Positive definite functions of noncommuting contractions, Hua-Bellman matrices, and a new distance metric

Suvrit Sra
LIDS, Massachusetts Institute of Technology, Cambridge, MA, USA

Abstract

We study positive definite functions on noncommuting strict contractions. In particular, we study functions that induce positive definite Hua-Bellman matrices (i.e., matrices of the form \( \det(I - A_i^*A_j)^{-\alpha}_{ij} \) where \( A_i \) and \( A_j \) are strict contractions and \( \alpha \in \mathbb{C} \)). We start by revisiting a 1959 work of Bellman (R. Bellman Representation theorems and inequalities for Hermitian matrices; Duke Mathematical J., 26(3), 1959) that studies Hua-Bellman matrices and claims a strengthening of Hua’s representation theoretic results on their positive definiteness (L.-K. Hua, Inequalities involving determinants; Acta Mathematica Sinica, 5(1955), pp. 463–470). We uncover a critical error in Bellman’s proof that has surprisingly escaped notice to date. We “fix” this error and provide conditions under which \( \det(I - A^*B)^{-\alpha} \) is a positive definite function; our conditions correct Bellman’s claim and subsume both Bellman’s and Hua’s prior results. Subsequently, we build on our result and introduce a new hyperbolic-like geometry on noncommuting contractions, and remark on its potential applications.

1 Introduction

We study an important positive definite function on strictly contractive complex matrices, namely, \( \det(I - Z)^{-\alpha} \), where \( \|Z\| < 1 \) and \( \alpha \in \mathbb{R} \). This function arises in a variety of contexts, including multivariable complex analysis (Hua, 1955), non-Euclidean geometry (Sra, 2016a), mathematical optimization (Sra and Hosseini, 2015), combinatorics (Brändén, 2012; Vere-Jones, 1988), probability theory (Shirai, 2007), and matrix analysis (Zhang, 2009; Xu et al., 2009; Ando, 2008), among others.

More specifically, we study conditions under which \( (A,B) \mapsto \det(I - A^*B)^{-\alpha} \) is a kernel function, i.e., a function that admits an inner-product representation \( \langle \Phi_\alpha(A), \Phi_\alpha(B) \rangle \). The study of such functions has a long history in mathematics, dating back at least to Szegő (1933), who studied the closely related question about positivity of Taylor series coefficients for \( P^{-\alpha} \) for the polynomial \( P(x) = \sum_{i=1}^{n} \prod_{j \neq i}(1 - x_j) \). This question for other polynomials such as the determinantal polynomial \( \tilde{P}(x) = \det(\sum_i x_i A_i) \) was revisited and closely studied by Scott and Sokal (2014), who in particular approached it via complete monotonicity; their results form the basis of Brändén (2012)’s results, which we will also build upon later in the paper.

There is an additional, different motivation behind our study. The starting point is an elegant block-matrix identity discovered by Loo-Keng Hua (Hua, 1955), which, for strict contractions \( A \) and \( B \) states that

\[
I - B^*B + (A - B)^*(I - A^*A)^{-1}(A - B) = (I - A^*B)(I - AA^*)^{-1}(I - B^*A).
\]
Hua’s identity immediately implies the positive definiteness of the block matrix
\[
\begin{bmatrix}
(I - A^*A)^{-1} & (I - A^*B)^{-1} \\
(I - B^*A)^{-1} & (I - B^*B)^{-1}
\end{bmatrix},
\]
a property that is otherwise not necessarily obvious. Subsequent to Hua’s discovery, various other works, e.g., (Marcus, 1958), (Bellman, 1959), (Ando, 1980), (Xu et al., 2009, 2011), and (Zhang, 2009), among others, continued the study of Hua-like block matrices and operator inequalities induced by them. Beyond Hua’s work (Hua, 1955), a common denominator of research on this topic has been Bellman (1959)’s work that studies Hua matrices through the lens of positive definite functions.

