WEIGHTED BOUND FOR COMMUTATORS

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Abstract. Let $K$ be the Calderón-Zygmund convolution kernel on $\mathbb{R}^d (d \geq 2)$. Define the commutator associated with $K$ and $a \in L^\infty(\mathbb{R}^d)$ by

\[ T_a f(x) = \text{p.v.} \int K(x - y)m_{x,y}a \cdot f(y)dy. \]

Recently, Grafakos and Honzík [5] proved that $T_a$ is of weak type $(1,1)$ for $d = 2$. In this paper, we show that $T_a$ is also weighted weak type $(1,1)$ with the weight $|x|^{-\alpha}$ ($-2 < \alpha < 0$) for $d = 2$. Moreover, we prove that $T_a$ is bounded on weighted $L^p(\mathbb{R}^d)$ ($1 < p < \infty$) for all $d \geq 2$.

1. Introduction

Suppose that $K$ is the Calderón-Zygmund convolution kernel on $\mathbb{R}^d \setminus \{0\} (d \geq 2)$, which means that $K$ satisfies following three conditions:

\[(1.1) \quad |K(x)| \leq C|x|^{-d}, \]
\[(1.2) \quad \int_{R < |x| < 2R} K(x) dx = 0, \text{ for all } R > 0, \]
\[(1.3) \quad |\nabla K(x)| \leq \frac{C}{|x|^{d+1}}. \]

In 1987, Christ and Journé [2] introduced a commutator associated with $K$ and $a \in L^\infty(\mathbb{R}^d)$ by

\[ T_a f(x) = \text{p.v.} \int K(x - y)m_{x,y}a \cdot f(y)dy, \quad f \in S(\mathbb{R}^d), \]

where $S(\mathbb{R}^d)$ denotes the Schwartz class and

\[ m_{x,y}a = \int_0^1 a((1 - t)x + ty)dt. \]

Note that when $d = 1$, then

\[ m_{x,y}a = \frac{\int_0^x a(z)dz - \int_0^y a(z)dz}{x - y}. \]

In this case, let $K(x) = \frac{1}{x}$ and $A(x) = \int_0^x a(z)dz$, then $A'(x) = a(x) \in L^\infty(\mathbb{R})$. So

\[ T_a f(x) = \text{p.v.} \int \frac{A(x) - A(y)}{x - y} \frac{f(y)}{x - y} dy, \]

\[ = \text{p.v.} \int \frac{A(x) - A(y)}{x - y} f(y) dy. \]
which is the famous Calderón commutator discussed in [1].

In [2], Christ and Journé showed that $T_a$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. In 1995, Hofmann [7] gave the weighted $L^p(\mathbb{R}^d)$ ($1 < p < \infty$) boundedness of $T_a$ when the kernel $K(x) = \Omega(x/|x||x|^{-d}$. Recently, Grafakos and Honzík [5] proved that $T_a$ is weak type $(1,1)$ for $d = 2$. Further, Seeger [9] showed that $T_a$ is still weak type $(1,1)$ for all $d \geq 2$. The purpose of this paper is to establish a weighted variety of Grafakos and Honzík’s results in [5].

In the sequel, for $1 \leq p \leq \infty$, $A_p$ denotes the Muckenhoupt weight class and $L^p(\omega)$ denotes the weighted $L^p(\mathbb{R}^d)$ space with norm $\| \cdot \|_{p,\omega}$. We also denote $\omega(E) = \int_E \omega(x)dx$ for a measurable set $E$ in $\mathbb{R}^d$. The main result obtained in the present paper is as follows.

**Theorem 1.1.** Suppose $K$ satisfies (1.1), (1.2) and (1.3) for $d = 2$. Let $a \in L^\infty(\mathbb{R}^2)$ and $\omega(x) = |x|^\alpha$ for $-2 < \alpha < 0$. Then there exists a constant $C > 0$ such that

$$\omega(\{x \in \mathbb{R}^2 : |T_a f(x)| > \lambda\}) \leq C\lambda^{-1}\|a\|_\infty\|f\|_{1,\omega}$$

for all $\lambda > 0$ and $f \in L^1(\omega)$.

We would like to point out that the proof of Theorem 1.1 follows the nice idea from [5]. However, there are some differences in proving $T_a$ is of weak type $(1,1)$ for the weighted case. In fact, the essential difficulties of proving Theorem 1.1 are to show the smoothness of kernels of $(T_j T_j)_\omega$ and $(T_j T_j)_\omega$ (see (3.2) and (3.13) below, respectively), these estimates are more complicated than no weight case, although we only consider power weight $|x|^\alpha$ for $-2 < \alpha \leq 0$. Our main innovations are further decomposition of power weight according to the dyadic decomposition. Note that $|x|^\alpha \in A_1(\mathbb{R}^2)$ if and only if $-2 < \alpha \leq 0$, but our method cannot be used to deal with the general $A_1$ weight. This is the reason why we now cannot get a similar result as Theorem 1.1 for general weight $w \in A_1(\mathbb{R}^2)$.

In order to prove Theorem 1.1 we need to establish the weighted $L^p$ boundedness of $T_a$ (actually we only need weighted $L^2$ boundedness). Although the $L^p(\omega)$ boundedness of $T_a$ given by [7, Theorem 2.15] for the homogeneous kernel $K(x) = \Omega(x/|x||x|^{-d}$, it seems that one cannot apply directly to $T_a$ with the kernel satisfying (1.1)-(1.3) discussed in this paper. However, Hofmann established a weighted $L^p$ boundedness criteria in [7] which is similar to $T1$ theorem. The proof of Theorem 1.2 given here is an application of that criteria. More precisely, $T_a$ is a special example of the general operators studied in [7].

**Theorem 1.2.** Suppose $K$ satisfies the conditions (1.1), (1.2) and (1.3). Let $a \in L^\infty(\mathbb{R}^d)$ ($d \geq 2$) and $\omega \in A_p$, $1 < p < \infty$. Then there exists a constant $C > 0$ such that

$$(1.4) \quad \|T_a f\|_{p,\omega} \leq C\|a\|_\infty\|f\|_{p,\omega}.$$ 

This paper is organized as follows. The proof of Theorem 1.2 is given in Section 4. In Section 2, we complete the proof of Theorem 1.1 based on Theorem 1.2 and Lemma 2.3. Moreover, in this section, we also state that the proof of Lemma 2.3 can be reduced to two key lemmas, their proofs will be given in Section 3. Throughout this paper the letter $C$ will stand for a positive
constant which is independent of the essential variables and not necessarily the same one in each occurrence.

2. Proof of Theorem 1.1

Let us begin by giving an analogous Calderón-Zygmund decomposition of \( f \in L^1(\omega) \). First, we recall the Whitney decomposition which can be found in [3]:

**Lemma 2.1. (Whitney decomposition)** Let \( F \) be an open nonempty proper subset of \( \mathbb{R}^d \). Then there exists a family of dyadic closed cubes \( \{Q_j\}_j \) such that

(a) \( \bigcup Q_j = F \) and \( Q_j \)'s have disjoint interior.
(b) \( \sqrt{d} \cdot l(Q_j) \leq \text{dist}(Q_j, F^c) \leq 4 \sqrt{d} \cdot l(Q_j) \), where \( l(Q_j) \) denotes the side’s length of \( Q_j \).

**Lemma 2.2.** Let \( \omega \in A_1 \) and \( f \in L^1(\omega) \). Set \( E := \{ Mf(x) > \frac{1}{|a|_{\infty}} \} \) where \( M \) is the Hardy-Littlewood maximal operator. Then for \( a \in L^\infty(\mathbb{R}^d) \) and \( \lambda > 0 \), we have the following conclusions:

(i) \( E = \bigcup_n Q_n \), \( Q_n \)'s are disjoint dyadic cubes.
(ii) \( \omega(E) \leq C \frac{\lambda}{\lambda} \| f \|_{1,\omega} \).
(iii) \( f = g + b \).
(iv) \( b = \sum b_n, \supp b_n \subset Q_n, \int b_n = 0, \| b_n \|_1 \leq C \frac{\lambda}{|a|_{\infty}} \| Q_n \|, \| b \|_{1,\omega} \leq C \| f \|_{1,\omega} \).
(v) \( \| g \|_{2,\omega} \leq C \frac{\lambda}{|a|_{\infty}} \| f \|_{1,\omega} \).

