Transient free convection in open-ended vertical channels

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Abstract. The aim of the present work is to investigate the heat transfer for laminar natural convection of a fully developed flow in a chimney consisting of two parallel vertical plates with open ends, occurring due to a sudden change in temperature of one of the plates. The momentum conservation equation was analytically solved by applying the Laplace transform technique and the Green’s function method, resulting in the evaluation of the velocity field with a very low computational effort. The proposed analytical solution was adopted to assess the effect of the Prandtl number on the fluid behaviour of the flow. The results revealed that the steady-state velocity distribution is reached when the Fourier number became comparable with the Prandtl number, except for low values of the Prandtl number.

1. Introduction

Unsteady laminar natural convection in vertical open-ended two-parallel-plate channel is of interest in several engineering applications, such as insulating air gaps in furnaces, ventilating solar passive heating, vertical circuits of electronic equipment cooled by parallel plates or external surfaces of electric transformers. Therefore, this phenomenon has been widely studied by adopting both numerical and analytical approaches.

In particular, the thermal behaviour of transient laminar natural convection in vertical channels was analytically investigated by Al-Nimr and El-Shaarawi [1] by adopting Green’s function method and by considering different thermal boundary conditions. The same issue was addressed by Paul et al. [2] and Singh et al. [3] who used the Laplace transform method and considered two different boundary conditions, i.e. constant heat flux and constant temperature, on the two walls. Moreover, Jha and Gambo [4] recently generalized the mathematical model proposed in [3] by incorporating Soret and Dufour effects. The Laplace transform method was adopted by Ajibade and Jha [5, 6] to investigate natural convection in vertical parallel plates by considering isothermal and adiabatic conditions on the plates.

Recently, Spiga and Vocale [7] proposed a solution for the temperature field, which was obtained even by resorting to the Laplace transform technique. The transient temperature distributions, presented in terms of sinusoidal and exponential functions, were characterized by a very fast convergence. Specifically, this solution was a complicated series of complementary error functions. However, an analytical solution for the velocity field, already available in literature, was assumed.

In the present work, an analytical analysis of the transient natural convection between vertical parallel plates, aimed at providing a new solution for the velocity profile, is presented. The proposed analytical solution, which required a small computational effort, was helpful to focus on the effect of the Prandtl number on the fluid behaviour of the flow.
2. Governing equations

A one-dimensional approach in transient conditions is considered, where the spatial coordinate \( \xi \), normal to the fluid velocity in continuum flow, varies in the range \( 0 \leq H \). The initial temperature of the fluid inside the channel is \( \theta_0 \); suddenly, one of the walls is heated up and its temperature increases to a constant value \( \theta_{\text{max}} \). Internal heat generation, viscous dissipation, slip flow and temperature jump are absent. The fluid is assumed to be Newtonian, in fully developed flow, with constant physical properties, but obeys the Boussinesq approximation, according to which the fluid density is constant except in the gravitational term of the Navier Stokes equation: \( \rho(\theta) = \rho_0[1 - \beta(\theta - \theta_0)] \).

Under these assumptions, the governing continuity, momentum and energy equations reduce to the following two dimensionless equations:

\[
\rho_0 \frac{\partial v}{\partial \tau} = \rho_0 \beta g \theta + \mu \frac{\partial^2 v}{\partial \xi^2} \tag{1}
\]

\[
\rho_0 c \frac{\partial \theta}{\partial \tau} = \lambda \frac{\partial^2 \theta}{\partial \xi^2} \tag{2}
\]

The initial and boundary conditions are \( v(\xi, \tau) = 0 \) for \( \tau = 0 \), in \( \xi = 0 \) and \( \xi = H \), \( \theta(\xi, \tau) = \theta_0 \) for \( \tau = 0 \), \( \xi = 0 \), \( \theta(\xi, \tau) = \theta_{\text{max}} \) at \( \xi = H \).

By adopting dimensionless space \( x = \xi/H \) and time \( (Fo = \alpha \tau/H^2) \) coordinates, the dimensionless velocity and temperature transient profiles can be obtained by solving the following equations:

\[
\frac{\partial \tilde{V}}{\partial Fo} - Pr \frac{\partial^2 \tilde{V}}{\partial x^2} = T(x, Fo) \tag{3}
\]

\[
\frac{\partial \tilde{T}}{\partial Fo} = \frac{\partial^2 \tilde{T}}{\partial x^2} \tag{4}
\]

where the dimensionless initial and boundary conditions are \( \tilde{V}(x, Fo = 0) = 0, V(x = 0, Fo) = 0, V(x = 1, Fo) = 0, T(x = 0, Fo) = 0, T(x = 1, Fo) = 1 \), being \( \tilde{V} = vH/(\nu \alpha) \) and \( T = (\theta - \theta_0)/(\theta_{\text{max}} - \theta_0) \), where the Rayleigh number is \( Pr Gr \), and the Grashof number is \( \beta g H^3 (\theta_{\text{max}} - \theta_0)/\nu^3 \).

