The possibility of **dissipatively** preparing interesting quantum states of matter in open quantum systems [1–7] provides a new paradigm in quantum engineering complementing both practically and conceptually ongoing efforts on the implementation of many-body Hamiltonians in closed quantum systems [8–12]. The guiding idea of the dissipative approach is to engineer a controlled interaction of a quantum system with its environment in order to realize an exotic state of quantum matter in a non-equilibrium fashion as the *unique* steady state of a quantum master equation, irrespective of the initial state. The degree of control over the system bath interaction determines the precision of the dissipative preparation. This is in contrast to the conventional Hamiltonian approach where the ability to reach sufficiently low temperatures is the key challenge in accessing a many-body ground state. The focus of this work is on inherent challenges which arise if one attempts to dissipatively prepare a particularly timely class of states, namely topological states of quantum matter [13–15]. Their generalization to the realm of open quantum systems out of thermal equilibrium is far from being conclusively understood despite its fundamental importance for realistic systems which are not perfectly isolated from their environment. Employing controlled dissipation to induce a topological state, a realization in terms of a dissipative master equation dynamics has been proposed [16, 17] for one-dimensional topological superconductors [18]. However, higher dimensional topological states, in particular states with a non-vanishing Chern number [19–21], have so far been elusive in this framework due to a fundamental interference between topology and locality. The degree of control over the system bath interaction determines the precision of the dissipative preparation. This is in contrast to the conventional Hamiltonian approach where the ability to reach sufficiently low temperatures is the key challenge in accessing a many-body ground state. The focus of this work is on inherent challenges which arise if one attempts to dissipatively prepare a particularly timely class of states, namely topological states of quantum matter [13–15]. Their generalization to the realm of open quantum systems out of thermal equilibrium is far from being conclusively understood despite its fundamental importance for realistic systems which are not perfectly isolated from their environment. Employing controlled dissipation to induce a topological state, a realization in terms of a dissipative master equation dynamics has been proposed [16, 17] for one-dimensional topological superconductors [18]. However, higher dimensional topological states, in particular states with a non-vanishing Chern number [19–21], have so far been elusive in this framework due to a fundamental interference between topology and the natural locality of the engineered system bath interaction.

Here, we report a novel mechanism coined “dissipative hole-plugging” to overcome this issue. Involving the interplay of two dissipative channels per degree of freedom in the system, our construction goes conceptually beyond intuition drawn from the Hamiltonian analog. We demonstrate that local system bath engineering can result in a master equation with a unique superfluid steady state that is characterized by a non-vanishing Chern number for fermions on a two-dimensional (2D) lattice. Quite remarkably, in our scheme, this topology from dissipation is robust, and exists as a stable topological phase in a non-equilibrium phase diagram. We demonstrate how our theoretical construction can be implemented in a microscopic model that is experimentally feasible with cold atoms in optical lattices.
Here, $\tau_j$ are the Pauli matrices in Nambu space, $a_k = \frac{1}{\sqrt{N}} \sum_j e^{ikj} \psi_j$ are the Fourier transforms of the field operators, and $|\tilde{n}_k| \leq 1$, where $|\tilde{n}_k| = 1$ holds for pure states. In the long time limit, the density matrix $\rho$ approaches a steady state $\rho^s$. The dissipative analog of an energy gap stabilizing a Hamiltonian ground state is a so called damping gap $\kappa$ [16, 17] (see supplementary material for technical details), which is defined as the smallest rate at which deviations from $\rho^s$ are damped out. In this sense, damping gapped steady states are analogous to insulating ground states in the Hamiltonian context.

