On the existence of solutions of a set-valued functional integral equation of Volterra–Stieltjes type and some applications

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Abstract
This paper is concerned with the existence of continuous solutions of a set-valued functional integral equation of Volterra–Stieltjes type. The continuous dependence of the solution on the set of selections of the set-valued function will be proven. As an application, we study the existence of solutions to an initial-value problem of arbitrary fractional-order differential inclusion.

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1 Introduction
Consider the set-valued functional integral equation of Volterra–Stieltjes type

\[ x(t) \in p(t) + \int_0^t F_1 \left( s, \int_0^s f_2 \left( \theta, x(\phi(\theta)) \right) d_0 g_2(s, \theta) \right) d_1 g_1(t, s), \quad t, s \in [0, T], \] (1.1)

and the initial-value problem

\[ \frac{dx(t)}{dt} \in I^\gamma F_1 \left( t, D^\gamma x(t) \right), \quad t \in (0, T], \gamma \in (0, 1], \] (1.2)
\[ x(0) = x_0. \] (1.3)

Here we study the existence of continuous solutions of the set-valued functional integral equation of Volterra–Stieltjes type (1.1). The continuous dependence of the solution on the set of selections of the set-valued function \( F_1 \) will be proven. As an application, we study the existence of solutions of the initial-value problem of arbitrary (fractional) order differential inclusion (1.2)–(1.3).

2 Preliminaries
This section is devoted to providing the notation, definitions, and preliminary facts from the set-valued analysis, which will be needed in our further study.
First, we establish some notation. We will denote by \( I = [0, T] \) a fixed interval, where \( T > 0 \) is arbitrarily fixed and by \( C(I) = C[0, T] \) the Banach space consisting of all continuous functions acting from the interval \( I \) into \( R \) with the standard norm

\[
\|x\|_C = \sup_{t \in I} |x(t)|.
\]

Define the Banach space \( X = C(I) \times C(I) \) with the norm

\[
\|(x, y)\|_X = \|x\|_C + \|y\|_C.
\]

**Definition 2.1** Let \( F \) be a set-valued map defined on a Banach space \( E \), \( f \) is called a selection of \( F \) if \( f(x) \in F(x) \), for every \( x \in E \) and we denote by

\[
S_F = \{ f : f(x) \in F(x), x \in E \}
\]

the set of all selections of \( F \) (for the properties of the selection of \( F \) see [1–3]).

**Definition 2.2** ([4]) A set-valued map \( F \) from \( I \times E \) to family of all nonempty closed subsets of \( E \) is called Lipschitzian if there exists \( k > 0 \) such that, for all \( t \in I \) and all \( x_1, x_2 \in E \), we have

\[
h(F(t,x_1), F(s,x_2)) \leq k(\lvert t-s \rvert + \lvert x_1 - x_2 \rvert),
\]

where \( h(A, B) \) is the Hausdorff distance between the two subsets \( A, B \in I \times E \).

(For properties of the Hausdorff distance see [5].)

The following theorem [5, Sect. 9, Chap. 1, Th. 1] assumes the existence of a Lipschitzian selection.

**Theorem 2.3** ([6]) Let \( M \) be a metric space and \( F \) be Lipschitzian set-valued function from \( M \) into the nonempty compact convex subsets of \( R^n \). Assume, moreover, that, for some \( \lambda > 0 \), \( F(x) \subset \lambda B \) for all \( x \in M \) where \( B \) is the unit ball on \( R^n \). Then there exist a constant \( c \) and a single-valued function \( f : M \to R^n \), \( f(x) \in F(x) \) for \( x \in M \); this function is Lipschitzian with constant \( k \).

In what follows, we discuss a few auxiliary facts concerning functions of bounded variation (cf. [7]). To this end assumes that \( x \) is a real function defined on a fixed interval \([a, b]\). By the symbol \( \int_a^b x \) we will denote the variation of the function \( x \) on the interval \([a, b]\).

In the case when \( \int_a^b x \) is finite we say that \( x \) is of bounded variation on \([a, b]\). In the case of a function \( u(t,s) = [a,b] \times [c,d] \to R \) we can consider the variation \( \int_{t=p}^{t=q} u(t,s) \) of the function \( t \to u(t,s) \) i.e., the variation of the function \( u(t,s) \) with respect to the variable \( t \) on the interval \([p,q] \subset [a,b]\). Similarly, we define the quantity \( \int_{s=p}^{s=q} u(t,s) \). We will not discuss the properties of the variation of functions of bounded variation, we refer to [7] for the mentioned properties. Furthermore, assume that \( x \) and \( \phi \) are two real functions defined on the interval \([a, b]\). Then, under some extra conditions (cf. [7]), we can define
the Stieltjes integral (more precisely, the Riemann–Stieltjes integral) of the function \( x \) with respect to the function \( \phi \) on the interval \([a, b]\) which is denoted by the symbol

\[
\int_{a}^{b} x(t) \, d\phi(t).
\]

In such a case, we say that \( x \) is Stieltjes integrable on the interval \([a, b]\) with respect to \( \phi \).

