Skewed Non-Fermi Liquids and the Seebeck Effect

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We consider non-Fermi liquids in which the inelastic scattering rate has an intrinsic particle-hole asymmetry and obeys $\omega/T$ scaling. We show that, in contrast to Fermi liquids, this asymmetry influences the low-temperature behaviour of the thermopower even when the impurity scattering dominates. Implications for the unconventional sign and temperature dependence of the thermopower in cuprates in the strange metal (Planckian) regime are emphasized.

\section{I. INTRODUCTION}

Besides its relevance to thermoelectricity, the Seebeck effect provides invaluable insights into the fundamental physics of materials\textsuperscript{[1–3]}. The Seebeck coefficient $S$ (thermopower) is sensitive to the balance between hole-like and electron-like excitations. It is negative when electrons dominate, positive when holes dominate and vanishes when particle/hole symmetry holds. In many cases, the particle-hole asymmetry originates in the bandstructure of the material: it is controlled by the number (density of states) and velocities of the two types of excitations. However, it has been recognized that another source of asymmetry may also influence the Seebeck coefficient: that of the lifetime (scattering rate) of these excitations. Although this has been discussed theoretically \textsuperscript{[4–12]} and put forward as a possible mechanism for materials in which the sign of $S$ is found to be opposite to that predicted by bandstructure \textsuperscript{[4, 13–15]}, it has received comparatively less attention. One of the reasons is that, as detailed below, the particle-hole asymmetry of the inelastic scattering rate does not influence $S$ at low-temperature for metals obeying Fermi liquid theory when impurity scattering is also present.

In this article, we show that the situation is entirely different in correlated metals which do not obey Fermi liquid theory. We consider a family of non-Fermi liquids in which the inelastic (electron-electron) scattering rate $\Gamma_\text{in}$ obeys $\omega/T$ scaling (with $\omega$ the energy of an excitation counted from Fermi level). We demonstrate that in ‘skewed’ non-Fermi liquids where the scaling function has an odd frequency component, this particle-hole asymmetry affects the low-$T$ behaviour of the Seebeck coefficient down to $T = 0$, even in the presence of impurity scattering. This is an unexpected finding because the impurity scattering is temperature independent, whereas the electron-electron scattering diminishes upon cooling down and vanishes at $T = 0$. The sign of $S$ can be reversed in comparison to that expected from bandstructure. The case of a ‘Planckian’ metal \textsuperscript{[16–22]} with $\Gamma_\text{in} \propto \omega, T$ turns out to be particularly interesting. In that case, $S/T$ ultimately diverges logarithmically at low temperature. However, the temperature dependence and sign of $S/T$ over an extended temperature range is strongly affected by the particle-hole asymmetry of $\Gamma_\text{in}$.

As discussed below, this may be relevant to the understanding of the Seebeck coefficient of cuprate superconductors, especially close to the critical doping where the pseudo-gap opens and a logarithmic dependence of the specific heat is observed \textsuperscript{[23]}.

The paper is organized as follows: In Sec. II we describe the Kubo-Boltzmann formalism used in our calculations of the Seebeck coefficients, taking explicit account of the skeweness (particle-hole asymmetry) of the inelastic scattering rate. In Sec. III we describe our main results that reveal the unusual effect of skewness in non-Fermi liquids and contrast it to the more moderate behavior in Fermi liquids. In Sec. IV we discuss the implications of our findings for experiments. Technical details are delegated to appendices: In Appendix A we give a derivation of the transport equations starting from the Kubo formalism, in Appendix B we discuss the $\omega/T$ scaling properties of the considered class of non-Fermi liquids, and in Appendix C we elaborate on the behavior of the Seebeck coefficient at higher temperatures.

\section{II. SEEBECK COEFFICIENT AND PARTICLE-HOLE ASYMMETRIC INELASTIC SCATTERING}

We recall that in metals, for non-interacting electrons and in the presence of elastic scattering only, the Seebeck coefficient at low temperature $T$ is given by:

$$ S = -\frac{k_B}{e} \begin{pmatrix} \pi^2 \Phi_0' \\ 3 \Phi_0 \end{pmatrix} $$

This expression involves the transport function $\Phi(\varepsilon) = 2 \int d^d k / (2\pi)^d v_k^2 \delta(\varepsilon - \varepsilon_k)$ at the Fermi level $\Phi_0 = \Phi(\varepsilon_F)$ and its derivative with respect to energy $\Phi_0' = \Phi'(\varepsilon_F)$. $v_k = (\nabla k \varepsilon_k) / \hbar$ denotes the electron velocity in the direction $\alpha = x, y, z$ considered (we set $\hbar = k_B = 1$ in most of the following). In this simplest description, $S/T$ does not depend on the magnitude of the scattering rate and its sign is determined by the particle-hole asymmetry of the bandstructure encoded in the transport function.
$S/T > 0$ for a hole-like bandstructure ($\Phi'_0 < 0$) and $S/T < 0$ for an electron-like bandstructure ($\Phi'_0 > 0$).

We now consider the effect of both elastic and inelastic scattering (that is, the impurity and the electron-electron scattering, respectively), writing the total scattering rate as:

$$\Gamma(T, \omega) = \gamma + \Gamma_{in}(T, \omega) \quad (2)$$

It is convenient for our purpose to decompose the scattering time into components which are even and odd in frequency:

$$\tau_{\pm}(T, \omega) = \frac{1}{2} \left[ \frac{1}{\Gamma(T, \omega)} \pm \frac{1}{\Gamma(T, -\omega)} \right] \quad (3)$$

Note that we assume that the elastic scattering rate $\gamma$ is isotropic and that the inelastic scattering rate only depends on frequency and not on momentum (i.e. does not vary along the Fermi surface). The isotropy assumption allows us to keep the discussion simple and is sufficient to reveal the main effects that we wish to emphasize. Likewise, in our calculation of transport we do not take into account vertex corrections. This simplification is exact in models in which both the self-energy $\Sigma$ and the many-particle vertex are local (momentum-independent) [24].

