The Structure of Corrections to Electroweak Interactions in Higgsless Models

R. Sekhar Chivukula and Elizabeth H. Simmons

Department of Physics and Astronomy, Michigan State University
East Lansing, MI 48824, USA
E-mail: sekhar@msu.edu, esimmons@msu.edu

Hong-Jian He

Department of Physics, University of Texas
Austin, TX 78712, USA
E-mail: hjhe@physics.utexas.edu

Masafumi Kurachi

Department of Physics, Nagoya University
Nagoya 464-8602, Japan
E-mail: kurachi@eken.phys.nagoya-u.ac.jp

Masaharu Tanabashi

Department of Physics, Tohoku University
Sendai 980-8578, Japan
E-mail: tanabash@tuhep.phys.tohoku.ac.jp
Abstract: Recently, “Higgsless” models of electroweak symmetry breaking have been proposed. Based on compactified five-dimensional gauge theories, these models achieve unitarity of electroweak boson self-interactions through the exchange of a tower of massive vector bosons rather than the exchange of a scalar Higgs boson. In this paper, using deconstruction, we analyze the form of the corrections to the electroweak interactions in a large class of these models, allowing for arbitrary 5-D geometry, position-dependent gauge coupling, and brane kinetic energy terms. We show that many models considered in the literature, including those most likely to be phenomenologically viable, are in this class. By analyzing the asymptotic behavior of the correlation function of gauge currents at high momentum, we extract the exact form of the relevant correlation functions at tree-level and compute the corrections to precision electroweak observables in terms of the spectrum of heavy vector bosons. We determine when nonoblique corrections due to the interactions of fermions with the heavy vector bosons become important, and specify the form such interactions can take. In particular we find that in this class of models, so long as the theory remains unitary, \( S - 4 \cos^2 \theta_W T > O(1) \), where \( S \) and \( T \) are the usual oblique parameters. We concur with the result of Cacciapaglia et.al. that small or negative \( S \) is possible – though only at the expense of substantial negative \( T \) at tree-level. Although we stress our results as they apply to continuum 5-D models, they apply also to models of extended electroweak gauge symmetries motivated by models of hidden local symmetry.

Keywords: Dimensional Deconstruction, Electroweak Symmetry Breaking, Higgsless Theories.
1. Introduction

The origin of electroweak symmetry breaking remains one of the most significant questions in elementary particle physics. Recently, “Higgsless” models of electroweak symmetry breaking have been proposed [1]. Based on five-dimensional gauge theories compactified on an interval, these models achieve unitarity of electroweak boson self-interactions through the exchange of a tower of massive vector bosons [2, 3, 4], rather than the exchange of a scalar Higgs boson [5]. Motivated by gauge/gravity duality [6, 7, 8, 9], models of this kind may be viewed as “dual” to more conventional models of dynamical symmetry breaking [10, 11] such as “walking technicolor” [12, 13, 14, 15, 16, 17], and they represent an exciting new avenue of investigation.

In this paper, using deconstruction [18, 19], we calculate the form of the corrections to the electroweak interactions in a large class of these models.* Our analysis applies to any Higgsless model which can be deconstructed to a chain of $SU(2)$ gauge groups adjacent to a chain of $U(1)$ gauge groups, with the fermions coupled to the $SU(2)$ group at the end of the chain and to the $U(1)$ group at the interface between the $SU(2)$ and $U(1)$ chains. We show that many models considered in the literature, including those most likely to be phenomenologically viable [21], are in this class. By analyzing the asymptotic behavior of the correlation function of gauge currents at high momentum in a deconstructed model of this sort, generalizing an argument for dual models of QCD [22, 23, 24], we extract the exact form of the relevant correlation functions at tree-level and compute the corrections to precision electroweak observables. We determine when nonoblique corrections due to the interactions of fermions with the heavy vector bosons become important, and specify the form such interactions can take. In particular we find that in this class of models, so long as the theory remains unitary, $S - 4 \cos^2 \theta_W T > \mathcal{O}(1)$, where $S$ and $T$ are the usual oblique parameters [25].

Precision electroweak constraints arising from corrections to the $W$ and $Z$ propagators in these models have previously been investigated [26, 27, 28, 29, 30, 21, 31, 32] in the continuum and, in the case of weak coupling, using deconstruction [33, 24]. These analyses begin by specifying a bulk gauge symmetry, “brane” kinetic energy terms for the gauge fields, 5-D geometry, and fermion couplings. In our analysis, generalizing [24], we leave the gauge couplings and $f$-constants of the deconstructed model arbitrary and parameterize the electroweak corrections in terms of the masses of the heavy vector bosons. We therefore obtain results which describe arbitrary 5-D geometry, position-dependent couplings, or brane kinetic energy terms.

As our results are computed in terms of the spectrum of heavy vector bosons, we can directly estimate the size of electroweak corrections through the constraints unitarity places

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*To be precise, in this analysis we are concerned with the corrections to electroweak processes at tree-level in the additional interactions. These corrections exist independent of any particular “high-energy” completion of the deconstructed theory. Recently, Perelstein [20] has concluded that the effects of a QCD-like high-energy completion – arising from higher-order contributions analogous to $L_{10}$ in the chiral Lagrangian – are potentially large as well.
Figure 1: Moose diagram for the class of models analyzed in this paper. SU(2) gauge groups are shown as open circles; U(1) gauge groups as shaded circles. The fermions couple to gauge groups 0 and N + 1. The values of the gauge couplings $g_i$ and f-constants $f_i$ are arbitrary.

on the masses of the heavy vector bosons. In contrast with other analyses, ours can determine when non-oblique corrections are relevant, and we can constrain the form of the non-oblique contributions. Unlike Barbieri et al. [28], we are not restricted to the case of spatially independent gauge coupling or the case in which the brane kinetic energy terms dominate those in the bulk. We concur with the result of Cacciapaglia et al. [21] that small or negative $S$ is possible – though only at the expense of negative $T$ consistent with $S - 4 \cos^2 \theta_W T > \mathcal{O}(1)$.

Although we stress our results as they apply to continuum 5-D models, they apply also far from the continuum limit when only a few extra vector bosons are present. As such, these results form a generalization of phenomenological analyses [34] of models of extended electroweak gauge symmetries [35, 36] motivated by models of hidden local symmetry [37, 38, 39, 40, 41].

2. The Model and Its Relatives

The model we analyze, shown diagrammatically (using “moose notation” [42, 43]) in Fig. 1, incorporates an SU(2)$^{N+1} \times$U(1)$^{M+1}$ gauge group, and N+1 nonlinear (SU(2) × SU(2))/SU(2) sigma models adjacent to M (U(1) × U(1))/U(1) sigma models in which the global symmetry groups in adjacent sigma models are identified with the corresponding factors of the gauge group. The Lagrangian for this model at $O(p^2)$ is given by

$$\mathcal{L}_2 = \frac{1}{4} \sum_{j=1}^{N+M+1} f_j^2 \text{tr} \left( (D_\mu U_j)^\dagger (D^\mu U_j) \right) - \sum_{j=0}^{N+M+1} \frac{1}{2g_j^2} \text{tr} \left( F_{j\mu\nu} F^{j\mu\nu} \right),$$

(2.1)

with

$$D_\mu U_j = \partial_\mu U_j - i A^{j-1}_\mu U_j + i U_j A^j_\mu,$$

(2.2)

where all gauge fields $A^j_\mu$ ($j = 0, 1, 2, \ldots, N + M + 1$) are dynamical. The first $N + 1$ gauge fields $(j = 0, 1, \ldots, N)$ correspond to SU(2) gauge groups; the other $M + 1$ gauge fields $(j = N + 1, N + 2, \ldots, N + M + 1)$ correspond to U(1) gauge groups. The symmetry breaking between the $A^{N}_\mu$ and $A^{N+1}_\mu$ follows an SU(2)$_L \times$SU(2)$_R$/SU(2)$_V$ symmetry breaking pattern with the U(1) embedded as the $T_3$-generator of SU(2)$_R$. 

