Super Efimov effect of resonantly interacting fermions in two dimensions

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We study a system of spinless fermions in two dimensions with a short-range interaction fine-tuned to a p-wave resonance. We show that three such fermions form an infinite tower of bound states of orbital angular momentum \( \ell = \pm 1 \) and their binding energies obey a universal doubly exponential scaling \( E_3^{(n)} \propto e^{-2\pi n/4+\theta} \) at large \( n \). This “super Efimov effect” is found by a renormalization group analysis and confirmed by solving the bound state problem. We also provide an indication that there are \( \ell = \pm 2 \) four-body resonances associated with every three-body bound state at \( E_4^{(n)} \propto e^{-2\pi n/4+\theta-0.188} \). These universal few-body states may be observed in ultracold atom experiments and should be taken into account in future many-body studies of the system.

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Introduction

Recently topological superconductors have attracted great interest across many subfields in physics [1, 2]. This is partially because vortices in topological superconductors bind zero-energy Majorana fermions and obey non-Abelian statistics, which can be of potential use for fault-tolerance topological quantum computation [3, 4]. A canonical example of such topological superconductors is a p-wave paired state of spinless fermions in two dimensions [3], which is believed to be realized in Sr\(_2\)RuO\(_4\) [8]. Previous mean-field studies revealed that a topological quantum phase transition takes place across a p-wave Feshbach resonance [7, 9].

In this Letter, we study few-body physics of spinless fermions in two dimensions right at the p-wave resonance. We predict that three such fermions form an infinite tower of bound states of orbital angular momentum \( \ell = \pm 1 \) and their binding energies obey a universal doubly exponential scaling

\[
E_3^{(n)} \propto e^{-2\pi n/4+\theta}
\]

at large \( n \). Here \( \theta \) is a nonuniversal constant defined modulo \( 3\pi/4 \). This novel phenomenon shall be termed a super Efimov effect, because it resembles the Efimov effect in which three spinless bosons in three dimensions right at an s-wave resonance form an infinite tower of \( \ell = 0 \) bound states whose binding energies obey the universal exponential scaling \( E_3^{(n)} \propto e^{-2\pi n/s_0} \) with \( s_0 \approx 1.00624 \) (see Table I for comparison). While the Efimov effect is possible in other situations [10, 11], it does not take place in two dimensions or with p-wave interactions [12–14]. We also provide an indication that there are \( \ell = \pm 2 \) four-body resonances associated with every three-body bound state at

\[
E_4^{(n)} \propto e^{-2\pi n/4+\theta-0.188},
\]

which also resembles the pair of four-body resonances in the usual Efimov effect [15, 16]. These universal few-body states of resonantly interacting fermions in two dimensions should be taken into account in future many-body studies beyond the mean-field approximation.

Renormalization group analysis

The above predictions can be derived most conveniently by a renormalization group (RG) analysis. The most general Lagrangian density that includes up to marginal couplings consistent with rotation and parity symmetries is

\[
\mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2} \right) \psi + \phi_4^\dagger \left( i\partial_t + \frac{\nabla^2}{4} - \xi_0 \right) \phi_a \\
+ g \phi_a^\dagger \psi (-i\nabla_a) \psi + g \psi^\dagger (-i\nabla_a) \psi^\dagger \phi_a \\
+ v_3 \psi^\dagger \phi_4^\dagger \phi_a \psi + v_4 \phi_a^\dagger \phi_4^\dagger \phi_a \phi_a + v'_4 \phi_a^\dagger \phi_4^\dagger \phi_a \phi_a. \tag{3}
\]

Here and below, \( \hbar = m = 1 \), \( \nabla_\pm \equiv \nabla_x \pm i\nabla_y \), and sums over repeated indices \( a = \pm \) are assumed. \( \psi \) and \( \phi_4 \) fields correspond to a spinless fermion and \( \ell = \pm 1 \) composite boson, respectively. The p-wave resonance is...
FIG. 1. Self-energy diagram of the φa field. Solid and dashed lines represent propagators of ψ and φa fields, respectively.

defined by the divergence of the two-fermion scattering amplitude at zero energy, which is achieved by tuning the bare detuning parameter at ε0 = g2Λ2/(2π) with Λ being a momentum cutoff.

