Edge states of photon pairs in cavity arrays with spatially modulated nonlinearity

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We study theoretically an extended Bose-Hubbard model with the spatially modulated interaction strength, describing a one-dimensional array of tunneling-coupled nonlinear cavities. It is demonstrated that the spatial modulation of the nonlinearity induces bound two-photon edge states. The formation of these edge states has been understood analytically in terms of nonlinear self-localization.

I. INTRODUCTION

The quantum simulations with arrays of coupled qubits are now rapidly developing with 51-qubit systems based on the cold atom platform being already available [1, 2]. The integrated quantum optical platform potentially promises robust scalable quantum simulations on a chip. It is now under active development [3–8], although still catches up with the cold atom and superconducting resonator networks [9]. Hence, it is specially instructive to re-examine the conceptual effects of interaction, that can be manifested already for a few interacting particles, and are potentially easier to implement.

One of the simplest interaction effects is the formation of spatially bound two-boson pairs (doublons), that can be present for both attractive and repulsive interaction [10]. Qualitatively, these states form because the energy of bound pairs is repelled either below or above the continuum of quasi-independent scattering states and hence the photons become co-localized. The doublon states arisen due to repulsive interaction have been already observed in a cold atom system [11, 12]. The one-dimensional two-particle Bose-Hubbard model, describing the formation of doublons, is very useful despite its conceptual simplicity. First, it can be readily emulated even classically by considering a mathematically equivalent two-dimensional array of coupled waveguides [13, 14] or even equivalent electric circuits [15]. Second, it can be generalized by including the spatial modulation of the nonlinearity, on-site energies and tunneling coefficients which significantly enriches the types of two-particle states available in the system. For instance, in Ref. [16] we recently demonstrated that the formation of bound two-particle states is a quite general phenomenon that can occur even when the interaction is dissipative. In Ref. [17] the Tamm-Hubbard two-photon edge states were found in the system, where the edge cavity has been detuned by energy from those in the bulk.

Recent interest to the localized biphoton states has been stimulated by the rapid progress of topological photonics that promises disorder-robust edge states of light [18, 19]. While the topological edge states of classical electromagnetic waves have already been known for a decade [20], the quest for nonlinear and quantum topological photonics systems has begun quite recently [6–8, 21–23]. The two-photon pairs in the arrays of coupled cavities allow one to study the interplay of topology and interactions [24]. It has been predicted that the spatial modulation of the tunneling constants in the array of nonlinear cavities can induce the formation of the two-photon edge states [25–28]. The physics of this system is unexpectedly rich. For instance, contrary to the classical Su-Schrieffer-Heeger model describing non-interacting particles in the array of cavities with modulated interaction strength [29], the edge states for the interacting photon pairs can arise both at the edges with weak and strong tunneling links [26]. Another interesting mechanism to realize the two-photon edge states is to implement the non-local interactions, when the photon energy is modified by the presence of the photon in the adjacent cavity [30].

Here, we examine one more extension of the Bose-Hubbard model by considering the array of cavities where all single-photon energies and tunneling links are the same, but the interaction strength is spatially modulated, see Fig. 1. We demonstrate, that this system also possesses bulk doublon states and edge doublon states. These edge states are an inherent feature of the spatially-modulated local on-site nonlinearity, qualitatively differing them from the topological edge states considered in Refs. [15, 26], where the single- and two-photon tunneling amplitudes have been modulated in space.

\begin{center}
\textbf{FIG. 1.} Schematics of the considered array of qubits with the single-photon resonance frequency $\omega_0$. The modulated interaction strength $2U$ is illustrated by the effective potentials with different anharmonicity.
\end{center}
II. MODEL FOR SCATTERING AND DOUBLON STATES

The structure under consideration is schematically illustrated in Fig. 1. It is described by the Hamiltonian

\[
H = H_0 + U = \hbar \omega_0 \sum_{j=1}^{N} a_j^\dagger a_j + J \sum_{j=1}^{N-1} (a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j) - U \sum_{j=1}^{N} (1 - (-1)^j) a_j^\dagger a_j (a_j^\dagger a_j - 1),
\]

where \(a_j^\dagger(a_j)\) are the photon creation (annihilation) operators, \(J\) is the tunneling parameter and \(U\) is the photon-photon interaction strength. The photon-photon interaction term \(2U\) is present only for every second cavity.

