Rotating black holes with scalar hair in three dimensions

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We examine the first law of thermodynamics in (2+1)-dimensional rotating hairy black holes and find that the first law of black hole thermodynamics can be protected when the scalar field parameter $B$ is constrained to relate to the black hole size. We disclose the Hawking-Page phase transition between the hairy black holes and the pure thermal radiation. Moreover, we find that the free energies of the rotating hairy black holes depend on the ratio between the horizon size to the scalar field parameter $B$. We also compare the free energies for the hairy black hole and the BTZ black hole when they have the same temperature and angular momentum, and find that when this ratio is large, the BTZ black hole has smaller free energy which is a thermodynamically more preferred phase; but when the ratio is small, the hairy black hole has smaller free energy and there exists the possibility for the BTZ black hole to dress up scalar field and become hairy.

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I. INTRODUCTION

Hairy black holes are interesting solutions in gravitational theories. They have been studied extensively over the past few years, mainly in connection with the no hair theorem. In general, the no-hair theorem rules out four dimensional black holes coupled to scalar field in asymptotically flat spacetimes [1, 3], because they are not physically acceptable since the scalar field can diverge on the horizon and black holes can become unstable [4]. In higher dimensional cases ($D > 4$), the hairy black hole solutions in asymptotically flat spacetimes simply do not exist [5, 6]. When the cosmological constant is taken into account, the no-hair theorem can usually be circumvented, and...

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then some hairy black holes have been found and the divergence of the scalar field can be hidden behind the horizon. In asymptotically de Sitter (dS) spacetimes, hairy black holes minimally coupled to scalar field were presented in [7], but it was found dynamically unstable. The exact black hole solutions with non-minimally coupled scalar field in dS spacetime have also been discussed in [8, 9], while they were also found unstable [10, 11].

When a negative cosmological constant is considered, four dimensional (MTZ) black hole solutions with minimal [12, 13] and non-minimal [14] scalar fields have been obtained. Thermodynamically, it was observed that when above some critical temperature \(T > T_c\) the black holes without scalar hair are more preferred, while below some critical temperature the hairy MTZ black holes are always thermodynamically favored. The phase transitions between the MTZ black holes and black holes without scalar hair have been found of the second order both in the canonical ensemble [15] and the grand canonical ensemble [16]. The hairy black holes in anti-de Sitter(AdS) spaces were found stable by examining quasi-normal modes in their backgrounds [17, 18]. Interestingly, the behavior of quasi-normal modes was disclosed useful to reflect the thermodynamical second-order phase transition [17, 18]. More discussions on the hairy black hole solutions in spherical AdS spacetimes can be found in [19–24]. Rotating hairy AdS black holes with complex, massive scalar fields have been addressed in [25, 26]. Further explorations of the AdS hairy black holes in higher order derivative gravity have been reported in [27–30].

In (2+1)-dimensional Einstein gravity, there exists static black holes with minimally [31] or non-minimally [20, 32] coupled scalar fields. Moreover, the thermodynamical properties of these hairy black holes were studied in [15, 33, 34], where the first law of black hole thermodynamics was found always hold. Recently, rotating black holes with non-minimally [35] and minimally [36] coupled scalar fields in the (2+1)-dimensional Einstein gravity were obtained and attracted attentions [37–39]. In this article, we will explore the thermodynamical properties of these rotating black holes with (non)minimally coupled scalar fields. We will examine the first law of thermodynamics in these black hole backgrounds. We will argue that when the parameter \(B\) of the scalar field is completely free [40], the first law of thermodynamics cannot be protected. To rescue the first law, we need to rewrite scalar potential \(V(\phi)\) which leads to a new form of the rotating hairy black hole in three dimensions. In this new black hole solution, the parameter \(B\) of the scalar field is related to the black hole horizon size through \(r_+ = \theta \times B\) where \(\theta\) is a dimensionless constant. We will show that this constraint on the parameter \(B\) is required to ensure the validity of the first law of thermodynamics. Furthermore, we will investigate the phase transitions between the (2+1)-dimensional rotating black holes (non)minimally coupled to scalar fields and the rotating
BTZ black holes.