We thus take $\Gamma_{in}(T, \omega) = -2m\Sigma(\omega + i\tau, T)$ (also neglecting possible distinctions between current and energy relaxation rates [25, 26]).

Starting from the Kubo formula, one can derive the following expression for the Seebeck coefficient in the low-temperature regime [27], as detailed in Appendix A:

$$S = -\frac{k_B}{e} \frac{I_1(T)}{I_0(T)} \quad (4)$$

In these expressions, the frequency dependence of the scattering rates is expressed in terms of the scaling variable $x = \omega/T$: $\tau_{\pm} = \tau_{\pm}(T,xT)$ and we use the notation: $(F(x)) \equiv \int_{-\infty}^{\infty} dx F(x)/4 \cosh^2 \frac{x}{2}$. $Z$ denotes the effective mass renormalisation, which for a local theory is related to the real part of the self-energy by: $1/Z(T, \omega) = 1/|\text{Re}\Sigma(0) - \text{Re}\Sigma(\omega)|/\omega$. Strictly speaking, the frequency dependence of $Z$ has to be kept in the integrals entering Eqs. (5,6). As we discuss in Appendix A, neglecting this effect is actually a good approximation, and we use it here to simplify the discussion. In a local Fermi liquid, $Z(T)$ coincides with the quasiparticle spectral weight and reaches a finite value at $T = 0$, while in a non-Fermi liquid $Z(T)$ may vanish as $T \to 0$ [28]. In the absence of interactions ($Z = 1$) and for elastic scattering only ($\tau_{\pm} = 1/\gamma, \tau_{-} = 0$) we recover from Eqs. (4,5,6) the simple expression Eq. (2) $\langle \Gamma \rangle = 1, \langle x^2 \rangle = \pi^2/3$. The effect of a possible temperature dependence of $Z(T)$ on both the Seebeck coefficient and the specific heat/entropy, for example near a quantum critical point, has been previously discussed in the literature, see e.g. Ref. [29]. In the present work, we focus on the effect of a particle hole asymmetry of the scattering rate, i.e on the terms involving $\langle x\tau_{-} \rangle$ in Eqs. (5,6).

III. FERMI LIQUIDS VS. NON-FERMI LIQUIDS

A. Fermi liquid

We first consider a (local) Fermi liquid with an inelastic scattering rate:

$$\Gamma_{in}(T, \omega) = -2m\Sigma(\omega + i\tau, T) \quad (5)$$

The key point is that the odd part of the scattering rate scales as $\omega^3, \omega T^2 \sim xT^3$ and hence is subdominant as compared to the even-frequency part: the inelastic scattering rate of conventional Fermi liquids is asymptotically particle-hole symmetric at low energy. Adding the elastic scattering rate, we see that there are two regimes. In the ‘elastic regime’ $\gamma$ dominates over the inelastic scattering rate: this holds for $\gamma \geq \lambda(T)^2$ or alternatively for $T \leq T^*$ with $T^* \sim (\gamma/\pi^2\lambda)^{1/2}$ a crossover temperature. In the ‘inelastic’ regime ($\gamma \leq \lambda(T)^2$ or $T > T^*$), inelastic scattering dominates [27]. Let us consider first the ‘inelastic’ regime, in which $\tau_{\pm}$ is of order $1/T^2$, and $\tau_{-}$ of order $1/T$. Hence, in the denominator $I_0$ of Eq. (4), $\langle \tau_{+} \rangle \sim 1/T^2$ dominates over $T \langle x\tau_{-} \rangle \sim \text{const}$. In contrast, in the numerator $I_1$, the odd term $\langle x\tau_{-} \rangle$ has the same $1/T$ temperature dependence as the even one $\langle x^2\tau_{+} \rangle$. Taking into account that, in a Fermi liquid,
the effective mass enhancement $1/Z = m^*/m$ reaches a constant at low temperature, we obtain in the inelastic limit $T \gg T^*$:

$$S^{FL}_{\text{in}} \simeq - \frac{k_B}{e} \left[ \frac{12}{\pi^2} \frac{1}{Z} \Phi_0 - \frac{12}{\pi^2} (c_a a + c_b b) \right]$$  \hspace{1cm} (8)

with $c_a = \langle x^4 \rangle / (1 + x^2/\pi^2)^2 \simeq 6.51$, $c_b = \langle x^2 (1 + x^2/\pi^2)^2 \rangle \simeq 1.09$. Remarkably, the odd-frequency terms of $\Gamma_{\text{in}}$ directly contribute to $S$ in this limit, on equal footing with the bandstructure term. This was, to our knowledge, first emphasized in Ref. [5] in which expression Eq. (8) was derived, and further discussed in Refs [6–10, 12]. We also note that the prefactor of the bandstructure term is modified (from $\pi^2/3 \simeq 3.29$ to $12 - \pi^2 \simeq 2.13$) as compared to the elastic (low-T) limit.

However, in the low-$T$ ‘elastic’ limit ($\gamma \gg \lambda \pi^2 T^2$ or $T \ll T^*$), this interesting effect disappears. Indeed, $\tau_+ \sim 1/\gamma$ is constant in this limit, while $\tau_-$ vanishes as $T^3$, leading to:

$$S^{FL}_{\text{el}} \simeq - \frac{k_B}{e} \frac{\pi^2}{Z} \Phi_0$$  \hspace{1cm} (9)

Hence, odd-frequency scattering does not contribute at low temperature. The conventional elastic value Eq. (2) is recovered, with the notable difference that the prefactor is enhanced by the effective mass $1/Z = m^*/m$. Indeed, it was emphasized in Ref. [30] that $S/T$ is proportional to the linear term in the specific heat ($\sim m^*/m$) in many materials.

The crossover between the low-$T$ elastic limit and the high-$T$ inelastic limit is illustrated on Fig. 1. The data in this plot as well as in Fig. 2 and Fig. 3 are obtained from evaluating the Seebeck coefficient with Eq. (4). In appendix A, we show that the results are unchanged when evaluated from the full Kubo formula. We see that, in a Fermi liquid, the band value of the Seebeck coefficient enhanced by the effective mass effect $Z$ is recovered below the crossover temperature $T^*$. In that regime, the particle-hole asymmetry of the inelastic scattering has no influence on the Seebeck coefficient. In practice, $T^*$ can be estimated as the characteristic temperature at which the measured $T$-dependent contribution to the resistivity becomes comparable in magnitude to the residual resistivity at low-$T$.

