Analysis on structured stability of highly nonlinear pantograph stochastic differential equations

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ABSTRACT

This paper investigates the structured stability and boundedness of highly nonlinear hybrid pantograph stochastic differential equations (PSDEs). The main contribution of this paper is to take the different structures into account to establish the structured robust stability and boundedness for highly nonlinear hybrid PSDEs. The theory established in this paper is applicable to hybrid PSDEs which may experience abrupt changes in both structures and parameters.

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1. Introduction

Stochastic delay differential equations (SDDEs) are widely used to model systems which do not only depend on present states but also involves past states. Robust stability and boundedness are two of most popular topics in the area of systems and controls, most of the papers can only be applied to delay systems where their coefficients are either linear or nonlinear but bounded by linear functions (see, e.g. Deng, Fei, Liang, & Mao, 2019; Wu, Tang, & Zhang, 2016). However, the linear growth condition is usually violated in many practical applications. Recently, there are some progress on stability for highly nonlinear stochastic delay systems (see, e.g. Deng, Fei, Liu, & Mao, 2019; Fei, Fei, Mao, Shen, & Yan, 2019; Fei, Fei, & Yan, 2019; Fei, Shen, Fei, Mao, & Yan, 2019; Fei, Hu, Mao, & Shen, 2019; Liu & Deng, 2017; Shen, Fei, & Mao, 2018). Particularly, Hu, Mao, and Zhang (2013) were first to investigate the robust stability and boundedness for SDDEs with Markovian switching without the linear growth condition. Fei, Hu, Mao, and Shen (2017) established stability criteria for delay dependent highly nonlinear hybrid stochastic systems. Pantograph stochastic differential equations (PSDEs) are unbounded delay stochastic differential equations which have been frequently applied in many practical areas, including biology, mechanic, engineering and finance. Baker and Buckwar (2000) established the existence and uniqueness of the solution for the linear stochastic pantograph equation. Shen, Fei, Mao, and Deng (2018) discussed exponential stability of highly nonlinear neutral PSDEs by Lyapunov functional and M-matrix. Liu and Deng (2018) investigated $p$th moment exponential stability of highly nonlinear neutral PSDEs which driven by Lévy noise. As we know, the hybrid systems driven by continuous-time Markov chains are often used to model systems that may experience abrupt changes in their structures and parameters caused by phenomena such as component failures or repairs (see, e.g. Mao & Yuan, 2006; Shen, Fei, Mao, & Liang, 2018; Zhou & Hu, 2016). The theory in Hu, Mao, and Zhang (2013) is good at dealing with the hybrid SDDEs that may experience abrupt changes in their parameters. You, Mao, Mao, and Hu (2015) show that a given stable hybrid PSDE can not only tolerate the linear perturbation but also the nonlinear perturbation without loss of the stability, while most of the papers could only cope with the linear perturbation. However, most of references on hybrid systems have dealt with subsytems with the same structures. In fact, stochastic systems may experience changes not only in their coefficients but also in their structures. Fei, Hu, Mao, and Shen (2018) first took the different structures in different modes to develop a new theory on the structured robust stability and boundedness for highly nonlinear hybrid SDDEs. But the theory of Fei et al. (2018) cannot applied directly to highly nonlinear hybrid PSDEs which experience abrupt changes in their structures. Motivated by the above discussion, this paper will study exponential stability of a class PSDEs which experience abrupt changes in their structures.

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2. Notation and assumptions

Throughout this paper, unless otherwise specified, we use the following notation. We denote by $|x|$ the Euclidean norm for $x \in \mathbb{R}^n$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$. If both $a$ and $b$ are real numbers, then $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. If $G$ is a set, its indicator function is denoted by $I_G$. That is, $I_G(x) = 1$ if $x \in G$ and $0$ otherwise. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. Let $(r(t), t \geq 0)$ be a right continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P[r(t + \Delta) = j | r(t) = i] = \begin{cases} \gamma_{ij} \Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii} \Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Consider an $n$-dimensional hybrid SDDE

$$dx(t) = f(x(t), x(\theta t), t, r(t)) \, dt + g(x(t), x(\theta t), t, r(t)) \, dB(t) \quad (1)$$

on $t \geq 0$, where the coefficients $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n \times m}$ are Borel measurable and $0 < \theta < 1$ with initial date $x(0) = x_0 \in \mathbb{R}^n$. Moreover, assume that $f(0, 0, t, i) = 0$ and $g(0, 0, t, i) = 0$ for all $(t, i) \in \mathbb{R}_+ \times S$.

For the convenience of the reader, let us cite some useful results on M-matrices. For a vector or matrix $A$, by $A > 0$ we mean all elements of $A$ are positive. A Z-matrix is a square matrix $A = (a_{ij})_{N \times N}$ which has non-positive off-diagonal entries.

Lemma 2.1: Let $A = (a_{ij})_{N \times N}$ be a Z-matrix. Then $A$ is a nonsingular M-matrix if and only if one of the following statements holds:

1. $A^{-1}$ exists and its elements are all nonnegative.
2. There exists $x > 0$ in $\mathbb{R}^N$ such that $Ax > 0$.

The well-known conditions imposed for the existence and uniqueness of global solution are the local Lipschitz condition and the linear growth condition (see, e.g., Mao, 2007). To be precise, let us state the local Lipschitz condition.

