Connection between the clique number and the Lagrangian of 3-uniform hypergraphs

Qingsong Tang¹, ⁴ · Yuejian Peng² · Xiangde Zhang¹ · Cheng Zhao³, ⁴

Abstract There is a remarkable connection between the clique number and the Lagrangian of a 2-graph proved by Motzkin and Straus (J Math 17:533–540, 1965). It would be useful in practice if similar results hold for hypergraphs. However, the obvious generalization of Motzkin and Straus’ result to hypergraphs is false. Frankl and Füredi conjectured that the \( r \)-uniform hypergraph with \( m \) edges formed by taking the first \( m \) sets in the colex ordering of \( \mathbb{N}^{(r)} \) has the largest Lagrangian of all \( r \)-uniform hypergraphs with \( m \) edges. For \( r = 2 \), Motzkin and Straus’ theorem confirms this conjecture. For \( r = 3 \), it is shown by Talbot that this conjecture is true when \( m \) is in certain ranges. In this paper, we explore the connection between the clique number and Lagrangians for 3-uniform hypergraphs. As an application of this connection, we confirm that Frankl and Füredi’s conjecture holds for bigger ranges of \( m \) when \( r = 3 \). We also obtain two weaker versions of Turán type theorem for left-compressed 3-uniform hypergraphs.

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1 Introduction

For a set $V$ and a positive integer $r$, let $V^{(r)}$ be the family of all $r$-subsets of $V$. An $r$-uniform graph or $r$-graph $G$ consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1, a_2, \ldots, a_r$. An $r$-graph $H$ is a subgraph of an $r$-graph $G$, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $K_t^{(r)}$ denote the complete $r$-graph on $t$ vertices, that is the $r$-graph on $t$ vertices containing all possible edges. A complete $r$-graph on $t$ vertices is also called a clique with order $t$. A clique is said to be maximal if there is no other clique containing it, while it is called maximum if it has maximum cardinality. The clique number of an $r$-graph $G$, denoted as $\omega(G)$, is defined as the cardinality of the maximum clique. Let $\mathbb{N}$ be the set of all positive integers. For an integer $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$. Let $[n]^{(r)}$ represent the complete $r$-graph on the vertex set $[n]$. When $r = 2$, an $r$-graph is a simple graph. When $r \geq 3$, an $r$-graph is often called a uniform hypergraph.

For an $r$-graph $G = (V, E)$, denote the $(r - 1)$-neighborhood of a vertex $i \in V$ by $E_i := \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, denote the $(r - 2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} := \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. Denote the complement of $E_i$ by $E^c_i := \{A \in V^{(r-1)} : A \cup \{i\} \notin V^{(r)}\}$. Also, denote the complement of $E_{ij}$ by $E^c_{ij} := \{B \in V^{(r-2)} : B \cup \{i, j\} \notin V^{(r)}\}$. Denote $E_{i\setminus j} := E_i \cap E^c_j$. An $r$-graph $G = ([n], E)$ is left-compressed if $j_1, j_2, \ldots, j_r \in E$ implies $i_1, i_2, \ldots, i_r \in E$ provided $i_p \leq j_p$ for every $p, 1 \leq p \leq r$. Equivalently, an $r$-graph $G = ([n], E)$ is left-compressed if $E_{j \setminus i} = \emptyset$ for any $1 \leq i < j \leq n$.

Definition 1 For an $r$-uniform graph $G$ with the vertex set $[n]$, edge set $E(G)$, and a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we associate a homogeneous polynomial in $n$ variables, denoted by $\lambda(G, x)$ as follows:

$$\lambda(G, x) := \sum_{i_1, i_2, \ldots, i_r \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$ 

Let $S := \{x = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. Let $\lambda(G)$ represent the maximum of the above homogeneous multilinear polynomial of degree $r$ over the standard simplex $S$. Precisely

$$\lambda(G) := \max \{\lambda(G, x) : x \in S\}.$$ 

The value $x_i$ is called the weight of the vertex $i$. A vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for $G$ iff $x \in S$. A vector $y \in S$ is called an optimal weighting for $G$ iff $\lambda(G, y) = \lambda(G)$. We call $\lambda(G)$ the graph-Lagrangian of hypergraph $G$, for abbreviation, the Lagrangian of $G$.

The following fact is easily implied by Definition 1.

Fact 1 Let $G_1, G_2$ be $r$-uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.

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The maximum clique problem is a classical problem in combinatorial optimization which has important applications in various domains. In [7], Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph.

**Theorem 1** ([7]) If $G$ is a 2-graph with clique number $t$ then $\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.

The obvious generalization of Motzkin and Straus’ result to hypergraphs is false because there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. Lagrangians of hypergraphs has been proved to be a useful tool, for example, it is useful to hypergraph extremal problems. Applications of Lagrangian method can be found in [3–6,11]. In most applications, an upper bound is needed. Frankl and Füredi [3] asked the following question. Given $m, t, \text{ and } r$ hypergraphs.

