LOGISTIC MODELS WITH TIME-DEPENDENT COEFFICIENTS AND SOME OF THEIR APPLICATIONS

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Dedicated to Doron Zeilberger on his 60th birthday.

Abstract. We discuss explicit solutions of the logistic model with variable parameters. Classical data on the sunflower seeds growth are revisited as a simple application of the logistic model with periodic coefficients. Some applications to related biological systems are briefly reviewed.

1. Introduction

We consider explicit solutions of logistic equations with time-dependent parameters. Particular attention is paid to the case of periodic coefficients which is especially important for many biological problems due to a natural periodicity of the Earth rotations. Models of this type can be used to mimic a population’s response to seasonal fluctuations in its environment or to mimic a population with several discrete life-cycle stages [5], [7], [8], [22]. As an example, sunflower plant growth data from the classical paper [21] are reviewed and analyzed from the viewpoint of the day-night periodicity. We dedicate this paper to Professor Doron Zeilberger on his 60th birthday — his nontraditional way of thinking has motivated one of the authors (S. K. S.) to review this classical topic from a modern perspective.

2. Solution of the Logistic Model with Time-Dependent Coefficients

The nonlinear equation for the logistic, or Verhulst’s, model [6], [7], [8], [24], [25], [26], [27]:

\[
\frac{dx}{dt} = \alpha(t) x \left(1 - \frac{x}{\beta(t)}\right)
\]

(2.1)

with time-dependent coefficients \(\alpha(t)\) and \(\beta(t)\) can be reduced to the linear equation

\[
z' + \alpha(t) z = \frac{\alpha(t)}{\beta(t)}
\]

(2.2)

by the standard substitution \(z = 1/x\). Integration of (2.2) with the help of an integrating factor,

\[
\frac{d}{dt} \left(z \exp \left(\int_{t_0}^t \alpha(s) \, ds\right)\right) = \frac{\alpha(t)}{\beta(t)} \exp \left(\int_{t_0}^t \alpha(s) \, ds\right),
\]

(2.3)
results in the explicit solution of the logistic model (in terms of a weighted harmonic mean):

\[
\frac{1}{x(t)} = e^{-\int_{t_0}^{t} \alpha(s) \, ds} \left( \int_{t_0}^{t} \frac{\alpha(s)}{\beta(s)} e^{\int_{t_0}^{s} \alpha(v) \, dv} \, ds + \frac{1}{x(t_0)} \right)
\]  

(2.4)
corresponding to the initial condition \(x(t_0) = x_0\). A traditional expression is given by

\[
x(t) = \frac{x_0 \exp \left( \int_{t_0}^{t} \alpha(s) \, ds \right)}{1 + x_0 \int_{t_0}^{t} \alpha(s) \exp \left( \int_{t_0}^{s} \alpha(v) \, dv \right) \, ds}
\]  

(2.5)
(see also [5], [6], [7], [8], [12], [13], [14], [23] and references therein). Here, we would like to consider the most general functions \(\alpha(t)\) and \(\beta(t)\) such that all integrals exist and

\[
\frac{d}{dt} \int_{t_0}^{t} f(s) \, ds = f(t)
\]  

(2.6)
by a corresponding version of the fundamental theorem of calculus (see, for example, [11], [18]).

The well-known solution with constant parameters \(\alpha\) and \(\beta\) can be written as

\[
\frac{1}{x(t)} = \frac{1}{\beta} + \left( \frac{1}{x(t_0)} - \frac{1}{\beta} \right) e^{-\alpha(t-t_0)},
\]  

(2.7)
or

\[
x(t) = \frac{\beta x(t_0)}{x(t_0) + (\beta - x(t_0)) e^{-\alpha(t-t_0)}},
\]  

(2.8)
where \(\beta = \lim_{t \to \infty} x(t)\) is the carrying capacity.

