SINGULAR SETS AND PARAMETERS OF GENERALIZED TRIANGLE ORBIFOLDS

MIKHAIL BELOLIPETSKY

ABSTRACT. We study groups generated by three half-turns in the Lobachevsky 3-space and their quotient orbifolds. These generalized triangle groups are closely related to the arbitrary 2-generator Kleinian groups. Our main result is a classification of the singular sets of the generalized triangle orbifolds. We also present a method to obtain the parameters defining a generalized triangle group from the structure of the singular set of its quotient orbifold and illustrate it by examples.

1. Introduction

After the famous lectures of W. Thurston [Thurston 80], the study of 3-dimensional manifolds was focused on geometric and, in particular, on hyperbolic manifolds and orbifolds. An orientable hyperbolic 3-orbifold can be obtained as a quotient of the Lobachevsky space $\mathcal{H}^3$ by the action of a discrete subgroup of the group of orientation preserving isometries $\text{Isom}^+\mathcal{H}^3 = \text{PSL}(2, \mathbb{C})$. This way hyperbolic 3-orbifolds are related to the discrete subgroups of $\text{PSL}(2, \mathbb{C})$ called the Kleinian groups. Since a subgroup of $\text{PSL}(2, \mathbb{C})$ is discrete if and only if each of its 2-generator subgroups is discrete (see e.g. [Berdon 83]), the class of the 2-generator Kleinian groups gains a special significance. These groups and, in particular, arithmetic 2-generator Kleinian groups were studied by F. Gehring, C. Maclachlan, G. Martin, J. Montesinos, A. Reid and others (see [Brooks, Matelski 78, Conder et al. 02, Gehring, Martin 94, Gehring et al. 98, Gehring et al. 97, Hilden et al. 92, Jørgensen 76, Maclachlan, Martin 99] and the references therein). It appears that the underlying spaces and singular sets of the orbifolds corresponding to the 2-generator Kleinian groups may have complicated structure. In a special case of the groups with real parameters the orbifolds were studied by J. Gilman (see [Gehring et al. 01]), E. Klimenko [Klimenko 89, Klimenko 90]; and have been later classified by E. Klimenko and N. Kopteva (see [Klimenko, Kopteva 05]).

In this article we consider the groups of hyperbolic isometries generated by three half-turns in $\mathcal{H}^3$. If the axes of the half-turns pairwise intersect then the group is isomorphic to a Fuchsian triangle group, so the groups generated by three half-turns can be considered as a generalization of the Fuchsian triangle groups. These
generalized triangle groups are closely related to the arbitrary 2-generator Kleinian groups (see Section 5) but have much more trackable geometric structure. Thus, since all the three generators have fixed points (axes) in $\mathcal{H}^3$, the fundamental groups of the underlying spaces of the corresponding 3-orbifolds are always trivial, so the underlying space of such an orbifold is a 3-sphere $S^3$ by the Poincaré conjecture (now proved by Perelman).

Let us briefly describe the contents of the paper. In Section 2 we give a partially conjectural classification of the singular structures of generalized triangle orbifolds. Namely, we show (Theorem 2.2) that there are eight possible types of singularities and we conjecture that under some mild assumptions on the group there are no any other possibilities. The set of the generalized triangle groups can be parameterized by three complex numbers that correspond to the complex distances [Fenchel 89] between the axes of the generators. In Section 3 we use W. Fenchel’s technics to study this parameterization and its connection with the matrix representation of the group in $\text{SL}(2, \mathbb{C})$. Section 4 shows how to find the parameters of the group for the eight types of singularities from Theorem 2.2. This gives an algorithm which provides a system of three polynomial equations defining parameters. An explicit connection between generalized triangle groups and 2-generator Kleinian groups with their usual parameterization is given in Section 5. In Section 6 we present examples of generalized triangle orbifolds with different types of singularities and in Section 7 we consider an exceptional example for which Property RE defined in Section 2 does not hold.

The results of this paper were previously available as a preprint [Belolipetsky 01], the current version contains only some changes in the exposition. While working on the paper I enjoyed the hospitality of MPIM in Bonn. I would like to thank Prof. Alexander D. Mednykh for suggesting this problem for my PhD thesis and for many helpful discussions.

The current version of the paper, except for the grammatical corrections and a few small comments, dates back to February of 2002. It was submitted for a publication in a journal but the referee did not find it sufficiently interesting. I also thought that the results were not sufficiently strong and decided not to resubmit the paper. The purpose of the present update is to improve the readability of the text.

2. Structure of the singular sets

The underlying space of a generalized triangle orbifold is the 3-sphere $S^3$, so the singular sets of the orbifolds are knotted graphs in $S^3$ with the orbifold fundamental groups generated by three involutions. While depicting the singular sets we follow a common notation and write the indexes of singularities near the corresponding components omitting the index 2. We shall employ the Wirtinger presentation of the fundamental group of a knotted graph [Fox 61], adapted for the groups of
the orbifolds as in [Haefliger, Quach 84], so the elementary arcs of the graph will correspond to the words in the group generators.

**Definition 2.1.** We call by a generalized triangle group a discrete subgroup $\Gamma$ of $\text{Isom}^+\mathcal{H}^3$ generated by three involutions (half-turns). The corresponding quotient orbifold $O = \text{Isom}^+\mathcal{H}^3/\Gamma$ is called a generalized triangle orbifold. We say that a generalized triangle orbifold (and group) has Property RE if the rank of the fundamental group of the complement of its singular set in $S^3$ is equal to the rank of $\Gamma$.

We shall mainly consider the groups with Property RE. In some situations this allows us to use the fundamental group of the complement of the singular set instead of the orbifold group and saves from certain difficulties which can arise while working with the groups that have many involutions. An example of an exceptional generalized triangle group that does not have Property RE will be given in the last section. It is easy to see that the class of groups with Property RE is closed with respect to the Reidemeister moves applied to the singular sets, hence we can safely work with the singular graphs remaining inside the class.

**Theorem 2.2.** (The Classification Theorem) The singular sets in $S^3$ that have a stratified structure determined by the signature $S(l_1, m_1, n_1, l_2, m_2, n_2, \ldots, l_k, m_k, n_k)$ (Figure 7) together with the central part $D_1$ of one of the types $A$–$H$ on Figure 2 correspond to generalized triangle orbifolds with Property RE.