**Conjecture 1** ([3]) The $r$-graph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all $r$-graphs with $m$ edges. In particular, the complete $r$-graph $[t]^{(r)}$ has the largest Lagrangian of all $r$-graphs with $(\binom{t}{r})$ edges.

This conjecture is true when $r = 2$ by Theorem 1. For the case $r = 3$, Talbot [12] proved the following.

**Theorem 2** ([12]) Let $m$ and $t$ be integers satisfying $(\binom{t-1}{3}) \leq m \leq (\binom{t-1}{3}) + (\binom{t-2}{2}) - (t - 1)$. Then Conjecture 1 is true for $r = 3$ and this value of $m$. Conjecture 1 is also true for $r = 3$ and $m = (\binom{t}{3}) - 1$ or $m = (\binom{t}{3}) - 2$.

Further evidence that supports Conjecture 1 can be found in [14,15]. In particular, Conjecture 1 is true for $r = 3$ and $(\binom{t}{3}) - 6 \leq m \leq (\binom{t}{3})$ (see [14,15]).

Although the obvious generalization of Motzkin and Straus’ result to hypergraphs is false, we attempt to explore the relationship between the Lagrangian of a hypergraph and its cliques number for hypergraphs when the number of edges is in certain ranges. In [9], it is conjectured that the following Motzkin and Straus type results are true for hypergraphs.

**Conjecture 2** Let $m, t, \text{ and } r \geq 3$ be positive integers satisfying $(\binom{t-1}{r}) \leq m \leq (\binom{t-1}{r}) + (\binom{t-2}{r-1})$. Let $G$ be an $r$-graph with $m$ edges and $G$ contain a clique of order $t - 1$. Then $\lambda(G) = \lambda([t - 1]^{(r)})$.

The upper bound $(\binom{t-1}{r}) + (\binom{t-2}{r-1})$ in this conjecture is the best possible. When $m = (\binom{t-1}{r}) + (\binom{t-2}{r-1}) + 1$, let $G_r,m$ be the $r$-graph with the vertex set $[t]$ and the edge set
We will impose one additional condition on any optimal weighting $x = (x_1, x_2, \ldots, x_n)$ for an $r$-graph $G$:

$$|\{i : x_i > 0\}|$$

is minimal, i.e. if $y$ is a feasible weighting for $G$ satisfying

$$|\{i : y_i > 0\}| < |\{i : x_i > 0\}|,$$

then $\lambda(G, y) < \lambda(G)$. \hspace{1cm} (1)

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G, x)$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, x)$ corresponds to the partial derivative of

$$\lambda(G, x) = \frac{1}{2r} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j (r-3).$$
\( \lambda(G, x) \) with respect to \( x_i \). The following lemma gives some necessary conditions of an optimal weighting for \( G \).

**Lemma 2** ([4]) Let \( G = (V, E) \) be an \( r \)-graph on the vertex set \([n]\) and \( x = (x_1, x_2, \ldots, x_n) \) be an optimal weighting for \( G \) with \( k \) (\( \leq n \)) non-zero weights \( x_1, x_2, \ldots, x_k \) satisfying condition (1). Then for every \( \{i, j\} \in [k]^{(2)} \), (a) \( \lambda(E_{ij}, x) = \lambda(E_{ji}, x) = r \lambda(G) \), (b) \( E_{ij} \neq \emptyset \).

**Remark 2** (a) In Lemma 2, part (a) implies that

\[
 x_j \lambda(E_{ij}, x) + \lambda(E_{i\setminus j}, x) = x_i \lambda(E_{ij}, x) + \lambda(E_{j\setminus i}, x).
\]

In particular, if \( G \) is left-compressed, then

\[
 (x_i - x_j) \lambda(E_{ij}, x) = \lambda(E_{i\setminus j}, x)
\]

for any \( i, j \) satisfying \( 1 \leq i < j \leq k \) since \( E_{j\setminus i} = \emptyset \).

(b) If \( G \) is left-compressed, then for any \( i, j \) satisfying \( 1 \leq i < j \leq k \),

\[
 x_i - x_j = \frac{\lambda(E_{i\setminus j}, x)}{\lambda(E_{ij}, x)}
\]

holds. If \( G \) is left-compressed and \( E_{i\setminus j} = \emptyset \) for \( i, j \) satisfying \( 1 \leq i < j \leq k \), then \( x_i = x_j \).

