$N = 2$ Hamiltonians with $sl(2)$ coalgebra symmetry and their integrable deformations

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Abstract

Two dimensional classical integrable systems and different integrable deformations for them are derived from phase space realizations of classical $sl(2)$ Poisson coalgebras and their $q-$deformed analogues. Generalizations of Morse, oscillator and centrifugal potentials are obtained. The $N = 2$ Calogero system is shown to be $sl(2)$ coalgebra invariant and the well-known Jordan-Schwinger realization can be also derived from a (non-coassociative) coproduct on $sl(2)$. The Gaudin Hamiltonian associated to such Jordan-Schwinger construction is presented. Through these examples, it can be clearly appreciated how the coalgebra symmetry of a hamiltonian system allows a straightforward construction of different integrable deformations for it.

1 Introduction

A Poisson coalgebra $(A, \Delta)$ is a Poisson algebra $A$ endowed with a Poisson map $\Delta$ between $A$ and $A \otimes A$. In other words, if we assume that $A$ is the dynamical algebra for a one-particle problem, the coproduct $\Delta$ provides a two-particle realization of the same dynamical symmetry. In fact, under certain conditions the coproduct $\Delta$ can be uniquely generalized to a Poisson map between $A$ and $A \otimes A \otimes \ldots A$, and the $N$-particle realization of the symmetry arises. By following this approach, a systematic construction of $N$-dimensional completely integrable Hamiltonians from any coalgebra with Casimir element $C$ has been introduced in [1]. Moreover, since $q-$ deformations (see, for instance, [2]-[5]) can be translated in a classical mechanical context as deformations of Poisson algebras preserving a coalgebra structure, such construction can be applied for them and leads to a systematic derivation of integrable deformations of Hamiltonian systems.
The aim of the present paper is two-fold. On one hand, we use the \( sl(2) \) Poisson coalgebra and its deformations to provide new classical mechanical examples of this general construction. In Section 2 we recall such Poisson \( sl(2) \) coalgebras; in Section 3 the general construction is reviewed through an example based on the Gelfan’d-Dyson realization \([1]\) of \( sl(2) \) and in Section 4 new \( N = 2 \) integrable systems related to Morse, oscillator and centrifugal problems are given. Another interesting example of two-dimensional coalgebra symmetry is provided by the \( N = 2 \) Calogero system \([7]\), that is shown to be canonically equivalent to one of the systems that have been previously derived. Therefore, the results here presented can be used to construct two new different integrable deformations of this remarkable Hamiltonian.

As a general fact, \( q \)-deformations introduce hyperbolic functions of the canonical variables, and the associated integrals of the motion have also such kind of hyperbolic terms depending on positions and/or momenta. Note also that the existence of the \( N \)-dimensional generalization of all these systems is ensured (although we will not describe their explicit form here) and that phase space realizations of coalgebras play an essential role in the formalism.

Secondly, in Section 5 we present the generalization of this construction to systems in which the dynamical symmetry algebra \( A \) is given in terms of an elementary canonical realization that already contains two pairs of conjugated variables. By using the classical Jordan-Schwinger (JS) realization \([8, 9]\) as an outstanding example of this kind of situation, we show how complete integrability is also preserved for the corresponding system, now with \( 2N \) degrees of freedom. Consequently, the deformation of such systems poses the interesting problem of deforming the JS map. It turns out that coalgebra symmetry is also essential at this point, since we show that the Jordan-Schwinger (JS) realization of \( sl(2) \) is canonically equivalent to a reducible representation of \( sl(2) \) given by a non-coassociative coproduct. From it, a standard deformation of the JS realization is given and related \( 2N \) dimensional integrable systems can be defined.

## 2 \( sl(2) \) Poisson coalgebras

Let us consider classical angular momentum variables \( J_3, J_\pm \) and the associated \( sl(2) \) Poisson-Lie algebra

\[
\{ J_3, J_\pm \} = \pm 2 J_\pm, \quad \{ J_+, J_- \} = J_3. \tag{2.1}
\]

with Casimir function

\[
C(J_3, J_+, J_-) = \frac{1}{4} J_3^2 + J_+ J_- \tag{2.2}
\]

The Poisson algebra (2.1) is endowed with a coalgebra structure by the usual “primitive” coproduct defined between \( sl(2) \) and \( sl(2) \otimes sl(2) \)

\[
\Delta(J_3) = 1 \otimes J_3 + J_3 \otimes 1, \quad \Delta(J_\pm) = 1 \otimes J_\pm + J_\pm \otimes 1. \tag{2.3}
\]

Compatibility between (2.3) and (2.1) means that \( \Delta \) is a Poisson map: the three functions defined through (2.3) close also a Poisson \( sl(2) \) algebra.

