On Properties of the Sturm-Liouville Operator with Degenerate Boundary Conditions

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Abstract

We consider spectral problems for the Sturm-Liouville operator with arbitrary complex-valued potential \( q(x) \) and degenerate boundary conditions. We solve corresponding inverse problem, and also study the completeness property and the basis property of the root function system.

1. Introduction. Consider the Sturm-Liouville equation

\[ u'' - q(x)u + \lambda u = 0 \]  \quad (1)

with two-point boundary conditions

\[ B_i(u) = a_{i1}u'(0) + a_{i2}u'(\pi) + a_{i3}u(0) + a_{i4}u(\pi) = 0, \]

\quad (2)

where the \( B_i(u) \) (\( i = 1, 2 \)) are linearly independent forms with arbitrary complex-valued coefficients and \( q(x) \) is an arbitrary complex-valued function of class \( L_1(0, \pi) \). It is convenient to write conditions (2) in the matrix form

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \]

and denote the matrix composed of the ith and jth columns of \( A \) (\( 1 \leq i < j \leq 4 \)) by \( A_{ij} \); we set \( A_{ij} = \det A(ij) \).

It is known that conditions (2) can be divided into two classes:

1) nondegenerate conditions;

2) degenerate conditions.
Boundary conditions (2) are called nondegenerate if they satisfy one of the following relations:

1) $A_{12} \neq 0$,  
2) $A_{12} = 0, A_{14} + A_{23} \neq 0$,  
3) $A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0$.

There is an enormous literature related to the spectral theory of the Sturm-Liouville operator with nondegenerate boundary conditions. In particular, the following assertion has been proved.

**Theorem ([1]).** For any nondegenerate conditions the spectrum of problem (1), (2) consists of a countable set $\{\lambda_n\}$ of eigenvalues with only one limit point $\infty$, and the dimensions of the corresponding root subspaces are bounded by one constant. The system $\{u_n(x)\}$ of eigen- and associated functions is complete and minimal in $L^2(0,1)$; hence, it has a biorthogonally dual system $\{v_n(x)\}$.

In this paper, we study eigenvalue problems for the Sturm-Liouville operator with degenerate boundary conditions. This type of boundary conditions has been investigated much less.

### 2. Preliminaries.

Let boundary conditions (2) be degenerate. According to [1, 2], this is equivalent to the fulfillment of the following conditions:

$$A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{34} = 0.$$  

According to [2], any boundary conditions of the considered class are equivalent to the boundary conditions determined by the matrix

$$A = \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & 0 & 1 & -b \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

If in the first case $b = 0$ then for any potential $q(x)$ we have the initial value problem (the Cauchy problem) which has no eigenvalues. The same situation takes place in the second case.
Further we will consider the first case if \( b = \pm 1 \). Then the boundary conditions can be written in more visual form

\[
\begin{align*}
    u'(0) + (-1)^\theta u'(\pi) &= 0, \\
    u(0) + (-1)^{\theta+1} u(\pi) &= 0.
\end{align*}
\]  
\( \theta = 0, 1 \). It is easily shown that if \( q(x) \equiv 0 \) then any \( \lambda \in \mathbb{C} \) is an eigenvalue of infinite multiplicity. This abnormal example illustrates the difficulty of investigation of problems with boundary conditions of the considered class.

Denote by \( c(x, \mu), s(x, \mu) \ (\lambda = \mu^2) \) the fundamental system of solutions to (1) with the initial conditions \( c(0, \mu) = s'(0, \mu) = 1, \ c'(0, \mu) = s(0, \mu) = 0 \). The following identity is well known

\[
c(x, \mu)s'(x, \mu) - c'(x, \mu)s(x, \mu) = 1.
\]  

Simple computations show that the characteristic equation of problem (1), (3) can be reduced to the form \( \Delta(\mu) = 0 \), where

\[
\Delta(\mu) = c(\pi, \mu) - s'(\pi, \mu).
\]  

By \( \Gamma(z, r) \) we denote the disk of radius \( r \) centered at a point \( z \). By \( PW_\sigma \) we denote the class of entire functions \( f(z) \) of exponential type \( \leq \sigma \) such that \( \|f(z)\|_{L_2(\mathbb{R})} < \infty \), and by \( PW^-\sigma \) we denote the set of odd functions in \( PW_\sigma \).