(c) By (2), if \( G \) is left-compressed, then an optimal feasible weighting \( x = (x_1, x_2, \ldots, x_n) \) for \( G \) must satisfy

\[
 x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.
\]

The following lemma implies that we only need to consider left-compressed \( r \)-graphs when Conjecture 1 is explored. Denote

\[
 \lambda_r^m := \max\{\lambda(G) : G \text{ is an } r \text{ - graph with } m \text{ edges}\}.
\]

**Lemma 3** ([12]) Let \( m, t \) and \( r \) be positive integers satisfying \( m \leq \binom{t}{r} - 1 \), then there exists a left-compressed \( r \)-graph \( G \) with \( m \) edges such that \( \lambda(G) = \lambda_r^m \).

### 3 Proof of Theorem 4

The following lemma showed in [10] implies that we only need to consider left-compressed 3-graphs \( G \) on \( t \) vertices to verify Conjecture 3 for \( r = 3 \).

**Lemma 4** ([10]) Let \( m \) and \( t \) be positive integers satisfying \( \binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} \). Then there exists a left-compressed 3-graph \( G \) on the vertex set \([t]\) with \( m \) edges and not containing a clique of order \( t - 1 \) such that \( \lambda(G) = \lambda_{(m,t-1)}^{3-} \).
Proof of Theorem 4. Let \((t-1)/3\) \(m\) \((t-2)/2 - (t - 1). Let \(G\) be a 3-graph with \(m\) edges without containing \([t-1]^{(3)}\) such that \(\lambda(G) = \lambda_{[m,t-1]}^{3-}\). To prove Theorem 4, we only need to prove \(\lambda_{[m,t-1]}^{3-} = \lambda(G) < \lambda([t-1]^{(3)})\).

By Lemma 4, we can assume that \(G\) is left-compressed. Let \(x = (x_1, x_2, \ldots, x_n)\) be an optimal weighting for \(G\). By Remark 2 (a), \(x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0\). If \(k \leq t - 1\), then \(\lambda(G) < \lambda([t-1]^{(3)})\) since \(G\) does not contain a clique order of \(t - 1\). So we assume \(k \geq t\). First we show that \(k = t\). We need the following lemma.

Lemma 5 ([12]) Let \(G = (V, E)\) be a left-compressed 3-graph with \(m\) edges such that \(\lambda(G) = \lambda_m^3\). Let \(x = (x_1, x_2, \ldots, x_k)\) be an optimal weighting for \(G\) satisfying \(x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0\). Let \(b := |E(k-1)k|\). Then

\[
|\left[k-1\right]^{(3)} \backslash E| \leq \left[ b \left( 1 + \frac{k - (b + 2)}{k - 3} \right) \right].
\]

Since \(G\) is left-compressed and \(1(k-1)k \in E\), then \(|[k-2]^{(2)} \backslash E_k| \geq 1\). If \(k \geq t + 1\), then applying Lemma 5, we have \(|[k-1]^{(3)} \backslash E| \leq k - 2\). Hence

\[
m = |E| = |E \backslash \left[k-1\right]^{(3)}| + |\left[k-2\right]^{(2)} \backslash E_k| + |E(k-1)k|
\geq \binom{t}{3} - (t - 1) + 2
\geq \binom{t-1}{3} + \binom{t-2}{2} + 1,
\]

which contradicts to the assumption that \(m \leq \binom{t-1}{3} + \binom{t-2}{2}\). Recalling that \(k \geq t\), so we have

\[k = t.\]

Hence we can assume \(G\) is on vertex set \([t]\).

Next we prove an inequality.

Lemma 6 Let \(G\) be a 3-graph on the vertex set \([t]\). Let \(x = (x_1, x_2, \ldots, x_i)\) be an optimal weighting for \(G\) satisfying \(x_1 \geq x_2 \geq \cdots \geq x_i \geq 0\). Then

\[
x_1 < x_{t-3} + x_{t-2} \text{ or } \lambda(G) \leq \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)} < \lambda([t-1]^{(3)}).
\]

Proof Assume that \(x_1 \geq x_{t-3} + x_{t-2}\), we show that \(\lambda(G) \leq \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)} < \lambda([t-1]^{(3)})\). Since \(x_1 \geq x_{t-3} + x_{t-2}\), then

\[3x_1 + x_2 + \cdots + x_{t-4} > x_1 + x_2 + \cdots + x_{t-4} + x_{t-3} + x_{t-2} + x_{t-1} + x_t = 1.\]
Recalling that \( x_1 \geq x_2 \geq \cdots \geq x_{t-4} \), we have \( x_1 > \frac{1}{t-2} \). Using Lemma 2, we have 
\[
\lambda(G) = \frac{1}{3} \lambda(E_1, x).
\]
Note that \( E_1 \) is a 2-graph with \( t-1 \) vertices and total weights \( x_2 + \cdots + x_t = 1 - x_1 \leq 1 - \frac{1}{t-2} \). Hence by Theorem 1 (replace the total weights 1 with \( 1 - \frac{1}{t-2} \)), we have
\[
\lambda(G) = \frac{1}{3} \lambda(E_1, x) 
\leq \frac{1}{3} \left( \frac{t-1}{2} \right) \left( 1 - \frac{1}{t-2} \right)^2 \text{ (by Theorem 1)}
\]
\[
= \frac{1}{6} \left( t-3 \right)^2 \left( t-2 \right) - \frac{1}{6} \left( t-3 \right) \left( t-2 \right) - \frac{3}{6} = \lambda \left( \left[ t-1 \right]^{(3)} \right).
\]

This completes the proof. \( \square \)

The following lemma is proved in [16].

