ON THE MONODROMY MAP FOR THE LOGARITHMIC DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the monodromy map for logarithmic $\mathfrak{g}$-differential systems over an oriented surface $S_0$ of genus $g$, with $\mathfrak{g}$ being the Lie algebra of a complex reductive affine algebraic group $G$. These logarithmic $\mathfrak{g}$-differential systems are triples of the form $(X, D, \Phi)$, where $(X, D) \in T_{g,d}$ is an element of the Teichmüller space of complex structures on $S_0$ with $d \geq 1$ ordered marked points $D \subset S_0 = X$ and $\Phi$ is a logarithmic connection on the trivial holomorphic principal $G$-bundle $X \times G$ over $X$ whose polar part is contained in the divisor $D$. We prove that the monodromy map from the space of logarithmic $\mathfrak{g}$-differential systems to the character variety of $G$-representations of the fundamental group of $S_0 \setminus D$ is an immersion at the generic point, in the following two cases:

1. $g \geq 2$, $d \geq 1$, and $\dim_{\mathbb{C}} G \geq d + 2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

The above monodromy map is nowhere an immersion in the following two cases:

1. $g = 0$ and $d \geq 4$;
2. $g \geq 1$ and $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

This extends to the logarithmic case the main results in [CDHL], [BD] dealing with nonsingular holomorphic $\mathfrak{g}$-differential systems (which corresponds to the case of $d = 0$).


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1. Introduction

The study of the Riemann-Hilbert mapping, which associates to a flat (algebraic or holomorphic) connection its monodromy morphism from the fundamental group is a classical topic in algebraic and analytical geometry (see, for instance, [De], [Ka] and references therein).

We recall the set-up and results of [CDHL] and [BD], the predecessors of this paper. Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$, with $\dim G > 0$, and let $\mathfrak{g}$ be the Lie algebra of $G$. A $\mathfrak{g}$–differential system is a pair of the form $(X, \Phi)$, where $X$ is a complex structure on a compact oriented smooth surface $S_0$ of genus $g$, and $\Phi$ is a holomorphic connection on the trivial holomorphic principal $G$–bundle $X \times G$ over the Riemann surface $X$. A $\mathfrak{g}$–differential system $(X, \Phi)$ is called irreducible if $\Phi$ is not induced by a holomorphic connection on $X \times P$ for some proper parabolic subgroup $P$ of $G$. Since any holomorphic connection on a Riemann surface is flat, associating the monodromy representation to a holomorphic connection we obtain a map from the space of irreducible $\mathfrak{g}$–differential systems to the irreducible $G$-character variety $\text{Hom}(\pi_1(S_0), G)^{ir}/G$. This monodromy map is actually holomorphic.

The main result of [CDHL] says that, if $g = 2$, this Riemann-Hilbert monodromy map is a local diffeomorphism from the space of irreducible $\mathfrak{g}$–differential systems into the irreducible $G$-character variety, for $G = \text{SL}(2, \mathbb{C})$. Being inspired by [CDHL], in [BD] it was shown that, for all $g \geq 2$, the above monodromy map is an immersion on an open dense subset of the space of irreducible $\mathfrak{g}$–differential systems, for all reductive groups $G$ with $\dim_{\mathbb{C}} G \geq 3$.

Our aim here is to study the Riemann-Hilbert monodromy mapping for logarithmic $\mathfrak{g}$–differential systems, where $\mathfrak{g}$ is as above. These logarithmic $\mathfrak{g}$–differential systems are defined by triples of the form $(X, D, \Phi)$, where $(X, D) \in T_{g,d}$ is an element of the Teichmüller space of complex structures on $S_0$ with $d$ ordered marked points $D \subset S_0 = X$ (see Section 3), and $\Phi$ is a logarithmic connection on the trivial holomorphic principal $G$-bundle $X \times G$ over $X$ whose polar part is contained in the divisor $D$.

We prove the following (see Theorem 4.4):

**Theorem 1.1.** Assume that $3g - 3 + d > 0$ and $d \geq 1$. The Riemann-Hilbert monodromy mapping from the above space of irreducible logarithmic $\mathfrak{g}$–differential systems to the character variety of irreducible $G$-representations of the fundamental group of $S_0 \setminus D$ is an immersion at the generic point in the following two cases:

1. $g \geq 2$ and $\dim_{\mathbb{C}} G \geq d + 2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

The Riemann-Hilbert monodromy mapping from the above space of irreducible logarithmic $\mathfrak{g}$–differential systems to the character variety of irreducible $G$-representations of the fundamental group of $S_0 \setminus D$ is nowhere an immersion in the following two cases:

1. $g = 0$. 

(2) \( g \geq 1 \) and \( \dim \mathbb{C} G < \frac{d+3g-3}{g} \) (in particular, when \( g = 1 \) and \( \dim \mathbb{C} G < d \)).

We note that Theorem 1.1 gives a complete answer only when \( g = 0 \) or \( g = 1 \). For given \( g \geq 2 \) and \( G \), there are finitely many cases of \( d \) that are not addressed in Theorem 1.1. When \( g = 1 \) and \( d = 0 \), from the first part of Theorem 1.1 it follows that the monodromy mapping from the space of irreducible logarithmic \( g \)-differential systems is an immersion at the generic point; see Remark 4.6.

Theorem 1.1 extends to the class of logarithmic \( g \)-differential systems, the main result in [BD] which deals with the nonsingular holomorphic \( g \)-differential systems (corresponding to the case \( d = 0 \)). Notice that the hypothesis \( 3g-3+d > 0 \) in Theorem 1.1 implies that the above Teichmüller space \( T_{g,d} \) has positive dimension.

Given a reductive complex affine algebraic group \( G_0 \), by setting \( G \) to be the product group \( G_0^m \), \( m \geq 1 \), we can make its dimension arbitrarily large.

The proof of Theorem 1.1 is based on a transversality result in the moduli space \( B_G \) of quadruples of the form \((X, D, E_G, \Phi)\), where

- \((X, D) \in T_{g,d}\),
- \(E_G\) is a holomorphic principal \( G \)-bundle on \( X \) such that \( E_G \) is topologically trivial, and
- \(\Phi\) is a logarithmic connection on \( E_G \) whose polar part is contained in \( D \).

A key ingredient of this transversality condition is proved in Lemma 4.7 which is an adaptation to the logarithmic case of Theorem 1.1 in [Gi] (where its proof is attributed to R. Lazarsfeld).

The article is organized as follows. Sections 2 and 3 are preparatory: they introduce the concept of logarithmic connections on holomorphic principal bundles, the above moduli space \( B_G \) of quadruples \((X, D, E_G, \Phi)\) and the \( G \)-character variety. We describe the infinitesimal deformation space of quadruples (the tangent space of \( B_G \)) as the first hypercohomology group of a certain 2-term complex (see Proposition 3.2). Section 4 is devoted to the proof of the main result (Theorem 4.4) and deals with the transversality, in the tangent space of \( B_G \), between the isomonodromy foliation and the subspace of logarithmic \( g \)-differential systems. This transversality condition, which is equivalent to the monodromy map being an immersion on the space logarithmic of \( g \)-differential systems, is proved by combining a criteria given in Lemma 4.3 (also Proposition 4.5), with Lemma 4.7 (dealing with the case \( g \geq 3 \)) and Lemma 4.9 (dealing with the case of \( g = 2 \)).

2. The logarithmic Atiyah bundle

Let \( X \) be a compact connected Riemann surface. Let

\[
D := \{x_1, \ldots, x_d\} \subset X
\]

be \( d \) distinct points, with \( d \geq 2 \). For notational convenience, the divisor \( x_1 + \ldots + x_d \) of degree \( d \) on \( X \) will also be denoted by \( D \). For a holomorphic vector bundle \( V \) on \( X \), the holomorphic vector bundles \( V \otimes \mathcal{O}_X(D) \) and \( V \otimes \mathcal{O}_X(-D) \) will be denoted by \( V(D) \)
and $V(-D)$ respectively. The holomorphic tangent and cotangent bundles of $X$ will be denoted by $TX$ and $K_X$ respectively.

Let $G$ be a connected complex affine algebraic group with $\dim G > 0$. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Let

$$p : E_G \longrightarrow X$$

be a holomorphic principal $G$-bundle over $X$. The action of $G$ on $E_G$ produces an action of $G$ on the holomorphic tangent bundle $TE_G$ of $E_G$. The quotient

$$At(E_G) := (TE_G)/G \longrightarrow X$$

is the Atiyah bundle for $E_G [At]$. Let $dp : TE_G \longrightarrow p^*TX$ be the differential of the map $p$ in (2.2). Let

$$ad(E_G) := \text{kernel}(dp)/G \subset (TE_G)/G$$

be the adjoint bundle for $E_G$. Note that this holomorphic vector bundle $\text{kernel}(dp)$ is identified with the trivial holomorphic vector bundle $E_G \times \mathfrak{g} \longrightarrow E_G$ using the action of $G$ on $E_G$. Hence $ad(E_G)$ coincides with the vector bundle $E_G \times^G \mathfrak{g} \longrightarrow X$ associated to $E_G$ for the adjoint action of $G$ on $\mathfrak{g}$.