We look for the two-photon solutions of the Hamiltonian (1) where the wavefunction has the form

\[
|\Psi\rangle = \sum_{j,j'=1}^{N} \Psi_{jj'} a_j^\dagger a_{j'}^\dagger |0\rangle,
\]

with \(\Psi_{jj'} = \Psi_{j'j}\), reflecting the bosonic nature of the excitations. We substitute the wavefunction in the Schrödinger equation \(E\Psi = H\Psi\) with the Hamiltonian (1) and obtain a system of linear equations for the coefficients \(\Psi_{jj'}\) in Eq. (2). As such, the interacting two-particle problem in one dimension is exactly mapped to the non-interacting single-particle problem in two dimensions.

The energy spectra dependence on the interaction strength is shown in Fig. 2. The entire continuum between \(E = -4\) and \(E = 4\), shown as green rectangle, is occupied by the scattering states, representing quasi-independent photons. For such states, the wave function is the tensor product of the wave functions of single photons and the energy is simply the sum of their energies, with a slight correction due to interaction term. Two branches emerge from this continuum. These are a band of bound photon pairs (red), that we will term as doublons from now on and the doublon edge state (blue), where both photons are localized at the same edge, respectively. In the finite system the edge state is clearly defined only in case of strong interaction (see Sec. IV), therefore in Fig. 2 relevant blue dashed line and dotted line are plotted up to \(U = 0.7\), which was found to be the lowest value, at which the edge state is preserved for \(N = 20\) cavities. The doublon band has a dispersion (8) derived below in Sec. III. The lower and higher edges of the doublon band correspond to \(k = \pm \pi\) and \(k = 0\), respectively. The spatial distribution for these three basic types of two-photon states are illustrated in Fig. 3. Here, the two axes correspond to the coordinates of the two photons. The scattering state in Fig. 3(a) corresponds to quasi-independent photons without strong correlations in their coordinates. The doublon state in Fig. 3(b) is very different with large occupation probability at only half of the diagonal cells. This corresponds to the two photons that are spatially bound together by the spatially modulated interaction. Both Fig. 3(c) and Fig. 3(d) depict the edge state of the photon pair in logarithmic scale. In case of strong interaction Fig. 3(c), the exponential decay away from the corner shows the pure edge state, whereas for weak interaction in a finite system, the edge state energy \(E \sim 2U\) lies within the scattering continuum and, therefore, it is hybridized with scattering states. In this case, the decay is exponential only in the close proximity to the corner, and the state is partially delocalized, see Fig. 3(d).

Figures 2,3 present a basic overview of the eigenstates depending on the photon-photon interaction strength. Now we present a more detailed analysis of bulk doublon states (Sec. III) and edge states of doublons (Sec. IV).

III. DOUBLON STATES

In this section we derive analytically the doublon dispersion for the infinite system, and compare it with the numerical calculation for a finite array of cavities. The results for these two approaches match exactly. In order to find the bound photon states we use the Bethe ansatz like in [17, 31], i.e. look for the wavefunction Eq. (2) in the form

\[
\Phi_{jj'} \equiv \begin{pmatrix} 
\Psi_{2j,2j'} \\
\Psi_{2j,2j'+1} \\
\Psi_{2j+1,2j'+1} \\
\Psi_{2j+1,2j'+2} 
\end{pmatrix} = \Phi e^{i(k(j+j')+iq(j'-j))},
\]

where \(\Phi\) is the tensor product of the wave functions of single photons, \(k\) is the wave vector of the doublon, \(q\) is the momentum of the state, and \(\Phi\) is a constant wavefunction that is independent of the momentum.
FIG. 3. The spatial distribution $|\Psi_{mn}|^2$ of four characteristic two-photon eigenmodes: (a) the scattering state, $E = 3J$; (b) doublon, $E = 5.16J$; (c) edge state, $E = 4.52J$; (d) edge state, $E = 2.98J$. The interaction strength is $U = 2$ for the panels (a–c) and $U = 0.9$ for the panel (d).

