LOWER BOUND AND OPTIMALITY FOR A NONLINEARLY DAMPED TIMOSHENKO SYSTEM WITH THERMOELASTICITY

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**Abstract.** In this paper, we consider a vibrating nonlinear Timoshenko system with thermoelasticity with second sound. We first investigate the stability of this system, then we devote our efforts to obtain the strong lower energy estimates using Alabau-Boussouira’s energy comparison principle introduced in \cite{3} (see also \cite{6}). We extend to our model the nice results achieved in \cite{6} for the case of nonlinearly damped Timoshenko system with thermoelasticity. The proof of our results relies on the approach in \cite{1, 2}.

1. Introduction

Mechanical structures such as beams and plates are a central part of life today, their vibration properties are extensively investigated by many researchers. These vibrations are undesirable because of their damaging and destructing nature. To reduce these harmful vibrations, several control mechanisms have been designed. In order to do that, it is natural to model and understand the corresponding equations of these problems.

In this article we are concerned with the following nonlinearly damped Timoshenko system in a one-dimensional bounded domain with thermoelasticity where the heat flux is given by the Cattaneo’s law:

\begin{equation}
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, &\text{in } & (0,1) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x + a(x)g(\psi_t) &= 0, &\text{in } & (0,1) \times \mathbb{R}_+, \\
\rho_3 \theta_t + q_x + \delta \psi_x &= 0, &\text{in } & (0,1) \times \mathbb{R}_+, \\
\tau \psi_t + \beta \theta + \theta_x &= 0, &\text{in } & (0,1) \times \mathbb{R}_+.
\end{aligned}
\end{equation}

(1.1)

We associate with (1.1) the following Dirichlet boundary conditions

\begin{equation}
\begin{aligned}
\varphi(0,t) &= \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, &\forall \ t \geq 0.
\end{aligned}
\end{equation}

(1.2)

Moreover, the initial conditions for the system (1.1) are given by :

\begin{equation}
\begin{aligned}
\varphi(x,0) &= \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), &\forall \ x \in & (0,1), \\
\psi(x,0) &= \psi_0(x), \psi_t(x,0) = \psi_1(x), &\forall \ x \in & (0,1), \\
\theta(x,0) &= \theta_0(x), \ q(x,0) = q_0(x), &\forall \ x \in & (0,1),
\end{aligned}
\end{equation}

(1.3)

where \( t \in (0,\infty) \) denotes the time variable and \( x \in (0,1) \) is the space variable, the function \( \varphi \) is the displacement vector, \( \psi \) is the rotation angle of the filament, the function \( \theta \) is the temperature difference, \( \varphi = q(x,t) \in \mathbb{R} \) is the heat flux, and \( \rho_1, \rho_2, \rho_3, b, k, \delta \) and \( \beta \) are positive constants.

The Timoshenko model describes the vibration of a beam when the transverse shear strain is significant. In 1920, Timoshenko \cite{26} introduced a purely conserved hyperbolic system given by

\begin{equation}
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, &\text{in } & (0,1) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) &= 0, &\text{in } & (0,1) \times \mathbb{R}_+.
\end{aligned}
\end{equation}

(1.4)

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The well understanding of this model was the goal of a great number of researchers, thus, an important amount of research has been devoted to the issue of the stabilization of the Timoshenko system by the use of diverse types of dissipative mechanisms aiming to obtain a solution which decays uniformly to the stable state as time goes to infinity. To achieve this goal several upper energy estimates have been derived. For an overview purpose, we shall mention some known results in this regard. Kim and Renardy [17], Messaoudi and Mustafa [18], Raposo et al. [23], and others, showed that the presence of damping terms on both equations (1.4) leads to uniform stability result regardless of the values of the damping coefficients. The situation is much different, when the damping term is only imposed on the rotation angle equation in the Timoshenko system. In this case, the exponential stability holds if and only if the propagation velocities are equal. It is worth noting that the first result including the linear and nonlinear indirect damping cases and showing polynomial stability for different speeds of propagation was established in [1] giving thus optimal results in the nonlinear damping case (and getting as a particular case the exponential decay for the same speeds of propagation \( \frac{k}{\rho_1} = \frac{b}{\rho_2} \)); see [1, 11, 12, 10, 19] and the references therein.

Concerning stabilization via heat effect, Rivera and Racke [20] investigated the following system

\[
\begin{align*}
\rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \gamma \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_3 \theta_t - k \theta_{xx} + \gamma \psi_{xt} &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\tau_0 q_x + q + \kappa \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+
\end{align*}
\]

where \( \varphi, \psi, \theta \) are functions of \((x, t)\) model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of \( \sigma, \rho_i, b, k, \gamma, \) they proved several exponential decay results for the linearized system and non-exponential stability result for the case of different wave speeds.

Concerning Timoshenko systems of thermoelasticity with second sound, Messaoudi et al. [21] studied

\[
\begin{align*}
\rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu \varphi_t &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \beta \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_3 \theta_t + \gamma q_x + \delta \psi_{tx} &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\tau_0 q_t + q + \kappa \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+
\end{align*}
\]

where \( \varphi = \varphi(x, t) \) is the displacement vector, \( \psi = \psi(x, t) \) is the rotation angle of the filament, \( \theta = \theta(x, t) \) is the temperature difference, \( q = q(x, t) \) is the heat flux vector, \( \rho_1, \rho_2, \rho_3, b, k, \gamma, \beta, \delta, \kappa, \tau_0 \) are positive constants. The nonlinear function \( \sigma \) is assumed to be sufficiently smooth and satisfy

\[
\sigma_{\varphi_x}(0, 0) = \sigma_{\psi}(0, 0) = k
\]

and

\[
\sigma_{\varphi_x \psi}(0, 0) = \sigma_{\varphi_x \psi}(0, 0) = \sigma_{\psi \psi} = 0.
\]

Several exponential decay results for both linear and nonlinear cases have been established in the presence of the extra frictional damping \( \mu \varphi_t \).

