EMBEDDINGS BETWEEN GRAND, SMALL AND VARIABLE LEBESGUE SPACES

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ABSTRACT. We give conditions on the exponent function $p(\cdot)$ that imply the existence of embeddings between the grand, small and variable Lebesgue spaces. We construct examples to show that our results are close to optimal. Our work extends recent results by the second author, Rakotoson and Sbordone [14].

1. Introduction

In this paper we consider the relationship between three Banach function spaces that generalize the classical Lebesgue spaces. Given a set $\Omega \subset \mathbb{R}^n$, $|\Omega| = 1$, $1 < p < \infty$, and $\theta > 0$, the generalized grand Lebesgue space $L^{p,\theta}(\Omega)$ consists of all measurable functions $f$ such that

$$
\|f\|_{p,\theta} = \sup_{0 < \epsilon < p^{-1}} \left( \epsilon^\theta \int_\Omega |f(x)|^{p-\epsilon} \, dx \right)^{\frac{1}{p-\epsilon}}.
$$

When $\theta = 0$ this reduces to the Lebesgue space $L^p(\Omega)$. When $\theta = 1$, this becomes the grand Lebesgue space $L^p(\Omega)$, which was introduced by Iwaniec and Sbordone [16]. The generalization for $\theta > 0$ was introduced in [15] and considered in a more systematic way in [3]. These spaces have proved very useful in proving limiting results in the study of partial differential equations: see [1, 9, 10, 15, 18].

The small Lebesgue space $L^{p,\theta}$ is defined as the associate space of $L^{p'}(\Omega)$, and so has the norm

$$
\|f\|_{(p,\theta)} = \sup \left\{ \int_\Omega f(x)g(x) \, dx : \|f\|_{p',\theta} \leq 1 \right\}.
$$

An intrinsic expression for the small Lebesgue space norm, when $\theta = 1$, was first found in [11] and for general $\theta > 0$ in [3]. These expressions were quite complicated, but much
simpler expressions were found in [7, 12]:

\[
\|f\|_{p,\theta} \approx \sup_{0 < t < 1} \log \left( \frac{e}{t} \right)^{-\frac{\theta}{p}} \left( \int_0^1 f_s(s)^p \, ds \right)^{\frac{1}{p}}
\]

\[
\|f\|_{(p,\theta)} \approx \int_0^1 \log \left( \frac{e}{t} \right)^{\frac{\theta}{p}-1} \left( \int_0^t f_s(s)^p \, ds \right)^{\frac{1}{p}} \, dt.
\]

The grand and small Lebesgue spaces are very “close” to the space \( L^p \). More precisely, we have for all \( 1 < p < \infty \) and \( \epsilon > 0 \) that

\[
L^{p+\epsilon}(\Omega) \subsetneq L^{(p,\theta)}(\Omega) \subsetneq L^p(\Omega) \subsetneq L^{p-\epsilon}(\Omega).
\]

The parameter \( \theta \) controls the “distance” of these spaces from \( L^p \): for instance, if \( \Omega = [0, 1] \), then we have that

\[
\left( \frac{1}{t} \right)^{\frac{1}{p}} \log \left( \frac{e}{t} \right)^{\frac{\theta+1}{p}} \in L^{p,\theta}.
\]

and for all non-negative \( \epsilon, \delta, \epsilon + \delta > 0 \),

\[
\left( \frac{1}{t} \right)^{\frac{1}{p}} \log \left( \frac{e}{t} \right)^{-(1+\epsilon)\frac{\theta}{p}-(1+\delta)\frac{\theta}{p'}} \in L^{(p,\theta)}.
\]

The first inclusion follows from the proof of [3, Proposition 5.6]. The second uses the proposition itself, which states that for \( \beta > 1 \), \( L^p(\log L)^{\beta(p-1)} \subset L^{(p,\theta)} \). For this embedding, see also [4].

The variable Lebesgue spaces generalize the classical Lebesgue spaces in a different way. Given a measurable function \( p(\cdot) : \Omega \to [1, \infty) \), we define \( L^{p(\cdot)}(\Omega) \) to be the collection of all measurable functions such that for some \( \lambda > 0 \),

\[
\rho(f/\lambda) = \int_\Omega \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty.
\]

\( L^{p(\cdot)}(\Omega) \) becomes a Banach function space with the norm

\[
\|f\|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho(f/\lambda) \leq 1 \}.
\]

These spaces were introduced by Orlicz in the 1930s and have been extensively studied for the past 25 years. (See [6] for more information on their history and applications.) If we define

\[
p_- = \text{ess inf}_{x \in \Omega} p(x), \quad p_+ = \text{ess sup}_{x \in \Omega} p(x),
\]

then

\[
L^{p_+}(\Omega) \subset L^{p(\cdot)}(\Omega) \subset L^{p_-}(\Omega).
\]
Equality holds if and only if \( p_- = p_+ \): i.e., when \( p(\cdot) \) is constant.