3. Logistic Models with Periodic Coefficients

If variable coefficients \(\alpha(t)\) and \(\beta(t)\) in the logistic equation (2.1) are periodic functions of time, namely,

\[
\alpha(t + T) = \alpha(t), \quad \beta(t + T) = \beta(t),
\]  

(3.1)
an important question arises about the basic dynamics and long-term behavior of a biological system under consideration. One may approach this matter in the following mathematical setting. If the period is \(T = t_1 - t_0\), the solution (2.4) reads

\[
\frac{1}{x(t_1)} = a + \frac{b}{x(t_0)},
\]  

(3.2)
where parameters are given in terms of integrals over the period:

\[
a = b \int_{t_0}^{t_1} \frac{\alpha(t)}{\beta(t)} \exp \left( \int_{t_0}^{t} \alpha(s) \, ds \right) \, dt, \quad b = \exp \left( - \int_{t_0}^{t_1} \alpha(t) \, dt \right).
\]  

(3.3)
Repeating this process indefinitely, we arrive at the following discrete map

\[
\frac{1}{x_{n+1}} = a + \frac{b}{x_n},
\]  

(3.4)
where by the definition \(x(t_n) = x_n\). This recurrence relation can be rewritten as follows

\[
x_{n+1} = x_n \left( 1 - b \right) x_{n+1} \left( 1 - \frac{a}{1 - b} x_n \right),
\]  

(3.5)
which can be thought of as a difference analog of Verhulst’s model (2.1) with constant parameters. We refer to (3.4) as a discrete logistic map (with a slight change of parameters it is also called the Beverton–Holt difference equation [4], [6], [23]).

But the first order nonhomogeneous difference equation of the form
\[ z_{n+1} = a + bz_n \]  
with constant coefficients \( a \) and \( b \) has the following explicit solutions [6], [7]:
\[ z_n = \frac{a}{1 - b} + b^n z_0, \quad \text{if} \quad b \neq 1, \]  
and
\[ z_n = an + z_0, \quad \text{if} \quad b = 1, \]  
which can be verified by a direct substitution. Therefore the discrete logistic map solution has the form
\[ \frac{1}{x_n} = \frac{a}{1 - b} + \left( \frac{1}{x_0} - \frac{a}{1 - b} \right) b^n \]  
for any initial data \( x_0 \). If \( b < 1 \), this sequence converges to the limit, say \( x_\infty := \lim_{n \to \infty} x_n \), given by
\[ \frac{1}{x_\infty} = \frac{a}{1 - b} \]  
and our solution (3.9) takes a more convenient form
\[ \frac{1}{x_n} = \frac{1}{x_\infty} + \left( \frac{1}{x_0} - \frac{1}{x_\infty} \right) b^n. \]  
Then
\[ \frac{x_n - x_\infty}{x_n} = \frac{x_0 - x_\infty}{x_0} b^n, \]  
which implies that \( x_n > x_\infty \) for all \( n \), if \( x_0 > x_\infty \) and vice versa. Finally, from (3.11) one gets
\[ -\frac{d}{dn} \left( \frac{1}{x_n} \right) = \frac{1}{x_n^2} \frac{dx_n}{dn} = \frac{x_0 - x_\infty}{x_0 x_\infty} b^n \ln b, \]  
which reveals monotonicity properties of this bounded sequence in general. Namely, our sequence is strictly increasing, when \( x_0 < x_\infty \), and strictly decreasing otherwise.

Traditionally, a special question which arises in the theory of equations with periodic coefficients is the existence of periodic solutions [2], [17]. In the case of the logistic model under consideration, when an explicit solution is available, one can only obtain a continuous periodic solution by letting \( x_1 = x_0 \) in our equation (3.2). Thus
\[ \frac{1}{x_0} = a + \frac{b}{x_0}, \]  
which corresponds to the special initial condition given by the expression (3.10). According to (3.11) the simple stability property holds, if \( b < 1 \), solutions with all other initial data converge to this periodic solution, or an attractor, as time goes to infinity with precise monotonicity properties described above. By (3.11)
\[ \frac{1}{x_n} - \frac{1}{y_n} = \left( \frac{1}{x_0} - \frac{1}{y_0} \right) b^n, \]  
where
which reveals the contracting map property for two different initial conditions $x_0 \neq y_0$ (see also [7] and [8]).