Since the other couplings (g, v3, v4, v′4) are all dimensionless, our effective field theory (3) is renormalizable, and its renormalization can be performed in a similar way to Ref. [17]. The self-energy diagram of the φa field depicted in Fig. 1 is logarithmically divergent and renormalizes the two-body coupling g. The RG equation that governs the running of g at a momentum scale e−sΛ is found to be

$$\frac{dg}{ds} = -\frac{g^3}{2\pi},$$ (4)

which is solved by

$$g^2(s) = \frac{1}{\pi + g(0)}.$$ (5)

Accordingly, the two-body coupling is marginally irrelevant; i.e., it gets weak toward the infrared limit s → ∞.

We now turn to the renormalization of the three-body coupling v3. Diagrams that renormalize v3 are depicted in Fig. 2. By taking into account the contribution from the φa field renormalization, the RG equation that governs the running of v3 is found to be

$$\frac{dv_3}{ds} = \frac{16g^4}{3\pi} - \frac{11g^2v_3}{3\pi} + \frac{2v_3^2}{3\pi}.$$ (6)

In the infrared limit s → ∞, where the two-body coupling scales as g2 → π/s [18], the analytic solution to Eq. (6) becomes

$$v_3(s) \to \frac{2\pi}{s} \left[ 1 - \cot \left( \frac{4}{3} \ln (s - \theta) \right) \right],$$ (7)

where the angle θ depends on the initial conditions of g and v3 at the ultraviolet scale s ~ 0.

We find that sv3 at large s is a periodic function of ln s and diverges at ln s = 3πn/4 + θ. These divergences in the three-body coupling indicate the existence of an infinite tower of energy scales in the three-body system in the ℓ = ±1 channels. Following the usual Efimov effect [19], we identify these scales with binding energies which leads to the prediction presented in Eq. (1). This identification will be confirmed later by solving the bound state problem.

Similarly, the four-body couplings v4 and v′4 are renormalized by diagrams depicted in Fig. 3. By taking into account the contribution from the φa field renormalization, the RG equations that govern the running of v4 and v′4 are found to be

$$\frac{dv_4}{ds} = -\frac{8g^4}{\pi} + \frac{2g^2v_3}{\pi} - \frac{4g^2v_4}{\pi} + \frac{2v_3^2}{\pi},$$ (8a)

$$\frac{dv_4'}{ds} = -\frac{4g^4}{\pi} + \frac{2g^2v_3}{\pi} - \frac{2g^2v_4'}{\pi} + \frac{2v_4'^2}{\pi}.$$ (8b)

FIG. 3. Diagrams to renormalize the four-body couplings v4 and v′4.

FIG. 4. RG evolutions of sv4, sv′4 (solid curves), and sv3 (dotted curve) with initial conditions g(0) = 10, v3(0) = i10−6, and v4(0) = v′4(0) = 0 [20].
respectively. Numerical solutions to these RG equations are shown in Fig. 4 and both $s v_4$ and $s v'_4$ at large $s$ are periodic functions of $\ln s$ with period $3 \pi/4$. We find that $v_4$ corresponding to the $\ell = 0$ channel diverges only at the same points as $v_3$. On the other hand, $v'_4$ corresponding to the $\ell = \pm 2$ channels has an additional divergence at $\ln s = 3 \pi n/4 + \theta - 0.188$ associated with every point where $v_3$ diverges. Following the RG study of spinless bosons in three dimensions [20], we identify these additional divergences with four-body resonances. We checked that this result is independent of the choice of initial conditions which leads to the prediction presented in Eq. (2).

