Robust asymptotic entanglement under multipartite collective dephasing

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Starting from the analytical description of the dynamics of a system of non-interacting atoms exposed to a homogeneous, fluctuating magnetic field, we specify field orientations which preserve any degree of atomic entanglement for all times, and families of states with entanglement properties which are time-invariant for arbitrary field orientations. Our formalism applies to arbitrary spectral distributions of the field’s fluctuations.

Control of the coherent evolution of quantum systems in noisy environments [1] is one of the crucial prerequisites for exploiting non-trivial quantum effects in composite systems of increasing complexity. Whether in the context of controlled molecular reactions [2], of many-particle quantum dynamics [3] or of quantum computers and simulators [4] – uncontrolled fluctuations and noise are detrimental to most purposes of optimal control. Various strategies may be followed to counteract the harmful influence of the environment – shielding the system degrees of freedom [5], correcting environment-induced errors [6], compensating dissipation by coherent dynamics [7], exploiting basins of attraction in dissipative systems [8]. In all these cases, a detailed understanding of the specific open system dynamics is of primordial importance, as it is often indicative of the optimal control strategy.

Here we consider an important class of environment-induced fluctuations, which are frequently encountered in state-of-the-art experiments [9]: they manifest in intensity fluctuations of spatially homogeneous experimental control fields [10], giving rise to an effective dephasing process, as well as to environment-induced interactions between the system constituents. We show how control of the external field’s orientation can lead to the complete preservation of entanglement in bipartite, as well as in multipartite settings, for arbitrary spectral characteristics of the control field fluctuations.

To set the stage, let us consider a collection of $N$ non-interacting atomic two-level systems with identical energy splitting, e.g., by a homogeneous magnetic field. Integration over the unavoidable fluctuations of the latter’s strength will induce a probability distribution $p(\omega)$ of the characteristic energy splitting, and the $N$-atom quantum state at time $t$ therefore needs to be described by the statistical operator

$$\rho(t) = \int p(\omega) U_\omega(t)^{\otimes N} \rho(0) U_\omega(t)^{\otimes N} \, d\omega,$$

provided the field fluctuations occur on time scales which are longer than the time $t$ over which the $N$-atom state is propagated by the unitary $U_\omega(t)$. In order to assess the open system time evolution of the quantum correlations inscribed into the $N$-atom system, it is convenient to derive an explicit expression for $\rho(t)$, in terms of the spectral distribution $p(\omega)$ characterizing the fluctuations.

The single-atom propagator $U_\omega(t) = e^{-iH_\omega t/\hbar}$ is generated by the time-independent single-atom Hamiltonian $H_\omega = (\hbar \omega/2) n \cdot \sigma$, with $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ the vector of Pauli matrices and $n$ the orientation of the field. $H_\omega$ describes atomic dipoles interacting with electromagnetic fields, as, e.g., the electronic qubits in trapped-ion quantum registers [9, 11]. Introducing pairs of orthogonal projectors $A_\pm = (I \pm n \cdot \sigma)/2$, we can rewrite the time evolution operator for a collection of $N$ atoms as

$$U_\omega(t)^{\otimes N} = \left(e^{-i\omega t/2} A_+ + e^{i\omega t/2} A_-\right)^{\otimes N} = \sum_{j=0}^{N} e^{i\omega(j-N/2)} \Theta_j,$$

where we have defined the operators

$$\Theta_j = \frac{1}{j!(N-j)!} \sum_{s \in \Sigma_j} V_s \left[ A_\ominus^j \otimes A_\ominus^{N-j} \right] V_s^\dagger,$$

with $V_s = \sum_{i_1, \ldots, i_N} |i_{(s)}\rangle \langle i\rangle |i_1\ldots i_N|$ representing permutations $s$ in the operator space of $N$ qubits. The ensemble-averaged state after time $t$,

