EXACT EXPONENTIAL TAIL ESTIMATIONS
IN THE LAW OF ITERATED LOGARITHM
FOR BOCHNER’S MIXED LEBESGUE SPACES

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Abstract.

We obtain the quite exact exponential bounds for tails of distributions of sums
of Banach space valued random variables uniformly over the number of summands
under natural for the Law of Iterated Logarithm (LIL) norming.

We study especially the case of the so-called mixed (anisotropic) Lebesgue-Riesz
spaces, on the other words, Bochner’s spaces, for instance, continuous-Lebesgue
spaces, which appear for example in the investigation of non-linear Partial Differen-
tial Equations of evolutionary type.

We give also some examples in order to show the exactness of our estimates.

\textit{Key words and phrases:} Law of Iterated Logarithm (LIL), Central Limit Theo-
rem (CLT), mixed (anisotropic) Lebesgue-Riesz spaces, norms, continuous-Lebesgue
spaces, partition, Hölder’s inequality, examples, Rosenthal constants and inequalities,
exponential upper tail estimates, triangle (Minkowsky) inequality, estimates,
moments, stationary sequences, superstrong mixingale, martingale.

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1 Notations. Statement of problem.

1. Let \((B, \| \cdot \| B)\) be separable non - trivial: \text{dim}(B) \geq 1 Banach space and \(\{\xi_j\}, \xi = \xi_1, \ j = 1, 2, \ldots\) be a sequence of centered in the weak sense: \(E(\xi_i, b) = 0 \ \forall b \in B^*\)
of independent identical distributed (i. i.d.) random variables (r.v.) (or equally
random vectors, with at the same abbreviation r.v.) defined on some non-trivial
probability space \((\Omega = \{\omega\}, F, \mathbf{P})\) with values in the space \(B\). Denote
We suppose also that the r.v. \( \xi \) has a weak second moment:

\[
\forall b \in B^* \Rightarrow (Rb, b) := \mathbf{E}(\xi, b)^2 < \infty.
\]

Let also \( v = v(n) \) be arbitrary non-random positive strictly increasing numerical sequence such that

\[
v(1) = 1, \quad \lim_{n \to \infty} v(n) = \infty,
\]

for instance,

\[
v_r(n) := \left[ \log(\log(n + e^e) - 1) \right]^r, \quad r = \text{const} \geq 1/2.
\]

We set

\[
\tau(n) = \frac{S(n)}{\sqrt{n} v(n)}, \quad \tau_r(n) = \frac{S(n)}{\sqrt{n} v_r(n)},
\]

\[
Q(u) = Q^{(\xi)}(u) := \mathbf{P}(\sup_n ||\tau|| > u),
\]

\[
Q_r(u) = Q^{(\xi)}_r(u) := \mathbf{P}(\sup_n ||\tau_r|| > u), \quad u = \text{const} \geq e.
\]

Our purpose in this article is obtaining some sufficient conditions for exponential decreasing as \( u \to \infty \) of the probabilities \( Q(u), Q_r(u) \), for example, of a view

\[
Q(u) \leq \exp \left(-C u^{\beta_1} \log^{\beta_2}(u)\right), \quad u > e, \quad C, \beta_{1,2} = \text{const}, \quad C, \beta_1 > 0.
\]

2. The problem of describing of necessary (sufficient) conditions for the infinite-dimensional LIL and closely related CLT in Banach space \( B \) has a long history; see, for instance, the monographs [22] - [31] and articles [33] - [47] (CLT); [56] - [62] (LIL); see also reference therein.

The case when \( B = C(T) \), where \( T \) is metrizable compact set, is considered in fact in [62], see also [31], chapter 2, section 2.7.

The applications of considered theorem in statistics and method Monte-Carlo see, e.g. in [48] - [52].

3. The lower bound for the probability \( Q(u) \) is very simple: as long as \( v(1) = 1 \),

\[
Q(u) \geq \mathbf{P}(\tau(1) > u) = \mathbf{P}(||\xi|| > u).
\]

Therefore, if we want to establish the inequality (1.7), we must adopt the condition

\[
\mathbf{P}(||\xi|| > u) \leq \exp \left(-C u^{\beta_1} \log^{\beta_2}(u)\right), \quad u > e, \quad C, \beta_{1,2} = \text{const}, \quad C, \beta_1 > 0.
\]
4. We recall here the definition and some important properties of the so-called anisotropic Lebesgue (Lebesgue-Riesz) spaces, which appeared in the famous article of Benedek A. and Panzone R. [2]. More detail information about this spaces with described applications see in the books of Besov O.V., Ilin V.P., Nikolskii S.M. [3], chapter 1,2; Leoni G. [11], chapter 11; [12], chapter 6.

Let $(X_k, A_k, \mu_k)$, $k = 1, 2, \ldots, l$ be measurable spaces with sigma-finite separable non-trivial measures $\mu_k$. The separability denotes that the metric space $A_k$ relative the distance

$$
\rho_k(D_1, D_2) = \mu_k(D_1 \Delta D_2) = \mu_k(D_1 \setminus D_2) + \mu_k(D_2 \setminus D_1)
$$

is separable.

Let also $p = (p_1, p_2, \ldots, p_l)$ be $l-$ dimensional vector such that $1 \leq p_j < \infty$.

