Distributed quantum metrology with a single squeezed-vacuum source

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Introduction and motivations. Quantum metrology aims at harnessing inherently quantum features such as entanglement, multiphoton interference, and squeezing, to develop novel quantum enhanced technologies for sensing and imaging beyond any classical capabilities [1–8]. More recently, a great deal of attention has been devoted to distributed quantum metrology, particularly on the problem of measuring a linear combination of several unknown phase shifts distributed over a linear optical network [9–14]. More explicitly, we will be interested in measuring a linear combination of \( M > 1 \) unknown distributed phases. This problem is of interest in a variety of settings: from the mapping of inhomogeneous magnetic fields [15–19], phase imaging [20–25], and quantum-enhanced nanoscale nuclear magnetic resonance imaging [9,26,27], to applications in precision clocks [28], geodesy, and geophysics [29–31].

A novel scheme was recently proposed to tackle distributed quantum metrology with Heisenberg-limited sensitivity [32]. However, its main limitation is the fact that it relies on two auxiliary interferometric channels up to the number \( M \) of unknown distributed phases, and photon-number-resolving detectors. Therefore, devising measurement schemes which can exhibit supersensitivity while making use of probe states which are simple to produce in the laboratory with current technology is a matter of great interest.

We overcome these limitations by introducing an interferometric scheme (Fig. 1) which employs only a single squeezed source and on-off photodetectors. Indeed, squeezed states of light are a natural candidate for Heisenberg-limited probing [33–35], on account of their experimental availability with a high mean photon number and their nonclassical character. While such states have been largely used to yield superstability in the estimation of a single unknown parameter, their quantum metrological advantage in the case of multiple distributed parameters has not yet been fully explored [13,14]. Here, we demonstrate how a simple \( M \)-channel linear optical interferometer with only a single squeezed-vacuum source and on-off photodetectors can achieve Heisenberg-limited sensitivity in distributed quantum metrology with \( M \) unknown phase delays. Remarkably, such a scheme can be implemented experimentally with present quantum optical technologies.

The optical interferometer. We describe here in details an interferometric setup (Fig. 1) able to estimate the combination

\[
\bar{\psi} = \sum_{j=1}^{M} w_j \psi_j, \tag{1}
\]

of \( M \) unknown phases \( \psi_j (j = 1, \ldots, M) \) for any given set of non-negative weights \( \{w_j\}_{j=1}^{M} \) [36]. Without loss of generality we will assume in the following the normalization \( \sum_j w_j = 1 \), so that the \( w_j \)’s are probability weights. The general situation will differ just by an immaterial factor.

The probe light at the input of our interferometer is prepared in the squeezed-vacuum state

\[
|\Psi_m\rangle = \bar{S}_1(z) |\Omega\rangle, \tag{2}
\]

where \( |\Omega\rangle = |0\rangle_1 \cdots |0\rangle_M \) is the vacuum state, \( \bar{S}_1(z) = e^{\frac{1}{2}(\alpha_1^2 - \alpha_2^2)} \) is the squeezing operator, \( \alpha_1 \) is the photonic annihilation operator of the first mode, and \( z \) is the squeezing parameter. The squeezing parameter \( z \) fixes the mean number...
the phase delays� are observable.

of input photons $\langle \tilde{N} \rangle = \langle \Psi_{\text{in}} | \tilde{N} | \Psi_{\text{in}} \rangle$, with $\tilde{N} = \sum_j \hat{a}_j^\dagger \hat{a}_j$, by the relation $\langle \tilde{N} \rangle = \bar{N}$, where

$$\bar{N} = \sinh^2(|z|).$$

The probe travels through the first linear optical transformation, described by the unitary operator $\hat{U}$ through the equation

$$\hat{U}^\dagger \hat{a}_i \hat{U} = \sum_{j=1}^{M} \mathcal{U}_{ij} \hat{a}_j,$$

where $\mathcal{U}$ is an $M \times M$ unitary matrix associated with the transition amplitudes from the channel $j$ to the channel $i$, with $i, j = 1, \ldots, M$. We set these amplitudes to

$$\mathcal{U}_{ij} = \sqrt{w_j},$$

where $j = 1, \ldots, M$ and the $w_j$‘s are the weights of Eq. (1). This can be always achieved with an appropriate combination of beam splitters [37]. More importantly, this step enables the estimation of $\varphi$ by making the output measurement explicitly dependent on it [see Eq. (12) later on], as well as creating useful entanglement [32], distributed across all channels containing the phase delays $\varphi_1, \ldots, \varphi_M$.