**Proof.** Since \( E \) is open, we can make a dyadic Whitney decomposition of the set \( E \). Thus \( E \) is the union of the disjoint dyadic cubes \( Q_n \) and we have

\begin{equation}
\sqrt{d} \cdot l(Q_n) < \text{dist}(Q_n, E^c) < 4 \sqrt{d} \cdot l(Q_n) \tag{2.1}
\end{equation}

By the weighted weak type \((1,1)\) of \( M \), we have

\begin{equation}
\omega(E) \leq C \frac{\lambda}{\lambda} \| f \|_{1,\omega} \tag{2.2}
\end{equation}

We write \( f = g + b \), where \( g = f \chi_E + \sum_n \frac{1}{|Q_n|} \int_{Q_n} f(x) dx \chi_{Q_n} \), \( b = \sum_n \{ f - \frac{1}{|Q_n|} \int_{Q_n} f(x) dx \} \chi_{Q_n} =: \sum b_n \). So, \( b_n \) supports in \( Q_n \) and \( \int b_n = 0 \). Let \( tQ_n \) denote the cube with \( t \) times the side length of \( Q_n \) and the same center. We first claim that

\begin{equation}
\frac{1}{|Q_n|} \int_{Q_n} |f(x)| dx \leq C \frac{\lambda}{|a|_{\infty}} \tag{2.3}
\end{equation}

In fact, by the Whitney decomposition’s property \( (2.1) \) we have \( 9 \sqrt{d} Q_n \cap E^c \neq \emptyset \). Thus by the definition of \( E \), there exists \( x_0 \in 9 \sqrt{d} Q_n \) such that \( Mf(x_0) \leq \frac{\lambda}{|a|_{\infty}} \). Using the property of maximal function, we have \( \frac{1}{|9 \sqrt{d} Q_n|} \int_{9 \sqrt{d} Q_n} |f(x)| dx \leq C \frac{\lambda}{|a|_{\infty}} \). Hence we have the estimate

\begin{equation}
\frac{1}{|Q_n|} \int_{Q_n} |f(x)| dx \leq \frac{1}{|Q_n|} \int_{9 \sqrt{d} Q_n} |f(x)| dx \leq C \frac{\lambda}{|a|_{\infty}} \tag{2.4}
\end{equation}

So, \( \int_{Q_n} |f(x)| dx \leq \frac{\lambda}{|a|_{\infty}} \| f \|_{1,\omega} \), and we conclude that \( \omega(E) \leq C \frac{\lambda}{\lambda} \| f \|_{1,\omega} \).
For $b_n$ and $b$, by (2.2) and (2.3) we have
\[ \|b_n\|_1 \leq 2 \int_{Q_n} |f(x)|dx \leq C \frac{\lambda}{\|a\|_\infty} |Q_n|, \]
\[ \|b\|_{1,\omega} \leq \|f\|_{1,\omega} + C \frac{\lambda}{\|a\|_\infty} \omega(E) \leq C \frac{\lambda}{\|a\|_\infty} \|f\|_{1,\omega}. \]

Note that if $x \in E^c$, it is obvious that $|f(x)| \leq \frac{\lambda}{\|a\|_\infty}$. Using this fact, (2.2) and (2.3), we have
\[ \|g\|_{2,\omega}^2 \leq \frac{\lambda}{\|a\|_\infty} \|f\|_{1,\omega} + C \left( \frac{\lambda}{\|a\|_\infty} \right)^2 \omega(E) \leq C \frac{\lambda}{\|a\|_\infty} \|f\|_{1,\omega}. \]

\[ \square \]

In the following we use Lemma 2.2 for $d = 2$ and $\omega(x) = |x|^\alpha$ with $-2 < \alpha < 0$. Denote $\Omega_k = \{Q_n : l(Q_n) = 2^k\}$ and let $B_k = \sum_{Q \in \Omega_k} b_Q$. Taking a smooth function $\phi$ on $[0, \infty)$ such that $\text{supp} \phi \subset \{ \frac{1}{4} \leq |x| \leq 1 \}$ and $\sum_j \phi_j(x) = 1$ for all $x \in \mathbb{R}^2 \setminus \{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$. Write $K = \sum_j K_j$, where $K_j(x) = \phi_j(x)K(x)$ and define the corresponding operators $T_j$ with the kernel $K_j(x-y)m_{x,y}a$. Clearly we have $T_n = \sum_j T_j$.

We now state a lemma, which plays an important role in the proof of Theorem 1.1

**Lemma 2.3.** There exists an $\varepsilon > 0$ such that for any integer $s \geq 10$,
\[ (2.4) \quad \| \sum_j T_j B_{j-s} \|_{2,\omega}^2 \leq C 2^{-\varepsilon s} \lambda \|a\|_\infty \|b\|_{1,\omega}, \]
where $C$ is a constant depended on $K$ only.

The proof of Lemma 2.3 will be stated below. We now explain that Theorem 1.1 can be obtained by Lemma 2.3 and Theorem 1.2. In fact, for any $f \in L^1(\omega)$ and $\lambda > 0$, by Lemma 2.2 we have
\[ \omega(\{|T_a f(x)| > \lambda\}) \leq \omega(\{|T_a g(x)| > \lambda/2\}) + \omega(\{|T_a b(x)| > \lambda/2\}). \]
Since $g \in L^2(\omega)$, by Theorem 1.2 we have $\|T_a g\|_{2,\omega} \leq C \|a\|_\infty \|g\|_{2,\omega}$. Hence, by Chebychev’s inequality and Lemma 2.2,
\[ \omega(\{|T_a g(x)| > \lambda/2\}) \leq 4 \|T_a g\|_{2,\omega}^2/\lambda^2 \leq C \frac{\|a\|_\infty^2 \lambda \|f\|_{1,\omega}}{\|a\|_\infty^2 \lambda^2} = C \|a\|_\infty \frac{\|f\|_{1,\omega}}{\lambda}. \]

Let $E^* = \bigcup 2^{11} Q_n$. Then we have
\[ \omega(\{|T_a b(x)| > \lambda/2\}) \leq \omega(E^*) + \omega(\{x \in (E^*)^c : |T_a b(x)| > \lambda/2\}). \]
Since $\omega$ satisfies the doubling condition, the set $E^*$ satisfies
\[ (2.5) \quad \omega(E^*) \leq C \omega(E) \leq C \frac{\|a\|_\infty}{\lambda} \|f\|_{1,\omega}. \]
We write
\[ T_a b(x) = \sum_{s \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} T_j B_{j-s}. \]
Note that $T_j B_{j-s}(x) = 0$, for $x \in (E^*)^c$ and $s < 10$. Therefore
\[
\omega \left( \left\{ x \in (E^*)^c : T_a b(x) > \frac{\lambda}{2} \right\} \right) = \omega \left( \left\{ x \in (E^*)^c : \sum_{s \geq 10} \sum_{j \in \mathbb{Z}} T_j B_{j-s}(x) > \frac{\lambda}{2} \right\} \right).
\]
From Lemma 2.3 we get
\[
\| \sum_{s \geq 10} \sum_{j \in \mathbb{Z}} T_j B_{j-s} \|_{2, \omega}^2 \leq \left( \sum_{s \geq 10} \| T_j B_{j-s} \|_{2, \omega} \right)^2 \leq C \lambda \| a \|_{\infty} \| b \|_{1, \omega}.
\]
By Chebychev’s inequality, we have
\[
\omega \left( \left\{ x \in (E^*)^c : \sum_{s \geq 10} \sum_{j \in \mathbb{Z}} T_j B_{j-s}(x) \right\} \right) \leq C \| a \|_{\infty} \frac{\| b \|_{1, \omega}}{\lambda} \leq C \| a \|_{\infty} \frac{\| f \|_{1, \omega}}{\lambda}.
\]
Hence we get the conclusion of Theorem 1.1. Thus, to complete the proof of Theorem 1.1, it suffices to show Lemma 2.3 and Theorem 1.2. The proof of Theorem 1.2 will be given in Section 4. Let us first state Lemma 2.3. We write
\[
\| \sum_{j \in \mathbb{Z}} T_j B_{j-s} \|_{2, \omega}^2 = \sum_{i, j \in \mathbb{Z}} \langle T_j B_{j-s}, T_i B_{i-s} \rangle_{\omega}
\]
\[
= \sum_{j \in \mathbb{Z}} \| T_j B_{j-s} \|_{2, \omega}^2 + 2 \sum_{i} \sum_{j=1}^{i-2} \langle T_j B_{j-s}, T_i B_{i-s} \rangle_{\omega} + 2 \sum_{i} \sum_{j=1}^{i-3} \sum_{j \leq j'} \langle T_j B_{j-s}, T_i B_{i-s} \rangle_{\omega}
\]
\[=: I + II + III,
\]
where $\langle u, v \rangle_{\omega} = \int u(x) v(x) \omega(x) dx$ for the real valued functions $u$ and $v$.

