An efficient phase-field model for fatigue fracture in ductile materials

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Abstract

Fatigue fracture in ductile materials, e.g., metals, is caused by cyclic plasticity. Especially regarding the high numbers of load cycles, plastic material models resolving the full loading path are computationally very demanding. Herein, a model with particularly small computational effort is presented. It provides a macroscopic, phenomenological description of fatigue fracture by combining the phase-field method for brittle fracture with a classic durability concept. A local lifetime variable is obtained, which degrades the fracture resistance progressively. By deriving the stress-strain path from cyclic material characteristics, only one increment per load cycle is needed at maximum. The model allows to describe fatigue crack initiation, propagation and residual fracture and can reproduce Paris behaviour.

Keywords: Phase-field, Fatigue, Notch strain concept, Ductile, Paris law

1. Introduction

Fatigue fracture is one of the most common causes of failure in structures, while still being insufficiently predictable. The fatigue life of a repeatedly loaded structure can be divided into three stages \cite{1}, \cite{2}: During the crack initiation stage slip bands are formed. Later, micro-structurally small cracks merge to macro cracks. The macro cracks first grow stably (crack propagation) until the residual cross section is overloaded and the structure fails (residual fracture). The influence of microstructure is especially significant for high cycle fatigue, because in this setting most of the lifetime is spent in the micro-structurally small regime, while for low cycle fatigue, plasticity is the determining factor \cite{2}.

For many applications it is sufficient to estimate the number of load cycles until crack initiation $N_{\text{frac}}$. For this purpose statistical durability concepts based on Wöhler experiments are employed. Component Wöhler curves describe $N_{\text{frac}}$ given a certain nominal stress amplitude. Beyond that, from strain-controlled experimental settings it is possible to derive the characteristic cyclic behaviour of the material itself. The strain Wöhler curves are e.g. utilised within the notch strain concept (NSC) \cite{3}, which ranks among the more advanced durability concepts and is able to consider the influence of plastic strains.

For highly optimised structures, on the other hand, the crack propagation stage has to be considered as well during lifetime prediction in order to design the structures economically. Usually, the defect tolerant approach \cite{4}, \cite{5} is adopted: Target quantity is the number of cycles it takes a crack to grow from a just visibly detectable range to failure.

Fracture mechanics provides tools to describe this crack propagation, such as the stress intensity factor $K$ \cite{6}. The Paris-Erdogan law \cite{7} applies this concept to fatigue: It links the cyclic crack propagation rate $da/dN$ with the amplitude stress intensity factor $\Delta K$ by the power law

$$\frac{da}{dN} = C\Delta K^m.$$ (1)
The Paris parameters $C$ and $m$ are meant to be material parameters and therefore independent of geometry and load. Noroozi et al. [8], [9] and Mikheevskiy and Glinka [10] combine fracture mechanics with the NSC, deriving a crack driving force for fatigue crack propagation. Instead of a sharp crack, which in an elastic setting produces a stress singularity, they consider plasticity at the crack tip, which is incorporated by equivalent residual stresses.

Besides statistical and analytical methods, crack propagation can be simulated numerically. Most commonly, the finite element method is used for the spatial discretisation on different scales [11]. The numerical approaches can be distinguished in terms of the geometrical representation of the crack. Sharp crack models generally use conforming meshes, i.e. the crack propagates along element edges or faces, respectively. The kinetics of crack growth are e.g. controlled in terms of nodal forces or cohesive zone models. As a consequence of the conforming meshes, the approaches are well suited for known crack paths but require topological updates for arbitrary unknown directions of crack growth. These restrictions have been partly relaxed by the extended finite element method (XFEM) [12] which allows for cracks within finite elements in terms of an enriched displacement approximation.

While the simulation of evolving cracks using a sharp crack topology proves difficult in particular in 3D settings, diffuse representations of the crack path may offer significant benefits from a computational point of view. Therefore, the phase-field method which is conceptually similar to gradient damage models is currently subject of intensive research. It is able to model both crack initiation and propagation, as well as arbitrary crack geometries in a straightforward manner using an additional field variable to represent the crack. For a description of the standard phase-field model for brittle fracture, which will also be used here, see Miehe et al. [13], [14]. A review on a wider range of phase-field formulations for fracture can be found in Ambati et al. [15].

Recently, several propositions to extend the phase-field method to fatigue [16, 17, 18, 19, 20] have been published. Representatively for the range of different approaches, two models which are able to reproduce Paris behaviour shall be highlighted here: Carrara et al. [19] introduce a phase-field model for fatigue fracture in brittle materials. The basic idea of their approach is that due to repetitive loading the crack resistance decreases, allowing cracks to evolve even far below the static crack resistance. The fracture toughness is modified depending on a measure of locally accumulated elastic strain energy density. Mesgarnejad et al. [20] follow a similar approach, but link the lowering of the fracture toughness also to the phase-field, localising the degradation to the vicinity of the crack tip.