3. Inverse problem.

The following two assertions provide necessary and sufficient conditions to be satisfied by the characteristic determinant \( \Delta(\mu) \).

**Theorem 1.**[3] If a function \( \Delta(\mu) \) is the characteristic determinant of a problem (1), (3), then

\[
\Delta(\mu) = \frac{f(\mu)}{\mu},
\]
where \( f(\mu) \in PW_\pi^- \).

**Theorem 2.** Let a function \( v(\mu) \) have the form

\[
v(\mu) = \frac{f(\mu)}{\mu},
\]

where \( f(\mu) \in PW_\pi^- \), and satisfies the condition

\[
\int_{-\infty}^{\infty} |\mu^m f(\mu)|^2 d\mu < \infty,
\]

where \( m \) is a nonnegative integer number. Then, there exists a function \( q(x) \in W_2^m(0, \pi) \) such that the characteristic determinant of problem (1), (3) with the potential \( q(x) \) satisfies \( \Delta(\mu) = v(\mu) \).

**Proof.** If \( m = 0 \) the theorem was proved in [3]. Further we will count that \( m > 0 \). Since [4]

\[
|f(\mu)| \leq C_1 \|f(\mu)\|_{L_2(R)} e^{\pi|\text{Im}\mu|},
\]

it follows that there exists an arbitrary large positive integer \( N \) such that

\[
|u(\mu)| < 1/10, \quad |f(\mu)| < 1
\]

on the set \( |\text{Im}\mu| \leq 1, \text{Re}\mu \geq N \). Let \( \mu_n \) (\( n = 1, 2, \ldots \)) be a strictly monotone increasing sequence of positive numbers such that \( |\mu_n - (N + 1/2)| < 1/10 \) if \( 1 \leq n \leq N \) and \( \mu_n = n \) if \( n \geq N + 1 \). Consider the function

\[
s(\mu) = \pi \prod_{n=1}^{\infty} \frac{\mu_n^2 - \mu^2}{n^2} = \frac{\sin \pi \mu}{\mu} \prod_{n=1}^{N} \frac{\mu_n^2 - \mu^2}{n^2 - \mu^2}.
\]

Obviously, all zeros of the function \( s(\mu) \) are simple, and, in addition, the inequality

\[
(-1)^n \dot{s}(\mu_n) > 0
\]

holds for any \( n \). Denote \( P(\lambda) = \prod_{n=1}^{N}(\mu_{n}^{2} - \lambda) \), \( Q(\lambda) = \prod_{n=1}^{N}(n^{2} - \lambda) \). Evidently,

\[
\frac{\lambda^{m} P(\lambda)}{Q(\lambda)} = \lambda^{m} + \sum_{j=1}^{m} \alpha_{j} \lambda^{m-j} + \frac{R(\lambda)}{Q(\lambda)},
\]

(12)

where \( R(\lambda) \) is a polynomial of degree \( N - 1 \) and \( \alpha_{j} \) are some constants.

It follows from (12) that

\[
\prod_{n=1}^{N} \frac{\mu_{n}^{2} - \lambda^{2}}{n^{2} - \lambda^{2}} = 1 + \sum_{j=1}^{l} \alpha_{j} \mu^{-2j} + \frac{R(\mu^{2})}{\mu^{2l}Q(\mu^{2})}.
\]

(13)

It follows from (10), (13) that

\[
\dot{s}(n) = \frac{\pi(-1)^{n}}{n}(1 + \sum_{j=1}^{m} \alpha_{j} n^{-2j} + O(n^{-2m-2})).
\]

(14)

