A Consistent Treatment of Strong Coupling in Disordered Superconducting Films

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We develop a consistent treatment of disorder effects in strong coupling superconductors. We use two different approaches, starting either from above or below the transition temperature, and show their equivalence. The normal state approach is a natural extension of a recently developed non-perturbative resummation technique. For the superconducting state we extend the standard Eliashberg theory to include disorder effects. We obtain a compact set of equations which are physically transparent and easy to use. They correctly reproduce the usual strong coupling results and disordered perturbation theory results in the appropriate limits. We numerically solve these equations to obtain transition temperature as a function of disorder for different phonon couplings.

The phenomenon of the suppression of the transition temperature in thin superconducting films by disorder has been extensively investigated in recent years [1]. The physical mechanism is easy to understand: disorder causes electrons to move diffusively, leading to a less efficient screening of the Coulomb repulsion between them. This manifests itself as a low-momentum singularity in the dynamically screened Coulomb interaction. Naively one would expect this to lead to a very strong suppression of superconductivity due to the enhancement of the effective Coulomb repulsion [2]. However this turns out not to be the case because of a cancellation between the various singular terms which arise, for example, in perturbation theory [3]. This apparently fortuitous cancellation is in fact due to gauge invariance [4], and the screened Coulomb potential can therefore be treated as if it were featureless. We note that all these subtleties are present even in the simplest possible calculation, namely first order perturbation theory assuming a featureless BCS attraction. However this model is too restrictive for the following reasons: (i) perturbation theory is not valid in the limit of strong disorder often seen experimentally; (ii) many of the experimentally relevant materials such as Pb and Pb-Bi are strong coupling superconductors for which the details of the phonon spectrum are important [5–10]. It is therefore desirable to extend the simple theory to accommodate these features in a consistent manner.

A method of going beyond perturbation theory has recently been developed by Oreg and Finkel'stein [11]. These authors use a ladder summation technique which treats the BCS attraction and impurity-dressed Coulomb repulsion on an equal footing. This is possible because both interactions can be treated as featureless, as explained above. The fact that this technique reproduces the renormalization group equation known from the interacting non-linear $\sigma$-model [12] proves its essential correctness. Its great advantage is its transparency and the fact that it is easy to use. The aim of this paper is to extend this technique to include strong coupling effects. Initially one might think that this would further complicate the problem. Counterintuitively the problem actually becomes simpler because one realises that the original technique is best understood in terms of strong coupling theory. Moreover this realisation allows us to naturally extend the non-perturbative treatment of disorder to below the transition temperature.

In this paper we present a consistent treatment of the effects of disorder and strong coupling on the transition temperature in superconductors. We use two different approaches which we show to be equivalent: (a) Starting from the normal state we derive the pair propagator and identify the transition temperature as the temperature at which it becomes singular; (b) following the Eliashberg theory of strong-coupling superconductors, we start from below the transition temperature and we derive the superconducting self-energy and identify the transition temperature as the point at which the order parameter goes to zero. The former approach has the advantage that it does not make any assumptions about the structure of the superconducting state and so is the natural way to initially address the problem. The latter approach has the advantage that it allows us to gain information about the superconducting state at all temperatures below the transition temperature at no extra cost. The equivalence of the results obtained by the two approaches justifies the assumptions we make about the superconducting state.

We begin by calculating the zero-momentum pair propagator, $\Gamma(\omega, \omega')$, shown in Fig. 1. This yields the equation

$$\Gamma(\omega, \omega') = \Gamma_0(\omega, \omega') + T \sum_{\omega''} \Gamma_0(\omega, \omega'') C(\omega'') \Gamma(\omega'', \omega')$$

where the $\omega$ are Fermi Matsubara frequencies, $\Gamma_0$ is the elementary interaction vertex and $C$ is the Cooperon impurity ladder. All momentum dependences have been removed by averaging over the Fermi surface. We identify the transition temperature as the temperature at which $\Gamma$ diverges. This will occur when there is an eigenvector $\phi(\omega)$ of the homogeneous part of Eq. (1), which leads to the equation
\[ T \sum_{\omega'} \Gamma(\omega, \omega') \phi(\omega') = \]
\[ T \sum_{\omega'} T \sum_{\omega''} \Gamma_0(\omega, \omega'') C(\omega'') \Gamma(\omega'', \omega') \phi(\omega'). \]

