ON THE ASYMPTOTIC PLATEAU PROBLEM IN HYPERBOLIC SPACE

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Abstract. In this paper, we solve the asymptotic Plateau problem in hyperbolic space for constant $\sigma_{n-1}$ curvature, i.e. the existence of a complete hypersurface in $\mathbb{H}^{n+1}$ satisfying $\sigma_{n-1}(\kappa) = \sigma \in (0, n)$ with a prescribed asymptotic boundary $\Gamma$. The key ingredient is the curvature estimates. Previously, this is only known for $\sigma_0 < \sigma < n$, where $\sigma_0$ is a positive constant.

1. Introduction

Let $\mathbb{H}^{n+1}$ be the hyperbolic space and let $\partial_{\infty}\mathbb{H}^{n+1}$ be the ideal boundary of $\mathbb{H}^{n+1}$ at infinity. The asymptotic Plateau problem in hyperbolic space asks to find a complete hypersurface of constant curvature in $\mathbb{H}^{n+1}$ with prescribed asymptotic boundary at infinity. More precisely, given a closed embedded smooth $(n-1)$-dimensional submanifold $\Gamma \subset \partial_{\infty}\mathbb{H}^{n+1}$, one seeks a complete hypersurface $\Sigma$ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa) = \sigma, \quad \partial\Sigma = \Gamma,$$

where $f$ is a smooth symmetric function of $n$ variables, $\kappa = (\kappa_1, \cdots, \kappa_n)$ are the principal curvatures of $\Sigma$ and $\sigma$ is a constant.

The asymptotic Plateau problem was first studied by Anderson [1, 2] and Hardt and Lin [10] for area-minimizing varieties using geometric measure theory. Their results were extended by Tonegawa [18] to hypersurfaces of constant mean curvature. The asymptotic Plateau problem for constant mean curvature was studied by Lin [13], Nelli and Spruck [14] and Guan and Spruck [4] using PDE methods. The asymptotic Plateau problem for constant Gauss curvature was studied by Labourie [12] in $\mathbb{H}^3$ and by Rosenberg and Spruck [17] in $\mathbb{H}^{n+1}$. For a broad class of $f$ defined in the positive cone $\Gamma_n$, the asymptotic Plateau problem was completely solved via works of Guan, Spruck, Szapiel and Xiao [7, 6, 8]. For general $f$ satisfying natural structure conditions, the asymptotic Plateau problem was solved by Guan and Spruck [5] if $\sigma > \sigma_0$, where $\sigma_0$ is a positive constant.

After the work of Guan and Spruck [5], it is of great interest to study the remaining case, i.e. $0 < \sigma \leq \sigma_0$. As pointed out in [5], the only missing piece

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is the curvature estimates. In a recent work by Wang [19], he was able to
obtain the curvature estimates for \( f = \frac{\sigma_k}{\sigma_{k-1}} \) following the method in [5], where
\( \sigma_k \) denotes the \( k \)-th elementary symmetric function. In contrast, the case for
\( f = \sigma_k \) remains open.

In this paper, we study the asymptotic Plateau problem for an important
case \( f = \sigma_{n-1} \). For \( n = 2 \), it reduces to the mean curvature case, which was
completely solved by previous works. Therefore we will restrict ourselves to
the case \( n \geq 3 \).

Before we state our main theorems, let us introduce some notation s. Let
\[ H^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\} \]
denote the upper half-space model of \( H^{n+1} \). Then \( \partial \infty H^{n+1} \) is naturally identi-
fied with \( \mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1} \).

Denote Garding’s \( \Gamma_k \) cone by
\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k \} \]
We now state our main theorems.

**Theorem 1.1.** For \( n \geq 3 \), let \( \Gamma = \partial \Omega \), where \( \Omega \) is a bounded smooth domain
in \( \mathbb{R}^n \) with nonnegative mean curvature and \( \sigma \in (0, n) \). Suppose \( \Sigma \) is a \( C^4 \)
vertical graph over \( \Omega \) in \( H^{n+1} \) satisfying
\[ \sigma_{n-1}(\kappa) = \sigma, \quad \partial \Sigma = \Gamma, \]
where \( \kappa \in \Gamma_{n-1} \). Then we have
\[ \max_{x \in \Sigma, 1 \leq i \leq n} |\kappa_i(x)| \leq C + C \max_{x \in \partial \Sigma, 1 \leq i \leq n} |\kappa_i(x)|, \]
where \( C \) is a constant depending only on \( n, \Omega \) and \( \sigma \).

