1. Introduction

In this paper we consider the eigenvalue problem, intersection theory of homogeneous spaces (in particular, the Horn problem) and the saturation problem for the symplectic and odd orthogonal groups. The classical embeddings of these groups in the special linear groups will play an important role. We deduce properties for these classical groups from the known properties for the special linear groups. The tangent space techniques from [B₁] play a crucial role. Another crucial ingredient is the relationship between the intersection theory of the homogeneous spaces for Sp(2ⁿ) and SO(2ⁿ + 1).

Let \( G \) be a semisimple, connected, complex algebraic group with a maximal compact subgroup \( K \). Let \( \mathfrak{h}^+ \) be the positive Weyl chamber of \( G \). Then, there is a bijection \( C : \mathfrak{h}^+ \rightarrow \mathfrak{k}/K, \quad x \mapsto ix \), where \( \mathfrak{k} \) is the Lie algebra of \( K \). Then \( K \) acts on \( \mathfrak{k} \) by the adjoint representation and \( ix \) denotes the \( K \)-conjugacy class of \( ix \). Define the eigencone (for any \( s \geq 1 \)):

\[
\Gamma(s, K) = \{ (\bar{k}_1, \ldots, \bar{k}_s) \in (\mathfrak{k}/K)^s : \exists k_j \in \bar{k}_j \text{ with } \sum_{j=1}^s k_j = 0 \}.
\]

For an algebraic group homomorphism \( G \rightarrow G' \) which takes \( K \rightarrow K' \), there is an induced map \( \Gamma(s, K) \rightarrow \Gamma(s, K') \). Therefore, \( \Gamma(s, K) \) is functorial. In the case of \( \text{Sp}(2n) \hookrightarrow \text{SL}(2n) \) (similarly \( \text{SO}(2n + 1) \leftrightarrow \text{SL}(2n + 1) \)), we have stronger properties described below:

Let \( \mathfrak{h}^{\text{Sp}(2n)}_+ \) be the Cartan subalgebra for \( \text{Sp}(2n) \) and similarly for \( \text{SO}(2n + 1), \text{SL}(2n) \) and \( \text{SL}(2n + 1) \). There are natural (linear) embeddings (see Sections 2.2 and 2.3):

\[
\mathfrak{h}^{\text{Sp}(2n)}_+ \hookrightarrow \mathfrak{h}^{\text{SL}(2n)}_+, \quad \text{and} \quad \mathfrak{h}^{\text{SO}(2n+1)}_+ \hookrightarrow \mathfrak{h}^{\text{SL}(2n+1)}_+.
\]

Identify \( \mathfrak{k}/K \) with \( \mathfrak{h}^+ \) under \( C \) as above. We state our first main theorem (cf. Theorem 33):

**Theorem 1.**

(a) For \( h_1, \ldots, h_s \in \mathfrak{h}^{\text{Sp}(2n)}_+ \),

\[
(h_1, \ldots, h_s) \in \Gamma(s, \text{Sp}(2n)) \iff (h_1, \ldots, h_s) \in \Gamma(s, \text{SU}(2n)).
\]
(b) For \( h_1, \ldots, h_s \in \mathfrak{h}^{\text{SO}(2n+1)}_+ \),
\[
(h_1, \ldots, h_s) \in \Gamma(s, \text{SO}(2n+1)) \Leftrightarrow (h_1, \ldots, h_s) \in \Gamma(s, \text{SU}(2n+1)).
\]

Our proof of this theorem is a consequence of a surprising result in intersection theory described below.

1.1. Intersection theory. Let \( V \) be a \( 2n \)-dimensional complex vector space equipped with a nondegenerate symplectic form \( \langle \cdot, \cdot \rangle \). Recall that if \( A = \{a_1 < a_2 < \cdots < a_m\} \) is a subset of \([2n] := \{1, \ldots, 2n\}\) of cardinality \( m \) and \( F_{\bullet} \) a complete flag on \( \mathbb{C}^{2n} \), then, by definition, the corresponding Schubert cell \( \Omega_{A_{\ell}}(F_{\bullet}) := \{ x \in \text{Gr}(m, \mathbb{C}^{2n}) : \dim X \cap F_u = \ell \text{ for } a_\ell \leq u < a_{\ell+1}, \ell = 1, \ldots, m \} \), where \( \text{Gr}(m, \mathbb{C}^{2n}) \) is the ordinary Grassmannian of \( m \)-dimensional subspaces of \( \mathbb{C}^{2n} \).

Let \( A_1, \ldots, A^s \) be subsets of \([2n] \) each of cardinality \( m \). Let \( E_{1\bullet}, \ldots, E_{s\bullet} \) be complete isotropic flags on \( V \) in general position. The following second main theorem (cf. Theorem 10) is a key technical result that underlies the proof of the first main theorem and, in fact, is central to our paper.

**Theorem 2.** The intersection \( \cap_{j=1}^s \Omega_{A_{\ell}}(E_{j\bullet}) \) of subvarieties of \( \text{Gr}(m, V) \) is proper (possibly empty) for complete isotropic flags \( E_{j\bullet} \) on \( V \) in general position.

We have the same result for \( \text{SO}(2n+1) \) (cf. Theorem 30). Let \( V' \) be a vector space of dimension \( 2n+1 \) equipped with a nondegenerate symmetric bilinear form. Let \( A_1, \ldots, A^s \) be subsets of \([2n+1] \) each of cardinality \( m \). Let \( E_{1\bullet}, \ldots, E_{s\bullet} \) be isotropic flags on \( V' \) in general position.

**Theorem 3.** The intersection \( \cap_{j=1}^s \Omega_{A_{\ell}}(E_{j\bullet}) \) of subvarieties of \( \text{Gr}(m, V') \) is proper.

We remark that the above intersection result is false for \( \text{SO}(2n) \), since Corollary 11 can easily seen to be violated in this case even for \( m = 1 \).

1.2. The saturation problem. Any dominant weight \( \lambda \) of \( \text{SL}(2n) \) restricts to a dominant weight \( \lambda_C \) of the symplectic group \( \text{Sp}(2n) \). Similarly, any dominant weight \( \lambda \) of \( \text{SL}(2n+1) \) restricts to a dominant weight \( \lambda_B \) of the orthogonal group \( \text{SO}(2n+1) \). The following theorem (cf. Theorem 41) is proved geometrically by the method of “theta sections”. We obtain invariants in tensor products by constructing divisors in the products of isotropic flag varieties (“the theta divisor”). This technique originates in our context in [B_2]. In addition to methods from [B_2], [B_1], we use some ideas of Schofield [S] in a crucial manner.

**Theorem 4.** Let \( V_{\lambda_1}, \ldots, V_{\lambda^s} \) be irreducible representations of \( \text{SL}(2n) \) (with highest weights \( \lambda_1, \ldots, \lambda^s \) respectively) such that their tensor product has a nonzero \( \text{SL}(2n) \)-invariant. Then, the tensor product of the representations of \( \text{Sp}(2n) \) with highest weights \( \lambda_{1C}, \ldots, \lambda_{sC} \) has a nonzero \( \text{Sp}(2n) \)-invariant, where \( \lambda_{jC} \) is the restriction of \( \lambda^j \) to the Cartan subalgebra of \( \text{Sp}(2n) \).

A similar property holds for the odd orthogonal group \( \text{SO}(2n+1) \).
Theorem 5. Let $V_{\lambda_1}, \ldots, V_{\lambda_s}$ be irreducible representations of $\text{SL}(2n+1)$ such that their tensor product has a nonzero $\text{SL}(2n+1)$-invariant. Then, the tensor product of the representations of $\text{SO}(2n+1)$ with highest weights $\lambda_B^1, \ldots, \lambda_B^s$ has a nonzero $\text{SO}(2n+1)$-invariant, where $\lambda_B^j$ is the restriction of $\lambda^j$ to the Cartan subalgebra of $\text{SO}(2n+1)$.

Using the saturation theorem of Knutson-Tao [KT], together with our Theorems 4 and 1, we obtain the following improvement of the general Kapovich-Millson saturation theorem in the case of $\text{Sp}(2n)$ (cf. Theorem 43).

Theorem 6. Given dominant integral weights $\mu^1, \ldots, \mu^s$ of $\text{Sp}(2n)$, the following are equivalent:

1. For some $N \geq 1$, the tensor product of representations of $\text{Sp}(2n)$ with highest weights $N\mu^1, \ldots, N\mu^s$ has a nonzero $\text{Sp}(2n)$-invariant.
2. The tensor product of representations with highest weights $2\mu^1, \ldots, 2\mu^s$ has a nonzero $\text{Sp}(2n)$-invariant.

By carrying out a similar analysis for the odd orthogonal groups, we obtain the following (cf. Theorem 44).

Theorem 7. Given dominant integral weights $\nu^1, \ldots, \nu^s$ of $\text{SO}(2n+1)$, the following are equivalent:

1. For some $N \geq 1$, the tensor product of representations with highest weights $N\nu^1, \ldots, N\nu^s$ has a nonzero $\text{SO}(2n+1)$-invariant.
2. The tensor product of representations with highest weights $2\nu^1, \ldots, 2\nu^s$ has a nonzero $\text{SO}(2n+1)$-invariant.

1.3. Horn’s problem for symplectic and odd orthogonal groups. There are two senses in which the term “Horn’s problem” is generally applied:

A. Determination of a system of inequalities for the eigencone $\Gamma(s,K)$, which is “cohomology free” (but perhaps recursive).

B. A system of inequalities characterizing nonvanishing structure coefficients in the cohomology $H^\ast(G/P)$ with the standard cup product and also $(H^\ast(G/P), \odot_0)$, where recall that $\odot_0$ is a deformation of the cup product in the cohomology of the flag variety $G/P$ introduced in [BK]. In this paper, we only consider the question in the case of $(H^\ast(G/P), \odot_0)$. The system of inequalities can reasonably be expected to be parameterized by the cohomology $(H^\ast(L/Q), \odot_0)$, where $L$ is a Levi subgroup of $G$ and $Q \subset L$ is a parabolic subgroup. It was suggested in [BK] that nonvanishing structure coefficients in the deformed cohomology can be characterized recursively. Recently, Richmond [R] showed that the cohomology $(H^\ast(SL_n/P), \odot_0)$ of partial flag varieties is strongly recursive (not just nonvanishing of structure coefficients, but also the actual structure coefficients).

We consider (A) and (B) for the groups $G = \text{Sp}(2n)$ and $\text{SO}(2n+1)$. Theorems 4 and 5 imply (A), because the eigenvalue problem for the special linear groups is recursive by
Horn's original conjecture, proved by the combined works of Klyachko and Knutson-Tao (see [F] for a survey).

It follows from [BK] that (B) implies (A). We reduce the Horn problem (B) for 
\((H^*((G/P), \odot_0))\) for the groups \(G = \text{Sp}(2n)\) and \(\text{SO}(2n + 1)\) and maximal parabolic subgroups \(P\) to the corresponding problem for the Lagrangian Grassmannians in Sections 8 and 9 (cf. Theorems 48 and 51). Now, the Horn problem for the Lagrangian Grassmannians was solved by Purbhoo-Sottile [PS]. It should be noted that the solution of (A) obtained through the solution for (B) is different from the one obtained from Theorems 4 and 5.

We make a conjecture (cf. Conjecture 47) generalizing many of the results in the paper for a diagram automorphism of \(G\).

We establish the basic notation in Section 2 and use them through the paper often without further explanation.

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2. Notation and Preliminaries

Let \(G\) be a connected semisimple complex algebraic group. We fix a Borel subgroup \(B\) and a maximal torus \(H \subset B\). Their Lie algebras are denoted by the corresponding Gothic characters: \(g, b, h\) respectively. Let \(W = W(H)/H\) be the Weyl group of \(G\), \(N(H)\) being the normalizer of \(H\) in \(G\). Let \(R^+ \subset h^*\) be the set of positive roots (i.e., the set of roots of \(b\)) and \(\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset R^+\) the set of simple roots. Let \(\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset h\) be the set of corresponding simple coroots and \(\{s_1, \ldots, s_n\} \subset W\) be the set of corresponding simple reflections. Let \(h_+ \subset h\) be the dominant chamber defined by

\[
h_+ = \{x \in h : \alpha_i(x) \in \mathbb{R}_+ \forall \alpha_i\},
\]

where \(\mathbb{R}_+\) is the set of nonnegative real numbers. For any \(1 \leq i \leq n\), let \(\omega_i \in h^*\) denote the \(i\)-th fundamental weight defined by

\[
\omega_i(\alpha_j^\vee) = \delta_{i,j}.
\]

Let \(B \subset P\) be a (standard) parabolic subgroup with the unique Levi subgroup \(L\) containing \(H\). We denote by \(W^P\) the set of minimal-length coset representatives in the \(W_P\)-cosets \(W/W_P\), where \(W_P\) is the Weyl group of \(P\) (which is, by definition, the Weyl group of the Levi subgroup \(L\)). For any \(w \in W^P\), we have the Bruhat cell

\[
\Lambda^P_w := BwP/P \subset G/P.
\]

This is a locally closed subset of the flag variety \(G/P\) isomorphic to the affine space \(A^{\ell(w)}\), \(\ell(w)\) being the length of \(w\). Its closure is denoted by \(\bar{\Lambda}^P_w\), which is an irreducible projective variety (of dimension \(\ell(w)\)). We denote by

\[
[\bar{\Lambda}^P_w] \in H^{2(\text{dim}_C(G/P) - \ell(w))}(G/P)
\]
the cycle class of the subvariety $\overline{\Lambda}_w^P$, where $H^*(G/P)$ is the singular cohomology of $G/P$ with integral coefficients. Then, by the Bruhat decomposition, $\{[\Lambda_w^P]\}_{w \in W^P}$ is an integral basis of $H^*(G/P)$. Define the basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{h}$ dual to the basis $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{h}^*$, i.e.,

$$\alpha_j(x_i) = \delta_{i,j}.$$  

Let $X(H)_+$ be the set of dominant characters of $H$. For any $\lambda \in X(H)_+$, let $V_\lambda$ be the finite dimensional irreducible $G$-module with highest weight $\lambda$. This sets up a bijective correspondence between $X(H)_+$ and the set of isomorphism classes of finite dimensional irreducible $G$-modules. Taking the derivative, we get an embedding $X(H)_+ \hookrightarrow D$, where

$$D := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{R}_+ \forall \alpha_i^\vee\}$$

is the set of dominant weights. If $G$ is simply-connected, $X(H)_+$ can be identified with

$$D_{\mathbb{Z}} := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+ \forall \alpha_i^\vee\},$$

where $\mathbb{Z}_+$ is the set of nonnegative integers.

We now give more specific details about the groups $SL(n+1), Sp(2n)$ and $SO(2n+1)$ below (see, e.g., [BL, Chapter 3]), since they will be of special interest to us in the paper.

2.1. **Special Linear Group** $SL(n+1)$. In this case we take $B$ to be the (standard) Borel subgroup consisting of upper triangular matrices of determinant 1 and $H$ to be the subgroup consisting of diagonal matrices (of determinant 1). Then, 

$$\mathfrak{h} = \{t = \text{diag}(t_1, \ldots, t_{n+1}) : \sum t_i = 0\},$$

and

$$\mathfrak{h}_+ = \{t \in \mathfrak{h} : t_i \in \mathbb{R} \text{ and } t_1 \geq \cdots \geq t_{n+1}\}.$$  

For any $1 \leq i \leq n$,

$$\alpha_i(t) = t_i - t_{i+1}; \alpha_i^\vee = \text{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0); \omega_i(t) = t_1 + \cdots + t_i,$$

where 1 is placed in the $i$-th place.

The Weyl group $W$ can be identified with the symmetric group $S_{n+1}$ which acts via the permutation of the coordinates of $t$. Let $\{r_1, \ldots, r_n\} \subset S_{n+1}$ be the (simple) reflections corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_n\}$ respectively. Then,

$$r_i = (i, i+1).$$

For any $1 \leq m \leq n$, let $P_m \supset B$ be the (standard) maximal parabolic subgroup of $SL(n+1)$ such that its unique Levi subgroup $L_m$ containing $H$ has for its simple roots $\{\alpha_1, \ldots, \alpha_m, \ldots, \alpha_n\}$. Then, $SL(n+1)/P_m$ can be identified with the Grassmannian $\text{Gr}(m, n+1) = \text{Gr}(m, \mathbb{C}^{n+1})$ of $m$-dimensional subspaces of $\mathbb{C}^{n+1}$. Moreover, the set of minimal coset representatives $W^{P_m}$ of $W/W_{P_m}$ can be identified with the set of $m$-tuples

$$S(m, n+1) = \{A := 1 < a_1 < \cdots < a_m \leq n+1\}.$$  

Any such $m$-tuple $A$ represents the permutation

$$v_A = (a_1, \ldots, a_m, a_{m+1}, \ldots, a_{n+1}),$$
where \(\{a_{m+1} < \cdots < a_{n+1}\} = [n+1] \setminus \{a_1, \ldots, a_m\}\) and
\[
[n+1] := \{1, \ldots, n+1\}.
\]

For a complete flag \(E_* : 0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{n+1} = \mathbb{C}^{n+1}\), and \(A \in S(m, n+1)\), define the corresponding shifted Schubert cell inside \(\text{Gr}(m, n+1)\):
\[
\Omega_A(E_*) = \{M \in \text{Gr}(m, n+1) : \text{for any } 0 \leq \ell \leq m \text{ and any } a_{\ell} \leq b < a_{\ell+1}, \dim M \cap E_b = \ell\},
\]
where we set \(a_0 = 0\) and \(a_{m+1} = n+1\). Then, \(\Omega_A(E_*) = g(E_*) A_{v_A}^{F_m}\), where \(g(E_*)\) is an element of \(\text{SL}(n+1)\) which takes the standard flag \(E^o\) to the flag \(E_*\). (Observe that \(g(E_*)\) is determined up to the right multiplication by an element of \(B\).) Its closure in \(\text{Gr}(m, n+1)\) is denoted by \(\overline{\Omega}_A(E_*)\) and its cycle class in \(H^*(\text{Gr}(m, n+1))\) by \([\overline{\Omega}_A]\). (Observe that the cohomology class \([\overline{\Omega}_A]\) does not depend upon the choice of \(E_*\).) For the standard flag \(E_* = E^o\), we thus have \(\Omega_A(E_*) = A_{v_A}^{F_m}\).

**Remark 8.** Note that, in the literature, it is more common to denote the Schubert cell by \(\Omega_\sigma(E_*)\) and its closure by \(\overline{\Omega}_\sigma(E_*)\). For notational uniformity we have denoted the Schubert cell by \(\Omega_A(E_*)\), and its closure by \(\overline{\Omega}_A(E_*)\).

