^2} \log \left(1 + \sum_{i=1}^{n} f_i(x,t)\right)$$ (7)

where $f_i(x,t)$ is a polynomial in $e^{p_j x - q_j t}$ for $j = 1..n$. We say that the equation is partially integrable if it has two-soliton solution in Hirota’s form. A remarkable fact is that some equations with soliton-type solutions (we call them soliton equations) form a hierarchies (Lax, SK, KK equations), i.e. they admits the same type of solitons. For example, famous KdV equation is the first element in Lax hierarchy.

Our primary goal is to investigate existence of high order $E^n_p$ equations with two-soliton solutions [1, 2, 3]. All of these equations have terms with $\partial_x$ of singular order $p = 2$ and only one term with $\partial_t$

$$E^3_2(u) = u_t + a_1 uu_x + u_{xxx}$$

$$E^5_2(u) = u_t + a_1 uu_{xxx} + a_2 uu_x + a_3 uu_{xx} + u_{xxxxx}$$

$$E^7_2(u) = u_t + a_1 u_x^3 + a_2 uu_x + a_3 uu_{xxx} + a_4 uu_{xxxx} + a_5 uu_{xxxxx} + a_6 uu_{xxx} + a_7 uu_{xxxxxxx} + \ldots$$

and so on, for arbitrary large $n$.

### 2 Soliton solutions for $E^n_2$

With $\tau$-function expansion

$$\tau = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \ldots$$ (8)
where \( \varepsilon \) is an arbitrary complex parameter, we obtain substitution for two-solition solution \([2, 3, 12]\):

\[
f_1(x, t) = \exp(p_1 x - q_1 t) + \exp(p_2 x - q_2 t) \tag{9}
\]

\[
f_2(x, t) = \alpha_{12} \exp((p_1 + p_2)x - (q_1 + q_2)t) \tag{10}
\]

\[
f_i(x, t) = 0, \, i \geq 3 \tag{11}
\]

\( \alpha_{12} = \frac{R_n(p_1, q_1, p_2, q_2)}{S_n(p_1, q_1, p_2, q_2)} \) is a rational function of complex parameters \( p_i, q_i \). After substituting \([9, 11]\) at \( E_n^2 \) we equal coefficients at \( \varepsilon \). The resulting algebraic system depends on \( \alpha_i, \alpha_{12} \) coefficients. Evaluation of linear part \( u_{nx} + u_t \) leads to \( q_i = p_i^n \), which is called a dispersion relation \([3]\).

Now we are going to obtain explicit form for denominator \( S_n \) of \( \alpha_{12} \). With

\[
\frac{\partial^n}{\partial x^n} + I \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad n = 3, 5, 7, \ldots \tag{12}
\]

where \( Iu = u \) and \( u(x, t) \) is 2-soliton substitution \([8]\) with 9-11, we equal coefficients at \( \varepsilon \). Evaluating \( \varepsilon^n \) coefficient from the expansion above, we get polynoms \( S_n \) of the form

\[
S_3 = 3(p_1 + p_2)^2
\]

\[
S_5 = 5(p_1 + p_2)^2(p_1^2 + p_2^2)
\]

\[
S_7 = 7(p_1 + p_2)^2(p_1^2 + p_2^2)^2
\]

\[
S_9 = 3(p_i + p_j)^2(3p_i^6 + 9p_i^5p_j + 19p_i^4p_j^2 + 23p_i^3p_j^3 + 19p_i^2p_j^4 + 9p_ip_j^5 + 3p_j^6)
\]

\[
S_{11} = 11(p_i + p_j)^2(p_i^2 + p_ip_j + p_j^2)(p_j^6 + 3p_ip_j^5 + 7p_i^2p_j^4 + 9p_i^3p_j^3 + 7p_i^4p_j^2 + 3p_i^5p_j + p_i^6)
\]

\[
S_{13} = 13(p_i + p_j)^2(p_i^2 + p_ip_j + p_j^2)^2(p_j^6 + 3p_ip_j^5 + 8p_i^2p_j^4 + 11p_i^3p_j^3 + 8p_i^4p_j^2 + 3p_i^5p_j + p_i^6)
\]

\[
S_{15} = (p_i + p_j)^2(15p_i^{12} + 15p_j^{12} + 90p_ip_i^{11} + 90p_j^{11}p_i + 365p_j^{10}p_i^2 + 1000p_j^{9}p_i^3 + 2003p_j^{8}p_i^4 + 3002p_j^{7}p_i^5 + 3433p_j^{6}p_i^6 + 3002p_j^{5}p_i^7 + 2003p_j^{4}p_i^8 + 1000p_j^{3}p_i^9 + 365p_j^{2}p_i^{10})
\]

\[
\vdots
\]