**Lemma 7** ([16], Lemma 5.3) Let \( G \) be a left-compressed 3-graph on the vertex set \([t]\). Let \( x = (x_1, x_2, \ldots, x_t) \) be an optimal weighting for \( G \). Then \( \|t-1\|^{(3)} \setminus E \| \leq t - 3 \), or \( \lambda(G) < \lambda \left( \left[ t-1 \right]^{(3)} \right) \).

**Remark 3** We can prove that \( \|t-1\|^{(3)} \setminus E \| \leq t - 3 \), or \( \lambda(G) < \lambda \left( \left[ t-1 \right]^{(3)} \right) \) under the condition of Lemma 7 through the method in [16].

Now we continue the proof of Theorem 4. Let \( D := [t-1]^{(3)} \setminus E \) and \( b := \|E_{(t-1)t}\| \). Since \( x_{t-1} \) and \( x_t > 0 \), then \( b = \|E_{(t-1)t}\| > 0 \) by Lemma 2 (b). By Lemma 5, we have \( |D| \leq 2b \). So \( \left\lfloor \frac{|D|}{2} \right\rfloor \leq b \) and the triples \( 1(t-1), \ldots, \left\lfloor \frac{|D|}{2} \right\rfloor (t-1)t \) are in \( G \).

Let \( G' = G \cup D \setminus \{1(t-1)t, \ldots, \left\lfloor \frac{|D|}{2} \right\rfloor (t-1)t \} \). If \( \lambda(G) < \lambda \left( \left[ t-1 \right]^{(3)} \right) \), we are done. Otherwise by Remark 3 we have \( |D| \leq t - 3 \). So
\[
|G'| = |G| + |D| - \left\lfloor \frac{|D|}{2} \right\rfloor \leq \left( t - \frac{1}{3} \right) + \left( t - \frac{2}{2} \right) - \frac{1}{2} (t - 1) + t - 3 = t - \frac{3}{2} + 1
\]
\[
= \left( t - \frac{3}{2} \right) + \left( t - \frac{2}{2} \right).
\]

Note that \( G' \) contains \( [t-1]^{(3)} \). By Theorem 3, we have \( \lambda(G', x) \leq \lambda(G') = \lambda \left( \left[ t-1 \right]^{(3)} \right) \).

Next we show that \( \lambda(G, x) < \lambda(G', x) \). By Remark 2 (b), \( x_1 = x_2 = \ldots = x_{\left\lfloor \frac{|D|}{2} \right\rfloor} \).

Hence
\[
\lambda(G', x) - \lambda(G, x) = \lambda(D, x) - \left\lfloor \frac{|D|}{2} \right\rfloor x_1 x_{t-1} x_t 
\geq |D| x_{t-3} x_{t-2} x_{t-1} - \left\lfloor \frac{|D|}{2} \right\rfloor x_1 x_{t-1} x_t 
\geq |D| x_{t-3} x_{t-2} x_{t-1} - \left\lfloor \frac{|D|}{2} \right\rfloor (x_{t-3} + x_{t-2}) x_{t-1} x_t.
\]
Let \( G \).

Theorem 6: Weaker versions of Turán type result for left-compressed 3-graphs.

In this section, we will confirm Conjecture 1 and Conjecture 3 for some left-compressed 3-graphs with specified structures. As an application, we also obtain two weaker versions of Turán type result for left-compressed 3-graphs.

Proof: The idea to prove Theorem 6 is similar to that in the proof of Lemma 6. Let \( G = (V, E) \) be a left-compressed 3-graph on vertex set \([t]\). If \( G \) does not contain a clique order of \( \lceil \frac{t-2}{2} \rceil \), then

\[
\lambda(G) \leq \frac{1}{6} \left( \frac{t-3}{t-2} \right)^2 < \lambda \left( \left[ t - 1 \right]^{(3)} \right).
\]

