ON DIVIDING BY TWO IN CONSTRUCTIVE MATHEMATICS

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Abstract. A classic result due to Bernstein states that in set theory with classical logic, but without the axiom of choice, for all sets \( X \) and \( Y \), if \( X \times 2 \cong Y \times 2 \) then also \( X \cong Y \). We show that this cannot be done in constructive mathematics by giving some examples of toposes where it fails.

1. Introduction

In classical set theory with the axiom of choice it is trivial to show that given two sets \( X \) and \( Y \), if there exists a bijection \( X \times 2 \cong Y \times 2 \), then there exists also a bijection \( X \cong Y \). We refer to this statement as “dividing by two.”

Surprisingly, it is possible to divide by two in even in absence of the axiom of choice, by explicitly defining a bijection between \( X \) and \( Y \), given a bijection between \( X \times 2 \) and \( Y \times 2 \). This was first proved by Bernstein in his thesis in 1905 [1]. Later, in 1922 in [8], Sierpiński gave a simplified proof of the same result. For a more recent exposition see the work of Conway and Doyle [3], who gave a proof of both this result and the more difficult problem of dividing by three.

More recently still, this construction has seen some attention on various forums online. In particular, the Mathematics Stack Exchange user, Hanno, raised the question that we will answer in this paper: whether division by two can be carried out in constructive mathematics [4].

Here, by constructive mathematics, we mean mathematics carried out without the use of excluded middle, in the style of, for instance, Bishop and Bridges in [2]. Toposes are categories analogous to the category of sets, wherein mathematical statements and constructive proofs can be interpreted. See e.g. [7] for an introduction to topos theory.

The key part of our proof that dividing by two is impossible in constructive mathematics is in fact the same idea used in proving that it can be done in \( \text{ZF} \).

Suppose that we are given two sets \( X \) and \( Y \) together with a bijection \( X \times 2 \cong Y \times 2 \). Following Sierpiński [8] we will think of the bijection as a permutation \( \theta \) of \( (X \times 2) + (Y \times 2) \) of order 2. Namely, \( \theta \) takes each element of \( X \times 2 \) to the corresponding element of \( Y \times 2 \) and vice versa.

Note that \( (X \times 2) + (Y \times 2) \cong (X + Y) \times 2 \). We will write \( Z \) for \( X + Y \), and so think of \( \theta \) as a permutation of \( Z \times 2 \).

There is another permutation of \( Z \times 2 \) that sends \((z, i) \in Z \times 2 \) to \((z, 1 - i)\). We denote this permutation \( \phi \).

Now we note that given any element \( z \) of \( Z \times 2 \), we can define a sequence \( \chi_z: Z \rightarrow 2 \), as follows. Writing \( \pi_1 \) for the projection \( Z \times 2 \rightarrow 2 \), we define \( \chi_z(n) \) to be \( \pi_1((\phi \theta)^n \cdot z) \). The construction of the bijection \( X \cong Y \) proceeds by dividing into cases depending on the properties of \( \chi \). We refer to [1], [8] or [3] for details.
The main idea for our proof is that in general we always have the computational information contained in $\chi$ available anyway. We can therefore clarify the situation by making the dependence on the signature explicit. We will take $X$ to be the even numbers, $2\mathbb{Z}$, and $Y$ to be the odd numbers $2\mathbb{Z} + 1$. Instead of starting with a bijection $\theta$ and then defining $\chi$, we will start off with an arbitrary sequence $\chi : \mathbb{Z} \to 2$, and then define a bijection $\theta_{\chi}$, as follows.

**Definition 2.1.** Suppose we are given $\chi : \mathbb{Z} \to 2$. We define a bijection $\theta_{\chi}$ from $2\mathbb{Z} \times 2$ to $(2\mathbb{Z} + 1) \times 2$ as follows.

$$
\theta_{\chi}(n, i) := \begin{cases} 
(n + 1, 1 - \chi(n + 1)) & i = \chi(n) \\
(n - 1, \chi(n - 1)) & i \neq \chi(n)
\end{cases}
$$

To see that $\theta_{\chi}$ is a bijection, note that we can use exactly the same definition for its inverse (although in that case the domain consists of odd numbers rather than even numbers).

Dividing by two gives us a way to transform each bijection $\theta_{\chi}$ between $2\mathbb{Z} \times 2$ and $(2\mathbb{Z} + 1) \times 2$ into a bijection between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$.