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The fermions in this model take their weak interactions from the $SU(2)$ group at $j = 0$ and their hypercharge interactions from the $U(1)$ group with $j = N + 1$, at the interface between the $SU(2)$ and $U(1)$ groups. The neutral current couplings to the fermions are thus written as

$$J_3^\mu A_\mu^0 + J_Y^\mu A_\mu^{N+1},$$

while the charged current couplings arise from

$$\frac{1}{\sqrt{2}} J_{\pm}^\mu A_\mu^{0\mp}.$$ 

The special case $M = 0$ is shown in Fig. 2. Foadi et al. [33] have made a detailed analysis of a version of the $M = 0$ model with a flat geometry (with $f_1 = f_2 = \ldots = f_{N+1}$), constant “bulk” coupling ($g_1 = g_2 = \ldots = g_N$), and weak coupling to fermions ($g_0, g_{N+1} \ll g_1$). They found that, so long as the model respects unitarity, $S = O(1)$. This result has since been generalized to arbitrary geometry and spatially dependent coupling in [24]. The continuum interpretation of this particular model might seem problematic, as the same fermions appear to couple at different positions in the discretized fifth dimension. However, this moose may be redrawn as shown in Fig. 3, and we see that it may equivalently be interpreted in terms of a bulk $SU(2) \times SU(2)$ gauge group. Motivated by models in a warped background, we interpret the two ends of the moose as ultraviolet (UV) and infrared (IR) “branes,” with the fermions coupled at the UV brane. In this case, the $SU(2) \times SU(2)$ bulk gauge group is broken by boundary conditions to a diagonal (custodial) $SU(2)$ at the IR brane, and to $SU(2) \times U(1)$ at the UV brane [1, 43].

Inspired by the AdS/CFT correspondence [3], the necessity for an approximate custodial $SU(2)$ symmetry has motivated the construction of models based on an $SU(2) \times SU(2) \times U(1)$ gauge group [44, 21]. The deconstructed version of the model discussed in [24, 21] is shown in Fig. 4. Here, the (light) fermions couple on the UV brane, where the boundary conditions break the bulk $SU(2) \times SU(2) \times U(1)$ gauge group to $SU(2) \times U(1)$; at the IR brane, the boundary conditions break the two $SU(2)$ bulk gauge groups to custodial $SU(2)$. With an appropriately chosen $U(1)$ IR brane gauge kinetic-energy term, it has been shown that this model may have a small, or even negative, $S$ parameter [21]. By “unfolding” this moose, we see that the deconstructed version of this model corresponds to a special case of that shown.
in Fig. 1. An IR $U(1)$ brane kinetic energy term corresponds, in this language, to having $M \geq 1$ with different $U(1)$ couplings appropriately chosen in the continuum limit.

3. The Propagator Matrices

All four-fermion processes, including those relevant for the electroweak phenomenology of our model, depend on the neutral gauge field propagator matrix

$$D(Q^2) \equiv [Q^2 \mathcal{I} + M_{N+M}^2]^{-1},$$

and the charged propagator matrix

$$\hat{D}(Q^2) \equiv [Q^2 \mathcal{I} + \hat{M}_N^2]^{-1}.$$  

Here, $M_{N+M}^2$ and $\hat{M}_N^2$ are, respectively, the mass-squared matrices for the neutral and charged gauge bosons and $\mathcal{I}$ is the identity matrix. Consistent with [24], $Q^2 = -q^2$ refers to the euclidean momentum.

The neutral vector meson mass-squared matrix is of rank $(N + M + 2) \times (N + M + 2)$

$$M_{N+M}^2 = \frac{1}{4} \left( \begin{array}{ccc} g_0^2 f_1^2 & -g_0 g_1 f_2 & \cdots \\ -g_0 g_1 f_2 & g_1^2 f_3 + f_2 \end{array} \right) \cdots \left( \begin{array}{ccc} -g_N g_{N+1} f_{N+2} \end{array} \right).$$

Figure 3: Moose diagram for a Higgsless model with bulk gauge group $SU(2) \times SU(2)$, which can be identified with identical to the model illustrated in Fig. 2. Motivated by the warping in an AdS space, we label the “IR” (infrared) and “UV” (ultraviolet) branes.
and the charged current vector bosons’ mass-squared matrix is the upper \((N + 1) \times (N + 1)\)
dimensional block of the neutral current \(\tilde{M}_N^2\) matrix

\[
\tilde{M}_N^2 = \frac{1}{4} \begin{pmatrix}
\begin{array}{cccc}
g_{N+2}f_{N+2} & -g_{N+2}f_{N+1} & \cdots & \cdots \\
-g_{N+1}f_{N+2} & g_{N+1}f_{N+1} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & g_{N+1}f_{N+1} \\
g_{N}f_{N+2} & -g_{N}f_{N+1} & \cdots & \cdots \\
\end{array}
\end{pmatrix}
\]  
(3.4)

It is convenient to define an additional \(M \times M\) mass-squared matrix \(\mathcal{M}_M\),

\[
\mathcal{M}_M = \frac{1}{4} \begin{pmatrix}
\begin{array}{cccc}
g_{K+2}(f_{K+2}^2 + f_{K+3}^2) & -g_{K+2}g_{K+3}f_{K+3} & \cdots & \cdots \\
-g_{K+1}g_{K+3}f_{K+3} & g_{K+1}g_{K+2}f_{K+2} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & g_{K+1}g_{K+2}f_{K+2} \\
g_{K+1}f_{K+2} & -g_{K+1}f_{K+1} & \cdots & \cdots \\
\end{array}
\end{pmatrix}
\]  
(3.5)

(where we define \(K \equiv N + M\)) because it allows us to express the neutral-current matrix

\[M_2^{N+M}\]
The mass eigenstates as follows: $W$ bosons. The heavier eigenstates of $W$ from the Lagrangian will be especially useful as we proceed. We also define the propagator matrix $D(Q^2) \equiv \left[ Q^2 I + M^2_M \right]^{-1}$. 

Recalling that fermions are charged under only a single SU(2) gauge group (at $j = 0$) and a single U(1) group (at $j = N + 1$), neutral current four-fermion processes may be derived from the Lagrangian

$$L_{nc} = -\frac{1}{2}g_0^2 D_{0,0}(Q^2) J^\mu J_{\mu} - g_0 g_{N+1} D_{0,N+1}(Q^2) J^\mu_{J_Y} - \frac{1}{2}g_{N+1}^2 D_{N+1,N+1}(Q^2) J^\mu_Y J_{\mu_Y},$$

and charged-current process from

$$L_{cc} = -\frac{1}{2}g_0^2 \tilde{D}_{0,0}(Q^2) J^\mu_+ J_{-\mu} .$$

where $D_{i,j}$ is the $(i,j)$ element of the gauge field propagator matrix. From direct inversion of the mass matrix (or see, e.g., [24]) we find that

$$2\sqrt{2} G_F \equiv \frac{1}{2}g_0^2 \tilde{D}_{0,0}(Q^2 = 0) = \sum_{j=1}^{N+1} \frac{2}{\sqrt{f_j}} \equiv \frac{2}{v^2}$$

where $v \approx 246$ GeV.

We should also consider the nature of the mass eigenstates of this model. The neutral vector boson mass matrix (3.3) is of a familiar form that has a vanishing determinant, due to a zero eigenvalue. Physically, this corresponds to a massless neutral gauge field – the photon. The lowest massive eigenstates of $M_{N+M}^2$ and $\tilde{M}_{N}^2$ are, respectively, identified as the $Z$ and $W$ bosons. The heavier eigenstates of $M_{N+M}^2$ are labeled by $Z_\ell$ ($\ell = 1, 2, \cdots, N$), while those of $\tilde{M}_{N}^2$ are labeled by $W_{\bar{m}}$ ($\bar{m} = 1, 2, \cdots, N$). It is also convenient to introduce the eigenvalues of $M_M^2$, which are labeled by $M_{\tilde{m}}^2$ ($\tilde{m} = 1, 2, \cdots, M$).

