DO THE $\pi N$ TOTAL CROSS SECTIONS INCREASE LIKE $\log \nu$ OR $(\log \nu)^2$ AT HIGH ENERGIES? \(^a\)

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We propose to use rich informations on $\pi p$ total cross sections below $N(\sim 10$ GeV) in order to investigate whether these cross sections increase like $\log \nu$ or $(\log \nu)^2$ at high energies. A finite-energy sum rule (FESR) which is derived in the spirit of the $P'$ sum rule as well as the $n = 1$ moment FESR have been required to constrain the high-energy parameters. We then searched for the best fit of $\sigma^{(+)}_{\text{tot}}$ above 70 GeV in terms of high-energy parameters constrained by these two FESR. We can conclude that our analysis strongly favours the $(\log \nu)^2$ behaviors satisfying the Froissart unitarity bound.

As is well-known, the sum of $\pi^+ p$ and $\pi^- p$ total cross sections has a tendency to increase above 70 GeV experimentally.\(^1\) It has not been known, however, if this increase behaves like $\log \nu$ or $\log^2 \nu$ consistent with the Froissart bound.\(^2\)

We would like to propose to use rich informations of $\pi p$ total cross sections at low and intermediate energy regions in order to investigate the high energy behaviours of $\pi p$ total cross sections above 70 GeV using new finite-energy sum rules (FESR) as constraints.

Such a kind of attempt has been initiated in Ref.\(^4\). The $s$-wave $\pi N$ scattering length $a^{(+)}$ of the crossing-even amplitude had been expressed as

\[
(1 + \frac{\mu}{M}) a^{(+)} = -\frac{g^2}{4\pi} \left( \frac{\mu}{2M} \right)^2 \frac{1}{M} \frac{1}{1 - \frac{\mu^2}{2M} + \left( \frac{\mu}{2M} \right)^2 + \int_0^\infty dk [\sigma^{(+)}_{\text{tot}}(k) - \sigma^{(+)}_{\text{tot}}(\infty)]
\]

with pion mass $\mu$ under the assumption that there are no singularities with the vacuum quantum numbers in the $J$ plane except for the Pomeron ($P$). The evidence that this sum rule had not

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been satisfied led us to the prediction of the $P'$ trajectory with $\alpha_{P'} \approx 0.5$, and soon the $f$ meson ($f_2(1275)$) has been uncovered on this $P'$ trajectory.

\textbf{(FESR(1))}: Taking into account the present situation of increasing total cross section data, we derive FESR in the spirit of the $P'$ sum rule. We consider the crossing-even (spin-averaged) forward scattering amplitude for $\pi p$ scattering:

$$f^{(+)}(\nu) = \frac{1}{4\pi}[A^{(+)}(\nu) + \nu B^{(+)}(\nu)].$$

We assume

$$\text{Im } f^{(+)}(\nu) \simeq \text{Im } R(\nu) + \text{Im } f_{P'}(\nu) = \frac{\nu}{\mu^2} \left( c_0 + c_1 \log \frac{\nu}{\mu} + c_2 \log^2 \frac{\nu}{\mu} \right) + \frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{P'}}$$

at high energies ($\nu \geq N$). Taking into account the amplitude $f^{(+)}(\nu)$ to be crossing-even, we can derive (for a detail see ref.\cite{ref})

$$\text{Re } f^{(+)}(\mu) = \text{Re } R(\mu) + \text{Re } f_{P'}(\mu) - \frac{g_{P}^2}{4\pi} \left( \frac{\mu}{2M} \right)^2 \frac{1}{M} \frac{1}{\left( \frac{\mu}{2M} \right)^2}$$

$$+ \frac{1}{2\pi^2} \int_0^N \sigma_{\text{tot}}^{(+)}(k) dk - \frac{2P}{\pi} \int_0^N \frac{\nu}{k^2} \left\{ \text{Im } R(\nu) + \frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{0.5} \right\} d\nu,$$

where $N \equiv \sqrt{N^2 - \mu^2} \simeq N$. Let us call Eq. \cite{ref} as the FESR(1) which we use as the first constraint. It is important to notice that Eq. \cite{ref} reduces to the $P'$ sum rule in ref.\cite{ref} if $c_1, c_2 \to 0$.