There are few deformations of the \( sl(2) \) algebra for which there exist compatible deformations of the coproduct (2.3). In fact, two relevant and distinct structures appeared in
quantum group literature during last years, and they can be realized as Poisson algebras as follows:

- The “standard” deformation \([2, 3, 10]\), \(sl_q(2) (q = e^z)\) given by the following deformed Poisson brackets

\[
\begin{align*}
\{\tilde{J}_3, \tilde{J}_+\} &= 2 \tilde{J}_+, \quad \{\tilde{J}_3, \tilde{J}_-\} = -2 \tilde{J}_-, \quad \{\tilde{J}_+, \tilde{J}_-\} = \frac{\sinh(z\tilde{J}_3)}{z}, \\
\end{align*}
\tag{2.4}
\]

which are compatible with the deformed coproduct

\[
\begin{align*}
\Delta_z(\tilde{J}_3) &= 1 \otimes \tilde{J}_3 + \tilde{J}_3 \otimes 1, \\
\Delta_z(\tilde{J}_+) &= e^{-\frac{z}{2}\tilde{J}_3} \otimes \tilde{J}_+ + \tilde{J}_+ \otimes e^{\frac{z}{2}\tilde{J}_3}; \\
\Delta_z(\tilde{J}_-) &= e^{-\frac{z}{2}\tilde{J}_3} \otimes \tilde{J}_- + \tilde{J}_- \otimes e^{\frac{z}{2}\tilde{J}_3}; \\
\end{align*}
\tag{2.5}
\]

in the sense that that \([2.5]\) is a Poisson algebra homomorphism with respect to \([2.4]\). The function

\[
C_z(\tilde{J}_3, \tilde{J}_\pm) = \left(\frac{\sinh(\frac{z}{2}\tilde{J}_3)}{z}\right)^2 + \tilde{J}_+ \tilde{J}_-,
\tag{2.6}
\]

is the deformed Casimir for this Poisson coalgebra.

- The “non-standard” deformation \([1, 12, 13]\), \(sl_h(2)\) whose defining relations are

\[
\begin{align*}
\{\tilde{J}_3, \tilde{J}_+\} &= 2 \frac{\sinh(h\tilde{J}_-)}{h}, \quad \{\tilde{J}_3, \tilde{J}_-\} = -2 \tilde{J}_- \cosh(h\tilde{J}_+), \quad \{\tilde{J}_+, \tilde{J}_-\} = \tilde{J}_3. \\
\end{align*}
\tag{2.7}
\]

\[
\begin{align*}
\Delta_h(\tilde{J}_+) &= 1 \otimes \tilde{J}_+ + \tilde{J}_+ \otimes 1, \\
\Delta_h(\tilde{J}_-) &= e^{-h\tilde{J}_+} \otimes \tilde{J}_- + \tilde{J}_- \otimes e^{h\tilde{J}_+}; \\
\Delta_h(\tilde{J}_3) &= e^{-h\tilde{J}_+} \otimes \tilde{J}_3 + \tilde{J}_3 \otimes e^{h\tilde{J}_+}; \\
\end{align*}
\tag{2.8}
\]

The non–standard deformed Casimir is now

\[
C_h(\tilde{J}_3, \tilde{J}_+, \tilde{J}_-) = \frac{4}{3}\tilde{J}^2_3 + \frac{\sinh(h\tilde{J}_+)}{h}\tilde{J}_-.
\tag{2.9}
\]

Equivalently, an isomorphic non-standard Poisson deformation is obtained by choosing \(\tilde{J}_-\) as the primitive generator. We then find that

\[
\begin{align*}
\{\tilde{J}_3, \tilde{J}_+\} &= 2 \tilde{J}_+ \cosh(h\tilde{J}_-), \quad \{\tilde{J}_3, \tilde{J}_-\} = -2\frac{\sinh(h\tilde{J}_-)}{h}, \quad \{\tilde{J}_+, \tilde{J}_-\} = \tilde{J}_3. \tag{2.10}
\end{align*}
\]

\[
\begin{align*}
\Delta_h(\tilde{J}_+) &= 1 \otimes \tilde{J}_+ + \tilde{J}_+ \otimes 1, \\
\Delta_h(\tilde{J}_-) &= e^{-h\tilde{J}_-} \otimes \tilde{J}_+ + \tilde{J}_+ \otimes e^{h\tilde{J}_-}; \\
\Delta_h(\tilde{J}_3) &= e^{-h\tilde{J}_-} \otimes \tilde{J}_3 + \tilde{J}_3 \otimes e^{h\tilde{J}_-}; \\
\end{align*}
\tag{2.11}
\]

The non–standard deformed Casimir is now

\[
C_h(\tilde{J}_3, \tilde{J}_+, \tilde{J}_-) = \frac{4}{3}\tilde{J}^2_3 + \tilde{J}_+ \frac{\sinh(h\tilde{J}_-)}{h}.
\tag{2.12}
\]

In what follows we shall make use of these coalgebra symmetries in order to construct new integrable systems. Note that the undeformed \(sl(2)\) structure is smoothly recovered when deformation parameters vanish. Jacobi identity can be also easily checked for \([2.4], [2.7]\) and \([2.10]\).
3 Gelfan’d-Dyson map, Gaudin magnet and deformations

Let us now recall the general construction [1] through an example provided by the classical phase space analogue of the one-boson (polynomial) Gelfan’d-Dyson (GD) realization of $sl(2)$:

$$J_3 = 2pq - b, \quad J_+ = p, \quad J_- = -pq^2 + bq,$$

(3.1)

where $b$ is a real constant that labels the representation through the Casimir \([2,2]\), which turns out to be $b^2/4$ under \([3,4]\).

