On the speed of domain walls in thin nanotubes: the transition from the Walker to the magnonic regime

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Abstract

Numerical simulations of domain wall propagation in thin nanotubes when an external magnetic field is applied along the nanotube axis have shown an unexpected behavior described as a transition from a Walker to a magnonic regime. As the applied magnetic field increases, the initial Walker regime of linear growth of the speed with the field is followed by a sudden change in slope accompanied by the emission of spin waves. In this work an analytical formula for the speed of the domain wall that explains this behavior is derived by means of an asymptotic study of the Landau Lifshitz Gilbert equation for thin nanotubes. We show that the dynamics can be reduced to a one dimensional hyperbolic reaction diffusion equation, namely, the damped double Sine Gordon equation, which shows the transition from the Walker to the magnonic regime as the domain wall speed approaches the speed of spin waves. This equation has been previously found to describe domain wall propagation in a weak ferromagnets with the mobility proportional to the Dzyaloshinskii-Moriya constant; for Permalloy nanotubes the mobility is proportional to the nanotube radius.

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Magnetic domain wall propagation is a subject of much current interest due to its possible applications in magnetic memory devices. Understanding and controlling the motion of domain walls is essential for applications. In the micromagnetic approach, the magnetization is governed by the Landau Lifshitz Gilbert (LLG) equation \[ 1, 2 \]

\[
\frac{d\vec{m}}{dt} = -\gamma_0 \vec{m} \times \vec{H}_{\text{eff}}(\vec{m}) + \alpha m \times \frac{d\vec{m}}{dt}
\]

where \( \vec{m} \) is the unit magnetization vector, that is, the magnetization \( \vec{M} = M_s \vec{m} \), where \( M_s \) is the constant saturation magnetization, a property of the material. The constant \( \gamma_0 = |\gamma|\mu_0 \), where \( \gamma \) is the gyromagnetic ratio of the electron and \( \mu_0 \) is the magnetic permeability of vacuum. The parameter \( \alpha > 0 \) is the dimensionless phenomenological Gilbert damping constant. The effective magnetic field \( \vec{H}_{\text{eff}} \) includes the physical interactions and the external applied field \( \vec{H}_a \). The different physical phenomena that must be included in the effective field and the geometry of the ferromagnetic material together with the intrinsic nonlinearity of the problem imply that exact analytical solutions are generally nonexistent so that numerical and approximate analytic methods have been developed to understand experimental results and predict new phenomena. The exact solution of Walker \[ 3, 4 \] developed for an infinite medium with an easy axis, a local approximation for the demagnetizing field, including exchange interaction and under the action of an external magnetic field along the easy axis, shows that when the applied field is small, the speed of the domain wall increases linearly with the field. When the applied field reaches a critical value, the Walker field \( H_w \) the magnetization enters into a precessing motion. This behavior, which is encountered even when additional physical effects and different geometries are studied, puts a limit to the maximum speed that a domain wall can achieve.

For applications it is desirable to have stable domain walls and to reach high propagation velocities. For such purpose different physical effects and geometries have been considered. Numerical simulations for thin Permalloy nanotubes under the action of an external field along the nanotube axis showed unexpected behavior \[ 5, 6 \]. For small fields the speed increases linearly with the field, reaching a plateau at relatively low applied field and very high velocity. No instability nor Walker breakdown of the domain wall was observed. This unexpected behavior occurs for right handed domain walls (see Fig. 4 in \[ 6 \]) for which the radial component of the magnetization remains small throughout the motion \[ 6 \].

The main result of this manuscript is the derivation of an analytical expression for the
speed of the domain wall which explains the linear increase at small fields, the reaching of a plateau and the high values of the velocity. For Permalloy, a material of negligible uniaxial anisotropy, we find that the speed is given by

\[ v = \frac{\gamma_0 R H_a}{\sqrt{\alpha^2 + \mu_0 R^2 H_a^2 / (2A)}}, \]  

(2)

where \( A \) is the exchange constant \[7\] and \( R \) the thin nanotube radius. For small \( H_a \) we recover the Walker regime \[8\],

\[ v_w = \frac{\gamma_0}{\alpha} R H_a, \]  

(3)

whereas for large applied field the speed tends to the constant value

\[ v_\infty = \gamma_0 \sqrt{\frac{2A}{\mu_0}} = 1006 \text{ ms}^{-1} \text{ for Permalloy} \]  

(4)

which we identify with the speed of spin waves. Following the notation used for weak ferromagnets, notice that Eqn. \([2]\) can be written as \( v = \mu H_a / \sqrt{1 + \mu^2 H_a^2 / v_\infty^2} \) with mobility \( \mu = \gamma_0 R H_a / \alpha \). For weak ferromagnets the mobility is proportional to Dzyalozhinski-Moriya constant \[9, 10\].

