A SURVEY TO NEVANLINNA-TYPE THEORY BASED ON HEAT DIFFUSIONS

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ABSTRACT. This is a survey to Nevanlinna-type theory of holomorphic mappings from a complete and stochastically complete Kähler manifold into compact complex manifolds with a positive line bundle. When some energy and Ricci curvature conditions are imposed, the Nevanlinna-type defect relations based on heat diffusions are obtained.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we mainly investigate the value distribution of holomorphic mappings from a complete and stochastically complete Kähler manifold into a compact complex manifold using Nevanlinna-type functions introduced by Atsuji [3]. In 2010, Atsuji [3] first introduced the notions of Nevanlinna-type functions \( \tilde{T}_x(t) \), \( \tilde{N}_x(t, a) \) and \( \tilde{m}_x(t, a) \) of meromorphic functions on a Kähler manifold based on heat diffusions. By using a technique of Brownian motion (see [1, 2, 3, 4, 6, 7]), Atsuji proved an analogy of the Second Main Theorem in Nevanlinna theory.

**Theorem 1.1** (Atsuji, [3]). Let \( f \) be a nonconstant meromorphic function on a complete and stochastically complete Kähler manifold \( M \), and \( a_1, \ldots, a_q \) be distinct in \( \mathbb{P}^1(\mathbb{C}) \). Assume that \( \tilde{T}_x(t) < \infty \) as \( 0 < t < \infty \) and \( \tilde{T}_x(t) \to \infty \) as \( t \to \infty \), and \( |\tilde{N}_x(t, \text{Ric})| < \infty \) as \( 0 < t < \infty \). Then

\[
\sum_{j=1}^{q} \tilde{m}_x(t, a_j) + \tilde{N}_1(t, x) \leq 2\tilde{T}_x(t) + 2\tilde{N}_x(t, \text{Ric}) + O(\log \tilde{T}_x(t)) + O(1)
\]

holds except for \( t \) in an exceptional set of finite length.

Furthermore, with certain energy and Ricci curvature conditions imposed, Atsuji proved some defect relations (see [3]). In this paper, we shall develop

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Atsuji’s technique based on heat diffusions. In doing so, first of all, we need to extend the notions of so-called Nevanlinna-type functions, see Section 2.1 in this paper. As a generalization of Theorem 1.1, we establish Theorem 1.3 below. Indeed, we develop some defect relations in Section 4.3.

Early in 1970s, Griffiths and his school (see [5, 11, 12]) made a significant progress in Nevanlinna theory. Carlson-Griffiths [5] established the Second Main Theorem and defect relations for holomorphic mappings from $\mathbb{C}^m$ into an algebraic variety $V$ intersecting simple divisors under dimension assumption $m = \dim_{\mathbb{C}} V$. The theory is referred to Griffiths’ equi-dimensional value distribution theory [12]. Later, Griffiths-King [11] generalized this theory to holomorphic mappings from complex affine algebraic varieties into $V$.

We consider an analogue of Griffiths’ equi-dimensional value distribution theory on stochastically complete Kähler manifolds based on heat diffusions. Our approach here is to combine the Logarithmic Derivative Lemma with a probabilistic method. So, the first task is to establish the Logarithmic Derivative Lemma for meromorphic functions on Kähler manifolds (see Theorem 1.2 below).

We state the main results of this paper.

**Theorem 1.2** (Logarithmic Derivative Lemma). Let $M$ be a complete and stochastically complete Kähler manifold. Let $\psi$ be a nonconstant meromorphic function on $M$ such that $\tilde{T}(t, \psi) < \infty$ as $0 < t < \infty$. Then for any $\delta > 0$, there exists a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure such that

$$\tilde{m}(t, \|\nabla M \psi\|) \leq (1 + \frac{1 + \delta}{2}) \log \tilde{T}(t, \psi) + O(1)$$

holds for $t \in (1, \infty)$ outside $E_\delta$.

**Theorem 1.3** (Second Main Theorem). Let $M$ be a complete and stochastically complete Kähler manifold. Let $L \to N$ be a positive line bundle over a compact complex manifold $N$, and $D \in |L|$ such that $D$ has only simple normal crossings. Assume that $f : M \to N$ $(\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} N)$ is a differentiably non-degenerate holomorphic mapping such that $\tilde{T}_f(t, L) < \infty$ and $|\tilde{T}(t, \mathcal{R}_M)| < \infty$ for $0 < t < \infty$. Then for any $\delta > 0$, we have

$$\tilde{T}_f(t, L) + \tilde{T}_f(t, K_N) + \tilde{T}(t, \mathcal{R}_M) \leq \tilde{N}_f(t, D) + O(\log \tilde{T}_f(t, L)) + O(1)$$

holds for $t \in (1, \infty)$ outside a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure.

**Theorem 1.4** (defect relation). Let $M$ be a complete and stochastically complete Kähler manifold of non-negative Ricci scalar curvature. Let $L \to N$ be a positive line bundle over a compact complex manifold $N$, and $D \in |L|$ such that $D$ has only simple normal crossings. Assume that $f : M \to N$
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(dim\(_C\) M ≥ dim\(_C\) N) is a differentiably non-degenerate holomorphic mapping satisfying

\[ \int_1^\infty e^{-\epsilon r^2} dr \int_{B_\epsilon(r)} e^{f^* c_1(L,h)}(x) dV(x) < \infty \]

for any \( \epsilon > 0 \). Then

\[ \tilde{\Theta}_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right]. \]

2. First Main Theorem

Let \( M \) be a \( m \)-dimensional complete Kähler manifold with Kähler metric form \( \alpha \). A Brownian motion in \( M \) (as a Riemannian manifold) is a Markov process generated by \( \frac{1}{2} \Delta_M \) with transition density function \( p(t, x, y) \) being the minimal positive fundamental solution of the heat equation

\[ \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta_M u(t, x). \]

Let \( X_t \) be the Brownian motion in \( M \) starting from \( o \in M \) with transition density function \( p(t, o, x) \), with law \( \mathbb{P}_o \) and expectation \( \mathbb{E}_o \). The Itô formula (see [13, 14]) asserts that

\[ u(X_t) - u(o) = B \left( \int_0^t \| \nabla_M u \|^2(X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \quad \mathbb{P}_o - a.s. \]

for \( u \in \mathcal{C}^2(M) \), where \( B_t \) is the one-dimensional standard Brownian motion in \( \mathbb{R} \) and \( \nabla_M \) is the gradient operator on \( M \). It follows the Dynkin formula

\[ \mathbb{E}_o[u(X_T)] - u(o) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^T \Delta_M u(X_t) dt \right] \]

for a stopping time \( T \) satisfying \( \mathbb{E}_o[T] < \infty \) such that each term in the above makes sense. Due to the expectation “\( \mathbb{E}_o \)”, the Dynkin formula works in the case when \( u \) is of a polar set of singularities if each term makes sense. \( M \) is said to be stochastically complete if

\[ \int_M p(t, o, x) dV(x) = 1 \]

holds for all \( o \in M \). By Grigor’yan’s criterion (see [10]), \( M \) is stochastically complete if \( R_M(x) \geq -cr^2(x) - c \) for some constant \( c > 0 \), where \( R_M \) is the pointwise lower bound of Ricci curvatures defined by

\[ R_M(x) = \inf_{\xi \in T_x M, \|\xi\|=1} \text{Ric}(\xi, \bar{\xi}). \]
2.1. Nevanlinna-type functions.

