On the Cauchy Problem for Weyl-Geometric Scalar-Tensor Theories of Gravity

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Abstract

In this paper, we analyse the well-posedness of the initial value formulation for particular kinds of geometric scalar-tensor theories of gravity, which are based on a Weyl integrable space-time. We will show that, within a frame-invariant interpretation for the theory, the Cauchy problem in vacuum is well-posed. We will analyse the global in space problem, and, furthermore, we will show that geometric uniqueness holds for the solutions. We make contact with Brans-Dicke theory, and by analysing the similarities with such models, we highlight how some of our results can be translated to this well-known context, where not all of these problems have been previously addressed.
I. INTRODUCTION

In this paper, we will analyse the initial value formulation for special types of geometric scalar-tensor theories, which are based on a Weyl integrable structure as a model for space-time. This type of structures have been proposed as a suitable model for space-time by several authors [1], [2], [3], [4], [5]. In fact, it is quite interesting that Weyl manifolds appear quite naturally as suitable models for space-time within the well-known axiomatic approach to space-time put forward by Elhers, Pirani and Schild (EPS) in [6]. There, it is shown that starting from basic assumptions about the behaviour of light and freely falling particles, space-time naturally acquires the structure of a general Weyl manifold. Then, it is shown that by adding some hypothesis about the behaviour of clocks, the Weyl structure should be integrable, i.e., Weyl's 1-form must be exact. A slightly different approach is followed in [7], where a rigorous discussion of the second-clock effect in the context of general Weyl structures is used to show that, without invoking any additional axioms, the rejection of such an effect as a realistic physical phenomenon, reduces the scenario to a Weyl integrable structure.

The ideas presented above, show that Weyl structures appear naturally as interesting models for space-time. In this context such structures have been explored by some cosmologists and, in particular, in [1] it has been pointed out that Weyl geometry might provide a natural geometric framework for some scalar-tensor theories. In such a context, the scalar field would appear as part of the geometric structure of space-time. For instance, in [1] it has been shown that applying a Palatini-type variational principle to the Brans-Dicke action, Weyl's compatibility condition naturally appears as the link between the metric, the scalar field and the connection. Thus, it has been proposed that such action should be posed on a Weyl integrable structure, and that in order to make the theory frame independent, it is argued that only geometrical quantities invariant under Weyl transformations should be regarded as containing physical information. This view may be supported by the fact that, following EPS, freely falling particles are shown to follow geodesics of the Weyl structure, which are invariant under Weyl transformations. Thus, each frame lies in equal footing when describing the motion of such particles. This point of view has been adopted in the study of some cosmological models [8], [9]. Also, a recent review on the use of such Weyl structures in modern physics can be found in [10].
Another important issue that has been raised, related with the above paragraph, is that the kind of model proposed in [1] is intimately related with Brans-Dicke theory. In fact, using different representatives in the equivalence class of Weyl integrable manifolds, the field equations take the same form as in the Brans-Dicke theory in either the Jordan frame or the Einstein frame, but, within the invariant interpretation presented in [1], both frames would be equivalent. (For a discussion on this matter, see [11]).

With the above considerations in mind, we will analyse whether the initial value formulation for the theory proposed in [1] is well-posed, within the invariant formulation presented therein. Since the initial value formulation of any physical theory is one of the things which lie in the core of its interpretation, giving it predictability, we regard this issue as fundamental if the theory is to be taken as realistic.

This paper will be organized as follows. We will first review the type of models we will be analysing and remind the reader the basic definitions concerning Weyl manifolds. Then, we will address the initial value formulation of such theories, and, finally, we will conclude with some remarks and future perspectives.

II. WEYL-GEOMETRIC SCALAR-TENSOR THEORIES (WGSTT)

Weyl geometry

A Weyl manifold is a triple $(M, g, \phi)$, where $M$ is a differentiable manifold, $g$ a semi-Riemannian metric on $M$, and $\phi$ a 1-form field on $M$. We assume that $M$ is endowed with a torsion-free linear connection $\nabla$ satisfying the following compatibility condition:

$$\nabla g = \phi \otimes g. \quad (1)$$

where, $\nabla g$ denotes the $(0,3)$-tensor field defined by $\nabla g(X,Y,Z) = (\nabla_X g)(Y,Z)$, where $X, Y$ and $Z$ are vector fields. Results concerning the existence and uniqueness of such a connection are straightforward and their proofs are analogous to those known for the Riemannian case [12]. It can easily be shown that, in local coordinates, the components of the Weyl connection $\nabla$ are given by:

$$\Gamma^u_{ac} = \frac{1}{2} g^{bu}(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca}) + \frac{1}{2} g^{bu}(\phi_b g_{ca} - \phi_a g_{bc} - \phi_c g_{ab}).$$
A Weyl integrable manifold is defined to be a Weyl manifold where Weyl’s 1-form $\varphi$ is exact, that is, $\varphi = d\phi$ for some (smooth) scalar function $\phi$ on $M$.

Let us now note that the compatibility condition (1) is invariant under the following group of transformations:

\[
\begin{align*}
\bar{g} &= e^{-f}g \\
\bar{\varphi} &= \varphi - df
\end{align*}
\]

(2)

where $f$ is an arbitrary smooth function defined on $M$. By this, we mean that if $\nabla$ is compatible with $(M, g, \varphi)$, then it is also compatible with $(M, \bar{g}, \bar{\varphi})$. It is easy to check that these transformations define an equivalence relation between Weyl manifolds. In this way, every member of the class is compatible with the same connection, hence has the same geodesics, curvature tensor and any other property that depends only on the connection. This is the reason why it is regarded more natural, when dealing with Weyl manifolds, to consider the whole class of equivalence $(M, [g, \varphi])$ rather than working with a particular element of this class. In this sense, it is argued that only geometrical quantities that are invariant under (2) are of real significance in the case of Weyl geometry. Following the same line of argument, it is also assumed that only physical theories and physical quantities presenting this kind of invariance should be considered of interest in this context.

An interesting fact to note about integrable Weyl structures, is that within each equivalence class $(M, [g, \phi])$, there is a representative for which its Weyl compatible connection coincides with the Riemannian connection related to this element. This is clear, since if we consider the element $(M, g', \phi')$ given by $g' = e^{-\phi}g$ and $\phi' = 0$, we see that the above argument holds.

**WGSTT**

In this section we will shortly review the main ideas about the models we will be considering in the remainder of the paper. The starting point of the model proposed in [1], is an action functional of the following type

\[
S(\nabla, g, \phi) = \int_V e^{-\phi}(R(\nabla) + \omega(d\phi, d\phi))\mu_g,
\]

(3)

where $V$ stands for a 4-dimensional space-time, $g$ for a Lorentzian metric, $\phi$ for a scalar field, $\omega$ is a coupling constant and $\mu_g$ the volume form associated with $g$. In this context the Ricci
scalar $R$ is associated to a torsionless connection $\nabla$.

It should be remarked that, if we consider that the background geometry is Riemannian, then (3) is equivalent to the Brans-Dicke action. The idea adopted in [1] is to consider a Palatini-type action principle for (3), and thus determine the relation between the connection and the other geometric objects (in this context $g$ and $\phi$) from the field equations. Interestingly enough, this procedure leads to the Weyl compatibility condition between the connection, the metric and the scalar field. Thus, we are led to a theory based on a Weyl integrable space-time. Written as in (3), this theory is posed in vacuum, but a prescription for coupling other fields is given [1].

It should be noted that, once it is established that the physical theory which comes from (3) should be posed on a Weyl integrable manifold, it is straightforward to try pose it on a Weyl integrable structure. This is a consequence of the natural invariance of the connection under Weyl transformations. Furthermore, following for instance the EPS axiomatic approach, we conclude that such structures appear as the most general mathematical models for space-time within the original scope proposed in [6]. In this scenario, freely falling point-like massive particles follow geodesics of the Weyl connection. This is a consequence of taking the equivalence principle as one of the axioms for space-time. Thus, the trajectories of massive particles will be invariant under Weyl transformations. These ideas point towards adopting the view that physical properties, in this context, should be invariant under Weyl transformations, and thus that the theory described by (3) should be posed on the equivalence class $(V, [g, \phi])$.