Assumption 2.2: For each integer $h \geq 1$ there is a positive constant $K_h$ such that

$$|f(x, y, t, i) - f(\tilde{x}, \tilde{y}, t, i)|^2 + |g(x, y, t, i) - g(\tilde{x}, \tilde{y}, t, i)|^2 \leq K_h(|x - \tilde{x}|^2 + |y - \tilde{y}|^2)$$

for those $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\tilde{x}| \vee |\tilde{y}| \leq h$ and all $(t, i) \in \mathbb{R}_+ \times S$.

However, we do not state the linear growth condition as we here is to study the structured robust stability and boundedness for highly nonlinear PSDEs which do not satisfy this condition.

For $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, define an operator $LV : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$ by

$$LV(x, y, t, i) = V_x(x, t, i)f(x, y, t, i) + \frac{1}{2} \text{trace}[g^T(x, y, t, i)V_{xx}(x, t, i)g(x, y, t, i)] + \sum_{j \in S} \gamma_{ij} V(x, j),$$

in which

$$V_x(x, t, i) = \left( \frac{\partial V(x, t, i)}{\partial x_1}, \ldots, \frac{\partial V(x, t, i)}{\partial x_n} \right)_{n \times n}$$

and

$$V_{xx}(x, t, i) = \left( \frac{\partial^2 V(x, t, i)}{\partial x_k \partial x_l} \right)_{n \times n}.$$  

Assumption 2.3: Without loss of any generality, assume that the state space $S$ of the Markov chain is divided into two proper sub-spaces $S_1$ and $S_2$, $S_1 = \{1, \ldots, N_1\}$ and $S_2 = \{N_1 + 1, \ldots, N\}$, where $1 \leq N_1 < N$. Let $q > p \geq 2$ and assume that for each $i \in S_1$, there are constants $\beta_{i2} \in R$ and $\beta_{i1}, \beta_{i3} \in R_+$ such that, for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$,

$$x^T f(x, y, t, i) + \frac{q - 1}{2} |g(x, y, t, i)|^2 \leq \beta_1 + \beta_2 |x|^2 + \beta_3 \theta \exp \left( \frac{-2}{p} (1 - \theta) t \right) |y|^2;$$

while for each $i \in S_2$, there are constants $\beta_{i2} \in R$, and $\beta_{i1}, \beta_{i3}, \beta_{i4}, \beta_{i5} \in R_+$ such that

$$x^T f(x, y, t, i) + \frac{p - 1}{2} |g(x, y, t, i)|^2 \leq \beta_1 + \beta_2 |x|^2 + \beta_3 \theta \exp \left( \frac{-2}{p} (1 - \theta) t \right) |y|^2 - \beta_{i4} |x|^{q-p+2} + \beta_{i5} \theta \exp \left( \frac{-q - p + 2}{q} (1 - \theta) t \right) |y|^{q-p+2}.$$  

Assumption 2.4: Under Assumption 2.3, assume furthermore that

$$A := -\text{diag}(p\beta_{i2}, \ldots, p\beta_{iN_2}) - \Gamma$$

and
\[ B := -\text{diag}(q_1\beta_{12}, \ldots, qN_{12}) - (\gamma_{ij})_{i,j \in S_1} \] 

are both nonsingular M-matrices.

### 3. Boundedness and stability

Set
\[ (\lambda_1, \ldots, \lambda_N)^T = A^{-1}(1, \ldots, 1)^T \]

and
\[ (\zeta_1, \ldots, \zeta_N)^T = B^{-1}(\rho, \ldots, \rho)^T, \]

where \( \rho \) is a free positive parameter. By the definitions of \( \lambda_i \) and \( \zeta_i \), we see that all \( \lambda_i \) and \( \zeta_i \) are positive.

**Assumption 3.1:** Choose \( \rho > 0 \) sufficiently small such that
\[ \beta_{ij} \geq \frac{\rho + \sum_{j \in S_1} \gamma_{ji}}{\lambda_i}, \quad \forall i \in S_2, \]

where \( \lambda_i \) and \( \zeta_i \) have been defined by (4) and (5). Assume also that
\[ \beta_{13} \leq \frac{1}{(p-2)\lambda_i + 2\zeta_i}, \quad \forall i \in S, \]
\[ \beta_{13} < \frac{\rho}{\zeta_i((q-2)\theta + q - p + 2)}, \quad \forall i \in S_1, \]

and
\[ \beta_{15} < \frac{\rho q}{\lambda_i p((q-2)\theta + q - p + 2)}, \quad \forall i \in S_2. \]

**Remark 3.2:** Let \( \tilde{b} \) be the maximum of the row sums of \( B^{-1} \) and \( \tilde{y} = \max_{i \in S_1} (\sum_{j \in S_1} \gamma_{ji}) \). Then \( \zeta_i \leq \rho \tilde{b} \) for all \( i \in S \) and \( \sum_{j \in S_1} \gamma_{ji} \tilde{y} \leq \rho \tilde{b}\tilde{y} \) for all \( i \in S_2 \). If we choose
\[ \rho = \frac{\min_{i \in S_1} p\lambda_i \beta_{14}}{1 + \tilde{b}\tilde{y}} \]

then condition (6) is guaranteed.