If we define \( \Delta(\omega) = T \sum_{\omega'} \Gamma(\omega, \omega') \phi(\omega') \) the above equation has the familiar Eliashberg form for the order parameter,
\[ \Delta(\omega) = T \sum_{\omega''} \Gamma_0(\omega, \omega'') C(\omega'') \Delta(\omega''). \]

The elementary interaction vertex \( \Gamma_0 \) contains the electron-phonon contribution, \( \Gamma_{0, ph} \), and the Coulomb contribution. The latter naturally splits into two parts: the bare Coulomb interaction, \( \Gamma_{0, C} \), for high momentum and frequency transfer, and the impurity-dressed Coulomb interaction, \( \Gamma_{0, dis} \) otherwise. The second contribution is important for low momentum transfers, \(|q| < 1/\ell\), and low frequency transfers, \(|\omega| < 1/\tau\), where \( \ell \) is the elastic mean free path and \( \tau \) is the elastic scattering time. This is because the disorder-screened Coulomb interaction keeps its long-range momentum singularity and its effective magnitude is therefore greatly enhanced. Gauge invariance leads to a cancellation of singular terms \( \frac{\Gamma_0}{\Gamma} \) which allows us to treat the disorder-screened interaction as featureless with magnitude \( V_{\text{dis}} = 1/2N(0) \), where \( N(0) \) is the electronic density of states at the Fermi surface.

The electron-phonon contribution, \( \Gamma_{0, ph} \), is given by
\[ \Gamma_{0, ph}(\omega, \omega') = \sum_{\lambda} \langle g_{k, k', \lambda} \rangle^2 D_\lambda(\mathbf{k} - \mathbf{k}', \omega - \omega') \rangle_{FS}, \]
\[ = \frac{1}{N(0)} \int_0^\infty dz \frac{2\alpha^2(z) F(z)}{(\omega - \omega')^2 + z^2} \equiv \frac{\lambda(\omega - \omega')}{N(0)}, \]
where we have averaged over the Fermi surface in the standard manner. In the above equation \( g_{k, k', \lambda} \) is the electron-phonon coupling for phonons of polarisation \( \lambda \), \( D_\lambda(k - k', \omega - \omega') \) is the phonon Green function and \( \alpha^2(z) F(z) = N(0) \sum_{\lambda} \langle g_{k, k', \lambda} \rangle^2 \delta(z - \omega - \omega') \rangle_{FS}. \) The bare Coulomb contribution, \( \Gamma_{0, C} \), is just the Coulomb potential averaged over the Fermi surface,
\[ \Gamma_{0, C} = V_C = \langle V(\mathbf{k} - \mathbf{k'}) \rangle_{FS}. \]

The disordered Coulomb contribution is given by
\[ \Gamma_{0, dis} = \pi t \log \left( \frac{1}{(\omega + |\omega'|)\tau} \right) \]
where we define the dimensionless measure of disorder, \( t = V_{\text{dis}}/4\pi^2 D = R_0/R_0 \), and \( R_0 = 2\pi h/e^2 = 162k\Omega \) is a quantum unit of resistance.

First we include only the bare Cooperon ladder, \( C_0 = \pi N(0)/|\omega| \), to yield the self-consistency equation

\[ \Delta(\omega) = \pi T \sum_{\omega''} \Gamma_0(\omega, \omega'') C(\omega'') \Delta(\omega''). \]

\[ \Gamma_{\text{dis}} = \quad \Gamma_0 \quad + \quad C \quad + \quad \Gamma \]

\[ \Gamma_0 \quad + \quad C \quad + \quad \Gamma_\Lambda \]

\[ \Gamma_\Lambda \quad + \quad \Gamma \]

\[ \Gamma \quad + \quad C \quad + \quad \Gamma \]

\[ \Sigma \quad + \quad C \quad + \quad \Gamma \]

