THE NON–ISENTPORIC RELATIVISTIC EULER SYSTEM WRITTEN IN
A SYMMETRIC HYPERBOLIC FORM

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This paper is dedicated to our friend Michael Reissig

Abstract. We cast the non–isentropic relativistic Euler system into a symmetric hyperbolic form. Such systems are very suited to treat initial value problems of hyperbolic type. We obtain this form by using the pressure \( p \) and not the density \( \rho \) as a variable. However, the system becomes degenerate when the pressure \( p \) approaches zero, and in these cases we regularise the system by replacing the pressure with an appropriate new matter variable, the Makino variable.

1. Introduction

Existence and uniqueness theorems of a class of solutions have been proved for the non–relativistic compressible Euler equations for the isentropic case by [Mak86], and later for the non-isentropic case by [MUK86].

The situation, however, for the relativistic compressible Euler equations is more involved. The equivalent to the result obtained by Makino [Mak86], has been proven, for a restricted setting by Rendall, [Ren92], which was later extended by the authors [BK14] and [BK11].

All those results had been obtained by casting, in one way or the other, the Euler equations into a symmetric-hyperbolic first-order system. Such systems had been introduced Friedrich in 1954 [Fri54], and has been one of the most effective approaches to prove the well–posedness (existence, uniqueness, and continuity of the flow map) for these systems.

The non-isentropic case is more complicated. Speck [Spe09] studied the Cauchy problem for the Nordström scalar gravitational field equation coupled to the non–isentropic Euler equations. He proved local existence, uniqueness and the continuity of the flow map, but since he claimed that the system could not be cast into symmetric hyperbolic form, he used Christodoulou’s theory of the energy current [Chr00] to obtain his results.

Choquet-Bruhat studied the Cauchy problem for both, the isentropic and the non–isentropic, Einstein–Euler system, using Leray hyperbolic systems [FB58]. Moreover, she also used a different method relying upon Leray-Ohya hyperbolic systems, see [CB66] and [CB09]. A different approach was proposed by Friedrich [Fri98], with the motivation to treat free initial boundary problems. So he was able to write the relativistic Euler equations in Lagrangian

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coordinates as a symmetric hyperbolic system by differentiating the equations in an appropriate manner. This leads to a system with constraint equations, whose propagation needs to be shown separately. The advantage of his system is the fact that it is more suited to deal with initial free-boundary problems since in Lagrangian coordinates the boundary is fixed.

Disconzi used Friedrich’s approach to derive local existence and uniqueness of classical solutions for the non–isentropic Einstein–Euler system [Dis15], using uniformly local Sobolev spaces, assuming the density to be strictly positive and a smooth equation of state. Another approach for the non–isentropic relativistic Euler equations was presented by Walton [Wal05], however, no local existence and uniqueness system is known using this approach.

The purpose of these notes is to generalize our approach as provided in [BK14] and present the non-isentropic relativistic Euler equations as a symmetric hyperbolic system, which would enable us to prove similar local existence and uniqueness theorem, therefore removing some of the restrictions posed in the results of [Dis15].

2. The relativistic Euler equations with entropy

We now briefly introduce the notion of a relativistic perfect, but and non-isentropic fluid. For more information and the thermodynamical background see for example [FR00], [Chr95], [CB09]. We consider the fluid in a prescribed Lorentzian manifold \((\mathcal{M}, g_{\alpha\beta})\), \(\alpha, \beta = 0, 1, 2, 3\), and we chose units such that the speed of light \(c = 1\). For a perfect fluid, the energy-momentum tensor takes the following form

\[
T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta},
\]  

(2.1)

where \(\epsilon\) is the proper energy density of the fluid, \(p\) is the pressure, and \(u^\alpha\) is the four-velocity, which is subject to the normalization constraint

\[
g_{\alpha\beta}u^\alpha u^\beta = -1.
\]  

(2.2)

The Euler equations for a perfect fluid are (see e.g. [Chr95])

\[
\nabla_{\alpha}T^{\alpha\beta} = 0 \quad (\beta = 0, 1, 2, 3)
\]  

(2.3)

\[
\nabla_{\alpha}(nu^\alpha) = 0,
\]  

(2.4)

where \(n\) is the proper number density and \(\nabla_{\alpha}\) denotes the covariant derivative induced by the spacetime metric \(g_{\alpha\beta}\). As we will discuss in section 3.2, the projection \(u_\beta \nabla_\alpha T^{\alpha\beta} = 0\) leads to the energy equation

\[
u^\nu \nabla_\nu \epsilon + (\epsilon + p) \nabla_\nu u^\nu = 0.
\]  