Fernández Sare and Racke [25] considered

\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \beta \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_3 \theta_t + \gamma q_x + \delta \psi_{tx} &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\tau_0 q_t + q + \kappa \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+
\end{align*}
\] (1.5)

and showed that, in the absence of the extra frictional damping \( \mu = 0 \), the coupling via Cattaneo’s law causes loss of the exponential decay usually obtained in the case of coupling via Fourier’s law
This surprising property holds even for systems with history of the form
\begin{align}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \int_0^{+\infty} g(s) \psi_{xx}(\cdot, t-s) ds + \beta \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_3 \theta_t + \gamma \psi_x + \delta \psi_{xx} &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\tau_0 \theta_t + q + \kappa \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+.
\end{align}

Precisely, it has been shown that both systems (1.5) and (1.6) are no longer exponentially stable even for equal-wave speeds \( \left( \frac{k}{\rho_1} = \frac{b}{\rho_2} \right) \). However, no other rate of decay has been discussed.

Very recently, Santos et al. [24] considered (1.5) and introduced a new stability number
\[ \chi = \left( \tau_0 - \frac{\rho_1}{k \rho_3} \right) \left( \rho_2 - \frac{\rho_1 b}{k} \right) - \frac{\rho_1 \beta^2 \rho_1}{k \rho_3} \]
and used the semigroup method to obtain exponential decay result for \( \chi = 0 \) and a polynomial decay for \( \chi \neq 0 \).

Later, in [7] the authors considered a vibrating nonlinear Timoshenko system with thermoelasticity with second sound. Precisely, they looked into the following system
\begin{align}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x + \alpha(t) h(\psi_t) &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_3 \theta_t + q_x + \delta \psi_{xx} &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\tau_0 \theta_t + \beta q + \theta_x &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\varphi_x(0, t) &= \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = q(0, t) = q(1, t) = 0, & \forall \ t \geq 0, \\
\varphi(x, 0) &= \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x), & \forall \ x \in (0, 1), \\
\psi(x, 0) &= \psi_0(x), \ \psi_t(x, 0) = \psi_1(x), & \forall \ x \in (0, 1), \\
\theta(x, 0) &= \theta_0(x), \ q(x, 0) = q_0(x), & \forall \ x \in (0, 1),
\end{align}
and they established an explicit and general decay result using a multiplier method for wide classe of systems. However, very few is known on lower energy estimates and optimality results. Let us mention the existing results in this regard. Haraux [13] examined the case of a one-dimensional wave equation subjected to polynomial globally distributed dampings, for some initial data in \( W^{2,\infty}(\Omega) \times W^{1,\infty}(\Omega) \). Haraux proved that
\[ \limsup_{t \to \infty} (t^{\frac{3}{p+1}} E(t)) > 0, \]
where \( E(t) \) is the energy associated with the damped wave equation, and,
\[ \limsup_{t \to \infty} \left( t^{\frac{1}{p}} \| u_t \|_{L^\infty(\Omega)} \right) > 0, \]
where \( g \) is a nondecreasing \( C^1 \) function which behaves essentially like \( k|s|^r s \) with \( k, r > 0 \) and the damping term \( g(x) \) grows as \( x^p \) near the origin. Since that time, this issue retains the attention of many other authors. We also refer to [5, Chapter1] for more details about the stabilization of wave-like equations.

More precisely, lower energy estimates have been previously studied in the articles [4], [3] for the scalar one-dimensional wave equations, the scalar Petrowsky equations in two-dimensions and \( (2 \times 2) \) Timoshenko systems.

Let us also quote the article of Alabau [6] for recent studies on strong lower energy estimates of the strong solutions of nonlinearly damped Timoshenko beams, Petrowsky equations, in two and three dimensions, and wave-like equations, in a bounded one-dimensional domain or annulus domains in two
or three dimensions. Note nevertheless that considering the system (1.1) makes our lower bound results more general from those considered so far in the literature.

The main objective of the present paper is to show how the energy $E$ (defined by (2.2) blow) associated with the nonlinearly damped Timoshenko system of thermoelasticity with second sound (1.1) satisfies the stability result. Once we have this stability result, one can use the expression of the energy $E$ (defined by 3.12 blow) and apply the comparison principle which allows us to give the strong lower estimates for the system (1.1).

The rest of the article is organized as follows. We start in Section 2 by giving a brief introduction, then we introduce some notations and material needed for our work. In Sections 3 we state and prove the stabilization result for (1.1). Then in Section 4 we derive the lower energy estimates for the Timoshenko system (1.1). Some examples are given in the last section.

2. Preliminaries

We formulate the following assumptions that would be required for the establishment of our results: $(H_0)$: we assume that $a$ is a smooth function and satisfies $a(x) \geq 0$, $x \in [0,1]$, $a > 0$ in a nonempty subset $]0,1[$ of $\omega$;

\[
\begin{align*}
(H_1) & \quad \left\{ \begin{array}{l}
g : \mathbb{R} \to \mathbb{R} \text{ is a nondecreasing } C^0 \text{-function} \\
such that for every } \epsilon \in (0,1), \text{ there exists positive constants } c_1, c_2, \\
and an increasing odd function } g_0 \in C^1(0, +\infty), g_0(0) = 0 \text{ such that} \\
\begin{array}{l}
g_0(|s|) \leq |g(s)| \leq g_0^{-1}(|s|), \quad \text{for all } |s| \leq \epsilon, \\
c_1 |s| \leq |g(s)| \leq c_2 |s|, \quad \text{for all } |s| \geq \epsilon.
\end{array}
\end{array} \right.
\]

In addition, we assume that, there exists $r_0 > 0$ such that $\Psi$ is a strictly convex $C^1$-function from $[0, r_0]$ on to $\mathbb{R}$, given by,

\[
\Psi(x) = \sqrt{x}g(\sqrt{x}).
\]

Remarks 2.1.