This paper is organized as follows: In Sec. [II] we will present the rotating black hole solution in Einstein gravity with non-minimally coupled scalar field. In Sec. [III] we will discuss the thermodynamical properties of this rotating hairy black hole and examine the first law of the black hole thermodynamics. Moreover, we will study the phase transition between the rotating hairy black hole and the rotating BTZ black hole. In Sec. [IV] we will extend the discussion to the rotating black hole with minimally coupled scalar field. Finally, we will present conclusions and discussions in Sec. [V].

II. ROTATING BLACK HOLES NON-MINIMALLY COUPLED TO SCALAR FIELD

We start with the action in (2 + 1)-dimensional Einstein gravity with non-minimally coupled scalar field [20, 35]

\[ I = \frac{1}{2} \int d^3x \sqrt{-g} \left( R - g^{\mu \nu} \nabla_{\mu} \phi \nabla_{\nu} \phi - \xi R \phi^2 - 2V(\phi) \right), \]  

(1)

where \( \xi \) equals to 1/8 signifying the coupling strength between gravity and the scalar field. With the scalar potential

\[ V(\phi) = -\frac{1}{l^2} + \frac{1}{512} \left( \frac{1}{l^2} + \frac{\beta}{B^2} \right) \phi^6 + \frac{1}{512} \left( \frac{a^2}{B^4} \right) \frac{\phi^6 - 40\phi^4 + 640\phi^2 - 4608}{(\phi^2 - 8)^5} \phi^{10}, \]  

(2)

we can have the rotating black hole solution [35]

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 (d\psi + \omega(r)dt)^2, \]  

(3)

\[ f(r) = 3\beta + \frac{2B\beta}{r} + \frac{(3r + 2B)^2 a^2}{r^4} + \frac{r^2}{l^2}, \quad \omega(r) = -\frac{(3r + 2B)a}{r^3}, \]  

(4)

where the scalar field takes the form

\[ \phi(r) = \pm \sqrt{\frac{8B}{r + B}}, \]  

(5)

and the parameters \( \beta, B \) and \( a \) are integration constants.

The hairy black hole solution (Eq. (4)) can reduce to the rotating BTZ black hole solution [41] if one takes \( B \to 0 \), which leads \( \beta = -\frac{M}{3} \) and \( a = \frac{J}{6} \). The rotating black hole solution in the gravity with non-minimally coupled scalar field can be rewritten into [35]

\[ f(r) = -M \left( 1 + \frac{2B}{3r} \right) + \frac{r^2}{l^2} + \frac{(3r + 2B)^2 J^2}{36r^4}, \quad \omega(r) = -\frac{(3r + 2B)J}{6r^3}. \]  

(6)
The thermodynamical quantities of this hairy black hole are [40]

\[ M = \frac{J^2 l^2 (2B + 3r_+)^2 + 36r_+^6}{12l^2 r_+^3 (2B + 3r_+)}, \] (7)

\[ T = \frac{f'(r_+)}{4\pi} = \frac{(B + r_+) \left[ 36r_+^6 - J^2 l^2 (2B + 3r_+)^2 \right]}{24\pi l^2 r_+^5 (2B + 3r_+)}, \] (8)

\[ S = \frac{A_H}{4G} \left( 1 - \xi \phi^2 (r_+) \right) = \frac{4\pi r_+^2}{B + r_+}, \] (9)

\[ \Omega_H = -\omega(r_+) = \frac{(3r_+ + 2B) J}{6r_+^3}, \] (10)

where the parameter \( B \) of the scalar field is set to be arbitrary [40]. We find

\[ \frac{\partial M}{\partial S} = \frac{(B + r_+) \left[ 36r_+^6 - J^2 l^2 (2B + 3r_+)^2 \right]}{8\pi l^2 r_+^5 (2B + 3r_+) (2B + r_+)}, \] (11)

which is different from the Eq. (8), so that the first law of black hole thermodynamics meets challenge in this rotating hairy black hole.