Consider the equation

\[
z^{2} - u(\mu_{n})z - 1 = 0.
\]

(15)

It has the roots

\[
c_{n}^{\pm} = \frac{u(\mu_{n}) \pm \sqrt{u^{2}(\mu_{n}) + 4}}{2}.
\]

(16)

It follows from (9) that for any \( n \) all numbers \( c_{n}^{+} \) lie in the disk \( \Gamma(1, 1/2) \) and all numbers \( c_{n}^{-} \) lie in the disk \( \Gamma(-1, 1/2) \). Let for even \( n \) \( c_{n} = c_{n}^{+} \), and for odd \( n \) \( c_{n} = c_{n}^{-} \). Then \( (-1)^{n}Rew_{n} > 0 \) for any \( n = 1, 2, \ldots \).

This, together with (11) implies that \( Rew_{n} > 0 \) for any \( n \), where

\[
w_{n} = \frac{c_{n}}{\mu_{n} \dot{s}(\mu_{n})}.
\]

(17)

We set \( F(x, t) = F_{0}(x, t) + \hat{F}(x, t) \), where

\[
F_{0}(x, t) = \sum_{n=1}^{N} \left( \frac{2c_{n}}{\mu_{n} \dot{s}(\mu_{n})} \sin \mu_{n}x \sin \mu_{n}t - \frac{2}{\pi} \sin nx \sin nt \right),
\]

5
\[ F(x, t) = \sum_{n=N+1}^{\infty} \left( \frac{2c_n}{\mu_n \dot{s}(\mu_n)} \sin \mu_n x \sin \mu_n t - \frac{2}{\pi} \sin nx \sin nt \right). \] (18)

One can readily see that \( F_0(x, t) \in C^\infty(R^2) \). Consider the function \( \hat{F}(x, t) \). If \( n \geq N + 1 \), then, by taking into account (8), (16) and the rule for choosing the roots of equation (15), we obtain

\[ c_n = (-1)^n + \frac{f(n)}{2n} + (-1)^n \left( \sum_{j=1}^{m} \beta_j \frac{f^{2j}(n)}{n^{2j}} + O(1/n^{2m+2}) \right). \] (19)

It follows from (8), (14), (18), and (19) that

\[
\hat{F}(x, t) = \sum_{n=N+1}^{\infty} \frac{2}{\pi} \left( \frac{1+(-1)^n f(n)}{2n} + \sum_{j=1}^{m} \beta_j \frac{f^{2j}(n)}{n^{2j}} + O(1/n^{2m+2}) \right) \sin nx \sin nt = \\
= \sum_{n=N+1}^{\infty} \frac{2}{\pi} \left( 1 + (-1)^n f(n) + \sum_{j=1}^{m} \beta_j \frac{f^{2j}(n)}{n^{2j}} + O(1/n^{2m+2}) \right) \times \\
\left( 1 + \sum_{j=1}^{m} \tilde{\alpha}_j n^{-2j} + O(n^{-2m-2}) \right) - 1 \right) \sin nx \sin nt = \\
= \frac{2}{\pi} \sum_{n=N+1}^{\infty} \left( \gamma_n f(n) + \sum_{j=1}^{m} \tilde{\alpha}_j n^{-2j} + O(n^{-2m-2}) \right) \sin nx \sin nt = \\
= (\hat{G}(x - t) - \hat{G}(x + t))/2,
\]

where

\[
\hat{G}(y) = \frac{2}{\pi} \sum_{i=1}^{3} G_i(y), \quad G_1(y) = \sum_{n=N+1}^{\infty} \gamma_n f(n) \cos ny,
\]

\[
G_2(y) = \sum_{n=N+1}^{\infty} \sum_{j=1}^{m} \tilde{\alpha}_j n^{-2j} \cos ny = \sum_{j=1}^{m} \tilde{\alpha}_j \sum_{n=N+1}^{\infty} n^{-2j} \cos ny,
\]