**Lemma 3.3:** Let Assumptions 2.2, 2.3, 2.4 and 3.1 hold. Define a Lyapunov function \( V : R^3 \times S \rightarrow R_+ \) by
\[ V(x, i) = \begin{cases} 
\lambda_i |x|^p + \zeta_i |x|^q & \text{if } i \in S_1; \\
\lambda_i |x|^p & \text{if } i \in S_2.
\end{cases} \]

Set
\[ c_1 := \min_{i \in S} \lambda_i, \quad c_2 := (\max_{i \in S} \lambda_i) \lor (\max_{i \in S_1} \zeta_i), \]
\[ c_3 := (\max_{i \in S} p\lambda_i \beta_{13}) \lor \left( \max_{i \in S_1} q\zeta_i \beta_{13} \right), \]
\[ \tilde{\rho} := \left( \max_{i \in S_1} q\zeta_i \beta_{13} \right) \lor \left( \max_{i \in S_2} \lambda_i \beta_{15} \right), \]
\[ \rho^1 := \frac{\rho}{2} - \frac{\tilde{\rho}((q-2)\theta + q - p + 2)}{2q}, \]
\[ \rho^2 := \frac{\tilde{\rho}(q-p+2)}{q}. \]

Then we have
\[ LV(x, y, t, i) \leq c_3(|x|^{p-2} + |x|^{q-2}) - 2\lambda_i \beta_{13} |x|^p \]
\[ + 2\lambda_i \beta_{13} \theta \exp(-(1-\theta)t)|y|^p \]
\[ - (2\rho_1 + \rho_2)|x|^q + \rho_2 \theta \exp(-(1-\theta)t)|y|^{q}. \]

**Proof:** By the defition of \( V(x, i) \), we can see that
\[ c_1 |x|^p \leq V(x, i) \leq c_2(|x|^p + |x|^q). \]

By the generalized Itô formula, we have that
\[ dV(x(t), y(t)) = LV(x(t), x(\theta t), t, y(t)) \, dt + dM(t) \]
on \( t \geq 0 \), where \( M(t) \) is a continuous local martingale with \( M(0) = 0 \).

For \( i \in S_1 \), we have
\[ LV(x, y, t, i) = p\lambda_i |x|^{p-2}x^T f(x, y, t, i) \]
\[ + \frac{1}{2} p\lambda_i |x|^{p-2} g(x, y, t, i)^2 \]
\[ + \frac{1}{2} p(p-2)\lambda_i |x|^{p-4} |x|^T g(x, y, t, i)^2 \]
\[ + q\zeta_i |x|^{q-2} x^T f(x, y, t, i) + \frac{1}{2} q\zeta_i |x|^{q-2} \]
\[ \times |g(x, y, t, i)|^2 \]
\[ + \frac{1}{2} q(q-2)\zeta_i |x|^{q-4} |x|^T g(x, y, t, i)^2 \]
\[ + \sum_{j \in S_1} \gamma_{ji} \lambda_j |x|^p + \sum_{j \in S_1} \gamma_{ji} \zeta_j |x|^q. \]

By inequality \( |x|^T g(x, y, t, i)^2 \leq |x| |g(x, y, t, i)|^2 \), we have
\[ LV(x, y, t, i) \leq p\lambda_i |x|^{p-2} \left( x^T f(x, y, t, i) \right) \]
\[ + \frac{p-1}{2} |g(x, y, t, i)|^2. \]
\[
+ q_\zeta_1|x|^{q-2} \left( x^T f(x, y, t, i) \right)
+ \frac{q - 1}{2} |g(x, y, t, i)|^2
+ \sum_{j \in S} \gamma_\lambda_j |x|^p
+ \sum_{j \in S_1} \gamma_\zeta_j |x|^q.
\]

By Assumption 2.3, we can get
\[
LV(x, y, t, i) \leq p \lambda_1 |x|^{p-2} \left[ \beta_1 + \beta_2 |x|^2 \right.
+ \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right) |y|^2
+ q_\zeta_1 |x|^{q-2} \left[ \beta_1 + \beta_2 |x|^2 \right.
+ \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right) |y|^2
+ \sum_{j \in S} \gamma_\lambda_j |x|^p + \sum_{j \in S_1} \gamma_\zeta_j |x|^q.
\]

By (4) and (5), we have
\[
p \beta_1 + \sum_{j = 1}^N \gamma_\lambda_j = -1, \quad q \beta_2 \theta + \sum_{j \in S_1} \gamma_\zeta_j = -\rho.
\]

Consequently,
\[
LV(x, y, t, i) \leq p \lambda_1 |x|^{p-2} - |x|^p
+ p \lambda_1 \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right) |x|^p
\times |x|^{p-2} |y|^2 + q_\zeta_1 \beta_1 |x|^{q-2}
- \rho |x|^q + q_\zeta_1 \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right)
\times |x|^{q-2} |y|^2.
\]

By the Young inequality, we have
\[
\exp \left( -\frac{2}{p} (1 - \theta) t \right) |x|^{p-2} |y|^2
\leq \frac{p - 2}{p} |x|^p + \frac{2}{p} \exp \left( -(1 - \theta) t \right) |y|^p
\]
and
\[
\exp \left( -\frac{2}{p} (1 - \theta) t \right) |x|^{q-2} |y|^2
\leq \frac{q - 2}{q} |x|^q + \frac{2}{q} \exp \left( -\frac{q}{p} (1 - \theta) t \right) |y|^q
\leq \frac{q - 2}{q} |x|^q + \frac{2}{q} \exp \left( -(1 - \theta) t \right) |y|^q.
\]

