All supersymmetric solutions of minimal supergravity in six dimensions

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Abstract

A general form for all supersymmetric solutions of minimal supergravity in six dimensions is obtained. Examples of new supersymmetric solutions are presented. It is proven that the only maximally supersymmetric solutions are flat space, $AdS_3 \times S^3$ and a plane wave. As an application of the general solution, it is shown that any supersymmetric solution with a compact horizon must have near-horizon geometry $\mathbb{R}^{1,1} \times T^4$, $\mathbb{R}^{1,1} \times K3$ or identified $AdS_3 \times S^3$.

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1 Introduction

The usual approach to finding supersymmetric solutions of supergravity theories is to make some physically motivated ansatz for the bosonic fields, and then seek examples of this ansatz that admit one or more supercovariantly constant “Killing” spinors. While this approach is often fruitful, it would be useful to obtain a more systematic method for finding supersymmetric solutions. In particular, given a supergravity theory, it is natural to ask whether one can obtain all supersymmetric solutions of that theory.

It turns out that this is possible for certain theories. Twenty years ago, following the derivation of a BPS inequality in [1], Tod managed to determine all supersymmetric solutions of minimal $N = 2, D = 4$ supergravity [2]. Starting from a Killing spinor $\epsilon$, his strategy was to construct bosonic objects quadratic in $\epsilon$, such as the vector $V^\alpha = \bar{\epsilon} \gamma^\alpha \epsilon$. Fierz identities imply algebraic relations between such quantities, and the supercovariant-constancy of $\epsilon$ yields differential relations. For example, in this case $V$ turns out to be a Killing vector field. Tod showed that these relations are sufficient to fully determine the local form of the solution. The solutions fall into two classes. In the first class, $V$ is timelike and the solutions are the Israel-Wilson-Perjes solutions of Einstein-Maxwell theory, which are specified by harmonic functions on $\mathbb{R}^3$. In the second class, $V$ is null and the solutions are certain pp-waves, specified by harmonic functions on $\mathbb{R}^2$. Some generalizations of this result to other $D = 4$ theories were presented in [3].

This method has recently been extended to certain $D = 5$ theories. In [4], all supersymmetric solutions of minimal $D = 5$ supergravity were obtained. Once again, one can construct a Killing vector field $V$ from a Killing spinor, and the solutions fall into a “timelike” and a “null” class. The solutions in the null class are plane-fronted waves, and are specified in terms of harmonic functions on $\mathbb{R}^3$. The solutions in the timelike class have metric

$$ds_5^2 = f^2 (dt + \omega)^2 - f^{-1} ds_4^2,$$

where $V = \partial/\partial t$ and $ds_4^2$ is the line element of an arbitrary hyper-Kähler 4-manifold $B$ referred to as the “base space”. $f$ and $\omega$ are a scalar and 1-form on $B$ that must obey

$$\nabla^2 f^{-1} = \frac{4}{9} (G^+)^2, \quad dG^+ = 0,$$

where $G^+$ is the self-dual part of $f d\omega$ with respect to the metric on $B$, and $\nabla^2$ the Laplacian on $B$. The solution for the gauge field is given in [4].

The analysis of [4] shows that these purely bosonic equations are necessary and sufficient conditions for supersymmetry. In contrast to the null class, and the $D = 4$ solutions, the general solution to these equations is not known, so the $D = 5$ timelike solutions are determined
somewhat implicitly. Ultimately, one still has to make an ansatz for \( f \) and \( \omega \) to solve these equations once \( \mathcal{B} \) has been chosen. However, this represents a significant advance over the usual approach of making an ansatz for the entire metric and gauge field.

Another \( D = 5 \) theory to which this method has been applied is minimal gauged supergravity \[5\]. The underlying algebraic structure is the same as for the ungauged theory, thus one still has the timelike and null classes to consider. However, the differential conditions obeyed by the spinorial bi-linears are different. \( V \) is still a Killing vector so that the metric of the timelike class can be written as above, whereas one finds that \( \mathcal{B} \) is now a Kähler manifold. \( f \) and \( \omega \) are determined essentially uniquely, although rather implicitly, once \( \mathcal{B} \) has been chosen. The null class is specified by solving certain nonlinear scalar equations on \( \mathbb{R}^3 \). Although the solutions are presented implicitly in terms of solutions of these equations, solving these equations only involves making ansätze for certain scalars rather than for the full metric and gauge field.

This strategy of determining the solution of a supergravity theory using the differential equations obeyed by certain spinorial bi-linears is closely related to the mathematical notions of \( G \)-structures and “intrinsic torsion” \[6\]. Given the existence of a spinor over a \( d \)-dimensional manifold \( M \), the spinor will be invariant under some isotropy group \( G \subset \text{Spin}(d) \). Thus the differential forms one constructs as bilinears enjoy the same invariance, and this defines, at least locally, a canonical reduction of the (spin cover of the) tangent bundle \( \text{Spin}(d) \to G \). The conditions obeyed by the differential forms can be shown to be equivalent to the Killing spinor equations.

In \[7\], the techniques used to analyse the four and five dimensional supergravities mentioned above were applied to \( D = 11 \) supergravity. It was shown that the existence of at least one Killing spinor implies that there is a Killing vector which is either timelike or null \[8\], corresponding to a \( SU(5) \) and \( (\text{Spin}(7) \ltimes \mathbb{R}^8) \ltimes \mathbb{R} \) structure respectively. The timelike case was examined in detail, and purely bosonic necessary and sufficient conditions for preservation of at least one supersymmetry were obtained. As might be expected, these conditions encode the full solution in a somewhat implicit manner. Using the same techniques, all static solutions of the \( D = 10 \) type II theories with only NS flux were analysed in \[9\]. In this case, a more refined analysis of solutions preserving different amount of supersymmetries was presented. Static solutions always preserve more than one supersymmetry. In the general case, when there is only one supersymmetry in \( D = 10 \), there is a null Killing vector field whose isotropy group is \( \text{Spin}(7) \ltimes \mathbb{R}^8 \).

Given the complexity of the results in \( D = 10 \) and \( D = 11 \), it appears that, as far as finding examples of new solutions is concerned, the “\( G \)-structures approach” is only significantly preferable to the usual ansatz-based approach in sufficiently simple theories. There are two natural candidates for theories in which this method might be particularly useful. The first is
minimal $N = 2$, $D = 4$ gauged supergravity. Understanding this case might lead to a better understanding of the results of [5]. The second is minimal $D = 6$ supergravity since, after the minimal $N = 2$, $D = 4$ and $N = 1$, $D = 5$ theories, this is the simplest ungauged supergravity theory with 8 supercharges. Furthermore, it is a natural generalization of these theories because they can be obtained from it by dimensional reduction and truncation.

In this paper, we shall apply the methods described above to minimal $D = 6$ supergravity. The bosonic sector of this theory consists of a graviton and self-dual 3-form. One novel feature that arises in this case is that the Killing vector $V$ obtained from the Killing spinor is always null and one has correspondingly an $SU(2) \rtimes \mathbb{R}^4$ structure [10], so there is no “timelike” case to consider. The $D = 6$ minimal theory arises as a consistent truncation of higher dimensional supergravities, which is reflected in the fact that $SU(2) \rtimes \mathbb{R}^4 \subset Spin(7) \rtimes \mathbb{R}^8 \subset (Spin(7) \rtimes \mathbb{R}^8) \rtimes \mathbb{R}$. The solutions can be trivially uplifted to solutions of $D = 10$ and $D = 11$ supergravities on flat tori.

In contrast to the $D = 5$ case [4], the null Killing vector in $D = 6$ is not hypersurface-orthogonal; this provides the main source of complication in our equations (similar complications would arise in general in $D = 10$ and in the null case in $D = 11$). Some insight into the nature of the $D = 6$ solutions can be obtained by noting that they must contain as subsets (the oxidation of) the timelike and null classes of the minimal $D = 5$ theory. The timelike class of the latter theory involves an arbitrary hyper-Kähler manifold, and the null class contains solutions with arbitrary dependence on a retarded time coordinate $u$. This suggests that there should be $D = 6$ solutions that exhibit both of these features, i.e., hyper-Kähler spaces whose moduli are arbitrary functions of some coordinate $u$. In fact the general supersymmetric solution with vanishing flux has precisely this form [10].

It turns out that supersymmetric solutions with flux are much more complicated. Coordinates can be introduced so that the solutions are expressed in terms of a four dimensional $u$-dependent base manifold $B$. In general $B$ exhibits a non-integrable hyper-Kähler structure. Necessary and sufficient conditions for supersymmetry can be expressed as equations for various bosonic quantities defined on $B$. These equations are more complicated than those encountered in $D = 5$. Nevertheless, as emphasized above, it is much easier to find supersymmetric solutions by substituting an ansatz into these equations than it is to start with an ansatz for the entire metric and 3-form.

There are two special cases in which the solutions simplify to yield an integrable hyper-Kähler structure. The first arises when the null Killing vector field is hyper-surface orthogonal (this include the case of vanishing flux). In this case, our solutions are closely related to the chiral null models of [11]. The second arises when there is no $u$-dependence, i.e., the solution admits a second Killing vector field. In this case the solutions are related to the generalized
chiral null models of [12]. In this case, the necessary and sufficient conditions for supersymmetry take a simple form similar to those for the timelike class of the minimal $D = 5$ theory and, with a few additional assumptions, can be solved explicitly.

In the minimal $N = 2$, $D = 4$ theory and the minimal $D = 5$ theory, supersymmetric solutions must preserve either $1/2$ or all of the supersymmetry, and the same is true for the minimal $D = 6$ theory (although not the minimal $D = 5$ gauged theory [5]). Determining which solutions of the $D = 5$ theory are *maximally* supersymmetric is rather involved [4]. Happily, this is much easier for the $D = 6$ theory, and we shall show that the only such solutions are flat space, $AdS_3 \times S^3$ and the plane wave solution of [13].

An example of the utility of the present approach was given in [14], where the analysis of [4] was exploited to prove a uniqueness theorem for supersymmetric black hole solutions of minimal $D = 5$ supergravity (non-supersymmetric $D = 5$ black holes are not unique [15] unless static [16]). It would be interesting to see if a similar uniqueness theorem could be proved for supersymmetric black strings in minimal $D = 6$ supergravity (non-supersymmetric black strings are not unique [19, 20, 21, 22]). Here we shall content ourselves with taking the first step towards such a proof, namely classifying the possible near-horizon geometries of supersymmetric solutions with compact horizons (e.g. a wrapped string). It turns out that there are just 3 possibilities: $\mathbb{R}^{1,1} \times T^4$, $\mathbb{R}^{1,1} \times K3$ and $AdS_3 \times S^3$. In the latter case, the solution must be identified so as to render the horizon compact.

This paper is organized as follows. In section 2 we construct bosonic objects quadratic in the Killing spinor and derive the algebraic and differential conditions they satisfy. In section 3 we show how these conditions lead to necessary and sufficient conditions for supersymmetry, formulated in terms of equations on $\mathcal{B}$. Section 4 discusses how our work fits into the general approach of classifying supersymmetric solutions using $G$-structures. In section 5 we study some special cases of the general solution, explain how these are related to previous work, and construct some examples of new solutions. Section 6 contains our classification of possible near-horizon geometries and section 7 the classification of maximally supersymmetric solutions. Finally, section 8 contains suggestions for future work.

## 2 Minimal six-dimensional supergravity

The field content of minimal six-dimensional supergravity is the graviton $g_{\mu \nu}$, a two-form $B_{\mu \nu}^+$ with self-dual field strength, and a symplectic Majorana-Weyl (left-handed i.e. $\gamma_7 \psi^A_\mu = -\psi^A_\mu$) gravitino $\psi^A_\mu$. $A$ is an $Sp(1)$ index which we will often suppress. This theory and the extensions coupled to various matter multiplets are described in [23, 24].

Writing a Lagrangian for this theory is notoriously complicated by the self-duality con-
straints, however the addition of a tensor matter multiplet allows one to write an action from which the equations of motion follow. This multiplet comprises of a two-form $B_{\mu\nu}$ with anti-self dual field strength, a right-handed symplectic Majorana-Weyl field $\chi$, and a scalar field $\varphi$. The Lagrangian, equations of motion, and supersymmetry variations can be written in terms of the field $G_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}^{+} + 3\partial_{[\mu}B_{\nu\rho]}^{-}$ and are given for instance in [24].

In this paper we will be interested in the minimal theory, so we consistently set the tensor multiplet to zero. The supersymmetry equation of the gravitino then can be written

$$\nabla_\mu \epsilon - \frac{1}{4} G_{\mu\rho\lambda} \gamma^{\rho\lambda} \epsilon = 0$$  \hspace{1cm} (2.1)

where here, and henceforth, we set $G = dB^+$. In this form, equation (2.1) has a clear geometrical interpretation; namely it implies that $\epsilon$ is a spinor parallel with respect to a modified spin connection $\nabla$ with torsion $G$ (see also [25]).