In the relevant literature, we may encounter a lot of conditions guaranteeing the Stieltjes integrability [7–9]. One of the most frequently exploited condition requires that \( x \) is continuous and \( \phi \) is of bounded variation on \([a, b]\).

Next, we recall a few properties of the Stieltjes integral which will be used in our considerations (cf. [7]).

**Lemma 2.4** Assume that \( x \) is Stieltjes integrable on the interval \([a, b]\) with respect to a function \( \phi \) of bounded variation. Then

\[
\left| \int_{a}^{b} x(t) \, d\phi(t) \right| \leq \int_{a}^{b} |x(t)| \, d\left( \frac{t}{a} \phi \right).
\]

**Lemma 2.5** Let \( x_1 \) and \( x_2 \) be Stieltjes integrable functions on the interval \([a, b]\) with respect to a nondecreasing function \( \phi \) such that \( x_1(t) \leq x_2(t) \) for \( t \in [a, b] \). Then the following inequality is satisfied:

\[
\int_{a}^{b} x_1(t) \, d\phi(t) \leq \int_{a}^{b} x_2(t) \, d\phi(t).
\]

In the sequel, we will also consider the Stieltjes integrals of the form

\[
\int_{a}^{b} x(s) \, d_s g(t, s),
\]

where \( g : [a, b] \times [a, b] \to R \) and the symbol \( d_s \) indicates the integration with respect to the variable \( s \). The details concerning the integral of such a type will be given later.

### 3 Existence of at least one continuous solution

Consider now the set-valued integral equation (1.1) under the following assumptions.

(i) \( p : I \to I \) is continuous function, where \( p^* = \sup_{t \in I} |p(t)| \).

(ii) \( F_1 : I \times R \to P(R) \) is a Lipschitzian set-valued map with a nonempty compact convex subset of \( 2^{R^+} \).

(iii) \( \varphi : I \to I \) is continuous function.

(iv) \( f_2 : I \times R \to R \) is continuous and there exist two constants \( a \) and \( b \) such that

\[
|f_2(t, x)| \leq a + b|x|, \quad \forall t \in [0, T] \text{ and } x \in R.
\]

(v) The function \( g_i \) is continuous on the triangle \( \Delta_i \), for \( i = 1, 2 \), where

\[
\Delta_1 = \{(t, s) : 0 \leq s \leq t \leq T\},
\]

\[
\Delta_2 = \{(s, \theta) : 0 \leq \theta \leq s \leq T\}.
\]
(vi) The function \( s \to g_i(t, s) \) is of bounded variation on \([0, t]\) for each \( t \in I \ (i = 1, 2) \).

(vii) For any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, for all \( t_1, t_2 \in I \) such that \( t_1 < t_2 \) and \( t_2 - t_1 \leq \delta \), the following inequality holds:

\[
\int_0^{t_1} \left[ g_i(t_2, s) - g_i(t_1, s) \right] \leq \epsilon
\]

for \( i = 1, 2 \).

(viii) \( g_i(t, 0) = 0 \) for any \( t \in I \ (i = 1, 2) \).

It is clear that, from Theorem 2.3 and assumption (ii), the set of Lipschitz selection of \( F_1 \) is non-empty. So, the solution of the single-valued integral equation

\[
x(t) = p(t) + \int_0^t f_1(s, \int_0^s f_2(\theta, x(\varphi(\theta))) \, d\theta \phi(\theta)) \, ds, \quad t, s \in [0, T],
\]

(3.1)

where \( f_1 \in S_{F_1} \), is a solution of inclusion (1.1).

It must be noted that \( f_1 \) satisfies the Lipschitz selection

\[
|f_1(t, x) - f_1(s, y)| \leq k (|t - s| + |x - y|).
\]

Obviously, we will assume that \( g_i \) satisfies assumptions (v)–(viii). For our purposes, we only need the following lemmas.

**Lemma 3.1** ([10]) The function \( z \to \bigvee_{s=0}^{t} g_i(t, s) \) is continuous on \([0, t]\) for any \( t \in I \ (i = 1, 2) \).

**Lemma 3.2** ([10]) Let the assumptions (v)–(vii) be satisfied. Then, for arbitrary fixed number \( 0 < t_2 \in I \) and for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( t_1 \in I; t_1 < t_2 \) and \( t_2 - t_1 \leq \delta \) then \( \bigvee_{s=0}^{t_2} g_i(t_2, s) \leq \epsilon \ (i = 1, 2) \).

**Lemma 3.3** ([10]) Under the assumptions (v)–(vii), the function \( t \to \bigvee_{s=0}^{t} g_i(t, s) \) is continuous on \( I \ (i = 1, 2) \).

Further, let us observe that based on Lemma 3.3 we infer that there exists a finite positive constant \( K_i \), such that

\[
K_i = \sup \left\{ \bigvee_{s=0}^{t} g_i(t, s) : t \in [0, T] \right\},
\]

where \( T > 0 \) is arbitrarily fixed and \( i = 1, 2 \).

