Abstract. In this paper we analyze Garden-of-Eden (GoE) states and fixed points of monotone, sequential dynamical systems (SDS). For any monotone SDS and fixed update schedule, we identify a particular set of states, each state being either a GoE state or reaching a fixed point, while both determining if a state is a GoE state and finding out all fixed points are generally hard. As a result, we show that the maximum size of their limit cycles is strictly less than \( \left( \frac{n}{\lfloor n/2 \rfloor} \right) \). We connect these results to the Knaster-Tarski theorem and the LYM inequality. Finally, we establish that there exist monotone, parallel dynamical systems (PDS) that cannot be expressed as monotone SDS, despite the fact that the converse is always true.

Keywords. Monotone dynamical system, Fixed point, Garden-of-Eden state, Limit cycle, Sequentialization, LYM inequality

AMS subject classifications. 68Q80, 06A06, 93C55

1 Introduction

The study of dynamical systems is central in modern mathematics and its applications in other fields such as physics, computer science, and biology. In this paper, we investigate discrete-time dynamical systems over graphs. Given a graph \( G \), each vertex of \( G \) has a state contained in some finite set \( P \). At each time step, some or all of the vertices update their states according to their respective local functions or updating rules to generate discrete-time dynamics. The precise manner in which the local functions assemble to a dynamical system map is called the update mechanism. Examples of classes of such dynamical systems include Boolean networks \([17]\), Cellular Automata (CA) \([24, 28]\), Hopfield networks \([13]\), and sequential dynamical systems (SDS) \([4, 5, 6, 7, 22]\).

For a graph \( G \) on \( n \) vertices there are \( |P|^n < \infty \) distinct states. As a result, the iterates of any system state will eventually cycle through a subset of system states, called

\[^{1}\text{We will write SDS in singular as well as plural form.}\]
periodic points. As a result, the system dynamics generates a directed graph, called the 
phase space, consisting of a collection of disconnected cycles with trees attached at cycle 
vertices. Accordingly, the dynamics can be characterized by these cycles (limit cycles), the 
leaves of the attached trees (Garden-of-Eden (GoE) states), and the non-leaf tree states 
(transient states) connecting GoE states and periodic points. A limit cycle consisting of 
only one periodic point is called a fixed point.

Systems composed of monotone functions were studied in [1, 3, 13, 14, 16, 25, 26], and 
systems of linear functions and monomial functions were analyzed in [10, 11] and [12], 
respectively. A number of studies are concerned with the existence and number of GoE 
states and fixed points [1, 2, 8, 20, 21], as well as the size of limit cycles [3, 23].

In this paper, we study dynamical systems having monotone, local functions. One key 
result characterizes a set of states that are either a GoE-state or that eventually reach a 
fixed point. Notably, this set depends exclusively on monotonicity and not the particular 
choice of monotone local functions.

The paper is organized as follows. In Section 2 we recall the notation of SDS and 
monotone SDS. In Section 3 we study monotone SDS in detail. Here we establish several 
key results and applications. For example, we show that in any monotone SDS on n 
vertices, the probability of a random state being either a GoE state or reaching a fixed 
point under a randomly chosen update schedule is at least \( \frac{n}{2^{n-1}} \). Furthermore, we prove 
that the maximum size of limit cycles of monotone SDS is strictly less than \( \binom{n}{\lfloor n/2 \rfloor} \). We 
also refine the LYM inequality and present a finite version of the Knaster-Tarski theorem, 
from which it follows that if there exists a non-trivial periodic state then there exist at 
least two trivial periodic states (i.e., fixed points.)

In Section 4 we discuss the sequentialization of monotone, parallel dynamical systems 
(PDS), that is, the construction of a monotone SDS whose graph \( G \) has the same number 
of vertices and which has exactly the same dynamics as a given monotone PDS. We prove 
that there exists monotone PDS that cannot be sequentialized, and we provide sufficient 
and necessary conditions for a monotone PDS to have a monotone sequentialization.

## 2 Basic Definitions

Let \( G \) be a simple graph with vertex set \( V(G) = \{1, 2, \ldots, n\} \) with each vertex having a 
state taken from a finite set \( P \). A state of the system is a tuple \( X = [x_1, x_2, \ldots, x_n] \in P^n \) 
where the \( i \)-th coordinate represents the state of the vertex \( i \). A function \( f_i \) is used to 
update the state of \( i \) based on the states of its neighbors and itself. A permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \) of the vertices in \( V(G) \) is called an update schedule. The system dynamics is 
generated by sequentially updating the states of all vertices according to \( \pi \), that is, the 
vertex labeled \( \pi_{i+1} \) is updated after the vertex labeled by \( \pi_i \). We denote the associated SDS 
by the triple \((G, f = (f_i)_i, \pi)\). The map resulting from a single application of this update 
procedure is denoted \( F_\pi \) (or \((G, f, \pi)\) by abuse of notation).

By inflation, each local function \( f_i \) induces the function 

\[
F_i : P^n \to P^n, \quad [x_1, \ldots, x_i, \ldots, x_n] \mapsto [x_1, \ldots, f_i, \ldots, x_n],
\]
where the arguments taken by $f_i$ are the current states of $i$ and its neighbors, and we have

$$F_\pi = F_{\pi_n} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_1}.$$ 

As mentioned above, the directed graph on $P^n$ with directed edges $(X, F_\pi(X))$ where $X \in P^n$ is called the phase space of the SDS. A state $X$ is called a Garden-of-Eden (GoE) state if there is no state $Y$ such that $F_\pi(Y) = X$. Directed cycles in the phase space are called limit cycles, and states contained in limit cycles are called attractors or periodic states. States contained in a limited cycle of length no less than two are called non-trivial periodic states. A state $X$ is called a fixed point if $F_\pi(X) = X$.