As an application, we solve the asymptotic Plateau problem for constant
\( \sigma_{n-1} \) curvature for all \( \sigma \).

**Theorem 1.2.** For \( n \geq 3 \), let \( \Gamma = \partial \Omega \), where \( \Omega \) is a bounded smooth domain
in \( \mathbb{R}^n \) with nonnegative mean curvature and \( \sigma \in (0, n) \). Then there exists a
complete hypersurface \( \Sigma \) in \( H^{n+1} \) satisfying
\[ \sigma_{n-1}(\kappa) = \sigma, \quad \partial \Sigma = \Gamma. \]

To obtain the curvature estimates, we adopt a new test function and make
full use of the structures of \( \sigma_{n-1} \) inspired by recent works of Guan, Ren and
Wang [9] and Ren and Wang [15]. It enables us to take care of the third order
terms and therefore obtain the desired estimates.
2. Preliminaries

In this section, we first collect some basic properties for hypersurfaces in $\mathbb{H}^{n+1}$. We will use upper half-space model of $\mathbb{H}^{n+1}$.

Let $\Sigma$ be a connected, orientable, complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity. Let $n$ be the unit normal vector of $\Sigma$ pointing to the unbounded region in $\mathbb{R}^{n+1} \setminus \Sigma$.

Let $X$ and $\nu$ be the position vector and Euclidean unit normal vector of $\Sigma$ in $\mathbb{R}^{n+1}$, define

$$u = X \cdot e, \quad \nu^{n+1} = e \cdot \nu,$$

where $e$ is the unit vector field in the positive $x_{n+1}$ direction in $\mathbb{R}^{n+1}$ and $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^{n+1}$.

Let $e_1, \cdots, e_n$ be a local frame, then the metric and second fundamental form of $\Sigma$ are given by

$$g_{ij} = \langle e_i, e_j \rangle, \quad h_{ij} = \langle D_{e_i}e_j, n \rangle,$$

where $D$ denotes the Levi-Civita connection of $\mathbb{H}^{n+1}$.

In the following, we will assume $e_1, \cdots, e_n$ are orthonormal. Consequently, $g_{ij} = \delta_{ij}$ and $h_{ij} = \kappa_i \delta_{ij}$, where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of $\Sigma$.

The Gauss and Codazzi equations are given by

$$R_{ijkl} = -(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk},$$

$$h_{ijk} = h_{ikj}.$$

The convention that $R_{ijkl}$ denotes the sectional curvature is used here.

The commutator formulas are given by

$$h_{klij} = h_{ijkl} - h_{ml}(h_{im}h_{kj} - h_{ij}h_{mk}) - h_{mj}(h_{mi}h_{kl} - h_{il}h_{mk}) - h_{ml}(\delta_{ij}\delta_{km} - \delta_{ik}\delta_{jm}) - h_{mj}(\delta_{il}\delta_{km} - \delta_{ik}\delta_{lm}).$$

Denote

$$\sigma_k^u = \frac{\partial \sigma_k}{\partial \kappa_i}, \quad \sigma_k^{u,ij} = \frac{\partial^2 \sigma_k}{\partial \kappa_i \partial \kappa_j}.$$

We now collect two lemmas relating to $\nu^{n+1}$.

Lemma 2.1.

$$\sum_i \frac{u_i^2}{u^2} = 1 - (\nu^{n+1})^2 \leq 1, \quad (\nu^{n+1})_i = \frac{u_i}{u}(\nu^{n+1} - \kappa_i),$$
\[ \sum_i \sigma_{n-1}^{ii} (\nu^{n+1})_{ii} = 2 \sum_i \sigma_{n-1}^{ii} \frac{u_i}{u} (\nu^{n+1})_i + (n-1)\sigma (1 + (\nu^{n+1})^2) \]

\[ - \nu^{n+1} \left( \sum_i \sigma_{n-1}^{ii} + \sum_i \sigma_{n-1}^{ii} \kappa_i^2 \right). \]

**Proof.** The first one follows from the definition of \( u \) and \( \nu_{n+1} \). The second one can be found in the proof of Theorem 3.1 in [8]. The third one can be found in (3.9) in [8]. Note that our equation is not normalized as in [8], thus the format is slightly different from (3.9) in [8]. \( \square \)

**Lemma 2.2.** For \( n \geq 2 \), let \( \Gamma = \partial \Omega \), where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \) with nonnegative mean curvature and \( \sigma \in (0, n) \). Suppose \( \Sigma \) is a \( C^4 \) vertical graph over \( \Omega \) in \( \mathbb{H}^{n+1} \) satisfying (1.1). Then we have

\[ \nu^{n+1} > a > 0, \]

where \( a \) is a constant depending only on \( n, \Omega \) and \( \sigma \).

