Some fully nonlinear problems on manifolds with boundary of negative admissible curvature

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1 Introduction

Let \((M^n, g)\) denote a compact smooth Riemannian manifold with no boundary of dimension \(n \geq 3\). The Yamabe problem is to search a metric \(\tilde{g}\) in the conformal class \([g]\) of \(g\) such that \(\tilde{g}\) has a constant scalar curvature \(R\tilde{g} = c\). Write \(\tilde{g} = u^{\frac{4}{n-2}}g\). The Yamabe problem is equivalent to solve

\[-L_g u = cu^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in} \quad M,\]

where \(L_g = \Delta_g - \frac{n-2}{4(n-1)}R_g\) is the conformal Laplacian of \(g\), and \(c = -1, 0,\) or \(1\).

Let \(\phi_1\) be a positive eigenfunction of the first eigenvalue \(\lambda_1\) of \(-L_g\), i.e.

\[\lambda_1 = \inf_{\phi \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla \phi|_g^2 + \frac{n-2}{4(n-1)}R_g \phi^2 \ dvol_g}{\int_M \phi^2 \ dvol_g},\]

and \(-L_g \phi_1 = \lambda_1 \phi_1\). A direct calculation yields that

\[R_{\phi_1^{\frac{n}{n-2}} g} = -\frac{n-2}{4(n-1)} \phi_1^{-\frac{n-2}{n-2}} L_g \phi_1 = \frac{n-2}{4(n-1)} \phi_1^{-\frac{4}{n-2}} \lambda_1.\]

After replacing \(g\) by \(\phi_1^{\frac{4}{n-2}} g\), we assume the scalar curvature of the background metric \(g\) has a definite sign, that is, either

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$R_g > 0$, or $R_g \equiv 0$, or $R_g < 0$.

Consider the functional

$$Q(\phi) = \frac{\int_M |\nabla \phi|^2_g + \frac{n-2}{4(n-1)} R_g \phi^2 \, dvol_g}{(\int_M \phi^{\frac{2n}{n-2}} \, dvol_g)^{\frac{n-2}{n}}}.$$ 

u is a solution of the equation (1), then u is a critical point of the above functional Q. It is a simple consequence of the Hölder inequality that

$$\lambda(M^n, g) := \inf_{\phi \in H^1(M) \setminus \{0\}} Q(\phi) > -\infty.$$ 

In [24], Yamabe approached the problem by attempting to prove that a minimizing sequence of Q will converge to a minimizer. Trudinger ([22]) pointed out that the convergence failed on the standard sphere $(S^n, g_{\text{round}})$, and Trudinger was able to fix Yamabe’s arguments when $\lambda(M^n, g) \leq 0$. In general, we know

$$\lambda(M^n, g) \leq \lambda(S^n, g_{\text{round}}).$$

In ([1]), Aubin proved the convergence of the minimizing sequence if

$$\lambda(M^n, g) < \lambda(S^n, g_{\text{round}}).$$

When the manifold $M^n$ is not locally conformally flat, it was proved by Aubin, for $n \geq 6$, and that by Schoen, for $n = 3, 4, 5$, that $\lambda(M^n, g) < \lambda(S^n, g_{\text{round}})$. When the manifold is locally conformally flat and not conformally diffeomorphic to the standard sphere, Schoen established the compactness result of the solutions to the equation (1) using a deep result of his joint work with Yau in [21], therefore confirmed the existence of the solutions.

For $(M^n, g)$, an $n$–dimensional($n \geq 3$) smooth Riemannian compact manifold with boundary, a similar problem is to look for a metric $\tilde{g} \in [g]$ having constant scalar curvature on $M^n$ and constant mean curvature on the boundary $\partial M$. Let $\tilde{g} = u^{\frac{4}{n-2}} g$. The problem is equivalent to searching a solution of the following equation

\[
\begin{align*}
-L_g u &= c_1 u^{\frac{n+2}{n-2}} \quad \text{on} \quad M^n \\
B_g u &= c_2 u^{\frac{n}{n-2}} \quad \text{on} \quad \partial M, 
\end{align*}
\]

where the boundary operator $B_g = \frac{n}{n-2} \frac{\partial}{\partial v} + h_g$, $h_g$ is the mean curvature of $g$ w.r.t. the unit outer normal $\frac{\partial}{\partial v}$, and $c_1, c_2$ denote two constants. When $c_2 = 0$, the
problem is variational. In fact, the equation (2) is the Euler-Lagrange equation of the functional

\[ F(\phi) = \frac{\int_M |\nabla \phi |^2_g + \frac{n-2}{4(n-1)} R_g \phi^2 \, dvol_g + \frac{n-2}{2} \int_{\partial M} h_g \phi^2 \, dS_g}{(\int_M \phi^{\frac{2n}{n-2}} \, dvol_g)^{\frac{n-2}{n}}} , \]

and we have

\[ \lambda(M, g) := \inf_{\phi \in H^1(M) \setminus \{0\}} F(\phi) > -\infty. \]

Cherrier ([3]) proved that the \( \inf F \) is achieved by a smooth positive function if \( \lambda(M, g) < \lambda(S^n_+, \text{round}) \), (3)

where \( (S^n_+, \text{round}) \) is the standard half sphere. When \( c_2 = 0 \) in the equation (2), Escobar ([6]) obtained the existence of the solution for a large class of manifolds by achieving (3). For the general constant \( c_2 \), let \( \phi_1 \) be a smooth positive function of the eigenvalue problem

\[
\begin{cases}
-L_g \phi_1 = \lambda_1 \phi_1 & \text{on } M^n \\
B_g \phi_1 = 0 & \text{on } \partial M,
\end{cases}
\]

where

\[ \lambda_1 := \inf_{\phi \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla \phi |^2_g + \frac{n-2}{4(n-1)} R_g \phi^2 \, dvol_g + \frac{n-2}{2} \int_{\partial M} h_g \phi^2 \, dS_g}{\int_M \phi^2 \, dvol_g} . \] (4)

Then

\[
\begin{cases}
R \frac{\phi_1}{\phi_1} = \frac{4(n-1)}{n-2} \lambda_1 \phi_1^{\frac{n-2}{n-4}} \quad & \text{on } M^n \\
h \frac{\phi_1}{\phi_1} = 0 & \text{on } \partial M.
\end{cases}
\]

Replacing \( g \) by \( \phi_1^\frac{n-2}{n-4} g \), we may assume one of the following three cases holds, i.e.,

\[ R_g > 0, \quad R_g < 0, \quad R_g = 0. \]

or

\[ h_g = 0 \quad \text{or} \quad h_g = 0 \quad \text{or} \quad h_g = 0 \]

We say the equation (2) is of positive/negative/zero type if \( \lambda_1 \) as defined in (4) is positive/negative/zero respectively (see [12] for more discussion). When \( c_2 = 0 \), by the Hopf lemma, the positive/negative/zero type implies that \( c_1 > 0 / c_1 < 0 / c_1 = 0 \). In [7], Escobar proved that the equation (2) is solvable for some \( c_2 > 0 \) and some
\( c_2 < 0 \) under certain hypothesis. In [12], and [13], Han and Li confirmed the existence of the solutions to the equation (2) when the manifold is of positive type and is locally conformally flat with umbilic boundary or with non totally umbilic boundary of dimension \( n \geq 5 \). In this paper, we will study the equation (2) of negative type. More generally, we will study a fully nonlinear version of the negative type being stated as follows.

Let \( \text{Ric}_g \) denote the Ricci tensor of \( g \). Consider the modified Schouten tensor of \( g \) as introduced in [18]

\[
A_t^g := \frac{1}{n-2} \left( \text{Ric}_g - \frac{t R_g}{2(n-1)} g \right), \quad t \leq 1.
\]

Note that \( A_0^g = \text{Ric}_g \) and \( A_1^g = A_g \) is the Schouten tensor (see [5]). Schouten tensor as a \((0,2)\) tensor appears in the decomposition of the Riemann tensor, i.e., the Riemann tensor can be decomposed as the direct sum of the Weyl tensor and the Kulkarni-Numizu product of \( A_g \) with \( g \). In [18], we introduced \( A_t^g \) up to a constant multiple. In fact, we introduced the tensor \( s A_g^t + \frac{(1-s) R_g}{2(n-1)} g = s A_g^t \) with \( t = n - 1 - \frac{n-2}{s} \).

Assume that

\[
\Gamma \subset R^n \quad \text{is an open convex symmetric cone with vertex at the origin} \quad (5)
\]

satisfying

\[
\Gamma_n := \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in R^n | \lambda_1 > 0, \cdots, \lambda_n > 0 \} \subset \Gamma \subset \Gamma_1 := \{ \lambda \in R^n | \sum_{i=1}^{n} \lambda_i > 0 \}, \quad (6)
\]

where \( \Gamma \) being symmetric means that

\[
(\lambda_1, \cdots, \lambda_n) \in \Gamma \iff (\lambda_{i_1}, \cdots, \lambda_{i_n}) \in \Gamma
\]

for any permutation \( (i_1, \cdots, i_n) \) of \( (1, \cdots, n) \).

For \( \alpha_0 \in (0,1) \), assume that

\[
f \in C^{2,\alpha_0}(\Gamma) \cap C^0(\overline{\Gamma}) \quad \text{is concave, homogeneous of degree 1 and symmetric in } \lambda_i, \quad (7)
\]

satisfying

\[
f|_{\partial \Gamma} = 0, \quad \nabla f \in \Gamma_n \quad \text{on } \Gamma, \quad (8)
\]

\[
\lim_{s \to \infty} f(s \lambda) = \infty, \quad \lambda \in \Gamma, \quad (9)
\]

and
\[ f(\lambda) \leq \frac{1}{\varepsilon} \sum_{i=1}^{n} \lambda_i, \quad \sum_{i=1}^{n} f(x_i(\lambda)) \geq \varepsilon \text{ on the level set } \{ f = 1 \} \quad (10) \]

for some constant \( \varepsilon > 0 \).

Notice that \( f \) is homogeneous of degree 1. Therefore \( f(x_i(\lambda)) \) is homogeneous of degree 0 and the above assumption (10) also holds in \( \Gamma \).

Let \( \lambda_g(A^t_g) \) denote the eigenvalues of \( A^t_g \) w.r.t. the metric \( g \). A fully nonlinear problem of negative admissible curvature is to look for a metric \( \tilde{g} \in [g] \) solving

\[
\begin{cases}
  f(-\lambda_{\tilde{g}}(A^t_{\tilde{g}})) = 1, & -\lambda_{\tilde{g}}(A^t_{\tilde{g}}) \in \Gamma \text{ on } M \\
  h_{\tilde{g}} = c & \text{on } \partial M,
\end{cases}
\]

if \(-\lambda_{g}(A^t_{g}) \in \Gamma \) on \( M \) and \( h_{g} \leq 0 \) on \( \partial M \), where \( c \) is a constant.

When \((f, \Gamma) = (\sigma^k, \Gamma_k)\), the problem is the \( k \)-th Yamabe problem of negative admissible type, where

\[ \sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \Gamma_k := \{ \lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0 \} \].

It is well-known that \((\sigma^k, \Gamma_k)\) satisfies assumptions (5)-(10). In particular, when \( k = 1 \), the problem (11) is equivalent to solving the equation (2) of negative type. This is because \( \sigma_1(-\lambda_{\tilde{g}}(A^t_{\tilde{g}}))) = -\frac{1}{n-2}(1 - \frac{m}{2(n-1)})R_{\tilde{g}} \), and the assumption \(-\lambda_{g}(A^t_{g}) \in \Gamma_1 \), \( h_{g} \leq 0 \) is to say that \( R_{\tilde{g}} < 0 \) and \( h_{\tilde{g}} \leq 0 \), which implies that \( \lambda_1 < 0 \) by taking \( \phi \equiv 1 \) in (4). Conversely, if the equation is of negative type, we can assume \( R_{g} < 0 \) and \( h_{g} = 0 \). Clearly the solution \( u \) of the equation (2) also gives a solution \( \tilde{g} = u^{\frac{4}{n-2}}g \) to the problem (11).

In [9], Gursky and Viaclovsky proved that, for \( t < 1 \), there exists a unique solution \( \tilde{g} \in [g] \) solving

\[ \sigma_k(-\lambda_{\tilde{g}}(A^t_{\tilde{g}})) = 1, \quad -\lambda_{\tilde{g}}(A^t_{\tilde{g}}) \in \Gamma_k \]

if the compact manifold of dimension \( n \geq 3 \) has no boundary and \(-\lambda_{g}(A^t_{g}) \in \Gamma_k \).

**Theorem 1.1** Let \((M^n, g)\) be an \( n \)-dimensional \( (n \geq 3) \) compact smooth Riemannian manifold with \( \partial M \neq \emptyset \), and let \((f, \Gamma)\) be a pair satisfying (5)-(10). Assume that \(-\lambda_{g}(A^t_{g}) \in \Gamma \) on \( M \) and \( h_{g} \leq 0 \) on \( \partial M \). Then, for \( c \leq 0 \) and for \( t < 1 \), there exists a unique solution \( \tilde{g} = e^{2v}g \) solving the problem (11). Moreover,

\[ \|v\|_{C^{4,\alpha_0}(M^n, g)} \leq C, \]

where \( C > 0 \) is a universal constant depending only on \((M^n, g), (f, \Gamma), \alpha, t, \) and \( |c| \).
The next theorem is a more general result.

**Theorem 1.2** Let $(M^n, g)$ be an $n$-dimensional ($n \geq 3$) compact smooth Riemannian manifold with $\partial M \neq \emptyset$, and let $(f, \Gamma)$ be a pair satisfying (5)-(10). Assume that $-\lambda_g(A^g_\phi) \in \Gamma$ on $M$ and $h_g \leq 0$ on $\partial M$. Given any $0 < \phi \in C^{2, \alpha_0}(M^n)$ and any $0 \geq \psi \in C^{3, \alpha_0}(\partial M)$, then, for $t < 1$, there exists a unique solution $\tilde{g} = e^{2v}g$ solving

\[
\begin{aligned}
&f(-\lambda_{\tilde{g}}(A^\tilde{g}_\phi)) = \phi, \\
&h_{\tilde{g}} = \psi \\
\end{aligned}
\]  

(12)

Moreover

$$\|v\|_{C^{4, \alpha_0}(M^n, g)} \leq C,$$

where $C > 0$ is a universal constant depending only on $(M^n, g)$, $(f, \Gamma)$, $\phi$, $\psi$, $\alpha$, and $t$.

In the above theorems, we do not assume the boundary $\partial M$ is umbilic or the manifold is locally conformally flat near $\partial M$, so when we establish the a-priori estimates on the boundary, we can not assume $\partial M$ is totally geodesic, which offers a very useful geodesic normal coordinates, i.e., locally, one direction of the geodesic normal coordinates is the normal direction and all the other directions of coordinates are the tangent directions of $\partial M$. On the general manifolds, the lack of such coordinates causes the a-priori estimates much more difficult to obtain. The Yamabe problem of the negative case can avoid this particular assumption on the boundary of the manifold since the problem is variational and the minimizing sequence is convergent. However, our problem (12) may not even be variational. To overcome this difficulty, we introduce a very useful coordinates near $\partial M$ in Section 4, called the tubular neighborhood normal coordinates. Such coordinates allow us get rid of the assumption of the umbilic boundary, which is very important in the following theorem. As an application of the above theorems, we affirm the existence of certain Riemannian metrics on a general compact smooth differential manifold with some boundary.

**Theorem 1.3** Let $(f, \Gamma)$, $\phi, \psi$ be as in Theorem 1.2. Any compact $n$-dimensional ($n \geq 3$) smooth differential manifold with some boundary always admits a smooth Riemannian metric $\tilde{g}$ with the negative Ricci tensor satisfying

\[
\begin{aligned}
&\det(-\text{Ric}_{\tilde{g}}) = 1 \\
h_{\tilde{g}} = 0 \\
\end{aligned}
\]  

on $M$, $\partial M$.

More generally, for $t < 1$, any compact $n$-dimensional ($n \geq 3$) smooth differential manifold with some boundary always admits a $C^{4, \alpha_0}$ Riemannian metric $\tilde{g}$ satisfying
We want to point out that a similar problem of positive admissible curvature has been studied by quite a few people and many important results have been obtained such as [2], [10], [11], [14], [17], [18], [20] and the references therein. If we write the equation (1.3) in \( v \) with \( \tilde{g} = e^{2v}g \). Then the equation becomes a fully nonlinear elliptic equation in \( v \) with the exact form being given in section 2. In general, fully nonlinear elliptic equations involving \( f(\lambda(D^2v)) \) have been studied by Caffarelli, Nirenberg and Spruck ([4]) and many others. Fully nonlinear equations involving \( f(\lambda(\nabla^2_gv + g)) \) have been investigated by Li ([16]), Urbas ([23]) and others.

We organize our paper as follows. In section 2, we present some prerequisites and prove the uniqueness of the solution. We establish the \( C^0 \) estimates in section 3. In section 4, we introduce the tubular neighborhood normal coordinates and discuss some of its properties. In the next two sections, we use such coordinates to derive the gradient estimates and the Hessian estimates. In section 7, we establish the existence of the solution to the equation (12). In the last section, we prove the Theorem 1.3.

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2 Uniqueness

For \( \tilde{g} \in [g] \), write \( \tilde{g} = e^{2v}g \). We have the conformal transformation

\[
\begin{align*}
A_{\tilde{g}} &= -W_g^v + e^{2v}A_g^y, \\
h_{\tilde{g}} &= (h_g + v) e^{-v},
\end{align*}
\]

where \( \frac{\partial}{\partial \nu} \) is the unit outer normal of \( g \) on \( \partial M \) and

\[
W_g^v := \nabla^2_g v + \frac{1 - t}{n - 2} (\Delta_g v) g + \frac{2 - t}{2} |\nabla v|_g^2 g - dv \otimes dv.
\]

The equation (12) is equivalent to solving

\[
\begin{align*}
\{ \frac{f(-\lambda_{\tilde{g}}(A_{\tilde{g}}^y))}{h_{\tilde{g}}} &= \phi, \quad -\lambda_{\tilde{g}}(A_{\tilde{g}}^y) \in \Gamma \quad \text{on} \quad M \\
\}
\end{align*}
\]

\[
\begin{align*}
h_{\tilde{g}} + v \ &= \ e^v \phi(x) \quad \text{on} \quad \partial M.
\end{align*}
\]

(13)
Proof of the Uniqueness. In this section, we give an independent proof of the uniqueness of the solution for $t \leq 1$ even though we can see this later from the method of continuity and a suitable homotopy for $t < 1$. Let $v_1, v_2$ be two solutions of the equation (13), and let $g_i = e^{2v_i}g$ for $i = 1, 2$. Write $g_2 = e^{2w}g_1$ with $w = v_2 - v_1$. Then $v_2$ is a solution of the equation (13) is to say

$$
\begin{align*}
\{ f(\lambda_{g_1}(W_{g_1}^w - A_{g_1}^t)) &= \phi(x)e^{2w}, \quad \lambda_{g_1}(W_{g_1}^w - A_{g_1}^t) \in \Gamma \quad \text{on } M \\
h_{g_1} + w_{\nu_1} &= e^w\psi(x) \quad \text{on } \partial M,
\end{align*}
$$

where $\frac{\partial}{\partial n_1}$ is the unit outer normal w.r.t. $g_1$ on $\partial M$.

Note that $v_1$ is also a solution of (13), so $h_{g_1} = \psi$ and the above equation becomes

$$
\begin{align*}
\{ f(\lambda_{g_1}(W_{g_1}^w - A_{g_1}^t)) &= \phi(x)e^{2w}, \quad \lambda_{g_1}(W_{g_1}^w - A_{g_1}^t) \in \Gamma \quad \text{on } M \\
w_{\nu_1} &= (e^w - 1)\psi(x) \quad \text{on } \partial M.
\end{align*}
$$

Let $w(x_0) = \max_M w$.

**Lemma 2.1** $w(x_0) \leq 0$.

**Proof of the Lemma 2.1**

**Case 1.** If $x_0$ is an interior point of $M$, then $\nabla_{g_1}w(x_0) = 0$, $\nabla_{g_1}^2w(x_0) \leq 0$, and

$$W_{g_1}^w(x_0) = \nabla_{g_1}^2w(x_0) + \frac{1}{n-2}(\Delta_{g_1}w)(x_0)g_1(x_0) \leq 0,$$

which, together with (8), implies that

$$\phi(x_0)e^{2w(x_0)} = f(\lambda_{g_1}(W_{g_1}^w - A_{g_1}^t)(x_0)) \leq f(\lambda_{g_1}(-A_{g_1}^t)(x_0)) = \phi(x_0),$$

therefore $e^{2w(x_0)} \leq 1$, i.e., $w(x_0) \leq 0$.

