VIRTUAL CROSSING REALIZATION

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ABSTRACT

We study virtual isotopy sequences with classical initial and final diagrams, asking when such a sequence can be changed into a classical isotopy sequence by replacing virtual crossings with classical crossings. An example of a sequence for which no such virtual crossing realization exists is given. A conjecture on conditions for realizability of virtual isotopy sequences is proposed, and a sufficient condition for realizability is found. The conjecture is reformulated in terms of 2-knots and knots in thickened surfaces.

Keywords: Virtual Knots

1. Introduction

In \cite{3}, it is observed that classical knot theory embeds in virtual knot theory, in the sense that if two classical knots are virtually isotopic, then they are classically isotopic; this follows from the fact that the fundamental quandle (or alternatively the group system) is preserved by virtual moves. Since the fundamental quandle is a complete invariant for classical knots, any two classical knot diagrams related by a sequence of virtual moves must have isomorphic quandles, and hence must be isotopic in the classical sense. This also follows from theorem 1 of \cite{5}, which says that every stable equivalence class of knots in thickened surfaces (which are equivalent to virtual knots) has a unique irreducible representative. In \cite{4}, it is suggested that a purely combinatorial proof for this fact may be instructive.

A naïve attempt at a constructive proof in terms of Gauss diagrams initially looks promising; virtual crossings (and hence virtual moves) do not appear in Gauss diagrams, which include only classical crossings. Thus, given a virtual isotopy sequence which begins and ends with classical diagrams, we may attempt to construct a classical isotopy sequence by simply translating the Gauss diagram sequence to knot diagrams.

This strategy fails because unlike the classical crossing-introducing type II move, the Gauss diagram type II move does not require the strands being crossed to be

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adjacent in the plane. Further, the Gauss diagram II move permits both the direct and reverse II moves on any pair of strands, regardless of the orientation of the strands, while at most one of these is realizable using only classical diagrams.

Figure 1: A realized virtual isotopy sequence.

All of these moves can be realized by introducing virtual crossings. Moreover, as these virtual crossings are removed by the end of the sequence, it is natural to ask under which circumstances these virtual crossings can be replaced with classical crossings throughout the sequence to yield a valid classical isotopy sequence.

An assignment of classical crossing type to each virtual crossing in a virtual isotopy sequence is a virtual crossing realization. A virtual crossing realization is valid if each classical knot diagram in the resulting move sequence differs from the previous diagram by a valid Reidemeister move, i.e., if every vII move becomes a II move and every v or vIII move becomes a III move. Figure I depicts a valid virtual crossing realization. Here we denote realized virtual crossings as circled classical crossings – these are ordinary classical crossings, the circle is retained only to indicate which crossings have been realized.

Initially we may hope that every virtual isotopy sequence admits a valid virtual
crossing realization; a combinatorial proof of the type suggested in [4] would then follow. However, as figure 8 shows, this is not the case. Invariance of the fundamental quandle with respect to virtual moves implies only the existence of some classical isotopy sequence between classical diagrams with isomorphic quandles; such a sequence might be very different from any given virtual sequence. In particular, for any pair of equivalent classical diagrams, there need only be one classical isotopy sequence with the given end diagrams to satisfy the theorem of [3], while virtual sequences with the same end diagrams are clearly not unique.

In this paper, we study the problem of when a virtual isotopy sequence may be realized. The paper is organized as follows: We begin with a definition of realizability for virtual isotopy sequences. We then identify the ways in which a sequence can fail to be realizable, which consist of two types of bad moves. We consider each of these types of bad moves in turn, proving that one type may always be avoided and analyzing circumstances in which the other arises. We then obtain our main result, theorem 2 which gives a sufficient condition for realizability of a virtual isotopy sequence. A conjecture is proposed, and we reformulate the conjecture in terms of 2-knots. Finally, the conjecture is reformulated in terms of knots in thickened surfaces.

2. Virtual Knots and Gauss Diagrams

A link diagram is a planar oriented 4-valent graph with vertices regarded as crossings and enhanced with crossing information. The edges are oriented so that each vertex has two incoming edges, one over and one under, and two outgoing edges, also one over and one under. We may regard knots and links combinatorially as equivalence classes of knot and link diagrams under the equivalence relation generated by the three Reidemeister moves, pictured in figure 2.

![Figure 2: Reidemeister moves.](image-url)
By enlarging the set of decorated graphs to include non-planar 4-valent graphs with vertices enhanced with crossing information and edges oriented as before, we obtain virtual links as equivalence classes under the equivalence relation generated by the three Reidemeister moves.

To draw non-planar graphs on planar paper, we must introduce virtual crossings, which we distinguish from the decorated vertices (or classical crossings) by denoting virtual crossings as circled intersections.

Since these virtual crossings are artifacts of representing non-planar graphs in the plane, we may obtain an equivalent diagram by replacing any arc containing only virtual crossings with any other arc containing only virtual crossings with the same endpoints. This breaks down into four virtual moves, one move for each type of thing we can move the arc past: the arc itself (move vI), another arc (move vII), a virtual crossing (move vIII) and a classical crossing (move v). We may then consider virtual knots as equivalence classes of virtual knot diagrams under the equivalence relation generated by moves I, II, III, vI, vII, vIII and v, known as virtual isotopy.