3. Analytical Solutions

In order to simplify eqs (4-5) by eliminating the dependency of the unknown quantities from the dimensionless time \( Fo \), the Laplace transform is adopted, thus obtaining:

\[
\frac{p}{Pr} \tilde{V}(x, p) - \frac{\partial^2 \tilde{V}}{\partial x^2} = \frac{1}{p} \tilde{T}(x, p) \tag{5}
\]

\[
\frac{\partial^2 \tilde{T}}{\partial x^2} + p \tilde{T}(x, p) = 0 \tag{6}
\]

with transformed boundary conditions \( \tilde{V}(x = 0, p) = 0, \tilde{V}(x = 1, p) = 0, \tilde{T}(x = 0, p) = 0, \tilde{T}(x = 1, p) = \frac{1}{p} \).

The analytical solution to the energy equation that satisfies the transformed boundary conditions is simply determined:

\[
\tilde{T}(x, p) = \frac{\sinh(x \sqrt{p})}{p \sinh(\sqrt{p})} \tag{7}
\]

The singular points of the complex function \( \tilde{T}(x, p) \) are the zeros of the denominator, i.e. \( p = 0 \) and \( p = -n^2 \pi^2/\beta \); all of them are simple poles \( i \) is the imaginary unit). Hence the back Laplace transform needs the evaluation of the infinite residues of the function \( \tilde{T}(x, p) \):

\[
\lim_{p \to 0} \frac{\sinh(x \sqrt{p})}{\sinh(\sqrt{p})} \exp(p Fo) = x
\]
\[ \lim_{p-p_n \to p} \frac{1}{\sinh(p)} \exp(pF_o) = \frac{2}{\pi} \frac{(-1)^n}{n} \sin(n\pi x) \exp(-n^2 \pi^2 F_o) \] (9)

All the poles are arranged along the negative real semi-axis of the Gauss plane. In the path of anti-transformation, it has to be chosen the abscissa of absolute convergence greater than 0. Hence, the solution to eq. (4) is:

\[ T(x, F_o) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) \exp(-n^2 \pi^2 F_o) \] (10)

Now, the transformed momentum equation, defined by eq. (11), must be solved:

\[ \frac{\partial^2 \tilde{V}}{\partial x^2} - \frac{p}{Pr} \tilde{V}(x, p) = -\frac{1}{Pr} \frac{\sin(x\sqrt{p})}{p \sinh(\sqrt{p})} \] (11)

The associated Green problem is then [8]:

\[ \frac{\partial^2 \tilde{G}}{\partial x^2} - \frac{p}{Pr} \tilde{G}(x, x', p) = \delta(x - x') \] (12)

where the boundary conditions for the transformed Green function are \( \tilde{G}(x, x', p) = 0 \) in \( x = 0 \) and \( x = 1 \). The ordinary differential equation for the transformed Green function is [8]:

\[ \frac{\partial^2 \tilde{G}}{\partial x^2} = \frac{p}{Pr} \tilde{G}(x, x', p) \] (13)

The solution for eq. (13) is expressed as follows:

\( \tilde{G}(x, x', p) = A \sinh(\sqrt{\frac{p}{Pr}} x) \quad 0 < x < x' \) (14)

\( \tilde{G}(x, x', p) = B \sinh(\sqrt{\frac{p}{Pr}} (1 - x)) \quad x' < x < 1 \) (15)

The continuity for the transformed Green function, coupled with the discontinuity for its derivative, allows to determine the constants \( A \) and \( B \) by solving the two following algebraic equations:

\[ A \sinh(\sqrt{\frac{p}{Pr}} x') = B \sinh(\sqrt{\frac{p}{Pr}} (1 - x')) \] (16)

\[ -B \frac{p}{Pr} \cosh(\sqrt{\frac{p}{Pr}} (1 - x')) - A \frac{p}{Pr} \cosh(\sqrt{\frac{p}{Pr}} x') = 1 \] (17)

