Lectures on Super Analysis

— Why necessary and What’s that?

Towards a new approach to a system of PDEs

By Atsushi INOUE
Notice: Commencement of a class

Syllabus

Analysis on superspace
— a construction of non-commutative analysis

3 October 2008 – 30 January 2009, 10.40-12.10, H114B at TITECH, Tokyo, A. Inoue

Roughly speaking, RA(=real analysis) means to study properties of (smooth) functions defined on real space, and CA(=complex analysis) stands for studying properties of (holomorphic) functions defined on spaces with complex structure.

On the other hand, we may extend the differentiable calculus to functions having definition domain in Banach space, for example, S. Lang “Differentiable Manifolds” or J.A. Dieudonné “Treatise on Analysis”. But it is impossible in general to extend differentiable calculus to those defined on infinite dimensional Fréchet space, because the implicit function theorem doesn’t hold on such generally given Fréchet space.

Then, if the ground ring (like $\mathbb{R}$ or $\mathbb{C}$) is replaced by non-commutative one, what type of analysis we may develop under the condition that newly developed analysis should be applied to systems of PDE or RMT(=Random Matrix Theory).

In this lectures, we prepare as a “ground ring”, Fréchet-Grassmann algebra having countably many Grassmann generators and we define so-called superspace over such algebra. On such superspace, we take a space of super-smooth functions as the main objects to study.

This procedure is necessary not only to associate a Hamilton flow for a given $2^d \times 2^d$ system of PDE which supports to resolve Feynman’s murmur, but also to make rigorous Efetov’s result in RMT.

(1) Feynman’s path-integral representation of the solution for Schrödinger equation
(2) Dirac and Weyl equations, Feynman’s murmur
(3) Why is such new algebra necessary? Differential operator representations of $2^d \times 2^d$-matrices
(4) Fréchet-Grassmann algebra $\mathcal{R}$ and superspace $\mathcal{R}^{m|n}$
(5) Elementary linear algebra on superspace, super-determinant and super-trace, etc
(6) Differential calculus on superspace; super smooth functions and implicit function theorem, etc
(7) Integral calculus on superspace; integration by parts, change of variables under integral sign, etc
(8) Fourier transformations on $\mathcal{R}^{m|n}$ and its application
(9) Path-integral representation of a fundamental solution of Weyl equation

Home Page: http://www.math.titech.ac.jp/~inoue/SLDE2-08.html
(Closed after my retirement from TITECH at 31th March 2009)

To audience: Please keep not only intellectual curiosity but also have patience to follow at least 3 lectures.
For what and why, this lecture note is written

I delivered 14 lectures, each 90 minutes, for graduate students at Tokyo Institute of Technology, started in Autumn 2008.

Since there is a special tendency based not only on Japanese culture but rather Asian aesthetic feeling, students hesitate to make a question during class. Reasons about this tendency seems based on modesty and timidity and feeling in his or her noviculate, or more frankly speaking, they are afraid of making a stupid question before friends which may probably exhibit their ignorance, which stems from their sense of guilty that they haven’t been studied sufficiently enough, or bother those appreciation for lectures by stopping by questions. In general, some one’s any questions on a lecture is very constructive that makes clear those which are not easy to understand for audience but also corrects miss understandings of speaker himself.

Though “Instantaneous response towards your uncomfortable feeling” is embodied by very young peoples saying “Why, Mamy?”, but it seems rather difficult to do so in student society. Even though, please make a question without hesitation in any time. Not only primitive stays near radical but also your slight doubt makes slow down the speed of speaker’s explanation which gives some time to other students to help consider and to follow up.

To make easy to pose such questions, I prepare pre-lecture note for a week before my lecture and corrected version of it after lecture with answers to questions if possible and necessary, those are posted on my home-page.

This lecture note is translated from them.

At the delivering time of my lectures, I haven’t yet clarified sufficiently the characterization of super-smooth functions but also the definition of integral on superspace which admits naturally the change of variables under integral sign. Therefore, I change significantly these representations in this notes from those original lectures. I owe much to my colleague Kazuo Masuda whose responses to my algebra related questions help me very much not only to my understanding but also to correct some defects in other’s publications with counter-examples. Concerning the characterization of super-smoothness, we need to prepare Cauchy-Riemann equation for them which is deeply connected with the countably infinite Grassmann generators.

In his naive definition of integral, Berezin’s formula of change of variables under integral sign holds only for integrand function having a compact support. This point is ameliorated when we modify the method of V.S. Vladimirov and I.V. Volovich [130] or A. Rogers [106] which relates to the different recognition of body part of “super space” from Berezin.

Leaving from syllabus, I give some application of super analysis to Random Matrix Theory (RMT) and of SUSYQM=Super SYmmetric Quantum Mechanics with Witten’s index.

I gather some facts which are not explained fully during these lectures in the last chapter named “Miscellaneous”.
(1) I give a precise proof of Berezin’s formula of change of variables under integral sign. I confess Rothstein’s paper is not understandable to me, even a question letter to him without response and an explanation in Rogers’ book.
(2) Function spaces on superspace and Fourier transformation for supersmooth functions are briefly introduced.

(3) As a typical simplest example of application of superanalysis, I give another proof of Qi’s result concerning an example of weakly hyperbolic equations, and

(4) I give also an example for a system version of Egorov’s theorem which begins with Bernardi’s question.

(5) I mentioned in the lecture that the problem posed by I.M. Gelfand at ICM congress address in 1954 concerning functional derivative equations related to QED or turbulence, which is more precisely explained there, and finally,

(6) in his famous paper, E. Witten introduced Supersymmetric Lagrangian, derived from his deformed $d\phi \ast d\phi + d\phi d\phi^\ast$. Here, we derive this as the supersymmetric extension of Riemannian metric $g_{ij}(q)dq^i dq^j$, which is merely my poor interpretation from physicists calculation.

(7) As an example where we need the countably infinite number of Grassmann generators, we consider Weyl equation with electro-magnetic external field. Besides whether it is physically meaningful, we solve the Hamilton equation corresponding to this equation by degree.

Though, references are delivered at each time in lectures, but I gathered them at the last part.

Finally, this note may be unique since I confess several times “their arguments are outside of my comprehension” which are not so proud of but I do so. Because not only I feel ashamed of my unmatredness of differential geometry and algebra’s intuition, but also I hope these confessions encourage to those young mathematicians who try to do something new!
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CHAPTER 1

A motivation of this lecture note

1.1. Necessity of the non-commutative analysis and its benefit

1.1.1. Another basic field? Is it truly necessary to introduce another ground ring in analysis except for $\mathbb{R}$ or $\mathbb{C}$? Why?

In the theory of linear PDEs (=Partial Differential Equations) of scalar type, the main problem is to reduce the non-commutativity inherited from the so-called Heisenberg’s uncertainty principle

\[(1.1.1) \quad \left[\frac{\hbar}{i} \frac{\partial}{\partial q_j}, q_k\right] = \frac{\hbar}{i} \delta_{jk}, \quad \text{i.e.} \quad \frac{\hbar}{i} \frac{\partial}{\partial q_j} (q_k u(q)) - q_k \frac{\hbar}{i} \frac{\partial}{\partial q_j} u(q) = \frac{\hbar}{i} \delta_{jk} u(q)\]

to the one where commutative algebraic calculation is available. This is done using Fourier transformations, that is, the non-commutativity caused by the Heisenberg’s uncertainty principle, is reduced to the commutative one with error terms on the phase space by Fourier transformation, and then this transformed one is analyzed there, and by the inverse Fourier transformation, it is transformed back to the original setting. This procedure is done modulo error terms with suitable estimates. These ideas and various devices are unified as the theory of ΨDO (=Pseudo Differential Operators) and FIO (=Fourier Integral Operators).

Whether this strategy is extendable also to a system of PDEs, is our main concern.

Since there exists another non-commutativity stems from matrices coefficients for a system of PDEs, it seems difficult to treat it as similar as scalar cases. But if we may diagonalize that system nicely then we may apply the standard method to its each component. Even if it is hard to diagonalize straightforwardly, then we impose certain conditions on the characteristic roots associated to that system in order to assure that we may essentially reduce that system to the scalar pseudo differential operators. But if this procedure fails, is there any detour? Especially, if we need a “Hamilton flow” for the given system, how do we associate that classical objects keeping matrix structure as it is?

On the other hand, if the phenomenon is describable only using a system of PDEs, it seems natural to abandon the idea of reducing it to the scalar case. Of course, treating that system of PDEs, we need new idea to overcome the non-commutativity of matrices.

This difficulty is clearly claimed by Feynman where he asks what is the corresponding classical mechanics and action integral for Dirac equation. Moreover, he proposes to use quarternion to resolve this difficulty.

Here, we propose a new idea to overcome the non-commutativity of matrices. This idea is essentially simple when we encounter $2^d \times 2^d$-matrices: Since those matrices are decomposed with elements in Clifford algebras and that algebras has the representation by differential operators on
Grassmann algebras, we extend the ground ring to the one having Grassmann character. Developing analysis on this ground ring, we may apply the standard process which are used in the scalar PDE. This idea is based on F.A. Berezin and M.S. Marinov \cite{10} of “Treat bosons and fermions on equal footing”. Therefore, my answer to Feynman’s proposal is,

“Mr. Feynman, if you use Fréchet-Grassmann algebras with countably infinite Grassmann generators instead of quaternions, then it goes well!”

1.1.2. Feynman’s path-integral representation of the solution for Schrödinger equation. More than seventy years ago, as a graduate student, R. Feynman \cite{42} has a primitive question why Schrödinger equation may be considered as the governing equation of quantum mechanics? In other word, though Bohr’s correspondence principle which is derived after many experiments and thoughts, that principle should be essential in Quantum Mechanics, but it seems difficult to derive it directly from the Schrödinger equation itself.

Mathematically, this question is interpreted as follows: Let $u(t, q): \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{C}$ satisfy the initial value problem for the Schrödinger equation
\begin{equation}
\begin{aligned}
&i\hbar \frac{\partial}{\partial t} u(t, q) = -\frac{\hbar^2}{2} \Delta u(t, q) + V(q) u(t, q), \\
&u(0, q) = u(q).
\end{aligned}
\end{equation}

How does the solution $u(t, q)$ depend on $\hbar$? Especially, can we deduce the Bohr’s correspondence principle from this?

On the other hand, about forty years before when I had been a student, main research subjects developing general theory of linear PDEs are “existence, uniqueness and regularity” of solutions for the given equation\footnote{Before advent of functional analytic approach to PDE, rather explicit solution is pursued at that time, therefore it is too hard to obtain a solution for a generally given PDE.}. Essential ingredients of these subjects is almost exhaustively studied and collected in L. Hörmander’s book \cite{60, 61} and one of the recent problems is to pick up special properties from governing equation or to represent the solution as explicit as possible by using known objects (for example, R. Beals \cite{8}). From this point of view, to make clear the dependence of the solution of Schrödinger equation on Planck’s constant $\hbar$ and to explain mathematically the appearance of Bohr’s correspondence principle is a good starting problem.

Therefore, we begin with retracing the heuristic procedure taken in Feynman’s doctor thesis (see also, S.A. Albeverio and R.J. Hoegh-Krohn \cite{53}) where he introduced his path-integral representation.

For the right-hand side of (1.1.2), we define the Hamiltonian operator on $C_0^\infty(\mathbb{R}^m)$ as
\[ \hat{H} = -\frac{\hbar^2}{2} \Delta + V(\cdot) = \hat{H}_0 + V, \quad \hat{H}_0 = \Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial q_j^2}. \]
If above $\hat{H}$ is essentially self-adjoint in $L^2(\mathbb{R}^m)$, applying Stone’s theorem, solution of (1.1.2) is written by

$$u(t, q) = (e^{-ih^{-1}t\hat{H}}u)(q).$$

Or generalizing a little, when and how the exponential function of a given operator $A$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

is well-defined? Guiding this problem, Hille-Yosida theory of semigroups is established.

[Report problem 1-1]: Check what is the Stone’s theorem. If the Hilbert space is finite-dimensional, what is the corresponding theorem in elementary linear algebra? It is also preferable to check what is the theory of Hille-Yosida.

On the other hand, Lie-Trotter-Kato’s product formula says that if $\hat{H} = \hat{H}_0 + V$, $e^{-ih^{-1}t\hat{H}}$ is given by

$$e^{-ih^{-1}t\hat{H}} = s-lim_{k \to \infty} \left(e^{-ih^{-1}\frac{t}{k}V} e^{-ih^{-1}\frac{t}{k}\hat{H}_0}\right)^k$$

even if $[\hat{H}_0, V] \neq 0$.

**Remark 1.1.1.** (i) In the above, if $[\hat{H}_0, V] = 0$, then $(\hat{H}_0 + V)^k = \sum_{j=0}^{k} \binom{k}{j} \hat{H}_0^j V^{k-j}$ and we have $e^{s(\hat{H}_0+V)} = e^{s\hat{H}_0} e^{sv}$, i.e. it isn’t necessary to apply above product formula.

(ii) There doesn’t exist the difference between strong and weak convergence in finite-dimensional vector spaces. Check the difference between the convergence of operators in “strong” or “uniform” sense in infinite-dimensional Banach space.

If the initial data $u$ belongs to $S(\mathbb{R}^m)$ (=a space of Schwartz’ rapidly decreasing functions), where

$$S(\mathbb{R}^m) = \{u \in C^\infty(\mathbb{R}^m : \mathbb{C}) \mid p_k, S(u) < \infty, \quad \forall k \in \mathbb{N}\}$$

with $p_k, S(u) = \sup_{q \in \mathbb{R}^m, \epsilon + |\beta| \leq k} \langle q \rangle^k |\partial^\beta u(q)|$, $\langle q \rangle = (1 + |q|^2)^{1/2}$, since we know

$$(e^{-ih^{-1}t\hat{H}_0}u)(q) = (2\pi iht)^{-m/2} \int_{\mathbb{R}^m} dq e^{ih^{-1}(q-q)^2/(2t)} u(q),$$

we have

$$(e^{-ih^{-1}t\hat{H}}u)(q) \sim (e^{-ih^{-1}tV} (e^{-ih^{-1}t\hat{H}_0}u))(q)$$

$$\sim (2\pi iht)^{-m/2} e^{-ih^{-1}tV(q)} \int_{\mathbb{R}^m} dq e^{ih^{-1}(q-q)^2/(2t)} u(q).$$

Therefore, we get

$$(e^{-ih^{-1}s\hat{H}(e^{-ih^{-1}t\hat{H}}u)})(q) \sim (2\pi ihs)^{-m/2} e^{-ih^{-1}sV(q)} \int_{\mathbb{R}^m} dq^{(1)} e^{ih^{-1}(q-q^{(1)})^2/(2s)} (e^{-ih^{-1}t\hat{H}}u)(q^{(1)})$$

$$\sim (2\pi i)^{-m} (ts)^{-m/2} e^{-ih^{-1}sV(q)} \int_{\mathbb{R}^m} dq^{(1)} e^{ih^{-1}(q-q^{(1)})^2/(2s)}$$

$$\times [e^{ih^{-1}tV(q^{(1)})} \int_{\mathbb{R}^m} dq e^{ih^{-1}(q^{(1)}-q)^2/(2t)} u(q)]$$

$$= (2\pi i)^{-m} (ts)^{-m/2} \int_{\mathbb{R}^m} dq$$

$$\times \left[ \int_{\mathbb{R}^m} dq^{(1)} e^{-ih^{-1}(sV(q)+tV(q^{(1)})]} e^{ih^{-1}(q-q^{(1)})^2/(2s)+ih^{-1}(q^{(1)}-q)^2/(2t)} \right] u(q).$$
Putting $t = s$ in the above, we have

$$-i\hbar^{-1}t(V(\bar{q}) + V(q^{(1)})) + i\hbar^{-1}[(\bar{q} - q^{(1)})^2 + (q^{(1)} - \bar{q})^2]/(2t)$$

$$= i\hbar^{-1}t\left[\frac{1}{2} \left(\frac{q^{(1)} - \bar{q}}{t}\right)^2 - V(\bar{q}) + \frac{1}{2} \left(\frac{q^{(1)} - \bar{q}}{t}\right)^2 - V(q^{(1)})\right].$$

Repeating this procedures $k$-times and denoting $q^{(k)} = \bar{q}^{(k)} = q^{(0)}$, we define

$$S_t(q^{(k)}, \ldots, q^{(0)}) = \sum_{j=1}^{k} \left[\frac{1}{2} \left(\frac{q^{(j)} - q^{(j-1)}}{t/k}\right)^2 - V(q^{(j)})\right] \frac{t}{k}$$

and we get

$$(e^{-i\hbar^{-1}tV} e^{-i\hbar^{-1}tH_0})^k u(q) \approx \int dq F_k(t, \bar{q}, q^{(k-1)}, \ldots, q^{(1)}, q)u(q).$$

Here, we put

$$F_k(t, \bar{q}, q^{(k-1)}, \ldots, q^{(1)}, q) = (2\pi i\hbar(t/k))^{-km/2} \int \cdots dq^{(1)} \cdots dq^{(k-1)} e^{i\hbar^{-1}S_t(q^{(k-1)}, \ldots, q^{(1)}, q)}.$$"
Then, if we could apply the stationary phase method to this representation when \( h \to 0 \), we got the main term which is obtained from the classical path \( \gamma_c \), i.e.

\[
\delta \int_0^t d\tau L(\gamma(\tau), \dot{\gamma}(\tau)) = \frac{d}{de} \int_0^t d\tau L((\gamma_c + e\phi)(\tau), (\dot{\gamma}_c + e\dot{\phi})(\tau))|_{e=0} = 0 \quad \text{for } \forall \phi \in C_{t,loop}
\]

In this sense, Bohr’s correspondence principle is derived! (Probably, Feynman yelled with delight “I did it”!)

The obstruction of this beautiful expression is the claim that “There doesn’t exist a non-trivial Lebesgue-like measure on any infinite-dimensional barrelled locally convex vector space”\(^3\).

[Report Problem 1-3 (Campbell-Hausdorff’s formula and its application)]:

1. Search “Campbell Hausdorff” in Google and check what it is.
2. Apply that formula to \( e^{tX} \) where

\[
X = \begin{pmatrix}
0 & \mu & 1 & 0 \\
-\mu & 0 & 0 & 1 \\
\Omega^2 - \mu^2 & 0 & 0 & \mu \\
0 & \Omega^2 - \mu^2 & -\mu & 0
\end{pmatrix}
\]

and get the concrete expression. Don’t use the diagonalization procedure but apply Campbell-Hausdorff formula to the suitable decomposition of \( X \). This matrix is derived from the Hamiltonian mechanics, for Lagrangian function \( L \) below, which is called Bateman-model.

\[
L(q, \dot{q}) = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \mu(q_1 \dot{q}_2 - q_2 \dot{q}_1) + \frac{\Omega^2}{2}(q_1^2 + q_2^2) \in C^\infty(T\mathbb{R}^2 : \mathbb{R}).
\]

3. Search also “Lie-Trotter-Kato formula”.

[Report Problem 1-4]: What is the meaning of AC function, what property it shares?

### 1.1.3. Non-existence of Feynman measure.

To “feel” the reason why there doesn’t exist Lebesgue-like measure (called Feynman measure), we give a simple theorem due to H.H. Kuo \([88]\).

Since that theorem is formulated in Hilbert space and the path space \( C_{t,loop} \) is not Hilbert one, those who don’t satisfy this explanation, consult the paper by O.G. Smolyanov and S.V. Fomin \([118]\).

For the sake of those who forget terminology, we recall the following:

**Definition 1.1.1 (Complete \( \sigma \)-algebra).** For a given space \( X \), a subset \( \mathcal{B} \) of all subsets \( \mathcal{P}^X \) satisfying

- \( \emptyset \in \mathcal{B} \),
- \( A \in \mathcal{B} \implies A^c = X \setminus A \in \mathcal{B} \),
- \( A_n \in \mathcal{B} \implies \sum_{n=1}^\infty A_n \in \mathcal{B} \)

is called complete \( \sigma \)-algebra.

**Definition 1.1.2 (measure).** A set function \( \mu \) defined on a complete \( \sigma \)-algebra \( \mathcal{B} \) of a space \( X \) is called a measure if it satisfies

- \( 0 \leq \mu(A) \leq \infty, \quad \mu(\emptyset) = 0 \),
- \( A_n \in \mathcal{B}, \quad A_j \cap A_k = \emptyset (j \neq k) \implies \mu(\sum_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n) \).

\(^3\) Though to construct Lebesgue’s integration theory, we are taught to prepare measure theory but is it truly necessary to do so? For example, Berezin integral below works without measure.
1. A MOTIVATION

DEFINITION 1.1.3 (Borel-algebra). A family \( \mathcal{B} \) of sets of a topological space \( X \) is called a Borel-algebra and denoted \( \mathcal{B} = \mathcal{B}(X) \) if it satisfies

- \( A \in \mathcal{B} \implies A^c = X \setminus A \in \mathcal{B} \),
- \( A_n \in \mathcal{B} \implies \sum_{n=1}^{\infty} A_n \in \mathcal{B} \),
- \( \mathcal{O}(X) \subset \mathcal{B} \),
- \( \mathcal{B} \) is the minimum in \( \mathcal{P}^X \) for the ordering by the set inclusion.

DEFINITION 1.1.4. A Borel measure\(^4\) satisfying below is called Lebesgue-like:

1. For any bounded Borel set, its measure is not only finite, but also positive if a set is not empty.
2. That measure is translation invariant\(^5\).

THEOREM 1.1.1. There exists no non-trivial Lebesgue-like Borel measure on an infinite dimensional separable Hilbert space.

Proof. Since \( H \) is separable, there exists a countable orthonormal base \( \{e_1, e_2, \ldots\} \)\(^6\).

Assume that there exists a non-trivial Lebesgue-like Borel measure \( \mu \) on \( \mathcal{B}(H) \). Define open sets as

\[
B_n = \{ u \in H \mid \|u - e_n\| < \frac{1}{2} \} \quad \text{and} \quad B = \{ u \in H \mid \|u\| < 2 \},
\]

then they satisfy

\[
B_n \cap B_m = \emptyset \quad \text{and} \quad \bigcup_{n=1}^{\infty} B_n \subset B.
\]

Since the measure is Lebesgue-like, we have

\[
0 < \mu(B_1) = \mu(B_2) = \cdots < \infty, \quad \infty = \sum_{n=1}^{\infty} \mu(B_n) \leq \mu(B) < \infty. \quad \text{Contradiction!} \quad \square
\]

[Report Problem 1-5]: What occurs if bases has continuous cardinality? By the way, check whether there exist non-separable Hilbert space. Check also the basis problem in general Banach space.

REMARK 1.1.2. Recently, I recognized very radical idea from Hung Cheng, a Professor of Applied Mathematics in the theoretical physics group of MIT, he is a physicist having job in math.department: He claimed in \([24]\),

The path integration approach is not only heuristic and non-rigorous; worse, it often leads to erroneous results when applied to non-Abelian gauge field.

REMARK 1.1.3 (Note added:2014.11). Though full Feynman measure doesn’t exist\(^7\), the objects represented formally using path-integration should be carefully researched. How to make rigorous the partial differentiation in “path-integral category” is now under-construction by Fujiiwara \([49]\), N. Kumano-go \([87]\). Their trials are done in \( (1 + 0) \)-dimension, how to generalize it to \( (1 + d) \)-dimensional case contains many interesting problem such as time-slicing should be replaced by something-like finite elements method corresponding to triangulation in topology, etc?.

---

\(^4\)measure defined on Borel algebra
\(^5\)assume the translation is defined on that topological space \( X \)
\(^6\)Hilbert-Schmidt’s procedure of orthogonalization holds for countable number of bases
\(^7\)recall also, there doesn’t exist full quantization, see R. Abraham and J.E. Marsden \([2]\)
Not as integrand by measure, but something new, my concern is how one can prove that this object is a solution of Functional Differential Equations. See, the last chapter!

1.1.4. Resume of known procedures. Assuming a certain convexity to apply Legendre transform, we have

\[
\begin{align*}
\text{Lagrangian Mechanics} & \quad \longleftrightarrow \quad \text{Hamilton Mechanics} \\
L(\gamma, \dot{\gamma}) & \quad \text{Legendre transform} \\
& \quad \longleftrightarrow \\
H(q, p).
\end{align*}
\]

Classical Mechanics

\[
\begin{align*}
\text{Hamilton equation} & = \begin{cases} 
\dot{q} = H_p(q, p), \\
\dot{p} = -H_q(q, p),
\end{cases} \\
\text{with} \quad \begin{pmatrix} q(0) \\
p(0) \end{pmatrix} & = \begin{pmatrix} q \\
p \end{pmatrix},
\end{align*}
\]

i.e.

\[
\frac{d}{dt} \begin{pmatrix} q(p) \end{pmatrix} = \mathbb{J} \begin{pmatrix} H_q \\
H_p \end{pmatrix} \quad \text{with} \quad \mathbb{J} = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}.
\]

Liouville equation \( \dot{\phi} = \{ \phi, H \} = \sum_{j=1}^m \left( \frac{\partial \phi}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial \phi}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \) with \( \phi(0, q, p) = \phi(q, p) \).

Quantum Mechanics

\[
\begin{align*}
\text{Liouville equation} & \quad \xrightarrow{\text{quantization}} \quad \text{Heisenberg picture} \\
\text{Hamilton equation} & \quad \xrightarrow{\text{quantization}} \quad \text{Schrödinger picture}
\end{align*}
\]

“\( L(\gamma, \dot{\gamma}) \) or \( H(q, p) \rightarrow \hat{H} = \hat{H}(q, -i\hbar \partial_q) \)”

(S) A description of the movement of the state vector \( u(t) \) w.r.t. time \( t \):

\[
\text{Schrödinger picture} \quad i\hbar \frac{\partial u(t)}{\partial t} = \hat{H} u(t) \quad \text{with} \quad u(0) = u,
\]

i.e. \( u(t) = e^{-i\hbar^{-1}t\hat{H}} u \).

(H) A description of the change of the kinetic operator \( \hat{F}(t) \) w.r.t. time \( t \):

\[
\text{Heisenberg picture} \quad i\hbar \frac{d}{dt} \hat{F}(t) = [\hat{F}(t), \hat{H}] \quad \text{with} \quad \hat{F}(0) = \hat{F}.
\]

(F) Path Integral method, clarifying Bohr’s correspondence principle:

\[
\text{Feynman picture} \quad u(t, q) = \int dq E(t, 0, q, \bar{q}) u(q)
\]

with

\[
E(t, 0, q, \bar{q}) = \int_{C_{t, q, \bar{q}}} D\gamma \exp \left(i\hbar^{-1} \int_0^t ds L(\gamma(s), \dot{\gamma}(s)) \right).
\]

Here,

\[
C_{t, q, \bar{q}} = \{ \gamma \in C([0, t] : \mathbb{R}^d) \mid \gamma(0) = q, \gamma(t) = \bar{q} \}
\]

and

\[
E(t, 0, q, \bar{q}) \sim D(t, 0, q, \bar{q})^{1/2} e^{i\hbar^{-1} S(t, 0, q, \bar{q})} \sim \delta_\delta(q).
\]
PROBLEM 1.1.1. Give a meaning to the symbolic representation
\[ \int D_F \gamma e^{ih^{-1} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau))d\tau} \]
for a wider class of Lagrangian \( L \).

(i) Concerning this question, D. Fujiwara [46] gives a rigorous meaning without notorious measure when the potential \( V \) satisfies \( |\partial_\alpha^\alpha V(x)| \leq C_\alpha (|\alpha| \geq 2) \).

(ii) At least in dimension 1, the essential selfadjointness of \( -\Delta + |q|^4 \) is proved by many methods (see, M. Reed and B. Simon vol I [102]). But we might not apply the procedure used by Fujiwara to construct a parametrix using classical quantities (but, see, S. Albeverio and S. Mazzucchi [4]).

(iii) How do we proceed when there exists many paths connecting points \( q \) and \( q' \) like the dynamics on the circle or sphere (see, L. Schulman [112])?

(iv) When \( |\partial_\alpha^\alpha V(q)| \leq C_\alpha (|\alpha| \geq 2) \), the above constructed parametrix converges in uniform operator norm. On the other hand, Lie-Trotter-Kato product formula assures only for the strong convergence.

How can one express the reason for this difference? In case of using polygonal line approximation for classical path to the harmonic oscillator, we get the strong but non-uniform convergence of parametrices. One possibility may to use non-standard analysis to check why there exists the difference of the convergence.

[Report Problem 1-6]: What is the meaning of essential adjointness? Check [102]!

PROBLEM 1.1.2. Fujiwara adapted the Lagrangian formulation in his procedure, stressing without Fourier transform. Does there exist the Hamiltonian object corresponding to this parametrix? (see for example, A. Intissar [78] and A. Inoue [68]):
\[ \int \int D_F q D_F p e^{ih^{-1} \int_0^t \dot{q}(\tau), \dot{p}(\tau))} \]

1.1.5. Feynman’s murmur. In p. 355 of their book [43], Feynman wrote as follows (underlined by the author):

\[ \cdots \text{path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part} \]

---

8 Rather recently, I find a paper by C.Grosche [54] where he claims this problem is solved by path integral method. But from his explanation, seemingly, we don’t have clearly the corresponding principle from their representation (2015.1.20)
of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

**Main Problem:** How do we treat this murmur as a mathematical problem?

Though for a given Schrödinger equation, we may associate a corresponding classical mechanics, but how do we define the classical mechanics corresponding to Dirac or Weyl equations? In other word, since Schrödinger equations is obtained from Lagrangian or Hamiltonian function by quantization, can we define a Hamiltonian function from which we get Dirac equation after quantization?

One of my objects to this lecture note, is to answer this main problem affirmatively by preparing new tools and giving sketchy explanation. There exists at least two problems for this:

1. How to define classical mechanics to Dirac or Weyl equations, or more generally for $2^d \times 2^d$ systems of LPDE?
2. Like Dirac or Weyl equations who have only first order derivatives in space variables, it seems impossible, even if there exist Hamilton equations, to assign initial and final positions in configuration space as is done in Schrödinger equation. How to get rid of this?

Finally, my answer is “yes”, it is possible with not only using superspace formulation but also re-interpreting the method of characteristics by Hamilton flow and Fourier transformation.

======== Mini Column 1: Stationary phase method ========

Consider the integral with parameter

$$I(\omega) = \int dq \, u(q)e^{i\omega \phi(q)}.$$

Study the asymptotic behavior of $I(\omega)$ when $\omega \to \infty$. Remembering Riemann-Lebesgue lemma, it seems natural to imagine the following fundamental fact holds.

**Lemma 1.1.1.** Let $\phi \in C^\infty(\mathbb{R}^d : \mathbb{R})$ and $u \in C^\infty_0(\mathbb{R}^d : \mathbb{R})$. Then,

$$\phi' \neq 0 \text{ on } \text{supp } u \implies I(\omega) = 0(\omega^{-k}) \text{ when } \omega \to \infty.$$

This is a fundamental fact for the stationary phase method. Therefore, further study is to study the behavior when “$\phi' \neq 0$ on supp $u$”. A typical answer for this is given

**Theorem 1.1.2** (Theorem 7.7.5, p.220 of Hörmander I of [60]). Let $K$ be a compact set of $\mathbb{R}^d$, $X$ an open neighborhood of $K$ and $k$ a positive integer. If $u \in C^{2k}_0(K)$, $\phi \in C^{3k+1}(X)$ and $\Im \phi \geq 0$ on $X$ and let there exists a point $q_0 \in K$ such that $\Im \phi(q_0) = 0$, $\phi'(q_0) = 0$, $\det \phi''(q_0) \neq 0$ and $\phi' \neq 0$ on $K \setminus \{q_0\}$. Then, we have

$$\left| \int_{\mathbb{R}^d} dq \, u(q)e^{i\omega \phi(q)} - e^{i\omega \phi(q_0)}(\det(\omega \phi''(q_0)/(2\pi i))^{-1/2} \sum_{j<k} \omega^{-j} L_j u \right| \leq C \omega^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u|.$$
Here, $C$ is bounded when $\phi$ stays in a bounded set in $C^{3k+1}(X)$ and $|q - q_0|/\phi'(q)$ has a uniform bound. With

$$g_{q_0}(x) = \phi(q) - \phi(q_0) - \frac{\langle \phi''(q_0)^{-1}(q - q_0), q - q_0 \rangle}{2}$$

which vanishes of third order at $q_0$, we have

$$L_j u = \sum_{\nu - \mu = j} \sum_{2\nu \geq 2\mu} i^{-j-\nu} \frac{\langle \phi''(q_0) D\nu (g_{q_0} u)(q_0) \rangle}{\mu! \nu!}.$$ 

This is a differential operator of order $2j$ acting on $u$ at $q_0$. The coefficients are rational homogeneous functions of degree $-j$ in $\phi''(q_0), \ldots, \phi^{(2j+2)}(q_0)$ with denominator $(\det \phi''(q_0))^{3j}$. In every term the total number of derivatives of $u$ and of $\phi''$ is at most $2j$.

**Remark 1.1.4.** In mathematics society, it is regarded as someone’s-conjecture if the statement “This is the main term” goes without precise estimates of error terms. But in papers of mathematical physics, seemingly there is not so many with estimating “error terms”. For example, the famous paper of E. Witten [134] doesn’t have estimates of “so-called small terms” with precise calculation. Or more frankly speaking, since there doesn’t exist Feynman measure, the representation using such measure seems a castle in the air, though it shows us the goal or a dream as it is so. Getting the main terms without error estimates, you may proceed very algebraically and geometrically, and you may have something-like solution, but it doesn’t mean its conclusion is true! Even if you may have experiments based on that calculation and if you may claim the data is inside measurement error, how you may assure the theory is correct even in mathematical sense!

====== End of Mini Column 1 ======

**1.1.6. Fujiwara’s procedure.** Since there doesn’t exist the so-called Feynman measure which guarantees the beautiful path-integral expression, how do we represent the solution of the Schrödinger equation?

As the operator

$$\hat{H} = \hat{H}(q, -i\hbar \partial_q) = -\frac{\hbar^2}{2} \Delta + V(q)$$

is essentially self-adjoint on $L^2(\mathbb{R}^m)$ under certain conditions on $V$, there exists a solution $e^{i\hbar^{-1}t\hat{H}} u$ (by Stone’s theorem) of the initial value problem

$$i\hbar \frac{\partial u(t,q)}{\partial t} = \hat{H} u(t,q) \quad \text{with} \quad u(0,q) = u_0(q).$$

Moreover, by L. Schwartz’s kernel theorem, we have a kernel $E(t,q,q') \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m)$ such that

$$\langle e^{i\hbar^{-1}t\hat{H}} u, \varphi \rangle = \langle E(t,q,q') u(q'), \varphi(q) \rangle = \langle E(t,q,q'), u(q') \rangle = \langle E(t,\cdot, \cdot), u \otimes \varphi \rangle.$$ 

On the other hand, for the heat case $e^{t\hat{H}} v$, the distributional kernel $H(t,q,q')$ has the representation by the “classical quantities”?
Method of Fujiwara: About 30 years before, there doesn’t exist a paper on the construction of a fundamental solution for the initial value problem of Schrödinger equation. Fujiwara adopts the argument of Feynman modifying mathematically.

(1) For given Lagrangian \( L(\gamma, \dot{\gamma}) = \frac{1}{2}|\dot{\gamma}|^2 - V(\gamma) \in C^\infty(TM)(M = \mathbb{R}^m) \), by Legendre transform, we have the Hamilton function \( H(q, p) = \inf_q [qp - L(q, \dot{q})] \in C^\infty(T^*M) \).

(2) For the Hamilton function \( H(q, p) = \frac{1}{2}|p|^2 + V(q) \), we construct a solution \( S(t, q, q) \) of the Hamilton-Jacobi equation

\[
S_t(t, q, q) + H(q, S_q(t, q, q)) = 0 \quad \text{with} \quad \lim_{t \to 0} S(t, q, q) = \frac{1}{2}|q - q'|^2.
\]

(3) For the action function \( S(t, q, q) \) obtained above, the amplitude function defined by

\[
D(t, q, q) = \det \left( \frac{\partial^2 S(t, q, q)}{\partial q \partial \bar{q}} \right) \quad \text{(van Vleck determinant)}
\]

satisfies the continuity equation

\[
D_t(t, q, q) + \partial_q (D(t, q, q)H_p(q, S_q(t, q, q))) = 0 \quad \text{with} \quad \lim_{t \to 0} D(t, q, q) = 1.
\]

(4) Then we define the integral transformation

\[
F(t)u(q) = (2\pi i)^{-m/2} \int_{\mathbb{R}^m} dq \, D(t, q, q)^{1/2} e^{i\hbar^{-1} S(t, q, q)} u(q).
\]

**Theorem 1.1.3** (Theorem 2.2 of Fujiwara [46]). Assume \( \sup_{q \in \mathbb{R}^m} |D^\alpha V(q)| \leq C_\alpha (|\alpha| \geq 2) \). Fix \( 0 < T < \infty \) arbitrarily. Put \( \mathbb{H} = L^2(\mathbb{R}^m : \mathbb{C}) \), \( B(\mathbb{H}) \) the set of bounded linear operators on \( \mathbb{H} \).

(i) \( F(t) \) defines a bounded linear operator in \( \mathbb{H} \)

\[
\|F(t)u\| \leq C\|u\| \quad \text{by Cotlar’s lemma}.
\]

(ii) For any \( u \in L^2(\mathbb{R}^m : \mathbb{C}) \), \( t, s, t + s \in [-T, T] \),

\[
\lim_{t \to 0} \|F(t)u - u\| = 0,
\]

\[
\left. i\hbar \frac{\partial}{\partial t} (F(t)u)(q) \right|_{t=0} = \hat{H}(q, -i\hbar \partial_q)u(q),
\]

\[
\|F(t + s) - F(t)F(s)\| \leq C(t^2 + s^2).
\]

(iii) Moreover, there exists a limit \( \lim_{k \to \infty} (F(t/k))^k = E(t) \) in \( B(\mathbb{H}) \), i.e. in the operator norm of \( L^2(\mathbb{R}^m : \mathbb{C}) \), which satisfies the initial value problem below:

\[
\left\{ \begin{array}{l}
i\hbar \frac{\partial}{\partial t} (E(t)u)(q) = \hat{H}(q, -i\hbar \partial_q)(E(t)u)(q), \\
(E(0)u)(q) = u(q).
\end{array} \right.
\]

Remark 1.1.5. The operator \( F(t) \) is said to be a parametrix and \( E(t) \) or its kernel is called the fundamental solution.

---

9 How to recognize Feynman’s idea of “put equal weight for each path”, I feel some difference between Fujiwara’s idea and mine.
Outline of the proof: In (2), for the construction of a solution of the Hamilton-Jacobi equation, he uses the Jacobi’s method.

(a) For the given $H(q, p)$ and the initial data $(q, p)$, there exists a unique Hamilton flow $(q(s), p(s)) = (q(s, q, p), p(s, q, p))$.

(b) For the given time interval $t$ which is sufficiently small, and for any given terminal position $\bar{q}$, applying the implicit function theorem to $\bar{q} = q(t, q, p)$, we get the unique $p$ denoted by $p = \xi(t, q, \bar{q})$.

(c) Using this function, we put $S(t, \bar{q}, q) = S_0(t, q, p)|_{p = \xi(t, q, \bar{q})}$. That is, there exists a unique path $\gamma_c$ in $C_{t, q, \bar{q}}$ such that

$$\inf_{\gamma \in C_{t, q, \bar{q}}} S_t(\gamma) = S_t(\gamma_c) = S(t, \bar{q}, q) \quad \text{with} \quad S_t(\gamma) = \int_0^t d\tau L(\gamma(\tau), \dot{\gamma}(\tau)).$$

Moreover, this function $S(t, \bar{q}, q)$ is a solution of the Hamiltonian-Jacobi equation.

Remark 1.1.6. By this construction, we have estimates of $S(t, \bar{q}, q)$ with respect to $\partial_t^k, \partial_q^\alpha, \partial_{\bar{q}}^\beta \{\}$. (3) is proved from (2) algebraically (see, Inoue and Maeda [74] or [76] even on superspace).

(4) Since we have estimates of $S(t, \bar{q}, q)$ or $D(t, \bar{q}, q)$ w.r.t. $(\bar{q}, q)$, we may prove the $L^2$-boundedness of the operator (1.1.5) applying Cotlar’s lemma. Since we take $D(t, \bar{q}, q)^{1/2}$ as the amplitude, the operator (1.1.5) is considered as acting on the half-density bundle (or the intrinsic Hilbert space) “$L^2(\mathbb{R}^m : \mathbb{C})$”. I regard this fact as corresponding to Copenhagen interpretation.

(5) Though above theorem is sufficient concerning the convergence of parametrix (1.1.3), but this convergence is not sufficient for the Feynman’s expression. Concerning this or the construction of the fundamental solution itself, there exists another paper by Fujiwara [48] which isn’t discussed in this lecture because I haven’t appreciated it fully.

Problem 1.1.3. In the above theorem, the momentum energy is restricted on the flat Riemannian metric $\frac{1}{2}|p|^2$ on $\mathbb{R}^m$. Whether this procedure works for the Riemannian metric is calculated by physicist (see for example, B. DeWitt [33]) and he suggests the desired Laplace-Beltrami operator but with the term $R/12$ where $R$ is the scalar curvature of $g_{ij}(q)dq^{i}dq^{j}$. In general, to prove the $L^2$-boundedness of the FIO with suitable phase and amplitude of order 0, Fujiwara applied Cotlar’ lemma which is formulated in flat space. Technically, we need new device to extend almost orthogonality in case the space is curved. Therefore, it is an open problem to associate a quantum mechanics for given Riemann metric $g_{ij}(q)$ on $\mathbb{R}^m$ following Fujiwara’s procedure.

On the other hand, above procedure of Fujiwara was used also by Inoue and Maeda [74] to explain mathematically the origin of the term $(1/12)R$, $R =$ the scalar curvature of the configuration manifold in the heat category, which appeared when one wants to “quantize with purely imaginary time” the Lagrangian on a curved manifold.

Problem 1.1.4. Feynman or Fujiwara used Lagrangian formulation. How do we connect the above procedure directly to the Hamiltonian without using Lagrangian?
1.2. The first step towards Dirac and Weyl equations

1.2.1. The origin of Dirac and Weyl equations. Why and how does P. Dirac introduce, now so-called, Dirac equation? We modify the description of Nishijima [98].

Assume that energy $E$ and momentum $p$ of the free particle with mass $m$ satisfy the Einstein relation

$$E^2 = c^2|p|^2 + c^4 m^2.$$  

Following the canonical quantization procedure of substitution

$$p_j \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial q_j}, \quad E \rightarrow ih \frac{\partial}{\partial t}$$

which we did to get the Schrödinger equation, we have the Klein-Gordon equation

$$\hbar^2 \frac{\partial^2}{\partial t^2} u - c^2 \hbar^2 \Delta u + c^4 m^2 u = 0.$$  

Unfortunately, the solution $u$ of this equation does not permit the Copenhagen interpretation, that is, the quantity $\rho = |u|^2$ is not interpreted as the probability density. In order to get rid of this inconvenience, it is claimed that it is necessary to have the first order derivative w.r.t. time in physics literature.

If this saying is accepted, the simplest prescription is to put

$$E = \pm c \sqrt{|p|^2 + c^2 m^2}.$$  

But the right-hand side of above defines a ΨDO, which doesn’t have local property. This gives us a certain conflict if we insist on that the physical law (which is assured by experiments in laboratory system) should satisfy local property. Therefore, it is not so nice to accept such ΨDO with symbol above, as the quantization of Einstein relation.

[Report problem 2-1]: For a ΨDO, it has pseudo-local property. Report on this subject.

In order to have the equation which stems from Einstein relation and admits probabilistic interpretation, we need to have

$$(ih \frac{\partial}{\partial t} - \hat{H})\psi = 0$$

which satisfies

$$(\hbar^2 \frac{\partial^2}{\partial t^2} + \hat{H}^2)\psi = 0.$$  

Assuming that this equation coincide with Klein-Gordon equation, we need that “the symbol corresponding to the operator $\hat{H}$” should satisfy

$$\hat{H}^2 = c^2 |p|^2 + c^4 m^2.$$  

Supposing that the state vector $\psi$ which satisfies the desired equation has multicomponents, then we may have the option such that

$$H = \sum_{j=1}^{3} \alpha_j p_j + mc^2 \beta.$$  

---

10 Remember, for $p = 0$, this gives the theoretical foundation of the possibility of atomic bomb!

11 If $P(q, p) = \sum_{|a| \leq N} a_a(q)p^a$, then roughly speaking, it is quantized as a PDE $P(q, -ih\partial_q)$. Then, it satisfies supp $P(q, -ih\partial_q)u \subset$ supp $u$ for any $u \in C_0^\infty(\mathbb{R}^m)$. This is the local property of PDE.
Here, above appeared letters \( \{ \alpha_j, \beta \} \) satisfy
\[
\alpha_j \alpha_k + \alpha_k \alpha_j = 2 \delta_{jk} I, \quad \alpha_j \beta + \beta \alpha_j = 0, \quad \beta^2 = I
\] (1.2.1)

Dirac gave an example of \( 4 \times 4 \) matrices satisfying the relation (1.2.1), which is now called Dirac matrices:
\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \sigma_3 \otimes I_2.
\]

Here, Pauli matrices \( \{ \sigma_j \}_{j=1}^3 \) are given by
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (1.2.2)
satisfying
\[
\sigma_j \sigma_k + \sigma_k \sigma_j = 2 \delta_{jk} I, \quad \sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_1 = i \sigma_2.
\] (1.2.3)

There are many representations satisfying (1.2.1), named Majorana rep, chiral rep, etc. Seek such representations as many as possible and check the relationship between them. By the way of checking these, study also the Lorentz invariance. If such relations are explained by unified manner, using the differential representation point of view, it will be good enough for master thesis, isn’t it?

Problem 1.2.1. For a given external electro-magnetic field, the IVP (= initial value problem) for the Dirac equation is given as follows: Find \( \psi(t,q) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4 \), for the given initial data \( \psi(q) \in C_0^\infty(\mathbb{R}^3 : \mathbb{C}^4) \), satisfying
\[
\begin{cases}
\left(i \hbar \frac{\partial}{\partial t}\right) \psi(t,q) = \mathbb{H}(t) \psi(t,q), \\
\psi(t_0,q) = \psi_0(q)
\end{cases}
\]
(1.2.4)

Here,
\[
\mathbb{H}(t) = c \sum_{k=1}^3 \alpha_k \left( \frac{\hbar}{i} \frac{\partial}{\partial q_k} - \frac{e}{c} A_k(t,q) \right) + mc^2 \beta + eA_0(t,q).
\] (1.2.5)

Though it is well-known that this IVP has a solution, we want to know a “good” parametrix or fundamental solution as Feynman desired. More explicitly, show the mathematical proof for the phenomena called Zitterbewegung (see, Inoue [67] for free case).

Seemingly H. Weyl had been at Dirac’s talk as an audience, he proposed \( 2 \times 2 \)-matrix representation (1.2.2) in stead of \( 4 \times 4 \)-one when the mass \( m = 0 \). From this, he derived the initial value problem of the free Weyl equation: For a “vector” \( \psi(t,q) \), it satisfies
\[
\begin{cases}
\left(i \hbar \frac{\partial}{\partial t}\right) \psi(t,q) = \mathbb{H} \psi(t,q), \\
\psi(0,q) = \psi_0(q)
\end{cases}
\]
(1.2.6)

In spite of the beauty of this equation, it is not accepted in physicists society for a while, because the parity is not preserved by this one. Its meaning is reconsidered after Lee-Yang’s theory and Wu’s experiment in weak interaction, which shows that the parity is not necessarily preserved for certain spinning particles.
Since Neutrino has been considered as the particle with mass 0, Weyl equation is believed to be the governing equation of Neutrino until the recent experiment of Kamiokande which suggests that at least certain Neutrino has non-zero mass.

[Report Problem 2-3]: Search “Weyl equation” in internet to check whether the usage of this equation in condensed matter physics, etc. Report things what you appreciate interesting.

Ordinary procedure: As a hint to get a result for Problem 1.2.1, we give a simple example. Though the equation (1.2.6) is a system but with constant coefficients, applying Fourier transform, we may have the solution rather by algebraic operation. In fact, defining Fourier transform as

\[
\hat{u}(p) = (2\pi \hbar)^{-m/2} \int_{\mathbb{R}^m} dq \ e^{-i\hbar^{-1}qp} u(q), \quad u(q) = (2\pi \hbar)^{-m/2} \int_{\mathbb{R}^m} dp \ e^{i\hbar^{-1}qp} \hat{u}(p),
\]

and applying this to \(q \in \mathbb{R}^3\) of (1.2.6), we get

(1.2.7)  
\[i\hbar \frac{\partial}{\partial t} \hat{\psi}(t, p) = -\hat{H} \hat{\psi}(t, p)\]

Here,  
\[\hat{H} = c \sum_{j=1}^{3} \sigma_j p_j = c \begin{pmatrix} p_3 & p_1 - ip_2 & -p_3 \\ p_1 + ip_2 & -p_3 \end{pmatrix}
\]
and  
\[\hat{H}^2 = c^2 |p|^2 \mathbb{I}_2.
\]

From this, we have

**Proposition 1.2.1.** For any \(t \in \mathbb{R}\) and \(\psi \in L^2(\mathbb{R}^3 : \mathbb{C}^2)\), we have

(1.2.8)  
\[e^{-i\hbar^{-1}t\hat{H}} \hat{\psi}(q) = (2\pi \hbar)^{-3/2} \int_{\mathbb{R}^3} dp \ e^{i\hbar^{-1}qp} e^{-i\hbar^{-1}t\hat{H}} \hat{\psi}(p).
\]

If \(\psi \in \mathcal{S}(\mathbb{R}^3 : \mathbb{C}^2)\), then

(1.2.9)  
\[\mathbb{E}(t, q) = (2\pi \hbar)^{-3} \int_{\mathbb{R}^3} dp \ e^{i\hbar^{-1}qp} \left[ \cos(ch^{-1}t|p|) \mathbb{I}_2 - \frac{i}{c|p|} \sin(ch^{-1}t|p|) \hat{H} \right] \in \mathcal{S}'(\mathbb{R}^3 : \mathbb{C}^2)
\]
and

(1.2.10)  
\[e^{-i\hbar^{-1}t\hat{H}} \hat{\psi}(q) = \mathbb{E} * \hat{\psi}(t, q) = \int_{\mathbb{R}^3} dq' \ \mathbb{E}(t, q - q') \hat{\psi}(q'),
\]

In spite of this, we may give another representation with action integral \(\mathcal{S}\) and amplitude \(\mathcal{D}^{1/2}\), which is proved in [66] and will be explained in later chapter.

**Remark 1.2.1.** Pauli said one day that “There exists no classical counterpart corresponding to quantum spinning particle”, so I had seen somewhere but I can’t remember where exactly. Therefore, such saying didn’t exist? Please give a look to the splendid book S. Tomonaga [123], written in Japanese.
In any way, it seems difficult to imagine the classical mechanics corresponding to the equation (1.2.6) from the formula (1.2.9). This is the one reason why I denote Feynman’s murmur as Feynman’s problem.

**Claim 1.2.1.** In spite of above, I claim that I may construct the classical mechanics corresponding to (1.2.6), which yields a path-integral-like representation [10] in Theorem 7.0.3 of it!

13Though I said on one hand that “There doesn’t exist path-integral” (more accurately, in path space, there doesn’t exist Lebesgue-like Borel measure) but here I mention path-integral-like. Therefore, it seems better to find more suitable nomination for path-integral-like representation.
1.2.2. The method of characteristics and Hamiltonian path-integral-like representation. Though Schrödinger equation has 2-times partial derivatives which guarantees to assign initial and final positions to the corresponding classical flow on configuration space, but there exists only 1-time partial derivatives w.r.t. the space variables in Dirac or Weyl equations, this is the very reason why we need Hamiltonian path-integral representation. We need to use phase space instead of configuration space.

Therefore, we want to give a simple example exhibiting “Hamiltonian path-integral-like representation”, which is a necessary device to resolve Feynman’s problem.

We may solve the following equation readily:

\[
\begin{aligned}
&i\hbar \frac{\partial}{\partial t} u(t, q) = a \frac{\hbar}{i} \frac{\partial}{\partial q} u(t, q) + b q u(t, q), \\
u(0, q) = u(q).
\end{aligned}
\]  

(1.2.11)

From the right-hand side of above, we get a Hamiltonian function

\[
H(q, p) = e^{-i\hbar^{-1}q p} \left( a \frac{\hbar}{i} \frac{\partial}{\partial q} + b q \right) e^{i\hbar^{-1}q p} \bigg|_{\hbar = 0} = a p + b q,
\]

then, the corresponding classical orbit is obtained easily from the Hamilton equation

\[
\begin{aligned}
\dot{q}(t) &= H_p = a, \\
\dot{p}(t) &= -H_q = -b
\end{aligned}
\]  

(1.2.12)

such as

\[
q(s) = q + as, \quad p(s) = p - bs.
\]

Using these, by applying the method of characteristics, we get

\[
U(t, q) = u(q) e^{-i\hbar^{-1}(a p t + 2^{-1} a b t^2)}.
\]

Using the inverse function \( q = y(t, \bar{q}) = \bar{q} - a t \) of \( \bar{q} = q(t, \bar{q}) \), the solution of (1.2.11) is given as

\[
u(t, \bar{q}) = U(t, q) \big|_{q=y(t,\bar{q})} = u(\bar{q} - a t) e^{-i\hbar^{-1}(a p t + 2^{-1} a b t^2)}.
\]

**Remark:** In the above procedure, the information from \( p(t) \) is not used.

[Report problem 2-4]: Study the method of characteristics for the first order PDE. Since from the information obtained from ODE(such as (non-linear) Hamilton equation), we get a solution of PDE(such as (linear) Louville equation), this is the core of the method of characteristics. What is the linear Liouville equation corresponding to the non-linear field equation, for example, the Hopf equation represented by functional derivatives is the Liouville equation corresponding to the Navier-Stokes equation.

**Another point of view from Hamiltonian path-integral-like method:** Put

\[
S_0(t, \bar{q}, p) = \int_0^t ds \left[ \dot{q}(s) p(s) - H(q(s), p(s)) \right] = -b \bar{q} t - 2^{-1} a b t^2,
\]

\[
S(t, \bar{q}, p) = \left( q \bar{q} + S_0(t, q, p) \right) \bigg|_{q=y(t,\bar{q})} = \bar{q} p - a p t - b \bar{q} t + 2^{-1} a b t^2.
\]

Then, the classical action \( S(t, \bar{q}, p) \) satisfies the Hamilton-Jacobi equation

\[
\begin{aligned}
\frac{\partial}{\partial t} S + H(\bar{q}, \partial_\bar{q} S) &= 0, \\
S(0, \bar{q}, p) &= \bar{q} p.
\end{aligned}
\]
On the other hand, the van Vleck determinant (though scalar in this case) is calculated as
\[ D(t, \bar{q}, p) = \frac{\partial^2 S(t, \bar{q}, p)}{\partial \bar{q} \partial p} = 1. \]
This quantity satisfies the continuity equation:
\[
\begin{aligned}
\frac{\partial}{\partial t} D + \frac{1}{2} \partial \bar{q}(DH_p) &= 0 \\
D(0, \bar{q}, p) &= 1.
\end{aligned}
\]
As an interpretation of Feynman’s idea, we regard that the transition from classical to quantum is to study the following quantity or the one represented by this (be careful, the term “quantization” is not so well-defined mathematically as functor, so ad-hoc):
\[
u(t, \bar{q}) = (2\pi \hbar)^{-1/2} \int_{\mathbb{R}} dp D^{1/2}(t, \bar{q}, p) e^{i\hbar^{-1}S(t, \bar{q}, p)} \hat{u}(p).
\]
That is, in our case at hand, we should study the quantity defined by
\[
u(t, \bar{q}) = (2\pi \hbar)^{-1/2} \int_{\mathbb{R}} dp e^{i\hbar^{-1}S(t, \bar{q}, p)} \hat{u}(p)
= (2\pi \hbar)^{-1} \int_{\mathbb{R}^2} dp dq e^{i\hbar^{-1}(S(t, \bar{q}, p) - \bar{q} \cdot p)} \hat{u}(q)
= \nu(\bar{q} - at) e^{i\hbar^{-1}(-b\bar{q} + 2^{-1}ab^2)}.
\]
Therefore, we may say that this second construction gives the explicit connection between the solution (1.2.11) and the classical mechanics given by (1.2.12). We feel the above expression “good” because there appears two classical quantities \( S \) and \( D \) and also explicit dependence on \( \hbar \).

Claim 1.2.2. Applying superanalysis, we may extend the second argument above to a system of PDOs e.g. quantum mechanical equations with spin such as Dirac, Weyl or Pauli equations, (and if possible, any other \( 2^d \times 2^d \) system of PDOs), after interpreting these equations as those on superspaces.

1.2.3. Decomposition of \( 2 \times 2 \) matrix by Clifford algebra.

How matrix does act on vectors?: Following matrices form a special class in \( 2 \times 2 \) matrices.
\[
A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.
\]
This set of matrices \( \{ A \} \) not only preserves their form under four rules of arithmetic but also is commutative each other. Moreover, we identify this matrix \( A \) with a complex number \( \alpha = a + ib \).
If we regard a vector \( \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix} \) with a complex number \( z = x + iy \), then the multiplication \( \alpha \) to \( z \) is considered as
\[
(a + ib)(x + iy) \sim \alpha \mathbf{z} \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} \sim (ax - by) + i(bx + ay) = (a + ib)(x + iy)
\sim \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.
\]
Then, may we find another interpretation of making act \( 2 \times 2 \) matrix to a column vector? Since above mentioned interpretation gives you many stand points, is it possible this idea generalize?
1. A MOTIVATION

Guided by the following theorem of C. Chevalley\footnote{Though I don’t know how to prove this theorem itself, but I’m satisfied by constructing the differential operator representation of 2 × 2-matrices using Pauli matrices. Oh, such a jerry-built attitude as a mathematician is allowed?!} below, we decompose a 2 × 2-matrix. Here,

**THEOREM 1.2.1** (C. Chevalley). Any Clifford algebra has the representation on Grassmann algebra.

(I) For any 2 × 2 matrix, we have

\[
\begin{pmatrix}
a & c \\
b & d
\end{pmatrix}
= \frac{a+b}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a-b}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{c+d}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{c-d}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

= \frac{a+b}{2} \mathbf{1}_2 + \frac{a-b}{2} \mathbf{e}_3 + \frac{c+d}{2} \mathbf{e}_1 + \frac{c-d}{2} \mathbf{e}_2.

Here, \( \{ \mathbf{e}_j \} \) satisfies not only (1.2.2) but also the relation (1.2.3).

This decomposition stands for that a set of all 2 × 2 matrices is spanned by Pauli matrices \( \{ \mathbf{e}_k \} \) having Clifford structure.

(II-1) Now, preparing a letter \( \theta \) satisfying \( \theta^2 = 0 \), we identify Pauli matrices with differential operators acting on Grassmann algebra \( \Lambda = \{ u(\theta) = u_0 + u_1 \theta \mid u_0, u_1 \in \mathbb{C} \} \), i.e. for

\[
u_0 + u_1 \theta \sim \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},
\]

define the action as

\[
\theta u(\theta) = u_0 \theta \sim \begin{pmatrix} 0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad \frac{\partial}{\partial \theta} u(\theta) = u_1 \sim \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.
\]

Then, we have

\[
\begin{aligned}
\left( \theta + \frac{\partial}{\partial \theta} \right) u(\theta) &= u_0 \theta + u_1 \sim \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\
\left( \theta - \frac{\partial}{\partial \theta} \right) u(\theta) &= u_0 \theta - u_1 \sim \begin{pmatrix} -u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\
\left( 1 - 2\theta \frac{\partial}{\partial \theta} \right) u(\theta) &= u_0 - u_1 \theta \sim \begin{pmatrix} u_0 \\ -u_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.
\end{aligned}
\]

This means that Pauli matrices are represented as differential operators acting on \( \Lambda \).

But such a representation is not unique!

(II-2) Here is another representation: Preparing 2 letters \( \theta_1, \theta_2 \) satisfying \( \theta_1 \theta_2 + \theta_2 \theta_1 = 0 \) for \( i, j = 1, 2 \), we put

\[
\Lambda_{ev} = \{ u = u_0 + u_1 \theta_1 \theta_2 \mid u_0, u_1 \in \mathbb{C} \}, \quad \Lambda_{od} = \{ v = v_1 \theta_1 + v_2 \theta_2 \mid v_1, v_2 \in \mathbb{C} \},
\]

and define differential operators acting on \( \Lambda_{ev} \) as

\[
\begin{aligned}
\sigma_1(\theta, \partial_\theta) &= \left( \theta \partial_\theta - \partial_\theta \partial_\theta \right) u(\theta) = u_0 \theta_1 \theta_2 + u_1 \sim \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\
- i \sigma_2(\theta, \partial_\theta) &= \left( \theta \partial_\theta + \partial_\theta \partial_\theta \right) u(\theta) = u_0 \theta_1 \theta_2 - u_1 \sim \begin{pmatrix} -u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\
\sigma_3(\theta, \partial_\theta) &= \left( 1 - \theta \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \theta} \right) u(\theta) = u_0 - u_1 \theta_1 \theta_2 \sim \begin{pmatrix} u_0 \\ -u_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.
\end{aligned}
\]
Remark 1.2.2. Above defined differential operators $\sigma_j(\theta, \partial_\theta)$ annihilate $\Lambda_{od}$. Moreover, the symbols corresponding to them are “even”. This evenness is crucial to derive Hamilton flow corresponding to Weyl or Dirac equations.
CHAPTER 2

Super number and Superspace

To explain symbols $\theta, \theta_1, \theta_2$ appeared in previous lectures, we prepare a set of countably infinite Grassmann generators. After introducing of these, we may consider the classical mechanics corresponding to PDE with spin, which is rather easily solved.

2.1. Super number

2.1.1. The Grassmann generators. Preparing symbols $\{\sigma_j\}_{j=1}^{\infty}$ which satisfy the Grassmann relation

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 0, \quad j, k = 1, 2, \cdots,$$

we put formally

$$\mathcal{C} = \{X = \sum_{I \in \mathcal{I}} X_I \sigma^I \mid X_I \in \mathbb{C}\}$$

and

$$\mathcal{C}^{(0)} = \mathcal{C}^{[0]} = \mathbb{C},$$

$$\mathcal{C}^{(j)} = \{X = \sum_{|I| \leq j} X_I \sigma^I\}$$

and

$$\mathcal{C}^{[j]} = \{X = \sum_{|I| = j} X_I \sigma^I\} = \mathcal{C}^{(j)}/\mathcal{C}^{(j-1)},$$

where the index set is defined by

$$\mathcal{I} = \{I = (i_k) \in \{0, 1\}^N \mid |I| = \sum_k i_k < \infty\},$$

$$\sigma^I = \sigma_1^{i_1} \sigma_2^{i_2} \cdots, \quad I = (i_1, i_2, \cdots), \quad \sigma^0 = 1, \quad \tilde{0} = (0, 0, \cdots) \in \mathcal{I}.$$

Remark 2.1.1. How do we construct symbols $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation? What is the meaning of summation appeared above? These will be soon explained.

In today’s lecture, we prove the following Proposition which guarantees that $\mathcal{C}$ (or $\mathfrak{A}$, defined later) plays the alternative role of $\mathbb{C}$ (or $\mathfrak{A}$) in analysis.

Proposition 2.1.1 (Inoue and Maeda [75]). $\mathcal{C}$ forms an $\infty$-dimensional Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.

Remark 2.1.2. There exist some papers using $\mathcal{C}$, for example, S. Matsumoto and K. Kakazu [92], Y. Choquet-Bruhat [19], P. Bryant [20]. But, seemingly, they didn’t try to construct “elementary and real analysis” on this “ground ring” $\mathcal{C}$ (or $\mathfrak{A}$).
2.1.2. Sequence spaces and their topologies. Following G. Köthe [85], we introduce the sequence spaces $\omega$ and $\phi$, (effm=except for finitely many)

\begin{equation}
\phi = \{x = (x_k) = (x_1, x_2, \ldots, x_k, \ldots) \mid x_k \in \mathbb{C} \text{ and } x_k = 0 \text{ effm } k\} ,
\end{equation}

\begin{equation}
\omega = \{u = (u_k) = (u_1, u_2, \ldots, u_k, \ldots) \mid u_k \in \mathbb{C}\}.
\end{equation}

For any sequence space $\mathcal{X}$ containing $\phi$, we define the space $\mathcal{X}^\times$ by

\[ \mathcal{X}^\times = \left\{ u = (u_k) \mid \sum_k |u_k||x_k| < \infty \text{ for any } x = (x_k) \in \mathcal{X} \right\}, \]

then, we get

\[ \phi^\times = \omega \quad \text{and} \quad \omega^\times = \phi. \]

We introduce the (normal) topology in $\mathcal{X}$ and $\mathcal{X}^\times$ by defining the seminorms

\begin{equation}
p_u (x) = \sum_k |u_k||x_k| = p_x (u) \quad \text{for } x \in \mathcal{X} \text{ and } u \in \mathcal{X}^\times.
\end{equation}

Especially, $x^{(n)}$ converges to $x$ in $\phi$, that is, $p_u (x^{(n)} - x) \to 0$ as $n \to \infty$ for each $u \in \omega$ if and only if for any $\epsilon > 0$, there exist $L$ and $n_0$ such that

\begin{equation}
(i) \quad x^{(n)}_k = x_k = 0 \quad \text{for } k > L \quad \text{when} \quad n \geq n_0, \quad \text{and}
(ii) \quad |x^{(n)}_k - x_k| < \epsilon \quad \text{for } k \leq L \quad \text{when} \quad n \geq n_0.
\end{equation}

Analogously, $u^{(n)}$ converges to $u$ in $\omega$, that is, $p_x (u^{(n)} - u) \to 0$ as $n \to \infty$ for each $x \in \phi$ if and only if for any $\epsilon > 0$ and each $k$, there exists $n_0 = n_0 (\epsilon, k)$ such that

\begin{equation}
|u^{(n)}_k - u_k| < \epsilon \quad \text{when} \quad n \geq n_0.
\end{equation}

Clearly, $\omega$ forms a Fréchet space because the above topology in $\omega$ is equivalent to the one defined by countable seminorms: $\{p_k (u)\}_{k \in \mathbb{N}}$ where $p_k (u) = |u_k|$ for $u = (u_1, u_2, \ldots) = \sum_{j=1}^{\infty} u_j \epsilon_j \in \omega$ with $\epsilon_j = (0, 0, \ldots, 0, 1, 0, \ldots) \in \omega$.

Now, we define the isomorphism (diadic-decomposition) from $\mathcal{I}$ onto $\mathbb{N}$ defined by

\begin{equation}
r : \mathcal{I} \ni I = (i_k) \to r (I) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} 2^k i_k \in \mathbb{N} \quad \text{where} \quad i_k = 0 \text{ or } 1.
\end{equation}

Using $r (I)$ in (2.1.7), we define a map

\[ T : \sigma^I \to e_{r (I)} \quad \text{for} \quad I = (i_k) \in \mathcal{I}. \]

Extending this map linearly, we put

\begin{equation}
T (X) = \sum_{|I| \leq j} x_r (I) e_{r (I)} \in \omega \quad \text{for} \quad X = \sum_{|I| \leq j} X_I \sigma^I \in \epsilon^{|j|}.
\end{equation}

More explicitly, we have the following first few terms:

\[ \sum x_r (I) e_{r (I)} = (X_{(0,0,0,\ldots)}, X_{(1,0,0,\ldots)}, X_{(0,1,0,\ldots)}, X_{(1,1,0,\ldots)}, X_{(0,0,1,\ldots)}, X_{(1,0,1,\ldots)}, X_{(0,1,1,\ldots)}, \ldots). \]

Then, since $T (\epsilon^{|j|})$ and $T (\epsilon^{|k|})$ are disjoint sets in $\omega$ if $j \neq k$, we have

\begin{equation}
\sum_{j=0}^{\infty} T (\epsilon^{|j|}) = \omega.
\end{equation}
Therefore, it is reasonable to write as in (2.1.2) and more precisely,
\begin{equation}
\mathcal{C} = \bigoplus_{j=0}^{\infty} \mathcal{C}[j], \quad \text{that is,} \quad X = \sum_{j=0}^{\infty} X[j] \quad \text{with} \quad X[j] = \sum_{|I|=j} X_I \sigma^I.
\end{equation}

Here, \(X[j]\) is called the \(j\)-th degree component of \(X \in \mathcal{C}\). By definition, we get
\begin{equation}
\left\{ \begin{array}{ll}
\mathcal{C}^{(j)} \subset \mathcal{C}^{(k)} & \text{for } j \leq k, \\
\mathcal{C} = \sum_{j=0}^{\infty} \mathcal{C}[j] & \text{with } \cap_{j=0}^{\infty} \mathcal{C}^{(j)} = \mathcal{C},
\end{array} \right.
\end{equation}
\begin{equation}
\mathcal{C}[j] \cdot \mathcal{C}[k] \subset \mathcal{C}[j+k] \quad \text{and} \quad \mathcal{C}^{(j)} \cdot \mathcal{C}^{(k)} \subset \mathcal{C}^{(j+k)}.
\end{equation}

Remark 2.1.3. The second relation with \(\mathcal{C}^{(+)}\) in (2.1.12) also holds for the Clifford algebras but the first one with \(\mathcal{C}^{(-)}\) is specific to the Grassmann algebras satisfying (2.1.11). Here, the Clifford relation for \(\{e_j\}\) is defined by
\begin{equation}
e_i e_j + e_j e_i = 2\delta_{ij}\mathbb{1} \quad \text{for any} \quad i, j = 1, 2, \cdots .
\end{equation}

Typical examples, though not countably many but finitely many elements, are the 2×2-Pauli matrices \(e_j = \{\sigma_j\}_{j=1,2,3}\) and the 4×4-Dirac matrices \(\{e_j\}_{j=0,1,2,3} = \{\beta, \alpha_j\}\).

2.1.3. Topology. We introduce the weakest topology in \(\mathcal{C}\) which makes the map \(T\) continuous from \(\mathcal{C}\) to \(\omega\), that is, \(X = \sum_{I \in \mathcal{I}} X_I \sigma^I \rightarrow 0\) in \(\mathcal{C}\) if and only if \(\text{proj}_I(X) \rightarrow 0\) for each \(I \in \mathcal{I}\) with \(\text{proj}_I(X) = X_I\); it is equivalent to the metric \(\text{dist}(X, Y) = \text{dist}(X - Y)\) defined by
\begin{equation}
\text{dist}(X) = \frac{1}{2^{r(I)}} \frac{1}{1 + |\text{proj}_I(X)|} \quad \text{for any} \quad X \in \mathcal{C}.
\end{equation}

For example, \(X^{(\ell)} = f(\ell)\sigma_1 \cdots \sigma_\ell \rightarrow 0\) in \(\mathcal{C}\) even if \(f(\ell) \rightarrow \infty\) because \(\text{dist}(X^{(\ell)}) \leq 2^{-2^\ell+1}\).

Comparison 2.1.1. The sequence space \(\omega\) is regarded as a formal power series ring of an interminate element \(X\) (see, for example p.25 or p.91 of F. Treves [126]). That is,
\begin{equation}
\mathbb{C}[[X]] = \{ u = u(X) = \sum_{n=0}^{\infty} u_n X^n \mid u_n \in \mathbb{C} \} \cong \{ u = (u_0, u_1, \cdots, u_n, \cdots) \mid u_n \in \mathbb{C} \}.
\end{equation}

Introducing “standard” algebraic operations and putting a fundamental neighbourhood system as
\begin{equation}
V_{m,n} = \{ u = u(X) = \sum_{p=0}^{\infty} u_p X^p \mid u_p \in \mathbb{C}, \ |u_p| \leq \frac{1}{m} \ for \ any \ p \leq n \},
\end{equation}
we may define a Fréchet topology on it.

2.1.4. Algebraic operations – addition and product. For any \(X, Y \in \mathcal{C}\), we define
\begin{equation}
X + Y = \sum_{j=0}^{\infty} (X + Y)[j] \quad \text{with} \quad (X + Y)[j] = X[j] + Y[j] \quad \text{for} \ j \geq 0
\end{equation}
and
\begin{equation}
XY = \sum_{j=0}^{\infty} (XY)[j] \quad \text{where} \quad (XY)[j] = \sum_{k=0}^{j} X[j-k]Y[k] = \sum_{|I|=j} (XY)_I \sigma^I.
\end{equation}
Here, \((XY)_I = \sum_{J+K}(-1)^{\tau(I;J,K)}X_J Y_K \in \mathbb{C}\) is well-defined because for any set \(I \in \mathcal{I}\), there exist only finitely many decompositions by sets \(J, K\) satisfying \(I = J \cup K\) (i.e. \(I = J \cup K, \ J \cap K = \emptyset\)). Here, the indeces \(\tau(I;J,K)\), or more generally \(\tau(I;J_1, \ldots, J_k)\) are defined by

\[
(2.1.17) \quad (-1)^{\tau(I;J_1,\ldots,J_k)} \sigma^{J_1} \cdots \sigma^{J_k} = \sigma^I \quad \text{with} \quad I = J_1 \cup J_2 \cup \cdots \cup J_k.
\]

But for notational simplicity, we will use \((-1)^{\tau(s)}\) without specifying the decomposition if there occurs no confusion.

**Exercise 2.1.1.** Prove that for sets \(J, K\) satisfying \(I = J + K\),

\[
(-1)^{|I|} (-1)^{\tau(I;J,K)} = (-1)^{\tau(I;K,J)}.
\]

Moreover, we get

**Lemma 2.1.1.** The product defined by \((2.1.6)\) is continuous from \(\mathcal{C} \times \mathcal{C} \to \mathcal{C}\).

**Proof.** It is simple by noting that there exist \(2^{|I|}\) elements \(J \in \mathcal{I}\) satisfying \(J \subset I\) and that

\[
| (XY)_I | \leq \sum_{I=J+K} |X_J||Y_K| \leq 2^{|I|}(\max_{J \subset I} |X_J|)(\max_{K \subset I} |Y_K|) \quad \text{for any} \ X, Y \in \mathcal{C}. \quad \square
\]

**Proof of Proposition (2.1.4).** Clearly, we get

\[
\begin{align*}
X(YZ) &= (XY)Z \quad \text{(associativity),} \\
X(Y+Z) &= XY + XZ \quad \text{(distributivity).}
\end{align*}
\]

Other properties have been proved. \(\square\)

**Remark 2.1.4.** We may consider that an element of \(X \in \mathcal{C}\) stands for the ‘state’ such that the position labeled by \(\sigma^I\) is occupied by \(X_I \in \mathbb{C}\). In other word, considering \(\{\sigma_i\}\) as the countable indeterminate letters, it seems reasonable to regard \(\mathcal{C}\) as the set of certain formal power series \(\mathcal{C}_{\text{power series}}\) with simple topology. Therefore, it is permitted to reorder the terms freely under ‘summation sign’. That is, the summation \(\sum_{I \in \mathcal{I}} X_I \tau(I)\) is ‘unconditionally (though not absolutely) convergent’ \(^1\) and so is \(\sum_{I \in \mathcal{I}} X_I \sigma^I\). We use such a big space \(\mathcal{C}\) with rather weak topology because this algebra is considered as the ambient space for reordering the places. We feel such a big ambient space will be prefarable and tractable for our future use.

**Remark 2.1.5.** (1) As \(\{\mathcal{C}^{(j)}\}\) forms a filter by \((2.1.11)\) and \((2.1.12)\), it gives a 0-neighbourhood base of the linear topology of \(\mathcal{C}\) which is equivalent to the above one defined by \((2.1.6)\). (See G. Köthe \(\textit{85}\) for the linear topology of vector spaces.)

(2) We may introduce a stronger topology in \(\mathcal{C}\) called the topology by degree, that is, \(X^{(n)} \rightarrow X\) in \(\mathcal{C}\) means that

(i) there exists \(\ell \geq 0\) such that \(X_I^{(n)} = X_I = 0\) for any \(n\) and \(I\) when \(|I| > \ell\) and

(ii) \(|X_I^{(n)} - X_I| \to 0\) as \(n \to \infty\) when \(|I| \leq \ell\).

\(^1\)with the special property that same letter appears only once in each monomials

\(^2\)diverting the terminology of the basis problem in the Banach spaces
2.1.5. The supernumber. The set $\mathcal{C}$ defined by (2.1.2) is called the (complex) supernumber algebra over $\mathbb{C}$ and any element $X$ of $\mathcal{C}$ is called (complex) supernumber.

**Parity:** We introduce the parity in $\mathcal{C}$ by setting

$$p(X) = \begin{cases} 0 & \text{if } X = \sum_{I \in \mathbb{Z}, |I|=\text{ev}} X_I \sigma^I, \\ 1 & \text{if } X = \sum_{I \in \mathbb{Z}, |I|=\text{od}} X_I \sigma^I. \end{cases}$$

(2.1.18) $X \in \mathcal{C}$ is called homogeneous if it satisfies $p(X) = 0$ or $= 1$. We put also

$$\mathcal{C}_{\text{ev}} = \bigoplus_{j=0}^{\infty} \mathcal{C}_{[2j]} = \{ X \in \mathcal{C} \mid p(X) = 0 \},$$

(2.1.19) $$\mathcal{C}_{\text{od}} = \bigoplus_{j=0}^{\infty} \mathcal{C}_{[2j+1]} = \{ X \in \mathcal{C} \mid p(X) = 1 \},$$

$$\mathcal{C} \cong \mathcal{C}_{\text{ev}} \oplus \mathcal{C}_{\text{od}} \cong \mathcal{C}_{\text{ev}} \times \mathcal{C}_{\text{od}}.$$ Moreover, it splits into its even and odd parts, called (complex) even number and (complex) odd number, respectively:

(2.1.20) $$X = X_\text{ev} + X_\text{od} = \sum_{|I|=\text{ev}} X_I \sigma^I + \sum_{|J|=\text{od}} X_J \sigma^J = \sum_{j=\text{ev}} X_j + \sum_{j=\text{od}} X_j.$$

Using (2.1.20), we decompose

(2.1.21) $$X = X_B + X_S$$

where $X_S = \sum_{1 \leq j < \infty} X_j$ and $X_B = X_0 = X^{[0]}$ and the number $X_B$ is called the body (part) of $X$ and the remainder $X_S$ is called the soul (part) of $X$, respectively. We define the map $\pi_B$ from $\mathcal{C}$ to $\mathbb{C}$ by $\pi_B(X) = X_B$, called the body projection (or called the augmentation map).

**Remark 2.1.6 (Important!** $\mathcal{C}$ does not form a field because $X^2 = 0$ for any $X \in \mathcal{C}_{\text{od}}$. But, it is easily proved that

(i) if $X$ satisfies $XY = 0$ for any $Y \in \mathcal{C}_{\text{od}}$, then $X = 0$, and

(ii) the decomposition of $X$ with respect to degree in (2.1.10) is unique.

These properties are shared only if the number of Grassmann generators is infinite. For example, if the number of Grassmann generators is finite, say $n$, then the number $\sigma_1 \sigma_2 \cdots \sigma_n$, which is not zero, is recognized 0 for the multiplication of any odd number generated by $\{\sigma_j\}_{j=1}^{n}$.

**Lemma 2.1.2 (the invertible elements).** Let $X \in \mathcal{C}$ with $X_B \neq 0$. Then there exists a unique element $Y \in \mathcal{C}$ such that $XY = 1 = YX$.

**Proof.** In fact, decomposing $X = X_B + X_S$ and $Y = Y_B + Y_S$, we should have

$$X_B Y_B = 1, \quad X_B Y_S + X_S Y_B + X_S Y_S = 0.$$ Therefore, putting $X_S = \sum_{|I|>0} X_I \sigma^I$ and $Y_S = \sum_{|J|>0} Y_J \sigma^J$ and noting that $\sigma^I \sigma^J = (-1)^{\tau(K, I, J)} \sigma^K$ for $K = I + J$, we have

$$Y_B = X_B^{-1}, \quad Y_K = -X_B^{-1} \sum_{K=I+J} (-1)^{\tau(K, I, J)} X_I Y_J.$$
For example,

for $|K| = 1$, then $Y_K = -X_B^{-1}X_KY_B, \cdots$

for $|K| = \ell$, then $Y_K = -X_B^{-1}\sum_{K=I+J} (-1)^{\tau(K,I,J)}X_IY_J$.

If $X_B = 0$, there exists no $Y$ satisfying $XY = 1$ or $YX = 1$. \hfill \square

Now, we define our \textit{(real) supernumber algebra} by

\[ \mathcal{R} = \pi^{-1}_B(\mathbb{R}) \cap \mathcal{C} = \left\{ X = \sum_{I \in I} X_I\sigma^I \mid X_B \in \mathbb{R} \text{ and } X_I \in \mathbb{C} \text{ for } |I| \neq 0 \right\}. \]

Defining as same as before, we have

\[ \mathcal{R} = \mathcal{R}_{\text{ev}} \oplus \mathcal{R}_{\text{od}}, \quad \mathcal{R} = \oplus_{j = 0}^{\infty} \mathcal{R}^{(j)}. \]

Analogous to $\mathcal{C}$, we put

\[ \mathcal{R} = \{ X \in \mathcal{C} \mid \pi_B X \in \mathbb{R} \}, \quad \mathcal{R}^{(j)} = \mathcal{R} \cap \mathcal{C}^{(j)}, \]

\[ \mathcal{R}_{\text{ev}} = \mathcal{R} \cap \mathcal{C}_{\text{ev}}, \quad \mathcal{R}_{\text{od}} = \mathcal{R} \cap \mathcal{C}_{\text{od}} = \mathcal{C}_{\text{od}}, \]

\[ \mathcal{R} \cong \mathcal{R}_{\text{ev}} \oplus \mathcal{R}_{\text{od}} \cong \mathcal{R}_{\text{ev}} \times \mathcal{R}_{\text{od}}. \]

Here, we introduced the body (projection) map $\pi_B$ by $\pi_B X = \text{proj}_B(X) = X_0 = X_B$.

$\mathcal{R}^{(j)}$ and other terminologies are analogously introduced.

\textbf{2.1.6. Conjugation.} We define the operation \textit{"{c}omplex} conjugation, denoted by $\overline{\cdot}$ as follows: Denoting the complex conjugation of $X_I \in \mathbb{C}$ by $X_I^\dagger$ and defining $\sigma^I = \sigma_{i_n}^1 \cdots \sigma_1^{i_n}$ for $I = (i_1, \ldots, i_n)$, we put

\[ \overline{X} = \sum_{I \in I} X_I\sigma^I = \sum_{I \in I} (-1)^{\frac{|I|(|I|+1)}{2}}X_I\sigma^I. \]

Then,

\textbf{Lemma 2.1.3.} For $X, Y \in \mathcal{C}$ and $\lambda \in \mathbb{C}$, we have

\[ \overline{\overline{X}} = X, \quad \overline{XY} = \overline{Y} \overline{X}, \quad \overline{\lambda X} = \overline{\lambda} \overline{X}. \]

\textit{Proof.} To prove the second equality, we remark $\overline{\sigma^I\sigma^J} = \overline{\sigma^J\sigma^I}$. In fact,

\[ \overline{\sigma^I\sigma^J} = (-1)^{\tau(K,L,M)}\sigma^K = (-1)^{\tau(K,L,M)}(-1)^{\frac{|K||L||M|}{2}}\sigma^K, \]

\[ \overline{\sigma^J\sigma^I} = (-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{\frac{|J|(|J|+1)}{2}}\sigma^J\sigma^I = (-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{\frac{|J|(|J|+1)}{2}}(-1)^{\tau(K,L,M)}\sigma^K, \]

\[ (-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{\tau(K,L,M)} = (-1)^{\tau(K,L,M)}. \]

Therefore, we get the desired result. \hfill \square

Moreover, we have, if $K = I + J$,

\[ (-1)^{\tau(K,L,M)}(-1)^{\frac{|K||L||M|}{2}} = (-1)^{\frac{|I|(|I|+1)}{2}}(-1)^{\frac{|J|(|J|+1)}{2}}(-1)^{\tau(K,L,M)}. \]

\textbf{Remark 2.1.7.} We may introduce \textit{"{r}eal" as} $\overline{X} = X$ for $X \in \mathcal{C}$, or from purely aethetical point of view, the set of \textit{"{reals}} may be defined by

\[ \mathcal{R}^\mathbb{R} = \{ X = \sum_{I \in I} X_I\sigma^I \mid X_I \in \mathbb{R} \}, \]
but we don’t use this “real” in the sequel. Because the analysis is really done for the body part and the soul part is used not only for reordering the places but also “imaginary”, therefore, we imagine that the set
\[ R_F = \left\{ x = \sum_{I \in I} x_I \sigma^I \mid x_B \in \mathbb{R} \text{ and } x_I \in F \right\} \]
would be more natural as our “supernumber algebra”. Here, \( F \) should be an associative algebra such that we may define seminorms analogously as before. This point of view will be discussed if necessity occurs.

**Remark 2.1.8.** There is another possible way of defining the conjugation: By Hahn-Banach extension theorem, we may define \( \overline{\sigma}_j \) as a linear mapping from \( \mathcal{C} \) to \( \mathbb{C} \) such that \( \langle \overline{\sigma}_j, \sigma_k \rangle = \delta_{jk} \), and by this, we may introduce the duality \( \langle \cdot, \cdot \rangle \) between \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) which is the Grassmann algebra generated by \( \{ \overline{\sigma}_j \} \), and whose Fréchet topology is compatible with the duality above. In this case, putting \( \overline{\sigma}^T = \overline{\sigma}_{i_n} \cdots \overline{\sigma}_{i_1} \) for \( I = (i_1, \cdots, i_n) \) and
\[ X^* = \sum_{I \in I} X_I \sigma^I = \sum_{I \in I} (-1) \frac{|I|!}{2} X_I \sigma^I, \]
we have also \( \overline{\mathcal{F}}_m = \mathcal{F}_m \).

### 2.2. Superspace

**Definition 2.2.1.** The super Euclidean space or (real) superspace \( \mathbb{R}^{m|n} \) of dimension \( m|n \) is defined by
\[ \mathbb{R}^{m|n} = \mathbb{R}^m_{ev} \times \mathbb{R}^n_{od} \ni X = (x, \theta), \]
where \( x = (x_1, \cdots, x_m) \) and \( \theta = (\theta_1, \cdots, \theta_n) \) with \( x_j \in \mathbb{R}_{ev}, \theta_s \in \mathbb{R}_{od} \).

Notation: In the following, we abbreviate the symbol ‘transposed’ \( (x_1, \cdots, x_m) \) and denote \( x = (x_1, \cdots, x_m) \), etc. unless there occurs confusion.

The topology of \( \mathbb{R}^{m|n} \) is induced from the metric defined by dist\(_{m|n}(X,Y) = \text{dist}_{m|n}(X-Y) \) for \( X,Y \in \mathbb{R}^{m|n} \), where we put
\[ \text{dist}_{m|n}(X) = \sum_{j=1}^m \left( \sum_{I \in I} \frac{1}{2^|I|} \frac{\| \text{proj}_I(x_j) \|}{1 + \| \text{proj}_I(x_j) \|} \right) + \sum_{s=1}^n \left( \sum_{I \in I} \frac{1}{2^|I|} \frac{\| \text{proj}_I(\theta_s) \|}{1 + \| \text{proj}_I(\theta_s) \|} \right). \]

Clearly, dist\(_{1|1}(X) = \text{dist}(X) \) for \( X \in \mathbb{R}^{1|1} \cong \mathbb{R} \subset \mathcal{C} \). Analogously, the complex superspace of dimension \( m|n \) is defined by
\[ \mathcal{C}^{m|n} = \mathcal{C}^m_{ev} \times \mathcal{C}^n_{od}. \]

We generalize the body map \( \pi_B \) from \( \mathbb{R}^{m|n} \) or \( \mathbb{R}^{m|0} \) to \( \mathbb{R}^m \) by \( \pi_B X = \pi_B(x_1, \cdots, \pi_B x_m) \in \mathbb{R}^m \) for \( X = (x, \theta) \in \mathbb{R}^{m|n} \). The (complex) superspace \( \mathcal{C}^{m|n} \) is defined analogously.

**Dual superspace.** We denote the superspace \( \mathbb{R}^{m|n}_X \) by \( \mathbb{R}^{m|n}_X \) whose point is presented by \( X = (x, \theta) = (x_1, \cdots, x_m, \theta_1, \cdots, \theta_n) \). We prepare another superspace \( \mathbb{R}^{m|n}_\Xi \) whose point is denoted by \( \Xi = (\xi, \pi) = (\xi_1, \cdots, \xi_m, \pi_1, \cdots, \pi_n) \), such that they are “dual” each other by
\[ \langle X|\Xi \rangle_{m|n} = \sum_{j=1}^m \langle x_j|\xi_j \rangle + \sum_{k=1}^n \langle \theta_k|\pi_k \rangle \in \mathbb{R}_{ev}. \]
Or, for any \( h \in \mathbb{R}^\times \) and \( k \in \mathbb{C}^\times \), we may put
\[
\langle X|\Xi \rangle_{h|k} = h^{-1} \sum_{j=1}^{m} \langle x_j|\xi_j \rangle + k^{-1} \sum_{k=1}^{n} \langle \theta_k|\pi_k \rangle \in \mathfrak{H}.
\]
We abbreviate above \( \langle \cdot | \cdot \rangle_{m|n} \) or \( \langle \cdot | \cdot \rangle_{h|k} \) by \( \langle \cdot | \cdot \rangle \) unless there occurs confusion.

