Finite-Time and Finite-Size Scaling of Relaxation Behavior

Mi Jin Lee, Su Do Yi, and Beom Jun Kim

Department of Physics, Sungkyunkwan University, Suwon 440-746, Korea

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Introduction – In the frame of statistical physics, it is important to find critical exponents in a system aiming at understanding of its critical behaviors. For a limited number of model systems, it might be possible to analytically obtain the exponents via the mean-field analysis, the transfer-matrix calculation, the renormalization group approach, and other analytic tools [1]. In most of realistic model systems, however, a rigorous analytic calculation of critical exponents is often a formidable task, making numerical approaches unavoidable and essential.

A phase transition manifests itself as a singularity of the free energy, which exists only in thermodynamic limit of the infinite system size. On the other hand, one can only simulate finite systems in any computational approach, which results in the so-called finite-size effects. The very finiteness of system sizes can be utilized in the form of the finite-size scaling (FSS), which has been successfully applied to further our understanding of critical behaviors [2]. In the conventional FSS, the system under study is asked to be in equilibrium, and thus the functional form of FSS lacks any time dependence. As the system size becomes larger, the time scale at criticality often diverges together with the correlation length, which gives rise to the problem of the critical slowing down. In order to circumvent this, the finite-size dynamical scaling approach [3, 4] has been introduced. The basic assumption in this approach of dynamical scaling is that one can extract critical behavior by observing the temporal relaxation of the system in early times even before reaching equilibrium.

In general, one can assume that a time-varying thermodynamic quantity \( Q \) is a function of the time \( t \), the linear size \( L \), and the coupling strength \( K \). As the critical point \( K = K_c \) is approached, the correlation length \( \xi \) diverges following the form \( \xi \sim (K - K_c)^{-\nu} \) and so does the relaxation time scale \( \tau \sim \xi^z \). If we further assume that \( Q \) is chosen in such a way that its anomalous dimension is null [4], the scaling form of \( Q \) is written as \( Q(t, L, K) = f \left( tL^{-z}, (K - K_c)L^{1/\nu} \right) \). In words, the first scaling variable \( tL^{-z} \) describes the competition of the two time scales, the finite observation time \( t \) and the relaxation time \( \tau \), while the second scaling variable \( (K - K_c)L^{1/\nu} \) is for the competition of the two length scales, the finite system size \( L \) and the correlation length \( \xi \). It is straightforward to extend the scaling form for the

\[ Q(t, L, K) = f \left( tL^{-z}, (K - K_c)L^{1/\nu} \right) \]

FIG. 1. (Color online) Finite-time-finite-size scaling (FTFSS): \( Q(t, N, K) = f \left( tN^2, (K - K_c)N^{1/\nu} \right) \) for the globally-coupled Kuramoto models with quenched disorder yields a smooth scaling surface with \( \bar{\nu} = 5/2, \bar{\nu} = 2/5, \) and \( K_c = 1.595769 \). The scaling collapse is good enough to make difference of surfaces obtained from different sizes \( N = 800, 1600, 3200, \) and 6400 almost invisible. The thick dashed and solid lines are two cross sections of the surface at \( K = K_c \) and at \( tN^{-z} = 1.2 \) displayed in Fig. 2(b) and (d), respectively.

\[ Q(t, L, K) = f \left( tL^{-z}, (K - K_c)L^{1/\nu} \right) \]
globally-coupled system, which reads

\[ Q(t, N, K) = f \left( tN^{-\bar{\nu}}, (K - K_c)N^{1/\nu} \right), \tag{1} \]

where \( N \) is the size of the system and \( \bar{\nu} \) and \( \bar{\nu} \) are correlation and dynamic exponents defined for globally-coupled system. Throughout the present study, we term the scaling in Eq. (1) as finite-time-finite-size scaling (FTFSS). Precisely speaking, the phase transition is defined only in the limit of infinite time \( t \to \infty \) and infinite system size \( N \to \infty \), whereas any numerical calculation is limited by finiteness of \( t \) and \( N \). Just like the conventional FSS aims to systematically utilizes the finite-size effect by using the scaling variable \( L/\xi \), our FTFSS uses \( t/\tau \) as well to utilize the finite-time effect introduced by the finiteness of the observation time \( t \). Figure 1 exhibits how the FTFSS can be used to produce a smooth scaling surface for the globally-coupled Kuramoto model (see below for details).