Obviously there are always bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$, for instance by adding 1. However, our proof will rest on the fact that $\theta_{\chi}$ depends on $\chi$ equivariantly and continuously, in a sense that we will define. We will see that these cannot both hold for families of bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$.

### 3. Equivariance

Let $D_\infty$ be the infinite dihedral group. There are a few different ways of defining this group. We will use the finite presentation below.

$$D_\infty := \langle t, r \mid r^2 = 1, rt = t^{-1} \rangle$$

It’s helpful to note that $D_\infty$ is also the group of isometries of $\mathbb{Z}$, with $t$ translation by 1, and $r$ reflection in the origin, say. For this reason, we will refer to elements of the form $t^a$ as translations, and elements of the form $rt^a$ as reflections.

In fact, we don’t use the usual action of $D_\infty$ on $\mathbb{Z}$, but instead define the action of translation to be translation by 2, so as to preserve odd and even numbers. Explicitly, we define $t \cdot n$ to be $n + 2$ and we define $r \cdot n$ to be $-n$.

We define an action on $2\mathbb{Z}$ as follows. Suppose we are given an element $\chi$ of $2\mathbb{Z}$. We define $t \cdot \chi$ to be $\lambda n \cdot \chi(n - 2)$ and we define $r \cdot \chi$ to be $\lambda n \cdot 1 - \chi(-n)$.

**Proposition 3.1.** The above specifies a well defined action of $D_\infty$ on $\mathbb{Z}$ which restricts to $2\mathbb{Z}$ and $2\mathbb{Z} + 1$, and a well defined action on $2\mathbb{Z}$.

**Proof.** To show these are well defined actions it suffices to check that they respect the equations in the finite presentation of $D_\infty$, which is straightforward. \qed

We also consider the trivial action on 2, which then defines an action on $2\mathbb{Z} \times 2$ and $(2\mathbb{Z} + 1) \times 2$.

**Lemma 3.2.** Let $g \in D_\infty$ and $\chi \in 2\mathbb{Z}$. Then $\theta_{g \cdot \chi}(g \cdot n) = g \cdot \theta_{\chi}(n)$.

**Proof.** It suffices to check this for the generators $t$ and $r$. Both are straightforward. \qed

**Proposition 3.3.**

1. The naive bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ defined by always adding 1 (or alternatively always subtracting 1) are not equivariant in $2\mathbb{Z}$.

2. There is an equivariant family of bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ (assuming the axiom of excluded middle).
Proof. We leave a direct proof of 1 as an exercise for the reader, although it will also follow from the arguments we will use in section 5.

For 2, we leave it as an exercise for the reader to give a direct proof based on Bernstein's argument, but we also give an abstract proof based on the ideas that we will see in section 6: note that if excluded middle holds, then it also holds internally in the topos of $D_\infty$-sets, i.e. $D_\infty - \text{Set}$ is a boolean topos. We can therefore carry out Bernstein’s argument in the internal logic of $D_\infty - \text{Set}$ to get an equivariant bijection. □

4. Continuity

Recall that Baire space is defined to be the topological space on the set $\mathbb{N}^\mathbb{N}$ with the product topology on $\mathbb{N}$ copies of $\mathbb{N}$ with the discrete topology. It’s often useful (in both constructive and classical mathematics) to give an explicit definition of continuity as follows.

Proposition 4.1. A function $F: \mathbb{N}^\mathbb{N} \to \mathbb{N}$ is continuous iff for all $\alpha \in \mathbb{N}^\mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $\beta \in \mathbb{N}^\mathbb{N}$ if $\alpha(n) = \beta(n)$ for all $n < N$, then $F(\alpha) = F(\beta)$.

Proposition 4.2. The bijections $(\theta_\chi)_{\chi \in 2^\mathbb{Z}}$ that we defined in section 3 are continuous when viewed as a single function from $2^\mathbb{Z} \times 2^\mathbb{Z}$ to $2^\mathbb{Z} \times (2^\mathbb{Z} + 1)$.

More explicitly, for every $\chi$ and every $n \in \mathbb{Z}$, there exists $N \in \mathbb{N}$ such that for all $\chi' \in 2^\mathbb{Z}$, if $\chi'(m) = \chi(m)$ for all $m$ with $|m| < N$, then $\theta_\chi(n) = \theta_{\chi'}(n)$.