### 2.3. Rogers’ construction of a countably infinite Grassmann generators

We borrow her construction in A. Rogers [104]. Denote by \( \mathcal{M}_L \) the set of integer sequences given by
\[
\mathcal{M}_L = \{ \mu \mid \mu = (\mu_1, \mu_2, \ldots, \mu_k), \ 1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq L \} \quad \text{and} \quad \mathcal{M}_\infty = \bigcup_{L=1}^{\infty} \mathcal{M}_L.
\]
We regard \( \emptyset \in \mathcal{M}_L \) and for any \( j \in \mathbb{N} \), we put \( (j) \in \mathcal{M}_\infty \). For each \( r \in \mathbb{N} \), we may correspond a member \( \mu \in \mathcal{M}_\infty \) by using
\[
(2.3.1) \quad r = \frac{1}{2}(2^{\mu_1} + 2^{\mu_2} + \cdots + 2^{\mu_k}).
\]
Conversely, for each \( \mu \in \mathcal{M}_\infty \), we define \( e_\mu \) as \( e_\mu = (0, \ldots, 0, 1, 0, \cdots) \) where \( r \) and \( \mu \) are related by \( (2.3.1) \). Then, \( w = \sum_\mu w_\mu e_\mu \). Now, we introduce the multiplication by
\[
(2.3.2) \quad \begin{cases} 
   e_\mu e_\emptyset = e_\emptyset e_\mu = e_\mu & \text{for } \mu \in \mathcal{M}_\infty, \\
   e_i e_j = -e_{(j)} e_{(i)} & \text{for } i, j \in \mathbb{N}, \\
   e_\mu = e_{(\mu_1)} e_{(\mu_2)} \cdots e_{(\mu_k)} \quad \text{where} \quad \mu = (\mu_1, \mu_2, \cdots, \mu_k).
\end{cases}
\]
That is, we identify
\[
\omega \ni w = (w_1, w_2, w_3, w_4, \cdots) = \sum_{j=1}^{r} w_j e_{(j)} \leftrightarrow (w_{(1)}, w_{(2)}, w_{(1,2)}, w_{(3)}, \cdots) = \sum_\mu w_\mu e_\mu
\]
where
\[
e_{(j)} \leftrightarrow \sigma_j, \ e_{(1)} e_{(2)} = e_{(1,2)} \leftrightarrow \sigma_1 \sigma_2 = \sigma^I, \ I_{(1,2)} = (1, 1, 0, \cdots),
\]
\[
e_\mu = e_{(\mu_1)} e_{(\mu_2)} \cdots e_{(\mu_k)} \leftrightarrow \sigma_{\mu_1} \sigma_{\mu_2} \cdots \sigma_{\mu_k} = \sigma^I, \ I_\mu = \left( \underbrace{0, \ldots, 0}_{\mu_1}, 1, 0, \cdots, 0, 1, 0, \cdots \right).
\]
Defining \( \sigma_j = e_{(j)} \), we have a countably infinite Grassmann algebra by \( (2.3.2) \).

In stead of the sequence space \( \omega \), Rogers uses \( \ell^1 \) to construct the real Banach-Grassmann algebra, which is the set of absolutely convergent sequences
\[
\|X\| = \sum_{\mathbf{i} \in \mathbb{Z}} |X_\mathbf{i}| < \infty \quad \text{for } X = \sum_{\mathbf{i} \in \mathbb{Z}} X_\mathbf{i} \sigma^I \text{ with } X_\mathbf{i} \in \mathbb{R}, \text{ such that } \|X\| \|Y\|.
\]

**Proposition 2.3.1 (Roger).** \( \ell^1 \) with the above multiplication forms a Banach-Grassmann algebra with countably infinite generators.

**Remark 2.3.1.** There are many papers treating super manifolds which are based on ground ring with Banach-Grassmann structure (for example, V.S. Vladimirov and I.V. Volovich [129], etc). This phenomenon is rather reasonable because inverse or implicit function theorems hold in Banach space as same as Euclidian case, but not so in general, in Fréchet space. But the condition \( \sum_{\mathbf{i} \in \mathbb{Z}} |X_\mathbf{i}| < \infty \) is too difficult to check in our concrete problem. This has the similarity to
indeterminate coefficients method to solve Cauchy-Kovalevsky theorem and to check its convergence by majorant test.

Remark 2.3.2. Concerning inverse or implicit function theorems in certain Fréchet space, see R. Hamilton [55] or J.T. Schwartz [115] for Nash’s implicit function theorem.
CHAPTER 3

Linear algebra on the superspace

In this chapter, we quote results from F.A. Berezin [9], B.S. deWitt [33] and D.A. Leites [90] with modifications if necessary.

Remark 3.0.3. Almost all papers prefixed “super”, treated the case of finite number of odd variables also with the finite number of Grassmann generators, or rather, they don’t distinguish odd variables and Grassmann generators. But in any way, after slight modification if necessary, algebraic operations not affected with the topology is borrowed from these papers.

3.1. Matrix algebras on the superspace

3.1.1. Super matrices.

Definition 3.1.1. A rectangular array \( M \), whose cells are indexed by pairs consisting of a row number and a column number, is called a supermatrix and denoted by \( M \in \text{Mat}((m|n) \times (r|s) : \mathfrak{C}) \), if it satisfies the following:

1. A \((m + n) \times (r + s)\) matrix \( M \) is decomposed blockwisely as \( M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \) where \( A, B, C \) and \( D \) are \( m \times r, n \times s, m \times s \) and \( n \times r \) matrices with elements in \( \mathfrak{C} \), respectively.

2. One of the following conditions is satisfied: Either
   - \( p(M) = 0 \), that is, \( p(A_{jk}) = 0 = p(B_{uv}) \) and \( p(C_{jv}) = 1 = p(D_{uk}) \) or
   - \( p(M) = 1 \), that is, \( p(A_{jk}) = 1 = p(B_{uv}) \) and \( p(C_{jv}) = 0 = p(D_{uk}) \).

We call \( M \) is even denoted by \( \text{Mat}_{ev}((m|n) \times (r|s) : \mathfrak{C}) \) (resp. odd denoted by \( \text{Mat}_{od}((m|n) \times (r|s) : \mathfrak{C}) \)) if \( p(M) = 0 \) (resp. \( p(M) = 1 \)). Therefore, we have

\[
\text{Mat}((m|n) \times (r|s) : \mathfrak{C}) = \text{Mat}_{ev}((m|n) \times (r|s) : \mathfrak{C}) \oplus \text{Mat}_{od}((m|n) \times (r|s) : \mathfrak{C}).
\]

Moreover, we may decompose \( M \) as \( M = M_B + M_S \) where

\[
M_B = \begin{cases} 
\begin{pmatrix} A_B & 0 \\ 0 & B_B \end{pmatrix} & \text{when } p(M) = 0, \\
\begin{pmatrix} 0 & C_B \\ D_B & 0 \end{pmatrix} & \text{when } p(M) = 1.
\end{cases}
\]

The summation of two matrices in \( \text{Mat}_{ev}((m|n) \times (r|s) : \mathfrak{C}) \) or in \( \text{Mat}_{od}((m|n) \times (r|s) : \mathfrak{C}) \) is defined as usual, but the sum of \( \text{Mat}_{ev}((m|n) \times (r|s) : \mathfrak{C}) \) and \( \text{Mat}_{od}((m|n) \times (r|s) : \mathfrak{C}) \) is not defined except at least one of them being zero matrix.
It is clear that if $M$ is the $(m+n) \times (r+s)$ matrix and $N$ is the $(r+s) \times (p+q)$ matrix, then we may define the product $MN$ and its parity $p(MN)$ as

$$(MN)_{ij} = \sum_k M_{ik}N_{kj}, \quad p(MN) = p(M) + p(N) \mod 2.$$ 

Moreover, we define $\text{Mat} [m|n : \mathbb{C}]$ as the algebra of $(m+n) \times (m+n)$ supermatrices.

### 3.1.2. Matrices as Linear Transformations. By definition of matrix operation to vector, we have

$$\text{Mat}_{\text{ev}}((m|n) \times (r|s) : \mathbb{C}) \ni M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} : \mathbb{R}^{r|s} \to \mathbb{R}^{m|n},$$

$$\text{Mat}_{\text{od}}((m|n) \times (r|s) : \mathbb{C}) \ni M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} : \mathbb{R}^{r|s} \to \mathbb{R}^{m|n},$$

$$\text{Mat}_{\text{od}}((n|m) \times (m|n) : \mathbb{C}) \ni \Lambda_{n,m} = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_m & 0 \end{pmatrix} : \mathbb{R}^m_{\text{od}} \times \mathbb{R}^n_{\text{ev}} \to \mathbb{R}^n_{\text{ev}} \times \mathbb{R}^m_{\text{od}} = \mathbb{R}^{m|n}.$$

For elements $X = (x, \theta) = (x_1, \cdots, x_m, \theta_1, \cdots, \theta_n)$ and $\Xi = (\xi, \pi) = (\xi_1, \cdots, \xi_m, \pi_1, \cdots, \pi_n)$ in $\mathbb{R}^{m|n}$, we define $\overline{X}$ and $\langle X|\Xi \rangle_{m|n}$ as in (2.1.25) and (2.2.4), respectively.

If we introduce the duality between $\mathbb{R}^{m|n}$ as in (2.2.4), we may define the transposed operator as

$$\langle MX|\Xi \rangle_{m|n} = \langle X|tM\Xi \rangle_{r|s} \text{ for any } M \in \text{Mat}_{\text{ev}}((m|n) \times (r|s) : \mathbb{C}),$$

for $X = (x, \theta) \in \mathbb{R}^{r|s}$ and $\Xi = (\xi, \omega) \in \mathbb{R}^{m|n}$. More precisely, we have

$$t^t M = t^t \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \begin{pmatrix} t^tA & t^tD \\ -t^tC & t^tB \end{pmatrix} \text{ and } tt^t M = M.$$

Analogously, defining the duality between $\mathbb{C}^{m|n}$ and $\mathbb{C}^{m|n}$ for $Z = (z, \theta) \in \mathbb{C}^{r|s}$, $Y = (w, \rho) \in \mathbb{C}^{m|n}$ by

$$\langle Z|Y \rangle_{m|n} = \sum_{j=1}^m z_j w_j + \sum_{k=1}^n \theta_k \rho_k, \text{ or } \sum_{j=1}^m z_j w_j + \sum_{k=1}^n \theta_k \rho_k,$$

we denote the conjugate (or adjoint) matrix of $A$ by $A^* = t^tA = t^tA$ etc. Then, we may introduce $M^*$, the conjugate (or adjoint) of matrix $M$, by

$$\langle MZ|Y \rangle_{m|n} = \langle Z|M^*Y \rangle_{r|s}.$$  

Therefore, we have

$$M^* = \begin{pmatrix} A & C \\ D & B \end{pmatrix}^* = \begin{pmatrix} A^* & D^* \\ C^* & B^* \end{pmatrix} \quad \text{and} \quad M^{**} = M.$$  

**Lemma 3.1.1.** For $M \in \text{Mat} ((m|n) \times (r|s) : \mathbb{C})$ and $N \in \text{Mat} ((r|s) \times (p|q) : \mathbb{C})$, we have

$$(MN)^t = N^t M^t, \quad (MN)^* = N^* M^*, \quad (M^t)^t = M \Lambda M^\Lambda, \text{ where } \Lambda = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}.$$  

If $M \in \text{Mat} [m|n : \mathbb{C}]$ is even, denoted by $M \in \text{Mat}_{\text{ev}}[m|n : \mathbb{C}]$, then $M$ acts on $\mathbb{R}^{m|n}$ linearly. Denoting this by $T_M$, we call it super linear transformation on $\mathbb{R}^{m|n}$ and $M$ is called the representative matrix of $T_M$. 
Proposition 3.1.1. Let $M \in \text{Mat}_{ev}[m|n : \mathcal{C}]$ and assume $\det M_B \neq 0$. Then, for given $Y \in \mathcal{R}^{m|n}$, 

\[(3.1.1) \quad T_M X = Y \]

has the unique solution $X \in \mathcal{R}^{m|n}$, which is denoted by $X = M^{-1}Y$.

Proof. Since $M_B$ has the inverse matrix $M_B^{-1}$, (3.1.1) is reduced to 

\[X + N_S X = Y', \quad Y' = M_B^{-1}Y\]

where $N_S = M_B^{-1}M_S$. Remark that $N_S X^{[j]} ∈ \sum_{k≥j+1} C^{[k]}$ for $j ≥ 0$. Decomposing by degree, we get 

\[X^{[j]} = Y'^{[j]} - (N_S X^{(j-1)})^{[j]} \quad \text{for} \quad j = 1, 2, \ldots.\]

As $X^{(0)} = X^{[0]} = Y'^{(0)}$, we get $X^{[j]}$ from $X^{(j-1)}$ for $j ≥ 1$ by induction. \(\square\)

Exercise 3.1.1. How about $M \in \text{Mat}_{od}((m|n) \times (n|m) : \mathcal{C})$ ?

Definition 3.1.2. $M \in \text{Mat}_{ev}[m|n : \mathcal{C}]$ is called invertible or non-singular if $M_B$ is invertible, i.e. $\det A_B \cdot \det B_B \neq 0$, and denoted by $M \in \text{GL}_{ev}[m|n : \mathcal{C}]$.

3.2. Supertrace, superdeterminant

3.2.1. Supertrace.

Lemma 3.2.1. Let $V, W$ be two rectangular matrices with odd elements, $m \times n$, $n \times m$, respectively. We have 

1. $\text{tr} (VW)^k = -\text{tr} (WV)^k$ for any $k = 1, 2, \cdots$.
2. $\det(\mathbb{I}_n + WV) = \det(\mathbb{I}_n + VW)^{-1}$.

Proof. Let $V = (v_{ij})$, $W = (w_{jk})$ with $v_{ij}, w_{jk} ∈ \mathcal{C}_{od}$. 

\[\text{tr} (VW)^k = \sum v_{ij_1} w_{j_1j_2} v_{j_2j_3} \cdots v_{j_{k-1}j_k} w_{j_ki} = -\text{tr} (WV)^k.\]

Using this, we have $\text{tr} ((WV)^{\ell-1}W) = -\text{tr} (V(WV)^{\ell-1}W)$ which yields 

\[\log \det(\mathbb{I}_n + WV) = \text{tr} \log (\mathbb{I}_n + WV) = \sum_{\ell} \frac{(-1)^{\ell+1}}{\ell} \text{tr} ((WV)^{\ell-1}W)\]

\[= \sum_{\ell} \frac{(-1)^{\ell+1}}{\ell} [- \text{tr} (V(WV)^{\ell-1}W)] = -\sum_{\ell} \frac{(-1)^{\ell+1}}{\ell} \text{tr} (WV)^\ell\]

\[= -\log \det(\mathbb{I}_m + VW). \quad \square\]

Comparison 3.2.1. If $A = (a_{ij}) ∈ \text{Mat} (m \times n : \mathcal{C}_{ev})$, $B = (b_{jk}) ∈ \text{Mat} (n \times m : \mathcal{C}_{ev})$, then we have 

1. $\text{tr} (AB)^k = \text{tr} (BA)^k$,
2. $\det(\mathbb{I}_m + AB) = \det(\mathbb{I}_n + BA)$.

Definition 3.2.1. Let $M = [A \quad C \quad D \quad B] ∈ \text{Mat} [m|n : \mathcal{C}]$. We define the supertrace of $M$ by 

\[\text{str} M = \text{tr} A - (-1)^{p(M)} \text{tr} B.\]
Using Lemma 3.2.1, we get readily

**Proposition 3.2.1.** (a) Let \( M, N \in \text{Mat}[m \times n : \mathbb{C}] \) such that \( p(M) + p(N) \equiv 0 \mod 2 \). Then, we have

\[
\text{str}(M + N) = \text{str} M + \text{str} N.
\]

(b) \( M \) is a matrix of size \((m + n) \times (r + s)\) and \( N \) is a matrix of size \((r + s) \times (m + n)\). Then,

\[
\text{str}(MN) = (-1)^{p(M)p(N)} \text{str}(NM).
\]

**Definition 3.2.2.** Let \( B = (B_{jk}) \) be \((\ell \times \ell)\)-matrix with elements in \( \mathcal{C}_{ev} \), denoted by, \( B \in \text{Mat}[\ell : \mathcal{C}_{ev}] \). As \( \mathcal{C}_{ev} \) is a commutative ring, we may define \( \det B \) as usual:

\[
\det B = \sum_{\rho \in \wp_\ell} \text{sgn}(\rho) B_{1\rho(1)} \cdots B_{\ell \rho(\ell)}.
\]

Then, we have, as ordinary case,

\[
(3.2.1) \quad \det(AB) = \det A \cdot \det B, \quad \det(\exp A) = \exp(\text{tr} A) \quad \text{for } A, B \in \text{Mat}[\ell : \mathcal{C}_{ev}].
\]

**3.2.2. Super determinant.**

**Definition 3.2.3.** Let \( M \) be a supermatrix. When \( \det B \neq 0 \), we put

\[
\text{sdet} M = \det(AB) \cdot (\det B)^{-1}
\]

and call it superdeterminant or Berezinian of \( M \).

**Comparison 3.2.2.** Let

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M = \begin{pmatrix} I_m & 0 \\ -A_{22}^{-1}A_{21} & I_n \end{pmatrix},
\]

be block matrices of even elements. Then, we have

\[
\det A = \det(AM) = \det \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \cdot \det A_{22}.
\]

**Corollary 3.2.1.** When \( \det B \neq 0 \) and \( \text{sdet} M \neq 0 \), then \( \det A_B \neq 0 \).

**Exercise 3.2.1.** Prove the above corollary.

**Remark 3.2.1.** It seems meaningful to cite here the result of F.J. Dyson \[38\].

**Theorem 3.2.1 (Dyson).** Let \( R \) be a ring with a unit element and without divisors of zero. Assume that on the matrix ring \( A \) with \( n > 1 \), a mapping \( D \) exists satisfying the following axioms:

Axiom 1. For any \( a \in A \), \( D(a) = 0 \) if and only if there is a non-zero \( w \in W \) with \( aw = 0 \). Here, \( W \) is the set of single-column matrices with elements in \( R \).

Axiom 2. \( D(ab) = D(ab) \).

Axiom 3. Let the elements of \( a \) be \( a_{ij} \), \( i, j = 1, \cdots, n \), and similarly for \( b \) and \( c \). If for some row-index \( k \) we have

\[
\begin{cases} 
    a_{ij} = b_{ij} = c_{ij}, & i \neq k \\
    a_{ij} + b_{ij} = c_{ij}, & i = k,
\end{cases}
\]

then

\[
D(a) + D(b) = D(c).
\]

Then, \( R \) is commutative.
This theorem states that if the elements of matrix are taken from non-commutative algebra, then it is impossible to define the determinant having above three properties. But, he claims a certain ‘determinant’ is defined for some class of matrices with elements in ‘quarternion’ requiring only one or two properties above (By the way, Moore’s point of view, is reconsidered significantly in that paper). In fact, we may define “superdeterminant” for “supermatrix” as above which satisfies the properties below.

Now, we continue to study the properties of super-determinant defined in the previous lecture.

Following decomposition of a even supermatrix $M$ will be useful:

\[
\begin{bmatrix}
A & C \\
D & B
\end{bmatrix} = \begin{bmatrix}
\mathbb{I}_m & CB^{-1} \\
0 & \mathbb{I}_n
\end{bmatrix} \begin{bmatrix}
A - CB^{-1}D & 0 \\
0 & B
\end{bmatrix} \begin{bmatrix}
\mathbb{I}_m & 0 \\
B^{-1}D & \mathbb{I}_n
\end{bmatrix} \text{ if } \det B \neq 0,
\]

\[
= \begin{bmatrix}
\mathbb{I}_m & 0 \\
DA^{-1} & \mathbb{I}_n
\end{bmatrix} \begin{bmatrix}
A & 0 \\
0 & B - DA^{-1}C
\end{bmatrix} \begin{bmatrix}
\mathbb{I}_m & A^{-1}C \\
0 & \mathbb{I}_n
\end{bmatrix} \text{ if } \det A \neq 0.
\]

Moreover, we have

\[
\begin{bmatrix}
\tilde{A} & \tilde{C} \\
\tilde{D} & \tilde{B}
\end{bmatrix} = \begin{bmatrix}
A & C \\
D & B
\end{bmatrix}^{-1} = \begin{bmatrix}
(A - CB^{-1}D)^{-1} & -A^{-1}C(B - DA^{-1}C)^{-1} \\
-B^{-1}D(A - CB^{-1}D)^{-1} & (B - DA^{-1}C)^{-1}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(\mathbb{I}_m - A^{-1}CB^{-1}D)^{-1}A^{-1} & -(\mathbb{I}_m - A^{-1}CB^{-1}D)^{-1}A^{-1}CB^{-1} \\
-(\mathbb{I}_n - B^{-1}DA^{-1}C)^{-1}B^{-1}DA^{-1} & (\mathbb{I}_n - B^{-1}DA^{-1}C)^{-1}B^{-1}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A^{-1}(\mathbb{I}_m - CB^{-1}DA^{-1})^{-1} & -A^{-1}CB^{-1}(\mathbb{I}_n - DA^{-1}CB^{-1})^{-1} \\
-B^{-1}DA^{-1}(\mathbb{I}_m - CB^{-1}DA^{-1})^{-1} & B^{-1}(\mathbb{I}_n - DA^{-1}CB^{-1})^{-1}
\end{bmatrix}
\]

\[
\text{sdet} \begin{bmatrix}
A & C \\
D & B
\end{bmatrix} = (\det A)(\det B)^{-1} \det(\mathbb{I}_m - A^{-1}CB^{-1}D)
\]

\[
= (\det A)(\det B)^{-1} \det(\mathbb{I}_m - CB^{-1}DA^{-1}) = (\det \tilde{A})^{-1}(\det B)^{-1}
\]

\[
= (\det A)(\det B)^{-1} \det(\mathbb{I}_n - B^{-1}DA^{-1}C)
\]

\[
= (\det A)(\det B)^{-1} \det(\mathbb{I}_n - DA^{-1}CB^{-1}) = (\det A)(\det \tilde{B}).
\]

As we have the following

\[
\begin{bmatrix}
A & C \\
D & B
\end{bmatrix} \begin{bmatrix}
A^{-1} & 0 \\
0 & B^{-1}
\end{bmatrix} \begin{bmatrix}
\mathbb{I}_m & 0 \\
0 & -\mathbb{I}_n
\end{bmatrix} \begin{bmatrix}
A & C \\
D & B
\end{bmatrix} \begin{bmatrix}
\mathbb{I}_m & 0 \\
0 & -\mathbb{I}_n
\end{bmatrix} = \begin{bmatrix}
A - CB^{-1}D & 0 \\
0 & B - DA^{-1}C
\end{bmatrix},
\]

we garantee the invertibility of matrices appeared above.

**Lemma 3.2.2.** (1) Let $L \in \text{Mat}_{ev}[\ell : \mathfrak{c}_{ev}]$ such that the product of any two entries of it is zero. Then

\[(\mathbb{I}_\ell + L)^{-1} = \mathbb{I}_\ell - L, \quad \det(\mathbb{I}_\ell + L) = 1 + \text{tr } L.\]

(2) Let $M \in \text{Mat}_{ev}[m|n : \mathfrak{c}]$ such that the product of any two entries of it is zero. Then

\[\text{sdet } (\mathbb{I}_m + n + M) = 1 + \str M.\]

**Proof.** (1) Remarking

\[(\mathbb{I}_\ell + L)^{-1} = \mathbb{I}_\ell - L + L^2 - L^3 + \cdots \quad \text{and} \quad \det(e^L) = e^{\text{tr } L},\]

we get the result readily.

(2) For $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$, satisfying $C(\mathbb{I}_n + B)^{-1}D = 0$ and $\text{tr } A \text{ tr } B = 0$ garanteed by the product
of any two entries of $M$ being zero,
\[
\text{sdet}(I_{m+n} + M) = \det(I_m + A - C(I_n + B)^{-1}D) \cdot \det(I_n + B)^{-1}
\]
\[
= \det(I_m + A) \cdot \det(I_n - B) = 1 + \text{tr} A - \text{tr} B = 1 + \text{str } M. \quad \square
\]

**Competition 3.2.3.** If $A = (a_{ij}) \in \text{Mat}(m \times n : \mathcal{C}_{ev})$, $B = (b_{jk}) \in \text{Mat}(n \times m : \mathcal{C}_{ev})$, then we have
(1) $\text{tr}(AB)^k = \text{tr}(BA)^k$,
(2) $\det(I_m + AB) = \det(I_n + BA)$.

**Definition 3.2.4.** Let $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix} \in \text{Mat}[m|n : \mathcal{C}]$. We define the supertrace of $M$ by
\[
\text{str } M = \text{tr } A - (-1)^{p(M)} \text{tr } B.
\]

Using Lemma 3.2.1, we get readily

**Proposition 3.2.2.** (a) Let $M, N \in \text{Mat}[m|n : \mathcal{C}]$ such that $p(M) + p(N) \equiv 0 \mod 2$. Then, we have
\[
\text{str } (M + N) = \text{str } M + \text{str } N.
\]
(b) $M$ is a matrix of size $(m+n) \times (r+s)$ and $N$ is a matrix of size $(r+s) \times (m+n)$. Then,
\[
\text{str } (MN) = (-1)^{p(M)p(N)} \text{str } (NM).
\]

**Definition 3.2.5.** Let $B = (B_{jk})$ be $(\ell \times \ell)$-matrix with elements in $\mathcal{C}_{ev}$, denoted by, $B \in \text{Mat}[\ell : \mathcal{C}_{ev}]$. As $\mathcal{C}_{ev}$ is a commutative ring, we may define $\det B$ as usual:
\[
\det B = \sum_{\rho \in \wp} \text{sgn } (\rho) B_{1 \rho(1)} \cdots B_{\ell \rho(\ell)}.
\]

Then, we have, as ordinary case,
\[
(3.2.5) \quad \det(AB) = \det A \cdot \det B, \quad \det(\exp A) = \exp(\text{tr } A) \quad \text{for } A, B \in \text{Mat}[\ell : \mathcal{C}_{ev}].
\]
Moreover, we have

**Comparison 3.2.4.** Let
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M = \begin{pmatrix} I_m & 0 \\ -A_{22}^{-1}A_{21} & I_n \end{pmatrix},
\]
be block matrices of even elements. Then, we have
\[
\det A = \det AM = \det \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \cdot \det A_{22}.
\]

Now, we define

**Definition 3.2.6.** Let $M$ be a supermatrix. When $\det B_{B} \neq 0$, we put
\[
\text{sdet } M = \det(A - CB^{-1}D) \cdot (\det B)^{-1}
\]
and call it superdeterminant or Berezinian of $M$.

**Corollary 3.2.2.** When $\det B_{B} \neq 0$ and $\text{sdet } M \neq 0$, then $\det A_{B} \neq 0$.

**Exercise 3.2.2.** Prove the above corollary.
Theorem 3.2.2. Let $M, N \in \text{Mat}[m|n : \mathbb{C}]$.

(1) If $M$ is invertible, then we have $\text{sdet } M \neq 0$. Moreover, if $A$ is nonsingular, then

$$\text{sdet } M^{-1} = (\text{det } A)^{-1} \cdot \text{det}(B - DA^{-1}C).$$

(2) Multiplicativity of sdet:

$$\text{sdet } (MN) = \text{sdet } M \cdot \text{sdet } N.$$

(3) str and sdet are matrix invariants. That is, if $N$ is invertible, then

$$\text{str } M = (-1)^{p(M)+p(N)} \text{str } (NMN^{-1}), \quad \text{sdet } M = \text{sdet } (NMN^{-1}).$$

Proof (due to Leites [90]). (1) By

$$\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ DA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{bmatrix} \begin{bmatrix} I_m & A^{-1}C \\ 0 & I_n \end{bmatrix},$$

if $\det B \neq 0$, we have readily by definition, $\text{sdet } M = \det A(\text{det}(B - DA^{-1}C))^{-1}$, which yields (3.2.6).

(2) [Step 1]: Let $\mathcal{G}_+, \mathcal{G}_0$ and $\mathcal{G}_-$ be subgroups of $\text{GL}[m|n : \mathbb{C}]$, given by

$$\mathcal{G}_+ = \left\{ \begin{bmatrix} I_m & C' \\ 0 & I_n \end{bmatrix} \right\}, \quad \mathcal{G}_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\}, \quad \mathcal{G}_- = \left\{ \begin{bmatrix} I_m & 0 \\ D & I_n \end{bmatrix} \right\}.$$

Then, we have, $M = M_+M_0M_-$ with $M_+ \in \mathcal{G}_+$, $M_0 \in \mathcal{G}_0$ and $M_- \in \mathcal{G}_-$, i.e., for any $M \in \text{GL}[m|n : \mathbb{C}]$,

$$M = \begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} I_m & CB^{-1} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A - CB^{-1}D & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_m & 0 \\ B^{-1}D & I_n \end{bmatrix},$$

if $\det B \neq 0$.

Remarkning that

$$\begin{bmatrix} I_m & C \\ 0 & I_n \end{bmatrix} \times \begin{bmatrix} I_m & C' \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_m & C + C' \\ 0 & I_n \end{bmatrix},$$

we introduce the notion of elementary matrices having the form

$$\begin{bmatrix} I_m & E \\ 0 & I_n \end{bmatrix}$$

where $E$ has only one non-zero entry.

[Step 2]: We claim $\text{sdet } (MN) = \text{sdet } M \cdot \text{sdet } N$ whenever $M \in \mathcal{G}_+$ or $M \in \mathcal{G}_0$, and similarly, whenever $N \in \mathcal{G}_0$ or $N \in \mathcal{G}_-$, i.e., for any $M \in \text{GL}[m|n : \mathbb{C}]$.

$$M = \begin{bmatrix} I_m & C' \\ 0 & I_n \end{bmatrix} \in \mathcal{G}_+, \quad N = \begin{bmatrix} A & C \\ D & B \end{bmatrix},$$

we have

$$\text{sdet } (MN) = \text{sdet } \begin{bmatrix} I_m & C' \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & C \\ D & B \end{bmatrix} = \text{sdet } \begin{bmatrix} A + C'D & C + C'B \\ D & B \end{bmatrix} = \det (A + C'D - (C + C'B)B^{-1}D)(\det D)^{-1} = \det (A - CB^{-1}D)(\det D)^{-1} = \text{sdet } M \cdot \text{sdet } N.$$

Exercise 3.2.3. Check other cases analogously.
Since we have by Step 1 and Step 2, we need to prove

\[ N = \begin{bmatrix} I_m & E \\ 0 & I_n \end{bmatrix} \in G_+ . \]

Since we have

\[
\text{sdet} (MN) = \text{sdet} (M_+ (M_0 M_-)) = \text{sdet} M_+ \cdot \text{sdet} (M_0 (M_-)) = \text{sdet} M_0 \cdot \text{sdet} (M_-) ,
\]

\[ \text{sdet} M \cdot \text{sdet} N = \text{sdet} M_0 \cdot \text{sdet} M_- \cdot \text{sdet} N , \]

by Step 1 and Step 2, we need to prove

\[ \text{sdet} (M_- N) = \text{sdet} M_- \cdot \text{sdet} N = 1 \]

when \( N \) is an elementary matrix. By definition,

\[
\text{sdet} \begin{bmatrix} I_m & 0 \\ D & I_n \end{bmatrix} = \text{sdet} \begin{bmatrix} I_m & E \\ D & I_n + DE \end{bmatrix} = \text{det}(1 - E(1 + DE)^{-1}D) \cdot \text{det}(1 + DE)^{-1}.
\]

As \( E \) has only one non-zero entry, the product of any two of the matrices \( E, DE, E(1 + DE)^{-1}D \)

is zero. Applying Lemma, we get, by \((1 + DE)^{-1} = 1 - DE \) and \( E \cdot DE = 0 ,
\]

\[ \text{sdet} (M_- N) = \text{det}(1 - DE) \cdot \text{det}(1 + DE)^{-1} = (1 - \text{tr} DE) (1 + \text{tr} DE)^{-1} . \]

As \( \text{tr} DE = - \text{tr} ED \), we have

\[ \text{sdet} (M_- N) = 1 = \text{sdet} M_- \cdot \text{sdet} N . \]

[Step 4]: Put

\[ G = \left\{ N \in \text{GL}[m|n : \mathbb{R}] \mid \text{sdet} (MN) = \text{sdet} M \cdot \text{sdet} N \quad \text{for any} \ M \in \text{GL}[m|n : \mathbb{R}] \right\} . \]

For \( N_1, N_2 \in G \), we have

\[
\text{sdet} (M \cdot N_1 N_2) = \text{sdet} ((MN_1) N_2) = \text{sdet} (MN_1) \cdot \text{sdet} N_2
\]

\[ = \text{sdet} M \cdot \text{sdet} N_1 \cdot \text{sdet} N_2 = \text{sdet} M \cdot \text{sdet} (N_1 N_2) , \]

which implies \( G \) froms a group. By Steps 2 and 3, \( G \) contains \( G_- \) and \( G_0 \) and all elementary matrices

\( N \in G_- \). By Step1, \( GL[m|n : \mathbb{C}] \) is generated by these matrices, we have \( G = GL[m|n : \mathbb{C}] \), that is,

\( \text{sdet} (MN) = \text{sdet} M \cdot \text{sdet} N . \)

(3) Let \( N, M \) be given. Then, using (3.2.11), we get

\[
\text{str} \ NMN^{-1} = (-1)^{p(N)p(MN^{-1})} \text{str} \ MN^{-1} N = (-1)^{p(N)+p(M)} \text{str} \ M ,
\]

since \( p(MN^{-1}) = p(M) + p(N^{-1}) \) mod 2 and \( 0 = p(NN^{-1}) = p(N) + p(N^{-1}) \) mod 2, we have \( p(N)p(MN^{-1}) = p(N) + p(M) \) mod 2.

Using (3.2.11), we have \( \text{sdet} (MN) = \text{sdet} (NM) \) which implies \( \text{sdet} (NMN^{-1}) = \text{sdet} (N^{-1} NM) = \text{sdet} M . \)

\[ \square \]

**Theorem 3.2.3** (Liouville’s theorem: Theorem 3.5 of [9]). Let \( M(t) \in \text{Mat} [m|n : \mathbb{C}] \) with a

real parameter \( t \). Let \( X(t) \in \text{Mat} [m|n : \mathbb{C}] \) satisfy

\[ \frac{d}{dt} X(t) = M(t) X(t) , \quad X(0) = \mathbb{I}_{m+n} . \]

Then \( X(t) \in \text{GL}[m|n : \mathbb{C}] \), and

\[ \text{sdet} X(t) = \exp\left\{ \int_0^t ds \text{str} M(s) \right\} . \]
3.2. SUPERTRACE, SUPERDETERMINANT

Proof (with slight modification of Berezin’s proof in [9]). Let \( \dot{X}(t) \) be a solution of
\[
\frac{d}{dt} \dot{X}(t) = -\dot{X}(t)M(t), \quad \dot{X}(0) = \mathbb{I}_{m+n}.
\]
Then, since
\[
\frac{d}{dt}(\dot{X}(t)X(t)) = 0 \quad \text{with} \quad \dot{X}(0)X(0) = \mathbb{I}_{m+n},
\]
we have \( \dot{X}(t)X(t) = \mathbb{I}_{m+n} \) which implies \( X(t) \in \text{GL}[m|n : \mathbb{C}] \).

Let
\[
M(t) = \begin{bmatrix} A(t) & C(t) \\ D(t) & B(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{bmatrix}.
\]
Then, we put \( Y(t) = X_{11}(t) - X_{12}(t)X_{22}^{-1}(t)X_{21}(t) \) and \( Z = X_{22}^{-1}(t) \). Differentiating \( X_{22}^{-1}X_{22} = \mathbb{I}_n \) w.r.t. \( t \) and substituting \( \dot{X}_{22} = DA_{12} + BX_{22} \) which is obtained from (3.2.12), we have,
\[
\frac{d}{dt}Z = -Z(DX_{12}X_{22}^{-1} + B).
\]
Analogously calculating, we get
\[
\frac{d}{dt}Y = (A - X_{12}X_{22}^{-1}D)Y.
\]
As all elements appeared in the above equations are even, we may apply the classical Liouville theorem to have
\[
\frac{d}{dt}\det Y = \text{tr} (A - X_{12}X_{22}^{-1}D)\det Y, \quad \frac{d}{dt}\det Z = -\text{tr} (DX_{12}X_{22}^{-1} + B)\det Z.
\]
Putting \( V = X_{12}X_{22}^{-1} \) and \( W = D \) in Lemma 3.2.1, we get \( \text{tr} (A - X_{12}X_{22}^{-1}D) = \text{tr} (A + DX_{12}X_{22}^{-1}) \), therefore, recalling the definition of super-determinant, we have
\[
\frac{d}{dt}\text{sdet} X = \frac{d}{dt}(\det Y \cdot \det Z) = \text{tr} (A - B) \cdot \det Y \cdot \det Z = \text{str} M \cdot \text{sdet} X \quad \text{with} \quad \text{sdet} X(0) = 1.
\]
This yields the desired result after integrating w.r.t. \( t \). \( \Box \)

**Corollary 3.2.3.** For \( M, N \in \text{Mat}_{ev}[m|n : \mathbb{C}] \) we have
\[
\text{sdet} (MN) = \text{sdet} M \cdot \text{sdet} N,
\]
(3.2.14)
\[
\exp(\text{str} M) = \text{sdet} (\exp M).
\]

**Proof.** (1) Put \( X(t) = (1 - t)\mathbb{I}_{m+n} + tM \) and \( Y(t) = (1 - t)\mathbb{I}_{m+n} + tN \). As \( X(t) \) and \( Y(t) \) are differentiable in \( t \) and invertible except at most one \( t \), we my define
\[
A(t) = \frac{dX(t)}{dt}X(t)^{-1}, \quad B(t) = \frac{dX(t)}{dt}Y(t)^{-1}.
\]
Then
\[
\frac{d}{dt}(X(t)Y(t)) = (A(t) + B_1(t))X(t)Y(t) \quad \text{where} \quad B_1(t) = X(t)B(t)X(t)^{-1}.
\]
Applying above theorem, we have
\[
\text{sdet} (MN) = \text{sdet} (X(1)Y(1)) = \exp \left\{ \int_0^1 ds \text{str} (A(t) + B_1(t)) \right\} = \exp \left\{ \int_0^1 ds \text{str} (A(t) + \text{str} B(t)) \right\}
\]
\[
= \text{sdet} X(1) \cdot \text{sdet} Y(1) = \text{sdet} M \cdot \text{sdet} N.
\]
(2) Putting \( M(t) = M, X(t) = e^{tM} \) and \( t = 1 \) in theorem above, we get the desired result. \( \Box \)
Comparison 3.2.5 (cited from “Encyclopaedia of Mathematics” ed. M. Hazewinkel). Liouville-Ostrogradski formula (or Liouville formula): A relation that connects the Wronskian of a system of solutions and the coefficients of an ordinary linear differential equation.

Let \( x_1(t), \ldots, x_n(t) \) be an arbitrary system of solutions of a homogeneous system of linear first-order equations

\[
(3.2.15) \quad x' = A(t)x, \quad x \in \mathbb{R}^n
\]

with an operator \( A(t) \) that is continuous on an interval \( I \), and let

\[
W(x_1(t), \ldots, x_n(t)) = W(t)
\]

be the Wronskian of this system of solutions. The Liouville-Ostrogradski formula has the form

\[
(3.2.16) \quad \frac{d}{dt} W(t) = A(t) \cdot \text{tr} A(t), \quad t \in I
\]
or, equivalently,

\[
(3.2.17) \quad W(x_1(t), \ldots, x_n(t)) = W(x_1(t), \ldots, x_n(t)) \cdot \exp \left\{ \int_{t}^{\underline{t}} ds \ \text{tr} \ A(s) \right\}, \quad t, \underline{t} \in I.
\]

Here, \( \text{tr} A(t) \) is the trace of the operator \( A(t) \). The Liouville-Ostrogradski formula can be written by means of the Cauchy operator \( X(t, \underline{t}) \) of the system (3.2.15) as follows:

\[
(3.2.18) \quad \det X(t, \underline{t}) = \exp \left\{ \int_{t}^{\underline{t}} ds \ \text{tr} \ A(s) \right\}, \quad t, \underline{t} \in I.
\]

The geometrical meaning of (3.2.18) (or 3.2.17) is that as a result of the transformation \( X(t, \underline{t}) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) the oriented volume of any body is increased by a factor \( \exp \left\{ \int_{t}^{\underline{t}} ds \ \text{tr} \ A(s) \right\} \).

3.3. An example of diagonalization

Definition 3.3.1. A supermatrix \( M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \in \text{Mat} [m|n : \mathbb{C}] \) is called generic if all eigenvalues of \( M_B \) as \( \text{Mat} [m+n : \mathbb{C}] \) are different each others.

Theorem 3.3.1 (Berezin). Let \( M \in \text{Mat} [m|n : \mathbb{C}] \) be generic. Then, there exists a matrix \( X \in \text{GL}[m|n : \mathbb{C}] \) such that \( E = XMX^{-1} \) is diagonal.

Proof. Decomposing the equality \( EX = XM \) with respect to the degree, we have

\[
(3.3.1) \quad (EX)^{[k]} = \sum_{j=0}^{k} E^{[j]} X^{[k-j]} = \sum_{j=0}^{k} X^{[j]} M^{[k-j]} = (XM)^{[k]}.
\]

From this, we want to construct \( X^{[k]} \) and \( E^{[k]} \): For \( k = 0 \), we have

\[
(3.3.2) \quad E^{[0]} X^{[0]} = X^{[0]} M^{[0]}.
\]

By the assumption, there exist \( X^{[0]}_{11}, E^{[0]}_{11} \) = diagonal matrix with \( (\lambda_1^{[0]}, \ldots, \lambda_m^{[0]}) \) and \( X^{[0]}_{22}, E^{[0]}_{22} \) = diagonal matrix with \( (\lambda_{m+1}^{[0]}, \ldots, \lambda_{m+n}^{[0]}) \) such that

\[
X^{[0]}_{11} A_B = E^{[0]}_{11} X^{[0]}_{11} \quad \text{and} \quad X^{[0]}_{22} B_B = E^{[0]}_{22} X^{[0]}_{22}.
\]

Defining

\[
X^{[0]} = \begin{pmatrix} X^{[0]}_{11} & 0 \\ 0 & X^{[0]}_{22} \end{pmatrix}, \quad E^{[0]} = \begin{pmatrix} E^{[0]}_{11} & 0 \\ 0 & E^{[0]}_{22} \end{pmatrix},
\]
we have the desired one satisfying (3.3.2).

Assume that there exist $X[j]$ and $E[j]$ for $0 \leq j \leq k - 1$ satisfying (3.3.1). Multiplying $(X[0])^{-1}$ from the right to (3.3.1) for $k$, we have

$$E[0]X[k](X[0])^{-1} - X[k](X[0])^{-1}E[0] + E[k] = K[k]$$

where

$$K[k] = \left( \sum_{j=0}^{k-1} X[j]M[k-j]\right)(X[0])^{-1} - \left( \sum_{j=1}^{k-1} E[j]X[k-j]\right)(X[0])^{-1}.$$

By inductive assumption, the matrix $K[k]$ is known and belongs to $\text{Mat}[m|n : \mathbb{C}]$. From (3.3.3), we have

$$(\lambda_i^{[0]} - \lambda_j^{[0]})(X[k](X[0])^{-1})_{ij} + \lambda_i^{[k]}\delta_{ij} = (K[k])_{ij}.$$ 

This equation is uniquely solvable since $\lambda_i^{[0]} \neq \lambda_j^{[0]}$ and

$$\begin{cases} 
\lambda_i^{[k]} = (K[k])_{ii}, \\
(X[k](X[0])^{-1})_{ij} = \frac{(K[k])_{ij}}{\lambda_i^{[0]} - \lambda_j^{[0]}}, \quad \text{for} \ i \neq j.
\end{cases}$$

Therefore, we define $X[j]$ and $E[j]$ for any $j \geq 0$. Since $X[0]$ is invertible, $X \in \text{GL}[m|n : \mathbb{C}]$. This implies $X$ and $E$ are defined as desired. \(\square\)

**Problem 3.3.1.** Find a condition for a supermatrix $M$ being diagonalizable? Is "generic" condition in Theorem 3.3.1 necessary?

3.3.1. A simple example. Let

$$Q = \begin{pmatrix} x_1 & \theta_1 \\
\theta_2 & ix_2 \end{pmatrix}$$

with $x_1, x_2 \in \mathfrak{R}_{ev}, \theta_1, \theta_2 \in \mathfrak{R}_{od},$

which maps $\mathfrak{R}^{1|1}$ to $\mathfrak{R}^{1|1}$ or $\mathfrak{R}_{od} \times i\mathfrak{R}_{ev}$ to $\mathfrak{R}_{od} \times i\mathfrak{R}_{ev}$. This supermatrix appears in Efetov’s calculation in Random Matrix Theory (see for example,K.B. Efetov [39], A. Inoue and Y. Nomura [77]).

3.3.2. Invertibility of $Q$. Find $Y$ for a given $V$ such that

$$QY = V \quad \text{with} \quad Y = \begin{pmatrix} y_1 \\
y_2 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\
v_2 \end{pmatrix} \in \mathfrak{R}^{1|1},$$

$$x_1y_1 + \theta_1\omega_2 = v_1, \quad \theta_2y_2 + 1x_2\omega_2 = v_2.$$

If $(x_1x_2)_B \neq 0$, we have readily

$$y_1 = \frac{ix_2v_1 - \theta_1\rho_2}{D_-}, \quad \omega_2 = \frac{x_1\rho_2 - \theta_2v_1}{D_+} \quad \text{with} \quad D_\pm = ix_1x_2 \pm \theta_1\theta_2.$$

Analogously, for

$$\tilde{Y} = \begin{pmatrix} \omega_1 \\
iy_2 \end{pmatrix} \in \mathfrak{R}_{od} \times i\mathfrak{R}_{ev}, \quad \tilde{V} = \begin{pmatrix} \rho_1 \\
v_2 \end{pmatrix} \in \mathfrak{R}_{od} \times \mathfrak{R}_{ev},$$

satisfying $Q\tilde{Y} = \tilde{V}$, we have

$$\omega_1 = \frac{ix_2\rho_1 - \theta_1v_2}{D_-}, \quad iy_2 = \frac{x_1v_2 - \theta_2\rho_1}{D_+}.$$
Then, from \( QY = I_2 \), we have
\[
\begin{align*}
x_1y_1 + \theta_1\omega_2 &= 1, & x_1\omega_1 + iy_2\theta_1 &= 0, \\
\theta_2y_1 + ix_2\omega_2 &= 0, & \theta_2\omega_1 - x_2y_2 &= 1.
\end{align*}
\]
Therefore, we have
\[
Y = \begin{pmatrix}
\frac{i x_2}{\rho_-} & -\frac{\theta_1}{\rho_-} \\
-\frac{\rho_-}{\rho_-} & \frac{x_1}{\rho_-} 
\end{pmatrix} = (\text{sdet } Q)^{-1} \begin{pmatrix}
\frac{1}{x_2} & \frac{\theta_1}{x_2} \\
\frac{x_1}{x_2} & -\frac{1}{x_2 + 2i \theta_1 \theta_2}
\end{pmatrix},
\]
which yields \( YQ = I_2 \) also. Here, we used
\[
s\text{det } Q = \text{det}(x_1 - \theta_1(i x_2)^{-1}\theta_2) \cdot (\text{det}(ix_2))^{-1} = \frac{ix_1x_2 - \theta_1\theta_2}{(ix_2)^2},
\]
\[
(\text{sdet } Q)^{-1} = \frac{ix_1x_2 + \theta_1\theta_2}{x_1^2}.
\]

Therefore,
\[
\begin{align*}
\begin{pmatrix}
y_1 \\
\omega_2
\end{pmatrix}
&= \frac{D_+}{x_1^2} \begin{pmatrix}
\frac{1}{x_2} & -\frac{\theta_1}{x_2} \\
-\frac{\theta_1}{x_2} & \frac{x_1}{x_2}
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\rho_2
\end{pmatrix}, \\
\begin{pmatrix}
\omega_1 \\
i y_2
\end{pmatrix}
&= \frac{D_+}{x_1^2} \begin{pmatrix}
\frac{1}{x_2} & -\frac{\theta_1}{x_2} \\
-\frac{\theta_1}{x_2} & \frac{x_1}{x_2}
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
v_2
\end{pmatrix}.
\end{align*}
\]

3.3.3. Eigenvalues of \( Q \). Let
\[
QU = \lambda U \quad \text{with } U = \begin{pmatrix} u \\ \omega \end{pmatrix}, \quad u \in \mathcal{R}_{\text{ev}}, \ \omega \in \mathcal{R}_{\text{od}}, \ \lambda \in \mathcal{R}_{\text{ev}}.
\]
Then,
\[
(x_1 - \lambda)u + \theta_1\omega = 0, \ \theta_2u + (ix_2 - \lambda)\omega = 0.
\]
Putting
\[
D_+ (\lambda) = (x_1 - \lambda)(ix_2 - \lambda) + \theta_1\theta_2, \quad D_- (\lambda) = (x_1 - \lambda)(ix_2 - \lambda) - \theta_1\theta_2,
\]
we have
\[
D_- (\lambda)u = 0, \quad D_+(\lambda)\omega = 0.
\]
To guarantee the existence of \( u_B \neq 0 \) satisfying above, we take \( \lambda \) satisfying
\[
D_- (\lambda) = \lambda^2 - (x_1 + ix_2)\lambda + ix_1x_2 - \theta_1\theta_2 = 0.
\]
This yields
\[
\lambda = x_1 + \frac{\theta_1\theta_2}{x_1 - ix_2} \quad \text{(or } \lambda = ix_2 - \frac{\theta_1\theta_2}{x_1 - ix_2}, \ \text{but } \pi_B (\lambda) \notin \mathbb{R})
\]
and
\[
U = \begin{pmatrix}
\frac{1}{\theta_2} \\
\frac{x_1 - ix_2}{\theta_2}
\end{pmatrix}, \quad QU = \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} \theta_1\theta_2 \\ x_1 - ix_2 \end{pmatrix} U.
\]
Analogously, we seek \( \lambda \in \mathcal{R}_{\text{ev}}, \quad \tilde{U} \in \mathcal{R}_{\text{od}} \times \mathcal{R}_{\text{ev}} \) satisfying \( \tilde{Q}\tilde{U} = \lambda\tilde{U} \) which is given
\[
\tilde{U} = \begin{pmatrix}
\frac{-\theta_1}{x_1 - ix_2} \\
\frac{1}{x_1 - ix_2}
\end{pmatrix}, \quad Q\tilde{U} = \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} \theta_1\theta_2 \\ x_1 - ix_2 \end{pmatrix} \tilde{U}.
\]
Therefore,
\[
Q \begin{pmatrix}
\frac{1}{\theta_2} & -\frac{\theta_1}{x_1 - ix_2} \\
\frac{x_1 - ix_2}{\theta_2} & 1
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\theta_2} & -\frac{\theta_1}{x_1 - ix_2} \\
\frac{x_1 - ix_2}{\theta_2} & 1
\end{pmatrix}
\begin{pmatrix}
x_1 + \frac{\theta_1\theta_2}{x_1 - ix_2} \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
x_2 + \frac{\theta_1\theta_2}{x_1 - ix_2}
\end{pmatrix}.
\]
3.3.4. Diagonalization of $Q$. We may diagonalize the matrix $Q$ by using the change of variables

$$(3.3.5) \quad \varphi^{-1}(x, \theta) = (y, \omega) = \begin{cases} y_1 = x_1 + \theta_1 \theta_2, & y_2 = x_2 - i \theta_1 \theta_2, \\ \omega_1 = \frac{\theta_1}{x_1 - ix_2}, & \omega_2 = -\frac{\theta_2}{x_1 - ix_2}, \end{cases}$$

or

$$(3.3.6) \quad \varphi(y, \omega) = (x, \theta) = \begin{cases} x_1 = y_1 + \omega_1 \omega_2(y_1 - iy_2), & x_2 = y_2 - i \omega_1 \omega_2(y_1 - iy_2), \\ \theta_1 = \omega_1(y_1 - iy_2), & \theta_2 = -\omega_2(y_1 - iy_2), \end{cases}$$

such that

$$(3.3.7) \quad GQG^{-1} = \begin{pmatrix} y_1 & 0 \\ 0 & iy_2 \end{pmatrix}, \quad GQ^2G^{-1} = \begin{pmatrix} y_1^2 & 0 \\ 0 & -y_2^2 \end{pmatrix}$$

where

$$G = \begin{pmatrix} 1 + 2^{-1} \omega_1 \omega_2 & \omega_1 \\ 1 & 1 - 2^{-1} \omega_1 \omega_2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 + 2^{-1} \omega_1 \omega_2 & -\omega_1 \\ -\omega_2 & 1 - 2^{-1} \omega_1 \omega_2 \end{pmatrix}.$$  

It is clear that

$$\text{str } Q = x_1 - ix_2 = y_1 - iy_2 = \text{str } GQG^{-1}, \quad \text{and}$$

$$\text{str } Q^2 = x_1^2 + x_2^2 + 2\theta_1 \theta_2 = y_1^2 + y_2^2 = \text{str } (GQG^{-1})^2.$$  

==== Mini Column 2: Tensor and exterior algebras, interior product ==

As a characteristic feature of mathematical thought, it sometimes happens that to generalize that situation makes it easier to understand. Though I have a tendency to feel bothered and sleepy following lengthy algebraic procedure, but I try to collect some terminology from T. Yokonuma [136](in Japanese).

Tensor algebras: Let $V$ be a $d$-dimensional vector space with inner product $(\cdot, \cdot)$. Put $V^*$ is a set of linear operators on $V$, then $V^*$ and $V$ is dual each other by $v \cdot (\cdot, v)$.

**Theorem 3.3.2.** Let $k$ be a field with characteristic 0, and let $V_1, V_2, \ldots, V_n$ be finite dimensional linear spaces on $k$. Then, we have a unique pair $(U_0, \iota_n)$ satisfying following properties $(\otimes)_1$, $(\otimes)_2$, here $U_0$ is a linear space on $k$ and $n$-times linear map $\iota_n \in \mathcal{L}(V_1, V_2, \ldots, V_n : U_0)$.

$(\otimes)_1$ $U_0$ is generated by image of $\iota_n$, $\iota_n(V_1 \times V_2 \times \cdots \times V_n : U_0)$.

$(\otimes)_2$ For any $\Phi \in \mathcal{L}(V_1, V_2, \cdots, V_n : U)$, there exists a linear map $F : U_0 \to U$ such that $\Phi = F \circ \iota_n$.

**Definition 3.3.2.** $(U_0, \iota)$ defined in the above theorem is called tensor product of $V_1, V_2, \ldots, V_n$, and denoted by

$$U_0 = V_1 \otimes V_2 \otimes \cdots \otimes V_n, \quad \iota_n(v_1, v_2, \ldots, v_n) = v_1 \otimes v_2 \otimes \cdots \otimes v_n \quad (v_i \in V_i).$$

**Definition 3.3.3.** For matrices $A = (\alpha_{ij}), \ B = (\beta_{ij})$, we define a matrix

$$\begin{pmatrix} \alpha_{11}B & \alpha_{12}B & \cdots & \alpha_{1n}B \\ \alpha_{21}B & \cdots & \cdots & \alpha_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}B & \alpha_{m2}B & \cdots & \alpha_{mn}B \end{pmatrix},$$

which is called the tensor (or Kronecker) product of $A$ and $B$ and denoted by $A \otimes B$. If $A$ is $(m, n)$-matrix and $B$ is $(m', n')$-matrix, then $A \otimes B$ is $(mn, mn')$-matrix.
Let $V$ be a linear space over $k$ and $V^*$ be the dual of $V$. Then,

$$T_q^p(V) = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

is denoted by $(p,q)$-tensor space. We put $T_0^0(V) = k$, $T_0^p(V) = T^p(V)$, $T_q^0(V) = T_q(V)$ and $T_0^q(V) = T^q(V) = T_0(k)$.

**Remark 3.3.1.** Here, we use identification $V \otimes V^* \equiv V^* \otimes V$, $V \otimes V^* \otimes V \otimes V^* \equiv V^* \otimes V^* \otimes V \otimes V$, etc.

$(p,0)$- or $(0,q)$- tensors are called $p$-th order contravariant or $q$-th order covariant tensor, respectively. An element in $T_0^1(V) = V$ is called contravariant vector, one in $T_1^0(V) = V^*$ is called covariant vector and one in $T_0^0(V) = k$ is scalar.

From Hom$(V, V) \equiv V^* \otimes V \equiv T_1^1(V)$, linear transformation on $V$ is regarded as $(1,1)$-tensor. Since $L(V, V : k) \equiv V^* \otimes V^* = T^2_0(V)$, bilinear form on $V$ is 2-th covariant tensor.

**Proposition 3.3.1 (contraction).** Take integers $p > 0$, $q > 0$ and consider a tensor space $T_q^p(V)$. For any integers $r$, $s$ satisfying $1 \leq r \leq p$ and $1 \leq s \leq q$, there exists a unique linear map $c_s^r : T_q^p(V) \to T_{q-1}^{p-1}(V)$ satisfying the following: For any $v \in V$, $\varphi \in V^*$,

$$c_s^r(v_1 \otimes \cdots \otimes v_r \otimes \varphi_1 \otimes \cdots \otimes \varphi_s) = \varphi_s(v_r)v_1 \otimes \cdots \otimes \tilde{v}_r \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \tilde{\varphi}_s \otimes \cdots \otimes \varphi_q.$$

**Remark.** In the above, $\tilde{v}_r$ or $\tilde{\varphi}_s$ stands for deleting that component, respectively. This map $c_s^r$ is called contraction w.r.t $r$-th contravariant index and $s$-th covariant index.

The permutation group with $p$ letters, $\{1,2,\cdots ,p\}$, is denoted by $\mathfrak{S}_p$ and each $\sigma \in \mathfrak{S}_p$ has the signature $\text{sgn} \sigma = \pm$.

**Proposition 3.3.2.** (1) For each $\sigma \in \mathfrak{S}_p$, there exists uniquely a linear transformation $P_\sigma$ of $T^p(V)$ satisfying

$$P_\sigma(v_1 \otimes \cdots \otimes v_p) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)} \quad (v_i \in V).$$

(2) For $\sigma, \tau, 1 \in \mathfrak{S}_p$, we have

$$P_\sigma P_\tau = P_{\sigma \tau}, P_1 = I.$$

**Definition 3.3.4.** An element $t \in T^p(V)$ is called a symmetric tensor when it satisfies for any $\sigma \in \mathfrak{S}_p$. All such elements is denoted by $S^p(V)$. In case $P_\sigma(t) = (\text{sgn} \sigma)t$ for any $\sigma \in \mathfrak{S}_p$, it is called alternating (=anti-symmetric?) tensor whose set is denoted by $A^p(V)$.

For $t \in S^p(V)$, $t' \in S^q(V)$, we define product of them as $t \cdot t' = S_{p+q}(t \otimes t')$.

For $t \in A^p(V)$, $t' \in A^q(V)$, we define exterior product of them as $t \land t' = A_{p+q}(t \otimes t')$.

**Definition 3.3.5.** We put

$$S_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} P_\sigma, \quad A_p = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (\text{sgn} \sigma)P_\sigma.$$

Unless there occurs confusion, we simply denote them as $S_p = S$, $A_p = A.$
3.3. AN EXAMPLE OF DIAGONALIZATION

Definition 3.3.6. In infinite direct sum

\[ T(V) = \bigoplus_{p=0}^{\infty} T^p(V), \]

is called tensor algebra, if we introduce addition and product as follows:

\[ t = \sum_{j=0}^{\infty} t_j, t' = \sum_{j=0}^{\infty} t'_j \in T(V), \quad t_j, t'_j \in T^j(V), \quad \alpha \in k \Rightarrow \]

\[ t + t' = \sum_{j=0}^{\infty} (t_j + t'_j), \quad \alpha t = \sum_{j=0}^{\infty} \alpha t_j, \quad t \otimes t' = \sum_{p=0}^{\infty} \left( \sum_{r+s=p} t_r \otimes t'_s \right). \]

Analogously, we put

Definition 3.3.7. Introducing product \( \cdot \) in

\[ S(V) = \bigoplus_{p=0}^{\infty} S^p(V) \]

as

\[ t \in S^p(V), \quad t' \in S^q(V) \Rightarrow t \cdot t' = S_{p+q}(t \otimes t'), \]

we have a symmetric algebra \( S(V) \) on \( V \).

Exterior algebra:

Definition 3.3.8. Noting \( A^p(V) = 0 \) for \( p > n \), we have

\[ A(V) = \bigoplus_{p=0}^{\infty} A^p(V) = \bigoplus_{p=0}^{n} A^p(V). \]

We define the exterior product \( \wedge \) as

\[ t \wedge t' = \sum_{p \cdot q} t_p \wedge t'_q = \sum_{k=0}^{n} A_k \left( \sum_{p+q=k} t_p \otimes t'_q \right). \]

\( A(V) \) is called the exterior algebra on \( V \).

Lemma 3.3.1. (1) \( v_i \in V \Rightarrow A(v_1 \otimes \cdots \otimes v_p) = v_1 \wedge \cdots \wedge v_p, \)

(2) \( \sigma \in \mathfrak{S}_n \Rightarrow v_{\sigma^{-1}} \wedge \cdots \wedge v_{\sigma^{-1}p} = \text{sign} \sigma(v_1 \wedge \cdots \wedge v_p). \)

(3) \( t \in A^p(V), t' \in A^q(V) \Rightarrow t' \wedge t = (-1)^{pq} t \wedge t'. \)

The \( p \)-th order covariant tensor space \( T^p_p(V) = T^p(V^*) \) with inner (or scalar) product

\[ \varphi_1 \otimes \cdots \otimes \varphi_p \in T^p_p(V), \quad v_1 \otimes \cdots \otimes v_p \in T^p_p(V) \]

\[ \rightarrow \langle v_1 \otimes \cdots \otimes v_p, \varphi_1 \otimes \cdots \otimes \varphi_p \rangle = \varphi_1(v_1) \cdots \varphi_p(v_p) \]

\[ (v_i \in V, \varphi_j \in V^*) \]

is regarded as the dual of \( T^p_p(V) \).

A bilinear form \( \langle \cdot | \cdot \rangle_p \) on \( A^p(V) \times A^p(V^*) \) is defined as

\[ \langle z | \xi \rangle_p = p! \langle z, \xi \rangle \quad (z \in A^p(V), \xi \in A^p(V^*)). \]

Then,
Proposition 3.3.3. (1) For \( z = v_1 \wedge \cdots \wedge v_p \) (\( v_i \in V \)) and \( \xi = \varphi_1 \wedge \cdots \wedge \varphi_p \) (\( \varphi_i \in V^* \)), we have
\[
\langle z | \xi \rangle_p = \det(\varphi_i(v_j)).
\]
(2) \( A_p(V) \) and \( A_p(V^*) \) are dual each other by the scalar product \( \langle \cdot | \cdot \rangle_p \).
(3) For \( z = \sum_{p=0}^n z_p \in A(V) \) (\( z_p \in A_p(V) \)) and \( \xi = \sum_{p=0}^n \xi_p \in A(V^*) \) (\( \xi_p \in A_p(V^*) \)), we define the scalar product
\[
\langle z | \xi \rangle = \sum_{p=0}^n \langle z_p | \xi_p \rangle_p
\]
then. \( A(V) \) and \( A(V^*) \) form dual spaces each other.

Definition 3.3.9. For \( \xi \in A(V^*) \), we define a linear transformation \( \delta(\xi) \) on \( A(V^*) \) as
\[
\delta(\xi)\zeta = \xi \wedge \zeta \quad (\zeta \in A(V^*))
\]
which is called (left)exterior multiplication. Transposed map of this \( \delta(\xi) \) is denoted by \( \partial(\xi) \) and called interior product (or multiplication) by \( \xi \):
\[
\langle \partial(\xi)z | \zeta \rangle = \langle z | \delta(\xi)\zeta \rangle = \langle z | \xi \wedge \zeta \rangle.
\]

Remark: Above defined operations are denoted also by \( \delta(\xi)\cdot = \xi \wedge \cdot, \partial(\xi)\cdot = \xi \cdot \).

======= End of Mini Column 2 =======
CHAPTER 4

Elementary differential calculus on superspace

On real Euclidean space \( \mathbb{R}^m \), to begin with, we consider a real-valued, continuous and smooth function. On the other hand if we work on complex space \( \mathbb{C}^m \) with a complex valued function, it seems natural to develop complex analytic functions. From these, what is a natural candidate for a function on superspace \( \mathcal{R}^{m|n} \). This chapter and the next one are rewritten rather significantly from the original lectures.

4.1. Gâteaux or Fréchet differentiability on Banach spaces

For the future use, we prepare the following lemma:

**Lemma 4.1.1** (see, Lemma 2.1.14 of Berger [11]). Let \( X_1, X_2, Y \) be Banach spaces. The Banach spaces \( L(X_1, X_2 : Y) \) and \( L(X_1 : L(X_2 : Y)) \) are identical up to a linear isometry.

**Definition 4.1.1.** Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be two Banach spaces. 

(i) A function \( \Phi : X \to Y \) is called Gâteaux (or \( G \)-) differentiable at \( x \in X \) in the direction \( h \in X \) if there exists an element \( \Phi'_G(x; h) \in Y \) such that

\[
\| \Phi(x + th) - \Phi(x) - t\Phi'_G(x; h) \|_Y \to 0 \quad \text{when} \quad t \to 0, \quad \text{i.e.} \quad \frac{d}{dt} \Phi(x + th)\bigg|_{t=0} = \Phi'_G(x; h).
\]

\( \Phi'_G(x; h) \) is also denoted by \( \Phi'_G(x)(h), \ d_G \Phi(x; h) \) or \( (d_G \Phi(x))(h) \). The second order Gâteaux-derivatives \( d_G^{(2)} \Phi(x; h) \) at \( x \in X \) in the direction \( h = (h_1, h_2) \in X^2 \) is defined by

\[
d_G^{(2)} \Phi(x; h) = d_G^{(2)} \Phi(x; h_1; h_2) = d_G(d_G \Phi(x; h_1; h_2)) = \frac{d}{dt} d_G \Phi(x + th_2; h_1)\bigg|_{t=0} = \frac{\partial^2}{\partial t_1 \partial t_2} \Phi(x + t_1 h_1 + t_2 h_2)\bigg|_{t_1=t_2=0}.
\]

Analogously, we may define \( N \)-th Gâteaux-derivatives \( d_G^{(N)} \Phi(x; h) \) (or \( \Phi^{(N)}_G(x; h) \)) with \( h = (h_1, \cdots, h_N) \in X^N \). If this \( d_G^{(N)} \Phi(x; h_1, \cdots, h_N) \) exists, then it is symmetric w.r.t. \( (h_1, \cdots, h_N) \).

(ii) \( \Phi : X \to Y \) is called Fréchet (or \( F \)-) differentiable at \( x \in X \) if there exist a bounded linear operator \( \Phi'_F(x) : X \to Y \) and an element \( \tau(x, h) \in Y \) such that

\[
\Phi(x + h) - \Phi(x) - \Phi'_F(x) h = \tau(x, h) \quad \text{with} \quad \| \tau(x, h) \|_Y = o(\|h\|_X).
\]

It is clear that if \( \Phi'_F(x) \) (or \( d_F \Phi(x) \)) exists, then \( \Phi'_G(x) \) exists also and \( \Phi'_G(x) = \Phi'_F(x) \). The second order Fréchet-derivatives \( \Phi''_F(x; h) \) at \( x \in X \) is defined if \( \Phi'_F : X \to L(X : Y) \) is differentiable at \( x \in X \) in the Fréchet sense. In this case, \( \Phi''_F \in L(X : L(X : Y)) \cong L_2(X : Y) = L(X, X : Y) \). It is denoted by \( \Phi \in C^2(U : Y) \) if (a) \( \Phi \) is twice Fréchet differentiable, and (b) \( \Phi''_F(x) : U \to L(X, X : Y) \) is continuous. We define analogously \( N \)-th Fréchet differentiables \( \Phi^{(N)}_F \) and a class of \( N \)-times Fréchet differentiable functions \( C^N(U : Y) \). That is, \( \Phi \in C^N_F(X : Y) \) means for each \( x \in U \subset X \), \( \Phi \) is
N-times Fréchet-differentiable and $\Phi^{(N)}_f(x)$ is a continuous map from $U$ to $L(X \times \cdots \times X : Y) = L_N(X : Y)$ w.r.t. $x$.

**Theorem 4.1.1** (see, Theorem 2.1.13 of Berger [11]). If $\Phi : X \to Y$ be Fréchet-differentiable at $x$, it is Gâteaux-differentiable at $x$. Conversely, if the Gâteaux derivative of $\Phi$ at $x$, $d_G\Phi(x,h)$, is linear in $h$ and is continuous in $x$ as a map from $X \to L(X : Y)$, then $\Phi$ is Fréchet-differentiable at $x$. In either case, we have $\Phi'_G(x) = \Phi'_F(x,y)$.

**Theorem 4.1.2** (see, Theorem 2.1.27 of Berger [11]). If $\Phi : X \to Y$ be $N$-times Fréchet-differentiable in a neighbourhood $U$ at $x$ and $\Phi^{(N)}_f(x)(h^{(1)}, \cdots, h^{(N)})$ denotes the $N$-th Fréchet derivative, the $\Phi$ is $N$-times Gâteaux-differentiable and

$$d^{(N)}_G\Phi(x; h^{(1)}, \cdots, h^{(N)}) = \Phi^{(N)}_f(x)(h^{(1)}, \cdots, h^{(N)}).$$

Conversely, if the $N$-th Gâteaux derivative $d^{(N)}_G\Phi(x; h^{(1)}, \cdots, h^{(N)})$ of $\Phi$ exists in a neighbourhood of $U$ of $x$, $d^{(N)}_G\Phi(x; h^{(1)}, \cdots, h^{(N)}) \in L_N(X : Y)$, and as a function of $x$, $d^{(N)}_G\Phi(x; h^{(1)}, \cdots, h^{(N)})$ is continuous from $U$ to $L_N(X : Y)$, then $\Phi$ is $N$-times Fréchet-differentiable and the two derivatives are equal at $x$.

**Problem 4.1.1.** How does one extend these notion of differentiability to those on functions on $\mathbb{R}^m|n$?

### 4.2. Gâteaux or Fréchet differentiable functions on Fréchet spaces

In this section, I borrow representations in R. Hamilton’s paper [55] which I overlooked when lecture had been prepared.

#### 4.2.1. Gâteaux-differentiability.

**Definition 4.2.1** (Gâteaux-derivative, -differential and -differentiability). (i) Let $X$, $Y$ be Fréchet spaces with countable seminorms $\{p_m\}$, $\{q_n\}$, respectively. Let $U$ be an open subset of $X$. For a function $f : U \to Y$, we say that $f$ is 1-time Gâteaux (or G-)differentiable at $x \in U$ in the direction $y \in X$ if there exists the following limit in $Y$:

$$\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \frac{df(x + ty)}{dt}\bigg|_{t=0} = d_Gf(x; y) = d_Gf(x)(y) = d_Gf(x)y = f'_G(x)y,$$

i.e., for given $x \in U$ and $y \in X$ there exists an element $d_Gf(x; y) \in Y$ such that for any $n \in \mathbb{N}$, we have

$$q_n(f(x + ty) - f(x) - td_Gf(x; y)) = o(t).$$

We call this $d_Gf(x; y)$ the G-differential of $f$ at $x$ in the direction $y$ and denoted as above, and $d_Gf(x)$ or $f'_G(x)$ are called the G-derivative. Moreover, $f$ is said to be G-differentiable in $U$ and denoted by $f \in C^1_G(U : Y)$ if $f$ has the G-differential $d_Gf(x; y)$ for every $x \in U$ and any direction $y \in X$. A map $f : U \to Y$ is said to be 1-time continuously G-differentiable on $U$, denoted by $f \in C^1_G(U : Y)$, if $f$ has G-derivative in $U$ and if $d_Gf : U \times X \ni (x, y) \to d_Gf(x; y) \in Y$ is jointly continuous.
(ii) If $X$, $Y$ are Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, then, $f$ has $G$-differential $df(x;y) \in Y$ at $x \in U$ in the direction $y \in X$ if and only if
\[
\|f(x+ty)-f(x)-td_Gf(x;y)\|_Y = o(|t|) \text{ as } t \to 0.
\]
Moreover, $f \in C^1_G(U : Y)$ if and only if $f$ is $G$-differentiable at $x$ and $d_Gf$ is continuous from $U \ni x \to d_Gf(x) \in L(X : Y)$.

**Proposition 4.2.1** (see, pp.76-77 of [55]). Let $X$, $Y$ be Fréchet spaces and let $U$ be an open subset of $X$. If $f \in C^1_G(U : Y)$, then $d_Gf(x;y)$ is linear in $y$.

**Remark 4.2.1** (see, p.70 of [55]). It should be remarked that even if $X, Y, Z$ are Banach spaces and $U \subset X$, there exists the difference between

"$L : U \times Y \to Z$ is continuous" and "$L : U \to L(Y : Z)$ is continuous".

**Definition 4.2.2** (Higher order derivatives, see, p.80 of [55]). Let $X$, $Y$ be Fréchet spaces.

(i) If the following limit exists, we put
\[
d_G^2f(x)\{y,z\} = d_G^2f(x; y, z) = \lim_{t \to 0} \frac{d_Gf(x + tz; y) - d_Gf(x; y)}{t}.
\]
Moreover, $f$ is said to be $C^2_G(U : Y)$ if $d_Gf$ is $C^1_G(U \times Y : Y)$, which happens if and only if $d_G^2f$ exists and is continuous, that is, $d_G^2f$ is jointly continuous from $U \times X \times X \to Y$.

(ii) Analogously, we define
\[
d_G^n f : U \times X \times \cdots \times X \ni (x, y_1, \cdots, y_n) \to d_G^nf(x)\{y_1, \cdots, y_n\} = d_G^n f(x; y_1, \cdots, y_n) \in Y.
\]

We have
\[
\frac{\partial^N}{\partial t_1 \cdots \partial t_N} \Phi(x + \sum_{j=1}^N t_j h_j) \bigg|_{t_1 = \cdots = t_N = 0} = \frac{d}{dt_N} d_G^{N-1} \Phi(x + t Nh_N; h_1, \cdots, h_{N-1}) \bigg|_{t_N = 0} = d_G \left( d_G^{N-1} \Phi(x; h_1, \cdots, h_{N-1}); h_N \right) = d_G \left( \Phi_G^{(N)}(x; h_1, \cdots, h_N) \right)
\]
for $f$ is said to be $C^0_G(U : Y)$ if and only if $d_G^n f$ exists and is continuous. We put $C^\infty_G(U : Y) = \cap_{n=0}^\infty C^n_G(U : Y)$.

**Definition 4.2.3** (Many variables case). (i) Let $X_1$, $X_2$, $Y$ be Fréchet spaces. For $x = (x_1, x_2) \in X_1 \times X_2$ and $z = (z_1, z_2) \in X_1 \times X_2$, we put
\[
\partial_{x_1} f(x)\{z_1\} = f_{x_1}(x; z_1) = f_{x_1}(x)z_1 = \lim_{t \to 0} \frac{f(x_1 + tz_1, x_2) - f(x_1, x_2)}{t},
\]
\[
\partial_{x_2} f(x)\{z_2\} = f_{x_2}(x; z_2) = f_{x_2}(x)z_2 = \lim_{t \to 0} \frac{f(x_1, x_2 + tz_2) - f(x_1, x_2)}{t}.
\]
They are called partial derivatives. We define the total $G$-derivative as
\[
d_G f(x)\{z\} = f'_x(x; z) = \lim_{t \to 0} \frac{f(x_1 + tz_1, x_2 + tz_2) - f(x_1, x_2)}{t}.
\]
For $f : X \to Y$ with $X = \prod_{i=1}^n X_i$, we define $\partial_{x_i} f(x)$ and $d_G f(x)$ for $x = (x_1, \cdots, x_n)$, analogously.

(ii) If $X_1$, $X_2$, $Y$ are Banach spaces, we may define analogously the above notion.

**Proposition 4.2.2.** Let $\{X_i\}_{i=1}^n$, $Y$ be Fréchet spaces and let $U$ be an open set in $X = \prod_{i=1}^n X_i$. 
(a) \( f \in C^1_G(U : Y) \), i.e. \( d_G f(x)\{y\} \) exists and is continuous, if and only if \( \partial_x f(x)\{\cdot\} \) exist and are continuous, and we have, for \( x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in X \),

\[
d_G f(x; y) = d_G f(x)\{y\} = \sum_{i=1}^n f_{x_i}(x; y_i) = \sum_{i=1}^n f_{x_i}(x)\{y_i\}. \tag{4.2.1}
\]

(b) [Taylor’s formula] Moreover, if \( f \in C^p_G(U : Y) \), we have

\[
f(x + y) = \sum_{k=0}^p \frac{1}{k!} d^k_G f(x)\{y, \ldots, y\} + R_p f(x, y)
\]

with

\[
\lim_{t \to 0} t^{-p} R_p f(x, ty) = 0 \quad \text{for } y \in X,
\]

where

\[
R_p f(x, y) = \int_0^1 (1 - s)^{p-1} \frac{d^p}{ds^p} f(x + sy)ds.
\]

**Proof.** \((4.2.1)\) is proved in Theorem 3.4.3 of [55] for \( N = 2 \). \((4.2.2)\) is given, for example, in p.101 of Keller [82], et al. \(\square\)

### 4.2.2. Fréchet-differentiability.

**Definition 4.2.4** (see, Definition 1.8. of Schwartz [115]). (i) Let \( X, Y \) be Fréchet spaces, and let \( U \) be an open subset of \( X \). A function \( \varphi : U \to Y \) is said to be horizontal (or tangential) at \( 0 \) if and only if for each neighbourhood \( V \) of \( 0 \) in \( Y \) there exists a neighbourhood \( U' \) of \( 0 \) in \( X \), and a function \( o(t) : (-1, 1) \to \mathbb{R} \) such that

\[
\varphi(tU') \subset o(t)V \quad \text{with } \lim_{t \to 0} \frac{o(t)}{t} = 0,
\]

i.e. for any seminorm \( q_n \) on \( Y \) and \( \epsilon > 0 \), there exists a seminorm \( p_m \) on \( E \) and \( \delta > 0 \) such that

\[
q_n(\varphi(tx)) \leq |t| \quad \text{for } p_m(x) < 1, \ |t| \leq \delta
\]

From \((4.2.1)\), putting \( V = \{ z \in Y \mid q_n(z) < 1 \}, U' = \{ x \in X \mid p_m(x) < 1 \} \), we may recover \((4.2.3)\).

(ii) For given Banach spaces \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \), “horizontal” implies

\[
\| \varphi(x) \|_Y \leq \| x \|_X \psi(x) \quad \text{with } \psi : X \to \mathbb{R}, \lim_{x \to 0} \psi(x) = 0 \quad \text{i.e. } \| \varphi(x) \|_Y = o(\| x \|_X) \quad \text{as } \| x \|_X \to 0.
\]

**Definition 4.2.5** (Fréchet differentiability). (i) (Definition 1.9. of [115]) Let \( X, Y \) be Fréchet spaces with \( U \) being an open subset of \( X \). We say that \( f \) has a Fréchet (or is \( F \)-)derivative (or \( f \) is \( F \)-differentiable) at \( x \) if there exists a continuous linear map \( A = A_x : X \to Y \) such that \( \varphi(x; y) \) is horizontal w.r.t \( y \) at \( 0 \), where \( \varphi(x; y) \) is defined by

\[
\varphi(x; y) = f(x + y) - f(x) - A_x y.
\]

We call \( A = A_x \) the \( F \)-derivative of \( f \) at \( x \), and we denote \( A_x y \) as \( d_F f(x; y) \). Moreover, we denote \( f \in C^1_F(U : Y) \) if \( f \) is \( F \)-differentiable and \( d_F f : U \times X \ni (x, y) \mapsto d_F f(x; y) \in Y \) is jointly continuous.

(ii) For Banach spaces, \( f \) is \( F \)-differentiable at \( x \) if there exists a continuous linear map \( A = A_x : X \to Y \) satisfying

\[
\| f(x + y) - f(x) - Ay \|_Y = o(\| y \|_X) \quad \text{as } \| y \|_X \to 0.
\]

Moreover, \( f \in C^1_F(U : Y) \) if \( f \) is \( F \)-differentiable and \( X \ni x \mapsto A_x \in L(X : Y) \) is continuous.
Remark 4.2.2. If $f$ is $F$-differentiable, then it is also $G$-differentiable. Moreover,

$$f'_G(x; y) = d_G f(x; y) = d_F f(x; y) = f'_F(x; y).$$

Definition 4.2.6 (Higher order derivatives). (i) Let $X, Y$ be Fréchet spaces with $U$ being an open subset of $X$. A $F$-differentiable function $f : U \to Y$ is twice $F$-differentiable at $x \in U$ if $d_F f : U \times X \ni (x, y) \to d_F f(x; y) \in Y$ is $F$-differentiable at $x \in X$. That is, the function

$$
\psi(x; y, z) = d_F f(x + z; y) - d_F f(x; y) - d^2_F f(x)(y, z),
$$

is horizontal w.r.t. $z$ at 0.

(ii) (p.72 of [11]) Let $X, Y$ be Banach spaces. A $F$-differentiable function $f : U \to Y$ is twice $F$-differentiable at $x \in U$ if $f''_F : X \to L(X : Y)$ is $F$-differentiable at $x \in X$ and $f''_F(x)$, the derivative of $f'_F(x)$, belongs to $L(X : L(X : Y)) = L(X \times X : Y)$. $f \in C^2(U : Y)$ if (a) $f$ is twice $F$-differentiable for each $x \in U$ and (b) $f''_F(x) : U \to L(X \times X : Y)$ is continuous.

(iii) Analogously $N$-times $F$-differentiability is defined.

Definition 4.2.7 (Many variables case). (i) Let $U = \prod_{i=1}^N U_i$ with each $U_i$ being a open subset of Fréchet spaces $X_i$. For $x = (x_1, \ldots, x_N) \in U$ with $x_i \in U_i$ and $h_i \in X_i$ s.t. $x_i + h_i \in U_i$, if there exists $F_i(x; h_i) \in Y$ such that

$$
\varphi_i(x; h_i) = f(x_1, \ldots, x_{i-1}, x_i + h_i, x_{i+1}, \ldots x_N) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots x_N) - F_i(x; h_i)
$$

is horizontal w.r.t. $h_i$. We denote $F_i(x; h_i)$ as $\partial_{x_i} f(x) h_i$, the partial derivative of $f$ w.r.t. $x_i$.

(ii) Let $U_i$ be Banach spaces, the partial derivative of $f$ w.r.t. $x_i$, $\partial_{x_i} f(x)$, is defined by

$$
f(x_1, \ldots, x_{i-1}, x_i + h_i, x_{i+1}, \ldots x_N) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots x_N) = \partial_{x_i} f(x) h_i + o(\|h_i\|).
$$

More generally, for each $x$ if there exists a continuous linear map $d_F f : X \ni h \to d_F f(x; h) \in Y$ such that

$$
\|f(x + h) - f(x) - d_F f(x)(h)\|_Y = o(\|h\|_X)
$$

for $h \in X.

We denote also $d_F f(x; h) = f'_F(x)\{h\}$ with $f'_F(x) \in L(X : Y)$. Moreover, there exist operators $\partial_{x_i} f(x) \in L(X_i : Y)$ such that

$$
f'_F(x; h) = f'_F(x)\{h\} = \sum_{i=1}^N \partial_{x_i} f(x) h_i = \sum_{i=1}^N \partial_{x_i} f(x; h_i) \quad \text{with} \quad h = (h_1, \ldots, h_N).
$$

4.3. Functions on superspace

4.3.1. Grassmann continuation. Let $\phi(q)$ be a $\mathbb{C}$-valued function on an open set $\Omega \subset \mathbb{R}^m$, that is,

$$
\phi(q) = \sum_{I \in \mathcal{I}} \phi_I(q) \sigma^I \quad \text{with} \quad \phi_I : \Omega \ni q \to \phi_I(q) \in \mathbb{C}.
$$

By the definition of the topology of $\mathbb{C}$, we have

$$
\lim_{q \to q_0} \phi(q) = \sum_{I \in \mathcal{I}} \left( \lim_{q \to q_0} \phi_I(q) \right) \sigma^I.
$$

The differentiation and integration of such $\phi(q)$ are defined by

$$
\frac{\partial}{\partial q_j} \phi(q) = \sum_{I \in \mathcal{I}} \frac{\partial}{\partial q_j} \phi_I(q) \sigma^I \quad \text{and} \quad \int_{\Omega} dq \phi(q) = \sum_{I \in \mathcal{I}} \left( \int_{\Omega} dq \phi_I(q) \right) \sigma^I.
$$
We say $\phi \in C^\infty(\Omega : \mathbb{C})$ if $\phi_I \in C^\infty(\Omega : \mathbb{C})$ for each $I \in \mathcal{I}$.

**Remark 4.3.1.** If we use Banach-Grassmann algebra instead of Fréchet-Grassmann algebra, we need to check whether $\sum_{I \in \mathcal{I}} |\phi_I(q)| < \infty$, etc., which seems cumbersome or rather impossible to check for applying it to concrete problems.

**Lemma 4.3.1.** Let $\phi(t)$ and $\Phi(t)$ be continuous $C$-valued functions on an interval $[a, b] \subset \mathbb{R}$. Then,

1. $\int_a^b dt \phi(t)$ exists,
2. if $\Phi'(t) = \phi(t)$ on $[a, b]$, then $\int_a^b dt \phi(t) = \Phi(b) - \Phi(a)$,
3. if $\lambda \in \mathcal{C}$ is a constant, then

$$\int_a^b dt (\phi(t) \cdot \lambda) = \left( \int_a^b dt \phi(t) \right) \cdot \lambda \quad \text{and} \quad \int_a^b dt (\lambda \cdot \phi(t)) = \lambda \cdot \int_a^b dt \phi(t).$$

Moreover, we may generalize above lemma for a $C$-valued function $\phi(q)$ on an open set $\Omega \subset \mathbb{R}^m$.

**Definition 4.3.1.** A set $U_{ev} \subset \mathbb{R}^n[0] = \mathbb{R}^n_{ev}$ is called an even superdomain if $U = \pi_B(U_{ev}) \subset \mathbb{R}^m$ is open and connected and $\pi_B^{-1}(U_{ev}) = U_{ev}$. When $U \subset \mathbb{R}^m[1]$ is represented by $U = U_{ev} \times \mathbb{R}_{od}$ with a even superdomain $U_{ev} \subset \mathbb{R}^n[0]$, $U$ is called a superdomain in $\mathbb{R}^m[1]$.

**Proposition 4.3.1.** Let $U_{ev} \subset \mathbb{R}^m[0]$ be an even superdomain. Assume that $f$ is a smooth function from $\mathbb{R}^m \supset U = \pi_B(U_{ev})$ into $\mathcal{C}$, denoted simply by $f \in C^\infty(U : \mathcal{C})$. That is, we have the expression

$$f(q) = \sum_{J \in \mathcal{I}} f_J(q) \sigma^J \quad \text{with} \quad f_J(q) \in C^\infty(U : \mathcal{C}) \quad \text{for each} \quad J \in \mathcal{I}. \quad (4.3.1)$$

Then, we may define a mapping $\tilde{f}$ of $U_{ev}$ into $\mathcal{C}$, called the Grassmann continuation of $f$, by

$$\tilde{f}(x) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial^\alpha_q f(x_B) x_S^\alpha \quad \text{where} \quad \partial^\alpha_q f(x_B) = \sum_J \partial^\alpha_q f_J(x_B) \sigma^J \quad (4.3.2)$$

Here, we put $x = (x_1, \ldots, x_m)$, $x = x_B + x_S$ with $x_B = (x_{1,B}, \ldots, x_{m,B}) = (q_1, \ldots, q_m) = q \in U$, $x_S = (x_{1,S}, \ldots, x_{m,S})$ and $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

**Proof.** [Since circulation of our paper Inoue and Maeda [75] is so-limited, I repeat here the proof whose main point is to check whether this mapping (4.3.2) is well-defined or not. Therefore, by using the degree argument, we need to define $\tilde{f}^{[k]}$, the $k$-th degree component of $\tilde{f}$.

Denoting by $x_{1,S}^{[k]}$, the $k_1$-th degree component of $x_{1,S}$, we get

$$(x_{1,S}^{[k]})^{[k]} = \sum_{\ell} (x_{1,S}^{[r_1]}y_{1,1} \cdots (x_{1,S}^{[r_\ell]}y_{1,\ell})$$

Here, the summation is taken for all partitions of an integer $\alpha_1$ into $\alpha_1 = p_{1,1} + \cdots + p_{1,\ell}$ satisfying $\sum_{i=1}^\ell r_ip_{1,i} = k_1$, $r_i \geq 0$. Using these notations, we put

$$\tilde{f}^{[k]}(x) = \sum_{|\alpha| \leq k, k_0 + k_1 + \cdots + k_m = k} \frac{1}{\alpha!} (\partial^\alpha_q f)^{[k_0]}(x_B) (x_{1,S}^{\alpha_1})^{[k_1]} \cdots (x_{m,S}^{\alpha_m})^{[k_m]} \quad (4.3.3)$$

where $k_1, \ldots, k_m$ are even.
where
\[
(\partial_\alpha^a f)^{[k_0]}(x_B) = \sum_{|\mathcal{J}|=k_0} \partial_\alpha^a f_\mathcal{J}(x_B) \sigma^\mathcal{J}.
\]

Or more precisely, we have
\[
\begin{align*}
\tilde{f}^{[0]}(x) &= f^{[0]}(x_B), \\
\tilde{f}^{[1]}(x) &= f^{[1]}(x_B), \\
\tilde{f}^{[2]}(x) &= f^{[2]}(x_B) + \sum_{j=1}^m (\partial_{q_j} f)^{[0]}(x_B)(x_j)^2, \\
\tilde{f}^{[3]}(x) &= f^{[3]}(x_B) + \sum_{j=1}^m (\partial_{q_j} f)^{[1]}(x_B)(x_j)^2, \\
\tilde{f}^{[4]}(x) &= f^{[4]}(x_B) + \sum_{j=1}^m (\partial_{q_j} f)^{[2]}(x_B)(x_j) + \sum_{j\neq k} (\partial_{q_j} f)^{[0]}(x_B)(x_j,x_k)^2, \\
&\quad + \frac{1}{2} \sum_{j=1}^m (\partial_{q_j} f)^{[0]}(x_B)(x_j)^2[4] + \sum_{j\neq k} (\partial_{q_j} f)^{[0]}(x_B)(x_j,x_k)^2(2)(k,x_k)^2, \text{ etc.}
\end{align*}
\]

Since \(\tilde{f}^{[j]}(x) \neq \tilde{f}^{[k]}(x)\) \((j \neq k)\) in \(\mathcal{C}\), we may take the sum \(\sum_{j=0}^\infty \tilde{f}^{[j]}(x) \in \mathcal{C} = \oplus_{k=0}^\infty \mathcal{C}^{[k]}\), which is denoted by \(\tilde{f}(x)\). Therefore, rearranging the above “summation”, we get rather the “familiar” expression as in [4.3.2].

