Exact dynamics and thermalization of an open bosonic quantum system in the presence of a quantum phase transition induced by the environment

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Abstract – We derive the exact out-of-equilibrium Wigner function of a bosonic mode linearly coupled to a bosonic bath of arbitrary spectral density. Our solution does not rely on any master equation approach and it therefore also correctly describes a bosonic mode which is initially entangled with its environment. It has been recently suggested that non-Markovian quantum effects lead to dissipationless dynamics in the case of a strong coupling to a bath whose spectral density has a support bounded from below. We show in this work that such a system undergoes a quantum phase transition at some critical bath coupling strength. The apparent dissipationless dynamics then correspond to the relaxation towards the new ground state.

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Introduction. – A precise understanding of the dynamics of open quantum systems is crucial nowadays in various domains of physics, ranging from quantum information and quantum optics to cold atoms and condensed-matter physics. The consequences which derive from the interaction of a quantum system with an environment raise fundamental questions about the governing principles of quantum mechanics. In particular, an external environment leads to a loss of quantum coherence. The decoherence of an open quantum system has been studied by a variety of methods in the past [1]. One frequently used approach is based on an exact (non-Markovian) master equation for the reduced density-matrix (RDM) of the system [2–5]. While its success is undeniable, all known master equations are however derived under the assumptions that the system and its environment are initially uncoupled. The question of how to describe a non-Markovian time evolution of open quantum systems, preferably via a closed equation for the RDM subject to arbitrary (i.e. entangled) initial conditions, is still under debate.

The objective of this letter is twofold. We first show how to derive the exact Wigner function of a bosonic mode linearly coupled to a bosonic bath in the general case of arbitrary, possibly non-factorizing initial conditions. Hence, our analysis will go beyond the range of validity of the master equations. We then focus on a particular environment which conserves the total number of excitations. We revisit the issue concerning “dissipationless dynamics” recently raised in the literature [5,6], where it has been argued that the relaxation is inhibited by strong non-Markovian effects in the bath. We show that this peculiar behavior is on the contrary related to a (static) quantum phase transition in the global system and hence it does not directly result from dynamic memory effects in the environment.

Model. – Let us consider the dynamics of a bosonic \((a^\dagger, a)\)-mode (of mass \(m\) and frequency \(\omega_b\)), the “system” \(S\), in contact with a “bath” \(B\) modeled as a set of \(N_B\) harmonic oscillators \((\hat{b}^\dagger, \hat{b})\) (of mass \(m_i\) and frequency \(\omega_i\)). The Hamiltonian is then given by (\(\hbar = k_B = 1\) throughout this letter)

\[
\hat{H} = \omega_0 a^\dagger a + \hat{H}_B + \hat{H}_c,
\]

(1)

where \(\hat{H}_B = \sum_{i=1}^{N_B} \omega_i \hat{b}_i^\dagger \hat{b}_i\) is the Hamiltonian of the bath and \(\hat{H}_c\) describes the \(SB\)-coupling. We assume \(\hat{H}_c\) to be quadratic in the ladder operators. Furthermore, all the
parameters appearing in $\hat{H}$ (and $\hat{H}_c$) can depend on time, implying that the whole system $U = \mathcal{S} \oplus \mathcal{B}$ may be driven out-of-equilibrium. The general dynamics after arbitrary initial conditions governed by eq. (1) will be analyzed in the following in terms of the Wigner function $W(t,q,p)$ of the RDM $\hat{\rho}$ of the system: $\hat{\rho}(t) = \text{Tr}_\mathcal{B}\{\hat{\rho}_d(t)\}$, with $\hat{\rho}_d(t) = \hat{U}(t)\hat{\rho}_0(0)\hat{U}^\dagger(t)$ the total density-matrix at time $t$ with initial condition $\hat{\rho}_0(0)$ (the evolution operator $\hat{U}(t)$ solves $i\hbar\partial_t \hat{U}(t) = \hat{H}(t)\hat{U}(t)$). $W_t(q,p)$ is defined as

$$W_t(q,p) = \frac{1}{2\pi} \int dq' e^{-iq'p} \langle q' + \frac{p}{2}\hat{\rho}(t) | q - \frac{p}{2} \rangle,$$

(2)

with $|q \pm u/2|$ position eigenstates.

