Sharp regularity of the Hartman-Grobman theorem in $C^0$ linearization

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Abstract

The classical Hartman-Grobman theorem states the existence of a $C^0$ topological conjugacy between the nonlinear system and its linear part. It is proved in the previous literature that the equivalent function $H$ and its inverse $G = H^{-1}$ are both Hölder continuous. Questions: is it possible to improve the regularity? Is the regularity sharp? This paper gives a positive answer if the linear system is contractive (or expansive) and the perturbation is nonlinear. We prove that $H$ is exactly Lipschitzian, but the inverse $G = H^{-1}$ is merely Hölder continuous. Furthermore, we present an example to illustrate the sharpness of the equivalent functions and we show that the regularity could not be improved anymore. It is the first time to observe these new facts.

keywords: Hartman-Grobman; conjugacy; regularity; Lipschitzian

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1 Introduction

1.1 History of linearization

The well-known Hartman-Grobman theorem \([1, 2]\) pointed out that nearby the hyperbolic equilibrium \(u^* = 0\), the dynamic behaviour of the nonlinear system \(u' = Au + f(u)\), where \(f \in C^1(\mathbb{R}^n)\), \(f(x) = o(\|x\|)\), is topologically conjugated to its linear problem \(v' = Av\). The global linearization study begins in 1969, Pugh \([3]\) presented a particular case of the Hartman-Grobman theorem as long as \(f\) satisfies some goodness conditions, like continuity, boundedness or being Lipschitzian. Moreover, the Hartman-Grobman theorem has been extended to many classes of differential equations such as retarded functional equations by Farkas \([4]\), scalar reaction-diffusion equations by Lu \([5]\), Cahn-Hilliard equation and phase field equations by Lu and Bates \([6]\) and semilinear hyperbolic evolution equations by Hein and Prüss \([7]\). Except for the \(C^0\) linearization of the differential equations, Sternberg \([8, 9]\) initially investigated \(C^r\) linearization for \(C^k\) diffeomorphisms. Sell \([10]\) extended the theorem of Sternberg. More mathematicians paid particular attention on the \(C^1\) linearization. Belitskii \([11]\), ElBialy \([12]\), Rodrigues and Solà-Morales \([13, 14]\) studied that \(C^1\) linearization of hyperbolic diffeomorphisms on Banach space independently. Recently, Zhang and Zhang \([15]\) proved that the derivatives of transformations in \(C^1\) linearization are Hölder continuous. A counter example was given to show that those estimates of Hölder exponent cannot be improved anymore. Zhang et al. \([16, 17, 18]\) studied the sharp regularity of linearization for \(C^1\) or \(C^{1,1}\) hyperbolic diffeomorphisms, and they proved that the derivatives of transformations in the linearization are Hölder continuous.

Efforts were made to the linearization of the nonautonomous dynamical system initially by Palmer \([19]\). To weakened the conditions of Palmer’s theorem, Jiang \([20]\) gave a version of linearization result by setting that the linear system admits a generalized exponential dichotomy. Recently, Barreira and Valls \([21, 22, 23, 24, 25, 26]\) proved several versions of Hartman-Grobman theorem provided the linear homogenous system admits a nonuniform contraction or nonuniform dichotomy. Huerta \([27, 28]\) constructed conjugacies between linear system and an unbounded nonlinear perturbation when nonautonomous linear system admits a nonuniform contraction. Other generalizations of Hartman-Grobman theorem were given for the instantaneous impulsive system \([29, 30, 31]\), dynamic systems on time scales \([32, 33]\), differential equations with piecewise constant argument \([34, 35]\).
1.2 Motivation and contributions

In this paper, we pay particular attention to the regularity of the equivalent function which transforms the solutions of nonlinear system onto its linear part. It is proved in the previous works that the equivalent function and its inverse are Hölder continuous for the Hartman-Grobman theorem in $C^0$ linearization. It is known that the equivalent function $H$ which linearises the nonlinear problem according to $x(t, H(x_0)) = H(e^{At}x_0), t \in \mathbb{R}, x_0 \in \mathbb{R}^n$, in general is not in $C^1$, see Chicone [36]. Questions: is it possible to improve the regularity to Lipschitz continuity? And is the regularity sharp?

In the present paper, we give a positive answer if the linear system admits a contraction (or expansion) and the perturbation is nonlinear. We show that $H$ is exactly Lipschitzian, but the inverse $G = H^{-1}$ is still Hölder continuous. Moreover, this is a sharp regularity for the Hartman-Grobman theorem in $C^0$ linearization. It is impossible to be improved anymore. And the equivalent function $H$ and its inverse $G$ do not have the same regularity. To the best of our knowledge, it is the first time to observe these facts in $C^0$ linearization. It was shown that both the equivalent function and its inverse are all Hölder continuous in the previous literature, see for examples [7, 21, 22, 23, 24, 26, 31, 37]. In order to demonstrate our results, we present an example to illustrates the sharpness of the equivalent functions and we show that the regularity could not be improved anymore. We obtain a unique concrete equivalent function $H$ and its inverse $G$ in the example (see Example 2.16 in next section). It is shown in this example that the unique equivalent function $H$ is exactly Lipschitzian, but not $C^1$; the unique inverse $G = H^{-1}$ is Hölder continuous, but not Lipschitzian. Therefore, this illustrative example shows that the sharpness of the regularity is indeed true.

Secondly, we also discuss two special cases of the perturbations. If the perturbation is linear homogeneous, $H$ is $C^1$ and the inverse $G$ is Hölder continuous; If the perturbation is constant, both $H$ and $G$ are $C^\infty$.

Thirdly, we also extend the Hartman-Grobman theorem in two aspects: (1) we may consider nonlinear term $f$ which may be unbounded or not Lipschitzian, (see Example 2.17); (2) we prove that it is enough to assume that the boundedness of the operator $K$ of the coefficients.

The rest of this paper is organized as follows: In Section 2, our main results are stated and the interesting examples are given to show the sharpness of the regularity. In Section 3, some
preliminary results are presented. In Section 4, rigorous proof is given to show the regularity of conjugacy.

2 Statement of main results and illustrative examples

Consider the following two nonautonomous systems:

\[ y' = A(t)y \quad (2.1) \]

and

\[ x' = A(t)x + f(t, x), \quad (2.2) \]

where \( A(t) \) is a bounded and continuous \( n \times n \) matrix and \( f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function.

Definition 2.1. (Topological Conjugacy) Suppose that there exists a function \( H : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

(i) for each fixed \( t \), \( H(t, \cdot) \) is a homeomorphism of \( \mathbb{R}^n \) into \( \mathbb{R}^n \);
(ii) \( H(t, x) - x \) is uniformly bounded with respect to \( t \);
(iii) \( G(t, \cdot) = H^{-1}(t, \cdot) \) also has property (ii);
(iv) if \( x(t) \) is a solution of Eq. (2.2), then \( H(t, x(t)) \) is a solution of Eq. (2.1); and if \( y(t) \) is a solution of Eq. (2.1), then \( G(t, y(t)) \) is a solution of Eq. (2.2).

If such a map \( H(t, x(t)) \) exists, then Eq. (2.2) is topologically conjugated to Eq. (2.1) and the transformation \( H(t, x) \) is said to be an equivalent function.