Globally-Coupled Kuramoto Model – Synchronization phenomena are ubiquitously observed in a variety of systems such as the neuronal network and epilepsy in the brain [3], circadian rhythm [4], the collapse of the Millennium Bridge [5], power grids [6], and social behavior of humans [8]. Kuramoto model has been most popularly used to describe such synchronization phenomena, and we first study the globally-coupled \( N \) oscillators with both quenched intrinsic frequency \( \omega_i \) and thermal noise \( \eta_i \) described by

\[ \frac{d\theta_i}{dt} = \omega_i - \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j) + \eta_i(t), \tag{2} \]

where \( K \) is the coupling strength and \( \theta_i \) is the phase of the \( i \)-th oscillator. We use the Gaussian distribution of the zero mean and the variance \( \sigma^2 \) for the distribution function for \( \omega_i \) and the thermal noise satisfies \( \langle \eta_i \rangle = 0 \) and \( \langle \eta_i(t)\eta_j(t') \rangle = 2T\delta_{ij}\delta(t - t') \) with \( T \) being the effective temperature and \( \langle \cdot \cdot \cdot \rangle \) the ensemble average. In the zero-temperature limit of \( T \to 0 \), the system corresponds to the conventional Kuramoto model for which the correlation exponent \( \bar{\nu} = 5/2 \) [10,11], and the dynamic exponent \( \bar{\nu} = 2/5 \) [11] have been found. In the limit of \( \sigma \to 0 \), on the other hand, the system behaves as the globally-coupled XY model for which \( \bar{\nu} = 2 \) [11,12], and \( \bar{\nu} = 1/2 \) [11] are known. The equation of motion (2) in a finite dimension can be viewed as the time-dependent Ginzberg Landau (TDGL) dynamics of the superconducting array with the dc current \( \omega_i \) and the thermal noise current \( \eta_i \) [13].

In order to carry out FTFSS, we use the initial condition \( \theta_i(t = 0) = 0 \) for all oscillators and measure the key quantity \( Q(t) \) defined by [4,14,15]

\[ Q(t) \equiv \left< \operatorname{sign} \left[ \sum_{i=1}^{N} \cos \theta_i(t) \right] \right>, \tag{3} \]

which satisfies \( Q(0) = 1 \) and \( Q(t \to \infty) = 0 \) at any parameter values of \( K, \sigma, \) and \( T \) due to the rotational \( U(1) \) symmetry in equilibrium. It is to be emphasized that the value of \( Q(t) \) is not directly related with the order parameter, but it detects how much initial information at \( t = 0 \) is lost as the system approaches equilibrium which is characterized by \( Q = 0 \). The advantage of using \( Q \) lies on that it does not have the anomalous dimension [4] which allows us to use the simple FTFSS form in Eq. (1). We also emphasize that our FTFSS method uses how \( Q(t) \) evolves in time before reaching equilibrium, and thus equilibration is not a requirement to extract critical exponents [3]. However, it is to be noted that before the ensemble average \( \langle \cdot \cdot \cdot \rangle \), each sample run returns the value either +1 or −1 at time \( t \) since only the sign is taken in Eq. (3). This means that in order to get a smooth continuous form for \( Q(t) \) as a function of time \( t \), the ensemble average must be performed over sufficiently many samples. We use the second-order algorithm to integrate equation of motion with the discrete time step \( \Delta t = 0.01 \).

We first report the scaling results obtained from 100000 samples for the usual Kuramoto model with only the quenched randomness in the intrinsic frequency, corresponding to the zero-temperature limit of Eq. (2) with the thermal noise term dropped. The full FTFSS with
the critical coupling strength $\nu$ with the critical exponents $\bar{\nu} = 2$ and $\bar{z} = 1/2$ are obtained at $K_c = 2$ (in units of the temperature $T$). The thick dashed and solid lines are two cross sections of the surface at $K = K_c$ and at $tN^{-\bar{z}} = 2.5$ displayed in (b) and (c), respectively.

