Generation of chiral asymmetry via helical magnetic fields

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It is well known that helical magnetic fields undergo a so-called inverse cascade by which their correlation length grows due to the conservation of magnetic helicity in classical ideal magnetohydrodynamics (MHD). At high energies above approximately 10 MeV, however, classical MHD is necessarily extended to chiral MHD and then the conserved quantity is $\langle H \rangle + 2\langle \mu_5 \rangle/\lambda$ with $\langle H \rangle$ being the mean magnetic helicity and $\langle \mu_5 \rangle$ being the mean chiral chemical potential of charged fermions. Here, $\lambda$ is a (phenomenological) chiral feedback parameter. In this paper, we study the evolution of the chiral MHD system with the initial condition of nonzero $\langle H \rangle$ and vanishing $\langle \mu_5 \rangle$. We present analytic derivations for the time evolution of $\langle H \rangle$ and $\langle \mu_5 \rangle$ that we compare to a series of laminar and turbulent three-dimensional direct numerical simulations. We find that the late-time evolution of $\langle H \rangle$ depends on the magnetic and kinetic Reynolds numbers $Re_M$ and $Re_K$. For a high $Re_M$ and $Re_K$ where turbulence occurs, $\langle H \rangle$ eventually evolves in the same way as in classical ideal MHD where the inverse correlation length of the helical magnetic field scales with time $t$ as $k_0 \propto t^{-2/3}$. For a low Reynolds numbers where the velocity field is negligible, the scaling is changed to $k_0 \propto t^{-1/2}\ln\left(t/t_{\text{eq}}\right)$. After being rapidly generated, $\langle \mu_5 \rangle$ always decays together with $k_0$, i.e. $\langle \mu_5 \rangle \approx k_0$, with a time evolution that depends on whether the system is in the limit of low or high Reynolds numbers.

I. INTRODUCTION

Natural systems can be modeled as fluids when their macroscopic spatial extension is much larger than the typical mean free path of particle collisions. This often applies in geophysics and astrophysics including planets, stars, the interstellar medium, and galaxies. Hydrodynamics is even applicable in cosmology, in particular, when modeling the plasma of the early Universe and the early stages of cosmic structure formation. Often, natural fluids are highly turbulent which is quantified by large hydrodynamic Reynolds numbers $Re_K$. The latter measures the ratio of advection and diffusion effects and is defined as $Re_K = u_{\text{rms}}/(k_0\nu)$ where $u_{\text{rms}}$ is the rms velocity, $k_0$ is the forcing wave number, and $\nu$ is the viscosity. Indeed, $Re_K \gg 1$ in many astrophysical fluids resulting from an efficient turbulent driving that has its origin, for instance in galaxies, in supernova explosions [1] and/or accretion flows [2]. Moreover, the local Universe appears to be permeated with magnetic fields on all length scales accessible to observations: They span from planets [3] and stars [4], including our Sun [5], to the interstellar medium [6], galaxies [7], up to galaxy clusters [8] and possibly cosmic voids [9]. The most established mechanisms to amplify and maintain magnetic fields in stars and galaxies are magnetohydrodynamical (MHD) dynamos that convert kinetic energy into magnetic energy [10]. In general, MHD turbulence describes the dynamics of many astrophysical and cosmological flows.

Without an energy input, turbulence and magnetic fields decay freely which can be of interest in various astrophysical applications. One such example of non-driven MHD turbulence is the evolution of magnetic fields in the very early Universe before the epoch of recombination. Scenarios of primordial magnetogenesis include specific models for inflation [e.g. 11] and the cosmological phase transitions [e.g. 12], but the subsequent evolution of primordial magnetic fields is governed by the laws of decaying MHD turbulence [67], [see e.g. 13, for a review]. Whether primordial magnetic fields can survive until they might e.g. serve as seed fields for galactic dynamos or remain as significant relics in present-day cosmic voids, depends on the prospect of transferring magnetic energy from small to large spatial scales. Such a transfer of magnetic energy is known as inverse cascade and is well studied within MHD turbulence [14, 15].

The scaling laws of the inverse cascade depend crucially on the magnetic helicity $\int_V \mathbf{A} \cdot \mathbf{B} \, dV$ which is a topological property of the magnetic field. Here, the volume integral is taken over the product of the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Since $\int_V \mathbf{A} \cdot \mathbf{B} \, dV$ is a conserved quantity in ideal MHD, a decay of the magnetic field necessarily results into an increase of its correlation length $\ell$. Using a phenomenological ap-
proach, similar to the Kolmogorov theory of nonmagnetized turbulence, magnetic helicity conservation implies scalings with time $t \propto t^{2/3}$ and $B \propto t^{-1/3}$ [16, 17]. It has been demonstrated in three-dimensional numerical simulations that inverse transfer of energy occurs even in nonhelical decaying MHD turbulence, however with less efficiency [18, 19]. The scaling laws of decaying MHD turbulence, estimated from phenomenological arguments or extracted from numerical simulations, have been applied, for instance, to the evolution of primordial magnetic fields from their generation until recombination [20, 21].

The assumption of conserved magnetic helicity, however, breaks down when charged fermions can be considered as being massless. This is the case at high energies that are reached in the very early Universe. In fact, above approximately 10 MeV [22], MHD necessarily needs to be extended to chiral MHD where only the sum of magnetic helicity and fermionic chirality is conserved. Fermionic chirality is the divergence of the chiral current $j_5$ which is not conserved due to the chiral anomaly, a pure quantum effect with macroscopic consequences. It can be quantified by the chiral chemical potential $\mu_5 \equiv \mu_L - \mu_R$ with $\mu_L$ and $\mu_R$ are the chemical potentials of right- and left-handed fermions, respectively. A non-zero $\mu_5$ in the presence of a magnetic field, leads to the chiral magnetic effect (CME) [23] which is a macroscopic quantum effect within the standard model of particle physics and implies an additional electric current along a magnetic field [24]. The CME leads to a magnetic field instability [25] which is the subject of many studies [26–42]. Recently, the nonlinear dynamics of a chiral plasma was studied also in direct numerical simulations (DNS) with a focus on chiral dynamos [43–45]. The energies necessary for chiral effects are reached in the early Universe and in proto-neutron stars, but also in heavy ion collisions [46]. Furthermore, chiral MHD is relevant for modeling the dynamics of electronic solid-state materials like Weyl semimetals [47].

The extension of MHD to chiral MHD raises the following questions: How is the inverse cascade affected by the new degree of freedom, the chiral chemical potential? How much chiral asymmetry can be generated from an initial helical magnetic field? This scenario has been explored by Hirono et al. [48] who considered a plasma composed of charged fermions with initial magnetic helicity and vanishing chiral asymmetry. Within their assumption of negligible velocity fields, they have identified a three-stage evolution: First, the magnetic helicity is transferred to fermionic chirality due to the conservation law of chiral MHD. Second, the total helicity is dominated by fermionic chirality which eventually leads to a CME-assisted inverse cascade of magnetic helicity. And third, at late times, Hirono et al. [48] report a self-similar evolution of $\mu_5$ and the peak of the magnetic energy spectrum proportional to $t^{-1/2}$. This self-similar evolution during the decay of a large $\mu_5$ has been observed in lattice simulations [42, 49]. A remaining open question is, however, how much an evolution of chiral MHD is modified in presence of turbulence, where there can be a strong coupling between the magnetic field and the velocity field. Understanding decaying chiral helical MHD turbulence and its differences to the classical MHD scenario is the goal of the present study.

