Reconciling and Validating the Ashworth-Davies Doppler Shifts of a Translating Mirror

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We simplify the Ashworth-Davies special relativistic theory of a uniformly translating mirror with an arbitrary angle of incidence and direction of propagation in the non-relativistic limit. We show that it is in good agreement with a more intuitive derivation that only considers the constancy of the speed of light. We experimentally confirm the theory predictions with phase-insensitive frequency measurements using a liquid crystal light valve.

INTRODUCTION

A Doppler shift is the frequency differential of a wave when a source and detector are in relative motion. Harnessing the Doppler effect has brought about great gains in scientific, engineering and society at large. Doppler shifts are used extensively in astronomy [1–3], remote weather monitoring [4–6], non-invasive medical diagnostics [7–9] and laser velocimetry (fluid flow) [10, 11] to name a few applications.

Einstein’s derivation of the Doppler shift of light from a uniformly translating mirror considered only the Doppler shift resulting from light reflected at an oblique angle [12]. While the angle of incidence of the beam relative to the surface normal was arbitrary, the direction of mirror propagation was in the same direction as the surface normal.

A derivation of the Doppler shift that considers both arbitrary mirror propagation direction and arbitrary incidence angle was derived by Ashworth and Davies [13]. Follow-on experiments demonstrated the intended prediction from the theory namely that there is no Doppler effect for transversely moving mirrors [14, 15]. However, no one has experimentally verified the results for the general case, even for nonrelativistic mirror velocities.

Understanding the general case of a Doppler shift from a moving mirror is important for our previous work on a Doppler-based gyroscope (see [16]). In that work, it was shown that Doppler shifts are fundamental in passive gyroscopes. However, it is impossible to derive a generalized theory about the Doppler gyroscope without an understanding of the Doppler shifts result from the movement of the mirror with respect to its surface normal.

When considering a rotating interferometer, each mirror in the interferometer can move in a nontrivial direction, depending on its position relative to the axis of rotation. To calculate the difference in frequency shift between the two optical paths in the interferometer, we need to understand what is the frequency shift that results from the reflection of each individual mirror.

Here, we show that the Ashworth-Davies result in the nonrelativistic domain can be written in an intuitive form that reconciles with a simple modification of a Doppler shift model by Ghurchinovski [17]. We verify the predictions of this derivation experimentally using a hypersensitive differential frequency measurement in a liquid crystal light valve [18].

THEORY

We consider the Doppler shift scenario as originally proposed by Ashworth and Davies as shown in Fig. 1. A mirror is translating at constant velocity along the x-axis. An incoming light beam with an angle of incidence $\beta$ is specularly reflected from the surface. Using a convention by Ashworth and Davies, $\alpha$ is the angle between the velocity vector and the surface normal and $\phi$ is the angle between the angle of incidence (from the opposite side of the mirror) and the velocity vector. We also define $\beta$ as the angle of incidence.

FIG. 1. Light reflecting from a mirror moving with velocity $v$ along the x-axis. We use the convention of Ashworth and Davies [13] in which $\alpha$ is the angle between the velocity vector and the surface normal and $\phi$ is the angle between the angle of incidence (from the opposite side of the mirror) and the velocity vector. We also define $\beta$ as the angle of incidence.
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In an effort to prove that transverse Doppler shifts do not exist in reflection from a mirror, Ashworth and Davies derived a generalized special relativistic Doppler shift formula

$$f_f = f_i \left[\frac{\tan \alpha}{\lambda} + \frac{\nu \sin \phi}{\lambda} \right]^2 + \left[1 - \frac{\nu \cos \phi}{\lambda} \right]^2,$$

where $f_f$ and $f_i$ are the frequency of the light after and before the reflection from the mirror, respectively, and $\nu$ is the speed of the mirror. In the limit $\nu \ll c$, we simplify the equation, namely

$$\Delta f = \frac{2\nu}{\lambda} \left(\sin \alpha \cos \phi - \cos \alpha \cos \phi\right).$$

We now make a change of variables. Instead of using $\phi$ (the angular difference between the angle of incidence and the velocity vector), we directly use the angle of incidence $\beta$ given by $\beta = \phi + \alpha$. This implies

$$\Delta f = \frac{2\nu}{\lambda} \left[\sin \alpha \cos \alpha \sin (\beta - \alpha) - \cos \alpha \cos (\beta - \alpha)\right].$$

Using the angle sum relations $\cos(\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha$ and $\sin(\beta - \alpha) = \sin \beta \cos \alpha - \cos \beta \sin \alpha$ as well as $\sin^2 \alpha + \cos^2 \alpha = 1$, we arrive at the relation

$$\Delta f = -\frac{2\nu}{\lambda} \cos \beta \cos \alpha.$$

This formula is more intuitive than the original Ashworth-Davies result and easier to use experimentally in describing a physical system as all angles are defined with respect to the mirror’s surface normal. The formula contains the familiar Einstein formula \[12\], in the low velocity limit, for $\alpha = 0$ (the scenario where the mirror surface normal also lies on the x-axis).

The result is also in good agreement with a simple addition to the intuitive derivation by Gjurchinovski \[17\]. Gjurchinovski considered sequential wavefronts reflecting with temporal separation $\Delta t$ from a mirror moving at velocity $\nu$ parallel to the surface’s normal. The first wavefront reflected from the mirror at the first time and after the time interval $\Delta t$ the mirror reflected the second wavefront. During the time interval $\Delta t$ the mirror propagated a distance $\nu \Delta t$. The distance between those two surfaces (the mirror surface at two different times) was $\nu \Delta t$ if the mirror propagated in the direction of the surface normal.