Let us now consider an arbitrary function $H(J_3, J_\pm)$. Since the coproduct \([2,3]\) is an algebra homomorphism, it is immediate to prove that the two-particle Hamiltonian that can be defined through the coproduct of $H$ in the form

$$H^{(2)} = \Delta(H(J_3, J_\pm)) = H(\Delta(J_3), \Delta(J_\pm)),$$

(3.2)

will commute with the coproduct $C^{(2)}$ of the Casimir element:

$$\left\{ H^{(2)}, C^{(2)} \right\} = \left\{ \Delta(H), \Delta(C) \right\} = \Delta(\{H, C\}) = 0.$$  

(3.3)

Therefore, any function of the generators defines a two-site integrable Hamiltonian. In particular, the Casimir itself can be taken as the function $H$. In that case, the other integral of motion will be given by the coproduct of any of the generators of the algebra. Explicitly, in the $sl(2)$ case we have

$$H^{(2)}_C := C(\Delta(J), \Delta(J_+), \Delta(J_-)) = \frac{1}{4}(1 \otimes J_3 + J_3 \otimes 1)^2 + (1 \otimes J_+ + J_+ \otimes 1)(1 \otimes J_- + J_- \otimes 1) = 1 \otimes C + C \otimes 1 + \frac{1}{2} J_3 \otimes J_3 + J_- \otimes J_+ + J_+ \otimes J_-,$$

(3.4)

which is just the two-site Gaudin magnet \([14, 15, 16]\). Once \(3.4\) is realized in terms of two copies of \([3,4]\) we shall obtain the integrable two-particle Hamiltonian

$$H^{(2)}_C(q_1, q_2, p_1, p_2) := \frac{q_1^2}{4} + \frac{q_2^2}{4} + \frac{1}{2}(2p_1q_1 - b)(2p_2q_2 - b) + (-p_1q_1^2 + bq_1)p_2 + p_1(-p_2q_2^2 + bq_2) = -p_1p_2(q_1 - q_2)^2 - b(p_1 - p_2)(q_1 - q_2) + b^2.$$  

(3.5)

Since $H^{(2)}_C$ is the coproduct of $C$, it will commute with, for instance, with $\Delta(J_\pm)$, which is just the total momentum $p_1 + p_2$ and gives us the additional integral of the motion. The generalization of this result to an $N$-site Gaudin magnet is straightforward by taking into account the appropriate $N$-th generalization of the coproduct. By following \([1]\), we obtain that

$$H^{(N)} = \sum_{i=1}^{N} C_i + \sum_{i<j} \left\{ \frac{1}{2} J_3^i J_3^j + J_+^i J_+^j + J_-^i J_-^j \right\}$$

(3.6)

$$= \sum_{i<j} \left\{ -p_i p_j (q_i - q_j)^2 - b(p_i - p_j)(q_i - q_j) \right\} + \frac{b^2}{4} N^2.$$  

(3.7)

The $m = 2, \ldots, N$ hamiltonians $H^{(m)}$, together with the total momentum $\Delta^{(N)}(J_\pm) = p_1 + p_2 + \ldots + p_N$ are again $N$ functionally independent integrals of motion in involution.
Note also that we could have taken a different realization on each lattice site through different \( b_i \) constants in (3.1), and the formalism will guarantee the integrability in the same manner (this kind of realizations will be relevant when the coalgebra symmetry of the Calogero system is analysed in Section 4).

Now, the same construction can be applied to deformed \( sl(2) \) coalgebras. We shall consider the non-standard one (2.7-2.9). The following deformed phase-space realization of (2.7) can be found

\[
\begin{align*}
\tilde{J}_+ &= p, \quad \tilde{J} = 2 \frac{\sinh(hp)}{h} q - b \cosh(hp), \\
\tilde{J}_- &= -\frac{\sinh(hp)}{h} q^2 + b \cosh(hp) q - b^2 4 \sinh(hp),
\end{align*}
\]  

which leads again the same \( b^2/4 \) constant when substituted in the deformed Casimir function (2.3). Therefore, an integrable deformation of the system (3.3) will be given by the deformed coproduct of the (also deformed) Casimir (2.9). In terms of two phase space realizations of the type (3.8), the coproduct (2.8) defines the two-particle functions

\[
\tilde{f}_+ = \Delta_h(\tilde{J}_+) = p_1 + p_2, \\
\tilde{f}_- = \Delta_h(\tilde{J}_-) = e^{-h p_1} \left\{ -\frac{\sinh(hp_2)}{h} q_2^2 + b \cosh(hp_2) q_2 - b^2 4 \sinh(hp_2) \right\} + \left\{ -\frac{\sinh(hp_1)}{h} q_1^2 + b \cosh(hp_1) q_1 - b^2 4 \sinh(hp_1) \right\} e^{h p_2}; \\
\tilde{f}_3 = \Delta_h(\tilde{J}_3) = e^{-h p_1} \left\{ 2 \frac{\sinh(hp_2)}{h} q_2 - b \cosh(hp_2) \right\} + \left\{ 2 \frac{\sinh(hp_1)}{h} q_1 - b \cosh(hp_1) \right\} e^{h p_2};
\]

The corresponding Hamiltonian is obtained as

\[
H^{(2)}_h(q_1, q_2, p_1, p_2) = \Delta_h \left( \frac{1}{2} \tilde{f}_3^2 + \frac{\sinh(h\tilde{J}_+)}{h} \tilde{J}_- \right) = \frac{1}{2} \tilde{f}_3^2 + \frac{\sinh(h\tilde{f}_+)}{h} \tilde{f}_-
\]

\[
= \frac{\sinh hp_1}{h} e^{-h(p_1-p_2)} \frac{\sinh hp_2}{h} (q_1 - q_2)^2 - b \frac{1 - e^{-2h(p_1-p_2)}}{2h} (q_1 - q_2) + \frac{q^2}{4} \left( 1 + e^{-2h p_1} + e^{2h p_2} + e^{-2h(p_1-p_2)} \right). 
\]

This Hamiltonian will commute, by construction, with the coproduct of \( \tilde{J}_+ \) (i.e., the total momentum \( p_1 + p_2 \) again) and the limit \( h \to 0 \) of this expression leads to (3.7). The \( N \)-th dimensional integrable generalization of this system is given by the \( N \)-th coproduct of the deformed Casimir, and the integrals of motion will be the lower degree coproducts of such Casimir and the \( N \)-th total momenta. Apart from the GD realization (3.1), the essential ingredients for the explicit formulation of such system can be extracted from [1]. In what follows, we shall restrict our study to the construction of two-body problems. However, we stress that, due to the underlying coalgebra symmetry, all the systems constructed in this way will have \( N \)-dimensional integrable generalizations.

