Regularity of distance functions from arbitrary closed sets

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Abstract

Given an arbitrary closed set \( K \subseteq \mathbb{R}^n \), we study on the complement of \( K \) the regularity properties of the distance function \( \delta^\phi_K \), computed with respect to an arbitrary uniformly convex \( C^2 \)-norm \( \phi \) on \( \mathbb{R}^n \). We prove that the gradient of \( \delta^\phi_K \) is a Lipschitz map on the complement of the \( \phi \)-cut-locus of \( K \). The available known results allow only to conclude that the gradient of \( \delta^\phi_K \) is locally Lipschitz on the interior part of the complement of the \( \phi \)-cut locus, which might be empty even if \( K \) is a \( C^{1,\alpha} \) hypersurface or \( K \) is the complement of a convex body with \( C^1 \) boundary. Then we study the set of points where \( \delta^\phi_K \) is second-order pointwise differentiable, providing an answer to a question by Hiriart-Urruty (1982).

1 Introduction

1.1 Background and motivation

In this paper we investigate the regularity of the distance function \( \delta^\phi_K \) from an arbitrary closed subset \( K \) in a finite-dimensional Banach space \((\mathbb{R}^n, \phi)\), where \( \phi \) a uniformly convex \( C^2 \)-norm in \( \mathbb{R}^n \). In particular, we identify the largest possible set outside \( K \) where a Lipschitz condition for the gradient of \( \delta^\phi_K \) holds and we prove a structural result for the set of points outside \( K \) where \( \delta^\phi_K \) is pointwise twice differentiable providing an answer to a question raised by Hiriart-Urruty in [HU82]. Our results give sharp generalisations of some classical results in the theory of distance functions and are motivated by examples for which these classical results give no meaningful, or restricted information.

The theory of distance functions has been developed, often independently, from many different points of view. Distance functions play a central role in the theory of partial differential equations, since \( \delta^\phi_K \) is the unique viscosity solution of the following Dirichlet problem for the \( \phi \)-Eikonal equation

\[
\begin{cases}
\phi^*(\nabla u) = 1 & \text{on } \mathbb{R}^n \sim K \\
u = 0 & \text{on } K,
\end{cases}
\]

where \( \phi^* \) denotes the conjugate norm given by \( \phi^*(u) = \inf \{ v \cdot u : \phi(v) = 1 \} \) for \( u \in \mathbb{R}^n \); see [Lio82] and [CS04]. In convex and non-smooth Analysis the distance function and the associated metric projection onto closed subsets satisfying various convexity-type assumptions (e.g. convexity, positive reach, proximal smoothness etc.) have been thoroughly investigated both in finite and infinite dimensional spaces; see [RW98]. In Riemannian geometry, besides more classical and well known applications, the distance function has recently emerged as a key tool for isoperimetric problems [GS21] and regularity of optimal transport [FRV12].

We denote with \( \Sigma^\phi(K) \) the set of points in \( \mathbb{R}^n \sim K \) where \( \delta^\phi_K \) is not differentiable. If \( K \) is at least a \( C^2 \)-submanifold, then \( \delta^\phi_K \) is at least of class \( C^2 \) on the open subset \( \mathbb{R}^n \sim (K \cup \text{Clos } \Sigma^\phi(K)) \) and \( \text{Clos } \Sigma^\phi(K) \) is a set of \( \mathcal{L}^n \)-measure zero; indeed if \( K \) is \( C^{2,1} \)-submanifold, then \( \text{Clos } \Sigma^\phi(K) \) is a set of locally finite \( \mathcal{H}^n-1 \)-measure. See [TT01, MM03, LX05a,LN05b and CM07].

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A sufficient condition that guarantees \( \mathcal{L}^n(\text{Clos } \Sigma(K)) = 0 \) for closed \( C^{1,1} \)-hypersurfaces \( K \) in terms of the inner radius of curvature is given in \[\text{Min16} \text{ Theorem 4.1} \].

It turns out that the \( C^2 \)-regularity is a critical hypothesis; indeed the second named author has shown in \[\text{San21} \] that for a convex open subset \( \Omega \) with \( C^{1,1} \)-boundary \( \text{Clos } \Sigma^\phi(\mathbb{R}^n) \sim \Omega \) might have non empty interior in \( \Omega \); moreover, for a typical (in the sense of Baire Category) convex open subset \( \Omega \) with \( C^1 \)-boundary we have that \( \Sigma^\phi(\mathbb{R}^n) \sim \Omega \) is dense in \( \Omega \). There exist even closed \( C^{1,\alpha} \)-hypersurfaces \( K \) such that \( \Sigma^\phi(K) \) is dense in all of \( \mathbb{R}^n \); see \[\text{San21 Corollary 2.9} \]. In all these examples one can choose \( \phi \) to be the Euclidean norm. On the other hand if \( K \) is a closed \( C^1 \)-hypersurface, then \[\text{Min16 Theorem 1.3} \] provides a necessary and sufficient condition for a point \( x \in \mathbb{R}^n \sim K \) to lie in \( \mathbb{R}^n \sim \text{Clos}(\Sigma(K)) \).

For a general closed subset \( K \) it is well known that \( \delta_K^\phi \) is a \textit{locally semiconcave function} on \( \mathbb{R}^n \sim K \); see \[\text{Asp73, Lio82, Zaj83a} \]. This implies that its gradient \( \text{grad } \delta_K^\phi \) is a function of bounded variation, see \[\text{CS04 2.3.1} \] (see also \[\text{HDLR11} \] for further results in this direction). Employing general results from the theory of Hamilton-Jacobi equations (see \[\text{Lio82 Theorem 15.1} \] or \[\text{Fat03} \]), one can also obtain the following \( C^{1,1} \)-\textit{regularity theorem}:

\[\textbf{1.1 Theorem.} \text{ If } K \subseteq \mathbb{R}^n \text{ is an arbitrary closed set, then } \delta_K^\phi \text{ is } C^1 \text{ with a locally Lipschitz gradient on the open subset } U = \mathbb{R}^n \sim \left( K \cup \text{Clos } \Sigma^\phi(K) \right) \].

For the Euclidean norm this statement can also be obtained using a purely geometric argument (see \[\text{Fed59 4.8} \]) and it plays a pivotal role in the development of the theory of curvature measures for sets of positive reach in the Euclidean space; see \[\text{Fed59} \]. However, as mentioned above, the set \( U \) might easily be empty even if \( K = \mathbb{R}^n \sim \Omega \) and \( \Omega \) is a convex open subset with \( C^1 \) boundary, or might reduce to a small tubular neighbourhood around \( K \) if \( \Omega \) has a \( C^{1,1} \) boundary. Consequently Theorem \[1.1 \] provides no (or very limited) information in these situations. On the other hand it is well known that the gradient of \( \delta_K^\phi \) is a continuous map on its domain \( \mathbb{R}^n \sim (K \cup \Sigma^\phi(K)) \). Therefore it is a natural question to ask to identify the largest set on which the gradient of \( \delta_K^\phi \) satisfies a Lipschitz condition. We answer this question in Theorem \[1.2 \] providing an effective sharp generalization of Theorem \[1.1 \] that is applicable in the aforementioned critical low-regularity cases.

Since for an arbitrary closed set \( K \) the function \( \delta_K^\phi \) is a locally-semiconcave viscosity solution of the Eikonal equation, several properties of the set \( \Sigma^\phi(K) \) can be deduced from the extensive literature available for abstract semi-convex functions or for semi-convex viscosity solutions of uniformly elliptic Hamilton-Jacobi equations. In particular \( \Sigma^\phi(K) \) can be covered by countably many \( C^2 \)-hypersurfaces (see \[\text{Zaj79} \]) and upper bounds on its Hausdorff measure are known (see \[\text{AAC92} \]). Lower bounds and results on the propagation of the non-differentiability points can be obtained from \[\text{AC92, AC02, CY09} \]. The topological properties of the set \( \Sigma^\phi(K) \) in a Euclidean or Riemannian setting are studied in \[\text{CP01, Lie04, ACNS13} \]. Being locally semiconcave, the function \( \delta_K^\phi \) is pointwise twice differentiable at \( \mathcal{L}^n \) almost all points in \( \mathbb{R}^n \sim K \) by the classical Alexandrov theorem \[\text{Ale39} \]. We recall that \( \delta_K^\phi \) is pointwise twice differentiable at \( x \) if and only if there exists a polynomial function \( P: \mathbb{R}^n \to \mathbb{R} \) of degree at most 2 such that

\[
P(x) = \delta_K^\phi(x) \quad \text{and} \quad \lim_{y \to x} \frac{\delta_K^\phi(y) - P(y)}{|y - x|^2} = 0.
\]

We denote with \( \Sigma^\phi_2(K) \) the set of points in \( \mathbb{R}^n \sim K \) where \( \delta_K^\phi \) is not pointwise twice differentiable. It is well known that \( \Sigma^\phi_2(K) \) corresponds to the set of points in \( \mathbb{R}^n \sim K \) where the \( \phi \)-nearest point projection \( \xi_K^\phi \) onto \( K \) (which is a multivalued map) is not differentiable (see \[\text{2.8, 2.21, 2.30(e)} \]).

If \( K \) is convex (in which case \( \Sigma^\phi(K) = \emptyset \)) then the following sharp structural result for \( \Sigma^\phi_2(K) \) may be derived from known facts.

\[\textbf{1.2 Theorem.} \text{ Suppose } K \subseteq \mathbb{R}^n \text{ is convex. Then there exists } Z \subseteq N^\phi(K) \text{ with } \mathcal{H}^{n-1}(Z) = 0 \text{ such that }
\]

\[
\Sigma^\phi_2(K) = \{ a + r\eta : 0 < r < \infty, \ (a, \eta) \in Z \}.
\]
This theorem essentially asserts propagation of pointwise twice differentiability of \( \delta^\phi_K \) along fibers of the normal bundle. The exceptional set \( Z \) cannot be excluded. In fact, even if \( \phi \) is the euclidean norm, there exist convex bodies \( K \) with \( C^{1,1} \) boundaries such that the set \( Z \) is dense in \( N^\phi(K) \) with Hausdorff dimension \( n - 1 \); see [5.1]. In the Euclidean setting Theorem 1.2 is a classical fact and plays a key role in the definition of the principal curvatures of \( K \); see [Sch14]. The general version of 1.2 appears to be missing in the literature; however, with the help of Theorem 1.1 it can be obtained by modifying the proof of the Euclidean case. In any case, the hypothesis of convexity is obviously strongly restrictive and it would be desirable to understand if and how this statement can be extended to arbitrary closed sets. Indeed the equivalent question about the structure of the set of differentiability points of the nearest point projection onto an arbitrary closed set has been posed in 1982 by Hiriart-Urruty in the Euclidean setting in [HUS2] (see last paragraph on page 458). Despite the long time-span, we are not aware of any contribution in this direction.

Our Theorem 1.4 extends Theorem 1.2 to arbitrary closed sets and provides an answer to this old question.

### 1.2 Results of the present paper

We start by introducing few definitions and facts. Let \( K \subseteq \mathbb{R}^n \) be closed. We consider the following unit \( \phi \)-normal bundle of \( K \)

\[
N^\phi(K) = \{(a, \eta) : a \in K, \eta \in \mathbb{R}^n, \phi(\eta) = 1, \delta^\phi_K(a + s\eta) = s \text{ for some } s > 0\}
\]

and we recall that \( N^\phi(K) \) is Borel and countably \((n-1)\)-rectifiable (in the sense of [Fed69, 3.2.14(2)]) subset of \( \mathbb{R}^{2n} \); cf. [DRKS2] Lemma 5.2. The normal \( \phi \)-distance function to the cut locus of \( K \) is the function \( r^\phi_K : N^\phi(K) \to (0, +\infty] \) given by

\[
r^\phi_K(a, \eta) = \sup\{s : \delta^\phi_K(a + s\eta) = s\} \quad \text{for } (a, \eta) \in N^\phi(K).
\]

Simple arguments show that \( r^\phi_K \) is upper semi-continuous; cf. [RSS21] Remark 5.6. We define \( \text{Cut}^\phi(K) \), the \( \phi \)-cut locus of \( K \), and \( \Sigma^\phi(K) \), the set of non-differentiability points of \( \delta^\phi_K \), by

\[
\begin{align*}
\text{Cut}^\phi(K) &= \{a + r^\phi_K(a, \eta)\eta : (a, \eta) \in N^\phi(K)\} \\
\Sigma^\phi(K) &= \{r^\phi_K \in (0, +\infty) : \text{Clos} \Sigma^\phi(K) \text{ is } \text{Borel and countably } (n-1)\text{-rectifiable}\}.
\end{align*}
\]

It is well known (and follows from 2.11, 2.39, and [Fre97 Theorem 3B]) that

\[
\Sigma^\phi(K) \subseteq \text{Cut}^\phi(K) \subseteq \text{Clos} \Sigma^\phi(K).
\]

Moreover, \( \mathcal{L}^n(\text{Cut}^\phi(K)) = 0 \) (see Remark 4.3) and

\[
\mathbb{R}^n \sim (\text{Cut}^\phi(K) \cup K) = \{a + \rho\eta : (a, \eta) \in N^\phi(K), 0 < \rho < r^\phi_K(a, \eta)\}.
\]

Our first result asserts that \( \text{grad} \delta^\phi_K \) is a Lipschitz map on every subset of \( \mathbb{R}^n \sim K \) with a positive normal \( \phi \)-distance function to the cut locus of \( K \). This is a sharp generalisation of Theorem 1.1.