(2.5)

A non-isentropic fluid contains a thermodynamic variable \(s\) that represents the Entropy, and satisfies the following thermodynamic relation, called Gibbs relation, [CB09]

\[
Td\left(\frac{\epsilon}{n}\right) + pd\left(\frac{1}{n}\right),
\]  

(2.6)

where \(T\) denotes the temperature. As it was proven by Pichon [Pic65], the energy equation (2.5), the rest-mass conservation equation (2.4) and the Gibbs relation (2.6) imply the following relation for the entropy

\[
u^\alpha \nabla_\alpha s = 0,
\]  

(2.7)

which just expresses the fact that it is conserved along the fluid lines.
The equation of state specifies the relations between the number density $n$, entropy $s$, and the mass density $\epsilon$. We assume an equation of state is given by a nonnegative function

$$\epsilon = \epsilon(n, s), \quad n, s \geq 0. \quad (2.8)$$

From laws of thermodynamics (see e.g. [FR00]) it follows that the pressure is given by

$$p = n \frac{\partial \epsilon}{\partial n} - \epsilon, \quad (2.9)$$

and the speed of sound is given by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon} = \frac{\partial p}{\partial n} \frac{\partial n}{\partial \epsilon}. \quad (2.10)$$

A fundamental thermodynamic assumption is that the right-hand side of (2.10) is positive, hence we require that

$$\frac{\partial \epsilon}{\partial n} > 0, \quad \frac{\partial p}{\partial n} > 0. \quad (2.11)$$

Another requirement is that $\sigma < 1$, which means that the sound speed is always less than the speed of light.

### 2.1. Energy conditions.

The General Relativity literature refers to three types of energy conditions (see e.g. [CB09]). The energy-momentum tensor $T_{\alpha\beta}$ satisfies:

1. The weak energy condition, if $T_{\alpha\beta}X^\alpha X^\beta \geq 0$ for all timelike vectors $X^\alpha$.
2. The strong energy condition, if $[T_{\alpha\beta} - T g_{\alpha\beta}]X^\alpha X^\beta \geq 0$ for all timelike vectors $X^\alpha$, where $T = g_{\mu\nu}T^{\mu\nu}$.
3. The dominant energy condition, if $-T_{\alpha\beta}X^\beta$ is timelike future-directed vector for all $X^\alpha$ future-directed timelike vector.

Whenever $\epsilon \geq 0$ and $p \geq 0$, the perfect fluid satisfies the weak and strong energy conditions. If $\epsilon \geq p$, then it satisfies also the dominant energy condition, see [CB09]. We shall see that the examples below meet all the three energy conditions.

### 2.2. Examples of an equation of state for the non–isentropic Euler equations.

A typical non-isentropic equation of state is given by (see also [GTZ99])

$$\epsilon = n + \frac{A(s)}{\gamma - 1} n^\gamma, \quad (2.12)$$

where $1 < \gamma < 2$ and $A(s)$ is a positive function. Equation (2.9) implies that $p = A(s)n^\gamma$, and from (2.10) we can compute the speed of sound as follows,

$$\sigma^2 = \frac{\gamma(\gamma - 1)A(s)n^{\gamma - 1}}{(\gamma - 1) + \gamma A(s)n^{\gamma - 1}}. \quad (2.13)$$

As a function of $n$, the speed of sound $\sigma$ is increasing and tends to $\sqrt{\gamma - 1}$ as $n$ tends to infinity. Hence the speed of sound is less than the speed of light. The equation of state (2.12) also satisfies the dominant energy condition, since

$$\epsilon - p = n + \frac{(2 - \gamma)A(s)n^\gamma}{\gamma - 1} \geq 0. \quad (2.14)$$
Another example is a polytropic equation of state with index \( \gamma = \frac{4}{3} \). We follow the convention of Choquet–Bruhat [CB09], here

\[
p = \frac{K}{3} \left( \frac{3s}{4K} \right) ^{\frac{4}{3}} n^{\frac{1}{3}} \quad \text{and} \quad \epsilon = 3p + n,
\]

where \( K \) is a positive constant. We see that \( \frac{\partial p}{\partial n} = \frac{4K}{3} \left( \frac{3s}{4K} \right) ^{\frac{4}{3}} n^{\frac{1}{3}} + 1 = \frac{p + \epsilon}{n} \), hence (2.9) is fulfilled. We also note that

\[
p = n + K \left( \frac{3s}{4K} \right) ^{\frac{4}{3}} n^{\frac{1}{3}},
\]

and hence it is a particular case of the equation of state (2.12). So this equation of state also satisfies the dominant energy condition.