Note that, at tree level, the model specified by the Lagrangian in eqn. (2.1) is determined by $2N + 2M + 3$ parameters: $N + M + 2$ couplings and $N + M + 1 f$-constants. In most of this paper we will specify the values of observable quantities in terms of the $2N + 2M + 3$ parameters $e^2, M_Z, M_{Z\ell}, M_W, M_{W\bar{m}}$, and $M_{\tilde{m}}$. In section 6, to facilitate comparison with experiment, we will exchange $M_W$ for the better-measured $G_F$ as an experimental input.

The propagator matrix elements may be written in a spectral decomposition in terms of the mass eigenstates as follows:

$$g_0^2 D_{0,0}(Q^2) \equiv [G_{NC}(Q^2)]_{WW} = \frac{[\xi_1]_{WW}}{Q^2} + \frac{[\xi_2]_{WW}}{Q^2 + M_{Z_1}^2} + \sum_{k=1}^{K} \frac{[\xi_{Zk}]_{WW}}{Q^2 + M_{Zk}^2},$$

\[\text{\textsuperscript{3.3}}\] more compactly as

$$M_{N+M}^2 = \begin{pmatrix}
\frac{\alpha}{N} & -g_N g_{N+1} f_{N+1/2} & -g_{N+N+1} f_{N+1/2} & -g_{N+2} g_{N+2} f_{N+2} \\
-g_N g_{N+1} f_{N+1/2} & \frac{\alpha}{2} & -g_{N+1} g_{N+2} f_{N+2} & -g_{N+2} g_{N+3} f_{N+3} \\
-g_{N+1} g_{N+2} f_{N+2} & -g_{N+1} g_{N+2} f_{N+2} & \frac{\alpha}{2} & -g_{N+2} g_{N+3} f_{N+3} \\
-g_{N+2} g_{N+3} f_{N+3} & -g_{N+2} g_{N+3} f_{N+3} & -g_{N+2} g_{N+3} f_{N+3} & \frac{\alpha}{2}
\end{pmatrix} .$$

This form will be especially useful as we proceed. We also define the propagator matrix $D(Q^2) \equiv \left[ Q^2 I + M^2_M \right]^{-1}$. 

for later use.
\[ g_0 g_{N+1} D_{0,N+1}(Q^2) \equiv [G_{NC}(Q^2)]_{WY} = \frac{[\xi]_{WY}}{Q^2} + \frac{[\xi]_{WZ}}{Q^2 + M_Z^2} + \sum_{k=1}^K \frac{[\xi]_{Zk}}{Q^2 + M_{Zk}^2}, \]  
\[ g_N^2 D_{N+1,N+1}(Q^2) \equiv [G_{NC}(Q^2)]_{YY} = \frac{[\xi]_{YY}}{Q^2} + \frac{[\xi]_{YZ}}{Q^2 + M_Z^2} + \sum_{k=1}^K \frac{[\xi]_{Zk}}{Q^2 + M_{Zk}^2}, \]  
\[ g_0^2 D_{0,0}(Q^2) \equiv [G_{CC}(Q^2)]_{WW} = \frac{[\xi]_{WW}}{Q^2 + M_W^2} + \sum_{n=1}^N \frac{[\xi]_{Wn}}{Q^2 + M_{Wn}^2}, \]  

All poles should be simple (i.e. there should be no degenerate mass eigenvalues) because, in the continuum limit, we are analyzing a self-adjoint operator on a finite interval.

Since the neutral bosons couple to only two currents, \( J_3^\mu \) and \( J_Y^\mu \), the three sets of residues in equations (3.11)–(3.13) must be related. Specifically, they satisfy the \( N+M+1 \) consistency conditions,

\[ [\xi]_{WW} [\xi]_{YY} = ([\xi]_{WY})^2, \quad [\xi]_{WW} [\xi]_{Y\ell} = ([\xi]_{Z\ell})^2. \]  

In the case of the photon, charge universality further implies

\[ e^2 = [\xi]_{WW} = [\xi]_{WY} = [\xi]_{YY}. \]  

We will find expressions for the other residues in the next sections of the paper.

As noted in the introduction, a number of models in the literature correspond to the special case of this theory where there is only a single \( U(1) \) group. In our notation, this corresponds to \( M = 0 \). Equation (3.6) for the neutral mass-squared matrix simplifies to

\[ M_{N+M}^2 = \begin{pmatrix} \tilde{\alpha}_h & -g_N g_{N+1} \frac{f_{N+1}}{f_{N+1}} \\ -g_N g_{N+1} \frac{f_{N+1}}{f_{N+1}} & \tilde{\alpha}_h \end{pmatrix}. \]  

The \( M_M^2 \) matrix does not arise, and all observables can be specified in terms of the \( 2N + 3 \) quantities \( e^2, M_Z, M_{Z\ell}, M_W, \) and \( M_{W\bar{n}} \). Hence, in the results that follow, the case of a single \( U(1) \) group will correspond to taking \( M = 0 \) and setting multiplicative factors involving the \( M_{\bar{n}} \) to unity.

### 4. Correlation Functions and Couplings

We now find the values of the coefficients \( \xi \) in the spectral representation of the neutral and charged-current correlation functions (3.11)–(3.14). These directly show the contributions of the various weak bosons to four-fermion processes. We can also write the couplings of the \( Z \) and \( W \) bosons in terms of the residues

\[ (\sqrt{[\xi]_{WW}} J_{3\mu} - \sqrt{[\xi]_{YY}} J_{Y\mu}) Z^\mu \]  
\[ \frac{[\xi]_{WW}}{2} J_{W^\mu} W^\pm \]  

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and see how the existence of the heavy gauge bosons alters these couplings from their standard model values. We will find that the $W$ and $Z$ couplings approach their tree-level standard model values in the limit $M_{Zk}, M_{W\hat{n}}, M_{\bar{m}} \to \infty$.

4.1 $[G_{\text{NC}}(Q^2)]_{YY}$ and its residues

We will start by determining the pure hypercharge element of the propagator matrix, $D(Q^2)_{N+1,N+1}$. This will lead us to the residues $[\xi]\_Y$ and $[\xi_Z]\_Y$ of $[G_{\text{NC}}(Q^2)]_{YY} = g_{N+1}^2 D(Q^2)_{N+1,N+1}$, which reveal the relative contributions of $Z$ and $Z_{\ell}$ exchange to four-fermion processes in the hypercharge channel.

Direct calculation of the matrix inverse involved in $D_{N+1,N+1}(Q^2)$ involves the computation of the cofactor related to the $(N+1, N+1)$ element of the matrix $[Q^2 I_{N+M+2} + M_{N+M}^2]$. Inspection of the neutral-boson mass matrix eqn. (3.6) shows that this cofactor is given by

$$\text{det}[\tilde{D}^{-1}(Q^2)] \text{det}[D^{-1}(Q^2)].$$

We therefore obtain

$$D(Q^2)_{N+1,N+1} = \frac{\text{det}[\tilde{D}^{-1}(Q^2)] \text{det}[D^{-1}(Q^2)]}{\text{det}[D^{-1}(Q^2)]}$$

$$= (Q^2 + M_W^2) \left( \prod_{\hat{n}=1}^{N} (Q^2 + M_{W\hat{n}}^2) \right) \left( \prod_{m=1}^{M} (Q^2 + \mathcal{M}_{m}^2) \right)$$

$$Q^2 (Q^2 + M_{Z_{\ell}}^2) \prod_{k=1}^{K} (Q^2 + M_{Zk}^2),$$

by writing the determinants in terms of the eigenvalues.