\[
G_3(y) = \sum_{n=N+1}^{\infty} \tilde{\gamma}_n n^{-2m-2} \cos ny
\]

where \(|\gamma_n| < C_2/n, |\tilde{\gamma}_n| < C_2|.
The relation
\[
\sum_{n=1}^{\infty} |n f(n)|^{2m} = \frac{1}{2} \| \mu^m f(\mu) \|_{L^2(R)},
\]
which follows from the Paley-Wiener theorem, together with the Parseval equality and (7), implies that \( G_1(y) \in W^{m+1}_2[0, 2\pi] \). It is known [5] that for any \( j = 1, 2, \ldots \) the series \( \sum_{n=N+1}^{\infty} n^{-2j} \cos ny \) are infinitely differentiable functions on the segment \([0, 2\pi]\). One can readily see that \( G_3(y) \in W^{m+1}_2[0, 2\pi] \). Therefore, we obtain the representation
\[
F(x, t) = F_0(x, t) + (\hat{G}(x - t) - \hat{G}(x + t))/2,
\]
where the functions \( F_0(x, t) \) and \( \hat{G}(y) \) belong to the above-mentioned classes.

Now let us consider the Gelfand-Levitan equation
\[
K(x, t) + F(x, t) + \int_0^x K(x, s)F(s, t)ds = 0
\]
and prove that it has a unique solution in the space \( L^2(0, x) \) for each \( x \in [0, \pi] \). To this end, it suffices to show that the corresponding homogeneous equation has only the trivial solution.

Let \( f(t) \in L^2(0, x) \). Consider the equation
\[
f(t) + \int_0^x F(s, t)f(s)ds = 0.
\]
Following [6], by multiplying the last equation by \( \bar{f}(t) \) and by integrating the resulting relation over the interval \([0, x]\), we obtain
\[
\int_0^x |f(t)|^2dt + \sum_{n=1}^{\infty} \frac{2c_n}{\mu_n s(\mu_n)} \int_0^x \bar{f}(t) \sin \mu_n tdt \int_0^x f(s) \sin \mu_n sds - \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^x f(t) \sin nt dt \int_0^x f(s) \sin nsds = 0.
\]
This, together with the Parseval equality for the function system \( \{\sin nt\}_1^\infty \)
on the interval \([0, \pi]\) implies that
\[
\sum_{n=1}^\infty w_n | \int_0^x f(t) \sin \mu_n t dt |^2 = 0,
\]
where the \(w_n\) are the numbers given by (17). Since \(\text{Re} w_n > 0\), we see that \(\int_0^x f(t) \sin \mu_n t dt = 0\) for any \(n = 1, 2, \ldots\). Since [7, 8] the system \(\{\sin \mu_n t\}_1^\infty\) is complete on the interval \([0, \pi]\), we have \(f(t) \equiv 0\) on \([0, x]\).

Let \(\hat{K}(x, t)\) be a solution of equation (21), and let \(\hat{q}(x) = 2 \frac{d}{dx} \hat{K}(x, x)\); then it follows [9] from (20) that \(\hat{q}(x) \in W_2^m(0, \pi)\). By \(\hat{s}(x, \mu), \hat{c}(x, \mu)\) we denote the fundamental solution system of equation (1) with potential \(\hat{q}(x)\) and the initial conditions \(\hat{s}(0, \mu) = \hat{c}'(0, \mu) = 0, \hat{c}(0, \mu) = \hat{s}'(0, \mu) = 1\). By reproducing the corresponding considerations in [6], we obtain \(\hat{s}(\pi, \mu) \equiv s(\mu)\), whence it follows that the numbers \(\mu_n^2\) form the spectrum of the Dirichlet problem for equation (1) with potential \(\hat{q}(x)\), and \(\hat{c}(\pi, \mu_n) = c_n\), which, together with identity (4), implies that \(\hat{s}'(\pi, \mu_n) = 1/c_n\).