1. The function $\Psi$ defined above is the same function $H$ introduced in [2].

2. In [6] Alabau assumed that $g$ is an odd, increasing function and has a linear growth at infinity.

In order to establish here the lower estimates, the hypotheses in [6] are only assumed for the function $g_0$ and not for $g$.

The energy associated with the system (1.1) is defined by

\[
E(\varphi, \psi, \theta; q)(t) := \frac{1}{2} \int_0^1 \left( \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b \psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau q^2 \right) dx.
\]

Differentiating (2.2) in time, it is easy to see that

\[
E'(t) = -\beta \int_0^1 q^2 dx - \int_0^1 a(x) \psi_t g(\psi_t) dx \leq 0,
\]

this relationship has been obtained by multiplying, formally, the first fourth equations of (1.1), respectively, by $\varphi_t, \psi_t, \theta$ and $q$, and using the integration by parts with respect to $x$ over $(0,1)$, the boundary and initial conditions, and the hypotheses $(H_0)$ and $(H_1)$.

Now, we define the function space associated with the problem (1.1) by

\[
\mathcal{H} = H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times (L^2(0,1))^3.
\]

We rewrite (1.1) as a first-order system. For that purpose, let $U = (\varphi; \varphi_t; \psi; \psi_t; \theta; q)^T$ and (1.1) becomes

\[
\begin{align*}
\frac{d}{dt} U(t) + (A + B) U(t) = 0, & \quad t > 0, \\
U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in \mathcal{H},
\end{align*}
\]
where $\mathcal{A}$ is an unbounded operator from $D(\mathcal{A})$ onto $\mathcal{H}$ defined by

$$
\mathcal{A} \begin{pmatrix} \varphi \\ w \\ \psi \\ z \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} -w \\ -b_1 \varphi_{xx} - b_2 \psi_x \\ -z \\ -b_2 \psi_{xx} + b_1 (\varphi_x + \psi) + \frac{\delta}{\rho_2} \theta_x \\ \frac{1}{\rho_3} \theta_x + \frac{\delta}{\rho_3} z_x \\ \frac{\beta}{\tau} q + \frac{\gamma}{\tau} \theta_x \end{pmatrix}.
$$

Here,

$$D(\mathcal{A}) = (H^2(0, 1) \cap H^1_0(0, 1)) \times H^1(0, 1)^2 \times H^1(0, 1) \times H^1_0(0, 1).$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Let $B$ be the damping nonlinear operator given by

$$B \begin{pmatrix} \varphi \\ w \\ \psi \\ z \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a(x)g(w) \\ 0 \\ 0 \end{pmatrix}.$$

Thanks to the theory of maximal nonlinear monotone operators (see [14]), we have the following existence and uniqueness result (see [7] for the proof).

**Theorem 2.2.** Assume that $(H_0)$ and $(H_1)$ are satisfied. Then for all initial data $U_0 \in \mathcal{H}$, the system $(1.1)$ has a unique solution $U \in C([0, \infty); \mathcal{H})$, the operator $\mathcal{A} + B$ generates a continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on $\mathcal{H}$. Moreover, for all initial data $U_0 \in D(\mathcal{A})$, the solution $U \in L^\infty([0, \infty); D(\mathcal{A})) \cap W^{1, \infty}([0, \infty); \mathcal{H})$.

**Remark 2.3.** As we already mentioned in the introduction, the exponential decay result $(1.5)$ depends on the stability number $\chi$ introduced in [24]. So, it is natural to wonder about the effects of the nonlinear dissipation mechanism $a(x)g(\psi_t)$ on the stability result of the system $(1.1)$. We recall that, in [7], the authors considered the same stability number $\chi$ and obtained a general decay of the system $(1.7)$ with a dissipation term of the form $\alpha(t)h(\psi_t)$ but no optimality result has been proved.

As a consequence, the following questions naturally arise:

- Is our system $(1.1)$ strongly stable?
- If we obtain a different equilibrium state ($E(t) \to \text{constant} \neq 0$ as $t \to \infty$), how can we characterize the decay rate of the energy?
- Can we obtain lower estimates for the new equilibrium state?

These questions will be investigated in the next sections.

### 3. Stability for Timoshenko System

In this section, we focus on the stability result for the energy. For that purpose, we follow the following steps.

We consider first the following conservative Timoshenko system:

$$
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) &= 0, & \text{in } (0, 1) \times \mathbb{R}_+.
\end{align*}
$$

We shall investigate these questions in the next sections.
Then, we assume the assumption below on the subset $\omega \subset [0,1],$

\[ (HS) \left\{ \begin{array}{ll}
\text{Let } (\varphi, \psi) \text{ be a weak solution of (3.1)} \\
\text{if } \psi_t \equiv 0 \text{ on } \omega \text{ then } (\varphi, \psi) \equiv (0,0).
\end{array} \right. \]

The assumption $(HS)$ is extracted from [6] and we note that we proceed as in [6] to extend the techniques there to our problem.

Now, we denote by $\omega(U_0)$ the $\omega-$limit set of $U_0$ and we consider $Z_0 \in \omega(U_0)$ such that $Z(t) = T(t)Z_0.$ Then we formulate the stability result for the energy of (1.1) in the following theorem.

**Theorem 3.1.** Assume that the hypotheses $(H_0)$ and $(H_1)$ hold. We assume in addition that $\omega$ satisfies $(HS).$ Then for all $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in \mathcal{H},$ the energy $E$ defined by (2.2) corresponding to the solution of (1.1), satisfies

\[ \lim_{t \to \infty} E(t, U) = E_\infty, \]

where $E_\infty$ is the energy of $Z \in \omega(U_0).$

Moreover, under the same assumptions we prove that the energy $\mathcal{E}(t)$ defined in (3.12) below, satisfies

\[ \mathcal{E}(t) \to 0, \text{ when } t \to \infty. \]

Before showing the proof of Theorem 3.1, we will state and prove two lemmas which will be useful to this end. The first lemma below proves the decreasing of the first order energy.

