A $\sigma_2$ PENROSE INEQUALITY FOR CONFORMAL ASYMPTOTICALLY HYPERBOLIC 4-DISCS

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Abstract. In this paper, we consider conformal flat metrics on $\mathbb{R}^4$ with an asymptotically hyperbolic (AH) end and possible isolated conic singularities. We define a mass term of the AH end. If the $\sigma_2$ curvature has lower bound $\sigma_2 \geq \frac{3}{2}$, we prove an inequality relating the mass and contributions from singularities. We also classify sharp cases, which is the standard hyperbolic 4-space $\mathbb{H}^4$ when no singularity occurs. It is worth noting that our curvature condition implies non-positive energy density.

1. Introduction

In this paper, we prove a sharp mass inequality for certain asymptotically hyperbolic (AH) 4-manifolds. Our work is motivated by research works in mathematical general relativity and conformal geometry of $\sigma_k$ curvature.

Positive mass theory is one of the central problems in geometry. Studies of asymptotically flat manifolds lead to positive mass theorems established by Schoen and Yau [55, 56], and then Witten [65]. When certain minimal surfaces, or black holes, are present, Riemannian Penrose inequalities have been established by Huisken-Ilmanen [50] and Bray [4, 5, 6]. In particular, for both positive mass theorems and Penrose inequalities, boundary cases are well understood and can be determined as Euclidean spaces and Schwarzschild spaces, respectively. These fundamental geometric results are based on an important assumption that comes from the physics consideration. Namely, the positive energy density condition in general relativity leads to proper local curvature constraints, which, in the Riemannian case, indicates non-negativity of the scalar curvature.

For universe models with a negative cosmological constant, the corresponding mathematical theory has also been considered. See, for example [42, 66]. When restricted to the Riemannian case, it involves studies of asymptotically hyperbolic manifolds of dimension $n$. Mainly, rigidity and positive mass theorems can be properly stated and proved for AH manifolds. See, for example, Min-Oo [51], Anderson-Dahl [2], Wang [64], Chruściel-Herzlich [7]. The positive energy density condition is also required in these results, which is equivalent to the geometric condition that the scalar curvature has a negative lower bound.

Conformal geometry regarding the so-called $\sigma_k$ curvature is another source of our motivation. As a natural extension of the scalar curvature, $\sigma_k$ curvature was first studied by Viaclovsky [59]. It has been then extensively studied as important type of fully non-linear PDEs with significant geometric applications. See [11, 12, 8, 9, 14, 17, 28, 20, 21, 27, 38, 32, 33, 34, 35, 36, 37, 25, 26, 40, 41, 46, 47, 44].

H.F.’s works is partially supported by a Simons Foundation research collaboration grant. W.W.’s works is partially supported by the Initiative postdoctoral fund of China.
In particular, works of Chang-Gursky-Yang [10, 13] explore properties of $\sigma_2$ curvature in closed 4-manifolds and give a conformal characterization of 4-spheres. Majority of geometric application in this direction requires a so-called positive $\sigma_k$ cone condition. In particular, this condition implies that the scalar curvature is point-wise positive.

To introduce our results, let us first fix notations. Let $\{x^1, \cdots, x^4\}$ be the standard coordinate system on $\mathbb{R}^4$ and $g_E$ be the Euclidean metric. Let $r = \sqrt{\sum(x^i)^2}$ and $\theta \in S^3$ be the standard polar coordinate of $\mathbb{R}^4$. Denote $D = \{r < 1\}$ to be the unit ball. For future use, we also define

$$s = \log \left(\frac{1}{r}\right),$$

which is a defining function of $\partial D$, the boundary of $D$. We consider a conformal metric on $D$ as $g = \exp(2u)g_E$. Let $R_g$, Ric$_g$, and $A_g$ be the corresponding scalar curvature, Ricci tensor and Schouten tensor, respectively. Let $\sigma_2(g) = \sigma_2(g^{-1}A_g)$ be the second symmetric polynomial of eigenvalues of $A_g$ with respect to $g$. For example, for the standard hyperbolic space $\mathbb{H}^4$, we have

$$u(x) = s - \log \sinh s,$$

$$\text{Ric}_g = -3g,$$

$$A_g = -\frac{1}{2}g,$$

$$R_g = 6\sigma_1(g^{-1}A_g) = -12,$$

(1.1) $$\sigma_2(g^{-1}A_g) = \frac{3}{2}.$$

**Definition 1.** Let $p_1, \cdots, p_k \in D$ be $k$ distinct points, $k \geq 0$. Let $(M, g) = (D \setminus \{p_1, \cdots, p_k\}, \exp(2u(x))g_E)$ be a conformal metric on $D$. It is called a conformal asymptotically hyperbolic space of dimension 4 with cone-like singularities if the conformal factor $u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\})$ satisfies the following conditions:

1. If $k > 0$, for each $i \in \{1, \cdots, k\}$ there exists $\beta_i > 0$ such that $|x - p_i|^2|\nabla^k u(x) - \beta_i \log |x - p_i|| is $C^{\alpha_j}(B_\delta(p_i))$ for $j = 0, 1, 2$, and some $\alpha_j \in (0, 1), \delta > 0$;
2. Near $\partial D$ we have the following asymptotic behavior of $u$,

$$u(x) = s - \log \sinh s + s^4 f(\theta) + h(r, \theta),$$

where $f \in C^2(S^3)$, and for some positive constant $C$

$$\lim_{s \to 0} \frac{|h| + s|\nabla_r h|}{s^4} = 0, \lim_{s \to 0} |\nabla_r h|s^0 \leq C,$nabla_{\hat{\alpha}} h| \leq C.$$

Note that for any metric satisfying Definition 1

$$g = \left(1 - \frac{2}{1 - |x|^2}\right)^2[1 + 2(1 - |x|^4)f(\theta) + \hat{h}(r, \theta)]g_E,$$

where $\lim_{r \to 1} \frac{\hat{h}(r, \theta)}{(1 - |x|^2)^2} = 0$. Thus, it is asymptotically hyperbolic near the boundary $\partial D$. Our definition is similar to those defined in the literature. See, for example, [15, 64].
In our setup, we allow the existence of isolated conic singularities. For each singular point \( p_i \), the tangent cone of the manifold is a cone of angle \( (1 + \beta_i)|S^3| \).

Now we define a mass quantity for the hyperbolic end.

**Definition 2.** For \((M, g)\) satisfying Definition 1, the mass for \( M \) is the following quantity

\[
m(M, g) = m(M) = \frac{1}{|S^3|} \int_{S^3} f(\theta) d\theta,
\]

where \( d\theta \) is the standard volume form for the unit sphere \( S^3 \), and \(|S^3|\) is the corresponding volume.

**Remark 3.** In Wang [64] and Chruściel-Herzlich [7], the mass of a general AH manifold is defined as the Minkowski norm of certain vector constructed via asymptotic of the metric near the AH end. The positive mass inequality is then established assuming \( R_g \geq -n(n-1) \) and the manifold being spin. The mass that we have defined is similar but in a conformally flat setting, which is more restrictive. In fact, we choose our sign convention so that up to a positive constant, it is one particular component of the mass vector in [64]. We use our notations out of mathematical convenience.

In this paper, we discuss conformally flat AH spaces of dimension 4 with a \( \sigma_2 \) curvature positive lower bound condition. Our main result is the following

**Theorem 4.** Assume that \((M, g)\) satisfies Definition 1. If we further assume that

\[
\sigma_2(g^{-1}A_g) \geq \frac{3}{2},
\]

then we have the following

\[
-m(M, g) \geq F(\beta_1, \cdots, \beta_k) \geq 0,
\]

where

\[
F(\beta_1, \cdots, \beta_k) = \tilde{\beta}^2 (\tilde{\beta}+2)^2 + (\frac{4}{3}\tilde{\beta}+4)(\sum_{i=1}^k \beta_i^2 - \tilde{\beta}^2) \text{ with } \tilde{\beta} := \left( \sum_{i=1}^k \beta_i^3 \right)^{1/3}.
\]

In particular, when \( k = 1 \), or \( M \) has exactly one singular point, we have

\[
-m(M, g) \geq \frac{1}{20}(\beta_1 + 2)^2 \beta_1^2.
\]

If the equality in (1.3) holds, then \( u \) is rotationally symmetric and \( k = 1 \); Furthermore, \((M, g)\) can be identified as the Chang-Han-Yang model, which will be discussed in Section 2.

As a consequence, we have the following special case

**Corollary 5.** Conditions are given as in Theorem 4. If \( M \) is smooth without singular points, then

\[
m(M, g) \leq 0.
\]

In particular, the equality holds if and only if \((M, g)\) is the standard hyperbolic space, \( \mathbb{H}^4 \).

We make some comments regarding our results.

Positive mass problems related to \( \sigma_k \) curvature have been considered for both asymptotic flat and asymptotic hyperbolic manifolds under different settings. See Ge-Wang-Wu [18, 19, 20, 21], Ge-Wang-Wu-Xia [22] and Li-Nguyen [48]. Our definitions and results are different in flavor. Also, we focus only on the \( \sigma_2 \) curvature in dimension 4 case.
The most interesting feature of our results is our curvature assumption. A simple computation shows that our assumption \( \sigma_2(g^{-1}A_g) \geq \frac{1}{4} \) leads to the scalar curvature condition \( R_g \leq -12 \), which is exactly the opposite comparing to that posed in previous works of [64] and Chruściel-Herzlich [7]. In a vague sense, we are considering a class of AH manifolds with negative or non-positive energy density. Theorem 4 and Corollary 5 should be viewed as negative mass theorems under these assumptions, which are reasonable. Furthermore, it is interesting to interpret the contribution of isolated singularities, which are right hand side terms of (1.3) and (1.4). We will, however, leave any possible physics implication of our results to experts.