**B. ‘Skewed’ Non-Fermi liquids.**

The washing out of the effects of the particle-hole asymmetry in the Fermi liquid case happens because the odd-frequency terms in the inelastic scattering rate are subdominant in comparison to the even ones, and hence do not contribute to $S$ at low-$T$. We consider now a class of non-Fermi liquids in which, in contrast, the odd-frequency terms are of the same order as the even-frequency ones, such that the inelastic scattering rate obeys a scaling form:

$$\Gamma_{\text{in}}(T, \omega) = \lambda (\pi T)^\nu g \left( \frac{\omega}{T} \right).$$  \hspace{1cm} (10)

Here, $\nu$ is an exponent (we focus on $\nu \leq 1$ in the following) and the scaling function $g(x)$ contains both an even and an odd component. It has a regular expansion at small $x$, so that for $\omega \ll T$: $\Gamma_{\text{in}} \sim \lambda (\pi T)^\nu g(0) + g'(0) T^{\nu-1} \omega + \cdots$, while $g(x) \sim c_+^\infty \cdot |x|^\nu$ at large $x$. The effects discussed in this article do not depend on the specific form of the scaling function $g(x)$. Systems obeying this scaling form with a non-even scaling function can be called ‘skewed non-Fermi liquids’.

It is of course expected that systems such as doped Mott insulators should display a particle-hole asymmetry (see e.g. [8]). However, while it is clear that such an asymmetry exists on energy scales comparable to electronic ones, it is a more demanding requirement that this asymmetry persists down to frequencies comparable to temperature itself, as assumed in (10). Indeed, this does not apply in a Fermi liquid. In contrast, we note that the scaling form Eq. (10) has been shown to apply in overscreened Kondo models controlled by a non-Fermi liquid fixed point [31]. It is also relevant to the proximity of the quantum critical point of doped random-exchange Hubbard models, related to Sachdev-Ye-Kitaev (SYK) models [32, 33], as recently studied in Refs. [34–41].

For such models obeying conformal invariance at low-energy, $g(x)$ is a universal scaling function that depends only on the exponent $\nu$ and on a ‘spectral asymmetry’ parameter $\alpha$. Its exact form was derived in Ref. [31] and reads (with the normalisation $g_{00}(0) = 1$):

$$g(x) = \left[ \Gamma \left( \frac{1 + \nu}{2} + i \frac{x + \alpha}{2\pi} \right) \right]^2 \frac{\cosh(x/2)}{\cosh(\alpha/2) \Gamma \left( (1 + \nu)/2 \right)^2},$$  \hspace{1cm} (11)

where $\Gamma(x)$ is the $\Gamma$-function. The $\omega/T$ scaling form is also relevant to the proximity of a quantum critical point associated with a strong-coupling fixed point [42].

The crossover temperature $T^*$ separating the ‘elastic’ and ‘inelastic’ limits now reads $(\pi T^*)^\nu = \gamma / \lambda$. In the low-$T$ elastic limit, $\tau_+ \sim 1/\gamma$ and $\tau_- \sim -\lambda^{-2} (\pi T)^\nu g_-(x) \sim -\gamma^{-1} (T/T^*)^\nu g_-(x)$. In this expression, $g_-(x) = [g(x) - g(-x)]/2$ is the odd-frequency component of the scaling function $g$. For the Seebeck coefficient at low $T$ one obtains

$$S^{NFL}_{\text{el}} = - \frac{k_B}{e} \left[ \frac{\pi^2}{3} \Phi_0 T - \frac{\lambda (\pi T)^\nu}{\gamma} c_- \right],$$  \hspace{1cm} (12)

in which $c_- = \langle x g_-(x) \rangle$ is a universal constant depending only on $g$. Ignoring at first the temperature dependence of quasiparticle weight $Z(T)$ one sees that whenever $\nu \leq 1$, the particle-hole asymmetry of the inelastic scattering influences $S^{NFL}_{\text{el}}$ even at the lowest temperatures, in sharp contrast to the Fermi liquid case discussed above.

What is the influence of the temperature dependence of $Z$? For $\nu < 1$ (we consider separately the $\nu = 1$ case
Expression Eq. (13) has several remarkable features. Firstly, we see that the odd-frequency inelastic scattering contributes to the low-\( T \) Seebeck on equal footing with the even-frequency/elastic contribution. Both terms in Eq. (13) have the same \( T \)-dependence \( \sim T^\nu \) but for different reasons: the first one because of the vanishing of \( Z(T) \), and the second one because of the \( T \)-dependence of \( \Gamma_B \). Secondly, in contrast to the former, this odd-frequency contribution is completely independent of the band-structure asymmetry: its sign is dictated by that of the constant \( c_- \), and thus by the intrinsic asymmetry of the inelastic rate scaling function. If this term dominates, the overall sign of the Seebeck can be opposite to that predicted by band-structure considerations in the low-\( T \) limit, even when elastic scattering is present. This is one of the main results of this work. The odd-frequency contribution dominates over the first term when the dimensionless ratio \( \eta = \gamma \Phi_0 / \Phi_0 \) is small, i.e. for clean-enough systems. Thirdly, we note that \( S \) depends on both the inelastic constant \( \lambda \) and the elastic rate \( \gamma \). This is an unusual situation in which the strength of the scattering does not drop out of the value of \( S \) at low-\( T \).