The field equations are

$$\nabla_\mu G^{\mu\nu\rho} = 0$$  \hspace{1cm} (2.2)

$$R_{\mu\nu} = G_{\mu\rho\sigma} G^{\rho\sigma}_{\nu}.$$  \hspace{1cm} (2.3)

Note that (2.2) is equivalent to the Bianchi identity $dG = 0$ as a consequence of the self-duality of $G$.

Solutions of minimal $D = 6$ supergravity can be trivially oxidized to yield solutions of type II supergravity in which only NS-NS sector fields are excited. The extra 4 spatial dimensions $z^i$ just form a flat torus:

$$ds^2_{10} = ds^2_6 - dz^2.$$  \hspace{1cm} (2.4)

In $D = 10$, the NS-NS field strength $H$ is given by $H = 2G$, so it is self-dual in the first six dimensions with vanishing components in the torus directions. Furthermore, the dilaton is constant. Roughly speaking, such solutions carry equal F-string and NS5-brane charge.

**Bi-linears and their Constraints**

We can now construct spinor bi-linears, and compute the differential conditions they obey, in order to reexpress the supersymmetry equation in terms of bosonic form fields defined on space-time. Mathematically, these encode information about the underlying $G$-structure, on which we comment in section 4. Given that we can set the conjugation matrix $C$ to unity, we always have $\tilde{\epsilon}^A = \epsilon^{AT}$. The non-zero bi-linears that we have are

$$V_{\mu}^{\hspace{5pt} AB} = \tilde{\epsilon}^A \gamma_{\mu} \epsilon^B$$  \hspace{1cm} (2.5)

$$\Omega^{\hspace{5pt} AB}_{\mu\rho} = \tilde{\epsilon}^A \gamma_{\mu\rho} \epsilon^B$$  \hspace{1cm} (2.6)
while the even forms vanish pulling through a $\gamma_7$. The three-forms are self-dual. Using (A.9) we can also check that the following reality properties hold

$$\Omega^{11*} = \Omega^{22}, \quad \Omega^{12*} = -\Omega^{21} = -\Omega^{12},$$

(2.7)

so that it will be convenient to work with three real self-dual three-forms $X^1$, $X^2$ and $X^3$ defined by

$$\Omega^{11} = X^1 + iX^2, \quad \Omega^{22} = X^1 - iX^2, \quad \Omega^{12} = -iX^3.$$  

(2.8)

### Algebraic Constraints

Simple algebraic relations between these bi-linears can be constructed using the Fierz identity, as explained in Appendix A. One obtains

$$V_\mu V^\mu = 0$$

(2.9)

so $V$ is null. We also find

$$i_V X^i = 0$$

(2.10)

where, for a vector $Y$ and $p$-form $A$, $i_Y A$ denotes the $(p - 1)$-form obtained by contracting $Y$ with the first index of $A$. From the self-duality of $X^i$ this is equivalent to

$$V \wedge X^i = 0.$$  

(2.11)

The 3-forms $\Omega^{AB}$ are found to obey an algebra (A.14). When expressed in terms of the real 3-forms $X^i$ this reads

$$X^i \rho^\nu \nu^\nu \lambda = \epsilon^{ijk}(X^k \sigma^\rho \nu^\nu V^\lambda - X^k \sigma^\nu \nu^\nu V^\lambda)$$

$$- \delta^{ij}(g^\sigma \lambda \nu^\nu V^\rho + g^\nu \nu^\nu V^\sigma - g^\sigma \nu^\nu V^\lambda - g^\rho \lambda \nu^\nu V^\sigma),$$

(2.12)

where $\epsilon^{123} = +1$. At this point it is useful to introduce a null orthonormal basis in which

$$ds^2 = 2e^+ e^- - \delta_{ab} e^a e^b$$

(2.13)

where $e^+ = V$ and $a, b = 1, 2, 3, 4$. We shall take orientation given by

$$\epsilon^{+-1234} = 1.$$  

(2.14)

Equation (2.10) and the self-duality of $X^i$ imply that

$$X^i = V \wedge I^i$$

(2.15)

where

$$I^i = \frac{1}{2} I^i_{ab} e^a \wedge e^b.$$  

(2.16)
The self-duality of $X^i$ implies that $I^i$ is anti-self dual with respect to the metric $\delta_{ab}e^a e^b$ with orientation $\epsilon^{abcd} = \epsilon^{+abcd}$. Substituting this expression for $X^i$ into the algebra defined by (2.12) we find that

$$(I^i)^a_c (I^j)^c_b = \epsilon^{ijk} (I^k)^a_b - \delta^{ij} \delta^a_b$$

(2.17)

where in the above, we have raised the indices of $I$ using $\delta^{ab}$. Hence, the $I^i$ satisfy the algebra of the imaginary unit quaternions on a 4-manifold equipped with metric $\delta_{ab}e^a e^b$.

Finally, the Fierz identity implies that the Killing spinor must obey the projection

$$V \cdot \gamma \epsilon = 0,$$

which, written in the above basis is

$$\gamma^+ \epsilon = 0.$$  

(2.19)

Differential Constraints

The vector $V$ and 3-forms $X^i$ satisfy differential constraints which hold because the spinor must satisfy the Killing spinor equation, namely it is parallel with respect to the connection $\nabla$ with torsion $G$. Explicitly, the constraint on $V$ is

$$\nabla_\alpha V_\beta = V^\lambda G_{\lambda\alpha\beta}$$

(2.20)

and the constraints on $X^i$ are

$$\nabla_\alpha X^i_{\beta\gamma\delta} = G_{\alpha\beta}^\rho X^i_{\rho\gamma\delta} + G_{\alpha\gamma}^\rho X^i_{\rho\delta\beta} + G_{\alpha\delta}^\rho X^i_{\rho\beta\gamma}.$$  

(2.21)

In particular, we note that because $X^i$ and $G$ are self-dual, it follows from (2.21) that

$$dX^i = d^\dagger X^i = 0.$$  

(2.22)

Furthermore, (2.20) implies that $\nabla (\alpha V_\beta) = 0$, so $V$ is a Killing vector, and also $dV = 2iV G$. An argument in [1] proves that $V$ (and $\epsilon$) cannot vanish anywhere, assuming analyticity; hence any supersymmetric solution will admit a globally defined null Killing vector. It is also useful to note that

$$\mathcal{L}_V X^i = \mathcal{L}_V G = 0,$$

(2.23)

using $dG = 0$. Hence $V$ generates a symmetry of the full solution.

To proceed, we shall rewrite the differential constraints (2.20) and (2.21) in an equivalent form. Let $\omega$ denote the spin connection. Then (2.20) is equivalent to

$$\omega_{\alpha\beta} = G_{\alpha\beta}.$$  

(2.24)
It is straightforward to show that (2.24) together with (2.23) can be used to simplify (2.21) and rewrite it as an equation for $I^i$:

$$\nabla_\alpha I^i_{bc} = G^{\alpha d} I^d_{ic} - G_{\alpha c} d I^i_{db}.$$  

(2.25)

To summarize, the differential constraints (2.20) and (2.21) are equivalent to (2.24) and (2.25).

**The Killing spinor**

The algebraic and differential constraints on $V$ and $I^i$ are clearly necessary conditions for the background to admit a supercovariantly constant spinor $\epsilon$ satisfying the projection (2.19). It turns out that these conditions are also sufficient. To see this it is convenient to choose the basis vectors $e^a$ so that the 2-forms $I^i$ have constant components. For example, one could choose a basis so that

$$I^1 = e^1 \wedge e^2 - e^3 \wedge e^4, \quad I^2 = e^1 \wedge e^3 + e^2 \wedge e^4, \quad I^3 = e^1 \wedge e^4 - e^2 \wedge e^3.$$  

(2.26)

Equation (2.25) then reduces to

$$\omega^{aab} - G^{aab} = 0$$  

(2.27)

where here $-$ denotes the anti-self-dual projection in the indices $a, b$. It can then be checked that equations (2.19), (2.27) and (2.24) imply that the Killing spinor equation reduces to

$$\partial_\mu \epsilon = 0.$$  

(2.28)

Hence, provided the above algebraic and differential conditions are satisfied then, in this basis, the Killing spinor equation is satisfied by any constant spinor obeying the projection (2.19). This is the only projection so the solution must preserve either 1/2 or all of the supersymmetry.

In summary, the above algebraic and differential conditions on $V$ and $I^i$ are necessary *and* sufficient to guarantee the existence of a $\nabla$-parallel chiral spinor obeying (2.19). Furthermore, all solutions must preserve either 1/2 or all of the supersymmetry. In the next section we shall introduce coordinates and examine further the conditions on $V$ and $I^i$ in order to obtain convenient forms for the necessary and sufficient conditions for supersymmetry.

**3 All supersymmetric solutions**

**Introduction of coordinates**

Coordinates can be introduced locally as follows. Pick a hypersurface $S$ nowhere tangent to $V$. Pick a 1-form $e^-$ that satisfies

$$e^- \cdot V = 1, \quad (e^-)^2 = 0.$$  

(3.1)
on $S$. Now propagate $e^-$ off $S$ by solving $\mathcal{L}_V e^- = 0$. Equations (3.1) continue to hold because $V$ is Killing. Let $e^+ = V$. Since $e^+$ and $e^-$ commute they must be tangent to two dimensional surfaces in spacetime. These 2-surfaces form a 4-parameter family $\Sigma_2(x^m)$ where $m = 1 \ldots 4$. Since $e^+$ is a null Killing vector field, we know that it must be tangent to affinely parametrized null geodesics. We can define a coordinate $v$ to be the affine parameter distance along these geodesics. Choose another coordinate $u$ so that $(u, v)$ are coordinates on the surfaces $\Sigma_2$. Then

\[ e^+ = \frac{\partial}{\partial v}, \tag{3.2} \]
\[ e^- = H \left( \frac{\partial}{\partial u} - \frac{\mathcal{F}}{2} \frac{\partial}{\partial v} \right), \tag{3.3} \]

for some functions $H$ and $\mathcal{F}$. $H$ must be non-zero because $e^+$ and $e^-$ are not parallel. $H$ and $\mathcal{F}$ must be independent of $v$ because $e^+$ and $e^-$ commute. We shall assume that $H > 0$, which can always be arranged by $u \rightarrow -u$. Other than these restrictions, $H$ and $\mathcal{F}$ are arbitrary and can be chosen to be anything convenient. However, we shall keep them arbitrary because different gauges are convenient for different solutions. This freedom in choosing $H$ and $\mathcal{F}$ means that our general solution will contain a lot of gauge freedom.

Using $(e^+)^2 = (e^-)^2 = 0$ and $e^+ \cdot e^- = 1$, the metric on the surfaces $\Sigma_2$ can be deduced to take the form

\[ ds^2_2 = H^{-1} \left( \mathcal{F} du^2 + 2 du dv \right). \tag{3.4} \]

We shall take $(u, v, x^m)$ as the coordinates on our six dimensional spacetime. Once the functions $x^m$ labelling the 2-surfaces have been chosen, the coordinates $u$ and $v$ are only defined up to transformations of the form

\[ u = u' + U(x), \quad v = v' + V(u', x). \tag{3.5} \]

In these coordinates, the six dimensional metric can be written

\[ ds^2 = 2H^{-1} \left( du + \beta_m dx^m \right) \left( dv + \omega_m dx^m + \frac{\mathcal{F}}{2} (du + \beta_m dx^m) \right) - H h_{mn} dx^m dx^n, \tag{3.6} \]

where the metric $h_{mn}$ will be referred to as the metric on the “base space” $B$ and $\omega$ and $\beta$ will be regarded as 1-forms on $B$. The functions $\mathcal{F}$ and $H$, the 1-forms $\omega$ and $\beta$ and the metric $h_{mn}$ all depend on $u$ and $x$ but not $v$ (because $V$ is Killing). Note that the only information we have used so far is that $V$ is a null Killing vector field.

In these coordinates we have

\[ e^+ = H^{-1} (du + \beta_m dx^m) \]
\[ e^- = dv + \omega_m dx^m + \frac{\mathcal{F} H}{2} e^+ \tag{3.7} \]
and we can complete $e^+$ and $e^-$ to a null basis by defining

$$e^a = H^2 \tilde{e}^a_m dx^m$$

where $\tilde{e}^a$ is a vierbein for $\mathcal{B}$, which we shall choose to be independent of $v$. Note that this basis need not be the same as that used in equations (2.27) and (2.28), so these equations will not hold in general.