In the last part of this letter we discuss the effects of a particular environment, the so-called resonant bath for which

$$\hat{H}_c = \hat{H}_t = \sum_i C_i (\hat{a} \hat{b}^\dagger_i + \hat{a}^\dagger \hat{b}_i).$$

(3)

Let us first analyze the physical content of the above model. The canonical model for dissipative quantum dynamics is the quantum Brownian motion (QBM) modeled by [7–9] $\hat{H}_c = \hat{H}_{QBM} = \sum_i C_i (\hat{a} + \hat{a}^\dagger)(\hat{b}_i + \hat{b}^\dagger_i)$. This Hamiltonian has been widely used to model decoherence [2,10,11]. With the methods presented in [2] one finds that the interference pattern of two wave packets gradually vanishes under the influence of the bath. Under certain circumstances one disregards the term $\hat{a} \hat{b}^\dagger_i + \hat{a}^\dagger \hat{b}_i$ which then leads to $\hat{H}_c$ defined in eq. (3). This is the well-known rotating wave approximation (RWA). Note however, that eq. (3) can be more than a simple approximation of QBM since it models an interaction which concerns the total number of particles $N = \hat{a}^\dagger \hat{a} + \sum_i \hat{b}^\dagger_i \hat{b}_i$, a symmetry that can exist right from the start to be described experimental setups, such as coupled resonator optical waveguides, see e.g. [12,13].

Note that the influence of the bath on the system is completely determined by the spectral function, which in the case of the resonant model takes the form $S(\omega) = 2\pi \sum_i C_i^2 \delta(\omega - \omega_i)$. In the present case the spectral function is non-zero only for $\omega \geq 0$. If the number of bath modes is infinite we can also write $S(\omega) = \eta \omega f(\omega/\omega_c)$, where $\eta \propto C_q^2$ characterizes the strength of the $\mathcal{SB}$-coupling, $f(x)$ is a cut-off function ($f(x) \to 0$ when $x \gg 1$), and $\omega_c$ is a characteristic cut-off frequency. Depending on the exponent $s$ which describes the low-$\omega$ behavior of $S(\omega)$ the bath is said to be Ohmic, super-Ohmic or sub-Ohmic ($s = 1$, $s > 1$ or $s < 1$, respectively).

Wigner function of a Gaussian initial state. – In order to obtain the exact Wigner function of $\mathcal{S}$, we analyze first $\hat{\rho}^G(t)$. The superscript $G$ indicates that we start from a Gaussian initial condition. First, we have

$$\langle e^{-i(y-x)\hat{p}^G(t)} e^{i\hat{q}^G(t)} \rangle_t \equiv \langle e^{-i(y-x)\hat{\rho}^G(t)} e^{i\hat{q}^G(t)} \rangle_t$$

$$= \text{Tr}_\mathcal{B}\{e^{-i(y-x)\hat{p}^G(t)} e^{i\hat{q}^G(t)} \hat{\rho}_d(t)\}$$

$$= \int dq' \langle q' | e^{-i(y-x)\hat{p}^G(t)} e^{i\hat{q}^G(t)} \hat{G}(t) | x' \rangle$$

$$= \int dq' e^{i(x' + y - y')\hat{q}^G(t)} \hat{G}_{x' + x, y' - y}(t),$$

(4)

where $\langle \cdots \rangle_t = \text{Tr}_\mathcal{B}\{ \hat{\rho}_d(t) \cdots \}$ is the time-dependent statistical average, $\text{Tr}_\mathcal{B}\{ \cdots \} = \int dq' \langle \cdots \rangle | q' \rangle | \hat{\rho}_d(t) = \langle \cdots | q' \rangle | q' \rangle | \hat{\rho}_d(t)$, and $\hat{q}^G = \sqrt{m \omega_c} (\hat{a} + \hat{a}^\dagger)$, $\hat{p}^G = i\sqrt{m \omega_c} (\hat{a}^\dagger - \hat{a})$ are the position and momentum operators of $\mathcal{S}$. We have introduced the matrix elements of the RDM $\rho_{x,y}(t) = \langle x | \hat{\rho}(t) | y \rangle$. If we multiply both sides of eq. (4) by $\int dq e^{-i\hat{q}_x x}$, and use the Baker-Campbell-Hausdorff formula, we find

$$\rho^G_{x,y}(t) = \int \frac{d\theta}{2\pi} e^{-\frac{i}{2}r(x+y)} \langle e^{-i(y-x)\hat{\rho}^G(t)} e^{i\hat{q}^G(t)} \rangle_0.$$ 

(5)

