Integrability of one bilinear equation: singularity analysis and dimension

SERGEI SAKOVICH

Institute of Physics, National Academy of Sciences of Belarus

sergsako@gmail.com

Abstract

The integrability of a four-dimensional sixth-order bilinear equation associated with the exceptional affine Lie algebra $D_4^{(1)}$ is studied by means of the singularity analysis. This equation is shown to pass the Painlevé test in three distinct cases of its coefficients, exactly when the equation is effectively a three-dimensional one, equivalent to the BKP equation.

1 Introduction

Most of the integrable nonlinear partial differential equations of modern mathematical physics are two-dimensional ones. There are very few three-dimensional integrable nonlinear equations in the literature. And even less is known about the integrability in dimension four and higher. In the present paper, we study one four-dimensional nonlinear equation and show that, unfortunately, this equation is integrable only if it is effectively three-dimensional.

We study the following four-dimensional sixth-order bilinear equation:

$$(D_x^6 + 36 D_x D_t - 10 D_y^2 - 10 D_z^2) \tau \cdot \tau + a (D_x^3 D_y + D_y^2 - D_z^2) \tau \cdot \tau + b (D_x^3 D_z - 2 D_y D_z) \tau \cdot \tau = 0,$$

where $D_x$, $D_t$, $D_y$ and $D_z$ denote Hirota’s bilinear differentiation operators, while $a$ and $b$ are parameters. This bilinear equation, associated with the exceptional affine Lie algebra $D_4^{(1)}$, appeared for the first time in [1] (see p. 207 there). The same equation appeared recently in [2], in a different form related to [1] by a linear transformation of independent variables. It is interesting to investigate whether there are such values of the parameters $a$ and $b$, for which this bilinear equation is a genuine four-dimensional integrable equation.

In Sections 2 we study the integrability of the four-dimensional sixth-order bilinear equation by the method of singularity analysis, in its formulation for partial differential equations [3,4]. In Section 3 we apply the same Painlevé test for integrability to other $D_4^{(1)}$-associated bilinear equations, initially three-dimensional ones. Section 4 contains discussion of the results.
2 Singularity analysis

In our experience, the Painlevé test for integrability is a reliable and easy-to-use method, especially convenient (if compared to other methods) for high-dimensional, high-order, non-evolutionary and multi-component nonlinear equations [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The reliability of the Painlevé test has been empirically verified by the integrability studies of fifth-order KdV-type equations [17], bilinear equations [18], coupled KdV equations [19, 20, 21, 22], coupled higher-order nonlinear Schrödinger equations [23], generalized Ito equations [24], sixth-order nonlinear wave equations [25], seventh-order KdV-type equations [26], etc. Some interesting new equations have been discovered by means of the singularity analysis [27, 28, 29, 30, 31, 32]. Also, the Painlevé property is very sensitive to the dimension and geometry of a studied nonlinear problem [12, 33] (see, however, a remark in [34] on why [33] was criticized).

In usual partial derivatives and with the notation 
\[ \tau \equiv u(x,y,z,t), \]
the bilinear equation (1) takes the form
\[ uu_{xxxxx} - 6u_xu_{xxxx} + 15u_{xx}u_{xxx} - 10u_{xxx} \]
\[ + 36u_xu_t - 36u_{yy} + 10u_y^2 - 10u_{yy}z + 10u^2_z \]
\[ + a(u_{xx}y - u_yu_{xx} - 3u_xu_{xy} + 3u_{xx}u_{xy} + uu_{yy} - u_y^2 - uu_z + u_z^2) \]
\[ + b(u_{xx}y - u_zu_{xx} - 3u_xu_{xz} + 3u_{xx}u_{xz} - 2u_{yz} + 2u_yu_z) = 0. \] (2)

A hypersurface \( \phi(x,y,z,t) = 0 \) is non-characteristic for this partial differential equation (2) if \( \phi_x \neq 0 \), and we take \( \phi_x = 1 \),
\[ \phi = x + \psi(y,z,t), \] (3)
without loss of generality. Substitution of the expansion
\[ u = u_0(y,z,t)\phi^p + \cdots + u_r(y,z,t)\phi^{p+r} + \cdots \] (4)
to the nonlinear equation (2) determines the admissible leading exponents \( p \) (the dominant behavior of solutions \( u \) near \( \phi = 0 \)), as well as the corresponding resonances \( r \) (the positions, where arbitrary functions can enter the expansion).