**Interference of locality and topology** – The major conceptual challenge in the dissipative preparation of topological states in spatial dimension $d \geq 2$ is due to the competition of topology and locality of the operators $L_j$. Drawing intuition from Hamiltonian ground states, a generic recipe for preparing pure Gaussian steady states is the following [16, 17]: Construct a so called parent Hamiltonian $H_p = \sum_j L_j^\dagger L_j$, from a complete set of anti-commuting Lindblad operators $L_j$. The ground state $|G \rangle \langle G|$ of this Hamiltonian is then the unique steady state of the corresponding master equation (1) since $|G \rangle$ is the only state vector annihilated by all $L_j$. In other words, the Lindblad operators are chosen as single particle operators that span the many-body ground state of $H_p$. For a lattice translation invariant parent Hamiltonian that defines a band structure, a set of Lindblad operators $L_j$ providing a real space representation of its many-body ground state corresponds to the Wannier functions of all occupied bands. However, non-trivial topological invariants characterizing the ground state impose fundamental constraints on the localization properties of the Wannier functions [29–32] (see Ref. [33] for a detailed recent discussion). The archetype of a topological invariant for band structures is the integer quantized first Chern number [19–21] distinguishing topologically inequivalent gapped 2D band structures. By its very definition, a non-vanishing Chern number implies an obstruction to finding a global smooth gauge for the associated family of Bloch functions [20, 21], or, equivalently, the impossibility [29–31] to find an exponentially localized set of Wannier functions. Hence, for the dissipative preparation of a gapped state with non-vanishing Chern number based on a parent Hamiltonian, long-ranged Lindblad operators with algebraic asymptotic decay properties would be inevitable.

**Mixed state topology** – To overcome this issue, going beyond the Hamiltonian analogy and the realm of pure steady states turns out to be crucial. For mixed states, topological properties over the lattice momentum Brillouin zone (BZ) are well defined as long as there is a finite purity gap $|\tilde{n}_k|^2$ [17] (see Eq. (2)), i.e., as long as $\rho_k$ has a finite polarization at all lattice momenta $k$. The first Chern number [19] which is the relevant topological invariant for our study is here given by

$$C = \frac{i}{2\pi} \int_{\text{BZ}} \text{Tr} \left\{ \rho_k \left[ (\partial_{k_x} \rho_k), (\partial_{k_y} \rho_k) \right] \right\} (2\text{Tr}(\rho_k^2) - 1)^{\frac{3}{2}} = \frac{1}{4\pi} \int_{\text{BZ}} \tilde{n}_k \cdot \left[ (\partial_{k_x} \tilde{n}_k) \times (\partial_{k_y} \tilde{n}_k) \right]. \quad (3)$$

This integer quantized topological invariant measures how often the normalized polarization vector $\tilde{n}_k = \frac{\tilde{n}_k}{|\tilde{n}_k|}$ covers the unit sphere and can only change if either the purity gap closes (rendering $\tilde{n}_k$ ill defined at some $k$) or if the damping gap closes (giving rise to discontinuities in $k \mapsto \rho_k$). In this sense the topological steady states we are concerned with are protected by both a finite damping gap and a finite purity gap, i.e., their topology is unchanged under continuous deformations as long as both gaps are maintained. To induce a steady state with non-vanishing Chern number (3), we proceed in two steps. First, we construct a set $L^C_\rho$ of compactly supported Lindblad operators that yield a critical Chern state as a steady state. Second, we devise a set of auxiliary Lindblad operators $L^D$ which is capable of lifting the topologically non-trivial critical point to an extended phase with a finite damping gap.

**Over-completeness and damping-criticality** – We define Lindblad operators corresponding to a set of non-orthonormal single particle states coined pseudo Wannier functions. These operators compactly supported but span only a critical topological state at fine-tuned parameters. Concretely, let us consider

$$L^C_j = \sum_i u^C_{i-j} \psi_i + \psi^C_{i-j} \hat{\psi}^\dagger \quad (4)$$

where the only non-vanishing coefficients are $v^C_0 = \beta$, $v^C_{\pm \hat{x}} = v^C_{\pm \hat{y}} = 1$ and $v^C_{\hat{y} \hat{x}} = \pm 1$, supported only on nearest neighbor sites of the square lattice $\mathbb{Z}^2$. The steady state of the associated master equation (1) is given by the ground state of the translation invariant parent Hamiltonian $H^C_p = \sum_j L^C_j \dagger L^C_j$. The Fourier transformed Nambu spinors