![Virtual moves](image)

Two potential moves are not allowed, the two forbidden moves $F_t$ and $F_h$ (depicted in figure), variants of the type III move with two classical crossings and one virtual crossing. Unlike the valid virtual moves, the forbidden moves alter the underlying graph of the diagram. Together, the two forbidden moves can be used to unknot any knot, virtual or classical.

Gauss diagrams provide another way of representing virtual knots combinatorially. A Gauss diagram for a knot is a circle with oriented chords representing crossings; if we think of the circle as the preimage of the knot diagram under the embedding into $\mathbb{R}^3$ and projection to the plane, then the chords join the two preimages of each crossing point. We orient the chords “in the direction of gravity,” that is, toward the preimage of the undercrossing, and we decorate these arrows with signs given by the local writhe number of the crossing. The Gauss diagram of a link
has one circle for each component of the link, and crossings between components correspond to arrows joining the circles.

Virtual knots and links may then be regarded as equivalence classes of Gauss diagrams under the Gauss diagram versions of the Reidemeister moves; a Gauss diagram determines a virtual knot diagram up to virtual moves (Vl, vII, vIII and v), while a virtual knot diagram determines a unique Gauss diagram. There are several instances of type III moves depending on the orientation and cyclic order of the three strands involved; only two of these are listed in figure 5.

3. Virtual Isotopy Sequences

**Definition 1.** A sequence of virtual knot diagrams $K_1 \to K_2 \to \ldots \to K_n$ where $K_i$ differs from $K_{i-1}$ by a single virtual move (and possibly a planar isotopy) is a *virtual isotopy sequence*. We will sometimes use the term *valid virtual isotopy sequence* to distinguish a virtual isotopy sequence from a sequence of virtual knot diagrams in which one or more pairs of diagrams $K_i$ and $K_{i-1}$ are not related by virtual moves; such a sequence may be called an *invalid sequence*.
The Gauss diagram type II move permits, on any two sections of the circle, both the direct type II move, in which both strands are oriented in the same direction, and the reverse type II move, in which the strands are oriented in opposite directions. Unlike the classical type II move, the Gauss diagram II move does not require the arcs being crossed to be adjacent in the plane. For any pair of strands, of the four possible Gauss diagram type II moves, at most two are classically realizable, and then only if the strands are adjacent. However, all of these non-classical moves are realizable in virtual knot diagrams with the addition of virtual crossings.

Figure 6: Examples of classically unrealizable type II moves.

Though a Gauss diagram sequence beginning and ending with realizable classical diagrams may not be classically realizable, each of the individual unrealizable moves is realizable as a sequence of classical moves on realizable classical diagrams if we introduce classical crossings in the moves pictured in figure 6 instead of virtual crossings.

Given a virtual isotopy sequence, we can name and follow each virtual crossing through the sequence. Introducing classical crossings in place of virtual crossings then means assigning classical crossing information to each virtual crossing throughout the sequence. For each individual move, it is clear that we can obtain a legitimate move in this way. However, a choice of classical crossing type which makes one move work may render another move invalid later in the sequence, since crossings cannot change type once introduced.

Definition 2. An assignment of a sign (+ or -) to each virtual crossing throughout the diagram is a virtual crossing realization. A virtual crossing realization is valid if it results in a sequence of valid classical moves.

There are two ways a virtual crossing realization can yield invalid moves. One of these is the three-crossing move with all three edges alternating, known as the \( \Delta \) move; the other is the two-crossing move with both edges alternating, called a
The $\Gamma$ move by analogy with the $\Delta$ move.\(^2\) Both the $\Delta$ move and $\Gamma$ move are invalid, meaning they are not realizable as ambient isotopies. The effect of the $\Delta$ move is studied in [6], and a $\Gamma$ move combined with a pair of type II moves effects a crossing change, resulting in unknotting.

![Invalid moves arising in virtual crossing realization.](image)

Thus, a virtual crossing realization with no $\Delta$ or $\Gamma$ moves yields a classical isotopy sequence from the initial diagram to the final diagram. Thus we ask, which virtual isotopy sequences admit a valid virtual crossing realization?

**Theorem 1.** Not every valid virtual isotopy sequence with classical end diagrams admits a valid virtual crossing realization.

**Proof.** Figure 8 depicts a virtual isotopy sequence with no valid virtual crossing realization; each of the eight possible assignments of signs to the virtual crossings yields either a $\Delta$ or a $\Gamma$ move. Moreover, this sequence may be spliced in to any other virtual isotopy sequence to yield a new sequence which does not admit a virtual crossing realization. \(\square\)

However, changing either of the classical crossings in the sequence in figure 8 yields a valid sequence which does admit a valid virtual crossing realization, and whose end diagrams are equivalent to the originals. Indeed, the sequence of figure 8 may be viewed as a truncation of a longer sequence which has been partially realized; in this scenario, the two classical crossings have been introduced, and are later removed, in realized type VI moves. Moreover, if we choose the opposite sign for either of these classical crossings, the resulting sequence admits a realization.

Indeed, let $S$ be a virtual isotopy sequence which admits a virtual crossing realization. Then we may obtain a virtual isotopy sequence which does not admit a virtual crossing realization by realizing some crossings in $S$ and then truncating the sequence provided the realizations chosen for the realized crossings yield a valid realization.

\(^2\)The $\Gamma$ move is sometimes called a “2-move.”
virtual isotopy sequence yet are incompatible with the remaining possible choices, as in Figure \ref{fig:8}.