### 2.2. Symplectic Group \(\text{Sp}(2n)\)

Let \(V = \mathbb{C}^{2n}\) be equipped with the nondegenerate symplectic form \(\langle \cdot, \cdot \rangle\) so that its matrix \((\langle e_i, e_j \rangle)_{1 \leq i, j \leq 2n}\) in the standard basis \(\{e_1, \ldots, e_{2n}\}\) is given by
\[
E = \begin{pmatrix}
0 & J \\
-J & 0
\end{pmatrix},
\]
where \(J\) is the anti-diagonal matrix \((1, \ldots, 1)\) of size \(n\). Let
\[
\text{Sp}(2n) := \{g \in \text{SL}(2n) : g \text{ leaves the form } \langle \cdot, \cdot \rangle \text{ invariant}\}
\]
be the associated symplectic group. Clearly, \(\text{Sp}(2n)\) can be realized as the fixed point subgroup \(G^\sigma\) under the involution \(\sigma: G \to G\) defined by \(\sigma(A) = E(A^t)^{-1} E^{-1}\), where \(G = \text{SL}(2n)\). The involution \(\sigma\) keeps both of \(B\) and \(H\) stable, where \(B\) and \(H\) are as in the \(\text{SL}(2n)\) case. Moreover, \(B^\sigma\) (respectively, \(H^\sigma\)) is a Borel subgroup (respectively, a maximal torus) of \(\text{Sp}(2n)\). We denote \(B^\sigma, H^\sigma\) by \(B^C = B_{v_A}^{C_{n}}, H^C = H_{v_A}^{C_{n}}\) respectively and (when confusion is likely) \(B, H\) by \(B_{A_{2n-1}}, H_{A_{2n-1}}\) respectively (for \(\text{SL}(2n)\)). Then, the Lie algebra of \(H^C\) (the Cartan subalgebra \(\mathfrak{h}^C\))
\[
\mathfrak{h}^C = \{\text{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) : t_i \in \mathbb{C}\}.
\]
Let \(\Delta^C = \{\beta_1, \ldots, \beta_n\}\) be the set of simple roots. Then, for any \(1 \leq i \leq n\), \(\beta_i = \alpha_i|_{\mathfrak{h}^C}\), where \(\{\alpha_1, \ldots, \alpha_{2n-1}\}\) are the simple roots of \(\text{SL}(2n)\). The corresponding (simple) coroots \(\{\beta_1^\vee, \ldots, \beta_n^\vee\}\) are given by
\[
\beta_i^\vee = \alpha_i^\vee + \alpha_{2n-i}^\vee, \quad \text{for } 1 \leq i < n
\]
and
\[
\beta_n^\vee = \alpha_n^\vee.
\]
Thus,
\[ \mathfrak{h}^C_+ = \{ \text{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) : \text{each } t_i \text{ is real and } t_1 \geq \cdots \geq t_n \geq 0 \}. \]
Moreover, \( \mathfrak{h}^{A_{2n-1}}_+ \) is \( \sigma \)-stable and
\[ (\mathfrak{h}^{A_{2n-1}}_+)^\sigma = \mathfrak{h}^C_+. \]
Let \( \{ s_1, \ldots, s_n \} \) be the (simple) reflections in the Weyl group \( W^C = W^{C_n} \) of \( \text{Sp}(2n) \) corresponding to the simple roots \( \{ \beta_1, \ldots, \beta_n \} \) respectively. Since \( H^{A_{2n-1}} \) is \( \sigma \)-stable, there is an induced action of \( \sigma \) on the Weyl group \( S_{2n} \) of \( \text{Sl}(2n) \). The Weyl group \( W^C \) can be identified with the subgroup of \( S_{2n} \) consisting of \( \sigma \)-invariants:
\[ \{(a_1, \ldots, a_{2n}) \in S_{2n} : a_{2n+1-i} = 2n + 1 - a_i \forall 1 \leq i \leq 2n \}. \]
In particular, \( w = (a_1, \ldots, a_{2n}) \in W^C \) is determined from \( (a_1, \ldots, a_n) \).
Under the inclusion \( W^C \subset S_{2n} \), we have
\[ s_i = r_i r_{2n-i}, \text{ if } 1 \leq i \leq n - 1 \]
\[ = r_n, \text{ if } i = n. \]
Moreover, for any \( u, v \in W^C \) such that \( \ell^C(uv) = \ell^C(u) + \ell^C(v) \), we have
\[ \ell^{A_{2n-1}}(uv) = \ell^{A_{2n-1}}(u) + \ell^{A_{2n-1}}(v), \]
where \( \ell^C(w) \) denotes the length of \( w \) as an element of the Weyl group \( W^C \) of \( \text{Sp}(2n) \) and similarly for \( \ell^{A_{2n-1}}. \)
For \( 1 \leq r \leq n \), we let \( \text{IG}(r, 2n) = \text{IG}(r, V) \) to be the set of \( r \)-dimensional isotropic subspaces of \( V \) with respect to the form \( \langle \cdot, \cdot \rangle \), i.e.,
\[ \text{IG}(r, 2n) := \{ M \in \text{Gr}(r, 2n) : \langle v, v' \rangle = 0, \forall v, v' \in M \}. \]
Then, it is the quotient \( \text{Sp}(2n)/P^C_r \) of \( \text{Sp}(2n) \) by the standard maximal parabolic subgroup \( P^C_r \) with \( \Delta^C \setminus \{ \beta_r \} \) as the set of simple roots of its Levi component \( L^C_r \). (Again we take \( L^C_r \) to be the unique Levi subgroup of \( P^C_r \) containing \( H^C_r \).) It can be easily seen that the set \( W^C_r \) of minimal-length coset representatives of \( W^C/P^C_r \) is identified with the set
\[ \mathfrak{S}(r, 2n) = \{ I := 1 \leq i_1 < \cdots < i_r \leq 2n \text{ and } I \cap \bar{I} = \emptyset \}, \]
where
\[ \bar{I} := \{ 2n + 1 - i_1, \ldots, 2n + 1 - i_r \}. \]
Any such \( I \) represents the permutation \( \omega_I = (i_1, \ldots, i_n) \in W^C \) by taking \( \{ i_{r+1} < \cdots < i_n \} = [n] \setminus (I \sqcup \bar{I}). \)

**Definition 9.** A complete flag
\[ E_* : 0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{2n} = V \]
is called an isotropic flag if \( E_a^\perp = E_{2n-a} \), for \( a = 1, \ldots, 2n \). (In particular, \( E_n \) is a maximal isotropic subspace of \( V \).)
For an isotropic flag $E_\bullet$ as above, there exists an element $k(E_\bullet) \in \text{Sp}(2n)$ which takes the standard flag $E_\bullet^r$ to the flag $E_\bullet$. (Observe that $k(E_\bullet)$ is determined up to the right multiplication by an element of $B^C$.)

For any $I \in \mathcal{S}(r,2n)$ and any isotropic flag $E_\bullet$, we have the corresponding shifted Schubert cell inside $\text{IG}(r,V)$:

$$\Phi_I(E_\bullet) = \{ M \in \text{IG}(r,V) : \text{for any } 0 \leq \ell \leq r \text{ and any } i_\ell < a < i_{\ell+1}, \dim M \cap E_a = \ell \},$$

where we set $i_0 = 0$ and $i_{r+1} = 2n$. Clearly, set theoretically,

$$\Phi_I(E_\bullet) = \Omega_I(E_\bullet) \cap \text{IG}(r,V);$$

this is also a scheme theoretic equality (cf. Proposition 16 (4)). Moreover, $\Phi_I(E_\bullet) = k(E_\bullet)\Lambda_{\varpi_I}^{PC}$. Denote the closure of $\Phi_I(E_\bullet)$ inside $\text{IG}(r,V)$ by $\Phi_I(E_\bullet)$ and its cycle class in $H^*(\text{IG}(r,V))$ (which does not depend upon the choice of the isotropic flag $E_\bullet$) by $[\Phi_I]$.

For the standard flag $E_\bullet = E_\bullet^r$, we have $\Phi_I(E_\bullet) = \Lambda_{\varpi_I}^{PC}$.

2.3. Special Orthogonal Group $\text{SO}(2n+1)$. Let $V' = \mathbb{C}^{2n+1}$ be equipped with the nondegenerate symmetric form $\langle \cdot, \cdot \rangle$ so that its matrix $E = (\langle e_i, e_j \rangle)_{1 \leq i,j \leq 2n+1}$ (in the standard basis $\{e_1, \ldots, e_{2n+1}\}$) is the $(2n+1) \times (2n+1)$ antidiagonal matrix with 1's all along the antidiagonal except at the $(n+1, n+1)$-th place where the entry is 2. Note that the associated quadratic form on $V'$ is given by

$$Q(\sum t_i e_i) = t^2_{n+1} + \sum_{i=1}^{n} t_i t_{2n+2-i}.$$

Let

$$\text{SO}(2n+1) := \{ g \in \text{SL}(2n+1) : g \text{ leaves the quadratic form } Q \text{ invariant} \}$$

be the associated special orthogonal group. Clearly, $\text{SO}(2n+1)$ can be realized as the fixed point subgroup $C^\theta$ under the involution $\theta : G \to G$ defined by $\theta(A) = E^{-1}(A^t)^{-1}E$, where $G = \text{SL}(2n+1)$. The involution $\theta$ keeps both of $B$ and $H$ stable. Moreover, $B^\theta$ (respectively, $H^\theta$) is a Borel subgroup (respectively, a maximal torus) of $\text{SO}(2n+1)$. We denote $B^\theta, H^\theta$ by $B^B, H^B$ respectively. Then, the Lie algebra of $H^B$ (the Cartan subalgebra $\mathfrak{h}^B$)

$$\mathfrak{h}^B = \{ \text{diag}(t_1, \ldots, t_n, 0, -t_n, \ldots, -t_1) : t_i \in \mathbb{C} \}.$$ 

This allows us to identify $\mathfrak{h}^C$ with $\mathfrak{h}^B$ under the map

$$\text{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) \mapsto \text{diag}(t_1, \ldots, t_n, 0, -t_n, \ldots, -t_1).$$

Let $\Delta^B = \{ \delta_1, \ldots, \delta_n \}$ be the set of simple roots. Then, for any $1 \leq i \leq n$, $\delta_i = \alpha_i|_{\mathfrak{h}^B}$, where $\{\alpha_1, \ldots, \alpha_{2n}\}$ are the simple roots of $\text{SL}(2n+1)$. The corresponding (simple) coroots $\{\delta_1^\vee, \ldots, \delta_n^\vee\}$ are given by

$$\delta_i^\vee = \alpha_i^\vee + \alpha_{2n+1-i}^\vee, \text{ for } 1 \leq i < n$$

where $\gamma^\vee$ denotes the coroot of $\gamma$. Note that $\delta_i^\vee$ is the highest coroot among all $\delta_j^\vee$. Then, the expression

$$\Phi_I(E_\bullet) = \Omega_I(E_\bullet) \cap \text{IG}(r,V);$$

this is also a scheme theoretic equality (cf. Proposition 16 (4)). Moreover, $\Phi_I(E_\bullet) = k(E_\bullet)\Lambda_{\varpi_I}^{PC}$. Denote the closure of $\Phi_I(E_\bullet)$ inside $\text{IG}(r,V)$ by $\Phi_I(E_\bullet)$ and its cycle class in $H^*(\text{IG}(r,V))$ (which does not depend upon the choice of the isotropic flag $E_\bullet$) by $[\Phi_I]$.

For the standard flag $E_\bullet = E_\bullet^r$, we have $\Phi_I(E_\bullet) = \Lambda_{\varpi_I}^{PC}$.
and
\[ \delta_n^\vee = 2(\alpha_n^\vee + \alpha_{n+1}^\vee). \]
Thus, under the above identification,
\[ \mathfrak{h}_+^B = \mathfrak{h}_+^C. \]
Moreover, \( \mathfrak{h}_+^{A_{2n}} \) is \( \theta \)-stable and
\[ (\mathfrak{h}_+^{A_{2n}})^\theta = \mathfrak{h}_+^B. \]
Let \( \{s'_1, \ldots, s'_n\} \) be the (simple) reflections in the Weyl group \( W^B = W^{B_n} \) of \( \text{SO}(2n+1) \) corresponding to the simple roots \( \{\delta_1, \ldots, \delta_n\} \) respectively. Since \( H^{A_{2n}} \) is \( \theta \)-stable, there is an induced action of \( \theta \) on the Weyl group \( S_{2n+1} \) consisting of \( \theta \)-invariants:
\[ \{(a_1, \ldots, a_{2n+1}) \in S_{2n+1} : a_{2n+2-i} = 2n + 2 - a_i \forall 1 \leq i \leq 2n + 1\}. \]
In particular, \( w = (a_1, \ldots, a_{2n+1}) \in W^B \) is determined from \( (a_1, \ldots, a_n) \). (Observe that \( a_{n+1} = n + 1 \).) We can identify the Weyl groups \( W^C \simeq W^B \) under the map
\( (a_1, \ldots, a_{2n}) \mapsto (a_1, \ldots, a_n, n+1, a_{n+1}+1, \ldots, a_{2n}+1). \)
Under the inclusion \( W^B \subset S_{2n+1} \), we have
\[ s'_i = r_ir_{2n+1-i}, \quad \text{if} \quad 1 \leq i \leq n-1 \]
\[ = r_nr_{n+1}r_n, \quad \text{if} \quad i = n. \]
(5)
For \( 1 \leq r \leq n \), we let \( \text{OG}(r, 2n+1) = \text{OG}(r, V') \) to be the set of \( r \)-dimensional isotropic subspaces of \( V' \) with respect to the quadratic form \( Q \), i.e.,
\[ \text{OG}(r, 2n+1) := \{ M \in \text{Gr}(r, V') : Q(v) = 0, \forall v \in M \}. \]
Then, it is the quotient \( \text{SO}(2n+1)/P_r^B \) of \( \text{SO}(2n+1) \) by the standard maximal parabolic subgroup \( P_r^B \) with \( \Delta^B \setminus \{\delta_r\} \) as the set of simple roots of its Levi component \( L_r^B \). (Again we take \( L_r^B \) to be the unique Levi subgroup of \( P_r^B \) containing \( H^B \).) It can be easily seen that the set \( W_r^B \) of minimal-length coset representatives of \( W^B/W_{P_r^B} \) is identified with the set
\[ \mathcal{S}'(r, 2n+1) = \{ J := 1 \leq j_1 < \cdots < j_r \leq 2n+1, j_p \neq n+1 \text{ for any } p \text{ and } J \cap J' = \emptyset \}, \]
where
\[ J' := \{ 2n+2-j_1, \ldots, 2n+2-j_r \}. \]
Any such \( J \) represents the permutation \( w'_j = (j_1, \ldots, j_n) \in W^B \) by taking \( \{ j_{r+1} < \cdots < j_n \} = [n] \setminus (J \cup J'). \)
Similar to the Definition 9 of isotropic flags on \( V \), we have the notion of isotropic flags on \( V' \). Then, for an isotropic flag \( E'_i \), there exists an element \( k(E'_i) \in \text{SO}(2n+1) \) which takes the standard flag \( E''_i \) to the flag \( E'_i \). (Observe that \( k(E'_i) \) is determined up to the right multiplication by an element of \( B^B \).)
For any $J \in \mathcal{S}(r, 2n + 1)$ and any isotropic flag $E'_j$, we have the corresponding shifted Schubert cell inside $\text{OG}(r, V')$:

$$\Psi_J(E'_j) = \{ M \in \text{OG}(r, V') : \text{for any } 0 \leq \ell \leq r \text{ and any } j_0 \leq a < j_{\ell+1}, \dim M \cap E'_a = \ell \},$$

where we set $j_0 = 0$ and $j_{r+1} = 2n + 1$. Clearly, set theoretically, (6) $\Psi_J(E'_j) = \Omega_J(E'_j) \cap \text{OG}(r, V')$; this is also a scheme theoretic equality. Moreover, $\Psi_J(E'_j) = k(E'_j) \Lambda^{PB}_{w'_j}$. Denote the closure of $\Psi_J(E'_j)$ inside $\text{OG}(r, V')$ by $\bar{\Psi}_J(E'_j)$ and its cycle class in $H^*(\text{OG}(r, V'))$ (which does not depend upon the choice of the isotropic flag $E'_j$) by $[\bar{\Psi}_J]$. For the standard flag $E' = E^o$, we have $\Psi_J(E'_j) = \Lambda^{PB}_{P Br}$. Denote the closure of $\Psi_J(E'_j)$ inside $\text{OG}(r, V')$ by $\bar{\Psi}_J(E'_j)$ and its cycle class in $H^*(\text{OG}(r, V'))$ (which does not depend upon the choice of the isotropic flag $E'_j$) by $[\bar{\Psi}_J]$. For the standard flag $E' = E^o$, we have $\Psi_J(E'_j) = \Lambda^{PB}_{P Br}$. Denote the closure of $\Psi_J(E'_j)$ inside $\text{OG}(r, V')$ by $\bar{\Psi}_J(E'_j)$ and its cycle class in $H^*(\text{OG}(r, V'))$ (which does not depend upon the choice of the isotropic flag $E'_j$) by $[\bar{\Psi}_J]$. For the standard flag $E' = E^o$, we have $\Psi_J(E'_j) = \Lambda^{PB}_{P Br}$.

3. **Isotropic flags and proper intersection of Schubert cells in $\text{Gr}(m, V)$**

Fix a positive integer $s$. Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$ as in Section 2, and let $1 \leq m \leq n$ be a positive integer. Let $A^1, \ldots, A^s \in S(m, 2n)$. The following theorem is a key technical result that underlies the proof of our theorem on the comparison of eigencone for $\text{Sp}(2n)$ with that of $\text{SL}(2n)$.

**Theorem 10.** Let $E'_1, \ldots, E'_s$ be isotropic flags on $V$ in general position. Then, the intersection of subvarieties $\cap_{j=1}^s \Omega_{A^j}(E'_j)$ inside $\text{Gr}(m, V)$ is proper (possibly empty).

As an immediate consequence of the above theorem, we get the following:

**Corollary 11.** Let $1 \leq m \leq n$ and let $I^1, \ldots, I^s \in \mathcal{S}(m, 2n)$ be such that

$$\prod_{j=1}^s [\Phi_{I^j}] \neq 0 \in H^*(\text{IG}(m, 2n)).$$

Then, $\prod_{j=1}^s [\Omega_{I^j}] \neq 0 \in H^*(\text{Gr}(m, 2n)).$

**Proof.** Observe that by [F$_2$, Proposition 7.1 and Section 12.2] and [B$_1$, Proposition 1.1],

$$\prod_{j=1}^s [\Phi_{I^j}] \neq 0 \text{ if and only if } \cap_{j=1}^s \Phi_{I^j}(E'_j) \neq \emptyset$$

for isotropic flags $\{E'_j\}$ such that the above intersection is proper. Thus, by assumption, $\cap_{j=1}^s \Phi_{I^j}(E'_j) \neq \emptyset$ for such flags $\{E'_j\}$. By the above theorem and Equation (4), we conclude that $\cap_{j=1}^s \Omega_{I^j}(E'_j) \neq \emptyset$. From this and using Equation (7) for $\text{Gr}(m, V)$, the corollary follows. 

Before we can prove the above theorem, we need the following preliminary work.

Given subsets $I$ and $K$ of $[2n]$, we denote by $|I > K|$ the number of pairs $(i, k)$ with $i \in I$, $k \in K$ and $i > k$. (We set $|A > \emptyset| = 0$.) For $K$ a singleton $\{k\}$, we abbreviate
Let us take an integer $1 \leq r \leq n$. For an $I \in S(r, 2n)$, we define

$$\tilde{I} = [2n] \setminus (I \sqcup \bar{I}).$$

(See Equation (3) for the definition of $\bar{I}$.) We also set (for any $I \in S(r, 2n)$)

$$\text{sym}^2(I) = \frac{1}{2}(|I > \tilde{I}| + \mu(w_I)),$$

and

$$\wedge^2(I) = |I > \tilde{I}| - \text{sym}^2(I),$$

where $\mu(w_I)$ represents the number of times the simple reflection $s_n$ appears in any reduced decomposition of $w_I$.

With this notation, we have the following proposition.

**Proposition 12.** For any $I \in S(r, 2n)$ and any isotropic flag $E_*$,

$$\dim \Phi_I(E_*) = |I > \tilde{I}| + \text{sym}^2(I).$$

**Proof.** Under the canonical inclusion $W^C \hookrightarrow S_{2n}$,

$$w_I \mapsto \hat{w}_I = (i_1, \ldots, i_r, j_1, \ldots, j_{n-r}, 2n+1-j_{n-r}, \ldots, 2n+1-j_1, 2n+1-i_r, \ldots, 2n+1-i_1),$$

where $\{j_1 < \cdots < j_{n-r}\} := [n] \setminus (I \sqcup \bar{I})$. Now, by Equations (1), (2) and [F₁, §10.2],

$$\ell^C(w_I) = \frac{1}{2}(\ell^{A_{2n-1}}(\hat{w}_I) + \mu(w_I))$$

$$= \frac{1}{2}(|I > \tilde{I}| + |I > \bar{I}| + |\bar{I} > \tilde{I}| + \mu(w_I))$$

$$= \frac{1}{2}(|I > \tilde{I}| + |I > \bar{I}| + |\tilde{I} > \bar{I}| + \mu(w_I))$$

$$= |I > \tilde{I}| + \text{sym}^2(I).$$

This proves the proposition. \qed

### 3.1. Tangent space of isotropic Grassmannians.

We calculate now the tangent space $T(X)_M$ to $X = \text{IG}(r, V)$ at a point $M$. Because of the natural embedding $\text{IG}(r, V) \subseteq \text{Gr}(r, V)$, we have $T(X)_M \subseteq T\text{Gr}(r, V)_M = \text{Hom}(M, V/M)$.

