Field Theory for Fractional Quantum Hall States

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Abstract

We develop a field theory description of fractional quantum Hall (FQH) states. We show that in the leading approximation in a gradient expansion, Laughlin states are described by a Gaussian free field theory with a background charge which is identified with the anomalous viscosity of the states. The background charge increases the central charge of the corresponding conformal field theory above 1, similar to that of the theory of 2D quantum gravity. Gradient corrections to the Gaussian field theory arising from ultraviolet regularization reflect the gravitational anomaly. They are also related to the Liouville theory of quantum gravity. We show how the gradient expansion of the field action encodes the universal features of the FQH effect beyond that of Hall conductance. This method provides a more transparent and useful alternative for computing the gravitational anomaly and correlation functions of the FQHE in a curved space than the method of iterating the Ward identity employed by the authors in previous papers.
1. Introduction Since the work of Laughlin [1], a common approach to analyzing the physics of the fractional quantum Hall effect (FQHE) is to start by writing a trial ground state wave function as a function of \( N \) electron coordinates. Despite its success, this approach is an impractical framework for studying the collective behavior of a large number of particles, say \( N \sim 10^6 \) which is typical in samples exhibiting the FQHE. As a result, some subtle properties of FQHE states, such as the anomalous (or odd) viscosity [2–6] and the gravitational anomaly [7–11], were computed only recently.

Previously, the authors showed that the universal features of the FQHE can be understood through geometry [7] and employed a Ward identity to extract transport coefficients. In this paper, we reformulate this geometric approach to the FQH states through a field theory. This method seems more appropriate for further inquiry than the Ward identity. Its advantage is that it naturally captures anomalies, the most subtle aspects ultimately responsible for the universal features of the effect.

The wave function for Laughlin states in radial gauge reads
\[
\Psi(\xi_1, \ldots, \xi_N) = Z^{-1/2} \prod_{1 \leq i < j \leq N} (z_i - z_j)^\beta e^{-\frac{1}{4\pi} \sum_{i=1}^{N} |z_i|^2}.
\]  
(1)

Here \( \xi_i = (z_i, \bar{z}_i) \) are coordinates of electrons, \( l = \sqrt{\hbar/eB} \) is the magnetic length, the integer \( \beta = \nu^{-1} \) is the inverse filling fraction, and \( Z \) is the normalization factor.

In order to study the expectation value of a symmetric operator over the ground state, we must compute the \( N \)-multiple integral over particle coordinates
\[
\langle O \rangle = \int \Psi^* O \Psi \, d^2\xi_1 \ldots d^2\xi_N
\]
and then proceed with the limit \( N \to \infty \). Instead, the field theory approach assumes that the appropriate variables are collective coordinates (or modes)
\[
a_{-k} = \sum_{i=1}^{N} z_i^k, \quad k = 1, \ldots
\]
rather than coordinates of individual particles, and that the proper measure of integration is

$$D\varphi = \prod_{k>0} da_{-k}d\bar{a}_{-k}.$$  

Such a measure is commonly referred to as a functional integration over the field $\varphi$ defined as a generating function of the modes $a_{-k}$. A particularly convenient definition is

$$\varphi(z, \bar{z}) \equiv \beta N \left( \pi |z|^2/V - \log(\pi |z|^2/V) \right) - \beta \sum_{k>0} \frac{1}{k} (a_{-k}z^{-k} + \bar{a}_{-k}\bar{z}^{-k}), \quad (3)$$

where $V$ is the volume of the droplet, defined as the support of the particle density. As such, it is an expansion at infinity of a function defined everywhere on a plane except positions of particles $\varphi = \beta N \left( \pi |z|^2/V - \frac{1}{N} \sum_i \log(\pi |z - z_i|^2/V) \right)$. Expectation values are obtained by an integral over fields with the appropriate action

$$\langle O \rangle = \frac{\int O[\varphi] e^{-\Gamma[\varphi]} D\varphi}{\int e^{-\Gamma[\varphi]} D\varphi} \quad (4)$$

as opposed to the multiple integral in (2). In this paper we develop this approach.

This approach is closely connected to the hydrodynamic theory of quantum Hall states of Ref [5, 6] and to the collective field theory approach of Sakita developed for the related problem of Calogero-Sutherland model in [12] and extended in [13, 14].

We will show that most universal properties of the FQHE are captured by the approximation that the field $\varphi$ is the Gaussian free field (GFF) characterized by the background charge. We will compute it and show that it is a function of the filling fraction.

