Quantum nonlocality of Heisenberg XX model with Site-dependent Coupling Strength

Chunfeng Wu†, Jing-Ling Chen†, D. M. Tong†, L. C. Kwek‡, and C. H. Oh†
† Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542
‡ Nanyang Technological University, National Institute of Education, 1, Nanyang Walk, Singapore 637616
E-mail: g0201819@nus.edu.sg

Abstract. We show that the generalized Bell inequality is violated in the extended Heisenberg model when the temperature is below a threshold value. The threshold temperature values are obtained by constructing exact solutions of the model using the temperature-dependent correlation functions. The effect due to the presence of external magnetic field is also illustrated.

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1. Introduction

An intriguing aspect of quantum mechanics is the lack of a local realistic description that could reproduce the necessary correlations for the experimental outcomes in composite systems\[1\]. This lack of local realism can be investigated using the entangled state as discussed in the original seminal paper by Einstein, Podolsky and Rosen. Nowadays, we recognize the importance of entanglement as a valuable resource for quantum information processing and communication. Its usefulness has since been demonstrated clearly in processes like quantum teleportation\[2, 3\], quantum computation\[4\], and quantum cryptography\[5\].

However, concepts such as entanglement and its implications concerning the non-existence of a local realism in quantum mechanics have a more fundamental role in quantum mechanics. The issue of "locality" as well as notion of quantum measurements has given rise to some of the recent and modern interpretations of quantum mechanics as well as a better understanding of quantum phenomena\[6\]. It is also amidst all these theoretical constructs that Bell proposed an inequality that could rule out the hidden variable description of quantum mechanics\[7\]. Since then, several variants of Bell inequalities, some of which were more amenable for experimental investigations, have been derived for two-body correlation functions to investigate the existence of local realism\[8\].

Recently there has been much work on the implementation of quantum processing on solid state devices. In this paper, we study the thermal states in a system of interaction spins and investigate its quantum “nonlocality”. An interesting type of entanglement, thermal entanglement, was studied in the context of the Heisenberg XXX\[9, 10\], XX\[11\], and XXZ\[12\] models. The Heisenberg model has been shown to have a potential candidate as a model for spin-spin interaction in a solid state quantum computer\[13\]. Being the large Coulomb repulsion limit of the Hubbard model, it has been partially realized in quantum dots\[13\], nuclear spins\[14\], and optical lattices\[15\]. In a recent work, Imamoglu et al\[16\] have realized quantum information processing using quantum dot spins and cavity QED, and obtained an effective interaction Hamiltonian based on the XY spin chain between two quantum dots. The effective Hamiltonian was shown to be capable of constructing the Controlled-Not gate\[16\]. The XY Hamiltonian is given by

$$H = \sum_{n=1}^{N} (J_1 S^x_n S^x_{n+1} + J_2 S^y_n S^y_{n+1})$$  \hspace{1cm} (1)

where $S^i = \sigma^i/2$ ($i = x, y, z$) and $\sigma^i$ are Pauli operators. When $J_1 = J_2$, the XY model becomes XX model. In the XY model, the interaction strength between neighboring sites is usually assumed to be independent of the sites. In most solid state models however, the inter-site coupling strength is site dependent. In this paper we consider an extended quantum XX model in which the interaction strength assumes a particular site dependent form.

This paper is organized as follows. In Sec\[2\] solutions of the extended XX model for 4 particles are given. In Sec\[3\] we construct the temperature dependent correlation functions in terms of thermal equilibrium state and investigate the violation of Bell inequality for the thermal state. The threshold temperature is given. We also point out that the eigenstates of the extended XX model do not realize maximal violation of Bell inequality. Effect of external magnetic field is discussed in Sec\[5\] and we end with some discussions in the final section.
2. Solution of the extended XX model

The extended XX Heisenberg model is described by the Hamiltonian

\[ H = 2 \sum_{n=1}^{N-1} J_{n,n+1} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y \right) \]

\[ = \sum_{n=1}^{N-1} J_{n,n+1} \left( \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ \right) \]

(2)

where \( J_{n,n+1} = \sqrt{n(N-n)} \) is the coupling strength between lattices \( n \) and \( n+1 \). Obviously, the Hamiltonian \( H \) describes a nearest-neighbor interaction spin chain.