Let $L \to N$ be a holomorphic line bundle over a compact complex manifold $N$. Denoted by $H^0(N, L)$ the vector space of holomorphic global sections of $L$ over $N$, and by $|L|$ the complete linear system of effective divisors $D_s$ for $s \in H^0(M, L)$. Given $D \in |L|$, let
\[ f : M \to N \]
be a holomorphic mapping into $N$ such that $f(M) \not\subset \text{Supp}D$. Endow $L$ with Hermitian metric $h$, which defines the Chern form $c_1(L, h) = -dd^c \log h$. We use the standard notations
\[ d = \partial + \overline{\partial}, \quad dd^c = \frac{-1}{4\pi} (\overline{\partial} - \partial), \quad dd^c = \frac{-1}{2\pi} \partial \overline{\partial}. \]

Lemma 2.1 ([8]). $\Delta_M \log(h \circ f)$ is globally defined on $M$ and
\[ \Delta_M \log(h \circ f) = -4m f^* c_1(L, h) \wedge \alpha^{m-1} / \alpha^m, \]
where $\alpha = \frac{\sqrt{-1}}{2\pi} \sum i_j g_{ij} dz_i \wedge d\bar{z}_j$ is the Kähler metric form on $M$.

Let $\{(U_\alpha), \{e_\alpha\}\}$ be a local trivialization covering of $(L, h)$. For $0 \neq s \in H^0(N, L)$, write $s = \bar{s} e_\alpha$ locally on $U_\alpha$. Note that $\Delta_M \log |\bar{s} \circ f|^2$ is globally defined on $M$ and
\[ \Delta_M \log ||s \circ f||^2 = \Delta_M \log(h \circ f) + \Delta_M \log |\bar{s} \circ f|^2. \]
The similar argument as in the proof of Lemma 2.1 shows that
\[ \Delta_M \log |\bar{s} \circ f|^2 = 4m dd^c \log |\bar{s} \circ f|^2 \wedge \alpha^{m-1} / \alpha^m. \]

Lemma 2.2 ([8]). For $s \in H^0(N, L)$ with $D = (s)$, we have

(i) $\log ||s \circ f||^2$ is locally the difference of two plurisubharmonic functions, hence $\log ||s \circ f||^2 \in \mathcal{L}_{\text{loc}}(M)$ and $\log ||s \circ f||^2 \in \mathcal{L}(S_o(r))$.

(ii) $dd^c \log ||s \circ f||^2 = f^*D - f^*c_1(L, h)$ in the sense of currents.

We now introduce the notion of the so-called Nevanlinna-type functions of holomorphic mappings into a compact complex manifold. For a continuous (1,1)-form $\omega$ on $N$, we use the following convenient symbols
\[ e_f \omega(x) = 2m f^* \omega \wedge \alpha^{m-1} / \alpha^m, \quad g_t(o, x) = \int_0^t p(s, o, x)ds. \]
If $\omega > 0$, we call $e_f \omega(x)$ the energy density function of $f$ with respect to the metrics $\alpha$, $\omega$. The characteristic function of $f$ with respect to $\omega$ is defined by
\[ \mathcal{T}_f(t, \omega) = \frac{1}{2} E_o \left[ \int_0^t e_f \omega(X_s)ds \right] = \frac{1}{2} \int_M g_t(o, x)e_f \omega(x)dV(x). \]
Apply Lemma 2.1 we have
\[ e_{f^{*}c_{1}(L,h)}(x) = \frac{1}{4} \Delta_{M} \log(h \circ f(x)). \]

It well defines
\[ \tilde{T}_{f}(t, L) := \tilde{T}_{f}(t, c_{1}(L, h)) \]
up to a constant term, due to the compactness of \( N \).

The conditions for \( \tilde{T}(t, L) < \infty \) as \( 0 < t < \infty \) and \( \tilde{T}(t, L) \to \infty \) as \( t \to \infty \) provided \( L > 0 \) will be discussed in Section 4.3 (see Lemma 4.5 and Lemma 4.6).

Assume that \( L > 0 \) and let \( s_{D} \) be the canonical section defined by \( D \). We may have \( \|s_{D}\| < 1 \) since the compactness of \( N \). Noticing that \( \log \|s_{D} \circ f(x)\| \) is locally the difference of two plurisubharmonic functions from Lemma 2.2 and is thus integrable on \( S_{o}(r) \). The proximity function of \( f \) with respect to \( D \) is defined by
\[ \tilde{m}_{f}(t, D) = \mathbb{P}_{o} \left[ \sup_{0 \leq s \leq t} \log \frac{1}{\|s_{D} \circ f(X_{s})\|} > \lambda \right]. \]

As for counting function, we use the expression
\[ \tilde{N}_{f}(t, D) = \frac{\pi^{m}}{(m-1)!} \int_{M \cap f^{-1}D} g_{t}(o, x) \alpha^{m-1} \]
\[ = \frac{\pi^{m}}{(m-1)!} \int_{M} g_{t}(o, x) dd^{c} \log |s_{D} \circ f(x)|^{2} \alpha^{m-1} \]
\[ = \frac{1}{4} \int_{M} g_{t}(o, x) \Delta_{M} \log |s_{D} \circ f(x)|^{2} dV(x). \]

We have \( \tilde{N}_{f}(t, D) = 0 \) if \( f \) omits \( D \). If
\[ (2) \quad \frac{1}{4} \mathbb{E}_{o} \left[ \int_{0}^{t} \Delta_{M} \log |s_{D} \circ f(X_{s})|^{2} ds \right] < \infty, \quad 0 < t < \infty \]
which is hence equal to \( \tilde{N}_{f}(t, D) \) by using Fubini theorem due to the absolute convergence of (2), then \( \tilde{N}_{f}(t, D) \) has an alternative expression
\[ (3) \quad \tilde{N}_{f}(t, D) = \lim_{\lambda \to \infty} \mathbb{P}_{o} \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_{D} \circ f(X_{s})\|} > \lambda \right). \]

To see that, one can apply the arguments appeared in [9] related to the local martingales and use Dynkin formula. It is shown that the above limit exists
and equals

\[
\lim_{\lambda \to \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \frac{1}{\| s_D \circ f(X_s) \|} > \lambda \right)
\]

\[
= -\frac{1}{2} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \frac{1}{s_D \circ f(X_s)} \, ds \right]
\]

\[
= \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log |s_D \circ f(X_s)|^2 ds \right]
\]

\[
= \frac{1}{4} \int_M \left[ \int_0^t p(s, o, x) ds \right] \Delta_M \log |s_D \circ f(x)|^2 dV(x)
\]

\[
= \bar{N}_f(t, D).
\]

Another proof of (3) will be given in Section 2.2 below. We remark that (2) can be guaranteed by \( \bar{T}_f(t, L) < \infty \) as \( 0 < t < \infty \), since Theorem 2.3 below.

2.2. First Main Theorem.

We adopt the same notations as in Section 2.1.