To protect the first law, we turn to define \( \beta = \mu B^2 \) and \( a = \alpha B^2 \) where \( \mu \) and \( \alpha \) are constants and the scalar potential \( V(\phi) \) (Eq. (2)) can be changed to

\[ \tilde{V}(\phi) = -\frac{1}{l^2} + \frac{1}{512} \left( \frac{1}{l^2} + \mu \right) \phi^6 + \frac{\alpha^2 (\phi^6 - 40\phi^4 + 640\phi^2 - 4608)}{512(\phi^2 - 8)^5} \phi^{10}. \] (12)

Inserting Eq. (12) into the action, we can find the rotating hairy black hole with the metric coefficient

\[ f(r) = \mu B^2 \left( 3 + \frac{2B}{r} \right) + \frac{(3r + 2B)^2 \alpha^2 B^4}{r^4} + \frac{r^2}{l^2}, \quad \omega(r) = -\frac{\alpha B^2 (3r + 2B)}{r^3}. \] (13)

When \( \alpha \to 0 \), the rotating black hole solution \( f(r) \) reduces to the nonrotating black hole solution in \((2 + 1)\)-dimensional Einstein gravity with non-minimally coupled scalar field [20].

To see the horizon structures of this new form of rotating hairy black hole, we define

\[ X = \frac{3r_+ + 2B}{r_+^3}, \] (14)

so that \( f(r_+) = 0 \) gives

\[ \alpha^2 B^4 X^2 + \mu B^2 X + \frac{1}{l^2} = 0, \] (15)

which leads to

\[ X_1 = -\frac{\mu - \sqrt{\mu^2 - 4\alpha^2 l^2}}{2\alpha^2 B^2} = \frac{X_1}{B^2}, \]

\[ X_2 = -\frac{\mu + \sqrt{\mu^2 - 4\alpha^2 l^2}}{2\alpha^2 B^2} = \frac{X_2}{B^2}. \] (16)
Here $X_{\nu}, (\nu = 1, 2)$ are both real positive when $-\mu \geq \frac{2\alpha}{l}$, but they are both imaginary when $-\mu < \frac{2\alpha}{l}$. We focus on the case when $-\mu \geq \frac{2\alpha}{l}$, thus $\tilde{X}_\nu > 0$.

Equating $X$ to $X_\nu$, Eq. (14) becomes

$$H_\nu(r_+) = X_\nu r_+^3 - 3r_+ - 2B = \tilde{X}_\nu r_+^3 - 3B^2 r_+ - 2B^3 = 0. \quad (17)$$

The only positive solution of Eq. (17) exists when $\frac{108B^6(\tilde{X}_\nu-1)}{X_\nu^3} > 0$, namely $\tilde{X}_\nu \geq 1$, which is given by

$$r_+^\nu = \frac{1}{X_\nu} \left[ \left( \tilde{X}_\nu^2 - \sqrt{(\tilde{X}_\nu - 1) \tilde{X}_\nu^3} \right)^{1/3} + \left( \tilde{X}_\nu^2 - \sqrt{(\tilde{X}_\nu - 1) \tilde{X}_\nu^3} \right)^{1/3} \right] B = \hat{\theta}(\tilde{X}_\nu) B. \quad (18)$$

When $\frac{108B^6(\tilde{X}_\nu-1)}{X_\nu^3} < 0$, namely $0 < \tilde{X}_\nu < 1$, there exists three roots for the solution of Eq. (17), but we only have the interest on the positive one

$$r_+^\nu = \frac{B}{\sqrt{\tilde{X}_\nu}} \left( \cos \eta + \sqrt{3} \sin \eta \right) = \hat{\theta}(\tilde{X}_\nu) B, \quad \eta = \frac{\arccos(-\sqrt{\tilde{X}_\nu})}{3}. \quad (19)$$

From the horizon structure, we find the constraint condition for the parameter of the scalar field $B$, $r_+^\nu = \theta(\tilde{X}_\nu) B$ when $\tilde{X}_\nu > 0$. The parameter $B$ is no longer free. The coupling constant $\theta(\tilde{X}_\nu)$ is plotted in Fig.1. For simplicity, we write the constraint relation into

$$r_+ = \theta \times B \quad (20)$$

in the following.