$$\rho(t) = \sum_{j=0}^{N} M_{jk}(t) \Theta_k \rho(0) \Theta_j,$$

is then fully characterized by the Toeplitz matrix $M(t)$, whose elements $M_{jk}(t) = \varphi((j-k)t)$ are generated by the characteristic function $\varphi(t) = \int p(\omega) e^{i\omega t} \, d\omega$ of the probability distribution $p(\omega)$. Bochner’s theorem [12] ensures that $M(t)$ is a Hermitian semi-positive definite matrix for all $t$. Diagonalisation leads to the canonical Kraus form [13]

$$\rho(t) = \sum_{i=0}^{N} A_i(t) \rho(0) A_i^\dagger(t),$$

where the Kraus operators $A_i(t) = \sum_{j=0}^{N} \sqrt{\lambda_i(t)} \Theta_j$ contain the eigenvalues $\lambda_i(t)$ and the components of the eigenvectors $\Lambda_i(t)$ of $M(t)$. Note, from the structure of $A_i(t)$, that the above defined Kraus operators mediate an effective interaction between the individual qubits – with its origin in the spatial homogeneity of the external field. These environment-induced interactions are able to create discord-type quantum correlations [11], and, as we will show in this article, given appropriate control of $n$, can uphold multipartite entanglement at all times, for arbitrary intensity fluctuations.
Using the fact that the operators $\Lambda_{\pm}$ are orthogonal projectors in $\mathbb{C}^2$, it is immediate to show that both the operators $\Theta_j$ and $(A_i(t))$, satisfy the condition $\sum_i A_i^\dagger(t)A_i(t) = \sum_i \Theta_i^\dagger \Theta_i = I$, which ensures that the map $\epsilon_{\rho j}$ defined in (5) – from now on called the “collective dephasing” map – is not only completely positive but also trace-preserving for all $t$ [13, 14].

For absolutely continuous distribution functions [15], the characteristic function vanishes asymptotically, i.e., $\lim_{t\to\infty} \Phi(t) = 0$. We then have that $\lim_{t\to\infty} M(t) = \mathbb{I}_{2N+1}$ and thus the Kraus operators reduce to $\lim_{t\to\infty} A_i(t) = \Theta_i$. The asymptotic $N$-qubit state is thus given by $\rho_\epsilon = \lim_{t\to\infty} \rho(t) = \sum_i \Theta_i \rho(0) \Theta_i$. Because the operators $\Theta_i$ depend exclusively on the magnetic field direction $\mathbf{n}$, the latter completely determines the properties of the asymptotic state.

To gain some intuition on the time evolution of the entanglement properties, e.g., of an $N$-ion quantum register under the action of the collective dephasing map, we first consider two-qubit states with maximally mixed reduced density matrices (also called Bell-diagonal states). Such states allow for a simple geometric representation, since they are fully characterized by the matrix $\beta_{ij} = \text{tr}(\rho \sigma_i \otimes \sigma_j)$ [18]. There always exist unitary operations $U_A$ and $U_B$ such that $U_A U_B^\dagger$ has a diagonal $\beta$ matrix, $\beta = \text{diag}(d_1, d_2, d_3)$, while $\rho$ and $U_A U_B^\dagger$ have the same separability properties. This allows to associate to each density matrix a point $\mathbf{d} = (d_1, d_2, d_3)^T \in \mathbb{R}^3$ and, because of positivity, any such point must lay inside a tetrahedron [Fig. 1 (a)] of vertices $(-1, -1, -1)^T$, $(-1, 1, 1)^T$, $(1, -1, 1)^T$ and $(1, 1, -1)^T$, which represent the four Bell states [18]. Inside this tetrahedron we distinguish an inner octahedron, which contains the separable states, from the four remaining corners, which consist of the entangled states [18], and are labelled by the Bell state they contain (e.g. $|\Psi^>_T$-corner). In this setting Wootters’ concurrence [19] is simply the distance from the faces of the octahedron: $C(d) = 1/2 \max(0, \sum_i |d_i| - 1)$. Equidistant points, parallel to the surfaces of the octahedron, form the “isocorrespondence” planes.