Note that the estimate of $II$ can be reduced to $I$:
\[
2 \left| \sum_{i} \sum_{j=1}^{i-2} \langle T_j B_{j-s}, T_i B_{i-s} \rangle_{\omega} \right| \leq \sum_{i} \sum_{j=1}^{i-2} \left( \| T_j B_{j-s} \|_{2, \omega}^2 + \| T_i B_{i-s} \|_{2, \omega}^2 \right)
\]
\[
\leq 4 \sum_{i} \| T_i B_{i-s} \|_{2, \omega}^2.
\]
Hence, if we can establish the following lemma, then we may get the estimate of $I$ and $II$.

**Lemma 2.4.** There exists an $\varepsilon > 0$ such that for any fixed $s \geq 10$,
\[
\| T_j B_{j-s} \|_{2, \omega}^2 \leq C 2^{-\varepsilon s} \lambda \| a \|_{\infty} \| B_{j-s} \|_{1, \omega},
\]
where $C$ is a constant dependent on the properties of $K$.

To handle the cross terms $III$, we need the following conclusion:

**Lemma 2.5.** There exist $C$, $\varepsilon > 0$ such that
\[
\left| \sum_{i} \sum_{j \leq i-3} \langle T_j B_{j-s}, T_i B_{i-s} \rangle_{\omega} \right| \leq C 2^{-\varepsilon s} \lambda \| a \|_{\infty} \| b \|_{1, \omega}
\]
for any $s \geq 10$.

So, to get Lemma 2.3, it remains to prove Lemma 2.4 and Lemma 2.5 which will be given in the following section.
3. Proofs of Lemma 2.4 and Lemma 2.5

3.1 Proof of Lemma 2.4

First let us consider Lemma 2.4. For any \( i, j \in \mathbb{Z} \), we write
\[
\langle T_j B_{j-s}, T_i B_{i-s} \rangle = \langle (T_i^* T_j)_\omega B_{j-s}, B_{i-s} \rangle,
\]
where \((T_i^* T_j)_\omega\) has the kernel
\[
K_{i,j}(y, x) = \int K_i(z - y) K_j(z - x) m_{x,z} a \cdot m_{y,z} a \cdot \omega(z) dz.
\]
Hence we can write
\[
\|T_j B_{j-s}\|_{2,\omega}^2 = \langle (T_i^* T_j)_\omega B_{j-s}, B_{j-s} \rangle,
\]
It is easy to see that the following two lemmas are the key to proving Lemma 2.4.

**Lemma 3.1.** For \(|y| > 2^{j+1}\) or \(|y| < 2^{j-3}\), there exist \( C, \varepsilon > 0 \) such that
\[
|(T_i^* T_j)_\omega B_{j-s}(y)| \leq C 2^{-\varepsilon s} \|a\|_\infty \omega(y)
\]
for any integer \( s \geq 10 \).

**Lemma 3.2.** For \( 2^{j-3} \leq |y| \leq 2^{j+1} \), there exist \( C, \varepsilon > 0 \) such that
\[
|(T_i^* T_j)_\omega B_{j-s}(y)| \leq C 2^{-\varepsilon s} \|a\|_\infty \omega(y)
\]
for any integer \( s \geq 10 \).

**Proof of Lemma 3.1.** We claim that the kernel \( K_{j,j} \) which is given by (3.1) has the Hölder smoothness:
\[
|K_{j,j}(y, x) - K_{j,j}(y, x')| \leq C 2^{\frac{3j}{2^s}} |x - x'|^{\frac{3j}{2^s} - 3} \|a\|_\infty^2 \omega(y),
\]
for any \(|x - x'| \leq 2^{j-10}\). Once we establish (3.2), we can get Lemma 3.1. In fact, write
\[
\left| \int K_{j,j}(y, x) B_{j-s}(x) dx \right| = \left| \sum_{Q_n \in \Omega_{j-s}} \int K_{j,j}(y, x) b_n(x) dx \right|
\]
\[
\leq \left| \sum_{Q_n \in \Omega_{j-s}} \int_{|y-x|>102^j |x-x_Qn|^{\frac{3j}{2^s}}} (K_{j,j}(y, x) - K_{j,j}(y, x_{Q_n})) b_n(x) dx \right|
\]
\[
+ \left| \sum_{Q_n \in \Omega_{j-s}} \int_{|y-x|<102^j |x-x_Qn|^{\frac{3j}{2^s}}} (K_{j,j}(y, x) - K_{j,j}(y, x_{Q_n})) b_n(x) dx \right|
\]
\[
=: J_1 + J_2,
\]
where \( x_{Q_n} \) denotes the center of \( Q_n \). For \( J_1 \), by using the Hölder smoothness (3.2) we have
\[
J_1 \leq \sum_{Q_n \in \Omega_{j-s}} C 2^{\frac{3j}{2^s}} |a|_\infty^2 \omega(y) \int |x - x_{Q_n}|^{\frac{3j}{2^s}} |b_n(x)| dx
\]
\[
\leq C 2^{-2j} 2^{-\frac{3j}{2^s} \omega(y)} |a|_\infty \sum_{\text{dist}(Q_n,y) \leq 2^{j+1}} |Q_n|
\]
\[
\leq C 2^{-\frac{3j}{2^s} \omega(y)} |a|_\infty.
\]
For $J_2$, we have

$$J_2 \leq C2^{-2j}\omega(y)\lambda\|a\|_{\infty}\sum_{\text{dist}(Q_n,y) \leq 10 \cdot 2^{j-\frac{5}{4}}} |Q_n| \leq C2^{-\frac{3}{2}}\|a\|_{\infty}\lambda\omega(y).$$

Then we can choose $\varepsilon = \frac{1}{2^j}$. To obtain the kernel’s H"{o}lder smoothness (3.2), write

$$K_{j,j}(y,x) - K_{j,j}(y,x') =: A_1 + A_2,$$

where

$$A_1 = \int (K_j(z-x) - K_j(z-x'))K_j(z-y)m_{x,y}a \cdot m_{y,z}a \cdot \omega(z)dz$$

and

$$A_2 = \int K_j(z-x')K_j(z-y)(m_{x,y}a - m_{x',y}a)m_{y,z}a \cdot \omega(z)dz.$$

Note that $2^{j-2} \leq |z - y| \leq 2^j$. Then $|z| \geq C \cdot \max\{|y|, 2^{j-3}\}$ when $|y| < 2^{j-3}$ or $|y| > 2^{j+1}$. Thus, $\omega(z) \leq C\omega(y)$. Since $K_j$ is a smooth function with compact support, we have

$$|K_j(z-x) - K_j(z-x')| = \int_0^1 \langle x' - x, \nabla K_j(z - (1-s)x' - sx)\rangle ds \leq C2^{-3j}|x-x'|.$$

Therefore

$$|A_1| \leq C2^{-3j}\|a\|_{\infty}^2\omega(y)|x-x'|.$$

To estimate $A_2$, we switch to polar coordinates $z = y + r\theta$, then

$$A_2 = \int_{S^1} \int_{2^{j-2}}^{2^j} \psi(r)(m_{x,y+\theta}a - m_{x',y+\theta}a)\omega(y + r\theta)drd\theta,$$

where

$$\psi(r) = K_j(y-x' + r\theta)K_j(r\theta)m_{y,y+\theta}a \cdot r = K_j(y-x' + r\theta)K_j(r\theta)\int_0^r a(y + s\theta)ds.$$

It is easy to see that $\|\psi\|_{\infty} \leq C2^{-3j}\|a\|_{\infty}$ and $\|\psi'\|_{\infty} \leq C2^{-4j}\|a\|_{\infty}$.