Unless stated otherwise, we will in the following assume $P = F_2 = \{0, 1\}$ with ordering $0 < 1$. Let ‘$\leq$’ be the partial order on $F_2^q$ ($q \geq 1$) given by

$$[x_1, x_2, \ldots, x_q] \leq [y_1, y_2, \ldots, y_q],$$

iff $x_i \leq y_i$ in $F_2$ for all $1 \leq i \leq q$. We shall denote the minimum $[0, 0, \ldots, 0]$ by $0$ and the maximum $[1, 1, \ldots, 1]$ by $1$.

**Definition 2.1.** A function $g : F_2^q \to F_2$ is called monotone if for any $X, Y \in F_2^q$ with $X \leq Y$ one has $g(X) \leq g(Y)$. The function $g$ is called a simple threshold function if for some fixed $k \geq 0$

$$\forall X = [x_1, x_2, \ldots, x_q] \in F_2^q, \quad g(X) = \begin{cases} 1, & \text{if } \sum_j x_j \geq k, \\
0, & \text{otherwise.} \end{cases}$$

It is easy to check that the binary functions ‘AND’ and ‘OR’ are simple threshold functions, and that simple threshold functions are monotone. We will call a (sequential) dynamical system with monotone, local functions a monotone (sequential) dynamical system.

### 3 GoE States and Fixed Points

In this section we analyze GoE states and fixed points of monotone SDS. We shall identify a set of states that are either GoE states or reach a fixed point, regardless of the particular choice of the (local) monotone functions.

**Lemma 3.1.** Let $(G, f, \pi)$ be a monotone SDS and $X, Y$ be two states. Then we have:

(a) $X \leq Y$ implies $F_\pi(X) \leq F_\pi(Y)$.

(b) if $X \leq F_\pi(X)$ or $X \geq F_\pi(X)$, then $X$ reaches a fixed point.

(c) the states contained in a limit cycle form an anti-chain.

**Proof.** Using the fact that the functions $F_{\pi_j}$ are monotone, the relation $F_{\pi_1} \circ \cdots \circ F_{\pi_2} \circ F_{\pi_1}(X) \leq F_{\pi_1} \circ \cdots \circ F_{\pi_2} \circ F_{\pi_1}(Y)$ for $1 \leq i \leq n$ follows by induction on $i$, establishing (a).
To prove (b), suppose $X \leq F_\pi(X)$ or $X \geq F_\pi(X)$, then (a) gives rise to the diagram

$$
\begin{array}{c}
\xrightarrow{}
X \xrightarrow{} F_\pi(X) \xrightarrow{} \cdots \xrightarrow{} F^{h}_\pi(X) \\
\downarrow \\
F_\pi(X) \xrightarrow{} F^{2}_\pi(X) \xrightarrow{} \cdots \xrightarrow{} F^{h+1}_\pi(X)
\end{array}
$$

from which we derive the chain $X, F_\pi(X), F^{2}_\pi(X), \ldots$ in the poset $\mathbb{F}^n_2$. Since $\mathbb{F}^n_2$ is a finite poset, the chain becomes stationary after a finite number of steps, that is, $X$ reaches a fixed point. Note if $X \leq F^k_\pi(X)$ or $X \geq F^k_\pi(X)$, then $X$ reaches a fixed point w.r.t. iteration of the map $F^k_\pi$ according to (b). Thus, any two different states on a limit cycle of $F_\pi$ cannot be comparable, whence Part (c). \qed

**Proposition 3.2.** Let $(G, f, \pi)$ be a monotone SDS. Then the following statements hold.

(a) Suppose that the state $X$ satisfies $X \leq Z$ or $X \geq Z$ and reaches the fixed point $Z$. Then any state $Y$ satisfying $X \leq Y \leq Z$ or $X \geq Y \geq Z$ reaches the fixed point $Z$.

(b) Suppose $X$ satisfies $X \leq Z_\pi$ or $X \geq Z_\pi$ and reaches $Z_\pi$. If $Z_\pi$ is a fixed point of $(G, f, \pi)$, then for any $\sigma$-schedule and $k \geq 0$ we have $F^k_\sigma(X) \leq Z_\pi$ or $F^k_\sigma(X) \geq Z_\pi$.

**Proof.** To prove (a) we may without loss of generality assume that $X \leq Z$. Lemma 3.1 guarantees that $X \leq Y \leq Z$ implies that, for any $k \geq 0$, $F^k_\pi(X) \leq F^k_\pi(Y) \leq F^k_\pi(Z) = Z$, where $F^0_\pi = \text{id}$. Since $X$ reaches $Z$, there exists $k_0$ such that $F^k_\pi(X) = Z$ for any $k \geq k_0$. Consequently, $Y$ reaches $Z$ and (a) follows. As for (b), we note, that if $Z_\pi$ is a fixed point for $(G, f, \pi)$, then $Z_\pi$ is a fixed point for $(G, f, \sigma)$ for any $\sigma$, and we therefore have $F^k_\sigma(X) \leq F^k_\sigma(Z_\pi) = Z_\pi$, completing the proof. \qed

The paper [8] considers the states $0$ and $1$ in order to probe for fixed points in monotone systems and in addition discusses the existence of GoE. From our previous observations we can immediately conclude:

**Proposition 3.3.** Let $(G, f, \pi)$ be a monotone SDS. Then $0$ and $1$ are either a GoE state reaching a fixed point or a fixed point.