In the tetrahedron, the collective dephasing evolution is always constrained onto a plane defined by $\text{Tr} \beta(t) = \text{Tr} \beta(0)$ [14]. In the $|\Psi^>_T$-corner, these planes coincide with isocorrespondence planes, which implies that entanglement is preserved for all of these states, for arbitrary directions of the magnetic field. This leads to a finite-measure set of states with time-invariant concurrence, despite the fact that those states do evolve in time, $\rho(t) \neq \rho(0)$ [20]. For entangled states outside the $|\Psi^>_T$-corner, we can use Eq. (4) to predict the final concurrence as $C_f(n) = 1/2 \max(0, \sum_i (1 - 2n_i^2) d_i - 1)$, where $\mathbf{n} = (n_1, n_2, n_3)^T$, and $\mathbf{d} = (d_1, d_2, d_3)^T$ characterizes the initial state [14]. Thus, by solving for $\mathbf{n}$, we can always find a field direction such that the entanglement is preserved at all times. This can be seen from the long-time limit in Fig. 1 (b), whereas the transient time evolution depends on $\rho(\omega)$, as we will discuss further down.

Collective interactions become particularly relevant in multipartite settings, where decoherence and dissipation can be strongly enhanced [23–26]. To analyze the effect of collective dephasing on multipartite entanglement, analytic expressions à la Wootters [19] are not available. Intricate hierarchies of multipartite entanglement [27] can, however, be characterized efficiently by resorting to separability criteria based on inequalities [28, 29]. An $N$-partite state $\rho$ is called $k$-separable if it can be written as a mixture of states of the form $\rho = \rho_{A_1} \otimes \cdots \otimes \rho_{A_k}$, where $A_1 \ldots A_k$ label a division of the $N$ parties into $k$ subgroups. For instance, the matrix elements in an arbitrary basis of any $k$-separable $N$-qubit density matrix $\rho$ satisfy $\sum_{0S|S|N-1} |\rho_{n[12]+12}^{S+2+12}|^2 \leq \sum_{0S|S|N-1} \sqrt{\rho_{n[12]+12}^{S+2+12}}^2 + (N-k)/2$ [30]. Defining $k_{\text{eff}}$ as the largest integer $k$ satisfying this inequality provides an upper bound to the state’s $k$-separability class, as $k \leq k_{\text{eff}}$. When $k_{\text{eff}} = 1$ the state certainly contains genuine multipartite entanglement, i.e., it is not even $2$-separable, while the state can be fully separable ($N$-separability) only if $k_{\text{eff}} \geq N$.

We consider the initial ($N$-partite entangled) $W$-state, $|W\rangle = (|10\ldots0\rangle + |01\ldots0\rangle + \cdots + |00\ldots1\rangle)/\sqrt{N}$, where $|1\rangle$ and $|0\rangle$ denote eigenstates of $\sigma_z$. Since the collective dephasing map (5) is invariant under the operation $\mathbf{n} \rightarrow -\mathbf{n}$ and, additionally, this class of states exhibits rotational symmetry around the $z$-
axis, the polar angle $\theta \in [0, \pi/2]$ between $n$ and the $z$-axis (which is defined by the local eigenbasis of the initial state) fully determines the evolution of the state under (5). Fig. 2 (a) displays the entanglement properties of the resulting asymptotic state, characterized by $k_{\text{eff}}$ as a function of $\theta$ [31]. In general, there are relatively small angle intervals that lead to a fully separable state, and typically $k_{\text{eff}}$ shows non-monotonic dependence on $\theta$.