Clearly, $M^\perp$ is a $2n - r$ dimensional subspace of $V$ that contains $M$ and there is a canonical isomorphism $V/M^\perp \simeq M^*$ (induced from the symplectic form). Hence, we have an exact sequence

$$0 \to \text{Hom}(M, M^\perp/M) \to \text{Hom}(M, V/M) \xrightarrow{\phi} \text{Hom}(M, V/M^\perp) = \text{Hom}(M, M^*) \to 0,$$

obtained from the inclusions $M \subseteq M^\perp \subseteq V$. It is clear that $M^\perp/M$ is a $2n - 2r$ dimensional space that possesses a nondegenerate symplectic form. Let $P_M \subseteq \text{Sp}(2n)$ be the stabilizer of $M$ and $\text{sym}^2 M^*$ the space of symmetric bilinear forms on $M$. 
Lemma 13. $T(X)_M = \phi^{-1}(\text{sym}^2 M^*)$ and hence there is an exact sequence of $P_M$-modules.

(8) $0 \to \text{Hom}(M, M^\perp/M) \xrightarrow{\xi} T(X)_M \xrightarrow{\phi} \text{sym}^2 M^* \to 0.$

Proof. Let $\psi : M \to V/M$ be a linear map (viewed as a deformation of the trivial map). The deformed $M$ (obtained from $\psi$) needs to be isotropic. So, up to the first order, we have

$$\langle v + \epsilon \psi(v), v' + \epsilon \psi(v') \rangle = 0, \forall v, v' \in M.$$ 

Hence,

$$\langle v, \psi(v') \rangle + \langle \psi(v), v' \rangle = 0,$$

or that

$$\langle v, \psi(v') \rangle = \langle v', \psi(v) \rangle.$$ 

This gives us the symmetric bilinear form $\phi(\psi)(v, v') = \langle v, \psi(v') \rangle$. Hence, $T(X)_M \subseteq \phi^{-1}(\text{sym}^2 M^*)$. But,

$$\dim T(X)_M = \dim \text{IG}(r, V) = \frac{r}{2}(4n - 3r + 1) = \dim \phi^{-1}(\text{sym}^2 M^*),$$

and hence $T(X)_M = \phi^{-1}(\text{sym}^2 M^*)$. $\square$

Let $E_\bullet$ be an isotropic flag on $V$. This induces flags on $M, M^\perp$ and hence on $M^\perp/M.$

Lemma 14. For any isotropic flag $E_\bullet$ on $V$, the induced flag on $M^\perp/M$ is isotropic with respect to the nondegenerate symplectic form on $M^\perp/M$.

Proof. Consider an element of the induced flag

$$E_a \cap M^\perp / E_a \cap M = (E_{2n-a} + M) / E_a \cap M.$$ 

Its perpendicular (in $M^\perp/M$) is therefore $(E_{2n-a} + M) \cap M^\perp / M = E_{2n-a} \cap M^\perp / E_{2n-a} \cap M$, which is again a member of the induced flag on $M^\perp/M$. $\square$

3.2. Some crucial inequalities derived from nonvanishing intersection product.

For any $I \in \mathcal{G}(r, 2n)$, isotropic flag $E_\bullet$ on $V$ and $M \in \Phi_I(E_\bullet)$, we now calculate the tangent space $T(\Phi_I(E_\bullet))_M \subseteq T(\Omega_I(E_\bullet))_M$ (with the notation as in Section 2).

Lemma 15. $M^\perp \in \Omega_{I\cup\bar{I}}(E_\bullet)$.

Proof. By definition, $I \cup \bar{I} = [2n] \setminus \bar{I}$. Now, $\dim(E_a \cap M^\perp) = \dim((E_{2n-a} + M)^\perp) = 2n - (r + 2n - a - \dim(E_{2n-a} \cap M)) = a - r + \dim(E_{2n-a} \cap M)$. Hence, $E_a \cap M^\perp \neq E_{a-1} \cap M^\perp$ if and only if $E_{2n-a} \cap M = E_{2n-a+1} \cap M$, i.e., $2n + 1 - a \notin I \Leftrightarrow a \notin \bar{I}$. $\square$

For any $a \in [r]$, let $\lambda_a := |i_a \geq \bar{I}|$, where $I = \{1 \leq i_1 < \cdots < i_r \leq 2n\}$. Let $X = \text{IG}(r, V)$.

$^{1}$There is exactly one way of inducing a complete flag on $M^\perp/M$ from $E_\bullet$. The elements of the flag are $(M^\perp/M) \cap (E_a + M/M)$. This can be written as $((E_a + M) \cap M^\perp)/M = E_a \cap M^\perp/E_a \cap M$. 


Proposition 16.  

(1) \( \text{Hom}(M, M^\perp/M) \cap T(\Omega_I(E_*))_M = \)

\[ \{ \gamma \in \text{Hom}(M, M^\perp/M) : \gamma(E(M)_a) \subset E(M^\perp/M)_{\lambda_a}, \forall a \in [r] \}, \]

where \( E(M)_* \) and \( E(M^\perp/M)_* \) are the induced complete flags on \( M \) and \( M^\perp/M \) respectively with the changed labels \( * \) by the dimension. The dimension of this vector space is \( |I \geq \bar{I}| \).

(2) \( \text{Hom}(M, M^\perp/M) \cap T(\Phi_I(E_*))_M = \text{Hom}(M, M^\perp/M) \cap T(\Omega_I(E_*))_M. \)

(3) \( \phi(T(\Phi_I(E_*))_M) = \phi(T(\Omega_I(E_*))_M \cap T(X)_M) \)

\[ = \{ \gamma \in \text{sym}^2 M^* : \gamma(E(M)_a, E(M)_{t_a}) = 0, \forall a \in [r] \}, \]

where \( t_a = |\bar{I} \geq i_a| \). Moreover, the dimension of this vector space is \( \text{sym}^2(I) \).

(4) We have an equality of schemes \( \Phi_I(E_*) = \Omega_I(E_*) \cap X. \)

Proof. We first prove Part (1): From the known description of \( T(\Omega_I(E_*))_M \) as the space of maps \( \gamma : M \to V/M \) so that \( \gamma(M \cap E_b) \subset (E_b + M)/M \) for all \( b \), we see that for \( \gamma \) to also be in \( \text{Hom}(M, M^\perp/M) \), the condition is

\[ \gamma(M \cap E_b) \subset ((E_b + M) \cap M^\perp)/M \text{ for any } b \in [2n]. \]

By Lemma 15, \( E_b \cap M^\perp/E_b \cap M \) is of dimension \( |b \geq I \cap I| - |b \geq I| = |b \geq \bar{I}| \) for any \( b \in [2n] \). Putting \( b = i_a \), the condition given in Equation (9) can be rewritten as

\[ \gamma(E(M)_{t_a}) \subseteq E(M^\perp/M)_{\lambda_a} \forall a \in [r]. \]

The dimension of this vector space is clearly \( |I \geq \bar{I}| \). This proves Part (1). In particular,

\[ \text{dim} \left( \text{Hom}(M, M^\perp/M) \cap T(\Phi_I(E_*))_M \right) \leq \text{dim} \left( \text{Hom}(M, M^\perp/M) \cap T(\Omega_I(E_*))_M \right) = |I \geq \bar{I}| = |I > \bar{I}|, \]

where the last equality follows since \( I \cap \bar{I} = \emptyset \). We have (by the description of \( T(\Omega_I(E_*))_M \) given above)

\[ \phi(T(\Phi_I(E_*))_M) \subseteq \phi(T(\Omega_I(E_*))_M \cap T(X)_M) \]

\[ \subseteq \{ \gamma \in \text{sym}^2 M^* : \gamma(E_b \cap M, E_b^\perp \cap M) = 0, \forall b \in [2n] \} \]

\[ = \{ \gamma \in \text{sym}^2 M^* : \gamma(E_b \cap M, E_{2n-b} \cap M) = 0, \forall b \in [2n] \} \]

\[ = \{ \gamma \in \text{sym}^2 M^* : \gamma(E(M)_{t_a}, E(M)_{t_a}) = 0, \forall a \in [r] \}. \]

(In the above, we have used \( \text{dim}(E_{2n-i_a} \cap M) = t_a \). Furthermore, if \( E_b \cap M = E_{b+1} \cap M \), then the condition \( \gamma(E_b \cap M, E_{2n-b} \cap M) = 0 \) implies the condition \( \gamma(E_{b+1} \cap M, E_{2n-b-1} \cap M) = 0. \)

Moreover, the last space has dimension \( \text{sym}^2(I) \) by the following calculation. As above, express the vector space under consideration in the more symmetric form

\[ V_1 = \{ \gamma \in \text{sym}^2 M^* : \gamma(E_b \cap M, E_{2n-b} \cap M) = 0, \forall b \in [2n] \}. \]
Form the “analogous” space
\[ V_2 = \{ \gamma \in \wedge^2 M^* : \gamma(E_b \cap M, E_{2n-b} \cap M) = 0, \forall b \in [2n] \}. \]

Clearly,
\[ V_1 \oplus V_2 = \{ \gamma \in (M \otimes M)^* : \gamma((E_b \cap M) \otimes (E_{2n-b} \cap M)) = 0, \forall b \in [2n] \}, \]
which, in turn, can be written as
\[ \{ \gamma \in \text{Hom}(M, M^*) : \gamma(E_b \cap M) \subset \text{ann}(E_{2n-b} \cap M), \forall b \in [2n] \}, \]
where \( \text{ann}(E_{2n-b} \cap M) \) is the annihilator of \( E_{2n-b} \cap M \) in \( M^* \). The last space is of dimension \( |I > \bar{I}| \). For this note that \( \text{ann}(E_{2n-i_a} \cap M) \) is of dimension \( r - |I \leq 2n - i_a| = |I > 2n - i_a| = |I > 2n + 1 - i_a| = |i_a > \bar{I}| \), since \( I \cap \bar{I} = \emptyset \).

Choose a basis of \( M \) compatible with the filtration \( \{E_b \cap M\}_{b \in [2n]} \). Then, in this basis, the vector spaces \( V_1 \) and \( V_2 \) are subspaces of symmetric and skew-symmetric matrices respectively. The difference \( \dim V_1 - \dim V_2 \) is the number of diagonal terms allowed in \( V_1 \), i.e.,
\[ \dim V_1 - \dim V_2 = |\{ a \in [r] : E_{i_a} \cap M \nsubseteq E_{2n-i_a} \cap M \}| = |\{ a \in [r] : i_a > 2n - i_a \}| = |I > n|. \]

Therefore, we conclude
\[ \dim V_1 + \dim V_2 = |I > \bar{I}| \]
\[ \dim V_1 - \dim V_2 = |I > n|. \]

Using Equation (16), we obtain \( \dim V_1 = \text{sym}^2 I \) and \( \dim V_2 = \wedge^2 I \). This proves the last assertion in Part (3).

Combining Equations (10) and (11), and using Lemma 13, we get
\[ \dim(T(\Phi_I(E_*)))_M) \leq \text{sym}^2(I) + |I > \bar{I}|. \]

But, by Proposition 12, the above inequality is an equality. Hence, all the inclusions and inequalities in Equations (10)- (11) are equalities. This proves Parts (2)-(3).

By the definition, set theoretically \( \Phi_I(E_*) = \Omega_I(E_*) \cap X \). Moreover, by Parts (2)-(3), the tangent space \( T(\Phi_I(E_*))_M = T(\Omega_I(E_*))_M \cap T(X)_M \). This proves Part (4) and thus completes the proof of the proposition. \( \square \)

Let \( V \) be as in the beginning of this section and let \( r \leq n \). Fix \( M \in \text{IG}(r, V) \) and \( I \in \mathcal{S}(r, 2n) \). Let \( U = U_I(M) \) be the set of isotropic flags \( E_* \) on \( V \) such that \( M \in \Phi_I(E_*) \). A flag \( E_* \in U \) induces a complete flag \( E_*(M) \) on \( M \), and a complete isotropic flag \( E_*(M^⊥/M) \) on \( M^⊥/M \) (Lemma 14). The following lemma follows by choosing basis elements appropriately.

**Lemma 17.** The map \( U \rightarrow \text{Fl}(M) \times \text{IFl}(M^⊥/M) \) is a surjective fiber bundle with irreducible fibers, where \( \text{Fl}(M) \) is the variety of all the complete flags on \( M \) and \( \text{IFl}(M^⊥/M) \) is the variety of all the isotropic flags on \( M^⊥/M \).
Now, let $V$ be a $2r$-dimensional vector space with a nondegenerate symplectic form containing $M$ of dimension $r$ as an isotropic subspace. Let $I^1, \ldots, I^s \in \mathcal{S}(r, 2r)$ be written as $I^j = \{i^j_1 < \cdots < i^j_s\}$. Define, for any $j \in [s]$ and $a \in [r]$, $t^j_a = |I^j \geq i^j_a|$.  

Lemma 18. The following are equivalent:  

(a) $\prod_{j=1}^s [\Phi_{I^j}] \neq 0 \in H^*(\mathbb{G}(r, 2r))$.  

(b) For some (and hence generic) complete flags $F^1_\bullet, \ldots, F^s_\bullet$ on $M$, the vector space\footnote{\textit{Note: Lemma 18 is presented with a formula that appears to be missing or incorrectly formatted here. The correct formula is expected to be $\{\gamma \in \text{sym}^2 M^* \mid \gamma(F^j_a, F^j_{\bar{a}}) = 0, \ a \in [r], j \in [s]\}$, but it is not clearly shown here.}}
\begin{equation}
\{\gamma \in \text{sym}^2 M^* \mid \gamma(F^j_a, F^j_{\bar{a}}) = 0, \ a \in [r], j \in [s]\}
\end{equation}

is of the expected dimension
\[ r(r + 1)/2 - \sum_{j=1}^s \cosym^2(I^j), \]
where $\cosym^2(I^j) := \frac{r(r+1)}{2} - \text{sym}^2(I^j)$.

Proof. (a) $\Rightarrow$ (b): Choose generic isotropic flags $E^1_\bullet, \ldots, E^s_\bullet$ on $V$. Because of the assumption (a), $\cap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$ is nonempty and, by simultaneously translating each $E^j_\bullet$ by a single element of $\text{Sp}(2r)$, we can assume that $M$ belongs to this intersection. Let $F^1_\bullet, \ldots, F^s_\bullet$ be the induced flags on $M$.

The vector space (13) is the tangent space to the scheme theoretic intersection $\cap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$ at $M$ (by Proposition 16 (3)). Again applying Proposition 16 (3), the transversality of the intersection implies that (b) holds.

Now assume (b). Choose (generic) complete flags $F^1_\bullet, \ldots, F^s_\bullet$ on $M$ such that (b) is satisfied. Now choose isotropic flags $E^1_\bullet, \ldots, E^s_\bullet$ on $V$ such that $M \in \cap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$ and the induced flags on $M$ are $F^1_\bullet, \ldots, F^s_\bullet$ respectively. This is possible by Lemma 17. Using (b) and Proposition 16 (3), we see that $\cap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$ is transverse at $M$. Therefore, using standard facts from intersection theory on a homogenous space, we see that (a) holds. $\square$

Let $\{\tilde{\epsilon}_i\}_{1 \leq i \leq n}$ be the basis of $\mathfrak{h}^C$, where $\tilde{\epsilon}_i$ is the diagonal matrix
\[ \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0), \]
where 1 is placed in the $i$-th slot and $-1$ in the $(2n+1-i)$-th slot. Recall from Section 2 that we have identified $\mathfrak{h}^C$ with $\mathfrak{h}^B$ and denote either of them by $\mathfrak{h}$. Let $\{\epsilon_i\}_{1 \leq i \leq n}$ be the dual basis (of $\mathfrak{h}^*$). Let $\rho^C$ (respectively, $\rho^B$) be half the sum of positive roots of $\text{Sp}(2n)$ (respectively, $\text{SO}(2n+1)$). Then,
\[ 2\rho^B = (2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \cdots + 3\epsilon_{n-1} + \epsilon_n, \]
and
\[ 2\rho^C = (2n)\epsilon_1 + (2n-2)\epsilon_2 + \cdots + 4\epsilon_{n-1} + 2\epsilon_n. \]
Thus,
\[ 2(\rho^C - \rho^B) = \epsilon, \] where $\epsilon := \epsilon_1 + \cdots + \epsilon_n$. 

Let \( \{x_1^C, \ldots, x_n^C\} \) be the basis of \( h^C \) as in Section 2 and similarly for \( x_i^B \). It is easy to see that
\[
x_i^B = x_i^C = \bar{\epsilon}_1 + \cdots + \bar{\epsilon}_i, \quad \text{if} \quad 1 \leq i \leq n - 1 \quad \text{and} \quad x_n^B = 2x_n^C = \bar{\epsilon}_1 + \cdots + \bar{\epsilon}_n.
\]
\[\tag{14}\]

**Lemma 19.** For any \( I \in S(r, 2n) \),
\[
w_I(\epsilon_{r+1} + \cdots + \epsilon_n) = \epsilon_{j_1} + \cdots + \epsilon_{j_{n-r}},
\]
for some \( 1 \leq j_1 < j_2 < \cdots < j_{n-r} \leq n \).

**Proof.** This follows easily from the definition of \( S(r, 2n) \) by observing that the action of \( W^C \subset S_{2n} \) on \( h^* = \bigoplus_{i=1}^n \mathbb{C}\epsilon_i \) is given by the standard action of \( S_{2n} \) on \( \bigoplus_{i=1}^{2n} \mathbb{C}\epsilon_i \) and making the identification \( \epsilon_j = -\epsilon_{2n+1-j} \), for any \( 1 \leq j \leq 2n \). \( \square \)

**Lemma 20.** For any \( w \in W^C \),
\[
we \in \pm \epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_n.
\]
Moreover,
\[\tag{15}\]
\[
\mu(w) = \# \text{ negative signs in the above} = \frac{1}{2}(n - \langle we, \epsilon \rangle),
\]
where \( \langle \, , \rangle \) is the normalized \( W^C \)-invariant form on \( h^* \) given by \( \langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j} \).

In particular, for any \( I \in S(r, 2n) \),
\[\tag{16}\]
\[
\mu(w_I) = |I| - n.
\]

**Proof.** Since the symmetric group \( S_n \subset W^C \) (acting as the permutation group of \( \{\epsilon_1, \ldots, \epsilon_n\} \)) acts trivially on \( \epsilon \), the first part follows easily. Applying \( s_n \) to \( \pm \epsilon_1 \pm \cdots \pm \epsilon_n \), the number of minus signs either increases by 1 or decreases by 1, whereas the application of any \( s_i \) to \( \pm \epsilon_1 \pm \cdots \pm \epsilon_n \) (for any \( i < n \)) keeps the number of minus signs unchanged. Thus, the number of negative signs in \( we \leq \mu(w) \), for any \( w \in W^C \). Further, the last equality in Equation (15) is trivially satisfied. Now, the longest element \( w_0 \in W^C \) makes all the signs in \( w_0 \epsilon \) negative and \( \mu(w_0) = n \). From this Equation (15) follows for \( w_0 \) and hence for any \( w \in W^C \). Otherwise, taking a reduced decomposition of \( w \) and extending to a reduced decomposition of \( w_0 \), we would get a contradiction.

Equation (16) follows from Equation (15) immediately. \( \square \)

**Corollary 21.** For any \( I \in S(r, 2n) \),
\[
(w_I^{-1}\epsilon)(x_r^B) = r - 2\mu(w_I).
\]

**Proof.**
\[
(w_I^{-1}\epsilon)(x_r^B) = \langle w_I^{-1}\epsilon, \epsilon \rangle - \langle w_I^{-1}\epsilon, \epsilon_{r+1} + \cdots + \epsilon_n \rangle
\]
\[
= n - 2\mu(w_I) - \langle \epsilon, w_I(\epsilon_{r+1} + \cdots + \epsilon_n) \rangle, \quad \text{by Equation (15)}
\]
\[
= n - 2\mu(w_I) - (n - r), \quad \text{by Lemma 19}
\]
\[
= r - 2\mu(w_I).
\]
Fix $I^1, \ldots, I^s \in \mathcal{G}(r, 2n)$ and define functions $\theta^C, \theta^B : \mathcal{G}(r, 2n) \to \mathbb{Z}$ by

$$\theta^C(I) = (\chi^C_{w_I} - \sum_{j=1}^{s} \chi^C_{w_{Ij}})(x^C_r),$$

and

$$\theta^B(I) = (\chi^B_{w_I} - \sum_{j=1}^{s} \chi^B_{w_{Ij}})(x^B_r),$$

where $\chi^C_{w_I} := (\rho^C + w_{I}^{-1}\rho^C)$ and $\chi^B_{w_I}$ is defined similarly.