As noted earlier, charge universality for the photon tells us that the residue of $[G_{\text{NC}}(Q^2)]_{YY}$ at $Q^2 = 0$ is $e^2$. Using equations (3.13) and (4.4) we find

$$e^2 = g_{N+1}^2 \frac{M_W^2}{\prod_{\hat{n}=1}^{N} M_{W\hat{n}}^2} \left( \prod_{m=1}^{M} \mathcal{M}_{m}^2 \right) \prod_{k=1}^{K} M_{Zk}^2,$$

and therefore

$$[G_{\text{NC}}(Q^2)]_{YY} = \frac{e^2 [Q^2 + M_W^2] M_{Z_{\ell}}^2}{Q^2 M_W^2} \left( \prod_{k=1}^{K} \frac{M_{Zk}^2}{M_{W\hat{n}}^2} \right) \left( \prod_{\hat{n}=1}^{N} \frac{Q^2 + M_{W\hat{n}}^2}{Q^2 + M_{W\hat{n}}^2} \right) \left( \prod_{m=1}^{M} \frac{Q^2 + \mathcal{M}_{m}^2}{\mathcal{M}_{m}^2} \right).$$

Reading off the residues of the poles in Eq.(4.6), we obtain

$$[\xi]\_Y = e^2 \frac{M_{Z_{\ell}}^2 - M_{W\hat{n}}^2}{M_W^2} \left( \prod_{k=1}^{K} \frac{M_{Zk}^2}{M_{Zk}^2 - M_{Z_{\ell}}^2} \right) \left( \prod_{\hat{n}=1}^{N} \frac{M_{W\hat{n}}^2 - M_{Z_{\ell}}^2}{M_{W\hat{n}}^2} \right).$$

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\[ \times \left( \prod_{m=1}^{M} \frac{M_{m}^{2} - M_{\tilde{Z}}^{2}}{M_{\tilde{m}}^{2}} \right), \quad (4.7) \]

\[ [\xi_{\tilde{Z}}]_{YY} = -e^{2} \left[ M_{Z_{\ell}}^{2} - M_{W}^{2} \right] M_{Z}^{2} \left( \prod_{k \neq \ell} \frac{M_{Z_{k}}^{2}}{M_{Z_{k}}^{2} - M_{\tilde{Z}}^{2}} \right) \left( \prod_{\bar{n}=1}^{N} \frac{M_{\tilde{W}_{\bar{n}}}^{2} - M_{Z_{\bar{n}}}^{2}}{M_{\tilde{W}_{\bar{n}}}^{2}} \right) \times \left( \prod_{m=1}^{M} \frac{M_{m}^{2} - M_{\tilde{Z}}^{2}}{M_{\tilde{m}}^{2}} \right), \quad (4.8) \]

Note that, in the limit \( M_{Z_{\ell}}, M_{\tilde{W}_{\bar{n}}, M_{\tilde{m}}} \to \infty \), \( [\xi_{\tilde{Z}}]_{YY} \to e^{2}(M_{Z}^{2} - M_{W}^{2})/M_{\tilde{W}}^{2} \), so that the correct standard model tree-level value of the Z-boson coupling to hypercharge is recovered.

### 4.2 \([G_{NC}(Q^{2})]_{WY}\) and its residues

Next, we turn to the weak-hypercharge interference term in equation (3.3). Direct calculation of the matrix inverse involved in \( D_{0,N+1}(Q^{2}) \) involves the computation of the cofactor related to the \((0, N + 1)\) element of the matrix \( Q^{2}I + M_{N+M}^{2} \). Inspection of the neutral-boson mass matrix eqn. (3.3) shows that the cofactor is the determinant of a matrix whose upper \((N + 1)\times(N + 1)\) block is upper-diagonal, with diagonal entries \{-g_{0}g_{1}F_{1}^{2}/4, -g_{1}g_{2}F_{2}^{2}/4, \ldots, -g_{N}g_{N+1}F_{N+1}^{2}/4\} that are independent of \( Q^{2} \); its determinant is therefore also constant. The lower \( M \times M \) block of the cofactor is just \( M_{M}^{2} \). Therefore

\[ [G_{NC}(Q^{2})]_{WY} = C \left[ \frac{\det[D^{-1}(Q^{2})]}{\det[D^{-1}(Q^{2})]} \right] = C \frac{\prod_{\bar{n}=1}^{N} (Q^{2} + M_{\tilde{W}_{\bar{n}}}^{2})}{Q^{2}(Q^{2} + M_{Z_{\ell}}^{2})} \prod_{k=1}^{K} (Q^{2} + M_{Z_{k}}^{2}) \quad (4.9) \]

where \( C \) is a constant.

Requiring the residue of the photon pole at \( Q^{2} = 0 \) to equal \( e^{2} \) determines the value of \( C \) to be \( [e^{2}M_{Z_{\ell}}^{2} \prod_{k=1}^{K} M_{Z_{k}}^{2}]/[\prod_{\bar{n}=1}^{N} M_{\tilde{W}_{\bar{n}}}^{2}] \). We thus obtain

\[ [G_{NC}(Q^{2})]_{WY} = \frac{e^{2}M_{Z}^{2}}{Q^{2}(Q^{2} + M_{Z}^{2})} \left( \prod_{k=1}^{K} \frac{M_{Z_{k}}^{2}}{Q^{2} + M_{Z_{k}}^{2}} \right) \left( \prod_{\bar{n}=1}^{N} \frac{Q^{2} + M_{\tilde{W}_{\bar{n}}}^{2}}{M_{\tilde{W}_{\bar{n}}}^{2}} \right). \quad (4.10) \]

Reading off the residues of the poles in Eq. (4.10), we find

\[ [\xi_{Z}]_{WY} = -e^{2} \left( \prod_{k=1}^{K} \frac{M_{Z_{k}}^{2}}{M_{\tilde{Z}}^{2} - M_{Z_{k}}^{2}} \right) \left( \prod_{\bar{n}=1}^{N} \frac{M_{\tilde{W}_{\bar{n}}}^{2} - M_{Z_{\bar{n}}}^{2}}{M_{\tilde{W}_{\bar{n}}}^{2}} \right), \quad (4.11) \]

\[ [\xi_{\tilde{Z}}]_{WY} = e^{2} \frac{M_{Z}^{2}}{M_{Z_{\ell}}^{2} - M_{\tilde{Z}}^{2}} \left( \prod_{k \neq \ell} \frac{M_{Z_{k}}^{2}}{M_{Z_{k}}^{2} - M_{\tilde{Z}}^{2}} \right) \left( \prod_{\bar{n}=1}^{N} \frac{M_{\tilde{W}_{\bar{n}}}^{2} - M_{Z_{\bar{n}}}^{2}}{M_{\tilde{W}_{\bar{n}}}^{2}} \right). \quad (4.12) \]

Note that, in the limit \( M_{Z_{\ell}}, M_{\tilde{m}} \to \infty \), \( [\xi_{Z}]_{WY} \to -e^{2} \) as at tree-level in the standard model.
Figure 5: Leading diagram at high-$Q^2$ ($Q^2 \gg M_{Zk}^2, M_{\ell Z}^2$) which distinguishes $[G_{NC}(Q^2)]_{WW}$ from $[G_{CC}(Q^2)]_{WW}$, resulting in the behavior of eqn. (4.16). The open circles at either end represent the weak current $J_3^\mu$ (see eqn. (2.3)), the solid circles represent the off-diagonal vector-boson mass matrix elements in eqn. (3.3), and the gauge-bosons are represented by the zigzag lines. The difference between $[G_{NC}(Q^2)]_{WW}$ and $[G_{CC}(Q^2)]_{WW}$ arises from the couplings of the $U(1)$ factors, and the leading contribution comes from the factor with coupling $g_{N+1}$.