with $j < j'$, where we write four values of the wave function in the unit cell as a one column-vector. Here, $k$ is the center-of-mass wave vector and the wave vector $q$ describes the relative motion of the photons. Inserting this ansatz into the Hamiltonian Eq. (1) for $j \neq j'$, i.e. in cavities outside of main diagonal with red cells in Fig. 4, we obtain the dispersion of the scattering states:

$$E = \left\{ \begin{array}{l} \pm 4J \cos \frac{q}{2} \cos \frac{k}{4} \\ \pm 4J \cos \frac{q}{2} \sin \frac{k}{4} \end{array} \right. \quad (4)$$

Equation (4) indicates, that two different wave vectors $q$ are possible for given eigenmode energy $E$ and the center-of-mass wave vector $k$:

$$\cos \frac{q_1}{2} = \frac{E}{4\cos k/4}, \quad \cos \frac{q_2}{2} = \frac{E}{4\sin k/4} \quad (5)$$

Due to this degeneracy in $q$ we need to include a superposition of two waves with $q_1, q_2$ in the ansatz (3) in order to describe the edge state of doublons

$$\Phi_{jj'} = e^{ik(j+j')} \left[ \Phi_1 e^{i\eta_1(j-j')} + \Phi_2 e^{i\eta_2(j-j')} \right], \quad (6)$$

where

$$\Phi_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}. \quad (7)$$

Next, we examine the effect of the photon-photon interaction on the solution Eq. (6). Namely, we substitute the (6) into the Hamiltonian (1) with $j = j'$, and consider complex values of $q_1, q_2$ with the positive imaginary part, so that the wavefunction would decay with increase of the photon-photon distance. Rigorous solution results in the following implicit doublon dispersion law:

$$\frac{1}{U} = \frac{1}{\sqrt{E^2 - 16J^2 \cos^2 \frac{k}{4}}} + \frac{1}{\sqrt{E^2 - 16J^2 \sin^2 \frac{k}{4}}} \quad (8)$$

Despite the fact that doublons are present in the spectrum for all $U > 0$, the smaller is the $U$ the smaller is the photon-photon binding, as has been illustrated by the gradient shading of the doublon band in Fig. 2. This can be shown by rewriting Eq. (8) as

$$\frac{E}{U} = \frac{1}{\tanh q_1''} + \frac{1}{\tanh q_2''} \quad (9)$$

where $q_{1,2}'' = \text{Im}(q_{1,2})$ are the inverse localization lengths.

In the case of vanishing interaction, $U \to 0$, we find that either $q_1'' \to 0$ or $q_2'' \to 0$, which quenches the binding between the two photons. The two-photon band structure for two different regimes is illustrated in Fig. 5.

We will now proceed to obtain the band structure of doublons in the infinite lattice. To this end, we first fixed the center-of-mass wave vector $k$, using the ansatz (6), and then numerically solved the problem for a one-dimensional effective model describing relative motion of the two photons, shown in Fig. 4. The unit cell has a shape of a stripe (green and pink lines in the Fig. 4) passing along the line of constant center of mass of two
photons. The Hamiltonian, describing the relative photon motion, reads

$$H = \begin{pmatrix} H_A & V \\ V^\dagger & H_B \end{pmatrix}$$  \hspace{1cm} (10)$$

