Noncommutative Finite–Dimensional Manifolds
Spherical Manifolds and Related Examples

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NONCOMMUTATIVE FINITE-DIMENSIONAL MANIFOLDS
I. SPHERICAL MANIFOLDS AND RELATED EXAMPLES

Alain CONNES\(^1\) and Michel DUBOIS-VIOLETTE\(^2\)

Abstract

We exhibit large classes of examples of noncommutative finite-dimensional manifolds which are (non-formal) deformations of classical manifolds. The main result of this paper is a complete description of noncommutative three-dimensional spherical manifolds, a noncommutative version of the sphere \(S^3\) defined by basic K-theoretic equations. We find a 3-parameter family of deformations of the standard 3-sphere \(S^3\) and a corresponding 3-parameter deformation of the 4-dimensional Euclidean space \(\mathbb{R}^4\). For generic values of the deformation parameters we show that the obtained algebras of polynomials on the deformed \(\mathbb{R}^4\) are isomorphic to the algebras introduced by Sklyanin in connection with the Yang-Baxter equation. Special values of the deformation parameters do not give rise to Sklyanin algebras and we extract a subclass, the \(\theta\)-deformations, which we generalize in any dimension and various contexts, and study in some details. Here, and this point is crucial, the dimension is not an artifact, i.e. the dimension of the classical model, but is the Hochschild dimension of the corresponding algebra which remains constant during the deformation. Besides the standard noncommutative tori, examples of \(\theta\)-deformations include the recently defined noncommutative 4-sphere \(S^4_\theta\) as well as \(m\)-dimensional generalizations, noncommutative versions of spaces \(\mathbb{R}^m\) and quantum groups which are deformations of various classical groups. We also show that the hermitian projections corresponding to the noncommutative 2\(n\)-dimensional spherical manifolds \(S^{2n}_\theta\) have differential self-duality properties which generalize the self-duality of the round instanton.

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1 Introduction

Our aim in this paper is to describe large classes of tractable concrete examples of noncommutative manifolds. Our original motivation is the problem of classification of spherical noncommutative manifolds which arose from the basic discussion of Poincaré duality in $K$-homology [14], [16].

The algebra $\mathcal{A}$ of functions on a spherical noncommutative manifold $S$ of dimension $n$ is generated by the matrix components of a cycle $x$ of the $K$-theory of $\mathcal{A}$, whose dimension is the same as $n = \dim (S)$. More specifically, for $n$ even, $n = 2m$, the algebra $\mathcal{A}$ is generated by the matrix elements $e_j^i$ of a self-adjoint idempotent

$$e = [e_j^i] \in M_q(\mathcal{A}), \quad e = e^2 = e^*, \quad (1.1)$$

and one assumes that all the components $\text{ch}_k(e)$ of the Chern character of $e$ in cyclic homology satisfy,

$$\text{ch}_k(e) = 0 \quad \forall k = 0, 1, \ldots, m - 1 \quad (1.2)$$

while $\text{ch}_m(e)$ defines a non zero Hochschild cycle playing the role of the volume form of $S$.

For $n$ odd the algebra $\mathcal{A}$ is similarly generated by the matrix components $U_j^i$ of a unitary

$$U = [U_j^i] \in M_q(\mathcal{A}), \quad UU^* = U^*U = 1 \quad (1.3)$$

and, with $n = 2m + 1$, the vanishing condition (1.2) becomes

$$\text{ch}_{k+\frac{1}{2}}(U) = 0 \quad \forall k = 0, 1, \ldots, m - 1. \quad (1.4)$$

The components $\text{ch}_k$ of the Chern character in cyclic homology are the following explicit elements of the tensor product

$$\mathcal{A} \otimes (\mathcal{A})^\otimes 2k \quad (1.5)$$
where $\tilde{A}$ is the quotient of $A$ by the subspace $C_1$,

$$\text{ch}_k(e) = \left(e^{i_0}_{i_1} - \frac{1}{2} e^{i_0}_{i_1}\right) \otimes e^{i_1}_{i_2} \otimes e^{i_2}_{i_3} \otimes \cdots \otimes e^{i_2k}_{i_0}$$  \hspace{1cm} (1.6)$$

and

$$\text{ch}_{k+\frac{1}{2}}(U) = U^{i_0}_{i_1} \otimes U^{i_1}_{i_2} \otimes U^{i_2}_{i_3} \otimes \cdots \otimes U^{i_2k+1}_{i_0} - U^{i_0}_{i_1} \otimes \cdots \otimes U^{i_2k+1}_{i_0}$$  \hspace{1cm} (1.7)$$

up to an irrelevant normalization constant.

It was shown in [14] that the Bott generator on the classical spheres $S^n$ give solutions to the above equations (1.2), (1.4) and in [16] that non trivial noncommutative solutions exist for $n = 3, q = 2$ and $n = 4, q = 4$.

In fact, as will be explained in our next paper (Part II), consistency with the suspension functor requires a coupling between the dimension $n$ of $S$ and $q$. Namely $q$ must be the same for $n = 2m$ and $n = 2m + 1$ whereas it must be doubled when going from $n = 2m - 1$ to $n = 2m$. This implies that for dimensions $n = 2m$ and $n = 2m + 1$, one has $q = 2^m q_0$ for some $q_0 \in \mathbb{N}$. Furthermore the normalization $q_0 = 1$ is induced by the identification of $S^2$ with one-dimensional projective space $P_1(\mathbb{C})$ (which means $q = 2$ for $n = 2$). We shall take this convention (i.e. $q = 2^m$ for $n = 2m$ and $n = 2m + 1$) in the following.

The main result of the present paper is the complete description of the noncommutative solutions for $n = 3$ ($q = 2$). We find a three-parameter family of deformations of the standard three-sphere $S^3$ and a corresponding 3-parameter deformation of the 4-dimensional Euclidean space $\mathbb{R}^4$. For generic values of the deformation parameters we show that the obtained algebras of polynomials on the deformed $\mathbb{R}^4_u$ are isomorphic to the algebras
introduced by Sklyanin in connection with the Yang-Baxter equation. It is however worth noticing that there are specific values (non generic) of the parameters for which these algebras do not coincide with Sklyanin algebras. In particular, Sklyanin algebras do not include the $\theta$-deformation of the classical algebra of polynomials on $\mathbb{R}^4$. We expect a similar analysis to go over to the description of 4-dimensional spheres as well. It turns out that one can extract from the above multiparameter deformation $R^4_\mu$ of $\mathbb{R}^4$ a one-parameter deformation $C^2_\mu$ of $\mathbb{C}^2$ (identified with $\mathbb{R}^4$) which is well suited for simple higher dimensional generalizations (i.e. $\mathbb{C}^2$ replaced by $\mathbb{C}^n \simeq \mathbb{R}^{2n}$). We shall describe and analyse them in details to exemplify what we intend to do in the general case. First we shall show that, unlike most deformations used to produce noncommutative spaces from classical ones, the above deformations do not alter the Hochschild dimension. The latter is the natural generalization of the notion of dimension to the noncommutative case and is the smallest integer $m$ such that the Hochschild homology of $\mathcal{A}$ with values in a bimodule $\mathcal{M}$ vanishes for $k > m$ ($H^k(\mathcal{A}, \mathcal{M}) = 0 \forall k > m$). Second we shall describe the natural notion of differential forms on the above noncommutative spaces and obtain the natural quantum groups of symmetries as “$\theta$-deformations” of the classical groups $GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$ and $GL(n, \mathbb{C})$.

The algebraic versions of differential forms on the above quantum groups turn out to be graded involutive differential Hopf algebras, which implies in particular that the corresponding differential calculi are bicovariant in the sense of [52].

Finally we shall come back to the metric aspect of the construction which was the original motivation for the definition of spherical manifolds from the
polynomial operator equation fulfilled by the Dirac operator.

We shall check in detail that $\theta$-deformations of Riemannian spin geometries fulfill all axioms of noncommutative geometry, thus completing the path, in the special case of $\theta$-deformations, from the crudest level of the algebra $C_{\text{alg}}(S)$ of polynomial functions on $S$ to the full-fledged structure of noncommutative geometry [13]. Needless to say our goal in Part II will be a similar analysis for general spherical manifolds.

In the course of the paper it will be shown that the self-duality property of the round instanton on $S^4$ extends directly to the self-adjoint idempotent identifying $S^4_\theta$ as a noncommutative 4-dimensional spherical manifold and that, more generally, the self-adjoint idempotents corresponding to the noncommutative $2n$-dimensional spherical manifolds $S^{2n}_\theta$ defined below satisfy a differential self-duality property which is a direct extension of the one satisfied by their classical counterparts as explained in [19].

In conclusion the above examples appear as an interesting point of contact between various approaches to noncommutative geometry. The original motivation came from the operator equation of degree $n$ fulfilled by the Dirac operator of a $n$-dimensional spin manifold [14]. The simplest way of “quantizing” the corresponding Hochschild cycle, namely $c = \text{ch}(e)$ led to the definition of spherical manifolds. What we show here is that in the simplest nontrivial case ($n = 3, \ q = 2$) the answer is intimately related to the Sklyanin algebras which play a basic role in noncommutative algebraic geometry.

Many algebras occurring in this paper are finitely generated and finitely
presented. These algebras are viewed as algebras of polynomials on the corresponding noncommutative space $S$ and we denote them by $C_{\text{alg}}(S)$. With these notations $C_{\text{alg}}(S)$ has to be distinguished from $C^\infty(S)$, the algebra of smooth functions on $S$ obtained as a suitable completion of $C_{\text{alg}}(S)$. Basic algebraic properties such as Hochschild dimension are not necessarily preserved under the transition from $C_{\text{alg}}(S)$ to $C^\infty(S)$. The topology of $S$ is specified by the $C^*$ completion of $C^\infty(S)$.

The plan of the paper is the following. After this introduction, in section 2, we give a complete description of noncommutative spherical manifolds for the lowest non trivial dimension : Namely for dimension $n = 3$ and for $q = 2$. These form a 3-parameter family $S_3^q$ of deformations of the standard 3-sphere as explained above and correspondingly one has a homogeneous version which is a three-parameter family $R_4^q$ of deformations of the standard 4-dimensional Euclidean space $R^4$. We then define their suspensions and show that the suspension $S_4^q$ of $S_3^q$ is a four-dimensional noncommutative spherical manifold (with $q = 4 = 2^2$). In Section 3, we show that for generic values of the parameters, the algebra $C_{\text{alg}}(R_4^q)$ of polynomial functions on the noncommutative $R_4^q$ reduces to a Sklyanin algebra [45], [46]. These Sklyanin algebras have been intensively studied [38], [47], from the point of view of noncommutative algebraic geometry but we postpone their analysis to Part II of this paper. We concentrate instead on a one-parameter family of non-generic values which do not give rise to Sklyanin algebras, as easily seen using the geometric data associated [38] [1][2] to such algebras. In Section 4 we define a noncommutative deformation $R_2^{2n}$ of $R^{2n}$ for $n \geq 2$ which is coherent with the identification $C^n = R^{2n}$ as real spaces and is also consequently a noncommutative deformation $C_2^{2n}$ of $C^n$. For $n = 2$, $R_2^4$ reduces to the above one-parameter family of deformations of $R^4$ which is included
in the multiparameter deformation \( \mathbb{R}^4 \) of Section 2. We introduce in this section a deformation of the generators of the Clifford algebra of \( \mathbb{R}^{2n} \) which will be very useful for the computations of Sections 5 and 12. In Section 5 we define noncommutative versions \( \mathbb{R}^{2n+1}_\theta \), \( S^{2n}_\theta \) and \( S^{2n-1}_\theta \) of \( \mathbb{R}^{2n+1} \), \( S^{2n} \) and \( S^{2n-1} \) for \( n \geq 2 \). For \( n = 2 \), \( S^{2n}_\theta \) reduces to the noncommutative 4-sphere \( S^4_\theta \) of [16] whereas \( S^{2n-1}_\theta \) reduces to the one-parameter family \( S^3_\theta \) of deformation of \( S^3 \) associated to the non-generic values of \( u \). We generalize the results of [16] on \( S^4_\theta \) to \( S^{2n}_\theta \) for arbitrary \( n \geq 2 \) and we describe their counterpart for the odd-dimensional cases \( S^{2n-1}_\theta \) showing thereby that these \( S^m_\theta \) (\( m \geq 3 \)) are noncommutative spherical manifolds. Furthermore, it will be shown later (in Section 12) that the defining hermitian projections of \( S^{2n}_\theta \) possess differential self-duality properties which generalize the ones of their classical counterpart (i.e. for \( S^{2n} \)) as explained in [19]. In Section 6, we define algebraic versions of differential forms on the above noncommutative spaces. These definitions, which are essentially unique, provide dense subalgebras of the canonical algebras of smooth differential forms defined in Sections 11, 12, 13 for these particular cases. These differential calculi are diagonal [25] which implies that they are quotients of the corresponding universal diagonal differential calculi [21]. In Section 7 we construct quantum groups which are deformations (called \( \theta \)-deformations) of the classical groups \( GL(m, \mathbb{R}) \), \( SL(m, \mathbb{R}) \) and \( GL(n, \mathbb{C}) \) for \( m \geq 4 \) and \( n \geq 2 \). The point of view for this construction is close to the one of [32] which is itself a generalization of a construction described in [33], [34]. It is pointed out that there is no such \( \theta \)-deformation of \( SL(n, \mathbb{C}) \) although there is a \( \theta \)-deformation of the subgroup of \( GL(n, \mathbb{C}) \) consisting of matrices with determinants of modulus one (\( |\det_{\mathbb{C}}(M)|^2 = 1 \)). In Section 8 we define the corresponding deformations of the groups \( O(m) \), \( SO(m) \) and \( U(n) \). As above there is no \( \theta \)-deformation
of $SU(n)$ which is the counterpart of the same statement for $SL(n, \mathbb{C})$. All the quantum groups $G_\theta$ considered in Section 7 and in Section 8 are matrix quantum groups [51] and in fact as coalgebra $C_{\text{alg}}(G_\theta)$ is undeformed i.e. isomorphic to the classical coalgebra $C_{\text{alg}}(G)$ of representative functions on $G$ [20], (only the associative product is deformed). In Section 9, we analyse the structure of the algebraic version of differential forms on the above quantum groups. These graded-involutive differential algebras turn out to be graded-involutive differential Hopf algebras (with coproducts and counits extending the original ones) which implies in particular that the corresponding differential calculi are bicovariant in the sense of [52], (see also the end of the conclusion of [21]). In Section 10, we define the splitting homomorphisms mapping the polynomial algebras $C_{\text{alg}}$ on the various $\theta$-deformations introduced previously into the polynomial algebras on the product of the corresponding classical spaces with the noncommutative $n$-torus $T^n_\theta$. In Section 11 we use the splitting homomorphisms to produce smooth structures on the previously defined noncommutative spaces, that is the algebras of smooth functions and of smooth differential forms. In Section 12 we describe in general the construction which associates to each finite-dimensional manifold $M$ equipped with a smooth action $\sigma$ of the $n$-torus $T^n$ a noncommutative deformation $C^\infty(M_\theta)$ of the algebra $C^\infty(M)$ of smooth functions on $M$ (and of the algebra of smooth differential forms) which defines the noncommutative manifold $M_\theta$ and we explain why the Hochschild dimension of the deformed algebra remains constant and equal to the dimension of $M$. The deformation $C^\infty(M_\theta)$ of the algebra $C^\infty(M)$ is a special case of Rieffel’s deformation quantization [42] and close to the form adopted in [43]. It is worth noticing here that at the formal level deformations of algebras for actions of $\mathbb{R}^n$ have been also analysed in [35]. It is however crucial to consider (non formal)
actions of \( T^n \); our results would be generically wrong for actions of \( \mathbb{R}^n \).

In Section 13 we analyse the metric aspect of the construction showing that the deformation is isospectral in the sense of [16] and that our construction gives an alternate setting for results like Theorem 6 of [16]. We use the splitting homomorphism to show that when \( M \) is a compact riemannian spin manifold equipped with an isometric action of \( T^n \) the corresponding spectral triple ([16]) \( (C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta) \) satisfies the axioms of noncommutative geometry of [13]. We show moreover (theorem 9) that any \( T^n \)-invariant metric on \( S^m \), \((m = 2n, 2n - 1)\), whose volume form is rotation invariant yields a solution of the original polynomial equation for the Dirac operator on \( S^m_\theta \).

Section 14 is our conclusion.

Throughout this paper \( n \) denotes an integer such that \( n \geq 2, \theta \in M_n(\mathbb{R}) \) is an antisymmetric real \((n, n)\)-matrix with matrix elements denoted by \( \theta_{\mu\nu} \) \((\mu, \nu = 1, 2, \ldots, n)\) and we set \( \lambda^{\mu\nu} = e^{i\theta_{\mu\nu}} = \lambda_{\mu\nu} \). The reason for this double notation \( \lambda^{\mu\nu}, \lambda_{\mu\nu} \) for the same object \( e^{i\theta_{\mu\nu}} \) is to avoid ambiguities connected with the Einstein summation convention (of repeated up down indices) which is used throughout. The symbol \( \otimes \) without other specification will always denote the tensor product over the field \( \mathbb{C} \). A self-adjoint idempotent or a hermitian projection in a *-algebra is an element \( e \) satisfying \( e^2 = e = e^* \). By a graded-involutive algebra we here mean a graded \( \mathbb{C} \)-algebra equipped with an antilinear involution \( \omega \mapsto \bar{\omega} \) such that \( \bar{\omega_\omega'} = (-1)^{pp'} \bar{\omega}' \bar{\omega} \) for \( \omega \) of degree \( p \) and \( \omega' \) of degree \( p' \). A graded-involutive differential algebra will be a graded-involutive algebra equipped with a real differential \( d \) such that \( d(\bar{\omega}) = \bar{d(\omega)} \) for any \( \omega \). Given a graded vector space \( V = \oplus_n V^n \), we denote by \( (-I)^{\text{gr}} \) the endomorphism of \( V \) which is the identity mapping on \( \oplus_k V^{2k} \) and minus the identity mapping on \( \oplus_k V^{2k+1} \). If \( \Omega' \) and \( \Omega'' \) are graded algebras one can
equip $\Omega' \otimes \Omega''$ with the usual product $(x' \otimes x'')(y' \otimes y'') = x' y' \otimes x'' y''$ or with the graded twisted one $(x' \otimes x'')(y' \otimes y'') = (-1)^{|x''||y'|} x' y' \otimes x'' y''$ where $|x''|$ is the degree of $x''$ and $|y'|$ is the degree of $y'$; in the latter case we denote by $\Omega' \otimes_{gr} \Omega''$ the corresponding graded algebra. If furthermore $\Omega'$ and $\Omega''$ are graded differential algebras $\Omega' \otimes_{gr} \Omega''$ will denote the corresponding graded algebra equipped with the differential $d = d' \otimes 1 + (-1)^{|y|} \otimes d''$. A bimodule over an algebra $A$ is said to be diagonal if it is a subbimodule of $A^I$ for some set $I$. Concerning locally convex algebras, topological modules, bimodules and resolutions we use the conventions of [8]. All our locally convex algebras and locally convex modules will be nuclear and complete. Finally we shall need some notations concerning matrix algebras $M_n(A) = M_n(\mathbb{C}) \otimes A$ with entries in an algebra $A$. For $M \in M_n(A)$, we denote by $\text{tr}(M)$ the element $\sum_{\alpha=1}^n M_{\alpha}^\alpha$ of $A$. If $M$ and $N$ are in $M_n(A)$, we denote by $M \otimes N$ the element of $M_n(A \otimes A)$ defined by $(M \otimes N)_{\alpha}^\beta = M_{\alpha}^\alpha \otimes M_{\beta}^\beta$.

2 Noncommutative 3-spheres and 4-planes

Our aim in this section is to give a complete description of noncommutative spherical three-manifolds. More specifically we give here a complete description of the class of complex unital $*$-algebras $\mathcal{A}^{(1)}$ satisfying the following conditions $(I_1)$ and $(II)$:

$(I_1)$ $\mathcal{A}^{(1)}$ is generated as unital $*$-algebra by the matrix elements of a unitary $U \in M_2(\mathcal{A}^{(1)}) = M_2(\mathbb{C}) \otimes \mathcal{A}^{(1)}$,

$$(\text{II}) \ U \text{ satisfies } \text{ch}_2(U) = U_j^i \otimes U_i^* - U_j^* \otimes U_i^j = 0$$

(i.e. with the notations explained above $\text{tr}(U \otimes U^* - U^* \otimes U) = 0$).
We shall show in Part II that generically an algebra of this class gives rise to a noncommutative spherical smooth 3-manifold.

It is convenient to consider the corresponding homogeneous problem, i.e. the class of unital $\dagger$-algebras $\mathcal{A}$ such that

(I) $\mathcal{A}$ is generated by the matrix elements of a $U \in M_2(\mathcal{A}) = M_2(\mathbb{C}) \otimes \mathcal{A}$ satisfying $U^*U = UU^* \in 1_2 \otimes \mathcal{A}$ where $1_2$ is the unit of $M_2(\mathbb{C})$ and,

(II) $U$ satisfies $\tilde{\chi}_2(U) = U_j^i \otimes U_i^{*j} - U_j^{*i} \otimes U_i^j = 0$

i.e. $\text{tr}(U \otimes U^* - U^* \otimes U) = 0$.

Notice that if $\mathcal{A}^{(1)}$ satisfies Conditions (I$_1$) and (II) or if $\mathcal{A}$ satisfies Conditions (I) and (II) with $U$ as above, nothing changes if one makes the replacement

$$U \mapsto U' = uV_1UV_2$$

where $u = e^{i\varphi} \in U(1)$ and $V_1, V_2 \in SU(2)$. This corresponds to a linear change in generators, $(\mathcal{A}^{(1)}, U')$ satisfies (I$_1$) and (II) whenever $(\mathcal{A}^{(1)}, U)$ satisfies (I$_1$) and (II) and $(\mathcal{A}, U')$ satisfies (I) and (II) whenever $(\mathcal{A}, U)$ satisfies (I) and (II).

Let $\mathcal{A}$ be a unital $\dagger$-algebra and $U \in M_2(\mathcal{A})$. We use the standard Pauli matrices $\sigma_k$ to write $U$ as

$$U = 1_2 z^0 + i\sigma_k z^k$$

where $z^\mu$ are elements of $\mathcal{A}$ for $\mu = 0, 1, 2, 3$. In terms of the $z^\mu$, the transformations (2.1) reads

$$z^\mu \mapsto uS^\mu z^\nu$$

with $u \in U(1)$ as above and where $S^\mu_{\nu}$ are the matrix elements of the real rotation $S \in SO(4)$ corresponding to $(V_1, V_2) \in SU(2) \times SU(2) = \text{Spin}(4)$. 

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The pair \((A, U)\) fulfills (I) if and only if \(A\) is generated by the \(z^\mu\) as unital \(*\)-algebra and the \(z^\mu\) satisfy
\[
zh^k z^0 z^k z^\ell z^m z^* = 0 \quad (2.4)
\]
\[
z^0 z^k z^* - z^k z^0 + \epsilon_{k\ell m} z^\ell z^m z^* = 0 \quad (2.5)
\]
\[
\sum_{\mu=0}^{3} (z^\mu z^{\mu*} - z^{\mu*} z^\mu) = 0 \quad (2.6)
\]
for \(k = 1, 2, 3\), where \(\epsilon_{k\ell m}\) is completely antisymmetric in \(k, \ell, m \in \{1, 2, 3\}\) with \(\epsilon_{123} = 1\). Condition (I) is satisfied if and only if one has in addition \(\sum_{\mu} z^{\mu*} z^\mu = 1\). The following lemma shows that there is no problem to pass from (I) to (I\(_1\)) just imposing the relation \(\sum_{\mu} z^{\mu*} z^\mu - 1 = 0\).