We now introduce some functions that will be useful in our further studies:

\[
N_i(\epsilon) = \sup \left\{ \bigvee_{s=0}^{t_1} (g_i(t_2, s) - g_i(t_1, s)) : t_1, t_2 \in [0, T], t_1 < t_2; t_2 - t_1 \leq \epsilon, i = 1, 2 \right\}.
\]

In our considerations, we will examine the double Stieltjes integral of the form

\[
\int_{\epsilon}^{d} \left( \int_{\epsilon}^{d} f(t, x) g_2(x, y) \right) g_1(t, s) = \int_{\epsilon}^{d} \int_{\epsilon}^{d} f(t, x) d_1 g_2(x, y) d_1 g_1(t, s),
\]
where \( g_i : [a, b] \times [c, d] \rightarrow R(i = 1, 2) \) and the symbol \( d_j \) indicates the integration with respect to the variable \( y \) (similarly, we define the symbol \( d_j \)).

Now, let
\[
y(t) = \int_0^t f_2(s, x(y(s))) \, d_j g_2(t, s), \quad t \in [0, T],
\]
then the nonlinear functional integral equation (3.1) can be written in the form
\[
x(t) = p(t) + \int_0^t f_1(s, y(s)) \, d_j g_1(t, s), \quad t \in [0, T].
\]

Hence, the functional integral equation (3.1) is equivalent to the coupled system (3.2) and (3.3).

Now, we study the existence of a continuous solution of the functional integral equation (3.1), which is a solution of the functional integral inclusion (1.1), by getting the continuous solution of the coupled system (3.2) and (3.3).

**Definition 3.4** By a solution of the coupled system (3.2), (3.3) we mean the functions \( x, y \in C[0; T] \) satisfying (3.2), (3.3).

**Remark 3.5** From the Lipschitz condition of \( f_1 \), we have
\[
|f_1(t, x) - f_1(t, 0)| \leq |f_1(t, x) - f_1(t, 0)| \leq k|x|,
\]
i.e.,
\[
|f_1(t, x)| \leq k|x| + \sup_{t \in [0, T]} |f_1(t, 0)| \leq k|x| + f_1^x,
\]
where
\[
f_1^x = \sup_{t \in [0, T]} |f_1(t, 0)|.
\]

Now for the existence of at least one solution \( u = (x, y), x, y \in C[0; T] \) of the coupled system (3.3), (3.2) we have the following theorem.

**Theorem 3.6** Under assumptions (i)--(viii), there exists at least one solution \( u = (x, y), x, y \in C[0; T] \) of the coupled system (3.3), (3.2).

**Proof** Define the set \( Q_r \) by
\[
Q_r = \left\{ u = (x, y) \in R^2, \|x\| \leq r_1, \|y\| \leq r_2, \|(x, y)\| \leq r_1 + r_2 = r \right\},
\]
where \( r = p + \frac{aK_2}{1 - bK_1} \) with \( bK_1 < 1, bK_2 < 1 \).

It is clear that the set \( Q_r \) is nonempty, bounded, closed and convex. Let \( A \) be any operator defined by
\[
Au(t) = A(x, y)(t) = (A_1 y(t), A_2 x(t)),
\]
\[ A_1 y(t) = p(t) + \int_0^t f_1(s, y(s)) \, d_x g_1(t, s), \quad t \in [0, T], \]

and

\[ A_2 x(t) = \int_0^t f_2(s, x(s)) \, d_x g_2(t, s), \quad t \in [0, T], \]

where for \( u = (x, y) \in Q_r \), and from Remark 3.5 we have

\[
|A_1 y(t)| = \left| p(t) + \int_0^t f_1(s, y(s)) \, d_x g_1(t, s) \right|
\leq |p(t)| + \int_0^t |f_1(s, y(s))| \, d_x g_1(t, s)
\leq p^* + \int_0^t (k|y| + f_1^*) \, d_x \left( \sqrt{g_1(t, p)} \right).
\]

Then

\[
\|A_1 y\| \leq p^* + (kr_1 + f_1^*) \left( \sqrt{\int_{s=0}^t g_1(t, s)} \right)
\leq p^* + (kr_1 + f_1^*) \sup_{t \in I} \left( \sqrt{g_1(t, s)} \right)
\leq p^* + (kr_1 + f_1^*) K_1 = r_1, \quad r_1 = \frac{p^* + f_1^* K_1}{1 - k K_1}.
\]

Also

\[
|A_2 x(t)| = \left| \int_0^t f_2(s, x(s)) \, d_x g_2(t, s) \right|
\leq \int_0^t |f_2(s, x(s))| \, d_x g_2(t, s)
\leq \int_0^t \left[ a + b|\psi(s)| \right] \, d_x \left( \sqrt{g_2(t, p)} \right).
\]

Then

\[
\|A_2 x\| \leq (a + br_2) \left( \sqrt{\int_{s=0}^t g_2(t, s)} \right)
\leq (a + br_2) \sup_{t \in I} \left( \sqrt{g_2(t, s)} \right)
\leq (a + br_2) K_2 = r_2, \quad r_2 = \frac{a K_2}{1 - b K_2}.
\]
From the above estimate we derive the following inequality:

\[ \|Au\|_X = \|A_1y\|_C + \|A_2x\|_C \leq r_1 + r_2 \]
\[ = \frac{p^* + f^*_K + 1}{1 - kK_1} + \frac{aK_2}{1 - bK_2} = r. \]

Hence, \( AQ_r \subset Q_r \) and the class \( \{Au\}, u \in Q_r \) is uniformly bounded.

