1. INTRODUCTION

It is generally believed that QCD with massless quarks undergoes a chiral phase transition (see [1] for a review). This leads to important observable signatures in the real world with two light quarks. The order parameter of the chiral phase transition is the chiral condensate $\langle \bar{\psi} \psi \rangle$, which is directly related to the average spectral density of the Dirac operator [2]. However, the eigenvalues of the Dirac operator fluctuate about their average position. The question we wish to address in this lecture is to what extent such fluctuations are universal. If that is the case, they do not depend on the full QCD dynamics and can be obtained from a much simpler chiral Random Matrix Theory (chRMT) with the global symmetries of the QCD partition function.

This conjecture has its origin in the study of spectra of complex systems [3]. According to the Bohigas conjecture, spectral correlations of classically chaotic quantum systems are given by RMT. A first argument in favor of the universality in Dirac spectra came from the analysis of the finite volume QCD partition function. In particular, for box size $L$ in the range $1/\Lambda \ll L \ll 1/m_\pi$, ($\Lambda$ is a typical hadronic scale and $m_\pi$ is the pion mass) we expect that the global symmetries determine its mass dependence [4]. This implies that the fluctuations of Dirac spectra are constrained by Leutwyler-Smilga sum rules.

Recently, it has become possible to obtain all eigenvalues of the lattice QCD Dirac operator on reasonably large lattices [5]. This makes a direct verification
of the above conjecture possible. This is the main objective of this lecture.

At nonzero chemical potential the QCD Dirac operator is nonhermitean with eigenvalues scattered in the complex plane. The possibility of a new type of universal behavior in this case will be discussed at the end of this lecture.

2. THE DIRAC SPECTRUM

The order parameter of the chiral phase transition, \(\langle \bar{\psi} \psi \rangle\), is nonzero only below the critical temperature. As was shown in \(\langle \bar{\psi} \psi \rangle\) is directly related to the eigenvalue density of the QCD Dirac operator per unit four-volume

\[
\Sigma \equiv |\langle \bar{\psi} \psi \rangle| = \frac{\pi}{\langle \rho(0) \rangle V}.
\]

It is elementary to derive this relation. The Euclidean Dirac operator for gauge field configuration \(A_\mu\) is given by \(D = \gamma_\mu (\partial_\mu + i A_\mu)\). For Hermitean gamma matrices \(D\) is anti-hermitean with purely imaginary eigenvalues, \(D\phi_k = i\lambda_k \phi_k\), and spectral density given by \(\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k)\). Because \(\{\gamma_\mu, D\} = 0\), nonzero eigenvalues occur in pairs \(\pm \lambda_k\). In terms of the eigenvalues of \(D\) the QCD partition function for \(N_f\) flavors of mass \(m\) can then be written as

\[
Z(m) = \langle \prod_k (\lambda_k^2 + m^2)^{N_f} \exp(-S_{YM}) \rangle,
\]

where the average \(\langle \cdot \rangle\) is over all gauge field configurations.

The chiral condensate follows immediately from the partition function \(\langle \bar{\psi} \psi \rangle\),

\[
\langle \bar{\psi} \psi \rangle = \frac{1}{V N_f} \partial_m \log Z(m) = \frac{1}{V} \langle \sum_k \frac{2m}{\lambda_k^2 + m^2} \rangle.
\]

If we express the sum as an integral over the spectral density, and take the thermodynamic limit before the chiral limit so that we have many eigenvalues less than \(m\) we recover \(\langle \bar{\psi} \psi \rangle\) (Notice the order of the limits.).

An important consequence of the Bank-Casher formula \(\langle \bar{\psi} \psi \rangle\) is that the eigenvalues near zero virtuality are spaced as \(\Delta \lambda = 1/\rho(0) = \pi/\Sigma V\). This should be contrasted with the eigenvalue spectrum of the non-interacting Dirac operator. Then \(\rho_{\text{free}}(\lambda) \sim V\lambda^3\) which leads to an eigenvalue spacing of \(\Delta \lambda \sim 1/V^{1/4}\). Clearly, the presence of gauge fields leads to a strong modification of the spectrum near zero virtuality. Strong interactions result in the coupling of many degrees of freedom leading to extended states and correlated eigenvalues. On the other hand, for uncorrelated eigenvalues, the eigenvalue distribution factorizes and we have \(\rho(\lambda) \sim \lambda^{2N_f+1}\), i.e. no breaking of chiral symmetry.