**LEMMA 1** Let \(A, U\) satisfy (I) as above. Then \(\sum_{\mu=0}^{3} z^{\mu*} z^\mu\) is in the center of \(A\).

This result is easily verified using relations (2.4), (2.5), (2.6) above.

Let us now investigate condition (II). In terms of the representation (2.2), for \(U\), condition (II) reads
\[
\sum_{\mu=0}^{3} (z^{\mu*} \otimes z^\mu - z^\mu \otimes z^{\mu*}) = 0 \quad (2.7)
\]
for the \(z^\mu \in A\). One has the following result.

**LEMMA 2** Condition (II), i.e. equation (2.7), is satisfied if and only if there is a symmetric unitary matrix \(\Lambda \in M_4(\mathbb{C})\) such that \(z^{\mu*} = \Lambda^\nu z^\nu\) for \(\mu \in \{0, \cdots, 3\}\).

The existence of \(\Lambda \in M_4(\mathbb{C})\) as above clearly implies Equation (2.7). Conversely assume that (2.7) is satisfied. If the \((z^\mu)\) are linearly independent, the
existence and uniqueness of a matrix $\Lambda$ such that $z^{\mu*} = \Lambda^{\mu}_{\nu} z^\nu$ is immediate, and the symmetry and unitarity of $\Lambda$ follow from its uniqueness. Thus the only difficulty is to take care in general of the linear dependence between the $(z^\mu)$. We let $I \subset \{0, \ldots, 3\}$ be a maximal subset of $\{0, \ldots, 3\}$ such that the $(z^i)_{i \in I}$ are linearly independent. Let $I'$ be its complements; one has $z^{i'} = \bar{L}^{i'}_i z^i$ for some $L^i_i \in \mathbb{C}$. On the other hand one has $z^{i*} = C^i_j z^j + E^i_A y^A$ where the $y^A$ are linearly independent elements of $\mathcal{A}$ which are independent of the $z^i$ and $C^i_j, E^i_A$ are complex numbers. This implies in particular that $z^{i'*} = L^{i'}_j C^i_j z^j + L^{i'}_i E^i_A y^A$. By expanding Equation (2.7) in terms of the linearly independent elements $z^i \otimes z^j, z^i \otimes y^A, y^A \otimes z^i$ of $\mathcal{A} \otimes \mathcal{A}$ one obtains

$$(1 + L^* L) C = (1 + L^* L) C^j_i \quad (2.8)$$

for the complex matrices $L = (L^i_j)$ and $C = (C^i_j)$ ($C$ is a square matrix whereas $L$ is generally rectangular) and

$$(1 + L^* L) E_A = 0$$

for the $E^i_A$ which implies $E^i_A = 0$ (since $1 + L^* L > 0$). Thus one has $z^{i*} = C^i_j z^j$ which implies $\bar{C} C = 1$ for the matrix $C$, $z^{i'} = \bar{L}^{i'}_i z^i$, $z^{i'*} = L^{i'}_j C^i_j z^j$ together with Equation (2.8). This implies $z^{\mu*} = \Lambda^{\mu}_{\nu} z^\nu$ together with $\Lambda^{i}_{\nu} = \Lambda^{\nu}_{i}$ for $\Lambda \in M_4(\mathbb{C})$ given by

$$\begin{align*}
\Lambda^{i}_{j} &= C^i_j - \sum_{n'} C^m_i L^n_m \bar{L}^{n'}_j \\
\Lambda^{i'}_{j'} &= L^n_m C^m_i = \Lambda^{j}_{i'} \\
\Lambda^{i'}_{j} &= 0
\end{align*}$$

With an obvious relabelling of the $z^\mu$, one can write $\Lambda$ in block from

$$\Lambda = \begin{pmatrix}
C - C^\mu L^\mu \bar{L} & C^\mu L^\mu \\
LC & 0
\end{pmatrix}$$
The equality \( \Lambda z = z^* \) and the symmetry of \( \Lambda \) show that \( \Lambda^* z^* = z \) so that \( \Lambda^* \Lambda z = z \). Let \( \Lambda = U|\Lambda| \) be the polar decomposition of \( \Lambda \). Since \( \Lambda \) is symmetric, the matrix \( U \) is also symmetric (symmetry means \( \Lambda^* = J\Lambda J^{-1} \) where \( J \) is the antilinear involution defining the complex structure, one has \( \Lambda = |\Lambda^*|U \) so that \( \Lambda^* = U^*|\Lambda^*| \) and the uniqueness of the polar decomposition gives \( U^* = JUJ^{-1} \)). Moreover the equality \( \Lambda^* \Lambda z = z \) shows that

\( (1) \quad \Lambda z = Uz, \quad Pz = 0 \) where \( P = (1 - U^*U) \)

One has \( (1 - UU^*) = JPJ^{-1} \) and with \( e_j \) an orthonormal basis of \( PC^4 \), \( f_j = Je_j \) the corresponding orthonormal basis of \( JPC^4 \) one checks that the following matrix is symmetric,

\( (2) \quad S = \sum |f_j\rangle\langle e_j| \)

Let now \( \tilde{\Lambda} = U + S \). By (1) one has \( \tilde{\Lambda}z = z^* \) since \( Sz = 0 \) and \( Uz = \Lambda z = z^* \). Since \( \tilde{\Lambda} \) is symmetric and unitary we get the conclusion.

Under the transformation (2.3), \( \Lambda \) transforms as

\[ \Lambda \mapsto u^2 \Lambda S \Lambda S \]

so one can diagonalize the symmetric unitary \( \Lambda \) by a real rotation \( S \) and fix its first eigenvalue to be 1 by choosing the appropriate \( u \in U(1) \) which shows that one can take \( \Lambda \) in diagonal form

\[ \Lambda = \begin{pmatrix} 1 & e^{-2i\varphi_1} & & \\ e^{-2i\varphi_2} & e^{-2i\varphi_2} & & \\ & & & e^{-2i\varphi_k} \end{pmatrix} \] (2.9)
i. e. one can assume that $z^0 = x^0$ and $z^k = e^{i\varphi_k} x^k$ with $e^{i\varphi_k} \in U(1) \subset \mathbb{C}$ for $k \in \{1, 2, 3\}$ and $x^{\mu*} = x^\mu (\in \mathcal{A})$ for $\mu \in \{0, \cdots, 3\}$.

Setting $z^0 = x^0 = x^{0*}$ and $z^k = e^{i\varphi_k} x^k$, $x^k = x^{k*}$ relations (2.4) and (2.5) read

\[
\begin{align*}
\cos(\varphi_k)[x^0, x^k]_- &= i \sin(\varphi_\ell - \varphi_m)[x^\ell, x^m]_+ \quad (2.10) \\
\cos(\varphi_\ell - \varphi_m)[x^\ell, x^m]_- &= -i \sin(\varphi_k)[x^0, x^k]_+ \quad (2.11)
\end{align*}
\]

for $k = 1, 2, 3$ where $(k, \ell, m)$ is the cyclic permutation of $(1, 2, 3)$ starting with $k$ and where $[x, y]_\pm = xy \pm yx$. Let $u$ be the element $(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$ of $T^3$, we denote by $\mathcal{A}_u$ the complex unital $*$-algebra generated by four hermitian elements $x^\mu, \mu \in \{0, \cdots, 3\}$, with relations (2.10), (2.11) above. It follows from the above discussion that all $\mathcal{A}$ satisfying (I) and (II) are quotient of $\mathcal{A}_u$ for some $u$. However it is straightforward that the pair $(\mathcal{A}_u, U_u)$ with $U_u = 1_2 x^0 + i \sum_{k=1}^3 e^{i\varphi_k} \sigma_k x^k$ satisfies (I) and (II) so the $\mathcal{A}_u$ are the maximal solutions of (I), (II) and any other solution is a quotient of some $\mathcal{A}_u$. In particular each maximal solution of (I), (II) is the quotient $\mathcal{A}^{(1)}_u$ of $\mathcal{A}_u$ by the ideal generated by $\sum_\mu (x^\mu)^2 - 1$ for some $u$. This quotient does not contain other relations since $\sum_\mu (x^\mu)^2$ is in the center of $\mathcal{A}_u$ (Lemma 1). In summary one has the following theorem.

**Theorem 1** (i) For any $u \in T^3$ the complex unital $*$-algebra $\mathcal{A}_u$ satisfies conditions (I) et (II). Moreover, if $\mathcal{A}$ is a complex unital $*$-algebra satisfying conditions (I) and (II) then $\mathcal{A}$ is a quotient of $\mathcal{A}_u$ (i.e. a homomorphic image of $\mathcal{A}_u$) for some $u \in T^3$.

(ii) For any $u \in T^3$, the complex unital $*$-algebra $\mathcal{A}^{(1)}_u$ satisfies conditions (I) and (II). Moreover, if $\mathcal{A}^{(1)}$ is a complex unital $*$-algebra satisfying conditions (I) and (II) then $\mathcal{A}^{(1)}$ is a quotient of $\mathcal{A}^{(1)}_u$ for some $u \in T^3$.  

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By construction the algebras $A_u^{(1)}$ are all quotients of the universal Grassmannian generated by $(I_i)$ i.e. by the matrix components of a two by two unitary matrix. We shall prove in the appendix that there are non-trivial relations fulfilled in all the quotients $A_u^{(1)}$ independently of $u$, thus settling a question left open in [16].

There is another way to write relations (2.10) and (2.11) which will be useful for the description of the suspension below, it is given in the following lemma.

**LEMMA 3** Let $\gamma_\mu = \gamma_\mu^* \in M_4(\mathbb{C})$ be the generators of the Clifford algebra of $\mathbb{R}^4$, that is $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \mathbb{1}$, and let $\tilde{\gamma}_\mu$ be defined by $\tilde{\gamma}_0 = \gamma_0$, $\tilde{\gamma}_k = e^{\frac{i}{2} \varphi_k} \gamma_k e^{-\frac{i}{2} \varphi_k}$ for $k \in \{1, 2, 3\}$ with $\gamma = \gamma_0 \gamma_1 \gamma_2 \gamma_3 (= \gamma_5)$. Then the relations (2.10) and (2.11) defining $A_u$ are equivalent to the relation

$$(\tilde{\gamma}_\mu x^\mu)^2 = \mathbb{1} \otimes \sum_\mu (x^\mu)^2$$

in $M_4(A_u) = M_4(\mathbb{C}) \otimes A_u$.

This is easy to check using $\gamma \gamma_\mu = -\gamma_\mu \gamma$ and $\gamma^2 = \mathbb{1}$. On the right-hand side of the above equality appears the central element $\sum_\mu (x^\mu)^2$ of $A_u$; the algebra $A_u$ has another central element described in the following lemma.

**LEMMA 4** The element $\sum_{k=1}^3 \cos(\varphi_k - \varphi_\ell - \varphi_m) \cos(\varphi_k) \sin(\varphi_k) (x^k)^2$ is in the center of $A_u$, where in the summation $(k, \ell, m)$ is the cyclic permutation of $(1, 2, 3)$ starting with $k$ for $k \in \{1, 2, 3\}$.

This can be checked directly using (2.10), (2.11). So one has two quadratic elements in the $x^\mu$ which belong to the center $Z(A_u)$ of $A_u$. In fact, for
generic $u$, the center is generated by these two quadratic elements.

By changing $x_k$ in $-x_k$ one can replace $\varphi_k$ by $\varphi_k + \pi$ and by a rotation of $SO(3)$ one can permute the $\varphi_k$ without changing the algebra $\mathcal{A}_u$ nor the algebra $\mathcal{A}^{(1)}_u$. It follows that it is sufficient to take $u$ in the 3-cell $\Sigma^3$ defined by

$$\Sigma^3 = \{(e^{i\varphi}) \in T^3 \mid \pi > \varphi_1 \geq \varphi_2 \geq \varphi_3 \geq 0\} \quad (2.12)$$

to cover all the $\mathcal{A}_u$ and $\mathcal{A}^{(1)}_u$.

It is apparent that $\mathcal{A}_u$ is a deformation of the commutative $*$-algebra $C_{\text{alg}}(\mathbb{R}^4)$ of complex polynomial functions on $\mathbb{R}^4$; it reduces to the latter for $\varphi_1 = \varphi_2 = \varphi_3 = 0$ that is for $u = e$ where $e = (1, 1, 1)$ is the unit of $T^3$. We shall denote $\mathcal{A}_u$ by $C_{\text{alg}}(\mathbb{R}^4_u)$ defining thereby the noncommutative 4-plane $\mathbb{R}^4_u$ as dual object. Similarly, the quotient $\mathcal{A}_u^{(1)}$ of $\mathcal{A}_u$ by the ideal generated by $\sum_\mu (x^\mu)^2 - 1$ is a deformation of the $*$-algebra $C_{\text{alg}}(S^3)$ of polynomial functions on $S^3$ that is of functions on $S^3$ which are restrictions to $S^3 \subset \mathbb{R}^4$ of elements of $C_{\text{alg}}(\mathbb{R}^4)$; we shall denote this quotient $\mathcal{A}_u^{(1)}$ by $C_{\text{alg}}(S^3_u)$ defining thereby the noncommutative 3-sphere $S^3_u$ by duality.

Let $C_{\text{alg}}(\mathbb{R}^5_u)$ be the unital $*$-algebra obtained by adding a central hermitian generator $x^4$ to $C_{\text{alg}}(\mathbb{R}^4_u) = \mathcal{A}_u$, i.e. $C_{\text{alg}}(\mathbb{R}^5_u)$ is the unital $*$-algebra generated by hermitian elements $x^\mu$, $\mu \in \{0, \ldots, 3\}$, and $x^4$ such that the $x^\mu$ satisfy (2.10), (2.11) and that one has $x^\mu x^4 = x^4 x^\mu$ for $\mu \in \{0, \ldots, 3\}$; the noncommutative 5-plane $\mathbb{R}^5_u$ being defined by duality. Define $C_{\text{alg}}(S^4_u)$ to be the unital $*$-algebra quotient of $C_{\text{alg}}(\mathbb{R}^5_u)$ by two-sided ideal generated by the hermitian central element $\sum_{\mu=0}^3 (x^\mu)^2 + (x^4)^2 - 1$. The noncommutative 4-sphere $S^4_u$ defined as dual object is in the obvious sense the suspension of
$S^3_u$. This is a 3-parameter deformation of the sphere $S^4$ which reduces to $S^4_\theta$ for $\varphi_1 = \varphi_2 = -\frac{1}{2}\theta$ and $\varphi_3 = 0$, (see below). We denote by $u^\mu$, $u$ the canonical images of $x^\mu$, $x^4 \in C_{\text{alg}}(\mathbb{R}^5_u)$ in $C_{\text{alg}}(S^4_u)$ and by $v^\mu$ the canonical images of $x^\mu \in C_{\text{alg}}(\mathbb{R}^4_u)$ in $C_{\text{alg}}(S^3_u)$, i.e. one has $\sum (u^\mu)^2 + u^2 = 1$ and $\sum (v^\mu)^2 = 1$, etc.. It will be convenient for further purpose to summarize some important points discussed above by the following theorem.

**THEOREM 2** (i) One defines a hermitian projection $e \in M_4(C_{\text{alg}}(S^4_u))$ by setting $e = \frac{1}{2}(1 + \tilde{\gamma}_u u^4 + \gamma u)$. Furthermore one has $\text{ch}_0(e) = 0$ and $\text{ch}_1(e) = 0$.

(ii) One defines a unitary $U \in M_2(C_{\text{alg}}(S^3_u))$ by setting $U = 1v^0 + i\tilde{\sigma}_k v^k$ where $\tilde{\sigma}_k = \sigma_k e^{i\varphi_k}$. Furthermore one has $\text{ch}_{\frac{1}{2}}(U) = 0$.

Statement (ii) is just a reformulation of what has be done previously. Concerning Statement (i), the fact that $e$ is a hermitian projection with $\text{ch}_0(e) = 0$ follows directly from the definition and Lemma 3 whereas $\text{ch}_1(e) = 0$ is a consequence of $\text{ch}_{\frac{1}{2}}(U) = 0$ in (ii).

We shall now compute $\tilde{\text{ch}}_{\frac{1}{2}}(U)$ and check that, except for exceptional values of $u$ for which $\tilde{\text{ch}}_{\frac{1}{2}}(U) = 0$, it is a non trivial Hochschild cycle on $A_u$.

One has by construction

$$\tilde{\text{ch}}_{\frac{1}{2}}(U_u) = \text{tr}(U_u \otimes U_u^* \otimes U_u \otimes U_u^* - U_u \otimes U_u \otimes U_u^* \otimes U_u)$$

which is an element of $A_u \otimes A_u \otimes A_u \otimes A_u$ and can be considered as a $A_u$-valued Hochschild 3-chain. One obtains using $(2.10)$, $(2.11)$

$$\tilde{\text{ch}}_{\frac{1}{2}}(U_u) = - \sum_{3 \geq \alpha, \beta, \gamma, \delta \geq 0} \epsilon_{\alpha \beta \gamma \delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta) x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta + i \sum_{3 \geq \mu, \nu \geq 0} \sin(2(\varphi_\mu - \varphi_\nu)) x^\mu \otimes x^\nu \otimes x^\mu \otimes x^\nu$$

$(2.13)$
where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$ and where we have set $\varphi_0 = 0$. Using (2.13), (2.10), (2.11) one checks that $\tilde{\text{ch}}_3(U_u)$ is in fact a Hochschild cycle, i.e. $b(\tilde{\text{ch}}_3(U_u)) = 0$. Actually, this follows on general grounds from the fact that $\text{ch}_3(U_u) = 0$ and that $U_u^*U_u = U_uU_u^*$ is an element of the center $\mathbb{1}_2 \otimes Z(A_u)$ of $M_2(A_u)$ in view of Lemma 1. In fact the $A_u$-valued Hochschild 3-cycle $\text{ch}_3(U_u)$ is trivial (i.e. is a boundary) if and only if it vanishes (which means that all coefficients vanish in formula (2.13)). Indeed $A_u$ is a $\mathbb{N}$-graded algebra with $A_u^0 = \mathbb{C} \mathbb{1}$ and $A_u^1 = \text{linear span of the } \{x^\mu | \mu \in \{0, \ldots, 3\}\}$ and the Hochschild boundary preserves the degree. It follows that $\tilde{\text{ch}}_3(U_u)$ can only be the boundary of linear combinations of terms which are in $\otimes^5 A_u$ of total degree 4 and contain therefore at least one tensor factor equal to $\mathbb{1}$. Among these terms, the $\mathbb{1} \otimes x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta$ are the only ones which contain in their boundaries tensor products of four $x^\mu$.

One has for these terms

$$b(\mathbb{1} \otimes x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta) = x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta + x^\delta \otimes x^\alpha \otimes x^\beta \otimes x^\gamma$$

$$- \mathbb{1} \otimes (x^\alpha x^\beta \otimes x^\gamma \otimes x^\delta - x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta + x^\alpha \otimes x^\beta \otimes x^\gamma x^\delta)$$

however the $x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta + x^\delta \otimes x^\alpha \otimes x^\beta \otimes x^\gamma$ cannot produce by linear combination a term with the kind of generalized antisymmetry of $\text{ch}_3(U_u)$ excepted of course if $\text{ch}_3(U_u) = 0$. Thus $\text{ch}_3(U_u)$ is non trivial if not zero.

The $A_u^{(1)}$-valued Hochschild 3-cycle $\text{ch}_3(U)$ on $A_u^{(1)}$ is the image of $\text{ch}_3(U_u)$ by the projection of $A_u$ onto $A_u^{(1)}$. In particular $\text{ch}_3(U)$ vanishes if $\text{ch}_3(U_u)$ vanishes which occurs on $\Sigma^3$ for $\varphi_1 = \varphi_2 = \varphi_3 = \frac{\pi}{2}$ and for $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \varphi_3 = 0$. For these two values of $u$, the algebras $A_u$ are isomorphic, one passes from $\varphi_1 = \varphi_2 = \varphi_3 = \frac{\pi}{2}$ to $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \varphi_3 = 0$ by the exchange of $x^0$ and $x^1$; this is of course the same for $A_u^{(1)}$. One can furthermore check that the Hochschild dimension of $A_u^{(1)}$ for these values of $u$ is one.

To obtain the Hochschild 4-cycle on $A_u$ corresponding to the volume
form on the noncommutative 4-plane $\mathbb{R}^4_u$, we shall just apply to $\tilde{\text{ch}}^2_2(U_u)$ the natural extension of the de Rham coboundary in the noncommutative case, namely the operator $B : \mathcal{A}_u \otimes \mathcal{A}_u^\circ \to \mathcal{A}_u \otimes \mathcal{A}_u^\circ$ ([8] [30]). Since $\tilde{\text{ch}}^2_2(U_u)$ is not only a Hochschild cycle but also fulfills the cyclicity condition, it follows that, up to an irrelevant normalization $B$ reduces there to the tensor product by $1$, thus

$$B\tilde{\text{ch}}^2_2(U_u) = 1 \otimes \tilde{\text{ch}}^2_2(U_u)$$

and the Hochschild 4-cycle $B\tilde{\text{ch}}^2_2(U_u)$ which plays the role of the volume form of $\mathbb{R}^4_u$ is thus given by

$$v = - \sum_{3 \geq \alpha, \beta, \gamma, \delta \geq 0} \epsilon_{\alpha, \beta, \gamma, \delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta) 1 \otimes x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta$$

$$+ i \sum_{3 \geq \mu, \nu \geq 0} \sin(2(\varphi_\mu - \varphi_\nu)) 1 \otimes x^\mu \otimes x^\nu \otimes x^\mu \otimes x^\nu$$

(2.14)

It turns out that this 4-cycle is non-trivial whenever it does not vanish as can be verified by evaluation at the origin which is the classical point of $\mathbb{R}^4_u$. The nontriviality of $\text{ch}^2_2(U)$ follows since $B\tilde{\text{ch}}^2_2(U_u)$ is its suspension.

3 Relation with Sklyanin algebras

Let us assume that for all $i, k, l$,

$$\varphi_i \notin \{n\frac{\pi}{2} | n \in \mathbb{Z}\}, \quad \varphi_k - \varphi_\ell \notin \{\frac{\pi}{2} + n\pi | n \in \mathbb{Z}\}$$

(3.1)

We shall refer to this as generic values of $u$. Then the relations (2.10), (2.11) can be written

$$[S_0, S_k]_- = iJ_{lm}[S_\ell, S_m]_+$$

(3.2)

$$[S_\ell, S_m]_- = \bar{i}[S_0, S_k]_+$$

(3.3)

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where $J_{\ell m} = - \tan(\varphi_\ell - \varphi_m) \tan(\varphi_k)$ for any cyclic permutation $(k, \ell, m)$ of $(1, 2, 3)$ and where

$$S_\mu = s^\mu x^\mu$$

(3.4)

with $s^\mu \in \mathbb{C}$, $\mu \in \{0, \cdots, 3\}$, solution of

$$
\begin{align*}
    s^0 s^1 \cos(\varphi_2 - \varphi_3) + s^2 s^3 \sin(\varphi_1) &= 0 \\
    s^0 s^2 \cos(\varphi_3 - \varphi_1) + s^3 s^1 \sin(\varphi_2) &= 0 \\
    s^0 s^3 \cos(\varphi_1 - \varphi_2) + s^1 s^2 \sin(\varphi_3) &= 0
\end{align*}
$$

(3.5)

So defined the three real numbers $J_{k\ell}$ satisfy the relation

$$J_{12} + J_{23} + J_{31} + J_{12} J_{23} J_{31} = 0$$

(3.6)

as easily verified. The relations (3.2), (3.3) together with (3.6) for the constants $J_{k\ell}$ characterize the algebra introduced by Sklyanin in connection with the Yang-Baxter equation [45], [46].

