BASIC SYSTEMS OF ORTHOGONAL FUNCTIONS FOR SPACE-TIME MULTIVECTORS

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Abstract

Space-time multivectors in Clifford algebra (space-time algebra) and their application to nonlinear electrodynamics are considered. Functional product and infinitesimal operators for translation and rotation groups are introduced, where unit pseudoscalar or hyperimaginary unit is used instead of imaginary unit. Basic systems of orthogonal functions (plane waves, cylindrical, and spherical) for space-time multivectors are built by using the introduced infinitesimal operators. Appropriate orthogonal decompositions for electromagnetic field are presented. These decompositions are applied to nonlinear electrodynamics. Appropriate first order equation systems for cylindrical and spherical radial functions are obtained. Plane waves, cylindrical, and spherical solutions to the linear electrodynamics are represented by using the introduced orthogonal functions. A decomposition of a plane wave in terms of the introduced spherical harmonics is obtained.

1 Introduction

Multivectors are scalars, vectors, and fully asymmetric tensors. The maximal rank of non-zero multivectors equals the dimension of space\(^1\). So called essential components of a fully asymmetric tensor define all its components. In particular, such tensor of maximal rank has one essential component: each its component is \(\pm \text{some number} \text{ or } 0\).

Space-time multivectors are scalars, vectors, fully asymmetric second-rank tensors or bivectors, fully asymmetric third-rank tensors or three-vectors or pseudovectors, and fully asymmetric fourth-rank tensors or pseudoscalars. Bivector has six essential components and pseudovector has four ones. There are \(1 + 4 + 6 + 4 + 1 = 2^4 = 16\) essential components of all four-dimensional space-time multivectors.

Space-time multivectors have wide application for physics, in particular, for electrodynamics, where the electromagnetic field is described as bivector space-time function. Electromagnetic potential is space-time vector and dual potential is pseudovector.

There is a very useful mathematical tool for manipulations with multivectors. This tool is based on Clifford algebra. The dimension of Clifford algebra is equal to the quantity of all multivector essential components, i.e. 16 for

\(^1\)If the number of tensor indices (the rank of tensor) exceeds the dimension of space, then each tensor component has at least two equal indices. Such fully asymmetric tensor is null tensor.
four-dimensional space-time. Having an appropriate multiplication table for non-commutative product we can make algebraic manipulations with multivectors. The members of the Clifford algebra are represented as hypercomplex numbers and their non-commutative product is defined by the multiplication table.

The main object of this work is a building of basic systems of orthogonal multivector functions and obtaining the multiplication tables for these functions.

2 Space-time hypernumbers

Let us call hypercomplex numbers as hypernumbers. The general form of space-time hypernumber is (see also my papers [3, 4])

\[ C = (1 C^0 + i C^{IV}) + (1 C^i_\mu + i C^{III}_\mu) b^\mu + (1 C^{III}_i + i C^{IV}_i) b^i, \]  

(2.1)

where 1 and \( i \) are hyperunit and hyperimaginary unit, \( b^\mu \) are basis vectors (Greek indices take on a value 0, 1, 2, 3), \( b^i \) are basis bivectors (Latin indices take on a value 1, 2, 3), and \( C^{I...IV} \) are connected with components of multivectors.

If we take \( C^I_\mu \equiv C^{III}_\mu \equiv 0 \) in (2.1), then we have by definition an even space-time hypernumber. On the contrary, if \( C^0 \equiv C^{IV} \equiv C^{III}_i \equiv C^{IV}_i \equiv 0 \), then the hypernumber is odd.

The first bracketed expression in (2.1) will be called hyperscalar.

The expressions of type \( (1 C^I_\mu + i C^{III}_\mu) \) will be called quasi-hyperscalars

\[ \text{2.2} \] .

They differ from hyperscalars because of existing tensor indices. Quasi-hyperscalars are transform coupled with coordinate system transformation: \( C = C'_\mu b^\mu = C''_\mu b'^\mu \). But a transformation of space-time rotation type for geometrical objects leaves quasi-hyperscalars to be invariable: \( C' = \Lambda C \Lambda^{-1} = C''_\mu \Lambda b'^\mu \Lambda^{-1} \), where \( \Lambda \) is an even hypernumber realizing the space-time rotation (see, for example, [5]) and \( C'_\mu \) are quasi-hyperscalars which are permutable with even hypernumbers (see late (2.7)).

Let us designate a hyperconjugate hypernumber as

\[ ^*C = (1 C^0 - i C^{IV}) + (1 C^i_\mu - i C^{III}_\mu) b^\mu + (1 C^{III}_i - i C^{IV}_i) b^i. \]  

(2.2)

\[ ^2\text{The prefix “quasi-” can be omitted} \]
Hyperreal and hyperimaginary parts of hypernumber \( C \) are defined as
\[
\Re C \doteq \frac{1}{2} (C + \ast C) = C^0 1 + C^\mu_\mu b^\mu + C^i_i b^i ,
\]
\[
\Im C \doteq \frac{1}{2} (\ast C - C) = C^{IV} 1 + C^\mu_\mu b^\mu + C^i_i b^i .
\]

Hyperimaginary unit is the coordinate-free form of unit fully asymmetric fourth-rank tensor or space-time pseudoscalar:\n\[
i \doteq \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} b^\mu b^\nu b^\rho b^\sigma ,
\]
where \( \epsilon_{\mu\nu\rho\sigma} \) are its components: \( \epsilon_{0123} = -\sqrt{|m|} \), \( \epsilon^{0123} = \sqrt{|m|}^{-1} \), \( m \doteq \det(m_{\mu\nu}) \), and \( m_{\mu\nu} \) are components of metric tensor.

It is convenient to divide the non-commutative but associative and distributive hypernumber product to symmetrical and asymmetrical parts:
\[
1 C 2 C = 1 C \otimes 2 C + 1 C \wedge 2 C ,
\]
\[
1 C \otimes 2 C \doteq \frac{1}{2} \left( C 2 C + 2 C 1 C \right) ,
\]
\[
1 C \wedge 2 C \doteq \frac{1}{2} \left( C 2 C - 2 C 1 C \right) .
\]

Introduced operations \( \otimes \) and \( \wedge \) will be called symmetrical and asymmetrical products\(^4\). They are non-associative but distributive. Consecutive use of these operations without brackets, such that \( 1 C \otimes 2 C \otimes 3 C \) or \( 1 C \wedge 2 C \wedge 3 C \), must be considered as symmetrization or alternation respectively by all co-factors. For example
\[
1 C \wedge 2 C \wedge 3 C \doteq \frac{1}{3!} \left( C 2 C 3 C - C C 2 C + C 1 C 2 C - C 1 C 3 C + C 2 C 1 C - C 3 C 1 C \right)
\]
\[\neq 1 C \wedge \left( 2 C \wedge 3 C \right) \neq \left( 1 C \wedge 2 C \right) \wedge 3 C .
\]

\(^3\) Here I use designations some differing from used in my preceding papers [3, 4]. All expressions of these my papers can be rewritten in the current designations by substitutions \( \epsilon_{\mu\nu\rho\sigma} \rightarrow -\epsilon_{\mu\nu\rho\sigma} \), \( \epsilon^{\mu\nu\rho\sigma} \rightarrow -\epsilon^{\mu\nu\rho\sigma} \), \( i \rightarrow -i \).

\(^4\) The designations used here for symmetrical and asymmetrical products (\( \otimes \) and \( \wedge \)) differs from ones used in my preceding papers [3, 4] (\( \cdot \) and \( \wedge \)). Customary meanings of the symbols \( \cdot \) and \( \wedge \) are internal and exterior products. Really we must distinguish the symmetrical and asymmetrical products from internal and exterior ones. There is the coincidence of these product pairs for vectors (it is clear, also for scalars) but not for the general case. The use of symmetrical and asymmetrical products for hypernumbers is preferably, because the definition of these products by using non-commutative one (2.5b) is independent of types of multiplied multivectors, in contrast to appropriate definitions for internal and exterior products. All expressions of my papers [3, 4] can be rewritten in the current designations by substitutions \( \cdot \rightarrow \otimes \) and \( \wedge \rightarrow \wedge \).
Using designations (2.5b), let us write the following multiplication table (see also [3][4]):

\[
\begin{align*}
C \cdot 1 &= C = C , \quad \mathbf{z} = -1 , \quad \mathbf{z} b^\mu = -b^\mu \mathbf{z} , \quad \mathbf{z} b^i = b^i \mathbf{z} , \\
\mathbf{b}^\mu \otimes \mathbf{b}^\nu &= 1 m_{\mu \nu} , \quad \mathbf{b}^i \wedge \mathbf{b}^0 = b^i , \quad \mathbf{b}^i \wedge \mathbf{b}^j = i^{0ij} \mathbf{b}_i , \quad \mathbf{b}_\nu = m_{\nu \mu} b^\mu , \\
\mathbf{b}^0 \otimes \mathbf{b}^i &= 0 , \quad \mathbf{b}^i \otimes \mathbf{b}^j = -i^{0ij} \mathbf{b}_k , \quad \mathbf{b}^i \wedge \mathbf{b}_j = -\delta^i_j \mathbf{b}_0 , \quad \mathbf{b}^0 \wedge \mathbf{b}^i = b^i , \\
\mathbf{b}^i \otimes \mathbf{b}^j &= -m^{00} m^{ij} + m^{i0} m^{j0} , \quad \mathbf{b}^i \wedge \mathbf{b}_j = b^i \wedge \mathbf{b}_j , \\
\mathbf{b}^i &= (-m^{00} m^{ij} + m^{i0} m^{j0}) \mathbf{b}_j .
\end{align*}
\]

(2.7)

There are so called zero divisors in this hypercomplex system. Zero divisor is a hypernumber \( C \) such that \( C \mathbf{X} = 0 \) (left) or \( \mathbf{X} C = 0 \) (right) for some hypernumber \( \mathbf{X} \neq 0 \). There is no inverse element for zero divisor. Really, if \( C \mathbf{X} = 0 \), then \( C^{-1} C = 1 \Rightarrow C^{-1} C \mathbf{X} = \mathbf{X} \Rightarrow 0 = \mathbf{X} \). For example, if \( (\mathbf{b}_1)^2 = 1 \), then \( (1 - \mathbf{b}_1)(1 + \mathbf{b}_1) = 1 - (\mathbf{b}_1)^2 = 0 \).

