Generalized measurements by linear elements

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I give a first characterization of the class of generalized measurements that can be exactly realized on a pair of qudits encoded in indistinguishable particles, by using only linear elements and particle detectors. Two immediate results follow from this characterization. (i) The Schmidt number of each POVM element cannot exceed the number of initial particles. This rules out any possibility of performing perfect Bell-measurements for qudits. (ii) The maximum probability of performing a generalized incomplete Bell-measurement is $\frac{1}{2}$.

I. INTRODUCTION

In the last years there has appeared very important contributions to the field of quantum information processing with linear elements (see below). Linear elements provide the means to exploit symmetry and interference effects associated with indistinguishable particles. This raises many interesting questions from the fundamental point of view, but it is also highly relevant technologically. Quantum information processing has a wide range of striking applications. Many of these applications have been first implemented in optical systems, where the lack of interaction at the single photon level makes the indistinguishability of the photons a crucial feature. Photons are ideal qudit ($d$-dimensional counterpart of a qubit) carriers, because of their low decoherence rates, and linear optical elements are extremely simple devices which allow one to perform certain operations on the encoded photons in a controlled fashion. It is therefore very desirable to know the capabilities and limitations of those operations. In a recent work E. Knill et al. make an important breakthrough in this direction showing that fault-tolerant computation with linear optics is in principle possible. Starting from the idea of teleportation of quantum gates, they develop a method to perform any quantum operation with a probability that asymptotically approaches unity with a growing number of highly entangled auxiliary photons. The preparation of the required auxiliary states is, however, far beyond the current technological possibilities. Moreover, their work makes a big step forward by presenting a proof of principle, but it does not elucidate the role played by the particle symmetry and indistinguishability, nor does exclude the idea that for specific applications one can find simpler protocols with less technological restrictions. For example, recent research shows how to purify entangled photons in noisy communication channels or from parametric down-conversion sources, reject bit flip errors in quantum communication, perform optimal unambiguous state discrimination, and efficiently eavesdrop a quantum key distribution, by using only a few beam splitters and particle detectors. Thus, the prospect of new applications and the need for a deeper understanding warrants further research on the power of linear elements.

In this communication I address the question of performing generalized measurements on indistinguishable particles by linear elements and particle detectors. For brevity, I term the qudits that are encoded in indistinguishable particles $i$-qudits, $b$-qudits for bosons, and $f$-qudits for fermions. At first sight it seems that linear elements cannot realize non-trivial generalized measurements, for they are unable to provide interaction between the particles. However, this argument has its roots in the concept of locality, which becomes vague when dealing with indistinguishable particles. Assigning a notion of locality to the $i$-qudits is only possible if each $i$-qudit occupies a different set of modes, so that Hilbert space of the whole system is the tensor product of the local Hilbert spaces of the individual qudits. At a descriptive level there is essentially no difference between $i$-qudits and qudits. Things change dramatically when one explores the technological possibilities of performing quantum operations on the $i$-qudits. As soon as the mixing of modes becomes possible the notion of locality vanishes, and realizing some non-trivial measurement becomes possible.

Until now, the measurement on $i$-qudits has only been studied in the context of unambiguous discrimination of given set of states. In Refs., the impossibility of performing a complete Bell-measurement on two $i$-qudits was proved and the optimum efficiency of the incomplete Bell-measurement was given in Ref. Later Carollo et al. showed that for a particular set of states, which are product states, discrimination without error is also impossible. The general approach in these papers was to feed the linear device with the states to be discriminated and check under what conditions the particle detectors at the output produce different “click” combinations that could identify the input state. Here I will adopt a different approach. Given that the measurement outcomes are of a known form, namely a “click” pattern, we will find
the POVM (see below) on the input i-qudits induced by this type of measurement. This approach is much more general and sets a suitable framework to arrive at the full characterization of the class of POVMs implemented by linear elements.

