Attainability of maximum work and the reversible efficiency from minimally nonlinear irreversible heat engines

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Abstract
We use the general formulation of irreversible thermodynamics and study the minimally nonlinear irreversible model for heat engines operating between a time varying hot heat source of finite size and a cold heat reservoir of infinite size. We explicitly calculate the condition for obtaining optimized work output for this model once the system reaches the final thermal equilibrium state with that of the cold heat reservoir. We find that our condition resembles with the generalized condition to achieve an optimized work output for generalized irreversible heat engines in the nonlinear regime [Y. Wang, Phys. Rev. E 93, 012120 (2016)]. We also find that the optimized efficiency obtained by this minimally nonlinear irreversible heat engine can reach the reversible efficiency under the tight coupling condition in which there is no heat leakage between the system and the reservoirs. Under this condition, we find that the reversible efficiency is obtain for any finite time interval with arbitrary power. We also calculate the efficiency at maximum power from the minimally nonlinear irreversible heat engine under the non-tight coupling condition. We find that the efficiency at maximum power is equal to the half of the reversible efficiency and the corresponding maximum work is half of the exergy for a specific choice of the heat leakage term. Our result matches exactly with the efficiency and the work at maximum power obtained in Ref. [Y. Izumida and K. Okuda, Phys. Rev. Lett. 112, 180603 (2014)] for the exergy study of linear irreversible heat engines under the tight coupling condition. Our study also shows that the efficiency and the work at maximum power obtained from the linear irreversible heat engines under the tight coupling is a special case of the efficiency at maximum power obtained from the minimally nonlinear irreversible heat engine under the non-tight coupling condition.

1 Introduction

The theory of irreversible thermodynamics [1, 2, 3] nowadays attracts more interest towards the formulation of a new theoretical framework as well as experimental studies of the biological systems and bio-inspired artificial nanosystems [4, 5]. Most of these systems are highly nonlinear and working under the general
principle of a heat engine operating in nonequilibrium conditions [6]. A heat engine is a thermodynamic system operating between two heat reservoirs which consumes heat $Q_h$ from the hot heat source at a given temperature $T_h$ and converts part of it into useful work $W$ and the remaining heat $Q_c$ is delivered into the cold heat reservoir at a given temperature $T_c$.

Traditional studies of heat engine are based on the reversible thermodynamics formulation of a linear system operating between the hot and cold reservoirs of infinite size. For an irreversible thermodynamics, most of the studies on heat engine are formulated for linear system operating between the hot and cold heat source of infinite size [7, 8]. These studies are focused mainly on obtaining the efficiency $\eta = \frac{W}{Q_h}$, at maximum power in a finite time and its universality behavior $\eta_U \equiv \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + O(\eta^3)$, [9] [10] [11] [12] where $\eta_C = 1 - T_c/T_h$ is the Carnot efficiency of the reversible heat engine with zero power. The Carnot efficiency, also called as the reversible efficiency, is the maximum efficiency that can be obtained in the quasi static process taking an infinite time for completion.

For an optimized thermal engine in the endoreversible limit, the efficiency at maximum power is given by $\eta_{CA} = 1 - \sqrt{1 - \eta_C}$ [13] [14] usually called as the Curzon-Ahlborn efficiency. When the temperature difference between two reservoirs are small, the Taylor expansion of $\eta_{CA}$ gives $\eta_U$ [15] [16] which is bounded below the Carnot efficiency of the reversible heat engines. It has been shown that, the efficiency at maximum power does not show universality behavior even in the linear response region of certain systems [6]. Since the efficiency of reasonably larger values obtained by the practical heat engines are not working in the regime of maximum power output [17], a recent study showed that universal bounds on efficiency can also be derived for an arbitrary power [18].