**Bound state problem**

The above predictions derived by the RG analysis can be confirmed by solving bound state problems. We employ a model Hamiltonian

$$H = \sum_{k, p} \int \frac{dk}{(2\pi)^2} \frac{k^2}{2} \psi_k^p \psi_{k-p} - V_0 \sum_{a, \pm} \int \frac{dkdpdq}{(2\pi)^6} \chi_a(p) \chi_{a}(q) \psi_{k-p}^p \psi_{k}^q \psi_{k+p}^{\pm} \psi_{k-p-q}^q,$$

(9)

which describes spinless fermions interacting by a separable $p$-wave potential. The form factor

$$\chi_a(p) = p_\alpha e^{-p^2/(2\Lambda^2)}$$

(10)

is introduced to regularize the ultraviolet behavior. By summing diagrams depicted in Fig. 5, the two-fermion scattering $T$ matrix with incoming $q$ and outgoing $p$ relative momenta is obtained as

$$T(E; p, q) = \frac{16\pi \alpha_{pq} \cos(\varphi_p - \varphi_q) \exp(-p^2 + q^2)/(2\Lambda^2)}{2 \omega_0 - \Lambda^2 - E \exp(-E/\Lambda^2) E_1(-E/\Lambda^2)},$$

(11)

where $E = \omega - k^2/4 + i0^+$ is the total energy in the center-of-mass frame and $E_1(u) \equiv \int_{-\infty}^{\infty} dt e^{-tu}/t$ is the exponential integral. Accordingly, the $p$-wave resonance is achieved by tuning the bare two-body coupling at $\omega_0 = 2\pi/\Lambda^2$.

A three-fermion scattering problem can be solved in a similar way to the corresponding problem in three dimensions [21, 22]. The $T$-matrix elements satisfy coupled integral equations depicted in Fig. 6. When the total energy approaches a binding energy $E \to -\kappa^2 < 0$, the $T$ matrix factorizes as $T_{ab}(E; p, q) \to Z_a(p) Z_b(q)/(E + \kappa^2)$, and the resulting residue function satisfies

$$Z_a(p) = -\int dq \left( \frac{p + 2q}{2\pi} \right) e^{-\left(5p^2 + 5q^2 + 8pq q^2/(8\Lambda^2)\right)} \times \sum_{b = \pm} (2p + q)_b Z_b(q)$$

$$\left( \frac{1}{4} q^2 + \kappa^2 \right) e^{\left(q^2 + \kappa^2\right)/\Lambda^2} E_1(|q^2 + \kappa^2|/\Lambda^2).$$

(12)

It can be seen that $Z_+(p) = e^{i(\ell - 1)\pi \kappa} z_+(p)$ couples with $Z_-(p) = e^{i(\ell + 1)\pi \kappa} z_-(p)$. We focus on the $\ell = +1$ channel in which the super Efimov effect is expected to emerge.

We first solve the two coupled integral equations [12] analytically within the leading-logarithm approximation [23]. We assume that the integral is dominated by the region where $\kappa < q < \Lambda$ and split the integral into two parts, $\kappa < q < p$ and $p < q < \Lambda$, in which a sum of $p$ and $q$ in the integrand is replaced with whichever is larger. This approximation simplifies Eq. (12) to

$$\frac{3}{4} \zeta(x) = -\int_p^q \frac{dz_+(q)}{q \ln \Lambda/q} - \int_p^{\kappa \Lambda} \frac{dz_+(q) + z_-(q)}{q \ln \Lambda/q},$$

