Shifted insertion algorithms for primed words

Eric Marberg

Abstract

This article studies new insertion algorithms that associate pairs of shifted tableaux to finite integer sequences in which certain terms may be primed. When primes are ignored in the input word these algorithms reduce to known correspondences, namely, a shifted form of Edelman-Greene insertion, Sagan-Worley insertion, and Haiman’s shifted mixed insertion. The latter maps have the property that when the input word varies such that one of the output tableaux is fixed, the other output tableau ranges over all (semi)standard tableaux of a given shape with no primed diagonal entries. Our algorithms have the same feature, but now with primes allowed on the diagonal. One application of this is to give another Littlewood-Richardson rule for products of Schur $Q$-functions. It is hoped that there will exist set-valued generalizations of our bijections that can be used to understand products of $K$-theoretic Schur $Q$-functions.

Contents

1 Introduction ........................................... 2
  1.1 Outline ........................................... 2
  1.2 Motivation ........................................ 3

2 Preliminaries ............................................. 5
  2.1 Involution words .................................... 6
  2.2 Primed words ...................................... 6
  2.3 Tableaux ........................................... 7

3 Insertion algorithms ..................................... 8
  3.1 Shifted Edelman-Greene insertion ................. 8
  3.2 Orthogonal Coxeter-Knuth equivalence .......... 13
  3.3 Modifying Sagan-Worley insertion .............. 18
  3.4 Extending shifted mixed insertion ............. 24

4 Remaining proofs ........................................ 28
  4.1 Bumping paths .................................... 28
  4.2 Cycle sequences ................................... 29
  4.3 Some reductions ................................... 34
  4.4 Final arguments ................................... 41
1 Introduction

This article studies some new insertion algorithms that generate pairs of shifted tableaux from finite integer sequences in which certain terms may be primed. We summarize our main results below, and then discuss the problems that motivate our constructions.

1.1 Outline

Let $S_Z$ be the group of permutations of the integers with finite support, and set $s_i := (i, i + 1) \in S_Z$ for $i \in \mathbb{Z}$. A reduced word for $w \in S_Z$ is a minimal length integer sequence $a_1 a_2 \cdots a_n$ with $\sigma = s_{a_1} s_{a_2} \cdots s_{a_n}$. Write $\mathcal{R}(\sigma)$ for the set of reduced words for $\sigma \in S_Z$.

Suppose $a = a_1 a_2 \cdots a_n$ is a reduced word for an element of $S_Z$. Then there is a unique subword $a_1 a_2 \cdots a_{i_m}$ of maximal length such that $\hat{a} := a_{i_1} \cdots a_{i_k} a_1 a_2 \cdots a_n$ is also a reduced word. We refer to the indices in $\{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_m\}$ as the commutations in $a$.

One can show that $\hat{a}$ is always a reduced word for an involution $z = z^{-1} \in S_Z$. If $a$ is of minimal length such that $\hat{a} \in \mathcal{R}(z)$, then $a$ is an involution word for $z$. Write $\mathcal{R}_{inv}(z)$ for the set of involution words for $z = z^{-1} \in S_Z$. For a more constructive definition of this set, see Section 2.1.

Involution words have been studied previously in different forms and under various names, for example, in [5, 11, 15, 17, 30]. We are concerned here with the following slight generalization. A primed involution word for $z = z^{-1} \in S_Z$ is any word formed by adding primes to the entries indexed by a subset of commutations in some $a \in \mathcal{R}_{inv}(z)$. Such a word is a sequence of letters in the primed alphabet $\{\cdots < 1' < 1 < 2' < 2 < \cdots\}$. Write $\mathcal{R}_{inv}^\prime(z)$ for the set of primed involution words for $z$.

For any word $a$, let $\text{Incr}_\infty(a)$ denote the set of sequences $(a^1, a^2, a^3, \ldots)$ where each $a^i$ is a weakly increasing word such that $a = a^1 a^2 a^3 \cdots$. For a set of words $\mathcal{A}$, let $\text{Incr}_\infty(\mathcal{A}) = \bigcup_{a \in \mathcal{A}} \text{Incr}_\infty(a)$. In Section 3.1 we describe a specific map $a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a))$ that takes an element of $\mathcal{R}_{inv}^\prime(z)$ or $\text{Incr}_\infty(\mathcal{R}_{inv}^\prime(z))$ as its input and gives a pair of shifted tableaux as its output. Our first main result is the following theorem concerning this operation.

**Theorem** (See Theorem 3.6 and Corollary 3.9). Let $z = z^{-1} \in S_Z$. The map $a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a))$ is a bijection from $\mathcal{R}_{inv}^\prime(z)$ (respectively, $\text{Incr}_\infty(\mathcal{R}_{inv}^\prime(z))$) to the set of pairs $(P, Q)$ where $P$ is a shifted tableau with increasing rows and columns and no primed diagonal entries whose row reading word is in $\mathcal{R}_{inv}^\prime(z)$, and $Q$ is a standard (respectively, semistandard) shifted tableau of the same shape.

Here, a semistandard shifted tableau is allowed to have primed entries in diagonal positions; for the precise definition, see Section 2.3. A pair $(P, Q) \in \{(P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \colon a \in \mathcal{R}_{inv}^\prime(z)\}$ belongs to \{(P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \colon a \in \mathcal{R}_{inv}(z)\} if and only if $P$ has no primed entries in any position and $Q$ has no primed entries on the diagonal.

Restricting $a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a))$ to unprimed words gives the map called involution Coxeter-Knuth insertion in [12, 23] and orthogonal Edelman-Greene insertion in [24]. The latter, in turn, is a special case of the shifted Hecke insertion algorithm from [14, 29]. Our correspondence is the “orthogonal” counterpart to a “symplectic” shifted insertion algorithm studied in [16, 23, 24]; see Remark 2.2.

It is an open problem to find a “primed” generalization of shifted Hecke insertion that extends our bijection $a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a))$. The image of such a map should consist of pairs of shifted...
tableaux \((P, Q)\) of the same shape, in which \(P\) has increasing rows and columns with no primed entries on the diagonal, and \(Q\) is an arbitrary \textit{(semistandard) set-valued shifted tableau} in the sense of [19, §9.1]. It is less clear what superset of \(\mathcal{R}_{inv}^+(z)\) should be the domain of such a correspondence. As discussed in the next subsection, generalizing shifted Hecke insertion in this way would have interesting applications.

Besides constructing the map \(a \mapsto (P^O_{EG}(a), Q^O_{EG}(a))\), we also seek to understand how \(a\) can vary when \(P^O_{EG}(a)\) is held constant, and how such changes affect \(Q^O_{EG}(a)\). Our second set of results, sketched below and explained more thoroughly in Section 3.2, fully solves this problem.

**Theorem** (See Theorem 3.16 and Corollary 3.17). There are explicit operators \(\text{ock}_i\) on primed words which act by changing at most three consecutive letters, along with operators \(\delta_i\) on standard shifted tableaux which act by changing at most three consecutive entries, such that if \(a\) is a primed involution word then \(Q^O_{EG}(\text{ock}_i(a)) = \delta_i(Q^O_{EG}(a))\), and if \(a \text{ and } b\) are both primed involution words then \(P^O_{EG}(a) = P^O_{EG}(b)\) if and only \(a = \text{ock}_{i_1}\text{ock}_{i_2} \cdots \text{ock}_{i_k}(b)\) for some sequence \(i_1, i_2, \ldots, i_k\).

We use the theorems above to derive some additional results. In Section 3.3, we describe a variation of \textit{Sagan-Worley insertion} from [31, 34] whose domain is the set of all \textit{primed biwords}. Section 3.4 investigates two related extensions of Haiman’s \textit{shifted mixed insertion algorithm} [9].

### 1.2 Motivation

The \textit{Schur P-function} of a strict partition \(\lambda\) is the generating function \(P_\lambda = \sum_T x^T\) for all semistandard shifted tableaux \(T\) of shape \(\lambda\) with no primed entries on the diagonal. The \textit{Schur Q-function} \(Q_\lambda\) is defined in the same way but without excluding primes from the diagonal, or more directly as the scalar multiple \(Q_\lambda = 2^{\ell(\lambda)} P_\lambda\). It is well-known that both power series are symmetric functions that are Schur positive, and that the set of all \(P_\lambda\)’s (respectively, all \(Q_\lambda\)’s) is a \(Z\)-basis for a ring with nonnegative integer structure constants [33].

Ikeda and Naruse introduce \(K\)-theoretic analogues \(GP_\lambda\) and \(GQ_\lambda\) for the Schur \(P\)-functions and \(Q\)-functions in [19]. These power series are also symmetric, and may be defined similarly as the generating functions for all semistandard \textit{set-valued shifted tableaux} of a given shape, where for \(GP_\lambda\) primed entries are again prohibited from appearing in diagonal positions.\(^1\) One recovers \(P_\lambda\) and \(Q_\lambda\) by taking the homogeneous terms of lowest degree in \(GP_\lambda\) and \(GQ_\lambda\), respectively.

It is conjectured in [19] that the set of all \(GP_\lambda\)’s (respectively, all \(GQ_\lambda\)’s) is a basis for a ring with nonnegative structure constants. For the \(GP_\lambda\)’s this follows from results in [6]. For the \(GQ_\lambda\)’s, surprisingly, Ikeda and Naruse’s conjecture is technically still unresolved, though only just. It follows from [19] that each product \(GQ_\lambda GQ_\mu\) is a formal sum of \(GQ_\nu\)’s. However, in general, it remains to show that this expansion has finitely many terms and to give an interpretation of its (positive) coefficients.\(^2\) These difficulties have to do with the fact that \(GQ_\lambda\) is no longer a scalar multiple of \(GP_\lambda\).

There is a bijective approach to proving that the \(K\)-theoretic Schur \(P\)-functions generate a ring, which we sketch below. The results in this article are a first step toward extending this strategy to handle the \(K\)-theoretic Schur \(Q\)-functions.

\(^1\)The precise definition involves a bookkeeping parameter \(\beta\), which makes both power series homogeneous if one sets \(\text{deg}(\beta) = -1\) and \(\text{deg}(x_i) = 1\). For simplicity, we take \(\beta = 1\) in our discussion here.

\(^2\)There is at least a Pieri rule to expand \(GQ_\lambda GQ_\mu\) into \(GQ_\nu\)’s when \(\mu = (p)\) has a single part [4, Cor. 5.6]. There is also a recent formula for the expansion of \(GQ_\lambda GQ_\mu\) for any strict \(\lambda, \mu\) into monomials [22, Cor. 7.8].
For each even integer $n > 0$, let $I_n^{\text{pf}}$ denote the set of fixed-point-free involutions in the symmetric group $S_n := \langle s_1, s_2, \ldots, s_{n-1} \rangle$. Each element $z \in I_n^{\text{pf}}$ has an associated set of symplectic Hecke words $\mathcal{H}_{\text{Sp}}(z)$ defined in [23, §1.3]. This set is infinite unless $z$ is $1_{\text{pf}} := (1,2)(3,4)\cdots(n-1,n)$. Each word in $\mathcal{H}_{\text{Sp}}(z)$ is a finite integer sequence that does not begin with an odd letter. The shortest words in $\mathcal{H}_{\text{Sp}}(z)$ are the minimal length sequences $a_1 a_2 \cdots a_k$ with $z = s_{a_k} \cdots s_{a_2} s_{a_1} 1_{\text{pf}} s_i s_2 \cdots s_{a_k}$.

Given $z \in I_n^{\text{pf}}$ and a strict partition $\lambda$, define $KP_z := \sum_{\phi \in \text{Incr}(\mathcal{H}_{\text{Sp}}(z))} x^\phi$ where $x^\phi := \prod_i x_i^\ell(a_i)$ for $\phi = (a_1, a_2, a_3, \ldots)$ and let $KP_\lambda := \sum_T x^T$ where the sum is over all (semistandard) weak set-valued shifted tableaux of shape $\lambda$ with no primed entries on the diagonal, as described in [14, Def. 3.1] (and with $x^T$ as defined in the same place). These power series are related to $GP_\lambda$ by the identity $GP_\lambda = \omega(KP_\lambda)$, where $\omega$ is the automorphism of the algebra of symmetric functions sending each Schur function $s_\lambda \mapsto s_\lambda^\dagger$ [21, Cor. 6.6]. In turn, each $KP_z$ is related to $KP_\lambda$ by:

**Theorem** (See [23, Thm. 4.5]). Let $z \in I_n^{\text{pf}}$. There is a bijection $\phi \mapsto (P_\phi(z), Q_\phi(z))$ from $\text{Incr}(\mathcal{H}_{\text{Sp}}(z))$ to the set of pairs $(P, Q)$ where $P$ is a shifted tableau with increasing rows and columns whose row reading word is in $\mathcal{H}_{\text{Sp}}(z)$, and $Q$ is a weak set-valued shifted tableau of the same shape with no primed entries on the diagonal. Moreover, it always holds that $x^\phi = x^{Q_\phi(\phi)}$.

This bijection is called symplectic Hecke insertion in [23]. If $a = a_1 a_2 \cdots a_k \in \mathcal{H}_{\text{Sp}}(z)$ then we set $P_{\phi}(a) = P_{\phi}(\phi)$ and $Q_{\phi}(a) = Q_{\phi}(\phi)$ for $\phi = (a_1, a_2, \ldots, a_k, 0, 0, \ldots)$. The value of $P_{\phi}(\phi)$ depends only on the underlying word, but not on its division into weakly increasing factors. All letters in a symplectic Hecke word for $z \in I_n^{\text{pf}}$ are in $\{1, 2, \ldots, n-1\}$, so there are only finitely many shifted tableaux with increasing rows and columns that can have row reading words in $\mathcal{H}_{\text{Sp}}(z)$. It follows that $KP_z$ is the finite sum $\sum_{T \in BP(z)} K_{P_{\lambda}(T)}$.

Assume $y \in I_n^{\text{pf}}$ and $z \in I_n^{\text{pf}}$ for even integers $m, n \geq 0$. Let $y \times z \in I_{m+n}^{\text{pf}}$ be the permutation mapping $i \mapsto y(i)$ for $1 \leq i \leq m$ and $i + m \mapsto z(i) + m$ for $1 \leq i \leq n$. Next, for $\phi = (a_1, a_2, \ldots) \in \text{Incr}(\mathcal{H}_{\text{Sp}}(y))$ and $\psi = (b_1, b_2, \ldots) \in \text{Incr}(\mathcal{H}_{\text{Sp}}(z))$, let $\phi \oplus \psi = (a_1^{c_1}, a_2^{c_2}, \ldots)$ where $c_i$ is formed by adding $m$ to each letter of $b_i$.

It is clear from the results about symplectic Hecke words in [23, §1.3] that $(\phi, \psi) \mapsto \phi \oplus \psi$ is a bijection $\text{Incr}(\mathcal{H}_{\text{Sp}}(y)) \times \text{Incr}(\mathcal{H}_{\text{Sp}}(z)) \rightarrow \text{Incr}(\mathcal{H}_{\text{Sp}}(y \times z))$. Therefore $KP_{y \times z} = KP_y \circ KP_z$. In turn, if the largest part of $\lambda$ is less than $n - 1$, then there exists $z_\lambda^{\text{pf}} \in I_n^{\text{pf}}$ (with an explicit formula) such that $KP_\lambda = KP_{z_\lambda^{\text{pf}}}$ [27, Thm. 4.17]. Since $\omega$ is an algebra automorphism, we have

$$ KP_\lambda KP_\mu = \sum_{\nu} e^\nu_{\lambda\mu} KP_\nu \quad \text{and} \quad GP_\lambda GP_\mu = \sum_{\nu} e^\nu_{\lambda\mu} GP_\nu \quad (1.1) $$

where $e^\nu_{\lambda\mu}$ is the number of tableaux in $\{P(a) : a \in \mathcal{H}_{\text{Sp}}(z_\lambda^{\text{pf}} \times z_\mu^{\text{pf}})\}$ of shape $\nu$.3

Here is how one could try to adapt this argument to show that the $GQ_\lambda$’s likewise generate a ring with positive structure constants. The appropriate analogue of $KP_\lambda$ is the generating function $KQ_\lambda := \sum_T x^T$ for all weak set-valued shifted tableaux $T$ of shape $\lambda$, now with primed entries allowed on the diagonal. We have $GQ_\lambda = \omega(KQ_\lambda)$ by [21, Cor. 6.6].

There is a natural candidate for the $Q$-form of $KP_\lambda$. When $n$ is even, the symplectic group $\text{Sp}_n(\mathbb{C})$ acts on the type $A_{n-1}$ flag variety $\text{Fl}_n$ with finitely many orbits indexed by $I_n^{\text{pf}}$. The closures of these orbits have canonical representatives in the connective $K$-theory ring of $\text{Fl}_n$ satisfying

---

3This becomes a Littlewood-Richardson rule for the symmetric functions $GP_\lambda^{(\beta)}$ defined in [19], which involve a formal parameter $\beta$, via the identity $GP_\lambda^{(\beta)} = \beta^{-|\lambda|} GP_\lambda(\beta x_1, \beta x_2, \beta x_3, \ldots)$. 

4
a certain stability property [36]. These representatives are polynomials \( \mathfrak{S}_z^{\text{Sp}} \in \mathbb{Z}[\beta][x_1, x_2, \ldots] \), and their “stable limits” give certain symmetric functions \( GP_z^{\text{Sp}} \) that satisfy \( KP_z = \omega(GP_z^{\text{Sp}}|_{\beta=1}) \) (compare [26, Cor. 4.6] with the results in [23, §5]).

For any positive integer \( n \), the orthogonal group \( O_n(\mathbb{C}) \) likewise acts on \( \text{Fl}_n \) with finitely many orbits, now indexed by \( I_n := \{ z \in S_n : z = z^{-1} \} \). The closures of these orbits again have canonical representatives in the connective \( K \)-theory ring of \( \text{Fl}_n \) satisfying a certain stability property [26]. These are inhomogeneous polynomials \( \mathfrak{S}_z^O \in \mathbb{Z}[\beta][x_1, x_2, \ldots] \) indexed by \( z \in I_n \). Mimicking the properties of \( KP_z \), one would like to define the “stable limit” \( GQ_z^O := \lim_{m \to \infty} \mathfrak{S}_m^O_{1m \times z} \) for \( z \in I_n \), where \( 1^m \) is the identity permutation in \( S_m \), and then set \( KQ_z := \omega(GQ_z^O|_{\beta=1}) \). These definitions would be appropriate because if \( z \) is vexillary, that is, 2143-avoiding, then the resulting power series \( KQ_z \) converges, the resulting power series \( KQ_z \) is equal to \( KQ_\lambda \) for a certain strict partition \( \lambda \), and any \( KQ_\lambda \) can be attained in this way [26, Thm. 4.11]. Some difficulties remain, however:

(a) No proof is yet known that \( \lim_{m \to \infty} \mathfrak{S}_1^{O \times z} \) converges if \( z \) is not vexillary [26, Prob. 5.3].

(b) There should exist a set of orthogonal Hecke words \( \mathcal{H}_O(z) \), analogous to \( \mathcal{H}_{\text{Sp}}(z) \), such that \( KQ_z = \sum_{\phi \in \text{Incr}_{\text{inv}}(\mathcal{H}_O(z))} x^\phi \) and \( KQ_y KQ_z = KQ_{y \times z} \) for all \( y \in I_m \) and \( z \in I_n \). It is not known how to define this set even when \( z \) is vexillary.

(c) If the first two issues can be addressed, then to prove that the \( GQ_\lambda \)'s generate a ring, it remains only to find an appropriate orthogonal Hecke insertion algorithm. This should bijectively map elements of \( \text{Incr}_\infty(\mathcal{H}_O(z)) \) to pairs \((P, Q)\) of shifted tableaux with the same shape, where now \( Q \) is weak set-valued but with primed entries allowed on the diagonal.

The results in this paper provide a base case for the last item.

Specifically, \( \mathcal{H}_O(z) \) should be a superset of \( \mathcal{R}_{\text{inv}}^+(z) \) and the definition of orthogonal Hecke insertion should be an extension of our map \( a \mapsto (P_{\text{Sp}}^O(a), Q_{\text{Sp}}^O(a)) \). This is because if we replace the inhomogeneous polynomial \( \mathfrak{S}_{1m \times z}^O \) by its terms of lowest degree, then the desired stable limit does always converge as \( m \to \infty \) (see [11, §1.5]), so at least the lowest degree terms of \( GP_z^O \) and \( KQ_z \) are well-defined. Both of these give the same homogeneous symmetric function (by [12, Cor. 4.62], since \( \omega \) fixes every Schur \( Q \)-function), which we denote by \( Q_z \).

As explained in Section 3.1, it further holds that \( Q_z = \sum_{\phi \in \text{Incr}_\infty(\mathcal{R}_{\text{inv}}^+(z))} x^\phi \) and \( Q_y Q_z = Q_{y \times z} \) for all \( y \in I_m \) and \( z \in I_n \). This resolves the “homogeneous” forms of (a) and (b), and our first main theorem gives a homogeneous version of the correspondence desired in (c). As an application, this leads to another Littlewood-Richardson rule for the Schur \( Q \)-functions (see Corollary 3.12). One hopes that this rule can be generalized to the \( GQ_\lambda \)'s in future work.

Acknowledgments

This work was partially supported by Hong Kong RGC grants ECS 26305218 and GRF 16306120. I am especially grateful to Travis Scrimshaw for many useful conversations, and for hosting a productive visit to the University of Queensland. I also thank Zach Hamaker, Joel Lewis, and Brendan Pawlowski for helpful discussions.

2 Preliminaries

In this section we review some basic facts and background material. Throughout, we write \( \mathbb{Z} \) for the set of integers. When \( n \in \mathbb{Z} \) is nonnegative, we let \([n] := \{i \in \mathbb{Z} : 0 < i \leq n\} \).
2.1 Involution words

We use the term *word* to mean a finite sequence of integers $a = a_1 a_2 \cdots a_n$. We write $\ell(a) := n$ for the length of a word. Recall from the introduction that $R(\sigma)$ denotes the set of reduced words for a permutation $\sigma \in S_\mathbb{Z} := \{s_i : i \in \mathbb{Z}\}$.

Let $\approx$ be the equivalence relation on words that has $aX(X + 1)Xb \approx a(X + 1)X(X + 1)b$ and $aXYb \approx aYXb$ for all words $a$, $b$ and all $X, Y \in \mathbb{Z}$ with $|X - Y| > 1$. Each set $R(\sigma)$ for $\sigma \in S_\mathbb{Z}$ is an equivalence class under $\approx$, and an arbitrary word belongs to $R(\sigma)$ for some $\sigma \in S_\mathbb{Z}$ if and only if its $\approx$-equivalence class contains no words with equal adjacent letters [2, §3.3].