Let \(\hat{\Delta}(\mu)\) be the characteristic determinant of problem (1), (3) with potential \(\hat{q}(x)\)). Let us prove that \(\hat{\Delta}(\mu) \equiv v(\mu)\). By theorem 1, the function \(\hat{\Delta}(\mu)\) admits the representation
\[
\hat{\Delta}(\mu) = \frac{\hat{f}(\mu)}{\mu},
\]
where \(\hat{f}(\mu) \in PW_\pi^-\). By taking into account relation (4) and the fact that the numbers \(c_n\) are roots of equation (15), we have
\[
\hat{\Delta}(\mu_n) = \hat{c}(\pi, \mu_n) - \hat{s}'(\pi, \mu_n) = c_n - c_n^{-1} = v(\mu_n).
\]
It follows that the function
\[ \Phi(\mu) = \frac{u(\mu) - \hat{\Delta}(\mu)}{s(\mu)} = \frac{f(\mu) - \hat{f}(\mu)}{\mu s(\mu)} \]
is an entire function on the complex plane. Since the function \( g(\mu) = f(\mu) - \hat{f}(\mu) \) belongs to \( PW_\pi^- \), it follows from (8) that
\[ |g(\mu)| \leq C_3 e^{\pi |Im\mu|}. \tag{22} \]
From (10), we find that if \( |Im\mu| \geq 1 \), then
\[ |\mu s(\mu)| \geq C_4 e^{\pi |Im\mu|} \tag{23} \]
(\( C_4 > 0 \)). If \( |Im\mu| \geq 1 \), then we obtain the estimate \( |\Phi(\mu)| \leq C_3/C_4 \).

By \( H \) we denote the union of the vertical segments \( \{ z : |Rez| = n + 1/2, |Imz| \leq 1 \} \), where \( n = N + 1, N + 2, \ldots \). It follows from (10) that if \( \mu \in H \), then \( |\mu s(\mu)| \geq C_5 > 0 \). The last inequality, together with (22), (23), and the maximum principle for the absolute value of an analytic function, implies that \( |\Phi(\mu)| \leq C_6 \) in the strip \( |Im\mu| \leq 1 \). Consequently, the function \( \Phi(\mu) \) is bounded on the entire complex plane; therefore, by the Liouville theorem, it is a constant.

It follows from the Paley-Wiener theorem and the Riemann lemma [1] that if \( |Im\mu| = 1 \), then \( \lim_{|\mu| \to \infty} g(\mu) = 0 \), whence, we obtain \( \Phi(\mu) \equiv 0 \).

4. Completeness and the basis property.

Completeness of the root function system of problem (1), (3) was investigated in [10]. In particular, it was shown that if \( q(x) \in C^k[0, \pi] \) for some \( k \geq 0 \), and \( q^{(k)}(0) \neq (-1)^k q^{(k)}(\pi) \), then the root function system is complete in \( L_2(0, \pi) \). If there exists an \( \varepsilon > 0 \) such that \( q(x) - q(\pi - x) = 0 \) for almost all \( x \in [0, \varepsilon] \), then the mentioned system is not complete in \( L_2(0, \pi) \). In was established in [3], [11]
that there exist potentials \( q(x) \) such that the root function systems of corresponding problems (1), (3) are complete in \( L_2(0, \pi) \) and contain associated functions of arbitrary high order, i.e. the dimensions of root subspaces infinitely grow.

Since for a wide class of potentials \( q(x) \) the root function system of problem (1), (3) is complete in \( L_2(0, \pi) \) one can set a question whether the mentioned system forms a basis.

Let \( \lambda_n = \mu_n^2 \) (\( \text{Re}\mu_n \geq 0, n = 1, 2, \ldots \)) be the eigenvalues of problem (1), (3) numbered neglecting their multiplicities in nondecreasing order of absolute value. By \( m(\lambda_n) \) we denote the multiplicity of an eigenvalue \( \lambda_n \). In addition, assume that the function \( q(x) \) is continuous on the interval \((0, \pi)\).