The orbifolds obtained from the types $A$, $B$, $C$ (with $1/t_1 + 1/t_2 + 1/t_3 > 1$) and $D$ (with $1/t_1 + 1/t_2 > 1/2$ and $1/t_2 + 1/t_3 > 1/2$) are compact; while the orbifolds corresponding to the types $C$ (with $1/t_1 + 1/t_2 + 1/t_3 = 1$), $D$ (with $1/t_1 + 1/t_2 = 1/2$ or $1/t_2 + 1/t_3 = 1/2$), $E$, $F$, $G$ and $H$ are non-compact.

**Proof.** The proof is straightforward. We can apply Wirtinger’s algorithm to see that in each of the cases the group has a presentation with the generators $a, b, c$ corresponding to the arcs labeled by the same symbols on Figure 1. Applying the same steps to the complement of the singular set in $S^3$ we see that the fundamental group of the complement can be generated by the elements corresponding to the same arcs, so Property RE is satisfied automatically. To distinguish between the compact and non-compact cases for each of the vertices of the singular graph we can check whether the boundary of the neighborhood of the vertex is a sphere or a plane, depending on the corresponding subgroup of the monodromy group. □

**Conjecture 2.3.** The classification theorem describes all possible singular sets of the hyperbolic generalized triangle orbifolds with Property RE.

In [Belolipetsky 01] this conjecture was a part of the classification theorem however the proof given there is incomplete and contains serious gaps. Let us briefly recall the argument since it still can be considered as a basis for Conjecture 2.3.
Figure 1. The structure of the singular set of signature $S(l_1, m_1, n_1, l_2, m_2, n_2, \ldots, l_k, m_k, n_k)$, $l_i, m_i, n_i \in \mathbb{Z}$ for $i = 1, \ldots, k$. The boxes denote the integral tangles with the inscribed number of (oriented) crossings.

Figure 2. Singular structures inside $D_1$ (here $t_i \in \mathbb{N} \cup \{\infty\}$).

We begin with a singular graph of a generalized triangle orbifold with Property RE. Let us choose three arcs of the graph which correspond to the generators of the group. Using the Reidemeister moves we can push the arcs outside of the disc which contains the plane projection of the singular set. This is the first step to obtain the stratified structure. Then we can show that since we are not allowed to
add more generators, the singular arcs can not be knotted with themselves, and we also can prove that by the Reidemeister moves it is possible to gather all the vertices of the singular graph in the central part of the diagram. This essentially leads to the stratified structure of the singular set depicted on Figure 1. It remains to prove that all the tangles on Figure 1 are integral or, equivalently, that the tangles of the form $1/n, n \geq 2$ cannot arise. This may seem obvious from the first view but I do not know how to prove this fact. It may require a different approach. Finally, it remains to understand the possible singular structures in the center of the diagram and here Property RE is essential, as it is confirmed by the examples from Section 7.

**Remark 1.** Some relatively simple singular structures from Theorem 2.2 correspond to non-hyperbolic orbifolds (see Examples 6C, 6D). All such geometric orbifolds can be found in [Dunbar 88] and we shall mainly be interested in the hyperbolic case.

**Remark 2.** By the rigidity argument, a generalized triangle orbifold is uniquely determined by the signature and the central part from the classification theorem, however, different signatures may correspond to the isomorphic orbifolds.

### 3. The Matrix Representation and the Parameters

In order to obtain a matrix representation of a group generated by three half-turns one has to fix the axes of the generators, which can be done by means of the complex distances between the hyperbolic lines (see [Fenchel 89, p. 67-70]). We define the complex distance between two oriented hyperbolic lines as follows: its real part equals the length of the common perpendicular to the lines and the imaginary part is given by the angle between the lines, taken from the first to the second line respecting orientations.

We call three complex distances between the axes of the generators by the parameters of the group $\Gamma = \langle a, b, c \rangle$ (this denotes a group generated by $a, b, c$ which has relations which are not specified). Let us note that starting from here by group $\langle a, b, c \rangle$ we mean the group with the given generators taken in a fixed order, so we work with the marked groups generated by $a, b, c$.

We have:

$$\text{par}(\langle a, b, c \rangle) = (\mu(a, b), \mu(a, c), \mu(c, b)) = (\mu_0, \mu_1, \mu_2),$$

where $\mu(\cdot, \cdot) = x + iy, x \in \mathbb{R}^+, y \in [0; 2\pi)$ denotes the complex distance between the oriented axes of the half-turns.

**Remark.** Since the directions of the axes of the half-turns are undefined, the parameters are defined up to a change of the orientation. We can canonically link the orientation of a hyperbolic line with the matrix of the half-turn in $\text{SL}(2, \mathbb{C})$ (see [Fenchel 89, p. 63]). This means that fixing the directions of the axes of generators is equivalent to choosing the inverse image of $\Gamma < \text{PSL}(2, \mathbb{C})$ under the canonical
projection \( P : \text{SL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C}) \). Following a usual agreement we shall call this inverse image by a representation of our group in \( \text{SL}(2, \mathbb{C}) \).

**Lemma 3.1.** The complex vector \( \text{par}(\langle a, b, c \rangle) \) determines the group \( \langle a, b, c \rangle \) uniquely up to a conjugation in \( \text{PSL}(2, \mathbb{C}) \), i.e. if \( \text{par}(\langle a, b, c \rangle) = \text{par}(\langle a', b', c' \rangle) \), then there exists \( h \in \text{PSL}(2, \mathbb{C}) \) such that \( a' = hah^{-1}, b' = hbh^{-1}, c' = hch^{-1} \).

*Proof.* Consider the axes of the half-turns \( a, b, c \) in \( \mathcal{H}^3 \). We endow the axes by the orientations and join them in couples by the common perpendiculars. The result is a hyperbolic right-angled hexagon (see Figure 3).

![Figure 3. The axes of the half-turns a, b, c together with their common perpendiculars define a right-angled hyperbolic hexagon.](image)

The vector \( \text{par}(\langle a, b, c \rangle) \) represents the three complex lengths of the sides of our hexagon which are common perpendiculars of the given axes. By a theorem from [Fenchel 89, p. 94] three pairwise non-adjacent sides define right-angled hexagon uniquely up to a simultaneous change of orientations of the other three sides. It means that any other hexagon defined by another group with the same parameters can be translated to the given one by a hyperbolic isometry. This isometry obviously defines the required conjugation \( h \in \text{PSL}(2, \mathbb{C}) \).