Since we assume in this paragraph $\hat{\rho}_d(0)$ to be Gaussian, the average in eq. (5) can be readily done by realizing that within a path-integral formalism [14]—$\hat{p}$ and $\hat{q}$ become Gaussian random variables. One then obtains

$$\langle e^{-i(y-x)\hat{p}^G(t)} e^{i\hat{q}^G(t)} \rangle_t = e^{i\hat{q}^G - \frac{C_{pp}}{2}(x-y)^2 - \frac{C_{qq}}{2}(y-q)^2 - i\hat{p}(x-y)}$$

(6)

where we denote the mean value by $\hat{A}(t) = \langle \hat{A} \rangle_t$, and the correlation functions by $C_{AB}(t) = \frac{1}{2} \langle \hat{A} \hat{B} + \hat{B} \hat{A} \rangle_t - \langle \hat{A}(t) \hat{B}(t) \rangle$ for any two operators $\hat{A}$ and $\hat{B}$. These correlation functions can be determined by using path-integral methods [9] or by averaging the solution of the equations of motion over the initial density-matrix $\hat{\rho}_d(0)$ (see a more detailed discussion below).

From eq. (5) we find in conjunction with eq. (6) the final expression for the RDM of a Gaussian initial condition after the time evolution under a quadratic (possibly time-dependent) Hamiltonian:

$$\rho^G_{x,y}(t) = \frac{e^{\tilde{m}(x-q)(y-q) - \frac{C_{pp}}{2}(x-q)^2 - \frac{C_{qq}}{2}(y-q)^2 - i\hat{p}(x-y)}}{\sqrt{2\pi C_{xx}}}.\quad (7)$$

with

$$\tilde{m} = C_{pp} - \frac{1}{C_{qq}} \left( \frac{1}{4} + C_{qq}^2 \right),$$

$$m = C_{pp} + \frac{1}{C_{qq}} \left( \frac{1}{2} i C_{qq} \right)^2.\quad (8)$$

Note that this result also allows the study of quantum quenches by setting $\hat{\rho}_d(0) = e^{-\frac{\hat{H}_\text{init}}{T}/T} \text{Tr}_\mathcal{B}\{ e^{-\frac{\hat{H}_\text{init}}{T}/T} \hat{\rho}_d(t) \}$ with $\hat{H}_\text{init}$, a quadratic (possibly interacting) Hamiltonian different from $\hat{H}$, and $T$, the initial temperature of the system.
Let us now determine the Wigner function associated with the Gaussian density-matrix (7) by using the definition (2). After a straightforward calculation one finds

\[ W^G_i(z) = \frac{e^{-\frac{1}{2}(z-\tilde{z})^T \mathcal{A}^{-1}_i (z-\tilde{z})}}{2\pi \sqrt{\det \mathcal{A}^G_i}}. \tag{9} \]

We introduced the vector notation \( \tilde{z}^T = (q, p) \) and \( \tilde{\psi}_i = (q_i, p_i) \) (which we shall use later) for the mode \( i \) of \( \mathcal{B} \), as well as the Euclidean scalar product \( \cdot \) of \( \mathbb{R}^2 \). Furthermore,

\[ \mathcal{A}^G_i = \frac{1}{2} (\hat{z} \cdot \hat{z}) , \tag{10} \]

is the covariance matrix (note that \( \hat{z} \cdot \hat{z}^T \neq (\hat{z} \cdot \hat{z})^T \) since \( \hat{q} \) and \( \hat{p} \) do not commute which can be recast as

\[ \mathcal{A}^G_i = \begin{pmatrix} C_{qq}(t) & C_{qp}(t) \\ C_{pq}(t) & C_{pp}(t) \end{pmatrix} . \tag{11} \]

The various correlators can be computed by using the equations of motion of the operators \( \hat{z} \) and \( \hat{\psi}_i \):

\[ i\partial_t \hat{z}(t) = M_0(t) \cdot \hat{z}(t) + \sum_i \hat{\psi}_i(0) \cdot \hat{\psi}_i(t), \tag{12} \]

\[ i\partial_t \hat{\psi}_i(t) = M_i(t) \cdot \hat{\psi}_i(t) + \hat{\psi}_i(0) \cdot \hat{\psi}_i(t), \tag{13} \]

where \( M_0, M_i, \hat{\psi}_i, \) and \( \hat{\psi}_i \) are two-dimensional matrices, the details of which depend on the model. By solving first the equations of the bath operators we find the solutions

\[ \hat{\psi}_i(t) = \mathcal{L}_i(t) \cdot \hat{\psi}_i(0) + \int_0^t d\tau \mathcal{L}_i(t - \tau) \cdot \hat{\psi}_i(\tau), \tag{14} \]