Our numerical data [Fig. 2 (b)] suggest that the asymptotic state is certainly entangled (i.e. $k_{\text{eff}} < N$) as long as $\theta < \theta_1$, where

$$\theta_1(N) = \arctan\left(1/\sqrt{N}\right).$$  \hspace{1cm} (6)

Conversely, when we choose a magnetic field that is close to the $z$-direction, the initial $N$-partite entanglement of the $W$ state will be preserved during the dephasing process, since $W$ is part of an eigenspace of the Hamiltonian for $n = (0,0,1)^T$. Again, we find a critical angle

$$\theta_{\text{NP}}(N) = \arctan\left(1/\sqrt{N(N-1)}\right),$$  \hspace{1cm} (7)

such that for $\theta < \theta_{\text{NP}}$, the asymptotic state will contain genuine multipartite entanglement ($k_{\text{eff}} = 1$). Conditions (6) and (7) provide a finite range of orientations that ensure preservation of entanglement properties. However, as the number of qubits gets larger, higher accuracy is required to maintain $N$-partite entanglement ($\theta_{\text{NP}}$) or at least some type of entanglement ($\theta_1$). Moreover, the fast decay of $\theta_{\text{NP}}$ with the number of qubits confirms that genuine $N$-partite entanglement is much more fragile than bipartite entanglement [27, 32], which is able to resist a larger range of field directions. We remark here that in order to modify $\theta$ in a trapped-ion experiment it is much more natural to apply unitary pulses to the initial state to shift its relative orientation to the field, instead of actually changing the orientation of the external field [11].

Furthermore, we notice that states displaying time-invariant entanglement properties can be found in the multipartite case, too. One example is given by a specific family of $W$ states, labelled $\bar{W}$, whose single-excited states carry the relative phases $\{e^{2i\pi k/N}\}_{k=1}^N$ in an arbitrary order. As shown in Fig. 2 (c), this state remains $N$-partite entangled throughout the whole evolution, but the state itself evolves into a stationary state, as is displayed by the trace distance $D_{tr}(t) = ||\rho(t) - \rho_s||/2$, where $||X|| = \text{Tr} \sqrt{X^\dagger X}$ denotes the trace norm. The question remains whether this state is part of a finite-measure set of states whose multipartite entanglement properties are conserved, similarly to the $|\psi_-\rangle$-corner in the bipartite case. Such states would constitute ideal candidates for quantum computations by exhibiting invariance under collective dephasing effects.

Let us finally characterize the family of time-invariant states, for arbitrarily many qubits. Using Eq. (4), it can be shown that any state of the form $\rho_W = \sum_{x\in X_N} c_x V_x$, where $c_x$ are arbitrary coefficients and $V_x$ are the permutation operators defined above, satisfies $\rho(t) = \rho(0)$ for all times [14]. These states, known as multipartite Werner states [33], are also characterized by their invariance under arbitrary local unitary transformations $U^{\otimes N}$ [34]. Since such transformations describe collective changes of the local qubit coordinate systems, it is quite intuitive that these states are time-invariant for arbitrary directions of the external field. This identifies a $(N! - 1)$-parameter family of states that always span a decoherence-free subspace [24, 35]. In the geometric picture of Fig. 1 (a), these states lie on the line passing through the origin of the tetrahedron and the $|\psi_-\rangle$ state.

We conclude by some remarks on the transient evolution towards the asymptotic state. To determine how close the evolved state is to its asymptotic state, we again employ the trace distance $D_{tr}(t)$ which has a clear interpretation in terms of the distinguishability of quantum states [36]. In our present context, the trace distance is employed as an auto-correlation

\hspace{1cm} FIG. 2. (Color online) Influence of collective dephasing (5) on $N$-partite entangled $W$-states. (a) Upper bound $k_{\text{eff}}$ to the state’s separability vs. polar angles $0 \leq \theta \leq \pi/2$ (measured from the $z$-axis to the $x,y$-plane) of the fluctuating magnetic field’s direction. The radii of the quarter circles indicate the number $N = 2, \ldots, 10$ of qubits. The radial lines represent the angles $\theta_{\text{NP}}(N)$ within which $N$-partite entanglement is preserved (i.e. $k_{\text{eff}} = 1$). (b) Dependence of the critical angles $\theta_1$ and $\theta_{\text{NP}}$, Eqs. (6) and (7), on $N$. The dots are numerically extracted from (a) and correspond to the smallest angle where $k_{\text{eff}}$ changes from 1 to 2 (defining $\theta_{\text{NP}}$) or from $N - 1$ to $N$ (defining $\theta_1$). The uncertainty on this angle, due to the finite bin width of our sampling, is covered by the dot size. The lines represent the empirical expressions (6) and (7). (c) Time evolution of the trace distance $D_{tr}$ between $\rho(t)$ and the asymptotic state $\rho_s$ (dashed lines), and of the state’s separability bound $k_{\text{eff}}$ (continuous), respectively, for $N = 8$ qubits, $W$ (red) and $W$ (blue) initial states (definition see text), box-distributed noise fluctuations $B_{\text{box}}(\alpha)$, and a polar angle $\theta = \pi/8$.\hspace{1cm}
function, which reveals the monotonicity of the quantum evolution.