Taking into account the remarks above, we should stress that the second term in (3) is not invariant under Weyl transformations. Thus, we must expect the field equations derived from such action to be frame-dependent. Therefore, from (3) we will get a set of frame-dependent field equations, whose solutions define equivalence classes of the form $(V, [g, \phi])$, and, from the physical point of view, any member in the equivalence class is on equal footing, and thus any physical property described by such a solution must be invariant under Weyl transformations. This point of view has been used to describe some cosmological scenarios in [8],[9].

Finally, we should stress that, as might be expected, there is a strong connection between the theory described above and Brans-Dicke theory. In fact, another interesting aspect of this geometric model, is that the frame transformations connected with the so called Einstein
and Jordan frames in Brans-Dicke theory, can be understood as Weyl transformation within this context [11].

The issue of the equivalence of frames in Brans-Dicke theory has been treated (from the classical point of view) in some previous works [13, 14]. In particular, [13] recaptures the arguments of Dicke’s original paper [15], in which the notion of running units is addressed and applied to analyse the equivalence between frames. In this line of argument, it is claimed that any measurement of a physical observable is performed by considering the ratio between such a quantity, derived in a certain frame, and some chosen unit. Being that this ratio is what is actually observed, it is this quantity which should be invariant under frame transformations. Therefore, although the field equations are not invariant, the measured physical observable would be.

Contrasting with the above interpretation, the kind of models proposed in [1] also consider that different frames should be treated on equal footing, but this idea is translated into the mathematical structure on which the models are based. This is accomplished by considering space-time, and physical phenomena, described by an equivalence class \((V, [g, \phi])\) defining a Weyl integrable structure, and proposing that physical observables should be invariant under Weyl transformations. It is worth noticing that a simple way of defining invariant quantities in this formulation, consists in using the invariant metric \(\gamma = e^{-\phi}g\) to perform contractions on invariant tensors.

To conclude this section, we will write down the \(d\)-dimensional version of the action functional (3) and the field equations we get from it. We will pose the problem on a Weyl integrable manifold from the beginning. Thus, the curvature terms are to be understood as computed with the Weyl connection. For quantities computed with a Riemannian connection we will take as a notational convention to put the symbol ‘\(^{\circ}\)’ on the upper left corner of such quantities.

Our \(d\)-dimensional action is the following:

\[
S(g, \phi) = \int_V e^{-\frac{d-2}{2}\phi}(R + \omega(d\phi, d\phi) - V(\phi))\mu_g, \tag{4}
\]

where, in contrast to (3), we have included a potential for the scalar field in order to take into account some cosmological scenarios, such as the ones discussed in [9]. Straightforward
computations show that the Euler-Lagrange equations for such an action are the following:

\[
G_{\alpha\beta} = -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} g_{\alpha\beta} (\omega \nabla_\sigma \phi \nabla^\sigma \phi - V(\phi)) + \frac{d - 2}{2} g_{\alpha\beta} \left( \frac{d + 2}{2} \omega \nabla_\sigma \phi \nabla^\sigma \phi + \frac{d - 2}{2} R + V'(\phi) - \frac{d - 2}{2} V(\phi) \right) = 0.
\]

These equations can be rewritten in a more convenient way by taking the trace of the first equations and using this information in the second one. After a few computations we get the following:

\[
G_{\alpha\beta} = -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} g_{\alpha\beta} (\omega \nabla_\sigma \phi \nabla^\sigma \phi - V(\phi)),
\]

\[
g^{\sigma\mu} \nabla_\sigma \nabla_\mu \phi + \frac{1}{2\omega} (V'(\phi) + V(\phi)) = 0.
\]

Recall that the point of view that we are taking is that a solution of these field equations defines an equivalence class \((V, [g, \phi])\), where each element is related to another by a Weyl transformation, and that there is no preferred element in the class to describe physical properties. Thus, it will be instructive to look at the field equations satisfied by a generic element in the class \((V, [g, \phi])\) defined by a solution of \((6)\). Since any element \((V, g', \phi')\) in the equivalence class of \((V, g, \phi)\) is of the form \((V, e^{-f} g, \phi - f)\) for some \(f \in C^\infty(V)\), then \((g, \phi) = (e^f g', \phi' + f)\). Replacing these expressions in \((6)\), we see that \((g', \phi')\) satisfy the following system:

\[
G_{\alpha\beta}(g', \phi') = -\omega \nabla_\alpha (\phi' + f) \nabla_\beta (\phi' + f) + \frac{\omega}{2} g'_{\alpha\beta} g'{}^{\mu\nu} \nabla_\mu (\phi' + f) \nabla_\nu (\phi' + f) - \frac{1}{2} e^f g'_{\alpha\beta} V(\phi' + f),
\]

\[
g'{}^{\mu\nu} \nabla_\mu \nabla_\nu (\phi' + f) + \frac{e^f}{2\omega} (V'(\phi' + f) + V(\phi' + f)) = 0.
\]

Where we have used the fact that the Einstein tensor is invariant under Weyl transformations, thus \(G_{\alpha\beta}(e^f g', \phi' + f) = G_{\alpha\beta}(g', \phi') \forall f\). Also, note that \(V'\) stand for the ordinary derivative of the function \(V\), it does not represent any transformation law for the potential induced by the Weyl transformations.
III. THE INITIAL VALUE FORMULATION FOR WGSTT

In this section we will analyse the initial value formulation for the theory described in the previous section. As usual, we will consider space-time to be globally hyperbolic. Thus, \( V = M \times \mathbb{R} \), where \( V \) is an \((n+1)\)-dimensional space-time and \( M \) an \( n \)-dimensional hypersurface. On this setting we have the standard \((n+1)\)-splitting for space-time. By this we mean that we can pick local adapted frames of the form

\[
\begin{align*}
  e_0 & = \partial_t - \beta, \\
  e_i & = \partial_i, \quad i = 1, \ldots, n,
\end{align*}
\]

where \( \partial_i \) denote the coordinate vector fields associated to the coordinates \( x^i \) of any adapted coordinate system to the product structure \((x,t)\). In this context, the vector field \( \beta \) is the standard shift vector, which merely represents the tangent component of \( \partial_t \) to each hypersurface \( M \times \{t\} \cong M \). Thus, \( e_0 \) is, by construction, orthogonal to each \( M \times \{t\} \).

It should be noted that since all the metrics in an equivalence class \([g, \phi]\) are conformally related, they produce the same orthogonal splitting on each tangent space, and thus \( \beta \) does not depend on the choice of representative in the class.

It is straightforward to see that the dual frame to (8) is given by

\[
\begin{align*}
  \theta^0 & = dt \\
  \theta^i & = dx^i - \beta^i dt, \quad i = 1, \ldots, n.
\end{align*}
\]

Using these adapted frames, we see that we can write any space-time metric in the conformal class in the following way:

\[
g = -N^2 \theta^0 \otimes \theta^0 + g_{ij} \theta^i \otimes \theta^j,
\]

where the function \( N > 0 \) is referred to as the lapse function, which is frame-dependent. From now on, when we write coordinate expressions, unless explicitly stated, we will be referring to coordinates with respect to adapted frames of the type described above.

In what follows, our strategy will be to first analyse the well-posedness of the initial value formulation in a frame, that is, to analyse the well-posedness of the set of equations (6), and
then to show that this is enough to guarantee the well-posedness for the class \((V, [g, \phi])\). We will now be more specific on what we mean by saying that the Cauchy problem is well-posed on the equivalence class.