The FESR \cite{ref,ref,ref}

$$\int_0^N d\nu \nu^n \text{Im } f(\nu) = \sum_i \beta_i \frac{N^\alpha_i + n + 1}{\alpha_i + n + 1}$$

holds for even positive integer $n$ when $f(\nu)$ is crossing odd, and holds for odd positive integer $n$ when $f(\nu)$ is crossing even. We can also derive negative-integer moment FESR. The only significant FESR is a one for $f^{(+)}(\nu)/\nu$ corresponding to $n = -1$. FESR(1) belongs to this case.

\textbf{(FESR(2))}: The second FESR corresponding to $n = 1$ is:

$$\pi \mu \left( \frac{g_{P}^2}{4\pi} \right) \left( \frac{\mu}{2M} \right)^3 + \frac{1}{4\pi} \int_0^N dk k^2 \sigma_{\text{tot}}^{(+)}(k)$$

$$= \int_0^N \nu \text{Im } R(\nu) d\nu + \int_0^N \nu \text{Im } f_{P'}(\nu) d\nu.$$

We call Eq. \cite{ref} as the FESR(2). It is to be noticed that the contribution from higher energy regions is enhanced.

\textbf{(Data)} The numerical value of $-\frac{g_{P}^2}{4\pi} \left( \frac{\mu}{2M} \right)^2 \frac{1}{M} \frac{1}{\left( \frac{\mu}{2M} \right)^2} = -0.0854 \text{GeV}^{-1}$, $\pi \mu \frac{g_{P}^2}{4\pi} \left( \frac{\mu}{2M} \right)^3 = 0.0026 \text{GeV}$ have been evaluated using $\frac{g_{P}^2}{4\pi} = 14.4$. Re $f^{(+)}(\mu) = (1 + \frac{\mu}{\nu}) \alpha^{(+)} = (1 + \frac{\mu}{\nu}) \frac{1}{2}(a_1 + 2a_2) = -(0.014 \pm 0.026) \text{GeV}^{-1}$ was obtained from $a_{\pm} = (0.171 \pm 0.005) \mu^{-1}$ and $a_{\mp} = -(0.088 \pm 0.004) \mu^{-1}$.

We have used rich data\cite{ref} of $\sigma^{+\pi}$ and $\sigma^{-\pi}$ to evaluate the relevant integrals of cross sections appearing in FESR(1) and (2). We have obtained $\frac{1}{2\pi^2} \int_0^N dk \sigma_{\text{tot}}^{(+)}(k) = 38.75 \pm 0.25 \text{GeV}^{-1}$,
\[ \frac{1}{4\pi} \int_{0}^{\mathcal{N}} dk \, k^2 \sigma_{\text{tot}}^{(+)}(k) = 1817 \pm 31 \text{ GeV} \text{ for } \mathcal{N} = 10 \text{ GeV}. \text{ For a detail, see ref. [1].} \]

**Analysis**  The FESR(1) and (2) are our starting points. Armed with these two, we expressed high-energy parameters \( c_0, c_1, c_2, \beta_{P'} \) in terms of the Born term and the \( \pi N \) scattering length \( a^+(+) \) as well as the total cross sections up to \( N \). We then attempt to fit the \( \sigma_{\text{tot}}^{(+)} \) above 70 GeV. We set \( N = 10 \text{ GeV}. \)

Let us first define the \( \log^2 \nu \) model and the \( \log \nu \) model. The \( \log^2 \nu \) model is a model for which the imaginary part of \( f^+(\nu) \) behaves as \( a + b \log \nu + c(\log \nu)^2 \) as \( \nu \) becomes large. The \( \log \nu \) model is a model for which the imaginary part of \( f^+(\nu) \) behaves as \( a' + b' \log \nu \) for large \( \nu \). So we generally assume that the \( \text{Im} \, f^+(\nu) \) behaves as Eq. (3) at high energies \( (\nu > N) \).