Now, for \( u = (x, y) \in Q_r \), for all \( \epsilon > 0, \delta > 0 \) and for each \( t_1, t_2 \in [0, T], t_1 < t_2 \), such that \( |t_2 - t_1| < \delta \), we have

\[ |A_1y(t_2) - A_1y(t_1)| = \left| p(t_2) + \int_0^{t_2} f_1(s, y(s)) \, ds \, g_1(t_2, s) \right| - p(t_1) - \int_0^{t_1} f_1(s, y(s)) \, ds \, g_1(t_1, s) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_2} f_1(s, y(s)) \, ds \, g_1(t_2, s) \]
\[ - \int_0^{t_1} f_1(s, y(s)) \, ds \, g_1(t_1, s) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_2} f_1(s, y(s)) \, ds \, g_1(t_2, s) \]
\[ - \int_0^{t_1} f_1(s, y(s)) \, ds \, g_1(t_1, s) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_2} f_1(s, y(s)) \, ds \, g_1(t_2, s) \]
\[ - \int_0^{t_1} f_1(s, y(s)) \, ds \, g_1(t_1, s) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_2} [k|y(s)| + f_1(s, 0)] \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) \right) \]
\[ + \int_0^{t_1} [k|y(s)| + f_1(s, 0)] \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) - g_1(t_1, p) \right) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_1} \left( k + f_1^* \right) \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) \right) \]
\[ + \int_0^{t_1} \left( k + f_1^* \right) \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) - g_1(t_1, p) \right) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_1} \left( k + f_1^* \right) \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) - g_1(t_1, p) \right) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_1} \left( k + f_1^* \right) \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) + N_1(\epsilon) \right) \]
\[ \leq |p(t_2) - p(t_1)| + \int_0^{t_1} \left( k + f_1^* \right) \, ds \, \left( \int_{p=0}^{s} g_1(t_2, p) + N_1(\epsilon) \right) \]
and

\[
\begin{align*}
&\quad |A_2 x(t_2) - A_2 x(t_1)| \\
&\leq \left| \int_0^{t_2} f_2(s, x(x(s))) \, d_2 g_2(t_2, s) - \int_0^{t_1} f_2(s, x(x(s))) \, d_2 g_2(t_1, s) \right| \\
&\leq \left| \int_0^{t_2} f_2(s, x(x(s))) \, d_2 g_2(t_2, s) - \int_0^{t_1} f_2(s, x(x(s))) \, d_2 g_2(t_1, s) \right| \\
&\quad + \left| \int_0^{t_2} f_2(s, x(x(s))) \, d_2 g_2(t_2, s) - \int_0^{t_1} f_2(s, x(x(s))) \, d_2 g_2(t_1, s) \right| \\
&\leq \int_0^{t_2} |f_2(s, x(x(s)))| \, d_2 g_2(t_2, s) \\
&\quad + \int_0^{t_2} |f_2(s, x(x(s)))| \left| [d_2 g_2(t_2, s) - d_2 g_2(t_1, s)] \right| \\
&\leq \int_0^{t_2} [a + b|x(x(s))|] d_2 \left( \sqrt{g_2(t_2, p)} \right) \\
&\quad + \int_0^{t_2} [a + b|x(x(s))|] d_2 \left( \sqrt{[g_2(t_2, p) - g_2(t_1, p)]} \right) \\
&\leq (a + b r_2) \left[ \int_0^{t_2} d_2 \left( \sqrt{g_2(t_2, p)} \right) + \int_0^{t_2} d_2 \left( \sqrt{[g_2(t_2, p) - g_2(t_1, p)]} \right) \right] \\
&\leq (a + b r_2) \left[ \sqrt{g_2(t_2, s)} - \sqrt{g_2(t_1, s)} \right] + \sqrt{g_2(t_2, s) - g_2(t_1, s)} \\
&\leq (a + b r_2) \left[ \sqrt{g_2(t_2, s) + N_2(e)} \right].
\end{align*}
\]

Further, for the operator \( A \) and \( u \in Q \), we have

\[
Au(t_2) - Au(t_1) = A(x, y)(t_2) - A(x, y)(t_1)
= (A_1 y(t_2), A_2 x(t_2)) - (A_1 y(t_1), A_2 x(t_1))
= (A_1 y(t_2) - A_1 y(t_1), A_2 x(t_2) - A_2 x(t_1)).
\]

Then

\[
\begin{align*}
\|Au(t_2) - Au(t_1)\|_x &= \| (A_1 y(t_2) - A_1 y(t_1), A_2 x(t_2) - A_2 x(t_1)) \|_x \\
&= \|A_1 y(t_2) - A_1 y(t_1)\|_C + \|A_2 x(t_2) - A_2 x(t_1)\|_C \\
&= \|p(t_2) - p(t_1)\| + \left[ k r_1 + f^*_s \right] \left[ \int_0^{t_2} g(t_2, s) + N_1(e) \right] \\
&\quad + (a + b r_2) \left[ \sqrt{g(t_2, s) + N_2(e)} \right].
\end{align*}
\]

(*)
This means that the class of functions \( Au \) is equi-continuous on \( Q_r \). Then by the Arzela–Ascoli theorem \([11]\) the operator \( A \) is compact.