Before presenting our main results, we need a particular case of Palmer’s result as follows:

Proposition 2.2. Suppose that

(A) there exist positive constants \( k > 0 \) and \( \alpha > 0 \) such that the transition matrix \( U(t, s) = U(t)U^{-1}(s) \) of Eq. (2.1) satisfies:

\[ \|U(t, s)\| \leq k \exp\{-\alpha(t - s)\}, \quad \text{for} \ t \geq s; \quad (2.3) \]

or (A) there exist positive constants \( k > 0 \) and \( \alpha > 0 \) such that \( U(t, s) \) of Eq. (2.1) satisfies:

\[ \|U(t, s)\| \leq k \exp\{\alpha(t - s)\}, \quad \text{for} \ t \geq s; \]
The perturbation \( f \) is continuous and bounded, i.e., for any \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \), \( \| f(t, x) \| \leq \mu < +\infty \);

(A3) for any \( t \in \mathbb{R} \) and \( x_1, x_2 \in \mathbb{R}^n \), \( \| f(t, x_1) - f(t, x_2) \| \leq L_f \| x_1 - x_2 \| ; \)

(A4) the Lipschitz constant \( L_f \) satisfies: \( L_f \leq \alpha/k \).

Then there exists a unique function \( H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) such that Eq. \( (2.2) \) is topologically conjugated to Eq. \( (2.1) \). Moreover, \( \| H(t, x) - x \| \leq k\mu\alpha^{-1} \).

Remark 2.3. The original Palmer’s linearization theorem requires that Eq. \( (2.1) \) admits an exponential dichotomy. The assumption \((A_1)\) (or \((\tilde{A}_1)\)) is a particular case of the exponential dichotomy, which means that Eq. \( (2.1) \) is exponential stable (or instability) as \( t \to +\infty \).

In what follows, we always consider the case that \( U(t, s) \) is contractive, i.e., \( U(t, s) \) satisfies \( (2.3) \). For the case that it is expansive, the results obtained by time symmetric transformation are completely similar.

We introduce the operator \( K \) given by
\[
K(\varphi)(t) = \int_{-\infty}^{t} U(t, s)\varphi(s)ds, \quad t \in \mathbb{R},
\]
where \( \varphi : \mathbb{R} \to \mathbb{R}^n \) is a function, \( \| K(\varphi) \| \leq \mathcal{L}(\| \varphi \|) \) with
\[
\mathcal{L}(b)(t) = \int_{-\infty}^{t} \exp\{-\alpha(t-s)\}b(s)ds,
\]
for \( b : \mathbb{R} \to (0, \infty) \) a continuous function.

Now, we are in a position to present our main results. Firstly, we state a theorem on the existence of topological conjugacy between Eq. \( (2.2) \) and Eq. \( (2.1) \).

Theorem 2.4. Suppose that Eq. \( (2.1) \) satisfies condition \((A_1)\). If there exist nonnegative local integrable functions \( \mu(t), r(t) \) such that for any \( t, x, x_1, x_2 \), the nonlinear term \( f(t, x) \) satisfies
\[
\| f(t, x) \| \leq \mu(t), \quad (2.4)
\]
\[
\| f(t, x_1) - f(t, x_2) \| \leq r(t) \| x_1 - x_2 \| , \quad (2.5)
\]
where \( \mu(t), r(t) \) satisfy
\[
\sup_{t \in \mathbb{R}} \mathcal{L}(\mu)(t) < \infty \quad \text{and} \quad \sup_{t \in \mathbb{R}} \mathcal{L}(r)(t) = \theta < k^{-1}. \quad (2.6)
\]
Then Eq. \( (2.2) \) is topologically conjugated to Eq. \( (2.1) \). Moreover, the equivalent function \( H(t, x) \) and \( G(t, y) \) satisfy
\[
\| H(t, x) - x \| \leq k\| \mathcal{L}(\mu) \| _{\infty}, \quad \| G(t, y) - y \| \leq k\| \mathcal{L}(\mu) \| _{\infty}.
\]
Corollary 2.5. In the nonuniform case, the result is true for
\[ b(t) = r(t) \exp\{-\epsilon|t|\}. \]

Lemma 2.6 (Coppel, [38]). If \( \mu(t), r(t) \) are nonnegative local integrable functions, and \( C_1 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \mu(s)ds, C_2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} r(s)ds \), then we have
\[ \mathcal{L}(\mu)(t) \leq (1 - \exp\{-\alpha\})^{-1} C_1, \quad \mathcal{L}(r)(t) \leq (1 - \exp\{-\alpha\})^{-1} C_2. \quad (2.7) \]

Thus, we have the following result:

Corollary 2.7. Suppose that Eq. (2.1) satisfies condition (A1). If the perturbation \( f(t, x) \) satisfies (2.4), (2.5) and
\[ 2k \cdot (1 - \exp\{-\alpha\})^{-1} C_2 < 1, \quad (2.8) \]
then Eq. (2.2) is topologically conjugated to Eq. (2.1).

Remark 2.8. Lemma 2.6 shows that a big class of functions \( \mu, r \) satisfy conditions (2.4), (2.5). \( r, \mu \in L^p, 1 \leq p \leq \infty, r \) with \( \| r \|_p \) small enough. If \( r \) is uniformly bounded, it is possible to choose \( \alpha = 0 \). Note when \( \mu(t) = \mu \) and \( r(t) = r \) are constants, Theorem 2.4 and Corollary 2.7 reduces to the classical Palmer linearization theorem. We note that Palmer did not give a conclusion on the Lipschitz nor the Hölder continuity of \( H \). It should be noted that \( f(t, x) \) in our theorem could be unbounded or not uniformly Lipschitzian. Note \( \mu(t), r(t) \) are locally integrable, so they could be unbounded.

Next three theorems are the regularity of the topological equivalent function \( H \) and its inverse \( G = H^{-1} \). According to the different classification of perturbation \( f \), different regularity for three cases is presented as follows.

Case 1: the perturbation \( f(t, x) \) is not linear with respect to \( x \). That is, \( f(t, x) \neq B(t)x + C(t) \).

Theorem 2.9. Suppose that the perturbation \( f(t, x) \) is not linear with respect to \( x \). If all conditions of Corollary 2.7 hold, then the equivalent function \( H(t, x) \) and its inverse \( G(t, y) \) fulfill:
(i) \( H \) is exactly Lipschitzian, that is, for any \( x, \bar{x} \in \mathbb{R}^n \), there exists a positive constant \( p_1 > 0 \) such that
\[ \|H(t, x) - H(t, \bar{x})\| \leq p_1 \|x - \bar{x}\|; \]
(ii) the inverse $G = H^{-1}$ is Hölder continuous, that is, for any $y, \bar{y} \in \mathbb{R}^n$, there exist positive constants $p_2 > 0$ and $0 < q < 1$ such that

$$\|G(t, y) - G(t, \bar{y})\| \leq p_2 \|y - \bar{y}\|^q.$$ 

(iii) $H$ is Lipschitzian, but not $C^1$; $G$ is Hölder continuous, but not Lipschitzian. The regularity of $H$ and $G$ can not be improved any more.

**Remark 2.10.** In the following, we present an illustrative example (Example 2.16) to show that $H$ is exactly Lipschitzian, but not $C^1$; $G$ is Hölder continuous, but not Lipschitzian. The regularity of $H$ and $G$ can not be improved any more. We show in details in Remark 4.2 to why the inverse $G$ can not be Lipschitzian.

**Corollary 2.11.** If $\mu(t) = \mu$ and $r(t) = r$ are constants, then all the conclusions in Theorem 2.9 are also true.

Now we pay particular attention to two special cases:

**Case 2: $f$ is linear homogeneous perturbation.**

If $f(t, x) = B(t)x$, then equation (2.2) reduces to

$$x' = A(t)x + B(t)x,$$

(2.9)

where $B(t)$ is a continuous $n \times n$ matrix.

We need the following definition.

**Definition 2.12 ([39]).** The exponential dichotomy spectrum of equation (2.1) is the set

$$\Sigma(A(t)) = \{\gamma \in \mathbb{R} | x' = (A(t) - \gamma I)x \text{ admits no exponential dichotomy}\},$$

and the resolvent set $\rho(A(t)) = \mathbb{R} \setminus \Sigma(A(t))$.