**Lemma 22.** For $r < n$ and any $I \in \mathcal{G}(r, 2n)$,

$$\theta^C(I) = \theta^B(I) + \bar{\mu}(I) - \sum_{j=1}^{s} \bar{\mu}(I^j),$$

where $\bar{\mu}(I) := r - \mu(w_I)$. Similarly, for $r = n$,

$$2\theta^C(I) = \theta^B(I) + \bar{\mu}(I) - \sum_{j=1}^{s} \bar{\mu}(I^j).$$

**Proof.** Assume first that $r < n$. In this case

$$\theta^C(I) - \theta^B(I) = \frac{1}{2}((\epsilon + w_{I}^{-1}\epsilon) - \sum_{j=1}^{s} (\epsilon + w_{Ij}^{-1}\epsilon))(x^C_r)$$

$$= r - \mu(w_I) - \sum_{j=1}^{s} (r - \mu(w_{Ij})), \text{ by Corollary 21}$$

$$= \bar{\mu}(I) - \sum_{j=1}^{s} \bar{\mu}(I^j).$$

Now, we consider the case $r = n$. In this case,

$$2\theta^C(I) - \theta^B(I) = (\epsilon + w_{I}^{-1}\epsilon) - \sum_{j=1}^{s} (\epsilon + w_{Ij}^{-1}\epsilon))(x^C_r)$$

$$= \bar{\mu}(I) - \sum_{j=1}^{s} \bar{\mu}(I^j).$$

This proves the lemma. \qed

**Lemma 23.** Let $I^1, \ldots, I^s \in \mathcal{G}(r, 2r)$ be such that

$$\prod_{j=1}^{s} [\bar{\Phi}_{Ij}] \neq 0 \in H^*(IG(r, 2r)).$$
Then,
\[ \sum_{j=1}^{s} \bar{\mu}(I^j) \geq r - \left( \dim IG(r, 2r) - \sum_{j=1}^{s} \codim(\Phi_{I^j}) \right). \]

Observe that, by Equation (16),
\[ \bar{\mu}(I^j) = |r \geq i |. \]

Proof. Let \( I \in \mathcal{S}(r, 2r) \) be such that \([ \bar{\Phi}_{I + } ]\) appears with nonzero coefficient in the product \( \prod_{j=1}^{s} [ \bar{\Phi}_{I^j} ] \). The maximal parabolic subgroup \( P^C_r \) is minuscule and hence by [BK, Lemma 19], \( \theta^C(I) = 0 \). Thus, by Lemma 22,
\[ \theta^B(I) = -\mu(I) + \sum_{j=1}^{s} \bar{\mu}(I^j). \]

But, clearly,
\[ \mu(w_I) \leq \dim \Phi_I = \dim IG(r, 2r) - \sum_{j=1}^{s} \codim(\Phi_{I^j}). \]

Hence,
\[ \bar{\mu}(I) = r - \mu(w_I) \geq r - \dim IG(r, 2r) + \sum_{j=1}^{s} \codim(\Phi_{I^j}). \]

Also, by Theorem 55 and [BK, Proposition 17(a)],
\[ \theta^B(I) \geq 0. \]

Combining Equations (18)-(20), we get the lemma.

**An alternative proof of the lemma:** The following is a quick (though less conceptual) way to prove the lemma. Let \( I^o = \{ r, r + 2, \ldots, 2r \} \). Then, by Proposition 12, \( \codim \Phi_{I^o} = 1 \) and \([ \bar{\Phi}_{I^o} ]\) is ample on \( IG(r, 2r) \). So, by cupping with a sufficient number of \([ \bar{\Phi}_{I^o} ]\), we are reduced to the case \( \dim IG(r, 2r) - \sum_{j=1}^{s} \codim(\Phi_{I^j}) = 0 \).

Suppose now by way of contradiction that \( \sum_{j=1}^{s} \bar{\mu}(I^j) < r \). Choose generic isotropic flags \( E^1_1, \ldots, E^s_1 \) on a \( 2r \)-dimensional vector space \( V \) with a nondegenerate symplectic form, and let \( M \in \cap_{j=1}^{s} \bar{\Phi}_{I^j}(E^j_1) \). Let \( F^j_1 \) be the induced flags on \( M \). Let \( t^j_a := |I^j \geq i_j^a| \).

Since the intersection \( \cap_{j=1}^{s} \Phi_{I^j}(E^j_1) \) is transverse at \( M \), by Proposition 16 (3), there are no nonzero symmetric bilinear forms \( \phi \) on \( M \), so that
\[ \phi(F^j_a, F^j_{t^j_a}) = 0, \quad j \in [s], \ a \in [r]. \]

Let \( M' := \sum_{j=1}^{s} F^j_{\bar{\mu}(I^j)} \), which is a proper subspace of \( M \) since, by assumption, \( \sum_{j=1}^{s} \bar{\mu}(I^j) < r \). Thus, \( \text{sym}^2(M/M')^* \hookrightarrow \text{sym}^2 M^* \). It is easy to see that any nonzero element \( \phi \) in \( \text{sym}^2(M/M')^* \) satisfies the forbidden possibility above (since at least one of \( a \) or \( t^j_a \leq \bar{\mu}(I^j) \) for any \( a \in [r] \) and \( j \in [s] \)). \( \square \)
Lemma 24. Let \( I^1, \ldots, I^s \in \mathfrak{S}(r, 2n) \) be such that
\[
\prod_{j=1}^s [\Phi_I] \neq 0 \in H^*(\text{IG}(r, 2n)).
\]
Then, the following inequalities hold:
\[
(21) \quad r(r + 1)/2 - \sum_{j=1}^s \text{cosym}^2(I^j) \geq 0.
\]
\[
(22) \quad \sum_{j=1}^s \bar{\mu}(I^j) \geq r - (\dim \text{IG}(r, 2n) - \sum_{j=1}^s \text{cosym}^2(I^j)),
\]
and
\[
(23) \quad r(r - 1)/2 - \sum_{j=1}^s \text{co}^2(I^j) \geq 0,
\]
where recall that \( \text{sym}^2(I) \) and \( \wedge^2(I) \) are defined above Proposition 12, \( \text{cosym}^2(I) \) is defined in Lemma 18, and
\[
\text{co}^2(I) := r(r - 1)/2 - \wedge^2(I).
\]

Proof. Choose generic isotropic flags \( E^1_1, \ldots, E^s_1 \) on \( V \), and let \( M \in \bigcap_{i=1}^s \Phi_{I^j}(E^j_1) \). Let \( F^j_1 \) be the induced flags on \( M \). Since the intersection \( \bigcap_{i=1}^s \Phi_{I^j}(E^j_1) \) is transverse at \( M \), the images of the tangent spaces of \( \Phi_{I^j}(E^j_1) \) via the map \( \phi : T(\text{IG}(r, 2n))_M \to \text{sym}^2 M^* \) (defined in Lemma 13) are transverse in \( \text{sym}^2 M^* \), as can be easily seen (see [PS] for a similar argument). This gives us Equation (21) in view of Proposition 16 (3).

Let \( \beta_j \) be the order preserving bijection of \( I^j \sqcup I^\bar{j} \) with \([2r]\) and let \( I^j_0 \) be the subset \( \beta(I^j) \) of \([2r]\) (of cardinality \( r \)). Hence, we see using Lemma 18 and Proposition 16 that
\[
\prod_{j=1}^s [\Phi_{I^j_0}] \neq 0 \in H^*(\text{IG}(r, 2r)).
\]
Observe next that \( |n \geq I^j| = |r \geq I^j_0| \) and hence \( |I^j > n| = |I^j_0 > r| \). Thus, by Equation (17),
\[
\bar{\mu}(I^j) = \bar{\mu}(I^j_0) \text{ and } \text{sym}^2(I^j) = \text{sym}^2(I^j_0).
\]

Now, we can use Lemma 23 and Proposition 12 to obtain Equation (22). Equation (23) follows from Equation (22) and the equality
\[
\text{cosym}^2(I^j) - \text{co}^2(I^j) = r - (\text{sym}^2(I^j) - \wedge^2(I^j)) = \bar{\mu}(I^j).
\]
\( \square \)
3.3. Isotropic flags and transversality. For any \( r \leq m \leq n \) define
\[
\mathcal{L}(m, r, 2n) = \{ X \in \text{Gr}(m, V) : \dim(X \cap X^\perp) = r \},
\]
where \( V \) is as in the beginning of Section 3.

Lemma 25. (1) \( \text{Sp}(2n) \) acts transitively on \( \mathcal{L}(m, r, 2n) \).

(2) \( \mathcal{L}(m, r, 2n) \) is a fiber bundle over \( IG(r, V) \) via the map \( \pi \) defined by \( X \mapsto M = X \cap X^\perp \).

(3) For a fixed \( m \), the union of the subvarieties \( \mathcal{L}(m, r, 2n) \) over all choices of \( 0 \leq r \leq m \) is \( \text{Gr}(m, V) \).

(4) The action of \( \text{Sp}(2n) \) on \( \text{Gr}(m, V) \) has only finitely many orbits.

Proof. Let \( X, X' \in \mathcal{L}(m, r, 2n) \), \( M = X \cap X^\perp \) and \( M' = X' \cap X'^\perp \). We need to produce \( g \in \text{Sp}(2n) \) such that \( gX = X' \). Since \( \text{Sp}(2n) \) acts transitively on \( IG(r, 2n) \), we may assume that \( M = M' \). Let
\[
G_M := \{ g \in \text{Sp}(2n) : gM = M \}.
\]

We easily see that \( G_M \) surjects onto the symplectic group of \( M^\perp/M \) by choosing subspaces \( N \) and \( Y \) of \( V \) such that \( V = M \oplus N \oplus Y, \dim N = \dim M \) and \( (M \oplus N, Y) = 0 \). Now, \( \pi^{-1}(\{M\}) \) can be identified with the set of subspaces \( X \subset M^\perp/M \) of dimension \( m - r \) which are nondegenerate with respect to the induced (nondegenerate) symplectic form on \( M^\perp/M \). This reduces us to the case of \( r = 0 \) (for the proof of the first part of the lemma). So, we need to show that \( \text{Sp}(2n) \) acts transitively on the set of subspaces \( N \) of a given dimension \( m \) such that \( N \cap N^\perp = (0) \). Let \( X \) be one such subspace and \( Y = X^\perp \). Clearly, \( V = X \oplus Y \) and the induced symplectic forms are nondegenerate on \( X \) and \( Y \). Choose “symplectic bases” \( \{e_i, f_i\} \) of \( X \) and \( \{u_j, v_j\} \) of \( Y \). Map them appropriately to the standard basis vectors in \( V \). This transformation is in \( \text{Sp}(2n) \). This proves Part (1). Part (2) follows from the above description of \( \pi^{-1}(\{M\}) \). Part (3) of course is clear and Part (4) follows from Parts (1) and (3). \( \square \)

For an isotropic flag \( E_\bullet \) on \( V \), and subsets \( A \in S(m, 2n) \) and \( I \in \mathcal{S}(r, 2n) \), define the subvariety:
\[
\Phi_{I, A}(E_\bullet) = \{ X \in \text{Gr}(m, V) : X \in \Omega_A(E_\bullet), X \cap X^\perp \in \Phi_I(E_\bullet) \} \subset \mathcal{L}(m, r, 2n).
\]

Lemma 26. Set \( L := A \setminus (I \sqcup \bar{I}) \) and \( T := [2n] \setminus (I \sqcup L \sqcup \bar{I}) \). Assume that \( \Phi_{I, A}(E_\bullet) \) is nonempty. Then, it is irreducible and
\[
\dim \Phi_{I, A}(E_\bullet) = |I > L| + |I > T| + \text{sym}^2(I) + |L > T|.
\]

Moreover, \( I \subset A \) and \( \bar{I} \cap A = \emptyset \).

Proof. This follows from considering the fibration \( \pi_{I, A} : \Phi_{I, A}(E_\bullet) \to \Phi_I(E_\bullet) \), where \( \pi_{I, A} \) is the restriction of \( \pi \) to \( \Phi_{I, A}(E_\bullet) \). (This is a surjective fibration because of the \( B^C \)-equivariance.) By Proposition 12, the dimension of \( \Phi_I(E_\bullet) \) is
\[
|I > L \sqcup T| + \text{sym}^2(I) = |I > L| + |I > T| + \text{sym}^2(I).
\]
So, we just need to prove that the fiber dimension of $\pi_{I,A}$ is $|L > T|$.

Let $M \in \Phi_I(E_*)$, and $\tilde{X} \in \text{Gr}(m - r, M^\perp/M)$ be such that the induced form on $\tilde{X}$ is nondegenerate. Then, $X := q^{-1}(\tilde{X}) \in \Phi_{I,A}(E_*)$ if and only if $X \in \Omega_A(E_*)$, where $q : M^\perp \to M^\perp/M$ is the quotient map. By definition, $X \in \Omega_A(E_*)$ if and only if $\dim(E_a \cap X) = |a \geq A|$ for all $a$. Now, by Lemma 15, the image of $E_a \cap M^\perp$ in $M^\perp/M$ is of dimension $|a \geq [2n] \setminus \tilde{I}| - |a \geq I| = |a \geq \tilde{I}|$ (see the footnote (1)). Moreover, for any subspace $X$ of $M^\perp$ containing $M$,

$$\dim(E_a \cap X) = \dim(E_a \cap M) + \dim((E_a + M/M) \cap \tilde{X}).$$

Thus, for any $\tilde{X} \in \text{Gr}(m - r, M^\perp/M)$, $X := q^{-1}(\tilde{X})$ belongs to $\Phi_{I,A}(E_*)$ if and only if

$$\dim(E(M^\perp/M)_{\varphi(a)} \cap X) = |a \geq A| - |a \geq I|, \forall a \in [2n],$$

where $\varphi(a) := |a \geq \tilde{I}|$.

We next claim that $I \subset A$. Write $A = \{1 \leq a_1 < \cdots < a_m \leq 2n\}$. If possible, let $a \in I$ be such that $a < a_i < a_{i+1}$, for some $0 \leq i \leq m$ (where we set $a_0 = 0$ and $a_{m+1} = 2n + 1$). Then,

$$|a \geq A| - |a \geq I| < |(a - 1) \geq A| - |(a - 1) \geq I|.$$

By Equation (24), this gives

$$\dim(E(M^\perp/M)_{\varphi(a)} \cap X) < \dim(E(M^\perp/M)_{\varphi(a-1)} \cap X),$$

which is a contradiction. Hence the claim $I \subset A$ is established.

We further claim that $\tilde{I} \cap A = \emptyset$. If possible, take $a \in I$ such that $a^* := 2n + 1 - a \in A$. Then,

$$|a^* \geq A| - |a^* \geq I| > |(a^* - 1) \geq A| - |(a^* - 1) \geq I|.$$

Thus, by Equation (24),

$$\dim(E(M^\perp/M)_{\varphi(a^*)} \cap X) < \dim(E(M^\perp/M)_{\varphi(a^*-1)} \cap X).$$

However, $\varphi(a^*) = \varphi(a^* - 1)$, which is a contradiction. This proves that $\tilde{I} \cap A = \emptyset$. Thus, for any $\tilde{X} \in \text{Gr}(m - r, M^\perp/M)$, $X = q^{-1}(\tilde{X})$ belongs to $\Phi_{I,A}(E_*)$ if and only if for any $1 \leq p \leq 2n - 2r$,

$$\dim(E(M^\perp/M)_p \cap X) = |j_p \geq L|,$$

where $\tilde{I} = \{1 \leq j_1 < \cdots < j_{2n-2r} \leq 2n\}$. This shows that the fiber of $\pi_{I,A}$ is an open subset of a Schubert cell in $\text{Gr}(m - r, 2n - 2r)$ of dimension $|L > T|$ since $T := \tilde{I} \setminus L$. In particular, $\Phi_{I,A}(E_*)$ is irreducible. This proves the lemma. \hfill $\square$

**Lemma 27.**

(1) The dimension of $\mathcal{L}(m, r; 2n)$ is $\dim(\text{Gr}(m, 2n)) - r(r - 1)/2$.

(2) If nonempty, $\Phi_{I,A}(E_*)$ is of dimension no greater than $\dim(\Omega_A(E_*)) - \wedge^2(I)$. 

Proof. By Lemma 25, $\mathcal{L}(m, r, 2n)$ is a fiber bundle over $\text{IG}(r, V)$. Moreover, as in the proof of Lemma 25, the fiber is an open subset of $\text{Gr}(m - r, 2n - 2r)$. Therefore, $\dim \mathcal{L}(m, r, 2n)$ equals

$$
\dim \text{IG}(r, V) + (m - r)(2n - 2r - (m - r)) = r(4n - 3r + 1)/2 + (m - r)(2n - m - r) = \dim(\text{Gr}(m, 2n)) - r(r - 1)/2.
$$

This proves the part (1) of the lemma.

We next prove part (2). By Lemma 26,

$$
\dim \Phi_{I, A}(E_\bullet) = |I > L| + |I > T| + \text{sym}^2(I) + |L > T|
$$

$$= |I > T| + |L > I| + |L > T| + |I > I| - \wedge^2(I) + |I > L| - |L > I|
$$

(26)

$$= \dim \Omega_A(E_\bullet) - \wedge^2(I) + |I > L| - |L > I|.
$$

The lemma will therefore follow from the inequality

$$
|I > L| \leq |L > I|,
$$

(27)

which is proved below.

We first show

$$
|a \geq L| \leq |L \geq a^*| \text{ for any } a \in [2n], \text{ where } a^* := 2n + 1 - a.
$$

(28)

Take $X \in \Phi_{I, A}(E_\bullet)$ and let $\pi(X) = M$. Let $Y := X \cap E_a \supset M \cap E_a$. We look at the image of $X \to Y^*$ (induced by the symplectic form), where $Y^*$ is the dual of $Y$. The image has dimension

$$
\dim Y - \dim(Y \cap M) = |a \geq A| - |a \geq I| = |a \geq L|.
$$

But, clearly, $(X \cap E_{a^*-1}) + M$ goes to zero under the map $X \to Y^*$ and hence the image of $X \to Y^*$ is of dimension no greater than $|L \geq a^*|$. This proves Equation (28). Hence,

$$
|I > L| \leq |L \geq I|.
$$

But since $I \cap L = \emptyset$ and also $I \cap L = \emptyset$, we get Equation (27) and hence Part (2) of the lemma is proved. \qed

3.4. Proof of Theorem 10. Choose isotropic flags $\{E_\bullet^j\}_{1 \leq j \leq s}$ such that the intersection $\cap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$ is transverse and dense in $\cap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$ for all $I^j \in \mathcal{S}(r, 2n)$ and all $1 \leq r \leq m$ (cf. [BK, Proposition 3]). For any irreducible component $C$ of $\cap_{j=1}^s \Omega_{A^j}(E_\bullet^j)$, there exists a dense open subset $U$ of $C$ and an $r$ together with subsets $\{I^j\}_{1 \leq j \leq s} \subset \mathcal{S}(r, 2n)$ such that $U \subset \mathcal{L}(m, r, 2n)$ and $\pi(U) \subset \cap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$.

It suffices to show that the dimension of any irreducible component of $\cap_{j=1}^s \Phi_{I^j, A^j}(E_\bullet^j)$ is no greater than $\dim \text{Gr}(m, 2n) - \sum_{j=1}^s \text{codim}(\Omega_{A^j})$.

Take (generic) isotropic flags $\{E_\bullet^j\}_{1 \leq j \leq s}$ such that the intersection $\cap_{j=1}^s \Phi_{I^j, A^j}(E_\bullet^j)$ in $\mathcal{L}(m, r, 2n)$ is proper for all the choices of $I^j \in \mathcal{S}(r, 2n)$ and $A^j \in S(m, 2n)$. This is possible since $\text{Sp}(2n)$ acts transitively on $\mathcal{L}(m, r, 2n)$. 


Therefore, using Lemma 27, any irreducible component of \( \bigcap_{j=1}^{s} \Phi_{I_j, A_j}(E_j') \) has dimension no greater than
\[
\dim \text{Gr}(m, 2n) - \frac{r(r - 1)}{2} - \sum_{j=1}^{s} \left( \dim \text{Gr}(m, 2n) - r(r - 1)/2 - (\dim \Omega_{A_j} - \wedge^2(I_j)) \right)
\]
\[
= \dim \text{Gr}(m, 2n) - \left( \sum_{j=1}^{s} \text{codim} \Omega_{A_j} \right) - \left( r(r - 1)/2 - \sum_{j=1}^{s} \text{co} \wedge^2(I_j) \right).
\]
Using Equation (23), this last quantity is no greater than
\[
\dim \text{Gr}(m, 2n) - \sum_{j=1}^{s} \text{codim} \Omega_{A_j},
\]
as desired. This finishes the proof of the theorem. \( \square \)

3.5. Analogue of Theorem 10 for \( \text{SO}(2n + 1) \). We now prove the analogue of Theorem 10 for \( \text{SO}(2n + 1) \). The proof is very similar and hence we will only give a brief sketch. First we need the following preparation. As in Subsection 2.3, \( V' := \mathbb{C}^{2n+1} \) is equipped with the quadratic form \( Q \).