Proof: The idea to prove Theorem 6 is similar to that in the proof of Lemma 6. Let \( G = (V, E) \) be a left-compressed 3-graph with \( m \) edges and \( \omega(G) \leq \lceil \frac{t-2}{2} \rceil \). Recall \( \omega(G) \) is the clique number of \( G \). If \( t \leq 5 \), Theorem 6 clearly holds. Next we assume \( t \geq 6 \). Let \( x = (x_1, x_2, \ldots, x_t) \) be an optimal weighting for \( G \) satisfying \( x_1 \geq x_2 \geq \cdots \geq x_t \). The clique number of \( E_{t-3} \) must be smaller than \( \frac{t-2}{2} \), otherwise \( \omega(G) > \lceil \frac{t-2}{2} \rceil \) since \( G \) is left-compressed. Assume that \( \lambda(G) > \frac{1}{6} \left( \frac{t-3}{t-2} \right)^2 \) for a contradiction. Since \( \lambda(G) > \frac{1}{6} \left( \frac{t-3}{t-2} \right)^2 \), by Lemma 6, we have \( x_{t-3} > \frac{x_1}{2} \geq \frac{1}{2t} \). Using Lemma 2, we have \( \lambda(G) = \frac{1}{3} \lambda(E_{t-3}, x) \). Note that \( E_{t-3} \) is a 2-graph with \( t - 1 \) vertices and total weights \( x_1 + x_2 + \cdots + x_{t-4} + x_{t-2} + x_{t-1} + x_t = 1 - x_{t-3} < \frac{1}{2t} \). Hence by Theorem 1 (replace the total weights 1 with \( 1 - \frac{1}{2t} \)), we have

\[
\lambda(G) = \frac{1}{3} \lambda(E_{t-3}, x)
\]

\[
< \frac{1}{3} \left( \left\lceil \frac{t-2}{2} \right\rceil \right)^2 \left( 1 - \frac{1}{2t} \right)^2 \quad \text{(by Theorem 1)}
\]

\[
\leq \frac{1}{6} \frac{t-4}{t-2} \left( \frac{2t-4}{t-2} \right)^2
\]

\[
< \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)}
\]

which contradicts to \( \lambda(G) > \frac{1}{6} \left( \frac{t-3}{t-2} \right)^2 \). This completes the proof. \( \square \)

4 Connection between the clique number and the Lagrangians of some left-compressed 3-graphs

In this section, we will confirm Conjecture 1 and Conjecture 3 for some left-compressed 3-graphs with specified structures. As an application, we also obtain two weaker versions of Turán type result for left-compressed 3-graphs.

Theorem 6: Let \( G = (V, E) \) be a left-compressed 3-graph on vertex set \([t]\). If \( G \) does not contain a clique order of \( \lceil \frac{t-2}{2} \rceil \), then

\[
\lambda(G) < \frac{1}{6} \left( \frac{t-3}{t-2} \right)^2 \lambda \left( \left[ t - 1 \right]^{(3)} \right).
\]
Corollary 1 Let $G = (V, E)$ be a left-compressed 3-graph with $t$ vertices and $m$ edges. If $m \geq \frac{(t-3)^2}{6(t-2)(t-1)}$, then $G$ contains a clique order of $\lfloor \frac{t-2}{2} \rfloor$.

Proof Let $G = (V, E)$ be a 3-graph with $t$ vertices and $m$ edges. Assume that $m \geq \frac{(t-3)^2}{6(t-2)(t-1)}$. Clearly, $x_1 = x_2 = \ldots = x_t = \frac{1}{t}$ is a feasible weighting for $G$. Hence $\lambda(G) \geq \frac{t-3}{6(t-2)(t-1)} \geq \frac{(t-3)^2}{6(t-2)(t-1)}$. However by Theorem 6 we know that $\lambda(G) < \frac{(t-3)^2}{6(t-2)(t-1)}$ if $G$ does not contain a clique order of $\lfloor \frac{t-2}{2} \rfloor$. This completes the proof.

For the case of forbidding a clique of order 4, we have the following result.

Proposition 1 Let $G$ be a left-compressed 3-uniform graph on $[t]$ with $m$ edges. If $G$ does not contain a clique of order 4, then $m \leq \frac{2}{27}t^3$.

Proof Let $x = (x_1, x_2, \ldots, x_t)$ be an optimal vector of $G$. We claim that all edges in $G$ must contain vertex 1. Otherwise, 234 is an edge of $G$ and $G$ contains the clique $[4]$ since $G$ is left-compressed. So

$$\lambda(G) \leq \frac{1}{2}x_1 \left( x_2 + x_3 + \ldots + x_t \right)^2$$

$$= \frac{1}{2}x_1 (1-x_1)^2 \leq \frac{1}{2} \times \frac{4}{27} \left( x_1 + \frac{1-x_1}{2} + \frac{1-x_1}{2} \right)^3 = \frac{2}{27}.$$

Let $y = (y_1, y_2, \ldots, y_t)$ given by $y_i = \frac{1}{t}$ for each $i$, $1 \leq i \leq t$. Then $\frac{2}{27} \geq \lambda(G) = \lambda(G, y) = \frac{m}{t}$. Therefore, $m \leq \frac{2}{27}t^3$.

In [2], Rota Buló and Pelillo proved the following theorem.