Interestingly, such a Hamiltonian has been shown to be useful for perfect state transfer in quantum spin networks [17]. The Hamiltonian \( H \) possesses \( 2^N \) complete and orthonormal eigenstates.

When spin chains are subjected to environmental disturbance, they inevitably become thermal equilibrium states. The state of a system at finite temperature \( T \) is given by the Gibb’s density operator \( \rho(T) = \exp(-H/kT)/Z \), where \( Z = \text{Tr}[\exp(-H/kT)] \) is the partition function, \( H \) is the system Hamiltonian and \( k \) is the Boltzmann constant, which is set to unity for convenience in this paper. At high temperature, the thermal state becomes maximally mixed and do not violate Bell inequalities of any kind. It is therefore interesting to consider the critical temperature at which a Bell inequality will be violated. For a two-qubit system, we have the original Bell inequality. For arbitrary number of qubits, we have the Zukowski-Brukner inequality[8].

Unfortunately it is not possible to test Zukowski-Brukner inequality for three qubits in this case since the correlation functions defined below are zero. Therefore, in this paper, we first focus on the next non-trivial case of a 4-qubit system and test the violation of local realistic description using the Zukowski-Brukner inequality. The extension to arbitrary number of sites, albeit complicating, can also be done in the same manner. The Hamiltonian has sixteen eigenvalues

\[ E_0 = E_7 = E_8 = E_{15} = 0, \]
\[ E_3 = E_{13} = -1, \quad E_4 = E_{14} = 1, \]
\[ E_6 = -2, \quad E_9 = 2, \]
\[ E_1 = E_{11} = -3, \quad E_2 = E_{12} = 3, \]
\[ E_5 = -4, \quad E_{10} = 4. \]

(3)

The corresponding eigenstates \( \{|\phi_0\rangle, |\phi_1\rangle, \ldots |\phi_{15}\rangle \} \) can be computed easily and can be found in appendix [Appendix A]. The above eigenvalues and eigenstates completely determine the thermal states. The density operator \( \rho(T) \) at the temperature \( T \) can be written as

\[ \rho(T) = \frac{1}{Z} \sum_{\mu=0}^{15} e^{-\beta E_\mu} |\phi_\mu\rangle \langle \phi_\mu| \]

(4)

where \( \beta = 1/T \) and the partition function

\[ Z = \text{Tr}(e^{-\beta H}) = \sum_{\mu=0}^{15} e^{-\beta E_\mu} \]

\[ = 4 + 4 \cosh(3\beta) + 4 \cosh \beta + 2 \cosh(4\beta) + 2 \cosh(2\beta) \]

(5)
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3. Violation of 4-qubit Bell inequality and the threshold temperature

To test quantum nonlocality for the state $\rho(T)$, correlation function $Q_{ijkl}$ should be computed. From the definition of $Q_{ijkl}$, we have

$$Q_{ijkl} = \text{Tr}[\rho \langle \hat{n}_i \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_j \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_k \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_l \cdot \hat{\sigma} \rangle]$$

$$= \frac{1}{Z} \sum_{\mu=0}^{15} e^{-\beta E_\mu} \text{Tr}[\phi_\mu \langle \hat{n}_i \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_j \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_k \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_l \cdot \hat{\sigma} \rangle]$$

$$= \frac{1}{Z} \sum_{\mu=0}^{15} e^{-\beta E_\mu} Q^\mu_{ijkl} \quad (6)$$

where $\hat{n}_\alpha = (\sin \theta_\alpha, 0, \cos \theta_\alpha)$, $\alpha = i, j, k, l$. $Q^\mu_{ijkl}$ is the correlation function for the eigenstate $|\phi_\mu\rangle$,

$$Q^\mu_{ijkl} = \text{Tr}[\phi_\mu \langle \hat{n}_i \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_j \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_k \cdot \hat{\sigma} \rangle \otimes \langle \hat{n}_l \cdot \hat{\sigma} \rangle] \quad (7)$$