It should be noted that the coefficients \( C^\mu..^\nu \) suppose to be real numbers. Thus there is not the imaginary unit in this hypernumber system. But the hyperimaginary unit \( \mathbf{z} \) can be used as customary imaginary unit for even space-time hypernumbers, because of \( \mathbf{z} \mathbf{z} = -1 \) and \( \mathbf{z} \mathbf{b}^i = b^i \mathbf{z} \).

The symmetrical and asymmetrical products for bivectors corresponds with scalar and vector products respectively for space vectors with quasi-hyperscalar components:

\[
\begin{align*}
1 \otimes 2 &= 1 2 = m^{ij} C^i C^j 1 , \\
1 \wedge 2 &= \mathbf{z} C \times C = C \times C , \quad 1 \times 2 \equiv (i^{0ijk} C^i C^j C^k) b^i ,
\end{align*}
\]

(2.8a)

(2.8b)

where \( C = C^i b^i \), \( C^i \) are quasi-hyperscalars, and it is supposed \( m_{00} = -1 \), \( m_{i0} = 0 \).

Since customary complex numbers do not use in the mathematical tool under consideration, we can, in principle, omit the prefix “hyper” in some words. But at the present paper I use the original long words.

### 3 Functional product

Because the hyperimaginary unit \( \mathbf{z} \) is permutable with even space-time hypernumbers, any manipulations with these hypernumbers are more convenient than for odd ones. But odd hypernumber can be transformed to even one with multiplication by the basis vector \( \mathbf{b}_0 \). Thus we will build basic systems of orthogonal functions for even hypernumbers.
Let us define functional product for hyperscalar or bivector space-time functions:
\[
\langle \mathbf{C} | \tilde{\mathbf{C}} \rangle \equiv \int_{\mathcal{M}} (\mathbf{C} \otimes \tilde{\mathbf{C}}) \, d\mathcal{M},
\]
where \( \mathcal{M} \) is some space-time volume and \( d\mathcal{M} \) is its element, \( \mathbf{C}(x) \) and \( \tilde{\mathbf{C}}(x) \) are both hyperscalar or bivector functions. The functional product for a hyperscalar function with a bivector one is zero by definition. The functional product takes constant hyperscalar or quasi-hyperscalar values.

It is evident that the functional product defined as (3.1) has all traditional properties of functional product defined for conventional complex functions.

Let us define the following norm of functional vector:
\[
\| \mathbf{C} \| \equiv \sqrt{\langle \mathbf{C} | \mathbf{C} \rangle}.
\]
(3.2)

Also we us define the module of hyperscalar or bivector:
\[
|\mathbf{C}| \equiv \sqrt{\mathbf{C} \otimes \mathbf{C}^*}.
\]
(3.3)

Adjoint operator is defined in the regular way:
\[
\langle \mathbf{Q} \mathbf{C} | \tilde{\mathbf{C}} \rangle \equiv \langle \mathbf{C} | \mathbf{Q} \tilde{\mathbf{C}} \rangle.
\]
(3.4)

Eigenvalues for operators take on quasi-hyperscalar values.

For self-adjoint (\( \mathbf{Q} = \mathbf{Q}^* \)) and anti-self-adjoint (\( \mathbf{Q} = -\mathbf{Q}^* \)) operators there are two useful properties: its eigenvalues are hyperreal and hyperimaginary accordingly and its eigenfunctions with different eigenvalues are orthogonal. The appropriate proof is very simple, it is fully analogous to the case of functional product defined for conventional complex functions.

## 4 Infinitesimal operators

for translation and rotation groups

A self-adjoint infinitesimal shift operators has the form
\[
\mathcal{F}_\mu \equiv -i \frac{\partial}{\partial x^\mu}.
\]
(4.1)

We have the appropriate invariant self-adjoint infinitesimal operator:
\[
\mathcal{F}_\mu \mathcal{F}^\mu = -\frac{\partial^2}{\partial x^\mu \partial x_\mu}.
\]
(4.2)
A self-adjoint infinitesimal rotation operators can be obtained in space-time algebra formalism. These operators have the form

$$\mathcal{J}^i \doteq -i \epsilon^{ijk} x_j \frac{\partial}{\partial x^k} + \mathfrak{b}^i \wedge .$$

(4.3)

Thus we have also the appropriate anti-self-adjoint operators $$-i \mathcal{J}_i$$ and $$-i \mathcal{J}^i$$. These operators realize group infinitesimal transformations. For example, we have the infinitesimal shift and rotation for some space-time hyper-number function $$C(x^0, x^1, x^2, x^3)$$ about the axis $$x^3$$:

$$C' = C(x^0, x^1, x^2, x^3 - \delta a) = C(x^0, x^1, x^2, x^3) - \delta a \frac{\partial C}{\partial x^3}$$

(4.4a)

$$C'' = C(x^0, x^1 + x_2 \delta \varphi, x^2 - x_1 \delta \varphi, x^3) + C \wedge (i \mathfrak{b}^3) \delta \varphi$$

(4.4b)

If it is possible to permute the order of differentiation, that is outside of singular sets, then we can obtain the following customary commutation relations for the self-adjoint infinitesimal operators:

$$[\mathcal{J}_\mu | \mathcal{J}_\nu ] = 0 , \quad [\mathcal{J}^i | \mathcal{J}^j ] = i^{ijk} i \mathcal{J}_k .$$

(4.5)

In the regular way (see, for example, [7, 6]) we obtain the appropriate raising and reducing operators

$$\mathcal{J}^+ = \mathcal{J}^1 + i \mathcal{J}^2 , \quad \mathcal{J}^- = \mathcal{J}^1 - i \mathcal{J}^2 .$$

(4.6)

with the customary commutation relations

$$[\mathcal{J}^+ | \mathcal{J}^3 ] = -\mathcal{J}^+ , \quad [\mathcal{J}^- | \mathcal{J}^3 ] = \mathcal{J}^- , \quad [\mathcal{J}^+ | \mathcal{J}^- ] = 2 \mathcal{J}^3 .$$

(4.7)

We have the invariant self-adjoint infinitesimal operator for rotation

$$\mathcal{J}^2 \doteq (\mathcal{J}^1)^2 + (\mathcal{J}^2)^2 + (\mathcal{J}^3)^2$$

$$= \mathcal{J} - \mathcal{J}^+ + (\mathcal{J}^3)^2 + \mathcal{J} = \mathcal{J}^+ \mathcal{J}^- + (\mathcal{J}^3)^2 - \mathcal{J}^3$$

(4.8a)

and the appropriate commutation relations

$$[ (\mathcal{J})^2 | \mathcal{J}^j ] = [ (\mathcal{J})^2 | \mathcal{J}^- ] = [ (\mathcal{J})^2 | \mathcal{J}^+ ] = 0 .$$

(4.9)

\(^5\text{See also the accordance between } \wedge \text{ and } \times \text{ operations for the case of bivector functions}.)
5 Electrodynamics

Two antisymmetric tensors or bivectors of electromagnetic field are represented in the form (see my paper [3])

\[ F \doteq \frac{1}{2} F_{\mu\nu} b^\mu b^\nu = E_i b_i + B^i \mathbf{b}_i = E + i B , \quad (5.1a) \]

\[ G \doteq \frac{1}{2} G_{\mu\nu} b^\mu b^\nu = D_i b_i + H^i \mathbf{b}_i = D + i H , \quad (5.1b) \]

where \( E \) and \( H \) are electric and magnetic field intensities, \( D \) and \( B \) are electric and magnetic inductions.

A hypernumber form of nonlinear electrodynamics was obtained in my work [3]. Let us write this equation with electromagnetic current (see [4]):

\[ \partial \otimes F + \partial \wedge G = -4\pi j , \quad (5.2) \]

where

\[ \partial \doteq b^\mu \partial_\mu \doteq b^\mu \frac{\partial}{\partial x^\mu} . \quad (5.3) \]

Here the operator of coordinate differentiation \( \partial_\mu \) is considered as scalar, it operate only to expression being on the right.

It is convenient to introduce the following two quasi-bivectors of electromagnetic induction and intensity (see also my paper [2]):

\[ Y \doteq D + i B , \quad Z \doteq E + i H . \quad (5.4) \]

The quasi-bivectors are invariant only for space transformations of coordinate system but not for transformations affecting time.

Let us multiply equation (5.2) by \( b_0 \) on the left. Using designations (5.4) and taking into consideration (5.1) and (2.7), we obtain the following equation for electromagnetic quasi-bivectors:

\[ \partial_0 Y + \partial \otimes Y + \partial \wedge Z = -4\pi b_0 j , \quad (5.5a) \]

\[ \partial_0 Y + \text{Div} Y + i \text{Curl} Z = -4\pi b_0 j , \quad (5.5b) \]

where \( \partial_\mu \doteq \frac{\partial}{\partial x^\mu} \) and

\[ \partial \doteq b_0 b^i \partial_i \quad (5.6) \]

(\( i = 1, 2, 3 \)) is an operator of space differentiation and

\[ \text{Div} Y \doteq \partial \otimes Y , \quad \text{Curl} Z \doteq -i \partial \wedge Z \doteq \partial \times Z . \quad (5.7) \]
For coordinate systems with $m_{00} = -1$ and $m_{0i} = 0$ we have $\partial = b^i \partial_i$.

As we can see, the hyperimaginary unit $i$ is permutable with $\partial_0$, $Y$, $Z$, $\partial$, and $b_0 j$, which are contained into equation (5.5). Thus the hyperimaginary unit can be used here as conventional imaginary unit.