The scalar field $\phi(r)$ at the black hole horizon becomes

$$\phi(r_+) = \pm \sqrt{\frac{8B}{r_+ + B}} = \pm \sqrt{\frac{8}{1 + \theta}}, \quad (21)$$

which is independent of the horizon radius $r_+$. 

![FIG. 1: $\theta(\tilde{X}_\nu)$ versus $\tilde{X}_\nu$.](image)
III. THERMODYNAMICS OF THE ROTATING BLACK HOLE WITH NON-MINIMAL SCALAR HAIR

In this rotating hairy black hole, the mass and angular momentum can be calculated by adopting the Brown-York method [42]. The quasilocal mass $m(r)$ at $r$ takes the following form [43–45]

$$m(r) = \sqrt{f(r)}E(r) - j(r)\omega(r), \quad (22)$$

where $E(r) = 2(\sqrt{f_0(r)} - \sqrt{f(r)})$ is the quasilocal energy at $r$, and $j(r) = \frac{d\omega(r)}{dr}r^3$ is the quasilocal angular momentum. As a result, the mass and angular momentum of the black hole can be obtained as

$$M \equiv \lim_{r \to \infty} m(r) = -3\mu B^2, \quad J \equiv \lim_{r \to \infty} j(r) = 6\alpha B^2. \quad (23)$$

Thus the black hole metric coefficient (Eq. (13)) can be rewritten into

$$f(r) = -M \left(1 + \frac{2B}{3r}\right) + \frac{r^2}{l^2} + \frac{(3r + 2B)^2J^2}{36r^4}, \quad \omega(r) = -\frac{(3r + 2B)J}{6r^3}. \quad (24)$$

The condition $-\mu \geq \frac{2\Omega}{l}$ is needed to protect the cosmic censorship, which puts the constraint $\frac{M}{J} \geq \frac{1}{r}$ for this black hole.

A. The first law of black hole thermodynamics

With Eq. (20), the mass of the rotating hairy black hole reads

$$M = \frac{J^2l^2(2B + 3r_+)^2 + 36r_+^6}{12l^2r_+^3(2B + 3r_+)} = \frac{J^2l^2(2 + 3\theta)^2 + 36r_+^4\theta^2}{12l^2r_+^3\theta(2 + 3\theta)}, \quad (25)$$

and the entropy is

$$S = \frac{A_H}{4G} \left(1 - \xi \phi^2(r_+)\right) = \frac{4\pi r_+^2}{B + r_+} = \frac{4\pi \theta r_+}{1 + \theta}. \quad (26)$$

The temperature of this rotating hairy black hole can be derived as

$$T = \frac{f'(r_+)}{4\pi} = \frac{(B + r_+)[36r_+^6 - J^2l^2(2B + 3r_+)^2]}{24\pi l^2r_+^5(2B + 3r_+)} = \frac{1 + \theta}{24\pi l^2r_+^3\theta(2 + 3\theta)} \left[36r_+^4\theta^2 - J^2l^2(2 + 3\theta)^2\right]. \quad (27)$$
For $T = 0$, we can obtain the radius of extremal rotating hairy black hole $r_{\text{ext}} = \left[ \frac{\mathcal{J}l(2+3\theta)}{6\theta} \right]^{1/2}$.