From a geometric point of view, we study metrics in the so-called negative cone, which means that in our settings the scalar curvature, \( R_g \), is strictly negative. Comparing to results in the positive cone case, there are relatively few works for the negative cone case. See [54, 33, 23, 31]. The difficulty for the negative cone is mainly due to the lack of interior \( C^2 \) estimate, which plays a significant role in fully nonlinear elliptic equations including the \( \sigma_k \) Yamabe problem. The counterexample of interior \( C^2 \) estimate has been constructed by Sheng-Trudinger-Wang [57]. Our result can be viewed as a necessary condition in further study of general \( \sigma_2 \) Nirenberg type problem in a similar setting.

Also, the geometry of sharp cases of our inequalities is first described in Chang-Han-Yang [8]. In particular, we are able to characterize the standard hyperbolic space, \( \mathbb{H}^4 \), among smooth conformal AH balls in dimension 4 with a positive \( \sigma_2 \) curvature condition. Comparing Corollary 5 to the Chang-Gursky-Yang’s conformal 4-sphere theorem [10], it is interesting to see that \( \sigma_2 \) curvature in dimension 4 carries particularly strong conformal geometric information to characterize space forms.

From an analytical point of view, our approach to prove Theorem 4 is heavily relying on our previous work [17], where the \( \sigma_2 \) Yamabe problem is studied for conic 4-spheres. Instead of \( \sigma_2 \) curvature being a positive constant, which is discussed in [17], we find out that in the current negative cone setup, \( \sigma_2 \) curvature positive lower bound condition (1.2) can be used to construct a quasi-local mass along level sets of the conformal factor, which is similar to geometric flow methods considered by in [50]. The monotonicity of the new quasi-local mass is established using the delicate divergence structure of the \( \sigma_2 \) curvature in dimension 4 and the iso-perimetric inequality for Euclidean spaces. Generalization to non-conformally flat and higher dimensional cases will be difficult but interesting.

The rest of the paper is organized as follows. In Section 2, we discuss the Chang-Han-Yang ODE model for constant \( \sigma_2 \) curvature metric and derive our main result in the special case where rotational symmetry is assumed. In Section 3, we follow construction in [17] to define a quasi-local mass along level sets of the conformal factor and prove its monotonicity. In Section 3, we study asymptotic behaviors of our quasi-local mass near naked singular points as well as near the hyperbolic end. As a consequence, we derive our main theorems.

The first named author would like to thank Xiao Zhang for valuable discussion on topics in general relativity. Both authors would like to thank Pedro Valentin De Jesus, Mijia Lai and Biao Ma for discussion. Both authors would thank anonymous referees for suggestions and corrections that have improved the accuracy and readability of our article.
2. Chang-Han-Yang model and related analysis

In this section, we first discuss the Chang-Han-Yang model of asymptotically hyperbolic manifolds with constant $\sigma_2$ curvature. Then, we briefly discuss a special case of our main result, when the conformal factor $u$ is rotationally symmetric with respect to the origin. And under this case, our non-linear problem gets greatly simplified. This serves also as the model case of our analysis. Note that the symmetry assumption on $u$ indicates the existence of possible one singular point at the origin. It also includes the case when no singularity exists.

First we have the following computation of $\sigma_2$ curvature under the rotational symmetry condition $u(x) = u(r)$. Using the variable $s = \log \frac{1}{r}$, $s > 0$, and $v(s) = u(s) - s$, we have the following local metric

$$g = \exp(2v)(ds^2 + dg_{S^3})$$

with

$$\sigma_2(g^{-1}A_g) = \frac{3}{2}(v_s^2 - 1)v_{ss}e^{-4v}.$$  

The Chang-Han-Yang model, first given in [8], is represented by the solution of $\sigma_2 = \frac{3}{2}$ under this setting, which can be written as

$$v_s^2 - 1)v_{ss} = e^{4v}.  \tag{2.1}$$

It is easy to see that (2.1) has the following first integral:

$$v_s^2 - 1) - e^{4v} = k^2,  \tag{2.2}$$

for some non-negative constant $k$. We summarize properties of Chang-Han-Yang solution in the following

Lemma 6. (2.2) has a solution, called the Chang-Han-Yang AH solution, such that

- (1) for $s \in [0, \infty)$, $v_s < -1$ and $v_{ss} > 0$;
- (2) when $s \to 0^+$, $v(s) \to \infty$, and $v_s \to -\infty$. Furthermore, when $s \to 0^+$, there is $k \geq 0$ such that

$$v(s) = -\log \sinh s - \frac{k^2}{20} s^4 + O(s^5);$$

- (3) when $s \to \infty$, $v(s) \to -\infty$, and $v_s \to -\sqrt{k + 1}$;
- (4) $(D, e^{2u}g_e)$ is asymptotically hyperbolic and has $m(M, e^{2u}g_e) = -\frac{k^2}{20}$.

Now we discuss a rotationally symmetric metric satisfying Definition 1. Assume that $\sigma_2 \geq \frac{3}{2}$, which means

$$v_s^2 - 1)v_{ss} \geq e^{4v}.  \tag{2.3}$$

Considering the fact that $g$ is asymptotically hyperbolic, it is clear to see that

$$v_s^2 > 1, \quad v_{ss} > 0.$$  

Noting the asymptotic behavior of $v_s$ as $s \to \infty$,

$$\lim_{s \to \infty} v_s(s) = -(\beta + 1),$$

where $\beta \geq 0$. Since $v_{ss} > 0$, for $s > 0$,

$$v_s \leq -\beta - 1 \leq -1.  \tag{2.4}$$
Lemma 7. Notations as above. If \( u \) be the critical set of \( \sigma_2 \) -Penrose inequality, then for any \( \sigma_2 \) -Penrose inequality, we have

\[
4v_s(v_s^2 - 1)v_{ss} \leq 4v_se^{4v_s}
\]

Integrating (2.6) and considering the boundary condition, we get

\[
(v_s^2 - 1)^2 - e^{4v_s} \geq \beta^2(\beta + 2)^2.
\]

Due to the rotational symmetry of \( v, f \) in (2) is now a constant. Thus,

\[
v(s) = -\log \sinh s + fs^4 + o(s^4).
\]

Now consider the following

\[
m(s) = \frac{1}{20}[(v'^2 - 1)^2 - e^{4v}]
\]

and its asymptotic behavior as \( s \to 0 \). A direct computation shows that

\[
\lim_{s \to 0^+} m(s) = \lim_{s \to 0} \frac{1}{20} \left\{ \left[ \frac{\cosh s}{\sinh s} + 4fs^4 \right] - e^{4fs^4} \right\} = -f = -m(M, g).
\]

Thus, from (2.5) we have the following

\[
(2.8) \quad -m(M, g) \geq \frac{\beta^2(\beta + 2)^2}{20}.
\]

It is then clear to see that when the equality in (2.8) holds, \( u \) has to satisfy (2.1). In particular, when \( \beta = 0 \), we have obtained the standard metric on \( \mathbb{H}^4 \).

3. Quasi-local mass via level sets

In this section we discuss general conformally flat AH spaces in dimension 4. In particular, on manifolds with positive \( \sigma_2 \) curvature lower bound, we define a quasi-mass quantity and prove that it is monotone. This construction has been actually discussed in our earlier work [17] in a different setup. In the asymptotically hyperbolic case, the corresponding Schouten curvature falls into the so-called negative cone, which means \( R_g < 0 \) everywhere. However, basic ideas in [17] still apply here.

We start by discussing about the critical set of the conformal factor. In this section, we suppose that \( u(x) \in C^2(D \setminus\{p_1, \ldots, p_k\}) \). Let

\[
C = \{x \in D \setminus\{p_1, \ldots, p_k\}; \nabla u(x) = 0\}
\]

be the critical set of \( u \).

Lemma 7. Notations as above. If \( \sigma_2(g^{-1}A_{g_0}) \) is never vanishing, then \( C \) has at most Hausdorff dimension 2.

Proof. For any \( P \in C \), we pick a local coordinate \( \{y^1, \ldots, y^4\} \) on an small open set \( U \) such that \( P \in U \subset \mathbb{R}^4 \), and denote \( u_i = \frac{\partial u}{\partial y^i} \) and \( u_{ij} = \frac{\partial^2 u}{\partial y^i \partial y^j} \). Since \( \sigma_2(g^{-1}A_g)(P) \neq 0 \) and \( |\nabla u|(P) = 0 \), we have \( \sigma_2(\nabla^2 u)(P) \neq 0 \). Thus, there exist \( i, j \in \{1, 2, 3, 4\}, i \neq j \), such that \( u_{ij}u_{ij} - u^2_{jj} \neq 0 \). Hence, \( \nabla u_i(P) \) and \( \nabla u_{ij}(P) \) are linearly independent. By the implicit function theorem, the set \( \{x, \frac{\partial u}{\partial y^i} = 0 \text{ and } \frac{\partial u}{\partial y^j} = 0\} \) is locally smooth and of dimension 2 near \( P \). Shrinking \( U \) if necessary, we have \( C \cap U \subset \{x, \frac{\partial u}{\partial y^i} = 0 \text{ and } \frac{\partial u}{\partial y^j} = 0\} \). We have thus concluded our proof.
For the rest of this section, consider a smooth function $u(x) \in C^2(D\setminus\{p_1, \cdots, p_k\})$ that satisfies conditions posed in Definition 1. We also assume that $\sigma_2(\tilde{g}_u^{-1}A_{p_k})$ is non-vanishing. We define $t_0 = \inf_{D\setminus\{p_1, \cdots, p_k\}} u$. Note that when $k \geq 1$, $t_0 = -\infty$. For any $t > t_0$, define the following
\[
S(t) = \{ x \in D\setminus\{p_1, \cdots, p_k\}, \ u(x) < t \},
\]
\[
S^*(t) = S(t) \cup \{ p_1, \cdots, p_k \}.
\]
It is clear that, by Definition 1, both $S(t)$ and $S^*(t)$ are non-empty, open and bounded. We fix the following notations for the rest of the article:
\[
\partial S(t) = \overline{S(t)} \setminus S(t),
\]
\[
\partial S^*(t) = S^*(t) \setminus S^*(t),
\]
\[
L(t) = \{ x \in D, \ u(x) = t \},
\]
\[
L(t)_0 = L(t) \setminus \mathcal{C}.
\]

Lemma 8. Notations and assumptions as above. For any $t > t_0$, $\partial S(t), \partial S^*(t)$ and $L(t)$ differ by 0-measure sets in 3-dimensional Hausdorff measure ($\mathcal{H}^3$) sense.