Before we proceed we comment on the role of geometry in the QHE and recall the definition the central object of this study - the generating functional. Transport properties of the FQHE are determined by low energy excitations. In QH states such
excitations only exist at the edge of the sample. On a closed manifold entirely covered by the FQH droplet, there is no boundary and all of the excitations are gapped. However, due to a correspondence between excited edge states and ground states on closed manifolds with variable geometry, the transport properties are determined by the ground state on a closed manifold. In a recent papers [7, 15], we quantified how the transport properties are encoded in the response of the ground state to a variation of the spatial metric.

As such, geometry provides a general framework for understanding transport in the QHE. For that reason, we studied the FQH states on a manifold with the Riemannian metric $g$ and showed that the transport coefficients fundamental to the FQHE, such as the Hall conductance and odd viscosity, are encoded in the metric dependence of the generating functional. It is a functional of the metric whose definition we recall below. We focus on the field theoretical representation of the generating functional and the computation of its anomalous part.

We will show that in the leading $1/N$ approximation (the bulk part of ) the action $\Gamma[\varphi]$ can be approximated by the GFF theory

$$\Gamma_G[\varphi] = \frac{1}{8\pi} \int \left[ \nu (\nabla \varphi)^2 - \varphi R \right] dV,$$

where $R$ is the scalar curvature. The action also contains higher order gradient corrections suppressed by vanishing magnetic length, which are essential for computing the gravitational anomalies. The total action including the leading gradient corrections to (5) is given by (30).

The second term in the action, $-\frac{1}{8\pi} \int R \varphi dV$, is called a background charge. It has the same structure as that of the Liouville theory of gravity (see e.g., [16]), and is responsible for increasing the central charge. We will show that the background charge is directly related to the odd (or anomalous) viscosity introduced in [2].
2. Quantum Hall states on a Riemann surface

We recall some basic aspects of FQH states on a compact surface with a metric \( g \). We assume that the surface is of genus zero and work in complex coordinates. We write the metric in the conformal gauge \( ds^2 = \sqrt{g} dz d\bar{z} \) and choose the coordinates such that \( \sqrt{g} \to (V/\pi)^2 |z|^{-4} \). We will need the Kähler potential defined through the relation \( \partial \bar{\partial} K = \frac{\pi}{V} \sqrt{g} \), assuming that at infinity \( K = \log |z|^2 + O(|z|^{-2}) \) (\( V \) is the area of the surface). In these coordinates, the Laughlin wave function \( \Psi \) reads \[ \Psi = \frac{1}{\sqrt{Z[g]}} \prod_{i<j} (z_i - z_j)^\beta e^{-N_\phi \sum_i K(\xi_i)/2}. \] (6)

Here, \( \beta = \nu^{-1} \) is the inverse filling fraction, and \( N_\phi \) is the total magnetic flux in units of the flux quantum. The formula (6) generalizes (1) to Riemann surfaces of genus zero. The Kähler potential for the flat metric is \( K = (\pi/V)|z|^2 \).

The normalization factor \( Z[g] \) encodes the structure of the underlying geometry. We call it the generating functional

\[ Z[g] = \int \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{i=1}^N e^{-N_\phi K(\xi_i)} dV_i, \quad dV_i = \sqrt{g(\xi_i)} d^2 \xi_i. \] (7)

The generating functional is independent of the choice of coordinates and depends only on the geometry of the surface.

The integral converges if the number of particles \( N \leq \nu N_\phi + 1 \) (more generally, \( N \leq \nu N_\phi + \chi/2 \), where \( \chi \) is the Euler characteristic of the manifold). This condition follows for genus zero by requiring the wave function to be regular at infinity.

We are interested in the case with

\[ N = \nu N_\phi + 1. \] (8)

which describes a state completely covering the surface. An important consequence of the absence of a boundary is that correlation functions in the large \( N \) limit will be supported on the entire surface and will be related to the intrinsic geometry.
The generating functional is mathematically equivalent to the partition function of a one component 2D Coulomb plasma (or a 2D Dyson gas). On a manifold with constant curvature, such as a sphere, torus, or a plane, the statistical mechanics of these systems has been studied extensively (see [17] for a comprehensive review).