In addition, there exists two commuting Killing vector fields for the metric (Eq. (3))

$$\xi_{(t)} = \frac{\partial}{\partial t}, \quad \xi_{\psi} = \frac{\partial}{\partial \psi}. \quad (28)$$

The various scalar products of these Killing vectors can be expressed through the metric components

$$\xi_{(t)} : \xi_{(t)} = g_{tt} = -f(r),$$
$$\xi_{(t)} : \xi_{(\psi)} = g_{t\psi} = r^2\omega(r),$$
$$\xi_{(\psi)} : \xi_{(\psi)} = g_{\psi\psi} = r^2. \quad (29)$$

To examine physical processes near such a black hole, we introduce a family of locally non-rotating observers. The angular velocity for these observers that move on orbits with constant $r$ and with a four-velocity $u^\mu$ satisfying $u \cdot \xi_{(\psi)} = 0$ is given by \[46, 47\]

$$\Omega = -\frac{g_{t\psi}}{g_{\psi\psi}} = -\omega(r) = \frac{(3r + 2B) J}{6r^3}. \quad (30)$$

When approaching the black hole horizon, the angular velocity $\Omega_H$ turns to be

$$\Omega_H = -\omega(r_+) = \frac{(3r_+ + 2B) J}{6r_+^3} = \frac{(2 + 3\theta) J}{6\theta r_+^2}. \quad (31)$$

Combining these quantities $M,T,\Omega_H$, we can verify that the first law of thermodynamics holds in this case

$$dM = TdS + \Omega_H dJ \quad (32)$$

and the Smarr relation can be found

$$M - \Omega_H J = \frac{1}{2} TS. \quad (33)$$

In general, the local thermodynamic stability is determined by the specific heat. The positive specific heat guarantees the stability of the black hole. When the specific heat becomes negative, it indicates that the black hole will be destroyed when it encounters a small perturbation. The specific heat can be calculated through $C_J = T \left( \frac{\partial S}{\partial T} \right)_J$, which reads

$$C_J = \frac{32\pi^2l^2r_+^4\theta^3(2+3\theta)T}{(1+\theta)^2 \left[ (2+3\theta)^2 J^2l^2 + 12r_+^4 \theta^2 \right]^2}. \quad (34)$$

Apparently, $C$ is always positive when $T > 0$, which implies that the rotating hairy black hole is locally stable when $r_+ > r_{\text{ext}}$. 
B. Phase transition between the rotating black hole with non-minimal scalar hair and the BTZ black hole

Now we examine the behavior of free energy $F$, which can be calculated as

$$F = M - TS = \frac{J^2 l^2 (2 + 3\theta)^2 - 12 r_+^4 \theta^2}{4 l^2 r_+^2 \theta (2 + 3\theta)}.$$  \hspace{1cm} (35)

When the black hole horizon $r_+ = r_c = \left(\frac{Jl(2 + 3\theta)}{2\sqrt{3}\theta}\right)^{1/2}$, the free energy vanishes. At $r_c$, the black hole temperature $T = T_c$, where

$$T_c = \frac{J(1 + \theta)}{3^{1/4} l \pi} \sqrt{\frac{\theta}{2(2 + 3\theta)Jl}}.$$  \hspace{1cm} (36)

The critical temperature $T_c$ depends on the values of $\theta$. We show the $F - T$ relation with different values of $\theta$ in Fig. 2, where we see that $T_c$ increases with the growth of $\theta$. Moreover, $F$ changes its sign at temperature $T_c$. In the region of $0 < T < T_c$, the free energy of this hairy black hole is always positive so the black hole phase evaporates completely to pure thermal radiation phase. When $T > T_c$, the free energy of this hairy black hole will be less than that of pure thermal radiation, and then this hairy black hole is thermodynamically favored. Hence there exist the Hawking-page phase transition between the black hole with non-minimally coupled scalar field and the pure thermal radiation \cite{[48]}. 