Recall that the anisotropic (mixed) Lebesgue - Riesz space $L_p^\rho$ consists on all the total measurable real valued function $f = f(x_1, x_2, \ldots, x_l) = f(\vec{x})$

$$
f : \otimes_{k=1}^l X_k \to R
$$

with finite norm $|f|_p \overset{def}{=} \left( \int_{X_1} \mu_1(dx_1) \left( \int_{X_2} \mu_2(dx_2) \cdots \left( \int_{X_l} |f(\vec{x})|^{p_1} \mu_1(dx_1) \right)^{p_2/p_1} \cdots \right)^{1/p_l} \right)^{1/p_l}.$ 

(1.11)

In particular, for the r.v. $\xi$

$$
|\xi|_p = [E|\xi|^p]^{1/p}, \; p \geq 1.
$$

Note that in general case $|f|_{p_1, p_2} \neq |f|_{p_2, p_1}$, but $|f|_{p,p} = |f|_p$.

Observe also that if $f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2)$ (condition of factorization), then $|f|_{p_1, p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2}$, (formula of factorization).

Note that under conditions of separability (1.6) on the measures $\{\mu_k\}$ this spaces are also separable Banach spaces.

These spaces arise in the Theory of Approximation, Functional Analysis, theory of Partial Differential Equations, theory of Random Processes etc.

5. Let for example $l = 2$; we agree to rewrite for clarity the expression for $|f|_{p_1, p_2}$ as follows:

$$
|f|_{p_1, p_2} := |f|_{p_1, X_1; p_2, X_2}.
$$

Analogously

$$
|f|_{p_1, p_2, p_3} = |f|_{p_1, X_1; p_2, X_2; p_3, X_3}.
$$

(1.12)

Let us give an example. Let $\eta = \eta(x, \omega)$ be bi-measurable random field, $(X = \{x\}, A, \mu)$ be measurable space, $p = \text{const} \in [1, \infty)$. As long as the expectation $E$ is also an integral, we deduce

$$
E|\eta(\cdot, \cdot)|_{p,X}^p = E \int_X |\eta(x, \cdot)|^p \mu(dx) =
$$
\[
\int_X E|\eta(x, \cdot)|^p \, \mu(dx) = \int_X \mu(dx) \left[ \int_\Omega |\eta(x, \omega)|^p \, P(d\omega) \right];
\]

\[
|\eta|_{p,X} |\eta|_{p,\Omega} = \left\{ \int_X \mu(dx) \left[ \int_\Omega |\eta(x, \omega)|^p \, P(d\omega) \right] \right\}^{1/p} = |\eta(\cdot, \cdot)|_{1,X,p,\Omega}. \tag{1.13}
\]

6. Constants of Rosenthal-Dharmadhikari-Jogdeo- ...

Let \( p = \text{const} \geq 1 \), \( \{\zeta_k\} \) be a sequence of numerical centered, i. i. d. r.v. with finite \( p^{th} \) moment \(|\zeta|_p < \infty\). The following constant, more precisely, function on \( p \), is called constants of Rosenthal-Dharmadhikari-Jogdeo-Johnson-Schechtman-Zinn-Latala-Ibragimov-Pinelis-Sharachmedov-Talagrand-Utev...:

\[
K_R(p) \overset{\text{def}}{=} \sup_{n \geq 1} \sup_{\{\zeta\}} \left[ \frac{\left| n^{-1/2} \sum_{k=1}^n \zeta_k \right|_p}{|\zeta|_p} \right]. \tag{1.14}
\]

We will use the following ultimate up to an error value \( 0.5 \cdot 10^{-5} \) for the constant \( C_R \) estimate for \( K_R(p) \), see [18] and reference therein:

\[
K_R(p) \leq \frac{C_R p}{e \cdot \log p}, \quad C_R = \text{const} := 1.77638. \tag{1.15}
\]

Note that for the symmetrical distributed r.v. \( \zeta \) the constant \( C_R \) may be reduced up to a value 1.53572.

7. Suppose the sequence \( \{\xi_i\} \) satisfies the LIL in the classical statement under our norming function:

\[
\lim_{n \to \infty} \frac{||S(n)||}{\sqrt{n} v(n)} =: \zeta = \zeta(\omega),
\]

where \( P(0 < \zeta < \infty) = 1 \) and wherein the r.v. \( \zeta \) may be non-random constant. Then

\[
\sup_n \frac{||S(n)||}{\sqrt{n} v(n)} < \infty
\]

almost everywhere and following

\[
\lim_{u \to \infty} Q(u) = 0.
\]

8. The natural norming sequence in Banach space valued LIL may be essentially greatest as the classical sequence \( v_{1/2}(n) = \sqrt{n} \left( \log(\log(n + e^e - 1))^2 \right) \). Namely, in the article [59] there is (for any number value \( r > 1/2 \)) an example of a separable Banach space \( B, || \cdot || \) and a sequence of i.i.d. centered r.v. \( \{\xi_i\} \) with finite weak second moment such that

\[
0 < \lim_{n \to \infty} \left[ \frac{||S(n)||}{\sqrt{n} \left( \log(\log(n + e^e - 1))^2 \right)} \right] < \infty \tag{1.16}
\]

almost surely.
2 Main result: exponential estimates for LIL in ordinary Lebesgue - Riesz spaces.

We study in this section the exponential tail estimates for the normed sums of centered, i.i.d. r.v. in the separable Banach space $L_p$, $2 \leq p < \infty$, having the weak second moment.

In detail, $\xi(x) = \xi(x, \omega)$ be bi - measurable centered random field, $(X = \{x\}, A, \mu)$ be measurable space with separable sigma - finite measure $\mu$, and $\{\xi_k(x, \omega)\} = \{\xi_k\}$ be independent copies of $\xi = \xi(x) := \xi(x, \omega)$.