![Diagram](image)

Thus, in order to find a classical isotopy sequence given a virtual sequence with classical end diagrams, we must avoid situations like the one in Figure \ref{fig:8}. We begin by considering which classical crossings may be replaced with virtual crossings, yielding an equivalent sequence.

4. Classical Crossing Virtualization

**Definition 3.** A classical crossing in a virtual knot diagram is *virtualizable* if the virtual knot diagram obtained by replacing the crossing with a virtual crossing is virtually isotopic to the original diagram. More generally, a set of crossings in a virtual knot diagram is virtualizable if the diagram obtained by replacing each crossing in the set with a virtual crossing is equivalent to the original diagram. Similarly, a set of classical crossings is *switchable* if the knot diagram obtained by switching the crossing type of all crossings in the set is equivalent to the original diagram.

Switchability does not imply virtualizability; if we simultaneously switch all the crossings in a square knot, for example, we obtain an equivalent diagram, but
virtualizing all the crossings yields an unknot. Similarly, virtualizability does not imply switchability. Figure 9 shows a virtual knot diagram with a pair of crossings which are virtualizable but not switchable.

Figure 9: Virtualizable crossings need not be switchable.

A crossing which appears in neither the initial nor the final diagram in a virtual isotopy sequence is temporary. Temporary crossings are both introduced and removed during the course of the isotopy, and may be classical or virtual. In a virtual isotopy sequence whose initial and final diagrams contain only classical crossings, every virtual crossing is temporary; such a sequence may also contain classical temporary crossings.

Virtualizability and switchability are properties of sets of crossings, and may be defined both for individual diagrams and for sequences of diagrams. Intuitively, a set of crossings is virtualizable (respectively, switchable) in a diagram if virtualizing (resp., switching) the crossings in the diagram results in an equivalent diagram. Similarly, a set of crossings in virtualizable (respectively, switchable) in a virtual isotopy sequence if virtualizing (resp., switching) the crossings in the diagram results in an equivalent sequence. More formally, we have the following:

**Definition 4.** Let \( K = K_1 \rightarrow \ldots \rightarrow K_n \) be a virtual isotopy sequence and let \( J \) be a set of crossings in \( K \). Then the set \( J \) is sequentially virtualizable if the move sequence \( K' \) obtained by replacing each crossing in \( J \) throughout \( K \) with a virtual crossing is a valid virtual isotopy sequence, with \( K'_1 \) equivalent to \( K_1 \) and \( K'_n \) equivalent to \( K_n \).

**Proposition 1.** A set of crossings \( J \) in a virtual isotopy sequence \( K \) is sequentially virtualizable if and only if \( J \) satisfies the following conditions:

(i) The subset of non-temporary crossings in \( J \) is virtualizable in each end diagram, and

(ii) No crossing in \( J \) appears in a type III move with two crossings not in \( J \).

**Proof.** Condition (i) says that the sequence \( K' \) obtained from \( K \) by replacing the crossings in \( J \) with virtual crossings has end diagrams which are equivalent to the end diagrams of \( K \). Condition (ii) says that type III moves in \( K \) get replaced with type \( v \) or \( vIII \) moves in \( K' \); in particular, \( K' \) is free from forbidden moves.
Conversely, if virtualizing the crossings in $J$ results in a valid virtual isotopy sequence, the absence of forbidden moves implies condition (ii), and $K_1$ equivalent to $K'_1$ and $K_n$ equivalent to $K'_n$ imply that any non-temporary crossings are virtualizable. \(\Box\)

Similarly, we have

**Definition 5.** Let $K = K_1 \rightarrow \ldots \rightarrow K_n$ be a virtual isotopy sequence and let $J$ be a set of classical crossings in $K$. Then the set $J$ is *sequentially switchable* if the move sequence $K'$ obtained by switching each crossing in $J$ throughout $K$ is a valid virtual isotopy sequence, with $K'_1$ equivalent to $K_1$ and $K'_n$ equivalent to $K_n$.

**Proposition 2.** A set of crossings is sequentially switchable iff

(i) Every non-temporary crossing in the set is switchable in the end diagrams and
(ii) If one crossing in the set lies on an over-arc in a III move, the other crossing on the over-arc is also in the set.

*Proof.* Switching one crossing on an over-arc changes a III move to a $\Delta$ move, but switching both keeps it a type III. \(\Box\)

**Definition 6.** A virtual isotopy sequence is *maximally virtualized* if it contains no sequentially virtualizable classical crossings. A virtual isotopy sequence has *switch-free ends* if every sequentially switchable set of crossings is virtualizable.

A union of sequentially virtualizable sets of crossings is sequentially virtualizable; hence, in any virtual isotopy sequence there is a maximal sequentially virtualizable set. Given a virtual isotopy sequence $K$, we may obtain a maximally virtualized sequence by replacing every crossing in the maximal sequentially virtualizable set with a virtual crossing. Note that a subsequence of a maximally virtualized sequence need not be maximally virtualized.

**Remark 1.** Though our interest in classical crossing virtualization is motivated by virtual crossing realization, it may occasionally be desirable to reduce the number of crossings needed in Gauss diagram isotopy sequences. A maximally virtualized isotopy sequence has the minimal number of classical crossings among virtual isotopy sequences with the specified underlying planar graph sequence.