**Proof.** For any update schedule $\pi$ it is clear that $F_\pi(0) \geq 0$. Lemma 3.1 guarantees that if $0$ is a GoE state, it will reach a fixed point. Otherwise, there exists $X > 0$ and $F_\pi(X) = 0$. Again, by Lemma 3.1 $0$ must be a fixed point since the formed decreasing chain $X > 0 \geq \cdots$ must become stationary at $0$. The argument for the case of $1$ is analogous and the proposition follows. \qed

For a finite system, all fixed points form a sub-poset of the poset $\mathbb{F}^n_2$. In view of Proposition 3.2 and Proposition 3.3, we obtain the following variation of the Knaster-Tarski theorem [18, 27]: the set of fixed points of a monotone function on a complete lattice is a complete lattice. To this end, it suffices to establish the existence of a unique maximal and minimal element (by definition of complete lattice).

**Proposition 3.4.**

(a) The set of fixed points of a monotone SDS $(G, f, \pi)$ is a complete lattice $\mathbb{L}_\pi$.

(b) In $\mathbb{F}^n_2$, any periodic point $X$ is comparable to the $\mathbb{L}_\pi$-maximum (MAX) and $\mathbb{L}_\pi$-minimum (MIN), and we have $\text{MIN} \leq X \leq \text{MAX}$. 

4
Proof. If 1 is a fixed point, then it is certainly the unique maximum. Otherwise, suppose that in the poset of fixed points the maximal elements are \(Z_1, \ldots, Z_p\) where \(p > 1\). Since the element 1 is not a fixed point, Proposition 3.3 guarantees that 1 reaches a fixed point.

Claim 1. The fixed point reached by 1 is maximal.

Suppose 1 reaches the fixed point \(Z_0\). If \(Z_0\) is not comparable with any \(Z_i\) for \(i \neq 0\), then \(Z_0\) is contained in the set of maximal elements. Otherwise we have \(Z_0 < Z_i < 1\) for some \(1 \leq i \leq p\). From Proposition 3.2 \(Z_i\) reaches the fixed point \(Z_0\) which implies that \(Z_i\) itself is not a fixed point, a contradiction, proving the claim.

Suppose 1 reaches the fixed point \(Z_1\). Since \(1 > Z_i (1 < i \leq p)\), monotonicity implies \(F_{\pi}^k(1) \geq F_{\pi}^k(Z_i) = Z_i\) for \(k \geq 1\). As 1 reaches the fixed point \(Z_1\), we arrive at \(Z_1 \geq Z_0\), which is impossible since \(Z_1\) and \(Z_i\) are distinct, maximal elements. Accordingly, \(Z_1\) is the unique maximal element. The argument in case of 0 is completely analogous.

Next, suppose that \(X\) is a periodic point. Then \(F_{\pi}^{mk}(X) = X\) for some \(k > 0\) and any \(m > 0\). Since \(0 \leq X \leq 1\), Lemma 3.1 implies that \(F_{\pi}^{l}(0) \leq F_{\pi}^{l}(X) \leq F_{\pi}^{l}(1)\) for any \(l \geq 0\). Since 0 and 1 reach MIN and MAX as fixed points, we have, for a sufficiently large \(m > 0\)

\[
\text{MIN} = F_{\pi}^{mk}(0) \leq F_{\pi}^{mk}(X) = X \leq F_{\pi}^{mk}(1) = \text{MAX},
\]

completing the proof. \(\square\)

Remark 3.5. Proposition 3.4 (b) appears to have not been addressed before and does not hold in the general case of systems with infinite phase space. The above proposition also implies that the phase space of a monotone system has specific properties that do not depend on the particular choice of local functions: if there exists a non-trivial periodic point, then there exist at least two fixed points.

We denote the symmetric group on the set \([n] = \{1, 2, \ldots, n\}\) by \(S_n\). Let \(g \in S_n\) and \(X = [x_1, x_2, \ldots, x_n] \in F_n^n\). The group \(S_n\) acts on \(F_n^n\) by

\[
g \cdot X = [x_{g^{-1}(1)}, x_{g^{-1}(2)}, \ldots, x_{g^{-1}(n)}].
\]

Moreover, \(S_n\) acts on the set of update schedules via \(g \cdot \pi = \pi_{g^{-1}(1)} \pi_{g^{-1}(2)} \cdots \pi_{g^{-1}(n)}\).

Two SDS phase spaces are called cycle equivalent \([23]\) iff there exists an isomorphism between their sets of limit cycles. Let \(\tau = (n \ n - 1 \ \cdots \ 1)\) be a cyclic permutation. For \(\pi = \pi_1 \pi_2 \cdots \pi_n\) we set

\[
\pi_{\tau^k} = \tau^k \cdot \pi = \pi_{k+1} \cdots \pi_n \pi_1 \pi_2 \cdots \pi_k.
\]

There is a relation between the phase space of the \(\pi\)-system \(F_{\pi} = F_{\pi_n} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_1}\) and the \(\pi_{\tau^k}\)-system \(F_{\pi_{\tau^k}} = F_{\pi_{k+1}} \circ F_{\pi_{k}} \circ \cdots \circ F_{\pi_1}\).

Proposition 3.6. The map \(h = F_{\pi_{\tau^k}} \circ F_{\pi_{k-1}} \circ \cdots \circ F_{\pi_1}\) is a homomorphism from the phase space of \((G, f, \pi)\) to the phase space of \((G, f, \pi_{\tau^k})\). Furthermore, restricted to the limit cycles \(h\) induces an isomorphism.