We first split the integral over $S^1$ as a sum over the arc

$$\left| \theta \pm \frac{x - y}{|x - y|} \right| < t_0$$

and its complement, $t_0$ will be chosen later as $C2^{-\frac{5}{4}}|x-x'|$. Therefore the part of the integral in (3.4) over this arc is bounded by $C2^{-2j}|a|_{\infty}^2 t_0\omega(y)$.

Now we reduce $A_2$ to estimate the part of the outer integral in (3.4) over the set

$$\left| \theta \pm \frac{x - y}{|x - y|} \right| \geq t_0.$$

By a rotation, without loss of generality, we can assume that $\theta = (1,0)$,

$$N = \int_{2^{j-2}}^{2^j} \psi(r)\int_0^1 a(x + s((r,0) - x + y))\omega(y + r(1,0))dsdr$$

and

$$- \int_{2^{j-2}}^{2^j} \psi(r')\int_0^1 a(x' + s((r',0) - x' + y))\omega(y + r'(1,0))dsdr'.$$
We make a coordinate transform. For the first term in the above integral, set
\[ u = x_1 + s(r - x_1 + y_1) \quad \text{and} \quad v = x_2 + s(-x_2 + y_2), \]
then \( r = \frac{u-x_1}{v-2} (y_2 - x_2) - y_1 + x_1. \) For the second term, let
\[ u = x_1' + s(r' - x_1' + y_1) \quad \text{and} \quad v = x_2 + s(-x_2' + y_2), \]
then \( r' = \frac{u-x_1'}{v-x_2'} (y_2 - x_2') - y_1 + x_1'. \) Therefore, we have
\[ N = \iint_A \psi(r) a(u,v) \omega(y + r(1,0)) \frac{dudv}{|x_2 - v|} - \iint_{A'} \psi(r') a(u,v) \omega(y + r'(1,0)) \frac{dudv}{|x_2 - v|}, \]
where \( A \) is the triangle with vertices \( \{ y + (2j-2,0), y + (2j,0), x \} \) and \( A' \) is the triangle with vertices \( \{ y + (2j-2,0), y + (2j,0), x' \} \). By symmetric, we may assume \( x_2 > y_2 \). Observe that
\[ \iint_A \frac{dudv}{|x_2 - v|} = \int_{y_2}^{x_2} \frac{3/4}{x_2 - y_2} \frac{dv}{x_2 - v} = \frac{3}{4} 2^{j}. \]
Now we assume that
\[ |x - x'| \leq 2^{j-10} \quad \text{and} \quad |x - y| > 10|x - x'| \frac{2^j}{2^{10}} \]
and set \( t_0 = 10|x - x'| \frac{2^j}{2^{10}}. \) Since \( |(1,0) \pm \frac{x-y}{x-y}| \geq t_0 \) and \( t_0 \) is small relative to \( 2^j \), we have
\[ \frac{|x_2 - y_2|}{|x - y|} > \frac{1}{10} t_0 = |x - x'| \frac{2^j}{2^{10}}. \]
Then \( |x_2 - y_2| \geq 10|x - x'| \frac{2^j}{2^{10}}. \) By using an analogous method we obtain \( |x_2 - y_2| \geq 9|x - x'| \frac{1}{2} 2^j. \)
Using polar coordinate transform we get
\[
\left| \iint_{A \cap B(x,2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}})} \frac{dudv}{|x_2 - v|} \right| \leq \iint_0^{2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}}} \frac{drd\theta}{|\sin \theta|} \leq C 2^j |x - x'| \frac{2^j}{|x_2 - y_2|} \leq C 2^{10j} |x - x'|^{\frac{1}{2^{10}}},
\]
where the angle \( \theta \) is between vector \( v - x \) and \( (1,0) \) and the second inequality comes from the geometry estimate \( |\sin \theta| \geq C \frac{2^{j-10}}{2^{10}}. \) So we have
\[
\left(3.5\right) \quad \left| \iint_{A \cap B(x,2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}})} \psi(r) a(u,v) \omega(y + (r,0)) \frac{dudv}{|x_2 - v|} \right| \leq C 2^{-3j} \| a \|_{L^\infty}^2 \omega(y) 2^{10j} |x - x'|^{\frac{1}{2^{10}}}. 
\]
Since \( A' \cap B(x, 2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}}) \subseteq A' \cap B(x', 2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}}) \), by an analogous method we also have the estimate
\[
\left(3.6\right) \quad \left| \iint_{A' \cap B(x,2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}})} \psi(r') a(u,v) \omega(y + (r',0)) \frac{dudv}{|x_2 - v|} \right| \leq C 2^{-41j} \| a \|_{L^\infty}^2 \omega(y) |x - x'|^{\frac{1}{2^{10}}}. 
\]
Now we denote \( A' \Delta A = (A' \setminus A) \cup (A \setminus A') \). We claim that
\[
\left(3.7\right) \quad \iint_{(A \Delta A)' \cap B(x,2^{\frac{j}{2}}|x-x'|^{\frac{1}{2}})} \frac{dudv}{|x_2 - v|} \leq C 2^{10j} |x - x'|^{\frac{1}{2^{10}}}. 
\]
Then by (3.7) we have
\[ \left| \int_{(A \cap A') \setminus B(x, 2^{\frac{2}{3}}|x - x'|)} \psi(r) a(u, v) \omega(y + (r, 0)) \frac{dudv}{|x_2 - v|} \right| \leq C 2^{-\frac{30}{11}j} \|a\|_{\infty} \omega(y) |x - x'|^{\frac{1}{30}}. \]

Now we come back to prove (3.7) first in the case \( x'_2 = x_2 \). By similar triangles, we obtain that
\[ |(A' \triangle A) \cap \{ (a_1, a_2) : a_2 = v \}| \leq \frac{2|x - x'||v - y_2|}{x_2 - y_2}. \]

Then the integral in (3.7) has an estimate
\[ \int_{(A \cap A') \setminus B(x, 2^{\frac{2}{3}}|x - x'|)} \frac{dudv}{|x_2 - v|} \leq \int_{\gamma_2} \frac{2|x - x'| v - y_2}{x_2 - y_2} dv, \]

where \( c_0 \) is the minimum sine of the angle between vector \( y + (2^j - 2, 0) - x \) and \( (1, 0) \) and the angle between vector \( y + (2^j, 0) - x \) and \( (1, 0) \). By a geometry estimate we have \( c_0 \geq C \frac{|x_2 - y_2|}{2^j} \). So (3.8) is controlled by
\[ \frac{2|x - x'|}{|x_2 - y_2|} \int_{\gamma_2} \frac{x_2 - y_2 - v}{v} dv \leq C 2^{\frac{10}{30}j} |x - x'|^{\frac{1}{30}}. \]

Now we consider the case where \( x'_2 > x_2 \) or \( x_2 > x'_2 \). By symmetry we only look at the case \( x_2 > x'_2 > 0 \). We extend one of the sides of the shorter triangle \( A' \) to make it have the same height as \( A \). Then we find a point \( x'' \) at the extend side such that \( x_2 = x''_2 \). Since
\[ |x'' - x'| < C 2^{\frac{5}{3}j} |x - x'|^{\frac{4}{3}}, \]
then
\[ |x - x''| \leq C 2^{\frac{5}{3}j} |x - x'|^{\frac{4}{3}}. \]