These polynomials can be derived by horizontal summation of neighbour elements in a binomial triangle, as shown below.
For every integer $n$ roots of $S_n$ lie on curve defined this way: $Re(z) \in (-1, -\frac{1}{2})$ implies unit circle centered at $(0, 0)$, $Re(z) \in (-\frac{1}{2}, 0)$ implies unit circle centered at $(-1, 0)$, $Re(z) = -\frac{1}{2}$ implies line $Re(z) = \frac{1}{2}$ minus segment bounded by the first two circles. At last, $Re(z) = -1, 0$ are limit points when $n \to \infty$.

Figure 1: Root geometry of $S_n$ and $R_n$ polynomials on complex plane

Now we should define numerator $R_n$ of $\alpha_{12}$. Let

$$S_n = (p_1 + p_2)^2(p_1^2 + p_1p_2 + p_2^2)^kT_n(p_1, p_2), k = 0, 1, 2$$

Direct computation gives two variants

$$R_n(p_1, p_2) = S_n(-p_1, p_2)$$
and
\[ R_n(p_1, p_2) = \frac{S_n(-p_1, p_2)(p_1^2 + p_1 p_2 + p_2^2)T_n(p_1, p_2)}{(p_1^2 - p_1 p_2 + p_2^2)^l}T_n(-p_1, p_2), l \leq k \] (15)

These selections keep symmetry property of \( \alpha_{12} \), so its zeros are reflected poles by \( Im(z) \) axis. This principle allows equation \((E^n_{p})_{\alpha_{12}}(u)\) with two-soliton solution in form (8) to be determined explicitly. It is uniquely defined by \( \alpha_{12} \).

In case of (14) this equation has no hierarchy. It can be constructed for any \( n \geq 9 \). Another case (15) leads to widely known completely integrable Lax and SK hierarchies \( (l = 2, 1 \) respectively \)[4]. When \( l = 0 \), we get new hierarchy \( (n = 7, 13, 19 \ldots) \). For \( n = 7 \), for example, this is equation (16). (17) represents its flux \( J_1 \), associated with conserving density \( \rho_1 \) \[3, 7\]. We also got that new and tail equations do not have a three-soliton solution.

Summary result about classification of equations and its properties are represented at table 1.

| \( n \) | Lax | SK | new | tail |
|-------|-----|----|-----|-----|
| 3     | *   | –  | –   | –   |
| 5     | *   | *  | –   | –   |
| 7     | *   | *  | *   | –   |
| 9     | *   | –  | –   | *   |
| 11    | *   | *  | –   | *   |
| 13    | *   | *  | *   | –   |
| 15    | *   | –  | –   | *   |
| 17    | *   | *  | –   | *   |
| 19    | *   | *  | *   | *   |
| 21    | *   | –  | –   | *   |

Table 1: \( E^n_2 \) classification; * denotes existence of an equation at given \( n \), * denotes existence of given property