It is evident that $\partial_0$ is anti-self-adjoint operator about functional product (3.1). Using multiplication table (2.7) we can verify directly for flat space-time that $\partial$ is also anti-self-adjoint operator for even hypernumber functions, i.e. $\partial = -\partial$. Thus $\pm i \partial_0$ and $\pm i \partial$ are self-adjoint operators.

In addition to equations (5.2) or (5.5) we must have relations connecting $F$ with $G$ or $Y$ with $Z$ respectively. They can be called constitutive relations for the general case including also nonlinear vacuum electrodynamics (see my articles [2, 5]) and electrodynamics in medium.

In particular, for the case of linear vacuum or simplest constitutive relations we have $F = G = Y = Z$, and we can write instead of (5.5a) the following equation:

$$\partial_0 Y + \partial Y = -4\pi b_0 j. \quad (5.8)$$

### 6 Plane Waves

Eigenfunctions of invariant self-adjoint operator (4.2) are $\exp (i k^\mu x_\mu)$. Representation for even hypernumber function in terms of this system of orthogonal functions have the form:

$$Q(x) = \int A^j \mathcal{Q} e^{ik^\mu x_\mu} dA, \quad (6.1)$$

where $A^j$ is unlimited volume in space of wave vectors, $dA$ is its element, $\mathcal{Q}$ is the Fourier transform for the function $Q(x)$:

$$\mathcal{Q}(k) = \frac{1}{(2\pi)^4} \left\langle Q(x) e^{ik^\mu x_\mu} \right\rangle = \frac{1}{(2\pi)^4} \int Q(x) e^{-ik^\mu x_\mu} dA, \quad (6.2)$$

where $A$ is unlimited space-time volume.

Let us substitute representation of type (6.1) for $Y$ and $Z$ into equation (6.5) without right-hand part or current. We have:

$$k \otimes Y + k \wedge Z = \omega Y, \quad (6.3)$$

where $\omega \equiv -k_0$ is cyclic frequency, $k \equiv b_0 b^i k_i$ is space wave vector.

Hyperscalar part of equation (6.3) gives the condition of transverse waves $k \otimes Y = 0$. 
For the case of simplest constitutive relations \( Y = Z \) we have
\[
k Y = \omega Y.
\] (6.4)

Since the hyperreal bivector \( k \) have inverse one \( k^{-1} = k / k^2 \), where \( k \equiv |k| \), we also obtain from (6.4) that \( Y = \omega k^{-1} Y \). Combining this relation with (6.4) we have
\[
\omega k^{-1} Y = \frac{k}{\omega} Y \implies \omega Y = \frac{k k}{\omega} Y \implies \omega^2 = k^2.
\] (6.5)

7 Cylindrical waves

Let us consider a cylindrical coordinate system \( \{x^0, \rho, \varphi, x^3\} \) in flat space-time \( x^0 = -x_0, x^i = x_i \): \( x_1 = \rho \cos \varphi, x_2 = \rho \sin \varphi \).

Definition 7.1 The angle cylindrical functions are called the even space-time hypernumber eigenfunctions depending on polar angle \( \varphi \) for the operator \( \mathcal{J}_3 \) and are denoted by \( C^m_j \):
\[
\mathcal{J}_3 C^m_j = m C^m_j.
\] (7.1)

where \( C^m_j = C^m_j(\varphi) \).

Infinitesimal operator for space rotation (4.3) about \( x_3 \) axis can be written in the form:
\[
\mathcal{J}_3 = -i \frac{\partial}{\partial \varphi} + b_3 \wedge .
\] (7.2)

There are the following hyperscalar and bivector angle cylindrical functions:

\[
C^m_s = e^{im \varphi}, \quad C^m_1 = e^{i(m-1) \varphi} b_1, \quad C^m_{-1} = e^{i(m+1) \varphi} b_{-1}, \quad C^m_0 = e^{im \varphi} b_3,
\] (7.3a)

where we take \( j = s \) for hyperscalars and \( j = 0, \pm 1 \) for bivectors,
\[
b_+ = \frac{1}{\sqrt{2}} b_1 + i b_2, \quad b_- = \frac{1}{\sqrt{2}} b_1 - i b_2.
\] (7.4)

According to (2.7) and (7.3) we have the following multiplication table for these bivectors (metric tensor is pseudo-Euclidean):

\[
b_+ b_+ = b_+ b_- = 0, \quad b_+ \otimes b_+ = 2 \cdot 1, \quad b_+ \wedge b_- = 2 b_3, \quad (7.5a)
b_+ \otimes b_1 = b_- \otimes b_1 = 1, \quad b_+ \otimes b_2 = -b_- \otimes b_2 = i, \quad (7.5b)
b_+ \wedge b_3 = b_- \otimes b_3 = 0, \quad (7.5c)
b_+ \wedge b_1 = -b_- \wedge b_1 = b_3, \quad b_+ \wedge b_2 = b_- \wedge b_2 = i b_3, \quad b_+ \wedge b_3 = b_- \wedge b_3 = b_-
\]
As we see in (7.5a), bivectors $b_-$ and $b_+$ have zero squares. But since $^*b_- = b_+$ and $^*b_+ = b_-$, six bivectors $b_-, b_+, b_3, z b_-, z b_+, z b_3$ are bivector basis about functional product (3.1).

For representation an arbitrary even hypernumber space-time function let us consider also plane waves propagating along $x^3$ axis. Thus we have:

$$Q(x^0, \rho, \varphi, x^3) = \sum_j \sum_{m=-\infty}^{+\infty} \mathcal{Q}_j^m \int_{-\infty}^{\infty} \mathcal{Q}_j^m e^{i(k_3 x^3 - \omega x^0)} d\omega dk_3, \quad (7.6)$$

where $\mathcal{Q}_j^m$ are quasi-hyperscalar functions:

$$\mathcal{Q}_j^m (\rho, \omega, k_3) = \frac{1}{(2\pi)^2 \|\mathbf{C}_j^m\|^2} \left( \mathbf{Q}(x^0, \rho, \varphi, x^3) | \mathbf{C}_j^m e^{i(k_3 x^3 - \omega x^0)} \right)_{x^3=0}. \quad (7.7)$$

Here the limits on integrals in functional product are $0 < \varphi < 2\pi$, $-\infty < x^3 < \infty$, and $-\infty < x^0 < \infty$, there is not a summation by $j$ index, $\|\mathbf{C}_1^m\|^2 = \|\mathbf{C}_{-1}^m\|^2 = 4\pi$, $\|\mathbf{C}_0^m\|^2 = 2\pi$.

The operator of space differentiation $\partial$ (5.6) has the following form in cylindrical coordinate system:

$$\partial = b^\rho \frac{\partial}{\partial \rho} + b^\varphi \frac{\partial}{\partial \varphi} + b^3 \frac{\partial}{\partial x^3} \equiv \partial^\rho + \partial^\varphi + \partial^3, \quad (7.8)$$

where $b^\rho \equiv b^\rho \wedge b^0 = b^\rho$, $b^\varphi \equiv b^\varphi \wedge b^0$. In view of the metric for cylindrical coordinate system ($m_{\rho\rho} = m_{33} = 1$, $m_{\varphi\rho} = \rho^2$) we have $b_\varphi = \rho^2 b^\varphi$. Being guided by geometrical consideration, multiplication table (7.5), and definition (7.3b) we have

\begin{align*}
    b_\rho &= b_1 \cos \varphi + b_2 \sin \varphi = \frac{1}{2} (C_0^0 - C_1^0), \quad (7.9a) \\
    b^\varphi &= \frac{1}{\rho} (b_1 \sin \varphi + b_2 \cos \varphi) = \frac{1}{2\rho} (C_0^0 - C_1^0), \quad (7.9b) \\
    b_3 &= C_0^0. \quad (7.9c)
\end{align*}

According to (7.3), (7.5) there is the following multiplication and hyper-conjugation table for the cylindrical angle functions:

\begin{align*}
    C_1^m \bullet C_1^m &= C_{-1}^m \bullet C_{-1}^m = 0, \quad C_1^m \circ C_0^m = C_{-1}^m \circ C_0^m = 0, \\
    C_0^m \circ C_{-1}^m &= 2 C_{-1}^{m+n}, \quad C_0^m \bullet C_0^m = C_{-1}^{m+n}, \quad C_1^m \circ C_0^m = C_s^m \circ C_j^m = C_j^m \circ C_s^m = C_{j+n}^m, \\
    C_1^m \wedge C_{-1}^m &= 2 C_0^{m+n}, \quad C_1^m \wedge C_0^m = -C_{-1}^{m+n}, \quad C_{-1}^m \wedge C_0^m = C_{-1}^{m+n}, \\
    *C_s^m &= C_s^{-m}, \quad *C_0^m = C_{-m}^0, \quad *C_1^m = C_{-1}^m, \quad *C_{-1}^m = C_{1}^{-m}. \quad (7.10)
\end{align*}
Taking into consideration (7.9b) and (7.10) we can obtain the following table for application of the angle differentiation operator \( \partial_\varphi \div \mathbf{b} \varphi \partial_\varphi \) to the cylindrical angle functions:

\[
\begin{aligned}
2 \rho \partial_\varphi \otimes C^m_s &= m \mathbf{u} (C^m_{-1} - C^m_1) , & \partial_\varphi \otimes C^m_s &= 0 , \\
\partial_\varphi \otimes C^m_0 &= 0 , & 2 \rho \partial_\varphi \otimes C^m_0 &= -m (C^m_{-1} + C^m_1) , \\
\rho \partial_\varphi \otimes C^m_{\pm 1} &= (1 \mp m) C^m_s , & \rho \partial_\varphi \otimes C^m_{\pm 1} &= (m \mp 1) C^m_0 .
\end{aligned}
\] (7.11)

Let us substitute representation of type (7.6) for two electromagnetic quasivectors \( \mathbf{Y} \) and \( \mathbf{Z} \) \((1)\) into equation \((3.3)\) without current. Using \((7.9)\), \((7.10)\), \((7.11)\) and extracting coefficients of the elementary cylindrical waves \( C^m_j e^{i(k_3 x_3 - \omega t)} \) and \( C^m_j e^{i(k_3 x_3 + \omega t)} \), we can obtain a system of equations for hyperscalar radial functions \( Y^m_j \) and \( Z^m_j \). Then simplifying and introducing new unknown functions

\[
\begin{aligned}
Y^m_+ &= Y^m_1 + Y^m_{-1} , & Z^m_+ &= Z^m_1 + Z^m_{-1} , \\
Y^m_- &= Y^m_1 - Y^m_{-1} , & Z^m_- &= Z^m_1 - Z^m_{-1} ,
\end{aligned}
\] (7.12)

we obtain the following system of equations:

\[
\begin{aligned}
\mathbf{Z}^m_0 \rho + \mathbf{u} (\omega Y^m_- - k_3 Z^m_+) &= 0 ,
(\rho Y^m_+ \rho) + \mathbf{u} k_3 \rho Y^m_0 - m Y^m_- &= 0 ,
(\rho Z^m_+ \rho) + \mathbf{u} \omega Y^m_0 - m Z^m_+ &= 0 ,
\mathbf{u} \omega Y^m_- - k_3 Z^m_0 &= 0 .
\end{aligned}
\] (7.13)

where \((...)\rho\) is the derivative on cylindrical radius \( \rho \).

