Permutation Symmetry, Tri-Bimaximal Neutrino Mixing and the $S_3$ Group Characters

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Abstract

We postulate that the neutrino mass matrix in the lepton flavour basis is an $S_3$ group matrix in the natural representation of $S_3$. This immediately requires one neutrino to be trimaximally mixed, as suggested by the solar neutrino data. We go on to postulate that the charged-lepton mass matrix in the neutrino mass-basis is an $S_3$ class matrix in the natural representation of the $S_3$ class-algebra, leading to exact tri-bimaximal mixing, which is compatible with data overall. The tri-bimaximal mixing matrix is seen to be closely related to the $S_3$ character table, and is properly the $S_3 \supset S_2$ table of induction coefficients, where the $S_2$ corresponds to symmetry under $\mu - \tau$ interchange in the lepton flavour basis.

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1. Tri-Bimaximal Mixing: A Problem for Democracy

We previously emphasised [1] the phenomenological promise of the so-called [2] [3] [4] [5] tri-bimaximal hypothesis, defined by the lepton mixing matrix [1] [2]:

\[
U = \begin{pmatrix}
\frac{\nu_1}{\sqrt{2}} & \frac{\nu_2}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
\]  

Since then, several new experimental results, especially the SNO [6] flux-independent solar neutrino result, updated measurements from SAGE [7] and GALLEX/GNO [8] and the recent KAMLAND [9] reactor result, have considerably strengthened the case. Given that we already know that \( |U_{e3}|^2 \lesssim 0.03 \) from reactors [10], one might even say that the evidence for the solar neutrino being trimaximally mixed, in particular \( |U_{e2}|^2 \simeq 0.34 \pm 0.05 \) from SNO [6], is now better than the evidence for the atmospheric neutrino to be bimaximally mixed, cf. \( |U_{\mu 3}|^2 \simeq 0.50 \pm 0.11 \) in SUPER-K [11].

Tri-bimaximal mixing is sometimes incorrectly linked (eg. Ref. [12]) with the original rank-one democratic mass matrix (defined by all mass-matrix elements equal [13]). In fact, as we will see, the mass matrices associated with tri-bimaximal mixing are very far from the democratic form (Sections 4-5 below). It is true that the democratic mass matrix has always one trimaximal eigenvector \((1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\) [14] [15] [16], but the problem is that it is always the heaviest mass-eigenstate (or in fact, more generally, the non-degenerate mass-eigenstate, see below) which ends up trimaximally mixed, ie. not what is needed phenomenologically (cf. Eq. 1). In particular, the democratic neutrino mass matrix can in no way be taken as a zeroth-order approximation for a mixing scenario where it is the solar neutrino (normally the intermediate mass neutrino \(\nu_2\)) which is trimaximally mixed, as in the case of tri-bimaximal mixing Eq. 1.

\( S_3 \) symmetry remains interesting. It has been remarked [17] that the \( S_3 \) invariance of the democratic matrix is unbroken under the addition of any multiple of the identity matrix to the purely democratic form (if the democratic component has negative sign then an inverted hierarchy results, still with full \( S_3 \) invariance). Indeed, taking any polynomial function of a matrix preserves all the symmetries, and in general gives any eigenvalues associated with the original eigenvectors in any order (the Vandermonde matrix [18] formed from the original eigenvalues provides the transformation between the required polynomial coefficients and the desired eigenvalues). The problem with the democratic mass matrix here is that (up to a factor) it is ‘idempotent’, ie. its square is not an independent matrix and so we have in effect only two polynomial coefficients available (with two degenerate eigenvalues, not only are the two
corresponding eigenvectors undefined, but the Vandermonde matrix has no inverse). In this paper, we ‘solve’ the problem indicated above, by suggesting that the democratic mass matrix be dropped, in favour of an $S_3$ ‘group matrix’ [19] (more precisely, see below, by an element of the $S_3$ group algebra, in the sense of representation theory) in the natural representation of the $S_3$ group. Later in the paper we extend our argument to give a new and succinct prescription leading to tri-bimaximal mixing itself.