with \( i\partial_t \mathcal{L}_i(t) = M_i(t) \cdot \mathcal{L}_i(t) \) and \( \mathcal{L}_i(0) = 1 \). By inserting these solutions into the equations of motion of \( \hat{z} \) we further obtain

\[ \hat{z}(t) = \Phi(t) \cdot \hat{z}(0) + \sum_i M_i(t) \cdot \hat{\psi}_i(0), \tag{15} \]

with

\[ \mathcal{M}_i(t) = \int_0^t d\tau \Phi(t - \tau) \cdot \hat{\psi}_i(\tau), \tag{16} \]

and \( \Phi(t) \) the solution of

\[ 0 = i\partial_t \Phi(t) - M_0(t) \cdot \Phi(t) - \sum_i \int_0^t d\tau \hat{\psi}_i(t - \tau) \cdot \hat{\psi}_i(\tau) \cdot \Phi(t), \tag{17} \]

with \( \Phi(0) = 1 \). Note also that \( \mathcal{M}_i(0) = 0 \). From eq. (15) the covariance matrix \( \mathcal{A}^G_i \) can be readily computed by averaging over \( \hat{\psi}_i(0) \) (see footnote 1).

Footnote 1: Note that \( C_{AB}(t) \) is an equal-time correlator and that \( \langle \hat{A} \hat{B} \rangle_t = \langle A(t) B(t) \rangle_0 \).

**Wigner function of an arbitrary initial state.** – We show now how the density-matrix of an arbitrary non-Gaussian initial condition can be constructed from the density-matrix of a coherent-state initial condition. Note that coherent states are Gaussian states such that we can make the link with the previous section.

It is well known that any density-matrix can be written as a diagonal matrix in the coherent-state basis by using the Glauber-Sudarshan \( P \)-function (see e.g. [15]):

\[ \hat{P}(0) = \int d^2 \alpha \prod_i d^2 \beta_i P_0(\alpha; \{ \beta_i \}) |\alpha; \{ \beta_i \} \rangle \langle \alpha; \{ \beta_i \} | , \tag{18} \]

where a state of \( \mathcal{U} \) is written as \( |\alpha; \{ \beta_i \} \rangle \) with \( \alpha \) the state of \( \mathcal{S} \) and \( \{ \beta_i \} = \{ \beta_1, \beta_2, \ldots \} \) the state of \( \mathcal{B} \). By taking a partial trace over the \( \mathcal{B} \)-states we find the RDM

\[ \rho_{x,y}(t) = \int d^2 \alpha \prod_i d^2 \beta_i P_0(\alpha; \{ \beta_i \}) \rho^G_{x,y}(t), \tag{19} \]

where \( \rho^G_{x,y}(t) \) is now associated with the particular Gaussian initial condition \( \hat{P}^{\alpha;\beta}(0) = |\alpha; \{ \beta_i \} \rangle \langle \alpha; \{ \beta_i \} | \).

Hence, by using eqs. (19) and (2) the Wigner function corresponding to an arbitrary initial state is given by

\[ W_i(z) = \int d\tilde{z} \prod_i d\tilde{\psi}_i W^G_i(z) P_0(\tilde{z}, \tilde{\psi}_i), \tag{20} \]

where \( W^G_i(z) \) is given in eq. (9) with the initial condition \( \hat{P}^{\alpha;\beta} \).

Note that the \( P \)-function contains all the information on the non-Gaussian initial condition. However, the \( P \)-function is in general highly singular and therefore not suited for concrete applications. We therefore proceed by eliminating \( P_0(\tilde{z}, \tilde{\psi}_i) \). Let us first express the problem solely in terms of the variables \( \tilde{z} \) and \( \{ \tilde{\psi}_i \} \). By definition, we can write

\[ \hat{\psi}_i = \frac{\hat{\psi}_i(0)}{\sqrt{2m\omega_0}} \text{Re}(\alpha), \quad \text{Im}(\alpha) \]

and similar relations between the \( \tilde{\psi}_i \) and \( \beta_i \). Hence, \( W^G_i(z) \) now depends via its initial condition \( \hat{P}^{\alpha;\beta}(0) = |\tilde{z}; \{ \tilde{\psi}_i \} \rangle \langle \tilde{z}; \{ \tilde{\psi}_i \} | \) on the new variables \( \tilde{z} \) and \( \tilde{\psi}_i \), too. Note that we have relabeled the coherent state \(|\alpha; \{ \beta_i \} \rangle\) by virtue of the above relation between \( z^T \), \( \text{Re}(\alpha) \) and \( \text{Im}(\alpha) \). In the following we use the convention for the Fourier transform \( \hat{F}(k) = \int dz e^{-ikz} F(z) \) of a function \( F(z) \), defining analogously the Fourier transform of functions of many variables.