3. The non–isentropic equations in symmetric hyperbolic form

The equation of state (2.8) and the explicit formula of the pressure (2.9) allows us to express the pressure \( p \) as a function of \( n \) and \( s \), which leads to consider \( U = (n, w^\alpha, s) \), \( \alpha = 0, 1, 2, 3 \) as the unknowns for the Euler equations (2.3) and (2.4).

However, such an equation of state implies also that \( \nabla_\alpha p = \frac{\partial p}{\partial n} \nabla_\alpha n + \frac{\partial p}{\partial s} \nabla_\alpha s \), which destroys the symmetry of the corresponding matrices and makes it almost impossible to cast the Euler equations in symmetric hyperbolic form. The same problem occurs for the non-relativistic case, and there the solution consists in using the pressure \( p \) as a matter variable instead of the density \( n \).

That is why we take a similar approach here for the relativistic equations and cast the equations in symmetric hyperbolic form.

Moreover, the resulting system is a more convenient starting point to introduce the regularizing Makino variable.

3.1. Symmetric Hyperbolic Systems. We recall the definition of symmetric hyperbolic systems.

**Definition 1** (Symmetric hyperbolic system). A first order quasi–linear \( k \times k \) system is symmetric hyperbolic system in a region \( G \subset \mathbb{R}^k \), if it is of the form

\[
L[U] = A^\alpha(U) \partial_\alpha U + B(U) = 0,
\]

where the matrices \( A^\alpha(U) \) are symmetric and for every arbitrary \( U \in G \), and there exists a covector \( \xi_\alpha \) such that

\[
\xi_\alpha A^\alpha(U)
\]

is positive definite. The covectors \( \xi_\alpha \) for which (3.2) is positive definite, are called spacelike with respect to equation (3.1).

**Remark 1.** In most applications, and in particular, for initial value problems, it is essential that \( A^0(U) \) is positive definite, and then system (3.1) takes the form

\[
A^0(U) \partial_t U = \sum_{i=1}^{3} A^i(U) \partial_{x^i} U + B(U).
\]

To derive equation (3.1) in the above form requires to show that $(1, 0, 0, 0)$ is spacelike with respect to the equation. Under the assumption that the speed of sound is less than one, we shall prove that the covector $(1, 0, 0, 0)$ belongs the future sound cone, and hence it is spacelike with respect to the equation (3.1).

3.2. Fluid decomposition. First, we apply the well known fluid decomposition (see for example [BK14]) to equation (2.3). We project $\nabla_\nu T^{\nu\beta}$ along the flow lines $u^\nu$, by $u_\beta \nabla_\nu T^{\nu\beta}$, and on the orthogonal subspace to the flow lines $O$, by $P_{\alpha\beta} \nabla_\nu T^{\nu\beta}$, where

$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \quad (3.4)$$

These projections result in

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p) \nabla_\nu u^\nu = 0 \quad (3.5)$$

$$(\epsilon + p) P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^{\nu}_{\alpha} \nabla_\nu p = 0, \quad (3.6)$$

which together with the continuity equation (2.4) form a system of equations. As we already pointed out the energy equation (3.5), together with the continuity equation (2.4) and the thermodynamical relation (2.6) imply the conservation of the entropy (2.7). Moreover, we will also need that fact, that thanks to equation (2.11), we can express $n$ as a function of $p$. All these considerations allow us to consider the following system of equations:

$$u^\nu \nabla_\nu n + n \nabla_\nu u^\nu = 0 \quad (3.7)$$

$$(\epsilon + p) P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^{\nu}_{\alpha} \nabla_\nu p = 0 \quad (3.8)$$

$$u^\alpha \nabla_\alpha s = 0. \quad (3.9)$$

3.3. Modification of the fluid decomposed system. In order to obtain a symmetric hyperbolic system we modify the coupled equations (3.7)-(3.9) the following way. The normalisation condition (2.2) implies that

$$u_\beta u^\nu \nabla_\nu u^\beta = 0. \quad (3.10)$$

So we add $nu_\beta u^\nu \nabla_\nu u^\beta = 0$ to equation (3.7), $u_\alpha u_\beta u^\nu \nabla_\nu u^\beta = 0$ to (3.8) and we obtain finally that

$$u^\nu \nabla_\nu n + n P^{\nu\beta}_{\beta} \nabla_\nu u^\beta = 0 \quad (3.11)$$

$$(\epsilon + p) \Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^{\nu}_{\alpha} \nabla_\nu p = 0, \quad (3.12)$$

where

$$\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_\alpha u_\beta = g_{\alpha\beta} + 2u_\alpha u_\beta \quad (3.13)$$

is a reflection with respect to the hyperplane $O$.