So, in this section

$$||\xi|| = ||\xi(\cdot, \cdot)|| = |\xi|_p = \sqrt{\int_X |\xi(x)|^p \mu(dx)} = |\xi|_{p, X}. \quad (2.0)$$

Some additional notations. $m = \text{const} \geq 1, L = \text{const} = pm \geq p$,

$$\Sigma_n(x) := n^{-1/2} \sum_{k=1}^n \xi_k(x) = n^{-1/2} S(n;x), \quad (2.1)$$

$$g(L) = g_p(L) = 2 K_R(L) \cdot \sqrt{\left[ \int_X E|\xi(x)|^L \mu(dx) \right]^{p/L}}, \quad (2.2)$$

$$L_0 = L_0(p) = \sup\{L : g_p(L) < \infty\}, \quad (2.3)$$

$$h(z) = h_p(z) := \inf_{L \in (p, L_0)} \left\{ \frac{g_L(L)}{z^L} \right\}, \quad z > 1. \quad (2.4)$$

We clarify the role of a function $h_p = h_p(z)$ : if for the real valued r.v. $\zeta$

$$\left[ E|\zeta|^L \right]^{1/L} \leq g(L),$$

then it follows from Tchebychev-Markov inequality after optimisation over $L :$

$$P(|\zeta| > z) \leq h(z). \quad (2.5)$$

It will be presumed hereinafter $L_0 > p$; may be $L_0 = \infty$; otherwise our propositions are trivial.

Further, let $Z_+ = (1, 2, \ldots)$ be positive integer semi - axis; we introduce as ordinary a partition $\Delta$ of the set $Z_+$:

$$\Delta = \{\Delta(k)\}, \quad k = 1, 2, \ldots; \quad \Delta(k) = Z_+ \cap [A(k), A(k + 1)), \quad A(1) = 1,$$

$$A(k + 1) \geq A(k) + 2, \quad \lim_{k \to \infty} A(k) = \infty,$$

so that $Z_+ = \bigcup_{k=1}^{\infty} \Delta(k)$.

Let $w = \text{const} > 1$; we will say that the partition $\Delta = \{\Delta(k)\}$ belongs to the class $Y(w)$, iff
\[
\inf_k \left[ \frac{A(k+1) - 1}{A(k)} \right] \geq w^2. \tag{2.6}
\]

For instance, \(A(k) = d^k - d + 1\) for some fixed \(d = 2, 3, \ldots\).

Let at last the partition \(\Delta\) be from the set \(Y(w)\), \(w > 1\) be a given. Define an important function
\[
G(u) = G_\xi(\Delta, p, v; u) \overset{\text{def}}{=} \sum_{k=1}^{\infty} h_p(u v(A(k))/w), \tag{2.7}
\]
if of course \(G(u)\) there exists and \(\lim_{u \to \infty} G(u) = 0\).

Theorem 2.1. Main result.
\[
Q^{(\xi)}(u) \leq G_\xi(\Delta, p, v; u), \ u \geq e. \tag{2.8}
\]
As a slight consequence:
\[
Q^{(\xi)}(u) \leq \inf_{w > 1} \inf_{\Delta \in Y(w)} G_\xi(\Delta, p, v; u), \ u \geq e. \tag{2.9}
\]

Proof. used standard arguments for investigation of LIL in the (separable) Banach space.

Step 1. Lemma 2.1. Let \(m = \text{const} \geq 1\) be number, not necessary to be integer, for which
\[
E|\xi(\cdot)|_{pm,X}^{pm} < \infty. \tag{2.10}
\]

or equally
\[
|\xi(\cdot)|_{p,X;mp,\Omega} < \infty. \tag{2.10a}
\]

or equally
\[
E \int_X |\xi(x)|_{pm}^m \mu(dx) = \int_X E|\xi(x)|_{pm}^m \mu(dx) < \infty. \tag{2.10b}
\]
Then
\[
\sup_n E|\xi(\cdot)|_{p,X}^{pm} \leq \left[ \int_X [E|\xi(x)|_{pm}^{1/m}]^m \mu(dx) \right]^m, \tag{2.11}
\]
or equally
\[
|\xi|_{p,X;mp,\Omega} \leq |\xi|_{pm,\Omega;p,X}. \tag{2.11a}
\]
Recall that \(pm = L\).

Proof of Lemma 2.1.
Denote $\eta(x) = |\xi(x)|^p$. Note that the space $L_m(\Omega)$ is a Banach space; and we can apply the generalized triangle (Minkowsky) inequality:

$$\left[ E \left( \int_X |\xi(x)|^p \mu(dx) \right)^m \right]^{1/m} =$$

$$\left[ E \left( \int_X \eta(x) \mu(dx) \right)^m \right]^{1/m} = \left| \int_X \eta(x) \mu(dx) \right|_{m,\Omega} \leq$$

$$\int_X |\eta(x)|_{m,\Omega} \mu(dx) = \int_X \sqrt[m]{E|\eta(x)|^m} \mu(dx) = \int_X \sqrt[m]{E|\xi|^{pm}(x)} \mu(dx),$$

which is equivalent to the assertions (2.11) - (2.11a).

We used theorem of Fubini-Tonelli.

**Step 2.** We apply the assertion of Lemma 2.1. to the random field $\Sigma_n(x)$ instead $\xi(x)$:

$$E \left[ \int_X |\Sigma_n(x)|^p \mu(dx) \right]^m \leq \left\{ \int_X [E|\Sigma_n(x)|^{pm}]^{1/m} \mu(dx) \right\}^m. \quad (2.13)$$

We obtain by means of Rosenthal’s inequality (recall that we consider here the case only when $p \geq 2$):

$$E|\Sigma_n(x)|^{pm} \leq K^{pm}_R (pm) \cdot |\xi(x)|^{pm}_{pm,\Omega}.$$  

It remains to substitute into (2.13):

$$\sup_n E|\Sigma_n(\cdot)|^{pm}_{p,X} \leq K^{pm}_R (pm) \left[ \int_X \left[ E|\xi(\cdot)|^{pm} \right]^{1/m} \mu(dx) \right]^m, \quad (2.14)$$

or equally

$$\sup_n |\Sigma_n|_{p,X,mp,\Omega} \leq K_R(pm) |\xi|_{pm,\Omega,p,X}. \quad (2.14a).$$