**Lemma 3.2.** Let $E_*(t)$ be the energy defined as follows:

\[ E_*(t) := \frac{1}{2} \int_0^1 (\rho_1 \varphi_{tt}^2 + \rho_2 \varphi_t^2 + b \psi_t^2 + k(\varphi_{tx} + \psi_t)^2 + \rho_3 \theta_t^2 + \tau q_t^2) dx. \]

Then, $E_*$ is a nonincreasing function. We shall call $E_*(t)$ the first order energy associated with (1.1).

**Proof.** We set $p = \varphi_t, z = \psi_t, u = \theta, d = q,$ then we have

\[ \begin{aligned}
\rho_1 p_t - k(\varphi_x + \psi)_x &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
\rho_2 z_t - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x + a(x)g(z) &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
\rho_3 u_t + q_x + \delta z_x &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
\tau d_t + \beta q + \theta_x &= 0, &\text{ in } (0,1) \times \mathbb{R}_+.
\end{aligned} \]

Differentiating the above equations with respect to time, we obtain

\[ \begin{aligned}
\rho_1 p_{tt} - k(\varphi_x + \psi)_x &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
\rho_2 z_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta u_x + a(x)g(z)z_t &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
\rho_3 u_t + q_x + \delta z_x &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
\tau d_t + \beta d + u_x &= 0, &\text{ in } (0,1) \times \mathbb{R}_+, \\
p(0,t) = p(1,t) = z(0,t) = z(1,t) = d(0,t) = d(1,t) = 0, &\forall t \geq 0.
\end{aligned} \]

We remark that if we formally multiply the equations in (3.6), respectively, by $p_t, z_t, u$ and $d,$ integrate over $(0,1)$ and use the integration by parts with respect to $x,$ the boundary conditions, and the hypotheses $(H_0)$ and $(H_1),$ we obtain the following inequality

\[ E_*'(t) = -\beta \int_0^1 d^2 dx - \int_0^1 a(x)g'(z)z_t^2 dx \leq 0. \]

Thus we deduce that $E_*$ is nonincreasing, hence, we have

\[ E_*(t) \leq E_*(0), \quad \forall t \geq 0. \]

□

We start by establishing the compactness of the orbit in the following lemma.

**Lemma 3.3.** For the initial data $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in D(A),$ the orbit of $U_0$ given by $\gamma(U_0) = \cup_{t \geq 0} T(t)U_0$ is relatively compact in $\mathcal{H}.$
Proof. Thanks to the first equation of (1.1), we have
\[ \varphi_{xx} = -\frac{\rho_1}{k}(\varphi_t + \psi_t), \]
and we get
\[ \int_0^1 \varphi_{xx}^2 dx \leq \left( \frac{\rho_1}{k} \right)^2 \left( \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx \right). \]
Using Lemma 3.2 which proves that \( E_* \) is bounded on \( \mathbb{R}_+ \), we deduce that the set \( \{ \varphi_t(t, \cdot); t \geq 0 \} \) is bounded in \( L^2(0, 1) \). In addition to the fact that \( E \) is bounded uniformly on \( \mathbb{R}_+ \), we deduce that the set \( \{ \psi_t(t, \cdot); t \geq 0 \} \) is also bounded in \( L^2(0, 1) \).
Applying the Poincare’s inequality and the Rellich-Kondrochov theorem, we obtain that the set
\[ \{ \varphi(t, \cdot); t \geq 0 \} \text{ is relatively compact in } H^1_0(0, 1). \]
Thanks to (3.7) the energy \( E_* \) is bounded on \( \mathbb{R}_+ \), then, the set \( \{ \varphi_t(t, \cdot); t \geq 0 \} \) is bounded in \( L^2(0, 1) \). Furthermore, we apply the Poincare’s inequality for \( \varphi_t \in H^1_0(0, 1) \),
\[ \| \varphi_t \|_{H^1_0(0, 1)} \leq (1 + c_\rho) \| \varphi_t \|_{L^2(0, 1)}. \]
Hence, we easily obtain that the set \( \{ \varphi_t(t, \cdot); t \geq 0 \} \) is bounded in \( H^1_0(0, 1) \) which implies, using the Rellich theorem, that the set
\[ \{ \varphi_t(t, \cdot); t \geq 0 \} \text{ is relatively compact in } L^2(0, 1). \]
From (1.1), we have
\[ \theta_x = -\tau q_t - \beta q, \]
and the sets \( \{ q(t, \cdot), t \geq 0 \} \) and \( \{ q_t(t, \cdot), t \geq 0 \} \) are bounded in \( L^2(0, 1) \). Hence, we deduce that \( \{ \theta_x(t, \cdot), t \geq 0 \} \) is bounded in \( L^2(0, 1) \). Moreover, using the equation
\[ b \psi_{xx} = \rho_2 \psi_t + k(\varphi_x + \psi) + \delta \theta + a(x)g(\psi_t), \]
and the hypotheses \( (H_0) \) and \( (H_1) \) on \( a \) and \( g \), we obtain that \( \{ \psi_{xx}(t, \cdot), t \geq 0 \} \) is bounded in \( L^2(0, 1) \), \( \{ \psi(t, \cdot), t \geq 0 \} \) is bounded in \( H^2(0, 1) \) and again applying the Rellich-Kondrochov theorem, we deduce that the set
\[ \{ \psi(t, \cdot); t \geq 0 \} \text{ is relatively compact in } H^1_0(0, 1). \]
Since, we have the set \( \{ \psi_t(t, \cdot), t \geq 0 \} \) is bounded in \( H^1_0(0, 1) \) we easily deduce from the Rellich theorem that
\[ \{ \psi_t(t, \cdot), t \geq 0 \} \text{ is relatively compact in } L^2(0, 1). \]
Using the fact that \( E_* \) is bounded and
\[ q_x = -\delta \psi_{tx} - \rho_3 \psi_t, \]
we infer that the \( \{ q_x(t, \cdot), t \geq 0 \} \) is bounded in \( L^2(0, 1) \). Applying the Poincare’s inequality, than the set \( \{ q(t, \cdot), t \geq 0 \} \) is bounded in \( H^1_0(0, 1) \), which infers that
\[ \{ q(t, \cdot), t \geq 0 \} \text{ is relatively compact in } L^2(0, 1). \]
Using the fourth equation of (1.1), we deduce that \( \{ \theta_x(t, \cdot), t \geq 0 \} \) is bounded in \( L^2(0, 1) \), as well, \( \forall t \geq 0, \theta_x(t, \cdot) \in H^1_0(0, 1) \), then \( \{ \theta_x(t, \cdot), t \geq 0 \} \) is bounded in \( H^1_0(0, 1) \). Therefore we conclude that
\[ \{ \theta(t, \cdot), t \geq 0 \} \text{ is relatively compact in } L^2(0, 1). \]
\[ \Box \]