Second, let us introduce the Wigner function \( W_0(\tilde{z}, \{ \tilde{\psi}_i \}) \) of the (non-Gaussian) initial condition \( \hat{P}_0(0) \). According to [15] it can be written in terms of \( P_0(\tilde{z}, \tilde{\psi}_i) \) in the Fourier domain as

\[ \hat{W}_0(k; \{ \kappa_i \}) = \hat{P}_0(k; \{ \kappa_i \}) e^{-\frac{i}{\hbar}k \cdot \mathcal{A}_0 k - \sum_i \frac{i}{2} \mathcal{A}_i \cdot \kappa_i^T \cdot \mathcal{A}_i \cdot \kappa_i}, \tag{21} \]

where

\[ \mathcal{A}_0 = \begin{pmatrix} \frac{1}{2}m\omega_0 & \frac{1}{2m\omega_0} & 0 \\ 0 & \frac{1}{2}m\omega_0 & \frac{1}{2m\omega_0} \end{pmatrix}, \quad \mathcal{A}_i = \begin{pmatrix} 0 & \frac{1}{2m\omega_0} & 0 \\ \frac{1}{2m\omega_0} & 0 & \frac{1}{2m\omega_0} \end{pmatrix}, \tag{22} \]

are the covariance matrices of a coherent state.
Third, the Fourier transform of eq. (20) with respect to \( z \) (using eq. (9)) now reads
\[
\hat{W}_t(k) = \int \frac{d\zeta}{i} e^{-\frac{1}{2}k T \cdot \zeta} \mathcal{A}_G \cdot \zeta - ik \cdot \zeta \cdot z \cdot P_0(\hat{\zeta}, \{ \hat{\zeta}_i \}).
\] (23)

From eq. (15) one easily shows that \( \hat{z}(t) = \Phi(t) \cdot \hat{z} + \sum_t \mathcal{M}_t(t) \cdot \hat{z} \), and eq. (23) can be recast as
\[
\hat{W}_t(k) = e^{-\frac{1}{2}k T \cdot \mathcal{A}_G \cdot \Phi - \sum_t \mathcal{M}_t \cdot \frac{\mathcal{M}_t}{k}} \cdot k \times \hat{W}_0(\Phi(t) \cdot k, \{ \mathcal{M}_t \cdot k \}).
\] (24)

Finally, with the definition of \( \mathcal{A}_G \), see eq. (10), and by noting that \( \mathcal{A}_G \) has to be calculated with \( \phi^{(0)} \) (see above) one can show after some algebra that
\[
\mathcal{A}_G = \Phi \cdot \mathcal{A}_0 \cdot \Phi^T - \sum_i \mathcal{M}_i \cdot \mathcal{M}_i^T
\] (25)
the exponent in the rhs of eq. (24) thus cancels out exactly and
\[
W_t(z) = \int \frac{dk}{2\pi} e^{i k T \cdot \hat{z}} \hat{W}_0(\Phi(t) \cdot k, \{ \mathcal{M}_i^T \cdot k \}).
\] (26)

Note that eq. (26) holds regardless of the entanglement in the initial condition\(^2\), and that it has all the expected properties such as normalization at all times, \( \int dz W_t(z) = 1 \).

In the particular case where the initial condition is factorized, \( \hat{\rho}_0(0) = \hat{\rho}_S \otimes \hat{\rho}_B \), we further have \( \hat{W}_0(z, \{ \zeta_i \}) = W_S(z) W_B(\{ \zeta_i \}) \). By defining the propagator \( K_t(z) = \int \frac{dk}{2\pi} e^{i k T \cdot \hat{z}} \hat{W}_B(\{ \mathcal{M}_i^T(t) \cdot k \}) \) we can write
\[
W_t(z) = \int \frac{d\zeta}{i} K_t(z - \Phi(t) \cdot \hat{z}) W_S(\hat{z}).
\] (27)

Since \( K_{t=0}(z) = \delta(z) \) (by definition \( \mathcal{M}_i(0) = 0 \) and by normalization \( \hat{W}_B(\{ \zeta_i = 0 \}) = 1 \)) the term "propagator" is particularly well suited for \( K_t \). Furthermore, in the case where both \( \Phi(t \to \infty) \to 0 \) and \( \mathcal{M}_i(t \to \infty) \to \text{const.} \) one has \( W_{t \to \infty}(z) \simeq K_{t \to \infty}(z) \). This relation is universal in the sense that it does not depend on the initial state of the system \( S \). This has to be contrasted with the case \( \Phi(t \to \infty) \to \text{const.} \neq 0 \) (that we will discuss below) where one can recover some information about the initial state.