Although the authors use different approaches, they are not yet overcoming one key challenge inherent to fatigue crack initiation and propagation: The immense computational effort related to the high number of load cycles. This issue is addressed in the present paper. In particular, we introduce an efficient phase-field model of fatigue fracture in ductile materials, such as metals. Analogously to [19], it is based on the reduction of the critical fracture energy, but uses a different local fatigue measure. In contrast to brittle materials, fatigue crack propagation in ductile materials is caused by cyclic plastic deformations. The straightforward way to treat the problem with an elasto-plastic material model is numerically expensive. Therefore, a different approach is chosen. With the help of the NSC, a local lifetime variable is introduced, accounting for cumulative elasto-plastic deformations. Since the stress-strain path within a load cycle is derived from material curves from cyclic experiments, the explicit simulation of each load cycle would be redundant and can therefore be avoided, saving computational costs. In other words, instead of introducing a ductile phase-field model, an elastic, brittle phase-field formulation which considers the elasto-plastic origins of fatigue is presented. As cracks are described on a macroscopic scale, microscopic effects are not resolved explicitly, but are represented statistically in the cyclic material characteristics. Conveniently, the utilised characteristics are derived from standardised experiments for which a large data base is already available.

This combination of the phase-field method with the NSC is considerably more flexible than the mere NSC, which itself is not applicable to crack propagation at all. The newly introduced model also incorporates static fracture with prior fatigue damage due to the general character of the phase-field approach.

This paper is organised as follows: After the introduction, the new model is introduced in Section 2 beginning with the standard phase-field formulation for brittle fracture, followed by its novel extension to fatigue. Furthermore, a scheme for numerical implementation is described. The method is applied in one-dimensional and two-dimensional examples in Section 3. Finally, conclusions are drawn in Section 4.
# Nomenclature

| Symbol | Definition |
|--------|------------|
| \(\alpha\) | Fatigue degradation function |
| \(\alpha_0\) | Fatigue degradation threshold |
| \(\sigma\) | Stress tensor |
| \(\varepsilon\) | Strain tensor |
| \(n\) | Normal |
| \(u\) | Displacement |
| \(\Delta K\) | Amplitude of stress intensity factor |
| \(\ell\) | Characteristic length |
| \(\Gamma\) | Crack surface |
| \(\lambda\) | 1st Lamé constant |
| \(G_c\) | Fracture toughness |
| \(H\) | Crack driving force |
| \(\mu\) | 2nd Lamé constant |
| \(\Omega\) | Domain |
| \(\partial\Omega\) | Boundary of \(\Omega\) |
| \(\Pi\) | Energy functional |
| \(\Pi_\ell\) | Energy functional for regularised crack |
| \(\psi^e\) | Elastic energy density |
| \(\psi^e_+\) | Tensile part of \(\psi^e\) |
| \(\psi^e_-\) | Compressive part of \(\psi^e\) |
| \(\sigma\) | Revaluated equivalent stress |
| \(\sigma'_f\) | Parameter of strain Wöhler curve |
| \(\sigma_a\) | Cycle amplitude of \(\sigma\) |
| \(\sigma_{el}\) | Equivalent stress |
| \(\sigma_m\) | Cycle mean of \(\sigma\) |
| \(\tilde{t}\) | Boundary traction |
| \(\tilde{u}\) | Boundary displacement |
| \(\tilde{F}\) | Force boundary condition |
| \(\tilde{F}_a\) | Force amplitude |
| \(\tilde{u}_a\) | Boundary displacement amplitude |

| Symbol | Definition |
|--------|------------|
| \(\varepsilon\) | Strain corresponding to \(\sigma\) |
| \(\varepsilon'_f\) | Parameter of strain Wöhler curve |
| \(\varepsilon_{a,el}\) | Elastic strain part corresponding to \(\sigma_a\) |
| \(\varepsilon_{a,pl}\) | Plastic strain part corresponding to \(\sigma_a\) |
| \(\varepsilon_a\) | Strain corresponding to \(\sigma_a\) |
| \(\varepsilon_{el}\) | Strain corresponding to \(\sigma_{el}\) |
| \(\xi\) | Fatigue degradation exponent |
| \(A\) | Area |
| \(a\) | Crack length |
| \(b\) | Parameter of strain Wöhler curve |
| \(C\) | Paris parameter |
| \(c\) | Parameter of strain Wöhler curve |
| \(D\) | Lifetime variable |
| \(d\) | Phase-field |
| \(E\) | Young’s modulus |
| \(g\) | Degradation function |
| \(h_{min}\) | Minimum element size |
| \(i\) | Increment number |
| \(K\) | Stress intensity factor |
| \(K'\) | Cyclic hardening coefficient |
| \(L\) | Length |
| \(m\) | Paris parameter |
| \(N\) | Number of load cycles |
| \(n'\) | Cyclic hardening exponent |
| \(N_{frac}\) | Load cycles until failure |
| \(P_{SWT}\) | Damage parameter |
| \(s\) | Control variable for time |
| \(T\) | Thickness of CT specimen |
| \(t\) | Time |
| \(W\) | Dimension of CT specimen |
2. Phase-field model for fatigue fracture

Since the newly developed method is based on the standard phase-field method for brittle fracture, the underlying formulation is introduced first. Then the extension of the model to fatigue is described, which makes use of a local lifetime variable. It is shown how this variable is determined using the NSC and how the model can be implemented.