**Theorem 3.** Suppose a subsequence of eigenvalues \( \lambda_{n_k} \) satisfies the following two conditions:

1. \( |\text{Im}\mu_{n_k}| < M \);
2. \( \lim_{k \to \infty} \frac{m(\lambda_{n_k})}{\ln|\lambda_{n_k}|} = 0 \);

Then the system of eigenfunctions and associated functions of problem (1), (3) is not a basis in \( L_2(0, \pi) \).

**Proof.** Let us calculate the Green function \( G(x, \xi, \mu) \) of operator (1), (3). By [9], \( G(x, \xi, \mu) = H(x, \xi, \mu)/\Delta(\mu), \) where \( H(x, \xi, \mu) = \Phi(x, \xi, \mu)/2 + g(x, \xi)\Delta(\mu), \) where

\[
\Phi(x, \xi, \mu) = s(x, \mu)\{c'(\pi, \mu)[-c(\xi, \mu)s(\pi, \mu) - s(\xi, \mu)(-1 - c(\pi, \mu))] - \\
- [1 - c(\pi, \mu)][c(\xi, \mu)(-1 + s'(\pi, \mu)) - s(\xi, \mu)c'(\pi, \mu)] - \\
- c(x, \mu)\{[1 + s'(\pi, \mu)][-c(\xi, \mu)s(\pi, \mu) - s(\xi, \mu)(-1 - c(\pi, \mu))] + \\
+ s(\pi, \mu)[c(\xi, \mu)(-1 + s'(\pi, \mu)) - s(\xi, \mu)c'(\pi, \mu)]\},
\]

\( g(x, \xi) = \pm(s(x, \mu)c(\xi, \mu) - c(x, \mu)s(\xi, \mu))/2, \) the sigh “ + ” is used for \( x > \xi \), and the sigh ” − ” is used for \( x < \xi \). Combining like terms
in (24) gives

$$\Phi(x, \xi, \mu) = 2[s(x, \mu)c(\xi, \mu) - c(x, \mu)s(\xi, \mu)] -$$
$$- [c(\pi, \mu) + s'(\pi, \mu)][s(x, \mu)c(\xi, \mu) + c(x, \mu)s(\xi, \mu)] +$$
$$+ 2[c'(\pi, \mu)s(x, \mu)s(\xi, \mu) + s(\pi, \mu)c(x, \mu)c(\xi, \mu)].$$

Let $e(x, \mu)$ be the solution of equation (1) satisfying the initial conditions $e(0, \mu) = 1$, $e'(0, \mu) = i\mu$, and let $K(x, t)$, $K^+(x, t) = K(x, t) + K(x, -t)$, and $K^-(x, t) = K(x, t) - K(x, -t)$ be the transformation kernels [1] realizing the representations

$$e(x, \mu) = e^{i\mu x} + \int_{-x}^{x} K(x, t)e^{i\mu t} dt,$$

$$c(x, \mu) = \cos \mu x + \int_{0}^{x} K^+(x, t) \cos \mu t dt,$$

$$s(x, \mu) = \frac{\sin \mu x}{\mu} + \int_{0}^{x} K^-(x, t) \frac{\sin \mu t}{\mu} dt. \quad (26)$$

It was shown in [12] that

$$c(\pi, \mu) = \cos \pi \mu + \frac{\pi}{2} \langle q \rangle \frac{\sin \pi \mu}{\mu} - \int_{0}^{\pi} \frac{\partial K^+(\pi, t)}{\partial t} \frac{\sin \mu t}{\mu} dt, \quad (27)$$

$$s'(\pi, \mu) = \cos \pi \mu + \frac{\pi}{2} \langle q \rangle \frac{\sin \pi \mu}{\mu} + \int_{0}^{\pi} \frac{\partial K^-(\pi, t)}{\partial x} \frac{\sin \mu t}{\mu} dt, \quad (28)$$