![FIG. 2: The free energy $F$ of the rotating hairy black hole relates to the temperature $T$ for different $\theta$ when we fix $J = 1$ and $l = 1.$](image)

It is worth comparing the free energies for the hairy rotating black hole and the BTZ black hole. For a fixed value of $M$, from Eq. (24) we see that the scalar field parameter $B$ cannot be switched off naively to reduce the hairy black hole into the BTZ black hole. The rotating hairy black hole and the BTZ black hole cannot be smoothly deformed into each other, they belong to different disconnected phases.
In the limit $\phi = 0$, the scalar potential $V(\phi)$ reduces to $-\frac{1}{l^2}$ and the action Eq.(1) admits the rotating BTZ black hole \[11\]

$$
\begin{align*}
    ds^2 &= -\left(\hat{M} + \frac{\rho^2}{l^2} + \frac{j^2}{4\rho^2}\right) dt^2 + \left(\hat{M} + \frac{\rho^2}{l^2} + \frac{j^2}{4\rho^2}\right) d\rho^2 + \rho^2 \left(d\psi - \frac{j^2}{2\rho^2} dt\right)^2.
\end{align*}
$$

(37)

The thermodynamic quantities, such as the temperature $\hat{T}$, mass $\hat{M}$, entropy $\hat{S}$ and the free energy $\hat{F}$ of the rotating BTZ black hole are given by

$$
\begin{align*}
    \hat{M} &= \frac{\rho^2}{l^2} + \frac{j^2}{4\rho_+^2}, & \hat{T} &= \frac{4\rho_+^4 - j^2 l^2}{8\pi l^2 \rho_+^4}, & \hat{S} &= 4\pi \rho_+,
    \\
    \hat{F} &= \frac{3j^2 l^2 - 4\rho_+^4}{4l^2 \rho_+^2}, & \Omega_H &= \frac{j}{2\rho_+^2}.
\end{align*}
$$

(38)

To compare the free energies of the hairy black hole and the BTZ black hole, we need to match the temperature $T = \hat{T}$ and the angular momentum $J = \hat{J}$ of these two black holes,

$$
\begin{align*}
    (1 + \theta) \left[36r_+^4 \theta^2 - J^2 l^2 (2 + 3\theta)^2\right] &= \frac{4\rho_+^4 - j^2 l^2}{8\pi l^2 \rho_+^4}.
\end{align*}
$$

(39)

In Fig. 3(a), we plot the the free energies $F$ of rotating hairy and BTZ black holes for different values of $\theta$. For large values of $\theta$, we find that the free energies $F$ of rotating hairy black holes are always larger than that of the rotating BTZ black hole when $T > 0$. This means that thermodynamically the BTZ black hole phase is more thermodynamically preferred. There exists possible thermodynamical phase transition for the hairy black hole to become BTZ black hole provided that there are some thermal fluctuations. For small values of $\theta$, we get that the free energies $F$ of the rotating hairy black holes are always smaller than that of the rotating BTZ black hole. This tells us that there exists possibility for the BTZ black hole undergoes a spontaneous dressing up nontrivial scalar hair, and the rotating hairy black hole is thermodynamically favored. In addition, the free energies $F$ of the rotating hairy black hole and rotating BTZ black hole share the same values when taking some certain value of $\theta$.

We have also evaluated the free energies of both black holes for different values of $J$ by assuming the same temperature $T$ and angular momentum $J$ in these two phases. We display the results in Figs.3(b)(c). We find $F < \hat{F}$ for small values of $\theta$ and $F > \hat{F}$ for large values of $\theta$. We observe that the relationships between the free energies of the rotating BTZ and hairy black holes are not affected by the values of the angular momentum $J$. The strength of $\theta$ is the key factor to determine which phase is more thermodynamically preferred.
IV. ROTATING BLACK HOLE WITH MINIMALLY COUPLED SCALAR FIELD