It is convenient to define anti-self dual 2-forms on $\mathcal{B}$ by

$$J^i = H^{-1} I^i,$$  

(3.9)

because one then finds

$$(J_i)^m_p (J_j)^p_n = \epsilon^{ijk} (J_k)^m_n - \delta^{ij} \delta^m_n,$$

(3.10)

where the indices $m, n \ldots$ have been raised with $h^{mn}$. Hence, these 2-forms yield an almost hyper-Kähler structure on $\mathcal{B}$.

We should emphasize that our introduction of coordinates is purely local, valid only in some open subset of spacetime. In particular, there is no reason why the notion of a base space should be valid globally. In general, the only globally well-defined objects are $V$ and $X^i$.

**Conditions for supersymmetry**

We shall now express the necessary and sufficient conditions for supersymmetry in these coordinates. It is convenient to define a restricted exterior derivative $\tilde{d}$ acting on $p$-forms defined on $\mathcal{B}$ as follows; suppose $\Phi \in \Lambda^p(\mathcal{B})$ with

$$\Phi = \frac{1}{p!} \Phi_{m_1 \ldots m_p}(x, u) dx^{m_1} \wedge \ldots \wedge dx^{m_p},$$

(3.11)

then let

$$\tilde{d}\Phi \equiv \frac{1}{(p + 1)!} (p + 1)! \frac{\partial}{\partial x^a} \Phi_{m_1 \ldots m_p} dx^a \wedge dx^{m_1} \wedge \ldots \wedge dx^{m_p}.$$  

(3.12)

Next, we define the operator $\mathcal{D}$ acting on such $p$-forms as

$$\mathcal{D}\Phi = \tilde{d}\Phi - \beta \wedge \dot{\Phi}$$

(3.13)

where $\dot{\Phi}$ denotes the Lie derivative of $\Phi$ with respect to $\frac{\partial}{\partial u}$. Note that

$$d\Phi = \mathcal{D}\Phi + H e^+ \wedge \dot{\Phi}$$

(3.14)

and

$$\mathcal{D}^2\Phi = -\mathcal{D}\beta \wedge \dot{\Phi}.$$  

(3.15)
With this choice of notation, we note that
\[\begin{align*}
d e^+ &= H^{-1} D \beta + e^+ \wedge (H^{-1} D H + \dot{\beta}) \\
d e^- &= D \omega + \frac{\mathcal{F}}{2} D \beta + H e^+ \wedge (\dot{\omega} + \frac{\mathcal{F}}{2} \dot{\beta} - \frac{1}{2} D \mathcal{F}).
\end{align*}\]  
Using these expressions, it is straightforward to compute the components of the spin connection. In particular, \(L_V e^\alpha = 0\) and (2.24) imply
\[\omega_{\alpha\beta} = -\omega_{\alpha\beta} = \frac{1}{2} (d e^+)_\alpha^\beta \]
and the remaining components are given in Appendix C. Equation (2.24) implies that
\[G^{-+}_{ab} e^a = \frac{1}{2} (H^{-1} D H + \dot{\beta}),\]  
and
\[\frac{1}{2} G^{-ab} e^a \wedge e^b = \frac{1}{2} H^{-1} D \beta.\]  
The self-duality of \(G\) now implies that
\[\frac{1}{6} G_{abc} e^a \wedge e^b \wedge e^c = \frac{1}{2} \star_4 (D H + H \dot{\beta}),\]  
and
\[D \beta = \star_4 D \beta,\]  
where \(\star_4\) denotes the Hodge dual defined on \(\mathcal{B}\). Hence \(D \beta\) is self-dual on \(\mathcal{B}\). This implies that equation (2.25) holds for \(\alpha = -\).  
The remaining components of \(G\) are \(G^{-ab}\). These are obtained using the \(\alpha = +\) component of (2.25). It is straightforward to show that
\[\frac{1}{2} G_{+ab} e^a \wedge e^b = H \psi - \frac{1}{2} (D \omega)^-\]  
where \(^\pm\) denotes the self-dual (anti-self-dual) projection on \(\mathcal{B}\), and
\[\psi = \frac{H}{16} \varepsilon^{ijk} (J^i)^pq (\dot{J}^j)_{pq} J^k.\]  
Using a coordinate transformation of the form \(x \rightarrow x(u, x')\) it would be possible to reach a gauge in which \(\psi = 0\) but we shall keep things general here.

To summarize, (2.24) and the \(\alpha = +\) component of (2.25) together with the self-duality of \(G\), fix \(G\) to be
\[\begin{align*}
G &= \frac{1}{2} \star_4 (D H + H \dot{\beta}) + e^+ \wedge \left( H \psi - \frac{1}{2} (D \omega)^- \right) \\
&\quad + \frac{1}{2} H^{-1} e^- \wedge D \beta - \frac{1}{2} e^+ \wedge e^- \wedge \left( H^{-1} D H + \dot{\beta} \right).
\end{align*}\]
The remaining differential constraints are the $\alpha = c$ components of (2.25) which constrain the covariant derivatives of $J$ on $\mathcal{B}$. In fact, it suffices to note that from the closure of $X^i$, we obtain
\[ \tilde{d}J^i = \partial_u (\beta \wedge J^i) \] (3.25)
where $\partial_u$ denotes the Lie derivative with respect to $\partial/\partial u$. This, together with the fact that the $J^i$ satisfy the quaternionic algebra, implies the $\alpha = c$ components of (2.27) (see for instance section 2 of [9]). Equation (3.25) shows that the almost hyper-Kähler structure of $\mathcal{B}$ is not integrable in general.

We have now exhausted the content of the algebraic and differential constraints satisfied by $V$ and $I^i$ hence, as explained above, the existence of a Killing spinor is guaranteed. Therefore, in these coordinates, the necessary and sufficient conditions for the existence of a Killing spinor are that the field strength be given by (3.24), that $\beta$ obey the self-duality condition (3.21), and that the complex structures obey (3.25). We are interested in obtaining supersymmetric solutions so we now turn to the field equations.

**The Bianchi identity**

Having obtained an expression for $G$, we need to solve the Bianchi identity $dG = 0$ (which is also the equation of motion for $G$ because $G$ is self-dual). Using (3.16) the Bianchi identity reduces to
\[ \mathcal{D} \left( *_4 (\mathcal{D} H + H \beta) \right) + \mathcal{D} \beta \wedge \mathcal{G}^+ = 0 , \] (3.26)
and
\[ \tilde{d} \left( \mathcal{G}^+ + 2 \psi \right) = \partial_u \left[ \beta \wedge (\mathcal{G}^+ + 2 \psi) + *_4 \left( \mathcal{D} H + H \beta \right) \right] , \] (3.27)
where we have introduced a self-dual 2-form
\[ \mathcal{G}^+ \equiv H^{-1} \left( (\mathcal{D} \omega)^+ + \frac{1}{2} F \mathcal{D} \beta \right) . \] (3.28)

**The Einstein equation**

It remains to consider the Einstein equations. In fact, as we show in Appendix B, the only component of the Einstein equations not implied by the Killing spinor and gauge equations is the $++$ component. We must therefore compute $R_{++}$ using the spin connection components given in Appendix C. It is useful to define
\[ L = \tilde{\omega} + \frac{1}{2} F \beta - \frac{1}{2} \mathcal{D} F \] (3.29)
so that $\omega_{++} = -HL_a$. Then we obtain

$$R_{++} = \ast_4 D(\ast_4 L) + 2\dot{\beta}_m L^m + \frac{1}{2} H^{-2} \left( D\omega + \frac{1}{2} F D\beta \right)^2$$

$$- \frac{H}{2} h^{mn} \partial^2_a (H h_{mn}) - \frac{1}{4} \partial_a (H h^{mn}) \partial_a (H h_{mn}), \quad (3.30)$$

where for $\Phi \in \Lambda^2 (\mathcal{B})$, $\Phi^2 \equiv (1/2) \Phi_{mn} \Phi^{mn}$. Hence the Einstein equation reduces to

$$\ast_4 D(\ast_4 L) = \frac{1}{2} H h^{mn} \partial^2_a (H h_{mn}) + \frac{1}{4} \partial_a (H h^{mn}) \partial_a (H h_{mn}) - 2\dot{\beta}_m L^m$$

$$+ \frac{1}{2} H^{-2} ((D\omega)^- - 2H\psi)^2 - \frac{1}{2} H^{-2} \left( D\omega + \frac{1}{2} F D\beta \right)^2. \quad (3.31)$$

**Summary**

We have obtained a general local form for all supersymmetric solutions of minimal $D = 6$ supergravity. The metric is given by (3.6) and the necessary and sufficient conditions for supersymmetry can be expressed as a set of equations on the base manifold $\mathcal{B}$. This must admit an almost hyper-Kähler structure with almost complex structures obeying equation (3.25). The 1-form $\beta$ must obey the self-duality condition (3.21). In terms of the basis (3.7), (3.8), the field strength $G$ is given by (3.24). Finally, the Bianchi identity and Einstein equation must be satisfied, which gives equations (3.26), (3.27) and (3.31).

**4 The $G$-structure**

We have shown that any solution to the supersymmetry equation (2.1) is characterized by the existence of a set of forms which obey algebraic and differential constraints. This fact is related to the notion of $G$-structures. The relevance of $G$ structures for classifying supersymmetric geometries in supergravity theories was put forward in [6] (see also [26]) and subsequently used to analyse and classify supersymmetric solutions in various supergravity theories in [4, 27, 28, 7, 29, 9, 30, 5, 31].

A $G$-structure is a global reduction of the frame bundle, whose structure group is generically $GL(n, \mathbb{R})$, to a sub-bundle with structure group $G$. This reduction is equivalent to the existence of certain tensors whose isotropy group is $G$. When these tensors are globally defined over the manifold, then their isotropy group is promoted to the structure group of the bundle. Here we assume that six-dimensional space-time is equipped with a Lorentzian metric $g$ and a spin structure, hence it has generically a $Spin(1, 5)$ structure. The existence of a globally defined spinor with isotropy group $G \subset Spin(1, 5)$ defines the $G$-structure of relevance here. Equivalently, this is defined by the spinorial bi-linears we have discussed above.
According to [10] there are four different types of stabilizer groups for a spinor in $\text{Spin}(1,5)$. The one relevant here is that associated to a chiral spinor and turns out to be the group $\text{SU}(2) \ltimes \mathbb{R}^4 \subset \text{Spin}(1,5)$ (see also [8]). Notice that in contrast to five dimensions [4] we have here only one possible isotropy group of the spinor: since the corresponding Killing vector is everywhere null (and non-zero), we have a globally defined $\text{SU}(2) \ltimes \mathbb{R}^4$ structure on spacetime. Indeed the $D = 5$ “timelike” and “null” cases discussed in [4] correspond to the $\text{SU}(2)$ and $\mathbb{R}^3$ subgroups of $\text{SU}(2) \ltimes \mathbb{R}^4$ respectively. Note that these $D = 5$ structures are only defined locally since a timelike Killing vector may become null somewhere. However, they admit a unified global description in six dimensions. In section 5.3 we will describe explicitly the reduction from six to five dimensions.

In terms of the Killing spinor, one can give an explicit demonstration of the $\text{SU}(2) \ltimes \mathbb{R}^4$ structure exploiting the isomorphism $\text{Spin}(1,5) \simeq \text{SL}(2,\mathbb{H})$ [10], namely one can realize $\text{Spin}(1,5)$ as $2 \times 2$ quaternion-valued matrices $A$ with unit determinant. They act on the space of spinors identified with $\mathbb{H}^2 \oplus \mathbb{H}^2$ as $A \cdot (s_+, s_-) = (A s_+, (A^*)^{-1} s_-)$, with $s_{+/-}$ corresponding to the positive/negative chirality spinor. A chiral spinor $s_+$ therefore has stabilizer group

$$s_+ = \begin{pmatrix} r \\ 0 \end{pmatrix}, \quad G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} | a \in \mathbb{H}, \ b \in \text{SU}(2) \right\} \simeq \text{SU}(2) \ltimes \mathbb{R}^4.$$(4.1)

An alternative way to derive this is, following [8], to write explicit representations for the Clifford algebra $\text{Cl}(1,5) \simeq \text{Cl}(0,4) \otimes \text{Cl}(1,1)$ and the corresponding spinor on which they act. From this it is not difficult to show that the algebra which leaves the spinor invariant has generators

$$\frac{1}{2} a_{ab} \gamma_{ab} + b_c \gamma_c$$

where $b \in \mathbb{R}^4$ and $a_{ab}$ are such that $a_{ab} \gamma_{ab}$ fixes the spinor in $\text{Cl}(0,4)$, i.e. they span the $\text{su}(2)$ algebra.