The two scaling variables in Eq. (1) is shown in Fig. (a) with $\sigma = 1$. All surfaces obtained for various sizes $N = 800, 1600, 3200,$ and 6400 collapse into a single smooth surface with the critical exponents $\bar{\nu} = 5/2$, $\bar{z} = 2/5$, and the critical coupling strength $K_c = \sqrt{8/\pi} = 1.595769$, as expected from previous studies [11, 12]. The dynamic scaling [11] is easily obtained by fixing the second scaling variable in the FTFSS (1) to null by putting $K = K_c$ as shown in Fig. (b). All the curves in Fig. (a) are shown to collapse nicely into a single smooth curve. Interesting application of the FTFSS is achieved by fixing the first scaling variable $tN^{-\bar{z}}$ in Eq. (1), to make the FTFSS form identical to the conventional FSS form. As an example, we use $tN^{-\bar{z}} = 1.2$ to plot Fig. (d). It is clearly seen that by fixing the first scaling variable, one can successfully obtain the critical exponent $\bar{\nu} = 5/2$ and $K_c = 1.595769$. In order to get such a finite-size scaling as in Fig. (d), it is important to choose the observation time $t$ systematically for the given system size $N$ to keep the value $tN^{-\bar{z}}$ as constant.

In other extreme case with only thermal noise, corresponding to $\sigma = 0$ in Eq. (2), the dynamics is effectively identical to the mean-field version of the TDGL equation, for which it is known that $\bar{\nu} = \nu d_u = 2$ and $\bar{z} = z/d_u = 1/2$ beyond upper critical dimension $d_u = 4$ [12]. In parallel to Fig. (2) where $K$ is in units of $\sigma$, we now measure $K$ in units of the temperature $T$. From the well-known result that the critical value of $T/K$ is 1/2 for the globally-coupled XY model [10], we expect that $K_c = 2$ in units of the temperature $T$. In the presence of thermal noise, integration of equation of motion takes much longer time since generation of random number is needed at each time step. We use $N = 400$, 800, and 1600 and the ensemble averages are performed for 20000 samples. As expected from known results of $K_c = 2$, $\bar{\nu} = 2$, and $\bar{z} = 1/2$, our FTFSS gives us again a good quality of scaling surface as shown in Fig. (a). As in Fig. (2) we also make cross sections of the scaling surface to construct scaling collapses in Fig. (b) and (c), in which it is shown clearly that the use of $\bar{\nu} = 2$ and $\bar{z} = 1/2$ with $K_c = 2$ yields scaling collapses as expected.

We also apply our FTFSS to the globally-coupled Kuramoto oscillators in the presence of both quenched and thermal disorder, which again yields results compatible with existing studies [see Supplemental Material (SM)].

**Kuramoto Model in the Small-World Network** – We next investigate the Kuramoto oscillators in the Watts-Strogatz (WS) small-world network structure [17, 18], for which equations of motion read

$$\frac{d\theta_i}{dt} = \omega_i - \frac{K}{\langle k \rangle} \sum_{j=1}^{N} A_{ij} \sin(\theta_i - \theta_j)$$

with the element of the adjacency matrix $A_{ij} = 1(0)$ if $i$ and $j$ are (dis)connected. Here, the thermal noise term...
\( \eta \) in Eq. (2) is neglected and the average degree \( \langle k \rangle = 6 \) is used for the construction of the WS network. Figure 3b displays the FTFSS surface of \( Q(t) \) in Eq. (3) produced for the WS network at the rewiring probability \( p = 0.5 \) with \( N = 100, 200, 400, \) and 800, confirming \( \tilde{\nu} = 2/5 \) and \( \nu = 5/2 \). We show in SM that \( \nu = 5/2 \) gives better scaling than \( \tilde{\nu} \approx 2.0 \) suggested in [18].

Globally-Coupled q-State Clock Model – We finally apply our FTFSS method to the Monte-Carlo (MC) dynamics of the globally-coupled \( q \)-state clock model [19] with the Hamiltonian

\[
\mathcal{H} = -J \sum_{i,j=1}^{N} S_i \cdot S_j,
\]

where \( S_i = (\cos \theta_i, \sin \theta_i) \) with \( \theta_i = 2\pi n_i/q \) (\( n_i = 0, ..., q-1 \)), and \( J \) is the coupling strength. In the limit of \( q \to \infty \), the \( q \)-state clock model becomes identical to the XY model, while the clock model at \( q = 2 \) is exactly the same as the Ising model. It is well-known that except for \( q = 3 \) the system shows the continuous phase transition of the mean-field nature [19]. In the MC simulations we use the heat-bath update algorithm [20], and the time \( t \) is measured in units of one MC step for which the number of tries of spin flipping is \( N \). Our key quantity \( Q(t) \) in Eq. (3) is measured near the critical point for different system sizes \( N = 100, 200, 400, 800, \) and 1600 to construct the scaling plots in Fig. 5 for \( q = 2, 4, 6, \) and 16, confirming the well-known mean-field exponents \( \tilde{\nu} = 1/2 \) and \( \nu = 2 \) (see SM for more details).