To this end, we investigate the evolution of the magnetic field $B$, the velocity field $U$ and the chiral chemical potential $\mu_5$ in both the laminar and the turbulent regime. Our initial conditions are a vanishing chiral asymmetry and velocity field, $U = \mu_5 = 0$ and a maximally helical magnetic field. These initial conditions are realized, for instance, in various inflationary magnetogenesis models where a pseudo-scalar field generates magnetic fields in a parity violating manner [50–56], but does not introduce a chemical potential for fermions. Note that if the fermion masses are negligible, not only the helical magnetic field but also the chiral asymmetry can be generated during inflation such that the net helicity plus chirality is conserved precisely due to the chiral anomaly [57]. This alternative initial condition is beyond the scope of the present study but a target of our future work.

The paper is structured as follows. In Section II we briefly review the inverse cascade in classical MHD and the system of equations in chiral MHD and introduce our numerical methods. In Section III we present an analytical derivation of the self-similar inverse cascade in chiral MHD with a vanishing velocity field and confirm the validity of our analytical results with three-dimensional numerical simulations. The transition from a system with vanishing velocity field to a regime where turbulence is driven efficiently via the Lorentz force exerted by the helical magnetic field, is presented in Section IV. For the limit of large Reynolds numbers, we use a phenomenological approach to find solutions for the evolution of $\mu_5$. The analytical solutions are compared to results from turbulent DNS. We draw our conclusions in Section V.

II. THEORETICAL BACKGROUND AND METHODS

A. Review of the classical inverse cascade

The dynamics of magnetized fluids in the one-fluid magnetohydrodynamical limit is described by the following set of equations:

$$\frac{\partial B}{\partial t} = \nabla \times [U \times B - \eta (\nabla \times B)],$$  \hspace{1cm} (1)

$$\frac{\rho D U}{D t} = (\nabla \times B) \times B - \nabla p + \nabla (2 \nu S),$$  \hspace{1cm} (2)

$$\frac{D \rho}{D t} = -\rho \nabla \cdot U.$$  \hspace{1cm} (3)

Here, $B$ is the magnetic field, $t$ is time, $U$ is the velocity field, and $\rho$ is the mass density. Furthermore, $p$ is the hydrodynamic pressure, $S_{ij} = 1/2(U_{i,j} + U_{j,i}) - 1/3 \delta_{ij} \nabla \cdot U$.
are the components of the tracefree strain tensor $S$, where commas denote partial spatial derivatives, and $D/DT = \partial/\partial t + U \cdot \nabla$ is the advective derivative. The Ohmic resistivity is denoted by $\eta$ and $\nu$ is the viscosity. The set of equations is closed by an isothermal equation of state, meaning that the pressure is related to the density via $p = c_s^2 \rho$, where $c_s$ is the sound speed.

An important role for the evolution of magnetic fields is played by magnetic helicity which is defined as $\int_V \mathcal{H} \, dV$ with $\mathcal{H} \equiv A \cdot B$. The integral is taken over a periodic volume $V$ or over an unbounded volume with the fields falling off sufficiently rapidly at spatial infinity so that a boundary term can be neglected. In these cases, magnetic helicity is gauge invariant. Its evolution equation can be derived by multiplying Faraday’s law with its uncurled version for the vector potential and yields

$$\frac{d}{dt} \int_V \mathcal{H} \, dV = -2\eta \int_V J \cdot B \, dV.$$  

(4)

A remarkable consequence of magnetic helicity conservation at $\eta \to 0$ (faster than the current helicity $\int_V J \cdot B \, dV$ may possibly diverge) is the inverse cascade of energy for a fully helical magnetic field [14, 58].

The highly nonlinear evolution of helical decaying MHD turbulence has been studied intensely with DNS. For incompressible 3D magnetohydrodynamic turbulence at relatively high $Re_m$, the energy decay [16] as well as scaling relations of the energy power spectrum have been analyzed [17]. The role of magnetic helicity in the inverse cascade was investigated by Christensson et al. [15]. With their DNS, Christensson et al. [15] found evidence for a self-similar evolution of magnetic energy spectrum with a development of a power law of roughly $k^{-2.5}$ beyond the peak and analyzed decay laws for both the kinematic and magnetic energy. The scaling relations of a helical magnetic field can, in the limit of high Reynolds numbers, be derived by using a Kolmogorov-type phenomenological approach [see e.g. 59]. In particular, the magnetic energy evolves as $\langle B^2 \rangle / 2 \propto t^{-2/3}$ and the correlation length of the magnetic field as $\xi \propto t^{2/3}$ which has been confirmed by DNS [e.g. 20]. An inverse transfer of magnetic energy has also been found for nonhelical magnetic fields, however, it is less efficient than in the fully helical case [18–20, 60, 61].

### B. Chiral MHD equations

At high energies, MHD necessarily needs to be generalized to chiral MHD in which the chiral asymmetry appears as a new degree of freedom. Here, an asymmetry between the number densities of left- and right-handed fermions gives rise to the CME that results in an electric current proportional to the magnetic field and a chiral chemical potential

$$\mu_5^{\text{phys}} \equiv \mu_L - \mu_R,$$  

(5)

where $\mu_L$ and $\mu_R$ are the chemical potentials of left- and right-handed fermions, respectively. In the following, we will replace $\mu_5^{\text{phys}}$ by a normalized version, $\mu_5 \equiv (4\alpha_{\text{em}}/\hbar c)\mu_5^{\text{phys}}$, that has the same units as a wave number.

Chiral asymmetry is coupled to magnetic helicity and significantly modifies the phenomenology of the plasma. The set of chiral MHD equations is given by [41]:

$$\frac{\partial B}{\partial t} = \nabla \times \left[ U \times B - \eta \left( \nabla \times B - \mu_5 B \right) \right],$$  

(6)

$$\frac{D\mu}{Dt} = (\nabla \times B) \times B - \nabla p + \nabla \cdot (\mu_5 V),$$  

(7)

$$D\mu_5^\nu = -D\mu_5^\nu \cdot U,$$  

(8)

$$D\mu_5^\nu = D_5 \Delta \mu_5 + \lambda \eta \left[ B \cdot (\nabla \times B) - \mu_5 B^2 \right].$$  

(9)

Here, the chiral vortical effect, the chiral separation effect, and chirality flipping are neglected. The latter is well justified at sufficiently high temperatures while the former might not be a very good approximation in the case of high Reynolds number where large vortical velocities can be generated. The new equation (9) includes a diffusion term with the diffusion constant $D_5$ that is only relevant when $\mu_5$ is strongly varying in space. In this study, the evolution of $\mu_5$ will be mostly affected by the electromagnetic field via the second term on the right-hand side of Equation (9). The strength of the feedback is controlled by the chiral feedback parameter

$$\lambda = 3hc \left( \frac{8\alpha_{\text{em}}}{k_B T} \right)^2,$$  

(10)

which is valid for $k_B T \gg \max(|\mu_L|, |\mu_R|)$ [62]. In the following, $\lambda$ will be considered constant, yet one should keep in mind that it scales with temperature $T^{-2}$ [see, e.g. 22].