It remains to prove the continuity of \( A : Q_r \to Q_r \). Let \( u_n = (x_n, y_n) \) is a sequence in \( Q_r \) with \( x_n \to x \) and \( y_n \to y \) and since \( f_1(t, y(t)) \) and \( f_2(t, x(t)) \) is continuous in \( \mathbb{C}[0, T] \times \mathbb{R} \) then \( f_1(t, y_n(t)) \) and \( f_2(t, x_n(t)) \) converge to \( f_1(t, y(t)) \) and \( f_2(t, x(t)) \), thus \( f_2(t, x_n(\psi(t))) \) converges to \( f_2(t, x(\psi(t))) \) (see assumption (iii)). Using assumption (iii) and applying Lebesgue dominated convergence theorem, we get

\[
\lim_{n \to \infty} \int_0^t f_2(s, x_n(\psi(s))) \, ds \, g_2(t, s) = \int_0^t f_2(s, x(\psi(s))) \, ds \, g_2(t, s)
\]

and

\[
\lim_{n \to \infty} \int_0^t f_1(s, y_n(s)) \, ds \, g_1(t, s) = \int_0^t f_1(s, y(s)) \, ds \, g_1(t, s);
\]

then

\[
\lim_{n \to \infty} A_1 y_n(t) = \lim_{n \to \infty} \int_0^t f_1(s, y_n(s)) \, ds \, g_1(t, s)
\]

\[
= p(t) + \int_0^t f_1(s, y(s)) \, ds \, g_1(t, s) = A_1 y(t), \quad t \in [0, T],
\]

\[
\lim_{n \to \infty} A_2 x_n(t) = \lim_{n \to \infty} \int_0^t f_2(s, x_n(\psi(s))) \, ds \, g_2(t, s)
\]

\[
= \int_0^t f_2(s, x(\psi(s))) \, ds \, g_2(t, s) = A_2 x(t), \quad t \in [0, T],
\]

\[
\lim_{n \to \infty} Au_n(t) = \lim_{n \to \infty} \left( A_1 y_n(t), A_2 x_n(t) \right)
\]

\[
= \left( \lim_{n \to \infty} A_1 y_n(t), \lim_{n \to \infty} A_2 x_n(t) \right) = (A_1 y(t), A_2 x(t)) = Au(t).
\]

Since all conditions of the Schauder fixed-point theorem \([12]\) hold, \( A \) has a fixed point \( u \in Q_r \), and then the system \((3.3), (3.2)\) has at least one continuous solution \( u = (x, y) \in Q_r \), \( x, y \in \mathbb{C}[0, T] \).

Consequently, the functional integral equation \((3.1)\) has at least one solution \( x \in \mathbb{C}[0, T] \). \( \square \)

### 4 Existence of a unique solution

In this section, we study the uniqueness of the solutions \( x \in \mathbb{C}[0, T] \) of the functional integral inclusion \((1.1)\).

**Theorem 4.1** Consider the assumptions of Theorem 3.6 satisfied with replacing condition (iv) by assuming that the \( f_2 \) satisfies the Lipschitz condition with respect to the second variable; that is, there exists a constant \( c \) such that

\[
|f_2(t, x) - f_2(t, y)| \leq c|x - y|.
\]

If \( k K_1 K_2 < 1 \), then the functional integral inclusion \((1.1)\) has a unique solution \( x \in \mathbb{C}[0, T] \), where \( k \) is Lipschitz constant of functions \( f_1 \) and \( K_i \) \((i = 1, 2)\) as defined in Lemma 3.3.
Proof Let \( x_1 \) and \( x_2 \) be two solutions of Eq. (3.1), then
\[
|x_1(t) - x_2(t)| \leq \int_0^t \left| f_1 \left( s, \int_0^s f_2 \left( \theta, x_1(\varphi(\theta)) \right) d_0 g_2(s, \theta) \right) - f_1 \left( s, \int_0^s f_2 \left( \theta, x_2(\varphi(\theta)) \right) d_0 g_2(s, \theta) \right) \right| d_1 g_1(t, s).
\]

Using the Lipschitz condition for \( f_1 \), we obtain
\[
|x_1(t) - x_2(t)| \leq k \int_0^t \int_0^s \left| f_2 \left( \theta, x_1(\varphi(\theta)) \right) - f_2 \left( \theta, x_2(\varphi(\theta)) \right) \right| d_0 g_2(s, \theta) \int_0^s d_1 g_1(t, s)
\]
\[
\leq k \int_0^t \int_0^s \left( |x_1(\varphi(\theta)) - x_2(\varphi(\theta))| \right) d_0 \left( \int_0^s g_2(s, p) \right) d_1 \left( \int_0^s g_1(t, q) \right)
\]
\[
\leq k \|x_1 - x_2\| \int_0^t \int_0^s \sup_{\varphi(\theta) \in [0, T]} \int_0^s g_2(s, p) \int_0^s g_1(t, q)
\]
\[
\leq k \|x_1 - x_2\| \sup_{t \in [0, T]} \int_0^t g(t, s) \sup_{s \in [0, T]} \int_0^s g(s, \theta)
\]
\[
\leq k b K_1 K_2 \|x_1 - x_2\|
\]
Then
\[
\|x_1(t) - x_2(t)\| \leq k b K_1 K_2 \|x_1 - x_2\|.
\]

This proves the uniqueness of the solution of the functional integral equation (3.1). \( \square \)

### 4.1 Continuous dependence

**Theorem 4.2** The solution of the inclusion (1.1) depends continuously on the \( S_{F_1} \) of all Lipschitzian selections of \( F_1 \).