Given the embeddings \((1.3)\) and \((1.5)\), it is a natural question to ask if the stronger embeddings \( L^{p(\cdot)}(\Omega) \subset L^{p_-,\theta}(\Omega) \) and \( L^{p_+,\theta}(\Omega) \subset L^{p(\cdot)}(\Omega) \) are possible. Our first result gives a sufficient condition for these inclusions to hold.

To state it, we introduce some notation. Let \( \mathcal{P}(\Omega) \) denote the set of all measurable exponent functions \( p(\cdot) : \Omega \to [1, \infty) \). Given \( p(\cdot) \in \mathcal{P}(\Omega) \), let \( p_+(\cdot) : [0,1] \to [1,\infty) \) denote the decreasing rearrangement of \( p(\cdot) \). More precisely, define the distribution function

\[
\mu_{p(\cdot)}(t) = |\{ x \in \Omega : p(x) > t \}|
\]

and define the decreasing rearrangement by

\[
p_+(t) = \inf\{ \lambda \geq 0 : \mu_{p(\cdot)}(\lambda) \leq t \},
\]

where the infimum of the empty set is defined to be \(+\infty\). Let \( p^\ast(\cdot) \) denote the increasing rearrangement, defined by

\[
p^\ast(t) = -(p(\cdot))^\ast(t) = p_+(1-t).
\]

Note that if we modify \( p(\cdot) \) on a set of measure zero, we may assume without loss of generality that \( p_- = p^\ast(0) = p_+(1) \) and \( p_+ = p^\ast(0) = p^\ast(1) \). Moreover, we have that \( p_+(t) \to p_+(0) \) as \( t \to 0^+ \).

Given \( p(\cdot) \), \( 1 < p_- \leq p_+ < \infty \), define the conjugate exponent function \( q(\cdot) = p'(\cdot) \) by

\[
\frac{1}{p(\cdot)} + \frac{1}{q(\ast)(\cdot)} = 1.
\]

By taking rearrangements we see that

\[
\frac{1}{p_+(t)} + \frac{1}{q^\ast(t)} = 1,
\]

and the same equality holds for \( p^\ast \) and \( q_\ast \). We also have that \( p'_\ast = (p_-)' = p^\ast(0)' = q_\ast(0) \).

**Theorem 1.1.** Given an exponent \( p(\cdot) \in \mathcal{P}(\Omega) \), \( 1 < p_- \leq p_+ < \infty \), and \( \theta > 0 \), suppose that there exists \( 0 < t_0 \leq 1 \) and \( \epsilon > 0 \) such that for all \( t \in [0,t_0] \),

\[
(1.6) \quad \frac{1}{p^\ast(0)} - \frac{1}{p^\ast(t)} \geq \left( \frac{\theta}{p'_\ast} + \epsilon \right) \frac{\log \log \left( \frac{\Omega}{\epsilon} \right)}{\log \left( \frac{\Omega}{\epsilon} \right)}.
\]

Then

\[
(1.7) \quad L^{p(\cdot)}(\Omega) \hookrightarrow L^{p_-,\theta}(\Omega).
\]

As a consequence of Theorem 1.1 and the abstract properties of Banach function spaces we get the second desired inclusion.

**Theorem 1.2.** Given an exponent \( p(\cdot) \in \mathcal{P}(\Omega) \), \( 1 < p_- \leq p_+ < \infty \), and \( \theta > 0 \), suppose that there exists \( 0 < t_0 \leq 1 \) and \( \epsilon > 0 \) such that for all \( t \in [0,t_0] \),

\[
(1.8) \quad \frac{1}{p_+(t)} - \frac{1}{p_+(0)} \geq \left( \frac{\theta}{p'_\ast} + \epsilon \right) \frac{\log \log \left( \frac{\Omega}{\epsilon} \right)}{\log \left( \frac{\Omega}{\epsilon} \right)}.
\]
Then
\begin{equation}
L^{p^+},\theta(\Omega) \hookrightarrow L^{p^()}(\Omega).
\end{equation}

We can adapt the proof of Theorem 1.1 to get another scale of weaker continuity conditions on the exponent \( p(\cdot) \) for the desired embedding to hold. We will discuss this immediately after the proof: see Remark 2.1 below. However, the continuity conditions in Theorem 1.1 and 1.2 are in some sense sharp, as the next result shows.