The system of Equations (6)–(9) implies a conservation law:

$$\frac{\partial}{\partial t} \left( \frac{\lambda}{2} \mu_5 + \mu_5 \right) + \nabla \cdot \left( \frac{\lambda}{2} (E \times A + B \Phi) - D_5 \nabla \mu_5 \right) = 0,$$  

(11)

where $E = -c^{-1} \left[ U \times B + \eta (\mu_5 B - \nabla \times B) \right]$ is the electric field and $\Phi$ is the electrostatic potential. Hence, the total chirality $\langle \mathcal{H} \rangle + 2\langle \mu_5 \rangle / \lambda$ is a conserved quantity, where $\langle \mu_5 \rangle$ is the mean value of the chiral chemical potential and $\langle \mathcal{H} \rangle \equiv V^{-1} \int A \cdot B \, dV$ is the mean magnetic helicity density in the volume $V$.

### C. Numerical methods

To go beyond the limitations of analytical calculations, we use the PENCIL CODE [68] for solving Equations (1)–(3) for classical MHD and Equations (6)–(9) for chiral
TABLE I: Summary of all runs presented in this paper. The reference runs for laminar (R1 and R1mhd) and turbulent (R8 and R8mhd) simulations which are presented in detail in Figures 1 and 10, respectively, are highlighted by bold font. The amplitude of the initial magnetic power spectrum is exactly the same for all runs R1–R7. Runs R7 and R7mhd have, however, a larger initial rms magnetic field strength $B_0$ which is due to the higher resolution, meaning the larger number of modes available in R7 (the exponentially suppressed tail extends to higher wave numbers). For runs R8, R8b, and R8mhd, a larger initial amplitude has been set.

| Name     | MHD resolution | Input parameters: $B_{rms,0}$ | $k_{p,0}$ | $\mu_5,0$ | $\frac{B_{rms,0}}{\eta}$ | $\lambda \frac{B^2_{rms,0}}{k_{p,0}}$ | Measured parameters: $R_{\eta}^{min}$ | $R_{M}^{k=1}$ | $R_{M}^{max}$ |
|----------|----------------|------------------------------|-----------|-----------|--------------------------|---------------------------------|-------------------|----------------|----------------|
| R1a      | chiral 320³    | 1.153                        | 85        | 0         | 11.53                    | 16.62                           | 7.2 x 10^{-6}       | 1.9 x 10^{-5}  | 9.6 x 10^{-4} |
| R1       | chiral 320³    | 1.153                        | 85        | 0         | 11.53                    | 16.618                          | 7.2 x 10^{-5}       | 1.4 x 10^{-1}  | 1.6 x 10^{-1} |
| R1b      | chiral 320³    | 1.153                        | 85        | 0         | 11.53                    | 166.176                         | 1.6 x 10^{-3}       | 2.3 x 10^{-1}  | 2.3 x 10^{-1} |
| R1mhd    | classic 320³   | 1.153                        | 85        | 0         | 11.53                    | 2.4 x 10^{-6}                  | 2.9 x 10^{-5}       | 1.0 x 10^{-3}  |                |
| R2       | chiral 320³    | 1.153                        | 85        | 0         | 23.06                    | 16.618                          | 1.2 x 10^{-4}       |                  |                |
| R3       | chiral 320³    | 1.153                        | 85        | 0         | 115.3                    | 16.618                          | 2.6 x 10^{-3}       | 3.4 x 10^{0}   |                |
| R4       | chiral 320³    | 1.153                        | 85        | 0         | 230.6                    | 16.618                          | 9.8 x 10^{-3}       | 7.1 x 10^{0}   |                |
| R5       | chiral 320³    | 1.153                        | 85        | 0         | 576.5                    | 16.618                          | 8.1 x 10^{-2}       |                  |                |
| R6       | chiral 320³    | 1.153                        | 85        | 0         | 1153.3                   | 16.618                          | 1.6 x 10^{0}        |                  |                |
| R7       | chiral 320³    | 1.153                        | 85        | 0         | 2800.0                   | 24.5                            | 1.4 x 10^{1}        |                  |                |
| R7mhd    | classic 512²   | 1.400                        | 85        | 0         | 2800.0                   |                                 | 8.3 x 10^{0}        |                  |                |
| R8       | chiral 512³    | 4.667                        | 85        | 0         | 9333.6                   | 24.501                          | 1.1 x 10^{2}        | 3.2 x 10^{2}   | 3.2 x 10^{2}   |
| R8b      | chiral 512³    | 4.667                        | 85        | 0         | 9333.6                   | 2450.140                       | 5.5 x 10^{1}        |                  |                |
| R8mhd    | classic 512³   | 4.667                        | 85        | 0         | 9333.6                   | 2.6 x 10^{2}                    | 2.7 x 10^{2}        |                  |                |

MHD, respectively. The system of equations is solved in a three-dimensional periodic domain of size $L^3 = (2\pi)^3$ via a third-order accurate time-stepping method of [63] and sixth-order explicit finite differences in space [64, 65]. The time step is specified as the Courant time step, however, for our reference runs we initially use very small manually set time steps to resolve the very early time evolution in more detail. After the initial phase, the automatic Courant time step is used in the reference runs. The resolution is varied between 320³ and 512³. The smallest wave number covered in the numerical domain is $k_1 = 2\pi/L = 1$ which we use as normalization of length scales. All velocities are normalized to the sound speed $c_s = 1$ and the mean fluid density to $\bar{\rho} = 1$. Time is normalized by the diffusion time $t_\eta = \eta^{-1}$, where $\eta$ is the Ohmic resistivity.

In this study, all runs are initialized with vanishing chiral chemical potential $\mu_5$ and a strong helical random magnetic field [69]. In practice, the magnetic field is set up via the vector potential $A(x)$ which is constructed from a random and $\delta$-correlated three-dimensional vector field in real space. The magnetic field is calculated from the Fourier transform of $A(x)$ via $B(k) = i k \times A(k)$. Then the magnetic field is scaled by functions of $k$ to adjust the shape of the magnetic energy spectrum $E_M(k) = 2\pi B^2(k)k^2$ for which we use a slope proportional to $k^4$ between 1 and the initial wave number of the energy-carrying eddies $k_{p,0} = 85$, i.e., the initial peak of the magnetic energy spectrum. For $k > k_{p,0}$ in all runs, the spectrum is suppressed. Multiplication by the operator $P_{ij}(k) = i \delta_{ij} k_i k_j$, where $P_{ij} = \delta_{ij} - k_i k_j$ is the projection operator, ensures a fully helical magnetic field for $\sigma_M = \pm 1$. Finally, the energy spectrum is normalized such that $\langle B^2 \rangle/2 = B_{rms}^2/2 = \int_0^{k_{max}} E_M(k) \, dk$, where the integration is performed over the entire numerical domain, i.e., from $k = 1$ up to the maximally resolved wave number $k_{max}$. We note that these initial conditions of the magnetic field are similar to the ones used in previous studies of decaying MHD turbulence, like in Brandenburg et al. [21].