We now use the equation of state (2.8) and (2.9), which allow us to express $p$ as a function of $n$ and $s$, that is, $p = p(n, s)$. Hence,

$$\nabla_\nu p = \frac{\partial p}{\partial n} \nabla_\nu n + \frac{\partial p}{\partial s} \nabla_\nu s, \quad (3.14)$$

and by the conservation of the entropy (2.7), we conclude that

$$u^\nu \nabla_\nu p = \frac{\partial p}{\partial n} u^\nu \nabla_\nu n + \frac{\partial p}{\partial s} u^\nu \nabla_\nu s = \frac{\partial p}{\partial n} u^\nu \nabla_\nu n. \quad (3.15)$$
So we finally obtain the system
\[ u^\nu \nabla_\nu p + n \frac{\partial p}{\partial n} \nabla_\beta p^\nu u^\beta = 0 \]  
(3.16)
\[ (\epsilon + p) \Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu_{\alpha} \nabla_\nu p = 0 \]  
(3.17)
\[ u^\alpha \nabla_\alpha s = 0. \]  
(3.18)

**Remark 2** (The pressure as a matter variable). The idea of using the pressure as a matter variable instead of the density is widely used in the non-relativistic case, see for example [Smo83]. In the relativistic case, Guo and Tahvildar-Zadeh [GTZ99] presented the following system for the variables \((p, u^\alpha, s)\)

\[ \frac{1}{(\epsilon + p)\sigma} u^\nu \partial_\nu p + \sigma \partial_\nu u^\nu = 0 \]  
(3.19)
\[ \sigma P^{\mu\nu} \partial_\nu p + (\epsilon + p)\sigma u^\nu \partial_\nu u^\mu = 0 \]  
(3.20)
\[ u^\nu \partial_\nu s = 0. \]  
(3.21)

It should be pointed out, that this system, however, is not symmetric hyperbolic as it can be easily checked.

### 3.4. Symmetric hyperbolic form.

We now write system (3.16)-(3.18) in matrix form

\[
\begin{pmatrix}
    u^\nu & n \frac{\partial p}{\partial n} P^\nu_{\beta} & 0 \\
    P^\nu_{\alpha} & (\epsilon + p) \Gamma_{\alpha\beta} u^\nu & 0 \\
    0 & 0 & u^\nu
\end{pmatrix} \nabla_\nu \begin{pmatrix}
p \\
u^\alpha_s
\end{pmatrix} = 0. \tag{3.22}
\]

These matrices are not symmetric, but they can be cast into a symmetric form by choosing an appropriate multiplier, for example, we multiply the second row of the matrices by \(n \frac{\partial p}{\partial n}\), and then we obtain

\[
\begin{pmatrix}
    u^\nu & n \frac{\partial p}{\partial n} P^\nu_{\beta} & 0 \\
    n \frac{\partial p}{\partial n} P^\nu_{\alpha} & n \frac{\partial p}{\partial n} (\epsilon + p) \Gamma_{\alpha\beta} u^\nu & 0 \\
    0 & 0 & u^\nu
\end{pmatrix} \nabla_\nu \begin{pmatrix}
p \\
u^\alpha_s
\end{pmatrix} = 0, \tag{3.23}
\]

which are symmetric matrices.

In fact, it turns out that system (3.23) is a symmetric hyperbolic system. The following theorem gives a precise statement.

**Theorem 1.** Let \(\epsilon\) in (2.8) be nonnegative density function, the pressure \(p\) be defined by (2.9) and assume conditions (2.11). Then the Euler equations (2.3)-(2.4) coupled with the constraint (2.2) can be written as a symmetric hyperbolic system. Moreover, under the assumption that the speed of sound is less than the speed of light, the matrix \(A^0\) is positive definite and therefore the Euler equations (2.3)-(2.4) form are symmetric hyperbolic system as specified in equation (3.3).