The rise of the speed with the field is very fast, for a Permalloy nanotube of radius \( R = 55 \) nanometers, for an external field field \( B = \mu_0 H_a = 2 \) mT, Eqn. \([2]\) yields \( v = 893 \) m s\(^{-1}\).

Although simulations have been carried out for Permalloy in the derivation below we will allow a material with non negligible uniaxial anisotropy for greater generality.

Consider a thin nanotube with an easy direction along the nanotube axis direction which we choose as the \( z \) axis. An external external field \( H_a \) is applied along this direction. For sufficiently thin tubes the demagnetizing field can be approximated by a local expression with the saturation magnetization acting as an effective anisotropy that penalizes the radial component of the magnetization \([8, 11, 12]\). Using a coordinate system \((\rho, \varphi, z)\) so that the magnetization \( \vec{m} = m_\rho(\rho, \varphi, z) \hat{\rho} + m_\varphi(\rho, \varphi, z) \hat{\varphi} + m_z(\rho, \varphi, z) \hat{z} \), the micromagnetic energy is given by \([7, 8]\)

\[ E = \int_{\Omega} d^3x (A |\nabla \vec{m}|^2 + K_u (1 - m_z^2) + \frac{\mu_0 M_s^2}{2} m_\rho^2 - H_a m_z), \]

where \( A \) is the exchange constant and \( K_u \) the uniaxial anisotropy. The effective field is given by

\[ \vec{H}_{\text{eff}} = -\frac{1}{\mu_0 M_s \delta \vec{m}} \delta E. \]
In a very thin nanotube one may neglect variations of the magnetization with radius, so that the unit magnetization depends only on the polar coordinate \( \varphi \) and the axial position \( z \). With \( \vec{m} = \vec{m}(\varphi, z) \), the effective magnetic field can be written as

\[
\vec{H}_e = \frac{2A}{\mu_0 M_s} \left[ \frac{1}{R^2} \frac{\partial \vec{m}}{\partial \varphi^2} + \frac{\partial \vec{m}}{\partial z^2} \right] + \frac{2K_u}{\mu_0 M_s} m_z \hat{z} - M_s m_\rho \hat{\rho} + H_a \hat{z} \quad (5)
\]

Introducing \( M_s \) as unit of magnetic field, and introducing the dimensionless space and time variables \( \xi = z/R \) and \( \tau = \gamma_0 M_s t \) we rewrite equations (1) and (5) in dimensionless form as

\[
\frac{d\vec{m}}{d\tau} = -\vec{m} \times \vec{h}_{\text{eff}} + \alpha \vec{m} \times \frac{d\vec{m}}{d\tau} \quad (6)
\]

with

\[
\vec{h}_{\text{eff}} = A_0 \left[ \frac{\partial \vec{m}}{\partial \varphi^2} + \frac{\partial \vec{m}}{\partial \xi^2} \right] + k_u m_z \hat{z} - m_\rho \hat{\rho} + h_a \hat{z} \quad (7)
\]

where \( h_a \) is the dimensionless applied field and the dimensionless numbers that have appeared are \( k_u = 2K_u/(\mu_0 M_s^2) \) and \( A_0 = 2A/(\mu_0 M_s^2 R^2) \). Equations (6) and (7) describe the dynamics of the problem.