**Case 2.** If $x_0 \in \partial M$, then $w_{\nu_1}(x_0) \geq 0$. By the second equation in (14), we know that

$$0 \leq w_{\nu_1}(x_0) = (e^w - 1)(x_0)\psi(x_0),$$

so either $(e^w - 1)(x_0) \leq 0$ when $\psi(x_0) < 0$, or $w_{\nu_1}(x_0) = 0$ when $\psi(x_0) = 0$, that is, when $\psi(x_0) < 0$, we have $w(x_0) \leq 0$, and when $\psi(x_0) = 0$, $w_{\nu_1}(x_0) = 0$ gives $\nabla_{g_1}w(x_0) = 0$, therefore $\nabla_{g_1}^2w(x_0) \leq 0$. We can proceed as in case 1 to conclude that $w(x_0) \leq 0$. Lemma 2.1 has been established. ♦

Let $w(y_0) = \min_M w$.

**Lemma 2.2** $w(y_0) \geq 0$. 

Proof of the Lemma 2.2

Case 1. If $y_0$ is an interior point of $M$, then $\nabla_{g_1} w(y_0) = 0$, $\nabla^2_{g_1} w(y_0) \geq 0$, and

$$W_{g_1}^w(y_0) = \nabla^2_{g_1} w(y_0) + \frac{1 - t}{n - 2} (\Delta_{g_1} w)(y_0) g_1(y_0) \geq 0,$$

which implies that

$$\phi(y_0) e^{2w(y_0)} = f(\lambda_{g_1} (W_{g_1}^w - A_{g_1}^t)(y_0)) \geq f(\lambda_{g_1} (-A_{g_1}^t)(y_0)) = \phi(y_0),$$

therefore $e^{2w(y_0)} \geq 1$, i.e., $w(y_0) \geq 0$.

Case 2. If $y_0 \in \partial M$, then $w_{\nu_1}(y_0) \leq 0$ and

$$0 \geq w_{\nu_1}(y_0) = (e^w - 1)(y_0) \psi(y_0),$$

so either $\psi(y_0) < 0$, we have $w(y_0) \geq 0$, or $\psi(y_0) = 0$, then $w_{\nu_1}(y_0) = 0$, which implies that $\nabla_{g_1} w(y_0) = 0$, therefore $\nabla^2_{g_1} w(y_0) \geq 0$. We can proceed as in case 1 to conclude that $w(y_0) \geq 0$. Lemma 2.2 has been established. ♣

Combining Lemma 2.1 and Lemma 2.2, we have $w \equiv 0$, that is, $v_1 \equiv v_2$. The uniqueness of the solution of the equation (12) has been proved. ♣

Remark 2.1 When $k = 1$, $c = 0$, the uniqueness of the solution has been obtained by Cheerier in [3] and it implies that the solution must be the unique minimum point of $F$.

3 $C^0$ estimates

When the manifold has some boundary, the $C^0$ estimate is not a trivial consequence of the maximum principle anymore. In this section, we obtain $C^0$ estimates by establishing the upper bounds and the lower bounds individually.

Lemma 3.1 Let $(M^n, g)$ and $(f, \Gamma)$ be as in Theorem 1.2. For $t \leq 1$, let $v$ be a $C^2$ solution of the equation (12). Then there exists a universal constant $C > 0$ depending only on $(M^n, g, t)$, $(f, \Gamma)$, $\phi$ and $\psi$ such that

$$v \leq C.$$
of the Lemma 3.1, but may change from line to line. Since \( \lambda_g(A^t_g) \in \Gamma \subset \Gamma_1 \) and \( h_g \leq 0 \), we have \( R_g < 0 \) and \( h_g \leq 0 \), from which, we know that \((M^n, g)\) is of negative type. Hence we can find \( g_0 = e^{2v_0} g \) such that

\[
\begin{align*}
R_{g_0} < 0 & \quad \text{on } M \\
h_{g_0} = 0 & \quad \text{on } \partial M.
\end{align*}
\]

Write \( e^{2v} g = e^{2\tilde{v}} g_0 \) with \( \tilde{v} = v - v_0 \). Then \( \tilde{v} \) satisfies

\[
\begin{cases}
f(\lambda_{g_0}(W_{g_0}^{\tilde{v}} - A^{t}_{g_0})) = \phi(x)e^{2\tilde{v}} & \lambda_{g_0}(W_{g_0}^{\tilde{v}} - A^{t}_{g_0}) \in \Gamma \quad \text{on } M \\
e^{\tilde{v}}\psi(x) & \lambda_{g_0}(W_{g_0}^{\tilde{v}} - A^{t}_{g_0}) \in \Gamma \quad \text{on } \partial M,
\end{cases}
\]

where \( \frac{\partial}{\partial v_0} \) is the unit outer normal of \( g_0 \) on \( \partial M \).

Let \( \tilde{v}(x_0) = \max_{M} \tilde{v} \).

**Case 1.** If \( x_0 \) is an interior point of \( M \), then \( \nabla_{g_0} \tilde{v}(x_0) = 0 \), \( \nabla^{2}_{g_0} \tilde{v}(x_0) \leq 0 \) and therefore \( W_{g_0}^{\tilde{v}}(x_0) \leq 0 \). Hence

\[
\lambda_{g_0}(W_{g_0}^{\tilde{v}} - A^{t}_{g_0})(x_0) \in \Gamma
\]

implies that \( \lambda_{g_0}(-A^{t}_{g_0}) \in \Gamma \). Thus by (9) and (10),

\[
e^{2\tilde{v}(x_0)}\phi(x_0) = f(\lambda_{g_0}(W_{g_0}^{\tilde{v}} - A^{t}_{g_0})(x_0)) \leq f(\lambda_{g_0}(-A^{t}_{g_0})(x_0)) \leq C\sigma_1(\lambda_{g_0}(-A^{t}_{g_0})(x_0)) \leq C \max_{M}(\lambda_{g_0}(x_0)) = C,
\]

so we have \( \tilde{v}(x_0) \leq C \).

**Case 2.** If \( x_0 \in \partial M \), then \( \psi(x_0) = 0 \). If not, then at \( x_0 \), the second equation in (15) implies that

\[
0 \leq \tilde{v}_{v_0}(x_0) = e^{\tilde{v}}(x_0)\psi(x_0) < 0,
\]

which is a contradiction. Thus \( \tilde{v}_{v_0}(x_0) = e^{\tilde{v}}(x_0)\psi(x_0) = 0 \), \( \nabla_{g_0} \tilde{v}(x_0) = 0 \), and \( \nabla^{2}_{g_0} \tilde{v}(x_0) \leq 0 \). We can proceed as in case 1 to obtain \( v(x_0) \leq C \).

Combining the above two cases, we have \( \tilde{v} \leq C \), which means \( v \leq C \). Lemma 3.1 has been established.

**Lemma 3.2** Let \((M^n, g)\) and \((f, \Gamma)\) be as in Theorem 1.2. For \( t \leq 1 \), let \( v \) be a \( C^2 \) solution of the equation (12). Then there exists a universal constant \( C > 0 \) depending only on \((M^n, g, t), (f, \Gamma), \phi \) and \( \psi \) such that

\[ v \geq -C. \]
Proof of the Lemma 3.2. Let \( \bar{w} \) be a smooth function such that \( \bar{w} \) is the distance function to \( \partial M \) near the boundary and \( \bar{w} \) takes value in \([0, 1]\) in general. Then \( \bar{w}_{\nu}|_{\partial M} \equiv -1 \). Let \( g_0 = e^{2\epsilon_0 \bar{w}} g \) with \( \epsilon_0 > 0 \) being a constant to be chosen later. We have

\[
h_{g_0} = (h_g + \epsilon_0 \bar{w}_\nu) e^{-\epsilon_0 \bar{w}} \leq -\epsilon_0 e^{-\epsilon_0} < 0, \tag{16}
\]

and

\[
-\lambda_{g_0}(A_{g_0}^t) = \lambda_g \left( \epsilon_0 [\nabla_g^2 \bar{w}] + \frac{1}{n - 2} (\Delta_g \bar{w}) g + \frac{2 - t}{2} \epsilon_0 |\nabla \bar{w}|^2 g - \epsilon_0 d \bar{w} \otimes d \bar{w} - A_g^t \right),
\]

so we can take \( \epsilon_0 \ll 1 \) depending only on \((M^n, g, t, f, \Gamma)\) such that

\[
-\lambda_{g_0}(A_{g_0}^t) \in \Gamma \quad \text{and} \quad f(-\lambda_{g_0}(A_{g_0}^t)) \geq \frac{1}{2} \min_M f(-\lambda_g(A_g^t)). \tag{17}
\]

Let \( \tilde{v} = v - \epsilon_0 \bar{w} \). Then \( e^{2\tilde{v}} g = e^{2\tilde{v}} g_0 \) and \( \tilde{v} \) solves

\[
\begin{align*}
\{ & f(\lambda_{g_0}(W_{g_0}^\tilde{v} - A_{g_0}^t)) = \phi(x) e^{2\tilde{v}}, \quad \lambda_{g_0}(W_{g_0}^\tilde{v} - A_{g_0}^t) \in \Gamma \quad \text{on} \ M, \\
\tilde{v}_{\nu_0} + h_{g_0} = e^{\tilde{v}}(x) \quad & \text{on} \ \partial M, \tag{18}
\end{align*}
\]

Let \( \tilde{v}(y_0) = \min_M \tilde{v} \).

**Case 1.** If \( y_0 \) is in the interior of \( M \), then \( \nabla_{g_0} \tilde{v}(y_0) = 0 \), \( \nabla_{g_0}^2 \tilde{v}(y_0) \geq 0 \) and \( W_{g_0}^\tilde{v}(y_0) \geq 0 \). Hence by (9), (17) and (18),

\[
e^{2\tilde{v}(y_0)} \phi(y_0) = f(\lambda_{g_0}(W_{g_0}^\tilde{v} - A_{g_0}^t)(y_0)) \geq f(\lambda_{g_0}(-A_{g_0}^t)(y_0)) \geq \frac{1}{2} \min_M f(\lambda_g(-A_g^t)),
\]

i.e.,

\[
\tilde{v}(y_0) \geq \frac{1}{2} \min_M \left( \frac{1}{2} \phi \min_M f(\lambda_g(-A_g^t)) \right) \geq -C.
\]

**Case 2.** If \( y_0 \in \partial M \), then \( \tilde{v}_{\nu_0}(y_0) \leq 0 \). By (16) and (18),

\[
-\epsilon_0 e^{-\epsilon_0} \geq h_{g_0}(y_0) + \tilde{v}_{\nu_0}(y_0) = e^{\tilde{v}(y_0)} \psi(y_0) \geq -Ce^{\tilde{v}(y_0)},
\]

so

\[
\tilde{v}(y_0) \geq \ln \frac{\epsilon_0}{C} - \epsilon_0 \geq -C.
\]

Combining the above two cases, we know \( \tilde{v} \geq -C \), hence \( v \geq -C \). Lemma 3.2 has been proved. ♣


4 Tubular Neighborhood Normal Coordinates

The main issue of the gradient and the Hessian estimates is the bounds on the boundary of \( M \). For this reason, we need to introduce certain coordinates near \( \partial M \).

Let \( g|_{\partial M} \) be the induced metric of \( g \) on \( \partial M \), and let \( \delta_1 > 0 \) be the minimum of the injectivity radius of \( (M^n, g) \) and the injectivity radius of \( (\partial M, g|_{\partial M}) \). Consider the map \( E: \partial M \times [0, \delta_1) \to M \) by \( E(y, t) = \exp_y (-t \frac{\partial}{\partial t}) \). Since \( E(y, 0) = y \) implies that, for any \( y \in \partial M \), \( dE|_{(y,0)}(X) = X \) for \( X \in T_y(\partial M) \), and \( dE|_{(y,0)}(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial y} \neq 0 \). That is, \( dE|_{(y,0)} \) is an isomorphism from \( T_{(y,0)}(\partial M \times [0, \delta_1)) \to T_yM \).

By the Implicit Function Theorem, there exists some constant \( \delta_0 \) such that, for any \( y \in \partial M \), there exists a unique \( \bar{y} \in (0, \delta_1) \) such that \( E \) is a smooth diffeomorphism on \( (\partial M \cap B_{\delta_0}(y)) \times [0, \delta_0) \), where \( B_{\delta_0}(y) \) is the open geodesic ball of \( (M^n, g) \) centered at \( y \) with radius \( \delta_0 \). By shrinking \( B_{\delta_0}(y) \), we can also assume the exponential map of \( (\partial M, g|_{\partial M}) \) at \( y \) is a smooth diffeomorphism in \( B_{\delta_0}(y) \cap \partial M \). Now we extend \( \frac{\partial}{\partial y} \) to the interior of \( M \), still denoted by \( \frac{\partial}{\partial y} \) such that \( \frac{\partial}{\partial y}|_{E(z,t)} = -\frac{\partial}{\partial t}|_{E(z,t)} \) for any \( z \in \partial M \cap B_{\delta_0}(y) \).

Then \( \frac{\partial}{\partial y} \) is a smooth unit vector field in \( E((\partial M \cap B_{\delta_0}(y)) \times [0, \delta_0)) \).

**Proposition 4.1** For any \( y_0 \in \partial M \),

\[
B_{\delta_{y_0}}(y_0) \subset E((\partial M \cap B_{\delta_{y_0}}(y_0)) \times [0, \delta_{y_0})).
\]

and for any \( y \in B_{\delta_{y_0}}(y_0) \), there exists a unique \( \bar{y} \in \partial M \) such that \( d(y, \bar{y}) = d(y, \partial M) \). Moreover \( \bar{y} \in B_{\delta_{y_0}}(y_0) \cap \partial M \).

**Proof of the Proposition 4.1** For any \( y \in B_{\delta_{y_0}}(y_0) \),

\[
s := d(y, \partial M) \leq d(y, y_0) < \frac{\delta_{y_0}}{2}.
\]

For any \( z \in \partial M \setminus B_{\delta_{y_0}}(y_0) \),

\[
d(y, z) \geq d(z, y_0) - d(y, y_0) > \delta_{y_0} - \frac{\delta_{y_0}}{2} = \frac{\delta_{y_0}}{2}.
\]

Thus if \( d(y, \partial M) = d(y, \bar{y}) \) for some \( \bar{y} \in \partial M \), then \( \bar{y} \in \partial M \cap B_{\delta_{y_0}}(y_0) \). Let \( r(t) \) be the normalized geodesic connecting \( y \) and \( \bar{y} \) such that \( r(0) = \bar{y} \) and \( r(s) = y \). Then \( \frac{dr}{dt}|_{t=0} = -\frac{\partial}{\partial y}|_{y} \), that is \( y = E(\bar{y}, s) \). Therefore \( y \in E((\partial M \cap B_{\delta_{y_0}}(y_0)) \times [0, \delta_{y_0})) \),

and \( B_{\delta_{y_0}}(y_0) \subset E((\partial M \cap B_{\delta_{y_0}}(y_0)) \times [0, \delta_{y_0})) \). Recall \( E \) is a smooth diffeomorphism in \( (\partial M \cap B_{\delta_{y_0}}(y_0)) \times [0, \delta_{y_0}) \) and \( \bar{y} \in \partial M \cap B_{\delta_{y_0}}(y_0) \). Thus \( \bar{y} \) is uniquely determined by \( y \). The Proposition 4.1 has been proved. \( \clubsuit \)
By the Proposition 4.1, $\frac{\partial}{\partial v} = -\frac{dE}{dt}$ is a smooth unit vector field in $B_{\frac{\delta y_0}{2}}(y_0)$. Moreover, in $B_{\frac{\delta y_0}{2}}(y_0)$, the parameter $t$ in $E(y, t)$ is the distance parameter to the boundary of $M$, which can be derived more precisely as in establishing (19). Let $\{y_j\}_{j=1}^{n-1}$ be the geodesic normal coordinates w.r.t. the metric $g|_{\partial M}$ at $y_0$. Then $\{y_j\}_{j=1}^{n-1}$ is smooth and well-defined in $\partial M \cap B_{\delta y_0}(y_0)$. For any $y \in B_{\frac{\delta y_0}{2}}(y_0)$, there is a unique $\bar{y} \in \partial M$ such that $d(y, \bar{y}) = d(y, \partial M)$. By the Proposition 4.1, $\bar{y} \in \partial M \cap B_{\delta y_0}(y_0)$. Let $(y_1, \ldots, y_{n-1})$ be the geodesic normal coordinates of $\bar{y}$ w.r.t. the metric $g|_{\partial M}$ at $y_0$. Define $(y_1, \ldots, y_{n-1}, y_n)$ as the coordinates of $y$ with $y_n = d(y, \partial M)$. Such coordinates are uniquely determined and smooth in $B_{\frac{\delta y_0}{2}}(y_0)$. The reason is that $\bar{y}$ is uniquely determined and $\bar{y} \in \partial M \cap B_{\delta y_0}(y_0)$, which implies that $y = E(\bar{y}, y_n)$. Hence the map from $y$ to $(\bar{y}, y_n)$ is the inverse of the smooth diffeomorphism $E$, therefore is also a smooth diffeomorphism, that is to say $(y_1, \ldots, y_n)$ is well-defined and smooth in $B_{\frac{\delta y_0}{2}}(y_0)$. We call such coordinates the tubular neighborhood normal coordinates of $y$ at $y_0$. Observe that $g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = \delta_{ij}$ for $1 \leq i, j \leq n-1$ at $y_0$. Moreover, such coordinates has the following proposition.

**Proposition 4.2** For $1 \leq j \leq n-1$,

$$\frac{\partial}{\partial y_n} = -\frac{\partial}{\partial v} - g(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_n}) = 0,$$ in $B_{\frac{\delta y_0}{2}}(y_0)$.

**Proof of the Proposition 4.2** For any $y \in B_{\frac{\delta y_0}{2}}(y_0)$ with $(a_1, \ldots, a_n)$ as its tubular neighborhood normal coordinates at $y_0$. Let $\bar{y} \in \partial M \cap B_{\delta y_0}(y_0)$ be the unique point such that $d(y, \bar{y}) = a_n < \frac{\delta y_0}{8}$. Clearly $\bar{y} \in B_{\frac{\delta y_0}{2}}(y_0)$ since

$$d(y, z) \geq d(z, y_0) - d(y, y_0) > \frac{\delta y_0}{4} - \frac{\delta y_0}{8} = \frac{\delta y_0}{8}$$ for any $z \in \partial M \setminus B_{\delta y_0}(y_0)$.

