Power Series Expansions of Modular Forms
and Their Interpolation Properties

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Abstract

We define a power series expansion of a holomorphic modular form $f$ in the $p$-adic neighborhood of a CM point $x$ of type $K$ for a split good prime $p$. The modularity group can be either a classical congruence group or a group of norm 1 elements in an order of an indefinite quaternion algebra. The expansion coefficients are shown to be closely related to the classical Maass operators and give $p$-adic information on the ring of definition of $f$. By letting the CM point $x$ vary in its Galois orbit, the $r$-th coefficients define a $p$-adic $K^\times$-modular form in the sense of Hida. By coupling this form with the $p$-adic avatars of algebraic Hecke characters belonging to a suitable family and using a Rankin-Selberg type formula due to Harris and Kudla along with some explicit computations of Watson and of Prasanna, we obtain in the even weight case a $p$-adic interpolation for the square roots of a family of twisted special values of the automorphic $L$-function associated with the base change of $f$ to $K$.

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Introduction

The idea that the power series expansion of a modular form at a CM point with respect to a well-chosen local parameter should have an arithmetic significance goes back to the author’s thesis, [33]. The goal of the thesis was to prove an expansion principle, namely a characterization of the ring of algebraic $p$-adic integers of definition of an elliptic modular form in terms of the coefficients of the expansion. Such a result would be analogous to the classical $q$-expansion principle based on the Fourier expansion (e.g. [24]), with the advantage of being generalizable in principle to groups of modularity without parabolic elements where Fourier series are not available. The simplest such situation is that of a Shimura curve attached to an indefinite non-split quaternion algebra $D$ over $\mathbb{Q}$ (quaternionic modular forms).

The basic idea in [33] was to consider a prime $p$ of good reduction for the modular curve that is split in the quadratic field of complex multiplications $K$ and use the Serre-Tate deformation parameter to construct a local parameter at the CM point $x$ corresponding to a fixed embedding of $K$ in the split quaternion algebra. The coefficients of the resulting power series are related to the values obtained evaluating the $C^\infty$-modular forms $\delta_k^{(r)} f$ at a lift $\tau$ of $x$ in the complex upper half-plane, where $k$ is the weight of $f$ and $\delta_k^{(r)}$ is the $r$-th iterate, in the automorphic sense, of the basic Maass operator.
Our first goal in this paper is to prove a version of the expansion principle valid also for quaternionic modular forms without making use of the local complex geometry and completely p-adic in nature. The realization of modular forms as global sections of a line bundle \( \mathcal{L} \) suitable for the Serre-Tate theory is subtler in the non-split case because for Shimura curves the Kodaira-Spencer map \( \text{Sym}^2 \omega \to \Omega_\mathcal{X} \) is not an isomorphism (for a trivial reason: the push-forward \( \omega = \pi_* \Omega_{A/\mathcal{X}} \) for the universal family of “false elliptic curves” has rank 2). This motivates the introduction of \( p \)-ordinary test triples (definition 1.8) that require moving to an auxiliary quadratic extension. The abelian variety of dimension \( \leq 2 \) corresponding to the CM point \( x \) defined over the ring of \( p \)-adic algebraic integers \( \mathcal{O}_v \) is either a CM curve \( E \) with \( \text{End}_0(E) = K \) or an abelian surface isogenous to a twofold product \( E \times E \) of such a CM curve. To it we associate a complex period \( \Omega \) of the GL\(_1\)-adic lift of \( \pi \). A key observation (proposition 4.10) is that the set of values \( c_v^{(r)}(x) \) for \( x \) ranging in a full set of representatives of the copy of the generalized ideal class group \( K^\times / K^\times \mathcal{O}_v^\times \) embedded in the modular (or Shimura) curve extends to a Hida [17] \( p \)-adic GL\(_2\)-(K)-modular form \( \hat{c}_r \), which is essentially the \( r \)-th moment of a \( p \)-adic measure on \( \mathbb{Z}_p \) with values in the unit ball of the \( p \)-adic Banach space of such \( p \)-adic forms. The scalar obtained by coupling the form \( \hat{c}_r \) with the \( p \)-adic avatar of a Grössencharakter \( \xi_r \) for \( K \) trivial on \( \mathcal{O}_v^\times \) and of suitable weight twisted by a power of the idelic norm is proportional to the integral

\[
J_r(f, \xi_r, \tau) = \int_{K^\times / K^\times \mathbb{R}^\times} \phi_r(t d_\infty) \xi_r(t) \, dt
\]

(2)

where \( \phi_r \) is the adelic lift of \( \delta_{2r}^{(v)}(f) \), \( \tau \in \mathfrak{H} \) represents \( x \) and \( d_\infty \in \text{SL}_2(\mathbb{R}) \) is the standard parabolic matrix such that \( d_\infty i = \tau \). When \( \xi_r \) is of the form \( \xi_r = \chi \xi^r \) and satisfies some technical conditions the value so obtained is essentially the \( r \)-th moment of a \( p \)-adic measure \( \mu(f, x; \chi, \xi) \) on \( \mathbb{Z}_p \).
On the other hand, the square of the integral (2) is a special case of the generalized Fourier coefficients $L_E(\Phi)$ studied by Harris and Kudla in [13]. Building on results of Shimizu [42] and refining the techniques of Waldspurger [46], Harris and Kudla use the seesaw identity associated with the theta correspondence between the similitude groups $GL_2$ and $GO(D)$ and the splitting $D = K \oplus K^\perp$ to express the generalized Fourier coefficients $L_E(\theta_{\varphi}(F))$ where $F \in \pi$ and $\varphi$ is a split primitive Schwartz-Bruhat function on $D_{K}$ as a Rankin-Selberg Euler product. Thus, we can use the explicit version of Shimizu’s theory worked out by Watson [47], the local non-archimedean computations of Prasanna [38] together with some local archimedean computations to obtain a formula relating the square of the $r$-th moment of $\mu(f, x; \chi, \xi)$ to the values $L(\pi_K \otimes \chi \xi^r, \frac{1}{2})$ whose local correcting terms are explicit outside the primes dividing the conductor of the Grössencharakter and the primes dividing the non square-free part of the level (Theorem 4.21).

Some natural questions arise. First of all, one would like to compute the special values of the $p$-adic $L$-function attached to the measure $\mu(f, x; \chi, \xi)$. Secondly, one may ask if the methods can be extended to treat different or more general families of Grössencharakters, in particular if one can control the interpolation as the ramification at $p$ increases. Proposition 4.15 implies that, if anything, this cannot be achieved without moving the CM point. Thus, some kind of geometric construction in the modular curve may be in order, with a possible link to the question of the determination of the action of the Hecke operators on the Serre-Tate expansions. Another question is whether the reinterpretation of the integral (2) as inner product in the space of $p$-adic $GL_1(K)$-modular forms can be used to obtain an estimate of the number of non-vanishing special values $L(\pi_K \otimes \xi \frac{1}{2})$. We hope to be able to attack these problems in a future paper.

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Notations and Conventions. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{F}_q$ denote, as usual, the integer, the rational, the real, the complex numbers and the field with $q$ elements respectively. We fix once for all an embedding $\mathcal{Q} \rightarrow \mathcal{C}$ and by a number field we mean a finite subextension of the field $\mathcal{Q}$ of algebraic numbers. If $L$ is a number field, we denote $\mathcal{O}_L$ its ring of integers and $\delta_L$ its discriminant. If $L = \mathbb{Q}(\sqrt{d})$ is a quadratic field, for each positive integer $c$ we denote $\mathcal{O}_{L,c} = \mathbb{Z} + c\mathcal{O}_L = \mathbb{Z}[\omega_d]$ its order of conductor $c$, with $\omega_d = \sqrt{d}$ if $d \equiv 2, 3 \pmod{4}$ or $\omega_d = (1 + \sqrt{d})/2$ if $d \equiv 1 \pmod{4}$. If $[L : \mathbb{Q}] = n$ we denote $I_L = \{\sigma_1, \ldots, \sigma_n\}$ the set of embeddings $\sigma_i : L \rightarrow \mathbb{C}$ and we assume $\sigma_1 = \iota_L$.

If $p$ is a rational prime we denote $\mathbb{Z}_p$ and $\mathbb{Q}_p$ the $p$-adic integers and $p$-adic numbers respectively. By analogy, $\mathbb{Q}_\infty = \mathbb{R}$. If $v|p$ is a place of the number field $L$ corresponding to the maximal ideal $p_v \subset \mathcal{O}_L$, we denote $\mathcal{O}_v, L_v, \mathcal{O}_v/k(v)$ the localization of $\mathcal{O}_L$ at $p_v$, the $v$-adic completion of $L$, the ring of $v$-adic integers in $L_v$ and the residue field respectively. The maximal ideal in $\mathcal{O}_v$ is still denoted $p_v$. Also, we denote $L_v^{ur}$ the maximal unramified extension of $L_v$ and $\mathcal{O}_v^{ur}$ its ring of integers.

We denote $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ and for each $\mathbb{Z}$-module $M$ we let $\hat{M} = M \otimes \hat{\mathbb{Z}}$. We denote $\mathbb{A}$ the ring of rational adeles and $\mathbb{A}_f$ the finite adeles, so that $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f = \mathbb{Q}\mathbb{R}\hat{\mathbb{Z}}$. For a number field $L$ we denote $\mathbb{A}_L = \mathbb{A} \otimes L$ and $L^\times$ the
corresponding ring of adeles and group of ideles respectively. If \( n \subseteq \mathcal{O}_K \) is an ideal, we let \( L_n^\times = \{ \lambda \in L^\times \text{ such that } \lambda \equiv 1 \mod n \} \) and denote \( \mathcal{I}_n \) the group of fractional ideals of \( L \) prime with \( n \), \( P_n \) the subgroup of principal fractional ideals generated by the elements in \( L_n^\times \) and \( U_n \) the subgroup of finite ideles product of local units congruent to 1 mod \( n \).

We fix an additive character \( \psi \) of \( \mathbb{A}/\mathbb{Q} \), by asking that \( \psi_{\infty}(x) = e^{2\pi i x} \) and \( \psi_p \) is trivial on \( \mathbb{Z}_p \) with \( \psi_p(x) = e^{2\pi i x} \) for \( x \in \mathbb{Z}[p^{-1}] \) and finite \( p \). On \( \mathbb{A} \) we fix the Haar measure \( dx = \prod_{p \leq \infty} dx_p \) where the local Haar measures \( dx_p \) are normalized so that the \( \psi_p \)-Fourier transform is autodual. For a quaternion algebra \( D \) with reduced norm \( \nu \), we fix on \( D \) the Haar measure \( dx = \prod_{p \leq \infty} dx_p \) where the local Haar measures \( dx_p \) are normalized so that the Fourier transform with respect to the norm form is autodual. Let \( (V, (\cdot, \cdot)) \) be a quadratic space of dimension \( d \) over \( \mathbb{Q} \). We denote \( S_\delta(V) = \bigotimes_{p \leq \infty} S_p \) the adelic Schwartz-Bruhat space, where for \( p \) finite, \( S_p \) is the space of Bruhat functions on \( V \otimes \mathbb{Q}_p \) and \( S_\infty \) is the space of Schwartz functions on \( V \otimes \mathbb{R} \) which are finite under the natural action of a (fixed) maximal compact subgroup of the similitude group \( GO(V) \). The Weil representation \( r_\psi \) is the representation of \( SL_2(\mathbb{A}) \) on \( S_\delta(V) \) which is explicitly described locally at \( p \leq \infty \) by

\[
 r_\psi \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \varphi(x) = \psi_p \left( \frac{1}{2} (bx, x) \right) \varphi(x), \quad (3a)
\]

\[
 r_\psi \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \varphi(x) = \chi_V(a) |a|^{d/2} \varphi(ax), \quad (3b)
\]

\[
 r_\psi \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \varphi(x) = \gamma_V \hat{\varphi}(x), \quad (3c)
\]

where \( \gamma_V \) is an eighth root of 1 and \( \chi_V \) is a quadratic character that are computed in our cases of interest in [21] (see also the table in [38, §3.4]), while the Fourier transform \( \hat{\varphi}(x) = \int_{V \otimes \mathbb{Q}_p} \varphi(y) \psi_p((x, y)) dy \) is computed with respect to a \( (\cdot, \cdot) \)-self dual Haar measure on \( V \otimes \mathbb{Q}_p \).

If \( R \) is a ring and \( M \) a \( R \)-module we denote \( M^\vee = \text{Hom}(M, R) \) the dual of \( M \). The same notation applies to a sheaf of modules over a scheme. If \( G \) is a subgroup of units in \( R \) we say that non-zero elements \( x, y \in M \) are \( G \)-equivalent and write \( x \sim_M y \) if there exists \( r \in G \) such that \( rx = y \).

The group \( SL_2(\mathbb{R}) \) acts on the complex upper half-plane \( \mathfrak{H} \) by linear fractional transformations, if \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) then \( g \cdot z = \frac{az + b}{cz + d} \). The automorphy factor is defined to be \( j(g, z) = cz + d \). The action extends to an action of the group \( GL_2^+(\mathbb{R}) \). If \( \Gamma \leq SL_2(\mathbb{R}) \) is a Fuchsian group of the first kind we shall denote \( M_\delta(\Gamma) \) the space of modular forms of weight \( k \in \mathbb{Z} \) with respect to \( \Gamma \) i.e. the holomorphic functions \( f \) on \( \mathfrak{H} \) such that

\[
 f(\gamma z) = f(z)j(\gamma, z)^k \quad \text{for all } z \in \mathfrak{H} \text{ and } \gamma \in \Gamma
\]

and extend holomorphically to a neighborhood of each cusp (when cusps exist). The subspace of cuspforms, i.e. those modular forms that vanish at the cusps, will be denoted \( S_\delta(\Gamma) \). The request that a holomorphic function on \( \mathfrak{H} \) extends holomorphically to a neighborhood of a cusp \( s \) is equivalent to a certain growth condition as \( z \to s \). Relaxing holomorphy but maintaining the growth condition yields the much bigger spaces of \( C^\infty \)-modular and \( \infty \)-cuspforms, which will be denoted \( M_k^\infty(\Gamma) \) and \( S_k^\infty(\Gamma) \) respectively. We will denote

\[
 M_{k, \epsilon}(\Delta, N), \quad S_{k, \epsilon}(\Delta, N), \quad M_k^\infty(\Delta, N), \quad S_k^\infty(\Delta, N)
\]
the above spaces of modular or cuspforms with respect to the groups \( \Gamma = \Gamma_1(\Delta, N) \), \( \varepsilon \in \{0, 1\} \), defined in section 1.2. It is a well-known fact that \( M_{k, \varepsilon}(\Delta, N) \) is always finite-dimensional and trivial for \( k < 0 \).

## 1 Modular and Shimura curves

### 1.1 Quaternion algebras.

Let \( D \) be a quaternion algebra over \( \mathbb{Q} \) with reduced norm \( \nu \) and reduced trace \( \text{tr} \). For each place \( \ell \) of \( \mathbb{Q} \) let \( D_\ell = D \otimes \mathbb{Q}_\ell \). Let \( \Sigma_D \) be the set of places at which \( D \) is ramified, i.e. \( D_\ell \) is the unique, up to isomorphism, quaternion division algebra over \( \mathbb{Q}_\ell \). If \( \ell \notin \Sigma_D \) the algebra \( D \) is split at \( \ell \), i.e. \( D_\ell \cong \mathbb{M}_2(\mathbb{Q}_\ell) \). The set \( \Sigma_D \) is finite and even and determines completely the isomorphism class of \( D \). Moreover, every finite and even subset of places of \( \mathbb{Q} \) is the set of ramified places of some quaternion algebra over \( \mathbb{Q} \) (for these and the other basic results on quaternion algebras the standard reference is [45]). In particular, \( M_2(\mathbb{Q}) \) is the only quaternion algebra up to isomorphism which is split, i.e. split at all places. The discriminant \( \Delta = \Delta_D \) of \( D \) is the product of the finite primes in \( \Sigma_D \) if \( \Sigma_D \neq \emptyset \), or \( \Delta = 1 \) otherwise. We shall henceforth assume that \( D \) is indefinite, i.e. split at \( \infty \), and fix an isomorphism \( \Phi_\infty : D_\infty \sim \mathbb{M}_2(\mathbb{R}) \) which will be often left implicit. There is a unique conjugacy class of maximal orders in \( D \). Once for all, choose a maximal order \( \mathcal{R}_1 \) and fix isomorphisms \( \Phi_\ell : D_\ell \sim \mathbb{M}_2(\mathbb{Q}_\ell) \) for \( \ell \notin \Sigma_D \) so that \( \Phi_\ell(\mathcal{R}_1) = \mathbb{M}_2(\mathbb{Z}_\ell) \). For an integer \( N \) prime to \( \Delta \) let \( \mathcal{R}_N \) be the level \( N \) Eichler order of \( D \) such that

\[
\mathcal{R}_N \otimes \mathbb{Z}_\ell = \Phi_\ell^{-1} \left( \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_\ell) \text{ such that } c \equiv 0 \mod N \right\} \right)
\]

for \( \ell \notin \Sigma_D \), and \( \mathcal{R}_N \otimes \mathbb{Z}_\ell \) is the unique maximal order in \( D_\ell \) for \( \ell \in \Sigma_D \). If \( D = M_2(\mathbb{Q}) \) we take \( \mathcal{R}_1 = M_2(\mathbb{Z}) \) and \( \mathcal{R}_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}) \text{ such that } c \equiv 0 \mod N \right\} \).

There are exactly two homomorphisms \( \text{or}_1^1, \text{or}_2^1 : \mathcal{R}_N \otimes \mathbb{F}_\ell \rightarrow \mathbb{F}_\ell^2 \) for each prime \( \ell | \Delta \), and two homomorphisms \( \text{or}_1^2, \text{or}_2^2 : \mathcal{R}_N \otimes \mathbb{F}_\ell \rightarrow \mathbb{F}_\ell^2 \) for each prime \( \ell | N \). These maps are called \( \ell \)-orientations and the two \( \ell \)-orientations are switched by the non-trivial automorphism of either \( \mathbb{F}_\ell^2 \) or \( \mathbb{F}_\ell \). An orientation for \( \mathcal{R}_N \) is the choice of an \( \ell \)-orientation \( \text{or}_\ell \) for all primes \( \ell | N \Delta \).

An involution \( d \mapsto d^1 \) in \( D \) is positive if \( \text{tr}(d d^1) > 0 \) for all \( d \in D \). By the Skolem-Noether theorem

\[
d^1 = t^{-1} d t
\]

where \( t \in D \) is some element such that \( t^2 \in \mathbb{Q}^{>0} \) and \( d \mapsto d^\ast \) denotes quaternionic conjugation, \( d + d^\ast = \text{tr}(d) \). If \( t \in D \) is such an element, let \( B_t \) be the bilinear form on \( D \) defined by

\[
B_t(a, b) = \text{tr}(a b d^\ast) = \text{tr}(a t b^1) \quad \text{for all } a, b \in D.
\]

If \( \mathcal{R} \subset D \) is an order, the involution \( d \mapsto d^1 \) is called \( \mathcal{R} \)-principal if \( \mathcal{R}^1 = \mathcal{R} \) and the bilinear form \( B_t \) is skew-symmetric, non-degenerate and \( \mathbb{Z} \)-valued on \( \mathcal{R} \times \mathcal{R} \) with pfaffian equal to 1. When \( \Delta > 1 \) an explicit model for the triple \( (D, \mathcal{R}_N, d \mapsto d^1) \) can be constructed as follows. The condition \( (n, -N \Delta)_\ell = -1 \) for all \( \ell \in \Sigma_D \) on Hilbert symbols defines for \( n \) a certain subset of non-zero congruence classes modulo \( N \Delta \). Passing to classes modulo \( 8 N \Delta \) and taking \( n > 0 \) we may assume that \( (n, -N \Delta)_{\infty} = (n, -N \Delta)_p = 1 \) for all primes \( p \) dividing \( N \) and also \( (n, -N \Delta)_2 = 1 \) if \( \Delta \) is odd. By Dirichlet’s theorem
of primes in arithmetic progressions there exists a prime $p_o$ satisfying these conditions and the product formula easily implies that

$$(p_0, -N\Delta)t = -1 \quad \text{if and only if} \quad \ell \in \Sigma_D.$$  

Let $a \in \mathbb{Z}$ such that $a^2N\Delta \equiv -1 \mod p_o$.

**Theorem 1.1** (Hashimoto, [16]). Let $D$ be a quaternion algebra over $\mathbb{Q}$ of discriminant $\Delta$ and let $t \in D$ such that $t^2 \in \mathbb{Q}^{<0}$. Then:

1. $D$ is isomorphic to the quaternion algebra $D_H = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$, where $i^2 = -N\Delta$, $j^2 = p_o$ and $ij = -ji$;
2. the order $\mathcal{R}_{H,N} = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \mathbb{Z}\epsilon_3 \oplus \mathbb{Z}\epsilon_4$, where $\epsilon_1 = 1$, $\epsilon_2 = (1+j)/2$, $\epsilon_3 = (i+ij)/2$ and $\epsilon_4 = (aN\Delta j + ij)/p_o$ is an Eichler order of level $N$ in $D_H$;
3. the skew symmetric form $B_t$ on $D_H$ is $\mathbb{Z}$-valued on $\mathcal{R}_{H,N}$ if and only if $t \in \mathcal{R}_{H,N}$. Moreover, it defines a non-degenerate pairing on $\mathcal{R}_{H,N} \times \mathcal{R}_{H,N}$ if and only if $t \in \mathcal{R}_{H,N}^\times$;
4. let $t = i^{-1}$. Then the elements $\eta_1 = \epsilon_3 - \frac{1}{2}(p_o - 1)\epsilon_4$, $\eta_2 = aN\Delta - \epsilon_4$, $\eta_3 = 1$ and $\eta_4 = \epsilon_2$ are a symplectic $\mathbb{Z}$-basis of $\mathcal{R}_{H,N}$.

We call Hashimoto model of a quaternion algebra endowed with an Eichler order $\mathcal{R}$ of level $N$ and a $\mathcal{R}$-principal positive involution the triple $(D_H, \mathcal{R}_{H,N}, i^{-1})$ given in the above theorem. We can fix the isomorphism $\Phi_\infty$ for the Hashimoto model by declaring that

$$\Phi_\infty(i) = \begin{pmatrix} 0 & -1 \\ N\Delta & 0 \end{pmatrix}, \quad \Phi_\infty(j) = \begin{pmatrix} \sqrt{p_o} & 0 \\ 0 & -\sqrt{p_o} \end{pmatrix}.$$ 

### 1.2 Moduli spaces.

Fix a $\mathcal{R}_1$-principal positive involution $d \mapsto d^\dagger$ as in (4). We shall consider the groups

$$\Gamma_0(\Delta, N) = \mathcal{R}_N^1 = \{ \gamma \in \mathcal{R}_N \text{ such that } \nu(\gamma) = 1 \}$$

and

$$\Gamma_1(\Delta, N) = \{ \gamma \in \Gamma_0(\Delta, N) \text{ such that } \mathfrak{c}_\ell(\gamma r) = \mathfrak{c}_\ell(r) \text{ for all } r \in \mathcal{R}_N, \ell|N, \epsilon = 1, 2 \}.$$ 

When $\Delta = 1$, $\Gamma_0(1, N)$ and $\Gamma_1(1, N)$ are the classical congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ such that } c \equiv 0 \mod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ such that } a, d \equiv 1 \text{ e } c \equiv 0 \mod N \right\}$$

respectively. Since $D$ is indefinite $\Gamma_\varepsilon(\Delta, N)$ for $\varepsilon \in \{0, 1\}$ is, via $\Phi_\infty$, a discrete subgroup of $\text{SL}_2(\mathbb{R})$ acting on the complex upper half plane $\mathfrak{H}$. When $\Delta > 1$ the quotient $X_\varepsilon(\Delta, N) = \Gamma_\varepsilon(\Delta, N)/\mathfrak{H}$ is a compact Riemann surface, [43, proposition 9.2]. When $\Delta = 1$ let $X_\varepsilon(N)$ be the standard cuspidal compactification of $Y_\varepsilon(N) = \Gamma_\varepsilon(N)/\mathfrak{H}$.