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FIG. 1. Diagrammatic equation for the pair propagator \( \Gamma \) in terms of the elementary interaction vertex \( \Gamma_0 \) and the screened Cooperon \( C \). \( \Gamma_0 \) consists of electron-phonon interaction (zig-zag line), the bare Coulomb interaction (wiggly line) and the disorder-dressed Coulomb interaction block \( t\Lambda \). \( \Sigma \) is the self-energy correction to the Cooperon due to the impurity-screened Coulomb interaction.

\[ \Delta(\omega) = \pi T \sum_{\omega''} \frac{\Delta(\omega'')}{|\omega''|} \left[ \lambda(\omega - \omega'') \right] \]
\[ - \mu - t \ln \left( \frac{\omega}{\omega + |\omega'|} \right), \]
where \( \mu = N(0)V_C \). The sums over the Matsubara frequencies in the above equation are cut off at the Fermi energy. However the phonon interaction is effective only for energies of the order the Debye frequency. We therefore follow the standard procedure of introducing a frequency \( \omega_c \) above which only the bare Coulomb term contributes. The equation for \( \Delta(\omega) \) with \( \omega > \omega_c \) then decouples and the only effect of these frequencies is to renormalize \( \mu \) to
\[ \mu' = \frac{\mu}{1 + \mu \ln (e_F/\omega_c)}. \]

We must therefore replace \( \mu \) by \( \mu' \) in Eq. (7), and cut off the sums at frequency \( \omega_c \).

Next we consider self-energy corrections to the impurity-dressed electron Green function. Having averaged out the momentum dependence, the self-energy has the form \( \Sigma(\omega) = i\omega[1 - Z(\omega)] \). The dressing of the electron Green function leads to the Cooperon being modified to \( C(\omega) = \pi N(0)Z(\omega)/|\omega| \), as shown in Fig. (1). At this point we note the difference between the effect of including the impurity self-energy in the original electron Green function, and the effects of including the phonon and Coulomb terms. The impurity correction, \( \Sigma_{\text{imp}} = (i/2\tau)\text{sgn}(\omega) \), is exactly cancelled in the process of constructing the impurity ladder – this is essentially the content of Anderson’s theorem which says that
non-magnetic impurities have no effect on superconductivity at the mean field level. The contributions to the self-energy are

\[
i\omega(1 - Z(\omega)) = -i\pi T \sum_{\omega'} \frac{\lambda(\omega - \omega') \text{sgn}(\omega')}{|\omega| + |\omega'|}
+ it\omega T \sum_{\omega'} \frac{1}{|\omega| + |\omega'|} .
\]  

(9)

The presence of \( Z(\omega) \) in the denominator of \( C(\omega) \) means that we should redefine \( \Delta(\omega) = T \sum_{\omega'} \Gamma(\omega, \omega') \delta(\omega')/Z(\omega) \) in Eq. (2) so that \( \Delta(\omega) \) is given by

\[
Z(\omega)\Delta(\omega) = \pi T \sum_{\omega'' < \omega_c} \frac{\Delta(\omega'')}{|\omega''|} \left[ \lambda(\omega - \omega'') - \mu^* - t \text{ln} \left( \frac{1}{(|\omega| + |\omega''|)\pi T} \right) \right] .
\]

(10)

Equations (9) and (10) constitute the main result of this paper. They are the Eliashberg equations for strong coupling superconductors including the effects of Coulomb interaction and disorder.

The above equations have been derived using the pair propagator approach, whereas they are normally derived in the framework of the Nambu-Gor’kov self-energy formalism. We will now extend the latter formalism to include the effects of disorder. The various contributions to the superconducting self-energy,

\[
\Sigma(\omega) = i\omega[1 - Z(\omega)]\tau_0 + Z(\omega)\Delta(\omega)\tau_1 ,
\]

(11)

are shown in Fig. (2). The phonon and bare Coulomb contributions are

\[
\Sigma_{\text{ph}}(\omega) = -\pi T \sum_{\omega'} \frac{\lambda(\omega - \omega')i\omega'\tau_0 - \Delta(\omega')\tau_1}{W'} ,
\]

\[
\Sigma_C(\omega) = U_C \pi T \sum_{\omega'} \frac{\Delta(\omega')}{W'} ,
\]