$$B^C_k \equiv (\tilde{u}^C_k, \tilde{v}^C_k)^T = (2i(\sin(k_x) + i \sin(k_y)), \beta + 2(\cos(k_x) + \cos(k_y)))^T \quad (5)$$

are coined pseudo Bloch functions and are non-vanishing all over the BZ except at isolated values of $\beta$. At $\beta = -4$, for example, $(\tilde{u}^C_0, \tilde{v}^C_0) = (0, 0)$. Such a zero gives rise to a closing of the damping gap $\nu^C_0 = |\tilde{u}^C_0|^2 + |\tilde{v}^C_0|^2$, i.e., a critical damping behavior. Yet, the normalized polarization vector $\tilde{n}_k$ of the steady state $\rho^s_k$ continuously approaches the value $\tilde{n}_0 = e_3$ as $k \to 0$ and can thus be defined all over the BZ even at the critical point $\beta = -4$. Direct calculation of the associated Chern number using Eq. (3) yields $C = -1$. However, $\tilde{n}_k$ is not real analytic in $k = 0$ but only finitely differentiable. As a consequence $\rho^s_k$ at $\beta = -4$ corresponds to a critical Chern-state that can have algebraically decaying correlations. The pseudo Wannier operators (4) form an overcomplete set of that critical state since the vanishing pseudo Bloch function $B^C_0$ (see Eq. (5)) is ill defined as long as there is a finite purity gap $|\tilde{n}_k|^2$ [17] (see Eq. (2))...
n state. With the Lindblad operators given by Eq. (4), we have is to consider the third component $\hat{n}_3$ may be viewed as tearing a hole into the smooth winding of dissipative hole-plugging mechanism a occupation number $\langle \hat{n}_3 \rangle$ of the pseudo Bloch functions, e.g., Ref. [30]). For the same topological reason, the above deviation removes the essential zero from the pseudo Bloch functions, in turn rendering the state topologically trivial. We will now devise an auxiliary set of Lindblad operators $L_j^A$ which is capable of lifting the critical state resulting only from the fine-tuned $L_j^C$ operators to a gapped extended phase. Dissipative hole-plugging mechanism – Deviating by $\delta$ from the topologically non-trivial critical point, i.e., $\beta = -4 - \delta$ in (4) may be viewed as tearing a hole into the smooth winding of $\hat{n}_k$ as a function of $k$. The simplest way to see this is to consider the third component $\hat{n}_3^A$ of $\hat{n}_k$ in the steady state. With the Lindblad operators given by Eq. (4), we have $\hat{n}_3^A = |\tilde{u}_k^C|^2 - |\tilde{v}_k^C|^2$. Indeed, a finite value of $\tilde{v}_k^C = \delta$ enforces that $\hat{n}_0 = -\hat{n}_3$ since $\tilde{u}_0^C = 0$ (see Eq. (5)). In contrast, at small $k > \delta$, $\hat{u}_k = \tilde{v}_k^C = \tilde{u}_k^C = 2i(k_x + ik_y) + O(k^2)$. This rapid change in $\hat{n}_3^A$ compensates the almost complete smooth winding over the rest of the BZ (compare Fig. 1 left and central panel or the red dashed and green dotted left plots in Fig. 2) rendering the state topologically trivial even at infinitesimal $\delta$. From a more physical perspective, $(1 - \hat{n}_3^A)/2$ measures the occupation number $\langle a_k^\dagger a_k \rangle$ which is rapidly changing around $k = 0$ for finite $\delta$ (see red dashed plot in the left panel of Fig. 2).