**Theorem 2.3** (FMT). Let \( L \to N \) be a positive line bundle over a compact complex manifold \( N \), and \( D \in |L| \). Let \( f : M \to N \) be a holomorphic mapping such that \( f(M) \not\subset \text{Supp} D \). Then

\[
\bar{T}_f(t, L) = \bar{m}_f(t, D) + \bar{N}_f(t, D) + O(1).
\]

**Proof.** Take a Hermitian metric \( h \) on \( L \) such that \( \omega := c_1(L, h) > 0 \). Set

\[
T_\lambda = \inf \left\{ t > 0 : \sup_{0 \leq s \leq t} \frac{1}{\| s_D \circ f(X_s) \|} > \lambda \right\}.
\]

Let \( \{ U_\alpha \}, \{ e_\alpha \} \) be a local trivialization covering of \( (L, h) \). Write \( s_D = \tilde{s}_D e_\alpha \) locally on \( U_\alpha \). Then, we get

(4) \[ \log \| s_D \circ f \|^2 = \log |s_D \circ f|^2 + \log (h \circ f). \]

Note that \( s_D \circ f \) is holomorphic and \( h \circ f > 0 \) is smooth, thereby the Dynkin formula is applicable to \( \log \| s_D \circ f \|^{-1} \). Consequently,

\[
\mathbb{E}_o \left[ \log \left\| s_D \circ f(X_t \wedge T_\lambda) \right\| \right]
\]

\[
= \frac{1}{2} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log \frac{1}{\| s_D \circ f(X_s) \|} \, ds \right] + \frac{1}{\| s_D \circ f(o) \|},
\]

where \( t \wedge T_\lambda = \min\{ t, T_\lambda \} \). Since \( \log \| s_D \circ f(X_s) \|^{-1} \) has no singularities as \( 0 \leq s \leq T_\lambda \) due to the definition of \( T_\lambda \), it concludes by (4) that

\[
\Delta_M \log \| s_D \circ f(X_s) \| = -\frac{1}{2} \Delta_M \log (h \circ f(X_s))
\]
as \( 0 \leq s \leq T_\lambda \), where we use the fact that \( \log |\tilde{s}_D \circ f| \) is harmonic on \( M \setminus f^*D \). Hence, (5) turns to
\[
\mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_t)\|} \right] = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log (h \circ f(X_s))ds \right] + O(1).
\]
Since \( f^* \omega = -dd^c \log (h \circ f) \), then by Lemma 2.1
\[
e_f \omega = -2m \frac{dd^c \log (h \circ f) \wedge \alpha^{m-1}}{\alpha^m} = -\frac{1}{2} \Delta_M \log (h \circ f).
\]
By the monotone convergence theorem, it yields from (6) that
\[
(7) \quad -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} \Delta_M \log (h \circ f(X_s))ds \right] = \frac{1}{2} \mathbb{E}_o \left[ \int_0^{t \wedge T_\lambda} e_f \omega(X_s)ds \right] \rightarrow \tilde{T}(t, L)
\]
as \( \lambda \rightarrow \infty \), where we use the fact that \( T_\lambda \rightarrow \infty \) a.s. as \( \lambda \rightarrow \infty \) for that \( f^*D \) is polar. Write the first term appeared in (5) as two parts
\[
I + II = \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_t)\|} : t < T_\lambda \right] + \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{T_\lambda})\|} : T_\lambda \leq t \right].
\]
Apply the monotone convergence theorem,
\[
(8) \quad I \rightarrow \tilde{m}_f(r, D)
\]
as \( \lambda \rightarrow \infty \). Now we look at II. By the definition of \( T_\lambda \), it is not hard to see
\[
(9) \quad II = \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right) \rightarrow \tilde{N}(t, D)
\]
as \( \lambda \rightarrow \infty \). By (7)-(9), we have the desired result. \(\square\)

**Another proof of (3).** Since \( f^*D \) is polar, we use Dynkin formula to get
\[
\mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_t)\|} \right] + O(1) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \frac{1}{\|s_D \circ f(X_s)\|} ds \right].
\]
This yields that
\[
\tilde{T}(t, L) + O(1) = \tilde{m}_f(t, D) + \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log |\tilde{s}_D \circ f(X_s)|^2 ds \right].
\]
On the other hand, the argument in the proof of Theorem 2.3 implies that
\[
\tilde{T}(t, L) + O(1) = \tilde{m}_f(t, D) + \lim_{\lambda \rightarrow \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq s \leq t} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right).
\]
By a comparison, we deduce (3).

3. **Logarithmic Derivative Lemma**
3.1. Logarithmic Derivative Lemma.

Let \((M, g)\) be a complete and stochastically complete Kähler manifold of complex dimension \(m\), with Kähler metric form \(\alpha\) and gradient operator \(\nabla_M\) associated to \(g\). Let \(X_t\) be the Brownian motion in \(M\) with generator \(\frac{1}{2}\Delta_M\), started at a fixed point \(o \in M\) with transition density function \(p(t, o, x)\).

**Lemma 3.1 (Calculus Lemma).** Let \(k\) be a non-negative function on \(M\) so that \(\mathbb{E}_o[k(X_t)] < \infty\) and \(\mathbb{E}_o[\int_0^t k(X_s)ds] < \infty\) for \(0 < t < \infty\). Then for any \(\delta > 0\), there exists a set \(E_\delta \subset [0, \infty)\) with finite Lebesgue measure such that

\[
\mathbb{E}_o[k(X_t)] \leq \left( \mathbb{E}_o \left[ \int_0^t k(X_s)ds \right] \right)^{1+\delta}
\]

holds for \(r \not\in E_\delta\).

**Proof.** Set \(\gamma(t) := \mathbb{E}_o[\int_0^t k(X_s)ds]\) and \(E_\delta := \{ t \in (0, \infty) : \gamma'(t) > \gamma^{1+\delta}(t) \}\), then \(\gamma'(t) = \mathbb{E}_o[k(X_t)]\). The claim clearly holds for \(k \equiv 0\). If \(k \not\equiv 0\), suppose that \(\gamma(1) \not\equiv 0\) without loss of generality. Note that

\[
\int_{E_\delta} dt \leq 1 + \int_1^\infty \frac{\gamma'(t)}{\gamma^{1+\delta}(t)} dt \leq 1 + \delta^{-1}\gamma^{-\delta}(1) < \infty.
\]

This finishes the proof. \(\square\)

Let \(\psi\) be a meromorphic function on \(M\). The norm of the gradient of \(\psi\) is defined by

\[
\|\nabla_M \psi\|^2 = \sum_{i,j} g^{\overline{z}_i z_j} \frac{\partial \psi}{\partial z_i} \overline{\frac{\partial \psi}{\partial z_j}},
\]

where \((g^{\overline{z}_i z_j})\) is the inverse of \((g_{z_i z_j})\). Locally, we write \(\psi = \psi_1/\psi_0\), where \(\psi_0, \psi_1\) are holomorphic functions so that \(\text{codim}_C(\psi_0 = 0) \geq 2\) if \(\text{dim}_C M \geq 2\).

Identify \(\psi\) with a meromorphic mapping into \(\mathbb{P}^1(\mathbb{C})\) by \(x \mapsto [\psi_0(x) : \psi_1(x)]\).

The characteristic function of \(\psi\) with respect to the Fubini-Study form \(\omega_{FS}\) on \(\mathbb{P}^1(\mathbb{C})\) is defined by

\[
\hat{T}_\psi(t, \omega_{FS}) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \left( |\psi_0(X_s)|^2 + |\psi_1(X_s)|^2 \right) ds \right].
\]

Let \(i : \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})\) be an inclusion defined by \(z \mapsto [1 : z]\). Via the pull-back by \(i\), we have a \((1,1)\)-form \(i^*\omega_{FS} = dd^c \log(1 + |\zeta|^2)\) on \(\mathbb{C}\), where \(\zeta := w_1/w_0\) and \([w_0 : w_1]\) is the homogeneous coordinate system of \(\mathbb{P}^1(\mathbb{C})\).