The Coulomb potential in the neutralized background is the Green function of the Laplace-Beltrami operator on a compact surface

$$-\Delta_g G(\xi, \xi') = \frac{1}{\sqrt{g}} \delta^{(2)}(\xi - \xi') - \frac{1}{V}, \quad \Delta_g = \frac{4}{\sqrt{g}} \partial_z \partial_{\bar{z}} \; \; (9)$$

In complex coordinates, it reads

$$G(\xi, \xi') = -\frac{1}{2\pi} \log |\xi - \xi'| + \frac{1}{4\pi} (K(\xi) + K(\xi')) - A^{(2)}[g]. \; \; (10)$$

The metric dependent constant $A^{(2)}[g]$ is a geometric functional, which does not depend on the choice of coordinates. It can alternatively be defined as the asymptote of the Green function at large separation of points, according to (10). Its explicit form follows from the condition $\int G(\xi, \xi')dV_\xi = 0$. For genus zero surfaces it is

$$A^{(2)}[g] = \frac{1}{4\pi} \left( \frac{1}{V} \int KdV - 1 \right). \; \; (11)$$

In mathematics, the functional is referred to as the Aubin-Yau functional (for a review see e.g., [18]).

With the help of (10), and the condition (8), we write the amplitude of the Laughlin state in a compact form

$$|\Psi|^2 = Z[g]^{-1} e^{-2\pi NN_0 A^{(2)}[g]} \prod_{i\neq j} e^{-2\pi \beta G(\xi_i, \xi_j)}. \; \; (12)$$

More accurately, the geometric functional is defined with respect to some reference metric $g_0$ of equal genus and equal area as $A^{(2)}[g, g_0] = -\frac{1}{4\pi V} \left( \int KdV - \int K_0 dV_0 \right)$. With a choice the reference metric to be a round sphere $A^{(2)}[g, g_0]$ is given by (11).
The natural definition of the field that generalizes (3) is therefore

$$\varphi(\xi) = 4\pi\beta \sum_i G(\xi, \xi_i)$$  \hspace{1cm} (13)

If the density is a smooth function we replace the rhs of (13) by the integral 

$$4\pi\beta \int G(\xi, \xi') \rho(\xi') dV'$$

This field is globally defined on the surface, such that 

$$\int \varphi dV = 0$$ \hspace{1cm} and \hspace{1cm} $$\varphi \rightarrow -4\pi\beta NA(2)$$ \hspace{1cm} at \hspace{1cm} $$z \rightarrow \infty$$

It is related to the particle density \(\rho = \frac{1}{\sqrt{g}} \sum_i \delta(\xi - \xi_i)\) via the Poisson equation

$$-\Delta_g \varphi = 4\pi\beta (\rho - N/V).$$ \hspace{1cm} (14)

Our goal is to represent the integral (7) in terms of the field \(\varphi\). The procedure consists of two steps - computing the integrand \(|\Psi|^2\) and the measure \(D\varphi\). The former is specific to FQHE, while the latter is not.

3. Boltzmann weight

The first step is to represent the amplitude (12) in terms of the field. The amplitude consists of the factor

$$\prod_{i \neq j} e^{-2\pi\beta G(\xi_i, \xi_j)}$$

which we treat as the Boltzmann weight of the plasma \(e^{-\beta E}\), such that (12) is written

$$|\Psi|^2 = Z[g]^{-1} e^{-2\pi NN_0 A(2)[g]} e^{-\beta E},$$ \hspace{1cm} (15)

where \(E = 2\pi \sum_{i \neq j} G(\xi_i, \xi_j)\) is the energy of the plasma. In this interpretation the filling fraction \(\nu = \beta^{-1}\) plays the role of the temperature.

In the continuum limit, we have to replace the sums over particles by integrals over the density \(\beta E \rightarrow 2\pi\beta \int \rho(\xi) G(\xi, \xi') \rho(\xi') dV dV'\). This formula, however, ignores the excluded self-energy at \(i = j\). To take this into account, we have to subtract the Green function at coincident points. The latter, however diverges as a logarithm of the distance between points. However, there is a unique covariant way to regularize it using the regularized Green function \(G^R(\xi)\), defined below. Thus

$$\beta E \rightarrow 2\pi\beta \int \int \rho(\xi) G(\xi, \xi') \rho(\xi') dV dV' - 2\pi\beta \int G^R \rho dV.$$
The regularized Green function covariantly removes the logarithmic divergency of the Green function by subtracting the logarithm of the geodesic distance between merging points

\[ G^R(\xi) = \lim_{\xi \to \xi'} \left( G(\xi, \xi') + \frac{1}{2\pi} \log \frac{d(\xi, \xi')}{l[\rho]} \right), \] (16)

where \( l[\rho] \) is a length scale representing a minimal distance between particles. This scale is \( \sim 1/\sqrt{\rho} \) up to a constant factor and depends on the particle configuration.

