FAULT-TOLERANT ANTI-SYNCHRONIZATION CONTROL FOR
CHAOTIC SWITCHED NEURAL NETWORKS WITH TIME
DELAY AND REACTION DIFFUSION

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ABSTRACT. This paper is concerned with the issue of fault-tolerant anti-synchro-
nization control for chaotic switched neural networks with time delay and
reaction-diffusion terms under the drive-response scheme, where the response
system is assumed to be disturbed by stochastic noise. Both arbitrary switching
signal and average dwell-time limited switching signal are taken into account.
With the aid of the Lyapunov-Krasovskii functional approach and combin-
ing with the generalized Itô formula, sufficient conditions on the mean-square
exponential stability for the anti-synchronization error system are presented.
Then, by utilizing some decoupling methods, constructive design strategies on
the desired fault-tolerant anti-synchronization controller are proposed. Finally,
an example is given to demonstrate the effectiveness of our design strategies.

1. Introduction. Chaos is a concept that defines a state of disorder. In Math-
ematics and Science, it is often used to refer to a nonlinear dynamic behavior which
is deterministic in nature but not predictable in advance. A remarkable character-
istic of chaotic system is the high sensitivity to the initial condition, suggesting any
small change or disturbance in the initial values may lead to significant differences
in state evolution. Such a characteristic is known as the “butterfly effect”. Since
American meteorologist Edward Lorenz discovered chaotic behavior in his simula-
tion study on weather prediction in 1963 [18], chaotic systems have been extensively
studied in various areas including mathematics, physics, biology, and economics. In
1990, a drive-response scheme was introduced by Pecora and Carroll to study the synchronization between two identical chaotic systems [23]. It was found that the chaos synchronization has broad application prospects in secure communication [16], parameter estimation [13], and other fields. Accordingly, a lot of research on the subject have been conducted and many different types of chaos synchronization have been explored in the literature, such as the phase synchronization [8], lag synchronization [34], network-based synchronization [14], finite-time synchronization [35], anti-synchronization [26] and so on [6]. Among them, anti-synchronization describes an interesting behavior that states of the asynchronous systems have the same amplitude but opposite sign. In the 17th century, in the pendulum clock experiment, Huygens found anti-synchronization phenomenon for the first time. After that, in the experiments of saltwater oscillators [21] and chaotic semiconductor lasers [29], the anti-synchronization phenomenon was also noticed.

Neural networks (NNs) are highly complex large-scale nonlinear systems with rich dynamic behaviors, which have been widely used in the fields of artificial intelligence, automatic control, statistics, and other information processing. It is well known that in the circuit implementation of NNs, time delays are unavoidable, which are able to induce complex chaotic behaviors. During the past decade, the chaotic anti-synchronization problem of time-delay NNs (TDNNs) has attracted the attention of many scholars and some theoretical results have been achieved. In 2009, a new adaptive $H_{\infty}$ control scheme was employed to study the anti-synchronization behavior of chaotic TDNNs with unknown parameters in [2], where analytical expressions of the controller and the corresponding updating law were given. In the same year, the anti-synchronization problem of chaotic TDNNs with stochastic disturbance was studied in [24], where, by using the Lyapunov functional method, two sufficient conditions regarding the mean-square exponential anti-synchronization were established. Recently, the anti-synchronization issue for TDNNs with reaction-diffusion was considered in [10, 27], where pinning control and non-fragile control strategies were developed, respectively.

It is worth noting that in [2, 10, 24, 27], the parameter switching problem was not considered; that is to say, the network mode was assumed to be single therein. However, as indicated by [3], in practice a NN may have a limited number of modes that can switch from one mode to another depending on a switching signal. Therefore, it is not surprising to see research efforts on switching TDNNs during the past few years. For example, state estimation of discrete-time switched NNs with time-varying delays was considered in [36], where a mode-dependent estimator was designed to ensure that, under the average dwell-time (ADT) switching, the error system is exponentially stable with a predefined $L_2$ gain from the noise signal to the estimation error. Switched coupled reaction-diffusion NNs subject to non-delayed and delayed couplings were considered in [11], where some criteria for the input and output strict passivity problems were proposed by utilizing several inequality techniques. However, to our knowledge, there is no result available on the anti-synchronization of switched NNs with both time delay and reaction-diffusion (TDRDNNs) yet. In addition, it is worth mentioning the existing studies of switched TDRDNNs have not taken the problem of actuator failures into account, while such a problem may cause the control performance to be significantly reduced. Therefore, a question arises naturally, is it feasible to design a controller affected by possible actuator failures, so that a switched TDRDNN to be anti-synchronized?
In this paper, we would like to address such an issue under the drive-response configuration. The response system under consideration allows being disturbed by stochastic noise. Besides, both arbitrary switching signal and limited switching signal are taken into account. Specifically, the limited switching refers to the ADT constraint. With the aid of the Lyapunov-Krasovskii functional approach and combining with the generalized Itô formula, sufficient conditions on the mean-square exponential stability are derived for the anti-synchronization error system. Then, considering the possible actuator failures, design strategies on the desired fault-tolerant anti-synchronization controller are developed by utilizing some decoupling methods. It is seen that the control gain is able to be achieved by solving several linear matrix inequalities (LMIs) that can be realized readily via the popular computing software MATLAB. Finally, the effectiveness of the developed design methods is illustrated by an example. The rest of this paper is organized as follows. In Section 2, the switched drive-response TDRDNNs models are given, and some useful assumptions, definitions, and Lemmas are provided. Section 3 presents the anti-synchronization analysis and fault-tolerant control synthesis under arbitrary switching and limited switching, respectively. In Section 4, a numerical example is given. Finally, there is a brief summary in Section 5.

2. Preliminaries. Throughout the present study, denote by \( \mathbb{R}^n \) the normal \( n \)-dimensional Euclidean space with norm \( |\cdot| \), by \( \mathbb{R}^{n \times m} \) real matrices, by \( \mathcal{E} \) the mathematical expectation, and by \( (\mathcal{Y}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}) \) the Lebesgue probability space [5]. Let \( \mathcal{C}^m(Z, \mathbb{R}^n) \) be the set of \( m \)-times continuously-differentiable functions \( \varphi \) which map \( Z \) into \( \mathbb{R}^n \), \( \mathcal{L}^2_0([-\mu, 0] \times \Omega, \mathbb{R}^n) \) be the set of \( \mathcal{F}_0 \)-measurable \( \mathcal{C}([-\mu, 0] \times \Omega, \mathbb{R}^n) \)-random variables \( \chi = \{\chi(\theta, x)\} \) and \( \Omega = \{x = [x_1 \cdots x_l]^T | \phi_k \leq x_1 \leq \psi_k, \phi_k, \psi_k \in \mathbb{R}, k = 1, \cdots, l\} \) be a closed set in \( \mathbb{R}^l \). The boundary and the volume of \( \Omega \) are defined as \( \partial \Omega \) and \( v_0 = \Pi_{k=1}^l(\psi_k - \phi_k) \), respectively. A function \( \kappa : [0, \infty) \to [0, \infty) \) is said to be of class \( \mathcal{K}_\infty \) if it is unbounded and belongs to class \( \mathcal{K} \) [15]. Denote by \( tr(M) \) the trace of a real square matrix \( M \), by \( He(M) \) the sum of \( M \) and \( MT \), by \( \lambda_M(M) \) and \( \lambda_m(M) \) the maximum and minimum eigenvalues of \( M \) respectively, by \( \ast \) the symmetry blocks in a square matrix, and by \( I \) and \( 0 \) the unity and zero real matrices, respectively.