The behavior of Eq. 4 with temperature is investigated analytically in Appendix C. At high \( T \), we obtain

\[
S \sim - \frac{k_B}{e} \left( \frac{c_2}{c_0} \frac{T}{\gamma} + \eta c_2 \frac{T^\nu}{(T^* \gamma)^{\nu/2} + c_1/c_0} \right)
\]

with \( c_n = (x^n/g) \). Hence, as long as \( \eta = \gamma \Phi_0 / \Phi_0 \neq 0 \), the behavior of \( S \) at high temperature is dominated by the first term of Eq. (14) (with a linear dependence in \( T \), the same sign as the band-structure result, but with a slope that is corrected by \( c_2/c_0 \)). In contrast to the low-\( T \) limit, the sign of the Seebeck coefficient is thus given by the band-structure term. Remarkably, in the absence of elastic scattering or band asymmetry \( (\eta = 0) \), \( S \) instead tends to a constant value \(-c_1/c_0 k_B/e \), which depends only on the universal scaling function and its asymmetry. A related finding was reported in Ref. [37] (see also [40]) in the context of SYK models, where \( S \) was shown to be constant and determined by the spectral asymmetry \( \alpha \) (related by holography to the electric field/charge of the black hole and to the ground-state entropy of these models).

On Fig. 2(a,b), we display the temperature dependence of \( S \) and \( S/T \), respectively, plotted vs. \( T/T^* \) in the non-Fermi liquid case for several values of the asymmetry parameter \( \alpha \) in Eq. (11). We see that at low-\( T \) the sign of \( S \) can be changed by the scattering rate asymmetry, while it is set by the bandstructure term at high-\( T \), except for vanishing \( \eta \) where the Seebeck approaches a constant given by the last term of Eq. (14), instead.

The overall behavior at larger \( T \) reveals a \( \text{const} \pm T^\nu + T \) behavior at higher \( T \) (with \( \nu = 1,2 \) in Fig. 2), with a different coefficient of the \( T^\nu \) term in the high-\( T \) regime from that in the low-\( T \) regime. We see that the larger the asymmetry, the larger the temperature at which the sign change occurs as compared to the symmetric case.
A remark is in order here. In microscopic models or materials realizations, the universal scaling form of the scattering rate (10) is only expected to apply below a certain cutoff, which is of order of the bare electronic energy scales, e.g. the bandwidth. The physics discussed here obviously can only apply for temperatures below that cutoff. Hence, for our analysis of the T-dependence to be valid, the system should be clean enough that the crossover scale $T^*$ is smaller than such bare electronic energies at which non-universal effects will appear.

C. Skewed Planckian metal.

We now discuss the case of a ‘Planckian’ metal [16–22] ($ν = 1$) with a particle-hole asymmetry scaling with $ω/T$, so that $Γ_{in} = πλg(0)T + πλg′(0)ω + \cdots$ at low $ω$ and $T$ (‘skewed’ Planckian metal). Setting $ν = 1$ in expression Eq. (13) would predict a linear dependence of $S$ at low-$T$, with a slope that is given in terms of mutual effect of the band-structure and inelastic scattering rate asymmetries. This behavior corresponds to a case where quasiparticle strength is approximated by a constant. However, the quasiparticle weight actually vanishes logarithmically: $1/Z(T) \sim 1 + πλc_Z^2/λ$, with $λ$ a high-energy cutoff and $c_Z$ a universal constant depending on the scaling function $g$, see Appendix B (note that $λ$ is dimensionless in the Planckian case). Hence, in the low-$T$ ‘elastic’ limit $T ≪ T^*$:

$$S|^{SPM}_{el} \sim -\frac{k_B}{e} \left[ \frac{π^2}{3c_Z} \frac{Φ_0}{Φ_0} \ln \frac{λ}{T} + \frac{π^2}{3} \frac{Φ'_0}{Φ_0} - c_¬\frac{πλ}{γ} + \cdots \right]$$

(15)

Thus, in a Planckian metal, $S/T$ ultimately diverges logarithmically at very low-$T$. This corresponds to the logarithmic divergence of the effective mass (specific heat coefficient). The sign of the logarithmic term in $S$ is dictated by bandstructure. In contrast, the last term in Eq. (15) is controlled by the odd-frequency part of the inelastic scattering, and has a sign which can counteract the conventional bandstructure effect corresponding to the second term in Eq. (15). Interestingly, we note that the ‘skewed’ term dominates over the conventional one for cleaner systems $η/λ \lesssim 3c_¬/π$. The full temperature dependence is again given by Eq. (C5), setting $ν = 1$ and replacing $c_Z$ by $c_Z/ln λ/T$.

To illustrate these effects, we display on Fig 3 the $T$-dependence of $S/T$ for a ‘skewed’ Planckian metal. Two opposite signs of the bandstructure term $Φ_0/Φ_0$ are considered. In the absence of a scattering rate asymmetry ($α = 0$) it is seen that $S/T$ has a rather weak $T$-dependence (except at low-$T$ where the logarithmic term becomes relevant), while it acquires significant $T$-dependence for $α \neq 0$. Importantly, in the presence of an asymmetry, the sign of the Seebeck coefficient can be reversed in comparison to its bandstructure value over a wide range of temperature.

IV. SUMMARY AND DISCUSSION

In summary, our work reveals the importance of a particle-hole asymmetry of the inelastic scattering rate in a class of non-Fermi liquids. This asymmetry influences the Seebeck coefficient even in the asymptotic low-temperature regime, when inelastic scattering events are comparatively rare as compared to elastic scattering on impurities.

A number of materials display non-Fermi liquid behaviour and an unconventional $T$-dependence of the Seebeck coefficient. This has been addressed in previous theoretical work, such as Refs [29] and [43] which have emphasized the logarithmic divergence of $S/T$ in metals with a $T$-linear scattering rate. The $T$-dependence of $S$ close to a Fermi surface Lifshitz transition has been considered in [44, 45]. However, to our knowledge, the key role of a particle-hole asymmetry of the inelastic scattering rate in non-Fermi liquid metals with $ω/T$ scaling has not been discussed before. The theory presented here may be relevant when both the temperature dependence and the sign of the Seebeck coefficient are found to be unconventional.