It may seem that we have missed the possibility of changing the orientations of the axes but we have already noted that the orientations affect only representation of the group in \( \text{SL}(2, \mathbb{C}) \) but not the isometries themselves. So in this situation we can change the orientations as it is needed. \( \square \)

The next step is to find some convenient representation of the group \( \langle a, b, c \rangle \) in \( \text{SL}(2, \mathbb{C}) \) and describe it in terms of \( \text{par}(\langle a, b, c \rangle) \). We denote the matrices corresponding to the half-turns about \( a, b, c \) by the capitals \( A, B, C \).

A half-turn about an oriented hyperbolic line \( m \) is represented by the matrix \( M \in \text{SL}(2, \mathbb{C}) \) of the form \( \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & -m_{11} \end{pmatrix} \), which is characterized by \( M^2 = -I \) or, equivalently, \( \text{tr}(M) = 0 \). This matrix \( M \) is called the normalized line-matrix of the line \( m \). By extending the transformation \( M \) to the boundary \( \overline{\mathcal{C}} = \partial \mathcal{H}^3 \), it is easy to find the set of its fixed points \( \text{fix}(M) \subset \overline{\mathcal{C}} \). Geometrically, the fixed points
\{z_1, z_2\} = \text{fix}(M)\) are the ends of the arc orthogonal to \(\mathbb{C}\) which represents the line \(m\) in the upper half-space model of \(\mathcal{H}^3\).

After a suitable conjugation of the group \(\langle A, B, C \rangle\) in \(\text{SL}(2, \mathbb{C})\) one can suppose that \(\text{fix}(A) = \{1/\beta; -1/\beta\}\) and \(\text{fix}(B) = \{\beta; -\beta\}\) with \(\beta \in \mathbb{C}, |\beta| \geq 1\).

We have:

\[
A = \begin{pmatrix} 0 & i/\beta \\ i\beta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i\beta \\ i/\beta & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & -c_{11} \end{pmatrix},
\]

where \(-c_{11}^2 + c_{12}c_{21} = 1\).

To find \(\beta\) and \(c_{ij}\) in terms of \(\text{par}(\langle a, b, c \rangle)\) we use the following basic formula allowing us to express the complex distance between the hyperbolic lines \(m_1, m_2\) in terms of the matrices \(M_1, M_2\) of the lines (see [Fenchel 89, p. 68]):

\[
\cosh(\mu(m_1, m_2)) = -\frac{1}{2} \text{tr}(M_1M_2).
\]

Denote \(\rho_k = -2\cosh(\mu_k), k = 0, 1, 2\). We have

\[
\rho_0 = \text{tr}(AB) = \text{tr} \left( \begin{array}{cc} -1/\beta^2 & 0 \\ 0 & -\beta^2 \end{array} \right);
\]

\[
\rho_1 = \text{tr}(AC) = \text{tr} \left( \begin{array}{cc} ic_{21}/\beta & -ic_{11}/\beta \\ ic_{11}/\beta & ic_{12}/\beta \end{array} \right);
\]

\[
\rho_2 = \text{tr}(BC) = \text{tr} \left( \begin{array}{cc} ic_{21}/\beta & -ic_{11}/\beta \\ ic_{11}/\beta & ic_{12}/\beta \end{array} \right);
\]

Hence we obtain a system of equations on \(\beta\) and \(c_{ij}\):

\[
\begin{align*}
-1/\beta^2 - \beta^2 &= \rho_0; \\
ic_{21}/\beta + ic_{12}/\beta &= \rho_1; \\
ic_{21}/\beta + ic_{12}/\beta &= \rho_2; \\
c_{11} &= i\sqrt{c_{12}c_{21} + 1}.
\end{align*}
\]

Solving these equations we can find all the parameters in our representation. Let us note that non-uniqueness of the solution of the system in general does not preoccupy us because different solutions will give different representations of the group \(\Gamma = \langle a, b, c \rangle\) which are conjugate in \(\text{SL}(2, \mathbb{C})\). The latter statement is a direct consequence of Lemma 3.1 since by the construction different solutions correspond to the same \(\text{par}(\Gamma)\). For the sake of concreteness we can always fix the analytic branches of the square roots in the following formulas.

We now can write down the representation of \(\langle a, b, c \rangle\) in \(\text{SL}(2, \mathbb{C})\):

\[
A = \begin{pmatrix} 0 & i/\beta \\ i\beta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i\beta \\ i/\beta & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & -c_{11} \end{pmatrix};
\]
\[ \beta = \sqrt{-\rho_0 + \sqrt{\rho_0^2 - 4}}, \]

(1)

\[ c_{21} = \frac{\rho_1/\beta - \rho_2/\beta}{i/\beta^2 - i\beta^2}, \quad c_{12} = -\frac{\rho_1/\beta + \rho_2/\beta}{i/\beta^2 - i\beta^2}, \quad c_{11} = i\sqrt{c_{12}c_{21} + 1}; \]

\[ \rho_k = -2 \cosh(\mu_k) \text{ for } k = 0, 1, 2. \]

To obtain the representation of \( \langle a, b, c \rangle \) in \( \text{PSL}(2, \mathbb{C}) \) one has to consider the image of the \( \text{SL}(2, \mathbb{C}) \)-representation under the canonical projection \( P : \text{SL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C}) \).

4. Deducing parameters from the singular structure

Knowing the structure of the singular set of a generalized triangle orbifold we can obtain a presentation of its fundamental group and a representation of the group in \( \text{SL}(2, \mathbb{C}) \). This representation is defined by three complex parameters \( \rho_0, \rho_1, \) and \( \rho_2 \), so it is possible to find the parameters corresponding to a given singular structure. In this section and in Section 6 we shall show how this can be carried out for the generalized triangle groups with Property RE.