**1 log \nu model**: This model has three parameters \( c_0, c_1 \) and \( \beta_{P'} \) with two constraints FESR (1), (2). We set \( N = 10 \text{ GeV} \) and expressed both \( c_0, \beta_{P'} \) as a function of \( c_1 \) using the FESR(1) and (2). We obtained

\[ c_0(c_1) = 0.0879 - 4.94c_1, \quad \beta_{P'}(c_1) = 0.1290 - 7.06c_1. \]  

We then tried to fit 12 data points of \( \sigma_{\text{tot}}^{(+)}(k) \) between 70 GeV and 340 GeV. The best fit we obtained is \( c_1 = 0.00185 \) which gives \( c_0 = 0.0787 \) and \( \beta_{P'} = 0.142 \) with the bad “reduced \( \chi^2 \);” \( \chi^2/(N_{\text{data}} - N_{\text{param}}) = 29.03/(12 - 1) \approx 2.6 \). Therefore it turned out that this model has difficulties to reproduce the experimental increase of \( \pi p \) total cross sections above 70 GeV.

**2 log \( \nu \) model**: This model has four parameters \( c_0, c_1, c_2 \) and \( \beta_{P'} \) with two constraints FESR(1),(2). We again set \( N = 10 \text{ GeV} \) and required both FESR(1) and (2) as constraints. Then \( c_0, \beta_{P'} \) are expressed as functions of \( c_1 \) and \( c_2 \) as

\[ c_0(c_1, c_2) = 0.0879 - 4.94c_1 - 21.50c_2, \quad \beta_{P'}(c_1, c_2) = 0.1290 - 7.06c_1 - 41.46c_2. \]

We then searched for the fit to 12 data points of \( \sigma_{\text{tot}}^{(+)}(k) \) above 70 GeV. The best fit in terms of two parameters \( c_1 \) and \( c_2 \) led us to greatly improved value of “reduced \( \chi^2 \);” \( \chi^2/(N_{\text{data}} - N_{\text{param}}) = 0.746/(12 - 2) \approx 0.075 \) for \( c_1 = -0.0215 < 0 \) and \( c_2 = 0.00182 > 0 \) which give \( c_0 = 0.155 \) and \( \beta_{P'} = 0.0574 \). This is an excellent fit to the data.

It is remarkable to notice that the wide range of data \( (k \geq 5 \text{ GeV}) \) have been reproduced within the error even in the region where the fit has not been made (see Fig. 1 (a) and (b)). It is also important to note that the results do not change so much for the value of \( N \). The increase of \( \sigma_{\text{tot}}^{(+)} \) above 50 GeV is explained via \( \log^2 \nu/\mu \) \((c_2 > 0)\) and the decrease between 5 \( \sim \) 50 GeV is explained by \( \log \nu/\mu \) \((c_1 < 0)\).

Therefore, we can conclude that our analysis based on the FESR(1),(2) strongly favours the \( \log^2 \nu/\mu \) behaviours satisfying the Froissart unitarity bound.

**Notes added in proof**—After completing the manuscript, we were informed by Dr. Jurgen Englert that the SELEX collaboration, U. Dersch et al. [ Nucl. Phys. B579 (2000) 277 ] had a datum for \( \pi^- N \) at 610 GeV. Our \( \log^2 \nu \) model predicts 25.9mb for \( \sigma_{\text{tot}}^{(+)} \) at 610 GeV which is consistent with their value on \( \pi^- N \), \((26.6 \pm 0.9)\)mb. We were also informed by Dr. Bararab Nicolescu that COMPETE collaboration, J. R. Cudell et al. [hep-ph/0107219] also reached a similar conclusion that the Froissart bound seemed favoured by completely different approach. We also came to know from a talk at the 37th Moriond Conference (March 16-23, 2002) by Dr. F. D. Steffen that the gluon saturation leads to \( \log^2 \nu \) behaviours at high energy [ A. I. Shoshi, F. D. Steffen and H. J. Pirner, hep-ph/0202012 ].
Figure 1: Fit to the \( \sigma^{(+)}_{\text{tot}} \) data above 70GeV by the \( \log^2 \nu \) model. The dashed line represents the contribution from \( \text{Im } R(\nu) \) with \( c_2 > 0 \).

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