Since we are regarding that every meaningful quantity, which can in fact be observed, must be invariant under Weyl transformation, then, certainly, the initial value formulation should satisfy this condition if this interpretation is to hold. By this we mean that if we have an initial data set for the evolution problem, which has a development into a space-time \((V, g, \phi)\), then any other equivalent space-time \((V, g', \phi')\) related to \((V, g, \phi)\) by a Weyl transformation, must also be the product of the evolution of initial data on \(M\), and, in a specific sense, both initial data sets should be equivalent. Similarly, given two equivalent initial data sets, if one of them has a development into a space-time satisfying (6), then the other must also have a development into an equivalent space-time, thus satisfying (7) for some gauge function \(f\). Pictorially, what we intend to show is that the following commutative diagram holds.

\[
\begin{array}{ccc}
(V, g, \phi) & \xrightarrow{W} & (V, g', \phi') \\
\downarrow{C} & & \downarrow{C} \\
(M, \bar{g}, \bar{\phi}, K, \bar{\phi}_t) & \xleftarrow{W} & (M, \bar{g}', \bar{\phi}', K', \bar{\phi}'_t)
\end{array}
\]

FIG. 1: Commutativity of performing a Weyl transformation on the initial data sets and their respective space-time evolutions

In the above diagram \(C\) stands for Cauchy evolution and \(W\) for Weyl transformation; the quantities \(\bar{g}, \bar{\phi}\) define a Weyl structure on the initial hypersurface; \(K\) stands for a symmetric \((0, 2)\)-tensor field that, after getting the embedding, represents the extrinsic curvature of the embedded hypersurface \(M \cong M \times \{0\}\), and \(\bar{\phi}_t\) represents the initial data for \(\partial_t \phi(\cdot, 0)\).

We should now make sense of what we mean by a Weyl transformation on the initial data set. This is because, even though it is clear how a Weyl transformation has to act on \(\bar{g}\) and \(\bar{\phi}\), the objects \(K\) and \(\bar{\phi}_t\) are extrinsic objects, and the way they should transform comes from looking at the embedded manifold \(M \hookrightarrow M \times \mathbb{R}\) and analysing their transformation.
behaviour. Thus, consider a space-time Weyl transformation:

\[ g' = e^{-f} g, \tag{11} \]
\[ \phi' = \phi - f. \tag{12} \]

Using the definition of the extrinsic curvature [16], we have that

\[ K'(X, Y) \overset{\text{def}}{=} g'\langle \nabla_X Y, n' \rangle, \]

where \( n' \) stands for the future-pointing unit normal vector field, in the metric \( g' \), to each slice \( M \times \{t\} \). It is clear that using the adapted frames, we have that \( N'^2 = e^{-f} N^2 \), where \( N \) and \( N' \) are the lapse functions in the frames \((V, g, \phi)\) and \((V, g', \phi')\). Then, we have that, by definition of the lapse,

\[ n' = \frac{1}{N'} e_0 = e^f n. \]

Then, clearly, we have that

\[ K' = e^{-\frac{f}{2}} K(X, Y) \tag{13} \]

That is, under a space-time Weyl transformation, the extrinsic curvature of each hypersurface \( M \times \{t\} \) transforms as \( K' = e^{-\frac{f}{2}} K \). Thus, on the initial slice \( M \cong M \times \{0\} \), setting \( h(\cdot) \equiv f(\cdot, 0) \in C^\infty(M) \), we have the following induced Weyl transformation on \( M \)

\[ K'|_{t=0} = e^{-\frac{h}{2}} K|_{t=0}. \]

Also, with the same line of argument, we have that \( \partial_t \phi'(\cdot, 0) = \partial_t \phi(\cdot, 0) - \partial_t f(\cdot, 0) \). Thus, since \( \eta(\cdot) \equiv \partial_t f(\cdot, 0) \) represents an arbitrary smooth function on \( M \), we get the following transformations on the initial data set

\[ \bar{g}' = e^{-h} \bar{g}, \quad \bar{\phi}' = \bar{\phi} - \eta, \]
\[ \bar{K}' = e^{-\frac{h}{2}} \bar{K} \quad \bar{\phi}_t = \bar{\phi}_t - \eta. \tag{14} \]

Thus, we will take as a definition that a Weyl transformation on the initial data set
\((\bar{g}, \bar{\phi}, \bar{K}, \bar{\phi}_t)\) is given by \([14]\), where \(h\) and \(\eta\) are arbitrary smooth functions on \(M\).

It will be important to remark that, from the way \(K\) transforms under a space-time Weyl transformation, we can easily compute its coordinate expression in an adapted frame. In order to do this, we simply use the fact that for the Riemannian element \((V, e^{-\phi}g, 0)\) in an equivalence class \((V, [g, \phi])\), we know what the coordinate expression of \(K'\) looks like, since this is a standard object in Riemannian geometry. Explicitly,

\[
K'_{ij} = -\frac{1}{2N}(\partial_t g'_{ij} - (\nabla_i \beta_j + \nabla_j \beta_i)).
\]

Since we have found that \(K(g, \phi) = e^{\frac{\phi}{2}} K'(g', 0)\), and we know that \(N^{-\frac{\phi}{2}}N\), we get that

\[
K_{ij}(g, \phi) = -\frac{e^\phi}{2N}(e^{-\phi} \partial_t g_{ij} - e^{-\phi} g_{ij} \partial_t \phi - (\nabla_i \beta_j + \nabla_j \beta_i)),
\]

\[
= -\frac{1}{2N}(\partial_t g_{ij} - g_{ij} \partial_t \phi - e^{\phi}(\nabla_i \beta_j + \nabla_j \beta_i)).
\]

Thus, on the initial slice \(M \cong M \times \{0\}\), we get

\[
\bar{K}_{ij}(g, \phi) = -\frac{1}{2\bar{N}}(\partial_t g_{ij}|_{t=0} - \bar{g}_{ij} \bar{\phi}_t - e^{\bar{\phi}}(\bar{\nabla}_i \bar{\beta}_j + \bar{\nabla}_j \bar{\beta}_i)),
\]

where all the quantities with a “bar” on top are taken on \(t = 0\).

**Constraint equations**

Just as in the case of general relativity (GR), the set of equations \([6]\) impose a set of constraint equations for the initial data. Before writing these equations, we will first establish some notation. The field equation can be rewritten as

\[
G_{\alpha\beta} = T_{\alpha\beta},
\]

\[
g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + \frac{1}{2\omega}(V'(\phi) + V(\phi)) = 0,
\]

where

\[
T_{\alpha\beta} = -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} g_{\alpha\beta}(\omega \nabla_\sigma \phi \nabla^\sigma \phi - V(\phi)).
\]

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The origin of the constraint equations is quite simple: If the above equations equations are to hold on space-time, they must, in particular, hold on the initial hypersurface \( M \cong M \times \{0\} \). Thus, the equations

\[
G(e_0, e_0) = T(e_0, e_0),
\]
\[
G(e_0, e_i) = T(e_0, e_i). 
\]

must be satisfied on the initial space-like slice. We will see that these equation depend only on the initial data set. This implies that the initial data set cannot be chosen arbitrarily. Only those initial data sets satisfying the constraint equations (19) can have a development into a space-time satisfying the above field equations.