We hence obtain from (13) that, for \(i \in S_1\),
\[
LV(x, y, t, i) \leq p \lambda_1 \beta_1 |x|^{p-2} - (1 - (p - 2) \lambda_1 \beta_3 \theta) |x|^p
+ 2 \lambda_1 \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right) |y|^p
+ q_\zeta_1 |x|^{q-2} - \beta_3 |x|^q
+ (q - 2) \lambda_1 \beta_3 \theta |x|^q + 2 \zeta_1 \beta_3 \theta
\times \exp \left( -(1 - \theta) t \right) |y|^q.
\]

Similarly, for \(i \in S_2\), we can show that
\[
LV(x, y, t, i) \leq p \lambda_1 \beta_1 |x|^{p-2} - |x|^p
+ p \lambda_1 \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right) |x|^p
\times |x|^{p-2} |y|^2
+ \left( -p \lambda_1 \beta_4 + \sum_{j \in S_1} \gamma_\zeta_j \right) |x|^q
+ p \lambda_1 \beta_5 \theta \exp \left( -\frac{q - p + 2}{q} (1 - \theta) t \right)
\times |x|^{p-2} |y|^{q-p+2}.
\]

But, by condition (6), we have
\[
-p \lambda_1 \beta_4 + \sum_{j \in S_1} \gamma_\zeta_j \leq -\rho.
\]

Hence
\[
LV(x, y, t, i) \leq p \lambda_1 \beta_1 |x|^{p-2} - |x|^p - \rho |x|^q
+ p \lambda_1 \beta_3 \theta \exp \left( -\frac{2}{p} (1 - \theta) t \right) |x|^p
\times |x|^{p-2} |y|^2
+ \lambda_1 \beta_5 \theta \exp \left( -\frac{q - p + 2}{q} (1 - \theta) t \right)
\times |x|^{p-2} |y|^{q-p+2}.
\]

By condition (7) and the Young inequality, we then obtain that, for \(i \in S_2\),
\[
LV(x, y, t, i) \leq p \lambda_1 \beta_1 |x|^{p-2} - (1 - (p - 2) \lambda_1 \beta_3 \theta) |x|^p
+ 2 \lambda_1 \beta_3 \theta \exp \left( -(1 - \theta) t \right) |y|^p
+ \frac{p(p - 2)}{q} \lambda_1 \beta_5 \theta |x|^q
- \rho |x|^q + \frac{p(q - p + 2)}{q} \lambda_1 \beta_5 \theta
\times \exp \left( -(1 - \theta) t \right) |y|^q.
\]

Combining (7), (14) and (17), we see that, for all \(i \in S\),
\[
LV(x, y, t, i) \leq c_5 (|x|^{p-2} + |x|^{q-2})
\]
Theorem 3.4: Let the conditions of Lemma 3.3 hold. Then we have the following assertions:

(i) For any initial data \( x_0 \in \mathbb{R}^n \), there is a unique global solution \( x(t) \) to the hybrid PSDE (1) on \( t \in [0, \infty) \).

(ii) The solution has the properties that

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t E|x(s)|^q \, ds \leq H_1, \tag{19}
\]

and

\[
\limsup_{t \to \infty} E|x(t)|^p \leq H_2, \tag{20}
\]

where \( H_1 \) and \( H_2 \) are positive constants.

Proof: Since the coefficients of the hybrid PSDE (1) are locally Lipschitz continuous, for any given initial date \( x(0) = x_0 \) there is a unique maximal local solution \( x(t) \) on \( t \in [0, \tilde{\sigma}_\infty) \), where \( \tilde{\sigma}_\infty \) is the explosion time. Let \( k_0 > 0 \) be a sufficiently large integer such that \( |x_0| < k_0 \). For each integer \( k \geq k_0 \), define the stopping time

\[
\tau_k = \inf\{t \geq 0 : |x(t)| \geq k\},
\]

where throughout this paper we set \( \inf \emptyset = \infty \). Clearly, \( \tau_k \) is increasing as \( k \to \infty \) and \( \tau_\infty = \lim_{k \to \infty} \tau_k \leq \tilde{\sigma}_\infty \) a.s.

If we can show that \( \tau_\infty = \infty \), then \( \tilde{\sigma}_\infty = \infty \) a.s. and the assertion (i) follows.

Inequality (11) can be rearranged as

\[
LV(x, y, t, i) \leq c_3(|x|^{p-2} + |y|^{q-2}) - \rho_1 |x|^q - 2\lambda_i \beta_3 |x|^p + 2\lambda_i \beta_3 \theta \exp(-(1 - \theta)t)|y|^p - (\rho_1 + \rho_2) |x|^q + \rho_2 \theta \exp(-(1 - \theta)t)|y|^q \tag{21}
\]

for all \((x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times R_+ \times S\). Set

\[
c_4 := \sup_{x \in \mathbb{R}^n} (c_3(|x|^{p-2} + |x|^{q-2}) - \rho_1 |x|^q) < \infty.
\]

Substituting this into (21) yields

\[
LV(x, y, t, i) \leq c_4 - 2\lambda_i \beta_3 |x|^p + 2\lambda_i \beta_3 \theta \exp(-(1 - \theta)t)\lambda_i |y|^p - (\rho_1 + \rho_2) |x|^q + \rho_2 \theta \exp(-(1 - \theta)t)|y|^q \tag{22}
\]