The characteristic function of \(\psi\) with respect to \(i^*\omega_{FS}\) is defined by

\[
\hat{T}_\psi(t, \omega_{FS}) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \left( 1 + |\psi(X_s)|^2 \right) ds \right].
\]

Clearly,

\[
\hat{T}_\psi(t, \omega_{FS}) \leq \hat{T}_\psi(t, \omega_{FS}).
\]
We adopt the spherical distance \( \| \cdot , \cdot \| \) on \( \mathbb{P}^1(\mathbb{C}) \), the proximity function of \( \psi \) with respect to \( a \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{ \infty \} \) is defined by

\[
\hat{m}_\psi(t, a) = \mathbb{E}_0 \left[ \log \frac{1}{\| \psi(X_t), a \|} \right].
\]

Again, set

\[
\hat{N}_\psi(t, a) = \lim_{\lambda \to \infty} \lambda \mathbb{P}_a \left( \sup_{0 \leq s \leq t} \log \frac{1}{\| f(X_s), a \|} > \lambda \right).
\]

Similar to Theorem 2.3, we can show that

\[
\tilde{T}(t, \psi) = \tilde{m}(t, \psi, \infty) + \tilde{N}(t, \psi, \infty),
\]

where

\[
\tilde{m}(t, \psi, \infty) = \mathbb{E}_0 \left[ \log^+ |\psi(X_t)| \right],
\]

\[
\tilde{N}(t, \psi, \infty) = \lim_{\lambda \to \infty} \lambda \mathbb{P}_a \left( \sup_{0 \leq s \leq t} \log^+ |f(X_s)| > \lambda \right).
\]

Since \( \tilde{N}(t, \psi, \infty) = \hat{N}_\psi(t, \infty) \) and \( \tilde{m}(t, \psi, \infty) = \hat{m}_\psi(t, \infty) + O(1) \), whence

\[
\tilde{T}(t, \psi) = \hat{T}_\psi(t, \omega_{FS}) + O(1), \quad \tilde{T}(t, \psi - a) = \tilde{T}(t, \psi) + O(1).
\]

On \( \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{ \infty \} \), we define a singular metric

\[
\Phi = \frac{1}{|\zeta|^2(1 + \log^2 |\zeta|)} \sqrt{-1} d\zeta \wedge d\overline{\zeta}.
\]

A direct computation shows that

\[
\int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 2m \pi \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m} = \frac{\| \nabla_M \psi \|^2}{\psi^2(1 + \log^2 |\psi|)}.
\]

Define

\[
\tilde{T}_\psi(t, \Phi) = \frac{1}{2} \mathbb{E}_0 \left[ \int_0^t e_{\psi^* \Phi}(X_s) ds \right], \quad e_{\psi^* \Phi}(x) = 2m \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m}.
\]

According to (11), we obtain

\[
\tilde{T}_\psi(t, \Phi) = \frac{1}{2 \pi} \mathbb{E}_0 \left[ \int_0^t \frac{\| \nabla_M \psi \|^2}{\psi^2(1 + \log^2 |\psi|)}(X_s) ds \right].
\]

**Lemma 3.2.** Assume that \( \tilde{T}(t, \psi) < \infty \) for \( 0 < t < \infty \), then

\[
\tilde{T}_\psi(t, \Phi) \leq \tilde{T}(t, \psi) + O(1).
\]
Proof. By Fubini theorem and the First Main Theorem

\[
\tilde{T}_\psi(t, \Phi) = \frac{1}{2} \mathbb{E}_0 \left[ \int_0^t e^{\psi^\ast \Phi}(X_s) ds \right] \\
= \frac{1}{2} \int_M \left[ \int_0^t p(s, o, x) ds \right] e^{\psi^\ast \Phi(x)} dV(x) \\
= \frac{\pi^m}{(m-1)!} \int_M g_t(o, x) \psi^\ast \Phi \wedge \alpha^{m-1} \\
= \frac{\pi^m}{(m-1)!} \int_{\mathcal{P}_1(C)} \Phi \int_{M \cap \psi^{-1}(\zeta)} g_t(o, x) \alpha^{m-1} \\
= \int_{\mathcal{P}_1(C)} \tilde{N}(t, \psi, \zeta) \Phi \leq \tilde{T}(t, \psi) + O(1).
\]

This completes the proof. \(\Box\)

Lemma 3.3. For any \(\delta > 0\), there exists a set \(E_\delta \subset (1, \infty)\) of finite Lebesgue measure such that

\[
\mathbb{E}_0 \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \leq (1 + \delta) \log \tilde{T}(t, \psi) + O(1)
\]

holds for \(r \in (1, \infty)\) outside \(E_\delta\).

Proof. By Jensen inequality,

\[
\mathbb{E}_0 \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \\
\leq \mathbb{E}_0 \left[ \log \left( 1 + \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right) \right] \\
\leq \log^+ \mathbb{E}_0 \left[ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] + O(1).
\]

Lemma 3.1 and Lemma 3.2 with (12) imply

\[
\log^+ \mathbb{E}_0 \left[ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] \\
\leq (1 + \delta) \log^+ \mathbb{E}_0 \left[ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \right] + O(1)
\]

This completes the proof. \(\Box\)

Define

\[
\bar{m} \left( t, \frac{\|\nabla_M \psi\|}{|\psi|} \right) = \mathbb{E}_0 \left[ \log^+ \frac{\|\nabla_M \psi\|}{|\psi|}(X_t) \right].
\]
Proof of Theorem 1.2

Proof. We have on the one hand

\[
\tilde{m}(t, \frac{\|\nabla_M \psi\|}{|\psi|}) = \frac{1}{2} E_o \left[ \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_t)(1 + \log^2 |\psi(X_t)|) \right) \right]
\]

\[
\leq \frac{1}{2} E_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_t) \right] + \frac{1}{2} E_o \left[ \log^+ (1 + \log^2 |\psi(X_t)|) \right]
\]

\[
\leq \frac{1}{2} E_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_t) \right] + E_o \left[ \log \left( 1 + \log^+ |\psi(X_t)| + \log^+ \frac{1}{|\psi(X_t)|} \right) \right].
\]

Lemma 3.3 implies that

\[
E_o \left[ \log^+ \frac{\|\nabla_M \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_t) \right] \leq (1 + \delta) \log \tilde{T}(t, \psi) + O(1).
\]

On the other hand, by Jensen inequality

\[
E_o \left[ \log \left( 1 + \log^+ |\psi(X_t)| + \log^+ \frac{1}{|\psi(X_t)|} \right) \right] \leq \log E_o \left[ 1 + \log^+ |\psi(X_t)| + \log^+ \frac{1}{|\psi(X_t)|} \right]
\]

\[
\leq \log \left( \tilde{m}(t, \psi, \infty) + \tilde{m}(t, \psi, 0) \right) + O(1)
\]