Proof. This is clear from the definition. □

We say that $\theta_\chi$ is continuous in $2^\mathbb{Z}$.

We make the following observation. A direct proof is left as an exercise for the reader, although it will also follow as a corollary from the arguments in the next section.

Proposition 4.3. The bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ resulting from Bernstein's construction are not continuous in $2^\mathbb{Z}$.

5. The Non-Existence of a Continuous and Equivariant Family of Bijections

We aim towards a proof that there is no continuous and equivariant family of bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ in the sense that we defined in previous sections (this will be theorem 5.2). It will be important for future sections that this proof is entirely constructive (and so valid in the internal logic of a topos).

Our first observation is that we do not need to consider all elements of $2^\mathbb{Z}$, but only those $\chi$ that are decreasing. We use the following notation for some of these elements. We write $-\infty$ for the sequence which is constantly 0, $\infty$ for the sequence which is constantly equal to 1, and given $n \in \mathbb{Z}$, we write $\underline{n}$ for the sequence defined as below.

$$\underline{n}(i) := \begin{cases} 1 & i < n \\ 0 & i \geq n \end{cases}$$

We will write the set of decreasing sequences as $\mathbb{Z}_\infty$.

Although we don’t formally need the following two observations, they help illustrate the motivation for this definition and notation.

(1) For any $m < n$ in $\mathbb{Z}$, in the pointwise ordering on $2^\mathbb{Z}$, $-\infty < \underline{m} < \underline{n} < \infty.$
(2) In classical logic, every decreasing binary sequence on $\mathbb{Z}$ is either of the form $-\infty, \infty$ or $\underline{0}$ for some $n \in \mathbb{Z}$. In fact this is equivalent to one of Brouwer’s omniscience principles, the \textit{limited principle of omniscience (LPO)}, which states that for every binary sequence $\alpha: \mathbb{N} \to 2$, either $\alpha(n) = 0$ for every $n \in \mathbb{N}$, or $\alpha(n) = 1$ for some $n \in \mathbb{N}$.

Throughout this section, we assume that we are given a family of bijections $\varphi: 2\mathbb{Z} \to 2\mathbb{Z} + 1$ indexed by elements of $\mathbb{Z}_{\infty}$ that are both continuous and equivariant in $\mathbb{Z}_{\infty}$, in the sense that we defined in the previous sections. Clearly if we are given a continuous equivariant family of bijections indexed by all elements of $2\mathbb{Z}$, we could just restrict to get a continuous equivariant family of bijections indexed over $\mathbb{Z}_{\infty}$.

We will often view the family of bijections $\varphi$ as a function $\mathbb{Z}_{\infty} \times 2\mathbb{Z} \to \mathbb{Z}_{\infty} \times (2\mathbb{Z} + 1)$ that forms part of the following commutative triangle.

\[
\begin{array}{ccc}
\mathbb{Z}_{\infty} \times 2\mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}_{\infty} \times (2\mathbb{Z} + 1) \\
\pi_0 \downarrow & & \downarrow \pi_0 \\
\mathbb{Z}_{\infty} & \xrightarrow{\varphi} & \mathbb{Z}_{\infty}
\end{array}
\]

**Lemma 5.1.** There exists $k \in \mathbb{Z}$ and $N > 0$ such that for all $n > N$, $\varphi(\underline{0}, n) = (\underline{0}, n + k)$, and for all $n < -N$, $\varphi(\underline{0}, n) = (\underline{0}, n - k)$.

**Proof.** Let $k$ be such that $\varphi(-\infty, 0) = (-\infty, k)$.

Since $\varphi$ is a continuous function, $\pi_1 \circ \varphi$ is also continuous as a function $\mathbb{Z}_{\infty} \times 2\mathbb{Z} \to \mathbb{Z}_{\infty}$. Hence there exists $N > 0$ such that for all $\chi$, if $\chi(i) = 0$ for $|i| < N$ then we have $\pi_1(\varphi(\chi, 0)) = \pi_1(\varphi(-\infty, 0))$. It clearly follows that $\varphi(\underline{0}, 0) = \varphi(-\infty, 0)$ for $n \leq -N$.