For instance, the quantum correlation for the ground state $|\phi_0\rangle$ is given by

$$Q^5_{ijkl} = \cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + \frac{\sqrt{3}}{2} \cos \theta_k \cos \theta_l \sin \theta_i \sin \theta_j$$

$$- \frac{\sqrt{3}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k + \frac{1}{2} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k$$

$$+ \frac{1}{2} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l - \frac{\sqrt{3}}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l$$

$$+ \frac{\sqrt{3}}{2} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l + \sin \theta_i \sin \theta_j \sin \theta_k \sin \theta_l. \quad (8)$$

Other quantum correlation functions can also be calculated in a similar way. The correlation function for the thermal state $\rho(T)$ are computed using Eq. (6). Based on the calculated values of $Q_{ijkl}$, we construct Bell quantity $B$

$$B = Q_{1111} - Q_{1112} - Q_{1121} - Q_{1122} - Q_{1211} - Q_{1212}$$

$$- Q_{1221} + Q_{1222} - Q_{2111} - Q_{2112} - Q_{2121} + Q_{2122}$$

$$- Q_{2211} + Q_{2212} + Q_{2221} + Q_{2222} \quad (9)$$

For a local realistic description, we require $-4 \leq B \leq 4$. In Figure 1 we have numerically computed the Bell quantity as a function of temperature. The results show that violation of the Bell inequality occurs at $T \leq T_0 = 0.626$. We call this critical value $T_0$ the threshold temperature. The maximum value of $B$ for the state $\rho(T)$ approaches 7.917 at temperature close to zero.

We have also evaluated the Bell quantity $B(|\phi_\mu\rangle)$ in terms of correlation functions of each pure state $|\phi_\mu\rangle$. The maximum value of $B(|\phi_\mu\rangle)$ are

$$B_{\text{max}}(|\phi_\mu\rangle) = 4 \quad \text{for} \ |\phi_{0,15}\rangle$$

$$6.112 \quad \text{for} \ |\phi_{1,2,3,4,11,12,13,14}\rangle$$

$$7.917 \quad \text{for} \ |\phi_{0,10}\rangle$$

$$5.657 \quad \text{for} \ |\phi_{0,9}\rangle$$

$$4.866 \quad \text{for} \ |\phi_{7}\rangle$$

$$4.060 \quad \text{for} \ |\phi_{8}\rangle \quad (10)$$
We can explain qualitatively why the maximum value of $B$ for the thermal state should be 7.917 by noting that the thermal state $\rho(T)$ is the linear combination of $|\phi_{\mu}\rangle\langle\phi_{\mu}|$ weighted with the factors $e^{-\beta E_{\mu}}$. For eigenvalue $E_5 = -4$, $B_{\max}(|\phi_5\rangle) = 7.917$, the power is $e^{4\beta}$ and when $\beta$ is large enough, the Bell quantity $B$ is totally determined by the contribution of state $|\phi_5\rangle$. Another thing worth noting is that the eigenstates of extended XX model do not lead to highest value of $B_{\max}$. We check the maximum value of the Bell quantities consist of correlation functions for the following three general states

\begin{align}
|\varphi'\rangle &= \cos \alpha_1 |1000\rangle + \sin \alpha_1 \cos \alpha_2 |0100\rangle \\
&\quad + \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 |0010\rangle + \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 |0001\rangle \\
|\varphi''\rangle &= \cos \alpha_1 |1110\rangle + \sin \alpha_1 \cos \alpha_2 |1101\rangle \\
&\quad + \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 |1011\rangle + \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 |0111\rangle \\
|\varphi'''\rangle &= \cos \alpha_1 |1100\rangle + \sin \alpha_1 \cos \alpha_2 |1010\rangle + \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 |1001\rangle \\
&\quad + \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \cos \alpha_4 |0110\rangle + \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \sin \alpha_4 \cos \alpha_5 |0101\rangle \\
&\quad + \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \sin \alpha_4 \sin \alpha_5 |0011\rangle \\
\end{align}