The short distance expansion of the geodesic distance is known to be \( \log d(\xi, \xi') = \log |z - z'| + \frac{1}{2} \log \sqrt{g} \) (see e.g. [18]). This gives

\[ G^R = \frac{1}{4\pi} \log(\rho \sqrt{g}) + \frac{1}{2\pi} K - A^{(2)}[g]. \] (17)

To proceed further we define two additional geometric functionals

\[ A_M = A^{(1)} + 8\pi A^{(2)}, \quad A^{(1)}[g] = \frac{2}{V} \int \log \sqrt{g} dV + 4. \] (18)

The functional \( A_M \) is referred to as the Mabuchi K-energy (see e.g., [18] for a review).^2

In order to compute the energy we make use of the identities

\[ 2\pi \beta \int \int \rho(\xi)G(\xi, \xi')\rho(\xi')dV\xi dV\xi' = \frac{1}{8\pi \beta} \int (\nabla \varphi)^2 dV, \]

\[ \int \rho \log \sqrt{g} dV = \frac{1}{4\pi \beta} \int \varphi R dV + \frac{1}{2} NA^{(1)}[g] - 2N, \]

\[ \int K\rho dV = 4\pi NA^{(2)}[g] + N, \]

\[ 2\pi \beta \int G^R \rho dV = \frac{\beta}{2} \int \rho \log \rho + \frac{1}{8\pi} \int \varphi R dV + \frac{\beta}{4} NA_M[g]. \]

Then the energy of local equilibrium (modulo the metric independent terms of the order of \( O(N) \)) reads

\[ \beta E = -\frac{1}{4} \beta NA_M[g] + \frac{1}{8\pi \beta} \int [(\nabla \varphi)^2 - \beta \varphi R] dV - \frac{\beta}{2} \int \rho \log \rho dV. \] (19)

^2 Similar to the Aubin-Yau functional, the \( A^{(1)}[g] \) is defined with respect to a reference metric as \( A^{(1)}[g, g_0] = \frac{1}{V} \left( \int \log \sqrt{g} dV - \int \log \sqrt{g_0} dV \right) \). If the reference metric is a round sphere, \( A^{(1)}[g, g_0] \) is given by [18].
Eqs. (15) with (19) give the field theoretical representation of the wave function in terms of the collective coordinate $\varphi$.

4. *Entropy* The next step is to pass from integration over coordinates of individual particles to integration over the macroscopic density. This is a standard procedure and is not specific to the FQHE. The transformation involves the Boltzmann entropy $S_B[\rho] = -\int \rho \log \rho \, dV$

$$\prod_i \sqrt{g(\xi_i)} d^2 \xi_i \to e^{S_B} D\rho.$$ Combining the Boltzmann weight and the entropy together we obtain the probability density $dP = |\Psi|^2 dV_1 \ldots dV_N$ in the form

$$dP = Z[g]^{-1} e^{-2\pi N N_s A^{(2)}[g]} e^{-\beta E[\rho] + S_B[\rho]} D\rho.$$ (20)

Here, the free energy of local equilibrium is

$$\beta E - S_B = -\frac{1}{4} \beta N A_M [g] + \Gamma_C[\varphi]$$

where

$$\Gamma_C[\varphi] = \Gamma_G[\varphi] - \left( \frac{\beta}{2} - 1 \right) \int \rho \log(\rho) dV,$$ (21)

is the free energy of the Coulomb plasma at local equilibrium and

$$\Gamma_G[\varphi] = \frac{1}{8\pi} \int \left[ \nu (\nabla \varphi)^2 - \varphi R \right] dV$$ (22)

is the GFF action with the background charge. We observe that the Boltzmann entropy and the short distance regularization of the energy combine to form the last term in (21).

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3 Note a similarity between this procedure and that of Dyson used in Random Matrix theory [19].
5. **Ghosts**  The last (and subtle) step is to determine the measure $D\rho$. The relation (14) connects the density and the field $\varphi$ and also their measures. The Jacobian of this transformation $\rho \rightarrow \varphi$ is the spectral determinant of the Laplace-Beltrami operator

$$D\rho \sim \text{Det}(-\Delta_{g'}) D\varphi. \quad (23)$$

The determinant can be represented by (1, 0) Faddeev-Popov ghosts $\text{Det}(-\Delta_{g'}) = \int e^{-\int \bar{\eta}(-\Delta_{g'}) \eta dV} D\eta D\bar{\eta}$, where $\eta$ are complex fermionic modes. The full action then consists of the Coulomb action (21) plus a ghost contribution, and reads

$$\Gamma_C[\varphi] + \log \text{Det}(-\Delta_{g'}). \quad (24)$$

**Gravitational anomaly**  The ghost part and the Coulomb part of the action in (24) seem decoupled. In fact they are not. The subtlety lies in the short-distance regularization scheme in computing the spectral determinant of the Laplacian in (23). The regularization occurs at a typical distance between particles, which is on the order of $l[\rho] \sim 1/\sqrt{\rho}$. The determinant is essentially controlled by the short distance regularization [20]. As a result, the determinant depends on the density. We emphasize this by denoting the metric $g'$ for the Laplacian $\Delta_{g'}$ in (23) as distinct from the background metric $g$.