No external forcing is applied to drive turbulence in our simulations, i.e., the velocity field is purely driven via the Lorentz force that is exerted on the flow through the magnetic field. The transition to a turbulent plasma occurs when the magnetic and kinetic Reynolds numbers, $R_{\eta} = u_{rms}/(k_{\eta} \eta)$ and $R_{M} = u_{rms}/(k_{\nu} \nu)$, respectively, become much larger than unity. Here, $u_{rms}$ is the rms velocity and $k_\eta$ is the wave number on which kinetic energy is injected in the system. For magnetically driven turbulence, $k_\eta$ corresponds to the inverse correlation length of the magnetic field and we will use $k_\eta = k_{p}(t)$. Viscosity $\nu$ and Ohmic resistivity $\eta$ are implemented explicitly in the code. To explore systems with different Reynolds numbers, the values of $\nu$ and $\eta$ are systematically changed, while their ratio, i.e., the magnetic Prandtl number $Pr_{M}$, is set to unity for all of the simulations. We note that the choice of $Pr_{M} = 1$ does not reflect the situation in most astrophysical applications. However, $Pr_{M} \ll 1$ or $Pr_{M} \gg 1$ are notoriously difficult to treat in DNS, since that requires a large separation of scales. In what follows, we therefore only mention the magnetic Reynolds number which in our settings equals the kinetic one.

An overview of the input parameters and characteristic numbers of all runs discussed in this work is presented in Table I.
III. INVERSE CASCADE IN CHIRAL MHD WITH A VANISHING VELOCITY FIELD

In this section, we discuss the evolution of a decaying helical magnetic field for simulations where the velocity field can be neglected throughout the entire simulation time. We note, however, that Equation (7) is nevertheless included in the DNS.

A. Analytical solutions in the limit of dynamically insignificant velocity fields

Provided that helical magnetic fields are nonzero but the chiral chemical potential is vanishing \( \mu_5(t_0) \equiv \mu_{5,0} = 0 \) at the initial time, the second term in the right-hand side in Equation (9) sources \( \mu_5 \). \( |\mu_5| \) may grow until it reaches the maximum value at which the second term and the third term cancel each other,

\[
|\mu_5(t)| \simeq \left| \frac{B \cdot (\nabla \times B)}{B^2} \right| \simeq k_p(t),
\]

(12)

where we ignore the diffusion term \( \mathcal{D}_5 \Delta \mu_5 \) in Equation (9). Due to the conservation law \( \langle \mathcal{H} \rangle \rightarrow 0 \) of magnetic helicity, we note, however, that Equation (7) is nevertheless included in the DNS.

\[
|\mu_5(t)| \lesssim \lambda \langle \mathcal{H}(t_0) \rangle \simeq \lambda B_0^2 / k_{p,0},
\]

(13)

where \( B_0 \) is the strength of the initial magnetic field with maximum energy on the wave number \( k_{p,0} \). Therefore, the chiral chemical potential generated from magnetic helicity is given by

\[
|\mu_5(t)| \simeq \min \left( k_p(t), \lambda B_0^2 / k_{p,0} \right).
\]

(14)

Since \( k_{p,0} > \lambda B_0^2 / k_{p,0} \) the system evolves in three phases: (i) Production of \( \langle \mu_5 \rangle \) at the expense of magnetic helicity \( \langle \mathcal{H} \rangle \), with an efficiency depending on \( \lambda \) up to \( \lambda B_0^2 / k_{p,0} \). (ii) Once \( k_p(t) \) has decayed to \( k_{p,0} \), a chiral dynamo, the so-called \( \nu_c^2 \) dynamo as discussed in [41], leads to an exponentially fast restoration of \( \langle \mathcal{H} \rangle \). (iii) A self-similar inverse cascade sets in with \( k_p \approx \mu_5 \), during which \( \langle \mu_5 \rangle \) is converted into \( \langle \mathcal{H} \rangle \) according to the conservation law.

In the other case with \( k_{p,0} < \lambda B_0^2 / k_{p,0} \), the growth of \( \langle \mu_5 \rangle \) stops when it becomes comparable to \( k_p(t) \) during the phase (i) and the system immediately enters the phase (iii) by skipping (ii). In this case, the magnetic helicity always dominates the conserved quantity and the evolution of magnetic fields is not significantly altered by \( \langle \mu_5 \rangle \).

As we show in Appendix A that during this self-similar evolution in the phase (iii), \( \mu_5 \) and \( k_p \) evolve as

\[
|\mu_5(t)| \approx k_p(t) \approx \left[ \frac{3 + n}{4n} \ln \left( \frac{t}{t_{\text{log}}} \right) \right]^{1/2}, \tag{15}
\]

(15)

where \( n \) denotes the slope of the initial magnetic helicity spectrum \( \langle \mathcal{H}(t_0) \rangle \propto k^n \). Our DNS are initiated with \( n = 4 \). We found that the logarithmic correction time is roughly given by \( t_{\text{log}} = (2k_{p,0}^2)^{-1} \tau_\mu \) when using explicit units. This result up to the logarithmic correction term has also been found in [48].

B. Comparison of classical MHD with a three-phase chiral MHD scenario in DNS

The reference run for a three phase scenario of a decaying magnetic field in chiral MHD is R1. In Figure 1, R1 is compared to a classical MHD analog (R1mhd). The parameters and initial conditions in both runs are the same but in R1 the plasma evolves according to Equations (6)–(9), in R1mhd according to Equations (1)–(3). R1 is presented in the right panels of Figure 1 and R1mhd is presented in the left panels.

The time evolution of the mean magnetic helicity, \( \langle \mathcal{H} \rangle \), and the wave number on which the magnetic energy spectrum has its maximum, \( k_p \), are presented in the top row of Figure 1. In the classical nonideal MHD case the magnetic helicity decreases by approximately eight orders of magnitude during one resistive time, i.e. until \( t \approx 1 \). Resistivity acts on small spatial scales, e.g. large wave numbers \( k \). This leads to a decrease of the magnetic energy on large \( k \) and therefore a move of the peak scale of the magnetic energy spectrum, \( k_p \), to smaller \( k \). Note, that \( k_p \) has discrete values only, leading to steps in its time evolution that become more evident at late times when \( k_p \) approaches 1. Since the velocity field is negligible during the entire run, there is no inverse transfer of magnetic energy, as can be seen in the evolution of the magnetic energy spectrum, see Figure 1c. The time evolution of \( k_p \), normalized to the theoretically predicted value for chiral MHD given in Equation (15), is presented in the middle row of Figure 1. For R1mhd, Equation (15) is not valid and therefore the orange dashed line in Figure 1b moves away from 1 with increasing time. The time evolution of \( \langle \mathcal{H} \rangle \) in R1 is significantly different from the one in classical MHD, as can be seen in Figure 1d. First, \( \langle \mathcal{H} \rangle \) decreases by roughly two orders of magnitude. At the same time, a mean chiral magnetic potential \( \langle \mu_5 \rangle \) is generated, such that the sum \( \langle \mathcal{H} \rangle + 2 \langle \mu_5 \rangle / \lambda \) is conserved during the entire run. The value of \( k_p \) decreases in time in R1, but not as quickly as in R1mhd. The three phases described in Section III A can be clearly distinguished in Figure 1d: Phase (i) during which \( \langle \mathcal{H} \rangle \)
For chiral MHD, we also show the mean chiral chemical potential \( \langle \mu_5 \rangle \) over the chiral feedback parameter \( \lambda \) as well as the conserved quantity \( \langle H \rangle \) and \( 2 \langle \mu_5 \rangle / \lambda \). The time \( t \) is normalized to the resistive time \( t_\eta \). For chiral MHD, we also show the mean chiral chemical potential \( \langle \mu_5 \rangle \) over the chiral feedback parameter \( \lambda \) as well as the conserved quantity \( \langle H \rangle \) and \( 2 \langle \mu_5 \rangle / \lambda \). The solid vertical lines indicate the time when \( k_p \) has reached the minimum wave number possible in the numerical domain, \( k = 1 \), and the dotted vertical lines indicates the time at which the minimum of magnetic helicity is reached. Middle: \( k_p \) and \( \langle H \rangle \) normalized by their theoretically expected scaling in the self-similar evolution phase (see Section III A). Bottom: Evolution of the magnetic energy spectrum. The thick dotted black lines show the initial spectra and the thick solid black lines show the final spectra of the simulations.
decreases and which ends at $t \approx 4 \times 10^{-2}$ is followed by a phase of dynamo amplification, phase (ii). For $t \geq 4 \times 10^{-1}$, $(\mu_5)$ and $k_p$ evolve in a self-similar way, what was defined as phase (iii). During this phase, the evolution of $(\mu_5)$ and $k_p$ is reasonably well described by Equation (15), as can be seen in Figure 1e. For comparison with the scaling of $k_p \propto t^{-1/2}$, we have added the black dashed line in Figure 1e from which the simulation data clearly deviates in phase (iii). The time evolution of the magnetic energy spectrum in run R1 (right bottom panel) is very different from the one in R1mhd (Figure 1c). The main difference occurs at late times, where in R1mhd, the magnetic energy first grows on $k \approx 5$ and then moves to smaller wave numbers in an CME-assisted inverse cascade.