By the generalized Itô formula, we have

\[
EV(x(t \wedge \tau_k), r(t \wedge \tau_k)) \leq EV(x(0), r(0)) + E \int_0^{t \wedge \tau_k} \left[ c_4 - 2\lambda_i \beta_3 |x|^p + 2\lambda_i \beta_3 \theta \exp(-(1 - \theta)s) \lambda_i |y|^p - (\rho_1 + \rho_2) |x|^q + \rho_2 \theta \exp(-(1 - \theta)s)|y|^q \right] ds \tag{23}
\]

for all \( t \geq 0 \). By \( \exp(-(1 - \theta)t) \leq 1 \) for all \( t \geq 0 \), we can get that

\[
2\lambda_i \beta_3 E \int_0^{t \wedge \tau_k} \theta \exp(-(1 - \theta)s) |x(\theta s)|^p \, ds \leq 2\lambda_i \beta_3 E \int_0^{t \wedge \tau_k} |x(u)|^p \, du \leq 2\lambda_i \beta_3 E \int_0^{t \wedge \tau_k} |x(u)|^p \, du \tag{24}
\]

and

\[
\rho_2 E \int_0^{t \wedge \tau_k} \theta \exp(-(1 - \theta)s) |x(\theta s)|^q \, ds \leq \rho_2 E \int_0^{t \wedge \tau_k} |x(u)|^q \, du \leq \rho_2 E \int_0^{t \wedge \tau_k} |x(u)|^q \, du.
\]

This, along with (12), implies easily that

\[
c_1 E|x(t \wedge \tau_k)|^p \leq c_5 + c_4 t - \rho_1 E \int_0^{t \wedge \tau_k} |x(s)|^q \, ds \tag{24}
\]

where

\[
c_5 = c_2(|x(0)|^p + |x(0)|^q).
\]

Consequently

\[
c_1 k^p \P(\tau_k \leq t) \leq c_5 + c_4 t.
\]

Letting \( k \to \infty \) gives that \( \P(\tau_\infty \leq t) = 0 \). This means that \( \tau_\infty > t \) a.s. Letting \( t \to \infty \), we get the desired result \( \tau_\infty = \infty \) a.s.
We now show assertion (19). It follows from (24) that
\[
\rho_1 E \int_0^{t \wedge \tau_k} |x(s)|^q \, ds \leq c_5 + c_4 t.
\]
Letting \( k \to \infty \) and then using the Fubini theorem, we get
\[
\rho_1 \int_0^t E|x(s)|^q \, ds \leq c_5 + c_4 t. \tag{25}
\]
Dividing both sides by \( \rho_1 t \) and then letting \( t \to \infty \), we see
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t E|x(s)|^q \, ds \leq \frac{c_4}{\rho_1},
\]
which is the desired assertion (19). Choose a positive constant \( \epsilon \) sufficiently small for
\[
\epsilon < \frac{\rho_1}{c_2} \quad \text{and} \quad \epsilon \leq 1. \tag{26}
\]
By the generalized Itô formula again, we have that for any \( t \geq 0 \),
\[
E[e^{\epsilon(t \wedge \tau_k)}V(x(t \wedge \tau_k), r(t \wedge \tau_k))] = EV(x(0), r(0))
+ \int_0^{t \wedge \tau_k} e^{\epsilon s} \left[ \epsilon V(x(s), r(s)) + LV(x(s), x(\theta(s), s, r(s))) \right] \, ds.
\]
By (12) and (22), we then have
\[
c_1 E(e^{\epsilon(t \wedge \tau_k)}|x(t \wedge \tau_k)|^p) \leq c_2 (|x(0)|^p + |x(0)|^q)
+ \int_0^{t \wedge \tau_k} e^{\epsilon \theta} \left[ \epsilon c_2 (|x(s)|^p + |x(s)|^q) + c_4 \right.
- 2\lambda_1 \beta_3 |x(s)|^p + 2\lambda_1 \beta_3 \theta \exp(-(1 - \theta) \tau_k) |x(s)|^p
- (\rho_1 + \rho_2) |x(s)|^q + \rho_2 \theta \exp(-(1 - \theta) \tau_k) |x(s)|^q \big] \, ds.
\]
By 0 < \( \theta < 1 \) and \( \epsilon \leq 1 \), we can get \((\epsilon - 1 + \theta)/\theta \leq \epsilon\), so that
\[
2\lambda_1 \beta_3 \int_0^{t \wedge \tau_k} \theta \exp((\epsilon - 1 + \theta)/\theta |x(s)|^p \, ds
\]
\[
= 2\lambda_1 \beta_3 \int_0^{t \wedge \tau_k} \exp\left( \frac{\epsilon - 1 + \theta}{\theta} u \right) |x(u)|^p \, du
\]
\[
\leq 2\lambda_1 \beta_3 \int_0^{t \wedge \tau_k} \exp(\epsilon u) |x(u)|^p \, du, \tag{28}
\]
and similarly,
\[
\rho_2 \int_0^{t \wedge \tau_k} \theta \exp((\epsilon - 1 + \theta)/\theta |x(s)|^p \, ds
\]
\[
\leq \rho_2 \int_0^{t \wedge \tau_k} \exp(\epsilon u) |x(u)|^p \, du, \tag{29}
\]
thus, we can get
\[
c_1 E(e^{\epsilon(t \wedge \tau_k)}|x(t \wedge \tau_k)|^p) \leq c_5 + \int_0^t e^{\epsilon s} D(|x(s)|) \, ds,
\]
where \( D : R_+ \to R \) is defined by
\[
D(u) = c_4 + \epsilon c_2 u^p - (\rho_1 - \epsilon c_2) u^q.
\]
By (26), we can obtain that
\[
c_6 = \sup_{u \geq 0} D(u) < \infty.
\]
Letting \( k \to \infty \), we have
\[
c_1 e^{\epsilon t} E|x(t)|^p \leq c_5 + \frac{c_6}{\epsilon} e^{\epsilon t},
\]
which yields
\[
\limsup_{t \to \infty} E|x(t)|^p \leq c_6/(c_1 \epsilon).
\]
The proof is complete. \( \blacksquare \)