In this way, we obtain the following two branches to be studied:
\[ p = 1, \quad r = -1, 0, 1, 2, 3, 10, \] (5)
and
\[ p = 2, \quad r = -2, -1, 0, 1, 5, 12, \] (6)
with \( u_0(y,z,t) \) being an arbitrary function in both cases. Let us note that the nonlinear equation (2) does not possess branches with negative values of \( p \) (this is typical for bilinear equations [18]), and that the expansions (4) with (5) or (6) are not covered by the Cauchy–Kovalevskaya theorem because the Kovalevskaya form of the partial differential equation (2) is singular at \( u = 0 \).

The branch (5) is the generic one. In this case, the expansion (4) represents the general solution of (2) and (potentially) contains six arbitrary functions of three variables: \( \psi(y,z,t) \), which corresponds to \( r = -1 \), and \( u_i(y,z,t) \) with \( i = 0, 1, 2, 3, 10 \). The branch (6) represents only a class of special solutions,
because the expansion (4) can contain only five arbitrary functions of three variables in this case: \( u_0, u_1, u_5, u_{12} \) and \( \psi \). For this reason, it seems natural to study the generic branch (5) first.

We substitute the expansion

\[
 u = \sum_{i=0}^{\infty} u_i(y, z, t) \phi^{i+1},
\]

with \( \phi \) given by (3), to the nonlinear equation (2), and collect terms with \( \phi^{n-4}, n = 0, 1, 2, \ldots \). In this way, we obtain recursion relations, which, for any given \( n \), either determine an expression for \( u_n(y, z, t) \) if \( n \) is not a resonance, or lead to a compatibility condition if \( n \) is a resonance. For \( n = 0, 1, 2, 3 \), which are resonances, we find that the compatibility conditions are satisfied identically and the functions \( u_0(y, z, t), u_1(y, z, t), u_2(y, z, t), u_3(y, z, t) \) remain arbitrary. Then, for \( n = 4, 5, 6, 7, 8, 9 \), which are not resonances, the recursion relations give us explicit expressions for the coefficients \( u_4, u_5, u_6, u_7, u_8, u_9 \). The expressions are very complicated, not suitable for presentation in the paper, and one definitely needs to use computer algebra tools to obtain them. Finally, at the highest resonance \( n = 10 \), where the function \( u_{10}(y, z, t) \) remains arbitrary, a huge non-trivial compatibility condition appears, which contains 337 terms and involves the parameters \( a \) and \( b \), the functions \( u_0(y, z, t), u_1(y, z, t), u_2(y, z, t), u_3(y, z, t) \) and their derivatives, and derivatives of the function \( \psi(y, z, t) \).

The analysis of this compatibility condition at the highest resonance of the branch (5) is hard but possible with the help of computer algebra tools. We find that the compatibility condition is satisfied identically, for arbitrary functions \( u_0(y, z, t), u_1(y, z, t), u_2(y, z, t), u_3(y, z, t) \) and \( \psi(y, z, t) \), in only three distinct cases of values of the parameters \( a \) and \( b \):

\[
 a = -10, \quad b = 0,
\]

or

\[
 a = 5, \quad b = \pm 5\sqrt{3}.
\]

For any other values of the parameters \( a \) and \( b \), the compatibility condition is not satisfied identically, the expansion (7) must be modified by additional logarithmic terms (starting from the term proportional to \( \phi^{11} \log \phi \)), and this is a clear symptom of non-integrability.

