ON LEFT $\phi$-BIPROJECTIVITY AND LEFT $\phi$-BIFLATNESS OF CERTAIN BANACH ALGEBRAS

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Abstract. In this paper, we study left $\phi$-biflatness and left $\phi$-biprojectivity of some Banach algebras, where $\phi$ is a non-zero multiplicative linear functional. We show that if the Banach algebra $A^{**}$ is left $\phi$-biprojective, then $A$ is left $\phi$-biflat. Using this tool we study left $\phi$-biflatness of some matrix algebras. We also study left $\phi$-biflatness and left $\phi$-biprojectivity of the projective tensor product of some Banach algebras. We prove that for a locally compact group $G$, $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is finite. We show that $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is compact.

1. Introduction and Preliminaries

Banach homology theory have two important notions, biflatness and biprojectivity which have key role in studying the structure of Banach algebras. A Banach algebra $A$ is called biflat (biprojective), if there exists a bounded $A$-bimodule morphism $\rho : A \rightarrow (A \otimes_p A)^{**}$ ($\rho : A \rightarrow A \otimes_p A$) such that $\pi^*_A \circ \rho$ is the canonical embedding of $A$ into $A^{**}$ ($\rho$ is a right inverse for $\pi_A$), respectively. It is well known that for a locally compact group $G$, the group algebra $L^1(G)$ is biflat (biprojective) if and only if $G$ is amenable (compact), respectively. We have to mention that a biflat Banach algebra $A$ with a bounded approximate identity is amenable and vice versa, see [13].

A Banach algebra $A$ is called left $\phi$-amenable, if there exists a bounded net $(a_\alpha)$ in $A$ such that $aa_\alpha - \phi(a)a_\alpha \rightarrow 0$ and $\phi(a_\alpha) \rightarrow 1$ for all $a \in A$, where $\phi \in \Delta(A)$. For a locally compact group $G$, the Fourier algebra $A(G)$ is always left $\phi$-amenable. Also the group algebra $L^1(G)$ is left $\phi$-amenable if and only if $G$ is amenable, for further information see [8] and [1].

Following this course, Essmaili et. al. in [3] introduced and studied a biflat-like property related to a multiplicative linear functional, they called it condition $W$ (which we call it here right $\phi$-biflatness). The Banach algebra $A$ is called left $\phi$-biflat, if there exists a bounded linear map $\rho : A \rightarrow (A \otimes_p A)^{**}$ such that

$$\rho(ab) = \phi(b)\rho(a) = a \cdot \rho(b)$$

and

$$\tilde{\phi} \circ \pi^*_A \circ \rho(a) = \phi(a),$$

for each $a, b \in A$. We followed their work and showed that the Segal algebra $S(G)$ is left $\phi$-biflat if and only if $G$ is amenable see [16]. also we defined a notion of left $\phi$-biprojectivity for Banach algebras. In fact $A$ Banach algebra is left $\phi$-biprojective if there exists a bounded linear map $\rho : A \rightarrow A \otimes_p A$ such that

$$\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a), \, \phi \circ \pi_A \circ \rho(a) = \phi(a), \, (a, b \in A).$$

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We showed that the Lebesgue-Fourier algebra $LA(G)$ is left $\phi$-contractible if and only if $G$ is compact. Also the Fourier algebra $A(G)$ is left $\phi$-contractible if and only if $G$ is discrete, see \cite{15}.

In this paper, We show that if the Banach algebra $A^{**}$ is left $\phi$-biflat, then $A$ is left $\phi$-biflat. Using this tool we study left $\phi$-biflatness of some matrix algebras. We also study left $\phi$-biflatness and left $\phi$-biprojectivity of the projective tensor product of some Banach algebras. We prove that for a locally compact group $G$, $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is finite. We show that $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is compact.

We remark some standard notations and definitions that we shall need in this paper. Let $A$ be a Banach algebra. If $X$ is a Banach $A$-bimodule, then $X^*$ is also a Banach $A$-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

Throughout, the character space of $A$ is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on $A$. Let $\phi \in \Delta(A)$. Then $\phi$ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let $A$ be a Banach algebra. The projective tensor product $A \otimes_p A$ is a Banach $A$-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

For Banach algebras $A$ and $B$ with $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$, we denote $\phi \otimes \psi$ for a multiplicative linear functional on $A \otimes_p B$ given by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for each $a \in A$ and $b \in B$. The product morphism $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$, for every $a, b \in A$. Let $X$ and $Y$ be Banach $A$-bimodules. The map $T : X \to Y$ is called $A$-bimodule morphism, if

$$T(a \cdot x) = a \cdot T(x), \quad T(x \cdot a) = T(x) \cdot a, \quad (a \in A, x \in X).$$

2. SOME GENERAL PROPERTIES

Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. $A$ is called approximate left $\phi$-biprojective if there exists a net of bounded linear maps from $A$ into $A \otimes_p A$, say $(\rho_\alpha)_{\alpha \in I}$, such that

(i) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \overset{\|\|}{\longrightarrow} 0$,
(ii) $\rho_\alpha(ba) - \phi(a) \rho_\alpha(b) \overset{\|\|}{\longrightarrow} 0$,
(iii) $\phi \circ \pi_A \circ \rho_\alpha(a) - \phi(a) \rightarrow 0$,

for every $a, b \in A$, see \cite{14}.