2. Development of Our Approach with a Familiar Example

Although the original trimaximal mixing scheme [20] [21] seems now essentially ruled-out by experiment, it cannot be denied that trimaximal mixing occupied a special place in the space of all possible mixings. In the briefest terms, one had only to require that the neutrino mass matrix in the lepton flavour basis (or the charged-lepton mass matrix in the neutrino mass basis) was a $C_3$ group matrix in the natural representation of $C_3$ (see below), and the lepton mixing matrix was completely determined to be the trimaximal mixing matrix, identically the $C_3$ group character table (see Appendix A) up to an overall normalisation factor $1/\sqrt{3}$ [22]. ($C_3$ here is the cyclic group on three objects, while $S_3$ above is the corresponding symmetric group).

Explicitly, a $C_3$ group matrix [19] is just an element of the $C_3$ group algebra, ie. an arbitrary linear combination of the three $C_3$ group elements, with arbitrary (complex) coefficients. (In the context of group representation theory, multiplication of group elements by scalars and addition of group elements are understood in the obvious way, within a given matrix representation). In the natural representation of $C_3$ (using cycle notation and taking $I$ to denote the identity) the $C_3$ group elements may be written:

\[ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  

\[ P(\alpha\beta\gamma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad P(\gamma\beta\alpha) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]  

(3)

From the physics point of view, if we restrict consideration to left-handed fields only, we may (as usual) take our mass matrices to be hermitian ($M \rightarrow M^2 := MM^\dagger$), whereby there is nothing to be gained by considering $C_3$ group matrices which are more general than the hermitian combination:

\[ M^2 = aI + bP(\alpha\beta\gamma) + b^*P(\gamma\beta\alpha) \]  

(4)

where $a$ is real, $b$ is complex, and $b^*$ is the complex conjugate of $b$ (ie. making $a$ complex and replacing $b^*$ by an arbitrary complex parameter $c$, simply yields the same form Eq. 4, on taking the hermitian square, $M \rightarrow MM^\dagger$).
We see immediately that Eq. 4 is just the familiar $3 \times 3$ circulant mass matrix [20]:

$$
M^2 = \begin{pmatrix}
    a & b & b^* \\
    b^* & a & b \\
    b & b^* & a
\end{pmatrix}
$$

(5)

invariant under cyclic ($C3$) permutations of the generation indicies. Diagonalising Eq. 5 is equivalent to reducing the $C3$ group algebra to independent idempotents [19] (the projection operators of Ref. [23]) and leads directly to trimaximal mixing [21]:

$$
U = \frac{1}{\sqrt{3}} \begin{pmatrix}
    1 & \omega & \bar{\omega} \\
    1 & \bar{\omega} & \omega \\
    1 & 1 & 1
\end{pmatrix}
$$

(6)

where $\omega = \exp(i2\pi/3)$ and $\bar{\omega} = \exp(-i2\pi/3)$ are the complex cube roots of unity.

We may say that the trimaximal mixing matrix Eq. 6 is the unitary matrix which reduces the natural representation of $C3$ to its irreducible form. The eigenvectors of the matrix Eq. 5 (appearing as the columns (or the complex-conjugated rows) of the matrix Eq. 6) are simply the character vectors corresponding to the three inequivalent (1-dimensional) irreducible representations of $C3$ (in which $P(\alpha\beta\gamma)$ (or $P(\gamma\beta\alpha)$) acts like $1$, $\omega$, $\bar{\omega}$ respectively, see Appendix A).