and find that

\begin{align}
B_{\max}(|\varphi'_0\rangle) &= 6.217 \\
B_{\max}(|\varphi''_0\rangle) &= 6.217 \\
B_{\max}(|\varphi'''_0\rangle) &= 8.485 \\
\end{align}

for $|\varphi'_0\rangle = 1/2(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$, $|\varphi''_0\rangle = 1/2(|1110\rangle + |1101\rangle + |1011\rangle)$ and $|\varphi'''_0\rangle = 1/\sqrt{6}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle)$ respectively. It is easy to see that the degree of violation of Bell inequality for state $|\varphi'_0\rangle$ is higher than that for the eigenstates $|\phi_{\mu}\rangle$, ($\mu = 1, 2, 3, 4$) listed in Eq. (13). The same results also happen for the eigenstates $|\phi_{\mu}\rangle$, ($\mu = 11, 12, 13, 14$) and $|\varphi''_0\rangle$, ($\mu = 5, 6, 7, 8, 9, 10$) respectively. We see that among all possible $B_{\max}$, the state $|\varphi'''_0\rangle$ yields the largest violation.
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4. The effect of external magnetic field

In this section, we would like to study the effect of magnetic field on the nonlocality property of thermal state in a general way, for which the Hamiltonian becomes

$$H' = 2 \sum_{n=1}^{N-1} J_{n,n+1}(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+) + B \sum_{n=1}^{N} \sigma_z$$  \hspace{1cm} (15)$$

where $B$ is the strength of the magnetic field. It is easy to verify that the eigenstates of $H'$ are identical with the ones listed in expression(A.2) of $H$, but with different eigenvalues.

$$E'_0 = 4B, \quad E'_1 = -3 + 2B, \quad E'_2 = 3 + 2B, \quad E'_3 = -1 + 2B,$$
$$E'_4 = 1 + 2B, \quad E'_5 = -4, \quad E'_6 = -2, \quad E'_7 = 0,$$
$$E'_8 = 0, \quad E'_9 = 2, \quad E'_{10} = 4, \quad E'_{11} = -3 - 2B,$$
$$E'_{12} = 3 - 2B, \quad E'_{13} = -1 - 2B, \quad E'_{14} = 1 - 2B, \quad E'_{15} = -4B. \hspace{1cm} (16)$$

and hence, a new correlation function and Bell quantity $B'$ are given

$$Q'_{ijkl} = \frac{1}{Z'} \sum_{\mu=0}^{15} e^{-\beta E'_\mu} Q'_{ijkl}$$  \hspace{1cm} (17)$$

$$B' = Q'_{1111} - Q'_{1112} - Q'_{1121} - Q'_{1122} - Q'_{1211} - Q'_{1212} - Q'_{1221} + Q'_{1222} + Q'_{2111} - Q'_{2112} - Q'_{2121} + Q'_{2122} - Q'_{2211} + Q'_{2212} + Q'_{2221} + Q'_{2222} \hspace{1cm} (18)$$

where $Z' = \text{Tr}(e^{-\beta H'})$. Now the violation of Bell inequality depends not only on the temperature, but also on external magnetic field. Our numerical calculations are shown in Fig 2.