**Proof.** By Proposition 4.1 in [5], we have

\[ w = \sqrt{1 + |Du|^2} \leq C. \]

Since \( \nu^{n+1} = \frac{1}{w} \), it follows that \( \nu^{n+1} \geq \frac{1}{C} \). \( \square \)

We now state some properties relating to \( \sigma_k \) function.

**Lemma 2.3.** Assume \( \kappa \in \Gamma_k \) and \( \kappa_1 \geq \cdots \geq \kappa_n \). Then

1. \[ \sum_i \sigma_k^{ii} \kappa_i^2 \geq \frac{k}{n} \sigma_1 \sigma_k. \]
2. If \( \kappa_i \leq 0 \), then

\[ -\kappa_i \leq \frac{n-k}{k} \kappa_1. \]

**Proof.** For (1), by Proposition 2.2 in [11], we have

\[ \sum_i \sigma_k^{ii} \kappa_i^2 = \sigma_1 \sigma_k - (k+1)\sigma_{k+1}. \]

The inequality now follows by a Newton-Maclaurin inequality.

For (2), see Lemma 10 in [10]. \( \square \)

We now state an important inequality by Ren and Wang (Theorem 11 in [15]), which is crucial to our estimate.

**Lemma 2.4.** Assume \( \kappa \in \Gamma_{n-1} \) and \( \kappa_1 \geq \cdots \geq \kappa_n \). For any constant \( \epsilon > 0 \), there exists a large constant \( K \) depending only on \( \epsilon \) such that

\[ \kappa_1 \left( K (\sigma_{n-1})^2_1 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pp} h_{qq} \right) + (1 + \epsilon) \sum_{i \neq 1} \sigma_{n-1}^{ii} h_{ii1}^2 - \sigma_{n-1}^{11} h_{111}^2 \geq 0. \]
3. Proof of main theorems

Proof of Theorem 3.1

Proof. Consider the quantity
\[ Q = \ln \kappa_1 - N \ln \nu^{n+1}, \]
where \( \kappa_1 \) is the largest principle curvature and \( N \) is a large constant to be determined later.

Suppose \( Q \) attains maximum at an interior point \( X_0 \). If \( \kappa_1 \) has multiplicity more than 1, then \( Q \) is not smooth at \( X_0 \). To overcome this difficulty, we apply a standard perturbation argument, see for instance [3]. Let \( g \) be the first fundamental form of \( \Sigma \) and \( D \) be the corresponding Levi-Civita connection. Choose an orthonormal frame \( e_1, \ldots, e_n \) near \( X_0 \) such that at \( X_0 \), we have
\[ D_{e_i} e_j = 0, \quad h_{ij} = \delta_{ij} \kappa_i, \quad \kappa_1 \geq \cdots \geq \kappa_n. \]