4. Two-Stage Compound Logistic Models

In general, splitting a time interval $[t_0, t_2]$ as the union of two successive subintervals, say $[t_0, t_2] = [t_0, t_1] \cup [t_1, t_2]$, one can write

$$\frac{1}{x(t_1)} = a_0 + \frac{b_0}{x(t_0)}, \quad \frac{1}{x(t_2)} = a_1 + \frac{b_1}{x(t_1)},$$  \hspace{1cm} (4.1)

where the corresponding coefficients are given by our general expressions (3.3). Then

$$\frac{1}{x(t_2)} = a + \frac{b}{x(t_0)},$$  \hspace{1cm} (4.2)

where the ‘compound’ coefficients are given by the following analog of the addition formula

$$a = a_1 + a_0 b_1, \quad b = b_0 b_1,$$  \hspace{1cm} (4.3)

which can also be derived directly from (3.3). It is worth noting the following group property

$$\left(1, \frac{1}{x(t_0)}\right) \left(1, \frac{1}{x(t_1)}\right) = \left(1, \frac{1}{x(t_2)}\right), \quad \left(1, \frac{1}{x(t_1)}\right) \left(1, \frac{1}{x(t_1)}\right) = \left(1, \frac{1}{x(t_2)}\right)$$  \hspace{1cm} (4.4)

and

$$\left(1, \frac{1}{x(t_0)}\right) \left(1, \frac{1}{x(t_1)}\right) = \left(1, \frac{1}{x(t_2)}\right).$$  \hspace{1cm} (4.5)

(Another applications of the group of upper triangular matrices can be found in Refs. [16], [19] and [28].)

With a slight change of notations, consider two step functions on the interval $[0, t_1] :

$$\alpha(t) = \begin{cases} 
\alpha_0, & \text{if } 0 \leq t < t_0 \\
\alpha_1, & \text{if } t_0 \leq t < t_1
\end{cases} \quad \beta(t) = \begin{cases}
\beta_0, & \text{if } 0 \leq t < t_0 \\
\beta_1, & \text{if } t_0 \leq t < t_1
\end{cases}$$  \hspace{1cm} (4.6)

and then define them by ‘periodicity’ for all $t \geq 0$ with the period $T = t_1$. We shall refer to this case as a two-stage compound logistic model (see also [12] for a special case). Evaluation of the integrals in (3.3) gives

$$a = \frac{1 - \tau_1}{\beta_1} + \frac{(1 - \tau_0) \tau_1}{\beta_0}, \quad b = \tau_0 \tau_1,$$  \hspace{1cm} (4.7)

where by definition

$$\tau_0 = e^{-\alpha_0 t_0}, \quad \tau_1 = e^{-\alpha_1 (t_1 - t_0)}.$$  \hspace{1cm} (4.8)

As a result

$$x_\infty = \lim_{n \to \infty} x_n = \frac{\beta_0 \beta_1 (1 - \tau_0 \tau_1)}{\beta_0 + \beta_1 \tau_1 - (\beta_0 + \beta_1 \tau_0) \tau_1}$$  \hspace{1cm} (4.9)

provided $\tau_0 \tau_1 < 1$.

Two special cases are of particular interest. If $\alpha_1 = 0$, then $\tau_1 = 1$ and

$$a = \frac{1 - \tau_0}{\beta_0}, \quad b = \tau_0 = e^{-\alpha_0 t_0}, \quad x_\infty = \beta_0.$$  \hspace{1cm} (4.10)

This will be useful in section 8. Another limit, $\beta_1 \to \infty$, corresponds to a resetting mechanism discussed in [13] (see also [23]).
5. **n-Stage Compound Logistic Model**

Results from the previous section can be easily generalized in the following fashion. Consider a partition $t_0 = s_0 < s_1 < ... < s_n = t_1$ of a time interval $[t_0, t_1] = \bigcup_{i=0}^{n-1} [s_i, s_{i+1}]$. Then with the help of (4.3)–(4.5):

$$\frac{1}{x(s_n)} = a + \frac{b}{x(s_0)},$$  \hspace{1cm} (5.1)

where

$$a = \sum_{i=0}^{n-1} a_i \prod_{k=i+1}^{n-1} b_k, \quad b = \prod_{i=0}^{n-1} b_i,$$  \hspace{1cm} (5.2)

or in the matrix form

$$\prod_{i=0}^{n-1} \begin{pmatrix} 1 & a_i \\ 0 & b_i \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & b_0 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & a_{n-1} \\ 0 & b_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sum_{i=0}^{n-1} a_i \prod_{k=i+1}^{n-1} b_k \\ 0 & \prod_{i=0}^{n-1} b_i \end{pmatrix}$$  \hspace{1cm} (5.3)

by induction. It is worth noting that this multiplication property holds for any numbers $\{a_i\}_{i=0}^{n-1}$ and $\{b_i\}_{i=0}^{n-1}$, thus generating solution of the following difference equation