**Proposition 2.1.** Let $A$ be a left $\phi$-biflat Banach algebra. Then $A$ is approximate left $\phi$-biprojective.

**Proof.** Since $A$ is left $\phi$-biflat, there exists a bounded linear map $\rho : A \to (A \otimes_p A)^{**}$ such that $\rho(ab) = a \cdot \rho(b) = \phi(b)\rho(a)$ and $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$. Since $\rho \in B(A, (A \otimes_p A)^{**})$, there exists a net $\rho_\alpha \in B(A, A \otimes_p A)$ such that $\rho_\alpha \overset{W^{**}}{\longrightarrow} \rho$. Thus for each $a \in A$ we have $\rho_\alpha(a) \overset{w^*}{\longrightarrow} \rho(a)$. Then

$$a \cdot \rho_\alpha(b) \overset{w^*}{\longrightarrow} a \cdot \rho(b) = \rho(ab), \quad \rho_\alpha(ab) \overset{w^*}{\longrightarrow} \rho(ab), \quad \phi(b) \rho_\alpha(a) \overset{w^*}{\longrightarrow} \phi(b) \rho(a) = \rho(ab).$$

On the other hand, the map $\pi_A^{**}$ is a $w^*$-continuous map, so $\pi_A^{**} \circ \rho_\alpha(a) \overset{w^*}{\longrightarrow} \pi_A^{**} \circ \rho(a)$, for each $a \in A$. Then

$$\phi \circ \pi_A \circ \rho_\alpha(a) = \tilde{\phi} \circ \pi_A^{**} \circ \rho_\alpha(a) = \pi_A^{**} \circ \rho_\alpha(a)(\phi) \rightarrow \pi_A^{**} \circ \rho(a)(\phi) = \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a).$$
Also for each \( a, b \in A \), we have
\[
a \cdot \rho_a(b) \xrightarrow{w^*} a \cdot \rho(b) = \rho(ab), \quad \rho_a(ab) \xrightarrow{w^*} \rho(ab), \quad \phi(b)\rho_a(a) \xrightarrow{w^*} \phi(b)\rho(a).
\]
So
\[
a \cdot \rho_a(b) - \rho_a(ab) \xrightarrow{w^*} 0, \quad \phi(b)\rho_a(a) - \phi(b)\rho_a(a) \xrightarrow{w^*} 0.
\]
Put \( F = \{a_1, a_2, ..., a_n\} \) and \( G = \{b_1, b_2, ..., b_m\} \) for finite subsets of \( A \). Define
\[
M = \{(a_1 \cdot T(b_1) - T(a_1b_1), a_2 \cdot T(b_2) - T(a_2b_2), ..., a_n \cdot T(b_n) - T(a_nb_n)) : T \in B(A, A \otimes_P A)\},
\]
it is easy to see that \( M \) is a convex subset of \( \prod_{i=1}^n (A \otimes_p A) \oplus 1 \prod_{i=1}^n \mathbb{C} \) and \((0,0,0,0) \in \overline{M}^w = \overline{M}^{**} \). It follows that, there exists a net \( \xi_{(e,F,G)} \in B(A, A \otimes_P A) \) such that
\[
\|a_i \cdot \xi_{(e,F,G)}(b_i) - \xi_{(e,F,G)}(a_ib_i)\| < \epsilon, \quad \|\xi_{(e,F,G)}(a_ib_i) - \phi(b_i)\xi_{(e,F,G)}(a_i)\| < \epsilon
\]
for each \( i \in \{1, 2, ..., n\} \). It follows that the net \( \xi_{(e,F,G)} \), for each \( a, b \in A \), satisfies
\[
a \cdot \xi_{(e,F,G)} - \xi_{(e,F,G)}(ab) \xrightarrow{0}, \quad \phi(b)\xi_{(e,F,G)}(a) - \xi_{(e,F,G)}(ab) \xrightarrow{0}
\]
and
\[
\phi \circ \pi_A \circ \xi_{(e,F,G)}(a) - \phi(a) \xrightarrow{0}.
\]
Therefore \( A \) is approximately left \( \phi \)-biprojective.

**Lemma 2.2.** If \( A \) is an approximately left \( \phi \)-biprojective with bounded net \( \rho_a \), then \( A \) is left \( \phi \)-biflat.