Let $r(t) = E(\bar{y}, t)$. Then $r$ is smooth and well-defined for $t \in [0, \delta_{y_0})$. For $t \in [0, \frac{\delta_{y_0}}{8})$, by

$$d(r(t), y_0) \leq d(r(t), \bar{y}) + d(\bar{y}, y_0) < \frac{\delta y_0}{8} + \frac{\delta y_0}{4} < \frac{\delta y_0}{2},$$

there exists a unique $\tilde{y} \in \partial M$ such that

$$d(r(t), \tilde{y}) = d(r(t), \partial M) =: d^t \leq d(r(t), \bar{y}) \leq t.$$
By the Proposition 4.1, $\tilde{y} \in B_{\delta_{y_0}}(y_0) \cap \partial M$ and $E(\tilde{y}, d') = r(t) = E(\bar{y}, t)$. Therefore $\tilde{y} = \bar{y}$ and $d' = t$ since $E$ is a smooth diffeomorphism on $(\partial M \cap B_{\delta_{y_0}}(y_0)) \times [0, \delta_{y_0})$. From which, we know that $(a_1, \ldots, a_{n-1}, t)$ is the tubular neighborhood normal coordinates of $r(t)$ at $y_0$ for $t \in [0, \frac{\delta_{y_0}}{8})$. Hence, for $t \in [0, \frac{\delta_{y_0}}{8})$,

$$\frac{\partial}{\partial y_n}|_{r(t)} = \frac{dr}{dt}|_t = \frac{dE}{dt}|_{(\bar{y}, t)} = -\frac{\partial}{\partial \nu}|_{E(\bar{y}, t)} = -\frac{\partial}{\partial \nu}|_{r(t)}.$$

In particular,

$$\frac{\partial}{\partial y_n}|_y = \frac{\partial}{\partial y_n}|_{r(a_n)} = -\frac{\partial}{\partial \nu}|_{r(a_n)} = -\frac{\partial}{\partial \nu}|_y.$$

To prove the second statement in the proposition, we consider the set

$$\mathcal{S} := \{z \in B_{\delta_{y_0}}(y_0) | d(z, \partial M) = a_n\}.$$

Clearly, $y \in \mathcal{S} \neq \emptyset$. For any $z \in \mathcal{S}$, let $r(t) = E(\bar{z}, t)$ for some $\bar{z} \in B_{\delta_{y_0}}(y_0) \cap \partial M$ such that $r(0) = \bar{z}$ and $r(a_n) = z$. As derived earlier, $d(r(t), \partial M) = t$ for any $t \in [0, \frac{\delta_{y_0}}{8})$, which implies that $r([0, \frac{\delta_{y_0}}{8}))$ intersects $\mathcal{S}$ at a single point $z = r(a_n)$. Moreover, we claim that

$$d(r(t), \mathcal{S}) = t - a_n, \quad \forall \ t \in [a_n, \frac{\delta_{y_0}}{8}). \quad (19)$$

Notice that, for $t \in [a_n, \frac{\delta_{y_0}}{8})$, $d(r(t), \mathcal{S}) \leq d(r(t), r(a_n)) \leq t - a_n$. If (19) does not hold, then $d(r(t), \mathcal{S}) < t - a_n$, which implies that there exists some $\bar{z} \in \mathcal{S}$ such that $d(r(t), \bar{z}) < t - a_n$. Therefore

$$t = d(r(t), \partial M) \leq d(r(t), \bar{z}) + d(\bar{z}, \partial M) < t - a_n + a_n = t,$$

which is a contradiction. Next, we claim

$$d(r(t), \mathcal{S}) = a_n - t, \quad \forall \ t \in [0, a_n). \quad (20)$$

If not, then $d(r(t), \mathcal{S}) < a_n - t$ since $d(r(t), \mathcal{S}) \leq d(r(t), r(a_n)) \leq a_n - t$, so there exists some $\hat{z} \in \mathcal{S}$ such that $d(r(t), \hat{z}) < a_n - t$, which implies that

$$a_n = d(\hat{z}, \partial M) \leq d(r(t), \hat{z}) + d(r(t), \partial M) < a_n - t + t = a_n,$$

which is a contradiction.
By (19) and (20), we know \( r(a_n) \) is a point in \( S \) such that \( d(r(a_n), r(t)) = \frac{1}{2}(r(t), S) \) for \( t \in [0, \frac{\delta_{y_0}}{16}] \), and \( r \) is the normalized geodesic connecting \( r(t) \) and \( r(a_n) \), so \( \frac{d}{dt} |_{a_n} = \frac{dE}{dt}(\varepsilon, a_n) \) is the unit normal vector of \( S \) at \( r(a_n) = z \), i.e., \( \frac{\partial}{\partial y_0} = -\frac{\partial}{\partial v} = \frac{dE}{dt} \) is the unit normal vector of \( S \) at \( r(a_n) = z \). Let \( (b_1, \ldots, b_{n-1}, a_n) \) be the tubular neighborhood normal coordinates of \( z \) at \( y_0 \). Observe that, for \( 1 \leq k \leq n - 1 \), since \( z \) is an interior point of \( B_{\delta_{y_0}}(y_0) \), the curve

\[
\{(y_1, \ldots, y_k, \ldots, y_n) = (b_1, \ldots, b_{k-1}, y_k, b_{k+1}, \ldots, a_n)\} \quad \text{for } y_k \text{ near } b_k
\]
is contained in \( S \), which implies that \( \{\frac{\partial}{\partial y_k} \} \in T_zS \). Hence \( g(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_n}) = 0 \) at \( z \) for \( 1 \leq k \leq n - 1 \) since \( \frac{\partial}{\partial y_k} = -\frac{\partial}{\partial v} \) is the normal vector of \( S \) at \( z \). \( z \in S \) is arbitrary and \( y \in S \), so, at \( y \), we also have

\[
g(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_n}) = 0 \quad \text{for } 1 \leq k \leq n - 1.
\]

The Proposition 4.2 has been proved. ♦

As a simple consequence, we have \( g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = \delta_{ij} \) at \( y_0 \) for \( 1 \leq i, j \leq n \).

**Proposition 4.3**

\[
\{(y_1, \ldots, y_n) | \sqrt{y_1^2 + \cdots + y_n^2} < \frac{\delta_{y_0}}{16}, \quad y_n \geq 0\} \subset B_{\frac{\delta_{y_0}}{16}}(y_0),
\]

where \((y_1, \ldots, y_n)\) is the tubular neighborhood normal coordinates at \( y_0 \).

**Proof of the Proposition 4.3** For any \((y_1, \ldots, y_n)\) with \( \sqrt{y_1^2 + \cdots + y_n^2} < \frac{\delta_{y_0}}{16} \) and \( y_n \geq 0 \), there exists a unique \( \bar{y} \in B_{\frac{\delta_{y_0}}{16}}(y_0) \) such that \((y_1, \ldots, y_{n-1})\) is the geodesic normal coordinates of \( \bar{y} \) w.r.t. the metric \( g|_{\partial M} \) at \( y_0 \). Consider \( r(t) = E(\bar{y}, t) \). Then \( r(t) \) is smooth for \( t \in [0, \frac{\delta_{y_0}}{16}] \) and \( r([0, \frac{\delta_{y_0}}{16}]) \subset B_{\frac{\delta_{y_0}}{16}}(y_0) \). Moreover \( d(r(t), \partial M) = t \) for \( t \in [0, \frac{\delta_{y_0}}{16}] \) as shown earlier. In particular, by \( y_n < \frac{\delta_{y_0}}{16} \), \( y = E(\bar{y}, y_n) \) has \((y_1, \ldots, y_n)\) as its tubular neighborhood normal coordinates at \( y_0 \). The Proposition 4.3 has been proved. ♦

Denote \( B_{\frac{\delta_{y_0}}{16}}^T(y_0) := \{(y_1, \ldots, y_n) | \sqrt{y_1^2 + \cdots + y_n^2} < \frac{\delta_{y_0}}{16}, \quad y_n \geq 0\} \), which is different from the geodesic ball \( B_{\frac{\delta_{y_0}}{16}}(y_0) \). The Proposition 4.3 says that \( B_{\frac{\delta_{y_0}}{16}}^T(y_0) \subset B_{\frac{\delta_{y_0}}{8}}(y_0) \). Since \( \cup_{y_0 \in \partial M} B_{\frac{\delta_{y_0}}{16}}^T(y_0) = \partial M \) and \( \partial M \) is compact, we can find \( \{y_i^N\}_{i=1}^{N} \subset \partial M \) such that \( \cup_{i=1}^{N} (B_{\frac{\delta_{y_0}}{16}}^T(y_i^N) \cap \partial M) = \partial M \).
5 Gradient estimates

Lemma 5.1 Under the same assumptions as in Theorem 1.2, for \( t < 1 \), let \( v \) be a \( C^3 \) solution of the equation (12). Then there exists a universal constant \( C > 0 \) depending only on \( (M^n, g, t), (f, \Gamma), \phi, \) and \( \psi \), such that

\[ |\nabla v|_g \leq C \quad \text{on } \partial M. \]

Proof of the Lemma 5.1. Extend \( h_g \) to a smooth function on \( M \), and \( \psi \) to a \( C^{3,\alpha} \) function on \( M \). More explanation is given in section 7. We still denote the extended functions by \( \psi, h_g \) respectively. For each \( 1 \leq i_0 \leq N \), let \( \{y_j\}_{j=1}^n \) be the tubular neighborhood normal coordinates at \( y^{i_0} \). Let \( \rho = \rho(y_1^2 + \cdots + y_n^2) \) be a smooth cut-off function satisfying

\[ \rho(y) = \begin{cases} 1 & \text{if } y \in B_{\frac{\delta y}{16}}^{\frac{\delta y}{16}}(y^{i_0}) \\ \in [0, 1] & \text{if } y \in B_{\frac{\delta y}{16}}^{\frac{\delta y}{16}}(y^{i_0}) \backslash B_{\frac{\delta y}{32}}^{\frac{\delta y}{32}}(y^{i_0}) \\ 0 & \text{otherwise,} \end{cases} \]

and let \( \beta(y) \) be a smooth function in \( B_{\frac{\delta y}{16}}^{\frac{\delta y}{16}}(y^{i_0}) \) satisfying

\[ \beta(y) = \begin{cases} y_n, & \text{if } y_n < \delta_0, \\ \in [0, 2\delta_0], & \text{o.w.,} \end{cases} \]

where \( 0 < \delta_0 < \frac{\delta y}{32} \) is a very small constant such that \( 1 + 2\delta_0\psi e^v > \frac{1}{2} \) on \( M \) and to be chosen later. Then in \( B_{\frac{\delta y}{16}}^{\frac{\delta y}{16}}(y^{i_0}) \cap \partial M \),

\[ \beta \equiv 0, \quad \beta_{\nu} \equiv -1. \quad \text{(21)} \]

Let

\[ \gamma := (\psi e^v - h_g)\beta, \]

In the following, we use subindices to denote the covariant derivatives w.r.t. \( \frac{\partial}{\partial y_{j_k}} \), e.g.,

\[ (v + \gamma)_k = \left( \nabla(v + \gamma) \right)(\frac{\partial}{\partial y_k}), \quad (v + \gamma)_{k\nu} = \left( \nabla^2(v + \gamma) \right)(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial \nu}). \]
Consider
\[ G := \rho \sum_k (v + \gamma)_k^2 \alpha \left( \frac{v + \gamma + L}{L^2} \right), \]
where \( L > 0 \) is a constant satisfying \( 1 < v + \gamma + L < 2L \) and \( \alpha : R^+ \rightarrow R^+ \) is a smooth positive function to be chosen later.

Notice that \( \frac{\partial}{\partial \nu} = -\frac{\partial}{\partial y_n} \) in \( B_T^{\frac{y_{10}}{16}}(y_{10}) \) and
\[ B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M = \{ y \in B_T^{\frac{y_{10}}{16}}(y_{10}) \mid y_n = 0 \text{ and } \sqrt{y_1^2 + \cdots + y_{n-1}^2} < \frac{\delta_{y_{10}}}{16} \}. \]
Hence in \( B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M, \)
\[ \rho \nu = -\frac{\partial \rho}{\partial y_n}|_{y_n=0} = 0. \quad (22) \]

Claim 5.1 In \( B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M, G_\nu \equiv 0. \)

Proof of the Claim 5.1. In \( B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M, \) by (21) and the second equation in (12),
\[ (v + \gamma)_\nu = v_\nu + ((\psi e^v - h_g)\beta)_\nu = (\psi e^v - h_g)_\nu \beta + (\psi e^v - h_g)\beta_\nu 
= (\psi e^v - h_g) - (\psi e^v - h_g) = 0 \]
Therefore in \( B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M, \)
\[ (v + \gamma)_{k,\nu} = -(v + \gamma)_{k,n} = -(v + \gamma)_{n,k} = (v + \gamma)_{\nu,k} = 0, \quad \forall \quad 1 \leq k \leq n - 1, \quad (24) \]
where in the last equality, we used the fact that \( \frac{\partial}{\partial y_k} \) is a tangent vector field of \( B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M. \)

In \( B_T^{\frac{y_{10}}{16}}(y_{10}) \cap \partial M, \) by (22) and (23),
\[ G_\nu = 2\rho \alpha \frac{(v + \gamma + L)}{L^2} \sum_{k=1}^n (v + \gamma)_k (v + \gamma)_{k,\nu} 
= 2\rho \alpha \frac{(v + \gamma + L)}{L^2} (v + \gamma)_n (v + \gamma)_{n,\nu} \quad \text{by 24) \quad} 
= 2\rho \alpha \frac{(v + \gamma + L)}{L^2} (v + \gamma)_\nu (v + \gamma)_{\nu,\nu} = 0. \]
Claim 5.1 has been proved. ♣.
Let $G(x_0) = \max_{B^T_{\frac{1}{16}}(y^{i_0})} G$ for some $x_0 \in B^T_{\frac{1}{16}}(y^{i_0})$. W.l.o.g., $G(x_0) \geq 1$. By the

Claim 5.1, we have

$$\nabla G(x_0) = 0, \quad \nabla^2 G(x_0) \leq 0.$$

In $B^T_{\frac{1}{16}}(y^{i_0})$

$$G_i = \rho_i \alpha \sum_k (v + \gamma)^2_k + 2 \rho \alpha (v + \gamma)_{k,i}(v + \gamma)_k + \frac{\alpha' L^2}{2\rho} (v + \gamma)_i \sum_k (v + \gamma)_k^2$$

so at $x_0$,

$$(v + \gamma)_k(v + \gamma)_{k,i} = -\frac{\alpha' L^2}{2\rho} \sum_k (v + \gamma)_k^2(v + \gamma)_i - \frac{\rho_i}{2\rho} \sum_k (v + \gamma)_k^2, \quad (25)$$

and

$$G_{ij}(x_0) = 2 \rho \alpha (v + \gamma)_{k,i}(v + \gamma)_{k,j} + 2 \rho \alpha (v + \gamma)_k(v + \gamma)_{k,ij} + \frac{\alpha' L^2}{2\rho} (v + \gamma)_i(v + \gamma)_j + \frac{\alpha'' L^2}{\rho^2} (v + \gamma)_{ij}$$

At $x_0$, assume $e_i = a^i_j \frac{\partial}{\partial y_j}$.

Then $g(e_i, e_j) = \delta_{ij}$ is to say that $A^T A = G^{-1}$, where

$A = (a^i_j), \quad G^{-1} = (g_{ij})^{-1}$, and $g_{ij} = g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$. Denote $B = (F_{ij})$ and $D = (G_{ij})$. By $\nabla^2 G(e_i, e_j) \leq 0$, we have $\sum_i \nabla^2 G(e_i, e_i) = g^{ij}G_{ij}(x_0) \leq 0$ and

$$0 \geq F_{ij} \nabla^2 G(e_i, e_j)(x_0) = F_{ij} a^i_r a^j_s G_{rs} = \text{tr}(BA^T DA) = \text{tr}(BA^T AD)$$

i.e., we have $L^{ij}G_{ij}(x_0) \leq 0$.

In the following, we use $C_1 > 0$ to denote a universal constant depending only on $(M^n, g, t)$, $\psi$, $\delta_{y0}$, $L$, $\alpha$, and we use $C_2 > 0$ to denote a universal constant depending only on $(M^n, g, t)$, $\psi$, $\delta_{y0}$, $L$, $\delta_0$, $\alpha$, $\beta$. We also use $O_1(1)$ to denote a quantity bounded by $C_1$, and $O_2(1)$ to denote a quantity bounded by $C_2$. Observe that $\frac{1}{C_1} \delta^{ij} \leq G^{-1} \leq C_1(\delta^{ij})$ in $B^T_{\frac{1}{16}}(y^{i_0})$. We will use this fact without mentioning.
At $x_0$, 

\[
0 \geq \bar{L}^{ij}G_{ij} = 2\rho\alpha\bar{L}^{ij}(v + \gamma)_{k,i}(v + \gamma)_{k,j} + 2\rho\alpha \bar{L}^{ij}(v + \gamma)_{k}(v + \gamma)_{k,ij} \\
+ \bar{L}^{ij} \left( \frac{\alpha(\rho\alpha - 2\rho_0\rho)}{L_\alpha} + \frac{\rho(\alpha\alpha'' - 2(\alpha')^2)}{L_\alpha^2} (v + \gamma)_i (v + \gamma)_j \right) \\
+ \frac{\rho\alpha}{L_\alpha} (v + \gamma)_{ij} - \frac{\rho\alpha}{L_\alpha} (\rho_j (v + \gamma)_i + \rho_i (v + \gamma)_j) \sum_k (v + \gamma)_k^2 \\
\geq 2\rho\alpha \bar{L}^{ij}(v + \gamma)_{k,i}(v + \gamma)_{k,j} + 2\rho\alpha \bar{L}^{ij}(v + \gamma)_{k}(v + \gamma)_{k,ij} \\
+ \frac{\rho(\alpha\alpha'' - 2(\alpha')^2)}{L_\alpha^2} \sum_k (v + \gamma)_k^2 \bar{L}^{ij}(v + \gamma)_i (v + \gamma)_j \\
+ \frac{\rho\alpha}{L_\alpha} (v + \gamma)_{ij} - C_1 \sqrt{\rho} \sum_{k,l} F^{il}(v + \gamma)_k^3 - C_1 \sum_{k,l} F^{il}(v + \gamma)_k^2 \tag{26}
\]

where in the last inequality, we used $G(x_0) \geq 1$, therefore $\sqrt{\rho} \sum_k |(v + \gamma)_k| \geq \frac{1}{C_1}$.

In general, 

\[
(v + \gamma)_{i,j,k} = \frac{\partial}{\partial y_{ij}} ((v + \gamma)_{i,j} - \Gamma_{ji}^l (v + \gamma)_l) \\
= (v + \gamma)_{i,j,k} - \Gamma_{ji}^l (v + \gamma)_l - \frac{\partial}{\partial y_{ik}} (v + \gamma)_l,
\]

so 

\[
(v + \gamma)_{k,i,j} = \nabla \frac{\partial}{\partial y_{ij}} (v + \gamma)_k = (v + \gamma)_{k,i,j} - \Gamma_{ji}^l (v + \gamma)_k,l \\
= (v + \gamma)_{i,j,k} + \frac{\partial}{\partial y_{ik}} (v + \gamma)_l,
\]

and 

\[
0 \geq \bar{L}^{ij}G_{ij}(x_0) \\
\geq 2\rho\alpha \bar{L}^{ij}(v + \gamma)_{k,i}(v + \gamma)_{k,j} + 2\rho\alpha \bar{L}^{ij}(v + \gamma)_{k}(v + \gamma)_{k,ij,k} \\
+ \frac{\rho(\alpha\alpha'' - 2(\alpha')^2)}{L_\alpha^2} \sum_k (v + \gamma)_k^2 \bar{L}^{ij}(v + \gamma)_i (v + \gamma)_j \\
+ \frac{\rho\alpha}{L_\alpha} (v + \gamma)_{ij} - C_1 \sqrt{\rho} \sum_{k,l} F^{il}(v + \gamma)_k^3. \tag{27}
\]

Recall that $\gamma = (\psi e^v - h_g)\beta$. At $x_0$, 

\[
(v + \gamma)_k = (1 + \psi\beta e^v)v_k + e^v \beta \psi_k + \psi e^v \beta_k - (h_g \beta)_k \\
= av_k + O_2(1) \quad \text{with} \quad a := 1 + \psi\beta e^v.
\]

\[
\sum_k (v + \gamma)_k^2 = a^2 \sum_k v_k^2 + O_2(1) \sum_k |v_k|. \tag{29}
\]
(v + \gamma)_{ij} = av_{ij} + e^v(\psi_i \psi_j + \psi_i + \psi_j) + e^v(\psi_i \psi_j + \psi_i \psi_j) + e^v(\psi_i \psi_j + \psi_i \psi_j) \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) - (h_\beta)_{ij} \\
= av_{ij} + O_2(1) \sum_k |v_k| + O_1(1) \beta \sum_k v_k^2. \tag{30}

The above identity (30) also holds for (v + \gamma)_{ij} after a slight modification, i.e., we only need to change \(v_{ij}, \psi_{ij}, \beta_{ij}, (h_\beta)_{ij} \) to \(v_{ij}, \psi_{ij}, \beta_{ij}, (h_\beta)_{ij} \) respectively.

\( (v + \gamma)_{ij,k} = av_{ij,k} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) + e^v(\psi_i \psi_j + \psi_i \psi_j) \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} \\
+ e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} + e^v(\psi_i \psi_j + \psi_i \psi_j) v_{ij} \\
+ O_2(1) \sum_k v_k^2. \tag{31}

By (25) and (28-30), at \(x_0\),

\[
(v + \gamma)_{k} = av_{k} + O_2(1) \sum_l |v_l| + O_2(1) \beta \sum_l v_l^2
\]

\[
= -\frac{a'}{2L^2 \alpha} \left( a^2 \sum_l v_l^2 + O_2(1) \sum_l |v_l| \right) (av_{i} + O_2(1)) \\
- \frac{\rho}{2 \rho} \left( a^2 \sum_l v_l^2 + O_2(1) \sum_l |v_l| \right),
\]

therefore

\[
a(v + \gamma)_{k}v_{k,i} + (av_{k} + O_2(1)) \left( O_2(1) \sum_l |v_l| + O_1(1) \beta \sum_l v_l^2 \right) \\
= -\frac{a'}{2L^2 \alpha} a^3 \sum_l v_l^2 v_i + O_2(1) \frac{1}{\sqrt{\rho}} \sum_l v_l^2,
\]

which implies that

\[
(v + \gamma)_{k}v_{k,i} = -\frac{a'}{2L^2 \alpha} a^2 \sum_l v_l^2 v_i + O_1(1) \beta \sum_l |v_l|^3 + O_2(1) \frac{1}{\sqrt{\rho}} \sum_l v_l^2, \tag{32}
\]

where we used \(a = 1 + \psi \beta e^v \in [\frac{1}{2}, 1]\).