Then we have the following for every even number $2n$ with $2n > N$.

\[
\pi_1(\varphi(\underline{0}, 2n)) = \pi_1(\varphi(t^n \cdot (-2n, 0)))
\]
\[
= t^n \cdot \pi_1(\varphi(\underline{0}, 0))
\]
\[
= t^n \cdot \pi_1(\varphi(-\infty, 0))
\]
\[
= t^n \cdot \pi_1(-\infty, k)
\]
\[
= 2n + k
\]

For $2n < -N$ we have $2n + 1 \leq -N$, and hence,

\[
\pi_1(\varphi(\underline{0}, 2n)) = \pi_1(\varphi(t^n \cdot (-2n, 0)))
\]
\[
= \pi_1(\varphi(t^n r \cdot (2n + 1, 0)))
\]
\[
= t^n r \cdot \pi_1(\varphi(2n + 1, 0))
\]
\[
= t^n r \cdot \pi_1(-\infty, 0)
\]
\[
= t^n r \cdot \pi_1(-\infty, k)
\]
\[
= 2n - k
\]

\[\square\]

**Theorem 5.2.** There is no family of bijections between $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ that is continuous and equivariant in $\mathbb{Z}_{\infty}$.

**Proof.** In lemma 5.1 we showed that there exists $N > 0$ (which is even without loss of generality) and $k \in \mathbb{Z}$ (which is necessarily odd) such that for $n > 0$, $\varphi(2n) = n + k$ and for $n < -N$, $\varphi(2n) = n - k$. Furthermore, without loss of generality $N > |k|$.

We deduce that the image of the set $((-\infty, -N - 2] \cup [N + 2, \infty)) \cap 2\mathbb{Z}$ under $\varphi_{2\mathbb{Z}}$ must be $((-\infty, N - k - 2] \cup [N + k + 2, \infty)) \cap (2\mathbb{Z} + 1)$. Since $\varphi_{2\mathbb{Z}}$ is a bijection,
it follows that the image of the set \([-N, N] \cap 2\mathbb{Z}\) is \([-N - k, N + k] \cap (2\mathbb{Z} + 1)\). However, the cardinality of \([-N, N] \cap 2\mathbb{Z}\) is odd, because for each even number \(n\) with \(0 < n \leq N\) it contains \(n\) and \(-n\), and it also contains 0. On the other hand the cardinality of \([-N - k, N + k] \cap 2\mathbb{Z} + 1\) is even because we can still pair up each element with its negation, but it does not contain 0.

\[\square\]

6. Construction of the Topos

We first consider the topos \(D_\infty - \mathbf{Set}\) of sets with \(D_\infty\)-action where \(D_\infty\) is the infinite dihedral group. Recall that an object is a set \(X\) together with an action of \(D_\infty\) on \(X\), and a morphism is a function that preserves the action.

Our first steps will look a little strange to readers unfamiliar with constructive mathematics.

**Lemma 6.1.** Suppose that every function from \(\mathbb{N}^\mathbb{N}\) to \(\mathbb{N}\) is continuous (this is sometimes referred to as Brouwer’s continuity principle or just Brouwer’s principle). Then the same is true for every retract of \(\mathbb{N}^\mathbb{N}\).

**Proof.** Suppose that \(S\) is a retract of \(\mathbb{N}^\mathbb{N}\). Then the inclusion \(i: S \to \mathbb{N}^\mathbb{N}\) is continuous, and by definition there is a continuous map \(p: \mathbb{N}^\mathbb{N} \to S\) such that \(p \circ i = 1_S\).

Let \(f\) be any continuous function from \(S\) to \(\mathbb{N}\). Note that \(f \circ p\) is a function from \(\mathbb{N}^\mathbb{N}\) to \(\mathbb{N}\), and so continuous. But then \(f = f \circ p \circ i\), and so \(f\) must also be continuous. \[\square\]

**Lemma 6.2.** Suppose that every function \(\mathbb{N}^\mathbb{N} \to \mathbb{N}\) is continuous. Then the slice category \(D_\infty - \mathbf{Set} / \mathbb{Z}_\infty\) contains two objects \(X\) and \(Y\) such that \(X \times 2 \cong Y \times 2\) but there is no isomorphism between \(X\) and \(Y\).

**Proof.** We consider the example from theorem 5.2. Note that the category of equivariant families of maps over \(\mathbb{Z}_\infty\) is equivalent to the slice category \(D_\infty - \mathbf{Set} / \mathbb{Z}_\infty\).