4.3 Consistency Conditions and $[G_{NC}(Q^2)]_{WW}$

Having determined the YY and WY residues for the neutral boson poles, we can use the consistency conditions of equation (3.15) to deduce the WW residues as well

$$\xi Z_{WW} = \frac{e^2 M_W^2}{M_Z^2 - M_W^2} \left( \prod_{k=1}^K \frac{M_{Zk}^2}{M_{Zk}^2 - M_Z^2} \right) \left( \prod_{\hat{n}=1}^N \frac{M_{W\hat{n}}^2}{M_{W\hat{n}}^2 - M_Z^2} \right) \times \left( \prod_{\hat{m}=1}^M \frac{M_{\hat{m}\hat{m}}^2 - M_Z^2}{M_{\hat{m}\hat{m}}^2} \right), \quad (4.13)$$

and

$$\xi Z_{\ell W} = -\frac{e^2 M_W^2 M_Z^2}{(M_Z^2 - M_W^2)(M_{Z\ell}^2 - M_{Z\ell}^2)} \left( \prod_{k\neq \ell} \frac{M_{Zk}^2}{M_{Zk}^2 - M_{Z\ell}^2} \right) \left( \prod_{\hat{n}=1}^N \frac{M_{W\hat{n}}^2}{M_{W\hat{n}}^2 - M_{Z\ell}^2} \right) \times \left( \prod_{\hat{m}=1}^M \frac{M_{\hat{m}\hat{m}}^2 - M_{Z\ell}^2}{M_{\hat{m}\hat{m}}^2} \right). \quad (4.14)$$

In the limit $M_{Zk}, M_{W\hat{n}}, M_{\hat{m}\hat{m}} \rightarrow \infty$, one finds $\xi Z_{WW} \rightarrow e^2 M_W^2/(M_Z^2 - M_W^2)$, so that the correct standard model tree-level form of the $Z$ boson coupling to weak charge is recovered.

4.4 Custodial Symmetry and $[G_{CC}(Q^2)]_{WW}$

Finally, we need to determine the charged gauge boson correlation functions. Here, we have no consistency conditions to guide us, but we can exploit custodial symmetry.

The value of the difference of correlation functions

$$[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW}. \quad (4.15)$$

vanishes in the limit of exact custodial symmetry, $g_{N+1} \rightarrow 0$. Since the violation of the custodial symmetry $g_{N+1} \neq 0$ is localized at the $(N + 1)$th site, Eq.(4.15) must fall at large
\[ Q^2 \text{ like} \]

\[
[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW} \propto \frac{g_{N+1}^2}{(Q^2)^{2N+3}},
\]

for \( Q^2 \gg M_{W\bar{W}}^2, M_{Z\bar{Z}}^2 \), as shown in fig. (3).

Consider using the spectral decompositions of the correlation functions from Eq.(3.11) and Eq.(3.14) to evaluate eqn. (4.15). Forming a common denominator, we see that the only way to achieve the high energy behavior of Eq.(4.16) is if the numerator includes a polynomial in \( Q^2 \) of order \( M \). Denoting that polynomial as \( R \), we can write

\[
[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW} = \frac{e^2 M_W^2 M_Z^2 R(Q^2)}{Q^2|Q^2 + M_W^2||Q^2 + M_Z^2|} \left( \prod_{k=1}^{K} \frac{M_{Z_k}^2}{Q^2 + M_{Z_k}^2} \right) \left( \prod_{\bar{n}=1}^{N} \frac{M_{W\bar{n}}^2}{Q^2 + M_{W\bar{n}}^2} \right).
\]

The residues \([\xi_Z]_{WW}\) and \([\xi_{Z\bar{Z}}]_{WW}\) extracted from Eq.(4.17) must satisfy the \( N + M + 1 \) consistency conditions of eqn. (3.13). The number of consistency conditions is \( N + M + 1 \), while the polynomial \( R \) possesses \( M \) parameters – thus the polynomial \( R \) is (over) determined solely by the consistency conditions.

To evaluate \( R \), we begin by defining a more compact notation

\[
R_Z(Q^2) \equiv \frac{1}{Q^2|Q^2 + M_W^2||Q^2 + M_Z^2|} \prod_{\bar{\ell}=1}^{K} \frac{M_{Z\bar{\ell}}^2}{Q^2 + M_{Z\bar{\ell}}^2},
\]

\[
R_W(Q^2) \equiv \frac{M_W^2}{Q^2 + M_W^2} \prod_{\bar{n}=1}^{N} \frac{M_{W\bar{n}}^2}{Q^2 + M_{W\bar{n}}^2},
\]

\[
R_m(Q^2) \equiv \prod_{\bar{m}=1}^{M} \frac{M_{W\bar{m}}^2}{Q^2 + M_{W\bar{m}}^2}.
\]

We can rewrite the correlation functions in this language as

\[
[G_{NC}(Q^2)]_{YY} = e^2 \frac{R_Z(Q^2)}{R_W(Q^2)R_m(Q^2)},
\]

\[
[G_{NC}(Q^2)]_{WW} = e^2 \frac{R_Z(Q^2)}{R_m(Q^2)},
\]

\[
[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW} = e^2 R(Q^2) R_Z(Q^2) R_W(Q^2)
\]

where \( R \) is still our unknown polynomial.

Likewise, the residues can be written rather simply in this notation. In particular, we can evaluate \([\xi_Z]_{WW}\) as the residue of the \( Z \)-boson pole of equation (4.23). Since \( R_Z(Q^2) \) includes a simple pole at \( Q^2 = -M_Z^2 \), the unknown polynomial \( R \) will be analytic at that point.

\[
[\xi_Z]_{WW} = e^2 R(-M_Z^2) R_W(-M_Z^2) \left[ (Q^2 + M_Z^2) R_Z(Q^2) \right]_{Q^2 = -M_Z^2}
\]
Alternatively, we can rewrite in this compact notation the expression (4.13) for $[\xi_Z]_{WW}$ which we derived earlier from $[\xi_Z]_{YY}$ and $[\xi_Z]_{WY}$ and the consistency conditions (3.21)

$$[\xi_Z]_{WW} = e^2 \frac{R_W(-M^2_Z) [(Q^2 + M^2_Z)R_Z(Q^2)]_{Q^2 = -M^2_Z}}{R_m(-M^2_Z)}$$  \hspace{1cm} (4.25)

Comparing expressions (4.23) and (4.25) for $[\xi_Z]_{WW}$ and making similar comparisons for the two ways of finding the residues of the $Z_\ell$ poles shows that

$$R(Q^2) = \prod_{m=1}^M \frac{Q^2 + M^2_{m}}{M^2_{m}} = \frac{1}{R_m(Q^2)}.$$  \hspace{1cm} (4.26)

is the polynomial we seek.

As a result, we find

$$[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW} = e^2 \frac{R_Z(Q^2)R_W(Q^2)}{R_m(Q^2)}$$  \hspace{1cm} (4.27)

with the interesting corollary from comparing this with equations (4.21) and (4.22)

$$[G_{NC}(Q^2)]_{YY} = [G_{NC}(Q^2)]_{YY} \cdot ([G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW}) .$$  \hspace{1cm} (4.28)

Note that the $W$ poles in $[G_{CC}(Q^2)]_{WW}$ are cancelled by the corresponding zeros in $[G_{NC}(Q^2)]_{YY}$.