After the linear optical transformation $\hat{U}$ the probe undergoes phase shifts $\varphi_1, \ldots, \varphi_M$ through the respective channels, and finally evolves through the inverse linear optical transformation $\hat{U}^\dagger$. Reversing the linear optical transformation will allow us to effectively project the output state onto the input state (see below). Thus, given the generator of the phase shifts,

$$\hat{G} = \sum_{j=1}^{M} \varphi_j \hat{a}_j^\dagger \hat{a}_j,$$

the state at the output of our interferometer is

$$|\Psi_{\text{out}}\rangle = \hat{U}^\dagger e^{-i\hat{G}} \hat{U} |\Psi_{\text{in}}\rangle.$$

Heisenberg-limited estimation. We now demonstrate Heisenberg-limited sensitivity [in Eq. (16)] by means of the observable

$$\hat{O} = |\Psi_{\text{in}}\rangle \langle \Psi_{\text{in}}|,$$

associated with the projection of the output state over the input state, i.e., to the probability that the probe leaves the interferometer with its state unaltered. Since the expectation value of $\hat{O}$ is

$$\langle \hat{O} \rangle_{\text{out}} = \langle \Psi_{\text{out}} | \hat{O} | \Psi_{\text{out}} \rangle = |\langle \Psi_{\text{in}} | \hat{U}^\dagger e^{-i\hat{G}} \hat{U} |\Psi_{\text{in}}\rangle|^2$$

$$= |\langle \Omega | \hat{S}_i(z) |\Psi_{\text{out}}\rangle|^2,$$

(9)

the measurement of $\hat{O}$ is equivalent to projecting onto the vacuum $|\Omega\rangle$ after the action of an antisqueezing operation on the first channel, described by $\hat{S}_i(z)$. This can be experimentally achieved, for instance, by retroreflecting the down-converted photons onto the crystal generating the original squeezed light [38–42], and then using on-off photodetectors.

Since $\langle \hat{O} \rangle_{\text{out}} = |\langle \Psi_{\text{in}} | \Psi_{\text{out}} \rangle|^2$ is the probability of the output state to coincide with the input state, if the phases are small, the total interferometric operator should be close to the identity, and therefore $\langle \hat{O} \rangle_{\text{out}}$ should be close to one. More precisely, since $-|\varphi_{\text{max}} N| \leq \tilde{G} \leq |\varphi_{\text{max}} N|$, with $|\varphi_{\text{max}}| = \max_i |\varphi_i|$, and the interferometer preserves the total number of photons, if

$$|\varphi_{\text{max}} \langle \tilde{N} \rangle | \ll 1,$$

(10)

we can perform an expansion of $\langle \hat{O} \rangle_{\text{out}}$ in powers of $\tilde{G}$.

By using the notation $(\hat{G}^m)_{ij}$ for the expectation value of the operator $\hat{G}^m$ with $m = 1, 2$ taken at the state $|\Psi_U\rangle = \hat{U} |\Psi_{\text{in}}\rangle$, and $\Delta \hat{G}_U^2 = (\hat{G}_U^2)_{ij} - (\hat{G}_U^2)_{ij}$, for the variance of $\tilde{G}$, we obtain

$$\langle \hat{O} \rangle_{\text{out}} = |\langle \Psi_{\text{in}} | \hat{U}^\dagger e^{-i\hat{G}} \hat{U} |\Psi_{\text{in}}\rangle|^2 \simeq \left| (1 - i\tilde{G} - \frac{1}{2} \hat{G}^2)_{ij} \right|^2$$

$$\simeq 1 - \Delta \hat{G}_U^2,$$

(11)

up to fourth-order terms (see the first section of the Supplemental Material [43]).