The anti-synchronization problem under consideration is based on the drive-response scheme, where the drive (or master) system is represented by the following switched TDRDNN:

\[
\begin{aligned}
\frac{dz_1(t,x)}{dt} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_k \frac{\partial z_1(t,x)}{\partial x_k} \right) - A_{r(t)} z_1(t,x) + W_{0,r(t)} f(z_1(t,x)) \\
y_1(t,x) &= C_{r(t)} z_1(t,x),
\end{aligned}
\]

(1)

where \( r(t) \) is a switching signal and takes its values in \( \mathcal{M} = \{1, \cdots, m\} \), \( z_1(t,x) = [z_{11}(t,x) \cdots z_{1n}(t,x)]^T \) with \( z_{1i}(t,x) \in \mathcal{C}^2(\mathbb{R} \times \Omega, \mathbb{R}) \) denoting the state variable of the \( i \)-th neuron at \( t \) and \( x = [x_1 \cdots x_l]^T \). \( D_k \) denotes the transmission diffusion coefficient along the \( i \)-th neuron, \( A_{r(t)} = diag\{a_{1,r(t)}, \cdots, a_{n,r(t)}\} \) with \( \dot{a}_{i,r(t)} > 0 \) denoting the rate at which the \( i \)-th neuron resets the potential to the resting state in the case when disconnected from the network and external inputs, \( W_{0,r(t)} = (W_{0,j,r(t)})_{n \times n}, W_{1,r(t)} = (W_{ij,r(t)})_{n \times n} \) with \( W_{ij,r(t)} \) and \( W_{ij,r(t)}^1 \) representing the connection strengths, \( f(z_1(t,x)) = \left[ f_1(z_{11}(t,x)) \cdots f_n(z_{1n}(t,x)) \right]^T \) with \( f_i(z_{1i}(t,x)) \) representing the activation function.
of the $i$th neuron which is odd for realizing anti-synchronization as in [2], $\mu$ is a positive constant representing the transmission delay, $y_1(t, x) = [y_{11}(t, x) \cdots y_{1p}(t, x)]^T$ stands for the system output, and $C_{r(t)} \in \mathbb{R}^{p \times n}$ denotes a known constant matrix for any $r(t) \in \mathcal{M}$.

The response (or slave) system is assumed to be disturbed by stochastic noise and is represented by:

$$
\begin{cases}
    dz_2(t, x) = \left[ \sum_{k=1}^l \frac{\partial}{\partial x_k} \left( D_k \frac{dz_2(t, x)}{\partial x_k} \right) - A_r(t)z_2(t, x) + W_{0, r(t)}f(z_2(t, x)) \\
    + W_{1, r(t)}f(z_2(t - \mu, x)) + B_{r(t)}u^F(t, x) \right] dt \\
    + \iota(t, z_1(t, x) + z_2(t, x), z_1(t - \mu, x) + z_2(t - \mu, x)) d\omega(t),
\end{cases}
$$

where $z_2(t, x) = [z_{21}(t, x) \cdots z_{2n}(t, x)]^T$ and $u^F(t, x) = [u^F_1(t, x), \cdots, u^F_p(t, x)]^T$ denote the system output and control input with possible actuator failures, respectively; $B_{r(t)}$ corresponds to a real constant matrix for any $r(t) \in \mathcal{M}$; $\omega(t)$ corresponds to a Brownian motion defined on $(\mathbb{T}, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0})$ with $\mathbb{E}\{\omega(t)\} = 0$ and $\mathbb{E}\{\omega(t)^2\} = dt$ [42]; $\iota(t, z_1(t, x) + z_2(t, x), z_1(t - \mu, x) + z_2(t - \mu, x))$ denotes the noise intensity that satisfies $\iota(t, 0, 0) = 0$. Without loss of generality, the initial and boundary conditions of the drive-response systems are described as

$$
\begin{align*}
    z_j(s, x) & = \varphi_j(s, x), \quad (s, x) \in [-\mu, 0] \times \Omega, \\
    z_j(t, x) & = 0, \quad (t, x) \in [0, +\infty) \times \partial \Omega,
\end{align*}
$$

respectively [1], in which $\varphi_j \in \mathbb{L}^2_{\mathbb{P}, 0}((-\mu, 0] \times \Omega, \mathbb{R}^n), j = 1, 2$.

In this paper, the actuator failure model is described as

$$
u^F(t, x) = Fu(t, x),$$

where $F = diag\{F_1, \cdots, F_q\} \in \mathbb{R}^{q \times q}$ is an unknown matrix with $0 \leq \hat{F}_i \leq F_i \leq \bar{F}_i (i = 1, \cdots, q)$. Here, $\hat{F}_i$ and $\bar{F}_i$ are known scalars denoting the lower and upper bounds of $F_i$, respectively. Note that there is a loss of efficiency in controller $u(t, x)$ when $0 < \hat{F}_i = \bar{F}_i < 1$, and $u(t, x)$ is out of service when $\hat{F}_1 = \cdots = \bar{F}_q = 0$ [32].

Denote

$$
\begin{align*}
    & F_0^i = \frac{\hat{F}_i + \bar{F}_i}{2}, \quad \Theta_i = \frac{F_i - F_0^i}{\bar{F}_i}, \quad \mathcal{H}_i = \frac{\bar{F}_i - F_i}{\hat{F}_i + \bar{F}_i}, \quad (i = 1, \cdots, q) \\
    & F_0 = diag\{F_0^1, \cdots, F_0^q\}, \\
    & \mathcal{H} = diag\{\mathcal{H}_1, \cdots, \mathcal{H}_q\}, \\
    & \Theta = diag\{\Theta_1, \cdots, \Theta_q\}.
\end{align*}
$$

Then, $u^F(t, x)$ can be written by a concise form as follows

$$
u^F(t, x) = (F_0 + F_0\Theta)u(t, x), \quad \Theta \leq \mathcal{H} \leq I.
$$

Considering that the states of a controlled system are often not available [4, 37], the controller to be used is based on the system output and takes the form as

$$
u(t, x) = K (y_1(t, x) + y_2(t, x)),
$$

in which $K$ denotes the controller gain to be fixed.
Let us set \( \zeta(t,x) = z_1(t,x) + z_2(t,x) \). Then by (5) and (6) one can write the anti-synchronization error system as follows:

\[
d\zeta(t,x) = \left[ \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_k \frac{\partial \zeta(t,x)}{\partial x_k} \right) - \left( A_{r(t)} - B_{r(t)} (F_0 + F_0 \Theta) KC_{r(t)} \right) \zeta(t,x) + W_{0r(t)} g(\zeta(t,x)) + W_{1r(t)} g(\zeta(t-\mu,x)) \right] dt
\]

\[
+ \iota(t,\zeta(t,x),\zeta(t-\mu,x)) d\omega(t),
\]

where \( g(\zeta(t,x)) = f(\zeta(t,x) - z_1(t,x)) + f(z_1(t,x)) \). To simplify the symbols, we denote \( r(t) \) in terms of \( i (i \in \mathcal{M}) \) when possible, and then write system (7) as

\[
d\zeta(t,x) = \delta_i (\zeta(t,x),\zeta(t-\mu,x)) dt + \iota(t,\zeta(t,x),\zeta(t-\mu,x)) d\omega(t),
\]

where

\[
\delta_i (\zeta(t,x),\zeta(t-\mu,x))
\]

\[
= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_k \frac{\partial \zeta(t,x)}{\partial x_k} \right) - A^K_i \zeta(t,x) + W_{0i} g(\zeta(t,x)) + W_{1i} g(\zeta(t-\mu,x))
\]

with

\[
A^K_i = A_i - B_i (F_0 + F_0 \Theta) KC_i
\]

In the present study, the activation functions and the noise intensity matrix are assumed to satisfy the following conditions:

**Assumption 1.** For any \( \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 \neq \alpha_2 \), activation function \( f_j(\cdot) \) (\( j = 1, \ldots, n \)) satisfies

\[
|f_j(\alpha_2) - f_j(\alpha_1)| \leq \gamma_j |\alpha_2 - \alpha_1|,
\]

where \( \gamma_j > 0 \) is known constant scalar.

**Assumption 2.** There exists positive scalars \( \iota_1 \) and \( \iota_2 \) ensuring that

\[
\text{tr} \left( \iota^T(t,\zeta(t,x),\zeta(t-\mu,x)) \iota(t,\zeta(t,x),\zeta(t-\mu,x)) \right) \leq \iota_1 \zeta^T(t,x) \zeta(t,x) + \iota_2 \zeta^T(t-\mu,x) \zeta(t-\mu,x).
\]

It may be worth pointing out that Assumptions 1 and 2 have been extensively employed in the literature; see, e.g. [30, 28, 38, 19, 25]. Note that the constraint on the activation functions in (10) is weaker than those imposed in [22, 17, 33, 12, 7].

At the end of this section, let us prepare several useful definitions and lemmas.

**Definition 2.1.** The systems in (1) and (2) are said to be stochastically exponentially anti-synchronized if system (8) is exponentially stable in mean square.

**Definition 2.2.** [9] Denote by \( N_{r(t)}(s_1,s_2) \) the number of its switching times in any open time interval \((s_1,s_2)\). Suppose that there are two constants \( T_a \) and \( N_c > 0 \) so that

\[
N_{r(t)}(s_1,s_2) \leq N_c + \frac{s_2 - s_1}{T_a}.
\]

Then, \( T_a \) is said to be the ADT and \( N_c \) the chatter bound.