Proof. It is obvious that $\partial S(t) \setminus \partial S^*(t) = \{ p_1, \cdots, p_k \}$, which has $\mathcal{H}^3$ measure 0. Since $u$ is $C^2$ smooth, it is easy to see that
(3.1)
\[
\partial S^*(t) \subset L(t).
\]
(3.2)
\[
L(t)_0 \subset \partial S^*(t).
\]
By (3.1) and (3.2),
\[
(\partial S^*(t) \setminus L(t)) \cup (L(t) \setminus \partial S^*(t)) \subset \mathcal{C},
\]
which, by Lemma 7, has measure 0 in $\mathcal{H}^3$ sense. We have finished the proof. \qed

Remark 9. By Definition 1 for $t > t_0$, $S^*(t)$ is non-empty, open and bounded. By the isoperimetric inequality [52], $\partial S^*(t)$ has non-trivial 3-dimensional Minkovski content. By Lemma 7, $\mathcal{C}$ is locally a subset of 2-dimensional surface with trivial 3-dimensional Minkovski content. Thus, for all $t > t_0$, $L(t)_0 = L(t) \setminus \mathcal{C}$ has non-trivial 3-dimensional Minkovski content, which means that it is non-empty. Since $L(t)_0$ is locally a 3-dimensional hypersurface, it has non-trivial $\mathcal{H}^3$ measure.

We proceed to discuss a local coordinate near a generic point $P \in L(t)_0$ near which $L(t)$ is smooth. We first define one particular coordinate function
\[
y^4(Q) = \text{sgn}(u(Q) - t) \text{dist}(Q, L(t))
\]
for $Q$ near $P$. Note that this is well defined since $L(t)$ is smooth near $P$. We also define local normal coordinate functions $y^1, y^2, y^3$ on an open set $V \subset L(t)$ near $P$ and then extend them smoothly to an open set $U \subset D$. Thus, we have got a local coordinate system $\{ y^i \}, i = 1, \cdots, 4$ of $\mathbb{R}^4$ near $P$ such that $\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle |_V = 1$ and $\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle |_V = 0$ for $i = 1, 2, 3$.

We use $\nabla$ to denote the Levi-Civita connection of $g_E$ and write $u_i = \nabla_{\frac{\partial}{\partial y^i}} u$ and $u_{ij} = \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} u$. By the definition of $y^4$, $\frac{\partial}{\partial y^4} |_V = \frac{\nabla u}{|\nabla u|}$. We note that $u_{44}$ is independent of choices of $y^1, y^2$ and $y^3$ and well defined for generic points in $L(t)$. Let $\nabla_{ab}^L u$ be the Hessian of $u$ with respect to the induced metric on $L(t)$. In the following, $\alpha, \beta$ range from 1 to 3. By the definition of $L(t)$, we have $\nabla_{\alpha}^L u = 0$. 
Let $h_{\alpha\beta}$ be the second fundamental form of the level set $L(t)$ with respect to the outward normal vector $\frac{\nabla u}{|\nabla u|}$. We have the following Gauss-Weingarten formula

\begin{equation}
\nabla_{\alpha\beta} u = \nabla^L_{\alpha\beta} u + h_{\alpha\beta} u_4.
\end{equation}

We may now describe the Schouten tensor using our choice of local coordinates near $P$. Recall that for the Schouten tensor $A = A_g$,

\begin{equation}
A_{ij} = -u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2} \delta_{ij},
\end{equation}

and near $P$, we write $g^{-1}_E A_g$ locally as a symmetric matrix

\begin{equation}
g^{-1}_E A_g = \begin{pmatrix}
-|\nabla u|^2 - h_{\alpha\beta} \delta_{\alpha\beta} & -\nabla 41 u & -\nabla 42 u & -\nabla 43 u & -\nabla 44 u + \frac{|\nabla u|^2}{2} \\
-\nabla 41 u & -\nabla 42 u & -\nabla 43 u & -\nabla 44 u & \frac{|\nabla u|^2}{2} \\
\end{pmatrix}.
\end{equation}

For simplicity, we define a local symmetric $3 \times 3$ matrix

\[ \tilde{A}(P) := (-h_{\alpha\beta} |\nabla u| - \frac{|\nabla u|^2}{2} \delta_{\alpha\beta}). \]

We also define, for $L(u(P))_0$ at $P$, we define the unit normal vector $\nu = \frac{\nabla u}{|\nabla u|}$ and a direct computation shows that the corresponding mean curvature

\begin{equation}
H = \text{div}(\frac{\nabla u}{|\nabla u|}).
\end{equation}

For future use, we establish the following key point-wise results, which is a consequence of the asymptotic hyperbolic condition.

**Lemma 10.** Let $(M, g) = (D \setminus \{p_1, \cdots, p_k\}, \exp(2u(x))g_E)$ be a conformal asymptotically hyperbolic space of dimension $4$ with cone-like singularities defined as in Definition 7 if $\sigma_2(A_g) > 0$, then for any $P \in D \setminus \{p_1, \cdots, p_k\} \setminus C$, we have

\[ \text{div}(|\nabla u|^2 \nabla u) = 3|\nabla u|^2 [\nabla 44 u + \frac{H}{3} |\nabla u|] > 0, \]

\[ \sigma_1(\tilde{A}) = -H |\nabla u| - \frac{3}{2} |\nabla u|^2 < 0. \]

**Proof.** We may use (3.3) and (3.4) to establish identities by direct computation. Furthermore, we compute

\begin{equation}
0 < \sigma_2(A) = \sigma_2(\tilde{A}) + (-\nabla 44 u + \frac{|\nabla u|^2}{2}) \sigma_1(\tilde{A}) - \sum_{a=1}^{3} (\nabla a u)^2 \leq \sigma_2(\tilde{A}) + (-\nabla 44 u + \frac{|\nabla u|^2}{2}) \sigma_1(\tilde{A}).
\end{equation}

Note also that

\begin{equation}
\sigma_2(\tilde{A}) \leq \frac{\sigma_2^2(\tilde{A})}{3}.
\end{equation}
We then combine (3.6) and (3.7) to conclude

$$\sigma_2(A) \leq \frac{\sigma_1(\tilde{A}) \sigma_1(\tilde{A})}{3} + (-\nabla_44 u + \frac{|\nabla u|^2}{2}) \sigma_1(\tilde{A})$$

\[(3.8)\]

By Lemma 7, \(D \setminus \{p_1, \cdots, p_k\} \setminus C\) is open and connected, which means both factors at the right hand side of (3.8) do not change signs. Using condition (2) of Definition 1, we may directly verify that both quantities are negative when \(P\) is near the boundary of \(D\). Thus, they are always negative, which leads to inequality parts of the lemma.

Next, we define some integral quantities. From now on we use \(\int_{L(t)}^{}\), \(\int_{S(t)}^{}\) to represent \(\frac{1}{|S^3|} \int_{L(t)}^{}\), \(\frac{1}{|S^3|} \int_{S(t)}^{}\), respectively, where \(|S^3|\) is the volume of the unit 3-sphere and \(|B^4|\) is the volume of the unit ball in \(\mathbb{R}^4\). Note that \(\frac{|S^3|}{4} = |B^4|\).

We will use the standard Euclidean measure \(dx\), and its induced hyper-surface measure, \(dl = dl_t\) on \(L(t)_0\). We also omit them if no confusion arises. We now define following quantities for all \(t > t_0\),

\[A(t) = \int_{S(t)} e^{4u} dx,\]

\[B(t) = \int_{S(t)} dx,\]

\[C(t) = e^{4t}B(t).\]

We remark here that we use a slightly different sign convention here comparing to definitions given in [17], but the construction is essentially same. When \(t_0 = \inf u\) is finite, then \(u \in C^2(D)\) has no interior singular points, and we define \(A(t_0) = B(t_0) = C(t_0) = 0\). Thus, \(A, B, C\) are defined for all \(t \in [t_0, \infty)\).

**Lemma 11.** Notations as above. Functions \(A(t), B(t), C(t)\) are absolutely continuous.

**Proof.** We follow similar arguments in [16] [17]. For completeness, we sketch a proof here. By the co-area formula (see Lemma 2.3 in [3]) and Lemma 7 for any \(t_2 > t_1 \geq t_0\),

\[B(t_2) - B(t_1) = \frac{|C \cap u^{-1}(t_1, t_2)|}{|S^3|} + \int_{t_1}^{t_2} \int_{L(\tau)_0}^{} \frac{1}{|\nabla u|^2} dH_d\tau = \int_{t_1}^{t_2} \int_{L(\tau)_0}^{} \frac{1}{|\nabla u|^2} dH_d\tau,\]

which by the fundamental theorem of Lebesgue integral means that \(B(t)\) is absolutely continuous. Thus, \(C(t) = e^{4t}B(t)\) is also absolutely continuous. A similar argument shows that \(A(t)\) is absolutely continuous. \(\Box\)
Corollary 12. Notations and assumptions as above. There exists a dense set \( T \subset [t_0, +\infty) \) such that \([t_0, +\infty) \setminus T \) has Lebesgue measure 0 and for any \( t \in T \),

\[
A'(t) = e^{L(t)} \int_{L(t)} \frac{1}{|\nabla u|},
\]

\[
B'(t) = \int_{L(t)} \frac{1}{|\nabla u|},
\]

\[
C'(t) = 4C + A'.
\]

In particular, \( 0 < |L(t)| \) is finite for all \( t \in T \).