The existence of the $\text{SU}(2) \ltimes \mathbb{R}^4$ structure implies that one can introduce a local null frame \{\(e^+, e^-, e^a\)\} in which the metric, one-form and three-forms are written as

$$ds^2 = 2e^+e^- - \delta_{ab}e^a e^b$$

and

$$V = e^+, \quad X^i = e^+ \wedge I^i,$$(4.4)

where $I^i$ obey the algebra of the quaternions. One can indeed check that these are invariant under the $\text{SU}(2) \ltimes \mathbb{R}^4$ action on the tangent space given by

$$e^+ = e^+$$

$$e^- = e^- + q^a q^a e^+ + \sqrt{2} q^a M^a_b e^b$$

$$e^a = M^a_b e^b + \sqrt{2} e^+ q^a$$

$$e^i =$$
for any \( q^a \in \mathbb{R}^4 \) and \( M^a_b \in SO(3) \).

The type of \( G \)-structure is determined completely by covariant derivatives of the spinor, or of the forms, and is characterized in terms of its intrinsic torsion which lies in the space \( \Lambda^1 \otimes g^\perp \) and decomposes under irreducible \( G \)-modules. Thus if all of the components vanish the Levi-Civita connection has holonomy contained in \( G \). Manifolds with \( SU(2) \ltimes \mathbb{R}^4 \) are discussed in [8, 10]. In general, when we have a non-trivial \( G \) field turned on, the holonomy of the Levi-Civita connection is not in \( SU(2) \ltimes \mathbb{R}^4 \), and departure from special holonomy is measured by the intrinsic torsion.

In our context, a convenient way to express the constraints on the intrinsic torsion is in terms of geometrical data on the base manifold \( B \). Although this is clearly not globally defined, in each patch the local form of the metric is given by (3.6) and instead of the globally defined objects \( \{V, X^i\} \) one can equivalently express the differential conditions in terms of \( \{\beta, J^i\} \). These encode information about the \((u\text{-dependent})\) almost hyper–Kähler structure of \( B \). We have shown that supersymmetry and self-duality of the \( G \) field are equivalent to the following constraints on the base manifold

\[
\begin{align*}
\tilde{d}J^i &= \partial_a (\beta \wedge J^i) \\
\mathcal{D} \beta &= \ast_d \mathcal{D} \beta.
\end{align*}
\]

(4.8)

Once these two (coupled) conditions are fulfilled, then the \( G \) field is explicitly determined by equation (3.24). Note that in general the almost hyper–Kähler structure is completely generic in terms of the three irreducible components of its intrinsic torsion \( (2 + 2) + (2 + 2) + (2 + 2) \) (see e.g. [9]), thus for instance \( B \) is not a complex or a Kähler manifold. This is certainly an unpleasant complication when it comes to seeking general examples. In the remainder of the paper we will discuss in detail some interesting and rather general cases where we do have some control over the base space. Although (4.8) (together with (3.24)) are sufficient to ensure supersymmetry, we recall that we must also impose the Bianchi identity and the Einstein equation to get solutions of the supergravity theory.

## 5 Special cases

### 5.1 Non-twisting solutions

If \( V \wedge dV \) vanishes everywhere then the congruence of null geodesics tangent to \( V \) has vanishing twist. Such solutions will therefore be referred to as non-twisting. For non-twisting solutions, \( V \) is hypersurface orthogonal, and hence there exist functions \( H \) and \( u \) such that

\[
V = H^{-1} du.
\]

(5.1)
Therefore, in the non-twisting case, there is a preferred definition of $H$ and $u$, in contrast with the general (twisting) case where the definition is gauge-dependent. Comparing with equation (3.7) we see that non-twisting solutions have

$$\beta = 0.$$  \hspace{1cm} (5.2)

Some gauge freedom remains in the definition of the coordinates $v$ and $x^m$: one is still free to perform coordinate transformations of the form $v \to v - V(u, x)$ and $x^m \to x^m(u, x')$. This freedom could be used to set $\omega = 0$ or $\psi = 0$, for example, but we shall keep $\omega$ and $\psi$ general here. The metric of a non-twisting solution takes the form of a plane-fronted wave:

$$ds^2 = 2H^{-1}du \left( dv + \omega_m dx^m + \frac{F}{2} du \right) - H h_{mn} dx^m dx^n. \hspace{1cm} (5.3)$$

Then (3.25) implies that

$$\tilde{d}J_i = 0 \hspace{1cm} (5.4)$$

so the $J^i$ define an integrable hyper-Kähler structure on $\mathcal{B}$, i.e., $\mathcal{B}$ is hyper-Kähler. From (3.24) we obtain

$$G = \frac{1}{2} \ast_4 (\tilde{d}H) + e^+ \wedge \left( H\psi - \frac{1}{2}(\tilde{d}\omega)^- \right) - \frac{1}{2} H^{-1} e^+ \wedge e^- \wedge \tilde{d}H. \hspace{1cm} (5.5)$$

There is also considerable simplification to the Bianchi and Einstein equations. In particular, from (3.26) we find

$$\tilde{\nabla}^2 H = 0, \hspace{1cm} (5.6)$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\mathcal{B}$. Hence $H$ is harmonic on $\mathcal{B}$. Equation (3.27) simplifies to

$$\tilde{d} \left( H^{-1}(\tilde{d}\omega)^+ + 2\psi \right) = \partial_u \left( \ast_4 (\tilde{d}H) \right) \hspace{1cm} (5.7)$$

and the Einstein equation (3.31) becomes

$$\tilde{\nabla}^m(\tilde{\omega})_m - \frac{1}{2} \tilde{\nabla}^2 F = - \frac{1}{2} H h^{mn} \partial_u (H h_{mn}) - \frac{1}{4} \partial_u (H h^{mn}) \partial_u (H h_{mn})$$

$$- \frac{1}{2} H^{-2} \left( (\tilde{d}\omega)^- - 2H\psi \right)^2 + \frac{1}{2} H^{-2} (\tilde{d}\omega)^2. \hspace{1cm} (5.8)$$

Note that these equations can be solved successively: first one picks a ($u$-dependent) hyper-Kähler base space $\mathcal{B}$, then a ($u$-dependent) harmonic function $H$ on $\mathcal{B}$, then one seeks a 1-form $\omega$ that solves equation (5.7) and finally a function $F$ satisfying equation (5.8).
Flat base space

To construct examples of non-twisting solutions with a flat base space, we take the base space to be flat $\mathbb{R}^4$ with the metric written in terms of either left-invariant $\sigma^i_R$ or right-invariant $\sigma^i_L$ one-forms on the three-sphere:

$$ds^2 = dr^2 + \frac{1}{4} r^2 \left( (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right)$$  \hspace{1cm} (5.9)

where $d\sigma^i = \frac{1}{2} \eta \epsilon^{ijk} \sigma^j \wedge \sigma^k$, and $\eta = 1$ if $\sigma = \sigma_R$, $\eta = -1$ if $\sigma = \sigma_L$. We take an orthonormal basis on $\mathbb{R}^4$ given by

$$e^1 = dr, \quad e^2 = \frac{r}{2} \sigma^1, \quad e^3 = \frac{r}{2} \sigma^2, \quad e^4 = \frac{r}{2} \sigma^3$$  \hspace{1cm} (5.10)

with positive orientation defined with respect to $e^1 \wedge e^2 \wedge e^3 \wedge e^4$. We take hyper-Kähler structures $J^i$ defined via

$$J^i = -\eta \frac{4}{r^2} (1 + \eta) d\left( \frac{r}{2} \sigma^i \right);$$

hence $\psi = 0$. We assume that the harmonic function $H$ depends only on $u$ and $r$, so

$$H = P(u) + \frac{Q(u)}{r^2}$$  \hspace{1cm} (5.11)

for arbitrary functions $P$, $Q$ to be fixed. It remains to solve for the remainder of the Bianchi identity and the Einstein equation. These simplify to

$$\tilde{d} \left( H^{-1}(\tilde{d}\omega)^+ \right) = \partial_u \left( \star_4 (\tilde{d}H) \right)$$  \hspace{1cm} (5.12)

and

$$\tilde{\nabla}^m (\tilde{\omega})_m - \frac{1}{2} \tilde{\nabla}^2 \mathcal{F} = -2 H \tilde{H} - \tilde{H}^2 + \frac{1}{2} H^{-2} \left( (\tilde{d}\omega)^+ \right)^2.$$  \hspace{1cm} (5.13)

To find a solution to these equations, we assume that

$$\omega = W(u,r) \sigma^3$$  \hspace{1cm} (5.14)

and $\mathcal{F} = \mathcal{F}(u,r)$. Substituting into (5.12) we obtain

$$\dot{Q} = 0,$$  \hspace{1cm} (5.15)

so $Q$ is constant, together with

$$W = \alpha_2(u) r^{-2\eta} + \frac{1}{2} \alpha_1(u) \left( \frac{P}{2\eta} r^{2\eta} + \frac{Q}{2\eta - 1} r^{2\eta - 2} \right)$$  \hspace{1cm} (5.16)

for arbitrary functions $\alpha_1(u)$, $\alpha_2(u)$. Lastly, we solve (5.13) for $\mathcal{F}$. We obtain

$$\mathcal{F} = \alpha_3(u) + \alpha_4(u) r^2 + \frac{1}{2} \left( \tilde{P} \bar{P} + \tilde{\dot{P}}^2 \right) r^2 - \frac{\alpha_1(u)^2}{4\eta(2\eta - 1)} r^{4\eta - 2} + 2QP \log r$$  \hspace{1cm} (5.17)
for arbitrary functions $\alpha_3(u)$, $\alpha_4(u)$. Observe that $P$ must be linear in $u$ in order for the logarithmic term in (5.17) to vanish.

It is clear that this treatment can be extended to other examples of hyper-Kähler base space, for example Eguchi-Hanson or Taub-NUT space with $u$-dependent parameters; and a large family of new solutions can be constructed in this manner. We shall however not pursue this here.

**pp-waves**

Our general non-twisting solution describes a pp-wave if $du$ is covariantly constant, which happens if, and only if, $H = \text{constant}$. By rescaling the coordinates we can take $H \equiv 1$ so the solution becomes

$$ds^2 = 2du \left( dv + \omega_m dx^m + \frac{F}{2} du \right) - h_{mn} dx^m dx^n ,$$

$$G = du \wedge (\psi - \frac{1}{2} (d\omega)^{-})$$

with $h_{mn}$ a hyper-Kähler metric. As mentioned above, we can always change coordinates $x \to x(u, x')$ so that $\psi = 0$ in the new coordinates. Alternatively, the same type of coordinate transformation could be used to make $\omega$ vanish. However, in general it is not possible to find a gauge in which both $\psi$ and $\omega$ vanish. As an illustration of this point we will derive the maximally supersymmetric plane wave solution in two ways: first in the gauge $\psi = 0$, and then in the gauge $\omega = 0$, using a flat base space in both cases.

When $\psi = 0$ we can just consider a special case of the flat base solution derived above. In particular, set $P = 1$, $\alpha_2 = 1/2$, $\alpha_1 = \alpha_3 = \alpha_4 = Q = 0$ and $\eta = -1$. Converting to Cartesian coordinates on $\mathbb{R}^4$ (see e.g. [4]) this is

$$\omega = \frac{1}{2} r^2 \sigma_3^2 = x^1 dx^2 - x^2 dx^1 - x^3 dx^4 + x^4 dx^3 .$$

Performing the following change of variables

$$x^1 = \cos u y^1 - \sin u y^2$$
$$x^2 = \sin u y^1 + \cos u y^2$$
$$x^3 = \cos u y^3 + \sin u y^4$$
$$x^4 = - \sin u y^3 + \cos u y^4$$

we obtain the maximally supersymmetric plane wave as given in [13]

$$ds^2 = 2du \left( dv + \frac{1}{2} y^i y^i du \right) - dy^2$$

$$G = -du \wedge (dy^1 \wedge dy^2 - dy^3 \wedge dy^4) .$$

18
Let us now derive this solution directly in the gauge $\omega = 0$. On flat space we have the standard complex structures

$$K^1 = dx^1 \wedge dx^2 - dx^3 \wedge dx^4, \quad K^2 = dx^1 \wedge dx^3 + dx^2 \wedge dx^4, \quad K^3 = dx^1 \wedge dx^4 - dx^2 \wedge dx^3. \quad (5.22)$$

We clearly cannot obtain the above solution by taking $J^i = K^i$ so instead we shall take a triplet of $u$-dependent complex structures defined by

$$J^1 = K^1$$
$$J^2 = \cos 2u K^2 + \sin 2u K^3 \quad (5.23)$$
$$J^3 = -\sin 2u K^2 + \cos 2u K^3.$$

Thus we have

$$\psi = -K^1. \quad (5.24)$$

With this choice, the Bianchi identity holds automatically, and from the Einstein equation we obtain

$$\tilde{\nabla}^2 F = 4\psi^2 = 8 \quad (5.25)$$

which is solved by taking $F = x^i x^i$. The solution is then the same as (5.21) with $y^i = x^i$.