Thus we have a short exact sequence of holomorphic vector bundles on $X$

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow At(E_G) \xrightarrow{dp} TX \longrightarrow 0,$$

where $At(E_G)$ is defined in (2.3), and the projection $dp$ is induced by $dp$; the sequence in (2.5) is known as the Atiyah exact sequence. Define

$$At(E_G)(-\log D) := (d'p)^{-1}(TX(-D)) \subset At(E_G),$$

where $d'p$ is the homomorphism in (2.5). So, from (2.3) we have the logarithmic Atiyah exact sequence

$$0 \longrightarrow \text{ad}(E_G) \xrightarrow{\iota_0} At(E_G)(-\log D) \xrightarrow{\tilde{dp}} TX(-D) \longrightarrow 0,$$

where $\tilde{dp}$ is the restriction of the homomorphism $d'p$ to $At(E_G)(-\log D)$, and $\iota_0$ is given by the homomorphism $\text{ad}(E_G) \longrightarrow At(E_G)$ in (2.5). We have the following commutative diagram of homomorphisms

$$\begin{array}{cccccc}
0 & \longrightarrow & \text{ad}(E_G) & \xrightarrow{\iota_0} & At(E_G)(-\log D) & \xrightarrow{\tilde{dp}} & TX(-D) & \longrightarrow & 0 \\
\| & & \downarrow{} & & \downarrow{} & & \downarrow{} & & \downarrow{} \\
0 & \longrightarrow & \text{ad}(E_G) & \longrightarrow & At(E_G) & \xrightarrow{d'p} & TX & \longrightarrow & 0 \\
\end{array}$$

where $\iota$ and $\iota'$ are the natural inclusion maps.

A logarithmic connection on $E_G$ with polar part in $D$ is a holomorphic homomorphism

$$\Phi : TX(-D) \longrightarrow At(E_G)(-\log D)$$

such that

$$\tilde{dp} \circ \Phi = \text{Id}_{TX(-D)},$$

where $\tilde{dp}$ is the surjective homomorphism in (2.7).
Since we have \( \iota' \circ \hat{dp} = (d'p) \circ \iota \) (see (2.8)), and \( \iota'(y)(TX(-D))_y = 0 \) for every point \( y \in D \) in (2.1), for a logarithmic connection \( \Phi \) on \( E_G \), from (2.9) we have

\[
\iota' \circ \hat{dp} \circ \Phi(TX(-D))_y = \iota'(y)(TX(-D))_y = 0
\]

for every \( y \in D \). Consequently, from the commutativity of (2.8) we conclude that \( (d'p) \circ \iota \circ \Phi(TX(-D))_y = 0 \). This implies that

\[
\iota \circ \Phi(TX(-D))_y \subset \text{ad}(E_G)_y \subset \text{At}(E_G)_y
\]

(2.10) (see (2.8)). On the other hand, \( TX(-D)_y = \mathbb{C} \) by the Poincaré adjunction formula [GH, p. 146]; for any holomorphic coordinate function \( z \) on \( X \) around \( y \) with \( z(y) = 0 \), the map \( \mathbb{C} \rightarrow TX(-D)_y \) defined by \( \lambda \mapsto (\lambda \frac{dz}{z})(y) \) is actually independent of the choice of the coordinate function \( z \). The element

\[
(\iota \circ \Phi)(y)(1) \in \text{ad}(E_G)_y
\]

(see (2.10)) is called the residue of \( \Phi \) at \( y \); see [De].

Fixing \( X \), the infinitesimal deformations of the principal \( G \)-bundle \( E_G \) are parametrized by \( H^1(X, \text{ad}(E_G)) \) [Do].

We recall that the infinitesimal deformations of the \( d \)-pointed Riemann surface \( (X, D) \) are parametrized by \( H^1(X, TX(-D)) \). The infinitesimal deformations of the above triple \( (X, D, E_G) \) are parametrized by \( H^1(X, \text{At}(E_G)(-\log D)) \) [BHH], [Ch1], [Ch2], [Hu], [Do].

The following lemma is standard (see [BHH] Section 2.2 and [Hu]).

**Lemma 2.1.**

1. The homomorphism of cohomologies

\[
\hat{dp}_* : H^1(X, \text{At}(E_G)(-\log D)) \rightarrow H^1(X, TX(-D)),
\]

induced by the projection \( \hat{dp} \) in (2.7), corresponds to the forgetful map from the infinitesimal deformations of the triple \( (X, D, E_G) \) to the infinitesimal deformations of the pair \( (X, D) \) obtained by simply forgetting the principal \( G \)-bundle.

2. The homomorphism of cohomologies

\[
\iota_0* : H^1(X, \text{ad}(E_G)) \rightarrow H^1(X, \text{At}(E_G)(-\log D)),
\]

induced by the homomorphism \( \iota_0 \) in (2.7), coincides with the map from the infinitesimal deformations of the principal \( G \)-bundle \( E_G \) to the infinitesimal deformations of the triple \( (X, D, E_G) \) obtained by keeping the pair \( (X, D) \) fixed.

3. **Logarithmic connections and isomonodromy**

3.1. **Logarithmic Atiyah bundle.** Since \( \text{At}(E_G) := (TE_G)/G \) (see (2.3)), the subsheaf \( \text{At}(E_G)(-\log D) \subset \text{At}(E_G) \) corresponds to a subsheaf of the sheaf of \( G \)-invariant holomorphic vector fields on \( E_G \). We will have occasions to use the following description of this subsheaf of the sheaf of \( G \)-invariant holomorphic vector fields on \( E_G \).
Let
\[ \tilde{D} := p^{-1}(D) \subset E_G \]
be the divisor, where \( p \) is the projection in (2.2). Let
\[ TE_G(-\log \tilde{D}) \subset TE_G \]
be the corresponding logarithmic tangent bundle. We recall that this subsheaf is characterized by the following property: A holomorphic vector field \( v \), defined on an open subset \( U \subset E_G \), is a section of \( TE_G(-\log \tilde{D}) \) if and only if for every holomorphic function \( f \) on \( U \) that vanishes on \( \tilde{D}\cap U \), the function \( v(f) \) also vanishes on \( \tilde{D}\cap U \). Since the divisor \( \tilde{D} \) is smooth, it follows that \( TE_G(-\log \tilde{D}) \) is a locally free \( \mathcal{O}_{E_G} \)-submodule of \( TE_G \). Consequently, \( TE_G(-\log \tilde{D}) \) is a holomorphic vector bundle on \( E_G \). The above characterizing property of \( TE_G(-\log \tilde{D}) \) immediately implies that the Lie bracket operation of locally defined holomorphic vector fields on \( E_G \) preserves the subsheaf \( TE_G(-\log \tilde{D}) \).

To describe \( TE_G(-\log \tilde{D}) \) locally, take a point \( x \in \tilde{D} \). Let \( (z_1, z_2, \cdots, z_m) \) be holomorphic coordinate functions on \( E_G \) defined around \( x \) such that \( z_1 = p \circ z \) for some holomorphic coordinate function \( z \) on \( X \) around \( p(x) \), and also \( z_i(x) = 0 \) for all \( 1 \leq i \leq m \); here \( p \) denotes the projection in (2.2). Then \( TE_G(-\log \tilde{D}) \) around \( x \) is generated by the holomorphic vector fields \( z_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_m} \).

The action of \( G \) on \( TE_G \), induced by the action of \( G \) on \( E_G \), actually preserves the subsheaf \( TE_G(-\log \tilde{D}) \). It is now straightforward to check that
\[ At(E_G)(-\log D) = TE_G(-\log \tilde{D})/G. \tag{3.1} \]

Let
\[ \Phi : TX(-D) \longrightarrow At(E_G)(-\log D) \tag{3.2} \]
be a logarithmic connection on \( E_G \). Let
\[ \tilde{\Phi} : At(E_G)(-\log D) \longrightarrow \text{ad}(E_G) \tag{3.3} \]
be the holomorphic homomorphism uniquely determined by the following conditions:

(1) \( \tilde{\Phi} \circ \iota_0 = \text{Id}_{\text{ad}(E_G)} \), where \( \iota_0 \) is the injective homomorphism in (2.7), and
(2) \( \text{kernel}(\tilde{\Phi}) = \Phi(TX(-D)) \).