Proof. The existence of \( T \) and identities in the claim can be derived from the co-area formula and direct computations. See also [16, 17]. In particular, for \( t \in T \), \( B'(t) \) is finite. Noting that \( \frac{1}{|\nabla u|} \) is uniformly bounded from below on the closed set \( L(t) \supset L(t)_0 \), we conclude that \( |L(t)| = |L(t)| = \int_{L(t)} 1 \) is finite for \( t \in T \). \( \Box \)

Now we define a few more quantities. For any \( t \in T \), by Lemma 8, Remark 9 and Corollary 12, \( L(t)_0 = L(t) \setminus C \) has finite, non-trivial \( H^3 \) measure, and we define:

\[
z(t) = \left( \int_{L(t)_0} |\nabla u|^3 dl \right)^2,
\]

\[
\mathcal{D}(t) = \frac{1}{4} \int_{L(t)_0} [2H|\nabla u|^2 + 2|\nabla u|^3] dl.
\]

Lemma 13. Notations as above. Suppose \( \sigma_2(g^{-1}A_u) \) never vanishes. For all \( t_1, t_2 \in T \) such that \( t_1 > t_2 \), we have

\[
(3.9) \quad \mathcal{D}(t_1) = \mathcal{D}(t_2) + \int_{S(t_1) \setminus S(t_2)} \sigma_2(g^{-1}A_g) e^{4u} dx,
\]

\[
(3.10) \quad z^3(t_1) = z^3(t_2) + \int_{S(t_1) \setminus S(t_2)} \text{div}(|\nabla u|^2 \nabla u) \ dx.
\]

Proof. We follow a similar argument in [17]. For any \( t \in T \), by Remark 9 and Corollary 12, \( 0 < H^3(L(t)_0) < \infty \). By the generalized Gauss-Green theorem (see Theorem 1 in Chapter 5.8 of [12]), Lemma 7, Lemma 8, and the divergence structure of \( \sigma_2 \), we know that for \( t_1 > t_2, t_1, t_2 \in T \),

\[
\int_{S(t_1) \setminus S(t_2)} \sigma_2 \left( g^{-1}A_g \right) e^{4u} dx
\]

\[
= \frac{1}{4} \int_{L(t_1)_0} [2H|\nabla u|^2 + 2|\nabla u|^3] - \frac{1}{4} \int_{L(t_2)_0} [2H|\nabla u|^2 + 2|\nabla u|^3]
\]

\[
= |S^3| (\mathcal{D}(t_1) - \mathcal{D}(t_2)),
\]

where for each \( i = 1, 2 \), on \( L(t_i)_0 \), \( \nu = \frac{\nabla u}{|\nabla u|} \) and \( H|\nabla u|^2 = \text{div}(\frac{\nabla u}{|\nabla u|})|\nabla u|^2 \) is well defined. We have proved (3.9).
Similarly, by the generalized Gauss-Green theorem, for \( t_1 > t_2, t_1, t_2 \in T \), we compute that
\[
\int_{S(t_1) \setminus S(t_2)} \text{div}(|\nabla u|^2 \nabla u) = \int_{L(t_1)_0} |\nabla u|^2 < \nabla u, \nu > - \int_{L(t_2)_0} |\nabla u|^2 < \nabla u, \nu > \\
= \int_{L(t_1)_0} |\nabla u|^3 - \int_{L(t_2)_0} |\nabla u|^3 \\
= |S^3|(z(t_1)^3 - z(t_2)^3),
\]
where for each \( i = 1, 2 \), on \( L(t_i)_0, \nu = \frac{\nabla u}{|\nabla u|} \). We have proved (3.10). \( \square \)

We may now extend definitions of functions \( z \) and \( D \) to all \( t \in [t_0, +\infty) \). Again, if \( t_0 > -\infty \), we define \( z(t_0) = D(t_0) = 0 \). Consider any \( t > t_0 \) and \( t \not\in T \). Since \( T \) is dense in \([t_0, \infty)\), there exists \( t' \in T \) such that \( t' < t \). We define
\[
D(t) = D(t') + \int_{S(t) \setminus S(t')} \sigma_2(g^{-1}A_g)e^{4u} dx; \\
z(t) = \left[z(t')^3 + \int_{S(t) \setminus S(t')} \text{div}(|\nabla u|^2 \nabla u) dx \right]^\frac{1}{3}.
\]

**Remark 14.** Due to Lemma 13, the above definitions are independent of choice of \( t' \in T \). In particular, (3.9) and (3.10) now hold for all \( t_1 > t_2, t_1, t_2 \in [t_0, +\infty) \).

**Lemma 15.** Notations as above. Functions \( D(t) \) and \( z(t) \) are absolutely continuous. Furthermore, for a.e. \( t \in [t_0, +\infty) \)
\[
D'(t) = \int_{L(t)_0} \frac{\sigma_2(g^{-1}A_g)e^{4u}}{|\nabla u|}, \\
z'(t) = \frac{1}{3z^2} \int_{L(t)_0} (H|\nabla u| + 3\nabla_{44}u)|\nabla u|,
\]
where \( H \) is the mean curvature of \( L(t)_0 \).

**Proof.** We apply (3.10), the co-area formula, Lemma 7 and Lemma 10 to get, for any \( t_1 > t_2 \geq t_0 \)
\[
z_3(t_1) - z_3(t_2) \\
= \int_{S(t_1) \setminus S(t_2)} \text{div}(|\nabla u|^2 \nabla u) \\
= \int_{\{t_2 < u \leq t_1\} \cap \mathcal{C}} \text{div}(|\nabla u|^2 \nabla u) + \int_{t_2}^{t_1} \int_{(L_r)_0} \frac{\text{div}(|\nabla u|^2 \nabla u)}{|\nabla u|} dH d\tau \\
= \int_{t_2}^{t_1} \int_{(L_r)_0} \frac{\text{div}(|\nabla u|^2 \nabla u)}{|\nabla u|} dH d\tau = \int_{t_2}^{t_1} \int_{(L_r)_0} (H|\nabla u| + 3\nabla_{44}u)|\nabla u| dH d\tau \geq 0.
\]
Thus by the fundamental theorem of Lebesgue integral calculus, \( z^3(t) \) is absolutely continuous and non-decreasing. To prove that \( z(t) \) is absolutely continuous, we only need to show that
\[
z(t) > 0
\]
for all \( t > t_0 \). By Remark 9, \( z(t) > 0 \) for \( t \in T \), which, combined with the monotonic property of \( z \), leads to (3.12).

Using (3.9), a similar argument can be made to establish the absolute continuity of \( D(t) \) and compute its derivative.

\[ \square \]

**Lemma 16.** Notations as above. If we further assume that \( \sigma_2(g^{-1}A_g) \geq \frac{3}{2} \), then for a.e. \( t \in [t_0, +\infty) \), \( D'(t) \geq \frac{3}{2}A'(t) \).

**Proof.** This is a direct consequence of Corollary 12 and Lemma 15. \( \square \)

Finally we are ready to define our quasi-local mass.

**Definition 17.** For any \( t \in [t_0, \infty) \), we define

\[
m(t) = \frac{1}{5} \frac{2}{3} D(t) + \frac{4}{9} D(t)z(t) + \frac{1}{36} z^4(t) - C(t).
\]

**Proposition 18.** Let \((M, g)\) be a conformally flat asymptotically hyperbolic 4-manifold with singularities. Suppose \( \sigma_2(g_u^{-1}A_u) \) never vanishes in \( D \). Then \( m(t) \) is absolutely continuous.

**Proof.** This follows from Definition 17, Lemma 11 and Lemma 15. \( \square \)

We proceed to establish the following crucial monotonic result, which is a slight modification of Theorem 15 in [17].

**Theorem 19.** Let \((M, g)\) be a conformally flat asymptotically hyperbolic 4-manifold with possible singularities. We use notations given as above. Suppose \( \sigma_2(g_u^{-1}A_u) \geq \frac{3}{2} \). Then \( m(t) \) is non-decreasing with respect to \( t \). That is, we have a.e. \( t \in [t_0, \infty) \),

\[
m'(t) \geq 0.
\]

**Proof.** This proof is similar to that of Theorem 15 in [17], with some key changes. For completeness, we provide an argument here. By Proposition 18 we only need to compute \( m'(t) \) for generic \( t \) such that (3.9), (3.10) and formulae in Corollary 12 and Lemma 15 hold.

From local estimates (3.8) and the fact that \( \sigma_2(A) \geq \frac{3}{2}e^{4u} \), we derive the following using Cauchy inequality:

\[
\begin{align*}
\int_{L(t)} \sigma_1(\tilde{A}) \cdot |\nabla u| dl \int_{L(t)} &\frac{H}{3} |\nabla u| + \nabla_{44} u \cdot |\nabla u| \\
\geq & \left( \int_{L(t)} \sqrt{-\sigma_1(\tilde{A}) |\nabla u| \left( \frac{H}{3} |\nabla u| + \nabla_{44} u |\nabla u| \right)} \right)^2 \\
\geq & \left( \int_{L(t)} \sqrt{\frac{3}{2} e^{4u} |\nabla u|^2} \right)^2 \\
\geq & \frac{3}{2} e^{4u} \left( \int_{L(t)} |\nabla u|^2 \right)^2.
\end{align*}
\]

(3.13)
Noting that Corollary 12 leads to
\[(A')^2 \int_{L(t)} \sigma_1(\tilde{A})|\nabla u| \cdot \left| \frac{1}{3} \frac{d}{dt} (z^3) \right| \]
\[\geq \frac{3}{2} \left( \int_{L(t)} |\nabla u|^2 \cdot e^{8t} \right)^2 \int_{L(t)} \frac{1}{|\nabla u|^2} \]
\[\geq \frac{3}{2} e^{12t} \int_{L(t)} |\nabla u|^2 \left( \int_{L(t)} \frac{1}{|\nabla u|^2} \right)^2 \]
\[\geq \frac{3}{2} e^{12t} \left( L(t)^2 \right)^2 \frac{1}{|S^3|^4} \]
\[\geq \frac{3}{2} e^{12t} B(t)^{\frac{3}{4}} |B|^4 \frac{1}{|S^3|^4} \]
\[= \frac{3}{2} (4C(t))^3, \]
where the third inequality is due to Cauchy inequality and the fourth inequality holds because of the iso-perimetric inequality. By the inequality of arithmetic and geometric means, we then derive from (3.14)
\[(3.15) \quad 4C \leq \frac{1}{3} (2zA' + \frac{2}{3} z' \int_{L(t)} \sigma_1(\tilde{A})|\nabla u|).\]