where matrices $H_A, H_B, V$ are infinite and their central part is

$$H_A = \begin{pmatrix} 0 & J & 0 & 0 & 0 \\ J & 0 & J & 0 & 0 \\ 0 & J & 2U & J & 0 \\ 0 & 0 & J & 0 & J \\ 0 & 0 & 0 & J & 0 \end{pmatrix}, \quad H_B = \begin{pmatrix} 0 & J & 0 & 0 \\ J & 0 & J & 0 \\ 0 & J & 0 & J \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & J & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & J e^{-ik} & 0 & 0 & 0 \\ J & 0 & J & 0 & 0 \\ 0 & J e^{-ik} & 0 & J e^{-ik} & 0 \\ 0 & 0 & J & 0 & J \\ 0 & 0 & 0 & J e^{-ik} & 0 \end{pmatrix}.$$  \hspace{1cm} (11)

Here, $H_A, H_B$ are Hamiltonians for the sites located at pink and green “stripes” in Fig. 4 correspondingly, and $V$ represents their interaction. Our calculations were performed for the matrices of the finite size $61 \times 61$, and the result is shown in Fig. 5. The green area and the red and black dashed lines represent scattering continuum and doublon band, calculated via (4) and (8), and the edge state, respectively. Blue lines are numerical eigenstates. For scattering states and doublon band numerical and analytical results perfectly agree. The doublon band Eq. (8) overlaps spectrally with the scattering states for $U < \sqrt{2}J$ and is spectrally separated for $U > \sqrt{2}J$. This is directly seen from the comparison of Fig. 5(a) and Fig. 5(b) and also agrees with the dependence on the interaction strength calculated in Fig. 2.

IV. EDGE STATE OF THE BOUND PHOTON PAIR

The interaction-induced two-photon edge states in the system have been found numerically in Figs. 2, 3. In this section we consider the limit of strong interaction regime ($U \gg J$), where the presence of edge states can be understood qualitatively by applying the perturbation theory. Visually, this can be illustrated in Fig. 4: the lattice nodes constitute the unperturbed Hamiltonian, red circles stand for the photon-photon interaction, and the tunneling constants $J$ (links between the nodes) represent a small perturbation.

We start with the separation of the Hamiltonian (1) into the unperturbed part $\tilde{H}_0$, determined only by the photon-photon interaction, and a perturbation $\tilde{V}$, including the tunneling effects:

$$\tilde{H}_0 = \sum_{j=1}^{N} \mathcal{E}_0 a_j^\dagger a_j +$$

$$(1 - (-1)^j)U \sum_{j=1}^{N} (-1)^j a_j^\dagger a_j (a_j^\dagger a_j - 1)$$

$$\tilde{V} = J \sum_{j=1}^{N-1} (a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}).$$  \hspace{1cm} (13)

From now we will consider only the case of odd number of cavities in the array $N$, so that both edge sites have a nonzero nonlinear interaction term and array has mirror symmetry. The eigenstates of Eq. (12) have then the $(N + 1)/2$-degeneracy with $E^{(0)} = 2\hbar \omega_0 + 2U$. Since we focus only on these states, we will enumerate them in a
new way by using only one index:

$$\Psi_n \equiv \Psi_{2n, 2n}, \quad E_n^{(0)} = 2U .$$

(14)

The eigenenergies are enumerated correspondingly. This $N/2$ degeneracy is lifted when the photon tunneling described by the operator $\hat{V}$ is taken into account. The photon tunneling can be visualized as quantum walks along the ribs of the graph in Fig. 4. The second-order processes include two ribs. Since the minimal distance between the nearest red cells in Fig. 4 is equal to four, the nearest red cells are not hybridized in the second order in $\hat{V}$. On the other hand, there exist nonzero diagonal elements $V_{nn}^{(2)}$ that describe the shifts of the state energies induced by the tunneling (blue arrows in Fig. 4). These energy corrections are proportional to the number of nearest neighbours of the $n^{th}$ red cell, so the resulting energy correction is

$$E_n^{(2)} = \begin{cases} 
J^2/U & n = 1; (N + 1)/2 \ (\text{edge state}) \\
2J^2/U & 1 < n < (N + 1)/2 \ (\text{doublons})
\end{cases}$$

