A Decision Problem for ultimately periodic Sets in non-standard Numeration Systems

Emilie Charlier  Michel Rigo

Department of Mathematics
University of Liège

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Definition
A *numeration system* is given by a (strictly) increasing sequence $U = (U_i)_{i \geq 0}$ of integers such that $U_0 = 1$ and $C_U := \sup_{i \geq 0} \left\lceil U_{i+1}/U_i \right\rceil$ is finite.

The *greedy $U$-representation* of a positive integer $n$ is the unique finite word $\text{rep}_U(n) = w_\ell \cdots w_0$ over $A_U := \{0, \ldots, C_U - 1\}$ satisfying $n = \sum_{i=0}^{\ell} w_i U_i$, $w_\ell \neq 0$ and $\sum_{i=0}^{t} w_i U_i < U_{t+1}$, $\forall t = 0, \ldots, \ell$. We set $\text{rep}_U(0) = \varepsilon$.

If $x = x_\ell \cdots x_0$ is a word over a finite alphabet of integers, then the *$U$-numerical value* of $x$ is $\text{val}_U(x) = \sum_{i=0}^{\ell} x_i U_i$. 
Non standard Numeration Systems

Definition

A *numeration system* is given by a (strictly) increasing sequence $U = (U_i)_{i \geq 0}$ of integers such that $U_0 = 1$ and $C_U := \sup_{i \geq 0} \left\lceil \frac{U_{i+1}}{U_i} \right\rceil$ is finite.

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Definition

A set $X \subseteq \mathbb{N}$ of integers is *$U$-recognizable* if the language $\text{rep}_U(X)$ over $A_U$ is regular (i.e., accepted by a finite automaton).
Definition
A numeration system $U = (U_i)_{i \geq 0}$ is said to be linear (of order $k$), if the sequence $U$ satisfies a homogenous linear recurrence relation like

$$U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i, \quad i \geq 0,$$

for some $k \geq 1$, $a_1, \ldots, a_k \in \mathbb{Z}$ and $a_k \neq 0$. 
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U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i, \quad i \geq 0,
\]

for some \( k \geq 1, \ a_1, \ldots, a_k \in \mathbb{Z} \) and \( a_k \neq 0 \).

Example (Fibonacci System)
Consider the sequence defined by \( F_0 = 1, F_1 = 2 \) and \( F_{i+2} = F_{i+1} + F_i, \ i \geq 0 \). The Fibonacci (linear numeration) system is given by \( F = (F_i)_{i \geq 0} = (1, 2, 3, 5, 8, 13, \ldots) \). For instance, \( \text{rep}_F(15) = 100010 \) and \( \text{val}_F(101001) = 13 + 5 + 1 = 19 \).
Motivation

Definition
Two integers $p, q \geq 2$ are *multiplicatively independant* if $p^k = p^\ell$ and $k, \ell \in \mathbb{N} \implies k = \ell = 0$.

Notation
If $p \geq 2$ and $U = (p^i)_{i \geq 0}$, a set $X \subseteq \mathbb{N}$ of integers is said $p$-recognizable if the language $\text{rep}_U(X)$ over $A_U = \{0, \ldots, p - 1\}$ is regular.

Theorem (Cobham, 1969)
Let $X \subseteq \mathbb{N}$ be a set of integers. If $p$ and $q$ are two multiplicatively independant integers, $X$ is $p$-recognizable and $q$-recognizable if and only if $X$ is ultimately periodic.

Theorem (J. Honkala, 1985)
Let $p \geq 2$. It is decidable whether or not a $p$-recognizable set is ultimately periodic.
A Decision Problem

Proposition
Let $U = (U_i)_{i \geq 0}$ be a (linear) numeration system such that $\mathbb{N}$ is $U$-recognizable. If $X \subseteq \mathbb{N}$ is ultimately periodic, then $X$ is $U$-recognizable, and a DFA accepting $\text{rep}_U(X)$ can be effectively obtained.

Problem
Given a linear numeration system $U$ and a $U$-recognizable set $X \subseteq \mathbb{N}$. Is it decidable whether or not $X$ is ultimately periodic, i.e., whether or not $X$ is a finite union of arithmetic progressions?
**Definition**

Let $X \subseteq \mathbb{N}$ be a set of integers. The *characteristic word of $X$* is an infinite word $x_0x_1x_2\cdots$ over \{0, 1\} defined by $x_i = 1$ if and only if $i \in X$. 

