Channel Estimation in Broadband Millimeter Wave MIMO Systems with Few-Bit ADCs

Jianhua Mo, Student Member, IEEE, Philip Schniter, Fellow, IEEE, and Robert W. Heath Jr., Fellow, IEEE

Abstract—We develop a broadband channel estimation algorithm for millimeter wave (mmWave) multiple input multiple output (MIMO) systems with few-bit analog-to-digital converters (ADCs). Our methodology exploits the joint sparsity of the mmWave MIMO channel in the angle and delay domains. We formulate the estimation problem as a noisy quantized compressed-sensing problem and solve it using efficient approximate message passing (AMP) algorithms. In particular, we model the angle-delay coefficients using a Bernoulli-Gaussian-mixture distribution with unknown parameters and use the expectation-maximization (EM) forms of the generalized AMP (GAMP) and vector AMP (VAMP) algorithms to simultaneously learn the distributional parameters and compute approximately minimum mean-squared error (MSE) estimates of the channel coefficients. We design a training sequence that allows fast, FFT-based implementation of these algorithms while minimizing peak-to-average power ratio at the transmitter, making our methods scale efficiently to large numbers of antenna elements and delays. We present the results of a detailed simulation study that compares our algorithms to several benchmarks. Our study investigates the effect of SNR, training length, training type, ADC resolution, and runtime on channel estimation MSE, mutual information, and achievable rate. It shows that our methods allow one-bit ADCs to perform comparably to infinite-bit ADCs at low SNR, and 4-bit ADCs to perform comparably to infinite-bit ADCs at medium SNR.

Index Terms—Low resolution analog-to-digital converter, millimeter wave, channel estimation, approximate message passing

I. INTRODUCTION

Millimeter wave (mmWave) communication is a promising technology for future outdoor cellular systems due to its potential to use very high bandwidth channels [2]. But larger bandwidths place difficult demands on the receiver’s analog-to-digital converters (ADCs). For example, at rates above 100 Msamples per second, ADC power consumption increases quadratically with sampling frequency [3]. High-precision ADCs (e.g., ≥ 6 bits) with bandwidths sufficient for mmWave systems (e.g., ≥ 1 Gsamples/s) are either unavailable or may be too costly and power-hungry for portable devices [4]. One possible solution is to employ low-resolution ADCs, which enjoy low power consumption and simple hardware implementations. Low resolution ADCs have further benefits in MIMO systems, where a large number of ADCs are needed when digital baseband processing of all antenna outputs is desired. For example, in massive MIMO systems, it has been suggested to equip the base station with dozens of antennas and the same number of 1-bit ADCs [5]–[11].

In this paper, we consider a receiver architecture based on few-bit (i.e., 1-4 bit) ADCs, which act to quantize the inphase and quadrature baseband received signals. The achievable rate of the quantized MIMO channel was studied in [12]–[14] assuming channel-state information at the receiver (CSIR) was perfect but channel-state information at the transmitter (CSIT) was absent. In the latter case, and assuming equal transmission power at each antenna, [13] showed that QPSK is the optimum signaling strategy at low SNR. The perfect-CSIT case was studied in our previous work [15], where constellation design methods were proposed to maximize the achievable rate. These methods achieve much higher rates than QPSK signaling, especially at high SNR, but require CSIT.

Due to the nonlinear nature of quantization, channel estimation with few-bit ADCs is challenging. To estimate broadband SISO channel coefficients, the work [16], [17] proposed to transmit periodic bursty training sequences, dither the ADCs, and estimate each tap separately. MIMO channel estimation is even more challenging because the linear combination of transmitted signals from different antennas is quantized. In [18] and [6], the MIMO channel was estimated using least squares (LS) methods. In particular, the quantization error was treated as additive white Gaussian noise. As a result, a large estimation error was introduced.

To take into account quantization effects, iterative channel estimation methods using Expectation Maximization (EM) were proposed and analyzed in [1], [19], [20]. The proposed methods have high complexity, however, since each EM iteration computes a matrix inverse and many iterations are needed for convergence. In addition, [19], [20] considered the MIMO channel with small antenna arrays in the lower frequency UHF (ultra high frequency) band, and thus did not take into account the sparsity of mmWave channels [19]–[23].

In recent work [2], [24]–[26], Approximate Message Passing (AMP) algorithms were used for channel estimation and (or) symbol detection in the massive MIMO channel with few-bit ADCs. These works assumed that the channel coefficients follow an IID Gaussian distribution and did not exploit the sparsity inherent in mmWave channels. (Delay-domain sparsity was exploited in [26] but not angle-domain sparsity.) Also, since they were based on the massive-MIMO channel, these works did not exploit structures present in the broadband mmWave model that can be exploited for significant complexity reduction.

Jianhua Mo and Robert W. Heath Jr. are with Wireless Networking and Communications Group, The University of Texas at Austin, Austin, TX 78712, USA (Email: {jhmno, rheath}@utexas.edu). Their work was supported in part by the National Science Foundation under Grant CCF-1527079.

Philip Schniter is with The Ohio State University, Columbus, OH 43210, USA (Email: {schniter.1@osu.edu}). His work was supported in part by the National Science Foundation under Grant CCF-1527162.

The material in this paper was presented in part at the 2014 Asilomar Conference on Signals, Systems and Computers [1].
In this paper, we propose high-performance AMP-based channel estimation schemes for broadband mmWave MIMO channels with few-bit ADCs. The main contributions of our work are summarized as follows.

- We formulate the problem of estimating broadband mmWave channels under few-bit ADC as a noisy, quantized, compressed-sensing problem, leveraging sparsity in both the angle and delay domains. To our knowledge, our work is the first to take this approach.
- For channel estimation, we consider two AMP algorithms: the Generalized AMP (GAMP) algorithm from [27] and the Vector AMP (VAMP) algorithm from [28], [29]. For both, we use EM extensions [30], [31] to avoid the need to specify a detailed prior on the channel distribution.
- We propose a novel training sequence design that results in low channel-estimation error, low complexity, and low peak-to-average power ratio. Low complexity is achieved through a novel FFT-based implementation that will be described in the sequel.
- We undertake a detailed experimental study of important design choices, such as ADC precision, the type and length of training sequence, and the type of estimation algorithm. When evaluating algorithms, we consider both performance and complexity. For performance, we consider several metrics: mean-square error (MSE), mutual information, and achievable rate.

From our experimental study, our main findings are as follows.

- Compared to other algorithms of which we are aware, the EM-VAMP algorithm has a superior performance-complexity tradeoff.
- Our FFT-based implementation facilitates low-complexity estimation of channels with large antenna numbers (e.g., 64 × 64) and delay spreads (e.g., 16 symbol intervals).
- Relative to infinite-bit ADCs, 1-bit ADCs incur only small performance losses at low SNR, and 3-4-bit ADCs incur only small losses up to medium SNRs.
- The MSEs of the EM-GAMP and EM-VAMP algorithms decay exponentially with training length, and the achievable rate is usually maximized by sending a relatively short training sequence.

A simple version of our proposed methodology was first published in [1]. Relative to [1], our current work expands from one-bit to few-bit ADCs, expands from narrowband to broadband channels, considers leakage effects in sparse-channel modeling, considers four training sequence designs instead of one, considers the VAMP algorithm as well as the GAMP algorithm, proposes an FFT-based implementation that facilitates many more antennas and delays, considers Gaussian-mixture (GM) as well as Bernoulli-Gaussian priors, and incorporates EM learning of the prior parameters. Also, relative to [1], the experimental study in our current work is much more elaborate.

The paper is organized as follows. In Section II, we describe the broadband MIMO system model with few-bit ADCs. The channel characteristics of mmWave communications are then summarized in Section III. In Section IV, we present our channel-estimation algorithms and training-sequence designs. Simulation results are presented in Section V, followed by conclusions in Section VI.

**Notation**: \( a \) is a scalar, \( \mathbf{a} \) is a vector and \( \mathbf{A} \) is a matrix. \( \text{tr}(\mathbf{A}) \), \( \mathbf{A}^\top \), \( \mathbf{A}^* \) and \( ||\mathbf{A}||_F \) represent the trace, transpose, conjugate transpose and Frobenius norm of a matrix \( \mathbf{A} \). \( \mathbf{A} \otimes \mathbf{B} \) denotes the Kronecker product of \( \mathbf{A} \) and \( \mathbf{B} \). \( \text{vec}(\mathbf{A}) \) is a vector stacking all the columns of \( \mathbf{A} \). \( \text{unvec}(\cdot) \) is the opposite operation of \( \text{vec}(\cdot) \). \( \mathcal{CN}(\mu, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-\mu|^2}{2\sigma^2}\right) \) is the probability density function of the circularly symmetric complex-Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \). Finally, \( j \triangleq \sqrt{-1} \).