C. Dependence on the chiral feedback parameter $\lambda$

The generation of a chiral asymmetry and the subsequent evolution of the plasma depends strongly on the chiral feedback parameter $\lambda$ or, in dimensionless units, $\lambda B_0^2/k_{p,0}$ as compared to the value of $k_{p,0}$. As discussed in Section III A, for $\lambda B_0^2/k_{p,0} \ll k_{p,0}$, we expect a two-phase evolution, while for $\lambda B_0^2/k_{p,0} \gg k_{p,0}$, our models suggest a 2-phase scenario. In this section, we present a comparison of R1 with a run with a smaller value of $\lambda$, R1a, and a run with a larger value of $\lambda$, R1b.

The time evolution of $\mu_5$ in runs R1a, R1, and R1b is presented in Figure 2. The largest maximum value of $\mu_5 \approx k_{p,0} = 85$ is reached in run R1b, for which $k_{p,0} = 85 < \lambda B_0^2/k_{p,0} \approx 166.2$. This run, almost instantaneously, enters the self-similar phase where both $\mu_5$ and $k_p$ decay proportional to $t^{-1/2}$ $(\ln(t/t_{log}))^{1/2}$. This scaling is indicated as a grey line in Figure 2. During the entire run time of R1b, magnetic helicity is sourced by $\mu_5$ and, therefore, a CME-assisted inverse cascade of magnetic energy occurs.

The situation is different in the reference run R1, where $k_{p,0} = 85 > \lambda B_0^2/k_{p,0} \approx 16.6$. In R1, a value of $\mu_5 \approx 8$ is generated quickly and stays constant up to $t \approx 0.3$. At that time the value of $(H)$ is up to two orders of magnitude less than its initial value; see the time evolution of
evolves similar as in R1b. Initially, also $k_\mu$ decreases constantly at a rate that is only slightly larger than for R1a. At $t \approx 0.45$, the peak of the magnetic energy spectrum in R1a reaches the minimum wave number in the simulation domain, $k_p = 1$. A dynamo instability for $\mu_5 = 0.8$ would occur at $k_5 = \mu_5/2 = 0.4$. Since this is outside of the numerical domain, a dynamo and a subsequent CME-assisted inverse cascade is not seen in R1a. We stress, however, that this is purely caused by the finiteness of the numerical domain. For infinite systems, a three-phase scenario is expected for all high-energy plasmas with parameters such that $k_{p,0} > \lambda B_0^2/k_{p,0}$.

The time evolution of the peak scale of the magnetic energy spectrum $k_p$ in R1, R1a, R1b, and R1mhd is presented in Figure 4. Here, $k_p$ is normalized by the analytical solution in the self-similar phase, Equation (15). For R1b, $k_p / (7/(4t) \ln(t/t_{\log}))^{1/2}$ has a constant value of $\approx 0.9$ for $t \gtrsim 3 \times 10^{-3}$ which is equivalent to a few timesteps of the simulation. Hence, Equation (15) describes the evolution during the CME-assisted inverse cascade well. We show the direct comparison with the scaling $k_p \propto t^{-1/2}$ as the black dashed line in Figure 4. The deviation from the $k_p \propto t^{-1/2}$ scaling is clearly visible in our simulations once they have entered phase (iii) in which $k_p$ and $\langle H \rangle$ evolve self-similarly. This is strong evidence for the need of a logarithmic correction that emerges naturally in our analytic derivation that is given in the appendix.

For the classical MHD simulation, R1mhd, $k_p / (7/(4t) \ln(t/t_{\log}))^{1/2}$ is, at maximum, 0.5 for $t \approx 3 \times 10^{-3}$ and later decreases as $k_p \propto t^{-1/2}$. In R1a, which has the lowest chiral feedback parameter, $k_p / (7/(4t) \ln(t/t_{\log}))^{1/2}$ evolves very similar to the MHD case, R1mhd. Initially, also $k_p / (7/(4t) \ln(t/t_{\log}))^{1/2}$ in R1 evolves similar as in R1mhd. But at $t \approx 0.2$, the transition to phase (iii) occurs and $k_p / (7/(4t) \ln(t/t_{\log}))^{1/2}$ in R1 evolves similar as in R1b.

### IV. INVERSE CASCADE IN CHIRAL MHD WITH TURBULENCE

In this section we explore the transition from laminar to turbulent flows. In particular, we are interested in how turbulence modifies the three-phase scenario of a decaying helical magnetic field in chiral MHD that was established in Section III. Therefore we run a series of simulations where the viscosity and Ohmic resistivity are systematically decreased. In the limit of large Reynolds numbers, analytical estimates can be compared to the results from DNS.

#### A. Reynolds numbers in DNS of decaying (chiral) MHD turbulence

During decaying (chiral) MHD, the magnetic Reynolds number $Re_M$ is a function of time because (i) the decaying magnetic field drives a velocity field which changes in time and (ii) the characteristic wave number on which magnetic forcing occurs corresponds to the correlation length of the magnetic field. The latter increases in time due to the inverse cascade which occurs when the magnetic field is helical. In the following, we approximate the correlation length of the magnetic field by the scale at which the magnetic energy spectrum reaches its maximum, $k_p$, and define the time-dependent magnetic Reynolds number as

$$Re_M(t) = \frac{u_{rms}(t)}{k_p(t)\eta}. \quad (16)$$

The time evolution of $Re_M$ in the majority of simulations (all except R1a, R1b, and R8b) from this study is shown in Figure 5. Especially for the DNS with high diffusion, $Re_M$ changes significantly during the simulation time. In our reference run for chiral MHD with a vanishing velocity field, R1, $Re_M$ decreases from a value of $Re_M \approx 10^{-3}$ at the beginning to $Re_M \approx 10^{-5}$ at $t \approx 0.1$, and then increases again, reaching $Re_M \approx 10^{-1}$ at the final time of the simulation $t \approx 4$. The time dependence in runs where $Re_M$ is larger than unity in the beginning are less dramatic. In the most turbulent run, R8, the magnetic Reynolds number increases only by a factor of approximately 10.