There is a unique associative product $\circ : S_\mathbb{Z} \times S_\mathbb{Z} \to S_\mathbb{Z}$ such that $\sigma \circ s_i = \sigma$ if $\sigma(i) > \sigma(i + 1)$, and $\sigma \circ s_i \neq \sigma s_i$ if $\sigma(i) < \sigma(i + 1)$ for each $i \in \mathbb{Z}$ [18, Thm. 7.1]. A reduced word for $\sigma \in S_\mathbb{Z}$ is thus a word $a_1 a_2 \cdots a_n$ of shortest possible length such that $\sigma = 1 \circ a_1 \circ a_2 \cdots \circ a_n$. Analogously, an *involution word* for $z \in S_\mathbb{Z}$ is a word $a_1 a_2 \cdots a_n$ of shortest possible length such that

$$z = s_{a_n} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ 1 \circ s_{a_1} \circ s_{a_2} \cdots \circ s_{a_n}.$$ 

Note that the right hand expression is equal to $s_{a_n} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ b \cdots \circ s_{a_n}$ if $n > 0$. This definition is equivalent to the one given in the introduction. Any such sequence is itself a reduced word for some permutation.

Let $I_z := \{\sigma \in S_\mathbb{Z} : \sigma = \sigma^{-1}\}$ and $I_n := S_n \cap I_z$ when $0 < n \in \mathbb{Z}$. If $z \in I_z$ and $i \in \mathbb{Z}$ then $s_i \circ z \circ s_i = z$ when $z(i) > z(i + 1)$, while $s_i \circ z \circ s_i = z s_i$ when $i$ and $i + 1$ are fixed points of $z$, and otherwise $s_i \circ z \circ s_i = s_i z s_i$. This implies that $z \in S_\mathbb{Z}$ has an involution word if and only if $z \in I_z$. If $z = s_{a_n} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ 1 \circ s_{a_1} \circ s_{a_2} \cdots \circ s_{a_n}$ then $a_1 a_2 \cdots a_n \in R_{inv}(z)$ if and only if $(s_{a_{i-1}} \circ \cdots \circ s_{a_1} \circ 1 \circ s_{a_1} \circ s_{a_2} \cdots \circ s_{a_{i-1}})(i) < (s_{a_{i-1}} \circ \cdots \circ s_{a_1} \circ 1 \circ s_{a_1} \circ \cdots \circ s_{a_{i-1}})(i + 1)$ for each $i \in [n]$.

As in the introduction, we denote the set of involution words for $z \in I_z$ by $R_{inv}(z)$. For example, we have $R_{inv}(3412) = \{132, 312\}$ and $R_{inv}(4231) = \{123, 231, 213, 321\}$.

Let $\equiv$ be the transitive closure of $\approx$ and the relation with $XYa \equiv YXa$ for all words $a$ and all letters $X, Y \in \mathbb{Z}$. Hu and Zhang derive the following result in [17]:

**Proposition 2.1** ([17, Thm. 3.1]). Each set $R_{inv}(z)$ for $z \in I_z$ is an equivalence class under $\equiv$. An arbitrary word is an involution word for some element of $I_z$ if and only if its $\equiv$-equivalence class contains no words with equal adjacent letters.

2.2 Primed words

Let $\mathbb{Z}' := \mathbb{Z} - \frac{1}{2}$ and given $i \in \mathbb{Z}$ define $i' := i - \frac{1}{2} \in \mathbb{Z}$. This convention means that $(i + 1)' = i' + 1$ and $[i'] = [i] = i$ for all $i \in \mathbb{Z}$, and that $\mathbb{Z} \sqcup \mathbb{Z}' = \{\cdots < 0' < 0 < 1' < 1 < 2' < 2 < \cdots\} = \frac{1}{2}\mathbb{Z}$. We refer to elements of $\mathbb{Z}'$ as *primed letters*.

“Removing the prime” from $x \in \mathbb{Z} \sqcup \mathbb{Z}'$ means to replace $x$ by $[x]$. “Reversing the prime” on $x \in \mathbb{Z} \sqcup \mathbb{Z}'$ means to replace $x$ by the unique element of $\{[x] - \frac{1}{2}, [x]\} \setminus \{x\}$, so that $i \in \mathbb{Z}$ becomes $i' \in \mathbb{Z}'$ and vice versa. A *primed word* (sometimes called a *colored word*) is a finite sequence $a = a_1 a_2 \cdots a_n$ with letters $a_i \in \mathbb{Z} \sqcup \mathbb{Z}'$. The *unprimed form* of $a$ is the word $\text{unprime}(a) := [a_1] [a_2] \cdots [a_n]$ obtained by removing the primes from all letters.

Suppose $z \in I_z$. An index $i$ is a *commutation* for a word $a_1 a_2 \cdots a_n \in R_{inv}(z)$ if both $a_i$ and $1 + a_i$ are fixed points of $s_{a_{i-1}} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ 1 \circ s_{a_1} \circ s_{a_2} \cdots \circ s_{a_{i-1}}$. A *primed involution word* for $z$ is a primed word whose unprimed form is in $R_{inv}(z)$ and whose primed letters occur only at indices that are commutations. We denote the set of such words by $R_{inv}^+(z)$. For example,

$$R_{inv}^+(321) = \{12, 1'2, 21, 2'1\} \quad \text{and} \quad R_{inv}^+(3412) = \{132, 13'2, 1'32, 1'3'2, 312, 31'2, 3'12, 3'1'2\}.$$
The number of commutations in each involution word for an element \( z \in I_{Z} \) is the absolute length \( \ell_{\text{abs}}(z) := \{|i \in \mathbb{Z} : i < z(i)\} \). Thus \( |R_{\text{inv}}^{+}(z)| = 2^{\ell_{\text{abs}}(z)}|R_{\text{inv}}(z)| \).

**Remark 2.2.** As explained in [35, §2.2-2.3] or [10, §8.1], the set \( I_{n} \subset S_{n} \) indexes the orbits of the orthogonal group \( O_{n}(\mathbb{C}) \) acting on the type \( A_{n-1} \) flag variety \( Fl_{n} := GL_{n}(\mathbb{C})/B \). In [3], Brion derives a formula for the cohomology classes of the closures of these orbits, involving a certain directed graph on the set of orbits. The directed paths that arise in Brion’s cohomology formula (from the orbit indexed by \( z \) to the unique dense orbit) are in bijection with \( R_{\text{inv}}^{+}(z) \). This is one motivation for studying these sets. This is also why we will often include the adjective “orthogonal” with constructions involving \( R_{\text{inv}}^{+}(z) \). There is a parallel “symplectic” story for a different analogue of reduced words corresponding to the orbits of \( Sp_{2n}(\mathbb{C}) \) acting on \( Fl_{2n} \) (see, e.g., [13, 23, 27, 36]).

In a few places we will need the following basic properties of commutations from [25].

**Proposition 2.3 ([25, Prop. 8.2]).** Suppose \( a_{1}a_{2}\cdots a_{n} \in R_{\text{inv}}^{+}(z) \) for some \( z \in I_{Z} \).

(a) If \( |a_{i}| = |a_{i+1}| \pm 1 \) for \( i \in [n-1] \) then at most one of \( a_{i} \) or \( a_{i+1} \) is in \( Z' \).

(b) If \( |a_{i}| = |a_{i+2}| \) for \( i \in [n-2] \) then \( a_{i+1} = [a_{i}] \pm 1 \) in \( Z \) and at most one of \( a_{i} \) or \( a_{i+2} \) is in \( Z' \).

Write \( \equiv \) for the transitive closure of the relation with \( aXYb \equiv aXYb \) for all \( X,Y \in Z \sqcup Z' \) such that \( |[X] - [Y]| \neq 1 \), as well as with \( aXYb \equiv aXYb \) and \( aX'YXb \equiv aYX'b \) for unprimed numbers \( X,Y \in Z \) such that \( |X - Y| = 1 \), and finally with \( Xa \equiv Xa \) and \( XYa \equiv YXa \) for unprimed numbers \( X,Y \in Z \). In these relations \( a \) and \( b \) are arbitrary primed words. For example, we have \( 1^{'2}3^{'2} \equiv 1'3'23 \equiv 13'23 \equiv 3'123 \equiv 3123 \equiv 1323 \equiv 1232 \equiv 2132 \equiv 2312 \equiv 3212 \equiv 3121 \). The following extension of Proposition 2.1 is a corollary of more general results in [25].

**Proposition 2.4 ([25, Cor. 8.3]).** Each set \( R_{\text{inv}}^{+}(z) \) for \( z \in I_{Z} \) is an equivalence class under \( \equiv \).

### 2.3 Tableaux

A *partition* of an integer \( n \geq 0 \) is a finite sequence of integers \( \lambda = (\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{k} > 0) \) that sum to \( n \). In this event we set \( \ell(\lambda) := k, \lambda_{i} := 0 \) for \( i > \ell(\lambda) \), and \( |\lambda| := \sum \lambda_{i} = n \). A partition is strict if the parts \( \lambda_{i} \) are all distinct. The *diagram* of a partition \( \lambda \) is the set of positions \( D_{\lambda} := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq j \leq \lambda_{i}\} \). The *shifted diagram* of a strict partition \( \mu \) is the set \( SD_{\mu} := \{(i,i+j-1) : (i,j) \in D_{\mu}) \} \).

In this article, a *tableau* of shape \( \lambda \) means an arbitrary map \( D_{\lambda} \to \mathbb{Z} \) and a *shifted tableau* of shape \( \mu \) means an arbitrary map \( SD_{\mu} \to \mathbb{Z} \sqcup Z' \). If \( T \) is a (shifted) tableau then we write \( T_{ij} \) for the value assigned to some position \((i,j)\) in its domain. We draw tableaux in French notation, so that row indices increase from bottom to top and column indices increase from left to right. If

\[
S = \begin{array}{cccc}
3 & 5 & 7 \\
1 & 2 & 4 & 6 \\
\end{array}
\quad \text{and} \quad
T = \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6 \\
\end{array}
\quad (2.1)
\]

then \( S \) is a tableau and \( T \) is a shifted tableau of shape \( \lambda = (4,3) \), and \( S_{24} = 7 \) while \( T_{23} = 5' \). The *(main) diagonal* of a shifted tableau is the set of positions \((i,j)\) in its domain with \( i = j \). We sometimes refer to positions \((i,j)\) in the domain of a tableau as its *boxes*.

A (shifted) tableau is *increasing* if its rows and columns are strictly increasing. An increasing (shifted) tableau of shape \( \lambda \) is *standard* if it contains an entry equal to \( i \) or \( i' \) for each \( i \in |\lambda| \). In
our examples $S$ and $T$ are both standard. A (shifted) tableau is semistandard if its entries are all positive and its rows and columns are weakly increasing, such that no primed entry is repeated in a row and no unprimed entry is repeated in a column. It is sometimes required that a semistandard shifted tableau not have primed diagonal entries, but we do not adopt this convention.

Suppose $T$ is a map from a finite subset of $\mathbb{Z} \times \mathbb{Z}$ to a totally ordered set. The row reading word of $T$ is the sequence $\text{row}(T)$ formed by listing the values $T_{ij}$ as $(i, j)$ ranges over the domain of $T$ in the order that makes $(-i, j)$ increase lexicographically. The column reading word of $T$ is the sequence $\text{col}(T)$ formed by listing the values $T_{ij}$ as $(i, j)$ ranges over the elements of the domain of $T$ in the order that makes $(j, -i)$ increase lexicographically. Above, we have $\text{row}(S) = 3571246$, $\text{col}(S) = 3152746$, $\text{row}(T) = 35'71'2'4'6$, and $\text{col}(T) = 1'32'5'4'76$.

When $T$ is a shifted tableau, we form $\text{unprime}(T)$ by removing all primes from $T$’s entries.

**Proposition 2.5.** Suppose $T$ is a shifted tableau such that $\text{row}(T)$ is a primed involution word for an element of $I_\mathbb{Z}$. Then $T$ is increasing if and only $\text{unprime}(T)$ is increasing.

**Proof.** If $\text{unprime}(T)$ is increasing then $T$ is clearly increasing. Assume that $T$ is increasing. Since $\text{row}(\text{unprime}(T))$ is an involution word and therefore reduced, no row of $T$ can contain both $x \in \mathbb{Z}$ and $x' \in \mathbb{Z}'$, so the rows of $\text{unprime}(T)$ are increasing. It remains to show that this property also applies to the columns of $T$. Arguing by contradiction, suppose there is an integer $x \in \mathbb{Z}$ such that $x'$ and $x$ occur as consecutive entries in some column of $T$. After possibly omitting some of the initial rows of $T$, we may assume without loss of generality that all such entries appear in the first and second rows. Let $j \geq 2$ be the first column with $T_{1j} = x'$ and $T_{2j} = x$ for some $x \in \mathbb{Z}$.

Let $l \geq 0$ be maximal such that $(2, j + l)$ is occupied in $T$ and $[T_{2,j+l}] \leq x + l$. Then we must have $T_{1,j+k} = x + k'$ and $T_{2,j+k} = x + k$ for each $0 \leq k \leq l$ since $T$ is increasing and $\text{unprime}(T)$ has increasing rows. If $l > 0$ then $\text{row}(T)$ is equivalent under $\equiv$ to a word containing $(x + l)(x + l − 1)'(x + l)'$ as a consecutive subsequence, which is impossible by Proposition 2.3.

Thus $l = 0$, so if cell $(2, j + 1)$ is occupied in $T$ then it must contain an entry greater than $x + 1$. This implies that $[T_{1,j−1}] = x − 1$ as otherwise $\text{row}(T)$ would be equivalent under $\equiv$ to a primed involution word with adjacent letters equal to $x$ and $x'$, which is impossible. We now reach one of two contradictions. If $j = 2$ then $\text{row}(\text{unprime}(T))$ is equivalent under $\equiv$ to a word starting with $(x − 1)x$, contradicting Proposition 2.1. But if $j > 2$, then since we cannot have $T_{2,j−1} = x'$, the inequalities $x' − 1 \leq T_{1,j−1} < T_{2,j−1} < T_{2j} = x$ can only hold if $T_{1,j−1} = x' − 1$ and $T_{2,j−1} = x − 1$, which contradicts the minimality of $j$. We conclude that $\text{unprime}(T)$ is increasing. \hfill $\square$

### 3 Insertion algorithms

This section contains our main results, which are organized around several insertion algorithms mapping primed (bi)words to pairs of shifted tableaux. The domains of these maps are similar to various super-RSK correspondences that exist in the literature (see, e.g., [20, 28, 32]). The algorithm to which we will devote the most attention is a shifted version of Edelman-Greene insertion [7]. The others are variants of Sagan-Worley insertion [31, 34] and shifted mixed insertion [9].

#### 3.1 Shifted Edelman-Greene insertion

The insertion algorithm described below is central to what follows. For this reason, we include a number of examples within our definitions, which incorporate some auxiliary data that will not be used until Section 4.
Definition 3.1. Suppose $T$ is an increasing shifted tableau and $u \in \mathbb{Z} \sqcup \mathbb{Z}'$ is such that $\text{row}(T)u$ is a primed involution word for some element of $I_{\mathbb{Z}}$. We define sequences of
shifted tableaux $T = T_0, T_1, T_2, \ldots, T_m,$
numbers $u = u_0, u_1, u_2, \ldots, u_{m-1} \in \mathbb{Z} \sqcup \mathbb{Z}'$, and
pairs $(x_1, y_1), (\tilde{x}_1, \tilde{y}_1), (x_2, y_2), (\tilde{x}_2, \tilde{y}_2), \ldots, (x_m, y_m), (\tilde{x}_m, \tilde{y}_m) \in \mathbb{Z} \times \mathbb{Z},$
according to the following inductive procedure:

1. Assume $i \geq 1$ and that $T_{i-1}$ and $u_{i-1}$ are given. On the $i$th iteration, the number $u_{i-1}$ is inserted into either the $i$th row or $i$th column of $T_{i-1}$. We insert into the $i$th column if any of $(x_1, y_1), \ldots, (x_{i-1}, y_{i-1})$ belong to $\{(n, n) : n \in \mathbb{Z}\}$, and otherwise we insert into the $i$th row.

2. If $\lceil u_{i-1} \rceil$ is not less than any entry in the current (possibly empty) row or column, then we set $m := i$ and let $(x_i, y_i) = (\tilde{x}_i, \tilde{y}_i)$ be the first position $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with $1 \leq x \leq y$ that is unoccupied in the current row or column. We then form $T_i$ from $T_{i-1}$ by filling this position by $\lceil u_{i-1} \rceil$ if $x_i = y_i$ or by $u_{i-1}$ if $x_i \neq y_i$. For example, we could have

$$T_{i-1} = \begin{array}{c|c|c} & & \\
2 & 3 & \\
\hline 1 & & 4 \end{array} \quad \xleftarrow{7'} \sim \quad T_i = \begin{array}{c|c|c} & & \\
2 & 3 & 7 \\
\hline 1 & & 4 \end{array},$$

when inserting into an empty row with $u_{i-1} = 7'$, or

$$T_{i-1} = \begin{array}{c|c} 5 & \\
\hline & 6 \end{array} \quad \xleftarrow{7'} \sim \quad T_i = \begin{array}{c|c} 5 & 7' \\
\hline & 6 \end{array},$$

when inserting into a nonempty row with $u_{i-1} = 7'$. We say that our insertion process ends in column insertion if on this final iteration we are inserting into a column, or if $x_i = y_i$ and $u_{i-1}$ is primed. Otherwise, we that the process ends in row insertion.

3. Suppose instead that $\lceil u_{i-1} \rceil$ is less than some entry in the current row or column. Let $v_i$ and $w_i$ be the smallest entries in the current row or column with $\lceil u_{i-1} \rceil \leq \lceil v_i \rceil$ and $\lceil u_{i-1} \rceil < \lceil w_i \rceil$. Define $(x_i, y_i)$ and $(\tilde{x}_i, \tilde{y}_i)$ to be the respective positions of $v_i$ and $w_i$.

(a) Suppose $(x_i, y_i) \neq (\tilde{x}_i, \tilde{y}_i)$. Then we set $u_i := u_{i-1} + 1$. If $v_i$ and $w_i$ are both primed or both unprimed, then we define $T_i := T_{i-1}$. Otherwise, we form $T_i$ from $T_{i-1}$ by reversing the primes on $v_i$ and $w_i$. For example, we could have

$$T_{i-1} = \begin{array}{c|c|c|c} & & & \\
4 & 5 & & \\
\hline & & 4' & \\
\hline 1 & 2 & 3 & \\
\hline \end{array} \quad \xleftarrow{4'} \sim \quad T_i = \begin{array}{c|c|c|c} & & & \\
4 & 5 & & \\
\hline & & 4' & \\
\hline 1 & 2 & 3 & \\
\hline \end{array} \quad \xleftarrow{5'}$$

when inserting into a row with $u_{i-1} = 4'$, $v_i = 4$, $w_i = 5$, and $u_i = 5'$, or

$$T_{i-1} = \begin{array}{c|c} 5 & \\
\hline 4' & \\
\hline 1 & 2 & 3 \\
\hline \end{array} \quad \xrightarrow{4'} \sim \quad T_i = \begin{array}{c|c} 5' & \\
\hline 4 & \\
\hline 1 & 2 & 3 \\
\hline \end{array} \quad \xrightarrow{5}$$

when inserting into a column with $u_{i-1} = 4$, $v_i = 4'$, and $u_i = w_i = 5$. 

9
(b) Suppose \((x_i, y_i) = (\bar{x}_i, \bar{y}_i)\), so that \(v_i = w_i\). If \(v_i\) is on the diagonal and \(u_{i-1}\) is primed then we set \(u_i := v'_i\) and form \(T_i\) from \(T_{i-1}\) by replacing \(v_i\) by \([u_{i-1}]\). Otherwise, we set \(u_i := v_i\) and form \(T_i\) from \(T_{i-1}\) by replacing \(v_i\) by \(u_{i-1}\). For example, we could have

\[
T_{i-1} = \begin{array}{ccc}
3 & & \\
& & \\
& & \\
& & \\
\end{array} \quad \xrightarrow{1'} \quad T_i = \begin{array}{ccc}
1 & & \\
& & \\
& & \\
& & \\
\end{array}
\]

when \(u_{i-1} = 1', v_i = w_i = 3\), and \(u_i = 3'\), or

\[
T_{i-1} = \begin{array}{ccc}
2 & 4 & \\
& & \\
& & \\
& & \\
\end{array} \quad \xrightarrow{3'} \quad T_i = \begin{array}{ccc}
2 & 3' & \\
& & \\
& & \\
& & \\
\end{array}
\]

when \(u_i = 3'\) and \(v_i = w_i = u_i = 4\).

Let \(T \triangleleft u := T_m\) and define \(\text{path}^\leq(T, u) := \{(x_i, y_i)\}_{i \in [m]}\) and \(\text{path}^\prec(T, u) := \{(\bar{x}_i, \bar{y}_i)\}_{i \in [m]}\). We refer to these sequences as the weak and strict bumping paths that result from inserting \(u\) into \(T\).

**Example 3.2.** Suppose \(T = \begin{array}{ccc}
4 & 5 & \\
1 & 3 & 4' \\
\end{array}\) and \(u = 2\). Then we have

\[
T_0 = \begin{array}{ccc}
4 & 5 & \\
1 & 3 & 4' \\
\end{array} \quad \xrightarrow{2} \quad T_1 = \begin{array}{ccc}
4 & 5 & \\
1 & 2 & 4' \\
\end{array} \quad \xrightarrow{3} \quad T_2 = \begin{array}{ccc}
3 & 5 & \\
1 & 2 & 4' \\
\end{array} \quad \xrightarrow{5} \quad T_3 = \begin{array}{ccc}
3 & 5' & \\
1 & 2 & 4 \\
\end{array} \quad \xrightarrow{4} \quad T_4 = \begin{array}{ccc}
3 & 5' & \\
1 & 2 & 4 \\
\end{array} = T \triangleleft u
\]

along with \(u_0 = 2 < u_1 = 3 < u_2 = 4 < u_3 = 5\) and \(v_1 = w_1 = 3 < v_2 = w_2 = 4 < v_3 = 4' < w_3 = 5\). The corresponding bumping paths are

\[
\text{path}^\leq(T, u) = \{(x_i, y_i)\}_{i \in [m]} = \{(1, 2), (2, 2), (1, 3), (1, 4)\},
\]

\[
\text{path}^\prec(T, u) = \{(x_i, y_i)\}_{i \in [m]} = \{(1, 2), (2, 2), (2, 3), (1, 4)\}.
\]

Proposition 3.14 will show that if \(\text{row}(T)u\) is a primed involution word then so is \(\text{row}(T \triangleleft u)\). We can therefore iterate the above insertion process as follows:

**Definition 3.3.** Given a primed involution word \(a = a_1 a_2 \cdots a_n\) for some element of \(I_Z\), let \(P_\text{EG}^\triangleright(a)\) be the shifted tableau \(\emptyset \triangleleft a_1 \triangleleft a_2 \triangleleft \cdots \triangleleft a_n\) and let \(Q_\text{EG}^\triangleright(a)\) be the standard shifted tableau with the same shape as \(P_\text{EG}^\triangleright(a)\) that has \(i\) (respectively, \(i'\)) in the box added by inserting \(a_i\) into \(P_\text{EG}^\triangleright(a_1 a_2 \cdots a_{i-1})\) when this ends in row insertion (respectively, column insertion).
We refer to \( a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \) as orthogonal Edelman-Greene insertion. There is a similar correspondence called symplectic Edelman-Greene insertion, with a different domain containing only unprimed words, which is denoted \( a \mapsto (P_{\text{EG}}^S(a), Q_{\text{EG}}^S(a)) \) in [24, Def. 3.23]. For more about the connection between these maps and the orthogonal and symplectic groups, see Remark 2.2.