Each of these complete curves $X$ has a canonical model over $\mathbb{Q}$, [43]. In fact, each $X$ can be reinterpreted as the set of complex points of a scheme $\mathcal{X}$ which is the solution
of a moduli problem, defined over \(\mathbb{Z}[1/N\Delta]\), e.g. [1, 6, 7, 32, 39]. When \(D = M_2(\mathbb{Q})\) and \(N > 3\), the functor \(F_1(N): \mathbb{Z}[1/N]-\text{Schemes} \to \text{Sets}\) defined by

\[
F_1(N)(S) = \begin{cases} \text{Isomorphism classes of generalized elliptic curves } E = E|_S, \\ \text{with a section } P: S \to E \text{ of exact order } N \end{cases}
\]

is represented by a proper and smooth \(\mathbb{Z}[\frac{1}{N}]-\text{scheme} \mathcal{X}_1(N)\) such that \(\mathcal{X}_1(N)(\mathbb{C}) = X_1(N)\). The complex elliptic curve with point \(P\) of exact order \(N\) corresponding to \(z \in \mathfrak{H}\) is the torus \(E_z = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\) with \(P = 1/N \text{ mod } \mathbb{Z}\). Denote

\[
\pi_N: \mathcal{E}_N \longrightarrow \mathcal{X}_1(N)
\]

the universal generalized elliptic curve attached to the representable functor \(F_1(N)\). The scheme \(\mathcal{X}_0(N)\) quotient of \(\mathcal{X}_1(N)\) by the action of the group of diamond operators \(\langle a \rangle: \mathcal{X}_1(N) \to \mathcal{X}_1(N)(\mathbb{C})\) is the coarse moduli scheme attached to the functor

\[
F_0(N)(S) = \begin{cases} \text{Isomorphism classes of generalized elliptic curves } E = E|_S, \\ \text{with a cyclic subgroup } C \subset E \text{ of exact order } N \end{cases}
\]

and a smooth \(\mathbb{Z}[1/N]-\text{model}\) for the curve \(\mathcal{X}_0(N)\).

When \(\Delta > 1\) and \(N > 3\), \(\mathcal{X}_1(\Delta, N) = \mathcal{X}_1(\Delta, N)(\mathbb{C})\) for the proper and smooth \(\mathbb{Z}[1/N\Delta]-\text{scheme} \mathcal{X}_1(\Delta, N)\) representing the functor \(F_1(\Delta, N): \mathbb{Z}[1/N\Delta]-\text{Schemes} \to \text{Sets}\) defined by

\[
F_1(\Delta, N)(S) = \begin{cases} \text{Isomorphism classes of compatibly principally polarized abelian surfaces } A = A|_S \text{ with a ring embedding } \\ \mathcal{R}_1 \hookrightarrow \text{End}(A) \text{ and an equivalence class of } \\ \mathcal{R}_N\text{-orientation preserving level } N \text{ structures} \end{cases}
\]

A level \(N\) structure on an abelian surface \(A\) with \(\mathcal{R}_1 \subset \text{End}(A)\) is an isomorphism of (left) \(\mathcal{R}_1\)-modules \(A[N] \cong \mathcal{R}_1 \otimes (\mathbb{Z}/N\mathbb{Z})\). Two such structures are declared equivalent if they coincide on \(\mathcal{R}_N \otimes (\mathbb{Z}/N\mathbb{Z})\) and induce the same \(\ell\)-orientations on \(\mathcal{R}_N\) for all \(\ell \mid N\). The principal polarization is compatible with the embedding \(\mathcal{R}_1 \subset \text{End}(A)\) if the involution \(d \mapsto d^!\) is the Rosati involution. The abelian surfaces in \(F_1(\Delta, N)(S)\) are called abelian surfaces with quaternionic multiplications (QM-abelian surfaces, for short) or false elliptice curves. The complex QM-abelian surface corresponding to \(z \in \mathfrak{H}\) is

\[
A_z = D^\times_\infty /\mathcal{R}_1,
\]

where \(D^\times_\infty\) is the real vector space \(D_\infty\) endowed with the \(\mathbb{C}\)-structure defined by the identification \(\mathbb{C}^2 = \Phi_\infty(D_\infty) \begin{pmatrix} 1 \\ i \end{pmatrix}\), i.e. \(A_z = \mathbb{C}^2 / \Phi_\infty(\mathcal{R}_1) \begin{pmatrix} 1 \\ i \end{pmatrix}\). The complex uniformization (8) defines a level structure \(N^{-1}\mathcal{R}_1 / \mathcal{R}_1 = (D/\mathcal{R}_1)[N] \sim (\mathbb{Z} / (\mathbb{Z} \cap D))[N]\) and the skew-symmetric form \(\langle \Phi_\infty(a) \begin{pmatrix} 1 \\ i \end{pmatrix}, \Phi_\infty(b) \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle = B_t(a, b)\) for all \(a, b \in D\), where \(B_t\) is as in (5), extended to \(\mathbb{C}^2\) by \(\mathbb{R}\)-linearity is the unique Riemann form on \(A_z\) with Rosati involution \(d \mapsto d^!\), [32, lemma 1.1]. Denote

\[
\pi_{\Delta, N}: A_{\Delta, N} \longrightarrow \mathcal{X}_1(\Delta, N)
\]

the universal QM abelian surface attached to the representable functor \(F_1(\Delta, N)\).

As with the split case, a smooth \(\mathbb{Z}[1/N\Delta]-\text{model} \mathcal{X}_0(\Delta, N)\) of \(X_1(\Delta, N)\) can be obtained as quotient of \(\mathcal{X}_1(\Delta, N)\) by a suitable action of \(\Gamma(\Delta, N) / \Gamma(\Delta, N) \cong (\mathbb{Z} / (\mathbb{Z} \cap D))[N]\). It is the
coarse moduli space for the functor

\[ F_0(\Delta, N)(S) = \begin{cases} 
\text{Isomorphism classes of compatibly principally polarized} \\
\text{abelian surfaces } A = A_{1|S} \text{ with a ring embedding } \mathcal{R}_1 \hookrightarrow \text{End}(A) \\
\text{and an } \mathcal{R}_N\text{-equivalence class of level } N \text{ structures}
\end{cases} \]

where two level \( N \) structures are \( \mathcal{R}_N\)-equivalent if they coincide on \( \mathcal{R}_N \otimes (\mathbb{Z}/N\mathbb{Z}) \).

**Remark 1.2.** In order to study the reduction of the modular and Shimura curves at primes dividing \( N\Delta \) one has to extend the moduli problems described above to moduli problems defined over \( \mathbb{Z} \), see [2, 28]. The \( \mathbb{Z} \)-schemes thus obtained are proper but not smooth. We shall not deal with primes of bad reduction and for the purposes of this paper the above descriptions will suffice.

### 1.3 Subfields and CM points.

Let \( \mathbb{Q} \subseteq L' \subset L \) be a tower of fields with \( [L : L'] = 2 \) and assume that \( L \) splits \( D \), i.e. \( D \otimes \mathbb{Q} \cong M_2(L) \) or, equivalently, that \( L \) admits an embedding in \( D \otimes \mathbb{Q} L' \). An embedding \( j : L \hookrightarrow D \otimes \mathbb{Q} L' \) endows \( D \otimes \mathbb{Q} L' \) with a structure of \( L \)-vector space. Scalar multiplication by \( \lambda \in L \) is left multiplication by \( j(\lambda) \). The opposite algebra \( D^{\text{op}} \) acts \( L \)-linearly on \( D \) by right multiplication, providing a direct identification

\[ D^{\text{op}} \otimes L \xrightarrow{\sim} \text{End}_L(D \otimes L'). \tag{10} \]

Let \( \sigma \) be the non-trivial element in \( \text{Gal}(L/L') \) and \( j^\sigma(\lambda) = j(\lambda^\sigma) \) for all \( \lambda \in L \). By the Skolem-Noether theorem there exists \( u \in (D \otimes \mathbb{Q} L')^\times \), well defined up to a \( L^\times \)-multiple, such that \( u j(\lambda) = j^\sigma(\lambda) u \) for all \( \lambda \in L \) and \( u^2 \in L' \). Thus, with a slight abuse of notation, the embedding \( j \) defines a splitting

\[ D \otimes L' = L \oplus Lu \tag{11} \]

which can be more intrinsically seen as the eigenspace decomposition under right multiplication by \( j(L^\times) \). Also, there is an isomorphism

\[ D \xrightarrow{\sim} D^{\text{op}}, \quad \lambda_1 + \lambda_2 u \mapsto \lambda_1 + \lambda_2^\sigma u. \tag{12} \]

Let \( L = L'(\alpha) \) with \( \alpha^2 = A \in L' \). The element

\[ e_j = \frac{1}{2} \left( 1 \otimes 1 + \frac{1}{A} j(\alpha) \otimes \alpha \right) \in D \otimes L' \tag{13} \]

is an idempotent which is easily seen to be, under (10), (11) and (12), the projection onto \( L \) with kernel \( Lu \). If \( L \subseteq \mathbb{C} \) the idempotent \( e_j \) defines a projector in \( D^*_\infty \) for all \( z \in \mathfrak{g} \) by scalar extension.

An involution \( d \mapsto d^\dagger \) in \( D \) extends by linearity to \( D \otimes L' \). If \( j^\dagger \) is the embedding \( j^\dagger(\lambda) = j(\lambda)^\dagger \), the explicit description (13) implies at once that \( e_j^\dagger = e_j \) and in particular

\[ e_j^\dagger = e_j \text{ if and only if } j(L)^\dagger = j(L) \text{ pointwise.} \]

When the involution is positive a fixed idempotent can be constructed as follows. As an element of \( \text{End}(D) \) the involution (4) has determinant \(-1\). Since \( 1^\dagger = 1 \) and \( \text{tr}(d^\dagger) = \text{tr}(d) \) for all \( d \in D \) its \((-1)\)-eigenspace is a subspace of trace 0 elements of dimension either 1 or 3. If the dimension is 3 then the involution is the quaternionic conjugation,
contradicting the positivity assumption. Therefore there exist a non-zero element \( d \) of trace 0 fixed by the involution. The subalgebra \( F = \mathbb{Q}(d) \subset D \) is a quadratic field fixed by the involution and the corresponding idempotent \( e \in D \otimes \mathbb{Q} F \) has the desired property. Note that the positivity of the involution implies further that \( F \) is real quadratic.

The conductor of an embedding \( j : L \to D \) of the quadratic field \( L \) relative to the order \( \mathcal{R}_N \) is the integer \( c = c_N > 0 \) such that \( j(\mathcal{O}_{L,c}) = j(L) \cap \mathcal{R}_N \). Denote \( \bar{c} \) the minimal conductor, i.e. the conductor relative to the maximal order \( \mathcal{R}_1 \). It is clear that \( c \) is a multiple of \( \bar{c} \), in fact \( c/\bar{c} \) is a divisor of \( N \) because \( \mathcal{O}_{L,\bar{c}}/\mathcal{O}_{L,c} \) injects into \( \mathcal{R}_1/\mathcal{R}_N \simeq \mathbb{Z}/N\mathbb{Z} \). In the following result the embedding is left implicit to simplify the notation.

**Proposition 1.3.** Let \( L \subset D \) be a quadratic subfield with associated decomposition \( D = L \oplus Lu \). Let \( \Lambda = L \cap \mathcal{R}_N \) and \( N = Lu \cap \mathcal{R}_N \). Then:

1. \( D \) is split at the prime \( p \) if and only if \( (u^2, \delta_L)_p = 1 \);
2. if \( p \) is unramified in \( L \) and \( \gcd(p, c) = 1 \) then \( \mathcal{R}_N \otimes \mathbb{Z}_p = \Lambda \otimes \mathbb{Z}_p \oplus N \otimes \mathbb{Z}_p \). Moreover, \( \mathcal{R} \otimes \mathbb{Z}_p = \mathcal{J} u \) for some fractional ideal \( \mathcal{J} \subset L \otimes \mathbb{Q}_p \) such that \( N(\mathcal{J})^p(u) = (q^e) \) with \( e = 1 \) if \( q \mid N\Delta \) and \( e = 0 \) otherwise.

**Proof.** Let \( L = \mathbb{Q}(\sqrt{d}) \). Then \( \{1, \sqrt{d}, u, \sqrt{du}\} \) is a \( \mathbb{Q} \)-basis of \( D \) and the local invariants of the norm form are \( \det = 1 \) and \( e_p = (-1, -1)_p(u^2, d)_p = (-1, -1)_p(u^2, \delta_L)_p \), thus proving the first part.

For the second part, choose \( u \) so that \( u^2 \in \mathbb{Z} \). Then there is an inclusion of orders \( \mathcal{R}' = \mathcal{O}_{L,c} \oplus \mathcal{O}_{L,c}u \subset \Lambda \oplus \mathcal{J} \subset \mathcal{R}_N \). The elements \( \{1, c, u, cu\} \) are a \( \mathbb{Z} \)-basis of \( \mathcal{R}' \), so that \( \mathcal{R}' \) has reduced discriminant \( \delta_L c u^2 \). We are thus reduced to check that when \( p → u^2 \) and \( \gcd(p, c, \delta_L) = 1 \) then there is no element \( x \in \mathcal{R}_N \) of the form \( x = (r + r'u)/p \) with \( r, r' \in \mathcal{O}_{L,c} - p\mathcal{O}_{L,c} \). For such an element \( x \) one must have \( p | \text{tr}(r) \) and \( p | N(r) \) from which one derives quickly a contradiction.

The last claim follows from the very same discriminant computation since \( \mathcal{R}_N \) has reduced discriminant \( N\Delta \).

Fix a quadratic imaginary field \( K \) that splits \( D \). Exactly one of the two embeddings \( j, j' \) is normalized in the sense of [43, (4, 4.5)]. The normalized embeddings correspond bijectively to a special subset of points \( \tau \in \bar{\mathcal{H}} \). More precisely, there is a bijection

\[
\begin{align*}
\text{normalized embeddings} & 
\{ j : K \to D \} 
\leftrightarrow 
\text{CM}_{\Delta, K} = 
\left\{ \tau \in \bar{\mathcal{H}} \text{ such that } \Phi_\infty(j(K^\times)) = 
\text{im\Phi}_\infty(\mathcal{O}_{L,c}) \cap \text{GL}_2^+(\mathbb{R}) \mid \gamma \cdot \tau = \tau \right\}.
\end{align*}
\]

The bijection is \( \Gamma_0(\Delta, N) \)-equivariant where \( \Gamma_0(\Delta, N) \) acts by conjugation on the left set and on \( \text{CM}_{\Delta, K} \) via its action on \( \bar{\mathcal{H}} \). Also, the correspondence \( j \leftrightarrow \tau \) is characterized by the fact that the complex structure on \( D_\infty \) induced by the embedding \( j \) coincides with that of \( D'_\infty \). In the split case \( \text{CM}_{1, K} = K \cap \bar{\mathcal{H}} \).

We shall denote \( c_\tau = c_{\tau, N} \) the conductor relative to the order \( \mathcal{R}_N \) of the embedding associated to the point \( \tau \in \text{CM}_{\Delta, K} \) and \( \bar{c}_\tau \) its minimal conductor.

**Proposition 1.4.** Let \( \tau \) and \( \tau' \in \text{CM}_{\Delta, K} \) such that \( \tau' = \gamma \cdot \tau \) for some \( \gamma \in \Gamma_0(\Delta, N) \). Then \( c_{\tau', N} = c_{\tau, N} \).

**Proof.** Let \( j \) and \( j' \) be the embeddings corresponding to \( \tau \) and \( \tau' \) respectively. Then \( j' = \gamma j \gamma^{-1} \) and so \( j'(\mathcal{O}_{c_{\tau', N}}) = j'(K) \cap \mathcal{R}_N = \gamma j(K) \gamma^{-1} \cap \mathcal{R}_N = \gamma j(K) \cap \mathcal{R}_N = \gamma j(K) \cap \mathcal{R}_N \gamma^{-1} = j'(\mathcal{O}_{c_{\tau, N}}) \gamma^{-1} = j'(\mathcal{O}_{c_{\tau, N}}) \).
Defnition 1.5. A point \( x \in X_0(\Delta, N) \) is a CM point of type \( K \) and conductor \( c = c_x \) if it is represented by a \( \tau \in \text{CM}_{\Delta,K} \) with \( c_{\tau,N} = c \). Denote

\[
\text{CM}(\Delta, N; \mathcal{O}_{K,e}) = \{ \text{CM points of } X_0(\Delta, N) \text{ of type } K \text{ and conductor } c \}.
\]

The following result is [4, Lemma 4.17]

Proposition 1.6. Let \( c > 0 \) be an integer such that \( \gcd(c, N\Delta) = 1 \). Then the set \( \text{CM}(\Delta, N; \mathcal{O}_{K,e}) \) is non-empty if and only if

- all primes \( l | \Delta \) are inert in \( K \), and

- all primes \( l | N \) are split in \( K \).

For \( \tau \in \text{CM}_{1,K} \) the elliptic curve \( E_\tau \) has complex multiplications in the field \( K \). When \( \Delta > 1 \) and \( \tau \in \text{CM}_{\Delta,K} \) the QM abelian surface \( A = A_\tau = \mathcal{A}(\mathbb{C}) \) contains the elliptic curve \( E = K \otimes \mathbb{R} / \mathcal{O}_{K,E} \) and in fact is isogenous to the product \( E \times E \). In particular there is an identification \( \text{End}^\sigma(A) \simeq D \otimes K \). Consider the left ideal \( \mathcal{E} = \text{End}(A) \cap \text{End}^\sigma(A)(1 - e_\tau) \) where \( e_\tau \) is the idempotent (13) attached to the embedding \( \mathcal{E} : K \rightarrow D \) associated to \( \tau \) and let \( \mathcal{E} = \mathcal{A}[\mathcal{E}] \) be the connected component of the subgroup scheme of \( \mathcal{A} \) killed by \( \mathcal{E} \). Note that since \( J(\mathcal{O}_{K,E}) \) and \( e \) commute, the order \( \mathcal{O}_{K,E} \) acts on \( \mathcal{E} \).

Proposition 1.7. \( A = \mathcal{E} \otimes_{\mathcal{O}_{K,E}} \mathcal{R}_1 \) as group schemes.

Proof. Let \( S \) be any scheme of definition for \( A \). Over any \( S \)-scheme \( T \) there is an obvious map \( (\mathcal{E} \otimes_{\mathcal{O}_{K,E}} \mathcal{R}_1)(T) \rightarrow \mathcal{A}(T) \) which is surjective because \( \mathcal{E} \otimes_{\mathcal{O}_{K,E}} \mathcal{R}_1 \) contains two independent abelian schemes of dimension 1, \( \mathcal{E} \) and any translate of it by an \( r \in \mathcal{R}_1 - \mathcal{O}_{K,E} \). To show that the map is injective, it is enough to do so over an algebraically closed field. Over \( \mathbb{C} \) we have \( \mathcal{E}(\mathbb{C}) = E \) and thus \( (\mathcal{E} \otimes_{\mathcal{R}_1} \mathcal{R}_1)(\mathbb{C}) = E \otimes_{\mathcal{O}_{K,E}} \mathcal{R}_1 = (K \otimes \mathbb{R} \otimes_{\mathcal{O}_{K,E}} \mathcal{R}_1)/\mathcal{R}_1 = A \). ■

Defnition 1.8. Let \( p \) be an odd prime number, \( \gcd(p, N\Delta) = 1 \). A \( p \)-ordinary test triple for \( \Gamma_\epsilon(\Delta, N) \) is a triple \((\tau, v, e)\), where \( \tau \in \text{CM}_{\Delta,K} \), \( v \) is a finite place dividing \( p \) in a finite extension \( L \supset \mathbb{Q} \) and \( e \in D \otimes F \) is the idempotent associated to a real quadratic subfield \( F \subset D \) pointwise fixed by the positive involution, such that

1. \( FK \subseteq L \);

2. the CM curve \( E_\tau \) or QM-abelian surface \( A_\tau \) has ordinary good reduction modulo \( p \); and

3. if \( w \) is the restriction of \( v \) to \( F \) then \( e \in \mathcal{R}_1 \otimes \mathbb{Z} \mathcal{O}_{(w)} \).

Furthermore, a \( p \)-ordinary test triple \((\tau, v, e)\) is said split if \( p \) splits in \( F \).

Let us observe that:

1. the ordinarity hypothesis implies that \( p \) splits in \( K \);

2. the idempotent \( e \) plays no role in the split case and can be omitted in that case;

3. the explicit description (13) of \( e \) shows that the third condition above is equivalent to \( \gcd(p, \bar{c} \delta_F) = 1 \) where \( \bar{c} \) is the minimal conductor of \( F \).
4. for a $p$-ordinary triple $(\tau, v, e)$ for $\Gamma_1(\Delta, N)$ the point $x \in X_1(\Delta, N)$ represented by $
abla$ is a smooth point in $X_1(\Delta, N)$. This is clear for $D$ split and follows for instance from [22, Theorem 1.1] in the non-split case.

**Proposition 1.9.** Let $p$ be an odd prime number, $\gcd(p, N\Delta) = 1$. There exist split $p$-ordinary triples for $\Gamma_1(\Delta, N)$.

**Proof.** Since any two positive involutions (4) are conjugated in $D$, up to a different choice of maximal order we are reduced to the Hashimoto model. Up to replacing $p_0$ in (6) in its congruence class modulo $8N\Delta p$, we may assume also that $\left(\frac{p_0}{p}\right) = 1$. Thus the subfield $F = \mathbb{Q} \oplus \mathbb{Q} j \subset D_H$ is pointwise fixed by the involution, has discriminant prime to $p$ and $p$ splits in it. Finally, the minimal conductor of the embedding $\sqrt{p_0} \mapsto j \in F$ is prime to $p$ since $j \in R_{H,N}$.

The decomposition $D = K \oplus Ku$ associated to a choice of $\tau \in \text{CM}_{\Delta,K}$ is also an orthogonal decomposition under the non-degenerate pairing $(x, y)_D = \text{tr}(xjy)$. Note that here $u^2 > 0$ since the norm is indefinite. We shall be concerned with the algebraic group of similitudes of $(\cdot, \cdot)_D$, i.e.

$$GO(D) = \{ g \in \text{GL}(D) \text{ such that } (gx, gy)_D = \nu_0(g)(x, y)_D \text{ for all } x, y \in D \}.$$

The structure of the group $GO(D)$ is well understood, e.g. [12, §1.1], [13, §7]. Let $t \in GO(D)$ be the involution $t(d) = \bar{d}$. Then $GO(D) = GO^0(D)^{\perp} < t >$, where $GO^0(D)$ is the Zariski connected component described by the short exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow D^\times \times D^\times \longrightarrow GO^0(D) \longrightarrow 1 \tag{14}$$

where $\mathbb{G}_m$ is embedded diagonally and $\rho(d_1, d_2)(x) = d_1 x d_2^{-1}$. The norm $\nu$ restricts to $N_{K/\mathbb{Q}}$ and $-u^2 N_{K/\mathbb{Q}}$ on $K$ and $Ku$ respectively, and $GO(K)^\perp \cong GO^0(Ku) \cong R_{K/\mathbb{Q}} \mathbb{G}_m$ where the isomorphism is given by left multiplication. Thus, the subgroup of $GO^0(D)$ that preserves the splitting $D = K \oplus Ku$ can be identified with the group

$$G(O(K) \times O(Ku))^\alpha = \{(k_1, k_2) \in (R_{K/\mathbb{Q}} \mathbb{G}_m)^2 \text{ such that } N_{K/\mathbb{Q}}(k_1 k_2^{-1}) = 1\}$$

and there is a commutative diagram

$$\begin{array}{ccc}
K^\times \times K^\times & \xrightarrow{\alpha} & G(O(K) \times O(Ku))^\alpha \\
\downarrow \times j & & \downarrow \\
D^\times \times D^\times & \xrightarrow{\rho} & GO^0(D) \end{array} \tag{15}$$

where $\alpha(k_1, k_2) = (k_1 k_2^{-1}, k_1^{-1} k_2^{-1})$.