We will now explicitly write down the constraint equations. In order to compute the components of the Einstein tensor, we can take advantage of the fact that we know their form in the Riemannian context, and that within each class \( (V, [g, \phi]) \) there is a Riemannian element \( (V, e^{-\phi}g, 0) \), i.e., for this element the Weyl connection coincides with its Riemannian connection. Thus, since we know that the Einstein tensor is invariant in the equivalence class, then, using the usual Gauss-Codazzi equations from Riemannian geometry, we see that for

\[
g' = e^{-\phi}g, \\
K' = e^{-2\phi}K,
\]

the following relations hold:

\[
G(e_0, e_0) = \frac{N^2}{2}((K'_l)_{ij}^2 - K'^{ij}K'_{ij} + R(\bar{g}')), \\
G(e_0, e_i) = N'(\bar{\nabla}_iK'_l^j - \bar{\nabla}_jK'_l^i),
\]

where \( \bar{\nabla} \) denotes the induced Weyl connection on \( M \), compatible with the induced Weyl structure \( (M, [\bar{g}, \bar{\phi}]) \). Using (14), we get

\[
G(e_0, e_0) = \frac{N^2}{2}((K'^{ij}K_{ij} + R(\bar{g}, \bar{\phi})), \\
G(e_0, e_i) = N\bar{g}^{ij}(\bar{\nabla}_iK_{uj} - \bar{\nabla}_jK_{ui}) + \frac{N}{2}(\bar{\nabla}_j\phi K'^{ij} - \bar{\nabla}_i\phi K'^{ij}).
\]

Now, in order to write down the constraints, we need to compute the same components
for the tensor $T$. Notice that, using an adapted frame, we have

$$T_{g, \phi} = \{-\omega e_{\alpha}(\phi)e_{\beta}(\phi) + \frac{1}{2}\omega g_{\alpha\beta}g^{\mu\nu}e_{\mu}(\phi)e_{\nu}(\phi) - \frac{1}{2}g_{\alpha\beta}V(\phi)\}\theta^\alpha \otimes \theta^\beta.$$  

From this expression it is straightforward to see that

$$T(e_0, e_0) = \frac{N^2}{2}\{-\omega(\pi^2 + |\nabla \phi|_g^2) + V(\phi)\}$$

$$T(e_0, e_i) = -\omega N\pi \nabla_i \phi,$$

where $\pi = e_0(\phi), \bar{g}$ denotes the induced metric on each slice $M \times \{t\}$ and $\nabla$ denotes the induced Weyl connection. From these computations we get that the constraint equations are the following

$$(K^i_j)^2 - K^{ij}K_{ij} + R = -\omega\{\pi^2 + |\nabla \phi|_g^2\} + V(\phi),$$

$$\bar{g}^{u_j}(\nabla_i K_{u_j} - \nabla_j K_{ui}) + \frac{1}{2}(\nabla_j \phi K^j_i - \nabla_i \phi K^i_j) = -\omega \pi \nabla_i \phi.$$  

The above expressions, on $M \cong M \times \{0\}$, depend only on the initial data set, as previously remarked. Since these constraints are to be posed for the initial data on the initial hypersurface $M$, it becomes clear that, just as discussed for $K$, the way that $\pi$ transforms under a Weyl transformation on the initial data set, does not come from the intrinsic Weyl transformation on $M$, but is induced from the way it transforms under space-time Weyl transformations. Also note that $\pi$ is a quantity that we get by analysing the right-hand side of (6), which is frame-dependent. Thus, we should expect $\pi$ to be frame-dependent. In fact, the above computations applied to the transformed system (7), give us the following constraints, valid in each frame:

$$(K^*_i_j)^2 - K^{*ij}K^{*ij} + R(\bar{g}^*, \bar{\phi}^*) = -\omega\left\{\left(\frac{e^{f(\cdot,0)}}{N}e_0(\bar{\phi}^* + f)\right)|_{t=0}^2 + |\nabla(\bar{\phi}^* + f(\cdot,0))|_{\bar{g}^*}^2\right\}$$

$$+ e^{f(\cdot,0)}V(\phi^* + f)|_{t=0},$$

$$\bar{g}^{*u_j}(\nabla_i K^*_{u_j} - \nabla_j K^*_{ui}) + \frac{1}{2}(\nabla_j \bar{\phi}^* K^*_{ji} - \nabla_i \bar{\phi}^* K^*_{ij}) = -\omega\left(\frac{e^{f(\cdot,0)}}{N}e_0(\bar{\phi}^* + f)\right)|_{t=0} \nabla_i (\bar{\phi}^* + f(\cdot,0)).$$

(23)
where the notation comes from considering the space-time Weyl transformation
\[ g^* = e^{-f} g, \]
\[ \phi^* = \phi - f. \]

From the above expression, we finally get that in an arbitrary frame,
\[ \pi^* = \frac{e^{ \frac{f}{2} } \pi}{N} e_0 (\phi^* + f) = \frac{e^{ \frac{f}{2} } e_0}{N} (\phi - f + f) = e^{ \frac{f}{2} } \pi. \]

Thus, we can write the constraints of the transformed system (7) as
\[
(K^*)^2 - K^{*ij} K^*_{ij} + R(\bar{\phi}^*, \bar{\phi}^*) = -\omega \left\{ \pi^{*2} + |\bar{\nabla} (\bar{\phi}^* + f(\cdot, 0))|^2_{\bar{g}^*} \right\} + e^{f(0)} V(\phi^* + f)|_{t=0},
\]
\[
\bar{g}^* u_i (\bar{\nabla}_i K^*_{uj} - \bar{\nabla}_j K^*_{ui}) + \frac{1}{2} (\bar{\nabla}_j \bar{\phi}^* K^*_{ij} - \bar{\nabla}_i \bar{\phi}^* K^*_{ij}) = -\omega \pi^* \bar{\nabla}_i (\bar{\phi}^* + f(\cdot, 0)).
\]

(24)

Now, just as instead of \( \partial_t \bar{g}|_{t=0} \), we actually consider \( K \) as part of the geometric initial data, we will consider \( \pi \) instead of \( \bar{\phi}_t \) as part of the geometric initial data. In this way, when given the initial data set \((\bar{g}, \bar{\phi}, K, \pi)\), we will have two gauge choices: one for the lapse-shift and another one for the values of \( f, \partial_t f|_{t=0} \). In this way we get that an initial data set for the system (6) is a set of the form \((\bar{g}, \bar{\phi}, K, \pi)\), which under a Weyl transformation on the initial slice \( M \), transforms according to
\[
\bar{g}' = e^{-h} \bar{g}, \quad \bar{\phi}' = \bar{\phi} - h, \quad K' = e^{-\frac{h}{2}} K, \quad \pi' = e^{\frac{h}{2}} \pi.
\]

(25)

Clearly, given a solution of (22) and applying these transformations to it, we get that, under the choice \( h \doteq f(\cdot, 0) \), the transformed quantities \((\bar{g}^*, \bar{\phi}^*, K^*, \pi^*)\) satisfy the system (24), which is the constraint system for the space-time equations (7).
The evolution problem

We will now analyse the set of equations \[ (6) \]. We will follow the same line of argument typically applied to the vacuum system in GR (Thorough reviews on this topic can be found in [17] and [18]). With this in mind, we should begin by making the following remarks. First, note that the geometric identities implied by the symmetries of the curvature tensor are vital in this problem. We will need to use analogue identities in our new setting. There is an easy way to show that there is a version of the contracted Bianchi identities valid in this new context. To see this, just note that given a Weyl integrable structure \((V, [g, \phi])\), for the Riemannian element in the class \((V, e^{-\phi}g, 0)\) we have the usual contracted Bianchi identities:

\[
e^\phi g^{\mu\nu} \nabla_\mu G_{\nu\sigma} = 0,
\]

where \(G_{\nu\sigma}\) is the Riemannian Einstein tensor associated with \(e^{-\phi}g\). Since the Einstein tensor is invariant under Weyl transformations, then \(G_{\nu\sigma}\) represents the Einstein tensor of all the elements in the class \((V, [g, \phi])\). Thus, we easily see that

\[
g^{\mu\nu} \nabla_\mu G_{\nu\sigma}(g, \phi) = e^{-\phi} \left( e^\phi g^{\mu\nu} \nabla_\mu G_{\mu\sigma}(e^{-\phi}g, 0) \right) = 0.
\]

Thus,

\[
g^{\mu\nu} \nabla_\mu G_{\nu\sigma}(g, \phi) = 0,
\]

and this geometric identity is valid for every element in the class \((V, [g, \phi])\). This identity clearly imposes a condition on the right-hand side of \[ (6) \], which is typically referred to as a conservation equation. In order for the system \[ (6) \] to be sensible, this geometric condition must be compatible with the equation posed for \(\phi\). It is a simple matter of computation to show that both conditions are equivalent.