Conclusion and Summary – We have suggested the finite-time-finite-size scaling (FTFSS) method which takes into account the finiteness of the observation time scale as well as the finiteness of the system size. We have applied the FTFSS to construct the scaling surface defined on the two-dimensional parameter space. Proper cross sections of the FTFSS surface give us conventional finite-size scaling curves. We have applied our FTFSS to the globally-coupled Kuramoto model with quenched and thermal randomness, to the Kuramoto oscillators in the Watts-Strogatz small-world network, and to the Monte-Carlo dynamics of the globally-coupled \( q \)-state clock models. We have found that our FTFSS successfully and unanimously produces scaling surfaces with previously known correlation exponent \( \tilde{\nu} \) and the dynamic critical exponent \( \tilde{\nu} \). Although we have focused only on globally-coupled systems in the present study, our FTFSS can directly be applied for systems in a finite dimension.

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* Corresponding author: beomjun@skku.edu

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Finite-Time and Finite-Size Scaling of Relaxation Behavior:
Supplemental Material

Mi Jin Lee, Su Do Yi, and Beom Jun Kim

Department of Physics, Sungkyunkwan University, Suwon 440-746, Korea

FIG. 1. FTFSS for the globally-coupled Kuramoto oscillators with both quenched and thermal disorder: $\bar{\nu} = 2.3$ and $\bar{\varepsilon} = 0.45$ are obtained at $K_c = 5$ with $\sigma = 2.28006$ (both in units of the temperature $T$). The thick dashed and solid lines are two cross sections of the surface at $K = K_c$ and at $tN^{-\bar{\varepsilon}} = 0.4$, which are displayed in (b) and (c), respectively.
FIG. 2. The standard FSS of the order parameter $r$ for the globally-coupled Kuramoto oscillators at zero temperature. With $\beta = 1/2$ and $K_c = 1.595769$, a scaling collapse is obtained at $\bar{\nu} = 2.3$.

I. GLOBALLY-COUPLED KURAMOTO MODEL WITH QUENCHED AND THERMAL DISORDER

Motivated by the success of our FTFSS method for the Kuramoto model in the presence of either purely quenched disorder (see Figs. 1 and 2 in the main paper) or purely thermal disorder (see Fig. 3 in the main paper), we hereby study the Kuramoto systems with both types of disorder [$\sigma \neq 0$ and $T \neq 0$ in Eq. (2) in the main paper]. In Fig. 1 we show the FTFSS of $Q$ at $\sigma/T = 2.28006$ gives us a good quality of scaling collapse with $\bar{\nu} = 2.3$, $\bar{z} = 0.45$, and $K_c/T = 5$ (the values for $\sigma/T$ and $K_c/T$ are taken from the phase boundary in [1]). It has been suggested that the upper critical dimension $d_u = 5$ for the Kuramoto model with quenched randomness [2], and $d_u = 4$ has been agreed for the globally-coupled $XY$ model [3], which explains the values obtained in the main paper ($\bar{\nu} = 5/2, \bar{z} = 2/5$ for quenched, and $\bar{\nu} = 2, \bar{z} = 1/2$ for thermal disorder, respectively) on equal footing via $\bar{\nu} = \nu d_u$ and $\bar{z} = z/d_u$ with $\nu = 1/2$ and $z = 2$. We apply the FTFSS to our key quantity $Q$ in the same way as in the main paper to find the critical exponents along the phase boundary, and compare with the results obtained from the standard finite-size scaling (FSS) of the Kuramoto order parameter $r$ defined by

$$r \equiv \left\langle \left| \frac{1}{N} \sum_i e^{i \theta_i} \right| \rightangle,$$  \hspace{1cm} (1)
where $\langle \cdots \rangle$ is for both the sample average and the temporal average in the steady state. The standard FSS form $r = N^{-\beta/\nu} g((T - T_c)N^{1/\nu})$ is then used with the critical exponent $\beta = 1/2$ \cite{1} for the order parameter. In Fig. 2 as an example, we display the standard FSS of $r$ measured in equilibrium at zero temperature, corresponding to Figs. 1 and 2 in the main paper. A good quality of the scaling collapse is achieved at $\bar{\nu} \approx 2.3$ at $\beta = 1/2$.