**II. BROADBAND MIMO WITH FEW-BIT ADC**

We consider a MIMO system with few-bit ADCs, as illustrated in Fig. [1]. The transmitter is equipped with \( N_t \) antennas and the receiver is equipped with \( N_r \) antennas. A total of \( 2N_t \) few-bit ADCs separately quantize the real and imaginary parts of the received signal of each antenna. Assuming that the delay spread of the channel is limited to \( L \) symbol intervals and that carrier and symbol synchronization have already been performed, the quantizer output \( \mathbf{y}[i] \in \mathbb{C}^{N_r \times 1} \) at time \( i \) can be written as

\[
\mathbf{y}[i] = \mathbf{Q} \left( \sum_{\ell=0}^{L-1} \mathbf{H}[\ell] \mathbf{t}[\ell] + \mathbf{w}[i] \right),
\]

where \( \mathbf{H}[\ell] \in \mathbb{C}^{N_r \times N_t} \) is the baseband channel impulse response at lag \( \ell \), \( t[i] \in \mathbb{C}^{N_t \times 1} \) is the transmitted symbol at time \( i \) with average transmit power \( \mathbb{E}[|t[i]|^2] = P_t \), and \( \mathbf{w}[i] \sim \mathcal{CN}(\mathbf{0}, \sigma_w^2 \mathbf{I}_{N_r}) \) is additive white Gaussian noise. Furthermore, \( \mathbf{Q}(\cdot) \) denotes the quantization function, which is applied component-wise and separately to the real and imaginary parts.

In this paper, we assume that uniform mid-rise quantization is used. In particular, if \( x \) is a complex-valued scalar \( x \), then \( y = \mathbf{Q}(x) \) means

\[
y = \text{sign} (\text{Re}(x)) \left( \min \left( \left| \frac{|\text{Re}(x)|}{\Delta_{\text{Re}}} \right|, 2^{b-1} - \frac{1}{2} \right) \Delta_{\text{Re}} + j \text{sign} (\text{Im}(x)) \left( \min \left( \left| \frac{|\text{Im}(x)|}{\Delta_{\text{Im}}} \right|, 2^{b-1} - \frac{1}{2} \right) \Delta_{\text{Im}},
\]

(2)

where \( \Delta_{\text{Re}} \triangleq \left( \mathbb{E}[|\text{Re}(x)|^2] \right)^{\frac{1}{2}} \Delta_b \) and \( \Delta_{\text{Im}} \triangleq (\mathbb{E}[|\text{Im}(x)|^2])^{\frac{1}{2}} \Delta_b \), and where \( \Delta_b \) is a stepsize that...
will be discussed in the sequel. In the special case of one-bit quantization, (2) becomes

\[ y = \text{sign} \left( \text{Re}(x) \right) \sqrt{\frac{2}{\pi}} \left( \mathbb{E} \left[ |\text{Re}(x)|^2 \right] \right)^{\frac{1}{2}} 
+ j \text{sign} \left( \text{Im}(x) \right) \sqrt{\frac{2}{\pi}} \left( \mathbb{E} \left[ |\text{Im}(x)|^2 \right] \right)^{\frac{1}{2}}. \]

(3)

The average powers \( \mathbb{E} \left[ |\text{Re}(x)|^2 \right] \) and \( \mathbb{E} \left[ |\text{Im}(x)|^2 \right] \) can be easily measured by analog circuits before the ADC, as in automatic gain control (AGC). In this paper, \( x \) is circularly symmetric, and so \( \mathbb{E} \left[ |\text{Re}(x)|^2 \right] = \mathbb{E} \left[ |\text{Im}(x)|^2 \right] = \frac{1}{2} \mathbb{E} \left[ |x|^2 \right] \), implying that \( \Delta_{\text{re}} = \Delta_{\text{im}} \).

The quantization stepsize \( \Delta_{\ell} \) is usually chosen to minimize the quantization MSE assuming a Gaussian input signal (see, e.g., [22]). These values of \( \Delta_{\ell} \) are given in Table I assuming a unit-power input. The normalized MSE (NMSE), defined as \( \eta_{\ell} \equiv \mathbb{E} \left[ \left( Q(x) - x \right)^2 \right] / \mathbb{E} \left[ |x|^2 \right] \), and the signal to quantization ratio (SQNR), defined as \( 10 \log_{10} \frac{1}{\eta_{\ell}} \), are also listed in Table I. From the table it can be seen that \( \Delta_{\ell} \approx 2^{-b} \), \( \eta_{\ell} \approx 2^{-2b} \) and SQNR \( \approx 5b \) dB. Notice that 4-bit ADCs yield quantization noise power around 20 dB below the signal power, which suggests that increasing ADC resolution beyond 4 bits should yield negligible performance improvement in the low and medium SNR regimes. This intuition will be verified by our simulations in Section V.

III. SPARSITY OF THE MMWAVE CHANNEL MODEL

In this section, we present the mmWave channel model assumed in our paper. The characteristics of the mmWave channel, especially the sparsity in the angle-delay domain, will be exploited by our proposed channel estimation algorithm.

A. Clustered MIMO Channel Model

The mmWave channel can be modeled using \( N_{\ell} \) multipath clusters, where the \( n \)th cluster comprises \( N_{\ell}^n \) paths [22]. For the \( n \)th path of the \( m \)th cluster, we use \( \alpha_{n,m}, \tau_{n,m}, \varphi_{n,m}^r, \varphi_{n,m}^i \) to denote the complex gain, delay, azimuth (or zenith) angle of arrival, and azimuth (or zenith) angle of departure, respectively. Using these quantities, the channel impulse response from (1) can be written as

\[
H[\ell] = \sum_{n=1}^{N_{\ell}} \sum_{m=1}^{N_{\ell}^n} \alpha_{n,m} a_r^* \left( \varphi_{n,m}^r, \varphi_{n,m}^i \right) h_{\ell} (\ell T - \tau_{n,m}) \text{, } 0 \leq \ell < L,
\]

(4)

where \( p(t) \) includes the effects of pulse shaping and analog/digital filtering, and where \( a_r \) and \( a_i \) are the array response vectors of the receive and transmit antenna arrays, respectively.

We assume that the receive and transmit arrays are each configured as a uniform planar array (UPA). This assumption is quite common. For example, in the massive MIMO systems described in [33], [34], UPAs with 100 or more antennas were tested. In the mmWave 5G cellular system prototype described in [35], there is a 256-element \( (16 \times 16) \) UPA array panel at the base station, and two sets of 1 \( \times \) 16 UPA arrays in the top and bottom portions of the mobile phone.

B. Angle-Delay Representation

The MIMO channel coefficients \( H[\ell] \in C^{N_r \times N_t} \) are expressed in what is known as the “antenna aperture domain.” We find it convenient to instead work with “angle domain” coefficients \( X[\ell] \in C^{N_r \times N_t} \), as proposed in [36]. The two representations are connected through

\[
H[\ell] = B_N[X[\ell]] B_N^*, \quad 0 \leq \ell < L, \tag{5}
\]

where \( B_{N_r} \in C^{N_r \times N_r} \) and \( B_{N_t} \in C^{N_t \times N_t} \) are the steering matrices for the transmitter and receiver arrays, respectively. With an \( N_r^e \times N_t^e \) receive UPA and an \( N_r^a \times N_t^a \) transmit UPA, we have that \( N_r = N_r^e N_r^a \), \( N_t = N_t^e N_t^a \), and [37]

\[
H[\ell] = (F_{N_r^e} \otimes F_{N_t^e}) X[\ell] (F_{N_r^a} \otimes F_{N_t^a})^*, \tag{6}
\]

where \( F_{N_r^e} \in C^{N_r^e \times N_r^e} \), \( F_{N_t^e} \in C^{N_t^e \times N_t^e} \), \( F_{N_r^a} \in C^{N_r^a \times N_r^a} \), and \( F_{N_t^a} \in C^{N_t^a \times N_t^a} \) are unitary Fourier Transform (DFT) matrices. The \( (i,j) \)th entry of the matrix \( X[\ell] \) can be interpreted as the channel gain between the \( j \)th discrete transmit angle and the \( i \)th discrete receive angle [36].