Replacing \( A' \) by the larger triangle \( A'' \) with the vertex \( \{ x'', y + (2^j - 2, 0), y + (2^j, 0) \} \), then \( A \triangle A'' \) contains \( A \triangle A' \), and the ball \( B(x, 2^{\frac{2}{3}j}|x - x'|^{\frac{1}{3}}) \supset B(x, C 2^{\frac{11}{30}j}|x - x'|^{\frac{5}{30}}) \) for some constant \( C \). Therefore, we have
\[ \int_{(A \cap A') \setminus B(x, 2^{\frac{2}{3}j}|x - x'|^{\frac{1}{3}})} \frac{dudv}{|x_2 - v|} \leq \int_{(A \cap A'') \setminus B(x, C 2^{\frac{11}{30}j}|x - x'|^{\frac{5}{30}})} \frac{dudv}{|x_2 - v|}. \]

Use the same method as the case \( x_2 = x'_2 \), we can get that the right side of (3.9) is bounded by
\[ C 2^{\frac{10}{30}j} |x - x'|^{\frac{1}{30}}. \]

The remaining part is \((A \setminus A') \setminus B(x, 2^{\frac{2}{3}j}|x - x'|^{\frac{1}{3}})\). By straightforward computation we have
\[ |r - r'| \leq |x_1 - x_2| + \frac{|x_1 - x'_1||x_2 - x'_2||v - x'_2|}{|x_2 - v||x'_2 - v|} + \frac{|x_1 - x'_1||y_2 - x_2||v - x'_2| + |u - x'_1||y_2 - x'_2||x_2 - x'_2|}{|x_2 - v||x'_2 - v|} \]
\[ \leq C 2^{\frac{10}{30}j} |x - x'|^{\frac{1}{30}}. \]
Write
\[ \int_{A \cap A' \setminus B(x, 2^j |x - x'|^4)} \left[ \frac{\psi(r)a(u, v)\omega(y + r\theta)}{|x_2 - v|} - \frac{\psi(r')a(u, v)\omega(y + r'\theta)}{|x_2' - v|} \right] dudv =: G_1 + G_2 + G_3, \]
where
\[ G_1 = \int_{A \cap A' \setminus B(x, 2^j |x - x'|^4)} (\psi(r) - \psi(r'))a(u, v)\omega(y + r\theta) \frac{dudv}{|x_2 - v|}, \]
\[ G_2 = \int_{A \cap A' \setminus B(x, 2^j |x - x'|^4)} \psi(r')a(u, v)(\omega(y + r\theta) - \omega(y + r'\theta)) \frac{dudv}{|x_2 - v|}, \]
and
\[ G_3 = \int_{A \cap A' \setminus B(x, 2^j |x - x'|^4)} \psi(r')a(u, v)\omega(y + r'\theta) \left( \frac{1}{|x_2 - v|} - \frac{1}{|x_2' - v|} \right) dudv. \]

For \( G_1 \), by (3.10) and \(|z|^\alpha = |y + r\theta|^\alpha \leq \omega(y)\), we have
\[ |G_1| \leq C 2^{19j} |x - x'|^4 \omega(y)|a|_2^2 \int_{\mathbb{R}^n} |dudv|_{x_2 - v}| \leq C 2^{19j} |x - x'|^4 \omega(y)|a|_2^2 2^{-3j}. \]

For \( G_2 \), by (3.10), \(|y + r\theta| \geq C \cdot \max\{|y|, 2^j\}\) and \(|y + r'\theta| \geq C \cdot \max\{|y|, 2^j\}\), we have
\[ |G_2| \leq C 2^{19j} |x - x'|^4 \|a\|_2^2 2^{-3j} \int_{\mathbb{R}^n} \left| \nabla \omega(y + (sr + (1 - s)r')\theta) \right| ds \frac{dudv}{|x_2 - v|} \leq C 2^{19j} |x - x'|^4 \|a\|_2^2 \omega(y). \]

For \( G_3 \), we also get
\[ |G_3| \leq C 2^{19j} |x - x'|^4 \|a\|_2^2 \omega(y). \]
Combining above estimates, we have
\[ |K_{j,j}(y, x) - K_{j,j}(y, x')| \leq C 2^{19j} |x - x'|^4 2^{-3j} \|a\|_2^2 \omega(y) \]
for any \(|x - x'| \leq 2^j \frac{1}{1000}\). Hence, we complete the proof of Lemma 3.1 \( \square \)

**Proof of Lemma 3.2** Fix \( 2^{j-3} \leq |y| \leq 2^{j+1} \). Let \( K_j = K_j^1 + K_j^2 \), where
\[ K_j^1(z - y) = K_j(z - y) \chi(z - y), \]
and \( \chi \) is the characteristic function of the set \( \{x : \rho(x, y) \leq 2^{-\beta s}\} \) with \( \rho(x, y) = \frac{|y|}{|y - x|} \) and \( \beta \) will be chosen later. Since \((T_j^* T_j) \omega\) has kernel \( K_{j,j} \), we write \( K_{j,j} = K_{j,j}^1 + K_{j,j}^2 \), where
\[ K_{j,j}^1(y, x) = \int K_j(z - x)K_j^1(z - y) m_{x,z,a} m_{y,z,a} \omega(z) dz \]
and
\[ K_{j,j}^2(y, x) = \int K_j(z - x)K_j^2(z - y) m_{x,z,a} m_{y,z,a} \omega(z) dz. \]
Then we only need to prove Lemma 3.2 corresponding to \( K_{j,j}^1 \) and \( K_{j,j}^2 \). It is easy to check the term corresponding to \( K_{j,j}^1 \). Indeed, we have
\[ |K_{j,j}^1(y, x)| \leq C 2^{-dj} \|a\|_2^2 \int_{|z| \leq C|y|} \chi(z)|z + y|^\alpha dz \leq C 2^{-2j} \|a\|_2^2 |y|^{\alpha - 2^\beta s}, \]
Then the corresponding term

Applying the same method as proving Lemma 3.1 we may obtain

where we use the following inequality (see (0.3) in [6, p. 425])

\[ |a|^\alpha y^{\alpha+1} \leq \begin{cases} C |y|^{\alpha+1} \left( 1 + \frac{|x|}{|y|} + \frac{|y|}{|x|} \right) & \alpha \neq -1, \\ C \cdot \log^+ \left( \frac{|x|}{|y|} + \frac{|y|}{|x|} \right) & \alpha = -1. \end{cases} \]

Then the corresponding term

\[ (3.12) \quad \int K_{j,j}^1(y,x)B_{j-s}(x)dx \leq C2^{-2j} \|a\|_\infty^2 |y|^{2-\beta s} \int B_{j-s}(x)dx \leq C \|a\|_\infty \lambda(y) 2^{-\beta s}. \]

Now we consider the remaining term corresponding to \( K_{j,j}^2 \). By the definition of \( \lambda, \rho(y-z,y) \geq 2^{-\beta s} \). Since \( \rho(y-z,y) = |\frac{y-x}{|y-x|} - \frac{y}{|y|}| \leq 2\frac{|x|}{|y|} \), we have \( |y| \geq C2^{-\beta s+j} \).

\[ \omega(z) \leq C2^{-\alpha \beta s} \omega(y). \]

Applying the same method as proving Lemma 3.1 we may obtain

\[ \left| \int K_{j,j}^2(y,x)B_{j-s}(x)dx \right| \leq C \|a\|_\infty \lambda(y) 2^{\beta s(1-\alpha)-\frac{\epsilon}{\gamma}} \]

as long as we choose \( \beta = \frac{1}{10(1-\alpha)}. \)

3.2 Proof of Lemma 2.5

We write

\[ \sum_{i \in \mathbb{Z}} \sum_{j \leq i-3} \langle T_j B_{j-s}, T_i B_{i-s} \rangle \omega = \sum_{i \in \mathbb{Z}} \left( \sum_{j \leq i-3} (T_i^* T_j) \omega B_{j-s}, B_{i-s} \right). \]

To prove Lemma 2.5, it is easy to see that it suffices to prove the following lemma:

**Lemma 3.3.** For a fixed \( i \), there exist \( C, \varepsilon > 0 \) such that

\[ \left| \sum_{j \leq i-3} (T_i^* T_j) \omega B_{j-s}(y) \right| \leq C2^{-\varepsilon s} \|a\|_\infty \omega(y) \]

for any \( s \geq 10 \).