**Example 1.3.** Given \( \theta > 0 \), there exists an increasing function \( p(\cdot) \in \mathcal{P}([0, 1]), 1 < p_\leq p_+ < \infty \), such that for \( t \in [0, e^{-2}] \),
\begin{equation}
\frac{1}{p(0)} - \frac{1}{p(t)} \leq \frac{\theta \log \log \left( \frac{e^t}{e} \right)}{p_+ \log \left( \frac{e^t}{e} \right)},
\end{equation}
and there exists \( f \in L^{p^+}([0, 1]) \) such that \( f \not\in L^{p_-, \theta}([0, 1]) \).

We could also consider the reverse inclusions: for which \( p(\cdot) \) do we have that \( L^{p^-(\cdot)}(\Omega) \subset L^{p^+}(\Omega) \) or \( L^{p^+}(\Omega) \subset L^{p^-(\cdot)}(\Omega) \)? However, while both inclusions are true if \( p(\cdot) \) is constant, if \( p_- < p_+ \), then neither can hold. By the same associate space argument as we use to prove Theorem 1.2 below, it suffices to show that the second inclusion can never hold. In this case, if \( p_- < p_+ \), there exists a set \( E \subset \Omega \), \(|E| > 0\), such that \( p_+(E) = \text{ess sup}_{x \in E} p(x) < p_+ \). But then there exists a function \( f \) such that \( \text{supp}(f) \subset E, f \in L^{p_+(E)}(E) \), and such that for any \( \delta > 0 \), \( f \not\in L^{p_+(E)+\delta}(E) \). Hence, \( f \in L^{p^+}(\Omega) \) (see [6, Corollary 2.50]), but not in \( L^{p^-, \theta} \): by definition, if \( f \in L^{p_+, \theta} \), then \( f \in L^{p^-+\epsilon}, 0 < \epsilon < p_+ - 1 \).

We can, however, prove a weaker result if we pass to the “rearranged” variable exponent spaces considered in [13, 14]. They showed the following result.

**Theorem 1.4.** Given \( p(\cdot) \in \mathcal{P}(\Omega), 1 < p_- \leq p_+ < \infty \), there exist constants \( c_1, c_2 > 0 \) such that for every \( u \in L^{p^+}(\Omega) \),
\begin{equation}
c_1 \|u_*\|_{p^*(\cdot)} \leq \|u\|_{p(\cdot)} \leq c_2 \|u_*\|_{p^*(\cdot)} \leq \infty.
\end{equation}

Note that the last inequality can be an equality: \( u \in L^{p^+}(\cdot) \) does not imply \( u_* \in L^{p_+}(\cdot) \); see [13, Remark after proof of Theorem 3]. But with this stronger hypothesis we have the following embedding.

**Theorem 1.5.** Given \( p(\cdot) \in \mathcal{P}(\Omega), 1 < p_- \leq p_+ < \infty \), and \( \theta \geq 1 \), suppose there exist \( A \in \mathbb{R} \) and \( 0 < t_0 \leq 1 \) such that for all \( t \in [0, t_0] \),
\begin{equation}
\frac{1}{p_*(t)} - \frac{1}{p_*(0)} \leq \frac{A}{\log \left( \frac{e^t}{e} \right)} + \frac{\theta - 1}{p_*(0)} \frac{\log \left( \frac{e^t}{e} \right)}{\log \left( \frac{e^t}{e} \right)}.
\end{equation}
Then for all \( u \in L^{p^+}(\Omega) \) such that \( u_* \in L^{p_+}(\cdot) ([0, 1]), u_* \in L^{p^+, \theta}([0, 1]) \).
Remark 1.6. When $\theta = 1$, Theorem 1.5 was proved in [14, Theorem 1]. Our proof generalizes and simplifies theirs. Note that in this case we must assume $A > 0$.

Remark 1.7. We conjecture that some version of Theorem 1.5 is true for $0 < \theta < 1$, but we have not been able to prove it. Note that for $\theta < 1$ the condition (1.11) is never possible: the left-hand side is positive, but the right-hand side is negative for all $t$ sufficiently close to 0.