where $\langle q \rangle = \frac{1}{\pi} \int_{0}^{\pi} q(x) dx$. By differentiating the second of equalities (26) and taking into account [12] that $K^+(\pi, \pi) = \frac{\pi}{2} \langle q \rangle$, we obtain

$$c'(\pi, \mu) = -\mu \sin \pi \mu + \frac{\pi}{2} \langle q \rangle \cos \pi \mu + \int_{0}^{\pi} \frac{\partial K^+(\pi, t)}{\partial x} \cos \mu t dt. \quad (29)$$
By substituting the right-hand sides of (26-29) in (25), we get
\[
\Phi(x, \xi, \mu) = 2(\sin \mu x \cos \mu \xi - \cos \mu x \sin \mu \xi)/\mu - 2\cos \pi \mu(\sin \mu x \cos \mu \xi + \cos \mu x \sin \mu \xi)/\mu + 2(- \sin \pi \mu \sin \mu x \sin \mu \xi + \sin \pi \mu \cos \mu x \cos \mu \xi)/\mu + o(\mu^{-1})e^{\pi |\text{Im}\mu|} = 2[\sin \mu(x - \xi) + \sin \mu(\pi - (x + \xi))]/\mu + o(\mu^{-1})e^{\pi |\text{Im}\mu|}.
\]

Throughout the following we assume that $|\text{Im}\mu| < M$. Then the last equality implies that
\[
G(x, \xi, \mu) = \frac{R(x, \xi, \mu)}{\Delta(\mu)} + g(x, \xi), \quad (30)
\]
where
\[
R(x, \xi, \mu) = [\sin \mu(x - \xi) + \sin \mu(\pi - (x + \xi))]/\mu + o(\mu^{-1}). \quad (31)
\]

Let us study the function $G(x, \xi, \mu)$ in the neighborhood of the eigenvalues $\lambda_n$. It follows from [3] that each root subspace contains one eigenfunction and possibly associated functions. Let \{\(h_u n(x)\)\} (\(h = 0, m(\lambda_n)\)) be an arbitrary canonical system of eigenfunctions and associated functions of problem (1), (3), and let \{\(h_v n(x)\)\} be appropriately normalized canonical system of eigenfunctions and associated functions of the adjoint boundary value problem [13], i.e. \(0 u_n(x)\) and \(0 v_n(x)\) are eigenfunctions, and \(h u_n(x)\) and \(h v_n(x)\) (\(h \geq 1\)) are associated functions of order \(h\), where
\[
(h u_n(x), g v_k(x))_{L_2(0,\pi)} = \delta_{n,k}\delta_{h,m(\lambda_n)-1-g}.
\]

Further we consider only root subspaces corresponding the mentioned-above subsequence of the eigenvalues $\lambda_{n_k}$. Denote
\[
R_{n_k}(x, \xi) = u_{n_k}(x) v_{n_k}(\xi),
\]
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\[ m(\lambda_{nk})^{-1} \mathcal{R}_{nk}(x, \xi) = \sum_{p=0}^{m(\lambda_{nk})-1} \frac{p}{u_{nk}(x)} m(\lambda_{nk})^{-1-p} v_{nk}(\xi). \]

Since the function \( f(\mu) \) has a root of multiplicity \( m(\lambda_{nk}) \) at the point \( \mu_{nk} \), then
\[ f(\mu) = \sum_{l=m(\lambda_{nk})}^{\infty} c_l (\mu - \mu_{nk})^l = (\mu - \mu_{nk})^{m(\lambda_{nk})} \sum_{l=0}^{\infty} c_{m(\lambda_{nk})+l} (\mu - \mu_{nk})^l. \]

Obviously, \( c_{m(\lambda_{nk})} = \frac{f^{(m(\lambda_{nk}))}(\mu_{nk})}{m(\lambda_{nk})!} \). Relations (30) and (32), together with [13] imply the equality
\[ 0 R_{nk}(x, \xi) = \lim_{\mu \to \mu_{nk}} (\mu^2 - \mu_{nk}^2)^{m(\lambda_{nk})} G(x, \xi, \mu) = \frac{2^{m(\lambda_{nk})} (\mu_{nk})^2}{f^{(m(\lambda_{nk}))}(\mu_{nk})} G(x, \xi, \mu_{nk}). \]