Secondly, we will rewrite \[ (6) \] merely in terms of the Ricci tensor. In order to achieve this, notice that taking the trace of this equation, we get

\[
R = -\omega g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{n + 1}{n - 1} V(\phi).
\]
Using this information, we can rewrite our system in the following way:

\[
R_{\alpha\beta} = -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{n-1} g_{\alpha\beta} V(\phi),
\]

\[
g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + \frac{1}{2\omega} (V'(\phi) + V(\phi)) = 0.
\] (26)

Now, some straightforward computations show that we can write down the Ricci tensor as:

\[
R_{\alpha\beta} = \circ R_{\alpha\beta} + \frac{n-1}{2} \nabla_\alpha \nabla_\beta \phi + \frac{n-1}{4} \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{n-1}{2} \nabla_\mu \phi \nabla_\mu \phi).
\]

It is worth noting that, again, using the invariance of the Ricci tensor under Weyl transformations, we can pick the Riemannian element in a class \((V, [g, \phi])\), for which its “Riemannian” Ricci tensor coincide with the Ricci tensor of the class, i.e., \(R_{\alpha\beta}(g, \phi) = \circ R(e^{-\phi} g)\), and then make a conformal transformation, mapping \(e^{-\phi} g \mapsto g\). Since it is well-known how \(\circ R(e^{-\phi} g)\) transforms under conformal transformations on the metric, that is, the relation between \(\circ R(e^{-\phi} g)\) and \(\circ R(g)\) is well-known, we can use such relation in \(R_{\alpha\beta}(g, \phi) = \circ R(e^{-\phi} g)\), and we will get the right decomposition for the Ricci tensor of the Weyl connection of the class in terms of the Ricci tensor associated with the Riemannian connection of a particular element in the class, and the extra terms involving the scalar field.

Furthermore, a straightforward computation gives that

\[
g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{n-1}{2} \nabla_\mu \phi \nabla_\mu \phi + \frac{1}{2} \omega (V'(\phi) + V(\phi)) = 0.
\] (28)

The above field equations are posed for \((g, \phi)\). Looking at (28), it is clear that this is a quasilinear wave equation which does not pose any difficulties from the point of view of the
theory of partial differential equations. In contrast, in the set of equations (27), the second
order term $\nabla_\alpha \nabla_\beta \phi$ brings some difficulties. If this term was not present, the same approach
towards the vacuum Einstein equations, which shows that such equations can be written as
a system of non-linear wave equations for the metric, could be applied here, and the lower
order terms in $\phi$ would not cause any difficulties. This approach does not work here. It
is interesting to remark that the very same problem appears when studying, in the Jordan
frame, the Cauchy problem for Brans-Dicke theory and other scalar tensor theories. This
problem has been studied before, for instance, in [19], [20], [21], [22]. In [19] and [20], their
considerations are limited to analytic initial data. In [21], the local well-posedness of the
initial value formulation for the smooth case (or even $C^k$) is addressed. By local we mean that
the problem is studied within a coordinate system, which is chosen by using wave-coordinates,
and this condition is not invariant under coordinate changes. Thus, the well-posedness is
both local in space and in time. In [22] more general type of scalar-tensor theories are
studied and they are shown to have a well-posed initial value formulation by rewriting them
as a first order hyperbolic system where classical theorems on partial differential equations
(PDE) can be applied. In this approach, they also treat the local in space and time problem,
using a generalized wave-gauge condition.

In contrast to the above mentioned approaches, we will address the global in space initial
value formulation for the set of equations (27)-(28), and, furthermore, we will do this in such
a way that the invariance of the problem under Weyl transformations becomes transparent.
This will be done by following the standard approach to the global in space problem in GR,
but adapting our gauge conditions to our new setting. In this direction, let us begin by
defining the following properly Riemannian metric on space-time:

$$\hat{e} = dt \otimes dt + e,$$

where $e$ is some smooth Riemannian metric on $M$. Now define the following vector field on
space-time

$$F^\lambda = g^{\alpha\beta} (\Gamma^\lambda_{\alpha\beta} - \hat{\Gamma}^\lambda_{\alpha\beta}),$$

where $\Gamma^\lambda_{\alpha\beta}$ are connection components of the Weyl connection associated with the space-time
Weyl structure $(g, \phi)$, and $\hat{\Gamma}^\lambda_{\alpha\beta}$ are the components of the Riemannian connection associated
with the metric $\hat{e}$. The fact that $F$ defines a vector field on space-time, comes from the fact
that the difference of two connections defines a tensor field. We can rewrite this vector field in the following way:

\[ F^\lambda = \varphi F^\lambda + g^{\alpha\beta}(\Gamma^\lambda_{\alpha\beta} - \varphi \Gamma^\lambda_{\alpha\beta}), \]

where \( \varphi F \) is defined as \( g^{\alpha\beta}(\Gamma^\lambda_{\alpha\beta} - \varphi \Gamma^\lambda_{\alpha\beta}) \). The vector field \( \varphi F \) has been used to analyse the global in space problem in GR [17]. Now, since \( g^{\alpha\beta}(\Gamma^\lambda_{\alpha\beta} - \varphi \Gamma^\lambda_{\alpha\beta}) = \frac{n-1}{2} \nabla^\lambda \phi \), we get that

\[ F^\lambda = \varphi F^\lambda + \frac{n-1}{2} \nabla^\lambda \phi. \quad (29) \]

At this point, we will take advantage of some useful expressions known from the Cauchy problem in GR. For instance, it is known that the Ricci tensor \( \varphi R_{\alpha\beta} \) has the following decomposition (see, for instance, chapter 6 in [17]):

\[ \varphi R_{\alpha\beta} = R^{(\hat{e})}_{\alpha\beta} + \frac{1}{2} \left( g_{\alpha\lambda} D_\beta F^\lambda + g_{\beta\lambda} D_\alpha F^\lambda \right), \]

\[ R^{(\hat{e})}_{\alpha\beta} = -\frac{1}{2} g^{\lambda\mu} D_\lambda D_\mu g_{\alpha\beta} + f_{\alpha\beta}(g, Dg), \]

where \( D \) represents the \( \hat{e} \)-covariant derivative and \( f_{\alpha\beta} \) has smooth dependence on both \( g \) and \( Dg \). Using this expressions together with \( \varphi R_{\alpha\beta} \), we can rewrite

\[ \varphi R_{\alpha\beta} = R^{(\hat{e})}_{\alpha\beta} + \frac{1}{2} \left( g_{\alpha\lambda} D_\beta F^\lambda + g_{\beta\lambda} D_\alpha F^\lambda \right) - \frac{n-1}{4} \left( g_{\alpha\lambda} D_\beta g^{\lambda\mu} D_\mu \phi + g_{\beta\lambda} D_\alpha g^{\lambda\mu} D_\mu \phi \right) - \frac{n-1}{2} D_\alpha D_\beta \phi. \quad (30) \]

Now, the idea is to rewrite (27) in terms of the \( \hat{e} \)-covariant derivatives, and to use the previous decomposition for the Riemannian Ricci tensor. The interesting thing is that the last term in (30) will exactly cancel the problematic term in (27). In this way, we get the following system:

\[ \varphi R_{\alpha\beta} - \rho_{\alpha\beta} + \frac{1}{2} \left( g_{\alpha\lambda} D_\beta F^\lambda + g_{\beta\lambda} D_\alpha F^\lambda \right) = 0, \]

\[ g^{\alpha\beta} D_\alpha D_\beta \phi - \varphi F^\sigma D_\sigma \phi - \frac{n-1}{2} D_\alpha \phi D^\sigma \phi + \frac{1}{2\omega} \left( V'(\phi) + V(\phi) \right) = 0, \quad (31) \]
where
\begin{align*}
\mathcal{R}_{\alpha\beta} &= R_{\alpha\beta}^{(e)} - \frac{n-1}{4} (g_{\alpha\lambda} D_{\beta} g^{\lambda\mu} D_{\mu} \phi + g_{\beta\lambda} D_{\alpha} g^{\lambda\mu} D_{\mu} \phi) - \frac{n-1}{2} \Delta^\sigma \Gamma^\sigma_{\alpha\beta} D_\sigma \phi \\
&\quad + \frac{n-1}{4} D_\alpha \phi D_\beta \phi - \frac{1}{4\omega} g_{\alpha\beta} (V'(\phi) + V(\phi)), \\
\rho_{\alpha\beta} &= -\omega D_\alpha \phi D_\beta \phi + \frac{1}{n-1} g_{\alpha\beta} V(\phi), \\
\Delta^\sigma \Gamma^\sigma_{\alpha\beta} &= \nabla^\sigma \Gamma^\sigma_{\alpha\beta} - \hat{\nabla}^\sigma_{\alpha\beta}.
\end{align*}