For generic values of $u$, the solution of (3.5) is unique (up to an overall normalization and choices of sign) and can be written in the form,

$$
\begin{align*}
    s^0 &= i^{a_0} (\prod_j \sin(\varphi_j))^{1/2} \\
    s^k &= i^{a_k} (\sin \varphi_k \prod_{\ell \neq k} \cos(\varphi_k - \varphi_\ell))^{1/2}
\end{align*}
$$

for appropriate powers $a_k$ of $i = \sqrt{-1}$. This allows to keep track of the involution under the change of variables (3.4). In the case when the $s_j$ are real, the transformation (3.4) preserves the involution which on the Sklyanin algebra $S(J_{k\ell})$ is given by

$$S^*_\mu = S_\mu \quad \mu = 0, 1, 2, 3.$$
In general, however, one cannot choose the $s^{\mu}$'s to be real and the involutive algebra $A_u$ gives a different real form of the Sklyanin algebra.

The Sklyanin algebras $S(J_{k\ell})$ have been extensively studied from the point of view of noncommutative algebraic geometry. An important role is played by the associated geometric data

$$\{E, \sigma, \mathcal{L}\}$$

consisting of an elliptic curve $E \subset P_3(\mathbb{C})$, an automorphism $\sigma$ of $E$ and an invertible $\mathcal{O}_E$-module $\mathcal{L}$ (cf. [1], [2], [38], [47]). This geometric data is invariantly defined for any graded algebra and in the above case of $S(J_{k\ell})$, it degenerates when one of the parameters $J_{k\ell}$ vanishes (or in the case $J_{k\ell} = 1, J_{tr} = -1$, cf. [47] for a careful discussion). The degenerate case $J_{k\ell} = 0$ corresponds in our case to $\varphi_k = \varphi_\ell$. Let us take $k = 1, \ell = 2, \varphi_1 = \varphi_2 = \psi, \varphi_3 = \varphi$ generic for definiteness. The corresponding Sklyanin algebra is then isomorphic to a graded version of the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ (cf. e.g. [47]). More precisely, and keeping track of the involution, the appropriate change of variables with our notation is

$$K_\pm = x^0 \pm \frac{\cos(\psi - \varphi)}{\sin \psi} itx^3$$

$$Y = x^1 + ix^2$$

where $t^2 = \tan(\psi) \tan(\psi - \varphi)$. One has $q = \frac{1 + it}{1 - it}$ and $q$ is either real, in which case $K_\pm$ is hermitian, or complex of modulus one, in which case $K_- = K^*_+$. Moreover (cf. e.g. [47]) $K_+ K_- = K_- K_+$ is central and fixing its non zero value, one obtains in our case the quantum enveloping algebra

$$U_q(\mathfrak{sl}_2(\mathbb{R})) \text{ or } U_q(\mathfrak{su}_2)$$
depending on the sign of \( \tan(\varphi) \sin(2\psi) \). We postpone the analysis of the algebras \( A_u \) in the general case to Part II of this paper and concentrate instead on the degenerate values \( \varphi_1 = \varphi_2 = \psi, \varphi_3 = 0 \) of our parametrization. This does not correspond to a Sklyanin algebra but can be viewed as a limiting case of the generic values \( (\psi, \psi, \varphi) \) discussed above.

In our situation the geometric data degenerates to the union of 6 projective lines \( P_1(\mathbb{C}) \), with \( \sigma \) given by multiplication by products of two conjugates of \( e^{i\varphi} \). The algebra becomes a very simple \( \theta \)-deformation of \( \mathbb{R}^4 \) (with \( \psi = -\frac{1}{2} \theta \) as explained below) which lends itself to easy higher dimensional generalization.

In spite of its apparent simplicity, the careful treatment of this special case will allow us to clearly define our task for Part II.

4 The \( \theta \)-deformed \( 2n \)-plane \( \mathbb{R}_\theta^{2n} \) and its Clifford algebra

In the previous sections, we have defined a multiparameter noncommutative deformation \( C_{\text{alg}}(\mathbb{R}_u^4) \) of the graded algebra \( C_{\text{alg}}(\mathbb{R}^4) \) of polynomial functions on \( \mathbb{R}^4 \) which induces a corresponding deformation \( C_{\text{alg}}(S_3^u) \) of the algebra of polynomial functions on \( S^3 \) in such a way that all dimensions are preserved as will be shown in Part II. Moreover this is the generic deformation under the above conditions. It turns out that one can extract from this multiparameter deformation of \( C_{\text{alg}}(\mathbb{R}^4) \) a one-parameter deformation \( C_{\text{alg}}(\mathbb{R}_\theta^4) \) of \( C_{\text{alg}}(\mathbb{R}^4) \) which is also a one-parameter deformation \( C_{\text{alg}}(\mathbb{C}_\theta^2) \) of \( C_{\text{alg}}(\mathbb{C}^2) \) whence \( \mathbb{C}^2 \) is identified with \( \mathbb{R}^4 \) through (for instance) \( z^1 = x^0 + ix^3, z^2 = x^1 + ix^2 \). The parameter \( \theta \) corresponds to the curve \( \theta \mapsto u(\theta) \) defined by \( u_1 = u_2 = e^{-\frac{i}{2} \theta} \) and \( u_3 = 1 \), i.e. to \( \varphi_1 = \varphi_2 = -\frac{1}{2} \theta \) and \( \varphi_3 = 0 \) in terms of the previous parameters. Indeed for these values of \( u \), the relations (2.10), (2.11) for
read in terms of $z^1 = x^0 + ix^3, \bar{z}^1 = x^0 - ix^3, z^2 = x^1 + ix^2, \bar{z}^2 = x^1 - ix^2$, (one has $z^{1*} = \bar{z}^1$ and $z^{2*} = \bar{z}^2$) $z^1 z^2 = \lambda z^2 z^1, \bar{z}^1 \bar{z}^2 = \lambda \bar{z}^2 \bar{z}^1, z^1 \bar{z}^1 = \bar{z}^1 z^1, z^2 \bar{z}^2 = \bar{z}^2 z^2, z^1 \bar{z}^2 = \lambda^{-1} z^2 \bar{z}^1, z^1 z^2 = \lambda \bar{z}^2 \bar{z}^1$ where we have set $\lambda = e^{i\theta}$. This one-parameter deformation is well suited for simple higher-dimensional generalizations (i.e. $\mathbb{C}^2$ is replaced by $\mathbb{C}^n$ and $\mathbb{R}^4$ by $\mathbb{R}^{2n}$, $n \geq 2$). In the following we shall describe and analyze them in details.

For this we shall generalize $\theta$ as explained at the end of the introduction as an antisymmetric matrix $\theta \in M_n(\mathbb{R})$, the previous one being identified as $
abla \left( \begin{array}{cc} 0 & \theta \\ -\theta & 0 \end{array} \right) \in M_2(\mathbb{R})$, and we shall use the notations explained at the end of the introduction.

Let $\mathcal{C}_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ be the complex unital associative algebra generated by $2n$ elements $z^\mu, \bar{z}^\mu$ ($\mu, \nu = 1, \ldots, n$) with relations

\begin{align*}
    z^\mu z^\nu &= \lambda^{\mu\nu} z^\nu z^\mu \quad (4.1) \\
    \bar{z}^\mu \bar{z}^\nu &= \lambda^{\mu\nu} \bar{z}^\nu \bar{z}^\mu \quad (4.2) \\
    \bar{z}^\mu z^\nu &= \lambda^{\mu\nu} z^\nu \bar{z}^\mu \quad (4.3)
\end{align*}

for $\mu, \nu = 1, \ldots, n$ ($\lambda^{\mu\nu} = e^{i\theta_{\mu\nu}}, \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}$). Notice that one has $\lambda^{\nu\mu} = 1/\lambda^{\mu\nu} = \overline{\lambda^{\mu\nu}}$ and that $\lambda^{\mu\mu} = 1$. We equip $\mathcal{C}_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ with the unique $\mathbb{C}$-algebra involution $x \mapsto x^*$ such that $z^{\mu *} = \bar{z}^\mu$. Clearly the $*$-algebra $\mathcal{C}_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ is a deformation of the commutative $*$-algebra $\mathcal{C}_{\text{alg}}(\mathbb{R}^{2n})$ of complex polynomial functions on $\mathbb{R}^{2n}$, (it reduces to the latter for $\theta = 0$). The algebra $\mathcal{C}_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ will be refered to as the algebra of complex polynomials on the noncommutative $2n$-plane $\mathbb{R}_{\theta}^{2n}$.

In fact the relations (4.1), (4.2), (4.3) define a deformation $\mathbb{C}_{\theta}$ of $\mathbb{C}^n$ and we can identify $\mathbb{C}_{\theta}$ and $\mathbb{R}_{\theta}^{2n}$ by writing $\mathcal{C}_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) = \mathcal{C}_{\text{alg}}(\mathbb{C}_{\theta})$. Correspondingly, the unital subalgebra $H_{\text{alg}}(\mathbb{C}_{\theta})$ generated by the $z^\mu$ is a deformation of
the algebra of holomorphic polynomial functions on \( \mathbb{C}^n \).

There is a unique group-homomorphism \( s \mapsto \sigma_s \) of the abelian group \( T^n \) into the group \( \text{Aut}(C_{\text{alg}}(\mathbb{R}^{2n}_\theta)) \) of unital \(*\)-automorphisms of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) which is such that \( \sigma_s(z^\nu) = e^{2\pi i s \cdot \nu} z^\nu \), \( (\sigma_s(\bar{z}^\nu) = e^{-2\pi i s \cdot \nu} \bar{z}^\nu) \). This definition is independent of \( \theta \), in particular \( s \mapsto \sigma_s \) is well defined as a group-homomorphism of \( T^n \) into \( \text{Aut}(C_{\text{alg}}(\mathbb{R}^{2n}_\theta)) \) where it is induced by a smooth action of \( T^n \) on the manifold \( \mathbb{R}^{2n} \). It follows from the relations (4.1), (4.2) and (4.3) that the \( z^\mu \bar{z}^{\ast} = z^\mu z^{\ast} \ (1 \leq \mu \leq n) \) are in the center of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \). Furthermore these hermitian elements generate the center as unital subalgebra of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) whenever \( \theta \) is generic, i.e. for \( \theta_{\mu \nu} \) irrational \( \forall \mu, \nu \) with \( 1 \leq \mu < \nu \leq n \). On the other hand these elements \( \bar{z}^\mu z^\mu \) generate the subalgebra \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta)^\sigma \) of elements of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) which are invariant by the action \( \sigma \) of \( T^n \). Thus \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta)^\sigma \) is contained in the center of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \). This is not an accident, moreover the subalgebra of invariant elements of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) is not deformed (i.e. does not depend on \( \theta \)) and is canonically isomorphic to \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta)^\sigma \).

**Remark.** In view of the centrality of \( z^\mu \bar{z}^{\ast} = \bar{z}^\mu z^\mu \), one can fix them in particular to 1 to obtain the noncommutative \( n \)-torus \( T^n_\theta \). More precisely the quotient \( C_{\text{alg}}(T^n_\theta) \) of \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) by the ideal generated by the \( z^\mu \bar{z}^{\ast} - 1 \ (\mu = 1, \ldots, n) \) is a deformation of the \(*\)-algebra of the polynomial (complex) functions on \( T^n \). By completion for the greatest \( C^*\)-semi-norm which is a norm one obtains the \( C^*\)-algebra \( C(T^n_\theta) \) which is a deformation of the algebra of continuous functions on \( T^n \).

Let \( \text{Cliff}(\mathbb{R}^{2n}_\theta) \) be the unital associative \( \mathbb{C} \)-algebra generated by \( 2n \) ele-
ments $\Gamma^\mu$, $\Gamma^{\nu*}$ ($\mu, \nu = 1, \ldots, n$) with relations

\begin{align}
\Gamma^\mu \Gamma^\nu + \lambda^{\mu\nu} \Gamma^\nu \Gamma^\mu &= 0 \\
\Gamma^{\mu*} \Gamma^{\nu*} + \lambda^{\mu\nu} \Gamma^{\nu*} \Gamma^{\mu*} &= 0 \\
\Gamma^{\mu*} \Gamma^\nu + \lambda^{\mu\nu} \Gamma^\nu \Gamma^{\mu*} &= \delta^{\mu\nu} \mathbb{1}
\end{align}

(4.4) (4.5) (4.6)

where $\mathbb{1}$ denotes the unit of the algebra. For $\theta = 0$ one recovers the usual Clifford algebra of $\mathbb{R}^{2n}$; the familiar generators $\gamma^a$ ($a = 1, 2, \ldots, 2n$) associated to the canonical basis of $\mathbb{R}^{2n}$ being then given by $\gamma^\mu = \Gamma^\mu + \Gamma^{\mu*}$ and $\gamma^{\mu+n} = -i(\Gamma^\mu - \Gamma^{\mu*})$. There is a unique involution $\Lambda \mapsto \Lambda^*$ such that $(\Lambda^\nu)^* = \Lambda^{\nu*}$ for which $\text{Cliff}(\mathbb{R}^{2n}_\theta)$ is a unital complex $*$-algebra. One also endows $\text{Cliff}(\mathbb{R}^{2n}_\theta)$ with a $\mathbb{Z}_2$-grading of algebra by giving odd degree to the $\gamma^\mu$; $\Gamma^{\nu*}$. The relations (4.4), (4.5) and (4.6) imply that the hermitian element $[\Gamma^{\mu*}, \Gamma^\mu] = \Gamma^{\mu*} \Gamma^\mu - \Gamma^\nu \Gamma^{\nu*}$ anticommutes with $\Gamma^\mu$ and $\Gamma^{\nu*}$ whereas it commutes with $\Gamma^\nu$ and $\Gamma^{\nu*}$ for $\nu \neq \mu$ and that furthermore one has $([\Gamma^{\mu*}, \Gamma^\mu])^2 = \mathbb{1}$. It follows that $\gamma \in \text{Cliff}(\mathbb{R}^{2n}_\theta)$ defined by

$$
\gamma = [\Gamma^{1*}, \Gamma^1] \cdots [\Gamma^{n*}, \Gamma^n] = \prod_{\mu=1}^n [\Gamma^{\mu*}, \Gamma^\mu]
$$

(4.7)

is hermitian ($\gamma = \gamma^*$) and satisfies

$$
\gamma^2 = 1, \quad \gamma \Gamma^\mu + \Gamma^\mu \gamma = 0, \quad \gamma \Gamma^{\mu*} + \Gamma^{\mu*} \gamma = 0
$$

(4.8)

in fact $\Lambda \mapsto \gamma \Lambda \gamma$ is the $\mathbb{Z}_2$-grading. The very reason why we have imposed the relations (4.4), (4.5) and (4.6) is the following easy lemma.

**LEMMA 5** In the algebra $\text{Cliff}(\mathbb{R}^{2n}_\theta) \otimes C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$, the $\Gamma^{\mu*} z^\mu$ and the $\Gamma^\rho \bar{z}^\rho$ = $\Gamma^\rho z^{\rho*}$, $\mu, \rho = 1, \ldots, n$, satisfy the following anticommutation relations

$$
\Gamma^{\mu*} z^\mu \Gamma^\rho \bar{z}^\rho + \Gamma^\rho z^\rho \Gamma^{\mu*} z^\mu = 0 \quad (\Gamma^\mu \bar{z}^\rho \Gamma^\rho \bar{z}^\rho + \Gamma^\rho z^\rho \Gamma^{\mu*} \bar{z}^\rho \bar{z}^\rho = 0)
$$

and $\Gamma^{\mu*} z^\mu \Gamma^\rho \bar{z}^\rho + \Gamma^\rho z^\rho \Gamma^{\mu*} \bar{z}^\rho \bar{z}^\rho = \delta^{\mu\rho} z^\mu \bar{z}^\rho$ which do not depend on $\theta$. 26
This straightforward result is a key to reduce lots of computations to the classical case $\theta = 0$, (see below). The next result shows that $\text{Cliff}(\mathbb{R}^{2n})$ is isomorphic to the usual $\text{Cliff}(\mathbb{R}^{2n})$ as $*$-algebra and as $\mathbb{Z}_2$-graded algebra.

**PROPOSITION 1** The following equality gives a faithful $*$-representation $\pi$ of $\text{Cliff}(\mathbb{R}^{2n})$ in the Hilbert space $\otimes^n \mathbb{C}^2$,

$$\pi(\Gamma^\mu) = \left( \begin{array}{cc} -\lambda^1\mu & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{cc} -\lambda^{n-1}\mu & 0 \\ 0 & 1 \end{array} \right) \otimes 1_2 \otimes \cdots \otimes 1_2 = \pi(\Gamma^\mu)^*$$

and $\pi$ is the unique irreducible $*$-representation of $\text{Cliff}(\mathbb{R}^{2n})$ up to a unitary equivalence.

The proof is straightforward. Note that $\otimes^n \mathbb{C}^2$, viewed as the graded tensor product of $\mathbb{C}^2$ graded by $\lambda$, is a $\mathbb{Z}_2$-graded $\text{Cliff}(\mathbb{R}^{2n})$-module. One has $\pi(\gamma) = \otimes^n \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$. In the following we will use the above representation to identify $\text{Cliff}(\mathbb{R}^{2n})$ with $M_{2^n}(\mathbb{C})$.

### 5 Spherical property of $\theta$-deformed spheres

Let $C_{\text{alg}}(\mathbb{R}_\theta^{2n+1})$, the algebra of polynomial functions on the noncommutative $(2n+1)$-plane $\mathbb{R}_\theta^{2n+1}$, be the unital complex $*$-algebra obtained by adding an hermitian generator $x$ to $C_{\text{alg}}(\mathbb{R}_\theta^{2n})$ with relations $x\mu = \mu x$ ($\mu = 1, \ldots, n$), i.e. $C_{\text{alg}}(\mathbb{R}_\theta^{2n+1}) \simeq C_{\text{alg}}(\mathbb{R}_\theta^{2n}) \otimes \mathbb{C}[x] \simeq C_{\text{alg}}(\mathbb{R}_\theta^{2n}) \otimes C_{\text{alg}}(\mathbb{R})$. One knows that the $z^\mu \bar{z}^\mu = \bar{z}^\mu z^\mu$ and $x$ are in the center so $\sum_{\mu=1}^n z^\mu \bar{z}^\mu + x^2$ is also in the center $C_{\text{alg}}(\mathbb{R}_\theta^{2n+1})$. We define the $*$-algebra $C_{\text{alg}}(S_\theta^{2n})$ to be the quotient of $C_{\text{alg}}(\mathbb{R}_\theta^{2n+1})$ by the ideal generated by $\sum_{\mu=1}^n z^\mu \bar{z}^\mu + x^2 - 1$. In the following, we shall denote by $u^\mu$, $\bar{u}^\mu = u^{\mu\ast}$, $u$ the canonical images of $z^\mu$, $\bar{z}^\mu$, $x$ in $C_{\text{alg}}(S_\theta^{2n})$. On the unital complex $*$-algebra $C_{\text{alg}}(S_\theta^{2n})$ there is a greatest $C^*$-seminorm which is a norm; the $C^*$-algebra $C(S_\theta^{2n})$ obtained by completion
will be referred to as the algebra of continuous functions on the noncommutative 2n-sphere $S^{2n}_\theta$.

It is worth noticing that the noncommutative 2n-sphere $S^{2n}_\theta$ can be viewed as “one-point compactification” of the noncommutative 2n-plane $\mathbb{R}^{2n}_\theta$. To explain this, let us slightly enlarge the $\ast$-algebra $C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ by adjoining a hermitian central generator $(1 + \sum_{\mu=1}^{n} \bar{z}^\mu z^\mu)^{-1} = (1 + |z|^2)^{-1}$ with relation 

$$(1 + \sum_{\mu=1}^{n} \bar{z}^\mu z^\mu)(1 + |z|^2)^{-1} = (1 + |z|^2)^{-1}(1 + \sum_{\mu=1}^{n} \bar{z}^\mu z^\mu) = 1.$$ 

As will become clear $(1 + |z|^2)^{-1}$ is smooth so that in fact we are staying in the algebra $C^\infty(\mathbb{R}^{2n}_\theta)$ of smooth functions on $\mathbb{R}^{2n}_\theta$. By setting 

$$\tilde{u}^\mu = 2z^\mu(1 + |z|^2)^{-1}, \quad \tilde{u}^\nu, \tilde{u}^{\nu_*} = 2\bar{z}^\nu(1 + |z|^2)^{-1}, \quad \tilde{u} = (1 - \sum_{\mu=1}^{n} \bar{z}^\mu z^\mu)(1 + |z|^2)^{-1},$$

one sees that the $\tilde{u}^\mu, \tilde{u}^{\nu_*}, \tilde{u}$ satisfy the same relations as the $u^\mu, u^{\nu_*}, u$. The “only difference” is that the classical point $u^\mu = 0, \bar{u}^\mu = 0, u = -1$ of $S^{2n}_\theta$ does not belong to the spectrum of $\tilde{u}^\mu, \tilde{u}^{\nu_*}, \tilde{u}$. In the same spirit, one can cover $S^{2n}_\theta$ by two “charts” with domain $\mathbb{R}^{2n}_\theta$ with transition on $\mathbb{R}^{2n}_\theta \setminus \{0\}$, $(z^\mu = 0, \bar{z}^\nu = 0$ being a classical point of $\mathbb{R}^{2n}_\theta$).

Let $C_{\text{alg}}(S^{2n-1}_\theta)$ be the quotient of the $\ast$-algebra $C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ by the two-sided ideal generated by the element $\sum_{\mu=1}^{n} z^\mu \bar{z}^\mu - 1$ of the center of $C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$. This defines by duality the noncommutative $(2n-1)$-sphere $S^{2n-1}_\theta$. In the following, we shall denote by $v^\mu, \bar{v}^\nu$ the canonical images of $z^\mu, \bar{z}^\nu$ in $C_{\text{alg}}(S^{2n-1}_\theta)$. Again there is a greatest $C^*$-seminorm which is a norm on $C_{\text{alg}}(S^{2n-1}_\theta)$; the $C^*$-algebra obtained by completion will be referred to as the algebra of continuous functions on the noncommutative $(2n-1)$-sphere $S^{2n-1}_\theta$. It is clear that, in an obvious sense, $S^{2n}_\theta$ is the suspension of $S^{2n-1}_\theta$. 

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As for the case of $\mathbb{R}^{2n}_\theta$, one has an action $\sigma$ of $T^n$ on $\mathbb{R}^{2n+1}_\theta$, $S^{2n}_\theta$ and $S^{2n-1}_\theta$ which is induced by an action on the corresponding classical spaces. More precisely the group-homomorphism $s \mapsto \sigma_s$ of $T^n$ into $\text{Aut}(\mathbb{C}^{\text{alg}}(\mathbb{R}^{2n}_\theta))$ extends as a group-homomorphism $s \mapsto \sigma_s$ of $T^n$ into $\text{Aut}(\mathbb{C}^{\text{alg}}(\mathbb{R}^{2n+1}_\theta))$ and these group-homomorphisms induce group homomorphisms $s \mapsto \sigma_s$ of $T^n$ into $\text{Aut}(\mathbb{C}^{\text{alg}}(\mathbb{S}^{2n-1}_\theta))$ and of $T^n$ into $\text{Aut}(\mathbb{C}^{\text{alg}}(\mathbb{S}^{2n}_\theta))$. As for $\mathbb{R}^{2n}_\theta$, one checks that the subalgebras of $\sigma$-invariant elements are in the respective centers, are not deformed, and are isomorphic to the subalgebras of $\sigma$-invariant elements of $\mathbb{C}^{\text{alg}}(\mathbb{R}^{2n+1}_\theta)$, $\mathbb{C}^{\text{alg}}(\mathbb{S}^{2n}_\theta)$ and $\mathbb{C}^{\text{alg}}(\mathbb{S}^{2n-1}_\theta)$ respectively.