Proof. It is sufficient to consider the case \(k = 1\), and we shall prove that \(h = F_{\pi_1}\).
Claim 1. If $F_\pi(X) = Y$, then $F_{\pi_\tau}(F_{\pi_1}(X)) = F_{\pi_1}(Y)$.

We compute
\[
F_{\pi_\tau}(F_{\pi_1}(X)) = F_{\pi_1} \circ F_{\pi_\alpha} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_2}(F_{\pi_1}(X))
\]
\[
= F_{\pi_1}(F_{\pi_\alpha} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_2} \circ F_{\pi_1}(X)) = F_{\pi_1}(Y).
\]

Thus, $h = F_{\pi_1}$ is a homomorphism as it maps the directed edge $X \to Y$ into the directed edge $F_{\pi_1}(X) \to F_{\pi_1}(Y)$. It remains to prove that $h$ induces an isomorphism on limit cycles. To prove this it is crucial that $F_\pi(X) \neq F_\pi(Y)$ implies $F_{\pi_1}(X) \neq F_{\pi_1}(Y)$. We conclude from this: suppose for a sequence $(X_1, X_2, \ldots, X_k)$, we have $F_{\pi_1}(X_i) = X_{i+1}$ for $1 \leq i \leq k-1$ and $X_i \neq X_j$ for any $2 \leq i < j \leq k$, then for the sequence
\[
(F_{\pi_1}(X_1), F_{\pi_1}(X_2), \ldots, F_{\pi_1}(X_k)),
\]
we have $F_{\pi_1}(X_i) \neq F_{\pi_1}(X_j)$ for $2 \leq i < j \leq k$ and $F_{\pi_\tau}(F_{\pi_1}(X_i)) = F_{\pi_1}(X_{i+1})$ for $1 \leq i \leq k-1$. Clearly, $X_k = X_1$ implies $F_{\pi_1}(X_1) = F_{\pi_1}(X_k)$. As a result, $h$ preserves both: directed paths and limit cycles. Thus, each limit cycle of $(G, f, \pi)$ has a unique isomorphic copy under $(G, f, \pi_\tau)$. Let $\text{Cyc}(F_\pi)$ denote the set consisting of limit cycles of $F_\pi$. Note that $\pi = [\pi_\tau]_{n-1}$. Then we have the following diagram

\[
\begin{array}{ccc}
\pi & \xrightarrow{\tau} & \pi \\
\downarrow & & \downarrow \\
\text{Cyc}(F_\pi) & \xrightarrow{h_\tau} & \text{Cyc}(F_{\pi_\tau}) \\
\downarrow & & \downarrow \\
\text{Cyc}(F_\pi) & \rightarrow & \text{Cyc}(F_\pi)
\end{array}
\]

which implies that $(G, f, \pi)$ and $(G, f, \pi_\tau)$ have the same number of limit cycles. In particular, restricted to the limit cycles, $h$ induces an isomorphism.

As an immediate application of Proposition 3.6, we can recover the cycle equivalence result of SDS in [23], which was proved differently there.

**Corollary 3.7.** [23] The SDS $(G, f, \pi)$ and $(G, f, \pi_{\tau^k})$ are cycle-equivalent.

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ and $\pi' = \pi'_1 \pi'_2 \cdots \pi'_n$ be two update schedules. Let $\pi \sim_\alpha \pi'$ if there exists some $k$ such that (i) $\pi_j = \pi'_j$, $j \neq k, k+1$, and (ii) the vertices $\pi_k$ and $\pi_{k+1}$ are not adjacent in $G$. The transitive and reflexive closure of $\sim_\alpha$ gives an equivalence relation on the set of all update schedules [4]. We denote the equivalence class of $\pi$ by $[\pi]_\alpha$, and for any update schedules $\pi$ and $\pi'$ for which $\pi \sim_\alpha \pi'$, we have $F_\pi = F_{\pi'}$ by construction.

For $X = [x_1, x_2, \ldots, x_n]$ and $\pi = \pi_1 \pi_2 \cdots \pi_n$, let
\[
[X]_\pi = \{\sigma \cdot X \mid \sigma \in [\pi]_\alpha\},
\]
where we consider update schedules as permutations using one-line representation. Let $S_{0,k} = [0,0,\ldots,0,1,1,\ldots,1] \in \mathbb{F}_2^n$ where $x_i = 0$ for $i \leq k$ and $x_j = 1$ otherwise, and define $S_{1,k} = [1,1,\ldots,1,0,0,\ldots,0]$ analogously.
Corollary 3.10. Let \( P \) be a poset not comparable to \( X \).

Remark 3.9. Theorem 3.8 can be generalized from the vertex state set being \( \mathbb{F}_2 \) to any poset \( P \) as long as \( P \) has a minimum and a maximum.

Corollary 3.10. Let \( (G, f, \pi) \) be a monotone SDS. If \( X \in [S_{0,k}]_\pi \cup [S_{1,k}]_\pi \) and \( F_\pi(X) \) is not comparable to \( X \), then \( X \) is a GoE state. If \( X \) is a GoE state, then \( X \) is a GoE state.