Now, we recall the definition of the \( \omega - \)limit that we borrow from [5].

Definition 3.4. Let \( (T(t))_{t \geq 0} \) be a continuous semigroup on a Banach \( X \). We recall that the \( \omega - \)limit set of \( z_0 \) in \( X \), is defined by
\[ \omega(z_0) = \{ z \in X, \exists (t_n)_n \in [0, \infty) \text{ such that } t_n \to \infty, \text{ as } n \to \infty, \text{ and } z = \lim_{n \to \infty} T(t_n)z_0 \}. \]
Now, we are ready to give the proof of Theorem 3.1.

**Proof of Theorem 3.1.**

We aim to apply the Dafermos strong stabilization technique based on Lasalle invariance principle (see Proposition 1.3.6 in [5]).

For that purpose, let $U_0 = (\varphi_0, \psi_0, \psi_1, \theta_0, \eta_0) \in D(A)$, and $U = (\varphi, p, \psi, z, \theta, \eta) = T(t)U_0$. Then, we define the Lyapunov function $L$ for $(T(t))_{t \geq 0}$ on $\mathcal{H}$ by

$$L(U) = \frac{1}{2} \int_0^1 (\rho_1 p^2 + \rho_2 z^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau \eta^2) dx, \quad \forall \ U \in \mathcal{H}.$$ 

Now, let $\omega(U_0)$ be the $\omega$-limit of $U_0$ (see Definition 3.4). Thanks to the Lasalle invariance principle, we show that for each $W_0$ in $\omega(U_0)$, the function $t \to L(T(t)W_0)$ is constant. In particular, let $Z_0 \in \omega(U_0)$ be given and we set $Z(t) = (w, r, z, p, u, \eta)(t) = T(t)Z_0$. Since $L(Z(\cdot))$ is constant, we deduce that $(w, z, u, \eta)$ is a solution of a conservative system. Then, the dissipation inequality will be equal to zero which yields

$$-\beta \int_0^1 \eta^2 dx - \int_0^1 a(x)pg(p)dx = 0 \Rightarrow \eta \equiv 0 \text{ and } a(x)g(p) = 0, \forall \ x \in (0, 1), \forall \ t \in \mathbb{R}_+.$$

Hence, the conservative system can be written as follows:

$$\begin{cases}
\rho_1 w_{tt} - k(w_x + z)_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 z_{tt} - bz_{xx} + k(w_x + z) = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
u_t + \delta p_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
u_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
z_t = 0, & \text{on } \{x \in \Omega, a(x) \neq 0\} \supset \omega.
\end{cases} \quad (3.8)$$

This yields,

$$\begin{cases}
\rho_1 w_{tt} - k(w_x + z)_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 z_{tt} - bz_{xx} + k(w_x + z) = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
u_t = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
u_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
z_t = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
q = 0, & \text{in } (0, 1) \times \mathbb{R}_+,
\end{cases} \quad (3.9)$$

as well as, we can infer from (3.9) that,

$$\begin{cases}
\rho_1 w_{tt} - k(w_x + z)_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 z_{tt} - bz_{xx} + k(w_x + z) = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
u = c = \theta_0(0), & \text{in } (0, 1) \times \mathbb{R}_+, \\
q = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
z_t = 0, & \text{in } (0, 1) \times \mathbb{R}_+,
\end{cases} \quad (3.10)$$

Using the assumption $(H\ S)$, we have that $(w, z) = (0, 0)$. This allows us to identify $Z(t)$ the element of $\omega(U_0)$ in the form $Z(t) = (0, 0, \theta(0), 0)$. Hence we conclude that,

$$\lim_{t \to \infty} E(t, U_0) = E(Z) = E_\infty, \quad \forall \ U_0 \in D(A).$$

Indeed, since $D(A)$ is dense in $\mathcal{H}$, we obtain

$$\lim_{t \to \infty} E(t, U) = E_\infty.$$
Moreover, since $\mathcal{E}$ is the energy of the difference between the solution $U \in \mathcal{H}$ and $Z = (0, \theta_0(0), 0) \in \omega(U_0)$, we obtain

$$E(t) = E(t, (\varphi, \psi, \theta - \theta_0(0), q)) = \frac{1}{2} \int_0^1 \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b \psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 (\theta - \theta_0(0))^2 + \tau q^2 dx.$$  

(3.12)

Thanks to the dissipation inequality (2.3) and (2.1), we have

$$\mathcal{E}'(t, U) = -\beta \int_0^1 q^2(t, x) dx - \int_0^1 a(x) \psi^2(t, x) \tilde{\Psi}(\psi^2(t, x)) dx.$$  