The space \(\mathbb{Z}_\infty \times \mathbb{Z}\) is evidently a retract of \(\mathbb{N}^\mathbb{N}\), and so every function to \(\mathbb{Z}\) continuous by lemma 6.1. Hence every function \(\mathbb{Z}_\infty \times 2\mathbb{Z} \to \mathbb{Z}_\infty \times (2\mathbb{Z} + 1)\) is continuous, and in particular any family of bijections \(\varphi\). We can now apply theorem 5.2. \[\square\]

**Theorem 6.3.** There is topos \(E\) containing objects \(X\) and \(Y\) such that \(X \times 2 \cong Y \times 2\) but \(X \not\cong Y\).

**Proof.** Let \(E\) be any topos that satisfies Brouwer’s continuity axiom that all functions \(\mathbb{N}^\mathbb{N} \to \mathbb{N}\) are continuous. This includes a couple of well known toposes in realizability, the effective topos and the function realizability topos (see Proposition 3.1.6 and Proposition 4.3.4 respectively in [9]). As shown by Van der Hoeven and Moerdijk in [5], it is also possible to construct a topos of sheaves with this property.

A well known result in topos theory is that one can construct a category of internal \(G\)-sets from an internal group \(G\) and that this category is again a topos. See [7] Section V.6] for more details.

Note that we can construct the infinite dihedral group \(D_\infty\) in the internal logic of \(E\), to obtain an internal group \((D_\infty)_E\) in \(E\). We then apply the construction of internal \(G\)-sets to \((D_\infty)_E\) and refer to the resulting topos as \((D_\infty - \mathbf{Set})_E\).

Next we define \(\mathbb{Z}_\infty\) and its action internally in \(E\) to obtain an object of \((D_\infty - \mathbf{Set})_E\) and take our topos \(F\) to be the slice category \((D_\infty - \mathbf{Set})_E / \mathbb{Z}_\infty\). It is again well known that every slice category of a topos is again a topos (see [7] Section IV.7).

Finally, we note that our proof of lemma 6.2 is entirely constructive, so we can carry it out in the internal logic of \(E\). However, we can now remove the assumption that all functions \(\mathbb{N}^\mathbb{N} \to \mathbb{N}\) are continuous, since we chose \(E\) so that it holds in the internal logic.
This then gives us two objects $X$ and $Y$ in $(D_\infty \text{-Set})_{/\mathbb{Z}_\infty}$ such that $X \times 2 \cong Y \times 2$ but $X \not\cong Y$, as required. \hfill \Box

7. Strengthenings of the Main Theorem

We now consider two slightly stronger versions of the main theorem. There are two issues that we address.

The first is that one might expect that it becomes possible to construct the bijection $X \cong Y$ if we add the extra requirement that $X$ and $Y$ have decidable equality. We check that in fact decidable equality of $X$ and $Y$ already holds in the topos we have constructed, and so it does not help.

**Proposition 7.1.** The objects $X$ and $Y$ considered in theorem 6.3 have decidable equality.

**Proof.** Note that since $\mathbb{Z}$ has decidable equality (provably in constructive mathematics), we can show internally in $\mathcal{E}$ that there is a decision function $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_\infty \times 2$ and similarly for $Y$. Since equality is preserved by the action of a group, the decision functions are equivariant, and so witness the decidable equality of $X$ and $Y$ in $(D_\infty \text{-Set})_{/\mathbb{Z}_\infty}$. \hfill \Box

**Corollary 7.2.** There is topos $\mathcal{F}$ containing objects $X$ and $Y$ with decidable equality such that $X \times 2 \cong Y \times 2$ but $X \not\cong Y$.

The next issue is a little subtle. Essentially, it might happen that even though there is no isomorphism $X \cong Y$ in the topos, the statement “$X$ and $Y$ are isomorphic” still holds in the internal logic of the topos. One way of looking at this is that we can construct the collection of bijections between $X$ and $Y$ in the internal logic of the topos, to give an object $\text{Iso}(X,Y)$. External isomorphisms then correspond to global sections of $\text{Iso}(X,Y)$, i.e., to maps $1 \rightarrow \text{Iso}(X,Y)$. Meanwhile, the internal truth of the statement “$X$ and $Y$ are isomorphic” corresponds to the unique map $\text{Iso}(X,Y) \rightarrow 1$ being an epimorphism, which is weaker. In fact in our topos $\mathcal{F}$, exactly this happens. In this case $\text{Iso}(X,Y)$ consists of all bijections from $X$ to $Y$, where the isomorphisms are precisely the equivariant elements.