In this section we generalize the above discussions to the (2+1)-dimensional rotating black hole with minimally coupled scalar field [36]. This black hole was obtained from the action
\[ I = \frac{1}{2} \int d^3x \sqrt{-g} \left( R - \nabla_\mu \phi \nabla^\mu \phi - 2V(\phi) \right); \] (40)

where \( \mu \) and \( \alpha \) are constants. Taking the scalar potential
\[
V(\phi) = -\frac{1}{l^2} \cosh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) + \frac{1}{l^2} \left( 1 + \mu l^2 \right) \sinh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) - \frac{\alpha^2}{64} \sinh^{10} \left( \frac{1}{2\sqrt{2}} \phi \right) \cosh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) \tan^6 \left( \frac{1}{2\sqrt{2}} \phi \right) - 5 \tanh^4 \left( \frac{1}{2\sqrt{2}} \phi \right) + 10 \tanh^2 \left( \frac{1}{2\sqrt{2}} \phi \right) - 9, \tag{41}
\]
we have the rotating hairy black hole solution [36]
\[
ds^2 = -\left( \frac{H(r)}{H(r) + B} \right)^2 f(H(r)) dt^2 + \left( \frac{H(r) + B}{H(r) + 2B} \right)^2 \frac{dr^2}{f(H(r))}
+ r^2 \left( d\psi - \omega(H(r)) dt \right)^2, \tag{42}
\]
\[
f(H(r)) = 3\mu B^2 + \frac{2\mu B^3}{H(r)} + \frac{\alpha^2 B^4(3H(r) + 2B)^2}{H(r)^4} + \frac{H(r)^2}{l^2}, \tag{43}
\]
\[
\omega(H(r)) = \frac{\alpha B^2(3H(r) + 2B)}{H(r)^3}, \quad H(r) = \frac{1}{2} \left( r + \sqrt{r^2 + 4Br} \right), \tag{44}
\]
and the scalar field is described by
\[
\phi(r) = 2\sqrt{2} \text{arctanh} \sqrt{\frac{B}{H(r) + B}}, \tag{45}
\]

The event horizon is located at [36]
\[
H(r_+) = h \times B, \tag{46}
\]
We require $h$ only takes the real positive value of the three roots of $f(H(r_+)) = 0$

$$h^{(1)} = \frac{X_1}{X_0} + \frac{1}{X_1},$$

$$h^{(2,3)} = -\frac{1}{2} \left( \frac{X_1}{X_0} + \frac{1}{X_1} \right) \pm \frac{\sqrt{3}}{2} \left( \frac{X_1}{X_0} - \frac{1}{X_1} \right)$$

(47)

with

$$X_1 = \left[ 1 + \sqrt{-1 + \frac{X_0}{X_0}} \right] \left( \hat{X}_0 \right)^2 \frac{1}{3}, \quad (i = 1, 2),$$

$$X_0^{(1)} = -\frac{\mu \ell + \sqrt{\mu^2 \ell^2 - 4 \alpha^2}}{2 \alpha^2 \ell B^2} = \frac{\hat{X}_0^{(1)}}{B^2},$$

$$X_0^{(2)} = -\frac{\mu \ell - \sqrt{\mu^2 \ell^2 - 4 \alpha^2}}{2 \alpha^2 \ell B^2} = \frac{\hat{X}_0^{(2)}}{B^2}.$$  

(48)

(49)

(50)

We require $\mu \leq -\frac{2a}{\ell} \Rightarrow X_0^i > 0, (i = 1, 2)$.

The thermodynamical quantities of the rotating black hole with minimally coupled scalar field are given by [36]

$$M = \frac{J^2 l^2 (3h + 2)^2 + 36h^2 H(r_+)^4}{12l^2 h (3h + 2) H(r_+)^2}, \quad T = \frac{(1 + h) \left[ 36H(r_+)^4h^2 - J^2 l^2 (2 + 3h)^2 \right]}{24\pi l^2 h^2 (2 + 3h) H(r_+)^3},$$

$$S = \frac{4\pi h H(r_+)}{(h + 1)}, \quad \Omega_H = \frac{(3h + 2)J}{6h H(r_+)^3}, \quad F = \frac{J^2 l^2 (3h + 2)^2 - 12h^2 H(r_+)^4}{4l^2 h (3h + 2) H(r_+)^2}. \quad (51)$$