The general theory of linear irreversible heat engines working between a finite sized hot heat source and an infinite sized cold reservoir has been formulated recently [19]. This formulation was based on the extraction of maximum work called Exergy [20] obtained from the finite sized hot heat source of time dependent temperature $T$ until the system reaches the final equilibrium state of cold reservoir. More general formulation of optimized maximum work output and the universal feature of the efficiency at maximum power for the irreversible heat engines operated between finite sized heat reservoirs beyond linear regime has been studied very recently [21]. It has been shown that the reversible efficiency can never be reached at finite power for linear irreversible systems [22]. This may raise the question whether it can be reachable for a nonlinear irreversible system at finite power [21]. In order to answer this question we have taken the minimally nonlinear irreversible thermodynamic model [23] [24] in our study.

Even though the minimally nonlinear model can be used to obtain various results derived earlier [25] [26] this model has been criticized in Ref. [27] since the dissipation term present explicitly in this model appears naturally in the linear irreversible Thermodynamic models. Apart from finding the condition for obtaining the reversible efficiency, we try to overcome this criticism by considering this model in exergy calculation. Although this model has been studied partially in Ref. [21] by using perturbation method, we use this model in our exergy study and explicitly calculate the condition to obtain the optimized work and the maximum efficiency without using any approximation methods. We find that the condition derived resembles with the Eq.(26) of Ref. [21]. This may
not be obtained in the linear system even for the tight coupling condition in which there is no heat leakage between the system and the reservoirs [19, 28].

This paper has been organized as follows. In section 2, we introduce the minimally nonlinear irreversible model for exergy calculation. In section 3, we incorporate thermodynamical optimization procedure and calculate the optimized efficiency under the tight coupling condition. We also calculate the efficiency at maximum power under the non-tight coupling condition in section 4 and finally conclude with the main results.

2 Minimally nonlinear irreversible model

Incorporating the Onsager relation in the study of heat engines [7, 29], a minimal model for a nonlinear heat engine has been introduced by Izumida and Okuda [23, 24] which is given by

\[ J_1 = L_{11}X_1 + L_{12}X_2, \]
\[ J_2 = L_{21}X_1 + L_{22}X_2 - r_h J_1^2, \]

where \( J_i \) is thermodynamic flux and \( X_i \) is its conjugate thermodynamic force defined as

\[ J_1 \equiv \dot{x}, \]
\[ X_1 \equiv F/T, \]
\[ J_2 \equiv \dot{Q}_h, \]
\[ X_2 \equiv \frac{1}{T_c} - \frac{1}{T}, \]

and \( L_{ij}'s \) are the Onsager coefficients with the reciprocity relation \( L_{12} = L_{21} \) [19]. For the nonnegativity of the entropy production rate [19, 23], the possible values of \( L_{ij} \) are restricted as \( L_{11} \geq 0, L_{22} \geq 0 \) and \( L_{11}L_{22} - L_{12}L_{21} \geq 0 \). The non linear term \( r_h J_1^2 \) was introduced to account for the dissipation effect with \( r_h > 0 \) in the Onsager relation [7, 23]. In the above equations, \( F \) denotes the time independent external generalized force and \( \dot{x} \) denotes the time derivative of its conjugate variable \( x \). The terms \( \dot{Q}_h, Q_c \) and \( W \) are the time derivatives of \( Q_h, Q_c \) and \( W \) respectively.

Although the dissipation effects due to friction on the heat devices have not been taken into account in the minimally nonlinear irreversible model, recent study showed that a clear interpretation of the global performance of generic heat devices has been obtained by this model [25]. For more detail studies on this model see, Refs. [23, 24, 25, 26].

In order to find out the optimized efficiency, we use the extended Onsager relation as described in Eqs. (1) and (2) and study the exergy of nonlinear irreversible heat engines operating between a time varying hot heat source of finite size and a cold heat reservoir of infinite size. The system finally reaches the thermal equilibrium state with a uniform temperature of the cold heat reservoir. We calculate the condition for obtaining the optimized efficiency as follows.