In particular, Bellman (1959) claims that for contractive real matrices \( A_1, \ldots, A_m \) of size \( n \times n \), the Hua-Bellman matrix
\[
H_\alpha := [\det(I - A_i^T A_j)^{-\alpha}]_{i,j=1}^m,
\]
is positive definite for \( \alpha \) being a half-integer \( \alpha = j/2 \) and for real \( \alpha > \frac{1}{2}(n-1) \). This positive definiteness is tantamount to \( \det(I - A^*B)^{-\alpha} \) being a positive definite kernel function for the noted choice of \( \alpha \). Bellman claims to offer a significant simplification and generalization to Hua (1955)’s work that had already shown \( H_\alpha \succeq 0 \) for \( n \times n \) complex contractions for \( \alpha > n - 1 \) (Hua’s result is based on representation theoretic ideas combined with multivariable complex analysis).

While Bellman’s investigation via integral representations of \( \det(A)^{-\alpha} \) is foundational, unfortunately, his proof contains an error that also invalidates subsequent works that build on his claims. We uncover this old error below, and study the function \( \det(I - A^*B)^{-\alpha} \) afresh, ultimately leading to Theorem 2 that presents the most general conditions (known to us) ensuring its positive definiteness. Subsequently, in Theorem 7 we present a closely related hyperbolic-like geometry on noncommuting strict contractions that is suggested by Hua-Bellman matrices.

1.1 Uncovering an old error

Claim 1 (Theorem 3 in Bellman (1959)). Let \( A_1, \ldots, A_m \) be strictly contractive real matrices. Then, the matrix
\[
H_{k/2} = \left[ \frac{1}{\det(I - A_i^T A_j)^{k/2}} \right]_{i,j=1}^m,
\]
is positive semidefinite for all integers \( k \geq 1 \).

To prove this claim Bellman begins with the Gaussian integral representation \( \frac{1}{\det(A)^{n/2}} \alpha \int_{\mathbb{R}^n} e^{-x^T Ax} dx \) for a real symmetric positive definite matrix \( A \). Then, on (Bellman, 1959, pg. 488, (7.6)) Bellman recalls an inequality of Ostrowski and Taussky (1951):
\[
\det(I - A_i^T A_j) \geq \det(I - (A_i^T A_j)\frac{s}{2}),
\]
where \( X_s := \frac{1}{2}(X + X^T) \). Subsequently, he states that (sic. Theorem) Claim 1 “\( \ldots \) will be demonstrated if we prove that \( [\det(I - (A_i^T A_j)\frac{s}{2})]^{-k/2} \) is positive semidefinite.”

This location is where the error lies: just because inequality (1.3) holds, from positive definiteness of \( [\det(I - (A_i^T A_j)\frac{s}{2})]^{-k/2} \) we cannot conclude that \( H_{k/2} \) is also positive definite, since the former matrix only dominates \( H_{k/2} \) entrywise, not in terms of its eigenvalues. The following counterexample makes this explicit.
Counterexample

\[ (A_j)_{j=1}^6 = \left( \begin{bmatrix} -2 & -9 \\ -5 & -10 \end{bmatrix}, \begin{bmatrix} 9 & -5 \\ 6 & 3 \end{bmatrix}, \begin{bmatrix} -10 & -3 \\ -6 & -10 \end{bmatrix}, \begin{bmatrix} -8 & -8 \\ 1 & -10 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 10 & -6 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 10 & -6 \end{bmatrix} \right) \]

\[ A_j \leftarrow \frac{1}{2 \|A_j\|} \text{ for } 1 \leq j \leq 6, \]

\[ \lambda_{\text{min}}((\det(I - A_i^T A_j)^{-1/2})) \approx -1.2066 \times 10^{-3}. \]

Remark: We found that the larger the number \( m \) of matrices, the easier it is to generate a counterexample; however, we could not yet generate a counterexample for \( m = 5 \).

Given this counterexample, the immediate question is whether Bellman’s motivation to generalize Hua’s claim \((\alpha > n - 1 \text{ ensures that } H_{\alpha} \succeq 0)\) can be still rescued? We answer this question below and show that it is indeed possible to generalize Hua’s result, and thereby uncover a deep and rich relation of \( \det(I - A^T B)^{-\alpha} \) to combinatorics and the theory of positive definite functions. While we provide sufficient conditions on \( \alpha \), we believe that they might also be necessary.