The same is true when one integrates over $\varphi$ with the action $\Gamma_C[\varphi]$. The Gaussian part of the Coulomb action $\frac{1}{8\pi\beta} \int (\nabla \varphi)^2 dV = -\frac{1}{8\pi\beta} \int \varphi \Delta_{g'} \varphi dV$ brings the factor $[\text{Det}(-\Delta_{g'})]^{-1/2}$. Together with the ghost contribution it yields the factor $[\text{Det}(-\Delta_{g'})]^{1/2}$.

Thus adding the factor $\sqrt{\frac{\text{Det}(\Delta_{g'})}{\text{Det}(-\Delta_{g'})}} = e^{-\Gamma_L[\rho]}$ to the measure $D\varphi \rightarrow e^{-\Gamma_L[\rho]} D\varphi$ allows us to treat the short-distance cut-off as a constant of the order of the magnetic length. This amounts to another (and final) contribution to the action

$$\Gamma[\varphi] = \Gamma_C[\varphi] + \Gamma_L[\rho]. \quad (25)$$
The dependence of the spectral determinant on a spatially varying short-distance cut-off is well-studied and commonly referred as gravitational anomaly. It is an essential part of the theory of quantum gravity (see e.g., [20]). Briefly, the argument for this is as follows: because of conformal covariance of the 2D Laplacian, a scale transformation of the short distance cut-off is equivalent to a Weyl transformation of the metric. Thus the transformation \( l[\rho] \to l[\rho] \sqrt{\rho} \) which brings the cut-off to a constant can be seen as a transformation of the metric \( \sqrt{g} \to \sqrt{g'} = \rho \sqrt{g} \). Under the Weyl transformation the spectral determinant transforms by the Polyakov-Liouville action according to the formula

\[
\frac{1}{2} \log \frac{\text{Det} (-\Delta_{g'})}{\text{Det} (-\Delta_g)} = -\frac{1}{96\pi} \left( A^{(0)}[g'] - A^{(0)}[g] \right),
\]

where

\[
A^{(0)}[g] = \int R \log \sqrt{g} \, dV + 16\pi
\]

is another geometric functional. Thus \( \Gamma_L[\rho] = \frac{1}{96\pi} \left( A^{(0)}[g'] - A^{(0)}[g] \right) \). Explicitly,

\[
\Gamma_L[\rho] = -\frac{c_0}{48\pi} \int \left[ \frac{1}{2} (\nabla \log \rho)^2 + R \log \rho \right] dV, \quad c_0 = -1.
\]

is the Polyakov-Liouville action. The coefficient \( c_0 = -1 \) is comprised of the ghost contribution \(-2\) and the ‘matter’ contribution \(+1\).

This part of the action alone is identical to the Liouville theory of gravity if the density \( \rho \) is identified as a random metric. From this point of view, the field \( \varphi \) plays the role of a random Kähler potential. Notice that we adopt the measure of integration \( D\varphi \) rather than the conformal factor \( D \log \rho \) as commonly adopted in the theory of quantum gravity [21]. Also, notice that since the number of particles

\[^4\] Here as before, we are defining \( A^{(0)} \) using the round metric \( g_0 \) as a reference, such that \( A^{(0)}[g] = \int R \log \sqrt{g} \, dV - \int R_0 \log \sqrt{g_0} \, dV_0 \)
is fixed, the term with the “cosmological constant” $\int \rho dV$ is a metric-independent constant of order $N$.