According to the measurement results reported in [22], the number of clusters \( N_{\ell} \) tends to be relatively few in the mmWave band as compared to lower-frequency bands. Also, the number of antenna elements used tends to be large in order to counteract the effects of path loss, which is much more severe in the mmWave band as compared to lower-frequency bands. Hence, in mmWave applications, the number of channel clusters is usually much fewer than the number of scalar coefficients in \( \{H[\ell]\}_{L_{\ell}=0}^{L_{\ell}-1} \) or \( \{X[\ell]\}_{L_{\ell}=0}^{L_{\ell}-1} \), i.e., \( N_{\ell} \ll N_r N_t L_{\ell} \).

Figure 2 shows an example of a broadband mmWave channel with delay spread \( L = 16 \) and \( N_{\ell} = 2 \) multipath clusters, each consisting of 10 paths and 7.5 degrees of azimuth and elevation angular spread, as seen through 8 \( \times \) 8 UAPs at both ends of the link (i.e., \( N_r = N_t = 64 \)) and raised-cosine pulse-shape filtering with roll-off factor = 0. Figure 2(a) plots the mean-squared channel coefficient magnitude in the antenna aperture domain, i.e., \( \sqrt{\sum_{\ell} |H[\ell]|^2} \), while subfigure (b) plots the mean-squared channel coefficient magnitude in the angle domain, i.e., \( \sqrt{\sum_{\ell} |X[\ell]|^2} \). Figure 2(b) shows that, in the angle domain, the channel energy is concentrated in two locations, each corresponding to one multipath cluster.

| Resolution \( b \) | 1-bit | 2-bit | 3-bit | 4-bit | 5-bit | 6-bit | 7-bit | 8-bit |
|------------------|------|------|------|------|------|------|------|------|
| Stepsize \( \Delta_b \) | \( \sqrt{\frac{2}{\pi}} \) (\( \approx 1.5958 \)) | 0.9957 | 0.586 | 0.3352 | 0.1881 | 0.1041 | 0.0569 | 0.0308 |
| NMSE \( \eta_b \) | \( \sqrt{\frac{2}{\pi}} \) (\( \approx 0.3634 \)) | 0.1188 | 0.03744 | 0.03154 | 0.001035 | 0.0002999 | 0.00008543 | 0.0001538 |
| SQNR (dB) | 10 log_{10} \left( \frac{1}{\eta_b} \right) (\( \approx 4.4 \)) | 9.25 | 14.27 | 19.38 | 24.55 | 29.85 | 35.23 | 40.68 |
measurements and a known training sequence. Approximately sparse angle-delay domain channel from few-bit

We now develop an efficient algorithm to estimate the approximate nature of the sparsity is also visible in Fig. 3(a). Our approach based on sparsity is illustrated in Fig. 3(a) for a sparse in the angle and delay domains. This joint clusters, as seen through a link with 8 × 8 UPAs at both ends. Subfigure (a) plots the mean-squared coefficient magnitude in the antenna aperture domain √∑∥[H(ℓ)k]∥2, and subfigure (b) plots the same for the angle domain √∑∥[X(ℓ)k]∥2. The angle domain representation in (b) clearly shows two groups of large-magnitude entries, each of which corresponds to one multipath cluster.

We emphasize that the angle-delay channel \( \{X(ℓ)\}_{ℓ=0}^{L-1} \) is jointly sparse in the angle and delay domains. This joint sparsity is illustrated in Fig. 3(a) for a 4 × 16 MIMO channel with delay spread \( L = 16 \) (see the details in Section V). However, \( \{X(ℓ)\}_{ℓ=0}^{L-1} \) is not exactly sparse, exhibiting what is known as leakage, in that none of its coefficients are expected to be exactly zero valued. The approximate nature of the sparsity is also visible in Fig. 3(a). Our approach based on GAMP will be robust to leakage effects.

IV. PROPOSED CHANNEL ESTIMATION ALGORITHM

Motivated by the channel model given in the last section, we now develop an efficient algorithm to estimate the approximately sparse angle-delay domain channel from few-bit measurements and a known training sequence.

A. Problem Formulation

We assume that the training consists of a block transmission of length \( N_p \), denoted as \( T \in \mathbb{C}^{N_t \times N_r} \), with a cyclic prefix of length \( L \). After discarding the cyclic prefix, the measurements take the form

\[
Y = Q(Z + W),
\]

where \( W \) is additive Gaussian noise and \( Z \in \mathbb{C}^{N_t \times N_r} \) is the unquantized noiseless received signal, of the form

\[
Z = \sum_{ℓ=0}^{L-1} H[ℓ]TJ_ℓ
\]

\[
= \sum_{ℓ=0}^{L-1} B_{N_r} X[ℓ] B_{N_t}^* T J_ℓ
\]

\[
= B_{N_r} [X[0] X[1] X[2] \cdots X[L-1]]
\]

\[
\triangleq X
\]

\[
= I_K \otimes B_{N_t}^* \begin{bmatrix} B_{N_t}^* & B_{N_t}^* T J_0 \\ & & \vdots \\ & & B_{N_t}^* T J_{K-1} \\ & & \vdots \\ & & B_{N_t}^* T J_{L-1} \end{bmatrix} \triangleq C
\]

where \( J_ℓ \in \mathbb{R}^{N_p \times N_p} \) is the ℓ- circulant-delay matrix.

We now rewrite \( Z \) in vector form \( z \triangleq \text{vec}(Z) \) as

\[
z = \left(C^T \otimes B_{N_t} \right) \text{vec}(X).
\]

Defining \( y \triangleq \text{vec}(Y) \), \( x \triangleq \text{vec}(X) \), and \( w \triangleq \text{vec}(W) \), the quantized noisy output becomes

\[
y = Q \left( \left(C^T \otimes B_{N_r} \right) x + w \right).
\]

The channel-estimation problem reduces to the following: estimate the angle-delay channel coefficients \( x \in \mathbb{C}^{N_t \times N_r} \) from the noisy quantized received signal \( y \in \mathbb{C}^{N_p \times N_r} \) under the known linear transform \( A \in \mathbb{C}^{N_p \times N_r \times N_r} \). For use in the sequel, we define \( N_y \triangleq N_t N_p \) and \( N_x \triangleq N_t N_r \).

Because the angle-delay channel is sparse (approximately due to leakage), this channel estimation problem can be viewed as an instance of noisy quantized compressed sensing. Several methods have been proposed for noisy quantized compressed sensing, including quantized iterative hard thresholding (QIHT) [38, 39], convex relaxation [40, 41], and GAMP [42, 43]. It has also been proposed to treat the quantization error as if it were additive white Gaussian noise [6, 18]. All of these methods are strongly dependent on the level of sparsity assumed for the channel, either explicitly or through
the specification of a regularization or prior distribution. But, in practice, the channel sparsity is apriori unknown.

B. EM-AMP Algorithms

For channel estimation, we propose to apply two approaches that combine expectation maximization (EM) with AMP as a means of avoiding the need to specify a prior. The first is based on a combination of generalized AMP (GAMP) [27] and EM, which was proposed in [30] but, to our knowledge, has never been applied to noisy few-bit quantized compressive sensing. The second is based on a combination of the recently proposed vector AMP (V AMP) [28], [29] and EM [31]. To our knowledge, neither V AMP nor EM-V AMP have been applied to noisy few-bit quantized compressive sensing. We focus on these AMP approaches because they offer nearly minimum MSE (MMSE) performance while being computationally efficient. We now provide background on these methods.

1) EM-GAMP: Suppose that a random vector $x$ with IID components $x_j \sim p_X$ is linearly transformed to produce $z = Ax$, which then propagates through a probabilistic measurement channel $p(y|z) = \prod_i p_{Y_i|Z}(y_i|z_i)$. Given knowledge of $y$, $A$, $p_X$, and $p_{Y|Z}$, we would like to compute the MMSE estimate of $x$. The GAMP algorithm [27] approaches this computationally difficult problem through a sequence of simple scalar estimation problems and matrix multiplies. Remarkably, when $A$ is very large with IID (sub)Gaussian entries, the behavior of GAMP is rigorously characterized by a scalar state evolution [27], [44]. When this state evolution has a unique fixed point, GAMP converges to the MMSE solution. In practice, $A$ may not be a very large IID (sub)Gaussian matrix, in which case the theoretical guarantees of GAMP do not hold. Still, the estimates it provides after a few (e.g., < 25) iterations are often very close to MMSE (see, e.g., [30]).