We assume that $\mathcal{E}'(t) = 0, \forall \ t \geq 0$ and that the hypothesis ($HS$) holds, we obtain the following system

$$\begin{aligned}
\rho_1 w_{tt} - k(w_x + z)_x &= 0, \quad \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 z_{tt} - b z_{xx} + k(w_x + z) &= 0, \quad \text{in } (0, 1) \times \mathbb{R}_+, \\
u - \theta_0(0) &= 0, \quad \text{in } (0, 1) \times \mathbb{R}_+, \\
u_t &= 0, \quad \text{in } (0, 1) \times \mathbb{R}_+, \\
u_x &= 0, \quad \text{in } (0, 1) \times \mathbb{R}_+, \\
z_t &= 0 \ \{x \in \Omega, a(x) \neq 0\} \supset \omega.
\end{aligned}$$  

(3.13)

Then for this case the set $\omega(U_0 - Z) = \{(0, 0, 0, 0)\}$. Applying the Deformos’ strong stabilisation technique as before, we obtain that

$$\lim_{t \to \infty} \mathcal{E}(t) = 0$$  

(3.14)

A straightforward consequence of the stabilisation result given by Theorem 3.1 is stated in the following lemma.

**Lemma 3.5.** For any $r_0 > 0$, there exists $T_0 > 0$ such that

$$\mathcal{E}(t) \leq \left(\frac{r_0^2}{\gamma}\right)^2, \quad \forall \ t \geq T_0.$$  

(3.15)

*Proof.* Using the strong stability result given by Theorem 3.1, the energy $\mathcal{E}(t)$ converges to 0 when $t$ tends to $\infty$. Then, energy $\mathcal{E}$ is bounded uniformly on $\mathbb{R}^+$. In particular, we take the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0)$ such that $\mathcal{E}(0) \leq \left(\frac{r_0^2}{\gamma}\right)^2$, where $\gamma$ is defined later in (4.1). Hence, we deduce (3.15). \qed

4. **Lower energy estimates**

The aim of this section is to establish a lower bound of the energy of the one-dimensional nonlinearly damped Timoshenko system with thermoelasticity and also to prove that the method based on the comparison principle, expressed through the energy of the solutions, can be extended to our case.

First, we define (as in [2]) the function $\Lambda$ as follows

$$\Lambda(x) = \frac{\Psi(x)}{x \Psi'(x)}.$$  

(4.1)
and we introduce the following assumption (which is the hypothesis (H2) in [6])

\[
(H_2) \begin{cases}
\exists r_0 > 0 \text{ such that the function } \Psi : [0, r_0^2] \to \mathbb{R} \text{ defined by (2.1)} \\
\text{is strictly convex on } [0, r_0^2], \\
\text{and either } 0 < \liminf_{x \to 0} \Lambda(x) \leq \limsup_{x \to 0} \Lambda(x) < 1, \\
\text{or there exists } \mu > 0 \text{ such that} \\
0 < \liminf_{x \to 0} \left(\frac{\Psi'(x)}{\mu x} \int_{x}^{1} \frac{1}{\Psi(y)} dy\right), \text{ and } \limsup_{x \to 0} \Lambda(x) < 1, \\
\text{for some } z_1 \in (0, z_0] \text{ and for all } z_0 > 0.
\end{cases}
\]

Then, we state in the sequel our main result.

**Theorem 4.1.** Assume that \((H_0), (H_1) \text{ and } (H_2)\) hold. Then for all non vanishing smooth initial data, there exist \(T_0 > 0\) and \(T_1 > 0\) such that the energy \(E\) of (1.1) satisfies the following lower estimate

\[
(4.2) \quad \frac{1}{\gamma^2 C_a^2} \left(\Psi^{-1} \left(\frac{1}{t - T_0}\right)\right)^2 \leq E(t), \quad \forall \ t \geq T_1 + T_0.
\]

**Remark 4.2.** The result of Theorem 4.1 holds true without any assumption on the wave speeds corresponding to the first two equations in (1.1), see e.g [4, 5, 6].

The proof of Theorem 4.1 relies on the next proposition together with Lemma 4.4 which is proved in [3, Lemma 2.4] and based on the approach of [2]. We reproduce here the details for the sake of completeness.

**Proposition 4.3.** Let

\[ U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_0) \in D(A). \]

We assume that the hypotheses of Theorem 2.2 hold and \(\lim_{t \to \infty} E(t) = 0\). Moreover, we assume that

\[
(4.3) \quad \widetilde{\Psi}(x) = \frac{\Psi(x)}{x}, \quad \widetilde{\Psi}(0) = 0, \quad \forall \ x > 0,
\]

where \(\Psi\) is a nondecreasing function on \([0, r_0^2]\) for \(r_0 > 0\) sufficiently small. Then there exists \(T_0 \geq 0\), depending on \(E_1(0)\) such that, defining \(K\) by

\[
(4.4) \quad K(\chi) = \int_{\chi}^\gamma \frac{1}{\Psi(y)} dy, \quad \chi \in (0, \gamma \sqrt{E(T_0)}),
\]

the energy \(E\) satisfies the following lower estimate

\[
(4.5) \quad \frac{1}{\gamma^2} \left(K^{-1}(\sigma(t - T_0))\right)^2 \leq E(t).
\]

Here, \(\sigma\) is a positive constant given by \(\sigma = \frac{\alpha_a}{\rho_2} + \frac{b \mu_0}{\tau C_1}\), where \(\alpha_a\) and \(\gamma\) are defined, respectively, by (4.9) and (4.6) below. Moreover, if \(\lim_{\chi \to 0^+} K(\chi) = \infty\), then

\[
\lim_{t \to \infty} K^{-1}(\sigma(t - T_0)) = 0.
\]

**Proof.** We assume that the initial data \(U_0 \in D(A)\). Then, thanks to the smoothness of the solution, we have

\[
2 \int_{0}^{x} \psi_t(t, z) \psi_{tx}(t, z) dz = \psi_t^2(t, x) - \psi_t^2(t, 0).
\]