2 Positive definite functions on noncommuting contractions

In this section we present results characterizing the positive definiteness of Hua-Bellman matrices. To that end, we study the corresponding function \( \det(I - A_i^T B)^{-\alpha} \) on \( A, B \in \mathcal{C}_n \), the class of \( n \times n \) strictly contractive (in operator norm) complex matrices. We provide sufficient conditions on \( \alpha \) to ensure its positive-definiteness that are defined using the following two sets of possible exponents:

\[ D_{\mathbb{R}} := \{ -(m + 1) \mid m \in \mathbb{N} \} \cup \{ \frac{m+1}{2} \mid m \in \mathbb{N} \} \cup \{0\}, \]

\[ D_{\mathbb{C}} := \{ \pm(m + 1) \mid m \in \mathbb{N} \} \cup \{0\}. \]

Our definitions of \( D_{\mathbb{R}} \) and \( D_{\mathbb{C}} \) follow (Brändén, 2012), though after inverting the elements to align with our presentation better. The first main result of this section is:

**Theorem 2.** Let \( A, B \in \mathcal{C}_n \). Then, \( \det(I - A^* B)^{-\alpha} \) is a positive definite function for \( \alpha \in D_{\mathbb{C}} \cup \{ x \in \mathbb{R} \mid x > n - 1 \} \). If \( A \) and \( B \) are in addition real, then \( \det(I - A^T B)^{-\alpha} \) is positive definite for \( \alpha \in D_{\mathbb{R}} \cup \{ x \in \mathbb{R} \mid x > n - 1 \} \).

The key ingredient in our proof of Theorem 2 is the \( \alpha \)-permanent, which we now recall.

2.1 \( \alpha \)-permanents

The \( \alpha \)-permanent generalizes the matrix permanent and determinant, and was introduced by Vere-Jones (1988). Let \( \alpha \in \mathbb{C} \) and \( A = (a_{ij}) \) be an \( n \times n \) matrix. Then, the \( \alpha \)-permanent is defined as

\[ \text{per}_{\alpha}(A) := \sum_{\sigma \in \mathcal{S}_n} \alpha^{#\sigma} \prod_{i=1}^n a_{i,\sigma(i)}, \tag{2.1} \]

where \( #\sigma \) denotes the number of disjoint cycles in the permutation \( \sigma \). This object interpolates between the determinant and permanent, and enjoys a variety of applications and connections; see e.g., (Shirai, 2007; Crane, 2013; Frenkel, 2009).
The first key property of $\alpha$-permanents that we will need is an inner-product based representation derived in Lemma 3.

**Lemma 3.** Let $A$, $B$ be arbitrary square $n \times n$ matrices and $\alpha \in \mathbb{C}$. Then, $\text{per}_\alpha(A^*B)$ can be written as a linear combination of inner products, i.e., there exist constants $c_1, c_2, \ldots$, such that $\text{per}_\alpha(A^*B) = \sum_i c_i \langle \psi_i(A), \psi_i(B) \rangle$ for some nonlinear maps $\{\psi_i\}_{i \geq 1}$.

**Proof.** Our proof relies on the (known) observation that the $\alpha$-permanent can be written in terms of immanants. To see how, let $\lambda \vdash n$ denote that $\lambda$ is a partition of the integer $n$. The *immanant* of $A$ indexed by $\lambda$ is defined as (see (Merris, 1997) for details)

$$d_\lambda(A) := \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^n a_{\sigma(i)i},$$

(2.2)

where $\chi_\lambda(\sigma)$ is the character associated to the irreducible representation $S^\lambda$, the Specht module corresponding to $\lambda \vdash n$—see (Fulton and Harris, 2013) for details. Then, Crane (2013, Eq. (12)) shows that for each $\lambda$, there exist constants $c_\lambda^\alpha$ such that:

$$\text{per}_\alpha(A) = \sum_{\lambda \vdash n} c_\lambda^\alpha d_\lambda(A).$$

(2.3)

Further, we know from multilinear matrix theory (Merris, 1997) that for each $\lambda \vdash n$, there exists a projection $P_\lambda$ such that $d_\lambda(X) = \text{tr} P_\lambda A^{\otimes n} P$. Thus, we may write

$$d_\lambda(A^*B) = \text{tr} P^*(A^*B)^{\otimes n} P = \text{tr} P^*(A^{\otimes d})^* (A^{\otimes d}) P = \langle \psi_\lambda(A), \psi_\lambda(B) \rangle.$$  

(2.4)

Plugging in representation (2.4) into identity (2.3) we obtain the identity

$$\text{per}_\alpha(A^*B) = \sum_{\lambda \vdash n} c_\lambda^\alpha \langle \psi_\lambda(A), \psi_\lambda(B) \rangle,$$

which shows that $\text{per}_\alpha(A^*B)$ is indeed a weighted sum of inner products.  