Ultimately periodic Sets
**Definition**

Let $X \subseteq \mathbb{N}$ be a set of integers. The *characteristic word of $X$* is an infinite word $x_0 x_1 x_2 \cdots$ over \{0, 1\} defined by $x_i = 1$ if and only if $i \in X$.

If now $X \subseteq \mathbb{N}$ is ultimately periodic, its characteristic word is an infinite word over \{0, 1\} of the form

$$x_0 x_1 x_2 \cdots = u v^\omega$$

where $u$ and $v$ are chosen of minimal length. We say that $|u|$ (resp. $|v|$) is the *preperiod* (resp. *period*) of $X$. 
Idea of Honkala’s Decision Procedure
The input is a finite automaton accepting $\text{rep}_U(X)$.

First, he gives an upper bound for the possible periods of $X$, by showing that, if $Y$ is a ultimately periodic set of integers, then the number of states of any deterministic automaton accepting $\text{rep}_U(Y)$ grows with the period of $Y$.

Then, once the period of $X$ is bounded, he gives an upper bound for the possible preperiods of $X$, in a similar way.
An upper Bound for the Period

Notation

For a sequence $U = (U_i)_{i \geq 0}$ of integers and an integer $m \geq 2$, $N_U(m) \in \{1, \ldots, m\}$ denotes the number of values that are taken infinitely often by the sequence $(U_i \mod m)_{i \geq 0}$.
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For a sequence $U = (U_i)_{i \geq 0}$ of integers and an integer $m \geq 2$, $N_U(m) \in \{1, \ldots, m\}$ denotes the number of values that are taken infinitely often by the sequence $(U_i \mod m)_{i \geq 0}$.

Example (Fibonacci System, continued)
$(F_i \mod 4) = (1, 2, 3, 1, 0, 1, 1, 2, 3, \ldots)$ and $N_F(4) = 4$.
$(F_i \mod 11) = (1, 2, 3, 5, 8, 2, 10, 1, 0, 1, 1, 2, 3, \ldots)$ and $N_F(11) = 7$. 
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For a sequence $U = (U_i)_{i \geq 0}$ of integers and an integer $m \geq 2$, $N_U(m) \in \{1, \ldots, m\}$ denotes the number of values that are taken infinitely often by the sequence $(U_i \mod m)_{i \geq 0}$.

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Proposition
Let $U = (U_i)_{i \geq 0}$ be a numeration system satisfying
$$\lim_{i \to +\infty} U_{i+1} - U_i = +\infty.$$ If $X \subseteq \mathbb{N}$ is an ultimately periodic $U$-recognizable set of period $|v|$, then any deterministic finite automaton accepting $\text{rep}_U(X)$ has at least $N_U(|v|)$ states.
An upper Bound for the Period

Corollary

Let $U = (U_i)_{i \geq 0}$ be a numeration system satisfying

\[ \lim_{i \to +\infty} U_{i+1} - U_i = +\infty. \]

Assume that $\lim_{m \to +\infty} N_U(m) = +\infty$. Then the period of an ultimately periodic set $X \subseteq \mathbb{N}$ such that $\text{rep}_U(X)$ is accepted by a DFA with $d$ states is bounded by the smallest integer $s_0$ such that for all $m \geq s_0$, $N_U(m) > d$, which is effectively computable.
Corollary

Let $U = (U_i)_{i \geq 0}$ be a numeration system satisfying
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Assume that $\lim_{m \to +\infty} N_U(m) = +\infty$. Then the period of an ultimately periodic set $X \subseteq \mathbb{N}$ such that $\text{rep}_U(X)$ is accepted by a DFA with $d$ states is bounded by the smallest integer $s_0$ such that for all $m \geq s_0$, $N_U(m) > d$, which is effectively computable.

Lemma

If $U = (U_i)_{i \geq 0}$ is a linear numeration system satisfying a recurrence relation of order $k \geq 1$ of the kind
\[ U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i, \quad i \geq 0, \]
with $a_k = \pm 1$, then $\lim_{m \to +\infty} N_U(m) = +\infty$. 