Combine (28-32). At \(x_0\),
\[ 2\alpha_\rho(v + \gamma)_k L^{ij}(v + \gamma)_{ij,k} \geq \]
\[ 2\alpha_\rho(v + \gamma)_k \bar{L}^{ij} v_{ij,k} + 2\alpha_\rho v^v(v_k + \gamma_k)(\psi_k \beta + \psi_\beta v_k) \bar{L}^{ij} v_{ij} + 4\alpha_\rho v^v \psi_\beta (v_k + \gamma_k) v_{i,k} + 4\alpha_\rho v^v \psi_\beta \mathcal{L}^{ij} v_j (v_k + \gamma_k) v_{i,k} + 2\alpha_\rho \psi_\beta (v_k + \gamma_k) \bar{L}^{ij} v_{ij} v_k - C_2\rho \sum_{k,l} F^{il} |v_k|^3 \]
\[ \geq 2\alpha_\rho(v + \gamma)_k \bar{L}^{ij} v_{ij,k} + 2\alpha_\rho v^v(v + \gamma)_k(\psi_k \beta + \psi_\beta + \psi_\beta v_k) \bar{L}^{ij} v_{ij} + 4\alpha_\rho v^v \psi_\beta (v + \gamma)_k(\psi_k \beta + \psi_\beta v_k) \bar{L}^{ij} v_{ij} + 2\alpha_\rho \psi_\beta (av_k + O_2(1)) \bar{L}^{ij} v_{ij} v_k - C_2\sum_{k,l} F^{il} |v_k|^3 \]
\[ \geq 2\alpha_\rho(v + \gamma)_k \bar{L}^{ij} v_{ij,k} + 2\alpha_\rho v^v(v + \gamma)_k(\psi_k \beta + \psi_\beta v_k) \bar{L}^{ij} v_{ij} - C_1\beta \rho \sum_{k,l} F^{il} v_k^4 - C_2\rho \sum_{k,l} F^{il} |v_k|^3, \]
that is,
\[
\bar{L}^ij v_{ij} = F^{ir} g^{jr} v_i v_j - \frac{2-t}{2} v_k v_l g^{kl} \sum_i F^{il} + O_1(1) \sum_{k,l} F^{il} |v_k|.
\]
From which, we have
\[
\frac{\alpha' \rho}{L^2} \sum_k (v + \gamma)^2 \bar{L}^ij (v + \gamma)_{ij}
= \frac{\alpha' \rho}{L^2} \left( a^2 \sum_k v_k^2 + O_2(1) \sum_k |v_k| \right) \bar{L}^ij v_{ij}
+ O_1(1) \beta \sum_k v_k^2 \bar{L}^ij v_{ij}
\geq \frac{a \alpha' \rho}{L^2} \left( a^2 \sum_k v_k^2 + O_2(1) \sum_k |v_k| \right) \bar{L}^ij v_{ij}
- C_1 \beta \rho \sum_k v_k^2 - C_2 \rho \sum_{k,l} F^{il} |v_k|^3
\geq \frac{a \alpha' \rho}{L^2} \left( a^2 \sum_k v_k^2 + O_2(1) \sum_k |v_k| \right) \left( F^{ir} g^{jr} v_i v_j - \frac{2-t}{2} v_k v_l g^{kl} \sum_i F^{ii} \right)
+ O_1(1) \sum_{k,l} F^{il} |v_k| - C_1 \beta \rho \sum_{k,l} F^{il} v_k^4 - C_2 \rho \sum_{k,l} F^{il} |v_k|^3
\geq \frac{a^3 \alpha' \rho}{L^2} \sum_k v_k^2 \left( F^{ir} g^{jr} v_i v_j - \frac{(2-t) a^2 \alpha' \rho}{2L^2} v_k v_l g^{kl} \sum_{i,j} F^{ii} v_j^2 \right)
- C_2 \rho \sum_{k,l} F^{il} |v_k|^3
\]
and
\[
2\alpha \rho e^{\nu} (v + \gamma)_k (\psi_k \beta + \psi \beta_k + \psi v_k) \bar{L}^ij v_{ij} \geq -C_1 \beta \rho \sum_{k,l} F^{il} v_k^4 - C_2 \rho \sum_{k,l} F^{il} |v_k|^3.
\]
which implies, by (33), that
\[
2\alpha \rho (v + \gamma)_k \bar{L}^ij (v + \gamma)_{ij,k} \bar{L}^ij v_{ij} \geq 2a \alpha \rho (v + \gamma)_k \bar{L}^ij v_{ij,k}
- C_1 \beta \rho \sum_{k,l} F^{il} v_k^4 - C_2 \rho \sum_{k,l} F^{il} |v_k|^3,
\]
Differentiate the equation \(F(\bar{W}_{ij} g^{jr}) = \phi e^{2\nu}\) along the \(y_k - th\) direction and evaluate at \(x_0\).
\[ \phi_k e^{2v} + 2\psi e^{2v} v_k = F^{ir} \left( g^{jr} \bar{W}_{ij} \right)_k = F^{ir} g^{jr} \left( \bar{W}_{ij} \right)_k + \frac{\partial g^{jr}}{\partial y_k} F^{ir} \bar{W}_{ij} \]

\[ = F^{ir} g^{jr} \left( v_{ij,k} + \frac{1}{n-2} (\Delta g) v_{ij} \right) + \frac{1}{n-2} \frac{\partial g^{jr}}{\partial y_k} (\Delta g) v + \frac{1}{n-2} \frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} (2v_{m,k} v_l g^{ml} + v_m v_l \frac{\partial g^{ml}}{\partial y_k}) g_{ij} + 2 \frac{1}{2} \frac{1}{2} v_m v_l g^{ml} \frac{\partial g_{ij}}{\partial y_k} - 2 v_{i,k} v_{j} - (A^t g)_{ij,k} + \frac{\partial g^{jr}}{\partial y_k} F^{ir} \left( v_{ij} + \frac{1}{n-2} (\Delta g) v_{ij} + \frac{1}{n-2} |\nabla v|^2 g_{ij} - v_{i,j} - (A^t g)_{ij} \right) \]

\[ = F^{ir} g^{jr} v_{ij,k} + \frac{1}{n-2} (\Delta g) v_{ij} \sum_i F^{ii} - 2 F^{ir} g^{jr} v_{i,k} v_{j} + \frac{\partial g^{jr}}{\partial y_k} F^{ir} \left( v_{ij} + \frac{1}{n-2} (\Delta g) v_{ij} \right) + O_1(1) \sum_{ij} F^{ii} v_{j}^2 \]

\[ = F^{ir} g^{jr} v_{ij,k} + \frac{1}{n-2} (\Delta g) v_{ij} \sum_i F^{ii} + (2 - t) v_{m,k} v_l g^{ml} \sum_i F^{ii} - 2 F^{ir} g^{jr} v_{i,k} v_{j} + O_1(1) \sum_{ij} F^{ii} |v_{j}| + O_1(1) \sum_{ij} F^{ii} (v_{j}^2 + 1) \]

\[ = \bar{L}^{ij} v_{ij,k} + (2 - t) v_{m,k} v_l g^{ml} \sum_i F^{ii} - 2 F^{ir} g^{jr} v_{i,k} v_{j} + O_1(1) \sum_{ij} F^{ii} |v_{j}| + O_1(1) \sum_{ij} F^{ii} v_{j}^2 \]

\[ = \bar{L}^{ij} v_{ij,k} + (2 - t) v_{m,k} v_l g^{ml} \sum_i F^{ii} - 2 F^{ir} g^{jr} v_{i,k} v_{j} + O_1(1) \sum_{ij} F^{ii} |v_{j}| + O_1(1) \sum_{ij} F^{ii} v_{j}^2. \]

Multiply both sides by \( 2\alpha a\rho(v + \gamma)_k \) and solve it for \( 2\alpha a\rho(v + \gamma)_k \bar{L}^{ij} v_{ij,k}. \)

\[ 2\alpha a\rho(v + \gamma)_k \bar{L}^{ij} v_{ij,k} = -2(2 - t)\alpha a\rho(v + \gamma)_k v_{m,k} v_l g^{ml} \sum_i F^{ii} + 4\alpha a\rho(v + \gamma)_k v_{i,k} F^{ir} g^{jr} v_{j} + O_1(1) \rho \sum_{i,j,k} F^{ii} |v_{j}| + O_1(1) \rho \sum_{i,j,k} F^{ii} v_{j}^2 |(v + \gamma)_k| \]

\[ = -2(2 - t)\alpha a\rho \left( \frac{\alpha'}{2L^2} a^2 \sum_j v_j^2 v_m + O_1(1) \beta \sum_j |v_j|^3 \right) \]
Substitute the above inequality into (35).

\[ +O_2(1) \frac{1}{\sqrt{\rho}} \sum_j v_j^2 \nu \gamma^{ml} \sum_i F^{ii} + 4\alpha a \rho \left( - \frac{\alpha'}{2L^2 \alpha} \sum_i v_i^2 \right) \]

\[ +O_1(1) \beta \sum_i |v_i|^3 + O_2(1) \frac{1}{\sqrt{\rho}} \sum_i v_i^2 \nu^{ijr} \nu_j \] by (32)

\[ +O_1(1) \rho \sum_{i,j} F^{ii} |v_{j,l}| |v_k| + O_2(1) \sqrt{\rho} \sum_{i,j} F^{ii} |v_j|^3 \]

\[ \geq \frac{(2 - t) a^3 \alpha' \rho}{L^2} \sum_{i,j} F^{ii} v_j^2 v_m v_g^{ml} - \frac{2a^3 \rho \alpha'}{L^2} \sum_i v_i^2 F^{ijr} v_j v_i \]

\[ -C_1 \beta \rho \sum_{i,j} F^{ii} v_j^4 - C_1 \rho \sum_{i,j,k,l} F^{ii} |v_{j,l}| |v_k| - C_2 \sqrt{\rho} \sum_{i,j} F^{ii} |v_j|^3. \] (36)

By (28) and (29),

\[ \frac{\rho (\alpha a - 2(\alpha')^2)}{L^2 \alpha} \sum_k (v + \gamma) \nu_k \nu^{ij}(v + \gamma)_i (v + \gamma)_j - C_1 \sqrt{\rho} \sum_{k,l} F^{ii} |(v + \gamma)_k|^3 \]

\[ \geq \frac{\rho (\alpha a - 2(\alpha')^2)}{L^2 \alpha} \left( a^2 v_k^2 + O_2(1) |v_k| \right) \nu^{ij}(av_i + O_2(1))(av_j + O_2(1)) \]

\[ -C_1 \sqrt{\rho} \sum_{k,l} F^{ii} (a^3 |v_k|^3 + O_2(1) v_k^2) \] (37)

\[ \geq \frac{\alpha a - 2(\alpha')^2}{L^2 \alpha} \rho a^4 \sum_k v_k^2 \nu^{ij} v_i v_j - C_2 \sqrt{\rho} \sum_{k,l} F^{ii} |v_k|^3, \]

and

\[ 2\rho \alpha \nu^{ij}(v + \gamma)_k (v + \gamma)_k \]

\[ = 2\rho \alpha \nu^{ij} \left( av_{k,i} + O_1(1) \beta \sum_i v_i^2 + O_2(1) \sum_i |v_i| \right) \left( av_{k,j} \right) \]

\[ + O_1(1) \beta \sum_i v_i^2 + O_2(1) \sum_i |v_i| \]

\[ \geq 2a^2 \rho \alpha \nu^{ij} v_i v_{k,j} - C_1 \beta \rho \sum_{i,j,k,l} F^{ii} v_j^2 |v_{k,l}| - C_2 \rho \sum_{i,j,k,l} F^{ii} |v_{j,l}| \]

\[ -C_1 \beta \rho \sum_{i,j} F^{ii} v_j^4 - C_2 \rho \sum_{i,j} F^{ii} |v_j|^3. \] (38)

Substitute (34), (36), (37), and (38) into (27). We have,
\[
0 \geq L^{ij}_{ij}(x_0) \geq \frac{(2-t)\alpha^2\rho}{2L^2} \sum_{i,j} F^{ii}_{ij} v_j^2 v_m v_l g^{ml} - \frac{\rho\alpha'}{L^2} \sum_l v_l^2 F^{ir} g^{jr} v_j v_j \\
+ \frac{\rho(\alpha'' - (\alpha')^2)}{L^4} \sum_k v_k^2 (F^{ir} g^{jr} + \frac{1-t}{n-2} (\sum_l F^{ll}) g^{ij}) v_i v_j \\
+ 2\alpha^2 \rho (F^{ir} g^{jr} + \frac{1-t}{n-2} (\sum_l F^{ll}) g^{ij}) v_k, i v_k, j - C_1 \beta \rho \sum_{i,j} F^{ii}_{ij} v_j^4 \\
- C_1 \beta \rho \sum_{i,j,k,l} F^{ii}_{ij} v_j^2 |v_{k,l} - C_2 \rho \sum_{i,j} F^{ii}_{ij} |v_{j}|^3 \\
\geq \rho \left( \frac{(2-t)\alpha'}{2L^2} + \frac{(1-t)(\alpha\alpha'' - 2(\alpha')^2)}{(n-2)L^4} \right) \sum_{i,j} F^{ii}_{ij} v_j^2 v_m v_l g^{ml} \\
+ \rho \frac{\alpha'' - (\alpha')^2}{L^4} - \frac{\alpha'}{L^2} \sum_l v_l^2 F^{ir} g^{jr} v_j v_j \\
+ \frac{2(n-2)\alpha^2 \rho}{n-2} \sum_l F^{ll} (\frac{1}{C_1} \delta^{ij}) v_k, i v_k, j - C_1 \beta \rho \sum_{i,j} F^{ii}_{ij} v_j^4 \\
- C_1 \beta \rho \sum_{i,j,k,l} F^{ii}_{ij} v_j^2 |v_{k,l} - C_2 \rho \sum_{i,j} F^{ii}_{ij} |v_{j}|^3 \\
\geq \rho \left( \frac{(2-t)\alpha'}{2L^2} + \frac{(1-t)(\alpha\alpha'' - 2(\alpha')^2)}{(n-2)L^4} \right) \sum_{i,j} F^{ii}_{ij} v_j^2 v_m v_l g^{ml} \\
+ \rho \frac{\alpha'' - (\alpha')^2}{L^4} - \frac{\alpha'}{L^2} \sum_l v_l^2 F^{ir} g^{jr} v_j v_j \\
+ \frac{(1-t)\alpha^2 \rho}{(n-2)C_1} \sum_l F^{ll} v_k, i v_k, j - C_1 \beta \rho \sum_{i,j} F^{ii}_{ij} v_j^4 - C_2 \sqrt{\rho} \sum_{i,j} F^{ii}_{ij} |v_{j}|^3 \\
\geq \rho \left( \frac{(2-t)\alpha'}{2L^2} + \frac{(1-t)(\alpha\alpha'' - 2(\alpha')^2)}{(n-2)L^4} \right) \sum_{i,j} F^{ii}_{ij} v_j^2 v_m v_l g^{ml}
\]

Recall \( a = 1 + \psi \beta e^v \). We can replace it by \( 1 + O_1(1) \beta \). Meanwhile we replace \( L^{ij} \) by \( F^{ir} g^{jr} + \frac{1-t}{n-2} (\sum F^{ll}) g^{ij} \) in the above inequality. We have,
\begin{align*}
&+ \rho \left( \frac{2\alpha' - 2(\alpha')^2}{L^4 \alpha} - \frac{\alpha'}{L^2} \right) \sum_i v_i^2 F^{ij} g^{jr} v_i v_j \\
&- C_1 \beta \rho \sum_{i,j} F^{ij} v_j^4 - C_2 \sqrt{\rho} \sum_{i,j} F^{ij} |v_j|^3.
\end{align*}

(39)

It is enough to find a smooth function \( \alpha : [\frac{1}{L^2}, \frac{2}{L}] \to \mathbb{R}^+ \) satisfying

\[ \{ \begin{array}{l}
\alpha' > 0 \\
\alpha'' - 2(\alpha')^2 - L^2 \alpha > 0.
\end{array} \]  

(40)

since the above inequalities imply that

\[
\frac{2\alpha' - 2(\alpha')^2}{L^4 \alpha} - \frac{\alpha'}{L^2} = \frac{1}{L^4 \alpha} (\alpha'' - 2(\alpha')^2 - L^2 \alpha) > 0,
\]

and

\[
\alpha'' - 2(\alpha')^2 > L^2 \alpha > 0,
\]

and

\[
\frac{(2-t)\alpha'}{2L^2} + \frac{1-t}{n-2} \frac{2\alpha'' - 2(\alpha')^2}{L^4 \alpha} > 0,
\]

i.e., the coefficients of the two leading terms in the inequality (39) are both positive, which will lead the preferred gradient bound.

Let \( \alpha = e^\eta \). The two inequalities in (40) are equivalent to

\[ \{ \begin{array}{l}
\eta' > 0 \\
\eta'' - (\eta')^2 - L^2 \eta > 0.
\end{array} \]  

To find \( \alpha \), let \( \eta(s) = s^r \) with \( r \gg 1 \) being chosen later. Clearly, \( \eta' > 0 \) and

\[
\eta'' - (\eta')^2 - L^2 \eta' = rs^{r-2} \left( (r-1) - rs^r - L^2 s \right)
\geq rs^{r-2} \left( (r-1) - r \left( \frac{2}{L} \right)^r - L^2 \left( \frac{2}{L} \right) \right)
= rs^{r-2} \left( (r-1) - r \left( \frac{2}{L} \right)^r - 2L \right)
\geq rs^{r-2} \left( (r-1) - \frac{r}{2} - 2L \right) \quad \text{by choosing} \quad L > 4
= rs^{r-2} \left( \frac{r}{2} - 1 - 2L \right) \geq rs^{r-2} > 0 \quad \text{by choosing} \quad r > 4 + 4L.
\]

Pick \( L > |v + \gamma| + 4 \) and \( r > 4 + 4L \). Then we have \( \frac{v + \gamma + L}{L^2} \in \left[ \frac{1}{L^2}, \frac{2}{L} \right] \) and there exists a universal constant \( C_3 > 0 \) independent of \( \beta \) such that (40) holds. By (39),
\[
0 \geq \bar{L}^{ij}G_{ij}(x_0) \geq C_3\rho \sum_{i,j,k,l} v_i^2 F^{kk}(\frac{1}{C_3} s^{ij}) v_i v_j - C_1\beta \rho \sum_{i,j} F^{ii} v_i^4 - C_2 \sqrt{\rho} \sum_{i,j} F^{ii} |v_j|^3 \\
\geq C_3\rho \sum_{k,l} v_k^2 F^{kk} - C_1\beta \rho \sum_{i,j} F^{ii} v_i^4 - C_2 \sqrt{\rho} \sum_{i,j} F^{ii} |v_j|^3 \\
\geq \frac{C_3}{2} \rho \sum_{k,l} v_k^2 F^{kk} - C_2 \sqrt{\rho} \sum_{i,j} F^{ii} |v_j|^3,
\]
where in the last inequality, we used \( \beta \in [0, 2\delta_0] \), therefore we can pick \( \delta_0 \ll 1 \) such that \( C_1\beta \leq \frac{C_3}{2} \).

We conclude that \( 0 \geq \frac{C_3}{2} \rho \sum_{k,l} v_k^2 F^{kk} - C_2 \sqrt{\rho} \sum_{i,j} F^{ii} |v_j|^3 \geq C_3 \rho \sum_{i,j} F^{kk}(\sum_i v_i^2)^2 - C_2 \sum_{i,j} F^{ij}(\sum_i v_i^2)^2 \frac{3}{2} \)

which implies that \( \sum_i v_i^2 \leq C \), therefore \( G(x_0) \leq C \). In particular \( \sum_i v_i^2 \leq C \) in \( B^T_{\frac{1}{32}}(y^{i_0}) \). From which, we have, in \( B^T_{\frac{1}{32}}(y^{i_0}) \),

\[ |\nabla v|^2_g = v_k v_l g^{kl} \leq C \sum_k v_k^2 \leq C. \]

By \( \bigcup_{i_0=1}^N \left( B^T_{\frac{1}{32}}(y^{i_0}) \cap \partial M \right) = \partial M \), \( |\nabla v|^2_g \leq C \) on \( \partial M \). The Lemma 5.1 has been established. ♣

**Remark 5.1** When the manifold \((M^n, g)\) is umbilic on the boundary, the above lemma and therefore the next lemma also hold for \( t = 1 \). The above proof still works after a slight modification.