Because the QCD Dirac spectrum is symmetric about zero, we have two different types of eigenvalue correlations: correlations in the bulk of the spectrum and spectral correlations near zero virtuality. In the context of chiral symmetry we wish to study the spectral density near zero virtuality. Because
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the eigenvalues are spaced as $1/\Sigma V$ it is natural to introduce the microscopic spectral density

$$
\rho_S(u) = \lim_{V \to \infty} \frac{1}{V \Sigma} \rho \left( \frac{u}{V \Sigma} \right).
$$

The dependence on the macroscopic variable $\Sigma$ has been eliminated and therefore $\rho_S(u)$ is a perfect candidate for a universal function.

3. SPECTRAL UNIVERSALITY

Spectra for a wide range of complex quantum systems have been studied both experimentally and numerically (see \[6\] for a review). One basic observation has been that the scale of variations of the average spectral density and the scale of the spectral fluctuations separate. This allows us to unfold the spectrum, i.e. we rescale the spectrum in units of the local average level spacing. The fluctuations of the unfolded spectrum can be measured by suitable statistics. We will consider the nearest neighbor spacing distribution, $P(S)$, the number variance, $\Sigma_2(n)$, and the $\Delta_3(n)$ statistic. The number variance is defined as the variance of the number of levels in a stretch of the spectrum that contains $n$ levels on average, and $\Delta_3(n)$ is obtained by a smoothening of $\Sigma_2(n)$.

These statistics can be obtained analytically for the invariant random matrix ensembles defined as ensembles of Hermitean matrices with independently distributed Gaussian matrix elements. Depending on the anti-unitary symmetry, the matrix elements are real, complex or quaternion real. The corresponding Dyson index is given by $\beta = 1, 2,$ and $4$, respectively. The nearest neighbor spacing distribution is given by $P(S) \sim S^{\beta} \exp(-a_{\beta} S^2)$. The asymptotic behavior of $\Sigma_2(n)$ and $\Delta_3(n)$ is given by $\Sigma_2(n) \sim (2/\pi^2) \log(n)$ and $\Delta_3(n) \sim \beta \Sigma_2(n)/2$. For uncorrelated eigenvalues one finds that $P(S) = \exp(-S)$, $\Sigma_2(n) = n$ and $\Delta_3(n) = n/15$. Characteristic features of random matrix correlations are level repulsion at short distances and a strong suppression of fluctuations at large distances.

The main conclusion of numerous studies of eigenvalue spectra is that spectral correlations of a classically chaotic systems are given by RMT \[7, 8\].

4. CHIRAL RANDOM MATRIX THEORY

In this section we will introduce an instanton liquid inspired RMT for the QCD partition function. In the spirit of the invariant random matrix ensembles we construct a model for the Dirac operator with the global symmetries of the QCD partition function as input, but otherwise Gaussian random matrix elements. The chRMT that obeys these conditions is defined by \[6, 9, 10\]

$$
Z^{\beta}_\nu = \int DW \prod_{f=1}^{N_f} \det(D + m_f) e^{-\frac{N g^2}{4} \nu W^+ W}, \quad \text{with} \quad D = \begin{pmatrix} 0 & iW \\ iW^t & 0 \end{pmatrix},
$$

(5)
and \( W \) is a \( n \times m \) matrix with \( \nu = |n - m| \) and \( N = n + m \). The matrix elements of \( W \) are either real (\( \beta = 1 \), chiral Gaussian Orthogonal Ensemble (chGOE)), complex (\( \beta = 2 \), chiral Gaussian Unitary Ensemble (chGUE)), or quaternion real (\( \beta = 4 \), chiral Gaussian Symplectic Ensemble (chGSE)).

This model reproduces the following symmetries of the QCD partition function: i) The \( U_A(1) \) symmetry. All nonzero eigenvalues of the random matrix Dirac operator occur in pairs \( \pm \lambda \). ii) The topological structure of the QCD partition function. The Dirac matrix has exactly \( |\nu| \equiv |n - m| \) zero eigenvalues. This identifies \( \nu \) as the topological sector of the model. iii) The flavor symmetry is the same as in QCD. iv) The chiral symmetry is broken spontaneously with chiral condensate given by \( \Sigma = \lim_{N \to \infty} \pi \rho(0)/N \). \( N \) is interpreted as the (dimensionless) volume of space time.) v) The anti-unitary symmetries. For fundamental fermions the matrix elements of the Dirac operator are complex for \( N_c \geq 3 \) (\( \beta = 2 \)) but can be chosen real for \( N_c = 2 \) (\( \beta = 1 \)). For adjoint fermions they can be arranged into real quaternions (\( \beta = 4 \)).

Note that spectral correlations of chRMT in the bulk of the spectrum are given by the invariant random matrix ensemble with the same value of \( \beta \). Both microscopic correlations near zero virtuality and in the bulk of the spectrum are stable against deformations of the ensemble. This has been shown by a variety of different arguments [11, 12, 13, 15, 14].