For any \( r \leq m \leq n \) define
\[
\mathcal{O}(m, r, 2n + 1) = \{ X \in \text{Gr}(m, V') : \dim(X \cap X^\perp) = r \}.
\]
We have the following analogue of Lemma 25 for \( \text{SO}(2n + 1) \). Its proof is similar and hence is omitted.

**Lemma 28.**

1. \( \text{SO}(2n + 1) \) acts transitively on \( \mathcal{O}(m, r, 2n + 1) \).
2. \( \mathcal{O}(m, r, 2n + 1) \) is a fiber bundle over \( \text{OG}(r, 2n + 1) \) via the map \( \pi' \) defined by \( X \mapsto M = X \cap X^\perp \).
3. For a fixed \( m \), the union of the subvarieties \( \mathcal{O}(m, r, 2n + 1) \) over all choices of \( 0 \leq r \leq m \) is clearly \( \text{Gr}(m, V') \). In particular, the action of \( \text{SO}(2n + 1) \) on \( \text{Gr}(m, V') \) has only finitely many orbits.

For an isotropic flag \( E' \) on \( V' \), and subsets \( A \in S(m, 2n + 1) \) and \( J \in \mathcal{S}'(r, 2n + 1) \) (cf. Subsection 2.3), define the subvariety of \( \mathcal{O}(m, r, 2n + 1) \):
\[
\Psi_{J, A}(E') = \{ X \in \text{Gr}(m, V') : X \in \Omega_A(E'), X \cap X^\perp \in \Psi_J(E') \}.
\]

**Lemma 29.**

1. The dimension of \( \mathcal{O}(m, r, 2n + 1) \) is \( \dim(\text{Gr}(m, 2n + 1)) - \frac{r(r + 1)}{2} \).
2. If nonempty, \( \Psi_{J, A}(E') \) is irreducible of dimension no greater than \( \dim(\Omega_A(E')) - \text{sym}^2(J) \).

**Proof.** The fiber of the bundle \( \pi : \mathcal{O}(m, r, 2n + 1) \to \text{OG}(r, V') \) is an open subset of the set of choices of \( m \)-dimensional spaces containing a fixed \( r \)-dimensional subspace and
contained in the perpendicular space of this $r$-dimensional space. Therefore, by a simple
calculation, $\dim \mathcal{O}(m, r, 2n + 1)$ equals

$$
\dim(\mathcal{O}(r, V')) + (m - r)(2n + 1 - 2r - (m - r))
= (r(r - 1)/2 + r(2n + 1 - 2r)) + (m - r)(2n + 1 - m - r)
= \dim(\text{Gr}(m, V')) - r(r + 1)/2.
$$

This proves Part (1).

We next observe that as in the symplectic case (Lemma 26), if nonempty,

$$
\dim \Psi_{J,A}(E'_s) = |J > L'| + |J > T'| + \Lambda^2(J) + |L' > T'|,
$$

where $L' := A \setminus (J \sqcup J')$, $T' := [2n + 1] \setminus (J \sqcup L' \sqcup J')$ and $J'$ is defined in Subsection 2.3.

Similar to Equation (26), we have

$$
\dim \Psi_{J,A}(E'_s) = \dim(\Omega_A(E'_s)) - \text{sym}^2(J) + |J > L'| - |L' > J'|.
$$

The second part of the lemma will therefore follow from the following inequality whose
proof is similar to that of Equation (27).

$$(29) \quad |J > L'| \leq |L' > J'|.
$$

\hspace{1cm} \square

**Theorem 30.** Let $A^1, \ldots, A^s$ be subsets of $[2n + 1]$ each of cardinality $m$. Let $E^1_s, \ldots, E^s_s$
be isotropic flags on $V'' = \mathbb{C}^{2n + 1}$ in general position. Then, the intersection
$\cap_{j=1}^s \Omega_{A^j}(E^j_s)$ of subvarieties of $\text{Gr}(m, V'')$ is proper.

**Proof.** For any irreducible component $C'$ of $\cap_{j=1}^s \Omega_{A^j}(E^j_s)$, there exists a dense open
subset $U' \subset C'$ and an $r$ together with subsets $\{J_p\}_{1 \leq j \leq s} \subset \mathcal{G}'(r, 2n + 1)$ such that
$U' \subset \mathcal{O}(m, r, 2n + 1)$ and $\pi'(U') \subset \cap_{j=1}^s \Psi_{J,A}(E^j_s)$.

It suffices to show that the dimension of any irreducible component of $\cap_{j=1}^s \Psi_{J,A}(E^j_s)$
is no greater than $\dim(\text{Gr}(m, 2n + 1)) - \sum_{j=1}^s \text{codim}(\Omega_A)$. The dimension of $\cap_{j=1}^s \Psi_{J,A}(E^j_s)$
is no greater than

$$
\dim(\text{Gr}(m, 2n + 1)) - r(r + 1)/2
- \sum_{j=1}^s \left( \dim(\text{Gr}(m, 2n + 1)) - r(r + 1)/2 - (\dim(\Omega_A) - \text{sym}^2(J)) \right)
= \dim(\text{Gr}(m, 2n + 1)) - r(r + 1)/2 - \sum_{j=1}^s (\text{codim}(\Omega_A) - \text{cosym}^2(J)).
$$

Using Theorem 55 and Equation (21), this last quantity is no greater than $\dim(\text{Gr}(m, 2n + 1)) - \sum_{j=1}^s \text{codim}(\Omega_A)$ as desired. \hspace{1cm} \square

As an immediate consequence of the above theorem (just as in the case of Sp(2n)), we get the following:
Corollary 31. Let $1 \leq m \leq n$ and let $J^1, \ldots, J^s \in \mathcal{G}(m, 2n+1)$ be such that

$$\prod_{j=1}^s [\Psi_{J^j}] \neq 0 \in H^*(OG(m, 2n+1)).$$

Then, $\prod_{j=1}^s [\Omega_{J^j}] \neq 0 \in H^*(Gr(m, 2n+1)).$

4. Comparison of eigencone for $Sp(2n)$ with that of $SL(2n)$

Let $G$ be a connected (complex) semisimple group. Choose a maximal compact subgroup $K$ of $G$ with Lie algebra $\mathfrak{k}$. Then, there is a natural homeomorphism $C : \mathfrak{k}/K \rightarrow h^+ +$, where $K$ acts on $\mathfrak{k}$ by the adjoint representation and $h_+$ is the positive Weyl chamber in $h$ as in Section 2. The inverse map $C^{-1}$ takes any $h \in h^+$ to the $K$-conjugacy class of $ih$.

For a positive integer $s$, the eigencone is defined as the cone: $\Gamma(s, K) :=$

$$\{(h_1, \ldots, h_s) \in h^+_+ \mid \exists (k_1, \ldots, k_s) \in \mathfrak{k}^s : \sum_{j=1}^s k_j = 0 \text{ and } C(k_j) = h_j \forall j = 1, \ldots, s\}.$$

Given a standard maximal parabolic subgroup $P$, let $\omega_P$ denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i) = 1$, if $\alpha_i \in \Delta \setminus \Delta(P)$ and 0 otherwise, where $\Delta(P)$ is the set of simple roots for the Levi subgroup $L$ of $P$ containing $H$. Then, $\omega_P$ is invariant under the Weyl group $W_P$ of $P$.

We recall the following theorem from [BeSj].

Theorem 32. Let $(h_1, \ldots, h_s) \in h^+_+$. Then, the following are equivalent:

(a) $(h_1, \ldots, h_s) \in \Gamma(s, K)$.

(b) For every standard maximal parabolic subgroup $P$ in $G$ and every choice of $s$-tuples $(w_1, \ldots, w_s) \in (W_P)^s$ such that

$$[\Lambda^P_{w_1}] \cdot \ldots \cdot [\Lambda^P_{w_s}] = d[\Lambda^P_e] \in H^*(G/P),$$

for some nonzero $d$, the following inequality holds:

$$\omega_P\left(\sum_{j=1}^s w_j^{-1}h_j\right) \leq 0.$$
Recall that $\mathfrak{h}^C_+$ (respectively, $\mathfrak{h}^B_+$) is the dominant chamber in the Cartan subalgebra of $\text{Sp}(2n)$ (respectively, $\text{SO}(2n + 1)$) as in Section 2.

The following theorem is our main result on the comparison of the eigencone for $\text{Sp}(2n)$ with that of $\text{SL}(2n)$ (and also for $\text{SO}(2n + 1)$ with that of $\text{SL}(2n + 1)$).

**Theorem 33.**

(a) For $h_1, \ldots, h_s \in \mathfrak{h}^C_+$,

\[(h_1, \ldots, h_s) \in \Gamma(s, \text{Sp}(2n)) \iff (h_1, \ldots, h_s) \in \Gamma(s, \text{SU}(2n)).\]

(b) For $h_1, \ldots, h_s \in \mathfrak{h}^B_+$,

\[(h_1, \ldots, h_s) \in \Gamma(s, \text{SO}(2n + 1)) \iff (h_1, \ldots, h_s) \in \Gamma(s, \text{SU}(2n + 1)).\]

(Observe that by Section 2, $\mathfrak{h}^C_+ \subset \mathfrak{h}^{A_{2n-1}}_+$ and $\mathfrak{h}^B_+ \subset \mathfrak{h}^{A_{2n}}_+$.)

**Proof.** Clearly, $\Gamma(s, \text{Sp}(2n)) \subseteq \Gamma(s, \text{SU}(2n))$. Conversely, we need to show that if $h = (h_1, \ldots, h_s) \in (\mathfrak{h}^C_+)^s$ is such that $h \in \Gamma(s, \text{SU}(2n))$, then $h \in \Gamma(s, \text{Sp}(2n))$. Take any $1 \leq m \leq n$ and any $I^1, \ldots, I^s \in \mathcal{S}(m, 2n)$ such that

\[\bar{\Phi}_{I^1} \cdot \ldots \cdot \bar{\Phi}_{I^s} = d[\bar{\Phi}_s] \in H^*(\text{IG}(m, 2n))\text{ for some nonzero } d.\]

By Corollary 11,

\[[\bar{\Omega}_{I^1}] \cdot \ldots \cdot [\bar{\Omega}_{I^s}] \neq 0 \in H^*(\text{Gr}(m, 2n)).\]

In particular, by Theorem 32 applied to $\text{SU}(2n)$,

\[\omega_m(\sum_{j=1}^s v_{I^j}^{-1} h_j) \leq 0,\]

where $\omega_m$ is the $m$-th fundamental weight of $\text{SU}(2n)$ and $v_{I^j} \in S_{2n}$ is the element associated to $I^j$ as in Subsection 2.1. It is easy to see that the $m$-th fundamental weight $\omega_m^C$ of $\text{Sp}(2n)$ is the restriction of $\omega_m$ to $\mathfrak{h}^C$. Moreover, even though the elements $v_{I^j} \in S_{2n}$ and $w_{I^j} \in W^C$ are, in general, different, we still have

\[\omega_m(v_{I^j}^{-1} h_j) = \omega_m(w_{I^j}^{-1} h_j).\]

Applying Theorem 32 for $\text{Sp}(2n)$, we get the (a)-part of the theorem.

The proof for $\text{SO}(2n + 1)$ is similar. (Apply Corollary 31 instead of Corollary 11.) \qed

5. A BASIC TRANSVERSALITY RESULT

Let $M$ be a $r$-dimensional space and $F = (F^1_\bullet, \ldots, F^s_\bullet)$ an $s$-tuple of complete flags on $M$. As earlier, let $V$ be a $2n$-dimensional vector space equipped with a nondegenerate symplectic form, and $G = (G^1_\bullet, \ldots, G^s_\bullet)$ an $s$-tuple of (complete) isotropic flags on $V$. Let $\mu = (\mu^1, \ldots, \mu^s)$, where $\mu^j$ is an ordered sequence with $r$ elements $2n \geq \mu^1_j \geq \cdots \geq \mu^j_r \geq 0, j \in [s]$. We fix $\mu$ in this section once and for all.

Make the definition:

\[\mathcal{H}_\mu(F, G) = \{\phi \in \text{Hom}(M, V) : \phi(F^a_i) \subset G^j_{2n - \mu^j_i}, a \in [r], j \in [s]\}.\]
(In the next section, we will need to use $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$ in the case when $G^i$ is an arbitrary complete flag on $V$, not necessarily an isotropic flag.)

Now assume that $(\mathcal{F}, \mathcal{G})$ is a generic point of $\text{Fl}(M)^s \times \text{Fl}(V)^s$, where $\text{Fl}(V)$ is the full isotropic flag variety of $V$ and $\text{Fl}(M)$ is the full flag variety of $M$. Then,

**Theorem 34.** The following are equivalent:

(A) $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$ is of the expected dimension $2nr - \sum_{j=1}^s |\mu^j|$, where $|\mu^j| := \sum_{a=1}^r \mu_a^j$.

(B) For any $1 \leq d \leq r$ and subsets $B^1, \ldots, B^s$ of $[r]$ each of cardinality $d$ such that the product $\prod_{j=1}^s [\Omega_{B^j}] \neq 0 \in H^*(\text{Gr}(d, r))$, the inequality $\sum_{j=1}^s \sum_{a \in B^j} \mu_a^j \leq 2dn$ holds.

We record the following corollary.

**Corollary 35.** Suppose that $2nr - \sum_{j=1}^s |\mu^j| = 0$. Then (A) (or (B)) is equivalent to the condition

$$\left( V_{\mu^1} \otimes V_{\mu^2} \otimes \cdots \otimes V_{\mu^s} \right)^{SL(r)} \neq 0,$$

where $V_{\mu^j}$ is the irreducible representation of $SL(r)$ as in Section 6.2.

**Proof.** The equivalence of the condition in the corollary and condition (B) in Theorem 34 is a consequence of the Knutson-Tao saturation theorem for $SL(n)$ [KT], together with Klyachko’s work [K]: these works together characterize the existence of invariants in a tensor product of $SL(r)$ representations by a system of inequalities, which is (B) (cf. [F4]).

5.1. (A) implies (B) in Theorem 34. This follows from [B1], Proposition 2.8 (1). The idea is that if (B) fails, pick an $S \in \cap_{j=1}^s \Omega_{B^j}(F^j) \subset \text{Gr}(d, M)$, compute the expected dimension of the vector space $\{ \phi \in \mathcal{H}_\mu(\mathcal{F}, \mathcal{G}) : \phi(S) = 0 \}$, and find it to be greater than the expected dimension of $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$.

5.2. (B) implies (A) in Theorem 34. We proceed by induction on $r$. To show that (B) implies (A), suppose $\phi^o$ is a general member of $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$, $S = \ker(\phi^o)$ and assume $S \in \cap_{j=1}^s \Omega_{B^j}(F^j) \subset \text{Gr}(d, M)$, for $B^j = \{ b^j_1 < \cdots < b^j_d \}$.

We will use ideas of Schofield [S] in the proof. We replace the Ext groups in [S] by the cohomology of suitable (2-step) complexes, and replace the long exact sequences in cohomology by the snake lemma. There are two other ingredients required (beyond the technique of Schofield).

- A critical dimension count (Proposition 36), for which we need Theorem 10.
- An idea from [B1] on genericity of induced structures (we could have used another technique of Schofield instead as well).

We were unable to use the strategy of [B1], Section 5 (essentially because we could not show that the induced flags on $S$ and $V/S$ were “mutually generic” in the situation here).
5.3. The set up, and the key inequality. Let \( L = \text{im}(\phi^o) \). Let \( \mathcal{A}^0(\text{Hom}(M, V)) = \text{Hom}(M, V) \), and

\[
\mathcal{A}^1(\text{Hom}(M, V)) = \oplus_{j=1}^{s} \text{Hom}(M, V)/P_j,
\]

where

\[
P_j = \{ \tau \in \text{Hom}(M, V) : \tau(F^j_a) \subset G^j_{2n-\mu^j_a}, a = 1, \ldots, r \}.
\]

There is a natural differential (a direct sum of projections) \( d : \mathcal{A}^0(\text{Hom}(M, V)) \to \mathcal{A}^1(\text{Hom}(M, V)). \) Call this (2-term) complex \( \mathcal{A}^* (\text{Hom}(M, V)) \). Clearly, \( H^0(\mathcal{A}^*(\text{Hom}(M, V))) \) is the same as \( H^0 \mu(M, V) \) and \( \chi(\mathcal{A}^*(\text{Hom}(M, V))) \) is the “expected dimension” of \( H^0 \mu(M, V) \), which is \( 2nr - \sum_{j=1}^{s} |\mu^j| \). To make the dependence on \( (F, G) \) and \( \mu \) precise, we sometimes write this complex as \( \mathcal{A}^*(M, V, F, G, \mu) \).

Hypothesis (A) is clearly equivalent to

\[
(C) \ h^1(\mathcal{A}^*(\text{Hom}(M, V))) = 0.
\]

We have surjections \( \text{Hom}(M, V) \xrightarrow{\tau} \text{Hom}(S, V) \xrightarrow{\pi} \text{Hom}(S, V/L) \). Let

\[
\mathcal{A}^0(\text{Hom}(S, V)) = \text{Hom}(S, V), \ \mathcal{A}^0(\text{Hom}(S, V/L)) = \text{Hom}(S, V/L)
\]

and

\[
\mathcal{A}^1(\text{Hom}(S, V)) = \oplus_{j=1}^{s} \text{Hom}(S, V)/\tau(P_j), \ \mathcal{A}^1(\text{Hom}(S, V/L)) = \text{Hom}(S, V/L)/\pi \circ \tau(P_j).
\]

There are natural differentials \( d : \mathcal{A}^0 \to \mathcal{A}^1 \) in each case. It is easy to see that

\[
\tau(P_j) = \{ \phi \in \text{Hom}(S, V) : \phi(F^j(S)_x) \subset G^j_{2n-\mu^j_{b^j_x}}, x = 1, \ldots, d \}.
\]

So, if we let \( \gamma^j_x = \mu^j_{b^j_x} \), the (2-term) complex \( \mathcal{A}^*(\text{Hom}(S, V)) \) is the same as \( \mathcal{A}^*(S, V, F(S), G, \gamma) \) where \( F(S) \) is the induced \( s \)-tuple of flags on \( S \).

\[
\pi \circ \tau(P_j) = \{ \phi \in \text{Hom}(S, V/L) : \phi(F^j(S)_x) \subset G^j_{2n-\gamma^j_x} + L/L, x = 1, \ldots, d \}.
\]

Define numbers \( \delta^j_x \) by

\[
2n - \dim(L) - \delta^j_x = 2n - \gamma^j_x - \dim(G^j_{2n-\gamma^j_x} \cap L)
\]

and hence, \( \mathcal{A}^*(\text{Hom}(S, V/L)) = \mathcal{A}^*(S, V/L, F(S), G(V/L), \delta) \).

The following critical result will be proved in Section 5.6 by a dimension count.

**Proposition 36.**

\[
h^1(\mathcal{A}^*(\text{Hom}(M, V))) \leq -\chi(\mathcal{A}^*(\text{Hom}(S, V/L))).
\]
5.4. **Proof of (B) implies (A) in Theorem 34 assuming Proposition 36.** The map of (2-term) complexes $A^*(\text{Hom}(M,V)) ightarrow A^*(\text{Hom}(S,V/L))$ is surjective in each degree. Therefore, the map

$$\theta : H^1(A^*(\text{Hom}(M,V))) \rightarrow H^1(A^*(\text{Hom}(S,V/L))) \tag{30}$$

is a surjection.

Assume Proposition 36. Hence,

$$h^1(A^*(\text{Hom}(M,V))) \leq -\chi(A^*(\text{Hom}(S,V/L))) \leq h^1(A^*(\text{Hom}(S,V/L))). \tag{31}$$

The surjection (30) and the Inequality (31) imply that $\theta$ is an isomorphism. Thus, equality holds in all the inequalities in (31); in particular, $h^0(A^*(\text{Hom}(S,V/L))) = 0$.