Let us remark here that even though the formal solution of the dynamics of \( N \) bosonic modes (equivalent to \( U = S + B \) discussed here) interacting through a quadratic Hamiltonian is known (see for instance [16] and references therein), a closed expression for the reduced Wigner function for arbitrary (non-factorizing) initial conditions has to our knowledge not been derived before. If in principle the trace over the bath can be performed it is a priori a non-trivial task for entangled initial conditions (see also footnote 2).

Equation (26) and in particular eq. (27) will be used in the following to analyze the relaxation dynamics of the resonant model.

Quantum phase transition and equilibration in the resonant model. — Let us come back to the resonant model with time-independent coupling. With the main result of the previous paragraph, in particular eq. (27), we are now in the position to analyze the "dissipationless dynamics" of the resonant model and to demonstrate that previous interpretations of these dynamics were incomplete. The central aspect of our argumentation is the analysis of the time evolution of \( \hat{\rho}_t(0) = |1; \{0\} \rangle \langle 1; \{0\} | \) which describes a factorizing initial condition between the system in its first Fock state and the bath ground state (i.e. at zero temperature). The equation of motion (15) is now readily solved and (at zero \( T \)) its solution is totally determined by the Green function \( \Phi(t) \) [6]. More precisely, the dynamics is given by (see below)
\[
\hat{u}(\lambda) = [\lambda + i\omega_0 + \Sigma(\lambda)]^{-1} \equiv \Phi_{11}(t) - i m \omega_0 \Phi_{12}(t).
\]
We have in the Laplace domain
\[
\Sigma(\lambda) = \int_0^{\infty} d\omega \frac{S(\omega)}{2\pi \lambda + i\omega}.
\] (29)

The relaxation of the system is then completely determined by the long-time behavior of \( u(t) \). Note that \( \Sigma(\lambda) \) has a branch cut on the imaginary half-axis for \( \text{Im} \lambda < 0 \). Generically, for sufficiently strong interactions, \( \eta \geq \eta_c \), where \( \eta_c \) depends on the details of \( S(\omega) \), an isolated pole \( \lambda_1 \) appears in the denominator of \( \hat{u}(\lambda) \) with \( \text{Re} \lambda_1 = 0 \) and \( \text{Im} \lambda_1 > 0 \). This pole is defined by the equation
\[
\lambda_1 + i\omega_0 + \Sigma(\lambda_1) = 0.
\] (30)

In real time such a pole gives rise to a purely oscillatory mode. One thus has \( u(t) = \mathcal{Z} e^{i \lambda_1 t} + \cdots \) in the long-time limit where the ellipsis stands for decaying terms. Here, \( \mathcal{Z} = [1 + \Sigma(\lambda_1)]^{-1} \) is the residue of the pole (see ref. [6] for a detailed discussion). Accordingly, for \( \eta > \eta_c \) it has been argued that the system’s relaxation is inhibited by the emergence of this isolated pole and the resulting dynamics have been called “dissipationless” in the recent literature [5,6].

Let us further interpret these formal equations by analyzing the spectrum of \( \hat{H} \). Consider in particular the eigenstates of \( \hat{H} \) with zero and one total excitation (the number of which is conserved), which we write as \( |\phi_0\rangle = |0; \{0\} \rangle \) and
\[
|\phi_1\rangle = c_0 |1; \{0\} \rangle + \sum_i c_i |0; \{i\} \rangle,
\] (31)
where \( |0; \{i\} \rangle = \hat{b}_i^\dagger |0; \{0\} \rangle \). The vacuum state \( |0; \{0\} \rangle \) has zero energy, \( c_0 = 0 \), regardless of the \( SB \)-coupling. When \( \eta \) is very small the vacuum is obviously the ground state of \( U \). Let us denote the energies of the one-excitation eigenstates by \( e_1^{(j)} \). By construction one has \( \hat{H} |\phi_j\rangle = e_1^{(j)} |\phi_j\rangle \).