**Proof.** Let \( A \) be approximately left \( \phi \)-biprojective with bounded net \( \rho_a \). So \( \rho_a \in B(A, (A \otimes_p A)^{**}) \cong (A \otimes_p (A \otimes_p A)^*)^* \) has a \( w^* \)-limit-point, say \( \rho \). Since
\[
a \cdot \rho_a(b) - \rho_a(ab) \xrightarrow{0}, \quad \phi(b)\rho_a(a) - \rho(ab) \xrightarrow{0}, \quad \phi \circ \pi_A \circ \rho_a(a) - \phi(a) \xrightarrow{0}.
\]
It follows that
\[
a \cdot \rho(b) = \rho(ab) = \phi(b)\rho(a), \quad \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a), \quad (a \in A).
\]
for each \( a \in A \). \( \square \)

**Proposition 2.3.** Let \( A \) be a Banach algebra with an approximate identity and let \( \phi \in \Delta(A) \). If \( A^{**} \) is approximately biflat, then \( A \) is left \( \phi \)-biflat.

**Proof.** Since \( A \) has an approximate identity \( \overline{\text{ker } \phi} = \text{ker } \phi \). Thus by [11] Theorem 3.3 \( A \) is left \( \phi \)-amenable. So there exists an element \( m \in A^{**} \) such that \( \rho m = \phi(a)m \) and \( \hat{m} = 1 \) for every \( a \in A \). Define \( \rho : A \to A^{**} \otimes_p A^{**} \) by \( \rho(a) = \phi(a)m \otimes m \). Clearly \( \rho \) is a bounded linear map such that
\[
a \cdot \rho(b) = \rho(ab) = \phi(b)\rho(a), \quad \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a), \quad (a \in A).
\]
There exists a bounded linear map \( \psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**} \) such that for \( a, b \in A \) and \( m \in A^{**} \otimes_p A^{**} \), the following holds;

(i) \( \psi(a \otimes b) = a \otimes b \),
(ii) \( \psi(m \cdot a) = \psi(m \cdot a) \),
(iii) \( \pi_A^{**}(\psi(m)) = \pi_A^{**}(m) \),

see [4] Lemma 1.7. Set \( \eta = \psi \circ \rho : A \to (A \otimes_p A)^{**} \). It is easy to see that \( a \cdot \eta(b) = \eta(ab) = \phi(b)\eta(a) \)
\[
\tilde{\phi} \circ \pi_A^{**} \circ \eta(a) = \tilde{\phi} \circ \pi_A^{**} \circ \psi \circ \rho(a) = \tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a), \quad (a \in A).
\]
So \( A \) is left \( \phi \)-biflat. \( \square \)
Let $A$ be a Banach algebra and $I$ be a totally ordered set. By $UP_I(A)$ we denote the set of $I \times I$ upper triangular matrices which its entries come from $A$ and

$$||(a_{i,j})_{i,j \in I}|| = \sum_{i,j \in I} ||a_{i,j}|| < \infty.$$  

With matrix operations and $|| \cdot ||$ as a norm, $UP_I(A)$ becomes a Banach algebra.

**Proposition 2.4.** Let $I$ be a totally ordered set with the greatest element. Also let $A$ be a Banach algebra with left identity and $\phi \in \Delta(A)$. Then $UP(I, A)^{**}$ is left $\psi_\phi$-biflat if and only if $|I| = 1$ and $A$ is left $\phi$-biflat.

**Proof.** Suppose $UP_I(A)$ is left $\psi_\phi$-biflat. Let $i_0 \in I$ be the greatest element of $I$ with respect to $\leq$. Since $A$ has a left unit, $UP_I(A)$ has a left approximate identity. By [16, Lemma 2.1] left $\psi_\phi$-amenability of $UP_I(A)^{**}$ implies that $UP_I(A)$ is left $\psi_\phi$-amenable.

Define

$$J = \{(a_{i,j})_{i,j \in I} \in UP_I(A) | a_{i,j} = 0 \text{ for } j \neq i_0\}.$$  

Clearly $J$ is a closed ideal of $UP_I(A)$ with $\psi_{i_0} | J \neq 0$. Applying [6, Lemma 3.1] gives that $J$ is left $\psi_{i_0}$-amenable. So by [6, Theorem 1.4] there exists a bounded net $(j_\alpha)$ in $J$ which satisfies

$$jj_\alpha - \psi_\phi(j)j_\alpha \to 0, \quad \psi_\phi(j_\alpha) = 1 \quad (j \in J).$$