There are five curves corresponding to $B = 0.1, 0.5, 1.0, 1.5$, and 2 respectively. When $B = 0.1$, the Bell quantity shows a similar variation of the violation of Bell inequality as a function of $T$ in the absence of magnetic field. With the increasing value of external magnetic field, the maximum value of the Bell quantity approaches the value 2 for which the $B$ field is about 1.5. The variation of the Bell quantity as a function of magnetic field can be explained qualitatively as follows. The $\rho'(T)$
 cases. For 2-qubit extended XX model, the Bell quantity approach has discussed the violation of Bell inequality for thermal state for 2-qubit and 3-qubit temperature. We restrict ourselves to the 4-qubit case. However, we could also imply that quantum "nonlocality" could be effectively controlled by magnetic field of the external magnetic field that for the violation of Bell inequalities. Our results strengths of magnetic field. For a fixed temperature, we can find the optimal value state violates Bell inequality. The effects of temperature are also studied at different critical temperatures are different from those needed for the optimization of magnetic fields. In the latter case, $B_{\text{max}}$ is totally determined by the contribution of state with the largest weight or factor for sufficiently large $\beta$. In the former case, depending on the value of the external magnetic field, the eigenstates contributing to the optimization changes and so the optimization is determined using a combination of the correlation functions from different states. In short, the variation of $T_0$ with $B$ is different from that of $B_{\text{max}}$ with $B$.

## 5. Conclusion

In this paper, we consider the extended Heisenberg XX model, modeling the nearest-neighbor interaction spin chain. For the 4-qubit extended XX model, it is shown that since the correlation functions depend on the temperature and the magnetic field, the violation of Bell inequality for the thermal state depends critically on these two parameters. The effect of temperature for a local realistic description of quantum theory is determined by the threshold value of $T$ below which the thermal state violates Bell inequality. The effects of temperature are also studied at different strengths of magnetic field. For a fixed temperature, we can find the optimal value of the external magnetic field that for the violation of Bell inequalities. Our results imply that quantum “nonlocality” could be effectively controlled by magnetic field and temperature. We restrict ourselves to the 4-qubit case. However, we could also have discussed the violation of Bell inequality for thermal state for 2-qubit and 3-qubit cases. For 2-qubit extended XX model, the Bell quantity approaches $2\sqrt{2}$ which is the

| $B$ | 0   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $T_0$ | 0.626 | 0.611 | 0.556 | 0.447 | 0.248 | $\text{None}$ | 0.122 | 0.243 |

| $B$ | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 and above |
|-----|-----|-----|-----|-----|-----|-----|-----|----------------|
| $T_0$ | 0.351 | 0.427 | 0.467 | 0.472 | 0.436 | 0.343 | 0.18 | $\text{None}$ |

Table 1. Threshold temperatures for different strengths of the external magnetic field. When $B = 0.5$ and $B = 1.5$ and above, the values of Bell quantity are no greater than 4 at all times. Therefore, no threshold temperatures exist for these cases.

is a different combination of $|\phi_\mu\rangle\langle\phi_\mu|$ compared with $\rho(T)$. The largest contribution of all the states $|\phi_\mu\rangle$ is determined by the value of $B$. When $B < 0.5$, it is the eigenstate, $|\phi_5\rangle$, which ultimately determines the maximal value of the Bell quantity ($B_{\text{max}} = 7.917$) since $e^{-\beta E_i^5} = e^{4\beta}$ is the largest power among all the factors. When $0.5 < B < 1.5$, $|\phi_{11}\rangle$ takes the place of $|\phi_5\rangle$ with power $e^{(3+2B)\beta}$ and $B_{\text{max}} = 6.112$ at $B = 1.0$, for example. When $B > 2$, $e^{-\beta E_i^5} = e^{4\beta}$ is the one with largest contribution and $B_{\text{max}} = 4$. But there are two singular values of $B = 0.5$ and 1.5. In these two cases, $B_{\text{max}} < 4$. The reason for this is that the largest factors of $e^{-\beta E_i^5}$ are $e^{-\beta E_i^5} = e^{-\beta E_i^{11}} = e^{4\beta}$ for $B = 0.5$, $e^{-\beta E_i^{15}} = e^{-\beta E_i^{11}} = e^{6\beta}$ for $B = 1.5$, respectively. Thus the Bell quantity is determined principally using a combinations of these two elements of $Q_{ijkl}^\mu$, namely, $e^{4\beta}(Q_{ijkl}^5 + Q_{ijkl}^{11})$ and $e^{6\beta}(Q_{ijkl}^{15} + Q_{ijkl}^{11})$. Note that the maximum values of the Bell quantity for the latter two correlation functions are 2.228 and 2.081 respectively.