**Step 3.** Denote

$$||\Sigma_n||^* = \max_{k=1,2,\ldots,n} ||\Sigma_k||. \quad (2.15)$$

Since the sequence $\{\Sigma_n\}$ under natural filtration is a (Banach value) martingale, we can use the Doob’s inequality; the Banach space valued martingale version is done by G. Pisier [40], [41]:

$$||\Sigma_n||^*_{L,\Omega} \leq \frac{L}{L-1} \cdot ||\Sigma_n||_{L,\Omega} \leq 2 \cdot ||\Sigma_n||_{L,\Omega}, \quad (2.16),$$

since $L \geq 2$.

**Step 4.** Now everything is ready for the final stage. Actually, let $\Delta$ be any partition from the class $Y(w)$. We get consequently:
$$Q(u) = P\left( \bigcup_{k=1}^{\infty} \max_{A(k) \leq n < A(k+1)} \frac{||S(n)||}{\sqrt{n} \ v(n)} > u \right) \leq$$

$$\sum_{k=1}^{\infty} P\left( \max_{A(k) \leq n < A(k+1)} \frac{||S(n)||}{\sqrt{n} \ v(n)} > u \right) \leq$$

$$\sum_{k=1}^{\infty} P\left( \max_{A(k) \leq n < A(k+1)} ||S(n)|| > u \sqrt{A(k) \ v(A(k))} \right) \leq$$

$$\sum_{k=1}^{\infty} P\left( \max_{A(k) \leq n < A(k+1)} ||\Sigma_n|| > u \sqrt{A(k)/A(k+1) \ v(A(k))} \right) \leq$$

$$\sum_{k=1}^{\infty} P\left( ||\Sigma_{A(k)+1}|| > 0.5 u \ v(A(k))/w \right) \leq$$

$$\sum_{k=1}^{\infty} h_p(u \ v(A(k))/w) = G_\xi(\Delta, p, v; u), \quad (2.17)$$

we used the inequality (2.5).

This completes the proof of theorem 2.1.

**Examples 2.1.**

Suppose

$$P (|\xi|_{p,X} > z) \leq e^{-z^\beta}, \ z > 0, \ \beta = \text{const} > 0;$$

then of course

$$\forall m > 0 \ \Rightarrow E|\xi|_{p,X}^m < \infty.$$ 

The case $\beta = \infty$ implies the boundedness of a r.v. $|\xi|_p$:

$$\underset{\omega \in \Omega}{\text{vraisup}} |\xi|_{p,X} < \infty.$$ 

Denote

$$r_0 = r_0(\beta) = \frac{\beta + 1}{\beta}, \ r_0(\infty) = 1.$$ 

We deduce after simple calculations:

$$Q_{r_0}(u) \leq \exp \left(-C(\beta, p) \ u^{\beta/(\beta+1)}\right), \ C(\beta, p) > 0.$$ 

In particular, if $\beta = \infty$, then

$$Q_1(u) \leq \exp \left(-C \ u\right).$$
More generally, if
\[ P(|\xi|_{p,X} > z) \leq e^{-z^{\beta_1} \left(\log z\right)^{-\beta_2}}, \quad \beta_1 = \text{const} > 0, \ \beta_2 = \text{const}, \ z \geq e, \]
then for some positive value \( C_3 = C_3(\beta_1, \beta_2, p) \) and \( u > e \)
\[ Q_{r_0}(u) \leq \exp \left(-C_3(\beta_1, \beta_2, p) \ u^{\beta_1/(\beta_1+1)} \left(\log u\right)^{(-\beta_2-\beta_1)/(\beta_1+1)}\right). \]

We used some estimations from the monograph [31], chapter 2, section 3, p. 55 - 57; where are obtained, in particular, the exponential tail estimates for random variables via its moment estimates.

3 Exponential bounds for LIL in Mixed Lebesgue - Riesz spaces.

Let us return to the CLT in the space \( L_{\vec{p}}, \vec{p} = \{p_k\}, k = 1, 2, \ldots, l, \ l \geq 2; \) where \( 1 \leq p_k < \infty, \) described below.

Define
\[ \vec{p} := \max(p_1, p_2, \ldots, p_l). \]

and suppose everywhere further
\[ \vec{p} \geq 2. \quad (3.0) \]

Some additional notations. As in last section, \( m = \text{const} \geq 1, \ L = \text{const} = \vec{p} m \geq \vec{p}, \)
\[ \Sigma_n(x) := n^{-1/2} \sum_{k=1}^{n} \xi_k(x) = n^{-1/2} S(n; x), \quad (3.1) \]
\[ \Gamma(L) = \Gamma_{p}(L) = 2 K_{R}(L) \cdot \sqrt[\vec{p}]{\left[ \int_X E|\xi(x)|^{L}\mu(dx)\right]^{\frac{L}{\vec{p} L}}}, \quad (3.2) \]
\[ L_0 = L_0(\vec{p}) = \sup\{L : \Gamma_{p}(L) < \infty\}, \quad (3.3) \]
\[ \gamma(z) = \gamma_{\vec{p}}(z) := \inf_{L \in [\vec{p}, L_0)} \left\{ \frac{\Gamma_L(L)}{z^L} \right\}, \ z > 1. \quad (3.4) \]

Let the partition \( \Delta \) be from the set \( Y(w), \ w > 1 \) be a given. Define again an important function
\[ F(u) = F_\xi(\Delta, \vec{p}, v; u) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \gamma_\vec{p}(u v(A(k))/w), \quad (3.5) \]
if of course \( F(u) \) there exists and \( \lim_{u \to \infty} F(u) = 0. \)
Theorem 3.1.

\[ Q^{(\xi)}(u) \leq F_\xi(\Delta, \vec{p}, v; u), \ u \geq e. \quad (3.6) \]

As a slight consequence:

\[ Q^{(\xi)}(u) \leq \inf_{w > 1} \inf_{\Delta \in Y(w)} F_\xi(\Delta, \vec{p}, v; u), \ u \geq e. \quad (3.7) \]

We preface the following auxiliary proposition which has by our opinion of independent interest.