A virtual isotopy sequence which contains sequentially virtualizable classical crossings may be viewed as a partially completed virtual crossing realization problem. Such a sequence is “correctly” partially completed in the sense that no invalid moves have yet been introduced by the choice of realization for the virtualizable crossings; however, a virtual isotopy sequence like the one in figure [8] with virtualizable classical crossings which does not admit a virtual crossing realization might admit one if we are allowed to switch the sequentially virtualizable classical crossings.

**Conjecture 1.** Every maximally virtualized virtual isotopy sequence with switch-free classical initial and final diagrams admits a valid virtual crossing realization.
Note that a maximally virtualized sequence with classical end diagrams has no sequentially virtualizable non-temporary crossings. Since sequentially virtualizable crossings in a sequence are sequentially virtualizable in every subsequence, requiring the end diagrams to be classical and free of switchable virtualizable crossings avoids situations such as figure 8 in which a realizable sequence is made unrealizable by truncation.

5. ir Classes and Realization Sets

**Definition 7.** A realization set is a set of virtual crossings in a virtual isotopy sequence for which a valid virtual crossing realization exists.

Conjecture 1 says that if a virtual isotopy sequence is maximally virtualized and has switch-free classical end diagrams, then the set of all virtual crossings in the set is a realization set.

Clearly, if a set of virtual crossings is obtained by virtualizing a sequentially virtualizable set of classical crossings, it is a realization set; as figure 8 shows, not every set of virtual crossings is a realization set. We now consider when a set of virtual crossings in a virtual isotopy sequence is a realization set.

Let $X$ be the set of virtual and sequentially virtualizable crossings in a virtual isotopy sequence which starts and ends with realizable classical diagrams. For each $x \in X$, define $i(x) = x$ if $x$ is either present in the initial diagram or is introduced in a type I or vI move; otherwise, $x$ is introduced in a type II or vII with another crossing $y$, in which case set $i(x) = y$. Similarly, define $r(x) = x$ if $x$ is either present in the final diagram or removed in a type I or vI move; otherwise, $x$ is removed in a type II or vII with another crossing $y$, in which case set $r(x) = y$. We then have involutions $i : X \to X$ and $r : X \to X$ taking each crossing $x \in X$ to its introduction partner $i(x)$ and each crossing $x \in X$ to its removal partner $r(x)$. In particular, both $i$ and $r$ are injective. For any $x \in X$, we may have $i(x) = r(x)$ or $i(x) \neq r(x)$. Reversing the order of steps in the isotopy sequence interchanges $i$ and $r$.

The equivalence classes of sequentially virtualizable crossings under the equivalence relation generated by the relations $x \sim i(x)$ and $x \sim r(x)$ are ir classes. The set of ir classes forms a partition on the set $X$ of virtual and sequentially virtualizable crossings in a virtual isotopy sequence. We can represent an ir class graphically with an ir diagram as follows: an ir diagram is a graph with a vertex for each crossing in the ir class, an edge labeled $i$ joining $x$ to $i(x)$ and an edge labeled $r$ joining $x$ and $r(x)$ for each crossing $x$ in the ir class.

**Proposition 3.** Classical crossings may be virtualized to obtain a valid virtual isotopy sequence only by virtualizing ir classes. A union of ir classes of classical crossings is sequentially virtualizable only if no crossing in any class in the set appears in a type III move with two classical crossings not in any class in the set. In particular, a lone ir class is sequentially virtualizable only if no crossing in the class appears in a type III move with two classical crossings not in the class.
Proof. Virtualizing $x$ but not $i(x) \neq x$ (or $r(x) \neq x$) results in an invalid pseudo-II move with one classical and one virtual crossing. If any crossing in the set of $ir$ classes appears in a type III move with two classical crossings not in the set, virtualizing that crossing will change the III move into an invalid move, either one of the two forbidden moves $F_i$ or $F_h$ of figure 4 or an invalid move equivalent to one of the two forbidden move sequences $F_o$ or $F_s$ in [7].

Virtualizing a complete $ir$ class of temporary classical crossings changes the type I and II moves to valid vI and vII moves, and the condition that no crossing in the set of $ir$ classes being virtualized appears in a type III move with two crossings not in any class in the set implies that each type III move is virtualized to either a valid type v move or a valid type vIII move, that is, $\Delta$ moves are avoided. \qed

A realization set therefore must be a union of $ir$ classes. Moreover, a realization set must be “closed under type v and vIII” moves, in the sense that if any crossing in the realization set appears in a type v move, the $ir$ class of the other virtual crossing must also be included in the set, and no crossing in any class in the set may appear in a type vIII move with two crossings whose classes are not in the set, in order to avoid forbidden moves.

**Proposition 4.** An $ir$ class has either

(i) $i(x) = x = r(x)$, or

(ii) $i(x) = x, r(y) = y$ for a unique $x$ and $y$, $x \neq y$, or

(iii) $i(x) \neq x$ and $r(x) \neq x$ for all $x$ in the class.

The $ir$ diagram of (i) is a single vertex with two loops, one labeled $i$ and one labeled $r$. The diagram of (ii) is a sequence of vertices connected by single edges with a loop at each end. The diagram of (iii) is a closed loop with an even number of edges and an even number of vertices, with edges alternately labeled $r$ and $i$. 