In order to formulate the last part of the next theorem, let us notice that, in view of (4.7) and (4.8), there is an injective representation of $\text{Cliff}(\mathbb{R}^{2n}_\theta)$ for which $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where $1$ denotes the unit of $M_{2n-1}(\mathbb{C})$. In such a representation one has in view of (4.8)

$$
\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^{\mu*} & 0 \end{pmatrix}, \quad \Gamma^{\mu*} = \begin{pmatrix} 0 & \bar{\sigma}^{\mu*} \\ \sigma^{\mu*} & 0 \end{pmatrix}
$$

where $\sigma^\mu$ and $\bar{\sigma}^{\mu}$ are in $M_{2n-1}(\mathbb{C})$.

**THEOREM 3**

(i) One defines a hermitian projection $e \in M_{2^n}(\mathbb{C}^{\text{alg}}(\mathbb{S}^{2n}_\theta))$ by setting $e = \frac{1}{2}(1 + \sum_{\mu=1}^n (\Gamma^{\mu*}w^\mu + \Gamma^\mu w^{\mu*}) + \gamma u)$. Furthermore one has $\text{ch}_m(e) = 0$ for $0 \leq m \leq n - 1$.

(ii) One defines a unitary $U \in M_{2n-1}(\mathbb{C}^{\text{alg}}(\mathbb{S}^{2n-1}_\theta))$ by setting $U = \sum_{\mu=1}^n (\bar{\sigma}^{\mu}v^\mu + \sigma^{\mu}\bar{\sigma}^{\mu*})$, where $\sigma^\mu$ and $\bar{\sigma}^{\mu}$ are as above. Furthermore one has $\text{ch}_{m - \frac{1}{2}}(U) = 0$ for $1 \leq m \leq n - 1$.

The relation $e = e^*$ is obvious. It follows from Lemma 5 that

$$
\left( \sum_{\mu=1}^n (\Gamma^{\mu*}z^\mu + \Gamma^\mu z^{\mu*}) \right)^2 = \sum_{\mu=1}^n z^\mu \bar{z}^\mu,
$$
which in terms of the $\sigma^\mu$ reads

$$(\bar{\alpha}^\mu z^\mu + \sigma^\mu \bar{z}^\mu)(\bar{\alpha}^\mu z^\mu + \sigma^\mu \bar{z}^\mu)^* = (\bar{\alpha}^\mu z^\mu + \sigma^\mu \bar{z}^\mu)^*(\bar{\alpha}^\mu z^\mu + \sigma^\mu \bar{z}^\mu) = \sum_{\mu=1}^{n} z^\mu \bar{z}^\mu.$$  

On the other hand relations (4.8) imply then

$$\left( \sum_{\mu=1}^{n} (\Gamma^\mu z^\mu + \Gamma^\mu \bar{z}^\mu) + \gamma x \right)^2 = \sum_{\mu=1}^{n} z^\mu \bar{z}^\mu + x^2$$

which reduces to $\mathbb{1} \in M_{2^n}(C_{\text{alg}}(S^2_{\theta})).$ This is equivalent to $e^2 = e$. Using again Lemma 5, $\text{ch}_m(e) = 0$ for $m < n$ follows from the vanishing of the corresponding traces of products of the $\Gamma^\mu, \Gamma^\mu*, \gamma$ in the representation of Proposition 1. The unitarity of $U \in M_{2^{n-1}}(C_{\text{alg}}(S^2_{\theta}^{n-1}))$ is clear whereas one has

$$\text{ch}_{m-\frac{1}{2}}(U) = \text{tr} ((U \otimes U^*)^{\otimes m} - (U^* \otimes U)^{\otimes m}) \quad (5.1)$$

which implies

$$\text{ch}_{m-\frac{1}{2}}(U) = \text{tr} \left( \frac{1 + \gamma}{2} \Gamma^{\otimes 2m} - \frac{1 - \gamma}{2} \bar{\Gamma}^{\otimes 2m} \right) = \text{tr}(\gamma \Gamma^{\otimes 2m}) \quad (5.2)$$

where $\Gamma = \sum_{\mu}(\Gamma^\mu z^\mu + \Gamma^\mu \bar{z}^\mu) \in M_{2^n}(C_{\text{alg}}(S^2_{\theta}^{n-1}))$ and where in (5.2) tr and $\otimes$ are taken for $M_{2^n}$ instead of $M_{2^{n-1}}$ as in (5.1), (see the definitions at the end of the introduction). It follows from (5.2) that one has $\text{ch}_{m-\frac{1}{2}}(U) = 0$ for $1 \leq m \leq n - 1$ for the same reasons as $\text{ch}_m(e) = 0$ for $m \leq n - 1$.

This theorem combined with the last theorem of Section 12 and the last theorem of Section 13 implies that $S^m_\theta$ is an $m$-dimensional noncommutative spherical manifold.

It follows from $\text{ch}_m(e) = 0$ for $0 \leq m \leq n - 1$ that $\text{ch}_n(e)$ is a Hochschild cycle which corresponds to the volume form on $S^2_{\theta}$. In fact it is obvious
that the whole analysis of Section III and IV of [16] generalizes from $S^4$ to $S^{2n}_g$. This is in particular the case of Theorem 3 of [16] (with the appropriate changes e.g. $4 \mapsto 2n$ and $M_4(\mathbb{C}) \mapsto M_{2n}(\mathbb{C})$). The odd case is obviously similar. This will be discussed in more details in Section 13.

The projection $e$ is a noncommutative version of the projection-valued field $P_+$ on the sphere $S^{2n}$ described in Section 2.7 of [19]; one has $P_+ = e|_{\theta=0}$.

As was shown there, $P_+$ satisfies the following self-duality equation

$$*P_+(dP_+)^n = i^n P_+(dP_+)^n$$

(5.3)

where $*$ is the usual Hodge duality of forms on $S^{2n}$. Since $*$ is conformally invariant on forms of degree $n$, this equation is conformally invariant. The above equation generalizes to $e$ (i.e. on $S^{2n}_g$) once the appropriate differential calculus and metric are defined, (see Theorem 6 of section 12 below).

For $n$ even, Equation (5.3) describes an instanton (the “round” one) for a conformally invariant generalization of the classical Yang-Mills action on $S^{2n}$ (which reduces to the Yang-Mills action on $S^4$), [19]. The fact, which was pointed out and used in [22], that classical gauge theory can be formulated in terms of projection-valued fields is a direct consequence of the theorem of Narasimhan and Ramanan on the existence of universal connections [36], [37], (see also in [18] for a short economical proof of this theorem).

It is clear that by changing $(u^\mu, u)$ into $(-u^\mu, -u)$ one defines also a hermitian projection $e_- \in M_{2n}(C_{\text{alg}}(S^{2n}_g))$ satisfying $\text{ch}_m(e_-) = 0$ for $0 \leq m \leq n - 1$.

For $\theta = 0$, $e_-$ coincides with the projection-valued field $P_-$ on $S^{2n}$ of [19] which satisfies $*P_-(dP_-)^n = -i^n P_-(dP_-)^n$. What replaces $e \mapsto e_-$ for the odd-dimensional case is $U \mapsto U^*$. 

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6 The graded differential algebras $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^m)$ and $\Omega_{\text{alg}}(S_{\theta}^m)$

There are mainly unique differential calculi, $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ and $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n+1})$, on the noncommutative planes $\mathbb{R}_{\theta}^{2n}$ and $\mathbb{R}_{\theta}^{2n+1}$, which are deformations of the differential algebras of polynomial differential forms on $\mathbb{R}_{\theta}^{2n}$ and $\mathbb{R}_{\theta}^{2n+1}$ and which are such that the $z^\mu \bar{z}^\mu = \bar{z}^\mu z^\mu$ are in the center of $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ and $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n+1})$ as well as $x$ in the case $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n+1})$. Let us first give a detailed description of the graded differential algebra $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$.

As a complex unital associative graded algebra $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) = \bigoplus_{p \in \mathbb{N}}\Omega^p_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ is generated by $2n$ elements $z^\mu, \bar{z}^\nu$ of degree 0 with relations (4.1), (4.2), (4.3) and by $2n$ elements $dz^\mu, d\bar{z}^\nu$ of degree 1 with relations

\begin{align*}
    dz^\mu dz^\nu + \lambda^\mu\nu dz^\nu dz^\mu &= 0 \\
    d\bar{z}^\mu d\bar{z}^\nu + \lambda^\mu\nu d\bar{z}^\nu d\bar{z}^\mu &= 0 \\
    d\bar{z}^\mu dz^\nu + \lambda^\mu\nu d\bar{z}^\nu \bar{z}^\mu &= 0 \\
    z^\mu dz^\nu &= \lambda^\mu\nu dz^\nu z^\mu \\
    \bar{z}^\mu d\bar{z}^\nu &= \lambda^\mu\nu d\bar{z}^\nu \bar{z}^\mu \\
    \bar{z}^\mu dz^\nu &= \lambda^\mu\nu d\bar{z}^\nu \bar{z}^\mu \\
    z^\mu d\bar{z}^\nu &= \lambda^\mu\nu dz^\nu \bar{z}^\mu
\end{align*}

for any $\mu, \nu \in \{1, \ldots, n\}$. There is a unique differential $d$ of $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$, (i.e. a unique antiderivation $d$ satisfying $d^2 = 0$), which extends the mapping $z^\mu \mapsto dz^\mu$, $\bar{z}^\nu \mapsto d\bar{z}^\nu$. One extends $z^\mu \mapsto \bar{z}^\mu$, $dz^\nu \mapsto d\bar{z}^\nu = (dz^\nu)$ as an antilinear involution $\omega \mapsto \bar{\omega}$ of $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ such that $\bar{\omega\omega'} = (-1)^{pp'}\bar{\omega}'\bar{\omega}$ for $\omega \in \Omega^p_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ and $\omega' \in \Omega^{p'}_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$. One has $d\bar{\omega} = \bar{d}\omega$, $\forall \omega \in \Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$. Elements $\omega \in \Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})$ satisfying $\omega = \bar{\omega}$ will be referred to as real elements. Notice
that the $\bar{z}^\mu z^\nu$, $z^\mu d\bar{z}^\nu$, $z^\mu d\bar{z}^\mu$, $d\bar{z}^\mu dz^\mu$ for $\mu \in \{1, \ldots, n\}$ generate a graded differential subalgebra of the graded center of $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ which coincides with this graded center whenever $\theta$ is generic. Notice also that these elements are invariant by the canonical extension to $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ of the action $\sigma$ of $T^n$ on $C_{\text{alg}}(\mathbb{R}^{2n}_\theta) = \Omega^0_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ (see the end of this section).

There is another useful way to construct $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ which we now describe. Consider the graded algebra $C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \otimes \mathbb{R}^{2n} = C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \otimes \wedge_c \mathbb{R}^{2n}$ where $\wedge_c \mathbb{R}^{2n}$ is the complexified exterior algebra of $\mathbb{R}^{2n}$. The graded algebra $C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \otimes \wedge_c \mathbb{R}^{2n}$ is the unital complex graded algebra generated by $2n$ elements of degree zero, $z^\mu, \bar{z}^\nu$ ($\mu, \nu = 1, \ldots, n$) satisfying relations (4.1), (4.2), (4.3) and by $2n$ elements of degree one, $\xi^\mu, \bar{\xi}^\nu$ ($\mu, \nu = 1, \ldots, n$) with relations

$$\xi^\mu \bar{\xi}^\nu + \bar{\xi}^\nu \xi^\mu = 0, \bar{\xi}^\nu \bar{\xi}^\mu + \xi^\nu \xi^\mu = 0, \bar{\xi}^\mu \bar{\xi}^\nu + \xi^\mu \xi^\nu = 0$$

(6.8)

$$z^\mu \bar{\xi}^\nu = \xi^\nu z^\mu, \bar{z}^\mu \bar{\xi}^\nu = \bar{\xi}^\nu \bar{z}^\mu, z^\mu \xi^\nu = \bar{\xi}^\nu z^\mu, \bar{z}^\mu \bar{\xi}^\nu = \bar{\xi}^\nu \bar{z}^\mu$$

(6.9)

for $\mu, \nu \in \{1, \ldots, n\}$. The $2n$ elements $\xi^\mu, \bar{\xi}^\nu$ satisfying (6.8) generate the complexified exterior algebra $\wedge_c \mathbb{R}^{2n}$. One define an involution $\omega \mapsto \bar{\omega}$ of graded algebra on $C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \otimes \wedge_c \mathbb{R}^{2n}$ by setting $\bar{z}^\mu = \bar{z}^\mu, \bar{\xi}^\mu = \bar{\xi}^\mu$ as before and by setting $\bar{\xi}^\mu = \bar{\xi}^\mu, \bar{\xi}^\nu = \xi^\nu$. There is a unique differential $d$ on the graded differential algebra $C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \otimes \wedge_c \mathbb{R}^{2n}$ such that

$$dz^\mu = z^\mu \xi^\nu \quad d\bar{z}^\mu = \bar{z}^\mu \bar{\xi}^\nu$$

(6.12)

(6.13)

for $\mu = 1, \ldots, n$. One then has $d\bar{\omega} = \bar{d}\omega$ for any $\omega \in C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \otimes \wedge_c \mathbb{R}^{2n}$. It is readily verified that the $dz^\mu, d\bar{z}^\mu$ defined by (6.12) and (6.13) satisfy relations (6.1) to (6.7). In other words $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ is the differential subalgebra
of \( C_{\text{alg}}(\mathbb{R}^2_\theta) \otimes \wedge_c \mathbb{R}^{2n} \) generated by the \( z^\mu, \bar{z}^\nu \) \((\mu, \nu = 1, \ldots, n)\). Furthermore, the involution \( \omega \mapsto \bar{\omega} \) of \( C_{\text{alg}}(\mathbb{R}^2_\theta) \otimes \wedge_c \mathbb{R}^{2n} \) induces on \( \Omega_{\text{alg}}(\mathbb{R}^2_\theta) \) the previously defined involution. As \( C_{\text{alg}}(\mathbb{R}^2_\theta) \)-bimodule, one has \( \Omega^p_{\text{alg}}(\mathbb{R}^2_\theta) \subset C_{\text{alg}}(\mathbb{R}^2_\theta) \otimes \wedge^p \mathbb{R}^{2n} \) that is \( \Omega^p_{\text{alg}}(\mathbb{R}^2_\theta) \subset (C_{\text{alg}}(\mathbb{R}^2_\theta))^C_{2n} \), thus the \( \Omega^p_{\text{alg}}(\mathbb{R}^2_\theta) \) are diagonal bimodules over \( C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) [25]. This implies in particular that \( \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) is a quotient of the graded differential algebra \( \Omega_{\text{Diag}}(C_{\text{alg}}(\mathbb{R}^2_\theta)) \) [21].

The differential algebra \( C_{\text{alg}}(\mathbb{R}^2_\theta) \otimes \wedge_c \mathbb{R}^{2n} \) has the following interpretation. Let us “suppress” the classical points \( z^\mu = 0 \) \((\mu = 1, \ldots, n)\) of \( \mathbb{R}^2_\theta \) by adjoining \( n \) real (hermitian) central generators of degree zero \( jz^\mu j^{-2} \) to \( \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) with relations

\[
\bar{z}^\mu z^\mu jz^\mu j^{-2} = jz^\mu j^{-2} \bar{z}^\mu z^\mu = \mathbf{1}
\]

for \( \mu = 1, \ldots, n \). This becomes a graded differential algebra \( \tilde{\Omega}_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) if one sets \( d|z^\mu|^2 = -(|z^\mu|^2)^2 d(\bar{z}^\mu z^\mu) \) for \( \mu = 1, \ldots, n \). Then the algebra \( C_{\text{alg}}(\mathbb{R}^2_\theta) \otimes \wedge_c \mathbb{R}^{2n} \) is the subalgebra generated by the \( z^\mu, \bar{z}^\nu \) and the \( \xi^\mu = |z^\mu|^2 \bar{z}^\mu dz^\mu, \xi^\nu = |z^\nu|^2 \bar{z}^\nu d\bar{z}^\nu \) and it is a graded differential subalgebra of \( \tilde{\Omega}_{\text{alg}}(\mathbb{R}^{2n}_\theta) \). The algebra \( \tilde{\Omega}_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) is the \( \theta \)-deformation of the algebra of complex polynomial differential forms on \((\mathbb{C}\setminus\{0\})^n \subset \mathbb{R}^{2n} \).

The complex unital associative graded algebra \( \Omega_{\text{alg}}(\mathbb{R}^{2n+1}_\theta) \) is defined as the graded tensor product \( \Omega_{\text{alg}}(\mathbb{R}^2_\theta) \otimes_{gr} \Omega_{\text{alg}}(\mathbb{R}) \). More concretely one adjoins to \( \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) one generator \( x \) of degree zero and one generator \( dx \) of degree one with relations

\[
xdx = dxx \quad (6.14)
\]
\[
x\omega = \omega x \quad (6.15)
\]
\[
dx\omega = (-1)^p \omega dx \quad (6.16)
\]

for \( \omega \in \Omega^p_{\text{alg}}(\mathbb{R}^{2n}_\theta) \). One extends the differential \( d \) of \( \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \) as the unique
differential $d$ of $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ mapping $x$ on $dx$. The graded involution of $\Omega_{\text{alg}}(\mathbb{R}^{2n})$ is extended into a graded involution $\omega \mapsto \bar{\omega}$ of $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ by setting $\bar{x} = x$ and $\bar{dx} = dx$. One has again $d\bar{\omega} = \bar{d}\omega$ for $\omega \in \Omega_{\text{alg}}(\mathbb{R}^{2n+1})$.

Again $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ is the differential subalgebra of $C_{\text{alg}}(\mathbb{R}^{2n+1})$ generated by the $z^\mu, \bar{z}^\nu, x$ where the $(2n+1)$-th basis element of $\mathbb{R}^{2n+1}$ is identified with $dx$. Thus again the $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ are diagonal bimodules over $C_{\text{alg}}(\mathbb{R}^{2n+1})$ which implies that $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ is a quotient of $\Omega_{\text{Diag}}(C_{\text{alg}}(\mathbb{R}^{2n+1}))$. Notice that these identifications are compatible with the involutions of the corresponding graded differential algebras.

Define now the graded differential algebra $\Omega_{\text{alg}}(S_{\theta}^{2n-1})$ to be the quotient of $\Omega_{\text{alg}}(\mathbb{R}^{2n})$ by the differential two-sided ideal generated by $\sum_{\mu=1}^{n} z^\mu \bar{z}^\mu - 1$ and the graded differential algebra $\Omega_{\text{alg}}(S_{\theta}^{2n})$ to be the quotient of $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ by the differential two-sided ideal generated by $\sum_{\mu=1}^{n} z^\mu \bar{z}^\mu + x^2 - 1$. These are again graded-involutive algebras with real differentials. Furthermore, it will be shown using the splitting homomorphism that they are diagonal bimodules over $C_{\text{alg}}(S_{\theta}^{2n-1})$ and over $C_{\text{alg}}(S_{\theta}^{2n})$ respectively from which it follows that they are quotient of $\Omega_{\text{Diag}}(C_{\text{alg}}(S_{\theta}^{2n-1}))$ and of $\Omega_{\text{Diag}}(C_{\text{alg}}(S_{\theta}^{2n}))$ respectively.

Let $m = 2n$ or $2n + 1$. The actions $s \mapsto \sigma_s$ of $T^n$ on $C_{\text{alg}}(\mathbb{R}^m)$ and $C_{\text{alg}}(S_{\theta}^{m-1})$ extend canonically to actions of $T^n$ as automorphisms of graded-involutive differential algebras, $s \mapsto \sigma_s \in \text{Aut}(\Omega_{\text{alg}}(\mathbb{R}^m)), s \mapsto \sigma_s \in \text{Aut}(\Omega_{\text{alg}}(S_{\theta}^{m-1}))$. The differential subalgebras $\Omega_{\text{alg}}(\mathbb{R}^m)^\sigma$ and $\Omega_{\text{alg}}(S_{\theta}^{m-1})^\sigma$ of $\sigma$-invariant elements are in the graded centers of $\Omega_{\text{alg}}(\mathbb{R}^m)$ and $\Omega_{\text{alg}}(S_{\theta}^{m-1})$.
and they are undeformed, i.e. isomorphic to the corresponding subalgebras $\Omega_{\text{alg}}(\mathbb{R}^m)^{\sigma}$ and $\Omega_{\text{alg}}(S^{m-1})^{\sigma}$ of $\Omega_{\text{alg}}(\mathbb{R}^m)$ and $\Omega_{\text{alg}}(S^{m-1})$.

7 The quantum groups $GL_\theta(m, \mathbb{R})$, $SL_\theta(m, \mathbb{R})$ and $GL_\theta(n, \mathbb{C})$

In this section we shall describe concretely the various quantum groups which are deformations (called $\theta$-deformations) of the classical groups $GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$, $GL(n, \mathbb{C})$ as well as their actions on the noncommutative spaces $\mathbb{R}_\theta^m$ and $\mathbb{C}_\theta^n$ for $m \geq 4$ and $n \geq 2$. It is worth noticing here that there is no $\theta$-deformation of $SL(n, \mathbb{C})$; the reason is that $dz^1 \cdots dz^n$ is not central and not $\sigma$-invariant in $\Omega_{\text{alg}}(\mathbb{C}_\theta^n) = \Omega_{\text{alg}}(\mathbb{R}_\theta^{2n})$. However, there is a $\theta$-deformation of the subgroup of $GL(n, \mathbb{C})$ consisting of matrices with determinants of modulus one because $dz^1 \cdots dz^n d\zbar^1 \cdots d\zbar^n$ is $\sigma$-invariant and (consequently) central.