The isomorphism $\theta$ factors as

$$H^1(A^*(\text{Hom}(M,V))) \xrightarrow{\theta} H^1(A^*(\text{Hom}(S,V))) \rightarrow H^1(A^*(\text{Hom}(S,V/L))),$$

where the map $\bar{\tau}$ is induced from the surjective map of complexes

$$A^*(\text{Hom}(M,V)) \rightarrow A^*(\text{Hom}(S,V)),$$

and is hence surjective. We conclude that $\bar{\tau}$ is an isomorphism.

It follows from [B1] that the induced flags $(\mathcal{F}(S), \mathcal{G})$ can also be assumed to be suitably generic (see Section 5.5). Now, by induction on the dimension of $M$, assume the validity of Theorem 34 with $M$ replaced by $S$ and $\mu^j$ replaced by $\gamma^j$. The hypothesis (B) holds because of Lemma 37 below.

Therefore, by conclusion (C) (valid by induction), we find that $h^1(A^*(\text{Hom}(S,V/L))) = 0$. Hence $h^1(A^*(\text{Hom}(M,V))) = 0$ as desired. This completes the proof of Theorem 34.

**Lemma 37.** Let $1 \leq d' \leq d$. In the above situation, suppose $C^1, \ldots, C^s$ are subsets of $[d]$ each of cardinality $d'$ such that $\prod_{j=1}^s [\Omega_{C^j}] \neq 0 \in H^*(\text{Gr}(d',d))$. Then, if $T^j = \{b^j_a : a \in C^j\}$,

(i) $\prod_{j=1}^s [\Omega_{T^j}] \neq 0 \in H^*(\text{Gr}(d',r))$.

(ii) $\sum_{j=1}^s \sum_{a \in C^j} \gamma^j_a = \sum_{j=1}^s \sum_{b \in T^j} \mu^j_b \leq 2d'n$.

**Proof.** Since $S \subseteq \bigcap_{j=1}^s \Omega_{B^j}(F^j) \subseteq \text{Gr}(d,M)$, and $F^j$ are generic flags on $M$, the product $\prod_{j=1}^s [\Omega_{B^j}] \neq 0 \in H^*(\text{Gr}(d',d))$. Now using the hypothesis $\prod_{j=1}^s [\Omega_{C^j}] \neq 0 \in H^*(\text{Gr}(d',d))$, and [F3], Proposition 1, we conclude that (i) holds. Because of our assumption (B), (i) implies the inequality in (ii). The equality in (ii) is trivial. □

5.5. **Genericity of induced structures.** Let $U \subseteq \text{Fl}(M)^s \times \text{IFl}(V)^s$ be a nonempty open subset of the space of pair of flags which is generic for our dimension calculations. There is a nonempty open subset $\hat{U} \subseteq \text{Fl}(S)^s \times \text{IFl}(V)^s$ of points $(\mathcal{H}, \mathcal{G}')$ for which $(\mathcal{H}, \mathcal{G}')$ is generic for the application of induction on $A^*(S,V,\mathcal{H}, \mathcal{G}', \gamma)$. The projection of $\hat{U}$ to $\text{IFl}(V)^s$ is a (diagonal) $\text{Sp}(V)$-invariant open subset of $\text{IFl}(V)^s$ which does not depend upon the choice of $S \subseteq \text{Gr}(d,M)$. 


We obtain a point \((\mathcal{F}(S), \mathcal{G}) \in \text{Fl}(S)^s \times \text{Fl}(V)^s\) as in the previous section. Since \(\mathcal{G}\) is generic, it may be assumed to be in the projection of \(\hat{U}\) to \(\text{Fl}(V)^s\). Let \(\mathcal{H} \in \text{Fl}(S)^s\) be close to \(\mathcal{F}(S)\) (in the complex topology) such that \((\mathcal{H}, \mathcal{G}) \in \hat{U}\). Now find \(\mathcal{F}' \in \text{Fl}(M)^s\) so that \(\mathcal{F}'(S) = \mathcal{H}\), \(\mathcal{F}'(M/S) = \mathcal{F}(M/S)\) and \(S \in \cap_{j=1}^s \Omega_{B_j}(\mathcal{F}'^j)\) (see Lemma 2.4 in [B1]). Clearly \((\mathcal{F}', \mathcal{G})\) is close to \((\mathcal{F}, \mathcal{G})\) and therefore is in \(\text{Fl}(M)^s \times \text{Fl}(V)^s\) as well (and hence belongs to \(U\)). Now \(\phi^0 \in H^0(\mathcal{A}^*(M, V, \mathcal{F}', \mathcal{G}, \mu))\) and we replace \((\mathcal{F}, \mathcal{G})\) by \((\mathcal{F}', \mathcal{G})\).

### 5.6. Proof of Proposition 36

Since we already know the Euler characteristic of \(\mathcal{A}^*(\text{Hom}(M, V))\), we want to give a formula for \(h^0(\mathcal{A}^*(\text{Hom}(M, V))) = \mathcal{H}_\mu(\mathcal{F}, \mathcal{G})\).

If we drop the isotropy condition on the flags \(\mathcal{G}\), then we are in a situation where Schofield’s original argument can be applied. We will compute \(\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})\) by a procedure that “essentially” does not see the isotropy condition on the flags \(\mathcal{G}\), and hence Schofield’s set up generalizes.

To calculate the dimension of \(\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})\), we can fiber the universal parameter space of triples \((\phi, \mathcal{F}, \mathcal{G})\) with \(\phi\) generic in \(\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})\), over the parameter space of pairs \((\text{im}(\phi), \mathcal{G})\) so that \(\text{im}(\phi)\) is in its generic Schubert state with respect to the flag \(\mathcal{G}\). The dimension of the choices of such pairs is the expected one, thanks to our main transversality result Theorem 10.

### 5.7. Calculation of \(\chi(\mathcal{A}^*(\text{Hom}(S, V/L)))\)

First, as in Subsection 5.3, \(\pi \circ \tau(P^j)\) is the same as

\[
\{\theta \in \text{Hom}(S, V/L) : \theta(F^j(S)_x) \subset G^j_{2n-\gamma^j_x} + L/L, \ x = 1, \ldots, d\},
\]

which has dimension \(\sum_{x=1}^d (2n - \gamma^j_x - \dim(L \cap G^j_{2n-\gamma^j_x}))\). Therefore, \(\chi(\mathcal{A}^*(\text{Hom}(S, V/L)))\) equals

\[
d(2n - r + d) - \sum_{j=1}^s (d(2n - r + d) - \sum_{x=1}^d (2n - \gamma^j_x - \dim(L \cap G^j_{2n-\gamma^j_x})))
\]

\[
= d(2n - r + d) + \sum_{j=1}^s \sum_{x=1}^d (r - d - \gamma^j_x - \dim(L \cap G^j_{2n-\gamma^j_x}))
\]

### 5.8. The dimension of \(\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})\):

Assume that

\(L \in \cap_{j=1}^s \Omega_{A^j}(G^j) \subset \text{Gr}(r - d, V)\).

Let \(U_A(V, s) \subset \text{Fl}(V)^s\) be the open subset consisting of isotropic flags \(E^1_\bullet, \ldots, E^s_\bullet\) such that the intersection \(\cap_{j=1}^s \Omega_{A^j}(E^j)\) inside \(\text{Gr}(r - d, V)\) is proper. By Theorem 10, this is nonempty. Let \(\mathcal{U}\) denote the parameter space

\[
\{(\mathcal{F}, \mathcal{G}, \phi, S, L) : \mathcal{F} \in \text{Fl}(M)^s, \mathcal{G} \in U_A(V, s), L \in \cap_{j=1}^s \Omega_{A^j}(G^j), S \in \cap_{j=1}^s \Omega_{B^j}(F^j), \phi \in \mathcal{H}_\mu(\mathcal{F}, \mathcal{G}), S = \ker(\phi), L = \text{im}(\phi)\}.
\]
Now, the dimension of $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$ equals the dimension of a generic fiber of $\mathcal{U} \rightarrow \text{FL}(M)^s \times U_A(V, s)$. We will therefore need to give an upper bound for the dimension of irreducible components of $\mathcal{U}$.

Clearly, $\mathcal{U}$ maps to the parameter space

$$V = \{(\mathcal{G}, \phi, S, L) : \mathcal{G} \in U_A(V, s), L \in \cap_{j=1}^s \Omega_{A^j}(G^j), S \in \text{Gr}(d, M), \phi \in \text{Hom}(M, V),$$

$$S = \ker(\phi), L = \text{im}(\phi)\}.$$

5.8.1. The dimension of $V$. Clearly $V$ fibers over the space

$$W = \{(L, \mathcal{G}) : \mathcal{G} \in U_A(V, s), L \in \cap_{j=1}^s \Omega_{A^j}(G^j)\}$$

with irreducible fibers of dimension $d(r - d) + (r - d)^2$ (the first term is the dimension of the space of choices of $S$ and the second term is the dimension of the space of isomorphisms $M/S \xrightarrow{\phi} L$).

Now the fiber dimension of $W \rightarrow U_A(V, s)$ is $((r - d)(2n - r + d) - \sum_{j=1}^s \text{codim}(\Omega_{A^j}))$. Therefore, the dimension of any irreducible component of $V$ is no more than

$$\dim(\text{IFl}(V)^s) + ((r - d)(2n - r + d) - \sum_{j=1}^s \text{codim}(\Omega_{A^j})) + d(r - d) + (r - d)^2.$$

By simplifying the above expression, we find

$$\dim V \leq 2nr + (r - d - 2n)d + \dim(\text{IFl}(V)^s) - \sum_{j=1}^s \text{codim}(\Omega_{A^j}).$$

(33)

5.8.2. Codimension of $\Omega_{A^j}$.

Lemma 38. $\dim(\Omega_{A^j}) \leq \sum_{a \notin B^j}(2n - \mu_a^j - \dim(L \cap G^j_{2n-\mu_a^j}))$, and hence

$$\text{codim}(\Omega_{A^j}) \geq \sum_{a \notin B^j}(\mu_a^j + \dim(L \cap G^j_{2n-\mu_a^j}) - (r - d)).$$

Proof. Let $A^j = \{a_1 < \cdots < a_{r-d}\}$. Therefore, $\dim(\Omega_{A^j})$ equals $\sum_{b=1}^{r-d}(a_b - b)$. Let $[r] \setminus B^j = \{x_1 < \cdots < x_{r-d}\}$. It suffices therefore to show that, for any $1 \leq b \leq r - d$,

$$a_b - b \leq 2n - \mu_{x_b}^j - \dim(L \cap G^j_{2n-\mu_{x_b}^j}),$$

i.e.,

$$a_b \leq b + 2n - \mu_{x_b}^j - \dim(L \cap G^j_{2n-\mu_{x_b}^j}).$$

(34)

Set $\dim(L \cap G^j_{2n-\mu_{x_b}^j}) = b + y$, where $y \geq 0$ (since $L \cap G^j_{2n-\mu_{x_b}^j}$ contains $\phi(F_{x_b}^j)$ which is $b$-dimensional).

Hence, to prove Equation (34), we need to show that $L \cap G^j_{2n-\mu_{x_b}^j} - y$ is at least $b$-dimensional, which is now immediate because $L \cap G^j_{2n-\mu_{x_b}^j}$ is $(b + y)$-dimensional. □
5.8.3. The fiber dimension of $U \to V$. We need to do the following basic calculation. Let $S \subset M$ be a $d$-dimensional subspace, $L$ a subspace of $V$ of dimension $r - d$ and $B \in S(d, r)$.

**Lemma 39.** Let $G_\bullet$ be a fixed flag on $V$, and $\phi : M \to V$ a map with kernel $S$ and image $L \in \Omega_A(G_\bullet)$. The dimension of the space of flags $F_\bullet \in \text{Fl}(M)$ so that $S \in \Omega_B(F_\bullet)$ and $\phi(F_a) \subset G_{2n-\mu_a}$ for $a = 1, \ldots, r$, equals

$$\dim(\text{Fl}(M)) - (\text{codim}(\Omega_B) + \sum_{a \notin B} (r - d - \dim(L \cap G_{2n-\mu_a}))).$$

**Proof.** There are two sets of conditions imposed on $F_\bullet$. The first one is that $S \in \Omega_B(F_\bullet)$; the second is that $\phi(F_a) \subset G_{2n-\mu_a} \cap L$ for $a \notin B$. These conditions are independent. □

Therefore the fiber dimension of $U \to V$ equals

$$\dim(\text{Fl}(M)^*) - \sum_{j=1}^{s} (\text{codim}(\Omega_{B_j}) + \sum_{a \notin B_j} (r - d - \dim(L \cap G_{2n-\mu_a}))).$$

(35)

5.9. **Conclusion of the proof of Proposition 36.** The dimension of any irreducible component of $U$ is less than or equal to the sum of the fiber dimension of $U \to V$ (given by (35)) and the right hand side of Inequality (33). This sum equals

$$\dim(\text{Fl}(M)^*) - \sum_{j=1}^{s} (\text{codim}(\Omega_{B_j}) + \sum_{a \notin B_j} (r - d - \dim(L \cap G_{2n-\mu_a}))) + 2nr + (r - d - 2n)d + \dim(\text{Fl}(V)^*) - \sum_{j=1}^{s} \text{codim}(\Omega_{A_j}).$$

(36)

The dimension of $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$, which equals the dimension of the generic fiber of the map $U \to \text{Fl}(M)^* \times U_A(V, s)$, is therefore less than or equal to the integer (36) minus the dimension of $\text{Fl}(M)^* \times U_A(V, s)$. Therefore, using Lemma 38,

$$\dim(\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})) \leq - \sum_{j=1}^{s} (\text{codim}(\Omega_{B_j}) + \sum_{a \notin B_j} (r - d - \dim(L \cap G_{2n-\mu_a})))$$

$$+ 2nr + d(r - d - 2n) - \sum_{j=1}^{s} \sum_{a \notin B_j} (\mu_a + \dim(L \cap G_{2n-\mu_a}) - (r - d))$$

$$= 2nr + d(r - d - 2n) - \sum_{j=1}^{s} \text{codim}(\Omega_{B_j}) - \sum_{j=1}^{s} \sum_{a \notin B_j} \mu_a.$$
Now, $h^1(A^\ast (M, V))$ equals the dimension of $\mathcal{H}_{\mu}(\mathcal{F}, G)$ minus the Euler characteristic which is $2nr - \sum_{j=1}^{s} \sum_{a=1}^{s} \mu^j_a$. This yields that $h^1(A^\ast (M, V))$ is less than or equal to

$$d(r - d - 2n) - \sum_{j=1}^{s} \text{codim}(\Omega_{B^j}) + \sum_{j=1}^{s} \sum_{a \in B^j} \mu^j_a,$$

which can be written in an illuminating (and expected) form

$$(\dim \text{Gr}(d, r) - \sum_{j} \text{codim}(\Omega_{B^j})) + (\sum_{j=1}^{s} \sum_{a \in B^j} \mu^j_a) - 2nd.$$

Comparing Equation (32) with Equation (37), it suffices to show (for each $j$)

$$\sum_{x=1}^{d} (d - r + b^j_x - x) + \sum_{a \in B^j} \mu^j_a \leq -\sum_{x=1}^{d} ((r - d) - \gamma^j_x - \dim(L \cap G^j_{2n-\gamma^j_x})),$$

which rearranges to

$$\sum_{x=1}^{d} \dim(L \cap G^j_{2n-\gamma^j_x}) \geq \sum_{x=1}^{d} (b^j_x - x).$$

But, this is clear because $L \cap G^j_{2n-\gamma^j_x}$ contains the $b^j_x - x$ dimensional subspace $\phi(F_{b^j_x})$. This completes the proof of Proposition 36 and hence Theorem 34 is proved. \qed

6. Tensor product decomposition for \text{SL}(r) versus \text{Sp}(2n)

Let $G, B$ be as in the beginning of Section 2.

6.1. Line bundles on $G/B$. For a character $\chi : B \to \mathbb{C}^\ast$, let $\mathcal{L}_\chi$ be the corresponding $G$-equivariant (homogeneous) line bundle on $G/B$ associated to the principal $B$-bundle $G \to G/B$ via the character $\chi^{-1}$. Recall that its total space is the set of all pairs $\{(g, c) : g \in G, c \in \mathbb{C}\}$ modulo the equivalence

$$(g, c) \sim (gb, \chi(b)c).$$

If $\chi$ is a dominant weight, then

$$H^0(G/B, \mathcal{L}_\chi) = V_\chi^\ast,$$

where (as in Section 2) $V_\chi$ is the irreducible representation of $G$ with highest weight $\chi$.

If $f : G' \to G$ is a map of algebraic groups so that $f(B') \subset B$, where $B'$ and $B$ are Borel subgroups of $G'$ and $G$. Then, $f$ induces a map $\tilde{f} : G'/B' \to G/B$. It can be checked that $f^* \mathcal{L}_\chi = \mathcal{L}_{\chi \circ f}$, where $\chi \circ f$ is the composition of $\chi$ with $f_{|B'} : B' \to B$.

Specific identification in the $\text{SL}(r)$ case: Let $B$ be the standard Borel subgroup of $\text{SL}(r)$ consisting of upper triangular matrices as in Subsection 2.1. Then, the full flag variety $\text{Fl}(r) = \text{SL}(r)/B$ can be identified with the space of full flags $\mathcal{F}_r$ on a $r$-dimensional vector space $M$. As such it receives natural line bundles $\mathcal{L}_a = \det(F_a)^*$ ($1 \leq a \leq r$). This line bundle is isomorphic with the line bundle $\mathcal{L}_{\omega_a}$, where $\omega_a$ is the
character of \( B \) which takes the diagonal matrix \((t_1, \ldots, t_r)\) to the product \(\prod_{i=1}^r t_i\). Recall that \(\omega_a\) is the \(a^{th}\) fundamental weight.

6.2. **Representations of \( \text{Gl}(r) \).** Irreducible polynomial representations of the general linear group \( \text{Gl}(r) \) correspond to the sequences of integers \( \mu = (\mu_1 \geq \cdots \geq \mu_r \geq 0) \) (also called Young diagrams or partitions). Given a sequence of integers \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0) \), form the line bundle

\[
\mathcal{L}_\mu = \prod_{a=1}^r \mathcal{L}_a^{\mu_a - \mu_{a+1}},
\]

where \( (\mu_{r+1} = 0) \). Then, \( H^0(\text{Fl}(r), \mathcal{L}_\mu) = V_\mu^* \), where \( V_\mu \) is the irreducible representation of \( \text{Gl}(r) \) with highest weight \( \mu \). To any such \( \mu \), we associate the character \( \bar{\mu} \) of \( B \) by

\[
\bar{\mu}(t) = \prod_{i=1}^r t_i^{\mu_i}, \text{ where } t = \text{diag}(t_1, \ldots, t_r).
\]

Then, clearly, \( \mathcal{L}_\mu \cong \mathcal{L}_{\bar{\mu}} \).

6.3. **A Basic Calculation.** As earlier in Subsection 2.2, let \( V = \mathbb{C}^{2n} \) be equipped with the symplectic form \( \langle \cdot, \cdot \rangle \) and \( M \) a vector space of dimension \( r \).

We will now make a basic determinantal calculation (which appeared originally in a weaker form in [B2, Lemma 4.5]). Let \( P := \text{Hom}(M, V) \). Fix an integer \( m \leq 2n \). For a full flag \( F_\bullet \) on \( M \), a full flag \( G_\bullet \) on \( V \), and a partition \( \mu = (m \geq \mu_1 \geq \cdots \geq \mu_r \geq 0) \), let \( P_m(\mu, F_\bullet, G_\bullet) \) denote the subspace

\[
\{ \phi \in \text{Hom}(M, V) : \phi(F_a) \subset G_{m - \mu_a}, \forall a \in [r] \}.
\]

Consider the vector bundle \( T_m(\mu) \) (respectively, \( P_m(\mu) \)) on \( \text{Fl}(M) \times \text{Fl}(V) \) whose fiber over \( (F_\bullet, G_\bullet) \) is \( T_m(\mu, F_\bullet, G_\bullet) := \text{Hom}(M, V)/P_m(\mu, F_\bullet, G_\bullet) \) (respectively, \( P_m(\mu, F_\bullet, G_\bullet) \)).