Let us find solutions to system \((7.13)\) for the case \( \mathbf{Y} = \mathbf{Z} \). At first, we differentiate equation \((7.13a)\) by \( \rho \), substitute expressions for derivatives \( Y^m_+ \rho \) and \( Y^m_- \rho \) obtained from \((7.13b)\) and \((7.13c)\), and multiply the result by \( \rho \). Then we subtract equation \([m \cdot \frac{7.13d}{\rho}] / \rho\) from obtained one and add equation \((7.13a)\). After multiplying by \( \rho \) we obtain as result the equation which contains the \( Y^m_0 \) component only:

\[
\left[ \rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + k^2_\rho \rho^2 - m^2 \right] Y^m_0 = 0 ,
\] (7.14)

where \( k^2_\rho = \omega^2 - k^2_3 \).

As we see, for the case \( k_\rho \neq 0 \) equation \((7.14)\) corresponds to Bessel equation. We will use Hankel functions \( H^{(1)}_m \) and \( H^{(2)}_m \) (the designations correspond to handbook \([\Pi]\) for representation its solution. We have \( H^{(2)}_m(z) =\)
Thus we can use one function $H_m^{(1)}$ but with the argument taking as positive as negative values. Also we must substitute hyperimaginary unit $i$ for imaginary one. Let us designate this function by the symbol $\mathcal{C}^m$. According to [1] we have that $\mathcal{C}^m(z)$ corresponds to Hankel function $H_m^{(2)}(z)$. Thus for real argument $z$, the hyperreal part $\mathfrak{R}\mathcal{C}^m$ is the Bessel function of the first kind and the hyperimaginary part $\mathfrak{I}\mathcal{C}^m$ is the Bessel function of the second kind.

It is convenient to introduce also the following

Definition 7.2 The radial cylindrical functions are called the functions
\begin{equation}
\mathcal{E}^m_{k\rho} = k^m \mathcal{C}^m(k\rho \rho)
\end{equation}
where $\mathcal{C}^m(z)$ is the first Hankel function with hyperimaginary unit instead of imaginary one, $\rho$ is real and $\rho > 0$.

Using an asymptotic form for the Hankel functions as $z \to 0$ [1] we can write
\begin{equation}
\mathcal{E}^m_0 = -i \frac{2^m \Gamma(m)}{\pi \rho^m} \text{ for } m > 0,
\end{equation}
where $\Gamma(m)$ is Euler gamma-function. Here expression (7.15) is understood in the sense of the limit $k\rho \to 0$ for fixed $m$.

Thus the radial cylindrical functions give also finite at infinity ($\rho \to \infty$) solutions to equation (7.14) for $k\rho = 0$ in form (7.16).

Taking into account $\mathcal{C}^m(z) = -(-1)^m \mathcal{C}^m(-z)$ and using (7.15) we obtain
\begin{equation}
\mathcal{C}^m_{k\rho} = -\mathcal{C}^m_{-k\rho}.
\end{equation}

Using recurrent relations for the Bessel functions [1] we obtain the following relations for introduced radial cylindrical functions:
\begin{equationa}
d\frac{d}{d\rho} \mathcal{E}^m_{k\rho} = k^2 \mathcal{E}^{m-1}_{k\rho} - \frac{m}{\rho} \mathcal{E}^m_{k\rho} = -\mathcal{E}^{m+1}_{k\rho} + \frac{m}{\rho} \mathcal{E}^m_{k\rho},
\end{equationa}
\begin{equationb}
k^2 \mathcal{E}^{m-1}_{k\rho} + \mathcal{E}^{m+1}_{k\rho} = \frac{2m}{\rho} \mathcal{E}^m_{k\rho}.
\end{equationb}

According to asymptotic form for the Hankel functions as $|z| \to \infty$ [1] we have the following asymptotic form for the radial cylindrical functions as $|k\rho \rho| \to \infty$:
\begin{equation}
\mathcal{E}^m_{k\rho} \sim k^{m-\frac{1}{2}} \sqrt{\frac{2}{\pi \rho}} \exp \left[ i \left( k\rho - \frac{m \pi}{2} - \frac{\pi}{4} \right) \right].
\end{equation}
Using the introduced radial cylindrical functions we can write the following form of solution to equation (7.14):

$$f_Y^m = C_{k^2}^m k^2 \rho C_{k^2}^{m|} \rho,$$

(7.20a)

where $C_{k^2}^m$ are arbitrary hyperscalar constants, the factor $k^2$ is introduced for (partial) inclusion the case $k^2 = 0$ in the representation of solution to system (7.13).

Then for the case $\omega^2 \neq k^2$ (i.e. $k^2 \neq 0$) we can obtain directly solutions for $f_Y^m$ and $f_Y^{-m}$ from system of equations (7.13a) and (7.13d) ($f_Y^m = f_Y^{-m}$). By direct substitution we have that the obtained solution for $f_Y^m$ and $f_Y^{-m}$ satisfies also equations (7.13b) and (7.13c). Using relations (7.12) and (7.18a) we have

$$f_Y^m = \frac{C_{k^2}^m \rho}{2} (m \pm |m| E_{k^2}^{m|} \rho \mp E_{k^2}^{m|+1} \rho),$$

(7.20b)

where $k^2 = \pm \sqrt{\omega^2 - k^2}$ for $\omega^2 > k^2$ and $k^2 = \pm i \sqrt{k^2 - \omega^2}$ for $\omega^2 < k^2$.

Let us introduce the following designation:

$$\pm \omega_{k^3} = \left( \pm E_{\omega_{k^3}}^m \rho \pm E_{\omega_{k^3}}^{-m} \rho \right) e^{ik^3 x^3},$$

(7.21)

where $\pm E_{\omega_{k^3}}^m$ are taken from (7.20) with substitution $C_{k^2}^m = 1$, left-hand index $+$ or $-$ corresponds to one case from the two $k^2 = \pm \sqrt{\omega^2 - k^2}$ (the both cases $\omega^2 \geq k^2$ are considered, $\sqrt{-1} = i$).

Thus an elementary cylindrical solution to equation (5.14) without sources and for the case $Y = Z$ can be written in the form

$$\pm \omega_{k^3} = \pm \omega e^{i\omega x^0}.$$

(7.22)

It is evident that $\pm \omega_{k^3}$ are eigenfunctions of self-adjoint operator $-i \partial$:

$$-i \partial \pm \omega_{k^3} = \omega \pm \omega_{k^3}.$$

(7.23)

According to asymptotic (7.19) we have that the case of positive real $k^2 = \sqrt{\omega^2 - k^2}$ (function $\pm \omega_{k^3}$ into (7.22)) corresponds to a divergent radial wave (from $x_3$ axis in addition to a propagation along it) and the case of negative real $k^2 = -\sqrt{\omega^2 - k^2}$ (function $\pm \omega_{k^3}$ into (7.22)) corresponds to the convergent radial wave.

In general case both the divergent and convergent waves must be considered for each time harmonic $e^{-i \omega x^0}$ (two harmonics differing with a sign of cyclic frequency $\omega$ are considered as different). If we change $E_{k^2}^{m|} \rho$ to $\Re E_{k^2}^{m|} \rho$ into
then we have an everywhere regular solution for real \( k_{\rho} \). Thus in view of (7.17) we can write the following everywhere regular and finite (for the case \( \omega^2 > k_{\rho}^2 \)) functions:

\[
\epsilon_m^{\omega_{k_{\rho}}} \equiv \frac{1}{2} \left( \epsilon_m^{\omega_{k_{\rho}}} - \epsilon_m^{\omega_{k_{\rho}}} \right).
\] (7.24)

The appropriate elementary solutions

\[
\epsilon_m^{\omega_{k_{\rho}}} e^{-i \omega x^3}
\] (7.25)

are radial-undistorted waves propagating along the \( x^3 \) axis. For the case of \( \omega^2 > k_{\rho}^2 \) these solutions have the form of standing waves in \((\rho, \varphi)\) plane and also are called Bessel beams. A phase velocity of these beams exceeds the velocity of light \((1 < |\omega/k_{\rho}| < \infty)\) but their group velocity less then the velocity of light \((1 < |\partial\omega/\partial k_{\rho}| < 1)\).

The case of hyperimaginary \( k_{\rho} = \pm i \sqrt{k_{\rho}^2 - \omega^2} \) \((\omega^2 < k_{\rho}^2)\) also must be considered. This case is connected with Hankel functions of imaginary argument. The appropriate radius-infinity asymptotic is obtained from (7.19). This case can be interested in nonlinear theory and for problems with boundaries such that in waveguide. A phase velocity along \( x^3 \) axis of the solutions with \( \omega^2 < k_{\rho}^2 \) less then the velocity of light \((1 < |\omega/k_{\rho}| < 1)\) but their group velocity exceeds the velocity of light \((1 < |\partial\omega/\partial k_{\rho}| < \infty)\). Solutions with these unusual properties are known [10].