4 \( N = 2 \) systems and their integrable deformations

Let us now use realizations of \( sl(2) \) linked to well-known dynamical symmetries of physically relevant potentials like the Morse and the oscillator with centrifugal term in order
to obtain new systems with $sl(2)$ coalgebra symmetry.

### 4.1 Morse potential realization

If we consider the following $sl(2)$ phase space realization

\[
J_3 = 2p_1,
J_+ = \frac{1}{2}e^{-q_1},
J_- = -2p_1^2 e^{q_1} - a_1 e^{q_1},
\]  

(4.1)

The dynamical Hamiltonian

\[
H_m = \frac{1}{8}J_3^2 - 4s J_+ + 4s J_+^2
\]  

(4.2)

leads to the Morse one:

\[
H_m = \frac{1}{2}p_1^2 + s(e^{-2q_1} - 2e^{-q_1}).
\]  

(4.3)

The corresponding Casimir is

\[
C_m = \frac{1}{8}J_3^2 + J_+ J_- = -a/2.
\]  

(4.4)

The two-body system is obtained by applying the method described above. The coproduct of the Hamiltonian (4.2) in the realization (4.1) is

\[
H^{(2)}_m = \frac{1}{2}(p_1 + p_2)^2 + s(e^{-2q_1} - 2e^{-q_1}) + s(e^{-2q_2} - 2e^{-q_2}) + 2se^{-(q_1 + q_2)}. 
\]  

(4.5)

Note that we can take different realizations on each copy of the algebra ($a_1$ and $a_2$ have not to be the same). Therefore, the coproduct of the Casimir gives a two-parametric family of constants of the motion in the form:

\[
C^{(2)}_m = -\frac{1}{2}(a_1 + a_2) - (\frac{1}{2}a_1 + p_1^2)e^{q_1 - q_2} - (\frac{1}{2}a_2 + p_2^2)e^{-(q_1 + q_2)} + 2p_1 p_2. 
\]  

(4.6)

The Hamiltonian (4.5) can be diagonalized if we consider the canonical transformation

\[
P_1 = p_1 + p_2, \quad P_2 = p_2, \quad Q_1 = q_1 \quad Q_2 = q_2 - q_1.
\]  

(4.7)

This leads to a Hamiltonian in which $P_2$ does not appear, and consequently the relative position between both particles $Q_2 = q_2 - q_1$ is a constant of the motion. Namely

\[
H^{(2)}_{Q_2} = \frac{1}{2}P_1^2 + s(e^{-Q_1}(1 + e^{-Q_2})^2 - 2e^{-Q_2}(1 + e^{-Q_2})). 
\]  

(4.8)

Note that (4.8) is a Morse-type problem depending on the constant parameter $Q_2$. Actually, in the limit $Q_2 \to \infty$ we recover the original Morse potential. It is worth recalling that this kind of parameter-dependent dynamics was already observed in [17], where the classical motion on the Poisson-Lie $sl(2)$ group was considered. On the other hand, it is immediate to check that the $N$-dimensional generalization of the system (4.5) can be reduced to the same type of one dimensional Morse-type problem (now with $N - 2$ parameters) through a canonical transformation containing the new total momenta $P_1 = \sum_{i=1}^{N} p_i$. 


4.1.1 Deformed Morse systems

• The **standard** case. The phase space realization

\[
\begin{align*}
\tilde{J}_3 &= 2p_1, \\
\tilde{J}_+ &= \frac{1}{2} e^{-q_1}, \\
\tilde{J}_- &= -2 \left( \frac{\sinh(z p_1)}{z} \right)^2 e^{q_1} - a e^{q_1},
\end{align*}
\]

(4.9)

leads to the \(sl_q(2)\) Poisson algebra (2.4). Therefore, the one-particle Hamiltonian

\[
H_z = \frac{1}{8} \tilde{J}_3^2 - 4 s \tilde{J}_+ + 4 s \tilde{J}_+^2,
\]

(4.10)

does not change under deformation:

\[
H_z = \frac{1}{2} p_1^2 + s \left( e^{-2q_1} - 2 e^{-q_1} \right),
\]

(4.11)

and the deformed Casimir element (2.6) is just \(-a/2\).