GAMP requires specification of $p_X$, which is unknown in practice. To circumvent this problem, [30] proposed to approximate the true $p_X$ by a Gaussian-mixture with parameters $\theta$ learned by an EM algorithm. In the E-step of the EM algorithm, GAMP’s posterior approximation is used in place of the true posterior, which is NP-hard to compute. The resulting EM-GAMP algorithm was empirically analyzed in [30], where it was shown to give similar performance to true-$p_X$ GAMP for a wide range of true $p_X$ (e.g., sparse, heavy tailed, discrete). Generalizations of EM-GAMP were theoretically analyzed in [45] using the GAMP state evolution.

Algorithm 1 details the steps of EM-GAMP. Lines 1-13 are from the original GAMP algorithm and line 14 is the EM update. Lines 6-7 can be interpreted as computing the posterior mean and variance of $z_i$ under the likelihood $p_{Y_i|Z}(y_i|z_i)$ and a pseudo-prior $z_i \sim \mathcal{CN}(\hat{p}_i, \nu_p)$, where $\hat{p}_i$ and $\nu_p$ are updated in lines 4-5. Likewise, lines 12-13 can be interpreted as computing the posterior mean and variance of $x_i$ under the prior $p_X(x_i; \theta)$ and pseudo-measurement $\hat{r}_i = x_i + \mathcal{CN}(0, \nu_r)$, where $\hat{r}_i$ and $\nu_r$ are updated in lines 8-11. We note that, if $A$ has a fast (e.g., FFT-based) implementation, then EM-GAMP can leverage this in lines 5 and 11. For more details on EM-GAMP, we refer the reader to [27], [30].

2) EM-VAMP: The VAMP algorithm [28], [29] aims to solve exactly the same problem targeted as GAMP and—like GAMP—is rigorously characterized by a scalar state-evolution. But VAMP’s state evolution holds for a much broader class of matrices $A$: those that are right-rotationally invariant (RRI). For $A$ to be RRI, its SVD $A = U_A S_A V_A^*$ should have $V_A$ drawn uniformly from the set of unitary matrices; there is no restriction on $U_A$ and $S_A$. For MMSE estimation, VAMP must be given the prior $p_X$ and likelihood $p_{Y|Z}$. When they are not available, approximations can be learned through the EM methodology, as described in [31].

Algorithm 2 details the steps of EM-VAMP. As can be seen, it alternates between nonlinear scalar estimations (lines 4-13), linear vector estimations (lines 14-23), and an EM update (line 24). For scalar estimation, the conditional mean and variance of $x_i$ under prior $p_X(x_i)$ and pseudo-measurements $r_{i1} = x_i + \mathcal{CN}(0, \nu_1)$ are computed in lines 5-6, and the conditional mean and variance of $z_i$ under likelihood $p_{Y_i|Z}(y_i|z_i)$ and pseudo-prior $z_i \sim \mathcal{CN}(\hat{p}_i, \nu_p)$ are computed in lines 10-11. Lines 14 and 19 then compute the joint MMSE estimate of $x$ and $z$ under the pseudo-prior

$$
\begin{align*}
[x]_z \sim \mathcal{CN}\left(\begin{bmatrix} r_2 \\ \nu_2 I \end{bmatrix}, \begin{bmatrix} \nu_2 I \\ r_2 I \end{bmatrix}\right)
\end{align*}
$$

and the constraint $z = Ax$. We note that EM-VAMP can leverage fast implementations of $U_A$ and $V_A$ if they exist. For more details, we refer the reader to [28], [29], [31].

3) Application to mmWave channel-estimation with few-bit ADCs: When applying EM-GAMP and EM-VAMP to the problem of mmWave channel-estimation with few-bit ADCs, we choose the (approximating) prior family as either a Bernoulli Gaussian-mixture (GM) or Bernoulli-Gaussian...
Algorithm 2 The EM-VAMP algorithm

1: define: \( p_{2|Y,P} \) and \( p_{X|R} \) from (14)-(15).
2: initialize: \( r_1, p_1, r_1, \tau_1 \) and \( \theta \).
3: for \( k = 0, 1, \ldots, N_{\text{max}} \) do
4: \( \text{// Scalar estimation of } x_i \)
5: \( \bar{x}_1 \leftarrow \mathbb{E}_{X|R}[x_i | r_1; \nu_1, \theta] \), \( \forall i \)
6: \( \alpha_1 \leftarrow \nu_{1-1}N_x^{-1}\sum_{i=1}^{N_y} \text{Var}_{X|R}[x_i | r_1; \nu_1, \theta] \)
7: \( \nu_2 \leftarrow (\bar{x}_1 - \alpha_1 P_1)/(1 - \alpha_1) \), \( \nu_2 \leftarrow \nu_2 \nu_1/(1 - \alpha_1) \)
8: \( \text{// Scalar estimation of } z_i \)
9: \( z_1 \leftarrow \mathbb{E}_{Z|Y,P}[z_i | y_1; P_1, \nu_1] \), \( \forall i \)
10: \( \beta_1 \leftarrow \tau_1^{-1}N_y^{-1}\sum_{i=1}^{N_y} \text{Var}_{Z|Y,P}[z_i | y_1; P_1, \nu_1] \)
11: \( P_2 \leftarrow (\bar{z}_1 - \beta_1 P_1)/(1 - \beta_1) \), \( \nu_2 \leftarrow \tau_1 \beta_1/(1 - \beta_1) \)
12: \( \text{// LMMSE estimation of } x \)
13: \( \tilde{x}_2 \leftarrow V_A(S_A^* \Sigma_A P_2 r_2 + 1)^{-1}(S_A^* \Sigma_A P_2 r_2 + V_A r_2) \)
14: \( \alpha_2 \leftarrow N_x^{-1}\sum_{i=1}^{N_y} \text{Var}_{x}(\tilde{x}_2 - \alpha_2 P_2)/(1 - \alpha_2) \), \( \nu_1 \leftarrow \nu_2 \nu_2/(1 - \alpha_2) \)
15: \( \text{// LMMSE estimation of } z \)
16: \( \bar{z}_2 \leftarrow A \bar{x}_2 \)
17: \( \beta_2 \leftarrow (1 - \alpha_2)N_y/N_x \)
18: \( P_1 \leftarrow (\bar{z}_2 - \beta_2 P_2)/(1 - \beta_2) \), \( \tau_1 \leftarrow \tau_2 \beta_2/(1 - \beta_2) \)
19: update the parameters \( \theta \) using EM [30];
20: end for
21: return \( \bar{x}_1 \).

(BG), with unknown parameters \( \theta \). That is, the coefficients \( x_i \) of \( x \) are assumed to be drawn from one of the following:

GM: \( p_X(x_i;\theta) = \lambda_0 \delta(x_i) + \sum_i \lambda_i C N(x_i; \mu_i, \phi_i, \nu_i) \), \( \forall i \), (17)

BG: \( p_X(x_i;\theta) = \lambda_0 \delta(x_i) + (1 - \lambda_0) C N(x_i; 0, \phi_i) \), \( \forall i \), (18)

where \( \lambda_0 = \text{Prob}\{x = 0\} \) and \( \{\lambda_i\}, \{\mu_i\}, \{\phi_i\} \) are the weights, means, and variances of the Gaussian mixture, respectively (which are all included in \( \theta \)), and \( \delta(\cdot) \) is the Dirac delta distribution. We expect GM to have better estimation performance, but higher complexity, than BG. The expressions for \( \mathbb{E}_{X|Y}[x_i | r_1; \nu_1, \nu_2, \theta] \) and \( \text{Var}_{X|Y}[x_i | r_1; \nu_1, \nu_2, \theta] \) in lines 12-13 of Algorithm [1] and lines 5-6 of Algorithm [2] can be found in [30], as can the EM-update expressions for \( \theta \).

For the few-bit quantizer (19), the likelihood \( p_{Y|Z} \) is

\( p_{Y|Z}(y_i | z_i) \equiv \text{Prob}\{y_i = Q(z_i + w_i) | z_i\} \equiv C N(w; 0, \sigma^2_w) \).