Following an approach analogous to the standard approach to the Einstein equations, we will consider the reduced system:

\begin{align*}
\mathcal{R}_{\alpha\beta} - \rho_{\alpha\beta} &= 0, \\
g^{\alpha\beta} D_\alpha D_\beta \phi - \mathcal{F}^\sigma D_\sigma \phi - \frac{n-1}{2} D_\sigma \phi D^\sigma \phi + \frac{1}{2\omega} (V'(\phi) + V(\phi)) &= 0.
\end{align*}

Using the expression for $R_{\alpha\beta}^{(e)}$ in (32), we see that (32) and (33) form a system of hyperquasilinear wave equations for $(g, \phi)$, thus, given appropriate initial data, the system has one and only one solution on $M \times [0, T)$, for some $T > 0$. By appropriate initial data, we mean initial data belonging to some appropriate functional spaces. The regularity of the solution will be directly tied to this choice. In particular, for smooth data, we get smooth solutions. For a detailed review of these properties, and low regularity results, see, for instance, [17].

Now, since $T_{\alpha\beta} = \rho_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \rho_{\mu\nu}$, we get that, given a solution $(g, \phi)$ for (32)-(33), its Einstein tensor satisfies the following identity:

$$G_{\alpha\beta} - T_{\alpha\beta} = \frac{1}{2} (g_{\alpha\lambda} D_{\beta} F^\lambda + g_{\beta\lambda} D_{\alpha} F^\lambda - g_{\alpha\beta} D_{\lambda} F^\lambda).$$

Now, the contracted Bianchi identities for Weyl’s connection together with (33), imply that the divergence of the previous expression vanishes. This gives us a geometric identity satisfied by $F$. Some straightforward computations give us that $F$ satisfies the following homogeneous system of linear wave equations:

$$g^{\alpha\mu} D_\alpha D_\mu F^\lambda + A^\alpha_{\nu}(g, \phi, Dg, D\phi) D_\alpha F^\nu + g^{\beta\lambda} R_{\mu\beta}(\hat{e}) F^\mu = 0,$$

where $A^\alpha_{\nu}$ is a $(2, 1)$-tensor field depending on $(g, \phi, Dg, D\phi)$. Uniqueness of solutions for
such a system gives us that, if $F(\cdot, 0) = 0$ and $\partial_t F(\cdot, 0) = 0$, then $F \equiv 0$. Thus, if we can guarantee that the initial data for (32)-(33) satisfy these conditions, then the solution to this system actually satisfies the full system of equations (31).

**Lemma 1.** Suppose that $(g, \phi)$ satisfy the reduced system of equation (32)-(33) and that the initial data satisfies $F(\cdot, 0) = 0$. Then $\partial_t F(\cdot, 0) = 0$ iff the constraint equations (22) are satisfied.

**Proof.** We have the following decomposition:

$$G_{\alpha\beta} = \gamma G_{\alpha\beta} + \frac{1}{2} (g_{\alpha\lambda} D_\beta F^\lambda + g_{\beta\lambda} D_\alpha F^\lambda - g_{\alpha\beta} D_\lambda F^\lambda),$$

where $\gamma G_{\alpha\beta} = \gamma R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \gamma R_{\alpha\mu\nu})$. Note that if $F|_{t=0} = 0$, then $D_i F|_{t=0} = 0$. Then we have that:

$$G_{\alpha0}|_{t=0} = \gamma G_{\alpha0}|_{t=0} + \frac{1}{2} (g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Thus,

$$(G_{\alpha0} - T_{\alpha0})|_{t=0} = (\gamma G_{\alpha0} - T_{\alpha0})|_{t=0} + \frac{1}{2} (g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Then, since $(g, \phi)$ solve the reduced system, we get that

$$(G_{\alpha0} - T_{\alpha0})|_{t=0} = \frac{1}{2} (g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Thus we see that the constraint equations are satisfied iff

$$0 = (g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Setting $\alpha = 0$, the previous expression gives $\partial_t F^0 = 0$ and setting $\alpha = j$ gives $\partial_t F^j = 0$. □

Using this lemma, we see that if we give an initial data set satisfying the constraint equations plus the gauge condition $F|_{t=0} = 0$, then the system (32)-(33) will have a unique solution, which, in fact, solves the full system (27)-(28). Let us see that we can always pick initial data satisfying the gauge condition.
Our geometric initial data set for (32)-(33) is \(( M, \bar{g}, \bar{\phi}, K, \pi)\). We need to provide initial data for \((g, \phi)\). It is clear that \(\bar{g}, \bar{\phi}\) determine the initial data \(g_{ij}|_{t=0}, \phi|_{t=0}\). Also that, once we fix the initial data for the lapse and shift, \(K\) and \(\pi\) determine the initial data \(\partial_t g_{ij}|_{t=0}, \partial_t \phi|_{t=0}\). Then, we freely set \(g_{00}|_{t=0} = -1\) and \(g_{0i}|_{t=0} = 0\), which amounts to setting \(N|_{t=0} = 1\) and \(\beta|_{t=0} = 0\). We still need initial the data \(\partial_t g_{00}|_{t=0}, \partial_t g_{0i}|_{t=0}\). These initial data will come from the solving the gauge condition. Note that

\[
F^\lambda|_{t=0} = \{g^{\alpha\beta} \Gamma^\lambda_{\alpha\beta} + \frac{n-1}{2} \nabla^\lambda \phi - g^{\alpha\beta} \hat{\Gamma}^\lambda_{\alpha\beta}\}|_{t=0}.
\]

Then, since we have that

\[
g^{\alpha\beta} \Gamma^0_{\alpha\beta}|_{t=0} = \frac{1}{2} \partial_t g_{00}|_{t=0} - \bar{g}^{ij} K_{ij},
\]

\[
g^{\alpha\beta} \Gamma^i_{\alpha\beta}|_{t=0} = -\bar{g}^{ij} \partial_t g_{j\theta}|_{t=0} + \bar{g}^{kl} \Gamma^i_{kl},
\]

we can solve the algebraic equations \(F^\lambda|_{t=0} = 0\) for \(\partial_t g_{00}|_{t=0}, \partial_t g_{0i}|_{t=0}\). With these choices for \(\partial_t g_{00}|_{t=0}, \partial_t g_{0i}|_{t=0}\) we complete the initial data for the reduced system. Then, the theory of hyperbolic partial differential equations guarantees the existence of a unique solution for this system, which will be as regular as the initial data. Thus, we have the following theorem:

**Theorem 1 (Existence in a frame).** Given initial data \((M, \bar{g}, \bar{\phi}, K, \pi)\) satisfying the constraint equations (22), there is a Cauchy development of this initial data set into a space-time \((V, g, \phi)\) satisfying the field equations (27)-(28).

At this point, it is worth noting that the gauge condition we have chosen, i.e., \(F|_{t=0} = 0\), is similar to the one proposed in [22]. The difference between the two gauges is that, while in [22] the gauge condition is cleverly chosen so as to apply to quite general scalar-tensor theories, this condition, despite its simplicity, reveals a non-trivial insight presented by the author, and it is presented as a local condition, meaning that it represents a choice for a preferred coordinate system. On the other hand, our gauge condition comes as a natural consequence of following the standard approach to the global in space formulation of the Cauchy problem and exploiting the specific geometric properties of WGSTT.