The time varying hot heat source, initially in equilibrium at temperature \( T_h \), is assumed to be always in equilibrium for any other temperature \( T \) at later
The heat capacity at constant volume at any temperature is $C_v = C_v(T)$ and the initial internal energy and entropy are $U_h$ and $S_h$ respectively. When the hot heat source approaches the final temperature of the cold reservoir in a time interval $0$ to $\tau$ one can calculate the total work extracted by the heat engine as $W = \int dW = \int \eta T dQ_h$ where $dQ_h$ is the infinitesimal heat that can be transformed into the infinitesimal work $dW$ with the efficiency $\eta T$ at each $T$. This work can be bounded by the Carnot efficiency $\eta_c^T = 1 - T_c/T$ at each $T$ which is given by [19]

$$W \leq \int \eta_c^T dQ_h = - \int_{T_h}^{T_c} \eta_c^T C_v dT = (U_h - U_c) - T_c(S_h - S_c) \equiv E,$$  

where

$$U_h - U_c \equiv \int_{T_c}^{T_h} C_v dT,$$  

$$S_h - S_c \equiv \int_{T_c}^{T_h} \frac{C_v}{T} dT,$$  

$U_c$ and $S_c$ are respectively the internal energy and entropy of the final equilibrium state of the hot heat source and $E$ is the maximum work called as the exergy. The corresponding efficiency $\eta = W/Q_h = W/(U_h - U_c)$ is bounded below the maximum value as [19]

$$\eta \leq \frac{E}{U_h - U_c} = 1 - \frac{T_c(S_h - S_c)}{U_h - U_c} \equiv \eta_{\text{max}},$$  

where $\eta_{\text{max}}$ is the maximum efficiency attained by the engine. We call $\eta_{\text{max}}$ as the reversible efficiency which can be obtained naturally for any reversible heat engines operating quasi statically and taking an infinite time to complete the process.

Let $J_3$ denotes the heat flux of the cold reservoir which is given by [19, 23]

$$J_3 \equiv \dot{Q}_c = \dot{Q}_h - \dot{W} = J_2 + J_1 X_1 T_c.$$  

Using Eq.(1) one can obtain $X_1 = (J_1 - L_{12} X_2)/L_{11}$, then Eq.(2) and $J_3$ can be rewritten as

$$J_2 = \frac{L_{21}}{L_{11}} J_1 + L_{22}(1 - q^2) X_2 - r_h J_1^2,$$  

$$J_3 = \frac{L_{21} T_c}{L_{11} T} J_1 + L_{22}(1 - q^2) X_2 + r_c J_1^2,$$  

where $r_c = \frac{T_c}{T_h} - r_h$ and $q = \frac{L_{21} T_c}{L_{11} L_{22}}$ with $|q| \leq 1$ is the coefficient of the coupling strength [23]. Under the condition $|q| = 1$ called as the tight coupling, the second term $L_{22}(1 - q^2) X_2$ known as the heat leakage from the hot heat source to the cold heat reservoir vanishes [19, 23].
By using the above relations the entropy production rate, $\dot{S} = -\frac{J_1}{T} + \frac{J_3}{T_c}$ can be written as 
\[ \dot{S} = L_{22}(1 - q^2)X_2^2 + \left( \frac{r_h}{T} + \frac{r_c}{T_c} \right) J_1^2 \geq 0. \] (13)

Since \( r_h > 0 \) and also the first term in the above equation is greater than or equal to zero, one can naturally make an assumption that \( r_c > 0 \) such that which should ensure the non negativity of the entropy production. In our study, we did not make such an assumption that the value of \( r_c \) should be greater than zero. However, in order to make the positive entropy production rate, we impose the condition
\[ \left( \frac{r_h}{T} + \frac{r_c}{T_c} \right) \geq 0. \] (14)

This condition can be useful for making the correspondence between the minimally nonlinear heat engine model and the thermoelectric heat devices with zero magnetic field (see Eqs.(31 & 36) of Ref.[27]). In such a case the first and second term in Eq.(13) can be linked respectively with the heat bypass and the Joule heating which are always positive [23, 27]. Since \( r_c = \frac{L_{11}}{L_{11}} - r_h \), the above condition becomes,
\[ \frac{1}{X_{11}} - r_h \left( \frac{1}{T} - \frac{1}{T} \right) \geq 0. \] For time varying hot heat source, the above condition can be rewritten as
\[ X_2 L_{11} r_h \leq 1. \] (15)

Under this condition, the entropy production rate becomes zero when \( X_2 L_{11} r_h = 1 \) and \( |q| = 1 \).