#### 1.3 Theorem

Suppose \( \phi : \mathbb{R}^n \to \mathbb{R} \) is a uniformly convex norm of class \( \mathcal{C}^2 \) away from the origin, \( K \subseteq \mathbb{R}^n \) is closed, \( 1 < \lambda \leq \infty \), \( 0 < s < t < \infty \), and

\[
K_{\lambda, s, t} = \{a + \rho\eta : (a, \eta) \in N^\phi(K), s \leq \rho \leq t, \lambda \rho \leq \delta^\phi_K(a, \eta)\}.
\]

Set \( U = (\text{dmn} \text{grad} \delta^\phi_K) \sim K \). Then there exists \( \Gamma = \Gamma(\lambda, s, t, \phi) \in \mathbb{R} \) such that

\[
|\text{grad} \delta^\phi_K(x) - \text{grad} \delta^\phi_K(y)| \leq \Gamma |x - y| \quad \text{for } x \in K_{\lambda, s, t} \text{ and } y \in U \text{ with } \delta^\phi_K(y) \leq t.
\]

In particular, \( \text{grad} \delta^\phi_K|_{K_{\lambda, s, t}} \) is Lipschitz continuous.
The restriction “$\lambda \rho \leq r^\phi_K(a, \eta)$” cannot be avoided since the Lipschitz constant of $\text{grad } \delta^\phi_K$ may explode near points of $\text{Cut}^\phi(K)$; cf. 4.1. Observe that if $x \in \mathbb{R}^n \sim \text{Clos } \Sigma^\phi(K) \cup K$, then $x$ has positive distance from $\text{Cut}^\phi(K)$; hence, theorem 1.3 includes Theorem 1.1 as a special case; moreover it is sharp in terms of specifying the set of points where a Lipschitz condition for $\text{grad } \delta^\phi_K$ holds. There are two main difficulties in proving 1.3. The first one arises from the fact that $\text{Cut}^\phi(K)$ might be dense in $\mathbb{R}^n \sim K$ and consequently it does not seem to be possible to rely on general results for Hamilton-Jacobi equations as for Theorem 1.1. The second difficulty arises with working with a possibly non-Euclidean norm. In fact, if $\phi$ is the Euclidean norm then the proof of Theorem 1.3 follows rather directly from the geometric argument originally found by Federer for sets of positive reach in [Fed59], see [San20, 3.10(1)]. However this argument is not applicable if $\phi$ is not the Euclidean norm, in which case one needs a considerably more sophisticated approach, based on a careful analysis of the geometric properties of the $\phi$-balls (which occupies the entire section 3). In fact this analysis allows to show that the $\phi$-nearest point projection $\xi^\phi_K$ onto $K$, which is a multivalued map, satisfies the asserted Lipschitz property; see 2.8. Recalling a well-known relation between $\xi^\phi_K$ and $\text{grad } \delta^\phi_K$ (see Lemma 2.30(c)), we obtain the conclusion in 1.3.

We remark that uniform convexity of the norm for $n \geq 3$ is crucial to obtain the Lipschitz property in Theorem 1.3; see the last section in [BHS16].

Our second main result below proves propagation of pointwise twice differentiability of $\delta^\phi_K$ along the fibers of the normal bundle of an arbitrary closed subset $K$ of $\mathbb{R}^n$. Due to the presence of a possibly dense cut locus the description of the set $\Sigma^\phi_2(K)$ is obviously more involved than in the convex case considered in Theorem 1.2. Therefore, we first need to introduce few additional notions. Referring to 2.28 we define the Borel function $\rho^\phi_K : \mathbb{R}^n \sim K \to [t : 1 \leq t \leq \infty]$ by

$\rho^\phi_K(x) = \sup \{s \geq 0 : \delta^\phi_K(a + s(x - a)) = s\delta^\phi_K(x)\}$ for $x \in \mathbb{R}^n \sim K$ and some $a \in \xi^\phi_K(x)$;

next, we take its approximate lower-limit (see 4.7)

$\rho^\phi_K(x) = \text{ap lim inf}_{y \to x} \rho^\phi_K(y)$ for $x \in \mathbb{R}^n \sim K$;

finally, we introduce

$r^\phi_K(a, \eta) = r\rho^\phi_K(a + r\eta)$ for $(a, \eta) \in N^\phi(K)$ and some $0 < r < r^\phi_K(a, \eta)$,

which is well defined by 4.10.

1.4 Theorem. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}$ is a uniformly convex norm of class $C^2$ away from the origin and $K \subseteq \mathbb{R}^n$ is closed. Define

$P = \{a + r\eta : (a, \eta) \in N^\phi(K), \ 0 < r < r^\phi_K(a, \eta)\}$,

$Q = \{a + r\eta : (a, \eta) \in N^\phi(K), \ r^\phi_K(a, \eta) \leq r \leq r^\phi_K(a, \eta)\}$.

(Notice $P \cap Q = \emptyset$, $P \cup Q = \mathbb{R}^n \sim K$ and $\text{Cut}^\phi(K) \subseteq Q$.) Then the following statements hold.

(a) $\text{Cut}^\phi(K) \subseteq \Sigma^\phi_2(K)$,

(b) $\mathcal{H}^{n-1}(N^\phi(K) \cap \{(a, \eta) : r^\phi_K(a, \eta) < r^\phi_K(a, \eta)\}) = 0$,

(c) $\mathcal{L}^n(Q) = 0$,

(d) there exists $Z \subseteq N^\phi(K)$ with $\mathcal{H}^{n-1}(Z) = 0$ such that

$P \cap \Sigma^\phi_2(K) = \{a + r\eta : (a, \eta) \in Z, \ 0 < r < r^\phi_K(a, \eta)\}$.

In particular, for $\mathcal{H}^{n-1}$ almost all $(a, \eta) \in N^\phi(K)$ the distance function $\delta^\phi_K$ is pointwise twice differentiable at all points of the fibre $\{a + r\eta : 0 < r < r^\phi_K(a, \eta)\}$. 

4
2 Preliminaries

In principle, but with some exceptions explained below, we shall follow the notation of Federer (see [Fed69 pp. 669–671]). In particular, we write $x \cdot y$ for the inner product of $x, y \in \mathbb{R}^n$ and $|x| = (x \cdot x)^{1/2}$ for $x \in \mathbb{R}^n$. We also denote the set theoretic difference of sets by $A \sim B$. For the derivative of a function $f$ we write $Df$ and for its gradient $\gamma f$; cf. [Fed69 3.1.1]. We modify the notation of [Fed69] in the follow ways. We denote the topological boundary of a set $A$ by $\partial A$. In consequence, to avoid ambiguities in the notation, we chose to write $\nabla f(x)$ for the set of subgradients of a convex function $f$ at $x \in \text{dmn} f$; cf. [Fed69]. If $A$ and $B$ are subsets of a vectorspace we write $A + B = \{a + b : a \in A, b \in B\}$ for the algebraic sum of sets. To denote open and closed balls with respect to a norm $\phi$ on $X$ we use the symbols $U^{\phi}(x, r)$ and $B^{\phi}(x, r)$. In general, we shall define several notions depending on a norm and the name of the norm in the superscript shall be omitted whenever we deal with the standard Euclidean norm on $\mathbb{R}^n$. To denote the identity map on $X$ we write $I_X$. When we say that $X$ is a normed vectorspace without specifying the name for the norm we tacitly assume that $|x|$ is the norm of $x \in X$. Whenever $\Lambda : X \to Y$ is a linear map between vectorspaces $X$ and $Y$ and $x \in X$ we use alternative notations $\Lambda x$ or $\langle x, \Lambda \rangle$ to denote the value of $\Lambda$ at $x$. Whenever $T$ is a linear subspace of a Euclidean space we write $T_\bot$ for the orthogonal projection onto $T$.

2.1 Definition. We say that a norm $\phi : X \to \mathbb{R}$ is strictly convex if for all $a, b \in X$ $\phi(a + b) = \phi(a) + \phi(b)$ implies $\phi(b)a = \phi(a)b$.

2.2 Remark. In the sequel, unless otherwise specified, $n$ shall be a fixed positive integer, $X$ will be a vectorspace of dimension $n$, and $\phi : X \to \mathbb{R}$ will be a strictly convex norm on $X$ of class $C^2$ away from the origin. Of course, $X$ shall be isomorphic with $\mathbb{R}^n$ but, whenever we write $X$ instead of $\mathbb{R}^n$, we want to emphasise that there might not be a natural choice of a Euclidean structure on $X$.

2.3 Definition. Whenever $X$ is equipped with a scalar product and $\phi : X \to \mathbb{R}$ is a norm we define the conjugate norm $\phi^* : X \to \mathbb{R}$ by the formula $\phi^*(x) = \sup \{x \cdot y : y \in X, \phi(y) = 1\}$.

2.4 Definition (cf. [DRKS20 2.12, 2.13]). Assume $X$ is equipped with a Euclidean structure. We say that $\phi : X \to \mathbb{R}$ is a uniformly convex norm if it is a norm and there exists $\gamma > 0$ such that the function $[X \ni x = \phi(x) - \gamma|x|]$ is convex.
2.5 Remark (cf. [DRKS20, 2.32]). If $\phi$ is a uniformly convex norm of class $\mathcal{C}^2$ away from the origin, then $\phi^*$ is also a uniformly convex norm of class $\mathcal{C}^2$ away from the origin. Moreover, grad $\phi^*$ is the inverse of grad $\phi$, where $S = \partial B^\phi(0,1)$ and $S^* = \partial B^{\phi^*}(0,1)$.

2.6 Definition. Given a closed set $K \subseteq X$, we define the $\phi$-distance function $\delta^K_\phi$ to $K$ as

$$\delta^K_\phi(x) = \inf \{ \phi(a-x) : a \in K \} \quad \text{for } x \in X$$

and we set $S^\phi(K,r) = \{ x : \delta^K_\phi(x) = r \}$ for $r > 0$.

2.7 Definition. A map of the type $f : X \to 2^Y$ shall be called $Y$-multivalued. In case $x \in X$ and $f(x)$ is a singleton, we abuse the notation and write $f(x)$ to denote the unique member of $f(x)$. We also allow use the convention that if $x \notin X$, then $f(x) = \emptyset$. In particular, if $A \subseteq X$, then $f|A(x) = \emptyset$ for $x \in X \sim A$.

2.8 Definition. Suppose $K \subseteq X$ is closed. The $\phi$-nearest point projection onto $K$ is the multivalued map $\xi^K_\phi : X \to 2^K$ defined by

$$\xi^K_\phi(x) = K \cap \{ a : \phi(x-a) = \delta^K_\phi(x) \} \quad \text{for } x \in X.$$ 

The Cahn-Hoffman map of $K$ associated to $\phi$ is then defined by the formula

$$\nu^K_\phi(x) = \delta^K_\phi(x)^{-1}(x - \xi^K_\phi(x)) \quad \text{for } x \in X \sim K.$$ 

2.9 Remark. It will be useful to notice that $\xi^K_\phi(x)$ is a compact subset of $X$ for every $x \in X$.

2.10 Remark. Since $\phi$ is a norm, one readily checks that if $a \in K$, $v \in X$ and $\delta^K_\phi(a + v) = \phi(v)$, then $\delta^K_\phi(a + tv) = t \phi(v)$ for every $0 \leq t \leq 1$.

2.11 Remark. It has been observed in [DRKS20, 2.38(g)], using strict convexity of $\phi$, that if $a \in K$, $u \in \partial U^\phi(0,1)$, $0 < t < \infty$ and $\delta^K_\phi(a + tu) = t$, then $\xi^K_\phi(a + su)$ is a singleton and $\xi^K_\phi(a + su) = \{ a \}$ for every $0 < s < t$.

2.12 Definition (cf. [Roc70, p. 213]). Let $f : X \to \overline{R}$ and $x,v \in X$. The one-sided directional derivative of $f$ at $x$ with respect to $v$ is defined to be

$$f'(x;v) = \lim_{\lambda \to 0^+} \frac{f(x + \lambda v) - f(x)}{\lambda},$$

whenever the limit exists in $\overline{R}$.

2.13 Remark. If $f$ is a convex function and $x$ is a point with $f(x) \in R$, then $f'(x;v)$ exists for every $v \in X$; cf. [Roc70, Theorem 23.1].

2.14 Definition (cf. [Roc70, p. 214-215 and Theorem 23.2]). Suppose $f : X \to \overline{R}$ is convex and $x \in X$ is such that $f(x) \in R$. We say that $\zeta \in X$ is a subgradient of $f$ at $x$ if

$$f'(x;v) \geq \zeta \cdot v \quad \text{for } v \in X.$$ 

The set of all subgradients of $f$ at $x$ is denoted by $\nabla f(x)$.

2.15 Definition. Let $X, Y$ be normed vectorspaces and $f$ be a function mapping a subset of $X$ into $Y$. We say that $f$ is pointwise differentiable of order $k$ at $x$ if there exist: an open set $U \subseteq X$ such that $x \in U \subseteq \text{dmn } f$ and a polynomial function $P : X \to Y$ of degree at most $k$ such that $f(x) = P(x)$ and

$$\lim_{y \to x} \frac{|f(y) - P(y)|}{|y - x|^k} = 0.$$ 

Whenever this holds $P$ is unique and the pointwise differential of order $i$ of $f$ at $x$, for $i = 1, \ldots, k$, is defined by $\text{pt } D^i f(x) = D^i P(x)$. As usual $\text{pt } D^1 f(x) = \partial f(x)$.
2.16 Remark. The notion of pointwise differentiability of order 1 coincides with the classical notion of differentiability so pt \( D = D \); cf. [Fed69, 3.1]. A summary of known facts about pointwise differentiability for functions can be found, e.g., in [Men19, §2].

2.17 Remark. If \( f \) is a \( \mathbb{R} \)-valued convex function on an open subset \( U \) of \( X \) then \( \nabla f(x) \) is non empty for every \( x \in U \); cf. [Roc70, Theorem 23.4]. Moreover, \( f \) is differentiable of order 1 at \( x \) if and only if \( \nabla f(x) \) is a singleton; cf. [Roc70, 25.1].

We need to extend the concept of continuity and differentiability to multivalued maps.

2.18 Definition (cf. [Zaj83b, Definition 2]). Let \( X, Y \) be normed vectorspaces and \( T \) be a \( Y \)-multivalued map defined on a subset of \( X \). We say that \( T \) is weakly continuous at \( x \in X \) if and only if \( x \in dm\!n \) \( T \) and for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
T(y) \subseteq T(x) + U(0, \varepsilon) \quad \text{whenever } y \in dm\!n \ T \text{ and } |x - y| < \delta.
\]

If, additionally, \( T(x) \) is a singleton, then we say that \( T \) is continuous at \( x \).

2.19 Remark. It might happen that \( T(y) = \emptyset \) for \( y \in B(x, \delta) \sim \{x\} \); then \( T \) is continuous at \( x \). Fortunately, studying the map \( \xi^p_K \) we do not need to worry about such strange behaviour. In [2.30(4)] we prove that \( \xi^p_K \) is weakly continuous on the whole of \( X \). Obviously, \( \xi^p_K(x) \) is a singleton for all \( x \in X \) if and only if \( K \) is convex.