While different types of noise fluctuations lead to the same asymptotic state, as discussed earlier, the transient behavior can be qualitatively different, as displayed in Fig. 1. When the distribution \( p(\omega) \) is Lorentzian, \( C_{\omega_0}(\omega) = (\gamma/\pi)\left[(\omega - \omega_0)^2 + \gamma^2\right]^{-1} \), or Gaussian, \( N_{\omega_0}(x) = \exp\left(-2\omega_0^2 x^2/\sqrt{2}\right) \) (as suggested in Ref. [32]), the properties of the state, such as the concurrence, decay exponentially towards the asymptotic value [Fig. 1 (b)]. When we instead consider the box-distribution over the interval \([0, \omega_0]\), i.e., \( B_{\omega_0}(\omega) = [\Theta_{H}(\omega) - \Theta_{H}(\omega - \omega_0)]/\omega_0 \), where \( \Theta_{H}(\omega) \) is the Heaviside step function, we observe a non-monotonic approach of the quantum system to the asymptotic state (Fig. 1). In fact, for this distribution the characteristic function \( \varphi(t) \) is proportional to \( \sin(\omega_0 t)/t \), which asymptotically decreases on a significantly longer time scale than the exponential decay characterizing the Lorentzian or Gaussian distributions. This non-monotonic behavior also implies that the ensemble-averaged dynamics of non-interacting atoms in a fluctuating classical field cannot be modeled by an effective Markovian environment for certain noise distributions \( p(\omega) \) [25, 37]. These frequency fluctuations, thus, take on the role of the environment’s spectral density in a standard open-system description of decoherence [25, 26].

To summarize, we have provided a model for the dephasing dynamics of a collection of non-interacting atoms subject to a homogeneous external field of fluctuating intensity. The effective environment-induced interactions are described analytically by an exact solution in terms of a canonical Kraus map, able to describe the time evolution of multipartite systems under arbitrary intensity fluctuations. Our model applies to a variety of experiments in atomic physics, and describes one of the dominant error sources for state-of-the-art trapped ion experiments. Complete theoretical control on transient as well as asymptotic dynamics allows for the formulation of precise conditions to preserve relevant quantities, such as entanglement, as well as for the identification of families of states which are completely invariant under arbitrary directions of the external field.

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Supplementary Material

TRACE PRESERVING PROPERTY OF THE COLLECTIVE DEPHASING MAP

The operators $\Lambda_i = \frac{1}{2} (I_2 \pm n \cdot \sigma_i)$ form a complete set of orthogonal projectors on the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ of each qubit. These properties are inherited by the $\Theta_i$ operators, which are themselves orthogonal projectors: $\Theta_i = \Theta_i^\dagger$ and $\Theta_i \Theta_j = \Theta_i \delta_{i,j}$.

The trace preserving property of the map therefore reduces to $\sum_i \Theta_i \Theta_i^\dagger = \sum_i \Theta_i = I_{2^N}$. This is simply proven by putting $t = 0$ in Eq. (2) from the main text, and following the equalities from left to right:

$$\sum_i^{N} \Theta_i = (\Lambda_+ + \Lambda_-)^{\otimes N} = I_{2^N}. \quad (8)$$

For the $\Lambda_i(t)$ operators we instead have

$$\sum_i^N \Lambda_i(t) \Lambda_i(t) = \sum_{i,j} \Lambda_i(t) \Lambda_j(t) \Theta_i \Theta_j = \sum_{i,j} \Lambda_i(t) \Lambda_j(t) \Theta_j = \sum_j \Theta_j = I_{2^N}, \quad (9)$$

where we have used the spectral decomposition $M(t) = \sum_i \Lambda_i(t) \Lambda_i(t)^T(t)$, together with $M_{ij}(t) = 1, \forall t$.