Theorem 3.2. Let \( m = \text{const} \geq 1 \) be (not necessary to be integer) number for which

\[ \left[ E|\xi(\vec{x})|^m \vec{p} \right]^{1/(m \vec{p})} \in L_{\vec{p}}. \quad (3.8) \]

Then

\[ \sup_n \left[ E|\Sigma_n(\cdot)|^{m \vec{p}} \right]^{1/(m \vec{p})} \leq K_R(m \vec{p}) \times \left[ E|\xi(\vec{x})|^m \vec{p} \right]^{1/(m \vec{p})}. \quad (3.9) \]

Proof of theorem 3.2.

1. Auxiliary fact: permutation inequality.

We will use the so-called permutation inequality in the terminology of an article [1]; see also [3], chapter 1, p. 24 - 26. Indeed, let \((Z, B, \nu)\) be another measurable space and \( \phi : (\vec{X}, Z) = \vec{X} \otimes Z \to R \) be measurable function. In what follows \( \vec{X} = \otimes_k X_k \). Let also \( r = \text{const} \geq \vec{p} \). It is true the following inequality (in our notations):

\[ |\phi|_{\vec{p}, \vec{X}, r, Z} \leq |\phi|_{r, Z; \vec{p}, \vec{x}}. \quad (3.10) \]

In what follows \( Z = \Omega, \ \nu = \mathbf{P} \).

2. Auxiliary inequality.

It follows from permutation inequality (3.10) that

\[ n^{m \vec{p}} E|\xi|^{m \vec{p}} \leq n^{m \vec{p}} E|\xi|^{m \vec{p}} \quad m = \text{const} \geq 1. \quad (3.11) \]

3. We deduce applying the inequality (3.11) for the random field \( \Sigma_n(x, t) \) and using the Rosenthal’s inequality:

\[ n^{m \vec{p}} E|\Sigma_n|^{m \vec{p}} \leq K_R(m \vec{p}) \cdot n^{m \vec{p}} E|\xi|^{m \vec{p}} \quad m = \text{const} \geq 1, \quad (3.12) \]

which is equivalent to the assertion of theorem 3.2.
Proof of theorem 3.1 is now completely analogous to ones in theorem 2.1 and may be omitted.

Remark 3.1. The examples on the acting of theorems 3.1 and 3.2 may be considered at the same as in second section with replacement the power \( p \) on the power \( p \).

4 LIL in continuous - Lebesgue spaces.

0. Definition of continuous-Lebesgue (Lebesgue-Riesz) space \( CL(p) = C(T, L_p(X)), p \geq 1 \).

Let \( (X, A, \mu) \) be again measurable space with sigma - finite separable measure \( \mu \), \( T = \{ t \} \) be metrizable compact set.

We will say that the (measurable) function of two variables \( f = f(x, t), x \in X, t \in T \) belongs to the space \( CL(p) = C(T, L_p(X)) \), where \( p = \text{const} \geq 1 \), if the map \( t \rightarrow f(\cdot, t), t \in T \) is continuous in the \( C(T) \) sense:

\[
\lim_{\epsilon \to 0^+} \sup_{d(t,s) < \epsilon} \left( \int_X |f(x, t) - f(x, s)|^p \mu(dx) \right)^{1/p} = 0.
\] (4.1)

The norm of the function \( f(\cdot, \cdot) \) in this space is defined as follows:

\[
|\|f(\cdot, \cdot)||C(T, L_p(X)) = |\|f(\cdot, \cdot)||CL(p) = \sup_{t \in T} |f(t, \cdot)|_p.
\] (4.2)

These spaces are complete separable Banach function spaces. The detail investigation of these spaces see, e.g. in a monograph [8], p. 113 - 119.

They are used, for instance, in the theory of non-linear evolution Partial Differential Equations, see [5], [6], [7], [9], [10], [21].

1. Additional construction and conditions.

Assume in addition that \( p \geq 2 \) and that the r.f. \( \xi(x, t) \), and with it the r.f. \( \xi_i(x, t) \) are mean zero: \( E\xi_i(x, t) = 0 \), and denote

\[
\Sigma_n(x, t) = n^{-1/2} \sum_{i=1}^n \xi_i(x, t), \quad S_n(x, t) = \sum_{i=1}^n \xi_i(x, t) = n^{1/2} \Sigma_n(x, t),
\] (4.3)

\[
\tau^{(n)}_p(t) = \int_X |\Sigma_n(x, t)|^p \mu(dx) = |\Sigma_n(\cdot, t)|^p_{p,X}.
\] (4.4)

We intend as before to estimate uniformly over numbers of summand \( n \) first of all in this section the moments of the random variable

\[
\zeta^{(n)}_p = \zeta_p := \sup_{t \in T} \tau^{(n)}_p(t),
\] (4.5)

i.e. the values
\[ \delta_p(Z) = \sup_n \left| \sup_{t \in T} \tau_p^{(n)}(t) \right|_{Z, \Omega}. \]

Note that
\[ [\delta_p(Z)]^{1/p} = \sup_n |\Sigma_n(\cdot, \cdot)|_{p, X; \infty, T; Z, \Omega}. \]