We will normalize the complex coordinates in $D^\tau_{\infty} = (K \oplus Ku) \otimes \mathbb{R}$ as follows. The standard normalized embedding $j^\tau : \mathbb{Q}(\sqrt{-1}) \hookrightarrow M_2(\mathbb{Q})$ with fixed point $i \in \mathbb{I}$ defines a splitting $M_2(\mathbb{R})^i = \mathbb{C} \oplus \mathbb{C}^\perp$ with $\mathbb{C} = \mathbb{R} \left( \binom{1}{i} \right) \oplus \mathbb{R} \left( \binom{-1}{1} \right)$ and $\mathbb{C}^\perp = \mathbb{C} \left( \binom{i}{1} \right) = \mathbb{R} \left( \binom{1}{1} \right) \oplus \mathbb{R} \left( \binom{-1}{1} \right)$. Define standard complex coordinates $z^\tau_1, z^\tau_2$ in $D^\tau_{\infty}$ by the identity

$$\Phi_{\infty}(d) = z^\tau_1 + z^\tau_2 \left( \begin{array}{c} 1 \\ 1 \end{array} \right). \tag{16}$$

The $\mathbb{R}$-linear extensions of the embeddings $j^\tau$ and $\Phi_{\infty} \circ j$ are conjugated in $M_2(\mathbb{R})$, namely $\Phi_{\infty} \circ j = d_{\infty} j^\tau d_{\infty}^{-1}$ where $d_{\infty} = \left( \begin{array}{cc} y^{1/2} & sy^{1/2} \\ y^{-1/2} & y^{-1/2} \end{array} \right)$ and $\tau = s + iy$. So we define normalized coordinates $z_1$ and $z_2$ in $D^\tau_{\infty}$ by the identity

$$z_i(d) = z^\tau_i (\Phi_{\infty}^{-1}(d_{\infty}) \Phi_{\infty}^{-1}(d_{\infty})) \quad \text{for all } d \in D_{\infty}, \quad i = 1, 2.$$
2 Some differential operators

2.1 Preliminaries.

We briefly review some basic facts about the Kodaira-Spencer map and the Gauß-Manin connection. For more details see [23, 29]. The Kodaira-Spencer class of a composition of smooth morphisms of schemes \( X \to S \to T \) is the element in \( H^1(X, (\Omega^1_{X/S})^\vee \otimes \pi^*\Omega^1_{S/T}) \) arising from the canonical exact sequence.

\[
0 \longrightarrow \pi^*\Omega^1_{S/T} \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0
\]  

(17)

by local freeness of the sheaves \( \Omega^1 \). The Kodaira-Spencer map is the boundary map

\[
\text{KS} : \pi_*\Omega^1_X \longrightarrow R^1\pi_*(\pi^*\Omega^1_{S/T}) \cong \Omega^1_{S/T} \otimes R^1\pi_*\mathcal{O}_X
\]

in the long exact sequence of derived functors obtained from (17) by pushing down. Under the natural maps \( H^i(X, (\Omega^1_{X/S})^\vee \otimes \pi^*\Omega^1_{S/T}) \to H^0(S, R^1\pi_*((\Omega^1_{X/S})^\vee \otimes \pi^*\Omega^1_{S/T})) \to H^0(S, \Omega^1_{S/T}) \otimes R^1\pi_*\mathcal{O}_X \otimes (\pi_*\Omega^1_X)^\vee \) the Kodaira-Spencer class maps to the Kodaira-Spencer map.

The \( q \)-th relative de Rham cohomology sheaf of \( X/S \) is defined as \( \mathcal{H}^q_{\text{dr}}(X/S) = R^q\pi_*(\Omega^\bullet_{X/S}) \) (hypercohomology). Following [29], the Gauß-Manin connection

\[
\nabla : \mathcal{H}^q_{\text{dr}}(X/S) \longrightarrow \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{\text{dr}}(X/S).
\]

can be seen as the differential \( d_{1,q} : E_{1,q}^1 \to E_{1,q}^0 \) in the spectral sequence defined by the finite filtration \( F^i\Omega^\bullet_{X/T} = \text{Im}((\Omega^\bullet_{X/T})^\vee \otimes \mathcal{O}_X \longrightarrow \Omega^\bullet_{X/T}) \), with associated graded objects \( \text{gr}^i(\Omega^\bullet_{X/T}) = \Omega^\bullet_{X/T}^i \otimes \mathcal{O}_X \). If \( X/S = A/S \) is an abelian scheme with 0-section \( e_0 \) and dual \( A^t/S \), denote \( e = e_A/S = \pi_*\Omega^1_A \otimes \mathcal{O}_S \) the sheaf on \( S \) of translation invariant relative 1-forms on \( A \). The first de Rham sheaf \( H^1_{\text{dr}} = H^1_{\text{dr}}(A/S) \) is the central term in a short exact sequence

\[
0 \longrightarrow e \longrightarrow \mathcal{H}^1_{\text{dr}} \longrightarrow R^1\pi_*\mathcal{O}_A \longrightarrow 0
\]

(18)

called the Hodge sequence). By Serre duality

\[
\text{Hom}_{\mathcal{O}_S}(\pi_*\Omega^1_A, R^1\pi_*(\pi^*\Omega^1_{S/T})) \cong \text{Hom}_{\mathcal{O}_S}(e_A/S \otimes e^t_A/S, \Omega^1_{S/T})
\]

(19)

and the Kodaira-Spencer map can be seen as an element of the latter group. It can be reconstructed from the Gauß-Manin connection as the composition

\[
\omega_{A/S} \longrightarrow \mathcal{H}^1_{\text{dr}} \longrightarrow \Omega^1_{S/T} \longrightarrow e_A/S \otimes \Omega^1_{S/T}.
\]

(20)

In fact, when \( A/S \cong A^t/S \) is principally polarized, the Kodaira-Spencer map becomes a symmetric map \( \text{KS} : \text{Sym}^2(e) \rightarrow \Omega^1_{S/T} \) [9, Section III. 9].

Let \( (S', i_0) \) be a smooth closed reduced subscheme of \( S \) and consider the commutative pull-back diagram of \( T \)-schemes

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow \pi' & & \downarrow \pi \\
S' & \longrightarrow & S
\end{array}
\]
Since also $\pi'$ is smooth, we can consider the Kodaira-Spencer class, or map, $KS'$ attached to the morphisms $X' \xrightarrow{\varphi} S' \rightarrow T$. When $X = A$ is a principally polarized abelian scheme, $KS' \in \text{Hom}_{\mathcal{O}_S}(\omega_{\mathcal{O}_S}^{\otimes 2}, \Omega^1_{S/T})$ as in (19), where $\omega_{\mathcal{O}_S}^{\otimes 2} = \pi_*\Omega^1_{A/S'} = e^\ast_0\Omega^1_{A/S'}$. Since $i^*\pi^!\Omega^1_{S/T} = \pi^*i_0^!\Omega^1_{S/T}$ and $i^!\Omega_{X/S}$ isomorphically, applying $i^*$ to (17) yields an exact sequence

$$0 \rightarrow \pi^*i_0^!\Omega^1_{S/T} \rightarrow i^!\Omega^1_{X/S'} \rightarrow \Omega^1_{X/S'} \rightarrow 0,$$

hence an element $KS^* \in \text{Ext}^1_{\mathcal{O}_{X'}}(\Omega^1_{X'/S'}, \pi^*i_0^!\Omega^1_{S/T})$. The composition $S' \xrightarrow{i_0} S \rightarrow T$ defines a canonical surjective map $\pi^*i_0^!\Omega^1_{S/T} \rightarrow \pi^*\Omega^1_{S'/T}$. In the same way, we get a surjective map $i^!\Omega^1_{X/T} \rightarrow \Omega^1_{X'/T}$. These data define a commutative diagram of $\mathcal{O}_{X'}$-modules

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*i_0^!\Omega^1_{S/T} & \longrightarrow & i^!\Omega^1_{X/T} & \longrightarrow & \Omega^1_{X'/S'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\Omega^1_{S'/T} & \longrightarrow & \Omega^1_{X'/T} & \longrightarrow & \Omega^1_{X'/S'} & \longrightarrow & 0
\end{array}
\]

Standard diagram-chasing shows that $KS^* \rightarrow KS'$ under the canonical map of $\text{Ext}^1$ groups. The following result follows easily from the definitions.

**Proposition 2.1.** Let $A/S$ be an abelian scheme with $S$ smooth over $T$, $(S', i_0)$ a closed $T$-smooth subscheme of $S$ and $A' = A \times_S S'$. Let $KS: \omega_{\mathcal{O}_S}^{\otimes 2} \rightarrow \Omega^1_{S/T}$ and $KS': \omega_{\mathcal{O}_{S'}}^{\otimes 2} \rightarrow \Omega^1_{S'/T}$ be the corresponding Kodaira-Spencer maps. Then $KS' = i_0 \circ i_0^*KS$, where $i_0: i_0^!\Omega^1_{S/T} \rightarrow \Omega^1_{S'/T}$ is the canonical pull-back map.

Let again $X = A$ be an abelian scheme, and let $\phi: A \rightarrow A$ be an $S$-isogeny (i.e., a surjective endomorphism such that $\pi \circ s = \pi$). The pull-back $\phi^*\Omega^1_{A/T} \rightarrow \Omega^1_{A/T}$ respects filtrations. Thus we have maps $\phi^*(F^i/F^j) \rightarrow F^i/F^j$ for all $i \leq j$ because the sheaves $F^i$ are locally free. In particular, there is a map of short exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \phi^*\text{gr}^{p+1} & \longrightarrow & \phi^*(F^p/F^{p+2}) & \longrightarrow & \phi^*\text{gr}^p & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{gr}^{p+1} & \longrightarrow & F^p/F^{p+2} & \longrightarrow & \text{gr}^p & \longrightarrow & 0
\end{array}
\]

where the bottom row is the tautological exact sequence of graded objects and the top row is obtained applying $\phi^*$ to it (again, it remains exact because the sheaves are locally free). Since $\phi$ is surjective, $\pi_*\phi^* = \pi_*$ as functors and the previous diagram yields a map of derived functors long exact sequences

\[
\begin{align*}
\ldots & \longrightarrow R^{p+q}\pi_*\text{gr}^p & \longrightarrow & R^{p+q+1}\pi_*\text{gr}^{p+1} & \longrightarrow & \ldots \\
& \downarrow [\phi]_{p,q} & \downarrow [\phi]_{p+1,q} & & & \\
\ldots & \longrightarrow R^{p+q}\pi_*\text{gr}^p & \longrightarrow & R^{p+q+1}\pi_*\text{gr}^{p+1} & \longrightarrow & \ldots
\end{align*}
\]

(21)

**Proposition 2.2.** Let $A/S$ be an abelian scheme with $S$ smooth over $T$. The algebra $\text{End}_S(A)$ acts linearly on the sheaves $\mathcal{H}_{\text{dR}}^1(A/S)$. If $\phi \in \text{End}_S(A)$ acts as $[\phi]$, then

$$\nabla \circ [\phi] = (1 \otimes [\phi])\nabla.$$
Proposition 2.3. Let \( \phi \in \text{End}_S(A) \) be an isogeny. The endomorphism \([\phi]\) of \( \mathcal{H}^0_{\text{dR}}(A/S) \) attached to \( \phi \) is the vertical map \([\phi]_{0,q}\) in diagram (21) at \( R^q\pi_*\mathcal{G}^0 \). Under the identification \( R^{q+1}\pi_*\mathcal{G}^1 = R^{q+1}\pi_* (\pi^*\Omega^1_{S/T} \otimes \mathcal{O}_A \Omega^{-1}_{A/T}) = \Omega^1_{S/T} \otimes \mathcal{O}_S \mathcal{H}^0_{\text{dR}}(A/S) \), the Gauss-Manin connection is the connecting homomorphism for the tautological exact sequence of graded objects, i.e., either horizontal connecting homomorphism in (21) at \( p = 0 \) and also \([\phi]_{1,q} = 1 \otimes [\phi]_{0,q}\) since \( \phi \) acts trivially on \( S \).

The formula follows.

Let \( s \in S \) be a geometric point and \( A_s \) the fiber at \( s \). Without loss of generality we may assume that \( S \) is connected and Grothendieck’s rigidity lemma [35] implies that the canonical map \( \text{End}_S(A) \to \text{End}(A_s) \) is injective. It follows that there exist division algebras \( D_1, \ldots, D_r \) such that \( \text{End}_S(A) \) is identified to a subring of \( M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r) \). The latter algebra is spanned over \( \mathbb{Q} \) by the invertible elements, so the result follows by linearity. 

### 2.2 Computations over \( \mathbb{C} \)

In order to compute explicitly the Kodaira-Spencer map for a complex family, i.e., when \( T = \text{Spec}(\mathbb{C}) \), it is more convenient to appeal to GAGA principles, work in the analytic category and follow [25, 11]. If \( A/S \) is a principally polarized family of abelian varieties over the smooth complex variety \( S \) and \( U \subset S \) is an open set, the choice of a section \( \sigma \in H^0(U, (\Omega^1_{S/T})^\vee) \) defines a map \( \varrho_\sigma : H^0(U, \omega_{\mathcal{H}^1_{\text{dR}}}) \to H^0(U, \omega_{\mathcal{H}^1_{\text{dR}}}) \) by the composition

\[
H^0(U, \omega_{\mathcal{H}^1_{\text{dR}}}) \hookrightarrow H^0(U, \mathcal{H}^1_{\text{dR}}) \xrightarrow{\nabla} H^0(U, \mathcal{H}^1_{\text{dR}} \otimes \Omega^1_{S/T}) \xrightarrow{(\cdot, \cdot)_{\mathcal{H}^1_{\text{dR}}}} H^0(U, \mathcal{H}^1_{\text{dR}}) \xrightarrow{\varrho_\sigma} H^0(U, \omega_{\mathcal{H}^1_{\text{dR}}}),
\]

where \( \varrho_\sigma \) is induced by the polarization pairing \((\cdot, \cdot)_{\mathcal{H}^1_{\text{dR}}} : \mathcal{H}^1_{\text{dR}} \otimes \mathcal{H}^1_{\text{dR}} \to \mathcal{O}_S \). The association \( \sigma \mapsto \varrho_\sigma \) defines a map \((\Omega^1_{S/T})^\vee \to \text{Hom}(\omega_{\mathcal{H}^1_{\text{dR}}}, \omega_{\mathcal{H}^1_{\text{dR}}})\) whose dual is the Kodaira-Spencer map \( \text{KS} \). By étale-ness, the actual computation of the map \( \text{KS} \) can be obtained applying the above procedure to the pullback of the family \( A/S \) on the universal cover of \( S \). For instance, for the universal family (7)

\[
\text{KS}(d\zeta \otimes 2^{\otimes 2}) = \frac{1}{2\pi i} dz.
\]

where \( \zeta \) is the standard complex coordinate in the elliptic curve \( E_z = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}, z \in \mathbb{H} \).

We follow this approach to compute the Kodaira-Spencer map for the universal complex family of QM abelian surfaces over \( X_1 \) using the Shimura family \( \mathcal{A}^\text{sh}/\mathfrak{H} \) of (8) in terms of the arithmetic of the maximal order \( \mathcal{R}_1 \). Let \( \rho = \{r_1, \ldots, r_4\} \) be a symplectic basis of \( \mathcal{R}_1 \). By linear extension, the real dual basis \( \{r_1^\vee, \ldots, r_4^\vee\} \) of \( D_\infty^\vee \) is a basis of \( \text{Hom}(\mathcal{R}_1 \otimes \mathbb{C}, \mathbb{C}) \cong H^1_{\text{dR}}(D_\infty, \mathcal{R}_1) \). Thus, the elements \( r_1^\vee, \ldots, r_4^\vee \) define global \( C^\infty \)-sections of \( \mathcal{H}^1_{\text{dR}}(\mathcal{A}^\text{sh}/\mathfrak{H}) \) with constant periods, hence \( \nabla \)-horizontal. If \( H \) denotes the \( \mathbb{C} \)-span of these sections, there is an isomorphism \( \mathcal{H}^1_{\text{dR}}(\mathcal{A}^\text{sh}/\mathfrak{H}) = H \otimes_{\mathbb{C}} \mathcal{O}_S \). In terms of this trivialization, \( \nabla = 1 \otimes d \), where \( d \) is the exterior differentiation. Also,

\[
\langle r_i^\vee, r_j^\vee \rangle_{\text{dR}} = \frac{1}{2\pi i} B_i(r_j, r_i), \quad i, j = 1, \ldots, 4,
\]

(23)

where the \( 2\pi i \) factor accounts for the differences of Tate twists between singular and algebraic de Rham cohomology, e.g., [5, §1]. Let \( \zeta_1 \) and \( \zeta_2 \) denote the standard coordinates in \( \mathbb{C}^2 \).

Proposition 2.3. Let

\[
(\text{KS}(d\zeta_1 \otimes d\zeta_2))_{i,j=1,2} = \frac{1}{2\pi i} \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} dz.
\]
of the symplectic basis $\eta$ and finally $L_\omega$.

See [24] and also [28, section 10.13] where the extension property is discussed.

**Proof.** Write

$$
\left( \begin{array}{c} d\zeta_1 \\ d\zeta_2 
\end{array} \right) = \Pi_\omega(z) \left( \begin{array}{c} r_1 \\ \vdots \\ r_4 
\end{array} \right)
$$

where $\Pi_\omega(z)$ is the period matrix computed in terms of the basis $\omega$. Using (23) and the definitions we first obtain

$$
\varrho_\sigma \left( \frac{d\zeta_1}{dz}, \frac{d\zeta_2}{dz} \right) = \frac{1}{2\pi i} \frac{d\Pi_\omega(z)}{dz} \left( \begin{array}{c} 0 & I_2 \\ -I_2 & 0 
\end{array} \right) \left( \begin{array}{c} r_1 \\ \vdots \\ r_4 
\end{array} \right) \sigma(dz),
$$

and finally

$$
(KS(d\zeta_i \otimes d\zeta_j))_{i,j=1,2} = \frac{1}{2\pi i} \frac{d\Pi_\omega(z)}{dz} \left( \begin{array}{c} 0 & I_2 \\ -I_2 & 0 
\end{array} \right)^t \Pi_\omega(z) dz.
$$

To obtain the final formula, we make use of the Hashimoto model with $N = 1$. In terms of the symplectic basis $\eta = \{\eta_1, \ldots, \eta_4\}$ of theorem 1.1

$$
\Pi_\omega(z) = \left( \begin{array}{c} \frac{-1}{\sqrt{p_o}} (\alpha^* \alpha + 1) & -\frac{1}{\sqrt{p_o}} (\alpha^* \alpha + 1) & z & \frac{1}{2} \alpha^* z \\
\frac{1}{\sqrt{p_o}} (\alpha^* \alpha - 1) & \frac{1}{\sqrt{p_o}} (\alpha^* \alpha - 1) & -z & \frac{1}{2} \alpha^* z 
\end{array} \right)
$$

where $\alpha^* = 1 \pm \sqrt{p_o}$. Plugging these values into the previous formula yields the result. $\blacksquare$

### 2.3 Maass operators.

When $D$ is split, the universal family (7) defines the line bundle $\omega = \omega_{X_1(N)}$ on the Zariski open set $Y_1(N)$ complement of the cusp divisor $C$ in $X_1(N)$. The Kodaira-Spencer map $KS: \omega \otimes 2 \mapsto \Omega^1_{X_1(N)}$ is an isomorphism.

**Theorem 2.4.** The line bundle $\omega$ extends uniquely to a line bundle, still denoted $\omega$, on the complete curve $X_1(N)$ and the Kodaira-Spencer isomorphism extends to an isomorphism

$$
KS: \omega \otimes 2 \sim \Omega^1_{X_1(N)}(\log C).
$$

**Proof.** See [24] and also [28, section 10.13] where the extension property is discussed for a general representable moduli problem. $\blacksquare$

If $D$ is not split, the universal family (9) of QM-abelian surfaces defines the sheaf $\omega = \omega_{X_1(N)/X_1(\Delta, N)}$ and the Kodaira-Spencer map is a surjective map $KS: \text{Sym}^2 \omega \rightarrow \Omega^1_{X_1(\Delta, N)}$.

Let $p$ be a prime such that $(p, N\Delta) = 1$ and let $v$ be a place of a number field $L$ dividing $p$. The algebra $\mathcal{R}_1 \otimes \mathcal{O}_v$ acts contravariantly and $\mathcal{O}_v$-linearly on $\omega_v = \omega \otimes \mathcal{O}_v$ by pull-back. For any geometric point $s \in X_1(\Delta, N) \otimes \mathcal{O}_v$ and any non-trivial idempotent $e \in \mathcal{R}_1 \otimes \mathcal{O}_v$ there is a non-trivial decomposition $H^0(A_s, \Omega^1_{A_s/\mathcal{O}_v}) = eH^0(A_s, \Omega^1_{A_s/\mathcal{O}_v}) \oplus (1 - e)H^0(A_s, \Omega^1_{A_s/\mathcal{O}_v})$. Therefore the subsheaf $e\omega_v$ is a line subbundle. Let $e\omega_v \otimes e^1\omega_v \subseteq \text{Sym}^2 \omega_v$ be the line bundle image of $e\omega_v \otimes e^1\omega_v$ under the natural map $\omega_v \otimes 2 \rightarrow \text{Sym}^2 \omega_v$. 

Power series expansions of modular forms
Theorem 2.5. If \( p, v \) and \( e \) are as above, then the Kodaira-Spencer map defines an isomorphism
\[
\text{KS} : e \omega_v \otimes e^\dagger \omega_v \to \Omega^1_{X_1(\Delta,N)/\mathcal{O}_v}
\]
of line bundles on \( X_1(\Delta,N) \) defined over \( \mathcal{O}_v \).

Proof. We claim that the action of \( r \otimes \lambda \in \mathcal{R}_1 \otimes \mathcal{O}_v \) on the universal family (9) base-changed to \( \mathcal{O}_v \) gives rise to a commutative diagram
\[
\begin{align*}
\omega_v & \longrightarrow \Omega^1_{X_1(\Delta,N)/\mathcal{O}_v} \otimes \omega_v \ya \nu \\
\downarrow \otimes \lambda & \quad \downarrow \otimes \lambda \\
\omega_v & \longrightarrow \Omega^1_{X_1(\Delta,N)/\mathcal{O}_v} \otimes \omega_v \ya \nu
\end{align*}
\]
(25)

Indeed, under the Serre duality identification \( R^1 \pi^*_A \mathcal{O}_A \simeq \mathcal{O}_A^{SA} \) for a principally polarized abelian scheme \( A/S \) the actions of \( \text{End}_S(A) \) correspond up to Rosati involution. The commutativity of the diagram (25) follows from proposition 2.2 and (20).

For an idempotent \( e \in \mathcal{R}_1 \otimes \mathcal{O}_v \), diagram (25) defines a map \( e \omega_v \longrightarrow \Omega^1 \otimes e^\dagger (\omega_v \ya \nu) \) which can be shown to be an isomorphism by the same deformation theory argument in [8, Lemma 6]. This is enough to conclude, because the sheaves \( e^\dagger (\omega_v \ya \nu) \) and \( e \omega_v \) are dual of each other. \( \blacksquare \)

Remarks 2.6. 1. We proved theorem 2.5 for \( p \)-adically complete rings of scalars. In fact the projectors onto the quadratic subfields of \( D \) are defined over the \( p \)-adic localizations of their rings of integers for almost all \( p \). Thus, in these cases, \( e \omega \) and the Kodaira-Spencer isomorphism are defined over the subrings \( \mathcal{O}_v \subset \mathbb{C} \).

2. If the \( e' = ded^{-1} \) are conjugated in \( \mathcal{R}_1 \otimes \mathbb{B} \) for some ring \( \mathbb{B} \), then the action of \( d \) on \( \omega \otimes \mathbb{B} \) defines an isomorphism of \( e \omega \) with \( e' \omega \) over \( \mathbb{B} \).

3. In the complex case the isomorphism of theorem 2.5 can be checked by a straightforward application of the computation in section 2.2. For instance, in the Hashimoto model for \( N = 1 \) of theorem 1.1 let \( d = ai + bj + cij \in D_H \) with \( \delta = d^2 = -a^2 \Delta + b^2 p_o + c^2 \Delta p_o \in \mathbb{Q} \) and let \( e \in D \otimes \mathbb{Q} (\sqrt{\delta}) \) be the idempotent giving the projection onto \( \mathbb{Q}(d) \). Then
\[
e = \frac{1}{2\sqrt{\delta}} \begin{pmatrix}
\sqrt{\delta + b\sqrt{p_o}} & -a + c\sqrt{p_o} \\
(a + c\sqrt{p_o})\Delta & \sqrt{\delta - b\sqrt{p_o}}
\end{pmatrix}
\]
and
\[
e^\dagger = \frac{1}{2\sqrt{\delta}} \begin{pmatrix}
\sqrt{\delta + b\sqrt{p_o}} & a + c\sqrt{p_o} \\
(-a + c\sqrt{p_o})\Delta & \sqrt{\delta - b\sqrt{p_o}}
\end{pmatrix}.
\]
Therefore \( e \omega \circ e^\dagger \omega \) is generated over \( \mathcal{O} \) by the global section
\[
(\sqrt{\delta + b\sqrt{p_o}}) d\zeta_1 \circ d\zeta_1 + (c^2 p_o - a^2) d\zeta_2 \circ d\zeta_2 + 2(\sqrt{\delta + b\sqrt{p_o}}) c\sqrt{p_o} d\zeta_1 \circ d\zeta_2
\]
whose image under the Kodaira-Spencer map \( \text{KS} \) is, by proposition 2.3,
\[
\frac{1}{\pi i} (\sqrt{\delta + b\sqrt{p_o}}) dz \in \Gamma(\mathcal{O}, \Omega^1_{\mathcal{O}}).
\]
(26)

Since \( \delta \neq p_o b^2 \) (else \( p_o = (a/c)^2 \in \mathbb{Q} \) which is impossible) the section (26) does not vanish and the Kodaira-Spencer map is an isomorphism.
Notation 2.7. We will denote $\mathcal{L}$ either the line bundle $\omega_{\Delta}$ on $\mathcal{Y}_1(N)$ or the line bundle $e\omega_{\Delta}$ on $X_1(\Delta, N)$ for some choice of idempotent $e$ satisfying the hypotheses of theorem 2.5 and such that $e^\dagger = e$. In either case the Kodaira-Spencer map gives an isomorphism

$$\text{KS} : \mathcal{L} \otimes^2 \mathcal{L} \xrightarrow{\sim} \Omega^1.$$ 

With an abuse of notation we will denote also $\mathcal{L}$ the pullback of the complexified bundle to $\mathfrak{H}$ under the natural quotient maps.