(13a)

$$\frac{3}{4} \zeta(y) = -\int_p^q \frac{dz_+(q)}{q \ln \Lambda/q},$$

(13b)

where $\kappa < 1$ is a positive constant. By changing variables to $x = \ln \Lambda/p$ and $y = \ln \Lambda/q$ and defining $\lambda = \ln \Lambda/\kappa$, $\eta = \ln \Lambda/\epsilon$, and $\zeta_{\pm}(x) = z_{\pm}(p)$, we obtain

$$\frac{3}{4} \zeta_+(x) = -\int_x^{\lambda} dy \zeta_+(y) - \int_x^{\kappa \eta} dy \left[ \zeta_+(y) + \zeta_-(y) \right],$$

(14a)

$$\frac{3}{4} \zeta_-(x) = -\int_x^{\lambda} dy \zeta_+(y).$$

(14b)

Then the differentiation of Eqs. (14) with respect to $x$ results in two coupled differential equations

$$\frac{3}{4} \zeta_+(x) = -\zeta_-(x),$$

(15a)

$$\frac{3}{4} \zeta_+(x) = \zeta_+(x)$$

(15b)

with boundary conditions

$$\frac{3}{4} \zeta_+(\eta) = -\int_\eta^\lambda dy \zeta_+(y),$$

(16a)

$$\frac{3}{4} \zeta_-(\lambda) = 0.$$
TABLE II. Lowest six binding energies $E_3^{(n)} = -\kappa_n^2/m$ in the form of $\lambda_n = \ln \ln \Lambda / \kappa_n$.

| $n$ | $\lambda_n$ | $\lambda_n - \lambda_{n-1}$ |
|-----|-------------|-----------------------------|
| 0   | 0.5632      | -                           |
| 1   | 2.770       | 2.207                       |
| 2   | 5.078       | 2.308                       |
| 3   | 7.430       | 2.352                       |
| 4   | 9.785       | 2.355                       |
| 5   | 12.141      | 2.356                       |
| $\infty$ | -           | 2.35619                     |

The differential equations (15) with the boundary condition (16b) are solved by

$$
\zeta_+(x) = \cos\left(\frac{4}{3}(x-\lambda)\right), \quad (17a)
$$

$$
\zeta_-(x) = \sin\left(\frac{4}{3}(x-\lambda)\right). \quad (17b)
$$

The other boundary condition (16a) constrains the allowed value of $\lambda$, while we cannot determine its value within the present approximation, because it is sensitive to the ultraviolet physics at $q \sim \Lambda$. However, owing to the periodicity of solutions (17), if $\lambda = \theta$ is a solution, then $\lambda = 3n\pi/4 + \theta$ must be all solutions, which is consistent with our previous RG analysis. We note that the solutions for $\ell = -1$ are obtained simply by exchanging $+\leftrightarrow -$, while the same approximation does not yield any solution for $\ell \neq \pm 1$. The double-logarithm scaling of solutions was also found in studying a scattering problem of three fermions.

We also solved the two coupled integral equations (12) with $\ell = \pm 1$ numerically and obtained binding energies of three fermions shown in Table II. We find that they obey the universal doubly exponential scaling with period $3\pi/4 \approx 2.35619$, which indeed confirms our prediction in Eq. (11). While our prediction for $\ell = \pm 2$ four-fermion resonances in Eq. (2) can be tested in principle in a similar way to the corresponding problem in the usual Efimov effect, we defer this problem to a future study.

Conclusion

In this Letter, we predicted the super Efimov effect in which resonantly interacting fermions in two dimensions form an infinite tower of $\ell = \pm 1$ three-body bound states whose binding energies obey the universal doubly exponential scaling as in Eq. (11). We also argued that there are $\ell = \pm 2$ four-body resonances associated with every three-body bound state as in Eq. (2). It is possible to extend our analysis away from the $p$-wave resonance and draw a binding energy diagram as a function of the detuning in analogy to the usual Efimov effect. These universal few-body states should be taken into account in future many-body studies of the system such as the topological quantum phase transition. Our super Efimov states (hopefully the lowest one) may be observed in ultracold atom experiments with $p$-wave Feshbach resonances in the same way as the usual Efimov states through a measurement of three-body or four-body recombination atom loss.

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