For the cases \( \omega = \pm k_{\rho} \neq 0 \), a combination of equations (7.13a) and (7.13d) gives two first order equations for \( Y_0^m \). Then we solve them and substitute obtained expressions for \( Y_0^m \) into equations (7.13b) and (7.13c). We obtain solutions to these equations, the cases \( m = \omega/k_{\rho} \) are considered separately. As result we can write the following solutions:

\[
\omega = k_{\rho} \implies Y_0^m = \frac{C_{m0}}{\rho^m},
\]

\[
Y_1^m = \frac{C_{m0}}{2 (m - 1) \rho^{m-1}} + \frac{C_{m0}}{2 \omega \rho^{m+1}}, \quad Y_{-1}^m = \frac{C_{m0}}{2 \omega \rho^{m+1}};
\] (7.26a)

\[
\omega = -k_{\rho} \implies Y_0^m = \frac{C_{m0}}{\rho^m},
\]

\[
Y_1^m = \frac{C_{m0}}{2 \omega \rho^{1-m}}, \quad Y_{-1}^m = \frac{C_{m0}}{2 (m + 1) \rho^{m+1}} + \frac{C_{m0}}{2 \omega \rho^{m+1}}.
\] (7.26b)

Here for the special cases \( m = \omega/k_{\rho} = \pm 1 \) we must take \( \pm C_{m0} = 0 \). There is also a logarithmic solution (containing the term \( \log \rho \)) for these cases but it is not considered here.
Let us consider solutions (7.26) that are finite as $\rho \to \infty$. These solutions are infinite at $\rho = 0$. But they can be interested in nonlinear theory for asymptotic behaviour of solution as $\rho \to \infty$. We have these solutions if we take into (7.26) the following:

$$\pm C_0^m = 0 \quad \text{for} \quad \frac{\omega}{k_3} m \leq 1, \quad \pm C_0^m = 0 \quad \text{for} \quad \frac{\omega}{k_3} m \geq 2,$$  \hspace{1cm} (7.27)

where $\omega \neq 0$. For the case $m \omega/k_3 < 0$ ($\pm C_0^m = 0$) solution (7.26) is given by formulas (7.20) with $\omega = \pm k_3$. A connection between the free constants $\pm C_0^m$ in (7.20) and $C_0^m$ in (7.20) can be obtained with the help of (7.16). The case $m \omega/k_3 \geq 2$ ($\pm C_0^m = 0$) is not described by formula (7.20).

For the case $\omega = k_3 = 0$ we must take the sum of two solutions (7.26a) and (7.26b) with $\pm C_0^m = 0$ for $m \neq 0$. If we take $\pm C_0^m = 0$ for $m \geq 2$ and $\pm C_0^m = 0$ for $m \leq -2$ the solution is finite at $\rho$-infinity.

As we can see, for the case $m \omega/k_3 = 1$ ($\omega \neq 0$) in (7.26) we have the following two solutions to equation (5.8) in the form of plane waves with constant amplitudes:

$$C_1^1 e^{i(k_3 x^3 - \omega x^0)} \quad \text{for} \quad \omega = k_3,$$  \hspace{1cm} (7.28a)

$$C_{-1}^1 e^{i(k_3 x^3 - \omega x^0)} \quad \text{for} \quad \omega = -k_3.$$  \hspace{1cm} (7.28b)

In view of (7.3b) and (7.4) we conclude that for $\omega > 0$ solution (7.28a) is the clockwise polarized mode propagating in positive direction of $x^3$ axis but for $\omega < 0$ it is the counterclockwise polarized mode. And solution (7.28b) for $\omega > 0$ is the clockwise polarized mode propagating in negative direction of $x^3$ axis but for $\omega < 0$ it is the counterclockwise polarized mode. Thus the sign of circular frequency $\omega$ is the sign of rotation for circularly polarized waves but the indices of cylindrical angle functions connected with a direction of propagation for this case.

A plane wave of arbitrary polarization is represented with the sum of positive and negative frequency components (for (7.28a) or (7.28b) waves) with arbitrary quasi-hyperscalar coefficients. These two coefficients (four real numbers) define a polarization ellipse and an amplitude of wave.

8 Spherical waves

Now let us consider a spherical coordinate system \( \{ x^0, r, \vartheta, \varphi \} \) in flat space-time \( (x^0 = -x_0, x^i = x_i): x_1 = r \sin \vartheta \cos \varphi, x_2 = r \sin \vartheta \sin \varphi, x_3 = r \cos \vartheta, \) where $0 \leq \vartheta \leq \pi$ and $0 \leq \varphi < 2\pi$. Let us call $\vartheta$ vertical angle (counting off from vertical line) and $\varphi$ horizontal angle.
Definition 8.1 The angle spherical functions are called the even space-time hypernumber eigenfunctions depending on vertical $\vartheta$ and horizontal $\varphi$ spherical angles for the operators $\mathcal{J}_3$ and $(\mathcal{J})^2$, and are denoted by $\mathcal{C}_j^{lm}$:

\begin{align}
\mathcal{J}_3 \mathcal{C}_j^{lm} &= m \mathcal{C}_j^{lm}, \\
(\mathcal{J})^2 \mathcal{C}_j^{lm} &= l(l+1) \mathcal{C}_j^{lm},
\end{align}

where $\mathcal{C}_j^{lm} = \mathcal{C}_j^{lm}(\vartheta, \varphi)$.

We take $j = s$ for hyperscalar and $j = 0, \pm 1$ for bivector functions.

Raising and reducing operators (4.6) have the following form in spherical coordinates:

\begin{align}
\mathcal{J}_+ &= e^{i\varphi} \left( \frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} + \mathcal{C}_0^0 \wedge \mathcal{C}_1^0 \right), \\
\mathcal{J}_- &= e^{-i\varphi} \left( -\frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} + \mathcal{C}_0^0 \wedge \mathcal{C}_-^0 \right).
\end{align}

From (4.8b), (8.2), and (7.2) it follows:

\begin{align}
(\mathcal{J})^2 = -\frac{\partial^2}{\partial \vartheta^2} - \cot \vartheta \frac{\partial}{\partial \vartheta} - \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \\
+ \left( \mathcal{C}_-^0 - \mathcal{C}_1^0 \right) \wedge \frac{\partial}{\partial \vartheta} + i \left[ \cot \vartheta \left( \mathcal{C}_1^0 + \mathcal{C}_-^0 \right) - 2 \mathcal{C}_0^0 \right] \wedge \frac{\partial}{\partial \varphi} + 2.
\end{align}

Here the formula is divided for two parts (8.3a) and (8.3b). The first part acts to hyperscalar and bivector functions but the second part acts only to bivector functions. The second part contains a multiplication by 2. There is not this operation for the case of hyperscalar function.

By analogy with (7.8) we represent the operator of space differentiation $\partial$ (5.6) in spherical coordinates:

\begin{align}
\partial &= b_r \frac{\partial}{\partial r} + b_\vartheta \frac{\partial}{\partial \vartheta} + b_\varphi \frac{\partial}{\partial \varphi} + i \left( b_\vartheta \frac{\partial}{\partial \vartheta} + b_\varphi \frac{\partial}{\partial \varphi} \right),
\end{align}

where $b_r \triangleq b_r \wedge b_0 = b_r$, $b_\vartheta \triangleq b_\vartheta \wedge b_0$. The metric tensor for spherical coordinate system ($m_{rr} = 1$, $m_{\vartheta \vartheta} = r^2$, $m_{\varphi \varphi} = \rho^2 = r^2 \sin^2 \vartheta$) gives relations $b_\vartheta = r^2 b_\vartheta$, $b_\varphi = r^2 \sin^2 \vartheta b_\varphi$.

We have

\begin{align}
\mathbf{b}_r &= b_\rho \sin \vartheta + b_3 \cos \vartheta = \frac{1}{2} \left( \mathcal{C}_-^0 + \mathcal{C}_1^0 \right) \sin \vartheta + \mathcal{C}_0^0 \cos \vartheta, \\
\mathbf{b}_\vartheta &= \frac{1}{r} \left( b_\rho \cos \vartheta - b_3 \sin \vartheta \right) = \frac{1}{r} \left[ \frac{1}{2} \left( \mathcal{C}_-^0 + \mathcal{C}_1^0 \right) \cos \vartheta - \mathcal{C}_0^0 \sin \vartheta \right], \\
\mathbf{b}_\varphi &= \frac{2}{r \sin \vartheta} \left( \mathcal{C}_-^0 - \mathcal{C}_1^0 \right).
\end{align}
Using (8.3) we can rewrite operator \((\mathcal{J})^2\) in the form:

\[
(\mathcal{J})^2 = -\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} - \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} - 2i \rho \sin \vartheta \left[ \mathbf{b} \times \mathbf{b} \right. \frac{\partial}{\partial \vartheta} \left. - \mathbf{b} \times \mathbf{b} \right] \left. \frac{\partial}{\partial \varphi} \right] + 2 \quad (8.7a)
\]

Existing results relating to rotation group (see [7, 9]) give guide messages which allow to consider the spherical angle functions in the following form:

\[
\mathbf{C}^{lm}_{j} = \mathbf{Z}^{l}_{jm} \mathbf{S}^{m}_{j} \quad (\mathcal{J}) \quad (8.7b)
\]

where \(\mathbf{Z}^{l}_{jm} = \mathbf{Z}^{l}_{jm}(\vartheta)\), \(\vartheta \div \cos \vartheta\) but in general case \(\mathbf{S}^{m}_{j} = \mathbf{S}^{m}_{j}(\vartheta, \varphi)\), \(l\) will be called zonal index and \(m\) sectorial one, \(-l \leq m \leq l\).