If $\gamma \in \Gamma_1(\Delta, N)$ the identities $\mathbb{Z}^2 (\gamma, z) = j(\gamma, z)^{-1} \mathbb{Z}^2 (\gamma, z)$ and $\Phi_\infty(\mathcal{R}_1)(\gamma, z) = j(\gamma, z)^{-1} \Phi_\infty(\mathcal{R}_1)(\gamma, z)$ as subsets of $\mathbb{C}$ (in the split case) and of $\mathbb{C}^2$ (in the non-split case) respectively, show that the natural action of $\Gamma_1(\Delta, N)$ on $\omega_{\Delta}$ is scalar multiplication by the automorphy factor. Thus the $\Gamma_1(\Delta, N)$-action extends to a $\text{SL}_2(\mathbb{R})$-homogeneous structure on $\omega_{\Delta}$ and on $\text{Sym}^2(\omega)$ as well. Also, in the non-split case the fiber identifications induced by the action are $D \otimes \mathbb{C}$-contravariant and since the line bundle $\mathcal{L}$ is defined using the $D$ action on $\omega_{\Delta}$ it is an homogeneous line subbundle of $\text{Sym}^2(\omega)$.

Let $n \in \mathbb{Z}$ and let $V_n$ be the $1$-dimensional representation of $\mathbb{C}^\infty$ given by the character $\chi_n(z) = z^n$. Let $\mathcal{Y}_n = V_n \times \mathfrak{H}$ the homogenous line bundle on $\mathfrak{H}$ with action $g \cdot (v, z) = (\chi_n(j(g, z))v, g \cdot z)$. Since $-1 \notin \Gamma_1(\Delta, N)$ and $\Gamma_1(\Delta, N)$ has no elliptic elements, the quotient $\Gamma_1(\Delta, N)/\mathcal{Y}_n$ is a line bundle on $X_1(\Delta, N)$ which we shall denote $\mathcal{Y}_n$ again. Pick $v_n \in V_n$, $v_n \neq 0$, and let $\tilde{v}_n = (v_n, z)$ be the corresponding global constant section of $\mathcal{Y}_n$ over $\mathfrak{H}$. Also, let $s(z)$ be the global section of $\mathcal{L}$ over $\mathfrak{H}$, defined up to a sign, normalized so that

$$\text{KS}(s(z) \otimes^2) = 2\pi i dz. \quad (27)$$

Then $\mathcal{L} \otimes^k = \mathcal{O}_{\mathfrak{H}} s(z) \otimes^k$ for all $k \geq 1$ and there are identifications of homogeneous complex line bundles over both $\mathfrak{H}$ and $X_1(\Delta, N)$

$$\mathcal{V}_2 \xrightarrow{\sim} \Omega^1, \; \tilde{v}_2 \mapsto 2\pi i dz \quad \text{and} \quad \mathcal{V}_k \xrightarrow{\sim} \mathcal{L} \otimes^k, \; \tilde{v}_k \mapsto s(z) \otimes^k. \quad (28)$$

These identifications preserve holomorphy and are compatible with tensor products and the Kodaira-Spencer isomorphisms. Note that $s(z) = \pm 2\pi i d\zeta$ in the split case by (22), and see remark 2.6.3 for the non-split case.

Following [25], we shall define differential operators associated to splittings of the Hodge sequence (18) where $A/S$ is either the universal elliptic curve $\mathcal{E}_N/\mathcal{Y}_1(N)$ or the universal QM-abelian surface $\mathcal{A}_{\Delta,N}/\mathcal{Y}_1(\Delta, N)$.

Over the associated differentiable manifold, which amounts to tensoring with the sheaf of $\mathcal{O}$-algebras $O_S^\infty = \mathcal{C}^\infty(S^{an})$ and which will be denoted with an $\infty$ subscript, the Hodge decomposition $H^1_\infty = \omega_{\infty} \oplus \mathbb{Z}_\infty$ is a splitting of the Hodge sequence with projection $\text{Pr}_\infty : H^1_\infty \rightarrow \omega_{\infty}$. For each $k \geq 1$, let $\Theta_{k,\infty}$ be the operator defined by the composition

$$\begin{align*}
\text{Sym}^k(\omega_{\infty}) &\subset \text{Sym}^k(H^1_{\text{dR}})_\infty \xrightarrow{\nabla} \text{Sym}^k(H^1_{\text{dR}})_\infty \otimes \Omega^1 \xrightarrow{1\otimes \text{KS}^{-1}} \text{Sym}^k(H^1_{\text{dR}})_\infty \otimes \mathcal{L}_\infty \\
\downarrow \text{Pr}_\infty^\otimes \otimes 1 &\
\text{Sym}^k(\omega_{\infty}) \otimes \mathcal{L}_\infty
\end{align*}$$

where the Gauß-Manin connection $\nabla$ extends to $\text{Sym}^k$ by the product rule. The composition $\mathcal{L} \otimes^k \subset \omega_{\infty} \otimes^k \rightarrow \text{Sym}^k(\omega)$ is injective and let $\Theta_{k,\infty}$ be the restriction of $\Theta_{k,\infty}$ to $\mathcal{L} \otimes^k$.  

\[17\]
Proposition 2.8. \( \Theta_{k, \infty} \) is an operator \( \mathcal{L}^{\otimes k} \rightarrow \mathcal{L}^{\otimes k+2} \)

**Proof.** If \( D \) is split then \( \mathcal{L}^{\otimes k} = \omega^{k} = \text{Sym}^{k}(\omega) \) and there is nothing to prove. If \( D \) is non-split, the element \( e^{\otimes k} \in (\mathcal{R}_{1} \otimes_{Z} \mathcal{O}(v))^{\otimes k} \) acting componentwise defines a projection \( \omega^{\otimes k} \rightarrow \mathcal{L}^{\otimes k} \) which factors through \( \text{Sym}^{k}(\omega) \). By proposition 2.2 the Gauß-Manin connection \( \nabla \) commutes with \( e^{\otimes k} \) and also the Hodge projection \( \text{Pr}_{\infty} \) is the identity on \( \mathcal{L}_{\infty} \) (in fact on \( \omega_{\infty} \)). The result follows. ■

The operators \( \Theta_{k, \infty} \) can be computed in terms of the complex coordinate \( z = x + iy \in \mathcal{H} \) and the identifications (28). For any (say \( C^{\infty} \)) function \( \phi \) on \( \mathcal{H} \) let

\[
\delta_{k}(\phi) = \frac{1}{2\pi i} \left( \frac{d}{dz} + \frac{k}{2iy} \right) \phi.
\]

The operator \( \delta_{k} \) was introduced, together with its higher dimensional analogues by Maass [31] and later extensively studied by Shimura (see [18, Ch. 10] and the references cited therein).

**Proposition 2.9.** There are commutative diagrams of \( C^{\infty} \)-bundles and differential operators

\[
\begin{array}{ccc}
\mathcal{V}_{k} & \xrightarrow{\sim} & \mathcal{L}^{\otimes k} \\
\downarrow \delta_{k} & & \downarrow \Theta_{k, \infty} \\
\mathcal{V}_{k+2} & \xrightarrow{\sim} & \mathcal{L}^{\otimes k+2}
\end{array}
\]

where \( \delta_{n}(\phi \tilde{v}_{n}) = \delta_{n}(\phi)\tilde{v}_{n+2} \).

**Proof.** The diagram for \( D \) split is but the simplest case (dimension 1) of [11, Theorem 6.5]. The computation in the non-split case is very similar. Let \( s \) be the KS-normalized section of \( \mathcal{L} \) as in (27), \( \eta = \{ \eta_{1}, \ldots, \eta_{4} \} \) be Hashimoto’s symplectic basis of theorem 1.1 and \( \Pi = \Pi_{2}(z) \) the period matrix as in (24). Since the sections \( \eta_{1}, \ldots, \eta_{4} \) are \( \nabla \)-horizontal,

\[
\nabla \left( \begin{array}{c}
\frac{d\zeta_{1}}{dz} \\
\vdots \\
\frac{d\zeta_{4}}{dz}
\end{array} \right) = d\Pi \left( \begin{array}{c}
\eta_{1}^{\vee} \\
\vdots \\
\eta_{4}^{\vee}
\end{array} \right) = d\Pi \left( \begin{array}{c}
\zeta_{1} \\
\vdots \\
\zeta_{4}
\end{array} \right) = (I_{2}, I_{2}) \left( \begin{array}{c}
\frac{d\zeta_{1}}{dz} \\
\vdots \\
\frac{d\zeta_{4}}{dz}
\end{array} \right) \otimes dz.
\]

Since \( s \) is in the \( \mathbb{C} \)-span of \( d\zeta_{1} \) and \( d\zeta_{2} \), \( \nabla(s) = \left( \frac{1}{z-\bar{z}} s + s_{0} \right) \otimes dz \) with \( \text{Pr}_{\infty}(s_{0}) = 0 \). Plugging this into \( \Theta_{k, \infty}(\phi s^{\otimes k}) = \text{Pr}_{\infty}(1 \otimes \text{KS}^{-1}) \left( \frac{d\zeta}{dz} s^{\otimes k} \otimes dz + k\phi s^{\otimes k-1} \nabla(s) \right) \) yields the result. ■

Let \( B \) be a \( p \)-adic algebra with \( (p, \Delta N) = 1 \) and such that the \( e \) is defined over \( B \) and the isomorphism of theorem 2.5 holds for the sheaves base-changed to \( B \). Let \( \mathcal{O}(^{(p)}) \) be the structure sheaf of the formal scheme \( S^{(p)} = \lim \left( S \otimes_{B/p^{n}B} B^{p-\text{ord}} \right) \) obtained taking out the non-ordinary points in characteristic \( p \). Denote \( \mathcal{M}^{(p)} \) the tensorization with \( \mathcal{O}(^{(p)}) \) of the restriction to \( S^{(p)} \) of a sheaf \( \mathcal{M} \).

In the split case the Dwork-Katz construction [24, § A2.3] of the unique Frobenius-stable \( \nabla \)-horizontal submodule \( \mathcal{U} \subset \mathcal{H}_{dR}^{1} \otimes B \) defines a splitting \( (\mathcal{H}_{dR}^{1})^{(p)} = \omega^{(p)} \oplus \mathcal{U} \) with projection \( \text{Pr}_{p}: (\mathcal{H}_{dR}^{1})^{(p)} \rightarrow \mathcal{U}^{(p)} \). The construction can be carried out in the non split case as well. If \( B' \) is a \( B \)-algebra and \( A_{B'} \) is is a QM-abelian surface with ordinary
Power series expansions of modular forms

reduction and canonical subgroup $H$ (which, by ordinarity, is simply the Cartier dual of
the lift of the kernel of Verschiebung–an étale group), then $H \subset A[p]$ with $A[p]/H$
ilifying
an étale group and for every $\phi \in \End(A)$, $\phi(H) \subseteq H$ by connectedness. Thus $A/H$
$\Omega$-abelian surface with a canonical embedding $\mathcal{R}_1 \hookrightarrow \End(A/H)$ and the construction
of the Frobenius endomorphism of $(\mathcal{H}^1_{dR})^{(p)}$ and its splitting follows. Assuming that
the line bundle $\mathcal{L}$ is defined over $B$ and following the same procedure as in (29) with the
projection $\Pr_{\infty}$ replaced by $\Pr_p$ yields a differential operator

$$\Theta^\sigma_{k,p} : \Sym^k(\omega^{(p)}_r) \rightarrow \Sym^k(\omega^{(p)}_r) \otimes \mathcal{L}^{(p)}.$$ 

Let $\Theta_{k,p}$ be its restriction to $(\mathcal{L}^{\otimes k})^{(p)}$.

**Proposition 2.10.** $\Theta_{k,p}$ is an operator $(\mathcal{L}^{\otimes k})^{(p)} \rightarrow (\mathcal{L}^{\otimes k+2})^{(p)}$.

**Proof.** The argument is the same as in the proof of proposition 2.8. The action of the
endomorphisms commutes with the pullback of forms in the quotient $A \rightarrow A/H$ and
so with the Frobenius endomorphism. Since $\mathcal{U}$ is Frobenius-stable, the endomorphisms
commute with the projection $\Pr_p$.\hfill$

Let $* \in \{\infty, p\}$. The operators $\Theta_{k,*}$ can be iterated. For all $r \geq 1$ let

$$\Theta^{(r)}_{k,*} = \Theta_{k+2r-2,*} \circ \cdots \circ \Theta_{k,*}.$$ 

Since the kernel of the projector $\Pr_*$ is $\nabla$-horizontal one has in fact

$$\Theta^{(r)}_{k,*} = \Pr_* ((1 \otimes KS^{-1})\nabla)^r. \quad (30)$$

The operators $\Theta^{(r)}_{k,\infty}$ do not preserve holomorphy because the Hodge projection $\Pr_{\infty}$ is
not holomorphic. Similarly, the operators $\Theta^{(r)}_{k,p}$ are only defined over $p$-adically complete
ring of integers. Nonetheless, the operators $\Theta^{(r)}_{k,*}$ are algebraic over the CM locus, in
the following sense. Let $x \in X_1(\Delta, N)(\mathcal{O}_{(v)})$ be represented by a $\tau \in \mathfrak{H}$ belonging to a
$p$-ordinary test triple $(\tau, v, e)$. Let $\mathcal{L}(x) = \tau^* \mathcal{L}$ be the algebraic fiber at $x$. The choice
of an invariant form $\omega_o$ on $A_x$ which generates either $H^0(A_x, \Omega^1 \otimes \mathcal{O}_{(v)})$ (in the split
case) or $eH^0(A_x, \Omega^1 \otimes \mathcal{O}_{(v)})$ (in the non-split case) over $\mathcal{O}_{(v)}$ identifies $\mathcal{L}(x)$ with a copy
of $\mathcal{O}_{(v)}$.

**Proposition 2.11.** Let $x \in X_1(\Delta, N)(\mathcal{O}_{(v)})$ be a point represented by a $p$-ordinary test
triple and let $\omega_o$ be an invariant form on $A_x$ as above. Then, for all $r \geq 1$, the operators
$\Theta^{(r)}_{k,*}$ define maps

$$\Theta^{(r)}_{k,*}(x) : H^0(X_1(\Delta, N) \otimes \mathcal{O}_{(v)}), \mathcal{L}^{\otimes k} \rightarrow \mathcal{L}^{\otimes k+2r}(x) \simeq \mathcal{O}_{(v)} \omega_o^{\otimes k+2r}.$$ 

Moreover $\Theta^{(r)}_{k,\infty}(x) = \Theta^{(r)}_{k,p}(x)$.

**Proof.** The result follows, as in [26, theorem 2.4.5], from the following observation. Let
$A$ be an abelian variety isogenous over $\mathcal{O}_{(v)}$ to the $g$-fold product of elliptic curves with
complex multiplications in the field $K$ and ordinary good reduction modulo $v$. The CM
splitting of the first de Rham group of $A$ is the splitting $H_{dR}^1(A/\mathcal{O}_{(v)}) = H_{\sigma_1} \oplus H_{\sigma_2}$ where
$H_{\sigma_i}$ is the $\sigma_i$-eigenspace under the action of complex multiplications, $I_K = \{\sigma_1, \sigma_2\}$. The
Hodge decomposition $H_{dR}^1(A) \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ and the Dwork-Katz decomposition
$H_{dR}^1(A) \otimes B = H^0(A \otimes B, \Omega^1) \oplus U$ for some $p$-adic $\mathcal{O}(v)$-algebra $B$ are both obtained
from the CM splitting by a suitable tensoring. The result follows from the algebraicity of the Gauß-Manin connection and the Kodaira-Spencer map, using the expression (30).

For all $r \geq 1$ write
\[
\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k = \left( \frac{1}{2\pi i} \right)^r \left( \frac{d}{dz} + \frac{k + 2r - 2}{2iy} \right) \circ \cdots \circ \left( \frac{d}{dz} + \frac{k}{2iy} \right)
\]
and set $\delta_k^{(0)}(\phi) = \phi$.

3 Expansions of modular forms

3.1 Serre-Tate theory.

Let $k$ be any field, $(\Lambda, m)$ a complete local noetherian ring with residue field $k$ and $C$ the category of artinian local $\Lambda$-algebras with residue field $k$. Let $\tilde{A}$ be an abelian variety over $k$ of dimension $g$. By a fundamental result of Grothendieck [37, 2.2.1], the local moduli functor $M : C \to \text{Sets}$ which associates to each $B \in \text{Ob} C$ the set of deformations of $\tilde{A}$ to $B$, is pro-represented by $\Lambda[\tau_1, \ldots, \tau_g]$. When $k$ is perfect of characteristic $p > 0$ and $\Lambda = W_k$ is the ring of Witt vectors of $k$, deforming $\tilde{A}$ is equivalent to deforming its formal group, as precised by the Serre-Tate theory [27, §2]. If $k$ is algebraically closed and $\tilde{A}$ is ordinary, an important consequence of the Serre-Tate theory is that there is a canonical isomorphism of functors
\[
M \xrightarrow{\sim} \text{Hom}(T_p \tilde{A} \otimes T_p \tilde{A}^t, \hat{\mathbb{G}}_m),
\]
[27, theorem 2.1]. Write $M = \text{Spf}(\mathcal{R}^u)$ with universal formal deformation $A^u$ over $\mathcal{R}^u$. The isomorphism endows $M$ with a canonical structure of formal torus and identifies its group of characters $X(M) = \text{Hom}(M, \hat{\mathbb{G}}_m) \subset \mathcal{R}^u$ with the group $T_p \tilde{A} \otimes T_p \tilde{A}^t$. Denote $q_S$ the character corresponding to $S \in T_p \tilde{A} \otimes T_p \tilde{A}^t$. For a deformation $A_B$ of $\tilde{A}$ with $(B, m_B) \in \text{Ob} C$, let
\[
q(A_B; \cdot, \cdot) : T_p \tilde{A} \times T_p \tilde{A}^t \to \hat{\mathbb{G}}_m(B) = 1 + m_B
\]
be the corresponding bilinear form. When $k$ is not algebraically closed, the group structure on $M \otimes \overline{\mathbb{F}}$ descends to a group structure on $M$, for the details see [36, 1.1.14].

Let $N \subset M$ be a formal subgroup and $\rho : X(M) \to X(N)$ the restriction map. The $\mathbb{Z}_p$-module $N = \ker(\rho)$ is called the dual of $N$. Via Serre-Tate theory, $N \subset T_p \tilde{A} \otimes T_p \tilde{A}^t$. Then
\[
N \xrightarrow{\sim} \text{Hom} \left( \frac{T_p \tilde{A} \otimes T_p \tilde{A}^t}{N}, \hat{\mathbb{G}}_m \right)
\]
and
\[
N \text{ is a subtorus of } M \iff X(N) \simeq X(M)/N \text{ is torsion-free}
\]
\[
\iff N \text{ is a direct summand of } T_p \tilde{A} \otimes T_p \tilde{A}^t.
\]

To simplify some of the next statements, we shall henceforth assume that $p > 2$.
Proposition 3.1. Let \( \tilde{f}: \tilde{A} \rightarrow \tilde{B} \) be a morphism of ordinary abelian varieties over \( k \). The morphism \( f \) lifts to a morphism \( f: A \rightarrow B \) of deformations over \( B \) if and only if
\[
q(A/B; P, \tilde{f}(P)) = q(B/f; \tilde{f}(P), Q) \quad \text{for all } P \in T_p\tilde{A} \text{ and } Q \in T_p\tilde{B}.
\]
In particular, if \((\tilde{A}, \tilde{\lambda})\) is principally polarized, the formal subscheme \( \mathcal{M}^{pp} \) that classifies deformations of \( \tilde{A} \) with a lifting \( \lambda \) of the principal polarization is a subtorus whose group of characters is
\[
X(\mathcal{M}^{pp}) = \text{Sym}^2(T_p\tilde{A}).
\]

**Proof.** The first part of the statement is [27, 2.1.4]. For the second part, the principal polarization \( \tilde{\lambda} \) identifies \( T_p\tilde{A} \cong T_p\tilde{A}^t \). For a deformation \( A/B \) let \( q'(A/B; P, P') = q(A/B; P, \tilde{\lambda}(P')) \). Then \( \tilde{\lambda} \) lifts to \( A \) if and only if \( q' \) is symmetric, and the submodule of symmetric maps is a direct summand. \( \blacksquare \)

The last part of the proposition can be rephrased by saying that there is a commutative diagram
\[
T_p\tilde{A} \otimes T_p\tilde{A}^t \xrightarrow{\sim} X(\mathcal{M}) \xrightarrow{\text{restriction}} X(\mathcal{M}^{pp})
\]

Concretely, if \( \{P_1, \ldots, P_g\} \) and \( \{P_1^*, \ldots, P_g^*\} \) are \( \mathbb{Z}_p \)-bases of \( T_p\tilde{A} \) and of \( T_p\tilde{A}^t \) respectively, the \( g^2 \) elements \( q_{i,j} = q(A_i^{pp}; P_i, P_j^*) - 1 \) define an isomorphism \( R^{pp} \cong W_k[q_{i,j}] \).

If \( \tilde{A} \) is principally polarized we may take \( P_i = P_i^* \) under the identification \( T_p(\tilde{A}) \cong T_p(\tilde{A}^t) \). Then \( q_{i,j} = q_{j,i} \) on \( \mathcal{M}^{pp} = \text{Spf}(R^{pp}) \) and
\[
R^{pp} \cong W_k[q_{i,j}^{pp}], \quad \text{with } q_{i,j}^{pp} = q_{j,i}^{pp}. \quad 1 \leq i \leq j \leq g.
\]

More generally, if \( N = \text{Spf}(R_N) \) is a subtorus with \( n = \text{rk}_{\mathbb{Z}_p}(N) \), a \( \mathbb{Z}_p \)-basis \( \{S_1, \ldots, S_n\} \) of \( N \) can be completed to a basis \( \{S_1, \ldots, S_n, S_{n+1}, \ldots, S_{2g}\} \) of \( T_p\tilde{A} \otimes T_p\tilde{A}^t \). If \( q_i = q_{i,N}(A_i^{pp}); P_i, P_j^* \) and \( q_{i,N} = q_{i,N} \) then \( q_{i} \) is well-defined, \( 1 \leq i \leq 2g \) by construction and \( R_N \cong W_k[q_{i,j}^{pp}] \).

Since \( \tilde{A} \) is ordinary, there is a canonical isomorphism \( T_p\tilde{A} \cong \text{Hom}_B(\tilde{A}, \hat{G}_m) \) for any deformation \( A/B \) of \( \tilde{A} \). Composition with the pullback of the standard invariant form \( dT/T \) on \( \hat{G}_m \) yields a functorial \( \mathbb{Z}_p \)-linear homomorphism \( \omega: T_p\tilde{A} \rightarrow \omega_{A/B} \) which is compatible with morphisms of abelian schemes, in the sense that if the morphism \( f: A \rightarrow B \) lifts the morphism \( \tilde{f}: \tilde{A} \rightarrow \tilde{B} \) of abelian varieties over \( k \) then, [27, lemma 3.5.1],
\[
f^*(\omega(P')) = \omega(\tilde{f}(P')), \quad \text{for all } P' \in T_p\tilde{B}.'
\]

By functoriality, the maps \( \omega \) extend to a well-defined \( \mathbb{Z}_p \)-linear homomorphism
\[
\omega: T_p\tilde{A} \rightarrow \omega_{A/B}.
\]

whose \( \mathcal{M}^{pp} \)-linear extension \( T_p\tilde{A} \otimes O_M \rightarrow \omega_{A/B} \) is an isomorphism. Thus, a choice of a \( \mathbb{Z}_p \)-basis \( \{P_1, \ldots, P_g\} \) of \( T_p\tilde{A} \) yields an identification \( \omega_{A/B} = (\bigoplus_i \mathcal{M}^{pp}) \text{th} \) where \( \omega_i = \omega(P_i), i = 1, \ldots, g \) and the superscript \( (\text{th}) \) denotes the sheafified module.