Cuprate superconductors in the ‘strange metal’ regime display clear signatures of quantum criticality: T-linear resistivity (for reviews see e.g. [22, 46, 47]), a logarithmic divergence of the specific heat coefficient $C/T$ [23], and $ω/T$ scaling observed in optics [48] and angular-resolved photoemission (ARPES) [49] spectroscopies. An increase of the in-plane $S/T$ at low-$T$ reminiscent of
Fig. 3 has been reported for La$_{1.8-x}$Eu$_x$Sr$_2$CuO$_4$ (Eu-LSCO) [50] at hole doping $p \simeq 0.21$ and $p \simeq 0.24$ and for La$_{1.6-x}$Nd$_{0.4}$Sr$_2$CuO$_4$ (Nd-LSCO) [51, 52] at $p \simeq 0.24$, just above the critical doping $p^*$ at which the pseudo-gap phase terminates. Interestingly, $S$ was found to be positive at those doping levels, while simple considerations based on band-structure and isotropic elastic scattering yield a negative value [53]. It is thus tempting to infer from these observations that the quantum critical (strong metal) regime of those cuprate superconductors may be described as a ‘skewed Planckian metal’. We emphasize that experiments involving different controlled levels of disorder would play a decisive role in assessing the relevance of the mechanism proposed here, since the asymmetric term in Eq. (15) becomes larger for cleaner systems. For a discussion of the relevance of the present theory to the interpretation of recent measurements of the ab-plane and c-axis measurements of the Seebeck coefficient in Nd-LSCO, see Ref. [54].

In Eu-LSCO, Nd-LSCO and also Bi-2201 [55], the increase of $S/T$ at low-T observed experimentally appears consistent with a logarithmic dependence. Given the sign of the bandstructure term for Nd-LSCO, one would expect from Eq. (15) a negative coefficient of this logarithmic term, in contrast to the experimental observation. This may suggest that the asymmetry term is actually dominant (as on Fig. 3) in the range of temperature of the measurement, in which the increase of $C/T$ is moderate [23].

As emphasized recently by Jin and coworkers [56], in the overdoped regime the Seebeck coefficient of the LSCO family remains positive [57, 58]. These overdoped compounds behave as Fermi liquids at low enough temperature however [59–61]. Hence, in view of the discussion above (Fig. 1), a particle-hole asymmetry of the inelastic scattering rate is unlikely to be responsible for this unexpected sign of $S$. It is possible that a particle-hole asymmetric dependence of the elastic scattering rate on momentum and energy is relevant to explain this observation [46, 62, 63]. We also note that the Seebeck coefficient of other single-layer cuprates such as Hg1201 [64] and Bi2201 [65, 66] has been reported to be negative in the overdoped regime.

Recently, the present theory was applied to experimental measurements of the ab-plane and c-axis Seebeck coefficients in Nd-LSCO [54]. The scattering rates used in this analysis were extracted from angular-dependent magneto-resistance measurements, which reveal that the elastic (temperature-independent) rate is momentum dependent but that the inelastic one is not [21]. The theory was hence minimally extended as compared to the present work by using a momentum-dependent elastic lifetime and found to describe the experimental results well. We stress that the qualitative aspects (change of sign for the in-plane Seebeck coefficient due to inelastic scattering) are already well described by the simpler approach used here which neglects the momentum dependence, but that quantitative aspects require the momentum dependence to be taken into account, especially for the out-of-plane Seebeck coefficient.

Our results are relevant also in the context of the non-Fermi liquid quantum critical point separating a metallic spin-glass phase and a Fermi liquid metal in doped SYK models, which has attracted a lot of attention recently [38, 39, 41, 67–69]. Indeed, numerical results show strong spectral asymmetry in the quantum critical regime, possibly signalling a skewed non-Fermi liquid at the critical point [41]. We also note that in the context of transport in SYK models, the Seebeck coefficient has been emphasized as a probe of the universal ground-state entropy [37, 40].

Our work may also have relevance to twisted bilayer graphene and related systems, in which linear resistivity and Planckian behaviour have been observed, see e.g. [70]. A systematic investigation of thermoelectric effects and heat transport in these materials would be of great interest. We note that a recent study of twisted bilayer graphene reports a rich temperature dependence with changes of sign in the Planckian regime [71].

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Appendix A: Kubo formula: Thermopower and Conductivity

The conductivity $\sigma$, thermopower $S$ and (electronic) heat conductivity $\kappa$ are given by:

$$\sigma = e^2 L_{11}, \quad S = -\frac{L_{12}}{\epsilon L_{11}}, \quad \kappa = T \left[ L_{22} - \frac{L_{12}^2}{L_{11}} \right]$$ (A1)

in which $L$ are Onsager’s coefficients and $\epsilon$ is the absolute magnitude of the electron charge. In the following, we set for simplicity $\hbar = k_B = e = 1$ (except when restored in final results). Within the Kubo formalism, and neglecting vertex corrections, the Onsager coefficients are given by (denoting for simplicity $L_{11} \equiv L_0, L_{12} \equiv L_1, L_{22} \equiv L_2$):

$$L_n = \frac{1}{T^n} \int d\omega \left( -\frac{\partial f}{\partial \omega} \right) \omega^n T(\omega)$$ (A2)
in which \( f(\omega) = 1/[1 + e^{\omega/T}] \) is the Fermi function and

\[
T(\omega) = 2\pi \int \frac{d^3k}{(2\pi)^3} n_k^2 A(k,\omega)^2 \quad (A3)
\]

In this expression, \( n_k = (\delta_{k1}^k\varepsilon_k)/\hbar \) denotes the band velocity in the direction \( x = x, y, z \) being considered, and \( A(k,\omega) \) is the electronic spectral function, related to the self-energy \( \Sigma = \Sigma(\omega + i0^+,k) \) by:

\[
A(k,\omega) = \frac{1}{\pi} \frac{\Gamma/2}{(\omega + \mu - \varepsilon_k - \text{Re}\Sigma)^2 + (\Gamma/2)^2} \quad (A4)
\]

In this expression, \( \Sigma = \Sigma(\omega + i0^+,k) \) is the self-energy due to inelastic interactions between electrons and \( \Gamma \) is the full scattering rate including both elastic and inelastic terms:

\[
\Gamma(k,\omega) = \gamma_k - 2\text{Im}\Sigma(k,\omega + i0^+) \quad (A5)
\]

These expressions can be further simplified when the elastic scattering rate \( \gamma_0 \) and electron-electron self-energy \( \Sigma \) do not depend on momentum and when the vertex corrections can be neglected [24].