Summing all of the pieces together, we obtain the generating functional as the functional integral

$$Z[g] = e^{-2\pi N N_{\phi} A^{(2)}[g]} e^{\frac{1}{2} \beta N A_M [g]} \text{Det} (-\Delta_g) \int e^{-\Gamma[\phi]} D\phi,$$

(29)

with the action

$$\Gamma = \int \left( \frac{1}{8\pi \beta} \left[ (\nabla \phi)^2 - \beta \phi R \right] - \frac{1}{2} (\beta - 2) \rho \log \rho + \frac{1}{96\pi} \left[ (\nabla \log \rho)^2 + 2R \log \rho \right] \right) dV$$

(30)

6. Gradient Expansion of the generating functional

If the curvature varies slowly and its gradients are small (measured in units of magnetic length), then the generating functional admits a gradient expansion, or equivalently a $1/N$ expansion. That expansion is equivalent to the semiclassical expansion, where $1/N$ plays a role of the semiclassical parameter. To expand the generating functional, we find the minimum of the action (30) and compute fluctuations around the minimum. We denote the classical action as $\Gamma_c[g]$. Then the integral $\int e^{-\Gamma[\phi]} D\phi$ evaluated in the Gaussian approximation reads $[\text{Det}(-\Delta_g)]^{-1/2} e^{-\Gamma_c[g]}$. Combining it with the ghosts and other contributions contributions in (29) we obtain

$$\log Z[g] \approx -2\pi N N_{\phi} A^{(2)}[g] + \frac{1}{4} \beta N A_M [g] + \frac{1}{2} \log \text{Det}(-\Delta_g) - \Gamma_c[g].$$

(31)

The transformation formula (26) helps to express the spectral determinant through the one on a round sphere (with the metric $g_0$)

$$\frac{1}{2} \log \text{Det}(-\Delta_g) = \frac{1}{2} \log \text{Det}(-\Delta_{g_0}) - \frac{1}{16\pi} A^{(0)}[g].$$

The result for the latter is known (see e.g., [22])

$$\log \text{Det}(-\Delta_g) = -\frac{1}{4\pi} V \frac{1}{\epsilon^2} - \frac{\chi}{6} \log \frac{V}{\epsilon^2},$$

(32)

where $\epsilon$ is the cutoff scale of the theory. The cut-off is of the order of the typical distance between particles which in the approximation we consider is treated as a
constant scale of the order of the magnetic length. Then the first term in (32) is of
the order of $N$ with an ambiguous coefficient. However, the second term is universal,
and for large $N$ is $-\frac{\chi}{6} \log N$.

The remaining task is to determine the classical value of the action (30) and also
to compute the expectation value of the density. In the Gaussian approximation the
solution of the Euler-Lagrange equation $\delta \Gamma / \delta \rho = 0$ leads to

$$\langle \rho \rangle = \nu \frac{N_\phi}{V} + \frac{R}{8\pi} + \frac{1}{4\pi \beta} \Delta_g \left[ \left(1 - \frac{\beta}{2}\right) \log \langle \rho \rangle + \frac{1}{48\pi} \left( \frac{1}{\langle \rho \rangle} (\Delta_g \log \langle \rho \rangle + R) \right) \right]. \quad (33)$$

The terms in (33) come in at different orders in $N$. Only the first three leading
orders of the $1/N$ of the solution are relevant within the Gaussian approximation.
They are

$$\langle \rho \rangle = \nu \frac{N_\phi}{V} + \frac{R}{8\pi} + \frac{1}{8\pi} \left[ \frac{1}{2} \left(1 - \frac{1}{2\nu}\right) + \frac{1}{12} \right] (l^2 \Delta_g) R + \ldots \quad (34)$$

This is the result of Ref. [7].

In order to compute the value of the action at the minimum $\Gamma_c[g]$ in the relevant
approximation it is sufficient to minimize only the Gaussian part of the action (22). It
is achieved at $\langle \rho \rangle \approx \nu \frac{N_\phi}{V} + \frac{R}{8\pi}$, the first two orders in (34). Up to metric independent
terms proportional to $N$ we obtain

$$\Gamma_c[g] = -\frac{\beta}{32\pi} \int R(\xi) G(\xi, \xi') R(\xi') dV_\xi dV_{\xi'}. $$

The value of this integral is given by the identity which follows from (10),

$$\Gamma_c[g] = -\frac{\beta}{32\pi} \left( A^{(0)}[g] - 8\pi A_M[g] \right), $$

where the functional $A^{(0)}[g]$ is given by (27).

Using (18,27,32) we obtain the first three leading terms of the $1/N_\phi$ expansion
computed in [7]

$$\log \mathcal{Z}[g] = -2\pi\nu N_\phi^2 A^{(2)}[g] + \frac{1}{4} N_\phi A^{(1)}[g] + \frac{(3\beta - 1)}{96\pi} A^{(0)}[g] - \frac{\chi}{12} \log N + O(N^{-1}). \quad (35)$$

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The first coefficient of the expansion (in decreasing order) is proportional to the stiffness $\nu/4\pi$ of the Gaussian field, the second (at $O(N_{\phi}^{0})$) represents the background charge, the third (at $O(N_{\phi}^{3})$) is proportional to the central charge of the combined system of the Gaussian and ghost fields.