Remark 1.8. We conjecture that if (1.11) holds, then a “dual” result holds as well. More precisely, we conjecture that given any decreasing function $u_\ast \in L(q-\theta, \cdot)$, we have $u_\ast \in L^q(\cdot)$. However, unlike in the proof of Theorem 1.2, we cannot use associativity to prove this since we are not dealing with a subspace but rather the cone of decreasing functions. Moreover, our other techniques do not seem applicable to this case.

The condition (1.11) is close to optimal as the following example shows.

Example 1.9. Given $p(\cdot) \in P(\Omega)$, $1 < p_- \leq p_+ < \infty$, suppose there exists $\theta > 0$, $\epsilon > 0$ and $0 < t_0 \leq 1$, such that for all $t \in [0, t_0]$,

\begin{equation}
\frac{1}{p_+(t)} - \frac{1}{p_+(0)} \geq \left( \frac{\theta + \epsilon}{p_+(0)} \right) \log \log \left( \frac{t}{\epsilon \log(t)} \right).
\end{equation}

Then there exist a (decreasing) function $f_\ast \in L^{p_-}(\cdot) \setminus L^{p_+}(\cdot)$.

Remark 1.10. If we compare the two conditions (1.11) and (1.12) when $\theta = 1$, we see that they differ by a factor of $\log \log \left( \frac{t}{\epsilon} \right)$. It is not clear if this gap can be closed or if either condition is optimal.

We can extend Theorem 1.5 to the range $0 < \theta < 1$, and generalize it for $\theta \geq 1$, if we pass to a larger scale of spaces. Given a decreasing function $\sigma_\ast : [0, 1] \rightarrow \mathbb{R}$, define the function $\varphi : [0, 1] \times \mathbb{R} \rightarrow [0, \infty)$ by

$$\varphi(a, b) = b^{p_+(a)} \log(e + b)^{\sigma_+(a)}.$$  

We define the space $L^{\varphi}(\cdot)$ to consist of all measurable functions $f_\ast$ defined on $[0, 1]$ such that for some $\lambda > 0$,

$$\rho_{\varphi}(f/\lambda) = \int_0^1 \varphi \left( t, \frac{|f(t)|}{\lambda} \right) dt < \infty.$$  

With a norm defined as above for the variable Lebesgue spaces, $L^{\varphi}(\cdot)$ becomes a Banach function space, a particular case of the Musielak-Orlicz spaces, also referred to as generalized Orlicz spaces. With this definition of $\varphi$, these spaces were first considered in [5] and were later considered by other authors: see [6, 8] for details and further references.

Theorem 1.11. Given $\theta > 0$, let $\sigma_\ast(\cdot) : [0, 1] \rightarrow \mathbb{R}$ be a bounded, decreasing function such that $\sigma_\ast(0) \geq 1 - \theta$. Suppose further that there exists $B > 0$ and $0 < t_0 \leq 1$ such that for
Given \( p(\cdot) \in \mathcal{P}(\Omega), 1 < p_- \leq p_+ < \infty \), suppose there exists \( A \in \mathbb{R} \) such that for \( t \in [0, t_0] \),

\[
\frac{1}{p_+(t)} - \frac{1}{p_+(0)} \leq \frac{A}{\log(\frac{e}{t})} + \frac{\theta - 1 + \sigma_+(0) \log \log(\frac{e}{t})}{p_+(0) \log(\frac{e}{t})}.
\]

Let \( \varphi(a, b) = b^{p_+(a)} \log(e + b)^{\sigma_+(a)} \). Then, for all \( u \in L^{p(\cdot)}(\Omega) \) such that \( u_+ \in L^{p(\cdot)}([0, 1]) \), \( u_+ \in L^{p_+(\cdot)\theta}([0, 1]) \).

**Remark 1.12.** If \( \sigma_+(\cdot) \equiv 0 \), then Theorem 1.11 reduces to Theorem 1.5. Theorem 1.11 is a more general result: for example, when \( \theta > 1 \), if \( \sigma_+(\cdot) \equiv 1 - \theta \), then \( L^{p(\cdot)}([0, 1]) \subset \subset L^{\varphi(\cdot)}([0, 1]) \). (See [17, Chapter II.8].)