**Remark 1.** ADT describes a type of switching signal for which despite sometimes the distance between two consecutive switching instants may be less than \( T_a \), the average length between consecutive switching instants is at least \( T_a \). It is noteworthy that, in the light of (12), chatter bound \( N_c \) should be greater than 0 for any switched system.
Lemma 2.3. [39] Let \( \alpha(x) \in C^1(\Omega, \mathbb{R}) \) with \( \alpha(x)/\partial\Omega = 0 \), then
\[
\int_{\Omega} \alpha^2(x) dx \leq \left( \frac{\psi_k - \phi_k}{\pi} \right)^2 \int_{\Omega} \left( \frac{\partial \alpha(x)}{\partial x_k} \right)^2 dx, \quad k = 1, \cdots, l.
\]
(13)

Lemma 2.4. [40] For any real matrices \( \Lambda_1 \) and \( \Lambda_2 \) with suitable dimensions and any scalar \( \varepsilon > 0 \), one can write
\[
\Lambda_1 \Lambda_2^T + \Lambda_2 \Lambda_1^T \leq \frac{1}{\varepsilon} \Lambda_1 \Lambda_1^T + \varepsilon \Lambda_2 \Lambda_2^T.
\]

Lemma 2.5. [41] Suppose that the following condition concerning a real constant \( \zeta \) and real matrices \( W, U, V, \) and \( S \) holds:
\[
\begin{bmatrix}
W & U + \zeta V \\
* & He(-\zeta S)
\end{bmatrix} < 0.
\]
Then one can obtain
\[
W + He(U S^{-1} V^T) < 0.
\]
(14)

3. Main result.

3.1. Arbitrary switching. In this subsection, we consider the case that the drive and response systems in (1) and (2) are subject to arbitrary switching, i.e. switching signal \( r(t) \) takes its values arbitrary in the finite set \( M \).

Firstly, we give a analysis result in the following theorem:

**Theorem 3.1.** System (8) is exponentially stable in mean-square, if, there are scalars \( \rho_i > 0 \), diagonal matrices \( P_i > 0, H_{1i} = \text{diag}(h_{11}, \cdots, h_{n1}) > 0, H_{2i} = \text{diag}(h_{12}, \cdots, h_{n2}) > 0, \) and matrices \( Q_{1i} > 0, Q_{2i} > 0 \) such that
\[
P_i < \rho_i I,
\]
(15)
\[
\Phi_i = \begin{bmatrix}
\Xi_{1i} & 0 & P_i W_0i & P_i W_1i \\
* & \Xi_{22} & 0 & 0 \\
* & * & Q_{2i} - H_{1i} & 0 \\
* & * & * & -Q_{2i} - H_{2i}
\end{bmatrix} < 0
\]
(16)

hold for any \( i \in M \), where
\[
\Xi_{1i} = -2P_i D_{1i} - 2P_i A_{i}^K + Q_{1i} + \rho_i \nu_1 I + \Gamma H_{1i} \Gamma,
\]
\[
\Xi_{22} = -Q_{1i} + \rho_i \nu_2 I + \Gamma H_{2i} \Gamma, \quad \Gamma = \text{diag}(\gamma_1, \cdots, \gamma_n),
\]
\[
D_{1i} = \text{diag} \left\{ \sum_{k=1}^{l} \left( \frac{\pi}{\psi_k - \phi_k} \right)^2 D_{1k}, \cdots, \sum_{k=1}^{l} \left( \frac{\pi}{\psi_k - \phi_k} \right)^2 D_{nk} \right\}.
\]

**Proof.** Consider the following Lyapunov-Krasovskii functional:
\[
V_i(t, \zeta(t, x)) = \int_{\Omega} \left[ \zeta^T(t, x) P_i \zeta(t, x) \right] dx
\]
\[
+ \int_{\Omega} \left[ \int_{t-\mu}^{t} \left( \zeta^T(\theta, x) Q_{1i} \zeta(\theta, x) \right) d\theta \right] dx
\]
\[
+ \int_{\Omega} \left[ \int_{t-\mu}^{t} \left( g^T(\zeta(\theta, x)) Q_{2i} g(\zeta(\theta, x)) \right) d\theta \right] dx \quad (i \in M).
\]
(17)
and we can obtain
\[ V_i(t, \zeta(t, x)) \]
\[ \leq \int_{\Omega} \left[ \lambda_M(P_i)\zeta^T(t, x)\zeta(t, x) + \int_{t-\mu}^{t} \left( \lambda_M(Q_{1i})\zeta^T(\theta, x)\zeta(\theta, x) \right) d\theta \right] dx \]
\[ + \int_{\Omega} \left[ \int_{t-\mu}^{t} \left( \lambda_M(Q_{2i})g^T(\zeta(\theta, x))g(\zeta(\theta, x)) \right) d\theta \right] dx. \]  
(18)

Utilizing the generalized Itô formula \([20]\) to \(e^{kt}V_i(t)(k > 0)\) yields
\[ \mathcal{E}\{e^{kt}V_i(t, \zeta(t, x))\} \]
\[ = \mathcal{E}\{V_i(0, \zeta(0, x))\} + \mathcal{E}\left\{ \int_{0}^{t} e^{ks} [kV_i(s, \zeta(s, x)) + \mathcal{L}V_i(s, \zeta(s, x))] ds \right\}, \]
(19)
where
\[ \mathcal{L}V_i(s, \zeta(s, x)) \]
\[ = \frac{\partial V_i(s, \zeta(s, x))}{\partial s} + V_i,\zeta(s, \zeta(s, x))\delta_i(s, x, \zeta(s, x), \zeta(s - \mu, x)) \]
\[ + \frac{1}{2} \text{tr}\left( \zeta^T(s, x) Q_{1i} \zeta(s, x) - \zeta^T(s - \mu, x) Q_{1i} \zeta(s - \mu, x) \right) \]
with
\[ V_{i,\zeta}(s, \zeta(s, x)) = \begin{bmatrix} \frac{\partial V_i(s, \zeta(s, x))}{\partial \zeta_1} & \cdots & \frac{\partial V_i(s, \zeta(s, x))}{\partial \zeta_n} \end{bmatrix}, \]
\[ V_{i,\zeta\zeta}(s, \zeta(s, x)) = \left( \frac{\partial^2 V_i(s, \zeta(s, x))}{\partial \zeta_j \partial \zeta_{j'}} \right)_{n \times n}. \]

According to (8), (9), and (17), one has
\[ \mathcal{L}V_i(s, \zeta(s, x)) \]
\[ = 2 \int_{\Omega} \zeta^T(s, x) P_i \left[ \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left( D_k \frac{\partial \zeta(s, x)}{\partial x_k} \right) - A_i^K \zeta(s, x) \right] dx \]
\[ + W_{0i} g(\zeta(s, x)) + W_{1i} g(\zeta(s - \mu, x)) \]
\[ + \int_{\Omega} \left( \zeta^T(s, x) Q_{1i} \zeta(s, x) - \zeta^T(s - \mu, x) Q_{1i} \zeta(s - \mu, x) \right) dx \]
\[ + \int_{\Omega} \left( g^T(\zeta(s, x)) Q_{2i} g(\zeta(s, x)) - g^T(\zeta(s - \mu, x)) Q_{2i} g(\zeta(s - \mu, x)) \right) dx \]
\[ + \int_{\Omega} \text{tr}\left( \zeta^T(s, x) Q_i \zeta(s, x) - \zeta^T(s - \mu, x) Q_i \zeta(s - \mu, x) \right) dx. \]
(20)

By (4), for any \(j \in \{1, \cdots, n\}\), it can be obtained that
\[ \int_{\Omega} \zeta_j(s,x) \frac{\partial}{\partial x_k} \left( D_{jk} \frac{\partial \zeta_j(s,x)}{\partial x_k} \right) \, dx = - \int_{\Omega} D_{jk} \left( \frac{\partial \zeta_j(s,x)}{\partial x_k} \right)^2 \, dx. \]

Using Lemma 2.3, one has
\[ \int_{\Omega} \zeta_j(s,x) \frac{\partial}{\partial x_k} \left( D_{jk} \frac{\partial \zeta_j(s,x)}{\partial x_k} \right) \, dx \leq - \left( \frac{\pi}{\psi_k - \phi_k} \right)^2 D_{jk} \int_{\Omega} \zeta_j^2(s,x) \, dx, \]
which suggests
\[ 2 \int_{\Omega} \zeta^T(s,x) P_1 \sum_{k=1}^n \frac{\partial}{\partial x_k} \left( D_k \frac{\partial \zeta(s,x)}{\partial x_k} \right) \, dx \leq -2 \int_{\Omega} \zeta^T(s,x) P_2 D_{\pi} \zeta(s,x) \, dx. \quad (21) \]