We will now show that the problem is invariant under Weyl transformations. By this we mean that if we make a space-time Weyl transformation, then, the space-time described in this other frame results as the evolution of initial data equivalent to \((\bar{g}, \bar{\phi}, K, \pi)\). And
conversely, given an *equivalent* initial data set to \((\bar{g}, \bar{\phi}, K, \pi)\), then this initial data set admits a development into a space-time *equivalent* to the one generated by \((\bar{g}, \bar{\phi}, K, \pi)\).

Consider that we have an initial data set \((\bar{g}, \bar{\phi}, K, \pi)\) satisfying the constraint equations \(^{22}\) for the system \(^{6}\). Then, using **Theorem 1** we know that this initial data set admits a development into a space-time \((V, g, \phi)\) satisfying the field equations \(^{6}\). Then, perform a space-time Weyl transformation of the form

\[ g' = e^{-f}g, \quad \phi' = \phi - f \quad (35) \]

We know that \((g', \phi')\) will satisfy the system of equations \(^{7}\). Furthermore, since the gauge condition \(F = 0\) is Weyl-invariant, i.e., if it satisfied in one frame, then it is satisfied in all the equivalence class, then applying the same procedure described above for the system \(^{6}\) to the transformed system \(^{7}\), we see that \(^{7}\) is actually equivalent to the set of reduced equations that result from setting \(F(g', \phi') = 0\). Thus, we see that \((g', \phi')\) actually solve a system of non-linear wave equations in the space-time metric \(g'\) analogous to \(^{32}-^{33}\). Furthermore, this system takes initial data satisfying the constraints \(^{24}\). This shows that, considering a Weyl transformation on the initial data \((\bar{g}, \bar{\phi}, K, \pi)\) of the form

\[ \bar{g}' = e^{-h}\bar{g}, \quad \bar{\phi}' = \bar{\phi} - h, \quad K' = e^{-\frac{h}{2}}K, \quad \pi' = e^{\frac{h}{2}}\pi, \quad \text{(36)} \]

and setting \(h \equiv f(\cdot,0)\), then \((V, g', \phi')\) results as the evolution of the initial data set \((\bar{g}', \bar{\phi}', K', \pi')\), which is equivalent to \((\bar{g}, \bar{\phi}, K, \pi)\).

Conversely, if we make a Weyl transformation on the initial data \((\bar{g}, \bar{\phi}, K, \pi)\) of the form \(^{36}\), then the transformed initial data will satisfy the system \(^{24}\) if we consider \(f|_{t=0} = h\). Thus, if we pick a function \(f\) on space-time satisfying \(f|_{t=0} = h\), and then we perform a space-time Weyl transformation on \((g, \phi)\) of the form \(^{35}\), we see that \((g', \phi')\) on \(M_0\) satisfy the initial conditions \((\bar{g}', \bar{\phi}', K', \pi')\). Thus, any initial data set equivalent to \((\bar{g}, \bar{\phi}, K, \pi)\) has an evolution into a space-time equivalent to \((V, g, \phi)\). We summarize this with the following theorem.

**Theorem 2.** *Given an initial data set \((M, \bar{g}, \bar{\phi}, K, \pi)\) satisfying the constraint equations \(^{22}\), then the Cauchy problem for the WGSTT is well-posed.*
Geometric uniqueness

We will finally discuss a subtle point, which in the context of GR is referred to as geometric uniqueness. This is related to the arbitrariness in the choice of the initial data for the lapse and shift. As we have sated above, this is a gauge freedom which is already present in GR. In our present context, besides this gauge freedom, we have added Weyl transformations. We have already dealt with the latter, and we will now deal with the former.

The question that naturally arises in this context, is that different choices for the initial values of the lapse and shift will provide different initial data sets for our PDE system. Thus, we will get different solutions for each case. If this were the case, then the lapse and shift would not actually be working as gauge variables. It is a very well-known fact that the solution to this problem comes by taking into account that in space-time we regard diffeomorphic structures as equivalent. That is, given the group of diffeomorphisms on a given space-time, such group defines an action on the different tensor fields defined on space-time, acting via pull-back, and such an action defines an equivalence relation. With this in mind, within the context of GR, it can be shown, for instance for vacuum initial data sets, that two different choices of initial data for the lapse and shift give rise to initial data sets whose associated space-time evolutions are diffeomorphic, and thus equivalent (see, for instance, chapter 6 in [17]). We will now show that the very same procedure works in this context. In order to present this result we will first need to introduce some terminology.

In order to show that different choices of initial data for the lapse and shift give rise to diffeomorphic space-times, we will actually show how to construct such diffeomorphisms. For this purpose we need to introduce the notion of a wave map.

Given two pseudo-Riemannian manifolds \((V, g)\) and \((M, h)\), let \(u: V \to M\) be a smooth map. We can consistently define the vector bundle \(E \xrightarrow{\pi} V\), over \(V\), whose typical fibre over a point \(x \in V\) is given by \(T_x^*V \otimes T_{u(x)}M\). That is,

\[
E = \bigsqcup_{x \in V} T_x^*V \otimes T_{u(x)}M.
\]

We can now endow \(E\) with a natural connection, denoted by \(\nabla^E\), by considering local charts \((\mathcal{W}, X)\), \(\mathcal{W} \subset V\), and \((\tilde{W}, Y)\), \(\tilde{W} \subset M\), and applying Leibniz rule to a local section of \(E\) of the form \(f = f^\alpha dx^\alpha \otimes \partial_y^\alpha\). Now we define the action of the connection on an element
of $T^*_xV$ using the coefficients of the Riemannian connection at $x$ of the metric $g$, while the coefficients acting on $T_{u(x)}M$ are the pull-back by $u$ of the Riemannian connection 1-form of the metric $h$ at $u(x)$. Some simple computations then give us the following:

$$\nabla^E f^A = \partial f^A - g^{\alpha\beta} f^A + \partial u^C h^B_{\alpha\beta} f^B,$$  \hspace{1cm} (37)

where the connection coefficients $g^{\alpha\beta}$ and $h^B_{\alpha\beta}$ are the Christoffel symbols associated with the Riemannian connections of $g$ and $h$ respectively.

Furthermore, notice that the “gradient” $\partial u$ of the map $u$, given in local coordinates by $\partial u^A$, defines a section of the vector bundle $E$. With all this notations, it is said that the mapping $u : (V, g) \mapsto (M, h)$ is a wave map if it satisfies the following PDE:

$$\text{tr}_g \nabla^E \partial u = 0.$$  

For a detailed review on wave maps, see [23].

Returning to the problem of geometric uniqueness for the initial value formulation for the WGSTT, given two smooth solutions $(V_1, g_1, \phi_1)$ and $(V_2, g_2, \phi_2)$ to the Cauchy problem with the same geometric initial data $(M, \bar{g}, \bar{\phi}, K, \pi)$, we will construct a diffeomorphism from a neighbourhood $U_1$ of $M_0 \subset V_1$ onto a neighbourhood $U_2$ of $M_0 \subset V_2$. For this purpose we need the following lemma.

**Lemma 2.** Suppose that $(M \times \mathbb{R}, g)$ is a globally hyperbolic manifold with $g$ smooth. There exists a wave map $f$, which is a diffeomorphism from a strip $W_\tau := M \times (-\tau, \tau)$ onto a neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$, such that $f$ takes the following initial values:

$$f(\cdot, 0) = \text{Id}, \hspace{1cm} \partial f(\cdot, 0) = (a, b),$$  \hspace{1cm} (38)

where $a > 0$ is a specified scalar on $M$ and $b$ a specified tangent vector to $M$.