Let us further denote the smallest of these energies by
\[
e_1 = \min(e_1^{(j)})
\] which satisfies (with all the other \( e_1^{(j)} \).
the condition
\[ e_1 = \omega_0 + \sum_{i=1}^{N_R} \frac{C_i^2}{e_1 - \omega_i}. \]  
(32)

Also, \( c_0 \) is determined by
\[ c_0^2 = \left( 1 + \sum_{i=1}^{N_R} \frac{C_i^2}{(e_1 - \omega_i)^2} \right)^{-1} = 1 - \sum_{i=1}^{N_R} c_i^2. \]  
(33)

The last equation is a consequence of the normalization condition \( 1 = \langle \phi_1 | \phi_1 \rangle \). Moreover, upon inspecting eqs. (32) and (33) it is clear that
\[ \lambda_1 = -ie_1, \quad c_0^2 = \frac{1}{1 + \Sigma'(e_1)} = Z_1. \]  
(34)

Obviously, when \( e_1 > 0 \) the sum in eq. (33) diverges in the limit of an infinite bath \( (N_B \to \infty) \) and \( c_0 \sim 1/\sqrt{N_B} \to 0 \). However, if eq. (32) has a solution \( e_1 < 0 \) then \( c_0 \) remains finite which implies that \( \langle \phi_1 | \hat{a}^\dagger \hat{a} | \phi_1 \rangle = c_0^2 > 0 \). Moreover, in that case \( e_1 < c_0 \) and the ground state changes due to a level crossing. Such a behavior implies a quantum phase transition (QPT). The critical value \( \eta_c \) is found by setting \( e_1 = 0 \) in eq. (32), which translates into \( \Sigma(0) = -i\omega_0 \). The ground state changes from \( |\phi_0\rangle \) for \( \eta < \eta_c \) to \( |\phi_1\rangle \) for \( \eta > \eta_c \) and it can be characterized by the density \( n_a = \langle \hat{a}^\dagger \hat{a} \rangle \), which serves as an order parameter \( (n_a = 0 \text{ for } \eta < \eta_c \text{ and } n_a \neq 0 \text{ for } \eta > \eta_c) \). This phase transition is reminiscent of the QPT that occurs in other paradigmatic models such as the Dicke model (the well-known super-radiance transition, see for instance [18]) or the spin-boson model (where a QPT was recently discussed in [19]).

The one-excitation eigen-subspace of \( \hat{H} \) is spanned by the states \( |\phi_1\rangle, |\phi_1^2\rangle, |\phi_1^3\rangle, \ldots \) and for \( \eta > \eta_c \) it is straightforward to show that \( e_1 < 0 < e_1^{(j)} \). Let us now make the connection to the system which is initially described by \( \hat{\rho}_U(0) = |1; \{0\}\rangle \langle 1; \{0\}| \). In order to find the time evolution we expand \( |1; \{0\}\rangle \) into a sum over \( |\phi_1\rangle \) and all \( |\phi_j\rangle \):
\[ |1; \{0\}\rangle = d(\phi_1) + \sum_{j=2}^{N_R} d_j |\phi_j\rangle, \]  
(35)

from which we find \( d_j = c_j^{(j)}(\phi_0) \) by multiplying the previous equation by \( |\phi_1^j\rangle \). Hence, after a time lag \( t \)
\[ \langle 1 | \hat{\rho}(t) | 1 \rangle = d^2(1, \{0\} | \phi_1^1 | \phi_1 | \{0\} \rangle 
\[ + \sum_{j=2}^{N_R} d_j^2 (1, \{0\} | \phi_1^j | \phi_1 | \{0\} \rangle + \text{osc}, \]  
(36)

where “osc” stands for oscillating terms \( \sim e^{it(e_0^{(j)} - e_0^{(j')})} \), \( j \neq j' \), which cancel out for \( t \to \infty \). Since \( d_j = e_0^{(j)} = \langle \phi_1^j | 1, \{0\} \rangle \sim 1/\sqrt{N_B} \) [note that \( e_0^{(j)} > 0 \) and \( d = c_0 \) we have in the \( N_B \to \infty \) limit \( \langle 1 | \hat{\rho}(t) | 1 \rangle \approx \frac{c_0^2}{2} \),\( \{1, \{0\} \}|^2 = \frac{c_0^4}{2} \). Accordingly, the RDM relaxes towards
\[ \hat{\rho}(t \to \infty) = (1 - c_0^4) |0; \{0\}\rangle \langle 0 | + c_0^4 |1; \{0\}\rangle \langle 1 |. \]  
(37)