**Remark 4.3.2.** Concerning the summation in [4.3.3], summation w.r.t. \(\alpha\) is clearly finite, but that in \((x_{1,\mathcal{S}}^{a_1})^{[k_1]}\) w.r.t. \(\mathcal{I}\) is infinite for \(|\mathcal{J}| = k_1\).

**Corollary 4.3.1.** If \(f\) and \(\tilde{f}\) be given as above, then

(i) \(\tilde{f}\) is continuous and

(ii) \(\tilde{f}(x) = 0\) in \(\mathcal{U}_{ev}\) implies \(f(x_B) = 0\) in \(U\).

Moreover, if we define the partial derivative of \(\tilde{f}\) in the \(j\)-direction by
\[
(4.3.4) \quad \partial_{x_j} \tilde{f}(x) = \frac{d}{dt} \tilde{f}(x + te_{(j)}) \bigg|_{t=0} \quad \text{where} \quad e_{(j)} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathcal{R}^{m|0},
\]
then we get
\[
(4.3.5) \quad \partial_{x_j} \tilde{f}(x) = \partial_{q_j} f(x) \quad \text{for} \quad j = 1, \ldots, m.
\]

**Proof.** Let \(y_j = y_{j,B} + y_{j,S} \in \mathcal{R}_{ev}\). For \(y_{(j)} = y_{j,e_{(j)}} = y_{j,B}e_{(j)} + y_{j,S}e_{(j)} = y_{(j),B} + y_{(j),S} \in \mathcal{R}^{m|0}\), as
\[
\frac{d}{dt} \tilde{f}(x + ty_{(j)}) = \frac{d}{dt} \left\{ \sum_{\alpha} \frac{1}{\alpha!} \left( \sum_{\mathcal{J}} \partial_\alpha^a f_\mathcal{J}(x_B + ty_{(j),B}) \sigma^\mathcal{J} \right) (x_S + ty_{(j),S})^\alpha \right\},
\]
we get easily,
\[
\frac{d}{dt} \tilde{f}(x + ty_{(j)}) \bigg|_{t=0} = y_{(j),B} \sum_{\alpha} \frac{1}{\alpha!} \left( \sum_{\mathcal{J}} \partial_\alpha^a f_\mathcal{J}(x_B) \sigma^\mathcal{J} \right) x_S^\alpha + y_{(j),S} \sum_{\alpha} \frac{1}{\alpha!} \left( \sum_{\mathcal{J}} \partial_{q_j}^\alpha \partial_\alpha^a f_\mathcal{J}(x_B) \sigma^\mathcal{J} \right) x_S^\hat{\alpha}
\]
\[
= y_j \sum_{\alpha} \frac{1}{\alpha!} \partial_\alpha^a \partial_{q_j} f(x_B) x_S^\alpha = y_j \partial_{q_j} \tilde{f}(x).
\]
Here, $\tilde{a} = (\alpha_1, \cdots, \alpha_j, \alpha_j - 1, \alpha_{j+1}, \cdots, \alpha_m)$. Putting $y_j = y_{j,B} + y_{j,S} = 1$ in the above, we have (4.3.5).

**Remark 4.3.3.** (i) By the same argument as above, we get, for $y = (y_1, \cdots, y_m) \in R^{m|0}$,

\begin{equation}
\frac{d}{dt} \tilde{f}(x + ty) \bigg|_{t=0} = \sum_{j=1}^{m} y_j \sum_{\alpha} \left\frac{1}{\alpha!} \partial_{\alpha} q_{\alpha} f(x_{B}) x_{S}^{\alpha} \right. \sum_{j=1}^{m} y_j \partial_{x_j} \tilde{f}(x).
\end{equation}

(ii) Unless there occurs confusion, we denote $\tilde{f}$ simply by $f$.

### 4.3.2. Supersmooth functions and their derivatives.

How to define the continuity and differentiability of functions from $R^{m|n}$ to $C$?

**Problem 4.3.1.** Since $R$, $C$ and $R^{m|n}$ are Fréchet spaces, we may define $G$- or $F$-differentiable functions as before. By the way, how to take into account the ring structure of Fréchet-Grassmann algebra in the definition of total differentiability?

In order to answer this problem, we introduce “the desired or tractable form of functions on $R^{m|n}$” and called them as “supersmooth (or called superfield by physicist)”. In the next section, we study their properties and we characterize them.

**Definition 4.3.2.** (1) Let $U_{ev} \subset R^{m|0}$ be a even super domain. A mapping $F$ from $U_{ev}$ to $C$ is called supersmooth if there exists a smooth mapping $f$ from $U = \pi_{B}(U_{ev})$ to $C$ such that $F = \tilde{f}$. We denote the set of supersmooth functions on $U_{ev}$ as $C_{SS}(U_{ev} : C)$.

(2) Let $U$ be a superdomain in $R^{m|n}$. A mapping $f$ from $U$ to $C$ is called supersmooth if it is decomposed as

\begin{equation}
f(x, \theta) = \sum_{|a| \leq n} \theta^{a} f_{a}(x).
\end{equation}

Here, $a = (a_1, \cdots, a_n) \in \{0,1\}^{n}$, $\theta^{a} = \theta_1^{a_1} \cdots \theta_{n}^{a_n}$ and $f_{a}(x) \in C_{SS}(U_{ev} : C)$. Without mentioning it, we assume always that $f_{a}(x) \in C_{ev}(or \in C_{od})$ for all $a$, and call them as even (or odd) supersmooth functions denoted by $C_{SS}(U : C)$. Moreover,

$C_{SS} = \{ f(x, \theta) \in C_{SS}(U : C) | f_{a}(x_{B}) \in C \}$.

Therefore, if $f \in C_{SS}$, $f_{a}(x)$ may be put any side of $\theta^{a}$.

(3) Let $f \in C_{SS}(U : C)$. We put

\begin{equation}
\begin{cases}
F_{j}(X) = \sum_{|a| \leq n} \theta^{a} \partial_{x_j} f_{a}(x) \quad \text{for} \quad j = 1, \cdots, m, \\
F_{s+m}(X) = \sum_{|a| \leq n} (-1)^{l(a)} \theta_1^{a_1} \cdots \theta_{s-1}^{a_{s-1}} \theta_{s}^{a_{s-1}} \cdots \theta_{n}^{a_{n}} f_{a}(x) \quad \text{for} \quad s = 1, \cdots, n
\end{cases}
\end{equation}

with $l(a) = \sum_{j=1}^{s-1} a_j$ and $\theta_{s}^{-1} = 0$. In this case, $F_{a}(X)$ is the partial derivative of $f$ at $X = (x, \theta) = (X_{\mu})$ w.r.t. $X_{\kappa}$

\begin{equation}
\begin{cases}
F_{j}(X) = \frac{\partial}{\partial x_{j}} f(x, \theta) = \partial_{x_j} f(x, \theta) = f_{x_j}(x, \theta) \quad \text{for} \quad j = 1, \cdots, m, \\
F_{s+m}(X) = \frac{\partial}{\partial \theta_{s}} f(x, \theta) = \partial_{\theta} f(x, \theta) = f_{\theta_{s}}(x, \theta) \quad \text{for} \quad s = 1, \cdots, n
\end{cases}
\end{equation}

\begin{equation}
F_{\kappa}(X) = \partial_{X_{\kappa}} f(X) = f_{X_{\kappa}}(X) \quad \text{for} \quad \kappa = 1, \cdots, m + n.
\end{equation}
Rémark 4.3.4. (1) In this lecture, we use the left odd derivatives. This naming stemms from putting most left the variable w.r.t. which we differentiate. There are some authors (see, for example Vladimirov and Volovich [129]) who give the name right derivative to this.

Put

\[ C_{SS}^{(r)}(\mathbb{U} : \mathfrak{C}) = \{ f(x, \theta) = \sum_{|\alpha| \leq n} f_a(x)\theta^a \mid f_a(x) \in C_{SS}(\mathbb{U}ev : \mathfrak{C}) \}. \]

For \( f \in C_{SS}^{(r)}(\mathbb{U} : \mathfrak{C}) \) with \( j = 1, 2, \ldots, m \) and \( s = 1, 2, \ldots, n \), we note here the right-derivatives:

\[
\begin{align*}
F_j^{(r)}(X) &= \sum_{|\alpha| \leq n} \partial_{x^j} f_a(x) \theta^a, \\
F_{s+m}^{(r)}(X) &= \sum_{|\alpha| \leq n} (-1)^{r(\alpha)} f_a(x) \theta_{s+1}^{a_1} \cdots \theta_{s+n}^{a_n}
\end{align*}
\]

We put here \( r(\alpha) = \sum_{j=s+1}^{n} a_j \). \( F_{\kappa}^{(r)}(X) \) is called the (right) partial \( \kappa \)-derivative w.r.t. \( X_\kappa \) at \( X = (x, \theta) \) denoted by

\[ F_j^{(r)}(X) = \frac{\partial}{\partial x_j} f(x, \theta) = \partial_{x_j} f(x, \theta), \quad F_{s+m}^{(r)}(X) = f(x, \theta) \frac{\partial}{\partial \theta_s} = f(x, \theta) \frac{\partial}{\partial \theta_s}. \]

(2) Since we use a countably infinite Grassmann generators, the decomposition \( 4.3.8 \) is unique. In fact, if \( \sum_{\alpha} \theta^a f_a(x) = 0 \) on \( \mathbb{U} \), then \( f_a(x) \equiv 0. \) (see, p 322 in Vladimirov and Volovich [129].)

(3) The higher derivatives are defined analogously. For a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m \) and \( a = (a_1, \ldots, a_n) \in \{0, 1\}^n \), we put

\[ \partial^{a_1}_x \cdots \partial^{a_m}_x \quad \text{and} \quad \partial^{a_1}_\theta \cdots \partial^{a_n}_\theta. \]

Assume that for \( X = (x, \theta), Y = (y, \omega) \in \mathfrak{M}^{m|n} \), we have \( X + tY \in \mathbb{U} \) (for any \( t \in [0, 1] \)). Repeating the proof used in the proof of Corollary 4.3.1 for \( f \in C_{SS}(\mathbb{U} : \mathfrak{C}) \), the following holds:

\[
\frac{d}{dt} f(X + tY) \bigg|_{t=0} = \sum_{j=1}^{m} y_j \sum_{i=1}^{n} \omega_i \partial_{x_j} f(X) + \sum_{s=1}^{m} \omega_s \partial_{\theta_s} f(X)
\]

Definition 4.3.3. A function \( f \) from the super domain \( \mathbb{U} \subset \mathfrak{M}^{m|n} \) to \( \mathfrak{C} \), is called G-differentiable at \( X = (x, \theta) \) if

\[ f(x + y, \theta + \omega) - f(x, \theta) = \sum (y_i F_i + \omega_s F_s) + \sum (y_i R_i + \omega_s R_s). \]

Here,

\[ d(R_i, 0) \to 0, \quad d(R_s, 0) \to 0, \quad d_{m|n}((y, \omega), 0) \to 0. \]

4.3.2.1. Taylor’s Theorem. For \( f \in C_{SS}(U : \mathfrak{C}) \), we have

\[
\frac{d}{dt} f(X + tY) \bigg|_{t=0} = \sum_{j=1}^{m} y_j \sum_{i=1}^{n} \omega_i \partial_{x_j} f(X) + \sum_{s=1}^{m} \omega_s \partial_{\theta_s} f(X).
\]

From this, we define

Definition 4.3.4. For a supersmooth function \( f \), we define its differential \( df \) as

\[ df(X) = d_X f(X) = \sum_{\kappa=1}^{m+n} dX_\kappa \frac{\partial f(X)}{\partial X_\kappa}, \]
or
\[
df(x, \theta) = \sum_{j=1}^{m} dx_j \frac{\partial f(x, \theta)}{\partial x_j} + \sum_{s=1}^{n} d\theta_s \frac{\partial f(x, \theta)}{\partial \theta_s}.
\]

From the before mentioned Definition 5.2.1 we have

**Proposition 4.3.2.** Let \( \mathcal{U} \) be a superdomain in \( \mathcal{R}^{m|n} \). For any \( f, g \in \mathcal{C}_{\mathcal{SS}}(\mathcal{U} : \mathcal{C}) \), the product \( fg \) belongs to \( \mathcal{C}_{\mathcal{SS}}(\mathcal{U} : \mathcal{C}) \) and their differentials \( d_Xf(X) \) and \( d_Xg(X) \) are continuous linear maps from \( \mathcal{R}^{m|n} \) to \( \mathcal{C}^{m+n} \).

Moreover,

(1) For any homogeneous elements \( \lambda, \mu \in \mathcal{C} \),
\[
(4.3.13) \quad d_X(\lambda f + \mu g)(X) = (-1)^{p(\lambda)p(X)} \lambda d_X f(X) + (-1)^{p(\mu)p(X)} \mu d_X g(X).
\]

(2) (Leibnitz’ formula)
\[
(4.3.14) \quad \partial_X[f(X)g(X)] = (\partial_X f(X))g(X) + (-1)^{p(X)p(f(X))} f(X)(\partial_X g(X)).
\]

**Proof.** (4.3.13) is trivial. For \( f, g \in \mathcal{C}_{\mathcal{SS}}(\mathcal{U} : \mathcal{C}) \), we have
\[
\left. \frac{d}{dt} f(X + tY)g(X + tY) \right|_{t=0} = \left( \sum_{j=1}^{m} y_j \frac{\partial f}{\partial x_j} + \sum_{s=1}^{n} \omega_s \frac{\partial f}{\partial \theta_s} \right) g(X)
\]
\[
+ f(X) \left( \sum_{j=1}^{m} y_j \frac{\partial g}{\partial x_j} + \sum_{s=1}^{n} \omega_s \frac{\partial g}{\partial \theta_s} \right).
\]

Therefore, we get the desired result. \( \square \)

**4.3.3. Characterization of supersmooth functions.** In previous lecture, we introduce a class \( \mathcal{C}_{\mathcal{SS}}(\mathcal{U} : \mathcal{C}) \) of functions on super domain \( \mathcal{U} \subset \mathcal{R}^{m|n} \). But such introduction is reasonable or it is stable under rather ordinary operations? Or how may we characterize it?

Though there exists multiplication in \( \mathcal{C} \) but not in \( \mathcal{R}^2 \). How the ring structure of the definition domain affects the total-differentiability of functions? How do we characterize such functions?

(a) We decompose a function \( f(z) \) from \( \mathcal{C} \) to \( \mathcal{C} \) as
\[
\mathcal{C} \ni z = x + iy \rightarrow f(z) = u(x, y) + iv(x, y), \quad u(x, y) = \Re f(z) \in \mathcal{R}, \quad v(x, y) = \Im f(z) \in \mathcal{R}
\]
For \( z = x + iy \) and \( z_0 = x_0 + iy_0 \), since \( |z| = \sqrt{x^2 + y^2} \), we have
\[
|f(z) - f(z_0)| = |u(x, y) + iv(x, y) - (u(x_0, y_0) + iv(x_0, y_0))|
= \sqrt{(u(x, y) - u(x_0, y_0))^2 + (v(x, y) - v(x_0, y_0))^2}.
\]

Therefore, if \( f(z) \) is continuous at \( z = z_0 \), \( u(x, y), v(x, y) \) are continuous at \( (x_0, y_0) \) as real-valued functions with 2 real variables.

(b) A function \( f(z) \) from \( \mathcal{C} \) to \( \mathcal{C} \) is called total differentiable at \( z = z_0 \) if there exists a number \( \gamma \in \mathcal{C} \cong L(\mathcal{C} : \mathcal{C}) \) such that it satisfies
\[
(4.3.16) \quad |f(z_0 + w) - f(z_0) - \gamma w| = o(|w|) \quad (|w| \to 0).
\]
This number \( \gamma = \alpha + i\beta \) (\( \alpha, \beta \in \mathcal{R} \)) is denoted by \( f'(z_0) \). We check a little bit more precisely. putting \( w = h + ik \) (\( h, k \in \mathcal{R} \)), then \( \gamma w = (\alpha + i\beta)(h + ik) = (h\alpha - k\beta) + i(k\alpha + h\beta) \), we
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have

\[ |f(z_0 + w) - f(z_0) - \gamma w| \]

\[ = |u(x_0 + h, y_0 + k) - u(x_0, y_0) - (h\alpha - k\beta) \]

\[ + i(v(x_0 + h, y_0 + k) - v(x_0, y_0) - (k\alpha + h\beta))| \]

\[ = \left( |(u(x_0 + h, y_0 + k) - u(x_0, y_0) - (h\alpha - k\beta))^2 \right) \]

\[ + |(v(x_0 + h, y_0 + k) - v(x_0, y_0) - (k\alpha + h\beta))^2|^{1/2} \]

Therefore, when \((h, k) \to 0\) (i.e. \(\sqrt{h^2 + k^2} \to 0\)),

\[ |u(x_0 + h, y_0 + k) - u(x_0, y_0) - (h\alpha - k\beta)| = o(\sqrt{h^2 + k^2}), \]

\[ |v(x_0 + h, y_0 + k) - v(x_0, y_0) - (k\alpha + h\beta)| = o(\sqrt{h^2 + k^2}). \]

From the first equation above, putting \(h = 0\) and \(k \to 0\), we get \(\beta = -u_y(x_0, y_0)\), and putting \(k = 0\) and \(h \to 0\) then \(\alpha = u_x(x_0, y_0)\). From the second one above, we have \(\alpha = v_y(x_0, y_0)\) and \(\beta = v_x(x_0, y_0)\). Therefore, we get a system of PDE

\[ u_x = v_y, \quad u_y = -v_x \]

called Cauchy-Riemann equation. If real valued functions \(u, v\) with two real variables satisfy Cauchy-Riemann equation, then they belong to \(C^\infty\), moreover, \((x, y) + iv(x, y)\) is shown as a convergent power series\(1\) in \(z = x + iy\) which is written \(f(z)\), and called analytic. Without confusion, we write

\[ \frac{\partial}{\partial z} f(z) = f'(z) = \alpha + i\beta = u_x - iu_y = v_y + iv_x. \]

(c) Using above notation, we consider a map \(\Phi\) from \(\mathbb{R}^2\) to \(\mathbb{R}^2\)

\[ \Phi : \mathbb{R}^2 \ni (x, y) \to \left( \begin{array}{c} u(x, y) \\ v(x, y) \end{array} \right) \in \mathbb{R}^2. \]

Denoting \((x_0, y_0)\) by \((x_0, y_0)\), \(\Phi\) is said to be totally differentiable at \((x_0, y_0)\) if there exists \(\Phi'_F(x_0, y_0) \in L(\mathbb{R}^2 : \mathbb{R}^2)\) such that

\[ \|
\Phi(x_0 + h, y_0 + k) - \Phi(x_0, y_0) - \Phi'_F(x_0, y_0) \left( \begin{array}{c} h \\ k \end{array} \right) \| = o(\|
\left( \begin{array}{c} h \\ k \end{array} \right) \|) \]

Representing \(\Phi'_F(x, y)\) as

\[ \Phi'_F(x, y) = \left( \begin{array}{cc} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{array} \right), \]

we have

\[ \left( |(u(x_0 + h, y_0 + k) - u(x_0, y_0) + hu_x(x_0, y_0) + kuy_y(x_0, y_0))^2 \right) \]

\[ + |(v(x_0 + h, y_0 + k) - v(x_0, y_0) + hv_x(x_0, y_0) + kv_y(x_0, y_0))^2|^{1/2} = o(\sqrt{h^2 + k^2}), \]

i.e.

\[ |u(x_0 + h, y_0 + k) - u(x_0, y_0) - (hu_x(x_0, y_0) + kuy_y(x_0, y_0))| = o(\sqrt{h^2 + k^2}), \]

\[ |v(x_0 + h, y_0 + k) - v(x_0, y_0) - (hv_x(x_0, y_0) + kv_y(x_0, y_0))| = o(\sqrt{h^2 + k^2}). \]

Identifying \(\mathbb{R}^2\) as \(\mathbb{C}\), we seek a condition that \(\Phi'_F(x_0, y_0) \in L(\mathbb{R}^2 : \mathbb{R}^2)\) is regarded as a multiplication in \(\mathbb{C}\). When an element \(a + ib \in \mathbb{C}\) acts as a multiplication operator, then a linear operator in \(\mathbb{R}^2\), that is, a matrix is identified with \(a + ib \sim \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \) . Since

\[ \Phi'_F(x, y) = \left( \begin{array}{cc} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{array} \right), \]

we have \(u_x = v_y, u_y = -v_x\). In another presentation, \(\Phi'_F(x_0, y_0) \in L(\mathbb{R}^2 : \mathbb{R}^2)\) is not only \(\mathbb{R}\) but also \(\mathbb{C}\)-linear, that is, for any \(a, b \in \mathbb{R}\),

\[ \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \Phi'_F(x_0, y_0) = \Phi'_F(x_0, y_0) \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right). \]

---

1In general, this is proved by applying Cauchy’s integral representation
hold. Here, \( b \neq 0 \) is essential.

(d) Generalizing above to a map \( \phi \) from \( \mathbb{C}^m \) to \( \mathbb{C} \),

\[
\phi : \mathbb{C}^m \ni z = (z_1, \cdots, z_m) \to \phi(z) \in \mathbb{C}.
\]

This is totally differentiable if there exists \( \phi'(z) \in L(\mathbb{C}^m : \mathbb{C}) \) at \( z \) such that

\[
\|\phi(z + w) - \phi(z) - \phi'(z)w\| = o(\|w\|).
\]

By same calculation, \( \phi'(z) \) is

\[\phi'(z) = \left( \frac{\partial \phi(z)}{\partial z_1}, \cdots, \frac{\partial \phi(z)}{\partial z_m} \right)\]

and each component of \( \phi \) is analytic w.r.t. each variable. From Hartogs’ theorem, \( \phi \) is holomorphic having a convergent power series expansion.

(c) A map \( \Phi \) from \( \mathbb{C}^m \) to \( \mathbb{C}^n \) is totally differentiable at \( z \) if there exists \( \Phi'(z) \in L(\mathbb{C}^m : \mathbb{C}^n) \)

\[
\Phi'(z) = \begin{pmatrix}
\frac{\partial \Phi_1(z)}{\partial z_1} & \cdots & \frac{\partial \Phi_1(z)}{\partial z_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial \Phi_n(z)}{\partial z_1} & \cdots & \frac{\partial \Phi_n(z)}{\partial z_m}
\end{pmatrix}
\]

such that

\[
\|\Phi(z + w) - \Phi(z) - \Phi'(z)w\| = o(\|w\|).
\]

**Problem 4.3.2.** Does there exist Cauchy-Riemann equation corresponding to supersmooth functions?

### 4.4. Super differentiable functions on \( \mathbb{R}^{m|n} \)

#### 4.4.1. Superdifferentiability of functions on \( \mathbb{R}^{m|n} \).

**Definition 4.4.1** (see, Yagi [135]). Let \( f \) be a \( \mathcal{C} \)-valued function on a superdomain \( \mathcal{U} \subset \mathbb{R}^{m|n} \). Then, a function \( f \) is said to be super \( C^1_G \)-differentiable, denoted by \( f \in \mathcal{G}^1_{SD}(\mathcal{U} : \mathcal{C}) \) or simply \( f \in \mathcal{G}^1_{SD} \) if there exist \( \mathcal{C} \)-valued continuous functions \( F_A \) (1 \( \leq A \leq m+n \)) on \( \mathcal{U} \) such that

\[
\frac{d}{dt} f(X + tH) \bigg|_{t=0} = f'_G(X,H) = \sum_{A=1}^{m+n} H_A F_A(X)
\]

or

\[f(X + tH) - f(X) - tf'_G(X,H) \to 0 \text{ in } \mathcal{C}, \text{ when } H \to 0 \text{ in } \mathcal{C},\]

for each \( X \in \mathcal{U} \) and \( H \in \mathbb{R}^{m|n} \) where \( f(X + tH) \) is considered as a \( \mathcal{C} \)-valued function w.r.t. \( t \in \mathbb{R} \).

We denote \( F_A(X) \) by \( f_{X,A}(X) \). Moreover, for \( r \geq 2 \), \( f \) is said to be in \( \mathcal{G}^r_{SD} \) if \( F_A \) are \( \mathcal{G}^{r-1}_{SD} \). \( f \) is said to be \( \mathcal{G}^\infty_{SD} \) or superdifferential if \( f \) is \( \mathcal{G}^r_{SD} \) for all \( r \geq 1 \).

**Definition 4.4.2.** Let \( f \) be a \( \mathcal{C} \)-valued function on a superdomain \( \mathcal{U} \subset \mathbb{R}^{m|n} \). A function \( f \) is said to be super \( C^1_F \)-differentiable, denoted by \( f \in \mathcal{F}^1_{SD}(\mathcal{U} : \mathcal{C}) \) or simply \( f \in \mathcal{F}^1_{SD} \) if there exist \( \mathcal{C} \)-valued continuous functions \( F_A \) (1 \( \leq A \leq m+n \)) on \( \mathcal{U} \) and functions \( \rho_A : \mathcal{U} \times \mathbb{R}^{m|n} \to \mathcal{C} \) such that

\[
f(X + H) - f(X) = \sum_{j=1}^{m+n} H_A F_A(X) + \sum_{A=1}^{m+n} H_A \rho_A(X;H) \quad \text{for } X \in \mathbb{R}^{m|n},
\]

\[
\rho_A(X,H) \to 0 \text{ in } \mathcal{C} \quad \text{when } H \to 0 \text{ in } \mathbb{R}^{m|n},
\]
for each $X \in \mathfrak{U}$ and $X + H \in \mathfrak{U}$. $f$ is said to be super $C^2_F$-differentiable, when $F_A \in \mathfrak{F}_SD^1(\mathfrak{R}^{m|n} ; \mathfrak{C})$ $(1 \leq A \leq m + n)$. Analogously, we may define super $C^r_F$-differenitability and we say it superdifferentiable if it is super $C^\infty_F$-differentiable, denoted by $\mathfrak{F}_SD^\infty$.

**QUESTION 4.4.1.** Does there exist the difference between $\mathfrak{F}_SD^1$ and $\mathfrak{F}_SD^1$, or between $\mathfrak{F}_SD^\infty$ and $\mathfrak{F}_SD^\infty$?

**REMARK 4.4.1.** Let $\mathfrak{B}$ be an open set $\mathfrak{C}^{m|n}$. When $f : \mathfrak{B} \to \mathfrak{C}$ is in $\mathfrak{F}_SD^\infty$, $f$ is also said to be superanalytic.

### 4.4.2. Remarks on Grassmann continuation

From Taylor’s expansion formula (4.2.2) mentioned before in general Fréchet space, we get

**LEMMA 4.4.1.** For $f(q) \in C^\infty(\mathbb{R}^m)$, its Grassmann continuation $\tilde{f}$ has the following Taylor’s expansion formula: For any $N$, there exists $\tilde{\tau}_N(x, y) \in \mathfrak{C}$ such that

\[
\tilde{f}(x + y) = \sum_{|\alpha| = 0}^{N} \frac{1}{\alpha!} \partial_{\alpha}^0 \tilde{f}(x)y^\alpha + \tilde{\tau}_N(f; x, y). \tag{4.4.3}
\]

Here,

\[
\tilde{\tau}_N(f; x, y) = \sum_{|\alpha| = N+1} y^\alpha \int_0^1 dt \frac{1}{N!} (1-t)^N \partial_{\alpha}^0 \tilde{f}(x + ty).
\]

**Proof:** Putting $q = x_B$ and $q' = y_B$ into Taylor’s expansion formula (4.2.2), we have

\[
f(q + q') = \sum_{|\alpha| = 0}^{N} \frac{1}{\alpha!} \partial_{\alpha}^0 f(q)q^\alpha + \sum_{|\alpha| = N+1} q^\alpha \int_0^1 dt \frac{1}{N!} (1-t)^N \partial_{\alpha}^0 f(q + tq').
\]

Taking Grassmann continuation of both sides, and remarking $\partial_{\alpha}^0 f(x) = \partial_{\alpha}^0 \tilde{f}(x)$, $\tilde{q}^\alpha = y^\alpha$, we get the desired equality (4.4.3). \qed

**COROLLARY 4.4.1.** For $f(q) \in C^\infty(\mathbb{R}^m)$, $\tilde{f}$ is super $F$-differentiable.

**Proof:** We prove the case $m = 1$. From Lemma, we have

\[
\tilde{f}(x + y) - \tilde{f}(x) = y\tilde{f}'(x) + \tau(x, y), \quad \tau(x, y) = y \int_0^1 dt (1-t)\tilde{f}''(x + ty)
\]

and because when $y \to 0$ in $\mathfrak{R}_{ev}$, then $\tau(x, y) \to 0$ in $\mathfrak{C}$. \qed

**EXERCISE 4.4.1.** Prove more precisely, the statement above “when $y \to 0$ in $\mathfrak{R}_{ev}$, then $\tau(x, y) \to 0$ in $\mathfrak{C}$”.

**COROLLARY 4.4.2.** For $f(q) \in C^\infty(\mathbb{R}^m)$, its Grassmann continuation $\tilde{f}(x) \in \mathfrak{F}_SD^1(\mathfrak{R}^{m|n})$.

**Proof:** Putting $N = 1$ in Taylor’s expansion formula (4.2.2), we get

\[
\tilde{f}(x + y) = \tilde{f}(x) + \sum_{j=1}^{m} y_j \partial_{x_j} \tilde{f}(x) + \sum_{j=1}^{m} y_j \rho_j(x, y)
\]
Remark

\[ \frac{\partial}{\partial x_j} \tilde{f}(x) = \tilde{\partial}_{q_j} f(x) \in \mathcal{C}, \]

(4.4.4)  
\[ \rho_j(x, y) = \sum_{k=1}^{m} y_k g_{j,k}(x, y), \quad \varrho_j(x, y) = \int_{0}^{1} (1 - t) \frac{\partial^2}{\partial x_k \partial y} \tilde{f}(x + ty) dt, \]

we need to prove

**Claim 4.4.1.** When \( y \to 0 \) in \( \mathfrak{R}^{m}_{ev} \), then \( \rho_j(x, y) \to 0 \) for each \( x \in \mathfrak{R}^{m}_{ev} \) and \( j = 1, \ldots, m \).

That is, for any \( \epsilon > 0 \), \( j \) and \( x \in \mathfrak{R}^{m}_{ev} \), there exists \( \delta > 0 \) such that if \( \text{dist}_{m|0}(y) < \delta \), then \( \text{dist}_{m|0}\rho_j(x, y) < \epsilon \).

**Proof.** Take any \( I \in \mathcal{I}_{ev} \) and decompose \( I = J + K \). Remarking Corollary 4.4.1, we have

\[ \frac{\partial^2}{\partial x_k \partial y} \tilde{f}(x + ty) = \sum_{|\alpha| = 0}^{\infty} \frac{\partial^2}{\partial x_k \partial y} f(x + ty) \frac{\alpha!}{\alpha!} (xS + tys)\alpha. \]

If \( I = 0 = (0, 0, 0, \ldots) \), then

\[ |\text{proj}_{0}\rho_j(x, y)| \leq \sum_{k=1}^{m} |y_kB||\piBg_{j,k}(x, y)| \to 0 \quad \text{for } yB \to 0. \]

For fixed \( I \neq 0 \), the family of index sets \( \{K \subset I\} \) has finite elements, \( \{x_B + tyB \mid t \in [0, 1], |yB| \leq 1\} \) is compact, if taking \( \alpha \) such that \( 2|\alpha| > |K| \), then \( \text{proj}_{K}(xS + tys)\alpha = 0 \). Therefore, there exists a constant \( C_K = C_K(xS, ys) \) such that

\[ \left| \text{proj}_{K} \left( \sum_{|\alpha| = 0}^{\infty} \frac{\partial^2}{\partial x_k \partial y} f(x + ty) \frac{\alpha!}{\alpha!} (xS + tys)\alpha \right) \right| \leq \sum_{\alpha} \left| \frac{\partial^2}{\partial x_k \partial y} f(x + ty) \frac{\alpha!}{\alpha!} \right| \text{proj}_{K}(xS + tys)\alpha \leq C_K. \]

In fact, if \( 2|\alpha| > |K| \) then \( \text{proj}_{K}(xS + tys)\alpha = 0 \), then

\[ \left| \text{proj}_{K}(g_{j,k}(x, y)) \right| \leq \int_{0}^{1} dt (1 - t) \left| \text{proj}_{K} \left( \frac{\partial^2}{\partial x_k \partial y} \tilde{f}(x + ty) \right) \right| \leq \sum_{2|\alpha| \leq |I|} \frac{\max_{t} \left| \frac{\partial^2}{\partial x_k \partial y} f(x + ty) \right|}{\alpha!} \int_{0}^{1} dt \left| \text{proj}_{K}(xS + tys)\alpha \right|, \]

and

\[ C_K = C_K(xS, ys) = \int_{0}^{1} dt \left| \text{proj}_{K}(xS + tys)\alpha \right| \to 0 \quad \text{when } ys \to 0 \quad \text{in } \mathcal{C}, \]

therefore

\[ \left| \text{proj}_{I}(\rho_j(x, y)) \right| \leq \sum_{k=1}^{m} \sum_{I=J+K} \left| \text{proj}_{J}(y_k) \right| \left| \text{proj}_{K}(g_{j,k}(x, y)) \right| \leq \sum_{k=1}^{m} \sum_{I=J+K} \left| \text{proj}_{J}(y_k) \right| C_K. \]

This finite sum tends to 0 when \( y \to 0 \), this implies \( \tilde{f}(x) \in \mathfrak{R}^{1}_{SD}(\mathfrak{R}^{m|0}). \)

**Remark 4.4.2.** By the way, concerning the Grassmann continuation \( \tilde{f} \) of \( f \) de Witt\cite{de Witt} claimed in p.7 as follows:
"The presence of a soul in the independent variable evidently has little practical effect on the variety of functions with which one may work in applications of the theory. In this respect $\mathbb{R}_{ev}$ is a harmless generalization of its own subspace $\mathbb{R}$, the real line."

Though he didn’t give more explanation of this intuitional claim in [34], but we interpret his saying as

**Proposition 4.4.1.** Let $F \in C^\infty_G(\mathbb{R}^{m|0} : \mathbb{C})$. Putting $f(q) = F(q)$ for $q \in \mathbb{R}^m$, we have $\tilde{f} = F$.

We rephrase this as

**Claim 4.4.2.** Let a $C^\infty_G$ differentiable function $H = H(x) = \sum_{J \in I} H_J(x)\sigma^J$ be given as a map from $\mathbb{R}^{m|0}$ to $\mathbb{C}$ such that it is 0 on $\mathbb{R}^m$. That is, for any $\alpha$ and $q \in \mathbb{R}^m$, if $\partial^\alpha q H_J(q) = 0$, then $H$ equals to 0 on $\mathbb{R}^{m|0}$, i.e. for any $I \in I$, $\text{proj}_I(H(x)) = 0$.

**Proof:** We apply Taylor’s expansion formula (14.22) once more: For any $N$ and $J$, remarking $\partial^\alpha q H_J(q) = 0$, we have

$$H_J(x_B + x_S) = \tau_N(H_J; x_B, x_S) = \sum_{|\alpha| = N+1} x_S^\alpha \int_0^1 dt \frac{1}{N!} (1-t)^N \partial^\alpha x H_J(x_B + tx_S).$$

We need to show that for any $I$, $\text{proj}_I(\tau_N(H_J; x_B, x_S)) = 0$. Since all terms consisting of $x_S^\alpha$ have at least $2|\alpha|$ as the degree of Grassmann generators, if $2|\alpha| > |I| \geq 0$ then $\text{proj}_I(x_S^\alpha) = 0$. Taking $N$ sufficiently large such that $\text{proj}_I(x_S^\alpha) = 0$ for all $\alpha$ with $|\alpha| = N + 1$, then for any $J$, $\text{proj}_I(\tau_N(H_J; x_B, x_S)) = 0$. Therefore,

$$\text{proj}_I(\sum_{J \in I} \sigma^J \tau_N(H_J; x_B, x_S)) = 0.$$ //

The proof of Proposition 4.4.1 is given by applying above Claim to $H(x) = F(x) - \tilde{f}(x)$.  

Following is claimed as (1.1.17) in [34] without proof and cited as Theorem 1 in [92].

**Claim 4.4.3.** Let $f$ be an analytic function from an open set $V \subset \mathbb{C}$ to $\mathbb{C}$. Then, we have an unique Grassmann continuation $\tilde{f} : \mathbb{C}^{1|0} \rightarrow \mathbb{C}$ which is super analytic.

$$\tilde{f}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_B)z_S^n \quad \text{for } z = z_B + z_S \text{ with } z_B \in V. \quad (4.4.5)$$

That is,

$$\tilde{f}(z + w) = \tilde{f}(z) + wF(z) + w\rho_j(z, w) \quad \text{when } w \rightarrow 0 \text{ in } \mathbb{C}^{1|0}, \text{ then } \rho_j(z, w) \rightarrow 0 \text{ in } \mathbb{C}.$$

**Remark 4.4.3.** Above Claim itself is proved, since $F$-differentiability of $\tilde{f} \in \mathcal{F}_SD^1(\mathbb{C}^{1|0} : \mathbb{C})$ from $\mathcal{C}$ to $\mathbb{C}$ is shown, by applying Corollary 4.4.2. Moreover, from Theorem 4.4.2 below, $\tilde{f} \in \mathcal{F}_SD^{\infty}(\mathbb{C}^{1|0} : \mathbb{C})$, i.e. $\tilde{f}$ is super analytic.

But I want to point out the argument in the proof of Proposition 4.3.1 in Matsumoto-Kakazu [92] which seems not transparent. They claim the convergence of the right-hand side of (4.4.5) and using this, they proceed as follows:

$$\tilde{f}(z + w) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_B + w_B)(z_S + w_S)^n$$
From this expression, they conclude that $\tilde{f}$ is super analytic. Surely, from this expression, putting $F(z) = \tilde{f}(1)(z)$, $\rho(z, w) = \sum_{n=2}^{\infty} \frac{f(n)}{n!} (z w)^{n-1}$, they have

$$F(z) + w \rho(z, w).$$

But we need to show that $F(z)$ is continuous w.r.t. $z$ and $\rho(z, w)$ is horizontal w.r.t $w$ to claim $\tilde{f}$ is super analytic. This horizontality is not so clear from their last argument. To clarify this, I propose to use the analogous proof in Claim 4.4.2.

Remark 4.4.4. If $f$ is real analytic on $\mathbb{R}^n$, there exists a function $\delta(q) > 0$ such that for $|q'| \leq \delta(q)$, $f(q + q')$ has Taylor expansion at $q$. From above proof, $\tilde{f}(x + y)$ is Pringsheim regular w.r.t $|y_B| \leq \delta(x_B)$. Here, those who is not familiar with Pringsheim regular, please check it in inter-net.

4.4.3. Super smooth functions on superdomain. For future use, we prepare some algebraic lemmas.

Lemma 4.4.2. Suppose that there exist elements $\{A_i\}_{i=1}^{\infty} \subset \mathfrak{R}_{od}$ satisfying

\begin{equation}
\sigma_j A_i + \sigma_i A_j = 0 \quad \text{for any } i, j \in \mathbb{N}
\end{equation}

Then there exists a unique element $F \in \mathfrak{R}$ such that $A_i = \sigma_i F$ for $i = 1, \cdots, \infty$.

Proof. We follow the argument in Lemma 4.4 of [135]. Since $A_i$ is represented by $A_i = \sum_{J \in \mathcal{I}} a^J_i \sigma^J$ with $a^J_i \in \mathbb{C}$ and $\sigma_i A_i = 0$, we have $\sum_{\{J \mid j_i = 0\}} a^J_i \sigma^J = 0$. Therefore, each $A_i$ can be written uniquely as $A_i = (\sum_{\{J \mid j_i = 0\}} b^J_i \sigma^J) \sigma_i$ for some $b^J_i \in \mathbb{C}$. From the condition (4.4.6), we have $b^J_i = b^J_j$ for $J$ with $j_i = j_j = 0$. Letting $b^J_i = b^J_j$ for $\{J \mid j_i = 0\}$, we put

$$F = \sum_{J \in \mathcal{I}} b^J_i \sigma^J = \sum_{i=1}^{\infty} \left( \sum_{\{J \mid j_i = 0\} \in \mathcal{I}} b^J_i \sigma^J \right)$$

which is well-defined and further more $A_i = \sigma_i F$ holds for each $i$. Since we may change the order of summation freely in $\mathfrak{R}$, we have

$$F = \sum_{\{J \mid j_i = 0\}} b^J_i \sigma^J + \sum_{\{J \mid j_i \neq 0\}} b^J_i \sigma^J = \sum_{\{J \mid j_i = 0\}} b^J_i \sigma^J + \sum_{\{J \mid j_i \neq 0\}} b^J_i \sigma^J. \quad \square$$

Repeating above argument, we have
Corollary 4.4.3 (Lemma 4.4 of [135]). Let \( \{ A_J \in \mathcal{R} \mid |J| = \text{od} \} \) satisfy
\[
\sigma^K A_J + \sigma^J A_K = 0 \quad \text{for } J, K \in \mathcal{I}_{\text{od}}.
\]
Then there exists a unique element \( F \in \mathcal{R} \) such that \( A_J = \sigma^J F \) for \( J \in \mathcal{I}_{\text{od}} \).

**Definition 4.4.3.** We denote the set of maps \( f : \mathcal{R}_{\text{od}} \to \mathcal{R} \) which are continuous and \( \mathcal{R}_{\text{ev}} \)-linear (i.e. \( f(\lambda X) = \lambda f(X) \) for \( \lambda \in \mathcal{R}_{\text{ev}}, X \in \mathcal{R}_{\text{od}} \)) by \( f \in L_{\mathcal{R}_{\text{ev}}} (\mathcal{R}_{\text{od}} : \mathcal{R}) \).

Corollary 4.4.4 (The self-duality of \( \mathcal{R} \)). For \( f \in L_{\mathcal{R}_{\text{ev}}} (\mathcal{R}_{\text{od}} : \mathcal{R}) \), there exists an element \( u_f \in \mathcal{R} \) satisfying
\[
f(X) = X u_f \quad \text{for } X \in \mathcal{R}_{\text{od}}.
\]

**Proof.** Since \( f : \mathcal{R}_{\text{od}} \to \mathcal{R} \) is \( \mathcal{R}_{\text{ev}} \)-linear, we have \( f(XYZ) = XY f(Z) = -XZ f(Y) \) for any \( X, Y, Z \in \mathcal{R}_{\text{od}} \). By putting \( X = \sigma_k, Y = \sigma_j, Z = \sigma_i \) and \( f_i = f(\sigma_i) \in \mathcal{R} \) for \( i = 1, \ldots, \infty \), we have \( \sigma_k(\sigma_j f_i + \sigma_i f_j) = 0 \) for any \( k \). Therefore, \( \sigma_j f_i + \sigma_i f_j = 0 \), and by Lemma above, there exists \( u_f \in \mathcal{R} \) such that \( f_i = \sigma_i u_f \) for \( i = 1, \ldots, \infty \).

For \( I = (i_1, \ldots) \in \{0, 1\}^\mathbb{N} \) and \( |I| = \text{odd} \), then \( i_k = 1 \) for some \( k \). Rewrite \( I = (-1)^{i_1 + \cdots + i_k-1} \sigma_k \sigma_i \) with \( \tilde{I}_k = (i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots) \), by \( \mathcal{R}_{\text{ev}} \)-linearity of \( f \), then we have \( f(\sigma^I) = (-1)^{i_1 + \cdots + i_k - 1} \sigma_k f(\sigma_i) \).

Then, this map is well-defined because of \( \sigma_j f_i + \sigma_i f_j = 0 \), that is, it doesn’t depend on other decomposition of \( I \).

: \( \vdash \) Put \( \tilde{I} = (i_1, \ldots, i_{j-1}-1, 0, i_{j+1}, \ldots, i_k-1, 0, i_{k+1}, \ldots) \), for \( I = (i_1, \ldots, i_{j-1}, 1, i_{j+1}, \ldots, i_k, 1, i_{k+1}, \ldots) \).

Then, for \( \ell = i_1 + \cdots + i_{j-1} + i_{j+1} + \cdots + i_k-1 + 1, i_{k+1}, \ldots ) \in \mathbb{N} \), remarking \( |I| = \text{odd} \), we have
\[
\sigma^I = \begin{pmatrix}
\sigma_\ell \\
\sigma_j 
\end{pmatrix} = \begin{pmatrix}
(-1)^{i_1} \sigma_k \sigma_j \\
(-1)^i \sigma_k \sigma_j 
\end{pmatrix},
\]

\[
f((-1)^{i_1} \sigma_k \sigma_j f^I) = (-1)^{i_1} \sigma_j f(\sigma_i),
\]

\[
f((-1)^{i_1} \sigma_k \sigma_j f^I) = (-1)^{i_1} \sigma_k f(\sigma_j).
\]

By \( \sigma_j f_i + \sigma_k f_j = 0 \), we have \( f((-1)^i \sigma_k \sigma_j f^I) - f((-1)^{i_1} \sigma_k \sigma_j f^I) = (-1)^{i_1} \sigma_l [\sigma_j f_k + \sigma_k f_j] = 0 \).

We extend \( f \) as \( \tilde{f}(X) = \sum_{I \in \mathbb{Z}} X_I f(I^I) \) for \( X = \sum_{I \in \mathbb{Z}} X_I \sigma^I \in \mathcal{R}_{\text{od}} \). Then, since \( X_I \in \mathbb{C} \), \( \tilde{f}(X) = \sum_{I \in \mathbb{Z}} X_I \sigma^I u_f = X u_f \). In fact, if \( I \) with \( |I| = \text{odd} \) with \( i_k \neq 0 \), then, by \( f_k = f(\sigma_k) = \sigma u_f \) and \( \mathcal{R}_{\text{ev}} \)-linearity,
\[
\tilde{f}(\sigma^I) = f(\sigma^I) = (-1)^{i_1 + \cdots + i_k - 1} \sigma_k f(\sigma_k) = (-1)^{i_1 + \cdots + i_k - 1} \sigma_k \sigma_k u_f = \sigma^I u_f.
\]

Clearly \( \tilde{f}(X) = f(X) \). \( \square \)

**Remark 4.4.5.** K. Masuda gives the following example which exhibits that \( \mathcal{B}_L \) is not necessarily self-dual.

A counter-example\(^2\): Let \( L = 2 \). Define a map \( f \) as
\[
f(X_1 \sigma_1 + X_2 \sigma_2) = X_1 \sigma_2 \quad \text{for any } X_1, X_2 \in \mathbb{R}.
\]
Then, remarking that \( (b_0 + b_1 \sigma_1 \sigma_2)(X_1 \sigma_1 + X_2 \sigma_2) = b_0(X_1 \sigma_1 + X_2 \sigma_2) \), we have readily \( f \in L_{\mathcal{B}_{2,\text{ev}}}(\mathcal{B}_{2,\text{od}} : \mathcal{B}_2) \). If we assume that there exists a \( u_f \in \mathcal{B}_2 \) such that \( f(X) = X u_f \), then \( \sigma_1 f(\sigma_1) = \sigma_1 \cdot \sigma_1 u_f = 0 \) but \( \sigma_1 f(\sigma_1) = \sigma_1 \cdot \sigma_2 \neq 0 \), contradiction! Hence, there exists no \( u_f \in \mathcal{B}_2 \) such that \( f(X) = X u_f \).

\(^2\)Though I don’t recognize at first reading, but analogously examples are considered in [15] or [80]
Repeating the argument in proving Corollary 4.3.1 we get

\[(4.4.7) \quad f \in C_{SS}(\mathcal{U} : \mathcal{C}) \implies \frac{d}{dt}f(X + tY) \bigg|_{t=0} = \sum_{j=1}^{m} y_j \frac{\partial f(X)}{\partial x_j} + \sum_{s=1}^{m} \omega_s \frac{\partial \theta_s}{\partial \theta_s} f(X)\]

where \(X = (x, \theta), Y = (y, \omega) \in \mathbb{R}^{m,n}\) such that \(X + tY \in \mathcal{U}\) for any \(t \in [0, 1]\). That is,

**Corollary 4.4.5.** \(C_{SS}(\mathcal{U} : \mathcal{C}) \implies \mathfrak{G}_{SD}^1(\mathcal{U} : \mathcal{C})\).

To relate the definitions \(C_{SS}\) and \(\mathfrak{G}_{SD}^\infty\) or \(\mathfrak{S}_{SD}^\infty\), we need the following notion.

**Definition 4.4.4 (p.246 of [135]).** Let \(\mathcal{U}\) be an open set in \(\mathbb{R}^{m,n}\) and \(f : \mathcal{U} \to \mathbb{R}\) (or \(\to \mathbb{C}\)). \(f\) is said to be admissible on \(\mathcal{U}\) if there exists some \(L \geq 0\) and a \((\mathbb{R}\text{ or } \mathbb{C})\)-valued function \(\phi\) defined on \(\mathcal{U}_L = p_L(\mathcal{U})\) such that \(f(X) = \phi \circ p_L(X) = \phi(p_L(X))\). For \(r\) with \((0 \leq r \leq \infty)\), \(f\) is said to be admissible \(C^r\) (or simply \(f \in C^r(\mathcal{U} : \mathcal{C})\)) if \(\phi \in C^r(\mathcal{U}_L : \mathbb{R})\) or \(C^r(\mathcal{U}_L : \mathbb{C})\).

Let \(f(X) = \sum_{I \in \mathcal{I}} \sigma^I \cdot f_I(X)\) with \(f_I\) is \((\mathbb{R}\text{ or } \mathbb{C})\)-valued on \(\mathcal{U}\). For each \(I \in \mathcal{I}\), if \(f_I\) is admissible \(C^r\) (or simply \(f_I \in C^r(\mathcal{U} : \mathcal{C})\)) on \(\mathcal{U}\), \(f \in C^r(\mathcal{U} : \mathcal{C})\) is called admissible on \(\mathcal{U}\). More precisely, there exists some \(L_I \geq 0\) and a \((\mathbb{R}\text{ or } \mathbb{C})\)-valued function \(\phi_I\) defined on \(\mathcal{U}_L = p_L(\mathcal{U})\) such that \(f_I(X) = \phi_I \circ p_L(X) = \phi_I(p_L(X))\). Moreover, we define its partial derivatives by

\[
\frac{\partial f}{\partial X_{A,K}} = \sum_{J} \sigma^J \cdot \frac{\partial f_J}{\partial X_{A,K}} = \begin{cases} |K| = \text{ev} & \text{if } 1 \leq A \leq m, \\ |K| = \text{od} & \text{if } m + 1 \leq A \leq m + n. \end{cases}
\]

**Definition 4.4.5 (p.246 of [135]).** A \((\mathbb{R}\text{ or } \mathbb{C})\)-valued function \(f\) on \(\mathcal{U}\) is said to be projectable if for each \(L \geq 0\), there exists a \((\mathbb{R}\text{ or } \mathbb{C})\)-valued function \(f_L\) defined on \(\mathcal{U}_L \subset \mathbb{R}^{m,n}\) such that \(p_L \circ f = f_L \circ p_L\) on \(\mathcal{U}\).

**Claim 4.4.4.** A projectable function on \(\mathcal{U}\) is also admissible on \(\mathcal{U}\).

**Proof.** We use the map \(\text{proj}_I : \mathbb{R} \ni X = \sum_{I \in \mathcal{I}} X_I \sigma^I \to X_I \in \mathbb{R}(\text{or } \mathbb{C})\) introduced in §2. Then, for each \(I \in \mathcal{I}\), taking \(L\) such that \(I \in \mathcal{I}_L\), we have

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{f} & \mathbb{R} \\
\text{proj}_I & \downarrow & \downarrow \text{proj} \circ f \\
\mathcal{U}_L & \xrightarrow{f_L} & \mathbb{R}_L
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{f} & \mathbb{R} \\
\text{proj}_I & \downarrow & \downarrow \text{proj} \circ f \\
\mathcal{U}_L & \xrightarrow{f_L} & \mathbb{R}_L & \square
\end{array}
\]

**Theorem 4.4.1 (Theorem 1 of [135]).** Let \(\mathcal{U}\) be a convex open set in \(\mathbb{R}^{m,n}\). If \(f : \mathcal{U} \to \mathbb{R}\) is in \(\mathfrak{G}_{SD}^1\), then \(f\) is projectable and \(C^1\) on \(\mathcal{U}\).

**Proof.** Since \(\frac{4}{3} f(X + tH) = \sum_{A=1}^{m+n} H_A F_A(X + tH)\), we have

\[
f(X + H) - f(X) = \int_{0}^{1} \frac{d}{dt} f(X + tH) dt = \sum_{A=1}^{m+n} H_A \int_{0}^{1} F_A(X + tH) dt.
\]

This means that if \(p_L(H_A) = 0\), then \(p_L(f(X + H) - f(X)) = 0\). Therefore if we define \(f_L : \mathcal{U}_L \to \mathcal{U}_L\) by \(f_L(p_L(Z)) = p_L(f(Z))\), then it implies that \(f\) is projectable and so admissible. For
\[ E_{A,K} = \sigma^K e_A \in \mathbb{R}^{m+n} \text{ with } e_A = \underbrace{0, \cdots, 0, 1, 0, \cdots, 0}_{m+n}, \] we have
\[
\frac{\partial}{\partial X_{A,K}} f(X) = \frac{d}{dt} f(X + t E_{A,K}) \bigg|_{t=0} = \sigma^K F_A(X), \quad |A| = |K|.
\]
f_L is \(C^1\) on \(\mathcal{U}_L\), thus the function \(f\) is admissible \(C^1\) on \(\mathcal{U}\). \(\square\)

### 4.4.4. Cauchy-Riemann relation

To understand the meaning of supersmoothness, we consider the dependence with respect to the “coordinate” more precisely.

**Proposition 4.4.2 (Theorem 2 of [135])**. Let \(f(X) = \sum f_1(X) \sigma^I \in G^\infty_{SD}(\mathcal{U} : \mathcal{C})\) where \(\mathcal{U}\) is a superdomain in \(\mathbb{R}^{m|n}\). Let \(X = (X_A)\) be represented by \(X_A = \sum X_{A,0} \sigma^I\) where \(A = 1, \ldots, m+n\), \(X_{A,1} \in \mathbb{C}\) for \(|I| \neq 0\) and \(X_{A,0} \in \mathbb{R}\). Then, \(f(X)\), considered as a function of countably many variables \(\{X_{A,1}\}\) with values in \(\mathcal{C}\), satisfies the following (Cauchy-Riemann type) equations.

\[
\begin{align*}
\frac{\partial}{\partial X_{A,1}} f(X) &= \sigma^I \frac{\partial}{\partial X_{A,0}} f(X) \quad \text{for } 1 \leq A \leq m, \quad |I| = \text{ev}, \\
\sigma^K \frac{\partial}{\partial X_{A,j}} f(X) + \sigma^I \frac{\partial}{\partial X_{A,K}} f(X) &= 0 \quad \text{for } m+1 \leq A \leq m+n, \quad |J| = \text{od} = |K|.
\end{align*}
\]

Here, we define

\[
\frac{\partial}{\partial X_{A,1}} f(X) = \frac{d}{dt} f(X + t E_{A,1}) \bigg|_{t=0} \quad \text{with } E_{A,1} = \sigma^I e_A = \underbrace{0, \cdots, 0, 1, 0, \cdots, 0}_{m+n} \in \mathbb{R}^{m|n}.
\]

Conversely, let a function \(f(X) = \sum f_1(X) \sigma^I\) be given such that \(f_1(X + t Y) \in C^\infty([0,1] : \mathbb{C})\) for each fixed \(X, Y \in \mathcal{U}\) and \(f(X)\) satisfies above (4.4.8) with (4.4.9). Then, \(f \in G^\infty_{SD}(\mathcal{U} : \mathcal{C})\).

**Proof.** Replacing \(Y\) with \(E_{A,j}\) with \(1 \leq A \leq m\) and \(|J| = \text{ev}\) in (4.4.7), we get readily the first equation of (4.4.8). Here, we have used (4.3.5). Considering \(E_{A,j}\) or \(E_{A,K}\) for \(m+1 \leq A \leq m+n\) and \(|J| = \text{od} = |K|\) in (4.4.7) and multiplying \(\sigma^K\) or \(\sigma^J\) from left, respectively, we have the second equality in (4.4.8) readily.

To prove the converse statement, we have to construct functions \(F_A(1 \leq A \leq m+n)\) which satisfies

\[
\frac{d}{dt} f(X + t H) \bigg|_{t=0} = \sum_{A=1}^{m+n} H_A F_A(X)
\]

for \(X \in \mathcal{U}\) and \(H = (H_A) \in \mathbb{R}^{m|n}\).

For \(1 \leq A \leq m\), we put \(F_A(X) = \frac{\partial}{\partial X_{A,0}} f(X)\) for \(X \in \mathcal{U}\).

On the other hand, from the second equation of (4.4.8) and Lemma 4.4.2, we have an element \(F_A(X)(m+1 \leq A \leq m+n)\) such that \(\sigma^J F_A(X) = \frac{\partial}{\partial X_{A,j}} f(X)\).

Using these \(\{F_A(X)\}\) defined above, we claim that (4.4.10) holds following Yagi’s argument.

Since \(f\) is admissible, for any \(L \geq 0\), \(p_L \circ f\) is so also, therefore there exist some \(N \geq 0\) and a \(\mathbb{R}_L\)-valued \(C^\infty\) function \(f_N\) such that \(p_L \circ f(X) = f_N \circ p_N(X)\) on \(X \in \mathcal{U}\). By natural imbedding
from $\mathcal{R}_L$ to $\mathcal{R}_N$, we may assume $N \geq L$. Then, we can show that

$$\frac{\partial}{\partial X_{A,K}} f_N(p_N(X)) = \begin{cases} p_L \left( \frac{\partial}{\partial X_{A,K}} f(X) \right) & \text{if } K \in \mathcal{I}_N, \\ 0 & \text{if otherwise.} \end{cases}$$

Therefore, for any $L \geq 0$,

$$p_L \left( \frac{d}{dt} f(X + tH) \bigg|_{t=0} \right) = p_L \left( f(X + tH) \bigg|_{t=0} \right) \quad \left( \because p_L \left( \frac{d}{dt} g(t) \bigg|_{t=0} \right) = p_L \left( g(t) \bigg|_{t=0} \right) \right),$$

$$= p_L \left( f_N(p_N(X + tH)) \bigg|_{t=0} \right) \quad \left( \because f_N(p_N(X)) \right),$$

$$= \sum \sum \left( \frac{\partial}{\partial X_{A,K}} f_N(p_N(X)) \right) \quad \left( \because \text{finite dimensional case} \right),$$

$$= \sum \sum \left( \frac{\partial}{\partial X_{A,K}} f(X) \right) \quad \left( \because \text{by (4.4.8)} \right),$$

$$= \sum \sum \left( \frac{\partial}{\partial X_{A,K}} \phi \right) \quad \left( \because \text{by (4.4.8)} \right),$$

$$= \sum \sum \left( \frac{\partial}{\partial X_{A,K}} \phi \right) \quad \left( \because \text{by (4.4.8)} \right),$$

$$= \sum \sum \left( \frac{\partial}{\partial X_{A,K}} \phi \right) \quad \left( \because \text{by (4.4.8)} \right),$$

Thus, we have (4.4.10). The continuity of $F_A(X)$ is clear. \hfill \square

**Remark 4.4.6.** For function with finite number of independent variables, it is well-known how to define its partial derivatives. But when that number is infinite, it is not so clear whether the change of order of differentiation affects the result, etc. Therefore, we reduce the calculation to the cases with finite number $L$ of generators and making that $L$ to infinity.

**Theorem 4.4.2** (Theorem 3 of [135]). Let $f$ be a $\mathcal{C}$-valued $C^\infty$ function on an open set $\mathcal{U} \subset \mathcal{R}^{m,n}$. If $f \in \mathcal{G}_{SD}(\mathcal{U} : \mathcal{C})$, then $f$ is $\mathcal{G}_{SD}$ on $\mathcal{U}$.

**Proof.** Since $f \in \mathcal{G}_{SD}$, it satisfies Cauchy-Riemann equation. As $f$ is $C^\infty$ on $\mathcal{U}$, $g(X) = \frac{\partial}{\partial X_{A,0}} f(X)$ also satisfies the C-R equation, for $1 \leq A \leq m$. In fact, for $1 \leq B \leq m$, $|J| = \text{even},$

$$\frac{\partial}{\partial X_{B,J}} g(X) = \frac{\partial}{\partial X_{B,J}} \frac{\partial}{\partial X_{A,0}} f(X) = \frac{\partial}{\partial X_{A,0}} \frac{\partial}{\partial X_{B,J}} f(X)$$

$$= \frac{\partial}{\partial X_{A,0}} \sigma^J \frac{\partial}{\partial X_{B,J}} f(X) = \sigma^J \frac{\partial}{\partial X_{B,J}} \frac{\partial}{\partial X_{A,0}} f(X) = \sigma^J \frac{\partial}{\partial X_{B,J}} g(X).$$

And for $m + 1 \leq A \leq m + n$, $|J| = |K| = \text{odd},$

$$\sigma^K \frac{\partial}{\partial X_{B,J}} g(X) + \sigma^J \frac{\partial}{\partial X_{B,K}} g(X) = \sigma^K \frac{\partial}{\partial X_{B,K}} \frac{\partial}{\partial X_{A,0}} f(X) + \sigma^J \frac{\partial}{\partial X_{B,K}} \frac{\partial}{\partial X_{A,0}} f(X)$$

$$= \frac{\partial}{\partial X_{A,0}} \left( \sigma^K \frac{\partial}{\partial X_{B,J}} f(X) + \sigma^J \frac{\partial}{\partial X_{B,K}} f(X) \right) = 0.$$
Hence $\frac{\partial}{\partial x_A} f$ for $1 \leq A \leq m$ is $\mathcal{G}_{SD}^1$ on $\mathcal{U}$.

Analogously, for $m + 1 \leq A \leq m + n$, $\frac{\partial}{\partial x_A} f = \sigma^{J_i} \frac{\partial}{\partial x_A} f$ is also $\mathcal{G}_{SD}^1$ on $\mathcal{U}$. In fact, we have, for $|K| = \text{even},$

**Lemma 4.4.3** (Lemma 5.1 of [135]). Let $f \in \mathcal{G}_{SD}^\infty(\mathbb{R}^{m|n})$. Then

$$f(\theta) = f(\theta_1, \cdots, \theta_n) = \sum_{|a| \leq n} \theta^a f_a \quad \text{with } f_a \in \mathcal{C}.$$

**Proof.** For $n = 1$ and $|J| = \text{odd}$, we have,

$$\frac{d}{dt} f(\theta + t\sigma^J)|_{t=0} = \frac{\partial}{\partial \theta^J} f(\theta) = \sigma^J \frac{d}{d\theta} f(\theta) \quad \text{with } \theta = \sum_{1 \in I_{\text{od}}} \theta_1 \sigma^1, \theta_I \in \mathbb{C}.$$ 

Hence

$$\frac{\partial}{\partial \theta_K} \frac{\partial}{\partial \theta_J} f(\theta) = \frac{\partial}{\partial s} \frac{\partial}{\partial t} f(\theta + t\sigma^J + s\sigma^K)|_{t=s=0} = \sigma^K \cdot \sigma^J \cdot \frac{d}{d\theta} \frac{d}{d\theta} f(\theta).$$

Since $|J|, |K|$ are odd, we have $\sigma^J \cdot \sigma^K = -\sigma^K \cdot \sigma^J$ and therefore

$$\frac{\partial}{\partial \theta_K} \frac{\partial}{\partial \theta_J} f(\theta) = \sigma^K \cdot \sigma^J \cdot \frac{d}{d\theta} \frac{d}{d\theta} f(\theta) = -\sigma^J \cdot \sigma^K \cdot \frac{d}{d\theta} \frac{d}{d\theta} f(\theta) = -\frac{\partial}{\partial \theta_K} \frac{\partial}{\partial \theta_J} f(\theta).$$

Since $f$ is $C^\infty$ as a function of infinite variables $\{\theta_J\}$ and its higher derivatives are symmetric, we have therefore

$$\frac{\partial}{\partial \theta_K} \frac{\partial}{\partial \theta_J} f(\theta) = 0.$$

By representing $f(\theta) = \sum_K \sigma^K f_k(\theta)$, the each component $f_k(\theta)$ is a polynomial of degree 1 with variables $\{\theta_J \mid J \in I_{\text{od}}\}$. Then $\sigma^J \cdot \frac{d}{d\theta} f(\theta) = \frac{\partial}{\partial \theta^J} f(\theta)$ is constant for any $|J| = \text{odd}$. Thus $\frac{d}{d\theta} f(\theta)$ is constant denoted by $a \in \mathcal{C}$. Then, $\frac{d}{d\theta} (f(\theta) - \theta a) = 0$. Therefore there exists $b \in \mathcal{C}$ such that $f(\theta) = \theta a + b$.

We proceed by induction w.r.t. $n$. Let $f$ be a $\mathcal{G}_{SD}^\infty$ function on an open set $U \subset \mathbb{R}^{m|n}$. Fixing $\theta_1, \cdots, \theta_{n-1}$, $f(\theta_1, \cdots, \theta_{n-1}, \theta_n)$ is a $\mathcal{G}_{SD}^\infty$ function with one variable $\theta_n$. Thus, we have

$$f(\theta_1, \cdots, \theta_{n-1}, \theta_n) = \theta_n g(\theta_1, \cdots, \theta_{n-1}) + h(\theta_1, \cdots, \theta_{n-1}) \quad \text{with } \frac{\partial}{\partial \theta_n} f(\theta) = g(\theta_1, \cdots, \theta_{n-1}).$$

Therefore $g$ is $\mathcal{G}_{SD}^\infty$ w.r.t. $(\theta_1, \cdots, \theta_{n-1})$, $h$ is also $\mathcal{G}_{SD}^\infty$ w.r.t. $(\theta_1, \cdots, \theta_{n-1})$. $\square$

**Remark 4.4.7.** Though this Lemma with a sketch of the proof is announced in [34] and is cited in [32] without proof, but I feel some ambiguity of his proof. This point is ameliorated by [135] as above.

**Lemma 4.4.4** (Lemma 5.2 of [135]). Let $f \in \mathcal{G}_{SD}^\infty(\mathbb{R}^{m|0})$ on a convex open set $U \subset \mathbb{R}^{m|0}$ which vanishes identically on $U_B = \pi_B(U)$. Then, $f$ vanishes identically on $U$.

**Proof.** It is essential to prove the case $m = 1$. Take an arbitrary point $t \in U_B$ and we consider the behavior of $f$ on $\pi_B^{-1}(t)$. Let $X \in \pi_B^{-1}(t)$ and $X_L = p_L(X)$. Then $\{X_K \mid K \in I_L, |K| = \text{ev} \geq 2\}$ is a coordinate for $(\pi_B^{-1}(t))_L$ as the ordinary space $\mathbb{C}$. Let $f_L$ be the $L$-th projection of $f$. Then,

$$\frac{\partial}{\partial X_K} f_N(X_L) = \sigma^K \cdot \frac{\partial}{\partial X_0} f_L(X_L) \quad \text{for } K \in I_L \quad \text{and } |K| = \text{ev}.$$
If \( K_1, \ldots, K_h \in I_L, |K_j| = \text{even} > 0 \) and \( 2h > L \), then \( \sigma^{K_1} \cdots \sigma^{K_h} = 0 \) and 
\[
\frac{\partial}{\partial x^{K_1}} \cdots \frac{\partial}{\partial x^{K_h}} f_L(X_L) = 0.
\]
This implies that \( f_L \) is a polynomial on \((\pi_B^{-1}(t))_L\). Moreover, for any \( h \geq 0 \),
\[
\frac{\partial}{\partial x^{K_1}} \cdots \frac{\partial}{\partial x^{K_h}} f_L(t) = \sigma^{K_1} \cdots \sigma^{K_h} \left( \frac{\partial}{\partial X_0} \right)^h f_L(t).
\]
Since \( f \) vanishes on \( U_B \), we have
\[
\left( \frac{\partial}{\partial X_0} \right)^h f_L(t) = 0 \quad \text{on} \quad U_B
\]
and hence
\[
\frac{\partial}{\partial x^{K_1}} \cdots \frac{\partial}{\partial x^{K_h}} f_L(t) = 0 \quad \text{for any} \quad h \geq 0 \quad \text{and} \quad K_1, \ldots, K_h \in I_L \quad \text{with} \quad |K_j| = \text{ev} > 0.
\]
Thus the polynomial \( f_L|_{\pi_B^{-1}(t)} \) must vanish identically and hence \( f_L \equiv 0 \) on \( U_L \). This holds for any \( L \geq 0 \). Thus \( f \equiv 0 \) on \( U \). \( \square \)

### 4.4.5. Proof of Main Theorem 4.4.3

**Theorem 4.4.3.** Let \( U \) be a superdomain in \( R^{m|n} \) and let a function \( f : U \to C \) be given. Following conditions are equivalent:

(a) \( f \) is super Fréchet (\( F_{-} \), in short) differentiable on \( U \), i.e. \( f \in \mathfrak{F}_{SD}^1(U; C) \),

(b) \( f \) is super Gâteaux (\( G_{-} \), in short) differentiable on \( U \), i.e. \( f \in \mathfrak{G}_{SD}^1(U; C) \),

(c) \( f \) is \( \infty \)-times G-differentiable and \( f \in \mathfrak{G}_{SD}^1(U; C) \),

(d) \( f \) is \( \infty \)-times G-differentiable and its G-differential \( df \) is \( R_{ev} \)-linear,

(e) \( f \) is \( \infty \)-times G-differentiable and its G-differential \( df \) satisfies Cauchy-Riemann type equations,

(f) \( f \) is supersmooth, i.e. it has the following representation, called superfield expansion, such that

\[
f(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha \tilde{f}_a(x) \quad \text{with} \quad f_a(q) \in C^\infty(\pi_B(U)) \quad \text{and} \quad \tilde{f}_a(x) = \sum_{|\alpha| = 0}^{\infty} \frac{1}{\alpha!} \frac{\partial^\alpha f_a(q)}{\partial q^\alpha} \bigg|_{q=x_B} x_0^\alpha.
\]

**Remark 4.4.8.** In the above, \( f \) stands for the “algebraic” nature and \( a \) claims the “analytic” nature of “superfields”. Yagi [135] proves essentially the equivalence \((b) \iff (e) \iff (f)\).

It is clear from outset that \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)\). From Proposition 4.4.3 \((d) \Rightarrow (a)\). Lastly, the equivalence of \((d) \) and \((e) \) is given by

**Theorem 4.4.4 (Theorem 4 of [135].)** Let \( f \) be a \( \mathfrak{G}_{SD}^\infty \) function on a convex open set \( U \subset R^{m|n} \). Then, there exist \( R \)-valued \( C^\infty \) functions \( u_a \) on \( U_B \) such that

\[
f(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha u_a(x).
\]

Moreover, this expression is unique.

**Proof.** \( \Rightarrow \) For fixed \( x \), by Lemma 4.4.3 \( f(x, \theta) \) has the representation \( f(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha \varphi_a(x) \)
with \( \varphi_a(x) \in C \). Since \( f \in \mathfrak{G}_{SD}^\infty \), it is clear that for each \( a \), \( \varphi_a(x) \in C \) on \( R^{m|0} \) and moreover \( \varphi_a(x_B) \) is in \( C^\infty(R^m) \). Denoting the Grassmann continuation of it by \( \tilde{\varphi}_a(x) \), we should have \( \tilde{\varphi}_a(x) = f_a(x) \) by Lemma 4.4.3

\( \Leftarrow \) Since the supersmoothness leads the C-R relation, we get the superdifferentiability. \( \square \)
4.5. Inverse and implicit function theorems

4.5.1. Composition of supersmooth functions. Following is the slight modification of the arguments in Inoue and Maeda [75].

**Definition 4.5.1.** Let \( \mathcal{U} \subset \mathbb{R}^{m|n} \) and \( \mathcal{V} \subset \mathbb{R}^{p|q} \) be superdomains and let \( \varphi \) be a continuous mapping from \( \mathcal{U} \) to \( \mathcal{V} \), denoted by \( \varphi(X) = (\varphi_1(X), \ldots, \varphi_p(X), \varphi_{p+1}(X), \ldots, \varphi_{p+q}(X)) \in \mathbb{R}^{p|q} \). \( \varphi \) is called a supersmooth mapping from \( \mathcal{U} \) to \( \mathcal{V} \) if each \( \varphi_A(X) \in C_{SS}(\mathcal{U} : \mathcal{R}) \) for \( A = 1, \ldots, p + q \) and \( \varphi(\mathcal{U}) \subset \mathcal{V} \).

**Proposition 4.5.1 (Composition of supersmooth mappings).** Let \( \mathcal{U} \subset \mathbb{R}^{m|n} \) and \( \mathcal{V} \subset \mathbb{R}^{p|q} \) be superdomains and let \( F : \mathcal{U} \to \mathbb{R}^{p|q} \) and \( G : \mathcal{V} \to \mathbb{R}^{r|s} \) be supersmooth mappings such that \( F(\mathcal{U}) \subset \mathcal{V} \). Then, the composition \( G \circ F : \mathcal{U} \to \mathbb{R}^{r|s} \) gives a supersmooth mapping and

\[
(d_X G(F(X)) = [d_X F(X)][d_Y G(Y)]|_{Y=F(X)}).
\]

Or more precisely,

\[
(d_X G(F(X)) = (\partial_X (G \circ F)_B(X)) = \left( \sum_{C=1}^{r+s} \partial_X G_C(X)(\partial_Y F_B(Y)) \right)|_{Y=F(X)}).
\]

**Proof.** Put \( F(Y) = (G_B(Y))_{B=1}^{r+s} \), \( F(X) = (F_C)_{C=1}^{p+q} \), \( X = (X_A)_{A=1}^{m+n} \), and \( Y = (Y_C)_{C=1}^{r+s} \). By smooth \( G \)-differentiability of the composition of mappings between Fréchet spaces, we have the smoothness of \( \Phi(X + tH) \) w.r.t \( t \). Moreover, we have (1.5.2).

By the characterization of supersmoothness, we need to say \( \mathcal{R}_{ev} \)-linearity of \( d_F \Phi \), i.e. \( d_F \Phi(X)(\lambda H) = \lambda d_F \Phi(X)(H) \) for \( \lambda \in \mathcal{R}_{ev} \) which is obvious from

\[
d(G \circ F)(X)(\lambda H) = \frac{d}{dt} G(F(X + tH)) \bigg|_{t=0}
\]

\[
= \left( \sum_{C=1}^{r+s} \lambda H_A \partial_X G_C(X)(\partial_Y F_B(Y)) \right)|_{Y=F(X)}
\]

\[
= \lambda \frac{d}{dt} G(F(X + tH)) \bigg|_{t=0} = \lambda d(G \circ F)(X)(H). \quad \Box
\]

**Definition 4.5.2.** Let \( \mathcal{U} \subset \mathbb{R}^{m|n} \) and \( \mathcal{V} \subset \mathbb{R}^{p|q} \) be superdomains and let \( \varphi : \mathcal{U} \to \mathcal{V} \) be a supersmooth mapping represented by \( \varphi(X) = (\varphi_1(X), \ldots, \varphi_{p+q}(X)) \) with \( \varphi_A(X) \in C_{SS}(\mathcal{U} : \mathcal{R}) \).

(1) \( \varphi \) is called a supersmooth diffeomorphism if

(i) \( \varphi \) is a homeomorphism between \( \mathcal{U} \) and \( \mathcal{V} \) and

(ii) \( \varphi \) and \( \varphi^{-1} \) are supersmooth mappings.

(2) For any \( f \in C_{SS}(\mathcal{V} : \mathcal{R}) \), \( (\varphi^* f)(X) = (f \circ \varphi)(X) = f(\varphi(X)) \), called the pull back of \( f \), is well-defined and belongs to \( C_{SS}(\mathcal{U} : \mathcal{R}) \).

**Remark 4.5.1.** It is easy to see that if \( \varphi \) is a supersmooth diffeomorphism, then \( \varphi_B = \pi_B \circ \varphi \) is an (ordinary) \( C^\infty \) diffeomorphism from \( \mathcal{U}_B \) to \( \mathcal{V}_B \).

**Remark 4.5.2.** If we introduce the topologies in \( C_{SS}(\mathcal{V} : \mathcal{E}) \) and \( C_{SS}(\mathcal{U} : \mathcal{E}) \) properly, \( \varphi^* \) gives a continuous linear mapping from \( C_{SS}(\mathcal{V} : \mathcal{E}) \) to \( C_{SS}(\mathcal{U} : \mathcal{E}) \). Moreover, if \( \varphi : \mathcal{U} \to \mathcal{V} \) is a supersmooth diffeomorphism, then \( \varphi^* \) defines an automorphism from \( C_{SS}(\mathcal{V} : \mathcal{R}) \) to \( C_{SS}(\mathcal{U} : \mathcal{R}) \).
4.5.2. Inverse and implicit function theorems. We recall

**Proposition 4.5.2 (Inverse function theorem on $\mathbb{R}^m$).** Let $U$ be an open set in $\mathbb{R}^m$. Let $f : U \ni x \rightarrow y = f(x) \in \mathbb{R}^m$ be a $C^k$ ($k \geq 1$) mapping such that $f'(x_0) \neq 0$ for some $x_0 \in U$.

Then, there exist a neighbourhood $W$ of $y_0 = f(x_0)$ and a neighbourhood $U_0 \subset U$ of $x_0$, such that $f$ maps $U_0$ injectively onto $W$. Therefore, $f|_{U_0}$ has its inverse $\phi = (f|_{U_0})^{-1} : W \rightarrow U_0 \subset C^k(W)$. Moreover, for any $y = f(x) \in W$ with $x \in U_0$, we have $\phi'(y) = f'(x)^{-1}$.

Applying this, we have

**Theorem 4.5.1 (Inverse function theorem on $\mathbb{R}^{m|n}$).** Let $F = (f,g) : \mathbb{R}^{m|n} \ni X \rightarrow Y = F(X) \in \mathbb{R}^{m|n}$ be a supersmooth mapping on some superdomain containing $\tilde{X}$. That is,

$$f(X) = (f_i(X))_{i=1}^m \in \mathbb{R}^m_{ev}, \quad g(X) = (g_k(X))_{k=1}^n \in \mathbb{R}^n_{od},$$

More precisely, we put

$$f(X) = (f_i(X))_{i=1}^m \in \mathbb{R}^m_{ev}, \quad g(X) = (g_k(X))_{k=1}^n \in \mathbb{R}^n_{od},$$

such that

$$f_i(x,\theta) = \sum_{|a|=ev\leq n} \theta^a f_{ia}(x), \quad g_k(x,\theta) = \sum_{|b|=od\leq n} \theta^b g_{kb}(x) \in \mathcal{C}_{SS}(\mathbb{R}^{m|n})$$

(4.5.3)

with

$$\begin{cases} f_{ia}(x_B), \quad g_{kb}(x_B) \in \mathbb{C} & \text{if } |a|, |b| \neq 0, \\ f_{ia}(x_B) \in \mathbb{R} & \text{if } |a| = 0. \end{cases}$$

We assume the super matrix $[d_X F(X)]$ is invertible at $\tilde{X}$, i.e. $\pi_B(\det [d_X \Phi(X)]_{X=\tilde{X}}) \neq 0$. Then, there exist a superdomain $\mathcal{U}$, a neighbourhood of $\tilde{X}$ and another superdomain $\mathcal{V}$, a neighbourhood of $\tilde{Y} = F(\tilde{X})$ such that $F : \mathcal{U} \rightarrow \mathcal{V}$ has a unique supersmooth inverse $\Phi = F^{-1} = (\phi, \psi) : \mathcal{V} \rightarrow \mathcal{U}$ satisfying

$$\phi(Y) = (\phi_i(Y))_{i=1}^m \in \mathbb{R}^m_{ev}, \quad \psi(Y) = (\psi_k(Y))_{k=1}^n \in \mathbb{R}^n_{od}$$

(4.5.4)

and

$$\Phi(F(X)) = X \quad \text{for } X \in \mathcal{U} \quad \text{and} \quad F(\Phi(Y)) = Y \quad \text{for } Y \in \mathcal{V}.$$
Here,
\[
A = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m}
\end{pmatrix}, \quad C = \begin{pmatrix}
\frac{\partial g_{m+1}}{\partial x_1} & \frac{\partial g_{m+1}}{\partial x_2} & \cdots & \frac{\partial g_{m+1}}{\partial x_m} \\
\frac{\partial g_{m+2}}{\partial x_1} & \frac{\partial g_{m+2}}{\partial x_2} & \cdots & \frac{\partial g_{m+2}}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{m+n}}{\partial x_1} & \frac{\partial g_{m+n}}{\partial x_2} & \cdots & \frac{\partial g_{m+n}}{\partial x_m}
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
\frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} & \cdots & \frac{\partial g_1}{\partial \theta_n} \\
\frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} & \cdots & \frac{\partial g_2}{\partial \theta_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_m}{\partial \theta_1} & \frac{\partial g_m}{\partial \theta_2} & \cdots & \frac{\partial g_m}{\partial \theta_n}
\end{pmatrix}, \quad D = \begin{pmatrix}
\frac{\partial g_{m+1}}{\partial \theta_1} & \frac{\partial g_{m+1}}{\partial \theta_2} & \cdots & \frac{\partial g_{m+1}}{\partial \theta_n} \\
\frac{\partial g_{m+2}}{\partial \theta_1} & \frac{\partial g_{m+2}}{\partial \theta_2} & \cdots & \frac{\partial g_{m+2}}{\partial \theta_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{m+n}}{\partial \theta_1} & \frac{\partial g_{m+n}}{\partial \theta_2} & \cdots & \frac{\partial g_{m+n}}{\partial \theta_n}
\end{pmatrix}.
\]

Therefore, \(d_X G(X)\) gives an even super matrix.

**Proof of Theorem 4.5.1.** (1) To make clear the point, we consider the case \(m = 1, n = 2\), that is, \(\mathcal{U}, \mathcal{W} \subseteq \mathbb{R}^{1\times 2}\). Let
\[
F(X) = F(x, \theta) = (f(x, \theta), g_1(x, \theta), g_2(x, \theta)) : \mathcal{U} \rightarrow \mathcal{W}
\]
with
\[
\begin{align*}
f(x, \theta) &= f(0)(x) + f(12)(x)\theta_1 \theta_2, \\
g_1(x, \theta) &= g_1(1)(x)\theta_1 + g_1(2)(x)\theta_2, \quad \text{and} \quad f(0)(x_B) \in \mathbb{R}, f(12)(x_B), g_k(x_B) \in \mathbb{C}.
\end{align*}
\]
In this case, we have
\[
d_X F(X) = \begin{pmatrix}
f'(0)(x) + f'(12)(x)\theta_1 \theta_2 & g'_1(1)(x)\theta_1 + g'_1(2)(x)\theta_2 & g'_2(1)(x)\theta_1 + g'_2(2)(x)\theta_2 \\
f'(12)(x)\theta_2 & g_1(1)(x) & g_1(2)(x) \\
-f'(12)(x)\theta_1 & g_1(1)(x) & g_1(2)(x)
\end{pmatrix} = \begin{pmatrix} A & C \\ D & B \end{pmatrix}
\]
with
\[
\text{sdet}(d_X F(X)) = \det(A - CB^{-1}D)(\det B)^{-1} \quad \text{and} \quad \det B = \beta(x) = g_1(1)(x)g_2(2)(x) - g_1(2)(x)g_2(1)(x).
\]
Therefore,
\[
(4.5.7) \quad \pi_B(\text{sdet}(d_X F(X))) = f'(0)(x_B)\beta(x_B)^{-1}.
\]

We need to find \(\Phi = (\phi, \psi_1, \psi_2)\) such that
\[
\begin{align*}
\phi(f(x_B, \theta), g_1(x_B, \theta), g_2(x_B, \theta)) &= x_B, \\
\psi_1(f(x_B, \theta), g_1(x_B, \theta), g_2(x_B, \theta)) &= \theta_1, \quad \text{with} \quad \psi_1(y, \omega) = \psi_1(1)(y)\omega_1 + \psi_1(2)(y)\omega_2, \\
\psi_2(f(x_B, \theta), g_1(x_B, \theta), g_2(x_B, \theta)) &= \theta_2, \\
\psi_2(y, \omega) &= \psi_2(1)(y)\omega_1 + \psi_2(2)(y)\omega_2.
\end{align*}
\]
To state more precisely, we have
\[
\begin{align*}
y_B &= f(0)(x_B), \quad y_S = f(12)(x_B)\theta_1 \theta_2, \\
\phi(0)(y_B + y_S) &= \phi(0)(y_B) + \phi'(0)(y_B)y_S, \\
\phi(12)(y_B + y_S) &= \phi(12)(y_B) + \phi'(12)(y_B)y_S,
\end{align*}
\]
and from the first equation of (4.5.8),
\[
(4.5.9) \quad \begin{align*}
\phi(0)(y_B) &= \phi(0)(f(0)(x_B)) = x_B, \\
\phi'(0)(y_B)y_S + \phi'(12)(y_B)\beta(x_B)\theta_1 \theta_2 &= [\phi'(0)(y_B)f(12)(x_B) + \phi(12)(y_B)\beta(x_B)]\theta_1 \theta_2 = 0.
\end{align*}
\]
Since \( f_0'(\tilde{x}_B) \neq 0 \), there exists a neighborhood \( U_0 \subset \mathbb{R} \) of \( \tilde{x}_B \) and \( V_0 \subset \mathbb{R} \) of \( f_0(\tilde{x}_B) \) where we find a function \( \phi_0(y_B) \) satisfying the first equation \([4.5.9]\). Moreover, since \( \beta(x_B) \neq 0 \), taking the smaller neighborhood if necessary, we define

\[
\phi(12)(y_B) = -\phi_0(y_B) f(12)(x_B) \beta(x_B)^{-1} |_{x_B=\phi_0(y_B)}.
\]

On the other hand, putting

\[
\omega_1 = g_1(1)(x_B) \theta_1 + g_1(2)(x_B) \theta_2,
\]

\[
\omega_2 = g_2(1)(x_B) \theta_1 + g_2(2)(x_B) \theta_2, \quad \omega_1 \omega_2 = \beta(x_B) \theta_1 \theta_2
\]

and remarking \( y_j \omega_j = 0 \) for \( j = 1, 2 \), from the last two equations of \([4.5.8]\), we should have

\[
\psi_1(1)(y_B)(g_1(1)(x_B) \theta_1 + g_2(1)(x_B) \theta_2) + \psi_1(2)(y_B)(g_2(1)(x_B) \theta_1 + g_2(2)(x_B) \theta_2) = \theta_1,
\]

\[
\psi_2(1)(y_B)(g_1(1)(x_B) \theta_1 + g_2(1)(x_B) \theta_2) + \psi_2(2)(y_B)(g_2(1)(x_B) \theta_1 + g_2(2)(x_B) \theta_2) = \theta_2,
\]

that is,

\[
\begin{pmatrix}
\psi_1(1)(y_B) & \psi_1(2)(y_B) \\
\psi_2(1)(y_B) & \psi_2(2)(y_B)
\end{pmatrix}
\begin{pmatrix}
g_1(1)(x_B) \\
g_2(1)(x_B)
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Therefore, we have \( \psi_*(y_B) \), which satisfy the desired property.

(II) Do analogously as above for general \( m, n \) by putting

\[
f_i(x, \theta) = \sum_{|a|=ev \leq n} f_{i,a}(x) \theta^a, \quad g_k(x, \theta) = \sum_{|b|=od \leq n} g_{k,b}(x) \theta^b,
\]

and

\[
\phi_\nu(y, \omega) = \sum_{|a'|=ev \leq n} \phi_{\nu,a'}(y) \omega^{a'}, \quad \psi_\nu(y, \omega) = \sum_{|b'|=od \leq n} \psi_{\nu,b'}(y) \omega^{b'},
\]

but with more patience. \( \square \)

**Remark 4.5.4.** Above theorem holds for functions \( f_i \in C_{SS}(\mathcal{R}^m|n : R_{ev}) \) and \( g_k \in C_{SS}(\mathcal{R}^m|n : R_{od}) \).

Moreover, we have

**Proposition 4.5.3** (Implicit function theorem). Let \( \Phi(X, Y) : \mathcal{U} \times \mathcal{V} \to \mathcal{C}^{pq} \) be a supersmooth mapping and \((\tilde{X}, \tilde{Y}) \in \mathcal{U} \times \mathcal{V}\), where \( \mathcal{U} \) and \( \mathcal{V} \) are superdomains of \( \mathcal{R}^m|n \) and \( \mathcal{R}^p|q \), respectively. Suppose \( \Phi(\tilde{X}, \tilde{Y}) = 0 \) and \( \partial_Y \Phi = [\partial_{y_j} \Phi, \partial_{y_\omega} \Phi] \) is a continuous and invertible supermatrix at \((\tilde{X}_B, \tilde{Y}_B) \in \pi_B(\mathcal{U}) \times \pi_B(\mathcal{V})\). Then, there exist a superdomain \( \mathcal{V} \subset \mathcal{U} \) satisfying \( \tilde{X}_B \in \pi_B(\mathcal{V}) \) and a unique supersmooth mapping \( Y = f(X) \) on \( \mathcal{V} \) such that \( \tilde{Y} = f(\tilde{X}) \) and \( \Phi(X, f(X)) = 0 \) in \( \mathcal{V} \). Moreover, we have

\[
\partial_X f(X) = - [\partial_X \Phi(X, Y)][\partial_Y \Phi(X, Y)]^{-1} |_{Y=f(X)}.
\]

**Proof.** \([4.5.10]\) is easily obtained by

\[
0 = \partial_X \Phi(X, f(X)) = (\partial_X \Phi(X, Y) + \partial_X f(X) \partial_Y \Phi(X, Y)) |_{Y=f(X)}.
\]

The existence proof is omitted here because the arguments in proving Proposition \([4.5.1]\) work well in this situation. \( \square \)

**Problem 4.5.1.** Use Ekeland’s idea \([40]\) to give another proof of above Theorems, if possible.
4.5.3. **Global inverse function theorem.** We have the following theorem of Hadamard type:

**Proposition 4.5.4 (Global inverse function theorem on \( \mathbb{R}^m \)).** Let \( f : \mathbb{R}^m \ni x \to y = f(x) \in \mathbb{R}^m \) be a smooth mapping on \( \mathbb{R}^m \). We assume the Jacobian matrix \( [d_x f(x)] \) is invertible on \( \mathbb{R}^m \), and \( \|\text{det}[d_x f(x)]\| \geq \delta > 0 \) for any \( x \). Then, \( f \) gives a smooth diffeomorphism from \( \mathbb{R}^m \) onto \( \mathbb{R}^m \).