Using to the Dirichlet boundary conditions (1.2) at \(x = 0\), we have

\[
\psi_t^2(t, x) = 2 \int_{0}^{x} \psi_t(t, z) \psi_{tx}(t, z) dz.
\]
Applying the Cauchy-Schwarz inequality, we have
\[
\psi_t^2(t, x) \leq 2 \sqrt{\left( \int_0^1 \psi_t^2(t, \cdot) dx \right) \sqrt{\left( \int_0^1 \psi_{tx}^2(t, \cdot) dx \right)}} \\
\leq \frac{4}{\rho_2} \sqrt{E(t) \sqrt{E_*(0)}}, \quad \forall \ x \in (0, 1), \ \forall \ t \geq 0.
\]

Using (3.7) and the fact that \( E_*(t) \leq E_*(0) \), we deduce that
\[
\psi_t^2(t, x) \leq \gamma \sqrt{E(t)}, \quad \forall \ t \geq 0, \ \forall \ x \in (0, 1),
\]
where \( \gamma \) is given by
\[
\gamma = \frac{4}{\rho_2} \sqrt{E_*(0)}. \tag{4.6}
\]

Thanks to Theorem 3.1, we have \( \psi_t \in W^{1, \infty}(0, \infty, L^2(0, 1)) \). From (4.6) and the above regularity of \( \psi_t \), we have
\[
\| \psi_t^2(t, \cdot) \|_{L^\infty(0, 1)} \leq \gamma \sqrt{E(t)}, \quad \forall \ t \geq 0. \tag{4.7}
\]

Thanks to the dissipation inequality (2.3) and using (2.1), we have
\[
E'(t) = -\beta \int_0^1 q^2(t, x) dx - \int_0^1 a(x) \psi_t^2(t, x) \frac{\partial}{\partial t} \psi_t^2(t, x) dx.
\]

On the other hand, from the expression of the energy \( E \) we have the following relation between \( E' \) and \( E' \)
\[
E'(t) = E'(t, U) - \rho_3 \theta_0(0) \frac{d}{dt} \left( \int_0^1 \theta(t, x) dx \right).
\]

Using (1.1) and the boundary conditions, we have \( \frac{d}{dt} \left( \int_0^1 \theta(x, t) dx \right) = 0, \ E'(t) = E'(t, U). \)

Moreover, using the Dafermos strong stabilization result, that is \( \lim_{t \to \infty} E(t) = 0 \), we deduce that there exists \( T_0 \geq 0 \) such that \( \psi_t^2 \) has values in which \( \tilde{\psi} \) is increasing.

Hence, we have
\[
\tilde{\psi}(\| \psi_t^2(t, \cdot) \|) \leq \tilde{\psi}(\gamma \sqrt{E(t)}), \quad \forall \ t \geq T_0, \ \forall \ x \in (0, 1).
\]

Using the last inequality we obtain
\[
\int_0^1 a(x) \psi_t^2(t, x) \tilde{\psi}(\psi_t^2(t, x)) dx \leq \frac{2\alpha_a}{\rho_2 \gamma} \sqrt{E(t) \tilde{\psi}(\gamma \sqrt{E(t)}),} \quad \forall \ t \geq T_0, \tag{4.8}
\]

where
\[
\alpha_a = \| a \|_{L^\infty(0, 1)}. \tag{4.9}
\]

Moreover, using Lemma 3.5, we obtain
\[
E(t) \leq \left( \frac{r_0}{\gamma^{1/2}} \right), \quad \forall \ t \geq T_0. \tag{4.10}
\]

**First case.** Let \( g_0 \) be a linear function on \( [0, \epsilon] \), the hypothesis \( (H_1) \) becomes
\[
c_1 |s| \leq |g(s)| \leq c_2 |s|, \quad \text{for all} \ s \in \mathbb{R}.
\]

In particular, for \( s = \gamma \frac{r}{2}(E(t))^\frac{1}{4} \), and, note that \( g : \mathbb{R} \to \mathbb{R} \), then we have
\[
\gamma \frac{r}{2}(E(t))^\frac{1}{4} \leq \frac{1}{c_1} g(\gamma \frac{r}{2}(E(t))^\frac{1}{4}) \leq \frac{1}{c_1^2} \psi(\gamma \sqrt{E(t)}), \quad \forall \ t \geq T_0. \tag{4.10}
\]
Second case.

Let \(g_0\) be a nonlinear function on \([0, \varepsilon]\). We assume that \(\max(r_0, g_0(r_0)) < \varepsilon\). Let \(\varepsilon_1 = \min(r_0, g_0(r_0))\), we deduce from the hypothesis \((H_1)\) that

\[
\frac{g_0(\varepsilon_1)}{\varepsilon} |s| \leq \frac{g_0(|s|)}{|s|} |s| \leq |g(s)| \leq \frac{g_0^{-1}(|s|)}{|s|} |s| \leq \frac{g_0^{-1}(\varepsilon)}{\varepsilon_1} |s|,
\]

for all \(s\) satisfying \(\varepsilon_1 \leq |s| \leq \varepsilon\).

Using the fact that \(\gamma \frac{1}{2} \Omega(t)^{\frac{1}{2}} \leq \Psi(\gamma \sqrt{\Omega(t)})\), \(\forall t \geq T_0\).

Now, thanks to (4.10) and (4.11), we deduce that for the two cases we obtain the following estimate

\[
\beta \int_0^1 q^2(t, x) dx \leq \frac{2\beta}{\tau} \Omega(t) \leq \frac{2\beta r_0 \sqrt{\Omega(t)}}{\tau C_1} \Psi(\gamma \sqrt{\Omega(t)}), \quad \forall t \geq T_0,
\]

where \(C_1\) is a positive constant.