Near \( X_0 \), define a new tensor \( B \) by
\[ B(V_1, V_2) = g(V_1, V_2) - g(V_1, e_1)g(V_2, e_1), \]
for tangent vectors \( V_1 \) and \( V_2 \).
Denote \( B_{ij} = B(e_i, e_j) \) and define
\[ \tilde{h}_{ij} = h_{ij} - B_{ij}. \]
Let \( \tilde{\kappa}_1 \geq \cdots \geq \tilde{\kappa}_n \) be the corresponding eigenvalues of \( \tilde{h}_{ij} \).
It follows that \( \kappa_1 \geq \tilde{\kappa}_1 \) near \( X_0 \) and at \( X_0 \), we have
\[ \tilde{\kappa}_i = \begin{cases} \kappa_1, & i = 1, \\ \kappa_i - 1, & i > 1. \end{cases} \]
Now consider the new test function
\[ \tilde{Q} = \ln \tilde{\kappa_1} - N \ln \nu^{n+1}. \]
It also attains maximum at \( X_0 \). Moreover, \( \tilde{\kappa}_1 \) has multiplicity 1, thus \( \tilde{Q} \) is smooth at \( X_0 \).
At \( X_0 \), we have
\[ 0 = \frac{(\tilde{\kappa}_1)_i}{\tilde{\kappa}_1} - N \frac{(\nu^{n+1})_i}{\nu^{n+1}}, \quad (3.1) \]
\[ 0 \geq \frac{(\tilde{\kappa}_1)_{ii}}{\tilde{\kappa}_1} - \frac{(\tilde{\kappa}_1)^2}{\tilde{\kappa}_1^2} - N \frac{(\nu^{n+1})_{ii}}{\nu^{n+1}} + N \frac{(\nu^{n+1})^2}{(\nu^{n+1})^2}, \quad (3.2) \]
By the definition of \( B \) and the fact that \( D_{e_i} e_j = 0 \) at \( X_0 \), we have
\[ (\tilde{\kappa}_1)_i = \tilde{h}_{11i} = h_{11i}, \]
\[
(\bar{\kappa}_1)_{ii} = \bar{h}_{11ii} + 2 \sum_{p \neq 1} \frac{\bar{h}_{1pi}^2}{\kappa_1 - \bar{\kappa}_p} = h_{11ii} + 2 \sum_{p \neq 1} \frac{h_{1pi}^2}{\kappa_1 - \bar{\kappa}_p}.
\]

Plug into (3.2), we have
\[
0 \geq \frac{h_{11ii}}{\kappa_1} + 2 \sum_{p \neq 1} \frac{h_{1pi}^2}{\kappa_1 (\kappa_1 - \bar{\kappa}_p)} - \frac{h_{11ii}^2}{\kappa_1^2} - N \frac{(\nu^{n+1})_{ii}}{\nu^{n+1}}.
\]

By commutator formula (2.1), we have
\[
h_{11ii} = h_{ii11} + \kappa_1^{-2} \kappa_i - \kappa_1 \kappa_i^2 - \kappa_1 + \kappa_i.
\]

It follows that
\[
0 \geq \frac{h_{ii11}}{\kappa_1} + 2 \sum_{p \neq 1} \frac{h_{1pi}^2}{\kappa_1 (\kappa_1 - \bar{\kappa}_p)} - \frac{h_{ii11}^2}{\kappa_1^2} - N \frac{(\nu^{n+1})_{ii}}{\nu^{n+1}}
+ \kappa_1 \kappa_i - \kappa_i^2 - 1 + \frac{\kappa_i}{\kappa_1}.
\]

Contract with \(F_{ii} = \sigma_{ii}^{n-1}\), we have
\[
(3.3) \quad 0 \geq \sum_i \frac{F_{ii} h_{i11}}{\kappa_1} + 2 \sum_i \sum_{p \neq 1} \frac{F_{ii} h_{1pi}^2}{\kappa_1 (\kappa_1 - \bar{\kappa}_p)} - \sum_i F_{ii} h_{11ii}^2
- N \sum_i \frac{F_{ii} (\nu^{n+1})_{ii}}{\nu^{n+1}} - \sum_i F_{ii} \kappa_i^2 - \sum_i F_{ii},
\]
we have used the fact \(\sum_i F_{ii} \kappa_i = (n - 1) \sigma > 0\) in the above inequality.

By Lemma 2.1, we have
\[
\sum_i F_{ii} (\nu^{n+1})_{ii} = 2 \sum_i F_{ii} \frac{u_i}{u} (\nu^{n+1})_i + (n - 1) \sigma (1 + (\nu^{n+1})_i^2)
- \nu^{n+1} \left( \sum_i F_{ii} + \sum_i F_{ii} \kappa_i^2 \right).
\]

Plug into (3.3), we have
\[
0 \geq \sum_i \frac{F_{ii} h_{i11}}{\kappa_1} + 2 \sum_i \sum_{p \neq 1} \frac{F_{ii} h_{1pi}^2}{\kappa_1 (\kappa_1 - \bar{\kappa}_p)} - \sum_i F_{ii} h_{11ii}^2
- 2N \sum_i \frac{F_{ii} \frac{u_i}{u} (\nu^{n+1})_i}{\nu^{n+1}}
+ (N - 1) \left( \sum_i F_{ii} + \sum_i F_{ii} \kappa_i^2 \right) - N(n - 1) \sigma \frac{1 + (\nu^{n+1})_i^2}{\nu^{n+1}}.
\]
By Lemma 2.2, we have