While Lemma 3 shows that $\text{per}_\alpha(A^*B)$ can be written as a weighted sum of inner products, in general, this sum need not correspond to a usual (positive definite) inner product. In particular, it is not obvious when is $\text{per}_\alpha(A^*A) \geq 0$. This rather nontrivial property was characterized by (Brändén, 2012) by building on the complete monotonicity theory established in (Scott and Sokal, 2014); we recall the relevant result below.

**Theorem 4** (Theorem 2.3 (Brändén, 2012)). Let $\alpha \in \mathbb{R}$. Then, $\text{per}_\alpha(A) \geq 0$ if and only if (i) $\alpha \in D_\mathbb{R}$ for $A$ real, symmetric positive definite; or (ii) $\alpha \in D_\mathbb{C}$ for $A$ Hermitian positive definite.

This theorem sheds light on the choice of $\alpha$ presented in Theorem 2. In addition to Lemma 3 and Theorem 4, our proof of Theorem 2 relies on the following generalization of MacMahon’s Master Theorem.

**Theorem 5** ((Vere-Jones, 1988; Brändén, 2012)). Let $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $A = [a_{ij}]_{i,j=1}^n$. Let $A[m]$ be the $|m| \times |m|$ matrix with $|m| = \sum_{i=1}^n m_i$, obtained by replacing the $(i,j)$-entry of $A$ by the $m_i \times m_j$ matrix $a_{ij} 1_{m_i} 1_{m_j}^T$. Let $X = \text{Diag}(x_1, \ldots, x_n)$ and $\alpha \in \mathbb{C}$. Then,

$$\det(I - XA)^{-\alpha} = \sum_{m \in \mathbb{N}^n} x^m m! \text{per}_\alpha(A[m]),$$

(2.5)

where $x^m = x_1^{m_1} \cdots x_n^{m_n}$ and $m! = m_1! \cdots m_n!$.

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1(Crane, 2013, Theorem 2.4) notes that $c_\lambda^\alpha = \frac{1}{n!} \sum_{\sigma \in S_n} a_n^{\sigma^*} \chi_\lambda(\sigma)$. 

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Theorem 5 writes the Taylor series of \( \det(I - XA)^{-a} \) using \( \alpha \)-permanents, and the conditions on \( \alpha \) stipulated by Theorem 4 ensure that all the coefficients of this series are nonnegative (provided \( A \) is positive semidefinite), and therefore help establish the desired positive definiteness property, as elaborated in the proof below.

### 2.2 Proof of Theorem 2

First, recall from (Hua, 1955) that if \( \alpha > n - 1 \), then \( \det(I - A^*B)^{-\alpha} \) is a positive definite function, a result that holds for complex contractions (and thus a fortiori also for real ones). What remains to prove is the extended range of \( \alpha \) values claimed in Theorem 2. The main idea to establish this extended range is to invoke Theorem 5, and use it to represent \( \det(I - A^*B)^{-\alpha} \) as the inner-product \( \langle \Phi_\alpha(A), \Phi_\alpha(B) \rangle \), where \( \Phi_\alpha \) is a nonlinear map that maps its argument to some Hilbert space. In other words, we build on Theorem 5 to prove that \( \det(I - A^*B)^{-\alpha} \) is a kernel function, and as a result, obtain positive definiteness of associated Hua-Bellman matrices.