Suppose in contradiction that $I$ has at least two elements. Let $a_0$ be an element in $A$ such that $\phi(a_0) = 1$. Set $j = \begin{pmatrix} \cdots & 0 & \cdots & 0 & a_0 \\ \cdots & 0 & \cdots & 0 & a_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & \cdots & 0 & a_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$. Clearly for each $\alpha$ the net $j_\alpha$ has a form

$$\begin{pmatrix} \cdots & 0 & \cdots & 0 & j^\alpha \\ \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & \cdots & 0 & j^\alpha \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $(j^\alpha_i)$, $(j^\alpha_j)$ and $(j^\alpha_{i_0})$ are some nets in $A$. Put $j$ and $j_\alpha$ in (2.1)

$$j_\alpha - \psi_\phi(j)j_\alpha \to 0, \quad \psi_\phi(j_\alpha) = 1 \quad (j \in J).$$

we have $j^\alpha_{i_0}a_0 \to 0$. Since $\phi$ is continuous, we have $\phi(j^\alpha_{i_0}) \to 0$. On the other hand $\psi_\phi(j_\alpha) = \phi(j^\alpha_{i_0}) = 1$ which is a contradiction. So $I$ must be singleton and the proof is complete. \hfill \Box

**Corollary 2.5.** Let $I$ be a totally ordered set with the greatest element. Also let $A$ be a Banach algebra with left identity and $\phi \in \Delta(A)$. Then $UP_I(A)^{**}$ is approximately biflat if and only if $|I| = 1$ and $A$ is approximately biflat.

**Example 2.6.** We give a Banach algebra which is not left $\phi$-biflat but it is approximate left $\phi$-biprojective. So the converse of Proposition [2.1] is not always true. Let denote $\ell^1$ for the set of all sequences $a = (a_n)$ of complex numbers equipped with $||a|| = \sum_{n=1}^{\infty} |a_n| < \infty$ as its norm.

With the following product:

$$(a * b)(n) = \begin{cases} a(1)b(1) & \text{if } n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \text{if } n > 1 \end{cases},$$

$A = (\ell^1, || \cdot ||)$ becomes a Banach algebra. Clearly $\Delta(\ell^1) = \{\phi_1, \phi_2 + \phi_3\}$, where $\phi_0(a) = a(n)$ for every $a \in \ell^1$. We claim that $\ell^1$ is not left $\phi_1$-biflat but $\ell^1$ is approximately left $\phi_1$-biprojective for some $\phi \in \Delta(\ell^1)$. We assume conversely that $\ell^1$ is left $\phi_1$-biflat. One can see that $(1, 0, 0, ...)$ is a unit for $\ell^1$. Therefore by [16, Lemma 2.1] left $\phi_1$-biflatness of $\ell^1$ implies that $\ell^1$ is left $\phi_1$-biflat.
φ₁-amenable. On the other hand by [9] Example 2.9, ℓ₁ is not left φ₁-amenable which is a contradiction.

Applying [9] Example 2.9, gives that ℓ₁ is approximate left φ₁-amenable. So [14] Proposition 2.4] follows that that ℓ₁ is approximate left φ₁-biprojective.

3. LEFT φ-BIPROJECTIVITY OF THE PROJECTIVE TENSOR PRODUCT BANACH ALGEBRAS

Theorem 3.1. Let A and B be Banach algebras which φ ∈ Δ(A) and ψ ∈ Δ(B). Suppose that A has a unit and B has an identity x₀ such that ψ(x₀) = 1. If A ⊗ₚ B is left φ ⊗ ψ-biflat, then A is left φ-amenable. Proof. Let ρ : A ⊗ₚ B → ((A ⊗ₚ B) ⊗ₚ (A ⊗ₚ B))** be a bounded linear map such that
\[ ρ(xy) = x · ρ(y) = φ ⊗ ψ(y)ρ(x), \]
\[ ρ(x) = φ ⊗ ψ(ρ(x)) = φ ⊗ ψ(x)(x, y ∈ A ⊗ₚ B). \]

For idempotent x₀ ∈ B and elements a₁, a₂ ∈ A we have
\[ a₁a₂ ∗ x₀ = a₁a₂ ∗ x₀ = a₁a₂ ∗ x₀² = (a₁ ∗ x₀)(a₂ ∗ x₀). \]

We denote e for the unit of A. So we have
\[ ρ(a₁a₂ ∗ x₀) = ρ((a₁ ∗ x₀)(a₂ ∗ x₀)) = (a₁ ∗ x₀) · ρ(a₂ ∗ x₀) = a₁(e ∗ x₀) · ρ(a₂ ∗ x₀) = a₁ρ(ea₂ ∗ x₀²), \]
also
\[ ρ(a₁a₂ ∗ x₀) = ρ((a₁ ∗ x₀)(a₂ ∗ x₀)) = φ ⊗ ψ(a₂ ∗ x₀)ρ(a₁ ∗ x₀) = φ(a₂)ρ(a₁ ∗ x₀) \]
and
\[ φ ⊗ ψ ⊗ π_{A⊗ₚB}^* ⊗ ρ(a₁ ∗ x₀) = φ ⊗ ψ(a₁ ∗ x₀) = φ(a₁), \]
for each a₁, a₂ ∈ A. Put ξ : (A ⊗ₚ B) ⊗ₚ (A ⊗ₚ B) → A ⊗ₚ A for a bounded linear map which is given by ξ((a ⊗ b) ∗ (c ⊗ d) = ψ(bd)a ⊗ c, for each a, c ∈ A and b, d ∈ B. Clearly
\[ π_{A}^* ⊗ ξ^* = (id_A ⊗ ψ)^* ⊗ π_{A⊗ₚB}^*. \]