Let us now construct the associated two-body integrable deformation. By applying the deformed coproduct onto (4.10) and with the aid of two copies of the realization (4.9) we get

\[
H^{(2)}_z = \frac{1}{2} (p_1 + p_2)^2 + s \left( e^{-2(q_1-z p_2)} - 2 e^{-(q_1-z p_2)} \right) + s \left( e^{-2(q_2+z p_1)} - 2 e^{-(q_2+z p_1)} \right) + 2 s e^{-((q_1-z p_2)+(q_2+z p_1))}.
\]

(4.12)

If we substitute again in terms of \(P_1\), we see that \(P_2\) does appear within the Hamiltonian, and the deformation would imply that \(Q_2\) is no longer a constant of motion (however, note the persistent coupling of the type \(q_1 - z p_2\) and \(q_2 + z p_1\)). The additional integral of motion for (4.12) is obtained as the phase space realization of

\[
C^{(2)}_z = \Delta_z(C_z) = \left( \frac{\sinh(z \Delta_z(\tilde{J}_3))}{z} \right)^2 + \Delta_z(\tilde{J}_+) \Delta_z(\tilde{J}_-).
\]

(4.13)

Explicitly, (4.13) gives the following two-particle function:

\[
C^{(2)}_z = \left( \frac{\sinh(z (p_1 + p_2))}{z} \right)^2 - \left( \frac{\sinh z p_2}{z} \right)^2 + \frac{a_2}{2} \left( e^{-2z p_1} + e^{-(q_1-q_2)-z (p_1-p_2)} \right) - \left( \frac{\sinh z p_1}{z} \right)^2 + \frac{a_1}{2} \left( e^{2z p_2} + e^{(q_1-q_2)-z (p_1-p_2)} \right).
\]

(4.14)

Note that the limit \(z \to 0\) of (4.14) leads to (4.6).

• The **non standard** case. The deformed phase space realization corresponding to the \(-a/2\) value of the deformed Casimir is now

\[
\begin{align*}
\tilde{J}_3 &= 2 \sinh \left( \frac{h e^{-q_1/2}}{2} \right) p_1, \\
\tilde{J}_+ &= \frac{1}{2} e^{-q_1}, \\
\tilde{J}_- &= -2 e^{q_1} \sinh \left( \frac{h e^{-q_1/2}}{2} \right) p_1^2 - h \frac{a_1}{\sinh(h e^{-q_1/2})}.
\end{align*}
\]

(4.15)
Therefore, the one-particle Hamiltonian
\[ H = \frac{1}{8} J_3^2 - 4 s \tilde{J}_+ + 4 s \tilde{J}_+^2, \] (4.16)
leads to:
\[ H^{(2)}_{\hbar} = \frac{1}{16} \left( \frac{\sinh(\hbar e^{-q_1/2})}{(\hbar e^{-q_1/2})} \right)^2 p_1^2 + s (e^{-2q_1} - 2 e^{-q_1}). \] (4.17)

The two-particle Hamiltonian is obtained from the deformed coproduct (2.8) of (4.16):
\[ H^{(2)}_{\hbar} = \frac{1}{16} \left( e^{-\frac{\hbar}{2} e^{-q_1/2}} \frac{\sinh(\hbar e^{-q_2/2})}{(\hbar e^{-q_2/2})} p_2 + 2 \frac{\sinh(\hbar e^{-q_1/2})}{(\hbar e^{-q_1/2})} e^{\frac{\hbar}{2} e^{-q_2} p_1} \right)^2 + 
+ s (e^{-2q_1} - 2 e^{-q_1}) + s (e^{-2q_2} - 2 e^{-q_2}) + 2 s e^{-(q_1+q_2)}. \] (4.18)

Again, the role of the \( p_1 + p_2 \) dynamical variable is no longer relevant under deformation. The two-particle Casimir would be obtained as the phase space realization of
\[ \Delta_{\hbar}(C_{\hbar}) = \frac{1}{4} (\Delta_{\hbar}(\tilde{J}_3))^2 + \frac{\sinh(\hbar \Delta_{\hbar}(\tilde{J}_+))}{\hbar} \Delta_{\hbar}(\tilde{J}_-) \] (4.19)
in terms of two copies of the non-standard deformation (4.15).

4.2 The potential \( x^2 + 1/x^2 \)

The following realization of \( sl(2) \)
\[ J_3 = p_1 q_1, \]
\[ J_+ = \frac{1}{\hbar} p_1^2 + \frac{c_1}{q_1^2}, \]
\[ J_- = -\frac{1}{\hbar} q_1^2, \] (4.20)
underlies the \( sl(2) \) dynamical symmetry of the above potential, since by defining
\[ H_c = J_+ - \omega^2 J_- \] (4.21)
we obtain that
\[ H = \frac{1}{2} (p_1^2 + \omega^2 q_1^2) + \frac{c_1}{q_1^2}. \] (4.22)

Note that the Casimir function is related to the centrifugal potential:
\[ C_c = \frac{1}{4} J_3^2 + J_+ J_- = -c_1/2. \] (4.23)

A two-particle Hamiltonian with coalgebra symmetry can be immediately derived by computing the coproduct of (4.21). Since this dynamical Hamiltonian is linear in the generators, we have that
\[ H^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 (q_1^2 + q_2^2) + \frac{c_1}{q_1^2} + \frac{c_2}{q_2^2}, \] (4.24)
in case we assume both phase space representations not to be the same \( (c_1 \neq c_2) \). The Casimir function for this well-known Hamiltonian is again the coproduct of \( C_c \) in the actual representation, which reads
\[ \Delta(C_c) = C^{(2)}(q_1, q_2, p_1, p_2) = -\frac{1}{4} (p_1 q_2 - p_2 q_1)^2 - \frac{(q_1^2 + q_2^2)}{2} \left( \frac{c_1}{q_1^2} + \frac{c_2}{q_2^2} \right). \] (4.25)
4.2.1 Deformed $x^2 + 1/x^2$ systems

The very same deformation machinery can be now used provided suitable deformed realizations generalizing (4.20) are found. Hereafter it will be useful to consider the function

$$I_{\{1\over 2\}}[tx] := \frac{\sinh(t x/2)}{tx}. \quad (4.26)$$

Note that $\lim_{t \to 0} I_{\{1\over 2\}}[tx] = 1/2$.