The rate of decrease of temperature \( T \) of the hot heat source when the heat engine operates from the initial temperature \( T_h \) to the final temperature \( T_c \) is given by [19, 21]
\[ J_2 = -C_v \dot{T}. \] (16)

The above equation also provides the relation that connects the temperature \( T \) to the time \( t \) with \( \dot{T} = \frac{dT}{dt} \neq 0 \) in general. Then Eq. (11) can be written as
\[ r_h J_1^2 - \frac{L_{21}}{L_{11}} J_1 - L_{22}(1 - q^2)X_2 - C_v \dot{T} = 0. \] (17)

The above equation can be written simply as
\[ r_h J_1^2 - a_0 J_1 - g - C_v \dot{T} = 0. \] (18)

where \( g = L_{22}(1 - q^2)X_2 \) is the heat leakage term and \( a_0 = \frac{L_{21}}{L_{11}} \). In terms of \( g \) and \( a_0 \), Eq. (12) can be written as
\[ J_3 = \frac{T_c}{T} a_0 J_1 + g + \left( \frac{T_c}{L_{11}} - r_h \right) J_1^2. \] (19)

Using Eq. (18) in the above equation for \( J_1^2 \) and after simplification one can get,
\[ J_3 = a_0(\beta - X_2 T_c) J_1 + \beta g + (\beta - 1) C_v \dot{T}, \] (20)
where $\beta = \frac{T_c}{r_{11} r_h} = \frac{T_c}{r_{11} r_h} + 1$ [26]. Here $r_c/r_h$ is the ratio of power dissipation between the cold and hot reservoirs. Using Eq. (15), $\beta = \frac{X_2 T_c}{X_2 L_{11} r_h}$ can takes value $\geq X_2 T_c$ and equality holds when $X_2 L_{11} r_h = 1$.

Since Eq. (18) is quadratic in $J_1$, it has two roots $J_1^+$ and $J_1^-$ which are given by

$$J_1^\pm = \frac{a_0}{2r_h} \left[ 1 \pm \sqrt{1 + \frac{4r_h}{a_0^2}(g + C_v \dot{T})} \right].$$

(21)

where $a_1 = 2r_h/a_0^2$. We consider only the physically acceptable solution of $J_1^-$ and discarded the other solution $J_1^+$ since $J_1^+ \neq 0$ as $J_2 = 0$ [21]. Using $J_1 = J_1^-$, Eq. (20) can be expressed as a function of $T$ and $\dot{T}$ as $J_3(T, \dot{T}) = a_0(\beta - X_2 T_c) J_1^- + \beta g + (\beta - 1)C_v \dot{T}$. The above equation can be rewritten as

$$J_3(T, \dot{T}) = k \left[ 1 - \sqrt{p} \right] + \beta g + (\beta - 1)C_v \dot{T}.$$  

(23)

Here, we have taken $k = \frac{(\beta - X_2 T_c)}{a_1^2(T)}$ and $p = 1 + 2a_1(g + C_v \dot{T})$ for notational convenience. In our further calculation, we assume that $\beta, a_0, a_1, C_v$ and the Onsager coefficients depends only on the temperature. Therefore, the leakage term $g$ and $k$ depends only on $T$ but $p$ depends on both $T$ and $\dot{T}$. Thus, $k(T) = \frac{\beta(T) - X_2 T_c T_c}{a_1^2(T)}$ and $p(T, \dot{T}) = 1 + 2a_1(T)(g(T) + C_v(T) \dot{T})$.