Define θ : A → (A ⊗ₚ A)** by θ(a) = ξ** ⊗ ρ(a ∗ x₀). Clearly θ is a bounded linear map. We have
\[ a ∗ θ(b) = a ∗ ξ** ∗ ρ(b) = ξ** ∗ ρ(ab) = φ(b)ξ** ∗ ρ(a) = φ(b)θ(a), \]
also
\[ ϕ ∗ π_{A}^* ∗ ϕ(a) = ϕ ∗ π_{A}^* ∗ ξ** ∗ ρ(a ∗ x₀) = ϕ ∗ (id_A ⊗ ψ)^* ⊗ π_{A⊗ₚB}^* ⊗ ρ(a ∗ x₀) \]
\[ = ϕ ⊗ ψ ∗ π_{A⊗ₚB}^* ⊗ ρ(a ∗ x₀) = φ ⊗ ψ(a ∗ x₀) = ϕ(a), \]
for each a ∈ A. It follows that A is left φ-biflat. Since A has a unit by ... A is left φ-amenable. □

Note that previous theorem is also valid in the left φ-biprojective case. In fact we have

Corollary 3.2. Let A and B be Banach algebras which φ ∈ Δ(A) and ψ ∈ Δ(B). Suppose that A has a unit and B has an identity x₀ such that ψ(x₀) = 1. If A ⊗ₚ B is left φ ⊗ ψ-biprojective, then A is left φ-contractive.

Proposition 3.3. Suppose that A is a Banach algebra and φ ∈ Δ(A). Let A** be left φ-biprojective. Then A is left φ-biflat.
Proof. Let $A^{**}$ be $\tilde{\phi}$-biprojective. Then there exists a bounded linear map $\rho : A^{**} \to A^{**} \otimes_{\mathcal{P}} A^{**}$ such that $\rho(ab) = a \cdot \rho(b) = \tilde{\phi}(b) \rho(a)$ and $\tilde{\phi} \circ \pi_{A^{**}} \circ \rho(a) = \tilde{\phi}(a)$, for each $a, b \in A^{**}$. There exists a bounded linear map $\psi : A^{**} \otimes_{\mathcal{P}} A^{**} \to (A \otimes_{\mathcal{P}} A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_{\mathcal{P}} A^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a)$, $a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi_{A}^{**}(\psi(m)) = \pi_{A^{**}}(m)$,

see [4, Lemma 1.7]. Set $\eta = \psi \circ \rho_{A} : A \to (A \otimes_{\mathcal{P}} A)^{**}$. Clearly $\eta$ is a bounded linear map which satisfies $\eta(ab) = \psi \circ \rho_{A}(ab) = \psi(a \cdot \rho_{A}(b)) = a \cdot \psi \circ \rho_{A}(b)$ and

$$
\phi(b)\eta(a) = \phi(b)\psi \circ \rho_{A}(a) = \psi(\phi(b)\rho_{A}(a)) = \psi \circ \rho_{A}(ab) = \eta(ab).
$$

Also we have

$$
\tilde{\phi} \circ \pi_{A}^{**} \circ \eta(a) = \tilde{\phi} \circ \pi_{A}^{**} \circ \psi \circ \rho_{A}(a) = \tilde{\phi} \circ \pi_{A^{**}} \circ \rho_{A}(a) = \phi(a),
$$

for each $a \in A$. It follows that $A$ is left $\phi$-biflat. \hfill \qedsymbol

Theorem 3.4. Let $A$ and $B$ be Banach algebra with $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. If $A$ left $\phi$-biprojective and $B$ is $\psi$-biprojective, then $A \otimes_{\mathcal{P}} B$ is left $\phi \otimes \psi$-biprojective.