Combining the above, we are led to the assertion. \qed

4. Second Main Theorem and defect relations

4.1. Preparation.

Let \( \psi : M \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping, and let \([w_0 : \cdots : w_n]\) be the homogeneous coordinate system of \( \mathbb{P}^n(\mathbb{C}) \). Assume that \( w_0 \circ \psi \neq 0 \). Let \( i : \mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C}) \) be an inclusion defined by

\[
(z_1, \cdots, z_n) \mapsto [1 : z_1 : \cdots : z_n].
\]

Clearly, the Fubini-Study form \( \omega_{FS} = dd^c \log \|w\|^2 \) on \( \mathbb{P}^n(\mathbb{C}) \) induces a \((1,1)\)-form \( i^* \omega_{FS} = dd^c \log(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2) \) on \( \mathbb{C}^n \), where \( \zeta_j := w_j/w_0 \) for
$0 \leq j \leq n$. The characteristic function of $\psi$ with respect to $i^*\omega_{FS}$ is defined by

$$\hat{T}_\psi(t, \omega_{FS}) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \left( \sum_{j=0}^n |\zeta_j \circ \psi(X_s)|^2 \right) ds \right].$$

Clearly,

$$\hat{T}_\psi(t, \omega_{FS}) \leq \hat{T}_{\zeta_j \circ \psi}(t, \omega_{FS}) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \|\psi(X_s)\|^2 ds \right].$$

As is noted, the Dynkin formula works for a set of singularities which is polar. Note that the indeterminacy set and pole divisors of $\zeta_j \circ \psi$ for $1 \leq j \leq n$ are polar. Assume that $o$ is not in both indeterminacy set and pole set. Hence,

$$\hat{T}_\psi(t, \omega_{FS}) = \frac{1}{2} \mathbb{E}_o \left[ \log \left( \sum_{j=0}^n |\zeta_j \circ \psi(X_t)|^2 \right) \right] - \frac{1}{2} \log \left( \sum_{j=0}^n |\zeta_j \circ \psi(o)|^2 \right)$$

$$\hat{T}_{\zeta_j \circ \psi}(t, \omega_{FS}) = \frac{1}{2} \mathbb{E}_o \left[ \log \left( 1 + |\zeta_j \circ \psi(X_t)|^2 \right) \right] - \frac{1}{2} \log \left( 1 + |\zeta_j \circ \psi(o)|^2 \right).$$

**Theorem 4.1.** We have

$$\max_{1 \leq j \leq n} \hat{T}(t, \zeta_j \circ \psi) + O(1) \leq \hat{T}_\psi(t, \omega_{FS}) \leq \sum_{j=1}^n \hat{T}(t, \zeta_j \circ \psi) + O(1).$$

**Proof.** On the one hand,

$$\hat{T}_\psi(t, \omega_{FS}) = \mathbb{E}_o \left[ \log \left( \sum_{j=0}^n |\zeta_j \circ \psi(X_t)|^2 \right) \right] - \frac{1}{2} \log \left( \sum_{j=0}^n |\zeta_j \circ \psi(o)|^2 \right)$$

$$\leq \frac{1}{2} \sum_{j=1}^n \left( \mathbb{E}_o \left[ \log \left( 1 + |\zeta_j \circ \psi(X_t)|^2 \right) \right] - \log \left( 1 + |\zeta_j \circ \psi(o)|^2 \right) \right) + O(1)$$

$$= \sum_{j=1}^n \hat{T}(t, \zeta_j \circ \psi) + O(1).$$

On the other hand,

$$\hat{T}(t, \zeta_j \circ \psi) = \hat{T}_{\zeta_j \circ \psi}(t, \omega_{FS}) + O(1)$$

$$= \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \left( 1 + |\zeta_j \circ \psi(X_s)|^2 \right) ds \right] + O(1)$$

$$\leq \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \left( \sum_{j=0}^n |\zeta_j \circ \psi(X_s)|^2 \right) ds \right] + O(1)$$

$$= \hat{T}_\psi(t, \omega_{FS}) + O(1).$$

The claim is certified. \qed
Corollary 4.2. We have
\[
\max_{1 \leq j \leq n} \tilde{T}(t, \zeta_j \circ f) \leq \tilde{T}_\psi(t, \omega_{FS}) + O(1).
\]

Let \( V \) be a complex projective algebraic variety and \( \mathbb{C}(V) \) be the field of rational functions defined on \( V \) over \( \mathbb{C} \). Let \( V \hookrightarrow \mathbb{P}^N(\mathbb{C}) \) be a holomorphic embedding, and let \( H_V \) be the restriction of hyperplane line bundle \( H \) over \( \mathbb{P}^N(\mathbb{C}) \) to \( V \). Denoted by \([w_0 : \cdots : w_N]\) the homogeneous coordinate system of \( \mathbb{P}^N(\mathbb{C}) \) and assume that \( w_0 \neq 0 \) without loss of generality. Notice that the restriction \( \{\zeta_j := w_j/w_0\} \) to \( V \) gives a transcendental base of \( \mathbb{C}(V) \). Thereby, any \( \phi \in \mathbb{C}(V) \) can be represented by a rational function in \( \zeta_1, \cdots, \zeta_N \)
(13)
\[
\phi = Q(\zeta_1, \cdots, \zeta_N).
\]

Theorem 4.3. Let \( f : M \to V \) be an algebraically non-degenerate holomorphic mapping such that \( \tilde{T}_f(t, H_V) < \infty \) as \( 0 < t < \infty \). Then for \( \phi \in \mathbb{C}(V) \), there is constant \( C > 0 \) such that
\[
\tilde{T}(t, \phi \circ f) \leq C\tilde{T}_f(t, H_V) + O(1).
\]

Proof. Assume \( f_0 \neq 0 \). Pulling back (13) by \( f \), \( \phi \circ f = Q(\zeta_1 \circ f, \cdots, \zeta_N \circ f) \). Since \( Q_j \) is rational, then there exists a constant \( C' > 0 \) such that
\[
\tilde{T}(t, \phi \circ f) \leq C' \sum_{j=1}^{N} \tilde{T}(t, \zeta_j \circ f) + O(1).
\]
By Corollary 4.2, \( \tilde{T}(t, \zeta_j \circ f) \leq \tilde{T}_f(t, H_V) + O(1) \). This proves the theorem. \( \square \)

Corollary 4.4. Let \( f : M \to V \) be an algebraically non-degenerate holomorphic mapping such that \( \tilde{T}_f(t, \omega) < \infty \) as \( 0 < t < \infty \). Let \( \omega \) be a positive \((1,1)\)-form on \( V \). Then for \( \phi \in \mathbb{C}(V) \), there is constant \( C > 0 \) such that
\[
\tilde{T}(t, \phi \circ f) \leq C\tilde{T}_f(t, \omega) + O(1).
\]

Proof. Since \( V \) is compact, then for two positive \((1,1)\)-forms \( \omega_1, \omega_2 \) on \( V \), we have \( c_1 \omega_1 \leq \omega_2 \leq c_2 \omega_1 \) for two constants \( c_1, c_2 > 0 \). Hence, the claim follows from Theorem 4.3. \( \square \)

