Bending of Functionally Graded Nanobeams using Hyperbolic Nonlocal Theory

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Abstract. Hyperbolic nonlocal theory is applied in this paper to calculate deflections in functionally graded nanobeams under uniform load. The theory is developed using the work done principle in conjunction with Eringen's theory. Free stress conditions on the upper and lower surfaces are achieved by the current hypothesis. Deflection of beam is estimated using Navier's approach considering simple ends of the nanobeam. Solutions found in the literature are taken for the comparison purpose and found that the present findings are matching with the existing solutions.

Keywords: Bending analysis, hyperbolic nonlocal theory, Eringen’s theory, functionally graded nanobeams.

1. Introduction

Nowadays functionally graded (FG) nanobeams are widely used in many industrial applications where those are subjected to bending actions due to transverse loads. Hence, it is important to accurately analyse the nanobeams under the bending action. The most widely used approach to model the nanobeams is Eringen's nonlocal theory [1]. Many researchers have developed nonlocal beam theories to analyse the isotropic nanobeams under bending action in conjunction with the Eringen's theory [2]. However, research on FG nanobeams is limited. In FG material, elastic constants are gradually varying along with the depth or length of the beam and hence it is treated an advanced composite material. Deflection of an isotropic beam utilizing classical and refined theories has been reported by Reddy [3]. Thai and Vo [4] have used sinusoidal function in the kinematics of nanobeams theory. Thai [5] has used third-order polynomial function for the modeling of nanobeams theory. Sayyad and Ghugal [6] have introduced inverse trigonometric nanobeams theory. Garg and Chalak [7] have developed novel higher order zigzag theory for analysis of laminated sandwich beams. Chalak, Zenkour, et al. [8] used zigzag theory for free vibration and modal stress analysis of FG-CNTRC beam under hygrothermal conditions. Belarbi, et al. [9] have used a novel shear deformation theory for nonlocal finite element model for the bending and buckling analysis of functionally graded nanobeams.
Garg et al. [10] studied hygro-thermo-mechanical based bending analysis of symmetric and unsym- 
metrical power-law, exponential and sigmoidal FG sandwich beams.

In this work, a hyperbolic nonlocal beam theory is applied to calculate central deflections in FG 
nanobeams under the action of uniform load on the top surface. The theory is formulated based on the 
work done principle and Eringen's model. Maximum deflections of nanobeams under uniform load 
are presented for isotropic material and functionally graded material. The present results are checked 
with previous research results and found closely matching with them for all nonlocal parameter val-
ues as well as the power-law index.

2. Materials and Methods

Geometry: Length of nanobeam is L, width is b and thickness is h (see Fig. 1)

![Fig. 1. FG nanobeams under consideration.](image)

Material: The nanobeam is made up of functionally graded material. Elastic constants of the material 
are varied using the power-law of gradation stated in Eq. (1).

\[
el(z) = e_m + (e_c - e_m)(0.5 + z/h)^p
\]

where \( e \) is the elastic constant \( (E, \mu, G) \); metal and ceramic materials are denoted by the letters \( m \) and \( c \), respectively; and \( p \) is the power-law coefficient whose values range from 0 to \( \infty \). The traction-
free boundary conditions on the top and bottom surfaces of the beam are obtained using the hyperbol-
ic shape function in the current nonlocal beam theory.

\[
u(x,z) = u_0(x) - z\frac{\partial w_0}{\partial x} + \left[ z \cosh\left(\frac{z}{2}\right) - \frac{h}{\xi} \sinh\left(\frac{z\xi}{h}\right)\right] \theta(x)
\]

\[
w(x) = w_0(x)
\]

where, \( \xi = 2.634 \), \( u \) and \( w \) are the displacements in \( x \)- and \( z \)- directions; \( \theta \) is the rotation. To ac-
count for the impacts of transverse shear deformation, hyperbolic shape function is employed in the 
axial displacement. The nonzero strains are obtained from the theory of elasticity.

\[
\varepsilon = \varepsilon_1 + z\varepsilon_2 + \left[ z \cosh\left(\frac{z}{2}\right) - \frac{h}{\xi} \sinh\left(\frac{z\xi}{h}\right)\right] \varepsilon_3
\]

\[
\gamma = \left[ \cosh\left(\frac{z}{2}\right) + \cosh\left(\frac{z\xi}{h}\right)\right] \gamma_1
\]

where

\[
\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = -\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_3 = \frac{\partial \theta}{\partial x}, \quad \gamma_1 = \theta
\]