Suppose we are given a singular set of a generalized triangle orbifold corresponding to one of the cases of Theorem 2.2. Using Wirtinger’s algorithm we can start from the given three arcs \( a, b, c \) and passing through the tangles obtain words in \( a, b, c \) corresponding to all the other arcs. It is clear that until we reach the central part of the diagram we do not obtain any equations on the generators, so all the words in the presentation of the group come from the central part of the diagram. Equations defining parameters are then obtained by taking traces of corresponding elements in \( \text{SL}(2, \mathbb{C}) \). With the help of the well known formulas

\[ \begin{align*}
\text{tr}(W_1W_2) &= \text{tr}(W_2W_1), \\
\text{tr}(W_1W_2W_1^{-1}) &= \text{tr}(W_2), \\
\text{tr}(W_1W_2) &= \text{tr}(W_1)\text{tr}(W_2) - \text{tr}(W_1^{-1}W_2), \\
\text{tr}(H^{-1}) &= -\text{tr}(H) \quad \text{if } H \text{ is a matrix of a half – turn},
\end{align*} \]

(2)

the traces of the expressions in \( A, B, C \) and their inverses can be written in terms of \( \text{tr}(AB) = \rho_0, \text{tr}(AC) = \rho_1, \text{tr}(BC) = \rho_2 \) and \( \text{tr}(ABC) \) (an explicit expression for \( \text{tr}(ABC) \) in terms of \( \rho_i \) can be obtained using (1)). Since we have three complex parameters, we require three independent equations to define them. In the remaining part of this section we shall give the presentations of the groups and corresponding equations for each of the types \( A–H \) from Theorem 2.2.
Group: \[ \langle a, b, c \mid a^2, b^2, c^2, w_1w_2, w_3w_4 \rangle, \]
equations:
\[
\begin{align*}
\text{tr}(W_1W_2) &= 2, \\
\text{tr}(W_3W_4) &= 2, \\
\text{tr}(W_5W_6) &= 2
\end{align*}
\]
(We denote by \( w_i \) the words in \( a, b, c \) and by \( W_i \) — corresponding products in \( \text{SL}(2, \mathbb{C}) \), \( i = 1, \ldots, 6 \).)

It follows from [Fox 61] that one of the equations \( w_1w_2, w_3w_4, w_5w_6 \) in the fundamental group is always a corollary of the two others. However, by the rigidity argument in the matrix group all the three words are independent and give three independent equations on the parameters (see Example 6A for an illustration).

Group: \[ \langle a, b, c \mid a^2, b^2, c^2, (w_1w_2)^t, (w_3w_4)^t, (w_5w_6)^t \rangle, \]
equations:
\[
\begin{align*}
\text{tr}(W_1W_2) &= 2, \\
\text{tr}(W_3W_4) &= -2\cos(\pi/t), \\
\text{tr}(W_5W_6) &= -2\cos(\pi/t).
\end{align*}
\]

Group: \[ \langle a, b, c \mid a^2, b^2, c^2, (w_1w_2)^{t_1}, (w_3w_4)^{t_2}, (w_5w_6)^{t_3} \rangle, \]
equations:
\[
\begin{align*}
\text{tr}(W_1W_2) &= -2\cos(\pi/t_1), \\
\text{tr}(W_3W_4) &= -2\cos(\pi/t_2), \\
\text{tr}(W_5W_6) &= -2\cos(\pi/t_3).
\end{align*}
\]

Group: \[ \langle a, b, c \mid a^2, b^2, c^2, (w_1w_2)^{t_1}, (w_3w_6)^{t_2}, (w_1w_2w_3)^{t_3} \rangle, \]
equations:
\[
\begin{align*}
\text{tr}(W_1W_2) &= -2\cos(\pi/t_1), \\
\text{tr}(W_5W_6) &= -2\cos(\pi/t_3), \\
\text{tr}(W_1W_2W_3) &= -2\cos(\pi/t_2).
\end{align*}
\]

It can be seen that the word \( W_1W_2W_3 \) has an odd length, which implies that in this case we require the expression for \( \text{tr}(ABC) \). In fact, this is the only type for which we may need this expression.
Group: \langle a, b, c \mid a^2, b^2, c^2, (w_1w_2)^2, (w_3w_6)^2 \rangle,

equations:
\begin{align*}
\text{tr}(W_1W_2) &= 0, \\
\text{tr}(W_5W_6) &= 0, \\
\text{tr}(W_3W_4) &= -2.
\end{align*}

The third equation follows from the fact that the multiple of two half-turns with parallel axes is a parabolic isometry, so $W_3W_4$ is a parabolic corresponding to the cusp. We choose the sign $'$ for the trace of the parabolic element because the axes of the half-turns have the same directions.

Group: \langle a, b, c \mid a^2, b^2, c^2, w_1w_2 \rangle,

equations:
\begin{align*}
\text{tr}(W_1W_2) &= 2, \\
\text{tr}(W_3W_4) &= -2, \\
\text{tr}(W_5W_6) &= -2.
\end{align*}

Group: \langle a, b, c \mid a^2, b^2, c^2, (w_3w_4w_5)^2 \rangle,

equations:
\begin{align*}
\text{tr}(W_3W_4W_5) &= 0, \\
\text{tr}(W_1W_2) &= -2, \\
\text{tr}(W_3W_4) &= -2.
\end{align*}

Here, as in the case $D$, we again have a word $W_3W_4W_5$ of an odd length, but since $\text{tr}(W_3W_4W_5) = 0$, in this case we can easily avoid using the explicit expression for $\text{tr}(ABC)$. Indeed, using formulas (2) we can present $\text{tr}(W_3W_4W_5)$ as $\text{tr}(ABC)P[\text{tr}(AB), \text{tr}(AC), \text{tr}(BC)]$, where $P[...]$ denotes a polynomial, and since $\text{tr}(ABC) \neq 0$ we obtain the equation $P[\text{tr}(AB), \text{tr}(AC), \text{tr}(BC)] = 0$. The same procedure can be applied for the type $H$ as well.

Group: \langle a, b, c \mid a^2, b^2, c^2, (w_1w_2)^t, (w_1w_2w_3)^2 \rangle,

equations:
\begin{align*}
\text{tr}(W_1W_2) &= -2\cos(\pi/t), \\
\text{tr}(W_5W_6) &= -2, \\
\text{tr}(W_1W_2W_3) &= 0.
\end{align*}
5. Relation to the two-generator Kleinian groups

Let $\Gamma_0 = \langle f, g \rangle$ be a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. There is a natural way to associate to $\Gamma_0$ a generalized triangle group $\Gamma$ which is equal to $\Gamma_0$ or contains it as a subgroup of index 2. The latter is controlled by the first homology group of $\Gamma$ having rank 2 or 3, and it can be seen that both cases are possible. It follows that such properties as discreteness and arithmeticity of two-generator subgroups of $\text{PSL}(2, \mathbb{C})$ can be studied using the generalized triangle groups.

The group $\Gamma$ is constructed as follows (cf. [Gilman 97]): If neither $f$ nor $g$ are parabolic then let $N$ be a common perpendicular to the axes of $f$ and $g$. In the case of a parabolic generator, the corresponding end of the line $N$ is the fixed point of the parabolic. Clearly, this construction uniquely determines the line $N$ for any two-generator Kleinian group $\Gamma_0 = \langle f, g \rangle$. Let $c$ be the half-turn about $N$. Then there exist such half-turns $a$ and $b$ that $f = ac$ and $g = cb$ (by [Fenchel 89, p. 47]). Therefore, we obtain a group $\Gamma = \langle a, b, c \rangle$ which contains $\Gamma_0$ as a subgroup of index at most 2.