From the Bernstein inequality [4], we obtain
\[ |f^{(m(\lambda_{nk}))}(\mu_{nk})| \leq C_1 \pi^{m(\lambda_{nk})}. \]

It follows from (33) and (34) that
\[ |0 R_{nk}(x, \xi)| \geq C_2 \frac{2^m(\lambda_{nk})m(\lambda_{nk})!|\mu_{nk}|^{m(\lambda_{nk})+1}||R(x, \xi, \mu_{nk})|}{2^m(\lambda_{nk})m(\lambda_{nk})!|\mu_{nk}|^{m(\lambda_{nk})+1}||R(x, \xi, \mu_{nk})|}, \]
where \( C_2 > 0 \), hence,
\[ \|0 R_{nk}(x, \xi)\|_{L^2((0, \pi) \times (0, \pi))}^2 \geq C_3 [(2/\pi)^{m(\lambda_{nk})} m(\lambda_{nk})!^2 |\mu_{nk}|^{2m(\lambda_{nk})+2}||R(x, \xi, \mu_{nk})|_{L^2((0, \pi) \times (0, \pi))}^2, \]
where \( C_3 > 0 \).
If $v_{nk} (\xi) \neq 0$, then the function $m(\lambda_{nk})^{-1} R^{nk} (x, \xi)$ is an associated function of order $m(\lambda_{nk}) - 1$, corresponding to the eigenvalue $\mu^2_{nk}$ and the eigenfunction $u_{nk} (x)$. It follows from [14] that

$$\| R_{nk}^0 (x, \xi) \|^2_{L_2(\pi/3, \pi/2)} \leq [C_4 m(\lambda_{nk}) |\mu_{nk}|]^{2m(\lambda_{nk})-2} \| R^{nk} (x, \xi) \|^2_{L_2(\pi/4, 3\pi/4)}, \quad (36)$$

where $C_4$ is a constant independent of $\xi$. If $v_{nk} (\xi) = 0$, then the validity of (36) is obvious. It follows from (36) and [15] that

$$\| R_{nk}^0 (x, \xi) \|^2_{L_2(0, \pi)} \leq [C_5 m(\lambda_{nk}) |\mu_{nk}|]^{2m(\lambda_{nk})-2} \| R^{nk} (x, \xi) \|^2_{L_2(0, \pi)}, \quad (37)$$

By integrating inequality (37) with respect to $\xi$, we have

$$\| R_{nk}^0 (x, \xi) \|^2_{L_2((0, \pi) \times (0, \pi))} \leq [C_6 m(\lambda_{nk}) |\mu_{nk}|]^{2m(\lambda_{nk})-2} \| R^{nk} (x, \xi) \|^2_{L_2((0, \pi) \times (0, \pi))}. \quad (38)$$

It follows from (31) that

$$\| R(x, \xi, \mu_{nk}) \|^2_{L_2((0, \pi) \times (0, \pi))} \geq C_7 |\mu_{nk}|^{-2}, \quad (39)$$

where $C_7 > 0$.

Relations (35), (38), (39), and the Stirling formula imply that

$$\| R^{nk} (x, \xi) \|^2_{L_2((0, \pi) \times (0, \pi))} \geq \frac{(m(\lambda_{nk}) |\mu_{nk}|)^2}{(C_8 m(\lambda_{nk}) m(\lambda_{nk}))^{2m(\lambda_{nk})}} \geq \frac{|\mu_{nk}|^2}{C_9 m(\lambda_{nk})},$$

where $C_8, C_9 > 0$. By the conditions of the theorem, the right-hand side of the last inequality tends to infinity as $k \to \infty$. This, combined with a resonance type theorem [16] implies the validity of theorem 3.
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