2.1. Phase-field model for brittle fracture

The phase-field method for brittle fracture is based on the Griffith-criterion \[21\] of linear elastic fracture mechanics. It implies that a brittle crack can only propagate if the fracture energy, which is released during the formation of new crack surface, equals the critical energy release rate or fracture toughness \( G_c \).

A variational formulation for this criterion was proposed by Francfort and Marigo \[22\]. Applying it by minimizing the total energy with regard to the displacement field and the crack geometry, arbitrary crack paths as well as crack initiation can be modelled without any further criteria. The total energy functional \( \Pi \) for a domain \( \Omega \) with a fracture surface \( \Gamma \), see Fig. 1a, can be given by

\[
\Pi = \int_{\Omega} \psi^e(\varepsilon) \, dV + \int_{\Gamma} G_c \, dA,
\]

excluding volume forces and boundary tractions for the sake of brevity. Assuming linear elasticity and small strains \( \varepsilon \), the elastic strain energy density can be written as \( \psi^e = \frac{1}{2} \lambda \text{tr}^2(\varepsilon) + \mu \text{tr}(\varepsilon^2) \) with the elastic constants \( \lambda \) and \( \mu \).

Figure 1: Fractured domain \( \Omega \) with crack surface \( \Gamma \). (a) Sharp representation of crack topology. (b) Regularized representation: The crack is described by the phase-field variable \( d = 1 \), while \( d = 0 \) represents undamaged material. The crack is regularized over the length scale \( \ell \).

In order to enable a convenient numerical implementation, Bourdin et al. \[23\] proposed a regularisation. To describe the crack topology, an additional field variable \( d \in [0,1] \) is introduced, smoothly bridging the entirely intact \( (d = 0) \) and totally broken \( (d = 1) \) state. Kuhn and Müller \[24\] interpreted the states of the material as phases and applied the term phase-field. Approximating the sharp crack by a crack density \( \gamma_\ell \) depending on a length scale parameter \( \ell \), see Fig. 1b), the regularized energy functional can be written as

\[
\Pi_\ell = \int_{\Omega} g(d) \psi^e(\varepsilon) \, dV + \int_{\Omega} \frac{G_c}{2\ell} (d^2 + \ell^2|\nabla d|^2) \, dV.
\]

Inspired by damage mechanics, a degradation function \( g(d) = (1 - d)^2 \) is introduced which models the loss of stiffness due to the developing crack \[14\]. Besides, it couples the mechanical field \( u \) and the phase-field \( d \). The stress is given by

\[
\sigma = g(d) \frac{\partial \psi^e}{\partial \varepsilon}.
\]

From the variation \( \delta \Pi_\ell = 0 \) one can derive the governing equations

\[
\vec{0} = \text{div} \, \sigma \quad d - \ell^2 \Delta d = (1 - d) \frac{2\ell}{G_c} \psi^e(\varepsilon)
\]
subject to the boundary conditions \( n \cdot \sigma = \tilde{t}, u = \tilde{u} \) and \( n \cdot \nabla d = 0 \) with \( \tilde{t} \) and \( \tilde{u} \) being the prescribed boundary tractions and displacements. In Eq. (5), the crack driving force \( \mathcal{H} \), which controls the evolution of the phase-field, can be identified. Miehe et al. [13] interpreted the phase-field variable in the sense of damage and therefore introduced the crack driving force as the maximum energy density in time history

\[
\mathcal{H} = \frac{2\ell}{G_c} \max_{s\in[0,t]} \psi^\sigma(\varepsilon, s),
\]

ensuring local irreversibility for the phase-field. The effects thereof are discussed in [25]. Physically motivated, the degradation can also solely be applied to the tensile part of the energy density

\[
\psi^\sigma(\varepsilon, d) = g(d) \psi^\sigma_+(\varepsilon) + \psi^\sigma_d(\varepsilon),
\]

while the compressive range remains unaffected. The strain is thereby split into a volumetric and a deviatoric part [26], according to its principal components [14] or the crack orientation [27].

2.2. Extension to fatigue

The following section addresses the question how fatigue can be modelled within the phase-field framework. Analogously to Carrara et al. [19], the fracture toughness \( G_c \) is reduced when the material degradation due to repetitive stressing precedes. Here, this process is described by a local lifetime variable \( D \). An additional scalar fatigue degradation function \( \alpha(D) \in [\alpha_0, 1] \) with \( 0 < \alpha_0 < 1 \) is introduced, which lowers the fracture toughness \( G_c \) locally. The energy functional then reads

\[
\Pi_\ell = \int_\Omega g(d) \psi^\sigma(\varepsilon) \, dV + \int_\Omega \alpha(D) G_c \frac{1}{2\ell} (d^2 + \ell^2 |\nabla d|^2) \, dV.
\]