FIG. 2. Consider a mirror moving at nonrelativistic speed $\nu$ in an angle $\alpha$ relative to its surface normal, during a time interval $\Delta t$. The mirror velocity in the direction of its normal is $v \cos \alpha$, so the distance between the mirror before and after the time interval will be $v \cos \alpha \Delta t$. Repeating Gjurchinovski derivation in \[17\] but replacing $v \Delta t$ with $v \cos \alpha \Delta t$ will yield Eqn. 5

However, we note that if the velocity of the mirror were not in the direction of the surface normal, but at an angle $\alpha$, the mirror would have only moved $\nu \Delta t \cos \alpha$ between sequential wavefronts, as seen in Fig 2. If we only care about the velocity in order to calculate the distance between the mirror before and after a time interval $\Delta t$ and consider only nonrelativistic velocities (and thus ignoring length contraction), we can easily replace $v$ with $v \cos \alpha$ and ignore second order of $\frac{\nu}{c}$ in Eqn. 5, thus reproducing Eqn. 5

**EXPERIMENTAL SETUP**

To experimentally test these results, we use a liquid crystal light valve (LCLV) \[18\] (see Supplementary Materials for more information). While any experimental system that can precisely measure Doppler shifts can be used, we find the LCLV to be exceptionally ideal. As shown in \[18\], a LCLV can measure down to $\mu Hz/Hz^{1/2}$
FIG. 3. The experimental setup. A displaced Mach-Zehnder interferometer, where one of the mirrors is mounted to an adjustable moving platform, controlled by a piezo-electric crystal (PZT). A slight angle between the beams is fabricated in order to create a wave mixing where the beams meet at the Liquid Crystal Light Valve (LCLV). The two primary diffracted output orders of the LCLV are focused on a balanced detector.

meaning that it is several orders of magnitude more sensitive than any other system per measurement time. This means that the movement of the mirror can be made to be very small, which has the effect of minimizing alignment issues as we rotate through the various measurement angles.

The experimental setup is shown in Figure 3. Collimated light from a laser at 532 nm is split on a 50/50 beamsplitter. One of the mirrors is mounted to an adjustable moving platform that is controlled by a piezo-actuator (PZT) so that the beam that’s reflected from the mirror experiences a Doppler shift relative to the other beam. A Mach-Zehnder type setup is used to create a slight angle between the two beams on the order of 0.01 radians, and both beams are then incident on a LCLV. A small angle between the beams is needed in order to create two-wave mixing in the Raman-Nath regime of the LCLV (see Supplementary Materials for more information). The two primary diffracted output orders of the LCLV are focused on a balanced detector. The difference in the intensities of the two beams hitting the balance detector is proportional to the difference in frequency between the two beams, $\Delta f$.

To test the Doppler shift dependence on the direction of propagation of the mirror, the PZT direction was changed while the angle of incidence was fixed. With each iteration of the experiment, the moving platform was tuned to a different angle, and the mirror sitting on it was tuned to be with a fixed angle relative to the incoming beam at $\beta = 45^\circ$. This caused the movement of the platform to be in a different direction relative to the mirror’s surface normal (which corresponds to $\alpha$ as defined above). The PZT was driven by an arbitrary waveform generator producing triangle waves at 20 mHz and a peak-to-peak voltage of 15V. The PZT response was measured to be approximately 60 nm/V. The experiment was repeated 36 times, with alpha ranging from $-180^\circ$ to $-180^\circ$. The absolute value of the mirror’s velocity was constant through all the iterations and equal to $V = 36$ nm/s.

The errors in the experiment are primarily from seismic and acoustic vibrations as well as from fluctuating air currents. While we had various forms of active and passive noise reduction, the light valve is particularly sensitive to noise. Further, the experiments are run at slow speeds meaning that 1/f noise is significant.

FIG. 4. Experimental results. The Doppler signal amplitude vs velocity angle $\alpha$ relative to the surface normal is shown in the blue points along with the error bars. We used a fixed angle of incidence of $\beta = 45^\circ$. The orange curve is the cosine curve from Eq. 5 when plugging the parameters of our experiment ($\lambda = 532$ nm, $V = 36$ nm/s).

RESULTS

The experimental results are shown in Fig. 4. The blue data points are the measured amplitude of the Doppler shift from the moving mirror. The X-axis is the angle between the direction of the mirror’s propagation to the mirror’s normal ($\alpha$). The orange curve is a least-squares fitted cosine function. Hence the experimental results are in excellent agreement with the main result of this paper (Eqn. 5). We have thus shown that there is not only a cosine term that arises from the angle of incidence (measured relative to the surface normal) but also from the direction of mirror propagation relative to the surface normal.
DISCUSSION AND CONCLUSION

We have shown that there is a simple nonrelativistic Doppler correction to the frequency of light that accounts for both the angle of incidence and the angle of mirror propagation relative to the surface normal. We showed that the more cumbersome Ashworth-Davies result simplifies to this result in this limit. We further showed using precision differential frequency measurements that the cosine behavior is observed. These results provide a novel simplified form of the Doppler shift in the nonrelativistic limit and can have a large impact in practical domains of application.

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