II. I-QUDITS

An arbitrary one qudit state $|\alpha\rangle = \sum_{i=1}^{d} \alpha_i|i\rangle$ is encoded in a single excitation, say, a photon, occupying $d$ field modes, $|\alpha\rangle = \sum_{i=1}^{d} \alpha_i a_i^\dagger|0\rangle$. Here $|0\rangle$ denotes the vacuum state and $a_i^\dagger$ are bosonic (fermionic) creation operators. Whenever needed, I will give the results corresponding to each of the particle statistics. In order to encode a two qudit state $|C\rangle = \sum_{i,j=1}^{d} C_{ij}|i\rangle|j\rangle$ a second particle is used occupying $d$ extra modes $\{a_{d+1}^\dagger, \ldots, a_{2d}^\dagger\}$, $|C\rangle = \sum_{i,j=1}^{d} C_{ij} a_{d+i}^\dagger a_{d+j}^\dagger|0\rangle$. Any two-boson (fermion) state can be defined with a bilinear form,

$$|\Psi\rangle = \sum_{i,j=1}^{n} N_{ij} a_i^\dagger a_j^\dagger = A^T N a(0),$$

where $N$ is an $n \times n$ symmetric (antisymmetric) matrix and $a = (a_1^\dagger, \ldots, a_n^\dagger)^T$. In particular, the bilinear form of the two i-qudit state $|C\rangle$ is

$$N = \frac{1}{2} \begin{pmatrix} 0 & C & \ast \\ (-)C^T & 0 & \ast \\ \ast & \ast & 0 \end{pmatrix}.$$  

The $d \times d$ matrix $C$ is defined using the correspondence between the state vectors $|C\rangle = \sum_{i,j=1}^{d} C_{ij}|i\rangle|j\rangle$ and the $n \times n$ complex matrix $C$ with elements $C_{ij}$. Matrix analysis theory [10] renders this representation into a very convenient one for studying bipartite quantum systems [11]. Some useful relations between both representations are

$$A \otimes B|C\rangle = |ACB^T\rangle,$$  

$$\langle A|B\rangle = \text{Tr}(A^T B),$$

$$\text{Tr}_1(|A\rangle\langle B|) = AB^T$$ and $\text{Tr}_2(|A\rangle\langle B|) = A^T B^*$.  

I write matrices and vectors in the canonical basis. Thereby, the correspondence between matrices and bipartite state vectors (which is obviously basis dependent) is always well defined.

The action of the linear elements is defined by a linear mapping of the input creation operators $\{a_{d+1}^\dagger, \ldots, a_{2d}^\dagger\}$ to the output creation operators $\{c_{d+1}^\dagger, \ldots, c_n^\dagger\}$

$$c_i^\dagger = \sum_{j=1}^{n} U_{ij} a_j^\dagger.$$  

To implement this operation one can use a series of beam splitters and phase shifters [17] or multiports [18].

III. GENERALIZED MEASUREMENTS ON I-QUDITS

A generalized measurement is described by a positive operator valued measure (POVM) [19] given by a collection of positive operators $F_k$ with $\sum_k F_k = 1$. Each operator $F_k$ corresponds to one classically distinguishable measurement outcome (e.g. a given combination of “clicks” in the output detectors). The probability $p(k|\rho)$ for the outcome $k$ to occur, conditioned to an input density matrix $\rho$, is $p(k|\rho) = \text{Tr}(F_k \rho)$.