To study the branch (6), we use the expansion

\[
 u = \sum_{i=0}^{\infty} u_i(y, z, t) \phi^{i+2},
\]

with \( \phi \) given by (3), substitute it to the nonlinear equation (2), collect the terms with \( \phi^{n-2}, n = 0, 1, 2, \ldots \) and obtain the recursion relations which determine the coefficients \( u_n(y, z, t) \) of the expansion (outside the resonances) and the compatibility conditions (at the resonances). We do not initially impose the already obtained conditions (8) and (9) on the parameters \( a \) and \( b \), because exactly the same conditions for \( a \) and \( b \) follow from the compatibility condition at the resonance 5 of this branch (6), whereas the corresponding expressions are sufficiently simple for presentation in the paper.
For \( n = 0 \) and \( n = 1 \) of the branch (6), we have resonances, the corresponding compatibility conditions are satisfied identically, and the functions \( u_0(y, z, t) \) and \( u_1(y, z, t) \) remain arbitrary. For \( n = 2, n = 3 \) and \( n = 4 \), we obtain, respectively, the following expressions for the coefficients of the expansion (10): 

\[
\begin{align*}
  u_2 &= \frac{u_1^2}{2u_0} - \frac{u_0}{60} (a\psi_y + b\psi_z), \\
  u_3 &= \frac{u_1^3}{6u_0^2} - \frac{u_1}{60} (a\psi_y + b\psi_z)
\end{align*}
\]

and 

\[
\begin{align*}
  u_4 &= \frac{u_1^4}{24u_0^3} + \frac{u_0\psi_t}{40} - \frac{u_1^2}{120u_0} (a\psi_y + b\psi_z) \\
  &\quad - \frac{u_1}{240u_0} (au_{0,y} + bu_{0,z}) + \frac{1}{240} (au_{1,y} + bu_{1,z}) \\
  &\quad + \frac{u_0}{1440} \left[ (a - 10)\psi_y^2 - 2b\psi_y\psi_z - (a + 10)\psi_z^2 \right].
\end{align*}
\]

The case of \( n = 5 \) is a resonance, the function \( u_5(y, z, t) \) remains arbitrary, and the following nontrivial compatibility condition appears:

\[
(a^2 + 5a - 50) \psi_{yy} + (2ab - 10b) \psi_{yz} + (b^2 - 5a - 50) \psi_{zz} = 0.
\]

This compatibility condition is satisfied identically, for any function \( \psi(y, z, t) \), if and only if the parameters \( a \) and \( b \) satisfy the system of equations

\[
(a + 10)(a - 5) = 0, \quad (a - 5)b = 0, \quad b^2 = 5(a + 10),
\]

that is, exactly in the three distinct cases given by (8) and (9).

It is nice that we have found an easier way to the conditions (8) and (9), suitable for presentation in the paper. However, we still have to do a lot of computational work for the branch (6). For \( n = 6, 7, 8, 9, 10, 11 \), which are not resonances, the recursion relations give us expressions (very complicated, indeed) for the coefficients \( u_6, u_7, u_8, u_9, u_{10}, u_{11} \) of the expansion (10). Then, at the highest resonance of this branch, that is for \( n = 12 \), where the function \( u_{12}(y, z, t) \) remains arbitrary, we obtain a huge nontrivial compatibility condition, which contains 1861 terms. Finally, we verify that this compatibility condition is satisfied identically, for arbitrary functions \( u_0(y, z, t), u_1(y, z, t), u_5(y, z, t) \) and \( \psi(y, z, t) \), whenever the parameters \( a \) and \( b \) satisfy (8) or (9).

We can conclude now that the four-dimensional bilinear equation (1) passes the Painlevé test for integrability in only three cases of its coefficients, when the parameters \( a \) and \( b \) are given by (8) or (9). These integrable cases, however, are neither new nor four-dimensional, actually. In the case of (8), we get from (1) the three-dimensional bilinear equation

\[
(D_x^6 - 10D_x^3D_y - 20D_y^2 + 36D_xD_t) \tau \cdot \tau = 0,
\]

which is the well-known BKP equation [35]. In the two cases with \( a \) and \( b \) given by (9), we can write the bilinear equation (1) in the form

\[
\left( D_x^6 + 5D_x^3(D_y \pm \sqrt{3}D_z) - 5(D_y \pm \sqrt{3}D_z)^2 + 36D_xD_t \right) \tau \cdot \tau = 0,
\]

and
where the ± signs correlate with the sign in (11). The linear transformation of two independent variables

\[ y' = y \mp \sqrt{3}z, \quad z' = z \mp \sqrt{3}y \]  