An upper Bound for the Period

Proposition
Let $U = (U_i)_{i \geq 0}$ be a numeration system satisfying condition 
\[ \lim_{i \to +\infty} U_{i+1} - U_i = +\infty \] and $X \subseteq \mathbb{N}$ be an ultimately periodic 
$U$-recognizable set of period $|v|$. If 1 occurs infinitely many times 
in $(U_i \mod |v|)_{i \geq 0}$ then any deterministic finite automaton 
accepting $\text{rep}_U(X)$ has at least $|v|$ states.
Idea of the Proof with the Fibonacci System

Definition
Let $L \subseteq \Sigma^*$ be a language over a finite alphabet $\Sigma$ and $x$ be a finite word over $\Sigma$. We set $x^{-1}.L = \{z \in \Sigma^* \mid xz \in L\}$. The Myhill-Nerode congruence $\sim_L$ is defined as follows. Let $x, y \in \Sigma^*$. We write $x \sim_L y$ if $x^{-1}.L = y^{-1}.L$.

Proposition
A language $L$ over a finite alphabet $\Sigma$ is regular if and only if $\sim_L$ has a finite index, being the number of states of the minimal automaton of $L$. 
Definition
Let \( L \subseteq \Sigma^* \) be a language over a finite alphabet \( \Sigma \) and \( x \) be a finite word over \( \Sigma \). We set \( x^{-1}.L = \{ z \in \Sigma^* \mid xz \in L \} \).

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Proposition
A language \( L \) over a finite alphabet \( \Sigma \) is regular if and only if \( \sim_L \) has a finite index, being the number of states of the minimal automaton of \( L \).

Example (Fibonacci System, continued)
For all \( m \geq 2 \), the sequences \((F_i \mod m)_{i \geq 0}\) is purely periodic. So \( F_0 = 1 \) appears infinitely often in \((F_i \mod m)_{i \geq 0}\).
Let \( X \subseteq \mathbb{N} \) be an ultimately periodic \( F \)-recognizable set of period \( |v| \) and preperiod \( |u| \).
Idea of the Proof with the Fibonacci System

**Example (Fibonacci System, continued)**
There exist $n_1, \ldots, n_{|v|}$ such that for all $t = 0, \ldots, |v| - 1,$

$$10^{n_{|v|}} 10^{n_{|v|-1}} \cdots 10^{n_1} 0^{\text{rep}_U(|v|-1)} - |\text{rep}_U(t)| \text{rep}_U(t)$$

is a greedy $F$-representation.
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Example (Fibonacci System, continued)

There exist $n_1, \ldots, n_{|v|}$ such that for all $t = 0, \ldots, |v| - 1,$

$$10^{n_{|v|}} 10^{n_{|v|}-1} \ldots 10^{n_1} 0^{\text{rep}_U(|v| - 1)} - |\text{rep}_U(t)| \text{rep}_U(t)$$

is a greedy $F$-representation. Moreover $n_1, \ldots, n_{|v|}$ can be chosen such that, for all $j = 1, \ldots, |v|,$

$$\text{val}_U(10^{n_j} \ldots 10^{n_1} + |\text{rep}_U(|v| - 1)|) \equiv j \mod |v|$$

and $\text{val}_U(10^{n_1} + |\text{rep}_U(|v| - 1)|) > |u|$. 
Example (Fibonacci System, continued)

There exist \( n_1, \ldots, n_{|v|} \) such that for all \( t = 0, \ldots, |v| - 1 \),

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is a greedy \( F \)-representation. Moreover \( n_1, \ldots, n_{|v|} \) can be chosen such that, for all \( j = 1, \ldots, |v| \),

\[
\text{val}_U(10^{n_j} \ldots 10^{n_1} 0^{\text{rep}_U(|v| - 1) - \text{rep}_U(t)}) \equiv j \mod |v|
\]

and \( \text{val}_U(10^{n_1} 0^{\text{rep}_U(|v| - 1) - \text{rep}_U(t)}) > |u| \). For \( i, j \in \{1, \ldots, |v|\} \), \( i \neq j \), the words

\[
10^{n_i} \ldots 10^{n_1} \text{ and } 10^{n_j} \ldots 10^{n_1}
\]

are nonequivalent for \( \sim_{\text{rep}_U}(X) \). This can be shown by concatenating some word of the kind \( 0^{\text{rep}_U(|v| - 1) - \text{rep}_U(t)} \text{rep}_U(t) \) with \( t < |v| \).
An upper Bound for the Preperiod