Below we will discuss the microscopic spectral density. For \( N_c = 3 \), \( N_f \) flavors and topological charge \( \nu \) it is given by

\[
\rho_S(u) = \frac{u}{2} (J^2_a(u) - J_{a+1}(u)J_{a-1}(u)),
\]

where \( a = N_f + \nu \). The more complicated result for \( N_c = 2 \) is given in [16].

Together with the invariant random matrix ensembles, the chiral ensembles are part of a larger classification scheme. As pointed out in [17], there is a one to one correspondence between random matrix theories and symmetric spaces.

5. LATTICE QCD RESULTS

Recently, Kalkreuter [5] calculated all eigenvalues of the lattice Dirac operator both for Kogut-Susskind (KS) fermions and Wilson fermions for lattices as large as \( 12^4 \). In the case of \( SU(2) \) the anti-unitary symmetry of the KS and the Wilson Dirac operator is different [18]. For KS fermions the Dirac matrix can be arranged into real quaternions, whereas the Hermitean Wilson Dirac matrix \( \gamma_5 D^{Wilson} \) can be chosen real. Therefore, we expect that the eigenvalue correlations are described by the GSE and the GOE, respectively [19]. In Fig. 1 we show results for \( \Sigma_2(n) \), \( \Delta_3(n) \) and \( P(S) \). The results for KS fermions are for 4 dynamical flavors with \( ma = 0.05 \) on a \( 12^4 \) lattice. The results for Wilson fermion were obtained for two dynamical flavors on a \( 8^3 \times 12 \) lattice. Other statistics are discussed in [20].
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Recent lattice studies of the microscopic spectral density for quenched $SU(2)$ show perfect agreement between the microscopic spectral density and random matrix theory [21]. These calculations were performed at moderately strong coupling. Result for couplings in the scaling regime are in progress.

However, an alternative way to probe the Dirac spectrum is via the valence quark mass dependence of the condensate [22], i.e. $\Sigma(m) = \frac{1}{N} \int d\lambda \rho(\lambda) 2m/(\lambda^2 + m^2)$, for a fixed sea quark mass. In the mesoscopic range, $\Sigma(m)$ can be obtained analytically from the microscopic spectral density [1, 23],

$$\frac{\Sigma(x)}{\Sigma} = x(I_a(x)K_a(x) + I_{a+1}(x)K_{a-1}(x)),$$

where $x = mV\Sigma$ is the rescaled mass and $a = N_f + \nu$. In Fig. 2 we plot this ratio as a function of $x$ for lattice data of two dynamical flavors with mass $ma = 0.01$ and $N_c = 3$ on a $16^3 \times 4$ lattice. We observe that the lattice data for different values of $\beta$ fall on a single curve. Moreover, in the mesoscopic range this curve coincides with the random matrix prediction for $N_f = \nu = 0$. 

Fig. 1. — Spectral correlations of Dirac eigenvalues for Wilson fermions (upper) and KS-fermions (lower).
6. CHIRAL RANDOM MATRIX THEORY AT $\mu \neq 0$

At nonzero temperature $T$ and chemical potential $\mu$ a schematic random matrix model of the Dirac operator in (5) is given by \cite{24, 25, 27}

$$D = \begin{pmatrix} 0 & iW + i\Omega_T + \mu \\ iW^\dagger + i\Omega_T + \mu & 0 \end{pmatrix}, \quad (8)$$

where $\Omega_T = T \otimes_n (2n+1)\pi 1$. Below, we will discuss a model with $\Omega_T$ absorbed in the random matrix and $\mu \neq 0$. For the three values of $\beta$ the eigenvalues of $D$ are scattered in the complex plane.

In the quenched approximation the eigenvalue distribution for $\beta = 2$ was obtained analytically \cite{27} from the $N_f \to 0$ limit of a partition function with the determinant replaced by its absolute value. The same analysis can be carried out for $\beta = 1$ and $\beta = 4$. In the normalization (5), it turns out that the average spectral density does not depend on $\beta$ \cite{26}. However, the fluctuations of the eigenvalues are $\beta$ dependent. In particular, one finds a very different behavior close to the imaginary axis and not too large values of $\mu$. This regime of almost Hermitean random matrices was first identified by Fyodorov et al. \cite{28}. For $\beta = 1$ we observe an accumulation of purely imaginary eigenvalues, whereas for $\beta = 4$ we find a depletion of eigenvalues in this domain. These results explain quenched instanton liquid calculations \cite{24} and lattice QCD simulations with Kogut-Susskind fermions \cite{30} at $\mu \neq 0$ (both for $SU(2)$), respectively.

7. CONCLUSIONS

We have shown that microscopic correlations of the QCD Dirac spectrum can be explained by RMT and have obtained an analytical understanding of the
distribution of the eigenvalues near zero. An extension of this model to nonzero chemical potential explains previously obtained lattice QCD results.

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