These expressions are similar to their counterparts (Eqs. (25)(26)(27)(31)(35)) for the rotating black hole non-minimally coupled with scalar field. It is easy to show that the first law of black hole thermodynamics is well protected in this black hole background. In [36], it was found that the free energy $F$ of this hairy black hole changes its sign at the critical temperature $T_c = \frac{J(1+h)}{3^{1/4} \pi \sqrt{2(2+3h)h}}$. In the region of $T > T_c$ the free energy $F$ of this hairy black hole is always less than that of pure thermal radiation, and then this hairy black hole is thermodynamically favored. On the other hand, the free energy $F$ of this hairy black hole is positive when $0 < T < T_c$, this hairy black hole phase evaporate completely into the pure thermal radiation. Therefore, there exists the so-called Hawking-Page phase transition between the black hole with minimally coupled scalar field phase and the pure thermal radiation phase [48].

Following the above discussions, we can further examine the phase transitions between the rotating BTZ and this kind of rotating hairy black holes for the same temperature and angular momentum values

$$\frac{(1 + h) \left[ 36H(r_+)^4h^2 - J^2 l^2 (2 + 3h)^2 \right]}{24\pi l^2 h^2 (2 + 3h) H(r_+)^3} = \frac{4r_+^4 - J^2 l^2}{8\pi l^2 \rho_+^3}. \quad (52)$$
Calculating the free energy, we find that the free energy of rotating black hole with the minimally coupled scalar field depends on the value of $h$ as shown in Fig. 4. This is similar to the behavior of the free energy for the rotating black hole with non-minimally coupled scalar field. For large values of $h$, we see that the BTZ black hole has smaller free energy than the hairy black hole with minimally coupled scalar field, which means that the BTZ black hole is a more thermodynamically preferred phase for large $h$. For small values of $h$, we see the opposite relation, $F < \hat{F}$, which tells us that in this case the hairy black hole with minimally coupled scalar field is thermodynamically more stable and there is probability for the BTZ black hole to dress up nontrivial scalar field. The thermodynamical behavior we observed here is very similar to the hairy black hole counterpart with non-minimally coupled scalar field. Moreover, the free energies $F$ of the rotating hairy black hole and rotating BTZ black hole share the same values when taking some certain value of $h$.

![FIG. 4: The free energies $F$ of (2+1)-dimensional rotating hairy black hole with minimally scalar field (dashed line) and BTZ black hole (solid line) versus the temperature $T$ when $l = 1$.](image)

V. CONCLUSIONS AND DISCUSSIONS

In the backgrounds of $(2 + 1)$-dimensional rotating hairy black holes, we found that when the scalar field parameter $B$ is arbitrary, the first law of thermodynamics cannot be protected. This problem can only be overcome in a new $(2+1)$-dimensional hairy black hole solution where the scalar field parameter is constrained to relate to the black hole size.

We have further calculated the free energy in the $(2+1)$-dimensional rotating hairy black hole backgrounds. Comparing the free energies of the hairy black hole and the pure AdS space, we found that the Hawking-Page phase transition exists between the hairy black hole and the pure AdS space. Moreover, we have computed the free energies for the hairy black hole and the BTZ black hole when they have the same temperature and angular momentum. We disclosed that the free energy depends on the ratio between the horizon size to the scalar field parameter. When
this ratio is large, the BTZ black hole has smaller free energy which is a thermodynamically more preferred phase. But when the ratio is small, the hairy black hole has smaller free energy and there exists the possibility for the BTZ black hole to dress up scalar field and become hairy. This property is general no matter whether the rotating black hole is minimally coupled or non-minimally coupled to scalar field.

Recently some new hairy black hole solutions have been found, such as the charged black hole with non-minimally coupled scalar field [49, 50], the Born-Infeld hairy black hole [52], rotating charged hairy black hole with infinitesimal electric charge and rotation parameters [53] in the Einstein gravity with non-minimally coupled scalar field, and black hole dressed by a (non)minimally scalar field in New Massive Gravity [54] etc. It would be interesting to extend our discussion to explore the thermodynamical properties and phase transitions of these new black hole solutions and see whether the results obtained here are general.

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