If we send an i-qudit $|\alpha\rangle$ through a linear device, the state in the output is $|\alpha_{out}\rangle = \sum_{j=1}^{n} U_{ji} \alpha_j c_i^\dagger|0\rangle = |U^T \alpha\rangle$, where the vector $\alpha$ is padded with extra zeros whenever $n > d$. Notice that the number of modes involved in the transformation can be larger than the number of modes occupied by the i-qudit. This provides a straightforward extension of our input Hilbert space, $\mathcal{H}_1 \oplus \mathcal{H}_1'$, where $\mathcal{H}_1$ is the i-qudit Hilbert space and $\mathcal{H}_1'$ is spanned by single particle states occupying modes $\{a_{d+1}^\dagger, \ldots, a_n^\dagger\}$. The particle detectors in the output modes effectively perform a von Neumann measurement in the canonical base of the extended Hilbert space. According to Neumark’s theorem [19] any POVM can be realized following this prescription, i.e. unitary transformation of the initial states in an extended Hilbert space followed by a projection measurement. Explicitly, the event of one “click” in mode $c_i$ is associated to the POVM element $F_i$,

$$\text{Tr}(F_i|\alpha\rangle\langle \alpha|) = p(i|\alpha) = |\langle 0|c_i U \alpha\rangle|^2 = \text{Tr}(|v_i\rangle\langle v_i|\alpha\rangle\langle \alpha|) \forall \alpha \rightarrow F_i = |v_i\rangle\langle v_i|.$$

where the $d$-dimensional vector $v_i = (U_{1i}^T, \ldots, U_{di}^T)^T$. Naturally this generalized measurement is destructive in the sense that the detectors absorb the measured particle. Nevertheless many protocols in quantum information conclude with a measurement which can be destructive; thus, by allowing postselection, one can use linear elements to perform any generalized measurement on a single qudit within the protocol. This is the reason why protocols such as optimal unambiguous state discrimination [3] or some particular entanglement transformation [20] can be successfully carried out in optical implementations.

As already mentioned, the situation is quite different in the two i-qudits case. A two i-qudit state $|C\rangle$ described, according to Eq. (1), by a matrix $N$ will be transformed to a two-particle state with the following matrix representation in terms of the output modes,

$$|C\rangle = c^T M c|0\rangle$$ with $M = U^T N U$,  

(8)
where \( c = (c_1, \ldots, c_d)^T \). Notice that the only mode transformations which leave the i-qudits Hilbert space invariant are

\[
U_{\text{sep}} = \begin{pmatrix}
U_1 & U_2 \\
U_2 & U_3
\end{pmatrix}, \quad U_{\text{sw}} = \begin{pmatrix}
0 & I_d \\
I_d & 0
\end{pmatrix}
\]

and compositions of both \((U_1 \text{ and } U_2)\) are \(d \times d\) unitary matrices. From Eq. (3) it follows that the first transformation corresponds to a separable operation in the i-qudits Hilbert space \( U_1 \otimes U_2(C) \), while the second transforms the i-qudit \(|C\rangle\) to \((-)|C^T\rangle\), i.e. performs the non-separable swap operation. I now define,

\[
U = \begin{pmatrix}
A & B \\
B^T & D
\end{pmatrix},
\]

(9)

where \( A \) and \( B \) are \(d \times n\) matrices. From Eqs. (2) and (8) it is clear that the output state will not depend on the values of matrix elements of \( D \), i.e. on how the initially unoccupied modes transform. Now we are in position to calculate the resulting POVM on the i-qudits when particle detectors are placed in the output modes. Linear elements preserve the number of particles; therefore each measurement outcome is associated to the absorption of two particles at modes \( c_i, c_j \). Given an arbitrary two i-qudit state \(|C\rangle\), the probability amplitude of such an event is \((i \neq j)\)

\[
\langle 0|c_ic_j|C\rangle = \langle 0|M_{ij}|j\rangle = 2\langle i|M_j\rangle = \langle i|A^T CB \pm B^T C^T A|j\rangle = \operatorname{Tr}(A^T CB(|i\rangle\langle j|) + |j\rangle\langle i|A^T)) = \operatorname{Tr}(CB(|i\rangle\langle j|) + |j\rangle\langle i|A^T)) = \operatorname{Tr}(CF^{ij}),
\]