For the case of hyperscalar spherical function we have that \(\mathbf{Z}_{s0}^{l}\) is the \(l\)-th Legendre polynomial and \(\mathbf{Z}_{zm}^{l}\) are connected with the associated Legendre polynomials. I call the functions \(\mathbf{Z}_{jm}^{l}\) zonal spherical functions or harmonics. This appellation looks justified because these functions divide a sphere into zones with the borders in the form of equator parallel circles\(^6\). Within a zone the sign of each multivector component of function \(\mathbf{Z}_{jm}^{l}\) is invariable. We consider quasi-hyperscalar zonal spherical harmonics.

For the case of hyperscalar angle spherical functions we have the following sectorial harmonics\(^7\):

\[
\mathbf{S}_{s}^{m} = e^{i m \varphi} \div \mathbf{C}^{m}_{s} \quad (8.8a)
\]

Sectorial harmonics divide a sphere into zones with the borders in the form of circles passing through poles. Within a sector the sign of each multivector component of function \(\mathbf{S}_{j}^{m}\) is invariable.

Sectorial bivector function depends on both vertical and horizontal angles:

\[
\mathbf{S}_{j}^{m} = \mathbf{S}_{j}^{m}(\vartheta, \varphi), \quad j = 0, \pm 1.\]

It is convenient to use the following sectorial spherical harmonics:

\[
\mathbf{S}_{0}^{m} \triangleq \mathbf{b}, \quad e^{i m \varphi} \frac{1}{2} \left( \mathbf{C}^{m}_{-1} + \mathbf{C}^{m}_{1} \right) \sin \vartheta + \mathbf{C}^{m}_{0} \cos \vartheta, \quad (8.8b)
\]

\[
\mathbf{S}_{\pm 1}^{m} \triangleq \left( r \mathbf{b} \times \pm r \sin \vartheta \mathbf{b} \times \mathbf{b} \right) e^{i m \varphi}
\]

\[
= \mathbf{C}^{m}_{\pm 1} \cos^2 \frac{\vartheta}{2} - \mathbf{C}^{m}_{\mp 1} \sin^2 \frac{\vartheta}{2} - \mathbf{C}^{m}_{0} \sin \vartheta. \quad (8.8c)
\]

\(^6\)The appellation “zonal harmonics” occurs also in literature for scalar angle spherical functions with \(m = 0\). But here we will use this appellation for the functions \(\mathbf{Z}_{jm}^{l}(\vartheta)\).

\(^7\)The appellation “sectorial harmonics” occurs also in literature for scalar angle spherical functions with \(m = l\). But here we will use this appellation for the functions \(\mathbf{S}_{j}^{m}\).
Accordingly, we have the following expression for basis bivectors of spherical coordinate system:

\[ \mathbf{b}_r = S_0^0, \quad \mathbf{b}^\vartheta = \frac{1}{2r} \left( S_1^0 + S_{-1}^0 \right), \quad \mathbf{b}^\varphi = \frac{2}{r \sin \vartheta} \left( S_{-1}^0 - S_1^0 \right). \tag{8.9} \]

Connection between cylindrical and spherical angle functions can be also represented as a local space rotation through angle \( \vartheta \) about the unit bivector \( r \sin \vartheta \mathbf{b}^\varphi \). In general, space-time rotation for multivectors in space-time Clifford algebra is realized by means of some even hypernumber \( \Lambda \) such that \( \mathbf{C}' = \Lambda \mathbf{C} \Lambda^{-1} \) (see, for example, \([8]\)). Thus we have the following representation for expressions (8.8):

\[ S_j^m = A_S C_j^m A_S^{-1} \triangleq \exp \left( -i \mathbf{b}^\varphi \frac{\vartheta}{2} \right) C_j^m \exp \left( i \mathbf{b}^\varphi \frac{\vartheta}{2} \right), \tag{8.10} \]

where \( j = s, 0, 1, -1 \), \( \mathbf{b}^\varphi \triangleq \mathbf{b}_r \mathbf{r} \mathbf{b}^\varphi \mathbf{r}^{-1} \mathbf{b}^\varphi \).

\[ A_S^{\pm 1} = \exp \left[ \frac{1}{4} \left( (C_0^0 \pm 1) - (C_0^0 \pm 1) i \vartheta \right) \right] = \exp \left[ \frac{1}{4} \left( (S_0^0 \pm 1) - (S_0^0 \pm 1) i \vartheta \right) \right] \]

\[ = \cos \frac{\vartheta}{2} + \frac{1}{2} \left( C_0^0 \pm 1 \right) \sin \frac{\vartheta}{2} = \cos \frac{\vartheta}{2} + \frac{1}{2} \left( S_0^0 \pm 1 \right) \sin \frac{\vartheta}{2}. \tag{8.11} \]

In view of formula (8.10), a multiplication table for the sectorial harmonics has the same form as for cylindrical angle functions (7.10):

\[ S_1^m S_1^n = S_{-1}^{-m} S_{-1}^{-n} = 0, \quad S_1^m \otimes S_0^n = S_{-1}^{-m} \otimes S_0^n = 0, \]

\[ S_1^m \otimes S_{-1}^{-n} = 2 S_{s}^{m+n}, \quad S_0^m \otimes S_0^n = S_{m+n}, \quad S_j^m S_s^n = S_n^m S_j^n = S_j^{m+n}, \]

\[ S_1^m \wedge S_{-1}^{-n} = 2 S_{0}^{m+n}, \quad S_0^m \wedge S_0^n = -S_{m+n}, \quad S_{-1}^{-m} \wedge S_0^n = S_{m-1}^{-n}, \]

\[ *S_3^m = S_3^{-m}, \quad *S_0^m = S_0^{-m}, \quad *S_1^m = S_1^{-m}, \quad *S_{-1}^{-m} = S_{-1}^{-m}. \tag{8.12} \]

Obviously (because of (8.8)), spherical angle functions are eigenfunctions with eigenvalue \( m \) for operator \( \mathbf{J}_3 \) (7.2).

By substituting (8.7) into (8.15), taking into account (8.6) and (8.8), and using new variable \( z = \cos \vartheta \), we obtain the following equation for zonal harmonics:

\[ \left( 1 - z^2 \right) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{j^2 + m^2 - 2m j z}{1 - z^2} + l(l + 1) \right] \mathbf{Z}_{jm}^l = 0. \tag{8.13} \]

Here for the cases of hyperscalar and bivector spherical functions we have \( j = s, 0, 1, -1 \). (we take \( s = 0 \) into operator). Thus \( \mathbf{Z}_{jm}^l \propto \mathbf{Z}_{0m}^l \) but \( S_s^m \neq S_0^m \).
Solutions to equation (8.13) are known. The appropriate functions are described by Gelfand with co-authors [7] and investigated in detail by Vilenkin [9] (where the designation $P_{nm}$ is used for these functions). According to [9] we have the following formula for zonal harmonics (the imaginary unit is changed by the hyperimaginary one):

$$
Z_{jm}(z) = \frac{(-1)^{l-j} v^{j-m}}{2^l} \sqrt{\frac{(l+m)!}{(l-j)!(l+j)!(l-m)!}} \cdot (1 + z)^{-\frac{1}{2}(m+j)} (1 - z)^{\frac{1}{2}(j-m)} \frac{d^{l-m}}{dx^{l-m}} [(1 - z)^{l-j} (1 + z)^{l+j}] ,
$$

where $l \geq 0$, $|j| \leq l$, $|m| \leq l$. Here we consider the case when all indices are whole numbers. Thus the minimal value of zonal index $l$ for the zonal harmonics $Z_{zm}$ and $Z_{0m}$ is 0 but for the zonal harmonics $Z_{1m}$ and $Z_{-1m}$ is 1.

The functions defined as (8.14) have, in particular, the following properties:

$$
Z_{jm} = Z_{j,-m} , \quad *Z_{jm} = (-1)^{j-m} Z_{jm} , \quad Z_{jm} = Z_{mj} .
$$

Also we have the recurrent formulas

$$
\int_{-1}^{1} Z_{jm}^{l} Z_{jm}^{l'} \, dz = \frac{2}{2l + 1} \delta_{ll'} .
$$

and multiplication table for the zonal harmonics:

$$
Z_{j_1 m_1}^{l_1} Z_{j_2 m_2}^{l_2} = \sum_{l_{\text{min}}}^{l_1 + l_2} \epsilon_{j_1 j_2}^{l_1 l_2} \epsilon_{m_1 m_2}^{l_1 l_2} Z_{j_1 + j_2, m_1 + m_2}^{l_1 + l_2} ,
$$

where $l_{\text{min}} = \max(|l_1 - l_2|, |j_1 + j_2|, |m_1 + m_2|)$ and $\epsilon_{m_1 m_2}^{l_1 l_2}$ are Clebsch-Gordan coefficients for rotation group. According to [9] we have the following

\[\text{Formula (8.14) gives also finite at the points } z = \pm 1 \text{ solutions to equation (8.13) for the case when all indices at functions } Z_{jm} \text{ be half-integer numbers [9]. This case is connected with double-valued representations for rotation group. Even hypernumber functions of type } A_{S} \text{ using for a local rotation of multivectors (see (8.10)) realizes such representation. But at the present paper this case is not considered.}\]
formula:

\[ C_{l_1 m_1}^{l_2 m_2} = \sqrt{(2l + 1)} \cdot \frac{(l_1 + m_1)!(l - m_1 - m_2)!(l - l_1 + l_2)!(l_1 + l_2 - l)!}{(l_1 - m_1)!(l_2 + m_2)!(l - m_2)!(l + m_1 + m_2)!(l + l_1 - l_2)!} \cdot \sum_{l' = l'_{\text{min}}}^{l} \frac{(-1)^{l_1 + m_2 - l'} (l + l')!(l_2 + l' - m_1)!}{(l - l')!(l' - m_1 - m_2)!(l - l_1 + l_2)!(l_1 + l_2 + l' + 1)!}, \quad (8.18) \]

where \( l'_{\text{min}} = \max(m_1 + m_2, l_1 - l_2) \).