**Proof.** To show that the system (3.23) is symmetric hyperbolic we need to show that \(\xi^\alpha A^\alpha(U)\) is positive definite for some covectors \(\xi^\alpha\). For that we slightly rewrite system (3.23). Using equations (2.10) and (2.10) we see that

\[
n \frac{\partial p}{\partial n} = \frac{\partial p}{\partial \epsilon} n \frac{\partial \epsilon}{\partial n} = \sigma^2 (\epsilon + p), \tag{3.24}\]

hence (3.23) is equivalent to system
\[
\begin{pmatrix}
\sigma^2 (\epsilon + p) P_{\nu}^\alpha & \sigma^2 (\epsilon + p) P_{\beta}^\nu & 0 \\
0 & \sigma^2 (\epsilon + p) P_{\nu}^\beta & 0 \\
0 & 0 & u^\nu
\end{pmatrix}
\begin{pmatrix}
\nabla_\nu \left( \frac{p}{u^\alpha} \right) \\
\end{pmatrix} = 0. \quad (3.25)
\]

Now we compute the principal symbol of system (3.25). For each \( \xi_\alpha \in T^*_x V \) the principal symbol is a linear map from \( \mathbb{R} \times E_x \) to \( \mathbb{R} \times F_x \), where \( E_x \) is a fiber in \( T_x V \) and \( F_x \) is a fiber in the cotangent space \( T^*_x V \). In local coordinates \( \nabla_\nu = \partial_\nu + \Gamma \), where \( \Gamma = \Gamma(g^{\gamma \delta}, \partial g_{\alpha \beta}) \) denotes the Christoffel symbols, hence the principal symbol of system (3.25) is
\[
\xi_\nu A^\nu = \begin{pmatrix}
(u^\nu \xi_\nu) & \sigma^2 (p + \epsilon) P_{\nu}^\beta \xi_\nu & 0 \\
\sigma^2 (p + \epsilon) P_{\nu}^\alpha \xi_\nu & \sigma^2 (p + \epsilon) (u^\nu \xi_\nu) \Gamma_{\alpha \beta} & 0 \\
0 & 0 & (u^\nu \xi_\nu)
\end{pmatrix}. \quad (3.26)
\]
The characteristics are the set of covectors \( \xi_\nu \) for which \((\xi_\nu A^\nu)\) is not an isomorphism. Hence the characteristics are the zeros of
\[
Q(\xi) \overset{\text{def}}{=} \det(\xi_\nu A^\nu). \quad (3.27)
\]
The geometric advantages of fluid decomposition are the following. The operators in the blocks of the matrix (3.26) are the projection \( P_{\nu}^\alpha \), on the hyperplane \( \mathcal{O} \) that is orthogonal to the flow lines, and the reflection \( \Gamma_{\alpha \beta} \), with respect to the same hyperplane. Therefore, the following relations hold:
\[
\Gamma^{\alpha \gamma} \Gamma_{\gamma \beta} = \delta^\alpha_\beta, \quad \Gamma^{\alpha \gamma} P_{\gamma}^\nu = P^{\alpha \nu} \quad \text{and} \quad P_\beta^\alpha P_{\alpha}^\nu = P^{\nu \beta},
\]
which yields
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \Gamma^{\alpha \gamma} & \xi_\nu A^\nu \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
(u^\nu \xi_\nu) & \sigma^2 (p + \epsilon) P_{\nu}^\beta \xi_\nu & 0 \\
\sigma^2 (p + \epsilon) P_{\nu}^\alpha \xi_\nu & \sigma^2 (p + \epsilon) (u^\nu \xi_\nu) \left( \delta^\alpha_\beta \right) & 0 \\
0 & 0 & (u^\nu \xi_\nu)
\end{pmatrix}. \quad (3.28)
\]
It is now fairly easy to calculate the determinant of the right-hand side of (3.28) and we have
\[
\det \begin{pmatrix}
(u^\nu \xi_\nu) & \sigma^2 (p + \epsilon) P_{\nu}^\beta \xi_\nu & 0 \\
\sigma^2 (p + \epsilon) P_{\nu}^\alpha \xi_\nu & \sigma^2 (p + \epsilon) (u^\nu \xi_\nu) \left( \delta^\alpha_\beta \right) & 0 \\
0 & 0 & (u^\nu \xi_\nu)
\end{pmatrix}
= \sigma^2 (p + \epsilon)^2 (u^\nu \xi_\nu)^4 \left\{ (u^\nu \xi_\nu)^2 - \sigma^2 P^{\alpha \nu} \xi_\nu P_{\alpha}^\nu \xi_\nu \right\}.
Since \( P^\alpha_\beta \) is a projection,
\[
P^\alpha_\beta \xi_\nu P^\alpha_\beta \xi_\nu = g^\nu_\beta \xi_\nu P^\alpha_\beta \xi_\nu = g^\nu_\beta \xi_\nu P^\nu_\beta \xi_\nu = P^\nu_\beta \xi_\nu \xi_\beta,
\]
and since \( \Gamma_\beta^\gamma \) is a reflection,
\[
\det \begin{pmatrix}
1 & 0 & 0 \\
0 & \Gamma^\alpha_\gamma & 0 \\
0 & 0 & 1
\end{pmatrix} = \det \left( g^{\alpha\beta} \Gamma^\beta_\gamma \right) = - \left( \det (g_{\alpha\beta}) \right)^{-1} > 0. \tag{3.29}
\]
Consequently,
\[
Q(\xi) = \det(\xi_\nu A^\nu) = -\sigma^2 (p + \epsilon)^2 \det(g_{\alpha\beta}) (u^\nu_\nu \xi_\nu)^4 \left\{ (u^\nu_\nu \xi_\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi_\beta \right\} \tag{3.30}
\]
and therefore the characteristic covectors are given by two simple equations:
\[
\xi_\nu u^\nu = 0 \tag{3.31}
\]
\[
(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi_\beta = 0. \tag{3.32}
\]