Proof. Suppose \( X \in [S_{0,k}]_\pi \). It suffices to consider \( X = S_{0,k} \) under \( (G, f, id) \) by relabeling. Suppose \( F_\pi(Y) = X \) and \( F_\pi(X) = Z \neq X \). Note from the phase space of \( (G, f, \pi_{k+1}) \) to the phase space of \( (G, f, \pi) \), \( h = F_{k+1} \cdots F_1 \) gives the homomorphism according to Proposition 3.6. It can be checked that \( h([x_1, \ldots x_k, y_{k+1}, \ldots, y_n]) = X \) and \( h([z_1, \ldots, z_k, x_{k+1}, \ldots, x_n]) = Z \). By assumption, \( h \) as compositions of monotone
functions is monotone and \([x_1, \ldots, x_k, y_{k+1}, \ldots, y_n] < [z_1, \ldots, z_k, x_{k+1}, \ldots, x_n]\). Therefore, \(X < Z = F_\pi(X)\). Hence, if \(X\) is not a GoE state and \(F_\pi(X) \neq X\), then \(X < F_\pi(X)\), which implies the corollary.

GoE states have been analyzed extensively \([8, 20, 21]\), see for instance the Garden of Eden theorem of Moore and Myhill in the context of (infinite) cellular automata \([20, 21]\). Determining if a particular state is a GoE state is generally hard \([9]\). Given an update schedule, Theorem 3.8 and Corollary 3.10 allow one to identify states which are either GoE states or reach a fixed point. It is worth pointing out that the framework presented facilitates identification of fixed points in monotone dynamical systems where vertices are not updated sequentially since fixed points do not depend on the order of the updates.

Example 3.1. Considers Figure 1 and let \(\pi = 241635\). It is easy to check that

\[
[\pi]_\alpha = \{241635, 241365, 243165, 214635, 214365, 213465, 234165, 231465, 421635, 421365, 423165\},
\]

and we obtain

\[
[S_{0,3}]_\pi = \{001011, 100011, 000111\}.
\]

Suppose all local functions are simple threshold functions, where the threshold values for the vertices 1, 2, 3, 4, 5, and 6 are 1, 2, 1, 2, 2, and 3, respectively. Then for \(X = [100011] \in [S_{0,3}]_\pi\) we have \(F_\pi(X) = [111111]\) which is a fixed point, that is, \(X\) reaches a fixed point. Next, suppose the threshold values are 2, 4, 1, 2, 2, and 3, respectively. Then we have \(F_\pi(X) = [001111]\) which is incomparable with \(X\), and we can conclude that in view of Corollary 3.10 \(X\) is a GoE state.

For a state \(X\) and \(i \in \{0, 1\}\) we set \(\vartheta_i(X) = \{\pi \mid \pi^{-1} \cdot X = S_{i,k}\ \text{for some} \ k\}\).

Lemma 3.11. Suppose \(X\) is a non-trivial periodic point in the phase space \(\mathbb{G}\) of some dynamical system. Then, for any \(\pi \in \vartheta_0(X) \cup \vartheta_1(X)\), no monotone SDS \((G, f, \pi)\) can generate \(\mathbb{G}\).
Proof. Suppose $G$ is the phase space of a monotone SDS $(G, f, \pi)$ with $\pi \in \vartheta_0(X) \cup \vartheta_1(X)$. According to Theorem 3.8, $X$ is either a GoE state or reaches a fixed point of $(G, f, \pi)$, which implies that $X$ is not a non-trivial periodic point. \qed

Lemma 3.11 gives a sufficient condition for a phase space to not be generated by some monotone SDS. Namely, the existence of a subset of states $A$ such that all states in $A$ are non-trivial periodic states and $\bigcup_{X \in A} \vartheta_i(X) = S_n$.

Lemma 3.12. If two states $X$ and $Y$ are not comparable, then $\vartheta_0(X) \cap \vartheta_0(Y) = \emptyset$.

Proof. Suppose there exists $\pi = \pi_1 \pi_2 \cdots \pi_n \in \vartheta_0(X) \cap \vartheta_0(Y)$. We have $\pi^{-1} \cdot Z = S_{0,k}$ if and only if $z_{\pi_i} = 0$ for $1 \leq i \leq k$ and $z_{\pi_i} = 1$ otherwise. By assumption, $\pi^{-1} \cdot X = S_{0,k_X}$ and $\pi^{-1} \cdot Y = S_{0,k_Y}$, and since $k_X \leq k_Y$ or $k_Y \leq k_X$ it follows that $X$ and $Y$ are comparable, which is impossible. \qed

Lemma 3.12 immediately implies that if $A$ is a set of states which are mutually incomparable, then

$$\sum_{X \in A} |\vartheta_0(X)| = \left| \bigcup_{X \in A} \vartheta_0(X) \right| \leq n!.$$  

An immediate consequence of this is the celebrated LYM inequality [19] and Sperner’s lemma which estimate the sizes of incomparable sets within the power set of a finite set of size $n$.

Proposition 3.13. In $(\mathbb{F}_2^n, \leq)$ let $A \subset \mathbb{F}_2^n$ be a set of states whose elements are mutually incomparable. Suppose there are exactly $a_k$ states of $A$ having $k$ coordinates for which $x_i = 0$. Then we have

$$\sum_k \frac{a_k}{\binom{n}{k}} \leq 1.$$  

In particular, we have $|A| \leq \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right)$.