![Figure 10: ir class diagrams.](image)
Proof. The maps $i$ and $r$ are injective, so every vertex meets one $i$ edge and one $r$ edge. If $i(x) = x$ or $r(x) = x$ for a vertex $x$, the $ir$ diagram has a loop at $x$; the other edge at $x$ can be another loop or it can connect to another vertex. This next vertex can either have a loop or connect to another new vertex, but it cannot connect back to a previous vertex since each vertex already listed in the diagram has met both an $i$ and $r$ edge. After some number of steps, we reach the vertex representing the final crossing in the $ir$ class, which must therefore meet a loop.

If an $ir$ class has no crossing with $i(x) = x$ or $r(x) = x$, then the graph is a closed loop, since every vertex meets one $i$ and one $r$ edge. Thus the number of vertices in the $ir$ diagram equals the number of edges; this number is even since the number of $i$ edges equals the number of $r$ edges. Note that the edges must alternate between $r$ and $i$ labels. □

Corollary 1. An $ir$ class with an odd number $n \geq 3$ of crossings must have two crossings either present in the end diagrams or introduced or removed in type I moves.

Definition 8. An $ir$ class realization is a choice of sign for each crossing in an $ir$ class. An $ir$ class realization is valid if the resulting sequence does not contain any $\Gamma$ moves. A virtual set realization is a choice of crossing sign for each virtual crossing in a set.

Proposition 5. Each virtual $ir$ class admits two $ir$ class realizations without $\Gamma$-moves.

Proof. In a $\Gamma$ move, the crossings have the same sign, while in both the direct and reverse $\Pi$ moves, the crossing pairs have opposite signs. Thus, if we assign alternating signs to distinct crossings connected by edges in a virtual $ir$ class diagram, the resulting virtual crossing realization contains no $\Gamma$ moves between crossings in that $ir$ class. For each type of $ir$ class diagram, we can make alternating sign assignments consistently. If $i(x) = x$ is assigned $\epsilon = \pm 1$, then $r(x)$ is assigned $-\epsilon$, and $i(r(x))$ gets $(-1)^2 \epsilon = \epsilon$, etc., until we reach the other end of the $ir$ class diagram. In a closed loop $ir$ class diagram with $2k$ vertices, choose a starting vertex $x$ and assign it $\epsilon$, then assign $-\epsilon$ to $i(x)$, $(-1)^2 \epsilon = \epsilon$ to $r(i(x))$ and continue around the loop; when we reach $x$ again, it gets assigned $(-1)^{2k} \epsilon = \epsilon$, and the assignment is consistent.

A choice of sign for one crossing in an $ir$ class thus determines the signs for the whole class; hence for each $ir$ class there are two alternating assignments of signs, and thus two $ir$ class realizations which do not contain $\Gamma$ moves within the $ir$ class. □

Corollary 2. A union of $n$ $ir$ classes has $2^n$ virtual set realizations without $\Gamma$ moves. In particular, if a virtual isotopy sequence has $n$ virtual $ir$ classes there are $2^n$ virtual crossing realizations which do not contain $\Gamma$ moves.

Say that a virtual set realization whose restriction to each $ir$ class is one of the two valid $ir$ class realizations respects $ir$ classes. Then any valid virtual crossing realization must respect $ir$ classes. For a given set of virtual crossings in a virtual
isotopy sequence, to determine whether the set is a realization set it suffices to check only the valid \( ir \) class realizations.

**Remark 2.** We can now see why the virtual isotopy sequence in figure 8 has no valid virtual crossing realization: it has only one \( ir \) class, and hence only two of the eight virtual crossing realizations are free of \( \Gamma \) moves. Inspection reveals that both of these realizations make the edge connecting crossing \( A \) and crossing \( C \) alternating; either choice then makes one of the two following \( v \) moves a \( \Delta \). Note that both classical crossings in this sequence are sequentially virtualizable, so this example does not contradict conjecture 1.

6. \( \Delta \) moves

In this section we fix a virtual isotopy sequence with a choice of virtual crossing realization and consider when the realized sequence includes \( \Delta \) moves.

**Definition 9.** An edge in a virtual link diagram whose endpoints belong to the same \( ir \) class is an **intra-class** edge. An edge joining crossings in distinct \( ir \) classes is an **inter-class** edge. A type \( v \) or \( vIII \) move is **inter-class** if any of the three edges joining the crossings in the move is an inter-class edge; otherwise, the move is **intra-class**.

If an intra-class edge is alternating for one choice of valid \( ir \)-class realization, it is also alternating for the other choice; if an intra-class edge is non-alternating for a choice of valid \( ir \)-class realization, it is non-alternating for the other choice. If a \( \Delta \) move involves a pair of \( i \)- or \( r \)-partners in a virtual crossing realization which respects \( ir \) classes, the edge connecting the pair in the move cannot be one of the edges originally joining the pair, since these are non-alternating for both class-respecting realizations. However, \( ir \) partners may be connected in a move by non-original edges, and an intra-class move need not contain \( ir \) partners, only a pair of crossings from the same \( ir \) class.

If an alternating intra-class edge appears in a type \( v \) move, then only one of the two \( ir \) class realizations makes the move a valid \( III \) move; in this case, the move determines a choice of realization for the class. Moreover, one determined \( ir \) class may determine another, if a crossing from the determined class appears opposite an alternating intra-class edge in a type \( vIII \) move. Note that switching any crossing in a \( \Delta \) move makes the move valid, while switching either of the two crossings on the over-arc in a type \( III \) move changes it to a \( \Delta \).

These observations enable us to classify possible counterexamples to conjecture 1 into three types.