To compensate for this "topological leak", we propose a dissipative hole-plugging mechanism which stabilizes the smooth winding of $\hat{n}_k$ even at finite deviations from the critical point. More specifically, we introduce the auxiliary Lindblad operators $L_j^A = \tilde{u}_k^A a_k$ in momentum space which have only an annihilation part $(\tilde{u}_k^A = 0)$ with, e.g., a Gaussian weight function $\tilde{u}_k^A = ge^{-k^2/2}$. These operators act selectively in momentum space: they prevent the unwanted occupation of the $k = 0$ mode as long as $g > \delta$, but their action becomes irrelevant for $k \gg d$. The Gaussian weight function in the definition of $L_j^A$ makes the favorable exponential localization properties manifest also in real space, alternative choices are, however, readily conceivable. The key qualitative point of the dissipative hole-plugging mechanism is the momentum selective depletion around $k = 0$ which can also be achieved with a finite number of Fourier modes, i.e., with Lindblad operators that are compactly supported in real space. Our dissipative hole-plugging mechanism is illustrated in terms of the Berry curvature $\mathcal{F} = \frac{1}{2} \hat{n}_k \cdot [\partial_{k_x} \hat{n}_k \times (\partial_{k_y} \hat{n}_k)]$, i.e., the integrand of Eq. (3) in Fig. 1. In the left panel, $\mathcal{F}$ is shown exactly at the topological critical point $\beta = -4$ where $\mathcal{C} = -1$. In the central panel, $\mathcal{F}$ is plotted somewhat away from the critical point at $\beta = -3$. The peak (dark region) around $k = 0$ compensates the negative curvature away from $k = 0$ thus giving rise to $\mathcal{C} = 0$. In the right panel this dark center region is suppressed by the action of the $L_j^A$ jump operators thus maintaining $\mathcal{C} = -1$ even away from the critical point.

Efficiency of dissipative hole plugging – We now demonstrate that our hole plugging mechanism leads to a topologically non-trivial steady state in a finite parameter range around the critical point $\beta = -4$. We calculate the Chern number of the steady state obtained from the interplay of the Lindblad operators $L_j^C$ and $L_j^A$ while monitoring both its damping gap $\delta k$ [17] and its purity gap $|\hat{n}_k|^2$. We find that the auxiliary jump operators $L_j^A$ are capable of lifting the isolated points at which the $L_j^C$ become topologically non-trivial to an extended phase (see inset in right panel of Fig. 2). If we start in the absence of $L_j^A$ with a topologically trivial $\beta = -4 - \delta$, i.e., detuned from the critical point by $\delta > 0$ and switch on $L_j^A$ by ramping up $g$, we observe a topological transition.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{(Color online) Density plot of the Berry curvature $\mathcal{F}$ (half of integrand in Eq. (3)) as a function of momentum for $\beta = -4.0, d = g = 0$ (left) $\beta = -3.0, d = g = 0$ (center) and $\beta = -3.0, d = 0.7, g = 2.0$ (right). Performing the integral in Eq. (3) for the plotted function gives $\mathcal{C} = -1$ (left), $\mathcal{C} = 0$ (center) and $\mathcal{C} = -1$ (right), respectively. In the central plot, the peak (dark region) around $k = 0$ compensates the smooth negative curvature away from the center. In the right plot the dissipative hole-plugging mechanism depletes the central peak thus maintaining the non-vanishing Chern number.}
\end{figure}
associated with a purity gap closing at $q = \delta$ (see Fig. 2 right panel). At $q > \delta$, the purity gap reopens and the steady state has Chern number $C = -1$. The damping gap stays finite throughout this procedure.