The reduction of the total energy due to \( \alpha(D) \) is meant to model the dissipation due to local cyclic plasticity. Demanding again \( \delta \Pi_\ell = 0 \), the evolution equation of the phase-field extends to

\[
\alpha(D) d - \nabla \alpha(D) \cdot \ell^2 \nabla d - \alpha(D) \ell^2 \Delta d = (1 - d) \psi^\sigma(\varepsilon) \frac{2\ell}{G_c}.
\]

The lifetime variable \( D \in [0, 1] \) is a history variable that is accumulated strictly locally. It can be interpreted as a damage variable with a special linear character: For \( D = 0 \) a material point has experienced no fatigue loads at all, while \( D = 1 \) means it has undergone all load cycles it can possibly bear before loosing its integrity, linearly spanning the lifetime in between. Its computation according to the NSC is described in Section 2.3.

On this basis, the fatigue degradation function

\[
\alpha(D) = (1 - \alpha_0)(1 - D)^\xi + \alpha_0
\]

with the parameters \( \alpha_0 \) and \( \xi \) is proposed, see Fig. 2 for an illustration. For \( D = 0 \), the material has experienced no cyclic loads at all and therefore must have full fracture toughness, consequently \( \alpha(0) = 1 \) must hold. The parameter \( \xi \) controls the relation between \( D \) and \( d \). The threshold \( \alpha_0 \), on the other hand, offers a link to experiments on residual fracture in which the remaining fracture toughness of fatigued components is measured. It has to be larger than zero, since even cyclically damaged material at the end of its lifetime offers a certain resistance against crack propagation. The influence of both parameters \( \alpha_0 \) and \( \xi \) on crack initiation and propagation is studied in Section 3.

Since the phase-field variable is here interpreted as physical damage instead of as a marker for a potential crack, the irreversibility condition (6) is adopted. However, a tension-compression split is not necessary, as will be explained further in Section 2.3.
2.3. Notch strain concept (NSC)

The computation of the lifetime variable $D$ follows the NSC [3], which is generally used for service life prediction of structures. Therein, experimental data can be applied to arbitrary specimen because of the assumption that same strains generate same damage. In contrast to other fatigue concepts, the NSC is a material concept as it considers the local stress-strain path instead of a nominal stress quantity. It is therefore especially suitable for application in combination with the finite element method. In the classical sense, the NSC is evaluated at the point of the structure with the highest stress which is assumed to be decisive for component failure. Here, it is applied to each material point instead.

Due to the local formulation of the NSC crack propagation can be described as crack initiation at a number of material points. Interestingly, this approach is similar to the crack propagation model of Noroozi et al. [8], who describe crack growth by applying the NSC blockwise to elements, interpreting crack growth as successive crack initiation.

The lifetime variable $D$ is computed load cycle wise, each load cycle $i$ contributing $\Delta D_i$. The computation scheme, adapted to application within the introduced model, is illustrated in Fig. 3. It is based on the loading path. For each reversal point, marking a minimum or a maximum load of a cycle, a linear elastic simulation determines the stress-strain state. The NSC generally uses only scalar stresses and strains, as it is based on experiments in which only axial quantities are considered. It is common practice to use the von Mises equivalent stress in case of multiaxial stress states [28]. The elastically determined equivalent stress $\sigma_{el}$ as well as all stress and strain quantities derived from it are denoted by non-bold symbols.

First of all, stresses and strains are revaluated to elasto-plastic values using the cyclic stress-strain curve (CSSC) of the material. This curve is a material characteristic which is determined experimentally, e. g. by incremental step tests [29]. As shown in Fig. 5, the CSSC describes stress-strain relationship of the reversal points of a cyclic experiment. It is approximated by the Ramberg-Osgood model [30]

$$\varepsilon_n = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'} \quad (11)$$

with the cyclic hardening coefficient $K'$ and the cyclic hardening exponent $n'$.

According to the Neuber rule [28], the product of stress and strain has to be the same before and after the reevaluation

$$\sigma_{el}\varepsilon_{el} = \sigma\varepsilon. \quad (12)$$

Using the Ramberg-Osgood ansatz (11)

$$\sigma_{el} = \sigma \left(\frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'}\right), \quad (13)$$
Figure 3: Scheme for determination of the local lifetime variable \( D \) using the notch strain concept (shaded). The elastically determined stress-strain state is revaluated to the cyclic stress-strain curve (CSSC) using the Neuber rule. For each load cycle, the CSSC and amplitude values of stress and strain \( \sigma_a \) and \( \varepsilon_a \) define the stress-strain hysteresis, from which the damage parameter \( P_{SWT} \) can be determined. The theoretically bearable number of load cycles to fracture \( N \) is then drawn from \( P_{SWT} \)-Wöhler curves (\( P_{SWT} \leftarrow WC \)), which are derived from strain Wöhler curves (SWC). It determines the lifetime contribution \( \Delta D \) which can be accumulated linearly over all load cycles.