**Example 3.4.** The words \( a = 134524' \), \( b = 5'431'4'2 \), and \( c = 41'354'2 \) all have

\[
P_{\text{EG}}^O(a) = P_{\text{EG}}^O(b) = P_{\text{EG}}^O(c) = \begin{array}{cccc}
3 & 5' & \\
1 & 2 & 4 & 5
\end{array}
\]

while \( Q_{\text{EG}}^O(a) = \begin{array}{cccc}
5 & 6 & \\
1 & 2 & 3 & 4
\end{array} \), \( Q_{\text{EG}}^O(b) = \begin{array}{cccc}
5' & 6' & \\
1' & 2' & 3' & 4'
\end{array} \), and \( Q_{\text{EG}}^O(c) = \begin{array}{cccc}
3' & 5 & \\
1 & 2' & 4 & 6'
\end{array} \).

The original Edelman-Greene correspondence \( a \mapsto (P_{\text{EG}}(a), Q_{\text{EG}}(a)) \) from [7], sending reduced words \( a \in R(\sigma) \) for \( \sigma \in S_n \) to pairs of (unshifted) tableaux, may be embedded in Definition 3.3 in the following way. Take any involution word \( b \) for \( z = (0, n)(-1, n-1)(-2, n-2) \cdots (-n+1, 1) \in I_Z \). Then \( a \mapsto ba \) is an injective map \( R(\sigma) \hookrightarrow R_{\text{inv}}(\sigma^{-1}Z) \) and we recover \( P_{\text{EG}}(a) \) from \( P_{\text{EG}}^O(ba) \) by omitting the first \( n \) columns, while \( Q_{\text{EG}}(a) \) is given by omitting the same columns of \( Q_{\text{EG}}^O(ba) \) and subtracting \( \ell(b) \) from the remaining entries, which are all unprimed numbers.

As noted in the introduction, \( a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \) restricted to unprimed involution words reduces to a map previously studied in [12, 23, 24]. Our inclusion of primes may seem like a minor generalization. However, there seems to be no simple way to derive our main results as corollaries of what is known about the unprimed form of orthogonal Edelman-Greene insertion.

**Remark 3.5.** Since \( a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \) restricted to unprimed words coincides with [24, Def. 3.20], Propositions 2.3 and 2.5 along with [24, Rem. 3.25] imply the following properties concerning the process to construct \( T \leftarrow O u \), stated in the notation in Definition 3.1:

- Each of \( T = T_0, T_1, \ldots, T_m = T \leftarrow O u \) is increasing with no primes on the diagonal.
- If \( (x_i, y_i) \neq (\tilde{x}_i, \tilde{y}_i) \) then \( \lfloor u_{i-1} \rfloor = \lfloor v_i \rfloor = w_i - 1 \in Z \) and \( |x_i - \tilde{x}_i| + |y_i - \tilde{y}_i| = 1 \).
- If \( (x_i, y_i) \neq (\tilde{x}_i, \tilde{y}_i) \) and \( x_i = y_i \), then \( x_i = y_i = i \) and \( T_{ii} + 1 = T_{i,i+1} = T_{i+1,i+1} - 1 \in Z \).
- It always holds that \( u_{i-1} < u_i \) and \( \lfloor u_i \rfloor = \lfloor w_i \rfloor \).

We may now state our first main result about orthogonal Edelman-Greene insertion.

**Theorem 3.6.** For each \( z \in I_Z \), the map \( a \mapsto (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \) is a bijection from the set of primed involution words \( R_{\text{inv}}^+(z) \) to the set of pairs \( (P, Q) \) of shifted tableaux of the same shape, in which \( P \) is increasing with no primes on the diagonal, \( Q \) is standard, and \( \text{row}(P) \in R_{\text{inv}}^+(z) \).

The theorem remains true on replacing the two instances of \( R_{\text{inv}}^+(z) \) by \( R_{\text{inv}}(z) \) if we further require \( Q \) to have no primes on the diagonal [12, Thm. 5.19]. It is routine, following [23, §3.3] or [29, §5.3], to describe a reverse insertion algorithm that gives the inverse map \( (P_{\text{EG}}^O(a), Q_{\text{EG}}^O(a)) \mapsto a \). However, we will end up deriving Theorem 3.6 by another method at the end of Section 4.2.

If \( T \) is a shifted tableau, then we construct \( \text{unprime} \) from \( T \) by removing the primes from all entries, and we form \( \text{unprime}_{\text{diag}}(T) \) by removing the primes from just the diagonal entries. Clearly

\[
P_{\text{EG}}^O(\text{unprime}(a)) = \text{unprime}(P_{\text{EG}}^O(a)) \quad \text{and} \quad Q_{\text{EG}}^O(\text{unprime}(a)) = \text{unprime}_{\text{diag}}(Q_{\text{EG}}^O(a)) \tag{3.1}
\]
for any primed involution word \( a \).

An integer \( i \) is a descent of a standard shifted tableau \( T \) if either (a) \( i \) and \( i + 1 \) both appear in \( T \) with \( i + 1 \) in a row strictly after \( i \), (b) \( i \) and \( i + 1 \) both appear in \( T \) with \( i + 1 \) in a column strictly after \( i \), or (c) \( i \) and \( i + 1 \) both appear in \( T \). Let \( \text{Des}(T) \) denote the set of descents of \( T \). If \( T \) is as in (2.1), then \( \text{Des}(T) = \{1, 3, 6\} \). If \( a = a_1a_2 \cdots a_n \) is a primed word then let \( \text{Des}(a) := \{i \in [n - 1] : a_i > a_{i+1}\} \). These descent sets are related as follows:

**Proposition 3.7.** Let \( a \in \mathcal{R}_{\text{inv}}^+(z) \) for some \( z \in I_Z \). Then \( \text{Des}(a) = \text{Des}(Q_{\text{EG}}^O(a)) \).

*Proof.* [14, Prop. 2.24] asserts that \( \text{Des}(\text{unprime}(a)) = \text{Des}(Q_{\text{EG}}^O(\text{unprime}(a))) \). We have \( \text{Des}(a) = \text{Des}(\text{unprime}(a)) \) since \( \text{unprime}(a) \in \mathcal{R}_{\text{inv}}^+(z) \), so it suffices by (3.1) to check \( \text{Des}(\text{unprime}_{\text{diag}}(T)) = \text{Des}(T) \) for any standard shifted tableau \( T \). This is a straightforward exercise. \( \square \)

When \( a \) is a word in a totally ordered alphabet and \( N \) is a nonnegative integer, we let \( \text{lnrc}_N(a) \) denote the set of \( N \)-tuples of weakly increasing, possibly empty subwords \((a_1, a_2, \cdots, a_N)\) such that \( a = a_1^2 a_2^2 \cdots a_N^2 \). We again define \( \text{lnrc}_\infty(a) \) to be the set of infinite sequences \((a_1, a_2, \cdots)\) of weakly increasing words such that \( a = a_1^2 a_2^2 \cdots \); here, all but finitely many \( a_i \) must be empty. If \( \mathcal{A} \) is a set of words and \( N \in \{0, 1, 2, \ldots\} \cup \{\infty\} \) then we let \( \text{lnrc}_N(\mathcal{A}) = \bigsqcup_{a \in \mathcal{A}} \text{lnrc}_N(a) \).

**Definition 3.8.** Given \( \phi = (a_1^2, a_2^2, \cdots) \in \text{lnrc}_N(\mathcal{R}_{\text{inv}}^+(z)) \) for \( z \in I_Z \), let \( P_{\text{EG}}^O(\phi) := P_{\text{EG}}^O(a_1^2 a_2^2 \cdots) \) and form \( Q_{\text{EG}}^O(\phi) \) from \( Q_{\text{EG}}^O(a_1^2 a_2^2 \cdots) \) by replacing each entry \( j \in Z \) (respectively, \( j' \in Z' \)) by \( i \) (respectively, \( i' \)), where \( i > 0 \) is minimal with \( j < \ell(a^1) + \ell(a^2) + \cdots + \ell(a^n) \).

For example, if \( \phi = (0, 4, 13, 0, 5, 0, 4', 2) \in \text{lnrc}_8(413542) \) then

\[
P_{\text{EG}}^O(\phi) = \begin{array}{c|c|c}
3 & 5' & \\
1 & 2 & 4 & 5
\end{array} \quad \text{and} \quad Q_{\text{EG}}^O(\phi) = \begin{array}{c|c|c}
3' & 7 & \\
2 & 3' & 5 & 8'
\end{array}
\]

If \((a^1, a^2, \cdots) \in \text{lnrc}_N(\mathcal{R}_{\text{inv}}^+(z)) \) for \( z \in I_Z \) then each \( a^i \) is strictly increasing by Proposition 2.1.

**Corollary 3.9.** For each \( z \in I_Z \), the map \( \phi \mapsto (P_{\text{EG}}^O(\phi), Q_{\text{EG}}^O(\phi)) \) is a bijection from \( \text{lnrc}_\infty(\mathcal{R}_{\text{inv}}^+(z)) \) to the set of pairs \((P, Q)\) of shifted tableaux of the same shape in which \( P \) is increasing with no primes on the diagonal, \( Q \) is semistandard, and \( \text{row}(P) \in \mathcal{R}_{\text{inv}}^+(z) \).

*Proof.* Let \( T \) be a standard shifted tableau whose shape is a strict partition of \( m \) and let \( \alpha = (\alpha_1, \alpha_2, \ldots) \) be a weak composition of \( m \) such that \( I(\alpha) := \{\alpha_1 + \alpha_2 + \cdots + \alpha_i : i \geq 1\} \) \( \setminus \{m\} \) contains \( \text{Des}(T) \). Such pairs \((T, \alpha)\) are in bijection with semistandard shifted tableaux via the map that replaces \( j \) (respectively, \( j' \)) in \( T \) by \( i \) (respectively, \( i' \)) where \( i > 0 \) is minimal with \( j < \ell(a^1) + \ell(a^2) + \cdots + \ell(a^n) \). Proposition 3.7 shows that if \( \phi = (a_1^2, a_2^2, \cdots) \in \text{lnrc}_\infty(\mathcal{R}_{\text{inv}}^+(z)) \), then we may obtain \( Q_{\text{EG}}^O(\phi) \) by applying this bijection to \((T, \alpha)\) for \( T = Q_{\text{EG}}^O(a_1^2 a_2^2 \cdots a^n) \) and \( \alpha = (\ell(a^1), \ell(a^2), \ldots) \). Given this observation, we deduce that \( \phi \mapsto (P_{\text{EG}}^O(\phi), Q_{\text{EG}}^O(\phi)) \) is injective and surjective from Theorem 3.6. \( \square \)

We discuss an application mentioned in Section 1.2. Let \( x_i \) for \( i \in \mathbb{Z} \) be commuting indeterminates. Given a shifted tableau \( T \), let \( x^T := \prod_{i \in \mathbb{Z}} x_i^{c_i} \) where \( c_i \) is the number of entries in \( T \) equal to \( i \) or \( i' \). The Schur \( Q \)-function of a strict partition \( \lambda \) is the formal power series

\[
Q_{\lambda} := \sum_T x^T \in \mathbb{Z}[x_1, x_2, \ldots]\n
\]

where \( T \) ranges over all semistandard shifted tableaux of shape \( \lambda \). The Schur \( Q \)-functions are symmetric in the \( x_i \) variables and linearly independent [33]. We present another proof that they span a ring with nonnegative integer structure coefficients.

For a strict partition \( \lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0) \), the dominant involution of shape \( \lambda \) is the permutation \( z_\lambda := (k, k + \lambda_1)(k - 1, k + \lambda_2) \cdots (1, k + \lambda_k) \in S_{k + \lambda_1} \). If \( \ell(\lambda) = 0 \) then set \( z_\lambda := 1 \).
Lemma 3.10. If \( \lambda \) is a strict partition, then \( P^Q_{\text{EG}}(a) \) for all \( a \in \mathcal{R}^{+}_{\text{inv}}(\lambda) \) is equal to the increasing shifted tableau \( T \) of shape \( \lambda \) with \( T_{ij} = i + j - 1 \) for all \( (i,j) \in SD_{\lambda} \).

Proof. One can check that \( \text{row}(T) \in \mathcal{R}^{\text{inv}}(\lambda) \) directly, or this is equivalent to \([11, \text{Prop. 3.9}]. \) \([24, \text{Lem. 5.5}] \) asserts that \( P^Q_{\text{EG}}(a) = T \) for all \( a \in \mathcal{R}^{\text{inv}}(\lambda) \subset \mathcal{R}^{+}_{\text{inv}}(\lambda) \), so \( |\mathcal{R}^{\text{inv}}(\lambda)| \) is the number of standard shifted tableau of shape \( \lambda \) with no primes on the diagonal. Thus, by Theorem 3.6 and (3.1), the number of primed words \( a \in \mathcal{R}^{+}_{\text{inv}}(\lambda) \) with \( P^Q_{\text{EG}}(a) = T \) must be \( 2^{|\mathcal{R}^{\text{inv}}(\lambda)|} \). But this is the size of \( \mathcal{R}^{+}_{\text{inv}}(\lambda) \) since \( \ell(\lambda) = \ell_{\text{abs}}(\lambda) \) is the number of nontrivial cycles in \( \lambda_\lambda \). \( \square \)

For \( z \in \mathcal{I}_{Z} \), let \( Q_z := \sum_{\phi \in \text{Incr}_{\infty}(\mathcal{R}^{+}_{\text{inv}}(z))} z^\phi \), where \( x^\phi := x_1^{\ell(a_1)} x_2^{\ell(a_2)} \cdots \) if \( \phi = (a_1, a_2, \ldots) \). These power series are denoted \( G_z \) in \([12, \S 4.5]\). The following is immediate from Corollary 3.9.

Corollary 3.11 ([12, Cor. 4.62]). We have \( Q_z = \sum_{T \in \{P^Q_{\text{EG}}(a); a \in \mathcal{R}^{+}_{\text{inv}}(z)\}} Q_{\text{shape}(T)} \) and \( Q_z = Q_{\lambda} \).

As in the introduction, given elements \( v \in S_m \) and \( w \in S_n \), let \( v \times w \in S_{m+n} \) be the permutation mapping \( i \mapsto v(i) \) for \( i \in [m] \) and \( m + j \mapsto m + w(j) \) for \( j \in [n] \).

Corollary 3.12. If \( \lambda \) and \( \mu \) are strict partitions then \( Q_{\lambda} Q_{\mu} = \sum_{\nu} g_{\lambda \nu}^{\mu} Q_\nu \), where the sum is over strict partitions \( \nu \) and \( g_{\lambda \nu}^{\mu} \) is the number of elements in \( \{ P^Q_{\text{EG}}(a) : a \in \mathcal{R}^{+}_{\text{inv}}(\lambda \times \mu) \} \) of shape \( \nu \).

Proof. Let \( y \in \mathcal{I}_{Z} \cap S_m \) and \( z \in \mathcal{I}_{Z} \cap S_n \). It follows from Proposition 2.4 that \( \text{Incr}_{\infty}(\mathcal{R}^{+}_{\text{inv}}(y \times z)) \) is in bijection with the product \( \text{Incr}_{\infty}(\mathcal{R}^{+}_{\text{inv}}(y)) \times \text{Incr}_{\infty}(\mathcal{R}^{+}_{\text{inv}}(z)) \) via the map \( ((a^1, a^2, \ldots), (b^1, b^2, \ldots)) \mapsto (a^1 c^1, a^2 c^2, \ldots) \) where \( c^i \) is formed by adding \( m \) to each letter of \( b^i \). This implies that \( Q_y Q_z = Q_{y \times z} \), and so the result follows from Corollary 3.11. \( \square \)

### 3.2 Orthogonal Coxeter-Knuth equivalence

An essential property of orthogonal Edelman-Greene insertion is that the fibers of \( a \mapsto P^Q_{\text{EG}}(a) \) are equivalence classes for a simple relation on primed words, which we define below. Let \( \text{ock} \) denote the operator that acts on 1- and 2-letter primed words by interchanging

\[
X \leftrightarrow X', \quad XY \leftrightarrow YX, \quad X'Y \leftrightarrow Y'X, \quad XY' \leftrightarrow YX', \quad \text{and} \quad X'Y' \leftrightarrow Y'X'
\]

for all \( X, Y \in \mathcal{I}_{Z} \). In addition, let \( \text{ock} \) act on 3-letter primed words as the involution interchanging

\[
XYX \leftrightarrow YXY, \quad X'YX \leftrightarrow YX'Y, \quad ACB \leftrightarrow CAB, \quad \text{and} \quad BCA \leftrightarrow BAC
\]

for all \( X, Y \in \mathcal{I}_{Z} \) and all \( A, B, C \in \mathcal{I}_{Z} \). While fixing any 3-letter words not of these forms. Given a primed word \( a = a_1 a_2 a_3 \cdots a_n \) and \( i \in [n-2] \), we define

\[
\text{ock}_{i-1}(a) := \text{ock}(a_1) a_2 a_3 \cdots a_n, \\
\text{ock}_0(a) := \text{ock}(a_1 a_2) a_3 \cdots a_n, \\
\text{ock}_i(a) := a_1 \cdots a_{i-1} \text{ock}(a_i a_{i+1} a_{i+2}) a_{i+3} \cdots a_n,
\]

while setting \( \text{ock}_i(a) := a \) for \( i \in \mathcal{I}_{Z} \) with \( i + 2 \notin [\ell(a)] \). For example, if \( a = 45'7121' \) then

\[
\text{ock}_{-1}(a) = 4'5'7121', \quad \text{ock}_0(a) = 5'4'7121', \quad \text{ock}_1(a) = 45'7121', \\
\text{ock}_2(a) = 45'1721', \quad \text{ock}_3(a) = 45'1721', \quad \text{ock}_4(a) = 45'72'12.
\]

The abbreviation “ock” is for orthogonal Coxeter-Knuth operator.
Lemma 3.13. If \( i \in \mathbb{Z} \) and \( a \) is a primed involution word then \( \text{unprime}(\text{ock}_i(a)) = \text{ock}_i(\text{unprime}(a)) \).

Proof. This is clear unless \( i \in [\ell(a) - 2] \) and \([a_i] = [a_{i+2}]\), but if this happens then Proposition 2.3 tells us that \( i + 1 \) is not a commutation in \( a \) and that at most one of \( i \) or \( i + 2 \) is a commutation. \( \square \)

The transitive closure of the relation on unprimed words with \( a \sim \text{ock}_i(a) \) for all \( i > 0 \) is often called **Coxeter-Knuth equivalence** [7, Def. 6.19]. We define **orthogonal Coxeter-Knuth equivalence** \( \text{\textasciitilde} \) to be the transitive closure of the relation on primed words with \( a \text{\textasciitilde} \text{ock}_i(a) \) for all \( i \in \mathbb{Z} \). If \( a \in \mathcal{R}_{\text{inv}}^+(z) \) for some \( z \in I_Z \), then \( a \text{\textasciitilde} b \) implies that \( b \in \mathcal{R}_{\text{inv}}^+(z) \) by Proposition 2.4.

Proposition 3.14. Suppose \( T \) is an increasing shifted tableau and \( u \in \mathbb{Z} \sqcup \mathbb{Z}' \) is such that \( \text{row}(T)u \) is a primed involution word for an element of \( I_Z \). Then \( \text{row}(T)u \text{\textasciitilde} \text{row}(T \leftarrow u) \). Consequently, if \( a \) is a primed involition word then \( a \text{\textasciitilde} \text{row}(P_{\text{EG}}^O(a)) \).

The claim that \( a \text{\textasciitilde} \text{row}(P_{\text{EG}}^O(a)) \) for (unprimed) involution words is [23, Cor. 4.12].

Proof. Let \( T_i, u_i, v_i, w_i, (x_i, y_i), (\tilde{x}_i, \tilde{y}_i) \) be as is Definition 3.1. Form \( \tilde{T}_i \) by adding \( u_i \) to the end of column \( i + 1 \) if \( x_p = y_p \) for some \( p \leq i \), and to the end of row \( i + 1 \) otherwise. For example if \( T = \begin{array}{ccc}
5 & 3 & 4 \\
3 & 4 & 7' \\
\end{array} \) and \( u = 1' \), then we would have

\[
\begin{array}{ccc}
5 & 3 & 4 & 7' \\
3 & 4 & 7' & 1 \\
\end{array} = T_0, \quad
\begin{array}{ccc}
5 & 3 & 4 \\
3 & 4 & 7' & 1 \\
\end{array} = T_1, \quad
\begin{array}{ccc}
5 & 3 & 4 \\
1 & 3' & 7' \\
\end{array} = T_2, \quad
\begin{array}{ccc}
5 & 3 & 4 \\
1 & 3' & 7' \\
\end{array} = T_3 = T \text{\textasciitilde} u.
\]

If there is no index \( p \) with \( x_p = y_p \), then it is easy to see that

\[
\text{row}(T)u = \text{row}(T_0) \text{\textasciitilde} \text{row}(T_1) \text{\textasciitilde} \text{row}(T_2) \text{\textasciitilde} \ldots = \text{row}(T \leftarrow u).
\]

Suppose \( p \) is the first index with \( x_p = y_p \). Then it is also easy to see that \( \text{row}(T)u = \text{row}(T_0) \text{\textasciitilde} \text{row}(T_1) \text{\textasciitilde} \ldots \text{row}(T_p) \text{\textasciitilde} \text{col}(\tilde{T}_{p+1}) \text{\textasciitilde} \ldots = \text{row}(T \leftarrow u) \). It follows from [23, Lem 2.7] or as an exercise that \( \text{row}(T \leftarrow u) \text{\textasciitilde} \text{col}(T \leftarrow u) \), so it just remains to show that \( \text{row}(T_p) \text{\textasciitilde} \text{col}(\tilde{T}_{p+1}) \). There are two cases to consider, according to whether \( y_p \) and \( \tilde{y}_p \) are equal.

First assume that \( y_p = \tilde{y}_p \). Let \( D = \{(2i,2j) : (i,j) \in T\} \). Then define \( U : D \cup \{(2p - 1,2p - 1)\} \to Z \sqcup Z' \) to be the map with \( U_{2i,2j} = (\tilde{T}_{p-1})_{ij} \) and \( U_{2p-1,2p-1} = u_{p-1} \), and define \( V : D \cup \{(2p + 1,2p + 1)\} \to Z \sqcup Z' \) to be the map with \( V_{2i,2j} = (T_{p})_{ij} \) and \( V_{2p+1,2p+1} = u_{p} \). In our example, we have \( p = 1, u_{p-1} = 1', \) and \( u_{p} = 3' \), along with

\[
U = \begin{array}{ccc}
\ldots & 5 & \ldots \\
1' & 3 & 4 & 7' \\
\ldots & \ldots & \ldots \\
\end{array} \quad \text{and} \quad
V = \begin{array}{ccc}
\ldots & 5 & \ldots \\
1 & 4 & 7' \\
\ldots & \ldots & \ldots \\
\end{array}.
\]

Since \( \text{row}(\tilde{T}_{p-1}) = \text{row}(U) \) and \( \text{col}(\tilde{T}_p) = \text{col}(V) \), it suffices to show that then \( \text{row}(U) \text{\textasciitilde} \text{col}(V) \). Form the **northeast** (respectively, **southwest**) **diagonal reading words** of \( U \) (and similarly for \( V \)) by listing the entries \( U_{ij} \) in the order that makes \((j - i, i)\) (respectively, \((j - i, -i)\)) increase lexicographically.
In our example, these words for $U$ are $1'3547'$ and $531'47'$, respectively. Then define $\widetilde{U}$ and $\widetilde{V}$ by removing the main diagonals from $U$ and $V$. It is easy to see that $\widetilde{U} = \widetilde{V}$.