- The standard case. The deformed realization is

$$\tilde{J}_3 = p_1 q_1,$$

$$\tilde{J}_+ = I_{\{1\over 2\}}[z p_1 q_1] p_1^2 + \frac{1}{2 I_{\{1\over 2\}}[z p_1 q_1]} c_1 q_1^2,$$

$$\tilde{J}_- = -I_{\{1\over 2\}}[z p_1 q_1] q_1^2. \quad (4.27)$$

The one-particle Hamiltonian derived from

$$H_z = \tilde{J}_+ - \omega^2 \tilde{J}_- \quad (4.28)$$

in this realization becomes:

$$H_z = I_{\{1\over 2\}}[z p_1 q_1] (p_1^2 + \omega^2 q_1^2) + \frac{1}{2 I_{\{1\over 2\}}[z p_1 q_1]} c_1 q_1^2. \quad (4.29)$$

The deformed Casimir function is again

$$C_z = \left( \frac{\sinh(z \tilde{J}_3/2)}{z} \right)^2 + \tilde{J}_+ \tilde{J}_- = -c_1/2. \quad (4.30)$$

When the deformed coproduct of (4.28) is realized in terms of two phase space realizations of the type (4.27) leads to the following integrable two-particle Hamiltonian:

$$H_z^{(2)} = \left( I_{\{1\over 2\}}[z p_1 q_1] (p_1^2 + \omega^2 q_1^2) + \frac{1}{2 I_{\{1\over 2\}}[z p_1 q_1]} c_1 q_1^2 \right) e^{z p_2 q_2/2} +$$

$$\left( I_{\{1\over 2\}}[z p_2 q_2] (p_2^2 + \omega^2 q_2^2) + \frac{1}{2 I_{\{1\over 2\}}[z p_2 q_2]} c_2 q_2^2 \right) e^{-z p_1 q_1/2}$$

$$= H^{(2)} + z \left\{ p_2 q_2 \left( \frac{1}{2} p_1^2 + \frac{1}{2} \omega^2 q_1^2 + \frac{c_1}{q_1^2} \right) - p_1 q_1 \left( \frac{1}{2} p_2^2 + \frac{1}{2} \omega^2 q_2^2 + \frac{c_2}{q_2^2} \right) \right\} + o(z^2). \quad (4.31)$$

Note that this power series expansion shows how the undeformed one-particle Hamiltonians arises in the first order of the perturbation. As usual, the constant of the motion will be given by (4.13) where we should use two copies of the proper realization (4.27). As a result, we obtain

$$C_z^{(2)} = \left( \frac{\sinh(z(p_1 + p_2)/2)}{z} \right)^2$$

$$- \left\{ e^{-z p_1 q_1} \left( I_{\{1\over 2\}}[z p_2 q_2] (p_2^2 + \frac{c_2}{q_2^2}) \right) + e^{z p_2 q_2} \left( I_{\{1\over 2\}}[z p_1 q_1] (p_1^2 + \frac{c_1}{q_1^2}) \right) \right\} \quad (4.32)$$

$$- I_{\{1\over 2\}}[z p_1 q_1] I_{\{1\over 2\}}[z p_2 q_2] \left\{ (p_1^2 q_2^2 + p_2^2 q_1^2) + \frac{1}{2} \left( \frac{c_1}{q_1^2} \frac{q_2^2}{q_1^2} + \frac{c_2}{q_2^2} \frac{q_1^2}{q_2^2} \right) \right\} \quad (4.33)$$
A straightforward computation shows that the limit $z \to 0$ of this integral gives the undeformed one (4.23).

- The **non-standard** case. The deformed realization for the non-standard $sl_h(2)$ Poisson algebra with $\tilde{J}_-$ as primitive generator (2.10) is:

$$
\begin{align*}
\tilde{J}_3 &= 2 I_{\{1/2\}}[h q_1^2] q_1 p_1, \\
\tilde{J}_+ &= I_{\{1/2\}}[h q_1^2] p_2^2 + \frac{c_1}{2 I_{\{1/2\}}[h q_1^2] q_1}, \\
\tilde{J}_- &= -\frac{i}{2} q_1^2.
\end{align*}
$$

(4.33)

If we consider again the dynamical one-particle Hamiltonian

$$
H_h = \tilde{J}_+ - \omega^2 \tilde{J}_-
$$

(4.34)

we obtain the following deformation of (4.22):

$$
H_h = p_1^2 I_{\{1/2\}}[h q_1^2] + \frac{\omega}{2} q_1^2 + \frac{c_1}{2 I_{\{1/2\}}[h q_1^2] q_1}.
$$

(4.35)

Note that, as a particular feature of this case, if we consider the dynamical Hamiltonian

$$
\tilde{H}_h = \left(\sinh h \tilde{J}_- / \tilde{J}_-\right)^{-1} H_h,
$$

we obtain a “natural” Hamiltonian (with the usual kinetic term $p_1^2$) in which the deformation implies that the motion is not expected to be always periodic (in fact, as long as $h$ is increasing, non-periodic motions become more relevant). Under such deformed realization (4.33), the Casimir (2.9) turns into $-c/2$.