4.2. Second Main Theorem.

Let \( M \) be a Kähler manifold with Kähler metric
\[
g = \sum_{i,j} g_{ij} dz_i \otimes d\overline{z}_j.
\]
The Ricci curvature tensor \( \text{Ric} = \sum_{i,j} R_{ij} dz_i \otimes d\overline{z}_j \) on \( M \) is given by
\[
R_{ij} = -\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \det(g_{st}).
\]
A well-known theorem by S. S. Chern asserts that the Ricci curvature form

\[ R_M := -dd^c \log \det(g_{st}^g) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} R_{ij} dz_i \wedge d\bar{z}_j \]

is a real and closed smooth \((1,1)\)-form which represents the first Chern class of \(M\) in de Rham cohomology group \(H^2_{dR}(M, \mathbb{R})\). Let \(s_M\) denote the Ricci scalar curvature of \(M\), which is defined by

\[ s_M = \sum_{i,j} g_{ij}^g R_{ij}, \]

where \((g_{ij}^g)\) is the inverse of \((g_{ij}^g)\). Combining the above, we have

\[ s_M = -\frac{1}{4} \Delta_M \log \det(g_{st}^g). \]

Let \((L, h) \to N\) be a positive line bundle over a compact complex manifold \(N\) of complex dimension \(n\), it defines a smooth volume form \(\Omega = \bigwedge^n c_1(L, h)\) on \(N\). Let \(D = \sum_{j=1}^q D_j \in |L|\) be the union of irreducible components so that \(D\) has only simple normal crossings. Endowing each \(L_{D_j}\) for \(1 \leq j \leq q\) with Hermitian metric such that the induced Hermitian metric on \(L = \bigotimes_{j=1}^q L_{D_j}\) is \(h\). Take \(s_j \in H^0(V, L_{D_j})\) with \((s_j) = D_j\) and \(\|s_j\| < 1\). On \(N\), one defines a singular volume form

\[ (14) \quad \Phi = \frac{\Omega}{\prod_{j=1}^q \|s_j\|^2}. \]

Set \(f^* \Phi \wedge \alpha^{m-n} = \xi \alpha^m\). Note that

\[ \alpha^m = m! \det(g_{ij}^g) \prod_{j=1}^m \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j. \]

A direct computation leads to

\[ dd^c \log \xi \geq f^* c_1(L, h) - f^* \text{Ric} \Omega + R_M - \text{Supp} f^* D\]

in the sense of currents, where \(\text{Ric} \Omega\) is the Ricci form of \(\Omega\). This follows that

\[ \frac{dd^c \log \xi \wedge \alpha^{m-1}}{\alpha^m} \geq \frac{f^* c_1(L, h) \wedge \alpha^{m-1}}{\alpha^m} - \frac{f^* \text{Ric} \Omega \wedge \alpha^{m-1}}{\alpha^m} \]

\[ + \frac{R_M \wedge \alpha^{m-1}}{\alpha^m} - \frac{\text{Supp} f^* D \wedge \alpha^{m-1}}{\alpha^m}. \]

Thus,

\[ (15) \quad \frac{1}{4} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \xi(X_s) ds \right]\]

\[ \geq \tilde{T}_f(t, L) + \tilde{T}_f(t, K_N) + \tilde{T}(t, R_M) - \tilde{N}_f(t, D) + O(1). \]

Proof of Theorem \ref{thm:1.3}
Proof. Identify $N$ with a complex projective algebraic manifold. To use Ru-Wong’s argument (see [17], Page 231-233), there exists a finite open covering \{\{U_\lambda\}\} of $N$ and rational functions $w_{\lambda 1}, \ldots, w_{\lambda n}$ on $N$ for every $\lambda$ such that $w_{\lambda 1}, \ldots, w_{\lambda n}$ are holomorphic on $U_\lambda$ satisfying
\[ dw_{\lambda 1} \wedge \cdots \wedge dw_{\lambda n}(y) \neq 0, \forall y \in U_\lambda; \quad U_\lambda \cap D = \{w_{\lambda 1} \cdots w_{\lambda h} = 0\}, \exists h_\lambda \leq n. \]
In addition, one may require $L_D |U_\lambda \cong U_\lambda \times \mathbb{C}$ for $\lambda, j$. On $U_\lambda$, we have
\[ \Phi = \frac{\phi_\lambda \circ f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda h}|^2} \bigwedge_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} df_{\lambda k} \wedge d\overline{f}_{\lambda k}, \]
where $\Phi$ is given by (14) and $\phi_\lambda > 0$ is a smooth function. Set $f_{\lambda k} = w_{\lambda k} \circ f$, then
\[ (16) \quad f^* \Phi = \frac{\phi_\lambda \circ f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda h}|^2} \bigwedge_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} df_{\lambda k} \wedge d\overline{f}_{\lambda k} \]
on each $U_\lambda$. Set $f^* \Phi \wedge \alpha^{m-n} = \xi \alpha^m$, we have (15). Again, set
\[ (17) \quad f^* c_{1}(L, h) \wedge \alpha^{m-1} = \varrho \alpha^m. \]
It follows that
\[ (18) \quad \varrho = \frac{1}{2m} \epsilon f^* \omega. \]
For each $\lambda$ and any $x \in f^{-1}(U_\lambda)$, take a local holomorphic coordinate system $z$ around $x$. Since $N$ is compact, then it is not very hard to compute by (16) and (17) that $\xi$ is bounded from above by $P_\lambda$, where $P_\lambda$ is a polynomial in
\[ \varrho, g^j_i \frac{\partial f_{\lambda k}}{\partial z_i} \frac{\partial f_{\lambda k}}{\partial z_j} / |f_{\lambda k}|^2, 1 \leq i, j \leq m, 1 \leq k \leq n. \]
This yields that
\[ \log^+ \xi \leq O \left( \log^+ \varrho + \sum_k \log^+ \frac{\| \nabla_M f_{\lambda k} \|}{|f_{\lambda k}|} \right) + O(1) \]
on each $f^{-1}(U_\lambda)$. Let \{\{\phi_\lambda\}\} be a partition of unity subordinate to \{\{f^{-1}(U_\lambda)\}\}. Then we have $0 \leq \phi_\lambda \leq 1$ and $\phi_\lambda \log^+ (\| \nabla_M f_{\lambda k} \| / |f_{\lambda k}|) = 0$ outside $f^{-1}(U_\lambda)$ for all $k, \lambda$. Hence, we have in further
\[ (19) \quad \log^+ \xi = \sum_\lambda \phi_\lambda \log^+ \xi \leq O \left( \log^+ \varrho + \sum_{k, \lambda} \phi_\lambda \log^+ \frac{\| \nabla_M f_{\lambda k} \|}{|f_{\lambda k}|} \right) + O(1) \]
\[ \leq O \left( \log^+ \varrho + \sum_{k, \lambda} \log^+ \frac{\| \nabla_M f_{\lambda k} \|}{|f_{\lambda k}|} \right) + O(1) \]
on $M$, where all $f_{\lambda_k}$ in the last inequality are understood as global functions on $M$ since all $w_{\lambda_k}$ are globally defined on $N$. By means of Dynkin formula,

$$\frac{1}{2} \mathbb{E}_o \left[ \int_0^t \Delta_M \log \xi(X_s) ds \right] = \mathbb{E}_o \left[ \log \xi(X_t) \right] - \log \xi(o).$$

It yields from (15) that

$$\frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_t) \right] \geq \tilde{T}_f(t, L) + \tilde{T}_f(t, K_N) + \tilde{T}(t, \mathcal{R}_M) - \tilde{N}_f(t, D) + \frac{1}{2} \log \xi(o).$$