The expressions for \( \mathbb{E}_{Z|Y,P}[z_i | y_i, P; \nu_p] \) and \( \text{Var}_{Z|Y,P}[z_i | y_i, P; \nu_p] \) needed in lines 6-7 of Algorithm [1] and lines 9-10 of Algorithm [2] can be obtained by following the procedures in [46] Chapter 3.9. Further details can be found in [9] Appendix A.

C. Computational Issues and Training Sequence Design

In practical mmWave applications, the dimensions \( N_y = N_y N_p \) and \( N_x = N_y N_L \) of the matrix \( A \) are expected to be very large. For example, in our simulations, we consider

\( N_y = N_x = 64, N_p = 1024, \) and \( L = 16 \), which yield \( N_y = N_x = 65536 \). At these dimensions, storing \( A \) as an explicit matrix can be a problem. Even when \( A \) fits in memory, the computational complexity of EM-GAMP and EM-VAMP (or any known algorithm, for that matter) will be impractical if \( A \) is treated as an explicit matrix, due to, e.g., per-iteration matrix-vector multiplications with \( A \) and \( A^* \).

For EM-VAMP, these problems are avoided if \( A \) and \( A^* \) can be represented as implicit fast operators. Likewise, for EM-VAMP, \( U_A, V_A, V_A, \) and \( V_A \) should be fast operators.

Recalling from [13] that \( A = C^T \otimes B_N \) with Fourier \( B_N \) and \( T \)-dependent \( C^T \), we see that the training sequence \( T \) will determine whether \( A \) is a fast operator. Thus, the design of the training signal plays a vital role in the practical implementability of mmWave channel estimation.

There are, in fact, three considerations for the design of the training signal \( T \):

1) For GAMP to be computationally efficient, \( A \) and \( A^* \) should be fast operators, and for VAMP to be computationally efficient, \( U_A, V_A, V_A, \) \( V_A \) should be.

2) For GAMP to converge to a good solution, \( A \) should be sufficiently dense and have sufficiently low peak-to-average squared-singular-value ratio [47].

3) To improve the efficiency of the power amplifier at the transmitter, the elements of \( T \) should have low peak-to-average power ratio (PAPR).

1) Training Structure: To satisfy these three considerations, we propose to structure \( T \in \mathbb{C}^{N_t \times N_p} \) as follows. Denote the first row of \( T \), i.e., the signal sent by the first transmit antenna, as \( t^T \equiv [t[0], \ldots, t[N_p - 1]] \). We fix \( N_p \) at an integer multiple of \( N_t L \) and construct \( T \) such that its \( n \)-th row (for \( n \leq 0 \) is the \( n \)-th channel of \( t \). That is, each antenna sends a circularly shifted version of the signal transmitted by the first antenna. The elements of \( T \) (for \( m \geq 0 \) are then

\( T_{n,m} = t[(m-n)N_p], \)

where \( \langle \cdot \rangle_{N_p} \) is the modulo-\( N_p \) remainder. With the construction in [21], \( T_{J,0} \) from [10] contains the \{0, L, 2L, ..., \( (N_t - 1)L \) \} shifts of \( t \). Likewise, \( T_{J,1} \) contains the \{1, L + 1, 2L + 1, ..., \( (N_t - 1)L + 1 \) \} shifts of \( t \). and \( T_{J,L-1} \) contains the \{L - 1, 2L - 1, 3L - 1, ..., \( N_t L - 1 \) \} right shifts of \( t \). Altogether, the rows of \( T \) in [10] will consist of the first \( N_t L \) circular shifts of the sequence \( t \). We can thus re-order the rows in \( T \) to make it a Toeplitz matrix, which we will denote by \( T \) in the sequel.

2) Fast Implementation of \( \tilde{A}X \) and \( A^* \tilde{s} \): Using the structure above, \( \tilde{A}X \) and \( A^* \tilde{s} \) can be efficiently computed. Notice

\( \text{unvec}(A \tilde{x}) = B_N \tilde{X} \left( I_L \otimes B_N^* \right) T \)

\( \text{(a)} \)

\( = B_N \tilde{X} \left( I_L \otimes B_N^* \right) K(L,N_t) T \)

\( \text{(b)} \)

\( = B_N \tilde{X} K(L,N_t) \left( I_L \otimes B_N^* \right) K(L,N_t) T \)

\( \text{(c)} \)

\( = B_N \tilde{X} K(L,N_t) \left( B_N^* \otimes I_L \right) T \)

\( \text{(d)} \)

\( = B_N \left( I_L \otimes B_N^* \right) \tilde{X} T \),

In this paper, we model \( x \) using an IID sparse prior. We do not exploit possible correlation within \( x \), which may improve estimation accuracy.
where (a) follows from commutation matrix\(^2\) \(K^{(L,N_L)} \in \mathbb{R}^{N_L \times N_L \times N_L}; (b) follows from \(K^{(L,N_L)}(K^{(N_L,L)}) = I_{N_L,L}; (c) follows from \(K^{(N_L,L)}(I_L \otimes B^*_N) K^{(L,N_L)} = B^*_N \otimes I_L; \) and (d) follows from \(X \triangleq (XK^{(L,N_L)})^T, (A \otimes B)^T = A^T \otimes B^T, \) and the symmetry of \(B^*_N.\) Note that \(XK^{(L,N_L)}\) is merely a reordering of the columns in \(X.\)

We now show that (26) has a fast implementation. First, for any \(v \in \mathbb{C}^{L \times N_x \times 1},\) notice that
\[
(B^*_N \otimes I_L) v = \text{vec}(VB^*_N) = \text{vec}((B_N, V^*)^T),
\]
where \(V \in \mathbb{C}^{L \times N_x} \) is the column-wise matricization of \(v.\) Second, for any \(u \in \mathbb{C}^{N_x \times 1},\) notice that
\[
B_N u = (F_N^* \otimes F_N^*) u = \text{vec}(F_N^* UF_N^*),
\]
where \(U \in \mathbb{C}^{N_x \times N_x} \) is the column-wise matricization of \(u.\) Therefore, each of the \(N_t\) columns of the multiplication \((B^*_N \otimes I_L)\) \(\tilde{X}\) in (26) can be accomplished by \(N_t^*\)-point and \(N_t^0\)-point FFTs. The result of that multiplication is then left-multiplied by \(T,\) which can be performed via fast convolution using an \(N_p\)-point FFT (since \(T^T\) contains the first \(N_t L\) columns of an \(N_p \times N_p\) circulant matrix). Finally, the left-multiplication by \(B_N\) can be performed, as in (28), using \(N_p^0\)-point and \(N_p^*\)-point FFTs. In summary, there are a total of \(N_t L\) FFTs of length \(N_p,\) one inverse-FFT of length \(N_p,\) \(N_p^0 N_p\) FFTs of length \(N_e^0,\) \(N_e^0 N_p\) FFTs of length \(N_t^0,\) \(N_e^0 N_p\) FFTs of length \(N_t^*\), and \(N_e^0 N_p\) FFTs of length \(N_p.\) Multiplications of the form \(A^*\)'s can be computed similarly. Note also that the memory footprint of \(A\) reduces to that of \(t \in \mathbb{C}^{N_p}.\)

3) Choice of the training sequence \(t: \) For GAMP to work well, we want the measurement matrix \(A\) to have sufficiently low peak-to-average squared-singular-value ratio \([47].\) Likewise, for VAMP to be fast, we want the singular-vector matrices \(U_A^*\) and \(V_A\) of \(A\) to be fast operators. We now show that both concerns can be addressed through the design of \(t.\)

From (10), (11), and (13), the matrix \(A\) can be written as
\[
A = ((I_L \otimes B^*_N) T^T \otimes B_N,)
\]
Since \(I_L \otimes B^*_N\) and \(B_N\) are unitary matrices, the singular values of \(A\) have the form
\[
\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_{N_t, L}, \ldots, \lambda_{N_t, L},
\]
where \(\lambda_1, \lambda_2, \ldots, \lambda_{N_t, L}\) are the singular values of \(T.\) Therefore, we want \(T\) to have sufficiently low peak-to-average squared-singular-value ratio. One way to ensure this property is to choose \(t\) as a Zadoff-Chu (ZC) sequence \([48].\)

A length-\(N_p,\) ZC sequence is defined as
\[
t[k] = \begin{cases} 
\sqrt{\frac{N_p}{N_p}} \exp \left( j \frac{\pi k (k+1)}{N_p} \right) & \text{if } N_p \text{ is odd}, \\
\sqrt{\frac{N_p}{N_p}} \exp \left( j \frac{\pi k^2}{N_p} \right) & \text{if } N_p \text{ is even}.
\end{cases}
\]

The periodic autocorrelation of a ZC sequence equals a scaled Kronecker delta. As a result, \(TT^T = \frac{N}{N_p} I_{N_t, L},\) and so \(\lambda_1 = 1 = \lambda_2 = \cdots = \lambda_{N_t, L} = \sqrt{\frac{N}{N_p}}.\) Consequently, \(A\) will have the minimum possible peak-to-average squared-singular-value ratio, which is good for GAMP. Furthermore, this singular-value structure implies that the singular-vector matrices can be chosen as \(V_A = I\) and \(U_A = \sqrt{\frac{N_p}{N_t}} A.\) Since \(V_A\) and \(U_A\) have fast implementations, they are good for VAMP. Note that, due to their good autocorrelation properties, ZC sequences are currently used as reference signals in LTE \([49].\)

Because the ZC sequence \(t\) has constant modulus, \(T\) will also have an optimum peak-to-average power ratio PAPR.