Suppose that \( \tilde{A} \) is principally polarized. Let \( N \subset \mathcal{M}^{pp} \) be a subtorus with dual \( N \) and let \( A_N \) be the restriction over \( N \) of the universal deformation \( A^*/M \). Let
\( \{ S_1, \ldots, S_{g(g+1)/2} \} \) be a \( \mathbb{Z}_p \)-basis of \( \text{Sym}^2(T_p \tilde{A}^t) = \text{Sym}^2(T_p \tilde{A}) \) such that \( N = \bigoplus_{j=1}^n \mathbb{Z}_p S_j \).

Let \( \omega_i \) be the pullback to \( \mathcal{A}_i \) of \( \text{Sym}^2(\omega^u)(S_i) \), \( q_i^{pp} \) and \( q_i^N \) be the restriction of the local parameters \( q_i \), constructed above, \( i = 1, \ldots, g(g+1)/2 \).

**Proposition 3.2.**

\[
\text{KS}_{\mathcal{A}_i}(\omega_i^{(2)}) = \begin{cases} 
0 & \text{for } i = 1, \ldots, n \\
d \log(q_i^{pp} + 1)|N & \text{for } i = n + 1, \ldots, g(g+1)/2 
\end{cases}
\]

**Proof.** It follows from proposition 3.1 that \( \omega_i \) is a 1-dimensional torus and \( \text{Sym}^2(\omega^u)(P) \) is identified with the set of the endomorphisms given by elements of the maximal order \( \mathcal{R}_1 \).

The maximal order \( \mathcal{R}_1 \) acts naturally on \( T_p \tilde{A} \) and since \( e \in \mathcal{R}_1 \otimes \mathbb{Z}_p \) we can find a \( \mathbb{Z}_p \)-basis \( \{ P, Q \} \) of \( T_p \tilde{A} \) such that \( eP = P \) and \( eQ = 0 \).

**Proposition 3.3.**

1. \( \mathcal{N} = \text{Spf}(\mathcal{R}_1) \) is a 1-dimensional subtorus of \( \mathcal{M}^{pp} \);

2. \( \mathcal{R}_1 = W_k[[q - 1]] \), where \( q = q_{pp}^p \);

3. if \( \omega_u \) denotes the pullback of \( \omega^u(P) \) to \( \mathcal{A}_1 \), then \( \text{KS}(\omega_u^{(2)}) = d \log(q) \).

**Proof.** It follows from proposition 3.1 that \( \mathcal{N}(B) \) is identified with the set of the symmetric bilinear forms \( q: T_p \tilde{A} \times T_p \tilde{A} \rightarrow \tilde{G}_m(B) \) such that \( q(P, r^t Q) = q(rP, Q) \) for all \( r \in \mathcal{R}_1 \). This makes clear that \( \mathcal{N} \) is a subgroup, and that its dual \( N \) is the \( \mathbb{Z}_p \)-submodule generated by the elements

\[
\begin{cases} 
P_1 \otimes P_2 - P_2 \otimes P_1 & \text{for all } P_1, P_2 \in T_p(\tilde{A}) \text{ and } r \in \mathcal{R}_1. 
\end{cases}
\]

Choose \( u \) in the decomposition (11) for the subfield \( F \subset D \) so that \( u \in \mathcal{R}_1, u^t = -u \) and \( |v(u)| \) is minimal. In particular \( u^2 = -v(u) \) is a square-free integer. Pick a basis of \( D \) in \( \mathcal{R}_1 \) of the form \( \{ 1, r, u, ru \} \) with \( r^2 \in \mathbb{Z} \). From our choice of test triple we can assume that \( \mathbb{Z}[r] \) is an order in \( F \) of conductor prime to \( p \), in particular \( p \) does not
divide $r^2$. The submodule $\mathcal{R} = \mathbb{Z} \oplus \mathbb{Z} r \oplus \mathbb{Z} u \oplus \mathbb{Z} ru$ is actually an order of discriminant $-16r^4u^4$ such that $\mathcal{R}^\perp = \mathcal{R}$.

Suppose that $p|u^2$ and let $y = \frac{1}{p}(a + br + cu + dru)$ with $a$, $b$, $c$ and $d \in \mathbb{Z}$ be an element in $\mathcal{R}_1 - \mathcal{R}$ such that $\bar{y} = y + \mathcal{R}$ generates the unique subgroup of order $p$ in $\mathcal{R}/\mathcal{R}_1$. The conditions $\text{tr}(y) \in \mathbb{Z}$, $\nu(y) \in \mathbb{Z}$, and $\bar{y}^\perp = \pm \bar{y}$ easily imply that the coefficients $a$, $b$, $c$ and $d$ are all divisible by $p$ and this is a contradiction. Thus $\mathcal{R} \otimes \mathbb{Z}_p = \mathcal{R}_1 \otimes \mathbb{Z}_p$ and this reduces the set of generators for $N$ to

$$
P \otimes Q - Q \otimes P, \quad rP \otimes Q - P \otimes rQ, \quad uP \otimes P + P \otimes uP,
$$

$$
uP \otimes Q + Q \otimes uQ, \quad uQ \otimes Q + Q \otimes uQ, \quad ruP \otimes Q - P \otimes ruQ.$$

From the relations $re = er$ and $ue = (1 - e)u$ in $\mathcal{R}_1 \otimes \mathbb{Z}_p$ we get that the elements $r$ and $u$ act on the basis $\{P, Q\}$ as the matrices $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ and $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ respectively. Since $\alpha$, $\beta$ and $\gamma$ are $p$-units, finally $N$ turns out to be the $\mathbb{Z}_p$-module generated by

$$
P \otimes Q - Q \otimes P, \quad P \otimes Q + \beta Q \otimes Q, \quad P \otimes Q + Q \otimes P$$

and

$$T_p \bar{A} \otimes T_p \bar{A} = N \oplus \mathbb{Z}_p(P \otimes P).$$

This proves points 1 and 2, and the last part follows at once from proposition 3.2.

**Remark 3.4.** The fact that $N$ is actually a subtorus can be reinterpreted, as in [36], in the more general context of Hodge and Tate classes.

### 3.2 Power series expansion

We shall now use the Serre-Tate theory to write a power series expansion around an ordinary CM point of a modular form $f \in \text{M}_{k,1}(\Delta, N)$ and compute the coefficients of this expansion in terms of the Maass operators studied in section 2.3. We assume that $N > 3$.

Let $\text{Sp}_D$ the full subcategory of the category of rings consisting of the rings $B$ such that $\mathcal{R}_1 \otimes B \simeq M_2(B)$. Note that if $(r, v, e)$ is a $p$-ordinary test triple, then $\mathcal{O}_v \in \text{ObSp}_D$. For a KS-normalized section $s(z)$ in (27) the assignment

$$f(z) \mapsto f^*(z) = f(z)s(z)^\otimes k$$

sets up an identification

$$\text{M}_{k,1}(\Delta, N) \simeq H^0(X_1(\Delta, N), \mathcal{L}^\otimes k)$$

defined up to a sign (the ambiguity obviously disappears for $k$ even). The identification extends naturally to an identification of the bigger space $\text{M}_{k, \infty}(\Delta, N)$ of $C^\infty$-modular forms with the global sections of the associated $C^\infty$-bundle $\mathcal{L}^\otimes k$. This “geometric” interpretation of modular forms can be used to endow the space $\text{M}_{k,1}(\Delta, N)$ with a canonical $B$-structure for any subring $B \subset C$ of definition for $\mathcal{L}$ in $\text{Sp}_D$. In fact for any ring $B$ in $\text{Sp}_D$ such that $\mathcal{L}$ is defined over $B$ the space of modular forms defined over $B$ may be defined as

$$\text{M}_{k,1}(\Delta, N; B) = H^0(X_1(\Delta, N) \otimes B, \mathcal{L}^\otimes k).$$


Remark 2.6.2 shows that this $B$-structure does not depend on the choice of $\mathcal{L}$, i.e. on the choice of idempotent $e$. If $B'$ is a flat $B$-algebra, the identification $M_{k,1}(\Delta, N; B) \otimes B' = M_{k,1}(\Delta, N; B')$ follows from the usual properties of flat base change. By smoothness, if $1/N\Delta \in B \subset \mathbb{C}$ then $M_{k,1}(\Delta, N; B) \otimes \mathbb{C} = M_{k,1}(\Delta, N; \mathbb{C}) = M_{k,1}(\Delta, N)$. In fact the assignment (32) is normalized so that $f \in M_{k,1}(N; B)$ if and only if its Fourier coefficients belong to $B$ (q-expansion principle, e.g. [24, Ch. 1] [11, theorem 4.8]).

Let $\mathcal{A}/\mathcal{X}$ be either universal family (7) or (9). Let $x \in \mathcal{X}(O(v))$ be represented by a point $\tau \in CM_{\Delta, K}$ in a split $p$-ordinary test triple $(\tau, v, e)$. Denote $A_x$ the fiber of $\mathcal{A}/\mathcal{X}$ over $x$ and $A_x$ the corresponding complex torus. We will implicitly identify the ring $W_{k(v)}$ of Witt vectors for the algebraic closure of $k(v)$ with $O_v^{nr}$. For each $n \geq 0$, let $J_{x,n} = \mathcal{O}_{\mathcal{X},x} / m_x^{n+1}$ and $J_{x,\infty} = \lim_{\rightarrow} J_{x,n} = O_{\mathcal{X},x}$. By smoothness, there is a non-canonical isomorphism $J_{x,\infty} \simeq O_v[[u]]$. For $n \in \mathbb{N} \cup \{\infty\}$ the family $\mathcal{A}/\mathcal{X}$ restricts to abelian schemes $A_{x,n}/J_{x,n}$. Tautologically $A_x = A_{x,0}$ and $A_{x,n} = A_{x,\infty} \otimes J_{x,n}$ with respect to the canonical quotient map $J_{x,\infty} \to J_{x,n}$. Also, let $J_{x,n}^{nr} = J_{x,n} \otimes O_v^{nr}$ and $A_{x,n}^{nr} = A_{x,n} \otimes J_{x,n}^{nr}$.

Let $\mathcal{M} = \text{Spf}(\mathcal{R})$ be either the full local moduli functor (in the split case) or its subfunctor described in proposition 3.3 (in the non-split case) associated with the reduction $\tilde{A}_x = A_x \otimes \mathbb{F}_v$ with universal formal deformation $A_x/\mathcal{M}$. In either case $\mathcal{M} \simeq \text{Hom}(T, \mathbb{G}_m)$ where $T$ is a free $\mathbb{Z}_p$-module of rank 1. Since the rings $J_{x,n}^{nr}$ are pro-$p$-Artinian, there are classifying maps

$$\phi_{x,n}: \mathcal{R} \to J_{x,n}^{nr}, \quad \text{for all } n \in \mathbb{N} \cup \{\infty\}$$

such that $A_{x,n}^{nr} = A_x \otimes_{\phi_{x,n}} J_{x,n}^{nr}$. Since the abelian schemes $A_{x,n}$ are the restriction of the universal (global) family, the map $\phi_{x,\infty}$ is an isomorphism. We will use it to transport the Serre-Tate parameter $q_S - 1 \in \mathcal{R}$ and the formal sections $\omega_0$ constructed in section 3.1 out of a choice of a $Z_p$-generator $S$ of $T$ to the $p$-adic disc of points in $\mathcal{X}$ that reduce modulo $p_k$ to the same geometric point in $\mathcal{X} \otimes \mathbb{F}_v$. Also, we can pull back the parameter along the translation by $x^{-1}$ in $\mathcal{M}$ to obtain a local parameter $u_x$ at $x$ (depending on $S$), namely

$$J_{x,\infty}^{nr} = O_v^{nr}[[u_x]], \quad \text{with } u_x = q_S(x)^{-1} q_S - 1.$$ The complex uniformization of $A_x$ associated with the choice of $\tau$ can be used to define transcendental periods. For any $\omega_0 \in H^0(A_x(\mathbb{C}), \mathcal{L}(x))$ write

$$\omega_0 = p(\omega_0, \tau) s(\tau) \quad p(\omega_0, \tau) \in \mathbb{C},$$

under the isomorphism $A_x(\mathbb{C}) \simeq A_\tau$. For $f \in M_{k,1}(\Delta, N)$ define complex numbers

$$c^{(r)}(f, x, \omega_0) = \frac{\delta_k^{(r)}(f)(\tau)}{p(\omega_0, \tau)^{k+2r}} \quad r = 0, 1, 2, \ldots$$

The use of $x$ in the definition (33) is justified by the following fact.

**Proposition 3.5.** Suppose that $f \in M_{k}(\Gamma)$ for some Fuchsian group of the first kind $\mathbb{R}_1^+ \geq \Gamma \geq \Gamma_1(\Delta, N)$. Then the numbers $c^{(r)}(f, x, \omega_0) \in \mathbb{C}$ do not depend on the choice of $\tau$ in its $\Gamma$-orbit.

**Proof.** For any $\gamma \in \Gamma$, multiplication by $j(\gamma, \tau)^{-1}$ induces an isomorphism of complex tori $A_x \rightarrow A_{\gamma x}$. Since $s$ is a global constant section of $\mathcal{L}$ over $\mathcal{O}_x$ under the standard identifications of invariant forms, $s(\gamma \tau) = s(\tau) j(\gamma, \tau)^{-1}$. The assertion follows at once.\[\blacksquare\]
The periods \( p(\omega_0, \tau) \) (and consequently the numbers \( c^{(r)}(f, x, \omega_0) \)) can be normalized by choosing \( \omega_0 \) as in proposition 2.11. For such a choice, defined up to a \( v \)-unit, set
\[
\Omega_\infty = \Omega_\infty(\tau) = p(\omega_0, \tau), \quad c_v^{(r)}(f, x) = c^{(r)}(f, x, \omega_0).
\]
Also, define the \( p \)-adic period \( \Omega_p = \Omega_p(x) \in \mathcal{O}_{\mathcal{V}}^{nr,x} \) (again defined up to a \( v \)-unit) as
\[
\omega_0 = \Omega_p \omega_0(x).
\]

Let \( f \in M_{k,1}(\Delta, N; \mathcal{O}_V^{nr}) \). Over \( \mathrm{Spf}(J_{\mathcal{V}}^{nr}) \) write \( f^* = f_x \omega_k^x \) and
\[
\text{jet}_x(f^*) = x^* \text{jet}(f_x) \otimes \omega_u(x)^{\otimes k} = \left( \sum_{n=0}^{\infty} \frac{b_n(f, x)}{n!} U_x^n \right) \omega_u(x)^{\otimes k}
\]
where \( f_x \) is expanded at \( x \) in terms of the formal local parameter \( U_x = \log(1 + u_x) \).

**Theorem 3.6.** Let \( x \in X_{\Delta, N}(\mathcal{O}_V) \) be represented by a split \( p \)-ordinary test triple \((\tau, v, e)\) and \( f \in M_k(\Delta, N; \mathcal{O}_V) \). Then, for all \( r \geq 0 \),
\[
\frac{b_r(f, x)}{\Omega_p^{k+2r}} = c_v^{(r)}(f, x) \in \mathcal{O}_V.
\]

**Proof.** The case \( r = 0 \) is clear, so let us assume that \( r \geq 1 \).

We have \( \nabla(f^*) = \nabla(f_x \otimes \omega_k^x) = df_x \otimes \omega_k^x + k f_x \omega_k^{x-1} \nabla(\omega_u) \). Since \( \nabla(\omega_u(P)) \in H^0(\mathcal{M}, \mathcal{U}) \) for each \( P \in T_p(A) \), [27, theorem 4.3.1], the term containing \( \nabla(\omega_u) \) is killed by the projection \( \text{pr}_p \). Also, \( dU_x = d \log(u_x + 1) = d \log(q + 1) \) doesn’t depend on \( x \) and we obtain \( \Theta_{k,p}(f^*) = (df_x/dU_x) \omega_k^{x-2} \). Iterating the latter computation \( r \) times and evaluating the result at \( x \) yields
\[
\Theta_{k,p}^{(r)}(f^*)(x) = \frac{d^r f_x}{dU_x^r}(x) \omega_k^{x+2r}(x) = \frac{b_r(f, x)}{\Omega_p^{k+2r}} \omega_k^{x+2r}.
\]

On the other hand, applying \( r \) times the proposition 2.9 and evaluating at \( \tau \) yields
\[
\Theta_{k,\infty}^{(r)}(f^*)(x) = \delta_k^{(r)}(f)(\tau)^{\otimes k+2r} = c^{(r)}(f, x, \omega_0) \omega_k^{x+2r}.
\]

The result follows from proposition 2.11. \( \blacksquare \)

This result has a converse. For, we need the following preliminary discussion. Let \( \mathcal{D} \) be any domain of characteristic 0 and field of quotients \( \mathcal{K} \). The formal substitution \( u = e^U - 1 = U + \frac{1}{2!} U^2 + \frac{1}{3!} U^3 + \cdots \) defines a bijection between the rings of formal power series \( \mathcal{K}[[u]] \) and \( \mathcal{K}[[U]] \). Under this bijection the ring \( \mathcal{D}[[u]] \) is identified with a subring of the ring of Hurwitz series, namely power series of the form
\[
\sum_{n=0}^{\infty} \frac{\beta_n}{n!} U^n, \quad \text{with } \beta_n \in \mathcal{D} \text{ for all } n = 0, 1, 2, \ldots.
\]

We say that a power series \( \Phi(U) \in \mathcal{K}[[U]] \) is \( u \)-integral if \( \Phi(U) = F(e^U - 1) \) for some \( F(u) \in \mathcal{D}[[u]] \). Denote \( c_{n,r} \) the coefficients defined by the polynomial identity
\[
n! \binom{X}{n} = X(X-1) \cdots (X-n+1) = \sum_{r=0}^{n} c_{n,r} X^r.
\]

The following possibly well known result is closely related to [41, Théorème 13].

---

**Power series expansions of modular forms**
Theorem 3.7. A Hurwitz series $\Phi(U) = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} Z^n$ is $u$-integral if and only if 
\[ \frac{1}{d!} (c_{d, 1}\beta_1 + c_{d, 2}\beta_2 \pm \cdots + c_{d, d}\beta_d) \in \mathcal{D} \text{ for all } d = 1, 2, \ldots. \]

Proof. Let $F(u) = \Phi((1 + u)) \in \mathcal{K}[[u]]$. For any polynomial $P(X) = p_0 + p_1 X + \cdots + p_d X^d \in \mathcal{K}[X]$ of degree $d$, an immediate chain rule computation yields

\[ F(u) = p_0 \beta_0 + p_1 \beta_1 + \ldots + p_d \beta_d. \]

Also $P \left( (u + 1) \frac{d}{du} \right) F(u) \bigg|_{u=0} = P \left( (u + 1) \frac{d}{du} \right) F_d(u) \bigg|_{u=0}$ where $F_d(u)$ is the degree $d$ truncation of $F(u)$, i.e. $F(u) = F_d(u) + U^{d+1} H(u)$. On the space of polynomials of degree $\leq d$, the substitution $u = v - 1$ is defined over $\mathcal{D}$ and $P \left( (u + 1) \frac{d}{du} \right) F_d(u) \bigg|_{u=0} = P \left( (v + 1) \frac{d}{dv} \right) F_d(v - 1) \bigg|_{v=1}. \]

Since $P \left( (v + 1) \frac{d}{dv} \right) v^k \bigg|_{v=1} = P(k) v^k$, the argument shows that if $\Phi(U)$ is $u$-integral, then the expression $p_0 \beta_0 + p_1 \beta_1 + \ldots + p_d \beta_d$ is a $\mathcal{D}$-linear combination of the values $P(0), P(1), \ldots, P(d)$.

On the other hand, the argument also shows that

\[ \frac{1}{d!} (c_{d, 1}\beta_1 + c_{d, 2}\beta_2 \pm \cdots + c_{d, d}\beta_d) = \left( \frac{v^d dv}{d} \right) F_d(u - 1) \bigg|_{v=1} \]

is the coefficient of $U^d$ in $F$.

Therefore, we obtain that $\Phi(U)$ is $u$-integral if and only $p_0 \beta_0 + p_1 \beta_1 + \ldots + p_d \beta_d \in \mathcal{D}$ for every polynomial $P(X) = p_0 + p_1 X + \cdots + p_d X^d \in \mathcal{K}[X]$ such that $P(0), P(1), \ldots, P(d) \in \mathcal{D}$.

We conclude observing that a degree $d$ polynomial $P(X) \in \mathcal{K}[X]$ such that $P(0), P(1), \ldots, P(d) \in \mathcal{D}$ is necessarily numeric, i.e. $P(\mathbb{N}) \subseteq \mathcal{D}$ and that the $\mathcal{D}$-module of numeric polynomials is free, generated by the binomial coefficients. 

Note that when $\mathcal{D}$ is a ring of algebraic integers, or one of its non-archimedean completions, the conditions of the theorem can be readily rephrased in terms of congruences, known as Kummer-Serre congruences.

Denote $L^{v, sc}$ the compositum of all finite extensions $L \subseteq F$ such that $v$ splits completely in $F$ and let $\mathcal{O}^s_v$ be the integral closure of $\mathcal{O}(v)$ in $L^{v, sc}$.

Theorem 3.8 (Expansion principle). Let $f \in M_{k, 1}(\Delta, N)$ and $x \in \mathcal{X}(\Delta, N)(\mathcal{O}(v))$ be represented by a split $p$-ordinary test triple $(\tau, v, \epsilon)$ such that the numbers $c_{v}^{(r)}(f, x) \in \mathcal{O}(v)$ for all $r \geq 0$ and the $p$-adic numbers $\Omega_{p}^{2r} c_{v}^{(r)}(f, x)$ satisfy the Kummer-Serre congruences. Then $f$ is defined over $\mathcal{O}(v)$.

Proof. Choose a field embedding $i: \mathbb{C} \rightarrow \mathbb{C}_p$ to view $f \in M_{k, 1}(\Delta, N; \mathbb{C}_p)$. For all $r \geq 0$ set $c_r = c_{v}^{(r)}(f, x)$ and $\beta_r = c_r \Omega_{p}^{2r+2k} \in \mathbb{C}_p$. Unwinding the computations that led to the equality in theorem 3.6 shows that $\text{jet}_x(f^*) = \left( \sum_{r \geq 0} \frac{\beta_r}{r!} U^r \right) \omega_u(x) \otimes \mathbb{C}_p \otimes \mathbb{C}_p \otimes \mathbb{C}_p$. We claim that $\text{jet}_x(f^*)$ is defined over $\mathcal{O}(v)$. Write

\[ \text{jet}_x(f^*) = \left( \sum_{r \geq 0} \frac{\beta_r}{r!} U^r \right) \omega_u(x) \otimes \mathbb{C}_p \otimes \mathbb{C}_p \otimes \mathbb{C}_p. \]

Since the $\beta_r$ are $v$-integral and satisfy the Kummer-Serre congruences, the first equality shows that $\text{jet}_x(f^*)$ is $v$-integral. Since the formal substitution $u = e^U - 1$ preserves the field of definition, the claim follows from the second equality if we check that the formal local parameter $\Omega_{p}^{2r} U_x$ is defined over $L_v$. The group $\text{Aut}(\mathbb{C}_p/L_v)$ acts on the
section \( \omega_u \) via the action of its quotient \( \text{Gal}(L_{v}^u/L_v) \simeq \text{Gal}(\overline{F}_v/k_v) \) on \( T_p(\overline{A}_F) \) which is scalar because \( A_F \) is either an elliptic curve or isogenous to a product of elliptic curves. Thus, the section \( \Omega_{p\omega_u} \), whose restriction at \( x \) is defined over \( L_v \), is itself defined over \( L_v \). Therefore \( \Omega_p^2 dU_x = KS(O_{p\omega_u}[2]) \) is defined over \( L_v \), and so is \( \Omega_p^2 dU_x \) because it is a priori defined over \( L_{v}^u \) and its value at the point \( x \), defined over \( L_v \), is 0.

We can now use the very same arguments of Katz’s proof [24] of the \( q \)-expansion principle to conclude that the section \( f^* \) is defined over \( \mathcal{O}_v \). For, observe that the \( q \)-expansion of a modular form \( f \) at the cusp \( s \) multiplied by the right power of the canonical Tate form is \( \text{jet}_s(f^*) \). The specific nature of a cusp in the modular curve plays no role in Katz’s proof, which works as well when the former are replaced by any point in a smooth curve.