According to (8.7), (8.12), and (8.15a) we have the following rule of hyperconjugation for the angle spherical functions:

\[ ^*C_{lm}^{jm} = (-1)^{j-m} C_{l, -m}^{l, m}. \quad (8.19) \]

According to (8.12) and (8.15b) we have relation of orthogonality on sphere for the angle spherical functions

\[ \left\langle C_{lj}^{lm} | C_{lj'}^{lm'} \right\rangle_{\theta, \varphi} = 4 \frac{\pi (1 + j^2)}{2l + 1} \delta_{ll'} \delta_{jj'} \delta_{mm'} , \quad (8.20) \]

where \( j = s, 1, 0, -1 \) and we take \( s = 0 \) into right-hand part.

Expansion of an even hypernumber function in the spherical angle harmonics has the following form:

\[ Q(x^0, r, \vartheta, \varphi) = \sum_{j = s}^{0, 1, -1} \sum_{l = |j|}^{\infty} \sum_{m = -l}^{l} C_{l}^{lm} \int_{-\infty}^{+\infty} Q_{j}^{lm} e^{-i \omega x^0} d\omega , \quad (8.21) \]

where for \( j = s \) we take \( |s| = 0 \), \( Q_{j}^{lm} \) are quasi-hyperscalars:

\[ Q_{j}^{lm}(r, \omega) = \frac{1}{2\pi \| C_{j}^{lm} \|^2} \left\langle Q(x^0, r, \vartheta, \varphi) \left| C_{j}^{lm} e^{i \omega x^0}\right\rangle_{\vartheta, \varphi x^0} , \quad (8.22) \]

where \( \| C_{j}^{lm} \|^2 = [4 \pi (1 + j^2)]/(2l + 1) \) according to (8.20).
Using (8.23), (8.24), (8.26), and (8.27) we obtain the following table:

\[
\begin{align*}
    r \vartheta_\theta \otimes w^{lm}_s &= \frac{\pi}{2} \sqrt{l(l+1)} \left( w^{lm}_1 + w^{lm}_{-1} \right), & r \vartheta_\theta \wedge w^{lm}_s &= 0, \\
    r \vartheta_\phi \otimes w^{lm}_0 &= \frac{\pi}{2} \sqrt{l(l+1)} \left( w^{lm}_{-1} - w^{lm}_1 \right), \\
    r \vartheta_\phi \otimes w^{lm}_{\pm 1} &= \pm \sqrt{l(l+1)} w^{lm}_{\pm 1}, \\
    r \vartheta_\phi \wedge w^{lm}_{\pm 1} &= \pm \left[ w^{lm}_{\pm 1} - \frac{\pi}{2} \sqrt{l(l+1)} w^{lm}_0 \right],
\end{align*}
\]

where \( \vartheta_\theta \vartheta_\phi + \vartheta_\phi \vartheta_\phi \) is the angle part of space differentiation operator (8.24).

We have also the table for multiplication of the radial basis bivector by the angle spherical functions:

\[
\begin{align*}
    b_r \otimes w^{lm}_s &= w^{lm}_0, & b_r \otimes w^{lm}_0 &= w^{lm}_s, & b_r \wedge w^{lm}_s &= b_r \wedge w^{lm}_0 &= 0, \\
    b_r \otimes w^{lm}_{\pm 1} &= 0, & b_r \wedge w^{lm}_{\pm 1} &= \pm w^{lm}_{\pm 1}.
\end{align*}
\]

Let us substitute expansion of type (8.21) for electromagnetic quasibivectors \( Y \) and \( Z \) (8.11) to equation (5.5) without right-hand part. Using (8.23), (8.24) and extracting coefficients for elementary spherical waves \( w^{lm}_j e^{-i\omega x^j} \) and \( w^{lm}_j e^{-i\omega x^j} \), we obtain a system of equations for quasi-hyperscalar radial functions \( Y^{lm}_j \) and \( Z^{lm}_j \). Making simplifying transformations for this system and introducing new unknown functions

\[
\begin{align*}
    X^{lm}_+ &= X^{lm}_1 + X^{lm}_{-1}, & Z^{lm}_+ &= Z^{lm}_1 + Z^{lm}_{-1}, \\
    X^{lm}_- &= X^{lm}_1 - X^{lm}_{-1}, & Z^{lm}_- &= Z^{lm}_1 - Z^{lm}_{-1},
\end{align*}
\]

we obtain the following system of equations:

\[
\begin{align*}
    (r^2 X^{lm}_{0,j})_{,r} &= r \omega Y_{0,j}^{lm} + \frac{\pi}{2} \sqrt{l(l+1)} X^{lm}_{1,j} = 0, \quad (8.26a) \\
    (r Z^{lm}_{1,j})_{,r} - r \omega Y_{0,j}^{lm} - \frac{\pi}{2} \sqrt{l(l+1)} Z^{lm}_{1,j} = 0, \quad (8.26b) \\
    (r Z^{lm}_{-j})_{,r} &= r \omega Y_{0,j}^{lm} = 0, \quad (8.26c) \\
    r \omega Y_{0,j}^{lm} + \frac{\pi}{2} \sqrt{l(l+1)} Z^{lm}_{-j} = 0. \quad (8.26d)
\end{align*}
\]

where \((...)_{,r}\) is the derivative with respect to radius \( r \).

Let us find solutions to equation system (8.26) for the case \( Y = Z \). After differentiation (8.26a) and making necessary substitutions we obtain the following second-order equation for \( X^{lm}_{0,j} \):

\[
\left\{ r^2 \frac{d^2}{dr^2} + 2 r \frac{d}{dr} + \frac{r^2 \omega^2 - l(l+1)}{r} \right\} \left( r X^{lm}_{0,j} \right) = 0.
\]
If we have function \( Y_{lm}^0 \) as a solution to equation (8.27) then functions \( Y_{lm}^+ \) and \( Y_{lm}^- \) are found directly from (8.26a) and (8.26d) \( (iY_j^m = Z_j^m) \) respectively.

At first let us consider the static case \( \omega = 0 \). Thus according to (8.26d) we have \( Y_{lm}^- = 0 \). For \( l = 0 \) system (8.26) has only one solution. For \( l \geq 1 \) we take a solution which is finite as \( r \to \infty \). As result we have for \( \omega = 0 \):

\[
Y_{lm}^0 = \frac{C_{lm}}{r^{l+1}}, \quad Y_{lm}^+ = -\sqrt{\frac{l}{l+1}} \frac{C_{lm}}{r^{l+1}}, \quad Y_{lm}^- = 0 ,
\]

(8.28)

where \( C_{lm} \) are quasi-hyperscalar constants, \( l \geq 0 \).

For the case \( \omega \neq 0 \) equation (8.27) has the form of equation for so-called spherical Bessel functions. These function are expressed by the Bessel functions of half-integer order (see. [1]). We will use the spherical Bessel function of the third kind \( h_l^{(1)} \) and \( h_l^{(2)} \) for the representation of solutions to equations (8.27). These functions are expressed by the Hankel functions of half-integer order. Because of relation \( h_l^{(2)}(z) = (-1)^l h_l^{(1)}(-z) \) we can use only the functions \( h_l^{(1)} \) but with its argument taking as positive as negative values. We will use only real values of the argument. Hyperimaginary unit \( \mathbf{ı} \) must be substituted for imaginary one in these functions. Let us introduce the designation \( S^l \) for spherical Bessel functions \( h_l^{(1)} \). Hyperreal part of this function \( \Re S^l \) is the spherical Bessel function of the first kind and hyperimaginary part \( \Im S^l \) is the spherical Bessel function of the second kind. Hyperconjugate function \( \ast S^l \) corresponds to the spherical Bessel function of the third kind \( h_l^{(2)} \).

According to [1] we have the following useful formulas:

\[
S^l(z) = \frac{e^{iz}}{i^{l+1}} \sum_{l'=0}^{l} \frac{(l + l')!}{(l - l')! l'!} (-2 i z)^{-l'} ,
\]

(8.29)

\[
e^{iz} \cos \vartheta = \sum_{l=0}^{\infty} (2l + 1) i^l \left[ \Re S^l(z) \right] \left[ Z_{l0}^l(\cos \vartheta) \right] .
\]

(8.30)

It is convenient to introduce also the following

**Definition 8.2** The radial spherical functions are called the functions

\[
S^l_{kr} = k^{l+1} S^l(kr) .
\]

(8.31)

where \( S^l(z) \) is the first spherical Bessel function of the third kind with hyperimaginary unit instead of imaginary one.
According to (8.29) we have the following form of radial spherical functions (8.31) for \( k_r = 0 \):

\[
S_l^0 = -\frac{(2l)!}{2^l l!} \frac{r}{l+1} . \tag{8.32}
\]

In view of (8.32) the using of the functions \( S_l^k \) gives a possibility to represent solutions to equation system (8.26) for the cases \( \omega \neq 0 \) and \( \omega = 0 \) with an united formula (see below (8.35)).

Taking into account \( *S_l^k(z) = (-1)^l S_l^k(-z) \) and using (8.31) we obtain

\[
*S_l^k = -S_l^{-k} . \tag{8.33}
\]

Using recurrent relations for the spherical Bessel functions [1] we obtain the following relations for the radial spherical functions:

\[
\frac{d}{dr} S_l^k = k_r^2 S_l^{k-1} - \frac{l+1}{r} S_l^k = -S_l^{k+1} + \frac{l}{r} S_l^k , \tag{8.34a}
\]

\[
k_r^2 S_l^{k-1} + S_l^{k+1} = \frac{2l+1}{r} S_l^k . \tag{8.34b}
\]

Using the introduced radial spherical functions we can write the following form of solution to equation (8.27):

\[
\mathbf{Y}_l^{lm} = \frac{C_{k_r}^{lm}}{r} S_l^k , \tag{8.35a}
\]

where \( C_{k_r}^{lm} \) are free hyperscalar constants, \( k_r^2 = \omega^2 \).