**Proposition 4.5.5 (Global inverse function theorem on \( \mathbb{R}^{m|n} \)).** Let \( F = (f, g) : \mathbb{R}^{m|n} \ni X \to Y = F(X) \in \mathbb{R}^{m|n} \) be a supersmooth mapping on \( \mathbb{R}^{m|n} \). We assume the super matrix

\[
d_X F(X) = \begin{pmatrix}
\frac{\partial f_i}{\partial x_j} & \frac{\partial g_i}{\partial x_j} \\
\frac{\partial f_i}{\partial \theta_l} & \frac{\partial g_i}{\partial \theta_l}
\end{pmatrix}
\]

is invertible at any \( X \in \mathbb{R}^{m|n} \), and there exists \( \delta > 0 \) such that for any \( x \)

\[
\|\pi_B(\text{sdet} \frac{\partial f_i}{\partial x_j})\| \geq \delta > 0, \quad \text{and} \quad \{x \mid \pi_B(\text{det} \frac{\partial g_k}{\partial \theta_l}) = 0\} = \emptyset.
\]

Then, \( F \) gives a supersmooth diffeomorphism from \( \mathbb{R}^{m|n} \) onto \( \mathbb{R}^{m|n} \).

**Proof.** From the proof of above Theorem 4.5.1 it is obvious. \( \square \)
CHAPTER 5

Elementary integral calculus on superspace

As is well-known, to study a scalar PDE by applying functional analysis, we use essentially the following tools: Taylor expansion, integration by parts, the formula for the change of variables under integral sign and Fourier transformation. Therefore, beside the elementary differential calculus, it is necessary to develop the elementary integral calculus on superspace $\mathbb{R}^{m|n}$. But as is explained soon later, we have the relations

\begin{align*}
\left\{ \begin{array}{ll}
dx_j \wedge dx_k = -dx_k \wedge dx_j & \text{for even variables } \{x_j\}_{j=1}^m, \\
d\theta_j \wedge d\theta_k = d\theta_k \wedge d\theta_j & \text{for odd variables } \{\theta_k\}_{k=1}^n,
\end{array} \right.
\end{align*}

which differs from ordinary one. Therefore, the integration containing odd variables doesn’t follow our conventional intuition.

5.1. Integration w.r.t. odd variables – Berezin integral

It seems natural to put formally

\[ d\theta_j = \sum_{I \in \mathcal{I}, |I| = \text{od}} d\theta_{j,I} \sigma^I \text{ for } \theta_j = \sum_{I \in \mathcal{I}, |I| = \text{od}} \theta_{j,I} \sigma^I. \]

**Remark 5.1.1.** Since above sum $\sum_I$ stands for the position in the sequence space $\omega$ of Köthe and the element of it is given by $d\theta_{j,I}$ for $|I|$ is finite, we may give the meaning to $d\theta_j$.

Then, rather formally, since for $J$ and $K$ with $|J| = \text{od} = |K|$, $d\theta_{J,I}$, $d\theta_{K,J}$ and $\sigma^I$, $\sigma^K$ are anticommutative, minus signs cancel out each other and the second equality in (5.0.11) holds. Anlogously the first one in (5.0.11) holds. This make us imagine that even if there exists the notion of integration, it differs much from the standard one on $\mathbb{R}^m$.

Here, we borrow explanation of Vladimirov and Volovich. Since the supersmooth functions on $\mathcal{R}^{0|n}$ are characterized as the polynomials with value in $\mathcal{C}$, we need to define the integrability for those under the conditions that

(i) integrability of all polynomials,

(ii) linearity of an integral, and

(iii) invariance of the integral w.r.t. shifts.

Put $\mathcal{P}_{n} = \mathcal{P}_{n}(\mathcal{C}) = \{u(\theta) = \sum_{a \in \{0,1\}^n} \theta^a u_a \mid u_a \in \mathcal{C}\}$.

We say a mapping $J_n : \mathcal{P}_{n} \to \mathcal{C}$ is an integral if it satisfies

(1) $\mathcal{C}$-linearity (from the right): $J_n(u\alpha + v\beta) = J_n(u)\alpha + J_n(v)\beta$ for $\alpha, \beta \in \mathcal{C}$, $u, v \in \mathcal{P}_{n}$.

(2) translational invariance: $J_n(u(\cdot + \omega)) = J_n(u)$ for all $\omega \in \mathcal{R}^{0|n}$ and $u \in \mathcal{P}_{n}$.
THEOREM 5.1.1. For the existence of the integral \( J_n \) satisfying above conditions (1) and (2), it is necessary and sufficient that

\[
J_n(\phi) = 0 \quad \text{for} \quad \phi(\theta) = \theta^a, \quad |a| \leq n - 1.
\]

Moreover, we have

\[
J_n(u) = \frac{\partial}{\partial \theta_0^1} \cdots \frac{\partial}{\partial \theta_0^n} u(\theta) \bigg|_{\theta = 0} J_n(\phi) \quad \text{where} \quad \phi_1(\theta) = \theta^1 = \theta_1 \cdots \theta_0.
\]

**Proof.** If there exists \( J_n \) satisfying (1) and (2), then we have

\[
J_n(v) = \sum_{|a| \leq n} I(\phi_a)v_a \quad \text{for} \quad v(\theta) = \sum |a| \leq n \theta^a v_a = \sum |a| \leq n \phi_a(\theta)v_a.
\]

As

\[(\theta + \omega)^a = \theta^a + \sum_{|a| - |b| \geq 1} (-1)^s \theta^b \omega^{a-b},\]

\[
J_n(v(\theta + \omega)) = \sum_{|a| \leq n} J_n(\phi_a(\theta + \omega))v_a = \sum_{|a| \leq n} J_n(\phi_a)v_a + \sum_{|a| \leq n} \sum_{|a| - |b| \geq 1} (-1)^s J_n(\phi_b)v_b \omega^{a-b},
\]

by virtue of (2), we have

\[
\sum_{|a| \leq n} \sum_{|a| - |b| \geq 1} (-1)^s J_n(\phi_b)v_b \omega^{a-b} = 0.
\]

Here, \( v_a \in \mathcal{C} \) and \( \omega \in \mathfrak{R}_{od}^n \) are arbitrary, we have (5.1.1). Converse is obvious. \( \square \)

**Definition 5.1.1.** We put \( J_n(\phi) = 1 \), i.e.,

\[
(5.1.2) \quad \int_{\mathfrak{R}^n_0} d\theta_0 \cdots d\theta_1 \theta_1 \cdots \theta_n = 1.
\]

Therefore, we put, for any \( v = \sum_{|a| \leq n} \theta^a v_a \in \mathcal{P}_n(\mathcal{C}) \)

\[
(5.1.3) \quad J_n(v) = \int_{\mathfrak{R}^n_0} d\theta v(\theta) = \int_{\mathfrak{R}^n_0} d\theta_0 \cdots d\theta_1 v(\theta_1, \cdots, \theta_n) = (\partial_{\theta_0} \cdots \partial_{\theta_1} v)(0)
\]

\[
= v_1 = \int_{\text{Berezin}} d^n \theta v(\theta).
\]

This is called the (Berezin) integral of \( v \) on \( \mathfrak{R}^n_0 \).

Then, we have

**Proposition 5.1.1.** Given \( v, w \in \mathcal{P}_n(\mathcal{C}) \), we have the following:

(1) (\( \mathcal{C} \)-linearity) For any homogeneous \( \lambda, \mu \in \mathcal{C} \),

\[
(5.1.4) \quad \int_{\mathfrak{R}^n_0} d\theta (\lambda v + \mu w)(\theta) = (-1)^{np(\lambda)} \lambda \int_{\mathfrak{R}^n_0} d\theta v(\theta) + (-1)^{np(\mu)} \mu \int_{\mathfrak{R}^n_0} d\theta w(\theta).
\]

(2) (Translational invariance) For any \( \rho \in \mathfrak{R}^n_0 \), we have

\[
(5.1.5) \quad \int_{\mathfrak{R}^n_0} d\theta v(\theta + \rho) = \int_{\mathfrak{R}^n_0} d\theta v(\theta).
\]

(3) (Integration by parts) For \( v \in \mathcal{P}_n(\mathcal{C}) \) such that \( p(\nu) = 1 \) or 0, we have

\[
(5.1.6) \quad \int_{\mathfrak{R}^n_0} d\theta v(\theta) \partial_{\theta_0} w(\theta) = -(1)^{p(\nu)} \int_{\mathfrak{R}^n_0} d\theta (\partial_{\theta_0} v(\theta)) w(\theta).
\]
(4) (Linear change of variables) Let $A = (A_{jk})$ with $A_{jk} \in \mathfrak{R}_{ev}$ be an invertible matrix. Then,

$$
\int_{\mathfrak{R}^n_{0}} \theta_{\nu} d\theta = (\text{det } A)^{-1} \int_{\mathfrak{R}^n_{0}} d\omega v(A \cdot \omega).
$$

(5) (Iteration of integrals)

$$
\int_{\mathfrak{R}^n_{0}} d\theta v(\theta) = \int_{\mathfrak{R}^n_{0}} d\omega v(\theta(\omega)) \det \frac{\partial \theta(\omega)}{\partial \omega}.
$$

(6) (Odd change of variables) Let $\theta = \theta(\omega)$ be an odd change of variables such that $\theta(0) = 0$ and $\det \frac{\partial \theta(\omega)}{\partial \omega} |_{\omega=0} \neq 0$. Then, for any $v \in \mathcal{P}_n(\mathfrak{C})$,

$$
\int_{\mathfrak{R}^n_{0}} d\theta v(\theta) = \int_{\mathfrak{R}^n_{0}} d\omega v(\theta(\omega)) \det \frac{\partial \theta(\omega)}{\partial \omega}.
$$

(7) For $v \in \mathcal{P}_n(\mathfrak{C})$ and $\omega \in \mathfrak{R}^n_{0}$,

$$
\int_{\mathfrak{R}^n_{0}} d\theta (\omega_1 - \theta_1) \cdots (\omega_n - \theta_n) v(\theta) = v(\omega).
$$

Proof. We follow the arguments in pp.755-757 of Vladimirov and Volovich with slight modifications if necessary.

By definition, we have (1).

(2) Remarking the top term of $v(\theta + \rho)$ containing the term $\nu_1 \cdots \nu_n$ is same as $v(\theta)$, combining (5.1.1) we get the result.

(3) Using $\partial_{\theta_j}(v\nu) = \partial_{\theta_j} v \cdot w + (-1)^{\nu(\nu)} v \cdot \partial_{\theta_j} w$ with (2), we get (3).

(4) As $(A\theta)_j = \sum_{k=1}^{n} a_{jk} \theta_k$ with $a_{jk} \in \mathfrak{R}_{ev}$,

$$
J_n(v(A\theta)) = \partial_{\theta_1} \cdots \partial_{\theta_n} v(\theta) \big|_{\theta=0} = \sum_{\sigma \in \mathcal{P}_n} \partial_{\theta_{\sigma(1)}} \cdots \partial_{\theta_{\sigma(n)}} v(0) a_{\sigma(1)} \cdots a_{\sigma(n)}
$$

$$
= \partial_{\theta_1} \cdots \partial_{\theta_n} v(0) \sum_{\sigma \in \mathcal{P}_n} \text{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(n)} = J_n(v) \det A.
$$

(5) Obvious.

(6) We prove by induction w.r.t. $n$. When $n = 1$, for $\theta_1 = a\omega_1$ where $a \in \mathfrak{R}_{ev}$ with $\pi_B a \neq 0$, it is clear that (5.1.9) holds. In fact,

$$
J_1(v) = a^{-1} J(v(a\omega_1)) = J_1((\frac{\partial \theta_1}{\partial \omega_1})^{-1} v(a\omega_1)).
$$

Assuming (5.1.9) holds for $J_{n-1}$, we prove it for $J_n$. Let $\theta_j = \theta_j(\omega_1, \cdots, \omega_n)$ for $j = 1, \cdots, n$.

Without loss of generality, we may assume that $\frac{\partial \theta_1(\omega)}{\partial \omega_1} |_{\omega=0}$ is invertible. By the property (5), we have

$$
J_n(v) = J_1(J_{n-1}(v)).
$$

Putting $\tilde{\omega} = (\omega_2, \cdots, \omega_n)$, we solve the equation $\theta_1 = \theta_1(\omega_1, \tilde{\omega})$ w.r.t. $\omega_1$, having

$$
\omega_1 = \tilde{\omega}_1(\theta_1, \tilde{\omega}) \quad \text{with} \quad \theta_1 = \tilde{\theta}_1(\theta_1(\omega_1, \tilde{\omega}), \tilde{\omega}), \quad \omega_1 = \tilde{\omega}_1(\theta_1(\omega_1, \tilde{\omega}), \tilde{\omega}).
$$

We put this relation into $\theta_j = \theta_j(\omega)$ to have

$$
\theta_j = \theta_j(\omega_1(\theta_1, \tilde{\omega}), \tilde{\omega}) = \theta_j(\theta_1, \tilde{\omega}) \quad \text{for} \quad j = 2, \cdots, n.
$$
By (5) and the induction hypothesis, we have
\[ J_{n-1}(v) = J_{n-1}(\det \frac{\partial}{\partial \omega} v(\theta_1, \theta'(\theta_1, \tilde{\omega}))). \]
Changing the order of integration w.r.t. $\theta_1$ and $\tilde{\omega}$, we get
\[ J_n(v) = J_{n-1}(J_1(\det \frac{\partial}{\partial \omega} v(\theta_1, \theta'(\theta_1, \tilde{\omega}))). \]
Using $\theta_1 = \theta_1(\omega_1, \tilde{\omega})$, we have
\[ J_n(v) = J_{n-1}(J_1(\det \frac{\partial}{\partial \omega} (\frac{\partial \theta_1}{\partial \omega_1})^{-1} v(\theta_1(\omega_1, \tilde{\omega}), \theta'(\theta_1(\omega_1, \tilde{\omega}), \tilde{\omega}))). \]
\[ = J_n((\frac{\partial \theta_1}{\partial \omega_1})^{-1} \det \frac{\partial}{\partial \omega} v(\theta(\omega))). \]
On the other hand, if we have
\[ (\det \frac{\partial \theta_1}{\partial \omega_1}) \frac{\partial \theta_1}{\partial \omega_1} = \det \frac{\partial \theta}{\partial \omega}, \]
then we prove \[5.1.9\]. In fact, from \[5.1.11\], we get
\[ \frac{\partial \theta_1}{\partial \omega_k} = \frac{\partial \omega_1}{\partial \omega_k} \frac{\partial \theta_1}{\partial \omega_1} + \frac{\partial \theta_1}{\partial \omega_k} \quad \text{for} \ k = 2, \ldots, n, \]
and
\[ \frac{\partial \theta'}{\partial \omega_k} = \frac{\partial \theta_1}{\partial \omega_k} + \frac{\partial \theta_1}{\partial \omega_1} \frac{\partial \theta_1}{\partial \omega_k} = \frac{\partial \theta_1}{\partial \omega_k} - \frac{\partial \theta_1}{\partial \omega_1} \frac{\partial \theta_1}{\partial \omega_k} (\frac{\partial \theta_1}{\partial \omega_1})^{-1}. \]
Subtracting from $k$-row ($k \geq 2$) in \((\frac{\partial \theta}{\partial \omega})\) the first row multiplied by $\frac{\partial \theta_1}{\partial \omega_k}(\frac{\partial \theta_1}{\partial \omega_1})^{-1}$, we obtain, using above relation,
\[ \det \frac{\partial \theta}{\partial \omega} = \begin{vmatrix} \frac{\partial \theta_1}{\partial \omega_1} & \frac{\partial \theta_2}{\partial \omega_2} & \cdots & \frac{\partial \theta_n}{\partial \omega_n} \\ \frac{\partial \theta_1}{\partial \omega_2} & \frac{\partial \theta_2}{\partial \omega_2} & \cdots & \frac{\partial \theta_n}{\partial \omega_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial \theta_n}{\partial \omega_n} \end{vmatrix} = \frac{\partial \theta_1}{\partial \omega_1} \cdot \det \frac{\partial \theta'}{\partial \omega} |_{\theta_1 = \theta_1(\omega)}. \]
(7) is clear. \[\square\]

**Remark 5.1.2.** Above Berezin integration is defined without measure but using inner-multiplication in exterior algebra\[1\] such that
\[ \frac{\partial}{\partial z_j} |dz_k = \delta_{jk} \sim \int d\theta_j \theta_k = \delta_{jk}. \]

**Remark 5.1.3.** (i) We get the integration by parts formula, without the fundamental theorem of elementary analysis.
(ii) Moreover, since in conventional integration we get $\int dyf(y) = a \int dx f(ax)$, therefore the formula in \[5.1.11\] is very different from usual one. Analogous difference appears in \[5.1.9\].
(iii) \[5.1.10\] allows us to put
\[ \delta(\theta - \omega) = (\theta_1 - \omega_1) \cdots (\theta_n - \omega_n), \]
though $\delta(-\theta) = (-1)^n \delta(\theta)$. 

\[\text{1} \text{See, Mini-column 2 in Chapter 3}\]
(iv) For the future use, we give a Lie group theoretic proof of (5.1.9) due to Berezin [9]. Let a transformation $T$ may be included in a one-parameter family $T_t$ of transformations $\theta = \theta(\omega)$, that is, $T_1 \omega = \theta(\omega)$ and $T_0 \omega = \omega$ with $T_{t+s} = T_t T_s$. Set

$$\theta_k(t) = (T_1 \omega)_k \quad \text{and} \quad g(t) = \int_{\mathcal{R}_{od}} d\omega \ det^{-1} \frac{\partial \theta(t)}{\partial \omega} v(\theta(t)).$$

CLAIM 5.1.1. $g(t)$ is an analytic function w.r.t. $t$ and $g(t) = g(0)$.

In fact, using the multiplicativity of determinant, we have

$$g(t + s) = \int_{\mathcal{R}_{od}} d\omega \ det^{-1} \left( \frac{\partial \theta(t + s)}{\partial \theta(t)} \right) \det^{-1} \left( \frac{\partial \theta(t)}{\partial \omega} \right) v(\theta(t + s)).$$

Putting $R(s) = \left( \frac{\partial \theta(t + s)}{\partial \theta(t)} \right) \in \mathcal{R}_{ev}$, $\Delta(s) = \det^{-1} R(s) \in \mathcal{R}_{ev}$, we get

$$g'(t) = \frac{d}{ds} g(t + s) \big|_{s=0} = \int d\omega \ det^{-1} \left( \frac{\partial \theta(t)}{\partial \omega} \right) \frac{d}{ds} \left[ \Delta(s) v(\theta(t + s)) \right]_{s=0}.$$
5.2. Berezin integral w.r.t. even and odd variables

5.2.1. A naive definition and its problem. Because of Remark 4.4.2 in §4 of Chapter 4, we are inclined to “define”
\[ \int_a^b d q f(q) = \int_a^b d x \tilde{f}(x) \quad \text{where} \quad \tilde{f}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_q^n f(q) x^n \quad \text{with} \quad x = q + x_S. \]
Therefore,

**Definition 5.2.1.** For a set \( U \subset \mathbb{R}^m \), we define \( \pi_B^{-1}(U) = \{ X \in \mathcal{R}^{m|0} \mid \pi_B(X) \in U \} \). A set \( \mathfrak{U}_{ev} \subset \mathfrak{R}^{m|0} \) is called an “even superdomain” if \( U = \pi_B(\mathfrak{U}_{ev}) \subset \mathbb{R}^m \) is open, connected and \( \pi_B^{-1}(U) = \mathfrak{U}_{ev} \). \( U \) is denoted also by \( \mathfrak{U}_{ev,B} \). When \( \mathfrak{U} \subset \mathcal{R}^{m|n} \) is represented by \( \mathfrak{U} = \mathfrak{U}_{ev} \times \mathfrak{R}_{od} \) with an even superdomain \( \mathfrak{U}_{ev} \subset \mathfrak{R}^{m|0} \), \( \mathfrak{U} \) is called a “superdomain” in \( \mathfrak{R}^{m|n} \).

**Definition 5.2.2** (A naive definition of Berezin integral). For a super domain \( \mathfrak{U} = \mathfrak{U}_{ev} \times \mathfrak{R}^{0|n} \) and a supersmooth function \( u(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha u_\alpha(x) : \mathfrak{U} \to \mathfrak{R} \), we “define” its integral as
\[ \mathcal{B} \int_{\mathfrak{U}} d x d \theta u(x, \theta) = \int_{\mathfrak{U}_{ev}} d x \left( \int_{\mathfrak{R}^{0|n}} d \theta u(x, \theta) \right) = \int_{\pi_B(\mathfrak{U}_{ev})} dq u_1(q), \]
(5.2.1)
where
\[ \int_{\mathfrak{R}^{0|n}} d \theta u(x, \theta) = \left. \frac{\partial}{\partial \theta_1} \cdots \frac{\partial}{\partial \theta_1} u(x, \theta) \right|_{\theta_1=\cdots=\theta_n=0} = u_1(x) \quad \text{and} \quad \tilde{\mathfrak{I}} = (1, \ldots, 1). \]
In the above, \( u_1(x) \) is the Grassmann continuation of \( u_1(q) \).

Desiring that the standard formula of the change of variables under integral sign \((=\text{CVF})\) holds by replacing standard Jacobian with super Jacobian (= super determinant of Jacobian matrix) on \( \mathcal{R}^{m|n} \), we have

**Theorem 5.2.1.** Let \( \mathfrak{U} = \mathfrak{U}_{ev} \times \mathfrak{R}^{0|n} \subset \mathfrak{R}^{m|n} \) and \( \mathfrak{W} = \mathfrak{W}_{ev} \times \mathfrak{R}^{0|n} \subset \mathfrak{R}^{m|n} \) be given. Let
\[ \varphi : \mathfrak{W} \ni Y = (y, \omega) \to X = (x, \theta) = (\varphi_0(y, \omega), \varphi_1(y, \omega)) \in \mathfrak{U} \]
be a supersmooth diffeomorphism from \( \mathfrak{W} \) onto \( \mathfrak{U} \), that is,
\[ \text{sdet} \ J(\varphi)(y, \omega) \neq 0 \quad \text{and} \quad \varphi(\mathfrak{W}) = \mathfrak{U} \quad \text{where} \quad J(\varphi)(y, \omega) = \begin{pmatrix} \frac{\partial \varphi_0(y, \omega)}{\partial y} & \frac{\partial \varphi_1(y, \omega)}{\partial y} \\ \frac{\partial \varphi_0(y, \omega)}{\partial \omega} & \frac{\partial \varphi_1(y, \omega)}{\partial \omega} \end{pmatrix}. \]
(5.2.3)
Then, for any function \( u \in C_{SS}(\mathfrak{U} : \mathfrak{C}) \) with “compact support”, that is, \( u(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha u_\alpha(x) \) where \( u_\alpha(x_B) \in C_0^\infty(\mathfrak{U}_{ev,B} : \mathfrak{R}) \) for all \( a \in \{0, 1\}^n \) except \( a = \tilde{\mathfrak{I}} \), we have CVF
\[ \mathcal{B} \int_{\mathfrak{U}} d x d \tilde{\mathfrak{I}} u(x, \theta) = \mathcal{B} \int_{\varphi^{-1}(\mathfrak{U})} d y d \omega \text{sdet} \ J(\varphi)(y, \omega) u(\varphi(y, \omega)). \]
(5.2.4)

**Remark 5.2.1.** Seemingly, this theorem implies that Berezin “measure” \( D_0(x, \theta) \) is transformed by \( \varphi \) as
\[ (\varphi^* D_0(x, \theta))(y, \omega) = D_0(y, \omega) \cdot \text{sdet} \ J(\varphi)(y, \omega), \]
(5.2.5)
where
\[ D_0(x, \theta) = d x_1 \wedge \cdots \wedge d x_m \otimes \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} = d x_1 \cdots d x_m \cdot \partial_\theta_n \cdots \partial_\theta_1 = d x \partial_\theta, \quad D_0(y, \omega) = d y \partial_\omega. \]
But this assertion is shown to be false in general by the following examples. Moreover, we remark also that the condition of “the compact supportness of integrands” above seems not only cumbersome from conventional point of view but also fatal in holomorphic category.
Remark 5.2.2. Though we give some examples which show the immaturity of the above naive definition, but we give a precise proof of this theorem in §1 of Chapter 9 for future use.

Example 5.2.1. Let $\mathcal{U} = \pi^{-1}_B(\Omega) \times \mathcal{R}^2_{od} \subset \mathcal{R}^{1|2}$ with $\Omega = (0, 1)$, $\pi_B : \mathcal{R}^{1|0} \to \mathbb{R}$ and let $u$ be supersmooth on $\mathcal{R}^{1|2}$ with value in $\mathcal{R}$ such that $u(x, \theta) = u_0(x) + \theta_2 u_1(x)$. Then, we have

$$\text{B} - \int_{\mathcal{U}} dxd\theta u(x, \theta) = \int_{\Omega} dx \int d\theta u(x, \theta) = \int_{\pi^{-1}_B(\Omega)} dx u_1(x) = \int_{\Omega} dq u_1(q).$$

But, if we use the coordinate change

$$(5.2.6)\quad \varphi : (y, \omega) \to (x, \theta) \quad \text{with} \quad x = y + \omega_1 \omega_2 \phi(y), \quad \theta_k = \omega_k : \mathcal{U} \to \mathcal{U}$$

whose Berezinian is

$$\text{Ber}(\varphi)(y, \omega) = \text{sdet} \, J(\varphi)(y, \omega) = 1 + \omega_1 \omega_2 \phi'(y)$$

where $J(\varphi)(y, \omega) = \begin{pmatrix} 1 + \omega_1 \omega_2 \phi'(y) & 0 & 0 \\ \omega_2 \phi(y) & 1 & 0 \\ -\omega_1 \phi(y) & 0 & 1 \end{pmatrix}$, and if we assume that the formula (5.2.4) holds, then since

$$u(\varphi(y, \omega)) = u_0(y + \omega \omega_2 \phi(y)) + \omega_2 \omega_1 u_1(y + \omega \omega_2 \phi(y)) = u_0(y) + \omega_1 \omega_2 \phi(y) u_0'(y) + u_1(y),$$

and

$$(1 + \omega_1 \omega_2 \phi'(y)) u(\varphi(y, \omega)) = u_0(y) + \omega_1 \omega_2 \phi(y) u_0'(y) + \phi'(y) u_0(y) + u_1(y),$$

we have

$$\text{B} - \int_{\varphi^{-1}(\mathcal{U})} dyd\omega (1 + \omega_1 \omega_2 \phi'(y)) u(\varphi(y, \omega)) = \int_{\pi^{-1}_B(\Omega)} dy (\phi(y) u_0'(y))' + \int_{\pi^{-1}_B(\Omega)} dx u_1(x).$$

Therefore, if $\int_{\pi^{-1}_B(\Omega)} dy (\phi(y) u_0'(y))' \neq 0$, then $\int_{\mathcal{U}} D_0(x, \theta) u(x, \theta) \neq \int_{\varphi^{-1}(\mathcal{U})} D_0(y, \omega) u(\varphi(y, \omega))$. This implies that if we apply (5.2.1) as definition, the change of variables formula doesn’t hold when, for example, the integrand hasn’t compact support.

Example 5.2.2. [Inconsistency related to $Q$-integration where matrix $Q$ is mentioned in Chapter 3] Let a set of matrix $Q$ be given by

$$Q = \left\{ Q = \begin{pmatrix} x_1 & \theta_1 \\ \theta_2 & ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathcal{R}_{ev}, \theta_1, \theta_2 \in \mathcal{R}_{od} \right\} \cong \mathcal{R}^{2|2}$$

and let regard $Q$ as a variable with its “volume element” $dQ = \frac{dx_1 dx_2}{2\pi} d\theta_2 d\theta_1$. Then, we have

$$(5.2.7)\quad \int_{\Omega} dQ e^{-str Q} = \left( \int_{\mathcal{R}^{2|2}} \frac{dx_1 dx_2}{2\pi} d\theta_2 d\theta_1 e^{-(x_1^2 + x_2^2 + 2\theta_1 \theta_2)} \right) = 1.$$

We apply change of variables to a super matrix $Q$ as

$$(5.2.8)\quad \left\{ \begin{array}{l} y_1 = x_1 + \frac{\theta_1 \theta_2}{x_1 - ix_2}, \quad y_2 = x_2 - \frac{i \theta_1 \theta_2}{x_1 - ix_2}, \\ \omega_1 = \frac{\theta_1}{x_1 - ix_2}, \quad \omega_2 = -\frac{\theta_2}{x_1 - ix_2}, \end{array} \right.$$ \quad or \quad

$$(5.2.9)\quad \left\{ \begin{array}{l} x_1 = y_1 + \omega_1 \omega_2 (y_1 - iy_2), \quad x_2 = y_2 - i \omega_1 \omega_2 (y_1 - iy_2), \\ \theta_1 = \omega_1 (y_1 - iy_2), \quad \theta_2 = -\omega_2 (y_1 - iy_2), \end{array} \right.$$ to make it diagonal. Then,

$$(5.2.10)\quad G Q Q^{-1} = \begin{pmatrix} y_1 & 0 \\ 0 & iy_2 \end{pmatrix}, \quad G Q^2 G^{-1} = \begin{pmatrix} y_1^2 & 0 \\ 0 & -y_2^2 \end{pmatrix}$$

where

$$G = \begin{pmatrix} 1 + \frac{1}{2} \omega_1 \omega_2 & \frac{1}{2} \omega_1 \\ \frac{1}{2} \omega_2 & 1 - \frac{1}{2} \omega_1 \omega_2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 + \frac{1}{2} \omega_1 \omega_2 & -\frac{1}{2} \omega_1 \\ -\frac{1}{2} \omega_2 & 1 - \frac{1}{2} \omega_1 \omega_2 \end{pmatrix}. $$
Clearly, 
\[ x_1 - ix_2 = y_1 - iy_2, \quad \text{and} \quad \text{str } Q^2 = x_1^2 + x_2^2 + 2\theta_1\theta_2 = y_1^2 + y_2^2, \]
and their Jacobian (called Berezian) is
\[ dQ = \frac{dx_1dx_2}{2\pi}d\theta_2d\theta_1 = -\frac{dy_1dy_2}{2\pi}d\omega_2d\omega_1 (y_1 - iy_2)^{-2}. \]
This implies
\[ -\int \frac{dy_1dy_2}{2\pi}d\omega_2d\omega_1 (y_1 - iy_2)^{-2}e^{-(y_1^2+y_2^2)} = 0 \]
which contradicts to (5.2.7).

5.2.2. Integration of Gaussian type and Pfaffian. In spite of above immatureness of
the naive definition, we may give examples which mention the relation of Gaussian type integral,
determinant and Pfaffian.

**Definition 5.2.3.** For \( n \times n \)-anti symmetric matrix \( \tilde{B} = (\tilde{B}_{jk}) \) with even elements, we define
the Pfaffian \( \text{Pf}(\tilde{B}) \) of \( B \) as
\[
\text{Pf}(\tilde{B}) = \frac{1}{(n/2)!} \sum_{\rho \in \varphi_n} \text{sgn}(\rho) \tilde{B}_{\rho(1)} \tilde{B}_{\rho(2)} \cdots \tilde{B}_{\rho(n-1)} \tilde{B}_{\rho(n)}.
\]
Here, \( \varphi_n \) is the permutation group of degree \( n \), \( \text{sgn}(\rho) \) is the signature of \( \rho \in \varphi_n \).

**Remark 5.2.3.** Let \( n \) be even, and let \( A = (A_{ij}) \) be anti symmetric matrix. Then, we have
\[
\int d\theta \exp\left(-\frac{1}{2}A_{ij}\theta_i\theta_j\right) = \text{Pf}(A).
\]
Moreover, \( \text{Pf}(A)^2 = \det(A) \) holds.

**Definition 5.2.4.** A even super matrix \( M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \) is called positive-definite if the follow-
ing conditions are satisfied:

(gs.1): \( A \) has the body part \( A_B \) which is regular positive definite symmetric matrix.
(gs.2): \( B \) is a regular anti-symmetric matrix.
(gs.3): \( C \) and \( D \) satisfies \( tC + D = 0 \).

For above super matrix \( M \), we define the corresponding bilinear form as
\[
\langle X, MX \rangle = ^tXMX
\]
\[
= \sum_{j,k=1}^m x_jA_{jk}x_k + \sum_{j=1}^m \sum_{s=1}^n x_jC_{j+m+s} \theta_s + \sum_{k=1}^m \sum_{t=1}^n \theta_tD_{m+t+k}x_k + \sum_{s,t=1}^n \theta_sB_{m+s+m+t}\theta_t.
\]

**Lemma 5.2.1.** Let \( M \) be a even, positive definite matrix. Then,
\[
G(\lambda, M) = \int_{\mathbb{R}^{m+n}} dX e^{-\lambda^{-1/2}-1}(X, MX) \quad \text{for } \lambda > 0
\]
\[
= \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
(2\pi\lambda)^{m/2}(2\lambda)^{-n/2}(\det A)^{-1/2} \text{Pf}(B - DA^{-1}C) & \text{if } n \text{ is even}.
\end{cases}
\]

**Comparison 5.2.1.** For a positive definite symmetric real matrix \( H \), we have
\[
\int_{\mathbb{R}^m} e^{-\lambda x \cdot H x/2} dx = \left(\frac{2\pi}{\lambda}\right)^{m/2} (\det H)^{-1/2}.
\]
Exercise 5.2.1. Prove the following by Berezin integral:
(i) \( \det(A) = \text{Pf}(A)^2 \) for antisymmetric matrix \( A \),
(ii) For any \( 2n \times 2n \) antisymmetric matrix \( A \) and any \( 2n \times 2n \) matrix \( B \), \( \det(B^tAB) = \det(B) \text{Pf}(A) \),
(iii) For any \( n \times n \) matrix \( B \),
\[
\text{Pf} \begin{pmatrix}
0 & B \\
-B^t & 0
\end{pmatrix} = (-1)^{n(n-1)/2} \det(B).
\]

5.3. Contour integral w.r.t. an even variable

To overcome the inconsistency in above examples, we need to reconsider the meaning of “body part”, that is, not to insist on Remark 4.4.2 of de Witt (§4 of Chapter 4).

We recall the idea of the contour integral noted in Rogers [109].

Contour integrals are a means of “pulling back” an integral in a space that is algebraically (as well as possibly geometrically) more complicated than \( \mathbb{R}^m \). A familiar example, of course, is complex contour integration; if \( \gamma : [0, 1] \to \mathbb{C} \) is piecewise \( C^1 \) and \( f : \mathbb{C} \to \mathbb{C} \), one has the one-dimensional contour integral
\[
\int_{\gamma} f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 dt \gamma'(t)\cdot f(\gamma(t)).
\]

This involves the algebraic structure of \( \mathbb{C} \) because the right-hand side of above includes multiplication \( \cdot \) of complex numbers.

We follow this idea to define the integral of a supersmooth function \( u(x) \) on an even superdomain \( \mathcal{U}_{ev} \subset \mathbb{R}^{m|0} = \mathbb{R}^m_{ev} \) (see also, Rogers [106, 107, 109] and Vladimirov and Volovich [130]).

Definition 5.3.1. Let \( u(x) \) be a supersmooth function defined on an even superdomain \( \mathcal{U}_{ev} \subset \mathbb{R}^{1|0} \) such that \( [a, b] \subset \pi_B(\mathcal{U}_{ev}) \). Let \( \lambda = \lambda_B + \lambda_S, \mu = \mu_B + \mu_S \in \mathcal{U}_{ev} \) with \( \lambda_B = a, \mu_B = b, \) and let a continuous and piecewise \( C^1 \)-curve \( \gamma : [a, b] \to \mathcal{U}_{ev} \) be given such that \( \gamma(a) = \lambda, \gamma(b) = \mu \). We define
\[
\int_{\gamma} dx u(x) = \int_a^b dt \dot{\gamma}(t)\cdot u(\gamma(t)) \in \mathcal{C}
\]
and call it the integral of \( u \) along the curve \( \gamma \).

Using the integration by parts for functions on \( \mathbb{R} \), we get the following fundamental result.

Proposition 5.3.1 (p.7 of de Witt [34]). Let \( u(t) \in C^\infty([a, b] : \mathcal{C}) \) and \( U(t) \in C^\infty([a, b] : \mathcal{C}) \) be given such that \( U'(t) = u(t) \) on \([a, b] \). We denote the Grassmann continuations of them as \( \check{u}(x) \) and \( \check{U}(x) \). Then, for any continuous and piecewise \( C^1 \)-curve \( \gamma : [a, b] \to \mathcal{U}_{ev} \subset \mathbb{R}^{1|0} \) such that \( [a, b] \subset \pi_B(\mathcal{U}_{ev}) \) and \( \gamma(a) = \lambda, \gamma(b) = \mu \) with \( \lambda_B = a, \mu_B = b, \) we have
\[
\int_{\gamma} dx \check{u}(x) = \check{U}(\lambda) - \check{U}(\mu).
\]

Proof. By definition, we get
\[
\int_a^b dt \dot{\gamma}(t)u(\gamma(t)) = \int_a^b dt (\dot{\gamma}_B(t) + \dot{\gamma}_S(t)) \sum_{\ell \geq 0} \frac{1}{\ell!} u^{(\ell)}(\gamma_B(t))\gamma_S(t)^\ell
\]
\[ \int_a^b dt \cdot \hat{\gamma}_B(t)u(\gamma_B(t)) + \int_a^b dt \cdot \hat{\gamma}_B(t) \sum_{k \geq 1} \frac{1}{k!} u^{(k)}(\gamma_B(t)) \gamma_S(t)^k \]
\[ + \int_a^b dt \sum_{\ell \geq 0} \frac{1}{\ell!} u^{(\ell)}(\gamma_B(t)) \gamma_S(t)^\ell \]
\[ = U(b) - U(a) + \sum_{\ell \geq 0} \frac{1}{(\ell + 1)!} \left\{ U^{(\ell + 1)}(b) \mu_\ell^+ - U^{(\ell + 1)}(a) \lambda_\ell^+ \right\} \]
\[ = \hat{U}(\mu) - \hat{U}(\lambda). \]

Here, we used the integration by parts formula for functions on \( \mathbb{R} \) with value in Fréchet space:
\[ \int_a^b dt \cdot u^{(\ell)}(\gamma_B(t)) \gamma_S(t)^\ell \]
\[ = \int_a^b dt \cdot u^{(\ell)}(\gamma_B(t)) \frac{d \gamma_S(t)^{\ell + 1}}{d t} \]
\[ = - \int_a^b dt \cdot \hat{\gamma}_B(t)u^{(\ell + 1)}(\gamma_B(t)) \frac{\gamma_S(t)^{\ell + 1}}{\ell + 1} + U^{(\ell + 1)}(b) \mu_\ell^+ - U^{(\ell + 1)}(a) \lambda_\ell^+. \]

**Problem 5.3.1.** How do we extend Proposition 5.3.1 to the case when \( u(t) \in C([a,b] : \mathcal{E}) \)?

**Lemma 5.3.1** (Lemma 3.9 in [106] on \( \mathcal{B}_L \)). (a) (reparametrization of paths) Let \( \gamma : [a,b] \to \mathcal{R}_{ev} \) be a path in \( \mathcal{R}_{ev} \) and let \( c, d \in \mathbb{R} \). Also let \( \phi : [c,d] \to [a,b] \) be \( C^1 \) with \( \phi(c) = a, \phi(d) = b \) and \( \phi'(s) > 0 \) for all \( s \in [c,d] \). Then
\[ \int_\gamma dx u(x) = \int_{\gamma \circ \phi} dx u(x). \]

(b) (sum of paths) Let \( \gamma_1 : [a,b] \to \mathcal{R}_{ev} \) and \( \gamma_2 : [c,d] \to \mathcal{R}_{ev} \) be two paths with \( \gamma_1(b) = \gamma_2(c) \) Also define \( \gamma_1 + \gamma_2 \) to be the path \( \gamma_1 + \gamma_2 : [a, b + d - c] \to \mathcal{R}_{ev} \) defined by
\[ \gamma_1 + \gamma_2(t) = \begin{cases} 
\gamma_1(t), & a \leq t \leq b, \\
\gamma_2(t - b + c), & b \leq t \leq b + d - c.
\end{cases} \]

Then if \( \mathcal{U}_{ev} \) is open in \( \mathcal{R}_{ev} \), \( u : \mathcal{U}_{ev} \to \mathcal{R} \) is in \( C_{ss} \) and \( \gamma_1([a,b]) \subset \mathcal{U}_{ev} \), \( \gamma_2([c,d]) \subset \mathcal{U}_{ev} \),
\[ \int_{\gamma_1 + \gamma_2} dx u(x) = \int_{\gamma_1} dx u(x) + \int_{\gamma_2} dx u(x). \]

(c) (inverse of a path) Let \( \gamma : [a,b] \to \mathcal{R}_{ev} \) be a path in \( \mathcal{R}_{ev} \). Define the curve \( -\gamma : [a,b] \to \mathcal{R}_{ev} \) by
\[ -\gamma(t) = \gamma(a + b - t) \]

Then if \( \mathcal{U}_{ev} \) is open in \( \mathcal{R}_{ev} \) with \( \gamma([a,b]) \subset \mathcal{U}_{ev} \) and \( u : \mathcal{U}_{ev} \to \mathcal{R} \) is supersmooth,
\[ \int_{-\gamma} dx u(x) = - \int_\gamma dx u(x). \]

**Proof.** Applying CVF on \( \mathbb{R} \) for \( t = \phi(s) \) and \( dt = \phi'(s)ds \), we have
\[ \int_{\gamma \circ \phi} dx u(x) = \int_c^d ds \cdot (\gamma(\phi(s)))' u(\gamma(\phi(s))) = \int_c^d ds \cdot \phi'(s) [\gamma'(\phi(s)) u(\gamma(\phi(s)))] \]
\[ = \int_a^b dt \cdot \gamma'(t)u(\gamma(t)) = \int_\gamma dx u(x). \]

Others are proved analogously. \( \square \)
COROLLARY 5.3.1 (Corollary 3.7 in [106] on $\mathfrak{B}_L$). Let $u(x)$ be a supersmooth function defined on a even superdomain $\mathfrak{U}_{ev} \subset \mathfrak{R}^{1|0}$ into $\mathcal{C}$.

(a) Let $\gamma_1, \gamma_2$ be continuous and piecewise $C^1$-curves from $[a, b] \rightarrow \mathfrak{U}_{ev}$ such that $\lambda = \gamma_1(a) = \gamma_2(a)$ and $\mu = \gamma_1(b) = \gamma_2(b)$. If $\gamma_1$ is homotopic to $\gamma_2$, then

\begin{equation}
\int_{\gamma_1} dx u(x) = \int_{\gamma_2} dx u(x).
\end{equation}

(b) If $u : \mathfrak{R}_{ev} \rightarrow \mathcal{R}$ is $C_{SS}$ on all $\mathfrak{R}_{ev}$, one can write “unambiguously”

\begin{equation}
\int_{\lambda}^{\mu} dx u(x) = \int_{\gamma} dx u(x).
\end{equation}

Here, $\gamma : [a, b] \rightarrow \mathfrak{R}_{ev}$ is any path in $\mathfrak{R}_{ev}$ with $\gamma(a) = \lambda, \gamma(b) = \mu$.

PROPOSITION 5.3.2. For a given change of variable $x = \varphi(y)$, we define the pull-back of 1-form $\nu_x = dx \rho(x)$ by $(\varphi^*v)_y = dy \frac{\partial \varphi(y)}{\partial y} \rho(\varphi(y))$. Then, for paths $\gamma : [a, b] \rightarrow \mathfrak{R}_{ev}, \varphi^{-1} \circ \gamma : [a, b] \rightarrow \mathfrak{R}_{ev}$ and $u$, we have

\begin{equation}
\int_{\gamma} \nu u = \int_{\gamma} dx \nu_x \rho(x) u(x) = \int_{\varphi^{-1} \circ \gamma} dy (\varphi^*v)_y \rho(\varphi(y)) u(\varphi(y)) = \int_{\varphi^{-1} \circ \gamma} \varphi^*v \varphi^* u.
\end{equation}

Proof. By definition, we have not only

\begin{equation}
\int_{\gamma} \nu_x u(x) = \int_{a}^{b} dt \gamma'(t) \rho(\gamma(t)) u(\gamma(t)),
\end{equation}

but also

\begin{equation}
\int_{\varphi^{-1} \circ \gamma} (\varphi^*v)_y \varphi^* u(y) = \int_{\varphi^{-1} \circ \gamma} dy \frac{\partial \varphi(y)}{\partial y} \rho(\varphi(y)) u(\varphi(y))
= \int_{a}^{b} dt (\varphi^{-1}(\gamma(t))) \frac{\partial \varphi(y)}{\partial y} \rho(\varphi(y)) u(\varphi(y))\bigg|_{y=\varphi^{-1} \circ \gamma(t)}
= \int_{a}^{b} dt \gamma'(t) \rho(\gamma(t)) u(\gamma(t)).
\end{equation}

Here, we used $y = \varphi^{-1}(\varphi(y)), x = \gamma(t), y = \varphi^{-1}(\gamma(t))$ with

\begin{equation}
1 = \varphi(y) \varphi^{-1}(\varphi(y)), \quad \varphi(y) = \frac{1}{\varphi^{-1}(\varphi(y))} = \frac{\partial \varphi(y)}{\partial y}.
\end{equation}

EXAMPLE 5.3.1 (Translational invariance). Let $I = (a, b) \subset \mathcal{R}$. We identify $q \in I$ as $\gamma(q) = x \in \mathfrak{R}_{ev}$. We put $\mathfrak{M} = \mathfrak{M}(I) = \{x \in \mathfrak{R}_{ev} \mid \pi_B(x) = q \in I\} \subset \mathfrak{R}_{ev}$. Taking a non-zero nilpotent element $\nu \in \mathfrak{R}_{ev}$, we put $\tau_\nu : \mathfrak{R}_{ev} \ni y \rightarrow x = \varphi(y) = \tau_\nu(y) = y - \nu \in \mathfrak{R}_{ev}$,

\begin{equation}
\mathfrak{M}_1 = \tau_\nu^{-1}(\mathfrak{M}) = \{x + \nu \mathfrak{R}_{ev} \mid \pi_B(x) = q \in I\}, \quad \gamma_1(q) = \tau_\nu^{-1}(\gamma(q)).
\end{equation}

Then, we have

\begin{equation}
\int_{\mathfrak{M}} dx u(x) = \int_{a}^{b} dq \gamma'(q) u(\gamma(q)) = \int_{a}^{b} dq \gamma_1'(q) u(\gamma(q)) = \int_{\mathfrak{M}_1} dy u(y - \nu).
\end{equation}

REMARK 5.3.1. Above identification $\gamma(q) = x \in \mathfrak{R}_{ev}$ is obtained as the Grassmann continuation $\hat{\iota}$ of a function $\iota(q) = q \in C^\infty(I : \mathcal{R})$. In fact,

\begin{equation}
\hat{\iota}(x) = \sum_{\alpha} \frac{\partial^\alpha \iota(x)}{\partial q^\alpha} (x_B)x_S^\alpha = x_B + x_S = x.
\end{equation}
5.4. Modification of Rogers, Vladimirov and Volovich's approach

Now, we modify arguments of V.S. Vladimirov and I.V. Volovich [130] suitably to get a new definition.

**Definition 5.4.1 (Parameter set, paths and integral).** For any domain \( \Omega \) in \( \mathbb{R}^m \), we denote \( \tilde{\Omega} = \Omega \times \mathbb{R}^n_{\text{od}} \) as a parameter set.

1. A smooth map \( \gamma \) from \( \tilde{\Omega} \) to \( \mathcal{R}^{m|n} \) belongs to \( C^\infty(\tilde{\Omega} : \mathcal{R}^{m|n}) \) when

\[
\gamma(q, \vartheta) = (\gamma_0(q, \vartheta), \gamma_1(q, \vartheta)) = (\gamma_{0,j}(q, \vartheta), \gamma_{1,k}(q, \vartheta))_{j=1, \ldots, m, k=1, \ldots, n},
\]

and

\[
\gamma_{0,j}(q, \vartheta) = \sum_{|\alpha| \leq n} \partial^\alpha \gamma_{0,j,\alpha}(q) \in \mathcal{R}_{\text{ev}}, \quad \gamma_{1,k}(q, \vartheta) = \sum_{|\alpha| \leq n} \partial^\alpha \gamma_{1,k,\alpha}(q) \in \mathcal{R}_{\text{od}}.
\]

Here,

\[
\gamma_{0,j,\alpha}(q) = \sum_{|J|=|\alpha|+1} \gamma_{0,j,a}(q) \sigma^J, \quad \gamma_{1,k,a}(q) = \sum_{|J|=|\alpha|+1} \gamma_{1,k,a}(q) \sigma^J
\]

and

\[
\gamma_{0,a,1}(q), \gamma_{1,a,1}(q) \in C^\infty(\tilde{\Omega} : \mathbb{C}^n) \text{ and } \gamma_{0,\bar{0},0}(q) \in C^\infty(\tilde{\Omega} : \mathbb{R}^m)
\]

with \( \bar{0} = \text{even} \) or \( \bar{0} = (0, \ldots, 0) \in \{0, 1\}^n, \bar{0} = (0, \ldots) \in \mathcal{I} \).

Moreover, if

\[
\text{sdet } J(\gamma)(q, \vartheta) \neq 0 \text{ where } J(\gamma)(q, \vartheta) = \frac{\partial \gamma(q, \vartheta)}{\partial (q, \vartheta)} = \left( \frac{\partial \gamma_{0}(q, \vartheta)}{\partial q} \frac{\partial \gamma_{0}(q, \vartheta)}{\partial \vartheta} \frac{\partial \gamma_{1}(q, \vartheta)}{\partial q} \frac{\partial \gamma_{1}(q, \vartheta)}{\partial \vartheta} \right),
\]

then, this \( \gamma \) is called a path from \( \tilde{\Omega} \) to \( \mathcal{R}^{m|n} \), whose image is said to be \( \text{FSM(=foliated singular manifold)} \) denoted by

\[
\mathcal{M} = \mathcal{M}(\gamma, \Omega) = \gamma(\Omega) = \{(x, \theta) \in \mathcal{R}^{m|n} \mid x = \gamma_0(q, \vartheta), \theta = \gamma_1(q, \vartheta), q \in \Omega, \vartheta \in \mathcal{R}^n_{\text{od}} \}.
\]

2. Let \( \mathcal{M} \) be given as above. For a supersmooth function \( u(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha u_{\alpha}(x) \) defined on \( \mathcal{M} \), we define the integration of \( u(x, \theta) \) on \( \mathcal{M} \) as follows:

\[
\text{RVW} \int_{\mathcal{M}} dx \, d\theta \, u(x, \theta) = \int_{\mathcal{R}^n_{\text{od}}} d\vartheta \left[ \int_{\mathcal{R}^m_{\text{od}}} dq \, \text{sdet } J(\gamma)(q, \vartheta) u(\gamma(q, \vartheta)) \right].
\]

Here, we assume that for each \( \vartheta \in \mathcal{R}^n_{\text{od}} \), integrands in the bracket \( \cdots \) above are integrable on \( \Omega \).

**Remark 5.4.1.** The reason for nomination “singular” is explained in A. Kharenkov [83]. It stems from the difference from the naive Definition 5.2.2 of Berezin i.e., their definition domain are not \( \mathcal{M}_{\text{ev}} = \{ x \in \mathcal{R}^{m|0} \mid \pi_B(x) \in \Omega \} \) though defined via \( \Omega \) in \( \mathbb{R}^m \).

We need to check the well-definedness of (5.4.1) in Definition 5.4.1. First of all, we remark that by the algebraic nature of integration w.r.t. odd variables, we may interchange the order of
In case when $\gamma_0(q, \vartheta)$ doesn’t depend on $\vartheta$, putting $\bar{\vartheta} = \gamma_1(q, \vartheta)$ and $\bar{q} = \gamma_0(q)$, we get

$$
\int_{\Omega} dq \frac{\partial}{\partial \vartheta_n} \cdots \frac{\partial}{\partial \vartheta_1} \left( \det J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right) \bigg|_{\vartheta = 0} = \int_{\Omega} dq \frac{\partial}{\partial \vartheta_n} \cdots \frac{\partial}{\partial \vartheta_1} \left( \det J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right) \bigg|_{\vartheta = 0} = \int_{\Omega} dq \left[ \int_{\mathfrak{R}_{\text{od}}} d\vartheta \det J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right].
$$

That is, putting $\bar{\Omega} = \Omega \times \mathfrak{R}_{\text{od}}$ and $\mathfrak{R} = \gamma(\bar{\Omega})$, we have $\gamma(q, \vartheta) = (\gamma_0(q), \gamma_1(q, \vartheta))$ and

$$
\text{RVV}-\iint_{\mathfrak{R}} dx d\theta u(x, \theta) = \int_{\mathfrak{R}_{\text{od}}} d\theta \left[ \int_{\gamma_0(\Omega)} dx u(x, \theta) \right] = \int d\theta \left( \int dx u(x, \theta) \right) = \int_{\gamma_0(\Omega)} dx \left[ \int_{\mathfrak{R}_{\text{od}}} d\theta u(x, \theta) \right] = \int dx \left( \int d\theta u(x, \theta) \right).
$$

Moreover, we need the following definition:

**Definition 5.4.2.** Let two FSM $\mathfrak{R} = \gamma(\bar{\Omega})$ and $\mathfrak{R}' = \gamma'(\bar{\Omega}')$ be given. They are called superdiffeomorphic each other if there exist diffeomorphisms $\phi : \bar{\Omega}' \rightarrow \bar{\Omega}$ and $\varphi : \mathfrak{R}' \rightarrow \mathfrak{R}$, such that the following diagram holds with $\gamma' = \varphi^{-1} \circ \gamma \circ \phi$:

$$
\begin{array}{ccc}
\bar{\Omega}' & \xrightarrow{\gamma_1} & \mathfrak{R}' = \gamma'(\bar{\Omega}') \\
\downarrow \phi & & \downarrow \varphi \\
\bar{\Omega} & \xrightarrow{\gamma} & \mathfrak{R} = \gamma(\bar{\Omega})
\end{array}
$$

Using this notion, we have the desired result:

**Proposition 5.4.1 (Reparametrization invariance).** Let $\Omega$ and $\Omega'$ be domains in $\mathbb{R}^m$ and we put $\bar{\Omega}$ and $\bar{\Omega}'$ as above. We assume $\bar{\Omega}$ and $\bar{\Omega}'$ are superdiffeomorphic each other, that is, there exist a diffeomorphism $\phi_0 : \Omega' \rightarrow \bar{\Omega}$ such that $\frac{\partial \phi_0(q')}{\partial q}$ which is continuous in $\Omega'$ and $\text{det} \left( \frac{\partial \phi_0(q')}{\partial q} \right) > 0$ and a map $\phi_1 : \Omega' \times \mathfrak{R}_{\text{od}} \ni (q', \eta') \rightarrow \phi_1(q', \eta') \in \mathfrak{R}_{\text{od}}$ which is smoothness w.r.t. $\eta'$ with $\text{det} \left( \frac{\partial \phi_1(q', \eta')}{\partial q'} \right) \neq 0$. Put

$$
\mathfrak{R}' = \{ X' = (x', \theta') \mid X' = \gamma \circ \phi(q', \eta'), \ (q', \eta') \in \bar{\Omega}' \} \quad \text{where} \quad \phi(q', \eta') = (\phi_0(q'), \phi_1(q', \eta')).
$$

For a given path $\gamma : \bar{\Omega} \rightarrow \mathfrak{R}_{\text{od}}$, we define a path $\gamma \circ \phi : \bar{\Omega}' \rightarrow \mathfrak{R}_{\text{od}}$. Then, we have

$$
\text{RVV}-\iint_{\gamma(\bar{\Omega})} dx d\theta u(x, \theta) = \text{RVV}-\iint_{\gamma \circ \phi(\bar{\Omega}')} dx' d\theta' u(x', \theta').
$$
Proof. By definition, we have
\[
\text{RVV-} \int_{\gamma(\bar{\Omega})} dxd\theta u(x, \theta) = \int_{\mathbb{R}^n_{od}} d\eta \left( \int_{\Omega} dq \text{ sdet } J(\gamma)(q, \eta)u(\gamma(q, \eta)) \right)
\]
and
\[
\text{RVV-} \int_{\gamma \circ \phi(\bar{\Omega})} dxd\theta u(x', \theta') = \int_{\mathbb{R}^n_{od}} d\eta' \left( \int_{\Omega'} dq' \text{ sdet } J(\gamma \circ \phi)(q', \eta')u(\gamma \circ \phi(q', \eta')) \right).
\]

Using
\[
\gamma \circ \phi(q', \eta') = (\gamma_0(\phi_0(q'), \phi_1(q', \eta')), \gamma_1(\phi_0(q'), \phi_1(q', \eta'))),
\]
\[
J(\gamma \circ \phi)(q', \eta') = J(\gamma)(\phi(q', \eta'))J(\phi)(q', \eta'),
\]
\[
\text{sdet } J(\phi)(q', \eta') = \det^{-1} \left( \frac{\partial \phi_1(q', \eta')}{\partial \eta'} \right) \det \left( \frac{\partial \phi_0(q')}{\partial q'} \right),
\]
we have
\[
\text{sdet } J(\gamma \circ \phi)(q', \eta') = \det \left( \frac{\partial \phi_0(q')}{\partial q'} \right) \det^{-1} \left( \frac{\partial \phi_1(q', \eta')}{\partial \eta'} \right) \text{sdet } J(\gamma)(q, \eta) \bigg|_{q=\phi_0(q')} \bigg|_{\eta=\phi_1(q', \eta')}.
\]

Remark the order of integration, we have
\[
\int_{\mathbb{R}^n_{od}} d\eta' \left( \int_{\Omega'} dq' \text{ sdet } J(\gamma \circ \phi)(q', \eta')u(\gamma \circ \phi(q', \eta')) \right)
\]
\[
= \int_{\Omega'} dq' \det \left( \frac{\partial \phi_0(q')}{\partial q'} \right) \left[ \int_{\mathbb{R}^n_{od}} d\eta' \det^{-1} \left( \frac{\partial \phi_1(q', \eta')}{\partial \eta'} \right) \text{sdet } J(\gamma)(q, \eta)u(\gamma(q, \eta)) \right]_{q=\phi_0(q')} \bigg|_{\eta=\phi_1(q', \eta')}
\]
\[
= \int_{\Omega} dq \left[ \int_{\mathbb{R}^n_{od}} d\eta \text{sdet } J(\gamma)(q, \eta)u(\gamma(q, \eta)) \right] = \int_{\Omega} dq d\eta \text{sdet } J(\gamma)(q, \eta)u(\gamma(q, \eta)). \quad \square
\]

Finally, we prove our goal:

**Theorem 5.4.1** (CVF=change of variable formula). Let a supersmooth diffeomorphism \( \varphi \) be given from a foliated singular manifold \( \mathfrak{M}(\delta, \Omega) \subset \mathbb{R}^{m|n} \) onto a neighbourhood \( \Omega \) of another foliated singular manifold \( \mathfrak{M}(\gamma, \Omega) \subset \mathbb{R}^{m|n} \):

\[
(5.4.3) \quad \varphi : (y, \omega) \to (x, \theta) \quad \text{with} \quad x = \varphi_0(y, \omega), \quad \theta = \varphi_1(y, \omega)
\]

That is, \( \mathfrak{M} = \varphi(\mathfrak{M}) \) and \( \text{sdet } J(\varphi) \neq 0 \). Moreover, we assume that \( \delta = \varphi^{-1} \circ \gamma \) is sdet \( J(\gamma) \neq 0 \).

Then, for any integrable function \( u \in C_{\text{SS}}(\mathfrak{M} : \mathbb{R}) \), CVF holds.

\[
(5.4.4) \quad \text{RVV-} \int_{\mathbb{R}^{m|n}_{L}} dxd\theta u(x, \theta) = \text{RVV-} \int_{\mathbb{R}^{m|n}_{L}} dyd\omega \text{sdet } J(\varphi)(y, \omega)u(\varphi(y, \omega)).
\]

**Remark 5.4.2.** Analogous result is proved on superspace \( \mathbb{B}^{m|n}_{L} \) based on Banach-Grassmann algebra assuming the set \( \{ x \in \mathbb{B}^{m|n}_{L0} \mid x = \gamma(q, \vartheta), q \in \Omega \} \) is independent from each \( \vartheta \in \mathbb{B}^{n|1}_{L} \) in [130].

**Remark 5.4.3.** Formulas (5.2.4) and (5.4.4) have the same form but their definitions (5.4.1) and (5.4.2) are very different each other! This difference is related to the primitive question “How to consider the body of supermanifolds?” (see, Catenacci, Reina and Teofilatto [22]). Though we don’t develop supermanifolds theory with charts based on \( \mathbb{R}^{m|n} \) in this note, but in Chapter 8, we consider the simplest case \( \mathbb{R}^{m|n} \).
5.4.0.1. Proof of Theorem 5.4.1 - change of variable formula under integral sign. We want to prove the following diagram:

\[
\begin{array}{ccc}
(\tilde{\Omega}, dqd\eta) & \xrightarrow{\gamma} & (\mathcal{M}, dx \times d\theta) \\
\varphi & \uparrow & \varphi \circ u(y, \omega) \\
(\bar{\Omega}, dqd\eta) & \xrightarrow{\delta} & (\mathcal{N}, dyd\omega)
\end{array}
\]

\[\text{RVV-} \int_{\mathcal{M}} dx \times d\theta \ u(x, \theta) \in \mathbb{R}\]

By definition, we have paths

\[
\begin{align*}
\Omega \times \mathbb{R}^n_{od} \ni (q, \eta) & \rightarrow \gamma(q, \eta) = (x, \theta), \\
\Omega \times \mathbb{R}^n_{od} \ni (q, \eta) & \rightarrow \gamma_1(q, \eta) = (y, \omega),
\end{align*}
\]

which are related each other

\[
(x, \theta) = \gamma(q, \eta) = \varphi(y, \omega) = \varphi(\gamma_1(q, \eta)), \quad \gamma_1 = \varphi^{-1} \circ \gamma.
\]

We define pull-back of a “superform” as

\[
v = dx \times d\theta \ u(x, \theta) \rightarrow \varphi^* v = dyd\omega \ sdet \ J(\varphi)(y, \omega) \ u(\varphi(y, \omega)).
\]

Then, we have

Claim 5.4.1.

\[
(5.4.5) \quad \text{RVV-} \int_{\varphi^{-1} \circ \gamma(\tilde{\Omega})} \varphi^* v = \text{RVV-} \int_{\gamma(\tilde{\Omega})} v.
\]

Proof. Since \( J(\varphi^{-1} \circ \gamma) = J(\gamma) \cdot J(\varphi^{-1}) \) which yields

\[
sdet \ J(\varphi^{-1} \circ \gamma)(q, \eta)(sdet \ J(\varphi)(y, \omega)) \bigg|_{(y, \omega) = \varphi^{-1} \circ \gamma(q, \eta)} = sdet \ J(\gamma)(q, \eta),
\]

and by the definitions of path(contour) and integral, we have

\[
\text{RVV-} \int_{\varphi^{-1} \circ \gamma(\tilde{\Omega})} \varphi^* v
\]

\[
= \int_{\mathbb{R}^n_{od}} d\theta \left[ \int_\Omega dq \ sdet \ J(\varphi^{-1} \circ \gamma)(q, \theta)(sdet \ J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega))) \right]_{(y, \omega) = \varphi^{-1} \circ \gamma(q, \eta)}
\]

\[
= \int_{\mathbb{R}^n_{od}} d\theta \left[ \int_\Omega dq \ sdet \ J(\gamma)(q, \theta) \cdot u(\gamma(q, \theta)) \right] = \text{RVV-} \int_{\gamma(\tilde{\Omega})} v.
\]

we have the claim.  

Now, we interpret (5.4.5) as change of variables: Since we may denote integrals as

\[
\text{RVV-} \int_{\gamma(\tilde{\Omega})} v = \text{RVV-} \int_{\mathcal{M}} dx \times d\theta \ u(x, \theta),
\]

and

\[
\text{RVV-} \int_{\varphi^{-1} \circ \gamma(\tilde{\Omega})} \varphi^* v = \text{RVV-} \int_{\phi^{-1} \mathcal{M}} dyd\omega \ sdet \ J(\varphi)(y, \omega) \ u(\varphi(y, \omega)),
\]

we have

\[
\text{RVV-} \int_{\mathcal{M}} dx \times d\theta \ u(x, \theta) = \text{RVV-} \int_{\phi^{-1} \mathcal{M}} dyd\omega \ sdet \ J(\varphi)(y, \omega) \ u(\varphi(y, \omega)). \quad \square
\]
Remark 5.4.4. It is fair to say that new definitions without inconsistency for CVF are introduced M.J. Rothstein [111] or M.R. Zirnbauer [138], but they are not so easy to understand at least for me.

Resolution of inconsistency: Here, we resolve the inconsistency derived from the naive definition of Berezin integral by applying modified Rogers, Vladimirov and Volovich’s definition above.

Resolution of inconsistency in Example 5.2.1 From Theorem 5.3.1 we interpret as follows:

For $\Omega = (0, 1)$, we are given $\tilde{\Omega} = \Omega \times \mathbb{R}^2_{\text{od}}$, Defining a map $\gamma : \tilde{\Omega} \to \mathcal{M}$ as

$$\gamma : \tilde{\Omega} \ni (q, \vartheta) \to (x, \theta) = (\gamma_0(q, \vartheta), \gamma_1(q, \vartheta)) = \gamma(q, \vartheta),$$

then we may regard $\mathcal{M} = \{ (x, \theta) \in \mathbb{R}^{1|2} \mid \pi_B(x) \in \Omega, \theta \in \mathbb{R}^2_{\text{od}} \}$ as a foliated singular manifold $\gamma(\tilde{\Omega})$ in $\mathbb{R}^{1|2}$. We are given another foliated singular manifold $\mathfrak{N} = \delta(\tilde{\Omega})$ in $\mathbb{R}^{1|2}$ such that they are super-differentiably isomorphic

$$\varphi : \delta(\tilde{\Omega}) \ni (y, \omega) \to \varphi(y, \omega) = (x, \theta) \in \gamma(\tilde{\Omega}),$$

with

$$\begin{cases}
x &= \varphi_0(y, \omega) = y + \omega_1 \omega_2 \phi(y), \\
\theta_1 &= \varphi_{1,1}(y, \omega) = \omega_1, \theta_2 = \varphi_{1,2}(y, \omega) = \omega_2,
\end{cases}$$

and moreover

$$\delta = \varphi^{-1} \circ \gamma : (q, \vartheta) \to (q - \vartheta_1 \vartheta_2 \phi(q), \vartheta) = (\delta_0(q, \vartheta), \delta_1(q, \vartheta)) = (y, \omega).$$

Then, $\mathfrak{N} = \varphi^{-1}(\mathcal{M})$ and

$$J(\varphi)(y, \omega) = \begin{pmatrix} 1 + \omega_1 \omega_2 \phi'(y) & 0 & 0 \\ \omega_2 \phi(y) & 1 & 0 \\ -\omega_1 \phi(y) & 0 & 1 \end{pmatrix}, \quad J(\gamma)(q, \vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J(\delta)(q, \vartheta) = \begin{pmatrix} 1 - \vartheta_1 \vartheta_2 \phi'(q) & 0 & 0 \\ -\vartheta_2 \phi(q) & 1 & 0 \\ \vartheta_1 \phi(q) & 0 & 1 \end{pmatrix}. $$

In this case, for $u(x, \theta) = u_0(x) + \theta_1 \theta_2 u_1(x)$, we have

$$\text{RVV-}\int\int_{\mathfrak{N}} dxd\theta u(x, \theta) = \int_{\mathbb{R}^{1|2}_{\text{od}}} dq d\vartheta \left[ \int_{\Omega} dq \text{ sdet } J(\gamma)(q, \vartheta) u(\gamma(q, \vartheta)) \right]$$

$$= \int_{0}^{1} dq \int_{\mathbb{R}^{1|2}_{\text{od}}} d\vartheta u(q, \vartheta) = \int_{0}^{1} dq \frac{\partial}{\partial \vartheta_1} \frac{\partial}{\partial \vartheta_2} u(q, \vartheta) \bigg|_{\vartheta=0}$$

$$= \int_{0}^{1} dq \quad u_1(q),$$

and

$$\text{RVV-}\int\int_{\mathfrak{N}} dyd\omega (\varphi^* u)(y, \omega)$$

$$= \int\int_{\tilde{\Omega}} dq d\vartheta \text{ sdet } J(\delta)(q, \vartheta) \left[ \text{ sdet } J(\varphi)(y, \omega) u(\varphi(y, \omega)) \right]_{(y, \omega)=\delta(q, \vartheta)}$$

$$= \int_{0}^{1} dq \left[ \int_{\mathbb{R}^{1|2}_{\text{od}}} d\vartheta \text{ sdet } J(\delta)(q, \vartheta) \left[ \text{ sdet } J(\varphi)(y, \omega) u(\varphi(y, \omega)) \right]_{(y, \omega)=\delta(q, \vartheta)} \right]$$

$$= \int_{0}^{1} dq \int_{\mathbb{R}^{1|2}_{\text{od}}} d\vartheta u(q, \vartheta).$$

(5.4.6)
Therefore, without any condition on support of integrand \( u \), we get the following:

\[
\text{RVV-} \int_{\mathbb{M}} dxd\theta \ u(x, \theta) = \int \int_{\Omega} dqd\vartheta \ \text{sdet} \ J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) = \int \int_{\Omega} dqd\vartheta \ \text{sdet} \ J(\delta)(q, \vartheta) \left[ \text{sdet} \ J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) \right]_{(y, \omega) = \delta(q, \vartheta)} = \text{RVV-} \int_{\varphi^{-1}(\mathbb{M})} dyd\omega \ \text{sdet} \ J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) = \text{RVV-} \int_{\mathbb{M}} dyd\omega \ (\varphi^* u)(y, \omega). \quad \square
\]

**Remark 5.4.5.** In order to recognize this phenomena and for future use, we calculate more precisely (please sensitive to the underlined parts):

\[
s\text{det} \ J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) = (1 + \omega_1 \omega_2 \phi'(y))[u_0(y) + \omega_1 \omega_2 u_1(y)] = (1 + \omega_1 \omega_2 \phi'(y))[u_0(y) + \omega_1 \omega_2 u_1(y) + u_1(y)],
\]

and putting \((y, \omega) = \delta(q, \vartheta))\), then we have

\[
s\text{det} \ J(\delta)(q, \vartheta) \left[ \text{sdet} \ J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) \right]_{(y, \omega) = \delta(q, \vartheta)} = (1 - \vartheta_1 \vartheta_2 \phi'(q))[u_0(q) - \vartheta_1 \vartheta_2 \phi(q) u_0(q)] = (1 - \vartheta_1 \vartheta_2 \phi'(q))[u_0(q) + \vartheta_1 \vartheta_2 u_1(q)]
\]

Or, since \(u(\varphi(y, \omega))(y, \omega) = \delta(q, \vartheta))\) = \(u(q, \vartheta))\) and 

\[s\text{det} \ J(\delta)(q, \vartheta) \cdot s\text{det} \ J(\varphi)(\delta(q, \vartheta)) = (1 - \vartheta_1 \vartheta_2 \phi'(q))(1 + \vartheta_1 \vartheta_2 \phi'(q)) = 1,\]

we get the result.

From these, one reason of inconsistent term in (5.4.7) comes from \(\omega_1 \omega_2 \phi'(y)u_0(y)\).
and sdet $J(\tilde{\gamma})(q, \eta) = -(q_1 - iq_2)^{-2}$, therefore
\[
\iint dQ e^{-\left( x_1^2 + x_2^2 - 2\theta_1 \right)} = \iint dq \eta \text{sdet } J(\gamma)(q, \eta) e^{-\left( q_1^2 + q_2^2 - 2\eta n_2 \right)},
\]
\[
\iint d\tilde{Q} e^{-\left( \tilde{x}_1^2 + \tilde{x}_2^2 \right)} = \iint dq \eta \text{sdet } J(\tilde{\gamma})(q, \eta) \left( \text{sdet } (\varphi)(y, \omega) e^{-\left( y_1^2 + y_2^2 \right)} \right)_{(y, \omega) = \tilde{\gamma}(q, \eta)}.
\]
By this calculation, we have no inconsistency.

5.5. Supersymmetric transformation in superspace – as an example of change of variables

Stimulated probably from the success of QED, Berezin and Marinov claim in their paper [10] “Treat bosons and fermions on equal footing”, as mentioned before. As the object of this lecture notes, we give a basic idea of this “equal footing”. To do this, we need to develop an integration theory which admits the change of variables under integral sign. Especially we use transformations of mixing even and odd variables.

In the following, we assume $f_{a}(xB) \in \mathbb{C}$, that is, $f_{a}(x) \in \mathcal{C}_{ev}$, therefore, we denote also $\theta^a f_{a}(x)$ by $f_{a}(x)\theta^a$.

Let $x = (x_1, \cdots, x_m) \in \mathbb{R}^m \subset \mathbb{R}^{m|0}$ and $(\theta_1, \theta_2) \in \mathcal{C}_{od}^2$. Taking $\varepsilon \in \mathcal{C}_{od}$, $\gamma > 0$ and $\lambda, \mu \in \mathbb{R}^m \subset \mathbb{R}^{m|0}$, we define transformation $\tau(\lambda, \mu)$ as
\[
\begin{cases}
x \rightarrow y = x + 2\mu \varepsilon \theta_1 + 2\lambda \varepsilon \theta_2, \text{ that is, } y_j = x_j + 2\mu \varepsilon \theta_1 + 2\lambda \varepsilon \theta_2 \\
\theta_1 \rightarrow \omega_1 = \theta_1 + \gamma \lambda \cdot x \varepsilon, \text{ with } \lambda \cdot x = \sum_{j=1}^{m} \lambda_j x_j \\
\theta_2 \rightarrow \omega_2 = \theta_2 - \gamma \mu \cdot x \varepsilon
\end{cases}
\]
By the definition of Grassmann continuation, for any $f \in C^\infty(\mathbb{R}^m : \mathbb{C})$, we get
\[
f(x + 2\mu \varepsilon \theta_1 + 2\lambda \varepsilon \theta_2) = f(x) + 2\mu \cdot \nabla f \varepsilon \theta_1 + 2\lambda \cdot \nabla f \varepsilon \theta_2, \quad \text{with } \mu \cdot \nabla = \sum_{j=1}^{m} \mu_j \frac{\partial}{\partial x_j}.
\]
Therefore, (for $a = (a_1, a_2) \in \{0, 1\}^2$, putting $\bar{0} = (0, 0)$, $\bar{1} = (1, 0)$, $\bar{2} = (0, 1)$, $\bar{3} = (1, 1)$), for
\[
u(y, \omega) = \sum_{|a| \leq 2} u_{a}(y)\omega^a = u_{\bar{0}}(y) + u_{\bar{1}}(y)\omega_1 + u_{\bar{2}}(y)\omega_2 + u_{\bar{3}}(y)\omega_1\omega_2,
\]
remark the
\[
(\theta_1 + \gamma \lambda \cdot x \varepsilon)(\theta_2 - \gamma \mu \cdot x \varepsilon) = \theta_1 \theta_2 + \gamma \lambda \cdot x \varepsilon \theta_2 - \theta_1 \gamma \mu \cdot x \varepsilon
\]
and calculating slightly, we have
\[
(\tau^*(\lambda, \mu)u)(x, \theta) = u(\tau(\lambda, \mu)(x, \theta))
\]
\[
= u(x, \theta) + \left( (\gamma \lambda \cdot x u_1 - \gamma \mu \cdot x u_2) \right)
\]
\[
(5.5.1)
\]
DEFINITION 5.5.1. A function $u \in \mathcal{E}_{SS}(\mathbb{R}^{m|2} : \mathbb{C})$ is called supersymmetric, if for any $\lambda, \mu \in \mathbb{R}^m \subset \mathbb{R}^{m|0}$, it satisfies
\[
(\tau^*(\lambda, \mu)u)(x, \theta) = u(x, \theta)(= (\tau^*(0, 0)u)(x, \theta)).
\]
Proposition 5.5.1 (Proposition 4.1 of KLP [84]). Following conditions are equivalent for $u \in \mathcal{C}_{SS}(\mathbb{R}^{m|2} : \mathbb{C})$:

(i) $u \in \mathcal{C}_{SS}(\mathbb{R}^{m|2} : \mathbb{C})$ is supersymmetric.

(ii) $u_1(x) = u_2(x) = 0$ and moreover

\begin{equation}
\frac{2}{\gamma} \nabla u_0(x) = xu_3(x), \text{ i.e. } \frac{2}{\gamma} \frac{\partial}{\partial x_j} u_0(x) = x_j u_3(x).
\end{equation}

(iii) There exists a function $\phi(\cdot) \in C^\infty([0, \infty) : \mathbb{C})$ satisfying

$u(x, \theta) = \phi(x^2 - \frac{4}{\gamma} \theta_1 \theta_2) = \phi(x^2) - \frac{4}{\gamma} \phi'(x^2) \theta_1 \theta_2$.

Proof. [(i) $\implies$ (ii)] If $u$ is supersymmetric, then the coefficient of $\varepsilon$ of the right-hand side of (5.5.1) should be 0 for any $\lambda, \mu \in \mathbb{R}^m$. This implies (5.5.2).

[(ii) $\implies$ (iii)] Restricting (5.5.2) to $\mathbb{R}^m$, we get that $u_0(q)$ depends only on $|q|^2 = q \cdot q$, that is, there exists a function $\phi(\cdot) \in C^\infty([0, \infty) : \mathbb{C})$ such that $u_0(q) = \phi(|q|^2)$. Since the derivative of the Grassmann continued function equals to the Grassmann continued of the derivative, therefore (iii) follows.

[(iii) $\implies$ (i)] Obvious. \hfill \Box

Proposition 5.5.2. Let $u \in \mathcal{C}_{SS}(\mathbb{R}^{m|2} : \mathbb{C})$ with $u_d(\cdot)$ is integrable for each $a$. Then, for any $\tau = \tau(\lambda, \mu)$, we have

\[
\int_{\mathbb{R}^{m|2}} dx d\theta_2 d\theta_1 (\tau^* u)(x, \theta_1, \theta_2) = \int_{\mathbb{R}^{m|2}} dx d\theta_2 d\theta_1 u(x, \theta_1, \theta_2).
\]

Proof. Integrating w.r.t. $\theta$, we have

\[
\int_{\mathbb{R}^{m|2}} dx d\theta_2 d\theta_1 u(x, \theta_1, \theta_2) = -\int_{\mathbb{R}^{m|0}} dx u_3(x),
\]

\[
\int_{\mathbb{R}^{m|2}} dx d\theta_2 d\theta_1 (\tau^* u)(x, \theta_1, \theta_2) = -\int_{\mathbb{R}^{m|0}} dx u_3(x) - 2 \int_{\mathbb{R}^{m|0}} dx (\nabla u_1(x) \cdot \lambda - \nabla u_2(x) \cdot \mu) \varepsilon.
\]

On the other hand, from integrability,

\[
\int_{\mathbb{R}^{m|0}} dx \nabla u_j(x) = 0, \quad j = 1, 2
\]

we get the result. \hfill \Box

Lemma 5.5.1. Let $u \in \mathcal{C}_{SS}(\mathbb{R}^{2|2} : \mathbb{C})$ be supersymmetric and integrable. Then,

\[
\int_{\mathbb{R}^{2|2}} dx d\theta_2 d\theta_1 u(x, \theta_1, \theta_2) = \frac{4\pi}{\gamma} u_0(0).
\]

Proof. From previous Proposition, there exists a function $\phi(\cdot) \in C^\infty([0, \infty) : \mathbb{C})$ such that $u(x, \theta_1, \theta_2) = \phi(x^2 + \frac{4}{\gamma} \theta \theta)$, and especially $u_0(x) = \phi(x^2)$. Since this is integrable, it implies $\lim_{t \to \infty} \phi(t) = 0$ and

\[
\int_{\mathbb{R}^{2|2}} dx d\theta_2 d\theta_1 u(x, \theta_1, \theta_2) = \frac{4}{\gamma} \int_{\mathbb{R}^{2|0}} dx \phi'(x^2)
\]

\[
= -\frac{8}{\gamma} \int_0^\infty rdr \phi'(r^2) = -\frac{4}{\gamma} \int_0^\infty ds \phi'(s) = \frac{4}{\gamma} \phi(0) = \frac{4}{\gamma} u_0(0). \quad \Box
\]
CHAPTER 6

Efetov’s method in Random Matrix Theory and beyond

In 1950s, physicists get so many experimental data which are obtained after making neutron collided with Uranium 238, etc. and checking these data, they are embarrassed so much to grasp the meaning of them. At that time, E. Wigner poses a working hypothesis or Ansatz that high resonance of these experiments behaves like eigenvalues of large size matrix. In other word, we quote from M.L. Mehta \[ 95 \]

Consider a large matrix whose elements are random variables with given probability laws. Then, what can one say about the probabilities of a few of its eigenvalues or of a its eigenvectors?

We give another quotation from Fyodorov \[ 50 \]:

Wigner suggested that fluctuations in positions of compound nuclei resonances can be described in terms of statistical properties of eigenvalues of very large real symmetric matrices with independent, identically distributed entries. The rational behind such a proposal was the idea that in the situation when it is hardly possible to understand in detail individual spectra associated with any given nucleus composed of many strongly interacting quantum particles, it may be reasonable to look at the corresponding systems as "black boxes" and adopt a kind of statistical description, not unlike thermodynamics approach to classical matter.

We should remark that there exist mathematical works concerning the distribution of the zeros of the Riemann zeta function from this point of view.

In any way, in Random Matrix Theory (=RMT), Wigner’s semi-circle law gives a well-known corner-stone.

We report mathematical refinements of this law as an application of superanalysis. That is, using Efetov’s idea, we rewrite the average of the empirical measure of the eigenvalue distribution of the Hermitian matrices in a compact form. Careful calculations give not only the precise convergence rate of that law, but also the precise rate of the edge mobility.

\[ \text{Remark 6.0.1. The usage of the method of steepest descent by physicists are not so mathematically rigorous in the sense of de Bruijn’s criteria, because we have no general method of choosing the steepest descent path for the integral considered.} \]
6.1. Results – outline

Let $\mathcal{H}_N$ be a set of Hermitian $N \times N$ matrices, which is identified with $\mathbb{R}^{N^2}$ as a topological space. In this set, we introduce a probability measure $d\mu_N(H)$ by

$$d\mu_N(H) = \prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk})d(\Im H_{jk})P_{N,J}(H),$$

(6.1.1)

$$P_{N,J}(H) = Z_{N,J}^{-1} \exp \left[ -\frac{N}{2J^2} \text{tr} H^*H \right]$$

where $H = (H_{jk})$, $H^* = (H^*_{jk})$, and $\prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk})d(\Im H_{jk})$ being the Lebesgue measure on $\mathbb{R}^{N^2}$, and $Z_{N,J}^{-1}$ is the normalizing constant given by $Z_{N,J} = 2^{N^2/(J^2\pi/N)^{3N/2}}$.

Let $E_\alpha = E_\alpha(H)$ ($\alpha = 1, \ldots, N$) be real eigenvalues of $H \in \mathcal{H}_N$.

We put

$$\rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^N \delta(\lambda - E_\alpha(H)),$$

(6.1.2)

where $\delta$ is the Dirac’s delta. Denoting for a function $f$ on $\mathcal{H}_N$,

$$\langle f \rangle_N = \langle f(\cdot) \rangle_N = \int_{\mathcal{H}_N} d\mu_N(H) f(H),$$

we get

**Theorem 6.1.1 (Wigner’s semi-circle law).**

$$\lim_{N \to \infty} \langle \rho_N(\lambda) \rangle_N = w_{sc}(\lambda) = \begin{cases} (2\pi J^2)^{-1} \sqrt{4J^2 - \lambda^2} & \text{for } |\lambda| < 2J, \\ 0 & \text{for } |\lambda| > 2J. \end{cases}$$

(6.1.3)

**Remark.** By definition, the limit (6.1.3) is interpreted as

$$\lim_{N \to \infty} \langle \phi(\cdot), \int_{\mathcal{H}_N} d\mu_N(H) N^{-1} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H)) \rangle = \langle \phi, w_{sc} \rangle = \int_{\mathbb{R}} d\lambda \phi(\lambda)w_{sc}(\lambda)$$

for any $\phi \in C_0^\infty(\mathbb{R}) = D(\mathbb{R})$. $\langle \cdot, \cdot \rangle$ stands for the duality between $D(\mathbb{R})$ and $D'(\mathbb{R})$. We need more interpretation to give the meaning to $\int_{\mathcal{H}_N} d\mu_N(H) N^{-1} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H))$, which will be given in §2. Or, $H \to \rho_N(\lambda; H)d\lambda$ is considered a measure (on $\mathbb{R}$)-valued random variable on $\mathcal{H}_N$ and $\langle \rho_N(\lambda) \rangle_N d\lambda$ is considered a family of probability measures on $\mathbb{R}$ which is tight.

Though there exist several methods to prove this fact, we explain a new derivation of this fact using odd variables obtained by Efetov [39]. We follow mainly Fyodorov [50] and Brézin [16] (see also, Mello [96], Zirnbauer [138]).

Moreover, we get, as a byproduct of this new treatise,

**Theorem 6.1.2 (A refined version of Wigner’s semi-circle law).** For each $\lambda$ with $|\lambda| < 2J$, when $N \to \infty$, we have

$$\langle \rho_N(\lambda) \rangle_N = \frac{\sqrt{4J^2 - \lambda^2}}{2\pi J^2} \left(-\frac{(-1)^N \lambda}{\pi(4J^2 - \lambda^2)} + 2\arcsin\left(\frac{\lambda}{2J}\right)\right) N^{-1} + O(N^{-2}).$$

(6.1.4)
When \( \lambda \) satisfies \( |\lambda| > 2J \), there exist constants \( C_\pm(\lambda) > 0 \) and \( k_\pm(\lambda) > 0 \) such that

\[
(6.1.5) \quad \left| \langle \rho_N(\lambda) \rangle_N \right| \leq C_\pm(\lambda) \exp \{-k_\pm(\lambda)N\}
\]

with \( k_\pm(\lambda) \to 0 \) and \( C_\pm(\lambda) \to \infty \) for \( \lambda \not\to 2J \) or \( \lambda \not\to -2J \), respectively.

**Theorem 6.1.3** (The spectrum edge problem). Let \( z \in [-1,1] \). We have

\[
(6.1.6) \quad \langle \rho_N(2J - zN^{-2/3}) \rangle_N = N^{-1/3} f(z/J) + O(N^{-2/3}) \quad \text{as } N \to \infty,
\]

\[
\langle \rho_N(-2J + zN^{-2/3}) \rangle_N = -N^{-1/3} f(z/J) + O(N^{-2/3}) \quad \text{as } N \to \infty,
\]

where

\[
f(w) = \frac{1}{4\pi^2J} (Ai'(w)^2 - Ai''(w) Ai(w)), \quad Ai(w) = \int_R dx \exp \left[-\frac{i}{3} x^3 + iwx\right].
\]

(A) One of the key expression obtained by introducing new auxiliary variables, is

\[
(6.1.7) \quad \langle \rho_N(\lambda) \rangle_N = \pi^{-1} \Im \int_{\Omega} dQ \left\{ (\lambda - i0)I_2 - Q \right\}_{bb} \exp \left[-N\mathcal{L}(Q)\right]
\]

where \( I_n \) stands for \( n \times n \)-identity matrix and

\[
\mathcal{L}(Q) = \text{str} \left\{ (2J^2)^{-1}Q^2 + \log ((\lambda - i0)I_2 - Q) \right\},
\]

\[
(6.1.8) \quad \Omega = \left\{ Q = \begin{pmatrix} x_1 \\ \rho_2 \\ ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathfrak{g}_{ev}, \rho_1, \rho_2 \in \mathfrak{g}_{ad} \right\} \cong \mathfrak{g}^2_{ev}, \quad dQ = \frac{dx_1dx_2d\rho_1d\rho_2}{2\pi},
\]

\[
\left\{ (\lambda - i0)I_2 - Q \right\}_{bb} = \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}.\]

Here in (6.1.4), the parameter \( N \) appears only in one place. This formula is formidably charming but not yet directly justified, like Feynman’s expression of certain quantum objects applying his notorious measure.

(B) In physics literatures, for example in [50], [138], they claim without proof that they may apply the method of steepest descent to (6.1.7) when \( N \to \infty \). More precisely, as

\[
\delta\mathcal{L}(Q)\hat{Q} = \frac{d}{de} \mathcal{L}(Q + e\hat{Q}) \bigg|_{e=0},
\]

they seek solutions of

\[
\delta\mathcal{L}(Q) = \text{str} \left( \frac{Q}{f^2} - \frac{1}{\lambda - Q} \right) = 0.
\]

As a candidate of effective saddle points, they take

\[
Q_c = \left( \frac{1}{2} \lambda + \frac{1}{2} \sqrt{\lambda^2 - 4J^2} \right)I_2,
\]

and they have

\[
\lim_{N \to \infty} \langle \rho_N(\lambda) \rangle_N = \pi^{-1} \Im (\lambda - Q_c)^{-1} = w_{sc}(\lambda). \quad \square
\]

**Problem 6.1.1.** Not only the expression (6.1.7) nor the applicability of the saddle point method to it are not so clear. Though these formulas obtained are not yet well-defined mathematically but so charming. Unfortunately for the time being, we mathematicians may not use them directly. For example, the expression (6.1.7) is mathematically verified only for \( \lambda - i\varepsilon (\varepsilon > 0) \), but we need probably some new integration theory which admits taking limit \( \varepsilon \to 0 \) under integral sign. Can we
justify the physicist procedures by using this new integration theory? More precisely, under what condition, do we have the following equality?

\[
(6.1.9) \quad \lim_{N \to \infty} \lim_{\epsilon \to 0} \left\langle \frac{1}{N} \operatorname{tr} \left( \frac{1}{\lambda - i\epsilon} \mathbb{I}_N - H \right) \right\rangle_N = \lim_{\epsilon \to 0} \lim_{N \to \infty} \left\langle \frac{1}{N} \operatorname{tr} \left( \frac{1}{\lambda - i\epsilon} \mathbb{I}_N - H \right) \right\rangle_N.
\]

If this assertion is true, may we justify the physicists argument of “saddle point method”? 