Theorem 7 ([2]) An $r$-graph $G = (V, E)$ with $m$ edges and $t$ vertices, which contains no $p$-clique with $p \geq r$, then

$$m \leq \binom{t}{r} - \frac{t}{(r-1)r} \left( \frac{t}{p-1} \right)^{r-1} - 1].$$

In [1], Caen proved the following theorem.

Theorem 8 ([1]) An $r$-graph $G = (V, E)$ with $m$ edges and $t$ vertices, which contains no $p$-clique with $p \geq r$, then

$$m \leq \binom{t}{r} - \frac{t-p+1}{r\binom{p-1}{r-1}} \left( \frac{t-1}{r-1} \right).$$

Remark 4 (1) We note that Theorem 6 and Corollary 1 establish a connection between Lagrangian and clique number for 3-graphs. They also provide evidence for Conjecture 3.
(2) For the case \( r = 3 \) and \( p = \lfloor \frac{t-2}{3} \rfloor \), the upper bound in Theorem 7 is bigger than \( \frac{(t^4-11t^3+39t^2-72t+48)t}{6(t-4)^2} \). Since
\[
\frac{(t^4-11t^3+39t^2-72t+48)t}{6(t-4)^2} > \frac{(t-3)^2t^3}{6(t-2)(t-1)}
\]
when \( t \geq 38 \), the result in Corollary 1 is better than the result in Theorem 7 under the left-compressed condition. We would like to point out that, the result in Corollary 1 is not good as the result in Theorem 8 even under the left-compressed condition.

(3) Again, for the case \( r = 3 \) and \( p = 4 \), the upper bound in Theorem 7 is bigger than the bound in Proposition 1 under the left-compressed condition. In this case, the upper bound in Theorem 8 is \( \frac{1}{18} (6 - 5t - 3t^2 + 2t^3) \), also bigger than the bound in Proposition 1 under the left-compressed condition.

Next we give the following partial result to Conjecture 1.

**Theorem 9** Let \( m, t, \) and \( a \) be positive integers satisfying \( m = \binom{t-1}{3} + \binom{t-2}{2} + a \) where \( 1 \leq a \leq t - 2 \). Let \( G = (V, E) \) be a left-compressed 3-graph on the vertex set \([t]\) with \( m \) edges satisfying \( |E_{(t-1)t}| \leq \frac{2t+3a-4}{5} \). If \( G \) contains a clique of order \( t - 1 \), then \( \lambda(G) \leq \lambda(C_{3,m}) \).

**Proof** Let \( G \) be a 3-graph with \( m \) edges and contain a clique of order \( t - 1 \). Assume \( x = (x_1, x_2, \ldots, x_t) \) is an optimal weighting for \( G \) satisfying \( x_1 \geq x_2 \geq \cdots \geq x_t \geq 0 \). We will prove that \( \lambda(C_{3,m}, x) - \lambda(G, x) \geq 0 \). Therefore \( \lambda(C_{3,m}) \geq \lambda(C_{3,m}, x) \geq \lambda(G, x) = \lambda(G) \).

In order to prove \( \lambda(C_{3,m}, x) - \lambda(G, x) \geq 0 \), we need to show \( x_1 - x_{t-3} \leq x_{t-2} - x_{t-1} \).

Since \( G \) contains \( [t-1]^3 \) then every edge in \( E_{t-3}^c \) contains vertex \( t \). So \( \lambda(E_{t-3}^c, x) = x_t \lambda(E_{(t-1)t}^c, x) \). By Remark 2 (b), we have
\[
\begin{align*}
x_1 &= x_{t-3} + \frac{\lambda(E_{1\backslash(t-3)}, x)}{\lambda(E_{(t-3)}, x)} \\
&= x_{t-3} + \frac{\lambda(E_{t-3}^c, x)}{\lambda(E_{1\backslash(t-3)}, x)} \\
&= x_{t-3} + \frac{x_t \lambda(E_{(t-1)t}^c, x)}{\lambda(E_{1\backslash(t-3)}, x)}.
\end{align*}
\]

Similarly, since \( G \) contains \( [t-1]^3 \) then every edge in \( E_{(t-2)\backslash(t-1)} \) contains vertex \( t \). So
\[
\lambda(E_{(t-2)\backslash(t-1)}, x) = x_t \lambda(E_{(t-2)t}^c \bigcap E_{(t-1)t}^c, x).
\]
By Remark 2 (b), we have

\[
x_{t-2} = x_{t-1} + \frac{\lambda(E_{(t-2)\setminus(t-1)}, x)}{\lambda(E_{(t-2)(t-1)}, x)}
= x_{t-1} + \frac{x_t \lambda\left(E_{(t-2)t} \cap E_{(t-1)t}^c, x\right)}{\lambda(E_{(t-2)(t-1)}, x)}.
\]  

(4)