Let us define $M_\theta(2n, \mathbb{R})$ to be the unital associative $\mathbb{C}$-algebra generated by $4n^2$ elements $a^\mu_\nu, b^\mu_\nu, \bar{a}^\mu_\nu, \bar{b}^\mu_\nu$ ($\mu, \nu = 1, \ldots, n$) with relations such that the elements $y^\mu, \bar{y}^\mu, \zeta^\mu, \bar{\zeta}^\mu$ of $M_\theta(2n, \mathbb{R}) \otimes \Omega_{\text{alg}}(\mathbb{R}_\theta^{2n})$ defined by

$$y^\mu = a^\mu_\nu \otimes z^\nu + b^\mu_\nu \otimes \zbar^\nu, \quad \bar{y}^\mu = \bar{a}^\mu_\nu \otimes \zbar^\nu + \bar{b}^\mu_\nu \otimes z^\nu,$$

$$\zeta^\mu = a^\mu_\nu \otimes dz^\nu + b^\mu_\nu \otimes d\zbar, \quad \bar{\zeta}^\mu = \bar{a}^\mu_\nu \otimes d\zbar + \bar{b}^\mu_\nu \otimes dz^\nu$$

satisfy the relation

$$y^\mu y^\nu = \lambda^{\mu\nu} y^\nu y^\mu, \quad \bar{y}^\mu \bar{y}^\nu = \lambda^{\mu\nu} \bar{y}^\nu \bar{y}^\mu, \quad \bar{\zeta}^\mu \zeta^\nu + \lambda^{\mu\nu} \bar{\zeta}^\nu \zeta^\mu = 0, \quad \bar{\zeta}^\mu \bar{\zeta}^\nu + \lambda^{\mu\nu} \bar{\zeta}^\nu \bar{\zeta}^\mu = 0.$$

There is a unique $\ast$-algebra involution $a \mapsto a^\ast$ on $M_\theta(2n, \mathbb{R})$ such that $(a^\mu_\nu)^\ast = \bar{a}^\mu_\nu, (b^\mu_\nu)^\ast = \bar{b}^\mu_\nu$. The relations between the generators are easy to
write explicitly, they read

\begin{align*}
a^\mu_\nu a^\rho_\sigma &= \lambda^{\mu\nu} \lambda_{\rho\sigma} a^\mu_\nu a^\rho_\sigma \quad (7.1) \\
a^\mu_\nu \bar{a}^\rho_\sigma &= \lambda^{\mu\nu} \lambda_{\rho\sigma} a^\mu_\nu \bar{a}^\rho_\sigma \quad (7.2) \\
a^\mu_\nu b^\rho_\sigma &= \lambda^{\mu\nu} \lambda_{\rho\sigma} b^\mu_\nu a^\rho_\sigma \quad (7.3) \\
\bar{a}^\mu_\nu b^\rho_\sigma &= \lambda^{\mu\nu} \lambda_{\rho\sigma} \bar{a}^\mu_\nu a^\rho_\sigma \quad (7.4) \\
b^\mu_\nu b^\rho_\sigma &= \lambda^{\mu\nu} \lambda_{\rho\sigma} b^\mu_\nu b^\rho_\sigma \quad (7.5) \\
b^\mu_\nu \bar{b}^\rho_\sigma &= \lambda^{\mu\nu} \lambda_{\rho\sigma} \bar{b}^\mu_\nu b^\rho_\sigma \quad (7.6)
\end{align*}

plus the relations obtained by hermitian conjugation, where we have also used the notation \(\lambda_{\nu\rho}\) for \(\lambda^{\nu\rho}\) to indicate that there is no summation in the above formulas. This \(*\)-algebra becomes a \(*\)-bialgebra with coproduct \(\Delta\) and counit \(\varepsilon\) if we equip it with the unique algebra-homomorphism

\[\Delta : M_\theta(2n, \mathbb{R}) \to M_\theta(2n, \mathbb{R}) \otimes M_\theta(2n, \mathbb{R})\]

and the unique character \(\varepsilon : M_\theta(2n, \mathbb{R}) \to \mathbb{C}\) such that

\begin{align*}
\Delta a^\mu_\nu &= a^\mu_\nu \otimes a^\lambda_\nu + b^\mu_\nu \otimes \bar{b}^\lambda_\nu, \quad \varepsilon(a^\mu_\nu) = \delta^\mu_\nu \quad (7.7) \\
\Delta \bar{a}^\mu_\nu &= \bar{a}^\mu_\nu \otimes \bar{a}^\lambda_\nu + b^\mu_\nu \otimes b^\lambda_\nu, \quad \varepsilon(\bar{a}^\mu_\nu) = \delta^\mu_\nu \quad (7.8) \\
\Delta b^\mu_\nu &= a^\mu_\nu \otimes b^\lambda_\nu + b^\mu_\nu \otimes \bar{a}^\lambda_\nu, \quad \varepsilon(b^\mu_\nu) = 0 \quad (7.9) \\
\Delta \bar{b}^\mu_\nu &= \bar{a}^\mu_\nu \otimes \bar{b}^\lambda_\nu + \bar{b}^\mu_\nu \otimes a^\lambda_\nu, \quad \varepsilon(\bar{b}^\mu_\nu) = 0 \quad (7.10)
\end{align*}

for any \(\mu, \nu \in \{1, \ldots, n\}\). It is easy to verify that there is a unique algebra-homomorphism \(\delta : \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \to M_\theta(2n, \mathbb{R}) \otimes \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)\) such that \(\delta z^\mu = y^\mu, \delta \bar{z}^\mu = \bar{y}^\mu, \delta dz^\mu = \zeta^\mu, \delta d\bar{z}^\mu = \bar{\zeta}^\mu\) and that this is furthermore a graded-involutive algebra-homomorphism. In fact, this is another way to define \(\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)\) starting from \(C_{\text{alg}}(\mathbb{R}^{2n}_\theta)\) and from the \(\theta\)-twisted complexified exterior algebra \(\wedge_\theta \mathbb{R}^{2n}_\theta\) generated by the \(dz^\mu, d\bar{z}^\nu\) satisfying (6.1), (6.2), (6.3).
One has

\[(\Delta \otimes I) \circ \delta = (I \otimes \delta) \circ \delta, \quad (\varepsilon \otimes I) \circ \delta = I\]

and \(\delta \Omega^p_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) \subset M_\theta(2n, \mathbb{R}) \otimes \Omega^p_{\text{alg}}(\mathbb{R}_{\theta}^{2n}), \forall p \in \mathbb{N}\). One has of course \(\delta C_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) \subset M_\theta(2n, \mathbb{R}) \otimes C_{\text{alg}}(\mathbb{R}_{\theta}^{2n})\), (this is the previous result for \(p = 0\) since \(C_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) = \Omega^0_{\text{alg}}(\mathbb{R}_{\theta}^{2n})\)), and one also has \(\delta \wedge_c \mathbb{R}_{\theta}^{2n} \subset M_\theta(2n, \mathbb{R}) \otimes \wedge_c \mathbb{R}_{\theta}^{2n}\) with \(\delta \wedge_c \mathbb{R}_{\theta}^{2n} \subset M_\theta(2n, \mathbb{R}) \otimes \wedge_c \mathbb{R}_{\theta}^{2n}\) for any \(p \in \mathbb{N}\). Since \(\wedge_c \mathbb{R}_{\theta}^{2n}\) is of dimension 1 and spanned by \(d\bar{z}^1dz^1 \ldots d\bar{z}^ndz^n = \prod_{\mu=1}^{n}d\bar{z}^\mu dz^\mu\), it follows that one defines an element \(\text{det}_\theta \in M_\theta(2n, \mathbb{R})\) by setting

\[
\delta \prod_{\mu=1}^{n}d\bar{z}^\mu dz^\mu = \text{det}_\theta \otimes \prod_{\mu=1}^{n}d\bar{z}^\mu dz^\mu
\]

which satisfies

\[
\Delta \text{det}_\theta = \text{det}_\theta \otimes \text{det}_\theta
\]

\[
\varepsilon(\text{det}_\theta) = 1
\]

and from the fact that \(\prod_{\mu=1}^{n}d\bar{z}^\mu dz^\mu\) is central in \(\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n})\) and from the very definition of \(M_\theta(2n, \mathbb{R})\) it also follows that \(\text{det}_\theta\) belongs to the center of \(M_\theta(2n, \mathbb{R})\). The element \(\text{det}_\theta\) of \(M_\theta(2n, \mathbb{R})\) is clearly hermitian, \((\text{det}_\theta)^* = \text{det}_\theta\).

Let \(L = (L^r_s) = \left(\begin{array}{cc} A & B \\ \overline{B} & A \end{array}\right)\) be the \((2n, 2n)\) matrix defined by \(A = (a^\mu_\nu)\), \(B = (b^\mu_\nu)\), etc., that is by \(L^r_\nu = a^r_\nu, L^r_\nu = b^r_\nu, L^\overline{r}_\nu = \overline{b}^r_\nu\) and \(L^{\overline{r}}_\overline{\nu} = \overline{a}^\nu_\nu\) where \(\overline{\mu} = \mu + n\) for \(\mu \in \{1, \ldots, n\}\). Then Equations (7.7), (7.8), (7.9), (7.10) read

\[
\Delta L^r_s = L^r_p \otimes L^p_s, \quad \varepsilon(L^r_s) = \delta^r_s
\]

whereas the relations (7.1), (7.2), (7.3), (7.4), (7.5), (7.6) and their hermitian conjugate read

\[
L^r_s L^s_t, \quad \overline{R}^{r'_{pq}} = \overline{R}^{r'_{pq}} L^p_q L^q_{r'}
\]
where the “$R$-matrix” $\hat{R}$ is defined by setting

\[ \hat{R}^{\mu\nu}_{\gamma\rho} = \hat{R}^{\bar{\nu}\bar{\rho}}_{\bar{\gamma}\bar{\rho}} = \lambda^{\mu\nu} \delta_{\rho}^{\gamma} \delta_{\tau}^{\rho} \]  
\[ \hat{R}^{\rho\mu}_{\gamma\rho} = \hat{R}^{\rho\mu}_{\gamma\rho} = \lambda^{\mu\rho} \delta_{\rho}^{\gamma} \delta_{\tau}^{\rho} \]  

for $\mu, \nu, \tau, \rho \in \{1, \ldots, n\}$ and by setting the other components equal to zero, (e.g. $\hat{R}^{\mu\nu}_{\gamma\rho} = 0$, $\hat{R}^{\mu\nu}_{\gamma\rho} = 0$, etc.). The $R$-matrix satisfies the Yang-Baxter relation (in the braid form)

\[ (\hat{R} \otimes I)(I \otimes \hat{R})(\hat{R} \otimes I) = (I \otimes \hat{R})(\hat{R} \otimes I)(I \otimes \hat{R}) \]  

where $\hat{R} \otimes I$ and $I \otimes \hat{R}$ denote as usual the corresponding endomorphisms of $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. Furthermore one has

\[ (\hat{R})^2 = I \otimes I \]  

as endomorphism of $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. In other words, the bialgebra $M_{\theta}(2n, \mathbb{R})$ is the bialgebra of the $R$-matrix $\hat{R}$. Notice that (7.19) and (7.20) means that for any $N \in \mathbb{N}$ one defines a representation of the group of permutations $S_N$ in $\otimes^N \mathbb{C}^{2n}$ using $\hat{R}$ as generalized transposition of two consecutive factors.

**Remark.** The $R$-matrix $\hat{R}$ is generally not real (i.e. has nontrivial imaginary elements). Indeed, coming back to a real basis for $\mathbb{R}^{2n}$, one sees that $\hat{R}$ is real if and only if $\hat{\lambda}^{\mu\nu} = \lambda^{\mu\nu}$ which means $\lambda^{\mu\nu} = \pm 1$ for $1 \leq \mu < \nu \leq n$, ($\lambda^{\mu\nu} = 1$ is the classical situation for which $\hat{R}$ is the usual transposition of factors in tensor product).

Let $C_\text{alg}(GL_\theta(2n, \mathbb{R}))$ be the $*$-bialgebra obtained by adding to $M_{\theta}(2n, \mathbb{R})$ a hermitian central element $\det_{\theta}^{-1}$ with relation $\det_{\theta} \cdot \det_{\theta}^{-1} = 1 = \det_{\theta}^{-1} \cdot \det_{\theta}$ and by setting $\Delta \det_{\theta}^{-1} = \det_{\theta}^{-1} \otimes \det_{\theta}^{-1}$ and $\varepsilon(\det_{\theta}^{-1}) = 1$. It is not hard
(but cumbersome) to see that the introduction of $\det^{-1}_\theta$ allows to invert the $(2n,2n)$ matrix $L = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ in $M_{2n}(\mathcal{C}_{\text{alg}}(GL_\theta(2n, \mathbb{R})))$ and to define an antipode $S$ on $\mathcal{C}_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ which of course satisfies $S(\det_\theta) = \det_\theta^{-1}$ and $S(\det_\theta^{-1}) = \det_\theta$. Thus $\mathcal{C}_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ is a *-Hopf algebra and the quantum group $GL_\theta(2n, \mathbb{R})$ is defined to be the dual object.

The quotient $\mathcal{C}_{\text{alg}}(SL_\theta(2n, \mathbb{R}))$ of $M_\theta(2n, \mathbb{R})$ by the relation $\det_\theta = 1$ is also the quotient of $\mathcal{C}_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ by the two-sided ideal generated by $\det_\theta - 1$ and $\det_\theta^{-1} - 1$ which is a *-Hopf ideal. So $\mathcal{C}_{\text{alg}}(SL_\theta(2n, \mathbb{R}))$ is again a *-Hopf algebra which defines the quantum group $SL_\theta(2n, \mathbb{R})$ by duality.

Replacing $\Omega_{\text{alg}}(\mathbb{R}^{2n})$ by $\Omega_{\text{alg}}(\mathbb{R}^{2n+1})$ one defines similarly the *-bialgebra $M_\theta(2n+1, \mathbb{R})$, the *-Hopf algebras $\mathcal{C}_{\text{alg}}(GL_\theta(2n+1, \mathbb{R}))$, $\mathcal{C}_{\text{alg}}(SL_\theta(2n+1, \mathbb{R}))$ and therefore the quantum groups $GL_\theta(2n+1, \mathbb{R})$ and $SL_\theta(2n+1, \mathbb{R})$. It is easy to write down the expression of the $R$-matrix satisfying (7.19) and (7.20) defining $M_\theta(2n+1, \mathbb{R})$, (as above for $M_\theta(2n, \mathbb{R})$).

Finally, one defines $\mathcal{C}_{\text{alg}}(GL_\theta(n, \mathbb{C}))$ to be the quotient of $\mathcal{C}_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ by the ideal generated by the $b_\nu^\mu$ and the $\tilde{b}_\nu^\mu$ which is a *-Hopf ideal. The coaction of the corresponding Hopf algebra on $\Omega_{\text{alg}}(\mathbb{C}^n_\theta)$ is straightforwardly defined. This defines the quantum group $GL_\theta(n, \mathbb{C})$ and its action on $\mathbb{C}^n_\theta$. The ideal generated by the image of $\det_\theta - 1$ in $\mathcal{C}_{\text{alg}}(GL_\theta(n, \mathbb{C}))$ is a *-Hopf ideal and the corresponding quotient Hopf algebra defines by duality a quantum group which is a deformation ($\theta$-deformation) of the subgroup of $GL(n, \mathbb{C})$ which consists of matrices with determinants of modulus one.
8 The quantum groups $O_\theta(m), SO_\theta(m)$ and $U_\theta(n)$

Define $C_{\text{alg}}(O_\theta(2n))$ to be the quotient of $M_\theta(2n, \mathbb{R})$ by the two-sided ideal generated by

$$
\sum_{\mu=1}^{n}(\bar{a}_\alpha^\mu a_\beta^\mu + b_\alpha^\mu \bar{b}_\beta^\mu) - \delta_{\alpha\beta} \mathbb{1}
$$

$$
\sum_{\mu=1}^{n}(\bar{a}_\alpha^\mu b_\beta^\mu + b_\alpha^\mu \bar{a}_\beta^\mu)
$$

$$
\sum_{\mu=1}^{n}(\bar{b}_\alpha^\mu a_\beta^\mu + a_\alpha^\mu \bar{b}_\beta^\mu)
$$

for $\alpha, \beta = 1, \ldots, n$. This ideal is $\ast$-invariant and is also a coideal. It follows that $C_{\text{alg}}(O_\theta(2n))$ is again a $\ast$-bialgebra. Furthermore, one can show that $(\det_\theta)^2 - \mathbb{1}$ is in the above ideal (see below) so $C_{\text{alg}}(O_\theta(2n))$ is a $\ast$-Hopf algebra which is a quotient of $C_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$. One verifies that $\delta : \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \to M_\theta(2n, \mathbb{R}) \otimes \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ passes to the quotient to give a homomorphism

$$
\delta_R : \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta) \to C_{\text{alg}}(O_\theta(2n)) \otimes \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)
$$

of graded-involutive algebras. This defines the quantum group $O_\theta(2n)$ which is a deformation of the group of rotations in dimension $2n$ and its action on $\mathbb{R}^{2n}_\theta$. Indeed one has

$$
\delta_R(\sum_{\mu=1}^{n} \bar{z}_\mu z_\mu^\mu) = \mathbb{1} \otimes (\sum_{\mu=1}^{n} \bar{z}_\mu z_\mu^\mu)
$$

by the very definition of $C_{\text{alg}}(O_\theta(2n))$. One can notice here that $C_{\text{alg}}(O_\theta(2n))$ is a quotient of the Hopf algebra of the quantum group of the non-degenerate bilinear form $B$ on $\mathbb{C}^{2n}$ with matrix $\left( \begin{array}{cc} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{array} \right)$ defined in [23], the later bilinear form is equivalent to the metric of $\mathbb{R}^{2n}$, (the involution being defined accordingly). The coaction $\delta_R$ passes to the quotient to define the coaction

$$
\delta_R : \Omega_{\text{alg}}(S^{2n-1}_\theta) \to C_{\text{alg}}(O_\theta(2n)) \otimes \Omega_{\text{alg}}(S^{2n-1}_\theta)
$$

which is also a homomorphism of graded-involutive algebras. By taking a further quotient by the relation $\det_\theta = \mathbb{1}$, one obtains the $\ast$-Hopf algebra
$C_{\text{alg}}(SO_\theta(2n))$ defining the quantum group $SO_\theta(2n)$. Let $\rho : M_\theta(2n, \mathbb{R}) \to C_{\text{alg}}(O_\theta(2n))$ be the canonical projection. The algebra $C_{\text{alg}}(O_\theta(2n))$ is the unital *-algebra generated by the $4n^2$ elements $\rho(a^\mu_\alpha), \rho(b^\mu_\alpha), \rho(\bar{a}^\mu_\alpha), \rho(\bar{b}^\mu_\alpha)$ with relations induced by (7.1), (7.2), (7.3), (7.4), (7.5), (7.6) and the relations

$$\sum_{\mu} (\rho(\bar{a}^\mu_\alpha)\rho(a^\mu_\beta) + \rho(b^\mu_\alpha)\rho(\bar{b}^\mu_\beta)) = \delta_{\alpha\beta} 1 \quad (8.1)$$

$$\sum_{\mu} (\rho(\bar{a}^\mu_\alpha)\rho(b^\mu_\beta) + \rho(b^\mu_\alpha)\rho(\bar{a}^\mu_\beta)) = 0 \quad (8.2)$$

(for $\alpha, \beta = 1, \ldots, n$), together with $\rho(\bar{a}^\mu_\alpha) = \rho(a^\mu_\alpha)^*$ and $\rho(\bar{b}^\mu_\alpha) = \rho(b^\mu_\alpha)^*$. It follows from (8.1) that, for any $C^*$-semi-norm $\nu$ on $C_{\text{alg}}(O_\theta(2n))$ one has

$$\nu(a^\mu_\alpha) = \nu(\bar{a}^\mu_\alpha) \leq 1 \quad \text{and} \quad \nu(b^\mu_\alpha) = \nu(\bar{b}^\mu_\alpha) \leq 1$$

so that there is a greatest $C^*$-semi-norm on $C_{\text{alg}}(O_\theta(2n))$ which is a norm and the corresponding completion $C(O_\theta(2n))$ of $C_{\text{alg}}(O_\theta(2n))$ is a $C^*$-algebra. This defines $O_\theta(2n)$ as a compact matrix quantum group [51]. The same applies to $SO_\theta(2n)$ which is therefore also a compact matrix quantum group.

One proceeds similarly (with obvious modifications) to define the quantum groups $O_\theta(2n + 1)$ and $SO_\theta(2n + 1)$ which are again compact matrix quantum groups. One has also the coaction

$$\delta_R : \Omega_{\text{alg}}(\mathbb{R}^{2n+1}_\theta) \to C_{\text{alg}}(O_\theta(2n + 1)) \otimes \Omega_{\text{alg}}(\mathbb{R}^{2n+1}_\theta)$$

which passes to the quotient to define the coaction

$$\delta_R : \Omega_{\text{alg}}(S^{2n}_\theta) \to C_{\text{alg}}(O_\theta(2n + 1)) \otimes \Omega_{\text{alg}}(S^{2n}_\theta)$$

these coactions are homomorphisms of graded-involutive algebras. This defines the action of the quantum group $O_\theta(2n + 1)$ on the noncommutative 2n-sphere $S^{2n}_\theta$. One defines similarly the action of $SO_\theta(2n)$ on $S^{2n-1}_\theta$ and of
Finally one defines $C_{\text{alg}}(U_{\theta}(n))$ to be the quotient of $C_{\text{alg}}(O_{\theta}(2n))$ by the ideal generated by the $\rho(b^\mu_{\nu})$ and $\rho(b^\mu_{\nu'})$ which is also a $\ast$-Hopf ideal. The coactions $\delta_R$ of $C_{\text{alg}}(O_{\theta}(2n))$ on $\Omega_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) = \Omega_{\text{alg}}(\mathbb{C}_{\theta}^{n})$ and on $C_{\text{alg}}(S_{\theta}^{2n-1})$ pass to quotient to define corresponding coactions of $C_{\text{alg}}(U_{\theta}(n))$.

Again there is no $\theta$-deformation of $SU(n)$.

Let us denote by $z_{\mu}, z_{\nu} = z_{\nu}^*$ the generators of $C_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) = C_{\text{alg}}(\mathbb{C}_{\theta}^{n})$ satisfying $z_{\mu}z_{\nu} = \lambda_{\mu\nu}z_{\nu}z_{\mu}$ and $\tilde{z}_{\mu}z_{\nu} = \lambda_{\mu\nu}\tilde{z}_{\nu}\tilde{z}_{\mu}$. One verifies that one defines a unique $\ast$-homomorphism $\varphi$ of $M_{\theta}(2n, \mathbb{R})$ into $C_{\text{alg}}'(\mathbb{R}_{\theta}^{2n}) \otimes C_{\text{alg}}(\mathbb{C}_{\theta}^{n})$ by setting $\varphi(a^\mu_{\nu}) = z^\mu \otimes z_{\nu}$ and $\varphi(b^\mu_{\nu}) = z^\mu \otimes z_{\nu}^*$. This homomorphism is injective and its image is invariant by the action $\sigma \otimes \sigma$ of $T^n \times T^n$ on $C_{\text{alg}}(\mathbb{R}_{\theta}^{2n}) \otimes C_{\text{alg}}(\mathbb{C}_{\theta}^{n})$. We shall again denote by $\sigma \otimes \sigma$ the corresponding action of $T^n \times T^n$ on $M_{\theta}(2n, \mathbb{R})$, i.e. the group-homomorphism of $T^n \times T^n$ into $\text{Aut}(M_{\theta}(2n, \mathbb{R}))$, e.g. one writes $\sigma_s \otimes \sigma_t(a^\mu_{\nu}) = e^{2\pi i(s_{\mu}+t_{\nu})}a^\mu_{\nu}$, $\sigma_s \otimes \sigma_t(b^\mu_{\nu}) = e^{2\pi i(s_{\mu}-t_{\nu})}b^\mu_{\nu}$, etc. This induces a group-homomorphism (also denoted by $\sigma \otimes \sigma$) of $T^n \times T^n$ into the group of automorphisms of unital $\ast$-algebras (not necessarily preserving the coalgebra structure) of the polynomial algebra $C_{\text{alg}}$ on each of the quantum groups defined in this section and in Section 7. In each case, the subalgebra of $\sigma \otimes \sigma$-invariant elements is in the center and is undeformed, that is isomorphic to the corresponding subalgebra for $\theta = 0$.

9 The graded differential algebras $\Omega_{\text{alg}}(G_{\theta})$ as graded differential Hopf algebras

The relations (7.1) to (7.6) define the $\ast$-algebra $M_{\theta}(2n, \mathbb{R})$ as $C_{\text{alg}}(\mathbb{R}_{\theta}^{2N})$ with $N = 2n^2$ and where $\Theta \in M_N(\mathbb{R})$ is the appropriate antisymmetric matrix
(which depends on $\theta \in M_n(\mathbb{R})$). Let $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$ be the corresponding graded-involutive differential algebra as in Section 6.

**PROPOSITION 2** The coproduct $\Delta$ of $M_\theta(2n, \mathbb{R})$ has a unique extension as homomorphism of graded differential algebras, again denoted by $\Delta$, of $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$ into $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\text{gr}} \Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$. The counit $\varepsilon$ of $M_\theta(2n, \mathbb{R})$ has a unique extension as algebra-homomorphism, again denoted by $\varepsilon$, of $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$ into $\mathbb{C}$ with $\varepsilon \circ d = 0$. The coaction $\delta : C_{\text{alg}}(\mathbb{R}^{2n}_\theta) \to M_\theta(2n, \mathbb{R}) \otimes C_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ has a unique extension as homomorphism of graded differential algebras, again denoted by $\delta$, of $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\Theta)$ into $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\text{gr}} \Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$. The extended $\Delta$ is coassociative and the extended $\varepsilon$ is a counit for it and one has $(\Delta \otimes I) \circ \delta = (I \otimes \delta) \circ \delta$, $(\varepsilon \otimes I) \circ \delta = I$. These extended homomorphisms are real.