It is known that the exponential dichotomy spectrum $\Sigma(A(t))$ of (2.1) is the disjoint union of $k$ closed intervals where $0 \leq k \leq n$. i.e., $\Sigma(A(t)) = \bigcup_{i=1}^{k} [a_i, b_i]$, $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$, where $a_1$ could be $-\infty$ and $b_k$ could be $\infty$. If $k = 0$, then $\Sigma(A(t)) = \emptyset$ (see Sacker and Sell [39], Xia et al [40]). Notice that instead of $f(t, x)$ being Lipschitz continuous with respect to $x$, it is differentiable with respect to $x$. Thus, a better regularity will be proved. Namely, $H$ is $C^1$ and its inverse $G$ is Hölder continuous.
Theorem 2.13. Suppose that $f(t, x) = B(t)x$ and (2.1) satisfies condition (A_1) with $\Sigma(A(t)) = \bigcup_{i=1}^{k}[a_i, b_i]$, where $b_i < 0$. If there exist $\delta > 0$ and appropriately small $0 \leq \epsilon < -b_k$ such that $\|B(t)\| \leq \delta$ and $\Sigma(A(t) + B(t)) \subseteq \bigcup_{i=1}^{k}[a_i + \epsilon, b_i + \epsilon]$, then the equivalent function $H(t, x)$ is $C^1$ and its inverse $G(t, y)$ is Hölder continuous.

Remark 2.14. Since $\Sigma(A(t) + B(t)) \subseteq \bigcup_{i=1}^{k}[a_i + \epsilon, b_i + \epsilon]$, equation (2.9) is also contractive and its linearization is a simple inference of Theorem 2.4. In addition, if (2.1) satisfies condition (A_1) with $\Sigma(A(t)) = \bigcup_{i=1}^{k}[a_i, b_i]$, where $a_1 > 0$, and there exist $\tilde{\delta} > 0$ and appropriately small $0 \leq \epsilon < a_1$ such that $\|B(t)\| \leq \tilde{\delta}$ and $\Sigma(A(t) + B(t)) \subseteq \bigcup_{i=1}^{k}[a_i - \epsilon, b_i - \epsilon]$, then our result is similar to Theorem 2.13.

Case 3: $f$ is constant.

If the nonlinear term $f(t, x)$ is a constant function, then equation (2.2) reduces to a non-homogeneous linear equation. Clearly,

Corollary 2.15. If the nonlinear term $f(t, x)$ is a constant, then there is a $C^\infty$ coordinate transformation such that Eq. (2.2) is topologically conjugated to Eq. (2.1), namely, the equivalent function $H(t, x)$ and its inverse $G(t, y)$ are both $C^\infty$.

Example 2.16. This example is given to show the sharpness of the regularity. For the sake of readable and easily verification, we consider the scalar equation

$$x' = -x + f(x). \quad (2.10)$$

Here the nonlinear term $f$ is given by

$$f(x) = \begin{cases} 
\epsilon, & x \geq 1, \\
\epsilon x, & 0 \leq x < 1, \\
\epsilon x^3, & -1 < x < 0, \\
-\epsilon, & x \leq -1, 
\end{cases}$$

where $0 < \epsilon < \frac{1}{3}$. We see that for all $x, x_1, x_2$,

$$|f(x)| \leq \epsilon, \quad |f(x_1) - f(x_2)| \leq 3\epsilon|x_1 - x_2|.$$ 

Moreover, the linear equation $x' = -x$ has an uniform contraction with constant $k = 1$ and exponent $\alpha = 1$. Proposition 2.2 applies with $k = 1, L_f = 3\epsilon, \alpha = 1$ ($L_f K/\alpha = 3\epsilon < 1$) to
show that there exists a continuous function $H(x)$ ($H(t, x) = H(x)$ in the autonomous system) such that $|H(x) - x|$ is bounded, and if $x(t)$ is a solution of (2.10), then $H(x(t))$ is a solution of

$$x' = -x.$$  \hfill (2.11)

Moreover, $H$ is a homeomorphism and $G = H^{-1}$ has similar properties to $H$.

To show the sharpness of the regularity of $H$ and $G$. Firstly, we should give an explicit expressions of $H$ and $G$. Notice that

$$-x + f(x) = \begin{cases} < 0, & \text{if } x > 0, \\ = 0, & \text{if } x = 0, \\ > 0, & \text{if } x < 0. \end{cases}$$

Thus a solution is either always 0, always $< 0$ or always $> 0$. Now we divide the discussion into three cases.

(I) zero solution. $H(0) = 0$ since 0 is a solution of (2.10) and $H(0)$ is the unique solution of (2.11). But 0 is such a solution.

(II) positive solution $x(t) > 0$. Clearly, $x(t)$ is strictly decreasing, i.e., $x(t) \to 0$ as $t \to +\infty$; $x(t) \to \infty$ as $t \to -\infty$. Therefore, there must exists a unique time $t_0$ such that $x(t_0) = 1$. Without loss of generality, we take $t_0 = 0$. If $t < 0$, then $x(t) > 1$ and so $x'(t) = -x(t) + \epsilon$ with $x(0) = 1$. Hence,

$$x(t) = (1 - \epsilon)e^{-t} + \epsilon, \quad t \leq 0.$$  

If $t > 0$, then $0 < x(t) < 1$ and so $x'(t) = -x(t) + \epsilon x(t)$ with $x(0) = 1$. Hence,

$$x(t) = e^{(-1+\epsilon)t}, \quad t \geq 0.$$  

We need to find the unique solution $y(t)$ of (2.11) such that $|y(t) - x(t)|$ is bounded. Looking at $x(t)$ when $t \leq 0$, we see that $y(t) = (1 - \epsilon)e^{-t}$. Hence for all $t$,

$$H(x(t)) = (1 - \epsilon)e^{-t}.$$  

Then

$$H(1) = H(x(0)) = 1 - \epsilon.$$
If $\xi > 1$, then there exists a unique time $t < 0$ such that $x(t) = (1 - \epsilon)e^{-t} + \epsilon = \xi$. Then

$$H(\xi) = H(x(t)) = (1 - \epsilon)e^{-t} = x(t) - \epsilon = \xi - \epsilon.$$ 

If $0 < \xi < 1$, then there exists a unique time $t > 0$ such that $x(t) = e^{-(1+\epsilon)t} = \xi$. Then

$$H(\xi) = H(x(t)) = (1 - \epsilon)e^{-t} = (1 - \epsilon)x(t)^{\frac{1}{1 - \epsilon}} = (1 - \epsilon)\xi^{\frac{1}{1 - \epsilon}}.$$ 

Therefore,

$$H(x) = \begin{cases} (1 - \epsilon)x^{\frac{1}{1 - \epsilon}}, & 0 < x < 1, \\ x - \epsilon, & x \geq 1. \end{cases}$$

(III) negative solution $x(t) < 0$. Clearly, $x(t)$ is strictly increasing, i.e., $x(t) \to 0$ as $t \to \infty$; $x(t) \to -\infty$ as $t \to -\infty$. So there must exists a unique time $t_0$ such that $x(t_0) = -1$. Without loss of generality, we take $t_0 = 0$. If $t < 0$, then $x(t) < -1$ and so $x'(t) = -x(t) - \epsilon$ with $x(0) = -1$. Hence,

$$x(t) = (\epsilon - 1)e^{-t} - \epsilon, \quad t \leq 0.$$ 

If $t > 0$, then $-1 < x(t) < 0$ and so $x'(t) = -x(t) + \epsilon x^3(t)$ with $x(0) = -1$. Letting $z = x^{-2}$, then $z' = 2z - 2\epsilon$ with $z(0) = 1$. Hence, $z(t) = (1 - \epsilon)e^{2t} + \epsilon$, that is,

$$x(t) = -\left[(1 - \epsilon)e^{2t} + \epsilon\right]^{-\frac{1}{2}}, \quad t \geq 0.$$ 