Proof. Since $A$ left $\phi$-biprojective and $B$ is $\psi$-biprojective, there exist bounded linear maps $\rho_{A} : A \to A \otimes_{\mathcal{P}} A$ and $\rho_{B} : B \to B \otimes_{\mathcal{P}} B$ such that

$$
\rho_{A}(a_{1}a_{2}) = a_{1} \cdot \rho_{A}(a_{2}) = \phi(a_{2})\rho_{A}(a_{1}), \quad \phi \circ \pi_{A} \circ \rho_{A} = \phi, \quad (a_{1}, a_{2} \in A)
$$

and

$$
\rho_{B}(b_{1}b_{2}) = b_{1} \cdot \rho_{B}(b_{2}) = \phi(b_{2})\rho_{B}(b_{1}), \quad \psi \circ \pi_{B} \circ \rho_{B} = \psi, \quad (b_{1}, b_{2} \in B).
$$

Let $\theta$ be an isometrical isomorphism from $(A \otimes_{\mathcal{P}} A) \otimes_{\mathcal{P}} (B \otimes_{\mathcal{P}} B)$ into $(A \otimes_{\mathcal{P}} B) \otimes_{\mathcal{P}} (A \otimes_{\mathcal{P}} B)$ which is given by $\theta(a_{1} \otimes a_{2} \otimes b_{1} \otimes b_{2}) = a_{1} \otimes b_{1} \otimes a_{2} \otimes b_{2}$ for each $a_{1}, a_{2} \in A$ and $b_{1}, b_{2} \in B$. Define $\rho = \theta \circ (\rho_{A} \otimes \rho_{B})$. So

$$
\rho((a_{1} \otimes b_{1})(a_{2} \otimes b_{2})) = \theta \circ (\rho_{A} \otimes \rho_{B})((a_{1} \otimes b_{1})(a_{2} \otimes b_{2})) = \theta(\rho_{A}(a_{1}a_{2}) \otimes \rho_{B}(b_{1}b_{2}))
$$

$$
= \theta(a_{1} \cdot \rho_{A}(a_{2}) \otimes b_{1} \cdot \rho_{B}(b_{2}))
$$

$$
= \theta((a_{1} \otimes b_{1}) \cdot (\rho_{A}(a_{2}) \otimes \rho_{B}(b_{2}))
$$

$$
= (a_{1} \otimes b_{1}) \cdot \theta \circ (\rho_{A} \otimes \rho_{B})(a_{2} \otimes b_{2}),
$$

for each $a_{1}, a_{2} \in A$ and $b_{1}, b_{2} \in B$. It follows that $\rho(xy) = x \cdot \rho(y)$ for each $x, y \in A \otimes_{\mathcal{P}} B$. Also we have

$$
\phi \otimes \psi(a_{1} \otimes b_{1})\rho(a_{2} \otimes b_{2}) = \phi(a_{1})\psi(b_{1})\theta \circ (\rho_{A}(a_{2}) \otimes \rho_{B}(b_{2})) = \theta \circ (\phi(a_{1})\rho_{A}(a_{2}) \otimes \psi(b_{1})\rho_{B}(b_{2}))
$$

$$
= \theta \circ (\rho_{A}(a_{2}a_{1}) \otimes \rho_{B}(b_{2}b_{1}))
$$

$$
= \rho((a_{2} \otimes b_{2})(a_{1} \otimes b_{1})),
$$

for each $a_{1}, a_{2} \in A$ and $b_{1}, b_{2} \in B$. So for each $x, y \in A \otimes_{\mathcal{P}} B$ we have $\phi \otimes \psi(x)\rho(y) = \rho(xy)$. Note that

$$
\pi_{A \otimes_{\mathcal{P}} B} \circ \theta(a_{1} \otimes a_{2} \otimes b_{1} \otimes b_{2}) = \pi_{A \otimes_{\mathcal{P}} B}(a_{1} \otimes b_{1} \otimes a_{2} \otimes b_{2}) = \pi_{A}(a_{1} \otimes a_{2})\pi_{B}(b_{1} \otimes b_{2}),
$$

for each $a_{1}, a_{2} \in A$ and $b_{1}, b_{2} \in B$. Thus $A \otimes_{\mathcal{P}} B$ is left $\phi \otimes \psi$-biprojective.
it implies that $\pi_{A \otimes_p B} \circ \theta = \pi_A \otimes \pi_B$. Then

$$(\phi \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho(a \otimes b) = (\phi \otimes \psi) \circ \pi_{A \otimes_p B} \circ \theta \circ (\rho_A \otimes \rho_B)(a \otimes b)$$

$$= (\phi \otimes \psi) \circ (\pi_A \otimes \pi_B) \circ (\rho_A \otimes \rho_B)(a \otimes b)$$

$$= \phi \circ \pi_A \circ \rho_A(a) \psi \circ \pi_B \circ \rho_B(b)$$

$$= \phi(a)\psi(b) = \phi \otimes \psi(a \otimes b),$$

for each $a \in A$ and $b \in B$. Therefore $(\phi \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho(x) = \phi \otimes \psi(x)$ for every $x \in A \otimes_p B$. It follows that $A \otimes_p B$ is left $\phi \otimes \psi$-biprojective.