The critical temperatures under different magnetic fields have been found (Table 4). The variation of $T_0$ with increasing strengths of $B$ is more complicated. This complication arises mainly because the eigenstates contributing to the optimization of critical temperatures are different from those needed for the optimization of magnetic fields. In the latter case, $B_{\text{max}}$ is totally determined by the contribution of state with the largest weight or factor for sufficiently large $\beta$. In the former case, depending on the value of the external magnetic field, the eigenstates contributing to the optimization changes and so the optimization is determined using a combination of the correlation functions from different states. In short, the variation of $T_0$ with $B$ is different from that of $B_{\text{max}}$ with $B$.
maximal violation of 2-qubit Bell inequality and the corresponding threshold value of temperature is $T_0 = 0.667$ when $B = 0$. However, for 3-qubit case, the correlation function defined by this method is always equal to 0. The violation of Bell inequality for arbitrary number of qubit can also be done in the same manner.

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Appendix A. Eigenstates of the 4-qubit Hamiltonian

Corresponding to the sixteen eigenvalues of the Hamiltonian

$$
E_0 = E_7 = E_8 = E_{15} = 0,
E_3 = E_{13} = -1,
E_4 = E_{14} = 1,
E_6 = -2,
E_9 = 2,
E_1 = E_{11} = -3,
E_2 = E_{12} = 3,
E_5 = -4,
E_{10} = 4,
$$

(A.1)

the orthogonal eigenstates are

$$
|\phi_0\rangle = |0000\rangle
$$
$$
|\phi_1\rangle = \frac{1}{2\sqrt{2}}(-|1000\rangle + \sqrt{3}|0100\rangle - \sqrt{3}|0010\rangle + |0001\rangle)
$$
$$
|\phi_2\rangle = \frac{1}{2\sqrt{2}}(|1000\rangle + \sqrt{3}|0100\rangle + \sqrt{3}|0010\rangle + |0001\rangle)
$$
$$
|\phi_3\rangle = \frac{\sqrt{3}}{2\sqrt{2}}(|1000\rangle - \frac{1}{\sqrt{3}}|0100\rangle - \frac{1}{\sqrt{3}}|0010\rangle + |0001\rangle)
$$
$$
|\phi_4\rangle = \frac{\sqrt{3}}{2\sqrt{2}}(|1000\rangle - \frac{1}{\sqrt{3}}|0100\rangle + \frac{1}{\sqrt{3}}|0010\rangle + |0001\rangle)
$$
$$
|\phi_5\rangle = \frac{1}{4}(|1100\rangle - 2|1010\rangle + \sqrt{3}|1001\rangle + \sqrt{3}|0110\rangle - 2|0101\rangle + |0011\rangle)
$$
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|φ₆⟩ = \frac{1}{2}(-|1100⟩ + |1010⟩ - |0101⟩ + |0011⟩)
|φ₇⟩ = \frac{\sqrt{3}}{\sqrt{10}}(|1100⟩ - \frac{2}{\sqrt{3}}|1001⟩ + |0011⟩)
|φ₈⟩ = \frac{5}{2\sqrt{10}}(-\frac{\sqrt{3}}{5}|1100⟩ - \frac{3}{5}|1001⟩ + |0110⟩ - \frac{\sqrt{3}}{5}|0011⟩)
|φ₉⟩ = \frac{1}{2}(-|1100⟩ - |1010⟩ + |0101⟩ + |0011⟩)
|φ₁₀⟩ = \frac{1}{4}(|1100⟩ + 2|1010⟩ + \sqrt{3}|1001⟩ + \sqrt{3}|0110⟩ + 2|0101⟩ + |0011⟩)
|φ₁₁⟩ = \frac{1}{2\sqrt{2}}(-|1110⟩ + \sqrt{3}|1101⟩ - \sqrt{3}|1011⟩ + |0111⟩)
|φ₁₂⟩ = \frac{1}{2\sqrt{2}}(|1110⟩ + \sqrt{3}|1101⟩ + \sqrt{3}|1011⟩ + |0111⟩)
|φ₁₃⟩ = \frac{\sqrt{3}}{2\sqrt{2}}(|1110⟩ - \frac{1}{\sqrt{3}}|1101⟩ - \frac{1}{\sqrt{3}}|1011⟩ + |0111⟩)
|φ₁₄⟩ = \frac{\sqrt{3}}{2\sqrt{2}}(-|1110⟩ - \frac{1}{\sqrt{3}}|1101⟩ + \frac{1}{\sqrt{3}}|1011⟩ + |0111⟩)
|φ₁₅⟩ = |1111⟩