Using Corollary 12 and Lemma 16, (3.15) then implies that
\[C' \leq A' + \frac{1}{3} \left( 2zA' - \frac{2}{3} z' \int_{L(t)} \sigma_1(\tilde{A})|\nabla u| \right) \]
\[\leq \frac{2}{3} D' + \frac{4}{9} zD' - \frac{2}{9} z' \int_{L(t)} \sigma_1(\tilde{A})|\nabla u| \]
\[= \frac{2}{3} D' + \frac{4}{9} (zD)' - \frac{4}{9} zD - \frac{2}{9} z' \int_{L(t)} \sigma_1(\tilde{A})|\nabla u| \]
\[= \frac{2}{3} D' + \frac{4}{9} (zD)' - \frac{4}{9} z' \left( -\frac{1}{4} z^3 \right) \]
\[= \frac{2}{3} D' + \frac{4}{9} (zD)' + \frac{1}{36} (z^4), \]
which is equivalent to \(m'(t) \geq 0.\) Noting that by Lemma 16, \(z(t) > 0,\) we may then use Lemma 10 effectively to get the second inequality above. \(\square\)

**Remark 20.** Our proof of Theorem 19 and that of Theorem 10 in [17] are very similar with two key different points. First, we assume only the \(\sigma_2 \geq \frac{3}{2}\) here while in [17], \(\sigma_2\) curvature is fixed as \(\frac{3}{2}.\) Second, we are working in the negative cone case here while in [17], the positive cone condition is assumed. We are thus working on different type of asymptotic profiles. The negative cone condition here plays a crucial role to deal with the partial differential inequality condition.

## 4. Proof of Main Result

In this section, we prove our main result. Due to Theorem 19, it is clear that we just need to estimate limits of our quasi-local mass \(m(t)\) as \(t\) approaches extreme values. When singularity exists, due to Definition 11 when \(t\) is very small, the level
set $L(t)$ is near the singular set $\{p_i, \ i = 1, \cdots, k\}$. Correspondingly, when $t$ is very large, the level set $L(t)$ is close to the boundary of disc $D$. Again, without loss of generality, we may work only on the generic $t$.

First, we define the following:

**Definition 21.** Let $\beta = (\beta_1, \cdots, \beta_k)$. Define

$$\tilde{\beta} := \left( \sum_{i=1}^{k} \beta_i^3 \right)^{1/3}$$

and

$$F = F(\beta_1, \cdots, \beta_k) := \frac{1}{20} [\tilde{\beta}^2 (\tilde{\beta} + 2)^2 + \frac{8}{3} \tilde{\beta} + 4] \left( \sum_{i=1}^{k} \beta_i^2 - \tilde{\beta}^2 \right).$$

It particular, if $k = 1$, we have

$$F = \frac{1}{20} \beta_1^2 (\beta_1 + 2)^2.$$

When $k = 0$, we define $F = 0$.

We now state the following

**Theorem 22.** If $k \geq 1$, and $u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\})$ satisfies Condition (1) in Definition 1, we have the following

$$\lim_{t \to -\infty} m(t) = \frac{1}{20} F(\beta) \geq 0;$$

If $u(x) \in C^2(D)$, and $t_0 = \inf_D u$, we have

$$\lim_{t \to t_0} m(t) = 0.$$

The proof of Theorem 22 follows closely a similar argument in [17] with some subtle changes.

With the asymptotic of $u$ given near singular points, using the divergence structure of $\sigma_2$ curvature, we may prove that $\sigma_2(A_g)$ is locally integrable. Then, the argument in [17] may be used to prove Theorem 22. Here, we present an alternative proof which is more direct without using divergence properties of $\sigma_2(A_g)$.

From now on, we use $C$ to denote a universal constant that depends only on $f$, $h$ and other universal constants. We write

$$J = K + O(|1 - r_1|^{k})$$

to mean that for quantities $J$ and $K$, there exists a constant $C$ such that $|J - K| \leq C|1 - r_1|^{k}$. We write $K = o(1)$ to mean $\lim_{s \to 0} |K| = 0$.

First, near each singular point we have the following

**Lemma 23.** Denote $p_l = (p^1_l, p^2_l, p^3_l, p^4_l)$ for $1 \leq l \leq k$. Assume that $u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\})$ satisfies Condition (1) in Definition 1 and we use notations given as above. We have the following derivative estimates: for $i, j \in \{1, 2, 3, 4\}$, as $|x - p_l| \to 0$,

$$u_i(x) = \frac{\beta_i}{|x - p_l|^2} (x^i - p^i_l) + o\left(\frac{1}{|x - p_l|}\right),$$

$$u_{ij}(x) = \beta_i \frac{\delta_{ij}}{|x - p_l|^2} - 2\beta_i \frac{(x^i - p^i_l)(x^j - p^j_l)}{|x - p_l|^4} + o\left(\frac{1}{|x - p_l|^2}\right).$$
Let $\Omega$ in Definition (1) and standard polar coordinate computations. By Lemma 23 and Lemma 24, we have

$$H(x) = \frac{3}{|x - p_i|} + o\left(\frac{1}{|x - p_i|}\right),$$

where $H(x)$ is the mean curvature of level set $\{x, u(x) = t\}$ near $p_i$ and $t$ is sufficiently negative.

The proof of Lemma follows a similar argument in Lemma 7 of [17]. Note that, local asymptotic properties given in Definition 1 is sufficient to carry through computations and establish identities above. We omit details of the proof here.

To present our next lemma, we further fix some notations. Assume that $k \geq 1$. Let $\Omega_i$ be connected small domain in $\mathbb{R}^4$ such that: $p_l \in \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for any $i \neq j$. Define, for $t$ sufficiently negative, $L_i(t) = L(t) \cap \Omega_i$ which is closed. We localize the geometry near each singular point.

Fix $l \in \{1, \cdots, k\}$. We use a local polar coordinate system near $p_l$. That means, any $x \in \Omega_l$ can be written as $x = r_l \theta$, where $r_l = |x - p_l|$ for $l = 1, \cdots, k$ and $\theta \in S^3$. Let $\pi_l : \mathbb{R}^4 \setminus \{p_l\} \to S^3 : x \to \pi_l(x) = \frac{x - p_l}{r_l}$. Then we have the Euclidean volume form written as $dx = r_l^3 dr_l \wedge \pi_l^*(d\theta)$. For $t$ sufficiently negative, let $i_{l,t}$ be the inclusion map $i_{l,t} : L_i(t) \to \Omega_l \subset \mathbb{R}^4$. Let $dl_{l,t}$ be the volume form of $L_i(t)$ and $n_{l,t}$ be the outward normal vector of $L_i(t) \subset \mathbb{R}^4$. We then have

$$dl_l = i_{l,t}^*(i(n_{l,t})dx),$$

where $i$ is the contraction map.

We may now present the following

**Lemma 24.** Assume that $u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\})$ satisfies Condition (1) in Definition [17] and we use notations given as above. Let $n_{l,t} = \frac{x - p_l}{|x - p_l|}$. We have

$$dl_l = i_{l,t}^*(i(n_{l,t})(dx))(1 + o(1)) = i_{l,t}^*\pi_l^*(r_l^3 \, d\theta)(1 + o(1)).$$

**Proof.** A direct consequence of (4.22) is that $|n_{l,t} - n_{l,t}^*| = o(1)$ as $t \to -\infty$. Noting also that $dx = r_l^3 dr_l \wedge \pi_l^*(d\theta)$, we get our volume form estimate by Condition (1) in Definition [11] and standard polar coordinate computation.

We are now ready to prove Theorem [22]

**Proof.** By Lemma [23] and Lemma [24]

$$\lim_{t \to -\infty} \int_{L_i(t)} |\nabla u|^3 \, dl_l = \lim_{t \to -\infty} \int_{L_i(t)} \frac{1}{r_l^3} \cdot (|\nabla u| r_l)^3 \, dl_l$$

$$= \beta_1^3 \lim_{t \to -\infty} \int_{L_i(t)} \frac{1}{r_l^3} i_{l,t}^* \pi_l^*(r_l^3 \, d\theta)$$

$$= \beta_1^3 \int_{S^3} d\theta = \beta_1^3.$$

(4.6)

Similarly, using (4.4), we get

$$\lim_{t \to -\infty} \int_{L_i(t)} H|\nabla u|^2 \, dl_l = 3\beta_1^2.$$  

(4.7)
Theorem 25 then follows directly from Definition 17 and 18 when \( k \geq 1 \).
For \( k = 0 \), the proof is similar and simpler since \( u \) is smooth. We have thus finished the proof.

Second, we discuss the limit of quasi-mass as \( t \to +\infty \). It is clear that this corresponds to the limit when \( r = |x| \to 1^- \). Our result is summarized in the following

**Theorem 25.** Let \((M, g)\) be a conformally flat asymptotically hyperbolic 4-manifold with possible singularities. With notations given as in Section 1 and Section 2, we have

\[
\lim_{t \to \infty} m(t) = -m(M, g) = -\frac{1}{|S^3|} \int_{S^3} f(\theta) d\theta.
\]

It is clear that our main results, Theorem 4 and Corollary 5 are then consequences of Theorems 19, 22 and 25. In other words, the quasi-local mass connects the information of the mass of the manifold, \( m(M) \), and local geometric information of singular points. In particular, when no singularity exists, this gives the non-positive estimate of \( m(M) \). For the sharp case, we may examine the proof of Theorem 19, where all inequalities become equalities. In particular, the iso-perimetric inequality has to be sharp. This leads to obvious geometric and analytical consequences that all functions involved have to be rotationally symmetric and \( \sigma_2(g^{-1}A_g) = \frac{3}{2} \). Thus, we have obtained the Chang-Han-Yang model case. In the sharp case when no singularity exists, we have obtained the standard hyperbolic space \( \mathbb{H}^4 \).