The 1D Hooke's law is referred to calculate stresses within the domain of nanobeams.
where \( E(z) \) represents the FG beam's Young's modulus as it varies across thickness according to Eq. (1). To develop differential equations, the work done principle is combined with the present theory.

\[
b \int_{-h/2}^{h/2} \left( \sigma \delta \varepsilon + \tau \delta \gamma \right) dz dx - \int_0^L q(x) \delta w dx = 0 \tag{7}
\]

where \( q(x) \) represents the transverse load. The current theory's differential equations are generated by part wise integrating Eq. (7), collecting the multiplied differential terms of unknowns, and equating them with zero.

\[
A \frac{\partial^2 u_0}{\partial x^2} - B \frac{\partial^3 w_0}{\partial x^3} + C \frac{\partial^3 \theta}{\partial x^3} = 0
\]

\[
B \frac{\partial^3 u_0}{\partial x^3} - D \frac{\partial^4 w_0}{\partial x^4} + E \frac{\partial^4 \theta}{\partial x^4} = -q(x) \tag{8}
\]

\[
C \frac{\partial^2 u_0}{\partial x^2} - F \frac{\partial^3 w_0}{\partial x^3} + H \frac{\partial^3 \theta}{\partial x^3} - J \theta = 0
\]

where

\[
(A, B, C, D, F, H, J) = b \int_{-h/2}^{h/2} E(z) \left[ 1, z, f(z), z^2, zf(z), f(z)^2, (f'(z))^2 \right] \ d z \tag{9}
\]

The small-scale effect of nanobeams is explained using Eringen's theory. Nonlocal stresses may be calculated for functionally graded materials using the following relation.

\[
\sigma - n \frac{\partial^2 \sigma}{\partial x^2} = E(z) \varepsilon \quad \text{and} \quad \tau - n \frac{\partial^2 \tau}{\partial x^2} = E(z) \gamma \tag{10}
\]

Using Eq. (10) and Eq. (8); we can write final differential equations of nanobeams corresponding to the current theory.

\[
\delta u_0 : A \frac{\partial^2 u_0}{\partial x^2} - B \frac{\partial^3 w_0}{\partial x^3} + C \frac{\partial^3 \theta}{\partial x^3} = 0
\]

\[
\delta w_0 : B \frac{\partial^3 u_0}{\partial x^3} - D \frac{\partial^4 w_0}{\partial x^4} + E \frac{\partial^4 \theta}{\partial x^4} = -\left( q - n \frac{\partial^2 q}{\partial x^2} \right) \tag{11}
\]

\[
\delta \phi : C \frac{\partial^2 u_0}{\partial x^2} - F \frac{\partial^3 w_0}{\partial x^3} + H \frac{\partial^3 \theta}{\partial x^3} - J \theta = 0
\]

Solution Method: Deflections in simply supported functionally graded nanobeam is estimated using the Navier approach which satisfies the simply supported end conditions exactly i.e. \( N_x = w = M_y = M_x = 0 \). The unknown variables of the current theory and the transverse load are assumed in the single Fourier series.


\[ u_0 = \sum_{m=1}^{\infty} u_m \cos mx, \quad w_0 = \sum_{m=1}^{\infty} w_m \sin mx, \quad \theta = \sum_{m=1}^{\infty} \theta_m \cos mx, \quad q(x) = \sum_{m=1}^{\infty} \frac{4q_0}{m\pi} \sin mx \quad (12) \]

where \( r = m\pi / L \), \( (u_m, w_m, \theta_m) \) are unknown coefficients, \( q_0 \) is the maximum intensity of uniform loading. Eq. (12) substituted into Eq. (11) to give solutions for the bending problem stated in Eq. (13).

\[
\begin{bmatrix}
-Ar^2 & Br^3 & -Cr^2 \\
Br^3 & -Dr^4 & Fr^3 \\
-Cr^2 & Fr^3 & -(Hr^2 + J)
\end{bmatrix}
\begin{bmatrix}
u_m \\
w_m \\
\theta_m
\end{bmatrix} = -\frac{4q_0}{m\pi} \begin{bmatrix} 0 \\
1 + nr^2 \\
0 \end{bmatrix} \quad (13)\]

3. Results and Discussions

The current theory is applied for the transverse central deflection estimation of uniformly loaded isotropic and FG nanobeams simply-supported at ends. Material properties used for the beam are \( E = 1 \) GPa and \( \mu = 0.3 \) for isotropic material; and \( E_c = 380 \) GPa, \( E_m = 70 \) GPa, \( \mu_c = \mu_m = 0.3 \) for FG material.