In view of (2.4) and (3.1), the homomorphism \( \tilde{\Phi} \) in (3.3) produces a \( G \)-invariant surjective holomorphic homomorphism
\[ \Phi' : TE_G(-\log \tilde{D}) \longrightarrow \text{kernel}(dp), \tag{3.4} \]
where \( p \) is the projection in (2.2).

Let \( w \) be a holomorphic vector field on an open subset \( U \subset X \) that vanishes on \( U \cap D \). In view of (3.1), the section \( \Phi(w) \) of \( At(E_G)(-\log D)|_U \) corresponds to a unique \( G \)-invariant holomorphic section of \( TE_G(-\log \tilde{D})|_{p^{-1}(U)} \) satisfying the condition that
\[ dp(\Phi(w)) = p^*w. \]
(as sections of $p^*TX$); let
\[ \Phi(w)' \in H^0(p^{-1}(U), TE_G(-\log D)|_{p^{-1}(U)}) \] (3.5)
denote this section constructed from $w$.

**Lemma 3.1.** Let $v$ be a $G$–invariant holomorphic section of $TE_G(-\log D)|_{p^{-1}(U)}$. Then the following three hold:

1. The holomorphic section $\Phi'_0([\Phi(w)', v])$ of kernel($dp$) is $G$–invariant, where $\Phi'_0$ is the homomorphism in (3.4) and $\Phi(w)'$ is the section of $TE_G(-\log D)|_{p^{-1}(U)}$ constructed above from $w$.
2. For every holomorphic function $h$ on $U$,
\[ \Phi'_0([\Phi(h \cdot w)', v]) = (h \circ p) \cdot \Phi'_0([\Phi(w)', v]). \]
3. If $v = \Phi(v_1)'$ for some holomorphic section $v_1$ of $T(-D)|_U$, then
\[ \Phi'_0([\Phi(w)', v]) = 0. \]

**Proof.** As noted before, the Lie bracket operation of locally defined holomorphic vector fields on $E_G$ preserves the subsheaf $TE_G(-\log D)$. Since the homomorphism $\Phi'_0$ in (3.4) is $G$–invariant, and $\Phi(w)'$ is $G$–invariant, while $v$ is given to be $G$–invariant, it follows that $\Phi'_0([\Phi(w)', v])$ is also $G$–invariant.

To prove the second statement, consider the identity
\[ [\Phi(h \cdot w)', v] = (h \circ p) \cdot [\Phi(w)', v] - v(h \circ p) \cdot \Phi(w)'. \]
Since $\Phi'_0(\Phi(w)') = 0$, where $\Phi'_0$ is the homomorphism in (3.4), the second statement follows from this identity.

The third statement follows from the fact that any holomorphic one-dimensional distribution is integrable. \qed

In view of (2.4) and (3.1), from Lemma 3.1(2) we get a homomorphism
\[ \Phi : TX(-D) \otimes At(E_G)(-\log D) \longrightarrow ad(E_G). \]
Then the homomorphism
\[ \Phi \otimes \text{Id}_{TX(-D)} : At(E_G)(-\log D)TX(-D) \otimes TX(-D)^* \longrightarrow ad(E_G) \otimes TX(-D)^* \]
produces a homomorphism
\[ \Phi : At(E_G)(-\log D) \longrightarrow ad(E_G) \otimes TX(-D)^* = ad(E_G) \otimes K_X(D) \] (3.6)
using the duality pairing $TX(-D) \otimes TX(-D)^* \longrightarrow \mathcal{O}_X$.

Let $\mathcal{C}_\bullet$ be the 2–term complex of sheaves on $X$
\[ \mathcal{C}_\bullet : \mathcal{C}_0 := At(E_G)(-\log D) \xrightarrow{\Phi} \mathcal{C}_1 := ad(E_G) \otimes K_X(D), \] (3.7)
where $\Phi$ is the homomorphism constructed in (3.6), and $\mathcal{C}_i$ is at the $i$-th position.

From Lemma 3.1(3) we know that
\[ \Phi \circ \Phi = 0. \]
Consequently, the logarithmic connection $\Phi$ in (3.2) produces a homomorphism of complexes

$$\Phi^C : TX(-D) \longrightarrow C_\bullet,$$

where $TX(-D)$ is the one-term complex concentrated at the 0-th position, and $C_\bullet$ is the complex in (3.7). In other words, we have the commutative diagram

$$\begin{array}{ccc}
T_X(-D) & \longrightarrow & 0 \\
\Phi^C \downarrow & & \downarrow \Phi \\
C_\bullet & \longrightarrow & C_0 \longrightarrow C_1
\end{array}$$

(3.9)

3.2. Character variety. Let $\mathcal{T}_{g,d}$ denote the Teichmüller space of compact connected Riemann surfaces of genus $g$ with $d$ ordered marked points, where $d \geq 1$. We will always assume that $3g - 3 + d > 0$. This $\mathcal{T}_{g,d}$ is a complex manifold of dimension $3g - 3 + d$. We recall a description of $\mathcal{T}_{g,d}$ which will be used here. Let $S_0$ be an oriented $C^\infty$ surface of genus $g$, and let $D_0 \subset S_0$ be $d$ ordered distinct points. Let $C(S_0)$ denote the space of all $C^\infty$ complex structures on $S_0$ compatible with the given orientation of $S_0$. Let $\text{Diff}^0_{D_0}(S_0)$ denote the group of all orientation preserving diffeomorphisms $\beta : S_0 \longrightarrow S_0$ such that

- $\beta(x) = x$ for every $x \in D_0$, and
- $\beta$ is homotopic to the identity map of $S_0$ through a continuous family of diffeomorphisms $\beta_t$ of $S_0$, $0 \leq t \leq 1$, such that $\beta_t(x) = x$ for all $t$ and all $x \in D_0$.

The group $\text{Diff}^0_{D_0}(S_0)$ acts on $C(S_0)$ by pushing forward complex structures using diffeomorphisms. Then we have

$$\mathcal{T}_{g,d} = C(S_0)/\text{Diff}^0_{D_0}(S_0).$$

We now assume the complex connected affine algebraic group $G$ to be reductive. The complement $S_0 \setminus D_0$ will be denoted by $S_0'$. Let

$$\mathcal{R}_G(S_0') := \text{Hom}^\text{ir}(\pi_1(S_0'), G)/G$$

be the irreducible $G$–character variety for $S_0'$; the space $\text{Hom}^\text{ir}(\pi_1(S_0'), G)$ consists of all homomorphisms $\gamma : \pi_1(S_0') \longrightarrow G$ such that $\gamma(\pi_1(S_0'))$ is not contained in any proper parabolic subgroup of $G$. We note that $\mathcal{R}_G(S_0')$ does not depend on the choice of the base point needed to define the fundamental group of $S_0'$. Since $\pi_1(S_0')$ is finitely presented, the complex algebraic structure of $G$ produces a complex algebraic structure on $\mathcal{R}_G(S_0')$, so $\mathcal{R}_G(S_0')$ is a complex affine variety. It is in fact a smooth complex orbifold. We have

$$\dim \mathcal{R}_G(S_0') = (2g + d - 1) \cdot \dim_C G - \dim_C[G, G].$$

(3.11)

For more details of the above dimension count the reader is referred to [Go] and [Si, Proposition 49] (to which the monodromy around the poles should be added).
3.3. **Monodromy of logarithmic connections.** A logarithmic connection \( \Phi \) on a holomorphic principal \( G \)-bundle \( E_G \rightarrow X \) is called **irreducible** if there is no holomorphic reduction of structure group \( E_P \subset E_G|_{X \setminus D} \) to some proper parabolic subgroup \( P \subset G \), over the open subset \( X \setminus D \subset X \), such that \( \Phi \) is induced by a holomorphic connection on \( E_P \).

The above definition of irreducibility needs a clarification, because in the special case where \( E_G \) is the trivial holomorphic principal \( G \)-bundle, and \( D \) is the zero divisor — so the logarithmic connection \( \Phi \) is holomorphic, meaning it has no poles — this definition of irreducibility is, a priori, weaker than the definition, given in the Introduction, of irreducible holomorphic \( g \)-differential systems. More precisely, in the definition, given in the Introduction, of irreducible holomorphic \( g \)-differential systems, the principal \( P \)-bundle is required to be the trivial bundle \( X \times P \rightarrow X \), while the above definition does not impose any other condition on \( E_P \) apart from the condition that the logarithmic connection \( \Phi \) is induced by a logarithmic connection on \( E_P \). We will show the following:

Let \( \Phi \) be a holomorphic connection on the trivial principal \( G \)-bundle 
\[
\mathcal{E}_G^0 := X \times G \rightarrow X ,
\]
and let \( E_P \subset \mathcal{E}_G^0 \) be a holomorphic reduction of structure group to \( P \) over \( X \), such that \( \Phi \) is induced by a holomorphic connection on \( E_P \). Then \( E_P \) is the trivial principal \( P \)-bundle \( X \times P \rightarrow X \).