The first step is to realize that \( A[m] = Q_m^*(A \otimes 11^T)Q_m \) for matrices \( Q_m \) and \( 11^T \) of appropriate sizes. Let \( E_{ij} \) denote the \( m_i \times m_j \) matrix of all ones, and \( 11^T = E = [E_{ij}]_{i,j=1}^{n} \) the corresponding block matrix.\(^2\) Then, consider the Khatri-Rao product between conformally partitioned matrices \( A \) and \( E \):

\[
A * E = \begin{bmatrix}
    a_{11} \otimes E_{11} & \cdots & a_{1n} \otimes E_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} \otimes E_{n1} & \cdots & a_{nn} \otimes E_{nn}
\end{bmatrix},
\]

which is nothing but \( A[m] \) since \( a_{ij} \otimes E_{ij} = a_{ij}E_{ij} \). From basic properties of Khatri-Rao products (Liu et al., 2008) it then follows that there exists a matrix \( Q_m \) such that \( A[m] = Q_m^*(A \otimes E)Q_m \). To see this identification explicitly, let \( Q_m \) be the diagonal matrix \( \text{Diag}(U_1^m, \ldots, U_n^m) \), where the \( U_i^m \) are suitable subsets of the identity matrix such that \( U_i^m * EU_j^m = E_{ij} \). Then,

\[
Q_m^*(A \otimes E)Q_m = \begin{bmatrix} U_1^m \end{bmatrix} \begin{bmatrix} a_{11}E & \cdots & a_{1n}E \\
    \vdots & \ddots & \vdots \\
    a_{n1}E & \cdots & a_{nn}E \end{bmatrix} \begin{bmatrix} U_1^m \\
    \vdots \\
    U_n^m \end{bmatrix} = (A[m].
\]

Next, using (2.5) we see that \( \det(I - XA^*B)^{-\alpha} = \sum_{m \in \mathbb{N}} \frac{\alpha^m}{m!} \text{per}_\alpha((A^*B)[m]) \). Thus, identity (2.7) allows us to write

\[
(A^*B)[m] = Q_m^*(A^*B \otimes 11^T)Q_m = Q_m^*((A \otimes 1^T)(B \otimes 1^T))Q_m =: \tilde{A}_m^*B_m.
\]

Representation (2.9) combined with Lemma 3 immediately allows us to write

\[
\text{per}_\alpha((A^*B)[m]) = \sum_{\lambda \vdash n} c^\alpha_{\lambda}(\tilde{A}_m, \psi_\lambda(\tilde{B}_m)),
\]

\(^2\)To reduce notational burden, we let the dependency of blocks of \( E \) on \( m \) remain implicit.
which shows that $\text{per}_\alpha((A^*B)[m])$ is a linear combination of inner products. To complete the proof that (2.10) indeed defines a valid inner product, it remains to verify that this inner product is positive definite. Since the coefficients $c^\alpha_i$ can be negative, this property is not obvious from (2.10). In fact, this property is fairly nontrivial, but fortunately, it follows from a result of Brändén (2012). Indeed, since $(A^*A)[m] = \hat{A}_m^*\hat{A}_m$ is positive definite, nonnegativity of $\text{per}_\alpha((A^*A)[m])$ follows from Theorem 4. Thus, (2.10) is a true inner product, and depending on whether $A$ and $B$ are real or complex, the corresponding necessary and sufficient conditions on $\alpha$ are also obtained from Theorem 4.

Combined with Theorem 5, we have thus shown that $\det(I - A^*B)^{-\alpha}$ is a nonnegative sum of inner products. Consequently, we can write $\det(I - A^*B)^{-\alpha} = \langle \Phi_\alpha(A), \Phi_\alpha(B) \rangle$, for a suitable map $\Phi_\alpha$, thus obtaining the desired inner product formulation.

Corollary 6 finally answers the questions posed by Bellman and Hua, regarding conditions ensuring the positive-definiteness of Hua-Bellman matrices.

**Corollary 6.** Let $m \geq 1$ and $A_1, \ldots, A_m \in \mathcal{C}_n$. Then $[\det(I - A_i^*A_j)^{-\alpha}]_{i,j=1}^m$ is Hermitian positive definite if $\alpha \in D_C \cup \{x \in \mathbb{R} \mid x > n - 1\}$. Let $B_1, \ldots, B_m \in \mathcal{C}_n \cap \mathbb{R}^{n \times n}$, then $[\det(I - B_i^T B_j)^{-\alpha}]$ is symmetric positive definite if $\alpha \in D_R \cup \{x \in \mathbb{R} \mid x > n - 1\}$.