3 Thermodynamic Optimization

The heat $Q_h$ and the work output $W$ can obtain from $J_2$ and $J_3$ in the time interval 0 to $\tau$ is [19]

$$Q_h = \int_0^\tau J_2(t) dt = \int_0^{T_c} C_v dT = U_h - U_c,$$

(24)

$$W = \int_0^\tau \dot{W}(t) dt = U_h - U_c - \int_0^\tau J_3(t) dt,$$

(25)

where $\dot{W} = \frac{dW}{dt} = J_2 - J_3$. Hence, the total power $P$ and the efficiency $\eta$ can be obtained as [19]

$$P = \frac{W}{\tau} = \frac{U_h - U_c - \int_0^\tau J_3(t) dt}{\tau},$$

(26)

$$\eta = \frac{W}{Q_h} = 1 - \frac{\int_0^\tau J_3(t) dt}{U_h - U_c}.$$  

(27)

In order to maximize the work and hence obtain the maximum efficiency, we express $J_3(t)$ as a function of $T$ and $\dot{T}$ as in Eq. (23) and then minimize
the integral \( \int_0^\tau J_3(T, \dot{T}) \, dt \) in the above equation by solving the following Euler-Lagrange equation for \( T(t) \) \[19, 21\]

\[
\frac{d}{dt} \left( \frac{\partial J_3(T, \dot{T})}{\partial \dot{T}} \right) - \frac{\partial J_3(T, \dot{T})}{\partial T} = 0. \tag{28}
\]

After solving the above equation, we have obtained the optimization condition as (see, appendix),

\[
\frac{d}{dt} \left( \frac{\partial Y}{\partial \dot{T}} \right) - \frac{\partial Y}{\partial T} = 0, \tag{29}
\]

where \( Y(T, \dot{T}) = k[1 - \sqrt{p}] + \beta g \). Multiply throughout by \( \dot{T} \) in the above equation, we obtain

\[
\frac{d}{dt} \left( \dot{T} \frac{\partial Y}{\partial \dot{T}} - Y \right) = 0. \tag{30}
\]

After integrating the above equation one can get

\[
\dot{T} \frac{\partial Y(T, \dot{T})}{\partial \dot{T}} - Y(T, \dot{T}) = A, \tag{31}
\]

where \( A \) is a \( \tau \) dependent integration constant. As similar to Eq. (26) of Ref. \[21\], for any coupling strength \( |q| \leq 1 \), we have obtained the necessary condition to achieve an optimized work output from the minimally nonlinear irreversible model of heat engines. Since Eq. \[31\] can not be obtained in the linear regime \[21\], our study ensures that the minimally nonlinear irreversible model can be considered as a simplest and suitable model for studying heat engines in the nonlinear regime even though the dissipation term is present explicitly in this model as criticized earlier \[27\].

It should be noted that the condition in terms of \( Y(T, \dot{T}) \) obtained in our study is not for the entropy production rate as given in Ref. \[21\]. Eq. \[31\] is a highly nonlinear implicit differential equation \[30\]. it may be difficult to simplify this equation for further analysis and hence we do not try any other optimization \[31\] to minimize the integral \( \int_0^\tau J_3(t) \, dt \) further. Under this optimization condition Eq. \[23\] becomes

\[
J_3(T, \dot{T}) = \dot{T} \frac{\partial Y(T, \dot{T})}{\partial \dot{T}} - A + (\beta - 1)C_v\dot{T} \\
= -\frac{k}{\sqrt{p}}a_1C_v\dot{T} - A + (\beta - 1)C_v\dot{T}.
\]

The condition for positive entropy production rate (Eq.\[15\]) can be written in terms of \( \beta \) as \( X_2 L_{11} r_h = \frac{X_2 T_c}{\beta} \leq 1 \), then

\[
\beta \geq X_2 T_c. \tag{32}
\]

For the lowest value of \( \beta = X_2 T_c \), \( k = \frac{(\beta - X_2 T_c)}{a_1} = 0 \) and hence from Eqs. \[56\] and \[31\] with \( Y(T, \dot{T}) = k[1 - \sqrt{p}] + \beta g \) we get

\[
A = -\frac{k}{\sqrt{p}}a_1C_v\dot{T} - \left( k[1 - \sqrt{p}] + \beta g \right) \tag{33}
\]
Then, the optimized flux is given by
\[ J_3(T, \dot{T}) = X_2 T_c g + (X_2 T_c - 1) C_v T_c. \] (34)