2.20 Remark. Note that weakly continuous multivalued functions may carry connected sets into disconnected sets. Consider, e.g., the function \( f : \mathbb{R} \to 2^{\mathbb{R}} \) given by \( f(t) = [-1] \) if \( t < 0, f(t) = 0 \) if \( t > 0, \) and \( f(0) = [-1, 0, 1]; \) then, \( f \) is weakly continuous in the sense of 2.18. Another example is \( \xi^p_K \) which is weakly continuous on the whole of \( \mathbb{R}^n \) regardless of the choice of the closed set \( K \subseteq \mathbb{R}^n; \) in particular, when \( K \) is disconnected; cf. [2.30(1)]

2.21 Definition (cf. [Zaj83b, Definition 3]). Let \( X, Y \) be finite dimensional normed vectorspaces and \( T \) be a \( Y \)-multivalued map defined on a subset of \( X \). We say that \( T \) is differentiable at \( x \in X \) if and only if \( T(x) \) is a singleton and there exists a linear map \( L : X \to Y \) such that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) satisfying

\[
|w - T(x) - L(y - x)| \leq \varepsilon|y - x| \quad \text{whenever } |x - y| \leq \delta \text{ and } w \in T(x).
\]

The set of all such \( L \) is denoted by \( D\!T(x) \). In case \( D\!T(x) \) is a singleton, we say that \( T \) is strongly differentiable at \( x \).

2.22 Remark. Note that it might happen that \( T(y) = \emptyset \) for some \( y \in B(x, \delta) \). Actually, if \( T(x) \neq \emptyset \) and there exists \( \delta > 0 \) such that \( T(y) = \emptyset \) for \( y \in B(x, \delta) \sim \{x\} \), then \( T \) is differentiable at \( x \) with \( D\!T(x) = \hom(X, Y) \). On the other hand if, e.g., \( \dim X = n \) and \( \Theta^n(\mathcal{L}^n L\{y : T(y) = \emptyset\}, x) = 0 \), then \( D\!T(x) \) is a singleton.

2.23 Remark. Let \( P \) and \( Q \) be two multivalued functions and \( x \in \mathbb{R}^n \). If \( P \) is differentiable at \( x \) and \( Q \) is differentiable at \( P(x) \) then the multivalued function \( R \) given by

\[
R(y) = Q(P(y)) \supseteq \bigcup\{Q(w) : w \in P(y)\}
\]

is differentiable at \( x \).

The next simple lemma shows that if \( A \subseteq \mathbb{R}^n \) is a set of points at which a multivalued function \( f \) satisfies a Lipschitz condition, \( a \) is a density points of \( A \), and \( f|A \) is differentiable at \( a \), then \( f \) is differentiable at \( a \). It is a variant of a classical result stating that a Lipschitz function that is approximately differentiable at a point is classically differentiable at that point; see [Fed69, 3.1.5].

2.24 Lemma. Assume

\[
a \in A \subseteq \mathbb{R}^n, \quad C \in \mathbb{R}, \quad f : \mathbb{R}^n \to 2^{\mathbb{R}^n}, \quad \Theta^n(L^n, \{x : f(x) \sim \emptyset\}, a) = 0, \quad f(b) \text{ is a singleton for } b \in A, \quad f|A \text{ is differentiable at } a, \quad |f(b) - y| \leq C|b - c| \quad \text{whenever } b \in A, \ c \in \mathbb{R}^n, \ y \in f(c).
\]

Then \( f \) is strongly differentiable at \( a \).
Proof. Since $a$ is a density point of $A$ we see that $f|A$ is strongly differentiable at $a$ and $Df(a) = \{L\}$ for some $L \in \text{Hom}(\mathbb{R}^n, \mathbb{R})$; cf. 2.22. Let $\varepsilon > 0$. Choose $0 < \delta < \varepsilon$ such that 

$$\mathcal{L}^n(B(a, r) \sim A) < \varepsilon \cdot 4^{-n}||L||^{-n} \alpha(n) r^n$$

whenever $0 < r \leq 2\delta$.

Let $c \in B(a, \delta)$ and $y \in f(c)$. Set $r = |c - a|$ and choose $b \in A$ such that $|c - b| = \delta_A(c) \leq r$. Clearly $B(c, |c - b|) \subseteq B(a, 2r)$ and $\mathcal{L}^n B(c, |c - b|) = \alpha(n)|c - b|^n$; hence,

$$\langle \|L\| + C \rangle |c - b| \leq \frac{1}{2} \varepsilon |c - a|.$$

Since $b \in A$ we obtain

$$|y - f(a) - L(c - a)| \leq |y - f(b)| + |f(b) - f(a) - L(b - a)| + |L(b - c)|$$

$$\leq C|c - b| + \frac{1}{2} \varepsilon |b - a| + \|L\| \cdot |c - b| \leq \varepsilon |c - a|.$$

The following lemma is a classical result in convex analysis.

2.25 Lemma. If $U \subseteq \mathbb{R}^n$ is an open convex set, $f : U \rightarrow \mathbb{R}$ is a convex function and $x \in U$, then the following three statements are equivalent.

(a) $f$ is pointwise differentiable of order 2 at $x$.

(b) The multivalued map $\nabla f$ is differentiable at $x$.

(c) Any function $g : U \rightarrow \mathbb{R}^n$ such that $g(y) \in \nabla f(y)$ for every $y \in U$ is differentiable at $x$.

If (a), (b) and (c) hold, then

$$D\nabla f(x)(u) \cdot v = Dg(x)(u) \cdot v = pt D^2 f(x)(u, v) \quad \text{for } u, v \in \mathbb{R}^n.$$

Proof. Clearly $\nabla f(y) \neq \emptyset$ for all $y \in U$ because $f$ is convex and 2.17. The proof that (a) implies (c) is contained in [Aba80, p. 495] (and attributed to Fitzpatrick). For the proof that (c) implies (b) and (b) implies (a), one can look in [Ban79]. In fact, first we notice that $f$ is "zweimal differenzierbar in $p$" in the sense of [Ban79, 4.2] if and only if $\nabla f$ is differentiable at $p$ in the sense of 2.21, then we look at [Ban79, 4.3] and [Ban79, 4.8] respectively.

2.26 Definition. Suppose $U \subseteq \mathbb{R}^n$ is open. We say that a function $g : U \rightarrow \mathbb{R}$ is semiconcave if and only if there exists $\kappa \geq 0$ such that the function $g(y) - (\kappa/2)|y|^2$ is concave.

2.27 Definition. Let $K \subseteq \mathbb{R}^n$ be closed. For $x \in \mathbb{R}^n$ define

$$\rho^\phi_K(x) = \sup K \cap \{s : \delta^\phi_K(a + s(x - a)) = s \delta^\phi_K(x)\} \geq 1 \quad \text{whenever } a \in \xi^\phi_K(x).$$

2.28 Remark. Definition 2.27 is well posed, since 2.11 gives that if $\xi^\phi_K(x)$ is not a singleton, then

$$\sup \{s : \delta^\phi_K(a + s(x - a)) = s \delta^\phi_K(x)\} = 1 \quad \text{for every } a \in \xi^\phi_K(x).$$

2.29 Remark. Observe that $\rho^\phi_K$ is upper semi-continuous. Indeed, let $x_0, x_1, x_2, \ldots \in \mathbb{R}^n$ and

$$\beta \in \mathbb{R} \text{ be such that } \lim_{i \rightarrow \infty} x_i = x_0, \phi(x_i - x_0) < 1 \quad \text{for } i \in \mathcal{P}, \text{ and } \lim_{i \rightarrow \infty} \rho^\phi_K(x_i) > \beta.$$

Since $\delta^\phi_K$ is continuous we have $\lim_{i \rightarrow \infty} \delta^\phi_K(x_i) = \delta^\phi_K(x_0)$ and we may assume $\delta^\phi_K(x_i) < \delta^\phi_K(x_0) + 1$ for $i \in \mathcal{P}$. Choose $a_i \in \xi^\phi_K(x_i)$ for $i \in \mathcal{P}$. Since $\{a_i : i \in \mathcal{P}\} \subseteq B^{\phi}(x_0, \delta^\phi_K(x_0) + 2)$ we may, possibly choosing a subsequence, assume that $\lim_{i \rightarrow \infty} a_i = a_0$ and then $a_0 \in \xi^\phi_K(x_0)$ by continuity of both $\delta^\phi_K$ and $\phi$. Assume further that $\rho^\phi_K(x_i) \geq \beta$ for $i \in \mathcal{P}$. Recalling the definition of $\rho^\phi_K$ we obtain

$$\delta^\phi_K(a_0 + \beta(x_0 - a_0)) = \lim_{i \rightarrow \infty} \delta^\phi_K(a_i + \beta(x_i - a_i)) = \lim_{i \rightarrow \infty} \beta \delta^\phi_K(x_i) = \beta \delta^\phi_K(x_0);$$

hence, $\rho^\phi_K(x_0) \geq \beta$. Since this holds for any $\beta \in \mathbb{R}$ satisfying $\lim_{i \rightarrow \infty} \rho^\phi_K(x_i) > \beta$, we see that

$$\lim_{i \rightarrow \infty} \rho^\phi_K(x_i) \leq \rho^\phi_K(x_0).$$
The following lemma collects few facts on the continuity, differentiability, and convexity properties of $\delta_K^\phi$ and $\xi_K^\phi$ for an arbitrary closed set $K$.

2.30 Lemma. Let $K \subseteq \mathbb{R}^n$ be a closed set. Then the following statements hold.

(a) $(\delta_K^\phi)'(x; v) = \inf\{\grad \phi(x-y) \cdot v : y \in \xi_K^\phi(x)\}$ for every $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n \sim K$.

(b) For each $x \in \mathbb{R}^n \sim K$ there exists an open neighbourhood $U \subseteq \mathbb{R}^n \sim K$ of $x$ such that $\delta_K^\phi|U$ is semiconcave.

(c) $\delta_K^\phi$ is differentiable at $x \in \mathbb{R}^n \sim K$ if and only if $\xi_K^\phi(x)$ is a singleton, in which case

$$\grad \delta_K^\phi(x) = \grad \phi(x - \xi_K^\phi(x)),$$

$\xi_K^\phi(x) = x - \delta_K^\phi(x) \grad \phi^*(\grad \delta_K^\phi(x))$.

(d) If $\delta_K^\phi$ is differentiable at $x \in \mathbb{R}^n \sim K$ then $\delta_K^\phi$ is differentiable at $\xi_K^\phi(x) + t(x - \xi_K^\phi(x))$ for $0 < t < \rho_K^\phi(x)$ with

$$\grad \delta_K^\phi(x) = \grad \delta_K^\phi(\xi_K^\phi(x) + t(x - \xi_K^\phi(x)))$$.

(e) $\delta_K^\phi$ is pointwise differentiable of order 2 at $x \in \mathbb{R}^n \sim K$ if and only if $\xi_K^\phi$ is differentiable at $x$ in the sense of 2.25, in which case

$$\pt D^2 \delta_K^\phi(x)(u, v) = D(\grad \phi \circ \nu_K^\phi)(x)(u) \cdot v \quad \text{for } u, v \in \mathbb{R}^n.$$ 

(f) $\xi_K^\phi$ is weakly continuous in the sense of 2.18.

Proof. The assertions [(a) and (b)] correspond to [Zaj83a, Corollary to Theorem 3\textsuperscript{a}] and [Zaj83a, Theorem 5], respectively.

We prove [(e)]. If $\xi_K^\phi(x)$ is a singleton, then for every $v \in \mathbb{R}^n$ the partial derivative of $\delta_K^\phi$ at $x$ with respect to $v$ exists and equals $\grad \phi(x - \xi_K^\phi(x)) \cdot v$ by [(a)]. Since $\delta_K^\phi$ is Lipschitz continuous with Lipschitz constant 1 by [DRKS20, Lemma 2.38(a)] and $(\delta_K^\phi)'(x; \nu_K^\phi(x)) = 1$ by [DRKS20, Lemma 2.32(c)] we conclude that $\delta_K^\phi$ is differentiable at $x$ using [FIT84, 2.4, 2.5]. On the other hand if $\xi_K^\phi(x)$ is not a singleton then $\delta_K^\phi$ is not differentiable at $x$ by a result of Konjagin [Kon78] (see also [Zaj83a, Proposition 2]).

Assertion [(d)] follows from [(e)] and 2.11.

To prove [(f)] we observe that for $x \in \mathbb{R}^n \sim K$ there exist, by [(b)] a constant $\kappa > 0$, an open neighbourhood $U$ of $x$, and a convex function $V : U \to \mathbb{R}$ such that

$$V(y) = (\kappa/2)|y|^2 - \delta_K^\phi(y) \quad \text{for } y \in U.$$ 

Moreover, we observe, using [(a)] that if $\xi : U \to \mathbb{R}^n$ is a function such that $\xi(y) \in \xi_K^\phi(y)$ for every $y \in U$, then $\nabla y - \grad \phi(y - \xi(y)) = \nabla y - \grad \phi(\delta_K^\phi(y)^{-1}(y - \xi(y))) \in \nabla V(y)$. Therefore, we conclude from 2.25 that $\delta_K^\phi$ is pointwise differentiable of order 2 at $x$ if and only if $\xi_K^\phi$ is differentiable at $x$. The displayed equation in [(f)] also follows from the postscript of 2.25.

Finally we prove [(1)]. The argument used in [DRKS20, 2.38(b)], which proves the statement for the restriction of $\xi_K^\phi$ to the set of points where it is single-valued, also works in the general case of [(1)]. For completeness we provide a proof. By contradiction we assume there are $x \in \mathbb{R}^n$, $\varepsilon > 0$ and two sequences $x_i \in \mathbb{R}^n$ and $a_i \in K$ such that $x_i \to x$, $a_i \in \xi_K^\phi(x_i)$ and $|a_i - b| \geq \varepsilon$ for every $b \in \xi_K^\phi(x)$ and for every $i \geq 1$. Noting that

$$|\delta_K^\phi(x_i) - \delta_K^\phi(x)| \leq \phi(x_i - x)$$

and

$$\phi(a_i - x) \leq \delta_K^\phi(x_i) + \phi(x_i - x) \leq \delta_K^\phi(x) + 2F(x_i - x)$$

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Moreover, if $\delta \geq 2.33$ Lemma. Suppose we were not able to find this specific statement in the literature and we add a proof here for $A$

2.32 Lemma. If $T \in \mathbf{R}(n, n-1)$, $a \in T$, $f : T \to T^\perp$ is continuous at $a$, $a = a + f(a)$, $A = \{\chi + f(\chi) : \chi \in T\}$ and $\Tan(A, a) \subseteq T$, then $f$ is differentiable at $a$ with $Df(a) = 0$.