Remark 6.1.1. (i) For mathematical rigour, we dare to lose such a beautiful expression like (6.1.7), but we have the two formulae (6.2.21) and (6.2.22) below which lead to our conclusion.

(ii) It is not so simple even in the integral on \(\mathbb{R}^m\) to apply the saddle point method, that is, to take an appropriately deformed “path” in \(\mathbb{C}^m\), as is explained in de Bruijn [29].

(iii) The set \((\mathcal{U}_N, d\mu_N(\cdot))\) is called GUE=the Gaussian Unitary Ensemble. Other ensembles may be treated analogously as indicated in [138] but are not treated here.

6.2. The derivation of (6.1.7) with \(\lambda - i\epsilon (\epsilon > 0 \text{ fixed})\) and its consequences

It is well-known that

\[
\delta(q) = \frac{1}{\pi} \lim_{\epsilon \to 0} \Im \frac{1}{q - i\epsilon} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left[ \frac{1}{q - i\epsilon} - \frac{1}{q + i\epsilon} \right] = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{q^2 + \epsilon^2} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}),
\]

that is, for any \(\phi \in C_0^\infty(\mathbb{R})\),

\[
\pi^{-1} \Im \int_{\mathbb{R}} dq \frac{\phi(q)}{q - i\epsilon} = \pi^{-1} \int_{\mathbb{R}} dq \frac{\epsilon \phi(q)}{q^2 + \epsilon^2} = \pi^{-1} \int_{\mathbb{R}} dq \frac{\phi(\epsilon q)}{1 + q^2} \rightarrow \phi(0) = \langle \phi, \delta \rangle \quad \text{as} \quad \epsilon \to 0.
\]

Therefore, for any fixed \(\phi \in C_0^\infty(\mathbb{R})\), we have

\[
\int_{\mathcal{U}_N} d\mu_N(H) \langle \phi(\cdot), \frac{1}{N} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H)) \rangle_{\mathcal{U}_N} \stackrel{\text{def}}{=} \int_{\mathcal{U}_N} \mu_N(H) \lim_{\epsilon \to 0} \int_{\mathbb{R}} d\lambda \phi(\lambda) \frac{1}{N} \sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2 + \epsilon^2}
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathcal{U}_N} d\mu_N(H) \int_{\mathbb{R}} d\lambda \phi(\lambda) \frac{1}{\pi N} \sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2 + \epsilon^2} \quad \text{by Lebesgue’s dom.conv.theorem}
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} d\lambda \phi(\lambda) \int_{\mathcal{U}_N} d\mu_N(H) \frac{1}{\pi N} \sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2 + \epsilon^2} \quad \text{by Fubini’s theorem}.
\]

The second equality is garanteed by the fact that for any \(\phi \in C_0^\infty(\mathbb{R})\), we have, for any \(\epsilon > 0\) and \(H \in \mathcal{U}_N\),

\[
\left| \int_{\mathbb{R}} d\lambda \phi(\lambda) \frac{1}{\pi N} \sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2 + \epsilon^2} \right| \leq \max |\phi(\lambda)|.
\]

Here, we used the fact \(\int_{\mathbb{R}} d\lambda \epsilon (\lambda^2 + \epsilon^2)^{-1} = \pi\). The third equality holds because we have

\[
|\phi(\lambda) \frac{1}{\pi N} \sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2 + \epsilon^2}| \leq \epsilon^{-1} |\phi(\lambda)| \left( \frac{1}{\pi N} \frac{\epsilon}{\alpha^2 + \epsilon^2} \leq \frac{1}{\epsilon} \right),
\]

and the right hand side is integrable w.r.t. the product measure \(d\lambda d\mu_N(H)\) for any fixed \(\epsilon > 0\).

In order to check whether we may take the limit before integration w.r.t. \(d\lambda\) in the last line above, we calculate the following quantity as explicitly as possible:

\[(6.2.1) \quad g(\lambda, \epsilon, N) = \int_{\mathcal{U}_N} d\mu(H) \frac{1}{\pi N} \Im \sum_{\alpha=1}^N \frac{1}{\lambda - i\epsilon - E_\alpha(H)}.
\]
We claim in this section that
(i) \(g(\lambda, \epsilon, N)\) exists as a function of \(\lambda\) for any \(\epsilon > 0\) and \(N \in \mathbb{N}\) and
(ii) \(\lim_{\epsilon \to 0} g(\cdot, \epsilon, N)\) exists in \(\mathcal{D}'(\mathbb{R})\) for any \(N \in \mathbb{N}\) and it is denoted by \(\langle \rho_N(\lambda) \rangle\).

Now, we put
\[
\begin{align*}
z_j &= x_j + iy_j, \quad \bar{z}_j = x_j - iy_j, \quad x_j, y_j \in \mathcal{R}_{\text{ev}}; \quad \theta_k, \bar{\theta}_k \in \mathcal{R}_{\text{od}} = \mathcal{C}_{\text{od}}, \\
X &= t'(z, \theta), \quad z = t(z_1, \ldots, z_N), \quad \theta = t(\theta_1, \ldots, \theta_N), \\
X^* &= (z^*, \theta^*), \quad z^* = (\bar{z}_1, \ldots, \bar{z}_N), \quad \theta^* = (\bar{\theta}_1, \ldots, \bar{\theta}_N).
\end{align*}
\]

Here, \(\theta_k\) and \(\bar{\theta}_k\) are considered as two different odd variables.

The following is the key formula which is well known:

**Lemma 6.2.1.** Put \(\mu = \lambda - i\epsilon\) \((\epsilon > 0)\).

\[
\begin{align*}
\text{tr} \frac{1}{\mu I_N - H} &= \sum_{\alpha=1}^{N} \frac{1}{\mu - E_{\alpha}(H)} \\
(6.2.2) &= i \int_{\mathcal{C}[N:2]} \prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i} \prod_{k=1}^{N} d\theta_k d\bar{\theta}_k (z^* \cdot z) \exp \left[-iX^* (I_2 \otimes (\mu I_N - H) )X \right] \\
&= \sum_{j=1}^{N} \frac{1}{\gamma_j - i\epsilon}. \\
\end{align*}
\]

To prove this lemma, we need the following lemma.

**Lemma 6.2.2.** Let \(\Gamma=\text{the diagonal matrix with diagonal given by } (\gamma_1, \ldots, \gamma_N)\) where \(\gamma_j \in \mathbb{R}\).

Putting \((z^* \cdot z) = \sum_{j=1}^{N} z_j \bar{z}_j = |z|^2\), we have

\[
(6.2.3) \quad i \int_{\mathcal{C}[N:2]} \prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i} \prod_{k=1}^{N} d\theta_k d\bar{\theta}_k (z^* \cdot z) \exp \left[-iX^* (I_2 \otimes (\Gamma - i\epsilon I_N))X \right] = \sum_{j=1}^{N} \frac{1}{\gamma_j - i\epsilon}.
\]

**Proof.** Identifying \(z_j = x_j + iy_j, \quad \bar{z}_j = x_j - iy_j, \quad dz_j \wedge d\bar{z}_j = 2i dx_j \wedge dy_j,\) using the polar coordinates \((r_j, \omega_j) \in \mathbb{R}_+ \times S^1\) and denoting \(|z_j|^2 = x_j^2 + y_j^2 = r_j^2, 0 \leq \omega_j \leq 2\pi,\) we get

\[
i \int_{\mathcal{C}[0]} \prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i} \left|z_j\right|^2 e^{-i(\gamma_j - i\epsilon)|z_j|^2} = i \int_{0}^{2\pi} \prod_{j=1}^{N} r_j e^{-i(\gamma_j - i\epsilon)r_j^2} \\
&= \left(\frac{1}{\epsilon + i\gamma_j}\right)^2 = \frac{1}{\epsilon + i\gamma_j} \frac{1}{\gamma_j - i\epsilon}.
\]

Analogously,

\[
\int_{\mathcal{C}[N-1]} \prod_{k \neq j}^{N} \frac{dz_k d\bar{z}_k}{2\pi i} e^{-i(\gamma_k - i\epsilon)|z_k|^2} = \prod_{k \neq j} \frac{1}{\epsilon + i\gamma_k} = \prod_{k=1}^{N} \frac{1}{\epsilon + i\gamma_k}.
\]

**Remark.**

\[
i \int_{\mathcal{C}[N:0]} \prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i} (z^* \cdot z) \exp \left[-iz^* (\Gamma - i\epsilon I_N)z \right] = \left(\sum_{j=1}^{N} \frac{1}{\gamma_j - i\epsilon}\right) \prod_{j=1}^{N} \frac{1}{\epsilon + i\gamma_j},
\]

and

\[
\int_{\mathcal{C}[N:2]} \prod_{k=1}^{N} d\theta_k d\bar{\theta}_k \exp \left[-i\theta^* (\Gamma - i\epsilon I_N)\theta \right] = \prod_{k=1}^{N} (\epsilon + i\gamma_k),
\]

we get the result \((6.2.3)\) readily. \(\square\)
Proof of Lemma 6.2.1. By diagonalization of $\lambda_N - H$, we reduce Lemma 6.2.1 to Lemma 6.2.2. In fact, taking $G$ such that $GG^* = G^*G = I_N GHG^* = \Gamma$, and defining a change of variables

$$
(z^*, \theta^*) \left( G(\mu I_N - H)G^* \right)^0 = z^*G(\mu I_N - H)G^* z + \theta^*G(\mu I_N - H)G^* \theta
$$

we reduce it to the diagonal case (Even if we use the naive definition of integration, every linear transformation is permitted under integral sign, see §1 of the last chapter). 

\[ \Box \]

Lemma 6.2.3. For $\mu = \lambda - i\epsilon$ ($\epsilon > 0$),

$$
\langle \frac{1}{\mu I_N - H} \rangle_N = i \int \prod_{j=1}^N \frac{d\zeta_j d\zeta_j}{2\pi i} \prod_{k=1}^N d\bar{\theta}_k d\theta_k (z^* z) \exp (-iX^* (\bar{\mu}_2 \otimes \mu I_N) X) \\
\times \exp \left[ -\frac{J^2}{2N} \sum_{j,k=1}^N (\bar{\zeta}_j z_k + \bar{\theta}_j \theta_k)(\bar{\zeta}_k z_j + \bar{\theta}_k \theta_j) \right].
$$

Proof. By definition, we have

$$
\langle \frac{1}{\mu I_N - H} \rangle_N = \int d\mu_N (H) \left[ i \int \prod_{j=1}^N \frac{d\zeta_j d\zeta_j}{2\pi i} \prod_{k=1}^N d\bar{\theta}_k d\theta_k \\
\times (z^* z) \exp (-iX^* (\bar{\mu}_2 \otimes (\mu I_N - H) X) \right].
$$

As $X^* (\bar{\mu}_2 \otimes H) X = H_{jk}(\bar{\zeta}_j z_k + \bar{\theta}_j \theta_k)$, we have

$$
\langle \exp \left[ \pm i \sum_{j,k=1}^N H_{jk}(\bar{\zeta}_j z_k + \bar{\theta}_j \theta_k) \right] \rangle_N = \exp \left[ -\frac{J^2}{2N} \sum_{j,k=1}^N (\bar{\zeta}_j z_k + \bar{\theta}_j \theta_k)(\bar{\zeta}_k z_j + \bar{\theta}_k \theta_j) \right].
$$

After changing the order of integration and substituting (6.2.6) into (6.2.5), we get (6.2.4). 

\[ \Box \]

There are at least two approach from (6.2.4) to Wigner’s law: The method (I) permits us to make $\epsilon \to 0$ rather easily and leads us to a not so simple looking formula but which is calculable, the other one (II) yields the beautiful formula (6.1.4) formally, but in order to make $\epsilon \to 0$ in that formula rigorously, we reform it until it is represented by Hermite polynomials, and at that time the beauty of the formula (6.1.4) is lost.

(I) The following calculation is proposed by E. Brézin [16, 17]:

Using

$$
\sum_{j,k=1}^N (\bar{\zeta}_j z_k + \bar{\theta}_j \theta_k)(\bar{\zeta}_k z_j + \bar{\theta}_k \theta_j) = (z^* z)^2 + 2(\theta^* z)(z^* \theta) - (\theta^* \theta)^2,
$$

we have

$$
\exp \left[ \frac{J^2}{2N} (\theta^* \theta)^2 \right] = \left( \frac{N}{2\pi J^2} \right)^{1/2} \int_{-\infty}^{\infty} d\tau \exp \left[ -\tau (\theta^* \theta) - \frac{N}{2J^2} \tau^2 \right].
$$
6.2. THE DERIVATION OF (??) WITH $\lambda - i\epsilon$

\( : \) This equality is non-trivial if we adopt the naive definition of integration. Formally, putting \( \gamma = \frac{N}{J^2} \), by

\[-\tau(\theta^* \cdot \theta) - \frac{\gamma}{2} \tau^2 = -\frac{\gamma}{2} (\tau + \frac{1}{\gamma} (\theta^* \cdot \theta))^2 - \gamma^{-2} (\theta^* \cdot \theta)^2 \]

we get

\[
\int_{-\infty}^{\infty} d\tau \exp \left[ -\tau(\theta^* \cdot \theta) - \frac{\gamma}{2} \tau^2 \right] = \exp \left[ \frac{\gamma}{2} \tau^2 / (2\gamma) \right] \int_{-\infty}^{\infty} d\tau \exp \left[ -\frac{\gamma}{2} (\tau + \frac{\theta^* \cdot \theta}{\gamma}) \right]
\]

\[= \int_{-\infty}^{\infty} ds \exp \left[ -\frac{\gamma}{2} s^2 \right] \]

and therefore it seems the above equality (6.2.7) follows. But for \( s = \tau + \frac{1}{\gamma} (\theta^* \cdot \theta) \), do we consider as \( \int d\tau = ds ? \) (Though RVV-integral is applicable, we dare to prove this fact as follows only with naive definition.)

We may regard the integrand as \( \mathbb{C} \)-valued real \( \tau \)-variable function, that is, enclosing \( (\theta^* \cdot \theta) \), we may integrate. In fact, since \( (\theta^* \cdot \theta)^{N+1} = 0 \),

\[\sqrt{\frac{\gamma}{2\pi}} \int_{-\infty}^{\infty} d\tau e^{-\gamma \tau^2/2} \sum_{j=0}^{N} \frac{(-\tau \cdot \theta^* \cdot \theta)^j}{j!} = \sqrt{\frac{\gamma}{2\pi}} \sum_{j=0}^{N} \int_{-\infty}^{\infty} d\tau e^{-\gamma \tau^2/2} (-\tau)^j \frac{(\theta^* \cdot \theta)^j}{j!}.\]

Using

\[\int_{-\infty}^{\infty} d\tau \, \tau^{2\ell+1} e^{-\gamma \tau^2/2} = 0, \quad \int_{-\infty}^{\infty} d\tau \, \tau^{2\ell} e^{-\gamma \tau^2/2} = \frac{2\pi}{\gamma} \frac{2(2\ell - 1)!}{2^\ell (\ell - 1)!},\]

the right-hand side of above equation,

\[\sum_{\ell=0}^{[N/2]} \sqrt{\frac{\gamma}{2\pi}} \int_{-\infty}^{\infty} d\tau e^{-\gamma \tau^2/2} (-\tau)^{2\ell} \frac{(\theta^* \cdot \theta)^{2\ell}}{(2\ell)!} = \exp \left[ \frac{1}{2\gamma} (\theta^* \cdot \theta)^2 \right]. \]

For comparison, we remember a little for ordinary integral case: By \( x^2 + 2ix\xi = (x + i\xi)^2 + \xi^2 \), we have

\[\int_{\mathbb{R}} dx \, e^{-ix\xi} e^{-x^2/2} = e^{-\xi^2/2} \int_{\mathbb{R}} dx \, e^{-(x+i\xi)^2/2} = e^{-\xi^2/2} \int_{\mathbb{R}} dy \, e^{-y^2/2} = \left( \sum_{n=0}^{\infty} \frac{(-iy\xi)^n}{n!} \right) e^{-x^2/2} = \sum_{m=0}^{\infty} \frac{(-1)^m \xi^{2m}}{(2m)!} \int_{\mathbb{R}} dx \, x^{2m} e^{-x^2/2}.
\]

Even in this case, we need to explain why the second equality and the last one holds in the above, but we take these admitted.

Substituting this relation into (6.2.5), we get

\[\left\langle \text{tr} \, \frac{1}{\mu \mathbb{I}_N - \mathbb{H}} \right\rangle_N = i \prod_{j=1}^{N} \frac{d\xi_j}{2\pi i} \prod_{k=1}^{N} \frac{d\theta_k}{2\pi i} (z^* \cdot z) \]

\[
\times \exp \left[ -iX^* (\mathbb{I}_2 \otimes \mu \mathbb{I}_N) X - \frac{J^2}{2N} ((z^* \cdot z)^2 + 2(\theta^* \cdot z)(z^* \cdot \theta)) \right]
\]

\[\times \left( \frac{N}{2\pi J^2} \right)^{1/2} \int_{-\infty}^{\infty} d\tau \exp \left[ -\tau(\theta^* \cdot \theta) - \frac{N}{2J^2} \tau^2 \right].\]
Proposition 6.2.1. We have the following formula:

\[
\left\langle \text{tr} \left( \frac{1}{\lambda - i 0} \mathbb{I}_N - H \right) \right\rangle_N = i \frac{1}{(N-1)!} \left( \frac{N}{2 \pi J^2} \right)^{1/2} \left( \frac{N}{J^2} \right)^{N+1} \int_0^\infty ds \, s^N \exp \left\{ -\frac{N}{2 J^2} \left( 2 i \lambda s + s^2 \right) \right\} \times \int_{-\infty}^\infty d\tau \, (\tau + i \lambda)^{N-1} (\tau + i \lambda + s) \exp \left\{ -\frac{N}{2 J^2} \tau^2 \right\}
\]

(6.2.8)

\[
\left\langle \text{tr} \left( \frac{1}{\lambda - i 0} \mathbb{I}_N - H \right) \right\rangle_N = i \frac{1}{(N-1)!} \left( \frac{N}{2 \pi J^2} \right)^{1/2} \left( \frac{N}{J^2} \right)^{N+1} \int_{\mathbb{R}^+} ds \, d\tau \, (1 + (\tau + i \lambda)^{-1} s) \exp \left\{ -N \left( \frac{1}{2 J^2} (\tau^2 + 2 i \lambda s + s^2) - \log s (\tau + i \lambda) \right) \right\}.
\]

Proof. Since

\[-i \mu (\theta^* - \theta) - \frac{J^2}{N} (\theta^* \cdot z) (z^* \cdot \theta) - \tau (\theta^* \cdot \theta) = -\sum_{a,b} \theta_a ((\tau + i \mu) \delta_{ab} + \frac{J^2}{N} z_a z_b) \theta_b,
\]

using Lemma 2.3 and Lemma 6.2.5 below, we get

\[
\left\langle \text{tr} \left( \frac{1}{\mu \mathbb{I}_N - H} \right) \right\rangle_N = i \int_{\mathbb{R}^2} d\theta_k d\theta_k \exp \left\{ -i \mu (\theta^* \cdot \theta) - \frac{J^2}{N} (\theta^* \cdot z) (z^* \cdot \theta) \right\} = (\tau + i \mu)^{N-1} (\tau + i \mu + \frac{J^2}{N} (z^* \cdot z)).
\]

(6.2.9)

Using the expression (6.2.9), we have

\[
\left\langle \text{tr} \left( \frac{1}{\mu \mathbb{I}_N - H} \right) \right\rangle_N = i \int_{\mathbb{R}^2} d\theta_k d\theta_k \exp \left\{ -i \mu (\theta^* \cdot \theta) - \frac{J^2}{N} (\theta^* \cdot z) (z^* \cdot \theta) \right\} \times \left( \frac{N}{2 \pi J^2} \right)^{1/2} \int_{-\infty}^\infty d\tau \, (\tau + i \mu)^{N-1} (\tau + i \mu + \frac{J^2}{N} (z^* \cdot z)) \exp \left\{ -\frac{N}{2 J^2} \tau^2 \right\}.
\]

Identifying \( \mathbb{C}^N = \mathbb{R}^{2N} \) by \( z_j = x_j + i y_j, \bar{z}_j = x_j - i y_j, d\bar{z}_j \wedge dz_j = 2 i dx_j \wedge dy_j \) and using the polar coordinate \( (r, \omega) \in \mathbb{R}^+ \times S^{2N-1} \) with \( \prod_{j=1}^N \frac{dz_j \wedge dz_j}{2 \pi i} = \prod_{j=1}^N \frac{dx_j dy_j}{\pi} = \pi^{-N} r^{2N-1} d r \, d \omega, \int_{S^{2N-1}} d \omega = \text{vol}(S^{2N-1}) = 2 \pi^N / (N-1)! \), we get,

\[
\left\langle \text{tr} \left( \frac{1}{\mu \mathbb{I}_N - H} \right) \right\rangle_N = i \frac{1}{(N-1)!} \int_0^\infty dr^2 \, r^{2N} \exp \left\{ -i \mu r^2 - \frac{J^2}{2 N} r^4 \right\} \times \left( \frac{N}{2 \pi J^2} \right)^{1/2} \int_{-\infty}^\infty d\tau \, (\tau + i \mu)^{N-1} (\tau + i \mu + \frac{J^2}{N} r^2) \exp \left\{ -\frac{N}{2 J^2} \tau^2 \right\}.
\]

Changing the independent variables as \( r^2 = (N/J^2) \bar{r} \) and making \( \epsilon \to 0 \), i.e. \( \mu \to \lambda - i 0 \), we get the result. Here, this procedure of making \( \epsilon \to 0 \) under integral sign is admitted because of Lebesgue’s dominated convergence theorem. □

Remark 6.2.1. The formula (6.2.8) equals to Brézin’s equality (2.16) of [16] where he takes \( \lambda + i \epsilon \) instead of our choice \( \lambda - i \epsilon \). On the other hand, probably, he miscopies his equality in (44) of [17], more precisely, the term \( (1 + xy) \) in (44) should be \( (1 + (x - iz)^{-1} y) \).

Now, we prepare a technical lemma:
Lemma 6.2.4. For any \( \ell > 0 \) and \( n = 0, 1, 2, 3, \ldots \),
\begin{align*}
\int_{-\infty}^{\infty} dt e^{-\ell t^2} t^{2n+1} &= 0, \\
\int_{0}^{\infty} dt e^{-\ell t^2} t^{2n+1} &= \frac{n!}{2^{\ell n+1}}.
\end{align*}

Let \( \delta_0 > 0, d_0 > 0 \). For \( \tau_N = d_0 N^{\gamma} \) such that \( 0 \leq \gamma \leq 1/2 \), we have
\begin{equation}
\int_{\tau_N}^{\infty} dt e^{-\delta_0 N t^2} < (2\delta_0 d_0 N^{1-\gamma})^{-1} e^{-\delta_0 d_0^2 N^{1-2\gamma}}.
\end{equation}

Proof. The first two are well known. As \( t > \tau_N \), we get
\begin{equation}
\delta N \left( t^2 - \tau_N^2 \right) = \delta N (t - \tau_N)(t + \tau_N) > 2\delta N \tau_N (t - \tau_N).
\end{equation}
Therefore, we get
\begin{equation}
\int_{\tau_N}^{\infty} dt e^{-\delta_0 N t^2} < e^{-\delta_0 N \tau_N^2} \int_{\tau_N}^{\infty} dt e^{-2\delta_0 d_0 N^{1-\gamma}(t-\tau_N)} = (2\delta_0 d_0 N^{1-\gamma})^{-1} e^{-\delta_0 d_0^2 N^{1-2\gamma}}.
\end{equation}

Then, we have
\begin{equation}
\langle \rho_N(0) \rangle_N = \frac{1}{\pi J} \left[ 1 - (-1)^N \frac{1}{4} N^{-1} + \frac{1}{32} N^{-2} + (-1)^N \frac{5}{128} N^{-3} + O(N^{-4}) \right].
\end{equation}

Proof. Using (6.2.10), (6.2.11) to have
\begin{equation}
\langle \rho_N(0) \rangle_N = \frac{1}{\pi N} \left\langle \text{tr} \left[ \frac{1}{-i\hat{H} - N} \right] \right\rangle_N
\end{equation}
\begin{align*}
&= \frac{1}{\pi N} \left( \frac{N}{2\pi J^2} \right)^{1/2} \left( \frac{N}{J^2} \right)^{N+1} \int_{0}^{\infty} ds s^N \exp \left( -\frac{N}{2J^2} s^2 \right) \int_{-\infty}^{\infty} d\tau \tau^{N-1}(\tau + s) \exp \left( -\frac{N}{2J^2} \tau^2 \right) \\
&= \frac{1}{\pi N} \left( \frac{N}{2\pi J^2} \right)^{1/2} \left( \frac{N}{J^2} \right)^{N+1} \times \left\{ \begin{array}{ll}
\frac{1}{2} \left( \frac{N}{2J^2} \right)^{-N-1} \Gamma \left( \frac{N+1}{2} \right), & N\text{ even}, \\
\frac{1}{2} \left( \frac{N}{2J^2} \right)^{-N-1} \Gamma \left( \frac{N+2}{2} \right), & N\text{ odd},
\end{array} \right.
\end{align*}
we may calculate explicitly by the Stirling formula.

In proving (6.2.9) above, we have also used

Lemma 6.2.5. Let \( M = (M_{ab}) \) with \( M_{ab} = \alpha \delta_{ab} + \beta z_a \bar{z}_b \). Then, we have
\begin{align*}
\det M &= \alpha^{N-1}(\alpha + |z|^2), \\
|z|^2 &= z^* \cdot z.
\end{align*}

Proof. Let \( u \) satisfy \( Mu = \gamma u \). Then, we have
\begin{align*}
\bar{z} \cdot Mu &= \gamma \bar{z} \cdot u, \\
\bar{u} \cdot Mu &= \gamma \bar{u} \cdot u.
\end{align*}
From the first equation above, we get
\begin{equation}
(\gamma - \alpha - \beta |z|^2) \sum_{j=1}^{N} u_j \bar{z}_j = 0.
\end{equation}
If \( \sum_{j=1}^{N} u_j \bar{z}_j \neq 0 \), \( \gamma = \alpha + \beta |z|^2 \). On the other hand, if \( \sum_{j=1}^{N} u_j \bar{z}_j = 0 \), the second one in (6.2.14) implies that \( \gamma = \alpha \). Taking into account the multiplicity, we have the desired result.

\[ \square \]
(II) In order to get the expression (6.1.4), we proceed as follows: Putting
\[ A_X = \left( \sum_{j=1}^{N} \bar{\varphi}_j \varphi_j \sum_{j=1}^{N} \bar{\theta}_j \varphi_j \right), \]
we have
\[ \text{str } A_X^2 = \sum_{j,k=1}^{N} (\bar{\varphi}_j \varphi_k + \bar{\theta}_j \varphi_k)(\bar{\varphi}_k \varphi_j + \bar{\theta}_k \varphi_j). \]

On the other hand, the following is known as the Hubbard-Stratonovich formula:

**Lemma 6.2.6.** Let \( A \) be any even \( 2 \times 2 \) supermatrix. For \( Q \in \Omega \) given in (6.1.5), we have
\[ \exp \left[ -\frac{J^2}{2N} \text{str } A^2 \right] = \int_{\Omega} dQ \exp \left[ -\frac{N}{2J^2} \text{str } Q^2 \pm i \text{str } (Q A) \right]. \]

**Proof.** Let \( A = \begin{pmatrix} a & \theta_1 \\ \theta_2 & b \end{pmatrix} \) with \( a, b \in \mathcal{R}_{ov} \) and \( \theta_1, \theta_2 \in \mathcal{R}_{od} \). For any \( \gamma > 0 \), we claim
\[ \int_{\Omega} dQ \exp \left[ -\frac{1}{2} \text{str } (\gamma Q \pm i\gamma^{-1} A)^2 \right] = 1. \]

As we have readily
\[ \text{str } (\gamma Q \pm i\gamma^{-1} A)^2 = \gamma^2 (x_1^2 + x_2^2 + 2\rho_1 \rho_2) \pm 2i(x_1 a + \rho_1 \theta_2 - \rho_2 \theta_1 - ix_2 b) - \gamma^{-2}(a^2 - b^2 + 2\theta_1 \theta_2), \]
we get
\[ \int d\rho_1 d\rho_2 \exp \left[ -\gamma^2 \rho_1 \rho_2 \mp i(\rho_1 \theta_2 - \rho_2 \theta_1) + \gamma^{-2}\theta_1 \theta_2 \right] = (\gamma^2 - \theta_1 \theta_2)(1 + \gamma^{-2}\theta_1 \theta_2) = \gamma^2, \]
\[ \int \frac{dx_1 dx_2}{2\pi} \exp \left[ -\frac{\gamma^2}{2} (x_1^2 + x_2^2) \mp i(x_1 a - ix_2 b) + \frac{\gamma^{-2}}{2}(a^2 - b^2) \right] \]
\[ = \int \frac{dx_1 dx_2}{2\pi} \exp \left[ -\frac{1}{2}(\gamma x_1 \pm i\gamma^{-1} a)^2 - \frac{1}{2}(\gamma x_2 \pm i\gamma^{-1} b)^2 \right] = \gamma^{-2}. \]

Substituting (6.2.15) with \( A = A_X \) into (6.2.4), noting \( \text{str } (Q A_X) = X^*(Q \otimes \mathbb{I}_N)X \), taking the part of integral and changing the order of integration, we have
\[ i \int \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \prod_{k=1}^{N} d\bar{\theta}_k d\theta_k (z^* \cdot z) \exp \left[ -iX^*((\mu \mathbb{I}_2 - Q) \otimes \mathbb{I}_N)X \right] \]
\[ = \sum_{j=1}^{N} (\mu \mathbb{I}_2 - Q) \otimes \mathbb{I}_N)^{-1} \text{sdet}^{-1}(i(\mu \mathbb{I}_2 - Q) \otimes \mathbb{I}_N). \]

Therefore, we have

**Lemma 6.2.7.** For \( \mu = \lambda - i\epsilon \) (\( \epsilon > 0 \)),
\[ \langle \text{tr } \frac{1}{\mu \mathbb{I}_N - H} \rangle_N = \int_{\Omega} dQ \sum_{j=1}^{N} (\mu \mathbb{I}_2 - Q) \otimes \mathbb{I}_N)^{-1} \text{sdet}^{-1}(i(\mu \mathbb{I}_2 - Q) \otimes \mathbb{I}_N). \]

Here, \((C)_{bb, jj}\) is the \( j \)-th diagonal element of the boson-boson block of the (even) supermatrix \( C \).
6.2. THE DERIVATION OF (??) WITH $\lambda - i\epsilon$

Remark
\[
\left(\{\mu I_2 - Q\} \otimes I_N\right)^{-1} = \left(\{\mu I_2 - Q\}^{-1}\right)_{bb,jj} \quad \text{for any } j = 1, 2, \cdots, N, \\
sdet^{-1}(i(\mu I_2 - Q) \otimes I_N) = sdet^{-N}(\mu I_2 - Q),
\]
\[
\text{str}(AB) = \text{str}(BA), \quad \text{str}(A + B) = \text{str} A + \text{str} B, \quad \log(sdet \ell A) = \ell \log(\text{str} A) \quad \text{for } \ell \in \mathbb{Z},
\]
we have
\[
\langle \text{tr} \frac{1}{\mu I_N - H}\rangle_N = \int_\Omega dQ N\left(\{\mu I_2 - Q\}^{-1}\right)_{bb} \exp[-N\mathcal{L}(\mu; Q)]
\]
with
\[
\mathcal{L}(\mu; Q) = \text{str}[(2J^2)^{-1}Q^2 + \log(\mu I_2 - Q)],
\]
\[
\left(\{\mu I_2 - Q\}^{-1}\right)_{bb,11} = \frac{\mu - i\nu_2}{(\mu - x_1)(\mu - i\nu_2) - \rho_1 \rho_2} = \frac{(\mu - x_1)(\mu - i\nu_2) + \rho_1 \rho_2}{(\mu - x_1)^2(\mu - i\nu_2)},
\]
\[
\Omega = \{ Q = \left(\begin{array}{cc}
x_1 & \rho_1 \\
\rho_2 & i\nu_2
\end{array}\right) \mid x_1, x_2 \in \mathbb{R}_{\text{ev}}, \rho_1, \rho_2 \in \mathbb{R}_{\text{od}}\}.
\]

Remark 6.2.2. If we could make directly $\epsilon \to 0$ in (6.2.17), we had the formula (6.1.4). We claim at least symbolically we do that.

Lemma 6.2.8. For $\mu = \lambda - i\epsilon$ ($\epsilon > 0$),
\[
\frac{1}{N} \langle \text{tr} \frac{1}{\mu I_N - H}\rangle_N = \int_{\mathbb{R}^2} \frac{dx_1 dx_2}{2\pi} \frac{N(\mu - x_1 - i\nu_2)}{J^2(\mu - x_1)(\mu - i\nu_2)} \exp[-N\Phi(x_1, x_2; \mu)],
\]
where
\[
\Phi(x_1, x_2; \mu) = \frac{x_1^2 + x_2^2}{2J^2} + \log \frac{\mu - x_1}{\mu - i\nu_2}.
\]

Proof. As the integrand in (6.2.17) is represented by
\[
\frac{(\mu - x_1)(\mu - i\nu_2) + \rho_1 \rho_2}{(\mu - x_1)^2(\mu - i\nu_2)} \exp[-N\{\frac{1}{2J^2}(x_1^2 + x_2^2 + 2\rho_1 \rho_2) + \log \frac{\mu - x_1}{\mu - i\nu_2} - \frac{\rho_1 \rho_2}{(\mu - x_1)(\mu - i\nu_2)}\}],
\]
we have
\[
\int d\rho_1 d\rho_2 \frac{(\mu - x_1)(\mu - i\nu_2) + \rho_1 \rho_2}{(\mu - x_1)^2(\mu - i\nu_2)} \exp[-N\{\frac{1}{2J^2}(x_1^2 + x_2^2 + 2\rho_1 \rho_2) + \log \frac{\mu - x_1}{\mu - i\nu_2} - \frac{\rho_1 \rho_2}{(\mu - x_1)(\mu - i\nu_2)}\}]
\]
\[
= \frac{-1}{(\mu - x_1)^2(\mu - i\nu_2)} + \frac{N}{\mu - x_1} \left\{ \frac{1}{J^2} - \frac{1}{(\mu - x_1)(\mu - i\nu_2)} \right\}.
\]
Remarking $(\mu - x_1)^{-2} = \partial_{x_1}(\mu - x_1)^{-1}$, by integration by parts, we have
\[
\int \frac{dx_1 dx_2}{2\pi} \frac{-1}{(\mu - x_1)^2(\mu - i\nu_2)} \exp[-N\Phi(x_1, x_2; \mu)]
\]
\[
= \int \frac{dx_1 dx_2}{2\pi} \frac{-N}{(\mu - x_1)(\mu - i\nu_2)} \left(\frac{x_1}{J^2} - \frac{1}{\mu - x_1}\right) \exp[-N\Phi(x_1, x_2; \mu)],
\]
which yields (6.2.18). \[\square\]

Remark 6.2.3. As the right-hand side of (6.2.18) is rewritten
\[
\frac{1}{N} \langle \text{tr} \frac{1}{\mu I_N - H}\rangle_N = \frac{N}{2\pi J^2} \int_{\mathbb{R}^2} dx_1 dx_2 \frac{(\mu - x_1 - i\nu_2)(\mu - i\nu_2)^{N-1}}{(\mu - x_1)^{N+1}} \exp[-N\frac{x_1^2 + x_2^2}{2J^2}],
\]
there is no singularity in the integrand when $\Im \mu \neq 0$. \[\square\]
Using the fact that for any real smooth integrable function $f$,

$$\lim_{\epsilon \to 0} \Im \int_{\mathbb{R}} dx \frac{\lambda - i\epsilon - x}{(\lambda - x)^2 + \epsilon^2} f(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} dx \frac{\epsilon}{(\lambda - x)^2 + \epsilon^2} f(x) = \pi f(\lambda),$$

and integrating by parts based on $\partial_x^\ell (\mu - x)^{-1} = \ell! (\mu - x)^{-\ell - 1}$, we have,

$$\lim_{\epsilon \to 0} \Im \int_{\mathbb{R}} dx_1 (\lambda - i\epsilon - x_1)^{-\ell - 1} \exp[-N \frac{x_1^2}{2J^2}] = \frac{(\ell - 1)\ell}{\ell!} \partial_x^\ell \exp[-N \frac{\lambda^2}{2J^2}].$$

Using the Hermite polynomial $H_\ell(x)$ defined by

$$H_\ell(x) = (-1)\ell^2 e^{x^2/2} \partial_x^\ell e^{-x^2/2} = \sum_{k=0}^{[\ell/2]} \prod_{k} (-1)^{2-k} \frac{2k!}{(\ell - 2k)!} x^{\ell-2k} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-t^2/2} (x \pm it)^\ell,$$

with

$$H_\ell(\gamma x) = \frac{\gamma^{\ell+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-\gamma t^2/2} (x \pm it)^\ell,$$

we have

**Lemma 6.2.9.**

(6.2.21) \[ \pi \frac{(-1)^\ell}{\ell!} \partial_x^\ell \exp[-N \frac{\lambda^2}{2J^2}] = \frac{\pi}{\ell!} \left( \frac{N}{J^2} \right)^{\ell+1/2} \exp[-N \frac{\lambda^2}{2J^2}] \int_{\mathbb{R}} dt \exp (\lambda \pm it) \exp[-N \frac{t^2}{2J^2}]. \]

**Proof.** Using Bell's polynomial, we have

$$\pi \frac{(-1)^\ell}{\ell!} \partial_x^\ell \exp[-N \frac{\lambda^2}{2J^2}] = \pi \frac{\ell}{\ell!} \sum_{k=0}^{[\ell/2]} \frac{(-1)^{2-k} \lambda^{\ell-2k}}{k!^{\ell-2k}} \left( \frac{N}{J^2} \right)^{\ell-k} \lambda^{\ell-k} \exp[-N \frac{\lambda^2}{2J^2}]$$

$$= \frac{\pi}{\ell!} \left( \frac{N}{J^2} \right)^{\ell/2} H_\ell \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right) \exp[-N \frac{\lambda^2}{2J^2}]. \quad \square$$

On the other hand,

$$\int_{\mathbb{R}} dx_2 (\mu - ix_2)^\ell \exp[-N \frac{x_2^2}{2J^2}] = \sqrt{2\pi} \left( \frac{N}{J^2} \right)^{-(\ell+1)/2} H_\ell \left( \left( \frac{N}{J^2} \right)^{1/2} \mu \right),$$

and

$$\int_{\mathbb{R}} dx_2 (-ix_2)(\mu - ix_2)^\ell \exp[-N \frac{x_2^2}{2J^2}] = -\frac{J^2}{N} \int_{\mathbb{R}} dx_2 (\mu - ix_2)^{\ell-1} \exp[-N \frac{x_2^2}{2J^2}]$$

$$= -\frac{\ell}{\ell!} \sqrt{2\pi} \left( \frac{N}{J^2} \right)^{-(\ell+2)/2} H_{\ell-1} \left( \left( \frac{N}{J^2} \right)^{1/2} \mu \right).$$

Therefore, we have

$$\langle \rho_N(\lambda) \rangle_N = \frac{N}{2\pi J^2} \lim_{\epsilon \to 0} (\Im K_1 + \Im K_2),$$

where

$$K_1 = \int_{\mathbb{R}^2} dx_1 dx_2 (\mu - x_1)^{-N} (\mu - ix_2)^{N-1} \exp[-N \frac{x_1^2}{2J^2} + x_2^2],$$

$$K_2 = \int_{\mathbb{R}^2} dx_1 dx_2 (\mu - x_1)^{-N-1} (-ix_2)(\mu - ix_2)^{N-1} \exp[-N \frac{x_1^2}{2J^2} + x_2^2].$$
Moreover, we get
\[
\lim_{\epsilon \to 0} \Im K_1 = \frac{\pi}{(N-1)!} \left( \frac{N}{J^2} \right)^{(N-1)/2} H_{N-1} \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right) \exp \left[ -N \frac{\lambda^2}{2J^2} \right] \\
\times \sqrt{2\pi} \left( \frac{N}{J^2} \right)^{-N/2} H_{N-1} \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right),
\]
\[
\lim_{\epsilon \to 0} \Im K_2 = \frac{\pi}{N!} \left( \frac{N}{J^2} \right)^{N/2} H_N \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right) \exp \left[ -N \frac{\lambda^2}{2J^2} \right] \\
\times (-1)(N-1)\sqrt{2\pi} \left( \frac{N}{J^2} \right)^{-(N+1)/2} H_{N-2} \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right).
\]
Combining these, we have proved

**Proposition 6.2.2.** For any \( \lambda \in \mathbb{R} \), we have
\[
\langle \rho_N(\lambda) \rangle_N = \frac{1}{\sqrt{2\pi J(N-1)!}} \exp \left[ -\frac{N}{2J^2} \lambda^2 \right] \\
\times \left[ \sqrt{N} H_{N-1}^2 \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right) - \frac{N-1}{\sqrt{N}} H_N \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right) \right] \\
\times H_{N-2} \left( \left( \frac{N}{J^2} \right)^{1/2} \lambda \right),
\]
(6.2.22)
\[
= \left( \frac{N}{2\pi J^2} \right)^{1/2} \frac{1}{2\pi(N-1)!} \left( \frac{N}{J^2} \right)^N \\
\times \iint_{\mathbb{R}^2} dt \, ds \exp \left[ -N \phi_\pm(t, s, \lambda) \right] a_\pm(t, s; \lambda; N),
\]
where
\[
\phi_\pm(t, s, \lambda) = \frac{1}{2J^2} (t^2 + s^2 + \lambda^2) - \log (\lambda \pm it)(\lambda \mp is),
\]
(6.2.23)
\[
a_\pm(t, s, \lambda : N) = \frac{1}{(\lambda \mp it)(\lambda \mp is)} - \frac{1}{2} (1 - N^{-1}) \left[ \frac{1}{(\lambda \mp it)^2} + \frac{1}{(\lambda \mp is)^2} \right].
\]

**6.3. The proof of semi-circle law and beyond that**

To prove semi-circle law, we need to apply the method of saddle point which is only proved in the ordinary space \( \mathbb{R}^m \). As we mentioned before, it seems difficult even ordinary case to assure whether we may apply that method suitably. And this point is stressed by the old book of de Bruijn [29], pp.77-78:

The saddle point method, due to B. Riemann and P. Debye, is one of the most important and most powerful methods in asymptotics. ... (omission) ... 

Any special application of the saddle point method consists of two stages.

(i) The stage of exploring, conjecturing and scheming, which is usually the most difficult one. It results in choosing a new integration path, made ready for application of (ii).

(ii) The stage of carrying out the method. Once the path has been suitably chosen, this second stage is, as a rule, rather a matter of routine, although it may be complicated. It essentially depends on the Laplace method of Ch.4.
Therefore, we proved the first part of Theorem 1.2.

Now, we study the asymptotic behavior of the following integral w.r.t. $N$:

$$
\langle \text{tr} \left( \frac{1}{(\lambda-i0) I_N - H} \right) \rangle_N = i \frac{1}{(N-1)!} \left( \frac{N}{2\pi J^2} \right)^{1/2} \left( \frac{N}{J^2} \right)^{N+1} (I_1 + I_2),
$$

(6.3.1)\]

\begin{align*}
I_1 &= \int_0^\infty ds \, s^N \exp \left[ -\frac{N}{2J^2} (2i\lambda s + s^2) \right] \int_{-\infty}^\infty d\tau \, (\tau + i\lambda)^N \exp \left[ -\frac{N}{2J^2} \tau^2 \right], \\
I_2 &= \int_0^\infty ds \, s^{N+1} \exp \left[ -\frac{N}{2J^2} (2i\lambda s + s^2) \right] \int_{-\infty}^\infty d\tau \, (\tau + i\lambda)^{N-1} \exp \left[ -\frac{N}{2J^2} \tau^2 \right].
\end{align*}

THEOREM 6.3.1. Let $|\lambda| < 2J$. Putting $\theta = -\arg \tau_+, \tau_+ = 2^{-1}(-i\lambda + \sqrt{4J^2 - \lambda^2})$, we have

$$
I_1 + I_2 = 2\pi e^{-N} J^{2(N+1)} \left[ e^{-i\theta} N^{-1} - \frac{1}{12} \left( e^{-i\theta} - (-1)^N \frac{3e^{-iN(\sin 2\theta + 2\theta)}}{\cos^2 \theta} \right) N^{-2} + O(N^{-3}) \right].
$$

For the proof, see Appendix A.4 of [77].

Remark. The Stirling formula

$$(N-1)! = e^{-N} N^{-1/2} \sqrt{2\pi} (1 + \frac{1}{12} N^{-1} - \frac{1}{288} N^{-2} - \frac{139}{51840} N^{-3} - \frac{571}{2488320} N^{-4} + O(N^{-5})),$$

that is,

$$
\frac{1}{(N-1)!} = \frac{e^N N^{-N+1/2}}{\sqrt{2\pi}} (1 - \frac{1}{12} N^{-1} - \frac{1}{96} N^{-2} + O(N^{-3})),
$$

(6.3.3)

we get

$$
\langle \text{tr} \left( \frac{1}{(\lambda-i0) I_N - H} \right) \rangle_N = i \frac{1}{(N-1)!} \left( \frac{N}{2\pi J^2} \right)^{1/2} \left( \frac{N}{J^2} \right)^{N+1} (I_1 + I_2),
$$

$$
= \frac{N^2}{J} \left( e^{-i\theta} N^{-1} + \frac{1}{12} [e^{-i\theta} - (-1)^N \frac{3e^{-iN(\sin 2\theta+2\theta)}}{\cos^2 \theta}] N^{-2} + O(N^{-3}) \right)
$$

$$
\times (1 - \frac{1}{12} N^{-1} + O(N^{-2}))
$$

$$
= \frac{N^2}{J} \left( e^{-i\theta} - \frac{(-1)^N e^{-iN(\sin 2\theta+2\theta)}}{4 \cos^2 \theta} N^{-1} + O(N^{-2}) \right).
$$

Therefore, we proved the first part of Theorem 1.2. \quad \square

The relation (6.1.5) for $|\lambda| \geq 2J$ is proved analogously: That is, we have

THEOREM 6.3.2. Let $\lambda > 2J$. There exists constnt $k(\lambda) > 0$ and $C(\lambda) > 0$ such that

$$
I_1 + I_2 = J^{2N} e^{-N} K(N) + \text{pure imaginary part } \text{ with } |K(N)| \leq C(\lambda) N^{-1/4} e^{-k(\lambda)N}.
$$

See, Appendix A.4 of [77], for the proof.
6.4. Edge mobility

To study the asymptotic behavior of \( \langle \rho_N(2J - zN^{-2/3}) \rangle_N \), or \( \langle \rho_N(-2J + zN^{-2/3}) \rangle_N \) for \(|z| \leq 1\) as \( N \to \infty \), we use in this section the formula (6.2.22):

\[
\langle \rho_N(2J - zN^{-2/3}) \rangle_N = \frac{N^{N+1/2}}{(2\pi)^{3/2}(N-1)!J^{2N+1}} \int_{\mathbb{R}^2} dt ds a_+(t, s, 2J - zN^{-2/3}; N) \times \exp \left[ -N\phi_+(t, s, 2J - zN^{-2/3}) \right],
\]

(6.4.1)

\[
\langle \rho_N(-2J + zN^{-2/3}) \rangle_N = \frac{N^{N+1/2}}{(2\pi)^{3/2}(N-1)!J^{2N+1}} \int_{\mathbb{R}^2} dt ds a_-(t, s, -2J + zN^{-2/3}; N) \times \exp \left[ -N\phi_-(t, s, -2J + zN^{-2/3}) \right],
\]

(6.4.2)

where

\[
\phi_\pm(t, s, \lambda) = \frac{1}{2J^2}(t^2 + s^2 + \lambda^2) - \log (\lambda \mp it)(\lambda \mp is),
\]

(6.4.3)

\[
a_\pm(t, s; \lambda; N) = \frac{2(\lambda \mp it)(\lambda \mp is) - (1 - N^{-1})[(\lambda \mp it)^2 + (\lambda \mp is)^2]}{2(\lambda \mp it)^2(\lambda \mp is)^2}.
\]

**Proposition 6.4.1.** For \(|z| \leq 1\), we have

\[
\langle \rho_N(2J - zN^{-2/3}) \rangle_N = \frac{N^{-1/3}}{8\pi^2J^5} \int_{\mathbb{R}^2} dx dy (x - y)^2 \exp \left[ -\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x + y)) \right] + O(N^{-2/3}).
\]

(6.4.4)

The right-hand integral above, should be interpreted as the oscillatory one.

**Proof.** In this proof, we abbreviate the subscript + of \( \phi_+ \) and \( a_+ \).

Put \( u = N^{-1/3} \). For \( \lambda = 2J - zu^2 \), using the change of variables \( s = -iJ + yu, \ t = -iJ + xu, \) we have

\[
\varphi(x, y, z; u) = \phi(-iJ + xu, -iJ + yu, 2J - zu^2) = h(u) - \log g(u),
\]
where
\[ h(u) = \frac{1}{2J^2} \left( (-iJ + xu)^2 + (-iJ + yu)^2 + (2J - zu)^2 \right) \]
\[ = 1 - \frac{i(x+y)}{J} u + \frac{x^2 + y^2 - 4zJ}{2J^2} u^2 + \frac{z^2}{2J^2} u^4, \]
\[ g(u) = (2J - zu^2 - i(-iJ + xu))(2J - zu^2 - i(-iJ + yu)) \]
\[ = J^2 - iJ(x+y)u - (xy + 2J)u^2 + iz(x+y)u^3 + z^2 u^4. \]

Analogously, we put
\[ \alpha(x, y, z; u) = a(-iJ + xu, -iJ + yu, 2J - zu^2; u^{-3}) \]
\[ = \frac{(x - y)^2 u^2 + u^3 [2J^2 - 2iJ(x + y)u - (x^2 + y^2 + 4zJ)u^2 + 2i(x + y)zu^3 + 2z^2 u^4]}{2g^2(u)}. \]

Using Taylor’s expansion of \( \varphi(x, y, z; u) \) w.r.t. \( u \) at \( u = 0 \), we get
\[ \varphi(x, y, z; u) = 1 - \log J^2 + \frac{i}{3J^3} [x^3 + y^3 - 3zJ(x + y)] u^3 + R(u), \]
with
\[ R(u) = \frac{u^4}{3!} \int_0^1 d\tau (1 - \tau)^3 \varphi^{(4)}(x, y, z; \tau u), \]
\[ \varphi^{(4)}(x, y, z; u) = 4! z^2 g(u) - \frac{3(g''(u)^2 + g'(u)g'''(u))}{g(u)^2} + \frac{12g'(u)^2 g''(u)}{g(u)^3} - \frac{6g'(u)^4}{g(u)^4}, \]
\[ |\partial_x^k \partial_y^\ell R(u)| \leq C_{k, \ell} u^4 \quad \text{for } u \geq 0, \; x, y \in \mathbb{R}, \; |z| \leq 1, \; k + \ell \leq 2. \]

Moreover,
\[ e^{-N R(u)} = 1 + S(u), \quad S(u) = -u^{-2} \int_0^1 d\tau R'(\tau u), \]
\[ R'(u) = \frac{2u^3}{3} \int_0^1 d\tau (1 - \tau)^3 \varphi^{(4)}(x, y, z; \tau u) + \frac{u^5}{6} \int_0^1 d\tau (1 - \tau)^3 \varphi^{(5)}(x, y, z; \tau u). \]

Therefore, we have
\[ \exp \left[ -N \varphi(x, y, z; N^{-1/3}) \right] = e^{-N J^{2N}} \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] e^{-N R(N^{-1/3})}. \]

On the other hand, as we have
\[ g(u)^{-2} = J^{-4} - 2u \int_0^1 d\tau g'(\tau u) g(\tau u)^{-3}, \]
we get
\[ \alpha(x, y, z; u) = \frac{(x - y)^2}{2J^4} u^2 + A(u), \quad A(u) = -u^3 (x - y)^2 \int_0^1 d\tau g'(\tau u) g(\tau u)^{-3}. \]
with
\[ |\partial_x^k \partial_y^\ell A(u)| \leq C_{k, \ell} u^3 \quad \text{for } u \geq 0, \; x, y \in \mathbb{R}, \; |z| \leq 1, \; k + \ell \leq 2. \]
Combining these, we get
\[
\iint_{\mathbb{R}^2} dt \, ds \, \exp \left[ -N \phi(t, s, 2J - zN^{-2/3}) \right] a(t, s, 2J - zN^{-2/3}; N)
\]
\[= N^{-2/3} \iint_{\mathbb{R}^2} dx \, dy \exp \left[ -N \varphi(x, y, z; N^{-1/3}) \right] \alpha(x, y, z; N^{-1/3}) \]
\[= e^{-N} J^{2N} N^{-2/3} \iint_{\mathbb{R}^2} dx \, dy \exp \left[ -\frac{i}{2J^4} (x^3 + y^3 - 3zJ(x + y)) \right] \]
\[\times (1 + S(N^{-1/3})) \left( \frac{(x - y)^2}{2J^4} N^{-2/3} + A(N^{-1/3}) \right) \]
\[= e^{-N} J^{2N} \left[ N^{-4/3} \iint_{\mathbb{R}^2} dx \, dy \frac{(x - y)^2}{2J^4} \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] \right. \]
\[+ O(N^{-5/3}) \].

Here, we applied the lemma below to
\[f(x, y) = A(u), A(u)S(u), S(u)\frac{(x - y)^2}{2J^2} u^2,\]
for getting the last term \(O(N^{-5/3})\).

Moreover, we may rewrite the above \((6.4.5)\) using the Stirling formula to get
\[
\langle \rho_N(2J - zN^{-2/3}) N \rangle = \frac{N^{-1/3}}{8\pi^2 J^6} \iint_{\mathbb{R}^2} dx \, dy \times \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] (x - y)^2 \]
\[\quad \quad \quad + O(N^{-2/3}). \]

**Lemma 6.4.1.** If \(f\) satisfies
\[|\partial_x^k \partial_y^l f(x, y)| \leq C_{k,l},\]
we have
\[\left| \iint_{\mathbb{R}^2} dx \, dy \, f(x, y) \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] \right| \leq C < \infty. \]

**Proof.** We use Lax’s technique (combining integration by parts with \(\partial_q e^{iq}(q) = i\partial_q \phi(q) e^{iq}(q)\) where \(|\partial_q \phi(q)| \neq 0\)) to estimate the oscillatory integrals noting
\[(1 - \partial_x^2 - \partial_y^2) \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] = \Phi(x, y) \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] \]
with
\[\Phi(x, y) = 1 + 18 z^2 J^2 + \frac{2i(x+y)}{J^3} + \frac{6z(x^2 + y^2)}{J^2} + \frac{x^4 + y^4}{J^6}. \]
Therefore, we have
\[\iint_{\mathbb{R}^2} dx \, dy \, f(x, y) \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right] \]
\[= \iint_{\mathbb{R}^2} dx \, dy \, (1 - \partial_x^2 - \partial_y^2) \frac{f(x, y)}{\Phi(x, y)} \exp \left[ -\frac{i}{3J^3} (x^3 + y^3 - 3zJ(x + y)) \right]. \]
By the assumption, \((1 - \partial_x^2 - \partial_y^2)(f(x, y)/\Phi(x, y))\) is integrable w.r.t. \(dx \, dy\), we get the desired result.

Using the Airy function defined by
\[\text{Ai}(z) = \int_{\mathbb{R}} dx \, \exp \left[ -\frac{i}{3} x^3 + izx \right] = \int_{\mathbb{R}} dx \, \exp \left[ \frac{i}{3} x^3 - izx \right] = \overline{\text{Ai}(z)} \text{ for } z \in \mathbb{R},\]
we have
\[
\int_{\mathbb{R}} dx \exp \left[ -\frac{ix^3}{3J^2} + \frac{ix}{J} \right] = J \text{Ai}(\frac{z}{J}),
\]
\[
\int_{\mathbb{R}} dx x \exp \left[ -\frac{ix^3}{3J^2} + \frac{ix}{J} \right] = -iJ^2 \text{Ai}'(\frac{z}{J}),
\]
\[
\int_{\mathbb{R}} dx x^2 \exp \left[ -\frac{ix^3}{3J^2} + \frac{ix}{J} \right] = -J^3 \text{Ai}''(\frac{z}{J}).
\]
And we get
\[
\langle \rho_N(2J - z N^{-2/3}) \rangle_N = N^{-1/3} f(z J^{-1}) + O(N^{-2/3}),
\]
where
\[
 f(z) = \frac{1}{4\pi^2 J} \left( \text{Ai}'(z) \text{Ai}'(z) - \text{Ai}''(z) \text{Ai}(z) \right).
\]

**Corollary 6.4.1.** For \(|z| \leq 1\), we have
\[
\langle \rho_N(-2J + z N^{-2/3}) \rangle_N = -N^{-1/3} f(z J^{-1}) + O(N^{-2/3}).
\]

**Remark 6.4.1.** Though Brézin and Kazakov applied the Brézin formula (2.7) to obtain the analogous statement, but we can’t follow their proof (48) of [17].

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**Mini Column 3: On GOE, GUE and GSE**

In the following, to expect giving a trigger to audience having curiosity to RMT, we mention here Gaussian ensembles classified by F. Dyson. The Dyson index \(\beta\) is defined by numbers of real components in matrix elements belonging to each ensemble. Though I give a look to not only a book by Mehta [95] but also survey paper by J. Verbaarschot [128], it seems not sufficient to comprehend this interesting branch only “a look”.

**Definition 6.4.1.** A set of \(N \times N\) Hermitian matrix \(H = (H_{jk})\) whose matrix element is distributed with the probability
\[
P(H) dH = Z_{\beta N}^{-1} e^{-(N\beta/4) \text{tr} H^2} dH
\]
is called Wigner-Dyson Ensemble. Especially

1. When \(H = (H_{jk})\) is real symmetric matrix, then it’s index is \(\beta = 1\) with volume element \(dH = \prod_{k \leq j} dH_{kj}\). This is called Gaussian Orthogonal Ensemble (=GOE).
2. When \(H = (H_{jk})\) is complex Hermite matrix with \(H_{kj} = H_{kj}^{(0)} + iH_{ij}^{(1)}\), \(H_{ij}^{(0)}, H_{ij}^{(1)} \in \mathbb{R}\), then it’s index is \(\beta = 2\) with volume element \(dH = \prod_{k \leq j} dH_{kj}^{(0)} \prod_{k < j} dH_{kj}^{(1)}\). This is called Gaussian Unitary Ensemble (=GUE).
3. Let (real) quaternion with base \((1, i, j, k)\) be decomposed as \(H_{jk} = H_{jk}^{(0)} + H_{jk}^{(1)} i + H_{jk}^{(2)} j + H_{jk}^{(3)} k\), \(H_{jk}^{(\alpha)} \in \mathbb{R}\). When \(H = (H_{jk})\) is Hermite whose matrix element is quaternion, then \(\beta = 4\) with volume element \(dH = \prod_{k \leq j} dH_{kj}^{(0)} \prod_{k = 1}^{3} \prod_{k < j} dH_{kj}^{(4)}\). This is called Gaussian Symplectic Ensemble (=GSE).

**Remark 6.4.2.** We prepare letters \(i, j, k\) with relations \(i^2 = j^2 = k^2 = ijk = -1\). For example, using Pauli matrices \(\{\sigma_j\}_{j=1}^{3}\), we may put \(i = i\sigma_3, j = i\sigma_2, k = i\sigma_1\), then we have a matrix representation
\[
q = q^{(0)} + q^{(1)} i + q^{(2)} j + q^{(3)} k \sim \begin{pmatrix}
q^{(0)} + q^{(1)} i & q^{(2)} + q^{(3)} i & q^{(0)} - q^{(1)} i
\end{pmatrix}.
\]
Remark 6.4.3. Following facts are known:

1. GOE is invariant under the similarity transformation $T_1(O) : H \mapsto tOHO$ by $O$ belonging to $O(N)=\text{real orthogonal group}$,
2. GUE is invariant under the similarity transformation $T_2(U) : H \mapsto U^{-1}HU$ by $U$ belonging to $U(N)=\text{unitary group}$,
3. GSE is invariant under the similarity transformation $T_4(S) : H \mapsto tSHS$ by $S$ belonging to $Sp(N)=\text{symplectic group}$.

Let $\lambda_1, \cdots, \lambda_N$ be eigenvalues of $N \times N$ Hermite matrix. We check the relation between their differential $d\lambda_j$ and Lebesgue measure $dH$. In case $H \in \text{GOE}$, since the number of independent components of $H_{jk}$ equals to $N(N+1)/2$, we have $\ell = N(N+1)/2 - N = N(N-1)/2$ independent variables $\mu_m$ except $\{\lambda_j\}$. Because

$$\text{tr} \ H^2 = \sum_{j=1}^{N} \lambda_j^2$$

and putting Jacobian of change of variables as

$$J(\lambda, \mu) = \left| \frac{\partial(H_{11},H_{12},\cdots,H_{NN})}{\partial(\lambda_1,\cdots,\lambda_N,\mu_1,\cdots,\mu_\ell)} \right|,$$

we have

$$dH = J(\lambda, \mu) \prod_{j=1}^{N} d\lambda_j \prod_{k=1}^{\ell} d\mu_k.$$

We need to calculate

$$\int \int d\lambda d\mu J(\lambda, \mu) = \int d\lambda \left[ \int d\mu J(\lambda, \mu) \right].$$

In other word, find the reason for the appearance of difference product of eigenvalues (van der Monde determinant)?

[Report problem 6-1]: Give a precise description for above representation using eigenvalues (see, pp. 55-69 in Mehta [95] or Theorem 5.22 of Deift [31] for $\beta = 1$).

**Theorem 6.4.1** (Theorem 3.3.1 of [95]). Let $x_1, \cdots, x_N$ be eigenvalues for a hermite matrix belonging to GOE ($\beta = 1$), GUE ($\beta = 2$) or GSE ($\beta = 4$). Then, the joint probability density of $x_1, \cdots, x_N$ is given

$$P_{N\beta}(x_1, \cdots, x_N) = Z_{N\beta}^{-1} e^{-\langle \beta/2 \rangle \sum_{j=1}^{N} x_j^2} \prod_{j<k}^N |x_j - x_k|^{\beta/2}.$$

Here,

$$Z_{N\beta} = (2\pi)^{N/2} \beta^{-N/2 - \beta N(N-1)/4} \prod_{j=1}^{N} \Gamma(1 + \beta j/2).$$

**Theorem 6.4.2** (Theorem 17.1 of Mehta [95]). For any given positive integer $N$, we put $dx = dx_1 \cdots dx_N$,

$$\Delta(x) = \Delta(x_1, \cdots, x_N) = \begin{cases} \prod_{1 \leq j < \ell \leq N} (x_j - x_\ell) & \text{if } N > 1, \\ 1 & \text{if } N = 1. \end{cases}$$

\[\text{Frankly speaking, atom has not enough patience to understand these facts, therefore he leaves these explanation for younger people expecting young gives stimulation to old.}\]
and

\[ \Phi(x) = \Phi(x_1, \ldots, x_N) = |\Delta(x)|^{2\gamma} \prod_{j=1}^{N} x_j^{\alpha-1}(1 - x_j)^{\beta-1}. \]

Then, for \( \Re\alpha > 0, \Re\beta > 0, \Re\gamma > -\min \left( \frac{1}{N}, \frac{\Re\alpha}{N-1}, \frac{\Re\beta}{N-1} \right) \), we have

\[
I(\alpha, \beta, \gamma, N) = \int_0^1 \cdots \int_0^1 dx \Phi(x) = \prod_{j=0}^{N-1} \frac{\Gamma(1 + \gamma + j\gamma)\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)}{\Gamma(1 + \gamma)\Gamma(\alpha + \beta + (N+j-1)\gamma)}.
\]

6.5. Relation between RMT and Painlevé transcendents

It is shown rather recently that there is a mysterious connection between RMT, combinatorics and Painlevé functions. Borrowing the description of Tracy and Widom \[124, 125\], we explain our problem.

Let \( U_N \) be a set of unitary \( N \times N \) matrices with Haar measure. Denoting (real) eigenvalues of \( U \in U_N \) as \( \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N\} \) with its maximum \( \lambda_N = \lambda_N(U) \), we consider \( P_2(\lambda_N(U) < t) \). Then, we have

**Theorem 6.5.1** (\[125\]).

\[
\lim_{N \to \infty} P_2 \left( \frac{\lambda_N(U) - 2N}{N^{1/6}} < s \right) = F_2(s) = \exp \left( - \int_s^\infty ds \, (x-s)q^2(x) \right).
\]

Here, \( q(s) \) is Painleve II function satisfying

\[
q'' = sq + 2q^3 \quad q(s) \sim \text{Ai}(s) = \int_{\mathbb{R}} dx \exp \left[ -\frac{i}{3} x^3 + isx \right] \quad \text{when } s \to \infty.
\]

Following theorem, which has curious resemblance to above, is proved by J. Baik, P. Deift and K. Johansson \[6\]: Putting uniform probability measure \( P \) on symmetric group \( S_N \), we denote \( \ell_N = \ell_N(\sigma) \) the length of the longest increasing subsequence for each \( \sigma \in S_N \). Then, we have

**Theorem 6.5.2** (\[6\]).

\[
\lim_{N \to \infty} P \left( \frac{\ell_N - 2N}{N^{1/6}} \leq s \right) = F_2(s).
\]
Table of Painlevé equations.
(I) \( w'' = 6w^2 + s, \)

(II) \( w'' = 2w^3 + sw + \alpha, \)

(III) \( w'' = \frac{w'^2}{w} - \frac{w'}{s} + \frac{\alpha w^2 + \beta}{s} + \gamma w^3 + \frac{\delta}{w}, \)

(IV) \( w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} + 4sw^2 + 2(s^2 - \alpha)w + \frac{\beta}{w}, \)

(V) \( w'' = \frac{w'^2}{2w} \left( \frac{1}{2w} + \frac{1}{w-1} \right) - \frac{w'}{s} + \frac{(w-1)^2}{s^2} \left( \frac{\alpha w + \beta}{w} \right) + \frac{\gamma w}{s} + \frac{\delta w(w + 1)}{w - 1}, \)

(VI) \( w'' = \frac{w'^2}{2} \left( \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-s} \right) - w' \left( \frac{1}{s} + \frac{1}{s-1} + \frac{1}{w-s} \right) \)
\[ + \frac{w(w-1)(w-s)}{s^2(s-1)^2} \left( \frac{\alpha + \beta s}{w^2} + \frac{\gamma(s-1)}{(w-1)^2} + \frac{\delta s(s-1)}{(w-s)^2} \right). \]

**Problem 6.5.1.** (i) Can we find another explanation of above theorems by finding “slowness variables” as analogous as Efetov’s reproof of Wigner’s semi-circle law?

(ii) Moreover, the results, for example, C. Itzykson and J.B. Zuber \[79\], D. Bessis, C. Itzykson and J.B. Zuber \[12\] or A. Matytsin \[94\], should be viewed from our point of view, but I have not enough intelligence to appreciate these works. By the way, the article by A. Zvonkin \[139\] seems a nice guide in this direction.

**Remark 6.5.1.** (i) To finish this lecture notes, I am almost drowned by checking by internet searching papers on RMT. For young researchers, T. Tao \[122\] may be recommended. But this is also thick to begin with, therefore take a look to X. Zeng and Z. Hou \[137\].

(ii) There are also papers concerned about. Whether Matytsin’s procedure \[94\] which tries to generalize Itzykson-Zuber formula has some relations to Functional Derivative Equation or not. At least, my life in heaven or hell will be full of mathematical problems considered.
CHAPTER 7

Fundamental solution of Free Weyl equation à la Feynman

Because of the integration theory is not yet completed at lecture time, this chapter is chosen because here we only use the “naive” definition of integral on $\mathbb{R}^3$.

Let $V$ be a representation space and let a function $\psi(t, q) : \mathbb{R} \times \mathbb{R}^3 \to V$ be given satisfying

$$
\begin{cases}
    i\hbar \frac{\partial}{\partial t} \psi(t, q) = \mathbb{H} \psi(t, q), & \mathbb{H} = c\sigma_j \frac{h}{i} \frac{\partial}{\partial q_j}, \\
    \psi(0, q) = \psi(q).
\end{cases}
$$

Here, $c$ and $\hbar$ are positive constants, the summation with respect to $j = 1, 2, 3$ is abbreviated. Let $I$ as an identity map from $V$ to $V$, and define maps $\{\sigma_j : V \to V\}$ satisfying

$$
\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}I, \quad (\text{Clifford relation})
$$

Especially when we put $V = \mathbb{C}^2$ and $\psi(t, q) = \{\psi_1(t, q), \psi_2(t, q)\}$, we have so-called Pauli matrices:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

This equation (7.0.1) is called free Weyl equation and we want to construct a fundamental solution of this modifying Feynman’s procedure.

Before this, we give a very primitive and well-known method which is applicable in this simple situation.

Applying formally the Fourier transformation (which contains a parameter $\hbar$) with respect to $q \in \mathbb{R}^3$ to (7.0.1), we get

$$
\hat{\psi}(t, p) = \mathbb{H} \hat{\psi}(t, p) \quad \text{where} \quad \mathbb{H} = c\sigma_j p_j = c \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}.
$$

As $\mathbb{H}^2 = c^2|p|^2 I_2$ by $(7.0.2)$, we easily have

$$
e^{-i\hbar^{-1}t\mathbb{H}} \hat{\psi}(p) = \left[ \cos \left( c(h^{-1}t|p|)I_2 - ic^{-1}|p|^{-1}\sin(ch^{-1}t|p|)\mathbb{H} \right) \right] \hat{\psi}(p).
$$

Therefore, we have

**Proposition 7.0.1.** For any $t \in \mathbb{R}$,

$$
e^{-i\hbar^{-1}t\mathbb{H}} \psi(q) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} dp e^{i\hbar^{-1}q p} e^{-i\hbar^{-1}t\mathbb{H}} \hat{\psi}(p) = \int_{\mathbb{R}^3} dq' E(t, q, q') \psi(q'),
$$

with

$$
E(t, q, q') = (2\pi\hbar)^{-3} \int_{\mathbb{R}^3} dp e^{i\hbar^{-1}(q-q')p} \left[ \cos \left( c(h^{-1}t|p|)I_2 - ic^{-1}|p|^{-1}\sin(ch^{-1}t|p|)\mathbb{H} \right) \right].
$$
This calculation doesn’t work when (7.0.1) is changed to

\[ H(t) = \sum_{j=1}^{3} c_{j} \left( \frac{\hbar}{i} \frac{\partial}{\partial q_{j}} - \frac{e}{c} A_{j}(t, q) \right) + eA_{0}(t, q). \]

We don’t insist on this equation having physical meaning because we add minimally electromagnetic potential to (7.0.1) very formally. But it seems interesting to solve mathematically as a toy model.

To begin with, we give another method to solve (7.0.1) to get

**Theorem 7.0.3 (Path-integral representation of a solution for the free Weyl equation)**

\[ \psi(t, q) = \hat{\psi} \left( (2\pi\hbar)^{-3/2} \hbar \int_{\mathbb{R}^{3}} d\xi d\pi D^{1/2}(t, \bar{x}, \bar{\theta}, \xi, \pi) e^{i\hbar^{-1}S(t, \bar{x}, \bar{\theta}, \xi, \pi)} \right|_{\bar{x}_{N}=q}. \]

Here, \( S(t, \bar{x}, \bar{\theta}, \xi, \pi) \) and \( D(t, \bar{x}, \bar{\theta}, \xi, \pi) \) are solutions of the Hamilton-Jacobi and continuity equations, (7.2.10) and (7.2.12) respectively.

### 7.1. A strategy of constructing parametrices

Taking the free Weyl equation as a simplest model of constructing parametrices for Dirac (1.2.5) in Chapter 1 or Weyl (7.0.7) above, we explain our strategy which is a superly extended version of what we explained before.

1. Is it possible to extract “symbol” corresponding to a given system of PDO’s?
2. To define “symbol”, we need to represent the matrix structure as the differential operators acting on superspace. For example, to regard \( 2 \times 2 \) matrix structure as differential operators, we decompose

\[
\begin{pmatrix} a & c \\ d & b \end{pmatrix} = \frac{a + b}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a - b}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{c + d}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{c - d}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

and attach differential operators for each \( \sigma_{j} \) and we get the symbol of that system of PDO’s.
3. For the Hamiltonian \( H(x, \xi, \theta, \pi) \), we construct a solution \( S(t, x, \xi, \theta, \pi) \) of the Hamilton-Jacobi equation with given initial data. (4) Calculating the Hessian of \( S(t, x, \xi, \theta, \pi) \) w.r.t. \( \xi, \pi \), we define its determinant below, called super Van Vleck determinant:

\[ D(t, x, \xi, \theta, \pi) = sdet \left( \frac{\partial^{2} S(t, x, \xi, \theta, \pi)}{\partial \xi_{j} \partial \pi_{k}} \right) \]

(5) Finally, defining an operator

\[
(\mathcal{U}_{t})_{\xi_{j}, \pi_{j}}(x, \theta) = (2\pi\hbar)^{-3/2} \hbar \int_{\mathbb{R}^{3}} d\xi d\pi D^{1/2}(t, \bar{x}, \bar{\theta}, \xi, \pi) e^{i\hbar^{-1}S(t, \bar{x}, \bar{\theta}, \xi, \pi)} \mathcal{F}_{\xi_{j}, \pi_{j}},
\]

we check its properties and show that it’s the desired parametrix.
7.2. Sketchy proofs of the procedure mentioned above

(1) A “spinor” $\psi(t,q) = \{\psi_1(t,q), \psi_2(t,q)\}$ : $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^2$ is identified with an even super-smooth function $u(t,x,\theta) = u_0(t,x) + u_1(t,x)\theta_1\theta_2 : \mathbb{R} \times \mathbb{R}^{3|2} \to \mathbb{C}_{ev}$ as follows:

$$\mathbb{C}^2 \ni \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \frac{x}{\bar{y}} u(\theta) = u_0 + u_1\theta_1\theta_2 \in \mathbb{C}_{ev} \text{ with } u_0 = \psi_1, u_1 = \psi_2.$$  

Here, functions $u_0(t,x), u_1(t,x)$ are the Grassmann continuation of $\psi_1(t,q), \psi_2(t,q)$, respectively.

(2) Pauli matrices $\{\sigma_j\}$ have differential operator representations

$$\sigma_1\left(\theta, \frac{\lambda}{i} \frac{\partial}{\partial \theta}\right) = i\lambda^{-1}\left(\theta_1\theta_2 + \lambda^2 \frac{\partial^2}{\partial \theta_1\partial \theta_2}\right),$$  

$$\sigma_2\left(\theta, \frac{\lambda}{i} \frac{\partial}{\partial \theta}\right) = -\lambda^{-1}\left(\theta_1\theta_2 \lambda^2 \frac{\partial^2}{\partial \theta_1\partial \theta_2}\right),$$  

$$\sigma_3\left(\theta, \frac{\lambda}{i} \frac{\partial}{\partial \theta}\right) = 1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}.$$  

Here, we take an arbitrarily chosen parameter $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Remark 7.2.1. Only for $|\lambda| = 1$, $\{b\sigma_j(\theta, -i\lambda\partial_\theta)\}$ are unitary matrices.

(3) Since, using (2), the differential operator $\mathbb{H}$ (7.0.1) is identified with

$$\mathbb{H}(\frac{\hbar}{i} \frac{\partial}{\partial x}, \theta, \frac{\lambda}{i} \frac{\partial}{\partial \theta}) = c\lambda^{-1}h\left(\theta_1\theta_2 + \lambda^2 \frac{\partial^2}{\partial \theta_1\partial \theta_2}\right) \frac{\partial}{\partial x_1} + ic\lambda^{-1}h\left(\theta_1\theta_2 - \lambda^2 \frac{\partial^2}{\partial \theta_1\partial \theta_2}\right) \frac{\partial}{\partial x_2} - ich\left(1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}\right) \frac{\partial}{\partial x_3}.$$  

Therefore, the super-version of Weyl equation is given

$$\begin{cases} 
 i\hbar \frac{\partial}{\partial t} u(t,x,\theta) = \mathbb{H}(\frac{\hbar}{i} \frac{\partial}{\partial x}, \theta, \frac{\lambda}{i} \frac{\partial}{\partial \theta}) u(t,x,\theta), \\
 u(0,x,\theta) = u(x,\theta).
\end{cases}$$  

Moreover, “complete Weyl symbol” of the right-hand side of the differential operator (7.2.2) is given by

$$\mathcal{H}(\xi, \theta, \pi) = ic\lambda^{-1}(\xi_1 + i\xi_2)\theta_1\theta_2 - ic\lambda k^{-2}(\xi_1 - i\xi_2)\pi_1\pi_2 - ick^{-1}\xi_3(\theta_1\pi_1 + \theta_2\pi_2).$$  

Here, $k \in \mathbb{R}^\times$ or $\in i\mathbb{R}^\times$ ($\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$) is a parameter introduced for Fourier transformations of even and odd variables:

$$ (F_{ev})\xi = (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} dx \ e^{-ih^{-1}(x|\xi)} v(x), $$  

$$ (F_{ew})\xi = (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} d\xi \ e^{ih^{-1}(x|\xi)} w(x), $$  

$$ (F_{ov})(\pi) = k^{n/2} t_n \int_{\mathbb{R}^n} d\theta \ e^{-ik^{-1}(\theta|\pi)} v(\theta), $$  

$$ (F_{ow})(\theta) = k^{n/2} t_n \int_{\mathbb{R}^n} d\pi \ e^{ik^{-1}(\theta|\pi)} w(\pi), $$  

which are explained later in §2 of Chapter 9. Here,

$$ \langle \eta | y \rangle = \sum_{j=1}^{m} \eta_j y_j, \quad \langle \rho | \omega \rangle = \sum_{k=1}^{n} \rho_k \omega_k, \quad t_n = e^{-\pi in(n-2)/4}. $$
More explicitly, we have

\begin{align*}
(Fu)(\xi, \pi) &= c_{m,n} \int_{\mathbb{R}^n} dX e^{-ih^{-1}(X|\Xi)} u(X) = \sum_a [(F_a u_a)(\xi)]\theta^a(\pi), \\
(\mathcal{F} v)(x, \theta) &= c_{m,n} \int_{\mathbb{R}^n} d\Xi e^{ih^{-1}(X|\Xi)} v(\Xi) = \sum_a [(\mathcal{F}_a v_a)(x)]\theta^a(\theta)
\end{align*}

with

\begin{align*}
(X|\Xi) = \langle x|\xi \rangle + 
\end{align*}

Moreover, we have

\begin{align*}
\langle X|\Xi \rangle = \langle x|\xi \rangle + \hbar k^{-1}\langle \theta|\pi \rangle \in \mathcal{R}_{ev}, \quad c_{m,n} = (2\pi \hbar)^{-m/2}k^{n/2}t_n.
\end{align*}

More explicitly, we have

\begin{align*}
\left\{ \begin{array}{l}
\int_{\mathbb{R}^n} d\theta e^{-ik^{-1}\theta s}(u_0 + u_1 \theta) = u_1 - ik^{-1}u_0 \pi, \\
\int_{\mathbb{R}^n} d\pi e^{ik^{-1}\theta s}(u_1 - ik^{-1}u_0 \pi) = -ik^{-1}(u_0 + u_1 \theta).
\end{array} \right.
\end{align*}

(4) Consider classical mechanics or Hamilton flow corresponding to \(H(\xi, \theta, \pi)\)

\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dt} x_j = \frac{\partial H(\xi, \theta, \pi)}{\partial \xi_j}, \\
\frac{d}{dt} \xi_k = -\frac{\partial H(\xi, \theta, \pi)}{\partial x_k} = 0 \quad \text{for} \ j, k = 1, 2, 3, \\
\frac{d}{dt} \theta_l = -\frac{\partial H(\xi, \theta, \pi)}{\partial \theta_l}, \\
\frac{d}{dt} \pi_m = -\frac{\partial H(\xi, \theta, \pi)}{\partial \theta_m} \quad \text{for} \ l, m = 1, 2, 3.
\end{array} \right.
\end{align*}

**Proposition 7.2.1.** Under above setting, for any initial data \((x(0), \xi(0), \theta(0), \pi(0)) = (\bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) \in \mathcal{R}^6|4, (7.2.5) has a unique solution \((x(t), \xi(t), \theta(t), \pi(t))\).

**Remark 7.2.2.** (i) The solution of above Proposition is denoted by \(x(t)\) or \(x(t, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})\), etc.

(ii) Instead of \(\mathcal{R}^{3\frac{1}{2}} \times \mathcal{R}^{3\frac{1}{2}}\), we regard \(\mathcal{R}^6|4\) as the cotangent space \(T^*\mathcal{R}^{3\frac{1}{2}}\) of \(\mathcal{R}^{3\frac{1}{2}}\).

**Inverse mapping:**

**Proposition 7.2.2.** For any \((t, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})\), the map defined by

\begin{align*}
(\bar{x}, \bar{\theta}) \mapsto (\bar{x} = x(t, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\theta} = \theta(t, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}))
\end{align*}

is supersmooth from \(\mathcal{R}^{3\frac{1}{2}}\) to \(\mathcal{R}^{3\frac{1}{2}}\). The inverse map of this, defined by

\begin{align*}
(\bar{x}, \bar{\theta}) \mapsto (x = y(t, x, \xi, \theta, \pi), \theta = \omega(t, x, \xi, \theta, \pi)),
\end{align*}

satisfies

\begin{align*}
\left\{ \begin{array}{l}
\bar{x} = x(t, y(t, x, \xi, \theta, \pi), \bar{\xi}, \bar{\theta}, \bar{\pi}), \\
\bar{\xi} = \omega(t, x, y(t, x, \xi, \theta, \pi))
\end{array} \right.
\end{align*}

**Action integral:** Putting

\begin{align*}
S_0(t, x, \xi, \theta, \pi) = \int_0^t ds \{\langle \dot{x}(s)|\xi(s) \rangle + \langle \dot{\theta}(s)|\pi(s) \rangle - H(x(s), \xi(s), \theta(s), \pi(s))\},
\end{align*}

and

\begin{align*}
S(t, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) = \langle \bar{x}|\xi \rangle + \hbar k^{-1}\langle \bar{\theta}|\pi \rangle + S_0(t, x, \xi, \theta, \pi) \bigg|_{x=y(t, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\theta}=\omega(t, x, \xi, \theta, \pi)}
\end{align*}

we have
Proposition 7.2.3. Above defined $S(t, x, \xi, \bar{\theta}, \bar{\pi})$ is represented

$$S(t, x, \xi, \bar{\theta}, \bar{\pi}) = \langle \bar{x} | \xi \rangle + [\xi \cos (ck^{-1}t|\xi|) - i\xi_3 \sin (ck^{-1}t|\xi|)]^{-1} \times \left[ \hbar k^{-1}|\xi| \langle \bar{\theta} | \pi \rangle - i\lambda^{-1}k \sin (ck^{-1}t|\xi|)(\xi_1 + i\xi_2)\bar{\theta}_1\bar{\theta}_2 + i\lambda^{-1}(2\hbar k^{-1} - 1) \sin (ck^{-1}t|\xi|)(\xi_1 - i\xi_2)\bar{\pi}_1\bar{\pi}_2 \right].$$

(7.2.9)

Moreover, if $\lambda = i, h = k$, then it satisfies Hamilton-Jacobi equation:

$$\left\{ \frac{\partial}{\partial t} S(t, x, \xi, \bar{\theta}, \bar{\pi}) + H \left( \frac{\partial S}{\partial x}, \bar{\theta}, \frac{\partial S}{\partial \theta} \right) = 0, \right. $$

$$S(0, x, \xi, \bar{\theta}, \bar{\pi}) = \langle \bar{x} | \xi \rangle + \langle \bar{\theta} | \pi \rangle.$$  

(7.2.10)

Then, we put

$$\mathcal{D}(t, x, \xi, \bar{\theta}, \bar{\pi}) = \text{sdet} \begin{pmatrix} \frac{\partial^2 S}{\partial x \partial \xi} & \frac{\partial^2 S}{\partial x \partial \bar{\pi}} \\ \frac{\partial^2 S}{\partial \theta \partial \xi} & \frac{\partial^2 S}{\partial \theta \partial \bar{\pi}} \end{pmatrix}. $$

(7.2.11)

Here, “sdet” stands for super-determinant.

Proposition 7.2.4. By calculation, we have

$$\mathcal{D}(t, x, \xi, \bar{\theta}, \bar{\pi}) = (h^{-1}k)^2|\xi|^2|\xi_3^2|\sin (ck^{-1}t|\xi|) - i\xi_3 \sin (ck^{-1}t|\xi|)]^2. $$

Moreover, if $h = k$, then it satisfies the continuity equation (or the 0th part of transport equation):

$$\left\{ \frac{\partial}{\partial t} \mathcal{D} + \frac{\partial}{\partial x} \left( \mathcal{D} \frac{\partial \mathcal{H}}{\partial \xi} \right) + \frac{\partial}{\partial \bar{\theta}} \left( \mathcal{D} \frac{\partial \mathcal{H}}{\partial \bar{\pi}} \right) = 0, \right. $$

$$\mathcal{D}(0, x, \xi, \bar{\theta}, \bar{\pi}) = 1. $$

(7.2.12)

In the above, the independent variables of $\mathcal{D}$ are $(t, x, \xi, \bar{\theta}, \bar{\pi})$, those of $\partial \mathcal{H}/\partial \xi$ or $\partial \mathcal{H}/\partial \bar{\pi}$ are $(S_\xi, \bar{\theta}, S_\theta)$.

Quantization: Using these, we have a new representation of solution desired as follows:

In this paragraph, we rewrite variables from $(x, \xi, \bar{\theta}, \bar{\pi}) \in \mathbb{R}^{6|4} \sim \mathbb{R}^{3|2} \times \mathbb{R}^{3|2}$ to $(\bar{x}, \bar{\theta}, \bar{\pi})$ as variables in the cotangent space $T^*\mathbb{R}^{3|2}$. We define an operator

$$\langle \mathcal{U}(t)u(x, \theta) = (2\pi h)^{-3/2}k \int d\xi d\bar{\pi} D^{1/2}(t, \bar{x}, \bar{\theta}, \bar{\xi}, \bar{\pi}) e^{ih^{-1}S(t, x, \bar{x}, \xi, \bar{\xi}, \bar{\pi})} \mathcal{F} u(\xi, \bar{\pi}), $$

(7.2.13)

where $\mathcal{F}$ stands for the Fourier transformation defined for functions on the superspace. The function $u(t, x, \theta) = \langle \mathcal{U}(t)u(x, \theta) \rangle$ will be shown as a desired solution for (7.2.3) if $h = k$.

$$\mathcal{H} \left( \frac{h}{i} \frac{\partial}{\partial x}, \theta, \frac{\lambda}{i} \frac{\partial}{\partial \theta} \right) = \hat{\mathcal{H}} $$

(7.2.14)

where $\hat{\mathcal{H}}$ is a (Weyl type) pseudo-differential operator with symbol $\mathcal{H}(\xi, \theta, \pi)$ defined by

$$\mathcal{H}(x, \theta) = (2\pi h)^{-3}k^2 \int d\xi d\pi d\omega \ e^{ih^{-1}(x-y|\xi|+i\omega|\pi|)} \mathcal{H}(\xi, \theta + \omega, \pi) u(y, \omega).$$

(7.2.15)

Proposition 7.2.5. (1) For $t \in \mathbb{R}$, $\mathcal{U}(t)$ is a well defined unitary operator in $\mathcal{L}^2_{\text{SS, ev}}(\mathbb{R}^{3|2})$ if $h = k$ and $|\lambda| = 1$.

(2) (i) $\mathbb{R} \ni t \mapsto \mathcal{U}(t) \in \mathbb{B}(\mathcal{L}^2_{\text{SS, ev}}(\mathbb{R}^{3|2}))$ is continuous.
(ii) $\mathcal{U}(t)\mathcal{U}(s) = \mathcal{U}(t+s)$ for any $t, s \in \mathbb{R}$.

(iii) Put $\lambda = i$. For $u \in \mathcal{C}_{SS, ev,0}(\mathfrak{N}^{3|2})$, we put $u(t, \bar{x}, \bar{\theta}) = (\mathcal{U}(t)u)(\bar{x}, \bar{\theta})$. Then, it satisfies

$$
\begin{align*}
7.2.16 & \begin{cases}
\frac{i\hbar}{\partial t} u(t, \bar{x}, \bar{\theta}) = \hat{\mathcal{H}} u(t, \bar{x}, \bar{\theta}), \\
0 \bar{x}, \bar{\theta}) = \bar{u}(\bar{x}, \bar{\theta}).
\end{cases}
\end{align*}
$$

Finally, we interpret the above theorem with $\hbar = \hbar$ and $|\lambda| = 1$ using the identification maps

$$
7.2.17 \quad \sharp : L^2(\mathbb{R}^3 : \mathbb{C}^2) \rightarrow \mathcal{L}_{SS, ev}(\mathfrak{N}^{3|2}) \quad \text{and} \quad \flat : L^2(\mathbb{R}^3 : \mathbb{C}^2) \rightarrow L^2(\mathbb{R}^3 : \mathbb{C}^2).
$$

That is, remarking $\flat \hat{\mathcal{H}}\sharp \psi = \mathbb{H}\psi$ and putting $\mathcal{U}(t)\psi = \flat \mathcal{U}(t)\sharp \psi$, we have

**Proposition 7.2.6.** (1) For $t \in \mathbb{R}$, $\mathcal{U}(t)$ is a well defined unitary operator in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$.