Table 1. Non-dimensional transverse deflection \( \overline{w} = 100EIw / (q_0L^4) \) of simple end isotropic nanobeams \((L=1nm, b=1nm, h=1nm, m=100)\)

| Theory                      | Nonlocal parameter (n) |
|-----------------------------|-------------------------|
|                             | 0       | 1       | 2       | 3       | 4       |
| Present                     | 1.3351  | 1.4628  | 1.5904  | 1.7181  | 1.8457  |
| Sayyad and Ghugal [6]       | 1.3394  | 1.4675  | 1.5956  | 1.7237  | 1.8517  |
| Thai [5]                    | 1.3346  | 1.4622  | 1.5898  | 1.7173  | 1.8449  |
| Thai and Vo [4]             | 1.3345  | 1.4621  | 1.5897  | 1.7173  | 1.8449  |
| Timoshenko [6]              | 1.3124  | 1.4382  | 1.5641  | 1.6901  | 1.8157  |
| Bernoulli-Euler [6]         | 1.3020  | 1.4270  | 1.5522  | 1.6772  | 1.8022  |

Summary of transverse deflection in isotropic nanobeams for various values of the nonlocal parameters is present in Table 1. The present results are compared with Sayyad and Ghugal [6], Thai [5], Thai and Vo [4], Timoshenko, [6] and Bernoulli-Euler [6] This table illustrates that the current theory is in good accord with previous research results. The values of central transverse deflection for different values of power-law coefficients and nonlocal parameters are presented in Table 2. Theories of Sayyad and Ghugal [6] and Reddy [10] are used for the comparison. The present theory predicts the central deflection of FG nanobeams under uniform load with good accuracy.

4. Conclusions

This study highlights the use of hyperbolic theory function with arbitrary variables to predict the maximum central deflection of simply-supported nanobeams made up of homogeneous and functionally
graded materials. It is concluded that the present theory is in good accord with previous research results while predicting the central deflection of uniformly distributed nanobeams.

Table 2. Non-dimensional transverse deflection \( \bar{w} = 100E_nh^3w/(q_0L^4) \) of simple end FG nanobeams under uniform load \((L=10nm, h=1nm, m=100)\)

| Theory               | n | Power law coefficients (p) |
|----------------------|---|---------------------------|
|                      | 0 | 2.9501                    | 5.8958 | 9.0186 | 9.9391 |
|                      | 1 | 3.2321                    | 6.4599 | 9.8801 | 10.888 |
|                      | 2 | 3.5141                    | 7.0239 | 10.741 | 11.837 |
|                      | 3 | 3.7962                    | 7.5880 | 11.603 | 12.786 |
|                      | 4 | 4.0783                    | 8.1520 | 12.464 | 13.734 |
| Sayyad and Ghugal [6] | 0 | 2.9556                    | 6.0409 | 9.2826 | 10.229 |
|                      | 1 | 3.2345                    | 6.6093 | 10.154 | 11.189 |
|                      | 2 | 3.5134                    | 7.1776 | 11.025 | 12.148 |
|                      | 3 | 3.7922                    | 7.7462 | 11.897 | 13.108 |
|                      | 4 | 4.0711                    | 8.3145 | 12.769 | 14.068 |
| Reddy [10]           | 0 | 2.9501                    | 5.8959 | 9.0204 | 9.9403 |
|                      | 1 | 3.2322                    | 6.4599 | 9.8820 | 10.889 |
|                      | 2 | 3.5142                    | 7.0240 | 10.743 | 11.838 |
|                      | 3 | 3.7963                    | 7.5880 | 10.743 | 12.787 |
|                      | 4 | 4.0783                    | 8.1521 | 12.466 | 13.736 |

5. Conflict of Interest
The authors state that the publication of this paper does not include any conflicts of interest.

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