CONSERVED TRACE OF THE $\beta$ MATRIX

We now prove that the trace of the $\beta(t)$ matrix, defined by $\beta_i(t) = \text{tr} [\rho(t) \cdot \sigma_i \otimes \sigma_i]$, is a time-invariant quantity. From the definition we have

$$\text{tr} \beta(t) = \sum_{i=1}^{3} \beta_i(t) = \sum_{i=1}^{3} \text{tr} \left( \rho(t) \cdot \sigma_i \otimes \sigma_i \right) = \text{tr} \left( \rho(t) \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \right). \quad (10)$$

Notice now that the Bell state $|\Psi^-\rangle \langle \Psi^-|$ reads [1]

$$|\Psi^-\rangle \langle \Psi^-| = \frac{1}{4} \left( I_4 - \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \right), \quad (11)$$

which yields $\sum_i \sigma_i \otimes \sigma_i = I_4 - 4|\Psi^-\rangle \langle \Psi^-|$. Substituting back we then have

$$\text{tr} \beta(t) = \text{tr} \rho(t) - 4 \text{tr} [\rho(t) |\Psi^-\rangle \langle \Psi^-|] = 1 - 4 \text{tr} [\rho(t) |\Psi^-\rangle \langle \Psi^-|]. \quad (12)$$

To prove that $\text{tr} \beta(t)$ is conserved under time evolution, we compute its derivative and check whether it vanishes:

$$\frac{d}{dt} \text{tr} \beta(t) = -4 \text{tr} [\dot{\rho}(t) |\Psi^-\rangle \langle \Psi^-|]. \quad (13)$$

The time derivative of $\rho(t)$ reads

$$\dot{\rho}(t) = \mathcal{L}_t [\rho(t)] = \sum_{i,j}^{N} \dot{M}_{ij}(t) \Theta_i \rho(t) \Theta_j, \quad (14)$$

which implies that

$$\text{tr} [\dot{\rho}(t) |\Psi^-\rangle \langle \Psi^-|] = \text{tr} \left[ \sum_{i,j=0}^{N} \dot{M}_{ij}(t) \Theta_i \rho(t) \Theta_j |\Psi^-\rangle \langle \Psi^-| \right]$$

$$= \text{tr} \left[ \sum_{i,j=0}^{N} \dot{M}_{ij}(t) \rho(t) |\Psi^-\rangle \langle \Psi^-| \right]$$

$$= \text{tr} \left[ \left( \mathcal{L}_t [|\Psi^-\rangle \langle \Psi^-|] \right)^\dagger \rho(t) \right]. \quad (15)$$

However, the Bell state $|\Psi^-\rangle$ is an eigenstate of the Hamiltonian of the system for whichever choice of $n$, which means that it is itself unaffected by collective dephasing. This implies that $\mathcal{L}_t [|\Psi^-\rangle \langle \Psi^-|] = 0, \forall t$ and therefore

$$\frac{d}{dt} \text{tr} \beta(t) \equiv 0. \quad (16)$$

CONCURRENCE FOR BELL-DIAGONAL STATES

In the tetrahedron of Bell-diagonal states [1], Wotters’ concurrence [2] reads

$$C(d) = \frac{1}{2} \max \left\{ 0, -1 + \sum_i |d_i| \right\}, \quad (17)$$

where $d = (d_1, d_2, d_3)^T \in \mathbb{R}^3$ is the point representing the quantum state (see main text).