**Proof.** The proof is similar to that of Lemma 2.4. So we only give the difference. Consider two cases of \( y \) as Lemma 3.1 and Lemma 3.2 respectively. We first consider the case \( |y| < 2^{i-3} \) or \( |y| > 2^{i+1} \). We proceed with the proof as we do in Lemma 3.1. For the analogous term of \( A_2 \), we switch to the polar coordinates \( z = y + r\theta \)

\[ A_2 = \int_\lambda \int_{2^{-i-2}}^{2^i} \psi(r) (m_{x,y+r\theta} - m_{x',y+r\theta}) \omega(y + r\theta) dr d\theta, \]

where

\[ \psi(r) = K_j(y-x'+r\theta)K_i(r\theta)m_{y,y+r\theta} \cdot r \]

and \( \lambda \) is an arc in \( \mathbb{S}^1 \).

We claim that \( \lambda \) is an arc of length of about \( 2^{-i+j} \). Indeed, consider the support of \( K_j \) and \( K_i \), we have

\[ 2^{i-2} \leq |y - x' + r\theta| \leq 2^i \quad \text{and} \quad 2^{i-2} \leq r \leq 2^i. \]
Since \( j \leq i - 3 \), we get \( 2^{i-3} \leq |y - x'| \leq 2^{i+1} \). Let \( \Theta \) be the smallest cone with vertex at origin which contains the disc of radius \( 2^j \) at \( y - x' \). Then the angle of \( \Theta \) is at most a constant multiple of \( 2^{-i+j} \). Since \( |y - x'| - 2^j \leq r \leq |y - x'| + 2^j \), so the integrate area on \( r \) is
\[
[|y - x'| - 2^j, |y - x'| + 2^j] \cap [2^{i-2}, 2^i]
\]
and we set it as \([r_1, r_2]\). Using \( j \leq i - 3 \), we have the estimates \( \|\psi\|_\infty \leq C2^{-i}2^{-2j}\|a\|_\infty \) and \( \|\psi'\|_\infty \leq C2^{-i}2^{-3j}\|a\|_\infty \). After making a coordinate transform back, we get the integrate area \( A \) and \( A' \), where \( A \) is the triangle with vertices \( \{y + (r_1, 0), y + (r_2, 0), x\} \) and \( A' \) is the triangle with vertices \( \{y + (r_1, 0), y + (r_2, 0), x'\} \). Then we have
\[
\int_A \frac{dudv}{|x_2 - v|} = \int_{y_2}^{x_2} (r_2 - r_1) \frac{x_2 - v}{x_2 - y_2} \frac{dv}{x_2 - v} = r_2 - r_1 \leq 2^{j+1}.
\]
Since we have assumed that \( |x - x'| \leq 2^j \frac{1}{1000} \), so it is not necessary to restrict \( |x - y| > 10|x - x'|\frac{1}{1000}2^j \) anymore. We just need \( \frac{1}{2^j|x - y|} > \frac{1}{10}t_0 \) and \( 2^{i-3} \leq |y - x| \leq 2^{i+1} \). At last we obtain the Hölder smoothness estimate
\[
(3.13) \quad |K_{i,j}(y, x) - K_{i,j}(y, x')| \leq C2^{-2i}2^{-j/20}|x - x'|^{\frac{1}{20}}\omega(y).
\]
Now we take a cube \( Q_n \) with side length \( 2^{j-s} \) and use the Hölder smoothness estimate (3.13) to get
\[
\left| \int K_{i,j}(y, x)b_n(x)dx \right| = \left| \int K_{i,j}(y, x) - K_{i,j}(y, xQ_n)b_n(x)dx \right| \leq C2^{-s/20}2^{-2i}\omega(y)\|a\|_\infty\|b_n\|_1
\]
where \( xQ_n \) is the center of \( Q_n \). For a fixed \( y \) the kernel \( K_{i,j}(y, x) \) is supported in \( B(y, 2^{i+1}) \). Hence we have
\[
\left| \int K_{i,j}(y, x)B_{j-s}(x)dx \right| \leq C\|a\|_\infty^2 2^{-\frac{s}{20}}2^{-2i}\omega(y) \sum_{Q_n \subset B(y, 2^{i+1})} \|b_n\|_1.
\]
Then we sum over \( j \leq i - 3 \). Note that the cubes \( Q_n \) are disjoint each other, therefore we get
\[
(3.14) \quad \sum_{j \leq i - 3} \sum_{Q_n \subset B(y, 2^{i+1})} \|b_n\|_1 \leq \sum_{j \leq i - 3} \sum_{Q_n \subset B(y, 2^{i+1})} C\lambda |Q_n| \|a\|_\infty \|b_n\|_1 \leq C2^{2i}\lambda \|a\|_\infty.
\]
where the last inequality comes from all the cubes that appear in (3.14) are contained in a disc of radius \( 2^{i+1} \). Hence we have
\[
\left| \sum_{j \leq i - 3} (T^*_iT_j)\omega B_{j-s}(y) \right| = \sum_{j \leq i - 3} \left| \int K_{i,j}(y, x)B_{j-s}(x)dx \right| \leq C2^{-2s}\lambda \|a\|_\infty \omega(y).
\]
For the case \( 2^{i-3} < |y| < 2^{i+1} \). As in Lemma 3.2 we write \( K_i = K_i^1 + K_i^2 \) and denote
\[
K_i^{1,j} = \int K_j(z - x)K_i^{1}(z - y)m_{x,z}a \cdot \omega(z)dz
\]
and
\[
K_i^{2,j} = \int K_j(z - x)K_i^{2}(z - y)m_{x,z}a \cdot \omega(z)dz.
\]
A similar argument, which has been used to deal with the term corresponding to $K^1_{i,j}$ in proving Lemma 3.2, can be applied to estimate $K^1_{i,j}$. Combining the method we deal with $K^2_{i,j}$ in Lemma 3.2 and the method we handle $K_{i,j}$ in the case $|y| \geq 2^{i+1}$ or $|y| < 2^{j-3}$, we can get the proof of the term related to $K^2_{i,j}$. Therefore we complete the proof of Lemma 3.3

\[ \square \]

## 4. Proof of Theorem 1.2

In [7], Hofmann gave a weighted $L^p$ boundedness for general singular integral operators. We will show that the commutator $T_a$ discussed in this paper is an example of that general operator.

Before stating the theorem in [7], let us give some notations. For an open set $\Omega$ in $\mathbb{R}^d$, we denote by $C_c^\infty(\Omega)$ the set of functions with continuous derivatives of any order and compact support in $\Omega$. Let $\psi \in C_c^\infty(\{|x| < 1\})$ be radial, non-trivial, have mean value zero, and be normalized so that $\int_0^\infty |\hat{\psi}(s)|^2 \frac{ds}{s} = 1$, then we define $Q_s f = \psi_s * f$, where $\psi_s(x) = s^{-d} \psi(\frac{x}{s})$.

Let $\mathcal{D}$ denote the space of smooth functions with compact support in $\mathbb{R}^d$ and $\mathcal{D}'$ be its dual space. We assume that $T$ maps $\mathcal{D}$ to $\mathcal{D}'$ and $T$ is associated a kernel $K(x,y)$ in the sense that for $f, g \in C_c^\infty(\mathbb{R}^d)$ with disjoint support

\[ \langle Tf, g \rangle = \int \int K(x,y) f(y) g(x) dy dx. \]

We will introduce some conditions similar to the conditions of T1 Theorem. We first suppose the kernel $K$ satisfies the size condition:

\[ |K(x,y)| \leq C_1 |x - y|^{-d}. \]  

Let $\varphi \in C_c^\infty(\frac{1}{2}, 2)$. Then for $v \in (0, \infty)$ we set

\[ T_v f(x) = \int K(x,y) \varphi\left(\frac{|x - y|}{v}\right) f(y) dy. \]

We introduce the weak smoothness condition (WS):

\[ \|Q_s T_v\|_{op} + \|Q_s T_v^*\|_{op} \leq C_2 \|\psi\|_1 (\|\varphi\|_\infty + \|\varphi'\|_\infty) \left(\frac{s}{v}\right)^{\varepsilon_0}. \]  

for some $0 < \varepsilon_0 < 1$ and $s < v$, where $T^*$ is the adjoint operator of $T$ and $\| \cdot \|_{op}$ denotes the norm of operator mapping $L^2$ to $L^2$.