\[
0 \geq \sum_i \frac{F^{ii} h_{ii1}}{\kappa_1} + 2 \sum_{i \neq 1} \frac{F^{ii} h_{1pi}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_p)} - \sum_i \frac{F^{ii} h_{11i}^2}{\kappa_1^2} \\
- \sum_i \frac{F^{ii} h_{ii1}^2}{\kappa_1^2} - 2N \sum_i \frac{F^{ii} h_{ii1}^2}{\kappa_1^2} + (N - 1) \left( \sum_i F^{ii} + \sum_i F^{ii} \kappa_i^2 \right) - CN,
\]

where \( C \) is a universal constant depending only on \( n, \Omega \) and \( \sigma \). From now on, we will use \( C \) to denote a universal constant depending only on \( n, \Omega \) and \( \sigma \), it may change from line to line.

Differentiate (1.1), we have

\[
\sum_i F^{ii} h_{ii1} = - \sum_{p, q, r, s} F^{pq, rs} h_{pq1} h_{rs1} = - \sum_{p \neq q} F^{pp, qq} h_{pp1} h_{qq1} + \sum_{p \neq q} F^{pp, qq} h_{pq1}.
\]

Plug into (3.4), we have

\[
0 \geq - \sum_{p \neq q} \frac{F^{pp, qq} h_{pp1} h_{qq1}}{\kappa_1} + \sum_{p \neq q} \frac{F^{pp, qq} h_{pq1}^2}{\kappa_1} + \sum_i \frac{F^{ii} h_{1pi}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_p)} - \sum_i \frac{F^{ii} h_{11i}^2}{\kappa_1^2} - 2N \sum_i \frac{F^{ii} h_{ii1}^2}{\kappa_1^2} + (N - 1) \left( \sum_i F^{ii} + \sum_i F^{ii} \kappa_i^2 \right) - CN.
\]

Now

\[
2 \sum_i \sum_{p \neq 1} \frac{F^{ii} h_{1pi}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_p)} \geq \sum_{p \neq 1} \frac{F^{pp} h_{1pp}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_p)} + \sum_{p \neq 1} \frac{F^{11} h_{11p}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_p)}
\]

\[
= \sum_{i \neq 1} \frac{F^{ii} h_{ii1}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_i)} + \sum_{i \neq 1} \frac{F^{11} h_{11i}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_i)}.
\]

Without loss of generality, assume \( \kappa_1 \) has multiplicity \( m \), then we have

\[
\sum_{p \neq q} \frac{F^{pp, qq} h_{pq1}^2}{\kappa_1} \geq 2 \sum_{i > m} \frac{F^{ii, 11i} h_{11i}^2}{\kappa_1} \geq 2 \sum_{i > m} \frac{(F^{ii} - F^{11}) h_{11i}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_i)} \geq 2 \sum_{i > m} \frac{(F^{ii} - F^{11}) h_{11i}^2}{\kappa_1 (\kappa_1 - \tilde{\kappa}_i)},
\]

where we have used the fact \( \tilde{\kappa}_i = \kappa_i - 1 \) in the last inequality.
Plug the above two inequalities into (3.5), we have

\[
0 \geq -\sum_{p\neq q} \frac{F_{pp,qq}h_{pp1}h_{qq1}}{\kappa_1} + 2 \sum_{i \neq 1} \frac{F_{ii}h_{i11i}^2}{\kappa_1(\kappa_1 - \kappa_i)} + 2 \sum_{1 < i \leq m} \frac{F_{11}h_{111i}^2}{\kappa_1(\kappa_1 - \kappa_i)} + 2 \sum_{i > m} \frac{F_{ii}h_{111i}^2}{\kappa_1(\kappa_1 - \kappa_i)} - \sum_{i} F_{ii}^2 \frac{\nu^{n+1}_i}{\nu^{n+1}} - 2N \sum_{i} F_{ii}u_i \frac{(\nu^{n+1})_i}{\nu^{n+1}} - \sum_{i} F_{ii}^2 \kappa_i^2 - CN.
\]

In the following, we will always assume \(\kappa_1\) is sufficiently large, for otherwise we have obtained the estimate.