**Theorem 3.5:** Let all the conditions in Lemma 3.3 hold and, moreover, \( \beta_{i1} = 0 \) for all \( i \in S \). Then the unique global solution \( x(t) \) of the PSDE (1) has the property that
\[
\int_0^\infty E|x(t)|^q \, dt < \infty. \tag{30}
\]

**Proof:** Noting that \( c_2 = 0 \) in (11) given that \( \beta_{i1} = 0 \) for all \( i \in S \). Hence, (11) becomes
\[
LV(x, y, t, i) \leq -2\lambda_i \beta_3 |x|^p + 2\lambda_i \beta_3 \exp(-(1 - \theta) \tau_k) |y|^p
- (\rho_1 + \rho_2) |x|^q + \rho_2 \theta \exp(-(1 - \theta) \tau_k) |y|^q.
\]
It is then easy to show by the generalized Itô formula that
\[
2\rho_1 \int_0^t E|x(s)|^q \, ds \leq c_2 (|x(0)|^p + |x(0)|^q).
\]
Letting \( t \to \infty \) yields assertion (30). \( \blacksquare \)

**Theorem 3.6:** Let all the conditions in Lemma 3.3 hold except condition (7) is replaced by
\[
\beta_{i3} < \frac{1}{(p - 2) \lambda_i \theta + 2\rho_1}, \quad \forall i \in S,
\]
and, moreover, \( \beta_{i1} = 0 \) for all \( i \in S \). Then there is a positive number \( \delta \) such that for any initial data \( x(0) = x_0 \), the unique global solution \( x(t) \) of the PSDE (1) satisfies
\[
\limsup_{t \to \infty} \frac{1}{t} \log(E|x(t)|^p) \leq -\delta \tag{31}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|) \leq -\frac{\delta}{p} \quad \text{a.s.} \tag{32}
\]
**Proof:** In the same way that (11) was proved, we can show from (13) and (16) that

\[
LV(x, y, t, i) \leq p\lambda_i \beta_3 \theta \exp \left( \frac{-2}{p}(1 - \theta)t \right) |x|^p - 2 \lambda_i \beta_3 \exp(0 - (1 - \theta)t) |y|^q + \rho_2 \theta \\
\times \exp((1 - \theta)t) |y|^q.
\]

This implies

\[
LV(x, y, t, i) \leq -(1 - (p - 2\theta + 2\lambda_i \beta_3)\delta c_2)
\]

and (29), by (33) and (34), we can then show that

\[
\int_0^t e^{\delta s} \left( \delta V(x(s), r(s)) + LV(x(s), x(\theta s), s, r(s)) \right) ds + M(t)
\]

on \( t \geq 0 \), where \( M(t) \) is a continuous local martingale with \( M(0) = 0 \). As \( \beta_1 = 0 \), applying the same argument as (28) and (29), by (33) and (34), we can then show that

\[
\int_0^t e^{\delta s} \left( \delta V(x(s), r(s)) + LV(x(s), x(\theta s), s, r(s)) \right) ds \leq -((p - 2\theta + 2\lambda_i \beta_3 - \delta c_2))\int_0^t e^{\delta s} |x|^p ds \\
- (2\rho_1 - \delta c_2) \int_0^t e^{\delta s} |x|^q ds \leq 0.
\]

Applying condition (12) to Equation (35), we have

\[
c_1 e^{\delta t} |x(t)|^p \leq c_5 + M(t).
\]

where \( c_5 = c_2(|x(0)|^p + |x(0)|^q) \). Taking the expectations on both sides of (36) gives assertion (31) immediately. Moreover, by the nonnegative semimartingale convergence theorem, we have

\[
\limsup_{t \to \infty} \left( c_1 e^{\delta t} |x(t)|^p \right) < \infty \quad \text{a.s.}
\]

which implies another assertion (32).

---

4. Two special cases and an example

We will also assume that all coefficients of PSDEs in this section will satisfy the local Lipschitz condition and, moreover, \( q > p \geq 2 \). To make our cases more understandable, we assume that the given hybrid system is described by a hybrid differential equation

\[
dx(t) = f(x(t), t, r(t)) dt. \tag{37}
\]

Assume that this given hybrid differential equation is either asymptotically stable or bounded. Its structured differences and various stochastic perturbations will be discussed in the following two cases.

4.1. Case 1

Assume that for each \( i \in S_1 \), there is a number \( b_{11} < 0 \) such that

\[
x^T f(x, t, i) \leq b_{11} |x|^2
\]

while for each \( i \in S_2 \), there is a pair of numbers \( b_{11} \in R \) and \( b_{12} > 0 \) such that

\[
x^T f(x, t, i) \leq b_{11} |x|^2 - b_{12} |x|^{q-p+2}
\]

for \( (x, t) \in R^p \times R_+ \). This means that the differential equation in mode \( i \in S_1 \) is stable but may not in mode \( i \in S_2 \). In order for the hybrid Equation (37) to be stable, we assume moreover that

\[
\mathcal{A} := -\text{diag}(pb_{11}, \ldots, pb_{n1}) - \Gamma \tag{38}
\]