A counterexample to conjecture 1 is **type i** if its \( \Delta \) moves involve crossings from a single \( ir \) class. It might have an intra-class type \( vIII \) move in which all three edges are alternating; then the move is realized as a \( \Delta \) for both choices of \( ir \)-class realization. Another example of this type would have an \( ir \) class with two intra-class type \( v \) moves with an alternating edge in each move, so that the two moves
determine opposite realizations for the class. The example in figure 8 would be of this type, if the sequence were maximally virtualized.

A counterexample to conjecture 1 is type ii if its \( \Delta \) moves involve distinct determined \( ir \) classes with incompatible \( ir \) class realizations. A pair of crossings from distinct determined \( ir \) classes might meet in an inter-class \( v \) move which is realized as a \( \Delta \) by the determined \( ir \) class realizations, or a \( vIII \) move could have three crossings from determined classes which realize as a \( \Delta \). Another example might have two determined \( ir \) classes which determine opposite realizations for a third \( ir \) class.

A counterexample to conjecture 4 is type iii if its \( \Delta \) moves involve a chain of \( ir \) classes with inter-class \( vIII \) moves such that every realization respecting \( ir \) classes realizes at least one of these \( vIII \) moves as a \( \Delta \). That is, any attempt to fix the move by switching one \( ir \) class realization simply creates a new inter-class \( \Delta \) move. An example of this type might have a “cycle” of \( ir \) classes so that fixing a \( \Delta \) move by switching one \( ir \) class changes another \( III \) move to a \( \Delta \); fixing this move by switching the \( ir \) class of another crossing in the move then changes another \( III \) move to a \( \Delta \), and so on, until we eventually break the first \( \Delta \) move we fixed. A minimal example of this type would have three \( ir \) classes, say \( A, B, C \), and inter-class moves between each, such that all eight choices of realization for the triple \( \{ A, B, C \} \) realize at least one of the inter-class moves as a \( \Delta \).

Any of these situations in a maximally virtualized virtual isotopy sequence with realizable switch-free classical end diagrams would contradict conjecture 1. Conversely, proving that none of the three types of situation listed above can occur would establish conjecture 1.

7. Virtually Descending Diagrams

We now give a sufficient condition for when a virtual isotopy sequence with single-component classical end diagrams has a valid virtual crossing realization. In this section, \( K \) is a virtual knot diagram, i.e., a single-component virtual link diagram.

**Definition 10.** A virtual crossing realization of \( K \) is **virtually descending** with respect to a chosen base point and orientation if, starting at the base point and following the orientation, we encounter each realized virtual crossing first as an overcrossing. Say that a move fixes a base point if the base point lies outside the part of the diagram pictured in the move.

**Lemma 1.** If the base point is fixed by a type III move and \( K \) is virtually descending before the move, \( K \) is virtually descending after the move.

*Proof.* Consider a realized \( vIII \) move. For \( K \) to be virtually descending, the strands must be encountered in the order listed in the picture below. Inspection shows that
the diagrams are virtually descending both before and after the move.

\[ \begin{array}{ccc}
1 & 2 & \text{III} \\
3 & & 3
\end{array} \]

The other cases are similar. \(\square\)

This observation suggests a strategy for choosing virtual crossing realizations, namely find a base point fixed by all v, vI, vII and vIII moves, then realize the crossings to make the diagram virtually descending. Lemma 2 shows that this strategy is compatible with our previous work.

**Lemma 2.** If a virtual isotopy sequence fixes a base point, realizing the virtual crossings to make \(K\) virtually descending at each step in the sequence respects \(ir\) classes.

**Proof.** Choose an orientation and realize the crossings at each step to keep \(K\) virtually descending at each move. Since the sequence of moves fixes the base point, every edge joining \(i\) or \(r\) pairs in a vII move is made non-alternating; otherwise, the base point lies on one of the edges, contrary to assumption. Given such a base point and vII move, the two choices of orientation yield the two choices of valid \(ir\) class realization from proposition 4. \(\square\)

**Lemma 3.** If \(K\) is virtually descending before a realized type vIII move with respect to a base point fixed by the move, the move is realized as a valid III move.

**Proof.** If \(K\) is virtually descending, then the first strand encountered in the move meets both realized crossings going over, hence the edge connecting them is non-alternating and the move is valid. \(\square\)

**Lemma 4.** If \(K\) is virtually descending before a type \(v\) move with respect to a base point fixed by the move, the move is realized as a \(\Delta\) move if and only if the classical undercrossing is encountered before the second virtual crossing when following the knot from the chosen base point and orientation. That is, the move is a \(\Delta\) iff the strand with the classical undercrossing is not third in the cyclic ordering of strands determined by the chosen base point and orientation.

**Proof.** Consider a realized type \(v\) move. A virtually descending diagram in which both virtual crossings are encountered before the classical crossing has both virtual overcrossings adjacent; hence the move is not a \(\Delta\). A virtually descending diagram in which the classical overcrossing is on the first strand encountered likewise has a pair of adjacent overcrossings, and hence is not a \(\Delta\). Thus, for a virtually descending diagram to be in position for a \(\Delta\) move, the crossings must be encountered the order
illustrated, namely, classical undercrossing before the second virtual crossing.