**Microscopic implementation** – So far, we have generally analyzed how a superfluid steady state with a non-vanishing Chern number can occur at the level of a Gaussian Lindblad master equation (1). In the Hamiltonian case, a superconducting condensate arises at mean field level in the thermodynamic limit from an interacting particle number conserving microscopic Hamiltonian. Also in our present dissipative framework, an effective quadratic master equation with spontaneously broken $U(1)$ symmetry arises from a microscopic, particle number conserving model described by a master equation that is quartic in the field operators. In the following, we introduce such a model and argue how it can be experimentally implemented with cold atoms in optical lattices. As we confirm numerically, the phenomenology described above, in particular our dissipative hole-plugging mechanism is obtained in a mean field approximation analogous to the one introduced in Ref. [16]. Our model again consists of a near-critical and -topological set of Lindblad operators $\ell^C$, and an auxiliary set $\ell^A$. The Lindblad operators have the number conserving bilinear form $\ell^{\alpha}_i = C_i^{\alpha\dagger}A^\alpha_i$, $\alpha = C, A$ with the creation $C_i^{\alpha\dagger} = \sum_j v_j^{\alpha\dagger}v_j^{\alpha}$ and annihilation parts $A_i^{\alpha\dagger} = \sum_j u_j^{\alpha\dagger}u_j^{\alpha}$. For $\alpha = C$, the coefficients $v_j^{C}, v_j^{C}$ are the same as in Eq. (4). The experimental implementation of Lindblad operators of such a form has been discussed in Ref. [17]. The auxiliary operators $\ell^{\alpha}_j$ are chosen such that particles are pumped out of the central region of the Brillouin zone into the higher momentum states thus reflecting the depletion of low momenta which is at the heart of our hole plugging mechanism. For atoms in optical lattices this can be achieved by momentum selective pumping techniques as described in [22]. In momentum space, $\ell^{A}_k = \sum_q C^{A\dagger}_{q-k}A^A_q$, with $C^{A\dagger}_k = \tilde{u}_k^{A}\tilde{u}_k^{A}$.

The momentum selective functions are ideally of the form $\tilde{u}_k^{A} = g_\nu e^{-k^2/d_\nu^2}$ removing particles from the central region, and $\tilde{v}_k^{A} = g_v\sum_i e^{-(k-\pi_i)^2/d_\nu^2}$ describing their reappearance at high momenta $\pi_i \in \{(0,0), (\pi,0), (\pi,\pi)\}$. The key qualitative point to the form of $\ell^{A}_k$ that has to be reflected in an experimental realization is the dominance of processes taking a particle at the center of the Brillouin zone and transferring a momentum of order $\pi$.

Upon mean field decoupling, the product of the creation and annihilation part in $\ell^{C}_k$ can be linearized and transforms into its sum [16], yielding precisely the form displayed in Eq. (4) at half filling. This allows us to evaluate the stationary state of the master equation with both sets of Lindblad operators $\ell^{C}, \ell^{A}$ at mean field level (see Supplementary Material for technical details). The above general picture, in particular the efficiency of the dissipative hole-plugging mechanism to stabilize a state with non-vanishing Chern number is fully confirmed by the numerical analysis of this microscopic model on a lattice with $501 \times 501$ sites and periodic boundary conditions.

**Concluding remarks** – The target state of the explicit construction presented here resembles the $p+ip$ superconducting ground state introduced by Read and Green [37], i.e., a topological state with non-vanishing Chern number in symmetry class D [38]. The generalization to gapped quantum anomalous Hall states a la Chern insulators [39] in symmetry class A, however, is straightforward.

The criticality of the steady state supported by the $L^{C}$ operators is in some analogy to Ref. [40], where a tensor network state representing a critical Chern state is constructed. While it may be impossible to represent pure non-critical states with non-vanishing Chern number as tensor network states, resorting to mixed states that are uniquely associated with a pure state by replacing $\tilde{n}_k$ with $\tilde{n}_k$ in Eq. (2) might be fruitful for systematic approximations of such scenarios, where the deviation from the gapped pure state in the expectation value of any observable can be bounded in terms of the purity gap.

**Acknowledgements** – We acknowledge interesting discussions with Emil J. Bergholtz, Jens Eisert, and Tao Shi on related projects. Support from the ERC Synergy Grant UQUAM, the START Grant No. Y 581-N16, the SFB FoQuS (FWF Project No. F4006- N16) is gratefully acknowledged.