Proof. For $X \in A$ having $k$ coordinates for which $x_i = 0$, it is easy to see that $|\vartheta_0(X)| = k!(n-k)!$. Then

$$\sum_{X \in A} |\vartheta_0(X)| = \sum_k a_k k!(n-k)! \leq n!;$$  

which produces the LYM inequality. Using the fact that $\binom{n}{k} \leq \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right)$ we obtain Sperner’s lemma

$$|A| = \sum_k a_k \leq \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right).$$  

The proof in Lubell [19] is essentially the same as the one given here, but we have a different motivation of relating states (or subsets) to permutations (i.e., update schedules). Note that if $k \neq \lfloor \frac{n}{2} \rfloor$ and $k \neq n - \lfloor \frac{n}{2} \rfloor$, then $\binom{n}{k} < \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right) = \left( \binom{n}{n-\lfloor \frac{n}{2} \rfloor} \right)$. Hence, $|A| = \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right)$ only if $A$ is either the set of states having exactly $\lfloor \frac{n}{2} \rfloor$ coordinates, where $x_i = 1$ or the set having exactly $n - \lfloor \frac{n}{2} \rfloor$ such coordinates.
According to Sperner’s lemma, the maximum possible size of a limit cycle (as an anti-chain) of a monotone system is \( \binom{n}{\frac{n}{2}} \). Our next theorem shows that this maximum is not achievable for monotone SDS.

**Theorem 3.14.** Let \((G, f, \pi)\) be a monotone SDS. The size of any limit cycle of \((G, f, \pi)\) is strictly less than \( \binom{n}{\frac{n}{2}} \).

**Proof.** We prove by contradiction. Suppose a monotone SDS has a limit cycle of length \( \binom{n}{\frac{n}{2}} \). Since a limit cycle of a monotone system gives an anti-chain, the limit cycle of length \( \binom{n}{\frac{n}{2}} \) must be either the set \( A_1 \) of states having exactly \( \lfloor \frac{n}{2} \rfloor \) coordinates where \( x_i = 1 \), or the set \( A_2 \) of states having exactly \( n - \lfloor \frac{n}{2} \rfloor \) such coordinates. According to Lemma 3.12, we have

\[
\left| \bigcup_{X \in A_i} \vartheta_0(X) \right| = \sum_{X \in A_i} |\vartheta_0(X)| = \sum_{X \in A_i} \lfloor \frac{n}{2} \rfloor!(n - \lfloor \frac{n}{2} \rfloor)! = \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} \right) \lfloor \frac{n}{2} \rfloor!(n - \lfloor \frac{n}{2} \rfloor)! = n!,
\]

for any \( i \in \{1, 2\} \), that is, \( \bigcup_{X \in A_i} \vartheta_0(X) \) contains all update schedules. Without loss of generality, we may assume that \( A_1 \) is the limit cycle of \((G, f, \pi)\). Lemma 3.11 shows that a phase space with states in \( A_1 \) being non-trivial periodic points can not be realized by any monotone SDS \((G', f', \sigma)\) for any \( \sigma \in \bigcup_{X \in A_1} \vartheta_0(X) \), whence the theorem.

We proceed by presenting some implications of Theorem 3.8:

**Corollary 3.15.** Let \( G \) be a graph on \( n > 1 \) vertices with fixed local, monotone functions \( f_i \). Selecting a permutation \( \pi \) induces the SDS \((G, f, \pi)\), and the probability that under a random \( \pi \) having \( X \) as either a GoE state or reaching a fixed point is at least \( \frac{2}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \).

Furthermore, the probability of a random state being either a GoE state or reaching a fixed point under a random update schedule is at least \( \frac{n}{2n-1} \).

**Proof.** Let \( X \) be some fixed state with \( m \) coordinates such that \( x_j = 0 \) and \( n - m \) coordinates for which \( x_j = 1 \) where \( 0 < m < n \). Considering \( \pi^{-1} \cdot X = S_{0,m} \) and \( \pi^{-1} \cdot X = S_{1,n-m} \), we conclude that there are \( m!(n-m)! \) different update schedules such that \( X \) is of the form \( S_{0,m} \) and \( m!(n-m)! \) different update schedules for which \( X \) is of the form \( S_{1,n-m} \). By Theorem 3.8, for each such permutation, \( X \) is either a GoE state or reaches a fixed point. Thus we obtain the first probability to be at least

\[
\frac{2m!(n-m)!}{n!} = \frac{2}{\binom{n}{m}} \geq \frac{2}{\binom{n}{\lfloor \frac{n}{2} \rfloor}},
\]

where the last inequality follows from \( \binom{n}{m} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \). Proposition 3.3 guarantees that in case \( X = 0 \) or \( X = 1 \), \( X \) is either a GoE state or reaches a fixed point for any update schedule, whence in this case the probability is 1.

The total number of pairs \((X, \pi)\), where \( X \) is either a GoE state or reaches a fixed point of \((G, f, \pi)\) is at least \( \sum_{0 < m < n} \binom{n}{m} 2m!(n-m)! + 2n! = 2n! \cdot n! \). Thus, the second probability in question is at least \( \frac{2n!}{2n!} = \frac{n}{2n-1} \), completing the proof of the corollary.
Corollary 3.16. Let $G$ be a bipartite graph with vertex set $U \cup V$ where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. Let $\pi = u_1 \cdots u_m v_1 \cdots v_n$ and $(G, f, \pi)$ be a monotone SDS. Then, the probability of a random state being either a GoE state or reaching a fixed point in $(G, f, \pi)$ is at least $\frac{2^{m+1} + 2^{n+1} - 2^2}{2^{m+n}}$.

Proof. Suppose $X = [x_{u_1}, x_{u_2}, \ldots, x_{u_m}, x_{v_1}, x_{v_2}, \ldots, x_{v_n}]$. For any update schedule $\sigma$ of the form $\sigma = u_{k_1} u_{k_2} \cdots u_{k_m} v_l v_2 \cdots v_n$, we have $\sigma \in [\pi]_\alpha$, since no pair of $u$-vertices or $v$-vertices are adjacent in $G$.