Reciprocally, starting with the group $\Gamma$ generated by three half-turns $a$, $b$ and $c$ one can easily obtain its two generator subgroup $\Gamma_0 = \langle ab, ac \rangle$. Since $\Gamma = \Gamma_0 \cup a\Gamma_0 \ (a^2 = \text{id})$, the index of $\Gamma_0$ in $\Gamma$ is equal to 2 or 1 depending on whether or not the element $a$ is contained in $\Gamma_0$.

The set of two-generator Kleinian groups is usually parameterized by three complex numbers [Gehring, Martin 94]:

$$\beta(f) = \text{tr}^2(f) - 4, \quad \beta(g) = \text{tr}^2(g) - 4, \quad \gamma(f, g) = \text{tr}([f, g]) - 2.$$ 

A standard computation with traces shows that this parameters are related to our parameterization for the corresponding generalized triangle group by the following formulas:

$$\beta(f) = \rho_1^2 - 4, \quad \beta(g) = \rho_2^2 - 4, \quad \gamma(f, g) = \rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_0\rho_1\rho_2 - 4.$$ 

Since we have described the singular sets of the generalized triangle orbifolds, we can at least in theory present an algorithm which using equations from the previous section enumerates the parameters corresponding to the generalized triangle groups and discrete two-generator subgroups of $\text{PSL}(2, \mathbb{C})$. The main practical difficulty is that the complexity of the equations grows as the singular structures become more complicated. The other difficulty is that we have to cast away the redundant solutions of the equations which correspond to non-discrete subgroups of $\text{PSL}(2, \mathbb{C})$. In the next section we shall see how this can be done for the examples considered there. In general, if an orbifold is arithmetic the equations always have only one complex solution (up to a conjugation) and it defines the parameters, but in the non-arithmetic case there is no any general method to choose the right complex place.

Let us also note that the suggested algorithm shows only that our set of parameters is semi-recursive: we can enumerate the parameters of the discrete generalized
triangle groups with Property RE, but there is no known procedure to decide in a finite number of steps whether or not a given triple of complex numbers defines such a group.

6. Examples of the generalized triangle orbifolds.

We now consider examples for different types of singularities from Theorem 2.2 and find the parameters defining corresponding generalized triangle groups. We shall try to choose the examples which may deserve a particular interest or previously appeared in a different context. Some other examples of generalized triangle orbifolds can be found in [Belolipetsky 97] and [Belolipetsky 99].

Let \( \text{tr}(AB) = t_0 \), \( \text{tr}(AC) = t_1 \), \( \text{tr}(BC) = t_2 \).

By taking traces of the relations we obtain a system of equations on \( t_0 \), \( t_1 \) and \( t_2 \):

\[
\begin{aligned}
3t_0 - t_0^3 - 2t_0t_1^2 + t_0^2t_1^2 + 2t_0^2t_1t_2 - t_0t_1^2t_2 - 2t_0t_1t_2^2 + 2t_0^2t_2^2 - t_0^2t_2t_1^2 - t_0^2t_2t_1^2 - t_0^2t_1t_2^2 = 2, \\
3t_2 - t_2^3 - 2t_2t_0^2 + t_2^2t_0^2 + 2t_2t_0t_1 - t_2^2t_0t_1 - 2t_2t_0t_1^2 + 2t_2^2t_1^2 - t_2^2t_0^2t_1 - t_2^2t_0^2t_1 = 2, \\
3t_1 - t_1^3 - 2t_1t_2^2 + t_1^2t_2^2 + 2t_1^2t_2t_0 - t_1^2t_2t_0^2 - 2t_1t_2^2t_0 + 2t_1^2t_0 + t_1^2t_0^2 - t_1^2t_2t_0^2 - t_1^2t_2t_0^2 = 2.
\end{aligned}
\]

We are interested only in complex solutions of this system and by the symmetry of the singular graph all the three roots should be equal. The only such solutions are (approximately):

\[
(t_0, t_1, t_2) = (0.662359 \pm 0.56228i, 0.662359 \pm 0.56228i, 0.662359 \pm 0.56228i).
\]

\(^1\)We used the computer program <tracer>(C++, 242 lines) to calculate the trace polynomials for the generalized triangle groups.
So we have $\rho_0 = \rho_1 = \rho_2 = 0.662359... \pm 0.56228...i$. The signs of the imaginary parts correspond to the orientations of the generating arcs and do not affect the presentation of the group in $\text{PSL}(2, \mathbb{C})$.

The primitive polynomial of the root is $t^3 - t + 1$. Using the arithmeticity test for generalized triangle groups from [Belolipetsky 99] it can be easily verified that the orbifold is arithmetic. It is also interesting to note (see [Mednykh, Vesnin 98]) that the two-fold covering of this orbifold is the Weeks-Mateveev-Fomenko manifold — the hyperbolic 3-manifold of the smallest volume.

(Here we could also have used the symmetry of the singular graph from the beginning by initially assuming that $\text{tr}(AB) = \text{tr}(AC) = \text{tr}(BC)$. This would considerably simplify the calculations but we are mainly interested in demonstrating the general method.)

\begin{align*}
a_1 &= bab^{-1} \\
b_1 &= cbc^{-1} \\
c_1 &= aca^{-1} \\
a_{-1} &= c_1^{-1}ac_1 \\
b_{-1} &= a_1^{-1}ba_1 \\
a_{-1}^{-1}a_1 &= ac^{-1}a^{-1}ca^{-1}bab^{-1} \\
b_{-1}^{-1}c^{-1} &= ba^{-1}b^{-1}ab^{-1}c^{-1} \\
b_1c_1 &= cbc^{-1}aca^{-1}
\end{align*}

Equations on $t_0 = \text{tr}(AB)$, $t_1 = \text{tr}(AC)$, $t_2 = \text{tr}(BC)$:

\begin{align*}
t_0^2t_1^2 + t_0t_1t_2 - t_0^2 - t_1^2 + 2 &= 2, \\
t_0^2t_2^2 + t_0t_1t_2 - t_2 &= -2 \cos(\pi/n), \\
t_1^2t_2^2 + t_0t_1t_2 - t_2 &= -2 \cos(\pi/n).
\end{align*}

We shall consider only $n = 3$. Up to the symmetry and change of orientations, the system has only one complex solution which defines the complex place of the orbifold group:

\begin{align*}
\rho_0 &= \frac{i + \sqrt{3}}{2}, \\
\rho_1 &= \frac{i - \sqrt{3}}{2}, \\
\rho_2 &= 0.
\end{align*}

The squares of $\rho_0$, $\rho_1$ are algebraic integers and by the arithmeticity test [Belolipetsky 99] the group is arithmetic and defined over the field $\mathbb{Q}[\sqrt{-3}]$.