$$\frac{1}{x_{i+1}} = a_i + \frac{b_i}{x_i},$$  \hspace{1cm} (5.4)

which is obtained here with the help of the group property of upper triangular matrices (see also [7] and [23]). In this paper, we take $x_i = x(s_i)$ and use definitions (3.3), namely,

$$a_i = b_i \int_{s_i}^{s_{i+1}} \alpha(t) \exp \left( \int_{s_i}^{t} \alpha(s) \, ds \right) \, dt, \quad b_i = \exp \left( - \int_{s_i}^{s_{i+1}} \alpha(t) \, dt \right)$$  \hspace{1cm} (5.5)

for the corresponding subintervals $[s_i, s_{i+1}]$ with $i = 0, 1, ..., n-1$.

For step functions $\alpha(t) = \alpha_i$ and $\beta(t) = \beta_i$ on subintervals $s_i \leq t < s_{i+1}$ one gets

$$a_i = \frac{1 - \tau_i}{\beta_i}, \quad b_i = \tau_i = e^{-\alpha_i(s_{i+1}-s_i)} \quad (i = 0, 1, ..., n-1)$$  \hspace{1cm} (5.6)

and

$$a = \sum_{i=0}^{n-1} \frac{1 - \tau_i}{\beta_i} \prod_{k=i+1}^{n-1} \tau_k, \quad b = \prod_{i=0}^{n-1} \tau_i.$$  \hspace{1cm} (5.7)

Further details are left to the reader.

6. **An Extension**

The following generalization of the logistic model:

$$\frac{dx}{dt} = \alpha(t) x \left( 1 - \frac{x^\theta}{\beta(t)} \right), \quad \theta = \text{constant}$$  \hspace{1cm} (6.1)
has been considered in Refs. [1], [3] and [6]. This equation is integrable by the substitution \( z = 1/x^\theta \) and the corresponding extension of (2.4) takes the form

\[
\frac{1}{x^\theta(t)} = e^{-\theta \int_{t_0}^{t} \alpha(s) \, ds} \left( \theta \int_{t_0}^{t} \frac{\alpha(s)}{\beta(s)} e^{\theta \int_{t_0}^{s} \alpha(v) \, dv} \, ds + \frac{1}{x^\theta(t_0)} \right)
\]  

(6.2)

with \( x \to x^\theta \) and \( \alpha \to \theta \alpha \). The case of periodic coefficients is analyzed in a similar fashion with the help of this substitution. Further details are left to the reader.

7. Logistic Models and Riccati’s Equation

Considering a more general model

\[
\frac{dx}{dt} = \alpha(t) x \left( 1 - \frac{x}{\beta(t)} \right) + \gamma(t),
\]  

(7.1)

where \( \alpha(t), \beta(t) \) and \( \gamma(t) \) are suitable real valued functions of time only, one obtains the so-called Riccati equation, which can also be reduced to a linear second order differential equation by a certain substitution (see, for example, Refs. [9] and [10] for more details). The case of periodic coefficients will be discussed elsewhere.

8. An Application: Logistic Growth of Sunflower Seeds Revisited

Logistic models are well-known in autocatalysis, population dynamics, mathematical epidemiology and theory of fishing (see, for example, [4], [6], [7], [8], [20], [23] and references therein). Among other biological applications, we utilize the periodic logistic models, described above in detail, to sunflower plant growth data from the classical paper [21], where the sunflower (helianthus) was chosen because of the fact that it grows without producing branches and it was thought that measurements of height and weight represented the growth of the entire organism with a fair degree of accuracy. Original experimental results are presented in the following table:

Sunflower height versus growing days by Reed and Holland (1919)