Hence, \( A \) and \( B \) result:

\[ A = -\frac{\sinh(\sqrt{\frac{p}{Pr}} (1 - x'))}{\frac{p}{Pr} \sinh(\sqrt{\frac{p}{Pr}} x')} \] (18)

\[ B = -\frac{\sinh(\sqrt{\frac{p}{Pr}} x')}{\frac{p}{Pr} \sinh(\sqrt{\frac{p}{Pr}} x')} \] (19)

The transformed Green function is:

\[ \tilde{G}(x, x', p) = \tilde{G}_1(x, x', p) = -\frac{\sinh(\sqrt{\frac{p}{Pr}} (1 - x'))}{\frac{p}{Pr} \sinh(\sqrt{\frac{p}{Pr}} x')} \sinh(\sqrt{\frac{p}{Pr}} x) \quad 0 < x < x' \] (20)
\[ \bar{G}(x,x',p) = \bar{G}_2(x,x',p) = -\frac{\sinh \left( \frac{p}{\sqrt{Pr}} x' \right)}{\sqrt{Pr} \sinh \left( \frac{p}{\sqrt{Pr}} (1 - x) \right)} \sinh \left( \frac{p}{\sqrt{Pr}} (1 - x) \right) \quad x' < x < 1 \tag{21} \]

This complex function presents simple poles when \( \sqrt{p/Pr} = \pm n\pi i \), i.e. \( p_n = -n^2\pi^2 Pr \) (n = 1, 2, ...). The point \( p = 0 \) is not a pole nor a zero for the function, being \( \lim_{p \to 0} \bar{G}_1(x,x',p) = x(1 - x) \) and \( \lim_{p \to 0} \bar{G}_2(x,x',p) = x'(1 - x) \).

The residue in the pole \( p_n \) for the function \( \bar{G}_1 \) is:

\[
\lim_{p \to -n^2\pi^2 Pr} = \frac{\sinh[n\pi i(1 - x') \sinh[n\pi i x]}{n\pi i} \frac{p + n^2\pi^2 Pr}{\sinh \left( \frac{p}{\sqrt{Pr}} \right)} = 2 Pr \left( -1 \right)^n \sin[n\pi(1 - x')] \sin[n\pi x] \tag{22} \]

Analogously, the pole for the function \( \bar{G}_2 \) is:

\[
\lim_{p \to -n^2\pi^2 Pr} = \frac{\sinh[n\pi i(1 - x) \sinh[n\pi i x']}{n\pi i} \frac{p + n^2\pi^2 Pr}{\sinh \left( \frac{p}{\sqrt{Pr}} \right)} = 2 Pr \left( -1 \right)^n \sin[n\pi(1 - x)] \sin[n\pi x'] \tag{23} \]

Consequently, the inverse Laplace transform is the Green function \( G(x,x',Fo) \):

\[
G_1(x,x',Fo) = 2Pr \sum_{n=1}^{\infty} (-1)^n \sin(n\pi x) \sin[n\pi(1 - x')] \exp(-n^2\pi^2 Pr Fo) \tag{24} \]

for \( 0 < x < x' \),

\[
G_2(x,x',Fo) = 2Pr \sum_{n=1}^{\infty} (-1)^n \sin[n\pi(1 - x)] \sin(n\pi x') \exp(-n^2\pi^2 Pr Fo) \tag{25} \]

for \( x' < x < 1 \).

The unknown transformed transient velocity is then:

\[
\bar{V}(x,p) = -\frac{1}{Pr} \int_0^1 \bar{T}(x',p) \bar{G}(x,x',p) dx' \tag{26} \]

The solution for eq. (3) is then the convolution of the temperature and the Green function:

\[
V(x,Fo) = -\frac{1}{Pr} \int_0^1 dx' \int_0^{Fo} G(x,x',Fo - Fo') T(x',Fo') dFo' = -\frac{1}{Pr} \int_0^x dx' \int_0^{Fo} G_1(x,x',Fo - Fo') \left[ x' \right. + \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n \sin(n\pi x') \exp(-n^2\pi^2 Fo') dFo' + \left. \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n \sin(n\pi x') \exp(-n^2\pi^2 Fo') dFo' \right] \tag{27} \]

In steady-state condition, the velocity and temperature profiles are \( T = x \) and \( V = 1/(6Pr) x (1-x^2) \), showing that the velocity \( v \) is directly proportional to \( \beta g H^2 (\theta_{max} - \theta_0) \) and inversely proportional to \( \nu \).
4. Results