(18)

turns the (formally) four-dimensional equation (17) into the three-dimensional BKP equation (16), with \( y \) replaced by \( y' \) and without any derivative with respect to \( z' \) in it. In this sense, all the Painlevé-integrable cases of the four-dimensional bilinear equation (1) are effectively three-dimensional and equivalent to the BKP equation.

3 Extra equations

The \( D(4) \)-associated bilinear equations of the lowest degree 6 are linear combinations of the following three linearly independent equations [1, 2]:

\[ (D_x^6 + 36 D_x D_t - 10 D_y^2 - 10 D_z^2) \tau \cdot \tau = 0, \]  
\[ (D_x^3 D_y + D_y^2 - D_z^2) \tau \cdot \tau = 0, \]  
\[ (D_x^2 D_z - 2 D_y D_z) \tau \cdot \tau = 0. \]  

(19) (20) (21)

(It is not said in [1, 2] what are the coefficients of those linear combinations, and we guess that any.) In Section 2 we have studied the integrability of all the four-dimensional linear combinations (11). Now, for completeness, let us briefly consider the singularity analysis of the remaining linear combinations, initially three-dimensional ones. They are the bilinear equation

\[ (D_x^3 D_y + D_y^2 - D_z^2) \tau \cdot \tau + c (D_x^3 D_z - 2 D_y D_z) \tau \cdot \tau = 0, \]  

(22)

with a parameter \( c \), and the bilinear equation (21) itself.

For the three-dimensional fourth-order equation (22), there is one branch to be studied. It describes the first-order zeroes of solutions near any non-characteristic hypersurface, the resonances being \(-1, 0, 1, 6\). The compatibility conditions at the resonances 0 and 1 are satisfied identically. However, the nontrivial compatibility condition (with 100 terms) at the resonance 6 is satisfied identically for only two values of the parameter \( c \),

\[ c = \pm \frac{1}{\sqrt{3}}. \]  

(23)

For \( c \) given by (23), we can write the bilinear equation (22) in the form

\[ \left( D_x^3 \left( D_y \mp \frac{1}{\sqrt{3}} D_z \right) + \left( D_y \mp \frac{1}{\sqrt{3}} D_z \right) \left( D_y \mp \frac{1}{\sqrt{3}} D_z \right) \right) \tau \cdot \tau = 0, \]  

(24)

where the ± and \( \mp \) signs correlate with the sign in (23). The linear transformation of two independent variables

\[ y' = -\frac{1}{2} y \pm \frac{\sqrt{3}}{2} z, \quad z' = z \pm \sqrt{3} y \]  

(25)

turns the bilinear equation (24) into the bilinear equation (21), in which \( y \) and \( z \) are replaced by \( y' \) and \( z' \), respectively. Consequently, all the Painlevé-integrable
cases of the bilinear equation (22) are equivalent to the bilinear equation (21), which also possesses the Painlevé property therefore.

It only remains to note that the bilinear equation (21) appeared for the first time in [36] (see Appendix B there).

4 Discussion

In this paper, we have studied the integrability of the four-dimensional sixth-order bilinear equation (1) associated with the exceptional affine Lie algebra \( D_{(1)}^{(4)} \). We have shown that this equation passes the Painlevé test for integrability in only three cases of its coefficients, given by (8) and (9), exactly when the equation either coincides with the three-dimensional BKP equation (19) or is related to the BKP equation by the linear transformation (18) of two independent variables. Also, we have briefly considered the integrability of another \( D_{(1)}^{(4)} \)-associated bilinear equation, the initially three-dimensional equation (22), and transformed its integrable cases to the Ito equation (21).