**Lemma 5.2** Under the same assumptions as in Theorem 1.2, for \( t < 1 \), let \( v \) be a \( C^3 \) solution of the equation (12). Then there exists a universal constant \( C > 0 \) depending only on \((M^n, g, t), (f, \Gamma), \phi, \) and \( \psi \), such that

\[ |\nabla v|^g \leq C \quad \text{on} \ M. \]

**Proof of the Lemma 5.2.** Consider

\[ \bar{G} := |\nabla v|^2_g \tilde{\alpha}(\frac{v + L}{L^2}), \]
where $L > 0$ is a constant satisfying $1 < v + L < 2L$ and $\alpha : R^+ \to R^+$ is a smooth positive function to be chosen later. Let $\bar{G}(x_0) = \max_M G$. Let $\{x_j\}_{j=1}^n$ be a geodesic normal coordinates w.r.t. the metric $g$ at $x_0$. W.l.o.g., we can assume $x_0$ is an interior point of $M$. In the following, subindices are taken w.r.t. $\frac{\partial}{\partial x_j}$. Repeat the arguments in the proof of the Lemma 5.1. We arrive at

$$0 \geq \bar{L}^{ij} \bar{G}_{ij}(x_0) \geq \left(\frac{(2-t)\alpha'}{2L^2} + \frac{(1-t)(n\alpha''-2(\alpha')^2)}{(n-2)L^4 \alpha}\right)|\nabla v|_g^4 \sum_i F^{ii} \geq \left(\frac{\alpha''-2(\alpha')^2}{L^4 \alpha} - \frac{\alpha'}{L^2}\right)|\nabla v|_g^2 F^{ij} v_i v_j - C|\nabla v|_g^3 \sum_i F^{ii}.$$ 

Choose the same $\alpha$ as in the proof of the Lemma 5.1. We conclude that there exists some universal constant $C_3 > 0$ such that

$$0 \geq \bar{L}^{ij} \bar{G}_{ij}(x_0) \geq C_3|\nabla v|_g^4 \sum_i F^{ii} - C|\nabla v|_g^3 \sum_i F^{ii} \geq |\nabla v|_g^3 \sum_i F^{ii} (C_3|\nabla v|_g - C),$$

which implies that $|\nabla v|_g(x_0) \leq C$ and therefore $G(x_0) \leq C$. The Lemma 5.2 has been proved.

6 Hessian Estimates

The main issue of the Hessian estimates is to bound the Hessian of the solutions on the boundary of $M$.

**Lemma 6.1** Under the same assumptions as in Theorem 1.2, for $t < 1$, let $v$ be a $C^4$ solution of the equation (12). For any $1 \leq i_0 \leq N$, there exists a universal constant $C > 0$ depending only on $(M^n, g, t)$, $(f, \Gamma)$, $\phi$, $\psi$, and $\delta_{y_0}$ such that in $B^T_{\delta_{y_0}}(y^{i_0})$,

$$|v_{\tau\tau}| < C, \quad \text{for any unit direction } \frac{\partial}{\partial \tau} \text{ satisfying } g\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \nu}\right) = 0.$$ 

**Proof of the Lemma 6.1.** Consider

$$\bar{H}(y) := \rho e^{\beta_{y_0} \nu_n} \left(\{\max_{\tau \in T_y M, \|\tau\|_g = 1, g(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \nu}) = 0} (\nabla^2 v + a|\nabla (v + \gamma)|_g^2 g)(\tau, \tau)\} - s_0 v_{\nu}(y)\right).$$
where \((y_1, \cdots, y_n)\) is the tubular neighborhood normal coordinates of \(y \in B^T_{\frac{y_0}{16}}(y_0)\) at \(y_0\), \(\gamma, \rho\) are the same as in the proof of Lemma 5.1, and \(a > 0, \beta_0 > 0, s_0 > 0\) are constants to be chosen later.

Let \(\bar{H}(x_0) = \max_{B^T_{\frac{y_0}{16}}(y_0)} \bar{H} \) for some \(x_0 \in B^T_{\frac{y_0}{16}}(y_0)\).

**Claim 6.1** Either \(\bar{H}(x_0) < C\) or \(x_0\) is an interior point of \(B^T_{\frac{y_0}{16}}(y_0)\) by choosing \(\beta_0, s_0 \gg 1\).

**Proof of the Claim 6.1.** If not, we assume \(\bar{H}(x_0) \geq 1\) and \(x_0 \in B^T_{\frac{y_0}{16}}(y_0) \cap \partial M\). Let \(\{\bar{x}_1, \cdots, \bar{x}_n\}\) be a tubular neighborhood normal coordinates at \(x_0\). Then \(\{\bar{x}_1, \cdots, \bar{x}_n\}\) is well-definition and smooth near \(x_0\). Meanwhile, \(y_n = \bar{x}_n\) near \(x_0\) since they both represent the distance parameter to the boundary \(\partial M\), which is to say \(\frac{\partial}{\partial \nu}\) has the same definition near \(x_0\). Recall that \(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}\) at \(x_0\). W.l.o.g., we can assume \(\bar{H}(x_0) := \rho e^{\beta_0 y_0}(v_{11} + a|\nabla(v + \gamma)|^2 - s_0 v_\nu(x_0)),\) where and in the following subindices denote the covariant derivatives w.r.t. \(\frac{\partial}{\partial x_i}\). Let

\[
H(x) := \rho e^{\beta_0 y_0}(\frac{v_{11}}{g_{11}} + a|\nabla(v + \gamma)|^2 - s_0 v_\nu).
\]

By \(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = 0\) near \(x_0\), we know \(x_0\) is a local minimum point of \(H\). Moreover \(\frac{\partial}{\partial x_n} = -\frac{\partial}{\partial \nu}\) near \(x_0\) implies that

\[
(|\nabla(v + \gamma)|^2 g_\nu(x_0) = \left((v + \gamma)_k(v + \gamma)_l g^{kl}\right)_\nu = 2(v + \gamma)_k(v + \gamma)_k,\nu + (v + \gamma)_k(v + \gamma)_l g^{kl} = -2(v + \gamma)_k(v + \gamma)_k,\nu + (v + \gamma)_k(v + \gamma)_l g^{kl} = -2(v + \gamma)_k \frac{\partial^2(v + \gamma)}{\partial \bar{x}_n \partial \bar{x}_k} + (v + \gamma)_k(v + \gamma)_l g^{kl} = -2(v + \gamma)_k \frac{\partial^2(v + \gamma)}{\partial \bar{x}_k \partial \bar{x}_n} + (v + \gamma)_k(v + \gamma)_l g^{kl} = -2(v + \gamma)_k(v + \gamma)_n,k + (v + \gamma)_k(v + \gamma)_l g^{kl} = 2(v + \gamma)_k(v + \gamma)_\nu,k + (v + \gamma)_k(v + \gamma)_l g^{kl}.
\]

Since \((v + \gamma)_\nu|_{\partial M} = 0\) by (23), we have \((v + \gamma)_\nu,k(x_0) = 0\) for \(k \leq n - 1\), which implies that
\[ \sum_{k=1}^{n} (v + \gamma)_{k}(v + \gamma)_{\nu,k}(x_0) = (v + \gamma)_{\nu}(v + \gamma)_{\nu,n} = -(v + \gamma)_{\nu}(v + \gamma)_{\nu,n} = 0. \]

Thus \( |\nabla (v + \gamma)|_{g^{\nu}}^2(x_0) = (v + \gamma)_{k}(v + \gamma)_{l}g^{kl}_{\nu}. \) By (22), \( y_n = 0 \) at \( x_0 \), and \( \frac{\partial g_{\nu}}{\partial y_n} = -\frac{\partial g_{\nu}}{\partial y_n} = -1 \) in \( B^T_{\psi_0}(g^{\nu}) \),

\[ 0 \leq H_{\nu}(x_0) = \left( v_{11,\nu} + a(v + \gamma k)(v + \gamma)_{l}g^{kl}_{\nu} - v_{111,\nu} - s_0 v_{\nu,\nu} \right) \rho e^{\beta_0 y_n} - \beta_0(v_{11} + a|\nabla (v + \gamma)|^2_{g} - s_0 v_{\nu}) \rho e^{\beta_0 y_n} = \rho \left( v_{11,\nu} - \beta_0(v_{11} + a|\nabla (v + \gamma)|^2_{g} - s_0 v_{\nu}) - v_{111,\nu} - s_0 v_{\nu,\nu} \right) (v + \gamma)_{k}(v + \gamma)_{l}g^{kl}_{\nu} \]  

(41)

We need to interchange \( v_{11,\nu} \) to \( \frac{\partial^2 (v_{\nu})}{\partial x^1 \partial x^1} \) in the above equation so that we can use the boundary condition. Recall \( \{\bar{x}_1, \cdots, \bar{x}_{n-1}\} \) is the geodesic normal coordinates w.r.t. the metric \( g|_{\partial M} \) at \( x_0 \). Then \( \bar{\nabla} \frac{\partial}{\partial x^k} (x_0) = 0 \) for \( 1 \leq k, l \leq n - 1 \), where \( \bar{\nabla} \) is the covariant derivative of \( \partial M \) induced by \( g|_{\partial M} \). For \( 1 \leq k, l \leq n - 1 \),

\[ \Gamma_{kl}^{i}(x_0) \frac{\partial}{\partial x^i} = \bar{\nabla} \frac{\partial}{\partial x^k} (x_0) = \bar{\nabla} \frac{\partial}{\partial x^k} (x_0) + II(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})(x_0) \frac{\partial}{\partial x^i}. \]

Comparing both sides of the above equation, we have, at \( x_0 \),

\[ \Gamma_{kl}^{i} = 0 \] for \( 1 \leq i, k, l \leq n - 1 \), \( \Gamma_{kl}^{n} = -II(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}). \) (42)

Hence at \( x_0 \),

\[ v_{11,\nu} = -v_{11,\nu} = -\frac{\partial v}{\partial x^1} + (\Gamma_{11}^{i})v_i + \Gamma_{11,\nu}^{i}v_i. \]

\[ = -\frac{\partial^2 v}{\partial x^1 \partial x^1} + \frac{\partial}{\partial x^1}(\Gamma_{11}^{i})v_i + \Gamma_{11,\nu}^{i}v_i. \]

\[ = -\frac{\partial x_n}{\partial^2 x^1 \partial x^1} + \frac{\partial}{\partial x^1}(\Gamma_{11}^{i})v_i + \Gamma_{11,\nu}^{i}v_i. \]

\[ = -\frac{\partial^2 (v_{\nu})}{\partial x^1 \partial x^1} + \frac{\partial}{\partial x^1}((\Gamma_{11}^{i})v_i + \Gamma_{11,\nu}^{i}v_i). \]

\[ = \frac{\partial^2 (v_{\nu})}{\partial x^1 \partial x^1} + \frac{\partial}{\partial x^1}((\Gamma_{11}^{i})v_i + \Gamma_{11,\nu}^{i}v_i). \]

\[ + \frac{\partial}{\partial x^1}((\Gamma_{11}^{i})v_i + \Gamma_{11,\nu}^{i}v_i), \]

\[ = e^\nu \psi \frac{\partial^2 v}{\partial x^1 \partial x^1} + e^\nu \psi v_i^2 + 2e^\nu \psi v_i v_1 + e^\nu \frac{\partial^2 \psi}{\partial x^1 \partial x^1} + \frac{\partial^2 h_d}{\partial x^1 \partial x^1}. \]
where in the second to last equality, we used the fact that \( \partial_{\bar{x}_1} \) is a tangent vector field of \( \partial M \) near \( x_0 \), so we can replace \( v_\nu \) by \( e^\nu \psi - h_\nu \).

In the following, we use \( C > 0 \) to denote a universal constant independent of \( \beta_0 \). Substitute the above equation into the inequality (41).

\[
0 \leq H_\nu (x_0) = \rho (e^\nu \psi \frac{\partial^2 v}{\partial x_1 \partial \bar{x}_1} - \beta_0 (v_{11} + a |\nabla (v + \gamma)|_g^2 - s_0 v_\nu) \]
\[
- v_{11} g_{11,\nu} - (s_0 - \Gamma^n_{11}) v_{n,n} + C \]
\[
= \rho (e^\nu \psi v_{11} + e^\nu \psi \Gamma^l_{11} v_l - \beta_0 (v_{11} + a |\nabla (v + \gamma)|_g^2 - s_0 v_\nu) \]
\[
- v_{11} g_{11,\nu} - (s_0 - \Gamma^n_{11}) v_{n,n} + C \]
\[
\leq \rho ((e^\nu \psi - \beta_0 - g_{11,\nu}) (v_{11} + a |\nabla (v + \gamma)|_g^2 - s_0 v_\nu) \]
\[
- (s_0 - \Gamma^n_{11}) v_{n,n} + C \).
\]

(43)

Since \( \partial_{\bar{\nu}} \) is the tangent vector of geodesic curves, we have \( \nabla \frac{\partial}{\partial \nu} = 0 \) near \( x_0 \). In particular, we have

\[
v_{n,n}(x_0) = v_{\nu,\nu}(x_0) = v_{\nu\nu} = (\nabla \frac{\partial}{\partial \nu}) v = v_{\nu\nu} = v_{nn}(x_0).\]

Recall \( \Gamma^n_{11} = -II(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \bar{x}_1}) \) by (42). We can pick \( s_0 \gg 1 \) such that \( \frac{s_0}{2} \) is bigger than the largest absolute value of the principle curvatures of the second fundamental form on \( \partial M \). Then we have \( \frac{3s_0}{2} \geq s_0 - \Gamma^n_{11} \geq \frac{s_0}{2} > 0 \) at \( x_0 \). By \( \Gamma \subset \Gamma_1 \), we have

\[
(1 + \frac{(1-t)n}{n-2}) \Delta_g v + \frac{\frac{(2-t)n}{2} - 1}{n-2} |\nabla v|^2_g - \frac{(2-t)n - 2}{2(n-1)(n-2)} R_g > 0,
\]

which implies that \( \Delta_g v(x_0) \geq -C \). W.l.o.g., we assume \( v_{11}(x_0) > 1 \) and \( v_{kk}(x_0) \leq C v_{11}(x_0) \) for \( 1 \leq k \leq n - 1 \). Then

\[
-v_{n,n}(x_0) = -v_{nn}(x_0) \leq C + \sum_{k=1}^{n-1} v_{kk}(x_0) \leq C v_{11}(x_0),
\]

and

\[
-(s_0 - \Gamma^n_{11}) v_{n,n}(x_0) \leq C (s_0 - \Gamma^n_{11}) v_{11} \leq \frac{3Cs_0}{2} v_{11} \leq C s_0 v_{11}.
\]

Substitute the above inequality into (43).
\[0 \leq H_\nu(x_0) \leq \rho \left( (e^\psi - \beta_0 - g_{11,\nu})(v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu) + Cs_0v_{11} + C \right) \leq \rho \left( (e^\psi + Cs_0 - g_{11,\nu} - \beta_0)(v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu) + C \right) \leq \rho \left( (C - \beta_0)(v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu) + C \right) \leq \rho \left( - (v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu) + C \right) \]

by choosing \( \beta_0 > C + 1 \),

which implies that \( (v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu)(x_0) < C \) and \( H(x_0) < C \). The Claim 6.1 has been proved. \( \blacklozenge \).

Due to the above claim, we assume \( x_0 \) is an interior point of \( B^T_{\iota,\nu_0}(y^{i_0}) \). To continue the proof of the Lemma 6.1, we need to introduce a new coordinates near \( x_0 \). Let \( d_0 = d(x_0, \partial M) \), and let \( S_0 := \{ y \in B^T_{\iota,\nu_0}(y^{i_0}) | y_n = d_0 \} \). As shown in the proof of the Proposition 4.2, \( \frac{\partial}{\partial \nu} \) is still the unit normal vector field of \( S_0 \). For any \( x \in B^T_{\iota,\nu_0}(y^{i_0}) \) but near \( x_0 \) with \( (y_1, \cdots, y_n) \) as its tubular neighborhood normal coordinates of \( x \) at \( y^{i_0} \), then \( \sqrt{\sum_{j=1}^{n} y_j^2} < \frac{\delta_{y^{i_0}}}{16} \). We conclude that there exists a unique \( \hat{x} \in S_0 \) such that \( d(x, \hat{x}) = d(x, S_0) \). In fact for such \( x \), let \( \bar{x} = (y_1, \cdots, y_{n-1}, 0) \). Then \( \bar{x} \) is the unique point on \( \partial M \) such that \( d(\bar{x}, x) = d(x, \partial M) = y_n \). Consider \( r(t) = E(\hat{x}, t) \). Then \( r(t) \) is smooth and well defined for \( t \in [0, \delta_{y^{i_0}}) \), \( r(y_n) = x \), and

\[
\sqrt{\sum_{j=1}^{n-1} y_j^2 + (\max\{d_0, y_n\})^2} < \frac{\delta_{y^{i_0}}}{16}
\]

as long as \( x \) is close to \( x_0 \) enough since \( x \) is an interior point of \( B^T_{\iota,\nu_0}(y^{i_0}) \). Moreover for \( t \in [0, \max\{d_0, y_n\}] \), the tubular neighborhood normal coordinates of \( r(t) \) at \( y^{i_0} \) is \( (y_1, \cdots, y_{n-1}, t) \), which implies that the curve \( r([0, \max\{d_0, y_n\}]) \subset B^T_{\iota,\nu_0}(y^{i_0}) \), therefore intersects with \( S_0 \) at a unique point \( r(d_0) \). As shown in the proof of the Proposition 4.2, i.e., by (19) and (20), \( d(r(t), S_0) = |t - d_0| \) for \( t \in [0, \frac{\delta_{y^{i_0}}}{2}) \). In particular,

\[
d(x, S_0) = d(r(y_n), S_0) = d(r(y_n), r(d_0)) = |y_n - d_0|.
\]

(44)

Next, we want to show that there exists only one point \( \hat{x} \in S_0 \) such that \( d(x, \hat{x}) = d(x, S_0) \). This is because if \( (a_1, \cdots, a_n) \) is the tubular neighborhood normal coordinates of \( \hat{x} \) at \( y^{i_0} \), then \( \hat{x} := (a_1, \cdots, a_{n-1}, 0) \in B^T_{\iota,\nu_0}(y^{i_0}) \cap \partial M \subset B^T_{\iota,\nu_0}(y^{i_0}) \cap \partial M \) by
the Proposition 4.3. Thus \( \dot{r}(t) := E(\tilde{x}, t) \) is smooth and well-defined for \( t \in [0, \delta_{y_0}] \) and \( \dot{r}(a_n) = \tilde{x} \). Let \( \tilde{r}(t) \) be the shortest normalized geodesic connecting \( x \) with \( \tilde{x} \). Then \( \tilde{r} \) has \( \frac{\partial}{\partial \nu} \) as its tangent vector at \( \tilde{x} \). Since \( \frac{\partial}{\partial \nu} \) is also the tangent vector of \( \dot{r} \) at \( \dot{r}(a_n) = \tilde{x} \), we know \( \tilde{r} \) and \( \dot{r} \) coincide. Hence \( \tilde{r}(y_n) = x \), which implies that \( E(\tilde{x}, y_n) = x = E(\tilde{x}, y_n) \), and \( \tilde{x} = \tilde{x} \) since \( E \) is a diffeomorphism in \( B_{y_0}^T(y_0) \). Therefore \( \tilde{x} \) is uniquely determined by \( x \). Clearly \( r(d_0) \in S_0 \) is near \( x_0 \) as long as \( x \) is near \( x_0 \). Let \( \{x_1, \ldots, x_{n-1}\} \) be the geodesic normal coordinates w.r.t. the metric \( g \) at \( \tilde{x} \). We assume \( \tilde{x} \) is near \( x \) \( E \) and \( \tilde{x} \) well-defined for \( 1 \). Let \( \tilde{r} \) be the Levi-Civita connection induced by \( g \). Coordinates w.r.t. the metric \( g \) \( \tilde{x} \) are \( \tilde{x} \) smooth and well-defined near \( x_0 \) in \( S_0 \). For any \( x \in B_{y_0}^T(y_0) \) and near \( x_0 \), there exists a unique \( \tilde{x} \in S_0 \) such that \( d(\tilde{x}, x) = d(x, S_0) \).