Some new notations: \( \rho_{v, x}(t, s) := |\xi(x, t) - \xi(x, s)|_{v, \Omega} = \big[ |E|\xi(x, t) - \xi(x, s)||^v \big]^{1/v}, v = \text{const} \geq 1; \]

\[ W_\gamma(x) = \sup_{t \in T} |\xi(x, t)|_{\gamma, \Omega} = \sup_{t \in T} \big[ |E|\xi(x, t)\big]^{\gamma}, \gamma, \Omega = \sup_{t \in T} \{ |E|\xi(x, t)\}^{1/\gamma}, \] (4.6)

\[ J(t, s; p, Z; \alpha, \beta) = \int_X W_{(p-1)\beta Z}(x) \rho_{\alpha Z, x}(t, s) \mu(dx), \; \alpha, \beta > 1, 1/\alpha + 1/\beta = 1; \]

\[ r_{p, Z}(t, s) = 2^{p} \inf_{\alpha, \beta} \left[ K_R(\alpha Z) K_R^{p-1}((p-1)\beta Z) J(t, s; p, Z; \alpha, \beta) \right]. \] (4.7)

Evidently, \( r_{p, Z}(t, s) \) is the distance as the function on \((t, s)\), if it is finite. The minimum in the right-hand side (4.7) is calculated over all the values \((\alpha, \beta)\) for which \( \alpha, \beta > 1, 1/\alpha + 1/\beta = 1. \)

Denote also for arbitrary set \( T \) equipped with distance \( d = d(t, s) \) by \( N(T, d, \epsilon) \) the so-called covering number: the minimal number of closed balls of a radii \( \epsilon > 0 \) in the distance \( d \) covering the set \( T. \) Obviously, \( \forall \epsilon > 0 \) \( N(T, d, \epsilon) < \infty \) if and only if the set \( T \) is precompact set relative the distance \( d. \)

Further, define
\[ \sigma_{p, Z} \overset{\text{def}}{=} \sup_{t \in T} \int_X \big[ |E|\xi(x, t)|^{p Z} \big]^{1/Z} \mu(dx), \; \hat{\sigma}_{p, Z} := K_R^p(p Z) \sigma_{p, Z}; \] (4.8)

\[ \hat{r}_{p, Z}(t, s) := r_{p, Z}(t, s)/\hat{\sigma}_{p, Z}, \] (4.9)

\[ \nu_p(Z) \overset{\text{def}}{=} \hat{\sigma}_{p, Z} \cdot \inf_{\theta \in (0, 1)} \left[ \sum_{k=1}^\infty \theta^{k-1} N^{1/Z}(T, \hat{r}_{p, Z}, (\theta \hat{\sigma}_{p, Z})^k) \right]. \] (4.10)

2. **Theorem 4.1.** If for some \( Z = \text{const} \geq 1 \) \( \Rightarrow \nu_p(Z) < \infty, \) then
\[ \sup_n \big\{ |E|\Sigma_n(\cdot, \cdot)|^{p Z}_{p, \infty} \big\}^{1/p Z} \leq \nu_p(Z). \] (4.11)

**Proof of theorem 4.1.**

A. We need first of all to obtain the estimate (4.4). We have using the Rosenthal’s constants and the Minkowsky inequality:
\[ | \tau_p^{(n)}(t) |_{Z, \Omega} = \left| \int_X |\Sigma_n(x, t)|^p \mu(dx) \right|_{Z, \Omega} \leq \int_X |\Sigma_n(x, t)|^p |_{Z, \Omega} \mu(dx) \leq \int_X K^p_R(p Z) |\xi(x, t)|^p |_{Z, \Omega} \mu(dx) \leq K^p_R(p Z) \sigma_{p, Z} = \hat{\sigma}_{p, Z}. \] (4.12)

B. The estimation of a difference

\[ \Delta \tau(t, s) = \tau_p^{(n)}(t) - \tau_p^{(n)}(s) \]

is more complicated. We have consequently:

\[ \Delta \tau = \int_X [|\Sigma_n(x, t)|^p - |\Sigma_n(x, s)|^p] \mu(dx), \]

\[ |\Delta \tau|_{Z, \Omega} \leq \int_X |\Sigma_n(x, t)|^p - |\Sigma_n(x, s)|^p |_{Z, \Omega} \mu(dx) = \int_X [E |\Sigma_n(x, t)|^p - |\Sigma_n(x, s)|^p |^p]^{1/p} \mu(dx). \] (4.13)

We exploit the following elementary inequality:

\[ |x|^p - |y|^p | \leq p \cdot |x - y| \cdot |x|^{p-1} + |y|^{p-1}, \] (4.14)

and obtain after substituting into (4.12), where \( x = \Sigma_n(x, t), y = \Sigma_n(x, s) : |\Delta \tau|_{Z, \Omega}/p \leq \)

\[ \int_X \left| |\Sigma_n(x, t) - \Sigma_n(x, s)| \cdot \left[ |\Sigma_n(x, t)|^{p-1} + |\Sigma_n(x, s)|^{p-1} \right] \right|_{Z, \Omega} \mu(dx). \] (4.15)