From (11) and (15), one can write
\[ \text{tr} \left( \nu^T(s, \zeta(s,x), \zeta(s-\mu,x)) P_1 \nu(s, \zeta(s,x), \zeta(s-\mu,x)) \right) \]
\[ \leq \rho_{\nu_1} \zeta^T(s,x) \zeta(s,x) + \rho_{\nu_2} \zeta(s-\mu,x) \zeta(s-\mu,x). \quad (22) \]

Furthermore, noting \( g(\zeta) \) is an odd function and using (10), for diagonal matrices \( H_1 \) and \( H_2 \), one has
\[ g^T(\zeta(s,x)) H_{1i} g(\zeta(s,x)) \]
\[ \leq \sum_{j=1}^n \gamma_j^2 h_{j,1i} |\zeta(s,x) - z_1(s,x) - (-z_1(s,x))|^2 \]
\[ = \zeta^T(s,x) \Gamma H_{1i} \Gamma \zeta(s,x), \quad (23) \]
\[ g^T(\zeta(s-\mu,x)) H_{2i} g(\zeta(s-\mu,x)) \]
\[ \leq \sum_{j=1}^n \gamma_j^2 h_{j,2i} |\zeta(s-\mu,x) - z_1(s-\mu,x) - (-z_1(s-\mu,x))|^2 \]
\[ = \zeta^T(s-\mu,x) \Gamma H_{2i} \Gamma \zeta(s-\mu,x). \quad (24) \]

By (20) - (24), one obtains
\[ \mathcal{L}V_i(s, \zeta(s,x)) \]
\[ \leq \int_{\Omega} \zeta^T(s,x) \left( -2P_1 D_{\pi} - 2P_1 A^K + \rho_{\nu_1} I \right) \zeta(s,x) \, dx \]
\[ + \int_{\Omega} \zeta^T(s,x) \left( 2P_2 W_0 \right) g(\zeta(s,x)) \, dx \]
\[ + \int_{\Omega} \zeta^T(s,x) \left( 2P_2 W_{1i} \right) g(\zeta(s-\mu,x)) \, dx \]
\[ + \int_{\Omega} \zeta^T(s-\mu,x) \left( \rho_{\nu_2} I \right) \zeta(s-\mu,x) \, dx \]
\[ + \int_{\Omega} \left( \zeta^T(s,x) Q_{1i} \zeta(s,x) - \zeta^T(s-\mu,x) Q_{1i} \zeta(s-\mu,x) \right) \, dx \]
\[\begin{align*}
+ & \int_{\Omega} \left( g^T (\zeta(s, x)) Q_{2i} g (\zeta(s, x)) \right) dx \\
- & g^T (\zeta(s - \mu, x)) Q_{2i} g (\zeta(s - \mu, x)) \right) dx \\
+ & \int_{\Omega} \left( \zeta^T (s, x) \Gamma H_{1i} \zeta(s, x) - g^T (\zeta(s, x)) H_{1i} g (\zeta(s, x)) \right) dx \\
+ & \int_{\Omega} \left( \zeta^T (s - \mu, x) \Gamma H_{2i} \zeta(s - \mu, x) - g^T (\zeta(s - \mu, x)) H_{2i} g (\zeta(s - \mu, x)) \right) dx \\
= & \int_{\Omega} \varpi^T (s, x) \Phi \varpi(s, x) dx,
\end{align*}\]

where

\[\varpi(s, x) = [ \zeta^T(s, x) \quad \zeta^T(s - \mu, x) \quad g^T(\zeta(s, x)) \quad g^T(\zeta(s - \mu, x)) ]^T.\]

Then, using (16), one can write

\[\mathcal{L} V_i(s, \zeta(s, x)) \leq \lambda_M (\Phi_i) \int_{\Omega} \zeta^T(s, x) \zeta(s, x) dx. \quad (25)\]

Now, by (3), (17) - (19), (23), and (25), one has

\[\min_{1 \leq i \leq n} \lambda_m (P_i) e^{kt} \mathcal{E} \left\{ \int_{\Omega} |\zeta(t, x)|^2 dx \right\} \]

\[\leq \mathcal{E} \left\{ e^{kt} V_i(t, \zeta(t, x)) \right\} \]

\[\leq \mathcal{E} \left\{ \int_{\Omega} \left[ \lambda_M (P_i) \zeta^T(0, x) \zeta(0, x) + \int_{-\mu}^{0} \left( \lambda_M (Q_{1i}) \zeta^T(\theta, x) \zeta(\theta, x) \right. \right. \right. \]

\[+ \lambda_M (Q_{2i}) g^T(\zeta(\theta, x)) g(\zeta(\theta, x)) \bigg) d\theta \right] dx \bigg) \]

\[+ \mathcal{E} \left\{ \int_{\Omega} \left[ \int_{0}^{t} e^{ks} k \left( \lambda_M (P_i) \zeta^T(s, x) \zeta(s, x) + \int_{s-\mu}^{s} \left( \lambda_M (Q_{1i}) \zeta^T(\theta, x) \zeta(\theta, x) \right. \right. \right. \]

\[+ \lambda_M (Q_{2i}) g^T(\zeta(\theta, x)) g(\zeta(\theta, x)) \bigg) d\theta \right] ds \right] dx \bigg) \]

\[+ \lambda_M (\Phi_i) \mathcal{E} \left\{ \int_{\Omega} \left[ \int_{0}^{t} \left( e^{ks} \zeta^T(s, x) \zeta(s, x) \right) ds \right] dx \right\} \]

\[\leq C_0 \mathcal{E} \left\{ \int_{\Omega} \left( \int_{0}^{t} e^{ks} |\zeta(s, x)|^2 ds \right) dx \right\} \]

\[+ C_1 \mathcal{E} \left\{ \int_{\Omega} \sup_{-\mu \leq \theta \leq 0} |\varphi_1(\theta, x) + \varphi_2(\theta, x)|^2 dx \right\}, \quad (26)\]
where

\[
\begin{align*}
C_0 &= k\lambda_M(P_i) + (e^{k\mu} - 1) \left[ \lambda_M(Q_{1i}) + \lambda_M(Q_{2i})\Gamma^2 \right] + \lambda_M(\Phi_i), \\
C_1 &= \lambda_M(P_i) + \mu \lambda_M(Q_{1i}) + \mu \lambda_M(Q_{2i})\Gamma^2 \\
&\quad + \left( \lambda_M(Q_{1i}) + \lambda_M(Q_{2i})\Gamma^2 \right) \frac{1}{k}(e^{k\mu} - 1)(1 - e^{-k\mu}).
\end{align*}
\]

Choose \(k > 0\) small enough such that \(C_0 < 0\) and define

\[
\lambda_1 = \sup_{1 \leq i \leq n} C_1 \mathcal{E} \left\{ \int_\Omega \sup_{-\mu \leq \theta \leq 0} |\varphi_1(\theta, x) + \varphi_2(\theta, x)|^2 \, dx \right\} / \min_{1 \leq i \leq n} \lambda_m(P_i).
\]

Then, (26) gives

\[
\mathcal{E} \left\{ \int_\Omega |\zeta(t, x)|^2 \, dx \right\} \leq \lambda_1 e^{-kt},
\]

which means that system (8) is exponentially stable in mean square. \(\square\)

Next, we consider the case of \(u(t, x) \neq 0\). Based on Theorem 3.1 and with the aid of some decoupling methods, we can get the following result:

**Theorem 3.2.** Given a scalar \(\varsigma > 0\), system (1) and system (2) with the fault-tolerant controller in (5) are stochastically exponentially anti-synchronized, if there are scalars \(\rho_i > 0\) and \(\varepsilon > 0\), diagonal matrices \(P_i > 0\), \(H_{1i} = diag(h_{1,1i}, \ldots, h_{n,1i}) > 0\), \(H_{2i} = diag(h_{1,2i}, \ldots, h_{n,2i}) > 0\), matrices \(Q_{1i} > 0\), \(Q_{2i} > 0\), and \(X, Y\), such that (15) and the following

\[
\Phi_i = \begin{bmatrix}
\Delta_{11} & 0 & P_i W_{0i} & P_i W_{1i} & \Delta_{15} & P_i B_i F_0 \\
* & \Delta_{22} & 0 & 0 & 0 & 0 \\
* & * & Q_{2i} - H_{1i} & 0 & 0 & 0 \\
* & * & Q_{2i} - H_{2i} & 0 & 0 & 0 \\
* & * & * & \Delta_{55} & 0 & 0 \\
* & * & * & * & \varepsilon & I
\end{bmatrix} < 0 \tag{27}
\]

hold for any \(i \in \mathcal{M}\), where

\[
\begin{align*}
\Delta_{11} &= -2P_i D_\varsigma - 2P_i A_i + Q_{1i} + \rho_i \varsigma I + He(B_i Y C_i) + \Gamma H_{1i} \Gamma, \\
\Delta_{15} &= P_i B_i F_0 - B_i X + \varsigma C_i^T Y^T, \\
\Delta_{22} &= -Q_{1i} + \rho_i \varsigma I + \Gamma H_{2i} \Gamma, \Gamma = diag(\gamma_1, \ldots, \gamma_n), \\
\Delta_{55} &= -He(\varsigma X) + \varepsilon H^T \mathcal{H}, \\
D_\varsigma &= \text{diag} \left\{ \sum_{k=1}^l \left( \frac{\varsigma}{\phi_k - \phi_k} \right)^2 D_{1k}, \ldots, \sum_{k=1}^l \left( \frac{\varsigma}{\phi_k - \phi_k} \right)^2 D_{nk} \right\}.
\end{align*}
\]

In that case, the desired controller gain in (6) is capable to be fixed by

\[
K = X^{-1} Y. \tag{28}
\]

**Proof.** Using (9), (16) can be rearranged as

\[
\Phi_i \mathfrak{F} + \begin{bmatrix}
He(B_i Y C_i) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + He \left\{ \mathcal{U}_i X^{-1} Y^T \right\} < 0, \tag{29}
\]
and

\[ \Phi_{iF} = \begin{bmatrix} \tilde{\Delta}_{11} & 0 & P_iW_0i & P_iW_{1i} \\ * & \Delta_{22} & 0 & 0 \\ * & * & Q_{2i} - H_{1i} & 0 \\ * & * & * & -Q_{2i} - H_{2i} \end{bmatrix}, \]

\[ U_i = \begin{bmatrix} P_iB_i(F_0 + F_0\Theta) - B_iX \\ 0 \\ 0 \end{bmatrix}, \]

\[ V_i = \begin{bmatrix} C^TY_i^T \\ 0 \\ 0 \end{bmatrix}, \]

where

\[ \tilde{\Delta}_{11} = \Xi_{11} - 2P_iB_i(F_0 + F_0\Theta)KC_i. \]

According to Lemma 2.5, (29) holds true if

\[ \begin{bmatrix} \tilde{\Delta}_{11} + He(B_iYC_i) & 0 & P_iW_0i & P_iW_{1i} & \tilde{\Delta}_{15} \\ * & \Delta_{22} & 0 & 0 & 0 \\ * & * & Q_{2i} - H_{1i} & 0 & 0 \\ * & * & * & -Q_{2i} - H_{2i} & 0 \\ * & * & * & * & -He(\varsigma X) \end{bmatrix} < 0, \] (30)

where

\[ \tilde{\Delta}_{15} = P_iB_iF_0 + P_iB_iF_0\Theta - B_iX + \varsigma C_i^TY_i^T. \]

It is not difficult to re-express (30) into the following

\[ \bar{\Phi}_i + He \begin{bmatrix} 0 & 0 & 0 & 0 & P_iB_iF_0\Theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \] (31)

where

\[ \bar{\Phi}_i = \begin{bmatrix} \tilde{\Delta}_{11} + He(B_iYC_i) & 0 & P_iW_0i & P_iW_{1i} & \Delta_{15} \\ * & \Delta_{22} & 0 & 0 & 0 \\ * & * & Q_{2i} - H_{1i} & 0 & 0 \\ * & * & * & -Q_{2i} - H_{2i} & 0 \\ * & * & * & * & -He(\varsigma X) \end{bmatrix}. \]

Using Lemma 2.4 and (5), one has

\[ He \begin{bmatrix} 0 & 0 & 0 & 0 & P_iB_iF_0\Theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leq \frac{1}{\varepsilon} \begin{bmatrix} P_iB_iF_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P_iB_iF_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H}^T\mathcal{H} \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]
and, thus, it can be concluded that (31) can be ensured by
\[
\Phi_i + \frac{1}{\varepsilon} \begin{bmatrix}
P_i B_i F_0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}^T + \varepsilon \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & I \\
0 & I \\
\end{bmatrix} < 0. 
\] (32)

By Schur’s complement, (32) can be written as (27). This completes the proof of the theorem.

3.2. ADT switching. In this subsection, we consider the ADT switching. First, a lemma of the ADT property used to restrict the switching signal is given in the following.

Lemma 3.3. Given scalars $\beta > 0$, $\nu > 1$, system (8) is mean-square exponentially stable, if, for any switching signal $r(t)$ with the following ADT constraint
\[
T_a > T_a^* = \frac{\ln(\nu)}{\beta},
\] (33)
there exist Lyapunov functions $V_i(t, \zeta(t, x))$, and functions $\kappa_1$ and $\kappa_2$ of class $\mathcal{K}_\infty$ such that for any $i, j \in \mathcal{M}$, $i \neq j$,
\[
\kappa_1 \left( \mathcal{E} \left\{ \int_{\Omega} |\zeta(t, x)|^2 dx \right\} \right) \leq \mathcal{E} \left\{ V_i(t, \zeta(t, x)) \right\} \leq \kappa_2 \left( \mathcal{E} \left\{ \int_{\Omega} |\zeta(t, x)|^2 dx \right\} \right),
\] (34)
\[
\mathcal{E} \left\{ V_i(t, \zeta(t, x)) - \nu V_j(t, \zeta(t, x)) \right\} \leq 0,
\] (35)
\[
\mathcal{E} \left\{ \mathcal{L} V_i(t, \zeta(t, x)) + \beta V_i(t, \zeta(t, x)) \right\} \leq 0,
\] (36)
where $\mathcal{L}$ is the infinitesimal generator and $||\zeta(t, x)|| = \sqrt{\sup_{t - \mu \leq \theta \leq t} |\zeta(\theta, x)|^2}$.

Proof. Suppose that the switching points are $t_0(t_0 = 0)$, $t_1$, $\cdots$, $t_k$. Then, from the generalized Itô formula one obtains
\[
\mathcal{E} \left\{ V_{r(t_k)}(t, \zeta(t, x)) \right\} = e^{-\beta(t-t_k)} \mathcal{E} \left\{ V_{r(t_k)}(t_0, \zeta(t_0, x)) \right\} + \int_{t_k}^{t} e^{-\beta(t-s)} \left[ \beta \mathcal{E} \left\{ V_{r(t_k)}(s, \zeta(s, x)) \right\} \right] ds
\]
\[
\quad + \int_{t_k}^{t} e^{-\beta(t-s)} \mathcal{E} \left\{ \mathcal{L} V_{r(t_k)}(s, \zeta(s, x)) \right\} ds, t \in (t_k, t_{k+1}], k \in \mathbb{Z}_+.
\] (37)

In view of (36), it follows from (37) that
\[
\mathcal{E} \left\{ V_{r(t_k)}(t, \zeta(t, x)) \right\} \leq e^{-\beta(t-t_k)} \mathcal{E} \left\{ V_{r(t_k)}(t_k, \zeta(t_k, x)) \right\}.
\] (38)

Using (35) and (38), for any $t \in (t_k, t_{k+1}]$, $k \in \mathbb{Z}_+$, one has
\[
\mathcal{E} \left\{ V_{r(t_k)}(t, \zeta(t, x)) \right\} \leq \nu V_{r(t_k)}(t_0, \zeta(t_0, x)) e^{-\beta(t-t_0)} \mathcal{E} \left\{ V_{r(t_k)}(t_0, \zeta(t_0, x)) \right\}.
\] (39)

In fact, obviously (39) is satisfied for $t \in (t_0, t_1]$. Suppose that it is true for $t \in (t_k, t_{k+1}]$. Then, for any $t \in (t_i, t_{i+1}]$, from (35), (38), and (39), one can write that
\[
\mathcal{E} \left\{ V_{r(t_{i+1})}(t, \zeta(t, x)) \right\} \leq e^{-\beta(t-t_{i+1})} \mathcal{E} \left\{ V_{r(t_{i+1})}(t_{i+1}, \zeta(t_{i+1}, x)) \right\} \leq \nu e^{-\beta(t-t_{i+1})} \mathcal{E} \left\{ V_{r(t_i)}(t_{i+1}, \zeta(t_{i+1}, x)) \right\}
\]
\[
\begin{align*}
& \leq \nu e^{-\beta(t-t_{i+1})} \left( \nu^N e^{-\beta(t_{i+1}-t_0)} \mathcal{E} \left\{ V_{r(t_0)}(t_0, \zeta(t_0, x)) \right\} \right)^{\rho_i - \lambda_i} \\
& = \nu^N e^{-\beta(t-t_0)} \mathcal{E} \left\{ V_{r(t_0)}(t_0, \zeta(t_0, x)) \right\},
\end{align*}
\]

which implies that (39) is also true for \( t \in (t_{i+1}, t_{i+2}] \).