The above equation has been obtained by optimizing \( J_3 \) with \( \beta = X_2 T_c \) for any value of the coupling strength \( |q| \leq 1 \). For this minimum value of \( \beta \) the entropy production should be independent of time under the tight coupling condition, \( |q| = 1 \). In this condition the leakage term \( g = 0 \) and the equation (34) becomes
\[ J_3(T, \dot{T}) = (X_2 T_c - 1) C_v T_c = \frac{T_c}{T} C_v T_c. \] (35)

Integrating the above equation from 0 to \( \tau \) and using Eq.(9), we get
\[ \int_0^\tau J_3(T, \dot{T}) dt = -T_c \int_0^T \frac{C_p}{C_v} dT = T_c (S_h - S_c). \]
By using Eq.(27), the optimized efficiency \( \eta \) can be obtained as
\[ \eta = 1 - \frac{\int_0^\tau J_3(T, \dot{T}) dt}{U_h - U_c} \]
\[ = 1 - \frac{T_c (S_h - S_c)}{U_h - U_c} \]
\[ = \eta_{\text{max}}. \] (36)

Thus, the reversible efficiency has been obtained from the minimally nonlinear irreversible heat engine under the tight coupling condition. In the case of \( C_v \to \infty \), for an isothermal environment \( \eta_{\text{max}} \) recovers the usual Carnot efficiency \( \eta_C \) by the definition [19] \( \frac{U_h - U_c}{T_h} = \frac{Q_h}{T_h} = S_h - S_c \). The maximum work (Exergy) extracted and also the total power obtained from this nonlinear irreversible heat engine are obtained as
\[ W = U_h - U_c - T_c (S_h - S_c) \equiv E, \] (37)
\[ P = \frac{W}{\tau} = \frac{E}{\tau}. \] (38)

Our result showed that the reversible efficiency obtained from the nonlinear irreversible heat engines under the tight coupling condition is not necessarily to be in the regime of maximum or zero power output.

Even though various studies on heat engines favored the attainability of Carnot efficiency at finite power [32], it has been ruled out for large classes of systems which are in the linear response regime [3, 22, 33]. Based on non-zero entropy production rate, it was recently proved that Carnot efficiency at finite power is impossible for (a Markov process description of) a general thermodynamic system even in the nonlinear regime [34]. However, in our work, we have achieved the reversible (Carnot) efficiency at finite power as a special case of zero entropy production rate. Although our result is entirely based on the positive entropy production rate condition \( X_2 L_{11} r_h \leq 1 \) (Eq.15) or \( \beta \leq X_2 T_c \) (Eq.32), when tight coupling condition is considered, zero entropy production rate is determined by Eq.(15) holding an equality \( X_2 L_{11} r_h = 1 \) (\( \beta = X_2 T_c \)). This equation relates the dissipation constant \( r_h \) with the rate of decrease of the temperature of the hot finite reservoir, which in turn is related with the size of the reservoir and with the time scale of the heat exchange process.

The Onsager symmetry used in our analysis reduces the generality of the present result and it is valid only for the steady state heat engines. However,
for cyclic heat engines, this simplification is permitted only under the condition that the driving protocols are symmetric under time-reversal [3]. Our result (Eq. 36) showed that if we design a practical heat engine whose positive entropy production does not change with time, one can achieve Carnot efficiency at finite power. We may call this equality as steady entropy production condition if there is no heat leakage between the system and the reservoirs. Using the above equality, in the following section, we try to calculate the efficiency at maximum power from the minimally nonlinear irreversible heat engines under the non-tight coupling condition.

4 Efficiency at maximum power under the non-tight coupling condition

Under the non-tight coupling condition, \(|q| \neq 1\) and hence the leakage term becomes non-zero \((g \neq 0)\) for \(L_{ij} > 0\). Since the integration constant \(A = \beta g\) as obtained from Eq. (33) also depends on \(\tau\) and \(\beta = X_2 T_c\) is a function of \(T\) alone, one can expect \(g\) should also depends on \(\tau\) for a given \(\beta\). For the simplest choice, we take \(g = B / (\beta \tau^2)\) where \(B\) is a constant and using this value of \(g\neq 0\) in Eq. (34), we get

\[
J_3(T, \dot{T}) = \frac{B}{\tau^2} + (X_2 T_c - 1) C_v \dot{T}.
\]  