Since \( f^* \) is defined over \( \mathbb{C} \) and over \( \mathcal{O}_v \), the modular form \( f \) is defined over the integral closure of \( \mathcal{O}(v) \) in the largest subfield \( F \subset \mathbb{C} \) such that \( \mathfrak{i}(F) \subset L_v \). The assertion follows from the arbitrariness of the choice of \( \mathfrak{i} \), since \( L_{v}^c \) can be characterized as the largest subfield of \( \mathbb{C} \) whose image under all the embeddings \( \mathbb{C} \to \mathbb{C}_p \) is contained in \( L_v \). ■

4 \( p \)-adic interpolation

4.1 \( p \)-adic \( K^\times \)-modular forms

A \emph{weight} for the quadratic imaginary field \( K \) is a formal linear combination \( w = w_1 \sigma_1 + w_2 \sigma_2 \in \mathbb{Z}[I_K] \), which will be also written \( w = (w_1, w_2) \). Following our conventions, write \( z^w = z^{w_1}z^{w_2} \) for all \( z \in \mathbb{C} \). Also, let \( \bar{w} = (w_2, w_1) \) and \( |w| = |w_1| \) with \( \bar{1} = (1, 1) \).

**Definition 4.1** (Hida [17]). Let \( E \supseteq K \) be a subfield of \( \mathbb{C} \). The space \( \widetilde{S}_w(n; E) \) of \( K^\times \)-modular form of weight \( w \) and level \( n \) with values in \( E \) is the space of functions \( \tilde{f} : \mathcal{I}_n \to E \) such that

\[
\tilde{f}(\lambda I) = \lambda^w \tilde{f}(I)
\]

for all \( \lambda \in K^\times_n \).

A remarkable subset of \( \widetilde{S}_w(n; E) = \widetilde{S}_w(n; \mathbb{C}) \) is the set of algebraic Hecke characters of type \( \Lambda_0 \),

\[
\mathcal{Z}_w(n) = \widetilde{S}_w(n) \cap \text{Hom}(\mathcal{I}_n, \mathbb{C}^\times).
\]

A well-known property noted by Weil [48] is that for every \( \tilde{\xi} \in \mathcal{Z}_w(n) \) there exists a number field \( E_{\tilde{\xi}} \) such that \( \tilde{\xi} \in \widetilde{S}_w(n; E_{\tilde{\xi}}) \). A classical construction identifies the space \( \widetilde{S}_w(n) \) with the space \( S_w(n) \) of functions \( f : K^\times_n \to \mathbb{C}^\times \) such that

\[
f(\lambda z u) = z^{-\bar{w}} f(s) \quad \text{for all } \lambda \in K^\times, z \in \mathbb{C}^\times \text{ and } u \in U_n.
\]

If \( \tilde{f} \leftrightarrow f \) under this identification, then

\[
\tilde{f}(I) = f(s) \quad \text{whenever } I = [s] \text{ and } s_v = 1 \text{ for } v = \infty \text{ and } v|n.
\]

This relation can be used to recognize \( \widetilde{S}_w(n; E) \) in \( S_w(n) \). Since \( U \mathcal{O}_K < \hat{O}_K^\times \), the functions \( f : K^\times_n \to \mathbb{C}^\times \) satisfying the relation in (34) for all \( \lambda \in K^\times, z \in \mathbb{C}^\times \) and \( u \in \hat{O}_K^\times \) form a linear subspace \( S_w(\mathcal{O}_K, c) \subset S_w(c\mathcal{O}_K) \). The subspace \( S_w(\mathcal{O}_K, c) \) includes the
Hecke characters trivial on $\hat{\mathcal{O}}_{K,c}^\times$, namely $\Xi_w(\mathcal{O}_{K,c}) = S_w(\mathcal{O}_{K,c}) \cap \text{Hom}(K_{K,c}^\times/K^\times \hat{\mathcal{O}}_{K,c}^\times, \mathbb{C})$. Denote
$$C_n = K_{K,c}^\times/K^\times \mathbb{C}^\times \mathcal{I}_n \simeq \mathcal{I}_n/P_n, \quad C_c = K_{K,c}^\times/K^\times \mathbb{C}^\times \hat{\mathcal{O}}_{K,c}^\times,$$
and let $h_n = |C_n|$ and $h_c^\pm = |C_c^\pm|$. Clearly $h_c^\pm |h_c$.

**Lemma 4.2.**
1. $\Xi_w(\mathcal{O}_K) \neq \emptyset$ if and only if $(\mathcal{O}_K)_{\mathfrak{p}} = 1$.
2. $\Xi_w(\mathcal{O}_K, \mathfrak{a}) = \Xi_w(2\mathcal{O}_K)$ and they are non-trivial if and only if $|\mathfrak{a}|$ is even.
3. If $c > 2$ then $\Xi_w(c\mathcal{O}_K) \neq \emptyset$ and $\Xi_w(\mathcal{O}_{K,c}) \neq \emptyset$ if and only if $|\mathfrak{a}|$ is even.
4. $\Xi_w(\mathcal{O}_{K,c})$ and $\Xi_w(c\mathcal{O}_K)$ are bases for $S_w(\mathcal{O}_{K,c})$ and $S_w(c\mathcal{O}_K)$ respectively.

**Proof.** Let $U < U_1$ and $C_U = K_{K}^\times/K^\times \mathbb{C}^\times U$. Then there is a short exact sequence
$$1 \to \mathbb{C}^\times/U \to K_{K,c}^\times/U \to C_U \to 1,$$
where $H_U = \mathbb{C}^\times \cap K_{K}^\times U$. The first three points follow from the observation that
$$H_{U_c} = \begin{cases} \mathcal{O}_{K,c}^\times & \text{if } c = 1, \\ \{\pm 1\} & \text{if } c = 2, \\ \{1\} & \text{if } c > 2, \end{cases} \quad \text{and} \quad H_{\hat{\mathcal{O}}_{K,c}} = \begin{cases} \mathcal{O}_{K,c}^\times & \text{if } c = 1, \\ \{\pm 1\} & \text{if } c \geq 2. \end{cases}$$

For the last part, observe that multiplication by any $\xi \in \Xi_w(c\mathcal{O}_K)$ defines, for every weight $\mathfrak{a}$, an isomorphism $S_w(c\mathcal{O}_K) \xrightarrow{\sim} S_w(c\mathcal{O}_K)$ which identifies the respective sets of Hecke characters. When $\xi \in \Xi_w(\mathcal{O}_{K,c}) \neq \emptyset$, the isomorphism restricts to an isomorphism of the subspaces $S_w(\mathcal{O}_{K,c}) \xrightarrow{\sim} S_{w,\mathfrak{p}}(\mathcal{O}_{K,c})$. So, we are reduced to check the assertion in the case of the null weight $0 = (0,0)$, which is clear because $S_0(\mathcal{O}_K)$ and $\Xi_0(c\mathcal{O}_K)$ (respectively $S_0(\mathcal{O}_{K,c})$ and $\Xi_0(\mathcal{O}_{K,c})$) are the set of functions on the finite abelian group $C_c$ (respectively $C_c^\pm$) and its Pontryagin dual.

If $\mathfrak{m}|\mathfrak{n}$ the inclusion $\mathcal{I}_n < \mathcal{I}_m$ defines a natural restriction map
$$\tilde{S}_w(\mathfrak{m}) \to \tilde{S}_w(\mathfrak{n}). \quad (36)$$

**Lemma 4.3.** The restriction maps (36) are injective.

**Proof.** We can assume that $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$ with $\mathfrak{p}$ prime and $(\mathfrak{p}, \mathfrak{m}) = 1$. Let $\tilde{f} \in \tilde{S}_w(\mathfrak{m})$ and suppose that $\tilde{f}(I) = 0$ for all ideals $I \in \mathcal{I}_n$. Let $\lambda \in K_{\mathfrak{m}}^\times$ such that $\lambda\mathcal{O}_n = \mathfrak{p}\mathcal{O}_n$. Then $\mathfrak{p}[\lambda^{-1}] \in \mathcal{I}_n$ and $0 = \tilde{f}(\mathfrak{p}[\lambda^{-1}]) = \lambda^{-w}\tilde{f}(\mathfrak{p})$, i.e. $\tilde{f}(\mathfrak{p}) = 0$ proving that $\tilde{f}$ is 0 identically. ■

For $f \in S_w(\mathfrak{n})$ and $g \in S_w(\mathfrak{n})$ let
$$\langle f, g \rangle = \begin{cases} h_n^{-1} \sum_{\sigma \in C_n} f(s_\sigma)g(s_\sigma) = h_n^{-1} \sum_{\sigma \in C_n} \tilde{f}(I_\sigma)\tilde{g}(I_\sigma) & \text{if } w' = -w, \\ 0 & \text{if } w' \neq -w, \end{cases} \quad (37)$$
where $\{s_\sigma\}$ and $\{I_\sigma\}$ are full set of representatives of $C_n$ in $K_{K,c}^\times$ and in $\mathcal{I}_n$ respectively. The bilinear form $\langle \cdot, \cdot \rangle$ extends by linearity to a pairing on $\tilde{S}(\mathfrak{n}) = \bigoplus_{w \in \mathbb{Z}[I_K]} S_w(\mathfrak{n})$, or on the corresponding space $\tilde{S}(\mathfrak{n}) = \bigoplus_{w \in \mathbb{Z}[I_K]} \tilde{S}_w(\mathfrak{n})$, compatible with the restriction.
Proof. We may use Lemma 4.3 to assume that \( S = C^0(\mathcal{E}_n, F) \) of \( F \)-valued continuous functions on \( \mathcal{E}_n = \lim_{r \to 0} \mathbb{C}_{np^r} \).

It is a \( p \)-adic Banach space under the sup norm \( \| \phi \| = \sup_{x \in \mathcal{E}_n} |\phi(x)| \) and we denote \( \mathcal{S}(n; O_F) \) its unit ball. Assume that \( E \) is a subfield of \( F \) (e.g. \( F \) is the completion of \( E \) at a prime dividing \( p \)) and write \( \tilde{S}_\omega(n; F) = \tilde{S}_\omega(n; E) \otimes F \) and \( \tilde{S}_\omega(O_K, e; F) = \tilde{S}_\omega(O_K, e; E) \otimes F \).

Remark 4.4. It follows at once from the definition (37) that the pairing \( \langle \cdot, \cdot \rangle \) takes values in \( E \) on \( E \)-valued forms.

Proposition 4.5 ([44]). For every ideal \( m \mid n \) and for every ideal \( q \) with support included in the set of primes dividing \( p \) there is a natural embedding

\[
\tilde{S}(mq; F) = \bigoplus_{\omega \in \mathbb{Z}[IK]} \tilde{S}_\omega(mq; F) \hookrightarrow \mathcal{S}(n; F).
\]

Proof. We may use Lemma 4.3 to assume that \( mq = np^a \) for some \( a \geq 1 \). Since \( \bigcap_{r \geq 0} P_{np^r} = \{1\} \) the group \( I_{np} \) embeds as a dense subset in \( \mathcal{E}_n \). The restriction of \( f \in \tilde{S}_\omega(np^a; F) \) to a coset \( I \cdot P_{np^r} \) is the function \( I(\lambda) \mapsto f(I) \chi(w) \). Since \( \omega \in \mathbb{Z}[IK] \) the character \( \chi(w) \) is continuous for the \( p \)-adic topology on \( K^\times \) and so extends to a character \( \chi(w) \) of \( (K \otimes \mathbb{Q}_p)^\times \). Therefore \( f \) extends locally to cosets of \( 1 + p^a(R_K \otimes \mathbb{Z}_p) \) and globally to the whole of \( \mathcal{E}_n \). The injectivity of the direct sum space \( \tilde{S}(np^a; F) \) follows from the linear independence of characters.

We shall denote \( \hat{f} \) the \( p \)-adic modular form associated to the \( K^\times \)-modular form \( f \). If \( f = \xi \) is an Hecke character, the \( p \)-adic form \( \hat{\xi} \) is again a character which is sometimes called the \( p \)-adic avatar of \( \xi \) (or of \( \xi \)). The density of \( I_{np} \) in \( \mathcal{E}_n \) implies also that the image of \( \tilde{S}_\omega(np^a; F) \) in \( \mathcal{S}(n; F) \) is characterized by the functional relations \( \hat{f}(\lambda s) = \lambda \chi(w) \hat{f}(s) \) for all \( \lambda \equiv 1 \mod np^a \). Thus, the association \( f \mapsto \hat{f} \) identifies \( \tilde{S}_\omega(np^a; F) \) with the closed linear subspace

\[
\mathcal{S}_{\omega, a}(n; F) = \left\{ \phi \in \mathcal{S}(n; F) \text{ such that } \phi(sx) = \phi(s) \chi(w)(x) \right\}
\]

for all \( x \in 1 + p^a(\mathcal{O}_K \otimes \mathbb{Z}_p) \).

(38) (when \( a = 0 \) the domain for \( x \) is \( (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \)). Let \( \overline{\mathcal{S}(np^a; F)} = \bigoplus_{\omega \in \mathbb{Z}[IK]} \mathcal{S}_{\omega, a}(n; F) \) be the closure of \( \tilde{S}(np^a; F) \) in \( \mathcal{S}(n; F) \). Since \( \mathcal{S}_{\omega, a}(n; F) \) is closed the projection onto the \( w \)-th summand extends to a projection \( \tilde{\pi}_w : \tilde{S}(np^a; F) \to \mathcal{S}_{\omega, a}(n; F) \). Define a pairing

\[
\langle \cdot, \cdot \rangle : \overline{\mathcal{S}(np^a; F)} \times \overline{\mathcal{S}(np^a; F)} \longrightarrow F
\]

as the composition

\[
\overline{\mathcal{S}(np^a; F)} \times \overline{\mathcal{S}(np^a; F)} \overset{m}{\longrightarrow} \overline{\mathcal{S}(np^a; F)} \overset{\tilde{\pi}_w}{\longrightarrow} \mathcal{S}_{\omega, a}(n; F) \overset{\mu_H}{\longrightarrow} F
\]

where \( m \) is multiplication and \( \mu_H \) is the Haar distribution which is bounded on the space \( \mathcal{S}_{\omega, a}(n; F) \).
Proposition 4.6. The pairing $\langle \cdot, \cdot \rangle$ extends to a continuous pairing on $\mathcal{S}(mq; F)$ which coincides with $\langle [\cdot], [\cdot] \rangle$.

Proof. The pairing $[\cdot, \cdot]$ is continuous as composition of continuous mappings. Thus it is enough to check the identity $\langle f, g \rangle = [\hat{f}, \hat{g}]$ for $f \in \mathcal{S}(mq; np; F)$ and $g \in \mathcal{S}(mq; np; F)$.

Recall that a $p$-adic distribution on $\mathbb{Z}_p$ with values in the $p$-adic Banach space $W$ over $F$ is a linear operator $C^0(\mathbb{Z}_p, F) \to W$. Given two $p$-adic distributions on $\mathbb{Z}_p$ with values in $\mathcal{S}(np; F)$ we construct a new distribution $\mu_{[\mu_1, \mu_2]}$ with values in $F$ as the composition

$$C^0(\mathbb{Z}_p, F) \xrightarrow{\mu_1 * \mu_2} \mathcal{S}(np; F) \xrightarrow{\pi_\mathbb{Q}_p} \mathcal{S}(\mathbb{Q}_p; F) \xrightarrow{\pi_{\mathbb{Q}_p}} F,$$

where $\mu_1 * \mu_2$ is the convolution product of $\mu_1$ and $\mu_2$. If $\mu_1$ and $\mu_2$ are measures (bounded distributions) $\mu_{[\mu_1, \mu_2]}$ is not a measure in general since the map $\pi_{\mathbb{Q}_p}$ is not bounded. Denote $m_k(\mu) = \int_{\mathbb{Z}_p} x^k d\mu(x)$, $k \geq 0$ the $k$-th moment of the distribution $\mu$.

Lemma 4.7. Let $M \in \mathbb{N} \cup \{\infty\}$ and suppose that there exist pairwise distinct weights $\{w_k\}$ for $0 \leq k < M$ such that $m_k(\mu_1) \in \mathcal{S}_{\mathbb{Q}_p}(np; F)$ and $m_k(\mu_2) \in \mathcal{S}_{\mathbb{Q}_p}(np; F)$ for all $0 \leq k < M$. Then

$$m_k(\mu_{[\mu_1, \mu_2]}) = \begin{cases} 0 & \text{if } 0 \leq k < M \text{ is odd}, \\ \binom{k}{l} [m_l(\mu_1), m_{l}(\mu_2)] & \text{if } 0 \leq k = 2l < M \text{ is even}. \end{cases}$$

If $M = \infty$ the latter formulae characterize the distribution $\mu$ completely.

Proof. By direct computation $m_k(\mu_{[\mu_1, \mu_2]}) = \mu_H \circ \pi_{\mathbb{Q}_p} \left( \int_{\mathbb{Z}_p} (x + y)^k \mu_1(x) \mu_2(y) \right) = \sum_{i=0}^{k} \binom{k}{i} \mu_H \circ \pi_{\mathbb{Q}_p}(m_i(\mu_1) m_{k-i}(\mu_2))$. The formula follows at once from the orthogonality relations in (37) since $w_i = w_{k-i}$ only if $k = 2l$ is even and $i = l$. The final assertion is also clear.

Let $\mu$ be a $p$-adic distribution on $\mathbb{Z}_p$ with values in a $p$-adic space $S$ of continuous $F$-valued functions on a profinite space $T$. For every $t \in T$, evaluation at $t$ defines an $F$-valued distribution $\mu(t)$ on $\mathbb{Z}_p$, $\mu(t)(\phi) = \mu(\phi)(t)$. Conversely, a family $\{\mu_t\}_{t \in T}$ of $F$-valued distributions such that the function $\mu(\phi)(t) = \mu(t)$ is in $S$ for all $\phi \in C^0(\mathbb{Z}_p, F)$ defines a $p$-adic distribution $\mu$ on $\mathbb{Z}_p$ with values in $S$ and $\mu(t) = \mu_t$ for all $t \in T$, which is obviously unique for this property.

Lemma 4.8. Let $T$ be a profinite space, $S$ a $p$-adic space of continuous $F$-valued functions and $\mu$ a $p$-adic distribution on $\mathbb{Z}_p$ with values in $S$. Then $\mu$ is a $p$-adic measure if and only if $\mu(t)$ is a $p$-adic measure for all $t \in T$.

Proof. If $\mu$ is bounded, the distributions $\mu(t)$ are obviously bounded.

Suppose that $\mu(t)$ is bounded for all $t \in T$. Let $\{\phi_k\}_{k = 0, 1, 2, ...}$ be functions in $C^0(\mathbb{Z}_p, F)$ with $|\phi_k| = 1$ and let $\varphi_k = \mu(\phi_k)$. If $|\varphi_k(t)| = p^{s_k}$ choose $t_k \in T$ such that $|\varphi_k(t_k)|_p = p^{s_k}$. If the set of values $\{p^{s_k}\}$ is not bounded we may assume without loss of generality that $r_1 < r_2 < r_3 < \cdots$ and since each $\mu(t_k)$ is bounded also that $\{t_k\}$ is an infinite set. By compactness of $T$, there exists $t \in T$, $t \neq t_k$ for all $k$, such that every neighborhood of $t$ meets $\{t_k\}$. This contradicts the boundedness of $\mu(t)$ since $|\varphi(t)|_p$ is locally constant for all $\varphi \in S$. 

30
In particular, the sequence \( \mu \binom{\chi}{\xi} \) is bounded and \( \mu \) is a measure.

As an application, let \( \tilde{\chi} \in \mathbb{Z}_w(n_p^\alpha) \) and \( \tilde{\xi} \in \mathbb{Z}_w(n_p^\alpha) \) be Hecke characters taking values respectively in \( \mathcal{O}_F^\times \) and \( \mathcal{O}_F^\times \) where \( \mathcal{O}_p \subseteq F' \) is a totally ramified subextension of \( F \). For every \( x \in \mathfrak{c}_n \) the series

\[
\sum_{k=0}^{\infty} \frac{1}{k!} \tilde{\chi}(x) \tilde{\xi}(x), \quad T = e^x - 1 = Z + \frac{1}{2}Z^2 + \frac{1}{3!}Z^3 + \cdots ,
\]

has integral coefficients in the variable \( T \) (the corresponding measure is \( \tilde{\chi}(x) \tilde{\xi}(x) \), where \( \partial_t \) denotes the Dirac measure concentrated at \( t \)). Thus, there exists a unique measure \( \mu_{\chi,\xi} \) on \( \mathbb{Z}_p \) with values in \( \mathfrak{f}(n;\mathcal{O}_F) \) such that \( m_k(\mu_{\chi,\xi}) = \tilde{\chi} \tilde{\xi}^k \). When \( \mathfrak{w}' \neq 0 \) the moments’ weights are pairwise distinct.

### 4.2 Expansions as distributions

Let \( j: K \hookrightarrow D \) a normalized embedding of conductor \( c = c_{\tau,\mathfrak{n}} \) with corresponding \( \tau \in \text{CM}_{\Delta,K} \) and \( x \in \text{CM}(\Delta,N;\mathcal{O}_{K,c}) \). Let \( y = \text{Im}(\tau) \). The embedding \( j \) defines by scalar extension a diagram

\[
\begin{array}{cccc}
K^\times_A / K^\times \mathbb{R}^x & \longrightarrow & D^\times / D^\times_A / \mathbb{Z}^x \\
\downarrow & & \downarrow \\
K^\times_A / K^\times \mathbb{R}^x \hat{\mathcal{O}}^\times_{K,c} & \longrightarrow & D^\times / D^\times_A / \hat{\mathcal{R}}^x_N \\
\downarrow & & \downarrow \\
\mathbb{C}^\times_x \simeq K^\times_A / K^\times \mathbb{C}^x \hat{\mathcal{O}}^\times_{K,c} & \longrightarrow & D^\times / D^\times_A / j(\mathbb{C}^x) \hat{\mathcal{R}}^x_N \simeq \Gamma_0(\Delta,N) \setminus \mathfrak{d}
\end{array}
\]

(39)

where the vertical maps are the natural quotient maps and \( \mathbb{Z}^x \) is the center of \( D^\times_A \). Under the decomposition

\[
D^\times_A = D^\times_Q \text{GL}_2^+(\mathbb{R}) \hat{\mathcal{R}}^x_N
\]

(40)

the idele \( d = d_{\mathfrak{w}g_{\mathfrak{w}}u} \) corresponds to the point represented by \( g_{\mathfrak{w}} \). Classfield theory provides an identification \( \mathbb{C}^\times_x \simeq \text{Gal}(H_c/K) \) where \( H_c \) is the ray class field of conductor \( c \). It is also well-known that the points in the image of the bottom map in (39) are defined over \( H_c \), so that \( \text{Gal}(H_c/K) \) acts naturally on them, and that the two actions are compatible (Shimura reciprocity law, [43]). In particular, if \( s_\tau \in K^\times_A \) represents \( \tau \in \text{Gal}(H_c/K) \), then \( s_\tau \) maps to \( x^\tau \) and \( A_{x^\tau} = A_{s_\tau}^{(s_\tau^{-1})} \). Write \( A_x(\mathbb{C}) = A_{x^\tau} = \mathbb{C}^x / \Lambda_\tau \) with \( \Lambda_\tau = \Lambda \left( \begin{smallmatrix} 1 & \epsilon \\ \epsilon & 1 \end{smallmatrix} \right) \) where \( \epsilon = 1 \) and \( \Lambda = \mathbb{Z}^2 \subset \mathbb{C} \) if \( D \) is split and \( \epsilon = 2 \) and \( \Lambda = \Phi_\infty(\mathfrak{R}_1) \subset \mathbb{C}^2 \) if \( D \) is non-split. The theory of complex multiplication implies that \( A_{x^\tau}(\mathbb{C}) \simeq \mathbb{C} / \mathbb{Z}_\tau \) where \( s_\tau \Lambda_\tau = \Lambda d_{s_\tau}^{-1} \left( \begin{smallmatrix} 1 & \epsilon \\ \epsilon & 1 \end{smallmatrix} \right) \) if \( s_\tau^{-1} = d_{\mathfrak{w}g_{\mathfrak{w}}u} \). For a fixed prime \( p \) one can choose representatives \( \{ s_\sigma \} \subset K^\times_A \) normalized as follows:

\[
\begin{cases}
\quad \ s_{\sigma,\infty} = 1,
\quad \ s_{\sigma,v} \text{ is } v\text{-integral at all finite places } v \text{ and a } v\text{-unit at the places } v|pc.
\end{cases}
\]

For each such representative \( s \) there is a diagram of complex tori

\[
\begin{array}{cccc}
A_{x^\tau} = \mathbb{C}^x / \Lambda_\tau & \xrightarrow{j(\mathfrak{w},\tau)} & \mathbb{C}^x / s\Lambda_\tau & \xrightarrow{\pi_x} & \mathbb{C}^x / \Lambda_\tau \\
\end{array}
\]

\[
\begin{array}{cccc}
A_{x^\tau}(\mathbb{C}) & \xrightarrow{\mathfrak{w}} & A_x(\mathbb{C})
\end{array}
\]
where $s = dgu$ under (40) and $\pi_s$ is the natural quotient map arising from the inclusion $s\Lambda_r \subset \Lambda_r$. The element $g \in \text{GL}_1^+(\mathbb{R})$ is defined by $g\tau \in \mathcal{S}$ only up to an element in $\mathcal{O}_{K,c}$. Choose $p$ and a place $v$ over $p$ in a number field $L$ large enough so that for each $s$ the triple $(g\tau, v, c)$ is a $p$-ordinary test triple and that the isogenies $\pi_s$ are defined over $L$.