We now appeal to [23, §2.2], which contains several general lemmas that relate our various reading words. These lemmas are stated in terms of tableaux without primed entries, but are straightforward to adapt our situation. In particular, [23, Lem. 2.8] implies that $\text{row}(U)$ is related under Coxeter-Knuth equivalence to the southwest diagonal reading word of $U$, [23, Lem. 2.9] implies that $\text{col}(V)$ is related under Coxeter-Knuth equivalence to the northeast diagonal reading word of $V$, and [23, Cor. 2.10] implies that the two diagonal reading words of $\widetilde{U} = \widetilde{V}$ are Coxeter-Knuth equivalent. (Alternatively, it is not hard to check each of these claims directly.) It is therefore enough to show that reading the diagonal of $\widetilde{U}$ in the southwest diagonal reading order gives a word that is equivalent under $\overset{O}{\sim}$ to the word given by reading the diagonal of $V$ in the northeast diagonal reading order. This is straightforward since both words have at most one primed letter; for example, we have $531' \overset{O}{\sim} 351' \overset{O}{\sim} 31'5 \overset{O}{\sim} 13'5$.

We are left to consider the second case in which $y_p \neq \bar{y}_p$. By Remark 3.5 this can only occur when $[u_{p-1}] = T_{pp} = T_{p,p+1} = 1 = p_{p+1,p+1} = 0$. Moreover, it is easy to see that the index of $T_{pp}$ in $\text{row}(T_{p-1})$ is a commutation, so it follows from Proposition 2.3 that $u_{p-1}$ must be unprimed. Define $U : D \sqcup \{(2p+1,2p+1)\} \to \mathbb{Z} \cup \mathbb{Z}'$ to have $U_{2i,2j} = (T_{p-1})_{ij} = (T_{p})_{ij}$ and $U_{2p+1,2p+1} = u_{p-1} = 1$. Then $\text{row}(\hat{T}_{p-1}) \overset{O}{\sim} \text{row}(U)$ and $\text{col}(\hat{T}_p) = \text{col}(U)$, so it suffices to show that $\text{row}(U) \overset{O}{\sim} \text{col}(U)$. This follows by repeating the argument in the previous paragraph $V$ replaced by $U$. In this setting, the key claim is that reading the main diagonal of $U$ in either diagonal reading order gives words that are equivalent under $\overset{O}{\sim}$. This holds since the diagonal has no primed entries.

The converse of the preceding result also holds. The proof of this fact is more difficult, and requires us to understand precisely how the operators $\text{ock}_i$ interact with $P^{\text{O}}_{\text{EG}}$ and $Q^{\text{O}}_{\text{EG}}$.

Assume $T$ is a standard shifted tableau. Choose $q > 0$ such that the domain of $T$ fits inside $[q] \times [q]$. Let $C_i$ be the sequence of primed entries in column $i$ of $T$, read in order, and let $R_i$ be the sequence of unprimed entries in row $i$ of $T$, read in order. The shifted reading word of $T$ is

$$\text{shword}(T) := \text{unprime}(C_q R_q \cdots C_2 R_2 C_1 R_1).$$

For the standard shifted tableau

$$T = \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array}$$

the nonempty sequences $C_i R_i$ are $C_1 R_1 = 1'6$, $C_2 R_2 = 2'37$, $C_3 R_3 = 4'5'$, so the shifted reading word is $\text{shword}(T) = 4523716$. One can check that $i \in \text{Des}(T)$ if and only if $i+1$ appears before $i$ in the sequence $\text{shword}(T)$.

Let $n$ be the size of the domain of $T$. For each $i \in [n]$, write $\Box_i$ for the unique position of $T$ containing $i$ or $i'$, and define $s_i(T)$ to be the shifted tableau formed from $T$ as follows:

- If $\Box_i$ and $\Box_{i+1}$ are in the same row or column then reverse the primes on the entries of whichever of these positions is off the diagonal; then, if both $\Box_{i-1}$ and $\Box_{i+1}$ (respectively, $\Box_i$ and $\Box_{i+2}$) are on the diagonal when $i-1 \in [n]$ (respectively, $i+2 \in [n]$), and their entries are not both primed or both unprimed, also reverse the primes on these entries.

- Otherwise, swap $i$ with $i+1$ and $i'$ with $i+1'$.
If \( T \) is the standard shifted tableau in (3.3), then
\[
\varrho_1(T) = \varrho_2(T) = \begin{array}{cccc}
3' & 5' & 7 \\
1 & 2 & 4' & 6
\end{array}
\quad \varrho_3(T) = \begin{array}{cccc}
4 & 5' & 7 \\
1' & 2' & 3' & 6
\end{array}
\quad \text{and} \quad \varrho_4(T) = \begin{array}{cccc}
3 & 5 & 7 \\
1' & 2' & 4 & 6
\end{array}
\]

Next, for each \( i \in \mathbb{Z} \), we construct a shifted tableau \( \varrho_i(T) \) of the same shape from \( T \) as follows. If \( i + 2 \notin [n] \) then we set \( \varrho_i(T) := T \). Otherwise, if \( i \in \{-1, 0\} \) then we form \( \varrho_i(T) \) from \( T \) by swapping the entries \( i + 2 \) with \( i + 2' \), so that for example
\[
\varrho_{-1} \left( \begin{array}{cccc}
3' & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \begin{array}{cccc}
3 & 5' & 7 \\
1 & 2' & 4' & 6
\end{array}
\quad \text{and} \quad \varrho_0 \left( \begin{array}{cccc}
3' & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array}
\]

Finally, if \( i \in [n-2] \) then we set
\[
\varrho_i(T) := \begin{cases} 
\varrho_i(T) & \text{if } i + 2 \text{ is between } i \text{ and } i + 1 \text{ in } \text{shword}(T) \\
\varrho_{i+1}(T) & \text{if } i \text{ is between } i + 1 \text{ and } i + 2 \text{ in } \text{shword}(T) \\
T & \text{if } i + 1 \text{ is between } i \text{ and } i + 2 \text{ in } \text{shword}(T).
\end{cases}
\]

For the tableau \( T \) in (3.3), this gives
\[
\varrho_1 \left( \begin{array}{cccc}
3' & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \begin{array}{cccc}
3' & 5' & 7 \\
1 & 2' & 4' & 6
\end{array} = \varrho_1(T),
\]
\[
\varrho_2 \left( \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \varrho_3 \left( \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \begin{array}{cccc}
4 & 5' & 7 \\
1' & 2' & 3' & 6
\end{array} = \varrho_3(T),
\]
\[
\varrho_4 \left( \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} = T,
\]
\[
\varrho_5 \left( \begin{array}{cccc}
3 & 5' & 7 \\
1' & 2' & 4' & 6
\end{array} \right) = \begin{array}{cccc}
3 & 6' & 7 \\
1' & 2' & 4' & 5
\end{array} = \varrho_5(T).
\]

Restricted to standard shifted tableaux with no primes on the diagonal, the operators \( \varrho_i \) for \( i > 0 \) coincide with the maps \( \psi_{i+1} \) defined in [1, §6]. The definitions of \( \varrho_i \) and \( \psi_{i+1} \) diverge for tableaux with primed entries on the diagonal. However, [1, Thm. 6.3] (stating that \( \{\psi_i\}_{1<i<n} \) is a dual equivalence for standard shifted tableaux) should still be true if one replaces \( \psi_i \) by \( \varrho_{i-1} \).

We note a few useful, general properties of these operators. Recall that \( \text{unprime}_{\text{diag}}(T) \) is formed by removing all primes from the diagonal of a shifted tableau \( T \). Let \( \text{primes}(T) \) be the total number of primed positions in \( T \) and let \( \text{primes}_{\text{diag}}(T) \) be the number of primed diagonal positions in \( T \).

**Proposition 3.15.** Suppose \( T \) is a standard shifted tableau with \( n \) boxes. Let \( \square_j \) for \( j \in [n] \) denote the unique box of \( T \) containing \( j \) or \( j' \). For each \( i \in \mathbb{Z} \), the following properties hold:

(a) \( \varrho_i(\varrho_i(T)) = T \).

(b) If \( i \in [n-2] \) then \( \varrho_i(T) \) only differs from \( T \) in its entries in positions \( \square_i, \square_{i+1}, \) and \( \square_{i+2} \).
(c) If $i = 0$, or if $i \in [n - 2]$ but $\square_i$ and $\square_{i+2}$ are not both on the diagonal, then the diagonal positions with primed entries in $\mathfrak{d}_i(T)$ are the same as those in $T$.

(d) If $i \neq -1$ then $\text{unprime}_{\text{diag}}(\mathfrak{d}_i(T)) = \mathfrak{d}_i(\text{unprime}_{\text{diag}}(T))$.

In addition, for each $i \in [n - 2]$, the following properties hold:

(e) $\text{primes}_{\text{diag}}(T) = \text{primes}_{\text{diag}}(\mathfrak{d}_i(T))$.

(f) $\text{primes}(T) = \text{primes}(\mathfrak{d}_i(T))$ if $\square_i$ and $\square_{i+2}$ are not both on the diagonal.

(g) $\text{primes}(T) = \text{primes}(\mathfrak{d}_i(T)) \pm 1$ if $\square_i$ and $\square_{i+2}$ are both on the diagonal.

Proof. We prove each part in turn. First let $i \in \{-1, 0, \ldots, n - 2\}$.

(a) This identity is obvious if $i \in \{-1, 0\}$ or if $i \in [n - 2]$ and $i + 1$ is between $i$ and $i + 2$ in $\text{shword}(T)$. Suppose $i \in [n - 2]$ and $i + 2$ is between $i$ and $i + 1$ in $\text{shword}(T)$. If $\square_i$ and $\square_{i+1}$ are not in the same row or column, then $\text{shword}(\mathfrak{d}_i(T))$ is formed from $\text{shword}(T)$ by reversing the positions of $i$ and $i + 1$, so $i + 2$ is also between $i$ and $i + 1$ in $\text{shword}(\mathfrak{d}_i(T))$ and we have $\mathfrak{d}_i(\mathfrak{d}_i(T)) = \mathfrak{d}_i(\mathfrak{d}_i(T)) = T$. If $\square_i$ and $\square_{i+1}$ are both in the same row or column, but neither box is on the diagonal, then one box must be primed and the other must be unprimed for $i + 2$ to appear between $i$ and $i + 1$ in $\text{shword}(T)$. In this case, reversing the primes on these boxes preserves the property that $i + 2$ appears between $i$ and $i + 1$ in the shifted reading word, so again $\mathfrak{d}_i(\mathfrak{d}_i(T)) = T$. Assume further that $\square_i$ and $\square_{i+1}$ appear in the same row or column and at least one of the boxes is on the diagonal. Then $\square_i$, $\square_{i+1}$, and $\square_{i+2}$ must be arranged in $T$ as

\[
\begin{array}{c|c|c|c|c}
& & & & \\
& i & & & \\
& i+1' & & & \\
& & i & & \\
& & & i+2' & \\
& & i+1 & & \\
& & & i+2 & \\
\end{array}
\text{ or }
\begin{array}{c|c|c|c|c}
& & & & \\
& & & i & \\
& & & i+1 & \\
& & & i+2 & \\
& i' & & & \\
& i+1' & & & \\
\end{array}
\]

for $i + 2$ to appear between $i$ and $i + 1$ in $\text{shword}(T)$. In each case we have $\mathfrak{d}_i(\mathfrak{d}_i(T)) = \mathfrak{d}_i(\mathfrak{d}_i(T)) = T$. The argument to show that $\mathfrak{d}_i(\mathfrak{d}_i(T)) = T$ when $i \in [n - 2]$ is between $i + 1$ and $i + 2$ in $\text{shword}(T)$ is similar.

(b) Assume $i \in [n - 2]$. If $i + 2$ is between $i$ and $i + 1$ in $\text{shword}(T)$, then $\square_{i+1}$ and $\square_{i+3}$ cannot both be on the diagonal, and if $i$ is between $i + 1$ and $i + 2$ in $\text{shword}(T)$ then $\square_{i+1}$ and $\square_{i+3}$ cannot both be on the diagonal. This shows that $\mathfrak{d}_i(T)$ can only differ from $T$ in its entries in positions $\square_i$, $\square_{i+1}$, and $\square_{i+2}$.

(c) This property is clear if $i = 0$. Let $i \in [n - 2]$. One checks that if $i + 2$ is between $i$ and $i + 1$ in $\text{shword}(T)$ but $\square_i$ and $\square_{i+1}$ are in the same row or column, then (1) $\square_{i+1}$ cannot be on the diagonal and (2) if $\square_i$ is on the diagonal then $\square_{i+2}$ must also be on the diagonal. Likewise, if $i$ is between $i + 1$ and $i + 2$ in $\text{shword}(T)$ but $\square_{i+1}$ and $\square_{i+2}$ are in the same row or column, then (3) $\square_{i+1}$ cannot be on the diagonal and (4) if $\square_{i+2}$ is on the diagonal then $\square_i$ must also be on the diagonal. The desired assertions now follow from the definition of $\mathfrak{d}_i$.  

17
(d) Because $\Box_2$ is never on the diagonal, the definition of $\partial_0$ implies that $\text{unprime}_{\text{diag}}(\partial_0(T)) = \partial_0(\text{unprime}_{\text{diag}}(T))$. When $i > 0$, the desired identity is again clear from the definition of the operator $\partial_i$ (now in terms of $s_i$) since $\text{shword}(T) = \text{shword}(\text{unprime}_{\text{diag}}(T))$.

For the rest of this proof assume $i \in [n - 1]$.

(e) This holds since applying $\partial_i$ to $T$ either preserves which diagonal positions are primed or reverses the primes on exactly two diagonal boxes that are not both primed or both unprimed.

(f) Assume $\Box_i$ and $\Box_{i+2}$ are not both on the diagonal. If $i + 2$ is between $i$ and $i + 1$ in $\text{shword}(T)$ and $\Box_i$ and $\Box_{i+1}$ are in the same row or column, then exactly one of these boxes must have a primed entry, so $\text{primed}(T) = \text{primed}(s_i(T))$. Likewise, if $i$ is between $i+1$ and $i+2$ in $\text{shword}(T)$ and $\Box_{i+1}$ and $\Box_{i+2}$ are in the same row or column, then exactly one of these boxes must have a primed entry, so $\text{primed}(T) = \text{primed}(s_{i+1}(T))$. Therefore $\text{primed}(T) = \text{primed}(\partial_i(T))$.

(g) Observe that if $\Box_i$ and $\Box_{i+2}$ are both on the diagonal, then we must have $\Box_i = (q - 1, q - 1)$, $\Box_{i+1} = (q - 1, q)$, and $\Box_{i+2} = (q, q)$ for some $q > 1$. No matter how the entries in these boxes are primed, we have $\partial_i(T) = s_i(T) = s_{i+1}(T)$. This tableau is formed from $T$ by reversing the prime on the entry in position $(q - 1, q)$, and then interchanging the primes on the entries in positions $(q - 1, q - 1)$ and $(q, q)$, so $\text{primed}(T) = \text{primed}(\partial_i(T)) \pm 1$.

This completes the proof of the proposition.

Our proof of the following theorem occupies all of Section 4.

**Theorem 3.16.** Suppose $i \in \mathbb{Z}$ and $a$ is a primed involution word for an element of $I_\mathbb{Z}$. Then it holds that $P_{E_G}^O(\text{ock}_i(a)) = P_{E_G}^O(a)$ and $Q_{E_G}^O(\text{ock}_i(a)) = \partial_i(Q_{E_G}^O(a))$.

When $a = \text{unprime}(a)$, this theorem is equivalent to results in [24]; see Lemma 4.1. Extending these identities to primed involution words is surprisingly involved. The proof of the unprimed version of Theorem 3.16 in [24] relies heavily on the *involution Little map*, which gives a family of bijections $\bigcup_{z \in X} R_{\text{inv}}(z) \leftrightarrow \bigcup_{z \in Y} R_{\text{inv}}(z)$ for certain finite subsets $X, Y \subseteq I_\mathbb{Z}$. Describing a “primed involution Little map” does not appear to be straightforward; one difficulty is that with primes allowed, the unions $\bigcup_{z \in X} R_{\text{inv}}^+(z)$ and $\bigcup_{z \in Y} R_{\text{inv}}^+(z)$ often have different sizes. As such, proving Theorem 3.16 requires a quite different strategy compared to [24].

**Corollary 3.17.** Two primed involution words satisfy $a \overset{O}{\sim} b$ if and only if $P_{E_G}^O(a) = P_{E_G}^O(b)$.

**Proof.** This is immediate from Proposition 3.14 and Theorem 3.16.

### 3.3 Modifying Sagan-Worley insertion

Before commencing the proof of Theorem 3.16, we discuss some related insertion algorithms sending primed (bi)words to pairs of shifted tableaux. A *biword* is a two-line array of positive integers

$$
\phi = \begin{bmatrix}
i_1 & i_2 & \ldots & i_n \\
a_1 & a_2 & \ldots & a_n
\end{bmatrix}
$$

(3.4)
where the entries in the top row are weakly increasing such that if \( i_j = i_{j+1} \) then \( a_j \leq a_{j+1} \). A **primed biword** is a two-line array satisfying the same conditions, but where we allow entries in the bottom row to have \( 0 < a_j \in \mathbb{Z} \sqcup \mathbb{Z}' \) as long as no column \[
\begin{bmatrix}
  i \\
  a
\end{bmatrix}
\] with \( a \in \mathbb{Z}' \) is repeated. Thus
\[
\begin{bmatrix}
  1 & 1 & 2 & 2 & 3 \\
  4 & 4 & 5 & 5 & 6 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  1 & 1 & 2 & 2 & 3 \\
  6 & 4' & 5' & 6 & 1
\end{bmatrix}
\]
are primed biwords while the following are not:
\[
\begin{bmatrix}
  1 & 1 & 1 & 2 & 2 & 3 \\
  4 & 4' & 5 & 5 & 6 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  1 & 1 & 1 & 2 & 2 & 3 \\
  5 & 4' & 5 & 6 & 1
\end{bmatrix}
\]

We identify a (primed) word \( a = a_1a_2\cdots a_n \) with the (primed) biword whose second row is \( a \) and whose first row is \( 1, 2, 3, \ldots, n \). If we never have \( a_i = a_{i+1} \in \mathbb{Z}' \) then the elements of \( \text{Incr}_N(a) \) for \( N \in \{0, 1, 2, \ldots \} \sqcup \{\infty\} \) are in bijection with the (primed) biwords of the form (3.4) whose entries in the first row are all at most \( N \).

**Definition 3.18.** Suppose \( \phi \) is a primed biword of the form (3.4). We construct a sequence of shifted tableaux \( \emptyset = P_0, P_1, \ldots, P_n \) in which \( P_j \) is formed from \( P_{j-1} \) as follows:

1. On each iteration, an entry \( u \in \mathbb{Z} \sqcup \mathbb{Z}' \) is inserted into a row or column of a shifted tableau. The process begins with \( a_j \) inserted into the first row of \( P_{j-1} \).
2. If inserting into a row when \( u \in \mathbb{Z} \), or into a column when \( u \in \mathbb{Z}' \), locate the first entry \( v \) in the row or column such that \( u < v \); otherwise, locate the first entry \( v \) such that \( u \leq v \). When such an entry exists, we say that \( u \) "bumps" \( v \) from its position.
3. If no such \( v \) exists then \( u \) is added to the end of the row or column to form \( P_j \). If \( u \) is primed and the added position is on the diagonal, then we change its value to \( \lceil u \rceil \) and say that the insertion process ends in column insertion. We also say that the process ends in column insertion if we are inserting into a column; otherwise, the process ends in row insertion.
4. If \( v \) is off the diagonal, replace \( v \) by \( u \) and insert \( v \) into the next row (respectively, column).
5. Assume \( v \) is on the diagonal; then \( v \in \mathbb{Z} \) is unprimed. If \( \lceil u \rceil = v \) then continue by inserting \( v \) into the next column. If \( \lceil u \rceil \neq v \) then replace \( v \) by \( \tilde{u} \) and insert \( \tilde{v} \) into the next column, where if \( u \) and \( v \) are both unprimed or both primed then \( \tilde{u} := u \) and \( \tilde{v} := v \), and otherwise \( \tilde{u} \) and \( \tilde{v} \) are given by reversing the primes on \( u \) and \( v \), respectively.

Now define \( P_{\text{SW}}(\phi) := P_n \) and let \( Q_{\text{SW}}(\phi) \) be the shifted tableau with the same shape whose entry in the unique box of \( P_j \) that is not in \( P_{j-1} \) is either \( i_j \) (when adding \( a_j \) to \( P_{j-1} \) ends in row insertion) or \( i'_j \) (when adding \( a_j \) to \( P_{j-1} \) ends in column insertion).

This slightly modifies the definition of Sagan-Worley insertion (sometimes called shifted RSK insertion) from [31, §8] or [34, §6.1]. The latter map, which we will denote by
\[
\phi \mapsto (P_{\text{SW}}(\phi), Q_{\text{SW}}(\phi)),
\]
is given by repeating Definition 3.18 with two changes:
• first, in step (3) we do not remove the prime from a newly added diagonal entry and we say that the insertion process ends in column insertion only if the last step inserts into a column;

• second, in step (5) we always take $\tilde{u} := u$ and $\tilde{v} := v$.

It is convenient to think of these maps as “orthogonal” and “symplectic” versions of the same algorithm. Lemma 3.21 will make the basis for this parallelism more precise. Entries with primes may occur on the diagonal of $P^{Sp}_{SW}(\phi)$ or $Q^{O}_{SW}(\phi)$ but not on the diagonal of $Q^{Sp}_{SW}(\phi)$ or $P^{O}_{SW}(\phi)$.