Let \( b := |E_{(t-1)t}| \). Since \( G \) contains the clique \([t - 1]^{(3)}\), we have \(|[t - 2]^{(2)} \setminus E_t| = b - a\). Note that if \( i(t - 2) \in E_t^c \) then \( i(t - 2) \in [t - 2]^{(2)} \setminus E_t \) except \( i = t - 1 \). So \( |E_{(t-2)t}^c| \leq |[t - 2]^{(2)} \setminus E_t| + 1 = b - a + 1 \) and \( |E_{(t-2)t}| = t - 2 - |E_{(t-2)t}^c| \geq t - 2 - (b - a) - 1 \). Therefore

\[
|E_{(t-2)t} \cap E_{(t-1)t}^c| \geq |E_{(t-2)t}| - |E_{(t-1)t}| \geq (t - 2) - (b - a) - 1 - b
= t - 2b + a - 1.
\]

We see that, if \( i(t - 2) \in E_t^c \) then \( i(t - 2) \in [t - 2]^{(2)} \setminus E_t \) except \( i = t - 1 \) and if \( i(t - 3) \in E_t^c \) then \( i(t - 3) \in [t - 2]^{(2)} \setminus E_t \) except \( i = t - 1 \), therefore, we have \( |E_{(t-3)t}^c| + |E_{(t-2)t}^c| \leq |[t - 2]^{(2)} \setminus E_t| + 2 = b - a + 2 \). Since \( G \) is left-compressed, then \( |E_{(t-3)t}^c| \leq |E_{(t-2)t}^c| \). Hence \( |E_{(t-3)t}| \leq b - a + 1 \). Recalling that \( b = |E_{(t-1)t}| \leq \frac{2t + 3a - 4}{5} \) and \( |E_{(t-2)t} \cap E_{(t-1)t}^c| \geq t - 2b + a - 1 \), we have

\[
|E_{(t-3)t}^c| \leq \frac{b - a}{2} + 1 \leq t - 2b + a - 1 \leq |E_{(t-2)t} \cap E_{(t-1)t}^c|.
\]

Let \( i \) be the minimum integer in \( E_{(t-2)t} \) and \( j \) be the minimum integer in \( E_{(t-2)t} \cap E_{(t-1)t}^c \). Because \( G \) is left-compressed, we have \( i \geq j \) and

\[
\lambda(E_{(t-3)t}^c, x) \leq \lambda(E_{(t-2)t} \cap E_{(t-1)t}^c, x).
\]  

(5)

In view of (3), (4) and (5), to show \( x_1 - x_{t-3} \leq x_{t-2} - x_{t-1} \), we only need to prove the following.

\[
\lambda(E_{1(t-3)}, x) - \lambda(E_{(t-2)(t-1)}, x) = x_{t-2} + x_{t-1} + x_t - x_1 - x_{t-3} \geq 0.
\]  

(6)

To verify (6), by Remark 2 (b) and similarly to (3) and (4), we have

\[
x_1 = x_{t-1} + \frac{\lambda(E_{1(t-1)}, x)}{\lambda(E_{(t-1)t}, x)} \leq x_{t-1} + \frac{(x_2 + \cdots + x_{t-2})x_t}{x_2 + \cdots + x_{t-2} + x_t} \leq x_{t-1} + x_t;
\]  

(7)
\[ x_1 = x_{t-2} + \frac{\lambda \left( E_{(t-2)} \setminus E_{(t-1)}, x \right)}{\lambda \left( E_{(t-2)}, x \right)} \]
\[ = x_{t-2} + \frac{\lambda \left( E_{(t-2)}^c, x \right)}{1 - x_1 - x_{t-2}} x_t \]
\[ \leq x_{t-2} + \frac{\lambda \left( E_{(t-2)}^c, x \right)}{1 - x_{t-3} - x_{t-1} - x_t} x_t; \] (8)

and
\[ x_{t-3} = x_{t-1} + \frac{\lambda \left( E_{(t-3)} \setminus E_{(t-1)}, x \right)}{\lambda \left( E_{(t-3)}, x \right)} \]
\[ = x_{t-1} + \frac{\lambda \left( E_{(t-3)}^c \setminus E_{(t-1)}^c, x \right)}{1 - x_{t-3} - x_{t-1} - x_t} x_t. \] (9)

Adding (8) and (9), we obtain that
\[ x_1 + x_{t-3} \leq x_{t-2} + x_{t-1} + \frac{\lambda \left( E_{(t-2)}^c, x \right) + \lambda \left( E_{(t-3)}^c \setminus E_{(t-1)}^c, x \right)}{1 - x_{t-3} - x_{t-1} - x_t} x_t. \]