To distinguish the level of turbulence in different simulations, we may use the maximum Reynolds number, respectively, which is given as

$$Re_M^{max} = \max\left(\frac{u_{rms}(t)}{k_p(t)\eta}\right). \quad (17)$$

However, $Re_M^{max}$ is not a useful characteristic of a simulation because it depends very much on the time at which the simulation is stopped. A more consistent way of comparing different simulations is by using Reynolds numbers that are defined at characteristic times during the evolution. In the following, we will use the value of the $Re_M$ at
FIG. 5: Magnetic Reynolds number $Re_M$ as a function of time. The colors refer to different simulations of classical and chiral MHD simulations as given in the legend; see Table I for details. In the same colors as the lines showing $Re_M$, we indicate as vertical dotted lines the time at which the minimal magnetic helicity is reached and as vertical solid lines the time when the peak of the magnetic energy spectrum $k_p$ has reached the minimal wave number possible in our numerical domain, $k_p = 1$, respectively. Arrows on the vertical lines indicate that these are characteristic times for classical MHD runs.

The time $t_{k_p=1}$ at which the peak of the magnetic energy spectrum reaches the minimum wave number within the numerical domain:

$$Re_M^{k_p=1} = Re_M(t = t_{k_p=1}). \quad (18)$$

Additionally, we will consider the Reynolds number at the time $t_{min(H)}$ when the magnetic helicity reaches its minimum:

$$Re_M^{min(H)} = Re_M(t = t_{min(H)}). \quad (19)$$

The values of $Re_M^{max}$, $Re_M^{k_p=1}$, and $Re_M^{min(H)}$ for all DNS presented in this work are listed in the last three columns of Table I.

**B. Transition from low to high $Re_M$ in DNS of decaying (chiral) MHD turbulence**

Starting from our reference run of chiral MHD with negligible velocity field, R1, we systematically decrease the values of $\eta$ and $\nu$ in runs R2–R8 in order to explore the transition to the turbulent regime where the velocity field is expected to impact significantly the evolution of the magnetic field. As a characteristic parameter for the degree of nonlinearity in the Equations (6)–(9), we list the ratio of the initial magnetic field strength $B_0$ over $\eta$ for all DNS in the 7th column of Table I. With $B_0$ being the Alfvén velocity within our unit system, $B_0/\eta$ can be considered as the initial Alfvénic Reynolds number. The time evolution of the $Re_M$ for runs R2–R8 is presented in Figure 5. The value of $Re_M$ is larger than 1 in the entire simulation time of R6, R7, and R8. In the latter run, the simulation reaches $Re_M \approx 270$.

When increasing the Reynolds number in the simulations, we observe two trends. First, the maximal ratio of magnetic over kinetic energy density decreases when $\eta$ and $\nu$ are decreased [70], see Figure 6. Second, the scaling of the peak scale of the magnetic energy with time changes from $k_p \propto (7/(4t) \log(t/t_0))^{1/2}$ at late times to $k_p \propto t^{-2/3}$, see Figure 7.

The changes in the plasma evolution at different
Reynolds numbers is also clearly visible in Figure 8. Different measured characteristics of the simulations are presented here as a function of ReM. According to our discussion in Section IV A, for each simulation these parameters are plotted as a function of ReM max (open dots), ReM k=1 (filled dots), and ReM minH (stars). In Figure 8a, the maximum value of 2<μ5>/λ over the conserved total chirality <H> + 2<μ5>/λ is presented. While 2<μ5>/λ is almost 100 percent of the total chirality at one time of the plasma evolution for ReM < 1, the maximum of the ratio (2<μ5>/λ) / ((<H> + 2<μ5>/λ) drops to about 1/2 once ReM becomes significantly larger than 1. The mean magnetic helicity, on the other hand, always dominates the total chirality at one point in time for all simulations regardless of their degree of turbulence as expected from the choice of your initial conditions. In simulations with ReM > 1, the maximum value of 2<μ5>/λ is never
reached, as is shown in Figure 8b. The difference between the logarithm of the maximum and the minimum of \(\langle H \rangle\), see Figure 9a, and also the time needed to reach the minimum of \(\langle H \rangle\) drops for \(\text{Re}_m > 1\), see Figure 9b. Overall, we observe a decrease of the maximal ratio of magnetic over kinetic energy in our simulations with increasing magnetic Reynolds number. This ratio is plotted in Figure 9c and decreases continuously with decreasing Ohmic resistivity and not suddenly at the transition \(\text{Re}_m \approx 1\).

C. Analytical estimates for the limit of large \(\text{Re}_m\)

Let us estimate analytically the time evolution of \(\mu_5\) for high magnetic Reynolds numbers. A high \(\text{Re}_m\) implies that the first term is more important than the second term in the right-hand side of Equation (6). In the case of the vanishing chiral chemical potential at the initial time, \(\mu_{5,0} = 0\), as we have seen in the previous section, the third term never overwhelms the second term in Equation (6), because \(\mu_5(t)\) would decay for \(k_p \ll \mu_5\). Therefore, the evolution of the magnetic field is governed by the interaction to the fluid velocity \(U\) in the same way as the (nonchiral) classical ideal MHD, and the magnetic fields undergo the classical inverse cascade.

As a simple model of the classical inverse cascade [see, e.g. 13], we consider the following behaviors,

\[
k_p(t) = \begin{cases} k_{p,0} & (t \leq t_1) \\ k_{p,0}(t/t_1)^{-2/3} & (t_1 < t \leq t_2) \\ 0 & (t > t_2) \end{cases}, \tag{20}
\]

\[
B(t) = \begin{cases} B_0 & (t \leq t_1) \\ B_0(t/t_1)^{-1/3} & (t_1 < t \leq t_2) \end{cases}, \tag{21}
\]

where \(t_1\) denotes the onset time of the classical inverse cascade. Inserting this into Equation (9) and solving for \(\mu_5\), we obtain

\[
\mu_5(t \leq t_1) = k_{p,0} \left[1 - e^{-t/t_1}\right], \tag{22}
\]

\[
\mu_5(t > t_1) = k_{p,0} \left[-3\xi(t/t_1)^{-1/3} + e^{-3\xi(t/t_1)^{1/3}} \left\{ e^\xi(1 + 3\xi) - 1 \right\} e^{3\xi} + 9\xi^2 \left( \text{Ei}(3\xi(t/t_1)^{1/3}) - \text{Ei}(3\xi) \right) \right], \tag{23}
\]

where \(\xi \equiv \lambda_B B_0^2 t_1\) and \(\text{Ei}(x)\) is the exponential integral function. It is interesting to consider the late time limit of this solution,

\[
\mu_5(t \gg \xi^{-3}t_1) \approx k_{p,0}(t/t_1)^{-2/3}, \tag{24}
\]

which exactly matches \(k_p(t)\) in Equation (20). Although \(\mu_5(t) \approx k_p(t)\) holds at late times irrespective of \(\text{Re}_m\), their time evolution Equations (15) and (24) are quite different. Note that since the sudden change of the behaviors in Equations (20) and (21) at \(t = t_1\) are crude approximations, we anticipate a slight deviation between the analytic estimates of \(\mu_5(t)\) and the DNS results there.