We assume \( x \) is close enough to \( x_0 \) such that the geodesic normal coordinates of \( \tilde{x} \) w.r.t. the metric \( g|_{S_0} \) at \( x_0 \) is smooth and well-defined. Let \( (x_1, \ldots, x_{n-1}) \) be such geodesic normal coordinates of \( \tilde{x} \) w.r.t. the metric \( g|_{S_0} \) at \( x_0 \). Define \( (x_1, \ldots, x_n) \) to be the new coordinates of \( x \) such that \( x_n = y_n - d_0 \). Then \( \{x_j\}_{j=1}^n \) is smooth and well-defined for \( x \) near \( x_0 \), and \( d(x, S_0) = |y_n - d_0| = |x_n| \) by (44). As shown in the proof of the Proposition 4.2, for \( x \) near \( x_0 \) and for \( 1 \leq k \leq n-1 \),

\[
\frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n} = -\frac{\partial}{\partial \nu}, \quad g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_n}) = 0.
\]

(45)

Let \( II_0 \) denote the second fundamental form of \( g \) w.r.t. \( \frac{\partial}{\partial \nu} \) on \( S_0 \) and let \( \tilde{\nabla} \) be the Levi-Civita connection induced by \( g|_{S_0} \). Recall on \( S_0 \), \( \{x_j\}_{j=1}^n \) is the geodesic normal coordinates w.r.t. the metric \( g|_{S_0} \) at \( x_0 \). Therefore \( g_{lm}(x_0) := g(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m})(x_0) = \delta_{lm} \) for \( 1 \leq l, m \leq n \), and \( \tilde{\nabla} \frac{\partial}{\partial x_i}(x_0) = 0 \) for \( 1 \leq i, j \leq n-1 \), which implies that

\[
\tilde{\nabla} \frac{\partial}{\partial x_i}(x_0) = \tilde{\nabla} \frac{\partial}{\partial x_i} + II_0(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \frac{\partial}{\partial \nu} = II_0(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \frac{\partial}{\partial \nu}.
\]

Thus for \( 1 \leq i, j, k \leq n-1 \),

\[
\frac{\partial}{\partial x_k} g_{ij}(x_0) = g(\tilde{\nabla} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}) + g(\frac{\partial}{\partial x_j}, \tilde{\nabla} \frac{\partial}{\partial x_k}) = g(II_0(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}), \frac{\partial}{\partial x_j}) + g(\frac{\partial}{\partial x_i}, II_0(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j})) = 0 \quad \text{by (45)}.
\]

(46)

Also by (45), we have, for \( 1 \leq i \leq n-1 \), \( g_{in} = 0 \) near \( x_0 \), which implies that

\[
\frac{\partial}{\partial x_k} g_{in}(x_0) = \frac{\partial}{\partial x_k} g_{ni}(x_0) = 0 \quad \text{for} \quad 1 \leq k \leq n.
\]

(47)

Notice that \( \frac{\partial}{\partial x_n} = -\frac{\partial}{\partial \nu} \) is a unit vector field. Therefore \( g_{nn} \equiv 1 \) near \( x_0 \), and for \( 1 \leq k \leq n \),
\[
\frac{\partial}{\partial x_k} g_{nn}(x_0) = 0.
\] (48)

Combine (46)-(48). We have
\[
\frac{\partial}{\partial x_k} g_{ij}(x_0) = 0 \quad \text{for } 1 \leq i, j \leq n \quad \text{and } 1 \leq k \leq n - 1.
\] (49)

Recall \( G = (g_{ij}) \). \( \mathcal{G} \mathcal{G}^{-1} = I_{n \times n} \) implies that \( \frac{\partial \mathcal{G}^{-1}}{\partial x_k} = -\mathcal{G}^{-1} \frac{\partial \mathcal{G}}{\partial x_k} \mathcal{G}^{-1} \). For 1 \leq k \leq n - 1,
\[
\frac{\partial}{\partial x_k} g_{ij}(x_0) = -g^{ij} \left( \frac{\partial}{\partial x_k} g_{rs} \right) g^{sj} = 0.
\] (50)

In the following, subindices denote the covariant derivatives w.r.t. \( \frac{\partial}{\partial x_i} \). Notice that \( g_{ij}(x_0) = \delta_{ij} \). W.l.o.g., we assume
\[
\tilde{H}(x_0) = \rho e^{\beta_0 \rho} \left( v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0 v_\nu \right),
\]
and \( v_{1,1}(x_0) \gg 1 \).

Let
\[
\tilde{H} = \rho e^{\beta_0 \rho_n} \left( \frac{v_{11}}{g_{11}} + a|\nabla(v + \gamma)|_g^2 - s_0 v_\nu \right).
\]

By (45), \( x_0 \) is a local maximum point of \( \tilde{H} \). Near \( x_0 \),
\[
\tilde{H}_i = \rho e^{\beta_0 \rho_n} \left( \frac{v_{11,i}}{g_{11}} - \frac{v_{11}}{g_{11,i}} g_{11,i} + 2a g_{kl}(v + \gamma)_k(v + \gamma)_l - a g_{kl}^2(v + \gamma)_k(v + \gamma)_l - s_0 v_{\nu,i} \right) + \left( \frac{\rho}{\rho} + \delta_{ni} \beta_0 \right) H.
\]

At \( x_0 \),
\[
v_{11,i} - g_{11,i} v_{11} + 2a(v_k + \gamma_k)(v_{k,i} + \gamma_{k,i}) + a g_{kl}^2(v_k + \gamma_k)(v_l + \gamma_l) - s_0 v_{\nu,i} = -\left( \frac{\rho}{\rho} + \delta_{ni} \beta_0 \right) (v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0 v_\nu),
\] (51)

and
\[
\tilde{H}_{ij}(x_0) = \rho e^{\beta_0 \rho_n} \left( v_{11,ij} - g_{11,j} v_{11,i} - g_{11,i} v_{11,j} + 2g_{11,i} g_{11,j} v_{11} - g_{11,i} v_{11} + 2a(v_k + \gamma_k)(v_{k,i} + \gamma_{k,i}) + a g_{kl}^2(v_k + \gamma_k)(v_l + \gamma_l) + a g_{kl}^2(v_k + \gamma_k)(v_{k,l} + \gamma_{k,l}) + a g_{kl}^2(v_k + \gamma_k)(v_{k,i} + \gamma_{k,i}) + a g_{kl}^2(v_k + \gamma_k)(v_{k,l} + \gamma_{k,l}) - s_0 v_{\nu,ij} \right) + \beta_0 \delta_{nj} \rho e^{\beta_0 \rho_n} \left( v_{11,i} - g_{11,i} v_{11} + 2a(v_k + \gamma_k)(v_{k,i} + \gamma_{k,i}) \right)
\]
so by (51),

\[
\rho^{-1}e^{-\beta_0 d_0} \tilde{H}_{ij}(x_0) = \\
\left(v_{11,ij} - g_{11,j}v_{11,i} - g_{11,i}v_{11,j} + 2g_{11,i}g_{11,j}v_{11} - g_{11,ij}v_{11}
+ 2a(v_{k,i} + \gamma_{k,j})(v_{k,j} + \gamma_{k,i}) + 2a(v_{k,j} + \gamma_{k,i})(v_{k,i} + \gamma_{k,j})
+ 2ag^k_{ij}(v_k + \gamma_k)(v_{i,j} + \gamma_{i,j}) + 2ag^k_{il}(v_k + \gamma_k)(v_{i,j} + \gamma_{i,j})
+ \left(\frac{\rho_i\rho_j - \rho_i\rho_j}{\rho}\right)(v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu)
\right) - C_2\rho^{-1} \sum_i F^{il}|v_{k,i}|,
\]

Recall in the proof of the Claim 6.1, the choice of \( \beta_0 \) depends on \( a \). We need to prove the choice of \( a \) is independent of \( \beta_0 \). For this reason, we let \( C_1 \) denote the universal constant depending only on \((M^n, g, t), (f, \Gamma), \phi, \psi, \delta_{y_0}, \) but independent of \( a, \beta_0 \), and let \( C_2 \) denote the universal constant depending on \((M^n, g, t), (f, \Gamma), \phi, \psi, \delta_{y_0}, \) and \( a, \beta_0 \).

Notice that \( g_{ij}(x_0) = \delta_{ij} \) and \( \tilde{L}^{ij}(x_0) = F^{ij} + \frac{1-t}{n-2}(\sum_l F^{il})\delta^{ij} \).

\[
0 \geq \rho^{-1}e^{-\beta_0 d_0} \tilde{L}^{ij} \tilde{H}_{ij}(x_0)
\geq \tilde{L}^{ij}
\begin{aligned}
&= \left(v_{11,ij} - 2g_{11,j}v_{11,i} + 2a(v_{k,j} + \gamma_{k,i})(v_{k,i} + \gamma_{k,j})
+ 2a(v_k + \gamma_k)(v_{k,i} + \gamma_{k,j}) - s_0v_{\nu,ij}
\right)
-C_2\rho^{-1} \sum_i F^{il}|v_{k,i}|
\end{aligned}
\] (52)

where we used \(|\nabla\rho| < C_1\sqrt{\rho}, \ |\nabla^2\rho| < C_1, \) and \( v_{1,1}(x_0) \geq 1 \).

By (51),

\[
v_{11,i}(x_0) = -2a(v_k + \gamma_k)(v_{k,i} + \gamma_{k,i}) - ag^k_{ij}(v_k + \gamma_k)(v_i + \gamma_i) - s_0v_{\nu,i}
+ g_{11,i}v_{11} - \left(\frac{\rho_i}{\rho} + \delta_{ni}\beta_0\right)(v_{11} + a|\nabla(v + \gamma)|_g^2 - s_0v_\nu),
\] (53)

Substitute the above into (52). Since \( v_{\nu,i} = -v_{n,i} \) and \( v_{11} = v_{1,1} - \Gamma_{11}^iv_i \),

\[
0 \geq \rho^{-1}e^{-\beta_0 d_0} \tilde{L}^{ij} \tilde{H}_{ij}(x_0)
\geq \tilde{L}^{ij}
\begin{aligned}
&= \left(v_{11,ij} + 2a(v_{k,j} + \gamma_{k,j})(v_{k,i} + \gamma_{k,i})
\right)
\end{aligned}
\]
\[ +2a(v_k + \gamma_k)(v_{k,ij} + \gamma_{k,ij}) - s_0 v_{v,ij} \] 
\[ - C_2 \rho^{-1} \sum_l F_{ll} |v_{k,i}| \]

\[ \geq \tilde{L}^{ij} \left( v_{11,ij} + 2a(1 + \psi \beta e^v)^2 v_{k,j} v_{k,i} + 2a(1 + \psi \beta e^v) (v_k + \gamma_k) v_{k,ij} - s_0 v_{v,ij} \right) - C_2 \rho^{-1} \sum_l F_{ll} |v_{k,i}| \]

(54)

At \( x_0 \),
\[ v_{ij,l} = \frac{\partial}{\partial x_j} \left( \nabla^2 u_l \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right) = \frac{\partial}{\partial x_j} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma^k_{ji} v_k \right) \]
\[ = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} - \Gamma^k_{ji} v_k, \]
so
\[ v_{1,ij} = \left( \nabla^2 u_l \right) \left( \frac{\partial^2 u}{\partial x_j \partial x_i} - \Gamma^k_{ji} \frac{\partial}{\partial x_k} \right) (v_l) \]
\[ = \frac{\partial^3 u}{\partial x_j \partial x_i \partial x_k} - \Gamma^k_{ji} v_{l,k} \]
\[ = v_{ij,l} + \frac{\partial (\Gamma^k_{ji})}{\partial x_k} v_k, \]
and
\[ v_{ij,11} = \left( \nabla^2 v_{ij} \right) \left( \frac{\partial^2 u}{\partial x_1 \partial x_1} \right) = \left( \nabla^2 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1} - \Gamma^{l}_{11} \frac{\partial}{\partial x_1} \right) v_{ij} \]
\[ = \frac{\partial^3}{\partial x_1 \partial x_1 \partial x_1} \left( \frac{\partial^2 u}{\partial x_j \partial x_i} - \Gamma^k_{ji} v_k \right) \]
\[ = \frac{\partial^4}{\partial x_1 \partial x_1 \partial x_1 \partial x_1} - \Gamma^k_{ji} v_k - 2 \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,1} - \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} - \Gamma^{l}_{11} v_{l,j} \]
\[ = \frac{\partial^4}{\partial x_1 \partial x_1 \partial x_1 \partial x_1} - \Gamma^k_{ji} v_k - 2 \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,1} - \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} - \Gamma^{l}_{11} v_{l,j} \]
\[ = -2 \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,1} + \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} - \Gamma^{l}_{11} v_{l,j} - \Gamma^{l}_{11} v_{l,j} - \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} \]
\[ = \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} - \Gamma^{l}_{11} v_{l,j} - \Gamma^{l}_{11} v_{l,j} - \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} \]
\[ = v_{ij,11} + \Gamma^{l}_{ji} v_{l,j} + \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,j} + 2 \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,1} + \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} \]
\[ + \Gamma^{l}_{11} v_{l,j} + \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1}, \]

therefore
\[ v_{11,ij} = \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} - \Gamma^{l}_{11} v_{l,j} + \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} \]
\[ = v_{ij,11} + \Gamma^{l}_{ji} v_{l,j} + \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,j} + 2 \frac{\partial (\Gamma^l_{ji})}{\partial x_1} v_{l,1} + \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} \]
\[ + \Gamma^{l}_{11} v_{l,j} + \frac{\partial^2 (\Gamma^l_{ji})}{\partial x_1 \partial x_1} v_{l,1} \]

(56)

Substitute (55) and (56) into (52). At \( x_0 \),
\[ 0 \geq \rho^{-1} e^{-\rho \delta_{0 \rho}} \tilde{L}^{ij} \tilde{H}_{ij}(x_0) \]
\[ \geq \tilde{L}^{ij} (v_{ij,11} + 2a(1 + \psi \beta e^v)^2 v_{k,j} v_{k,i} + 2a(1 + \psi \beta e^v)(v_k + \gamma_k)v_{ij,k} - s_0 v_{ij,0}) - C_2 \rho^{-1} \sum_i F^{il} |v_{k,i}| \]

(57)

Differentiate the equation \( F(W_{ij} g^{jr}) = \phi e^{2v} \) along the \( x_l \)-th direction.

\[
(\phi e^{2v})_l = F^{ir} (g^{jr} W_{ij,l} + \tilde{W}_{ij} g^{jr}) = F^{ir} g^{jr} (v_{ij,l} + \frac{1-t}{n-2} (\Delta_g v) g_{ij,l}) + F^{ir} g^{jr} (\frac{1-t}{n-2} (\Delta_g v) g_{ij,l} + (2-t) v_m v_k g^{km} g_{ij} + \frac{2+t}{2} v_k v_m g^{km} g_{ij} + \frac{2+t}{2} |\nabla v|^2 g_{ij,l}) - v_i v_j - (A' g)_{ij,l} + F^{ir} g^{jr} \tilde{W}_{ij},
\]

which implies that, at \( x_0 \), by \( g_{ij} = \delta_{ij} \)

\[
|F^{ij} v_{ij,l} + \frac{1-t}{n-2} (\Delta_g v) \sum_i F^{ii}| \leq C_1 \sum_i F^{rr} |v_{ij}|,
\]

(58)

where we used \( |\Delta_g v|(x_0) = \sum_k |v_{kk}| \leq C_1 \sum_l |v_{k,m}| \) and

\[
|\tilde{W}_{ij}(x_0)| = |v_{ij} + \frac{1-t}{n-2} (\Delta_g v) g_{ij} + \frac{2+t}{2} |\nabla v|^2 g_{ij} - v_i v_j - (A' g)_{ij}| \leq C_1 \sum_l |v_{k,m}|.
\]

Recall the Laplace-Beltrami operator \( \Delta_g = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_k} (\sqrt{|g|} g^{km} \frac{\partial}{\partial x_m}) \).

\[
(\Delta_g v)_l = \left( \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_k} (\sqrt{|g|} g^{km} v_{mk}) \right)_l
\]
\[
= \left( \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km}) v_{mk,l} + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km}) v_{km,l} \right)
\]
\[
= g^{km} v_{mk,l} + g^{km} v_{mk,l} + \left( \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km}) \right)_l v_m + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_l v_{km,l}
\]
\[
= g^{km} (v_{mk} + \Gamma^s_{km} v_s)_l + g^{km} (\Gamma^s_{km} v_s)_l + \left( \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km}) \right)_l v_m + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_l v_{km,l}
\]

Substitute the above identity into (58). At \( x_0 \),
Since we have already assumed $v_{1,1}(x_0) \geq 1$.

By the concavity of $f$ in $\Gamma$, we have $F^{ij,rs}(\bar{W}_{ij}g^{ij})_1(\bar{W}_{rk}g^{ks})_1 < 0$, and

\begin{align*}
-C_1 \leq & F^{ij}(\bar{W}_{ij}g^{ij})_1(x_0) \\
= & F^{ij}(\bar{W}_{ij,11} + 2\bar{W}_{ij,1}g^{ij} + \bar{W}_{ij}g^{ij})_1 \\
= & F^{ij}(\bar{W}_{ij,11} + \bar{W}_{ij}g^{ij})_1 \text{ by (50)} \\
\leq & F^{ij}\bar{W}_{ij,11} + C_1 \sum_{i,j,k} F^{ii}|v_{j,k}| \\
= & F^{ij} \left( v_{ij,11} + \frac{1-t}{n-2} (\Delta_g v)^{11} \right) \\
& + 2\frac{(1-t)}{n-2} (\Delta_g v) g_{ij,11} + (2-t) v_{k1} \delta_{ij} + 2(2-t) v_{k1} g_{ij} \delta_{ij} \\
& + 2\frac{1}{2} v_{k1} \Delta v g_{ij} + 2\frac{1}{2} |\Delta v|^2 g_{ij} \\
& + C_1 \sum_{i,j,k} F^{ii}|v_{j,k}| \\
= & F^{ij} \left( v_{ij,11} + \frac{1-t}{n-2} (\Delta_g v)^{11} \right) \\
& + 2\frac{(1-t)}{n-2} (\Delta_g v) g_{ij,11} + (2-t) v_{k1} \delta_{ij} \\
& + 2(2-t) v_{k1} g_{ij} \delta_{ij} + 2\frac{1}{2} v_{k1} \Delta v g_{ij} + 2\frac{1}{2} |\Delta v|^2 g_{ij} \\
& - 2v_i v_{j,11} - (A_{ij}^t)_{ij,11} + C_1 \sum_{i,j,k} F^{ii}|v_{j,k}|, \text{ by (49) and (50)}. \\
\end{align*}

Thus at $x_0$,

\begin{align*}
-C_1 \sum_{i,j,k} F^{ii}|v_{j,k}| \leq & F^{ij} \left( v_{ij,11} + \frac{1-t}{n-2} (\Delta_g v)^{11} \right) \\
& + 2\frac{(1-t)}{n-2} (\Delta_g v) g_{ij,11} + (2-t) v_{k1} \delta_{ij} \\
& - 2v_i v_{j,11} + (2-t) v_{k1} \delta_{ij} - 2v_i v_{j,1} \\
\leq & F^{ij} \left( v_{ij,11} + \frac{1-t}{n-2} (\Delta_g v)^{11} \right) \\
& + 2\frac{(1-t)}{n-2} (\Delta_g v) g_{ij,11} + (2-t) v_{k1} \delta_{ij} \\
& - 2v_i v_{j,11} + (2-t) v_{k1} \delta_{ij} - 2v_i v_{j,1} \\
& + C_1 \sum_{i,j,k} F^{ii}|v_{j,k}| \text{ by (55)} \\
\leq & F^{ij} \left( v_{ij,11} + \frac{1-t}{n-2} (\Delta_g v)^{11} \right) \\
& + 2\frac{1}{2} v_{p1} \sum_{i,j,k} F^{ii}|v_{j,k}| \text{ by (53)}
\end{align*}
i.e., at $x_0$

$$F^{ij}(v_{ij,11} + \frac{1-t}{n-2}(\Delta_g v)_{11} \delta_{ij}) \geq -(2-t) \sum_{k,i} F^{ii} v_{k,1}^2 + 2 F^{ij} v_{i,1} v_{j,1}$$

$$-C_2 \frac{1}{\sqrt{\rho}} \sum_{i,j,k} F^{ii} v_{j,k} \geq -C_1 \sum_{i,j,k} F^{ii} v_{j,k}^2 - C_2 \frac{1}{\sqrt{\rho}} \sum_{i,j,k} F^{ii} |v_{j,k}| \leq$$

For the term $(\Delta_g v)_{11}$ in the above inequality, we need to replace it by $\sum_k v_{kk,11}$.