4) Other choices of \(T: \) In Section V, we investigate other training designs, for example, \(T\) composed of Golay complementary sequences, IID QPSK entries, and IID Gaussian entries. Although their recovery performance is comparable to our proposed design, they lead to GAMP and VAMP algorithms with much higher complexity due to the lack of fast methods to compute the matrix-vector multiplications with \(A, A^*, U_A, U_A^*, V_A, V_A.\)

D. Benchmark Algorithms

In this section, we describe several other signal reconstruction approaches that we will use as benchmarks. The first approach is the LS estimator adopted in \([6, 18].\) In this case
\[
\hat{X}_{LS} = B^*_N Y C^\dagger,
\]
where \(C^\dagger\) is the Moore-Penrose pseudo-inverse of \(C.\) When \(C\) has full row-rank, we have
\[
\hat{X}_{LS} = B^*_N Y C^\dagger (C^\dagger C)^{-1}.
\]

Another approach is obtained by linearizing the quantizer and applying LMMSE \([10, 50.\) We explain the linearization procedure because it will be used later for achievable-rate analysis. Using Bussgang’s theorem, the quantizer output \(y\) can be decomposed into a signal component plus a distortion \(w_q\) that is uncorrelated with the signal component \([14, i.e.,
\[
y = Q (A x + w) = (1 - \eta_y) (A x + w) + w_q.\]

where \(\eta_y\) is the NMSE given in Table I. Under additional mild assumptions, \([14, eq.(30)]\) showed that the “effective” noise \(\tilde{w} = (1 - \eta_y) w + w_q\) in the linearized model
\[
y = (1 - \eta_y) A x + \tilde{w}
\]
has covariance \(\Sigma_{\tilde{w}}\) that is well approximated as
\[
\Sigma_{\tilde{w}} \approx (1 - \eta_y) \left( (1 - \eta_y) E [w w^*] + \eta Wy \text{diag}(\Sigma_{\tilde{w}}^*) \right) \triangleq \tilde{\Sigma}_{\tilde{w}},
\]

\(^2\)The commutation matrix \(K^{(m,n)}\) is the \(mn \times mn\) matrix which, for any \(m \times n\) matrix \(M,\) transforms \(\text{vec}(M)\) into \(\text{vec}(M^T), i.e., K^{(m,n)} \text{vec}(M) = \text{vec}(M^T).\)

\(^3\)To see this, note that each column of the multiplication \((B^*_N \otimes I_L)X\) can be represented as \([27]\) with appropriate \(v,\) and thus computed via \(B^*_N V^*\) due to \([27].\) Then, \(B^*_N V^*\) can be computed columnwise through subproblems of the form \([25]\) with appropriate \(u,\) which in turn can be tackled through \(N_e^0\)-point FFTs of the columns of \(U\) and \(N_e^0\)-point FFTs of the rows of \(U = \text{unvec}(u),\) as shown by the right side of \([28].\)
where diag(M) is formed by zeroing the off-diagonal elements of M. If we furthermore assume \( \mathbb{E}[xx^*] = \sigma_x^2 I \) with \( \sigma_x^2 \) given in (45), then, due to the independence of x and A, \( \mathbf{\Sigma}_{w} = (1 - \eta_b)\sigma_w^2 I_{N_r N_p} + \eta_b (1 - \eta_b)\sigma_x^2 \text{diag}(\mathbb{E}[\mathbf{A}^*]) \) (37)

\[
(1 - \eta_b)\sigma_w^2 I_{N_r N_p} + \eta_b (1 - \eta_b)\sigma_x^2 \text{diag}(\mathbb{E}[\mathbf{T}^H T]) \otimes I_{N_r},
\]

where (a) uses \( \mathbf{A}^* = (\mathbf{C}^T B_{N_r})^T \otimes I_{N_r} \), which follows from (13), the unitary property of \( \mathbf{B}_{N_r} \), and (10)- (11), and (38) uses \( \text{diag}(\mathbb{E}[\mathbf{T}^H T]) = P_L \sigma_p^2 I_{N_r} \), which follows from the transmitter power constraint. Thus, the effective noise \( \tilde{w} \) can be approximated as spectrally white with variance

\[
\sigma_w^2 = (1 - \eta_b)\sigma_w^2 + \eta_b P_L \sigma_p^2.
\]

The approximation (39) has been shown to be quite accurate for MIMO communication, especially at low SNR [14], [51]. Note, however, that \( \tilde{w} \) is non-Gaussian.

Assuming \( \mathbb{E}[xx^*] = \sigma_x^2 I \) (as above) and leveraging the result (38) that \( \mathbb{E}[\tilde{w}^* \tilde{w}] = \sigma_w^2 I \), one can straightforwardly derive the linear MMSE (LMMSE) estimator of x from \( \tilde{w} \) in (35). We will refer to it as the “approximate LMMSE (ALMMSE) estimator” due to the approximation (36). Expressed in terms of \( \mathbf{X} = \text{unvec}(x) \), it takes the form

\[
\hat{\mathbf{X}}_{\text{ALMMSE}} = \mathbf{B}_{N_r}^* \mathbf{Y} \mathbf{C}^* \left(1 - \eta_b \right) \mathbf{C} \mathbf{C}^* + \left( \sigma_w^2 + \eta_b P_L \right) I_{N_r L} \]

(40)

which is similar to (33) but with a regularized inverse.

A third benchmark algorithm follows by applying sparse reconstruction to the linearized model (35), which—unlike the methods above—leverages the fact that x is approximately sparse due to leakage. In particular, we used the “SPGL1” algorithm [52] to solve the basis pursuit denoising (BPDN) problem

\[
\text{arg min } \|x\|_1 \text{ s.t. } \|y - (1 - \eta_b)Ax\| \leq \sigma_\omega^2 N_p N_r,
\]

with \( \sigma_\omega^2 \) defined in (39)\(^4\). Similar to EM-GAMP and EM-VAMP, the computational complexity of SPGL1 is dominated by matrix-vector multiplications with \( \mathbf{A} \) and \( \mathbf{A}^* \), and so the fast implementation [26] is used.

A fourth benchmark algorithm is the quantized iterative hard thresholding (QIHT) algorithm proposed in [38], [39]. At iteration \( k = 1, 2, \ldots \), the update is estimated as

\[
\hat{x}^{(k)} = Q \left(y - A \hat{x}^{(k)}\right),
\]

(42)

\[
\hat{x}^{(k+1)} = h_K \left(\hat{x}^{(k)} + \tau A^* \hat{s}^{(k)}\right),
\]

(43)

where \( h_K(x) \) is the hard thresholding operator that zeros all but the K largest (in magnitude) components of x. For convergence, it is suggested that \( \tau < \|A\|_2^{-2} \) for spectral norm \( \|A\|_2 \). Our simulations used \( \hat{x}^{(1)} = 0 \), \( \tau = 0.1 \frac{N_r}{P_L N_p} \), and \( K = N_r / 100 \). That is, QIHT kept only the largest 1% of the elements in each iteration.

\(^4\) We also tried the \( \ell_1 \)-based method from [40], but we experienced numerical problems with their first-order method. Since our \( \mathbf{A} \) was too large to fit in memory, we could not use the CVX implementation of [40] nor [41].