Numerical simulations have been performed for Permalloy for which the exchange constant \( A = 1.3 \times 10^{-11} \text{ J m}^{-1} \), \( M_s = 8 \times 10^5 \text{A m}^{-1} \), \( K_u \approx 0 \) and the external applied field does not exceed \( 10^{-2} M_s \). The nanotube used in simulations has inner radius \( R \), and width \( w \) with \( w << R \). Here we neglect the variations with radius and adopt a typical value \( R = 55 \times 10^{-9} \text{ m} \). The vacuum permeability \( \mu_0 = 4\pi \times 10^{-7} \text{N A}^{-2} \) so that \( \mu_0 M_s \approx 1\text{T} \). We take the value \( \gamma_0 = 2.21 \times 10^5 \text{s}^{-1} \text{T}^{-1} \). For Permalloy the dimensionless parameters adopt the values \( A_0 \approx 10^{-2} \), \( k_u = 0 \) and the dimensionless applied field is in the range \( 0 < h_a < 10^{-2} \).

We are interested in right handed vortex walls for which the radial component of the magnetization is small, \( m_\rho \ll 1 \) [5, 6]. Introducing a small dimensionless parameter \( \epsilon \) we write this condition as

\[
m_\rho = \epsilon \vec{m}_\rho.
\]

The normalization condition \( \vec{m}^2 = 1 \) becomes

\[
m_\varphi^2 + m_z^2 = 1 - \epsilon^2 m_\rho^2.
\]

We will model a situation in which the dimensionless numbers that describe the problem and the applied field are of an order smaller in \( \epsilon \) than the radial magnetization, whereas the Gilbert constant is of the same order. Let then

\[
A_0 = \epsilon^2 \tilde{A}, \quad k_u = \epsilon^2 \tilde{k}_u, \quad h_a = \epsilon^2 \tilde{h}_a, \quad \alpha = \epsilon \tilde{\alpha}.
\]
It is found that a consistent asymptotic approach can be obtained if a new time scale \( s = \epsilon \tau \) is introduced. With these scalings, the components of the effective magnetic field can be written as

\[
(h_{\text{eff}})_{\rho} = -\epsilon \tilde{m}_{\rho} - 2\epsilon^2 \tilde{A} \frac{\partial m_{\varphi}}{\partial \varphi} + \epsilon^3 \tilde{A} \left( \nabla_s^2 \tilde{m}_{\rho} - \tilde{m}_{\rho} \right) = \epsilon H_0^\rho + \epsilon^2 H_1^\rho + \epsilon^3 H_2^\rho, \tag{9a}
\]

\[
(h_{\text{eff}})_{\varphi} = \epsilon^2 \tilde{A} \left( \nabla_s^2 m_{\varphi} - m_{\varphi} \right) + 2\epsilon^3 \tilde{A} \frac{\partial m_{\rho}}{\partial \varphi} = \epsilon^2 H_1^\varphi + \epsilon^3 H_2^\varphi, \tag{9b}
\]

\[
(h_{\text{eff}})_{z} = \epsilon^2 \left( \tilde{h}_a + \tilde{A} \nabla_s^2 m_z + \tilde{k}_u m_z \right) = \epsilon^2 H_1^z, \tag{9c}
\]

where \( \nabla_s^2 = \partial_{\xi\xi} + \partial_{\varphi\varphi} \) and where we grouped terms according to the power of \( \epsilon \) so that \( H_0^\rho = -\tilde{m}_{\rho}, \ H_1^\varphi = \tilde{A} \left( \nabla_s^2 m_{\varphi} - m_{\varphi} \right) \) and \( H_1^z = \tilde{h}_a + \tilde{A} \nabla_s^2 m_z \).

Introducing the scaling for \( \alpha \) and \( m_{\rho} \) in the LLG equation, we obtain at leading order in \( \epsilon \),

\[
\dot{m}_{\rho} = -(m_{\varphi} H_1^z - m_z H_1^\varphi) + \tilde{\alpha} (m_{\varphi} \tilde{m}_z - m_z \tilde{m}_{\varphi}) \tag{10a}
\]

\[
\dot{m}_{\varphi} = -m_z H_0^\rho \tag{10b}
\]

\[
\dot{m}_z = m_{\varphi} H_0^\rho \tag{10c}
\]

where a dot represents a derivative with respect to the scaled time variable \( s \) and the subindices represent the components of each vector.