**Remark 3.** The characteristics conormal cone is a union of two hypersurfaces in \( T^*_x V \). One of these hypersurfaces is given by the condition (3.31) and it is a three dimensional hyperplane \( \mathcal{O} \) with the normal \( u^\alpha \). The other hypersurface is given by the condition (3.32) and forms a three–dimensional cone, the so–called, sound cone.

Let us now consider the timelike vector \( u_\nu \) and insert the covector \( -u_\nu \) into the principal symbol (3.26), then
\[
-u_\nu A^\nu = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sigma^2 (p + \epsilon) \Gamma_{\alpha\beta} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
is a positive definite matrix . Indeed, \( \Gamma_{\alpha\beta} \) is a reflection with respect to a hyperplane having a timelike normal, and as in (3.29) we see that \( \det(\Gamma_{\alpha\beta}) > 0 \). Hence, \( -u_\nu \) is a spacelike covector with respect to the hydrodynamical equations (3.25). Herewith, we have shown relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric hyperbolic.

We want now to show that \( A^0 \) is positive definite. To do that it suffices to show that the covector \( \zeta_\nu = (1, 0, 0, 0) \) is also spacelike with respect to the system (3.25). Since \( P^\alpha_\beta u_\alpha = 0 \), the covector \( -u_\nu \) belongs to the sound cone
\[
(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi_\beta > 0. \tag{3.33}
\]
Inserting \( \zeta_\nu = (1, 0, 0, 0) \) the right-hand side of (3.33), yields
\[
(u^0)^2 (1 - \sigma^2) - \sigma^2 g^{00}. \tag{3.34}
\]
Under the assumption sound velocity is less than the speed of light, that is \( \sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1 \), we conclude that (3.34) is positive, and hence \( \zeta_\nu = (1, 0, 0, 0) \) also belongs to the sound cone (3.33). Hence, the vector \( -u_\nu \) can be continuously deformed to \( \zeta_\nu \) while condition (3.33) holds along the deformation path. Consequently, the determinant of (3.30) remains positive under this process and hence \( \zeta_\nu A^\nu = A^0 \) is also positive definite. \( \square \)
4. Symmetrization and regularization

In the case of a physical vacuum, that is, if the density or the pressure vanish in certain regions, or fall-off at infinity, the symmetrization we obtained in Section 3 breaks down. The reason for this can be seen easily by inspecting the matrix \( A^0(U) \) which is no longer uniformly positive definite if the pressure approaches zero. Makino symmetrised and regularised the Euler-Poisson system by introducing a new nonlinear matter variable \( w = M(\rho) \) [Mak86], so that the matrix \( A^0(U) \) remains uniformly positive even for \( \rho = 0 \). Later Makino generalised his regularisation to the nonisentropic Euler-Poisson system [MU87], starting with a system for \((p, u^\alpha, s)\). We follow this strategy but, naturally, have to modify it due to the more complicated character of our equations.