is a nonsingular \( M \)-matrix. It is then known (see, e.g. Hu, Mao, & Zhang, 2013) that Equation (37) is exponentially stable in \( p \)-th moment. Suppose that Equation (37) is subject to a stochastic perturbation and the perturbed system is described by

\[
dx(t) = f(x(t), t, r(t)) dt + G(x(\theta t), t, r(t)) dB(t), \tag{39}
\]

and the perturbation has its structured difference in the sense that

\[
|G(y, t, i)|^2 \leq b_{13} \exp \left( -\frac{2}{p}(1 - \theta t) \right) |y|^2, \quad i \in S_1
\]

and

\[
|G(y, t, i)|^2 \leq b_{13} \exp \left( -\frac{q - p + 2}{q}(1 - \theta t) \right) |y|^{q-p+2}, \quad i \in S_2
\]

for \( (y, t) \in R^p \times R_+ \), where \( b_{13} > 0 \) for all \( i \in S \). We wish to obtain upper bounds on \( b_{13} \)’s for the perturbed system (39) to remain stable. Noting that for \( i \in S_1 \)

\[
x^T f(x, t, i) + 0.5(q - 1)|G(y, t, i)|^2 \leq b_{11} |x|^2 + 0.5(q - 1)b_{13} \exp \left( -\frac{2}{p}(1 - \theta t) \right) |y|^2
\]

where \( b_{11} \) and \( b_{13} \) are the nonnegative constants. If we set

\[
b_{11} = b_{13} = b_{13} \exp \left( -\frac{2}{p}(1 - \theta t) \right) |y|^2,
\]

then we have

\[
|G(y, t, i)|^2 \leq b_{11} |x|^2 - b_{13} |x|^{q-p+2}, \quad i \in S_2
\]

for \( (y, t) \in R^p \times R_+ \), where \( b_{13} > 0 \) for all \( i \in S \). We wish to obtain upper bounds on \( b_{13} \)’s for the perturbed system (39) to remain stable. Noting that for \( i \in S_1 \)

\[
x^T f(x, t, i) + 0.5(q - 1)|G(y, t, i)|^2 \leq b_{11} |x|^2 + 0.5(q - 1)b_{13} \exp \left( -\frac{2}{p}(1 - \theta t) \right) |y|^2
\]

where \( b_{11} \) and \( b_{13} \) are the nonnegative constants. If we set

\[
b_{11} = b_{13} = b_{13} \exp \left( -\frac{2}{p}(1 - \theta t) \right) |y|^2,
\]

then we have

\[
|G(y, t, i)|^2 \leq b_{11} |x|^2 - b_{13} |x|^{q-p+2}, \quad i \in S_2
\]
Moreover, the matrix while we see that Assumption 2.3 is satisfied with Remark 3.2. Compute

Assume that

In this case we will discuss the robust boundedness. Hence the matrix $A$ defined by (2) is the same as the matrix $A$ defined by (38) and hence $A$ is a nonsingular M-matrix. Moreover, the matrix $B$ defined by (3) becomes

which is a nonsingular M-matrix too by Lemma 2.1. We choose $\rho$ by (10), so condition (6) is satisfied by Remark 3.2. Compute $\lambda_i$'s and by $\zeta_i$'s by (4) and (5), respectively. Conditions (7)–(9) then yield the following bounds

\[
\begin{align*}
\beta_{i1} &= 0, \quad \beta_{i2} = b_{i1}, \quad \text{for } i \in S_1; \\
\beta_{i3} &= \frac{(q-1)b_{i3}}{2\theta} \quad \text{for } i \in S_1; \\
\beta_{i4} &= b_{i2}, \quad \beta_{i5} = \frac{(p-1)b_{i3}}{2\theta} \quad \text{for } i \in S_2.
\end{align*}
\]

Hence, the matrices $A$ and $B$ in Assumption 2.4 become

\[
A = \text{diag}(\rho b_{12}, \ldots, \rho b_{n2}) - \Gamma
\]

and

\[
B = \text{diag}(q b_{12}, \ldots, q b_{n2}) - (\gamma y)_{ij} S_1.
\]
Consider a scalar stochastically perturbed hybrid system

\[ b_{13} \leq \frac{2\theta}{(q-1)((p-2)\lambda_i \theta + 2\lambda_i)}, \]

\[ b_{13} < \frac{2\theta \rho}{\zeta_i(q-1)((q-2)\theta + q-p+2)} \quad \text{for } i \in S_1 \tag{48} \]

and

\[ b_{15} \leq \frac{2\theta}{(q-1)((p-2)\lambda_i \theta + 2\lambda_i)}, \]

\[ b_{15} < \frac{2\theta \rho}{\rho \lambda_i(p-1)((q-2)\theta + q-p+2)} \quad \text{for } i \in S_2. \tag{49} \]

By Theorems 3.4, we can therefore conclude that if the perturbed parameters \( b_{13} \) satisfy (48) and (49), then the solution \( x(t) \) of the PSDE (45) has properties (19) and (20).