Example 3.19. Suppose $\phi = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 4 & 5' & 2' & 3 & 7' \end{bmatrix}$. Then in the notation of Definition 3.18

$p_1 = \begin{bmatrix} 4 \\ \end{bmatrix}$, $p_2 = \begin{bmatrix} 4 & 5' \end{bmatrix}$, $p_3 = \begin{bmatrix} 2 & 4' & 5' \end{bmatrix}$, $p_4 = \begin{bmatrix} 4 & 3 & 5' \end{bmatrix}$, $p_5 = \begin{bmatrix} 4 & 2 & 3 & 5' & 7' \end{bmatrix}$

so we have

$P^{O}_{SW}(\phi) = \begin{bmatrix} 4 & 2 & 3 & 5' & 7' \\ \end{bmatrix}$ and $Q^{O}_{SW}(\phi) = \begin{bmatrix} 2' \\ 1 & 1 & 2' & 2 \end{bmatrix}$

On the other hand, one can check that

$P^{Sp}_{SW}(\phi) = \begin{bmatrix} 4 & 2' & 3 & 5' & 7' \\ \end{bmatrix}$ and $Q^{Sp}_{SW}(\phi) = \begin{bmatrix} 2 \\ 1 & 1 & 2' & 2 \end{bmatrix}$

Similarly, if $\phi = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 5 \\ 4 & 4 & 5' & 2' & 2' & 2 & 3 & 3 & 7' & 7 \end{bmatrix}$ then

$P^{O}_{SW}(\phi) = \begin{bmatrix} 4 & 4 & 5' \\ 2 & 2 & 3 & 3 & 5 & 7' & 7 \end{bmatrix}$ and $Q^{O}_{SW}(\phi) = \begin{bmatrix} 3' & 3 & 5 \\ 1 & 1 & 1 & 3' & 5 & 5 \end{bmatrix}$

Finally, comparing with Example 3.4, if $c = 41'354'2$ then

$P^{O}_{SW}(c) = \begin{bmatrix} 3 & 4 \\ 1 & 2 & 4' & 5 \end{bmatrix}$ and $Q^{O}_{SW}(c) = \begin{bmatrix} 3' & 5 \\ 1 & 2' & 4' & 6' \end{bmatrix}$

Here we evaluate $P^{O}_{SW}(c)$ and $Q^{O}_{SW}(c)$ by converting $c$ to the biword with first row 1, 2, 3, 4, 5, 6.

Given a primed word $a = a_1a_2 \cdots a_n$, form double$(a)$ by applying the map with $i \mapsto 2i$ and $i' \mapsto (2i)'$ for $i \in \mathbb{Z}$ to the letters of $a$. If $\phi$ is a primed biword then define double$(\phi)$ by applying double to its second row. For a shifted tableau $T$, construct double$(T)$ by applying double to all of its entries.

We say that a primed word $a$ is a partial signed permutation if unprime$(a)$ has all distinct letters. Define a primed biword to be value-strict if its second row is a partial signed permutation.

Proposition 3.20. If $\phi$ is a value-strict primed biword, then the second row of double$(\phi)$ is a primed involution word, double $\circ P^{O}_{SW}(\phi) = P^{O}_{EG} \circ$ double$(\phi)$, and $Q^{O}_{SW}(\phi) = Q^{O}_{EG} \circ$ double$(\phi)$.

This terminology is motivated by the fact that if unprime$(a)$ is a permutation of 1, 2, 3, \ldots, $n$ then $a$ is the one-line representation of a signed permutation, that is, an element of the hyperoctahedral group.
To compute $Q_{	ext{EC}}^{O} \circ \text{double} (\phi)$, we view $\text{double} (\phi)$ as an element of $\text{lncr}_{\infty} (R_{\text{inv}}^{+} (z))$ for some $z \in I_{2}$. For example if $\phi = \begin{bmatrix} 1 & 1 & 3 & 3 & 3 \\ 4 & 5 & 2 & 3 & 7 \end{bmatrix}$ then $\text{double} (\phi) = (45', 0, 237')$ so $Q_{	ext{EC}}^{O} \circ \text{double} (\phi) = \begin{bmatrix} 3' \\ 1 & 1 & 3' & 3 \end{bmatrix}$

Proof. Let $\phi$ be as in (3.4). The first claim holds since $\text{unprime} (\text{double} (a_{1}a_{2} \cdots a_{n}))$ is an involution word where every index is a commutation. This ensures that $P_{EC}^{O} \circ \text{double} (\phi)$ and $Q_{\text{EC}}^{O} \circ \text{double} (\phi)$ are defined, and it is easy to see that the first tableau coincides with $P_{SW}^{O} \circ \text{double} (\phi)$ while the second coincides with $Q_{SW}^{O} \circ \text{double} (\phi) = Q_{SW}^{O} (\phi)$. \hfill \Box

Let $a = a_{1}a_{2} \cdots a_{n}$ be a primed word, and define

$$P_{SW}^{O} (a) := P_{SW} \left( \begin{bmatrix} 1 & 2 & \ldots & n \\ a_{1} & a_{2} & \ldots & a_{n} \end{bmatrix} \right).$$

For each $j \in [n]$, consider $P_{SW}^{O} (a_{1}a_{2} \cdots a_{j-1})$ and $P_{SW}^{O} (a_{1}a_{2} \cdots a_{j})$. If these tableaux have different numbers of rows or the same entries in all diagonal positions, then define $\tau_{j}^{SW} (a)$ to be the identity permutation of $Z$. Otherwise, there is a unique diagonal position with different entries in the two tableaux, and we let $\tau_{j}^{SW} (a)$ be the transposition interchanging these entries. If $a = 45'2'37'$ as in Example 3.19, then $\tau_{3}^{SW} (a) = (2, 4)$ and $\tau_{j}^{SW} (a) = 1$ for $j \in \{1, 2, 4, 5\}$. Let

$$\tau^{SW} (a) := \tau_{1}^{SW} (a) \tau_{2}^{SW} (a) \cdots \tau_{n}^{SW} (a) \quad \text{and} \quad \tau^{SW} (\phi) := \tau^{SW} (a_{1}a_{2} \cdots a_{n})$$

for a primed biword $\phi$ of the form (3.4).

We say that a position $(i, j)$ in a semistandard shifted tableau $T$ is free if $[T_{xy}] \neq [T_{ij}]$ whenever $x > i$ or $y < j$. Every diagonal position is free. One can freely add or remove primes from free positions without changing whether a shifted tableau is semistandard. Moreover, if $u$ and $v$ are the entries in distinct free positions of a semistandard shifted tableau, then $[u] \neq [v]$. Given a semistandard shifted tableau $T$, form $\text{unprime}_{\text{free}} (T)$ from $T$ by removing the primes from the entries in all free positions. This is called the \textit{canonical form} of $T$ in [8, Def. 2.6].

We say that $u \in Z$ is \textit{initially primed} (respectively, \textit{unprimed}) in a primed word if $u'$ (respectively, $u$) appears in the word and there is no earlier letter equal to $u$ (respectively $u'$). Form $\text{unprime}_{\text{init}} (a)$ from a primed word $a$ by unpriming the first appearance of $u'$ for each initially primed letter $u \in Z$. This is called the \textit{canonical form} of $a$ in [8, Def. 2.1], and we have $\text{unprime}_{\text{init}} (\text{row} (T)) = \text{row} (\text{unprime}_{\text{free}} (T))$ for any semistandard shifted tableau.

\textbf{Lemma 3.21.} Suppose $\phi$ is a primed biword of the form (3.4). Then $P_{SW}^{O} (\phi)$ and $P_{SW}^{Sp} (\phi)$ are semistandard shifted tableaux with the same free positions and it holds that

$$\text{unprime}_{\text{free}} (P_{SW}^{O} (\phi)) = \text{unprime}_{\text{free}} (P_{SW}^{Sp} (\phi)) \quad \text{and} \quad \text{unprime}_{\text{diag}} (Q_{SW}^{O} (\phi)) = Q_{SW}^{Sp} (\phi).$$

Let $(i, j)$ be a free position in $P_{SW}^{Sp} (\phi)$ and let $u \in Z$ be the value in this position with its prime removed. The entry of $P_{SW}^{Sp} (\phi)$ in this position is primed if and only if $u$ is initially primed in the second row of $\phi$. If $i \neq j$ (respectively, $i = j$), then the entry of $P_{SW}^{O} (\phi)$ (respectively, $Q_{SW}^{O} (\phi)$) in position $(i, j)$ is primed if and only if $\tau^{SW} (\phi) (u)$ is initially primed in the second row of $\phi$. 

21
Remark. The lemma shows that \( P_{SW}^O(a) = P_{SW}^{Sp}(a) \) for primed words with \( a = \text{unprime}_{unt}(a) \), and that both \( a \mapsto P_{SW}^O(a) \) and \( a \mapsto P_{SW}^{Sp}(a) \) descend to the same map from “equivalence classes” of words to “equivalence classes” of tableaux in the sense of [8, Defs. 2.1 and 2.6].

Proof. In the insertion process that defines \( P_{SW}^{Sp}(\phi) \), whenever a free position with entry \( v \) is bumped by a number \( u \), the position that \( v \) subsequently bumps (or the new position added to the tableau when \( v \) is placed at the end of a row or column) only depends on \([v]\). The latter position is also free unless \( v \) is on the diagonal and \([u] = [v]\), in which case the diagonal free entry is unchanged. If \( u = [a_j] \) and \( T = P_{SW}^{Sp}(a_1a_2 \cdots a_{j-1}) \) has no entries equal to \( u \) or \( u' \), then the position bumped by \( a_i \) in the first row of \( T \) is free in \( P_{SW}^{Sp}(a_1a_2 \cdots a_j) \) and contains \( a_j \).

Given these observations, it follows by induction on the number of columns of \( \phi \) that \( P_{SW}^{Sp}(\phi) \) contains \( u' \) in a free position for some \( u \in \mathbb{Z} \) if and only if \( u \) is initially primed in the second row of \( \phi \). Moreover, it is easy to see that \( P_{SW}^O(\phi) \) is formed from \( P_{SW}^{Sp}(\phi) \) by toggling the primes on certain free positions, and that the identities in the displayed equation hold. Thus \( P_{SW}^O(\phi) \) is semistandard, since \( P_{SW}^{Sp}(\phi) \) is known to be semistandard [31, Thm. 8.1].

For the last part of the lemma, consider a semistandard shifted tableau \( T \) and let \( \Box_u \) for \( u \in \mathbb{Z} \) denote the free position of \( T \) containing \( u \) or \( u' \), if this exists. If \( \Box_u \) and \( \Box_v \) are both defined, then let \((u,v) \in S_T \) act on \( T \) by reversing the primes on the entries in these positions if they are not both primed or both unprimed, and otherwise leaves \( T \) unchanged. This operation extends to an action of the group of permutations of the entries of \( \text{unprime}(T) \).

Let \( a = a_1a_2 \cdots a_n \) be the second row of \( \phi \). Form \( \tilde{P}_{SW}^O(a) \) from \( P_{SW}^O(a) \) by adding primes to all diagonal positions that are primed in \( Q_{SW}^O(a) \). Then \( \tilde{P}_{SW}^O(a) \) is constructed by the same insertion process as the one that defines \( P_{SW}^{Sp}(a) \), except that whenever an inserted number \( u \) is about to bump a diagonal entry \( v \) with \([u] < [v]\) and \( \{u,v\} \not\subset \mathbb{Z} \) and \( \{u,v\} \not\subset \mathbb{Z}' \), we reverse the primes on the entries in these positions if they are not both primed or both unprimed, and otherwise leaves \( T \) unchanged. This operation extends to an action of the group of permutations of the entries of \( \text{unprime}(T) \).

We may represent a primed biword \( \phi \) as the matrix \( A \) whose entry in position \((i,j)\) is the number of columns equal to \[
\begin{bmatrix}
i & j' \\
i' & j
\end{bmatrix}
\] plus \(-\frac{1}{2}\) if the column \[
\begin{bmatrix}
i & j'
\end{bmatrix}
\] appears. This gives a bijection between primed biwords and \([0 < 1' < 1 < 2' < 2 < \ldots]\)-valued matrices with finitely many nonzero entries. Following [31], we call the latter \textit{circled matrices}. For example,

\[
\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 1 \\
2' & 2 & 1 & 1 & 1 & 2' \\
\end{bmatrix}
\]

corresponds to the circled matrix

\[
\begin{bmatrix}
0 & 3' \\
2 & 1' \\
1 & 0
\end{bmatrix}
\].

For a circled matrix \( A \) with primed biword \( \phi \), we set \( P_{SW}^O(A) = P_{SW}^O(\phi) \) and \( Q_{SW}^O(A) = Q_{SW}^O(\phi) \) as well as \( P_{SW}^{Sp}(A) = P_{SW}^{Sp}(\phi) \) and \( Q_{SW}^{Sp}(A) = Q_{SW}^{Sp}(\phi) \).

Theorem 3.22. The map \( A \mapsto (P_{SW}^O(A), Q_{SW}^O(A)) \) is a bijection from circled matrices \( A = [A_{ij}] \) to pairs \((P, Q)\) of semistandard shifted tableaux of the same shape, where \( P \) has no primes on the diagonal and where the number of times that \( j \) or \( j' \) (for any \( j \in \mathbb{Z} \)) appear in \( P \) and in \( Q \) are the respective sums \( \sum_i [A_{ij}] \) and \( \sum_k [A_{jk}] \).
The same statement holds for the map \( A \mapsto (P_{SW}^P(a), Q_{SW}^P(a)) \) if one requires \( Q \) instead of \( P \) to have no primes on the diagonal (see \([31, \text{Thm. 8.1}]\) or \([34, \text{Thm. 6.1.1}]\)).

**Proof.** Let \( \phi \) be the primed biword associated to a circled matrix \( A \). Toggling whether a given number in the second row of \( \phi \) is initially primed or not has no effect on \( \tau_{SW}(\phi) \) by Lemma 3.21. The result is therefore clear from the same lemma and \([31, \text{Thm. 8.1}]\) or \([34, \text{Thm. 6.1.1}]\). \( \square \)

Finally, we discuss a conjectural analogue of Theorem 3.16. Let \( \text{okn} \) denote the operator that acts on 1- and 2-letter primed words by interchanging

\[
X \leftrightarrow X', \quad XX \leftrightarrow XX', \quad X'X \leftrightarrow X'X',
\]

\[
XY \leftrightarrow YX, \quad X'Y \leftrightarrow Y'X, \quad XY' \leftrightarrow YX', \quad \text{and} \quad X'Y' \leftrightarrow Y'X',
\]

for all distinct \( X, Y \in \mathbb{Z} \). Let \( \text{okn} \) act on 3-letter primed words as the involution interchanging

\[
ACB \leftrightarrow CAB \quad \text{and} \quad YXZ \leftrightarrow YZX
\]

for all \( A, B, C, X, Y, Z \in \mathbb{Z} \) with \([A] \leq B \leq [C] - \frac{1}{2} \) and \( X + \frac{1}{2} \leq [Y] \leq Z \), while fixing any 3-letter words not of these forms. For a primed word \( a = a_1a_2 \cdots a_n \) and \( i \in [n - 2] \), define

\[
\text{okn}_{i-1}(a) := \text{okn}(a_1)a_2a_3 \cdots a_n,
\]

\[
\text{okn}_0(a) := \text{okn}(a_1a_2)a_3 \cdots a_n,
\]

\[
\text{okn}_i(a) := a_1 \cdots a_{i-1}\text{okn}(a_ia_{i+1}a_{i+2})a_{i+3} \cdots a_n,
\]

while setting \( \text{okn}_i(a) := a \) for \( i \in \mathbb{Z} \) with \( i + 2 \not\in [\ell(a)] \). These *orthogonal Knuth operators* coincide with \( \text{ock}_i \) on partial signed permutations.

**Conjecture 3.23.** If \( i \in \mathbb{Z} \) then \( P_{SW}^O(\text{okn}_i(a)) = P_{SW}^O(a) \) and \( Q_{SW}^O(\text{okn}_i(a)) = \partial_i(Q_{SW}^O(a)) \).

It is trivial to verify these identities when \( i \in \{-1, 0\} \). As with Theorem 3.16, the difficulty lies in the case when \( 1 \leq i \in \ell(a) - 2 \). Let \( \text{shK} \) denote the transitive closure of the relation on primed words with \( a \sim \text{okn}_i(a) \) for all \( i \in \mathbb{Z} \). Checking the following is also straightforward:

**Proposition 3.24.** If \( a \) is a primed word then \( a \sim_{\text{shK}} \text{row}(P_{SW}^O(a)) \).

**Proof.** One can repeat the proof of Proposition 3.14, using the relation \( \sim_{\text{shK}} \) in place and \( \sim \), after rewriting Definition 3.18 in a form similar to Definitions 3.1 and 3.3. We omit the details. \( \square \)

Thus, Conjecture 3.23 would imply the following:

**Conjecture 3.25.** Two primed words satisfy \( a \sim_{\text{shK}} b \) if and only if \( P_{SW}^O(a) = P_{SW}^O(b) \).

A version of this property for the original “symplectic” form of Sagan-Worley insertion is already known. Modify the definition of \( \text{okn}_i \) by setting

\[
\text{spkn}_{i-1}(a) := a \quad \text{and} \quad \text{spkn}_0(a) := a_2a_1a_3a_4 \cdots a_n \text{ if } [a_1] \neq [a_2] \text{ and } n := \ell(a) \geq 2,
\]

while defining \( \text{spkn}_i(a) := \text{okn}_i(a) \) in all other cases. Write \( \sim \) for the transitive closure of the relation with \( a \sim \text{spkn}_i(a) \) for all \( i \in \mathbb{Z} \). Worley \([34, \text{Thm. 6.2.2}]\) shows that then two primed words satisfy \( a \sim b \) if and only if \( P_{SW}^S(a) = P_{SW}^S(b) \), so in particular \( P_{SW}^S(\text{spkn}_i(a)) = P_{SW}^S(a) \) for all \( i \). We do not know of a reference for this analogue of the second identity in Conjecture 3.23:
Conjecture 3.26. If \( i > 0 \) and \( a \) is any primed word then \( Q^{\text{Sp}}_{\text{SW}}(\text{spkn}_i(a)) = \alpha_i(Q^{\text{Sp}}_{\text{SW}}(a)) \).

The identity \( Q^{\text{Sp}}_{\text{SW}}(\text{spkn}_0(a)) = \alpha_0(Q^{\text{Sp}}_{\text{SW}}(a)) \) is easy to check directly.

Proposition 3.27. If \( Q^{\text{O}}_{\text{SW}}(\text{okn}_i(a)) = \alpha_i(Q^{\text{O}}_{\text{SW}}(a)) \) then \( Q^{\text{Sp}}_{\text{SW}}(\text{spkn}_i(a)) = \alpha_i(Q^{\text{Sp}}_{\text{SW}}(a)) \).

Proof. We may assume \( i > 0 \). Then \( \text{spkn}_i = \text{okn}_i \) so \( Q^{\text{Sp}}_{\text{SW}}(\text{spkn}_i(a)) = \text{unprime}_{\text{diag}}(Q^{\text{O}}_{\text{SW}}(\text{okn}_i(a))) \) and \( \alpha_i(Q^{\text{Sp}}_{\text{SW}}(a)) = \text{unprime}_{\text{diag}}(\alpha_i(Q^{\text{O}}_{\text{SW}}(a))) \) by Proposition 3.15 and Lemma 3.21.

If Conjecture 3.26 were known, then one could derive Conjectures 3.23 and 3.23 by (a simplified version of) the strategy we use in Section 4 to prove Theorem 3.16.

In more detail, suppose \( a \) is a primed word, \( i \in [\ell(a) - 2] \), and \( b := \text{okn}_i(a) \). The numbers that are initially primed in \( a \) are the same as in \( b \), so we have \( \text{unprime}_{\text{init}}(b) = \text{okn}_i(\text{unprime}_{\text{init}}(a)) \) and \( \text{unprime}_{\text{free}}(P^{\text{O}}_{\text{SW}}(a)) = P^{\text{Sp}}_{\text{SW}}(\text{unprime}_{\text{init}}(a)) = P^{\text{Sp}}_{\text{SW}}(\text{unprime}_{\text{init}}(b)) = \text{unprime}_{\text{free}}(P^{\text{O}}_{\text{SW}}(b)) \) by Lemma 3.21 and [34, Thm. 6.2.2]. To prove that \( P^{\text{O}}_{\text{SW}}(a) = P^{\text{O}}_{\text{SW}}(b) \) it suffices by Lemma 3.21 to show that \( \tau^{\text{SW}}(a) = \tau^{\text{SW}}(b) \). This can be achieved by proving appropriate versions of the lemmas in Sections 4.3 and 4.4. Then one can deduce \( Q^{\text{O}}_{\text{SW}}(b) = \alpha_i(Q^{\text{O}}_{\text{SW}}(a)) \) from \( Q^{\text{Sp}}_{\text{SW}}(b) = \alpha_i(Q^{\text{Sp}}_{\text{SW}}(a)) \) by an argument similar to the proof of Theorem 3.16 given at the end of Section 4.4.

For partial signed permutations, all of these conjectural results follow from Section 3.2:

Corollary 3.28. Suppose \( a \) and \( b \) are partial signed permutations. Then \( a \preceq b \) if and only if \( P^{\text{O}}_{\text{SW}}(a) = P^{\text{O}}_{\text{SW}}(b) \). Moreover, \( Q^{\text{O}}_{\text{SW}}(\text{okn}_i(a)) = \alpha_i(Q^{\text{O}}_{\text{SW}}(a)) \) for all \( i \).

Proof. This follows from Proposition 3.20 given Theorem 3.16 and Corollary 3.17, since the operators \( \text{okn}_i \) and \( \text{ock}_i \) coincide on partial signed permutations, as do the relations \( \sh^K \) and \( \sim \).

Our two forms of Sagan-Worley insertion do not coincide on partial signed permutations. However, because of Proposition 3.27, the previous corollary implies the following:

Corollary 3.29. If \( a \) is a partial signed permutation then \( Q^{\text{Sp}}_{\text{SW}}(\text{spkn}_i(a)) = \alpha_i(Q^{\text{Sp}}_{\text{SW}}(a)) \) for all \( i \).

3.4 Extending shifted mixed insertion

Finally, we discuss two extensions of Haiman’s \textit{shifted mixed insertion algorithm} [9, Def. 6.7] that are closely related to the forms of Sagan-Worley insertion analyzed in the previous section. Define a primed biword to be \textit{index-strict} if its first row is strictly increasing.

Definition 3.30. Suppose \( \phi \) is an index-strict primed biword of the form (3.4). We construct a sequence of shifted tableaux \( \emptyset = U_0, U_1, \ldots, U_n = U \) in which \( U_j \) is formed from \( U_{j-1} \) as follows:

1. Define \( \alpha \in \{\pm\} \times \mathbb{Z} \) to be \( (+, [a_j]) \) if \( a_j \in \mathbb{Z} \) or \( (-, [a_j]) \) if \( a_j \in \mathbb{Z}' \). Insert this pair into the first row of \( U_{j-1} \) according to the following procedure.

2. At each stage, a pair \( \beta_1 = (\epsilon_1, u_1) \) with \( u_1 \in \mathbb{Z} \cup \mathbb{Z}' \) is inserted into a row or column. If every pair \( (\epsilon_2, u_2) \) in the current row or column has \( u_1 \geq u_2 \) then \( \beta_1 \) is added to the end. Otherwise let \( \beta_2 = (\epsilon_2, u_2) \) be the first pair in the current row or column with \( u_1 < u_2 \).

3. If \( \beta_2 \) is on the diagonal, then it will always holds that \( u_2 \in \mathbb{Z} \), and we proceed by replacing \( \beta_2 \) with \( \beta_1 \) and inserting \( (\epsilon_2, u_2') \) into the next column.
(4) If $\beta_2$ is not on the diagonal, then replace $\beta_2$ with $(\epsilon_2, u_1)$ and insert $(\epsilon_1, u_2)$ into either the next row when $u_2 \in \mathbb{Z}$ or the next column when $u_2 \in \mathbb{Z}'$.

Form $P^O_{HM}(\phi)$ from $U$ by replacing each diagonal entry $(\epsilon, x)$ with $\epsilon = -$ by $x'$, and all other entries $(\epsilon, x)$ by $x$. Let $Q^O_{HM}(\phi)$ be the shifted tableau with the same shape whose entry in the box of $U_j$ that is not in $U_{j-1}$ is either $i_j$ or $i_j'$, with a primed number occurring precisely when this box is off the diagonal and its entry in $U_j$ has the form $(\epsilon, x)$ with $\epsilon = -$.