Now, from (12) and (39) we obtain
\[
\mathcal{E} \left\{ V_{r(t_0)}(t, \zeta(t, x)) \right\} \leq \nu^N e^{-\beta(t-t_0)} \mathcal{E} \left\{ V_{r(t_0)}(t_0, \zeta(t_0, x)) \right\},
\]

where \( \beta_1 = \beta - \ln(\nu)/T_a \). Since (33) implies \( \beta_1 > 0 \), it follows by (34) and (39) that
\[
\mathcal{E} \left\{ \int_{\Omega} \| \zeta(t, x) \|^2 dx \right\} \leq \kappa_i^{-1} \left( \nu^N e^{-\beta_1(t-t_0)} \mathcal{E} \left\{ \int_{\Omega} \| \zeta(t_0, x) \|^2 dx \right\} \right),
\]

which means that system (7) is exponentially stable in mean square. □

By Lemma 3.3, one can get the following theorem.

**Theorem 3.4.** For given scalars \( \beta > 0 \) and \( \nu > 1 \), system (8) is exponentially stable in mean square for any switching signal \( r(t) \) with \( T_a \) satisfying (33), if there are scalars \( \rho_i > 0 \), diagonal matrices \( P_i > 0 \), \( H_{1i} = \text{diag}(h_{1i,1}, \ldots, h_{1i,n}) > 0 \), \( H_{2i} = \text{diag}(h_{2i,2}, \ldots, h_{2i,2}) > 0 \), and matrices \( Q_{1i} > 0, Q_{2i} > 0 \) such that
\[
P_i < \rho_i I, \quad P_i < \nu P_{ij}, Q_{1i} < \nu Q_{1j}, Q_{2i} < \nu Q_{2j}, \quad (40, 41)
\]

hold for any \( i \in \mathcal{M} \), where
\[
\begin{align*}
\Pi_{11} &= -2P_i A_i - 2P_i A_i^K + \beta P_i + Q_{1i} + \rho_i t_i I + \Gamma H_{1i}, \\
\Pi_{22} &= \rho_i t_i I - e^{-\beta \mu} Q_{1i} + \Gamma H_{2i}, \\
\Pi_{33} &= Q_{2i} - H_{1i}, \\
\Pi_{44} &= -e^{-\beta \mu} Q_{2i} - H_{2i}, \\
D_z &= \text{diag} \left\{ \sum_{k=1}^{l} \left( \frac{\pi}{\psi_k - \phi_k} \right)^2 D_{1k}, \cdots, \sum_{k=1}^{l} \left( \frac{\pi}{\psi_k - \phi_k} \right)^2 D_{nk} \right\}.
\end{align*}
\]

**Proof.** Let us construct an Lyapunov-Krasovskii functional as follows:
\[
V_i(t, \zeta(t, x)) = \int_{\Omega} \left[ \zeta^T(t, x) P_i \zeta(t, x) + \int_{-\mu}^{0} e^{\beta \alpha} \left( \zeta^T(t + \alpha, x) Q_{1i} \zeta(t + \alpha, x) + g^T(\zeta(t + \alpha, x)) Q_{2i} g(\zeta(t + \alpha, x)) \right) d\alpha \right] dx.
\]

It can be written that
\[
\min_{1 \leq i \leq n} \lambda_i (P_i) \int_{\Omega} |\zeta(t, x)|^2 dx \leq V_i(t, \zeta(t, x))
\]
\[
\begin{align*}
&= \int_\Omega \left[ \zeta^T(t, x) P_i \zeta(t, x) + \int_{-\mu}^0 e^{\beta \alpha} (\zeta^T(t + \alpha, x) Q_{1i} \zeta(t + \alpha, x) \\
&\quad + g^T(\zeta(t + \alpha, x)) Q_{2i} g(\zeta(t + \alpha, x)) \right] da \right] dx \\
&\leq \int_\Omega \left[ \zeta^T(t, x) P_i \zeta(t, x) + \int_{t-\mu}^t \zeta^T(\alpha, x) (Q_{1i} + \Gamma Q_{2i} \Gamma) \zeta(\alpha, x) da \right] dx \\
&\leq \sup_{1 \leq i \leq n} \lambda_m (P_i + \mu [Q_{1i} + \Gamma Q_{2i} \Gamma]) \int_\Omega |\zeta(t, x)|^2 dx.
\end{align*}
\]

Thus (34) holds true with \( \kappa_2 = \sup_{1 \leq i \leq n} \lambda_m (P_i + \mu [Q_{1i} + \Gamma Q_{2i} \Gamma]) \) and \( \kappa_1 = \min_{1 \leq i \leq n} \lambda_m (P_i) \). Noting that (35) can be ensured by (41), one only needs to check the validity of (36). Using (20) and (22), it is not difficult to get that

\[
\begin{align*}
&\mathbf{V}_i(t, \zeta(t, x)) \\
&= 2 \int_\Omega \zeta^T(t, x) P_i \left[ \sum_{k=1}^i \frac{\partial}{\partial x_k} \left( D_k \frac{\partial \zeta(t, x)}{\partial x_k} \right) - A_i^K \zeta(t, x) \\
&\quad + W_{0i} g(\zeta(t, x)) + W_{1i} g(\zeta(t - \mu, x)) \right] dx \\
&\quad + \int_0^t \left( \zeta^T(t, x) Q_{1i} \zeta(t, x) - e^{-\beta \mu} \zeta^T(t - \mu, x) Q_{1i} \zeta(t - \mu, x) \\
&\quad - \beta \int_{t-\mu}^t e^{\beta \alpha} (\zeta^T(t + \alpha, x) Q_{1i} \zeta(t + \alpha, x)) da \right) dx \\
&\quad + \int_0^t \left( g^T(\zeta(t, x)) Q_{2i} g(\zeta(t, x)) - e^{-\beta \mu} g^T(\zeta(t - \mu, x)) Q_{2i} g(\zeta(t - \mu, x)) \\
&\quad - \beta \int_{t-\mu}^t e^{\beta \alpha} (g^T(\zeta(t + \alpha, x)) Q_{2i} g(\zeta(t + \alpha, x))) da \right) dx \\
&\quad + \int_0^t \left( \mathbf{t}(t, \zeta(t, x), \zeta(t - \mu, x)) P_i \mathbf{t}(t, \zeta(t, x), \zeta(t - \mu, x)) \right) dx \\
&\leq \int_\Omega \left[ \zeta^T(t, x) \left( -2 P_i D_i - 2 P_i A_i^K + Q_{1i} + \rho_{i1} I \right) \zeta(t, x) \\
&\quad + \zeta^T(t - \mu, x) (\rho_{i1} I - e^{-\beta \mu} Q_{1i}) \zeta(t - \mu, x) \\
&\quad + 2 \zeta^T(t, x) P_i W_{0i} g(\zeta(t, x)) + 2 \zeta^T(t, x) P_i W_{1i} g(\zeta(t - \mu, x)) \\
&\quad + g^T(\zeta(t, x)) Q_{2i} g(\zeta(t, x)) - e^{-\beta \mu} g^T(\zeta(t - \mu, x)) Q_{2i} g(\zeta(t - \mu, x)) \\
&\quad - \beta \int_{t-\mu}^t e^{\beta \alpha} (\zeta^T(t + \alpha, x) Q_{1i} \zeta(t + \alpha, x)) da \right] dx.
\end{align*}
\]
From (23), (24), and (43), one can write

\[-\beta \int^0_{-\mu} e^{\beta \alpha} \left( g^T (\zeta(t + \alpha, x)) Q_{2i} g (\zeta(t + \alpha, x)) \right) d\alpha \right) dx. \quad (43)\]