### 5.2 $u$-independent solutions

Another instance in which the general equations simplify considerably is when there is no dependence of the solution on the co-ordinate $u$. Geometrically, we can characterize this case by the existence of a second Killing vector field $K$ which commutes with $V$, and is not orthogonal to $V$. $V$ and $K$ are then tangent to timelike 2-surfaces so we can use these as the 2-surfaces $\Sigma_2$ in our introduction of coordinates, with $K = \partial/\partial u$. If we also assume that $K$ preserves the 3-forms $X^i$ then we can drop all $u$-dependence in our equations.

For such solutions, the base space $\mathcal{B}$ is hyper-Kähler since

$$\tilde{d} J^i = 0, \quad (5.26)$$

and $\beta$ has self-dual curvature on $\mathcal{B}$

$$\tilde{d} \beta = \ast_4 \tilde{d} \beta. \quad (5.27)$$

The Bianchi identity and Einstein equation reduce respectively to

$$\tilde{d} \ast_4 \tilde{d} H + \tilde{d} \beta \wedge G^+ = 0, \quad (5.28)$$
$$\tilde{d} G^+ = 0, \quad (5.29)$$

19
and
\[ \tilde{\nabla}^2 \mathcal{F} = - (\mathcal{G}^+)^2, \tag{5.30} \]
where \( \mathcal{G}^+ \) is given by
\[ \mathcal{G}^+ = H^{-1} \left( (\tilde{d}\omega)^+ + \frac{1}{2} \mathcal{F} d\beta \right). \tag{5.31} \]

An example

As an example of such a solution, take \( H = 1, \mathcal{F} = 0 \) and the base space to be flat with \( u \)-independent complex structures and line element
\[ ds^2 = dx^i dx^i, \tag{5.32} \]
and
\[ \beta = \frac{1}{\sqrt{2}} (x^1 dx^2 - x^2 dx^1 + x^3 dx^4 - x^4 dx^3), \tag{5.33} \]
\[ \omega = \frac{1}{\sqrt{2}} (x^1 dx^2 - x^2 dx^1 - x^3 dx^4 + x^4 dx^3). \tag{5.34} \]

Constants multiplying \( \beta \) and \( \omega \) can be absorbed into an overall scale by rescaling the coordinates.

Let \( u = \frac{1}{\sqrt{2}} (t + z) \) and \( v = \frac{1}{\sqrt{2}} (t - z) \) to obtain the metric
\[ ds^2 = (dt + x^1 dx^2 - x^2 dx^1)^2 - (dx^1)^2 - (dx^2)^2 - (dz + x^3 dx^4 - x^4 dx^3)^2 - (dx^3)^2 - (dx^4)^2. \tag{5.35} \]

The field strength is
\[ G = (dt + x^1 dx^2 - x^2 dx^1) \wedge dx^3 \wedge dx^4 - (dz + x^3 dx^4 - x^4 dx^3) \wedge dx^1 \wedge dx^2. \tag{5.36} \]

The metric is a direct product of the metric of a three dimensional Gödel universe (first constructed in [11]) with a three dimensional internal space. However, the field strength is not a direct product. The internal space is homogeneous with isometry group \( Nil \). The Gödel universe is also homogeneous.

It has been shown [32, 33] that supersymmetric Gödel universes can be related via T-duality to supersymmetric plane wave solutions. For the solution above, this works as follows. First, we write it as a solution of type II supergravity with constant dilaton and self-dual three form flux \( H = 2G \). In order to perform a T-duality along the \( z \) direction it is convenient to choose the following gauge for the \( B \) field
\[ B = (dz + x^3 dx^4 - x^4 dx^3) \wedge (dt + x^1 dx^2 - x^2 dx^1). \tag{5.37} \]

After T-duality we obtain the following metric and NS-NS field strength
\[ ds^2 = 2dz \left( dt + (x^1 dx^2 - x^2 dx^1) - \frac{1}{2} dz \right) - dx^2 - dz^2 \]
\[ H = -2dz \wedge dx^3 \wedge dx^4, \tag{5.38} \]
where $z^i$ are the 4 flat directions arising in the oxidation to $D = 10$. Note that $H$ is not self-dual, hence this does not give a solution of minimal $D = 6$ supergravity. Performing a change of variables similar to (5.20) in the $(x^1, x^2)$-plane the solution reads

$$
\begin{align*}
    ds^2 &= 2du' \left( dv' + \frac{1}{2}(y_1^2 + y_2^2)du' \right) - d\gamma^2 - dz^2 \\
    H &= -2du' \wedge dy^3 \wedge dy^4 
\end{align*}
$$

(5.39)

where we have defined $u' = z, v' = t - \frac{1}{2}z$. Thus the metric looks like $CW_4 \times \mathbb{R}^6$ and is a homogeneous plane wave. The field $H$ however breaks the symmetry of the solution because it has mixed components. This in particular shows that the solution is not a parallelizable plane wave [34, 35].

5.3 Dimensional reduction

Kaluza-Klein reduction of minimal six dimensional supergravity on a circle yields a five dimensional supergravity theory. The reduction yields 1 KK vector from the metric, 1 from the 2-form potential and 1 from dualizing the 3-form field strength. However, self-duality of this 3-form implies that only 2 of these vectors are independent. One also obtains a dilaton from the reduction of the metric. Hence the $D = 5$ theory consists of minimal $D = 5$ supergravity coupled to a $D = 5$ vector multiplet.

It is of interest to examine how the supersymmetric solutions of minimal five dimensional supergravity obtained in [4] arise from the six dimensional theory (this has already been done for some maximally supersymmetric solutions [36]). The details of the dimensional reduction are given in [36, 37]. It is convenient to consider the five dimensional timelike and null classes separately.

Timelike solutions

The $D = 5$ timelike class can be obtained by dimensional reduction of a subset of our $u$-independent (generically twisting) solutions as follows. Solutions with no $u$-dependence can be Kaluza-Klein reduced to $D = 5$ provided $\partial/\partial u$ is spacelike, i.e., provided $F$ is negative. The $D = 6$ line element can be written

$$
    ds_6^2 = H^{-1}F \left[ du + \beta + F^{-1}(dv + \omega) \right]^2 - H^{-1}F^{-1}(dv + \omega)^2 - Hds_4^2. 
$$

(5.40)

The minimal $D = 5$ theory does not contain a dilaton so we take $F = -H$. Consistency of equations (5.28) and (5.30) then requires $\tilde{d}\beta = G^+$ hence

$$
    \tilde{d}\beta = \frac{2}{3}H^{-1}(\tilde{d}\omega)^+. 
$$

(5.41)
Now introduce some notation: let $t = v$, $f = H^{-1}$ and $G^+ = f(\tilde{d}\omega)^+$ so $G^+ = (2/3)G^+$. The five dimensional metric is therefore
\[
d s_5^2 = f^2(dt + \omega)^2 - f^{-1}d s_4^2,
\] (5.42)
and equations (5.28) and (5.29) can be rewritten as
\[
\tilde{\nabla}^2 f^{-1} = \frac{4}{9}(G^+)^2 \tag{5.43}
\]
and
\[
\tilde{d}G^+ = 0. \tag{5.44}
\]
This reproduces the timelike class of supersymmetric solutions of minimal five dimensional supergravity as given in equations (1.11), (1.12). It can be verified that the reduction of the $D = 6$ field strength correctly reproduces the $D = 5$ field strength. Hence the $D = 5$ timelike solutions are obtained by taking $\mathcal{F} = - H$ and choosing $\beta$ to satisfy equation (5.41) (which is just a consistency condition for the truncation required in reducing the $D = 6$ theory to the minimal $D = 5$ theory). Note that $D = 5$ solutions with $G^+ = 0$ arise from six dimensional solutions with $\beta = 0$, i.e., $u$-independent non-twisting solutions.

**Null solutions**

The $D = 5$ and $D = 6$ metrics are related by
\[
d s_6^2 = d s_5^2 - \left(dz - \frac{2}{3}A\right)^2, \tag{5.45}
\]
where $F = dA$ is the five dimensional Maxwell field strength and $z$ the coordinate around the Kaluza-Klein circle. The five dimensional null solution is
\[
d s_5^2 = H^{-1}\left(\mathcal{F}_5 du^2 + 2dudv\right) - H^2(\mathbf{d}x + adu)^2,
\]
\[
F = -\frac{H^{-2}}{2\sqrt{3}}\epsilon_{ijk}\nabla_j(H^3a_k)du^i \wedge dx^i - \frac{\sqrt{3}}{4}\epsilon_{ijk}\nabla_k H dx^j \wedge dx^i, \tag{5.46}
\]
where bold letters denote quantities transforming as 3-vectors and $\nabla_i = \partial/\partial x^i$. $H$ is a $u$-dependent function harmonic on $\mathbb{R}^3$ that must also obey
\[
\partial_u \nabla H = \frac{1}{3}\nabla \times [H^{-2}\nabla \times (H^3a)], \tag{5.47}
\]
The function $\mathcal{F}_5$ satisfies a Poisson-like equation. Solving for $A$ gives
\[
A = A_u du - \frac{\sqrt{3}}{2}\chi_i dx^i, \tag{5.48}
\]
22
where \( \chi \) satisfies
\[
\nabla \times \chi = \nabla H, \tag{5.49}
\]
which admits solutions because \( H \) is harmonic. \( A_u \) is obtained by solving
\[
\nabla A_u = -\frac{\sqrt{3}}{2} \partial_u \chi + \frac{1}{2\sqrt{3}} H^{-2} \nabla \times (H^3 a), \tag{5.50}
\]
which admits solutions because the integrability condition (5.47) is satisfied. Using these results, the six dimensional metric is
\[
ds_6^2 = 2H^{-1}du \left[ dv + \omega_idx^i + \omega_zdz + \frac{1}{2} F du \right] - H \left[ Hdx^2 + H^{-1} (dz + \chi_i dx^i)^2 \right], \tag{5.51}
\]
where
\[
\omega_i = \frac{2}{\sqrt{3}} HA_u \chi_i - H^3 a_i, \tag{5.52}
\]
\[
\omega_z = \frac{2}{\sqrt{3}} HA_u, \tag{5.53}
\]
\[
F = F_5 - H^3 a^2 - \frac{4}{3} HA_u^2. \tag{5.54}
\]
The six dimensional solution belongs to our non-twisting family of solutions. The base space is a Gibbons-Hawking space \([38]\) with harmonic function \( H \). In general, this will be \( u \)-dependent.

In summary, we have shown how all supersymmetric solutions of minimal \( D = 5 \) supergravity arise from supersymmetric solutions of minimal \( D = 6 \) supergravity. The \( D = 5 \) timelike class arise from \( D = 6 \) solutions for which \( \partial/\partial u \) is a spacelike Killing vector field whereas the \( D = 5 \) null class arise from non-twisting \( D = 6 \) solutions for which the base space is a Gibbons-Hawking space (and therefore admits a Killing vector field appropriate for dimensional reduction). Some solutions can be reduced in both ways, for example the \( D = 6 \) maximally supersymmetric plane wave of \([13]\) can be reduced either to the (timelike) \( D = 5 \) Gödel solution of \([4]\) or the (null) \( D = 5 \) maximally supersymmetric plane wave (also given in \([13]\)).

### 5.4 Chiral null models

Our non-twisting solutions closely resemble the “chiral null models”, a class of exact classical string backgrounds obtained in \([11]\). (These generalize earlier classes of exact string backgrounds obtained in \([39, 40]\). Another family of solutions was obtained by duality in \([41, 42]\) but these solutions reduce to chiral null models when restricted to minimal \( D = 6 \) supergravity.) When \( H \) and the base space are independent of \( u \), our non-twisting solutions are chiral null models, provided only that the choice of base space corresponds to an exact “transverse” CFT. This is guaranteed by the hyper-Kähler nature of our base space \([13]\). Hence, using the results of the
previous subsection, all $D = 5$ timelike solutions with $G^+ = 0$ and all $D = 5$ null solutions with $u$-independent $H$ are exact classical string backgrounds. (Note that the latter family includes the entire null class of minimal $N = 2$, $D = 4$ supergravity, which can be obtained by dimensional reduction [4].)

In fact, there exist generalizations of chiral null models which describe exact string backgrounds even when $H$ depends on $u$ [14], and examples of exact string backgrounds with a $u$-dependent transverse space [15]. It would be interesting to know how large a class of exact string backgrounds is contained in our class of non-twisting solutions.

Chiral null models are always non-twisting (i.e. the null Killing vector field is hyper-surface orthogonal). However, an example of a twisting exact string background was presented in [11]. A large family of such “generalized chiral null models” was considered in [12]. It was suggested (although not proved) that these are all exact string backgrounds, again assuming an exact transverse CFT. Our $u$-independent solutions are examples of such solutions. If these are indeed exact classical string backgrounds then it follows that the entire timelike class of the minimal $D = 5$ theory must also be exact string backgrounds (and this in turn includes the timelike class of minimal $N = 2$, $D = 4$ supergravity [4]).