As our notation suggests, this definition has a “symplectic” variant.

Definition 3.31. Given an index-strict primed biword $\phi$ of the form (3.4), define shifted tableaux $\emptyset = U_0, U_1, \ldots, U_n = U$ by repeating the construction in Definition 3.30, but modifying step (3) so that the entry $\beta_2$ is replaced by $(\epsilon_2, u_1)$ while $(\epsilon_1, u_2')$ is inserted into the next column. Then:

- Form $P^{Sp}_{HM}(\phi)$ from $U$ by replacing all entries $(\epsilon, x)$ by $x$.
- Let $Q^{Sp}_{HM}(\phi)$ be the shifted tableau with the same shape whose entry in the box of $U_j$ that is not in $U_{j-1}$ is either $i_j$ or $i_j'$, with a primed number occurring precisely when the entry of $U_j$ in this box has the form $(\epsilon, x)$ with $\epsilon = -$.

When the first row of $\phi$ consists of the numbers 1, 2, 3, \ldots, $n$ and the second row of $\phi$ has no primed entries, both $\phi \mapsto (P^O_{HM}(\phi), Q^O_{HM}(\phi))$ and $\phi \mapsto (P^{Sp}_{HM}(\phi), Q^{Sp}_{HM}(\phi))$ reduce to shifted mixed insertion [9, Def. 6.7]. Neither extension seems to have appeared before in the literature. We refer to these maps as orthogonal and symplectic mixed insertion.

Example 3.32. Suppose our index-strict primed biword is

$$\phi = \left[ \begin{array}{cccccc} 2 & 3 & 4 & 5 & 7 \\ 2' & 1 & 1' & 2' \end{array} \right].$$

Then, writing $\pm x$ in place of $(\pm, x)$, the sequence of shifted tableaux $U_j$ in Definition 3.30 are

$$U_1 = \begin{array}{c} -2 \end{array}, \quad U_2 = \begin{array}{c} -2 +2 \\ +1 \\ -1 \end{array}, \quad U_3 = \begin{array}{c} -2 \\ +1 +2' \\ +1 \end{array}, \quad U_4 = \begin{array}{c} -2 \\ +1 +1 \end{array}, \quad U_5 = \begin{array}{c} -2 \\ +1 +1 \end{array},$$

so we have

$$P^O_{HM}(\phi) = \begin{array}{c} 2' \\ 1 \\ 1 \\ 2' \\ 2 \end{array} \quad \text{and} \quad Q^O_{HM}(\phi) = \begin{array}{c} 4 \\ 2 \\ 3' \\ 5' \\ 7' \end{array}.$$
There is a transpose operation \( \phi \mapsto \phi^\top \) on primed biwords given as follows: first move the primes from any primed elements in the second row of \( \phi \) to the entries directly above them, then interchange the two rows and reorder the columns to be lexicographically increasing. If

\[
\phi = \begin{bmatrix} 2 & 3 & 4 & 5 & 7 \\
2' & 2 & 1 & 1' & 2' \end{bmatrix}
\quad \text{then} \quad
\phi^\top = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\
4 & 5' & 2' & 3 & 7' \end{bmatrix},
\]

for example. In terms of the associated circled matrices, this operation is just the matrix transpose, so it interchanges index-strict biwords and value-strict biwords.

The following result, which relates our primed forms of shifted mixed insertion and Sagan-Worley insertion via the biword transpose, generalizes [9, Thm. 6.10].

**Theorem 3.33.** If \( \phi \) is index-strict, then it holds that \( P_{\text{HM}}^\phi(\phi) = Q_{\text{SW}}^\phi(\phi^\top) \) and \( Q_{\text{HM}}^\phi(\phi^\top) = P_{\text{SW}}^\phi(\phi^\top) \), and it also holds that \( P_{\text{HM}}^{\phi_{\text{sp}}}(\phi^\top) = Q_{\text{SW}}^{\phi_{\text{sp}}}(\phi^\top) \) and \( Q_{\text{HM}}^{\phi_{\text{sp}}}(\phi^\top) = P_{\text{SW}}^{\phi_{\text{sp}}}(\phi^\top) \).

One can observe this property by comparing Examples 3.19 and 3.32.

**Proof.** We will only prove the first two identities, as the argument for the symplectic case is similar and easier. Let \( \phi \) be a primed biword of the form (3.4). We first assume

\[
\{i_1 < i_2 < \cdots < i_n\} = \{[a_1], [a_2], \ldots, [a_n]\} = \{1, 2, \ldots, n\}. \tag{3.6}
\]

Then the result to prove is a generalization of [9, Thm. 6.10], which asserts that \( Q_{\text{HM}}^\phi(\text{unprime}(\phi)) = P_{\text{SW}}^\phi(\text{unprime}(\phi^\top)) \) and \( P_{\text{HM}}^\phi(\text{unprime}(\phi)) = Q_{\text{SW}}^\phi(\text{unprime}(\phi^\top)) \), or equivalently that

\[
\text{unprime}(Q_{\text{HM}}^\phi(\phi)) = \text{unprime}(P_{\text{SW}}^\phi(\phi^\top)),
\]

\[
\text{unprime}_{\text{diag}}(P_{\text{HM}}^\phi(\phi)) = \text{unprime}_{\text{diag}}(Q_{\text{SW}}^\phi(\phi^\top)). \tag{3.7}
\]

Choose a position \( \square_1 \) in the shared domain of these shifted tableaux. First assume \( \square_1 \) is off the diagonal, and let \( i \in \mathbb{Z} \) be its entry in \( \text{unprime}(Q_{\text{HM}}^\phi(\phi)) \). All diagonal entries of \( Q_{\text{HM}}^\phi(\phi) \) and \( P_{\text{SW}}^\phi(\phi^\top) \) are unprimed by definition, so to show that \( Q_{\text{HM}}^\phi(\phi) = P_{\text{SW}}^\phi(\phi^\top) \) it suffices to check that \( i' \) appears in \( Q_{\text{HM}}^\phi(\phi) \) if and only if \( i' \) appears in \( P_{\text{SW}}^\phi(\phi^\top) \).

Let \( \phi_{\mid i} \) denote the primed biword consisting of the first \( i \) columns of \( \phi \). If any diagonal entry of \( \text{unprime}_{\text{diag}}(P_{\text{HM}}^\phi(\phi_{\mid i-1})) \) differs from the corresponding entry of \( \text{unprime}_{\text{diag}}(P_{\text{HM}}^\phi(\phi_{\mid i})) \), and the last such entry occurs in position \( \square_2 \), then \( i' \) appears in \( Q_{\text{HM}}^\phi(\phi) \) if and only if the entry of \( P_{\text{HM}}^\phi(\phi_{\mid i-1}) \) in position \( \square_2 \) is primed. Otherwise \( i' \) appears in \( Q_{\text{HM}}^\phi(\phi) \) if and only if \( a_i \) is primed.

Note that \( (\phi_{\mid i})^\top \) is formed from \( \phi^\top \) by omitting all columns whose entries in the second row are greater than \( i \). Thus, the entry of \( P_{\text{SW}}^\phi((\phi_{\mid i})^\top) \) in position \( \square_1 \) is the same as in \( P_{\text{SW}}^\phi(\phi^\top) \). Moreover, if any diagonal entry of \( \text{unprime}_{\text{diag}}(Q_{\text{SW}}^\phi((\phi_{\mid i-1})^\top)) \) differs from the corresponding entry of \( \text{unprime}_{\text{diag}}(Q_{\text{SW}}^\phi((\phi_{\mid i})^\top)) \), and the last such entry occurs in position \( \square_2 \), then the entry of \( P_{\text{SW}}^\phi((\phi_{\mid i-1})^\top) \) in position \( \square_2 \) must bump \( i \) from the same position in \( P_{\text{SW}}^\phi((\phi_{\mid i})^\top) \). In this case, it follows that \( i' \) appears in \( P_{\text{SW}}^\phi(\phi^\top) \) if and only if the entry of \( Q_{\text{SW}}^\phi((\phi_{\mid i-1})^\top) \) in position \( \square_2 \) is primed. On the other hand, if no such position \( \square_2 \) exists then \( i \) or \( i' \) never reaches the diagonal as the successive entries of the second row of \( \phi^\top \) are inserted to form \( P_{\text{SW}}^\phi(\phi^\top) \), so \( i' \) appears in this shifted tableau if and only if \( i' \) appears in the second row of \( \phi^\top \), or equivalently if \( a_i \) is primed.

We may assume by induction that \( P_{\text{HM}}^\phi(\phi_{\mid i-1}) = Q_{\text{SW}}^\phi((\phi_{\mid i-1})^\top) \). Thus, it follows in view of (3.7) that the conditions in the previous paragraph for \( i' \) to appear in \( P_{\text{SW}}^\phi(\phi^\top) \) are equivalent to the conditions in the one before it for \( i' \) to appear in \( Q_{\text{HM}}^\phi(\phi) \). We conclude that \( Q_{\text{HM}}^\phi(\phi) = P_{\text{SW}}^\phi(\phi^\top) \).
Now suppose that the position □₁ is on the diagonal. Denote the value of \( \text{unprime}_{\text{diag}}(P_{\text{HM}}^O(\phi)) \) \( \text{unprime}_{\text{diag}}(Q_{\text{SW}}^O(\phi^\top)) \) in this position by \( u \in \mathbb{Z} \). To show that \( P_{\text{HM}}^O(\phi) = Q_{\text{SW}}^O(\phi^\top) \), it suffices to show that \( u' \) appears in \( P_{\text{HM}}^O(\phi) \) if and only if \( u' \) appears in \( Q_{\text{SW}}^O(\phi^\top) \). Let \( j \in [n] \) be the first index such that \( u \) appears in position □₁ of \( \text{unprime}_{\text{diag}}(P_{\text{HM}}^O(\phi_j)) \). Observe that \( a_j \leq u \) and that if \( \psi \) is the primed biword formed from \( \phi \) by removing all columns \( \begin{bmatrix} k \\ a_k \end{bmatrix} \) with \( u < a_k \), then \( P_{\text{HM}}^O(\psi) \) has \( u \) or \( u' \) in position □₁ and \( Q_{\text{SW}}^O(\psi) \) has \( j \) in position □₁.

Similar to above, if any diagonal entry of \( \text{unprime}_{\text{diag}}(P_{\text{HM}}^O(\phi_{j-1})) \) differs from the corresponding entry of \( \text{unprime}_{\text{diag}}(P_{\text{HM}}^O(\phi_j)) \), and the last such entry occurs in position □₂, then \( u' \) appears in \( P_{\text{HM}}^O(\phi) \) if and only if the entry of \( P_{\text{HM}}^O(\phi_{j-1}) \) in position □₂ is primed. Otherwise \( u' \) appears in \( P_{\text{HM}}^O(\phi) \) if and only if \( a_j \) is primed.

Likewise, if any diagonal entry of \( \text{unprime}_{\text{diag}}(Q_{\text{SW}}^O((\phi_{j-1})^\top)) \) differs from the corresponding entry of \( \text{unprime}_{\text{diag}}(Q_{\text{SW}}^O((\phi_j)^\top)) \), and the last such entry occurs in position □₂, then \( u' \) appears in \( Q_{\text{SW}}^O(\phi^\top) \) if and only if the entry of \( Q_{\text{SW}}^O((\phi_{j-1})^\top) \) in position □₂ is primed. Assume no such position □₂ exists. Since \( \psi^\top \) consists of the first \( u \) columns of \( \phi^\top \), as the successive entries of the second row of \( \phi^\top \) are inserted to form \( Q_{\text{SW}}^O((\phi_j)^\top) \), the first entry to reach position □₁ in \( Q_{\text{SW}}^O((\phi_j)^\top) \) will be \( j \) or \( j' \), and this will result from inserting the entry from column \( u \) of \( \phi^\top \). Thus \( u' \) appears in \( Q_{\text{SW}}^O((\phi_j)^\top) \) if and only if \( j' \) appears in the second row of \( \phi^\top \), or equivalently if \( a_j \) is primed.

As in the first case of our argument, the conditions in the last two paragraphs are equivalent given (3.7) since we may assume that \( P_{\text{HM}}^O(\phi_{j-1}) = Q_{\text{SW}}^O((\phi_{j-1})^\top) \) by induction. Thus, we also have \( P_{\text{HM}}^O(\phi) = Q_{\text{SW}}^O((\phi_j)^\top) \). This proves the theorem when (3.6) holds, and it is easy to see that the result still holds as long as \(|\{i_1 < i_2 < \cdots < i_n\}| = |\{a_1, a_2, \ldots, a_n\}| = n\).

To finish the proof, suppose \( \phi \) is any index-strict primed biword with \( n \) columns. Form \( \psi \) from \( \phi \) by taking its transpose, then replacing the first row by the consecutive numbers \( 1 < 2 < \cdots < n \), and then taking the transpose again. For example, if

\[
\phi = \begin{bmatrix} 2 & 3 & 4 & 5 & 7 \\ 2' & 2 & 1 & 1' & 2' \end{bmatrix}
\text{ then } \psi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5' & 2' & 3 & 7' \end{bmatrix}^\top = \begin{bmatrix} 2 & 3 & 4 & 5 & 7 \\ 3' & 4 & 1 & 2' & 5' \end{bmatrix}.
\]

It is clear that \( P_{\text{SW}}^O((\phi_j)^\top) = P_{\text{SW}}^O((\psi_j)^\top) \) and also not hard to see that \( Q_{\text{HM}}^O(\phi) = Q_{\text{HM}}^O(\psi) \). Let \( \mathcal{F} : \{i' \leq 1 < 2' < \cdots < n' < n\} \to \{i' \leq 1 < 2' < \cdots < \} \) be the map with \( \mathcal{F}(i) = j \) and \( \mathcal{F}(i') = j' \) if \( j' \) is the entry in the first row of \( \phi_j \) in column \( i \). Then \( \phi \) is formed by applying \( \mathcal{F} \) to the second row of \( \psi \), and we have \( \mathcal{F}(Q_{\text{SW}}^O((\psi_j)^\top)) = Q_{\text{SW}}^O((\phi_j)^\top) \) and \( \mathcal{F}(P_{\text{HM}}^O(\psi)) = P_{\text{HM}}^O(\phi) \). As we already know that \( Q_{\text{HM}}^O(\psi) = P_{\text{SW}}^O((\psi_j)^\top) \) and \( P_{\text{HM}}^O(\psi) = Q_{\text{SW}}^O((\psi_j)^\top) \), the theorem follows.

There may be a way to extend Definitions 3.30 and 3.31 so that this theorem holds for all primed biwords, similar to what is done in [32, §3.4] for (unshifted) mixed insertion. We will not pursue this here, however.

Recall that we identify \( a = a_1 a_2 \cdots a_n \) with the (primed) biword \( \begin{bmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} \).

**Corollary 3.34.** The map \( a \mapsto (P_{\text{HM}}^O(a), Q_{\text{HM}}^O(a)) \) (respectively, \( a \mapsto (P_{\text{HM}}^S(a), Q_{\text{HM}}^S(a)) \)) is a bijection from the set of primed words with all positive letters to the set of pairs \((P, Q)\) of shifted tableaux of the same shape, in which \( P \) is semistandard, \( Q \) is standard, and \( Q \) (respectively, \( P \)) has no primed entries on the main diagonal.
Proof. Primed words with positive letters correspond to circled matrices with exactly one nonzero entry, given by 1 or 1', in each of the first \( \ell(a) \) rows, and no other nonzero rows. By Theorem 3.22 and the remark which follows it, the maps \( A \mapsto (P_{SW}^O(A), Q_{SW}^O(A)) \) and \( A \mapsto (P_{SW}^{SP}(A), Q_{SW}^{SP}(A)) \) are bijections from the set of transposes of such matrices to the set of pairs of shifted tableaux with the desired properties, just in the reverse order. The result therefore holds by Theorem 3.33.

4 Remaining proofs

This section is devoted to proving Theorem 3.16. We will also end up deriving Theorem 3.6 as a corollary of our methods in Section 4.2. Underpinning everything along the way to these proofs is the following lemma, which says that Theorem 3.16 holds for unprimed words.

Lemma 4.1 ([24, Thms. 3.31 and 5.11]). If \( i \in \mathbb{Z} \) and \( a = \text{unprime}(a) \) is an (unprimed) involution word for an element of \( I_{\mathbb{Z}} \) then \( P_{EG}^O(\text{ock}_a(a)) = P_{EG}^O(a) \) and \( Q_{EG}^O(\text{ock}_a(a)) = \omega(Q_{EG}^O(a)). \)

Remark. Neither \( \text{ock}^{-1} \) nor \( \omega^{-1} \) is defined in [24], but the identities \( P_{EG}^O(\text{ock}^{-1}(a)) = P_{EG}(a) \) and \( Q_{EG}^O(\text{ock}^{-1}(a)) = \omega(Q_{EG}^O(a)) \) are trivial to check directly (even for primed involution words \( a \)).

In view of the lemma, to prove Theorem 3.16 we just need to precisely understand the relationship between the indices of the primed letters in \( a \) and the locations of the primed entries in \( P_{EG}^O(a) \) and on the main diagonal of \( Q_{EG}^O(a) \). To this end, we will first prove an analogue of Lemma 3.21, showing that the positions of the relevant primes are controlled by a permutation \( \tau(a) \) that can be read off from the successive tableaux \( P_{EG}^O(a_1a_2\cdots a_i) \) for \( i \in [\ell(a)] \). Then, in Sections 4.3 and 4.4, we will prove a series of lemmas clarifying the relationship between \( \tau(a) \) and \( \tau(\text{ock}_a(a)) \).

4.1 Bumping paths

We start by listing some properties of the bumping paths in Definition 3.1. For \( (x, y) \in \mathbb{Z} \times \mathbb{Z} \), let \( \nabla(x, y) := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : x \geq i \text{ and } y \geq j \} \) and \( \Delta(x, y) := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : x \leq i \text{ and } y \leq j \} \).

In this subsection, let \( T \) be an increasing shifted tableau with no primes on the diagonal and let \( u \in \mathbb{Z} \sqcup \mathbb{Z}' \) be such that \( \text{row}(T)u \) is a primed involution word for an element of \( I_{\mathbb{Z}} \). Write

\[
\text{path}^\leq(T, u) := \{(x_i, y_i)\}_{i \in [m]} \quad \text{and} \quad \text{path}^\geq(T, u) := \{(\bar{x}_i, \bar{y}_i)\}_{i \in [m]} 
\]

(4.1)

for the bumping paths specified in Definition 3.3. The following observations are straightforward to derive from the definitions and Remark 3.5. We omit a detailed proof.

Proposition 4.2. The sequences \( \text{path}^\leq(T, u) \) and \( \text{path}^\geq(T, u) \) each contain at most one diagonal position. Let \( q \) be the unique index such that \( x_q = y_q \), or set \( q := m \) if no such index exists. Define \( \nabla(T, u) := \bigcup_{1 \leq i \leq q} \nabla(x_i, y_i) \) and \( \Delta(T, u) := \bigcup_{q < k \leq m} \Delta(x_i, y_i) \). Then the following holds:

(a) If \( 1 \leq i \leq q \) then \( x_i = \bar{x}_i = i \) and \( y_i \in \{y_i, y_i + 1\} \), while

\[
y_1 \geq y_2 \geq \cdots \geq y_q \quad \text{and} \quad \bar{y}_1 \geq \bar{y}_2 \geq \cdots \geq \bar{y}_q.
\]

(b) If \( q < k \leq m \) then \( y_k = \bar{y}_k = k \) and \( \bar{x}_k \in \{x_k, x_k + 1\} \), while

\[
q \geq x_{q+1} \geq x_{q+2} \geq \cdots \geq x_m \quad \text{and} \quad q + 1 \geq \bar{x}_q+1 \geq \bar{x}_q+2 \geq \cdots \geq \bar{x}_m.
\]
If \( (x_q, y_q) \neq (\tilde{x}_q, \tilde{y}_q) \), then \( q < m \) and \( \nabla(T, u) \cap \Delta(T, u) = \{(q, q + 1)\} \) and
\[
(q, q) = (x_q, y_q),
\]
\[
(q, q + 1) = (\tilde{x}_q, \tilde{y}_q) = (x_{q+1}, y_{q+1}),
\]
\[
(q + 1, q + 1) = (\tilde{x}_{q+1}, \tilde{y}_{q+1}).
\]

If instead \( (x_q, y_q) = (\tilde{x}_q, \tilde{y}_q) \), then \( \nabla(T, u) \cap \Delta(T, u) = \emptyset \).

We sometimes treat \( \text{path}^\leq(T, u) \) and \( \text{path}^\leq(T, u) \) as sets. This practice is justified as Proposition 4.2 shows that the positions in each path are distinct and their order is uniquely determined.

Define \( q \) as above to be the index of the row containing the unique diagonal position in \( \text{path}^\leq(T, u) \) or set \( q = m \) if no such row exists. Write
\[
\text{rpath}^\leq(T, u) := \{(x_i, y_i)\}_{i \in [q]} \quad \text{and} \quad \text{rpath}^\leq(T, u) := \{(\tilde{x}_i, \tilde{y}_i)\}_{i \in [q]}
\]
for the first \( q \) terms of \( \text{path}^\leq(T, u) \) and \( \text{path}^\leq(T, u) \), and let
\[
\text{cpath}^\leq(T, u) := \{(x_i, y_i)\}_{i \in [m]\setminus[q]} \quad \text{and} \quad \text{cpath}^\leq(T, u) := \{(\tilde{x}_i, \tilde{y}_i)\}_{i \in [m]\setminus[q]}
\]
We think of these subsequences as the “row-bumping paths” and “column-bumping paths” from inserting \( u \) into \( T \). Finally, if \( a = a_1a_2\cdots a_n \) is a primed involution word and \( i \in [n] \), then we let \( \text{path}^\leq_i(a) := \text{path}^\leq(T, a_i) \) and \( \text{path}^\leq_i(a) := \text{path}^\leq(T, a_i) \) for \( T := P_{\mathbb{E}}^0(a_1a_2\cdots a_{i-1}) \). We define the sequences \( \text{rpath}^\leq_i(a) \), \( \text{cpath}^\leq_i(a) \), \( \text{rpath}^\leq_i(a) \), and \( \text{cpath}^\leq_i(a) \) analogously.

**Proposition 4.3.** Let \( a = a_1a_2\cdots a_n \) be a primed involution word and choose \( i \in [n - 1] \).

(a) Suppose \( a_{i+1} < a_i \). In each row where \( \text{rpath}^\leq_i(a) \) and \( \text{rpath}^\leq_{i+1}(a) \) both have positions, the position of \( \text{rpath}^\leq_i(a) \) is weakly after that of \( \text{rpath}^\leq_{i+1}(a) \). Consequently, if \( \text{path}^\leq_i(a) \) has a diagonal position, then \( \text{path}^\leq_{i+1}(a) \) has a non-terminal diagonal position.