It follows that

\[\mathcal{E} \{ \mathcal{L} V_i(t, \zeta(t, x)) + \beta V_i(t, \zeta(t, x)) \} \]

\[\leq \mathcal{E} \left\{ \int^0_{-\mu} \left[ \zeta^T(t, x) \left( -2P_iD + 2P_i + Q_{1i} + \rho_i I + \Gamma H_1 \Gamma \right) \zeta(t, x) \right. \right. \]

\[+ \zeta^T(t - \mu, x) (\rho_i I + \Gamma H_2 \Gamma - e^{-\beta \mu} Q_{1i}) \zeta(t - \mu, x) \]

\[+ 2\zeta^T(t, x) P_i W_0 g (\zeta(t, x)) + 2\zeta^T(t, x) P_i W_1 g (\zeta(t - \mu, x)) \]

\[+ g^T (\zeta(t, x)) (Q_{2i} - H_1) g (\zeta(t, x)) \]

\[+ g^T (\zeta(t - \mu, x)) (-e^{-\beta \mu} Q_{2i} - H_2) g (\zeta(t - \mu, x)) \]

\[- \beta \int^0_{-\mu} e^{\beta \alpha} \left( g^T (\zeta(t + \alpha, x)) Q_{2i} g (\zeta(t + \alpha, x)) \right) d\alpha \right] dx. \]

which, in conjunction with (42), guarantees (36).

The following theorem provides a design approach for the output-feedback controller. The proof is similar to Theorem 3.2 and thus omitted herein.

**Theorem 3.5.** For given scalars \( \beta > 0, \zeta > 0, \nu > 1 \), system (1) and system (2) are stochastically exponentially anti-synchronized for any switching signal \( r(t) \) with \( T_a \) satisfying (33), if there are scalars \( \rho_i > 0 \) and \( \varepsilon > 0 \), diagonal matrices \( P_i > 0 \), \( H_{1i} = \text{diag}(h_{1i1}, \cdots, h_{1i1}) > 0 \), \( H_{2i} = \text{diag}(h_{1i2}, \cdots, h_{2i2}) > 0 \), matrices
\[ Q_{1i} > 0, \ Q_{2i} > 0, \ \text{and} \ X, \ Y, \ \text{such that} \ (40), \ (41), \ \text{and the following} \]
\[
\begin{bmatrix}
\bar{\Pi}_{11} & 0 & P_1W_{0i} & P_1W_{1i} & P_1B_iF_0 - B_iX + \varsigma C_i^TY^T & P_1B_iF_0 \\
* & \bar{\Pi}_{22} & 0 & 0 & 0 & 0 \\
* & * & \bar{\Pi}_{33} & 0 & 0 & 0 \\
* & * & * & \bar{\Pi}_{44} & 0 & 0 \\
* & * & * & * & -He(\varsigma X) + \varepsilon H^T\mathcal{H} & 0 \\
* & * & * & * & * & -\varepsilon I
\end{bmatrix} < 0, \quad (44)
\]

hold for any \( i \in \mathcal{M} \), where
\[
\bar{\Pi}_{11} = -2P_1D_\pi - 2P_2A_i + \beta P_i + Q_{1i} + \rho t_i I + He(B_iYC_i) + \Gamma H_{1i}\Gamma, \\
\bar{\Pi}_{22} = \rho t_2 I - e^{-\beta\mu}Q_{2i} + \Gamma H_{2i}\Gamma, \ \\
\bar{\Pi}_{33} = Q_{2i} - H_{1i}, \\
\bar{\Pi}_{44} = -e^{-\beta\mu}Q_{2i} - H_{2i}, \\
D_\pi = \text{diag}\left\{ \sum_{k=1}^l \left( \frac{\pi - \phi_k}{\phi_k - \phi_k} \right)^2 D_{1k}, \ldots, \sum_{k=1}^l \left( \frac{\pi - \phi_k}{\phi_k - \phi_k} \right)^2 D_{nk} \right\}.
\]

In that case, the desired gain of (6) is capable to be fixed by (28).

4. **Numerical example.** In this section, we give an example to illustrate the applicability of our design strategies.

![Figure 1. ADT switching signal](image)

Considering switched chaotic TDRDNNs in (1) and (2) with two modes. The parameters are as follows:
\[
W_{01} = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 4.5 \end{bmatrix}, \quad W_{11} = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -3 \end{bmatrix},
\]
\[
W_{02} = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix},
\]
\[
A_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0.4 & 0.6 \\ -1 & -1 \end{bmatrix}.
\]
\[
D_1 = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}
, \quad \hat{F} = \begin{bmatrix}
0.8 & 0 \\
0 & 0.8
\end{bmatrix}
, \quad \tilde{F} = \begin{bmatrix}
0.2 & 0 \\
0 & 0.2
\end{bmatrix},
\]
\[
\nu(t) = \text{diag} \{ z_1(t, x) + z_2(t, x), z_1(t - \mu, x) + z_2(t - \mu, x) \},
\]
\[
\mu = 1, \Omega = \{ x | -2 \leq x \leq 2 \}, i = 1, 2.
\]
The activation functions are given by \( f_i(z_{ij}) = \tanh(z_{ij}) \) \( (i, j = 1, 2) \) as [31].
Furthermore, the initial conditions are given as
\[
\begin{align*}
    z_1(s, x) & = \begin{bmatrix}
        1.5 \\
        -2.0
    \end{bmatrix}, \quad z_2(s, x) = \begin{bmatrix}
        -2.5 \\
        3.0
    \end{bmatrix}, (s, x) \in [-1, 0] \times \Omega.
\end{align*}
\]

\textbf{Figure 2.} Phase plane plot of the drive system at \( x = 0.5 \)

\textbf{Figure 3.} State evolution of the unforced drive-response systems at \( x = 0.5 \)
Set $\nu = 1.05$, $\iota_1 = 1$. Then, it is found in the case of arbitrary switching, the LMIs in Theorem 3.2 possess solutions for $\iota_2 \in [0, 1.497]$. And, in the case of ADT switching, the LMIs in Theorem 3.5 possess solutions for $\iota_2 \in [0, 1.273]$. Especially, let $\iota_2 = 1.273$. Then, solving the LMIs in (40), (41), and (44) one get

$$X = \begin{bmatrix} 5470.6 & 4707.2 \\ 0 & 5470.6 \end{bmatrix}, \quad Y = \begin{bmatrix} 227200 & 1800 \\ 1172300 & 547100 \end{bmatrix}.$$ 

Thus, according to Theorem 3.5, the anti-synchronization of systems (1) and (2) can be ensured by the designed fault-tolerant output feedback controller with gain $K = X^{-1}Y$.
The switching signal is shown in Figure 1. When there is no control input, the state trajectories of the phase-plane plot of the drive system, the switched drive-response TDRDNNs, and the anti-synchronization error system at space point $x = 0.5$ are depicted in Figure 2, 3, and 4, respectively. Clearly, the drive system has a double-scroll-like attractor. With the aid of the designed controller, the state trajectories of the drive-response TDRDNNs and the anti-synchronization error system at space point $x = 0.5$ are depicted in Figure 5 and 6, respectively. From these figures, it can be found the present design strategy in Theorem 3.5 is effective, which can make the switched drive-response TDRDNNs anti-synchronized.

5. Conclusions. In this paper, the fault-tolerant anti-synchronization control for a class of chaotic switched TDRDNNs has been investigated under a drive-response scheme, where the response system is assumed to be disturbed by stochastic noise. Both arbitrary switching signal and ADT limited switching signal have been taken into account. By the Lyapunov-Krasovskii functional method and combining with the generalized Itô formula, several criteria on the mean-square exponential stability are derived for the anti-synchronization error system. Based on the analysis result, constructive design strategies on the desired fault-tolerant anti-synchronization controller are proposed by utilizing some decoupling methods. Finally, an example has been utilized to illustrate the effectiveness of our design strategies.