In this proposition, $N = 2n^2$ and $\Theta$ are as explained above and one equips $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\text{gr}} \Omega(\mathbb{R}^{2N}_\Theta)$ of the involution $\omega' \otimes \omega' \mapsto \overline{\omega'} \otimes \overline{\omega''} = \omega' \otimes \overline{\omega''}$. So equipped $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta) \otimes_{\text{gr}} \Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$ is a graded-involutive differential algebra and the reality of $\Delta$ means $\overline{\Delta(\omega)} = \Delta(\overline{\omega})$. The uniqueness in the proposition is obvious and the only thing to verify is the compatibility of the extension with the relations $a^\rho_\nu d_{\mu}^\tau + \lambda^\nu_\rho a^\tau_{\mu} d_{\mu}^\nu = 0, \ldots, a^\rho_\nu d_{\mu}^\tau = \lambda^\nu_\rho a^\tau_\mu d_{\mu}^\nu, \ldots$, etc. which is easy. One proceeds similarly for $\delta$. In short, $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$ is a graded-involutive differential bialgebra and $\Omega_{\text{alg}}(\mathbb{R}^{2n}_\theta)$ is a graded-involutive differential comodule over $\Omega_{\text{alg}}(\mathbb{R}^{2N}_\Theta)$. Notice that to say that $\Delta$ is a homomorphism of graded differential algebras means that $\Delta$ is a homomorphism of graded algebras and that one has the graded co-Leibniz rule $\Delta \circ d = (d \otimes I + (-I)^{\text{gr}} \otimes d) \circ \Delta$.

Let us define a graded differential Hopf algebra to be a graded differential bialgebra which admits an antipode; the antipode $S$ is then necessarily unique.
and satisfies $S \circ d = d \circ S$. The notion of graded-involutive differential Hopf algebra is clear. By adding $\det^{-1}_\theta$ to $M_\theta(2n, \mathbb{R}) = \Omega^0_{\text{alg}}(\mathbb{R}^N)$ as in Section 7 to obtain the Hopf algebra $C_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ and by setting
\[
[\det^{-1}_\theta, \omega] = 0, \quad \forall \omega \in \Omega_{\text{alg}}(\mathbb{R}^N)
\]
\[
d(\det^{-1}_\theta) = -(\det^{-1}_\theta)^2 d(\det_\theta)
\]
one defines the graded-involutive differential algebra $\Omega_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ (writing $\Omega^0_{\text{alg}}(GL_\theta(2n, \mathbb{R})) = C_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$, etc.) which is naturally a graded-involutive differential bialgebra and it is easy to show that the antipode $S$ of $C_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$ extends (uniquely) as an antipode, again denoted by $S$, of $\Omega_{\text{alg}}(GL_\theta(2n, \mathbb{R}))$. One proceeds similarly to define $\Omega_{\text{alg}}(GL_\theta(2n + 1, \mathbb{R}))$.

One has thus the following result.

**THEOREM 4** Let $m$ be either $2n$ or $2n + 1$. Then the differential algebra $\Omega_{\text{alg}}(GL_\theta(m, \mathbb{R}))$ is a graded-involutive differential Hopf algebra and $\Omega_{\text{alg}}(\mathbb{R}^m_{\theta})$ is canonically a graded-involutive differential comodule over $\Omega_{\text{alg}}(GL_\theta(m, \mathbb{R}))$.

Let $G_\theta$ be any of the quantum groups defined in Sections 7 and 8. Then $C_{\text{alg}}(G_\theta)$ is a $*$-Hopf algebra which is a quotient of $C_{\text{alg}}(GL_\theta(m, \mathbb{R}))$ by a real Hopf ideal $I(G_\theta)$ for $m = 2n$ or $m = 2n + 1$. Let $[I(G_\theta)]$ be the closed graded two-sided ideal of $\Omega_{\text{alg}}(GL_\theta(m, \mathbb{R}))$ generated by $I(G_\theta)$ and let $\Omega_{\text{alg}}(G_\theta)$ be the quotient of $\Omega_{\text{alg}}(GL_\theta(m, \mathbb{R}))$ by $[I(G_\theta)]$. The above result has the following corollary.

**COROLLARY 1** The differential algebra $\Omega_{\text{alg}}(G_\theta)$ is a graded-involutive differential Hopf algebra and $\Omega_{\text{alg}}(\mathbb{R}^m_{\theta})$ is graded-involutive differential comodule over $\Omega_{\text{alg}}(G_\theta)$.

Similarly the algebra $\Omega_{\text{alg}}(SO_\theta(m))$ is a graded-involutive differential comodule over $\Omega_{\text{alg}}(SO_\theta(m + 1))$. Notice also that for $GL_\theta(n, \mathbb{C})$, $m = 2n$ and it
is better to use the notation $\Omega_{\text{alg}}(C^n_\theta)$ instead of $\Omega_{\text{alg}}(R^n_2 \mu)$ which is the same thing for the corresponding graded-involutive differential comodule.

**Remark.** By a slight abuse of notations we used the same symbol $\delta$ for the comodule structure of $\Omega_{\text{alg}}(R_m^m \mu)$ over $\text{alg}(GL_\mu(m, R))$ in Section 7 and here for the differential comodule structure of $\Omega_{\text{alg}}(R_m^m \mu)$ over $\Omega_{\text{alg}}(GL_\mu(m, R))$.

One passes from the second $\delta$ to the first one by composition with $\pi^0 \otimes I$ where $\pi^0$ is the canonical projection $\pi^0 : \Omega_{\text{alg}}(GL_\mu(m, R)) \to \Omega^0_{\text{alg}}(GL_\mu(m, R)) = C_{\text{alg}}(GL_\mu(m, R))$.

### 10 The splitting homomorphisms

We let $C_{\text{alg}}(T^n_\theta)$ be the $*$-algebra of polynomials on the noncommutative $n$-torus $T^n_\theta$ i.e. the unital $*$-algebra generated by $n$ unitary elements $U^\mu$ with relations

$$U^\mu U^\nu = \chi^\mu{\nu}U^\nu U^\mu$$

for $\mu, \nu = 1, \ldots, n$. We denote by $s \mapsto \tau_s \in \text{Aut}(C_{\text{alg}}(T^n_\theta))$ the natural action of $T^n$ on $T^n_\theta$ ([7]) such that $\tau_s(U^\mu) = e^{2\pi i s_\mu}U^\mu \forall s \in T^n$ and $\mu \in \{1, \ldots, n\}$.

We let as in Section 4, $s \mapsto \sigma_s \in \text{Aut}(C_{\text{alg}}(R^{2n}_\theta))$ be the natural action of $T^n$ on $C_{\text{alg}}(R^{2n}_\theta)$. It is defined for any $\theta$ (real antisymmetric $(n, n)$-matrix) and in particular for $\theta = 0$. This yields two actions $\sigma$ and $\tau$ of $T^n$ on $R^{2n} \times T^n_\theta$ given by the group-homomorphisms $s \mapsto \sigma_s \otimes I$ and $s \mapsto I \otimes \tau_s$ of $T^n$ into $\text{Aut}(C_{\text{alg}}(R^{2n}) \otimes C_{\text{alg}}(T^n_\theta))$ with obvious notations.

The noncommutative space $R^{2n} \times T^n_\theta$ is here defined by duality by writing $C_{\text{alg}}(R^{2n} \times T^n_\theta) = C_{\text{alg}}(R^{2n}) \otimes C_{\text{alg}}(T^n_\theta)$. We shall use the actions $\sigma$ and the diagonal action $\sigma \times \tau^{-1}$ of $T^n$ on $R^{2n} \times T^n_\theta$, where $\sigma \times \tau^{-1}$ is defined by $s \mapsto \sigma_s \otimes \tau_{-s} = (\sigma \times \tau^{-1})_s$ (as group homomorphism of $T^n$ into
Aut(Calg(R2n × Tnθ))).

In the following statement, zμ(0) denotes the classical coordinates of Cn corresponding to zμ for θ = 0.

**THEOREM 5** a) There is a unique homomorphism of unital ∗-algebra

\[ st : C_{\text{alg}}(R_2) \to C_{\text{alg}}(R_2) \otimes C_{\text{alg}}(T_n) \]

such that \( st(z^μ) = z^{μ(0)} \otimes U^μ \) for \( μ = 1, \ldots, n \).

b) The homomorphism \( st \) induces an isomorphism of \( C_{\text{alg}}(R_2) \) onto the sub-algebra \( C_{\text{alg}}(R_2n × T_θ)^{σ×τ−1} \) of \( C_{\text{alg}}(R_2n × T_θ) \) of fixed points of the diagonal action of \( T^n \).

One has \( st(\tilde{z}^μ) = st(z^μ)^* \) and, using (10.1), one checks that \( st(z^μ) \), \( st(\tilde{z}^μ) \) fulfill the relations (4.1), (4.2), (4.3). On the other hand, it is obvious that the \( st(z^μ) \) are invariant by the diagonal action of \( T^n \). Thus the only non-trivial parts of the statement, which are not difficult to show, are the injectivity of \( st \) and the fact that \( C_{\text{alg}}(R_2n × T_θ)^{σ×τ−1} \) is generated by the \( z^μ \) as unital ∗-algebra.

This extends trivially to

\[ st : C_{\text{alg}}(R_2^{2n+1}) \to C_{\text{alg}}(R_2^{2n+1}) \otimes C_{\text{alg}}(T_θ) = C_{\text{alg}}(R_2^{2n+1} × T_θ) \]

with \( st(x) = x_{(0)} \otimes 1 \) and \( st(z^μ) = z^{μ(0)} \otimes U^μ \). This is again an isomorphism of \( C_{\text{alg}}(R_2^{2n+1}) \) onto \( C_{\text{alg}}(R_2^{2n+1} × T_θ)^{σ×τ−1} \).

The above homomorphisms \( st \) pass to the quotient to define homomorphisms of unital ∗-algebras \( (m = 2n, 2n + 1) \)

\[ st : C_{\text{alg}}(S_θ^m) \to C_{\text{alg}}(S^m) \otimes C_{\text{alg}}(T_θ^n) = C_{\text{alg}}(S^m × T_θ^n) \]

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which are isomorphisms of $C_{\text{alg}}(S^m_\theta)$ with $C_{\text{alg}}(S^m \times T^n_\theta)^{\sigma \times \tau^{-1}}$, the fixed points of the diagonal action $\sigma \times \tau^{-1}$ of $T^n$ (recall that $\sigma$ was previously defined for any $\theta$, in particular for $\theta = 0$).

We shall refer to the above homomorphisms $st$ as the splitting homomorphisms. They satisfy $st \circ \sigma_s = (\sigma_s \otimes I) \circ st$ for any $s = (s_1, \ldots, s_n) \in T^n$ and thus $st$ induce isomorphisms

$$st : C_{\text{alg}}(M_\theta)^\sigma \cong C_{\text{alg}}(M)^\sigma \otimes \mathbb{1}(\subset C_{\text{alg}}(M) \otimes C_{\text{alg}}(T^n_\theta))$$

for $M = \mathbb{R}^m$ and $S^m$.

In a similar manner, with $M$ as above, $st$ extends to isomorphisms of unital graded-involutive differential algebras

$$st : \Omega_{\text{alg}}(M_\theta) \rightarrow (\Omega_{\text{alg}}(M) \otimes C_{\text{alg}}(T^n_\theta))^{\sigma \times \tau^{-1}}$$

by setting

$$st(dz^n) = dz^n(0) \otimes U^n \text{ and } st(dx) = dx(0) \otimes \mathbb{1}$$

using the previously defined action $\sigma$ of $T^n$ on $\Omega_{\text{alg}}(M_\theta)$ for any $\theta$ (in particular $\theta = 0$).

The compatibility with the differential and the action of $T^n$ is explicitly given by,

$$st \circ d = (d \otimes I) \circ st \quad (10.2)$$

$$st \circ \sigma_s = (\sigma_s \otimes I) \circ st \quad (10.3)$$

Remark. The splitting homomorphisms $st$ do implicitly depend on the antisymmetric matrix $\theta$. Their construction easily extends replacing the value
\( \theta = 0 \) in the left hand side of the equality by another antisymmetric matrix \( \theta_0 \in M_\theta(\mathbb{R}) \). This yields an homomorphism

\[
\text{st}_{\theta, \theta_0} : C_{\text{alg}}(M_\theta) \to C_{\text{alg}}(M_{\theta_0}) \otimes C_{\text{alg}}(T^n_{\theta - \theta_0})
\]

We shall use the splitting homomorphisms \( \text{st} \) to reduce computations involving \( \theta \)-deformations to the classical case (\( \theta = 0 \)). For instance we shall later define the Dirac operator, \( D_\theta \), on \( M_\theta \) in such a way that satisfies with obvious notations \( \text{st} \circ \text{ad}(D_\theta) = (\text{ad}(D) \otimes I) \circ \text{st} \) where on the right-hand side \( D \) is the ordinary Dirac operator on the riemannian spin manifold \( M \), \( (M = \mathbb{R}^{2n}, \mathbb{R}^{2n+1}, S^{2n-1}, S^{2n}) \); this will imply the first order condition, the reality condition and the identification of the differential algebra \( \Omega_D \) with \( \Omega_{\text{alg}}(M_\theta) \), (see Section 13).

A similar discussion applies to the various \( \theta \)-deformed groups mentioned above. To be specific, we introduce the \( n \) unitary elements \( U_\mu \) with relations

\[
U_\mu U_\nu = \lambda_{\nu\mu} U_\nu U_\mu
\]

for \( \mu, \nu = 1, \ldots, n \), (recall that \( \lambda_{\mu\nu} = e^{i\theta_{\mu\nu}} = \chi_{\mu\nu}, \forall \mu, \nu \)) which generate \( C_{\text{alg}}(T^n_{\theta}) \), the opposite algebra of \( C_{\text{alg}}(T^n_{\theta}) \).

Let us consider for \( m = 2n \) or \( m = 2n + 1 \) the homomorphism

\[
r_{23} \circ (\text{st} \otimes \text{st}) : C_{\text{alg}}(\mathbb{R}^m) \otimes C_{\text{alg}}(\mathbb{R}^m) \to C_{\text{alg}}(\mathbb{R}^m) \otimes C_{\text{alg}}(\mathbb{R}^m) \otimes C_{\text{alg}}(T^n_{\theta}) \otimes C_{\text{alg}}(T^n_{\theta - \theta_0})
\]

where \( r_{23} \) is the transposition of the second and the third factors in the tensor product, (i.e. \( C_{\text{alg}}(T^n_{\theta}) \otimes C_{\text{alg}}(\mathbb{R}^m) \) is replaced by \( C_{\text{alg}}(\mathbb{R}^m) \otimes C_{\text{alg}}(T^n_{\theta}) \) there). This \( * \)-homomorphism restricts to give a homomorphism, again denoted by \( \text{st} \)

\[
\text{st} : M_\theta(m, \mathbb{R}) \to M(m, \mathbb{R}) \otimes C_{\text{alg}}(T^n_{\theta}) \otimes C_{\text{alg}}(T^n_{\theta - \theta_0})
\]

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which is again a homomorphism of unital $\ast$-algebras and will be also referred to as splitting homomorphism. For instance, for $m = 2n$, it is the unique unital $\ast$-homomorphism such that

$$st(a^\mu_\nu) = a^\mu_\nu \otimes U^\mu \otimes U_\nu \quad (10.5)$$

$$st(b^\mu_\nu) = b^\mu_\nu \otimes U^\mu \otimes (U_\nu)^* \quad (10.6)$$

for $\mu, \nu = 1, \ldots, n$ where $a^\mu_\nu$ and $b^\mu_\nu$ are the classical coordinates corresponding to $a^\theta_\nu$ and $b^\theta_\nu$ for $\theta = 0$. The counterpart of b) in Theorem 5 is that $st$ induces here an isomorphism of $M_\theta(m, \mathbb{R})$ onto the subalgebra of elements $x$ of $M(m, \mathbb{R}) \otimes \text{C}_{\text{alg}}(T^\theta n) \otimes \text{C}_{\text{alg}}(T_n \cdot \theta)$ which are invariant by the diagonal action $(\sigma \otimes \tau) \cdot (\tau \otimes \tau)^{-1}$ of $T^n \times T^n$ i.e. which satisfy $(\sigma_\tau \otimes \sigma_t)(\tau_s \otimes \tau_{-s})(x) = x$, $\forall (s, t) \in T^n \times T^n$ (with the notations of the end of last section). One has

$$(st \circ (\sigma_\tau \otimes \sigma_t) \otimes I \otimes I) \circ st$$

which then implies that $st$ induces an isomorphism of $M_\theta(m, \mathbb{R})^{\sigma \otimes \sigma}$ onto $M(m, \mathbb{R})^{\sigma \otimes \sigma} \otimes \mathbf{1} \otimes \mathbf{1}$ where $M_\theta(m, \mathbb{R})^{\sigma \otimes \sigma}$ denotes the subalgebra of elements which are invariant by the action of $T^n \times T^n$, (the same for $\theta = 0$ on the right-hand side). This in particular implies that $st(\det_\theta)$ is in $M(2n, \mathbb{R})^{\sigma \otimes \sigma} \otimes \mathbf{1} \otimes \mathbf{1}$; in fact one has $st(\det_\theta) = \det \otimes \mathbf{1} \otimes \mathbf{1}$ where $\det = \det_{\theta=0}$ is the ordinary determinant.

The above homomorphism passes to the quotient to define homomorphisms

$$st : \text{C}_{\text{alg}}(G_\theta) \to \text{C}_{\text{alg}}(G) \otimes \text{C}_{\text{alg}}(T^\theta n) \otimes \text{C}_{\text{alg}}(T_n^{\cdot \theta})$$

where $G$ is any of the classical groups $GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$, $O(m)$, $SO(m)$, $GL(n, \mathbb{C})$, $U(n)$ or the subgroup $GL(1)(n, \mathbb{C})$ of $GL(n, \mathbb{C})$ consisting of matrices with determinants of modulus one, $m = 2n$ or $m = 2n + 1$, and where
$G_\theta$ denote the corresponding quantum groups defined in Section 7 and in Section 8. These homomorphisms $\sigma$ which will still be referred to as the splitting homomorphisms, have the property that they induce isomorphisms of $C_{\text{alg}}(G_\theta)$ onto $(C_{\text{alg}}(G)\otimes C_{\text{alg}}(T^n_\theta)\otimes C(T^n_{-\theta}))^{(\sigma^\otimes \sigma^\otimes \tau^\otimes \tau)^{-1}}$ for these groups $G$.

Thus, one sees that the situation is the same for the above quantum groups as for the noncommutative spaces $M_\theta$ with $M = \mathbb{R}^m$, $S^m$ excepted that the action of $T^n$ is replaced by an action of $T^n \times T^n = T^{2n}$ and that the noncommutative $n$-torus $T^n_\theta$ is replaced by the noncommutative $2n$-torus $T^{2n}_{\theta \times (-\theta)}$ where $\theta \times (-\theta)$ is the real antisymmetric $(2n, 2n)$-matrix

$$
\begin{pmatrix}
\theta & 0 \\
0 & -\theta
\end{pmatrix}
\in M_{2n}(\mathbb{R});
$$

one has of course $C_{\text{alg}}(T^{2n}_{\theta \times (-\theta)}) = C_{\text{alg}}(T^n_\theta)\otimes C_{\text{alg}}(T^n_{-\theta})$.

11 Smoothness

Beside their usefulness for computations, the splitting homomorphisms give straightforward unambiguous notions of smooth functions on $\theta$-deformations.

The locally convex $*$-algebra $C^\infty(T^n_\theta)$ of smooth functions on the noncommutative torus $T^n_\theta$ was defined in [7]. It is the completion of $C_{\text{alg}}(T^n_\theta)$ equipped with the locally convex topology generated by the seminorms

$$
|u|_r = \sup_{\tau_1 + \cdots + \tau_n \leq r} \| X_1^{\tau_1} \ldots X_n^{\tau_n}(u) \|
$$

where $\| \cdot \|$ is the $C^*$-norm (which is the sup of the $C^*$-seminorms) and where the $X_\mu$ are the infinitesimal generators of the action $s \mapsto \tau_s$ of $T^n$ on $T^n_\theta$. They are the unique derivations of $C_{\text{alg}}(T^n_\theta)$ satisfying

$$
X_\mu(U^\nu) = 2\pi i \delta^\nu_\mu U^\nu
$$

(11.1)
for \( \mu, \nu = 1, \ldots, n \). Notice that these derivations are real and commute between themselves, i.e. \( X_\mu(u^*) = (X_\mu(u))^* \) and \( X_\mu X_\nu - X_\nu X_\mu = 0 \). This locally convex \(*\)-algebra is a nuclear Fréchet space and it follows from the general theory of topological tensor products that the \( \pi \)-topology and \( \varepsilon \)-topology coincide [28] on any tensor product, [48] i.e.

\[
E \otimes_\varepsilon C^\infty(T^n_\theta) = E \otimes_\pi C^\infty(T^n_\theta)
\]

so that on \( E \otimes C^\infty(T^n_\theta) \) there is essentially one reasonable locally convex topology and we denote by \( E\hat{\otimes}C^\infty(T^n_\theta) \) the corresponding completion.

It is then straightforward to define the function spaces \( C^\infty(M_\theta) \) (of smooth functions) and \( C^\infty_c(M_\theta) \) (of smooth functions with compact support) for any of the \( \theta \)-deformed spaces mentioned above, as the fixed point algebra of the diagonal action of \( T^n \) on the completed tensor product \( C^\infty(M)\hat{\otimes}C^\infty(T^n_\theta) \) (and on \( C^\infty_c(M)\hat{\otimes}C^\infty(T^n_\theta) \)).

Using the appropriate splitting homomorphisms, one defines in the same way the locally convex \(*\)-algebras \( C^\infty(G_\theta) \) and \( C^\infty_c(G_\theta) \) of smooth functions on the different quantum groups defined in Section 7 and in Section 8. The same discussion applies to the algebras \( \Omega(M_\theta) \) and \( \Omega_c(M_\theta) \) of smooth differential forms.

12 Differential forms, self-duality, Hochschild cohomology for \( \theta \)-deformations

Let \( M \) be a smooth \( m \)-dimensional manifold endowed with a smooth action \( s \mapsto \sigma_s \) of the compact abelian Lie group \( T^n \), (the \( n \)-torus). We also denote by \( s \mapsto \sigma_s \) the corresponding group-homomorphism of \( T^n \) into the
group Aut($C^\infty(M)$) (resp Aut($\Omega(M)$)) of automorphisms of the unital $*$-algebra $C^\infty(M)$ of complex smooth functions on $M$ with its standard topology (resp of the graded-involutive differential algebra $\Omega(M)$ of smooth differential forms).

Let $C^\infty(M_\theta)$ be the $\theta$-deformation of the $*$-algebra $C^\infty(M)$ associated by [42] to the above data. We shall find it convenient to give the following (trivially equivalent) direct description of $C^\infty(M_\theta)$ as a fixed point algebra.

The completed tensor product $C^\infty(M) \hat{\otimes} C^\infty(T^n_\theta)$ is unambiguously defined by nuclearity and is a unital locally convex $*$-algebra which is a complete nuclear space. We define by duality the noncommutative smooth manifold $M \times T^n_\theta$ by setting $C^\infty(M \times T^n_\theta) = C^\infty(M) \hat{\otimes} C^\infty(T^n_\theta)$; elements of $C^\infty(M \times T^n_\theta)$ will be referred to as the smooth functions on $M \times T^n_\theta$. Let $C^\infty(M \times T^n_\theta)^{\sigma \times \tau^{-1}}$ be the subalgebra of the $f \in C^\infty(M \times T^n_\theta)$ which are invariant by the diagonal action $\sigma \times \tau^{-1}$ of $T^n$, that is such that $\sigma_s \otimes \tau_s^{-1}(f) = f$ for any $s \in T^n$. One defines by duality the noncommutative manifold $M_\theta$ by setting $C^\infty(M_\theta) = C^\infty(M \times T^n_\theta)^{\sigma \times \tau^{-1}}$ and the elements of $C^\infty(M_\theta)$ will be referred to as the smooth functions on $M_\theta$. This definition clearly coincides with the one used before for the examples of the previous sections once identified using the splitting homomorphisms.