We need to find the unique solution $y(t)$ of (2.11) such that $|y(t) - x(t)|$ is bounded. Looking at $x(t)$ when $t \leq 0$, we see that $y(t) = (\epsilon - 1)e^{-t}$. Hence for all $t$,

$$H(x(t)) = (\epsilon - 1)e^{-t}.$$ 

Then

$$H(1) = H(x(0)) = \epsilon - 1.$$ 

If $\xi < -1$, then there exists a unique time $t < 0$ such that $x(t) = (\epsilon - 1)e^{-t} - \epsilon = \xi$. Then

$$H(\xi) = H(x(t)) = (\epsilon - 1)e^{-t} = x(t) + \epsilon = \xi + \epsilon.$$ 

If $-1 < \xi < 0$, then there exists a unique time $t > 0$ such that $x(t) = -[(1 - \epsilon)e^{2t} + \epsilon]^{-\frac{1}{2}} = \xi$. Then

$$H(\xi) = H(x(t)) = (\epsilon - 1)e^{-t} = -(1 - \epsilon)^{\frac{1}{2}}\left(\frac{1}{(-x(t))^2} - \epsilon\right)^{-\frac{1}{2}} = -(1 - \epsilon)^{\frac{1}{2}}\left(\frac{1}{(-\xi)^2} - \epsilon\right)^{-\frac{1}{2}}.$$
Thus we obtain that

$$H(x) = \begin{cases} 
- (1 - \epsilon)^{\frac{3}{2}} \left( \frac{1}{(1-x)^{x}} - \epsilon \right)^{-\frac{1}{2}}, & -1 < x < 0, \\
x + \epsilon, & x \leq -1.
\end{cases}$$

Summarizing we have found that

$$H(x) = \begin{cases} 
x - \epsilon, & x \geq 1, \\
(1 - \epsilon)x^{\frac{1}{x - \epsilon}}, & 0 < x < 1, \\
0, & x = 0, \\
-(1 - \epsilon)^{\frac{3}{2}} \left( \frac{1}{(1-x)^{x}} - \epsilon \right)^{-\frac{1}{2}}, & -1 < x < 0, \\
x + \epsilon, & x \leq -1,
\end{cases}$$

where $0 < \epsilon < \frac{1}{3}$. Next, we show that $H$ is a continuous function, but it is not $C^1$. In fact, we only say that $H$ is continuous at $0$ and $\pm 1$, but is not $C^1$ at $0$.

1. $H(x)$ is continuous at $x = 1$:

$$\lim_{x \to 1^-} (1 - \epsilon)x^{\frac{1}{x - \epsilon}} = 1 - \epsilon,$$

$$\lim_{x \to 1^+} x - \epsilon = 1 - \epsilon.$$

2. $H(x)$ is continuous at $x = 0$:

$$\lim_{x \to 0^+} (1 - \epsilon)x^{\frac{1}{x - \epsilon}} = 0,$$

$$\lim_{x \to 0^-} -(1 - \epsilon)^{\frac{3}{2}} \left( \frac{1}{(1-x)^{x}} - \epsilon \right)^{-\frac{1}{2}} = 0.$$

3. $H(x)$ is continuous at $x = -1$:

$$\lim_{x \to -1^+} -(1 - \epsilon)^{\frac{3}{2}} \left( \frac{1}{(1-x)^{x}} - \epsilon \right)^{-\frac{1}{2}} = -1 + \epsilon,$$

$$\lim_{x \to -1^-} x + \epsilon = -1 + \epsilon.$$

Hence, $H(x)$ is continuous, but the following fact proves that $H$ is not $C^1$ at $0$. Clearly, for $0 < x < 1$, $H'(x) = x^{\frac{1}{x - \epsilon}}$, and for $-1 < x < 0$, $H'(x) = (1 - \epsilon)^{\frac{3}{2}}(1 - \epsilon(-x)^{x})^{-\frac{3}{2}}$. Therefore,

$$\lim_{x \to 0^+} x^{\frac{1}{x - \epsilon}} = 0,$$

$$\lim_{x \to 0^-} (1 - \epsilon)^{\frac{3}{2}}(1 - \epsilon(-x)^{x})^{-\frac{3}{2}} = (1 - \epsilon)^{\frac{3}{2}},$$

$$\lim_{x \to -1^+} -(1 - \epsilon)^{\frac{3}{2}} \left( \frac{1}{(1-x)^{x}} - \epsilon \right)^{-\frac{1}{2}} = -1 + \epsilon,$$

$$\lim_{x \to -1^-} x + \epsilon = -1 + \epsilon.$$
it implies that \( H(x) \) is not in \( C^1 \).

Fortunately, \( H(x) \) is Lipschitz continuous, since \( H'(x) \) is continuous at \( x \) except for \( x = 0 \), and it is bounded with \( |H'| \leq 1 \). So function \( H(x) \) is globally Lipschitz continuous with Lipschitz constant \( L = 1 \), but is not in \( C^1 \).

However, the inverse function \( G = H^{-1} \) is

\[
G(y) = \begin{cases} 
  y + \epsilon, & y \geq 1 - \epsilon, \\
  \left( \frac{y}{1-\epsilon} \right)^{1-\epsilon}, & 0 < y < 1 - \epsilon, \\
  0, & y = 0, \\
  -((1-\epsilon)^3(-y)^{-2} + \epsilon)^{-\frac{1}{2}}, & -1 + \epsilon < y < 0, \\
  y - \epsilon, & y \leq \epsilon - 1.
\end{cases}
\]

Obviously, \( G(y) \) is continuous at \( 0 \) and \( \pm 1 \). So \( G(y) \) is a continuous function. However, this is not Lipschitz continuous since \( y^{1-\epsilon} \) is not Lipschitz, as \( 0 < 1 - \epsilon < 1 \).

![Fig. 1 The equivalent function \( H \) and its inverse \( G \), where \( \epsilon = \frac{1}{4} \).](image)

**Special Case 1.** For Eq. (2.10), we consider the nonlinear term \( f = \epsilon x \), \( 0 < \epsilon < 1 \). In this case, \( \Sigma(A + B) = -1 + \epsilon < 0 \) and \( |B| = \epsilon \). Similar to the procedure just shown, we have that

\[
H(x) = \begin{cases} 
  (1-\epsilon)x^{\frac{1}{1-\epsilon}}, & x > 0, \\
  0, & x = 0, \\
  (\epsilon - 1)(-x)^{\frac{1}{\epsilon}}, & x < 0.
\end{cases}
\]
This function $H(x)$ is at least in $C^1$. And the inverse function $G = H^{-1}$ is

$$G(y) = \begin{cases} \left(\frac{y}{1-\epsilon}\right)^{1-\epsilon}, & y > 0, \\ 0, & y = 0, \\ -\left(\frac{-y}{1-\epsilon}\right)^{1-\epsilon}, & y < 0. \end{cases}$$

This function $G(y)$ is Hölder continuous. Hence, this particular example illustrates the feasibility of Corollary 2.13.

**Special Case 2.** If $f(x)$ is a constant, then Eq. (2.10) reduces to

$$x' = -x + \delta.$$ 

In this case, linearization theorem is always satisfied. Moreover, it is easy to obtain that

$$H(x) = x - \delta, \quad G(y) = y + \delta.$$

Thus, $H, G$ are $C^\infty$.

The following example illustrates the feasibility of our linearization theorem. We construct a continuous function $f(t,x)$ which is unbounded, not uniformly Lipschitzian, but locally integrable.