**SUPPLEMENTARY MATERIAL: GAUSSIAN STEADY STATES**

Within the Gaussian approximation, all static information about the system state is contained in its Gaussian correlation matrix $\Gamma$ which can be conveniently represented in the basis of the Majorana operators $c_{j,1} = \psi_j + \psi_j^\dagger$, $c_{j,2} = i(\psi_j - \psi_j^\dagger)$.

\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{figure2.png}
    \caption{(color online) Left panel: $\tilde{n}^2_q$ as a function of $k_x$ at $k_y = 0$ for $\delta = 0, d = g = 0$ (green dotted), for $\delta = 0.5, d = g = 0$ (Red dashed), and for $\delta = 0.5, d = 0.7, g = 1.0$ (blue solid). Right panel: Purity gap $p = |\tilde{n}_k|^2$ of the steady state $p_k$ as a function of $k_x$ at $k_y = 0$, $\delta = d = 0.2$. Gap at $g = 0.1$ (blue dotted) in the topologically trivial phase, purity critical point at $g = 0.2$ (red dashed), gap at $g = 1.0$ (green solid) in the non-trivial phase. Inset: Phase diagram of the steady state as a function of $\delta = -4 - \beta$ and $g, d = 1.0$ is fixed. The purple dotted line has Chern number $C = -1$, while $C = 0$ in the bright region. The purity gap closes at the transition lines. The damping gap is finite everywhere except at the critical point $\delta = g = 0$.}
\end{figure}
\[ \Gamma_{ij}^{\alpha} = \frac{i}{2} \text{Tr} \{ \rho \left[ c_i \lambda, c_j, \mu \right] \}. \tag{6} \]

With the density matrix \( \rho \) evolving in time according to Eq. (1) with \( H = 0 \), \( \Gamma \) obeys the equation of motion \cite{35, 36}

\[ \dot{\Gamma} = \{ \Gamma, X \} - Y, \tag{7} \]

where the matrices \( X = M + M^T \) and \( Y = 2i(M - M^T) \) are determined in terms of the Majorana representation of the Lindblad operators \( L_j = i \beta \bar{c} \) via \( M = \sum_j \beta \bar{c}_j \). Since we are dealing with lattice translation invariant systems here, it is convenient to consider the equation of motion for the Fourier transform of the correlation matrix, \( \tilde{\Gamma}_{\lambda \mu}(k) = \frac{1}{2} \text{Tr} \{ \rho \left[ \tilde{c}_k \lambda, \tilde{c}_{-k}, \mu \right] \} \) which simply reads as

\[ \ddot{\tilde{z}}(k) = \left\{ \tilde{\Gamma}(k), \tilde{X}(k) \right\} - \dot{\tilde{Y}}(k), \tag{8} \]

where \( \tilde{X}, \tilde{Y} \) denote the Fourier transforms of \( X, Y \). Eq. (8) is a \( 2 \times 2 \) matrix equation for every lattice momentum \( k \). A steady state obeys \( \dot{\tilde{\Gamma}}_s(k) = 0 \). Plugging this into Eq. (8) yields the Sylvester equation

\[ \left\{ \dot{\tilde{\Gamma}}_s(k), \tilde{X}(k) \right\} = \tilde{Y}(k), \tag{9} \]

which has a unique solution if \( \tilde{X}(k) \) is invertible. The spectrum of the generally positive semidefinite matrix \( \tilde{X}(k) \) determines the damping rates towards the steady state. The minimum eigenvalue of \( \tilde{X}(k) \) all over the Brillouin zone is called the damping gap. For a finite damping gap, the steady state \( \dot{\tilde{\Gamma}}_s(k) \) and from that, via a simple basis transformation back to the Nambu basis, the Gaussian density matrix \( \rho^G_k \) itself is readily obtained by direct solution of Eq. (9).