The proofs of our results are given in the next section. Throughout, our notation is standard; for variable Lebesgue spaces we follow the notation established in [6]. Constants \( C, c, \ldots \) may vary in value from line to line. If we write \( A \lesssim B \), we mean that there exists a constant \( c \) such that \( A \leq cB \); the constant \( c \) can depend on the exponent function \( p(\cdot) \) and other fixed parameters, but it does not depend on any variables in functions or summations. If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \).
Further, if we exponentiate our hypothesis (1.6), for \( t \in [0, t_0] \) we get

\[
\left( \frac{e}{t} \right)^{\frac{q}{p} - 1} \geq \log \left( \frac{e}{t} \right)^{\theta + \epsilon}.
\]

Therefore, we have that

\[
\|f\|_{p(\cdot)} \int_{0}^{1} \| \chi_{(0,t)} \|_{r_{\ast}(\cdot)} \frac{dt}{t \log(t)^{1 - \frac{1}{p_{\ast}(t)}}} \leq \|f\|_{p(\cdot)} \int_{0}^{1} \frac{t}{t \log(t)^{1 - \frac{1}{p_{\ast}(t)}}} dt = \|f\|_{p(\cdot)} \int_{0}^{1} \frac{dt}{t \log(t)^{1 + \epsilon}} \lesssim \|f\|_{p(\cdot)};
\]

for the second to last inequality we use (2.1) for \( t \in [0, t_0] \); for \( t \in [t_0, 1] \) we use that all of these functions are bounded and bounded away from 0. Combining these two inequalities, we see that (1.7) holds.

\[\square\]

**Remark 2.1.** If we replace the hypothesis (1.6) with the weaker assumption that for some \( \epsilon > 0 \) and \( t \in [0, t_0] \),

\[
\frac{1}{p^*(0)} - \frac{1}{p^*(t)} \geq \frac{\theta}{p_{\ast}} \frac{\log \log(t)}{\log(t)} + (1 + \epsilon) \frac{\log \log(t)}{\log(t)};
\]

then the above argument goes through with almost no change, except that in the final inequality the last integral becomes

\[
\int_{0}^{1} \frac{dt}{t \log(t)^{1 + \epsilon}} < \infty.
\]

Note, however, that if we take \( \epsilon = 0 \) in (2.2), we do not recapture the sharp endpoint condition (1.10).

Even weaker sufficient conditions can be found by finding larger functions that are still in \( L^1([0, 1]) \) and working backwards through the proof. Details are left to the interested reader.

**Proof of Theorem 1.2.** This is an immediate consequence of Theorem 1.1 and the abstract properties of Banach function spaces. Given \( p(\cdot) \), recall that we let \( q(\cdot) = p'(\cdot) \) be the dual exponent. Suppose (1.8) holds; then a straightforward calculation shows that (1.6) holds for \( q(\cdot) \). Hence, we have that \( L^{q(\cdot)}(\Omega) \subset L^{(p_{\ast}, \theta)}(\Omega) \).

Given a Banach function space \( X \), let \( X' \) denote its associate space. Then \( L^{q(\cdot)}(\Omega)' = L^{p'(\cdot)}(\Omega) \) (6, Proposition 2.37) and since \( (L^{p_{\ast}, \theta})' = L^{p', \theta} = L^{q_{\ast}, \theta} \), we have that \( (L^{q(\cdot), \theta})' = (L^{p_{\ast}, \theta})' = L^{p_{\ast}, \theta} \) (2, Theorem 2.7). By [2, Proposition 2.10], if \( Y \) is another Banach function space such that \( X \subset Y \), then \( Y' \subset X' \). Hence, we get (1.9) as desired. \[\square\]
Construction of Example 1.3. For \( t \in [0, e^{-2}] \), define
\[
p(t) = 2 + 2\theta \log \log(\frac{e}{t}) \frac{\log \log(\frac{e}{t})}{\log(\frac{e}{t})}.
\]
(Note that we could replace 2 by any value \( q > 1 \); however, for clarity we will restrict ourselves to this special case.) Then we have that \( p(0) = p_- = 2 \) and \( p(\cdot) \) is increasing. To get an increasing exponent function on \([0, 1]\) we can extend \( p(\cdot) \) to be constant on \([e^{-2}, 1]\).