Figure 4: Scheme for numerical implementation. In a staggered algorithm, the fields \( u \) and \( d \) are solved separately. In between, at each quadrature point, the notch strain concept (shaded) is applied to the elastically simulated stress amplitude \( \sigma_{a,el} \). For constant load amplitudes, the boundary condition is kept to the amplitude value for all increments. Each increment covers \( \Delta N \) load cycles.
the reevaluated stress $\sigma$ can be derived by solving Eq. (13) numerically. This revaluation is not equivalent to a plastic material model, it rather incorporates plasticity as the cause of fatigue crack growth in an empirical way.

The CSSC approximates the stress-strain path also between the reversal points. The hysteresis curves are thereby assumed to have the shape of the CSSC scaled by factor 2 [31]. To characterise the damaging effect of one full hysteresis $i$, the damage parameter by Smith, Watson and Topper [32] is computed. It is associated with the area inside one hysteresis loop and is therefore connected to the occurring dissipation. $P_{\text{ SWT },i}$ only depends on the stress and strain amplitudes $\sigma_a$ and $\varepsilon_a$ and the mean stress $\sigma_m$, which shall not be considered here yet. One simulation with amplitude load is therefore enough to obtain $\sigma_i$ and $\varepsilon_i$ for each quadrature point, describing the damaging effect of the full hysteresis without a simulation of the entire loading and unloading path.

The material specific sensitivity to fatigue is included by strain Wöhler curves (SWC). See Fig. 6 for a schematic illustration. The relation between the strain amplitude $\varepsilon_a$ and the number of load cycles until fracture $N_{\text{frac}}$ is given by

$$ P_{\text{ SWT },i} = \sqrt{(\sigma_{a,i} + \sigma_{m,i})\varepsilon_{a,i}/E} $$

(14)

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$$ \varepsilon_a = \varepsilon_{a,\text{el}} + \varepsilon_{a,\text{pl}} = \frac{\sigma'_f}{E}(2N_{\text{frac}})^b + \varepsilon'_f(2N_{\text{frac}})^c $$

(15)

with parameters $\sigma'_f$, $\varepsilon'_f$, $b$ and $c$ according to Manson, Coffin and Morrow [33], [34], [35]. Due to the strain-controlled setting of the underlying material experiments, both the contributions of elastic and plastic strains can be considered. By reformulating Eq. (15) for the damage parameter $P_{\text{ SWT }}$

$$ P_{\text{ SWT },i}^2 = \sigma'_f^2(2N_i)^{2b} + \sigma'_f\varepsilon'_fE(2N_i)^{b+c}, $$

(16)

the damage parameter Wöhler curve ($P_{\text{ SWT }\text{ WC}}$) is derived. After equalling it to Eq. (14), the equation can be solved for $N_i$, which describes the number of load cycles the material could bear assuming continuous stressing with only the damage parameter $P_{\text{ SWT },i}$. The lifetime contribution $\Delta D_i$ of the single load cycle then is

$$ \Delta D_i = 1/N_i. $$

(17)

According to the linear damage accumulation hypothesis, i.e. Miner rule [36], [37], the lifetime contribution of all passed load cycles (5) can be added up to

$$ D = \sum_i \Delta D_i. $$

(18)
Hence, e.g. $D = 0.5$ implies that half of the bearable load cycles have been experienced, given constant stress amplitudes. It becomes clear that $D$, describing expired life of a material point as a strictly local history variable, is of a very different character than the phase-field variable $d$, which represents the evolution and the topology of the actual physical crack driven by $D$. This coupling is implemented in the phase-field model by the fatigue degradation function $\alpha(D)$.

Within the model no split is performed on the energy density. Due to the symmetry of the CSSC even material points which experience compressive stresses for $\sigma = \sigma_a$ (upper reversal point), would experience tensile stresses for $\sigma = -\sigma_a$ (lower reversal point) and should therefore contribute to crack propagation.

In conclusion, applying the NSC for recording the fatiguing effect of the load history on the material brings along two crucial benefits in the matter of computational effort: Firstly, due to the revaluation an elastic material model is applied instead of an elasto-plastic one. And secondly, the explicit loading path has not to be resolved. Instead, only the amplitude load of the cycle is applied.

2.4. Numerical implementation

In the following, a scheme for implementing the model in a standard finite element analysis programme for phase-field simulations is proposed. See Fig. 4 for an illustration. It follows the staggered approach by Hofacker and Miehe [38] which decouples displacement $u$ and phase-field $d$. In an iterative scheme, both are solved separately while freezing the other one, respectively. The computation of the lifetime variable $D$ can be inserted between the two, resulting in a three-step scheme. The damaging effect of the loading, described by $D$, is computed from the stresses and strains derived from the mechanical field. $D$ reduces the fracture toughness, which can again lead to crack evolution in the phase-field.