3. The Neutrino Mass Matrix as an $S3$ Group Matrix

Forced by experiment to renounce $C3$ invariance for leptons, we turn with renewed interest to the symmetric group $S3$. If we are to apply the foregoing argument to $S3$, we would expect to have to include, in addition, the odd $S3$ permutations:

$$
P(\alpha\beta) = \begin{pmatrix}
    0 & 1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 1
\end{pmatrix} \quad P(\beta\gamma) = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{pmatrix} \quad P(\gamma\alpha) = \begin{pmatrix}
    0 & 0 & 1 \\
    0 & 1 & 0 \\
    1 & 0 & 0
\end{pmatrix}.
$$

(7)

Constructing a general (hermitian) $S3$ group matrix amounts to adding an arbitrary (real) linear combination $M^2(\text{odd})$ of the odd group elements Eq. 7 to the previous linear combination $M^2(\text{even})$ ($:= M^2$ from Eq. 4) of the even ($C3$) operators, Eq. 3:

$$
M^2_\nu = M^2(\text{even}) + M^2(\text{odd})
= aI + bP(\alpha\beta\gamma) + b^* P(\beta\gamma\alpha) + xP(\beta\gamma) + yP(\gamma\alpha) + zP(\alpha\beta)
$$

(8)

where (as above) there is nothing to be gained by considering non-hermitian $S3$ group matrices, eg. with $x$, $y$, $z$ complex, which always yield the form Eq. 8 on taking the hermitian square ($M \rightarrow MM^\dagger$). Notice that, within the natural representation, the six $S3$ group operators (Eq. 3 together with Eq. 7) are not fully independent:

$$
I + P(\alpha\beta\gamma) + P(\gamma\beta\alpha) = P(\alpha\beta) + P(\beta\gamma) + P(\gamma\alpha)
$$

(9)
so that the effective number of (real) parameters in Eq. 8 is actually only five. ¹

The structure of Eq. 8 may perhaps be best appreciated by noting that the contribution of the odd operators (from Eq. 7) is ‘retrocirculant’ [24]:

\[ M^2(\text{odd}) = \begin{pmatrix} x & z & y \\ z & y & x \\ y & x & z \end{pmatrix}. \] (10)

The significant property of a retrocirculant here, is that it has non-degenerate eigenvalues in general, and (clearly) always one trimaximal eigenvector \((1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})!\) Futhermore, it is evident that these properties are not in general invalidated by the inclusion of the circulant (even) contribution already discussed. With \(S3\) being a non-abelian group, Eq. 8 is not invariant under \(S3\) permutations of the generation indices. We observe that it does, however, satisfy an \(S3\) invariant constraint:

\[ (M^2_\nu)_{\alpha\alpha} - (M^2_\nu)_{\beta\beta} = (M^2_\nu)_{\gamma\beta} - (M^2_\nu)_{\alpha\gamma} \quad (\alpha \neq \beta \neq \gamma) \] (11)

for all \((S3)\) permutations of the generation indices \((\alpha, \beta, \gamma = e, \mu, \tau)\).

If Eq. 8 is taken to be the neutrino mass matrix in the lepton-flavour basis (as was already anticipated by the subscript on \(M^2_\nu\) in Eq. 8 and by the introduction of explicit flavour indices in Eq. 11), then, for a suitable choice of the coefficients, we have:

\[ m^2_1 = a - \text{Re} b - \sqrt{3} (\text{Im} b)^2 + x^2 + y^2 + z^2 - xy - yz - zx \] (12)
\[ m^2_2 = a + 2\text{Re} b + x + y + z \] (13)
\[ m^2_3 = a - \text{Re} b + \sqrt{3} (\text{Im} b)^2 + x^2 + y^2 + z^2 - xy - yz - zx \] (14)

('suitable' only in the sense that the mass-eigenstates should turn out to be ordered appropriately, eg. \(m^2_1 < m^2_2 < m^2_3\) for a conventional neutrino mass hierarchy). Eq. 8 is seen to correspond to the (two-parameter) mixing scheme proposed phenomenologically in Ref. [2] (to interpolate between tri-\(\phi\)maximal and tri-\(\chi\)maximal mixing):

\[
U = \begin{pmatrix}
    e & \frac{\nu_1}{\sqrt{3}} & \frac{\nu_3}{\sqrt{6}} \\
    \mu & \frac{1}{\sqrt{2}} & \frac{\nu_2}{\sqrt{2}} \\
    \tau & -\frac{\nu_1}{\sqrt{3}} & \frac{\nu_2}{\sqrt{2}}
\end{pmatrix}
\] (15)