In Table I we compare $\bar{\nu}$ obtained from the FTFSS of $Q$ and from the FSS of $r$, along the phase boundary presented in Fig. 1 of \cite{1}. When the randomness in the system is purely thermal ($\sigma = 0$) for which the phase boundary crosses $T/K$ axis at $1/2$, the value $\bar{\nu} \approx 2$ is observed both from $Q$ and $r$. In the zero-temperature limit ($T/K = 0$), on the other hand, the correct value $\bar{\nu} = 5/2$ is well obtained from $Q$, but the value is rather inaccurate if $r$ is used instead. Another advantage of using the FTFSS of $Q$ is that it allows us to obtain other exponent $\bar{z}$, which is also listed in Table I. It is noteworthy that along the full phase boundary, $\bar{z} \approx 1/\bar{\nu}$ is found regardless of the value of $T/K$, which appears to suggest that $\bar{z}\bar{\nu} = (z/d_u)(\nu d_u) = z\nu = 2 \cdot (1/2) = 1$ remains constant along the phase boundary. In spite of the advantage of using the FTFSS of $Q$ that only initial stage of nonequilibrium short-time relaxation is enough to detect universality class, we point out that the calculation of $Q$ requires average over a larger number of samples. In contrast, the calculation of $r$, for large

| $T/K$ | $\bar{\nu}$ from $r$ | $\bar{\nu}$ from $Q$ | $\bar{z}$ from $Q$ |
|-------|---------------------|---------------------|-------------------|
| 0     | 2.3(1)              | 2.5(1)              | 0.40(1)           |
| 0.1   | 2.2(1)              | 2.4(1)              | 0.42(1)           |
| 0.2   | 2.1(1)              | 2.3(1)              | 0.45(1)           |
| 0.3   | 2.1(1)              | 2.1(1)              | 0.49(1)           |
| 0.4   | 2.0(1)              | 2.1(1)              | 0.50(1)           |
| 0.5   | 2.0(1)              | 2.0(1)              | 0.51(1)           |

TABLE I. The correlation exponent $\bar{\nu}$ is computed along the phase boundary in the $(T, \sigma)$ plane via the FTFSS of $Q$ and the FSS of $r$. The system sizes used here are 100, 200, 400, and 800 and the averages over 10000 samples are performed. We also include the values of $\bar{z}$ from the FTFSS of $Q$, which satisfy $\bar{z} \approx 1/\bar{\nu}$. We expect that if much bigger system sizes are used, $\bar{\nu}$ and $\bar{z}$ converge to 2.0 and 0.5, respectively, except at the zero temperature limit for which $\bar{\nu} = 2.5$ and $\bar{z} = 0.4$ are expected. The numbers in parentheses are errors in the last digits.
FIG. 3. The cross section scaling at $K = K_c$ of the FTFSS surface of the globally-coupled Kuramoto oscillators for $T/K = 0.2$. (a) For relatively small system sizes $N = 100, 200, 400,$ and $800$, $\bar{z} = 0.45$ gives us the best scaling collapse. (b) For larger system sizes $N = 8000, 10000, 20000,$ and $40000$, the quality of the scaling for the same value $\bar{z} = 0.45$ is shown to become worse, whereas the use of the larger value $\bar{z} = 0.47$ yields much better scaling collapse as shown in (c).

systems in particular, does not require extensive sample average, thanks to the self-averaging property.

Standard universality class argument suggests that as soon as thermal disorder is added to the Kuramoto system, the correlation exponent is expected to change from $\bar{\nu} = 5/2$ to $\bar{\nu} = 2$, and the dynamic exponent $\bar{z} = 2/5$ to $\bar{z} = 1/2$. Although our results in Table I do not show such abrupt changes of $\bar{\nu}$ and $\bar{z}$, we believe that this can be a finite-size artifact which might be remedied if much bigger system sizes are used. In order to check this, we fix $T/K = 0.2$ and compute $Q(t)$ at the critical point $K_c$ for larger sizes $N = 8000, 10000, 20000,$ and $40000$ to compute the dynamic critical exponent $\bar{z}$. As displayed in Fig. 3, the bigger system sizes yield the larger value of $\bar{z}$, which appears to suggest that the expected value $\bar{z} = 1/2$ can be obtained if much bigger system sizes are used for the FTFSS analysis.