Starting simply with a steady oscillating load with constant amplitude and no mean load, the following implementation scheme can be applied. Instead of explicitly simulating the loading path with loading and unloading stage, the boundary condition is kept to the amplitude value, e.g. $\tilde{F} = \tilde{F}_a$ for a force-controlled experiment. To begin with, each increment $i$ is associated with one load cycle. Within the increment, an iteration over the fields takes place. At first, the problem is solved for the mechanical field, yielding the local stress and strain amplitudes $\sigma_{a,el,i}$ and $\varepsilon_{a,el,i}$. At each quadrature point independently, those are revaluated to the pseudo elasto-plastic quantities $\sigma_{a,i}$ and $\varepsilon_{a,i}$, which are then used to compute the damage parameter $P_{SWT,i}$ and the lifetime contribution $D_i$. $D_i$ yields the local fatigue degradation factor $\alpha_i$. Considering the updated distribution of the fracture toughness $\alpha G_c$ accounting for the damage due to the load cycle $i$, the problem is solved for the phase-field. This iteration is stopped by relative and absolute convergence criteria for both fields.

For small to moderate crack propagation rates it is even possible to cover several load cycles in one increment: Because of the linear character of $D$, the lifetime contribution of $\Delta N$ load cycles is $\Delta D = \Delta N/N_i$. Here, a control for the simulated load cycles per increment, depending on the rate of change in the fracture energy, is applied in order to cover the slow crack evolution at the beginning as well as the abrupt development later in fatigue life properly. Alternatively, $\Delta N$ can be hinged on the number of staggered loops until convergence.

Since the NSC is designed for arbitrary loads with varying amplitudes, the proposed model can be generalised to those loading cases. See [39] for a description of the NSC for general loads and transient cyclic stress-strain behaviour. Due to the general character of the phase-field formulation, static loading is included as a special case $\Delta N = 0$. For the limit case of static load on previously unloaded material, the model exactly matches the standard static phase-field model for brittle fracture. Previous cyclic loads before the static loading stage, on the other hand, are also covered by the reduced fracture toughness.
Table 1: Parameters of the cyclic stress-strain curve (CSSC) according to the Ramberg-Osgood model and the strain Wöhler curve (SWC) according to the Manson, Coffin and Morrow approach both for 42CrMo4 steel [40].

| CSSC | SWC |
|------|-----|
| $K'$ | $n'$ |
| 2115 MPa | 0.195 |
| $\sigma_f'$ | $\varepsilon_f'$ | $b$ | $c$ |
| 1554 MPa | 1.447 | -0.086 | -0.710 |

Figure 7: One-dimensional setup: Bar with a quadratically reduced cross section in the middle. A constant displacement amplitude $\tilde{u}$ is applied.

3. Numerical examples

In the following, the validity of the method and its influencing factors are to be studied using various examples. For the sake of simplicity the method is applied to a one-dimensional bar first, demonstrating crack initiation. In order to study crack propagation, single-edge notched tests as well as a compact tension test are presented subsequently.

3.1. One-dimensional example

A one-dimensional bar of the length $L = 1$ mm is considered. As shown in Fig. 7, its cross section is reduced in the middle from $A = A_0 = 0.1$ mm$^2$ to $0.5A_0$. The Young’s modulus and the initial fracture toughness are set to $E = 210$ GPa and $G_c = 2.7$ N/mm, the characteristic length to $\ell = 0.01$ mm. The quadratic elements are of the size $h_{min} = 0.0013$ mm within the critical region. Unless otherwise stated, the parameters of the fatigue degradation function are chosen to $\xi = 1$ and $\alpha_0 = 1 \cdot 10^{-8} \approx 0$, whereat $\alpha_0$ is set to a small value instead of 0 to avoid numerical difficulties. The bar is virtually loaded with a displacement oscillation with an amplitude $\tilde{u}_a$ and no mean load, which entails a constant displacement boundary condition $\tilde{u} = \tilde{u}_a = 0.0013$ mm for the simulation. Due to an application project, a heat-treated steel, namely 42CrMo4, is used as an example here. The corresponding cyclic parameters are displayed in Tab. 1.

The results of the test are depicted in Fig. 8. Fig. 8a) shows the distribution of the lifetime variable $D$ over the length of the bar for increasing load cycles $N$. As expected, $D$ accumulates fastest in the range of the reduced cross section, where stresses and strains are higher and with them the damage parameter $P_{SWT}$. Furthermore, the linear character of $D$ over $N$ becomes obvious, which underlines the meaning of $D$ as expiring lifetime. By comparison, Fig. 8b) shows the evolution of the phase-field. Obviously, a crack initiates after $N \approx 4.6$ million load cycles. The results match observations in Wöhler experiments for metals: Until crack initiation, the component weakens continuously without losing much stiffness until a crack occurs abruptly. Accordingly, $D$ increases continuously, only degrading the crack resistance. Meanwhile $d$, which degrades the stiffness, develops much slower, until in the end it shows an abrupt increase. This is underlined by Fig. 8c).