It follows from the Hölder’s inequality

\[ |\eta_1 \eta_2|_{Z, \Omega} \leq |\eta_1|_{\alpha Z, \Omega} \cdot |\eta_2|_{\beta Z, \Omega}, \]

where as before \( \alpha, \beta > 1, 1/\alpha + 1/\beta = 1. \) Therefore

\[ |\Delta \tau|_{Z, \Omega}/p \leq \int_X \delta_1(t, s; \alpha, x) \cdot \delta_2(\beta, x) \mu(dx), \]

where

\[ \delta_1(t, s; \alpha, x) = \delta_1(t, s; \alpha, x; \Omega) = |\Sigma_n(x, t) - \Sigma_n(x, s)|_{\alpha Z, \Omega}, \] (4.16)

and

\[ \delta_2(\beta, x) = \delta_2(\beta, x; p, \Omega) = \sup_{t, s, \in \Omega} \left| \left[ |\Sigma_n(x, t)|^{p-1} + |\Sigma_n(x, s)|^{p-1} \right] \right|_{\beta Z, \Omega}. \] (4.17)

We estimate \( \delta_1(\cdot) \) using the Rosenthal’s inequality:
\[ \delta_1(t, s; \alpha, x; \Omega) \leq K_R(\alpha Z) |\xi(x, s) - \xi(x, s)|_{\alpha Z} = \rho_{\alpha Z, x}(t, s). \quad (4.18) \]

Further,

\[ \delta_2(\beta, x; p, \Omega) \leq 2 K_R^{p-1}(\beta(p - 1)) \sup_{t \in T} |\xi(x, t)|_{\beta(p - 1), \Omega}^{p-1} = 2 K_R^{p-1}(\beta(p - 1)) W_{\beta(p - 1)}^{p-1}(x). \quad (4.19) \]

We get after substituting into (4.16) and (4.17)

\[ |\Delta \tau|_{Z, \Omega} \leq r_{p, Z}(t, s). \quad (4.20) \]

C. The proposition of theorem 4.1. follows from the main result of article G.Pisier [40]; see also [20], [63], [64].

Remark 4.1. Perhaps, it may be used in this section the so-called method of "majorizing measures", or equally "generic chaining"; see e.g. [28], [53], [54], [55], [23], [24], [19], [20].

But, by our opinion, the offered here way is more convenient for declared aims.

3. We are ready now to formulate the main result of this section. Some new notations:

\[ L_0 = L_0(\vec{p}) = \sup\{L : \nu_p(L/p) < \infty\}, \quad (4.21) \]

\[ \zeta(z) = \zeta_{\vec{p}}(z) := \inf_{L \in (p, L_0)} \left\{ \frac{\nu_p(L/p)}{z^L} \right\}, \quad z > 1. \quad (4.22) \]

Let the partition \( \Delta \) from the set \( Y(w), \ w > 1 \) be a given. Define again an important function

\[ \Theta(u) = \Theta_{\xi}(\Delta, \vec{p}, v; u) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \zeta_{\vec{p}}(u v(A(k))/w), \quad (4.23) \]

if of course \( \Theta(u) \) there exists and \( \lim_{u \to \infty} \Theta(u) = 0 \).

Theorem 4.2.

\[ Q^{(\xi)}(u) \leq \Theta_{\xi}(\Delta, \vec{p}, v; u), \ u \geq e. \quad (4.24) \]

As a slight consequence:

\[ Q^{(\xi)}(u) \leq \inf_{w > 1} \inf_{\Delta \in Y(w)} \Theta_{\xi}(\Delta, \vec{p}, v; u), \ u \geq e. \quad (4.25) \]

The proof is at the same as before.

Example 4.1. Let \( T \) be in addition closure of bounded set in the Euclidean space \( \mathbb{R}^d, \ d = 1, 2, \ldots \) with ordinary distance \( ||t - s|| \). Assume that
\[
\rho_{v,x}(t, s) \leq B_v(x) \ |t - s|^l, \ 0 < l = \text{const} \leq 1,
\] (4.26)

where

\[
\int_X W_{2(p-1)Z}^1(x) B_{2Z}(x) \mu(dx) \leq C(p) \ Z^b, \ b = \text{const} \leq 1, \ Z > 2d/l; \ (4.27)
\]

we put \(\alpha = \beta = 1\). Therefore

\[
J(t, s; p, Z; 2, 2) \leq C_1(p) \ |t - s|^l \ Z^b.
\]

Further, suppose

\[
\Psi_{p,Z} \overset{\text{def}}{=} \sup_{t \in T} \int_X \left[ E|\xi(x, t)|^{pZ}\right]^{1/Z} \mu(dx) \simeq C_2(p) \ Z^b, \ Z > 2d/l, C_2(p) \in (0, \infty).
\] (4.28)

We deduce after simple calculations

\[
Q^{(\xi)}(u) \leq C_3(p) \exp\left(-C_4(p) \ u^b\right), \ u \geq 1.
\] (4.29)

5 Concluding remarks.

A. Lower estimates.

We give a lower slightly less trivial as in (1.8) estimate for \(Q_{1/2}(u)\). Namely, for all the non-trivial distribution \(\xi\)

\[
Q_{1/2}(u) \geq \max \left( \mathcal{P}(|\xi| > u), \ \exp\left(-C u^2 \log \log u\right) \right), \ u > e^e.
\]

It is sufficient to note that this estimate is true in the one-dimensional case; see [62], [31], chapter 2, section 2.6.

B. LIL in mixed Sobolev’s spaces.

The method presented here may be used by investigation of the Law of Iterated Logarithm as well as Central Limit Theorem in the so-called mixed Sobolev’s spaces \(W^{A}_{p} X\), see e.g. [61], [63].