### 3 A hyperbolic-like geometry on noncommuting contractions

In this section, we introduce a new (to our knowledge) hyperbolic-like geometry on the space of contractions. The distance function introduced is suggested by Hua-Bellman matrices, whose positive definiteness induces the nonnegativity of the proposed distance. The main result of this section is Theorem 7, which formally introduces the said geometry.

**Theorem 7.** Let $d : \mathcal{C}_n \times \mathcal{C}_n \to \mathbb{R}_+$ be defined as

$$d^2(A, B) := \log \frac{\left| \det(I - A^*B) \right|}{\sqrt{\det(I - A^*A) \det(I - B^*B)}}, \quad A, B \in \mathcal{C}_n.$$  

(3.1)

Then, $(\mathcal{C}_n, d)$ is a metric space.

Before proving Theorem 7, we first recall a closely related distance function.

**Theorem 8** (S-Divergence (Sra, 2016a)). Let $X, Y$ be Hermitian positive definite. Then,

$$\delta^2_S(X, Y) := \log \frac{\det \left( \frac{X + Y}{2} \right)}{\sqrt{\det(X) \det(Y)}},$$

(3.2)

is the square of a distance, i.e., $\delta_S$ is a distance.

We will use Theorem 8 in conjunction with Theorem 9 in our proof of Theorem 7. Theorem 9 is considerably more general than what we need, however, we believe that it may be of independent interest, so we state it in its more general form; it is the second key result of this section.

**Theorem 9.** Let $X, Y$ be arbitrary complex matrices, and $0 \leq p \leq 2$, then

$$\delta^2_p(X, Y) := \log \det(I + |X - Y|^p),$$

(3.3)

is the square of a distance, i.e., $\delta_p$ is a distance (here $|X| := (X^*X)^{1/2}$).
We will need the following simple observation in our proof of Theorem 9:

**Lemma 10.** Let $0 \leq p \leq 2$. The function $f(t) = \sqrt{\log(1+t^p)}$ is concave on $(0, \infty)$.

**Proof.** We prove that $f''(t) \leq 0$. Since $f''(t) = -\frac{pt^{p-2}(pt^p+2(t^p-p+1)\log(t^p+1))}{4(t^p+1)^2\log^2(t^p+1)}$, it suffices to verify that $(pt^p+2(t^p-p+1)\log(t^p+1)) \geq 0$. Writing $t^p = x$, this inequality is equivalent to $px^2 + 2(1 + x)\log(1+x) \geq 2p\log(1+x)$. Since $(1+x)\log(1+x) \geq x$, the l.h.s exceeds $px + 2x$ which combined with the inequality $\log(1+x) \leq x$ yields $px + 2x \geq (p+2)\log(1+x) \geq 2p\log(1+x)$ as desired since $p \leq 2$. \hfill $\Box$

**Proof of Theorem 9.** The key idea is to reduce the question to a setting where we can apply a subadditivity theorem of Uchiyama (2006) for singular values. In particular, for complex matrices $A$, $B$, $C$ of size $n \times n$, such that $C = A + B$, Uchiyama’s theorem states that for any concave function $f : \mathbb{R}_+ \to (0, \infty)$ such that $f(0) = 0$, we have

$$\{f(\sigma_j(C))\}_{j=1}^n \prec_w \{f(\sigma_j(A)) + f(\sigma_j(B))\}_{j=1}^n,$$

(3.4)

where $\prec_w$ denotes the weak-majorization partial order (see e.g., (Bhatia, 1997, Chapter 2)), and $\sigma_j(\cdot)$ denotes the $j$-th singular value (in decreasing order).