(b) Suppose \( a_i < a_{i+1} \). In each row where \( \text{rpath}^\leq_i(a) \) and \( \text{rpath}^\leq_{i+1}(a) \) both have positions, the position of \( \text{rpath}^\leq_i(a) \) is strictly before that of \( \text{rpath}^\leq_{i+1}(a) \). Consequently, if \( \text{path}^\leq_{i+1}(a) \) has a diagonal position, then \( \text{path}^\leq_i(a) \) has a non-terminal diagonal position.

**Proof.** Both parts are straightforward to check directly, using Remark 3.5 and Proposition 4.2. \( \square \)

### 4.2 Cycle sequences

A set of distinct integers \( \{j, k\} \) is a **cycle** of an element \( z \in I_Z \) if \( j \neq z(j) = k \). We denote the set of these pairs by \( \text{cyc}(z) \). For an unprimed word \( a = a_1a_2\cdots a_n \in \mathcal{R}_{\text{inv}}(z) \) and \( i \in [n] \), we define
\[
\gamma_i(a) := \begin{cases} 
    s_{a_{n-1}} \cdots s_{a_{i+2}} s_{a_{i+1}} \{a_i, a_{i+1}\} & \text{if } i \text{ is a commutation in } a \\
    \emptyset & \text{otherwise.} 
\end{cases}
\]

The map \( i \mapsto \gamma_i(a) \) is a bijection from the set of commutations in \( a \) to \( \text{cyc}(z) \). For primed involution words \( a \in \mathcal{R}_{\text{inv}}^+(z) \), let \( \gamma_i(a) := \gamma_i(\text{unprime}(a)) \) and \( \text{marked}(a) := \{\gamma_i(a) : i \in [\ell(a)] \text{ with } a_j \in Z'\} \).

For example, if \( z = 65431 \in I_6 \subset I_Z \), then \( \text{cyc}(z) = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\} \) and \( a = 513243541 \in \mathcal{R}_{\text{inv}}(z) \), and \( \gamma_1(a) = \{3, 4\}, \gamma_2(a) = \{2, 5\}, \gamma_3(a) = \{1, 6\} \), while \( \gamma_i(a) = \emptyset \) for \( i \in \{4, 5, 6, 7, 8, 9\} \).
Suppose $T$ is a shifted tableau and $b$ is a word with letters in $\mathbb{Z} \sqcup \mathbb{Z}'$ such that $\text{row}(T)b$ is an (unprimed) involution word. For each position $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, define

$$
\gamma_{ij}(T,b) := \begin{cases} 
\emptyset & \text{if } (i,j) \text{ is not in the domain of } T \\
\gamma_k(a) & \text{if } (i,j) \text{ is in the domain of } T,
\end{cases}
$$

where in the second case $a := \text{row}(T)b$ and $k := 1 + j - i + \lambda_{j+1} + \lambda_{j+2} + \cdots + \lambda_q$ is the index of the letter in $a$ corresponding to the entry of $T$ in box $(i,j)$. We also let $\gamma_{ij}(T) := \gamma_{ij}(T,\emptyset)$.

**Example 4.4.** If $T = P_{\text{EG}}^O(51324) = \begin{array}{ccc} 
1 & 2 & 4 \\
3 & 5 & \emptyset
\end{array}$ and $b = 3154$ then, abbreviating by writing $\gamma_{ij}$ in place of $\gamma_{ij}(T,b)$ and $pq$ in place of $\{p < q\}$, we have $\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} = \begin{bmatrix} 16 & 34 \\
25 & \emptyset & \emptyset \end{bmatrix}$.

In the next lemma, we suppose that $T$ is an increasing shifted tableau and $b$ is a word such that $\text{row}(T)b \in R_{\text{av}}(z)$ for some $z \in I_{\mathbb{Z}}$. Assume $b$ is nonempty with first letter $u$, and let $c$ be the subword formed by removing this letter. Denote the weak and strict bumping paths resulting from $a$ corresponding to the entry of $T$ in box $(i,j)$. We also let $\gamma_{ij}(T) := \gamma_{ij}(T,\emptyset)$.

**Example 4.5.** Let $T = P_{\text{EG}}^O(51324)$ and $b = 3154$ as in Example 4.4. Then $u = 3$ and

$$
\text{path}^\leq(T,u) = \{(x_i,y_i)\}_{i \in [m]} = \{(1,3),(2,3),(3,3)\},
\text{path}^\leq(T,u) = \{(\tilde{x}_i,\tilde{y}_i)\}_{i \in [m]} = \{(1,3),(2,3),(3,3)\},
$$

so $u_0 = 3$, $u_1 = 4$, and $u_2 = 5$, while $\theta_0 = \emptyset$, $\theta_1 = \emptyset$, and $\theta_2 = (3,4)$.

**Lemma 4.6.** For each position $(x,y)$ in the domain of $U := T \leftarrow^O u$, the following holds:

(a) If $(x,y) = (x_i,y_i) = (\tilde{x}_i,\tilde{y}_i)$ for some $i \in [m]$, then

$$
U_{xy} = u_{i-1} \quad \text{and} \quad \gamma_{xy}(U,c) = \begin{cases} 
\gamma_{xy}(T,b) & \text{if } x = y \text{ and } i < m \text{ and } u_{i-1} + 1 = u_i \\
\theta_{i-1} & \text{otherwise}.
\end{cases}
$$

(b) If $(x,y) \in \{(x_i,y_i) \neq (\tilde{x}_i,\tilde{y}_i)\}$ for some $i \in [m]$ with $x_i \neq y_i$ and $\tilde{x}_i \neq \tilde{y}_i$, then

$$
U_{xy} = T_{xy} \quad \text{and} \quad \gamma_{xy}(U,c) = \begin{cases} 
\gamma_{x_i,y_i}(T,b) & \text{if } (x,y) = (x_i,y_i) \\
\gamma_{\tilde{x}_i,\tilde{y}_i}(T,b) & \text{if } (x,y) = (\tilde{x}_i,\tilde{y}_i).
\end{cases}
$$

(c) If $(x,y) \in \{(i,i) , (i,i+1) , (i+1,i+1)\}$ for some $i \in [m]$ with $x_i = y_i \neq \tilde{y}_i$, then

$$
U_{xy} = T_{xy} \quad \text{and} \quad \gamma_{xy}(U,c) = \begin{cases} 
\gamma_{i+1,i+1}(T,b) & \text{if } (x,y) = (i,i) \\
\gamma_{i,i+1}(T,b) & \text{if } (x,y) = (i,i+1) \\
\gamma_{ii}(T,b) & \text{if } (x,y) = (i+1,i+1) \neq \emptyset.
\end{cases}
$$
(d) Otherwise, \((x, y) \notin \text{path}^\leq(T, u) \cup \text{path}^\leq(T, u)\), \(U_{xy} = T_{xy}\), and \(\gamma_{xy}(U, c) = \gamma_{xy}(T, b)\).

Proof. Suppose \(a\) is any involution word and \(i \in [\ell(a)]\). If \(\ell(a) \geq 2\) then

\[
\gamma_i(\text{ock}(a)) = \begin{cases} 
\gamma_2(a) & \text{if } i = 1 \text{ and } |a_1 - a_2| > 1 \\
\gamma_1(a) & \text{if } i = 2 \text{ and } |a_1 - a_2| > 1 \\
\gamma_i(a) & \text{otherwise},
\end{cases}
\]

while if \(j \in [\ell(a) - 2]\) then \(\gamma_i(\text{ock}_j(a)) = \begin{cases} 
\gamma_q(a) & \text{if } i = p \\
\gamma_p(a) & \text{if } i = q \\
\gamma_i(a) & \text{otherwise}
\end{cases}\) for \(p\) and \(q\) defined by

\[
\{p, q\} := \begin{cases} 
\{j, j + 1\} & \text{if } \text{ock}(a_ja_{j+1}a_{j+2}) = a_{j+1}a_ja_{j+2} \\
\{j + 1, j + 2\} & \text{if } \text{ock}(a_ja_{j+1}a_{j+2}) = a_ja_{j+2}a_{j+1} \\
\{j, j + 2\} & \text{if } \text{ock}(a_ja_{j+1}a_{j+2}) = a_{j+1}a_ja_{j+1} = a_ja_{j+1}a_{j+1} \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Given these observations and Proposition 2.3, it is straightforward to derive the lemma by following the proof of Proposition 3.14. We omit the explicit details. \(\square\)

Continuing our notation from above, define \(p\) to be the index of the unique diagonal position in \(\text{path}^\leq(T, u)\) or set \(p = m\) if not such index exists. Define the sequence

\[
\Delta^{\text{bump}}(T, b) := \{(y_i, \tilde{y}_i, u_{i-1}, \theta_{i-1})\}_{i \in [p]}.
\]

From Example 4.5, we see that if \(T = P^O_{\text{EG}}(51324)\) and \(b = 3154\) then \(p = 3\) and \(\Delta^{\text{bump}}(T, b) = \{(1, 1, 3, 0), (2, 2, 4, 0), (3, 3, 5, (3, 4))\}\). We think of this as a record of the change between \(T \leftarrow u\) and \(T\), and we can use it to compute successive values of \(\theta_i\) by the formula

\[
\theta_i = \begin{cases} 
\gamma_{i,y_i}(T, b) & \text{if } y_i = \tilde{y}_i \text{ and either } i \neq y_i \text{ or } u_{i-1} + 1 < u_i \\
\theta_{i-1} & \text{otherwise}
\end{cases}
\]

for \(i \in [p - 1]\). For any involution word \(a = a_1a_2 \cdots a_n\) and \(j \in [n]\), define \(\Delta^{\text{bump}}_j(a) := \Delta^{\text{bump}}(T, b)\) where \(T = P^O_{\text{EG}}(a_1a_2 \cdots a_{j-1})\) and \(b = a_ja_{j+1} \cdots a_n\).

Assume \(T\) is a shifted tableau with \(q\) rows and \(b\) is a word such that \(\text{row}(T)b\) is a (unprimed) involution word. We define the cycle sequence \(\text{cseq}(T, b)\) to be the two-line array

\[
\text{cseq}(T, b) := \begin{bmatrix} 
\gamma_{11}(T, b) & \gamma_{22}(T, b) & \cdots & \gamma_{qq}(T, b) \\
T_{11} & T_{22} & \cdots & T_{qq}
\end{bmatrix}.
\]

For involution words \(a = a_1a_2 \cdots a_n\) and \(0 \leq i \leq n\), we define \(\text{cseq}_i(a) := \text{cseq}(T, b)\) where \(T = P^O_{\text{EG}}(a_1a_2 \cdots a_i)\) and \(b = a_{i+1}a_{i+2} \cdots a_n\). If \(T = P^O_{\text{EG}}(51324)\) and \(b = 3154\) as in Example 4.4 then

\[
\text{cseq}(T, b) = \begin{bmatrix} 
\{2, 5\} & \{1, 6\} \\
1 & 3
\end{bmatrix} = \text{cseq}_5(513243154).
\]

When \(T\) is increasing and \(b\) is a word such that \(\text{row}(T)b \in \mathcal{R}_{\text{inv}}(z)\) for some \(z \in I_Z\), the second row of \(\text{cseq}(T, b)\) is strictly increasing and the elements in the first row are distinct cycles of \(z\), since the index of \(T_{ii}\) in \(\text{row}(T)b\) is a commutation for all diagonal positions \((i, i)\) in \(T\).
Lemma 4.7. Let \( a \) be an (unprimed) involution word and choose \( j \in [\ell(a)] \). Suppose

\[
cseq_{j-1}(a) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_q \\ c_1 & c_2 & \cdots & c_q \end{bmatrix} \quad \text{and} \quad \Delta_j^{\text{bump}}(a) = \{(y_i, \tilde{y}_i, u_{i-1}, \theta_{i-1})\}_{i \in [q]}.
\]

Exactly one of the following cases applies:

(a) The sequence \( \text{path}^\leq_j(a) \) ends before reaching the diagonal if and only \( p < y_p \). In this case \( i \) appears in \( Q^\sigma_{\text{EG}}(a) \) in an off-diagonal position and \( \text{cseq}_j(a) = \text{cseq}_{j-1}(a) \).

(b) The sequence \( \text{path}^\leq_j(a) \) terminates on the diagonal if and only \( p = y_p = \tilde{y}_p = q + 1 \). In this case \( i \) appears in \( Q^\sigma_{\text{EG}}(a) \) in position \((q + 1, q + 1)\) and

\[
\text{cseq}_j(a) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_q & \theta_q \\ c_1 & c_2 & \cdots & c_q & u_q \end{bmatrix}.
\]

(c) The sequences \( \text{path}^\leq_j(a) \) and \( \text{path}^\leq_i(a) \) reach (but do not terminate on) the diagonal in the same row if and only \( p = y_p = \tilde{y}_p \leq q \). In this case \( i' \) appears in \( Q^\sigma_{\text{EG}}(a) \) and we have

\[
u_{p-1} + 1 \leq c_p \quad \text{and} \quad \text{cseq}_j(a) = \begin{bmatrix} \gamma_1 & \cdots & \gamma_p-1 & \eta & \gamma_{p+1} & \cdots & \gamma_q \\ c_1 & \cdots & c_{p-1} & u_{p-1} & c_{p+1} & \cdots & c_q \end{bmatrix},
\]

where \( \eta := \gamma_p \) if \( u_{p-1} + 1 = c_p \) and \( \eta := \theta_{p-1} \) if \( u_{p-1} + 1 < c_p \).

(d) The sequences \( \text{path}^\leq_j(a) \) and \( \text{path}^\leq_i(a) \) reach the diagonal in different rows if and only \( p = y_p < \tilde{y}_p = p + 1 \leq q \). In this case \( i' \) appears in \( Q^\sigma_{\text{EG}}(a) \) and we have

\[
u_{p-1} = c_p \quad \text{and} \quad \text{cseq}_j(a) = \begin{bmatrix} \gamma_1 & \cdots & \gamma_{p-1} & \gamma_p & \gamma_{p+1} & \cdots & \gamma_q \\ c_1 & \cdots & c_{p-1} & c_p & c_{p+1} & \cdots & c_q \end{bmatrix}.
\]

Proof. The assertion that exactly one of these cases applies follows from Proposition 4.2. The claims about \( u_{p-1} \) in cases (c) and (d) are clear from how \( P^\sigma_{\text{EG}}(a_1a_2 \cdots a_{j-1}) \rightarrow a_j \) is defined. The description of \( \text{cseq}_j(a) \) is immediate from the formulas in Lemma 4.6. \( \square \)

Putting all of this together, we associate a permutation of \((\gamma_q^2) := \{(i, j) : i, j \in \mathbb{Z}, i < j\}\) to each involution word. This is similar to the definition of \( \tau_{\text{SW}}(a) : \mathbb{Z} \to \mathbb{Z} \) from Section 3.3. Let \( a = a_1a_2 \cdots a_n \) be an (unprimed) involution word for some \( z \in I_{\mathbb{Z}} \). For each \( i \in [n] \), let \( \tau_i(a) \) be the following permutation of \((\gamma_q^2) \) with support in \( \text{cyc}(z) \). If \( \text{cseq}_{i-1}(a) \) and \( \text{cseq}_i(a) \) are equal or have different lengths then \( \tau_i(a) := 1 \). Otherwise, writing

\[
cseq_{i-1}(a) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_q \\ c_1 & c_2 & \cdots & c_q \end{bmatrix} \quad \text{and} \quad \text{cseq}_i(a) = \begin{bmatrix} \eta_1 & \eta_2 & \cdots & \eta_q \\ d_1 & d_2 & \cdots & d_q \end{bmatrix},
\]

there is either a unique index \( j \in [q] \) with \( d_j < c_j \), or a unique index \( j \in [q - 1] \) with \( \gamma_{j+1} = \eta_j \neq \gamma_j = \eta_{j+1} \), and in both cases we define \( \tau_i(a) \) to be the transposition of \((\gamma_q^2) \) that swaps \( \eta_j \) and \( \gamma_j \) while fixing all other elements. We then let \( \tau(a) := \tau_1(a)\tau_2(a) \cdots \tau_n(a) \).

When \( a \) is a primed involution word for some element of \( I_{\mathbb{Z}} \), we set \( \text{cseq}_i(a) := \text{cseq}_i(\text{unprime}(a)) \) along with \( \tau_i(a) := \tau_i(\text{unprime}(a)) \) and \( \tau(a) := \tau(\text{unprime}(a)) \).
Example 4.8. Suppose $a = 513243154$. This word belongs to $\mathcal{R}_{inv}(z)$ for $z = (1,6)(2,5)(3,4) \in I_Z$. The successive values of $P_{EG}^O(a_1a_2 \cdots a_t)$ are

\[
\begin{array}{cccccccc}
5 & 1 & 5 & 1 & 3 & 1 & 2 & 5 \\
3 & 5 & 1 & 2 & 4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2 & 3 & 1 & 2 & 3 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

and the corresponding values of $\gamma_x(T, b)$ for $T = P_{EG}^O(a_1a_2 \cdots a_t)$ and $b = a_{i+1}a_{i+2} \cdots a_9$ are

\[
\begin{array}{cccccccc}
34 & 25 & 34 & 25 & 16 & 25 & 0 & 34 \\
34 & 16 & 34 & 25 & 0 & 34 & 25 & 0 \\
16 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Thus, we have

\[
cseq_1(a) = \left[ \begin{array}{c}
3, 4 \\
5
\end{array} \right], \quad cseq_2(a) = cseq_4(a) = \left[ \begin{array}{c}
2, 5 \\
1
\end{array} \right], \quad cseq_3(a) = \left[ \begin{array}{c}
2, 5 \\
1
\end{array} \right],
\]

\[
cseq_5(a) = \left[ \begin{array}{c}
1, 6 \\
3
\end{array} \right], \quad cseq_6(a) = \left[ \begin{array}{c}
2, 5 \\
1
\end{array} \right], \quad cseq_7(a) = cseq_8(a) = cseq_9(a) = \left[ \begin{array}{c}
1, 6 \\
3
\end{array} \right],
\]

which means that $\tau_1(a) = \tau_3(a) = \tau_5(a) = \tau_6(a) = \tau_8(a) = \tau_9(a) = 1$ while

$$
\tau_2(a) = (\{2, 5\} \leftrightarrow \{3, 4\}), \quad \tau_4(a) = (\{1, 6\} \leftrightarrow \{3, 4\}), \quad \text{and} \quad \tau_7(a) = (\{1, 6\} \leftrightarrow \{2, 5\}),
$$

so we have $\tau(a) = (\{1, 6\} \leftrightarrow \{3, 4\})$.

Lemma 4.9. Suppose $a \in \mathcal{R}_{inv}^+(z)$ is a primed involution word for some $z \in I_Z$. Let $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ and $\theta = \gamma_{ij}(P_{EG}^O(a))$. If $i \neq j$ (respectively, $i = j$), then the entry of $P_{EG}^O(a)$ (respectively $Q_{EG}^O(a)$) in position $(i, j)$ is primed if and only if $\theta \in cyc(z)$ and $\tau(a)(\theta) \in \text{marked}(a)$.

Proof. Assume $n = \ell(a)$ and $i \in \{-1, 0, 1, 2, \ldots, n-2\}$. It is easy to check that $\text{marked}(a) \neq \text{marked}(\text{ock}_{i}(a))$ if and only if (1) $i = -1$ or (2) $i = 0$ and $|[a_1] - [a_2]| > 1$ and exactly one of $a_1$ or $a_2$ is primed. In case (1) we have $\gamma_1(a) \in cyc(z)$ and $\text{marked}(\text{ock}_{-1}(a)) = \text{marked}(a) \triangle \{\gamma_1(a)\}$, where $\triangle$ denotes the symmetric set difference. In case (2) exactly one of $\gamma_1(a)$ or $\gamma_2(a)$ belongs to marked($z$) $\subseteq$ cyc($z$) and we have marked($\text{ock}_{0}(a)$) = marked($a$) $\triangle$ $\{\gamma_1(a), \gamma_2(a)\}$.

Form $P_{EG}^O(a)$ from $P_{EG}(a)$ by adding primes to the diagonal positions that are primed in $Q_{EG}^O(a)$. We will show that the entry in position $(x, y)$ of $P_{EG}^O(a)$ is primed if and only if $\theta := \gamma_{xy}(P_{EG}^O(a))$ has $\theta \in cyc(z)$ and $\tau(a)(\theta) \in \text{marked}(a)$. Define $T^j := P_{EG}^O(a_1a_2 \cdots a_j)$ and $b^j := a_{j+1}a_{j+2} \cdots a_n$ for $0 \leq j \leq n$, and abbreviate by writing marked($T^j, b^j$) := marked($\text{row}(T^j)b^j$). It suffices to check that marked($T^j, b^j$) $\subseteq$ $\{\gamma_j(a)(\theta) : \theta \in \text{marked}((T^{j-1}, b^{j-1})) \}$ for all $j \in [n]$, since this will imply that marked($\text{row}(P_{EG}^O(a))$) $\subseteq$ $\{\theta : \tau(a)(\theta) \in \text{marked}(a)\}$.

Let $\sim$ be the transitive closure of the relation on primed involution words that has $w \sim \text{ock}_i(w)$ for all $i \in \mathbb{Z}$ such that marked($w$) = marked($\text{ock}_i(w)$). In Lemma 4.7, if we are in case (a), (b), or (c) with $\eta = \gamma_p$, then $\tau_1(a) = 1$ and it follows by tracing through the proof of Proposition 3.14 that we have $\text{row}(T^{j-1})b^{j-1} \sim \text{row}(T^j)b^j$ as needed.
If we are in case (c) of Lemma 4.7 with \( \eta \neq \gamma_p \), then \( \tau_j(a) \) is the transposition of \( \text{cseq}_i \) interchanging \( \eta \leftrightarrow \gamma_p \), and it follows similarly that \( \text{marked}(T^j, b^j) \) is formed by applying this transposition to all elements of \( \text{marked}(T^{j-1}, b^{j-1}) \).

Finally, suppose we are in case (d) of Lemma 4.7, so that \( \tau_j(a) = (\gamma_p \leftrightarrow \gamma_{p+1}) \). Form \( U^j \) from \( T^j \) by reversing the primes on the diagonal entries in positions \((p, p)\) and \((p+1, p+1)\) if these entries are not both primed or both unprimed, and otherwise set \( U_j := T^j \). Then, again following the proof of Proposition 3.14, one can check that \( \text{row}(T^{j-1})b^{j-1} \sim \text{row}(U^j)b^j \) where \( \sim \) is the relation defined in the paragraph above. It follows that \( \text{marked}(T^{j-1}, b^{j-1}) = \text{marked}(U^j, b^j) = \{ \tau_j(a)(\theta) : \theta \in \text{marked}(T^{j-1}, b^{j-1}) \} \) as desired.