(39)

Integrating the above equation from 0 to \(\tau\) and using Eqs. (7), (9) and (26) we obtain

\[
\int_0^\tau J_3(T, \dot{T}) dt = \frac{B}{\tau} \int_0^\tau dt - T_c \int_0^T \frac{C_v}{T} dT.
\]

(40)

and the total power

\[
P = \frac{1}{\tau} \left( U_h - U_c - \frac{B}{\tau} - T_c (S_h - S_c) \right).
\]

(41)

It should be noted that the total power goes to zero in the quasi static limit \(\tau \rightarrow \infty\). In order to find out the value of \(\tau = \tau^*\) in which the total power is maximum, one can maximize Eq. (41) with respect to \(\tau\) as

\[
\frac{dP}{d\tau} = -\frac{E}{\tau^2} + \frac{2B}{\tau^3} = 0
\]

(42)

and obtain

\[
\tau^* = \frac{2B}{E}.
\]

(43)

With this value of \(\tau^*\), we obtain the maximum power

\[
P^* = \frac{E^2}{4B}.
\]

(44)
Using Eqs. (10) and (44), we obtain the work output and the efficiency at maximum power under the non-tight coupling condition as

\[ W^* = P^* r^* = \frac{E}{2}, \quad \text{(45)} \]

\[ \eta^* = \frac{W^*}{U_h - U_c} = \frac{1}{2} \eta_{\text{max}}. \quad \text{(46)} \]

This result shows that the efficiency at maximum power is equal to half of the reversible efficiency and the corresponding maximum work is half the exergy. Our final result is exactly the same as the one obtained earlier for the study of exergy [19] in the case of linear irreversible heat engines under the tight coupling condition. This shows that the efficiency and the work at maximum power obtained from the linear irreversible heat engines under the tight coupling [19] is a special case of the efficiency at maximum power obtained from the minimally nonlinear irreversible heat engine under the non-tight coupling condition for a specific value of \( g \).

The above result was obtained by using the equality \( X_2 L_{11} r_h = 1 \) (\( \beta = X_2 T_c \equiv \frac{1}{r_c} + 1 \) [26]). This indicates that the mapping of linear irreversible model by the nonlinear one can be possible if the ratio of power dissipation (\( r_c r_h \)) [25, 26] between the cold and hot reservoirs is related with the heat leakage between the system and the reservoirs by the relation \( g = B/(\beta r^2) \).

5 Conclusion

Using the general formulation of the irreversible thermodynamics we studied the optimized work and the efficiency of minimally nonlinear irreversible heat engines operating between finite sized hot and infinite sized cold reservoirs. We obtained the necessary condition to achieve an optimized work output. Our condition obtained in the case of minimally nonlinear irreversible model resembles with the one obtained recently [21] for the generalized study of the irreversible heat engines in the nonlinear regime. This condition can not obtained in linear irreversible models [21]. Our result supports the fact that the minimally nonlinear irreversible model can be considered as a simplest and consistent model to study heat engines in the nonlinear regime.

We used the optimization condition, Eq. (31), and calculated the maximum work and efficiency of the minimally nonlinear irreversible heat engines. Earlier studies for the irreversible heat engines showed that the tight coupling condition serves as an upper bound of the efficiency at maximum power. Even though the universal feature in the efficiency at maximum power has been obtained for the linear irreversible heat engine, the reversible efficiency can not be reached at finite power even for the tight coupling condition which in general provides the upper bound of the efficiency at the maximum power. Interestingly, our result showed that the reversible efficiency can be achieved at finite power for nonlinear irreversible heat engine under the tight coupling condition. Our results also showed that the reversible efficiency obtained from the nonlinear irreversible heat engines in the tight coupling condition is not necessarily to be in the regime of maximum or zero power output.
We have calculated the efficiency at maximum power from the nonlinear irreversible heat engine under the non-tight coupling condition for a specific value of $g$ and found that the efficiency at maximum power is equal to the half of the reversible efficiency and the corresponding maximum work is half the exergy. This result is exactly same as the efficiency and the work at maximum power obtained from the linear irreversible heat engines under the tight coupling condition [19].