**Lemma 4.9.** With the above notations, it is possible to choose for every $\sigma \in \text{Gal}(H_c/K)$ an invariant 1-form on $A_{x^r}$ that generates $\mathcal{L}(x^r) \otimes \mathcal{O}_c$ and for which

1. $\Omega_\infty(g\tau) \sim_{\mathcal{O}_c} j(g, \tau)\Omega_\infty(\tau)$;
2. $\Omega_p(x^r) \sim_{\mathcal{O}_c} \hat{x}_p(x)$.

**Proof.** Take $\omega_s \in H^0(A_{x^r}(\mathbb{C}), \mathcal{L}(x) \otimes \mathbb{C})$. The quotient map $\pi_s$ is the identity on (co)tangent spaces and commutes with the action of the endomorphisms. Thus $\omega_s = \pi_\ast^s(\omega_s) \in H^0(A_{x^r}(\mathbb{C}), \mathcal{L}(x^r))$ and $p(\omega_s, g\tau) = j(g, \tau)p(\omega_s, \tau)$. Furthermore, $p$ doesn't divide the degree of $\pi_s$ and so $\pi_\ast^s$ is an isomorphism between the natural $p$-adic structures on the spaces of invariant forms. This proves part 1.

For part 2 observe that the reduction mod $p$ of the dual map $\pi'^{s}_c$ gives an isomorphism of the rank 1 Tate module quotient $T$ of $\Omega_\infty$. Thus $\pi'^{s}_c(\omega_s(P))$ is a universal form on the deformations of $\hat{A}_{x^r}$ by formula (31) and the equality follows.

If $s$ and $s' = s\lambda z u$ with $\lambda \in K^\times$, $z \in \mathbb{C}^\times$ and $u \in \hat{O}_{K,c}^\times$ are two normalized representants of the same $\sigma \in \mathcal{C}_c^\dagger$ a comparison of the relations in lemma 4.9 for the decompositions $s = dgr$ and $s' = (\lambda d)(gz)(ru)$ shows that $\omega_{s'} \sim_{\mathcal{O}_{K,c}} z\omega_s$. Therefore the construction of $\omega_s$ can be extended modulo $\mathcal{O}_{K,c}^\times$-equivalence to all $s \in K^\times$ by setting

$$\omega_{s\lambda z u} \sim_{\mathcal{O}_{K,c}} z\omega_s \text{ for all } \lambda \in K^\times, z \in \mathbb{C}^\times, u \in \hat{O}_{K,c}^\times \text{ and } s \text{ normalized.}$$

(41)

Let $f \in M_{2k,0}(\Delta, N)$ and normalize the invariant form as in proposition 2.11. For all integers $r \geq 0$ such that $(\mathcal{O}_{K,c}^\times)^{2(k+r)} = 1$ define a function $c^{(r)}(f, x): K^\times_c \rightarrow \mathbb{C}$ as

$$c^{(r)}(f, x)(s) = \frac{\delta^{(r)}_s f(g\tau)}{p(\omega_s, g\tau)^{2(k+r)}}$$

where $s = dgu$ as above.

**Proposition 4.10.** Suppose that $f$ is defined over $\mathcal{O}_c$ and assume that $(\mathcal{O}_{K,c}^\times)^{2(k+r)} = 1$. Then $c^{(r)}(f, x) \in S_{2(k+r),0}(\mathcal{O}_{K,c}) \cap S_{cR_K,\mathcal{O}_c}$.

**Proof.** The modular relation for $c^{(r)}(f, x)$ follows at once from (41) and the definition since $g\tau = g\tau$. For an idèle $s$ satisfying the conditions (35) for $n = (pc)$ the invariant form $\omega_s$ satisfies proposition 2.11 and then theorem 3.6 together with lemma 4.9 shows that as a $p$-adic $K^\times$-modular form $c^{(r)}(f, x)$ has coefficients in $L_\nu$ and in fact belongs to the unit ball.

Assume that $\mathcal{O}_{K,c}^\times = \{\pm 1\}$. Let $\mu_{f,x}$ be the $p$-adic distribution on $\mathbb{Z}_p$ with values in $\mathcal{S}(c\mathcal{O}_K; L_\nu)$ such that $m_f(\mu_{f,x}) = c^{(r)}(f, x)$ and let $\mu_{c^{(r)},\xi}$ be the $p$-adic measure associated to a choice of Grössencharakters $\chi \in \Xi(-2k,0)(\mathcal{O}_{K,c})$, $\xi \in \Xi(-2,0)(\mathcal{O}_{K,c})$ as in the discussion after lemma 4.8.
Theorem 4.11. There exist a $p$-adic field $F$ and a $p$-adic measure $\mu(f, x; \chi, \xi)$ on $\mathbb{Z}_p$ with values in $\mathcal{O}_F$ such that

$$m_r(\mu_{|f,x;\mu_x,\xi}) = \begin{cases} 0 & \text{if } 0 \leq r \text{ is odd}, \\ (h_x^{2r})^{-1}\Omega_p^{-2(\kappa+l)}(2l)\cdot m_l(\mu(f, x; \chi, \xi)) & \text{if } 0 \leq r = 2l \text{ is even}, \end{cases}$$

Proof. Let $F$ be large enough to contain $L_\nu$, the field of values of $\chi$ and $\xi$ and the $p$-adic period $\Omega_p$. The expression follows from Lemma 4.7 and the fact that for a suitable choice of representants for $C^\infty_c$ we have, combining the definition (37) with theorem 3.6, proposition 4.6 and lemma 4.9, $\hat{\chi}$ and $\hat{\xi}$.

Finally, each term $\hat{\xi}(s)b_l(x^n)$ is the $l$-th moment of a suitable $p$-adic measure on $\mathbb{Z}_p$ because the identification $\sum_{n=0}^{\infty}(b_n(x^n)/n!)T^n_n = \sum_{n=0}^{\infty}a_nU^n$ with $a_n \in \mathcal{O}_F$ through the substitution $U = e^{T} - 1$ yields an identification $\sum_{n=0}^{\infty}(b_n(x^n)/n!)z^nT^n = \sum_{n=0}^{\infty}a_nV^n$ where $V = (U + 1)^z - 1$ and this substitution preserves $\mathcal{O}_F$-integrality when $z$ is a unit in a field with residue field $\mathbb{F}_p$. Conclude using the linearity of measures. ■

4.3 Special $\mathcal{L}$-values

For $f \in M_{2n,0}(\Delta, N)$, let $\phi_f \in \mathcal{L}^2(D^*_\mathbb{C})$ be the usual $\tilde{\mathcal{R}}^*_\mathbb{C}$-invariant $C^\infty$ lift of $f$ to $D^*_\mathbb{C}$. Namely, $\phi_f(d) = f(g_{\infty}, i)j(g_{\infty}, i)^{-2\pi}det(g_{\infty})^r$ if $d = d_0g_{\infty}u$ under (40). The Lie algebra $\mathfrak{g} = \mathfrak{gl}_2 \simeq \text{Lie}(D^*_\mathbb{C})$ acts on the $\mathbb{C}$-valued $C^\infty$ functions on $D^*_\mathbb{C}$ by $(A \cdot \varphi)(d) = \frac{d}{d_0}\varphi(de^{A})|_{d_0=0}$. By linearity and composition the action extends to the complexified universal enveloping algebra $\mathfrak{A}(\mathfrak{g})_\mathbb{C}$. Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X^\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$$

be the usual eigenbasis of $\mathfrak{g}_\mathbb{C}$ for the adjoint action of the maximal compact subgroup

$$\text{SO}(2) = \left\{ r(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \text{such that } \theta \in \mathbb{R} \right\}.$$ 

Since $\text{Ad}(r(\theta))X^\pm = e^{\pm 2\theta}X^\pm$, we have $X^\pm \cdot \varphi_f \in M^\infty_{2n,\pm 2,0}(\Delta, N)$. A standard computation (e.g. [3, §2.1–2]) links the Lie action to the Maass operators of §2.3, namely

$$X^\pm \cdot \phi_f = -4\pi \phi_{V^\pm f}.$$ 

For $r \geq 0$ let

$$\phi_r = \left( -\frac{1}{4\pi}X^+ \right)^r \cdot \phi_f = \phi_{\delta^r f}.$$ 

Definition 4.12. Let $f \in M_{2n,0}(\Delta, N)$, $\xi \in \Xi_u(c\mathcal{O}_K)$ for a weight $\mathfrak{w}$ such that $|\mathfrak{w}| = 0$ and $\tau = t + iy \in \text{CM}_{\Delta, K}$ with $c_{\tau, N} = c$ and associated normalized embedding $\iota$. For each $r \geq 0$, let

$$J_r(f, \xi, \tau) = \int_{K^*_\mathbb{A}/K^*_\mathbb{R}} \phi_r(j(t)d_\infty)\xi(t)\, dt$$

where $d_\infty = \left( \begin{smallmatrix} y^{1/2} & t^{1/2} \\ 0 & y^{-1/2} \end{smallmatrix} \right)$ and $dt$ is the Haar measure on $K^*_\mathbb{A}$ whose archimedean component is normalized so that $\text{vol}(\mathbb{C}^\times/\mathbb{R}^\times) = \pi$ and such that the local groups of units have volume $1$ (hence $m_c = \text{vol}(\hat{\mathcal{O}}_{K,c}^\times) = (\mathcal{O}_K/c\mathcal{O}_K)^\times : (\mathbb{Z}/c\mathbb{Z})^\times)^{-1}$).
We show that \( J_r(f, \xi, \tau) \) can be expressed in terms of the pairing introduced in §4.1. Write \( w_{K,c} = |\mathcal{O}_{K,c}^\times| \).

**Theorem 4.13.** Let \( f \in M_{2k,0}(\Delta, N) \) and \( \xi \in \Xi_{(w, -w)}(\mathcal{O}_{K,c}) \). Assume that \((\mathcal{O}_{K,c}^\times)^{2w} = 1\). Then

\[
J_r(f, \xi, \tau) = \frac{\pi mc}{w_{K,c}} h^*_\mathbb{C} \gamma^{-w} \Omega_\infty(\tau)^{-2w} \left( c(\tau) f, x, \xi \| N_{K/Q} \|^{-w} \right).
\]

**Proof.** Since the integrand function is right \( \hat{O}_{K,c}^\times \)-invariant, we have \( J_r(f, \xi, \tau) = mc \int_{K^\times/K \times \hat{\mathcal{O}}_{K,c}^\times} \phi_r(j(s_\sigma z_\tau) \xi_\sigma(z) \xi) dz \xi(z) dx \). Then

\[
K^\times/K \times \hat{\mathcal{O}}_{K,c}^\times = \bigcup_{\sigma \in \mathcal{O}_c^\times} \mathbb{C}^\times s_\sigma / \mathbb{R}^\times \mathcal{O}_{K,c}^\times \quad \text{(disjoint union)}.
\]

Therefore, \( J_r(f, \xi, \tau) = mc \int_{C/K \times \hat{\mathcal{O}}_{K,c}^\times} \phi_r(j(s_\sigma z_\tau) \xi_\sigma(z) \xi) dz \xi(z) dx \). Since the integrand is right \( \hat{O}_{K,c}^\times \)-invariant, we have \( J_r(f, \xi, \tau) = mc \int_{C/K \times \hat{\mathcal{O}}_{K,c}^\times} \phi_r(j(s_\sigma z_\tau) \xi_\sigma(z) \xi) dz \xi(z) dx \). Thus, we may now assume that

\[
J_r(f, \xi, \tau) \begin{cases} \pi mc w_{K,c}^{-1} \sum_{\sigma \in \mathcal{O}_c^\times} \xi(s_\sigma) \phi_r(j(s_\sigma d_\infty) \xi) d_\infty & \text{if } w = -k - r, \\ 0 & \text{otherwise} \end{cases}.
\]

Note that this proves the claimed formula when \( w \neq -k - r \) since the inner product in its right hand side vanishes in this case. Thus, we may now assume that \( w = -k - r \). Put \( I_\sigma = \xi(s_\sigma) \phi_r(s_\sigma d_\infty) \) and write \( s = s_\sigma = d_\tau g_\tau u_\tau \) under (40) and \( \tau_\sigma = g_\tau \tau_\tau \). Note that \( \| N_{K/Q}(s) \| = \det(g_\tau) \). Under the hypothesis \((\mathcal{O}_{K,c}^\times)^{2(k+r)} = 1\) we have

\[
I_\sigma = \xi(s) \delta_{2k} f(g_\tau d_\infty, i) j(g_\tau d_\infty, i)^{-2(k+r)} \det(g_\tau)^{k+r}.
\]

\[
y^{k+r} \xi(s) \delta_{2k} f(\tau_\tau) j(g_\tau, \tau)^{-2(k+r)} \| N_{K/Q}(s) \|^{k+r}.
\]

\[
y^{k+r} \xi(s) c(\tau) f(x, s) p(a_\tau, \tau_\tau)^{2(k+r)} j(g_\tau, \tau)^{-2(k+r)} \| N_{K/Q}(s) \|^{k+r}.
\]

\[
y^{k+r} \Omega_\infty(\tau)^{2(k+r)} \xi(s) c(\tau) f(x, s) \| N_{K/Q}(s) \|^{k+r}.
\]

where \( s_\sigma \) is a normalized representant. It is now clear that the formula follows. \( \blacksquare \)

**Definition 4.14.** Let \( M \) be a proper divisor of \( N \), \( x \in \text{CM}(\Delta, N; \mathcal{O}_{K,c}) \) and \( x \in \text{CM}(\Delta, M; \mathcal{O}_{K,c}) \) the image of \( x \) under the natural quotient map. A character \( \xi \in \Xi_{(w, -w)}(\mathcal{O}_{K,c}) \) is called \((x, M)\)-primitive if it is not trivial on \( \hat{\mathcal{O}}_{K,c}^\times \).

For a divisor \( d \) of \( N/M \) there is an embedding \( \iota_{\Delta,d} : M_{2k,0}(\Delta, M) \rightarrow M_{2k,0}(\Delta, N) \). When \( \Delta = 1 \) the embedding is simply \( f(z) \rightarrow f(dz) \). When \( \Delta > 1 \) the explicit description of \( \iota_{\Delta,d} \) is less immediate, e.g. \cite[§3]{34}. We denote \( M_{2k,0}(\Delta, N)^{M-\text{old}} \) the span of the images of the embeddings \( \iota_{\Delta,d} \) for all \( d \). After theorem 4.13 the following result can be read as an orthogonality statement between primitive characters and \( K^\times \)-modular forms arising from oldforms.
Proposition 4.15. Let $\tau \in \text{CM}_{\Delta,K}$ and $x \in \text{CM}(\Delta,N;\mathcal{O}_{K,c})$ be the point represented by $\tau$. Let $f \in M_{2\rho,0}(\Delta,N)$ be $M$-old and suppose that $\xi \in \Xi(\kappa,\tau;\mathcal{O}_{K,c})$ is $(x,M)$-primitive. Then $J_r(f,\xi,\tau) = 0$.

Proof. Consider again the first expression in (43). Let $x' \in \text{CM}(\Delta,M;\mathcal{O}_{K,c'})$ the point image of $x$ and choose a system of representants $\{s_{x'}\}$ of $C_{K,c'}$ and a system of representants $\{r_i\}$ of $\hat{O}_{K,c'/\hat{O}_{K,c}}$. Then the set of products $\{s_{x'}r_i\}$ is a system of representatives of $C_{K,c'}$ and since $d_{\infty}$ commutes with each $r_i$ and $f$ is $M$-old we obtain the expression

$$J_r(f,\xi,\tau) = \frac{\pi m}{w_{K,c}} \left( \sum_{\xi(r_i)} \xi(r_i) \right) \left( \sum_{\sigma' \in C_{K,c'}} \xi(s_{x'}) \phi_{r_i}(s_{x'},d_{\infty}) \right)$$

which vanishes because $\xi$ is non trivial on $\hat{O}_{K,c'/\hat{O}_{K,c}}$. $lacksquare$

We shall assume from now on that the modular form $f$ is a holomorphic newform with associated automorphic representation $\pi^D = \pi_f$. Let $\pi$ be the automorphic representation of $\text{GL}_2(\mathbb{A})$ corresponding to $\pi^D$ under the Jacquet-Langlands correspondence.

Other than the Weil representation $r_{\psi}$ of $\text{SL}_2(\mathbb{A})$, the adelic Schwartz-Bruhat space $S_{\mathbb{A}}(D) = \bigotimes_{p \leq \infty} S_p$ supports the unitary representation of $\text{GO}(D)(\mathbb{A})$ given by

$$L(h)\varphi(x) = \|\nu_0(h)\|_h^{-1} \varphi(h^{-1}x), \quad x \in D_{\mathbb{A}}.$$  

We assume that the archimedean space $\mathcal{S}_\infty$ consists only of the Schwartz functions on $D_\infty$ which are $K_{\infty}^1 \times K_\infty^1$-finite under the action of $D_\infty^\times \times D_\infty^\times$ via the group $\text{GO}(D)$ (§1.3). Here $K_\infty^1$ is the maximal compact subgroup of $j(K_\infty \otimes \mathbb{R}) \subset D_\infty^\times \simeq \text{GL}_2(\mathbb{R})$. As explained in [14, §5], the two representations mingle into one single representation, still denoted $r_{\psi}$, of the group $R(D) = \{(g,h) \in \text{GL}_2 \times \text{GO}(D) \text{ such that } \det(g) = \nu_0(h)\}$ given by $r_{\psi}(g,h)\varphi = r_{\psi}(g_1)L(h)\varphi$ where $g_1 = g \left( \begin{smallmatrix} 1 & 0 \\ 0 & \nu_0(h) \end{smallmatrix} \right)^{-1}$. Note that

- the assignment $(g,h) \mapsto (g_1,h)$ sets up an isomorphism $R(D) \sim \text{SL}_2 \times \text{GO}(D)$;
- the group $R(D)$ is naturally a subgroup of the symplectic group $\text{Sp}(W)$, where $W = P \otimes D$ with $P$ the standard hyperbolic plane, via $(g,h)x \otimes y = gx \otimes h^{-1}y$.

The groups $(\text{SL}_2, \text{O}(D))$ form a dual reductive pair in $\text{Sp}(W)$ and the extended Weil representation $r_{\psi}$ allows to realize the theta correspondence between the similitude groups. The theta kernel associated to a choice of $(g,h) \in R(D)$ and $\varphi \in \mathcal{S}_{\mathbb{A}}(D)$ is

$$\vartheta(g,h;\varphi) = \sum_{d \in D} r_{\psi}(g,h)\varphi(d).$$

The theta lift to $\text{GO}(D)$ of a cuspidal automorphic form $F$ on $\text{GL}_2(\mathbb{A})$ is the automorphic form on $\text{GO}(D)(\mathbb{A})$ given by

$$\theta_{\varphi}(F)(h) = \int_{\text{SL}_2(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{A})} \vartheta(gg';h;\varphi) F(gg') \, dg'$$  

(44)

where $\det(g) = \nu_0(h)$ and $dg'$ is induced by a choice of a Haar measure $dg = \prod dg_p$ on $\text{GL}_2(\mathbb{A})$. A straightforward substitution yields

$$\theta_{r_{\psi}(g_1,h_1)}(F)(h) = \vartheta(\pi(g_1^{-1}F)(hh_1), \quad \forall (g_1,h_1) \in R(D).$$  

(45)

An automorphic form $\Phi$ on $\text{GO}(D)(\mathbb{A})$ pulls back via the map $g$ of (15) to an automorphic form $\tilde{\Phi}$ on the product group $D^\times \times D^\times$. Let $\hat{\Theta}(\pi)$ be the space of automorphic forms on $D^\times \times D^\times$ which are pull-backs of theta lifts (44) with $F \in \pi$. If $\pi^D$ denotes the contragredient representation of $\pi^D$ the crucial result is, with a slight abuse of notation, the following, [42].
Theorem 4.16 (Shimizu). \( \wedge(\pi) = \pi_D \otimes \pi_D \).

Remarks 4.17. 1. In our case of interest \( \pi_D = \pi_D \).

2. The Schwartz functions, hence the theta lifts \( \bar{\theta}_p(F) \), are \( K_\infty^1 \times K_\infty^1 \)-finite. Thus, in Shimizu’s theorem the representation space \( \pi_D \) consists of \( K_\infty^1 \)-finite automorphic forms. Note that the functions \( \pi(d_\infty)\phi_r \) are \( K_\infty^1 \)-finite.

3. An explicit version of Shimizu’s theorem has been worked out by Watson [47], see also [38, §3.2] and [14, §12]. Namely, if \( \varphi = \otimes_{p \leq \infty} \varphi_p \) is chosen as

\[
\varphi_\infty(z_1, z_2) = \frac{(-1)^n}{\pi} \sqrt{2} \pi^{-2} \pi(z_1 \bar{z}_1 + z_2 \bar{z}_2), \quad \varphi_p = \frac{\chi_p}{\text{vol}(\mathbb{Q}_p \otimes \mathbb{Z}_p^\times)}
\]

(46)

where \( z_1 \) and \( z_2 \) are the complex coordinates in \( D_\infty \) of \( \S \). Thus the seesaw identity \([30]\) associated with the seesaw pair \( \theta \)

\[
\text{Let } \xi = (\xi, \xi') \in \mathbb{Z}_w(cO_K) \times \mathbb{Z}_w(cO_K) \text{ thought of as a character of the torus } K_\mathbb{A}^\times \times K_\mathbb{A}^\times. \text{ Let } \tilde{H}(t) \text{ be any function on } K_\mathbb{A}^\times \times K_\mathbb{A}^\times \text{ such that } \tilde{H}(t)(\xi)(t) \text{ is } (K_\mathbb{R}^\times)^2 \text{-invariant. Following } [13, \S 14] [12, \S 1.4] \text{ we let }
\]

\[
L_\xi(\tilde{H}) = \int_{(K \times \mathbb{R}^\times \backslash K_\mathbb{A}^\times)^2} \tilde{H}(t)(\xi)(t) \, dt.
\]

In particular, for \( \xi \) as in definition 4.12,

\[
L_{(\xi, \xi')}(\pi(d_\infty)\phi_r \otimes \pi(d_\infty)\phi_r) = J_\infty(f, \xi, \tau)^2.
\]

When \( \xi = \xi' \) is unitary, \( |w| = |w'| = 0 \), the integral \( L_\xi(\bar{\theta}_p(F)) \) can also be read, via the map \( \alpha \) of (15), as the Petersson scalar product of two automorphic forms on the similitude group \( T = \hat{G}(O(K) \times O(K_\infty^\perp)) \) associated with the decomposition \( D = K \oplus K_\infty^\perp \), namely

\[
L_{(\xi, \xi')}(\bar{\theta}_p(F)) = \int_{T(\mathbb{Q}) \times \mathbb{R} \backslash T(\mathbb{A})} \bar{\theta}_p(F)(\xi)(t) \, \text{d}^\times t \, \text{d}^\times b,
\]

where \( \alpha(t) = (a, b) \). Thus the seesaw identity [30] associated with the seesaw dual pair

\[
\begin{array}{ccc}
\text{GL}_2 \times \text{GL}_2 & \xrightarrow{\uparrow} & \text{GO}(D) \\
\uparrow & \times & \uparrow \\
\text{GL}_2 & \xrightarrow{\uparrow} & G(O(K) \times O(K_\infty^\perp))
\end{array}
\]

identifies, up to a renormalization of the Haar measures, the value \( L_{(\xi, \xi')}(\bar{\theta}_p(F)) \) with a scalar product on \( \text{GL}_2 \),

\[
L_\xi(\bar{\theta}_p(F)) = \int_{\text{GL}_2(\mathbb{Q}) \backslash \mathbb{A} \times \text{GL}_2(\mathbb{A})} F(g) \theta^t_p(1, \xi)(g, g) \, dg,
\]

(47)

where \( \theta^t_p \) denotes the theta lift to \( \text{GL}_2 \times \text{GL}_2 \). If \( \varphi \) is split and primitive, i.e. admits a decomposition \( \varphi = \varphi_1 \otimes \varphi_2 \) under \( D_\infty = (K \oplus K_\infty^\perp) \otimes \mathbb{R} \) and each component decomposes in a product of local factors, \( \varphi_i = \otimes_{p \leq \infty} \varphi_{i, p} \) for \( i = 1, 2 \), then \( \theta^t_p(1, \xi) \) splits as a product of two separate lifts. In fact

\[
\theta^t_p(1, \xi)(g_1, g_2) = E(0, \Phi, g_1) \theta_{\varphi_2}(\xi)(g_2)
\]

where:
• $E(0, \Phi, g)$ is the value at $s = 0$ of the holomorphic Eisenstein series attached to the unique flat section ([3, §3.7]) extending the function $\Phi(g) = r_\psi(g, k) \varphi_1(0)$ where $k \in K_h^\times$ is such that $N(k) = \det(g)$ and $r_\psi$ denotes here the extended adelic Weil representation attached to $K$ as a normed space (Siegel-Weil formula),

• $\theta_{\varphi_2}(\xi)(g)$ is a binary form in the automorphic representation $\pi(\xi)$ of $GL_2$ attached to $\xi$.