We consider the case \( \omega \neq 0 \). From (8.26a) we have that \( \mathbf{Y}_0^{00} = 0 \) for \( \omega \neq 0 \) and so we consider the values \( l \geq 1 \) for zonal index. We obtain the functions \( \mathbf{Y}_l^{lm} \) and \( \mathbf{Y}_l^{lm} \) from \( \mathbf{Y}_0^{lm} \) directly with the help of equations (8.26a) and (8.26d) respectively. Then we check that the obtained solution satisfies to equations (8.26b) and (8.26c). At last, using relations (8.25) and (8.34a) we obtain

\[
\mathbf{Y}_l^{lm} = \frac{C_{k_r}^{lm}}{2} \sqrt{\frac{l}{l+1}} \left[ \frac{2\omega^2}{l} S_{l-1}^{k_r} - \left( \frac{l \pm \omega}{r} \right) S_l^k \right] , \tag{8.35b}
\]

where \( l \geq 1 \), \( -l \leq m \leq l \), \( k_r = \pm \omega \).

For the case \( \omega = 0 \) (and, consequently, \( k_r = 0 \)) solution (8.35) coincides with the solution given by formula (8.28) for \( l \geq 1 \). The case \( \omega = 0 \), \( l = 0 \) also can be represented by formula (8.33), if we take by definition \( \omega^2/l = 0 \) for this case. According to (8.32) we can obtain a connection between the free constants \( C_0^{lm} \) and \( C_{k_r}^{lm} \) which used in formulas (8.35) and (8.28). Thus
formula (8.35) can be used both for static and time-periodical cases. It gives the solutions which decrease at $r \to \infty$.

Let us introduce the following designation:

$$± \theta \approx C_{lm} \omega = Y_l^0 \theta^0 + Y_l^1 \theta^1 + Y_{l-1}^1 \theta_{l-1}^1,$$  \hspace{1cm} (8.36)

where $Y_j^l \div Y_{j}^{lm}$ from (8.35) with substitution $C_{lm}^0 = 1$, left-hand index $+$ or $-$ corresponds to one case from the two $k_r = \pm \omega$.

Using (8.19) and (8.33)-(8.36) we obtain the rule for hyperconjugation

$$* \pm \theta^{\ell m} = (-1)^{m+1} \pm \theta^{l,-m}.$$  \hspace{1cm} (8.37)

Thus an elementary spherical solution to equation (5.5) without sources and for the case $Y = Z$ can be written in the form

$$\pm \theta^{\ell m} e^{-\imath \omega x^0}.$$  \hspace{1cm} (8.38)

It is evident that $\pm \theta^{\ell m}$ are eigenfunctions of self-adjoint operator $-\imath \partial$:

$$-\imath \partial \pm \theta^{\ell m} = \omega \pm \theta^{\ell m}.$$  \hspace{1cm} (8.39)

Using (8.29) we can conclude that the function $+ \theta^{\ell m} e^{-\imath \omega x^0}$ is a radially divergent elementary spherical wave and $- \theta^{\ell m} e^{-\imath \omega x^0}$ is a convergent one. The sign of circular frequency $\omega$ influences to its polarization but not to the direction of propagation.

In general case, for given time harmonic $e^{-\imath \omega x^0}$ (harmonics differing with the sign of the cyclic frequency $\omega$ is considered as different) we must take the sum of divergent and convergent waves with arbitrary quasi-hyperscalar coefficients. We have an everywhere regular solution if we take hyperreal parts of radial spherical functions $\Re S^l_{kr}$ in expressions (8.35). Thus in view of (8.33) we can write the following everywhere regular and finite functions:

$$\pm \theta^{\ell m} \div \frac{1}{2} (+ \theta^{\ell m} - - \theta^{\ell m})$$  \hspace{1cm} (8.40)

The appropriate elementary solutions

$$\pm \theta^{\ell m} e^{-\imath \omega x^0}$$  \hspace{1cm} (8.41)

are regular standing waves with descending amplitude as $r \to \infty$.

Any regular solution to equation (5.5) for the case $Y = Z$ and $j = 0$ can be represented as a sum (and integral with respect to $\omega$ from $-\infty$ to $\infty$) of elementary spherical waves (8.41) with some hyperscalar coefficients.
Let us find this representation for plane waves $C_1^1 e^{i \omega (x^3 - x^0)}$ (7.28a) and $C_{-1}^{-1} e^{-i \omega (x^3 + x^0)}$ (7.28b). Obviously, it will suffice to obtain the expansion of bivector function $C_1^1 e^{i \omega x^3}$ in term of bivector functions $\approx C_{lm}^{lm}$ (8.40). For the mode propagating in the negative direction of $x^3$ axis we can use the hyperconjugation operation $^* (C_1^1 e^{i \omega x^3}) = C_{-1}^{-1} e^{-i \omega x^3}$.

Using (8.30) and (8.31) we can write

$$C_1^1 e^{i \omega r \cos \vartheta} = \sum_{l=0}^{\infty} (2l + 1) i^{l-1} \omega^{-l-1} Z_{00}^l C_1^1 \mathbf{R} S^l_\omega .$$  

(8.42)

To expand $Z_{00}^l C_1^1$ let us use the multiplication tables which we have. Converting relation (8.10) and using (8.7) we have, in particular, the following relation:

$$C_1^1 = C_{11}^{11} + C_{1-1}^{1-1} - i \sqrt{2} C_{01}^{11}.$$  

(8.43)

According to (8.17) and (8.18) we can obtain

$$Z_{00}^l Z_{01}^l = \frac{1}{\sqrt{2}} \left[ -\sqrt{l(l-1)} Z_{01}^{l-1} + \sqrt{(l+1)(l+2)} Z_{01}^{l+1} \right],$$  

(8.44a)

$$Z_{00}^l Z_{\pm1,1}^l = \frac{1}{2} \left\{ \pm Z_{\pm1,1}^l + \frac{1}{2l + 1} \left[ (l-1) Z_{\pm1,1}^{l-1} + (l+2) Z_{\pm1,1}^{l+1} \right] \right\},$$  

(8.44b)

where $l \geq 1$.

Using (8.43), (8.44), and (8.7) we have

$$Z_{00}^l C_1^1 = \frac{1}{2} \left( C_{11}^{11} - C_{1-1}^{1-1} \right) + \frac{1}{2} (l-1) \left( C_{1-1,1}^{l-1,1} + C_{1,1}^{l-1,1} \right) + \frac{2}{2l + 1} \left[ \sqrt{l(l-1)} C_{01}^{l-1,1} - \sqrt{(l+1)(l+2)} C_{01}^{l+1,1} \right].$$  

(8.45)

Substituting (8.45) into (8.42), grouping the terms with identical $l$ indices, using (8.34b), and comparing with (8.35), (8.36), (8.40) we obtain the following expansion:

$$C_1^1 e^{i \omega r \cos \vartheta} = -\sum_{l=1}^{\infty} i^l \sqrt{l(l+1)} (2l + 1) \omega^{-l-2} \approx C_{lm}^{lm} \omega.$$  

(8.46)
9 Conclusions

Thus the basic systems of orthogonal functions for space-time multivectors are built at the present work. This work considers, in particular, cylindrical and spherical space-time multivector functions with special emphasis on their application to nonlinear electrodynamics. Obtained appropriate systems of first order differential equations (7.13) and (8.26) can be used for various problems of nonlinear and linear electrodynamics. Non-commutative multiplication tables for cylindrical and spherical bivector functions obtained in this work are of considerable importance in nonlinear and linear (with non-homogeneous medium) problems. Also these multiplication tables are very helpful in symbolic computing for the mathematical problems.
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# Main designations

| Symbol | Name | Appearance |
|--------|------|------------|
| $b^\mu$, $b^i$ | Basis vectors and bivectors | (2.1) |
| $1$, $i$, $\mu^\rho\rho$ | Hyperunit and hyperimaginary unit | (2.1), (2.4) |
| $^\ast C$ | Hyperconjugation | (2.2) |
| $\Re C$, $\Im C$ | Hyperreal and hyperimaginary parts | (2.3) |
| $m_{\mu\nu}$ | Metric tensor | (2.4) |
| $\Diamond$, $\wedge$ | Symmetrical and asymmetrical products | (2.5b) |
| $F$, $F_{\mu\nu}$; $G$, $G_{\mu\nu}$ | Bivectors of electromagnetic field | (5.1) |
| $E$, $E_i$; $H$, $H_i$ | Intensities of fields: electric and magnetic | (5.1) |
| $D$, $D^i$; $B$, $B^i$ | Inductions: electric and magnetic | (5.1) |
| $\vartheta$, $\vartheta'$ | Operator of space differentiation, its angle part | (5.3), (8.23) |
| $Y$, $Z$ | Quasi-bivectors of electromagnetic field | (5.4) |
| $\langle \hat{C} \mid \hat{C} \rangle$ | Functional product | (3.1) |
| $\mathcal{F}_{\mu i}$, $\mathcal{F}_i$ | Self-adjoint infinitesimal operators for shift and rotation | (4.1), (4.3) |
| $[A \mid B]$ | Commutator for operators | (4.5) |
| $Q$ | Fourier transform or coefficient | (6.2) |
| $C^m_j$ | Angle cylindrical functions | (7.1) |
| $C^m_k_{\mu}$, $S^l_{kr}$ | Radial functions: cylindrical and spherical | (7.15), (8.31) |
| $C^m_{\omega k_j}$, $S^l_{\omega}$ | Cylindrical and spherical bivector eigenfunctions of operator $-i \vartheta$ | (7.21), (8.56) |
| $E^m_{j, s_j}$, $S^m_{j m}$, $Z_{j m}$, $\pi$ | Angle spherical functions: general, sectorial, zonal, its argument | (8.1), (8.7) |
| $\bigtriangledown$ | Cancellation of summation on repeating indices | (8.7) |
| $\epsilon^{m_1m_2}_{12}$ | Clebsch-Gordan coefficients for rotation group | (8.17) |