(For $n = 2$ and $n = \infty$ the group has no any complex places and the orbifold is not hyperbolic. For large values of $n$ the group has more than one complex place.
induced by the Galois automorphisms \( \cos(\pi/n) \to \cos(k\pi/n) \), \((k, n) = 1\), and so it is not arithmetic.)

\[
\begin{align*}
b_{-2} &= b^{-1}aba^{-1}b \\
b_{-1} &= aba^{-1} \\
b_{1} &= b_{-2}bb_{-2}^{-1} \\
b_{2} &= cb_{1}c^{-1} \\
a_{1} &= b_{-1}^{-1}ab_{-1} \\
c_{1} &= ac\!a^{-1} \\
c_{1}a_{1} &= acb^{-1}a\!b^{-1} \\
b_{-2}^{-1}c^{-1} &= b^{-1}ab^{-1}a\!b^{-1} \\
b_{2}a^{-1} &= cb^{-1}aba^{-1} \\
a_{1}^{-1}a^{-1}b^{-1}c^{-1} &= cb^{-1}aba^{-1}b\!a^{-1}b^{-1} \\
abla_{2} &= \nabla_{0} \\
\end{align*}
\]

Equations on \( t_{0} = \text{tr}(AB), t_{1} = \text{tr}(AC), t_{2} = \text{tr}(BC) \):

\[
\begin{align*}
t_{0}t_{2} + t_{1} &= 0, \\
t_{0}^{2}t_{2} + t_{0}t_{1} - t_{2} &= -2\cos(\pi/n), \\
(t_{0}^{2} - 3t_{0})(t_{0}t_{1}t_{2} + t_{1}^{2} + t_{0}^{2} - 2) + t_{0}t_{1}^{2} + t_{1}t_{2} - t_{0} &= 0.
\end{align*}
\]

The complex solution \((n > 3)\):

\[
\begin{align*}
\rho_{0}^{2} &= \frac{1}{2}(1 + 4\sin^{2}(\pi/n)) \left(1 \pm \sqrt{1 - \frac{4}{1+4\sin^{2}(\pi/n)}}\right), \\
\rho_{1} &= -2\cos(\pi/n)\rho_{0}, \\
\rho_{2} &= 2\cos(\pi/n).
\end{align*}
\]

(Here \(\rho_{2}\) is always a real number; for \(n = 2, 3\), \(\rho_{0}\) and \(\rho_{1}\) are also real, so the corresponding orbifolds are not hyperbolic. According to [Dunbar 88], for \(n = 2\) the orbifold admits the spherical structure and for \(n = 3\) it is Euclidean.)

As it was shown in [Rasskazov, Vesnin 99], the orbifolds considered here can be obtained as the quotient orbifolds of the Fibonacci manifolds \(F(2n, n)\) by their full groups of isometries. Hence using the arithmeticity test from [Belolipetsky 99] we can now easily find all arithmetic Fibonacci manifolds (see also [Hilden et al. 90]). The field of definition of the orbifold group is \(\mathbb{Q}[\rho_{0}^{2}, \cos^{2}(\pi/n)]\). It has exactly one complex place only if \(n = 4, 5, 6, 8, 12, \infty\); and it is a matter of a direct verification
that these cases satisfy the remaining conditions of the arithmeticity test. Thus, the orbifolds and corresponding Fibonacci manifolds are arithmetic for \( n = 4, 5, 6, 8, 12, \infty \).

\[
\begin{align*}
  c_{-2} &= ac_{-1}a^{-1} \\
  c_{-1} &= b^{-1}cb \\
  b_{-1} &= c_{-1}^{-1}b^{-1}c \\
  a_{1} &= c_{-1}^{-1}ac_{-2} \\
  b_{1} &= cbc^{-1} \\
  c_{1} &= b_{1}cb_{1}^{-1} \\
  a_{1}b_{-1}^{-1} &= ab^{-1}c^{-1}ab^{-1}cba^{-1} \\
  c_{-2}c_{1} &= ab^{-1}c^{-1}ba^{-1}cb^{-1}c^{-1} \\
  a^{-1}a_{1}b_{-1}^{-1} &= b^{-1}c^{-1}ab^{-1}cba^{-1}
\end{align*}
\]

Equations on \( t_0 = \text{tr}(AB), t_1 = \text{tr}(AC), t_2 = \text{tr}(BC) \):

\[
\begin{align*}
  t_0 &= 0, \\
  4t_2^2 - t_1^2t_2^2 - t_2^4 + t_1^2 - 2 &= 0, \\
  4t_1^2 - t_2^2t_1^2 - t_1^4 &= 4\cos^2(\pi/n).
\end{align*}
\]

The equation \( \text{tr}(AB) = 0 \) geometrically means that the axes of the half-turns \( a \) and \( b \) intersect orthogonally in \( \mathcal{H}^3 \).

The system has exactly one pair of complex-conjugate roots for any \( n > 2 \) (for \( n = 2 \) the orbifold is Euclidean), these roots give the values of the parameters \( \rho_0(=0), \rho_1, \rho_2 \) defining the orbifold group. Since the analytic formulas for the solutions are rather complicated we do not present them here. One can see that the field of definition \( \mathbb{Q}[\rho_1^2, \rho_2^2] \) of the group has exactly one complex place iff \( n = 3, 4, 6 \), and it is easy to check the other conditions of the arithmeticity test [Belolipetsky 99] for these values of \( n \). So we obtain that the orbifolds are arithmetic for \( n = 3, 4, 6 \).