(A.2)

Appendix B. Quantum correlation functions for each pure states
The calculation of the quantum correlation functions is straightforward. In this appendix, we list all the correlation functions for each eigenstate of the 4-qubit Hamiltonian for easy reference.
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| correlation function | explicit expression |
|----------------------|---------------------|
| $Q_{ijkl}^{0} = Q_{ijkl}^{13}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l$ |
| $Q_{ijkl}^{1} = Q_{ijkl}^{11}$ | $-\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l - \frac{\sqrt{2}}{4} \cos \theta_k \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k - \frac{3}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
|                     | $- \frac{1}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l + \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_j \sin \theta_l$ |
|                     | $- \frac{1}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{2} = Q_{ijkl}^{12}$ | $-\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + \frac{\sqrt{2}}{4} \cos \theta_k \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k + \frac{3}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
|                     | $+ \frac{1}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l + \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_j \sin \theta_l$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{3} = Q_{ijkl}^{13}$ | $-\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l - \frac{\sqrt{2}}{4} \cos \theta_k \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $- \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k + \frac{3}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
|                     | $+ \frac{1}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l - \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_j \sin \theta_l$ |
|                     | $- \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{4} = Q_{ijkl}^{14}$ | $-\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + \frac{\sqrt{2}}{4} \cos \theta_k \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $- \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k - \frac{3}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
|                     | $- \frac{1}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l + \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_j \sin \theta_l$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{5} = Q_{ijkl}^{15}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + \cos \theta_j \cos \theta_i \sin \theta_k \sin \theta_j$ |
|                     | $- \cos \theta_j \cos \theta_i \sin \theta_l - \sin \theta_i \sin \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{6} = Q_{ijkl}^{16}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + 2 \sqrt{\frac{2}{3}} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_l \sin \theta_j + \frac{3}{4} \sin \theta_j \sin \theta_k \sin \theta_l \sin \theta_j$ |
| $Q_{ijkl}^{7} = Q_{ijkl}^{17}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + \sqrt{\frac{2}{3}} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $\times \cos \theta_i \cos \theta_k \sin \theta_l \sin \theta_j + \frac{3}{4} \sin \theta_j \sin \theta_k \sin \theta_l \sin \theta_j$ |
| $Q_{ijkl}^{8} = Q_{ijkl}^{18}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l - \cos \theta_j \cos \theta_i \sin \theta_k \sin \theta_j$ |
|                     | $+ \cos \theta_j \cos \theta_i \sin \theta_l - \sin \theta_i \sin \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{9} = Q_{ijkl}^{19}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l - \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $- \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k - \frac{3}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
|                     | $- \frac{1}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l - \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_j \sin \theta_l$ |
|                     | $- \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
| $Q_{ijkl}^{10} = Q_{ijkl}^{20}$ | $\cos \theta_i \cos \theta_j \cos \theta_k \cos \theta_l + \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_j$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_j \cos \theta_l \sin \theta_i \sin \theta_k - \frac{3}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |
|                     | $+ \frac{1}{4} \cos \theta_j \cos \theta_k \sin \theta_i \sin \theta_l + \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_k \sin \theta_j \sin \theta_l$ |
|                     | $+ \frac{\sqrt{2}}{4} \cos \theta_i \cos \theta_j \sin \theta_k \sin \theta_l$ |