Our result showed that the reversible efficiency at finite power is theoretically possible for heat engines working in the nonlinear regime. The validity of this result is based mainly on the assumption of presence of symmetry in the Onsager coefficient. Our future work will focus the alteration of the present analysis for the non-symmetric Onsager coefficient and the possibility of designing a heat engine to satisfy the steady entropy production condition.

Appendix:

$$k(T) = \frac{\beta(T) - X(\hat{T})}{a_1(T)}$$ and

$$p(T, \hat{T}) = 1 + 2a_1(T)(g(T) + C_v(T)\hat{T}).$$

The partial differentiation of $p$ with respect to $T$ and $\hat{T}$ is given by

$$\frac{\partial p}{\partial \hat{T}} = 2a_1C_v$$

$$\frac{\partial p}{\partial T} = 2 \frac{\partial (a_1g)}{\partial T} + 2\hat{T} \frac{\partial (a_1C_v)}{\partial T}. \quad (47)$$

By using Eqs. (23-48) one can calculate

$$\frac{\partial J_3}{\partial T} = \frac{\partial}{\partial T} \left( k[1 - \sqrt{p}] + \beta g \right)$$

$$+ T \frac{\partial}{\partial T} \left( (\beta - 1)C_v \right). \quad (49)$$

$$\frac{\partial J_3}{\partial \hat{T}} = -k \frac{a_1C_v}{\sqrt{p}} + (\beta - 1)C_v. \quad (50)$$

$$\frac{\partial}{\partial T} \left( \frac{\partial J_3}{\partial \hat{T}} \right) = \frac{k}{p^{3/2}} a_1^2 C_v^2. \quad (51)$$

$$\frac{\partial}{\partial \hat{T}} \left( \frac{\partial J_3}{\partial T} \right) = \frac{\partial}{\partial T} \left( -k \frac{a_1C_v}{\sqrt{p}} \right)$$

$$+ \frac{\partial}{\partial T} \left( (\beta - 1)C_v \right). \quad (52)$$

For optimization Eq. (28) can be rewritten in terms of $T(t)$ as

$$\hat{T} \frac{\partial}{\partial T} \left( \frac{\partial J_3}{\partial \hat{T}} \right) + \hat{T} \frac{\partial}{\partial T} \left( \frac{\partial J_3}{\partial T} \right) - \frac{\partial J_3}{\partial T} = 0. \quad (53)$$
By using Eqs. (49-52) in the above equation one can obtain

\[ \ddot{T} \frac{k a_1^2 C_v^2}{p^{1/2}} + \dot{T} \frac{\partial}{\partial T} \left( - \frac{k}{\sqrt{p}} a_1 C_v \right) - \frac{\partial}{\partial T} \left( k[1 - \sqrt{p}] + \beta g \right) = 0. \]  

(54)

Since

\[ \frac{d}{dt} \left( - \frac{k}{\sqrt{p}} a_1 C_v \right) = \ddot{T} \frac{k a_1^2 C_v^2}{p^{1/2}} + \dot{T} \frac{\partial}{\partial T} \left( - \frac{k}{\sqrt{p}} a_1 C_v \right) \]  

(55)

and

\[ \frac{\partial}{\partial T} \left( k[1 - \sqrt{p}] + \beta g \right) = - \frac{k}{\sqrt{p}} a_1 C_v, \]  

(56)

Eq. (54) can be rewritten as

\[ \frac{d}{dt} \left( \frac{\partial Y}{\partial \dot{T}} \right) - \frac{\partial Y}{\partial T} = 0, \]  

(57)

where \( Y(T, \dot{T}) = k[1 - \sqrt{p}] + \beta g \). We also get the same type of Eq. (57) for the other value of \( J_1 = J_1^\pm \) with \( Y(T, \dot{T}) = k[1 + \sqrt{p}] + \beta g \). Therefore for two different values of \( J_1 = J_1^\pm \), one can get the same equation.

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