To prove the above statement, first note that a holomorphic reduction of structure group \( E_P \subset \mathcal{E}_G^0 \) to \( P \) is given by a holomorphic map \( \phi : X \rightarrow G/P \). For this map \( \phi \), we have
\[
\phi^*T(G/P) = \text{ad}(\mathcal{E}_G^0)/\text{ad}(E_P). \tag{3.12}
\]
If \( E_P \) admits a holomorphic connection \( \Phi_P \), then \( \Phi_P \) induces holomorphic connections on both \( \text{ad}(\mathcal{E}_G^0) \) and \( \text{ad}(E_P) \). This implies that
\[
\text{degree}(\text{ad}(\mathcal{E}_G^0)) = 0 = \text{degree}(\text{ad}(E_P)) ,
\]
and hence from (3.12) it follows that
\[
\text{degree}(\phi^*T(G/P)) = 0 . \tag{3.13}
\]
Since the anti-canonical line bundle \( K_{G/P}^{-1} \) on \( G/P \) is ample, from (3.13) we conclude that \( \phi \) is a constant map. Consequently, \( P \) is the trivial principal \( P \)-bundle \( X \times P \rightarrow X \).

Let
\[
\varphi : \mathcal{B}_G \rightarrow \mathcal{T}_{g,d} \tag{3.14}
\]
be the moduli space of irreducible logarithmic connections on topologically trivializable holomorphic principal \( G \)-bundles. So \( \mathcal{B}_G \) is the moduli space of quadruples of the form \((X, D, E_G, \Phi)\), where
- \((X, D) \in \mathcal{T}_{g,d} ,
- \( E_G \) is a holomorphic principal \( G \)-bundle on \( X \) such that \( E_G \) is topologically trivial, and
• $\Phi$ is an irreducible logarithmic connection on $E_G$ whose polar part is contained in $D$.

The map $\varphi$ in (3.14) sends any $(X, D, E_G, \Phi)$ to the pair $(X, D)$. The moduli space $\mathcal{B}_G$ is a smooth complex orbifold.

Any logarithmic connection on a Riemann surface $X$ is flat because $\bigwedge^2 (TX)^* = 0$ (consequently, its curvature 2-form vanishes identically). So considering monodromy representation of logarithmic connections, we get a holomorphic map

$$\theta : \mathcal{B}_G \longrightarrow \mathcal{R}_G(S'_0), \quad (3.15)$$

where $\mathcal{R}_G(S'_0)$ is constructed in (3.10).

We will prove that a logarithmic connection is irreducible if and only if the corresponding monodromy representation is irreducible.

First, let $E_G$ be a holomorphic principal $G$–bundle on $X$ equipped with a logarithmic connection $\Phi$, such that $\Phi$ is not irreducible. So there is a proper parabolic subgroup $P \subset G$, a holomorphic reduction of the structure group of $E_G|_{X \setminus D}$ to $P$, given by a subbundle $E_P \subset E_G|_{X \setminus D}$, and a holomorphic connection $\Phi_P$ on $E_P$, such that the logarithmic connection on $E_G$ induced by $\Phi_P$ coincide with $\Phi$. Since the monodromy of $\Phi_P$ coincides with the monodromy of $\Phi$, the monodromy of $\Phi$ is contained in $P$, and hence the monodromy representation for $\Phi$ is not irreducible. To prove the converse, let $\Phi$ be an irreducible logarithmic connection on a holomorphic principal $G$–bundle $E_G$ on $X$. Take a point $x_0 \in X \setminus D$ and fix a point $z_0 \in (E_G)_{x_0}$ in the fiber of $E_G$ over $x_0$. Taking parallel translations of $z_0$ along all possible homotopy classes of loops based at $x_0$ we get the monodromy representation

$$H_\Phi : \pi_1(X \setminus D, x_0) \longrightarrow G$$

of $\Phi$. Assume that the image of $H_\Phi$ is contained in a parabolic subgroup $P \subset G$. Let $S \subset E_G|_{X \setminus D}$ be the subset obtained by taking parallel translations of $z_0$ along all possible homotopy classes of paths starting at $x_0$. Then

$$E_P := SP \subset E_G|_{X \setminus D}$$

(recall that $G$ acts on $E_G$) is a holomorphic reduction of the structure group of $E_G|_{X \setminus D}$ to $P$ over $X \setminus D$. The logarithmic connection $\Phi$ produces a holomorphic connection on the holomorphic principal $P$–bundle $E_P$, which in turn induces $\Phi$. Consequently, the logarithmic connection $\Phi$ is not irreducible. Thus, a logarithmic connection is irreducible if and only if the corresponding monodromy representation is irreducible.

3.4. Isomonodromy. Let

$$d\theta : T\mathcal{B}_G \longrightarrow \theta^* T\mathcal{R}_G(S'_0)$$

be the differential of the map $\theta$ in (3.15). The map $\theta$ is a holomorphic submersion, meaning $d\theta$ is surjective. The kernel of $d\theta$

$$\mathcal{I} := \text{kernel}(d\theta) \subset T\mathcal{B}_G \quad (3.16)$$
is a holomorphic foliation on $B_G$; it is known as the isomonodromy foliation.

For any point $(X, D) \in \mathcal{T}_{g,d}$, the restriction of $\theta$ to $\varphi^{-1}((X, D))$, where $\varphi$ is the projection in (3.14), is a holomorphic local diffeomorphism. Consequently, for any point $z \in B_G$, the differential of $\varphi$

$$d\varphi(z) : T_z B_G \longrightarrow T_{\varphi(z)}\mathcal{T}_{g,d},$$

when restricted to the subspace $\mathcal{I}_z \subset T_z B_G$ in (3.16), produces an isomorphism

$$\mathcal{I}_z \sim T_{\varphi(z)}\mathcal{T}_{g,d}.$$

Therefore, there is a unique holomorphic homomorphism

$$\mathbb{L} : \varphi^* T_{\mathcal{T}_{g,d}} \longrightarrow T_{B_G}$$

(3.17)

such that

- $d\varphi \circ \mathbb{L} = \text{Id}_{\varphi^* T_{\mathcal{T}_{g,d}}}$, and
- $\mathbb{L}(\varphi^* T_{\mathcal{T}_{g,d}}) \subset \mathcal{I}$, where $\mathcal{I}$ is constructed in (3.16).

Since for any point $(X, D) \in \mathcal{T}_{g,d}$, the restriction of $\theta$ to $\varphi^{-1}((X, D))$ is a holomorphic local diffeomorphism, it follows that $\mathbb{L}$ actually satisfies the condition that

$$\mathbb{L}(\varphi^* T_{\mathcal{T}_{g,d}}) = \mathcal{I}.\quad (3.18)$$

**Proposition 3.2.** Take any point $z = (X, D, E_G, \Phi) \in B_G$.

1. The tangent space to $B_G$ at $z$ is the first hypercohomology

$$T_z B_G = \mathbb{H}^1(X, C_\bullet),$$

where $C_\bullet$ is the complex in (3.7).

2. The homomorphism

$$\mathbb{L}(z) : T_{\varphi(z)}\mathcal{T}_{g,d} = H^1(X, TX(-D)) \longrightarrow T_z B_G = \mathbb{H}^1(X, C_\bullet)$$

in (3.17) coincides with the homomorphism of hypercohomologies

$$\Phi_C^\bullet : H^1(X, TX(-D)) \longrightarrow \mathbb{H}^1(X, C_\bullet)$$

induced by the homomorphism $\Phi_C$ in (3.8).

For the proof of Proposition 3.2 the reader is referred to [Ko, Proposition 3.8] (Proposition 3.4 of the arxiv version of [Ko]), [Ch2, p. 1417, Proposition 5.1], [Ch1], [In] and [BS].

4. Monodromy map on logarithmic differential systems

4.1. **Logarithmic differential systems.** As before, $G$ is a connected complex reductive affine algebraic group with $\dim G > 0$, and

$$d_s := \dim_{\mathbb{C}}[G, G].\quad (4.1)$$

Consider the moduli space $B_G$ in (3.14). Let

$$\mathbb{T}(G) \subset B_G\quad (4.2)$$
be the locus of all \((X, D, E_G, \Phi)\) such that the holomorphic principal \(G\)-bundle \(E_G\) on \(X\) is holomorphically trivial. Note that \(E_G\) is topologically trivial by the definition of \(B_G\); also by the definition of \(B_G\) the logarithmic connection \(\Phi\) is irreducible. The subset \(T(G)\) in (4.2) is a complex subspace.