A straightforward calculation shows that for \( t \in [0, e^{-2}] \), \( p(\cdot) \) satisfies (1.10):
\[
\frac{1}{p(\cdot)} - \frac{1}{p(t)} = \frac{\theta}{2\log(\frac{e}{t}) + \log(\frac{e}{t})} \leq \frac{\theta}{p_- \log(\frac{e}{t})}.
\]

Now fix \( 1 < b < 2 \) and for \( t \in [0, 1/e^2] \) define
\[
f(t) = \sum_{j=2}^{\infty} a_j \chi_{(e^{-j-1}, e^{-j}]}(t), \quad a_j = \left[ \frac{e^j}{j \log(j)^b} \right]^{1/2 + \theta \log(j+2)}.
\]

Extend \( f \) to be constant on \([e^{-2}, 1]\); then \( f \) is a decreasing function on \([0, 1]\). We will first show that \( f \in L^{p(\cdot)}([0, 1]) \) and then show that \( f \notin L^{p_- \theta}([0, 1]) \). We first estimate \( f \) on \([0, e^{-2}]\):
\[
\int_0^{e^{-2}} f(t)^p(t) dt = \sum_{j=2}^{\infty} \int_{e^{-j-1}}^{e^{-j}} a_j \frac{2^{2g \log(\frac{e}{t})}}{\log(\frac{e}{t})} dt
\]
\[
\leq \sum_{j=2}^{\infty} \int_{e^{-j-1}}^{e^{-j}} a_j \frac{2^{2g \log(j+2)}}{j+1} dt
\]
\[
< \sum_{j=2}^{\infty} \frac{1}{j \log(j)^b} < \infty.
\]

Since \( f \) and \( p(\cdot) \) are constant on \([e^{-2}, 1]\), it follows that \( f \in L^{p(\cdot)}([0, 1]) \). (This follows from the definition of the norm since \( p_+ < \infty \).)

We now prove that \( f \notin L^{p_- \theta}([0, 1]) \). Fix \( t \in (0, \frac{1}{e^2}] \); there exists \( j > 2 \) such that \( e^{-j} < t \leq e^{-j+1} \). But then we have that
\[
\|f \chi_{(0,t)}\|_2^2 = \int_t^\infty \left( \sum_{k=2}^{\infty} a_k \chi_{(e^{-k-1}, e^{-k}]}(s) \right)^2 ds
\]
\[
\sum_{k=2}^{\infty} \int_0^{e^{-j}} a_k^2 \chi_{(e^{-k-1}, e^{-k})}(s) \, ds
\]

\[
= \sum_{k=j}^{\infty} \int_{e^{-k-1}}^{e^{-k}} a_k^2 \, ds
\]

\[
\gg \sum_{k=j}^{\infty} a_k^2 e^{-k}
\]

\[
= \sum_{k=j}^{\infty} \frac{1}{k^{1+\theta \log(k+2)} \log(k) b^{(k+1)}}
\]

To estimate the final sum we will compare it to the corresponding integral. For \( x \geq 2 \),

\[
(\theta + b) \int_x^{\infty} \frac{dt}{t^{1+\theta \log(t)^b}} \geq \int_x^{\infty} \left( \frac{\theta}{t} + \frac{b}{t \log(t) \log(b)} \right) \frac{1}{t^{1+\theta \log(t)^b}} \, dt = \frac{1}{x^{\theta \log(x)^b}}.
\]

Combining these inequalities we get that

\[
\left\| f \chi_{(0,t)} \right\|_2^2 \geq \sum_{k=j}^{\infty} \frac{1}{k^{1+\theta \log(k)^b}} \geq \frac{1}{j^{\theta \log(j)^b}}.
\]

Therefore, by (1.2) we have that

\[
\left\| f \right\|_{L^p_{\theta \log}} \geq \int_0^{\infty} \left\| f \chi_{(0,t)} \right\|_2 \frac{1}{t \log(t)^{1-\frac{\theta}{2}}} \, dt
\]

\[
= \sum_{j=3}^{\infty} \int_{e^{-j}}^{e^{-j+1}} \left\| f \chi_{(0,t)} \right\|_2 \frac{1}{t \log(t)^{1-\frac{\theta}{2}}} \, dt
\]

\[
\gg \sum_{j=3}^{\infty} \int_{e^{-j}}^{e^{-j+1}} \frac{1}{j^{\frac{\theta}{2} \log(j)^{\frac{\theta}{2}}} e^{j \log(e^{j+1})^{1-\frac{\theta}{2}}}} \, dt
\]
\[ \sum_{j=3}^{\infty} \frac{1}{j \log(j)^{\frac{\theta}{2}}} = \infty. \]

The last sum is infinite because \( 1 < b < 2 \). Thus, we have that \( f \not\in L^{(p-\theta)}([0,1]). \)