Using the band transport function \( \Phi \) one can express

\[
T(\omega) = \pi \int d\varepsilon \Phi(\varepsilon) A(\varepsilon,\omega)^2 \quad (A6)
\]

We change the integration variable by setting \( \varepsilon = \omega + \mu - \text{Re}\Sigma + \frac{\Gamma}{2} y \), so that:

\[
T(\omega) = \frac{\pi}{2} \Gamma \int_{-\infty}^{+\infty} dy \Phi \left( \omega + \mu - \text{Re}\Sigma + \frac{\Gamma}{2} y \right) \left( \frac{1}{y^2 + 1} \right)^2 \quad (A7)
\]

We now perform an expansion of this expression for small \( \Gamma \) and retain only the most singular term, yielding:

\[
T(\omega) = \frac{1}{\Gamma} \Phi (\omega + \mu - \text{Re}\Sigma) + \cdots \quad (A8)
\]

where we have used \( \int_{-\infty}^{+\infty} dy 1/\pi^2 (y^2 + 1)^2 = 1/(2\pi) \). It is convenient to introduce the notations:

\[
\tau_F(T) \equiv \mu - \text{Re}\Sigma(\omega = 0, T)
\]

\[
1 - \frac{1}{Z(T,\omega)} \equiv \frac{1}{\omega} [\text{Re}\Sigma(\omega, T) - \text{Re}\Sigma(0, T)] \quad (A9)
\]

so that, to dominant order in \( \Gamma \):

\[
T(\omega) = \frac{1}{\Gamma(T,\omega)} \Phi \left[ \tau_F(T) + \frac{\omega}{Z(T,\omega)} \right] + \cdots \quad (A10)
\]

Inserting this expression in the Onsager coefficients \( L_n \) above, and changing variable to \( x = \omega/T \) in the integral over frequency, we obtain, with \( \tau = 1/\Gamma \):

\[
L_n = \int dx \frac{x^n}{4\cosh^2 \frac{x}{2}} \Phi \left[ \tau_F(T) + x \frac{T}{Z(T,xT)} \right] \tau(T,xT) \quad (A11)
\]

We now perform a low-\( T \) expansion of this expression. We note that for both the Fermi liquid and non-Fermi liquid cases considered in this article, \( T/Z(T,xT) \) vanishes at low-\( T \) (as \( \sim T \) and \( \sim T'^{\nu} \), respectively). We assume furthermore that the \( T \)-dependence of \( \tau_F(T) \) is subdominant and can be neglected. We thus obtain:

\[
L_n = \Phi_0 \langle x^n \tau(T,xT) \rangle + T \Phi_0 \left( \frac{x^{n+1} \tau(T,xT)}{Z(T,xT)} \right) + \cdots \quad (A12)
\]

with the notation:

\[
\langle F(x) \rangle = \int_{-\infty}^{+\infty} dx \frac{F(x)}{4\cosh^2 \frac{x}{2}} \quad (A13)
\]

Retaining just the lowest two terms in the Taylor expansion of \( \phi(\varepsilon) \) is sufficient at low temperatures as the transport function usually varies only on the bare electronic scales, e.g. the bandwidth. We have checked furthermore that neglecting the frequency \( (x) \) dependence of \( Z \) in these equations is a good approximation, so that we finally obtain the expressions Eqs. (4,5,6) of Sec.II.

We also note the expression of the conductivity:

\[
\sigma = e^2 \Phi_0 I_0(T) \simeq e^2 \Phi_0 \langle \tau_+ \rangle \quad (A14)
\]

Although the non-Fermi liquid case does not have conventional quasiparticles, the final expressions for \( S, \sigma \) and the Onsager coefficient \( L_n \)'s at low-\( T \) are formally identical to the ones obtained by applying a Boltzmann equation formalism to excitations with a lifetime \( \tau_{ex} = \tau/Z \) and dispersing as \( \varepsilon_{ex}^T = Z\varepsilon_k \), corresponding to a transport function of these excitations \( \Phi_{ex}(\varepsilon) = Z\Phi(\varepsilon/Z) \).

On Fig. 4, we compare the Seebeck coefficient evaluated with the full Kubo formula and with the simplified Boltzmann-like expressions. The data is for the sub-planckian \( \nu = 1/2 \) case. The results of the full-Kubo evaluation (dotted) agree well with the results of a simplified Boltzmann calculation that approximates \( Z(\omega,T) \) by its zero-frequency value \( Z(\omega,T) \rightarrow Z(0,T) \) (full lines). The result of such Boltzmann calculations are the data given in the main text. The remaining small discrepancy between the Kubo and the Boltzmann results can be remedied if one retains the frequency dependence of \( Z(\omega,T) \) (dashed lines that overlap with the dotted ones).

For application of our theory to cuprates it is essential to retain momentum dependence. However, as recently reported in an angular-dependent magnetoresistance (ADMR) study of cuprates [21], the inelastic scattering is found to be local and it is sufficient to retain the momentum dependence in the elastic scattering rate. In such a case, the theory can be minimally extended, with only quantitative but no qualitative changes, by allowing for the elastic rate to have momentum dependence. Such an extension is presented in [54].
Appendix B: Skewed non-Fermi liquids: $\omega/T$ scaling

1. Scaling function and conformally invariant case

The (local) non-Fermi liquids considered in this article have a scattering rate that obeys $\omega/T$ scaling ($\nu \leq 1$):

$$\Gamma_{in}(T,\omega) = \lambda(\pi T)^\nu g\left(\frac{\omega}{T}\right) \quad (B1)$$

The scaling function $g(x)$ has a regular expansion at small $x$: $g(x) = g(0) + x g'(0) + \cdots$, while at large $x$ it obeys $g(x \to +\infty) \sim c_x^+ x^\nu$ and $g(x \to -\infty) \sim c_x^- x^\nu$. The ‘skewed’ case with a particle-hole asymmetry will in general have $g'(0) \neq 0$ and $c_x^+ \neq c_x^-$. Hence:

$$\Gamma_{in}(T \gg |\omega|) \sim \lambda g(0) (\pi T)^\nu + \lambda g'(0) \frac{\pi |\omega|}{(\pi T)^{1+\nu}} + \cdots$$

$$\Gamma_{in}(T \ll |\omega|) \sim c_x^\pm \lambda |\omega|^{\nu} + \cdots \quad (B2)$$

Our qualitative results do not depend on the specific form of the scaling function $g(x)$ provided it obeys these general properties.