**Proof** Let \( f_1(t, x(t)) \) and \( f_2^*(t, x(t)) \) be two different Lipschitzian selections of \( F_1(t, x(t)) \) such that
\[
\left| f_1(t, x(t)) - f_2^*(t, x(t)) \right| < \delta, \quad \delta > 0, \ t \in [0, T],
\]
then for the two corresponding solutions $x_1(t)$ and $x_1^*(t)$ of (1.1) we have

$$
x_1(t) - x_1^*(t) = \int_0^t \left[ f_1(s, \int_0^s f_2(\theta, x_1(\psi(\theta))) \, d_0 g_2(s, \theta))
- f_1^*(s, \int_0^s f_2(\theta, x_1^*(\psi(\theta))) \, d_0 g_2(s, \theta)) \right] \, d_g(t, s) \, ds

\leq \int_0^t \left| f_1(s, \int_0^s f_2(\theta, x_1(\psi(\theta))) \, d_0 g_2(s, \theta))
- f_1^*(s, \int_0^s f_2(\theta, x_1^*(\psi(\theta))) \, d_0 g_2(s, \theta)) \right| \, d_g(t, s) \, ds

\leq \int_0^t \left| f_1(s, \int_0^s f_2(\theta, x_1(\psi(\theta))) \, d_0 g_2(s, \theta)) - f_1^*(s, \int_0^s f_2(\theta, x_1^*(\psi(\theta))) \, d_0 g_2(s, \theta)) \right| \, d_g(t, s) \, ds

\leq k \int_0^t \left| f_2(\theta, x_1(\psi(\theta))) - f_2(\theta, x_1^*(\psi(\theta))) \right| \, d_0 g_2(s, \theta) \, ds + \delta \int_0^t \, d_g(t, s) \, ds

\leq k \| x_1 - x_1^* \| \int_0^t \int_0^s \left| f_2(\theta, x_1(\psi(\theta))) - f_2(\theta, x_1^*(\psi(\theta))) \right| \, d_0 g_2(s, \theta) \, ds + \delta \int_0^t \, d_g(t, s) \, ds

\leq k b \| x_1 - x_1^* \| \left( \int_0^t \int_0^s \left( \int_0^r g_2(s, q) \, ds \right) \, d_0 g_2(t, p) \, dp + \delta \int_0^t \, d_g(t, p) \, dp \right)

\leq b k \int_0^t \left( \int_0^s \left( \int_0^r g_2(s, q) \, ds \right) \, d_0 g_2(t, p) \, dp + \delta \int_0^t \, d_g(t, p) \, dp \right)

\leq b k \int_0^t \left( \int_0^s \left( \int_0^r g_2(s, q) \, ds \right) \, d_0 g_2(t, p) \, dp + \delta \int_0^t \, d_g(t, p) \, dp \right)

\leq b k \| x_1 - x_1^* \| \left( \sup_{t \in [0, T]} \int_0^t \left( \int_0^s \left( \int_0^r g_2(s, q) \, ds \right) \, d_0 g_2(t, p) \, dp + \delta \int_0^t \, d_g(t, p) \, dp \right) \right)

\leq b k \| x_1 - x_1^* \| K_2 K_1 + \delta K_1,
Thus from last inequality, we get
\[ \| x_{f_1} - x_{f_1}^* \| \leq \epsilon. \]

This proves the continuous dependence of the solution on the set \( S_{f_1} \) of all Lipschitzian selections of \( F_1 \). This completes the proof. □

5 Volterra integral inclusion of fractional order
In this section, we will consider the fractional integral inclusion, which has the form
\[
\begin{align*}
x(t) & \in p(t) + \int_0^t \left( t - s \right)^{\alpha - 1} \Gamma(\alpha) F_1 \left( s, \int_0^s \left( s - \theta \right)^{\beta - 1} \Gamma(\beta) f_2 (\theta, x(\psi(\theta))) d\theta \right) ds,
\end{align*}
\]
where \( t \in I = [0, T] \) and \( \alpha \in (0, 1) \). Moreover, \( \Gamma(\alpha) \) denotes the gamma function. Let us mention that (5.1) represents the so-called nonlinear Volterra integral inclusion of fractional orders. Recently, the inclusion of such a type was intensively investigated in some papers [13–18].

Now, we show that the functional integral inclusion of fractional orders (5.1) can be treated as a particular case of the set-valued functional integral equation of Volterra–Stieltjes (1.1) studied in Sect. 3.