**SUPPLEMENTARY MATERIAL: SELF-CONSISTENT MEAN FIELD THEORY**

A self-consistent mean field theory can be derived along the lines of \cite{17}. We start from the master equation

\[ \partial_t \rho = \mathcal{L}^C[\rho] + \mathcal{L}^A[\rho], \tag{10} \]

where \( \mathcal{L}^\alpha[\rho] = \sum_j \left\{ \rho_j^{\alpha^\dag} \rho_j^{\alpha} - \frac{1}{2} \left\{ \rho_j^{\alpha}, \rho_j^{\alpha^\dag} \right\} \right\} \), \( \alpha = C, A \). We write out this quartic master equation in momentum space, and make the ansatz \( \rho = \prod_k \rho_k \), where \( \rho_k \) describes the mode pair \( \{ k, -k \} \) and obeys \( \text{Tr} \rho_k \rho_k = 1 \). \( \prod_k \) reminds that the product is taken over half of the Brillouin zone only, e.g. the upper half. We then focus on one particular momentum mode pair \( \{ p, -p \} \), and keep only terms which are quadratic in operators associated to this mode pair. By means of the prescription \( \rho_p = \text{Tr}_{-p} \rho \), we obtain an evolution equation for \( \rho_p \) with coefficients \( C^* \), which are governed by the mean fields of the remaining modes in the system,

\[ \partial_t \rho_p = \sum_{\alpha=C,A} \left\{ C^\alpha_p \bar{v}^{\alpha}_p \mathcal{L}^A_{\alpha,p} \rho_p + C^\alpha_p \bar{v}^*_{-p} \mathcal{L}^C_{\alpha,-p} \rho_p \right\} \\
- C^\alpha_p \bar{v}^*_{-p} \mathcal{L}^A_{\alpha,p} \rho_p - C^\alpha_p \bar{v}^*_p \mathcal{L}^C_{\alpha,-p} \rho_p \tag{11} \]

with abbreviation \( \mathcal{L}_{\alpha,b}[\rho] = \alpha p - \frac{1}{2} \{ b, \rho \} \) and

\[ C^\alpha_p = \sum_{q \neq p} |\bar{v}_q^\alpha|^2 \langle a^\dag_q a_q \rangle, \quad C^\alpha_p = \sum_{q \neq p} |\bar{v}_q^\alpha|^2 \langle \bar{a}^\dag_q a_q \rangle, \tag{12} \]

we note \( \langle a^\dag_q a_q \rangle^* = \langle a^\dag_q a_q \rangle \), and that the constraint on the sum can be neglected in the thermodynamic limit.

In the absence of \( \mathcal{L}^A \), the stationary state is known explicitly, using the equivalence of fixed number and fixed phase wavefunctions in the thermodynamic limit, and the exact knowledge of the fixed number state annihilated by the set \( E^C_q \) \cite{17}. It is given by the pure density matrix \( \rho_D = |\psi\rangle \langle \psi| \), where \( |\psi\rangle = \prod_q N_q (1 + \bar{v}_q^C \bar{a}^\dag_q a_q |0\rangle \) and \( N_q = 1 / \sqrt{1 + |\bar{v}_q^C|^2} \). In this case, the averages can be evaluated explicitly, \( \langle a^\dag_q a_q \rangle = |\bar{v}_q^C|^2, \langle \bar{a}^\dag_q a_q \rangle = |\bar{v}_q^C|^2, (a^\dag_q a_q) = \bar{v}_q^C \bar{v}_q^C = \bar{v}_q^C \bar{v}_q^C \), and \( \langle a^\dag_q a_q \rangle = \bar{v}_q^C \bar{v}_q^C \), where \( \bar{v}_q^C = \bar{v}_q^C / \sqrt{|\bar{v}_q^C|^2 + |\bar{v}_q^C|^2} \) and analogous for \( \bar{v}_q^C \). Note that, in this case, one common real number \( C_0 = C_1^C = C_2^C = C_3^C = C_5^C \) can be factored out of Eq. (11), and the resulting linearized Lindblad operators coincide with those defined in Eq. (4) in the main text (at half filling, and up to an irrelevant relative phase which reflects spontaneous symmetry breaking).