Fig. 8d) compares the numbers of load cycles until failure $N_{frac}$ in the same setting for varying displacement amplitudes $\tilde{u}_a$. As expected, smaller load amplitudes lead to longer fatigue lives. Fig. 8e) shows the influence of the parameter $\xi$ of the fatigue degradation function $\alpha(D) = (1 - \alpha_0)(1 - D)^\xi + \alpha_0$. $\xi$ has only a small impact on the total number of load cycles until fracture. Instead, it controls the suddenness of crack formation, which is associated with the brittleness of the material. This link also becomes apparent in Fig. 8f), where the relation between the phase-field variable and the lifetime variable, both represented by their current maximum $d_{max}$ and $D_{max}$, is shown. The higher $\xi$ is, the earlier in terms of the total life and the more gentle the phase-field, i. e. the crack, develops.

A fatigue test with a subsequent static load is shown in Fig. 9. After $9 \cdot 10^5$ load cycles at a displacement amplitude of $\tilde{u}_a = 0.0012$ mm, a static load is applied. Due to the prior damage by the fatigue load the crack initiates at a displacement which is only one fourth compared to undamaged material. Interestingly, the phase-field shows a more narrow profile due to the inhomogeneous distribution of the fracture toughness.
Figure 8: (a) Evolution of the distribution of the lifetime variable $D$ over the length of the bar $x$ for increasing load cycles $N$. $D$ accumulates in ranges of high stress. (b) Evolution of phase-field $d$. After $N_{\text{frac}} \approx 4.6$ million load cycles, the crack initiates ($d = 1$). (c) Evolution of maximum phase-field variable $d_{\text{max}}$ over $N$ shows the successively accelerating damage evolution. (d) Load cycles until fracture $N_{\text{frac}}$ over displacement load amplitude $\tilde{u}$. The decrease of $N_{\text{frac}}$ for increasing $\tilde{u}$ can be described by a power function. (e) Parameter study for $\xi$ of the fatigue degradation function $\alpha(D) = (1 - \alpha_0)(1 - D)^\xi + \alpha_0$. Maximum phase-field variable over number of load cycles. The total lifetime is hardly influenced by $\xi$. (f) Relationship of maximum phase-field variable and corresponding lifetime variable, which is primarily controlled by $\xi$. 
Figure 9: Two-stage simulation. After $9 \cdot 10^5$ load cycles (blue line), the load is increased statically, leading to further crack growth under the influence of the pre-damaged state. A static simulation with undamaged material (orange line) is displayed for comparison. (a) Maximum phase-field variable $d$ over the length of the bar. (b) Profile of $d$ in the moment of fracture.

3.2. Two-dimensional examples

3.2.1. Single-edge notched test

In a two-dimensional setting, the method is at first tested with the single-edge notched specimen subject to tensile and shear loading. A hierarchically refined quadrilateral mesh with 8 element size levels and quadratic ansatz functions is applied [41]. The minimum element size is $h_{\text{min}} = 0.003125$ mm within the area of the growing crack, which in combination with a characteristic length of $\ell = 0.01$ mm results in a ratio $\ell/h_{\text{min}} = 3.2$. Young’s modulus, Poisson’s ratio and fracture toughness are chosen to $E = 210$ GPa, $\nu = 0.3$ and $G_c = 2.7$ N/mm. The parameters of the fatigue degradation function are set to $\alpha_0 = 0.05$ and $\xi = 1$. No split is applied to the energy density. Any point with $d > 0.95$ is considered a crack. The load cycles simulated by one increment $\Delta N$ are adjusted according to the required number of staggered loops. The specimen is subject to plane strain condition and loaded with a constant displacement amplitude of $\tilde{u} = 0.0018$ mm in the tension test and $\tilde{u} = 0.003$ mm in the shear test.

The evolution of the phase-field and the lifetime variable are displayed in Fig. 10 (tension) and Fig. 11 (shear). Starting from the notch, a zone of $D = 1$ forms progressively. This area can be associated with the plastic zone, since $D$ is computed from the damage parameter $P_{\text{SWT}}$, which is again associated with the area inside a stress-strain hysteresis – a measure for dissipation. The crack forms in the corridor of material with reduced fracture toughness.

Interestingly, even in the shear test, the crack runs straight through the specimen. Due to the alternating load $D$ has to accumulate symmetrically. This is enabled through the absence of a split. However, with a static, brittle phase-field formulation without split the crack would branch into two symmetric cracks running towards the edges [14]. The straight crack observed here is caused by the ductile formulation. This also corresponds to results for a ductile phase-field model by Ambati et al. [42], where a single-edge notched specimen is subjected to static shear loading.

3.2.2. Compact tension test

In order to study the Paris behaviour of the method, a compact tension test is considered. Fig. 12 displays the setting of the test which follows the guidelines ASTM E647-05 [43] and ASTM E1820-01 [44]. The specimen has a sharp notch but, differing from the guidelines, no initial crack. In this way, crack initiation as well as propagation can be studied. For the sake of simplicity, the free ends behind the borehole are not included in the model, instead the axes of load application are evened out. The specimen is loaded with a constant force amplitude of $\tilde{F} = 0.0049$ kN and no mean load, which again entails a constant force...
Figure 10: Single-edge notched tension test with cyclic loading. Distribution of phase-field $d$ and lifetime variable $D$ for different numbers of load cycles $N$.