Let us now give a first construction of smooth differential forms on $M_\theta$ generalizing the one given before in the examples. Let $\Omega(M_\theta)$ be the graded-involutive subalgebra $(\Omega(M) \hat{\otimes} C^\infty(T^n_\theta))^{\sigma \times \tau^{-1}}$ of $\Omega(M) \hat{\otimes} C^\infty(T^n_\theta)$ consisting of elements which are invariant by the diagonal action $\sigma \times \tau^{-1}$ of $T^n$. This subalgebra is stable by $d \otimes I$ so $\Omega(M_\theta)$ is a locally convex graded-involutive differential algebra which is a deformation of $\Omega(M)$ with $\Omega^0(M_\theta) = C^\infty(M_\theta)$ and which will be referred to as the algebra of smooth differential forms on $M_\theta$.
\( M_\theta \). The action \( s \mapsto \sigma_s \) of \( T^n \) on \( \Omega(M) \) induces \( s \mapsto \sigma_s \otimes I \) on \( \Omega(M) \hat{\otimes} C^\infty(T^n_\theta) \) which gives by restriction a group-homomorphism, again denoted \( s \mapsto \sigma_s \), of \( T^n \) into the group \( \text{Aut}(\Omega(M_\theta)) \) of automorphisms of the graded-involutive differential algebra \( \Omega(M_\theta) \).

**PROPOSITION 3** The graded-involutive differential subalgebra \( \Omega(M_\theta)^\sigma \) of \( \sigma \)-invariant elements of \( \Omega(M_\theta) \) is in the graded center of \( \Omega(M_\theta) \) and identifies canonically with the graded-involutive differential subalgebra \( \Omega(M)^\sigma \) of \( \sigma \)-invariant elements of \( \Omega(M) \).

In other words the subalgebra of \( \sigma \)-invariant elements of \( \Omega(M_\theta) \) is not deformed (i.e. independent of \( \theta \)). In fact one has \( \Omega(M_\theta)^\sigma = \Omega(M)^\sigma \otimes 1 \) \((\subset \Omega(M) \hat{\otimes} C^\infty(T^n_\theta))\).

It is worth noticing that the notation \( M_\theta, C^\infty(M_\theta) \) introduced here are coherent with the standard ones \( T^n_\theta, C^\infty(T^n_\theta) \) used for the noncommutative torus. Indeed it is true that one has \( C^\infty(T^n_\theta) = (C^\infty(T^n) \hat{\otimes} C^\infty(T^n_\theta))^{\sigma \times \tau^{-1}} \) where here \( \sigma \) is the canonical action of \( T^n \) on itself. Furthermore there is a natural definition of the graded differential algebra of smooth differential forms on the noncommutative \( n \)-torus \( T^n_\theta \) [7] and it turns out that it coincides with the above one for \( M = T^n \), that is with \( \Omega(T^n_\theta) \), as easily verified.

Although simple and useful, the previous definition of smooth differential forms on \( M_\theta \) is not the most natural one. Indeed the construction has the following heuristical geometric interpretation. The noncommutative manifold \( M_\theta \) is the quotient of the product \( M \times T^n_\theta \) by the diagonal action of \( T^n \), that is one has a sort of noncommutative fibre bundle

\[
M \times T^n_\theta \xrightarrow{T^n} M_\theta
\]
with fibre $T^n$. In such a context it is natural to describe differential forms on $M_\theta$ as the basic forms on $M \times T^n_\theta$ for the operation of $\text{Lie}(T^n)$ corresponding to the infinitesimal diagonal action of $T^n$. More precisely, let $Y_\mu$, $\mu \in \{1, \ldots, n\}$ be the vector fields on $M$ corresponding to the infinitesimal action of $T^n$

$$Y_\mu(x) = \frac{\partial}{\partial s}\sigma_s(x) \mid_{s=0}$$

(12.1)

for $x \in M$. These vector fields are real and define $n$ derivations of $C^\infty(M)$, again denoted by $Y_\mu$, which are real and commute between themselves. The inner anti-derivations $Y_\mu \mapsto i_{Y_\mu}$ define an operation of the (abelian) Lie algebra $\text{Lie}(T^n)$ in the graded differential algebra $\Omega(M)$ [4], [27] and the corresponding Lie derivatives $L_{Y_\mu} = di_{Y_\mu} + i_{Y_\mu}d$ are derivations of degree zero of $\Omega(M)$ which extend the $Y_\mu$ and correspond to the infinitesimal action of $T^n$ on $\Omega(M)$. The natural graded differential algebra of smooth differential forms on $M \times T^n_\theta$ is $\Omega(M \times T^n_\theta) = \Omega(M) \otimes_{\text{gr}} \Omega(T^n_\theta)$, and the operation [4], [27] of $\text{Lie}(T^n)$ in $\Omega(M \times T^n_\theta)$ corresponding to the diagonal action of $T^n$ is described by the antiderivations $i_\mu = i_{Y_\mu} \otimes I - (-I)^{gr} \otimes i_{X_\mu}$ of $\Omega(M \times T^n_\theta)$ where $i_{X_\mu}$ is the antiderivation of degree -1 of $\Omega(T^n_\theta) = C^\infty(T^n_\theta) \otimes_{\mathbb{R}} \Lambda^{\mathbb{R}^n}$ [7] such that $i_{X_\mu}(\omega^\nu) = \delta_\mu^\nu$ with $\omega^\mu = \frac{1}{2\pi}U^{\mu*}dU^\mu$. The infinitesimal diagonal action of $T^n$ is described by the Lie derivatives $L_\mu = di_\mu + i_\mu d$ on $\Omega(M \times T^n_\theta)$ and the differential subalgebra $\Omega_B(M \times T^n_\theta)$ of the basic elements of $\Omega(M \times T^n_\theta)$, that is of the elements $\alpha$ satisfying $i_\mu(\alpha) = 0$ and $L_\mu(\alpha) = 0$ for $\mu \in \{1, \ldots, n\}$, is a natural candidate to be the algebra of smooth differential forms on $M_\theta$. Fortunately, it is not hard to show that one has the following result which allows to use one point of view or the other one.

**PROPOSITION 4** As graded-involutive differential algebra $\Omega_B(M \times T^n_\theta)$ is isomorphic to $\Omega(M_\theta)$. 

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The (first) construction of $\Omega(M_\theta)$ admits the following generalization. Let $S$ be a smooth complex vector bundle of finite rank over $M$ and let $C^\infty(M, S)$ be the $C^\infty(M)$-module of its smooth sections. One equips $C^\infty(M, S)$ with the obvious topology obtained by taking local trivializations over local charts of $M$, etc.; so equipped it is a complete nuclear space. The vector bundle $S$ will be said to be $\sigma$-equivariant if there is a group-homomorphism $s \mapsto V_s$ of $T^n$ into the group Aut($S$) of automorphisms of $S$ which covers the action $s \mapsto \sigma_s$ of $T^n$ on $M$. In terms of smooth sections this means that one has

$$V_s(f\psi) = \sigma_s(f)V_s(\psi) \quad (12.2)$$

for $f \in C^\infty(M)$ and $\psi \in C^\infty(M, S)$ with an obvious abuse of notations.

Let $C^\infty(M_\theta, S)$ be the closed subspace of $C^\infty(M, S) \hat{\otimes} C^\infty(T^n_\theta)$ consisting of elements $\Psi$ which are invariant by the diagonal action $V \times \tau^{-1}$ of $T^n$, i.e. which satisfy $V_s \otimes \tau_{-s}(\Psi) = \Psi$ for any $s \in T^n$. The locally convex space $C^\infty(M_\theta, S)$ is also canonically a topological bimodule over $C^\infty(M_\theta)$, or which is the same, a topological left module over $C^\infty(M_\theta) \otimes C^\infty(M_\theta)^{opp}$.

**PROPOSITION 5** The bimodule $C^\infty(M_\theta, S)$ is diagonal and (topologically) left and right finite projective over $C^\infty(M_\theta)$.

The proof of this proposition uses the equivalence between the category of $\sigma$-equivariant finite projective modules over $C^\infty(M)$ (i.e. of $\sigma$-equivariant vector bundles over $M$) and the category of finite projective modules over the cross-product $C^\infty(M) \rtimes_\sigma T^n$, the fact that one has $C^\infty(M) \rtimes_\sigma T^n \simeq C^\infty(M_\theta) \rtimes_\sigma T^n$ and finally the equivalence between the category of finite projective modules over $C^\infty(M_\theta) \rtimes_\sigma T^n$ and the category of $\sigma$-equivariant finite projective modules over $C^\infty(M_\theta)$ [29].
Let $D$ be a continuous $\mathbb{C}$-linear operator on $C^\infty(M, S)$ such that

$$DV_s = V_s D$$

(12.3)

for any $s \in T^n$. Then $C^\infty(M, S) \subset C^\infty(M, S) \widehat{\otimes} C^\infty(T^n_M)$ is stable by $D \otimes I$ which defines the operator $D_\theta (= D \otimes I | C^\infty(M, S))$ on $C^\infty(M, S)$. If $D$ is a first-order differential operator it follows immediately from the definition that $D_\theta$ is a first-order operator of the bimodule $C^\infty(M, S)$ over $C^\infty(M)$ into itself, [9], [24]. If $D$ is of order zero i.e. is a module homomorphism over $C^\infty(M)$ then it is obvious that $D_\theta$ is a bimodule homomorphism over $C^\infty(M)$. We already met this construction in the case of $S = \wedge T^* M$ and $D = d$ there, $D_\theta$ is the differential $d$ of $\Omega(M)$ which is a first-order operator on the bimodule $\Omega(M)$ over $C^\infty(M)$. Let $\omega \mapsto *\omega$ be the Hodge operator on $\Omega(M)$ corresponding to a $\sigma$-invariant riemannian metric on $M$. One has $* \sigma_s = \sigma_s *$ thus $*$ satisfies (12.3) from which one obtains an endomorphism $*\theta$ of $\Omega(M)$ considered as a bimodule over $C^\infty(M)$. We shall denote $*\theta$ simply by $*$ in the following. One has $*\Omega^p(M) \subset \Omega^{m-p}(M)$.}

**THEOREM 6** Let the $2n$-sphere $S^{2n}$ be equipped with its usual metric, let $*$ be defined as above on $\Omega(S^{2n})$ and let $e$ be the hermitian projection of Theorem 3. Then $e$ satisfies the self-duality equation $*e(de)^n = i^n e(de)^n$.

Indeed using the splitting homomorphism, $e$ identifies with

$$e = \frac{1}{2}(1 + \sum_{\mu=1}^n (u^\mu_{(0)} \hat{\Gamma}^{\mu*} + u^\mu_{(0)} \hat{\Gamma}^{\mu*} + u)$$

where $u^\mu_{(0)}, \ldots, u$ are now the classical coordinates of $\mathbb{R}^{2n+1}$ for $S^{2n} \subset \mathbb{R}^{2n+1}$ and where $\hat{\Gamma}^{\mu*} = \Gamma^{\mu*} \otimes U^\mu$, $\hat{\Gamma}^{\mu} = \Gamma^{\mu} \otimes U^{\mu*}$ with $\gamma$ identified with $\gamma \otimes 1 \in \mathbb{R}^{2n+1}$.
\( M_{2n}(C^\infty(T^n_\theta)) \). Now one verifies easily that the \( \tilde{\Gamma}^\mu, \tilde{\Gamma}^\nu \) satisfy the relations of the usual Clifford algebra of \( \mathbb{R}^{2n} \) so \( *e(de)^n = i^n e(de)^n \) follows from the classical relation (5.3) for \( P_+ = e \mid_{\theta=0} \) and from \( * = * \otimes I \) where on the right-hand side \( * \) is the classical one.

Similarly one has \( *e_-(de_-)^n = -i^n e_-(de_-)^n \). Notice that if one replaces the usual metric of \( S^{2n} \) by another \( \sigma \)-invariant metric which is conformally equivalent, the same result holds but that \( \sigma \)-invariance is a priori necessary for this.

Let us now compute the Hochschild dimension of \( M_\theta \). We first construct a continuous projective resolution of the left module \( C^\infty(M_\theta) \) over \( C^\infty(M_\theta) \otimes C^\infty(M_\theta) \).

**Lemma 6** There are continuous homomorphisms of left modules

\[
i_p : \Omega^p(M_\theta) \otimes C^\infty(M_\theta) \to \Omega^{p-1}(M_\theta) \otimes C^\infty(M_\theta)
\]

over \( C^\infty(M_\theta) \otimes C^\infty(M_\theta) \) for \( p \in \{1, \cdots, m\} \) such that the sequence

\[
0 \to \Omega^m(M_\theta) \otimes C^\infty(M_\theta) \to \cdots \to C^\infty(M_\theta) \rightarrow 0
\]

is exact, where \( \mu \) is induced by the product of \( C^\infty(M_\theta) \).

In fact as was shown and used in [8] one has continuous projective resolutions of \( C^\infty(M) \) and of \( C^\infty(T_\theta^n) \) of the form

\[
0 \to \Omega^n(M) \otimes C^\infty(M) \to \cdots \to C^\infty(M) \otimes C^\infty(M) \rightarrow 0
\]

\[
0 \to \Omega^n(T_\theta^n) \otimes C^\infty(T_\theta^n) \to \cdots \to C^\infty(T_\theta^n) \rightarrow 0
\]

which combine to give a continuous projective resolution of

\[
C^\infty(M) \otimes C^\infty(T_\theta^n) = C^\infty(M \times T_\theta^n)
\]
of the form
\[ 0 \to \Omega^{m+n}(M \times T^n_\theta) \otimes C^\infty(M \times T^n_\theta) \stackrel{\tilde{i}}{\to} \cdots \]
\[ \tilde{\Omega}(M \times T^n_\theta) \otimes C^\infty(M \times T^n_\theta) \to C^\infty(M \times T^n_\theta) \to 0 \]
where \( \tilde{\Omega}(M \times T^n_\theta) = \oplus_{p \geq k \geq 0} \Omega^k(M) \otimes \Omega^{p-k}(T^n_\theta) \) and where
\[ \tilde{i}_p = \sum_k (i^p_k \otimes I + (-I)^k \otimes j_{p-k}) \]

There is some freedom in the choice of the \( i^p_k, j^{\ell} \) and one can choose them equivariant (by choosing a \( \sigma \)-invariant metric on \( M \), etc.) in such a way that the \( \tilde{i}_p \) restrict as continuous homomorphisms
\[ i_p : \tilde{\Omega}_B^p(M \times T^n_\theta) \otimes C^\infty(M_\theta) \to \Omega_B^{p-1}(M \times T^n_\theta) \otimes C^\infty(M_\theta) \]
of \( C^\infty(M_\theta) \otimes C^\infty(M_\theta)^{opp} \)-modules which gives the desired resolution of \( C^\infty(M_\theta) \) using Proposition 5.

This shows that the Hochschild dimension \( m_\theta \) of \( M_\theta \) is \( \leq m \) where \( m \) is the dimension of \( M \).

Let \( w \in \Omega^m(M) \) be a non-zero \( \sigma \)-invariant form of degree \( m \) on \( M \) (obtained by a straightforward local averaging). In view of Proposition 3, \( w \otimes 1 = w_\theta \) is a \( \sigma \)-invariant element of \( \Omega^m(M_\theta) \), i.e. \( w_\theta \in \Omega^m(M_\theta)^{\sigma} \) which defines canonically a non-trivial invariant cycle \( v_\theta \) in \( Z_m(C^\infty(M_\theta), C^\infty(M_\theta)) \). Thus one has \( m_\theta \geq m \) and therefore the following result.

**THEOREM 7** Let \( M_\theta \) be a \( \theta \)-deformation of \( M \) then one has \( \dim(M_\theta) = \dim(M) \), that is the Hochschild dimension \( m_\theta \) of \( C^\infty(M_\theta) \) coincides with the dimension \( m \) of \( M \).

Note that the conclusion of the theorem fails for general deformations by actions of \( \mathbb{R}^d \) as described in [42]. Indeed, in the simplest case of the Moyal
deformation of $\mathbb{R}^{2n}$ the Hochschild dimension drops down to zero for non-degenerate values of the deformation parameter. It is however easy to check that periodic cyclic cohomology (but not its natural filtration) is unaffected by the $\theta$-deformation.

13 Metric aspect: The spectral triple

As in the last section we let $M$ be a smooth $m$-dimensional manifold equipped with a smooth action $s \mapsto \sigma_s$ of $T^n$. It is well-known and easy to check that we can average any riemannian metric on $M$ under the action of $\sigma$ and obtain one for which the action $s \mapsto \sigma_s$ of $T^n$ on $M$ is isometric. Let us assume moreover that $M$ is a spin manifold. Let $S$ be the spin bundle over $M$ and let $D$ be the Dirac operator on $C^\infty(M, S)$. The bundle $S$ is not $\sigma$-equivariant in the sense of the last section but is equivariant in a slightly generalized sense which we now explain. In fact the isometric action $\sigma$ of $T^n$ on $M$ does not lift directly to $S$ but lifts only modulo $\pm I$. More precisely one has a twofold covering $p : \tilde{T}^n \to T^n$ of the group $T^n$, and a group homomorphism $\tilde{s} \mapsto V_{\tilde{s}}$ of $\tilde{T}^n$ into the group $\text{Aut}(S)$ which covers the action $s \mapsto \sigma_s$ of $T^n$ on $M$. In terms of smooth sections, (12.2) generalizes here as

$$V_{\tilde{s}}(f \psi) = \sigma_s(f) V_{\tilde{s}}(\psi)$$

(13.1)

where $f \in C^\infty(M)$ and $\psi \in C^\infty(M, S)$ with $s = p(\tilde{s})$. The bundle $S$ is also a hermitian vector bundle and one has

$$(V_{\tilde{s}}(\psi), V_{\tilde{s}}(\psi')) = \sigma_{s}((\psi, \psi'))$$

(13.2)

for $\psi, \psi' \in C^\infty(M, S)$, $\tilde{s} \in \tilde{T}^n$ and $s = p(\tilde{s})$ where $(\ldots)$ denotes the hermitian scalar product. Furthermore, the Dirac operator $D$ commutes with the $V_{\tilde{s}}$. 

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To the projection $p : \tilde{T}^n \to T^n$ corresponds an injective homomorphism of $C^\infty(T^n)$ into $C^\infty(\tilde{T}^n)$ which identifies $C^\infty(T^n)$ with the subalgebra $C^\infty(\tilde{T}^n)_{\text{Ker}(p)}$ of $C^\infty(\tilde{T}^n)$ of elements which are invariant by the action of the subgroup $\text{Ker}(p) \simeq \mathbb{Z}_2$ of $\tilde{T}^n$. Let $\tilde{T}_\theta^n$ be the noncommutative $n$-torus $T^n_\theta$ and let $\tilde{s} \mapsto \tilde{\tau}_{\tilde{s}}$ be the canonical action of the $n$-torus $\tilde{T}^n$ that is the canonical group-homomorphism of $\tilde{T}^n$ into the group $\text{Aut}(C^\infty(\tilde{T}_\theta^n))$. The very reason for these notations is that $C^\infty(T^n_\theta)$ identifies with the subalgebra $C^\infty(\tilde{T}^n_\theta)_{\text{Ker}(p)}$ of $\tilde{T}^n_\theta$ of elements which are invariant by the action of $\tilde{\tau}_{\tilde{s}}$ for $\tilde{s} \in \text{Ker}(p) \simeq \mathbb{Z}_2$. Under this identification, one has $\tilde{\tau}_{\tilde{s}}(f) = \tau_s(f)$ for $f \in C^\infty(T^n_\theta)$ and $s = p(\tilde{s}) \in T^n$.

Define $C^\infty(M_\theta, S)$ to be the closed subspace of $C^\infty(M, S) \otimes C^\infty(\tilde{T}_\theta^n)$ consisting of elements $\Psi$ which are invariant by the diagonal action $V \times \tilde{\tau}^{-1}$ of $\tilde{T}^n$; this is canonically a topological bimodule over $C^\infty(M_\theta)$. Since the Dirac operator commutes with the $V_s$, $C^\infty(M_\theta, S)$ is stable by $D \otimes I$ and we denote by $D_\theta$ the corresponding operator on $C^\infty(M_\theta, S)$. Again, $D_\theta$ is a first-order operator of the bimodule $C^\infty(M_\theta, S)$ over $C^\infty(M_\theta)$ into itself. The space $C^\infty(M, S) \otimes C^\infty(\tilde{T}_\theta^n)$ is canonically a bimodule over $C^\infty(M) \otimes C^\infty(\tilde{T}_\theta^n)$ (and therefore also on $C^\infty(M) \otimes C^\infty(T^n_\theta)$). One defines a hermitian structure on $C^\infty(M, S) \otimes C^\infty(\tilde{T}_\theta^n)$ for its right-module structure over $C^\infty(M) \otimes C^\infty(\tilde{T}_\theta^n)$ [7] by setting

$$(\psi \otimes t, \psi' \otimes t') = (\psi, \psi') \otimes t^* t'$$

for $\psi, \psi' \in C^\infty(M, S)$ and $t, t' \in C^\infty(T^n_\theta)$. This gives by restriction the hermitian structure of $C^\infty(M_\theta, S)$ considered as a right $C^\infty(M_\theta)$-module; that is one has

$$(\psi f, \psi' f') = f^*(\psi, \psi') f'$$

for any $\psi, \psi' \in C^\infty(M_\theta, S)$ and $f, f' \in C^\infty(M_\theta)$. Notice that when dim($M$)
is even, one has a $\mathbb{Z}_2$-grading $\gamma$ of $C^\infty(M, S)$ as hermitian module which induces a $\mathbb{Z}_2$-grading, again denoted by $\gamma$, of $C^\infty(M_\theta, S)$ as hermitian right $C^\infty(M_\theta)$-module.

Let $J$ denote the charge conjugation of $S$. This is an antilinear mapping of $C^\infty(M, S)$ into itself such that

$$ (J\psi, J\psi) = (\psi, \psi) \quad (13.3) $$

$$ JfJ^{-1} = f^* \quad (13.4) $$

for any $\psi \in C^\infty(M, S)$ and for any $f \in C^\infty(M)$, $(f^*(x) = \overline{f(x)})$. Furthermore one has also

$$ JV_\tilde{s} = V_\tilde{s}J \quad (13.5) $$

for any $\tilde{s} \in \tilde{T}^n$. Let us define $\tilde{J}$ to be the unique antilinear operator on $C^\infty(M, S) \otimes C^\infty(T^n_\theta)$ satisfying $\tilde{J}(\psi \otimes t) = J\psi \otimes t^*$ for $\psi \in C^\infty(M, S)$ and $\tilde{t} \in C^\infty(T^n_\theta)$. The subspace $C^\infty(M_\theta, S)$ is stable by $\tilde{J}$ and we define $J_\theta$ to be the induced antilinear mapping of $C^\infty(M_\theta, S)$ into itself. It follows from (13.3), (13.4) and from the definition that one has

$$ (J_\theta\psi, J_\theta\psi) = (\psi, \psi) \quad (13.6) $$

$$ J_\theta f J^{-1}_\theta \psi = \psi f^* \quad (13.7) $$

for any $\psi \in C^\infty(M_\theta, S)$ and $f \in C^\infty(M_\theta)$. Thus left multiplication by $J_\theta f^* J^{-1}_\theta$ is the same as right multiplication by $f$. Obviously $J_\theta$ satisfies, in function of $\dim(M)$ modulo 8, the table of normalizations, commutations with $D_\theta$ and with $\gamma$ in the even dimensional case which corresponds to the reality conditions 7) of [13]. This follows of course from the same properties of $J, D, \gamma$ (i.e. the same properties for $\theta = 0$). So equipped $C^\infty(M_\theta, S)$ is in particular an involutive bimodule with a right-hermitian structure [41], [26].