**Proposition 4.1.** The complex space \(T(G)\) in (4.2) is a complex orbifold of dimension \((g + d - 1) \cdot \dim \mathcal{C} G - d_s + 3g - 3 + d\), where \(d_s\) is defined in (4.1).

**Proof.** Let \(\varpi : C_{g,d} \rightarrow \mathcal{T}_{g,d}\) be the universal Riemann surface equipped with the universal divisor \(D \subset C_{g,d}\) of relative degree \(d\) over \(\mathcal{T}_{g,d}\). Let \(\mathcal{K} \rightarrow C_{g,d}\) be the relative holomorphic cotangent bundle for the projection \(\varpi\). Let \(\varphi' : T(G) \rightarrow \mathcal{T}_{g,d}\) be the restriction of the map \(\varphi\) in (3.14).

For any Riemann surface \(X\), the space of all logarithmic connections on the trivial holomorphic principal \(G\)-bundle \(X \times G \rightarrow X\) with polar part contained in \(D \subset X\) is the vector space \(H^0(\mathcal{K}, K_X(D)) \otimes \mathfrak{g}\), where \(\mathfrak{g}\) is the Lie algebra of \(G\). Consequently, \(T(G)\) is the quotient of an open dense subset of the total space of \(\varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(D)) \otimes \mathfrak{g}\) by the adjoint action of \(G\); the group \(G\) acts trivially on \(\varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(D))\) and it has the adjoint action on \(\mathfrak{g}\).

Take a point
\[
w \in \varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(D)) \otimes \mathfrak{g}
\]
that defines an irreducible logarithmic connection on the trivial principal \(G\)-bundle. The adjoint action of the center of \(G\) on \(\mathfrak{g}\) is trivial. The isotropy subgroup of \([G, G]\) for the action of \([G, G]\) on \(w\) is a finite subgroup of \([G, G]\). The proposition follows from these.

Let
\[
\hat{\theta} : T(G) \rightarrow R_G(S'_0),
\]
be the restriction to \(T(G) \subset B_G\) of the monodromy map \(\theta\) in (3.15). We are interested in the following question: When is the map \(\hat{\theta}\) an immersion over an open dense subset of \(T(G)\)?

**Remark 4.2.** In [BD] it was proved that \(\hat{\theta}\) is an immersion over an open dense subset of \(T(G)\), if \(g \geq 2\), \(d = 0\) and \(\dim \mathcal{C} G \geq 3\). From this it can be deduced that \(\hat{\theta}\) is an immersion over an open dense subset of \(T(G)\), if \(g \geq 2\), \(d = 1\) and \(\dim \mathcal{C} G \geq 3\). To see this, first note that there is no logarithmic one-form on a compact Riemann surface \(X\) with exactly one pole, because the residue has to be zero. So the space \(T(G)\) in (4.3) for \(d = 1\) coincides with \(T(G)\) for \(d = 0\). On the other hand, the natural map from the space \(R_G(S'_0)\) in (4.3) for \(d = 0\) to the space \(R_G(S'_0)\) for \(d = 1\) is an embedding; this natural map corresponds to restricting any flat \(G\)-connection on \(S_0\) to the open subset \(S'_0 = S_0 \setminus D_0\) of it. Therefore, we conclude that the map \(\hat{\theta}\) is an immersion over an open dense subset of \(T(G)\), if \(g \geq 2\), \(d = 1\) and \(\dim \mathcal{C} G \geq 3\).
4.2. The main theorem. We first state a lemma of linear algebra which will be used in the proof of Theorem 4.4.

**Lemma 4.3.** Let $\beta : V \to W$ be a linear map between two finite dimensional complex vector spaces. Let $S_1$ and $S_2$ be two subspaces of $V$ such that

1. $\text{kernel}(\beta) \subset S_1$, and
2. the homomorphism $\beta|_{S_2} : S_2 \to W$ is injective.

Then $\dim S_1 \cap S_2 = \dim \beta(S_1) \cap \beta(S_2)$.

**Proof.** Since $\beta|_{S_2}$ is injective, the restriction $\beta|_{S_1 \cap S_2}$ is injective. For any $v \in S_2$ with $\beta(v) \in \beta(S_1)$, there is an element $w \in S_1$ such that $\beta(v) = \beta(w)$. But then $v - w \in \text{kernel}(\beta) \subset S_1$, and hence $v \in S_1$. Consequently, the restriction $\beta|_{S_1 \cap S_2}$ is realized as an isomorphism between $S_1 \cap S_2$ and $\beta(S_1) \cap \beta(S_2)$. \hfill $\square$

**Theorem 4.4.** Assume that $3g - 3 + d > 0$ and $d \geq 1$. The map $\hat{\theta}$ in (4.3) is an immersion over a nonempty open dense subset of $T(G)$ in the following two cases:

1. $g \geq 2$ and $\dim_{\mathbb{C}} G \geq d + 2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

The map $\hat{\theta}$ in (4.3) is nowhere an immersion in the following two cases:

1. $g = 0$;
2. $g \geq 1$ and $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

**Proof.** First assume that $g = 0$. The trivial holomorphic principal $G$–bundle over $\mathbb{CP}^1$ is rigid [Ra], [Ha]. In other words, in any holomorphic family of holomorphic principal $G$–bundle over $\mathbb{CP}^1$, parametrized by a complex manifold $Z$, the locus of points of $Z$ over which the principal $G$–bundle on $\mathbb{CP}^1$ is holomorphically trivial is an open subset of $Z$. Therefore, the map $\hat{\theta}$ in (4.3) is nowhere an immersion. We note that this also follows from the fact that

$$\dim T(G) - R_G(S'_0) = d - 3 > 0$$

if $g = 0$ (see (3.11) and Proposition 4.1).

So we assume that $g \geq 1$.

If $g \geq 1$ and $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$, then from (3.11) and Proposition 4.1, we have

$$\dim T(G) - R_G(S'_0) = 3g - 3 + d - g \cdot \dim_{\mathbb{C}} G > 0.$$  

Hence the map $\hat{\theta}$ in (4.3) is nowhere an immersion in this case also.

So we assume that at least one of the following two holds:

1. $g \geq 2$ and $\dim_{\mathbb{C}} G \geq d + 2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$. 


The map $\hat{\theta}$ in (4.3) is an immersion over the subset of $T(G)$ over which the homomorphism
\[
\bigwedge^e d\hat{\theta} : \bigwedge^e T\mathbb{T}(G) \longrightarrow \hat{\theta}^* \bigwedge^e T\mathcal{R}_G(S'_0)
\]
is fiber-wise nonzero, where $e = \dim_C T(G)$ and $d\hat{\theta}$ is the differential of the map $\hat{\theta}$. Therefore, to prove the theorem, it suffices to show that there is a point $z \in T(G)$ such that the differential at $z$
\[
d\hat{\theta}(z) : T_z T(G) \longrightarrow T_{\hat{\theta}(z)} \mathcal{R}_G(S'_0)
\]
is injective; recall that $T(G)$ is irreducible.

Take a point $z = (X, D, E_G, \Phi) \in T(G)$.

We recall that $I(z) = \ker((d\theta)(z))$ (see (3.16)). The homomorphism $d\hat{\theta}(z)$ (see (4.4)) is injective if and only if
\[
I(z) \cap T_z T(G) = 0 ;
\]
note that both $I(z)$ and $T_z T(G)$ are subspaces of the tangent space $T_z B_G$.

We will use Lemma 4.3 to prove that (4.6) holds when $z$ is chosen suitably.

We recall from Proposition 3.2(1) that $T_z B_G = \mathbb{H}^1(X, C\bullet)$, where $C\bullet$ is the complex in (3.7). Also, recall that the infinitesimal deformations of the triple $(X, D, E_G)$ are parametrized by $H^1(X, \text{At}(E_G)(-\log D))$. Let
\[
\rho : \mathbb{H}^1(X, C\bullet) \longrightarrow H^1(X, \text{At}(E_G)(-\log D))
\]
be the forgetful map that sends any infinitesimal deformation of the quadruple $z = (X, D, E_G, \Phi)$ in (4.5) to the infinitesimal deformation of the triple $(X, D, E_G)$ obtained from it by simply forgetting the logarithmic connection. We shall describe $\rho$ explicitly.

Let $A\bullet$ be the one-term complex with $\text{At}(E_G)(-\log D)$ at the 0-th position.