The main result of this paper can be reformulated in the following way: every genuinely four-dimensional case of the bilinear equation (1) fails to pass the Painlevé test for integrability. Therefore it could be an interesting problem to study multi-soliton solutions of the bilinear equation (1) with any values of \( a \) and \( b \) different from (8) and (9), in order to find any difference in the soliton dynamics between the genuinely four-dimensional cases and the three-dimensional integrable cases. Other integrability criteria, based on generalized symmetries, conservation laws, etc., could also be applied.

The last remark concerns the transformation (18) which effectively merges two independent variables of the formally four-dimensional equation (17) into one independent variable of the three-dimensional equation (16). Intentionally or not, the importance of such transformations is often ignored in the current literature. This generates a flood of “novel” higher-dimensional integrable equations which are nothing but the well-known old lower-dimensional ones with, say, \( \partial_x \) replaced everywhere by \( \partial_x + \partial_y \), or the like. Integrability of this kind could only be called “true integrability in fake dimensions”.

References

[1] V.G. Kac, M. Wakimoto, Exceptional hierarchies of soliton equations, Proc. Sympos. Pure Math. 49.1 (1989) 191–237.

[2] M. Cafasso, A. du Crest de Villeneuve, D. Yang, Drinfeld–Sokolov hierarchies, tau functions, and generalized Schur polynomials, Symmetry Integr. Geom. Methods Appl. 14 (2018) 104; arXiv:1709.07399

[3] J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. 24 (1983) 522–526.

[4] M. Tabor, Chaos and Integrability in Nonlinear Dynamics: An Introduction, Wiley, New York, 1989.

[5] S.Yu. Sakovich, Painlevé analysis of new soliton equations by Hu, J. Phys. A: Math. Gen. 27 (1994) L503–L505.
[6] S.Yu. Sakovich, Painlevé analysis and Bäcklund transformations of Doktorov–Vlasov equations, J. Phys. A: Math. Gen. 27 (1994) L33–L38.

[7] S.Yu. Sakovich, On zero-curvature representations of evolution equations, J. Phys. A: Math. Gen. 28 (1995) 2861–2869.

[8] S.Yu. Sakovich, Painlevé analysis of a higher-order nonlinear Schrödinger equation, J. Phys. Soc. Jpn. 66 (1997) 2527–2529.

[9] S.Yu. Sakovich, On integrability of a (2+1)-dimensional perturbed KdV equation, J. Nonlinear Math. Phys. 5 (1998) 230–233; arXiv:solv-int/9805012.

[10] A. Karasu-Kalkanlı, S.Yu. Sakovich, Bäcklund transformation and special solutions for the Drinfeld–Sokolov–Satsuma–Hirota system of coupled equations, J. Phys. A: Math. Gen. 34 (2001) 7355–7358; arXiv:nlin/0102001.

[11] A. Karasu-Kalkanlı, S.Yu. Sakovich, İ. Yurduçen, Integrability of Kersten–Krasil'shchik coupled KdV–mKdV equations: singularity analysis and Lax pair, J. Math. Phys. 44 (2003) 1703–1708; arXiv:nlin/0209046.

[12] A. Karasu-Kalkanlı, S. Sakovich, Singularity analysis of a spherical Kadomtsev–Petviashvili equation, J. Phys. Soc. Jpn. 74 (2005) 505–507; arXiv:nlin/0404037.

[13] S. Sakovich, Enlarged spectral problems and nonintegrability, Phys. Lett. A 345 (2005) 63–68; arXiv:nlin/0504037.

[14] S. Sakovich, Singularity analysis and integrability of a Burgers-type system of Frousov, Symmetry Integr. Geom. Methods Appl. 7 (2011) 002; arXiv:1010.5709.

[15] S. Sakovich, On two aspects of the Painlevé analysis, Int. J. Analysis 2013 (2013) 172813; arXiv:solv-int/9909027.

[16] S. Sakovich, Integrability study of a four-dimensional eighth-order nonlinear wave equation, Nonlinear Phenom. Complex Syst. 20 (2017) 267–271; arXiv:1607.05408.