Notation

For a sequence $U = (U_i)_{i \geq 0}$ of integers, if $(U_i \mod m)_{i \geq 0}$, $m \geq 2$, is ultimately periodic, we denote its (minimal) preperiod by $\iota_U(m)$ and its (minimal) period by $\pi_U(m)$. 
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$(F_i \mod 11) = (1, 2, 3, 5, 8, 2, 10, 1, 0, 1, 1, 2, 3, \ldots)$ and $\pi_F(11) = 10$.
We have $\iota_F(m) = 0$, for all $m \geq 2$. 
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We have $\iota_F(m) = 0$, for all $m \geq 2$.

Remark
If $U = (U_i)_{i \geq 0}$ is a linear numeration system of order $k$, then for all $m \geq 2$, we have $N_U(m) \geq k \sqrt[2]{\pi_U(m)}$. 
An upper Bound for the Preperiod

Proposition

Let $U = (U_i)_{i \geq 0}$ be a linear numeration system. Let $X \subseteq \mathbb{N}$ be an ultimately periodic $U$-recognizable set of period $|v|$ and preperiod $|u|$ such that $|\text{rep}_U(|u| - 1)| - \iota_U(|v|) > 0$. Then any deterministic finite automaton accepting $\text{rep}_U(X)$ has at least $|\text{rep}_U(|u| - 1)| - \iota_U(|v|)$ states.
Theorem (E. C., M. Rigo)

Let $U = (U_i)_{i \geq 0}$ be a linear numeration system such that $\mathbb{N}$ is $U$-recognizable and satisfying a recurrence relation of order $k$ of the kind

$$U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i, \ i \geq 0,$$

with $a_k = \pm 1$ and such that $\lim_{i \to +\infty} U_{i+1} - U_i = +\infty$.

It is decidable whether or not a $U$-recognizable set is ultimately periodic.
A Decision Procedure

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It is decidable whether or not a $U$-recognizable set is ultimately periodic.

Remark

Whenever $\gcd(a_1, \ldots, a_k) = g \geq 2$, for all $n \geq 1$ and for all $i$ large enough, we have $U_i \equiv 0 \mod g^n$ and $N_U(m)$ does not tend to infinity.
A Decision Procedure

**Theorem (E. C., M. Rigo)**

Let $U = (U_i)_{i \geq 0}$ be a linear numeration system such that $\mathbb{N}$ is $U$-recognizable and satisfying a recurrence relation of order $k$ of the kind

$$U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i, \quad i \geq 0,$$

with $a_k = \pm 1$ and such that $\lim_{i \to +\infty} U_{i+1} - U_i = +\infty$.

It is decidable whether or not a $U$-recognizable set is ultimately periodic.

**Remark**
Whenever $\gcd(a_1, \ldots, a_k) = g \geq 2$, for all $n \geq 1$ and for all $i$ large enough, we have $U_i \equiv 0 \mod g^n$ and $N_U(m)$ does not tend to infinity.

**Question**
What happen whenever $\gcd(a_1, \ldots, a_k) = 1$ and $a_k \neq \pm 1$?
Abstract Numeration Systems

Definition
An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where $L$ is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of $L$ with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\text{rep}_S : \mathbb{N} \to L \quad \text{val}_S = \text{rep}_S^{-1} : L \to \mathbb{N}. $$

Example
$L = a^*$, $\Sigma = \{a\}$

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|-----|---|---|---|---|---|--------|
| rep ($n$) | $\varepsilon$ | $a$ | $aa$ | $aaa$ | $aaaa$ | $\cdots$ |
Example

\( L = \{a, b\}^*, \ \Sigma = \{a, b\}, \ a < b \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | \( \cdots \) |
|--------|---|---|---|---|---|---|---|---|---------|
| \( \text{rep}(n) \) | \( \varepsilon \) | a | b | aa | ab | ba | bb | aaa | \( \cdots \) |

Example

\( L = a^* b^*, \ \Sigma = \{a, b\}, \ a < b \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | \( \cdots \) |
|--------|---|---|---|---|---|---|---|---------|
| \( \text{rep}(n) \) | \( \varepsilon \) | a | b | aa | ab | ba | bb | aaa | \( \cdots \) |
Abstract Numeration Systems