(15)

for the red cavities. Equation (15) is already sufficient to explain the formation of the edge states of the photon pairs. The energies of the edge sites are detuned from the energies of the bulk sites since these sites have less neighbors. As such, the hybridization of the edge sites with the bulk is suppressed and the localized edge states are formed. The blue line in Fig. 2 has been calculated according to Eq. (15) and well agrees with the result of numerical calculation of the edge state energy (black dots). Such formation mechanism is inherent for the few-particle quantum states in the one-dimensional lattices and can be understood as a quantum analogue of the nonlinear self-trapping at the edge [31, 32]. However, the distinctive ingredient of the considered model is the spatial modulation of the nonlinearity. If the nonlinearity were present both at the odd and even sites, the two-particle edge states would have not existed.

Similar approach can be extended to describe the formation of the bulk Bloch states of the doublons in the limit of strong interaction. This requires a consideration in the fourth order of perturbation theory. In this case the paths including four ribs should be included in Fig. 4, that can connect the neighboring red sites to each other. The secular equation has the form

$$\sum_{n'} (E_n^{(4)} \delta_{nn'} - V_{nn'}^{(4)} c_n^{(0)} c_{n'}^{(0)}) = 0,$$

(16)

where the indices $n, n'$ run through the states that have remained degenerate after the second order terms have been taken into account, and $V_{nn'}^{(4)}$ corresponds to a bulk expression including the terms, proportional to $\sum_{m,k} V_{nm} V_{mk} V_{kn'}/(2U)^3$. After the careful calculation of the matrix elements, we obtain the following system of equations for the sites $n$

$$(E - 2\hbar \omega_0 - J^2/U)\Psi_n = \frac{J^4}{2U^3} (\Psi_{n-1} + \Psi_{n+1}) ,$$

(17)

$$1 < n < N$$

$$(E - 2\hbar \omega_0 - 2J^2/U)\Psi_1 = \frac{J^4}{2U^3} \Psi_2 ,$$

(18)

$$(E - 2\hbar \omega_0 - 2J^2/U)\Psi_N = \frac{J^4}{2U^3} \Psi_{N-1} .$$

The system of equations Eq. (18) has a transparent interpretation. It describes an effective one-dimensional tight binding model for doubling with the tunneling constant $t = J^4/2U^3$, and the energy detuning for the first and last cavities is equal to $\Delta E = J^2/U$, leading to the formation of the edge states.

V. SUMMARY

To summarize, we have considered theoretically the two-photon energy spectrum in the array of nonlinear cavities with spatially modulated photon-photon interaction, when the interaction is nonzero only for the every second cavity. The eigenstates of the infinite system have been found analytically from the Bethe ansatz. They can be divided into the scattering states, where the two photons are quasi independent from each other and the doublons, i.e. the two-photon states, bound by the interaction. We demonstrate, that for sufficiently strong interaction the system features two-photon edge states, with both photons localized at the same edge of the array. The presence of such doublon edge states requires spatial modulation of the nonlinearity, they are absent if the interaction parameter is the same for all cavities. The formation of edge states has been interpreted analytically in the regime of strong interaction as a nonlinear self-localization at the edge. Our results might be useful to understand the energy spectrum and the mechanisms of edge state formation in various quantum systems, from structured polaritonic cavities to the arrays of superconducting qubits.

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[1] Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Pichler, Soon-won Choi, Alexander S. Zibrov, Manuel Endres, Markus
Greiner, Vladan Vuletić, and Mikhail D. Lukin, “Probing many-body dynamics on a 51-atom quantum simulator,” Nature 551, 579–584 (2017).