Consider the homomorphism $\mathcal{H}$ of complexes
\[
\begin{array}{ccc}
C\bullet & : & \text{At}(E_G)(-\log D) \xrightarrow{\Phi} \text{ad}(E_G) \otimes K_X(D) \\
\mathcal{H} | & & | \\
A\bullet & : & \text{At}(E_G)(-\log D) \longrightarrow 0
\end{array}
\]
where $C\bullet$ is the complex in (3.7). Let
\[
\mathcal{H}_* : \mathbb{H}^1(X, C\bullet) \longrightarrow \mathbb{H}^1(X, A\bullet) = H^1(X, \text{At}(E_G)(-\log D))
\]
be the homomorphism of hypercohomologies induced by this homomorphism of complexes. Then the homomorphism $\rho$ in (4.7) coincides with $\mathcal{H}_*$ in (4.9).

In Lemma 4.3, set $V = \mathbb{H}^1(X, C\bullet)$, $W = H^1(X, \text{At}(E_G)(-\log D))$, $\beta = \mathcal{H}_*$ (see (4.9)), $S_1 = T_z T(G)$ (see (4.2)) and $S_2 = I(z)$ (see (3.16)).

We will show that the hypotheses in Lemma 4.3 are satisfied.

**Proposition 4.5.** For the above data, the two conditions in Lemma 4.3 hold.
Proof of Proposition 4.5. The first condition in Lemma 4.3 says that
\[
\text{kernel}(H^\ast) \subset T_z T(G).
\] (4.10)

To prove (4.10), we will identify the kernel of $H^\ast$. For this, observe that the homomorphism of complexes $H$ in (4.8) fits in the following short exact sequence of complexes:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
A_\prime & 0 & \longrightarrow & \text{ad}(E_G) \otimes K_X(D) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{C}_\ast & \text{At}(E_G)(-\log D) & \xrightarrow{\Phi} & \text{ad}(E_G) \otimes K_X(D) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{H} & \text{At}(E_G)(-\log D) & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

This short exact sequence of complexes yields the following long exact sequence of hypercohomologies:

\[
\longrightarrow \mathbb{H}^1(X, A_\prime) = H^0(X, \text{ad}(E_G) \otimes K_X(D)) \xrightarrow{\nu} \mathbb{H}^1(X, \mathcal{C}_\ast) \\
\xrightarrow{\mathcal{H}} \mathbb{H}^1(X, \mathcal{A}_\ast) = H^1(X, \text{At}(E_G)(-\log D)) \longrightarrow \cdots
\]

The above homomorphism $\nu$ corresponds to moving the holomorphic connection on the trivializable holomorphic principal $G$-bundle $E_G$, keeping the triple $(X, D, E_G)$ fixed. This immediately implies that (4.10) holds.

The second condition in Lemma 4.3 says that the restriction of the homomorphism $\mathcal{H}_\ast$ to $\mathcal{I}(z)$

\[
\mathcal{H}_\ast|_{\mathcal{I}(z)} : \mathcal{I}(z) \longrightarrow H^1(X, \text{At}(E_G)(-\log D))
\] (4.11)

is injective.

To prove that the homomorphism in (4.11) is injective, from (3.18) we conclude that $\mathcal{H}_\ast|_{\mathcal{I}(z)}$ is injective if the composition of homomorphisms

\[
H^1(X, TX(-D)) \xrightarrow{\mathbb{L}(z)} \mathbb{H}^1(X, \mathcal{C}_\ast) \xrightarrow{\mathcal{H}} H^1(X, \text{At}(E_G)(-\log D))
\] (4.12)

is injective, where $\mathbb{L}(z)$ is the homomorphism in (3.17). From Proposition 3.2(2) we know that $\mathbb{L}(z) = \Phi^C$. Therefore, the composition of homomorphisms in (4.12) coincides with the homomorphism of cohomologies

\[
\Phi_\ast : H^1(X, TX(-D)) \longrightarrow H^1(X, \text{At}(E_G)(-\log D))
\]

induced by the logarithmic connection $\Phi : TX(-D) \longrightarrow \text{At}(E_G)(-\log D)$ in (4.5). But from the definition of a logarithmic connection we know that

- the homomorphism $\Phi$ is fiber-wise injective, and
- $\Phi(TX(-D))$ is a direct summand of $\text{At}(E_G)(-\log D)$.
Consequently, the above homomorphism $\Phi_*$ is injective. Hence the composition of homomorphisms in (4.12) is injective. This implies that the homomorphism in (4.11) is injective. This completes the proof of Proposition 4.5. □

Continuing with the proof of Theorem 4.4 in view of Proposition 4.5 from Lemma 4.3 we conclude that the statement in (4.6) is equivalent to the following statement:

$$\mathcal{H}_*(\mathcal{I}(z)) \cap \mathcal{H}_*(T_z\mathbb{T}(G)) = 0,$$

(4.13)

where $\mathcal{H}_*$ is the homomorphism in (4.9).

Fix a holomorphic trivialization of the principal $G$–bundle $E_G$ in (4.5). Using it we will identify $E_G$ with the trivial holomorphic principal $G$–bundle $X \times G \to X$. So $\text{ad}(E_G)$ is the trivial holomorphic vector bundle $X \times \mathfrak{g} \to X$, where $\mathfrak{g}$ is the Lie algebra of $G$, and also

$$\text{At}(E_G)(-\log D) = \text{ad}(E_G) \oplus TX(-D) = X \times \mathfrak{g} \oplus TX(-D).$$

(4.14)

Let $\Phi_0$ be the trivial logarithmic (in fact it is holomorphic) connection on the trivial holomorphic principal $G$–bundle $X \times G \to X$. Note that the trivial holomorphic connection on $E_G$ does not depend on the choice of the trivialization of $E_G$. The homomorphism $TX(-D) \to \text{At}(E_G)(-\log D)$ that defines $\Phi_0$ coincides with the inclusion map $TX(-D) \hookrightarrow \text{ad}(E_G) \oplus TX(-D) = \text{At}(E_G)(-\log D)$ (see (4.14)). So we have

$$\Phi = \Phi_0 + \delta,$$

(4.15)

where

$$\delta \in H^0(X, K_X(D) \otimes \mathfrak{g}) = H^0(X, K_X(D)) \otimes \mathfrak{g};$$

recall that $\text{ad}(E_G) = X \times \mathfrak{g}$.

Consider the infinitesimal deformations of the triple $(X, D, E_G)$ in (4.5) such the principal $G$–bundle remains trivial, but the pair $(X, D)$ moves. These correspond to the image of the homomorphism

$$H^1(X, TX(-D)) \to H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$

(see (4.14) for the decomposition) given by the identity map of $H^1(X, TX(-D))$ and the zero map of $H^1(X, TX(-D))$ to $H^1(X, \text{ad}(E_G))$. In other words, these correspond to the image of the homomorphism of cohomologies

$$H^1(X, TX(-D)) \to H^1(X, \text{At}(E_G)(-\log D))$$

induced by the inclusion map $TX(-D) \hookrightarrow \text{ad}(E_G) \oplus TX(-D)$ which is defined using (4.14).

Consequently, the subspace in (4.13)

$$\mathcal{H}_*(T_z\mathbb{T}(G)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$
coincides with the subspace
\[ 0 \oplus H^1(X, TX(-D)) = H^1(X, TX(-D)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)). \]

Consider the section \( \delta \) in (4.15). Using the natural duality pairing
\[ TX(-D) \otimes K_X(D) \rightarrow O_X \]
it produces a homomorphism
\[ \hat{\delta} : TX(-D) \rightarrow O_X \otimes g = \text{ad}(E_G). \] (4.16)

Let
\[ \hat{\delta}_* : H^1(X, TX(-D)) \rightarrow H^1(X, O_X) \otimes g = H^1(X, \text{ad}(E_G)) \] (4.17)
be the homomorphism of cohomologies induced by \( \hat{\delta} \) in (4.16).

We will now show that the subspace in (4.13)
\[ \mathcal{H}_s(\mathcal{I}(z)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)) \]
coincides with the subspace
\[ \{ (\hat{\delta}_*(v), v) \mid v \in H^1(X, TX(-D)) \} \subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)), \]
where \( \hat{\delta}_* \) is the homomorphism in (4.17).

To prove this, let
\[ \iota_{0*} : H^1(X, \text{ad}(E_G)) \rightarrow H^1(X, \text{At}(E_G)(-\log D)) \] (4.18)
be the homomorphism of cohomologies induced by the homomorphism \( \iota_{0*} \) of sheaves in (2.7). We note that \( \iota_{0*} \) coincides with the natural map that sends any infinitesimal deformation of \( E_G \) (keeping \((X, D)\) fixed) to the corresponding infinitesimal deformation of \((X, D, E_G)\) where only \( E_G \) is moving.