¹The constraint Eq. 9 is due to the fact that the natural representation of the \(S3\) group receives no contribution from the ‘alternating’ representation (as defined in Appendix A), whereby the corresponding idempotent of the \(S3\) group algebra (:= Eq. 9-LHS – Eq. 9-RHS) is identically the null matrix, in the natural representation.
In Eq. 15 we have used the abbreviations: \( c_\chi = \cos \chi \), \( s_\chi = \sin \chi \), \( c_\phi = \cos \phi \), \( s_\phi = \sin \phi \), where:

\[
\tan 2\phi = \frac{\sqrt{3}(z - y)}{z + y - 2x} \tag{16}
\]

\[
\tan 2\chi = \frac{\sqrt{3} \text{Im} b}{\left[ x^2 + y^2 + z^2 - xy - yz - zx \right]^{1/2}} \tag{17}
\]

The CP-violation parameter \( J \) [25] is given by:

\[
J = \frac{\text{Im} b}{6 \left[ 3 (\text{Im} b)^2 + x^2 + y^2 + z^2 - xy - yz - zx \right]^{1/2}} = \frac{\sin 2\chi}{6\sqrt{3}}. \tag{18}
\]

Clearly, imposing \( \text{Im} b = 0 \) in Eq. 8 (ie. \( \chi = 0 \)) would imply no CP violation, whereas imposing \( y = z \) instead (ie. \( \phi = 0 \)) implies ‘mu-tau reflection symmetry’ [26]. For both \( \text{Im} b = 0 \) and \( y = z \) (ie. \( \chi = 0 \) and \( \phi = 0 \)) the mixing matrix Eq. 15 evidently reduces to the tri-bimaximal form [1] [2], as does the mass matrix (Eq. 8), accordingly [4].

The six constants appearing in Eq. 8 may be expressed in terms of the three neutrino masses and the two mixing-matrix parameters as follows:

\[
a = \frac{m_1^2}{3} + \frac{m_2^2}{3} + \frac{m_3^2}{3} \tag{19}
\]

\[
b = \left( -\frac{m_1^2}{6} + \frac{m_2^2}{3} - \frac{m_3^2}{6} \right) + i \frac{m_3^2 - m_1^2}{2\sqrt{3}} \sin 2\chi \tag{20}
\]

\[
x = \frac{m_3^2 - m_1^2}{3} \cos 2\chi \left( -\cos 2\phi \right) \tag{21}
\]

\[
y = \frac{m_3^2 - m_1^2}{6} \cos 2\chi \left( \cos 2\phi - \sqrt{3} \sin 2\phi \right) \tag{22}
\]

\[
z = \frac{m_3^2 - m_1^2}{6} \cos 2\chi \left( \cos 2\phi + \sqrt{3} \sin 2\phi \right), \tag{23}
\]

where clearly (by virtue of Eq. 9) any arbitrary constant may be added to Eqs.19-20 provided that the same constant is subtracted from Eqs. 21-23.

Note that in this approach, in the case of S3 (cf. the case of C3, Section 2) the resulting mixing matrix (Eq. 15) is not directly the S3 character table. It is simply the generic unitary matrix which reduces the natural representation of S3 to irreducible form. The natural representation of S3 comprises the trivial 1-dimensional representation and a faithful 2-dimensional representation, which is determined only up to similarity transformations (hence the undetermined parameters appearing in Eq. 15). From its present derivation (and to distinguish it clearly from mixing an satze based on the ‘democratic’ mass matrix) we will refer to the mixing Eq. 15 as ‘S3 group mixing’. 

6
4. The Charged-Lepton Mass Matrix as an $S3$ Class Matrix

We have seen in Section 2 that a very succinct way to introduce trimaximal mixing is to demand that one or other of the mass-matrices is a $C3$ group matrix in the natural representation of $C3$. The mixing matrix is then essentially the $C3$ character table, with all states trimaximally mixed [22]. We went on to generalise the argument to $S3$, finding that it is enough to take one of the mass matrices to be an $S3$ group matrix in the natural representation of the $S3$ group, to obtain a mixing matrix where one (and in particular any one) of the eigenvectors is trimaximally mixed, thereby ‘solving’ the problem of the democratic mass matrix discussed in Section 1.