This expression yields a relation between the right hand side of (47) and the value at the centre of symmetry of a Rankin-Selberg convolution integral. If the Whittaker function $W$ of $F$ decomposes as a product of local Whittaker functions, the Rankin-Selberg integral admits an Euler decomposition [20] and $L(\xi, \xi)(\theta_\varphi(F))$ is equal to the value at $s = 1/2$ of the analytic continuation of

$$
\frac{1}{\prod_{q \leq \infty} L_q(\varphi_q, \xi_q, s)},
$$

where

$$
L_q(\varphi_q, \xi_q, s) = \int_{K_q} \int_{\mathbb{Q}_q^\times} W_{F, q}^\psi_0 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right) W_{\varphi_2, q}^\psi_0 \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k \right) \Phi_\varphi(\xi_q, s) \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right) \left| \frac{a}{q} \right|^{-s} d^\times a d\xi_q.
$$

The local measures are normalized so that $K_{\infty} = SO_2(\mathbb{R})$ has volume $2\pi$ and $K_q = GL_2(\mathbb{Z}_q)$ has volume $1$ for finite $q$. Also $W_{\varphi_2}$ is the Whittaker function and $\Phi^s(g) = \|a\|^{s-1/2} \Phi(g)$ if $g = nak$ under the NAK-decomposition where $|(a, b)|_q = |a/b|$. Since the local term (48) does not vanish and for almost all $q$ is the local Euler factor of some automorphic $L$-function, one obtains, as in [12, 13], a version of Waldspurger's result [46]. Namely,

$$
L_{\xi}(\theta_\varphi(F)) = \Lambda(\varphi, \xi, s)L(\pi_{K} \otimes \xi, s/2)L(\eta_{K}, 2s)^{-1}\bigg|_{s=1/2},
$$

where $\Lambda(\varphi, \xi, s)$ is a finite product of local integrals, $\pi_K$ is the base change to $K$ of the automorphic representation $\pi$ and $L(\eta_{K}, 2s)$ is the Dirichlet $L$-function attached to $\eta_K$, the quadratic character associated to $K$.

When $\varphi = \bigotimes_{p < \infty} \varphi_p$ and $F$ are chosen as in Remark 4.17.3 the local non-archimedean terms in the Rankin-Selberg integral have been explicitly computed by Prasanna [38, §3] under the simplifying assumptions that $N$ is squarefree, $c = 1$ and $\xi$ is unramified. The effect of these assumptions is that

1. the local component of $\xi$ can be written either as $\xi_q = (\xi_q^p, (\xi_q^p)^{-1})$ for some unramified character $\xi_q^e$ of $\mathbb{Q}_q^\times$ at a prime $q$ split in $K$ under the isomorphism $(K \otimes \mathbb{Q}_q^\times)^{\times} \simeq \mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$, or as $\xi_q = \xi_q^e \circ N_{K/K_q}$ for an unramified character $\xi_q^e$ of $\mathbb{Q}_q^\times$ at a prime $q$ inert in $K$, or as $\xi_q = \xi_q^e \circ N_{K/K_q}$ at a ramified prime $q$ where $\xi_q^e$ is the unramified character of $\mathbb{Q}_q^\times$ obtained by a trivial extension;

2. at a prime $q | N\Delta$ the local component $\pi_q$ is equivalent to the special representation $\sigma(|^{1/2} \xi_q, |^{1/2} \xi_q)$ with $q^{2\xi_q} = 1$. 

37
Lemma 4.19. \[ L_{\xi}(\theta_{\varphi_{\infty} \otimes \varphi_{l}}(F)) = \frac{V_N}{h_K} \lambda_{\infty}(\varphi_{\infty}, \xi, s) \left( \prod_{q \leq \infty} \nu_q(\xi_q) \right) \left( \sum_{\eta \leq \infty} \nu_q(\eta_q) \right) \eta_{\varsigma_{\infty}}(\xi, \frac{s}{2}) \frac{1}{L(\eta_{\varsigma}, 2s)} \left| \frac{1}{s=1/2} \right. \]

where \( V_N = \prod_q \text{vol}(\mathcal{R}_N \otimes \mathbb{Z}_q^{-}) \), \( \lambda_{\infty}(\varphi_{\infty}, \xi, s) = \left| \text{Nx}_{\infty} \right|_{\xi_{\infty}}(z_u)^{-1} L_{\infty}((\varphi_{\infty}, \xi_{\infty}, \hat{f}) (z_u) \) denotes the complex coordinate of \( u \) in the chosen identification \( (K \otimes \mathbb{R} \simeq \mathbb{C}) \) and

\[
\begin{align*}
\nu_q(\xi_q) &= \xi_{\mathbb{Q}_p}(q)^{n_1-n_2} & \text{if } q \text{ splits, } q \notin \Sigma, (q, N_{sf}) = 1 \ , \\
\nu_q(\xi_q) &= -\frac{1}{q+1} q^{-2} \xi_{\mathbb{Q}_p}(q)^{n_1-n_2} & \text{if } q | N_{sf}, \\
\nu_q(\xi_q) &= \xi_{\mathbb{Q}_p}(q)^{-2n} & \text{if } q \text{ is inert, } q \notin \Sigma, \\
\nu_q(\xi_q) &= \xi_{\mathbb{Q}_p}(-N_{sf}) & \text{if } q \text{ ramifies}, \\
\nu_{\infty}(\xi_{\infty}) &= \xi_{\mathbb{C}}(z_u) \end{align*}
\]

where the ideal \( \mathcal{J} \) of proposition 1.3 in \( K \otimes \mathbb{Q}_q \) is generated by \( q^{-n} \) when \( q \) is inert and decomposes as \( q^{-n} \mathbb{Z}_q \times q^{-n} \mathbb{Z}_q \) under \( K \otimes \mathbb{Q}_q \simeq \mathbb{Q}_q \times \mathbb{Q}_q \) when \( q \) is split.

Remark 4.18. It is clear that \( \nu_q(\xi_q) = 1 \) for almost all \( q \). The local terms \( \lambda_q(\nu_q) \) do depend on the choice of \( u \) in (11) (replacing \( u \) with \( xu \) the local Whittaker function \( W_{\lambda_{su}} \) gets modified by the factor \( |Nx|_{q^{1/2}} \xi_q(x)^{-1} \), but the quantity

\[ \nu(\xi, s) = \prod_{q \leq \infty} \nu_q(\xi_q) \]

depends only on \( \xi \) and the chosen embedding \( j : K \to D \).

For a pair of non-negative integers \((m, q)\) consider the function of two complex variables \( \varphi^{(m,q)}(z_1, z_2) = (z_1 z_2)^{1/2} \). \( 2 \pi (z_1 z_2)^{1/2} \).

Lemma 4.19. Let \( \varphi(z) = (z\bar{z})^{1/2} e^{-2\pi z \bar{z}}. \) Then the Fourier transform of \( \varphi \) is

\[ \hat{\varphi}(w_1 + w_2) = e^{-2\pi(w_1^2 + w_2^2)} \sum_{0 \leq \alpha, \beta \leq t} \gamma(\alpha, \beta; l) w_1^{2\alpha} w_2^{2\beta}, \]

where

\[ \gamma(\alpha, \beta; l) = \sum_{j+k=l} \left( \frac{(-4\pi)^{\alpha+\beta-l}}{\alpha, \beta \leq k} \right) \frac{1}{j} \binom{2j}{2\alpha} \frac{1}{2} \binom{2k}{2\beta} (2j - 2\alpha - 1)!!(2k - 2\beta - 1)!! \]

Proof. One has \( \hat{\varphi}(w_1 + w_2) = 2 \int_{\mathbb{R}^2} e^{2\pi i(w_1 x_1 + w_2 x_2)} (x_1^2 + x_2^2) e^{-2\pi(x_1^2 + x_2^2)} dx_1 dx_2 = 2 \sum_{j+k=l} \left( \frac{l}{j} \right) \int_{\mathbb{R}^2} e^{2\pi i w_1 x_1} x_1^j e^{-2\pi x_1^2} dx_1 \int_{\mathbb{R}^2} e^{2\pi i w_2 x_2} x_2^k e^{-2\pi x_2^2} dx_2 \) and the result follows from \( \int_{\mathbb{R}^2} e^{2\pi i x_2} x_2^2 dx_2 = \frac{1}{\sqrt{2}} e^{-2\pi x^2} \sum_{i=0}^n (-4\pi)^{-i} (2i)!! (2i - 1)! 2^{(2i)} \).
Lemma 4.20. Let \( r \geq l \geq 0 \) be integers, \( F \) the lift of a weight \( 2\kappa \) eigenform and \( \xi_{\infty} \) the character \( \xi_{\infty}(z) = (z/\bar{z})^{\kappa+r} \) of \( \mathbb{C}^\times \). Then

\[
L_{\infty}(\varphi^{l,2(\kappa+r)}, \xi_{\infty}, s) = \begin{cases} 
0 & \text{if } l < r, \\
\frac{(1-r)^{2r(\kappa-Nu^{-1}-\frac{1}{2})} \xi_{\infty}(z_e^{(s)^{r}})}{(4\pi)^{l+2(\kappa+r)}} \Gamma(s + 2\kappa + r - \frac{1}{2}) & \text{if } l = r.
\end{cases}
\]

Proof. It is well known that \( W_{F}^{\psi_{\infty}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \) is the character \( \xi_{\infty} \) and also \( \varphi_{\infty}(z) = (z/\bar{z})^{\kappa} \). We compute the other two terms in the integrand of (48) separately with \( \varphi_{1}(z) = (z_{1}z_{1})^{\kappa} \) and \( \varphi_{2}(z_{2}) = \frac{2^{2(\kappa+r)+1}}{\pi} e^{-2\pi z_{2}^2} \).

1. To compute \( \Phi_{\varphi_{1}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \) for a choice of \( \alpha \), we use the definitions (3) together with the decomposition

\[
r(\theta) = \left( \begin{array}{cc}
1 - \tan \theta & 0 \\
0 & 1 \\
1 - \sin \theta \cos \theta & 0 \\
1 - \cos \theta & 1 \\
\end{array} \right)
\]

Some straightforward passages yield \( r_{\psi_{\infty}}(r(\theta)) \varphi_{1}(0) = (-cos \theta) \varphi_{1}^{\psi}(0) \) where \( \varphi_{1}^{\psi}(z) \) is the Fourier transform of \( e^{-2\pi i \cos \theta \sin \theta} \varphi_{1}((\cos \theta)z) \).

2. To compute \( \Phi_{\varphi_{1}}^{\psi_{\infty}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \) we need to use again (3) together with the decomposition (50). For, it should be noted that this time the norm in \( \mathfrak{K}_{\psi} \otimes \mathbb{R} \cong \mathbb{C} \) is \( -N_{\psi} \) (in particular, definite negative) and the main involution is \( z \mapsto -z \). Thus, we get \( \Phi_{\varphi_{1}}^{\psi_{\infty}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) = e^{2(2\kappa+2r+1)} W_{\theta_{\psi_{\infty}}}^{\psi_{\infty}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \). On the other hand, for a choice of \( h \in \mathbb{C}^\times \) such that \( Nh = -aNu^{-1} > 0 \),

\[
W_{\theta_{\psi_{\infty}}}^{\psi_{\infty}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) = \frac{1}{2\pi} \int_{S_{1}} r_{\psi_{\infty}} \left( \left( \begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right)h \theta \right) \varphi_{2}(u) \xi_{\infty}(hu) d\theta
\]

\[
= \frac{(-aNu^{-1})^{\frac{1}{2}}}{2\pi} \int_{S_{1}} \varphi_{2}(-aNu^{-1}(h\theta)u) \xi_{\infty}(h\theta) d\theta
\]

\[
= \frac{(-aNu^{-1})^{\frac{1}{2}}}{2\pi} \int_{S_{1}} \varphi_{2}(\bar{h}\theta^{-1}u) \xi_{\infty}(h\theta) d\theta
\]

\[
= \frac{(-aNu^{-1})^{\frac{1}{2}}}{2\pi} \int_{S_{1}} (\bar{h}^{\theta-1}u)^{2(\kappa+r)} e^{-2\pi \kappa (h\theta)^{\kappa+r}(h\theta^{-1})^{-\kappa-r}} d\theta
\]

\[
= (-N^{-1})^{\frac{1}{2}} \xi_{\infty}(z_{u}) a^{\kappa+r+\frac{1}{2}} e^{-2\pi a}
Putting all the ingredients together

\[ L_\infty(\varphi^{(l,2}\kappa+r)), \xi_\infty, s) = \]

\[ = -(-Nu^{-1}) \frac{\pi}{r} \int_{s > 0} a^{s+2\kappa+r} e^{-\frac{1}{2}a^{-\frac{1}{2}}e^{-4\pi a}} e^{(2\kappa+r+1)} \theta \]

\[ \times \left( \sum_{0 \leq \alpha + \beta \leq l} \frac{\gamma(\alpha, \beta; l)(2\alpha - 1)!(2\beta - 1)!}{(-4\pi)^{\alpha + \beta}} (\cos \theta)^{\alpha + \beta} e^{-(\alpha + \beta + 1)} \theta \right) d^s a d\theta \]

\[ = -(-Nu^{-1}) \frac{\pi}{r} \int_{s > 0} a^{s+2\kappa+r} e^{-\frac{1}{2}a^{-\frac{1}{2}}e^{-4\pi a}} d^s a \]

\[ \times \sum_{0 \leq \alpha + \beta \leq l} \frac{\gamma(\alpha, \beta; l)(2\alpha - 1)!(2\beta - 1)!}{(-4\pi)^{\alpha + \beta}} \int_{S^1} (\cos \theta)^{\alpha + \beta} e^{-(\alpha + \beta + 1)} \theta d\theta \]

Since \( \alpha + \beta \leq l \leq r \) we have

\[ \int_{S^1} (\cos \theta)^{\alpha + \beta} e^{(2r - \alpha - \beta + 1)} \theta d\theta = \]

\[ \frac{1}{2^{\alpha + \beta}} \sum_{j=0}^{\alpha + \beta} \binom{\alpha + \beta}{j} \int_{S^1} e^{2i(r-j)\theta} d\theta = \begin{cases} 2^{-r+\pi} & \text{if } \alpha + \beta = l = r, \\ 0 & \text{otherwise}. \end{cases} \]

hence \( L_\infty(\varphi^{(l,2}\kappa+r)), \xi_\infty, s) = 0 \) if \( l < r \). When \( l = r \), since \( \gamma(j, r-j; r) = (r) \) and \( \sum_{j=0}^{r} \frac{(r)}{(2j-1)!(2r-2j-1)!} = 2^r r! \) as readily proved by induction, we have

\[ L_\infty(\varphi^{(r,2}\kappa+r)), \xi_\infty, s) = \frac{2\pi(-Nu^{-1}) \frac{\pi}{r} \int_{s > 0} a^{s+2\kappa+r} e^{-\frac{1}{2}a^{-\frac{1}{2}}e^{-4\pi a}} d^s a}{(-4\pi)^{r}} \]

\[ = \frac{(-1)^r 2\pi(-Nu^{-1}) \frac{\pi}{r} \int_{s > 0} a^{s+2\kappa+r} e^{-\frac{1}{2}a^{-\frac{1}{2}}e^{-4\pi a}} d^s a}{(4\pi)^{s+2\kappa+r}} \Gamma(s + 2\kappa + r - \frac{1}{2}). \]

\[ \blacksquare \]

We shall now state and prove the main result of this section.

**Theorem 4.21.** Let \( N \) be a positive integer and fix a decomposition \( N = \Delta N_o \) with \( \Delta \) a product of an even number of distinct primes and \( (\Delta, N_o) = 1 \). Let \( \pi \) be an automorphic cuspidal representation for \( \text{GL}_2 \) of conductor \( N \) such that

1. \( \pi_\infty \simeq \sigma(\mu_1, \mu_2) \), the discrete series representation with \( \mu_1 \mu_2^{-1}(t) = t^{2\kappa-1} \text{sgn}(t) \).

2. \( \pi_\ell \) is special for each \( \ell \mid \Delta \).

Let \( K \) be a quadratic imaginary field such that all \( \ell \mid \Delta \) are inert in \( K \) and all \( \ell \mid N_o \) are split in \( K \). Let \( c \) be a positive integer with \( (c, N) = 1 \) and \( p \) an odd prime number not dividing \( N \) that splits in \( K \). Assume that \( O_{K,c} = \{ \pm 1 \} \). Suppose that there exist Grössencharakter \( \chi \in \Xi_{-2c,0}(O_{K,c}) \) and \( \xi \in \Xi_{-2,0}(O_{K,c}) \) such that the \( p \)-adic avatar \( \hat{\xi} \) takes values in a totally ramified extension of \( \mathbb{Q}_p \).

Then, there exists \( x \in \text{CM}(\Delta, N; O_{K,c}) \) represented by \( \tau = t + yi \in \text{CM}_{\Delta, K} \) with
associated periods $\Omega_\infty$ and $\Omega_p$ such that for all $r \geq 0$

$$
\Omega_p^{-4(\kappa + r)} \int_{\mathbb{Z}_p} z^r \, d\mu(f, x; \chi, \xi) = \frac{2\varpi V_N u_{K,c}^4}{m_c^2 h_K} \frac{(-1)^{\kappa+r}!(2\kappa + r)!}{4^{2\kappa+3r} \pi^{2(\kappa+r+1)} y_r(2\kappa+r) \Omega_\infty^{4(\kappa+r)}} \nu(\xi_r, \tau_r, \frac{1}{2}) L(\pi_K \otimes \xi_r, \frac{1}{2}) L(\eta_K, 1)^{-1}
$$

where $\xi_r = \xi^r \| N_{K/q} \|^{-\kappa - r}$ and $\varpi$ is a (fixed) ratio of Petersson norms.

**Proof.** Let $D$ be the quaternion algebra with $\Delta_D = \Delta$. By hypothesis the representation $\pi$ is the image of an automorphic representation $\pi^D$ of $D^\times$ under the Jacquet-Langlands correspondence and let $f \in S_{2k,0}(\Delta, N)$ be a holomorphic newform in $\pi^D$. For all integers $r \geq 0$ let $\phi_r$ be as in (42).

By proposition 1.6 there exists $x \in \text{CM}(\Delta, N; \mathcal{O}_{K,c})$ and choose a split $p$-ordinary test triple $(\tau, v, e)$, $\tau = t + iy$, representing $x$ with corresponding $d_\infty \in \text{SL}_2(\mathbb{R})$. By taking $C_v$ and $F$ large enough, we can assume that $f$ is defined over $\mathcal{O}^{(v)} \subset \mathcal{O}_F$, and that the measure $\mu(f, x; \chi, \xi)$ has values in $\mathcal{O}_F$.

By remark 4.17.3 we can write $\pi(d_\infty) \phi_0 \otimes \pi(d_\infty) \phi_0 = \varpi \tilde{\theta}_p(F)$ with $\varphi = \varphi_\infty \otimes \varphi_f$ as in (46), $F$ the adelization of the normalized newform. We claim that for all $r \geq 0$

$$
\pi(d_\infty) \phi_r \otimes \pi(d_\infty) \phi_0 = \varpi \left( -1 \right)^{\kappa} \frac{\partial}{\partial t} \tilde{\theta}_{\varphi_\infty, \varphi_f} (F) + \sum_{l=0}^{r-1} a_{r,l} \left( \frac{\partial}{\partial t} \cdot \frac{\partial}{\partial t} \cdot \varphi_\infty, \varphi_f \right) (F)
$$

(51)

where $a_{r,l} \in \varpi \mathbb{Z}[\pi]$. For, the short exact sequence (14) gives a Lie algebras identification $\mathfrak{g}(D) \simeq (D_\infty \times D_\infty)/\mathbb{R}$ and in particular $\mathfrak{o}(D) = \{(A, B) \in D_\infty \times D_\infty \, | \, \text{tr}A = \text{tr}B \}/\mathbb{R} \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Under this identification, differentiating (45) yields

$$
\tilde{\theta}_H \varphi(F) = \left. \frac{d}{dt} \tilde{\theta}_\varphi(F)(h \exp(tH)) \right|_{t=0} \text{ with } H \varphi(x) = \left. \frac{d}{dt} \varphi(e^{-th_1} xe^{th_2}) \right|_{t=0}
$$

for all $H = (H_1, H_2) \in \mathfrak{Lie}(O(D))$. If $A \in \mathfrak{sl}_2$ a repeated application of the last formula with $A' = (A, 0)$ and $A'' = (0, A)$ shows that the diagonal action of $A$ on $\pi^D \otimes \pi^D$ corresponds to the action of the second order operator $A_2 = A' A'' = A' A' \in \mathfrak{L}(\mathfrak{Lie}(O(D)))$ on Schwartz functions, i.e.

$$
A_2 \varphi(x) = \left. \frac{\partial^2}{\partial u \partial v} \varphi \left( e^{-u A} x e^{v A} \right) \right|_{u=v=0}.
$$

We are interested in the expression of the operator $A_2$ in the normalized coordinates for $A = d_\infty X^+ d_\infty^{-1}$. Up to conjugation, this is the same as to compute the second order operator associated to $A = X^+$ under the standard coordinates (16). A straightforward computation using the obvious real coordinates associated to the underlying real decomposition $D_\infty = \mathbb{R} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ shows that $A' = -i \left( z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2} \right)$ and $A'' = i \left( z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2} \right)$, so that

$$
A_2 = z_2^2 \frac{\partial^2}{\partial z_1 \partial z_1} + z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} + z_1^2 \frac{\partial^2}{\partial z_2 \partial z_2} + z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial z_2}.
$$

41
Since

\[ A_2 \phi^{m,q} = \begin{cases} 
-2\pi \phi^{0,q+2} + 4\pi^2 \phi^{1,q+2} & \text{if } m = 0, \\
2m \phi^{m-1,q+2} - (4m - 2)\pi \phi^{m,q+2} + 4\pi^2 \phi^{m+1,q+2} & \text{if } m \geq 1,
\end{cases} \]

formula (51) follows from an \( r \)-fold iteration using the linearity of the theta lift and the definitions (42) and (46) of \( \phi_r \) and \( \varphi_\infty \) respectively.

Let \( \chi_r \) be a Grössencharakter of weight \( (-2(\kappa + r), 0) \) and trivial on \( \hat{R}_c^\times \) such that \( \xi_r = \chi_r \|N_{K/Q}\|^{-\kappa-r} \) is unitary. Combining (51) and (49) with lemma 4.20 we get

\[ J_r(f, \xi_r, \tau)^2 = \frac{(-1)^{\kappa+r}2\pi V_N r!(2\kappa + r)!}{4^{2\kappa+3r}h_K \pi^{2\kappa+2r}} \nu(\xi_r, \tau, \frac{1}{2}) L(\pi_K \otimes \xi_r, \frac{1}{2}) L(\eta_K, 1)^{-1}. \] (52)

On the other hand, from theorem 4.13,

\[ J_r(f, \xi_r, \tau)^2 = \frac{2^{\kappa} \pi^2}{w^2_{K,c}} (h_c^2)^2 y^{2(\kappa+r)} \Omega^{4(\kappa+r)} \langle \epsilon_r(f, x), \chi_r \rangle^2. \]

When \( \chi_r = \chi \xi_r \) we use proposition 4.6 to rewrite the last formula as

\[ \Omega_p^{-4(\kappa+r)} m_r(\mu(f, x; \chi, \xi))^2 = \frac{w^2_{K,c}}{m^2 \pi^2} \Omega^{4(\kappa+r)} y^{-2(\kappa+r)} J_r(f, \xi_r, \tau)^2. \]

Substituting (52) into the latter formula proves the theorem. \( \square \)

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Power series expansions of modular forms

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