Finally, we can write the difference of the neutral and charged current correlation functions in our original notation as

$$[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW} = \frac{e^2 M^2_W M^2_Z}{Q^2[Q^2 + M^2_W][Q^2 + M^2_Z]} \left( \prod_{k=1}^K \frac{M^2_{Zk}}{Q^2 + M^2_{Zk}} \right) \left( \prod_{\hat{n}=1}^N \frac{M^2_{W\hat{n}}}{Q^2 + M^2_{W\hat{n}}} \right) \times \left( \prod_{m=1}^M \frac{Q^2 + M^2_{m}}{M^2_{m}} \right).$$  \hspace{1cm} (4.29)

Reading the residues of the poles from Eq.(4.23), we obtain the same results as before (equations (4.13) and (4.14)) for the $Z$ and $Z_\ell$ residues. We also obtain the values of the last residues we need, those for the $W$ and $W_\ell$ poles

$$[\xi_W]_{WW} = \frac{e^2 M^2_Z}{M^2_Z - M^2_W} \left( \prod_{k=1}^K \frac{M^2_{Zk}}{M^2_{Zk} - M^2_W} \right) \left( \prod_{\hat{n}=1}^N \frac{M^2_{W\hat{n}}}{M^2_{W\hat{n}} - M^2_W} \right) \left( \prod_{m=1}^M \frac{M^2_{m} - M^2_W}{M^2_{m}} \right),$$  \hspace{1cm} (4.30)

$$[\xi_{W_\ell}]_{WW} = \frac{e^2 M^2_W M^2_Z}{(M^2_{W_\ell} - M^2_W)(M^2_{W_\ell} - M^2_Z)} \left( \prod_{k=1}^K \frac{M^2_{Zk}}{M^2_{Zk} - M^2_{W_\ell}} \right) \left( \prod_{\hat{n} \neq \ell} \frac{M^2_{W\hat{n}}}{M^2_{W\hat{n}} - M^2_{W_\ell}} \right) \times \left( \prod_{m=1}^M \frac{M^2_{m} - M^2_{W_\ell}}{M^2_{m}} \right).$$  \hspace{1cm} (4.31)
In the limit $M_{Zk}, M_{Wk}, M_m \to \infty$, we have $[\xi_W]_{WW} \to e^2 M_Z^2 / (M_Z^2 - M_W^2)$ and the correct standard model tree-level value of the $W$-boson coupling to weak charge is recovered.

Comparison of equations (4.13) with equation (4.30) shows that the formulae for $[\xi_Z]_{WW}$ and $[\xi_W]_{WW}$ are related by the simple exchange $[Z \leftrightarrow W]$. A similar relationship holds for $[\xi_W]_{W}^{\ell\ell}$ and $[\xi_Z]_{W}^{\ell\ell}$. While this relationship seems intuitive in hindsight, a first-principles proof starting from the relationship between $M_Z^2 + M$ and $\tilde{M}_N^2$ has not been found.

5. The Low-Energy $\rho$ Parameter

In discussing low-energy interactions it is conventional to rewrite the neutral-current Lagrangian of eqn. (3.8) in terms of weak and electromagnetic currents as

$$L_{nc} = -\frac{1}{2} A(Q^2) J_3^\mu J_{3\mu} - B(Q^2) J_3^\mu J_{Q\mu} - \frac{1}{2} C(Q^2) J_Q^\mu J_{Q\mu}, \quad (5.1)$$

where

$$A(Q^2) = [G_{NC}(Q^2)]_{WW} - 2[G_{NC}(Q^2)]_{WY} + [G_{NC}(Q^2)]_{YY} \quad (5.2)$$

$$B(Q^2) = [G_{NC}(Q^2)]_{WY} - [G_{NC}(Q^2)]_{YY} \quad (5.3)$$

$$C(Q^2) = [G_{NC}(Q^2)]_{YY}. \quad (5.4)$$

As required, the photon pole cancels in the expressions for $A(Q^2)$ and $B(Q^2)$. Charged-current interactions will continue to be expressed by eqn. (3.9)

$$L_{cc} = -\frac{1}{2} [G_{CC}(Q^2)]_{WW} J_+^\mu J_{-\mu} \quad (5.5)$$

It is informative to compare the overall strength of the low-energy neutral current interactions with that of the charged-current interactions. In assessing the strength of low-energy neutral currents, we must calculate $A(Q^2 = 0)$. This is conveniently done by evaluating the derivative

$$\left( \frac{d\{Q^2 A(Q^2)\}}{dQ^2} \right)_{Q^2=0}. \quad (5.6)$$

Similarly, since the charged-current propagator has no pole at $Q^2 = 0$, one finds

$$\left( \frac{d\{Q^2 [G_{CC}(Q^2)]_{WW}\}}{dQ^2} \right)_{Q^2=0} = [G_{CC}(0)]_{WW}. \quad (5.7)$$

Using eqn. (4.28) and the fact that the products $R_W(Q^2)$ and $R_m(Q^2)$ are well-behaved at $Q^2 = 0$, with $R_m(0) = R_W(0) = 1$, one finds that expression (5.6) is equal to $[G_{CC}(0)]_{WW}$. The low-energy $\rho$ parameter is identically 1:

$$\rho = \frac{1}{[G_{CC}(0)]_{WW}} \left( \frac{d\{Q^2 A(Q^2)\}}{dQ^2} \right)_{Q^2=0} = 1. \quad (5.8)$$
Therefore, in the sense defined here, the strengths of the charged-current and weak neutral-current interactions at low energy are the same.\footnote{Note that the strength of low-energy neutral and charged currents can be the same, while the ratio of \( W \) and \( Z \) masses can differ from the standard model value, if the contributions from heavy boson exchange are significant. This is the case in the model discussed in ref. \[1\], for example.} This is an extension of the result in \[34\] and is due to our specifying that the fermion hypercharges arise from the \( U(1) \) group with \( j = N + 1 \).

Finally, we note for future use that the phenomenologically relevant low-energy neutral current coupling for atomic parity violation (since \( Z \approx N \) for most nuclei) is \( B(Q^2) \), which couples \( J^a_\mu \) to the electromagnetic current \( J_{Q\mu} \). In this quantity, the photon pole cancels and we find that

\[
B(Q^2 = 0) \equiv \left( \frac{d\{Q^2 B(Q^2)\}}{dQ^2} \right)_{Q^2 = 0} = -\frac{e^2}{M_W^2} \left[ 1 + \sum_{l=1}^N \frac{M_W^2}{M_{W^l}^2} \right].
\]  

We will find in the next section that in some limits of the model \( B(Q^2 = 0) \) arises from \( Z \)-boson exchange alone, while in others the exchange of heavy neutral vector bosons must be included at low energies to give the correct value.

## 6. Electroweak Parameters

From the calculation of the propagator and corresponding residues, we see that the size of the couplings of the heavy resonances to fermions depends crucially on the isospin asymmetry of these states. In the isospin symmetric limit, \( g_{N+1} \to 0 \), \( M_Z \to M_W \) and \( M_{Zk} \to \{M_{W^\hat{n}} \text{ or } M_{m^l}\} \). The couplings of the heavy resonances depend on the size of \( \Delta M_{Zk}^2 \), the mass splitting between \( M_{Zk}^2 \) and the corresponding mass \( M_{W^\hat{n}}^2 \) or \( M_{m^l}^2 \) with which it becomes paired in the isospin symmetric limit. In the subsections below we examine limits with different values of the mass splitting \( \Delta M_{Zk}^2 \).

Note that up to this point we have used \( M_W \) as an input. In order to make contact with experiment, it will be necessary to shift to a scheme in which the more precisely measured \( G_F \) is instead used as an input. In this language, the standard-model weak mixing angle \( s_Z \) is defined in terms of \( \alpha, M_Z \) and \( G_F \) as

\[
c_Z^2(1 - c_Z^2) \equiv \frac{e^2}{4\sqrt{2}G_FM_Z^2}, \quad s_Z^2 \equiv 1 - c_Z^2.
\]  

The relationship between \( \alpha, G_F \) and \( M_Z \) is altered by the non-standard elements of our model; thus the weak mixing angle appearing in the amplitude for four-fermion processes is shifted from its standard model value

\[
e^2 = c_Z^2 + \Delta_Z.
\]  

The size of \( M_W \) relative to \( M_Z \) will likewise be shifted from its standard model value. We will calculate the values of \( \Delta_Z \) and \( M_W \) separately for each isospin limit considered below.
6.1 Moderate Isospin Violation

In this section, we will assume the small, but non-zero isospin splitting

\[ M_{W,n}^2 \text{ or } M_{m}^2 \gg \Delta M_{Zk}^2 \gg (M_{Z}^2 - M_{W}^2) \frac{M_{W}^2}{M_{W,n}^2 \text{ or } M_{m}^2}, \]

(6.3)

for all heavy vector mesons. An examination of eqns. (4.12), (4.8), and (4.14) for the \( \xi_{Zl} \) shows that the exchange of heavy resonances is negligible in this limit. The only corrections to electroweak processes at “low energies”, i.e. energies much less than the heavy boson masses, arise from corrections to the \( W \) and \( Z \) couplings and masses; all electroweak corrections are oblique.