E. Norm Estimation of the Channel

With a one-bit ADC, all amplitude information is lost during quantization. Therefore, it is difficult to precisely recover the channel norm \( \|x\| \), especially at high SNR. In previous work [8], [9], the variance of the channel was assumed to be known to avoid this issue. However, it is possible in practice to estimate the channel norm from the average signal power received by the antenna circuit before quantization, for example, the AGC circuit.

Let us use \( \tilde{z} \triangleq (\mathbf{C}^T \otimes \mathbf{B}_{N_r})x + w \) to denote the noisy unquantized signal. We assume that it is possible to measure the total received power across all antennas, which is \( \mathbb{E}[\|\tilde{z}\|^2] \).

Under the assumption that \( \mathbb{E}[xx^*] = \sigma_x^2 I \), it is straightforward to show that

\[
\mathbb{E}[\|\tilde{z}\|^2] = \|\mathbf{C}^T \otimes \mathbf{B}_{N_r}\|^2 \sigma_x^2 + N_p N_r \sigma_w^2.
\]

(44)

In the UPA case, we have \( \|\mathbf{C}^T \otimes \mathbf{B}_{N_r}\|^2 = N_r \|\mathbf{C}\|^2 \), and for ZC training we have \( \|\mathbf{C}\|^2 = P_L N_p L \), which implies that

\[
\sigma_x^2 = \frac{\mathbb{E}[\|\tilde{z}\|^2] - N_p N_r \sigma_w^2}{P_L N_p N_r}.
\]

(45)

Under \( \mathbb{E}[xx^*] = \sigma_x^2 I \), the law of large numbers then says that \( \|x\| \approx \sigma_x \sqrt{N_r N_L} \) for large \( N_r N_L \). Thus, for all channel estimators, we normalized the channel estimate \( \tilde{x} \) as follows,

\[
\tilde{x} = \frac{x}{\|x\|} \text{ for } \|x\| \approx \sigma_x \sqrt{N_r N_L},
\]

(46)

since it reduced the channel estimation error in all cases.

V. Simulation Results

We first provide an illustrative example to visualize the behavior of the proposed channel estimation procedure. In Fig. 3 we show the result of estimating a broadband \( N_p \times N_r = 4 \times 16 \) MIMO channel. The transmitter was equipped with a \( 4 \times 4 \) UP and the receiver with a \( 2 \times 2 \) UP. The channel had \( N_{\text{cl}} = 2 \) clusters and delay spread of at most \( L = 16 \) samples. 4-bit ADCs and raised-cosine pulse-shaping with roll-off factor zero was used. The training length was shifted-ZC of length \( N_p = 512 \). The true magnitudes of the angle-delay channel are plotted in Fig. 3(a) while the estimated magnitudes are plotted in Fig. 3(b). As seen in the figure, the EM-GM-VAMP algorithm can estimate the channel \( \mathbf{X} \) quite accurately: the two clusters can be easily identified in Fig. 3(b) and the estimation error in Fig. 3(c) is small. The joint angle-delay sparsity of the mmWave channel is visible in Fig. 3(a) as is the approximate nature of the sparsity.

In the following subsections, we consider a more practical scenario, where the transmitter and receiver are both equipped with \( 8 \times 8 \) UPAs. We consider a channel with 4 clusters, each of 10 paths, with azimuth and elevation angular spread of 7.5 degrees. The delay spread was at most 16 samples. These numbers were chosen according to the urban macro (UMa) NLOS channel measurement results at 28 GHz, as given in the white paper [22]. The results we report are the average of 100 different channel realizations. The channel was normalized so that \( \mathbb{E} \left[ \sum_k \|\mathbf{H}[\ell]\|^2 \right] = \mathbb{E} \left[ \|xx^*\|^2 \right] = N_r N_p \). We define

\[
\text{SNR} \triangleq \frac{\mathbb{E}[\|\tilde{z}\|^2]}{\mathbb{E}[\|w\|^2]} = \frac{\mathbb{E}[\text{tr} \{ Axx^* A^* \}]}{N_r N_p \sigma_w^2} = \frac{P_L N_p N_r}{N_p N_r \sigma_w^2} = \frac{P_L}{\sigma_w^2}.
\]
A. Choice of Training Matrix

We first investigate the performance of various choices of training matrix $T$. In addition to the circularly-shifted-ZC design proposed in Section IV-C we try $T$ constructed from Golay complementary sequences, IID random Gaussian entries, and IID random QPSK entries. The channel estimation NMSEs of EM-GM-VAMP with these $T$ are compared in Fig. 4 where

$$\text{NMSE}(\tilde{x}) \triangleq \mathbb{E} \left[ \frac{\| \tilde{x} - x \|^2}{\| x \|^2} \right], \quad (47)$$

for $\tilde{x}$ normalized according to (46). As seen in the figure, the NMSEs of the four training designs are very close. However, the Golay sequence, random QPSK, and ZC sequences are preferred because of their constant modulus property, which leads to low PAPR. Furthermore, the ZC-based design allows efficient implementation of the matrix-vector multiplications in VAMP, as discussed in Section IV-C.

The runtime of the matrix-vector multiplications $A\tilde{x}$ and $A^s\tilde{s}$ and their fast implementations (using MATLAB R2016a on a standard desktop computer) are shown in Fig. 5. In the baseline implementation, $A\tilde{x}$ was computed via vec($B_{N_T}C\tilde{X}$) to avoid generating and storing the high-dimensional matrix $A$, and the computation of $A^s\tilde{s}$ was done similarly. The fast implementation was described in Section IV-C. As seen in the figure, the fast implementation consumes much less time than the baseline implementation. Much faster runtimes would be possible if the FFTs were implemented in hardware.

B. Algorithm Complexity

We now investigate the computational complexity of the various algorithms under test. Fig. 6 shows NMSE versus iteration for EM-GAMP, EM-VAMP, and QIHT. The figure shows the VAMP algorithms converging in $\approx 3$ iterations, the GAMP algorithms converging in $\approx 8$ iterations, and QIHT converging in $\approx 35$ iterations.

To further investigate algorithm complexity, we plot NMSE versus runtime in Fig. 7. We controlled the runtime of the QIHT, EM-GAMP and EM-VAMP algorithms by varying the number of iterations. The other three (non-iterative) algorithms are each represented by a single point on the plot. The figure shows that EM-GM-VAMP gives the best NMSE-complexity.
C. Effect of SNR, ADC resolution, and Training Length

We now investigate estimation performance versus SNR, ADC resolution, and training length. Fig. 8 shows the NMSE of the VAMP algorithms versus SNR. The figure shows that, at low SNR (< 0 dB), the performance gap between 1-bit ADC and infinite-bit ADC is only 2 dB. But as the SNR increases, the gap between 1-bit and infinite-bit performance grows. Thus, higher resolution ADCs provide significant benefits only at higher SNRs. To reduce power consumption and cost, few-bit ADCs should be deployed when the SNR is low.

Fig. 9 shows NMSE versus ADC resolution for all algorithms under test when SNR = 10 dB. The figure shows that NMSE decreases with ADC resolution for all algorithms, but not in a uniform way. For example, SPGL1 and QIHT perform similarly with 1 and 2 bits of resolution, but SPGL1 benefits from higher ADC resolution while QIHT does not. In fact, SPGL1 performs as good as EM-BG-GAMP/V AMP for > 7 bits, but significantly worse with few bits, because it does not leverage the structure of the quantization error. The figure also shows that, when the EM-GAMP/V AMP are used, there is little benefit in increasing the ADC resolution above 4 bits at this SNR.

Fig. 10 shows the NMSE of the VAMP algorithms versus training length. We note that, as the training length varies from 1024 to 5120, the sampling rate relative to Nyquist, i.e., \( N_s \), varies from 1 to 5. The figure shows...
that the NMSE decays polynomially with the training length. In particular, \( \text{NMSE} \propto N_p^{-\alpha} \) with \( \alpha \approx 1/2 \).

**D. Mutual Information and Achievable Rate Bounds**

In this section, we investigate the effect of channel-estimation accuracy on mutual information and achievable rate, which are important metrics for communication systems. Our methodology is inspired by that in [25].