By using Eq. (6) and the canonical commutation relations (see the second section of the Supplemental Material [43]), we obtain that the exact expression for the variance of $\tilde{G}$ depends on $\varphi$ in Eq. (1), and on $\tilde{N}^2 = \sum j w_j \varphi_j^2$ as

$$\Delta \hat{G}_U^2 = \tilde{N}^2 (N^2) - (\tilde{N}^2) + (\varphi^2 - \tilde{N}) \langle \tilde{N} \rangle,$$

(12)

where $\langle \tilde{N}^2 \rangle = \langle \Psi_{\text{in}} | \tilde{N}^2 |\Psi_{\text{in}}\rangle$. The variance of $\tilde{G}$ is made of a contribution from number fluctuations and a contribution from the fluctuations of the phases $\varphi_i$ with respect to the weights $w_j$.

This result is valid for any (not necessarily Gaussian) $M$-boson state $|\Psi_{\text{in}}\rangle$ with all modes but the first in the vacuum. In our case, since the first mode is in a squeezed-vacuum state, its photon-number statistics is super-Poissonian [44], with the mean photon number $\langle \tilde{N} \rangle = \tilde{N}$ given by (3) and a variance

$$\langle \tilde{N}^2 \rangle - \langle \tilde{N} \rangle^2 = 2\tilde{N}(\tilde{N} + 1),$$

(13)

which scales as $\tilde{N}^2$. This scaling is unlike a coherent state which has a Poissonian photon-number statistics with variance equal to the mean $\langle \tilde{N} \rangle$. As we will see, this is an essential ingredient for obtaining a Heisenberg-limited sensitivity.

For large $\tilde{N}$ one gets $\Delta \hat{G}_U^2 \simeq 2\tilde{N}^2 \tilde{\varphi}^2$, from which the expectation value of our observable (11) reads

$$\langle \hat{O} \rangle_{\text{out}} \simeq 1 - 2\tilde{N}^2 \tilde{\varphi}^2,$$

(14)
and differs from 1 by a small quantity, as expected. Indeed, we are in the regime of large $\bar{N}$ and small $\overline{\varphi}$, such that $|\overline{\varphi}|\bar{N} \ll 1$, in accordance with Eq. (10). The sensitivity in the estimation of $\overline{\varphi}$ is obtained by the error propagation formula [7]

$$\delta \overline{\varphi}^2 = \frac{\langle \hat{O}_{\text{out}}^2 \rangle - \langle \hat{O}_\text{out} \rangle^2}{\langle \hat{O}_\text{out} \rangle^2}. \tag{15}$$

By using the fact that $\hat{O} = \hat{O}^2$ is a projection, and by virtue of Eq. (14), we easily get

$$\delta \overline{\varphi}^2 \approx \frac{1}{8\bar{N}^2}, \tag{16}$$

i.e., the sensitivity scales at the Heisenberg limit.

**Example.** Let us consider the $M = 2$ case, i.e., we wish to estimate

$$\overline{\varphi} = w_1 \psi_1 + w_2 \psi_2, \tag{17}$$

in a two-mode interferometer, for assigned weights $w_1$, $w_2 \geq 0$, $w_1 + w_2 = 1$. One possible choice for $U$ which satisfies Eq. (5) is

$$U = \left( \frac{\sqrt{w_1}}{\sqrt{w_2}}, -\sqrt{\frac{w_2}{w_1}} \right). \tag{18}$$

This is just the matrix describing a beam splitter of reflectivity $R = w_1$ and transmittivity $T = w_2$, therefore the interferometric setup is simply that of a Mach-Zehnder interferometer (see Fig. 2). Remarkably, here both phases $\psi_1$ and $\psi_2$ are unknown, differently from previous proposals where only one parameter is unknown [1,45].