Indeed, we can consider the functions \( g_i : \triangle \rightarrow \mathbb{R} \) \( (i = 1, 2) \) defined by the formulas
\[
\begin{align*}
g_1(t,s) &= \frac{t^\alpha - (t - s)^\alpha}{\Gamma(\alpha + 1)}, \\
g_2(s,\theta) &= \frac{s^\beta - (s - \theta)^\beta}{\Gamma(\beta + 1)}.
\end{align*}
\]
Note that the functions \( g_1 \) and \( g_2 \) satisfy assumptions (v)–(viii) in Theorem 3.6; see [10, 19].

Now, we can formulate the following existence results concerning with the Volterra integral inclusion of fractional order (5.1).

**Theorem 5.1** Under the assumptions (i)–(iv) of Theorem 3.6, the fractional integral inclusion (5.1) has at least one continuous solution \( x \in C[0, T] \).

**Theorem 5.2** Under the assumptions of Theorem 4.1, the fractional integral inclusion (5.1) has exactly one unique solution \( x \in C[0, T] \).

6 Existence of the maximal and minimal solutions
In this section, we establish the existence of the maximal and minimal solutions of the nonlinear Volterra integral inclusion of fractional order (5.1). It is clear that, from Theorem 2.3 and assumption (ii) of Theorem 3.6, the set of Lipschitz selections of \( F_1 \) is non-empty. So, the solution of the nonlinear functional integral equation of fractional order
\[
\begin{align*}
x(t) &= p(t) + \int_0^t f_1 \left( t, \int_0^s f_2 \left( t, x(\psi(t)) \right) \right) dt, \quad t, s \in [0, T],
\end{align*}
\]
where \( f_1 \in S_{f_1} \), is a solution of inclusion (5.1).
Definition 6.1 ([12]) Let \( m(t) \) be a solution of the non-linear functional integral equation (6.1), then \( m(t) \) is said to be a maximal solution of (6.1), if for every solution \( x \) of inclusion (6.1) existing on \([0, T]\) the inequality \( x(t) < m(t) \), \( t \in [0, T] \) holds. A minimal solution \( s(t) \) may be defined similarly by reversing the last inequality i.e. \( x(t) > s(t), \forall t \in [0, T] \).

Consider the following lemma.

Lemma 6.2 Let \( p(t), f_i(t; x) (i = 1, 2) \) and \( \varphi(t) \) satisfy the assumptions in Theorem 5.1 and let \( x(t), y(t) \) be two continuous functions on \([0, T]\) satisfying

\[
\begin{align*}
x(t) &\leq p(t) + \int_0^t (t - s)^{\mu - 1} \frac{1}{\Gamma(\mu)} f_1(s, I^\beta f_2(s, x(\varphi(s)))) ds, \\
y(t) &\geq p(t) + \int_0^t (t - s)^{\mu - 1} \frac{1}{\Gamma(\mu)} f_1(s, I^\beta f_2(s, y(\varphi(s)))) ds,
\end{align*}
\]

where one of them is strict.

Suppose \( f_1 \) and \( f_2 \) are monotonic nondecreasing functions in \( x \), then

\[
x(t) < y(t), \quad t > 0. \tag{6.2}
\]

Proof Let the conclusion (6.2) be false, then there exists \( t_1 \) such that

\[
x(t_1) = y(t_1), \quad t_1 > 0,
\]

and

\[
x(t) < y(t), \quad 0 < t < t_1, t \in [0, T].
\]

From the monotonicity of the functions \( f_1 \) and \( f_2 \) in \( x \), we get

\[
x(t_1) \leq p(t_1) + \int_0^{t_1} (t_1 - s)^{\mu - 1} \frac{1}{\Gamma(\mu)} f_1(s, I^\beta f_2(s, x(\varphi(s)))) ds < p(t_1) + \int_0^{t_1} (t_1 - s)^{\mu - 1} \frac{1}{\Gamma(\mu)} f_1(s, I^\beta f_2(s, y(\varphi(s)))) ds,
\]

\[
x(t_1) < y(t_1).
\]

This contradicts the fact that \( x(t_1) = y(t_1) \).

Then

\[
x(t) < y(t).
\]

□

Now, for the existence of the continuous maximal and minimal solutions of the non-linear functional integral inclusion (6.1) we have the following theorem.

Theorem 6.3 Consider the assumptions (i)–(iv) of Theorem 5.1 satisfied, furthermore, if \( f_1 \) and \( f_2 \) are monotonic nondecreasing functions in \( x \) for each \( t \in [0, T] \), then the non-linear functional integral inclusion (6.1) has maximal and minimal solutions.
Proof. Firstly, we shall prove the existence of the maximal solution of (6.1).