In Sec. III we considered $g(x)$ given by Eq. (11). It has been shown in [31], in the context of overscreened Kondo impurity models, that this specific form of the scaling function holds in models which have conformal invariance at low energy/temperature. It is also relevant in the context of SYK models [34–37, 41]. The parameter $\alpha$ controls the particle-hole asymmetry (see Fig. 5), with $\alpha = 0$ corresponding to particle-hole symmetry $g(x) = g(-x)$ and $g_{-\alpha,\nu}(x) = g_{\alpha,\nu}(-x)$. The normalisation $g_{\alpha=0,\nu}(0) = 1$ was chosen in the above expression. In the Planckian case $\nu = 1$, Eq. (11) can be cast in the explicit form:

$$g_{\alpha=1}(x) = \frac{(x + \alpha)/2}{\sinh[(x + \alpha)/2] \cosh(\alpha/2)} \quad (B3)$$

and this function has the following spectral representation ($\tau = \tau/\beta$):

$$\Sigma(\tau) \propto e^{\alpha(\tau - 1/2)} \left(\frac{\pi/\beta}{\sin \pi \tau/\beta}\right)^{1+\nu} \quad (B4)$$

with $C = \cosh(\alpha/2)(2\pi e^\alpha)/\Gamma((1 + \nu)/2)^2/(\pi \Gamma[1 + \nu])$.

Other form of the scaling functions $g$ can also be considered. In the absence of a particle-hole asymmetry, the following phenomenological scaling function has sometimes been considered in the literature (see e.g [49]):

$$g_{\alpha=0}(x) = \left[1 + x^2/\pi^2\right]^{\nu/2} \quad (B5)$$

Note that $\nu = 2$ reduces to the Fermi liquid scaling function $1 + x^2/\pi^2$ - this is also the case of Eq. (11) for $\nu = 2, \alpha = 0$. Starting with this form, one can introduce an asymmetry for example by deforming the scaling function in an analogous way to Eq. (11) [31]:

$$g_{\alpha}(x) = g_{\alpha,\nu}(x + \alpha) \frac{\cosh(x/2)}{\cosh[(x + \alpha)/2] \cosh(\alpha/2)} \quad (B6)$$

A simple calculation using the spectral representation of $\Sigma$ then shows that:

$$\Sigma_{\alpha}(\tau) = \frac{e^{\alpha(\tau - 1/2)}}{\cosh(\alpha/2)} \Sigma_{\alpha=0}(\tau) \quad (B7)$$
2. Scaling form of $Z(T, \omega)$

Here, we discuss the scaling form of the real part of the self-energy. We start from the spectral representation:

$$\text{Re}\Sigma(\omega) = P \int d\epsilon \frac{\sigma(\epsilon)}{\omega - \epsilon}$$

$$\sigma(\omega) = -\frac{1}{\pi} \text{Im}\Sigma(\omega + i0^+) = \frac{1}{2\pi} \Gamma_{\text{in}}(T, \omega)$$  \hspace{1cm} (B9)

in which $P$ denotes the principal part of the integral and we have recalled the relation (factor of two) between the imaginary part of the self-energy and the inelastic scattering rate for a local theory. The frequency and temperature dependent effective mass enhancement $1/Z(T, \omega)$ is given by:

$$1 - \frac{1}{Z(T, \omega)} \equiv \frac{1}{\omega} \left[\text{Re}\Sigma(\omega) - \text{Re}\Sigma(0)\right] = P \int d\epsilon \frac{\sigma(\epsilon)}{\epsilon(\omega - \epsilon)}$$  \hspace{1cm} (B10)

Because we have subtracted $\text{Re}\Sigma(0)$, we can substitute the scaling form of $\sigma = \Gamma_{\text{in}}/2$ in the integral without encountering divergencies for a NFL with $\nu < 1$, and thus obtain:

$$\frac{T}{Z(T, \omega = xT)} = T - \frac{\lambda}{\pi} (\pi T)^\nu P \int dy \frac{g(y)}{g(x - y)}$$  \hspace{1cm} (B11)

Hence, remarkably, $T/Z$ obeys a universal scaling form which depends only on the scaling function $g$. This is in contrast to a Fermi liquid in which $Z$ depends on all energy scales (as signalled by the fact that inserting the low-energy expression $\sigma(\omega) \propto \omega^2 + (\pi T)^2$ in Eq. (B9) would lead to a divergent integral).

For results given in Sec. III, we ignore the $\omega$ (i.e. $x$) dependence of $Z$ and replace $Z(T, \omega)$ by $Z(T, 0) = Z(T)$. Comparison with the full Kubo formula calculations given in Fig. 4 justify this approximation for transport calculations. The $x \to 0$ limit of the above expression requires some care because of the appearance of a double pole. This is resolved by integrating by parts and one obtains:

$$\frac{T}{Z(T, \omega = 0)} = T + \frac{\lambda}{\pi} (\pi T)^\nu \int dy \frac{g'(y)}{y}$$  \hspace{1cm} (B12)

so that, with the notation introduced in the text:

$$\frac{T}{Z(T)} = T + \frac{\lambda (\pi T)^\nu}{c_\gamma}, \quad \frac{1}{c_\gamma} = \frac{1}{2\pi} \int dy \frac{g'(y)}{y}$$  \hspace{1cm} (B13)