(10)

where \(|C\rangle\) is the \(d \times d\) matrix \( F^{ij} \) defined as

\[
P^{ij} = A^\ast \Delta^{ij} B \]

and the \(+(-)\) refers to the b-qudits (f-qudits) result. In the bosonic case a normalizing factor \(\frac{1}{\sqrt{2}}\) should be added to Eqs. (11) and (12) when \( i = j \). The isomorphism between bipartite pure states and matrices assigns the state \(|P^{ij}\rangle = \sum_{i,j} P^{ij} |i\rangle |j\rangle\) to this matrix. Moreover, Eq. (9) allows us to write the probability amplitude as

\[
\langle 0|c_ic_j|C\rangle = \langle P^{ij}|C\rangle.
\]

(12)

The relation of the probability amplitude of this event to the corresponding POVM element \(F^{ij}\) results in

\[
\operatorname{Tr}(|C\rangle\langle C|F^{ij}) = \operatorname{Tr}(P^{ij}\langle C|C\rangle) = |\langle P^{ij}|C\rangle|^2 = \operatorname{Tr}(|C\rangle\langle C|P^{ij}) = |P^{ij}|^2 \rightarrow F^{ij} = |P^{ij}\rangle\langle P^{ij}|.
\]

(13)

Making use of Eq. (8), we arrive at the central result,

\[
|P^{ij}\rangle = \sqrt{2} (A^\ast \otimes B^\ast) |\psi^{ij}\rangle \quad \text{or} \quad |P^{ij}\rangle = |a_i\rangle b_j + |a_j\rangle b_i,
\]

(14)

where we have introduced the normalized states \(|\psi^{ij}\rangle \propto (|i\rangle\langle j|) \pm (|j\rangle\langle i|))\), and \(a_i\) and \(b_i\) are the \(i\)th columns of \(A^\ast\) and \(B^\ast\) respectively. For b-qudits, double-clicks, i.e. \(i = j\), correspond to separable POVM elements, while for f-qudits, the Pauli exclusion principle prohibits these events. In the last equations we also see that each \(|P^{ij}\rangle\) has at most Schmidt rank 21 two. All other accessible POVM elements are convex combinations of those defined in Eq. (13). From here it follows that all POVM elements on two i-qudits realized with linear elements will have a Schmidt number 21 at most equal to two 22. This means that no analog of the incomplete Bell-measurement 23—or its usage in teleportation 24, entanglement swapping 25, quantum dense coding 26 or probabilistic implementation of non-local gates 27—can exist for i-qudits with \(d \geq 2\). Also from this result and a theorem by Carollo and Palma 28 it follows 12 that even with the aid of auxiliary photons and conditional dynamics, it is not possible to do a never failing Bell-measurement for qudits.

For \(d = 2\) it has been recently proven 13 that an incomplete Bell-measurement can at most unambiguously discriminate a Bell-state in half of the trials. The appeal of a Bell-measurement is not only its ability to discriminate unambiguously between the specific four Bell-states, but that it can project an unknown state into a maximally entangled state. Any generalized measurement in which every POVM element is maximally entangled, would have much the same appeal. Trivial modifications, consisting only in local operations, could make the teleportation, entanglement swapping or the probabilistic non-local gates function with such generalized Bell-measurement. It is characteristic of bipartite maximally entangled pure states, that each of its subsystems has a reduced density matrix proportional to the identity matrix. By Eq. (2), this means that, in the matrix representation, maximally entangled states are proportional to unitary matrices. For both, bosons and fermions, the POVM elements defined in Eq. (13) that have a contribution from the detection in a mode \(c_i\) can be written, using a local base transformation, as

\[
|\tilde{P}^{ij}\rangle \equiv W_i^\dagger \otimes V_i^\dagger |P^{ij}\rangle = |a_i| |x\rangle + |b_i| |y\rangle |1\rangle,
\]