**Example 2.17.** [41] Considering $t \in [0, \infty)$, for any positive constant $c$ and integer $n$, let

$$g(t) = \begin{cases} 0, & \text{if } t \in [0, 1), \\ cn^2t - cn^3, & \text{if } t \in [n, n + \frac{1}{2n}), \\ -cn^2t + cn^3 + cn, & \text{if } t \in [n + \frac{1}{2n}, n + \frac{1}{n}), \\ 0, & \text{if } t \in [n + \frac{1}{n}, n + 1). \end{cases}$$

Note that $g(t)$ is continuous on $[0, \infty)$. Let $\mu(t)$ the continuous function on $\mathbb{R}$:

$$\mu(t) = \begin{cases} g(t), & \text{if } t \geq 0, \\ g(-t), & \text{if } t < 0. \end{cases}$$

Hence,

$$f(t, x) = \mu(t) \sin(x)$$

is continuous on $\mathbb{R} \times \mathbb{R}$. It is easy to see that for any $(t, x), (t, \bar{x}) \in \mathbb{R} \times \mathbb{R}$,

$$\|f(t, x) - f(t, \bar{x})\| \leq \mu(t)\|x - \bar{x}\|, \quad \|f(t, x)\| \leq \mu(t).$$
and
\[ \int_t^{t+1} \mu(s) ds \leq c. \]

However, we see that \( \mu \) and \( f \) are unbounded functions, since

\[ \mu \left( n + \frac{1}{2n} \right) \to +\infty, \quad \text{as} \quad n \to \infty. \]

Consequently, \( f(t, x) \) is not only unbounded, but also \( f(t, x) \) is not uniformly Lipschitzian. This example shows that in some cases \( f(t, x) \) could be unbounded or not uniformly Lipschitzian, and still fulfill the conditions of our theorem. In this sense, we generalize and improve the linearization theorem.

3 Preliminary results

In order to prove Theorem 2.4, we will recall some facts of the Palmer’s proof tailored for our purpose. In what follows, we always suppose that the conditions of Theorem 2.4 are satisfied. Let \( X(t, t_0, x_0) \) be a solution of Eq. (2.2) satisfying the initial condition \( X(t_0) = x_0 \) and \( Y(t, t_0, y_0) \) is a solution of Eq. (2.1) satisfying the initial condition \( Y(t_0) = y_0 \).

Lemma 3.1. For each \( (\tau, \xi) \), the following equation

\[ z' = A(t)z - f(t, X(t, \tau, \xi)), \quad (3.1) \]

has a unique bounded solution \( h(t, (\tau, \xi)) \) with \( \|h(t, (\tau, \xi))\| \leq k \|L(\mu)\|_\infty \).

Proof. We show that Eq. (3.1) has a unique bounded solution for any fixed \( (\tau, \xi) \), even if \( f \) is locally integrable. In fact, let

\[ h(t, (\tau, \xi)) = -K(f(\cdot, X(\cdot, \tau, \xi)))(t) = -\int_{-\infty}^{t} U(t, s)f(s, X(s, \tau, \xi)) ds. \quad (3.2) \]

Differentiating it, it is easy to see that \( h(t, (\tau, \xi)) \) is a solution of Eq. (3.1). It follows from (A1) and (2.4) that

\[ \|h(t, (\tau, \xi))\| \leq \int_{-\infty}^{t} k\mu(s) \exp\{-\alpha(t - s)\} ds = kL(\mu)(t) \leq k \|L(\mu)\|_\infty, \]

which implies that \( h(t, (\tau, \xi)) \) is a bounded solution of Eq. (3.1). Since Eq. (3.1) is linearly inhomogeneous, we immediately obtain that the bounded solution is unique. \( \square \)
Lemma 3.2. For each \((\tau, \xi)\), the following equation

\[ z' = A(t)z + f(t, Y(t, \tau, \xi) + z), \tag{3.3} \]

has a unique bounded solution \(g(t, (\tau, \xi))\), and \(\|g(t, (\tau, \xi))\| \leq k \| L(\mu) \|_\infty\).

Proof. Let \(B\) be the complete metric space of all the continuous bounded functions \(z(t)\), provided of supremum metric, with \(\|z(t)\| \leq k \| L(\mu) \|_\infty\). For each \((\tau, \xi)\) and any \(z(t) \in B\), define a mapping \(T\) as follows,

\[ Tz(t) = \int_{-\infty}^{t} U(t, s)f(s, Y(s, \tau, \xi) + z(s))ds. \]

A simple computation leads to

\[ \| Tz(t) \| \leq kL(\mu)(t), \]

which implies that \(TB \subset B\). For any \(z_1(t), z_2(t) \in B\),

\[ \|Tz_1(t) - Tz_2(t)\| \leq kL(r)(t) \| z_1 - z_2 \|. \]

Now, by (2.6), \(k\theta < 1\), then \(T\) has a unique fixed point, namely \(z_0(t)\), and

\[ z_0(t) = \int_{-\infty}^{t} U(t, s)f(s, Y(s, \tau, \xi) + z_0(s))ds. \]

It is easy to see that \(z_0(t)\) is a bounded solution of Eq. (3.3). From standard argument, the bounded solution is unique. We may call the unique solution \(g(t, (\tau, \xi))\). From the above proof, it is easy to see that \(\|g(t, (\tau, \xi))\| \leq k \| L(\mu) \|_\infty\). \(\square\)

Similarly, we have:

Lemma 3.3. Let \(x(t)\) be any solution of Eq. (2.2). Then \(z(t) \equiv 0\) is the unique bounded solution of the equation

\[ z' = A(t)z + f(t, x(t) + z) - f(t, x(t)). \tag{3.4} \]

Note the importance in these results of the uniform boundedness of \(L(\mu)\).

Now we define two functions as follows

\[ H(t, x) = x + h(t, (t, x)), \tag{3.5} \]
\[ G(t, y) = y + g(t, (y)), \quad (3.6) \]

for \( g \) and \( h \) as in lemmas 3.1 and 3.2.

By differentiation and similar arguments, we have the following lemma.

**Lemma 3.4.** For any fixed \((t_0, x_0)\), \(H(t, X(t, t_0, x))\) is a solution of Eq. (2.1); and for any fixed \((t_0, y_0)\), \(G(t, Y(t, t_0, y))\) is a solution of Eq. (2.2).

**Lemma 3.5.** For any \(t \in \mathbb{R}, y \in \mathbb{R}^n\), \(H(t, G(t, y)) = y\).

**Proof.** Let \(y(t)\) be any solution of Eq. (2.1). From Lemma 3.4, \(G(t, y(t))\) is a solution of Eq. (2.2). Then by Lemma 3.4, we see that \(H(t, G(t, y(t)))\) is a solution of Eq. (2.1), written as \(\overline{y}(t)\). Let

\[ J(t) = \overline{y}(t) - y(t). \]

To prove this conclusion, we need to show that \(J(t) \equiv 0\). In fact, differentiating \(J\), we have

\[ J'(t) = \overline{y}'(t) - y'(t) = A(t)\overline{y}(t) - A(t)y(t) = A(t)J(t), \]

which implies that \(J\) is a solution of the system \(Z' = A(t)Z\). From Lemma 3.1 and Lemma 3.2, it follows that

\[ \|J(t)\| = \|\overline{y}(t) - y(t)\| = \|H(t, G(t, (t, y(t)))) - y(t)\| \leq \|H(t, G(t, (t, y(t)))) - G(t, (t, y(t)))\| + \|G(t, (t, y(t))) - y(t)\| \leq 2k \| L(\mu) \|_\infty. \]

This implies that \(J(t)\) is a bounded solution of the system \(Z' = A(t)Z\). However, the linear system \(Z' = A(t)Z\) has no nontrivial bounded solution. Hence \(J(t) \equiv 0\), that is, \(\overline{y}(t) = y(t)\). Thus, \(J(t) \equiv 0\), that is,

\[ \overline{y}(t) = y(t), \quad \text{or} \quad H(t, G(t, y(t))) \equiv y(t). \]

Since \(y(t)\) is an arbitrary solution of Eq. (2.1), the proof of Lemma 3.5 is complete.