We established the fact that some hierarchies are “holey”: for example, SK equation of 9th order does not exist.
\[
\text{(new)}_2^7(u) = u_x + 56 \frac{u u_x u_{x,x,x} x}{K} + 56 \frac{u u_x u_{x,x,x} x}{K} + 840 \frac{u^2 u_{x,x} K}{K^2} + 3360 \frac{u^3 u_x}{K^3} + 140 \frac{u_{x,x} u_{x,x}}{K} + 56 \frac{u_{x,x} x u_x x}{K} + 1680 \frac{u u_{x,x,x} u_x}{K^2} \]
\]
\[
J_1(u) = 56 \frac{u u_{x, x} x_{x}}{K} + 840 \frac{u^4}{K^3} + 840 \frac{u^2 u_{x,x}}{K^2} + 70 \frac{u_{x,x} x}{K} + u_{x,x,x,x,x,x} \]
\[
\text{(tail)}_2^7(u) = 420 u^3 + 90 \frac{u u_x u_{x,x,x} x}{K} + 90 \frac{u u_x u_{x,x,x} x}{K} + 90 \frac{u u_x u_{x,x,x} x}{K} + 12600 \frac{u x u_{x,x,x,x} x}{K^2} + 75600 \frac{u^2 u_{x,x} x}{K^3} + 5040 \frac{u u_{x,x} x_{x} x}{K^2} + 2520 \frac{u^2 u_{x,x} x_{x} x}{K^2} + 25200 \frac{u^3 u_{x,x} x}{K^3} + 75600 \frac{u^4 u_{x,x} x}{K^4} + 132 \frac{u u_x u_{x,x,x,x,x,x,x} x}{K} + 1995840 \frac{u^5 u_{x,x} x_{x} x}{K^5} + 332640 \frac{u^7 u_{x,x} x_{x} x}{K^7} + 55440 \frac{u^9 u_{x,x} x_{x} x}{K^9} + 69300 \frac{u^{11} u_{x,x} x_{x} x}{K^{11}} + 831600 \frac{u^{13} u_{x,x} x_{x} x}{K^{13}} + 831600 \frac{u^{15} u_{x,x} x_{x} x}{K^{15}} + 55440 \frac{u^{17} u_{x,x} x_{x} x}{K^{17}} + 5940 \frac{u^{19} u_{x,x} x_{x} x}{K^{19}} \]
\[
(\text{SK})_4^5(u) = u_t + u_{x,x,x,x,x} + 60 \frac{u x^3}{K^2} + 30 \frac{u x u_{x,x,x,x}}{K} \]
\[
(\text{Lax})_4^3(u) = u_{x,x,x} + 36 \frac{\int \infty}{K} + 12 \frac{\int \infty}{K} + 12 \frac{\int \infty}{K} \]
\]

3 Modified equations \(E^n_p\)

We obtained sufficient conditions for two-soliton solution when singularity order \(p = 2\). If \(p\) changes, we still have the same conditions (\(\tau\)-function conserves, but solution form changes). So at this approach classification (tab. 2) is \(p\)-independent. For example, we built modified 5th order SK equation with kink-type solitons \(\n\) \((p = 1)\)

\[
(\text{SK})_4^5(u) = u_t + u_{x,x,x,x,x} + 60 \frac{u x^3}{K^2} + 30 \frac{u x u_{x,x,x,x}}{K} \]
\]

From \(\text{(18)}\) SK equation can be derived by taking \(x\)-derivative and substituting \(u_x = v\). The same trick can be made for \(p > 2\), and that will lead us to differintegral equations. For example, modified 3rd order Lax equation \((p = 4)\).

\[
(\text{Lax})_4^3(u) = u_{x,x,x} + 36 \frac{\int \infty}{K} + 12 \frac{\int \infty}{K} + 12 \frac{\int \infty}{K} \]
\]

Solitary waves solutions for modified Lax and SK equations are represented at fig. below.

4 Conclusion

Using the \(\tau\)-function substitution we classified high order PDEs with polynomial nonlinearities that accept two-soliton solution in Hirota's form (tab. 2). All
Equations admit dispersion law in form \( q_i = p_i^n \). In integrable case ratio \( \alpha_{12} \) uniquely determines single equation or hierarchy of equations. In paper [6] were established similar conditions for two-soliton solution for bilinear type equations. But there was no information on general structure of these equations. We discovered, apart from known completely integrable cases (Lax and SK hierarchies, [3, 5, 4, 8, 9, 10, 11, 12]), one hierarchy of two-soliton equations (new) and countable family of isolated two-soliton equations (tail).

Furthermore, this algebraic structure does not depend on singularity order of an equation. So the transformation of singularities that preserves form of a soliton should not produce another partial or complete integrable PDEs.

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