Non-diagonal problem Hamiltonian for adiabatic quantum computation

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Overview

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Adiabatic quantum computation (AQC) – two-step procedure:
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$$H_t = (1 - \frac{t}{T})H_0 + \frac{t}{T}H_p$$

Adiabatic condition: $T \gg \frac{1}{\Delta^2}$
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adiabatic condition: $T_N \sim 1/\Delta_N^2$
Appeals of AQC
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- good implementation prospects
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- multiple interrelations with condensed matter physics
Challenges of AQC
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- there is strong evidence that $T$ can often scale unfavorably with the problem size $N$
Monotone not-all-equal 3-satisfiability (MNAE3SAT)

$N$ bits $z = (z_1, z_2, ..., z_N)$ (we take $z_i = \pm 1$)
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- set \( C \) of \( M \) clauses
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MNAE3SAT is NP-complete
MNAE3SAT as a binary optimization problem

MNAE3SAT = binary optimization problem with the cost function

\[ H_{cl}^p(z) = \sum_{(i,j,m) \in C} C_{ijm}^{cl}(z) \]

with

\[ C_{ijm}^{cl}(z) = \begin{cases} 1 & \text{if } z_i = z_j = z_k, \\ 0 & \text{otherwise.} \end{cases} \]
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\[ H_p^{cl}(z) \geq 0 \]

z is a satisfying assignment \[ \iff \] \[ H_p^{cl}(z) = 0 \]
Conventional $H_p$ for MNAE3SAT

$$H_p = \sum_{(i,j,m) \in C} C_{ijm}$$

with

$$C_{ijm} = \frac{1}{4} \left( 1 + \sigma_i^z \sigma_j^z + \sigma_j^z \sigma_k^z + \sigma_k^z \sigma_i^z \right)$$
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$$|z\rangle \equiv |z_1, z_2, \ldots, z_N\rangle, \quad \sigma_j^z |z\rangle = z_j |z\rangle$$
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$H_p$ is frustration-free:

$$H |z\rangle = 0 \iff \forall (i,j,m) \in C \quad C_{ijm} |z\rangle = 0.$$
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$z$ is a satisfying assignment \iff $z$ is a gs, i.e. $H_p |z\rangle = 0$
Bottleneck of AQC $\equiv$ avoided level crossings with $\Delta \sim e^{-N^\alpha}$
Two types of bottlenecks
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- Quantum phase transitions
Two types of bottlenecks

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- Many-body localised (glassy) phase [Altshuler, Krovi, Roland, 2010; Laumann *et al.* 2015; Knysh 2016; ...]
“Conventional” AQC

\[
H_t = (1 - \frac{t}{T})H_0 + \frac{t}{T}H_p
\]

\[
H_p = \sum_{i,j} J_{ij}\sigma_i^z\sigma_j^z + \sum_i h_i\sigma_i^z
\]

\[
\hat{H}_0 = \sum_i \sigma_i^x
\]
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- modify $H_P$
What is wrong with a conventional $H_p$?
Bottlenecks of AQC

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- In disordered systems, eigenstates can be many-body localised (MBL).
- MBL entails small energy gaps.
- Product states are ultimately localised.
- Eigenstates of $H_p$ are of product form, hence the evolution inevitably traverses MBL phase.
Non-diagonal problem Hamiltonian

The ground state of $H_p$ is of product form for a purpose – it should be easily measurable. However, excited states of $H_p$ are also product states – absolutely unnecessary for computation!

The idea is to introduce $H_{ent}$ with a product ground state and entangled excited states.
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\[ H_{\text{p}} = \sum_{(i, j, m) \in \mathbb{C}} C_{ijm} \]

\[ C_{ijm} = \frac{1}{4} (1 + \sigma_z^i \sigma_z^j + \sigma_z^j \sigma_z^k + \sigma_z^k \sigma_z^i) \]

A problem Hamiltonian (generically) non-diagonal in comp. basis:

\[ H_{\text{ent}} = \sum_{(i, j, m) \in \mathbb{C}} C_{ijm} A_{ijm} \]

\[ A_{ijm} \] arbitrary positive non-diagonal term

\[ A_{ijm} \] not necessarily acts on spins
Non-diagonal problem Hamiltonian

Reminder:

\[ H_P = \sum_{(i,j,m) \in \mathcal{C}} C_{ijm} \]

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A specific choice of \( A_{ijm} \):

\[ A_{ijm} = 2 + \sigma_i^x \sigma_j^x \sigma_m^x + \sigma_r^x \sigma_s^x \]

\( r \neq i, j, m \) and \( s \neq i, j, m \)
Non-diagonal problem Hamiltonian

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Locality issue: \( H_{\text{ent}}^p \) is 4-local (while \( H_p \) is only 2-local)
Entanglement of eigenstates of $H_p^{\text{ent}}$

Participation ratio – figure of merit for entanglement:

$$R(\Psi) = \left( \sum_{\mu=1}^{2^N} |\psi_{\mu}|^4 \right)^{-1}.$$
Entanglement of eigenstates of $H_{p}^{\text{ent}}$

Participation ratios of eigenstates of $H_{p}^{\text{ent}}$ (blue dots) compared to those of eigenstates of a nonintegrable Ising model.
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- some excited states have small $R \sim 1$
- entanglement of most of low lying excited states is comparable to that of a bona fide chaotic model
- work in progress...
Diagonal frustration-free problem Hamiltonian:

\[ H_p = \sum_{(i,j,m) \in \mathcal{C}} C_{ijm}, \quad C_{ijm} \geq 0 \]

Product ground state \( |z\rangle \) with zero energy: \( H_p |z\rangle = 0 \)
Non-diagonal problem Hamiltonian - generalisation

Diagonal frustration-free problem Hamiltonian:

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Product ground state |z⟩ with zero energy: \( H_p |z⟩ = 0 \)

Non-diagonal frustration-free problem Hamiltonian:

\[ H_{p}^{\text{ent}} = \sum_{(i,j,m)\in C} C_{nlq} A_{ijm}^{nlq} C_{ijm}, \quad A_{ijm}^{nlq} > 0 \]

has the same ground state, \( H_{p}^{\text{ent}} |z⟩ = 0 \) but (generically) entangled excited eigenstates.
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more work is needed to evaluate their performance
arXiv:1811.09453

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Thank you for your attention!