For \(1 < i \leq m\), we have

\[
2 \frac{F_{11}h_{111i}^2}{\kappa_1(\kappa_1 - \kappa_i)} - \frac{F_{ii}h_{i11i}^2}{\kappa_1^2} = (2\kappa_1 - 1) \frac{F_{ii}h_{111i}^2}{\kappa_1^2} \geq \frac{F_{ii}h_{111i}^2}{\kappa_1^2}.
\]

For \(i > m\), by Lemma 2.3 we have

\[
2 \frac{F_{ii}h_{111i}^2}{\kappa_1(\kappa_1 - \kappa_i)} - \frac{F_{ii}h_{111i}^2}{\kappa_1^2} = \frac{F_{ii}(\kappa_1 + \kappa_i)h_{111i}^2}{\kappa_1^2(\kappa_1 - \kappa_i)} \geq c(n) \frac{F_{ii}h_{111i}^2}{\kappa_1^2},
\]

where \(c(n)\) is a constant depending only on \(n\).

Plug into (3.6), we have

\[
0 \geq -\sum_{p\neq q} \frac{F_{pp,qq}h_{pp1}h_{qq1}}{\kappa_1} + 2 \sum_{i \neq 1} \frac{F_{ii}h_{i11i}^2}{\kappa_1(\kappa_1 - \kappa_i)} + c(n) \sum_{i \neq 1} \frac{F_{ii}h_{i11i}^2}{\kappa_1^2} - \frac{F_{11}h_{111i}^2}{\kappa_1^2} - 2N \sum_{i} F_{ii}u_i \frac{(\nu^{n+1})_i}{\nu^{n+1}} + (N - 1) \left(\sum_{i} F_{ii} + \sum_{i} F_{ii}^2 \kappa_i^2\right) - CN.
\]

By Lemma 2.3 the fact that \(n \geq 3\) and that \(\kappa_1\) is sufficiently large, we have

\[
\frac{2}{\kappa_1(\kappa_1 - \kappa_i)} \geq \frac{2}{\kappa_1 \left(\frac{n}{n-1}\kappa_1 + 1\right)} \geq \frac{7}{6} \frac{1}{\kappa_i^2}.
\]

Apply Lemma 2.4 we have

\[
0 \geq c(n) \sum_{i \neq 1} \frac{F_{ii}h_{111i}^2}{\kappa_1^2} - 2N \sum_{i} F_{ii}u_i \frac{(\nu^{n+1})_i}{\nu^{n+1}} + (N - 1) \left(\sum_{i} F_{ii} + \sum_{i} F_{ii}^2 \kappa_i^2\right) - CN.
\]

By Lemma 2.1 we have

\[
-2NF_{ii}u_i \frac{(\nu^{n+1})_i}{\nu^{n+1}} = -2NF_{ii}u_i^2 \frac{\nu^{n+1} - \kappa_i}{\nu^{n+1}}.
\]
It is negative only if \( \kappa_i < \nu^{n+1} \). Since \( \kappa_1 \) is sufficiently large, in particular \( \kappa_1 > \nu^{n+1} \), it follows that

\[
0 \geq c(n) \sum_{i \neq 1} F_{ii} h_{ii}^2 - 2N \sum_{i \neq 1} F_{ii} u_i (\nu^{n+1})_i + (N - 1) \left( \sum_i F_{ii} + \sum_i F_{ii} \kappa_i^2 \right) - CN.
\]

By critical equation (3.1), we have

\[
0 \geq c(n) N^2 \sum_{i \neq 1} F_{ii} \frac{(\nu^{n+1})_i^2}{(\nu^{n+1})^2} - 2N \sum_{i \neq 1} F_{ii} u_i (\nu^{n+1})_i
\]

\[
+ (N - 1) \left( \sum_i F_{ii} + \sum_i F_{ii} \kappa_i^2 \right) - CN
\]

\[
\geq - C \sum_{i \neq 1} F_{ii} u_i^2 + (N - 1) \left( \sum_i F_{ii} + \sum_i F_{ii} \kappa_i^2 \right) - CN.
\]

Choose \( N \) sufficiently large, together with Lemma 2.1 and Lemma 2.3, we have

\[
0 \geq (N - 1) \sum_i F_{ii} \kappa_i^2 - CN \geq (N - 1) \frac{n-1}{n} \sigma \sigma_1 - CN.
\]

It follows that \( \sigma_1 \leq C \). The theorem is now proved.

\[ \square \]

**Proof of Theorem 1.2**

**Proof.** As pointed out in [5], the only missing piece is the interior curvature estimates. In view of Theorem 1.1, Theorem 1.2 is now proved, see details in [5].

\[ \square \]

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