On the other hand, by (19) and Theorem 1.2 we have

$$\frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_t) \right] \leq O \left( \sum_{k, \lambda} \mathbb{E}_o \left[ \log^+ \frac{\|\nabla_M f_{\lambda_k}\|}{f_{\lambda_k}}(X_t) \right] \right) + O(1)$$

$$\leq O \left( \sum_{k, \lambda} \tilde{m} \left( t, \frac{\|\nabla_M f_{\lambda_k}\|}{f_{\lambda_k}} \right) \right) + O(1)$$

$$\leq O \left( \sum_{k, \lambda} \log \tilde{T}(t, f_{\lambda_k}) \right) + O(1)$$

$$\leq O \left( \log \tilde{T}_f(t, L) \right) + O(1),$$

where the last inequality is due to Corollary 4.4. Indeed, Lemma 3.1 and (18) imply

$$\log^+ \mathbb{E}_o \left[ \varrho(X_t) \right] \leq (1 + \delta) \log^+ \mathbb{E}_o \left[ \int_0^t \varrho(X_s) ds \right]$$

$$= \frac{(1 + \delta)}{2m} \log^+ \mathbb{E}_o \left[ \int_0^t e^{f_{c_1(L_1, h)}(X_s)} ds \right]$$

$$\leq \frac{(1 + \delta)}{m} \log \tilde{T}_f(t, L) + O(1).$$

By this with (20), the theorem is proved. \hfill \Box

4.3. Defect relations.

Let $L_1, L_2$ be holomorphic line bundles over a compact complex manifold $N$, we set

$$\frac{c_1(L_2)}{c_1(L_1)} = \sup \left\{ a \in \mathbb{R} : L_2 > aL_1 \right\}, \quad \frac{c_1(L_2)}{c_1(L_1)} = \inf \left\{ a \in \mathbb{R} : L_2 < aL_1 \right\}.$$
It is clear that
\[(21) \quad \frac{[c_1(L_2)]}{[c_1(L_1)]} \leq \liminf_{t \to \infty} \frac{\bar{T}_f(t, L_2)}{\bar{T}_f(t, L_1)} \leq \limsup_{r \to \infty} \frac{\bar{T}_f(t, L_2)}{\bar{T}_f(t, L_1)} \leq \frac{[c_1(L_2)]}{[c_1(L_1)]}.
\]

Let \( f : M \to N \) be a differentiably non-degenerate holomorphic mapping from a complete and stochastically complete Kähler manifold \( M \) into \( N \) such that \( \bar{T}_f(t, L) \to \infty \) as \( t \to \infty \). Let \((L, h) \to N\) be a positive line bundle over \( N \). The defect of \( f \) with respect to \( D \) is defined by
\[\tilde{\delta}_f(D) = 1 - \limsup_{t \to \infty} \frac{\bar{N}_f(t, D)}{\bar{T}_f(t, L)}\]

Then \( \tilde{\delta}_f(D) = 1 \) if \( f \) omits \( D \). Another defect \( \tilde{\Theta}_f(D) \) is defined by
\[\tilde{\Theta}_f(D) = 1 - \limsup_{t \to \infty} \frac{\bar{N}_f(t, D)}{\bar{T}_f(t, L)},\]

where \( \bar{N}_f(t, D) = \bar{N}(t, \text{Supp} f^* D) \). We have \( 0 \leq \tilde{\delta}_f(D) \leq \tilde{\Theta}_f(D) \leq 1 \).

In what follows, we give some conditions for \( \bar{T}_f(t, L) < \infty \) as \( 0 < t < \infty \) and \( \bar{T}_f(t, L) \to \infty \) as \( t \to \infty \) provided that \( f \) is differentiably non-degenerate and \( L > 0 \).

**Lemma 4.5.** Each of the following conditions ensures that \( \bar{T}_f(t, L) < \infty \) as \( 0 < t < \infty \).

(i) \( f \) has finite energy, i.e., \( \int_M e^{f^* c_1(L, h)}(x)DV(x) < \infty \);
(ii) the energy density function \( e^{f^* c_1(L, h)}(x) \) is bounded;
(iii) \( R_M(x) \geq -k(r(x)) \) for a nondecreasing function \( k \geq 0 \) on \([0, \infty)\) with \( k(r)/r^2 \to 0 \) as \( r \to \infty \) and \((1)\);
(iv) \( R_M(x) \geq -k \) for a constant \( k \geq 0 \) with
\[\int_1^\infty e^{-\epsilon r^2} \sup_{x \in B_o(r)} e^{f^* c_1(L, h)}(x) dr < \infty\]

for any \( \epsilon > 0 \).

**Proof.** (i) and (ii) are obvious. (iii) can be verified by applying the estimate of \( p(t, o, x) \) due to Li-Yau \[15\]. (iv) follows from (iii) since the boundedness of Ricci curvature implies that \( \text{Vol}(B_o(r)) \) has at most the exponential growth. The arguments here is also referred to Proposition 6 in \[3\]. \( \square \)

**Lemma 4.6.** Each of the following conditions ensures that \( \bar{T}_f(t, L) \to \infty \) as \( t \to \infty \).
(i) there exists no nonconstant bounded harmonic functions on \( M \);
(ii) \( M \) is parabolic, i.e., there exists no nonconstant bounded plurisubharmonic functions on \( M \);
(iii) \( \text{Ric}_M \geq 0 \).

Proof. Since \( L > 0 \), we can identify \( N \) with an algebraic subvariety of \( \mathbb{P}^K(\mathbb{C}) \) for some integer \( K > 0 \). Let \( H_N \) be the restriction of hyperplane line bundle \( H \) over \( \mathbb{P}^K(\mathbb{C}) \) to \( N \). Note that

\[
C_1 \tilde{T}_f(t, H_N) \leq \tilde{T}_f(t, L) \leq C_2 \tilde{T}_f(t, H_N)
\]

for some constants \( C_1, C_2 > 0 \). Denoted by \( w_0 : \cdots : w_K \) the homogeneous coordinate system of \( \mathbb{P}^K(\mathbb{C}) \). Assuming \( w_0 \circ f \neq 0 \) without loss of generality.

Then

\[
u := \log(1 + |\zeta_1 \circ f|^2 + \cdots + |\zeta_K \circ f|^2)
\]

is a plurisubharmonic function outside \( (w_0 \circ f = 0) \), where \( \zeta_j = w_j/w_0 \) for \( 1 \leq j \leq K \). Condition (i) implies that

\[
\tilde{T}_f(t, H_N) = \frac{1}{4} E_o \left[ \int_0^t \Delta u(X_s) ds \right] \to \infty
\]
as \( t \to \infty \). Since \( \tilde{T}_f(t, H_N) \leq \tilde{T}_f(t, H_N) \) and \( (22) \), we have (i) holds. (ii) and (iii) follow from (i).

Theorem 4.7 (defect relation). Assume the same conditions as in Theorem 1.3 and \( \tilde{T}_f(t, L) \to \infty \) as \( t \to \infty \). Then

\[
\tilde{\Theta}_f(D) \leq \left[ c_1(K^*_N) \right] - \left[ \frac{\mathcal{R}_M}{f^c c_1(L)} \right].
\]

Proof. By Theorem 1.3 it follows that

\[
1 - \frac{N_f(t, D)}{T_f(t, L)} \leq \frac{\tilde{T}_f(t, K^*_N)}{\tilde{T}_f(t, L)} - \frac{\tilde{T}(t, \mathcal{R}_M)}{\tilde{T}_f(t, L)}.
\]

Let \( t \to \infty \), we have the claim.