(16)

where \(x = V_i |b_j\rangle\) and \(y = W_i |a_j\rangle\), and \(V_i\) and \(W_i\) are unitary transformations. The matrix representation of this state is,

\[
\tilde{P}^{ij} = \begin{pmatrix}
|a_i| |x_1| + |b_i| |y_1| & |a_i| |x_2| \\
|b_i| |y_2| & 0
\end{pmatrix}.
\]

(17)

The following conditions have to hold if this matrix ought to be unitary (up to a constant \(\kappa_i\)),

\[
|a_i| |x_1| + |b_i| |y_1| = 0 \quad \text{and} \quad |a_i| |x_2| = |b_i| |y_2| = |\kappa_i|.
\]

(18)
Enforcing these conditions, we have $\tilde{P}_{ij} = \kappa_i (\frac{1}{2} \sigma^i \otimes \sigma^j)$, which, after switching back to the state representation, allows us to conclude that a detection in mode $c_i$ can only contribute to maximally entangled POVM elements of the form

$$P_{ij} = \kappa_i W_i \otimes V_i (|1\rangle|2\rangle + e^{i\phi} |2\rangle|1\rangle).$$

(19)

On the other hand, after some simple algebra and using $AB^\dagger = 0$, one can find the total contribution, in the resolution of the identity, of all the POVM elements where a detection in the $c_i$ mode is involved,

$$1 = \sum_{i \geq j=1}^n |P_{ij}\rangle\langle P_{ij}| = \sum_{i=1}^n \frac{\kappa_i}{2} W_i \otimes V_i (|a_i|^2 |1\rangle|1\rangle \otimes 1_2 + |b_i|^2 1_2 \otimes |1\rangle|1\rangle) W_i^\dagger \otimes V_i^\dagger
\sum_{j=1}^n |b_j|^2 1_2 \otimes |1\rangle|1\rangle) W_i^\dagger \otimes V_i^\dagger$$

(20)

The factor $\frac{1}{2}$ comes from the symmetry $P_{ij} = P_{ji}$ of the double counting of the terms with $i \neq j$. Comparing this result with Eq. (19) it is clear that not all POVMs involving a detection in $c_i$ can be maximally entangled; the space spanned by POVMs defined by Eq. (19) does not cover the whole support of the $i$th term in the sum in Eq. (20). An upper bound on the total weight of the maximally entangled POVMs in this term fixes the maximum probability of successfully projecting an unknown input state $\rho = \frac{1}{2} 1_2 \otimes 1_2$ onto a maximally entangled state,

$$p_{\text{succ}}^i \leq \frac{1}{2} \text{Tr} (W_i \otimes V_i (|a_i|^2 |1\rangle|2\rangle \otimes |b_i|^2 2_2 \otimes |1\rangle|1\rangle))$$

$$W_i^\dagger \otimes V_i^\dagger \rho = \frac{1}{8} (|a_i|^2 + |b_i|^2) = \frac{1}{8} \sum_{k=1}^4 |U_{ik}|^2,$$

(21)

where we employed the definition in Eq. (9). By adding up the contributions from all the detectors we obtain total probability of success,

$$\sum_{i=1}^n p_{\text{succ}}^i \geq \frac{1}{8} \sum_{k=1}^4 \sum_{i=1}^4 |U_{ik}|^2 = \frac{1}{8} \sum_{k=1}^4 |U_{ik}|^2 = \frac{1}{2}$$

(22)

This also sets to one half the ultimate efficiency of teleportation, entanglement swapping and the probabilistic implementation of non-local gates on i-qudits.

In this communication I have introduced a formalism to study the characterization of the generalized measurements on two i-qudits implemented by linear elements. This approach provides two otherwise non-trivial results concerning maximally entangled POVMs. It should also be very helpful in determining the viability or efficiency of other relevant POVM in quantum information.

As a last remark, even if the results for fermions and bosons are apparently similar, there is actually a large difference that is manifest in the asymptotic method that also uses auxiliary photons. In Ref. [27] it is proved that the analogue of the photonic efficient quantum computation [3] cannot exist for fermions.

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