II. KURAMOTO OSCILLATORS WITH QUENCHED RANDOMNESS IN THE WATTS-STROGATZ SMALL-WORLD NETWORK

As one of examples of complex network structures, we use the Watts-Strogatz(WS) small-world network as an underlying interaction structure of the Kuramoto oscillators. The WS network has drawn much interest by playing the role of a bridge between the regular
FIG. 4. (a) The FTFSS of $Q$ for the Kuramoto oscillators in the WS network at the rewiring probability $p = 0.5$. The two cross sections for $K = K_c$ and $tN^{-\bar{z}} = 2.54$, respectively, denoted as two thick lines in (a), exhibit scaling collapses in (b) and (c) with $\bar{z} = 2/5$ and $\bar{\nu} = 5/2$. (d) The scaling collapse at $tN^{-\bar{z}} = 1.28$ with $\bar{z} = 1/2$ and $\bar{\nu} = 2$, corresponding to $d_u = 4$ with $z = 2$ and $\nu = 1/2$, becomes much worse than in (c).

and random networks, which is controlled by the rewiring procedure [4]. For a rewiring probability $p = 0$, the WS network has a regular ring lattice structure, while it becomes a fully random network for $p = 1$. For the intermediate value of $p$ ($0 < p < 1$) the structure of the WS network is well described as the small-world network, characterized by the short path length and the large clustering coefficient [4]. It is well known that for any nonzero value of $p$, the universality class of the WS network becomes identical to the globally-coupled system [5]. We generate the WS networks at $p = 0.2, 0.5,$ and $0.8$ with the average degree fixed as $\langle k \rangle = 6$ and integrate equations of motion [Eq. (4) in the main text] for the Kuramoto oscillators. Our FTFSS method is then applied to $Q(t)$, as shown in Fig. 4(a). In Fig. 4(b)
and (c), we exhibit the cross sections [denoted as thick full and dotted lines in Fig. 4(a)]. As expected, the WS network and the globally-coupled structure are shown to have the same critical exponents (results for \( p = 0.2 \) and \( p = 0.8 \) are not included in SM): \( \bar{\nu} = 2.5 \) and \( \bar{z} = 0.4 \). In Fig. 4(d), we show the scaling along the cross section for \( tN^{-\bar{z}} = 1.28 \) similarly to Fig. 4(c), but with \( \bar{z} = 1/2 \) and \( \bar{\nu} = 2 \). Different from [5], we conclude that the use of \( \bar{z} = 2/5 \) and \( \bar{\nu} = 5/2 \) as in Fig. 4(c) yields better scaling collapse, confirming \( d_u = 5 \).

III. MONTE-CARLO DYNAMICS OF THE GLOBALLY-COUPLED \( q \)-STATE CLOCK MODEL

To confirm the validity of FTFSS, we carry out similar procedure for the globally-coupled \( q \)-state clock model, in which each planar spin \( S_i = (\cos \theta_i, \sin \theta_i) \) has a discrete direction \( \theta_i = 2\pi n_i/q \) with \( n_i = 0, \ldots, q - 1 \) and the Hamiltonian is written as

\[
\mathcal{H} = -J \sum_{i,j=1}^{N} S_i \cdot S_j
\]

with the coupling strength \( J \). In the limit of \( q \to \infty \), the model corresponds to the \( XY \) model, and when \( q = 2 \) it is exactly identical to the Ising model. In the \( q \)-state clock model, the competition between thermal fluctuation and ferromagnetic interaction produces the phase transition from disordered to ordered phase [6, 7]. It is well known that for \( q \neq 3 \) the system exhibits the continuous phase transition of the mean-field nature. When \( q = 3 \), on the other hand, there is a discontinuous phase transition [8].

In contrast to the Kuramoto model, dynamics of the \( q \)-state clock model is not solely specified by the Hamiltonian and the Monte-Carlo (MC) dynamics depends on the choice of the spin update rule. We use the the heat-bath algorithm and measure the time \( t \) in units of one MC step, which corresponds to \( N \) MC tries of the spin update. We present our FTFSS for the globally-coupled \( q \)-state clock model with \( q=2, 4, 6, \) and 16 in Fig. 5 of the main paper. In Fig. 5(b) in the present SM, we display the scaling for \( q = 6 \) along the cross section denoted in Fig. 5(a). The choice of the well-known mean-field exponents \( \bar{z} = 1/2 \) and \( \bar{\nu} = 2 \) gives us again a good quality of the scaling collapse, validating our FTFSS method also for
FIG. 5. (a) The FTFSS surface for the globally-coupled $q$-state clock model at $q = 6$. With $T_c = 1/2$ in units of $J$ and the well-known mean-field exponents $\bar{z} = 1/2$ and $\bar{\nu} = 2$, a good quality of scaling surface is obtained. (b) Cross section of the scaling surface at $T = T_c$ [denoted as thick full line in (a)].

MC dynamics.

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