Clearly, \( t - 3 \notin E_{(t-3)}^c \). Since \( G \) is left-compressed and \( G \neq C_{3,m} \), we have \( t - 2 \notin E_{(t-3)}^c \). On the other hand both \( t - 3 \) and \( t - 2 \) are in \( E_{(t-1)}^c \). Hence \( |E_{(t-2)}^c| + |E_{(t-3)}^c \setminus E_{(t-1)}^c| \leq |E_{(t-2)}^c| + |E_{(t-1)}^c| - 2 \). Note that \( |E^c| \leq t - 3 \) and there is exactly one edge in \( E^c \), namely \((t-2)(t-1)\), containing both \((t-2)t\) and \((t-1)t\). So we have \( |E_{(t-2)}^c| + |E_{(t-1)}^c| \leq |E^c| + 1 \leq t - 2 \) and \( |E_{(t-2)}^c| + |E_{(t-1)}^c| \leq |E_{(t-2)}^c| + |E_{(t-1)}^c| - 2 \leq t - 4 \). Recalling that \( x_1 \geq x_2 \geq \cdots \geq x_t \), we have
\[ \lambda \left( E_{(t-2)}^c, x \right) + \lambda \left( E_{(t-3)}^c \setminus E_{(t-1)}^c, x \right) \leq x_1 + x_2 + \cdots + x_t - 4 \]
and
\[ \frac{\lambda \left( E_{(t-2)}^c, x \right) + \lambda \left( E_{(t-3)}^c \setminus E_{(t-1)}^c, x \right)}{1 - x_{t-3} - x_{t-1} - x_t} \leq \frac{x_1 + x_2 + \cdots + x_t - 4}{x_1 + x_2 + \cdots + x_t - 4 + x_t - 2} \leq 1. \]

So, (6) is true. Combining (3), (4), (5) and (6), we obtain that \( x_1 - x_{t-3} \leq x_{t-2} - x_{t-1} \) and \( x_{t-3}x_{t-2}x_t - x_1x_{t-1}x_t \geq 0 \). Hence
\[ \lambda \left( C_{3,m}, x \right) - \lambda \left( G, x \right) = \lambda \left( [t-2] \setminus E_t, x \right) - |[t-2] \setminus E_t| x_1x_{t-1}x_t \]
\[ \geq |[t-2] \setminus E_t| (x_{t-3}x_{t-2}x_t - x_1x_{t-1}x_t) \]
\[ \geq 0. \]

This completes the proof. \[\square\]

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Connection between the clique number…

References

1. Caen, D.D.: Extension of a theorem of Moon and Moser on complete hypergraphs. Ars Combin. 16, 5–10 (1983)
2. Bulò Rota, S., Pelillo, M.: A generalization of the Motzkin-Straus theorem to hypergraphs. Optim. Lett. 3, 287–295 (2009)
3. Frankl, P., Füredi, Z.: Extremal problems whose solutions are the blow-ups of the small Witt-designs. J. Combin. Theory. Ser. A. 52, 129–147 (1989)
4. Frankl, P., Rödl, V.: Hypergraphs do not jump. Combinatorica. 4, 149–159 (1984)
5. Keevash, P.: Hypergraph Turán problems. http://www.maths.qmul.ac.uk/keevash/papers/turan-survey
6. Mubayi, D.: A hypergraph extension of Turán’s theorem. J. Combin. Theory. Ser. B. 96, 122–134 (2006)
7. Motzkin, T.S., Straus, E.G.: Maxima for graphs and a new proof of a theorem of Turán. Canad. J. Math. 17, 533–540 (1965)
8. Peng, Y., Tang, Q., Zhao, C.: On Lagrangians of $r$-uniform Hypergraphs. J. Comb. Optim. doi:10.1007/s10878-013-9671-3 (in press)
9. Peng, Y., Zhao, C.: A Motzkin-Straus type result for 3-uniform hypergraphs. Graphs Comb. 29, 681–694 (2013)
10. Peng, Y., Zhu, H., Zheng, Y., Zhao, C.: On cliques and Lagrangians of 3-uniform hypergraphs. arXiv preprint arXiv:1211.6508, (2012)
11. Sidorenko, A.F.: Solution of a problem of Bollobás on 4-graphs. Mat. Zametki. 41, 433–455 (1987)
12. Talbot, J.M.: Lagrangians of hypergraphs. Comb. Probab. Comput. 11, 199–216 (2002)
13. Turán, P.: On an extremal problem in graph theory. Mat. Fiz. Lapok. 48, 436–452 (1941)
14. Tang, Q., Peng, H., Wang, C., Peng Y.: On Frankl and Füredi’s conjecture for 3-uniform hypergraphs, submitted
15. Tang, Q., Peng, Y., Zhang, X., Zhao, C.: Some results on Lagrangians of Hypergraphs. Discrete Appl. Math. 166, 222–238 (2014)
16. Tang, Q., Peng Y., Zhang, X., Zhao,C.: On the graph-Lagrangians of 3-uniform hypergraphs containing dense subgraphs, J. Optim. Theory Appl. 163, 31–56 (2014)