We emphasize that the formula misses the terms $C \cdot N$, where $C$ is a metric independent constant. Although these terms are unknown (see [23] for further discussion), they are irrelevant for major applications. For surfaces of constant curvature the most interesting universal term in (35) is the last one. It depends only on the Euler-characteristic of the surface and represents the gravitational anomaly. It was correctly conjectured in early papers of Jancovici and coworkers [24, 25].

The functionals $A^{(2)}[g], A^{(1)}[g]$ are a subject of recent inquiry in Kähler geometry (see [18] and references therein). They are non-local integrals of curvature. In contrast, the functionals which appear as the coefficients of negative powers of $N_{\phi}$ are local integrals of curvature as well as covariant derivatives thereof.

Once the generating functional is obtained, computing physical quantities such as the particle density and static structure factor is straightforward. For example, one can check a general relation (valid in all orders) between the generating functional (35) and the density (34) obtained in Ref. [7]:

$$-\frac{l^2}{2} \pm \log \delta \log Z \log \delta g = \left(1 - \frac{1}{2} l^2 \Delta g \right) \langle \rho \rangle.$$  (36)

In summary, the Gaussian approximation of the action (5) and ghosts determine the scale invariant $(N_{\phi})^{0}$ term in the $1/N_{\phi}$ expansion of the generating functional and by virtue of (31) the three leading orders of the generating functional.

The GFF action shares fundamental similarities with the Liouville theory of quantum gravity. There too, the background charge (the $-\frac{1}{8\pi} \int R \varphi dV$ term), increases the central charge from 1 to $1 + 3\beta$. Following the notation in [16], the background charge is $Q = \sqrt{\beta/2}$. Ghost fields decrease the central charge by 2, resulting in
the central charge \(-1 + 3\beta\), which is proportional to the coefficient of \(A^{(0)}\) in the expansion of the generating functional \(35\). The Polyakov-Liouville action for the density, which appears as a higher order gradient correction to the action \(30\), is a necessary consequence of the short-distance regularization of the GFF theory.

In the remaining two sections we briefly discuss some applications.

8. Transport coefficients The action \(30\) determines the non-vanishing part of the generating functional as \(N \to \infty\), and therefore captures the universal features characteristic of FQH plateaus, such as the transport properties of FQH states.

A useful characteristic of the state is the linear response of the density to the curvature introduced in \([7]\). It is defined as

\[
\eta(\xi - \xi') = 2\pi \beta \frac{\delta \langle \rho(\xi) \rangle}{\delta R(\xi')} \quad \text{at} \quad R = 0.
\]

Most universal features of the FQHE can be derived solely from this characteristic. In \([7]\) we showed that \(\eta(k)\), the Fourier transform of \(\eta(\xi - \xi')\), determines the long-wave behavior of the Hall conductance \(\sigma_H(k)\), anomalous viscosity \(\eta^{(A)}\) of \([2]\), and the static structure factor \(S(k) = \frac{1}{N} \langle \rho_k \rho_{-k} \rangle_c\), where \(\rho_k\) is the Fourier transform of the density, via the general relations

\[
\frac{k^2 l^2}{2} + \frac{k^4 l^4}{2} \eta(k) = \left(1 + \frac{k^2 l^2}{2}\right) S(k),
\]

\[
\eta^{(A)} = \hbar \eta(0), \quad \sigma_H(k) = \frac{\nu e^2}{\hbar} \frac{2S(k)}{k^2 l^2}.
\]

From \(34\) we deduce

\[
\eta(k) = \frac{1}{4\nu} - \frac{1}{4\nu} \left[ \frac{1}{2} \left(1 - \frac{1}{2\nu}\right) + \frac{1}{12} \right] (kl)^2 + \ldots .
\]

and compute the gradient expansion of the transport coefficients via \(37,38\). In particular, we observe the relation between the anomalous viscosity and the background
charge of the Gaussian field given by the second term in the expansion of the density \( \eta^{(A)} = \hbar/(4\nu) \), which agrees with \([26, 27]\).