In general, supersymmetric solutions of minimal $D = 6$ supergravity are twisting and $u$-dependent. It would be interesting to know which of these solutions describe exact classical string backgrounds.

5.5 Solutions with Gibbons-Hawking base space

The equations satisfied by our $u$-independent twisting solutions are non-linear. It was argued in [12] that such solutions could not satisfy the superposition principle expected of BPS objects. Here we shall show that this reasoning is incorrect by considering solutions with a Gibbons-Hawking [38] base space, i.e, the most general hyper-Kähler 4-manifold admitting a Killing vector field $\partial/\partial z$ preserving the three complex structures [46]:

$$ds_4^2 = H_2^{-1} \sigma^2 + H_2 dx^2,$$

(5.55)

where

$$\sigma = dz + \chi_i dx^i,$$

(5.56)

$i = 1, 2, 3$, $H_2$ and $\chi_i$ are independent of $z$, and

$$\nabla^2 H_2 = 0, \quad \nabla \times \chi = \nabla H_2,$$  

(5.57)

where $\nabla_i \equiv \partial_i$ in this subsection.

We shall obtain all $u$-independent twisting solutions with a Gibbons-Hawking base space for which the Gibbons-Hawking Killing vector field $\partial/\partial z$ extends to a symmetry of the full
spacetime. This was done for the minimal \( D = 5 \) theory in [4], where it was shown that the solution is specified by four harmonic functions of \( x^i \). The analysis for the \( D = 6 \) is very similar so we shall just sketch the details here. Introduce an orthonormal basis on the base space

\[
e^0 = H_2^{1/2} \sigma, \quad e^i = H_2^{1/2} dx^i
\]  

so that the base space has orientation given by the volume form \( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \). Let

\[
\beta = \beta_0 \sigma + \beta_i dx^i, \quad \omega = \omega_0 \sigma + \omega_i dx^i.
\]  

The general solution of equation (5.27) is given by

\[
\beta_0 = H_1^{-1} H_3, \quad \nabla \times \beta = -\nabla H_3,
\]  

where \( H_3 \) is an arbitrary harmonic function of \( x \). Self-duality implies that \( G^+ \) must take the form

\[
G^+ = -\frac{1}{2} C_i e^0 \wedge e^i - \frac{1}{4} \epsilon_{ijk} C_k e^i \wedge e^j.
\]  

Solving equation (5.29) yields

\[
C = 2\nabla \left( H_2^{-1} H_4 \right),
\]  

where \( H_4 \) is an arbitrary harmonic function of \( x \). Substituting these results into (5.28) and (5.30) gives respectively

\[
H = H_1 + H_2^{-1} H_3 H_4, \quad \mathcal{F} = -H_5 - H_2^{-1} H_4^2,
\]  

where \( H_1 \) and \( H_5 \) are further arbitrary harmonic functions of \( x \). Finally the definition of \( G^+ \) yields an equation for \( \omega \):

\[
H_2 \nabla \omega_0 - \omega_0 \nabla H_2 - \nabla \times \omega = 2 \left( H_1 H_2 + H_3 H_4 \right) \nabla \left( H_2^{-1} H_4 \right) + \left( H_4^2 + H_2 H_5 \right) \nabla \left( H_2^{-1} H_3 \right).
\]  

Taking the divergence of this gives an integrability condition which can be solved to determine \( \omega_0 \):

\[
\omega_0 = H_2^{-2} H_3 H_4^2 + H_1 H_2^{-1} H_4 + \frac{1}{2} H_2^{-1} H_3 H_5 + H_6,
\]  

where \( H_6 \) is yet another arbitrary harmonic function of \( x \). Substituting this back into (5.65) gives an equation that determines \( \omega \) up to a gradient (which can be eliminated by shifting \( v \)).

We have obtained the most general \( u \)-independent solution with a Gibbons-Hawking base space whose Killing vector field extends to a symmetry of the full solution. It is determined by 6 arbitrary harmonic functions of \( x \). It is clear that such solutions can be freely superposed, as expected for solutions describing BPS states, in spite of the non-linearity of the equations derived above.

25
If $\mathcal{F} < 0$ then one can Kaluza-Klein reduce these solutions in the $u$ and $z$ directions to yield solutions of (non-minimal) $D = 4$ supergravity. In general, this reduction yields 2 KK vectors from the metric and 2 from the 2-form gauge potential, so the $D = 4$ solution is parametrized by 8 charges: 4 electric and 4 magnetic [17]. However, the requirement of self-duality of the 3-form field strength reduces the number of independent vectors to 3, so there are 6 charges, corresponding to our 6 independent harmonic functions. Taking (coincident) point sources for the harmonic functions generally leads to $D = 4$ solutions describing rotating naked singularities (supersymmetric rotating black holes in $D = 4$ apparently don’t exist). However, by demanding $\omega = 0$ one can obtain regular static supersymmetric black holes. These are related by duality to the generating solutions of [18, 19].

These solutions can also be reduced to $D = 5$ to obtain solutions of minimal $D = 5$ supergravity coupled to a vector multiplet. KK reduction in the $z$-direction yields $u$-independent null solutions in a similar manner to subsection 5.3. Reduction in the $u$-direction yields $D = 5$ timelike solutions with Gibbons-Hawking base space, generalizing those of [11] (to which they reduce when $H_5 = H_1$ and $H_4 = H_3$). This class of solutions includes supersymmetric rotating black holes [50, 51, 52] (assuming a flat base space: $H_2 = 1/|x|$).

If the $D = 6$ solution is non-twisting then $\beta = 0$ so one can set $H_3 = 0$, which simplifies matters considerably. According to the the discussion of the previous subsection, such solutions are chiral null models and have been well-studied. For example, if one also sets $\omega = 0$ then equation (5.65) imposes one further condition, reducing the number of independent harmonic functions to 4. These solutions are the subclass of the solutions of [53] corresponding to self-dual field strength and constant dilaton.

Note that the results of subsection 5.3 show that the oxidation of the $D = 5$ null solutions leads to $D = 6$ solutions with $u$-dependent Gibbons-Hawking base space. This suggests that it might be possible to extend the analysis of the present subsection to include $u$-dependence, although we shall not do so here.

Example: black string

Introduce spherical polar coordinates $(R, \theta, \phi)$ for $x$ on the base space, and take

$$H_2 = \frac{1}{R}, \quad \chi_i dx^i = \cos \theta d\phi.$$  \[5.67\]

The base space is then flat: let $R = \frac{1}{4} \rho^2$ to get

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4} \left( (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 \right),$$  \[5.68\]

where $\sigma_i$ are left-invariant 1-forms on $SU(2)$ - see [11] for details. In the above notation, we have $\sigma = \sigma_3 = d\psi + \cos \theta d\phi$ where $\psi = z$. $(\theta, \phi, \psi)$ are Euler angles on $S^3$. We shall look for a
solution for which all of the harmonic functions are described by monopole sources at the origin $R = 0$. Furthermore, assume that the solution is asymptotically flat as $R \to \infty$, i.e., $H \to 1$ and $\beta, \omega \to 0$. The only way of ensuring $\beta_0 \to 0$ is to take $H_3 = 0$, which implies that $\beta = 0$ (so the solution is non-twisting). Now also assume that $\omega = 0$. Then, integrating equation (5.66) and demanding $\omega_0 \to 0$ implies that $H_4 \propto H_2$. However, from the definition of $H_4$ (equation (5.62)) we see that $H_4$ is arbitrary up to addition of a multiple of $H_2$, so we can set $H_4 = 0$. Doing so we arrive at the solution

$$H = 1 + \frac{\mu}{\rho^2}, \quad \mathcal{F} = -1 - \frac{p}{\rho^2}, \quad \beta = 0, \quad \omega = \frac{j}{2\rho^2}\sigma_3,$$

(5.69)

where $\mu$, $p$ and $j$ are constants. The constant term in $\mathcal{F}$ can be adjusted by shifting $v \to v + cu$; the above choice will be convenient below. This solution describes a rotating momentum-carrying string. As we shall see below, it has a regular event horizon at $\rho = 0$. When oxidized to $D = 11$, this solution is a special case of a 4-charge solution constructed in [54].

6 Solutions with a horizon

6.1 Gaussian null coordinates

Supersymmetric solutions with event horizons are of special interest. If a solution has an event horizon then it must be preserved by $V$ hence $V$ must be tangent to the horizon, so the event horizon is a Killing horizon of $V$. In this section we shall consider all solutions with a Killing horizon of $V$.

We shall start by choosing a suitable gauge, corresponding to Gaussian null coordinates adapted to a Killing horizon $\mathcal{H}$ [55] [14]. Pick a partial Cauchy surface $\Sigma$. Let $\mathcal{H}_0$ denote the intersection of $\Sigma$ with the horizon. Introduce coordinates $x^m$ on $\mathcal{H}_0$. Define a coordinate $v$ on $\mathcal{H}$ to be the parameter distance of a point from $\mathcal{H}_0$ along the orbits of $V$. Let $n$ be the unique (past directed) null vector on $\mathcal{H}$ that obeys $V \cdot n = -1$ and $V \cdot X = 0$ for any vector $X$ tangent to surfaces of constant $v$ in $\mathcal{H}$. Consider the null geodesic from a point $p \in \mathcal{H}$ with tangent $n$. Let the coordinates of a point affine parameter distance $r$ along this geodesic be $(v, r, x^m)$, where $(v, x^m)$ are the coordinates of $p$. The line element must take the form

$$ds^2 = -2dr dv - 2rh_m(r, x)dv dx^m - \gamma_{mn}(r, x)dx^m dx^n,$$

(6.1)

with $V = \partial/\partial v$, the horizon is at $r = 0$ and $h_m$ and $\gamma_{mn}$ must be smooth functions of $r$ in a neighbourhood of the horizon $r = 0$.

As an example, consider the string solution [5.69]:

$$ds^2 = \left(1 + \frac{\mu}{\rho^2}\right)^{-1} \left[2du' dv' + \frac{j}{\rho^2} du' \sigma_3 - \left(1 + \frac{p}{\rho^2}\right) du'^2\right]$$

27
\[- \left( 1 + \frac{\mu}{\rho^2} \right) \left[ d\rho^2 + \frac{\rho^2}{4} \left( (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 \right) \right]. \tag{6.2} \]

We have included primes on some coordinates to avoid confusion with the coordinates of (6.1). It is convenient to identify \( u' \sim u' + l \) to render the length of the string finite. Consider a coordinate transformation defined by

\[
\begin{align*}
dv' &= dv - A(r) dr, & du' &= du - B(r) dr, & d\psi' &= d\psi' - C(r) dr, & \rho d\rho &= 2\sqrt{D(r)} dr, \\
(6.3)
\end{align*}
\]

where \( u \sim u + l \), and choose the functions \( A, B, C, D \) so that the metric takes the form of (6.1). One obtains

\[
D = \left( 1 + \frac{\mu}{\rho^2} \right) \frac{\rho^2}{4} - \left( 1 + \frac{\mu}{\rho^2} \right)^{-2} \frac{j^2}{4\rho^4}. \tag{6.4}
\]

This is required to be positive, which implies

\[
p > \frac{j^2}{\mu^2}. \tag{6.5}
\]

For small \( \rho \) we have

\[
r = \frac{1}{2} \left( p - \frac{j^2}{\mu^2} \right)^{-1/2} \rho^2 + O(\rho^4). \tag{6.6}
\]

In the form (6.1), the solution has

\[
r h_m dx^m = -\rho^2 (\rho^2 + \mu)^{-1} du, \tag{6.7}
\]

\[
\gamma_{mn}(r, x) dx^m dx^n = \frac{(\rho^2 + p)}{(\rho^2 + \mu)} \left( 1 - \frac{j^2}{(\rho^2 + p)(\rho^2 + \mu)^2} \right) du^2 \]
\[
+ \frac{1}{4} (\rho^2 + \mu) \left[ \left( d\psi' + \cos \theta d\phi - \frac{2j}{(\rho^2 + \mu)^2} du \right)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right]. \tag{6.8}
\]

For fixed \( r \) (i.e. fixed \( \rho \)), a shift \( \psi' \rightarrow \psi' + cu \) shows that this is just the standard product metric on \( S^1 \times S^3 \). The sizes of the \( S^1 \) and \( S^3 \) vary with \( r \), approaching constant non-zero values as \( r \rightarrow 0 \) or \( r \rightarrow \infty \).