**Proof of Theorem 1.5.** If we rearrange (1.11) as

\[ \left( \frac{1}{p_*(t)} - \frac{1}{p_*(0)} \right) \log \left( \frac{e}{t} \right) \leq A + \frac{\theta - 1}{p_*(0)} \log \log \left( \frac{e}{t} \right) \]

and exponentiate, we get that for \( 0 < t \leq t_0 \),

\[ \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} - \left( \frac{e}{t} \right)^{\frac{1}{p_*(0)}} \leq e^A \log \left( \frac{e}{t} \right)^{\frac{\theta - 1}{p_*(0)}}. \]

Since \( u_* \in L^{p_*}([0,1]) \), by the proof of [14, Theorem 1] we have that for \( 0 \leq t \leq 1 \),

\[ u_*(t) \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}}. \]

Therefore, if we combine these two inequalities, we get that for \( 0 < t \leq t_0 \),

\[ u_*(t) \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(0)}} \log \left( \frac{e}{t} \right)^{\frac{\theta - 1}{p_*(0)}}. \]

Since \( u_*(t) \) is bounded and the righthand side is bounded away from 0 for \( t_0 \leq t \leq 1 \), the same inequality holds (with a possibly larger constant \( C \)) for all \( t \). Therefore, by (1.4) we have that \( u_*(t) \in L^{p_*,\theta}([0,1]). \)

**Construction of Example 1.9.** Define the function \( f_* \) on \([0, t_0], 0 < t_0 < 1\), by

\[ f_*(t) = \left[ \log \log \left( \frac{t}{\xi} \right) \right]^{\frac{1}{p_*(0)}.} \]

Note that for all \( \theta > 0 \) there exists \( t_0 > 0 \) such that \( f_* \) is decreasing on \([0, t_0] \); extend \( f_* \) to be constant on \([t_0, 1]\) to get a decreasing function on \([0, 1]\). Without loss of generality we may assume that this is the same \( t_0 \) as in the hypotheses.

We will first show that \( f_* \not\in L^{p_*,\theta}([0,1]) \) and then prove that \( f_* \in L^{p_*}([0,1]) \). We may assume that \( t_0 \) is small enough that there exists \( c > 0 \) so that for \( t \in [0, t_0] \),

\[ \frac{c(1 + \log \log \left( \frac{t}{\xi} \right))}{t \log \left( \frac{t}{\xi} \right)^{1-\theta}} \leq \frac{\log \log \left( \frac{t}{\xi} \right)}{t \log \left( \frac{t}{\xi} \right)^{1-\theta}}. \]

But then for \( t < t_0 \) sufficiently close to 0,

\[ \int_t^{t_0} f_*(s)^{p_*(0)} ds \geq \int_t^{t_0} \frac{\log \log \left( \frac{t}{s} \right)}{s \log \left( \frac{t}{s} \right)^{1-\theta}} ds \geq \int_t^{t_0} \frac{c(1 + \log \log \left( \frac{t}{s} \right))}{s \log \left( \frac{t}{s} \right)^{1-\theta}}. \]
\[
= c \log \left( \frac{e}{t} \right) \log \left( \frac{e}{t} \right)^\theta - c \log \log \left( \frac{e}{t_0} \right) \log \left( \frac{e}{t_0} \right)^\theta \geq c \log \log \left( \frac{e}{t} \right) \log \left( \frac{e}{t} \right)^\theta.
\]

Hence, for all \( t \in [0, 1] \),
\[
\log \left( \frac{e}{t} \right)^{-\theta} \int_t^1 f(s)^{p_+(0)} \, ds \geq C \log \log \left( \frac{e}{t} \right),
\]
and so by (1.1), \( f_+(0) \notin \mathcal{L}_{p^+}^\varphi \).

To prove that \( f_* \in \mathcal{L}^{p_+}([0, 1]) \), first note that if we rearrange and exponentiate (1.12), we get
\[
\left( \frac{e}{t} \right)^{\frac{p_+(t)}{p_+(0)}} \leq \left( \frac{e}{t} \right)^{-\frac{\varphi(t,u(t))}{\varphi(0)}}.
\]
Since \( p_+(t) \to p_+(0) \) as \( t \to 0 \), if necessary by taking \( t_0 > 0 \) smaller, we may assume that there exist \( 0 < \sigma < \tau < \epsilon \) such that for all \( t \in [0, t_0] \),
\[
\frac{p_+(t)}{p_+(0)} (1 + \epsilon - \tau) > 1 + \sigma.
\]