In particular, any state $X$ for which $x_{u_i} = 1$ is contained in $[S_{1,d}]_\pi$ where $d \geq m$, and any state $X$ where $x_{u_i} = 0$ is contained in $[S_{0,d}]_\pi$ where $d \geq m$. Any state $X$ where $x_{v_i} = 1$ is clearly in $[S_{0,j}]_\pi$, $j \leq n$ and any state $X$ where $x_{v_i} = 0$ is contained in $[S_{1,j}]_\pi$, $j \leq n$.

By construction, note that the states 0, 1, the state $x_{u_i} = 1$ while $x_{v_j} = 0$, and the state $x_{u_i} = 0$ while $x_{v_j} = 1$ are counted twice. Thus, there are at least $2^{m+1} + 2^{n+1} - 2^2$ states each of which is either a GoE state or reaches a fixed point of $(G, f, \pi)$. Consequently, the probability in question is at least $\frac{2^{m+1} + 2^{n+1} - 2^2}{2^{m+n}}$ and the corollary follows. □

4 Sequentializing Monotone Parallel Systems

Let $G$ be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ where each vertex $i$ has a binary state and a monotone, local function $f_i$. Updating all vertex states in parallel produces the parallel monotone dynamical system of $G$ and $f$ denoted by $(G, f)$.

For certain systems, it might not be possible to maintain accurate synchronization of all vertices in the system as required under a parallel update. In such cases, a sequential update, possibly over a different graph $G'$ and monotone, local functions, $f_i'$ generating the same dynamics as the parallel system may be desirable.

Definition 4.1. Let $(G, f)$ be a parallel dynamical system and $(G', f', \pi)$ an SDS where $V(G) = V(G')$. Then $(G', f', \pi)$ is a sequentialization of $(G, f)$ and $(G, f)$ is a parallelization of $(G', f', \pi)$ iff $(G', f', \pi) = (G, f)$.

Any SDS has a parallelization. Namely, given an SDS, $(G, f, \pi)$, we may assume without loss of generality that $\pi = 12 \cdots n$ with the underlying local maps:

$$
x_1 \mapsto y_1 = f_1([x_1, x_2, x_3, \ldots, x_n]),
$$
$$
x_2 \mapsto y_2 = f_2([y_1, x_2, x_3, \ldots, x_n]),
$$
$$
x_3 \mapsto y_3 = f_3([y_1, y_2, x_3, \ldots, x_n]),
$$
$$
\vdots
$$
$$
x_n \mapsto y_n = f_n([y_1, y_2, y_3, \ldots, x_n]).
$$
Then the parallel system, \((G', f')\), whose local maps are given by

\[
x_1 \mapsto y_1 = f'_1([x_1, x_2, x_3, \ldots, x_n]) = f_1([x_1, x_2, x_3, \ldots, x_n]),
\]
\[
x_2 \mapsto y_2 = f'_2([x_1, x_2, x_3, \ldots, x_n]) = f_2([y_1, x_2, x_3, \ldots, x_n]),
\]
\[
x_3 \mapsto y_3 = f'_3([x_1, x_2, x_3, \ldots, x_n]) = f_3([y_1, y_2, x_3, \ldots, x_n]),
\]
\[
\vdots
\]
\[
x_n \mapsto y_n = f'_n([x_1, x_2, x_3, \ldots, x_n]) = f_n([y_1, y_2, y_3, \ldots, x_n])
\]

and where \(G'\) is implied by the dependencies of the \(f'_i\) on the \(x_j\) represents the parallelization of \((G, f, \pi)\). If \(f_1, \ldots, f_n\) are monotone, for \(X \leq X'\), we have for any \(i\), \((y_1, \ldots, y_{i-1}, x_i, \ldots, x_n) \leq (y'_1, \ldots, y'_{i-1}, x'_i, \ldots, x'_n)\) and hence

\[f'_i(X) = f_i(y_1, \ldots, y_{i-1}, x_i, \ldots, x_n) \leq f_i(y'_1, \ldots, y'_{i-1}, x'_i, \ldots, x'_n) = f'_i(X').\]

As a result, if \((G, f, \pi)\) is monotone, then its parallelization, \((G', f')\), is a monotone, parallel system.

We next analyze whether or not a parallel system \((G', f')\) can be sequentialized. For particular classes of systems, such a sequentialization is always possible, for instance, linear, parallel systems can always be sequentialized as linear SDS \([11]\). In the following we shall show that there exist monotone parallel systems for which there is no monotone sequentialization.

Let \((G, f)\) be a monotone, parallel dynamical system. Lemma \([3.11]\) implies that if \(X\) is a non-trivial periodic point of \((G, f)\), then for any \(\pi \in \vartheta_0(X) \cup \vartheta_1(X)\), the SDS \((G', f', \pi)\) is not a sequentialization of \((G, f)\).