**Lemma 3.6.** For any \(t \in \mathbb{R}, x \in \mathbb{R}^n\), \(G(t, H(t, x)) = x\).
Proof. The proof is similar to that in Lemma 3.5.

Lemma 3.7. If \( r(t) \) is a nonnegative local integrable function and \( C_2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} r(s)ds < \infty \), then we have

\[
\|X(t, 0, x_0) - X(t, 0, \bar{x}_0)\| \leq ke^{kC_2\|x_0 - \bar{x}_0\|} \cdot e^{(kC_2 - \alpha)t}.
\]  

(3.7)

Proof. Since \( r(t) \) is a local integrable function, i.e., \( C_2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} r(s)ds \), we have

\[
\int_0^t r(s)ds = \int_0^1 r(s)ds + \int_1^2 r(s)ds + \ldots + \int_{[t]}^t r(s)ds 
\]

\leq ([t] + 1)C_2,

where \([t]\) means taking the largest integer not great than \( t \). Note that

\[
X(t, 0, x_0) = U(t, 0)x_0 + \int_0^t U(t, s)f(s, X(s, 0, x_0))ds.
\]

Then for any \( x_0, \bar{x}_0 \in \mathbb{R}^n \), the above equality leads to

\[
\|X(t, 0, x_0) - X(t, 0, \bar{x}_0)\| \leq ke^{-\alpha t}\|x_0 - \bar{x}_0\| + \int_0^t ke^{-\alpha(s-t)}r(s)\|X(s, 0, x_0) - X(s, 0, \bar{x}_0)\|ds.
\]

Using the Bellman inequality, we have

\[
\|X(t, 0, x_0) - X(t, 0, \bar{x}_0)\| \leq k\|x_0 - \bar{x}_0\|e^{\int_0^t kr(s)ds - \alpha t} \\
\leq ke^{kC_2\|x_0 - \bar{x}_0\|} e^{(kC_2 - \alpha)t}.
\]

4 Proofs of main results

Now we are in a position to prove our results.

Proof of Theorem 2.4. Now we show that \( H(t, \cdot) \) satisfies the four conditions of Definition 2.1. For any fixed \( t \), it follows from Lemma 3.5 and Lemma 3.6 that \( H(t, \cdot) \) is homeomorphism and \( G(t, \cdot) = H^{-1}(t, \cdot) \). Thus, Condition (i) is satisfied.

In view of Lemma 3.1, we derive that \( \|H(t, x) - x\| = \|h(t, (t, x))\| \leq K \|L(\mu)\|_\infty \). Note \( \|H(t, x)\| \to \infty \) as \( \|x\| \to \infty \), uniformly with respect to \( t \). Thus, Condition (ii) is satisfied.
In view of Lemma 3.2, we derive that \(\|G(t, y) - y\| = \|g(t, (t, y))\|\). Note \(\|G(t, y)\| \to \infty\) as \(\|y\| \to \infty\), uniformly with respect to \(t\). Thus, Condition \((iii)\) is satisfied. From Lemma 3.4, we know that Condition \((iv)\) is true. Hence, Eq. (2.2) and Eq. (2.1) are topologically conjugated. This completes the proof of Theorem 2.4.

**Proof of Corollary 2.5.** It follows from Theorem 2.4 the Corollary is valid immediately.

**Proof of Corollary 2.7.** Replacing condition (2.6) with condition (2.7), one can easily prove Corollary 2.7 by similar argument to Theorem 2.4.

**Proof of Theorem 2.9.** We divide the proof of Theorem 2.9 into two steps.

**Step 1.** We are going to prove the Lipschitz continuity of the equivalent function \(H(t, x)\) by Lemma 3.7. That is, we will show that
\[
\|H(t, x) - H(t, \bar{x})\| \leq p_1\|x - \bar{x}\|,
\]
where
\[
p = \frac{1 + k^2C_2e^{kC_2} - e^{-kC_2}}{1 - e^{-kC_2}}.
\]
In fact, by uniqueness \(X(t, (\tau, \xi)) = X(t, (t, \tau, X(t, (\tau, \xi))))\). From Lemma 3.1 it follows that
\[
h(t, (t, \xi)) = -\int_{-\infty}^{t} U(t, s)f(s, X(s, t, \xi))ds.
\]
Thus we get
\[
h(t, (t, \xi)) - h(t, (t, \bar{\xi})) = -\int_{-\infty}^{t} U(t, s)(f(s, X(s, t, \xi)) - f(s, X(s, t, \bar{\xi})))ds. \quad (I_1)
\]
By condition (2.5) and Lemma 3.7 we have
\[
\|I_1\| \leq \int_{-\infty}^{t} ke^{-\alpha(t-s)}r(s) \cdot \|X(s, t, \xi) - X(s, t, \bar{\xi})\|ds
\]
\[
\leq \int_{-\infty}^{t} ke^{-\alpha(t-s)}r(s) \cdot ke^{kC_2}\|\xi - \bar{\xi}\|e^{(kC_2-\alpha)(s-t)}ds
\]
\[
\leq \frac{k^2C_2e^{kC_2}}{1 - e^{-kC_2}} \cdot \|\xi - \bar{\xi}\|.
\]
By the definition of $H(t,x)$,

$$\|H(t,x) - H(t,\bar{x})\| \leq \|x - \bar{x}\| + \frac{k^2 C_2 e^{kC_2}}{1 - e^{-kC_2}} \|x - \bar{x}\| \leq 1 + \frac{k^2 C_2 e^{kC_2} - e^{-kC_2}}{1 - e^{-kC_2}} \|x - \bar{x}\| \equiv p_1 \|x - \bar{x}\|,$$

which implies that $H$ is Lipschitzian. This completes the proof of Step 1.

**Remark 4.1.** We prove that $H = x + h$ is Lipschitzian above. However, the regularity of $H$ can not be improved any more. $H$ is not $C^1$. Thus, Lipschitzian is the best regularity of $H$ for $C^0$ linearization. In fact, $h$ is not $C^1$ because $f$ is Lipschitzian, not $C^1$.

**Step 2.** We claim that there exist positive constants $p_2 > 0$ and $0 < q < 1$ such that for all $t \in \mathbb{R}, y, \bar{y} \in \mathbb{R}^n$,

$$\|G(t,y) - G(t,\bar{y})\| \leq p_2 \|y - \bar{y}\|^q.$$

Usually this point is treated with successive approximations. From Lemma 3.2 we know that $g(t,(\tau,\xi))$ is a fixed point of the following map $\mathcal{T}$

$$(\mathcal{T} z)(t) = \int_{-\infty}^{t} U(t,s) f(s,Y(s,\tau,\xi) + z(s))ds. \quad (4.1)$$

Let $g_0(t,(\tau,\xi)) \equiv 0$, and by recursion define

$$g_{m+1}(t,(\tau,\xi)) = \int_{-\infty}^{t} U(t,s) f(s,Y(s,\tau,\xi) + g_m(s,(\tau,\xi)))ds.$$

It is not difficult to show that

$$g_m(t,(\tau,\xi)) \to g(t,(\tau,\xi)), \quad \text{as} \quad m \to +\infty,$$

uniformly with respect to $t, \tau, \eta$.