The normalization condition \( \text{(8)} \) implies that, at leading order, we may write,

\[
m_{\varphi} = \sin \theta(\xi, \varphi, s), \quad m_z = \cos \theta(\xi, \varphi, s).
\]

It follows then that equations \( \text{(10b)} \) and \( \text{(10c)} \) are equivalent and imply

\[
\dot{\theta} = -H_0^\rho = \tilde{m}_{\rho}. \tag{11}
\]

Replacing the value of \( \tilde{m}_{\rho} \) from \( \text{(11)} \) in \( \text{(10a)} \) together with the expressions for the effective field \( H_1^{\varphi}, H_1^z \), the evolution equation for \( \theta \) is found to be

\[
\ddot{\theta} + \tilde{\alpha} \dot{\theta} = \tilde{A} \left( \theta_{\xi\xi} + \theta_{\varphi\varphi} \right) - \sin \theta \left( \tilde{h}_a + (\tilde{A} + \tilde{k}_u) \cos \theta \right), \tag{12}
\]
where the subscripts in $\theta$ denote derivatives with respect to $\xi$ and $\varphi$ respectively. Notice that one may go back to the original unscaled variables and the small parameter $\epsilon$ cancels out.

In what follows we study cylindrically symmetric domain walls, for which $\theta \varphi = 0$ and identify the evolution equation with the damped double Sine Gordon equation, a particular case of hyperbolic reaction diffusion equation, for which the existence and stability of traveling waves have been studied rigorously in [13, 14].

$$\frac{\partial \theta}{\partial \tau^2} + \alpha \frac{\partial \theta}{\partial \tau} = A_0 \theta_{\xi \xi} - \sin \theta (h_a + (A_0 + k_u) \cos \theta).$$

(13)

This equation has been derived in the analysis of domain wall propagation in weak ferromagnets, which exhibit behaviour analogous to that encountered in Permalloy nanotubes [9, 10, 15, 16] and in systems with a strong easy plane [17, 18]. In [10] the dependence of mobility on the Dzyaloshinskii constant is derived with great detail. A common feature in these problems is the sudden decrease in the rate of increase of the speed with the applied field. As discussed in [19–21] when the domain wall velocity is close to the phase velocity of some wave in the material emission of Cherenkov radiation will occur.

This equation has the same traveling wave solutions as the reaction diffusion equation $\alpha \dot{\theta} = A_0 \theta_{\xi \xi} - \sin \theta (h_a + (A_0 + k_u) \cos \theta)$ but with velocity $c = c_t / \sqrt{1 + c_t^2 / A_0}$ where $c_t$ is the speed of the reaction diffusion equation [13]. We give the explicit expression for the head to head (HH) domain wall, the tail to tail solution is similar. The HH solution is given by the usual domain wall profile,

$$\theta(\xi, t) = 2 \arctan \left[ \exp \left( \frac{\xi - c t \Delta}{\Delta} \right) \right]$$

(14)

with the speed given by

$$c = \sqrt{\frac{A_0}{A_0 + k_u}} \frac{h_a}{\sqrt{\alpha^2 + (A_0 + k_u)^{-1} h_a^2}},$$

(15)

and the domain wall width $\Delta = \alpha c / h_a$. The leading order magnetization $\vec{m} = m_\varphi \hat{\varphi} + m_z \hat{z}$ is given by

$$\vec{m} = \text{sech} \left( \frac{\xi - c t \Delta}{\Delta} \right) \hat{\varphi} - \tanh \left( \frac{\xi - c t \Delta}{\Delta} \right) \hat{z}.$$

The external field is applied along the $z$ axis so the magnetization is a right handed ($m_\varphi \geq 0$) head to head domain wall as defined in [6]. The small radial component of the magnetization is calculated from (11).
For small applied field we recover the Walker regime, that is, the speed increases linearly with the field, and the domain wall width tends to a constant value, that is,
\[
\lim_{h_a \to 0} c = \sqrt{\frac{A_0}{A_0 + k_u}} \frac{h_a}{\alpha}, \quad \lim_{h_a \to 0} \Delta = \sqrt{\frac{A_0}{A_0 + k_u}}.
\]
In this limit the dynamics is primarily governed by the reaction diffusion equation \( \alpha \theta_\tau = A_0 \theta_{\xi \xi} - \sin \theta (h_a + (A_0 + k_u) \cos \theta) \) as already found in [8]. For large applied field the speed tends to a constant value and the domain wall width decreases as the field increases,
\[
\lim_{h_a \to \infty} c = \frac{\alpha \sqrt{A_0}}{h_a}, \quad \lim_{h_a \to \infty} \Delta = \frac{\alpha \sqrt{A_0}}{h_a}.
\]
The speed in this limit is determined by the wave equation \( \theta_{\tau \tau} = A_0 \theta_{\xi \xi} \). For large values of the magnetic field the dynamics is governed by (13) with a growing influence of the second order time derivative as the applied field increases. The evolution equation (13) contains the two extremes, the Walker regime and the magnonic regime, and shows the transition between them as the external field is varied. A different transition occurs at \( h_{aKPP} = 2A_0 \) when the speed of the reaction diffusion equation \( c_r \) changes from a pushed to a pulled or KPP front [22] and \( c_r \) becomes proportional to the square root of the applied field.