Let $\hat{G}$ be the dual group of $G$ which consists of all non-zero continuous homomorphism $\rho : G \to \mathbb{T}$. It is well-known that every character (multiplicative linear functional) $\phi \in \Delta(L^1(G))$ has the form $\phi(f) = \int_G \phi(x)f(x)dx$, where $dx$ is the normalized Haar measure and $\rho \in \hat{G}$, for more details see [5] Theorem 23.7. Note that, since $L^1(G)$ is a closed ideal of the measure algebra $M(G)$, each character on $L^1(G)$ can be extended to $M(G)$. Note that for a locally compact group $G$, we denote $A(G)$ for the Fourier algebra. The character space $\Delta(A(G))$ consists of all point evaluations $\phi_x$ for each $x \in G$, where

$$\phi_x(f) = f(x), \quad (f \in A(G)),$$

see [6] Example 2.6].

**Theorem 3.5.** Let $G$ be a locally compact group. Then $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is finite, where $\phi \in \Delta(L^1(G))$ and $\psi \in \Delta(A(G))$.

**Proof.** Let $M(G) \otimes_p A(G)$ be left $\phi \otimes \psi$-biprojective. Let $e$ be the unit of $M(G)$ and $a_0$ be the element of $A(G)$ such that $\psi(a_0) = 1$. Put $x_0 = e \otimes a_0$. Clearly $xx_0 = x_0x$ and $\phi \otimes \psi(x_0) = 1$, for every $x \in M(G) \otimes_p A(G)$. Now applying [13] Lemma 2.2 $M(G) \otimes_p A(G)$ is left $\phi \otimes \psi$-contractible. Now using [10] Theorem 3.14 $M(G)$ is left $\phi$-contractible, so by [10] Theorem 6.2 $G$ is compact. Also by [10] Theorem 3.14 $A(G)$ is left $\psi$-contractible. Thus by [10] Proposition 6.6] $G$ is discrete. Therefore $G$ is finite.

Converse is clear. \(\square\)

**Theorem 3.6.** Let $G$ be a locally compact group. Then $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is compact, where $\phi, \psi \in \Delta(L^1(G))$.

**Proof.** Suppose that $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective. Let $e$ be the unit of $M(G)$ and $e_\alpha$ be a bounded approximate identity of $L^1(G)$. Clearly $e \otimes e_\alpha$ is a bounded approximate identity. Thus by [13] Lemma 2.2 $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-contractible. So [10] Theorem 3.14 $L^1(G)$ is left $\psi$-contractible. Then by [10] Theorem 6.2 $G$ is compact.

For converse, suppose that $G$ is compact. Then by [10] Theorem 3.14 $M(G)$ is left $\phi$-contractible and by [10] Theorem 3.14 $L^1(G)$ is left $\psi$-contractible. Applying [10] Theorem 3.14 $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-contractible. So by [13] Lemma 2.1 $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective. \(\square\)

A Banach algebra $A$ is called left character biprojective (left character biflat) if $A$ is left $\phi$-biprojective (if $A$ is left $\phi$-biflat) for each $\phi \in \Delta(A)$, respectively.

**Theorem 3.7.** Let $G$ be a locally compact group. Then $M(G) \otimes_p L^1(G)$ is left character biprojective if and only if $G$ is finite.

**Proof.** Let $M(G) \otimes_p L^1(G)$ be left character biprojective. So $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective for each $\phi \in \Delta(M(G))$ and $\psi \in \Delta(L^1(G))$. So by similar arguments as in previous
Proposition, $M(G)$ left $\phi$-contractible for each $\phi \in \Delta(M(G))$. Since $M(G)$ is unital, by [10] Corollary 6.2 $G$ is finite.

Converse is clear. □

**Theorem 3.8.** Let $G$ be a locally compact group. Then $M(G) \otimes_p L^1(G)$ is left character biflat if and only if $G$ is a discrete amenable group.

**Proof.** Since $M(G)$ is unital and $L^1(G)$ has a bounded approximate identity, $M(G) \otimes_p L^1(G)$ has a bounded approximate identity. Thus by [16] Lemma 2.1 $M(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-amenable for each $\phi \in \Delta(M(G))$ and $\psi \in \Delta(L^1(G))$. So by [6] Theorem 3.3 $M(G)$ is left $\phi$-amenable for each $\phi \in \Delta(M(G))$. Since $M(G)$ is unital, $M(G)$ character amenable. Therefore by the main result of [8], $G$ is discrete and amenable.

For converse, let $G$ be discrete and amenable. Then $M(G) \otimes_p L^1(G) = \ell^1(G) \otimes_p \ell^1(G) \cong \ell^1(G \times G)$. Applying Johnson’s theorem (see [13] Theorem 2.1.18) that $\ell^1(G \times G)$ is an amenable Banach algebra. So by [13] Exercise 4.3.15 $\ell^1(G \times G)$ biflat. Then $\ell^1(G \times G)$ is left character biflat. □

**Proposition 3.9.** Let $G$ be an amenable group. Then $A(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-biprojective if and only if $G$ is finite.