The corresponding two-body system is, as usual, provided by the non-standard coproduct of the dynamical Hamiltonian (4.34), that reads

$$
H_h^{(2)} = 
\left(I_{\{1/2\}}[h q_1^2] p_1^2 + \frac{c_1}{2 I_{\{1/2\}}[h q_1^2] q_1^2}\right) e^{-h q_1^2/2} + 
\left(I_{\{1/2\}}[h q_2^2] p_2^2 + \frac{c_2}{2 I_{\{1/2\}}[h q_2^2] q_2^2}\right) e^{h q_2^2/2} + \frac{h}{4} \omega^2 (q_1^2 + q_2^2)
$$

$$
= H^{(2)} + \frac{h}{2} \left(q_1^2 \left(\frac{1}{2} p_1^2 + \frac{c_1}{q_1^2}\right) - q_2^2 \left(\frac{1}{2} p_2^2 + \frac{c_1}{q_2^2}\right)\right) + o(h^2).
$$

(4.36)

Note that this power series expansion shows the presence of “crossed” oscillator and centrifugal terms coming from the deformation. Finally, the associated constant of the motion can be deduced from the coproduct of the non-standard Casimir (2.12) under two realizations of the type (4.33).

### 4.2.2 Coalgebra symmetry of the $N = 2$ Calogero system

Let us now consider the $N = 2$ Calogero Hamiltonian [7]

$$
H_{\alpha,\beta}(Q, P) = \frac{1}{4} (P_1^2 + P_2^2) + \Omega^2 (Q_1^2 + Q_2^2) + \frac{\alpha}{(Q_1 - Q_2)^2} + \frac{\beta}{(Q_1 + Q_2)^2}.
$$

(4.37)

It turns out that the canonical transformation

$$
Q_1 := (q_1 + q_2)/2, \quad Q_2 := (q_2 - q_1)/2, \quad P_1 := p_1 + p_2, \quad P_2 := p_2 - p_1.
$$

(4.38)
leads to the identification
\[ H^{(2)}_{\alpha,\beta}(q,p) = 2 H^{(2)}(q,p), \quad (4.39) \]
where \( H^{(2)}(q,p) \) is given by (1.24) with \( \Omega^2 = \omega^2/2 \), \( \alpha = 2c_1 \) and \( \beta = 2c_2 \).

Therefore, the Calogero hamiltonian (4.37) does have \( sl(2) \) coalgebra symmetry, being canonically equivalent to the (non-deformed) coproduct \( \mathcal{H} := 2(J_+ - \omega^2 J_-) \) and provided two appropriate (and, in general, different) phase space realizations of \( sl(2) \) are considered. Moreover, an integral of the motion for (4.37) is immediately deduced from the coalgebra symmetry, since it will be given by the (canonically transformed) coproduct of the \( sl(2) \) Casimir given by (4.25), which takes the following form in terms of the Calogero variables:
\[ C^{(2)}_{\alpha,\beta}(Q,P) = -\frac{1}{4}(P_1 Q_2 - P_2 Q_1)^2 - (Q_1^2 + Q_2^2) \left( \frac{\alpha/2}{(Q_1 - Q_2)^2} + \frac{\beta/2}{(Q_1 + Q_2)^2} \right). \quad (4.40) \]

As a further consequence, the coalgebra symmetry provides a systematic procedure to get integrable deformations of the \( N = 2 \) Calogero system. In particular, a factor 2 times the deformed Hamiltonians (4.31) and (4.36) will give rise, (respectively, and by using the inverse of the canonical transformation (4.38)) to the standard and non-standard deformations of \( H^{(2)}_{\alpha,\beta}(Q,P) \). Once again, corresponding deformed integrals would be given by the inverse canonical transformation of the coproduct of the deformed Casimirs in the original \( (q,p) \) variables.

5 Two-particle realizations and 2 \( N \) dimensional systems

So far we have considered one-particle phase space realizations of coalgebras, but this is not the most general possibility. For instance, the classical analogue of the so called Jordan-Schwinger (JS) realization of \( sl(2) \) would be
\[ J_3 = a_1^* a_1 - b_1^* b_1 = N^a_1 - N^b_1, \]
\[ J_+ = b_1 a_1^*, \]
\[ J_- = a_1 b_1^*, \]
where \( \{a_1, a_1^*\} = \{b_1, b_1^*\} = 1 \). This realization can be used to construct integrable systems by using the coalgebra approach, but now each \( sl(2) \) copy will have two degrees of freedom and we would obtain a 2\( N \) dimensional system, whose complete integrability will be linked to the existence of 2\( N \) quantities in involution.

The previous formalism provides us with \( N \) of them (the \( m \)-th coproducts of the Casimir and the \( N \)-th coproduct of the dynamical Hamiltonian). However \( N \) more integrals are also available if we take into account that, under (5.1), the Casimir of \( sl(2) \) is no longer a numerical constant, but a two-particle function. Namely,
\[ C_i = \frac{1}{4}(J_3^2 + J_+^2 J_-^2) = \frac{1}{4}(a_1^* a_1 + b_1^* b_1)^2 = \frac{1}{4}(N^a_1 + N^b_1)^2. \quad (5.2) \]

And we have \( N \) of this quantities (in terms of classical number operators) that, by construction, will commute with the \( N \) integrals coming from the coproduct. For instance, from (5.1) we find that the JS classical Gaudin magnet is
\[ H^{(N)}_{JS} = \sum_{i=1}^{N} C_i + \sum_{i<j}^{N} \left( \frac{1}{2} J_3^i J_3^j + J_+^i J_-^j + J_+^j J_-^i \right) \]
\[
\frac{N}{2} \sum_{i=1}^{N} (N_i^a + N_i^b)^2 + \frac{1}{2} \sum_{i<j}^{N} (N_i^a - N_j^b)(N_j^a - N_i^b) + \sum_{i<j}^{N} (a_i^+ b_j^+ a_j b_i + b_i^+ a_j^+ b_j a_i).
\]

Therefore, this Hamiltonian is completely integrable, since it is in involution with both the \(C_i\) functions (5.3) and the “lower dimensional” Hamiltonians \(H_{JS}^{(n)}\). Finally, we remark that the last term in (5.3) is just the classical counterpart of a long-range interacting system with a four-wave interaction Hamiltonian mixing each pair of sites (see [18] for related quantum optical integrable systems).