Since \( \gamma_{mn} \) and \( h_m \) are smooth at \( r = 0 \), we have shown that this solution must have a regular horizon there. The condition (6.5) ensures that the identification of \( u \) is consistent with a regular horizon. Upon dimensional reduction, this solution gives a rotating black hole in \( D = 5 \), with (6.5) being the condition for this black hole to have a regular horizon instead of naked closed timelike curves \([50, 51, 52]\). For example, if \( p = \mu \) then the black hole is a solution of minimal \( D = 5 \) supergravity and (6.5) reduces to \( j^2 < \mu^3 \), as obtained in \([52]\).
6.2 Near-horizon geometry

It has recently been realized that the black hole uniqueness theorems do not extend to higher dimensions [15]. However, in [14], a uniqueness theorem was proved for supersymmetric black hole solutions of minimal $D = 5$ supergravity. It is of interest to ask whether a uniqueness theorem can also be proved for supersymmetric black strings in $D = 6$. We shall not attempt that here, but will instead just repeat the first step of [14], namely to determine all possible near-horizon geometries for supersymmetric solutions with a spatially compact Killing horizon, i.e., compact $\mathcal{H}_0$. The black string solution of the previous subsection is an example of such a solution but note that there exist black strings with regular horizons for which $\mathcal{H}_0$ cannot be rendered compact by making identifications. A simple example would be the solution of the previous subsection with $p = j = 0$. The naive identification of this solution destroys regularity of the horizon [56].

The Gaussian null coordinates of (6.1) give a coordinate system of the form discussed in section 3 if we identify $u = -r$, $\beta_m = uh_m$, $\omega = 0$, $\mathcal{F} = 0$ and $H = 1$. Hence it is straightforward to apply our general formalism to supersymmetric solutions with horizons. For example, one can easily read off the field strength for such a solution from the general case. To determine the possible near-horizon geometries, we just have to evaluate everything on $\mathcal{H}_0$, i.e., at $r = 0$. Evaluating equation (3.26) at $r = 0$ gives

$$\tilde{d} \ast_4 h = 0 \quad \text{on } \mathcal{H}_0. \quad (6.9)$$

Equation (3.25) becomes

$$\tilde{d} J^i = h \wedge J^i \quad \text{on } \mathcal{H}_0. \quad (6.10)$$

This implies that the 2-forms $J^i$ form an integrable hyper-hermitian structure on $\mathcal{H}_0$ with Lee form $h$ (see e.g. [57]) i.e. each of the three almost complex structures is integrable. The integrability condition of equation (6.10) tells us that $\tilde{d} h$ is self-dual on $\mathcal{H}_0$, which also follows from equation (3.21).

The special case in which $h$ vanishes on $\mathcal{H}_0$ is straightforward. In this case, $\mathcal{H}_0$ must be a compact hyper-Kähler space and hence either $\mathbb{T}^4$ or $K3$. The near-horizon limit of the solution is just the product $\mathbb{R}^{1,1} \times \mathcal{H}_0$ with vanishing flux.

Now we shall assume that $h \neq 0$ on $\mathcal{H}_0$, with $\mathcal{H}_0$ compact. In this case, integrating $\tilde{d} h \wedge \ast_4 \tilde{d} h$ over $\mathcal{H}_0$ using self-duality and Stokes’ theorem yields the result

$$\tilde{d} h = 0 \quad \text{on } \mathcal{H}_0. \quad (6.11)$$

Hence $h$ is closed and co-closed on $\mathcal{H}_0$. It follows that

$$I \equiv \int_{\mathcal{H}_0} \tilde{\nabla}_m h_n \tilde{\nabla}^m h^n = - \int_{\mathcal{H}_0} R_{mn} h^m h^n, \quad (6.12)$$

29
where $\tilde{\nabla}$ is the metric connection associated with $\gamma_{mn}$, indices have been raised with $\gamma^{mn}$, and $R_{mn}(x)$ is the Ricci tensor of $\mathcal{H}_0$. This can be obtained from an integrability condition for (6.10) but it is more easily obtained by the following trick. Note that (since $h$ is closed) locally we can write $h = -2\Omega^{-1}\tilde{d}\Omega$, and one then sees that $\Omega^2 J^i$ yield an integrable hyper-Kähler structure with metric $\tilde{\gamma}_{mn} = \Omega^2 \gamma_{mn}$. Since this metric must be Ricci flat, we can therefore calculate the Ricci tensor for $\gamma_{mn}$ in terms of $\Omega$. Now $\Omega$ is only defined locally, but $h$ is defined globally so we must rewrite the expression for the Ricci tensor using only $h$. The result is

$$R_{mn} = -\tilde{\nabla}_m h_n - \frac{1}{2} h_m h_n + \frac{1}{2} h^p h^r \gamma_{mn} - \frac{1}{2} \tilde{\nabla}_p h^r \gamma_{mn}, \quad (6.13)$$

which is symmetric in $m$ and $n$ because $\tilde{d}h = 0$ on $\mathcal{H}_0$. In the present case, the final term vanishes because of (6.9). Plugging this into equation (6.12) and integrating by parts yields

$$I = 0 \text{ hence } \tilde{\nabla}_m h_n = 0 \quad \text{on } \mathcal{H}_0. \quad (6.14)$$

So $\mathcal{H}_0$ admits a covariantly constant vector $h$. Now define a constant $L$ by $4L^{-2} = h_m h^m$ and define a coordinate $\alpha$ on $\mathcal{H}_0$ to be the parameter along the integral curves of $h$, so $h = \partial/\partial\alpha$. The metric on $\mathcal{H}_0$ must be

$$ds^2_4 = 4L^{-2} (d\alpha + \nu)^2 + \gamma_{ij} dx^i dx^j, \quad (6.15)$$

where $\nu$ and $\gamma_{ij}$ are independent of $\alpha$. $\tilde{d}h = 0$ implies that locally we can write $\nu = \tilde{d}\lambda$, which can be gauged away by shifting $\alpha$. Examining $R_{mn}$ reveals that the Ricci tensor of $\gamma_{ij}$ is $R_{ij} = 2L^{-2}\gamma_{ij}$ hence $\gamma_{ij}$ must be locally isometric to the metric on a round $S^3$ of radius $L$. So locally we have

$$ds^2_3 = L^2 (dZ^2 + d\Omega^2), \quad (6.16)$$

where we have performed a change of coordinates $\alpha = -(L^2/2)Z$, which gives $h = -2dZ$ on $\mathcal{H}_0$. So the metric on $\mathcal{H}_0$ is locally isometric to the standard metric on $S^1 \times S^3$.

Plugging these results back into the six dimensional metric yields

$$ds^2 = -2drd\nu + 4r d\nu dZ - L^2 (dZ^2 + d\Omega^2) + O(r^2) dx^m dx^m + O(r) dx^m dx^n, \quad (6.17)$$

where $x^m = \{Z, \Omega\}$. Taking the near horizon limit $r = \epsilon \hat{r}$, $\nu = \hat{\nu}/\epsilon$, $\epsilon \rightarrow 0$, and making the change of coordinates $\hat{r} = \hat{\omega}^{2Z}$ gives

$$ds^2 = -2e^{2Z} d\hat{\omega} d\hat{\nu} - L^2 (dZ^2 + d\Omega^2), \quad (6.18)$$

which is the metric of $AdS_3 \times S^3$. Of course, this result is only local since our discussion of the geometry of $\mathcal{H}_0$ was purely local. Globally, the solution will be some identification of $AdS_3 \times S^3$. 

30
the simplest possibility being just \( Z \sim Z + \text{constant} \). Note that, in these coordinates, the horizon we are studying is the one at \( \hat{u} = 0 \) (not \( Z = -\infty \) since compactness of \( \mathcal{H}_0 \) implies that \( Z \) is bounded).

In summary, we have shown that any supersymmetric solution of minimal six dimensional supergravity with a spatially compact Killing horizon of \( V \) must have a near-horizon geometry that is \( \mathbb{R}^{1,1} \times T^4 \), \( \mathbb{R}^{1,1} \times K3 \), or identified \( AdS_3 \times S^3 \).

7 Maximal Supersymmetry

In order to determine the maximally supersymmetric solutions of the theory, we observe that the integrability conditions (B.2) imply that

\[
R_{\mu\nu\rho\lambda} - \nabla_\nu G_{\mu\rho\lambda} + \nabla_\mu G_{\nu\rho\lambda} + 2G_{\mu\alpha[\rho}G_{\nu\alpha]\lambda] = 0 .
\]  

(7.1)

Hence, on antisymmetrizing on the indices \( \mu, \rho, \lambda \) and making use of (B.4) together with \( dG = 0 \), it is straightforward to show that

\[
\nabla G = 0
\]  

(7.2)

i.e. \( G \) is parallel with respect to the Levi-Civita connection. Substituting this into (7.1) we obtain

\[
R_{\mu\nu\rho\lambda} = -2G_{\mu\alpha[\rho}G_{\nu\alpha]\lambda] .
\]  

(7.3)

Observe, that as \( G \) is parallel, so is the Riemann tensor and hence the geometry must be locally symmetric. In addition, (B.3) implies that \( G \) satisfies an orthogonal Plücker-type relation. Hence, as a consequence of section 2.2 in [58], it is straightforward to show that we can write

\[
G = P + \ast P
\]  

(7.4)

where \( P \) is a decomposable 3-form. In addition \( \nabla G = 0 \) implies that \( \nabla P = 0 \). To proceed we shall make a modification of the reasoning used in section 3.3 in [59]. There are two cases to consider.

In the first case, the 3-form \( P \) is not null. Then \( P \) induces a local decomposition of the manifold into a product of two three dimensional symmetric spaces \( M = M_1 \times M_2 \) with \( P \propto d\text{vol}(M_1) \) and \( \ast P \propto d\text{vol}(M_2) \). Without loss of generality we can assume that \( P \) has positive norm and so \( M_1 \) is Lorentzian (with mostly minus signature). Then we have

\[
G = \chi[d\text{vol}(M_1) + d\text{vol}(M_2)]
\]  

(7.5)

for constant \( \chi \). If \( \chi = 0 \) then it is clear that the Riemann tensor vanishes and hence the geometry is flat. Otherwise, if \( \chi \neq 0 \), then on \( M_1 \) the components of the Riemann curvature tensor satisfy

\[
R_{ijmn} = \chi^2(g_{im}g_{jn} - g_{jm}g_{in})
\]  

(7.6)
so $M_1$ is isometric to $AdS_3$, and on $M_2$ the components of the Riemann curvature tensor satisfy
\[ R_{ijmn} = -\chi^2 (g_{im}g_{jn} - g_{jm}g_{in}) \] (7.7)
so $M_2$ is isometric to $S^3$. Both the $AdS_3$ and $S^3$ have the same radius of curvature.

In the second case, the 3-form $P$ is null. It is known from [60] that all Lorentzian symmetric 6-manifolds admitting parallel null forms are locally isometric to a product $M = CW_d(A) \times Q_{6-d}$ for $d = 3, 4, 5, 6$ where $Q_{6-d}$ is a Riemannian symmetric space and $CW_d(A)$ is a $d$-dimensional Cahen-Wallach space. As $P$ is null and decomposable, we must have
\[ P = dx^- \wedge \psi \] (7.8)
where $dx^-$ is a parallel null form which exists in every Cahen-Wallach space and $\psi$ is a parallel 2-form on $M$ with negative norm. It is straightforward to see that the components of the Riemann curvature of $Q$ must all vanish and hence $Q_{6-d} = \mathbb{R}^{6-d}$. The metric on $M = CW_d(A) \times \mathbb{R}^{6-d}$ can be written locally as
\[ ds^2 = 2dx^+dx^- + \sum_{i,j=1}^{4} A_{ij} \delta^i x^j (dx^-)^2 - \sum_{i=1}^{4} (dx^i)^2 \] (7.9)
where $A$ is a symmetric $4 \times 4$ matrix with constant coefficients, which is degenerate along the $\mathbb{R}^{6-d}$ directions. We can choose co-ordinates on $\mathbb{R}^4$ so that
\[ P = \mu dx^- \wedge dx^1 \wedge dx^2 . \] (7.10)
The maximally supersymmetric Cahen-Wallach type solutions have been examined in [13] and it is straightforward to show that the only possible solution is in fact $CW_6(A)$ with $A_{ij} = \mu^2 \delta_{ij}$.

To summarize, the only maximally supersymmetric solutions of the minimal six-dimensional supergravity are $\mathbb{R}^{1,5}$, $AdS_3 \times S^3$ and the $CW_6$ solution described above.

8 Outlook

We have presented a general form for all supersymmetric solutions of minimal supergravity in six dimensions. The solutions preserve either half or all of the supersymmetry. Our method relies on the analysis of the algebraic and differential constraints obeyed by certain differential forms constructed as spinor bilinears, and is related to the mathematical notion of $G$-structures. Our results, together with those of [4] and [5] which analysed minimal supergravities in five dimensions, provide encouraging evidence that our approach could be extended to other supergravity theories.
For instance, we expect that minimal $N = 2$, $D = 4$ gauged supergravity could be easily analysed using our methods. Recall that the ungauged theory was tackled some time ago by Tod [2], from which it is known that there are a timelike and a null case to analyse. In the gauged theory the algebraic structure will remain unchanged while new differential conditions will arise.