The rest of the section is now devoted to the proof of Theorem 25. By definition, it is clear that as \( t \to +\infty \), level set \( L(t) \) is convergent to \( \partial D \) in the Gromov-Hausdorff sense. We will analyze limits of geometric quantities during this procedure in detail.

For \( x = (x^1, \cdots, x^4) \in D \), we also use the corresponding polar coordinate \( x = r \theta \) where \( r = |x| \) and \( \theta \in S^3 \). Fix a \( t \in \mathbb{R} \) such that \( L(t) \) is smooth and pick \( x \in L(t) \). We define

\[
w(x) = w(r) = \log \frac{2}{1 - r^2} = s - \log \sinh s.
\]

By Definition 1

\[
u(x) = u(r, \theta) = w(r) + f(\theta)s^4 + h(x),
\]

where \( h(x) = o(s^2) \). We denote \( F(r, t, \theta) := t - (w(r) + f(\theta)s^4 + h(x)) \). Then \( F(r, t, \theta) = 0 \) on \( L(t) \). By Condition (2) in Definition 1, we see that

\[
F_r = -w_r - 4f(\theta)s^3s_r - h_r \\
= -\frac{2r}{1 - r^2} - \frac{4f(\theta)(\ln r)^3}{r} - h_r \neq 0
\]

near \( r = 1 \). By the implicit function theorem, we may present \( r \) as a local \( C^2 \) function of \( t \) and \( \theta \) near \( r = 1 \). We may then write \( r = r(t, \theta) \). We also define a rotationally symmetric comparison function

\[
u_1(x) = u_1(r) = w(r) + (\ln r)^4 \int_{S^3} f(\theta) d\theta.
\]

It is clear that \( u_1 \) satisfies similar asymptotic behavior as that of \( u \). Thus, we may, at least when \( s \) is small enough, define \( r_1 = r_1(t) \) to be the unique value such that
We get \( s' = |\ln r_1| \).

It is clear that \( s' \) is dependent only on \( t \) and independent of choice of \( \theta \). The following limits are clear

\[
\lim_{t \to \infty} r_1(t) = 1, \quad \lim_{t \to \infty} r(t, \theta) = 1.
\]

Furthermore, \( s \) and \( s' \) are bounded by \(|1 - r_1| \) and \(|1 - r_1| \), respectively.

We first establish the following basic estimates:

**Lemma 26.** Assume that \( u(x) \in C^2(D \setminus \{p_1, \ldots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. For \( t >> 1 \), we have,

\[
1 - r_1 = O(1 - r) = O(s'),
\]

\[
1 - r = O(s'), \quad \ln(r) = O(s'),
\]

\[
r - r_1 = O(s'^5).
\]

**Proof.** With a fixed \( t \) that is large enough, we have

\[
u_1(r_1) = t = u(r, \theta),
\]

which, according to (4.8) and (4.9), implies that

\[
-ln(1-r_1^2) + 2 + \int_{S^3} f(\theta) d\theta (\ln r_1)^4 = -ln(1-r^2) + 2 + f(\theta)(\ln r)^4 + h(x).
\]

We first claim that \(|r - r_1| \leq C|1 - r_1| \) for some positive constant \( C > 0 \). In fact,

\[
-ln(1-r_1^2) + \ln(1-r^2) = f(\theta)(\ln r)^4 - \int_{S^3} f(\theta)d\theta (\ln r_1)^4 + h(x) = o(1).
\]

Noting that

\[
\ln(1-r^2) - \ln(1-r_1^2) = \ln(1 + \frac{r_1^2 - r^2}{1 - r_1^2}),
\]

we get \( \frac{r_1^2 - r^2}{1 - r_1^2} = o(1) \), which implies that

\[
|r_1 - r| = o(1)(1 - r_1^2) \leq o(1)|1 - r_1|.
\]

Similarly, we obtain \(|r_1 - r| \leq o(1)|1 - r| \). Thus, \( 1 - r = 1 - r_1 + r_1 - r \leq C(1 - r_1) \),

Also \( 1 - r_1 \leq C(1 - r) \).

Observing that \(|\ln r_1| = s' \) and \( s = |\ln r| = O(1 - r) = O(1 - r_1) = O(s') \), we get

\[
f(\theta)(\ln r)^4 - \int_{S^3} f(\theta)d\theta (\ln r_1)^4 + h(x)
\]

\[
= \ln(1-r^2) - \ln(1-r_1^2)
\]

\[
= \int_{r_1}^{r} d\ln(1-\kappa^2) \frac{d\kappa}{dk} = -2\kappa_0 \frac{1}{1 - \kappa_0^2} (r - r_1)
\]

with \( \kappa_0 \) between \( r \) and \( r_1 \), which means that \( 1 - \kappa_0^2 = O(s) = O(s') \). We get from (4.13) that

\[
|r - r_1| = O(s'^5).
\]
It is clear that due to (4.14), we have $O(s^k) = O(s^k)$. In the following, we do not distinguish them.

For future use, we define

$$
\varepsilon(t, \theta) := r(t, \theta) - r(t, \theta),
$$

and obtain a sharper estimate of $\varepsilon$. To simplify the notation, we define

$$
\tilde{f}(\theta) = f(\theta) - \int_{S^3} f,
$$

which leads to

$$
\int_{S^3} \tilde{f} = 0.
$$

We now present the following consequence of Lemma 26:

**Corollary 27.** Assume that $u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\})$ satisfies Condition (2) in Definition 1 and we use notations given as above. We have

$$
\varepsilon(t, \theta) = -\frac{(1 - r^2)}{2r_1} \tilde{f}(\theta) s^4 + o(s^5).
$$

**Proof.** By Lemma 28,

$$
f(\theta)(\ln r)^4 - \int_{S^3} f(\theta)d\theta(\ln r_1)^4
= (\ln r)^4(f(\theta) - \int_{S^3} f(\theta)d\theta) + \int_{S^3} f(\theta)d\theta ((\ln r)^4 - (\ln r_1)^4)
= \tilde{f}(\theta)(\ln r)^4 + \int_{S^3} f(\theta)d\theta(\ln r - \ln r_1)((\ln r_1)^4 + (\ln r_1)^3 \ln r + \ln r_1(\ln r)^2 + (\ln r)^3)
= \tilde{f}(\theta)(\ln r)^4 + \int_{S^3} f(\theta)d\theta(\frac{r - r_1}{r}) + O(s^5) + O(s^3)
= \tilde{f}(\theta)(\ln r)^4 + O(s^8).
$$

Also $\ln(1 - r^2) - \ln(1 - r_1^2) = \ln(1 + \frac{r^2 - r^2}{1 - r_1^2}) = \frac{r^2 - r^2}{1 - r_1^2} + O(s^8)$. Noticing that $h = o(s^4)$, by (4.11) and (4.10), we get the conclusion. □

We now proceed to obtain derivative estimates.

**Lemma 28.** Assume that $u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\})$ satisfies Condition (2) in Definition 1 and we use notations given as above. For $r = r(t, \theta)$, we have $r_0 = O(s^5)$ and $r_{\theta \theta} = O(s^5)$.

**Proof.** Taking derivatives to both sides of (4.11), we have

$$
(4.16) \quad 0 = \frac{2r}{1 - r^2} r_0 + \nabla \theta f \cdot s^4 - \frac{f}{r} s^3 r_0 + h_r r_0 + h_\theta.
$$

Thus, noting the asymptotic behavior of $h$ and $h_\theta$ by Definition 11, we get $r_\theta = O(s^5)$. Taking further derivative to (4.16), we obtain

$$
0 = \left(\frac{2r}{1 - r^2}\right) r_\theta^2 + \frac{2r}{1 - r^2} r_{\theta \theta} + \nabla \theta f \cdot s^4 - \frac{f}{r} s^3 r_\theta - \frac{f}{r} s^3 r_{\theta \theta} + h_r r_\theta + h_{rr} r_\theta + h_{\theta \theta}.
$$
Thus

\[ r_{\theta \theta} = O(s)[O(s^4) + h_{rr}r^2 + h_{\theta \theta}] = O(s^5) \]

by Condition (2) in Definition 1. \qed

**Lemma 29.** Assume that \( u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. By the asymptotic behavior of \( u \), we have

\[
\nabla_r u = \nabla_r w + O(s^3) = \frac{1}{s}[1 + O(s)],
\]

\[
\nabla_\theta u = s^4 \nabla_\theta f + o(s^4) = O(s^4),
\]

\[
\nabla_{\theta,\theta} u = O(s^4),
\]

(4.17)

\[ |\nabla u| = |u_r|(1 + O(s^5)). \]

The proof of Lemma 29 is straightforward so we omit it. Lemma 29 implies that we may approximate \( |\nabla u| \) by \( |\partial_r u| \).

To discuss geometry near \( L(t) \), we define following maps: first let \( \pi \) be the projection map \( \mathbb{R}^4 \setminus \{O\} \to S^3 : x \to \pi(x) = \frac{x}{|x|} \) and \( i_t : L(t) \to \mathbb{R}^4 \setminus \{O\} \) be the inclusion map. Then for \( d\theta \) being the volume form on \( S^3 \), \( (\pi \circ i_t)^*(d\theta) \) is then a volume form on \( L(t) \) for \( t \) large. To simplify the notation, we simply write \( (\pi \circ i_t)^*(d\theta) \) as \( d\theta \) when no confusion arises. We are now ready to give the following estimate of geometric terms.