These expressions are valid for $\nu < 1$. The Planckian case $\nu = 1$ requires a slightly different analysis because in Eq. (B9) the integral would diverge logarithmically at large frequency. Hence, a cutoff $\Lambda$ must be kept. The low-temperature behavior of the zero-frequency limit of $Z(T)$ can be again obtained from an integration by part (up to the cutoff). We obtain:

$$1 - \frac{1}{Z(T)} = -\frac{\lambda}{2} \left[P \int_{-\Lambda/T}^{\Lambda/T} dy \frac{g'(y)}{y} - \frac{g(y)}{y}\right]$$  \hspace{1cm} (B14)

At large $y$, $g'$ tends to a constant in the Planckian case $g'(y \sim \pm\infty) \to g'(\pm\infty)$, so that the low temperature behavior is:

$$\frac{1}{Z(T)} = 1 + \frac{\lambda}{2} \left[g'(\pm\infty) - g'(-\infty)\right] \ln \left(\frac{\Lambda}{T}\right)$$  \hspace{1cm} (B15)

with $\lambda \approx 0.28$ a numerically determined constant. Hence, using the notation in the main text (and absorbing $\zeta$ in the definition of the cutoff $\Lambda$), we obtain: $2\pi/c\zeta = [g'(\pm\infty) - g'(-\infty)] (= 1$ for any $\nu$). In the Planckian case, the high-energy cutoff does not entirely disappear from the expression of $Z$, but we note that the prefactor of the $\ln T$ term depends only on the inelastic coupling constant and not on the cutoff.

Appendix C: Temperature dependence of the Seebeck coefficient

Setting $\theta = T/T^*$ and $\eta = \gamma \Phi_0/\Phi_0$, expressions Eqs. (5,6) can be written as:

$$I_1(\theta) = F_1(\theta) + \eta \left[\frac{T^*}{\gamma} \theta + \frac{1}{c_\gamma} \theta^\nu\right] F_2(\theta)$$  \hspace{1cm} (C1)

$$I_0(\theta) = F_0(\theta) + \eta \left[\frac{T^*}{\gamma} \theta + \frac{1}{c_\gamma} \theta^\nu\right] F_1(\theta)$$  \hspace{1cm} (C2)

with:

$$F_n(\theta) = \langle \frac{x^n}{1 + \theta^\nu g(x)} \rangle$$  \hspace{1cm} (C3)

The low-temperature expansion ($\theta \to 0$) of the functions $F_n$ reads:

$$F_0(\theta) = 1 - \langle g \rangle \theta^\nu + \langle g^2 \rangle \theta^{2\nu} + \cdots$$

$$F_1(\theta) = -\langle x g \rangle \theta^\nu + \langle x g^2 \rangle \theta^{2\nu} + \cdots$$

$$F_2(\theta) = \frac{\pi^2}{3} - \langle x^2 g \rangle \theta^\nu + \langle x^2 g^2 \rangle \theta^{2\nu} + \cdots$$  \hspace{1cm} (C4)

The key difference in the behavior of $F_n$ is that $F_1 \sim c_\gamma \theta^\nu$ ($c_\gamma = \langle x g \rangle$) while $F_{0,2} \sim \text{const.}$ As a result, $\theta^\nu F_2$ and $F_1$ have the same temperature dependence in this regime and hence both contribute to $I_1$ and to the thermopower. In contrast, in $I_0$, the term $\theta^\nu F_1 \sim \theta^{2\nu}$ can be neglected in comparison to $F_0$ in both the conductivity and the Seebeck coefficient, which can be hence written as:

$$S(T) = -\frac{k_B}{e} \left[\eta \left(\frac{1}{c_\gamma} \theta^\nu + \frac{T^*}{\gamma} \theta^\nu\right) \frac{F_2(\theta)}{F_0(\theta)} + \frac{F_1(\theta)}{F_0(\theta)}\right]$$

$F_1$ is much smaller in magnitude than $\theta^\nu F_2$, as shown on Fig. 7. Hence, the odd-frequency contribution $F_1$ in $I_1$ is comparable to the even one $\eta \theta F_2$ only when the parameter $\eta = \gamma \Phi_0/\Phi_0$ is small (of order $10^{-2}$). This is however typically the case since the elastic scattering rate is usually much smaller than electronic energy scales (the elastic scattering rate is usually much smaller than
FIG. 6. Coefficient $c_-$ vs. $\alpha$ for $\nu = 0.5$ (blue) and $\nu = 1$ (red).

FIG. 7. $\theta^\nu F_3(\theta)/F_1(\theta)$ vs. $\theta$ (red) and $F_0(\theta)/\theta^\nu F_1(\theta)$ vs. $\theta$ (blue) for $\nu = 0.5$ and $\alpha = 0.3$.

\[ S \sim -\frac{k_B}{e} \left( \frac{T}{T^*} \right)^\nu \left[ \frac{\pi^2}{3c_2} \eta - c_- \right] \]  

(C5)

The dependence of the coefficient $c_-$ on the asymmetry parameter $\alpha$ for the choice Eq. (11) of scaling functions is displayed on Fig. 6.

In the high-temperature limit ($\theta \to \infty$), all the functions $F_n$ have the same temperature dependence $\sim c_n/\theta^\nu$:

\[ F_n(\theta) = \frac{1}{\theta^\nu} \left( \frac{x^n}{g} \right) - \frac{1}{\theta^{2\nu}} \left( \frac{x^n}{g^2} \right) + \cdots \]  

(C6)

Hence formally, at very high $T$, $S$ tends to a constant $-c_2/c_1 k_B/e$ in the presence of odd frequency scattering. However, the term involving $F_1$ in $I_0$ remains small for most temperatures of interest as shown on Fig. 7. Hence, in practice, the relevant high-$T$ behaviour of $S$ is:

\[ S \sim -\frac{k_B}{e} \left[ \frac{c_2}{c_0} \Phi_0 \frac{T}{\Phi_0 T^*} + \eta \frac{c_2}{c_0} \epsilon Z \left( \frac{T}{T^*} \right)^\nu + \frac{c_1}{c_0} \right]. \]  

(C7)

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