For a short proof of this lemma see [17], and for a detailed treatment of this problem see [23]. We can prove the following:

**Lemma 3.** Let $f$ be the unique wave map corresponding to the data (38). Such map defines a diffeomorphism between neighbourhoods of $M_0$ in $V$. Let $g \doteq f^* g$ and $\phi \doteq f^* \phi$ be the metric and scalar field in the wave map pull back by $f$ of a Lorentzian metric $g$ and a scalar
field $\phi$ which result as the evolution of a specified initial data set $(\bar{g}, \bar{\phi}, K, \pi)$. Then, it is possible to choose “$a$” and “$b$” such that the initial values of the metric $\bar{g}$ and the scalar field $\bar{\phi}$, and their first time-derivatives, take preassigned specified values, depending only on the geometric data $(\bar{g}, \bar{\phi}, K, \pi)$.

**Proof.** Notice that some of the initial data for $(\bar{g}, \bar{\phi})$ are already specified by the geometric initial data set $(\bar{g}, \bar{\phi}, K, \pi)$. For instance, recalling that two Weyl manifolds $(M_1, g_1, \phi_1)$ and $(M_2, g_2, \phi_2)$ are called isometric if there is a diffeomorphism $f : M_1 \rightarrow M_2$ such that $g_1 = f^*g_2$ and $\phi_1 = f^*\phi_2$, then, by means of the wave map diffeomorphism we are defining an isometry between the two Weyl manifolds $(V, g, \phi)$ and $(V, \bar{g}, \bar{\phi})$, thus all the intrinsically geometric objects will be unaffected. That is, the Weyl connection of the two structures is the same, and so is the extrinsic curvature of $M_0$ (just think of the diffeomorphism as generating coordinate transformations). Thus, we already have the initial data $\phi(\cdot, 0) = \bar{\phi}(\cdot, 0)$ and $K = K$. Now, the diffeomorphism gives us the freedom to specify the initial values of the lapse $N$ and shift $\beta$. In particular we are interested in setting $N(\cdot, 0) = 1$ and $\beta(\cdot, 0) = 0$.

Taking into account (38), we get the following relations:

\[
\begin{align*}
\bar{g}(\partial_t, \partial_t)|_{t=0} &= g(f_*\partial_t, f_*\partial_t)|_{t=0}, \\
\bar{g}(\partial_t, \partial_i)|_{t=0} &= g(f_*\partial_t, f_*\partial_i)|_{t=0} = g(f_*\partial_t, \partial_i)|_{t=0}, \\
\bar{g}(\partial_i, \partial_j)|_{t=0} &= g(f_*\partial_i, f_*\partial_j)|_{t=0} = \bar{g}(\partial_i, \partial_j).
\end{align*}
\]

In order to satisfy our prescribed conditions on the initial data for the shift and lapse, we get that

\[
f_*\partial_t(\cdot, 0) = \frac{1}{N} e_0|_{t=0} = \frac{1}{N} (\partial_t - \beta)|_{t=0},
\]

from which we deduce that $\partial_t f(\cdot, 0) = (\frac{1}{N}|_{t=0}, -\frac{1}{N}|_{t=0})$, which implies the choices $a = \frac{1}{N}|_{t=0}$ and $b = -\frac{1}{N}|_{t=0}$.

Now, by means of the relation between $\partial_t g_{ij}^\alpha$ and $K_{ij}$, we can pick the initial data $\partial_t g_{ij}^\alpha|_{t=0}$. Similarly, once we have set $(a, b)$ in terms of the initial data for $N$ and $\beta$, we get the initial data for $\partial_t \bar{\phi}|_{t=0}$. Finally, the initial data for $\partial_t g_{0\alpha}|_{t=0}$ is again chosen by solving the initial wave gauge condition for the vector field $F$. 

**Theorem 3 (Short-time geometric uniqueness).** Let $(V_1, [g_1, \phi_1])$ and $(V_2, [g_2, \phi_2])$ be two smooth solutions of the Cauchy problem for the vacuum WGSTT satisfying the initial
data \((M, [\bar{g}, \bar{\phi}, \bar{K}, \bar{\pi}])\). There exists an isometry from \((U_1, [g_1, \phi_1])\) onto \((U_2, [g_2, \phi_2])\), where \(U_1\) and \(U_2\) are neighbourhoods of \(M_0\), respectively in \(V_1\) and \(V_2\).

**Proof.** Consider the representatives of the solution \((V_1, [g_1, \phi_1])\) and \((V_1, [g_2, \phi_2])\), given by \((V_1, g_1, \phi_1)\) and \((V_1, g_2, \phi_2)\), which satisfy the system \([4]\) and take the initial data \((M, \bar{g}, \bar{\phi}, \bar{K}, \bar{\pi})\). Using the above lemmas, we know that there are diffeomorphisms \(f_1\) and \(f_2\) from neighbourhoods \(U_1\) and \(U_2\) of \(M\) in \(V_1\) and \(V_2\), respectively, onto some neighbourhoods of \(M\), such that \((f_1^{-1} g_1, f_1^{-1} \phi_1)\) and \((f_2^{-1} g_2, f_2^{-1} \phi_2)\) take the same initial data on \(M_0\). Also, both \((f_i^{-1} g_i, f_i^{-1} \phi_i)\), \(i = 1, 2\), satisfy the reduced system \([32]-[33]\) and the gauge condition, since these are tensor relations. Thus, by uniqueness of solutions for the reduced system, both solutions agree. Thus, the composition \(f_2^{-1} \circ f_1\) defines an isometry from \((U_1, g_1, \phi_1)\) onto \((U_2, g_2, \phi_2)\), where \(U_1\) and \(U_2\) are neighbourhoods of \(M_0\) in \(V_1\) and \(V_2\) respectively. Thus, the equivalence classes \((U_1, [g_1, \phi_1])\) and \((U_2, [g_2, \phi_2])\) are isometric. 

With this theorem we end the local in time study of the initial value formulation of the WGSTT without non-geometric (matter) sources. The final result is that the Cauchy problem for such models is well-posed within the physical interpretation proposed for those theories. That is, not only the PDE systems have a well-posed initial value formulation, but the uniqueness holds within the physical equivalence class. In this sense, when we speak of the physical equivalence class \((V, [g, \phi])\), we should consider as equivalent not only elements linked by the action of Weyl transformations, but also by the action of the diffeomorphism group of \(V\), which act on tensor fields via pull-back.

**IV. DISCUSSION**

We would like to conclude making a few remarks about the results presented above. First, we have been able to show that the WGSTT presented in the first sections, have a well-posed initial value formulation for initial data satisfying a system of constraint equations. Furthermore, it is an interesting fact that the Weyl structure results as the evolution of initial data, where the space-time structure, in each frame, is generated as a solution of a system of non-linear wave equations in the space-time metric. This shows that in each frame we have that the speed of propagation of the gravitational interaction, now described by the whole Weyl structure, is determined by the null cones (of any) of the space-time metrics in the conformal class, which is an invariant property in Weyl structure.
We should stress that the results presented above should be interpreted according to Theorem 3, where we have shown the short-time geometric uniqueness for the initial value problem, taking into account both the invariance under the action of Weyl transformations and isometries. Again, it is worth stressing the relation between this problem and the initial value formulation for standard scalar-tensor theories. In that context, well-posedness has been studied in each frame independently, without analysing the (possible) equivalence between space-times evolving from equivalent geometric initial data. Such result should be of interest for someone willing to adopt Dicke’s interpretation regarding the equivalence of both frames [15]. In this line, by considering a geometric interpretation for such theories, where by means of Weyl integrable structures we can make precise mathematical sense of the physical equivalence of different frames, we have not only shown well-posedness in each frame independently, but also shown that equivalent initial data sets evolve into equivalent space-time structures. Furthermore, we have provided a global in space result, which, as far as we aware, is an issue which had not been addressed outside the Riemannian (Einstein) frame.

We leave as a future research perspective the study of the evolution problem in the presence of matter fields. Furthermore, it would be essential to analyse the well-posedness of the system of constraint equations, and all the related problems to this issue.

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