Conversely, one checks that the situation illustrated is indeed a $\Delta$ move. $\square$

**Theorem 2.** If a virtual isotopy sequence fixes a base point such that the over-crossing is always encountered before the undercrossing for all classical crossings involved in type $v$ moves for a choice of orientation of the knot, the virtual isotopy sequence admits a valid virtual crossing realization.

*Proof.* Realize all virtual crossings as they are introduced to make the diagrams virtually descending with respect to the given base point and orientation. The first possible $\Delta$ move is either of type $v$ or $v\text{III}$; if the former, lemma 4 implies the move is realized as a valid type III move, while if the latter, lemma 3 yields the same conclusion. Then lemma 1 implies that the next diagram is virtually descending; then the next $v$ or $v\text{III}$ move is also a valid III move, and we repeat until we reach the final diagram. Lemma 2 implies that the sequence is free from $\Gamma$ moves. Thus, the virtual crossing realization specified is valid. $\square$

**Remark 3.** The proof of theorem 2 relies on the fact that the entire diagram is virtually descending at every step. Thus, an attempt to apply this method to individual ir classes, e.g., considering distinct base points and orientations for each ir class, fails in general. If the set of virtual crossings in a given virtual isotopy sequence can be divided into disjoint non-interacting classes, then distinct base points and orientations satisfying the condition of theorem 2 may yield the result, though such a sequence may also be reduced after possible move reordering into shorter sequences.

8. **Knotted Surfaces**

A 2-knot is a compact smooth surface embedded in $\mathbb{R}^4$. A 2-knot diagram is a compact smooth surface $M$ immersed in $\mathbb{R}^3$ with singular set enhanced with crossing information. As with 1-knots, at a crossing curve we indicate which sheet goes over by drawing the undercrossing sheet “broken.” The preimage of the singular set is a set of closed curves and arcs in $M$, called the *double decker curves*; double decker arcs end at *branch points*. The double decker curves are divided into *upper decker curves* on the upper sheet at each crossing and *lower decker curves* on the lower sheet, analogous to upper and lower crossing point preimages in an ordinary 1-knot. Each double point curve is the image of an upper decker curve and a lower decker curve. When three sheets meet at a triple point, one sheet is highest, one between the others, and one lowest.
If we take a 2-dimensional slice of a 2-knot diagram in $\mathbb{R}^3$ by intersecting the 2-knot diagram with a plane missing any triple points, we obtain an ordinary link diagram; conversely, if we stack the diagrams in an isotopy sequence, letting the link diagram sweep out a broken surface diagram in $\mathbb{R}^3$, we obtain a portion of a 2-knot diagram connecting the initial and final diagrams. Triple points in the resulting 2-knot diagram correspond to Reidemeister III moves. Call the direction normal to the planes of the link diagrams *vertical* and the planes *horizontal*. Note that taking slices of an arbitrary 2-knot diagram does not typically yield an isotopy sequence, since local extrema in the vertical direction may result in differing numbers of link components in the resulting link diagrams.

Representing virtual crossings as undecorated self-intersections, a virtual isotopy sequence sweeps out an immersed broken surface diagram in $\mathbb{R}^3$ with singular set divided into classical or “decorated” (crossing information specified) and virtual or “undecorated” (no crossing information specified) parts. Conversely, an immersed surface diagram with some decorated and some undecorated double point curves represents a virtual isotopy sequence only if it has no local extrema in the vertical direction and no triple points with one undecorated and two decorated arcs.

An immersed surface which corresponds to a Reidemeister move sequence can be lifted to a 2-knot diagram, that is, we can choose crossing information along the undecorated singular set which makes the immersed surface a portion of an ordinary 2-knot diagram.

**Theorem 3.** Conjecture 1 is equivalent to the following: If $M$ is an immersed smooth broken surface in $\mathbb{R}^3$ satisfying

(i) $M$ has no local extrema in the vertical direction,
(ii) the boundary of $M$ is a pair of switch-free classical link diagrams,
(iii) the crossing information on the sheets of $M$ is compatible with the crossing information in the end diagrams and every triple point with three decorated curves has a highest, middle and lowest sheet,
(iv) no undecorated double point curve intersects two decorated double point curves at a triple point, and
(v) all double point curves not reaching the end diagrams and meeting at most one decorated curve are undecorated,

then $M$ lifts to a surface knot diagram.

**Proof.** Conditions (i) - (iv) guarantee that $M$ defines a virtual isotopy sequence: (i) says that the number of components stays constant, (ii) says that the end diagrams are switch-free classical link diagrams, (iii) says that the classical crossing information is consistent, and (iv) avoids forbidden moves. Then condition (v) says that the sequence is maximally virtualized. □

Examples of unliftable immersed surfaces are known (see [1] for some examples, such as a double cover of Boy’s Surface), but it is unknown whether any such

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[1] More precisely, we have a link concordance connecting the link diagrams.
unliftable surface satisfies the conditions in theorem \ref{thm:unliftable}.

Each undecorated double point curve corresponds to a virtual pair class, and sets of undecorated double point curves represent possible realization sets. While the surface $M$ has no local extrema in the vertical direction, the double-point curves typically do. Each portion of a double point curve joining a maximum or minimum on the curve (including endpoints) is the “trajectory” of a virtual crossing, and two portions of a curve meeting at a maximum or minimum correspond to $i$ or $r$ partners. Indeed, this observation provides another proof of proposition \ref{prop:virtual}.