Corollary 4.8. Assume the same conditions as in Theorem 4.7. If \( s_M \geq 0 \), then

\[
\tilde{\Theta}_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right].
\]

Proof. Since \( s_M = -\frac{1}{4} \Delta_M \log \det(g_{ij}) \geq 0 \) implies \( \mathcal{R}_M = -dd^c \log \det(g_{ij}) \geq 0 \), then the conclusion holds.
Corollary 4.9. Let $D_j \in |L|$ for $1 \leq j \leq q$ such that $\sum_{j=1}^{q} D_j$ has only simple normal crossings. Assume the same conditions as in Theorem 4.7. If $s_M \geq 0$, then

$$\sum_{j=1}^{q} \tilde{\Theta}_f(D_j) \leq \frac{1}{q} \left[ \frac{c_1(K^*_\mathbb{P}^n)}{c_1(L)} \right].$$

Corollary 4.10. Let $D_1, \cdots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_1, \cdots, d_q$ such that $\sum_{j=1}^{q} D_j$ has only simple normal crossings. Assume that $f : M \to \mathbb{P}^n(\mathbb{C})$ (dim$_\mathbb{C} M \geq n$) is a differentiably non-degenerate holomorphic mapping such that $\tilde{T}(t, \omega_{FS}) < \infty$ for $0 < t < \infty$. If $s_M \geq 0$ satisfies

$$E_o \left[ \int_{0}^{t} s_M(X_s) ds \right] < \infty$$

for $0 < t < \infty$. Then

$$q \sum_{j=1}^{q} d_j \Theta_f(D_j) \leq n + 1.$$

Proof. Since $s_M \geq 0$ implies that $R_M \geq 0$, then it follows $R_M \geq 0$. Thereby, $\tilde{T}(t, L) \to \infty$ as $t \to \infty$ due to Lemma 4.6. Note that

$$0 \leq T(t, R_M) = -\frac{1}{4} E_o \left[ \int_{0}^{t} \Delta_M \log(g_{\overline{\partial}})(X_s) ds \right] = E_o \left[ \int_{0}^{t} s_M(X_s) ds \right] < \infty.$$

It is an immediate consequence by using the facts $c_1(K^*_\mathbb{P}^n) = (n + 1)[\omega_{FS}]$ and $c_1(L_{D_j}) = d_j[\omega_{FS}]$. \qed

Corollary 4.11. Let $a_1, \cdots, a_q$ be different points in a compact Riemann surface $S$ of genus $g$. Assume that $f : M \to S$ is a differentiably non-degenerate holomorphic mapping such that $\tilde{T}(t, L_{a_1}) < \infty$ for $0 < t < \infty$. If $s_M \geq 0$ satisfies

$$E_o \left[ \int_{0}^{t} s_M(X_s) ds \right] < \infty$$

for $0 < t < \infty$. Then

$$q \sum_{j=1}^{q} \Theta_f(a_j) \leq 2 - 2g.$$

Proof. Apply $c_1(K^*_S) = (2 - 2g)c_1(L_{a_1})$, we can prove the assertion. \qed

In the case when $M$ is non-parabolic, we have
Theorem 4.12. Let $M$ be a parabolic complete Kähler manifold. Let $L 	o N$ be a positive line bundle over a compact complex manifold $N$, and $D \in |L|$ such that $D$ has only simple normal crossings. Assume that $f : M \to N$ ($\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} N$) is a differentiably non-degenerate holomorphic mapping. If

\begin{equation}
\int_M s_M(x) dV(x) < \infty.
\end{equation}

Then we have

(i) Let $R_M(x) \geq -cr^2(x) - c$ for a constant $c > 0$. If $f$ has finite energy, i.e., $E(f) := \int_M e^f c_1(L,h)(x) dV(x) < \infty$, then

\[ \tilde{\Theta}_f(D) \leq \frac{c_1(K_N^*)}{c_1(L)} + \frac{2 \int_M s_M(x) dV(x)}{E(f)}. \]

(ii) Let $R_M(x) \geq -k(r(x))$ for a nondecreasing function $k \geq 0$ such that $k(r)/r^2 \to 0$ as $r \to \infty$, and satisfies \[(1).\] If $f$ has infinite energy, then

\[ \tilde{\Theta}_f(D) \leq \frac{c_1(K_N^*)}{c_1(L)}. \]

Proof. (i) From Lemma [4.6] note that $\tilde{T}_f(t,L) \to \infty$ as $t \to \infty$. Ricci curvature assumption implies that $M$ is stochastically complete and parabolicity assumption implies that ratio ergodic theorem holds (see [16]). Apply ratio ergodic theorem, we get

\[ \frac{\tilde{T}(t,R_M)}{T_f(t,L)} = \frac{2 \mathbb{E}_0 \left[ \int_0^t s_M(X_s) ds \right]}{\mathbb{E}_0 \left[ \int_0^t e^{f^* c_1(L,h)}(X_s) ds \right]} \to \frac{2 \int_M s_M(x) dV(x)}{\int_M e^{f^* c_1(L,h)}(x) dV(x)} \]

as $t \to \infty$. Thus, $\tilde{T}(t,L) < \infty$ for $t < \infty$ and

\[ -\left[ \frac{R_M}{f^* c_1(L)} \right] \leq \frac{2 \int_M s_M(x) dV(x)}{E(f)}. \]

Invoking Theorem [4.7] (i) holds. For (ii), we first note that $T_f(t,L)$ makes sense by Lemma [4.5]. Assertion (ii) follows by applying ratio ergodic theorem provided $E(f) = \infty$. \qed

In the case when $M$ is non-parabolic, we have
Theorem 4.13. Let $M$ be a non-parabolic complete Kähler manifold with Ricci curvature satisfying \((23)\) and $R_M(x) \geq -k(r(x))$ for a nondecreasing function $k \geq 0$ such that $k(r)/r^2 \to 0$ as $r \to \infty$. Let $L \to N$ be a positive line bundle over a compact complex manifold $N$, and $D \in |L|$ such that $D$ has only simple normal crossings. Assume that $f : M \to N$ (dim$_C M \geq$ dim$_C N$) is a differentiably non-degenerate holomorphic mapping satisfying \((1)\) and $\bar{T}_f(t,L) \to \infty$ as $t \to \infty$. Then

$$\tilde{\Theta}_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right].$$

Proof. The non-parabolicity implies that

$$|\bar{T}(t,R_M)| \leq \mathbb{E}_o \left[ \int_0^\infty s_M(X_t)dt \right] \leq \mathbb{E}_o \left[ \int_0^\infty R_M(X_t)dt \right] < \infty.$$

Thus, the assertion follows from Theorem 4.7. \(\square\)

Proof of Theorem 1.4

Proof. $s_M \geq 0$ implies that $\bar{T}_f(t,L) \to \infty$ as $t \to \infty$, the energy assumption implies that $\bar{T}_f(t,L) < \infty$ for $t < \infty$ due to Lemma 4.5. Notice $s_M \geq mR_M$, where $m = \text{dim}_C M$. It yields that

$$|\bar{T}(t,R_M)| \leq \mathbb{E}_o \left[ \int_0^\infty s_M(X_t)dt \right] \leq m\mathbb{E}_o \left[ \int_0^\infty R_M(X_t)dt \right] < \infty,$$

which deduces the theorem by Theorem 4.7. \(\square\)

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