(2) (i) $\mathbb{R} \ni t \mapsto \mathcal{U}(t) \in \mathbb{B}(L^2(\mathbb{R}^3 : \mathbb{C}^2), L^2(\mathbb{R}^3 : \mathbb{C}^2))$ is continuous.

(ii) $\mathcal{U}(t)\mathcal{U}(s) = \mathcal{U}(t+s)$ for any $t, s \in \mathbb{R}$.

(iii) Put $\lambda = i$. For $\psi \in C^\infty_0(\mathbb{R}^3 : \mathbb{C}^2)$, we put $\psi(t, q) = \flat \mathcal{U}(t)\sharp \psi |_{\mathbb{B} = q}$. Then, it satisfies

$$
\begin{align*}
7.2.18 & \begin{cases}
\frac{i\hbar}{\partial t} \psi(t, q) = \mathbb{H}\psi(t, q), \\
\psi(0, q) = \bar{\psi}(q).
\end{cases}
\end{align*}
$$

**Corollary 7.2.1.** $\mathbb{H}$ is an essentially self-adjoint operator in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$.

**Remark 7.2.3.** Since the free Weyl equation is simple, it is not necessary to use Fréchet-Grassmann algebra with countably infinite Grassmann generators, because I can construct quantities explicitly which are necessary for quantization. Odd variables are symbolically important for presenting the position in matrix structure and for operations being consistent. In this case, odd and even variables are separated without interaction!

### 7.3. Another construction of the solution for H-J equation

In the above explanation, most essential part is to define “phase” function satisfying Hamilton-Jacobi equation. We introduce another method here. Moreover, we seek the reason why we need to put $\hbar = \hbar$ ( $\hbar$ has a physical meaning but $\hbar$ is artificially introduced, on the other hand, if Parseval equality holds for odd Fourier transform, we need $|\kappa| = 1$ in certain case. Curious?).

**Claim 7.3.1.** Let $\mathcal{H}(\xi, \theta, \pi)$ be given in (7.2.1), Hamiltonian function corresponding to free Weyl equation. A solution of the Hamilton-Jacobi equation

$$
7.3.1 \quad S_t + \mathcal{H}(S_x, \theta, S_\theta) = 0 \quad \text{with} \quad S(0, x, \xi, \theta, \pi) = \langle x | \xi \rangle + \hbar k^{-1} \langle \theta | \pi \rangle
$$

is constructed without solving Hamilton equation.

Assuming that the solution $S(t, x, \xi, \theta, \pi)$ of (7.3.1) is even supersmooth, we expand it as

$$
7.3.2 \quad S(t, x, \xi, \theta, \pi) = S_{00} + S_{01} \xi_1 \theta_1 + S_{02} \theta_2 + S_{11} \theta_1 \pi_1 + S_{12} \theta_2 \pi_2 + S_{22} \theta_2 \pi_2 + S_{21} \theta_2 \pi_1 + S_{21} \theta_1 \pi_1 + S_{21} \theta_1 \theta_2 \pi_1 \pi_2.
$$

Here, $(f_{\theta_1})_{\theta_2} = f_{\theta_2 \theta_1}, S_{\theta_2 \theta_1} = S_{\theta_1 \theta_2}(t, x, \xi) = S_{\theta_2 \theta_1}(t, x, \xi, 0, 0), \text{ etc. } \bar{0} = (0, 0), \bar{1} = (1, 1) \in \{0, 1\}^2, S_{11} = S_{\pi_2 \pi_1 \theta_1 \theta_1}$.
Lemma 7.3.1. Let $S$ be a solution of (7.3.1). Putting $\widetilde{H}_* = H_*(S_x(t, x, \xi, \theta, \pi), \theta, S_\theta(t, x, \xi, \theta, \pi))$ and $\widetilde{H}_0 = \widetilde{H}_*\big|_{\theta=\pi=0}$, then terms $S_{\pi x^1}$ in (7.3.2) satisfy

\begin{align}
S_{\theta^2 t^2} + \widetilde{H}_0 & = 0 \quad \text{with} \quad S_{\theta^2 t^2}(0, x, \xi) = \langle x | \xi \rangle, \\
S_{\theta^2 t^1} + \partial_{\theta^2 t^1} S_{\theta^2 t^1}^2 + (\widetilde{H}_{\pi^2^1}^0 + \widetilde{H}_{\pi^2^2}^0)S_{\theta^2 t^1} + \partial_{\theta^2 t^1}^0 = 0 \quad \text{with} \quad S_{\theta^2 t^1}(0, x, \xi) = 0, \\
S_{\pi^2 t^2} + (\widetilde{H}_{\pi^2^2}^0 + \partial_{\pi^2^2})S_{\pi^2 t^2}^0 = 0 \quad \text{with} \quad S_{\pi^2 t^2}(0, x, \xi) = 0, \\
S_{\pi^2 t^1} + (\partial_{\pi^2^1} + \partial_{\pi^2^1})S_{\pi^2 t^1}^0 = 0 \quad \text{with} \quad S_{\pi^2 t^1}(0, x, \xi) = 0, \\
S_{\pi^2 t^2} + (\partial_{\pi^2^2} + \partial_{\pi^2^2})S_{\pi^2 t^2}^0 = 0 \quad \text{with} \quad S_{\pi^2 t^2}(0, x, \xi) = 0, \\
S_{\pi^2 t^1} + (\partial_{\pi^2^1} + \partial_{\pi^2^1})S_{\pi^2 t^1}^0 = 0 \quad \text{with} \quad S_{\pi^2 t^1}(0, x, \xi) = 0.
\end{align}

Remark: Terms $S_{\pi^2^2}$ and $S_{\pi^2^3}$ will be checked soon later.

Proof: Restricting (7.3.1) to $\theta = \pi = 0$, we get (7.3.3). Differentiating (7.3.1) w.r.t $\theta$ and then $\theta$ and restricting to $\theta = \pi = 0$, we get Riccati type ODE (7.3.4) with parameter $(x, \xi)$. In fact, from the differential formula for composite functions and remarking $\partial_{\theta^j} \partial_{\theta^j} S = 0$, $H_{\xi_k \xi_j} = 0$, we get

$$
\partial_{\theta^j} \tilde{H} = \frac{\partial S_{\pi^1 t^1}}{\partial \theta^1} \tilde{H}_{\xi_j} + \frac{\partial S_{\pi^2 t^1}}{\partial \theta^1 \partial \theta^1} \tilde{H}_{\xi_j} + \frac{\partial S_{\pi^2 t^1}}{\partial \theta^1 \partial \theta^1} \tilde{H}_{\pi^1 \theta^1},
$$

and

$$
\partial_{\theta^j} \tilde{H}_{\xi_j} = \tilde{H}_{\theta^j \xi_j} + \frac{\partial S_{\pi^2 t^1}}{\partial \theta^1 \partial \theta^1} \tilde{H}_{\pi^1 \xi_j}.
$$

Remark also $\tilde{H}_{\xi_j^2} = 0$, $\tilde{H}_{\xi_j^2 \xi_k} = 0$, etc, and restricting $\partial_{\theta^j} \partial_{\theta^j} \tilde{H}$ to $\theta = \pi = 0$, we have (7.3.7).

Analogously, from

$$
\partial_{\pi^1} \tilde{H}_{\xi_j} = \frac{\partial^2 S_{\pi^1 t^1}}{\partial \pi^1 \partial \theta^1} \tilde{H}_{\xi_j} - \frac{\partial S_{\pi^1 t^1}}{\partial \pi^1} \partial_{\pi^1} \tilde{H}_{\xi_j} + \frac{\partial S_{\pi^2 t^1}}{\partial \pi^1 \partial \theta^1} \tilde{H}_{\xi_j} + \frac{\partial S_{\pi^2 t^1}}{\partial \theta^1 \partial \theta^1} \tilde{H}_{\pi^1 \xi_j},
$$

and $\tilde{H}_{\pi^2 t^1} = 0 = \tilde{H}_{\pi^1 \theta^1}$ with

$$
\partial_{\pi^1} \tilde{H}_{\xi_j} = \frac{\partial S_{\pi^1}}{\partial \pi^1} \tilde{H}_{\xi_j}, \quad \partial_{\pi^1} \tilde{H}_{\theta^1} = \frac{\partial S_{\pi^1}}{\partial \pi^1} \tilde{H}_{\xi_j \theta^1} + \frac{\partial S_{\pi^1}}{\partial \pi^1} \tilde{H}_{\pi^1 \theta^1},
$$

we get (7.3.8). Other equations (7.3.6)-(7.3.8) are obtained analogously. \qed

Lemma 7.3.2. Regarding $x$ and $\xi$ as parameters, we put

$$
z = \xi_1 + i \xi_2, \quad \bar{z} = \xi_1 - i \xi_2, \quad |\xi|^2 = \sum_{j=1}^3 \xi_j^2 = |z|^2 + \xi_3^2, \quad \gamma_t = tk^{-1}|\xi|, \quad \delta_t = |\xi| \cos \gamma_t - i \xi_3 \sin \gamma_t.
$$
Solving ODEs in the above Lemma, we get

\[ (7.3.9) \quad S_{00}(t, x, \xi) = \langle x | \xi \rangle, \]

\[ (7.3.10) \quad S_{\theta_2 \theta_1} = \frac{-k \sin \gamma t}{\delta t}, \]

\[ (7.3.11) \quad S_{\pi_1 \theta_1} = S_{\pi_2 \theta_2} = \frac{hk^{-1} \xi}{\delta t}, \]

\[ (7.3.12) \quad S_{\pi_1 \theta_2} = S_{\pi_2 \theta_1} = 0. \]

**Proof.** From (7.3.2), using \( S_{\theta} \mid \theta = \pi = 0 = 0 \), we get \( \tilde{H}^0 = 0 \) and so (7.3.9) is obvious. Putting this into (7.3.3), we get

\begin{align*}
\tilde{H}_{\pi_2 \pi_1}^0 &= k^{-2}(S_{x_1} - iS_{x_2}) \mid \theta = \pi = 0 = k^{-2}(\xi_1 - i\xi_2) = k^{-2}\bar{z}, \\
\tilde{H}_{\pi_1 \theta_1}^0 &= -ik^{-1}S_{x_3} \mid \theta = \pi = 0 = \tilde{H}_{\pi_2 \theta_2}^0 = -ik^{-1}\xi_3, \\
\tilde{H}_{\theta_2 \theta_1}^0 &= (S_{x_1} + iS_{x_2}) \mid \theta = \pi = 0 = \xi_1 + i\xi_2 = z
\end{align*}

and (7.3.3) becomes

\[ (7.3.13) \quad S_{\theta_2 \theta_1, t} + k^{-2}\bar{z}S_{\theta_2 \theta_1}^2 - 2ik^{-1}\xi_3S_{\theta_2 \theta_1} + z = 0 \quad \text{with} \quad S_{\theta_2 \theta_1}(0, x, \xi) = 0. \]

**Solving ODE of Riccati type:** For a given ODE

\[ y' = q_0(t) + q_1(t)y + q_2(t)y^2, \]

assuming \( q_2 \neq 0 \) and putting \( v = q_2(t)y \), we define

\[ P = q_1 + \frac{q'}{q_2}, \quad Q = q_2q_0 \]

then \( v' = (q_2(t)y)' \) is calculated as

\[ v' = v^2 + P(t)v + Q(t). \]

Moreover, differentiating \( v = -\frac{u'}{u} \) w.r.t. \( t \), we get

\[ u'' - P(t)u' + Q(t)u = 0. \]

Solving this and using \( u \), we put \( y = -\frac{u'}{q_2u} \), then this is a solution of ODE of Riccati type.

**Problem 7.3.1.** If \( q_2(t) \) has 0, then is there any explicit formula?

Using this, we calculate (7.3.13). Putting \( q_0 = -z, q_1 = 2ik^{-1}, q_2 = -k^{-2}\bar{z}, \) we get

\[ \ddot{u} - 2ik^{-1}\dot{u} + k^{-2}|z|^2u = 0 \]

and defining \( \lambda_{\pm} = ik^{-1}(\xi_3 \pm |\xi|) \), we have

\[ u(t) = \alpha e^{\lambda_+ t} + \beta e^{\lambda_- t}. \]

Therefore, using

\[ S_{\theta_2 \theta_1}(0) = \frac{\dot{u}}{k^{-2}\bar{z}u} = 0, \]

and

\[ \dot{u}(0) = 0 = ik^{-1}[\alpha(\xi_3 + |\xi|) + \beta(\xi_3 - |\xi|)] \]
we finally get the desired result (7.3.10),

\[
S_{\theta_2 \theta_1}(t) = \frac{\alpha(\xi_3 + |\xi|)e^{ik^{-1}|\xi|t} + \beta(\xi_3 - |\xi|)e^{-ik^{-1}|\xi|t}}{\alpha e^{ik^{-1}|\xi|t} + \beta e^{-ik^{-1}|\xi|t}} \frac{-kz \sin \gamma t}{|\xi| \cos \gamma t - i\xi_3 \sin \gamma t}.
\]

Using this result and putting

\[ w_0(t, x, \xi) = 0 \]

we get

\[
S_{\theta_2 \theta_1}(t, x, \xi) = \frac{\alpha(\xi_3 + |\xi|)e^{ik^{-1}|\xi|t} + \beta(\xi_3 - |\xi|)e^{-ik^{-1}|\xi|t}}{\alpha e^{ik^{-1}|\xi|t} + \beta e^{-ik^{-1}|\xi|t}} \frac{-kz \sin \gamma t}{|\xi| \cos \gamma t - i\xi_3 \sin \gamma t}.
\]

\[ w_0(t, x, \xi) = 0 \]

we get

\[
S_{\pi_1 \theta_1}(t, x, \xi) = \frac{\alpha(\xi_3 + |\xi|)e^{ik^{-1}|\xi|t} + \beta(\xi_3 - |\xi|)e^{-ik^{-1}|\xi|t}}{\alpha e^{ik^{-1}|\xi|t} + \beta e^{-ik^{-1}|\xi|t}} \frac{-kz \sin \gamma t}{|\xi| \cos \gamma t - i\xi_3 \sin \gamma t}.
\]

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7. FUNDAMENTAL SOLUTION OF FREE WEA YL EQUATION À LA FEYNMAN

\[ \partial_{\pi_1} \partial_{\pi_1} \hat{H}_{\xi_j} = S_{\pi_1 \theta_1} S_{\pi_2 \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j}, \]

\[ \partial_{\pi_2} (\hat{H}_{\theta_2 \xi_j} + \frac{\partial S_{\theta_2}}{\partial \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j}) = \frac{\partial S_{\theta_2}}{\partial \theta_2} \hat{H}_{\pi_2 \theta_2 \xi_j} + \frac{\partial S_{\theta_1}}{\partial \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j} + \cdots, \]

\[ \partial_{\pi_1} \hat{H}_{\xi_j \theta_1} = \frac{\partial S_{\theta_1}}{\partial \pi_1} \hat{H}_{\pi_2 \pi_1 \xi_j}, \]

\[ \partial_{\pi_2} \partial_{\pi_1} \hat{H}_{\theta_2 \theta_1} = \frac{\partial^2 S_{x_j}}{\partial \theta_2 \partial \pi_1} \hat{H}_{\xi_j \theta_2 \theta_1}, \]

\[ \partial_{\pi_2} \partial_{\pi_1} \hat{H}_{\theta_2 \theta_1} = \frac{\partial^2 S_{x_j}}{\partial \theta_2 \partial \pi_1} \hat{H}_{\pi_2 \pi_1 \xi_j}, \]

\[ \partial_{\pi_2} \partial_{\pi_1} \frac{\partial S_{\pi_1}}{\partial \pi_2 \partial \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j} + \frac{\partial S_{\theta_2}}{\partial \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j} \]

\[ = -\frac{\partial^2 S_{x_j}}{\partial \theta_2 \partial \pi_1} \hat{H}_{\pi_2 \pi_1 \xi_j} + \frac{\partial^2 S_{x_j}}{\partial \pi_2 \partial \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j} + \frac{\partial^3 S_{\theta_2}}{\partial \pi_2 \partial \pi_1} \hat{H}_{\pi_2 \pi_1 \xi_j} + \frac{\partial S_{\theta_1}}{\partial \theta_2} \frac{\partial^2 S_{x_j}}{\partial \pi_2 \partial \pi_1} \hat{H}_{\pi_2 \pi_1 \xi_j} + \cdots \]

and restricting to \( \theta = \pi = 0, \)

\[ \hat{H}_{11} = S_{\pi_1 \theta_1 x_j} S_{\pi_2 \theta_1 x_j} \hat{H}_{\pi_2 \pi_1 \xi_j} - S_{\pi_1 \theta_1 x_j} (S_{\pi_2 \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j} + S_{\theta_2 \theta_1} S_{\pi_2 \theta_2} \hat{H}_{\pi_2 \pi_1 \xi_j}) \]

\[ - S_{\pi_2 \theta_2 x_j} S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \theta_2 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} \]

\[ - S_{\pi_1 \theta_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} - (S_{\pi_2 \theta_2 x_j} S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \theta_2 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j}) \]

\[ = S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} - (S_{\pi_2 \theta_2 x_j} S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \theta_2 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j}) \]

\[ = S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} - S_{\pi_1 \theta_1 x_j} S_{\pi_2 \theta_2 x_j} S_{\pi_1 \theta_1} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \theta_2 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} \]

\[ - S_{\pi_1 \theta_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} - S_{\pi_1 \theta_1 x_j} S_{\pi_2 \theta_2 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \theta_2 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j} + S_{\pi_2 \pi_1 x_j} \hat{H}_{\pi_1 \pi_1 \xi_j}. \]

From (7.3.17) and (7.3.18),

\[ S_{\pi_2 \pi_1} (t, x, \xi) = -h^2 a^2 k^{-1} \xi \sin \gamma \frac{\delta t}{\delta t}, \quad S_{11} (t, x, \xi) = 0 \]

Finally

\[ S(t, x, \theta, \xi, \pi) = \langle x | \xi \rangle + \left[ | \xi \rangle \cos (k^{-1} t | \xi \rangle) - i \xi_j \sin (k^{-1} t | \xi \rangle) \right]^{-1} [h k^{-1} | \xi \rangle \langle \theta | \pi] \]

\[ - k \sin (k^{-1} t | \xi \rangle) (\xi_j + i \xi_2) \hat{\theta}_1 \hat{\theta}_2 - h^2 k^{-3} \sin (k^{-1} t | \xi \rangle) (\xi_1 - i \xi_2) \pi_1 \pi_2] \]

**Remark 7.3.1.** (1) From above calculation, we needn’t assume that \( h k^{-1} = 1. \) Since I’m not sure whether there exists mistakes there, but I try to report the reason in the revised version of lecture notes. (I suppose that to calculate classical quantities, we may move \( h \) and \( k \) independently, but to get the desired object after quantization, we need some relation between \( h \) and \( k. \))

(2) Theoretically, we may calculate even when \( A_j(q) \) are added, but we mayn’t have explicit formula of \( S_{11} (t, x, \xi). \) Especially when \( A_j(t, q) \) depends on \( t, \) we have only the existence of the solution of ODE of Riccati type.

From this, we get
Proposition 7.3.1. Van Vleck determinant is calculated as

\[ D(t, \bar{x}, \xi, \bar{\theta}, \pi) = \text{sdet} \left( \begin{array}{ccc} \frac{\partial^2 S}{\partial \bar{x} \partial \xi} & \frac{\partial^2 S}{\partial \bar{x} \partial \pi} \\ \frac{\partial^2 S}{\partial \bar{\theta} \partial \xi} & \frac{\partial^2 S}{\partial \bar{\theta} \partial \pi} \end{array} \right) = (h^{-1}k)^2 |\xi|^2 \left[ |\xi| \cos (ck^{-1}t|\xi|) - i\frac{\xi}{2} \sin(c k^{-1}t|\xi|) \right]^2. \]

Moreover, if \( h = k \), it satisfies

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} D + \frac{\partial}{\partial \bar{x}} (D \frac{\partial H}{\partial \xi}) + \frac{\partial}{\partial \bar{\theta}} (D \frac{\partial H}{\partial \pi}) = 0, \\
D(0, \bar{x}, \xi, \bar{\theta}, \pi) = 1.
\end{array} \right.
\]

Here, independent variables in \( D \) is \((t, \bar{x}, \xi, \bar{\theta}, \pi)\), those for \( \frac{\partial H}{\partial \xi} \) and \( \frac{\partial H}{\partial \pi} \) are \((S_{\bar{x}}, \bar{\theta}, S_{\bar{\theta}})\).

To quantize, it seems better to change the order of independent variables from \((\bar{x}, \xi, \bar{\theta}, \pi)\) to \((x, \bar{\theta}, \xi, \pi)\).

Defining operator

\[
(U(t)u)(x, \bar{\theta}) = (2\pi h)^{-3/2} k \int d\xi d\pi D^{1/2}(t, \bar{x}, \xi, \bar{\theta}, \pi) e^{i h^{-1}S(t, x, \bar{\theta}, \xi, \pi)} F u(\xi, \pi),
\]

then we get that \( u(t, x, \bar{\theta}) = (U(t)u)(x, \bar{\theta}) \) satisfies free Weyl equation when \( h = k \).
CHAPTER 8

Supersymmetric Quantum Mechanics and Its Applications

8.1. What is SUSYQM

8.1.1. Another interpretation of the Atiyah-Singer index theorem. Seemingly, being stimulated by a physicist E. Witten’s paper [134], a mathematician E. Gezler declared in the introduction of his paper [53] that

The Atiyah-Singer index theorem is nothing but the superversion of the Weyl’s theorem on the asymptotic behavior w.r.t. time \( t \) for \( e^{t\Delta_g/2} \).

Here, \((M, g)\) is a compact \( d \)-dimensional Riemannian manifold, \( \Delta_g \) is the Laplace-Beltrami operator corresponding to the Riemannian metric \( g = g_{jk}(q) dq^j dq^k \). Though, he declared this, but he didn’t try to demonstrate this assertion directly in that paper.

Our goal in this chapter. We interprete his declaration and calculate the index for the simplest example following prescription of Witten and Gezler.

Roughly speaking, his declaration is sketched as follows: Let \( K(t, q, q') \) be the kernel of the fundamental solution of IVP

\[
\frac{\partial}{\partial t} v(t, q) = \frac{1}{2} \Delta_g v(t, q) \quad \text{with} \quad \lim_{t \to 0} v(t, q) = \varphi(q).
\]

That is,

\[
v(t, q) = \int_M d_g q' K(t, q, q') \varphi(q') = (e^{t\Delta_g/2} \varphi)(q)
\]

where

\[
g(q) = \det(g_{ij}(q)), \quad d_g q = \sqrt{g(q)} dq, \quad \Delta_g = d^* d = \frac{1}{\sqrt{g(q)}} \partial_q^i (\sqrt{g(q)} g^{ij}(q) \partial_q^j).
\]

Then, the Weyl’s theorem states that \( e^{t\Delta_g/2} \) belongs to trace class and

\[
\text{tr} (e^{t\Delta_g/2}) = \int_M d_g q K(t, q, q) \to C t^{-d/2} \int_M d_g q 1 \quad \text{when} \quad t \to 0.
\]

Here, \( d \) is the exterior differential, \( d^* \) is the adjoint of \( d \) w.r.t. \( d_g q \) and \( \Delta_g = d^* d \) which is desired to be derived from \( \sum_{j,k=1}^d g^{jk}(q) p_j p_k \in C^\infty(T^* M : \mathbb{R}) \) by “quantization”.

His claim goes as follows: Extend \((M, g)\) superly denoted by \((\tilde{M}, \tilde{g})\) where \( \tilde{M} \) is a supermanifold corresponding to \( M \) and \( \tilde{g} \) is a super Riemannian metric of \( g \). In this case, \( \Delta_g \) corresponds to the form Laplacian \( dd^* + d^* d \) acting on differential forms on \((M, g)\), moreover, it has the supersymmetric structure. Therefore, calculating the trace of the kernel for \( e^{t(\Delta_g + d^* d)/2} \), we get the Witten index which gives us new proof of Atiyah-Singer index theorem.

\[\text{For simplicity, we only consider the case } M = \mathbb{R}^d \text{ in this section}\]

\[\text{For Riemann metric } g = g_{ij}(q) dq^i dq^j \text{ on } \mathbb{R}^d, \text{ we calculate its super-extension in §6 of Chapter 9}\]
Remark 8.1.1. (i) What occurs when we quantize Lagrangian \((1/2) \sum_{j,k=1}^{d} g_{jk}(\bar{q}) \dot{q}^j \dot{q}^k\) on \((M,g)\)? In case if we quantize following Feynman’s prescription with purely imaginary time, the quantized object deviate \((1/2)R\) from \((1/2)\Delta_g\) with \(R=\)the scalar curvature (see B. DeWitt [33], Inoue-Maeda [74]).

(ii) See also, the recent work of Y. Miyanishi [97], where he constructs a parametrix for the Schrödinger equation on \(S^2\) with action integral deformed with \((1/2)R\) from \((1/2)\Delta_g\), but \(R=2\) for \(S^2\). More precisely, he goes as follows;

Let \(\bar{q}, \bar{q}\) be 2 points on \(S^2\), let \(\gamma_0 \in C_{t,q,\bar{q}}\) be the shortest path between them with length \(d(q, \bar{q})\). Taking a bump function \(\chi\) with compact support contained in \(d(q, \bar{q}) < \pi\), he defines an integral operator

\[ U(t)u(q) = \frac{1}{2\pi i\hbar} \int_{S^2} dq \chi(d(q, \bar{q})) A(t, q, \bar{q}) e^{i(h(S(t, q, \bar{q}) + iRt/12)} u(q) \]

where

\[ S(t, q, \bar{q}) = \frac{d(q, \bar{q})^2}{2t} \quad \text{and} \quad A(t, q, \bar{q}) = \left[ g^{-1/2}(q) g^{-1/2}(\bar{q}) \det \left( \frac{\partial^2 S(t, q, \bar{q})}{\partial q \partial \bar{q}} \right) \right]^{1/2}. \]

Then, he asserts that taking the suitable products of these operators corresponding to time slicing method and restricting it to “lower energy” part of \((1/2)\Delta_g\), then it converges to the solution of

\[ i\hbar \frac{\partial u(t, \bar{q})}{\partial t} = \frac{\hbar^2}{2} \Delta_g u(t, \bar{q}) \quad \text{with} \quad \lim_{t \to 0} u(t, \bar{q}) = u(\bar{q}). \]

His definition of the integral operator is different from ours because he needs to introduce additionally the cut off and to use not only the action integral but also van Vleck determinant corresponding to the shortest path between two points [4]. By the way, how to recognize the claim “put equal weights for every possible paths” in physics literature? From my point of view, if we consider “weights” as amplitude, we need to use

\[ \left[ \det \left( \frac{\partial^2 S(\gamma)}{\partial q \partial \bar{q}} \right) \right]^{1/2} \quad \text{with} \quad \text{for any} \quad \gamma \in C_{t,q,\bar{q}} \]

or need another phase factor for each path as proposed in L. Schulman [112].

The usage of the projection to low energy part corresponding to the spectral decomposition for \((1/2)\Delta_g\) make us suspicious “Is his procedure truly quantization?”, because the quantization should be carried out only using classical quantities. To overcome this point, it seems reasonable to present such projector using classical objects like Fujiwara [47]. Moreover, he adds also the factor \(g^{-1/2}(q) g^{-1/2}(\bar{q})\) to permit Copenhagen interpretation, that is, consider time evolution in the intrinsic Hilbert space (=half density bundle).

In spite of these, we have [3]

\[ i\hbar \frac{\partial}{\partial t} \left[ \int_{S^2} dq \chi \left[ \det \left( \frac{\partial^2 S(t, q, \bar{q})}{\partial q \partial \bar{q}} \right) \right]^{1/2} e^{ih^{-1}S(t, q, \bar{q})} u(q) \right] \bigg|_{t=0} = \hbar^2 \left[ \frac{1}{2} \Delta_g + \frac{1}{12} R \right] u(\bar{q}). \]

That is, this guarantees us Feynman’s picture of quantization. Therefore, it seems more natural to consider separated two things, one is quantization process and another is construction of

4In our case considered, we only have the unique classical trajectory!
5no problem for integrability and differentiation under integral sign for this case
6From Feynman’s introduction and Fujiwara’s procedure, we, at least myself, insist too much to get quantized object from Feynman picture implies also to have directly fundamental solution of Schrödinger equation by his method
8.1. WHAT IS SUSYQM

the fundamental solution for evolution equation corresponding to that quantized object having as infinitesimal operator.

**Remark 8.1.2.** As mentioned before, how do we interpret the saying “put equal weights for every possible paths”: it is explained, for example, in D. V. Perepelitsa [100] as follows (with slight modification):

Feynman [42] posits that the contribution to the propagator from a particular trajectory is \( \exp[\i \hbar^{-1} S(\gamma)] \) where \( \gamma = \gamma(\cdot) \in C_{\vec{q}, \vec{q}} \). That is, every possible path contributes with equal amplitude to the propagator, but with a phase related to the classical action. Summing over all possible trajectories, we arrive at the propagator. The normalization constant \( A(t) \) is independent of any individual path and therefore depends only on time.

\[
U(\vec{q}, t; \vec{q}, 0) = A(t) \sum_{\gamma \in C_{\vec{q}, \vec{q}}} e^{\i \hbar^{-1} S(\gamma)}.
\]

As there doesn’t exist full Feynman measure, we “approximate” \( D_F \gamma \) on \( C_{\vec{q}, \vec{q}} \) by the measure on \( M \) with some density function, that is,

\[
D(\gamma) = \left[ \det \left( \frac{\partial^2 \tilde{S}(t, \vec{q}, \vec{q})}{\partial \vec{q} \partial \vec{q}} \right) \right]^{1/2} = D(t, \vec{q}, \vec{q}),
\]

where \( S(\gamma) = S(t, \vec{q}, \vec{q}) = \int_0^t dt L(\gamma(s), \dot{\gamma}(s)) \), but even taking the classical trajectory \( \gamma_c \in C_{\vec{q}, \vec{q}} \) in \( D(\gamma_c) \), it generally depends not only on \( t \) but also \( (\vec{q}, \vec{q}) \)?

### 8.1.2. What is SUSYQM?

In order to make clear what should be calculated, we cite the definition.

**Definition 8.1.1 (p.120, H.L. Cycon, R.G. Froese, W. Kirsh and B. Simon [27]).** Let \( \mathcal{H} \) be a Hilbert space and let \( H \) and \( Q \) be selfadjoint operators, and \( P \) be a bounded self-adjoint operator in \( H \) such that

\[
H = Q^2 \geq 0, \quad P^2 = I, \quad [Q, P]_+ = QP + PQ = 0.
\]

Then, we say that the system \( (H, P, Q) \) has supersymmetry or it defines a SUSYQM (=supersymmetric Quantum Mechanics).

Under this circumstance, we may decompose

\[
\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f \quad \text{where} \quad \mathcal{H}_f = \{ u \in \mathcal{H} | Pu = -u \}, \quad \mathcal{H}_b = \{ u \in \mathcal{H} | Pu = u \}.
\]

Using this decomposition and identifying an element \( u = u_b + u_f \in \mathcal{H} \) as a vector \( \begin{pmatrix} u_b \\ u_f \end{pmatrix} \), we have a representation

\[
P = \begin{pmatrix} I_b & 0 \\ 0 & -I_f \end{pmatrix} = (\text{or simply denoted by}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Since \( P \) and \( Q \) anti-commute and \( Q \) is self-adjoint, \( Q \) has always the form

\begin{equation}
(8.1.1) \quad Q = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix},
\end{equation}

\[\]

### \[\]

Recall also, there doesn’t exist full quantization, see Abraham-Marsden [2].
where $A$, called the annihilation operator, is an operator which maps $\mathcal{H}_b$ into $\mathcal{H}_f$, and its adjoint $A^*$, called the creation operator, maps $\mathcal{H}_f$ into $\mathcal{H}_b$. Thus, $P$ commutes with $H$, and $\mathcal{H}_b$ and $\mathcal{H}_f$ are invariant under $H$, i.e. $H\mathcal{H}_b \subset \mathcal{H}_b$ and $H\mathcal{H}_f \subset \mathcal{H}_f$. That is, there is a one-to-one correspondence between densely defined closed operators $A$ and self-adjoint operators $Q$ (supercharges) of the above form.

**Definition 8.1.2.** We define a supersymmetric index of $H$ if it exists by

$$\text{ind}_s(H) \equiv \dim(\text{Ker}(H|_{\mathcal{H}_b})) - \dim(\text{Ker}(H|_{\mathcal{H}_f})) \in \mathbb{Z} = \mathbb{Z} \cup \{±\infty\}.$$ 

On the other hand, we have

**Definition 8.1.3.** Let $X, Y$ be two Banach spaces and let $\mathcal{C}(X,Y)$ be a set of densely defined closed operators from $X$ to $Y$. $T \in \mathcal{C}(X,Y)$ is called Fredholm iff the range of $T$, $R(T)$, is closed in $Y$ and both $\ker T$ and $Y/R(T)$ are finite-dimensional. $T \in \mathcal{C}(X,Y)$ is called semi-Fredholm iff $R(T)$ is closed in $Y$ and at least one of $\ker T$ and $Y/R(T)$ is finite-dimensional. If the operator is semi-Fredholm, then the Fredholm index $\text{ind}_F(T) = \dim(\ker T) - \dim(Y/R(T))$ exists in $\overline{\mathbb{Z}}$.

**Corollary 8.1.1.** If the operator $A$ is semi-Fredholm, we have the relation

$$\text{ind}_s(H) = \text{ind}_F(A) \equiv \dim(\ker A) - \dim(\ker A^*).$$

In order to check whether the supersymmetry is broken or unbroken, E. Witten introduced the so-called Witten index.

**Definition 8.1.4.** Let $(H, P, Q)$ be SUSYQM with \textbf{(S.1.1)}.

(I) Putting, for $t > 0$

$$\Delta_t(H) = \text{tr}(e^{-tA^*A} - e^{-tAA^*}) = \text{str}e^{-tH},$$

we define, if the limit exists, the (heat kernel regulated) Witten index $\mathcal{W}_H$ of $(H, P, Q)$ by

$$\mathcal{W}_H = \lim_{t \to \infty} \Delta_t(H).$$

We define also the (heat kernel regulated) axial anomaly $\mathcal{A}_H$ of $(H, P, Q)$ by

$$\mathcal{A}_H = \lim_{t \to 0} \Delta_t(H).$$

(II) Putting, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$\Delta_z(H) = -z \text{ tr} [(A^*A - z)^{-1} - (AA^* - z)^{-1}] = -z \text{ str}(H - z)^{-1},$$

we define the (resolvent regulated) Witten index $\mathcal{W}_R$ of $(H, P, Q)$, if the limit exists, by

$$\mathcal{W}_R = \lim_{z \to 0} \Delta_z(H) \text{ for some } C_0 > 0.$$ 

Similarly, we define the (resolvent regulated) axial anomaly $\mathcal{A}_R$ by

$$\mathcal{A}_R = -\lim_{z \to \infty} \Delta_z(H) \text{ for some } C_1 > 0.$$ 

We have
theorem 8.1.1. Let $Q$ be a supercharge on $\mathfrak{g}$. If $\exp(-tQ^2)$ is trace class for some $t > 0$, then $Q$ is Fredholm and
\[ \text{ind}_t(Q) = \text{ind}_F(Q) = \text{ind}_s(H). \]
If $(Q^2 - z)^{-1}$ is trace class for some $z \in \mathbb{C} \setminus [0, \infty)$, then $Q$ is Fredholm and
\[ \text{ind}_z(Q) = \text{ind}_F(Q) = \text{ind}_s(H). \]

Concerning definitions used in above theorem:

Definition 8.1.5. Let $X$, $Y$ be two Banach spaces and let $C(X,Y)$ be the set of all densely defined closed operators from $X$ to $Y$. $T \in C(X,Y)$ is called Fredholm if it has closed range $R(T)$ in $Y$ and $\ker T$ and $Y/R(T)$ are finite dimension. $T \in C(X,Y)$ is called semi-Fredholm if $R(T)$ is closed in $Y$ and if at least one of $\ker T$ or $Y/R(T)$ is finite dimension.

If $T \in C(X,Y)$ is semi-Fredholm, then Fredholm index $\text{ind}_F(T) = \dim(\ker T) - \dim(Y/R(T))$ exists in $\mathbb{Z}$.

Corollary 8.1.2. If an operator $A$ is semi-Fredholm, we have
\[ \text{ind}_s(H) = \text{ind}_F(A) = S(\ker A) - \dim(\ker A^*). \]

8.1.3. Examples of SUSYQM.

Example 1. [Witten 134] Let $(M,g)$, $g = \sum_{i,j=1}^d g_{ij}(q) dq^i dq^j$ be a $d$-dimensional smooth Riemannian manifold. We put $\Lambda(M) = \bigcup_{k=0}^d \Lambda^k(M)$ or $\Lambda_0(M) = \bigcup_{k=0}^d \Lambda^k_0(M)$, where
\[ \Lambda^k(M) = \{ \omega = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \omega_{i_1 \cdots i_k}(q) dq^{i_1} \wedge \cdots \wedge dq^{i_k} \mid \omega_{i_1 \cdots i_k}(q) \in C^\infty(M : \mathbb{C}) \}, \]
\[ \Lambda^k_0(M) = \{ \omega \in \Lambda^k(M) \mid \omega_{i_1 \cdots i_k}(q) \in C^\infty_0(M : \mathbb{C}) \}, \]
\[ \Lambda_0^k(M) = \{ \omega \in \Lambda^k(M) \mid \| \omega \| < \infty \}. \]

Let $d$ be an exterior differential acting on $\omega_{i_1 \cdots i_k}(q) dq^{i_1} \wedge \cdots \wedge dq^{i_k}$ as
\[ d\omega = \sum_{j=1}^d \frac{\partial \omega_{i_1 \cdots i_k}(q)}{\partial q^j} dq^j \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_k}. \]

$P$ is defined by $P\omega = (-1)^k \omega$ for $\omega \in \Lambda^k(M)$.

Put $\mathfrak{g} = \Lambda(M)$ where $\Lambda(M) = \bigcup_{k=0}^d \Lambda^k(M)$ with $\overline{\Lambda^k(M)}$ is the closure of $\Lambda^k(M)$ in $L^2$-norm $\| \cdot \|$. Denoting the adjoint of $d$ in $\mathfrak{g}$ by $d^*$ and putting
\[ Q_1 = d + d^*, \quad Q_2 = i(d - d^*), \quad H = Q_1^2 = Q_2^2 = dd^* + d^*d, \]
we have that $(H, Q_\alpha, P)$ has the supersymmetry on $\mathfrak{g}$ for each $\alpha = 1, 2$.

Example 1'. [Witten’s deformed Laplacian 134] For any real-valued function $\phi$ on $M$, we put
\[ d_\lambda = e^{-\lambda \phi} de^\lambda \phi, \quad d^*_\lambda = e^{\lambda \phi} d e^{-\lambda \phi} \]
where $\lambda$ is a real parameter. We have $d_\lambda^2 = 0 = \lambda^2.$
\[ Q_{1\lambda} = d_\lambda + d^*_\lambda, \quad Q_{2\lambda} = i(d_\lambda - d^*_\lambda), \quad H_\lambda = d_\lambda d^*_\lambda + d^*_\lambda d_\lambda. \]

Defining $P$ as before, we have the supersymmetric system $(H_\lambda, Q_\alpha, P)$ on $\mathfrak{g}$ for each $\alpha = 1, 2$.

Now, we calculate $H_\lambda$ more explicitly:
\[ H_\lambda = dd^* + d^*d + \lambda^2 (d\phi)^2 + \sum_{i,j=1}^d \lambda \frac{D^2 \phi}{Dq^i Dq^j} [a^i, a^j]_-. \]
Here, the annihilation and creation operators \( a^j \) and \( a^j \), respectively, are defined as follows: For any \( 0 \leq \ell \leq d \) and \( q \in M \),

\[
\begin{align*}
a^j_q dq^{j_1} \land \cdots \land dq^{j_\ell} &= dq^j \land dq^{j_1} \land \cdots \land dq^{j_\ell}, \\
a^\ell_q dq^{j_1} \land \cdots \land dq^{j_\ell} &= \sum_{k=1}^\ell (-1)^k g^{j_k} (q) dq^{j_1} \land \cdots \land dq^{j_k-1} \land dq^{j_k+1} \land \cdots \land dq^{j_\ell}. 
\end{align*}
\]

Then, these give mappings from \( \Lambda(T^* M) \rightarrow \Lambda(T^* M) \), and we get

\[ [a^j_q, a^j_{q'}]_+ = 0, \quad [a^j_q, a^j_{q'}^*]_+ = g^{j_j}(q), \quad [a^j_{q'}, a^j_{q'}^*]_+ = 0. \]

Moreover,

\[ (d\phi)^2 = g^{ij}\frac{\partial \phi}{\partial q^i} \frac{\partial \phi}{\partial q^j}, \quad \frac{D^2 \phi}{Dq^i Dq^j} = \nabla_i \nabla_j \phi, \quad \frac{D\psi_i}{Dt} = \frac{d\psi_i}{dt} - \Gamma^i_{jk} \dot{q}^j \psi^k. \]

[Notation]: For \( \omega = \sum_{1 \leq i_1 < \cdots < i_k \leq d} \omega_{i_1 \cdots i_k} (q) dq^{i_1} \land \cdots \land dq^{i_k} \), we put

\[ (\nabla_j \omega)_{i_1 \cdots i_k} = \frac{\partial \omega_{i_1 \cdots i_k}}{\partial q_j} - \sum_{r,l} \Gamma^l_{ij} \omega_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_k}. \]

Then, we have

\[ \nabla_j = \partial_{q^j} - \sum_{k,l,m} \Gamma^k_{jkl} a^k a^l a^m, \quad d = \sum_{i=1}^d a^i \nabla_i = - \sum_{i=1}^d \nabla_i a^i, \quad d^* = - \sum_{i=1}^d a^i \nabla_i, \]

\[ d^* \omega = \sum_{j,r} (-1)^{r-1} g^{ij} \left[ \frac{\partial \omega_{i_1 \cdots i_k}}{\partial q^j} - \sum_{l,s} \Gamma^l_{ij} \omega_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_k} (q) \right] \times dq^{i_1} \land \cdots \land dq^{i_{r-1}} \land dq^{i_{r+1}} \land \cdots \land dq^{i_k}. \]

The most important thing is to consider the operator \( \mathbf{H}_\lambda \) as the quantized one from the Lagrangian

\[ \mathcal{L}_\lambda = \frac{1}{2} \int dt \left[ g_{ij} \left( \frac{dq^i}{dt} \frac{dq^j}{dt} + i \bar{\psi}^j \frac{D\psi^i}{Dt} \right) + \frac{1}{4} R_{ijkl} \bar{\psi}^i \bar{\psi}^j \psi^k \psi^l - \lambda^2 \left| g^{ij} \frac{\partial \phi}{\partial q^i} \frac{\partial \phi}{\partial q^j} \right|^2 - \lambda \frac{D^2 \phi}{Dq^i Dq^j} \bar{\psi}^i \psi^j \right]. \]

Here, we used the summation convention and \( \psi^j \) and \( \bar{\psi}^j \) are anti-commuting fields tangent to \( M \), which becomes the creation and annihilation operators after quantization. After representing the solution of

\[ \lambda^{-1} \frac{\partial}{\partial t} u(t) = \lambda^{-2} \mathbf{H}_\lambda u(t), \]

and applying the SUSYQM structure, he concludes that the principal term above when \( \lambda \rightarrow \infty (\lambda^{-1} \sim \hbar) \) is governed by the instantons or tunneling paths corresponding to \( \mathcal{L}_\lambda \). That is, those paths are defined by Lagrangean below;

\[ \tilde{\mathcal{L}}_\lambda = \frac{1}{2} \int dt \left[ g_{ij} \left( \frac{dq^i}{dt} \frac{dq^j}{dt} + \lambda \lambda \frac{\partial \phi}{\partial q^i} \frac{\partial \phi}{\partial q^j} \right) \right] \]

\[ = \frac{1}{2} \int dt \left[ \frac{d\phi}{dt} + \lambda \frac{\partial \phi}{\partial q^i} \frac{\partial \phi}{\partial q^j} \right]^2 \right] \right] + \lambda \int dt \frac{d\phi}{dt}. \]

Using these paths with physicists’ steepest descent method, he may calculate Witten index.
Example 2. (Deift [30] in p.123 Cycon et al. [27]). Let $\mathcal{S} = L^2(\mathbb{R}) \otimes \mathbb{C}^2 = L^2(\mathbb{R} : \mathbb{C}^2) = L^2(\mathbb{R})^2$, and $\phi$ be a polynomial in $q$. Set $A = d/dq + \phi(q)$ and $A^* = -d/dq + \phi(q)$ with domains $D(A) = D(A^*) = \{ u \in H^1(\mathbb{R}) \mid \phi u \in L^2(\mathbb{R}) \}$.

\[
Q = \begin{pmatrix} 0 & -d/dq + \phi \\ d/dq + \phi & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then, $D(AA^*) = D(A^*A) = \{ u \in L^2(\mathbb{R}) \mid u'', \phi^2 u, \phi'u \in L^2(\mathbb{R}) \}$,

\[
A^*A = -\frac{d^2}{dq^2} + \phi^2(q) - \phi'(q), \quad AA^* = -\frac{d^2}{dq^2} + \phi^2(q) + \phi'(q),
\]

and

\[
H = Q^2 = \begin{pmatrix} -\frac{d^2}{dq^2} + \phi^2(q) - \phi'(q) & 0 \\ 0 & -\frac{d^2}{dq^2} + \phi^2(q) + \phi'(q) \end{pmatrix}.
\]

This $(H, Q, P)$ forms a SUSYQM in $\mathcal{S}$.

(i) Especially $\phi(q) = q$, we have

\[
\dim(\text{Ker} A) = 1, \quad \dim(\text{Ker} A^*) = 0, \quad \text{ind}_F(A) = 1 = \text{ind}_s(H).
\]

(ii) (Bollé et al. [14]). For real-valued $\phi, \phi' \in L^\infty(\mathbb{R})$, we assume that

\[
\lim_{q \to \pm \infty} \phi(q) = \phi_\pm \in \mathbb{R}, \quad \phi_-^2 \leq \phi_+^2,
\]

\[
\int_\mathbb{R} dq (1 + |q|^2)|\phi'(q)| < \infty \quad \text{and} \quad \int_\mathbb{R} dq (1 + |q|^2)|\phi(q)-\phi_\pm| < \infty.
\]

Then, they assert that, for $z \in \mathbb{C} \setminus [0, \infty)$,

\[
\Delta_z(H) = \frac{1}{2} [\phi_+^2(z) - z]^{-1/2} - \phi_-(z)^{-1/2}], \quad W_R = \frac{1}{2} [\text{sign}(\phi_+) - \text{sign}(\phi_-)] \quad \text{and} \quad A_R = 0.
\]

Remark 8.1.3. Especially, physicists have the above result by calculating the quantity

\[
\Delta_t(H) = \int dq d\psi d\bar{\psi} \int_{\{t\text{-periodic}\}} \left[ dq |d\psi| [d\bar{\psi}] e^{-\int_0^t ds L(q(s),\dot{q}(s),\psi(s),\dot{\psi}(s))} \right],
\]

but it seems difficult to make rigorous their procedure mathematically.

8.2. Super extension of harmonic oscillator and its index

Though arguments so far in this chapter mainly treat (abstract) heat equation, but we need special trick if we use Fourier transformation within path-integral method. Because, if we naively proceed as before, we may have

\[
v(t, \vec{q}) = \frac{1}{\sqrt{2\pi}} \int dp D^{1/2}(t, \vec{q}, p) e^{S(t, \vec{q}, p)} \widehat{\mathcal{L}}(p)
\]

and by Fourier inversion formula, we need to have at $t = 0$,

\[
D^{1/2}(t, \vec{q}, p) \bigg|_{t=0} = 1, \quad e^{S(t, \vec{q}, p)} \bigg|_{t=0} = e^{i\phi}.
\]

Since the second requirement above seems strange because $S(t, \vec{q}, p) \in \mathbb{R}$, therefore we need to reconsider the procedure from scratch.
8.2.1. Harmonic oscillator of quantum type. We construct the fundamental solution of the following IVP:

\[ i\hbar \frac{\partial u}{\partial t} = Hu \quad \text{with} \quad H = H(q, \partial_q) = \frac{1}{2} \left( \frac{\hbar}{i} \frac{\partial}{\partial q} - aq \right)^2 - \frac{1}{2} \omega^2 q^2. \]

The Weyl symbol \( H(q, p) \) of \( H(q, \partial_q) \) is defined by

\[ H(q, p) = e^{-ih^{-1}qp} H(q, \partial_q) e^{ih^{-1}qp} \bigg|_{\hbar = 0} = \frac{1}{2} (p - aq)^2 - \frac{1}{2} \omega^2 q^2, \]

and the Hamilton flow corresponding to \( H(q, p) \) is defined by

\[
\begin{cases}
    \dot{q} = H_p(q, p) = p - aq, \\
    \dot{p} = -H_q(q, p) = \omega^2 q + a(p - aq)
\end{cases}
\]

with \( (q(0), p(0)) = (\underline{q}, \underline{p}) \).

Here, remarking

\[ X = \begin{pmatrix} -a & 1 \\ \omega^2 - a^2 & a \end{pmatrix}, \quad \begin{pmatrix} \omega^2 \Pi_2 \\ \begin{pmatrix} 1 \\ \omega \end{pmatrix} \end{pmatrix}, \quad e^{\omega t} = \cosh \omega t \Pi_2 + \sinh \omega t X, \]

and putting for notational simplicity,

\[ \cosh \omega s = \wp_s, \quad \sinh \omega s = \wp_s, \quad b = \omega^2 - a^2, \]

we have

\[ q(s) = \frac{1}{\omega} \left( (\omega \wp_s - a \wp_s q + \wp_s P) \right), \quad p(s) = \frac{1}{\omega} \left( b \wp_s q + (\omega \wp_s + a \wp_s) p \right). \]

On the other hand, action integral is defined by

\[ S_0(t, \underline{q}, \underline{p}) = \int_0^t ds \left[ q(s)p(s) - H(q(s), p(s)) \right] = \int_0^t ds \left[ \frac{(p(s))^2}{2} + \frac{b q(s)^2}{2} \right] \]

\[ = \frac{S(t)(\omega \wp_t + a \wp_t)}{2 \omega^2} \underline{P}^2 + \frac{b S(t)^2}{\omega^2} \underline{q}^2 + \frac{b S(t)(\omega \wp_t - a \wp_t)}{2 \omega^2} \underline{q}^2. \]

[Q]:L-formulation. In (8.2.3), putting \( q(t) = \underline{q} \) and solving w.r.t. \( \underline{p} \), we have

\[ \underline{p} = p(t, \underline{q}, \underline{q}) = \omega \frac{\wp_t}{\wp_t} \left( \underline{q} - \frac{\omega \wp_t - a \wp_t q}{\wp_t} \right) = \omega \underline{q} - \frac{\omega \wp_t - a \wp_t q}{\wp_t}, \]

Putting this into \( S_0(t, q, \underline{p}) \), we get

\[ S^L(t, \underline{q}, \underline{q}) = \frac{S_0(t, q, p(t, \underline{q}, \underline{q}))}{2 \wp_t} \left( \frac{\wp_t}{\wp_t} \right)^2 - \frac{\omega \wp_t q}{\wp_t} + \frac{\omega \wp_t - a \wp_t q}{2 \wp_t} \]

and also

\[ D^L(t, \underline{q}, \underline{q}) = -\frac{\partial^2 S(t, \underline{q}, \underline{q})}{\partial \underline{q} \partial \underline{q}} = \omega \wp_t. \]

Remark 8.2.1. In d-dimensional case, we have

\[ D^L(t, \underline{q}, \underline{q}) = (-1)^d \det \left( \frac{\partial^2 S(t, \underline{q}, \underline{q})}{\partial \underline{q} \partial \underline{q}} \right). \]

In fact, for free Schrödinger equation in \( \mathbb{R}^d \), as \( S(t, q, q) = \frac{|q - \bar{q}|^2}{2t} \), we have

\[ D^L(t, \underline{q}, \underline{q}) = (-1)^d \det \left( \frac{\partial^2 S(t, \underline{q}, \underline{q})}{\partial \underline{q} \partial \underline{q}} \right) = \left( \frac{1}{t} \right)^d. \]
It is easily checked that $S^L(t, \bar{q}, q)$ and $D^L(t, \bar{q}, q)$ satisfy

\begin{align*}
(8.2.8) \quad & S^L_t + H(\bar{q}, S^L_q) = 0 \quad \text{with} \quad \lim_{t \to 0} \left( S(t, \bar{q}, q) - \frac{(\bar{q} - q)^2}{2t} - \frac{a(q^2 - \bar{q}^2)}{2t} \right) = 0, \\
(8.2.9) \quad & D^L_t + \partial_q(D^L H_p) = 0 \quad \text{with} \quad \lim_{t \to 0} t D^L(t, \bar{q}, q) = 1 \quad \text{where} \quad H_p = H_p(\bar{q}, S^L_q),
\end{align*}

respectively. Then, we put

\begin{equation}
U^L_t \, \mathfrak{u}(\bar{q}) = \frac{1}{\sqrt{2\pi i \hbar}} \int_\mathbb{R} dq \, D^L(t, \bar{q}, q)^{1/2} e^{i\hbar^{-1}S^L(t, \bar{q}, q)} \mathfrak{u}(\bar{q}) = \int_\mathbb{R} dq \, U^L(t, \bar{q}, q) \mathfrak{u}(\bar{q})
\end{equation}

where

\begin{equation}
U^L(t, \bar{q}, q) = \sqrt{\frac{\omega}{2\pi i \hbar s_t}} e^{i\hbar^{-1}S^L(t, \bar{q}, q)} \sim \sqrt{\frac{1}{2\pi i \hbar t}} e^{i\hbar^{-1}(\bar{q} - q)^2/(2t) + a(q^2 - \bar{q}^2)/2} \quad (\omega \to 0).
\end{equation}

$u(t, q) = U^L_t \mathfrak{u}(\bar{q})$ satisfies (8.2.2) with initial data $u(0, q) = \mathfrak{u}(q)$.

**Remark 8.2.2.** In the above, the oscillatory integral is calculated as

\[
\int dx e^{i\alpha x^2} = \lim_{\epsilon \to 0} \int dx e^{-(\epsilon - i\alpha)x^2} = \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{\epsilon - i\alpha}} = \begin{cases} \sqrt{\frac{\pi e^{-\alpha^2}}{\alpha}} & a > 0, \\ \sqrt{\frac{\pi e^{\alpha^2}}{\alpha}} & a < 0. \end{cases}
\]

**[Q]: H-formulation.** On the other hand, putting $q(t) = \bar{q}$ in (8.2.3), we solve w.r.t. $q$ having

\begin{equation}
q = q(t, \bar{q}, p) = \frac{\omega}{\omega \xi_t - a s_t} \left( \bar{q} - \frac{s_t}{\omega} \right) = \frac{\omega \bar{q} - s_t p}{\omega \xi_t - a s_t},
\end{equation}

Using this, we put

\begin{equation}
S^H(t, \bar{q}, p) = \left[ qp + S_0(t, q, p) \right] \bigg|_{q=q(t,\bar{q},p)} = \frac{s_t}{2(\omega \xi_t - a s_t)} (b q^2 - p^2) + \frac{\omega}{\omega \xi_t - a s_t} \bar{p},
\end{equation}

\begin{equation}
D^H(t, \bar{q}, p) = \frac{\partial d S^H(t, \bar{q}, p)}{\partial q} = \frac{\omega}{\omega \xi_t - a s_t}.
\end{equation}

These satisfy

\begin{align*}
(8.2.14) \quad & S^H_t + H(\bar{q}, S^H_q) = 0 \quad \text{with} \quad \lim_{t \to 0} S^H(t, \bar{q}, p) = \bar{p}, \\
& D^H_t + \partial_q(D^H H_p) = 0 \quad \text{with} \quad \lim_{t \to 0} D^H(t, \bar{q}, p) = 1 \quad \text{where} \quad H_p = H_p(\bar{q}, S^H_q).
\end{align*}

Therefore, from these

\begin{equation}
U^H_t \, \mathfrak{u}(\bar{q}) = \frac{1}{\sqrt{2\pi i \hbar}} \int_\mathbb{R} dp \, D^H(t, \bar{q}, p)^{1/2} e^{i\hbar^{-1}S^H(t, \bar{q}, p)} \mathfrak{u}(p) = \int_\mathbb{R} dp \, U^H(t, \bar{q}, p) \mathfrak{u}(p)
\end{equation}

\begin{equation}
= \frac{1}{\sqrt{2\pi i \hbar}} \int_{\mathbb{R}^2} dp \, dq \, \frac{\omega}{\omega \xi_t - a s_t} e^{i\hbar^{-1}(S^H(t, \bar{q}, p) - q p)} \mathfrak{u}(q)
\end{equation}

with

\begin{equation}
U^H(t, \bar{q}, p) = \frac{1}{\sqrt{2\pi i \hbar}} \frac{\omega}{\omega \xi_t - a s_t} e^{i\hbar^{-1}S^H(t, \bar{q}, p)}.
\end{equation}

**Remark 8.2.3.** (1) In the last expression of (8.2.15), we may change the order of integration from $dp \, dq$ to $dp$ and then $dq$, if we regard the integral as oscillatory one. Moreover, remarking

\[
\frac{1}{2\pi i \hbar} \int dp \, \frac{\omega}{\omega \xi_t - a s_t} \exp \left[ i\hbar^{-1} \left( -\frac{s_t}{2(\omega \xi_t - a s_t)} p^2 + \frac{\omega}{\omega \xi_t - a s_t} \bar{p} q - p \bar{p} \right) \right] \exp \left[ i\hbar^{-1} \left( \frac{b s_t}{2(\omega \xi_t - a s_t)} \bar{q}^2 \right) \right]
\]

\[
= \frac{1}{2\pi i \hbar} \sqrt{\frac{\omega}{\omega \xi_t - a s_t}} \sqrt{\frac{2\pi(\omega \xi_t - a s_t)}{i\hbar^{-1}s_t}} e^{i\hbar^{-1}S^L(t, \bar{q}, q)} = \sqrt{\frac{\omega}{2\pi i \hbar s_t}} e^{i\hbar^{-1}S^L(t, \bar{q}, q)},
\]
we get \( U_t^H = U_t^L \).

(2) Here, we use explicitly the Fourier transformation. The origin of the prefactor \((2\pi h)^{-1/2}\) of (8.2.15) is clear by the Fourier inversion formula comparing with the one \((2\pi i h)^{-1/2}\) of (8.2.10) which should be decided such that the kernel \((2\pi i h)^{-1/2}D_{1/2}(t, \vec{q}, \vec{p})e^{i\hbar^{-1}S(t, \vec{q}, \vec{p})}\) converges to \(\delta(q) = \delta(q - \bar{q})\) when \(t \to 0\).

**Remark 8.2.4.** In the last term in (8.2.15), we have 
\[ e^{i\hbar^{-1}S_H(t, \vec{q}, \vec{p}) - \vec{p} \vec{q}}, \]
this gives the key of \(H\)-formulation in Schrödinger equation, but we feel something curious for \(H\)-formulation in heat equation case.

### 8.2.2. Harmonic oscillator of heat type.

Substituting \(h = -i\) and for simplicity, putting \(a = 0\) into (8.2.1), we consider the following heat type equation:

\[
(8.2.16) \quad \frac{\partial v}{\partial t} = Hv \quad \text{with} \quad H = H(q, \partial_q) = \frac{1}{2} \partial_q^2 - \frac{\omega^2}{2} q^2.
\]

**[H]: L-formulation.** “Weyl symbol” \(H(q, p)\) of (8.2.16) is obtained by substituting \(ip\) formally into \(\partial_q\),

\[
(8.2.17) \quad H(q, p) = -\frac{1}{2} p^2 - \frac{\omega^2}{2} q^2.
\]

Corresponding Hamilton equation is given by

\[
\begin{cases}
\dot{q} = -p, \\
\dot{p} = \omega^2 q,
\end{cases}
\]

i.e. \(\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = X \begin{pmatrix} q \\ p \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ \omega^2 & 0 \end{pmatrix} \) with \(X^2 = -\omega^2 id_2\).

Denoting \(C_s = \cos \omega s, S_s = \sin \omega s\), we have the Hamilton flow

\[
\begin{pmatrix} q(s) \\ p(s) \end{pmatrix} = e^{sX} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} C_s q - S_s p \\ C_s p + \omega S_s q \end{pmatrix}.
\]

As

\[
S_0(t, q, p) = \int_0^t ds (\dot{q}p - H(q, p)) = \int_0^t ds \left(-\frac{p^2}{2} + \frac{\omega^2}{2} q^2\right)
\]

\[
= \int_0^t ds \left[\frac{C_s^2 - S_s^2}{2}(-p^2 + \omega^2 q^2) - 2\omega C_s S_s qp\right]
\]

\[
= \frac{C_t S_t}{2\omega}((\omega^2 q^2 - p^2) - S_t^2 qp),
\]

substituting \(p = \frac{\omega}{S_t}(C_t q - \bar{q})\) into above, we get

\[
(8.2.18) \quad S^L(t, \bar{q}, q) = -\frac{\omega C_t}{2S_t} q^2 + \frac{\omega}{S_t} \bar{q}^2 - \frac{\omega C_t}{2S_t} \bar{q}^2 \quad \text{and} \quad D(t, \bar{q}, q) = \frac{\partial^2 S^L(t, \bar{q}, q)}{\partial \bar{q} \partial q} = \frac{\omega}{S_t}.
\]

Therefore, putting

\[
v(t, \bar{q}) = V_t^L v(q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dq D_{1/2}(t, \bar{q}, q) e^{S^L(t, \bar{q}, q)} v(q) = \int_{\mathbb{R}} dq K^L(t, \bar{q}, q) v(q)
\]

where \(K^L(t, \bar{q}, q) = \sqrt{\frac{\omega}{2\pi \sin \omega t}} e^{S^L(t, \bar{q}, q)}\).

we have the solution (8.2.16) of with initial data \(v(0, q) = \sigma(q)\). That is, from Hamilton function (8.2.17), by path-integral method quantization, we have the evolution operator \(V(t)\) with infinitesimal generator (8.2.17).
8.2. SUPER EXTENSION OF HARMONIC OSCILLATOR AND ITS INDEX

Problem 8.2.1. Whether we may represent the solution of a given PDE by using classical mechanical objects corresponding to it, is our problem.

On the other hand, we may compare this with the problem posed by Widom [133]: Let $A^W$ be the self-adjoint operator obtained from a real valued symbol $A(q,p)$ by Weyl quantization whose spectral resolution is given by $A^W = \int dE \lambda$. Taking a function $f$ in the suitable class, we may define, by functional calculus method, $f(A^W) = \int dE f(\lambda)$. In this case, whether $f(A^W)$ is a pseudo-differential operator and how its symbol is represented by, are discussed there. Moreover, let two self-adjoint operators $A^W$ and $B^W$ be given with a function with two variables. Whether $f(A^W, B^W)$ gives a pseudo-differential operator, this is considered in R.S. Strichartz [119]. These consideration is applied to to the system version of Egorov’s theorem in §4 of Chapter 9.

We need to remark the physicist’s usage of analytic continuation w.r.t. time $t$, because it is not so obvious whether the operator like $e^{itA^W}$ is analytically continued to $e^{itA^W}$.

(1) Let a non-negative function $H(q,p)$ with order 2 w.r.t. $p$ having functions $S^L(t, \bar{q}, \bar{q})$ and $D^L(t, \bar{q}, \bar{q})$ on configuration space as solutions of Hamilton-Jacobi and continuity equation, respectively. Taking the normalization constant $C^L$, and defining

$$(T^L_t \hat{v})(\bar{q}) = v^L(t, \bar{q}) = C^L \int \frac{dq}{\text{euc}} \sqrt{D^L(t, \bar{q}, \bar{q})} e^{S^L(t, \bar{q}, \bar{q})} \hat{v}(q),$$

have we a parametrix of corresponding heat type equation

$$\frac{\partial}{\partial t} v^L(t, \bar{q}) = H^W(\bar{q}, \partial_q) v^L(t, \bar{q}) \quad \text{with} \quad v^L(0, \bar{q}) = \hat{v}(q) ?$$

(2) Under the same setting as (1), we define functions $S^H(t, \bar{q}, \bar{p})$ and $D^H(t, \bar{q}, \bar{p})$ on phase space as solutions of Hamilton-Jacobi and continuity equation, respectively. Defining

$$(T^H_t \hat{v})(\bar{q}) = v^H(t, \bar{q}) = C^H \int \frac{dp}{\text{euc}} \sqrt{D^H(t, \bar{q}, \bar{p})} e^{S^H(t, \bar{q}, \bar{p})} \hat{v}(p),$$

whether we have a parametrix of corresponding heat type equation with suitable devices?

Remark 8.2.5. For (1), I show not only simple examples in the previous paragraph, but also construct a quantized operator of Riemann metric in Inoue and Maeda [74]. Concerning (2), I give two simple examples showing it seems hard to get the desired results without some devices.

- As is mentioned at the very beginning of this section, taking the Fourier transformation of

$$v_t = \frac{1}{2} \partial_q^2 v \quad \text{with} \quad v(0) = \hat{v},$$

we get readily

$$v(t, \bar{q}) = \frac{1}{\sqrt{2}} \int dp e^{i\bar{q}p - \frac{1}{2}p^2 t} \hat{v}(p).$$

But, except normalization constant, it seems impossible to have a solution $S(t, \bar{q}, \bar{p})$ of Hamilton-Jacobi equation satisfying

$$e^{i\bar{q}p - \frac{1}{2}p^2 t} = A(t, \bar{q}, \bar{p}) e^{S(t, \bar{q}, \bar{p})} \quad \text{with} \quad A \sim \left( \frac{\partial^2 S}{\partial q \partial p} \right)^{1/2}.$$
Therefore, multiplying imaginary unit “$i$” to both sides of heat equation,
\[ iv_t = i\frac{1}{2} \partial_q^2 v \quad \text{with} \quad v(0) = v \]
and substituting $p$ into $-i\partial_q$, we assign the Hamilton function $\tilde{H}(q, p) = -i\frac{1}{2} p^2$. Then, we have
\[ \tilde{S}_0(t, \tilde{q}, \tilde{p}) = -i\frac{1}{2} \tilde{p}^2 t, \quad \tilde{q} = q - ipt, \]
\[ \tilde{S}(t, \tilde{q}, \tilde{p}) = q\tilde{p} + \tilde{S}_0(t, \tilde{q}, \tilde{p}) \bigg|_{\tilde{q} = \tilde{q} + ipt} = \tilde{q}\tilde{p} + i\frac{1}{2} \tilde{p}^2 t. \]
Putting
\[ v(t, \tilde{q}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dp e^{i\tilde{S}(t, \tilde{q}, \tilde{p})} \tilde{v}(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dp e^{iq\tilde{p} - \frac{1}{2} \tilde{p}^2 t} \tilde{v}(p), \]
we have the desired expression. See also, Qi’s equation in §3, Chapter 9.

Unfortunately, this procedure doesn’t work for harmonic oscillator. In fact, since we have
\[ \hat{H}(q, p) = -\frac{i}{2} (p^2 + \omega^2 q^2), \]
solutions of Hamilton equation are given by
\[ \begin{cases} \dot{q} = \hat{H}_p(q, p) = -ip, \\ \dot{p} = -\hat{H}_q(q, p) = i\omega^2 q \end{cases} \quad (q(s), p(s)) = e^{sX} \left( \begin{array}{c} q \\ p \end{array} \right) \quad \text{with} \quad X = \left( \begin{array}{cc} 0 & -i \\ i\omega^2 & 0 \end{array} \right), \]
there appeared the terms with $\cosh \omega s, \sinh \omega s$. To get the desired one with $\cos \omega s, \sin \omega s$, we need to complexify $\omega$, but no philosophical evidence to do so. Only when $\omega \to 0$, we have the desired result obtained before.

**8.2.3. Spin addition.** Preparing a representation space $V$ with the scalar product $(\cdot, \cdot)$ and two bounded operators $b$ and $b^*$ such that
\[ b^2 = (b^*)^2 = 0, \quad bb^* + b^*b = 0. \]
In stead of ODE $\mathbb{H}(q, \partial_q)$ [§2.16] in $L^2(\mathbb{R})$, we put
\[ Q_- = \frac{1}{\sqrt{2}} (\partial_q - \omega q) b, \quad Q_+ = \frac{1}{\sqrt{2}} (\partial_q + \omega q) b^* \]
and we define
\[ \mathbb{H} = \mathbb{H}(q, \partial_q) = Q^2 = H(q, \partial_q)I + \frac{\omega}{2}[b, b^*], \]
\[ Q = Q_- + Q_+, \quad P = [b, b^*], \]
on $\mathcal{F} = L^2(\mathbb{R}) \otimes V$. Since
\[ P^2 = I, \quad [Q, P]_+ = QP + PQ = 0, \]
$(\mathbb{H}, P, Q)$ gives SUSYQM on $\mathcal{F} = L^2(\mathbb{R}) \otimes V$. Especially, taking
\[ V = \mathbb{C}^2 \quad \text{with} \quad (u, v) = u_1 \bar{v}_1 + u_2 \bar{v}_2 \quad \text{for} \quad u = t(u_1, u_2), v = t(v_1, v_2) \in \mathbb{C}^2, \]
\[ b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [b, b^*] = bb^* - b^*b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
we get an ordinary matrix representation of $\mathbb{H}$ in $\mathcal{F} = L^2(\mathbb{R}) \otimes V$.

We represent these by using an odd variable $\theta$.

Putting $\Lambda = \{u_0 + u_1 \theta \mid u_0, u_1 \in \mathbb{C}\}$, we decompose
\[ \Lambda_b = \{u_0 \mid u_0 \in \mathbb{C}\}, \quad \Lambda_f = \{u_1 \theta \mid u_1 \in \mathbb{C}\}, \quad \Lambda = \Lambda_b \oplus \Lambda_f. \]
Then
\[
\theta(u_0 + u_1 \theta) = u_0 \theta \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad \frac{\partial}{\partial \theta}(u_0 + u_1 \theta) = u_1 \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},
\]
and
\[
[b, b^*] \sim \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \theta} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Remark 8.2.6. Since
\[
(bu, v) = (u, b^* v) \quad \text{for} \quad u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathbb{C}^2,
\]
we may have scalar product \((\cdot, \cdot)\) in \(S = L^2(\mathbb{R}) \otimes V\) such that
\[
(u_0 + u_1 \theta, \theta(v_0 + v_1 \theta)) = \left( \frac{\partial}{\partial \theta}(u_0 + u_1 \theta), v_0 + v_1 \theta \right)
\]
which permits integration by parts w.r.t. \(\theta\). Please refer \(\text{§9.2.19}, \ \text{§2 in Chapter 9}\).

Classical Mechanics corresponding to \(\mathbb{H}(q, \partial_q)\): Prepare \(k \in \mathbb{C} \setminus \{0\}\) and define Fourier transformation for functions of \(\theta \in \mathcal{R}^{01}\) as
\[
\left\{ \begin{array}{l}
\int_{\mathbb{R}^{01}} d\theta e^{-k^{-1} \theta \pi} (u_0 + u_1 \theta) = u_1 - k^{-1} u_0 \pi, \\
\int_{\mathbb{R}^{01}} d\pi e^{-k^{-1} \theta \pi} (u_1 - k^{-1} u_0 \pi) = -k^{-1} (u_0 + u_1 \theta), \\
-k \int_{\mathbb{R}^{02}} d\pi d\theta' e^{k^{-1}(\theta - \theta') \pi} (u_0 + u_1 \theta') = u_0 + u_1 \theta \end{array} \right.
\]
then the Weyl symbol of \(\mathbb{H}(q, \partial_q)\) is given by
\[
\mathcal{H}(x, \xi, \theta, \pi) = -\frac{1}{2} \xi^2 - \frac{\omega}{2} x^2 - k^{-1} \omega \theta \pi.
\]
In fact, since
\[
-k \int_{\mathbb{R}^{01}} d\pi d\theta' e^{k^{-1}(\theta - \theta') \pi} \theta + \theta' \frac{\omega}{2} \pi (u_0 + u_1 \theta') = \frac{1}{2} (u_0 - u_1 \theta),
\]
we get
\[
\hat{\mathcal{H}}(x, \partial_x, \theta, \partial \theta)u(x, \theta) = \frac{-k}{2 \pi h} \int_{\mathbb{R}^{01}} d\xi d\pi d\theta' e^{ih^{-1}(x-x') \xi + k^{-1}(\theta - \theta') \pi} \times \mathcal{H}\left(\frac{x + x'}{2}, \xi, \frac{\theta + \theta'}{2}, \pi\right) (u_0(x') + u_1(x') \theta')
\]
\[
= \frac{1}{2 \pi h} \int_{\mathbb{R}^{01}} e^{ih^{-1}(x-x') \xi} H\left(\frac{x + x'}{2}, \xi\right) (u_0(x') + u_1(x') \theta) + \omega(u_0(x) - u_1(x) \theta)
\]
\[
\sim \left( H(q, \partial_q)I + \frac{\omega}{2} [b, b^*] \right) u.
\]

8.2.3.1. LH-formulation. In this case, since the Hamilton equation w.r.t. odd variables is given by
\[
\dot{\theta} = -k^{-1} \omega \theta, \quad \dot{\pi} = k^{-1} \omega \pi,
\]
we have
\[
\theta(s) = e^{-k^{-1} \omega s} \theta, \quad \pi(s) = e^{k^{-1} \omega s} \pi.
\]
Without using Fourier transform w.r.t. even variables, we may put
\[
S(t, \bar{x}, \bar{\theta}, \bar{x}, \bar{\pi}) = [\theta \pi + S_0(t, x, \xi)]_{\xi = \ldots, \theta = \ldots} = e^{k^{-1} \omega t} \bar{\theta} \bar{\pi} + S(t, \bar{x}, \bar{x}),
\]
where
\[
S(t, \bar{x}, \bar{x}) = \frac{\omega \xi}{2s^2} \bar{x}^2 - \frac{\omega \xi}{s} \bar{x} + \frac{\omega \xi}{2s^2} \bar{x}^2.
\]
Then (8.2.22) satisfies
\[
S_t + \mathcal{H}(\bar{x}, \bar{S}_\bar{\theta}, S_\theta) = 0 \quad \text{with} \quad \lim_{t \to 0} (S - \frac{(\bar{x} - x)^2}{2t} - \bar{\theta}_\pi) = 0.
\]

Moreover, putting
\[
D(t, \bar{x}, \bar{\theta}, x, \pi) = \text{sdet} \begin{pmatrix}
- \frac{\partial^2 S}{\partial x^2}
& 0

0
& - \frac{\partial^2 S}{\partial \theta^2}
\end{pmatrix} = \frac{\omega}{\sinh(\omega t)} e^{\omega t},
\]
we have
\[
D_t + \partial_\theta (D\mathcal{H}_\xi) + \partial_\bar{\theta} (D\mathcal{H}_\pi) = 0 \quad \text{with} \quad \lim_{t \to 0} tD(t, \bar{x}, \bar{\theta}, x, \pi) = 1
\]
where \(\mathcal{H}_\xi = \mathcal{H}_\xi(\bar{x}, S_x, \bar{\theta}, S_\theta)\) and \(\mathcal{H}_\pi = \mathcal{H}_\pi(x, S_x, \bar{\theta}, S_\theta)\).

We put
\[
V_t^{LH} u(t, \bar{x}, \bar{\theta}) = \frac{1}{\sqrt{2\pi}} \int_{[0,1]^4} d\bar{x} d\pi D_{1/2} e^{-S(\bar{x} - x) - l^{-1} \bar{u}_0(\bar{x})}
= e^{\omega t/2} V_t^{L} u_0(\bar{x}) + e^{-\omega t/2} V_t^{L} u_1(\bar{x}) \bar{\theta}.
\]

8.2.3.2. HH-formulation. We define
\[
S(t, \bar{x}, \bar{\theta}, x, \pi) = [\theta_\pi + S_0(t, x, \xi)]_{\bar{x} = \ldots, \bar{\theta} = \ldots} = e^{-\omega t} \bar{\theta}_\pi + S(t, \bar{x}, \xi),
\]
which satisfies
\[
S_t + \mathcal{H}(\bar{x}, S_x, \bar{\theta}, S_\theta) = 0 \quad \text{with} \quad \lim_{t \to 0} S = \bar{x} \xi + \bar{\theta}_\pi.
\]

Moreover, putting
\[
D(t, \bar{x}, \bar{\theta}, x, \pi) = \text{sdet} \begin{pmatrix}
- \frac{\partial^2 S}{\partial x^2}
& 0

0
& - \frac{\partial^2 S}{\partial \theta^2}
\end{pmatrix} = \frac{1}{\cosh(\omega t)} e^{\omega t},
\]
we have
\[
D_t + \partial_\theta (D\mathcal{H}_\xi) + \partial_\bar{\theta} (D\mathcal{H}_\pi) = 0 \quad \text{with} \quad \lim_{t \to 0} D(t, \bar{x}, \bar{\theta}, x, \pi) = 1
\]
where \(\mathcal{H}_\xi = \mathcal{H}_\xi(\bar{x}, S_x, \bar{\theta}, S_\theta)\) and \(\mathcal{H}_\pi = \mathcal{H}_\pi(x, S_x, \bar{\theta}, S_\theta)\).

(8.2.30) \(S(t, \bar{x}, \bar{\theta}, x, \pi) = [-\theta_\pi + S_0(t, x, \xi)]_{\bar{x} = \ldots, \bar{\theta} = \ldots} = -e^{-\omega t} \bar{\theta}_\pi + S(t, \bar{x}, \xi)\)

we put
\[
V_t^{LH} u(t, \bar{x}, \bar{\theta}) = e^{\omega t/2} V_t^{L} u_0(\bar{x}) + e^{-\omega t/2} V_t^{L} u_1(\bar{x}) \bar{\theta}.
\]

Putting \(v(t, x, \theta) = V_t^{LH} u(t, x, \theta)\), we have
\[
\frac{\partial}{\partial t} v(t, x, \theta) = \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} \omega^2 x^2 + \frac{\omega}{2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \right) v(t, x, \theta) \quad \text{with} \quad v(0, x, \theta) = v(x, \theta).
\]

From the kernel of \(V_t\), we may calculate the Witten index which is shown soon later.
8.2.4. A generalization. For

\[ H(q, p) = \frac{1}{2} \sum_{j=1}^{d} p_j^2 - \sum_{j=1}^{d} \frac{\omega_j^2 q_j^2}{2} \in C^\infty(T^*\mathbb{R}^d : \mathbb{R}), \]

we may extend it as

\[ \mathcal{H}(x, \theta, \xi, \pi) = \frac{1}{2} \sum_{j=1}^{d} \xi_j^2 - \sum_{j=1}^{d} \frac{\omega_j^2 \theta_j^2}{2} + \sum_{j=1}^{d} \omega_j \theta_j \pi_j, \]

(8.232) \[ S(t, \bar{x}, \bar{\theta}, \bar{\pi}) = \left[ -\langle \theta | \bar{\pi} \rangle + S_0(t, x, \xi) \right] \bigg|_{\theta = \ldots \xi = \ldots} = -\sum_{j=1}^{d} e^{-\omega_j t} \bar{\theta}_j \bar{\pi}_j + S(t, \bar{x}, \bar{\pi}) \]

with

\[ S(t, \bar{x}, \bar{\pi}) = \sum_{j=1}^{d} \left[ \frac{\omega_j \cosh(\omega_j t)}{2 \sinh(\omega_j t)} (\bar{x}_j^2 + \bar{\pi}_j^2) - \frac{\omega_j}{\sinh(\omega_j t)} \bar{x}_j \bar{\pi}_j \right]. \]

We define

\[ \mathcal{D}(t, \bar{x}, \bar{\theta}, \bar{\pi}) = \text{sdet} \begin{pmatrix} -\frac{\partial^2 S}{\partial \theta_j \bar{\pi}_j} & 0 \\ 0 & \frac{\partial^2 S}{\partial \bar{\theta}_j \bar{\pi}_j} \end{pmatrix} = \prod_{j=1}^{d} \frac{\omega_j}{\sinh(\omega_j t)} e^{\omega_j t} \]

and we have

\[ \mathcal{D}^{1/2} e^{-S} = e^{\omega t/2} \sqrt{\prod_{j=1}^{d} \frac{\omega_j}{\sinh(\omega_j t)}} e^{\sum_{j=1}^{d} e^{-\omega_j t} \bar{\theta}_j \bar{\pi}_j - S(t, \bar{x}, \bar{\pi})}, \quad \omega = \sum_{j=1}^{d} \omega_j. \]

8.2.4.1. [H]: LH-formulation for the case \( d = 2 \) with \( u(t, x, \theta) = u_0(t, x) + u_1(t, x) \theta_1 + u_2(t, x) \theta_2 + u_3(t, x) \theta_1 \theta_2 \). Define the Fourier transformations by

\[
\begin{align*}
\int d\theta e^{i\theta |\pi|} (u_0 + u_1 \theta_1 + u_2 \theta_2 + u_3 \theta_1 \theta_2) &= u_3 - u_2 \pi_1 + u_1 \pi_2 - u_0 \pi_1 \pi_2, \\
\int d\pi e^{-i\pi |\theta|} (u_3 - u_2 \pi_1 + u_1 \pi_2 - u_0 \pi_1 \pi_2) &= -(u_0 + u_1 \theta_1 + u_2 \theta_2 + u_3 \theta_1 \theta_2).
\end{align*}
\]

Then, we have

\[
\mathcal{V}_t u(\bar{x}, \bar{\theta}) = -\frac{1}{2\pi} \int d\bar{x} d\bar{\pi} \mathcal{D}^{1/2}(t, \bar{x}, \bar{\theta}, \bar{\pi}) e^{-S(t, \bar{x}, \bar{\theta}, \bar{\pi})} (u_3(\bar{x}) - u_2(\bar{x}) \pi_1 + u_1(\bar{x}) \pi_2 - u_0(\bar{x}) \pi_1 \pi_2)
\]

\[
= \int d\bar{x} \sqrt{\frac{\omega_1}{2\pi \sinh(\omega_1 t)} \frac{\omega_2}{2\pi \sinh(\omega_2 t)}} e^{-S(t, \bar{x}, \bar{\pi})} e^{\omega t/2}
\]

\[
\times (u_0(\bar{x}) - e^{-\omega_1 t} u_1(\bar{x}) \bar{\theta}_1 - e^{-\omega_2 t} u_2(\bar{x}) \bar{\theta}_2 + e^{-\omega_3 t} u_3(\bar{x}) \bar{\theta}_1 \bar{\theta}_2).
\]

Since

\[
\int d\bar{\theta} (e^{\omega t/2} \bar{\theta}_1 \bar{\theta}_2 + e^{(-\omega_1 + \omega_2) t/2} \bar{\theta}_1 \bar{\theta}_2 + e^{(\omega_1 - \omega_2) t/2} \bar{\theta}_1 \bar{\theta}_2) (u_0 + u_1 \theta_1 + u_2 \theta_2 + u_3 \theta_1 \theta_2)
\]

\[
= e^{\omega t/2} u_0 - e^{(-\omega_1 + \omega_2) t/2} u_1 \theta_1 - e^{(\omega_1 - \omega_2) t/2} u_2 \theta_2 + e^{-\omega t/2} u_3 \theta_1 \theta_2
\]

and

\[
P u = \left( 1 - 2 \theta_1 \frac{\partial}{\partial \theta_1} \right) \left( 1 - 2 \theta_2 \frac{\partial}{\partial \theta_2} \right) (u_0 + u_1 \theta_1 + u_2 \theta_2 + u_3 \theta_1 \theta_2) = u_0 - u_1 \theta_1 - u_2 \theta_2 + u_3 \theta_1 \theta_2,
\]

we have the corresponding SUSYQM with

\[ W(x) = \frac{1}{2} (\omega_1 x_1^2 + \omega_2 x_2^2), \quad b_j = \theta_j, \quad b_k^* = \frac{\partial}{\partial \theta_k}, \]

and

\[ -\sum_{j,k=1}^{2} \frac{\partial^2 W}{\partial x_j \partial x_k} [b_j, b_k^*] = \omega_1 \left( 1 - 2 \theta_1 \frac{\partial}{\partial \theta_1} \right) + \omega_2 \left( 1 - 2 \theta_2 \frac{\partial}{\partial \theta_2} \right) = \sum_{j=1}^{2} \omega_j \left[ \frac{\partial}{\partial \theta_j}, \theta_j \right]_. \]
Moreover, we have

$$\text{str } V_t = \int d\omega \frac{e^{\omega t/2} - e^{(-\omega_1 + \omega_2)t/2} - e^{(\omega_1 - \omega_2)t/2} + e^{-\omega t/2}}{\sqrt{2(\cosh(\omega_1 t) - 1)} \sqrt{2(\cosh(\omega_2 t) - 1)}} = 1.$$  

8.2.4.2. $[H]$: LH-formulation for general $d$ with $u(t, x, \theta) = \sum_{|a| \leq d} u_a(t, x) \theta^a$. Since we have

$$\prod_{j=1}^{d} (\theta_j \pi_j)^{b_j} = (-1)^{|b|(|b|-1)/2} \theta^b \pi^b$$

with $b = (b_1, \ldots, b_d) \in \{0, 1\}^d$, $|b| = \sum_{j=1}^{d} b_j$,

$$\theta^a \theta^b = (-1)^{\tau(a,b)} \theta^{a+b}$$

with $\tau(b,a) \equiv |a||b| + \tau(a,b)$ mod 2,

we get, with $b_j = 1 - a_j$, $a + b = 1 = (1, \ldots, 1)$, $d = |a| + |b|$,

$$\int d\theta e^{(\theta|\pi)} \theta^a = \int d\theta \frac{1}{\prod_{j=1}^{d} (1 + \theta_j \pi_j)} \prod_{j=1}^{d} (\theta_j \pi_j)^{b_j} = (-1)^{|b|(|b|-1)/2 + \tau(a,b)} \theta^b,$$

$$\int d\pi e^{-(\theta|\pi)} \pi^b = (-1)^{|a|(|a|-1)/2 + \tau(a,b)} \theta^a.$$  

Therefore

$$\int d\theta e^{(\theta|\pi)} \sum_{|a| \leq d} u_a \theta^a = \int d\theta \sum_{|a| \leq d} u_a \theta^a \prod_{j=1}^{d} (\theta_j \pi_j)^{b_j} = \sum_{|a| \leq d} (-1)^{|b|(|b|-1)/2 + \tau(a,b)} u_a \pi^b,$$

$$\int d\pi e^{-(\theta|\pi)} \sum_{|a| \leq d} (-1)^{|b|(|b|-1)/2 + \tau(a,b)} \theta^a \pi^b = (\sum_{|a| \leq d} u_a \theta^a).$$

On the other hand, as

$$\int d\pi \sum_{j=1}^{d} e^{-\omega_j t} \theta_j \pi_j (-1)^{|b|(|b|-1)/2 + \tau(a,b)} \pi^b = \int d\pi (-1)^{|b|(|b|-1)/2 + \tau(a,b)} \pi^b \prod_{j=1}^{d} (e^{-\omega_j t} \theta_j \pi_j)^{a_j}$$

$$= \prod_{j=1}^{d} e^{-a_j \omega_j t} (-1)^{(d-1)/2} (-1)^{|a| + \tau(a,b)} \theta^a,$$

we have

$$V_t u(x, \bar{\theta}) = \frac{(-1)^{(d-1)/2}}{(2\pi)^{d/2}} \int d\omega \int d\pi D^{1/2}(t, \bar{x}, \bar{\theta}, \pi) e^{-S(t, \bar{x}, \bar{\theta}, \pi)} \sum_{|a| \leq d} (-1)^{|b|(|b|-1)/2 + \tau(a,b)} u_a(x) \pi^b$$

$$= \int d\omega \prod_{j=1}^{d} \sqrt{\frac{\omega_j}{2\pi \sinh(\omega_j t)}} e^{\omega_j t/2} \sum_{|a| \leq d} \prod_{j=1}^{d} e^{-a_j \omega_j t} (-1)^{|a| + \tau(a,b)} u_a(x) \theta^a.$$  

Since

$$\int d\theta \sum_{|a| \leq d} \sum_{|a'| \leq d} e^{\sum_{j=1}^{d} (a_j - b_j) t/2} \theta^a \bar{\theta}^{a'} \sum_{|a| \leq d} u_a \theta^a$$

$$= \sum_{|a| \leq d} \prod_{j=1}^{d} e^{(b_j - a_j) \omega_j t} (-1)^{|a| + \tau(a,b)} u_a(x) \bar{\theta}^a,$$

and

$$Pu = \prod_{j=1}^{d} \left(1 - 2\theta_j \frac{\partial}{\partial \theta_j}\right) \sum_{|a| \leq d} u_a \theta^a = \sum_{|a| \leq d} (-1)^{|b|} u_a \theta^a,$$

we have the corresponding SUSYQM as before.
Moreover, we have
\[
\text{str } \mathcal{V}_t = \int dx dt \prod_{j=1}^{d} \left( e^{\omega_j t/2} - e^{-\omega_j t/2} \right) \partial_1 \cdots \partial_d e^{-S(t,x)}
\]
\[
= \prod_{j=1}^{d} \left( e^{\omega_j t/2} - e^{-\omega_j t/2} \right) \frac{1}{\sqrt{\prod_{j=1}^{d} 2(\cosh(\omega_j t)-1)}} = 1.
\]

Putting \( v(t,x,\theta) = \mathcal{V}_t \tilde{\psi}(t,x,\theta) \), we have
\[
(8.2.33) \quad \frac{\partial}{\partial t} v(t,x,\theta) = \left( \frac{1}{2} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j=1}^{d} \omega_j^2 x_j^2 + \sum_{j=1}^{d} \frac{\omega_j}{2} \left[ \frac{\partial}{\partial \theta_j}, \theta_j \right] \right) v(t,x,\theta)
\]
with \( v(0,x,\theta) = \tilde{\psi}(x,\theta) \). Here, we put \( \omega = \sum_{j=1}^{d} \omega_j \).

**Problem 8.2.2.** Extend the procedure in this chapter to the operator posed by M.S. Abdalla and U.A.T. Ramjit [1]:
\[
H(q,p,t) = \frac{1}{2m(t)} p^2 + \frac{m(t)}{2} \omega^2 q^2, \quad m(t) = m_0 e^{2\Gamma(t)},
\]
Show the difference of solution of above operator from harmonic oscillator represented by \( \Gamma(t) \) and how it changes adding spin?