However, a more predictive (and perhaps more interesting) way to generalise the trimaximal argument is to recognise that, with $C3$ being an abelian group, there is no distinction between the group elements and the group conjugacy classes in that case. Each $C3$ group element being individually a class, an arbitrary element of the $C3$ group algebra is also an arbitrary element of the $C3$ class algebra. It is therefore not obvious that the better generalisation to $S3$ should not simply postulate that one or other of the mass matrices should live in the natural representation of the $S3$ class algebra, rather than the $S3$ group algebra, which is certainly a significantly different idea.

Following this line of thought, we define (normalised) $S3$ class operators $c_i$:

\[
\begin{align*}
    c_1 &= I \\
    c_2 &= \frac{P(\xi\eta\zeta) + P(\zeta\eta\xi)}{\sqrt{2}} \\
    c_3 &= \frac{P(\xi\eta) + P(\eta\zeta) + P(\zeta\xi)}{\sqrt{3}}
\end{align*}
\]

(24) \hspace{1cm} (25) \hspace{1cm} (26)

where the precise physical meaning of $\xi$, $\eta$, $\zeta$ remains unclear. Evidently, the $S3$ class multiplication table (by definition commutative) then takes the form:

|     | $c_1$ | $c_2$ | $c_3$ |
|-----|-------|-------|-------|
| $c_1$ | $c_1$ | $c_2$ | $c_3$ |
| $c_2$ | $c_2$ | $c_1 + c_2/\sqrt{2}$ | $\sqrt{2}c_3$ |
| $c_3$ | $\sqrt{2}c_3$ | $c_1 + \sqrt{2}c_2$ |     |

The structure constants in the table themselves provide a matrix representation for
the $c_i$ (which is the natural representation of the $S3$ class algebra in terms of the $c_i$):

$$
c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \quad c_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 0 \end{pmatrix}
$$  

(27)
as is readily verified by direct multiplication of the matrices.

We now postulate that the charged-lepton mass matrix in the neutrino mass-basis is a suitable linear combination of the $S3$ class operators in the above representation:

$$
M_l^2 = pc_1 + qc_2 + rc_3,
$$  

(28)
i.e. explicitly:

$$
M_l^2 = \begin{pmatrix} p & q/\sqrt{2} & r \\ q & p + q/\sqrt{2} & \sqrt{2}r \\ r & \sqrt{2}r & p + \sqrt{2}q \end{pmatrix}.
$$  

(29)
From the usual argument (see eg. Section 2) the coefficients $p$, $q$, $r$ may be taken to be real. The eigenvalues of the matrix Eq. 29 are then the charged-lepton masses:

$$
m_e^2 = p - q/\sqrt{2} 
$$  

(30)
$$
m_\mu^2 = p + \sqrt{2}q - \sqrt{3}r 
$$  

(31)
$$
m_\tau^2 = p + \sqrt{2}q + \sqrt{3}r.
$$  

(32)
The coefficients (being 'suitable' only in that $0 < r/\sqrt{3} < q/\sqrt{2} < p$ to order the mass-eigenstates in accord with experiment) are expressible in terms of the masses by:

$$
p = \frac{m_\tau^2 + m_\mu^2}{6} + \frac{2}{3}m_e^2
$$  

(33)
$$
q = \sqrt{2}(\frac{m_\tau^2 + m_\mu^2}{6} - \frac{m_e^2}{3})
$$  

(34)
$$
r = \frac{m_\tau^2 - m_\mu^2}{2\sqrt{3}}.
$$  

(35)
The unitary matrix diagonalising Eq. 29 (independent of the values of $p$, $q$ and $r$) is directly the tri-bimaximal mixing matrix:

$$
U = e^\mu \tau \begin{pmatrix} \frac{\nu_1}{\sqrt{6}} & \frac{\nu_2}{\sqrt{3}} & \nu_3 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}
$$  