So, we start with system (3.16)–(3.18)

\[
\begin{align*}
    u^\nu \nabla_\nu p + n \frac{\partial p}{\partial n} P_\beta \nabla_\nu u^\beta &= 0 \\
    (\epsilon + p) \Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P_\alpha \nabla_\nu p &= 0 \\
    u^\alpha \nabla_\alpha s &= 0.
\end{align*}
\]

(4.1) (4.2) (4.3)

and replace \( p \) by \( w = w(p) \). Then we multiply equation (4.1) by \( \kappa^2(w, s) \frac{\partial w}{\partial p} \) where \( \kappa \) is a positive function we specify later in order to simplify our calculations. Moreover, we divide equation (4.2) by \( \epsilon + p \), then equations (4.1) and (4.2) written in matrix form, take the following form

\[
\begin{pmatrix}
    \kappa^2 u^\nu \\
    \frac{1}{\epsilon + p} \frac{\partial w}{\partial p} P_\alpha \\
    0
\end{pmatrix}
\begin{pmatrix}
    \frac{\partial n}{\partial \nu} \\
    \Gamma_{\alpha\beta} u^\nu \\
    0
\end{pmatrix}
\begin{pmatrix}
    \partial w \\
    0 \\
    u^\alpha
\end{pmatrix} = 0,
\]

(4.4)

The matrices (4.4) are symmetric provided that

\[
\kappa^2 n \frac{\partial w}{\partial n} = \kappa^2 n \frac{\partial w}{\partial p} \frac{\partial p}{\partial n} = \frac{1}{\epsilon + p} \frac{\partial p}{\partial w},
\]

(4.5)

which results in

\[
w = \int \frac{1}{\kappa} \left( \frac{1}{\epsilon + p} \right)^{\frac{1}{\gamma}} \left( \frac{\partial n}{\partial p} \right)^{\frac{1}{\gamma}} dp.
\]

(4.6)

We will now, in the subsection below, calculate an explicit form of this new variable using the equation of state (2.12) presented in section 2.2.

4.1. The Makino variable for the equation of state (2.12). For this equation of state we easily compute

\[
\epsilon + p = n + \frac{1}{\gamma - 1} A(s)n^{\gamma} + p = n + \frac{\gamma}{\gamma - 1} p,
\]

(4.7)

\[
n \frac{\partial p}{\partial n} = \gamma p
\]

(4.8)

and

\[
n = A^{-\frac{1}{\gamma}}(s)p^{\frac{1}{\gamma}}.
\]

(4.9)
This allows us to calculate

\[
\frac{1}{(\epsilon + p) n \frac{\partial \rho}{\partial n}} = \frac{1}{n + \frac{\gamma}{\gamma - 1} p} \frac{\partial}{\partial n} (p \gamma) = \frac{1}{\gamma \frac{1}{\gamma - 1} (s) p^{1 + \frac{\gamma}{\gamma - 1}} + \frac{\gamma}{\gamma - 1} p^{2}} = \frac{1}{\gamma \left( \frac{1}{A^{\gamma - 1}} (s) + \frac{\gamma}{\gamma - 1} p^{1 - \frac{\gamma}{\gamma - 1}} \right)} \frac{1}{p^{1 + \frac{\gamma}{\gamma - 1}}}. \]

Keeping in mind the symmetry condition (4.5), we see that setting

\[
\kappa^2 = \left( \frac{2\gamma}{\gamma - 1} \right)^2 \frac{1}{\gamma A^{\gamma - 1} (s) + \frac{\gamma}{\gamma - 1} w^2} \right),
\]

implies that \( \frac{\partial w}{\partial p} = \frac{\gamma - 1}{2} p^{-\frac{\gamma - 1}{2}} \), which leads to

\[
w = p^{\frac{\gamma - 1}{2}}
\]

and

\[
k^2 (w, s) = \left( \frac{2\gamma}{\gamma - 1} \right)^2 \frac{1}{\gamma A^{\gamma - 1} (s) + \frac{\gamma}{\gamma - 1} w^2} \right)
\]

So we conclude the Euler equations (2.3)-(2.4) coupled with the constraint (2.2) can be written in the form

\[
\left( \kappa^2 \frac{\partial \nu}{\partial \gamma} - \kappa^2 \frac{2\gamma}{\gamma - 1} w P\nu \gamma \right) + \begin{pmatrix}
\kappa^2 \frac{2\gamma}{\gamma - 1} w P\nu \alpha \\
0 \\
0
\end{pmatrix} \Gamma_{\alpha\beta}^{\nu} u^\beta 0 \nabla_\nu \begin{pmatrix}
w \\
u^\alpha \\
w^\nu
\end{pmatrix} = 0,
\]

which is symmetric and regular when \( p \), or equivalently \( w \) approaches zero.