Proof. It follows from the definition of tangent cone; cf. [Fed69, 3.1.21].

The first part of the following lemma is strictly related to the classical result of [GP72]. However, we were not able to find this specific statement in the literature and we add a proof here for reader's convenience.

2.33 Lemma. Suppose $K$ is a closed subset of $\mathbf{R}^{n+1}$, $r > 0$, $x \in S(K, r)$, $\delta \geq 2.33$ K is differentiable at $x$ and $T = \{v : v \cdot \grad \delta_K^\phi(x) = 0\}$.

Then $T = \Tan(S(K, r), x)$ and there exists an open neighbourhood $V$ of $x$ and a continuous function $f : T \to T^\perp$ such that $f$ is differentiable at $T_x$ with $Df(T_x) = 0$ and $V \cap S(K, r) = V \cap \{\chi + f(\chi) : \chi \in T\}$.

Moreover, if $\delta_K^\phi$ is pointwise differentiable of order 2 at $x$ then $f$ is pointwise differentiable of order 2 at $T_x$ and $pt D^2 f(T_x)(u, v) \cdot \grad \delta_K^\phi(x) = \inf_{u, v} pt D^2 \delta_K^\phi(x)(u, v)$ for $u, v \in T$.

Proof. To prove the first part we closely follow [GP72] Theorem 1 with minor modifications. We notice from [2.33(1)] that

$$
\phi'(\nu_K^\phi(x); \grad \delta_K^\phi(x)) = \lim_{t \to 0^+} t^{-1} \left( \phi(\nu_K^\phi(x) - t \grad \delta_K^\phi(x)) - \phi(\nu_K^\phi(x)) \right)
$$

$$
= -\grad \phi(\nu_K^\phi(x)) \cdot \grad \delta_K^\phi(x) = -|\grad \phi(\nu_K^\phi(x))|^2 < 0.
$$

For $z \in \mathbf{R}^n \sim K$, $\varepsilon > 0$ and $\alpha \in \mathbf{R}^n$ we define

$$
N(z) = \{v : z + \delta_K^\phi(z)v \in K, \phi(v) = 1\},
$$

$$
N(z, \varepsilon) = \{v : \phi(v) \geq 1, \phi(v - w) < \varepsilon \text{ for some } w \in N(z)\},
$$

$$
g(y, \alpha) = \lim_{t \to 0^+} \inf \{\phi'(v; -\alpha) : v \in N(y, t)\}.
$$

Notice that $N(x) = \{-\nu_K^\phi(x)\}$ by [GP72] Lemma 3 and the strict convexity of $\phi$. We use [Roc70, Theorem 23.1] to infer that $\phi'(\cdot; \alpha)$ is an upper semi-continuous function and to estimate

$$
\sup \{-\phi'(v, -\grad \delta_K^\phi(x)) : v \in N(x, \varepsilon)\} \leq \sup \{\phi'(v, \grad \delta_K^\phi(x)) : v \in N(x, \varepsilon)\} < 0
$$

for some $\varepsilon > 0$. It follows that $g(x, \grad \delta_K^\phi(x)) > 0$. Employing [GP72, Lemma 2] we notice that the set $U = \{y : g(y, \grad \delta_K^\phi(x)) > 0\}$ is an open neighbourhood of $x$ so there is a relatively open subset $Q$ of $T$ with $0 \in Q$ and an open interval $J \subseteq \mathbf{R}$ with $0 \in J$ such that $x + s \grad \delta_K^\phi(x) + \chi \in U$ whenever $(\chi, s) \in Q \times J$. For each $\chi \in Q$ let $h_\chi : J \to \mathbf{R}$ be given by $h_\chi(s) = \delta_K^\phi(x + s \grad \delta_K^\phi(x) + \chi)$ for $s \in J$. Notice that $h_\chi$ is Lipschitz continuous and if $s \in J$ is a point of differentiability of
Lemma 3.2

Let $\Lambda$ be a $\omega$-Euclidean structure on $X$. If $a, b \in \Lambda$ and $\phi$ is pointwise differentiable of order 2 at $0$ with $\phi(0) = 0$ and, setting $\delta_\lambda^\phi(0) = 0$, then setting $\delta_\lambda^\phi(0) = 0$ and, setting $\delta_\lambda^\phi(0) = 0$ and, setting $\delta_\lambda^\phi(0) = 0$.

Therefore, $\delta_\lambda^\phi$ is pointwise differentiable of order 2 at $x = 0$. We notice that $\lim_{x \to 0} f(x)/|x| = 0$ and, setting $\zeta = \chi + f(\chi)$,

$$
\frac{0 = \lim_{T \ni x \to 0} \delta_\lambda^\phi(\zeta) - \delta_\lambda^\phi(0) - \frac{1}{2} \frac{\partial^2 \delta_\lambda^\phi(0)(\zeta, \zeta)}{|\zeta|^2}}{= - \lim_{T \ni x \to 0} \frac{\partial^2 \delta_\lambda^\phi(0)(\zeta, \zeta)}{|\zeta|^2}}.
$$

which means that $f$ is pointwise differentiable of order 2 at 0 with

$$
\lim_{x \to 0} f(x)/|x| = 0.
$$

3 Proof of Theorem 1.3

In this section we consider an abstract Minkowski space $(X, \phi)$ of dimension $n$ and we are defining a Euclidean structure on $X$ to fit our problem. We set $d = n - 1$. If $k \in \mathcal{P}$, the operator norm of $\Lambda \in \bigotimes^k(X, X)$ with respect to $\phi$ is defined as in [Fed69, 1.10.5], i.e.,

$$
\|\Lambda\|_\phi = \sup \{\phi(\Lambda(x_1, \ldots, x_k)) : x_1, \ldots, x_k \in X, \phi(x_i) \leq 1 \text{ for } i \in \{1, 2, \ldots, k\}\}.
$$

Once the Euclidean structure on $X$ is defined we shall use the symbol $\|\Lambda\|_\phi$ to denote the operator norm of $\Lambda$ with respect to that Euclidean structure.

3.1 Definition. Let $K \subseteq X$ be closed. We define

$$
\text{Unp}^\phi(K) = X \cap \{x : \mathcal{H}^0(\xi_k^\phi(x)) = 1\}.
$$

3.2 Lemma. Assume

$$
\phi \text{ is strictly convex}, \quad K \subseteq X \text{ is closed}, \quad 1 < \lambda < \infty,
$$

Then there exists $\omega_\lambda : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t \to 0} \omega_\lambda(t) = 0$ and

$$
\phi(a - b) \leq \delta_\lambda^\phi(x) \omega_\lambda(\phi(x - y)/\delta_\lambda^\phi(x)) \text{ for } x \in K, y \in X, a \in \xi_k^\phi(x), b \in \xi_k^\phi(y).
$$

Proof. For $0 \leq t < \infty$ define

$$
K_\lambda(t) = X \times X \cap \{(a, b) : \phi(a) = \lambda, \phi(b) \geq \lambda, \phi((1 - 1/\lambda)a - b) \leq 1 + 2t\},
$$

$$
\omega_\lambda(t) = \sup \{\phi(a - b) : (a, b) \in K_\lambda(t)\}.
$$

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Observe that strict convexity of \( \phi \) yields
\[
\bigcap\{K_\lambda(t) : 0 < t < \infty\} = X \times X \cap \{(a, a) : \phi(a) = \lambda\}
\quad \text{and} \quad \lim_{t \to 0} \omega_\lambda(t) = 0.
\]

Indeed, assume \( \lim \sup_{t \to 0} \omega_\lambda(t) = \delta \). Find sequences \( X \cap \{a_j : j \in \mathcal{P}\} \) and \( X \cap \{b_j : j \in \mathcal{P}\} \) such that \((a_j, b_j) \in K_\lambda(1/j), \phi(a_j - b_j) \geq \delta - 1/j, \lim_{j \to \infty} a_j = a_0 \) and \( \lim_{j \to \infty} b_j = b_0 \) with \( \phi(a_0) = \lambda \), \( \phi(b_0) \geq \lambda, \phi(b_0 - a_0) \geq \delta, \phi(z_0 - (1 - 1/\lambda)a_0) \leq 1 \). Then
\[
\lambda \leq \phi(b_0) \leq \phi(b_0 - (1 - 1/\lambda)a_0) + \phi((1 - 1/\lambda)a_0) \leq 1 + \lambda - 1 = \lambda
\]
which implies that \( a_0 = b_0 \) and \( \delta = 0 \).

Let \( x \in K_\lambda \subseteq \Unp^\phi K, y \in X \). Choose
\[
\bar{a} \in \xi^\phi_K(x), \quad \bar{b} \in \xi^\phi_K(y), \quad c = \bar{a} + \lambda(x - \bar{a}),
\quad r = \delta^\phi_K(x), \quad a = (\bar{a} - c)/r, \quad b = (\bar{b} - c)/r, \quad t = \phi(x - y)/r.
\]

Clearly we have
\[
\phi(\bar{b} - x) \leq \phi(\bar{b} - y) + \phi(y - x)
\leq \phi(\bar{a} - y) + \phi(y - x) \leq \phi(\bar{a} - x) + 2\phi(x - y) = r(1 + 2t).
\]

Since \((x - c)/r = (1 - 1/\lambda)a\) we obtain
\[
\phi(\bar{b} - x) = \phi(\bar{b} - y) + \phi(y - x)
\leq \phi(\bar{a} - y) + \phi(y - x) \leq \phi(\bar{a} - x) + 2\phi(x - y) = r(1 + 2t).
\]

Because \( x \in K_\lambda \) we know also that \( \phi(\bar{a} - c) < \phi(\bar{b} - c) \); hence,
\[
r\phi(\bar{b}) = \phi(\bar{b} - c) + \phi(\bar{a} - c) = \phi(a) + \lambda\phi(x - \bar{a}) = \lambda r.
\]

This shows that \((a, b) \in K_\lambda(t)\) so \( \phi(a - b) \leq \omega_\lambda(t) \) and \( \phi(\bar{a} - \bar{b}) \leq r\omega_\lambda(t) \).

\[\text{3.3 Corollary.} \quad \text{Assume } \phi \text{ is strictly convex, } K \subseteq X \text{ is closed, } 0 < s < t < \infty, 1 < \lambda < \infty, \text{ and}
\]
\[K_{\lambda,s,t} = \{x : \rho^\phi_K(x) \geq \lambda, s \leq \delta^\phi_K(x) \leq t\}
\]

Then \( \xi^\phi_K|K_{\lambda,s,t} \) is uniformly continuous.

\[\text{3.4 Remark.} \quad \text{This provides an alternative proof that } \xi^\phi_K|\Unp^\phi(K) \text{ is continuous; cf. [DRKS20, 2.42].}
\]

\[\text{3.5 Remark.} \quad \text{Assume that } X \text{ is a finite dimensional vector space equipped with a strictly convex}
\]
\[\text{and continuously differentiable (away from the origin) norm } \phi : X \to \mathbb{R}. \quad \text{We define}
\]
\[\text{1) } S = \partial B^\phi(0, 1), \quad \xi : X \sim \{0\} \to S \quad \text{by} \quad \xi(x) = x\phi(x)^{-1} \quad \text{for } x \in X \sim \{0\},
\]
\[\pi : S \to \Hom(X, X) \quad \text{by} \quad \pi = D\xi|S.
\]

Note that whenever \( \eta \in S \) the map \( \pi(\eta) \) is a projection onto \( \Tan(S, \eta) \) such that
\[\text{2) } \pi(\eta) \circ \pi(\eta) = \pi(\eta), \quad \text{im } \pi(\eta) = \Tan(S, \eta), \quad \eta \in \ker \pi(\eta) \quad \text{for } \eta \in S.
\]

\[\text{3.6 Lemma.} \quad \text{Consider the situation as in } 3.5. \quad \text{Let } 0 < \varepsilon < 1 \text{ and set}
\]
\[R = \sup \mathbb{R} \cap \{r : 0 < r < 1, \eta, \zeta \in S, \phi(\eta - \zeta) \leq r \text{ implies } \|\pi(\eta) - \pi(\zeta)\|_\phi \leq 1 - \varepsilon\}.
\]

Then \( \pi(\eta)|S \cap B^\phi(\eta, R) \) is injective whenever \( \eta \in S. \)
Proof. Assume that for some \( \eta \in S \) the map \( \pi(\eta)|S \cap B^\phi(\eta, R) \) is not injective. Set
\[
D = S \cap B^\phi(\eta, R)
\]
and let \( \xi, \zeta \in D \) be such that \( \pi(\eta)\xi = \pi(\eta)\zeta; \) hence, \( \xi - \zeta \in \ker \pi(\eta) = \text{span}\{\eta\}. \) Assume \( \phi(\xi - \eta) \leq \phi(\zeta - \eta). \) If \( \eta = \xi, \) then \( \zeta = -\eta \) and \( \phi(\zeta - \eta) = 2 > 1 \) which cannot happen because \( \zeta \in D \) and \( R \leq 1. \) Let \( P = \text{span}\{\eta, \xi\}. \) Then \( \zeta = \xi + \lambda \eta \) for some \( \lambda \in \mathbb{R} \) and we get
\[
\eta, \xi, \zeta \in P \cap S.
\]
Let \( \gamma : \mathbb{R} \to S \cap P \) be such that
\[
\phi(\gamma'(t)) > 0 \quad \text{for } t \in \mathbb{R}, \quad \gamma(0) = \xi, \quad \gamma(1) = \zeta.
\]
Set \( A = \text{im} \gamma|[0, 1]. \) Since \( \xi - \zeta \in \text{span}\{\eta\} \) we see that both \( \xi \) and \( \eta \) are on the same side of the line \( \text{span}\{\eta\} \) in \( P. \) Therefore, the Monotonicity Lemma [MSW01 Proposition 31] yields that \([0, 1] \ni t \mapsto \phi(\gamma(t) - \eta) \) is a strictly increasing function and we know that \( \phi(\zeta - \eta) \leq R; \) thus, we have \( \phi(\gamma(t) - \eta) \leq R \) for all \( t \in [0, 1] \) and
\[
A \subseteq D.
\]
Let \( w \in P \) and \( \omega \in P^* \) be such that \( w \wedge \eta \neq 0, \omega(w) = 1, \) and \( \omega(\eta) = 0. \) Define the function \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(t) = \omega(\gamma(t)) \quad \text{for } t \in \mathbb{R}.
\]
Note that \( f(1) - f(0) = \omega(\zeta - \xi) = 0 \) so \( f(1) = f(0) \) and, by the mean value theorem, there exists \( t_0 \in [0, 1] \) such that
\[
0 = f'(t_0) = \omega(\gamma'(t_0)); \quad \text{hence, } \gamma'(t_0) = \lambda \eta \quad \text{for some } \lambda \in \mathbb{R} \sim \{0\}.
\]
Set \( \nu = \gamma(t_0). \) Since \( \gamma'(t_0) \in \text{Tan}(S, \nu) \) we see that \( \eta \in \text{Tan}(S, \nu) \) and \( \pi(\nu)\eta = \eta \) so
\[
\|\pi(\eta) - \pi(\nu)\|_\phi \geq \phi(\pi(\eta)\eta - \pi(\nu)\eta) = \phi(\eta) = 1
\]
but \( \nu \in A \subseteq D \) so this contradicts the choice of \( R. \) \( \square \)