Consider the homomorphism \( \Phi^C \) in (3.8) constructed from the connection \( \Phi \). Let
\[ \Phi^{0,C} : TX(-D) \rightarrow C^0_\bullet \] (4.19)
be the homomorphism as in (3.8) constructed for the trivial connection \( \Phi_0 \) on \( E_G \); here \( C^0_\bullet \) is the complex as in (3.7) for the trivial connection \( \Phi_0 \). From (4.15) it follows immediately that
\[ \mathcal{H} \circ \Phi^C - \mathcal{H}^0 \circ \Phi^{0,C} = \iota_{0*} \circ \hat{\delta}_*, \] (4.20)
where \( \hat{\delta}, \mathcal{H} \) and \( \iota_{0*} \) are the homomorphisms in (4.16), (4.8) and (2.7) respectively, while \( \mathcal{H}^0 \) is the homomorphism for the trivial holomorphic connection \( \Phi_0 \) constructed as in (4.8) (by substituting \( \Phi_0 \) in place of \( \Phi \) in the construction of \( \mathcal{H} \)). As in Proposition 3.2(2), let
\[ \Phi^{0,C}_s : H^1(X, TX(-D)) \rightarrow \mathbb{H}^1(X, C^0_\bullet) \] (4.21)
be the homomorphism of hypercohomologies induced by \( \Phi^{0,C} \) in (4.19). From (4.20) we conclude that
\[ \mathcal{H}_s \circ \Phi^C_s - \mathcal{H}^0_s \circ \Phi^{0,C}_s = \iota_{0*} \circ \hat{\delta}_s, \] (4.22)
where \( \hat{\delta}_s, \mathcal{H}_s, \Phi^{0,C}_s, \iota_{0s} \) and \( \Phi^C_s \) are the homomorphisms in \((4.17), (4.9), (4.21), (4.18)\) and Proposition 3.2(2) respectively, and
\[
\mathcal{H}^0_0 : \mathbb{H}^1(X, \mathcal{C}_0^0) \longrightarrow H^1(X, \text{At}(E_G)(- \log D))
\]
is the homomorphism of hypercohomologies induced by the homomorphism \( \mathcal{H}^0_0 \) in \((4.20)\). Note that both sides of \((4.22)\) are actually homomorphisms from \( H^1(X, TX(-D)) \) to \( H^1(X, \text{At}(E_G)(- \log D)) \). Also, note that from the decomposition in \((4.14)\) it follows immediately that the homomorphism \( \iota_{0s} \) in \((4.22)\) is injective. In fact, the decomposition in \((4.14)\) realizes \( H^1(X, \text{ad}(E_G)) \) as a direct summand of \( H^1(X, \text{At}(E_G)(- \log D)) \).

Now consider the homomorphism
\[
\mathbb{L}(z) : T_{(X,D)}T_{g,d} \longrightarrow T_z B_G
\]
in \((3.17)\) constructed for the connection \( \Phi \) in the expression of \( z \) in \((4.5)\). Let
\[
\mathbb{L}^0 : T_{(X,D)}T_{g,d} \longrightarrow T_{(X,D,E_G; \Phi_0)}B_G
\]
be the homomorphism as in \((3.17)\) constructed for the trivial connection \( \Phi_0 \). From Proposition 3.2(2) we know that
\[
\mathcal{H}_s \circ \mathbb{L} - \mathcal{H}^0_0 \circ \mathbb{L}^0 = \mathcal{H}_s \circ \Phi^C_s - \mathcal{H}^0_0 \circ \Phi^{0,C}_s,
\]
(4.24)
where \( \mathcal{H}_s \) and \( \mathcal{H}^0_0 \) are the homomorphisms in \((4.9)\) and \((4.23)\) respectively; recall that \( T_z B_G = \mathbb{H}^1(X, \mathcal{C}_s) \) and \( T_{(X,D,E_G; \Phi_0)}B_G = \mathbb{H}^1(X, \mathcal{C}_0^0) \).

Combining \((4.22)\) and \((4.24)\) it follows that
\[
\mathcal{H}_s \circ \mathbb{L} - \mathcal{H}^0_0 \circ \mathbb{L}^0 = \iota_{0s} \circ \hat{\delta}_s,
\]
(4.25)
where \( \hat{\delta}_s \) is the homomorphism in \((4.17)\). It was noted earlier that the decomposition in \((4.14)\) realizes \( H^1(X, \text{ad}(E_G)) \) as a direct summand of \( H^1(X, \text{At}(E_G)(- \log D)) \).

Therefore, from \((4.25)\) and \((4.18)\) we conclude that the subspace in \((4.13)\)
\[
\mathcal{H}_s(I(z)) \subset H^1(X, \text{At}(E_G)(- \log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))
\]
(see \((4.14)\) for the above decomposition) coincides with the subspace
\[
\{ (\hat{\delta}_s(v), v) \mid v \in H^1(X, TX(-D)) \} \subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)).
\]
On the other hand, it was shown earlier the subspace in \((4.13)\)
\[
\mathcal{H}_s(T_z \mathbb{T}(G)) \subset H^1(X, \text{At}(E_G)(- \log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))
\]
coincides with the subspace
\[
0 \oplus H^1(X, TX(-D)) = H^1(X, TX(-D)) \subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)).
\]
Combining these two we obtain an isomorphism
\[
\eta : \text{kernel}(\hat{\delta}_s) \xrightarrow{\sim} \mathcal{H}_s(I(z)) \cap \mathcal{H}_s(T_z \mathbb{T}(G))
\]
(4.26)
that sends any \( v \in \text{kernel}(\hat{\delta}_s) \subset H^1(X, TX(-D)) \) to
\[
(0, v) \in H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)) = H^1(X, \text{At}(E_G)(- \log D)).
\]
Consequently, (4.13) holds if and only if we have
\[ \text{kernel}(\hat{\delta}_*) = 0, \] (4.27)
where \( \hat{\delta}_* \) is the homomorphism constructed in (4.17).

Take any subspace
\[ V \subset H^0(X, K_X(D)). \]
Let \( H^1(X, TX(-D)) \otimes V \to H^1(X, O_X) \) be the homomorphism constructed using the duality pairing \( TX(-D) \otimes K_X(D) \to O_X \). Let
\[ F_V : H^1(X, TX(-D)) \to H^1(X, O_X) \otimes V^* \] (4.28)
be the homomorphism given by it. From the construction of \( \hat{\delta}_* \) in (4.17) we see that
\[ \text{kernel}(\hat{\delta}_*) = \text{kernel}(F_V), \]
where \( V \subset H^0(X, K_X(D)) \) is the image of the homomorphism
\[ H_\delta : g^* \to H^0(X, K_X(D)) \] (4.29)
given by \( \delta \) in (4.15); note that since \( \delta \in H^0(X, K_X(D)) \otimes g \), it produces a homomorphism \( H_\delta \) as in (4.29) by sending any \( w \in g^* \) to \( w(\delta) \in H^0(X, K_X(D)) \). Consequently, (4.27) holds if and only if
\[ \text{kernel}(F_{H_\delta(g^*)}) = 0, \] (4.30)
where \( H_\delta \) and \( F_{H_\delta(g^*)} \) are the homomorphism constructed in (4.29) and (4.28) respectively.

It is evident that there is an element \( z = (X, D, E_G, \Phi) \in T(G) \) such that (4.30) holds if and only if there is a subspace \( V \subset H^0(X, K_X(D)) \), with \( \dim V \leq \dim g \), satisfying the condition that the homomorphism \( F_V \) in (4.28) is injective. Indeed, choosing a homomorphism
\[ \delta' : g^* \to H^0(X, K_X(D)) \]
for which \( V \subset \delta'(g^*) \), consider the element \( \delta \in H^0(X, K_X(D)) \otimes g \) given by \( \delta' \). Then the logarithmic connection \( (X, D, X \times G, \Phi_0 + \delta) \) satisfies (4.30), where \( \Phi_0 \) is the trivial holomorphic connection on \( X \times G \to X \).

First assume that \( g = 1 \) (hence, by hypothesis, \( d \geq 1 \)) and \( \dim G \geq d \). This implies that
\[ \dim H^0(X, K_X(D)) = d \leq \dim G. \]
So in this case there is a subspace \( V \subset H^0(X, K_X(D)) \), with \( \dim V \leq \dim g \), for which the homomorphism \( F_V \) in (4.28) is injective, if the homomorphism
\[ F_{H^0(X, K_X(D))} : H^1(X, TX(-D)) \to H^1(X, O_X) \otimes H^0(X, K_X(D))^* \] (4.31)
is injective; if the homomorphism in (4.31) is injective, then we may take \( V \) to be \( H^0(X, K_X(D)) \) itself and the homomorphism \( F_V \) is injective.