As usual, we require the weak boundedness property (WBP):

\[ \langle Th, \tilde{h} \rangle \leq C_3 R^d (\|h\|_\infty + R \|\nabla h\|_\infty) (\|\tilde{h}\|_\infty + R \|\nabla \tilde{h}\|_\infty). \]

for all $h, \tilde{h} \in C_c^\infty(\mathbb{R}^d)$ with support in any ball of radius $R$.

To define T1, we impose the qualitative technical condition (QT):

\[ \int_{|x - u| > 2s} |\int \psi_s(x - z) K(z, u) dz| du < \infty, \]

\[ \int_{|x - u| > 2s} |\int \psi_s(x - z) K^*(z, u) dz| du < \infty, \]
where $K^*(z,u)$ is the kernel of $T^*$. Let $\tilde{\psi} \in C_c^\infty(B(x_0,s))$ and $\int \tilde{\psi}(x)dx = 0$. We write $1 = h + (1 - h)$, where $h \in C_c^\infty(B(x_0,4s))$ and $h \equiv 1$ for $|x - x_0| \leq 2s$. Then we define

$$\langle \tilde{\psi}, T \rangle = \langle \tilde{\psi}, Th \rangle + \langle T^* \tilde{\psi}, 1 - h \rangle,$$

where the second term is well defined by (4.4).

We consider truncations of $T$. Let $\Phi \in C_c^\infty(-1, 1)$ and $\Phi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. For $t < r$ and $f \in L^p$ with $1 < p < \infty$, $T_{(t,r)}$ is defined as follows:

$$T_{(t,r)}f(x) = \int K(x,y)\Phi\left(\frac{|x-y|}{r}\right) - \Phi\left(\frac{|x-y|}{t}\right) f(y)dy.$$

$T_{(0,r)}$ can be defined formally:

$$T_{(0,r)}f(x) = \int K(x,y)\Phi\left(\frac{|x-y|}{r}\right) f(y)dy.$$

Now we need two conditions to replace the usual condition $T1$, $T^*1 \in BMO$. One is the quasi-Carleson measure condition (QCM): For any ball $B$ of radius $10\sqrt{dt}$, $1 < q < \infty$, $t > 0$, we have

$$\left\| \left( \int_0^t |Q_sT_{(0,t)}1| \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^q(B, \frac{ds}{t})} \leq C_4,$$

where $\frac{ds}{t}$ denotes normalized Lebesgue measure.

The other is the local paraproduct type condition (LP): For all $r > 0$ and $1 < q < \infty$, for all $f \in C_c^\infty(\mathbb{R}^d)$ with support in any ball of radius $10\sqrt{dt}r$, we have

$$\|\pi_r f\|_{L^q(B, \frac{ds}{r})} < C_5\|f\|_\infty,$$

where

$$\pi_r f = \int_0^r \int_0^t Q_s T_{(t,r)} 1_{Q(t)} f \frac{ds}{s} \frac{dt}{t}.$$
where
\[ T_{a,v}f(x) = \int L(x,y)\varphi\left(\frac{|x-y|}{v}\right)f(y)dy. \]

The proof of Claim 1 is similar to the proof of Lemma 4.3 in [7]. We just give the difference by the following lemma.

**Lemma 4.1.** Let \( S_1 \) denote the convolution operator with the kernel \( H(x) = K(x)\varphi(|x|) \), where \( K \) is a Calderón-Zygmund convolution kernel. Then \( \|Q_sS_1\|_{op} \leq C\varepsilon_0 \) for \( s < 1 \) and \( 0 < \varepsilon_0 < 1 \).

**Proof.** In fact, by Plancherel theorem we only need to check \(|\hat{\psi}_s(\xi)\hat{H}(\xi)| \leq C s^{\varepsilon_0} \), for \( 0 < \varepsilon_0 < 1 \). We firstly give an estimate of \( \hat{H}(\xi) \). Write
\[ \hat{H}(\xi) = \int_{S^{d-1}} e^{-2\pi ir\theta \cdot \xi}K(r\theta)\varphi(r)r^{d-1}drd\theta. \]

By Van der Corput’s lemma, we have
\[ \left| \int e^{-2\pi ir\theta \cdot \xi}K(r\theta)\varphi(r)r^{d-1}dr \right| \leq \frac{C}{2\pi|\theta \cdot \xi|}, \]

On the other hand, by
\[ \left| \int e^{-2\pi ir\theta \cdot \xi}K(r\theta)\varphi(r)r^{d-1}dr \right| \leq C, \]

thus it is also dominated by \( C|\theta \cdot \xi|^{-\varepsilon_0} \) for any \( 0 < \varepsilon_0 < 1 \). So we have \(|\hat{H}(\xi)| \leq C|\xi|^{-\varepsilon_0} \). For \(|s\xi| > 1\), \(|\hat{\psi}_s(\xi)\hat{H}(\xi)| \leq \|\psi\|_{L^1}|\xi|^{-\varepsilon_0} \leq Cs^{\varepsilon_0} \). For \(|s\xi| \leq 1\), since
\[ |\hat{\psi}_s(\xi)| = |\hat{\psi}(s\xi)| = \left| \int (e^{-2\pi i\xi z} - 1)\hat{\psi}(z)dz \right| \leq C|s\xi|^{\varepsilon_0}, \]

we have \(|\hat{\psi}_s(\xi)\hat{H}(\xi)| \leq Cs^{\varepsilon_0} \) for \( 0 < \varepsilon_0 < 1 \). Hence we complete the proof. \( \square \)

**Claim 2:** The operator \( T_a \) satisfies the technical condition (4.3).

Since \( L^*(x,y) \) has the same form as \( L(x,y) \), it is sufficient to prove (4.3) for \( L(x,y) \). We need to use the following estimate (see Lemma 3 in [21 p. 68]):
\[ \left( \int_{x,y' \in B(x_0,R)} |(m_{x,y}a)^k - (m_{x,y'}a)^k|^2dydy' \right)^{1/2} \leq Ck^2\left( \frac{r}{R} \right)^\frac{k}{2}R^{2d}\|a\|_\infty^{2k}, \]

where \( 0 < r < R \), \( k \) is a positive integer. Since \( \psi \) has mean value zero, we have
\[ \int_{|x-u| > 2s} \left| \int \psi_s(x - z)K(z - u)m_{z,u}adz \right| du \leq H_1 + H_2, \]
where
\[ H_1 = \int_{|x-u| > 2s} \left| \int \psi_s(x - z)(K(z - u) - K(x - u))m_{z,u}adz \right| du, \]

and
\[ H_2 = \int_{|x-u| > 2s} \left| \int \psi_s(x - z)K(x - u)(m_{z,u}a - m_{x,u}a)dz \right| du. \]

For \( H_1 \), we have
\[ H_1 \leq \int_{|x - u| > 2s} \int_{|x - z| < s} \frac{|z - x|}{s^d} \cdot |x - u|^{d+1}dzdu \leq C. \]
For $H_2$, we have
\[
H_2 = \sum_{j=0}^{+\infty} \int_{2^{j+1}x > |x-u| > 2^j} \left| \int \psi_s(x-z)K(x-u)(m_{z,u}a - m_{x,u}a)dz \right| du =: \sum_{j=0}^{+\infty} m_j(x).
\]