3.7 Remark. Consider the situation as in 3.5 and assume \( \phi \) is of class \( \mathcal{C}^2 \) away from the origin. Let \( \varepsilon \in (0, 1) \) and \( \eta \in S. \) Set \( R = R(\varpi \varepsilon), \) \( T = \text{Tan}(S, \eta), \) and \( M = S \cap B^\phi(\eta, R). \) Since \( \pi(\eta)|M \) is injective and \( M \) is compact we see that \( \pi(\eta)|M \) is a homeomorphism between \( M \) and \( A = \pi(\eta)|M \) \( \subseteq T. \) Set
\[
H = (\pi(\eta)|M)^{-1} \circ \pi(\eta) \quad \text{and} \quad C = \pi(\eta)^{-1}|\text{Int} A.
\]
Since \( \phi \) is of class \( \mathcal{C}^2 \) we see that \( M \) is a manifold of class \( \mathcal{C}^2 \) and \( H : C \to M \) is of class \( \mathcal{C}^2, \)
\[
H(\zeta) = \xi(\zeta) \quad \text{and} \quad DH(\zeta)u = D\xi(\zeta)u \quad \text{for } \zeta \in S \cap C \text{ and } u \in \text{Tan}(S, \zeta).
\]
Differentiating the equation
\[
DH(\zeta) \circ \pi(\zeta)u = D\xi(\zeta) \circ \pi(\zeta)u \quad \text{which holds for } \zeta \in S \cap C \text{ and } u \in T
\]
we get
\[
D^2H(\eta)(u, v) + DH(\eta)(D\pi(\eta)uv) = D^2\xi(\eta)(u, v) + \pi(\eta)(D\pi(\eta)uv) \quad \text{for } u, v \in T;
\]
however, if \( u, v \in T = \text{im} \pi(\eta), \) then \( D\pi(\eta)uv \in \ker \pi(\eta) = \text{span}\{\eta\} \) by (2) and for all \( x \in S \cap C \) we also have \( DH(x)\eta = 0; \) hence
\[
D^2H(\eta)(u, v) = D^2\xi(\eta)(u, v) \quad \text{for } u, v \in T.
\]
Since \( T \) is tangent at \( \eta \in S \) to the level-set \( S \) of \( \phi \) we have \( D\phi(\eta)u = 0 \) whenever \( u \in T; \) thus, differentiating \( \{1\} \) twice and recalling that \( \phi(\eta) = 1 \) and \( \xi(\eta) = \eta \) we obtain
\[
D^2H(\eta)(u, v) = D^2\xi(\eta)(u, v) = -D^2\phi(\eta)(u, v)\eta \quad \text{for } u, v \in T.
\]
3.8 Remark. In we prove that $\xi^\phi_K$ is Lipschitz continuous on each of the sets $K_{\lambda,s,t} = \{ x : \rho^\phi_K(x) \geq \lambda, s \leq \delta^\phi_K(x) \leq t \}$ defined for $0 < s < t < \infty$ and $1 < \lambda < \infty$. Since the proof is a bit technical we briefly describe the main idea. For $x \in K_{\lambda,s,t}$ and $y \in \mathbb{R}^n$ we set $\omega = \xi^\phi_K(x)$ and choose any $b \in \xi^\phi_K(y)$. First we find a point $c$ for which $T = \nabla(\partial B^\phi(x, \delta^\phi_K(x)), a) = \nabla(\partial B^\phi(y, \delta^\phi_K(y)), c)$.

Then we choose $c \in \partial B^\phi(y, \delta^\phi_K(y))$ and $d \in \partial B^\phi(a + \lambda(x-a), \lambda \delta^\phi_K(x))$ which have the same orthogonal (with respect to the Euclidean structure induced by $D^2 \phi(a-x)$) projections onto $T$ as $a$ and $b$ respectively; see Figure 1. We represent $\partial B^\phi(a + \lambda(x-a), \lambda \delta^\phi_K(x))$ and $\partial B^\phi(y, \delta^\phi_K(y))$ locally around $a$ and $c$ as graphs over $T$ of functions $g_u$ and $g_v$ of class $C^2$ using 3.6. Employing 3.2 we can find $\epsilon > 0$ which guarantees that $d$, $e$, and $f$ fit on the graphs of $g_w$, $g_y$, and $g_b$ respectively. Let $q$ be the signed distance from $T$ such that $q(x-a) > 0$. The crucial point of the proof is in the estimates 10 and 11, where we use the second order Taylor formulas for $g_u$ and $g_v$ to compare (both ways) the heights $q(d-a)$, $q(e-c)$, and $q(b-c)$ with $\lambda^{-2}[T_1(d-a)]^2$, $|T_1(a-c)|^2$, and $|T_1(b-c)|^2$ respectively, up to errors expressed in terms of the modulus of continuity of $D^2 H$, where $H$ comes from 3.7. Analysing the situation presented on Figure 1 we obtain an estimate of the form

$$q(b-c) \leq q(d-a) + q(e-c),$$

which, using the comparison mentioned before, is translated into

$$|T_1(b-c)|^2 \leq \Delta_1 \lambda^{-1}|T_1(b-a)|^2 + |\Delta_2|T_2(a-c)|^2,$$

where $\Delta_1$ and $\Delta_2$ can be made arbitrarily close to 1 by adjusting $\epsilon$ depending on the modulus of continuity of $D^2 H$. This leads to the estimate 12 of the form

$$|T_1(b-a)| \leq |T_1(b-c)| + |T_2(c-a)| \leq \Delta_3|T_2(c-a)| + \lambda^{-1/2}|\Delta_4|T_2(b-a)|,$$

where, again, $\Delta_3$ is close to 1 given $\epsilon$ is small enough; hence, the last term may be absorbed on the left-hand side. Since $|T_1(b-a)| \approx |b-a|$ and $|T_2(c-a)| \approx |x-y|$ we get the conclusion.

3.9 Theorem. Consider the situation as in 3.8. Assume

$$\phi|X \sim \{0\} \quad \text{is of class } C^2, \quad K \subseteq X \quad \text{is closed}, \quad 1 < \lambda < \infty, \quad x, y \in X, \quad \rho^\phi_K(x) \geq \lambda, \quad a \in \xi^\phi_K(x), \quad b \in \xi^\phi_K(y), \quad \eta = \frac{a-x}{\phi(a-x)}, \quad D^2 \phi(\eta)(u, u) > 0 \quad \text{for } u \in \nabla(S, \eta) \sim \{0\}.$$

There exist $\epsilon = \epsilon(\lambda, \phi, \delta^\phi_K(x))$ and $\Gamma = \Gamma(\lambda, \phi)$ such that

$$\phi(x-y) \leq \epsilon \quad \text{implies} \quad \phi(a-b) \leq \Gamma \phi(x-y).$$

Proof. Clearly we can assume $a \neq b$ and $y \in X \sim K$. Define

$$r_x = \delta^\phi_K(x) = \phi(a-x), \quad r_y = \delta^\phi_K(y) = \phi(b-y),$$

$$c = y + \frac{r_y}{r_x}(a-x), \quad w = a + \lambda(x-a), \quad \eta = \frac{a-x}{\phi(a-x)}, \quad T = \nabla(S, \eta).$$

Note for the record (see Figure 1)

$$a \in \partial B^\phi(x, r_x) \cap \partial B^\phi(w, \lambda r_x) \sim B^\phi(y, r_y), \quad b, c \in \partial B^\phi(y, r_y), \quad b \notin B^\phi(x, r_x).$$

Define

$$R = R_{\frac{1}{2}}, \quad H = H_{\frac{1}{2}}, \quad M = M_{\frac{1}{2}}, \quad C = C_{\frac{1}{2}}, \quad \eta = \frac{1}{\phi(a-x)}.$$

Let $q \in X^*$ be such that $q(\eta) = -1$ and $\ker q = T$. Note that $D^2 \phi(\eta)(\eta, \eta) = 0$ by one-homogeneity of $\phi$. Define

$$B \in C^2 X \quad \text{by} \quad B(u, v) = D^2 \phi(\eta)(\eta(\eta)u, \eta(\eta)v) + qu \cdot qv \quad \text{for } u, v \in X.$$
By our assumption on $D^2\phi(\eta)$ the bilinear map $B$ defines a scalar product on $X$. In the sequel of this proof we shall assume the Euclidean structure on $X$ comes from $B$. In particular, we shall use the notations

$$T_\epsilon = \pi(\eta), \quad u \cdot v = B(u,v), \quad \text{and} \quad |u| = B(u,u)^{1/2} \quad \text{for} \ u,v \in X.$$ Let $\omega_\lambda$ be the map obtained from 3.2. Set

$${\tau}^\#_y = \pi(\eta), \quad u \cdot v = B(u,v), \quad \text{and} \quad |u| = B(u,u)^{1/2} \quad \text{for} \ u,v \in X.$$
Recall that $H = H \circ T_z$ and $a - x, c - y \in \ker T_z = T_z^\perp = \text{span}\{\eta\}$. Set
\[
d = g_w(T_z(b - a)) \quad \text{and} \quad e = g_y(T_z(a - c))
\]
and observe that
\[
T_z(b - a) = T_z(d - a), \quad T_z(a - c) = T_z(e - c),
\]
\[
z = g_y \circ T_z(z - c) \quad \text{if} \ z \in \text{im} \ g_y, \quad z = g_w \circ T_z(z - a) \quad \text{if} \ z \in \text{im} \ g_w,
\]
\[
b \in \partial \mathcal{B}_x^w(y, r_y) \sim \mathcal{B}_x^w(w, \lambda r_x) ; \quad \text{hence,} \quad q(b - a) < q(d - a),
\]
\[
a \in \partial \mathcal{B}_x^w(w, \lambda r_x) \sim \mathcal{B}_x^w(y, r_y) ; \quad \text{hence,} \quad q(a - c) < q(e - c).
\]
Recalling [3.7] we see that
\[
Dg_w(0) = DH(0)|T = I_T \quad \text{and} \quad D^2 g_w(0) = -\eta(\lambda r_x)^{-1}D^2 \phi(\eta);
\]
thus, since $|\eta| = 1$ and $q(-\eta) = 1$ the Taylor formula [Fed69, 3.1.11, p. 220] yields
\[
|q(d - a) - (2\lambda r_x)^{-1}|T_z(d - a)|^2| = |d - a - T_z(d - a) - (2\lambda r_x)^{-1}|T_z(d - a)|^2(-\eta)|
\]
\[
= \left|g_w(T_z(d - a)) - g_w(0) - \langle T_z(d - a), Dg_w(0) \rangle \frac{1}{2} - \langle T_z(d - a) \circ T_z(d - a), D^2 g_w(0) \rangle \right| \leq \left(2\lambda r_x\right)^{-1}|T_z(d - a)|^2 \sigma(T_z(d - a))(\lambda r_x)^{-1}.
\]
Repeating the above computation twice with $g_y, c, e$ and $g_y, c, b$ in place of $g_w, a, d$ we get
\[
|q(c - a) - (2\lambda r_y)^{-1}|T_z(a - c)|^2| \leq (2\lambda r_y)^{-1}|T_z(a - c)|^2 \sigma(T_z(a - c))(\lambda r_y)^{-1}
\]
and
\[
|q(b - c) - (2\lambda r_y)^{-1}|T_z(b - c)|^2| \leq (2\lambda r_y)^{-1}|T_z(b - c)|^2 \sigma(T_z(b - c))(\lambda r_y)^{-1}.
\]
Consequently, using (6), (7), and (8)
\[
|T_z(b - a)|^2 \leq \frac{2r_y |q(b - c)|}{1 - \sigma(T_z(b - c))(\lambda r_y)^{-1}}
\]
\[
\leq \frac{r_y (1 + \sigma(T_z(b - a))(\lambda r_x)^{-1})}{\lambda r_x (1 - \sigma(T_z(b - c))(\lambda r_y)^{-1})}|T_z(b - a)|^2 + \frac{1 + \sigma(T_z(a - c))(\lambda r_y)^{-1}}{1 - \sigma(T_z(b - c))(\lambda r_y)^{-1}}|T_z(a - c)|^2;
\]
hence,
\[
|T_z(b - a)| \leq |T_z(b - c)| + |T_z(c - a)|
\]
\[
\leq \left(1 + \left(\frac{1 + \sigma(T_z(a - c))(\lambda r_y)^{-1}}{1 - \sigma(T_z(b - c))(\lambda r_y)^{-1}}\right)^{1/2}\right)^{1/2}|T_z(c - a)| + \left(\frac{r_y (1 + \sigma(T_z(b - a))(\lambda r_x)^{-1})}{\lambda r_x (1 - \sigma(T_z(b - c))(\lambda r_y)^{-1})}\right)^{1/2}|T_z(b - a)|.
\]
Recalling (4), (5), $\phi(x - y) \leq \varepsilon$, $\rho^2_n(x) \geq \lambda$ and using (3.2) we obtain
\[
r_x^{-1}|T_z(a - b)| \leq r_x^{-1}\Delta_1 \phi(a - b) \leq \Delta_1 \omega_1(\varepsilon/r_x),
\]
\[
r_y^{-1}|T_z(a - c)| \leq r_y^{-1}\Delta_1 \phi(a - c) \leq r_x^{-1}2\Delta_1 \varepsilon/r_x \leq r_x^{-1}4\Delta_1 \varepsilon,
\]
\[
r_y^{-1}|T_z(b - a)| \leq r_x^{-1}|T_z(b - a)|r_x/r_y + r_y^{-1}|T_z(a - c)| \leq 4\Delta_1 (\omega_1(\varepsilon/r_x) + \varepsilon/r_x).
\]
Employing (3), (4), and noting that
\[
\frac{r_y}{r_x} \leq 1 + 2^{-5}\Delta_2, \quad \frac{1 + 2^{-5}\Delta_2}{1 - 2^{-5}\Delta_2} \leq 1 + \frac{\Delta_2}{32}, \quad \frac{r_y}{r_x} \frac{1 + 2^{-5}\Delta_2}{1 - 2^{-5}\Delta_2} \leq 1 + \frac{\Delta_2}{2}
\]
we obtain
\[
\frac{1}{\lambda} \frac{r_y}{r_x} \frac{1 + \sigma(T_2(b-a)(\|r\|_{x})^{-1})}{1 - \sigma(T_2(b-c)(\|r\|_{y})^{-1})} \leq \left(1 + \frac{\lambda - 1}{2}\right) = \frac{\lambda + 1}{2} \leq 1,
\]
\[
\frac{1 + \sigma(|T_2(a-c)|^{-1})}{1 - \sigma(|T_2(b-c)|^{-1})} \leq \frac{33}{32} \leq 4, \quad \left(1 - \left(\frac{\lambda + 1}{2\lambda}\right)^{1/2}\right)^{-1} = 2\lambda \frac{1 + (\frac{\lambda + 1}{2}\lambda)^{1/2}}{\lambda - 1} \leq 4\lambda - 1;
\]
hence; plugging these estimates to (12) yields
\[
|T_2(b-a)| \leq \frac{12\lambda}{\lambda - 1}|T_2(c-a)|.
\]
Note that \( |T_2(a-c)| \leq 2\varepsilon \Delta_1 \leq \min\{r_x, r_y\} \) by (3) and (4). In case \( q(a-c) \geq 0 \) we combine (13), (11), (10), (8) to get
\[
|q(b-a)| \leq q(b-c) + q(c-a) \leq |T_2(b-a)|^2 \frac{1 + 2^{-5}\Delta_2}{2\lambda r_x} + |T_2(a-c)|^2 \frac{1 + 2^{-5}\Delta_2}{r_y} \leq \left(2 + \frac{(12\lambda)^2}{(\lambda - 1)^2}\right)|T_2(a-c)|. \tag{14}
\]
If \( q(a-c) < 0 \), then \( q(b-a) = q(b-c) + q(c-a) \geq 0 \) and we get by (7), (9), (6), (13)
\[
|q(b-a)| = q(b-a) \leq q(d-a) \leq |T_2(b-a)|^2 \frac{1 + 2^{-5}\Delta_2}{2\lambda r_x} \leq \frac{(12\lambda)^2}{\lambda(\lambda - 1)^2}|T_2(a-c)|.
\]
As a result the final estimate of (14) holds regardless of the sign of \( q(a-c) \). Employing (5)
\[
|b-a| \leq |T_2(b-a)| + |q(b-a)| \leq \left(\frac{12\lambda}{\lambda - 1} + 2 + \frac{(12\lambda)^2}{(\lambda - 1)^2}\right)|T_2(c-a)| \leq \left(\frac{12\lambda}{\lambda - 1} + 2 + \frac{(12\lambda)^2}{(\lambda - 1)^2}\right) \Delta_1 \phi(c-a) \leq \Gamma \phi(x - y),
\]
where \( \Gamma = 2\Delta_1 \left(\frac{12\lambda}{\lambda - 1} + 2 + \frac{(12\lambda)^2}{(\lambda - 1)^2}\right) \).
\]