We now calculate the determinant line bundles of \( P_m(\mu) \) and \( T_m(\mu) \). We begin the computation by setting (for \( 1 \leq b \leq r \))

\[
P_m^b = \{ \phi \in \text{Hom}(M, V) : \phi(F_a) \subset G_{m - \mu_a}, a = 1, \ldots, b \}.
\]

There exist exact sequences for \( 0 \leq b \leq r - 1 \):

\[
0 \to P_m^{b+1} \to P_m^b \to \text{Hom}(F_{b+1}/F_b, V/G_{m - \mu_{b+1}}) \to 0,
\]

where \( P_m^0 := \text{Hom}(M, V) \). This permits us to write (\( \mathcal{L}_a := \det(F_a)^*, \mathcal{K}_b := \det(G_b)^* \))

\[
\det T_m(\mu) = \left( \prod_{a=1}^r \mathcal{L}_a^{\mu_a - \mu_{a+1}} \right) \otimes (\det M^*)^{2n-m} \otimes \left( \det V \right)^r \otimes \prod_{a=1}^r \mathcal{K}_{m - \mu_a},
\]

where \( \mu_{r+1} := 0 \). (Here we have used the fact that for vector spaces \( V_1, V_2, \det(\text{Hom}(V_1, V_2)) \cong ((\det V_1)^* \otimes (\det V_2))^{\dim V_2 \otimes \dim V_1} \). Let \( \lambda \) be the partition conjugate to the partition \( \lambda' := (m - \mu_1 \geq \cdots \geq m - \mu_r \geq 0) \). We call \( \lambda \) to be obtained from \( \mu \) by \( m\)-flip. Then, we may rewrite the above formula as

\[
\det T_m(\mu) = \left( \mathcal{L}_\mu \otimes (\det M^*)^{2n-m} \right) \otimes \left( \det V \right)^r \otimes \mathcal{L}_\lambda.
\]
6.4. The theta section. Let $M$ and $V$ be as in Section 6.3. We take $m = 2n$. Let \( \{ \mu^j \}_{1 \leq j \leq s} \) be partitions:

\[
2n \geq \mu^1_1 \geq \cdots \geq \mu^1_r \geq 0
\]
satisfying

\[
\sum_{j=1}^s |\mu^j| = 2nr,
\]

where \( |\mu^j| := \sum_{i=1}^r \mu^j_i \).

Consider two \( \text{GL}(M) \times \text{GL}(V) \)-equivariant bundles \( \mathcal{P} \) and \( \mathcal{Q} \) on \( \text{Fl}(M)^s \times \text{Fl}(V)^s \) (with the diagonal actions of \( \text{GL}(M) \) and \( \text{GL}(V) \) on \( \text{Fl}(M)^s \) and \( \text{Fl}(V)^s \) respectively). The fiber of \( \mathcal{P} \) over \( (\mathcal{F}, \mathcal{G}) \) is just the constant vector space \( \text{Hom}(M, V) \) (with the natural action of the group \( \text{GL}(M) \times \text{GL}(V) \)), where \( \mathcal{F} = (F^1_1, \ldots, F^s_1) \) is an \( s \)-tuple of full flags on \( M \) and similarly for \( \mathcal{G} \). The fiber of \( \mathcal{Q} \) over \( (\mathcal{F}, \mathcal{G}) \) is the direct sum

\[
\bigoplus_{j=1}^s T(\mu^j_1, F^1_j, G^1_j),
\]

where \( T(\mu^j_1, F^1_j, G^1_j) := T_{2n}(\mu^j_1, F^1_j, G^1_j) \). Consider the canonical \( \text{GL}(M) \times \text{GL}(V) \)-equivariant map \( S : \mathcal{P} \to \mathcal{Q} \) between vector bundles (of the same rank) on \( \text{Fl}(M)^s \times \text{Fl}(V)^s \) induced via the quotient maps. Thus, taking its determinant, we find a \( \text{GL}(M) \times \text{GL}(V) \)-equivariant global section \( \Theta \in \det(\mathcal{Q}) \otimes (\det \mathcal{P})^{-1} \) such that \( \Theta \) vanishes at \( (\mathcal{F}, \mathcal{G}) \) if and only if \( \mathcal{H}_\mu(\mathcal{F}, \mathcal{G}) \neq 0 \), where \( \mathcal{H}_\mu(\mathcal{F}, \mathcal{G}) \) is as defined in the beginning of Section 5 (extended for any, not necessarily isotropic, complete flags \( \mathcal{G} = (G^1_1, \ldots, G^s_1) \)). By Equation (38),

\[
\det(\mathcal{Q}) = L_{\mu^1_1} \otimes \cdots \otimes L_{\mu^s_1} \otimes (\det(V)^r \otimes L_{\lambda^1_1}) \otimes \cdots \otimes (\det(V)^r \otimes L_{\lambda^r_1}).
\]

Here \( \lambda^i \) is obtained from \( \mu^i \) by 2n-flip.

Fixing a point \( \mathcal{F} \in \text{Fl}(M)^s \), on restriction, we get (as in [B2]) a \( \text{SL}(V) \)-invariant section \( \Theta_\mathcal{F} \) in

\[
H^0(\text{Fl}(V)^s, L_{\lambda^1} \otimes \cdots \otimes L_{\lambda^s})^{\text{SL}(V)}.
\]

Restricting this to \( \text{IFl}(V)^s \) we find a section in

\[
H^0(\text{IFl}(V)^s, L_{\lambda^1_C} \otimes \cdots \otimes L_{\lambda^s_C})^{\text{Sp}(V)} = (M_{\lambda^1_C}^1 \otimes \cdots \otimes M_{\lambda^s_C}^1)^{\text{Sp}(2n)},
\]

where \( \text{IFl}(V) \) is the space of (full) isotropic flags on \( V \), \( \lambda^i_C \) is the weight of \( \text{Sp}(2n) \) obtained by the restriction of \( \lambda^i \) to the Cartan of \( \text{Sp}(2n) \) and \( M_{\nu} \) is the irreducible representation of \( \text{Sp}(2n) \) with highest weight \( \nu \). (Observe that for a dominant weight \( \lambda \) of \( \text{SL}(2n) \), its restriction \( \lambda^i_C \) is dominant for \( \text{Sp}(2n) \).)

We can therefore record the following consequence of Corollary 35.

**Corollary 40.** Let \( \{ \mu^j \}_{1 \leq j \leq s} \) be partitions: \( 2n \geq \mu^1_1 \geq \cdots \geq \mu^1_r \geq 0 \) such that \( \sum_j |\mu^j| = 2nr \). Assume further that

\[
(V_{\mu^1} \otimes \cdots \otimes V_{\mu^s})^{\text{SL}(r)} \neq 0.
\]
Then,
\[
(M_{\nu^1} \otimes \cdots \otimes M_{\nu^s})^{\text{Sp}(2n)} \neq 0,
\]
where \(\nu^j := \lambda^j_C\) and \(\lambda^j\) is obtained from \(\mu^j\) by 2n-flip.

We now come to the main theorem of this section.

**Theorem 41.** Let \(V_{\lambda^1}, \ldots, V_{\lambda^s}\) be irreducible representations of \(\text{SL}(2n)\) (with highest weights \(\lambda^1, \ldots, \lambda^s\) respectively) such that their tensor product has a nonzero \(\text{SL}(2n)\)-invariant. Then, the tensor product of the representations of \(\text{Sp}(2n)\) with highest weights \(\lambda^1_C, \ldots, \lambda^s_C\) has a nonzero \(\text{Sp}(2n)\)-invariant, where \(\lambda^j_C\) is the restriction of \(\lambda^j\) to the Cartan \(h^C\) of \(\text{Sp}(2n)\).

**Proof.** We would like to find some \(r\) and partitions \(\mu^1, \ldots, \mu^s\) such that
\[
2n \geq \mu^1 \geq \cdots \geq \mu^s \geq 0
\]
with \(\sum_{j=1}^{s} |\mu^j| = 2nr\), and such that \(\lambda^j\) is obtained from \(\mu^j\) by 2n-flip.

To achieve this, express \(\lambda^j\) as sequences of integers (with \(r\) large)
\[
r \geq \lambda^1 \geq \cdots \geq \lambda^{2n} \geq 0
\]
so that (since the tensor product of the corresponding representations of \(\text{SL}(2n)\) has a nonzero \(\text{SL}(2n)\)-invariant)
\[
\sum_{j=1}^{s} \sum_{a=1}^{2n} \lambda^j_a = 2nb' \text{ for some integer } b' \geq 0,
\]
i.e.,
\[
\sum_{j=1}^{s} \sum_{a=1}^{2n} (r - \lambda^j_a) = 2nr + 2nb,
\]
where \(b := r(s-1) - b'\). Taking \(r \geq \frac{2n}{s} \lambda^j_1\) (for every \(j\)), we can assume that \(b \geq 0\).

Rewrite the above equality as \(\sum_{j=1}^{s} \sum_{a=1}^{2n} (r + b - (b + \lambda^j_a)) = 2n(r + b)\). So, we replace \(r\) by \(r + b\) and \(\lambda^j_a\) by \(b + \lambda^j_a\) to assume \(b = 0\). Let \(\mu^j\) be obtained from \(\lambda^j\) by \(r\)-flip. Since, by assumption,
\[
(V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^s})^{\text{SL}(2n)} \neq 0,
\]
we obtain (using ordinary duality)
\[
(V_{\lambda^1'} \otimes \cdots \otimes V_{\lambda^s'})^{\text{SL}(2n)} \neq 0,
\]
where \(\lambda^j' = (r - \lambda^j_{2n}, \ldots, r - \lambda^j_1)\).

Now using Grassmann duality (see Section 6.5),
\[
(V_{\mu^1} \otimes \cdots \otimes V_{\mu^s})^{\text{SL}(r)} \neq 0.
\]
Now, apply Corollary 40. (Observe that 2n-flip of \(\mu^j\) is the partition \(\lambda^1 - \lambda^2_{2n} \geq \cdots \geq \lambda^j_{2n-1} - \lambda^j_{2n} \geq 0\).) This completes the proof of the theorem. \(\square\)
6.5. **Grassmann duality.** Let \( r \) and \( k \) be positive integers and let \( m = r + k \). Given a Young diagram \( \mu = (k \geq \mu_1 \geq \cdots \geq \mu_r \geq 0) \), one obtains naturally a cohomology class \( \omega_\mu \) of \( \text{Gr}(r, m) \), which is the cycle class of \( \Omega_{A(\mu)} \) with \( A(\mu) = \{m - r + a - \mu_a | a \in [r] \} \).

It is known that if \( \mu_1, \ldots, \mu^s \) are such Young diagrams with \( \sum_{j=1}^s |\mu_j| = kr \), then the dimension of \( \left( V_{\mu_1} \otimes \cdots \otimes V_{\mu^s} \right)^{SL(r)} \) equals the coefficient of the cycle class of a point in the cup product \( \prod_{j=1}^s \omega_{\mu_j} \in H^{2rk}(\text{Gr}(r, m)) \) (see, e.g., [F1], Chapter 9).

Clearly, under the natural duality map \( \text{Gr}(r, \mathbb{C}^m) = \text{Gr}(k, (\mathbb{C}^m)^*) \), the class \( \omega_\mu \) goes to \( \omega_{\tilde{\mu}} \), where \( \tilde{\mu} \) is the conjugate diagram to \( \mu \) (see, e.g., [F1], Exercise 20, page 152). We therefore obtain the following consequence, which we call the **Grassmann duality**.

**Lemma 42.** Suppose \( \sum_{j=1}^s |\mu_j| = kr \). Then,

\[
\dim \left( V_{\mu_1} \otimes \cdots \otimes V_{\mu^s} \right)^{SL(r)} = \dim \left( V_{\tilde{\mu}_1} \otimes \cdots \otimes V_{\tilde{\mu}^s} \right)^{SL(k)}.
\]

We clearly have another “ordinary duality”. Let \( \nu^j \) be the dual partition to \( \mu^j \) (so that \( \nu^j = (k - \mu^j_1, \ldots, k - \mu^j_r) \)). Then, clearly, \( V_{\nu^j} \) is the \( SL(r) \)-dual of \( V_{\mu^j} \) and hence

\[
\dim \left( V_{\mu_1} \otimes \cdots \otimes V_{\mu^s} \right)^{SL(r)} = \dim \left( V_{\nu_1} \otimes \cdots \otimes V_{\nu^s} \right)^{SL(r)}.
\]

**7. Saturation for \( \text{Sp}(2n) \) and \( \text{SO}(2n+1) \)**

Let \( X(H^C)_+ \) be the set of dominant characters of \( H^C \) (for the group \( \text{Sp}(2n) \)) and similarly let \( X(H^B)_+ \) be the set of dominant characters of \( H^B \) (for the group \( \text{SO}(2n+1) \)). We have the following saturation theorems for the groups \( \text{Sp}(2n) \) and \( \text{SO}(2n+1) \) respectively.

**Theorem 43.** Given \( \nu^1, \ldots, \nu^s \in X(H^C)_+ \), the following are equivalent:

1. For some positive integer \( N \), the tensor product of \( \text{Sp}(2n) \)-representations with highest weights \( N\nu^1, \ldots, N\nu^s \) has a nonzero \( \text{Sp}(2n) \)-invariant.
2. The tensor product of representations with highest weights \( 2\nu^1, \ldots, 2\nu^s \) has a nonzero \( \text{Sp}(2n) \)-invariant.

**Proof.** For any semisimple algebra \( g \) with Cartan subalgebra \( h \), we identify \( \kappa : h \cong h^* \) via the normalized invariant bilinear form \( \langle , \rangle \) on \( h^* \), normalized so that \( \langle \theta, \theta \rangle = 2 \) for the highest root \( \theta \). Under this identification, \( h_+ \cong D \) and, moreover, \( x_i \mapsto 2\omega_i/\langle \alpha_i, \alpha_i \rangle \).

For any \( s \geq 1 \) and \( G \) as in the beginning of Section 2, define

\[
\hat{\Gamma}(s, G) = \{(\lambda^1, \ldots, \lambda^s) \in (X(H)_+)^s: \text{ for some } N \geq 1, \ (V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^s})^G \neq 0 \}.
\]

Then, under the identification induced by \( \kappa \) (and still denoted by) \( \kappa : h_+^* \to D^s \), we have (cf., e.g., [Sj, Theorem 7.6])

\[
\kappa^{-1}(\hat{\Gamma}(s, G)) = \Gamma(s, K) \cap \kappa^{-1}((X(H)_+)^s),
\]

where the eigencone \( \Gamma(s, K) \) is defined in Section 4.
Recall from Subsection 2.2 the inclusion \( i : h^C \hookrightarrow h^{A_{2n-1}} \) and hence we get the dual (surjective) map \( i^* : (h^{A_{2n-1}})^* \twoheadleftarrow (h^C)^* \). We have the following commutative diagram:

\[
\begin{array}{ccc}
\rightarrow \hspace{1cm} & \hspace{1cm} \rightarrow \\
\downarrow \hspace{1cm} 2\kappa_C \hspace{1cm} & \hspace{1cm} \downarrow \kappa_A \\
(h^C)^* \hspace{1cm} & \hspace{1cm} (h^{A_{2n-1}})^*, \\
\end{array}
\]

where \( \kappa_C \) (respectively, \( \kappa_A \)) denotes \( \kappa \) for \( \text{Sp}(2n) \) (respectively, \( \text{SL}(2n) \)).

We only need to prove \((1) \Rightarrow (2)\). By assumption, \( \nu := (\nu^1, \ldots, \nu^s) \in \hat{\Gamma}(s, \text{Sp}(2n)) \) and hence \( \kappa^{-1}_C \nu \in \Gamma(s, \text{Sp}(2n)) \). But, clearly, \( \Gamma(s, \text{Sp}(2n)) \subset \Gamma(s, \text{SU}(2n)) \) and hence \( \kappa^{-1}_A i \kappa^{-1}_C (\nu) \in \hat{\Gamma}(s, \text{SL}(2n)) \). (Observe that \( \kappa^{-1}_A i \kappa^{-1}_C (\nu) \in Q^{A_{2n-1}} \) is the root lattice of \( \text{SL}(2n) \).) Moreover, since \( \kappa_A i \kappa^{-1}_C (\nu^j) \in Q^{A_{2n-1}} \), by the saturation theorem of Knutson-Tao [KT],

\[
(V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^s})^{\text{SL}(2n)} \neq 0,
\]

where \( \lambda^j := \kappa_A i \kappa^{-1}_C (\nu^j) \) and \( V_{\lambda^j} \) is the irreducible representation of \( \text{SL}(2n) \) with highest weight \( \lambda^j \).

Applying Theorem 41, we get

\[
(M_{\lambda_C^1} \otimes \cdots \otimes M_{\lambda_C^s})^{\text{Sp}(2n)} \neq 0,
\]

where \( \lambda_C^j := \lambda_C^j \) and \( M_{\lambda_C^j} \) is the irreducible representation of \( \text{Sp}(2n) \) with highest weight \( \lambda_C^j \). Moreover, by the commutativity of the diagram (*), we get \( \lambda_C^j = 2\nu^j \). This proves the theorem. \( \square \)

By using [KS, Section 4] and the above theorem, we get the corresponding result for the odd orthogonal groups.

**Theorem 44.** Given \( \nu^1, \ldots, \nu^s \in X(H^B)_+ \), the following are equivalent:

1. For some positive integer \( N \), the tensor product of \( \text{SO}(2n+1) \)-representations with highest weights \( N\nu^1, \ldots, N\nu^s \) has a nonzero \( \text{SO}(2n+1) \)-invariant.
2. The tensor product of representations with highest weights \( 2\nu^1, \ldots, 2\nu^s \) has a nonzero \( \text{SO}(2n+1) \)-invariant.

**Remark 45.** (a) The general saturation result proved by Kapovich-Millson [KM] specialized to the case of \( \text{Sp}(2n) \) requires saturation factor of 4 as opposed to our saturation factor of 2 proved in Theorem 43. Our Theorem 43 is optimal for general weights \( \nu^1, \ldots, \nu^s \in X(H^C)_+ \).

(b) We can prove an analogue of Theorem 34 for \( \text{SO}(2n+1) \) by a similar method and use it to give a proof of Theorem 44 analogous to that of Theorem 43.
As an immediate consequence of Theorem 44, we get the following special case of the general saturation result due to Kapovich-Millson [KM].

**Corollary 46.** Let \( \nu^1, \ldots, \nu^s \) be dominant characters for Spin\((2n + 1)\) (whose sum is not necessarily in the root lattice). Then, the following are equivalent:

1. For some positive integer \( N \), the tensor product of representations of Spin\((2n + 1)\) with highest weights \( N\nu^1, \ldots, N\nu^s \) has a nonzero Spin\((2n + 1)\)-invariant.
2. The tensor product of representations with highest weights \( 4\nu^1, \ldots, 4\nu^s \) has a nonzero Spin\((2n + 1)\)-invariant.

We would like to conjecture the following generalization of Theorems 10, 30 and 41. Let \( G \) be a connected simply-connected, semisimple complex algebraic group and let \( \sigma \) be a diagram automorphism of \( G \).

**Conjecture 47.** (a) For any standard parabolic subgroup \( P \) and any Bruhat cells \( \Lambda^P_{w_1}, \ldots, \Lambda^P_{w_s} \) in \( G/P \), there exist elements \( k_1, \ldots, k_s \in K \) such that the intersection \( \bigcap_{i=1}^s k_i\Lambda^P_{w_i} \) is proper.

(b) Let \( V_{\lambda^1}, \ldots, V_{\lambda^s} \) be irreducible representations of \( G \) (with highest weights \( \lambda^1, \ldots, \lambda^s \) respectively) such that their tensor product has a nonzero \( G \)-invariant. Then, the tensor product of irreducible \( K \)-modules with highest weights \( \lambda^1_K, \ldots, \lambda^s_K \) has a nonzero \( K \)-invariant, where \( \lambda^i_K \) is the restriction of \( \lambda^i \) to the Cartan subalgebra \( \mathfrak{h}_K := \mathfrak{h}^\sigma \) of \( K \).

(Observe that \( \lambda^i_K \) is dominant for \( K \) for any dominant weight \( \lambda \) of \( G \) with respect to the Borel subgroup \( B^K := B^\sigma \) of \( K \).)