For this reason, recall $\Delta_g = \frac{\partial}{\partial x_k} (\sqrt{|g|} g^{km} \frac{\partial}{\partial x_m})$,

$$(\Delta_g v)_{11}(x_0) = \left( \frac{1}{\sqrt{|g|}} \left( \frac{\partial}{\partial x_k} (\sqrt{|g|} g^{km} v_m) \right)_{11} \right)$$

$$= \left( g^{km} v_{m,k} + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_{11} \right)$$

$$= \left( g^{km} (v_{mk} + \Gamma^l_{km} v_l) + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_{11} \right)$$

$$= \left( g^{km} v_{mk} + g^{km} \Gamma^l_{km} v_l + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_{11} \right)$$

$$= \left( g^{km} v_{mk,11} + 2 g^{km} v_{m,k,1} + g^{km} v_{mk} + g^{km} \Gamma^l_{km} v_{l,11} \right.$$  

$+ 2 (g^{km} \Gamma^l_{km})_{11} v_{l,1} + (g^{km} \Gamma^l_{km})_{11} v_l + \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_{k,11} v_{m,11} \right.$  

$+ 2 (\frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_{k})_{11} v_{m,1} + (\frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{km})_{11} v_m \right) \leq$ (50).

Plug the above equation into (60). At $x_0$,

$$-C_1 \sum_{i,j,k} F^{ii} v_{j,k}^2 - C_2 \frac{1}{\sqrt{\rho}} \sum_{i,j,k} F^{ii} |v_{j,k}| \leq$$
\[
F_{ij}(v_{ij,11} + \frac{1-t}{n-2}v_{kk,11}\delta_{ij} + \frac{1-t}{n-2}\Gamma_{kk}^{l}v_{l,11}\delta_{ij} + \frac{1-t}{n-2}(\sqrt{|g|g^{km})_{k}v_{m,11}\delta_{ij}} + C_1\sum_{i,j,k}F^{ii}|v_{j,k}|)
\]
\[
\leq F_{ij}(v_{ij,11} + \frac{1-t}{n-2}v_{kk,11}\delta_{ij} + \frac{1-t}{n-2}\Gamma_{kk}^{l}v_{l,11}\delta_{ij} + \frac{1-t}{n-2}(\sqrt{|g|g^{km})_{k}v_{11,m}\delta_{ij}} + C_1\sum_{i,j,k}F^{ii}|v_{j,k}| \text{ by (55)}
\]
\[
\leq F_{ij}(v_{ij,11} + \frac{1-t}{n-2}v_{kk,11}\delta_{ij} + \frac{1}{\sqrt{\rho}}\sum_{i,j,k}F^{ii}|v_{j,k}| \text{ by (53)}
\]
\[
= \tilde{L}_{ij}v_{ij,11} + C_2\frac{1}{\sqrt{\rho}}\sum_{i,j,k}F^{ii}|v_{j,k}|,
\]
that is,
\[
\tilde{L}_{ij}v_{ij,11}(x_0) \geq -C_1\sum_{i,j,k}F^{ii}v_{j,k}^2 - C_2\frac{1}{\sqrt{\rho}}\sum_{i,j,k}F^{ii}|v_{j,k}|
\] (61)

Substitute (59) and (61) into (57). Notice that \(v_{ij,\nu} = -v_{ij,n}\) and \(1 + \psi\beta e^{\nu} \in [\frac{1}{2}, 1]\).

At \(x_0\),
\[
0 \geq 2a(1 + \psi\beta e^{\nu})^2\tilde{L}_{ij}v_{k,j}v_{k,i} - C_2\rho^{-1}\sum_{l,k,i}F^{il}|v_{k,i}| - C_1\sum_{l,k,i}F^{il}v_{k,i}^2
\]
\[
\geq \frac{a}{2}\tilde{L}_{ij}v_{k,j}v_{k,i} - C_2\rho^{-1}\sum_{l,k,i}F^{il}|v_{k,i}| - C_1\sum_{l,k,i}F^{il}v_{k,i}^2
\]
\[
\geq \frac{a(1-t)}{2(n-2)}\sum_{l,k,i}F^{il}v_{k,i}^2 - C_2\rho^{-1}\sum_{l,k,i}F^{il}|v_{k,i}| - C_1\sum_{l,k,i}F^{il}v_{k,i}^2
\]
\[
\geq \sum_{l,k,i}F^{il}v_{k,i}^2 - C_2\rho^{-1}\sum_{l,k,i}F^{il}|v_{k,i}| \text{ by taking } a > \frac{2(n-2)(C_1+1)}{(1-t)}.
\]

Multiply the above inequality by \(\rho^2\). At \(x_0\),
\[
0 \geq \sum_{l,k,i}F^{il}(\rho v_{k,i})^2 - C_2\rho|v_{k,i}|,
\]
which implies that \((\rho|v_{1,1}|)(x_0) < C_2\), therefore \(H(x_0) < C_2\). Lemma 6.1 has been established. ♠.

Remark 6.1 As a consequence of the Lemma 6.1, \(\forall y \in B_{y_0}^{T_yM} \subseteq (y^{\nu}), \) let \((e_1, \cdots, e_n)\) be an orthonormal basis of \(T_yM\) with \(e_n = \frac{\partial}{\partial y^{\nu}},\) and let subindices denote the covariant...
derivatives w.r.t. $e_j$. By $\Gamma \subset \Gamma_1$, we have $\Delta_g v(y) > -C$, which implies that $v_{\nu\nu}(y) > -C$, and for $1 \leq k \leq n - 1$,

$$v_{kk}(y) = \Delta_g v - \sum_{l \neq k, n} v_l - v_{nn} \geq -C - v_{nn} = -C - v_{\nu\nu}.$$

If $v_{\nu\nu}(y) \geq 0$, then $v_{\nu\nu} + C \geq C > v_{kk}(y) > -C - v_{\nu\nu}$ implies that $|v_{kk}(y)| \leq C + v_{\nu\nu}(y)$ for $1 \leq k \leq n - 1$. If $v_{\nu\nu}(y) < 0$, then $C > v_{kk}(y) > -C - v_{\nu\nu} > -C$ implies that $|v_{kk}(y)| \leq C \leq C + v_{\nu\nu}(y)$ for $1 \leq k \leq n - 1$ since $v_{\nu\nu}(y) \geq -C$. Hence, for any two vectors $X, Y \in T_yM$ with $g(X, \frac{\partial}{\partial \nu}) = g(Y, \frac{\partial}{\partial \nu}) = 0$,

$$|\nabla^2_g v(X, Y)|(y) = \frac{1}{4}(|\nabla^2_g v(X + Y, X + Y) - \nabla^2_g v(X, X) - \nabla^2_g v(Y, Y)|$$

$$\leq \frac{1}{4}(|\nabla^2_g v(X + Y, X + Y)| + |\nabla^2_g v(X, X)| + |\nabla^2_g v(Y, Y)|$$

$$\leq \frac{1}{4}(|X + Y|^2_g + |X|^2_g + |Y|^2_g)(v_{\nu\nu} + C)$$

$$\leq \frac{1}{2}(|X|^2_g + |Y|^2_g)(v_{\nu\nu} + C) \leq 2(|X|^2_g + |Y|^2_g)(v_{\nu\nu} + C)$$

$$\leq (|X|^2_g + |Y|^2_g)(2v_{\nu\nu} + C).$$

\textbf{Lemma 6.2} Under the same assumptions as in Theorem 1.2, for $t < 1$, let $v$ be a $C^4$ solution of the equation (12). Then there exists a universal constant $C > 0$ depending only on $(M^n, g, t), (f, \Gamma), \phi, \psi, \delta_{y_0}$ such that in $B^T_{\frac{1}{4}y_0}(y_0) \cap \partial M$,

$$v_{\nu\nu} < C.$$

\textbf{Proof of the Lemma 6.2.} Let $\{y_1, \ldots, y_n\}$ be the tubular neighborhood normal coordinates of $y \in B^T_{\frac{1}{4}y_0}(y_0)$ at $y_0$. Let $\{e_1, \ldots, e_n\}$ be a smooth orthonormal frame of $TM$ in $B^T_{\frac{1}{4}y_0}(y_0)$ with $e_n = \frac{\partial}{\partial \nu}$. In fact, we can obtain such frame by moving an orthonormal basis of $T_{y_0}(\partial M)$ parallelly along the geodesic of $(\partial M, g|_{\partial M})$ to get an orthonormal frame of $\Gamma(\partial M)$ in $B^T_{\frac{1}{4}y_0}(y_0)$, then moving such frame parallelly along the geodesic $r(t) = E\left(\frac{\partial}{\partial \nu}, t\right)$. In this way, we can get smooth orthonormal vector fields $\{e_j\}_{j=1}^{n-1}$ in $B^T_{\frac{1}{4}y_0}(y_0)$ with $g(e_j, \frac{\partial}{\partial \nu}) = 0$, and $\{e_j\}_{j=1}^{n}$ with $e_n = \frac{\partial}{\partial \nu}$ will be an orthonormal frame of $TM$ in $B^T_{\frac{1}{4}y_0}(y_0)$.

Observe $\frac{\partial}{\partial \nu}$ is the unit tangent vector of the geodesic. We have

$$\nabla^T_{\frac{\partial}{\partial \nu}} = 0 \quad \text{in} \quad B^T_{\frac{1}{4}y_0}(y_0).$$

(63)
In the following, subindices denote the covariant derivatives w.r.t. \( \{e_1, \ldots, e_n\} \). Differentiate the equation \( F(\tilde{W}_{ij}) = \phi e^{2u} \) along the normal direction \( e_n \),

\[
F_{ij}\left(v_{ij,\nu} + \frac{1-t}{n-2}(\Delta v)\nu \delta_{ij} + (2-t)v_k v_{k,\nu} \delta_{ij} - 2v_i v_{j,\nu} - (A^i_{g})_{ij,\nu}\right) = e^{2u}(\phi_{\nu} + 2\phi v_\nu) \tag{64}
\]

We need to interchange \( v_{ij,\nu} \) to \( v_{\nu,ij} \). For this reason, let \( e_i = a_i^j \frac{\partial}{\partial y_j} \). Then \( a_i^j \in C^\infty(B_{3/4}^{T_{\delta,\nu}}(y^n)) \), and

\[
g(e_i, e_j) = \delta_{ij} \iff a_i^k g_{kj} a_j^i = \delta_{ij}. \tag{65}
\]

Notice \( e_n = \frac{\partial}{\partial v} = -\frac{\partial}{\partial y_n} \) and \( g(e_i, e_n) = \delta_{in} \). We have

\[
a_n^i = -\delta_n^i, \quad a_i^n = -\delta_i^n. \tag{66}
\]

In \( B_{3/4}^{T_{\delta,\nu}}(y^n) \),

\[
v_{i,\nu} = -\frac{\partial}{\partial y_n}(a_i^r \frac{\partial}{\partial y_r}) = -a_i^r \frac{\partial^2 v}{\partial y_n \partial y_r} - \frac{\partial a_i^r}{\partial y_n} \frac{\partial v}{\partial y_r},
\]

so

\[
v_{ij,\nu} = \frac{\partial}{\partial v} \left( \nabla^2 v(e_i, e_j) \right) = -\frac{\partial}{\partial y_n}(a_i^r a_j^s \nabla^2 v(\frac{\partial}{\partial y_r}, \frac{\partial}{\partial y_s}))
\]

\[
= -a_i^r a_j^s \frac{\partial}{\partial y_n}(\nabla^2 v(\frac{\partial}{\partial y_r}, \frac{\partial}{\partial y_s})) - \frac{\partial(a_i^r a_j^s)}{\partial y_n}(\nabla^2 v(\frac{\partial}{\partial y_r}, \frac{\partial}{\partial y_s}))
\]

\[
= -a_i^r a_j^s \frac{\partial^2 v}{\partial y_n \partial y_r} - \frac{\partial(a_i^r a_j^s)}{\partial y_n}(\frac{\partial^2 v}{\partial y_r \partial y_s} - \Gamma^l_{sr} \frac{\partial v}{\partial y_l})
\]

\[
= -a_i^r a_j^s \frac{\partial^2 v}{\partial y_n y_r y_s} + a_i^r a_j^s \Gamma^l_{sr} \frac{\partial^2 v}{\partial y_l y_s y_r}
\]

\[
- \frac{\partial(a_i^r a_j^s)}{\partial y_n}(\frac{\partial^2 v}{\partial y_r \partial y_s} - \Gamma^l_{sr} \frac{\partial v}{\partial y_l})
\]

\[
: = v_{\nu,ij} + \Omega_{ij} = a_i^r a_j^s \frac{\partial v}{\partial y_n} + \frac{\partial(a_i^r a_j^s)}{\partial y_n} \Gamma^l_{sr},
\]

where

\[
\Omega_{ij} = -\frac{\partial(a_i^r a_j^s)}{\partial y_n}, \quad \text{and} \quad \Theta_{ij} = a_i^r a_j^s \frac{\partial v}{\partial y_n} + \frac{\partial(a_i^r a_j^s)}{\partial y_n} \Gamma^l_{sr},
\]

depend only on \( (M^n, g) \), and are smooth and bounded in \( B_{3/4}^{T_{\delta,\nu}}(y^n) \).

In particular,
Therefore (70) implies that

\[(\Delta v)_\nu = \Delta (v_\nu) + \Omega_{kk}^s \frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} + \Theta_{kk}^t \frac{\partial v}{\partial y_{ijl}}.\]  

(69)

Substitute (67), (68), and (69) into (64). We have

\[F^{ij} \{ v_{\nu,ij} + \Omega_{ij}^s \frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} + \Theta_{ij}^t \frac{\partial v}{\partial y_{ijl}} + \frac{1-t}{n-2} \Delta (v_\nu) \delta_{ij} + \frac{1-t}{n-2} \Omega_{kk}^s \frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} \delta_{ij} + \frac{1-t}{n-2} \Theta_{kk}^t \frac{\partial v}{\partial y_{ijl}} \delta_{ij} + (2-t) v_k (a_r^i \frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} - \frac{\partial^2 v}{\partial y_{ijl} \partial y_{s}}) \delta_{ij} - 2 v_i (a_r^i \frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} - \frac{\partial^2 v}{\partial y_{ijl} \partial y_{s}}) - (A^t_{g})_{ij,\nu} \}
\]

which can be written as

\[L^{ij} (v_\nu)_{ij} + \Lambda^{rs} \frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} = \Pi,\]  

(70)

where

\[L^{ij} = F^{ij} + \frac{1-t}{n-2} \sum_l F^{il} \delta^{ij},\]

\[\Lambda^{rs} = F^{ij} \{ \Omega_{ij}^s + \frac{1-t}{n-2} \Omega_{kk}^s \delta_{ij} - (2-t) v_k (a_r^i \delta_{ij} \delta_{sn} + 2 v_i a_r^i \delta_{sn}) \},\]

\[\Pi = e^{2v} (\phi_\nu + 2 \phi v_\nu) - F^{ij} \{ \Theta_{ij}^t \frac{\partial v}{\partial y_{ijl}} + \frac{1-t}{n-2} \Theta_{kk}^t \frac{\partial v}{\partial y_{ijl}} \delta_{ij} + (2-t) \frac{\partial v}{\partial y_{ijl} \partial y_{s}} v_i + 2 \frac{\partial v}{\partial y_{ijl} \partial y_{s}} \delta_{ij} - (A^t_{g})_{ij,\nu} \}\]

depend only on \((M^n, g), |\nabla v|_{C^1(M,g)}, t, \) and \(\phi,\) and are \(C^3\) and bounded by \(C \sum_l F^{il}.\)

For \(\frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}},\) we need to replace it by the partial derivatives of \(v\) w.r.t. \(e_i.\) Recall that \(e_i = a^i_i \frac{\partial}{\partial y_i}.\) Hence \(\frac{\partial}{\partial y_i} = b^i_l e_j\) with \((b^i_l) = (a^i_i)^{-1},\) which is also smooth in \(B^T_{y^i}(y^j).\) In \(B^T_{y^i}(y^j),\)

\[\frac{\partial^2 v}{\partial y_{ijr} \partial y_{s}} = \nabla^2 v (\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) + \Gamma^l_{rs} \frac{\partial v}{\partial y_l} = b^j_l b^i_r v_{ij} + \Gamma^l_{rs} \frac{\partial v}{\partial y_l},\]

therefore (70) implies that

\[L^{ij} (v_\nu)_{ij} + \Lambda^{rs} b^j_s b^i_r v_{ij} = \Pi - \Lambda^{rs} \Gamma^l_{rs} \frac{\partial v}{\partial y_l},\]

or

\[L^{ij} (v_\nu)_{ij} + \sum_{j=1}^n \Lambda^{rs} b^j_s b^i_r v_{ij} + \sum_{i=1}^{n-1} \Lambda^{rs} b^i_l b^j_r v_{i\nu} = \Pi - \Lambda^{rs} \Gamma^l_{rs} \frac{\partial v}{\partial y_l} - \sum_{i,j=1}^{n-1} \Lambda^{rs} b^i_l b^j_r v_{ij}.\]  

(71)
By

\[ v_{\nu j} = (\nabla_{e_j} \nabla_{e_j})(v) - (\nabla_{e_j}^\nu)(v) = v_{\nu j} - (\nabla_{e_j}^\nu)(v), \]

and

\[ v_{\nu i} = v_{\nu i} = v_{\nu,i} - (\nabla_{e_i}^\nu)(v), \]

(71) implies that

\[ L^{ij}(v_{\nu})_{ij} + \sum_{j=1}^{n} \Lambda^{rs} b^{n}_{s} b^{n}_{r} v_{\nu,j} + \sum_{i=1}^{n-1} \Lambda^{rs} b^{n}_{s} b^{n}_{r} v_{\nu,i} = \Pi - \Lambda^{rs} \Gamma_{rs}^{\nu} \frac{\partial v}{\partial x_i} \]

\[ + \sum_{j=1}^{n} \Lambda^{rs} b^{n}_{s} b^{n}_{r} (\nabla_{e_j}^\nu)(v) + \sum_{i=1}^{n-1} \Lambda^{rs} b^{n}_{s} b^{n}_{r} (\nabla_{e_i}^\nu)(v) - \sum_{i,j=1}^{n} \Lambda^{rs} b^{n}_{s} b^{n}_{r} v_{ij}. \]

Define an elliptic 2nd order linear differential operator in \( B^T_{\frac{y}{2}} (y^o) \) as follows.

\[ L(w) = L^{ij} w_{ij} + (b^i - \bar{s} \sum_l F^{il} \delta^{ni}) w_i, \]

where \( b^i = \begin{cases} \Lambda^{rs} b^{n}_{s} b^{n}_{r} + \Lambda^{rs} b^{n}_{s} b^{n}_{r} & \text{if } 1 \leq i \leq n - 1 \text{ and } \bar{s} > 0 \text{ is some constant} \\ \Lambda^{rs} b^{n}_{s} b^{n}_{r} & \text{if } i = n, \end{cases} \)

to be determined later. Then \( |b^i| \leq C \sum_l F^{il} \) in \( B^T_{\frac{y}{2}} (y^o) \), and

\[ L(v_{\nu}) = \Pi - \Lambda^{rs} \Gamma_{rs}^{\nu} \frac{\partial v}{\partial x_i} + \sum_{j=1}^{n} \Lambda^{rs} b^{n}_{s} b^{n}_{r} (\nabla_{e_j}^\nu)(v) + \sum_{i=1}^{n-1} \Lambda^{rs} b^{n}_{s} b^{n}_{r} (\nabla_{e_i}^\nu)(v) \]

\[ - \sum_{i,j=1}^{n} \Lambda^{rs} b^{n}_{s} b^{n}_{r} v_{ij} - \bar{s} \sum_l F^{il} v_{\nu,n} \]

\[ \leq C \sum_l F^{il} - \sum_{i,j=1}^{n} \Lambda^{rs} b^{n}_{s} b^{n}_{r} v_{ij} - \bar{s} \sum_l F^{il} v_{\nu,n} \]

\[ \leq C \sum_l F^{il} + C \sum_l F^{il} \sum_{i,j=1}^{n-1} |v_{ij}| - \bar{s} \sum_l F^{il} (v_{\nu,n} + (\nabla_{e_j}^\nu)(v) \]

\[ \leq C \sum_l F^{il} + C \sum_l F^{il} \sum_{i,j=1}^{n-1} |v_{ij}| - \bar{s} \sum_l F^{il} v_{\nu,n} \quad \text{by (63)} \]

\[ \leq C \sum_l F^{il} + C \sum_l F^{il} \sum_{i,j=1}^{n-1} 2|v_{ij}| - \bar{s} \sum_l F^{il} v_{\nu,n} \quad \text{by (62)} \]

\[ \leq C \sum_l F^{il} - (\bar{s} - C) \sum_l F^{il} v_{\nu,n} \]

\[ \leq C \sum_l F^{il} \quad \text{by taking } \bar{s} > C, \]

where, in the last inequality, we used \( v_{\nu,n} > -C \), therefore \( -v_{\nu,n} < C \).
From the equation $F(W) = \phi e^{2v}$, we know

$$L^{ij}v_{ij} = \phi e^{2v} + F^{ij}v_i v_j - \frac{2}{2} \nabla v_i^2 \sum_l F^{il} + F^{ij}(A^t_g)_{ij},$$

hence

$$|L(v)| \leq C \sum_l F^{il} \text{ in } B_{\delta y_0}^T(y_0). \quad (73)$$

For any $y_0 \in B_{\delta y_0}^T(y_0) \cap \partial M$, let $(a_1, \cdots, a_{n-1}, 0)$ be the tubular neighborhood normal coordinates of $y_0$ at $y_0$, and let

$$D := \{(y_1, \cdots, y_n) \mid y_n \geq 0, \sqrt{(y_1 - a_1)^2 + \cdots + (y_n - a_n - 1)^2 + y_n^2} < \frac{\delta y_0}{64}\}. \quad (74)$$

Then

$$\sqrt{y_1^2 + \cdots + y_n^2} \leq \sqrt{a_1^2 + \cdots + a_{n-1}^2} + \sqrt{(y_1 - a_1)^2 + \cdots + (y_n - a_n - 1)^2 + y_n^2} < \frac{\delta y_0}{64} + \frac{\delta y_0}{64} = \frac{\delta y_0}{32},$$

i.e., $D \subset B_{\delta y_0}^T(y_0)$.