Even more interestingly, the scheme in Fig. 2 is sensitive to the sum, rather than the difference, of the phases $\psi_1$ and $\psi_2$ with positive weights $w_1$ and $w_2$, respectively. To see how this is possible, let us set $w_1 = w_2 = 1/2$ for simplicity, and let us consider the optical unitary transformation describing the balanced Mach-Zehnder,

$$\hat{U}_{\text{MZ}} = e^{\frac{i}{2} (\psi_1 - \psi_2) \hat{a}} e^{\frac{i}{2} (\psi_1 + \psi_2) \hat{a}^\dagger}, \tag{19}$$

where $\hat{J}_z = -\frac{1}{2} (\hat{a}^\dagger \hat{a}^\dagger - \hat{a} \hat{a})$ [2]. As we can see, the output state $|\Psi_{\text{out}}\rangle = \hat{U}_{\text{MZ}} |\Psi_{\text{in}}\rangle$ does depend, in general, on both $\psi_1$ and $\psi_2$, however, the information on the sum of the phases can be “washed out” by the choice of the measurement protocol. Indeed, if $|\Psi_{\text{in}}\rangle$ is an eigenstate of the number operator $\hat{N}$, $|\Psi_{\text{out}}\rangle$ depends only on the relative phase $\psi_1 - \psi_2$, because the second exponential in Eq. (19) gives rise to a global complex phase, and the information on $\overline{\varphi} = (\psi_1 + \psi_2)/2$ is completely lost. Furthermore, if one measures an observable $\hat{O}$ which commutes with $\hat{N}$, then $\langle \hat{O} \rangle_{\text{out}}$, as well as all the higher moments $\langle \hat{O}^k \rangle_{\text{out}}$ [2], will again depend on $\psi_1 - \psi_2$ only, since $\hat{U}_{\text{MZ}}^* \hat{O} \hat{U}_{\text{MZ}} = e^{-\frac{i}{2} (\psi_1 - \psi_2) \hat{J}_y} \hat{O} e^{\frac{i}{2} (\psi_1 - \psi_2) \hat{J}_y}$.

**Discussion.** We have shown how squeezed light can be used to estimate an arbitrary superposition of phases with non-negative weights. Our protocol can overcome the limitations of Ref. [32], most notably we can achieve the Heisenberg limit with a single squeezed state rather than two Fock states. Furthermore, our protocol does not necessitate the use of auxiliary channels nor photon-number-resolving detectors. The interferometric setup is easily realizable for any set of weights by using only beam splitters and phase shifters [37]. The initially separable input state acquires entanglement across the various channels where the phase shifts are distributed owing to the linear optical network. The squeezed source can be produced in a number of ways, including spontaneous parametric down-conversion and four-wave mixing. Antisqueezing has already been achieved with high efficiency [38–42] by retroreflecting the down-converted photons and the pump back onto the crystal. We would like to mention that it is also possible to have, instead of the antisqueezer $S(z) = S(-z)$, an output squeezer $S(z')$ where $|z'| \neq |z|$, but $z$ and $z'$ have opposite complex phases. Analysis of this sort of interferometer protocol is beyond the scope of this Rapid Communication. Remarkably, given the parameter of the first squeezer, increasing the parameter of the second one can compensate for detection losses [33]. A final comment is in order. We have shown that by a simply implementable setup, with a single-mode squeezed state and on-off detectors, one can attain Heisenberg-limited sensitivity $\delta \overline{\varphi} \propto 1/\bar{N}$ in the presence of an arbitrary number of unknown phases. A detailed analysis of the quantum Fisher information matrix and of the quantum Cramér-Rao bound—that will be deferred to a more technical publication in order not to obscure the main point of the work—can reveal what is the optimal prefactor and whether it is attained already by our simple setup.

In conclusion, our protocol can achieve Heisenberg-limited sensitivity for distributed quantum metrology while being well within the realm of current quantum optical technologies.

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[1] C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D 23, 1693 (1981).
[2] B. Yurke, S. L. McCall, and J. R. Klauder, SU(2) and SU(1,1) interferometers, Phys. Rev. A 33, 4033 (1986).
[3] M. J. Holland and K. Burnett, Interferometric Detection of Optical Phase Shifts at the Heisenberg Limit, Phys. Rev. Lett. 71, 1355 (1993).

[4] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science 306, 1330 (2004).

[5] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum Metrology, Phys. Rev. Lett. 96, 010401 (2006).

[6] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photon. 5, 222 (2011).