In detail, let \((Y_k, B_k, \zeta_k)\), \(l = 1, 2, \ldots, l\) be again measurable spaces with separable sigma-finite measures \(\zeta_k\). Let \(A\) be closed unbounded operator acting from the space \(W^{A}_{p} X\) into the space \(W^{A}_{p} Y\), for instance, differential operator, Laplace’s operator or its power, may be fractional, for instance:

\[
A[u](x, y) = \frac{D^{(q)}u(x) - D^{(q)}u(y)}{|x - y|^b}, \ x, y \in \mathbb{R}^d, \ x \neq y,
\]

where

\[
\rho_{v,x}(t, s) \leq B_v(x) \ |t - s|^l, \ 0 < l = \text{const} \leq 1,
\] (4.26)
\[
\zeta(G) = \int \int_G \frac{dx \, dy}{|x - y|^{\alpha}} \quad \alpha, \beta = \text{const} \in [0, 1], \alpha + \beta p < d, \ G \subset \mathbb{R}^d,
\]

\[\vec{q} = \{q_1, q_2, \ldots, q_d\}, \quad q_s = 0, 1, \ldots,\]

\[D(\vec{q}) u(x) = \frac{\partial q_1}{\partial x_1} \frac{\partial q_2}{\partial x_2} \cdots \frac{\partial q_d}{\partial x_d} u(x).\]

The norm in this space may be defined as follows (up to closure):

\[|f|_{W^{A p}_p} \overset{\text{def}}{=} \max \left( |f|_p, |Af|_p \right). \tag{5.1}\]

Analogously to the proof of theorems 4.1, 3.2 may be obtained the following results.

**Theorem 5.1.** Let \(m = \text{const} \geq 1\) be (not necessary to be integer) number for which

\[\left[ \mathbb{E}|\xi(\vec{x})|^m \frac{p}{p} \right]^{1/(m \frac{p}{p})} \in L_{\frac{p}{p}}, \quad \left[ \mathbb{E}|A[\xi](\vec{x})|^m \frac{p}{p} \right]^{1/(m \frac{p}{p})} \in L_{\frac{p}{p}}. \tag{5.2}\]

Then

\[\sup_n \left[ \mathbb{E}|A[\Sigma_n](\cdot)|^m \frac{p}{p} \right]^{1/(m \frac{p}{p})} \leq K_R(\frac{p}{p} m) \times \left| \left[ \mathbb{E}|A[\xi](\vec{x})|^m \frac{p}{p} \right]^{1/(m \frac{p}{p})} \right|_p, \tag{5.3}\]

Hence the propositions of theorem 2.1, 3.1 remains true in this norm.

**B. LIL for dependent r.v. in anisotropic spaces.**

We refuse in this section on the assumption about independence of random vectors \(\{\xi_k(\cdot)\}\).

**Martingale case.**

We suppose as before that \(\{\xi_k(\cdot)\}\) are mean zero and form a strictly stationary sequence with values in Bochner’s (mixed) space \(L_{\frac{p}{p}}, \frac{p}{p} \geq 2,\)

Assume in addition that \(\{\xi_k(\cdot)\}\) form a martingale difference sequence relative certain filtration \(\{F(k)\}, \quad F(0) = \{\emptyset, \Omega\},\)

\[\mathbb{E}\xi_k/F(k) = \xi_k, \quad \mathbb{E}\xi_k/F(k - 1) = 0, \quad k = 1, 2, \ldots.\]

Then the propositions of theorems 2.1, 3.1 remains true; the estimate of theorem 3.2 is also true up to multiplicative absolute constant.

Actually, the Law of Iterated Logarithm for one-dimensional martingales see in the classical monograph of Hall P., Heyde C.C. [4], chapter 2; the Rosenthal’s constant for the sums of martingale differences with at the same up to multiplicative constant coefficient is obtained by A.Osekowski [14], [15]. See also [17].
Mixingale case.

We suppose again that \( \{ \xi_k(\cdot) \} \) are mean zero and form a strictly stationary sequence, \( \mathbb{P} \geq 2 \). This sequence is said to be mixingale, in the terminology of the book [4], if it satisfies this or that mixing condition.

We consider here only the superstrong mixingale. Recall that the superstrong, or \( \beta = \beta(F_1, F_2) \) index between two sigma - algebras is defined as follows:

\[
\beta(F_1, F_2) = \sup_{A \in F_1, B \in F_2, \mathbb{P}(A) \mathbb{P}(B) > 0} \left| \frac{\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)\mathbb{P}(B)} \right|
\]

Denote

\[
F^0_{-\infty} = \sigma(\xi_s, s \leq 0), \quad F^\infty_n = \sigma(\xi_s, s \geq n), \quad (5.4)
\]

\[
\beta(n) = \beta \left( F^0_{-\infty}, F^\infty_n \right),
\]

The sequence \( \{ \xi_k \} \) is said to be superstrong mixingale, if \( \lim_{n \to \infty} \beta(n) = 0 \).

This notion with some applications was introduced and investigated by B.S.Nachapetyan and R.Filips [13]. See also [17], [31], p. 84 - 90.

Introduce the so-called mixingale Rosenthal coefficient:

\[
K_M(m) = m \left[ \sum_{k=1}^{\infty} \beta(k) (k + 1)^{(m-2)/2} \right]^{1/m}, \quad m \geq 1. \quad (5.5)
\]

B.S.Nachapetyan in [13] proved that for the superstrong centered strong stationary strong mixingale sequence \( \{ \eta_k \} \) with \( K_M(m) < \infty \) the following estimate is true:

\[
\sup_{n \geq 1} \left| n^{-1/2} \sum_{k=1}^{n} \eta_k \right|_m \leq C \cdot K_M(m) \cdot |\eta_1|_m, \quad (5.6)
\]

so that the ”constant” \( K_M(m) \) play at the same role for mixingale as the Rosenthal constant for independent variables.

As a consequence: theorems 3.1 (and 3.2) remains true for the strong mixingale sequence \( \{ \xi_k \} \): theorem 3.1 under conditions: \( K_M(m\mathbb{P}) < \infty \), with replacing \( K_R(m \mathbb{P}) \) on the expression \( K_M(m \mathbb{P}) \).

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