(36)
(the neutrino mass-eigenstates having been already implicitly ordered in accord with experiment, by the labelling of the class operators, in Eqs. 24-26). The charged-lepton mass eigenstates (i.e. the eigenvectors of Eq. 29) appear as the rows of Eq. 36.
Clearly, the tri-bimaximal mixing matrix Eq. 36 is very closely related to the $S_3$ table of characters (cf. as displayed in Appendix A below). In fact, it differs only by the class-dependent normalisation factors introduced into Eqs. 24-26. Diagonalising an $S_3$ class matrix (such as Eq. 29) is entirely equivalent [27] to determining the $S_3$ group characters, ie. to finding all the irreducible representations of $S3$ by reducing the $S_3$ class algebra to independent idempotents. Explicitly:

$$U_{li} = \sqrt{\frac{g_i}{g}} \chi_i^{(l)}$$ (37)

where $\chi_i^{(l)}$ is the $i$-th component of the $l$-th character vector, $g_i$ is the order of the class ($g_i = 1, 2, 3$ for $i = 1, 2, 3$ for $S3$) and $g$ is the order of the group ($g = 6$ for $S3$). Individual character components are simply related [27] to the eigenvalues of the corresponding class operators, whereby the charged-lepton masses may also be expressed (equivalently to Eqs. 30-32) in terms of the $S3$ group characters and the constants $(p_1, p_2, p_3) := (p, q, r)$, as follows:

$$m_l^2 = \sum_i p_i \sqrt{\frac{g_i}{d_i}} \chi_i^{(l)}$$ (38)

where $d_l$ is the dimension of the irreducible representation corresponding to the lepton flavour $l$. The irreducible representations for $l = \tau$ and $l = \mu$ are the two mutually conjugate 1-dimensional representations (the trivial and alternating representations respectively), while the electron ($l = e$) is to be associated with the 2-dimensional faithful representation having a self-conjugate tableau. In the extreme hierarchical limit, $r/\sqrt{3} \to q/\sqrt{2} \to p$ in Eqs. 30-32, only the trivial representation has mass.

It is perhaps worth re-iterating at this point that ‘data on neutrino oscillations point strongly ... to tri-bimaximal mixing’ [1]. We note that from its present derivation, and in view of the need to distinguish it from ‘$S3$ group mixing’ (Section 3), tri-bimaximal mixing might reasonably be termed ‘$S3$ class mixing’.

5. The Neutrino Mass Matrix as an $S_3 \supset S2$ Class Operator.

Alerted to the relevance of class operators, we may now return to reconsider the neutrino mass matrix in the flavour basis. According to Section 3, the charged-lepton flavour basis ($\alpha, \beta, \gamma = e, \mu, \tau$) carries the natural representation of the $S3$ group (it was also noted that tri-bimaximal mixing requires a particular $S3$ group matrix with $\text{Im} \ b = 0$ and $y = z$). Clearly any representation of a group also provides a representation for the classes, and seeking consistency with the results of Section 4, we now postulate that the neutrino mass matrix in the flavour basis is a class operator.
for the canonical subgroup chain $S_1 \subset S_2 \subset S_3$ in the natural representation of the $S_3$ group (class operators for successive subgroups clearly commute). The individual class operators may be written:

$$C(1) = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C(2) = P(\mu\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$C(3) = P(e\mu) + P(\mu\tau) + P(\tau e) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(class normalisation factors are not needed here since they may be absorbed into the coefficients $s$, $t$, $u$ below, with no change of basis involved). The $S_2$ class operator $C(2)$ has been chosen to be the $\mu - \tau$ interchange operator [26]. Of course, $C(3)$ is familiar as the ‘democratic’ mass matrix.