Since all corrections are oblique, we may summarize the amplitude for low-energy four-fermion neutral weak current processes by

\[ \mathcal{M}_{NC} = e^2 \frac{QQ'}{Q^2} + \frac{(I_3 - s^2 Q)(I'_3 - s^2 Q')}{\left( e^2 c^2 - \frac{S}{16\pi} \right) Q^2 + \frac{1}{4\sqrt{2} G_F} - s^2 c^2 M_{Z}^2 T - \frac{1}{4\pi}}, \]

(6.4)

and charged-current processes by

\[ \mathcal{M}_{CC} = \frac{(I_+ I'_- + I_- I'_+)/2}{\left( e^2 - \frac{S + U}{16\pi} \right) Q^2 + \frac{1}{4\sqrt{2} G_F}}. \]

(6.5)

Here \( I_a^{(t)} \) and \( Q^{(t)} \) are weak-isospin and charge of the corresponding fermion, and the weak mixing angle is denoted by \( s^2 \) (\( c^2 \equiv 1 - s^2 \)). \( S, T, \) and \( U \) are the familiar oblique electroweak parameters. Given the form of eqn. (6.4) and the constraint from the low-energy \( \rho \)-parameter (eqn. (5.8)), we expect to find \( T = 0 \).

Eqs. (6.4) and (6.5) may be compared with our definitions (neglecting the heavy boson contributions) in eqns. (3.11), (3.12), (3.13), and (3.14) to obtain the pole residues

\[ [\xi_Z]_{WW} = \frac{e^4}{s^2 c^2} - \frac{S}{16\pi} \]

\[ [\xi_Z]_{WY} = \frac{-s^2 c^2}{e^2} \frac{S}{16\pi} \]

\[ [\xi_Z]_{YY} = \frac{s^4}{s^2 c^2} \frac{S}{16\pi} \]

(6.6)
The pole residues \([\xi_Z]_{WW}, [\xi_Z]_{WY}, [\xi_Z]_{YY}\) can be precisely determined from the LEP/SLC precision measurements on the Z-pole, since they can be related to the decay widths and the asymmetries of the Z boson.

Comparing equations (6.1), (6.2) and (6.7) gives us the value of \(\Delta_Z\) in this isospin limit,

\[
\Delta_Z = \frac{\alpha}{c_Z^2 - s_Z^2} \left[-\frac{1}{4}S + s_Z^2 c_Z^2 T\right].
\]

Knowing the shift in the weak mixing angle allows us to find the W mass and to rewrite the pole residues of the Z boson in terms of the tree-level standard model weak angle (\(c_Z\)). The W boson mass is calculated from Eq. (6.8),

\[
M_W^2 = c_Z^2 M_Z^2 \left[1 + \frac{\alpha}{c_Z^2 - s_Z^2} \left[-\frac{1}{2}S + c_Z^2 T + \frac{c_Z^2 - s_Z^2}{4s_Z^2} U\right]\right].
\]

The pole residues of the Z boson are given by

\[
\frac{1}{e^2} [\xi_Z]_{WW} = \frac{c_Z^2}{s_Z^2} + \frac{\alpha}{s_Z^2(c_Z^2 - s_Z^2)} \left[-\frac{1}{2}S + c_Z^2 T\right],
\]

\[
\frac{1}{e^2} [\xi_Z]_{WY} = -1 - \frac{\alpha}{4s_Z^2 c_Z^2} S.
\]

We can obtain expressions for \(S, T,\) and \(U\) by comparing the residues as calculated here with their values as calculated earlier in terms of the gauge boson spectrum. To do this self-consistently, we must first insert our calculated value of \(M_W\) (6.10) into the earlier expressions (4.13) and (4.11) for \([\xi_Z]_{WW}\) and \([\xi_Z]_{WY}\); this will align them with our current scheme of using \(G_F\) rather than \(M_W\) as an experimental input. The result of these calculations is

\[
\alpha S = 4s_Z^2 c_Z^2 M_Z^2 (\Sigma_Z - \Sigma_M),
\]

\[
\alpha T = s_Z^2 M_Z^2 (\Sigma_Z - \Sigma_W - \Sigma_M),
\]

\[
\alpha U = 0,
\]
to lowest nontrivial order in $1/M^2_{Zk}$, where we have defined the sums

$$\Sigma_Z \equiv \sum_{k=1}^{K} \frac{1}{M^2_{Zk}}, \quad \Sigma_W \equiv \sum_{n=1}^{N} \frac{1}{M^2_{Wn}}, \quad \Sigma_M \equiv \sum_{m=1}^{M} \frac{1}{M^2_{m}}.$$  
(6.14)

However, for the assumed level of isospin violation $\Delta M^2_{Zk} \ll \{M^2_{Wn} \text{ or } M^2_{m}\}$, we have

$$\Sigma_Z - \Sigma_W - \Sigma_M \approx 0,$$
(6.15)
and hence

$$\alpha_S \approx 4 s^2_Z c^2_Z M^2_Z \Sigma_W,$$
(6.16)
$$\alpha_T \approx 0.$$  
(6.17)

As a check of the consistency of these results, we may use the explicit form of the couplings of the $Z$-boson (4.1) to see if $Z$-exchange accounts for the low-energy neutral current relevant to atomic parity violation, eqn. (5.9). We find that, so long as eqn. (6.15) holds, $Z$-exchange suffices to reproduce eqn. (5.9).

### 6.2 Large Isospin Violation

We next consider the case in which the isospin violation in the spectrum of heavy resonance masses is large

$$\Delta M^2_{Zk} \simeq \{M^2_{Wn} \text{ or } M^2_{m}\}.$$  
(6.18)

In this case, examining eqn. (4.8), we expect that the exchange of heavy neutral resonances may be important and result in an extra four-fermion contribution at low-energies proportional to $J^\mu Y_J$. Given the constraint that the low-energy $\rho$-parameter must equal one, we see that eqn. (5.4) must be modified to

$$\mathcal{M}_{NC} = e^2 \frac{QQ'}{Q'^2} + \left( \frac{s^2 c^2}{e^2} - \frac{S}{16\pi} \right) Q^2 + \frac{1}{4\sqrt{2}G_F} - s^2 c^2 M^2_Z \frac{T}{4\pi}$$
$$- 8G_F^2 s^2 c^2 M^2_Z T \left[ \frac{(Q - I_3)(Q' - I'_3)}{\pi} \right].$$  
(6.19)

The calculations leading to eqns. (6.13) remain unchanged. Here, however, we cannot simplify further using eqn. (6.13). The eqns. (6.13) imply that

$$\alpha_S - 4 c^2_Z \alpha_T = 4 s^2_Z c^2_Z M^2_Z \Sigma_W.$$  
(6.20)

The explicit form of the couplings of the $Z$-boson (4.1) and the additional $J^\mu Y_J$ contribution in eqn. (6.13) together account for the low-energy neutral current relevant to atomic parity violation, eqn. (5.9).
6.3 Small Isospin Violation

Finally, we consider the case of small isospin violation in the heavy resonance masses

\[
\Delta M_{Zk}^2 \simeq (M_Z^2 - M_W^2) \frac{M_W^2}{M_W^2 - \hat{m}_n} \text{ or } M_{\tilde{m}}^2. \tag{6.21}
\]