Note that $g_0(t,(\tau,\xi)) = g_0(t,(t,Y(t,\tau,\xi))))$. Thus, by induction, it is clear that for all $m (m \in \mathbb{N})$,

$$g_m(t,(\tau,\xi)) = g_m(t,(t,Y(t,\tau,\xi))).$$

Choose $\lambda > 0$ sufficiently large and $q > 0$ sufficiently small such that

$$\begin{align*}
\lambda &> \frac{3kC_1}{1 - e^{-\alpha}} + \frac{3}{2} k^2 C_2, \\
q &< \frac{\alpha}{\alpha + 1} < 1, \\
0 &< \frac{C_2 k^3 + 1}{e^{\alpha} - \alpha q - 1} < \frac{1}{3}.
\end{align*} \quad (4.2)$$
Now we first show that if $0 < \|\xi - \bar{\xi}\| < 1$ for all $n$, we have

$$\|g_n(t, (t, \xi)) - g_n(t, (t, \bar{\xi}))\| < \lambda \|\xi - \bar{\xi}\|^q. \quad (4.3)$$

Note that inequality (4.3) holds when $n = 0$. Assume that inequality (4.3) holds when $n = m$. We now show that inequality (4.3) holds when $n = m + 1$. Note

$$g_{m+1}(t, (t, \xi)) - g_{m+1}(t, (t, \bar{\xi}))$$

$$= \int_{-\infty}^{t} U(t, s)[f(s, Y(s, t, \xi) + g_m(s, (t, \xi))) - f(s, Y(s, t, \bar{\xi}) + g_m(s, (t, \bar{\xi})))]ds. \quad (J_1)$$

We divide $J_1$ into two parts:

$$J_1 = \int_{-\infty}^{t-\tau} + \int_{t-\tau}^{t} \triangleq J_{11} + J_{12},$$

where $\tau = \frac{1}{\alpha + 1} \ln \frac{1}{\|\xi - \bar{\xi}\|}$. Then by conditions $(A_1)$ and (2.4), we deduce that

$$\|J_{11}\| \leq \int_{-\infty}^{t-\tau} ke^{-\alpha(t-s)}2\mu(s)ds$$

$$\leq \frac{2kC_1}{1 - e^{-\alpha}} \|\xi - \bar{\xi}\|^\frac{\alpha}{\alpha + 1}.$$

Since Eq. (2.1) satisfies condition $(A_1)$, we have

$$\|Y(s, t, \xi) - Y(s, t, \bar{\xi})\| \leq \|U(s, t)\|\|\xi - \bar{\xi}\| \leq ke^{-\alpha(t-s)}\|\xi - \bar{\xi}\|. \quad (4.4)$$

Furthermore, it follows from $(A_1)$, (2.4) and (4.4) that

$$\|g_m(s, (t, \xi)) - g_m(s, (t, \bar{\xi}))\| = \|g_m(s, (s, Y(s, t, \xi))) - g_m(s, (s, Y(s, t, \bar{\xi})))\|$$

$$\leq \lambda \|Y(s, t, \xi) - Y(s, t, \bar{\xi})\|^q$$

$$\leq \lambda k^q \|\xi - \bar{\xi}\|^q \cdot e^{-\alpha(q)(s-t)},$$
and
\[
\|J_{12}\| \leq \int_{t-\tau}^{t} ke^{-\alpha(t-s)} r(s) \left[ \|Y(s, t, \xi) - Y(s, t, \bar{\xi})\| + \|g_m(s, (t, \xi)) - g_m(s, (t, \bar{\xi}))\| \right] ds \\
\leq \int_{t-\tau}^{t} ke^{-\alpha(t-s)} r(s) \left[ ke^{-\alpha(s-t)} \|\xi - \bar{\xi}\| + \lambda k^q e^{-aq(s-t)} \|\xi - \bar{\xi}\|^q \right] ds \\
\leq \int_{t-\tau}^{t} k^2 r(s) \|\xi - \bar{\xi}\| ds + \int_{t-\tau}^{t} \lambda k^{q+1} r(s) e^{(a-q)(s-t)} \|\xi - \bar{\xi}\|^q ds \\
\leq \sum_{m \in [0, \tau]} k^2 \|\xi - \bar{\xi}\| \int_{t-\tau+m}^{t-\tau+m+1} r(s) ds \\
+ \sum_{m \in [0, \tau]} \lambda k^{q+1} \|\xi - \bar{\xi}\|^q \int_{t-\tau+m}^{t-\tau+m+1} r(s) e^{(a-q)(s-t)} ds \\
\leq k^2 C_2 \tau \|\xi - \bar{\xi}\| + \sum_{m \in [0, \tau]} \lambda k^{q+1} \|\xi - \bar{\xi}\|^q C_2 e^{(a-q)(\tau-1)} \left( \frac{1 - e^{(a-q)\tau}}{1 - e^{a-q}} \right) \\
\leq k^2 C_2 \|\xi - \bar{\xi}\| + \lambda C_2 k^{q+1} \frac{1}{e^{a-q} - 1} \|\xi - \bar{\xi}\|^q.
\]

Notice that \(\alpha - aq > 0\) implies that \(\exp\{\alpha - aq\} > 1\) and \(q < \frac{\alpha}{\alpha+1}\). Then
\[
\|J_{12}\| \leq \left( C_2 k^2 + \frac{\lambda C_2 k^{q+1}}{e^{a-q} - 1} \right) \|\xi - \bar{\xi}\|^q.
\]

Therefore, it follows from (4.2) that
\[
\|g_{m+1}(t, (t, \xi)) - g_{m+1}(t, (t, \bar{\xi}))\| \leq \|J_1\| + \|J_{12}\| \\
\leq \left( \frac{2kC_1}{1 - e^{-\alpha}} + C_2 k^2 + \frac{\lambda C_2 k^{q+1}}{e^{a-q} - 1} \right) \|\xi - \bar{\xi}\|^q \\
\leq \lambda \|\xi - \bar{\xi}\|^q.
\]

Thus inequality (4.3) holds for all \(n\). Setting \(n \to +\infty\), we have
\[
\|g(t, (t, \xi)) - g(t, (t, \bar{\xi}))\| \leq \lambda \|\xi - \bar{\xi}\|^q.
\]

Now by the definition of \(G(t, y)\), we have
\[
\|G(t, y) - G(t, \bar{y})\| \leq \|y - \bar{y}\| + \lambda \|y - \bar{y}\|^q \leq (1 + \lambda) \|y - \bar{y}\|^q = p_2 \|y - \bar{y}\|^q.
\]

This completes the proof of Step 2. \(\square\)
Remark 4.2. We now explain that the equivalent function \( G \) cannot be Lipschitzian. In fact, in proving the Hölder regularity of \( G \), we divide \( J_1 \) into two parts: \( J_{11}, J_{12} \). If we directly prove that \( G \) is Lipschitzian, then the following contradiction will appear:

\[
J_1 \leq \int_{-\infty}^{t} ke^{-\alpha(t-s)} r(s) \left[ \|Y(s, t, \xi) - Y(s, t, \bar{\xi})\| + \|g_m(s, (t, \xi)) - g_m(s, (t, \bar{\xi}))\| \right] ds
\]

\[
\leq \int_{-\infty}^{t} ke^{-\alpha(t-s)} r(s) [ke^{-\alpha(t-s)} \|\xi - \bar{\xi}\| + \lambda k q e^{-\alpha q(t-s)} \|\xi - \bar{\xi}\|^{q}] ds
\]

\[
\leq \int_{-\infty}^{t} k^2 r(s) \|\xi - \bar{\xi}\| ds + \int_{-\infty}^{t} \lambda k^{q+1} r(s) e^{(\alpha - \alpha q)(s-t)} \|\xi - \bar{\xi}\|^{q} ds.
\]

Obviously, \( J_1 \) is not Lipschitzian, since the integral \( \int_{-\infty}^{t} r(s) ds \) is divergent in the first right-hand estimation. Hence, the equivalent function \( G \) cannot be Lipschitzian.