**Example 4.1:** Consider a scalar stochastically perturbed hybrid system

\[ dx(t) = f(x(t), t, r(t)) \, dt + G(x(\theta t), t, r(t)) \, dB(t), \tag{50} \]

where \( B(t) \) is a scalar Brownian, \( r(t) \) is a Markov chain with the state space \( S = \{1, 2, 3, 4\} \) and the generator

\[ \Gamma = \begin{pmatrix} -8 & 4 & 2 & 2 \\ 4 & -6 & 1 & 1 \\ 2 & 3 & -6 & 0 \\ 1 & 2 & 1 & -4 \end{pmatrix}. \]

Let \( S_1 = \{1, 2\}, S_2 = \{3, 4\} \) and \( p = 2, q = 4, \theta = 0.1 \). The coefficients are defined by

\[ f(x, t, i) = \begin{cases} \cos(t) - 2x, & i = 1, \\ \sin(t) - 3x, & i = 2, \\ \cos(t) + x - 2x^3, & i = 3, \\ \sin(t) + x - 3x^3, & i = 4, \end{cases} \]

\[ G(y, t, i) = \begin{cases} \sigma_1 e^{-0.45t}y, & i = 1, \\ \sigma_2 e^{-0.45t}y, & i = 2, \\ \sigma_3 e^{-0.45t}y + \sigma_4, & i = 3, \\ \exp(-0.45t)y^2, & i = 4, \\ \sigma_5 e^{-0.45t}y + \sigma_6, & i = 3, \\ \exp(-0.45t)y^2, & i = 4. \end{cases} \]

It is then easy to show

\[ xf(x, t, i) = \begin{cases} x \cos(t) - 2x^2, & i = 1, \\ x \sin(t) - 3x^2, & i = 2, \\ x \cos(t) + x^2 - 2x^4, & i = 3, \\ x \sin(t) + x^2 - 3x^4, & i = 4. \end{cases} \]

and

\[ |G(y, t, i)|^2 = \begin{cases} |\sigma_1 e^{-0.45t}y|^2, & i = 1, \\ |\sigma_2 e^{-0.45t}y|^2, & i = 2, \\ |\sigma_3 e^{-0.45t}y + \sigma_4 e^{-0.45t}y^2|^2, & i = 3, \\ |\sigma_5 e^{-0.45t}y + \sigma_6 e^{-0.45t}y^2|^2, & i = 4 \end{cases} \]

where \( b_{1i}'s \) are all positive numbers but their values are of no further use so we do not specify them.

It is straightforward to show that conditions (43), (44), (46) and (47) are satisfied with

\[ b_{12} = -1.9, \quad b_{22} = -2.9, \quad b_{32} = 1, \]

\[ b_{42} = 1, \quad b_{34} = 1.9, \]

\[ b_{44} = 2.9, \quad \sigma_1^2 = \frac{3}{2} b_{13}, \quad \sigma_2^2 = \frac{3}{2} b_{23}, \quad \sigma_3^2 = \frac{1}{4} b_{33}, \]

\[ \sigma_4^2 = \frac{1}{4} b_{43}, \quad \sigma_5^2 = \frac{1}{4} b_{35}, \quad \sigma_6^2 = \frac{1}{4} b_{45}. \]

Then by (2), we have

\[ A = \begin{pmatrix} 11.8 & -4 & -2 & -2 \\ -4 & 11.8 & -1 & -1 \\ -2 & -3 & 4 & -1 \\ -1 & -2 & -1 & 2 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 15.6 & -4 \\ -4 & 17.6 \end{pmatrix}. \tag{1} \]

Noting that

\[ A^{-1} = \begin{pmatrix} 0.3478 & 0.3609 & 0.4528 & 0.7546 \\ 0.2133 & 0.3314 & 0.3248 & 0.5413 \\ 0.4921 & 0.6365 & 0.9803 & 1.3005 \\ 0.6332 & 0.8301 & 1.0413 & 2.0689 \end{pmatrix}, \tag{2} \]

and

\[ B^{-1} = \begin{pmatrix} 0.0681 & 0.0155 \\ 0.0155 & 0.0603 \end{pmatrix}. \tag{3} \]
we see, by Lemma 2.1, that both $A$ and $B$ are nonsingular M-matrices. We can then compute

$$
\lambda_1 = 1.916, \quad \lambda_2 = 1.4108, \quad \lambda_3 = 3.4094, \\
\lambda_4 = 4.5735, \quad \bar{\gamma} = 5, \quad \bar{d} = 0.0836, \quad \rho = 9.1366, \\
\xi_1 = 0.7633, \quad \xi_2 = 0.6926.
$$

Conditions (48) and (49) then become

$$
\sigma_1 \leq 0.1615, \quad \sigma_2 \leq 0.1883, \quad \sigma_3 < 0.0856, \\
\sigma_4 < 0.0739, \quad \sigma_5 < 0.7585, \quad \sigma_6 < 0.7962. \quad (51)
$$

We can therefore conclude that if the perturbed parameters $\sigma_i$ satisfy (51), then the solution $x(t)$ of the SDDE (50) has the properties that

$$
\limsup_{t \to \infty} \frac{1}{t} \int_0^t E|\xi(s)|^4 \, ds \leq H_1,
$$

and

$$
\limsup_{t \to \infty} E|x(t)|^2 \leq H_2,
$$

where $H_1$ and $H_2$ are positive constants.

To perform a computer simulation for the solution, we set $\sigma_1 = 0.15, \sigma_2 = 0.07, \sigma_3 = 0.07, \sigma_4 = 0.15, \sigma_5 = 0.07, \sigma_6 = 0.15, x(0) = 1$ and $r(0) = 1$. The computer simulations in Figure 1 show a single sample path of the Markov chain and that of the solution, from which we can see how the Markov chain jumps from one mode to another and also the solution evolves in a bounded domain.

5. Conclusion

In this paper, we have discussed robust stability and boundedness for highly nonlinear hybrid PSDEs with different structures. We have also discussed two special cases and an example to illustrate our theory.

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