9. Conformal dimension of Quasi-holes

Introduced by Laughlin \([1]\), a quasi-hole state with charge \(\alpha\nu\) reads

\[
\Psi(w, \{\xi\}) = Z[w; g]^{-1/2} \left[ \prod_{i=1}^{N} (z_i - w)^{a} e^{-\frac{a}{2} K(\xi_i)} \right] \Psi(\{\xi\}),
\]

where \(w\) is a holomorphic coordinate of the quasi-hole. The factor \(e^{-\frac{a}{2} K(\xi)}\) reflects that on a compact manifold a single quasi-hole at a fixed number of particles requires a change of the magnetic flux by the flux of the quasi-hole \(a\), i.e. \(N_\phi \to N_\phi + a\).

The generating functional in this case is related to the expectation value of the vertex operator \(e^{-\nu \varphi(w)}\) over the state unperturbed by the quasi-hole: a state with the total flux \(N_\phi\) and \(N = \nu N_\phi + 1\) particles, which completely covers the surface

\[
Z[w; g] = Z_{N_\phi}[g] e^{aN K(w) - 4\pi a N A^{(2)}[g]} \langle e^{-\nu \varphi(w)} \rangle.
\]

In the Gaussian approximation \(\log Z[w; g]/Z_{N_\phi}[g] \approx -4\pi a N A^{(2)}[g] + aN K(w) - a\nu \langle \varphi(w) \rangle + \langle \varphi(w)^2 \rangle / 2\). In the same order of the approximation, where \(\langle \rho \rangle - \frac{N}{1} \approx \frac{R - R_s}{8\pi}, \langle \varphi \rangle\) is given by the Ricci potential \(\nu \langle \varphi \rangle \approx \frac{1}{2} \int G(\xi, \xi') R(\xi') dV_{\xi'}\). With the help of (10) we compute \(\nu \langle \varphi \rangle \approx \frac{1}{2} \log \sqrt{g} + K - \frac{1}{4} A^{(1)} - 4\pi A^{(2)}\). Then using \(\langle \varphi(w)\varphi(w') \rangle = 4\pi \beta G(w, w')\) and \(\nu \langle \varphi^2 \rangle = 4\pi G R\) and with the help of Eq’s. \([10,17,18]\), we obtain

\[
Z[w; g] \approx Z_{N_\phi + a}[g] e^{a(N_\phi + a) K(w)} (\sqrt{g})^{-h_a}, \quad h_a = \frac{a}{2}(1 - a\nu).
\]

The first factor is the generating functional \((35)\) with \(N_\phi \to N_\phi + a\).

This formula suggests that the quasi-hole state can be represented by the vertex operator

\[
V_a(w) = \exp[-a\nu \varphi(w) + 2h_a K(w)].
\]
The quasi-hole operator transforms as a primary field under conformal transformation

\[ w \rightarrow f(w), \quad V_a \rightarrow |f'|^{2h_a} V_a. \]

The dimension \( h_a \) matches that of vertex operators in the Liouville theory of gravity with an identification of the charge \( a = \alpha \sqrt{2\beta} \) and the background charge \( Q = \sqrt{\beta/2} \) (\( h_a = \alpha(Q - \alpha) \) in the notations of Ref. \[16\]).

The relation (36) can be applied to the quasi-hole state under appropriately defined magnetic length \( 2\pi l^2 = V/(N_\phi + a) \). Applying this to (39), and passing to the flat case we compute the first two moments of the density around the quasi-hole. The result is

\[
\int \langle \rho_w \rangle dV = -\nu a, \quad \frac{1}{2l^2} \int (|z - w|^2 - 2l^2) \langle \rho_w \rangle dV = h_a. \tag{40}
\]

The formula (40) suggests that each quasi-hole is a puncture which fractional charge \( \nu a \). This is the known result going back to [1]. The result for the second moment seems new. It happens to be exactly the dimension of the quasi-hole state.

In summary, we formulate the theory of the Laughlin FQHE states as a field theory of a scalar Bose field. We demonstrate that this theory provides a simple way to capture the transport properties beyond the Hall conductance, and also clarifies the appearance of gravitational anomaly. We believe that the field theory can be extended to other FQHE states such as Pfaffian and parafermion states associated to the affine \( \hat{sl}_k(2) \) algebra. In this case, the field \( \varphi \) is likely to be replaced by the field representing \( sl(2) \) Lie algebra and the Gaussian action by the Wess-Zumino-Witten-Novikov action.

\textit{Note added in proof} \hspace{1em} After completion of this paper S. Klevtsov brought to our attention that the action similar to (30) has been considered in [21] as an admissible action for a random metric. The actions become analogous if one identifies...
the fluctuating density as a random metric and the field $\varphi$ as a fluctuating Kähler potential. The formula [23] also appeared in Ref. [21] as a relation between the invariant measures on conformal factors and on Kähler potentials.

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