Proof of Corollary 2.11. Step 1. proof of Lipschitz continuous of the map \( H \):

We start with auxiliary statement: since

\[
X(t, 0, x_0) = U(t, 0)x_0 + \int_0^t U(t, s)f(s, X(s, 0, x_0)) ds,
\]

for any \( x_0, \bar{x}_0 \in \mathbb{R}^n \), the above equality leads to

\[
\|X(t, 0, x_0) - X(t, 0, \bar{x}_0)\| \leq ke^{-\alpha t} \|x_0 - \bar{x}_0\| + \int_0^t ke^{-\alpha(t-s)} r(s) \|X(s, 0, x_0) - X(s, 0, \bar{x}_0)\| ds.
\]

Applying the Bellman inequality, we obtain

\[
\|X(t, 0, x_0) - X(t, 0, \bar{x}_0)\| \leq k \|x_0 - \bar{x}_0\| e^{(rk - \alpha)t}.
\]

We now continue with the proof of Lipschitzian of \( H \). From \( (I_1) \), we obtain that

\[
\|I_1\| = \|h(t, (t, \xi)) - h(t, (t, \bar{\xi}))\|
\]

\[
\leq \int_{-\infty}^{t} ke^{-\alpha(t-s)} r(s) \cdot \|X(s, t, \xi) - X(s, t, \bar{\xi})\| ds
\]

\[
\leq \int_{-\infty}^{t} ke^{-\alpha(t-s)} r(s) \cdot k \|\xi - \bar{\xi}\| e^{(rk - \alpha)(s-t)} ds
\]

\[
\leq k \|\xi - \bar{\xi}\|.
\]

By the definition of \( H(t, x) \), we have

\[
\|H(t, x) - H(t, \bar{x})\| \leq \|x - \bar{x}\| + k \|x - \bar{x}\| \leq (1 + k) \|x - \bar{x}\|.
\]
Step 2. proof of Hölder regularity of the map $G$:

Choose $\lambda > 0$ sufficiently large and $q > 0$ sufficiently small such that

$$
\begin{align*}
\lambda &> \frac{3}{2}rk^2 + 3\mu k, \\
n &< \frac{\alpha}{\alpha + 1} < 1, \\
0 &< rk^{q+1}(\alpha + 1) < \frac{1}{3}.
\end{align*}
$$

(4.5)

We also divide $J_1$ into two parts:

$$
J_1 = \int_{-\infty}^{t-\tau} + \int_{t-\tau}^{t} \triangle J_{11} + J_{12},
$$

where $\tau = \frac{1}{\alpha+1} \ln \frac{1}{\|\xi - \bar{\xi}\|}$. Then

$$
\|J_{11}\| \leq \int_{-\infty}^{t-\tau} ke^{-\alpha(t-s)}2\mu ds \leq 2\mu ke^{-\alpha\tau} \leq 2\mu k\|\xi - \bar{\xi}\|^{\frac{\alpha}{\alpha + 1}},
$$

and

$$
\|J_{12}\| \leq \int_{t-\tau}^{t} ke^{-\alpha(t-s)}r[ke^{-\alpha(s-t)}\|\xi - \bar{\xi}\| + \lambda k^q e^{-\alpha q(s-t)}\|\xi - \bar{\xi}\|_q]ds
$$

$$
\leq rk^2 \tau\|\xi - \bar{\xi}\| + \int_{t-\tau}^{t} \lambda r k^{q+1} e^{(\alpha - \alpha q)(s-t)}\|\xi - \bar{\xi}\|_q ds
$$

$$
\leq rk^2 e^\tau\|\xi - \bar{\xi}\| + \frac{\lambda r k^{q+1}}{\alpha - \alpha q}\|\xi - \bar{\xi}\|_q
$$

$$
\leq rk^2\|\xi - \bar{\xi}\|^{\frac{\alpha}{\alpha + 1}} + \frac{\lambda r k^{q+1}}{\alpha - \alpha q}\|\xi - \bar{\xi}\|^q.
$$

Notice that $\alpha - \alpha q > 0$ implies that $q < \frac{\alpha}{\alpha + 1}$. Then

$$
\|J_{12}\| \leq (rk^2 + \lambda r k^{q+1}(\alpha + 1))\|\xi - \bar{\xi}\|^q.
$$

Therefore, it follows from (4.2) that

$$
\|J_1\| \leq (2\mu k + rk^2 + \lambda r k^{q+1}(\alpha + 1))\|\xi - \bar{\xi}\|^q \leq \lambda\|\xi - \bar{\xi}\|^q.
$$

Thus inequality (4.3) holds for all $n$. Setting $n \to +\infty$, we have

$$
\|g(t, (t, \xi)) - g(t, (t, \bar{\xi}))\| \leq \lambda\|\xi - \bar{\xi}\|^q.
$$

Now by the definition of $G(t, y)$, we have

$$
\|G(t, y) - G(t, \bar{y})\| \leq \|y - \bar{y}\| + \lambda\|y - \bar{y}\|^q \leq (1 + \lambda)\|y - \bar{y}\|^2 = p_2\|y - \bar{y}\|^q.
$$
Proof of Corollary 2.13. Step 1. proof of the map $H \in C^1$: 
It follows from Lemma 3.1 that
\[ h(t, (t, \xi)) = - \int_{-\infty}^{t} U(t, s)B(s)X(s, t, \xi)ds. \]
Thus we can deduce that
\[ \frac{\partial h(t, (t, \xi))}{\partial \xi} = - \int_{-\infty}^{t} U(t, s)B(s)\frac{\partial X(s, t, \xi)}{\partial \xi}ds, \]
and this integral is convergent, i.e., it is well defined. In fact, we need the following auxiliary statement:
\[ \frac{\partial X(t, 0, \xi)}{\partial \xi} = U(t, 0) + \int_{0}^{t} U(t, \tau)B(\tau)\frac{\partial X(\tau, 0, \xi)}{\partial \xi}d\tau. \]
Since $\|U(t, \tau)\| \leq ke^{\alpha(t-\tau)}$ and $\|B(\tau)\| \leq \delta$, we have
\[ \left\| \frac{\partial X(t, 0, \xi)}{\partial \xi} \right\| \leq ke^{-\alpha t} + \int_{0}^{t} ke^{-(\alpha(t-\tau))} \cdot \delta \cdot \left\| \frac{\partial X(\tau, 0, \xi)}{\partial \xi} \right\|d\tau. \]
Using the Bellman inequality, we obtain
\[ \left\| \frac{\partial X(t, 0, \xi)}{\partial \xi} \right\| \leq ke^{(k\delta-\alpha)t}. \]
We now continue with the proof of $H \in C^1$. Since
\[ \left\| \frac{\partial h(t, (t, \xi))}{\partial \xi} \right\| \leq \int_{-\infty}^{t} ke^{-\alpha(t-\tau)} \cdot \delta \cdot \left\| \frac{\partial X(s, t, \xi)}{\partial \xi} \right\|ds \]
\[ \leq \int_{-\infty}^{t} k\delta e^{-(\alpha(t-\tau))} \cdot ke^{(k\delta-\alpha)(s-t)}ds \]
\[ \leq k, \]
$\frac{\partial h(t, (t, \xi))}{\partial \xi}$ is well defined and $h \in C^1$. Hence, it is clear that $H \in C^1$.

Step 2. proof of Hölder regularity of the map $G$: The proof is similar to that in Corollary 2.11.

5 Conflict of Interest

The authors declare that they have no conflict of interest.
6 Data Availability Statement

My manuscript has no associated data. It is pure mathematics.

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