**3.10 Corollary.** Assume \( \phi \) is uniformly convex, \( K \subseteq X \) is closed, \( 0 < s < t < \infty \), \( 1 < \lambda < \infty \), and
\[
K_{\lambda,s,t} = \left\{ x : \rho_K^\phi(x) \geq \lambda, s \leq \delta_K^\phi(x) \leq t \right\}.
\]
Then there exists \( \Gamma \in R \) depending only on \( s, t, \lambda \), and \( \phi \) such that
\[
\phi(x_K^\phi(a) - y) \leq \Gamma \phi(a - b) \text{ whenever } a \in K_{\lambda,s,t}, b \in R^n, y \in x_K^\phi(b), \text{ and } \delta_K^\phi(b) \leq t.
\]
In particular, \( x_K^\phi|_{K_{\lambda,s,t}} \) is Lipschitz continuous.

**Proof.** Assume \( a \in K_{\lambda,s,t}, b \in R^n, y \in x_K^\phi(b), x \in x_K^\phi(a), \text{ and } \delta_K^\phi(b) \leq t \). Let \( \varepsilon = \Gamma\phi(\lambda, \phi, s) \). If \( \phi(a-b) \leq \varepsilon \), then the conclusion follows from 3.9. In case \( \phi(a-b) > \varepsilon \), we have
\[
\phi(x - y) \leq \phi(x - a) + \phi(a - b) + \phi(b - y) \leq \phi(a - b) + 2t \leq \phi(a - b)(1 + 2t/\varepsilon).
\]

**3.11 Remark.** Observe that the bound for the Lipschitz constant of \( x_K^\phi|_{K_{\lambda,s,t}} \) obtained in 3.9 explodes with \( \lambda \to 1^+ \). This is in accordance with 4.1.

**Proof of (13).** Since \( \text{grad } \delta_K^\phi(x) = \text{grad } \phi(x - x_K^\phi(x)) \) for \( x \in \text{diam } \text{grad } \delta_K^\phi \sim K = \text{Unip } K \sim K \) by 2.3(6), we obtain the claim directly from 3.10.
4 Proof of Theorem 1.4

Recall the definitions of $N^\phi(K)$, $r^\phi_K$ and $\text{Cut}^\phi(K)$ from the introduction. We start with a remark.

4.1 Remark. Consider the parabola $K = \{(x, x^2) : x \in \mathbb{R}\}$ with centre of curvature at the point $a = (0, \frac{1}{2}) \in \mathbb{R}^2$. Then $a \in \text{Cut}(K) \cap \text{Unp}(K)$. We look at the behaviour of $\xi_K$ on the line \{$(x, \frac{1}{2}) : x \in \mathbb{R}$\}. Whenever $0 < x < 8^{-1/2}$, setting $b = (2x, \frac{1}{2})$, we have $\xi_K(b) = (\sqrt{x}, x)$; hence, $\xi_K$ is not differentiable at $a$ and $\delta_K$ is not pointwise differentiable of order 2 at $a$. Note also that $\xi_K$ is not even Lipschitz continuous in any neighbourhood of $a$. On the other hand 2.33[c] yields differentiability of $\delta_K$ at $a$ (which can also be checked by direct computation). We conclude $a \in \Sigma_2(K) \sim \Sigma(K)$. In 4.2 we prove that this is a generic situation for points in $\text{Cut}(K) \cap \text{Unp}(K)$.

4.2 Lemma. Assume $K \subseteq \mathbb{R}^n$ is closed, $x \in \mathbb{R}^n \sim K$, and $\delta^\phi_K$ is pointwise differentiable of order 2 at $x$.

Then $\rho^\phi_K(x) > 1$. In particular $\text{Cut}^\phi(K) \subseteq \Sigma_2^\phi(K)$.

Proof. Define $r = \delta^\phi_K(x)$, $\nu = \nu^\phi_K(x)$, $a = \xi^\phi_K(x)$, and $T = \mathbb{R}^n \cap \{v : \nu \cdot \text{grad} \delta^\phi_K(x)\}$. We use 2.33 to find $r > 0$ and a continuous function $f : T \to T^\perp$ which is pointwise twice differentiable at $T_x$ with $\text{D}(f(T_x)) = 0$ such that, defining $M = \{\chi + f(\chi) : \chi \in T\}$ and $U = \text{U}^\phi(x, r_1)$, it holds $U \cap S(K, r) = U \cap M$. Decreasing $r_1 > 0$ if necessary, we infer from the pointwise twice differentiability of $f$ in $T_x$ there exists a polynomial function $P : T \to T^\perp$ of degree at most 2 such that

\[ S(K, r) \cap U \subseteq \mathbb{R}^n \cap \{y : P(T_y) \cdot \text{grad} \delta^\phi_K(x) \geq y \cdot \text{grad} \delta^\phi_K(x)\}. \]

Decreasing $r_1 > 0$ even more, we can assume also that $U \sim S(K, r)$ is the union of two connected and disjointed open sets $U^-$ and $U^+$ such that

\[ \{\chi + P(\chi) : \chi \in T\} \cap U \subseteq \text{Clos}(U^+) \cap U. \]

Since $\text{U}^\phi(a, r) \cap S(K, r) = \emptyset$ we infer $U \cap \text{U}^\phi(a, r) \subseteq U^-$. Moreover, it follows from (15) that there exists $s > 0$ such that $\text{U}^\phi(x + s\nu, s) \subseteq U^+$ (notice $s < r_1$) and

\[ B^\phi(x + s\nu, s) \cap S(K, r) = \{x\}. \]

Choose $0 < \epsilon < \frac{r_1}{4}$. The continuity of $\xi^\phi_K$ and $\delta^\phi_K$ at $x$ implies that there exists $0 < \delta < \epsilon$ such that $\phi(b - a) < \epsilon$ and $\phi(b - y) < r + \epsilon$ for every $b \in \xi^\phi_K(y)$ and for every $y \in \text{U}^\phi(x, 2\delta)$. Define $y = x + \delta\nu$, choose $b \in \xi^\phi_K(y)$, and let $\tau = \sup\{t : 0 \leq t \leq 1, \phi(y + t(b - y) - a) > r\}$. Notice

\[ r - \epsilon \leq \phi(y + \tau(b - y) - a) - \phi(b - a) \leq \phi(y + \tau(b - y) - b) = (1 - \tau)\phi(b - y), \]

\[ \tau\phi(b - y) \leq \phi(b - y) - (r - \epsilon) \leq 2\epsilon, \]

\[ \phi(y + \tau(b - y) - x) \leq \phi(y - x) + \tau\phi(b - y) \leq \delta + 2\epsilon < r_1. \]

Therefore, $y + \tau(b - y) \in U \cap B^\phi(a, r) \subseteq \text{Clos}U^- \cap U$. Since $y \in U^+$ we infer there exists $0 < t \leq \tau$ such that $y + t(b - y) \in S(K, r)$. Defining $z = y + t(b - y)$ and noting that $\phi(z - b) = \phi(z - a)$ and $\phi(y - z) \geq \phi(y - x)$ by (16), we infer

\[ \phi(y - b) = \phi(y - z) + \phi(z - b) \geq \phi(y - x) + \phi(x - a) = \phi(y - a), \]

whence we conclude that $a \in \xi^\phi_K(y)$ and consequently $\rho^\phi_K(x) > 1$.

4.3 Remark. Since $\mathcal{L}^n(\Sigma_2^\phi(K)) = 0$ by the Alexandrov theorem [Ale39], it follows that

\[ \mathcal{L}^n(\text{Cut}^\phi(K)) = 0. \]

In a Riemannian setting a conclusion analogous to Lemma 4.2 is contained in [Alb15]. A proof of (17) along different lines can be found in the proof of [DRKS20] Theorem 5.9, Claim 1, see also [DRKS20] Remark 5.10.
4.4 Remark. The function $r^\phi_K$ is upper semi-continuous (cf. [RSS21, Remark 5.6]), but can fail to be continuous even if $K$ is a compact convex $C^{1,1}$ hypersurface. In fact in [San21] we show that there exists a compact and convex $C^{1,1}$-hypersurface $K$ such that $\text{Clos}(\Sigma(K))$ has non empty interior. Noting that $N(K)$ is the classical unit normal bundle of $K$ and consequently it is compact, we infer that if $r_K$ was continuous then $\text{Cut}(K)$ would be compact; consequently $\text{Clos}(\Sigma(K)) = \text{Cut}(K)$ and $\mathcal{L}^n(K) > 0$ which is incompatible with [17].

4.5 Remark. We notice that

$$r^\phi_K(a, \eta) = r\rho^\phi_K(a + r\eta) \quad \text{for} \quad (a, \eta) \in N^\phi(K) \text{ and any } 0 < r < r^\phi_K(a, \eta).$$

4.6 Lemma. Suppose $T \in \mathcal{G}(n, n-1)$, $f : T \to T^\perp$ is a function of class 2 such that $f(0) = 0$ and $Df(0) = 0$, $\Sigma = \{\chi + f(\chi) : \chi \in T\}$ and $\eta : \Sigma \to \mathbb{S}^{n-1}$ is a function of class 1 such that $\eta(x) \in \text{Nor}(\Sigma, x)$ for $x \in \Sigma$. Then

$$\text{D}\eta(0)(u) \cdot v = -D^2 f(0)(u, v) \cdot \eta(0) \quad \text{for} \quad u, v \in T.$$ 

Proof. Noting that $\eta(\chi + f(\chi)) \cdot (u + Df(\chi)(u)) = 0$ for $u \in T$ and $\chi \in T$, we differentiate this relation with respect to $\chi$ at 0.

4.7 Remark. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a Borel function and set $K_t = \mathbb{R}^n \cap \{y : \rho(y) \geq t\}$ for $t \in \mathbb{R}$. Recall the definition of the approximate lower limit [Fed69, 2.9.12]

$$\text{ap lim inf}_x \rho = \sup \mathbb{R} \cap \{t : \Theta^n(\mathcal{L}^n \mathbb{L}\{y : \rho(y) < t\}, x) = 0\}.$$ 

In particular, ap lim inf $\rho \geq \sigma \in \mathbb{R}$ if and only if

$$\Theta^n(\mathcal{L}^n \mathbb{L}\{K_t, x\}) = 1 \quad \text{whenever} \quad -\infty < t < \sigma.$$ 

4.8 Lemma. Suppose

$$K \subseteq \mathbb{R}^n \text{ is closed, } \quad x \in \mathbb{R}^n \sim K, \quad \delta^\phi_K \text{ is pointwise differentiable of order 2 at } x, \quad \quad T = \mathbb{R}^n \cap \{v : v \cdot \text{grad} \delta^\phi_K(x) = 0\}, \quad K_t = \mathbb{R}^n \cap \{y : \rho^\phi_K(y) \geq t\} \quad \text{for} \quad t \in \mathbb{R}, \quad \quad h_t(y) = ty + (1-t)\xi^\phi_K(y) \quad \text{for} \quad y \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$ 

Then the following statements hold.