D. Simulations of chiral helical MHD turbulence

In Figure 10, the run with lowest diffusion, hence highest \(\text{Re}_m\), \(R8\) (R8, left panels) is compared to a classical MHD analogue (R8mhd, right panels). The analysis is exactly the same as in Figure 1, except for the addition of the kinetic energy spectra in the last row of Figure 10.

Due to the small but finite value of the resistivity in R8mhd, \(\langle H \rangle\) decays by roughly a factor of three over the entire simulation time. Nevertheless, the magnetic helicity, in combination with turbulence leads to an efficient inverse cascade in energy which can be seen in the evolution of the magnetic energy spectrum in Figure 10c. The scaling of \(k_p\) proportional to \(t^{-2/3}\) as expected for the turbulent inverse cascade of helical MHD turbulence, sets in at time \(t \approx 10^{-5}\). This coincides roughly with the minimum of the eddy turn over time in the simulation, hence we will use

\[
t_1 \approx t_{\text{min}}^{\text{eddly}} \equiv \min \left( \frac{1}{k_p(t)u_{\text{rms}}(t)} \right).
\tag{25}
\]

The time evolution of \(k_p\) normalized by \(k_{p,0}(t/t_{\text{min}}^{\text{eddly}})^{-2/3}\) is presented in Figure 10b. The scaling with \(t^{-2/3}\) is observed in our DNS for times later than approximately \(t_{\text{min}}^{\text{eddly}}\). The value of \(k_p\) reaches the minimum value of the box after \(t \lesssim 0.2\).

The time evolution and energy spectra of the chiral MHD run with highest magnetic Reynolds number, R8, that are presented in the right panels of Figure 10 are very similar to the ones in the classical MHD run R8mhd. Up \(t \approx 10^{-3}\), \(k_p\), \(\langle H \rangle\), and \(t_{\text{eddly}}\) evolve identically in turbulent MHD and turbulent chiral MHD. However, in R8 a \(\langle \mu_5 \rangle\) is generated and restores a small amount of \(\langle H \rangle\). The energy spectra in the classical MHD run R8mhd and R8 are indistinguishable [71]. As expected from Equation (22), \(\langle \mu_5 \rangle\) increases linearly in time in the beginning. However the scaling proportional to \(t^{-2/3}\) as expected for late times according to Equation (24) is not observed in R8. This is caused by the fact that the peak of the magnetic energy spectrum has moved to the minimum wave number, \(k_p = 1\), before the scaling of \(\langle \mu_5 \rangle\) could converge to the one of \(k_p\).

To test the late time scaling of \(\langle \mu_5 \rangle \propto t^{-2/3}\) we have repeated run R8 with a larger value of \(\lambda\). For larger \(\lambda\), the condition \(t \gg \xi^{-3}t_1\) is fulfilled while the inverse cascade still proceeds within the numerical domain. Run R8b has a value of \(\lambda\) that is \(10^2\) times larger than the one in R8. We compare these two runs with the MHD analog, R8mhd, in Figure 11. The time evolution of \(k_p\) in all three runs is almost identical, reaching a scaling of \(k_p \propto t^{-2/3}\) at \(t \gtrsim 10^{-5}\). Except for the time around the onset of the inverse cascade \(t_1\), the time evolution of \(\langle \mu_5 \rangle\) measured in DNS (solid lines) agrees very well with the theoretically predicted curves (dotted lines). However, the value in DNS is approximately larger by a factor of 1.5 compared to the result from Equation (22) at early times. This behavior might be corrected when including
FIG. 10: Comparing runs with high Reynolds numbers: Classical MHD (left panels, run R8mhd) and chiral MHD (right panels, run R8). Top: Time evolution of the mean magnetic helicity \( \langle H \rangle \) and the wave number at which the magnetic energy spectrum has its maximum \( k_p \). For chiral MHD, we also show \( \langle \mu_5 \rangle / \lambda \) as well as the conserved quantity \( \langle H \rangle + 2 \langle \mu_5 \rangle / \lambda \). The solid vertical lines indicate the time when \( k_p \) has reached its minimum, \( k = 1 \), and the dotted vertical lines indicate the time at which \( \langle H \rangle \) is minimal. 2nd row: \( k_p \) and \( \langle \mu_5 \rangle \) normalized by the theoretically expected scaling during self-similar evolution (see Section III A). 3rd row: Evolution of the magnetic energy spectrum. The thick dotted black lines show the initial spectra and the thick solid black lines show the final spectra of the simulations. Bottom: The same but for the kinetic energy spectrum.
the exact shape of the initial magnetic energy spectrum. The evolution of $\langle \mu_5 \rangle$ in R8b is very well described by Equation (23) for $t \gtrsim 10^{-4}$.

V. CONCLUSION

In this study we have explored the evolution of a decaying fully helical magnetic field in a high-energy plasma in which the chiral magnetic effect can occur. The chiral magnetic effect is a macroscopic quantum effect that describes the emergence of an electric current in the presence of a chiral asymmetry, e.g., a difference between the chemical potential of left- and right-handed fermions, $\mu_5$, and a magnetic field. The dynamics of such a plasma is determined by the system of equations of chiral MHD. We have investigated how a magnetic field decays in chiral MHD and how a chiral asymmetry develops, from vanishing initial $\mu_5$.

When the velocity field plays no dynamically important role, we have identified a three phase evolution: (i) a mean chiral chemical potential $\langle \mu_5 \rangle$ is produced at the expense of the mean magnetic helicity $\langle \mathcal{H} \rangle$, (ii) once $\langle \mu_5 \rangle$ exceeds the inverse correlation length of the helical magnetic field $k_p$, a chiral dynamo reestablishes $\langle \mathcal{H} \rangle$, and (iii) a self-similar inverse cascade where $\langle \mu_5(t) \rangle \approx k_p(t) \approx (3 + n)/(4\eta t) \ln(t/t_0)_{\text{log}}^{1/2}$, where $n$ is the slope of the initial magnetic helicity spectrum. A similar scenario has been reported by Hiroto et al. [48]. However, they did not include the logarithmic correction in phase (iii) that agrees well with our DNS.

Our simulations, performed with the PENCIL CODE that has explicit viscosity and Ohmic resistivity, allow us to systematically explore a decaying magnetic field in chiral MHD and, in particular, to probe the transition from low to high magnetic Reynolds numbers. We find that the magnetic energy spectrum evolves more and more similar in chiral MHD and classical MHD when $\text{Re}_M$ is increased by lowering the dissipation in DNS. For $\text{Re}_M \gg 1$, we observe an inverse cascade of the magnetic field, where the peak of the magnetic energy spectrum $k_p \propto t^{-2/3}$, regardless of the existence of an additional degree of freedom in from of a $\mu_5$. In the simulations of chiral MHD, a $\mu_5$ is initially generated linearly in time. For $t \gg \xi^{-3}t_1$ with $\xi \equiv \lambda \eta B_0^2 t_1$ we find a scaling of $\mu_5 \approx k_p(t)^{-2/3}$, i.e., like for low Reynolds number, the evolution of $\mu_5$ follows the one of $k_p$, $k_p \approx \mu_5$.