This result can indeed be derived directly within our formalism (see, i.e., eq. (27)), as can be shown that for the initial condition \( \hat{\rho}_U(0) = |1; \{0\}\rangle \langle 1; \{0\}| \),
\[ W_t(z) = e^{-\frac{\sqrt{\pi}}{2} z^T A_0^{-1} z} \times (|u(t)|^2 z^T A_0^{-1} z - 2|u(t)|^2 + 1), \]  
(38)

which then, in the large-time limit, \( (\eta > \eta_c) \) note that \( u(t) \to \mathcal{Z} e^{-i \pi t} \) yields
\[ W_t(z) = e^{-\frac{\sqrt{\pi}}{2} z^T A_0^{-1} z} \times (e_0^4 z^T A_0^{-1} z + 1 - 2e_0^4). \]  
(39)

This is indeed the Wigner-function associated with eq. (37). On the other hand, if \( \eta < \eta_c \), \( u(t) \to 0 \) in the long-time limit and \( W_t(z) \to \frac{\sqrt{\pi}}{2} e^{-2|z|^2} \), which is the Wigner function of the vacuum. Note that, while the number of total excitations is conserved, the single occupation number of the system \( \hat{a}^\dagger \hat{a} \) is not. The RDM of the system can thus relax towards \( |0; \{0\}\rangle \) for \( \eta < \eta_c \).

The fact that for \( \eta > \eta_c \), the Wigner function (39) does not relax towards the Wigner function of the vacuum (which is reflected by the non-trivial dissipationless dynamics, i.e. the non-zero limit of \( u(t) \) at long time) has been associated in ref. [6] to strong non-Markovian effects (and hence strong non-equilibrium effects). However, we have shown here that these effects rather stem from a static quantum phase transition. The dynamics is dissipatioless in the long-time limit not because of strong non-Markovian effects. The system rather relaxes to its (non-trivial) ground state —within the constraints imposed by energy and total particle conservation— and more dissipation of energy is then not possible.

**Discussion and conclusion.**— Let us briefly summarize what we have achieved in this letter. Equation (26) gives the general form of the Wigner function in the case of a Gaussian evolution, but arbitrary initial conditions (in particular non-factorizing ones) which are encoded in the Fourier transform of their Wigner function \( W_0 \). Since the time evolution (given after eqs. (15)) is in principle exactly solvable (note that we assumed \( \hat{H} \) to be quadratic) a remaining difficulty may arise when calculating \( W_0 \) via eq. (2). However, in the case where a superposition of Gaussian states is considered (i.e. Schrödinger cat states), or for Fock states, the effort for determining \( W_0 \) is minimal.

In the case of a factorized initial condition, eq. (27) yields the evolution of the reduced initial Wigner function \( W_S \) through the convolution with the propagator

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$K_1$. Equation (27) can be considered as the solution to the corresponding master equations: For instance it is straightforward to show that with $\hat{\rho}_B = e^{-\hat{H}_B/T}/Z_B$ (with $Z_B = \text{Tr} e^{-\hat{H}_B/T}$), eq. (27) reproduces the solutions of the exact master-equation of the QBM, derived for instance in ref. [4], and of the resonant coupling, derived in ref. [5].

Finally, we have investigated the relaxation dynamics of the resonant model by analyzing its exact Wigner function. We have shown that the out-of-equilibrium Wigner functions of the state $|1; \{0\}\rangle$ is non-trivial in presence of a strong $SB$-coupling as the system does not relax to the vacuum. In contrast to previous interpretations, the so-called “dissipationless” dynamics are not caused by memory effects in the non-Markovian environment. It is rather a thermodynamic quantum phase transition in the whole system-bath ensemble which alters the dynamics. The emergence of the non-trivial ground state does not prevent the system from relaxing; on the contrary, we have shown that – within the constraints imposed by energy and total particle conservation – the system shows standard relaxation towards its non-trivial ground state. Non-trivial ground states appear in a number of physical systems ranging from condensed matter to atomic physics, and the “dissipationless dynamics” is a consequence of the stability of these non-trivial states (see, for instance, [20]). We want to stress that this peculiar behavior could not appear in the standard QBM, as the self-energy in that case is of the form $\Sigma_{QBM}(\lambda) \propto \int \frac{d\omega}{2\pi} S_{QBM}(\omega)/(\lambda^2 + \omega^2)$ (with a suitably defined spectral function $S_{QBM}(\omega)$ for the QBM), implying a branch cut on the whole imaginary axis. Therefore no stable pole can appear in QBM.

We emphasize that our approach can be easily generalized to the case of several bosonic modes and used to study entanglement in such extended systems. Also, eq. (27) can be used as a guide to derive more easily the correct master equations for other systems in the future.

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