Figure 11: Single-edge notched shear test with cyclic loading. Distribution of phase-field $d$ and lifetime variable $D$ for different numbers of load cycles $N$. 
boundary condition for this approach. The same mesh refinement and model parameters as in the single-edge notched specimen (Section 3.2.1) are applied, if not stated differently. The load cycles simulated by one increment are again controlled adaptively by the required staggered loops.

The resulting crack length \( a \) is displayed in Fig. 13a). After \( \approx 800 \) load cycles, a crack initiates and then propagates with an increasing crack propagation rate \( \Delta a / \Delta N \). This rate is plotted over the amplitude of the stress intensity factor \( \Delta K \) in a Paris plot in Fig. 13b). Although \( \Delta K \) is only valid for small plastic zones, it is here used for the sake of comparability. For this geometry, \( \Delta K \) is derived from the crack length \( a \) according to the mentioned ASTM guidelines

\[
\Delta K = \frac{\bar{F}}{T \sqrt{W}} \left( \frac{2 + a/W}{1 - a/W} \right)^{3/2} \left( 0.866 + 4.64 \frac{a}{W} - 13.32 \left( \frac{a}{W} \right)^2 + 14.72 \left( \frac{a}{W} \right)^3 - 5.6 \left( \frac{a}{W} \right)^4 \right).
\] (19)

After the stable crack propagation stage, to which the Paris parameters \( C \) and \( m \) of Eq. (1) are fitted, the crack proceeds instably. Fig. 13c) shows the influence of the threshold \( \alpha_0 \) of the fatigue degradation function. A higher \( \alpha_0 \) leads to delayed crack initiation. Since the fracture toughness is not reduced as much, the "plastic zone" has to grow bigger in order to initiate a crack. The same can be shown for a varying load amplitude \( \bar{F} \), displayed in Fig. 14a-b). As expected, for higher loads the crack initiates earlier. Nevertheless, the fitted Paris parameters are roughly the same. This shows that the model can reproduce Paris behaviour with the material parameters \( C \) and \( m \) independent from loading.

However, according to Paris theory, material properties can change the Paris parameters. The cyclic hardening coefficient and exponent \( K' \) and \( n' \) of the CSSC according to the Ramberg-Osgood model are varied in Fig. 14c-f). The corresponding CSSCs are shown within the figures. Apparently, the Paris line is shifted upwards by increasing \( n' \) and decreasing \( K' \), respectively. Both lead to a lower CSSC. Hence, the revaluation for same stress and strain yields a higher plastic strain and therefore a wider stress-strain hysteresis. This is associated with more dissipation. Consequently the higher damage parameter leads to a stronger degradation of the fracture toughness. This causes higher crack propagation rates and higher Paris lines. Moreover, since already small stresses lead to large plastic strains, the degraded zone as well as the phase-field show a wider profile for increasing \( n' \) and decreasing \( K' \) and cracks initiate earlier.
Figure 13: Cyclic compact tension test. Parameters: element size $h_{\text{min}}$, load cycles per increment $\Delta N$, characteristic length $\ell$ in mm. (a), (c), (d), (e) Crack length $a$ over number of load cycles $N$. (b), (f) Paris plot: Crack propagation rate $\Delta a/\Delta N$ over amplitude of stress intensity factor at crack tip $\Delta K$. Fitted with the Paris parameters $C$ and $m$ within the Paris range.
Figure 14: Cyclic compact tension test. Parameters: Load amplitude $\tilde{F}$, threshold of fatigue degradation function $\alpha_0$, parameters of cyclic stress-strain curve (CSSC) $K'$ and $n'$. (a), (c), (d) Crack length $a$ over number of load cycles $N$. (b), (e), (f) Paris plot: Crack propagation rate $\Delta a/\Delta N$ over amplitude of stress intensity factor at crack tip $\Delta K$. Fitted with the Paris parameters $C$ and $m$ within the Paris range. For $K'$ and $n'$, the CSSCs are schematically depicted.
4. Conclusion

A combination of the phase-field method for brittle fracture with a fatigue life concept is introduced. It can model cyclic crack initiation and propagation as is demonstrated in 1D- and 2D-examples. The fatigue effects are considered by degrading the fracture toughness depending on a local lifetime variable. This variable is determined with the so-called notch strain concept, considering plasticity as the cause of ductile fatigue fracture. Static fracture is included in the formulation as a special case. The model causes relatively small computational effort, since an elasto-plastic stress-revaluation is performed instead of using an elasto-plastic material model. Moreover, several load cycles can be simulated within one increment. It is shown that the model recovers Paris behaviour.

Further research will now concentrate on validation with experiments and the determination of the parameters of the fatigue degradation function. Moreover, the revaluation technique for elastic stresses has to be compared to more elaborate revaluation concepts and an elasto-plastic material model.

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Highlights

1. A new phase-field model for fatigue crack initiation and propagation is proposed.
2. It minimises computational effort due to a combination with a classic fatigue concept.
3. A revaluation technique replaces an elasto-plastic material model.
4. The model can reproduce Paris behaviour.

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