| days \( n \) | observed mean height \( x_n \) (cm) | constant \( K \) (this paper) | \( \ln \left( \frac{x_n}{x_\infty - x_n} \right) \) |
|---|---|---|---|
| 0 | 10.00 | 0.040593 | -3.1966 |
| 7 | 17.93 ± 0.14 | 0.04119 | -2.5798 |
| 14 | 36.36 ± 0.43 | 0.03852 | -1.7917 |
| 21 | 67.76 ± 0.78 | 0.033353 | -1.0137 |
| 28 | 98.10 ± 1.38 | 0.032672 | -0.46643 |
| 35 | 131.00 ± 1.73 | 0.032005 | 0.058956 |
| 42 | 169.50 ± 2.21 | 0.038430 | 0.6902 |
| 49 | 205.50 ± 2.92 | 0.042069 | 1.4336 |
| 56 | 228.30 ± 3.41 | 0.043129 | 2.1649 |
| 63 | 247.10 ± 3.80 | 0.052904 | 3.5083 |
| 70 | 250.50 ± 3.76 | 0.050188 | 4.1372 |
| 77 | 253.80 ± 3.99 | 0.059799 | 5.8932 |
| 84 | 254.50 ± 3.89 | not available | not available |
The following solution of the continuous Verhulst model, see (2.7)–(2.8),
\[ \log \frac{x_n}{254.5 - x_n} = K(n - t_1) \]  \hspace{1cm} (8.1)
with \( K = 0.0421 \) and \( t_1 = 34.2 \) was originally suggested in order to interpolate these data \[21\]. Here, the carrying capacity was chosen to be \( \beta = 254.5 \) and the time at which the growth has run halfway to equilibrium; that is the time at which \( x = \beta/2 \) and the maximum growth occurs; was estimated as \( t_1 = 34.2 \) days. It is worth noting that the average value of the constant \( K \), reevaluated here from our corrected version of the third column (without \( n = 0 \)), is given by \( K = 0.042205 \), which is slightly different from the value \( K = 0.0421 \) obtained in the original paper \[21\] (this average becomes \( K = 0.042071 \approx 0.0421 \) if our \( n = 0 \) data is included).

A simple assumption of the day-night periodicity of the plant growth results in our solution (3.11) of the discrete Verhulst map, which can be rewritten for the numerical analysis purposes in a more convenient form:
\[ \ln \left( \frac{x_n}{x_\infty - x_n} \right) = \ln \left( \frac{x_7}{x_\infty - x_7} \right) + (n - 7) \ln \left( \frac{1}{b} \right). \]  \hspace{1cm} (8.2)
Then we match the data available from the table on the seven day scale only from \( x_7 = 17.93 \) till \( x_84 = 254.50 \approx \infty \). Corresponding data from the last column have been first analyzed with the help of a least square linear curve fitting on the entire available time interval \[7, 77\]. As a result
\[ \ln \left( \frac{x_n}{254.50 - x_n} \right) = -3.612328764 + 0.112057018181818180 n \]  \hspace{1cm} (8.3)
\[ = 0.112057018181818180 (n - 32.237), \]
which gives the following modified values of parameters \( t_1 = 32.237 \) and \( K = 0.1120570181818180 / \ln 10 = 0.048666 \) in the original formula (8.1). Unfortunately, this formula does not match observed data better then the original expression in Ref. \[21\] (even when one uses a quadratic least square approximation for the last column; two (or several)-stage logistic models seem more appropriate). An elementary data analysis of the last column shows that the best linear approximation occurs during the time interval \[7, 49\], when by the least square method:
\[ \ln \left( \frac{x_n}{254.50 - x_n} \right) = -3.106504286 + 0.0922278367346938688 n \]  \hspace{1cm} (8.4)
\[ = 0.0922278367346938688 (n - 33.68293561) \]
with the best values of parameters \( t_1 = 33.68293561 \) and \( K = 0.04005404057 \) in the formula (8.1). Original data are approximated well by sigmoid shape function; the revised results are presented in the following graph (see Figure 1).

Finally, we would like to apply our periodic two-stage (day-night) compound logistic model to helianthus growth. One may speculate that the sunflower plant was growing about fourteen hours during an average California day from May to August of 1919 and was not growing at all during nights. Then by (4.10):
\[ b = \exp \left( -\frac{7}{12} \alpha_0 \right) = b_0^{7/12}, \quad b_0 = b^{12/7}. \]  \hspace{1cm} (8.5)
If the (sun)light would be provided for twenty four hours every day, say in a greenhouse, expression (8.1) should be modified as follows
\[ \log \frac{x_n}{254.5 - x_n} = K_0 (n - t_0) \]  \hspace{1cm} (8.6)
with \( K_0 = (12/7) K = 0.068664 \) and \( t_0 = (7/12) t_1 = 19.648 \). The corresponding graph is also presented on Figure 1.

![Graph](image)

**Figure 1.** Comparison of observed and calculated values for the mean height of helianthus.

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