[17] H. Harada, S. Oishi, A new approach to completely integrable partial differential equations by means of the singularity analysis, J. Phys. Soc. Jpn. 54 (1985) 51–56.

[18] B. Grammaticos, A. Ramani, J. Hietarinta, A search for integrable bilinear equations: The Painlevé approach, J. Math. Phys. 31 (1990) 2572–2578.

[19] A. Karasu-Kalkanlı, Painlevé classification of coupled Korteweg–de Vries systems, J. Math. Phys. 38 (1997) 3616–3622.

[20] S.Yu. Sakovich, Coupled KdV equations of Hirota–Satsuma type, J. Nonlinear Math. Phys. 6 (1999) 255–262; arXiv:solv-int/9901005.

[21] S.Yu. Sakovich, Addendum to: Coupled KdV equations of Hirota–Satsuma type, J. Nonlinear Math. Phys. 8 (2001) 311–312; arXiv:nlin/0104072.

[22] S. Sakovich, A note on the Painlevé property of coupled KdV equations, Int. J. Part. Diff. Equns. 2014 (2014) 125821; arXiv:nlin/0402004.

[23] S.Yu. Sakovich, T. Tsuchida, Symmetrically coupled higher-order nonlinear Schrödinger equations: singularity analysis and integrability, J. Phys. A: Math. Gen. 33 (2000) 7217–7226; arXiv:nlin/0006004.
[24] A. Karasu-Kalkanlı, A. Karasu, S.Yu. Sakovich, Integrability of a generalized Ito system: the Painlevé test, J. Phys. Soc. Jpn. 70 (2001) 1165–1166; arXiv:nlin/0102030.

[25] A. Karasu-Kalkanlı, A. Karasu, A. Sakovich, S. Sakovich, R. Turhan, A new integrable generalization of the Korteweg–de Vries equation, J. Math. Phys. 49 (2008) 073516; arXiv:0708.3247.

[26] G.Q. Xu, The integrability for a generalized seventh-order KdV equation: Painlevé property, soliton solutions, Lax pairs and conservation laws, Phys. Scr. 89 (2014) 125201.

[27] A. Karasu-Kalkanlı, A. Karasu, S.Yu. Sakovich, A strange recursion operator for a new integrable system of coupled Korteweg-de Vries equations, Acta Appl. Math. 83 (2004) 85–94; arXiv:nlin/0203036.

[28] S.Yu. Sakovich, T. Tsuchida, Coupled higher-order nonlinear Schrödinger equations: a new integrable case via the singularity analysis, arXiv:nlin/0002023.

[29] S.Yu. Sakovich, T. Tsuchida, A new integrable system of symmetrically coupled derivative nonlinear Schrödinger equations via the singularity analysis, arXiv:nlin/0004025.

[30] B.A. Kupershmidt, KdV6: An integrable system, Phys. Lett. A 372 (2008) 2634–2639; arXiv:0709.3848.

[31] S. Sakovich, Integrability of the vector short pulse equation, J. Phys. Soc. Jpn. 77 (2008) 123001; arXiv:0801.3179.

[32] S. Sakovich, A new Painlevé-integrable equation possessing KdV-type solitons, Nonlinear Phenom. Complex Syst. 22 (2019) 299–304; arXiv:1907.01324.

[33] E.V. Doktorov, S.Yu. Sakovich, Painlevé test and integrability of nonlinear Klein–Fock–Gordon equations, J. Phys. A: Math. Gen. 18 (1985) 3327–3334.

[34] S.Yu. Sakovich, The Painlevé property transformed, J. Phys. A: Math. Gen. 25 (1992) L833–L836.

[35] E. Date, M. Kashiwara, T. Miwa, Transformation groups for soliton equations. II. Vertex operators and \(\tau\) functions, Proc. Japan Acad. Ser. A Math. Sci. 57 (1981) 387–392.

[36] M. Ito, An extension of nonlinear evolution equations of the K-dV (mK-dV) type to higher orders, J. Phys. Soc. Jpn. 49 (1980) 771–778.