Now we consider
\[
\frac{1}{|B(x_0, 2^js)|} \int_{B(x_0, 2^js)} m_j(x)dx \\
\leq \frac{1}{(2^js)^d v_n} \int_{B(x_0, 2^js)} \int_{2^{j+1}x > |x-u| > 2^j} \int_{|x-z| < s} \frac{C}{s^d (2^js)^d} |m_{z,u}a - m_{x,u}a| dz du dx \\
\leq \frac{1}{(2^js)^d v_n} \frac{C}{s^d (2^js)^d} \int \int \int_{x,z,u \in B(x_0, 2^js)} |m_{z,u}a - m_{x,u}a| dz du dx \\
\leq C 2^{-\frac{1}{2}j},
\]

where the third inequality follows from Hölder’s inequality and (1.8). Note that the constant $C$ is independent of $s$, so $m_j(x) \leq M m_j(x) \leq C \cdot 2^{-\frac{1}{2}j}$. We hence get
\[
H_2 \leq \sum_{j=0}^{+\infty} C \cdot 2^{-\frac{1}{2}j} < C.
\]

**Claim 3:** The operator $T_a$ satisfies the quasi-Carleson measure condition (4.5).

By dilation invariance we may take $t = 1$. Suppose $B$ is a ball of radius $10\sqrt{d}$ with center $x_0$. We have
\[
\langle T_{(0,1)}f, g \rangle = \iint K(x-y)m_{x,y}(a)\Phi(|x-y|)f(y)g(x)dydx.
\]

Here and in the sequel we still use the notation $T_{(0,1)}$. Choose $\eta \in C_\infty_c(\mathbb{R}^d)$, such that $\eta(x) = 1$ on $2B(x_0, 10\sqrt{d})$ and $\eta(x) = 0$ on $(4B(x_0, 10\sqrt{d}))^c$. By the support of $Q_sT_{(0,1)}1$, we have
\[
(\int_B \left( \int_0^1 |Q_sT_{(0,1)}1(x)|^2 ds \right)^{\frac{q}{2}} dx)^{\frac{1}{q}} = \left( \int_B \left( \int_0^1 |Q_sT_{(0,1)}\eta(x)|^2 ds \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}}.
\]

By Littlewood-Paley theory (See [3]), (4.9) is majorized by $(\int_B (T_{(0,1)}\eta(x))^q dx)^{\frac{1}{q}}$. If the operator with kernel $K(\cdot)\Phi(|\cdot|)$ is bounded on $L^2$, then by Christ’s result in [2], $\|T_{(0,1)}\|_{q \rightarrow q} \leq C\|a\|\infty$ for all $1 < q < +\infty$. Hence (4.9) is bounded.

Indeed, it is easy to check that $K(\cdot)\Phi(|\cdot|)$ is still a Calderón-Zygmund convolution kernel. Note that
\[
\hat{K}\hat{\Phi}(\xi) = \hat{K} \ast \hat{\Phi}(\xi) = \int_{\mathbb{R}^n} \hat{K}(y)\hat{\Phi}(|\xi - y|)dy
\]
is bounded since $K$ is a Calderón-Zygmund convolution kernel and $\hat{\Phi}$ is a Schwartz function. So, the operator with the kernel $K(\cdot)\Phi(|\cdot|)$, initially defined on Schwartz class, has a bounded extension to an operator mapping $L^2(\mathbb{R}^n)$ to itself.

**Claim 4:** The operator $T_a$ satisfies the local paraproduct condition (4.6).
By dilation invariance, we may take $r = 1$. Let $f, g \in C^\infty_c(\mathbb{R}^d)$ with support in $B(x_0, 10\sqrt{d})$. Fix $1 < q < \infty$, by duality we only need to prove

$$\left| \int_{\varepsilon}^{1} \int_{\delta}^{\varepsilon} (Q_sT_{(t,1)}1Q_s^2f, Q_sg) \frac{ds}{s}\frac{dt}{t} \right| \leq C\|a\|_\infty \|f\|_\infty \|g\|_q'.$$

We write

$$\leq \eta \text{ for all } 2 \leq p < +\infty. \text{ Make a smooth partition of unity and write }$$

$$\Phi\left(\frac{\rho}{t}\right) - \Phi(\rho) = \sum_{j:t \leq 2^j t \leq 4} \tilde{\varphi}\left(\frac{\rho}{2^j t}\right) \left(\Phi\left(\frac{\rho}{t}\right) - \Phi(\rho)\right),$$

where $\tilde{\varphi} \in C^\infty_c(\frac{1}{2}, 2)$ and $\sum_{j=-\infty}^{\infty} \tilde{\varphi}(\frac{\rho}{2^j}) \equiv 1$ for all $\rho > 0$. We define

$$T_{a,j}f(x) = \int L(x, y) \tilde{\varphi}\left(\frac{|x-y|}{2^j t}\right) \left(\Phi\left(\frac{|x-y|}{t}\right) - \Phi(x-y)\right) f(y) dy,$$

then $T_{(t,1)} = \sum_{\frac{1}{4} \leq 2^j t \leq 2} T_{a,j}$. Applying Minkowski inequality and (4.12), we have

$$\left(\frac{1}{t^n} \int_{|x-z| \leq t} |Q_sT_{(t,1)}\eta(z)|^{q'_1} dz\right)^{\frac{1}{q'_1}} \leq \left(\frac{1}{t^n} \right)^{\frac{1}{q'_1}} \sum_{\frac{1}{4} \leq 2^j t \leq 2} \|Q_sT_{a,j}\eta\|_{q'_1}$$

$$\leq \left(\frac{1}{t^n} \right)^{\frac{1}{q'_1}} \sum_{\frac{1}{4} \leq 2^j t \leq 2} C\|a\|_\infty \left(\frac{s}{2^j t}\right)^{\varepsilon(q'_1)} \||\eta\|_{q'_1}$$

$$\leq C\|a\|_\infty \left(\frac{s}{t}\right)^{\varepsilon(q'_1)}.$$

By estimates (4.11), (4.13) and Hölder’s inequality, the left side of (4.10) is bounded by

$$\leq C\|a\|_\infty \int_{\varepsilon}^{1} \int_{\delta}^{\varepsilon} \left(\frac{s}{t}\right)^{\varepsilon(q'_1)} \|Qsg\|_{2,\frac{1}{2}} \|Q_tf\|_{2,\frac{1}{2}} \frac{ds}{s}\frac{dt}{t}.$$
By using Hölder’s inequality again, the last term above is majorized by

\[(4.15) \quad C \|a\|_\infty \left( \int_\varepsilon^1 \int_\delta^t \left( \frac{s}{t} \right)^{\varepsilon(q'_1)} |Q_t f|_{2,\omega}^2 \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{2}} \left( \int_\varepsilon^1 \int_\delta^t \left( \frac{s}{t} \right)^{\varepsilon(q'_1)} |Q_t g|_{2,\omega}^2 \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{2}}.\]

Firstly, let us consider the first integral factor in (4.15). Note that
\[
\int_\delta^t \left( \frac{s}{t} \right)^{\varepsilon(q'_1)} ds \leq C,
\]

hence by weighted Littlewood-Paley theory (See [8]), we have
\[
\left( \int_\varepsilon^1 \int_\delta^t \left( \frac{s}{t} \right)^{\varepsilon(q'_1)} \|Q_t f\|_{2,\omega}^2 \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{2}} \leq C \left( \int_\varepsilon^1 \|Q_t f\|_{2,\omega}^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq C \|f\|_{2,\omega}.
\]

Then using the same method for the other factor, we get
\[
\left( \int_\delta^t \int_\varepsilon^{s} \left( \frac{s}{t} \right)^{\varepsilon(q'_1)} \|Q_s g\|_{2,\omega}^2 \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{2}} \leq C \left( \int_\delta^t \|Q_s g\|_{2,\omega}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \leq C \|g\|_{2,\omega}.
\]

Therefore (4.10) is controled by \( \|a\|_\infty \|f\|_{2,\omega} \|g\|_{2,\omega} \). By extrapolation (See [4]), we can replace this bound by \( \|a\|_\infty \|f\|_{q,\omega} \|g\|_{q',\omega} \), for any \(1 < q < \infty\). Now since \(\omega\) is a Muckenhoupt weight, we can replace the bound by \( \|a\|_\infty \|f\|_q \|g\|_{q'} \) by setting \(\omega \equiv 1\). Since \(f \in C_c^\infty(\mathbb{R}^d)\) with compact support, we have \( \|f\|_q \leq C \|f\|_\infty \). Hence we complete the proof of Theorem 1.2.

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