Remark
This generalizes non-standard numeration systems $U = (U_i)_{i \geq 0}$ for which $\mathbb{N}$ is $U$-recognizable, like integer base $p$ systems or Fibonacci system.

\[ L = \{ \varepsilon \} \cup \{1, \ldots, p - 1\}\{0, \ldots, p - 1\}^* \text{ or } L = \{ \varepsilon \} \cup 1\{0, 01\}^* \]
Abstract Numeration Systems

Notation

If $S = (\mathcal{L}, \Sigma, <)$ is an abstract numeration system and if $M_L = (Q_L, q_{0,L}, \Sigma, \delta_L, F_L)$ is the minimal automaton of $L$, we denote by $u_j(q)$ (resp. $v_j(q)$) the number of words of length $j$ (resp. $\leq j$) accepted from $q \in Q_L$ in $M_L$. 
Abstract Numeration Systems

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If $S = (L, \Sigma, <)$ is an abstract numeration system and if $M_L = (Q_L, q_0, L, \Sigma, \delta_L, F_L)$ is the minimal automaton of $L$, we denote by $u_j(q)$ (resp. $v_j(q)$) the number of words of length $j$ (resp. $\leq j$) accepted from $q \in Q_L$ in $M_L$.

**Remark**

The sequences $(u_j(q))_{j \geq 0}$ (resp. $(v_j(q))_{j \geq 0}$) satisfy the same homogenous linear recurrence relation for all $q \in Q_L$. 
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If $S = (L, \Sigma, <)$ is an abstract numeration system and if $\mathcal{M}_L = (Q_L, q_{0,L}, \Sigma, \delta_L, F_L)$ is the minimal automaton of $L$, we denote by $u_j(q)$ (resp. $v_j(q)$) the number of words of length $j$ (resp. $\leq j$) accepted from $q \in Q_L$ in $\mathcal{M}_L$.

Remark
The sequences $(u_j(q))_{j \geq 0}$ (resp. $(v_j(q))_{j \geq 0}$) satisfy the same homogenous linear recurrence relation for all $q \in Q_L$.

Lemma
Let $w = \sigma_1 \cdots \sigma_n \in L$. We have

$$\text{val}_S(w) = \sum_{q \in Q_L} \sum_{i=1}^{\left|w\right|} \beta_{q,i}(w) u_{\left|w\right|-i}(q)$$  \hspace{1cm} (1)

where $\beta_{q,i}(w) := \# \{ \sigma < \sigma_i \mid \delta_L(q_{0,L}, \sigma_1 \cdots \sigma_{i-1}\sigma) = q \} + 1_{q,q_{0,L}}$, for $i = 1, \ldots, \left|w\right|$.
Abstract Numeration Systems

**Definition**
A set $X \subseteq \mathbb{N}$ of integers is *$S$-recognizable* if the language $\text{rep}_S(X)$ over $\Sigma$ is regular (i.e., accepted by a finite automaton).
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A set $X \subseteq \mathbb{N}$ of integers is $S$-recognizable if the language $\text{rep}_S(X)$ over $\Sigma$ is regular (i.e., accepted by a finite automaton).

Proposition
Let $S = (L, \Sigma, <)$ be an abstract numeration system built over an infinite regular language $L$. Any ultimately periodic set $X$ is $S$-recognizable and a DFA accepting $\text{rep}_S(X)$ can be effectively obtained.
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Problem
Given an abstract numeration system $S$ and a $S$-recognizable set $X \subseteq \mathbb{N}$. Is it decidable whether or not $X$ is ultimately periodic?
A Decision Procedure

Theorem

Let $S = (L, \Sigma, <)$ be an abstract numeration system and let $\mathcal{M}_L = (Q_L, q_0, L, \Sigma, \delta_L, F_L)$ the trim minimal automaton of $L$. Assume that

$$\forall q \in Q_L \lim_{j \to \infty} u_j(q) = +\infty;$$

$$\forall j \geq 0 \ u_j(q_0, L) > 0.$$

Assume moreover that $v = (v_i(q_0, L))_{i \geq 0}$ satisfies a linear recurrence relation of the form

$$U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i, \ i \geq 0$$

with $k \geq 1$, $a_1, \ldots, a_k \in \mathbb{Z}$ and $a_k = \pm 1$.

It is decidable whether or not a $S$-recognizable set is ultimately periodic.