We consider OFDM transmission with \( N_1 \) subcarriers. For the \( k \)th subcarrier, denote the true frequency-domain MIMO channel by \( \mathbf{G}_k \in \mathbb{C}^{N_r \times N_t} \), the transmitted signal by \( s_k \), and the variance-\( \sigma_w^2 \) additive Gaussian noise by \( w_k \). Using the linearized model (55), we can approximate its covariance (conditional on \( \{\mathbf{G}_l\}_{l=1}^{N_b} \)) by

\[
\mathbb{E}[\hat{\mathbf{w}}_k \hat{\mathbf{w}}_k^* | \{\mathbf{G}_l\}] \approx (1 - \eta_b) \mathbf{I}_{N_r} + \eta_b \text{diag} \left( \frac{1}{N_b} \sum_{l=1}^{N_b} \mathbf{G}_l \mathbf{R}_l \mathbf{G}_l^* \right) \triangleq \hat{\Sigma}_{\hat{\mathbf{w}}_k},
\]

where \( \mathbf{R}_k \triangleq \mathbb{E}[s_k s_k^*] \) and the averaging over \( N_b \) subcarriers occurs because quantization is performed in the time-domain.

Suppose that \( s_k \) is precoded based on the estimated frequency-domain channel \( \hat{\mathbf{G}}_k \), which is computed from the estimated time-domain channel \( \{\hat{\mathbf{H}}(f)\}_{f=1}^{L-1} \). In particular,

\[
s_k = \sum_{m=1}^{\min(N_t, N_r)} \hat{v}_{km} \sqrt{p_{km} d_{km}},
\]

where \( \hat{v}_{km} \) is the \( m \)th right-singular vector of \( \hat{\mathbf{G}}_k \), \( p_{km} \geq 0 \) are the powers allocated by the waterfilling algorithm, and \( d_{km} \) is a unit-variance message-bearing symbol.

If each stream is decoded independently and the effective noise \( \hat{\mathbf{w}}_k \) is treated as if it has the worst-case Gaussian distribution with covariance \( \hat{\Sigma}_{\hat{\mathbf{w}}_k} \), then the mutual information between \( \{s_k\}_{k=1}^{N_b} \) and \( \{y_k\}_{k=1}^{N_b} \) conditioned on \( \{\mathbf{G}_k\}_{k=1}^{N_b} \) (in units of bps/Hz) is lower bounded by [53]

\[
I \left( \{s_k\}_{k=1}^{N_b}, \{y_k\}_{k=1}^{N_b} \mid \{\mathbf{G}_k\}_{k=1}^{N_b} \right) \triangleq \mathbb{E} \left\{ \frac{1}{N_b} \sum_{k=1}^{N_b} \sum_{m=1}^{\min(N_t, N_r)} \log_2 \left( 1 + \frac{(1 - \eta_b)^2 \tilde{u}_{km}^2 \hat{G}_k \tilde{v}_{km}^2 p_{km}}{\hat{\Sigma}_{\hat{\mathbf{w}}_k} + \sum_{n \neq m} (1 - \eta_b)^2 \tilde{u}_{km}^2 \hat{G}_k \tilde{v}_{km}^2 p_{km}} \right) \right\}
\]

where \( \tilde{u}_{km} \) is the \( m \)th left-singular vector of \( \hat{\mathbf{G}}_k \). The expectation in (51) is taken over \( \{\mathbf{G}_k\}_{k=1}^{N_b} \) and the effective noise \( \{\hat{\mathbf{w}}_k\}_{k=1}^{N_b} \). (Because the channel estimates depend on \( \mathbf{G}_k \) and \( \hat{\mathbf{w}}_k \), so do \( \tilde{u}_{km} \) and \( \hat{\Sigma}_{\hat{\mathbf{w}}_k} \).) If the coherence time of the channel is \( N_{co} \) symbols, then a lower bound on the achievable rate (in bps/Hz) is given by [53]

\[
R \geq \frac{N_{co} - N_p}{N_{co}} \mathbb{E} \left\{ \frac{1}{N_b} \sum_{k=1}^{N_b} \sum_{m=1}^{\min(N_t, N_r)} \log_2 \left( 1 + \frac{(1 - \eta_b)^2 \tilde{u}_{km}^2 \hat{G}_k \tilde{v}_{km}^2 p_{km}}{\hat{\Sigma}_{\hat{\mathbf{w}}_k} + \sum_{n \neq m} (1 - \eta_b)^2 \tilde{u}_{km}^2 \hat{G}_k \tilde{v}_{km}^2 p_{km}} \right) \right\}
\]

where \( \frac{N_{co} - N_p}{N_{co}} \) represents loss due to training overhead.

Fig. 11 plots the mutual information lower bound [51] versus SNR under EM-GAMP/VAMP channel estimates and perfect CSI at various ADC resolutions, using Monte-Carlo to approximate the expectation. The figure shows that using EM-GAMP/VAMP channel estimates results in a relatively small loss in mutual information compared to perfect CSI. It also shows that, at low SNR, the mutual information loss of few-bit ADC relative to infinite-bit ADC is small.

Fig. 12 plots the mutual information lower bound [51] versus ADC resolution for all the channel-estimation algorithms under test at SNR = 10 dB. The figure shows that the EM-GM-GAMP/VAMP algorithms achieve the highest mutual information, with EM-BG-GAMP/VAMP close behind at low ADC resolutions. The figure also shows that the mutual information saturates when the ADC resolution is \( > 5 \) bits at this SNR.

Fig. 13 shows the achievable rate lower bound [52] versus training length \( N_p \) for the EM-VAMP algorithms. For this
experiment, the channel coherence time was assumed to be $N_{co} = 10240$ symbols. The figure shows that, under these conditions, the optimal training length is $N_p = 1536$ when the ADC has 1 or 2 bits of resolution. When the ADC resolution increases to 3 or 4 bits, a shorter training length (e.g., $N_p = 1024$) is preferred because the cost of each training symbol (in bps/Hz) is larger.

Fig. 14 shows the achievable rate lower bound versus the training length $N_p$ for all the channel estimation algorithms under test. For this experiment, the ADC has 2-bit resolution. The EM-GM-GAMP and EM-GM-VAMP algorithms provide highest achievable rate and their optimal training length is 1536. Note that, for the LS and ALMMSE algorithms, the optimal training length is 2048, which is longer than the other algorithms.

VI. CONCLUSION

In this paper, we propose a methodology for estimation of broadband mmWave MIMO channels at receivers with few-bit ADCs. The broadband mmWave MIMO channel is sparse in both angle and delay domains, making it natural to apply compressed sensing techniques. We propose to use computationally efficient AMP algorithms (i.e., EM-GAMP and EM-VAMP) that accurately estimate the channel in the absence of prior information about its distribution (e.g., sparsity), and we enhance those methods with separate channel-norm estimation. We also design a training scheme, based on shifted ZC sequences, that leads to accurate and computationally efficient estimation with minimal transmitter PAPR. Finally, we report the results of an extensive simulation study that tests various algorithms, ADC precisions, training sequences, training lengths, SNRs, and runtime limits. Our simulations investigated channel-estimation MSE as well as mutual information and achievable rate bounds.

From the results of our study, we draw several conclusions. First, it is important to exploit the available joint angle-delay domain sparsity in channel estimation. Second, the ADC precision should be chosen based on the SNR; at low SNR, the use of few-bit ADCs results in very small loss (in MSE or achievable rate) compared to infinite-bit ADCs. Third, the training length should be chosen based on the ADC precision; at lower ADC precision, the achievable rate is maximized by a longer training sequence.

In this paper, the correlation among angle-delay channel coefficients was neglected. A possible direction to improve the channel estimation accuracy is to exploit this correlation.

REFERENCES

[1] J. Mo, P. Schniter, N. G. Prelcic, and R. W. Heath Jr., “Channel estimation in millimeter wave MIMO systems with one-bit quantization,” in Proc. Asilomar Conf. Signals, Systems and Computers, Nov 2014, pp. 957–961.
[2] T. S. Rappaport, R. W. Heath Jr, R. C. Daniels, and J. N. Murdock, Millimeter Wave Wireless Communications. Englewood Cliffs, NJ, USA: Prentice Hall, 2015.
[3] B. Murmann, “Energy limits in A/D converters,” in Proc. IEEE Failure Tolerance Faulty Concom (FTTC), June 2013, pp. 1–4.
[4] B. Le, T. Rondeau, J. Reed, and C. Bostian, “Analogue-to-digital converters,” IEEE Signal Process. Mag., vol. 22, no. 6, pp. 69–77, 2005.