### 5.1 Jordan-Schwinger map and coalgebra structure

At this point, it becomes clear that a suitable deformation of the classical JS map (5.1) would give rise to a new class of \(2N\) dimensional \(sl_q(2)\) invariant Hamiltonians. The answer to this question can be directly related to the study of the reducibility properties of the classical JS map, and it will lead us to a new interesting type of coalgebra structures.

A straightforward computation shows that the \(\bar{J}_i\) functions

\[
\bar{J}_3 = \Delta(J_3) = 1 \otimes J_3 + J_3 \otimes 1,
\]
\[
\bar{J}_+ = \Delta(J_+) = 1 \otimes J_+ - J_+ \otimes 1,
\]
\[
\bar{J}_- = \Delta(J_-) = 1 \otimes J_- - J_- \otimes 1.
\]

close a \(sl(2)\) algebra. We could now consider a pair of “harmonic oscillator” realizations –given by (4.20) with \(c = 0\)– and obtain a realization of (5.4) on the two-particle phase space in the form

\[
\bar{J}_3 = p_1 q_1 + p_2 q_2,
\]
\[
\bar{J}_+ = \frac{1}{2}(p_1^2 - p_2^2),
\]
\[
\bar{J}_- = -\frac{1}{2}(q_1^2 - q_2^2).
\]

Now, the following canonical transformation can be defined

\[
p_1 = b_1 + \frac{1}{2} a_1^+,
\]
\[
q_1 = a_1 - \frac{1}{2} b_1^+,
\]
\[
p_2 = b_1 - \frac{1}{2} a_1^+,
\]
\[
q_2 = -a_1 - \frac{1}{2} b_1^+.
\]

By substituting (5.6) onto (5.5) we recover (5.1). Therefore, the JS realization is canonically equivalent to a “fermionic coproduct” (5.4) of two irreps of \(sl(2)\). Note that the map \(\Delta\) defined by (5.4) is not coassociative since

\[
(id \otimes \Delta) \circ \Delta \neq (\Delta \otimes id) \circ \Delta.
\]

The coalgebra structure allows us to perform the same trick in the deformed case, where we can obtain the deformed realization by setting

\[
\Delta_z(\bar{J}_3) = 1 \otimes \bar{J}_3 + \bar{J}_3 \otimes 1,
\]
\[
\Delta_z(\bar{J}_+) = e^{-\frac{z}{2} J_3} \otimes \bar{J}_+ - \bar{J}_+ \otimes e^{\frac{z}{2} J_3};
\]
\[
\Delta_z(\bar{J}_-) = e^{-\frac{z}{2} J_3} \otimes \bar{J}_- - \bar{J}_- \otimes e^{\frac{z}{2} J_3}.
\]

These expressions are compatible with \(sl_q(2)\) brackets (2.4) and define a non-cocommutative and non-coassociative homomorphism. From them, the \(sl_q(2)\) generators can be expressed in terms of two phase space realizations (4.27) with \(c = 0\):

\[
\tilde{f}_3 = \tilde{\Delta}_z(\tilde{J}_3) = p_1 q_1 + p_2 q_2,
\]
\[\begin{align*}
\tilde{f}_+ &= \Delta_z(\tilde{J}_+) = e^{-(\tilde{p}_1 q_1)} I_{\{\frac{1}{2}\}}[z q_2 p_2] p_2^2 - I_{\{\frac{1}{2}\}}[z q_1 p_1] p_1^2 e^{\tilde{p}_2 q_2}; \\
\tilde{f}_- &= \Delta_z(\tilde{J}_-) = -e^{-(\tilde{p}_1 q_1)} I_{\{\frac{1}{2}\}}[z q_2 p_2] q_2^2 + I_{\{\frac{1}{2}\}}[z q_1 p_1] q_1^2 e^{\tilde{p}_2 q_2};
\end{align*}\]  

(5.9)

Now, if we apply the canonical transformation (5.6) onto the functions \(\tilde{f}_i\), we shall obtain the appropriate standard deformation of the JS map (5.1), that can be recovered in the limit \(z \to 0\).

Note that the \(q\)-deformation of the JS map has been treated in the previous literature \cite{19} by making use of a deforming functional approach \cite{20, 21}. However, the discovery of its “internal” coalgebra structure makes it possible to give an answer in terms of the \(su_q(2)\) properties. It is also worth to stress that the non-standard deformation seems not to be compatible with such “fermionic” comultiplication.

Finally, we mention that the construction of a \(2N\) dimensional standard deformation of the Gaudin-JS system (5.3) is straightforward by considering \(N\) copies of the deformed JS map (5.9) and by representing through them the \(N\)-th deformed coproduct of the standard \(sl_q(2)\) Casimir (2.6). The integrals of motion will be given by the \(M\)-th coproducts \((M = 2, \ldots, N - 1)\) of such deformed Casimir and by the \(N\) different functions \(C_i^z\) defined by the expressions of the \(sl_q(2)\) Casimir on each copy of the JS realization.

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