**Proof.** Since $G$ is amenable, Leptin’s Theorem [13] Theorem 7.1.3 gives that $A(G)$ has a bounded approximate identity. It is well-known that $L^1(G)$ has a bounded approximate identity. Then by [15] Proposition 2.4, left $\phi \otimes \psi$-biprojectivity of $A(G) \otimes_p L^1(G)$ implies that $A(G) \otimes_p L^1(G)$ is left $\phi \otimes \psi$-contractible. So using [10] Theorem 3.14 gives that $A(G)$ is left $\phi$-contractible. Then by [10] Proposition 6.6 $G$ is discrete. Also by [10] Theorem 3.14 $L^1(G)$ is left $\psi$-contractible. Then [10] Theorem 6.1 implies that $G$ is compact. It follows that $G$ is finite.

Converse is clear. □

**Proposition 3.10.** Let $G$ be a locally compact group. Then $A(G) \oplus_1 L^1(G)$ is left character biprojective if and only if $G$ is finite.

**Proof.** Suppose that $A(G) \oplus_1 L^1(G)$ is left character biprojective. Let $\phi \in \Delta(A(G))$. Choose an element $a_0 \in A(G)$ such that $\phi(a_0) = 1$. Clearly the element $x_0 = (a_0, 0)$ belongs to $A(G) \oplus_1 L^1(G)$ which $xx_0 = x_0x$ and $\phi(x_0) = 1$. Using [15] Lemma 2.2, left character biprojectivity of $A(G) \oplus_1 L^1(G)$ implies that $A(G) \oplus_1 L^1(G)$ is left $\phi$-contractible. Since $A(G)$ is a closed ideal in $A(G) \oplus_1 L^1(G)$ and $\phi|_{A(G)} \neq 0$, by [10] Proposition 3.8 $A(G)$ is left $\phi$-contractible. So by [10] Proposition 6.6 $G$ is discrete. Thus $A(G) \oplus_1 L^1(G) = A(G) \oplus_1 \ell^1(G)$. We know that $\ell^1(G)$ has an identity $e$. Replacing $e$ with $a_0$ and $\psi$ with $\phi$ (for some $\psi \in \Delta(L^1(G))$) and following the same argument as above, we can see that $\ell^1(G)$ is left $\psi$-contractible. Thus by [10] Theorem 6.1 $G$ is compact. Therefore $G$ must be finite.

Converse is clear. □

A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on $G$ if it satisfies the following conditions

1. $S(G)$ is dense in $L^1(G)$,
2. $S(G)$ with a norm $||x||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,
3. for $f \in S(G)$ and $g \in G$, we have $L_g(f) \in S(G)$ the map $y \mapsto L_y(f)$ from $G$ into $S(G)$ is continuous, where $L_g(f)(x) = f(y^{-1}x)$,
4. $||L_g(f)||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $g \in G$.

For various examples of Segal algebras, we refer the reader to [12].

A locally compact group $G$ is called $SIN$, if it contains a fundamental family of compact invariant neighbourhoods of the identity, see [2] p. 86].
Proposition 3.11. Let \( G \) be a SIN group. Then \( S(G) \otimes_p S(G) \) is left \( \phi \otimes \psi \)-biprojective if and only if \( G \) is compact, for some \( \phi \in \Delta(S(G)) \).

Proof. Let \( S(G) \otimes_p S(G) \) be left \( \phi \otimes \phi \)-biprojective. Since \( G \) is a SIN group, the main result of [7] gives that \( S(G) \) has a central approximate identity. It follows that there exists an element \( x_0 \in S(G) \) such that \( xx_0 = x_0x \) and \( \phi(x_0) = 1 \), for each \( x \in S(G) \). Set \( u_0 = x_0 \otimes x_0 \). It is easy to see that \( uu_0 = u_0u \) and \( \phi \otimes \phi(u_0) = 1 \), for every \( u \in S(G) \otimes_p S(G) \). Using [15, Lemma 2.2] left \( \phi \otimes \phi \)-biprojectivity of \( S(G) \otimes_p S(G) \) follows that \( S(G) \otimes_p S(G) \) is left \( \phi \otimes \psi \)-contractible. By [10, Theorem 3.14] \( S(G) \) is left \( \phi \)-contractible. Thus [1, Theorem 3.3] gives that \( G \) is compact.

For converse, suppose that \( G \) is compact. Then by [1, Theorem 3.3] \( S(G) \) is left \( \phi \)-contractible. So by [10, Theorem 3.14] \( S(G) \otimes_p S(G) \) be left \( \phi \otimes \phi \)-contractible. Applying [15, Lemma 2.1] finishes the proof. \( \square \)

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