(a) $1 \leq \rho^\phi_K(x) = \sigma \leq \lambda = \rho^\phi_K(x) > 1.$

(b) $\text{im D}\rho^\phi_K(x) \subseteq T.$

(c) $\text{D}\rho^\phi_K(x)(\nu^\phi_K(x)) = \text{D}\xi^\phi_K(x)(\nu^\phi_K(x)) = 0.$

(d) The eigenvalues $\chi_1 \leq \ldots \leq \chi_{n-1}$ of $\text{D}\rho^\phi_K(x)|T$ are real numbers such that

$$\frac{1}{(1-\lambda)\delta^\phi_K(x)} \leq \chi_i \leq \frac{1}{\delta^\phi_K(x)}.$$ 

(e) $\xi^\phi_K|K_{y/h}$ is differentiable at $h_t(x)$ whenever $1 \leq \gamma \leq \lambda$ and $0 < t < \gamma.$

(f) If $\sigma > 1$, then $\delta^\phi_K$ is pointwise differentiable of order 2 at $h_t(x)$ whenever $0 < t < \sigma.$

Proof. From 4.2 follows that $\lambda > 1$ and upper semi-continuity of $\rho^\phi_K$ (see 2.29) gives $\sigma \leq \lambda$; hence, (a) is true.

Notice that $\rho^\phi_K$ is differentiable at $x$ by 2.3(c). We choose a function $\xi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\xi(y) \in \xi^\phi_K(y)$ for $y \in \mathbb{R}^n$ and we define

$$\nu(y) = \frac{y - \xi(y)}{\delta^\phi_K(y)} \quad \text{and} \quad \eta(y) = \frac{\text{grad} \phi(y - \xi(y))}{|\text{grad} \phi(y - \xi(y))|} \quad \text{for} \quad y \in \mathbb{R}^n \sim K.$$
Employing \( \text{2.30(e)(c)} \) we notice that \( \eta \) is differentiable at \( x \),

\[
\eta(x) = \frac{\|\delta^\phi_K(x)\|}{\|\delta^\phi_K(x)\|}, \quad \text{and} \quad \text{Im} \eta(x) \subseteq T
\]
since \( |\eta(y)| = 1 \) for every \( y \in \mathbb{R}^n \sim K \). Moreover, we compute

\[
D\eta(x)u \cdot v = \frac{D(\phi \circ \nu)(x)u \cdot v}{\|\phi(\nu(x))\|} = \frac{\text{pt} D^2 \delta^\phi_K(x)(u, v)}{\|\delta^\phi_K(x)\|} \quad \text{for } u, v \in T;
\]
whence we conclude that \( D\eta(x)T \in \text{Hom}(T, T) \) is self-adjoint. Recalling \( \text{2.5} \) we notice that

\[
\text{grad} \phi^*(\eta(y)) = \text{grad} \phi^*(\text{grad} \phi(y - \xi(y))) = \text{grad} \phi^* \left( \text{grad} \phi \left( \frac{y - \xi(y)}{\delta^\phi_K(y)} \right) \right) = \nu(y)
\]
for \( y \in \mathbb{R}^n \sim K \). Henceforth,

\[
D\nu(x) = D\text{grad} \phi^*(\eta(x)) \circ D\eta(x).
\]
Since \( D\text{grad} \phi^*(v)v = 0 \) and \( D\text{grad} \phi^*(v) \) is self-adjoint for \( v \in \mathbb{R}^n \sim \{0\} \), we conclude that

\[
D\text{grad} \phi^*(v)u \cdot v = u \cdot D\text{grad} \phi^*(v)v = 0 \quad \text{for } u, v \in \mathbb{R}^n, v \neq 0,
\]
whence we deduce that \( \text{Im} D\text{grad} \phi^*(v) \subseteq \{ u : u \cdot v = 0 \} \) for \( v \neq 0 \), \( \text{Im} D\nu(x) \subseteq T \), and \( D\text{grad} \phi^*(\eta(x))T \in \text{Hom}(T, T) \) is a positive definite self-adjoint linear map. In particular, we established \( \text{[b]} \) and, moreover, it follows from \( \text{[20]} \) and \( \text{DRKS20, 2.25} \) that the eigenvalues of \( D\nu(x)T \) are real numbers.

To prove \( \text{[c]} \) we notice by \( \text{2.11} \) that the equations

\[
\xi(\xi(x) + t(x - \xi(x))) = \xi(x) \quad \text{and} \quad \nu(\xi(x) + t(x - \xi(x))) = \nu(x)
\]
hold for \( 0 < t \leq 1 \) and we differentiate with respect to \( t \) at \( t = 1 \).

We now check the estimate claimed in \( \text{[d]} \) for the eigenvalues of \( D\nu^R_K(x)T \). Assume \( x = 0 \) and \( \lambda = \rho^R_K(0) > 1 \). Define

\[
r = \delta^\phi_K(0), \quad W_1 = U^\phi(\xi(0), r), \quad W_2 = U^\phi(\xi(0) - \lambda \xi(0), (\lambda - 1)r),
\]
and observe that \( W_1 \subseteq \{ y : \delta^\phi_K(y) < r \}, W_2 \subseteq \{ y : \delta^\phi_K(y) > r \} \) and \( 0 \in \partial W_1 \cap \partial W_2 \cap S(K, r) \). Notice that \( \text{Tan}(S(K, r), 0) = \text{Tan}(\partial W_1, 0) = \text{Tan}(\partial W_2, 0) \). Using \( \text{2.33} \) we find an open set \( V \) containing \( 0 \) and three functions \( f : \tilde{T} \to \tilde{T}, f_1 : T \to \tilde{T} \) and \( f_2 : T \to \tilde{T} \) such that \( f_1 \) and \( f_2 \) are of class \( \mathcal{C}^2 \), \( f \) is pointwise differentiable of order 2 at 0, \( f(0) = f_1(0) = f_2(0) = 0 \), \( Df(0) = Df_1(0) = Df_2(0) = 0 \),

\[
V \cap S(K, r) = V \cap \{ \chi + f(\chi) : \chi \in \tilde{T} \}, \quad V \cap \partial W_i = V \cap \{ \chi + f_i(\chi) : \chi \in \tilde{T} \} \quad \text{for } i \in \{1, 2\},
\]
and \( f_i(\chi) \cdot \eta(0) \leq f(\chi) \cdot \eta(0) \leq f_2(\chi) \cdot \eta(0) \) for \( \chi \in T \). In particular,

\[
D^2 f_1(0) \cdot \eta(0)(u, u) \leq \text{pt} D^2 f(0) \cdot \eta(0)(u, u) \leq D^2 f_2(0) \cdot \eta(0)(u, u) \quad \text{for } u \in T.
\]

Let \( \eta_i : \partial W_i \to S^{n-1} \) be the exterior unit normal function of \( W_i \) for \( i \in \{1, 2\} \). Then \( \eta_0(0) = \eta(0) = -\eta_2(0) \) and we use \( \text{2.33} \) in combination with \( \text{[18], [19], and [1.6]} \) to infer

\[
D\eta_1(0)u \cdot u = D\eta_2(0)u \cdot u \leq \text{D} f_1(0)u \cdot u = D\eta_1(0)u \cdot u \quad \text{for } u \in T.
\]
To conclude we use the argument from the third paragraph of the proof of [DRKS20, 2.34]. First, we find a positive definite self-adjoint map \( C \in \text{Hom}(T, T) \) such that \( \text{D\,grad}\,(\phi(0))|T = C \circ C \); then we observe, using [DRKS20, 2.33], that

\[
C \circ \text{D}\eta_2(0) \circ C = C^{-1} \circ \text{D\,grad}\,(\phi(0)) \circ \text{D}\eta_2(0) \circ C = (\rho^K(\nu) - 1)^{-1}r^{-1}1_T,
\]

whence we deduce, employing (21)

\[
r^{-1}|u|^2 = (C \circ \text{D}\eta_1(0) \circ C)u \cdot u = (\text{D}\eta_1(0) \circ C)u \cdot C(u)
\]

\[
\geq (\text{D}\eta_1(0) \circ C)u \cdot C(u) \geq -(\text{D}\eta_2(0) \circ C)u \cdot C(u)
\]

\[
= -(C \circ \text{D}\eta_2(0) \circ C)(u) \cdot u = (1 - \rho^K(\nu))^{-1}r^{-1}|u|^2 \quad \text{for } u \in T.
\]

Noting that \( C \circ \text{D}\eta(0) \circ C \) is a self adjoint map with the same eigenvalues as \( \text{D}\nu(0) = \text{D\,grad}\,(\phi(0)) \circ \text{D}\eta(0) \) we finally obtain the estimate in [d].

We turn to the proof of (e). If \( h_t(x) = x \). Assume \( 1 \leq \gamma < \lambda \), \( 0 < t < \gamma \), and \( t \neq 1 \). Note that \( h_t[K_\gamma \sim K] = K_{\gamma/t} \). Employing 2.11 we find that \( h_t[K_{\gamma} \sim K]^{-1} = 1 \), \( \text{D}\nu(0) = 1 \), and \( \text{D}\nu(0) = 1 \). Note that

\[
\eta_t^{-1}(z) = t^{-1}z + (1 - t^{-1})\xi^K_t(z) = h_{1/t}(z) \quad \text{for } z \in K_{\gamma/t}.
\]

For each \( 0 < s < \infty \) the multivalued map \( h_t \) is weakly continuous by 2.30(e) and is differentiable at \( x \) by 2.30(f). Moreover, \( \text{D}h_t(x) \) is an isomorphism of \( \mathbb{R}^n \). Indeed, if there was \( v \neq 0 \) with \( \text{D}h_t(x)v = 0 \), then it would follow that

\[
\frac{t}{t-1}v = \text{D}\xi^K_t(x)v
\]

and, noting that \( \nu^K_t(x) \cdot \text{grad}\,\delta^K_t(x) = \phi(\nu^K_t(x)) = 1 \) and \( v = w + \kappa \nu^K_t(x) \) for some \( w \in T \) and \( \kappa \in \mathbb{R} \), we could employ [c] to compute

\[
\frac{t}{t-1}(\kappa \nu^K_t(x) + w) = \text{D}\xi^K_t(x)w = w - \delta^K_t(x) \text{D}\nu^K_t(x)w,
\]

whence we would deduce from [b] that \( \kappa = 0, w \neq 0 \) and

\[
\text{D}h_t(x)w = (\delta^K_t(x)(1-t))^{-1}w,
\]

which would contradict one of the estimates in [d]. In case \( 0 < t < 1 \), we have \( (1-t)^{-1} > 1 \) and if \( 1 < t < \gamma < \lambda \), then \( (1-t)^{-1} < (1-\lambda)^{-1} \). Now, we apply [Zaj83b, Lemma 2] to conclude that \( h_t^{-1} \) is differentiable at \( h_t(x) \). Noting that \( \xi^K_t(z) = (1-t)^{-1}(z - t h_t^{-1}(z)) \) for \( z \in h_t[K_{\gamma} \sim K] = K_{\gamma/t} \), finishes the proof of (e).

Next, we prove (f). Let \( 0 < t < \sigma \) and \( 0 < \varepsilon < 2 \delta^K_t(x) \min(1, t) \). By 2.30(e) it suffices to show that \( \xi^K_t \) is differentiable at \( h_t(x) \). Recalling 3.10 we see that \( h_t[K_{\gamma} \cap B^\phi(x, \varepsilon)] \) is a bilipschitz map whose image equals \( K_{\gamma/t} \) for \( t \leq \gamma < \sigma \). From 1.7 we have

\[
\Theta^n(\mathcal{L}^n \cap K_{\gamma}, x) = 1 \quad \text{whenever } t \leq \gamma < \sigma.
\]

Employing [Buc92] we obtain

\[
\Theta^n(\mathcal{L}^n \cap K_{\gamma/t}, h_t(x)) = 1 \quad \text{whenever } t \leq \gamma < \sigma.
\]

Fix \( t < \gamma < \sigma \). From [e] we know that \( \xi^K_t[K_{\gamma/t}] \) is differentiable at \( h_t(x) \). Recalling 3.10 we may apply 2.24 with \( h_t(x) \) and \( \xi^K_t[K_{\gamma/t} \cap B^\phi(h_t(x), \varepsilon)] \) in place of \( a \) and \( f \) which completes the proof.

Keeping in mind Remark 4.5 we introduce the following definition.
4.9 Definition. Whenever $x \in \mathbb{R}^n \sim K$, $(a, \eta) \in N^\phi(K)$, and $0 < r < r^\phi_K(a, \eta)$ we set

$$
\rho^\phi_K(x) = \text{ap lim inf}_{y \to x} \rho^\phi_K(y) \quad \text{and} \quad r^\phi_K(a, \eta) = r\rho^\phi_K(a + r\eta).
$$

4.10 Remark. The definition of $r^\phi_K(a, \eta)$ is well posed for each $(a, \eta) \in N^\phi(K)$. In fact, firstly notice that $\rho^\phi_K(x) = t\rho^\phi_K(tx + (1-t)\xi^\phi_K(x))$ for each $x \in \text{Unp}^\phi(K)$ and $0 < t < 1$. Continuity of $\xi^\phi_K$ then yields

$$
\rho^\phi_K(x) = t\rho^\phi_K(tx + (1-t)\xi^\phi_K(x)) \quad \text{for} \quad x \in \text{Unp}^\phi(K) \quad \text{and} \quad 0 < t < 1.
$$

Consequently, if $(a, \eta) \in N^\phi(K)$ and $0 < s < r < r^\phi_K(a, \eta)$ then, writing $a + sn = a + s((a + r\eta) - a)$, we infer that $\frac{s}{r}\rho^\phi_K(a + sn) = \rho^\phi_K(a + r\eta)$. Moreover, $\rho^\phi_K$ is $\mathcal{L}^n$ measurable by [Fed69, 3.1.3(2)].