[2] Alexander Keesling, Ahmed Omran, Harry Levine, Hannes Bernien, Hannes Pichler, Soonwon Choi, Rhine Samajdar, Sylvain Schwartz, Pietro Silvi, Subir Sachdev, Peter Zoller, Manuel Endres, Markus Greiner, Vladan Vuletić, and Mikhail D. Lukin, “Quantum Kibble–Zurek mechanism and critical dynamics on a programmable Rydberg simulator,” Nature 568, 207 (2019).

[3] Iacopo Carusotto and Cristiano Ciuti, “Quantum fluids of light,” Rev. Mod. Phys. 85, 299–366 (2013).

[4] A. Peruzzo, M. Lobino, J. C. F. Matthews, N. Matsuda, A. Politi, K. Poulous, X.-Q. Zhou, Y. Lahini, N. Ismail, K. Worhoff, Y. Bromberg, Y. Silberberg, M. G. Thompson, and J. L. O’Brien, “Quantum walks of correlated photons,” Science 329, 1500–1503 (2010).

[5] Alexander S. Solntsev, Frank Setzpfandt, Alex S. Clark, Che Wen Wu, Matthew J. Collins, Chunle Xiong, Andreas Schreiber, Fabian Katzschmann, Falk Eilenberger, Roland Schieck, Wolfgang Sohler, Arnan Mitchell, Christine Silberhorn, Benjamin J. Eggleton, Thomas Pertsch, Andrey A. Sukhorukov, Dragomir N. Neshev, and Yuri S. Kivshar, “Generation of nonclassical biphoton states through cascaded quantum walks on a nonlinear chip,” Phys. Rev. X 4, 031007 (2014).

[6] S. Mittal, E. A. Goldschmidt, and M. Hafezi, “A topological source of quantum light,” Nature 561, 502–506 (2018).

[7] Sabyasachi Barik, Aziz Karasahin, Christopher Flower, Tao Cai, Hirokazu Miyake, Wade DeGottardi, Mohammad Hafezi, and Edo Waks, “A topological quantum optics interface,” Science 359, 666–668 (2018).

[8] Andrea Blanco-Redondo, Bryn Bell, Dikla Oren, Benjamin J. Eggleton, and Mordechai Segev, “Topological protection of biphoton states,” Science 362, 568–571 (2018).

[9] P. Roushan, C. Neill, A. Megrant, Y. Chen, R. Babbush, R. Barends, B. Campbell, Z. Chen, B. Chiaro, A. Dunsworth, A. Fowler, E. Jeffrey, J. Kelly, E. Lucero, J. Mutus, P. J. J. O’Malley, M. Neeley, C. Quintana, D. Sank, A. Vainsencher, J. Wenner, T. White, E. Kapit, H. Neven, and J. Martinis, “Chiral ground-state currents of interacting photons in a synthetic magnetic field,” Nature Phys. 13, 146–151 (2016).

[10] Daniel C. Mattis, “The few-body problem on a lattice,” Rev. Mod. Phys. 58, 361 (1986).

[11] K. Winkler, G. Thalhammer, F. Lang, R. Grimm, J. Hecker Denschlag, A. J. Daley, A. Kantian, H. P. Büchler, and P. Zoller, “Repulsively bound atom pairs in an optical lattice,” Nature 441, 853–856 (2006).

[12] P. M. Preiss, R. Ma, M. E. Tai, A. Lukin, M. Rispohl, P. Zupancic, Y. Lahini, R. Islam, and M. Greiner, “Strongly correlated quantum walks in optical lattices,” Science 347, 1229–1233 (2015).

[13] Andreas Schreiber, Aurél Gábris, Peter P. Rohde, Kaisa Laiho, Martin Štefaňák, Václav Potoček, Craig Hamilton, Igor Jex, and Christine Silberhorn, “A 2d quantum walk simulation of two-particle dynamics,” Science 336, 55–58 (2012).

[14] G. Corrielli, A. Crespi, G. Della Valle, S. Longhi, and R. Osellame, “Fractional Bloch oscillations in photonic lattices,” Nat. Commun. 4, 1555 (2013).