We note that this is the minimum size of isospin violation one expects in the limit \( g_{N+1} \to 0 \). In this case, examining eqns. (4.31) and (4.14), we expect that the exchange of heavy resonances may be important and result in an extra four-fermion contribution to both low-energy neutral and charged current exchange proportional to the square of the weak \( SU(2) \) currents \( \vec{J}_\mu \cdot \vec{J}_\mu \). The form of amplitudes for neutral-current processes are then given by

\[
M_{NC} = e^2 \frac{QQ'}{Q^2} + \frac{(I_3 - s^2 Q)(I'_3 - s^2 Q')}{(s^2 - S/16\pi)Q^2} \left[ 1 + \frac{e^2\delta/16\pi s^2 c^2}{4\sqrt{2}G_F} \right] + \sqrt{2}G_F \frac{e^2\delta}{4\pi s^2 c^2} I_3 I'_3, \tag{6.22}
\]

and charged-current processes by

\[
M_{CC} = \left( \frac{I_+ I'_+ + I_- I'_-}{2} \right) \frac{1 + (e^2\delta/16\pi s^2 c^2)}{4\sqrt{2}G_F} + \sqrt{2}G_F \frac{e^2\delta}{4\pi s^2 c^2} \frac{(I_+ I'_+ + I_- I'_-)}{2}, \tag{6.23}
\]

where isospin symmetry implies \( T = U = 0 \), and the dimensionless parameter \( \delta \) measures the size of the contributions from heavy resonance exchange. Note that, formally, eqns. (4.31) and (4.14) provide an expression for \( \delta \) in terms of the boson masses and \( e^2 \).

The expressions for the pole residues in eqns. (6.6) remain the same, while for the gauge-boson masses we find

\[
M_Z^2 = \frac{1}{4\sqrt{2}G_F \left( \frac{s^2 c^2 - S + \delta}{e^2 - S/16\pi} \right)}, \tag{6.24}
\]

\[
M_W^2 = \frac{1}{4\sqrt{2}G_F \left( \frac{s^2 c^2 - S + \delta/c^2}{e^2 - S/16\pi} \right)}. \tag{6.25}
\]

From eqns. (6.1) and (6.2), we obtain

\[
\Delta_Z = -\frac{\alpha}{4(c_Z^2 - s_Z^2)} (S + \delta), \tag{6.26}
\]

and consequently, from eqn. (6.25)

\[
M_W^2 = c_Z^2 M_Z^2 \left[ \frac{1}{2} + \frac{\alpha}{c_Z^2 - s_Z^2} \left[ -\frac{1}{2} S - \frac{\delta}{4c_Z^2} \right] \right]. \tag{6.27}
\]
To this order, we continue to find, from analyzing $[\xi Z]_{WY}$ for example,
\[ \alpha S = 4s_Z^2 c_Z^2 M_Z^2 (\Sigma_Z - \Sigma_M) = 4s_Z^2 c_Z^2 M_Z^2 \Sigma_W , \]
where the last equality follows from the isospin invariance of the heavy spectrum. In terms of $M_W^2$, $\delta$ is determined by eqn. (6.27)
\[ \alpha \delta = 4c_Z^2 (c_Z^2 - s_Z^2) \left( 1 - \frac{M_W^2}{c_Z^2 M_Z^2} \right) - 8s_Z^2 c_Z^2 M_Z^2 (\Sigma_Z - \Sigma_M) . \] (6.29)

In practice, however, $M_W^2$ is not precisely measured and eqn. (6.27) is used as just one constraint in a combined $S$ & $\delta$ fit.

7. Discussion and Conclusions

In this paper, using deconstruction, we have calculated the form of the corrections to the electroweak interactions in a large class of Higgsless models, allowing for arbitrary 5-D geometry, position-dependent gauge coupling, and brane kinetic energy terms. We have shown that many models considered in the literature, including those most likely to be phenomenologically viable, are in this class. By analyzing the asymptotic behavior of the correlation function of gauge currents at high momentum, we have extracted the exact form of the relevant correlation functions at tree-level and computed the corrections to precision electroweak observables in terms of the spectrum of heavy vector bosons. Although we have stressed our results as they apply to continuum 5-D models, they apply also to models of extended electroweak gauge symmetries motivated by models of hidden local symmetry. In a subsequent paper, we will address the consequences of our results for model-building in light of current precision electroweak data.

We have shown that the amount of isospin violation in the heavy boson spectrum (as measured by $\Delta M_{k}^2$) determines when nonoblique corrections due to the interactions of fermions with the heavy vector bosons become important. For moderate $\Delta M_{k}^2$, exchange of heavy resonances is negligible. For large $\Delta M_{k}^2$, heavy neutral resonance exchange becomes significant in the $J_{\nu}^{\mu} J_{\nu \mu}$ channel; for example, light Z-boson exchange no longer suffices to account for atomic parity violation. In the case of small $\Delta M_{k}^2$, exchange of both charged and neutral heavy resonances is important; while the small size of isospin violation makes $T$ and $U$ zero in this limit, and electroweak fits must include an additional parameter $\delta$, which measures the size of contributions from heavy resonance exchange.

Independent of the amount of isospin violation, we have found the following constraint\(^4\) on the oblique electroweak parameters $S$ and $T$
\[ \alpha S - 4 \cos^2 \theta_W \alpha T = 4 \sin^2 \theta_W \cos^2 \theta_W M_Z^2 \sum_{n=1}^{N} \frac{1}{M_{Wn}^2} , \]

\(^4\)Note that to the order we are working, it does not matter whether we write $\cos^2 \theta_W$ or $c_Z^2$ in this relationship because $S$, $T$ and the sum over $M_{Wn}^2$ are all small.
in the class of models§ studied here. In the limit of moderate or small isospin violation, one finds $T \approx 0$, which simplifies this relationship. In any unitary theory [2, 3], we expect the mass of the lightest vector $m_{W^1}$ to be less than $\sqrt{8\pi v}$ ($v \approx 246$ GeV) – the scale at which $WW$ spin-0 isospin-0 elastic scattering would violate unitarity in the standard model in the absence of a higgs boson [46, 47, 48, 49, 50, 51]. Evaluating eqn. (7.1) we see that we expect $S - 4\cos^2\theta_W T$ to be of order one-half or larger, generalizing the result of [21].

It is interesting to note how this bound applies to the model discussed in [21], the deconstructed version of which is sketched in Fig. 4. In the case with a $U(1)_{B-L}$ TeV brane kinetic term, these authors find

$$S \approx \frac{6\pi}{g^2 \log \frac{R'}{R}} - \frac{8\pi}{g^2} \left(1 - \left(\frac{g'}{g}\right)^2\right) \frac{\tau'^2}{(R \log R'/R)^2},$$

(7.2)

$$T \approx \frac{-2\pi}{g^2} \left(1 - \left(\frac{g'}{g}\right)^4\right) \frac{\tau'^2}{(R \log R'/R)^2},$$

(7.3)

$$U \approx 0,$$  

(7.4)

where $g$ and $g'$ are the usual $SU(2)_W$ and $U(1)_Y$ couplings, $\tau'$ is a measure of the strength of the brane kinetic term, and $R$ and $R'$ are the positions of the Planck and TeV branes respectively. Noting that $1 + (g'/g)^2 \approx 1/\cos^2\theta_W$, we immediately find

$$S - 4\cos^2\theta_W T \approx \frac{6\pi}{g^2 \log \frac{R'}{R}}.$$  

(7.5)

To ensure unitarity, one expects $1/R' \approx 1$ TeV; using $1/R \approx M_{\text{planck}}$ one obtains $S - 4\cos^2\theta_W T = \mathcal{O}(1)$ as expected. Note that although the individual values of $S$ and $T$ depend on $\tau'$, the relationship between them in eqn. (7.5) agrees with our result (7.1) independent of the strength of the brane kinetic energy term.

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§Recently a Higgsless model with vanishing $S$ has been proposed [45] – the simplest version of which was previously analyzed in [24]. However, the chiral symmetries of this model, like those of QCD-like technicolor, allow for an $O(p^4)$ term (not considered in [45]) which directly contributes to the $S$ parameter. When this term is included, the value of $S$ is expected to be of order 1.
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