### 8.3. A simple example of supersymmetric extension

We consider the simplest 1-dimensional example. Let
\[
(8.3.1) \quad H(q,p) = \frac{1}{2} (p - A(q))^2 + V(q) \in C^\infty(T^*\mathbb{R} : \mathbb{R})
\]
be given with \( A(q), V(q) \in C^\infty(\mathbb{R} : \mathbb{R}) \). Using Legendre transformation as
\[
(8.3.2) \quad \dot{q} = \frac{\partial H}{\partial p} = p - A(q), \quad L(q,\dot{q}) = \dot{q} p - H(q,p),
\]
we get a Lagrangian
\[
L(q,\dot{q}) = \frac{1}{2} \dot{q}^2 + A(q) \dot{q} - V(q) \in C^\infty(T\mathbb{R} : \mathbb{R}).
\]

Instead of a path \( q : \mathbb{R} \ni t \rightarrow q(t) \in \mathbb{R} \), we consider a generalized path
\[
(8.3.3) \quad \Phi : \mathbb{R} \ni t \rightarrow \Phi(t) = x(t) + i(\rho_1 \psi_2(t) - \rho_2 \psi_1(t)) + i\rho_1 \rho_2 F(t) \in \mathcal{R}_{ev}
\]
with \( \rho_\alpha \in \mathcal{R}_{od} (\alpha = 1,2) \) being odd parameters and
\[
\left\{ \begin{array}{l}
\text{x} : \mathbb{R} \ni t \rightarrow x(t) \in \mathcal{R}_{ev}, \\
\psi_\alpha : \mathbb{R} \ni t \rightarrow \psi_\alpha(t) \in \mathcal{R}_{od} \quad \text{with } \alpha = 1,2, \\
F : \mathbb{R} \ni t \rightarrow F(t) \in \mathcal{R}_{ev}.
\end{array} \right.
\]

Introducing operators
\[
D_\alpha = \frac{\partial}{\partial \rho_\alpha} - i\rho_\alpha \frac{\partial}{\partial t} \quad \text{with } \alpha = 1,2,
\]
and \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1 \), we extend \( L(q,\dot{q}) \) as
\[
(8.3.5) \quad \tilde{L}_0 = -\frac{1}{4} (D_\alpha \Phi) \epsilon_{\alpha\beta} (D_\beta \Phi) + \frac{i}{2} A_{\rho_\alpha} \epsilon_{\alpha\beta} D_\beta \Phi - iW(\Phi).
\]
In the above, $A(q)$ is extended from $q \in \mathbb{R}$ to $\Phi = x + i(\rho_1 \psi_2 - \rho_2 \psi_1) + i \rho_1 \rho_2 F \in \mathfrak{H}_v$ by the Grassmann extension as

\[ A(\Phi) = A(x) + iA'(x)(\rho_1 \psi_2 - \rho_2 \psi_1) + \rho_1 \rho_2 F + A''(x)\rho_1 \rho_2 \psi_1 \psi_2. \]

and $W(\Phi)$ is analogously extended from $W(q)$ to $W(\Phi)$ whose relation to $V(q)$ will be given later.

**Remark 8.3.1.** The following relation will be worth noticing:

\[ \left( \frac{\partial}{\partial \rho_{\alpha}} - i \rho_{\alpha} \frac{\partial}{\partial t} \right)^2 = -i \frac{\partial}{\partial t} \text{ for each } \alpha = 1, 2. \]

Now, we have

\[ L_0' \overset{\text{def}}{=} \int d\rho_2 d\rho_1 \hat{L}_0 \]

\[ = \frac{1}{2} \dot{x}^2 + A(x)\dot{x} + \frac{1}{2} F^2 + \frac{i}{2} \left( \psi_2 \dot{\psi}_2 - \dot{\psi}_1 \psi_1 \right) + W'(x)F - iW''(x)\psi_1 \psi_2. \]

Assuming that the “auxiliary field $F$” should satisfy

\[ 0 = \frac{\delta L_0'}{\delta F} = F' + W', \]

we arrived at

\[ L_0 = \frac{1}{2} \dot{x}^2 + A(x)\dot{x} + \frac{i}{2} \left( \psi_2 \dot{\psi}_2 - \dot{\psi}_1 \psi_1 \right) - \frac{1}{2} W'(x)^2 - iW''(x)\psi_1 \psi_2. \]

This is the desired Lagrangian with variables $x, \dot{x}, \psi_1, \dot{\psi}_1$, but variables $\psi_1, \dot{\psi}_1$ are not independent each other. In fact, they satisfy

\[ \{\psi_\alpha, \psi_\beta\} = \psi_\alpha \dot{\psi}_\beta + \dot{\psi}_\beta \psi_\alpha = 0, \quad \{\psi_\alpha, \dot{\psi}_\beta\} = 0 \quad \text{and} \quad \{\dot{\psi}_\alpha, \dot{\psi}_\beta\} = 0. \]

To find out “independent Grassmann variables” in (S.3.10), we introduce new variables by the following two methods:

(I) Defining new variables as

\[ \left\{ \begin{array}{l} \xi = \frac{\delta L_0}{\delta \dot{x}} = \dot{x} + ax, \\ \phi_{\alpha} = \frac{\delta L_0}{\delta \dot{\psi}_\alpha} = -i \frac{1}{2} \psi_\alpha \quad \text{for } \alpha = 1, 2, \end{array} \right. \]

we put

\[ H(x, \xi, \psi_1, \psi_2) = \dot{x} \xi + \dot{\psi}_\alpha \phi_\alpha - L_0 \]

\[ = \frac{1}{2} (\xi - ax)^2 + \frac{1}{2} W'(x)^2 + \frac{i}{2} W''(x)\epsilon_{\alpha\beta} \psi_\alpha \psi_\beta. \]

Rewriting the variables $\psi_1, \psi_2$ as $\theta, \pi$, respectively, we get

\[ H(x, \xi, \theta, \pi) = \frac{1}{2} (\xi - ax)^2 + \frac{1}{2} W'(x)^2 + iW''(x)\theta \pi. \]

(II) In the above, we use the “real” odd variables $\psi_\alpha$. We “complexify” these variables by putting

\[ \psi = \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2), \quad \bar{\psi} = \frac{1}{\sqrt{2}} (\psi_1 - i\psi_2), \quad \text{i.e.} \quad \psi_1 = \frac{1}{\sqrt{2}} (\psi + \bar{\psi}), \quad \psi_2 = \frac{1}{\sqrt{2}i} (\psi - \bar{\psi}), \]

and then we rewrite $L_0$ as

\[ \bar{L}_0 = \frac{1}{2} \dot{x}^2 + A(x)\dot{x} + \frac{i}{2} (\bar{\psi} \dot{\psi} + \bar{\psi} \dot{\psi}) - \frac{i}{2} W'(x)^2 - W''(x)\psi \bar{\psi}. \]
Introducing new variables as
\begin{equation}
\xi = \frac{\delta \bar{L}_0}{\delta \bar{\psi}} = \dot{x} + A(x), \quad \phi = \frac{\delta \bar{L}_0}{\delta \psi} = -\frac{i}{2} \bar{\psi}, \quad \bar{\phi} = \frac{\delta \bar{L}_0}{\delta \bar{\psi}} = -\frac{i}{2} \psi,
\end{equation}
we put
\[\mathcal{H}(x, \xi, \psi, \bar{\psi}) \overset{\text{def}}{=} \dot{x} \xi + \dot{\psi} \phi + \dot{\bar{\psi}} \bar{\phi} - \bar{L}_0
\]
\[= \frac{1}{2} (\xi - A(x))^2 + \frac{1}{2} W'(x)^2 + W''(x) \bar{\psi} \psi.\]
Rewriting \(\psi\) and \(\bar{\psi}\) by \(\theta\) and \(\pi\), respectively, we get finally a function
\begin{equation}
\mathcal{H}(x, \xi, \theta, \pi) = \frac{1}{2} (\xi - A(x))^2 + \frac{1}{2} W'(x)^2 - W''(x) \theta \pi \in \mathcal{C}_{\text{SS}}(\mathfrak{M}^{2|2} : \mathfrak{M}_v).
\end{equation}
Here, \((x, \theta) \in \mathfrak{M}^{1|1}, (\xi, \pi) \in \mathfrak{M}^{1|1}\).

\textbf{Remark 8.3.2.} (0) The difference between \((8.3.12)\) and \((8.3.16)\) is the existence of \(i\) in front of the term \(W''(x)\theta \pi\). This difference is rather significant when we consider Witten index for supersymmetric quantum mechanics using the kernel representation of the corresponding evolution operator.

1. As there is no preference at this stage to take \(\pi\) and \(\theta\) instead of \(\theta\) and \(\pi\), there is no significance of the sign \(\pm\) in front of the terms \(i W''(x) \theta \pi\) in \((8.2.27)\) or \(W''(x) \theta \pi\) in \((8.3.16)\) in these cases.

2. We may regard \(\mathcal{H}(x, \xi, \theta, \pi)\) as a Hamiltonian in \(\mathcal{C}_{\text{SS}}(T^* \mathfrak{M}^{1|1} : \mathfrak{M}_v)\).

3. These Hamiltonians \((8.3.12)\) and \((8.3.16)\) are called supersymmetric extensions of \((8.3.1)\) because they give supersymmetric quantum mechanics after quantization (see \S4). The procedure above is author’s unamused understanding of amalgam of physics papers such as Cooper and Friedman \[26\], Davis, Macfarlane, Popat and van Holten \[28\] etc. But supersymmetry in superspace \(\mathfrak{M}^{n|n}\) will be studied separately.

4. On the other hand, using the identification \((1.13)\), we have
\[\mathcal{H}_\pm^b(x, \partial_x, \theta, \partial_\theta) = \# \left( \begin{array}{cc} H_+^b & 0 \\ 0 & H_-^b \end{array} \right) b = \# H_\pm^b b .\]
Moreover, in this case, the “complete Weyl symbol of the above \(\mathcal{H}^b(x, \partial_x, \theta, \partial_\theta)\)” is calculated by
\begin{equation}
\mathcal{H}_\pm(x, \xi, \theta, \pi) = (e^{-i h^{-1}(x \xi + \theta \pi)} H_\pm^b(x, \partial_x, \theta, \partial_\theta) e^{i h^{-1}(x \xi + \theta \pi)})_{h=0} = \frac{1}{2} (\xi - a x)^2 + \frac{1}{2} b^2 x^2 + i b \theta \pi.
\end{equation}
\(\mathcal{H}_+\) equals to \((8.3.12)\) when \(A(q) = a q\) and \(W(x) = \frac{1}{2} b x^2\), and \(\mathcal{H}_-\) is obtained from \((8.3.16)\) with \(A(q) = a q\) and \(W(x) = -\frac{1}{2} b x^2\). These give the relation between \(W(q)\) and \(V(q)\). (See SUSYQM defined in \$1.\)

5. Witten \[134\] considered as a quantum mechanical operator
\begin{equation}
\mathbf{H}(q, \partial_q) = \left( \begin{array}{c} -\frac{1}{2} \partial_q^2 + v(q) \\ 1 \\ 0 \\ 1 \end{array} \right) - \frac{1}{2} w(q) \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \end{array} \right).
\end{equation}
This operator is supersymmetric when there exists a function \(\psi(q)\) such that
\[v(q) = \frac{1}{2} \psi'(q)^2, \quad w(q) = \psi''(q).\]

\textbf{Problem 8.3.1.} Though a trial to prove Atiyah-Singer index theorem applying super analysis by S. Rempel and T. Schmitt \[103\], was informed by A. Rogers long-time ago, but I feel shame that I haven’t comprehend well yet. Moreover, because I have stumbled before appreciate the naturalness of definition of weights of Douglas-Nirenberg for system of PDE, therefore I’m far from Rempel and Schmitt’s reformulation.
More recently, I find the paper of F.F. Voronov [132] though I haven’t appreciated it yet. Because Voronov [131, 132] uses Banach-Grassmann algebra, therefore besides algebraic calculation, estimates by inequalities seems far from my understanding.

Problem 8.3.2. The proofs of “analytic torsion=Reidemeister torsion” by J. Cheeger [23], D. Burghelea, L. Friedlander and T. Kappeler [21] are given. Reprove this in our context.
CHAPTER 9

Miscellaneous

9.1. Proof of Berezin’s Theorem 5.2.1

To be self-contained, we give a precise proof following Berezin [9] and Rogers [110] because their proofs are not so easy to understand at least for a tiny little old mathematician.

First of all, we prepare

**Lemma 9.1.1.** Let \( u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x) \) be supersmooth on \( U = U_{ev} \times R^n_{od} \). If \( \int_{U} dx u_a(x) \) exists for each \( a \), then we have

\[
B \iint_{U} dx d\theta u(x, \theta) = \int_{U} dx \left[ \int_{R^n_{od}} d\theta u(x, \theta) \right] = \int_{U} dx \left[ \int_{U_{ev}} dx u(x, \theta) \right].
\]

**Proof.** By the primitive definition of integral, we have

\[
B \iint_{U} dx d\theta u(x, \theta) = \int_{U} dx \left[ \int_{R^n_{od}} d\theta u(x, \theta) \right] = \int_{U} dx u_1(x),
\]

and

\[
\int_{R^n_{od}} d\theta \sum_{|a| \leq n} \left[ \int_{U_{ev}} dx \theta^a u_a(x) \right] = \int_{R^n_{od}} d\theta \sum_{|a| \leq n} \theta^a \left[ \int_{U_{ev}} dx u_a(x) \right] = \int_{U_{ev}} dx u_1(x). \quad \square
\]

(I) Now, we consider a simple case: Let a linear coordinate change be given by

\[
(x, \theta) = (y, \omega) M, \quad M = \begin{pmatrix} A & C \\ D & B \end{pmatrix},
\]

that is,

\[
x_i = \sum_{k=1}^{m} y_k A_{ki} + \sum_{\ell=1}^{n} \omega_\ell D_{\ell i} = x_i(y, \omega), \quad \theta_j = \sum_{k=1}^{m} y_k C_k j + \sum_{\ell=1}^{n} \omega_\ell B_{\ell j} = \theta_j(y, \omega)
\]

with \( A_{ki}, B_{\ell j} \in C_{ev} \) and \( C_{\ell i}, D_{kj} \in C_{od} \), and we have

\[
\text{sdet} \left( \frac{\partial(x, \theta)}{\partial(y, \omega)} \right) = \det A \det^{-1}(B - DA^{-1}C) = \det(A - CB^{-1}D)\det^{-1}B = \text{sdet} M. \quad (9.1.1)
\]

Interchanging the order of integration, putting \( \omega^{(1)} = \omega B \) and \( y^{(1)} = yA \), we get

\[
B \iint dy d\omega u(yA + \omega D, yC + \omega B) = \int dy \left[ \int d\omega u(yA + \omega D, yC + \omega B) \right]
\]

\[
= \int dy \left[ \int d\omega^{(1)} \det B \cdot u(yA + \omega^{(1)} B^{-1} D, yC + \omega^{(1)}) \right]
\]

\[
= \int d\omega^{(1)} \det B \left[ \int dy u(yA + \omega^{(1)} B^{-1} D, yC + \omega^{(1)}) \right]
\]

\[
= \int d\omega^{(1)} \det B \left[ \int dy^{(1)} \det A^{-1} \cdot u(y^{(1)} + \omega^{(1)} B^{-1} D, y^{(1)} A^{-1} C + \omega^{(1)}) \right],
\]

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that is, since

\[
\frac{\partial(y, \omega)}{\partial(y^{(1)}, \omega^{(1)})} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}, \quad \text{sdet} \left( \frac{\partial(y, \omega)}{\partial(y^{(1)}, \omega^{(1)})} \right) = \text{det} A^{-1} \cdot \text{det} B,
\]

we have

\[
B\int dyd\omega u(yA + \omega D, yC + \omega B) = B\int dyd\omega \text{sdet} \left( \frac{\partial(y, \omega)}{\partial(y^{(1)}, \omega^{(1)})} \right) u(y^{(1)} + \omega^{(1)}B^{-1}D, y^{(1)}A^{-1}C + \omega^{(1)}).
\]

(9.1.2)

Analogously, using Lemma 9.1.1 and by introducing change of variables as

\[
y^{(2)} = y^{(1)}, \omega^{(2)} = \omega^{(1)} + y^{(1)}A^{-1}C \implies \text{sdet} \left( \frac{\partial(y^{(1)}, \omega^{(1)})}{\partial(y^{(2)}, \omega^{(2)})} \right) = \text{sdet} \begin{pmatrix} 1 & 0 \\ 0 & -A^{-1}C^{'} \end{pmatrix} = 1,
\]

we get

\[
B\int dyd\omega u(y^{(1)} + \omega^{(1)}B^{-1}D, y^{(1)}A^{-1}C + \omega^{(1)})
\]

\[
= B\int dyd\omega \text{sdet} \left( \frac{\partial(y^{(1)}, \omega^{(1)})}{\partial(y^{(2)}, \omega^{(2)})} \right) u(y^{(2)} + (\omega - y^{(2)}A^{-1}C)B^{-1}D, \omega^{(2)}).
\]

Then by

\[
y^{(3)} = y^{(2)}(1 - A^{-1}CB^{-1}D), \omega^{(3)} = \omega^{(2)}
\]

\[
\implies \text{sdet} \left( \frac{\partial(y^{(2)}, \omega^{(2)})}{\partial(y^{(3)}, \omega^{(3)})} \right) = \text{sdet} \begin{pmatrix} (1 - A^{-1}CB^{-1}D)^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \text{det}^{-1}(1 - A^{-1}CB^{-1}D),
\]

we have

\[
B\int dyd\omega u(y^{(2)} + (\omega - y^{(2)}A^{-1}C)B^{-1}D, \omega^{(2)})
\]

\[
= B\int dyd\omega \text{sdet} \left( \frac{\partial(y^{(2)}, \omega^{(2)})}{\partial(y^{(3)}, \omega^{(3)})} \right) u(y^{(3)} + \omega^{(3)}B^{-1}D, \omega^{(3)}).
\]

(9.1.3)

Finally by

\[
x = y^{(3)} + \omega^{(3)}B^{-1}D, \theta = \omega^{(3)} \implies \text{sdet} \left( \frac{\partial(y^{(3)}, \omega^{(3)})}{\partial(x, \theta)} \right) = \text{sdet} \begin{pmatrix} 1 & 0 \\ -B^{-1}D & 1 \end{pmatrix} = 1,
\]

using \(\text{det} B \cdot \text{det}^{-1}(A - CB^{-1}D) \cdot (\text{det} A \cdot \text{det}^{-1}(B - DA^{-1}C)) = 1\) from (9.1.1), we have,

\[
B\int dyd\omega u(yA + \omega D, yC + \omega B) = \text{sdet} M^{-1} \cdot B\int dx d\theta \text{sdet} \left( \frac{\partial(y^{(3)}, \omega^{(3)})}{\partial(x, \theta)} \right) u(x, \theta).
\]

(9.1.5)

**Remark 9.1.1.** For the linear change of variables, it is not necessary to assume the compactness of support for integrand using primitive definition of integration.

(II) (ii-a) If \(H_1\) and \(H_2\) are superdiffeomorphisms of open subsets of \(\mathbb{R}^{m|n}\) with the image of \(H_1\) equals to the domain of \(H_2\), then

\[
\text{Ber}(H_1) \cdot \text{Ber}(H_2) = \text{Ber}(H_2 \circ H_1) \quad \text{where} \quad \text{Ber}(H)(y, \omega) = \text{sdet} J(H)(y, \omega).
\]

Here, for \(H(y, \omega) = (x_k(y, \omega), \theta_l(y, \omega)) : \mathbb{R}^{m|n} \to \mathbb{R}^{m|n}\), we put

\[
J(H)(y, \omega) = \begin{pmatrix} \frac{\partial x_k(y, \omega)}{\partial \omega_i} & \frac{\partial \theta_l(y, \omega)}{\partial \omega_i} \\ \frac{\partial x_k(y, \omega)}{\partial \omega_j} & \frac{\partial \theta_l(y, \omega)}{\partial \omega_j} \end{pmatrix} = \frac{\partial (x, \theta)}{\partial (y, \omega)}.
\]
(ii-b) Any superdiffeomorphism of an open subset of $\mathcal{R}^m$ may be decomposed as $H = H_2 \circ H_1$ where

\begin{equation}
(9.1.6)
\begin{aligned}
H_1(y, \omega) = (h_1(y, \omega), \omega) &= (\tilde{y}, \tilde{\omega}) \quad \text{with} \quad h_1 : \mathcal{R}^m \to \mathcal{R}^m, \\
H_2(\tilde{y}, \tilde{\omega}) &= (\tilde{y}, h_2(\tilde{y}, \tilde{\omega})) \quad \text{with} \quad h_2 : \mathcal{R}^m \to \mathcal{R}^m.
\end{aligned}
\end{equation}

Remark 9.1.2. (i) If $H(y, \omega) = (h_1(y, \omega), h_2(y, \omega))$ is given by $h_1(y, \omega) = yA + \omega D$ and $h_2(y, \omega) = yC + \omega B$ as above, putting $H_1(y, \omega) = (yA + \omega D, \omega) = (\tilde{y}, \omega)$ and $H_2(\tilde{y}, \omega) = (\tilde{y}, \tilde{y}A^{-1}C + \omega(B - DA^{-1}C))$, we have $H = H_2 \circ H_1$. In this case, we rewrite the procedures \((9.1.2), (9.1.5)\) as

$$
B\int dyd\omega u(yA + \omega D, yC + \omega B)
= B\int dyd\omega sdet \left( \frac{\partial(y, \omega)}{\partial(\tilde{y}, \omega)} \right) u(\tilde{y}, (\tilde{y} - \omega D)A^{-1}C + \omega B) \quad \text{with} \quad \tilde{y} = yA + \omega D
= \det A^{-1}B\int dx \int d\theta sdet \left( \frac{\partial(y, \theta)}{\partial(x, \theta)} \right) u(x, \theta) \quad \text{with} \quad x = \tilde{y}, \quad \theta = \tilde{y}A^{-1}C + \omega(B - DA^{-1}C)
= \det A^{-1}\det(B - DA^{-1}C)B\int dx \int d\theta u(x, \theta).
$$

(ii) Analogously, putting $H_1(y, \omega) = (y, yC + \omega B) = (y, \theta)$ and $H_2(y, \theta) = (y(A - CB^{-1}D) + \theta B^{-1}D, \theta)$, we have $H = H_2 \circ H_1$, and

$$
B\int dyd\omega u(yA + \omega D, yC + \omega B)
= B\int dyd\omega sdet \left( \frac{\partial(y, \omega)}{\partial(y, \theta)} \right) u(y(A - CB^{-1}D) + \theta B^{-1}D, \theta) \quad \text{with} \quad \theta = yC + \omega B
= \det B\cdot B\int dx \int d\theta sdet \left( \frac{\partial(y, \theta)}{\partial(x, \theta)} \right) u(x, \theta) \quad \text{with} \quad x = y(A - CB^{-1}D) + CB^{-1}\theta
= \det B\cdot \det^{-1}(A - CB^{-1}D)_{B}\int dx \int d\theta u(x, \theta).
$$

(iii) For any given superdiffeomorphism $H(y, \omega) = (h_1(y, \omega), h_2(y, \omega))$, defining $H_1(y, \omega) = (h_1(y, \omega), \omega) = (\tilde{y}, \omega)$ and $H_2(\tilde{y}, \omega) = (\tilde{y}, h_2(\tilde{y}, \omega))$ such that $h_2(y, \omega) = h_2(h_1(y, \omega), \omega)$, we have $H = H_2 \circ H_1$. Using the inverse function $y = g(\tilde{y}, \omega)$ of $\tilde{y} = h_1(y, \omega)$, we put $h_2(\tilde{y}, \omega) = h_2(\tilde{y}, \omega)$. We denote $h_1(y, \omega) = (h_{1\alpha}(y, \omega)) = (h_{11}, \ldots, h_{1n})$, etc. Then, for $k, \ell = 1, \ldots, n$,

$$
\frac{\partial \tilde{h}_{2\ell}}{\partial \omega_k} = \frac{\partial h_{2\ell}}{\partial \omega_k} + \sum_{i=1}^{m} \frac{\partial g_i}{\partial \omega_k} \frac{\partial h_{2\ell}}{\partial y_i}
$$

with

$$
0 = \frac{\partial \tilde{y}_j}{\partial \omega_k} = \frac{\partial h_{1j}(g(y, \omega), \omega)}{\partial \omega_k} = \sum_{j=1}^{m} \frac{\partial g_i}{\partial \omega_k} \frac{\partial h_{1j}}{\partial y_i} + \frac{\partial h_{1j}}{\partial \omega_k},
$$

we get

$$
\frac{\partial \tilde{h}_{2\ell}}{\partial \omega_k} = \frac{\partial h_{2\ell}}{\partial \omega_k} - \sum_{i,j=1}^{m} \frac{\partial h_{1j}}{\partial \omega_k} \left( \frac{\partial h_{1j}}{\partial y_i} \right)^{-1} \frac{\partial h_{2\ell}}{\partial y_i}.
$$

Therefore,

$$
\Ber H = \det \left( \frac{\partial h_{1}}{\partial y} \frac{\partial h_{2}}{\partial \omega} \frac{\partial h_{3}}{\partial \omega} \right) = \det \frac{\partial h_{1}}{\partial y} \cdot \det^{-1} \left( \frac{\partial h_{2}}{\partial \omega} - \frac{\partial h_{1}}{\partial \omega} \left( \frac{\partial h_{1}}{\partial y} \right)^{-1} \frac{\partial h_{2}}{\partial y} \right) = \det \frac{\partial h_{1}}{\partial y} \cdot \det^{-1} \frac{\partial h_{2}}{\partial \omega}.
$$
(III) For each type of superdiffeomorphisms \(H_1\) and \(H_2\), we prove the formula.

(III.1) Let \(H(y, \omega) = (h(y, \omega), \omega)\) where \(h = (h_j)_{j=1}^m : \mathbb{R}^m \to \mathbb{R}^m\). Then it is clear that

\[
\text{Ber}(H)(y, \omega) = \det \left( \frac{\partial h_j(y, \omega)}{\partial y_i} \right) = \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \prod_{i=1}^m \frac{\partial h_{\sigma(i)}(y, \omega)}{\partial y_i}.
\]

For any \(u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)\), we put

\[
B \int dx \theta u(x, \theta) = \int \mathfrak{U}_{ev,B} dx \left( \int_{\mathfrak{M}[n]} d\theta u(x, \theta) \right) = \int \mathfrak{U}_{ev,B} dx u_\mathfrak{I}(x).
\]

On the other hand, we have

\[
B \int \mathfrak{M} dy d\omega \text{Ber}(H)(y, \omega) (u \circ H)(y, \omega)
\]

\[
= \int_{\mathfrak{M}[n]} dy \frac{\partial}{\partial \omega_n} \cdots \frac{\partial}{\partial \omega_1} \left( \det \left( \frac{\partial h_j(y, \omega)}{\partial y_i} \right) u(h(y, \omega), \omega) \right) \bigg|_{\omega=0}
\]

\[
= \int_{\mathfrak{M}[n]} dy \left( \det \left( \frac{\partial h_j(y, 0)}{\partial y_i} \right) u_1(h(y, 0)) \right)
\]

\[
+ \int_{\mathfrak{M}[n]} dy \frac{\partial}{\partial \omega_n} \cdots \frac{\partial}{\partial \omega_1} \left( \sum_{|a| < n} \omega^a u_a(h(y, \omega)) \det \left( \frac{\partial h_j(y, \omega)}{\partial y_i} \right) \right) \bigg|_{\omega=0}.
\]

Applying the standard integration on \(\mathbb{R}^m\) to the first term of the right hand side above, we have

\[
\int_{\mathfrak{M}[n]} dy \left( \det \left( \frac{\partial h_j(y, 0)}{\partial y_i} \right) u_1(h(y, 0)) \right) = \int \mathfrak{U}_{ev,B} dx u_\mathfrak{I}(x) \quad \text{where} \quad \mathfrak{M} = H(\mathfrak{M}).
\]

**Claim 9.1.1.** The second term of the right hand side of (9.1.7) equals to the total derivatives of even variables. More precisely, we have, for \(u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)\),

\[
\frac{\partial}{\partial \omega_n} \cdots \frac{\partial}{\partial \omega_1} \left( \sum_{|a| < n} \omega^a u_a(h(y, \omega)) \text{Ber}(H)(y, \omega) \right) \bigg|_{\omega=0} = \sum_{j=1}^m \frac{\partial}{\partial y_j}(\star)
\]

As \(h_j(y, \omega) \in \mathfrak{U}_{ev}\), we have

\[
h_j(y, \omega) = h_j_0(y) + \sum_{|c| = ev \geq 2} \omega^c h_{j,c}(y),
\]

\[
u_a(h(y, \omega)) = u_a(h_0(y)) + \sum_{|c| = ev \geq 2} \omega^c h_{j,c}(y) u_{a,x_j}(h_0(y)) + \sum_{|a| \geq 2} \frac{\partial^a}{\partial x^a} u_a(h_0(y)) \left( \sum_{|c| = ev \geq 2} \omega^c h_{j,c}(y) \right)^a,
\]

\[
\text{Ber}(H)(y, \omega) = \det \left( \frac{\partial h_j(y, \omega)}{\partial y_i} \right) = \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \prod_{i=1}^m \frac{\partial h_{\sigma(i)}(y, \omega)}{\partial y_i}
\]

\[
= \det \left( \frac{\partial h_j(y, 0)}{\partial y_i} \right) + \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \sum_{j=1}^m \sum_{|c| = ev \geq 2} \omega^c \frac{\partial h_{\sigma(j),c}(y)}{\partial y_j} \prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\delta(y)}}{\partial y_i}
\]

\[
+ \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \sum_{j,k=1}^m \sum_{|c_j| = ev \geq 4} \omega^c \frac{\partial h_{\sigma(j),c_j}(y)}{\partial y_j} \frac{\partial h_{\sigma(k),c_k}(y)}{\partial y_k} \prod_{i=1, i \neq j,k}^m \frac{\partial h_{\sigma(i),\delta(y)}}{\partial y_i} + \text{etc.}
\]

Putting \(\mathfrak{I} - a = b\) or \(c_1 + c_2 = c_1 + c_2 + c_3\), etc, we have

\[
(9.1.8) \quad \text{the coefficient of } \omega^b \text{ of } f_a(h(y, \omega)) \text{Ber}(H)(y, \omega) = I + II + III
\]
where

\[ I = \sum_{j=1}^{m} h_{j,b}(y) u_{a,x_j}(h_0(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) \prod_{i=1}^{m} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i}, \]

\[ II = u_a(h_0(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) \prod_{j=1}^{m} \frac{\partial h_{\sigma(j),b}(y)}{\partial y_j} \sum_{i=1,i \neq j} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i}, \]

\[ III = u_a(h_0(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) \sum_{j,k=1}^{m} \sum_{\sigma \in \wp_m} \frac{\partial h_{\sigma(j),c_1}(y)}{\partial y_j} \frac{\partial h_{\sigma(k),c_2}(y)}{\partial y_k} \prod_{i=1,i \neq j,k} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i} + \text{etc.} \]

The term II is calculated as

\[ II = \sum_{j=1}^{m} \frac{\partial}{\partial y_j} \left[ f_a(h_0(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) h_{\sigma(j),b}(y) \prod_{i=1,i \neq j} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i} \right] - A - B \]

where

\[ A = \sum_{j=1}^{m} \left( \sum_{k=1}^{m} \frac{\partial h_{k,b}(y)}{\partial y_j} u_{a,x_k}(h_0(y)) \right) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) h_{\sigma(j),b}(y) \prod_{i=1,i \neq j} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i}, \]

\[ B = \sum_{j=1}^{m} u_a(h_0(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) h_{\sigma(j),b}(y) \frac{\partial}{\partial y_j} \left( \prod_{i=1,i \neq j} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i} \right). \]

We want to prove

**Claim 9.1.2.** (i) \( A = I \), (ii) \( B = 0 \) and (iii) \( III = 0 \).

(i) To prove \( A = I \), for each \( k = 1, \ldots, m \), we take all sums w.r.t. \( \sigma \in \wp_m \) and \( j \) such that \( \sigma(j) = k \). Then, relabeling in \( A \), we have

\[ \sum_{\sigma \in \wp_m} \sum_{j=1}^{m} \frac{\partial h_{\sigma(j),b}(y)}{\partial y_j} u_{a,x_k}(h_0(y)) \text{sgn}(\sigma) h_{k,b}(y) \prod_{i=1,i \neq j} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i} \]

\[ = u_{a,x_k}(h_0(y)) h_{k,b}(y) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) \prod_{i=1}^{m} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i}. \]

(ii) Take two permutations \( \sigma \) and \( \tilde{\sigma} \) in \( \wp_m \) such that

\[ \sigma(i) = \tilde{\sigma}(j), \ \sigma(j) = \tilde{\sigma}(i), \ \sigma(k) = \tilde{\sigma}(k) \quad \text{for} \quad k \neq i, j, \quad \text{and} \quad \text{sgn}(\sigma) \text{sgn}(\tilde{\sigma}) = -1. \]

Then,

\[ \text{sgn}(\sigma) h_{\sigma(j),b}(y) \frac{\partial}{\partial y_j} \left( \prod_{i=1,i \neq j} \frac{\partial h_{\sigma(i),b}(y)}{\partial y_i} \right) + \text{sgn}(\tilde{\sigma}) h_{\tilde{\sigma},b}(y) \frac{\partial}{\partial y_j} \left( \prod_{i=1,i \neq j} \frac{\partial h_{\tilde{\sigma}(i),b}(y)}{\partial y_i} \right) = 0. \]

(iii) Interchanging the role of \( j, k \) and \( c_1, c_2 \) in III, we have III = 0. Others are treated analogously.

Therefore,

\[ I + II + III = A + B = \sum_{j=1}^{m} \frac{\partial}{\partial y_j}(\ast) \]

and we have proved the claim above.

Now, if we assume the compactness of the support of \( u_a(x) \) for \( |a| \neq \tilde{1} \), then

\[ \int_{\pi_B(u)} dy \frac{\partial}{\partial y_i} \left( u_a(h(y, \omega)) \partial_{\omega}^{1-a} \text{Ber}(H(y, \omega)) \right) |_{\omega=0} = 0. \]
For \( H(y, \omega) = (y, \phi(y, \omega)) \) with \( \phi(y, \omega) = (\phi_1(y, \omega), \ldots, \phi_n(y, \omega)) \in \mathfrak{g}^{[n]} \), we want to claim
\[
\left( B - \iint_V dxd\theta u(x, \theta) \right) = \left( B - \iint_U dyd\omega \left( \det \left( \frac{\partial \phi_i}{\partial \omega_j} \right) \right)^{-1} u(y, \phi(y, \omega)) \right).
\]
By the analogous proof of (5.1.9) in Proposition 5.1.1, i.e. odd change of variables formula, we have the above readily. □

Remark 9.1.3. How we decompose a given superdiffeomorphism? Though Berezin decomposes it as (9.1.6) but Rothstein [111] introduces another decomposition and arguments which are outside of my comprehension. Moreover, there is another trial by Zirnbauer [138] which is not appreciated for me. I should be ashamed?! Therefore, I take the understandable arguments of Rogers, Vladimirov and Volvich with slight modification in Chapter 5. In any way, the following Rothstein’s decomposition is a key which is not proved here:

**Proposition 9.1.1 (Proposition 3.1 of Rothstein [111]).** Let superdiffeomorphism \( \varphi \) from \( \mathfrak{g} \) to \( U = \varphi(\mathfrak{g}) \) be given as
\[
x = x(y, \omega) = \varphi_0(y, \omega), \quad \theta = \theta(y, \omega) = \varphi_1(y, \omega).
\]
We assume that the following:
\[
\pi_B(\det J(\varphi)(y, \omega)) \neq 0 \quad \text{with} \quad J(\varphi)(y, \omega) = \begin{pmatrix}
\frac{\partial \varphi_0(y, \omega)}{\partial y} & \frac{\partial \varphi_1(y, \omega)}{\partial y} \\
\frac{\partial \varphi_0(y, \omega)}{\partial \omega} & \frac{\partial \varphi_1(y, \omega)}{\partial \omega}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x_1}{\partial y} & \frac{\partial \theta_1}{\partial y} \\
\frac{\partial x_1}{\partial \omega} & \frac{\partial \theta_1}{\partial \omega}
\end{pmatrix}.
\]
Then, \( \varphi \) is decomposed uniquely by \( \varphi^{(1)} : (y, \omega) \to (\tilde{y}, \tilde{\omega}) \) and \( \varphi^{(2)} : (\tilde{y}, \tilde{\omega}) \to (x, \theta) \) satisfying
(i) \( \varphi^{(1)} \) endows identical \( \mathbb{Z}_2 \)-gradings, that is, \( \varphi^{(1)} : (y, \omega) \to (\tilde{y}, \tilde{\omega}) = (\tilde{\varphi}_0^{(1)}(y), \tilde{\varphi}_1^{(1)}(y, \omega)) \),
and
(ii) \( \varphi^{(2)} \) is derived from the following even and degree increasing derivation \( \mathcal{Y}(\tilde{y}, \tilde{\omega}) \) by
\[
(x, \theta) = e^{\mathcal{Y}(\tilde{y}, \tilde{\omega})} (\tilde{y}, \tilde{\omega}) = \varphi^{(2)}(\tilde{y}, \tilde{\omega}) \quad \text{and} \quad (x, \theta) = \varphi^{(2)}(\varphi^{(1)}(y, \omega)) = \varphi^{(2)} \circ \varphi^{(1)}(y, \omega).
\]
Here, \( \lambda_j \in \mathcal{C}_{\text{SS}}(\varphi(\mathfrak{g}) : \mathfrak{g}^{[m]}) \), \( \gamma_k \in \mathcal{C}_{\text{SS}}(\varphi(\mathfrak{g}) : \mathfrak{g}^{[n]}) \) and
\[
(9.1.11) \quad \mathcal{Y}(\tilde{y}, \tilde{\omega}) = \sum_{j=1}^m \lambda_j(\tilde{y}, \tilde{\omega}) \frac{\partial}{\partial y_j} + \sum_{k=1}^n \gamma_k(\tilde{y}, \tilde{\omega}) \frac{\partial}{\partial \omega_k}.
\]

\footnote{Though following "Quick response to feel strange", I asked him by mail, but no response. I don’t know exactly, but he maybe changes from mathematician to different occupation}
9.2. Function spaces and Fourier transformations

9.2.1. Function spaces. Let \( U \subset \mathbb{R}^m \) be a domain. We introduce following function spaces:

**Definition 9.2.1** (Function spaces on \( U \) with values in \( \mathcal{C} \)).

\[
C^\infty(U : \mathcal{C}) = \{ u(q) = \sum_{i \in I} u_1(q)\sigma^i \mid u_1(q) \in C^\infty(U : \mathcal{C}) \text{ for any } I \in \mathcal{I} \},
\]

\[
C^\infty_0(U : \mathcal{C}) = \{ u(q) \in C^\infty(U : \mathcal{C}) \mid u_1(q) \in C^\infty_0(U : \mathcal{C}) \text{ for any } I \in \mathcal{I} \},
\]

\[
\mathcal{B}(U : \mathcal{C}) = \{ u(q) \in C^\infty(U : \mathcal{C}) \mid u_1(q) \in \mathcal{B}(U : \mathcal{C}) \text{ for any } I \in \mathcal{I} \},
\]

\[
\dot{\mathcal{B}}(\mathbb{R}^m : \mathcal{C}) = \{ u(q) \in C^\infty(U : \mathcal{C}) \mid u_1(q) \in \dot{\mathcal{B}}(\mathbb{R}^m : \mathcal{C}) \text{ for any } I \in \mathcal{I} \},
\]

\[
\mathcal{S}(\mathbb{R}^m : \mathcal{C}) = \{ u(q) \in C^\infty(\mathbb{R}^m : \mathcal{C}) \mid u_1(q) \in \mathcal{S}(\mathbb{R}^m : \mathcal{C}) \},
\]

\[
\mathcal{O}_M(\mathbb{R}^m : \mathcal{C}) = \{ u(q) \in C^\infty(\mathbb{R}^m : \mathcal{C}) \mid u_1(q) \in \mathcal{O}_M(\mathbb{R}^m : \mathcal{C}) \}. \]

**Remark 9.2.1.** In the above, we use the notation given in L.Schwartz [116]. That is,

\[
\dot{\mathcal{B}}(\mathbb{R}^m : \mathcal{C}) = \{ u(q) \in \mathcal{B}(\mathbb{R}^m : \mathcal{C}) \mid \lim_{|q| \to \infty} |\partial^\alpha_q u(q)| = 0 \text{ for any } \alpha \},
\]

\[
\mathcal{O}_M(\mathbb{R}^m : \mathcal{C}) = \{ u(q) \in C^\infty(\mathbb{R}^m : \mathcal{C}) \mid |\partial^\alpha_q u(q)| \leq C(1 + |q|^2)^{k/2} \}
\]

for some constants \( C > 0 \) and \( k > 0 \).

For a superdomain \( \mathcal{U} = \mathcal{U}_{ev} \times \mathcal{R}^n_{od} \subset \mathbb{R}^{m+n} \) with \( U = \pi_B(\mathcal{U}) = \pi_B(\mathcal{U}_{ev}) \subset \mathbb{R}^m \), we put:

**Definition 9.2.2** (Function spaces on \( \mathcal{U} \) with values in \( \mathcal{C} \)). Putting \( X = (x, \theta) \), \( X_B = q \in U \), \( \partial^\alpha_\theta u(x, 0) = u_a(x) \),

\[
\mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) = \{ u(X) = \sum_{|\alpha| \leq n} u_a(x)\theta^\alpha \mid u_a(q) \in C^\infty(U : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{C}_{SS,0}(\mathcal{U} : \mathcal{C}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) \mid u_a(q) \in C^\infty_0(U : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{B}_{SS}(\mathcal{U} : \mathcal{C}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) \mid u_a(q) \in \mathcal{B}(U_B : \mathcal{C}) \text{ for any } a \},
\]

\[
\dot{\mathcal{B}}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) \mid u_a(q) \in \dot{\mathcal{B}}(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{S}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) \mid u_a(q) \in \mathcal{S}(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{O}_{M,SS}(\mathcal{R}^{m+n} : \mathcal{C}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) \mid u_a(q) \in \mathcal{O}_M(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \}. \]

**Remark 9.2.2.** If we consider function spaces whose member are homogeneous, we denote them by adding subindices \( ev \) or \( od \), i.e. \( \mathcal{C}_{SS,ev}(\mathcal{U} : \mathcal{C}) = \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}_{ev}) \) or \( \mathcal{C}_{SS,od}(\mathcal{U} : \mathcal{C}) = \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}_{od}) \) etc.

**Definition 9.2.3** (Function spaces on \( \mathcal{U} \) with value ‘real’).

\[
\mathcal{C}_{SS}(\mathcal{U}_{ev}) = \{ u(x) \in \mathcal{C}_{SS}(\mathcal{U}_{ev} : \mathcal{C}) \mid u(x_B) \in C^\infty(\mathcal{U}_B : \mathcal{C}) \},
\]

\[
\mathcal{C}_{SS}(\mathcal{U}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) \mid \partial^\alpha_\theta u(x, 0) \in \mathcal{C}_{SS}(\mathcal{U}_{ev}) \text{ for any } a \},
\]

\[
\mathcal{C}_{SS,0}(\mathcal{U}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) \mid \partial^\alpha_\theta u(x_B) \in C^\infty_0(\mathcal{U}_B : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{B}_{SS}(\mathcal{U}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) \mid \partial^\alpha_\theta u(x_B) \in \mathcal{B}(\mathcal{U}_B : \mathcal{C}) \text{ for any } a \},
\]

\[
\dot{\mathcal{B}}_{SS}(\mathcal{R}^{m+n}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) \mid \partial^\alpha_\theta u(x_B) \in \dot{\mathcal{B}}(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{S}_{SS}(\mathcal{R}^{m+n}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) \mid \partial^\alpha_\theta u(x_B) \in \mathcal{S}(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \},
\]

\[
\mathcal{O}_{M,SS}(\mathcal{R}^{m+n}) = \{ u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m+n} : \mathcal{C}) \mid \partial^\alpha_\theta u(x_B) \in \mathcal{O}_M(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \}. \]
DEFINITION 9.2.4 (Topology of function spaces).

- We introduce seminorms in $C_{SS}(\mathfrak{U} : \mathfrak{C})$ for any integer $k$, $\mathfrak{I} \in \mathcal{I}$ and a compact set $K \subset \mathfrak{U}_B$, by defining

$$p_{k,\mathfrak{I}}(u) = \sup_{X_B \in K, |a| \leq k} |\text{proj}_\mathfrak{I}(D_X^a u(X_B))|.$$ 

$C_{SS}(\mathfrak{U} : \mathfrak{C})$ with this topology will be denoted by $E_{SS}(\mathfrak{U} : \mathfrak{C})$.

- We say that $u_j \to 0$ in $C_{SS,0}(\mathfrak{U} : \mathfrak{C})$ when $j \to \infty$ iff for any $\mathfrak{I} \in \mathcal{I}$, there exists a compact set $K_1 \subset U_B$ such that

(i) the support of $\text{proj}_\mathfrak{I}(\partial_0^a u_j(\cdot, 0))$ is contained in $K_1$ for any $a$ and $j$, and

(ii) $\sup_{X_B \in K_1} |\text{proj}_\mathfrak{I}(D_X^a u_j(X_B))| \to 0$ as $j \to \infty$ for any $a = (\alpha, \alpha)$.

We denote $D_{SS}(\mathfrak{U} : \mathfrak{C})$ the set $C_{SS,0}(\mathfrak{U} : \mathfrak{C})$ with this topology and call it as the space of test functions on $U$.

- We say that $u_j \to 0$ in $B_{SS}(\mathfrak{U} : \mathfrak{C})$ iff for any $\mathfrak{I} \in \mathcal{I}$ and $\mathfrak{a}$, $\text{proj}_\mathfrak{I}(D_X^a u_j(X_B))$ converges uniformly to 0 on any compact set $K_1$ and they are bounded on $\mathbb{R}^m$.

- For $u \in S_{SS}(\mathbb{R}^m) : \mathfrak{C})$ and any integer $k$ and $\mathfrak{I} \in \mathcal{I}$, we put

$$p_{k,\mathfrak{I}}(u) = \sup_{X_B \in \mathbb{R}^m, |a| + j \leq k} (1 + |X_B|^2)^{j/2} |\text{proj}_\mathfrak{I}(D_X^a u(X_B))|.$$ 

The topology of spaces $\mathcal{F}_{SS}(\mathfrak{U})$, $\mathcal{D}_{SS}(\mathfrak{U})$, $\mathcal{B}_{SS}(\mathfrak{U})$ and $\mathcal{S}_{SS}(\mathbb{R}^m)$ is defined accordingly as above.

9.2.2. Scalar products and norms.

DEFINITION 9.2.5 (Hermitian conjugation).

- For $x = (x_1, \ldots, x_m) \in \mathbb{R}_m$ and $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}_n$, we put

$$\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \text{ with } \bar{x}_j = \sum_{|I| = \text{ev}} x_I \sigma^I$$

where $\sigma^I$ is defined in (2.1.15) and $\bar{x}_j$ is the complex conjugate of $x_{j,\mathfrak{I}}$ in $\mathfrak{C}$.

(i) $\bar{\theta}^a = \theta_a^{\ast} \ldots \theta_1^{\ast} = (-1)^{|a|-1/2} \bar{\theta}^a$ and $\bar{\theta}_k = \sum_{|I| = \text{od}} \bar{\theta}_{k,I} \sigma^I$

where $\bar{\theta}_{k,I}$ is the complex conjugate of $\theta_{k,I}$ in $\mathfrak{C}$.

- Let $w(q) = \sum_{I \in \mathcal{I}} w_1(q) \sigma^I \in C^\infty(\mathbb{R}^m : \mathfrak{C})$. We put

$$\bar{w}(q) = \sum_{I \in \mathcal{I}} \bar{w}_1(q) \sigma^I \text{ with } \bar{w}_1(q) = \text{the complex conjugate of } w_1(q) \text{ in } \mathfrak{C}.$$ 

- For $\bar{w}(x) \in C_{SS}(U_{\text{ev}} : \mathfrak{C})$ with $w(q) \in C^\infty(\mathbb{R}^m : \mathfrak{C})$, we put

$$\bar{w}(x) = \bar{w}(\bar{x}).$$ 

- For $v(\theta) = \sum_{|a| \leq n} \theta^a v_a \in \mathcal{P}_n(\mathfrak{C})$, we put

$$\bar{v}(\theta) = \sum_{|a| \leq n} \bar{\theta}^a v_a^{\ast} = \bar{v}(\bar{\theta}) \text{ with } v_a^{\ast} = (-1)^{|a|-1/2} (\bar{v}_a^{\ast})_{\text{ev}} + (-1)^{|a|} (\bar{v}_a^{\ast})_{\text{od}}.$$
9.2. FUNCTION SPACES AND FOURIER TRANSFORMATIONS

- For \( u(X) \in \mathcal{C}_{SS}(\mathcal{U} : \mathcal{C}) \), we put
  \[ u(X) = \sum_{|a| \leq n} \partial^a u_a^*(\bar{x}) = \bar{u}(\bar{X}) \quad \text{where} \quad \bar{X} = (\bar{x}, \bar{\theta}) \]  
  \[ u_a(x) = \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha u_a(x_B) x^\alpha_S = \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha u_a(x_B) (x_B + \bar{x}), \quad \text{where} \quad \bar{x} = x_B + \bar{x}. \]

Remark 9.2.3. (1) The Grassmann continuation \( \tilde{w}(x) \) is expanded as
  \[ \tilde{w}(x) = \sum_H \tilde{w}_H(x) \sigma^H \]
  with
  \[ \tilde{w}_H(x) = \sum_{H=J+I^{(1)}+\ldots+I^{(m)}, I_j=I^{(1)}_j+\ldots+I^{(s)}_j, \alpha=(\alpha_1, \ldots, \alpha_m)} (-1)^s \frac{1}{\alpha!} \partial^\alpha w(x_B) x^\alpha_S = \tilde{w}(\bar{x}). \]

This guarantees the naturalness of (9.2.3).

(2) The definition of (9.2.4) and (9.2.5) follows from the relation \( \bar{v}_a v_a = v_a^a v_a \) for any \( v_a \in \mathcal{C} \).

Definition 9.2.6 (\( L^2 \) spaces).

- For \( u, w \in \mathcal{C}_{SS,0}(\mathcal{U}_{ev} : \mathcal{C}) \), we define the scalar product and the \( L^2 \)-norm by
  \[ (u, w) = \int_{\pi_B(\mathcal{U}_{ev})} dx_B u(x_B) \overline{w(x_B)} \in \mathcal{C} \quad \text{and} \quad \|u\|^2 = (u, u) \in \mathcal{R}. \]

- For \( v \in \mathcal{P}_n(\mathcal{C}) \), we put
  \[ (v, w) = \sum_{|a| \leq n} \overline{v_a} w_a \quad \text{and} \quad \|v\|^2 = (v, v). \]

- For \( u \in \mathcal{C}_{SS,0}(\mathcal{U} : \mathcal{C}) \), we define also
  \[ (u, w) = \sum_{|a| \leq n} \int_{\mathcal{U}_{ev}} dx_u a(x) w_a(x) = \sum_{|a| \leq n} \int_{\pi_B(\mathcal{U}_{ev})} dx_B u_a(x_B) w_a(x_B) \quad \text{with} \quad \|u\|^2 = (u, u). \]

Remark 9.2.4. (1) It is clear that if \( u, v \) are in \( \mathcal{C}_{SS,0}(\mathcal{U}) \) or \( \mathcal{F}_{SS}(\mathcal{R}^m|\mathcal{R}), (u, v) \in \mathcal{C} \) and \( (u, u) = \|u\|^2 \geq 0. \)

(2) In the following, we explain the derivation of the above scalar products:
(i) If \( f(q), g(q) \in C^\infty(\mathcal{U}_B : \mathcal{C}) \), using \( \delta \)-function symbolically, we may consider the standard scalar product as
  \[ (f, g) = \int_{\mathcal{U}_B} dq f(q) g(q) = \int_{\mathcal{U}_B, q'} dq' dq \delta(q-q') f(q) g(q') \quad \text{with} \quad U_B = \pi_B(\mathcal{U}). \]

On the other hand, for \( u(x) \in \mathcal{C}_{SS,0}(\mathcal{U}_{ev,x} : \mathcal{C}) \) and \( w(y) \in \mathcal{C}_{SS,0}(\mathcal{U}_{ev,y} : \mathcal{C}) \), we may regard \( u(x) w(y) \) as a function \( \overline{u}(\overline{X}) w(y) \in \mathcal{C}_{SS,0}(\mathcal{U}_{ev,\overline{X}} \times \mathcal{U}_{ev,y} : \mathcal{C}) \). Therefore, remarking \( \overline{u}(\overline{X}) w(y) = \overline{u}(x_B) w(x_B) = \sum_{I=J+K \in \mathcal{T}} (\overline{-1}^{(1, J, K)}) u_I(q) w_K(q) \sigma^I \) with \( \sigma^I \sigma^K = (\overline{-1}^{(1, J, K)}) \sigma^I \) for \( I = J + K \), we may define
  \[ (u, w) = \int_{\mathcal{U}_{ev,y} \times \mathcal{U}_{ev,\overline{X}}} dy \overline{dx} \delta(\bar{x} - y) u(x) w(y) = \int_{\mathcal{R}^m} dx_B u(x_B) w(x_B) \in \mathcal{C}. \]
(ii) We introduce here a constant $\tau(a, b)$ for any multi-indeces $a, b$ by

$$\theta^a \theta^b = (-1)^{\tau(a,b)} \theta^{a+b}$$  \hspace{1cm} (9.2.11)$$

from which we get easily

$$\tau(b,a) \equiv |a||b| + \tau(a,b) \mod 2.$$  \hspace{1cm} (9.2.12)$$

Moreover, for any $b$ and $\theta$, $\bar{\theta}$, $\pi \in \mathcal{R}_{od}$, we get, by induction with respect to $|b|$,  \hspace{1cm} (9.2.13)$$

$$\prod_{j=1}^n (\bar{\pi}_j \theta_j)^{b_j} = (-1)^{|b||b|-1}/2 \pi^b \theta^b$$ \quad \text{and} \quad \prod_{j=1}^n (\bar{\theta}_j \theta_j)^{b_j} = (-1)^{|b|(|b|-1)/2} \bar{\theta}^b \theta^b.$$  \hspace{1cm} (9.2.14)$$

(iii) By putting $\langle \bar{\theta} \theta \rangle = \sum_{j=1}^n \bar{\theta}_j \theta_j$ and

$$\bar{d}\theta = \bar{d}\theta_1 \cdots \bar{d}\theta_n = (-1)^{n(n-1)/2} \bar{d}\theta_n \cdots \bar{d}\theta_1,$$  \hspace{1cm} (9.2.14)$$

we get [9.2.8] from

$$\int \int_{\mathcal{R}^n_{od} \times \mathcal{R}^n_{ev}} d\bar{\theta} d\theta \frac{e^{\langle \bar{\theta} \theta \rangle}}{\bar{\theta}^a \theta^b} w_b = \int \int_{\mathcal{R}^n_{od} \times \mathcal{R}^n_{ev}} d\bar{\theta} d\theta \frac{e^{\langle \bar{\theta} \theta \rangle}}{\bar{\theta}^a \theta^b} w_b = \sum_a (u_a)_{ev} + (u_a)_{od} w_a = \sum_a u_a w_a.$$  \hspace{1cm} (9.2.15)$$

Here, we used equalities below and [9.2.13]:

$$\left\{ \begin{array}{l}
\bar{\theta}^a \theta^b w_b = \theta^a \theta^b ((u_a)_{ev} + (-1)^{|a||b|} (u_a)_{od}) w_b,

\int \int_{\mathcal{R}^n_{od} \times \mathcal{R}^n_{ev}} d\theta d\bar{\theta} \frac{e^{\langle \bar{\theta} \theta \rangle}}{\bar{\theta}^a \theta^b} = \delta_{ab} = \prod_{k=1}^n \delta_{a_k b_k}.
\end{array} \right.$$  \hspace{1cm} (9.2.16)$$

In fact, taking up the top term w.r.t. $\bar{\theta}$,

$$e^{\langle \bar{\theta} \theta \rangle} \frac{\bar{\theta}^a \theta^b}{\bar{\theta}^a \theta^b} \sim (-1)^{|a||a|-1}/2 \bar{\theta}^a \prod_{j=1}^n (\bar{\theta}_j \theta_j)^{\bar{a}_j} \theta^a = (-1)^{|a||a|-1}/2 \bar{\theta}^a (-1)^{|\bar{a}||\bar{a}|-1}/2 \bar{\theta}^a \theta^b$$

$$= (-1)^{|a||a|-1}/2 \bar{\theta}^a \theta^b,$$

and using [9.2.14], we have

$$\int \int_{\mathcal{R}^n_{od} \times \mathcal{R}^n_{ev}} d\bar{\theta} d\theta \frac{e^{\langle \bar{\theta} \theta \rangle}}{\bar{\theta}^a \theta^b} = (-1)^{n(n-1)/2+|a||a|-1}/2 \bar{\theta}^a \theta^b + \tau(a, \bar{a}) \delta_{ab} = \delta_{ab}$$

since $n(n-1)/2 + |a||a|-1)/2 + |\bar{a}||\bar{a}|-1)/2 + \tau(a, \bar{a}) \delta_{ab} = \delta_{ab}$.

(iv) Finally, as we have

$$\int \int_{\mathcal{U}_X \times \mathcal{U}_\chi} dX \bar{d}X \delta(\bar{x} - x) e^{\langle \bar{\theta} \theta \rangle} u(X) w(X)$$

$$= \int \int_{\pi_B(\mathcal{U}_ev) \times \pi_B(\mathcal{U}_ev)} dx d\bar{x} \delta(\bar{x} - x) \left\{ \int \int_{\mathcal{R}^n_{od} \times \mathcal{R}^n_{ev}} d\theta d\bar{\theta} e^{\langle \bar{\theta} \theta \rangle} u(x, \theta) w(x, \theta) \right\}$$

$$= \sum_{|a| \leq n} \int \pi_B(\mathcal{U}_ev) dx_B u_a(x_B) w_a(x_B) = \sum_{|a| \leq n} (u_a, w_a),$$  \hspace{1cm} (9.2.17)$$

our definition [9.2.9] seems canonical.
Lemma 9.2.1. Denoting by $A^*$ the dual of the operator $A$ in the above scalar product, we get easily

\begin{align}
\langle \frac{h}{i} \partial x_j \rangle^* &= \frac{h}{i} \partial x_j, \quad (x_j)^* = x_j, \hspace{1cm} (2.18) \\
\langle \theta_k \rangle^* &= \theta_k, \quad (\theta_k)^* = \partial \theta_k, \quad (D_\theta)^* = \theta^a, \quad (\partial_\theta)^* = \iota(\theta^a). \hspace{1cm} (2.19)
\end{align}

Proof. For $a, b \in \{0, 1\}^n$, we put $\hat{a}_k = (a_1, \ldots, a_k - 1, \ldots, a_n)$, $\hat{b}_k = (b_1, \ldots, b_k + 1, \ldots, b_n)$ where $\partial \theta_k \theta^a = 0$ with $a$ for $a_k = 0$. Let $v, w \in \mathcal{P}_n(\mathcal{C})$. Remarking (9.2.16), we have

\[
\partial \theta_k (\theta^a u_a)(\theta^b v_b) = \tilde{\theta} \theta^b ((\theta^a v_b) = \tilde{\theta} \theta^b ((\theta^a v_b + (-1)^{|a|+|b|}(\theta^a v_b)) v_b,
\]

and

\[
\overline{\theta^a u_a \theta_k v_b} = \overline{\theta^a} ((\theta^b v_b) = \tilde{\theta} ((\theta^a v_b + (-1)^{|a|+|b|}(\theta^a v_b)) v_b,
\]

which yield

\[
\langle \phi_k u, v \rangle = \int_{\mathbb{R}^n} d\theta \partial \theta e^{(\theta|)} \sum_a \partial \partial \theta_k (\theta^a u_a) \sum_b v_b \theta^b
\]

\[
= \sum_{a,b} (-1)^{|k|} \tilde{\delta}_{k,b} ((\theta^a v_b) + (-1)^{|a|+|b|}(\theta^a v_b)) v_b
\]

\[
= \sum_{a,b} (-1)^{|k|} \tilde{\delta}_{k,b} ((\theta^a v_b) + (-1)^{|a|+|b|+1}(\theta^a v_b)) v_b
\]

\[
= \int_{\mathbb{R}^n} d\theta \partial \theta e^{(\theta|)} \sum_a u_a \theta^a \sum_b \theta_k v_b \theta^b = \langle u, \theta_k v \rangle.
\]

Repeating this argument, we have other equalities readily. $\square$

9.2.3. Distributions.

Definition 9.2.7 (Distributions on $U \subset \mathbb{R}^m$ with values in $\mathcal{C}$).

- Let $\Phi$ be a linear functional defined on $\mathcal{D}_x(U : \mathcal{C})$ such that $\Phi(u_j) \to 0$ in $\mathcal{C}_{x,0}(U : \mathcal{C})$. Then, we call this functional as a distribution on $U$, and the set composed of these is denoted by $\mathcal{D}'_x(U : \mathcal{C})$.
- $\mathcal{E}'_x(U : \mathcal{C})$ stands for the set consisting of continuous linear functionals on $\mathcal{E}_x(U : \mathcal{C})$.
- $\mathcal{B}'_x(U : \mathcal{C})$ stands for the set consisting of continuous linear functionals on $\mathcal{B}_x(U : \mathcal{C})$.
- $\Phi \in \mathcal{S}'_x(\mathbb{R}^m : \mathcal{C})$ iff $\Phi$ is a continuous linear functional on $\mathcal{S}_x(\mathbb{R}^m : \mathcal{C})$.
- $\mathcal{D}'_x(U), \mathcal{E}'_x(U), \mathcal{B}'_x(U)$ and $\mathcal{S}'_x(\mathbb{R}^m : \mathcal{C})$ are defined analogously.

\[
\mathcal{D}'_x(U : \mathcal{C}) = \{ \Phi(q) = \sum_{I \in \mathcal{I}} \Phi_1(q) \sigma^I | \Phi_1(q) \in \mathcal{D}'(U : \mathcal{C}) \ \text{for any} \ I \in \mathcal{I} \},
\]

\[
\mathcal{E}'_x(U : \mathcal{C}) = \{ \Phi(q) \in \mathcal{D}_x(U : \mathcal{C}) | \Phi(q) = \text{proj}_1(\Phi(q)) \in \mathcal{E}'(U : \mathcal{C}) \ \text{for any} \ I \in \mathcal{I} \},
\]

\[
\mathcal{S}'_x(\mathbb{R}^m : \mathcal{C}) = \{ \Phi(q) \in \mathcal{D}_x(\mathbb{R}^m : \mathcal{C}) | \Phi(q) \in \mathcal{S}'(\mathbb{R}^m : \mathcal{C}) \ \text{for any} \ I \in \mathcal{I} \}.
\]

Here, $\Phi \in \mathcal{D}'_x(U : \mathcal{C})$ acts on $u \in \mathcal{D}_x(U : \mathcal{C})$ by

\[
\langle \Phi, u \rangle = \sum_{I \in \mathcal{I}} \left( \sum_{J+K} \langle \Phi_J, u_K \rangle \right) \sigma^I.
\]

Other dualities are defined analogously.
**Definition 9.2.8** (Distributions on $\mathcal{U}$ with values in $\mathcal{C}$).

$$D'_{SS}(\mathcal{U} : \mathcal{C}) = \{ \Phi(X) = \sum \Phi_a(x)\theta^a \mid \Phi_a(X_B) = \partial_a^\theta \Phi(x_B, 0) \in D'_S(\mathcal{U}_B : \mathcal{C}) \text{ for any } a \},$$

$$E'_{SS}(\mathcal{U} : \mathcal{C}) = \{ \Phi(X) \in D'_{SS}(\mathcal{U} : \mathcal{C}) \mid \Phi_a(X_B) \in E'_{SS}(\mathcal{U}_B : \mathcal{C}) \text{ for any } a \},$$

$$S'_{SS}(\mathbb{R}^m^n : \mathcal{C}) = \{ \Phi(X) \in D'_{SS}(\mathbb{R}^m^n : \mathcal{C}) \mid \Phi_a(X_B) \in S'_{SS}(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \}.$$

Action of $\Phi \in D'_{SS}(\mathcal{U} : \mathcal{C})$ on $u \in D_{SS}(\mathcal{U} : \mathcal{C})$ are defined by

$$\langle \Phi, u \rangle = \sum_{|a| \leq n} \langle \partial_a^\theta \Phi(x, \theta), \partial_a^\theta u(x, \theta) \rangle.$$

**Proposition 9.2.1.** Let $\Phi$ be a continuous linear functional on $D_{SS}(\mathcal{U} : \mathcal{C})$. Then, $\Phi(X)$ is represented by

$$\Phi(X) = \sum_{|a| \leq n} \Phi_a(x)\theta^a \text{ where } \text{proj}_I(\Phi_a(x_B)) \in D'(\mathcal{U}_B : \mathcal{C}) \text{ for each } I \in \mathcal{I}.$$

Analogous results hold for any element of $E'_{SS}(\mathcal{U} : \mathcal{C})$, $B'_{SS}(\mathcal{U} : \mathcal{C})$ or $S'_{SS}(\mathbb{R}^m^n : \mathcal{C})$.

**Proof** is omitted here.

**Definition 9.2.9** (Sobolev spaces). Let $k$ be a non-negative integer.

- We define, for $u, v \in C_{SS, 0}(\mathcal{U} : \mathcal{C})$,

$$(u, v)_k = \sum_{|a| \leq k} (D_X^a u, D_X^a v) \text{ and } ||u||_k^2 = ((u, u)_k).$$

- For any $u, v \in S_{SS}(\mathbb{R}^m^n : \mathcal{C})$, we define

$$(u, v)_k^2 = \sum_{|a| + |l| \leq k} ((1 + |X_B|^2)^{l/2} D_X^a u, (1 + |X_B|^2)^{l/2} D_X^a v)$$

and $||u||_k^2 = (((u, u)_k)_k).$

Now, we put

$$\tilde{L}^2_{SS}(\mathcal{U} : \mathcal{C}) = \{ u \in C_{SS, 0}(\mathcal{U} : \mathcal{C}) \mid ||u|| < \infty \},$$

$$\tilde{H}^k_{SS}(\mathcal{U} : \mathcal{C}) = \{ u \in C_{SS}(\mathcal{U} : \mathcal{C}) \mid ||u||_k < \infty \}$$

and taking the completion of these spaces with respect to corresponding norms, we get the desired spaces $L^2_{SS}(\mathcal{U} : \mathcal{C})$ and $H^k_{SS}(\mathcal{U} : \mathcal{C})$. The closure of $C_{SS, 0}(\mathcal{U} : \mathcal{C})$ in $H^k_{SS}(\mathcal{U} : \mathcal{C})$ is denoted by $H^k_{SS, 0}(\mathcal{U} : \mathcal{C})$. The spaces $L^2_{SS}(\mathcal{U})$ and $H^k_{SS}(\mathcal{U})$ are defined analogously.

**Remark.** We should consider the integral in the last form as the one in the Lebesgue sense.

**Definition 9.2.10.** Let $1 \leq r \leq \infty$.

$$D_{L^r, SS}(\mathbb{R}^m^n) = \{ u(X) \in S_{SS}(\mathbb{R}^m^n) \mid \partial_a^\theta u(X_B) \in D_{L^r}(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \}$$

where

$$D_{L^r}(\mathbb{R}^m : \mathcal{C}) = \{ u(q) \in C_\infty(\mathbb{R}^m : \mathcal{C}) \mid \partial_a^\theta u(q) \in L^r(\mathbb{R}^m : \mathcal{C}) \text{ for any } a \}.$$

The topology on $D_{L^r, SS}(\mathbb{R}^m^n)$ is defined by seminorms

$$p_{k,L^r}(u) = \sum_{|a| \leq k} ||\text{proj}_I(D_X^a u(X_B))||_{L^r}.$$
Definition 9.2.11.
\[
\mathcal{O}_c'(\mathbb{R}^m : \mathbb{C}) = \{ \Phi(q) \in \mathcal{D}'(\mathbb{R}^m : \mathbb{C}) \mid (1 + |q|^2)^k D^b_{\alpha} \Phi(q) \in \mathcal{B}'(\mathbb{R}^m : \mathbb{C}) \text{ for any } \alpha \text{ and some } k \},
\]
\[
\mathcal{O}_c'(\mathbb{R}^m : \mathbb{C}) = \{ \Phi \in \mathcal{D}'_{SS}(\mathbb{R}^m : \mathbb{C}) \mid \text{proj}_I(\Phi(q)) \in \mathcal{O}_c'(\mathbb{R}^m : \mathbb{C}) \},
\]
\[
\mathcal{O}_{c,SS}(\mathbb{R}^m|n : \mathbb{C}) = \{ \Phi \in \mathcal{D}'_{SS}(\mathbb{R}^m|n : \mathbb{C}) \mid D^b_{\alpha} \Phi(X_B) \in \mathcal{O}_c'(\mathbb{R}^m : \mathbb{C}) \}.
\]

Remark 9.2.5. (1) The following assertions follow directly from definitions above and the standard distribution theory of [116].

(i) \( \Phi \in \mathcal{D}_{c,SS}(\mathbb{R}^m|n : \mathbb{C}) \) iff for any \( \varphi \in \mathcal{D}(\mathbb{R}^m|n : \mathbb{C}) \), \( \Phi \ast \varphi \in \mathcal{S}_{SS}(\mathbb{R}^m|n : \mathbb{C}) \).

(ii) \( \mathcal{D}_{c,SS}(\mathbb{R}^m|n) \)' = \( \mathcal{B}'(\mathbb{R}^m|n) \).

(2) Though we don’t mention other properties on function or distribution spaces on \( \mathbb{U} \subset \mathbb{R}^m|n \), but they will be almost comparable to those in standard case treated in [116].

(3) Sobolev inequalities should be studied separately.

9.2.4. Fourier transformations, definitions and their basic properties. In this section, we borrow ideas from [9, 10, 34, 91] with necessary modifications.

9.2.4.1. Fourier transformations (even case). We introduce the Fourier and inverse Fourier transformations of functions with even variables. For \( u(x), v(\xi) \in \mathcal{S}_{SS}(\mathbb{R}^m|0 : \mathbb{C}) \), we define
\[
(F_e u)(\xi) = (2\pi \hbar)^{-m/2} \int_{\mathbb{R}^m|0} dx \, e^{-i \hbar^{-1} \langle x | \xi \rangle} u(x),
\]
\[
(F_e v)(\xi) = (2\pi \hbar)^{-m/2} \int_{\mathbb{R}^m|0} d\xi \, e^{i \hbar^{-1} \langle y | \xi \rangle} v(\xi).
\]

Remark 9.2.6. If \( F \) stands for the standard Fourier transformation on \( \mathcal{S}(\mathbb{R}^m : \mathbb{C}) \), then it acts on \( \mathcal{S}(\mathbb{R}^m : \mathbb{C}) \) by \( (Fu)(p) = \sum_{\lambda} \langle Fu_\lambda(p) | \lambda \rangle \) for \( u = \sum_{\lambda} u_\lambda(p) \) with \( u_\lambda(q) \in \mathcal{S}(\mathbb{R}^m : \mathbb{C}) \).

As \( u(x) \in \mathcal{S}_{SS}(\mathbb{R}^m|0 : \mathbb{C}) \) is the Grassmann continuation of \( u(q) \in \mathcal{S}(\mathbb{R}^m : \mathbb{C}) \), we need to say
\[
(Fu)(\xi) = (F_e \tilde{u})(\xi).
\]

Proposition 9.2.2. Let \( u, v \in \mathcal{S}_{SS}(\mathbb{R}^m|0 : \mathbb{C}) \).
\[
(F_e \partial_x^\alpha u)(\xi) = (ih)^{-|\alpha|} \xi^\alpha (F_e u)(\xi), \quad (F_e (x^\alpha u))(\xi) = (ih)^{|\alpha|} \partial_{\xi}^\alpha (F_e u)(\xi).
\]
\[
(F_e (e^{ih^{-1} \langle x | \xi \rangle} u))(\xi) = (F_e u)(\xi - \xi'), \quad (F_e (u(x - x')))(\xi) = e^{-ih^{-1} \langle x' | \xi \rangle} (F_e u)(\xi).
\]
\[
(F_e (u(tx)))(\xi) = |t|^{-m} (F_e u)(t^{-1} \xi) \quad \text{for } t \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}.
\]
\[
(F_e u = u \quad \text{and} \quad F_e \tilde{u} = v).
\]
\[
(u,v) = (F_e u, F_e v) \quad \text{and} \quad \|F_e u\| = \|u\|.
\]
\[
(F_e \delta)(\xi) = (2\pi \hbar)^{-m/2} \int_{\mathbb{R}^m|0} dx \, \delta(x) e^{-i \hbar^{-1} \langle x | \xi \rangle} = (2\pi \hbar)^{-m/2}.
\]

Moreover, \( F_e : \mathcal{S}_{SS}(\mathbb{R}^m|0 : \mathbb{C}) \to \mathcal{S}_{SS}(\mathbb{R}^m|0 : \mathbb{C}) \) mapping satisfying
\[
\| \text{proj}_I((F_e u)(\xi_B)) \|^2 \leq C_m \| \text{proj}_I(u(x_B)) \|^2 \quad \text{for any } I \in \mathcal{I}.
\]
Proof. As \( \partial_x^\alpha \tilde{u}(x) = (\partial_x^\alpha u)(x) \), if \( \xi = p \in \mathbb{R}^n \), we get the first part of (9.2.24) by
\[
(F_e(\partial_x^\alpha \tilde{u}))(\xi)|_{\xi=p} = (2\pi h)^{-m/2} \int_{\mathbb{R}^n} dq e^{-ih^{-1}(q|p)} \partial_q^\alpha u(q) = (ih^{-1})^{|\alpha|} p^\alpha (Fu)(p) = (ih^{-1})^{|\alpha|} \xi^\alpha (F_e \tilde{u})(\xi)|_{\xi=p}.
\]
The second equality in (9.2.24) is proved analogously and which shows that \( (F_e \tilde{u})(\xi) \in \mathcal{S}_{\mathbb{R}^{m+n}}(\mathbb{C}) \). Other equalities in (9.2.24)-(9.2.28) are proved as same as the standard case. (9.2.29) follows by defining \( \langle F_e \delta, \tilde{u} \rangle = \langle \delta, F_e \tilde{u} \rangle = \int_{\mathbb{R}^n} dp \delta(p)(F u)(p) = (F u)(0) = \int_{\mathbb{R}^n} dq u(q) = \int_{\mathbb{R}^n} dx \tilde{u}(x) = (F_e \tilde{u})(0) \). [9.2.30] is a direct consequence of the standard theory of Fourier transformation. \( \square \)

Remark 9.2.7. The Plancherel formula (9.2.28) stands for \( F_e^* = F_e = F_e^{-1} \).

9.2.4.2. Fourier transformations (odd case). For \( v(\theta), w(\pi) \in \mathcal{P}_n(\mathbb{C}) \), we define Fourier transformations with \( k \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \) as
\[
(F_o v)(\pi) = k^{n/2} \int_{\mathbb{R}^n} d\theta e^{-ik^{-1}(\theta|\pi)} v(\theta),
\]
\[
(F_o w)(\theta) = k^{n/2} \int_{\mathbb{R}^n} d\pi e^{ik^{-1}(\theta|\pi)} w(\pi)
\]
where we put
\[
t_n = e^{-i\pi n(n-2)/4} \quad \text{and} \quad t_n^2 = i^n (-1)^{n(n-1)/2} = \begin{cases} i & \text{when } n \equiv 1 \mod 4, \\
1 & \text{when } n \equiv 2 \mod 4, \\
i & \text{when } n \equiv 3 \mod 4, \\
1 & \text{when } n \equiv 0 \mod 4.
\end{cases}
\]

Remark 9.2.8. (1) Clearly, in (9.2.32), if we change the role of the variables \( \pi \) and \( \theta \), we get \( F_o = \tilde{F}_o \).
(2) Moreover, we may put differently as
\[
(\tilde{F}_o v)(\pi) = k^{n/2} \int_{\mathbb{R}^n} d\theta e^{-k^{-1}(\theta|\pi)} v(\theta),
\]
\[
\tilde{(F}_o w)(\theta) = k^{n/2} \int_{\mathbb{R}^n} d\pi e^{k^{-1}(\theta|\pi)} w(\pi).
\]
with
\[
J_n = e^{i\pi n(n-1)/2}, \quad J_n^2 = (-1)^{n(n-1)/2} = \begin{cases} 1 & \text{when } n \equiv 1 \mod 4, \\
1 & \text{when } n \equiv 0 \mod 4.
\end{cases}
\]

Proposition 9.2.3. Putting \( \tilde{1} = (1, \cdots, 1) \) and \( \tilde{a} = \tilde{1} - a \), we have, for \( v_a \in \mathcal{C} \),
\[
\int_{\mathbb{R}^n} d\theta e^{-i\theta^1(a|\pi)} \theta^a v_a = (-i)^{|a|} \{ (-1)^{|a|} / 2 \}^2 \pi^{\tilde{a}} v_a.
\]
Moreover, for \( v, w \in \mathcal{P}_n(\mathbb{C}) \), we have the following:
\[
(F_o(\partial^a v))(\pi) = (i^{\tilde{1}})^{|a|} (-1)^{|a|} \pi^{\tilde{a}} (F_o v)(\pi),
\]
\[
(F_o(\theta^a v))(\pi) = (i^{\tilde{1}})^{|a|} (-1)^{|a|} \partial^a (F_o v)(\pi).
\]
\[ (9.2.37) \quad (F_0(e^{ik^{-1}(\theta|\pi')v})(\pi) = (F_0v)(\pi - \pi'), \]
\[ (9.2.38) \quad (F_0(v(s\theta)))(\pi) = s^n(F_0v)(s^{-1}\pi) \quad \text{for } s \in \mathbb{C}^x. \]
\[ (9.2.39) \quad F_0F_0v = w \quad \text{and} \quad \tilde{F}_0F_0v = v. \]
\[ (9.2.40) \quad (v, w) = (F_0v, F_0w) \quad \text{and} \quad \|F_0v\| = \|v\| \quad \text{if } k = 1. \]
\[ (9.2.41) \quad (F_0\delta)(\pi) = k^{n/2}t_n \quad \text{for } \delta(\theta) = \theta^\top. \]

**Proof.** We get, by the definition of integration w.r.t. \( \theta \) and \( (9.2.13) \),
\[
\int_{\mathbb{R}^{l_n^n}} d\theta e^{-ik^{-1}(\theta|\pi)}\theta^a v_a = \int_{\mathbb{R}^{l_n^n}} d\theta \theta^a v_a \prod (-ik^{-1}\theta_j \pi_j)^{\bar{a}_j} = (-ik^{-1})^{\bar{a}} \bar{\theta} (-1)^{|\bar{a}|(|\bar{a}| - 1)/2 + \tau(a, \bar{a})} \pi^\bar{a} v_a. \]
Moreover, we get
\[
\int_{\mathbb{R}^{l_n^n}} d\pi e^{ik^{-1}(\theta|\pi)} \left\{ \int_{\mathbb{R}^{l_n^n}} d\theta' e^{-ik^{-1}(\theta'|\pi')}\theta'^a v_a \right\} = \int_{\mathbb{R}^{l_n^n}} d\pi e^{ik^{-1}(\theta|\pi)} \left[ (-ik^{-1})^{\bar{a}} \bar{\theta} (-1)^{|\bar{a}|(|\bar{a}| - 1)/2 + \tau(a, \bar{a})} \pi^\bar{a} v_a \right] = (-ik^{-1})^\bar{a} (-1)^{|\bar{a}|(|\bar{a}| - 1)/2 + \tau(a, \bar{a})} \theta^a v_a,
\]
where we used \( (9.2.13) \) with \( b = \bar{a} \). By the definition of \( t_n, \ t_n^2(-1)^n(n-1)/2 = 1 \) and \( (9.2.12) \), we have the Fourier inversion formula \( (9.2.39) \). Or, we may prove directly this by changing the order of integration:
\[
\tilde{F}_0F_0w(\theta) = k^{n/2}t_n^2 \int d\pi d\theta'^{-ik^{-1}(\theta'|\pi')}w(\theta') = k^{n/2}(-1)^n \int d\theta' \left\{ \int d\pi e^{-ik^{-1}(\theta|\pi')} \right\} w(\theta') \]
\[ = k^{n/2}(-1)^n \int d\theta'(ik^{-1})^\bar{a} (-1)^{(n(n-1)/2)(\theta' - \theta)\top}w(\theta') = k^{n/2} (\theta' - \theta) w(\theta'). \]

Remarking \( \partial_{\theta_j}(e^{-ik^{-1}(\theta|\pi)v}) = e^{-ik^{-1}(\pi|\theta)}(-ik^{-1}\pi_j v + \partial_{\theta_j} v), \) we get, after integration with respect to \( \theta, \)
\[
(F_0\delta(\theta) v(\pi) = ik^{-1} (-1)^n \pi_j F_0 v(\pi)
\]
which proves the first equality of \( (9.2.36) \) when \( |a| = 1 \). Assuming the first equality of \( (9.2.36) \) holds for any \( a \) satisfying \( |a| = l \), we apply the above for \( w(\theta) = (\partial_{\theta}^a v)(\theta) \). Then, we get
\[
(1)^{\ell_j(a)} F_0 \partial^\bar{a}_j w = F_0 \partial^\bar{a}_j w = ik^{-1} (-1)^n \pi_j F_0 w
\]
\[ = (ik^{-1})^{\bar{a}} (-1)^{(n(n-1)/2)(\pi_j \pi^\bar{a})} F_0 v = (ik^{-1})^{\bar{a}} (-1)^{(n(n-1)/2)(\pi_j \pi^\bar{a})} F_0 v.
\]
As before, we put \( \hat{a}_j = (a_1, \ldots, a_{j-1}, a_j + 1, a_{j+1}, \ldots, a_n) \) and \( \ell_j(a) = \sum_{k=1}^{j-1} a_k \).

To prove the Plancherel formula \( (9.2.40) \), remarking that
\[
(F_0 u(\pi) = k^{n/2}t_n \sum_{|a| \leq n} (ik^{-1})^{\bar{a}} (-1)^{|\bar{a}|(|\bar{a}| - 1)/2 + \tau(a, \bar{a})} \pi^\bar{a} u_a,
\]
\[ (F_0 u(\pi) = k^{n/2}t_n \sum_{|\bar{b}| \leq n} (-ik^{-1})^{\bar{b}} (-1)^{|\bar{b}|(|\bar{b}| - 1)/2 + \tau(b, \bar{b})} \pi^\bar{b} u_b, \]
we have
\[
(F_o u, F_o u') = k^n \sum_{|a| \leq n, |b| \leq n} \int d\pi d\bar{\pi} e^{\langle \pi | \bar{\pi} \rangle} (i k^{-1})^{|a|} (-1)^{|a||b|-1}/2 + \tau(a,\bar{a}) \pi^a \bar{a}_{\pi} u_a u'_b
\]
\[
= \sum_{|a| \leq n} k^{n-2|a|} u_a u'_a.
\]
This implies (9.2.1) for \( k = 1 \). Especially in case \( n = 2, |a| = 1 \), above holds for any \( k \). Other equalities are proved by the analogous fashion so omitted. \( \square \)

**Example 9.2.1** (\( n = 2 \)). For \( u(\theta) = u_0 + \theta_1 \theta_2 u_1 \) and \( v(\pi) = \pi_1 v_1 + \pi_2 v_2 \) with \( u_0, u_1, v_1, v_2 \in \mathcal{C} \), we have
\[
(F_o u)(\pi) = k \int_{\mathbb{S}^0 q_2} d\theta e^{-i k^{-1}(\theta | \pi)} u(\theta) = k(u_1 + k^{-2} \pi_1 v_2 u_0),
\]
\[
(F_o v)(\pi) = k \int_{\mathbb{S}^0 q_2} d\pi e^{i k^{-1}(\theta | \pi)} v(\pi) = k(-i k^{-1} \theta v_1 + i k^{-1} \theta v_2),
\]
\[
F_o(F_o u)(\pi) = k \int_{\mathbb{S}^0 q_2} d\pi e^{i k^{-1}(\theta | \pi)} [k(u_1 + k^{-2} \pi_1 v_2 u_0)] = u_0 + \theta_1 \theta_2 u_1 = u(\theta),
\]
\[
F_o(F_o v)(\pi) = k \int_{\mathbb{S}^0 q_2} d\theta e^{-i k^{-1}(\theta | \pi)} [k(-i k^{-1} \theta v_1 + i k^{-1} \theta v_2)] = \pi_1 v_1 + \pi_2 v_2 = v(\pi).
\]
Therefore, we get
\[
(F_o u, F_o u') = \int d\pi d\bar{\pi} e^{\langle \pi | \bar{\pi} \rangle} F_o u(\pi) F_o u'(\pi)
\]
\[
= \frac{\partial}{\partial \pi_1} \frac{\partial}{\partial \bar{\pi}_1} \frac{\partial}{\partial \pi_2} \frac{\partial}{\partial \bar{\pi}_2} [e^{\langle \pi | \bar{\pi} \rangle} (k_{\bar{\pi}_1} - k^{-1} u_0 \bar{\pi}_2)(k u_1 + k^{-1} \pi_2 u_0)] \bigg|_{\pi = \bar{\pi} = 0}
\]
\[
= k^2 u_1 + k^{-2} u_0,
\]
which implies that the Plancherel formula \((F_o u, F_o u') = (u, u')\) for above \( u, u' \) holds only when \( k = 1 \).

Analogously but for any \( k \neq 0 \),
\[
(F_o v, F_o v') = \int d\pi d\bar{\pi} e^{\langle \pi | \bar{\pi} \rangle} F_o v(\pi) F_o v'(\pi)
\]
\[
= \frac{\partial}{\partial \pi_1} \frac{\partial}{\partial \bar{\pi}_1} \frac{\partial}{\partial \pi_2} \frac{\partial}{\partial \bar{\pi}_2} [e^{\langle \pi | \bar{\pi} \rangle} (v_1 \bar{\pi}_1 - \bar{\pi}_2)(\pi_1 v'_1 - \pi_2 v'_2)] \bigg|_{\pi = \bar{\pi} = 0}
\]
\[
= \bar{v}_1 v'_1 + \bar{v}_2 v'_2 = (v, v'). \quad //
\]

**Example 9.2.2** (\( n = 3 \)). Let \( u(\theta) = u_0 + \theta_1 \theta_2 u_1 + \theta_2 \theta_3 u_2 + \theta_1 \theta_3 u_3, v(\theta) = \theta v_1 + \theta_2 v_2 + \theta_3 v_3 + \theta_1 \theta_2 \theta_3 v_4 \) with \( u_0, u_1, u_2, u_3, v_1, v_2, v_3, v_4 \in \mathcal{C} \), we have the following inversion formula:
\[
(F_o u)(\pi) = k^{3/2} t_3 \int_{\mathbb{S}^0 q_3} d\theta e^{-i k^{-1}(\theta | \pi)} u(\theta)
\]
\[
= k^{3/2} t_3 [ik^{-3} \pi_3 \pi_2 \pi_1 u_0 - ik^{-1} \pi_3 u_1 - ik^{-1} \pi_1 u_2 + ik^{-1} \pi_2 u_3],
\]
\[
F_o(F_o u)(\pi) = k^{3/2} t_3 \int_{\mathbb{S}^0 q_3} d\pi e^{i k^{-1}(\theta | \pi)} (F_o u)(\pi)
\]
\[
= k^{3/2} t_3 [ik^{-3} \pi_3 \pi_2 \pi_1 u_0 - ik^{-1} \pi_3 u_1 - ik^{-1} \pi_1 u_2 + ik^{-1} \pi_2 u_3]
\]
\[
= k^{3/2} t_3 [ik^{-3} u_0 + ik^{-3} \theta_2 \theta_1 u_1 + ik^{-3} \theta_2 \theta_3 u_2 + ik^{-3} \theta_3 \theta_1 u_3]
\]
\[
= u_0 + \theta_1 \theta_2 u_1 + \theta_2 \theta_3 u_2 + \theta_1 \theta_3 u_3.
\]
Since
\[ (F_0 u)(\pi) = \frac{k^3}{2t_c} \left[ -ik^{-3} u_0 \pi_2 \pi_3 + ik^{-1} u_1 \pi_3 + ik^{-1} u_2 \pi_1 - ik^{-1} u_3 \pi_2 \right], \]
and
\[ (F_0 u')(\pi) = \frac{k^3}{2t_c} \left[ ik^{-3} \pi_3 u_0' - ik^{-1} \pi_3 u_1' - ik^{-1} \pi_1 u_2' + ik^{-1} \pi_2 u_3' \right], \]
where the term \((\cdots)\) vanishes after integration w.r.t. \(d\pi d\bar{\pi}\), we get
\[ (F_0 u, F_0 u') = \iint_{\mathbb{R}^4} d\pi d\bar{\pi} e^{i\langle\pi, \bar{\pi}\rangle} (F_0 u)(\pi)(F_0 u')(\pi) \]
\[ = \kappa^{-2} u_0' u_0 + \kappa (u_1' u_1 + u_2' u_2 + u_3' u_3) = \sum_{|a|=0,2} \kappa^{-2|a|} u_a u_a'. \]

Therefore, the Plancherel formula holds for \(k = 1\).