REFERENCES

[1] A. Abdulle, Y. Bai and G. Vilmart, Reduced basis finite element heterogeneous multiscale method for quasilinear elliptic homogenization problems, *Discrete Contin. Dyn. Syst. Ser. S*, 8 (2015), 91–118.
[2] C. K. Ahn, Adaptive $H_{\infty}$ anti-synchronization for time-delayed chaotic neural networks, *Prog. Theoretical Phys.*, 122 (2009), 1391–1403.
[3] J. Cao, R. Rakkiyappan, K. Maheswari and A. Chandrasekar, Exponential $H_{\infty}$ filtering analysis for discrete-time switched neural networks with random delays using sojourn probabilities, *Sci. China Technol. Sci.*, 59 (2016), 387–402.
[4] X. Chang, R. Liu and J. H. Park, A further study on output feedback $H_{\infty}$ control for discrete-time systems, *IEEE Trans. Circuits Systems II: Express Briefs*, 67 (2020), 305–309.
[5] N. D. Cong and T. S. Doan, On integral separation of bounded linear random differential equations, *Discrete Contin. Dyn. Syst. Ser. S*, 9 (2016), 995–1007.

[6] Y. Fan, X. Huang, Y. Li, J. Xia and G. Chen, Aperiodically intermittent control for quasi-synchronization of delayed memristive neural networks: An interval matrix and matrix measure combined method, *IEEE Trans. Systems Man Cybernetics: Systems*, 49 (2019), 2254–2265.

[7] Y. Fan, X. Huang, H. Shen and J. Cao, Switching event-triggered control for global stabilization of delayed memristive neural networks: An exponential attenuation scheme, *Neural Networks*, 117 (2019), 216–224.

[8] J. Fell and N. Axmacher, The role of phase synchronization in memory processes, *Nature Rev. Neurosci.*, 12 (2011), 105–118.

[9] J. P. Hespanha and A. S. Morse, Stability of switched systems with average dwell-time, *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, AZ, 1999, 2655–2660.

[10] J. Hou, Y. Huang and S. Ren, Anti-synchronization analysis and pinning control of multi-weighted coupled neural networks with and without reaction-diffusion terms, *Neurocomputing*, 330 (2019), 78–93.

[11] Y.-L. Huang, S.-Y. Ren, J. Wu and B.-B. Xu, Passivity and synchronization of switched coupled reaction-diffusion neural networks with non-delayed and delayed couplings, *Int. J. Comput. Math.*, 96 (2019), 1702–1722.

[12] T. Jiao, J. H. Park, G. Zong, Y. Zhao and Q. Du, On stability analysis of random impulsive and switching neural networks, *Neurocomputing*, 350 (2019), 146–154.

[13] R. Konnur, Synchronization-based approach for estimating all model parameters of chaotic systems, *Phys. Rev. E*, 67 (2003), 1387–1396.

[14] T. H. Lee, C. P. Lim, S. Nahavandi and J. H. Park, Network-based synchronization of T-S fuzzy chaotic systems with asynchronous samplings, *J. Franklin Inst.*, 355 (2018), 5736–5758.

[15] X. Li, M. Bohner and C. K. Wang, Impulsive differential equations: Periodic solutions and applications, *Automatica J. IFAC*, 52 (2015), 173–178.

[16] T. L. Liao and N. S. Huang, An observer-based approach for chaotic synchronization with applications to secure communications, *IEEE Trans. Circuits Systems I: Fundamental Theory Appl.*, 46 (1999), 1144–1151.

[17] Y. Liu, J. H. Park and F. Fang, Global exponential stability of delayed neural networks based on a new integral inequality, *IEEE Trans. Systems Man Cybernetics: Systems*, 49 (2019), 2318–2325.

[18] E. N. Lorenz, Deterministic nonperiodic flow, *J. Atmospheric Sci.*, 20 (1963), 130–141.

[19] Q. Ma, S. Xu, Y. Zou and G. Shi, Synchronization of stochastic chaotic neural networks with reaction-diffusion terms, *Nonlinear Dyn.*, 67 (2012), 2183–2196.

[20] X. Mao, Razumikhin-type theorems on exponential stability of stochastic functional differential equations, *Stochastic Process. Appl.*, 65 (1996), 233–250.

[21] S. Nakata, T. Miyata, N. Ojima and K. Yoshikawa, Self-synchronization in coupled salt-water oscillators, *Phys. D*, 115 (1998), 313–320.

[22] N. Ozcan, M. S. Ali, J. Yogambigai, Q. Zhu and S. Arik, Robust synchronization of uncertain Markovian jump complex dynamical networks with time-varying delays and reaction-diffusion terms via sampled-data control, *J. Franklin Inst.*, 355 (2018), 1192–1216.

[23] L. M. Pecora and T. L. Carroll, Synchronization in chaotic systems, *Phys. Rev. Lett.*, 64 (1990), 821–824.

[24] F. Ren and J. Cao, Anti-synchronization of stochastic perturbed delayed chaotic neural networks, *Neural Comput. Appl.*, 18 (2009), 515–521.

[25] I. Stamova, T. Stamov and X. Li, Global exponential stability of a class of impulsive cellular neural networks with suprema, *Internat. J. Adapt. Control Signal Process.*, 28 (2014), 1227–1239.

[26] V. Sundarapandian and R. Karthikeyan, Anti-synchronization of Lü and Pan chaotic systems by adaptive nonlinear control, *European J. Sci. Res.*, 64 (2011), 94–106.

[27] W. Tai, Q. Teng, Y. Zhou, J. Zhou and Z. Wang, Chaos synchronization of stochastic reaction-diffusion time-delay neural networks via non-fragile output-feedback control, *Appl. Math. Comput.*, 354 (2019), 115–127.

[28] Z. Wang, L. Li, Y. Li and Z. Cheng, Stability and Hopf bifurcation of a three-neuron network with multiple discrete and distributed delays, *Neural Process. Lett.*, 48 (2018), 1481–1502.
[29] I. Wedekind and U. Parlitz, Synchronization and antisynchronization of chaotic power dropouts and jump-ups of coupled semiconductor lasers, *Phys. Rev. E*, 66 (2002).

[30] J. Xia, G. Chen and W. Sun, Extended dissipative analysis of generalized Markovian switching neural networks with two delay components, *Neurocomputing*, 260 (2017), 275–283.

[31] Z. Yan, X. Huang and J. Cao, Variable-sampling-period dependent global stabilization of delayed memristive neural networks via refined switching event-triggered control, *SCIENCE CHINA Information Sciences*, in progress.

[32] D. Ye and G. Yang, Adaptive fault-tolerant tracking control against actuator faults with application to flight control, *IEEE Trans. Control Systems Tech*, 14 (2006), 1088–1096.

[33] E. Yucel, M. S. Ali, N. Gunasekaran and S. Arik, Sampled-data filtering of Takagi–Sugeno fuzzy neural networks with interval time-varying delays, *Fuzzy Sets and Systems*, 316 (2017), 69–81.

[34] X. Zhang, X. Lv and X. Li, Sampled-data-based lag synchronization of chaotic delayed neural networks with impulsive control, *Nonlinear Dynam.*, 90 (2017), 1901–1912.

[35] W. Zhang, S. Yang, C. Li and Z. Li, Finite-time and fixed-time synchronization of complex networks with discontinuous nodes via quantized control, *Neural Process. Lett.*, 50 (2019), 2073–2086.

[36] D. Zhang, L. Yu, Q. G. Wang and C. J. Ong, Estimator design for discrete-time switched neural networks with asynchronous switching and time-varying delay, *IEEE Trans. Neural Networks Learning Systems*, 23 (2012), 827–834.

[37] J. Zhou, Y. Wang, X. Zheng, Z. Wang and H. Shen, Weighted $H_\infty$ consensus design for stochastic multi-agent systems subject to external disturbances and ADT switching topologies, *Nonlinear Dyn.*, 96 (2019), 853–868.

[38] Y. Zhou, J. Xia, H. Shen, J. Zhou and Z. Wang, Extended dissipative learning of time-delay recurrent neural networks, *J. Franklin Inst.*, 356 (2019), 8745–8769.

[39] J. Zhou, S. Xu, H. Shen and B. Zhang, Passivity analysis for uncertain BAM neural networks with time delays and reaction-diffusions, *Internat. J. Systems Sci.*, 44 (2013), 1494–1503.

[40] K. Zhou and P. P. Khargonekar, Robust stabilization of linear systems with norm-bounded time-varying uncertainty, *Systems Control Lett.*, 10 (1988), 17–20.

[41] J. Zhou, J. H. Park and Q. Ma, Non-fragile observer-based $H_\infty$ control for stochastic time-delay systems, *Appl. Math. Comput.*, 291 (2016), 69–83.

[42] G. Zhuang, Q. Ma, B. Zhang, S. Xu and J. Xia, Admissibility and stabilization of stochastic singular Markovian jump systems with time delays, *Systems Control Lett.*, 114 (2018), 1–10.

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