**Lemma 30.** Assume that \( u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. For the mean curvature \( H \) of \( L(t) \), we have, as \( t \to \infty \),

\[ |H - \frac{3}{r_1}| = O(s^5). \]

On the level set \( L(t) \), we have the volume form \( dl \)

\[ dl = r_1^3(t)(\pi \circ i_t)^*(d\theta)[1 + O(s^5)] = r^3(t, \theta)(\pi \circ i_t)^*(d\theta)[1 + O(s^5)]. \]

**Proof.** Using the polar coordinates, we denote \( x \in \mathbb{R}^4 \) also as \( x = r\theta, \theta \in S^3 \). Then for any point \( x \in L(t) \subset B_1 \), the outer normal vector of \( L(t) \), denoted as \( n \), can be computed using Lemma 29

\[
n = \frac{\nabla u}{|\nabla u|} = \frac{x}{r} + O(s^5) = \frac{x}{r_1} + O(s^5).
\]

(4.18)

According to Lemma 29, we may find local coordinate \( \{\eta^\alpha\}, \alpha = 1, 2, 3 \) near \( \theta \in S^3 \). Then \( \eta^\alpha \) can be extended to an open set \( W \subset B_1 \setminus \{O\} \) as \( \pi^* \eta^\alpha \), which we write as \( \eta^{\alpha} \) for simplicity. We have then \( \frac{\partial}{\partial \eta^{\beta}}(y) = |y| \cdot \frac{\partial}{\partial \eta^{\beta}}(\pi(y)) \) for \( y \in W \). Consider the natural orthogonal projection map \( p^\perp : T\mathbb{R}^4 \to TL(t) \). Now \( \{p^\perp(\frac{\partial}{\partial \eta^{\alpha}})\} \) is a local basis of \( TL(t) \). Furthermore, by Lemma 29

\[ |\frac{\partial}{\partial \eta^{\alpha}} - p^\perp(\frac{\partial}{\partial \eta^{\alpha}})| = O(s^5). \]
By Lemma 28 and 29, we estimate the first and second fundamental forms of \( L(t) \) as following

\[
\begin{align*}
    h_{\alpha\beta} &= \langle p^+(\frac{\partial}{\partial \eta^\alpha})p^+(\frac{\partial}{\partial \eta^\beta})x, n \rangle \\
    &= \langle \frac{\partial^2 r(t, \theta)}{\partial \eta^\alpha \partial \eta^\beta} \rangle + \frac{\partial r}{\partial \eta^\alpha} \frac{\partial \theta}{\partial \eta^\beta} + r \frac{\partial^2 \theta}{\partial \eta^\alpha \partial \eta^\beta}, n \rangle + o(s^5) \\
    &= r \langle \frac{\partial^2 \theta}{\partial \eta^\alpha \partial \eta^\beta}, \frac{x}{r} \rangle + o(s^5),
\end{align*}
\]

and similarly,

\[
\begin{align*}
    g_{\alpha\beta} &= \langle p^+(\frac{\partial}{\partial \eta^\alpha})x, p^+(\frac{\partial}{\partial \eta^\beta})x \rangle \\
    &= \langle \frac{\partial^2 \theta}{\partial \eta^\alpha \partial \eta^\beta}, \pi \circ i^* \rangle + o(s^5) \\
    \quad \text{(4.19)}
\end{align*}
\]

Therefore we estimate the mean curvature of \( L(t) \),

\[
H = g^{\alpha\beta}h_{\alpha\beta} = \frac{1}{r}H_{S^3} + O(s^5) = \frac{3}{r_1(t)} + O(s^5).
\]

Furthermore, note that \( dx = r^3 dr \pi^*(d\theta) \), and \( dl = i^*(\iota(n)dx) \). Using (1.18), we have \( |n - \xi| = O(s^5) \). We may then estimate the volume form of \( L(t) \) as follows

\[
dl = i^*(\iota(n)dx) = i^* r^3 \pi^*(d\theta)[1 + O(s^5)] = r^3_1(t)(\pi \circ i_1)^* d\theta[1 + O(s^5)].
\]

As a consequence, we also have

\[
|L(t)| = r^3_1(t)|S^3|[1 + O(s^5)].
\]

\( \square \)

With all local point-wise estimates in place, we are ready to compute integrals that have appeared in our quasi-local mass. First, we have

**Lemma 31.** Assume that \( u(x) \in C^2(D \setminus \{p_1, \cdots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. For any fixed \( t \) very large, let \( x = (r(t, \theta), \xi) \in L(t) \), and \( r_1 = r_1(t) \), then

\[
\int_{L(t)} r^3(t, \theta)(w'(r))^3(\pi \circ i_1)^* (d\theta) = r^3_1(t) w'(r_1))^3 + o(s).
\]

**Proof.** By definition of \( w' \),

\[
\begin{align*}
    w'(r)_{r=r(t, \theta)} &= \frac{1}{1-r} - \frac{1}{1+r} \\
    &= \frac{1}{1-r_1 - \varepsilon} - \frac{1}{1+r_1 + \varepsilon} \\
    &= \varepsilon \frac{1}{(1-r_1)(1-r_1 - \varepsilon)} + \varepsilon \frac{1}{(1+r_1)(1+r_1 + \varepsilon)} + w'(r_1).
\end{align*}
\]
We get
\begin{align*}
  r w' \big|_{r=r(t, \theta)} &= (r_1 + \varepsilon) \left\{ \frac{\varepsilon}{(1 - r_1)(1 - r_1 - \varepsilon)} + \frac{\varepsilon}{(1 + r_1)(1 + r_1 + \varepsilon)} + w'(r_1) \right\} \\
  = r_1 w'(r_1) + r_1 \left[ \frac{\varepsilon}{(1 - r_1)(1 - r_1 - \varepsilon)} + \frac{\varepsilon}{(1 + r_1)(1 + r_1 + \varepsilon)} \right] \\
  &\quad + \varepsilon \left[ \frac{\varepsilon}{(1 - r_1)(1 - r_1 - \varepsilon)} + \frac{\varepsilon}{(1 + r_1)(1 + r_1 + \varepsilon)} + w'(r_1) \right] \\
  = r_1 w'(r_1) + F_1 + F_2,
\end{align*}
where
\begin{align*}
  F_1 &= r_1 \left[ \frac{\varepsilon}{(1 - r_1)(1 - r_1 - \varepsilon)} + \frac{\varepsilon}{(1 + r_1)(1 + r_1 + \varepsilon)} \right] \\
  F_2 &= \varepsilon \left[ \frac{\varepsilon}{(1 - r_1)(1 - r_1 - \varepsilon)} + \frac{\varepsilon}{(1 + r_1)(1 + r_1 + \varepsilon)} + w'(r_1) \right].
\end{align*}

It is clear that \( r_1 w'(r_1) = O\left(\frac{1}{r_1}\right) \), \( F_1 = O(s^3) \) and \( F_2 = O(s^4) \) by Lemma 26 and Corollary 27. Thus,
\begin{align*}
  \int_{L(t)} r^3(w')^3(\pi \circ i_t)^*d\theta &= \int_{L(t)} [r_1 w'(r_1) + F_1 + F_2]^3(\pi \circ i_t)^*d\theta \\
  &= \int_{L(t)} \{ (r_1 w'(r_1))^3 + 3F_1[r_1 w'(r_1)]^2 \} (\pi \circ i_t)^*d\theta + O(s^2) \\
  &= (r_1 w'(r_1))^3 + 3[r_1 w'(r_1)]^2 \int_{L(t)} F_1(\pi \circ i_t)^*d\theta + O(s^2).
\end{align*}

We compute the second term. Noting that by Lemma 26 and Corollary 27,
\begin{align*}
  \int_{L(t)} F_1(\pi \circ i_t)^*d\theta &= \int_{L(t)} r_1 \left[ \frac{\varepsilon}{(1 - r_1)(1 - r_1 - \varepsilon)} + \frac{\varepsilon}{(1 + r_1)(1 + r_1 + \varepsilon)} \right] (\pi \circ i_t)^*d\theta \\
  &= \frac{r_1}{1 - r_1} \int_{L(t)} \left( \frac{\varepsilon}{1 - r_1 - \varepsilon} + \frac{\varepsilon}{1 + r_1} \right) (\pi \circ i_t)^*d\theta + O(s^2) \\
  &= \frac{r_1}{1 - r_1} \int_{L(t)} \left( \frac{\varepsilon}{1 - r_1} + \frac{\varepsilon^2}{(1 - r_1 - \varepsilon)(1 - r_1)} \right) (\pi \circ i_t)^*d\theta + O(s^3).
\end{align*}

Using Corollary 27 and (4.20), and noting that \( \int_{S^3} f d\theta = 0 \), we get
\begin{align*}
  \int_{L(t)} (\pi \circ i_t)^*d\theta &= \int_{L(t)} \left( \frac{1 - r_1^2}{2r_1} \bar{f} s^4 + o(s^5) \right) (\pi \circ i_t)^*(d\theta) = o(s^5).
\end{align*}

Therefore, by (4.25) and (4.26), we have
\begin{align*}
  \int_{L(t)} F_1 &= o(s^3).
\end{align*}
We then apply Lemma 26 and (4.24) to (4.27) and obtain
\[
(4.28) \quad \int_{L(t)} r^3 (w')^3 (\pi \circ i_t)^* d\theta = (r_1 w'(r_1))^3 + o(s).
\]
\[
\square
\]

Now we estimate \( z(t) \).