[7] L. Pezzè and A. Smerzi, Quantum theory of phase estimation, in Atom Interferometry, Proceedings of the International School of Physics “Enrico Fermi,” Course 188, Varenna, 2013, edited by G. M. Tino and M. A. Kasevich (IOS Press, Amsterdam, 2014), p. 691.

[8] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kolodyński, Quantum limits in optical interferometry, Prog. Opt. 60, 345 (2015).

[9] Z. Eldredge, M. Foss-Feig, J. A. Gross, S. L. Rolston, and A. V. Gorshkov, Optimal and secure measurement protocols for quantum sensor networks, Phys. Rev. A 97, 042337 (2018).

[10] T. J. Proctor, P. A. Knott, and J. A. Dunningham, Multiparameter Estimation in Networked Quantum Sensors, Phys. Rev. Lett. 120, 080501 (2018).

[11] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, Generalized Limits for Single-Parameter Quantum Estimation, Phys. Rev. Lett. 98, 090401 (2007).

[12] M. D. Lang and C. M. Caves, Optimal Quantum-Enhanced Interferometry Using a Laser Power Source, Phys. Rev. Lett. 111, 173601 (2013).

[13] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Distributed quantum sensing using continuous-variable multipartite entanglement, Phys. Rev. A 97, 032329 (2018).

[14] X. Guo, C. R. Breum, J. Borregaard, S. Izumi, M. V. Larsen, T. Gehring, M. Christandl, J. S. Neergaard-Nielsen, and U. L. Andersen, Distributed quantum sensing in a continuous variable entangled network, Phys. Rev. A, 042337 (2018).

[15] F. D. Stacey, The seismomagnetic effect, Pure Appl. Geophys. 70, 33ND (2005).

[16] T. J. Wright, B. E. Parsons, and Z. Lu, Toward mapping surface deformation in three dimensions using InSAR, Geophys. Res. Lett. 31, L01607 (2004).

[17] F. D. Stacey, The seismomagnetic effect, Pure Appl. Geophys. 58, 5 (1964).

[18] W. Ge, K. Jacobs, Z. Eldredge, A. V. Gorshkov, and M. Foss-Feig, Distributed Quantum Metrology with Linear Networks and Separable Inputs, Phys. Rev. Lett. 121, 043604 (2018).

[19] M. Manceau, F. Khalili, and M. Chekhova, Improving the phase super-sensitivity of squeezing-assisted interferometers by squeeze factor unbalancing, New J. Phys. 19, 013014 (2017).

[20] L. Maccone and A. Riccardi, Squeezing metrology, arXiv:1901.07482.

[21] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit, Phys. Rev. Lett. 104, 103602 (2010).

[22] This sign restriction could possibly be lifted by occupying more than a single input channel, but this is out of the scope of this Rapid Communication.

[23] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Experimental Realization of Any Discrete Unitary Operator, Phys. Rev. Lett. 73, 58 (1994).
[38] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Nonlinear Optics with Less Than One Photon, Phys. Rev. Lett. 87, 123603 (2001).
[39] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Quantum State Preparation and Conditional Coherence, Phys. Rev. Lett. 88, 113601 (2002).
[40] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Conditional-Phase Switch at the Single-Photon Level, Phys. Rev. Lett. 89, 037904 (2002).
[41] J. S. Lundeen and A. M. Steinberg, Experimental Joint Weak Measurement on a Photon Pair as a Probe of Hardy’s Paradox, Phys. Rev. Lett. 102, 020404 (2009).
[42] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, The elusive Heisenberg limit in quantum-enhanced metrology, Nat. Commun. 3, 1063 (2012).
[43] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.1.032024 for the details concerning the series expansion of $\hat{O}$ and variance of $\hat{G}$.
[44] M. C. Teich and B. E. A. Saleh, Squeezed states of light, Quantum Opt. 1, 153 (1989).
[45] M. Takeoka, K. P. Seshadreesan, C. You, S. Izumi, and J. P. Dowling, Fundamental precision limit of a Mach-Zehnder interferometric sensor when one of the inputs is the vacuum, Phys. Rev. A 96, 052118 (2017).