References

[BK11] Uwe Brauer and Lavi Karp, Well-posedness of the Einstein-Euler system in asymptotically flat spacetimes: the constraint equations, J. Differential Equations 251 (2011), no. 6, 1428–1446, URL: https://doi.org/10.1016/j.jde.2011.05.037, doi:10.1016/j.jde.2011.05.037. MR 2813883

[BK14] Local existence of solutions of self gravitating relativistic perfect fluids, Comm. Math. Phys. 325 (2014), no. 1, 105–141, URL: https://doi.org/10.1007/s00220-013-1854-3. MR 3182488

[CB66] Yvonne Choquet-Bruhat, Diagonalisation des systèmes quasi-linéaires et hyperbolicité non stricte, J. Math. Pures Appl. (9) 45 (1966), 371–386. MR 0216131 (35 #6966)

[CB09] General Relativity and the Einstein Equations, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2009. MR 2473363

[Chr95] Demetrios Christodoulou, Self-gravitating relativistic fluids: a two-phase model, Arch. Rational Mech. Anal. 130 (1995), no. 4, 343–400, URL: http://dx.doi.org/10.1007/BF00375144, doi:10.1007/BF00375144. MR 1346362

[Chr00] The action principle and partial differential equations, Annals of Mathematics Studies, vol. 146, Princeton University Press, Princeton, NJ, 2000, URL: https://doi.org/10.1515/9781400882687, doi:10.1515/9781400882687. MR 1739321

[Dis15] M. M. Disconzi, Remarks on the Einstein-Euler-entropy system, Reviews in Mathematical Physics 27 (2015), no. 6, 1550014 (45 pages), doi:10.1142/S0129055X15500142. MR 3370953

[FB58] Yvonne Fourès-Bruc, Théorèmes d’existence en mécanique des fluides relativistes, Bull. Soc. Math. France 86 (1958), 155–175, URL: http://www.numdam.org/item?id=BSMF_1958__86__155_0. MR 105294
[FR00] H. Friedrich and A.D. Rendall, *The cauchy problem for the einstein equations*, Einstein’s field equations and their physical implications (Lecture Notes in Phys., ed.), vol. 540, Springer, 2000, pp. 127–213.

[Fri54] K.O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure and Appl. Math 7 (1954), 345–392.

[Fri98] H. Friedrich, *Evolution equations for gravitating ideal fluid bodies in general relativity*, Phys. Rev. D 57 (1998), 2317–2322.

[GTZ99] Yan Guo and A. Shadi Tahvildar-Zadeh, *Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics*, Nonlinear partial differential equations (Evanston, IL, 1998), Contemp. Math., vol. 238, Amer. Math. Soc., Providence, RI, 1999, pp. 151–161, URL: http://dx.doi.org/10.1090/conm/238/03545, doi:10.1090/conm/238/03545. MR 1724661

[Mak86] Tetu Makino, *On a local existence theorem for the evolution equation of gaseous stars*, Patterns and Waves (Amsterdam) (T. Nishida, M. Mimura, and H. Fujii, eds.), North–Holland, 1986, pp. 459–479.

[MU87] Tetu Makino and Seiji Ukai, *Sur l’existence des solutions locales de l’équation d’Euler-Poisson pour l’évolution d’étoiles gazeuses*, J. Math. Kyoto Univ. 27 (1987), no. 3, 387–399, URL: http://dx.doi.org/10.1215/kjm/1250520654, doi:10.1215/kjm/1250520654. MR 910225

[MUK86] Tetu Makino, Seiji Ukai, and S. Kawashima, *Sur la solution à support compact de l’équation d’Euler compressible*, Japan Journal of Applied Mathematics 3 (1986), 249–257.

[Pic65] Guy Pichon, *Étude relativiste de fluides visqueux et chargés*, Ann. Inst. H. Poincaré Sect. A (N.S.) 2 (1965), 21–85. MR 0204029

[Ren92] A. D. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, Journal of Mathematical Physics 33 (1992), no. 2, 1047–1053.

[Smo83] Joel Smoller, *Shock waves and reaction–diffusion equations*, Grundlehren der Mathematischen Wissenschaften, vol. 258, Springer, Heidelberg, 1983.

[Spe09] Jared Speck, *Well-posedness for the Euler-Nordström system with cosmological constant*, Journal of Hyperbolic Differential Equations 6 (2009), no. 2, 313–358, URL: http://dx.doi.org/10.1142/S0219891609001885. MR 2543324 (2011a:35529)

[Wal05] R.A. Walton, *Symmetric hyperbolic euler equations for relativistic perfect fluids*, Arxiv: astro-ph/0502233 (2005).

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