8. **Horn’s problem for symplectic groups**

In this section we prove an analogue of Horn’s problem for symplectic groups. Fix \( r \leq n \) and let \( I^1, \ldots, I^s \in \mathfrak{S}(r, 2n) \) be such that

\[
\sum_{j=1}^s \text{codim}(\Phi_{I_j}) = \dim \text{IG}(r, 2n).
\]  

(39)

Let \( \beta_j \) be the order preserving bijection of \( I^j \sqcup \bar{I}^j \) with \([2r] \) and let \( I_0^j \) be the subset \( \beta(I^j) \) of \([2r] \) (of cardinality \( r \)). For any \( a \in [r] \) and \( j \in [s] \), let \( \lambda_a^j := |i_a^j \geq \bar{I}^j| \) and \( \mu_a^j := 2n - 2r - \lambda_a^j \), where \( I^j = \{ i_1^j < \cdots < i_s^j \} \). Recall the definition of the deformed product \( \odot_0 \) in the cohomology \( H^*(G/P) \) from [BK, Definition 18].

**Theorem 48.** Under the assumption (39), the following conditions \((\alpha)\) and \((\beta)\) are equivalent:

\((\alpha)\) \( \prod_{j=1}^s [\Phi_{I_j}] \neq 0 \in H^*(\text{IG}(r, 2n), \odot_0) \).

\((\beta)\) \( (\beta_1) \dim \text{IG}(r, 2r) = \sum_{j=1}^s \cosym^2(I^j) \).

\((\beta_2) (V_{\nu^1} \otimes \cdots \otimes V_{\nu^s})_{\text{SL}(r)} \neq 0 \).

\((\beta_3) \prod_{j=1}^s [\Phi_{I_0^j}] \neq 0 \in H^*(\text{IG}(r, 2r)) \).
Remark 49. (1) Assume that the condition \((\beta_1)\) is satisfied. Then, the condition \((\beta_2)\) is equivalent to the following condition.

For any \(1 \leq d \leq r\) and subsets \(B^1, \ldots, B^s\) of \([r]\) each of cardinality \(d\) such that the product \(\prod_{j=1}^s [\Omega_B] \neq 0 \in H^*(\text{Gr}(d,r))\), the following inequality holds

\[
\sum_{j=1}^s \sum_{a \in B^j} \mu_a^j \leq 2d(n-r).
\]

This follows from Corollary 35 applied to a space \(M\) of dimension \(r\) and \(V = M^\perp/M\) of dimension \(2n - 2r\) and Identity (41). In fact, we only need the equivalence of the assertion in Corollary 35 and the \((B)\)-part of Theorem 34, which is due to Klyachko and Knutson-Tao (cf., the proof of Corollary 35).

(2) The condition \((\beta_2)\) in the above theorem is recursive by the work of Purbhoo and Sottile [PS]. The Purbhoo-Sottile recursion says that \((\beta_2)\) is equivalent to a system of inequalities indexed by nonvanishing \(s\)-fold intersections of Schubert cycle classes in the Grassmannians \(\text{Gr}(d,r)\) where \(0 < d < r\).

8.1. Proof of Theorem 48. We first observe that, under the assumption (39), the following two conditions are equivalent.

\[
(\gamma_1) \prod_{j=1}^s [\Phi_{I_j}] \neq 0 \in H^*(\text{IG}(r,2n), \mathfrak{g}_0).
\]

\[
(\gamma_2) \prod_{j=1}^s [\Phi_{I_j}] \neq 0 \in H^*(\text{IG}(r,2n))\) and the condition \((\beta_1)\) is satisfied.
\]

To see this we use [BK, Theorem 15] and the following lemma.

Lemma 50. For \(I \in \mathfrak{S}(r,2n), \chi^C_{w_I}(x_I^C) = \text{codim}(\Phi_I) + \text{codim}(\Phi_{I_0}), \) where \(\chi^C_{w_I}\) is defined just before Lemma 22.

Proof. Consider the exact sequence (8). Let \(L_M = \text{GL}(M) \times \text{Sp}(M^\perp/M)\) be the Levi subgroup of the stabilizer \(P_M\) of \(M\) in \(\text{Sp}(2n)\). The connected center \(\mathbb{C}^*\) of \(L_M\) (which is the center of \(\text{GL}(M)\)) acts as \(t \mapsto t^{-1}\) (multiplication by \(t^{-1}\)) on \(\text{Hom}(M, M^\perp/M)\) and by \(t \mapsto t^{-2}\) on \(\text{sym}^2(M)^*\). We conclude the proof by using Proposition 16 and the formula for \(\chi^C_{w_I}\) given at the end of page 192 in [BK].  

Let \(V\) be a vector space of dimension \(2n\) with a nondegenerate symplectic form and let \(M \subset V\) be an isotropic subspace of dimension \(r\). Let \(X = \text{IG}(r,2n)\) and assume the condition \((\beta_1)\). Let \(E_\bullet = (E^1_\bullet, \ldots, E^s_\bullet)\) be an \(s\)-tuple of isotropic flags on \(V\) such that for all \(j \in [s], M \in \Phi_{I_j}(E^j_\bullet)\). Consider the exact sequence (as in Lemma 13):

\[
0 \to \text{Hom}(M, M^\perp/M) \xrightarrow{\xi} T(X)_M \xrightarrow{\phi} \text{sym}^2 M^* \to 0.
\]

Let us abbreviate

\[
\mathcal{A}^0 = \text{Hom}(M, M^\perp/M), \ \mathcal{B}^0 = T(X)_M, \ \mathcal{C}^0 = \text{sym}^2 M^*.
\]

For \(j = 1, \ldots, s,\) let

\[
P^j = \text{Hom}(M, M^\perp/M) \cap (T(\Phi_{I_j}(E^j_\bullet))_M), \ Q^j = T(\Phi_{I_j}(E^j_\bullet))_M, \ R^j = \phi(T(\Phi_{I_j}(E^j_\bullet))_M).
\]
Define

\[ A^1 = \bigoplus_{j=1}^s A^0 / P^j, \quad B^1 = \bigoplus_{j=1}^s B^0 / Q^j, \quad C^1 = \bigoplus_{j=1}^s C^0 / R^j. \]

There are natural differentials

\[ A^0 \rightarrow A^1, \quad B^0 \rightarrow B^1, \quad C^0 \rightarrow C^1, \]

and an exact sequence of two-term complexes

\[
\begin{align*}
0 \rightarrow A^* &\xrightarrow{\xi} B^* \xrightarrow{\phi} C^* \rightarrow 0.
\end{align*}
\]

Our assumption (39) implies that \( \chi(B^*) = 0 \), and the assumption \((\beta_1)\) (in view of Proposition 16 (3)) implies that \( \chi(C^*) = 0 \). Therefore, from the above exact sequence, \( \chi(A^*) = 0 \) as well. Thus, by Proposition 16 (1)-(2),

\[
2(n - r)r - \sum_{j=1}^s |\mu_j| = 0.
\]

Moreover, \( h^0(B^*) = 0 \) if and only if \( h^0(A^*) = 0 \) and \( h^0(C^*) = 0 \) from the above exact sequence (40).

We now prove:

- \((\alpha) \Rightarrow (\beta)\): Let \( E_\bullet = (E^1_\bullet, \ldots, E^s_\bullet) \) be a generic \( s \)-tuple of isotropic flags on \( V \) such that for all \( j \in [s], M \in \Phi_{d,l}(E^j_\bullet) \). This can be achieved by the virtue of \((\alpha)\) and by simultaneously translating each \( E^j_\bullet \) by the same element of \( \text{Sp}(2n) \). Since \( E_\bullet \) is generic, \( \Omega = \cap_{j=1}^s \Phi_{d,l}(E^j_\bullet) \) is a transverse intersection at \( M \) and hence \( h^1(B^*) = 0 \). Thus, \( h^0(B^*) = 0 \) as well and we get the following:
  - \( h^0(A^*) = 0 \). Hence, by Proposition 16 (1)-(2), Identity (41) and Corollary 35 applied to \( V = M^\perp / M \), condition \((\beta_2)\) holds.
  - \( h^0(C^*) = 0 \) which implies condition \((\beta_3)\) (using Lemma 18, Proposition 16 (3) and the condition \((\beta_1)\)).

- \((\beta) \Rightarrow (\alpha)\): Assume \((\beta)\). Find a \( s \)-tuple \( E_\bullet = (E^1_\bullet, \ldots, E^s_\bullet) \) of isotropic flags on \( V \) such that \( M \) lies in the intersection \( \Omega = \cap_{j=1}^s \Phi_{d,l}(E^j_\bullet) \). The parameter space of such \( E_\bullet \) is irreducible (Lemma 17); pick a generic such \( E_\bullet \). It follows that the induced flags on \( M \) and \( M^\perp / M \) are generic. Therefore, our assumptions \((\beta_1), (\beta_2), \text{Identity } (41), \text{Corollary } 35 \) applied to \( V = M^\perp / M \) and Proposition 16 (1)-(2) imply that \( h^0(A^*) = 0 \). Moreover, our assumption \((\beta_3), \text{Proposition } 16 (3) \) and Lemma 18 imply that \( h^0(C^*) = 0 \). Hence, \( h^0(B^*) = 0 \) and thus \( h^1(B^*) = 0 \). Therefore, \( \Omega \) is a transverse intersection at \( M \), and hence \((\alpha)\) holds by the equivalence of \((\gamma_1)\) and \((\gamma_2)\). This proves the theorem.

9. Horn’s problem for odd orthogonal groups

In this section we extend the results from the last section to the odd orthogonal groups. The proofs are similar, so we will be brief. Refer to the Section 2.3 for various notation.
Fix $r \leq n$ and let $J^1, \ldots, J^s \in \mathcal{G}(r, 2n + 1)$ be such that

\begin{equation}
\sum_{j=1}^{s} \text{codim}(\Psi_{J^j}) = \dim \text{OG}(r, 2n + 1) .
\end{equation}

Let $a_j$ be the largest integer $\leq n$ in $J^j \cup \overline{J}^j$ and set $b_j := 2n + 2 - a_j$. Let $\beta_j$ be the order preserving bijection of $(J^j \cup \overline{J}^j) \setminus \{a_j, b_j\}$ with $[2r - 2]$ and let $J^j_\beta$ be the subset $\beta_j(J^j \setminus \{a_j, b_j\})$ of $[2r - 2]$ (of cardinality $r - 1$). For any $a \in [r]$ and $j \in [s]$, let $\lambda_a := |\overline{\mu}_a^j| \geq J^j|$, and $\mu_a := 2n + 1 - 2r - \lambda_a^j$, where $J^j = \{i_1^j < \cdots < i_r^j\}$ and $\overline{J}^j := [2n + 1] \setminus (J^j \cup \overline{J}^j)$.

**Theorem 51.** Under the assumption (42), the following conditions (α) and (β) are equivalent.

(α) $\prod_{j=1}^{s} \Psi_{J^j} \neq 0 \in H^*(\text{OG}(r, 2n + 1), \Omega_0)$.

(β) (β1) $\dim \text{OG}(r, 2r) = \sum_{j=1}^{s} \text{c} \omega^2(J^j)$, where $\text{OG}(r, 2r)$ is the orthogonal Grassmannian of isotropic $r$-dimensional subspaces in the $2r$-dimensional space with a symmetric nondegenerate form.

(β2) $(V_{\mu_1} \otimes \cdots \otimes V_{\mu_s})^{SL(r)} \neq 0$.

(β3) $\prod_{j=1}^{s} \Phi_{J^j} \neq 0 \in H^*(\text{IG}(r - 1, 2r - 2))$.

Similar to Remark 49, we have the following:

**Remark 52.** Assume that the condition (β1) is satisfied. Then, the condition (β2) is equivalent to the following condition.

For any $1 \leq d \leq r$ and subsets $B^1, \ldots, B^s$ of $[r]$ each of cardinality $d$ such that the product $\prod_{j=1}^{s} \overline{\Omega}_{B^j} \neq 0 \in H^*(\text{Gr}(d, r))$, the following inequality holds

$$\sum_{j=1}^{s} \sum_{a \in B^j} \mu_a^j \leq d(2n + 1 - 2r).$$

Before we can complete the proof of the theorem, we need an interplay between the following three homogeneous spaces.

### 9.1. Fundamental triple of classical homogeneous spaces.

As is well known, the following three homogeneous spaces have ‘essentially’ equivalent intersection theory.

1. The isotropic Grassmannian $\text{IG}(r - 1, 2r - 2)$.
2. The orthogonal Grassmannian $\text{OG}(r - 1, 2r - 1)$.
3. The orthogonal Grassmannians $\text{OG}^{\pm}(r, 2r)$, where $\text{OG}^{\pm}(r, 2r)$ are the two components of the orthogonal Grassmannian $\text{OG}(r, 2r)$.

The relation between (1) and (2) is discussed in the appendix. The spaces (2) and (3) are actually homeomorphic. Let $V$ be a $2r$-dimensional vector space with a symmetric
nondegenerate bilinear form. Fix a subspace $T$ of $V$ of dimension $2r - 1$, such that the quadratic form is nondegenerate on $T$. Then, there are homeomorphisms

$$\varphi^\pm : \text{OG}^\pm(r, V) \xrightarrow{\sim} \text{OG}(r - 1, T),$$

which send $N$ to $N \cap T$. Notice that $N$ can not be a subspace of $T$ (because $T$ does not have isotropic subspaces of dimension $r$).

Fix one component and denote it by $\text{OG}^+(r, 2r)$. Then, as in [BL, §3.5], the Schubert cells in $\text{OG}^+(r, 2r)$ are parameterized by the subsets $I \subset [2r]$ of cardinality $r$ satisfying:

- $I \cap \bar{I} = \emptyset$, where $\bar{I} := \{2r + 1 - a | a \in I\}$.
- The number of elements in $I$ that are less than or equal to $r$ has the same parity as $r$.

One easily sees that the above set is in bijection with $S(r - 1, 2r - 2)$, the indexing set for the Schubert cells in $\text{IG}(r - 1, 2r - 2)$. This bijection takes $I$ to $\beta(I \{r, r + 1\})$, where $\beta$ is the order preserving bijection $[2r] \setminus \{r, r + 1\} \xrightarrow{\sim} [2r - 2]$. By Theorem 55 and the above homeomorphism $\varphi^+$, this bijection preserves non-zeroness of intersection numbers.

9.2. **Proof of Theorem 51.** Under the assumption (42), the following two conditions are equivalent.

$$(\gamma_1) \prod_{j=1}^s [\bar{\Psi}_{J_{j}}] \neq 0 \in H^*(\text{OG}(r, 2n + 1), \odot_0).$$

$$(\gamma_2) \prod_{j=1}^s [\bar{\Psi}_{J_{j}}] \neq 0 \in H^*(\text{OG}(r, 2n + 1))$$ and the condition $(\beta_1)$ is satisfied.

A straightforward reworking of the proof of Theorem 48 yields Theorem 51 with condition $(\beta_3)$ replaced by the following:

$$(\beta'_3) \text{ For generic complete flags } F^1, \ldots, F^s \text{ on a } r\text{-dimensional vector space } M, \text{ the vector space}$$

$$(43) \{ \gamma \in \wedge^2 M^* | \gamma(F^i_a, F^j_{t_a}) = 0, a \in [r], j \in [s] \}$$

is of the expected dimension

$$r(r - 1)/2 - \sum_{j=1}^s \text{co } \wedge^2(J^j),$$

where $J^j := \{i_1^j < \cdots < i_r^j\}$ and $t_a^j := |J^j_a| \geq i_a^j$.

In the following lemma, we will consider only those isotropic flags $F^*_r$ on $V$ such that $F^*_r \in \text{OG}^+(r, 2r)$. We have the following analogue of Proposition 16 (3) for the Grassmannian $\text{OG}^+(r, 2r)$.

**Lemma 53.** The tangent space at a point $P$ to the shifted Schubert cell in $\text{OG}^+(r, 2r)$ parameterized by $I = \{i_1 < \cdots < i_r\}$ and corresponding to an isotropic flag $F^*_r$ on $V$ is given by

$$(44) \{ \gamma \in \wedge^2 P^* | \gamma(F(P)_a, F(P)_{t_a}) = 0, \forall a \in [r] \},$$
where \( t_a := |I| \geq i_a \) and (as earlier) \( F(P)_* \) is the filtration on \( P \) induced from that of \( F_* \).

Let \( s_r \) be the permutation in the symmetric group \( S_{2r} \) which transposes \( r \) and \( r + 1 \). Then, observe that in the above lemma, the space (44) corresponding to \( I \) is the same as that for \( s_r(I) \).

Using an obvious analogue of Lemma 18 (with \( IG(r, 2r) \) replaced by \( OG^+(r, 2r) \)) and the relation between structure constants in \( OG^+(r, 2r) \) and \( IG(r - 1, 2r - 2) \) as given in Subsection 9.1, we conclude that the conditions (\( \beta_3 \)) and (\( \beta'_3 \)) are the same. This proves Theorem 51. \( \square \)

10. Appendix. Relation between intersection theory of homogenous spaces for \( \text{Sp}(2n) \) and \( \text{SO}(2n + 1) \)

For completeness, we include a proof of the following well-known Theorem 55 (cf., e.g., [BS, pages 2674-75]). As in Section 2, there is a canonical Weyl group equivariant identification between the Cartan subalgebras \( \mathfrak{h}^C \) and \( \mathfrak{h}^B \) (and also the Weyl groups \( W^C \) and \( W^B \)) of \( \text{Sp}(2n) \) and \( \text{SO}(2n + 1) \) respectively. We will make these identifications and denote them by \( \mathfrak{h} \) and \( W \). Let \( A_i^C \) (respectively, \( A_i^B \)) be the BGG operators for \( \text{Sp}(2n) \) (respectively, \( \text{SO}(2n + 1) \)) acting on \( S(\mathfrak{h}^*) = \mathbb{C}[\epsilon_1, \ldots, \epsilon_n] \) corresponding to the simple reflections \( s_i \), where \( \{\epsilon_1, \ldots, \epsilon_n\} \) is the basis of \( \mathfrak{h}^* \) as in [Bo, Planche II and III].

Lemma 54.

\[
A_i^B = A_i^C, \quad \text{for } 1 \leq i < n, \\
\frac{1}{2} A_n^B = A_n^C.
\]

For any \( w \in W \) with reduced decomposition \( w = s_{i_1} \cdots s_{i_p} \), define \( A_w^C = A_{i_1}^C \cdots A_{i_p}^C \) and similarly define \( A_w^B \). (They do not depend upon the choice of the reduced decomposition.) Consider the isomorphism

\[
\phi : H^*(\text{SO}(2n + 1)/B^B, \mathbb{C}) \to H^*(\text{Sp}(2n)/B^C, \mathbb{C}),
\]

induced from the Borel isomorphism via the identity map

\[
\frac{\mathbb{C}[\epsilon_1, \ldots, \epsilon_n]}{I^B} \xrightarrow{\sim} \frac{\mathbb{C}[\epsilon_1, \ldots, \epsilon_n]}{I^C},
\]

where \( I^B = I^C \) is the ideal generated by the positive degree \( W \)-invariants in \( \mathbb{C}[\epsilon_1, \ldots, \epsilon_n] \). Let \( \{[\Lambda_w(C)]\}_{w \in W} \subset H^*(\text{Sp}(2n)/B^C) \) be the Schubert basis and similarly for \( \{[\Lambda_w(B)]\}_{w \in W} \). Let \( p_e^C \in \mathbb{C}[\epsilon_1, \ldots, \epsilon_n] \) be a representative of \([\Lambda_e(C)]\) and similarly \( p_e^B \) for \([\Lambda_e(B)]\). Then,
by [BGG], we can take
\[ p^B_e = \frac{\left( \prod_{i=1}^n \epsilon_i \right) \left( \prod_{1 \leq i < j \leq n} (\epsilon_i^2 - \epsilon_j^2) \right)}{|W|}, \]
and
\[ p^C_e = \frac{2^n \left( \prod_{i=1}^n \epsilon_i \right) \left( \prod_{1 \leq i < j \leq n} (\epsilon_i^2 - \epsilon_j^2) \right)}{|W|}. \]

Thus,
\[ p^C_e = 2^n p^B_e. \]

By loc. cit., \( p^C_w := A^C_{w^{-1}} p^C_e \) represents the class \([\bar{\Lambda}_w(C)]\) and similarly \( p^B_w := A^B_{w^{-1}} p^B_e \) represents the class \([\bar{\Lambda}_w(B)]\). Hence
\[ p^C_w = 2^{n - \mu(w)} p^B_w, \]

where (as in Section 3) \( \mu(w) \) represents the number of times the simple reflection \( s_n \) appears in any reduced decomposition of \( w \). Thus, we have the following:

**Theorem 55.** The algebra isomorphism \( \phi \) satisfies:
\[ \phi([\bar{\Lambda}_w(B)]) = 2^{\mu(w) - n} [\bar{\Lambda}_w(C)], \]
for any \( w \in W \).

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