These groups and orbifolds first appeared in [Helling et al. 95] as an infinite one-parameter family extending one of the three regular tessellations of \( \mathcal{H}^3 \). It is interesting to note that the corresponding family for the one of the two remaining regular
tessellations is also well known and consists of the Fibonacci manifolds considered in the previous example. For the third tessellation the question of constructing such a family is still open.

\[
a_1 = bab^{-1}
\]
\[
b_1 = a_1ba_1^{-1}
\]
\[
c_1 = aca^{-1}
\]
\[
a_2 = ca_1c^{-1}
\]
\[
c_1b^{-1} = aca^{-1}b^{-1}
\]
\[
a_2a^{-1} = cbab^{-1}c^{-1}a^{-1}
\]
\[
b_1c^{-1} = baba^{-1}b^{-1}c^{-1}
\]

Equations on \( t_0 = \text{tr}(AB), t_1 = \text{tr}(AC), t_2 = \text{tr}(BC) \):

\[
\begin{aligned}
    t_0t_1 + t_2 &= 0, \\
    t_0t_1t_2 + t_0^2 + t_1^2 - 2 &= 0, \\
    -t_0^2t_2 - t_0t_1 + t_2 &= -2.
\end{aligned}
\]

The solution:

\[
\rho_0^2 = \rho_1^2 = 1 + i, \quad \rho_2^2 = 2i.
\]

This orbifold first appeared in Adams 90 as the third hyperbolic 3-orbifold with a non-rigid cusp. We see that the orbifold is arithmetic with the field of definition \( \mathbb{Q}[i] \) and so its group is commensurable in \( \text{PSL}(2, \mathbb{C}) \) with the Picard group \( \text{PSL}(2, \mathbb{Z}[i]) \).
SINGULAR SETS AND PARAMETERS OF GENERALIZED TRIANGLE ORBIFOLDS

\[ a_1 = bab^{-1} \]
\[ a_2 = ca_1c^{-1} \]
\[ b_1 = cbc^{-1} \]
\[ b_2 = a_2b_1a_2^{-1} \]
\[ b_3 = b^{-1}b_2b \]
\[ c_{-1} = b^{-1}cb \]
\[ c_{-2} = b_3^{-1}c_{-1}b_3 \]
\[ b_{-1} = a_2^{-1}ba_2 \]
\[ c^{-2}a_2 = b^{-1}cbab^{-1}a^{-1}b^{-1}c^{-1}baba^{-1}b^{-1}c^{-1}bcbab^{-1}c^{-1} \]
\[ b_{-1}^{-1}c = cba^{-1}b^{-1}c^{-1}b^{-1}cbab^{-1} \]
\[ a^{-1}b_3 = a^{-1}b^{-1}cbaba^{-1}b^{-1}c^{-1}b \]

Equations on \( t_0 = \text{tr}(AB) \), \( t_1 = \text{tr}(AC) \), \( t_2 = \text{tr}(BC) \):

\[
\begin{aligned}
& t_0^2t_1^2t_2^2 + t_0t_1^2t_2(2 - 2t_2^2) + t_0t_1t_2(5 - 4t_2^2 + t_2^4 + t_0(-1 + t_2)^2) + t_0t_1(3t_2^4 - 2t_2^2 - 2t_2^4 + t_0(-1 + t_2)^2 + t_0(-1 + t_2)^2) + t_1(-3 + 7t_2^4 - 4t_2^4 + t_2(7t_2^4 - 3t_2^2)) = 2, \\
& t_2 + (t_1 + t_0t_2)(t_0 - t_1t_2 - t_0t_2^2) = -2, \\
& t_0 + (t_1 + t_2t_0)(t_2 - t_1t_0 - t_2t_0^2) = -2.
\end{aligned}
\]

The complex solution:

\[ \rho_1 = 2, \quad \rho_0 = \rho_2 = \frac{-1 \pm \sqrt{-3}}{2}. \]

This example previously appeared as an orbifold which has the sister of the figure-eight knot as a two-fold cover. As we see, the orbifold is arithmetic with the field of definition \( \mathbb{Q}[\sqrt{-3}] \) (again by the arithmeticity test from Belolipetsky 99).
In our usual notation \( t_0 = \text{tr}(AB), t_1 = \text{tr}(AC), t_2 = \text{tr}(BC) \) we obtain
\[
\text{tr}(A_1B_1C_1) = \text{tr}(ABC)(t_0t_1t_2 + 1) = 0.
\]
Since \( \text{tr}(ABC) \neq 0 \), it gives \( t_0t_1t_2 + 1 = 0 \), and so we have the following system of equations on the parameters:
\[
\begin{align*}
& t_0t_1t_2 = -1, \\
& t_0t_2^2 + t_1t_2 - t_0 = -2, \\
& t_0^2t_2 + t_1t_0 - t_2 = -2.
\end{align*}
\]

The complex solution:
\[
\rho_0 = \rho_1 = \rho_2 = \frac{1 \pm \sqrt{-3}}{2}.
\]

Since \( \rho_0 = \rho_1 = \rho_2 \), the orbifold has an order 3 symmetry which cyclically permutes the generators of the group. This symmetry can be easily seen on the spatial representation of the singular graph but it is lost on the plane projection. According to [Belolipetsky 99], this orbifold is one of the three arithmetic non-compact generalized triangle orbifolds with equal parameters (the remaining two orbifolds correspond to Type C).

**H.** We have tried more than 20 different structures for this case but did not succeed in finding any arithmetic examples. The central part structure implies rather strong restrictions on the orbifold: it should be non-compact and it is essentially asymmetric. Let us leave as an open problem to find an arithmetic example for Type H or to prove that there do not exist such examples.

7. **Examples of generalized triangle orbifolds without Property RE.**

Here we shall consider examples of generalized triangle orbifolds whose singular sets do not correspond to any of the eight types of singularities from Theorem [2.2].

Our basic example is well known, it is the Picard orbifold. It was first shown in [Brunner 92] that the Picard group can be generated by only two elements or by three involutions. Let us recall this presentation.

The singular set of the Picard orbifold is:

\[
\begin{align*}
b_1 &= cbc^{-1} \\
b_{-1} &= aba^{-1}
\end{align*}
\]
We shall not write down a matrix representation of the Picard group because it is well known and can be easily obtained similarly to the examples considered in the previous section. Our goal is only to show that this group can be generated by three involutions, so we shall make all the calculations in the group.

We have:

\[ \Gamma = < a, b, c, x | a^2, b^2, c^2, x^2, (xac)^2, (xabc)^3, (bax)^3, (abax)^2 > \].

\((xac)^2 = 1\) implies \(acx = xca\);
\((abax)^2 = 1\) implies \(xaba = abax\).

Now consider the word \((xabc)^3 = 1\):

\[(xabc)^3 = xabcxabcxabc = xabcxabaa cxabc = xabc abax xca abc = xabcabacbc; \]

\(x = abcabacbc\).