Extend $h_g, \psi$ to a smooth and $C^{3,\alpha}$ function in $B_{\delta y_0}^T(y_0)$ independently, still denoted by $h_g, \psi$. In $D$, consider

$$\bar{w} = v_\nu - \psi e^v + h_g + a(1 - e^{-b_yn}) + \bar{c}\left((y_1 - a_1)^2 + \cdots + (y_n - a_n - 1)^2 + y_n^2\right),$$

where $a, b, \bar{c}$ are positive constants to be determined later.

Pick $\bar{c} > 0$ such that

$$v_\nu - \psi e^v + h_g + \bar{c}\left(\frac{\delta y_0}{64}\right)^2 \geq 0 \text{ in } B_{\delta y_0}^T(y_0).$$

Then

$$\bar{w}(x_0) = 0 \quad \text{and} \quad \bar{w} \geq 0 \text{ on } \partial D. \quad (74)$$

Denote $\mathcal{R} = (y_1 - a_1)^2 + \cdots + (y_n - a_n - 1)^2 + y_n^2$. By (73),

$$|L(-\psi e^v + h_g + \bar{c}\mathcal{R})| = |L(h_g + \bar{c}\mathcal{R}) - \psi e^v L(v) - e^v L(\psi) - 2L^{ij} \psi_i v_j - \psi e^v L^{ij} v_i v_j| \leq C \sum_l F^{il}. \quad (75)$$

To estimate $L(e^{-b_yn})$, recall $e_i = a_i^t \frac{\partial}{\partial y_0}$. In $D$,
$$|b^i (e^{-by_n})_{ij}| = |b^i a^i_j \frac{\partial}{\partial y_j} (e^{-by_n})| = |-bb^i a^i_j e^{-by_n} \delta_{jn}| \leq Cbe^{-by_n} \sum_l F^{ll},$$

where and in the following, $C > 0$ denotes a universal constants independent of $a$ and $b$.

$$L^{ij} (e^{-by_n})_{ij} = \begin{align*} &L^{ij}(\nabla^2 (e^{-by_n})(e_i, e_j)) = a^i_j a^k_j L^{ij}(\nabla^2 (e^{-by_n})(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})) \\
&= a^i_j a^k_j L^{ij}(\frac{\partial^2}{\partial y_i \partial y_j}(e^{-by_n}) - \Gamma^l_{st} \frac{\partial}{\partial y_i} (e^{-by_n})) \\
&= a^i_j a^k_j L^{ij}(b^2 e^{-by_n} \delta_{rn} \delta_{sn} + be^{-by_n} \Gamma^l_{st} \delta_{nl}) \\
&= b^2 e^{-by_n} L^{ij} a^i_j a^k_j + be^{-by_n} \Gamma^l_{st} a^i_j a^k_j \\
&\geq b^2 e^{-by_n} a^i_j a^k_j - Cbe^{-by_n} \sum_l F^{ll} \\
&\geq b^2 e^{-by_n} a^i_j a^k_j (F^{ij} + \frac{l+4}{n-2} \sum_l F^{ll} \delta_{ij}) - Cbe^{-by_n} \sum_l F^{ll} \\
&\geq \frac{l+4}{n-2} b^2 e^{-by_n} (a^i_j)^2 \sum_l F^{ll} - Cbe^{-by_n} \sum_l F^{ll} \\
&= \frac{l+4}{n-2} b^2 e^{-by_n} \sum_l F^{ll} - Cbe^{-by_n} \sum_l F^{ll} \text{ by (66).} \end{align*}$$

Thus in $D$,

$$L(e^{-by_n}) \begin{cases} \geq \frac{l+4}{n-2} b^2 e^{-by_n} \sum_l F^{ll} - Cbe^{-by_n} \sum_l F^{ll} \\
\geq be^{-by_n} (\frac{1-\alpha l}{n-2} - C) \sum_l F^{ll} \\
\geq be^{-by_n} \sum_l F^{ll}, \end{cases}$$

by choosing $b \gg 1$ such that $\frac{1-\alpha l}{n-2} - C \geq 1$.

Back to $L(\bar{w})$, we have in $D$,

$$L(\bar{w}) = L(v_{\nu} - \psi e^n + h_g + c\mathcal{R}) - aL(e^{-by_n}) \leq -abe^{-by_n} \sum_l F^{ll} + C \sum_l F^{ll} \leq 0,$$

by choosing $a \gg 1$ such that $abe^{-by_n} > C$.

Hence (74) implies that

$$\bar{w} \geq 0 \text{ in } D,$$

therefore we have $\bar{w}_\nu (y_0) \leq 0$, i.e., $v_{\nu \nu} (y_0) < C$. Since $y_0 \in B^T_{\frac{1}{\delta_0}} (y_0) \cap \partial M$ is arbitrary, Lemma 6.2 has been established. ♦
Remark 6.2 By the Lemma 6.1 and the Lemma 6.2 and \( \bigcup_{i_0=1}^{N} (B_{\delta y^0}^{T} \cap \partial M) = \partial M \), we know the Hessian of \( v \) on \( \partial M \) is upper bounded w.r.t. the metric \( g \). Thus \( \Gamma \subset \Gamma_1 \) implies that
\[
|\nabla^2 g|_g \leq C \quad \text{on} \quad \partial M.
\]

Lemma 6.3 Under the same assumptions as in Theorem 1.2, for \( t < 1 \), let \( v \) be a \( C^4 \) solution of the equation (12). Then there exists a universal constant \( C > 0 \) depending only on \((M^n, g, t), (f, \Gamma), \phi, \psi\) such that on \( M \),
\[
|\nabla^2 v| < C.
\]

Proof of the Lemma 6.3. Consider
\[
E(x) = \max_{e \in T_x M, \ g(e,e) = 1} (\nabla^2 v + a|\nabla v|^2_g)(e,e).
\]
Let \( E(x_0) = \max_{M} E \), and let \( \{x_j\}_{j=1}^{n} \) be a geodesic normal coordinates w.r.t. the metric \( g \) at \( x_0 \). In the following, subindices denote the covariant derivatives w.r.t. \( \frac{\partial}{\partial x_j} \). W.l.o.g, we assume \( x_0 \) is an interior point of \( M \), and \( E(x_0) = v_{11} + a|\nabla v|^2_g \). Consider \( \bar{E} = \frac{v_{11}}{g} + a|\nabla v|^2_g \). Then \( x_0 \) is a local maximum point of \( \bar{E} \). We can proceed as in the proof of the Lemma 6.1 to finish the proof of the Lemma 6.3.

7 Proof of the Theorem 1.2

Consider the homotopy equation \( H_s \), for \( 0 \leq s \leq 1 \),
\[
\begin{array}{ll}
\{ & f \left( -\lambda_g(s W + (1-s) \sigma_1(W) g) \right) - s \phi e^{2v} - (1-s) e^{2v} = 0 \quad \text{on} \quad M, \\
v_\nu + h_g - s e^v \psi = 0 \quad \text{on} \quad \partial M,
\end{array}
\]
(76)
where \( W = W^v - A^v \).

By the uniform \( C^2 \) estimates we established and the result of Lieberman and Trudinger ([19]), we have the uniform \( C^{2, \alpha_0} \) bounds for the solutions of the above equation. \( C^{4, \alpha_0} \) estimates follow from the Schauder estimates. By the direct computation, the linearized operator \( L_s(w) \) at a solution \( v \) is given by
\[
\begin{array}{ll}
\{ & (s L^{ij} + (1-s) L^{ij}) w_{ij} + \tilde{b}^i w_i - 2(s \phi + (1-s)) e^{2v} w \quad \text{on} \quad M, \\
w_\nu - s e^v \psi w \quad \text{on} \quad \partial M,
\end{array}
\]
(77)
where
\[
\bar{b} = s(2 - t)F^{ij}v_i - 2sF^{ij}v_j + (2n - nt - 2)(1 - s)F^{ij}v_i.
\]

By \(\phi > 0, \psi \leq 0\) and the maximum principle, the linearized operator is an elliptic invertible operator: \(C^{2,\alpha} \rightarrow C^{\alpha}\). Hence the equation of (76) for \(s = 1\) is uniquely solvable in \(C^{4,\alpha}\) if and only if the equation of (76) for \(s = 0\) is uniquely solvable in \(C^{4,\alpha}\). When \(s = 0\), the uniqueness and the existence of the solution has been confirmed in [3]. ♣

8 Proof of the Theorem 1.3

Take an arbitrary Riemannian metric \(g\) on \(M^n\). For instance, let \(\{U_i, x^{(i)}\}_{i=1}^N\) be a finite coordinate charts on \(M^n\) and let \(\phi^i\) be a partition of unity subordinate to \(U_i\). We can simply take \(g\) to be \(\sum_{i=1}^N \phi^i((dx_1^{(i)})^2 + \cdots + (dx_n^{(i)})^2)\). Let \(w(x)\) be a smooth function on \(M^n\) such that \(w(x)\) is the distance of \(x\) to \(\partial M\) w.r.t. the metric \(g\) when \(x\) is near \(\partial M\). Then \(\frac{\partial w}{\partial \nu}\) on \(\partial M\) is \(-1\), where \(\frac{\partial}{\partial \nu}\) is the unit outer normal of \(g\) on \(\partial M\).

Extend the mean curvature \(h_g\) to a smooth function defined on \(M^n\), still denoted by \(h_g\). We can obtain such extension by straightening the boundary and extending any function \(\bar{\psi}\) defined on \(\partial R^n_+\) to \(R^n_+\) using \(\bar{\psi}(x')(1 - x_n)\), where \(x = (x', x_n) \in R^n_+\). However, we want to mention a different way which seems more natural. In fact, we only need to extend \(h_g\) smoothly to the interior of \(M\) near \(\partial M\). Using the partition of unity, we can localize the extension to a small neighborhood of each \(x_0 \in \partial M\). Notice \(h_g\) is the trace of the second fundamental form of \(g\) on \(\partial M\) whose definition is, at every point \(x \in \partial M\),

\[
II(X, Y) = -g(\nabla_X^\partial, Y), \quad \forall X, Y \in T_x(\partial M).
\]

Let \(U\) be a small neighborhood where the tubular neighborhood normal coordinates of \(x \in U\) at \(x_0\) is smooth and well-defined. Let \(\{x_j\}_{j=1}^n\) be such coordinates. Then \(-\frac{\partial}{\partial x_n}\) is a smooth extension of \(\frac{\partial}{\partial \nu}\) to \(U\) and \(g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_n}) = 0\) in \(U\) for \(1 \leq k \leq n - 1\), so

\[
\frac{1}{n - 1} \sum_{i,j=1}^{n-1} g(\nabla_{\frac{\partial}{\partial x_l}} \frac{\partial}{\partial x_j}) g^{ij}
\]

is an extension of \(h_g\) to \(U\), where \((g^{ij})\) is the inverse of \((g_{ij}) = (g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}))\). From the linear algebra, we know \((g^{ij}) = \frac{1}{\det(g_{ij})} \text{adj}(g_{ij})\), hence \(g^{ij}\) is smooth and (78) gives a smooth extension of \(h_g\) to \(U\).

Let \(v = h_g w\). Consider \(g_1 = e^{2v} g\). Then on \(\partial M\),
\[ h_{g_1} = \left( \frac{\partial v}{\partial v} + h_y \right) e^{-v} = (h_y \frac{\partial w}{\partial v} + w \frac{\partial h_g}{\partial v} + h_y) e^{-v} = (h_y (-1) + (0) \frac{\partial h_g}{\partial v} + h_y) = 0. \]

For \( g_1 \), let \( w_1 \) be a smooth function such that, near \( \partial M \), \( w_1 \) is the distance function to \( \partial M \) w.r.t. \( g_1 \). We know that \( \frac{\partial w_1}{\partial v_1} \) at \( \partial M \) is the unit outer normal of \( g_1 \) on \( \partial M \). Take \( g_2 = e^{A(w_1)^2} g_1 \) with \( A > 0 \) being a constant to be chosen later.

Direct computations yield that on \( \partial M \)
\[ h_{g_2} = (2Aw_1 \frac{\partial w_1}{\partial v_1} + h_{g_1}) e^{A(w_1)^2} = 0, \]
and
\[ \text{Ric}_{g_2} = \text{Ric}_{g_1} - (n - 2)A \nabla^2 (w_1^2) - A(\Delta g_1 (w_1^2)) g_1 + (n - 2)A^2 d(w_1^2) \otimes d(w_1^2) \]
\[ \leq \text{Ric}_{g_1} - A(n - 2) \nabla^2_1 (w_1^2) - A(\Delta g_1 (w_1^2)) g_1, \]

where in the last inequality, we used a general fact that \( df \otimes df \leq |\nabla f|^2 g_1 \) for any \( C^1 \) function \( f \). The explanation is given as follows. At each \( x \), we take a geodesic normal coordinates \( \{x_i\}_{i=1}^n \) of \( g_1 \) at \( x \). At \( x \),
\[(df \otimes df)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}) = (\frac{\partial f}{\partial x_i})^2 \leq \sum_{i=1}^n (\frac{\partial f}{\partial x_i})^2 = (|\nabla f| g_1) (\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}),\]
which implies that \( df \otimes df \leq |\nabla f|^2 g \) since both \( df \otimes df \) and \( |\nabla f|^2 g \) are symmetric \((0, 2)\) tensors.

For any \( x_0 \in \partial M \), we take a tubular neighborhood normal coordinates \( \{x_j\}_{j=1}^n \) of \( g_1 \) at \( x_0 \). Then \( w_1 = x_n \) near \( x_0 \). At \( x_0 \), by \( x_n = 0 \)

\[ \nabla^2_{g_1} [(x_n)^2] (\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} [(x_n)^2] - (\nabla \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}) [(x_n)^2] \]
\[ = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} [(x_n)^2] - 2x_n (\nabla \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}) [x_n] \]
\[ = \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} [(x_n)^2] = \nabla_{\frac{\partial}{\partial x_j}} [2x_n \nabla \frac{\partial}{\partial x_i} (x_n)] \]
\[ = 2(\nabla \frac{\partial}{\partial x_j} x_n) (\nabla \frac{\partial}{\partial x_i} x_n) + 2x_n \nabla \frac{\partial}{\partial x_i} \nabla \frac{\partial}{\partial x_j} x_n \]
\[ = 2(\nabla \frac{\partial}{\partial x_j} x_n) (\nabla \frac{\partial}{\partial x_i} x_n) = 2 \delta^i_j \delta^n, \]
so at \( x_0 \),
\[
\nabla_{g_1}^2 [w_1^2] = \nabla_{g_1}^2 [(x_n)^2] = 2 dx_n \otimes dx_n \geq 0,
\]
and
\[
\Delta_{g_1}[w_1^2] = \Delta_{g_1}[(x_n)^2] = 2.
\]
Substitute the above two into (79). At \( x_0 \), we have
\[
\text{Ric}_{g_2} \leq \text{Ric}_{g_1} - (n-2) A \nabla_{g_1}^2 (w_1^2) - A (\Delta_{g_1} (w_1^2)) g_1
\]
\[
\leq \text{Ric}_{g_1} - 2 A g_1 \leq C_1 g_1 - 2 A g_1,
\]
where \( C_1 > 0 \) is a universal constant depending only on \( (M^n, g) \) and independent of \( x_0 \).

Choose \( A \geq \frac{C_1}{2} + \frac{1}{2} \). Then \( \text{Ric}_{g_2}(x_0) \leq -g_1(x_0) \), which implies that \( \text{Ric}_{g_2} \leq -g_1 \) on \( \partial M \), hence
\[
\text{Ric}_{g_2} < 0 \quad \text{in a tubular neighborhood of } \partial M.
\]
By the result in [15], there is a smooth metric \( g_3 \) on \( M \) such that
\[
g_3 \equiv g_2 \quad \text{in a smaller tubular neighborhood of } \partial M,
\]
and
\[
\text{Ric}_{g_3} < 0 \quad \text{on } M.
\]
Clearly, \( h_{g_3} = h_{g_2} = 0 \) on \( \partial M \), and \( \text{Ric}_{g_3} < 0 \) on \( M \) implies that
\[
-\lambda_{g_3}(A^t_{g_3}) \in \Gamma_n \subset \Gamma, \quad \forall t < 1.
\]
Thus, by the Theorem 1.2, there exists a unique \( C^{4,\alpha_0} \) metric \( g_4 \in [g_3] \) solving
\[
\begin{aligned}
\{ \ f(-\lambda_{g_4}(A^t_{g_4})) & = \phi, \quad -\lambda_{g_4}(A^t_{g_4}) \in \Gamma \quad \text{on } M \\
\ h_{g_4} & = \psi \quad \text{on } \partial M.
\end{aligned}
\]
In particular, we can take \( (f, \Gamma) = (\sigma^n, \Gamma_n), \ t = 0, \ \text{and } \phi \equiv 1, \ \psi \equiv 0 \). Theorem 1.3 has been established. ♠

From the arguments in the proof of Theorem 1.3, it is easy to see that for any smooth compact Riemannian manifold \( (M^n, g) \ (n \geq 3) \) with some boundary including those metrics with positive Ricci tensors, there exists some metric \( g_3 \) which is conformal to \( g \) near \( \partial M \) satisfying
\[
-\lambda_{g_3}(A^t_{g_3}) \in \Gamma_n \subset \Gamma \quad \text{on } M \quad \text{and} \quad h_{g_3} = 0 \quad \text{on } \partial M.
Thus we have the following result

**Theorem 8.1** Let $(M^n, g)$ be an $n-$dimensional ($n \geq 3$) compact smooth Riemannian manifold with $\partial M \neq \emptyset$ and let $f \in C^{2,\alpha_0}(\Gamma)$ ($0 < \alpha < 1$) satisfy (5)-(9). Given $0 < \phi \in C^{2,\alpha_0}(M^n)$, $0 \geq \psi \in C^{3,\alpha_0}(\partial M)$ and for any $t < 1$, there exists a $C^{4,\alpha_0}$ solution $\tilde{g}$ which is conformal to $g$ near $\partial M$ and solves

\[
\begin{aligned}
&f(-\lambda_{\tilde{g}}(A_{\tilde{g}}^t)) = \phi, \quad -\lambda_{\tilde{g}}(A_{\tilde{g}}^t) \in \Gamma \quad \text{on} \quad M \\
&h_{\tilde{g}} = \psi \quad \text{on} \quad \partial M.
\end{aligned}
\]

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