The homomorphism in (4.31) is injective if the dual homomorphism
\[ F_{H^0(X, K_X(D))}^* : H^0(X, K_X) \otimes H^0(X, K_X(D)) \to H^0(X, K_X^{\otimes 2}(D)) \] (4.32)
is surjective. Now the homomorphism in (4.32) is injective because \( \dim H^0(X, K_X) = 1 \). On the other hand, we have

\[
\dim H^0(X, K_X(D)) = d = \dim H^0(X, K_X^{\otimes 2}(D)),
\]

so the homomorphism in (4.32) is an isomorphism, in particular, it is surjective. This proves the theorem when \( g = 1 \) and \( \dim C G \geq d \).

Now assume that \( g \geq 2 \) and \( \dim C G \geq d + 2 \). Since \( \dim g \geq d + 2 \), we conclude that there is an element \( z = (X, D, E_G, \Phi) \in T(G) \) such that (4.30) holds if there is a subspace \( V \subset H^0(X, K_X(D)) \), with \( \dim V \leq d + 2 \), for which the homomorphism \( F_V \) in (4.28) is injective. From Lemma 4.7 and Lemma 4.9 (see also Remark 4.10) it follows that such a subspace \( V \) exists. This completes the proof of the theorem. □

Remark 4.6. From Theorem 4.4 it follows that when \( g = 1 \) and \( d = 0 \), the map \( \hat{\theta} \) in (4.3) is an immersion over a nonempty open dense subset of \( T(G) \). Indeed, from Remark 4.2 we know that \( T(G) \) for \( d = 0 \) coincides with \( T(G) \) for \( d = 1 \). On the other hand, \( R_G(S'_0) \) for \( d = 0 \) is embedded into \( R_G(S'_0) \) for \( d = 1 \). From Theorem 4.4 we know that the map \( \hat{\theta} \) in (4.3) is an immersion over a nonempty open dense subset of \( T(G) \) if \( g = 1 \) and \( d = 1 \). Therefore, the same holds when \( g = 1 \) and \( d = 0 \). Recall that \( \dim G > 0 \).

In view of Remark 4.2, we assume that \( d > 1 \) when \( g > 1 \).

Lemma 4.7. Take integers \( g > 1 \) and \( d > 1 \). Then for any compact connected non-hyperelliptic Riemann surface \( X \) of genus \( g \geq 3 \), and any effective divisor \( D \) on \( X \) of degree \( d \), there exists a subspace \( W \subset H^0(X, K_X(D)) \), with \( \dim W = d + 2 \), such that the homomorphism constructed in (4.28)

\[
F_W : H^1(X, TX(-D)) \rightarrow H^1(X, \mathcal{O}_X) \otimes W^*
\]

is injective.

Proof. For a compact Riemann surface \( X \) of genus \( g \), and an effective divisor \( D \) on \( X \) of degree \( d \), denote the holomorphic line bundle \( K_X^{\otimes 2} \otimes \mathcal{O}_X(D) \) by \( K_X^2(D) \). For any subspace \( V \subset H^0(X, K_X(D)) \), let

\[
F_V^* : H^0(X, K_X) \otimes V \rightarrow K^0(X, K_X^2(D))
\]

be the dual of the homomorphism \( F_V \) in (4.28).

We need to show that there is a \( W \) with \( \dim W = d + 2 \) such that the above homomorphism

\[
F_W^* : H^0(X, K_X) \otimes W \rightarrow H^0(X, K_X^2(D))
\]

(4.33)

is surjective.

Consider the natural homomorphism

\[
J : H^0(X, K_X) \otimes H^0(X, K_X(D)) \rightarrow H^0(X, K_X^2(D)).
\]

(4.34)
We will now show that under our assumptions, the homomorphism $J$ in (4.34) is surjective. To this end, we apply [Gr, Theorem (4.e.1)] and see that it suffices to prove that
\[ h^1(X, \mathcal{O}_X(D)) \leq g - 2. \tag{4.35} \]

When $D$ is non–special, (4.35) evidently holds. So we suppose that $D$ is special. In order to prove (4.35), first assume that $d \geq 4$. Then Clifford’s theorem (see [GH, p. 251]) says that $h^0(X, \mathcal{O}_X(D)) \leq d/2 + 1$. Now using Riemann-Roch theorem we get that
\[ d + 1 - g = h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) \leq \frac{d}{2} + 1 - h^1(X, \mathcal{O}_X(D)). \]
This implies that (4.35) holds, and hence $J$ is surjective in this case by [Gr, Theorem (4.e.1)].

Assume now that $d = 2$ or $d = 3$. Since $X$ is not hyperelliptic, if $d = 2$, then we have $h^0(X, \mathcal{O}_X(D)) = 1$. If $d = 3$, Clifford’s theorem implies that $h^0(X, \mathcal{O}_X(D)) \leq 2$. Then the Riemann-Roch theorem implies that (4.35) holds in both these cases. Applying [Gr, Theorem (4.e.1)], we infer that $J$ is surjective in these cases as well.

Consequently, we have obtained the surjectivity of the map $J$ in (4.34) for any pair $(X, D)$ as in the lemma.

From the commutative diagram
\[
\begin{array}{ccc}
H^0(X, K_X) \otimes H^0(X, K_X(D)) & \xrightarrow{J} & H^0(X, K_X^2(D)) \\
\downarrow & & \downarrow \\
H^0(X, K_X) \otimes H^0(X, K_X(D)) / H^0(X, K_X) & \rightarrow & H^0(X, K_X^2(D)) / H^0(X, K_X^2)
\end{array}
\]
we notice that the surjectivity of $J$ implies the surjectivity of the map
\[ H^0(X, K_X) \otimes (H^0(X, K_X(D))/H^0(X, K_X)) \rightarrow H^0(X, K_X^2(D))/H^0(X, K_X^2). \]

Consider $U \subset H^0(X, K_X(D))$ of dimension $(d - 1)$ such that $U \cap H^0(X, K_X) = \{0\}$ inside $H^0(X, K_X(D))$. Then the map
\[ U \rightarrow H^0(X, K_X(D))/H^0(X, K_X) \]
is an isomorphism and hence the induced map
\[ H^0(X, K_X) \otimes U \rightarrow H^0(X, K_X^2(D))/H^0(X, K_X^2) \tag{4.36} \]
is surjective.

On the other hand, since $X$ is non–hyperelliptic, [Gi, Theorem 1.1] (whose proof is attributed to Lazarsfeld) shows that for a general subspace $W_0 \subset H^0(X, K_X)$ of dimension 3 the multiplication map
\[ H^0(X, K_X) \otimes W_0 \rightarrow H^0(X, K_X^2) \tag{4.37} \]
is surjective. Set
\[ W = W_0 \oplus U \subset H^0(X, K_X(D)). \]
The surjectivity of the maps in (4.36) and (4.37) implies the surjectivity of

$$F^*W : H^0(X, K_X) \otimes W \rightarrow H^0(X, K_X^2(D))$$

which concludes the proof. □

**Remark 4.8.** Lemma 4.7 is optimal in the following sense. If $W \subset H^0(X, K_X(D))$ is a subspace such that the intersection $W \cap H^0(X, K_X)$ inside $H^0(X, K_X(D))$ is at least three-dimensional and $H^0(X, K_X) \otimes W \rightarrow H^0(X, K_X^2(D))$ is surjective, then $\dim W \geq d + 2$. This claim is easily obtained by reverting the argument in the proof of Lemma 4.7.

Lemma 4.7 excluded the case of $g = 2$. This is dealt with separately below.

**Lemma 4.9.** Let $X$ be a compact connected Riemann surface of genus two, and let $D$ be an effective divisor of degree $d > 1$ such that $D \notin |K_X|$. Then the multiplication map

$$H^0(X, K_X) \otimes H^0(X, K_X(D)) \rightarrow H^0(X, K_X^2(D))$$

is surjective.

**Proof.** We start with the short exact sequence

$$0 \rightarrow TX \rightarrow H^0(X, K_X) \otimes \mathcal{O}_X \rightarrow K_X \rightarrow 0,$$

twist it by $K_X(D)$ and take the corresponding long exact sequence of cohomologies

$$H^0(X, K_X) \otimes H^0(X, K_X(D)) \rightarrow H^0(X, K_X^2(D)) \rightarrow H^1(X, \mathcal{O}_X(D)) \rightarrow .$$

By the hypothesis, we have $H^1(X, \mathcal{O}_X(D)) = 0$ and hence from this exact sequence of cohomologies it follows that the multiplication map is surjective. □

**Remark 4.10.** Note that, under the hypotheses of Lemma 4.9 the Riemann-Roch theorem implies that $h^0(X, K_X(D)) = d + 1 < d + 2$.

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