So \(\Gamma = < a, b, c | a^2, b^2, c^2, (abcabacbc)^2, (abcabacbcac)^2, (abacb)^3, (acabacbc)^2 > \).

Let \(\tilde{X}\) be a 3-manifold obtained as a complement to the singular set of the Picard orbifold in \(S^3\). We have \(H_1(\tilde{X}) \cong \mathbb{Z}^4\), so the rank of \(\pi_1(\tilde{X})\) is greater than 3, and hence the Picard group does not have Property RE.

The Picard orbifold \(\mathcal{O}\) is a non-compact orbifold with a non-rigid cusp. It follows that we can obtain an infinite series of compact generalized triangle orbifolds without Property RE by \((n, 0)\)-Dehn fillings on the cusp of \(\mathcal{O}\).

In \cite{Belolipetsky 01} the generalized triangle orbifolds without Property RE were called “exceptional”. In fact, the examples considered here have a very special structure and we expect that there are either no any or very few other examples of generalized triangle groups without Property RE.

**References**

[Adams 90] C. Adams, Noncompact hyperbolic 3-orbifolds of small volume, In: Topology’90, edited by B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann, de Gruyter, Berlin (1992), 1–15.

[Belolipetsky 97] M. Belolipetsky, Arithmetic properties of the groups of isometries generated by three involutions (in Russian), Master Thesis, Novosibirsk State University, Novosibirsk (1997), 30 pp.

[Belolipetsky 99] M. Belolipetsky, On arithmetic Kleinian groups generated by three half-turns, arXiv:math/9912071v1, 13 pp.

[Belolipetsky 01] M. Belolipetsky, Singular sets and parameters of generalized triangle orbifolds, arXiv:math/0103015v3, 21 pp.

[Berdon 83] A. Berdon, The geometry of discrete groups, Springer-Verlag, 1983.

[Brooks, Matelski 78] R. Brooks, J. P. Matelski, The dynamics of 2-generator subgroups of \(\text{PSL}(2, \mathbb{C})\), Riemann surfaces and related topics: Proc. of the 1978 Stony Brook Conf. (I. Kra and B. Maskit, eds), 65–71.

[Brunner 92] A. M. Brunner, A two-generator presentation for the Picard group. Proc. of Amer. Math. Soc., 115, N 1 (1992), 45–46.
[Conder et al. 02] M. D. E. Conder, C. Maclachlan, G. Martin, E. A. O’Brien, 2-generator arithmetic Kleinian groups. III, Math. Scand., 90, N 2 (2002), 161–179.

[Dunbar 88] W. Dunbar, Geometric orbifolds, Rev. Mat. Univ. Complutense Madr., 1 (1988), 67–99.

[Fenchel 89] W. Fenchel, Elementary geometry in hyperbolic space, De Gruyter, 1989.

[Fox 61] R. Fox, A quick trip through knot theory, Topology of 3-manifolds, Proc. 1961 Top. Inst. Univ. Georgia, Prentice Hall.

[Gilman 97] J. Gilman, A discreteness condition for subgroups of PSL(2, C), Contemp. Math., 211 (1997), 261–267.

[Gehring, Martin 94] F. Gehring and G. Martin, Commutators, collars and the geometry of Möbius groups, J. d’Analyse Math., 63 (1994), 175–219.

[Gehring et al. 01] F. Gehring, J. Gilman, G. Martin, Discrete groups with real parameters, Commun. Contemp. Math. 3, N 2 (2001), 163–186.

[Gehring et al. 97] F. Gehring, C. Maclachlan, G. Martin, A. Reid, Arithmeticity, Discreteness and Volume, Trans. Am. Math. Soc., 349, N 9 (1997), 3611–3643.

[Gehring et al. 98] F. Gehring, C. Maclachlan, G. Martin, Two-generator arithmetic Kleinian groups II, Bull. Lond. Math. Soc., 30, N 3 (1998), 258–266.

[Haefliger, Quach 84] A. Haefliger, N. D. Quach, Une presentation d’un groupe fundamental d’une orbifold, Astérisque, 116 (1984), 98–107.

[Helling et al. 95] H. Helling, A. C. Kim, J. L. Mennicke, Some honey-combs in hyperbolic 3-space, Comm. in algebra, 23 (1995), 5169–5206.

[Helling et al. 98] H. Helling, A. C. Kim, J. L. Mennicke, A geometric study of Fibonacci groups, J. Lie Theory, 8, N 1 (1998), 1–23.

[Hilden et al. 90] H. M. Hilden, M.-T. Lozano, J. M. Montesinos, The arithmeticity of the figure-eight knot orbifolds, In: Topology’90, edited by B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann, de Gruyter, Berlin (1992), 169–183.

[Hilden et al. 92] H. M. Hilden, M.-T. Lozano and J. M. Montesinos, A characterization of arithmetic subgroups of SL(2, R) and SL(2, C), Math. Nachr., 159 (1992), 245–270.

[Jørgensen 76] T. Jørgensen, On discrete groups of Möbius transformations. Amer. J. Math., 78 (1976), 739–749.

[Klimenko 89] E. Klimenko, Discrete groups in three-dimensional Lobachevsky space generated by two rotations, Sib. Math. J., 30 (1989), 95–100.

[Klimenko 90] E. Klimenko, Some remarks on subgroups of PSL(2, C), Q & A in General Topology, 8, N 2 (1990), 371–381.

[Klimenko, Kopteva 05] E. Klimenko, N. Kopteva, All discrete $\mathbb{RP}$ groups whose generators have real traces, Internat. J. Algebra Comput., 15 (2005), 577–618.

[Maclachlan, Martin 99] C. Maclachlan, G. Martin, 2-generator arithmetic Kleinian groups, J. Reine Angew. Math., 511 (1999), 95–117.

[Mednykh, Vesnin 98] A. Mednykh, A. Vesnin, Visualization of the isometry group action on the Fomenko-Matveev-Weeks manifold, J. Lie Theory, 8, N 1 (1998), 51–66.

[Rasskazov, Vesnin 99] A. Rasskazov, A. Vesnin, Isometries of hyperbolic Fibonacci manifolds, Sib. Math. J., 40 (1999), 9–22.

[Thurston 80] W. Thurston, The geometry and topology of 3-manifolds, Lecture Notes, Princeton University, 1980.

Sobolev Institute of Mathematics, Koptyuga 4, 630090 Novosibirsk, Russia, and Max Planck Institute of Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

E-mail address: mbel@math.nsc.ru