Theorem 4.6 of \cite{Virtual} says that an immersed surface is liftable if and only if its decker set can be partitioned into two classes $A$ and $B$, where every singular curve is the image of one $A$ curve and one $B$ curve, at every branch point an $A$ curve and a $B$ curve meet, and at every triple point the preimage in $M$ consists of three intersections of decker curves, one involving two $A$ curves, one involving two $B$ curves, and one involving one $A$ and one $B$. Hence, To prove conjecture $\text{\ref{conj:virtual}}$ it would suffice to show that every immersed surface diagram meeting the conditions in the theorem \ref{thm:virtual} above admits such a coloring of the virtual double decker curves by $A$ and $B$ compatible with the crossing information specified for all classical decker curves intersecting the virtual decker curves in triple points.

A curve in an immersed surface diagram $M$ satisfying the condition of theorem \ref{thm:virtual} which misses the double point set and connects the end diagrams with no relative extrema in the vertical direction defines a base point fixed by the moves in the virtual isotopy sequence; call such a curve a base curve. If $M$ is connected, then the vertical slices which miss triple points are virtual knot diagrams; hence, if there is a base curve and a choice of direction of travel around the diagrams starting at the base point such that the preimage of every horizontal slice of $M$ meets the upper decker curve of each classical double curve intersecting two virtual curves at a triple point before the lower decker curve, then by theorem \ref{thm:virtual} the surface is liftable.

9. Knots in Surfaces

In \cite{Virtual}, virtual links are shown to be equivalent to links in thickened surfaces $S \times I$ with a stabilization operation consisting of adding or removing handles which miss the link. This corresponds to the intuitive concept of virtual knot diagrams as non-planar link diagrams; if we draw our link $L$ on a surface $S$ and then project $S$ onto $\mathbb{R}^2$, virtual crossings arise as the result of parts of the link in distinct handles or opposite sides of a handle projecting to the same point in the plane.

More specifically, a virtual link diagram $D \subset \mathbb{R}^2$ is the image of a link $L \subset S \times I$ under the composition $p = p_2 \circ p_1$ where $p_1 : S \times I \to S$ and $p_2 : S \to \mathbb{R}^2$. Virtual crossings and moves appear only in the projection $p_2$ and hence are dependent on the choice of embedding of $S \times I$ in $\mathbb{R}^3$. For a given $L \subset S \times I$, there may be many non-isotopic embeddings of $S \times I$ into $\mathbb{R}^3$, involving knotted and linked handles as well as Dehn twists. A sequence of Reidemeister and stabilization moves then corresponds to a sequence of choices of embedding $S \times I \to \mathbb{R}^3$ and projection.
\[ p = p_2 \circ p_1 \] which yields a virtual isotopy sequence, with the classical crossings occurring as double points of \( p_1 \) and the virtual crossings as double points of \( p_2 \).

In \([5]\), it is shown that every stable equivalence class of links in thickened surfaces has a unique irreducible representative; that is, given a link \( L \) in a thickened surface \( S \times I \), any two sequences of destabilization moves resulting in irreducible surfaces yield homeomorphic \((S \times I, L)\) pairs. In particular, any two sequences of Reidemeister moves and stabilization moves on \( L \subset S \times I \) which starts and ends with genus zero, i.e. classical, diagrams, result in diagrams which differ only by Reidemeister moves.

An embedding of a thickened surface containing a link into \( \mathbb{R}^3 \) is a virtual crossing realization for the corresponding virtual link diagram, since the choice of embedding for the surface includes a choice of over/under for each handle; we may view the virtual crossing realization as simply forgetting the distinction between crossings arising from \( p_1 \) and \( p_2 \). Conversely, stabilization moves in an embedded surface in \( \mathbb{R}^3 \) can result in virtualizing virtualizable crossings. A set of classical crossings is sequentially virtualizable if the crossings can be removed via stabilization.

**Definition 11.** Let \( L \subset S \times I \) be a link in a thickened surface. An embedding \( e : S \times I \hookrightarrow \mathbb{R}^3 \) is a lift of a virtual knot diagram \( D \) if \( D = p(L) \) where \( p = p_2 \circ p_1 : e(S \times I) \rightarrow e(S \times 0) \rightarrow \mathbb{R}^2 \).

With this definition, we end with another reformulation of conjecture \( \mathcal{H} \) namely:

**Theorem 4.** Conjecture \( \mathcal{H} \) is equivalent to: Every maximally virtualized virtual isotopy sequence with switch-free classical end diagrams \( K_1 \rightarrow \ldots \rightarrow K_n \) has a sequence of lifts \( e_i : S \times I \hookrightarrow \mathbb{R}^3, i = 1 \ldots n \) such that \( e_i \) is a lift of \( K_i \) and \( e_i(L) \) is ambient isotopic to \( e_{i+1}(L) \) in \( \mathbb{R}^3 \) for each \( i = 1 \ldots n \) by an isotopy which fixes the part of the link fixed by the corresponding virtual move.

**Proof.** If a maximally virtualized virtual isotopy sequence with switch-free classical end diagrams admits a virtual crossing realization, this realization tells us how to choose the lifts for each diagram so that the link \( L \) is changed only by ambient isotopies in \( \mathbb{R}^3 \) by specifying handle crossing and other embedding information. Conversely, if every such sequence is liftable, the specific lifts define a virtual crossing realization, and the condition that the links are ambient isotopic guarantees that the realization is valid. \( \square \)

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Virtual Crossing Realization

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