[15] Nikita A. Olekhno, Egor I. Kretov, Andrey A. Stepakonenko, Dmitry S. Filonov, Barbara Cappello, Ladislau Matekovits, and Maxim A. Gorlach, “Topological edge states of interacting photon pairs realized in a topological circuit,” In preparation (2019).

[16] Mark Lyubarchov and Alexander Poddubny, “Exceptional points for photon pairs bound by nonlinear dissipation in cavity arrays,” Optics Letters 43, 5917 (2018).

[17] S Longhi and G Della Valle, “Tamm–Hubbard surface states in the continuum,” J. Phys.: Cond. Mat. 25, 235601 (2013).

[18] Ling Lu, John D. Joannopoulos, and Marin Soljačić, “Topological states in photonic systems,” Nature Physics 12, 626–629 (2016).

[19] Tomoki Ozawa, Hannah M. Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikhail C. Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, and Iacopo Carusotto, “Topological photonics,” Rev. Mod. Phys. 91, 015006 (2019).

[20] Zheng Wang, Yidong Chong, J. D. Joannopoulos, and Marin Soljačić, “Observation of unidirectional backscattering-immune topological electromagnetic states,” Nature 461, 772–775 (2009).

[21] P. St-Jean, V. Goblot, E. Galopin, A. Lemaître, T. Ozawa, L. Le Gratiet, I. Sagnes, J. Bloch, and A. Amo, “Lasing in topological edge states of a one-dimensional lattice,” Nature Photonics 11, 651–656 (2017).

[22] Sergey Kruk, Alexander Poddubny, Daria Smirnova, Lei Wang, Alexey Slobozhanyuk, Alexander Sherokhov, Ivan Kravchenko, Barry Luther-Davies, and Yuri Kivshar, “Nonlinear light generation in topological nanostructures,” Nature Nanotechnology 14, 126–130 (2019).

[23] Jean-Luc Tambasco, Giacomo Corrielli, Robert J. Chapman, Andrea Crespi, Oded Zilberberg, Roberto Osellame, and Alberto Peruzzo, “Quantum interference of topological states of light,” Science Advances 4, eaat3187 (2018).

[24] Grazia Salerno, Marco Di Liberto, Chiara Menotti, and Iacopo Carusotto, “Topological two-body bound states in the interacting Haldane model,” Phys. Rev. A 97, 013637 (2018).

[25] M. Di Liberto, A. Recati, I. Carusotto, and C. Menotti, “Two-body physics in the Su-Schrieffer-Heeger model,” Phys. Rev. A 94, 062704 (2016).

[26] M. A. Gorlach and A. N. Poddubny, “Topological edge states of bound photon pairs,” Phys. Rev. A 95, 053866 (2017).

[27] M. Di Liberto, A. Recati, I. Carusotto, and C. Menotti, “Two-body bound and edge states in the extended SSH Bose-Hubbard model,” Eur. Phys. J. Special Topics 226, 2751–2762 (2017).

[28] Maxim A. Gorlach, Marco Di Liberto, Alessio Recati, Iacopo Carusotto, Alexander N. Poddubny, and Chiara Menotti, “Simulation of two-boson bound states using arrays of driven-dissipative coupled linear optical resonators,” Phys. Rev. A 98, 063625 (2018).

[29] B.A. Bernevig and T.L. Hughes, Topological Insulators and Topological Superconductors (Princeton University Press, 2013).

[30] M. A. Gorlach and A. N. Poddubny, “Interaction-induced two-photon edge states in an extended Hubbard model realized in a cavity array,” Phys. Rev. A 95, 033831 (2017).
[31] Maxim A. Gorlach and Alexander N. Poddubny, “Topological edge states of bound photon pairs,” Phys. Rev. A 95, 053866 (2017).

[32] Ricardo Pinto, Masudul Haque, and Sergej Flach, “Edge-localized states in quantum one-dimensional lattices,” Phys. Rev. A 79, 052118 (2009).