9.2.4.3. Fourier transformations (mixed case). Putting
\[ c_{m,n} = (2\pi)\kappa^{-2} \kappa_n^2 t_n \quad \text{and} \quad \langle X | \Xi \rangle = \kappa^{-1}(x|\xi) + \kappa^{-1}(\theta|\pi) \in \mathcal{R}_{ev}, \]
for any \(u(X) = \sum_a \theta^a u_a(x), \quad v(\Xi) = \sum_b \pi^b v_b(\xi) \in \mathcal{S}_{SS}(\mathbb{R}^m|n : \mathcal{C}), \) we define
\[ (Fu)(\xi, \pi) = c_{m,n} \int_{\mathbb{R}^m|n} dX e^{-i\langle X | \Xi \rangle} u(X) \]
\[ = (2\pi)\kappa^{-2} \kappa_n^2 t_n \int_{\mathbb{R}^m|n} dX d\theta e^{-i\langle x|\xi \rangle - ik^{-1}(\theta|\pi)} \sum_a \theta^a u_a(x) \]
\[ = \sum_{|a| \leq n} (2\pi)\kappa^{-2} \kappa_n^2 t_n \int_{\mathbb{R}^m|n} dx d\theta e^{-i\langle x|\xi \rangle} \kappa_n^2 t_n \int_{\mathbb{R}^m|n} d\theta e^{i\langle \theta|\pi \rangle} \theta^a u_a(x) \]
\[ = \sum_{|a| \leq n} [(F_0 \theta^a)(\pi)](F_0 u_a)(\xi). \]

(9.2.42)

\[ (Fv)(x, \theta) = c_{m,n} \int_{\mathbb{R}^m|n} d\Xi e^{i\langle X | \Xi \rangle} v(\Xi) \]
\[ = (2\pi)\kappa^{-2} \kappa_n^2 t_n \int_{\mathbb{R}^m|n} d\xi d\pi e^{i\langle x|\xi \rangle + ik^{-1}(\theta|\pi)} \sum_b \pi^b v_b(x) \]
\[ = \sum_{|b| \leq n} (2\pi)\kappa^{-2} \kappa_n^2 t_n \int_{\mathbb{R}^m|n} d\xi e^{i\langle x|\xi \rangle} \kappa_n^2 t_n \int_{\mathbb{R}^m|n} d\pi e^{i\langle \theta|\pi \rangle} \pi^b v_b(\xi) \]
\[ = \sum_{|b| \leq n} [(F_0 \pi^b)(\theta)](F_0 v_b)(x). \]

(9.2.43)

Proposition 9.2.4. For any \(u, v \in \mathcal{S}_{SS}(\mathbb{R}^m|n : \mathcal{C}), \)
\[ (F(D_X^a)u)(\Xi) = (ih^{-1})^{\alpha} (ik^{-1})^{\alpha} (-1)^{|a|} \Xi^a (Fu)(\Xi), \]
\[ (F(X^a)u)(\Xi) = (ih)^{\alpha} (ik)^{\alpha} (-1)^{|a|} D_\Xi^a (Fu)(\Xi). \]

(9.2.44)

(9.2.45)

(9.2.46)
If we define $\delta(X) = \delta(x)\delta(\theta)$, then

\[(9.2.49) \quad (F\delta)(\Xi) = (F_0\delta)(\xi)(F_0\delta)(\pi) = c_{m,n}. \]

$F : S_{SS}(\mathbb{R}^{m,n} : \mathbb{C}) \rightarrow S_{SS}(\mathbb{R}^{m,n} : \mathbb{C})$ gives a continuous linear mapping satisfying

\[(9.2.50) \quad ||\text{proj}_I((\partial^2_{\theta}F\bar{u})(B))|| \leq C'_{m,n} \quad ||\text{proj}_I((\partial^2_{\theta}u)(X_B))|| \quad \text{for each} \quad I \in \mathcal{I}. \]

**Proof.** Combining above results, we have readily these statements.  

**Remark 9.2.9.** Since by the formal definition of $\delta$-function, we have

\[
\int_{\mathbb{R}^m} dp \overline{Fu}(p) Fv(p) = (2\pi\hbar)^{-m} \int_{\mathbb{R}^m} dp \left[ \int_{\mathbb{R}^m} dq e^{-i\hbar^{-1}p\bar{u}(q)} \right] \left[ \int_{\mathbb{R}^m} dq' e^{-i\hbar^{-1}q'v(q')} \right] 
\]

\[
= \int_{\mathbb{R}^m} dq dq' \left[ (2\pi\hbar)^{-m} \int_{\mathbb{R}^m} dp e^{-i\hbar^{-1}(q-q'|p)v(q')} \right] u(q) v(q') = \int_{\mathbb{R}^m} dq u(q) v(q),
\]

and for $A = (A_{jk})$, putting

\[
\langle p, Ap \rangle = \sum_{j,k=1}^m p_j A_{jk} p_k, \quad \langle D_q, AD_q \rangle = \sum_{j,k=1}^m (-i\hbar\partial_{q_j})A_{jk}(-i\hbar\partial_{q_k}) \quad \text{with} \quad D_{q_j} = -i\hbar\partial_{q_j},
\]

we have

\[
(2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} dp \langle p, Ap \rangle \overline{u}(p) = \langle D_q, AD_q \rangle u(q).
\]

Therefore, we want to ask whether following claim holds or not:

**Claim 9.2.1.** Let $u, v \in S_{SS}(\mathbb{R}^{m,n} : \mathbb{C})$. Then, we have

\[
\int_{\mathbb{R}^{m,n}} dX u(X)v(X) = \int_{\mathbb{R}^{m,n}} d\Xi (\overline{F\bar{u}})(\Xi)(Fv)(\Xi).
\]

But Claim 9.2.1 doesn't hold in general: For example, take $u(\theta) = u_0 + \theta_1 \theta_2 u_1$, $u'(\theta) = u'_0 + \theta_1 \theta_2 u'_1$, then

\[
\int_{\mathbb{R}^2} d\theta (u_0 + \theta_1 \theta_2 u_1)(u'_0 + \theta_1 \theta_2 u'_1) = - \int_{\mathbb{R}^2} d\pi \overline{\pi}(\pi) u'(\pi).
\]

In fact

\[
\int_{\mathbb{R}^2} d\theta (u_0 + \theta_1 \theta_2 u_1)(u'_0 + \theta_1 \theta_2 u'_1) = u_0 u'_0 + u_1 u'_1,
\]

\[
\overline{u}(\cdot)(\theta) = u_0 + \theta_1 \theta_2 u_1 = u_0 + u_0 \overline{\theta_1 \theta_2} = \overline{u_0} - \overline{\theta_1 \theta_2 u_1} = \overline{u}(\theta),
\]

\[
k \int_{\mathbb{R}^2} d\theta e^{-ik^{-1}(|\theta|^2)} \overline{u}(\cdot)(\theta) = k^{-1} \overline{\pi}_1 \overline{\pi}_2 u_0 - k u_1,
\]

\[
\overline{\pi}(\pi) = -k^{-1} \pi_1 \pi_2 u_0 - k u_1,
\]

\[
\int_{\mathbb{R}^2} d\pi (-k^{-1} \pi_1 \pi_2 u_0 - k u_1)(-k^{-1} \pi_1 \pi_2 u'_0 + k u'_1) = -u_0 u'_1 - u_1 u'_0. \quad //
\]

But, for $v(\theta) = \theta_1 v_1 + \theta_2 v_2$, $v'(\theta) = \theta_1 v'_1 + \theta_2 v'_2$, Claim 9.2.1 does hold:

\[
\int_{\mathbb{R}^2} d\theta (\theta_1 v_1 + \theta_2 v_2)(\theta_1 v'_1 + \theta_2 v'_2) = \int_{\mathbb{R}^2} d\pi \overline{v}(\cdot)(\pi) v'(\pi).
\]
In fact

\[
\int_{\mathbb{R}^0/2} d\theta (\theta_1 v_1 + \theta_2 v_2)(\theta_1 v'_1 + \theta_2 v'_2) = ((v_1)_{ev} - (v_1)_{od})v'_2 - ((v_2)_{ev} - (v_2)_{od})v'_1,
\]

\[
\theta_1 v_1 + \theta_2 v_2 = \overline{v_1} \theta_1 + \overline{v_2} \theta_2,
\]

\[
k \int_{\mathbb{R}^0/2} d\theta e^{-i k^{-1}(\overline{\theta} \overline{\pi})} (\overline{v_1} \theta_1 + \overline{v_2} \theta_2) = -i \overline{\pi}_2 ((v_1)_{ev} - (v_1)_{od}) + i \overline{\pi}_1 ((v_2)_{ev} - (v_2)_{od}),
\]

\[
\overline{\pi}(\cdot)(\pi) = i((v_1)_{ev} - (v_1)_{od})\pi_2 - i((v_2)_{ev} - (v_2)_{od})\pi_1,
\]

\[
\int_{\mathbb{R}^0/2} d\pi [i((v_1)_{ev} - (v_1)_{od})\pi_2 - i((v_2)_{ev} - (v_2)_{od})\pi_1](-i\pi_2 v'_1 + i\pi_1 v'_2)
\]

\[
= ((v_1)_{ev} - (v_1)_{od})v'_2 - ((v_2)_{ev} - (v_2)_{od})v'_1. \quad //
\]
9.3. Qi’s example of weakly hyperbolic equation

In 1958, M-y. Qi considered the following IVP:

\[(9.3.1)\]
\[v_{tt} - L(t, \partial_q)v = 0 \quad \text{with} \quad L(t) = L(t, \partial_q) = t^2 \partial_q^2 + (4k + 1) \partial_q \quad \text{and} \quad v(0, q) = \varphi(q), \ v_t(0, q) = 0.\]

In Dreher and Witt [37], following claim is cited from Qi [101]:

**Claim 9.3.1.** For suitably chosen \(c_{jk} \neq 0\),

\[v(t, q) = \sum_{j=0}^{k} c_{jk} t^{2j} \varphi^{(j)} \left( q + \frac{t^2}{2} \right)\]

gives the solution of \((9.3.1)\).

Though Qi uses the knowledge of Euler-Poisson equation and the Riemann-Louville fraction integral, we generalize the method of characteristics to a system of PDOp using superanalysis to have readily

**Theorem 9.3.1.**

\[v(t, q) = \left(2\pi\right)^{-1/2} \int dp e^{iwp} e^{i\xi p/2} \sum_{\ell=0}^{k} \frac{2^{2\ell} k!}{(2\ell)! (k - \ell)!} t^{2\ell} (ip)^{\ell} \hat{\varphi}(p) = \sum_{\ell=0}^{k} \frac{2^{2\ell} k!}{(2\ell)! (k - \ell)!} t^{2\ell} \varphi^{(\ell)} \left( q + \frac{t^2}{2} \right).\]

**9.3.1. A systemization and superspace setting.** Putting \(w = v_t - tv_q\), we have

\[\partial_t w = v_{tt} - v_q - tv_{tq} = t^2 v_{qq} + (4k + 1)v_q - v_q - tv_q = -tw_q + 4kv_q,\]

then

\[(9.3.2) \quad i\partial_t U = \mathbb{H}(t, \partial_q)U, \quad U = U(t, q) = \begin{pmatrix} v(t, q) \\ w(t, q) \end{pmatrix} \quad \text{and} \quad U(0, q) = \begin{pmatrix} v(q) \\ w(q) \end{pmatrix},\]

with

\[\mathbb{H}(t, \partial_q) = i \begin{pmatrix} t\partial_q & 1 \\ 4k\partial_q & -t\partial_q \end{pmatrix}.\]

Preparing odd variables \(\theta_1, \theta_2\), we define an operator

\[(9.3.3) \quad \mathcal{H}(t, \partial_x, \theta_1, \theta_2) = it\partial_x \left( 1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2} \right) + 4ik\partial_x \theta_1 \theta_2 - i \frac{\partial^2}{\partial \theta_1 \partial \theta_2}\]

which acts on \(u(t, x, \theta) = v(t, x) + w(t, x)\theta_1 \theta_2\). Then, we reformulate \((9.3.2)\) as follows:

\[(9.3.4) \quad i\partial_t u(t, x, \theta) = \mathcal{H}(t, \partial_x, \theta_1, \theta_2)u(t, x, \theta) \quad \text{with} \quad u(0, x, \theta) = v(x) + w(x)\theta_1 \theta_2.\]

Introducing Fourier transformation w.r.t. odd variables, we have a supersmooth function

\[(9.3.5) \quad \mathcal{H}(t, \xi, \theta, \pi) = it\xi \langle \theta | \pi \rangle - 4k\xi \theta_1 \theta_2 + i\pi_1 \pi_2,\]

which is the Hamilton function corresponding to \(\mathcal{H}(t, \partial_x, \theta, \partial_\theta)\).
9.3.2. A solution of the Hamilton-Jacobi equation by direct method. We solve the following Hamilton-Jacobi equation directly:

\[
\begin{cases}
S_t + itS_x(\langle \theta \rangle) - 4kS_x\theta_1\theta_2 + iS_{\theta_1}S_{\theta_2} = 0, \\
S(0, x, \xi, \theta, \pi) = \langle x|\xi \rangle + \langle \theta|\pi \rangle.
\end{cases}
\]  

(9.3.6)

Decomposing \( S(t, x, \xi, \theta, \pi) \) as

\[
S(t, x, \xi, \theta, \pi) = S(t, x, \xi, 0, 0) + X(t, x, \xi)\theta_1\theta_2 + Y(t, x, \xi)\theta_1\pi_1 + \tilde{Y}(t, x, \xi)\theta_2\pi_2 + V(t, x, \xi)\theta_1\pi_2 + \tilde{V}(t, x, \xi)\theta_2\pi_1 + Z(t, x, \xi)\pi_1\pi_2 + W(t, x, \xi)\theta_1\theta_2\pi_1\pi_2,
\]

we calculate \( X, Y, \tilde{Y}, V, \tilde{V}, Z, W \) which are shown also independent of \( x \).

(0) Taking \( \theta = 0, \pi = 0 \) in (9.3.6), we have

\[
S(t, x, \xi, 0, 0) = \langle x|\xi \rangle,
\]

which gives

\[
S(t, x, \xi, 0, 0) = \langle x|\xi \rangle.
\]

(1) Differentiating (9.3.6) w.r.t \( \theta_1 \) and \( \theta_2 \) and restricting to \( \theta = \pi = 0 \), we have

\[
X_t = 4k\xi - 2it\xi X - iX^2 \quad \text{with} \quad X(0) = 0 \quad \text{where} \quad X = \partial_{\theta_2}\partial_{\theta_1}S(t, x, \xi, \theta, \pi)|_{\theta=\pi=0}.
\]

Moreover, differentiating (9.3.7) once more w.r.t \( x \), we put \( \tilde{X} = X_x \) which gives

\[
\tilde{X}_t + 2it\xi \tilde{X} + 2iX\tilde{X} = 0 \quad \text{with} \quad \tilde{X}(0) = 0.
\]

Therefore \( \tilde{X}(t) = 0 \) which implies \( X(t, x, \xi) \) is independent of \( x \).

To solve (9.3.7), we may associate the 2nd order ODE:

\[
\ddot{\phi} + 2it\xi \dot{\phi} - 4ik\xi \phi = 0 \quad \text{with} \quad \dot{\phi}(0) = 0,
\]

from which we have a solution \( X(t, \xi) = -i^{\phi(t)} \) of (9.3.7).

For the sake of notational simplicity, we rewrite (9.3.8) as

\[
\ddot{\phi} + \alpha t\phi + \beta \phi = 0 \quad \text{with} \quad \dot{\phi}(0) = 0. \quad \alpha = 2i\xi, \quad \beta = -4ik\xi.
\]

This equation is solvable in polynomial w.r.t. \( t \): Putting \( \phi(t) = \sum_{j=0}^{\infty} c_j t^j \), we have

\[
\ddot{\phi} = \sum_{j=1}^{\infty} j(j-1)c_j t^{j-2}, \quad \dot{\phi} = \sum_{j=2}^{\infty} j(j-1)c_j t^{j-2}.
\]

Then, the coefficients of \( t^j \) are given by

\[
\begin{align*}
& t^0 : 2c_2 + \beta c_0 = 0 \implies c_2 = -\frac{\beta}{2}c_0, \\
& t^1: 3 - 2c_3 + \alpha c_1 + \beta c_1 = 0 \implies c_3 = -\frac{1}{3\cdot 2}(\alpha + \beta)c_1, \\
& t^2: 4 - 3c_4 + 2\alpha c_2 + \beta c_2 = 0 \implies c_4 = -\frac{1}{4\cdot 3}(2\alpha + \beta)c_2 = (-1)^2\frac{1}{4!}(2\alpha + \beta)\beta c_0, \\
& t^3: 5 - 4c_5 + 3\alpha c_3 + 2\beta c_3 = 0, \\
& \ldots \\
& t^{2\ell-2}: 2\ell(2\ell - 1)c_{2\ell} + (2\ell - 2)\alpha c_{2\ell-2} + \beta c_{2\ell-2} = 0, \\
& t^{2\ell-1}: (2\ell + 1)2\ell c_{2\ell+1} + (2\ell - 1)\alpha c_{2\ell-1} + \beta c_{2\ell-1} = 0, \quad \text{etc.}
\end{align*}
\]
Since \( \dot{\phi}(0) = 0 \) implies \( c_1 = 0 \), we have \( c_{2\ell+1} = 0 \) for any \( \ell \). Moreover, putting \( c_0 = 1 \), i.e. \( \phi(0) = 1 \), we have
\[
c_{2\ell} = \frac{(-1)^\ell (2\alpha)\ell \beta(2\alpha)^{\ell}}{(2\ell)!} \quad \text{where} \quad (x)_\ell = x(x+1)\cdots(x+\ell-1).
\]
In our case, we have \( \beta(2\alpha)^{-1} = -k \) and \( (-k)_\ell = (-1)^\ell \frac{k!}{(k-\ell)!} \), therefore
\[
\phi(t) = \sum_{\ell=0}^{k} c_{2\ell} t^{2\ell} = \sum_{\ell=0}^{k} \frac{(-1)^\ell (4i\xi)^\ell (-k)\ell^{2\ell}}{(2\ell)!} t^{2\ell} = \sum_{\ell=0}^{k} \frac{4^\ell k!}{(2\ell)! (k-\ell)!} (i\xi)^\ell t^{2\ell}.
\]
(2) \( Y = Y(t,\xi) = \partial_{\pi_2} \partial_{\theta_2} S(t, x, \xi, 0, 0) \) satisfies
(9.3.10) \( Y_t + i\xi Y + iXY = 0 \) with \( Y(0) = 1 \).
From this, we have
\[
\frac{d}{dt} (\log(Y\phi)) = -i\xi, \quad Y(t) = \frac{e^{-i\xi^2 t/2}}{\phi(t)}.
\]
Same relation holds for \( \tilde{Y} = \partial_{\pi_2} \partial_{\theta_2} S(t, x, \xi, 0, 0) \).
(3) \( V = \partial_{\pi_2} \partial_{\theta_1} S(t, x, \xi, 0, 0) \) satisfies
\( V_t + i\xi V + XV = 0 \) with \( V(0) = 0 \).
From this, \( V(t) = 0 \). Analogously, \( \tilde{V} = \partial_{\pi_2} \partial_{\theta_2} S(t, x, \xi, 0, 0) = 0 \).
(4) For \( Z = Z(t,\xi) = \partial_{\pi_2} \partial_{\theta_1} S(t, x, \xi, 0, 0) \), we have
(9.3.11) \( Z_t + iY^2 = 0 \) with \( Z(0) = 0 \).
Therefore,
\[
Z(t) = -i \int_0^t ds Y^2(s) = -i \int_0^t ds \frac{e^{-i s^2 \xi}}{\phi^2(s)}.
\]
(5) Putting \( W = W(t,\xi) = \partial_{\pi_2} \partial_{\theta_1} \partial_{\theta_2} \partial_{\theta_3} S(t, x, \xi, 0, 0) \), and remarking \( Z_x = \partial_x \partial_{\pi_2} \partial_{\theta_1} S(t, x, \xi, 0, 0) = 0 \), we have
(9.3.12) \( W_t + 2i\xi W + 2iXW = 0 \) with \( W(0) = 0 \).
Therefore, \( W = 0 \).
Now, we define
\[
\mathcal{D}(t, \xi, \theta, \pi) = \text{sdet} \begin{pmatrix}
S_{x\xi} & S_{x\pi_1} & S_{x\pi_2} \\
S_{h_1} & S_{h_1\pi_1} & S_{h_1\pi_2} \\
S_{h_2} & S_{h_2\pi_1} & S_{h_2\pi_2}
\end{pmatrix} = Y^{-2}(t,\xi) = \mathcal{A}^2(t, \xi, \theta, \pi).
\]
9.3.3. Quantization. Using above defined \( S \) and \( \mathcal{A} \), we construct a function
\[
u(t, x, \theta) = (2\pi)^{-1/2} \int d\xi d\pi \mathcal{A}(t, x, \xi, \theta, \pi) e^{iS(t,x,\xi,\theta,\pi)} \tilde{\nu}(\xi, \pi).
\]
It is shown that this gives a solution of (9.3.14).
Since
\[
\int d\pi e^{iY(\theta,\pi)+iZ\pi_1\pi_2} (\tilde{\nu}(\xi)\pi_1\pi_2 + \tilde{\nu}(\xi)) = \tilde{\nu}(\xi) + (Y^2\theta_1\theta_2 + iZ)\tilde{\nu}(\xi),
\]
we have
\[(2\pi)^{-1/2} \int d\xi d\pi A e^{iS(\hat{u}(\xi)\pi_1\pi_2 + \hat{w}(\xi))} \]
\[= (2\pi)^{-1/2} \int d\xi A e^{i(x|\xi) + iX\theta_1\theta_2[\hat{u}(\xi) + (Y^2\theta_1\theta_2 + iZ)\hat{w}(\xi)]} \]
\[= (2\pi)^{-1/2} \int d\xi Y^{-1} e^{i(x|\xi)} (1 + iX\theta_1\theta_2)(i\hat{w}(\xi) + iZ\hat{w}(\xi) + Y^2\theta_1\theta_2\hat{w}(\xi)) \]
\[= (2\pi)^{-1/2} \int d\xi e^{i(x|\xi)} Y^{-1} (\hat{u}(\xi) + iZ\hat{w}(\xi)) \]
\[+ (2\pi)^{-1/2} \int d\xi e^{i(x|\xi)} Y^{-1} [iX(\hat{u}(\xi) + iZ\hat{w}(\xi)) + Y^2\hat{w}(\xi)]\theta_1\theta_2. \]

Since \(\hat{w} = 0\), we have
\[v(t, q) = (2\pi)^{-1/2} \int d\xi e^{i(x|\xi)} e^{it\frac{x^2\xi}{2}} \sum_{\ell=0}^{k} \frac{2^\ell k!}{(2\ell)! (k-\ell)!} \ell^{2\ell} (i\xi)^{\ell} \hat{\varphi}(\xi) \bigg|_{x=q}. \]
\textbf{9.4. An example of a system version of Egorov's theorem – Bernardi’s question}

It is well-known that Egorov’s theorem concerning the conjugation of $\Psi DO (=\text{pseudo-differential operator})$ with FIOs (=Fourier integral operators) is a very powerful tool for the study of $\Psi DOs$.

Using superanalysis, we extend that theorem to the $2 \times 2$ system of PDOs (=partial differential operators) or $\Psi DOs$. As a by-product, we give a new geometrical interpretation of the similarity transformations $e^{iH}pe^{-iH}$ for any $2 \times 2$-matrices $P$ and $H = H^*$.

\textbf{9.4.1. Bernardi’s question.} Remarking that

\begin{align*}
(-\alpha \partial_x^2 + \beta x^2)e^{-i\gamma x^2/2} &= [i\alpha \gamma + (\beta + \alpha \gamma^2)x^2]e^{-i\gamma x^2/2}, \\
(c(x)\partial_x + \partial_x c(x))e^{-i\gamma x^2/2} &= (c'(x) - 2ic(x)\gamma x)e^{-i\gamma x^2/2},
\end{align*}

we have

\begin{equation}
(9.4.1) \quad \begin{pmatrix} e^{i\gamma x^2/2} & 0 \\ 0 & e^{i\gamma x^2/2} \end{pmatrix} \begin{pmatrix} -\alpha \partial_x^2 + \beta x^2 & 2^{-1}i(c(x)\partial_x + \partial_x c(x)) \\ 2^{-1}(d(x)\partial_x + \partial_x d(x)) & -\alpha \partial_x^2 + \beta x^2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma x^2/2} & 0 \\ 0 & e^{-i\gamma x^2/2} \end{pmatrix} \\
= \begin{pmatrix} i\alpha \gamma + (\beta + \alpha \gamma^2)x^2 & 2^{-1}(c'(x) - 2i\gamma x c(x))e^{i(\gamma - \tilde{\gamma})x^2/2} \\ 2^{-1}(d'(x) - 2i\gamma x d(x))e^{-i(\gamma - \tilde{\gamma})x^2/2} & i\tilde{\alpha} \tilde{\gamma} + (\tilde{\beta} + \tilde{\alpha} \tilde{\gamma}^2)x^2 \end{pmatrix}.
\end{equation}

In February 2001, Bernardi (as a chairman of a session where I gave a talk) asked me whether it is possible to explain (9.4.1) using superanalysis. Especially, why appear the terms $c'(x)$ and $d'(x)$ in the off-diagonal part?

\textbf{9.4.2. An answer to Bernardi.} We re-interpret (9.4.1) as follows: For $u, v \in C^\infty(\mathbb{R}),$

\begin{equation}
(9.4.2) \quad \begin{pmatrix} e^{i\gamma x^2/2} & 0 \\ 0 & e^{i\gamma x^2/2} \end{pmatrix} \begin{pmatrix} \alpha D^2 + \beta x^2 & 2^{-1}i(c(x)D + Dc(x)) \\ 2^{-1}i(d(x)D + Dd(x)) & \tilde{\alpha} D^2 + \tilde{\beta} x^2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma x^2/2} & 0 \\ 0 & e^{-i\gamma x^2/2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
= \begin{pmatrix} (\beta + \alpha \gamma^2)x^2 + i\alpha \gamma + 2i\gamma x a\partial_x - \alpha \partial_x^2 & e^{i(\gamma - \tilde{\gamma})x^2/2}(2^{-1}c'(x) + c(x)\partial_x - i\tilde{\gamma} x c(x)) \\ e^{-i(\gamma - \tilde{\gamma})x^2/2}2^{-1}d'(x) + d(x)\partial_x - i\tilde{\gamma} x d(x) & (\tilde{\beta} + \tilde{\alpha} \tilde{\gamma}^2)x^2 + i\tilde{\alpha} \tilde{\gamma} + 2i\tilde{\gamma} x a\partial_x - \tilde{\alpha} \partial_x^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\end{equation}

Since (9.4.2) with $u = v = 1$ gives (9.4.1), we should explain the meaning of (9.4.2) instead of (9.4.1).

Since we have

\begin{align*}
((\beta + \alpha \gamma^2)x^2 + i\alpha \gamma + 2i\gamma x a\partial_x - \alpha \partial_x^2)u(x) &= (2\pi)^{-1} \int_{\mathbb{R}^2} d\xi dy e^{i(x-y)\xi} a \left( \frac{x+y}{2}, \xi \right) u(y), \\
e^{2^{-1}i(\gamma - \tilde{\gamma})x^2}(c'/2 + c\partial_x - i\tilde{\gamma} xc)u(x) &= (2\pi)^{-1} \int_{\mathbb{R}^2} d\xi dy e^{i(x-y)\xi} c \left( \frac{x+y}{2}, \xi \right) u(y)
\end{align*}

with Weyl symbols

\begin{align*}
a(x, \xi) &= \alpha(\xi - \gamma x)^2 + \beta x^2 = (\beta + \alpha \gamma^2)x^2 - 2\alpha \gamma x \xi + \alpha \xi^2, \\
c(x, \xi) &= ie^{2^{-1}i(\gamma - \tilde{\gamma})x^2}c(x) \left( \xi - \frac{\gamma + \tilde{\gamma}}{2} x \right),
\end{align*}

we get the Weyl symbol of the right-hand side of (9.4.2), given by
\[(9.4.3)\]
\[
\begin{pmatrix}
\alpha(\xi - \gamma x)^2 + \beta x^2 & ie^{i(\gamma - \bar{\gamma})x^2/2}c(x)(\xi - 2^{-1}(\gamma + \bar{\gamma})x) \\
-ie^{-i(\gamma - \bar{\gamma})x^2/2}d(x)(\xi - 2^{-1}(\gamma + \bar{\gamma})x) & \tilde{\alpha}(\xi - \bar{\gamma}x)^2 + \tilde{\beta}x^2 
\end{pmatrix}
\approx \frac{\alpha + \tilde{\alpha}}{2} \xi^2 - (\alpha\gamma + \tilde{\alpha}\bar{\gamma})x\xi + \frac{(\beta + \alpha\gamma^2) + (\tilde{\beta} + \tilde{\alpha}\bar{\gamma}^2)}{2}x^2
\]
\[-i \left[ \frac{\alpha - \tilde{\alpha}}{2} \xi^2 - (\alpha\gamma - \tilde{\alpha}\bar{\gamma})x\xi + \frac{(\beta + \alpha\gamma^2) - (\tilde{\beta} + \tilde{\alpha}\bar{\gamma}^2)}{2}x^2 \right] \langle \theta|\pi \rangle
\]
\[+ ie^{-i(\gamma - \bar{\gamma})x^2/2}d(x) \left( \xi - \frac{\gamma + \bar{\gamma}}{2}x \right) \theta_1 \theta_2 + ie^{i(\gamma - \bar{\gamma})x^2/2}c(x) \left( \xi - \frac{\gamma + \bar{\gamma}}{2}x \right) \pi_1 \pi_2.\]

**Superspace interpretation:** On the other hand, putting

\[\sigma(\mathbb{P})(x, \xi) = \begin{pmatrix} \alpha \xi^2 + \beta x^2 & ic(x)\xi \\ id(x)\xi & \tilde{\alpha} \xi^2 + \tilde{\beta} x^2 \end{pmatrix}, \quad \sigma(\mathbb{H})(x) = \begin{pmatrix} 2^{-1}\gamma x^2 & 0 \\ 0 & 2^{-1}\bar{\gamma} x^2 \end{pmatrix},\]

we have
\[(9.4.4)\]
\[
\sigma(\mathbb{H}_\mathbb{P})(x, \xi, \theta, \pi) = \sigma(\mathbb{P})(x, \xi, \theta, \pi) = \mathbb{P}(x, \xi, \theta, \pi)
\]
\[= \frac{\alpha + \tilde{\alpha}}{2} \xi^2 + \frac{\beta + \tilde{\beta}}{2} x^2 - i \left[ \frac{\alpha - \tilde{\alpha}}{2} \xi^2 + \frac{\beta - \tilde{\beta}}{2} x^2 \right] \langle \theta|\pi \rangle + id(x)\xi \theta_1 \theta_2 + ic(x)\xi \pi_1 \pi_2,
\]
\[\sigma(\mathbb{H}_\mathbb{D})(x, \theta, \pi) = \sigma(\mathbb{H})(x, \theta, \pi) = \mathbb{H}(x, \theta, \pi) = \frac{(\gamma + \bar{\gamma})x^2}{4} - i \left( \frac{\gamma - \bar{\gamma}}{4} x^2 \right) \langle \theta|\pi \rangle.
\]

Therefore, we have
\[
\begin{cases}
\dot{x} = \mathbb{H} \xi = 0, \\
\dot{\xi} = -\mathbb{H} x = -\frac{\gamma + \bar{\gamma}}{2} x + i \frac{\gamma - \bar{\gamma}}{2} x \langle \theta|\pi \rangle, \\
\dot{\theta}_j = -\mathbb{H} \pi_j = -i \frac{\gamma - \bar{\gamma}}{4} x^2 \theta_j, (j = 1, 2), \quad \text{with} \quad (x(0), \xi(0), \theta(0), \pi(0)) = (x, \xi, \theta, \pi)
\end{cases}
\]
which yields the Hamilton flow corresponding to \(\mathbb{H}\) as
\[
\mathcal{C}(t)(x, \xi, \theta, \pi) = (x(t), \xi(t), \theta(t), \pi(t)) \quad \text{with}
\]
\[x(t) = x(t, x, \xi, \theta, \pi) = x, \quad \xi(t) = \xi(t, x, \xi, \theta, \pi) = \xi - \frac{\gamma + \bar{\gamma}}{2} x t + i \frac{\gamma - \bar{\gamma}}{2} x t \langle \theta|\pi \rangle,
\]
\[\theta_j(t) = \theta_j(t, x, \xi, \theta, \pi) = e^{-i(\gamma - \bar{\gamma})x^2/4} \theta_j, \quad \pi_j(t) = \pi_j(t, x, \xi, \theta, \pi) = e^{i(\gamma - \bar{\gamma})x^2/4} \pi_j \quad (j = 1, 2).
\]

Putting operators
\[
\hat{A} = -i \partial_x - \frac{\gamma + \bar{\gamma}}{2} xt, \quad \hat{B} = \frac{\gamma - \bar{\gamma}}{2} xt, \quad \hat{\sigma}_3 = 1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}, \quad \hat{\xi}(t) = \hat{A} - \hat{B} \hat{\sigma}_3,
\]
with Weyl symbols
\[
\sigma(\hat{A})(x, \xi) = \xi - \frac{\gamma + \bar{\gamma}}{2} xt, \quad \sigma(\hat{B})(x, \xi) = \frac{\gamma - \bar{\gamma}}{2} xt, \quad \sigma(\hat{\sigma}_3)(\theta, \pi) = -i \langle \theta|\pi \rangle,
\]
\[
\sigma(\hat{\xi}(t))(x, \xi, \theta, \pi) = \xi - \frac{\gamma + \bar{\gamma}}{2} xt + i \frac{\gamma - \bar{\gamma}}{2} xt \langle \theta|\pi \rangle, \quad \langle \theta(t)\pi(t) \rangle = \langle \theta|\pi \rangle
\]
we have the following:
\[
\sigma(\hat{A}\hat{B} + \hat{B}\hat{A})(x, \xi) = (\gamma - \bar{\gamma}) xt \left( \xi - \frac{\gamma + \bar{\gamma}}{2} xt \right), \quad \sigma(\hat{\sigma}_3^2)(\theta, \pi) = 1.
\]
Remark 9.4.1. Let $a$, $b$ and $c$ be non-commutative or commutative “operators”. For monomials $p_2(x, y) = xy$ and $p_3(x, y, z) = xyz$, we define

$$p_{2,s}(a, b) = \begin{cases} \frac{1}{2}(ab + ba) & \text{if } [a, b] \neq 0, \\ ab & \text{if } [a, b] = 0, \end{cases}$$

$$p_{3,s}(a, b, c) = \begin{cases} \frac{1}{2}(abc + acb + bca + bac + cba + cab) & \text{if } a, b, c \text{ are non-commutative each other}, \\ \frac{1}{2}(abc + bca) & \text{if } [a, b] \neq 0, \text{ but } [a, c] = 0 \text{ and } [b, c] = 0, \\ abc & \text{if } a, b, c \text{ are commutative each other}. \end{cases}$$

From these, we get

$$\sigma(\hat{\xi}(t)^2)(x, \xi, \theta, \pi) = \xi^2 - (\gamma + \hat{\gamma})xt\xi + \frac{\gamma^2 + \hat{\gamma}^2}{2}x^2 t^2 + i\left(\xi - \frac{\gamma + \hat{\gamma}}{2}xt\right)(\gamma - \hat{\gamma})xt(\theta|\pi).$$

In fact,

$$\hat{\xi}(t)^2 = (\hat{A} - \hat{B}\sigma_3)(\hat{A} - \hat{B}\sigma_3) = \hat{A}^2 - (\hat{A}\hat{B} + \hat{B}\hat{A})\sigma_3 + \hat{B}^2\sigma_3^2,$$

with

$$\hat{A}^2 + \hat{B}^2 = -\partial^2_x + i\partial_x\left(\frac{\gamma + \hat{\gamma}}{2}xt\right) + \left(\frac{\gamma + \hat{\gamma}}{2}xt\right)^2 t^2,$$

and

$$\sigma(\hat{A}^2 + \hat{B}^2) = \xi^2 - i\xi(\gamma + \hat{\gamma})xt + \frac{\gamma^2 + \hat{\gamma}^2}{2}x^2 t^2.$$
9.4. AN EXAMPLE OF A SYSTEM VERSION OF EGOROV’S THEOREM – BERNARDI’S QUESTION

That is,

\[ p_{3,3}(d(x(t)), \xi(t), \theta_1(t), \theta_2(t)) (u_0 + u_1 \theta_1 \theta_2) = \frac{1}{2} e^{-i(\gamma - \tilde{\gamma}) \xi_2^2/2} \{ d(z)(-i \partial_z - \gamma x t) + (-i \partial_z - \gamma x t) d(z) \} u_0 \theta_1 \theta_2. \]

Analogous holds for \( p_{3,3}(c(x(t)) \xi(t) \pi_1(t) \pi_2(t)) \).

From these, we have

\[ \mathcal{P} [C(t)(x, \xi, \theta, \pi)] \]

\[ = \frac{\alpha + \tilde{\alpha}}{2} \sigma(\xi(t)^2) + \frac{\beta + \tilde{\beta}}{2} \sigma(x(t)^2) \]

\[ - i \frac{\alpha - \tilde{\alpha}}{2} \sigma(\xi(t)^2 \langle \theta(t) | \pi(t) \rangle) - i \frac{\alpha - \tilde{\alpha}}{2} \sigma(x(t)^2 \langle \theta(t) | \pi(t) \rangle) \]

\[ + i \sigma(d(x(t)) \xi(t) \theta_1(t) \theta_2(t)) + i \sigma(c(x(t)) \xi(t) \pi_1(t) \pi_2(t))(x, \xi, \theta, \pi) \]

\[ = \frac{\alpha + \tilde{\alpha}}{2} \xi_2^2 + \frac{\beta + \tilde{\beta}}{2} x_2^2 + \frac{\alpha \gamma^2 t^2 + \tilde{\alpha} \tilde{\gamma} t^2}{2} \xi_2^2 - (\alpha \gamma t + \tilde{\alpha} \tilde{\gamma} t) x_2 \xi_2 \]

\[ - i \left[ \frac{\alpha - \tilde{\alpha}}{2} \xi_2^2 + \frac{\beta - \tilde{\beta}}{2} x_2^2 - (\alpha \gamma t - \tilde{\alpha} \tilde{\gamma} t) x_2 \xi_2 + \frac{\alpha \gamma^2 t^2 - \tilde{\alpha} \tilde{\gamma}^2 t^2}{2} x_2^2 \right] \langle \theta | \pi \rangle \]

\[ + i d(x) \left( \xi - \frac{\gamma + \tilde{\gamma}}{2} x t \right) e^{-i(\gamma t - \tilde{\gamma} t) x_2^2/2 \theta_1 \theta_2} + i c(x) \left( \xi - \frac{\gamma t + \tilde{\gamma} t}{2} x \right) e^{i(\gamma t - \tilde{\gamma} t) x_2^2/2 \pi_1 \pi_2}. \]

This equals to (9.4.3) after replacing \( \gamma \rightarrow \gamma t \) and \( \tilde{\gamma} \rightarrow \tilde{\gamma} t \).

Therefore, denoting \( x \) simply by \( x \), etc, we have proved

\[ \sigma(ze^{itH}P e^{-itH}) (x, \xi, \theta, \pi) = \sigma(e^{it \tilde{H}} \tilde{P} e^{-it \tilde{H}})(x, \xi, \theta, \pi) = \mathcal{P} [C(t)(x, \xi, \theta, \pi)]. \]

Remark 9.4.2. In the above, we calculate the product of operators and find its symbol, rather directly. In the near future, we need to give a product formula for operators as analogous to “bosonic” case.
9.5. Functional Derivative Equations

9.5.1. Liouville equation. I mentioned, at 7 or 8-th lecture, a function with countably infinite independent variables. There, I regard a function with an odd variable $\theta$ as a function with countably infinite independent variables $\{\theta_j \in \mathbb{C}\}$ where $\theta = \sum_{j \in I} \theta_j \sigma^j$. This resembles to consider a functional

As is well-known, a non-linear system of ODEs may be regarded as a linear PDE and therefore ask what occurs when we have non-linear PDE on $\mathbb{R}^d$ instead of ODE.

Typically, the solution of Hamilton equation relates to the solution of Liouville equation by the method of characteristics. For $H(q,p) \in C^\infty(\mathbb{R}^n : \mathbb{R})$, Hamilton equation is written down as

$$\frac{\dot{q}_j}{\partial t} = H_{p_j}(q(t),p(t)), \quad \frac{\dot{p}_j}{\partial t} = -H_{q_j}(q(t),p(t)),$$

with $\left( \begin{array}{c} q(0) \\ p(0) \end{array} \right) = \left( \begin{array}{c} \underline{q} \\ \underline{p} \end{array} \right)$.

and the Liouville equation is

$$\frac{\partial}{\partial t} u(t,q,p) = \{u,H(q,p)\} \quad \text{with} \quad u(0,q,p) = \underline{u}(q,p).$$

Here, Poisson bracket is defined by

$$\{f,g\} = \sum_{j=1}^m \left( \frac{\partial f}{\partial q_j} \frac{\partial f}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial f}{\partial q_j} \right).$$

In general, Hamilton equation is a non-linear ODE and Liouville equation is a linear PDE. Even non-linear, applying Galerkin method, PDE may be regarded as ODE with infinitely many components in certain function spaces.

If we take as the special initial data in Liouville equation, for example, $\underline{u}(q,p)$ may be a measure $\delta_{\underline{q}}(q) dq$ or $\delta_{\underline{p}}(p) dp$, respectively, we get the solution of Hamilton equation. Conversely, putting $u(t,q,p) = \underline{u}(q(t),q,p),p(t,q,p))$ from the solution of Hamilton equation, we get the solution $u(t,q,p) dq dp$.

9.5.2. Hopf equation (H). [E. Hopf 57] As Liouville equation corresponds to Hamilton equation, Hopf equation written by functional derivatives corresponds to Navier-Stokes equation. More precisely,

9.5.2.1. Navier-Stokes equation: Let a domain $\Omega$ in $\mathbb{R}^n$ be given with smooth boundary $\partial \Omega$. Find a vector field $u(t,x) = \sum_{j=1}^m u_j(t,x) \frac{\partial}{\partial x_j}$ and a pressure $p(t,x)$ satisfying

$$\frac{\partial}{\partial t} u(t,x) - \nu \Delta u(t,x) + (u \cdot \nabla)(t,x) + \nabla p = 0,$$

$$\text{div} u = 0,$$

$$u(0,x) = u_0(x), \quad u(t,x) \big|_{\partial \Omega} = 0.$$

Assuming this has a solution, we denote it $(T_t u_0)(x) = u(t,x)$.

For the sake of simplicity, we take a Riemannian manifold $M$ with metric $g_{ij}(x) dx^i dx^j$. Let $L^2(\sigma|M)$ be a set of $L^2$-integrable solenoidal vector field on $M$, and let $\Lambda^2(\sigma|M)$ be a solenoidal vector field with compact support on $M$.\footnote{here, we misuse measure $\delta_{\underline{q}}(dq)$ and its density $\delta_{\underline{q}}(q)$ w.r.t. $dq$}
Put $H = L^2_0(M, g)$ and $\hat{H}$ its dual. Find a functional $W(t, \eta)$ on $[0, \infty) \times \hat{H}$, for $(t, \eta) \in (0, \infty) \times A^1_0(M)$, it satisfies

\begin{equation}
\tag{9.5.1}
\frac{\partial}{\partial t} W(t, \eta) = \int_M \left( -i(\overline{\hat{\mathcal{T}}} \eta)_{jk}(x) \frac{\delta^2}{\delta \eta_{ij}(x)} W(t, \eta) \right.
\left. + \nu(\Delta \eta)_{jk}(x) \frac{\delta}{\delta \eta_{ij}(x)} W(t, \eta) + i\eta_{ij}(x) f^j(x, t) W(t, \eta) \right) d_g x,
\end{equation}

\begin{equation}
\frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x_j} \left( \sqrt{g(x)} \frac{\delta}{\delta \eta_j(x)} W(t, \eta) \right) = 0,
\end{equation}

$W(t, 0) = 1$ and $W(0, \eta) = W_0(\eta)$.

and

\begin{equation}
W_0(0) = 1, \quad \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x_j} \left( \sqrt{g(x)} \frac{\delta}{\delta \eta_j(x)} W_0(\eta) \right) = 0,
\end{equation}

\begin{equation}
f(x, t) = f^j(x, t) \frac{\partial}{\partial x_j} \in L^2(0, \infty : V^{-1}).
\end{equation}

9.5.2.2. Hopf-Foiaš equation (HF). \cite{CF} Take $H = L^2_0(\Omega)$ as Hilbert space. Find a family of Borel measures $\{\mu(t, \cdot)\}_{t \in (0, \infty)}$ on $H$, such that for a suitable class of test functionals $\Phi(t, u)$, it satisfies

\begin{equation}
\tag{9.5.2}
- \int_0^\infty \int_H \frac{\partial \Phi(t, u)}{\partial t} \mu(t, du) dt - \int_H \Phi(t, 0) \mu_0(du)
= \int_0^\infty \int_H \int_M \left( (\nabla u)^2 - \nu(\Delta u)^2 - f^j(x, t) \right) \frac{\delta \Phi(t, u)}{\delta w^j(x)} d_g x \mu(t, du) dt
\end{equation}

Here $\mu_0(\cdot)$ is a given Borel measure on $H$. This equation is obtained, for any $\omega \in \mathcal{B}(H)$, putting $\mu(t, \omega) = \mu_0(T_t^{-1} \omega)$, calculate

\begin{equation}
\frac{\partial}{\partial t} \int_H \Phi(t, T_t u) \mu_0(du).
\end{equation}

Here, we assume $T_t u$ above is considered well-defined.

The solution of Hopf equation is obtained by putting $\Phi(t, u) = \rho(t) e^{i(u, \eta)}$ into (9.5.2), and

\begin{equation}
W(t, \eta) = \int_H e^{i(u, \eta)} \mu(t, du) = \int_H e^{i(T_t u, \eta)} \mu_0(du) \quad \text{with} \quad \langle u, \eta \rangle = \int_M u^j(x) \eta_{ij}(x) d_g x.
\end{equation}

What are functional derivatives, which is abruptly mentioned above. Let $\Omega$ be a domain in $\mathbb{R}^m$. We consider a functional $W : X(\Omega) \ni \eta \rightarrow W(\eta) \in \mathbb{C}$ on suitable functional space $X(\Omega)$. Taking a test function $\phi \in C^\infty_0(\Omega) \subset X(\Omega)$, if we have

\begin{equation}
\frac{d}{dt} W(\eta + t \phi)|_{t=0}
\end{equation}

then, we denote it as

\begin{equation}
D' \langle w, \phi \rangle_D = \int dx \ w(x) \phi(x), \quad w(x) = \frac{\delta W(\eta)}{\delta \eta(x)} \in D'(\Omega)
\end{equation}

and call this functional derivative. We need higher order functional derivatives for $W$. Formally we may put

\begin{equation}
\frac{\delta^2 W(\eta)}{\delta \eta(x) \delta \eta(y)} \in D'(\Omega \times \Omega)
\end{equation}

but what does this mean when $x = y$? See, Inoue \cite{63}. 

\[\text{Mini Column 4} \]
I feel the address of I.M. Gelfand at ICM Amsterdam conference in 1954 suggests beautiful and important problems and I believe as he mentioned that we need to develop a very new theory of differential equations to study quantum field theory or turbulence theory, for example, theory of functional derivative equations (FDE). The configuration space where functional lives is a function space which is infinite-dimensional, therefore no suitable Lebesgue-like measure. This means it is not yet possible to integrate functional freely, and no integration by parts, no Fourier transformation does exist. The tool which we are available now is Taylor expansion if it exists, therefore, only very algebraic treatise is possible. Concerning a simple model equation with removable $\infty$ by renormalization, see Inoue.

9.5.2.3. FDE representing turbulence? Though Hopf equation is related to the invariant measure w.r.t. the flow governed by Navier-Stokes equation, I suspect that the equation related to turbulence will be Fokker-Planck type FDE derived from Navier-Stokes equation:

Find a measure $P(t,v)\,d_Fv$ satisfying below:

$$
\frac{\partial}{\partial t}P(t,v) = \int_{\mathbb{R}^3} d^3x \frac{\delta}{\delta v^i(x)} \left\{ \left( v^j(x) \nabla_j v^i(x) + \frac{1}{\rho} \nabla^i p(x) - \nu \Delta v^i(x) - f^i(x) \right) P(t,v) \right\} \\
+ \frac{k_B T \nu}{\rho} \int_{\mathbb{R}^3} d^3x \left( \nabla_j \frac{\delta}{\delta v^i(x)} \right)^2 P(t,v).
$$

Here

$$p(x) = \frac{\rho}{4\pi} \int_{\mathbb{R}^3} d^3x' \frac{(\nabla_i v^j(x'))(\nabla'_j v^i(x')) - \nabla'_i f^i(x')}{|x-x'|}$$

and the functional derivatives are taken w.r.t. transversal velocity field

$$v^j(x) = \int_{\mathbb{R}^3} \frac{d^3\xi}{(2\pi)^3} \left( \delta^j_k - \frac{\delta^j \xi_k \xi^k}{|\xi|^2} \right) v^k(x)e^{i\xi x}.$$

**Problem 9.5.1.** Very recently, I make know the paper written by O.V. Troshkin, where he cited the “result” by W. Thomson (alias Lord Kelvin) such that W.T. obtained a wave equation for an incompressible fluid by averaging Euler’s equation. Troshkin claims that the formal analogy existing between waves of small disturbances of inviscous and incompressible turbulent medium and electromagnetic waves is established. Prove these facts mathematically using Reynolds equation by Foiaş.

In the above, seemingly Thomson assumes the intrinsic fluctuation associated to Euler flow and averaging w.r.t. this. On the other hand, we derive Navier-Stokes equation from Euler equation, by adding artificially white noise (extrinsic) fluctuation to each flow line of Euler equation.
9.5.3. Equation for QED?. As a functional derivative equation for QED=quantum electrodynamics), it might be the following forms a base?

\[
\begin{align*}
\square \frac{\delta Z(\eta, \bar{\eta}, J)}{\delta J_\mu(x)} &= -iJ_\mu(x)Z(\eta, \bar{\eta}, J) + ie\gamma^\mu \frac{\delta^2 Z(\eta, \bar{\eta}, J)}{\delta \eta(x) \delta \bar{\eta}(x)}, \\
(i\gamma^\mu \bar{\delta}_\mu - M) \frac{\delta Z(\eta, \bar{\eta}, J)}{\delta \eta(x)} &= -i\bar{\eta}(x)Z(\eta, \bar{\eta}, J) + ie\gamma^\mu \frac{\delta^2 Z(\eta, \bar{\eta}, J)}{\delta \eta(x) \delta J_\mu(x)}, \\
\frac{\delta Z(\eta, \bar{\eta}, J)}{\delta \bar{\eta}(x)}(i\gamma^\mu \bar{\delta}_\mu + M) &= i\eta(x)Z(\eta, \bar{\eta}, J) - ie\gamma^\mu \frac{\delta^2 Z(\eta, \bar{\eta}, J)}{\delta \eta(x) \delta J_\mu(x)}, \\
Z(0, 0, 0) &= 1.
\end{align*}
\]

Remark 9.5.1. This equation stems from functional method in QFT(Quantum Field Theory), more precisely, adjoining external forces to each component of Maxwell-Dirac equation, we get this equation. In finite dimensional case, I know vaguely a story, for given non-linear ODE, adding fluctuating external force to it to have Langevin equation, and solving that and taking average of solution w.r.t. fluctuation to get Fokker-Planck equation. In this story, if we replace this ODE with the coupled Maxwell-Dirac equation, what facts do we get? Considering like this, what type of classical property is inherited to the solution of quantum or statistical equation, and how the quantum or statistical effect is represented by “classical quantity”?

As the Feynman’s path-integral representation gives directly quantum object from Lagrangian without solving Schrödinger equation, physicists write down the quantum quantity using path integral with Feynman measure and no use of FDE etc. As is mentioned before explaining “stationary phase method”, how to make rigorous their arguments?
9.6. Supersymmetric extension of the Riemannian metric \( g_{jk}(q)dq^j dq^k \) on \( \mathbb{R}^d \)

In Witten’s paper, he writes down “classical object” or rather “quantity before quantization” corresponding to the deformed Laplace-Beltrami operator \( d_{\phi}^*d_{\phi} + d_{\phi}d_{\phi}^* \) as if it is evident. Here the function \( \phi \) is the Morse function on the manifold \( M \).

Mathematically it is not so clear what is the classical object for the given quantum operator. Therefore reversely, I try to give a prescription how one obtains the super symmetrically extended extension.

\[
\Phi = \Phi(x,\xi,\theta,\pi) = \psi(x) + i\epsilon_{\alpha\beta}\phi_{\alpha\beta}(x) + \frac{\sqrt{-1}}{2}\rho_\alpha\epsilon_{\alpha\beta}\rho_\beta F^j(t)
\]

where for a certain interval \( I \subset \mathbb{R} \)

\[
x^j(t) \in C^\infty(I : \mathcal{R}_{ev}), \quad \psi^j_\beta(t) \in C^\infty(I : \mathcal{G}_{od}), \quad F^j(t) \in C^\infty(I : \mathcal{R}_{ev})
\]

with \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1, j = 1, 2, \cdots, d \) and \( \alpha, \beta = 1, 2 \).

Introducing operators \( D^-_\alpha \) as

\[
D^-_\alpha = \frac{\partial}{\partial \rho_\alpha} - \sqrt{-1}\rho_\alpha \frac{\partial}{\partial t} \quad \text{for} \; \alpha = 1, 2,
\]

we put

\[
\mathcal{L}_0 = \mathcal{L}_0(\Phi, D^-_\alpha \Phi)
\]

\[
def \quad = -\frac{1}{4}g_{jk}(\rho_\alpha A^j_\alpha D^-_\beta \Phi^k + D^-_\alpha \Phi^j_\beta \epsilon_{\alpha\beta} \Phi^k) - \sqrt{-1}W,
\]

\( ^3 \)Semi-classical analysis is a study to get classical objects from the given quantum thing

\( ^4 \)In this section, as the dimension of the configuration space, we use \( d \) instead of \( m \),
where argument of \(g_{ij}, A_j\) and \(W\) is \(\Phi\). Here and what follows, for any smooth function \(u\) on \(\mathbb{R}^d\), we extend it (called Grassmann continuation) on \(\mathcal{M}^{m|0}\) as

\[
(9.6.5) \quad u(\Phi) = u(x) + \sqrt{-1} u_{,k}(x) \left( \rho_{\alpha \epsilon \alpha \beta} \psi_{\beta}^k + \frac{1}{2} \rho_{\alpha \epsilon \alpha \beta} \rho_{\beta} \psi_{\alpha}^j \right) - \frac{1}{2} u_{,kl}(x) \rho_{\alpha \epsilon \alpha \beta} \psi_{\beta}^k \rho_{\beta} \epsilon_{\alpha} \psi_{\alpha}^j
\]

for

\[
(9.6.6) \quad \Phi^j = x^j + \sqrt{-1} \rho_{\alpha \epsilon \alpha \beta} \psi_{\beta}^j + \sqrt{-1} \rho_{\alpha \epsilon \alpha \beta} F^j \quad \text{where} \quad x^j \in \mathcal{N}_{ev}, \psi_{\alpha}^j \in \mathcal{N}_{od} \text{ and } F^k \in \mathcal{N}_{ev}.
\]

Notations. We put

\[
(9.6.7) \quad [jk] \overset{\text{def}}{=} \psi_{\alpha}^j \epsilon_{\beta \alpha} \psi_{\beta}^k = \psi_{\alpha}^j \psi_{\beta}^k - \psi_{\beta}^j \psi_{\alpha}^k = [kj],
\]

\[
(9.6.8) \quad R_{ikjl} \overset{\text{def}}{=} \frac{1}{2} (g_{ij,kl} + g_{kl,ij} - g_{jk,il} - g_{il,jk}) + \Gamma_{ij}^{mn} \Gamma_{kl}^{mn} - \Gamma_{il}^{mn} \Gamma_{jk}^{mn}
\]

and

\[
(9.6.9) \quad \nabla_j W = W_{,j}, \quad \nabla^j W = g^{jk} W_{,k}, \quad \nabla_j \nabla_k W = \nabla_k \nabla_j W = W_{,jk} = W_{,jk} - \Gamma_{jk}^{mn} W_{,m}.
\]

Formulas. Following formulas are easily obtained using renumbering and symmetry of indices combining with the anticommutativity of \(\{\psi_{\alpha}^j\}\)’s.

\[
(9.6.10) \quad (g_{ij,kl} + g_{mn} \Gamma_{ij}^{mn} \Gamma_{kl}^{mn}) [ij][kl] = \frac{2}{3} R_{ikjl}[ij][kl],
\]

\[
(9.6.11) \quad g_{ij,kl}[ij][kl] = g_{ij,kl}(\psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l + \psi_{\alpha}^j \psi_{\gamma}^k \psi_{\beta}^l + \psi_{\beta}^j \psi_{\alpha}^k \psi_{\gamma}^l + \psi_{\beta}^j \psi_{\gamma}^k \psi_{\alpha}^l + \psi_{\gamma}^j \psi_{\alpha}^k \psi_{\beta}^l + \psi_{\gamma}^j \psi_{\beta}^k \psi_{\alpha}^l)
\]

\[
= 4 g_{ij,kl} \psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l = g_{k,lj}[ij][kl],
\]

\[
(9.6.12) \quad g_{ij,kl}[ij][kl] = g_{ij,kl}(\psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l + \psi_{\alpha}^j \psi_{\gamma}^k \psi_{\beta}^l + \psi_{\beta}^j \psi_{\alpha}^k \psi_{\gamma}^l + \psi_{\beta}^j \psi_{\gamma}^k \psi_{\alpha}^l + \psi_{\gamma}^j \psi_{\alpha}^k \psi_{\beta}^l + \psi_{\gamma}^j \psi_{\beta}^k \psi_{\alpha}^l)
\]

\[
= 2 g_{ij,kl} \psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l = g_{k,lj}[ij][kl],
\]

\[
(9.6.13) \quad g_{ij,kl} \psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l = -g_{ij,kl} \psi_{\beta}^j \psi_{\alpha}^k \psi_{\gamma}^l,
\]

\[
(9.6.14) \quad \Gamma_{ij}^{mn} \Gamma_{kl}^{mn} [ij][kl] = 4 \Gamma_{ij}^{mn} \Gamma_{kl}^{mn} \psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l,
\]

\[
(9.6.15) \quad \Gamma_{ij}^{mn} \Gamma_{kl}^{mn} [ij][kl] = -2 \Gamma_{ij}^{mn} \Gamma_{kl}^{mn} \psi_{\alpha}^j \psi_{\beta}^k \psi_{\gamma}^l.
\]

Simple but lengthy calculations yield that

\[
L_0'(x, \dot{x}, \dot{\psi}_\alpha, \dot{\bar{\psi}}_\alpha, F) \overset{\text{def}}{=} \int_{\mathbb{R}^d} d\rho \tilde{L}_0(\Phi(t, \rho), \mathcal{D}_\alpha \Phi(t, \rho))
\]

\[
= \frac{1}{2} g_{jk}(\dot{x}_j \dot{x}_k + F_j F_k + \sqrt{-1} \psi_{\alpha}^j \dot{\psi}_{\alpha}^k + \frac{1}{8} g_{ij,kl}[ij][kl]
\]

\[
+ A_j \dot{x}_j - \frac{\sqrt{-1}}{2} A_{ij,kl} \dot{\psi}_\alpha^j \psi_{\alpha}^k - \frac{\sqrt{-1}}{2} \Gamma_{ij}^{mn} \Gamma_{kl}^{mn} F^l + \sqrt{-1} \frac{1}{2} g_{ij,kl}[ij][kl]
\]

\[
+ \Gamma_{kl}^{mn} F^k - \frac{\sqrt{-1}}{2} W_{,kl}[ij].
\]

Assuming the auxiliary field \(F = (F^1, \ldots, F^d)\) satisfies

\[
(9.6.17) \quad \frac{\delta L_0'}{\delta F^k} = 0, \quad \text{i.e.} \quad F^k = \frac{\sqrt{-1}}{2} \Gamma_{ij}^{mn} [ij] - g^{kl} W_{,l}.
\]
we get

\[ \mathcal{L}_0(x, \dot{x}, \psi_\alpha, \dot{\psi}_\alpha) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k + \sqrt{-1} g_{jk} \psi_\alpha \frac{D}{dt} \psi_\alpha^k + \frac{1}{12} R_{ikjl}[ijkl] \]

(9.6.18)

\[ + A_j \dot{x}^j - \frac{\sqrt{-1}}{2} A_j,k \psi_\alpha \psi_\alpha^k - \frac{1}{2} g^{jk} W_{j,k} - \frac{\sqrt{-1}}{2} \nabla_j W_{j,k} [jk], \]

where

\[ \frac{D}{dt} \psi_\alpha^k \overset{\text{def}}{=} \dot{\psi}_\alpha^k + \Gamma^k_{pl} \dot{x}^p \psi_\alpha^l. \]

**Remark 9.6.1.** (a) Above derivation of [9.6.18] is essentially due to Davis et al. [28], though in their calculations the coefficient 1/8 in [9.6.16] is replaced by 1. Moreover, they didn’t mention the necessity of using the Grassmann algebra with infinite number of generators, which is necessary to define odd derivatives uniquely.

(b) To eliminate the auxiliary fields \( F^i \), we assume that the ‘equation of motion’ described in [9.6.17] holds. Though there is a work, for example Cooper and Freedman [26], which asserts that \( F^i \) is calculated out after integrating the partition function expressed by the Feynman measure, it seems curious to use the “quantum argument” when we are discussing the “classical objects”. (In any way, there does not exist the ‘Fubini theorem’ with respect to the ‘Feynman measure’.)

**9.6.2. A prescription for a supersymmetric extension of Hamiltonian \( H(q,p) \).** We restart from the Lagrangian \( \mathcal{L}_0 \) [9.6.18] ignoring the procedures itself. As the variables \( \{ \psi_\alpha^j \} \) are assumed to be “real” and anticommutative, we define from them the “complex” odd variables as follows:

\[ \psi^j = \frac{1}{\sqrt{2}} (\psi_1^j + \sqrt{-1} \psi_2^j), \quad \bar{\psi}^j = \frac{1}{\sqrt{2}} (\psi_1^j - \sqrt{-1} \psi_2^j), \]

(9.6.20)

that is

\[ \psi_1^j = \frac{1}{\sqrt{2}} (\psi^j + \bar{\psi}^j), \quad \psi_2^j = \frac{1}{\sqrt{2} \sqrt{-1}} (\psi^j - \bar{\psi}^j). \]

(9.6.21)

Then, clearly we have

\[ \{ \bar{\psi}^i, \psi^j \} = \{ \psi^i, \psi^j \} = \{ \psi^i, \bar{\psi}^j \} = \{ \psi^j, \bar{\psi}^i \} = 0 \]

(9.6.22)

\[ [ij] = \sqrt{-1} (\psi^i \bar{\psi}^j - \psi^j \bar{\psi}^i). \]

By the same calculation as before,

\[ R_{ikjl}[ijkl] = -6 (g_{ij,kl} + \Gamma^m_{ij} \Gamma^m_{kl} g_{mn}) \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l \]

(9.6.23)

\[ = 6 (g_{lk,jl} + \Gamma^m_{ik} \Gamma^m_{jl} g_{mn}) \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l. \]

So, we get

\[ \mathcal{L}(x, \dot{x}, \psi, \dot{\psi}, \bar{\psi}, \dot{\bar{\psi}}) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k + \sqrt{-1} g_{jk} (\psi^j \frac{D}{dt} \bar{\psi}^k + \bar{\psi}^j \frac{D}{dt} \psi^k) - \frac{1}{4} R_{ijkl} \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l \]

(9.6.24)

\[ + A_j \dot{x}^j - \frac{\sqrt{-1}}{2} B_{jk} \psi^j \bar{\psi}^k + \frac{1}{2} A_j A^j - \frac{1}{2} \nabla^j W \nabla_j W + \frac{1}{2} \nabla_i \nabla_j W (\psi^i \bar{\psi}^j - \bar{\psi}^i \psi^j). \]
Introducing new variables by

\[
\begin{align*}
\xi_i &= \frac{\delta \mathcal{L}}{\delta x^i} = g_{ij}(x)\dot{x}^j + \frac{\sqrt{-1}}{2} g_{ij,k}(\dot{\psi}^j \psi^k + \psi^j \dot{\psi}^k) + A_i(x), \\
\phi_i &= \frac{\delta \mathcal{L}}{\delta \psi^i} = -\frac{\sqrt{-1}}{2} g_{ij}(x)\dot{\psi}^j, \\
\bar{\phi}_i &= \frac{\delta \mathcal{L}}{\delta \bar{\psi}^i} = -\frac{\sqrt{-1}}{2} g_{ij}(x)\dot{\psi}^j,
\end{align*}
\]

we get

\[
\mathcal{H} = \mathcal{H}(x, \xi, \psi, \bar{\psi}) \overset{\text{def}}{=} \dot{x} \xi + \dot{\psi} \phi + \dot{\bar{\psi}} \bar{\phi} - \mathcal{L}
\]

\[
= \frac{1}{2} g^{ij}(\xi_j - \sqrt{-1} g_{ik,l}(\dot{\psi}^k \psi^l + \psi^k \dot{\psi}^l) - A_j)(\xi_j - \sqrt{-1} g_{jm,n}(\dot{\psi}^m \psi^n + \psi^m \dot{\psi}^n) - A_j)
\]

\[
- \frac{1}{2} R_{ikjl} \psi^i \psi^j \bar{\psi}^k \bar{\psi}^l + \frac{1}{2} g^{ik} W_{j,k} W_{i,j} + \frac{1}{2} W_{i,j} (\dot{\psi}^i \dot{\psi}^j - \psi^i \bar{\psi}^j).
\]

Now, rewriting the variables \(\psi^i, \bar{\psi}^i\) as \(\theta^i, \pi^i\), we get finally

\[
\mathcal{H} = \mathcal{H}(x, \xi, \theta, \pi)
\]

\[
= \frac{1}{2} g^{ij}(\xi_j - \sqrt{-1} g_{ik,l}(\dot{\theta}^k \theta^l + \theta^k \dot{\theta}^l) - A_j)(\xi_j - \sqrt{-1} g_{jm,n}(\dot{\theta}^m \pi^n + \theta^m \dot{\pi}^n) - A_j)
\]

\[
+ \frac{1}{2} R_{ikjl} \theta^i \theta^j \pi^k \pi^l + \frac{1}{2} g^{ik} W_{j,k} W_{i,j} - W_{i,j} \pi^i \pi^j,
\]

which is thought as the supersymmetric extension of \(H(q, p)\) when \(V = \frac{1}{2} g^{ik} W_{j,k} W_{i,j}\).

**9.6.3. Supersymmetry and supercharges.** Preparing a pair of “real” Grassmann parameters \(\varepsilon_\alpha \in \mathfrak{Re}_{ed}\) for \(\alpha = 1, 2\), we introduce a one parameter group of transformations \(T_s\) \((s \in \mathbb{R})\) from \((t, \rho_1, \rho_2) \in \mathfrak{Re}_{12}\) to \((t', \rho'_1, \rho'_2) \in \mathfrak{Re}_{12}\) defined by

\[
\begin{align*}
t' &= t - is(\varepsilon_1 \rho_2 - \varepsilon_2 \rho_1), \\
\rho'_1 &= \rho_1 - s\varepsilon_1, \\
\rho'_2 &= \rho_2 - s\varepsilon_2
\end{align*}
\]

and also two operators

\[
D^+_\alpha = \frac{\partial}{\partial \rho_\alpha} + i \rho_\alpha \frac{\partial}{\partial t} \quad \text{for} \quad \alpha = 1, 2.
\]

Clearly, the infinitesimal generator of transformations above is given by

\[
\frac{\partial}{\partial s} v(T_s(t, \rho_1, \rho_2))|_{s=0} = -(\varepsilon_1 D^+_2 - \varepsilon_2 D^+_1) v(t, \rho_1, \rho_2)
\]

for any smooth function \(v(t, \rho_1, \rho_2)\) from \(\mathfrak{Re}_{12}\) to \(\mathfrak{Re}_{ev}\). Here, we remark that

\[-(\varepsilon_1 D^+_1 + \varepsilon_2 D^+_2) v(t, \rho_1, \rho_2) = \delta t \frac{\partial v}{\partial t} + \delta \rho_1 \frac{\partial v}{\partial \rho_1} + \delta \rho_2 \frac{\partial v}{\partial \rho_2}\]

with

\[
\delta t = -i(\varepsilon_1 \rho_1 + \varepsilon_2 \rho_2), \quad \delta \rho_1 = -\varepsilon_1, \quad \delta \rho_2 = -\varepsilon_2.
\]

Moreover, \(v(t, \rho_1, \rho_2)\) is called supersymmetric if

\[
\delta v(t, \rho_1, \rho_2) = -(\varepsilon_1 D^+_1 + \varepsilon_2 D^+_2) v(t, \rho).
\]
Above relation implies the following: If $\Phi^j(t, \rho_1, \rho_2)$ given in (9.6.1) is supersymmetric, we have

$$\delta \Phi^j(t, \rho_1, \rho_2) \equiv \delta x^j(t) + i \rho_\alpha \epsilon_\alpha \beta \delta \psi^j_\beta(t) + \frac{i}{2} \rho_\alpha \epsilon_\alpha \beta \rho_\beta \delta F^j(t)$$

(9.6.32)

$$= -(\epsilon_1 D_1 + \epsilon_2 D_2) \Phi^j(t, \rho_1, \rho_2)$$

Since $\mathfrak{M}$ is infinite dimensional, if $\omega$ satisfies $\delta \rho_1 \omega = 0$ and $\rho_1 \omega = 0$ then $\omega = 0$. This yields that

$$\begin{cases}
\delta x^j = -i \epsilon_\alpha \epsilon_\alpha \beta \psi^j_\beta, \\
\delta \psi^j_\alpha = -\epsilon_\alpha F^j - \epsilon_\alpha \beta \epsilon_\beta \dot{x}^j, \\
\delta F^j = i \epsilon_\alpha \dot{\psi}^j_\alpha.
\end{cases}$$

(9.6.33)

Using the relation (??), we get

$$\begin{cases}
\delta x^j = -i \epsilon_\alpha \epsilon_\alpha \beta \psi^j_\beta, \\
\delta \psi^j_\alpha = -\epsilon_\alpha (\frac{i}{2} \Gamma^j_{kl}[kl] - \nabla^j W) - \epsilon_\alpha \beta \epsilon_\beta \dot{x}^j.
\end{cases}$$

(9.6.34)

From this, we have the following quantities, called supercharges,

$$Q_\alpha = \psi^j_\alpha g^j \dot{x}^j - \epsilon_\alpha \beta \psi^j_\beta \nabla_i W$$

(9.6.35)

which is conserved by the flow defined by the above Lagrangian.

On the other hand, the following supersymmetric Lagrangian is introduced by physicist:

$$L_0(x; \dot{x}, \psi_\alpha; \dot{\psi}_\alpha) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k + \frac{\sqrt{-1}}{2} g_{jk} \bar{\psi}^j \gamma^0 \frac{D}{dt} \psi^k + \frac{1}{12} R_{ijkl} \bar{\psi}^i \psi^j \bar{\psi}^k \psi^l$$

$$= \frac{1}{2} \nabla^j W \nabla_j W - \frac{1}{2} \nabla_i \nabla_j W [\bar{\psi}^i, \psi^j].$$

Here,

$$\psi^j = \begin{pmatrix} \psi^j_1 \\ \psi^j_2 \end{pmatrix}, \quad \bar{\psi}^j = \sqrt{-1} (\psi^j_2, -\psi^j_1) = t \begin{pmatrix} \psi^j_1 \\ \psi^j_2 \end{pmatrix}^*, \quad \gamma^0 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$$

Alvarez-Gaumé [5] used above with $W = 0$ on a general manifold $M$. 
9.7. Hamilton flow for Weyl equation with external electro-magnetic field

In this section, we consider the Weyl equation with external electro-magnetic field \( (7.0.7) \) with its symbol below:

\[
(9.7.1) \quad \mathcal{H}(t, x, \xi, \theta, \pi) = \sum_{j=1}^{3} c\sigma_j(\theta, \pi)(\xi_j - \frac{e}{c} A_j(t, x)) + eA_0(t, x).
\]

Corresponding Hamilton equation is, for \( j = 1, 2, 3, l = 1, 2 \),

\[
(9.7.2) \quad \frac{d}{dt} x_j = \frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \xi_j}, \quad \frac{d}{dt} \xi_j = -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial x_j}, \\
\frac{d}{dt} \theta_l = -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \pi_l}, \quad \frac{d}{dt} \pi_l = -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \theta_l}.
\]

We take this as a simple example for the necessity of the countably infinite Grassmann generators.

**Proposition 9.7.1.** Assume \( (A_0(t, q), A_1(t, q), A_2(t, q), A_3(t, q)) \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R}) \) in (0.7) in Chapter 7. Then, for any initial data \( (x(0), \xi(0), \theta(0), \pi(0)) = (x, \xi, \theta, \pi) \in \mathcal{O}^4 \cong \mathcal{T}^* \mathcal{O}^4 \), \( (9.7.2) \) has the unique, global in time, solution \( (x(t), \xi(t), \theta(t), \pi(t)) \).

**Remark 9.7.1.** We require only smoothness for \( \{A_j(t, q)\}_{j=0}^{3} \) without strict conditions on the behavior when \( |q| \to \infty \).

Odd variables part of \( (9.7.2) \) are rewritten as

\[
(9.7.3) \quad \frac{d}{dt} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \pi_1 \\ \pi_2 \end{pmatrix} = ich^{-1} \mathcal{K}(t) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \pi_1 \\ \pi_2 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \pi_1 \\ \pi_2 \end{pmatrix}.
\]

Here, we put

\[
(9.7.4) \quad \eta_j(t) = \xi_j(t) - \frac{e}{c} A_j(t, x(t)),
\]

and

\[
(9.7.5) \quad \mathcal{K}(t) = \begin{pmatrix} -\eta_3(t)\pi_2 & h^{-1}(\eta_1(t) - in_2(t))\sigma_2 \\ h(\eta_1(t) + in_2(t))\sigma_2 & \eta_3(t)\pi_2 \end{pmatrix}.
\]

Moreover, for \( \sigma_j(t) = \sigma_j(\theta(t), \pi(t)) \),

\[
(9.7.6) \quad \mathcal{Y}(t) = \begin{pmatrix} 0 & -\eta_3(t) & \eta_2(t) \\ \eta_3(t) & 0 & -\eta_1(t) \\ -\eta_2(t) & \eta_1(t) & 0 \end{pmatrix}.
\]

by simple calculation

\[
(9.7.7) \quad \frac{d}{dt} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 2ch^{-1}\mathcal{Y}(t) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \theta_1 \pi_2 + h^{-2}\pi_1 \pi_2 \\ i(\theta_1 \theta_2 - h^{-2}\pi_1 \pi_2) \\ -ih^{-1}(\theta_1 \pi_1 + \theta_2 \pi_2) \end{pmatrix}.
\]

Now we begin our proof. We decompose dependent variables \( (x, \xi, \theta, \pi) \) by degree:

\[
(9.7.8) \quad x_j(t) = \sum_{\ell=0}^{\infty} x_{j, \ell}^{[2\ell]}(t), \quad \xi_j(t) = \sum_{\ell=0}^{\infty} \xi_{j, \ell}^{[2\ell]}(t), \quad \theta_k(t) = \sum_{\ell=0}^{\infty} \theta_{k, \ell}^{[2\ell+1]}(t), \quad \pi_k(t) = \sum_{\ell=0}^{\infty} \pi_{k, \ell}^{[2\ell+1]}(t).
\]
For given $m = 0, 1, 2, \ldots$, 

\begin{align*}
\begin{cases}
\frac{d}{dt} \sigma_j^{[2m]} = c\sigma_j^{[2m]} & \text{where } \sigma_j^{[0]} = 0, \\
\frac{d}{dt} \xi_j^{[2m]} = e \sum_{\ell=1}^{m} \sum_{k=0}^{3} \sigma_k^{[2\ell]} \frac{\partial A_k^{[2m-2\ell]}}{\partial x_j} - e \frac{\partial A_0^{[2m]}}{\partial x_j} \end{cases} with \quad \left( x_j^{[2m]}(t) \right) = \left( x_j^{[2m]}(t) \right),
\end{align*}

(9.7.9)

\begin{align*}
\frac{d}{dt} \left( \frac{\theta^{[2m+1]}}{\pi_1^{[2m+1]}} \right) = ich^{-1} \sum_{\ell=0}^{m-1} \mathcal{X}^{[2\ell]}(t) \left( \frac{\theta^{[2m-2\ell]}}{\pi_1^{[2m-2\ell]}} \right) & \text{with } \left( \theta^{[2m+1]}(t) \right) = \left( \theta^{[2m+1]}(t) \right), \\
\frac{d}{dt} \left( \frac{\pi_1^{[2m+1]}}{\pi_2^{[2m+1]}} \right) \end{align*}

and

\begin{align*}
\frac{d}{dt} \left( \frac{\sigma_j^{[2m]} \pi_2^{[2m]}}{\sigma_3^{[2m]}} \right) = 2ch^{-1} \mathcal{Y}^{[2\ell]}(t) \left( \frac{\sigma_j^{[2m-2\ell]} \pi_2^{[2m-2\ell]}}{\sigma_3^{[2m-2\ell]}} \right) \quad \text{with } \left( \frac{\sigma_j^{[2m]} \pi_2^{[2m]}}{\sigma_3^{[2m]}} \right) = \left( \frac{\sigma_j^{[2m]} \pi_2^{[2m]}}{\sigma_3^{[2m]}} \right).
\end{align*}

(9.7.10)

Here, $\mathcal{X}^{[2\ell]}(t)$, $\mathcal{Y}^{[2\ell]}(t)$, $\sigma^{[2\ell]}(t)$ are degree $2\ell$ part of $\mathcal{X}(t)$, $\mathcal{Y}(t)$, $\sigma(t)$, respectively.

\begin{align*}
\eta_j^{[2\ell]}(t) = \xi_j^{[2\ell]}(t) - \frac{e}{c} A^{[2\ell]}(t, x),
\end{align*}

\begin{align*}
A^{[2\ell]}(t, x) = \sum_{|\alpha| \leq 2\ell} \frac{1}{\alpha!} \partial^\alpha_q A_k(t, x[0]) \cdot (x_1^{\alpha_1})(x_2^{\alpha_2})(x_3^{\alpha_3}),
\end{align*}

\begin{align*}
\frac{\partial A^{[2\ell]}_0(t, x_j)}{\partial x_j} = \sum_{|\alpha| \leq 2\ell} \frac{1}{\alpha!} \partial^\alpha_q A_0(t, x[0]) \cdot (x_1^{\alpha_1})(x_2^{\alpha_2})(x_3^{\alpha_3}).
\end{align*}

And

\begin{align*}
\sigma_1^{[2m]} &= \sum_{\ell=0}^{m-1} \left( \theta_1^{[2\ell+1]} \theta_2^{[2m-2\ell-1]} + \hbar^2 \pi_1^{[2\ell+1]} \pi_2^{[2m-2\ell-1]} \right), \\
\sigma_2^{[2m]} &= i \sum_{\ell=0}^{m-1} \left( \theta_1^{[2\ell+1]} \theta_2^{[2m-2\ell-1]} - \hbar^2 \pi_1^{[2\ell+1]} \pi_2^{[2m-2\ell-1]} \right), \\
\sigma_3^{[2m]} &= -ih^{-1} \sum_{\ell=0}^{m-1} \left( \theta_1^{[2\ell+1]} \pi_1^{[2m-2\ell-1]} \theta_2^{[2m-2\ell-1]} + \theta_2^{[2\ell+1]} \pi_2^{[2m-2\ell-1]} \right).
\end{align*}

[0] Putting $m = 0$ in (9.7.9), for $j = 1, 2, 3$, 

\begin{align*}
\frac{d}{dt} x_j^{[0]}(t) = 0 \quad \text{and} \quad \frac{d}{dt} \xi_j^{[0]}(t) = -e \frac{\partial A_0^{[0]}(t, x[0])}{\partial x_j} = -e \partial_q A_0^{[0]}(t, x[0]).
\end{align*}

Therefore, for any $t \in \mathbb{R}$ and $j = 1, 2, 3$, 

\begin{align*}
x_j^{[0]}(t) = x_j^{[0]} \quad \text{and} \quad \xi_j^{[0]}(t) = \xi_j^{[0]} - e \int_s^t ds \partial_q A_0^{[0]}(s, x[0]).
\end{align*}

[1] Putting these result into (9.7.10) with $m = 0$

\begin{align*}
\frac{d}{dt} \left( \frac{\theta_1^{[1]}}{\pi_1^{[1]}} \right) = ich^{-1} \mathcal{X}^{[0]}(t) \left( \frac{\theta_1^{[1]}}{\pi_1^{[1]}} \right) \quad \text{with } \left( \frac{\theta_1^{[1]}(t)}{\pi_1^{[1]}(t)} \right) = \left( \frac{\theta_1^{[1]}(t)}{\pi_1^{[1]}(t)} \right).
\end{align*}

(9.7.12)
Here, $X^0(t)$ is $4 \times 4$-matrix whose components are complex valued, depending on
\[(t, t, x^0_1, x^0_2, \xi^0_1, \xi^0_2, \partial_\theta^0 A_0, \partial_\theta^0 A) : |\alpha| = 0, |\beta| \leq 1).\]

More precisely, components of $X^0(t)$ are given as
\[\eta_j^0(t) = \xi_j^0(t) - \frac{e}{c} A_j(t, x^0) = \xi_j^0(t) - e \int_\theta^0 \partial_\theta^0 A_0^0(s, x^0) - \frac{e}{c} A_j^0(t, x^0).\]

ODE (9.7.12) has smooth coefficients w.r.t. $t$ with value in $(\mathbb{R}^0)^4$, which has a unique global in time solution depending on as follows: Putting $A = (A_1, A_2, A_3)$, we have
(9.7.13) \[\begin{align*}
\theta_j^1(t) &= \theta_j^1(t, x^0) \leq \xi_0^0 \leq \xi_1^0, \theta_j^1, \theta_j^2, \theta_j^3, A_0, A^0 A ; |\alpha| = 0, |\beta| \leq 1), \text{ linear w.r.t. } \theta_j^1, \theta_j^2, \\
\pi_j^1(t) &= \pi_j^1(t, x^0) \leq \xi_0^0 \leq \xi_1^0, \pi_j^1, \pi_j^2, \pi_j^3, A_0, A^0 A ; |\alpha| = 0, |\beta| \leq 1), \text{ linear w.r.t. } \theta_j^1, \theta_j^2.
\end{align*}\]

[2] Putting $m = 1$ in (9.7.10),
\[\begin{align*}
\frac{d}{dt} x_j^2 &= c \sigma_j^2, \\
\frac{d}{dt} \xi_j^2 &= e \sum_{k=1}^{3} \sigma_k^2 \frac{\partial A_0^0}{\partial x_j} - e \frac{\partial A_0^2}{\partial x_j} \text{ with } (x^2(t) = x^2),
\end{align*}\]

From $m = 1$ in (9.7.11) and (9.7.13), for $j = 1, 2, 3,$
\[\begin{align*}
\sigma_j^2 &= \theta_j^1, \theta_j^2 + h^2 \pi_j^1, \pi_j^2, \\
\sigma_j^2 &= i(\theta_j^1, \theta_j^2, \theta_j^3, \theta_j^4, \pi_j^1, \pi_j^2, \pi_j^3, \pi_j^4) - h^2 \theta_j^1, \theta_j^2, \\
\sigma_j^3 &= -ih^{-1}(\theta_j^1, \theta_j^2, \theta_j^3, \theta_j^4, \pi_j^1, \pi_j^2, \pi_j^3, \pi_j^4)
\end{align*}\]

Therefore, for $j = 1, 2, 3,$
\[\begin{align*}
x_j^2(t) &= x_j^2(t, x^0) \leq \xi_1^0 \pi_1^0, \pi_1^0, \pi_2^0, \pi_2^0, \pi_3^0, \pi_3^0, \pi_4^0, \pi_4^0, A_0, A^0 A ; 0 \leq \ell \leq 1, |\alpha| = 0, |\beta| \leq 1), \\
\xi_j^2(t) &= \xi_j^2(t, x^0) \leq \xi_1^0 \pi_1^0, \pi_1^0, \pi_2^0, \pi_2^0, \pi_3^0, \pi_3^0, \pi_4^0, \pi_4^0, A_0, A^0 A ; 0 \leq \ell \leq 1, |\alpha| = 0, |\beta| \leq 2).
\]

[3] Putting $m = 1$ in (9.7.10),
\[\begin{align*}
\frac{d}{dt} \left( \begin{array}{c}
\theta_j^1 \\
\theta_j^2 \\
\theta_j^3 \\
\theta_j^4
\end{array} \right) &+ ich^{-1}X^0(t) \left( \begin{array}{c}
\theta_j^1 \\
\theta_j^2 \\
\theta_j^3 \\
\theta_j^4
\end{array} \right) + ich^{-1}X^2(t) \left( \begin{array}{c}
\theta_j^1 \\
\theta_j^2 \\
\theta_j^3 \\
\theta_j^4
\end{array} \right) \text{ with } \left( \begin{array}{c}
\theta_j^1 \\
\theta_j^2 \\
\theta_j^3 \\
\theta_j^4
\end{array} \right) = \left( \begin{array}{c}
\theta_j^1 \\
\theta_j^2 \\
\theta_j^3 \\
\theta_j^4
\end{array} \right).
\end{align*}\]

Here, $4 \times 4$-matrix $X^2(t)$ has components, valued in $C_{ev}$ depending on
\[(t, x^0, x^0, \xi_1^0, \pi_1^0, \pi_2^0, \pi_3^0, \pi_4^0, A_0, A^0 A ; 0 \leq \ell \leq 1, |\alpha| \leq 1, |\beta| \leq 2).\]

From these, for $k = 1, 2,$
\[\begin{align*}
\theta_k^3(t) &= \theta_k^3(t, x^0, x^0, \xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0, \pi_1^0, \pi_2^0, \pi_3^0, \pi_4^0, A_0, A^0 A ; 0 \leq \ell \leq 1, |\alpha| \leq 1, |\beta| \leq 2), \\
\pi_k^3(t) &= \pi_k^3(t, x^0, x^0, \xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0, \pi_1^0, \pi_2^0, \pi_3^0, \pi_4^0, A_0, A^0 A ; 0 \leq \ell \leq 1, |\alpha| \leq 1, |\beta| \leq 2).
\end{align*}\]
Repeating this procedure, we get
\[
\begin{cases}
  x^{[2m]}(t) = x^{[2m]}(t, \pi^{[2]}_t, \xi^{[2]}_t, \theta_t, \pi_t; 0 \leq \ell \leq m, |\alpha| \leq m - 1, |\beta| \leq m), \\
  \xi^{[2m]}(t) = \xi^{[2m]}(t, \pi^{[2]}_t, \xi^{[2]}_t, \theta_t, \pi_t; 0 \leq \ell \leq m, |\alpha| \leq m - 1, |\beta| \leq m), \\
  \theta^{[2m+1]}(t) = \theta^{[2m+1]}(t, \pi^{[2]}_t, \xi^{[2]}_t, \theta_t, \pi_t; 0 \leq \ell \leq m, |\alpha| \leq m, |\beta| \leq m + 1), \\
  \pi^{[2m+1]}(t) = \pi^{[2m+1]}(t, \pi^{[2]}_t, \xi^{[2]}_t, \theta_t, \pi_t; 0 \leq \ell \leq m, |\alpha| \leq m, |\beta| \leq m + 1).
\end{cases}
\]
These prove the existence, moreover, since for each degree, the solution of (9.7.9) with (9.7.10) is unique, so follows the uniqueness of the solution of (9.7.2). □

Moreover, we get easily

**Corollary 9.7.1.** If \((x(t), \xi(t), \theta(t), \pi(t)) \in C^1(\mathbb{R} : T^*\mathbb{R}^3)\) is a solution of (9.7.2), then
\[
\frac{d}{dt} \mathcal{H}(t, x(t), \xi(t), \theta(t), \pi(t)) = \frac{\partial \mathcal{H}}{\partial t}(t, x(t), \xi(t), \theta(t), \pi(t)).
\]

Putting
\[
B_{jk}(t, x) = \frac{\partial A_k(t, x)}{\partial x_j} - \frac{\partial A_j(t, x)}{\partial x_k},
\]
and rewriting
\[
\begin{align*}
\frac{d}{dt} x_j &= c \sigma_j(\theta, \pi), \\
\frac{d}{dt} \eta_j &= \sum_{k=1}^{3} e \sigma_k(\theta, \pi) B_{jk}(t, x) - e \frac{\partial A_0(t, x)}{\partial x_j},
\end{align*}
\]
we have

**Corollary 9.7.2.** For \(\{A_j(t, q)\}_{j=0}^{3} \in C^{\infty}(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})\), putting the initial data
\[
(x(\bar{t}), \xi(t), \theta(t), \pi(t)) = (x, \eta, \theta, \pi) \quad \text{where} \quad \eta_j = \xi_j - \frac{e}{c} A_j(t, x)
\]
for (9.7.15) with (9.7.3), there exists a unique solution \((\bar{x}(t), \bar{\eta}(t), \bar{\theta}(t), \bar{\pi}(t)) \in C^1(\mathbb{R} : T^*\mathbb{R}^3)\). These are related with \((x(t), \xi(t), \theta(t), \pi(t))\) as
\[
\begin{align*}
  x_j(t, t; x, \xi, \theta, \pi) &= \bar{x}_j(t, t; x, \xi, \theta, \pi), \\
  \xi_j(t, t; x, \xi, \theta, \pi) &= \bar{\eta}_j(t, t; x, \xi, \theta, \pi) - \frac{e}{c} A_j(t, x), \\
  \theta_k(t, t; x, \xi, \theta, \pi) &= \bar{\theta}_k(t, t; x, \xi, \theta, \pi) - \frac{e}{c} A_k(t, x), \\
  \pi_k(t, t; x, \xi, \theta, \pi) &= \bar{\pi}_k(t, t; x, \xi, \theta, \pi).
\end{align*}
\]

**Remark 9.7.2.** The solution of the Hamilton flow corresponding to free Weyl equation in Proposition 9.7.1 is solved explicitly. Obtaining such an explicit solution is not necessarily happened frequently, in general we have only its existence abstractly. Fortunately, because of the countable degree stems from the countable Grassmann generators, we get rather easily the existence proof. But it is rather complicated to have the estimates w.r.t. the initial data.
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