The most general (hermitian) $S_1 \subset S_2 \subset S_3$ class operator may be written:

$$M_\nu^2 = sC(1) + tC(2) + uC(3).$$

Explicitly:

$$M_\nu^2 = \begin{pmatrix} s + t + u & u & u \\ u & s + u & t + u \\ u & t + u & s + u \end{pmatrix}$$

where $s$, $t$, $u$ are real. The eigenvalues of the matrix Eq. 41 are the neutrino masses:

$$m_1^2 = s + t$$

$$m_2^2 = s + t + 3u$$

$$m_3^2 = s - t.$$ 

The coefficients $0 \leq 3u \leq -2t \leq 2s$ (for $m_1^2 \leq m_2^2 \leq m_3^2$) are given in terms of the neutrino masses by:

$$s = \frac{m_1^2 + m_3^2}{2}$$

$$t = \frac{m_2^2 - m_3^2}{2}$$

$$u = \frac{m_2^2 - m_1^2}{3}$$

now with no arbitrary constant involved (cf. Eqs. 19-23). The extreme hierarchical limit for the neutrino masses is approached as $u \to 0$ and $t \to -s$, when only the $\nu_3$ has mass. It may be remarked that it is the ‘democratic’ component $C(3)$ which has the (numerically) smallest coefficient ($u$) in Eq. 41, vanishing in the hierarchical limit.
Diagonalising the mass matrix Eq. 42, the resulting mixing matrix takes the familiar (Eq. 1) tri-bimaximal form, which is also referred to here as ‘$S3 \supset S2$ mixing’:

\[
U = e^{\frac{\mu}{\tau}} \left( \begin{array}{ccc}
\nu_1 & \nu_2 & \nu_3 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{array} \right).
\]

The eigenvectors of Eq. 42 appear as the columns of Eq. 49.

Clearly, the mass matrix Eq. 42 may equally well be viewed as the particular ‘$S3$ group matrix’ [19] having \(\text{Im} \ b = 0\) and \(y = z\) (see Section 3). The \(\nu_2\) has the trimaximal eigenvector \((1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\) which is in effect the character vector of the trivial 1-dimensional (symmetric) representation of \(S3\). The \(\nu_1\) and \(\nu_3\) both belong to the self-conjugate (faithful) 2-dimensional representation of \(S3\), being distinguished here by their symmetry (\(\pm 1\) respectively) under \(\mu - \tau\) exchange (‘mutativity’ [26]). The tri-bimaximal mixing matrix is then properly regarded as nothing more than the table of induction coefficients for the \([2] \otimes [1] = [3] + [21]\) induced representation of \(S3\). It is simply the unitary matrix which reduces the natural representation of \(S3\), whilst simultaneously diagonalising the \(\mu - \tau\) interchange operator [26].

In retrospect, the original circulant mass matrix [20] leading to trimaximal mixing might have been seen as a class operator for the group chain \(S3 \supset C3\). Of course we now know that, for the neutrino mass matrix in the flavour basis, an \(S3 \supset S2\) class operator Eq. 41, is preferred experimentally (the ‘\(\supset S1\)’ in fact carries no additional symmetry information and is dropped here in accord with usual practice).

6. Discussion

We have been to a large degree logically led, from the original trimaximal hypothesis, first to ‘$S3$ group mixing’ Eq. 15, and then on to ‘$S3$ class mixing’ or ‘$S3 \supset S2$ mixing’, ie. to tri-bimaximal mixing. The two levels of generalisation are not inconsistent: the latter is clearly more restrictive, in that exact tri-bimaximal mixing requires the charged-lepton mass matrix to be an \(S3\) class matrix in the neutrino mass-basis, and also requires the neutrino mass matrix in the lepton flavour basis to be a particular \(S3\) group matrix (with \(\text{Im} \ b = 0\) and \(y = z\)), ie. an \(S3 \supset S2\) class operator. For a discussion of the forms of both mass-matrices in an intermediate basis see Ref. [2].