In what follows consider Permalloy for which \( k_u = 0 \). Going back to dimensional quantities, the speed of the domain wall for Permalloy is given by Eq. (2) given above together with the limiting values at low and high fields Eq. (3) and Eq. (4). In Fig. 1 the graph of the speed as a function of the applied field shows the gradual change from the Walker to the magnonic regime. We have used the values given above for Permalloy.

An approximate estimate of the field \( H_{a*}^s \) at which this transition occurs is obtained by the intersection \( v_w(H_{a*}^s) = v_\infty \) which yields
\[
H_{a*}^s = \frac{\alpha}{R} \sqrt{\frac{2A}{\mu_0}}.
\]
For Permalloy, we obtain \( v_\infty = 1006 \text{ m s}^{-1} \), and \( B_{a*}^s = \mu_0 H_{a*}^s = 0.001 \text{ T} \). The transition to the KPP regime occurs at a much higher field, \( B_{aKPP}^s = \mu_0 H_{aKPP}^s = 0.021 \text{ T} \) and is not associated to the transition from the Walker to the magnonic regime. The order of magnitude in this simple model, where the thickness of the tube is neglected and only a leading order calculation is performed, agrees with the order of magnitude of the numerical simulations. As discussed in [5, 6], the transition from the Walker to the magnonic regime is accompanied by the emission of spin waves by a spin-Cherenkov mechanism.
FIG. 1: Speed of the domain wall versus applied field in millitesla for a thin Permalloy nanotube of radius $R = 55$ nanometers. At low field the speed increases linearly with the field, after a sudden change in slope the speed tends to a constant value $v_\infty$ at large fields. The dashed lines show the limiting speeds $v_w$ and $v_\infty$.

In summary, the dynamics of a vortex domain wall in a thin nanotube, studied by means of an asymptotic study of the Landau-Lifshitz Gilbert equation, was reduced, in leading order, to a hyperbolic reaction diffusion equation, the damped double sine-Gordon equation. The exact analytic solution for the propagating domain wall shows that as the external field increases, the dynamics evolves from a Walker regime, where the speed increases linearly with the field, to the magnonic regime, where the wave nature of the dynamics becomes important and the speed tends to a high constant value which only depends on the exchange constant and corresponds to the speed of spin waves. The results obtained agree qualitatively and quantitatively with the numerical simulations reported for Permalloy which share features with the dynamics of antiferromagnets.

This equation has arisen in the study of domain walls in weak ferromagnets ([10] and references therein) and in ferromagnetic materials with a strong easy plane [18, 23]. In all this cases a transition from a Walker regime to one where the increase of velocity is damped by spin wave emission. For a nanotube the ratio of the exchange constant with the radius of the nanotube $A/R^2$ plays the role of an effective uniaxial anisotropy which leads to a mobility proportional to the nanotube radius. In constrast, for weak ferromagnets the mobility is proportional to de Dzyaloshinkya-Moriya constant. That the effect of curvature acts as an equivalent effective field was already studied in [8, 24], and more recently an analogy between the effect of DMI and curvature was found in the dispersion relation of spin waves [25].
results in this manuscript show that a similar effect arises when studying the transition from
the Walker to the magnonic regime in nanotubes. In the derivation of the results we have
assumed that the magnetic field is of comparable magnitude to other physical parameters,
further increases in the applied field lead to speeds beyond $v_\infty$ [5, 6] which are beyond the
range of validity of this model.

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