**Theorem 4.1.** There exists a monotone parallel dynamical system which does not have a monotone sequentialization.

**Proof.** By Theorem \([3.14]\) it is sufficient to find a monotone, parallel dynamical system which has a limit cycle of length \(\binom{n}{\lfloor \frac{n}{2} \rfloor}\). In \([3]\), such a monotone, parallel system is constructed: let \(A = \{X_0, X_1, \ldots, X_{p-1}\}\) be the set of states having exactly \(\lfloor \frac{n}{2} \rfloor\) coordinates \(x_i = 1, p = \binom{n}{\lfloor \frac{n}{2} \rfloor}\) and \(A^c = \mathbb{F}_2^n \setminus A\). We construct a parallel dynamical system

\[(G, f) : X_i \mapsto X_{i+1},\]

where the indices are taken modulo \(p\), and for \(Y \in A^c\),

\[(G, f) : Y \mapsto \begin{cases} 1, & \text{if there exists } X \in A \text{ such that } X < Y; \\ 0, & \text{if there exists } X \in A \text{ such that } X > Y. \end{cases}\]

By Proposition \([3.13]\) this map is well defined and monotone, whence the theorem. \(\square\)

In the following we shall further discuss the sequentialization of monotone, parallel systems. Suppose the monotone parallel system \((G', f')\) has a sequentialization \((G, f, \pi)\) for \(\pi = 12 \cdots 2^n\) as in (*), above.

\(^2\)Without loss of generality we may assume that \(\pi = 12 \cdots n\) by relabeling.
Suppose for \([0, x_2, \ldots, x_n] < [1, x'_2, \ldots, x'_n]\) where \(x_i < x'_i\) for some \(2 \leq i \leq n\), we have
\[
\begin{align*}
 f_1([1, x_2, \ldots, x_n]) &= f'_1([1, x_2, \ldots, x_n]) = 0, \\
 f_1([0, x'_2, \ldots, x'_n]) &= f'_1([0, x'_2, \ldots, x'_n]) = 1.
\end{align*}
\]
As \([1, x_2, \ldots, x_n]\) and \([0, x'_2, \ldots, x'_n]\) are not comparable, such a monotone function \(f'_1\) exists. By construction, \([0, x_2, \ldots, x_n] \leq [1, x_2, \ldots, x_n]\), and in view of (*), we have
\[
\begin{align*}
 f_2([0, x_2, \ldots, x_n]) &= f'_2([1, x_2, \ldots, x_n]), \\
 f_2([1, x'_2, \ldots, x'_n]) &= f'_2([0, x'_2, \ldots, x'_n]).
\end{align*}
\]
However, since \([1, x_2, \ldots, x_n]\) and \([0, x'_2, \ldots, x'_n]\) are not comparable, the monotonicity of \(f'_2\) does not necessarily imply that \(f'_2([1, x_2, \ldots, x_n]) \leq f'_2([0, x'_2, \ldots, x'_n])\), that is, we cannot conclude that \(f_2\) is monotone. Accordingly, monotonicity is not guaranteed, even if the underlying local maps of the parallel system are monotone.

**Lemma 4.2.** Suppose \(g: A \to \mathbb{F}_2\) is monotone where \(A \subset \mathbb{F}_n^2\). Then there exists a monotone function \(\hat{g}: \mathbb{F}_n^2 \to \mathbb{F}_2\) such that \(\hat{g}(X) = g(X)\) if \(X \in A\).

**Proof.** For \(X \in A\), we set \(\hat{g}(X) = g(X)\). We extend \(\hat{g}\) from \(A\) to \(\mathbb{F}_n^2\) inductively, using the following procedure:

**Step 1.** Set \(B = A\).

**Step 2.** Let \(Y \in \mathbb{F}_n^2 \setminus B\), and let \(\text{Max} = \max\{g(Z) : (Z > Y) \land (Z \in B)\}\) and \(\text{Min} = \max\{g(Z) : (Z < Y) \land (Z \in B)\}\).

**Step 3.** Set \(\hat{g}(Y) = \text{Min}\) and set \(B = B \cup \{Y\}\). If \(B \neq \mathbb{F}_n^2\), then go to Step 2.

The above procedure generates a monotone function \(\hat{g}: \mathbb{F}_n^2 \to \mathbb{F}_2\). \(\square\)

**Proposition 4.3.** Suppose the monotone parallel system \((G', f')\) has a sequentialization \((G, f, \pi)\) where \(\pi = 12\cdots n\). For \(1 \leq i \leq n\), let \(A_i = \{Z : Z = f'_i(X), X \in \mathbb{F}_n^2\}\). Then \((G', f')\) can be sequentialized as a monotone SDS with respect to \(\pi\) if and only if \(f_i\) is monotone on \(A_i \subseteq \mathbb{F}_n^2\) for any \(1 \leq i \leq n\).

**Proof.** First, if \((G, f, \pi)\) is a sequentialization of \((G', f')\), then any SDS \((\hat{G}, \hat{f}, \pi)\) is a sequentialization of \((G', f')\) as long as \(\hat{f}_i\) and \(f_i\) agree on \(A_i\), that is, the dynamics does not depend on the behavior of \(f_i\) on \(\mathbb{F}_n^2 \setminus A_i\).

Secondly, if \(f_i\) is monotone on \(A_i\) then, by Lemma 4.2, there exists \(\hat{f}_i\), which is monotone on \(\mathbb{F}_n^2\) and agrees with \(f_i\) on \(A_i\). Hence \((G', f')\) can be sequentialized as a monotone SDS.

Finally, if \((G', f')\) can be sequentialized via the monotone SDS \((\hat{G}, \hat{f}, \pi)\), both \((G', f')\) and \((G, f, \pi)\) as well as \((G', f')\) and \((\hat{G}, \hat{f}, \pi)\) satisfy (*). Comparing the two systems of equations, we observe that \(\hat{f}_i\) and \(f_i\) agree on \(A_i\). By assumption, \(\hat{f}_i\) is monotone on \(\mathbb{F}_n^2\), and thus \(f_i\) is monotone on \(A_i\). \(\square\)
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