Let $A = X - Z$, $B = Z - Y$, and $C = X - Y$; let $a_j$, $b_j$, and $c_j$ be their singular values. Since $f(t) = \sqrt{\log(1+t^p)}$ is concave (see Lemma 10), from (3.4) we thus obtain

$$\{f(c_j)\}_{j} \prec_w \{f(a_j) + f(b_j)\}_{j}.$$

(3.5)

We further know that if $x \prec_w y$ for $x, y \in \mathbb{R}_+$, then $x^s \prec_w y^s$ for $s \geq 1$ (see e.g., (Bhatia, 1997, Example II.3.5)). Consequently, for $s = 2$ from inequality (3.5) we obtain

$$\{f(c_j)^2\}_{j} \prec_w \{f(a_j) + f(b_j)\}_{j}^2.$$

(3.6)

Thus, in particular it follows from the majorization inequality (3.6) that

$$\sum_{j=1}^n \log(1+c_j^p) \leq \sum_j \left(\sqrt{\log(1+a_j^p)} + \sqrt{\log(1+b_j^p)}\right)^2,$$

(3.7)

so that upon taking square roots on both sides we obtain

$$\sqrt{\sum_{j=1}^n \log(1+c_j^p)} \leq \sqrt{\sum_j \left(\sqrt{\log(1+a_j^p)} + \sqrt{\log(1+b_j^p)}\right)^2}.$$

Applying Minkowski’s inequality on the right hand side we get

$$\sqrt{\sum_{j=1}^n \log(1+c_j^p)} \leq \sqrt{\sum_j \log(1+a_j^p)} + \sqrt{\sum_j \log(1+b_j^p)}.$$

(3.8)

But $\sum_j \log(1+c_j^p) = \log \det(I + |C|^p) = \log \det(I + |X - Y|^p) = \delta_p^2(X,Y)$. Thus, inequality (3.8) is nothing but the triangle inequality

$$\delta_p(X,Y) \leq \delta_p(X,Z) + \delta_p(Y,Z),$$

which concludes the proof. \hfill $\Box$
Before presenting the proof of Theorem 7, let us briefly note Corollary 11 that also explains why we call the geometry induced by distance (3.1) to be hyperbolic-like.

**Corollary 11.** After suitable rescaling, identify diagonal matrices $X, Y \in \mathbb{C}_n$ with contractive vectors $x, y \in \mathbb{C}^n$. Then, we have the distance

$$d^2(X, Y) = \log \frac{|1 - x^*y|}{\sqrt{(1 - \|x\|^2)}\sqrt{(1 - \|y\|^2)}},$$

which is similar to the Cayley-Klein-Hilbert distance (Deza and Deza, 2013, pg. 120).

To prove the hardest part of Theorem 7, namely, the triangle inequality, we will proceed by rewriting $d^2(A, B)$ is a more amenable form. To that end, we need Lemma 12.

**Lemma 12** (Möbius). Let $A, B \in \mathbb{C}_n$. There exist matrices $X$ and $Y$ such that

1. $I - A^*B = 2(I + X^*)^{-1}(X^* + Y)(I + Y)^{-1}$, and
2. $\Re(X) > 0$, and $\Re(Y) > 0$.

**Proof.** Part (i): Consider the Möbius transformation $A \mapsto (X - I)(X + I)^{-1}$, which leads to $X = (I - A)^{-1}(I + A)$, an object that is well-defined because $I - A$ is invertible due to $A$ being a strict contraction. Using this transformation on $A$ and $B$ we obtain

$$I - A^*B = I - (I + X^*)^{-1}(X^* - I)(Y - I)(Y + I)^{-1}$$

$$= I - (I + X^*)^{-1}[X^*Y - X^* - Y + I](Y + I)^{-1}$$

$$= (I + X^*)^{-1}[(I + X^*)(Y + I) - X^*Y + X^* + Y - I](Y + I)^{-1}$$

$$= (I + X^*)^{-1}[2X^* + 2Y](Y + I)^{-1}.$$ 

Part (ii): From Part (i), we have $I - A^*A = (I + X^*)^{-1}[2X^* + 2Y](I + X)^{-1}$. Thus,

$$\Re(X) = \frac{1}{2}(I + X^*)(I - A^*A)(I + X),$$

which is clearly strictly positive definite by assumption on $A$ and definition of $X$.

We are now ready to prove Theorem 7.