4.11 Remark. Notice that $1 \leq \rho^\phi_K(x) \leq r^\phi_K(x)$ for all $x \in \mathbb{R}^n \sim K$ and $0 < r^\phi_K(a, \eta) \leq r^\phi_K(a, \eta)$ for all $(a, \eta) \in N^\phi(K)$.

4.12 Remark. If $K$ is a convex body then $r^\phi_K(a, \eta) = r^\phi_K(a, \eta) = +\infty$ for all $(a, \eta) \in N^\phi(K)$.

Proof of [4.9]. First notice that item (a) follows from 4.2. Define $R_+ = \{t : 0 < t < \infty\}$ and for $t \in R_+$ the function $f_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by the formula $f_t(a, \eta) = a + t\eta$ whenever $a, \eta \in \mathbb{R}^n$. Next, define $g : \text{Unp}^\phi(K) \to N^\phi(K)$ by $g(x) = (\xi^\phi_K(x), r^\phi_K(x))$ for $x \in \text{Unp}^\phi(K)$. For each $\lambda > 1$ set $K_\lambda = \{x : \rho^\phi_K(x) \geq \lambda\} \subseteq \text{Unp}^\phi K$ and note that $\text{Unp}^\phi K \sim \text{Cut}^\phi(K) = \bigcup_{\lambda > 1} K_\lambda$. Moreover, Corollary 3.10 yields

$$
g[K_\lambda \cap S(K, t)] \text{ is Lipschitz continuous for each } t > 0 \text{ and } \lambda > 1.\]

Moreover, for every $t \in R_+$

$$N^\phi(K) \cap \{(a, \eta) : t < r^\phi_K(a, \eta)\} = \bigcup_{\lambda > 1} g[K_\lambda \cap S(K, t)].$$

We turn to the proof of (b). Set $B = \{(a, \eta) \in \mathbb{R}^n \sim K \cap \{x : \rho^\phi_K(x) < \rho^\phi_K(a, \eta)\}$. Since $\rho^\phi_K$ is a Borel function by 2.29 and $\rho^\phi_K$ is $\mathcal{L}^n$ measurable by 4.10, we see that $B$ is $\mathcal{L}^n$ measurable and it follows from [Fed69, 2.9.13] that $\mathcal{L}^n(B) = 0$; hence, the coarea formula [Fed69, 3.2.11] yields a set $J \subseteq R_+$ such that $\mathcal{L}^1[R_+ \sim J] = 0$ and

$$\mathcal{H}^{n-1}(B \cap S(K, t)) = 0 \quad \text{for} \quad t \in J.$$ 

For $t \in R_+$ define

$$W = N^\phi(K) \cap \{(a, \eta) : r^\phi_K(a, \eta) < r^\phi_K(a, \eta)\} \quad \text{and} \quad W_t = W \cap \{(a, \eta) : t < r^\phi_K(a, \eta)\}.$$ 

Since $f_t[W_t] \subseteq B \cap S(K, t)$ for $t \in R_+$ by 4.5, it follows that

$$\mathcal{H}^{n-1}(f_t[W_t]) = 0 \quad \text{for} \quad t \in J.$$ 

Since $g[K_\lambda \cap f_t[W_t]] = g[K_\lambda \cap S(K, t)] \cap W_t$ for $\lambda > 1$ and $t \in R_+$, we conclude from (23) and (22) that

$$\mathcal{H}^{n-1}(g[K_\lambda \cap S(K, t)] \cap W_t) = 0 \quad \text{for each} \quad \lambda > 1 \text{ and } t \in J.$$ 

Since $W_t = \bigcup_{\lambda > 1} W_t$ and $W_t \subseteq W_s$ for $s \leq t$, we infer $\mathcal{H}^{n-1}(W) = 0$.

Next, we prove (c). Observe that

$$Q \sim \text{Cut}^\phi(K) = \{a + r\eta : (a, \eta) \in N^\phi(K), r^\phi_K(a, \eta) \leq r < r^\phi_K(a, \eta)\} \subseteq B;$$

hence, $\mathcal{L}^n(Q \sim \text{Cut}^\phi(K)) = 0$. Recalling Remark 4.3 yields $\mathcal{L}^n(Q) = 0$. 

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Proof of (b) Define 
\[ Z = N^\phi(K) \cap \{ (a, \eta) : a + s\eta \in \Sigma^\phi_2(K) \text{ for all } 0 < s < r^\phi_K(a, \eta) \}. \]
If \((a, \eta) \in N^\phi(K) \sim Z\), then there exists \(0 < s < r^\phi_K(a, \eta)\) such that \(\delta^\phi_K\) is pointwise twice differentiable at \(a + s\eta\). Since \(r^\phi_K(a + s\eta) = s^{-1}r^\phi_K(a, \eta) > 1\) we infer from \(4.4[1]\) that \(\delta^\phi_K\) is pointwise twice differentiable at \(a + t\eta\) for all \(0 < t < r^\phi_K(a, \eta)\).

Consequently, it remains to show that \(\mathcal{H}^{n-1}(Z) = 0\). This can be done with an argument similar as in the proof of (b). Since \(\mathcal{L}^n(\Sigma^\phi_2(K)) = 0\) it follows from coarea formula \[\text{Fed69}, 3.2.11\] that there is \(I \subseteq \mathbb{R}_+\) such that \(\mathcal{L}^1(\mathbb{R}_+ \sim I) = 0\) and 
\[ \mathcal{H}^{n-1}(\Sigma^\phi_2(K) \cap S(K, t)) = 0 \quad \text{for } t \in I. \]
We define \(Z_t = Z \cap \{ (a, \eta) : t < r^\phi_K(a, \eta) \}\) for \(t \in \mathbb{R}_+\). Noting that \(f_1[Z_t] \subseteq \Sigma^\phi_2(K) \cap S(K, t)\) for \(t > 0\), we infer that 
\[ \mathcal{H}^{n-1}(f_1[Z_t]) = 0 \quad \text{for } t \in I. \]
Since \(g[K \cap f_1[Z_t]] = g[K \cap S(K, t)] \cap Z_t\) for \(\lambda > 1\) and \(t > 0\), we conclude that 
\[ \mathcal{H}^{n-1}(g[K \cap S(K, t)] \cap Z_t) = 0 \quad \text{for } \lambda > 1 \text{ and } t \in I. \]
It follows that \(\mathcal{H}^{n-1}(Z_t) = 0\) for \(\mathcal{L}^1\) almost all \(t > 0\). Noting that \(Z = \bigcup_{t > 0} Z_t\) and \(Z_t \subseteq Z_s\) for \(s \leq t\), we see that \(\mathcal{H}^{n-1}(Z) = 0\). \(\blacksquare\)

4.13 Remark. Here we construct a closed set \(K \subseteq \mathbb{R}^2\) such that \(Q \sim \text{Cut}(K)\) contains a half-line. Let \((e_1, e_2)\) be the standard basis of \(\mathbb{R}^2\) and set \(x_k = 2^{-k}\) for \(k \in \mathcal{P}\). Then 
\[ \frac{x_k - x_{k+1}}{x_k} = \frac{1}{2} = \frac{x_{k+1}}{x_k} \quad \text{for } k \in \mathcal{P}. \]
Define 
\[ K = \{ x_ke_1 : k \in \mathcal{P} \} \cup \{ 0 \} \subseteq \mathbb{R}^2. \]
Let \(a > 0\). Evidently \(\rho_K(ae_2) = \infty\) and \((0, e_2) \in N(K)\). We shall prove that \(\rho_K(ae_2) < \infty\) and infer that 
\[ r_K(0, e_2) = \infty > r_K(0, e_2). \]
Define \(c_k = \frac{x_k + x_{k+1}}{2}\) to be the centre of the segment in \(\mathbb{R}\) joining \(x_{k+1}\) and \(x_k\). Moreover, let \(Q_k\) be the 2-dimensional square with centre in \(c_ke_1 + ae_2\) and side-length \(\frac{x_k - x_{k+1}}{2} = 2^{-k-2}\) and let \(T_k\) be the 2-dimensional triangle with vertices in \(x_{k+1}e_1, x_ke_1,\) and \(c_ke_1 + 2ae_2\). Clearly, there exists \(k_0 \in \mathcal{P}\) depending only on \(a\) such that \(Q_k \subseteq T_k\) for \(k \geq k_0\) and \(Q_k \subseteq \mathcal{P}\). Letting 
\[ \{ c_ke_1 : k \geq 1 \} \times \mathbb{R} = \Sigma(K), \]
one infers that there is \(k_1 \geq k_0\) depending only on \(a\) such that \(k \geq k_1\) there holds 
\[ \delta_K(z)\rho_K(z) \leq |c_ke_1 + 2ae_2 - x_ke_1| \leq 3a \quad \text{for } k \geq k_1 \text{ and } z \in T_k, \]
\[ \delta_K(z) \geq a - \frac{x_k - x_{k+1}}{4} = a - 2^{-k-3} \geq \frac{1}{2}a \quad \text{for } k \geq k_1 \text{ and } z \in Q_k. \]
Consequently, 
\[ \rho_K(z) \leq 6 \quad \text{for } k \geq k_1 \text{ and } z \in Q_k. \]
Now, for \(k \geq k_1\) we get 
\[ x_k^{-2} \mathcal{L}^2(\mathbb{R}^2 \cap \{ z : |z \cdot e_1| \leq x_k, \ |(z \cdot e_2) - a| \leq x_k, \ \rho_K(z) \leq 6 \}) \]
\[ \geq x_k^{-2} \sum_{h \geq k} \mathcal{L}^2(Q_h) = 2^{2k} \sum_{h \geq k} 2^{-2h-4} = \frac{1}{12}. \]
and conclude that $\Theta^2(\mathcal{L}^2\{z: \rho_K(z) \leq 6\}, \alpha e_2) > 0$; hence, $\rho_K(\alpha e_2) \leq 6$.

Finally, we prove that $\xi$ is not differentiable at $ae_2$ for each $a > 0$. Assume that there exists $L = D|\xi| : \text{Hom}(\mathbb{R}^2, \mathbb{R}^2)$. Since $\xi(\alpha e_2) = 0$ and $\xi(\alpha e_2 + xe_1) = x e_1$ for $k \in \mathcal{P}$ it must be $Le_1 = e_1$. However, $\xi(\alpha e_2 + xe_1) = \{xe_1, xe_{k+1}\}$ for $k \in \mathcal{P}$ and

$$\frac{x_k - c_k}{c_k} = \frac{2^{-k-2}}{3} > 0.$$ 

4.14 Remark. The closed convex set constructed in Theorem 5.1 shows that $P \cap \Sigma_1^2(K)$ is a set of Hausdorff dimension $\eta$ (though the $\eta$ dimensional Lebesgue measure is zero by Alexandrov theorem). This proves that the statement (d) in Theorem 1.4 is sharp. Moreover, the closed sets constructed in Remark 4.13 show that $H^{n-1}(Q \sim \text{Cut}^\phi(K))$ might be infinity. Therefore, the following question naturally arises.

Is there a closed set $K \subseteq \mathbb{R}^n$ such that $\mathcal{H}^s(Q \sim \text{Cut}^\phi(K)) = \infty$ for each $s < n$?

For the closed sets constructed in Remark 4.13 we always have $Q \subseteq \Sigma_2^\phi(K)$. Consequently we ask

is there a closed set $K \subseteq \mathbb{R}^n$ such that $Q \sim \Sigma_2^\phi(K) \not= \emptyset$?

5 An example

5.1 Theorem. There exists a convex set $K \subseteq \mathbb{R}^2$ with $\mathcal{C}^{1,1}$ boundary and a dense set $S \subseteq \partial K$ such that

(a) $\mathcal{H}^1(S) = 0$ and $\mathcal{H}^s(S) = +\infty$ for all $s < 1$,

(b) $\mathcal{H}^1(N(K)|S) = 0$ and $\mathcal{H}^s(N(K)|S) = +\infty$ for all $s < 1$,

(c) $\xi_K|S(K,r)$ is not differentiable at $a + ru$ for all $r > 0$ and $(a, u) \in N(K)|S$.

In particular, both $\Sigma_2(K)$ and the set of points where $\delta_K$ is not directionally differentiable are dense in $\partial K$ with Hausdorff dimension 2.

Proof. We choose a dense $G_\delta$-set $G \subseteq \mathbb{R}$ with $\mathcal{L}^1(G) = 0$ and $\mathcal{H}^s(G) = +\infty$ for all $s < 1$. Then Zahorski theorem (see [Zah46] or [FP09]) ensures the existence of non-decreasing Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that $f$ is not differentiable at each point of $G$ and it is differentiable at each point of $\mathbb{R} - G$. Let $g$ be a primitive of $f$ and notice that $g$ is a $\mathcal{C}^{1,1}$ function. Let

$$K = (\mathbb{R} \times \mathbb{R}) \cap \{xy: g(x) \leq y\}, \quad S = (G \times \mathbb{R}) \cap \{xy: g(x) = y\}.$$ 

Evidently $K$ is a convex set with $\mathcal{C}^{1,1}$ boundary, $\mathcal{H}^1(S) = 0$ and $\mathcal{H}^s(S) = +\infty$ for every $s < 1$. If $\eta : \partial K \to S^1$ is the exterior unit normal of $K$ then

$$\eta(x, f(x)) = \frac{1}{\sqrt{1 + f(x)^2}}(f(x), -1) \quad \text{for } x \in \mathbb{R},$$

whence we infer that $\eta$ is not differentiable at each point of $S$ and it is differentiable at all points of $\partial K \sim S$. Moreover, we notice that the map $\phi : \partial K \to N(K)$, defined by $\phi(z) = (z, \eta(z))$ for $x \in \partial K$, is a bilipschitz homeomorphism and $N(K)|S = \phi(S)$. Therefore, $[b]$ holds.

To check $[c]$ we assume that $\xi_K|S(K,r)$ is differentiable at $a + ru$ for some $r > 0$ and $(a, u) \in N(K)|S$. We notice that $\xi_K|S(K,r)$ is a bilipschitz homeomorphism onto $\partial K$ with

$$(\xi_K|S(K,r))^{-1}(b) = b + ru \in \partial K.$$ 

Therefore, $D(\xi_K|S(K,r))(a + ru) : \text{Tan}(S(K,r), a + ru) \to \text{Tan}(\partial K, a)$ is a linear homeomorphism and $(\xi_K|S(K,r))^{-1}$ is differentiable at $a$. This contradicts the fact that $\eta$ is not differentiable at $a$. \qed
Declarations

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