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Let us now investigate the symbol of $D_\theta$. It is easy to see that the left universal symbol $\sigma_L(D_\theta)$ of $D_\theta$ (as defined in [24]) factorizes through a homomorphism

$$\tilde{\sigma}_L(D_\theta) : \Omega^1(M_\theta) \otimes_{C^\infty(M_\theta)} C^\infty(M_\theta, S) \to C^\infty(M_\theta, S)$$

of bimodules over $C^\infty(M_\theta)$. By definition, one has

$$[D_\theta, f] \psi = \tilde{\sigma}_L(D_\theta)(df \otimes \psi)$$

for $f \in C^\infty(M_\theta)$ and $\psi \in C^\infty(M_\theta, S)$ and $df \mapsto [D_\theta, f]$ extends as an injective linear mapping of $\Omega^1(M_\theta)$ into the continuous linear endomorphisms of $C^\infty(M_\theta, S)$.

**Lemma 7** Let $f_i, g_i$ be a finite family of elements of $C^\infty(M_\theta)$ such that $\sum_i f_i[D_\theta, g_i] = 0$. Then the endomorphism $\sum_i[D_\theta, f_i][D_\theta, g_i]$ is the left multiplication in $C^\infty(M_\theta, S)$ by an element of $C^\infty(M_\theta)$.

When no confusion arises, we shall summarize this statement by writing $\sum_i[D_\theta, f_i][D_\theta, g_i] \in C^\infty(M_\theta)$ whenever $\sum_i f_i[D_\theta, g_i] = 0$. Indeed, using the fact that $D_\theta$ is the restriction of $D \otimes I$ where $D$ is the classical Dirac operator on $M$ one shows that

$$\sum_i[D_\theta, f_i][D_\theta, g_i] + \sum_i f_i \Delta_\theta(g_i) = [D_\theta, \sum_i f_i[D_\theta, g_i]] = 0$$

where $\Delta_\theta$ is the restriction of $\Delta \otimes I$ to $C^\infty(M_\theta)$ with $\Delta$ being the ordinary Laplace operator on $M$ which is $\sigma$-invariant. This implies that $\sum_i f_i \Delta_\theta(g_i)$ is in $C^\infty(M_\theta)$ and therefore the result.

Concerning the particular case $M = \mathbb{R}^{2n}$ one shows the following result using the splitting homomorphism.
PROPOSITION 6 Let $z^\mu, \bar{z}^\nu \in C^\infty(\mathbb{R}^{2n}_\theta)$ be as in Section 4. Then the
$\hat{\Gamma}^\mu = [D_\theta, z^\mu], \hat{\Gamma}^\nu = [D_\theta, \bar{z}^\nu]$ satisfy the relations

$$
\hat{\Gamma}^\mu \hat{\Gamma}^\nu + \lambda^{\mu\nu} \hat{\Gamma}^\rho \hat{\Gamma}_\rho = 0
$$

where 1 is the identity mapping of $C^\infty(\mathbb{R}^{2n}_\theta, S)$ onto itself.

This $\theta$-twisted version of the generators of the Clifford algebra connected with the symbol of $D_\theta$ differs from the one introduced in Section 4 by the replacement $\lambda^{\mu\nu} \mapsto \lambda^{\nu\mu}$ and is the version associated with the $\theta$-twisted version $\wedge^\cdot \mathbb{R}^{2n}_\theta$ of the exterior algebra which is itself behind the differential calculus $\Omega(\mathbb{R}^{2n}_\theta)$. This is a counterpart for this example of the fact that $\Omega_{D_\theta} = \Omega(M_\theta)$.

We now make contact with the axiomatic framework of [13]. To simplify the discussion we shall assume now that $M$ is a compact oriented $m$-dimensional riemannian spin manifold equipped with an isometric action of $T^n$, (i.e. we add compacity). One defines a positive definite scalar product on $C^\infty(M, S)$ by setting

$$
\langle \psi, \psi' \rangle = \int_M (\psi, \psi') \mathrm{vol}
$$

where vol is the riemannian volume $m$-form which is $\sigma$-invariant and we denote by $\mathcal{H} = L^2(M, S)$ the Hilbert space obtained by completion. As an unbounded operator in $\mathcal{H}$, the Dirac operator $D : C^\infty(M, S) \to C^\infty(M, S)$ is essentially self-adjoint on $C^\infty(M, S)$. We identify $D$ with its closure that is with the corresponding self-adjoint operator in $\mathcal{H}$. The spectral triple $(C^\infty(M), \mathcal{H}, D)$ together with the real structure $J$ satisfy the axioms of [13]. The homomorphism $\bar{s} \mapsto V_\bar{s}$ uniquely extends as a unitary representation of the group $\bar{T}^n$ in $\mathcal{H}$ which will be still denoted by $\bar{s} \mapsto V_\bar{s}$. On the other
hand the action $\tilde{s} \mapsto \tilde{\tau}_s$ of $\tilde{T}^n$ on $C^\infty(\tilde{T}_\theta^n)$ extends as a unitary action again denoted by $\tilde{s} \mapsto \tilde{\tau}_s$ of $\tilde{T}^n$ on the Hilbert space $L^2(\tilde{T}_\theta^n)$ which is obtained from $C^\infty(\tilde{T}_\theta^n)$ by completion for the Hilbert norm $f \mapsto \|f\| = \text{tr}(f^*f)^{1/2}$ where tr is the usual normalized trace of $C^\infty(\tilde{T}_\theta^n) = C^\infty(T^n\theta)$. We now define the spectral triple $(C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta)$ to be the following one. The Hilbert space $\mathcal{H}_\theta$ is the subspace of the Hilbert tensor product $\mathcal{H} \otimes L^2(\tilde{T}_\theta^n)$ which consists of elements $\Psi$ which are invariant by the diagonal action of $\tilde{T}^n$, that is which satisfy $V_s \otimes \tilde{\tau}_{-s}(\Psi) = \Psi$, $\forall \tilde{s} \in \tilde{T}^n$. The operator $D_\theta$ identifies with an unbounded operator in $\mathcal{H}_\theta$ which is essentially self-adjoint on the dense subspace $C^\infty(M_\theta, S)$. We also identify $D_\theta$ with its closure that is with the self-adjoint operator which is also the restriction to $\mathcal{H}_\theta$ of $D \otimes I$. The antilinear operator $J_\theta$ canonically extends as anti-unitary operator in $\mathcal{H}_\theta$ (again denoted by $J_\theta$).

**THEOREM 8** The spectral triple $(C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta)$ together with the real structure $J_\theta$ satisfy all axioms of noncommutative geometry of [13].

Notice that axiom 4) of orientability is directly connected to the $\sigma$-invariance of the $m$-form vol on $M$. Consequently this form defines a $\sigma$-invariant $m$-form on $M_\theta$ in view of Proposition 3 which corresponds to a $\sigma$-invariant Hochschild cycle in $Z_m(A, A)$ for both $A = C^\infty(M)$ and $A = C^\infty(M_\theta)$. The argument for Poincaré duality is the same as in [16]. Finally, the isospectral nature of the deformation $(C^\infty(M), \mathcal{H}, D, J) \mapsto (C^\infty(M_\theta), \mathcal{H}_\theta, D_\theta, J_\theta)$ follows immediately from the fact that $D_\theta = D \otimes I$.

Coming back to the notations of sections 4 and 5, we can then return to the noncommutative geometry of $S^m_\theta$.

This geometry (with variable metric) is entirely specified by the projection $e$, the matrix algebra (which together generate the algebra of coordinates)
and the Dirac operator which fulfill a polynomial equation of degree $m$.

**THEOREM 9** Let $g$ be any $T^n$-invariant Riemannian metric on $S^m$, $m = 2n$ or $m = 2n - 1$, whose volume form is the same as for the round metric.

(i) Let $e \in M_{2n}(C^\infty(S^m_\theta))$ be the projection of Theorem 3. Then the Dirac operator $D_\theta$ of $S^m_\theta$ associated to the metric $g$ satisfies

$$\langle (e - \frac{1}{2})[D_\theta, e]^{2n} \rangle = \gamma$$

where $\langle \rangle$ is the projection on the commutant of $M_{2n}(C)$.

(ii) Let $U \in M_{2n-1}(C^\infty(S^{2n-1}_\theta))$ be the unitary of Theorem 3. Then the Dirac operator $D_\theta$ of $S^{2n-1}_\theta$ associated to the metric $g$ satisfies

$$\langle U[D_\theta, U^*][[D_\theta, U][D_\theta, U^*]]^{n-1} \rangle = 1$$

where $\langle \rangle$ is the projection on the commutant of $M_{2n-1}(C)$.

Using the splitting homomorphism as for Theorem 6 it is enough to show that this holds for the classical case $\theta = 0$, i.e. when $D$ is the classical Dirac operator associated to the metric $g$.

This result is of course a straightforward extension of results of [15], [16]. Since the deformed algebra $C^\infty(S^m_\theta)$ is highly nonabelian the inner fluctuations of the noncommutative metric ([13]) generate non-trivial internal gauge fields which compensate for the loss of gravitational degrees of freedom imposed by the $T^n$-invariance of the metric $g$.

**14 Further prospect**

We have shown that the basic $K$-theoretic equation defining spherical manifolds admits a complete solution in dimension 3. We then concentrated on
the subclass of $\theta$-deformations and identified as $m$-dimensional noncommutative spherical manifolds the noncommutative $m$-sphere $S^m_\theta$ for any $m \in \mathbb{N}$. For this class we completed the path from the crudest level of the algebra $C_{\text{alg}}(S)$ of polynomial functions on $S$ to the full-fledged structure of noncommutative geometry [13], as exemplified in theorem 9.

Needless to say our goal in part II will be a similar analysis for general spherical 3-manifolds including the smooth structure, the differential calculus and the metric aspect. Moreover it is desirable to classify noncommutative spherical manifolds in dimension greater than three, in particular in dimension four which is the first even dimension with noncommutative examples and for which we only described the suspensions of three-dimensional spherical manifolds. It is our aim to fill these gaps in Part II. There we shall in particular define more carefully the category of algebras corresponding to noncommutative spherical manifolds and the suspension functor which was used here in a relative unformal manner. We shall push further the classification of noncommutative spherical manifolds, analyse their differential calculi and the metric aspect. By the latter we mean the study of spectral triples and in particular the definition in this context of the Levi-Civita connection, the Riemannian and Ricci curvatures. In connection with the differential calculus we also wish to describe, in the spirit of what has been done here for $\theta$-deformations, the various related quantum groups. It is worth noticing here that there are other approaches to quantum groups which are symmetry of $S^4_\theta$ and some generalizations [44], [49]. In [44] the dual point of view is adopted and what is produced is the deformation of the universal enveloping algebra whereas in [49] the deformation is on the same side of the duality as developed here; both points of view are of course useful. However it must
be stressed again that, beside the fact that our approach is closely related to the differential calculus, the important point here is the observation that the quantum groups we have introduced arise with their expected dimensions that is the dimensions of the corresponding classical groups.

15 Appendix: Relations in the noncommutative Grassmannian

Let \( \mathcal{A} \) be the universal Grassmannian generated by the \( 2^2 \) elements \( \alpha, \beta, \gamma, \delta \) with the relations,

\[
U U^* = U^* U = 1, \quad U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

(15.1)

In this appendix we shall show that the intersection \( \mathcal{J} \) of the kernels of the representations \( \rho \) of \( \mathcal{A} \) such that \( \text{ch}_{\frac{1}{2}}(\rho(U)) = 0 \) is a non-trivial two sided ideal of \( \mathcal{A} \). Thus the odd Grassmanian \( \mathcal{B} \) which was introduced in [16] is a nontrivial quotient of \( \mathcal{A} \).

Given an algebra \( \mathcal{A} \) and elements \( x_j \in \mathcal{A} \) we let,

\[
[x_1, \ldots, x_n] = \sum \epsilon(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}
\]

(15.2)

where the sum is over all permutations and \( \epsilon(\sigma) \) is the signature of the permutation.

With the above notations, let \( \mu = [\alpha, \beta, \gamma, \delta] \). We shall check that,

**Lemma 8** In any representation \( \rho \) of \( \mathcal{A} \) for which \( \text{ch}_{\frac{1}{2}}(\rho(U)) = 0 \) one has, \( \rho([\mu, \mu^*]) = 0 \). Moreover \( [\mu, \mu^*] \neq 0 \) in \( \mathcal{A} \).

**Proof.** For \( y_i = \lambda_i^j x_j \), one has \( [y_1, \ldots, y_n] = \det \lambda [x_1, \ldots, x_n] \). This allows to extend the map \( a \otimes b \otimes c \otimes d \to [a, b, c, d] \) to a linear map \( c \),

\[
c: \wedge^4 \mathcal{A} \to \mathcal{A}.
\]

(15.3)
Let us now show that, for any representation $\rho$ of $\mathcal{A}$ for which
\[ \text{ch}_2(\rho(U)) = 0, \]
the following relation fulfilled by the matrix elements $\alpha = \rho(\alpha), \ldots, \delta = \rho(\delta)$,
\[ \alpha \otimes \alpha^* + \beta \otimes \beta^* + \gamma \otimes \gamma^* + \delta \otimes \delta^* = \alpha^* \otimes \alpha + \beta^* \otimes \beta + \gamma^* \otimes \gamma + \delta^* \otimes \delta \]  
(15.4)
implies,
\[ [\alpha, \beta, \gamma, \delta][\alpha, \beta, \gamma, \delta]^* = [\alpha, \beta, \gamma, \delta]^*[\alpha, \beta, \gamma, \delta]. \]  
(15.5)
It follows from (15.4) that,
\[ (\alpha \wedge \beta \wedge \gamma \wedge \delta) \otimes (\alpha^* \wedge \beta^* \wedge \gamma^* \wedge \delta^*) = (\alpha^* \wedge \beta^* \wedge \gamma^* \wedge \delta^*) \otimes (\alpha \wedge \beta \wedge \gamma \wedge \delta). \]  
(15.6)
Indeed we view $\tilde{\mathcal{A}} = \rho(\mathcal{A})$ as a linear space and consider the tensor product of exterior algebras,
\[ \wedge \tilde{\mathcal{A}} \otimes \wedge \tilde{\mathcal{A}} \]  
(ungraded tensor product).  
(15.7)
We then take the 4th power of (15.4) and get,
\[ 24 (\alpha \wedge \beta \wedge \gamma \wedge \delta) \otimes (\alpha^* \wedge \beta^* \wedge \gamma^* \wedge \delta^*) = 24 (\alpha^* \wedge \beta^* \wedge \gamma^* \wedge \delta^*) \otimes (\alpha \wedge \beta \wedge \gamma \wedge \delta). \]  
(15.8)
We can then apply $c \otimes c$ on both sides and compose with $m : \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$, the product, to get (15.5), that is
\[ \rho([\mu, \mu^*]) = 0 \]  
(15.9)
It remains to check that $[\mu, \mu^*] \neq 0$ in $\mathcal{A}$.

One has
\[ M_2(\mathbb{C}) \ast \mathbb{C} \mathbb{Z} = M_2(\mathcal{A}) \]  
(15.10)
where the free product in the left hand side is the free algebra generated by $M_2(\mathbb{C})$ and a unitary $U$, $U^* U = U U^* = 1$. As above $\mathcal{A}$ is generated by the matrix elements of $U$,
\[ U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \quad \alpha, \beta, \gamma, \delta \in \mathcal{A} \]  
(15.11)
As a linear basis of $M_2(\mathbb{C})$ we use the Pauli spin matrices, which we view as a projective representation of $\Gamma = (\mathbb{Z}/2)^2$,

$$
(0,0) \xrightarrow{\sigma} 1, \quad (0,1) \xrightarrow{\sigma} \sigma_1, \quad (1,0) \xrightarrow{\sigma} \sigma_2, \quad (1,1) \xrightarrow{\sigma} \sigma_3
$$

(15.12)

with $\sigma(a + b) = c(a, b) \sigma(a) \sigma(b)$ $\forall a, b \in (\mathbb{Z}/2)^2$.

Since we are dealing with a free product, we have a natural basis of $M_2(A)$ given by the monomials,

$$
\sigma_{i_1} U^{j_1} \sigma_{i_2} U^{j_2} \ldots \sigma_{i_k} U^{j_k}
$$

(15.13)

where $i_1$ and $j_k$ can be 0 but all other $i_\ell, j_\ell$ are $\neq 0$. The projection to $A$ is given by,

$$
P(T) = \frac{1}{4} \sum_{\Gamma} \sigma(a) T \sigma^{-1}(a).
$$

(15.14)

In particular the matrix components $\alpha, \beta, \gamma, \delta$, of $U$ are linear combinations of the four elements,

$$
x_a = P(\sigma(a) U), \quad a \in \Gamma = (\mathbb{Z}/2)^2.
$$

(15.15)

We want to compute $[\alpha, \beta, \gamma, \delta]$ or equivalently $[x_0, x_1, x_2, x_3]$.

Let us first rewrite the product $x_{a_1} x_{a_2} x_{a_3} x_{a_4}$, which is up to an overall coefficient $4^{-4}$,

$$
\sum_{b_i} \sigma(b_1) \sigma(a_1) U \sigma(b_1)^{-1} \sigma(b_2) \sigma(a_2) U \sigma(b_2)^{-1} \sigma(b_3) \sigma(a_3) U \sigma(b_3)^{-1}
$$

$$
\sigma(b_4) \sigma(a_4) U \sigma(b_4)^{-1}
$$

(15.16)

as a sum of terms of the form,

$$
\sigma(c_1) U \sigma(c_2) U \ldots U \sigma(c_4) U \sigma(c_4)^{-1} \sigma(c_3)^{-1} \sigma(c_2)^{-1} \sigma(c_1)^{-1}
$$

$$
\lambda(c_1, \ldots, c_4) \sigma(a_1) \sigma(a_2) \sigma(a_3) \sigma(a_4).
$$

(15.17)
where \( c_1 = b_1 + a_1, c_2 = b_2 - b_1 + a_2, c_3 = b_3 - b_2 + a_3, c_4 = b_4 - b_3 + a_4 \)

vary independently in \( \Gamma \) and \( \lambda(c_1, \ldots, c_4) \in U(1) \) can be computed using the trivial representation, \( U \to 1 \) by,

\[
\sigma(b_1) \sigma(a_1) \sigma(b_1)^{-1} \sigma(b_2) \sigma(a_2) \sigma(b_2)^{-1} \sigma(b_3) \sigma(a_3) \sigma(b_3)^{-1} \sigma(b_4) \sigma(a_4) \sigma(b_4)^{-1} = \lambda(c_1, \ldots, c_4) \sigma(a_1) \sigma(a_2) \sigma(a_3) \sigma(a_4). \tag{15.18}
\]

Each term in the reduced expansion of \([x_0, x_1, x_2, x_3]\) is the sum of the above expressions multiplied by \( \varepsilon(a) = \delta_{01}^{a_1} \delta_{12}^{a_2} \delta_{23}^{a_3} \delta_{34}^{a_4} \) the signature of the permutation \( \{0, 1, 2, 3\} \to \{a_1, a_2, a_3, a_4\} \).

To see that \([x_0, x_1, x_2, x_3] \neq 0\) we compute the terms in

\[
U^3 \sigma(c) U \sigma(c)^{-1}. \tag{15.19}
\]

Fixing \( c \) there is one contribution for each of the permutation of \( \{0, 1, 2, 3\} \) and in (15.16) we have

\[
b_1 = a_1, \ b_2 = a_1 + a_2, \ b_3 = a_1 + a_2 + a_3, \ b_4 = c + a_1 + a_2 + a_3 + a_4. \tag{15.20}
\]

In \((\mathbb{Z}/2)^2 = \Gamma\) one has \( a_1 + a_2 + a_3 + a_4 = 0 \) so that \( b_4 = c, \ b_3 = a_4 \). Since \( \sigma(x)^2 = 1 \) one can thus write (15.16) as

\[
U \sigma(a_1) \sigma(a_1 + a_2) \sigma(a_2) U \sigma(a_1 + a_2) \sigma(a_3) U \sigma(a_4) \sigma(c) \sigma(a_4) U \sigma(c) \tag{15.21}
\]

which we should multiply by \( \varepsilon(a) \) and sum over \( a \).

It is clear here that \( \sigma(a_1) \sigma(a_1 + a_2) \sigma(a_2) \) and \( \sigma(a_1 + a_2) \sigma(a_4) \sigma(a_3) \) are scalar and thus commute with \( U \) which allows to write (15.21) as follows,

\[
U^3 \sigma(a_1) \sigma(a_1 + a_2) \sigma(a_2) \sigma(a_1 + a_2) \sigma(a_3) \sigma(a_4) \sigma(c) \sigma(a_4) U \sigma(c). \tag{15.22}
\]

One has,

\[
\sigma(a) \sigma(a') \sigma(a)^{-1} \sigma(a')^{-1} = (-1)^{(a, a')} \quad \forall a, a' \in \Gamma \tag{15.23}
\]
using the bilinear form with $\mathbb{Z}/2$ values on $\Gamma$ given by,

$$\langle a, a' \rangle = \alpha \beta' - \alpha' \beta \quad \text{for} \quad a = (\alpha, \beta), \ a' = (\alpha', \beta') \in (\mathbb{Z}/2)^2. \quad (15.24)$$

Permuting $\sigma(a_1 + a_2)$ with $\sigma(a_2)$ and $\sigma(a_3)$ with $\sigma(a_4)$ introduces terms in $(-1)^n$ with $n = \langle a_1 + a_2, a_2 \rangle + \langle a_3, a_4 \rangle = \langle a_1, a_2 \rangle + \langle a_3, a_4 \rangle$. One has $0 \in \{a_1, a_2\}$ or $0 \in \{a_3, a_4\}$. In the first case $\langle a_1, a_2 \rangle = 0$ and $\langle a_3, a_4 \rangle = 1$ since they are distinct $\neq 0$. Similarly if $0 \in \{a_3, a_4\}$ we get $\langle a_1, a_2 \rangle + \langle a_3, a_4 \rangle = 1$ in all cases. We can thus replace (15.22) by

$$-U^3 \sigma(a_1) \sigma(a_2) \sigma(a_3) \sigma(c) \sigma(a_4) U \sigma(c). \quad (15.25)$$

Permuting $c$ with $a_4$ gives a $(-1)^{(c,a_4)}$. We have,

$$\sigma(a_1) \sigma(a_2) \sigma(a_3) \sigma(a_4) = (-1)^s \sigma_1 \sigma_2 \sigma_3 = i (-1)^s. \quad (15.26)$$

where $(-1)^s$ is the signature of the permutation of $\{1, 2, 3\}$ given by the non zero $a_j$'s. The coefficient of $U^3 \sigma(c) U \sigma(c)^{-1}$ is thus,

$$-4^{-4} \sum \varepsilon(a) (-1)^s (-1)^{(c,a_4)}. \quad (15.27)$$

Taking $c = (1, 0)$ we find 16 $-$ signs and 8 $+$ signs so that we get the term,

$$4^{-4}(-(-16 + 8)i) U^3 \sigma_2 U \sigma_2 = \frac{i}{32} U^3 \sigma_2 U \sigma_2. \quad (15.28)$$

Taking $c = (0, 1)$ we also find 16 $-$ signs and 8 $+$ signs which gives

$$\frac{i}{32} U^3 \sigma_1 U \sigma_1. \quad (15.29)$$

Thus if we let $\mu = [x_0, x_1, x_2, x_3]$ and compute $\mu \mu^*$ we get terms of the form,

$$\frac{-1}{(32)^2} U^3 \sigma_1 U \sigma_1 \sigma_2^{-1} \sigma_2^{-3} \quad (15.30)$$
which cannot be simplified and do not appear in the product $\mu^* \mu$ where we always have negative powers for the first $U$'s on the left followed by positive powers.

Thus we conclude that in the universal algebra $\mathcal{A}$ one has

$$[\mu, \mu^*] \neq 0. \quad (15.31)$$
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