Let $\epsilon > 0$ be given such that $0 < \epsilon < \frac{T}{2}$, and consider the nonlinear functional integral equation of fractional order

\[ x_\epsilon(t) = p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x_\epsilon(\phi(t)))) dt, \]

where

\[ f_1(t, \int_0^t I^\beta f_2(t, x_\epsilon(\phi(t)))) = f_1(t, \int_0^t I^\beta f_2(t, x_\epsilon(\phi(t)))) + \epsilon, \]
\[ f_2(t, x_\epsilon(\phi(t))) = f_2(t, x_\epsilon(\phi(t))) + \epsilon. \]

Clearly the functions $f_1(t, \int_0^t I^\beta f_2(t, x_\epsilon(\phi(t))))$ and $f_2(t, x_\epsilon(\phi(t)))$ satisfy the assumptions of Theorem 5.1 and therefore, Eq. (6.1) has a continuous solution $x_\epsilon$ according to Theorem 5.1. Let $\epsilon_1$ and $\epsilon_2$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

\[ x_{\epsilon_2}(t) = p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x_{\epsilon_2}(\phi(t)))) dt, \]

\[ x_{\epsilon_2}(t) = p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x_{\epsilon_2}(\phi(t)))) + I^\beta \epsilon_2 + I^\alpha \epsilon_2, \] (6.3)

also

\[ x_{\epsilon_1}(t) = p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x_{\epsilon_1}(\phi(t)))) dt, \]
\[ x_{\epsilon_1}(t) = p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x_{\epsilon_1}(\phi(t)))) + I^\beta \epsilon_1 + I^\alpha \epsilon_1, \]
\[ x_{\epsilon_1}(t) > p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x_{\epsilon_1}(\phi(t)))) + I^\beta \epsilon_2 + I^\alpha \epsilon_2. \] (6.4)

Applying Lemma 6.2, and (6.3) and (6.4), we have

\[ x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \quad \text{for } t \in [0, T]. \]

As shown before in the proof of Theorem 3.6, the family of functions $x_{\epsilon_i}(t)$ is uniformly bounded and equi-continuous. Hence by the Arzela–Ascoli theorem, there exists a decreasing sequence $\epsilon_n$ such that $\epsilon_n \to 0$ as $n \to \infty$, and $\lim_{n \to \infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and we denote this limit by $m(t)$. From the continuity of the functions $f_i\epsilon_n$ for $i = 1, 2$ and in the second argument, we get

\[ f_2(\epsilon_n, t, x_{\epsilon_n}(\phi(t))) \to f_2(t, x(\phi(t))), \quad \text{as } n \to \infty, \]
\[ f_1(\epsilon_n, t, \int_0^t I^\beta f_2(t, x_{\epsilon_n}(\phi(t)))) \to f_1(t, \int_0^t I^\beta f_2(t, x(\phi(t))))), \quad \text{as } n \to \infty, \]

and

\[ m(t) = \lim_{n \to \infty} x_{\epsilon_n} = p(t) + \int_0^t I^\alpha f_1(t, \int_0^t I^\beta f_2(t, x(\phi(t)))) dt, \]

which implies that $m(t)$ is a solution of the quadratic integral equation (6.1). Finally, we shall show that $m(t)$ is the maximal solution of (6.1).
To do this let \( x(t) \) be any solution of (6.1), then
\[
x(t) = p(t) + I^\alpha f_1(t, I^\beta f_2(t, x(\psi(t)))),
\]
and also
\[
x_*(t) = p(t) + I^\alpha f_1(t, I^\beta f_2(t, x_*(\psi(t)))),
\]
\[
x_*(t) = p(t) + I^\alpha f_1(t, I^\beta f_2(t, x_*(\psi(t)))) + I^\alpha \epsilon + I^\alpha \epsilon,
\]
\[
x_*(t) > p(t) + I^\alpha f_1(t, I^\beta f_2(t, x_*(\psi(t)))) + I^\alpha \epsilon + I^\alpha \epsilon.
\]

Applying Lemma 6.2 and (6.5) and (6.6), we get
\[
x(t) < x_*(t), \quad \forall t \in [0, T].
\]

From the uniqueness of the maximal solution (see [12]), it is clear that \( x_*(t) \) tends to \( m(t) \) uniformly in \([0, T]\) as \( \epsilon \to \infty \).

Similarly, we can prove the existence of the minimal solution. We set
\[
f_{1e}(t, I^\beta f_2(t, x_*(\psi(t)))) = f_1(t, I^\beta f_2(t, x_*(\psi(t)))) - I^\beta \epsilon - \epsilon,
\]
\[
f_{2e}(t, x_*(\psi(t))) = f_2(t, x_*(\psi(t))) - \epsilon,
\]
and thus we prove the existence of a minimal solution. \( \square \)

### 7 Differential inclusion

Consider now the initial-value problem of the differential inclusion (1.2) with the initial data (1.3).

**Theorem 7.1** Consider the assumptions of Theorem 5.1 satisfied, then the initial-value problem (1.2)–(1.3) has at least one positive solution \( x \in C([0, 1]) \).

**Proof** Let \( y(t) = \frac{dx(t)}{dt} \), then the inclusion (1.2) will be
\[
y(t) \in I^\beta F_1(t, I^{1-\tau}y(t)).
\]

Letting \( f_2(t, x) = x, \phi(t) = t, \) and \( \beta = 1 - \tau \), applying Theorem 5.2 to the functional inclusion (7.1) we deduce that there exists a continuous solution \( y \in C[0, T] \) of the functional inclusion (7.1) and this solution depends on the set \( S_{F_1} \).

This implies the existence of a solution \( x \in C[0, T] \),
\[
x(t) = x_0 + \int_0^t y(s) \, ds
\]
of the initial-value problem (1.2)–(1.3). \( \square \)

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