We have shown that the chiral magnetic effect not only modifies the inverse cascade of magnetic fields but it also leads to the generation of a chiral anomaly. Such an anomaly may manifest itself in the neutrino density in the late Universe and it might be relevant, e.g., for the the number of effective degrees of freedom, $N_{\text{eff}}$, at recombination. A study of this possibility is referred to future work. Our results may also have other important implications for the evolution of relativistic plasmas in the early Universe and protoneutron stars.

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Appendix A: Derivation of the self-similar evolution in the limit of a vanishing velocity field

Here we shall derive the self-similar solution of $\mu_5(t)$ and $k_p(t)$ in the phase (iii), Equation (15). Ignoring the velocity field in Equation (6), the equation of motion for
magnetic field reads
\[ \partial_t \mathbf{B} = \eta \nabla \times ( - \nabla \times \mathbf{B} + \mu_5 \mathbf{B}) . \]  
(A1)
The Fourier transformation of the magnetic field is written as
\[ \mathbf{B}(t, x) = \sum_{\lambda = \pm} \int \frac{d^3k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}} e_\lambda(k) \mathcal{H}_\lambda(t, k), \]  
(A2)
where \( \lambda = \pm \) is the label of the circular polarization and \( e_\lambda(k) \) is the circular polarization vector that satisfies \( i \mathbf{k} \times e_\pm(k) = \pm k e_\pm(k) \). Using the relation between the magnetic helicity and \( B_\perp \) in Fourier space, \( \mathcal{H}_k \equiv (B_2^2 - B_3^2)/k \), one can recast Equation (A1) into the equation of motion for the magnetic helicity,
\[ \partial_t \mathcal{H}_k + 2\eta k^2 \mathcal{H}_k - 4\eta \mu_5 \rho_B(k) = 0, \]  
(A3)
where the spatial fluctuations of \( \mu_5 \) are neglected (i.e. \( \mu_5 \rightarrow \langle \mu_5 \rangle \)) and \( \rho_B(k) \equiv (B_2^2 + B_3^2)/2 \) is the magnetic energy density in Fourier space. When the magnetic field is maximally helical, \( B_2^2 \gg B_3^2 \), \( \rho_B(k) \simeq \sigma_H k \mathcal{H}_k/2 \), where \( \sigma_H \equiv \text{sign}(|\mathcal{H}_k|) \) ensures \( \rho_B \) is always positive. In this case, the above equation is reduced into
\[ \partial_t \mathcal{H}_k + 2k\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k = 0, \]  
(maximal helical) \( \) (A4)
where we used \( \sigma_H \mu_5 = |\mu_5| \) because the magnetic helicity and \( \mu_5 \) have the same sign, if \( \mu_5 \) is produced by the magnetic helicity as we assume. This equation has the formal analytic solution
\[ \mathcal{H}_k(t) = \mathcal{H}_k(t_0) \exp[2k\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k + \theta(t)] \]  
with \( \theta(t) \equiv \int_{t_0}^t dt' |\mu_5(t')| \). In the phase (iii) the magnetic helicity dominates the conserved quantity and hence \( \mathcal{H}(\gg 2\mu_5/\lambda) \) is independently conserved
\[ \int dk k^2 \mathcal{H}_k(t_0) \exp[2k\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k + \theta(t)]] = \text{const}. \]  
(A6)
Taking the time derivative of this equation and dropping time dependent but nonvanishing factors, one finds
\[ \int_{k_\star}^{k_\star} dk k^{3+n} [\mu_5(t) - k] e^{-2\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k - \mu_5(t)/\lambda} = 0. \]  
(A7)
Here we assume a power-law helicity slope with an UV-cutoff at \( k_\star \)
\[ \mathcal{H}_k(t_0) = \mathcal{H}_{k_\star}(t_0) \left( \frac{k}{k_\star} \right)^n \Theta(k_\star - k), \]  
(A8)
where \( \Theta(x) \) is the Heaviside function. Changing the dummy variable from \( k \) into \( p \equiv \sqrt{2\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k} \), we obtain
\[ |\mu_5(t)| = \frac{1}{\sqrt{2\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k}} \int_0^p dp p^{3+n} e^{-\left(p - \sqrt{2\eta(\mathcal{H}_k - |\mu_5|) / \mathcal{H}_k} \right)^2} \]  
with \( p_* \equiv \sqrt{2\eta(t - t_0)} k_\star. \)
To simplify this expression, we make an additional approximation. For this, we restrict ourselves into a late time regime, \( t \gg t_0 \). Then the upper limit of the integrals \( p_\star \propto t^{1/2} \) can be approximated by \( \infty \), and \( \sqrt{\eta/2t} \theta \) is also considered as large, as we will confirm a posteriori. The integrals are computed for \( X \equiv \sqrt{\eta/(2(t - t_0))} \theta(t) \gg 1 \) as
\[ \int_0^\infty dp p^{2+n} \exp[-(p - X)^2] = X + \frac{3 + n}{X} + \mathcal{O}(X^{-3}). \]  
(A10)
Therefore Equation (A9) is simplified to
\[ \theta'(t) = |\mu_5(t)| \approx \frac{\theta(t)}{2t} + \frac{3 + n}{2\eta \theta(t)}. \]  
(A11)
Note that in order for \( \mathcal{H} \) not to develop an infrared singularity we must require \( n > -3 \) so that both terms in (A11) are always positive. The solution of this differential equation is
\[ \theta(t) \approx \sqrt{\frac{t}{\eta}} \sqrt{C + (3 + n) \ln \left( \frac{t}{t_c} \right)}, \]  
(A12)
\[ \mu_5(t) \approx \frac{1}{2\sqrt{\eta t}} \frac{3 + n + C + (3 + n) \ln(t/t_c)}{\sqrt{C + (3 + n) \ln(t/t_c)}}, \]  
(A13)
where \( C \) is an integration constant and \( t_c \) is degenerate with \( C \). The approximation used above, \( X = \sqrt{\eta/2t} \theta \gg 1 \), is valid for a sufficiently late time,
\[ \sqrt{\frac{\eta}{t}} \theta(t) \approx \sqrt{C + (3 + n) \ln \left( \frac{t}{t_c} \right)} \gg 1. \]  
(A14)
This also allows us to further simplify \( \mu_5 \) as
\[ \mu_5 \simeq \left[ \frac{3 + n}{4\eta t} \ln \left( \frac{t}{t_{\log}} \right) \right]^{1/2}, \]  
(A15)
where the integration constant is rewritten as \( C = (3 + n) \ln(t_c/t_{\log}) \), assuming \( n \neq -3 \). Note that this logarithmic correction which slightly slows down the decay of \( \mu_5 \) becomes more significant as \( n \) increases. It implies that also the inverse cascade (i.e. the transportation to larger scales) of the peak scale \( k_\star \) is slowed-down. This is because it takes more time for a large-scale helicity modes to grow large enough to ensure the conservation law, when the initial helicity has a bluer spectrum, i.e. more power on smaller scales. If the initial helicity is scale invariant, \( n = -3 \), the logarithmic correction vanishes.
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It is worth noting that in a flat expanding Universe, the equations of relativistic MHD simplify to the nonrelativistic MHD equations if all quantities are replaced by co-moving quantities and instead of physical time conformal time is used [66].

The simulations were restarted several time and, since they cover many orders of magnitude in time, the frequency of writing spectra is reduced at every restart to save computing power. The gap appears after a restart when the sampling time step was reduced slightly too much to homogeneously fill the logarithmically spaced array of example spectra.