Similarly, minimal $D = 6$ gauged supergravity could be analysed by generalizing our results for the ungauged theory. Here one adds the tensor multiplet mentioned at the beginning of section 2 and a vector multiplet whose bosonic field is a one-form potential $A$. From a Killing spinor one can construct a vector $V$ and 3-forms $X^i$ obeying the same algebraic relations as in the present paper, although the differential relations will be different. Given the results of [5] one anticipates that the resulting $SU(2) \ltimes \mathbb{R}^4$ structure will be some generalization of the one encountered here. A systematic analysis of this theory might address some of the questions recently raised in [61].

More generally, it is interesting to ask which combinations of vector and tensor multiplets can be added to the minimal theories for them to remain tractable using our techniques. Dimensional reduction of the minimal $D = 6$ theory yields the minimal $D = 5$ theory coupled to a vector multiplet, hence all supersymmetric solutions of the latter theory must arise as a subset of the solutions presented here. So the case of a single vector multiplet is certainly tractable. Similarly, reduction to $D = 4$ yields the minimal $N = 2$ $D = 4$ theory coupled to 3 vector multiplets, so this theory should also be tractable. These examples suggest that it might be fruitful to examine the cases in which arbitrary many vector multiplets are present. More ambitiously, one might hope that a similar analysis could be applied to non-abelian gauged supergravities, which in recent years have proved valuable tools for finding new solutions of interest in string theory.

The results of [7] and [9] have shown that the same techniques prove useful in classifying and analyzing supersymmetric solutions of higher dimensional supergravities. In particular in [7] the most general form of supersymmetric solutions admitting at least a “timelike” Killing spinor in $D = 11$ supergravity was given, while in [9] static solutions of $D = 10$ Type II theories with NS fields were analyzed in detail. Although in these theories the form of the solutions is determined somewhat implicitly, it is nevertheless useful to have the most general solutions catalogued. To complete such a catalogue, one would have to examine null solutions, which in general preserve 1/32 supersymmetry. The null Killing vector field will, as in $D = 6$, generally be twisting, so we hope that the analysis of such solutions presented here will be of some use in understanding how things work in $D = 10, 11$. 

33
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A Conventions and useful identities

We follow the spinor conventions of [23, 24]. The $8 \times 8$ Dirac matrices in six dimensions obey the Clifford algebra

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = (+, -, -, -, -, -) . \quad (A.1)$$

where $\alpha, \beta, \ldots$ are tangent space indices. Curved indices will be denoted by $\mu, \nu, \ldots$. The conjugation matrix $C$ is symmetric and can be set to unity. Hence, in this representation the $\gamma$ matrices are antisymmetric

$$\gamma^T_\alpha = -\gamma_\alpha . \quad (A.2)$$

The chirality projector is defined as

$$\gamma_7 = \gamma_0 \gamma_1 \cdots \gamma_5, \quad \gamma_7^2 = 1, \quad \gamma_7^T = -\gamma_7 . \quad (A.3)$$

The duality relation of the $\gamma$ matrices reads

$$\gamma^{\alpha_1 \ldots \alpha_n} = (-1)^{\lceil n/2 \rceil} \frac{(6-n)!}{(6-n)!(6-n-n)!} \epsilon^{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_6} \gamma^{\beta_1 \ldots \beta_6} \gamma_7 \gamma_7 \gamma_7, \quad (A.4)$$

with $\epsilon^{012345} = +1$. All the spinors are symplectic Majorana

$$\chi_A = \epsilon^{AB} \tilde{\chi}_B^T, \quad \tilde{\chi}_A = (\chi_A)\gamma_0 \quad (A.5)$$

which means that $\tilde{\chi}_A = \chi_A^T$.

For any given four symplectic Majorana-Weyl spinors $\psi_1, \ldots, \psi_4$ with chiralities $\gamma_7 \psi_2 = c_2 \psi_2$, $\gamma_7 \psi_4 = c_4 \psi_4$, the Fierz rearrangement formula reads

$$\bar{\psi}_1 \psi_2 \bar{\psi}_3 \psi_4 = \frac{1}{8} (1 + c_2 c_4) \left[ \bar{\psi}_1 \psi_4 \bar{\psi}_3 \psi_2 - \frac{1}{2} \bar{\psi}_1 \gamma^{rs} \psi_4 \bar{\psi}_3 \gamma_{rs} \psi_2 \right]$$

$$- \frac{1}{8} (1 - c_2 c_4) \left[ \bar{\psi}_1 \gamma^r \psi_2 \bar{\psi}_3 \gamma_r \psi_4 - \frac{1}{12} \bar{\psi}_1 \gamma^{rst} \psi_4 \bar{\psi}_3 \gamma_{rst} \psi_2 \right] . \quad (A.6)$$
Note that there is a change in sign with respect to \[24\] because we are using *commuting* spinors. Since \( C = 1 \) we can use \( \gamma_0 \) as the intertwining operator between a representation of gamma matrices and its complex conjugate, in particular
\[
\gamma_0 \gamma_\alpha \gamma_0 = -\gamma_\alpha^* .
\] (A.7)

Notice that this is indeed consistent with the simplectic Majorana condition, which in \( Sp(1) \) components can be written as
\[
\chi^1 = -\gamma_0 \chi^{2*} , \quad \chi^2 = \gamma_0 \chi^{1*} .
\] (A.8)

From these we obtain the following reality properties of spinor bi-linears
\[
(\epsilon^{1} \gamma_{\alpha_1...\alpha_n} \epsilon^{2})^* = (-1)^n \epsilon^{2} \gamma_{\alpha_1...\alpha_n} \epsilon^{1}
\]
\[
(\epsilon^{1} \gamma_{\alpha_1...\alpha_n} \epsilon^{1})^* = (-1)^{n+1} \epsilon^{2} \gamma_{\alpha_1...\alpha_n} \epsilon^{2} .
\] (A.9)

We also note the following useful gamma-matrix identity
\[
\gamma_\alpha \gamma^\beta \delta \gamma^\gamma = 0
\] (A.10)

### Algebraic identities satisfied by spinor bi-linears

The bi-linears \( V \) and \( \Omega^{AB} \) defined in the main text in (2.5) and (2.6) satisfy some lengthy but useful relations which arise from the Fierz identity (A.6). Setting \( \psi_1 = \epsilon^A, \psi_2 = \gamma_\alpha \epsilon^B, \psi_3 = \epsilon^C, \psi_4 = \gamma_\rho \epsilon^D \) in (A.6) we obtain
\[
(\epsilon^{AB} \epsilon^{CD} - \frac{1}{2} \epsilon^{AD} \epsilon^{CB}) V_\lambda V_\rho + \frac{1}{4} \epsilon^{AD} \epsilon^{CB} V_\mu V_\nu g_{\lambda \rho} = -\frac{1}{8} \Omega^{AB} \mu \nu \Omega^{CD} \mu \nu \lambda
\]
\[
- \frac{1}{4} \epsilon^{CB} U^\mu \Omega^{AD} \mu \lambda \rho + \frac{1}{4} \epsilon^{AD} V_\mu \Omega^{CB} \mu \lambda \rho .
\] (A.11)

Contracting this with \( g^\lambda \rho \) and setting \( A = B, C = D \) we find \( V_\mu V_\nu = 0 \), i.e., \( V \) is null. Now use (A.11), setting \( A = B, C = D \), to obtain
\[
\frac{1}{2} (\epsilon^{AB})^2 V_\lambda V_\rho = -\frac{1}{8} \Omega^{AB} \mu \rho \Omega^{AB} \mu \nu \lambda + \frac{1}{2} \epsilon^{AB} V_\mu \Omega^{AB} \mu \lambda \rho .
\] (A.12)

By anti-symmetrizing this on \( \lambda, \rho \) we obtain
\[
i V \Omega^{AB} = 0 .
\] (A.13)

Next use \( \psi_1 = \gamma_\nu \gamma_\rho \epsilon^A, \psi_2 = \epsilon^B, \psi_3 = \epsilon^C, \psi_4 = \gamma^\nu \gamma^\lambda \epsilon^D \) in the Fierz identity; note that \( \psi_2 \) and \( \psi_4 \) are of opposite chirality. Using (A.10) we obtain
\[
\Omega^{AB} \rho \sigma \Omega^{CD} \nu \lambda \mu = (2 \epsilon^{CB} \epsilon^{AD} - \epsilon^{AB} \epsilon^{CD}) (g^{\sigma \lambda} V_\mu V_\rho + g^{\rho \nu} V_\lambda V_\sigma - g^{\sigma \mu} V_\nu V_\lambda - g^{\rho \lambda} V_\mu V_\sigma)
\]
\[
- \epsilon^{AB} (\Omega^{CD} \sigma \lambda \mu V_\rho - \Omega^{CD} \rho \lambda \nu V_\sigma) - \epsilon^{CD} (\Omega^{AB} \rho \sigma \lambda \mu V_\mu - \Omega^{AB} \rho \sigma \mu V_\lambda)
\]
\[
- 2 \epsilon^{CB} (\Omega^{AD} \rho \lambda \nu V_\sigma - \Omega^{AD} \sigma \lambda \mu V_\rho) .
\] (A.14)
B  Integrability conditions

Note that the Killing spinor equation
\[ \nabla_\mu \epsilon = \frac{1}{4} G_{\mu \rho \lambda} \gamma^{\rho \lambda} \epsilon \] (B.1)
for self-dual \( G \) implies the following integrability condition
\[ (R_{\nu \mu \rho \lambda} - \nabla_\nu G_{\mu \rho \lambda} + \nabla_\mu G_{\nu \rho \lambda} + 2G_{\mu \sigma \rho} G_{\nu \sigma \lambda}) \gamma^{\rho \lambda} \epsilon = 0 \] (B.2)

On contracting this identity with \( \gamma^\mu \), and using the self-duality of \( G \), we obtain
\[ \frac{1}{3} dG_{\mu \nu \rho \lambda} \gamma^{\mu \rho \lambda} \epsilon + (-2R_{\nu \lambda} - 2\nabla^\mu G_{\mu \nu \lambda} + 2G_{\nu \mu \rho} G_{\lambda \mu \rho}) \gamma^\lambda \epsilon = 0 \] (B.3)
where we note that as a consequence of the self-duality of \( G \),
\[ G_{\mu \nu \rho \sigma} G^\rho_{\mu \lambda \sigma} = 0. \] (B.4)

On imposing the Bianchi identity \( dG = 0 \) we obtain
\[ E_{\mu \nu} \gamma^\mu \epsilon = 0 \] (B.5)
where
\[ E_{\mu \nu} \equiv R_{\mu \nu} - G_{\mu \rho \sigma} G^\rho_{\nu \sigma} \] (B.6)

In particular, we observe that (B.3) implies that, in the null basis, \( E_{-\alpha} = 0 \). In addition, (B.5) also implies that
\[ E^\mu_{\nu} E^\mu_{\nu} = 0 \] (B.7)
with no sum on \( \mu \), from which we also find that \( E^{+a} = E_{ab} = 0 \). Hence, the integrability of the Killing spinor equation is sufficient to imply that all except the ++ components of the Einstein equation hold automatically.

C  Spin connection

The spin connection is defined by
\[ \omega_{\mu \alpha \beta} = e^\rho_\alpha \nabla_\mu e_{\beta \nu}. \] (B.1)
In the basis (3.7), the components of the spin connection are given by (3.17) and
\[ \omega_{++a} = H \left( \frac{1}{2} D F - \frac{1}{2} F \beta - \dot{\omega} \right)_a, \] (B.2)
\[ \omega_{b+a} = \frac{1}{2} \left( D\omega + \frac{\mathcal{F}}{2} D\beta \right)_{ab} + \frac{1}{2} \partial_a (H h_{mn}) \tilde{e}_a^m \tilde{e}_b^n, \quad (B.3) \]

\[ \omega_{+ab} = -\frac{1}{2} \left( D\omega + \frac{\mathcal{F}}{2} D\beta \right)_{ab} - H \tilde{e}_a^m \partial_a \tilde{e}_b^m, \quad (B.4) \]

\[ \omega_{cab} = -H^{-1/2} \tilde{\omega}_{cab} + H^{-1}(DH)_{ab} \delta_{bc} + \frac{1}{2} H^{1/2} \left( (\beta \wedge \dot{\tilde{e}}_a)_{bc} - (\beta \wedge \dot{\tilde{e}}_c)_{ab} + (\beta \wedge \dot{\tilde{e}}_b)_{ca} \right) \quad (B.5) \]

where \( \tilde{\omega}_{cab} \) are the basis components of the spin connection of the base manifold with indices lowered by \( \delta_{ab} \), and similarly \( \tilde{e}_a = \tilde{e}^a \).

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