Given these two inequalities we can estimate as follows:
\[
\int_0^{t_0} f_*(t)^{p_+(t)} \, dt = \int_0^{t_0} \left[ \log \log \left( \frac{e}{t} \right) \right]^{\frac{p_+(t)}{p_+(0)}} dt \lesssim \int_0^{t_0} \left[ \frac{\log(\frac{e}{t})}{t \log(\frac{e}{t})^{1-\theta}} \right]^{\frac{p_+(t)}{p_+(0)}} dt \\
\lesssim \int_0^{t_0} \frac{dt}{t \log(\frac{e}{t})^{\left(\frac{\varphi(t,u(t))}{\varphi(0)}(\theta + 1 - \theta - \sigma)\right)}} \leq \int_0^{t_0} \frac{dt}{t \log(\frac{e}{t})^{1+\sigma}} < \infty.
\]

Given this, and since \( f_* \) is bounded on \([t_0, 1]\), we conclude that \( f_* \in \mathcal{L}^{p_+}([0, 1]) \). \( \square \)

**Proof of Theorem 1.11.** We begin by generalizing an argument in the proof of [14, Theorem 1]. First note that for \( b > 1 \), \( \varphi(a,b) \) is decreasing in \( a \), and that for fixed \( a \) it is increasing in \( b \) for all \( b \) sufficiently large. Second, we may assume without loss of generality that \( u_* \) is unbounded, since otherwise the desired inclusion holds trivially. But then there exists \( 0 < t_0 \leq 1 \) such that for all \( 0 < s \leq t \leq t_0 \), \( u_*(s) \geq 1 \) and \( \varphi(t,u(s)) \) is decreasing in \( s \). Therefore, since \( u_* \in L^\varphi([0, 1]) \), there exists a constant \( c > 0 \) such that for all such \( t \),
\[
c \geq \int_0^t u_*(s)^{p_+}(s) \log(e + u_*(s))^{\sigma_+(s)} \, ds \\
\geq \int_0^t u_*(s)^{p_+(t)} \log(e + u_*(s))^{\sigma_+(t)} \, ds \geq tu_*(t)^{\varphi_+(t)} \log(e + u_*(t))^{\sigma_+(t)}.
\]

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\[1.1\]
\[1.11\]
\[1.12\]
\[14\]
\[0\]
\[91\]
\[441\]
We want to rearrange this inequality to dominate \( u_*(t) \). If we fix \( a \), then the inverse of \( \varphi(a, b) \) as a function of \( b \) is

\[
\varphi^{-1}(a, b) \approx b^{\frac{1}{p_*(a)}} \log(e + b)^{-\frac{\sigma_*(a)}{p_*(a)}}.
\]

The implicit constants depend on \( p_*(a) \) and \( \sigma_*(a) \). Therefore, since these functions are bounded, we have that there exists an absolute constant \( C \) such that

\[
 u_*(t) \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} \log \left( e + \frac{c}{t} \right)^{-\frac{\sigma*(t)}{p_*(t)}} \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} \log \left( e \right)^{-\frac{\sigma*(t)}{p_*(t)}}.
\]

By possibly taking \( t_0 \) closer to 0, we have, by rearranging and exponentiating (1.13) and (1.14), that for all \( t \in [0, t_0] \),

\[
 \log \left( \frac{e}{t} \right)^{\sigma_*(0) - \sigma_*(t)} \leq e^{B},
\]

and

\[
 \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} \leq e^{A} \log \left( \frac{e}{t} \right)^{\frac{\theta - 1 + \sigma_*(0)}{p_*(0)}}.
\]

If we combine these three inequalities, we see that for \( t \in [0, t_0] \),

\[
 u_*(t) \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} \log \left( \frac{e}{t} \right)^{-\frac{\sigma*(t)}{p_*(t)}} \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} \log \left( \frac{e}{t} \right)^{-\frac{\theta - 1 + \sigma_*(0)}{p_*(0)}} \log \left( \frac{e}{t} \right)^{\frac{\sigma*(0) - \sigma_*(t)}{p_*(0)}} \log \left( \frac{e}{t} \right)^{-\frac{\sigma*(t)}{p_*(t)}} \leq C \left( \frac{e}{t} \right)^{\frac{1}{p_*(t)}} \log \left( \frac{e}{t} \right)^{\frac{\theta - 1 + \sigma_*(0)}{p_*(0)}} ;
\]

in the final inequality we used the fact that the exponent of the last log term is negative. Since \( u_*(t) \) is bounded for \( t_0 < t \leq 1 \), we conclude that \( u_* \in L^{p+\theta}([0, 1]) \). □

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