The proposed analytical solution for the velocity profile was evaluated by using the Symbolic Math Toolbox within the Matlab R2019a environment. The convergence analysis of such solution was performed by means of several iterations, in which the number of the converging series terms were varied in the range $2 \div 50$. Four fluids, namely liquid sodium, air, water and glycol, having Prandtl number equal to 0.02, 0.7, 5 and 199, respectively, were studied. In the first part of the analysis, the velocity profiles for three values of the Fourier number and for two values of the Prandtl number, i.e. the lowest ($Pr = 0.02$) and upper limit ($Pr = 199$) for the four considered Prandtl numbers are presented in Figs.1,2. It was observed that, for $n > 10$, the profiles did not change significantly. However, to better evaluate the velocity distributions for very low values of the Fourier and Prandtl numbers, at least 20 terms of the converging series were considered.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1a.png}
\caption{(a)}
\end{subfigure}\hfill
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1b.png}
\caption{(b)}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1c.png}
\caption{(c)}
\end{subfigure}\hfill
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1d.png}
\caption{(d)}
\end{subfigure}
\caption{Convergence analysis of the proposed analytical solution, for Fourier number equal to 0.01, 0.1 and 1, and for $Pr = 0.02$: $n = 2$ (a); $n = 4$ (b); $n = 10$ (c); $n = 50$ (d).}
\end{figure}
Figure 2. Convergence analysis of the proposed analytical solution, for Fourier number equal to 0.01, 0.1 and 1, and for Pr = 199: n = 2 (a); n = 4 (b); n = 10 (c); n = 50 (d).

The accuracy of the proposed solution was checked by comparing the results with those obtained by using a similar analytical solution, available in literature [7]. A good agreement was found for all the considered Fourier and Prandtl numbers, as appreciable by Fig. 3, in which the results for Pr = 0.7 are reported.

Figure 3. Comparison between present results and data available in literature [7], for Pr = 0.7.
The influence of the fluid properties on the fluid-dynamics of the laminar flow between two parallel plates was assessed. The transient velocity distributions are shown in Figure 4 for all the considered Prandtl numbers and for five values of the Fourier number. The velocity distribution, which has its maximum close to the heated wall (x=1), increases and moves towards the cold wall with the increase of Fo. The maximum decreases by increasing Pr, and reaches its steady-state (0.06415/Pr) value at the coordinate x=$\sqrt{3}$/3. It was observed that, for Pr = 5 and Pr = 199, the steady-state velocity distribution is reached for Fourier number $\approx$ 0.5, while such steady-state condition is achieved for higher values of Fo as Pr decreases, in accordance with [7].

5. Conclusions
Natural convection of a laminar fully developed flow between two parallel vertical plates, triggered by a sudden change in temperature of one of the plates, was analytically investigated by applying the Laplace transform technique and the Green’s function method to the governing equations. In particular, the solution for the velocity distribution required a very small computational effort, thus highlighting the advantage of such analytical resolution. The proposed solution was then adopted in the assessment of the influence of the fluid properties on the fluid-dynamics of the flow. The results, obtained for several fluids (liquid sodium, air, water and glycol), highlighted that, for Pr > 1, the velocity distribution reaches the steady-state profile for Fourier number almost equal to 0.5, while, as Pr decreases, the steady-state profile is reached at higher Fourier numbers (> 1).
Nomenclature

| Symbol | Quantity                          | SI unit |
|--------|----------------------------------|---------|
| c      | Specific heat                    | J/kgK   |
| Fo     | Fourier number                   | -       |
| g      | Gravity acceleration             | m/s²    |
| Gr     | Grashof number                   | -       |
| H      | Channel width                    | m       |
| p      | Complex coordinate in the Laplace transform | -     |
| Pr     | Prandtl number                   | -       |
| Ra     | Rayleigh number                  | -       |
| T      | Dimensionless temperature        | -       |
| v      | Fluid velocity                   | m/s     |
| V      | Dimensionless fluid velocity     | -       |
| x      | Dimensionless coordinate         | -       |
| α      | Thermal diffusivity              | m²/s    |
| β      | Thermal expansion coefficient    | K⁻¹     |
| δ      | Dirac delta function             | -       |
| λ      | Thermal conductivity             | W/mK    |
| μ      | Dynamic viscosity                | kg/ms   |
| ν      | Kinematic viscosity              | m²/s    |
| θ      | Fluid temperature                | K       |
| θ₀     | Constant initial temperature     | K       |
| ρ      | Fluid density                    | kg/m³   |
| ρ₀     | Fluid density at θ₀              | kg/m³   |
| τ      | Time                             | s       |
| ξ      | Spatial coordinate               | m       |

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