Thus, while ‘$S3$ group mixing’ is regarded as an interesting mixing ansatz in its own right [2], our main results relate to tri-bimaximal mixing, and the link to the \(S3\) group characters [19] via Eq. 37 (Section 4) and to the \(S3\) induction coefficients [27] (Section 5). In the first case the neutrino mass eigenstates are associated with the
normalised \( S_3 \) class operators Eqs. 24-26 \((\nu_i \sim c_i)\), while the charged-lepton mass-eigenstates are in correspondence with the \( S_3 \) irreducible representations. Then, in the flavour basis, the charged leptons \( e, \mu, \tau \) are in correspondence with the \( C_3 \) classes \( c_0, c_-, c_+ \) respectively (viewed as the coset representatives with respect to the \( \mu - \tau \) exchange subgroup) while the neutrino mass-eigenstates are in correspondence with the irreducible basis vectors of the corresponding induced representation of \( S_3 \). Clearly classes (and hence linear combinations of classes) are always permutation invariants.

Finally, we remark that the notion of the yukawa couplings here being related to the structure constants of a permutation class algebra, is not so different in character from the established notion of the couplings between gauge bosons being the structure constants of a lie algebra. Of course as always, experiment will be the ultimate judge, with the detailed experimental predictions of exact tri-bimaximal mixing (eg. \( P(e \to e) \to 5/9 \simeq 0.56 \) in KAMLAND [9], zero \( CP \) violation, no high-energy matter resonance etc.) being already documented in the literature [1].

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Appendix A: Group Character Tables for the \( C_3 \) and \( S_3 \) Groups
For ease of reference, the character tables for the cyclic group \( C_3 \) on three symbols, and for the corresponding symmetric group \( S_3 \), are given below.

For \( C_3 \) there are three irreducible representations, all 1-dimensional, where the generator of (say ‘clockwise’) cyclic permutations acts like 1, \( \omega \) or \( \bar{\omega} \), which are referred to here as the trivial, \( \omega \) and \( \bar{\omega} \)-representations respectively. The three classes \( (c_0, c_+, c_-) \) comprise the identity, clockwise and anti-clockwise cyclic permutations, respectively.

For \( S_3 \) there are likewise three irreducible representations, two of which are 1-dimensional. In the trivial representation all group elements act like \(+1\), while in the alternating representation, elements corresponding to odd permutations act instead like \((-1\)\). There is a faithful 2-dimensional representation which may be written [28]:

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
P(\alpha \beta \gamma) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \quad P(\gamma \beta \alpha) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}
\]

\[
P(\alpha \beta) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad P(\beta \gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad P(\gamma \alpha) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}
\]

(up to equivalence transformations). The three classes \( (c_i, i = 1 - 3) \) correspond to the identity, the even (ie. cyclic), and odd permutations, respectively.
The group character tables below give the traces $\chi^{(l)}_{ci}$ of the matrices comprising all the inequivalent irreducible representations ($l$) of the group, as a function of conjugacy class $c_i$. (Matrices representing different group elements within the same class have the same trace, and traces are unaltered by equivalence transformations).

| Cyclic Group C3  |
|------------------|
| (order of group: $g = 3$) |
| Irreducible Representations ↓ |
| $\omega$-rep. | 1 | $\omega$ | $\bar{\omega}$ |
| $\bar{\omega}$-rep. | 1 | $\bar{\omega}$ | $\omega$ |
| triv. | 1 | 1 | 1 |

| Conjugacy Classes $\rightarrow$ |
| (order of class, $g_i$) |
| $c_0$ | (1) |
| $c_+^{-}$ | (1) |
| $c_-$ | (1) |

| Symmetric Group S3 |
|-------------------|
| (order of group: $g = 6$) |
| Irreducible Representations ↓ |
| faith. | 2 | $-1$ | 0 |
| alt. | 1 | 1 | $-1$ |
| triv. | 1 | 1 | 1 |

| Conjugacy Classes $\rightarrow$ |
| (order of class, $g_i$) |
| $c_1$ | (1) |
| $c_2$ | (2) |
| $c_3$ | (3) |

The following abbreviations have been used: $\omega$-rep. = $\omega$-representation, triv. = trivial, alt. = alternating and faith. = faithful (representations). The complex cube roots of unity are given by: $\omega = \exp(i2\pi/3)$ and $\bar{\omega} = \exp(-i2\pi/3)$. 

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