For states in the $|\Phi^-\rangle$-corner, we have $d_i \leq 0, \forall i$. Hence, the concurrence in this corner can be rewritten as

$$C(d) = \frac{1}{2} \max \left\{ 0, -1 - \sum_i d_i \right\} = \frac{1}{2} \max \left\{ 0, -1 - k \right\}, \quad (18)$$

where we have used the fact that $\text{tr} \beta = \sum_i d_i = k$ is a constant. This explicitly proves that Bell-diagonal states in this corner have time-invariant concurrence.

In the other corners of the tetrahedron, only one of the coordinates is negative. If we suppose that $d_1 \leq 0$ (i.e. $|\Phi^-\rangle$-corner), we have

$$-1 + \sum_i |d_i| = -1 - d_1 + d_2 + d_3 = -1 + k - 2d_1.$$
Let $d$ represent the initial state, whereas $d'$ represents the final, asymptotic state of the system after collective dephasing. Furthermore, we denote the negative components of $d$ and $d'$ with a subscript $j$, i.e., we have $d_j \leq 0$ and $d'_j \leq 0$, respectively. Direct application of the collective dephasing map leads to
\[
d'_j = \sum_i d_i n_i^2.
\]

The concurrence in the final state is then
\[
C(d') = \frac{1}{2} \max \left\{ 0, -1 + \sum_i d_i - 2 \sum_i d_i n_i^2 \right\}
= \frac{1}{2} \max \left\{ 0, -1 + \sum_i (1 - 2n_i^2)d_i \right\}.
\]

From these formulae one can immediately notice that $C(d) = C(d') \iff n = \pm e_j$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$.

**TIME-INFRINGEMENT OF MULTIPARTITE WERNER STATES**

In this Section we prove that the multipartite Werner states are time-invariant under the action of the collective dephasing map. Let $s \in \Sigma_N$ be a permutation in the symmetric group, and $V_s$ its representation in the operator space of $N$ qubits:
\[
V_s = \sum_{i_1 \ldots i_N \in \{0,1\}} |i_{s(1)} \ldots i_{s(N)}\rangle \langle i_1 \ldots i_N|.
\]

The multipartite Werner states are then defined as $\rho_W = \sum_{s \in \Sigma_N} c_s V_s$, where $c_s$ are arbitrary coefficients, leading to a valid quantum state $\rho_W$.

Analogously, if we define $k_i = i!(N - i)!$ and $Q_i = \Lambda^{o_i}_i \otimes \Lambda^{o_{N-i}}$, we can rewrite the $\Theta_i$ operators as
\[
\Theta_i = \frac{1}{k_i} \sum_{s \in \Sigma_N} V_s Q_s V_s^\dagger.
\]

Direct application of the collective dephasing map yields
\[
\epsilon[\rho_W] = \sum_{i=0}^N \sum_{\pi,\sigma,\tau \in \Sigma_N} c_\pi \left( V_\sigma Q_\tau V_\sigma^\dagger \right) \left( V_\tau Q_\sigma V_\tau^\dagger \right).
\]

Because $\Sigma_N$ is a closed group, the concatenation of two permutations describes another permutation, and therefore $\exists \alpha \in \Sigma_N : V_\sigma = V_\pi V_\alpha$. The expression above can then be rewritten as
\[
\epsilon[\rho_W] = \sum_{i=0}^N \sum_{\pi,\sigma,\tau \in \Sigma_N} c_\pi \left( V_\sigma Q_\tau V_\sigma^\dagger \right) \left( V_\tau Q_\sigma V_\tau^\dagger \right) = \sum_{i=0}^N c_\pi V_\pi \sum_{\sigma \in \Sigma_N} \left( V_\sigma Q_\pi V_\sigma^\dagger \right) \sum_{\tau \in \Sigma_N} \frac{\left( V_\tau Q_\pi V_\tau^\dagger \right)}{k_i} = \sum_{i=0}^N c_\pi V_\pi \sum_{\sigma \in \Sigma_N} \Theta_i \Theta_i = \rho_W,
\]

where we have used the idempotency of the $\Theta_i$ operators and the closure relation $\sum_i \Theta_i = 1_{2^N}$.

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