To deal with this, we will show in the next lemma how to construct a new topos where $\text{Iso}(X,Y) \rightarrow 1$ is not an epimorphism.

**Lemma 7.3.** Suppose that we are given a topos $\mathcal{F}$ with objects $X$ and $Y$, and isomorphism $X \times 2 \cong Y \times 2$ such that there exists no (external) isomorphism $X \cong Y$. Then there is a topos $\mathcal{F}'$ with objects $X'$ and $Y'$ and an isomorphism $X' \cong Y'$ such that the internal logic of $\mathcal{F}'$ does not satisfy the statement “there exists a bijection from $X$ to $Y$”.

**Proof.** We take $\mathcal{F}'$ to be the Sierpiński cone of $\mathcal{F}$, which we recall is defined to be the comma category $(\text{Set} \downarrow \Gamma)$, where $\Gamma : \mathcal{F} \rightarrow \text{Set}$ is the global sections functor, $\mathcal{F}(1,-)$ (see e.g. [6, Example A2.1.12]).

We define $X'$ to be the unique map $0 \rightarrow \Gamma X$, and $Y'$ to be the unique map $0 \rightarrow \Gamma Y$. In $(\text{Set} \downarrow \Gamma)$, limits and colimits are computed levelwise. Hence $2$ is given by the canonical map $2 \rightarrow \Gamma 2$, and $X' \times 2$ is the canonical map $0 \times 2 \rightarrow \Gamma (X \times 2)$. Since $0 \times 2 \cong 0$, this means $X' \times 2$ is the unique map $0 \rightarrow \Gamma (X \times 2)$. Similarly, $Y' \times 2$ must be the unique map $0 \rightarrow \Gamma (Y \times 2)$. We can now clearly see that the isomorphism $X \times 2 \cong Y \times 2$ lifts to an isomorphism $X' \times 2 \cong Y' \times 2$.

Now $\text{Iso}(X',Y')$ has to be of the form $\text{Iso}(X',Y')_0 \rightarrow \Gamma(\text{Iso}(X',Y'))_1$. Since the projection from $(\text{Set} \downarrow \Gamma)$ to $\mathcal{F}$ is logical, we have that $\text{Iso}(X',Y')_0 \cong \text{Iso}(X,Y)$. So $\text{Iso}(X',Y')$ is of the form $\text{Iso}(X',Y')_0 \rightarrow \Gamma(\text{Iso}(X,Y))$. 
By assumption there are no isomorphisms from $X$ to $Y$. Hence $\text{Iso}(X,Y)$ has no global sections, which precisely says that $\Gamma(\text{Iso}(X,Y))$ is the empty set. Hence $\text{Iso}(X',Y')_0$ must also be empty.

Finally, we note that the statement “there exists an isomorphism from $X'$ to $Y'$” holds in the internal logic if and only if the unique map $\text{Iso}(X',Y') \to 1$ is an epimorphism. This is the case precisely when both of the maps $\text{Iso}(X',Y')_0 \to 1$ and $\text{Iso}(X',Y')_1 \to 1$ are epimorphisms. Although the latter might be epi (which is exactly why we need this lemma), the former is certainly not, since it is a function from the empty set to $1$.

□

Lemma 7.4. Suppose that $X$ and $Y$ have decidable equality in a topos $F$. Then $X'$ and $Y'$ have decidable equality in the topos $F'$ constructed in lemma 7.3.

Proof. It is straightforward to show that the decision morphism $X \times X \to 2$ lifts to a morphism $X' \times X' \to 2$ witnessing that $X'$ has decidable equality, and similarly for $Y'$.

□

Corollary 7.5. There is a topos $F'$ containing objects $X$ and $Y$, and an isomorphism $X \times 2 \cong Y \times 2$ such that the statement “there exists a bijection from $X$ to $Y$” does not hold in the internal logic of $F'$.

Furthermore $X$ and $Y$ have decidable equality.

Proof. Apply lemma 7.3 to theorem 6.3. For decidable equality, apply lemma 7.4 and proposition 7.1.

□

Corollary 7.6. It is not provable in constructive mathematics that if $X \times 2 \cong Y \times 2$ then $X \cong Y$, even if we require $X$ and $Y$ to have decidable equality.

Proof. If there was a constructive proof of this statement, then it would hold in the internal logic of any topos, contradicting corollary 7.5. □

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