**Lemma 32.** Assume that \( u(x) \in C^2(D \setminus \{p_1, \ldots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. We have
\[
z(t) = r_1 w'(r_1) + O(s^3),
\]
\[
z^3(t) = (r_1 w'(r_1))^3 + 12(r_1 w'(r_1))^2 \ln r_1^3 \int_{S^3} f(\theta) d\theta + o(s),
\]
and
\[
z^4(t) = (r_1 w'(r_1))^4 + 16(\ln r_1)^3 (r_1 w'(r_1))^3 \int_{S^3} f(\theta) d\theta + o(1).
\]

**Proof.** We use Lemma 30 to see that
\[
\int_{L(t)} |\nabla u|^3 d\theta = \int_{L(t)} |\nabla u|^3 (t, \theta)(\pi \circ i_t)^* d\theta (1 + O(s^5)).
\]
Then, by Lemma 29 we get
\[
\int_{L(t)} |\nabla u|^3 (t, \theta)(\pi \circ i_t)^* d\theta \cdot O(s^5) \leq O(s^2).
\]
We may then use (4.8) to compute \( z \). Let \( e(x) = -f(\theta)(\ln r)^4 - h(x) \). Then \( u = w - e \), and
\[
|\nabla u|^3 = ((w')^2 + |\nabla e|^2 - 2w_r \nabla_r e)^{\frac{3}{2}}
\]
\[
= (w')^3 \left\{ 1 + \frac{3}{2} \left( \frac{|\nabla e|^2 - 2w_r \nabla_r e}{(w')^2} \right) + O \left( \frac{3}{2} \left( \frac{|\nabla e|^2 - 2w_r \nabla_r e}{(w')^2} \right)^2 \right) \right\}
\]
\[
= (w')^3 + \frac{3}{2} w'(|\nabla e|^2 - 2w_r \nabla_r e) + O\left( \frac{|\nabla e|^2 - 2w_r \nabla_r e}{w'} \right),
\]
which leads to
\[
(4.29) \quad \int_{L(t)} r^3 |\nabla u|^3 (\pi \circ i_t)^* d\theta
\]
\[
(4.30) \quad = \int_{L(t)} r(t, \theta)^3 (w')^3 (\pi \circ i_t)^* d\theta + \frac{3}{2} r^3 w'(|\nabla e|^2 - 2w_r \nabla_r e)(\pi \circ i_t)^* d\theta + O(r^3 \left( \frac{|\nabla e|^2 - 2w_r \nabla_r e}{w'} \right)).
\]
By Condition (2) in Definition 1 \( O(r^3 \left( \frac{|\nabla e|^2 - 2w_r \nabla_r e}{w'} \right)) = O(s^5) \).
By \( \mathbf{1.22.} \) \( \mathbf{1.21.} \) and Condition (2) in Definition \( \mathbf{1.} \)

\[
\int_{L(t)} \frac{3}{2} r^3 w' (|\nabla e|^2 - 2w' \nabla_r e) (\pi \circ i_t)^* d\theta
\]

\[
= o(s) + \int_{L(t)} 3r^3 (w')^2 f(\theta) \nabla_r ((\ln r)^4) (\pi \circ i_t)^* d\theta
\]

\[
= o(s) + \int_{L(t)} 12 \left( r_1^2 (w'(r_1))^2 + O(s^2) \right) f(\theta)(\ln r)^3 (\pi \circ i_t)^* d\theta
\]

\[
= o(s) + \int_{L(t)} 12 \left( r_1^2 (w'(r_1))^2 + O(s^2) \right) f(\theta)(\ln r_1)^3 + O(s^7) (\pi \circ i_t)^* d\theta
\]

\[
= o(s) + 12 (r_1 w'(r_1))^2 (\ln r_1)^3 \int_{S^3} f(\theta) d\theta
\]

\[
= 12r_1^2 (w'(r_1))^2 (\ln r_1)^3 \int_{S^3} f(\theta) d\theta + o(s).
\]

By \( \mathbf{1.28.} \) \( \mathbf{1.29.} \) \( \mathbf{4.31.} \) and Lemma \( \mathbf{31.} \) we have

\[
z^3 = \int_{L(t)} r^3 |\nabla u|^3 (\pi \circ i_t)^* d\theta + O(s^2)
\]

\[
= (r_1 w'(r_1))^3 + 12 (r_1 w'(r_1))^2 (\ln r_1)^3 \int_{S^3} f(\theta) d\theta + o(s).
\]

Furthermore,

\[
z^4 = \left( \int_{S^3} r^3 |\nabla u|^3 \right)^{4/3}
\]

\[
= (r_1 w'(r_1))^4 \left( 1 + 16 \frac{(\ln r_1)^3}{r_1 w'(r_1)} \int_{S^3} f(\theta) d\theta + o(s^4) \right)
\]

\[
= (r_1 w'(r_1))^4 + 16 (\ln r_1)^3 (r_1 w'(r_1))^3 \int_{S^3} f(\theta) d\theta + o(1).
\]

And

\[
z = r_1 w'(r_1) \left( 1 + 4 \frac{(\ln r_1)^3}{r_1 w'(r_1)} \int_{S^3} f(\theta) d\theta + o(s^4) \right)
\]

\[
= r_1 w'(r_1) + O(s^3).
\]

Now let us compute \( \int_{L(t)} H |\nabla u|^2 dl. \)

**Lemma 33.** Assume that \( u(x) \in C^2(D \setminus \{p_1, \ldots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. For \( f \in C^2(S^3), \)

\[
\int_{L(t)} H |\nabla u|^2 dl = 3 (r_1 w'(r_1))^2 + O(s^2).
\]

**Proof.** By Lemma \( \mathbf{29.} \) and \( \mathbf{4.22.} \).
\[ H|\nabla u|^2 r^3 = r^3 \left( \frac{3}{r_1^2} + O(s^5) \right) \left( u_r^2 + \frac{1}{r_2^2} u_\theta^2 \right) \\
= 3r^2 u_r^2 + O(s^3) \\
= 3r^2 (w' + O(s^3))^2 + O(s^3) \\
= 3r^2 (w')^2 + O(s^2) \\
= 3(r_1 w'(r_1) + F_1 + F_2)^2 + O(s^2) \\
= 3(r_1 w'(r_1))^2 + O(s^2). \]

Combining with Lemma 30, we get

\[ \int_{L(t)} H|\nabla u|^2 dl = \int_{L(t)} H|\nabla u|^2 r^3 (\pi \circ i_t)^* d\theta (1 + O(s^5)) \\
= \int_{L(t)} [3(r_1 w'(r_1))^2 + O(s^2)] (\pi \circ i_t)^* d\theta (1 + O(s^5)) \\
= 3(r_1 w'(r_1))^2 + O(s^2). \]

\[ \square \]

From Lemma 32 and Lemma 33, we obtain

**Lemma 34.** Assume that \( u(x) \in C^2(D\setminus \{p_1, \cdots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. We have

\[ \frac{2}{9} \int_{L(t)} H|\nabla u|^2 dl = \frac{2}{9} (r_1 w'(r_1))^3 + O(s). \]

**Proof.** By Lemmas 32 and 33, we have

\[ \frac{2}{9} \int_{L(t)} H|\nabla u|^2 dl = \frac{2}{9} (r_1 w'(r_1) + O(s^3)) (3(r_1 w'(r_1))^2 + O(s^2)). \]

\[ \square \]

Next, we estimate \( C(t) \).

**Lemma 35.** Assume that \( u(x) \in C^2(D\setminus \{p_1, \cdots, p_k\}) \) satisfies Condition (2) in Definition 1 and we use notations given as above. We have

\[ C(t) = \frac{r_1^4 w(r_1)}{4} + \int_{S^2} f(\theta) d\theta \left( \frac{2}{1 - r_1^2} \right)^4 (\ln r_1)^4 r_1^4(t) + O(s). \]

**Proof.** For \( t = u_1(r_1(t)) = u(r(t, \theta), \theta) \), noting \( u_1(r_1) = w(r_1) + \frac{(\ln r_1)^4}{8} \int_{S^2} f(\theta) d\theta \), Lemma 27 and Lemma 26, we use the polar coordinate to compute the integrals over regions of \( \mathbb{R}^4 \),
(4.34)
\begin{equation}
C(t) = \frac{1}{|S^3|} e^{4t} \left| \{ u < t \} \right|
\end{equation}

(4.35)
\begin{align*}
&= \frac{1}{|S^3|} e^{4u(t)(r_1(t))} \int_{S^3} \int_{\gamma \leq r(t, \theta)} \gamma^3 d\gamma \pi^+(d\theta) \\
&= \frac{1}{|S^3|} e^{4w(r_1)} \int_{S^3} \frac{1}{4} r^4(t, \theta) d\theta \\
&= \frac{1}{|S^3|} e^{4w(r_1)} \left( 1 + 4 \int_{S^3} f(\theta) d\theta \left( \ln r_1 \right)^4 + O(s^8) \right) \left( \int_{S^3} \frac{1}{4} r_1^4(t) d\theta + O(\varepsilon(t, \theta)) \right) \\
&= \frac{r_1^4 e^{4w(r_1)}}{4} + \int_{S^3} f(\theta) d\theta \left( \frac{2}{1 - r_1^2} \right)^4 (\ln r_1)^4 r_1^4 + O(s). \quad \square
\end{align*}

Finally, we are ready to prove Theorem 25.

Proof. First, recall our quasi-local mass
\begin{equation}
m(t) = \frac{1}{5} \frac{1}{4} e^{-4} + \frac{2}{9} \int_{L(t)} H |\nabla u|^2 + \frac{1}{3} e^{-3} + \frac{1}{3} \int_{L(t)} H |\nabla u|^2 - C(t) \right].
\end{equation}

Second, recall \( w = \log \frac{2}{1 - r^2} \), which satisfies the following differential equation:
\begin{equation}
\frac{1}{4} (rw')^4 + (rw')^3 + (rw')^2 - \frac{r^4}{4} e^{4w(r)} = 0.
\end{equation}

When \( t \to \infty \), we have \( s \to 0^+ \), and \( r_1 \to 1 \). We use Lemma 32, Lemma 33, Lemma 34, Lemma 35 and (4.36) to compute
\[ \lim_{t \to \infty} m(t) = \frac{1}{5} \left[ -4 \int_{S^3} f(\theta) d\theta - \int_{S^3} f(\theta) d\theta \right] = -\int_{S^3} f(\theta) d\theta = -m(M, g). \]

We have thus finished the proof. \( \square \)

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