Hence, there exists \(T_0 > 0\) such that the following inequality holds

\[
\mathbf{K}(\gamma \sqrt{\Omega(t)}) \leq \left( \frac{\alpha_a}{\rho_2} + \frac{\beta r_0}{\tau C_1} \right) (t - T_0).
\]

Since \(\mathbf{K}\) is a nonincreasing function, this completes the proof of (4.5).

Now, we will use the following key comparison with the result borrowed from Lemma 2.4 in [3].

**Lemma 4.4.** Let \(\Psi\) be a given strictly convex set of \(C^1\) function from \([0, r_0^2]\) to \(\mathbb{R}\) such that \(H(0) = H'(0) = 0\), where \(r_0 > 0\) and sufficiently small. Let us define \(\Lambda\) on \([0, r_0^2]\) by

\[
\Lambda(x) = \frac{H(x)}{x H'(x)}.
\]

Let \(z\) be the solution of the following ordinary differential equation:

\[
z'(t) + \kappa H(z(t)) = 0, \quad z(0) = z_0, \forall t \geq 0,
\]

where \(z_0 > 0\) and \(\kappa > 0\) are given. Then \(z(t)\) is well defined for all \(t \geq 0\), and it decays to 0, as \(t \to \infty\).

Assume, in addition, that \((H_2)\) holds. Then there exists \(T_1 > 0\) such that for all \(R > 0\), there exists a constant \(C > 0\) such that

\[
z(t) \geq C(H')^{-1} \left( \frac{R}{t} \right), \quad \forall t \geq T_1.
\]

**Proof of Theorem 4.1.**

Let \(z(t)\) be the solution of the ordinary differential equation (4.14), where we assume that \(z_0 = \gamma \sqrt{\Omega(T_0)}\), \(H = \Psi\) and \(\kappa = \sigma\).

Hence, we have

\[
\mathbf{K}(z(t)) = \left( \frac{\alpha_a}{\rho_2} + \frac{\beta r_0}{\tau C_1} \right) t, \quad \forall t \geq 0.
\]

We set \(z(t) = z(t - T_0)\), then we have

\[
z(t) = K^{-1} \left( \left( \frac{\alpha_a}{\rho_2} + \frac{\beta r_0}{\tau C_1} \right) (t - T_0) \right), \quad \forall t \geq T_0.
\]
Thanks to (4.5), we have

\[(4.16)\]
\[
\frac{\tilde{z}(t)^2}{\gamma^2} \leq \mathcal{E}(t), \quad \forall \ t \geq T_0.
\]

We apply Lemma 4.4 to \(\Psi = H\) for \(R = 1\), then, we obtain the existence of a constant \(C_\gamma\) depending on \(\gamma\) and a positive constant \(T_1\), such that

\[(4.17)\]
\[
(\Psi')^{-1}\left(\frac{1}{t}ight) \leq C_\gamma \tilde{z}(t), \quad \forall \ t \geq T_1.
\]

By using (4.16) and (4.17), we deduce that

\[(4.15)\]
\[
(\Psi')^{-1}\left(\frac{1}{t - T_0}\right) \leq C_\gamma \tilde{z}(t), \quad \forall \ t \geq T_0 + T_1.
\]

Hence, we have (4.15).\(\Box\)

We conjecture that driving the lower estimates leads to optimal energy decay rates in general. However, the proof of such a result is open.

5. Examples

Throughout this section, we will first introduce some examples which allow us to illustrate the main advantages of our results. Let \(c'\) be a positive constant explicitly given here and it only depends on the constant \(\sigma\).

Example 1. Let \(g(s) = s^p, \forall s \in (0, r_0^2]\) for \(p > 1\).

We have \(\Psi(s) = \frac{s^{p+1}}{p+1}, \Psi\) is strictly convex, for \(s \in [0, r_0^2]\), and \(\Psi'(s) = \frac{p+1}{2} s^{\frac{p-1}{2}}\), then

\[
\tilde{\Psi}(s) = \frac{\Psi(s)}{s} = s^{p-1}, \text{ for } p > 1, \forall s \in ]0, r_0^2[.
\]

Thus, \(\tilde{\Psi}\) is nondecreasing on \(]0, r_0^2[\).

Since \(\Lambda(x) = \frac{2}{p+1} < 1\), this proves that \(g\) satisfies the first assumption of \((H_2)\).

By applying (4.2) of Theorem 4.1, we obtain

\[(5.1)\]
\[
\mathcal{E}(t) \geq c' t^{\frac{p}{p-1}}.
\]

Example 2. Let \(g(s) = \frac{1}{s} \exp(-\ln(s)^2), \text{ for all } s \in (0, r_0^2]\). This yields \(\Psi(s) = \exp\left(-\frac{1}{4} (\ln(s))^2\right)\), and

\[
\tilde{\Psi}(s) = \frac{\Psi(s)}{s} = \frac{1}{s} \exp\left(-\frac{1}{4} (\ln(s))^2\right), \quad s \in (0, r_0^2].
\]

Thus, \(\tilde{\Psi}\) is nondecreasing on \(]0, r_0^2[\).

In addition, \(\Lambda(s) = \frac{2}{\ln(s)}\), thus, \(\lim_{s \to 0} \Lambda(s) = 0 < 1\), and we get also for any \(\mu > 1\),

\[
\liminf_{s \to 0} \int_{s}^{r_1} \frac{1}{\Psi(y)} dy > 0.
\]
It is easy to see that $\Psi^\prime(t)$ is equivalent to $D(t)$, as $t$ goes to $\infty$, where $D(t) = \exp(-\frac{1}{4} (\ln(t))^2)$. So, we have $D^{-1}(t) = \exp(-2(\ln(t))^2)$; here we apply the result of the Theorem 4.1 and we obtain the following inequality
\[
E(t) \geq c' \exp(-4(\ln(t))^2).
\]

By these examples we obtain explicit lower bounds which characterize the decay rate of the energy $E(t)$, associated with the solution of (1.1), to the corresponding non-zero equilibrium state energy $E_\infty$.

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