**Proof of Theorem 7.** For $A, B, C \in \mathbb{C}_n$ we need to show that (i) $d(A, B) \geq 0$; (ii) $d(A, B) = d(B, A)$; and (iii) $d(A, B) \leq d(A, C) + d(B, C)$. From the $2 \times 2$ matrix version of Theorem 2 Hua’s classical inequality $|\det(I - A^*B)| \geq \sqrt{\det(I - A^*A)\det(I - B^*B)}$ follows immediately; here, the inequality holds strictly unless $A = B$. Thus, $d(A, B) \geq 0$ with equality if and only if $A = B$. Symmetry (ii) of $d$ is evident, so it only remains to prove the triangle inequality (iii).

From Lemma 12-(i) it follows that there exist $X, Y$ such that

$$d^2(A, B) = \log \frac{|\det(2X^* + 2Y)|\sqrt{\det(I + X^*)}(I + X)\sqrt{\det(I + Y^*)}(I + Y)}{|\det(I + X^*)||\det(I + Y)|\sqrt{|\det(2X^* + 2X)|}\sqrt{|\det(2Y^* + 2Y)|}} =: \delta^2(X, Y).$$

\[ (3.9) \]
By Lemma 12-(ii) \(X^* + X > 0\), so that we can rewrite (3.9) as

\[
\delta^2(X,Y) = \log \frac{|\det(X^* + Y)|}{\sqrt{\det(X^* + X)\det(Y^* + Y)}}. \tag{3.10}
\]

It remains to verify that \(\delta\) defined by (3.10) is a distance.

Write \(X = \text{Re}(X) + i\text{Im}(X)\) (similarly \(Y\)), so that \(|\det(X^* + Y)| = |\det(\text{Re}(X + Y) + i[\text{Im}(Y) - \text{Im}(X)])|\). The matrix \(\text{Re}(X + Y)\) is positive definite while \([\text{Im}(Y) - \text{Im}(X)]\) is Hermitian. Thus, there exists a matrix \(T\) such that \(T^* \text{Re}(X + Y) T = I\) and \(T^* (\text{Im}(Y) - \text{Im}(X)) T = D\) for some diagonal matrix \(D\). Moreover, since \(T^* \text{Re}(X) T + T^* \text{Re}(Y) T = I\), the matrices \(T^* \text{Re}(X) T\) and \(T^* \text{Re}(Y) T\) commute. As a result, they can be simultaneously diagonalized using a unitary matrix, say \(U\). Thus, we can write \(U^* T^* \text{Re}(X) T U = D_x\) and \(U^* T^* \text{Re}(Y) T U = D_y\) for diagonal matrices \(D_x, D_y\), and also \(U^* T^* \text{Im}(Y) T U = S_y\), and \(U^* T^* \text{Im}(X) T U = S_x\) for Hermitian matrices \(S_x, S_y\), which permits us to rewrite (3.10) as

\[
\delta^2(X,Y) = \log \frac{|\det(D_x + D_y + iU^*DU)|}{\sqrt{\det 2D_x \det 2D_y}}. \tag{3.11}
\]

Since \(D_x + D_y = I\), we can split (3.11) into two parts as follows:

\[
\delta^2(X,Y) = \log \frac{\det(D_x + D_y)}{\sqrt{\det 2D_x \det 2D_y}} + \log |\det(I + iU^*DU)|
\]

\[
= \log \frac{\det(D_x + D_y)}{\sqrt{\det D_x \det D_y}} + \log |\det(I + i(S_y - S_x))|
\]

\[
= \delta_5^2(D_x, D_y) + \frac{1}{2} \log \det(I + (S_y - S_x)^2)
\]

\[
= \delta_5^2(D_x, D_y) + \frac{1}{2} \delta_2^2(S_x, S_y),
\]

from which upon using Theorems 8 and 9, the triangle inequality for \(\delta(X,Y)\) follows. \(\square\)

Remarks: We believe that the distance functions (3.1) and (3.3) may find several applications in a variety of domains. Our belief is based on the diverse body of applications the related S-Divergence (3.2) has found, for instance in computer vision (Cherian et al., 2012), brain-computer interfaces and imaging (Yger et al., 2016), matrix means (Sra, 2016a; Chebbi and Moakher, 2012), geometric optimization (Sra and Hosseini, 2015; Boumal, 2020), numerical linear algebra (Sra, 2016b), signal processing (Bouchard et al., 2018), machine learning (Zern et al., 2018; Tiomoko et al., 2019), quantum information theory (Virosztek, 2021), among many others.

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