(12)

where \( W = \sqrt{\omega^2 + \Delta^2(\omega)} \) and \( W' = \sqrt{\omega'^2 + \Delta^2(\omega')} \). They can be found in Ref. [10]. The disorder-dressed Coulomb diagrams of Fig. (2d) and (2e) give

\[
\Sigma_{\text{dis},1}(\omega) = -t\pi T \sum_{\omega'} \frac{i\omega\tau_0 - \Delta(\omega)\tau_1}{W + W'} \left[ 1 - \frac{\omega'\Delta(\omega) - \omega\Delta(\omega')}{WW'} \right] .
\]

\[
\Sigma_{\text{dis},2}(\omega) = t\pi T \sum_{\omega'} \ln \left( \frac{1}{(W + W')\pi} \right) \left( \frac{\omega'\Delta(\omega) - \omega\Delta(\omega')}{WW'} \right) \left( \frac{i\Delta(\omega)\tau_0 + \omega\tau_1}{W} \right) .
\]

(13)

In the limit where \( T \to T_c \), the order parameter \( \Delta(\omega) \to 0 \), and we obtain the previously derived Eliashberg equations (3) and (4).

Having derived the Eliashberg equations in Matsubara frequencies, let us now proceed to their numerical solution. Following Refs. [10-14], we multiply Eq. (3) by \( \Delta(\omega) \) and equate it to Eq. (4). Noting that \( \omega = (2n + 1)\pi T , \omega' = (2n' + 1)\pi T \), and defining \( \Delta = \Delta(\omega)/(2n + 1)^{1/2} \), we obtain the matrix equation

\[
\sum_{n' = 0}^{n} S_{nn'} \Delta_{n'} = 0 ,
\]

(14)

where

\[
S_{nn'} = \left\{ \lambda(n - n') + \lambda(n + n' + 1) - 2\mu^* - 2t \text{ln} \left( \frac{M}{n + n' + 1} \right) \right\} \frac{1}{(2n + 1)(2n' + 1)^{1/2}} + \delta_{nn'} \left( 1 + \left( \lambda(0) + 2 \sum_{n'' = 1}^{n} \lambda(n'') \right) \frac{1}{(2n + 1)(2n' + 1)^{1/2}} + t \sum_{n'' = 0}^{n} \frac{1}{n + n'' + 1} \right) .
\]

(15)
FIG. 3. Relative suppression of the transition temperature $T_c$ vs $t \ln^3(1/2\pi T_c \tau)$ for different values of the electron-phonon coupling $\lambda$. All calculations are performed using an Einstein spectrum.

and $M = 1/2\pi T_c \tau$. The numerical strategy is to find the temperature at which the largest eigenvalue of the matrix $S$ becomes positive. We have to specify the phonon elements, $\lambda(n)$, the Coulomb pseudopotential, $\mu^*$, the effective disorder strength, $t = R_0/R$, and the elastic scattering rate, $1/\tau$. As an illustration, we solve the equation for an Einstein phonon spectrum, $\alpha^2 F(z) = \lambda \omega_E \delta(z - \omega_E)/2$, for which $\lambda(n) = \omega_E^2 / \left[ \omega_E^2 + (2n\pi T_c)^2 \right]$. We plot $T_c(\lambda, t)/T_c(\lambda, 0)$ vs $t \ln^3(1/2\pi T_c(\lambda, 0) \tau)$, the latter being the initial slope predicted from perturbation theory. This enables us to see if the shape of the curve deviates from the weak coupling prediction as $\lambda$ is increased. The only parameters we fix are $\mu^* = 0.1$ and $1/\omega_E \tau = 250$ which are typical values for lead films. The results are shown in Fig. 3. We see that there is a significant deviation from the weak coupling predictions which may be important in fitting experimental data. We intend to extend our numerical analysis to lower values of $\lambda$ and different phonon spectra.

In conclusion we have developed a consistent treatment of the effect of disorder and strong coupling on the transition temperature of superconducting films. Previous authors have considered similar systems within an exact eigenstate framework, but this approach remains somewhat opaque physically, and its relation to weak coupling theories is unclear. Our approach has the advantage of yielding a compact set of equations that are easy to use, and whose weak coupling and weak disorder limits reproduce the known results. In future work we will analytically continue these equations to real frequencies, and use them to analyse the effect of disorder on tunneling characteristics in strong coupling materials.

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