**FIG. 3.** \( \dot{n}_k^A = -\text{Tr} \{ \rho_{k^A} \} \) as a function of \( k_x \) at \( k_y = 0 \). Red dashed plot for \( \delta = -4 - \beta = 0.5 \) in the presence of \( E^C_q \) only. Blue solid plot shows the self-consistent solution of Eq. (15) for \( \delta = 0.5, \, q_u = g_u = 5.0, \, d_u = 0.5, \, d_v = 1.5 \) in the presence of both \( E^C_q \) and \( E^A_q \) on a lattice of 501 \times 501 sites.

When \( \mathcal{L}^A \) is added, the stationary state is no longer known explicitly. However, a self-consistent mean field theory can be constructed. To this end, we derive the evolution of the
covariances for the mode pair \( \{ p, -p \} \),

\[
\partial_t \left( \frac{\langle a^\dagger_p a_p \rangle}{\langle a^\dagger_{-p} a_{-p} \rangle} \right) =
\begin{pmatrix}
-\kappa_p & \nu_p & \nu_p \\
0 & -\kappa_p & 0 \\
0 & 0 & -\kappa_p \\
\end{pmatrix}
\begin{pmatrix}
\langle a^\dagger_p a_p \rangle \\
\langle a^\dagger_{-p} a_{-p} \rangle \\
\end{pmatrix} + \begin{pmatrix}
\mu_p \\
\lambda_p \\
\lambda_p^* \\
\end{pmatrix},
\]

where

\[
\kappa_p = \sum_\alpha (C_{\alpha}^2 |\tilde{\nu}_\alpha|^2 + C_{\alpha}^2 |\tilde{\nu}_\alpha^*|^2),
\]

\[
\nu_p = \frac{1}{2} \sum_\alpha C_{\alpha}^2 (\tilde{\nu}_\alpha^* \tilde{\nu}_\alpha + \tilde{\nu}_\alpha \tilde{\nu}_\alpha^*),
\]

\[
\mu_p = \sum_\alpha C_{\alpha}^2 |\tilde{v}_\alpha|^2, \lambda_p = \frac{1}{2} \sum_\alpha C_{\alpha}^2 (\tilde{v}_\alpha^* \tilde{v}_\alpha + \tilde{v}_\alpha \tilde{v}_\alpha^*) - \tilde{\nu}_p \tilde{\nu}_p^*.
\]

Here we have used the property \( |\tilde{\nu}_\alpha|^2 = |\tilde{\nu}_\alpha^*|^2 \) and analogous for \( \tilde{u}_p \), exhibited by our Lindblad operators. The coefficients \( C_{\alpha}^2 \) make these equations non-linear. The implicit equations for the stationary state read as

\[
\begin{pmatrix}
\langle a^\dagger_p a_p \rangle \\
\langle a^\dagger_{-p} a_{-p} \rangle \\
\end{pmatrix} = \frac{1}{\kappa_p} \begin{pmatrix}
1 & \nu_p & \nu_p \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
\mu_p \\
\lambda_p \\
\lambda_p^* \\
\end{pmatrix}.
\]

These equations can be solved iteratively, starting from the known solution of \( L^C \) alone. Qualitative properties of the solution can be discussed on the basis of the localization properties of the functions \( \tilde{u}_q \), \( \tilde{v}_q \) in momentum space. In particular, based on Eq. (11), we expect modifications of the solution for \( L^C \) only in the central and edge regions of the Brillouin zone, as clearly effective annihilation (creation) of particles takes place in the center (edges) of the Brillouin zone. In addition, the off-diagonal contributions to the effective non-topological Liouvillian are exponentially small.

In Fig. 3, we give an explicit example demonstrating how the self consistent solution of Eq. (15) is capable of achieving the dissipative hole-plugging mechanism. Our numerical simulations are done for a lattice of 501 × 501 sites with periodic boundary conditions.

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