We may now prove Theorem 2.6. Remark 3.5 and Proposition 3.14 imply that if \( a \in \mathcal{R}_{\text{inv}}^+(z) \) for some \( z \in I_Z \), then \( P_{\text{EG}}^O(a) \) is an increasing shifted tableau with no primes on the diagonal whose row reading word is in \( \mathcal{R}_{\text{inv}}^+(z) \); it follows by definition that \( Q_{\text{EG}}^O(a) \) is a standard shifted tableau of the same shape. Let \((P, Q)\) be an arbitrary pair of shifted tableaux with these properties. The unprimed form \([12, \text{Thm. 5.19}] \) of the result to prove asserts that there is a unique unprimed word \( a \in \mathcal{R}_{\text{inv}}(z) \) with \( P_{\text{EG}}^O(a) = \text{unprime}(P) \) and \( Q_{\text{EG}}^O(a) = \text{unprime}_{\text{diag}}(Q) \). Since we have \( \gamma_{ii}(P) \in \text{cseq}(z) \) for all diagonal positions \((i, i)\) in \( P \), Lemma 4.9 implies that there is a unique way to assign primes to the commutations in \( a \) to obtain a primed word \( b \in \mathcal{R}_{\text{inv}}^+(z) \) with \( P_{\text{EG}}^O(b) = P \) and \( Q_{\text{EG}}^O(b) = Q \). \( \square \)

### 4.3 Some reductions

In this section we prove three technical results constraining the values of \( \text{cseq}_i(a) \) and \( \tau_i(a) \). Let \( \text{entries}(T) \subset Z \sqcup Z' \) denote the set of entries in a shifted tableau \( T \). Let \( \text{diag}(T) \) denote the subset of entries appearing on the main diagonal of \( T \). Our first result is the following:

**Lemma 4.10.** Suppose \( a \) and \( b \) are involution words for elements of \( I_Z \). Fix \( 0 \leq i \leq \ell(a) - 2 \) with \( a_{i+1} < a_{i+2} \) and suppose \( 0 \leq j \leq \ell(b) - 2 \) is an index such that the following holds:

(a) \( \text{cseq}_{i+2}(a) = \text{cseq}_{j+2}(b) \).

(b) \( |\text{diag}(Q_{\text{EG}}^O(a)) \cap \{i + 1, i + 2\}| = |\text{diag}(Q_{\text{EG}}^O(b)) \cap \{j + 1, j + 2\}| \), and

(c) \( |\text{entries}(Q_{\text{EG}}^O(a)) \cap \{i + 1, i + 2\}| = |\text{entries}(Q_{\text{EG}}^O(b)) \cap \{j + 1, j + 2\}| \).

Then \( \tau_{i+1}(a)\tau_{i+2}(a) = \tau_{j+1}(b)\tau_{j+2}(b) \) as permutations of \( \binom{Z}{2} \).

**Proof.** Let \( s(a) := |\text{diag}(Q_{\text{EG}}^O(a)) \cap \{i + 1, i + 2\}| \in \{0, 1\} \) be the number of diagonal entries in \( Q_{\text{EG}}^O(a) \) equal to \( i + 1 \) or \( i + 2 \) and let \( r(a) := 2 - |\text{entries}(Q_{\text{EG}}^O(a)) \cap \{i + 1, i + 2\}| \in \{0, 1, 2\} \) be the number of (necessarily off-diagonal) entries in \( Q_{\text{EG}}^O(a) \) equal to \( i' + 1 \) or \( i' + 2 \). Similarly let \( s(b) \in \{0, 1\} \) be the number of diagonal entries in \( Q_{\text{EG}}^O(b) \) equal to \( j + 1 \) or \( j + 2 \) and let \( r(b) \in \{0, 1, 2\} \) be the number of entries in \( Q_{\text{EG}}^O(b) \) equal to \( j' + 1 \) or \( j' + 2 \).

Conditions (b) and (c) imply that \( r(a) = r(b) \) and \( s(a) = s(b) \). The key idea in the proof of this lemma is to observe how this fact combined with Lemma 4.7 limits the possible values of \( \text{cseq}_{i+1}(a) \) and \( \text{cseq}_{j+1}(b) \) once \( \text{cseq}_{i+2}(a) = \text{cseq}_{j+2}(b) \) are given. We will then deduce that \( \tau_{i+1}(a)\tau_{i+2}(a) = \tau_{j+1}(b)\tau_{j+2}(b) \) from these constraints.

From now on set \( r := r(a) = r(b) \) and \( s := s(a) = s(b) \). The desired equality clearly holds when \( r = 0 \) since then \( \tau_{i+1}(a) = \tau_{i+2}(a) = \tau_{j+1}(b) = \tau_{j+2}(b) = 1 \) by Lemma 4.7.
Instead assume \( r = 1 \). Then at least one of \( \tau_{i+1}(a) \) or \( \tau_{i+2}(a) \) is trivial, and likewise for \( \tau_{j+1}(b) \) or \( \tau_{j+2}(b) \). Suppose further that \( s = 0 \). Then \( c\text{seq}_i(a) = c\text{seq}_j(b) \) and \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) have the same number of columns, and we have at least one of \( c\text{seq}_i(a) = c\text{seq}_{i+1}(a) \) or \( c\text{seq}_{i+1}(a) = c\text{seq}_{i+2}(a) \), as well as at least one of \( c\text{seq}_j(b) = c\text{seq}_{j+1}(b) \) or \( c\text{seq}_{j+1}(b) = c\text{seq}_{j+2}(b) \). Write

\[
c\text{seq}_i(a) = c\text{seq}_j(b) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_q \\ c_1 & c_2 & \cdots & c_q \end{bmatrix}
\]

and suppose the first row of \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) is \( [\eta_1 \quad \eta_2 \quad \cdots \quad \eta_q] \). If this is equal to the first row of \( c\text{seq}_i(a) = c\text{seq}_j(b) \), then it is easy to see from Lemma 4.7 that \( \tau_{i+1}(a) = \tau_{i+2}(a) = \tau_{j+1}(b) = \tau_{j+2}(b) = 1 \). Otherwise, Lemma 4.7 implies that there is either a unique index \( p \in [q] \) with \( \gamma_p \neq \eta_p \), or a unique \( p \in [q-1] \) with \( \gamma_{p+1} = \eta_p \neq \gamma_p = \eta_{p+1} \), and in either case \( \tau_{i+1}(a) \tau_{i+2}(a) = \tau_{j+1}(b) \tau_{j+2}(b) \) is the permutation of \( \binom{q}{2} \) swapping \( \gamma_p \) and \( \eta_p \).

Next suppose \( r = s = 1 \). Consider the weak bumping paths \( \text{path}_{i+1}(a) \) and \( \text{path}_{i+2}(a) \) that result from inserting \( a_{i+1} \) and \( a_{i+2} \) successively into \( P_{\mathcal{E}}(a_1 a_2 \cdots a_i) \). Since \( a_{i+1} < a_{i+2} \), it follows from Proposition 4.3 that \( \text{path}_{i+1}(a) \) terminates at a diagonal position \( (q+1, q+1) \) and \( \text{path}_{i+1}(a) \) contains a unique non-terminal diagonal position \((p, p)\) for some \( p \in [q] \). Denote \( c\text{seq}_i(a) = c\text{seq}_j(b) \) as in (4.9). There are four possibilities for \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \), namely:

\[
\begin{align*}
\begin{bmatrix} \gamma_1 & \cdots & \gamma_p & \cdots & \gamma_q \\ c_1 & \cdots & c_p - 1 & \cdots & c_q \end{bmatrix} & \quad \text{or} \quad \begin{bmatrix} \gamma_1 & \cdots & \eta_p & \cdots & \gamma_q \\ c_1 & \cdots & d_p & \cdots & c_q \end{bmatrix} \\
\begin{bmatrix} \gamma_1 & \cdots & \gamma_p & \cdots & \gamma_q \\ c_1 & \cdots & c_p & \cdots & c_q \end{bmatrix} & \quad \text{or} \quad \begin{bmatrix} \gamma_1 & \cdots & \eta_p & \cdots & \gamma_q \\ c_1 & \cdots & d_p & \cdots & c_q \end{bmatrix}
\end{align*}
\]

(4.10)

where \( \eta_p, \eta_{q+1} \notin \{\gamma_1, \gamma_2, \ldots, \gamma_q\} \) and \( d_p < c_p - 1 \). In each case, it is straightforward to work out the unique possibility for \( c\text{seq}_{i+1}(a) \) from Lemma 4.7.

As we pass from \( c\text{seq}_j(b) \) to \( c\text{seq}_{j+1}(b) \) to \( c\text{seq}_{j+2}(b) \), it follows from Lemma 4.7 that one step must add an extra column and the other must alter the first \( q \) columns either by changing a single column or swapping adjacent entries in the first row. From this observation, we deduce that if \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) has one of the first three forms in (4.10), then there are two possibilities for \( c\text{seq}_{i+1}(a) \), but in either case the factors \( \tau_{i+1}(a) \) and \( \tau_{i+2}(a) \) commute and \( \tau_{i+1}(a) \tau_{i+2}(a) = \tau_{i+1}(b) \tau_{i+2}(b) \) is respectively either the identity permutation, the transposition \( (\gamma_p, \eta_p) \), or the transposition \( (\gamma_p, \gamma_{p+1}) \). If \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) has the last form in (4.10) then

\[
c\text{seq}_{i+1}(a) = c\text{seq}_{j+1}(b) = \begin{bmatrix} \gamma_1 & \cdots & \eta_p & \cdots & \gamma_q \\ c_1 & \cdots & d_p & \cdots & c_q \end{bmatrix}
\]

(4.11)

so \( \tau_{i+1}(a) = \tau_{j+1}(b) \) and \( \tau_{i+2}(a) = \tau_{j+2}(b) \).

Finally suppose \( r = 2 \) so that \( s = 0 \). Then \( c\text{seq}_i(a) = c\text{seq}_j(b) \) and \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) have the same number of columns but \( c\text{seq}_i(a) \neq c\text{seq}_{i+1}(a) \neq c\text{seq}_{i+2}(a) \) and \( c\text{seq}_j(b) \neq c\text{seq}_{j+1}(b) \neq c\text{seq}_{j+2}(b) \). Denote \( c\text{seq}_i(a) = c\text{seq}_j(b) \) as in (4.9) and consider the weak bumping paths \( \text{path}_{i+1}(a) \) and \( \text{path}_{i+2}(a) \) that result from inserting \( a_{i+1} \) and \( a_{i+2} \) successively into \( P_{\mathcal{E}}(a_1 a_2 \cdots a_i) \). Both paths now must contain unique non-terminal diagonal positions \((k, k)\) and \((l, l)\), and it follows from Proposition 4.3 that \( k < l \) since we assume \( a_{i+1} < a_{i+2} \). We may therefore list the possibilities for \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) as follows. To start, this array could be

\[5\]If \( p = q \) in this case, then Lemma 4.7 given \( c\text{seq}_i(a) = c\text{seq}_j(b) \) and \( c\text{seq}_{i+2}(a) = c\text{seq}_{j+2}(b) \) does not uniquely determine the first row of \( c\text{seq}_{i+1}(b) \). But considering the arrays’ second rows shows that (4.11) still must hold.
Lemma 4.11. Let \(a, b, c\) be unprimed words with \(n := \ell(a)\). Suppose \(u, v \in \mathbb{Z}\) are such that

(a) \(auvb\) and \(avuc\) are involution words for elements of \(I_Z\), and

(b) \(uvb\) and \(vuc\) are reduced words for the same permutation in \(S_Z\).

Let \(T := P_{EG}^O(a)\) and assume \(\text{path}^<(T, u) \cap \text{path}^<(T, v)\) contains an off-diagonal position. Then \(cseq_{n+1}(auvb) = cseq_{n+1}(avuc)\); also, \(n + 1\) is on the diagonal in \(Q_{EG}^O(auvb)\) if and only if \(n + 1\) is on the diagonal in \(Q_{EG}^O(avuc)\), while \(n' + 1\) is in \(Q_{EG}^O(auvb)\) if and only if \(n' + 1\) is in \(Q_{EG}^O(avuc)\).
Condition (b) implies that $auvb$ and $avuc$ are involution words for the same $z \in I_Z$.

**Proof.** Suppose $rpath^k(T, u) \cap rpath^k(T, v)$ is nonempty and the first position in this intersection is $(j, k)$. We claim that $(j, k)$ also belongs to $rpath^k(T, u) \cap rpath^k(T, v)$. To check this, write $u_0 := u < v_0 := v$ and let $u_i$ and $v_i$ be the entries of $T$ in the $i$th positions of $rpath^k(T, u)$ and $rpath^k(T, v)$ respectively. Then $u_{j-1} < v_{j-1}$ and the smallest entry of $T$ in row $j$ that is greater than both of these numbers is $u_j = v_j$ by definition. This means that row $j$ of $T$ cannot contain any entry $w$ with $u_{j-1} < w \leq v_{j-1}$, so by Remark 3.5, row $j$ of $T$ also cannot contain $u_{j-1}$. Hence $(j, k) \in rpath^k(T, u) \cap rpath^k(T, v)$ as desired.

It is clear from Definition 3.1 that $rpath^k(T, u)$ and $rpath^k(T, v)$ coincide after their first $j - 1$ positions, and it follows by our claim that $rpath^j(T, u)$ and $rpath^j(T, v)$ also coincide after their first $j - 1$ positions. If $j \neq k$, then all of these paths continue after row $j$, and we have $\gamma_{xy}(T, uvb) = \gamma_{xy}(T, vuc)$ for all positions $(x, y)$ since $uvb$ and $vuc$ are reduced words for the same permutation. Given these observations, the result follows from Lemma 4.7.

Our last result in this section requires a longer argument.

**Lemma 4.12.** Suppose $a, b$ are unprimed words and $u, v \in Z$ are such that $u + 1 < v$ and $auvb$ is an involution word for an element of $I_Z$. Let $T = P^0_{EG}(a)$ and $n = \ell(a)$, and assume $rpath^k(T, u)$ and $rpath^k(T, v)$ are disjoint. Then $cseq_{n+2}(auvb) = cseq_{n+2}(avub)$, and for each $\epsilon \in \{0, 1\}$, the number $n + 1 + \epsilon$ is on the diagonal in $Q_{EG}^0(auvb)$ if and only if $n + 2 - \epsilon$ is on the diagonal in $Q_{EG}^0(auvb)$, while $n' + 1 + \epsilon$ appears in $Q_{EG}^0(auvb)$ if and only if $n' + 2 - \epsilon$ appears in $Q_{EG}^0(auvb)$.

**Proof.** Again write $u_0 := u < v_0 := v$ and let $u_i$ and $v_i$ be the entries of $T$ in the $i$th positions of $rpath^k(T, u)$ and $rpath^k(T, v)$ respectively. Suppose $rpath^k(T, u)$ and $rpath^k(T, v)$ are disjoint. The first paragraph of the proof of Lemma 4.11 shows that $rpath^k(T, u)$ and $rpath^k(T, v)$ must also be disjoint. We argue that since $u + 1 < v$, it must further hold that $rpath^k(T, u)$ and $rpath^k(T, v)$ are disjoint. To see this, note that if $u_i = v_i - 1$ in some row $i > 0$ of $T$ occupied by both $rpath^k(T, u)$ and $rpath^k(T, v)$, then this row of $T$ must also contain $u_i - 1$ and we must have $u_{i-1} = u_i - 1$ and $v_{i-1} = v_i$, since otherwise $rpath^k(T, u)$ and $rpath^k(T, v)$ would intersect in the position of $u_i$ in row $i$. But this means that if $u_i = v_i - 1$ for any row $i > 0$ then we also have $u_0 = v_0 - 1$, which is a contradiction since $u_0 = u$ and $v_0 = v$.

From these properties, we deduce that in any given row occupied by all four paths, the position in $rpath^k(T, u)$ is weakly to the left of the position in $rpath^k(T, u)$, which is strictly to the left of the position in $rpath^k(T, v)$, which finally is weakly to the left of the position in $rpath^k(T, v)$. It follows that if $(i, i) \in rpath^k(T, u) \cap rpath^k(T, u)$ then any diagonal position $(j, j) \in rpath^k(T, v)$ must have $i < j$, while if $(i, i) \in rpath^k(T, u)$ and $(i, i + 1) \in rpath^k(T, u)$ then any diagonal position $(j, j) \in rpath^k(T, v)$ must have $i + 1 < j$.

In addition, $T_{xy}$ and $\gamma_{xy}(T)$ only differ from $(T \leftarrow w)_{xy}$ and $\gamma_{xy}(T \leftarrow w)$ at positions $(x, y) \in rpath^k(T, w) \cup \path^k(T, u)$ by Lemma 4.6. Since $\gamma_{n+1}(auvb) = \gamma_{n+2}(avub)$ and $\gamma_{n+1}(avub) = \gamma_{n+2}(auvb)$ as $u + 1 < v$, it follows in view of Proposition 4.2 that

$$\Delta_{n+1}^{bump}(auvb) = \Delta_{n+2}^{bump}(avub). \quad (4.12)$$

To prove the lemma, it suffices to show that $\Delta_{n+1}^{bump}(auvb)$ and $\Delta_{n+2}^{bump}(avub)$ end with the same tuple, or that $\rpath^k(T, u)$ and $\rpath^k(T \leftarrow u, v)$ both never reach the diagonal. In the former
situation Lemma 4.7 implies the desired result. In the latter situation Lemma 4.7 implies that
\[ cseq_n(auvb) = cseq_n(avub) = cseq_{n+1}(avub), \]
which means that \( cseq_{n+1}(avub) = cseq_{n+2}(avub) \) in view of (4.12), along with
\[ cseq_{n+1}(avub) = cseq_{n+2}(avub), \]
so \( cseq_{n+2}(avub) = cseq_{n+2}(avub) \) holds. The other assertions about the locations of \( n + 1, n + 2, n' + 1, \) and \( n' + 2 \) in \( Q'_E(auvb) \) and \( Q'_E(avub) \) are easy to deduce from Lemma 4.7.

To this end, recall the definitions of \( cpath^≤(T,u) \) and \( cpath^<(T,u) \) from (4.3). If the positions in \( cpath^≤(T,u) \cup cpath^<(T,u) \) are disjoint from \( rpath^≤(T,v) \cup rpath^<(T,v) \), then the latter union is disjoint from \( path^≤(T,u) \cup path^<(T,u) \), and so the stronger property \( \Delta^bump_{n+1}(avub) = \Delta^bump_{n+2}(avub) \) holds in view of Lemma 4.6.

Instead suppose that \( cpath^≤(T,u) \cup cpath^<(T,u) \) and \( rpath^≤(T,v) \cup rpath^<(T,v) \) are not disjoint. For each integer \( i > 1 \), let \( cpath^<(T,u,i) \) be the set of positions in \( cpath^<(T,u) \) in row \( i \), and let
\[ cpath^≤(T,u,i) := \{(i-1,j) \in cpath^≤(T,u) : (i,j) \in cpath^<(T,u)\}. \]

Then each position in \( cpath^≤(T,u) \cup cpath^<(T,u) \) belongs to \( cpath^≤(T,u,i) \cup cpath^<(T,u,i) \) for a unique value of \( i \), and every position in \( cpath^≤(T,u,i) \cup cpath^<(T,u,i) \) occurs in a column strictly to the left of every position in \( cpath^≤(T,u,i+1) \cup cpath^<(T,u,i+1) \) by Proposition 4.2.

Let \( i \) be minimal such that \( cpath^≤(T,u,i) \cup cpath^<(T,u,i) \) and \( rpath^≤(T,v) \cup rpath^<(T,v) \) intersec. Assume the leftmost position in \( cpath^≤(T,u,i) \cup cpath^<(T,u,i) \) is in column \( j + 1 \) while
\[ |cpath^≤(T,u,i)| = l \text{ and } |cpath^<(T,u,i)| = k + l \]
for some integers \( k, l \geq 0 \) with \( k + l > 0 \). If \( i = 1 \) then we must have \( l = 0 \) and \( j + k - 1 \) must be the length of the first row of \( T \). If \( i > 1 \) then we must have \( v_{j+k+t} = v_{j+k} + t \) for \( t \in [l] \). Finally, all positions in \( cpath^≤(T,u,i) \cup cpath^<(T,u,i) \) must be occupied in \( T \), except that when \( l = 0 \) the single position \((i,j+k)\) may be outside the domain of \( T \).

First assume all positions in \( cpath^≤(T,u,i) \cup cpath^<(T,u,i) \) are occupied in \( T \). Then we must have \( i > 1 \), so the entries of \( T \) in positions \( \{i-1,i\} \times \{j+1,j+2,\ldots,j+k+l\} \) are

\[
\begin{array}{cccccc}
  & u_{j+1} & u_{j+2} & \cdots & u_{j+k} & u_{j+k} + 1 & u_{j+k} + 2 & \cdots & u_{j+k} + l \\
T_{i-1,j+1} & T_{i-1,j+2} & \cdots & T_{i-1,j+k} & u_{j+k} & u_{j+k} + 1 & \cdots & u_{j+k} + l - 1
\end{array}
\]
while the corresponding entries of $T \overset{O}{\prec} u$ are\(^6\)

| $u_j$ | $u_{j+1}$ | \cdots | $u_{j+k-1}$ | $u_{j+k}$ | $u_{j+k+1}$ | \cdots | $u_{j+k+l}$ |
|-------|--------|--------|-------------|---------|-----------|--------|-------------|
| ?     | ?      | \cdots | ?           | $u_{j+k}$| $u_{j+k+1}$| \cdots | $u_{j+k+l-1}$|

In this case one of the following holds:

1. $i = j$ and $T_{ii} = u_j$,
2. $i = j + 1$ and $k = 0$ and $T_{i-1,i-1} + 1 = T_{i-1,i} = T_{ii} - 1 = u_j$, or
3. $i < j$ and $u_j$ appears in column $j$ of $T$ above row $i$.

Position $(i - 1, j + k + l + 1)$ in $T$ must be unoccupied or contain an entry greater than $u_{j+k+l}$, so position $(i, j + k + l + 1)$ in $T$ is unoccupied or contains an entry greater than $u_{j+k+l+1}$. This implies that neither $(i - 1, j + k + l)$ nor $(i, j + k + l)$ can belong to $rpath \preceq(T,v) \setminus rpath \preceq(T,v)$. Therefore if $(x,y)$ is in the intersection of $rpath \preceq(T,v)$ and $cpath \preceq(T,u,i) \cup cpath \preceq(T,u,i)$ then $(x,y)$ or $(x,y + 1)$ must be in the intersection of $rpath \preceq(T,v)$ and $cpath \preceq(T,u,i) \cup cpath \preceq(T,u,i)$. Furthermore, if $(i - 1, y) \in rpath \preceq(T,v) \cap cpath \preceq(T,u,i)$ then $(i, y) \in rpath \preceq(T,v) \cap cpath \preceq(T,u,i)$.

So we may assume that $(i, j + \delta) \in rpath \preceq(T,v) \cap cpath \preceq(T,u,i)$ for some $\delta \in [k+1]$. If $k < \delta \leq l$ then we also have $(i - 1, j + \delta) \in rpath \preceq(T,v) \cap cpath \preceq(T,u,i)$. In view of the minimality of $i$, apart from these one or two positions there are no other elements in the intersection of $rpath \preceq(T,v)$ and $cpath \preceq(T,u,i) \cup cpath \preceq(T,u,i)$, since $rpath \preceq(T,v)$ contains at most one position in each row, and since all positions of $rpath <(T,v)$ above row $i$ contain entries of $T$ that are greater than $u_{j+\delta}$ while all positions $cpath \preceq(T,u) \cup cpath <(T,u)$ above row $i$ contain entries of $T$ that are at most $u_j$. To proceed, we divide our analysis into six subcases:

(a) If $k + 1 < \delta \leq l$ then Lemma 4.6 implies $\Delta_{n+1}^{bump}(avub) = \Delta_{n+2}^{bump}(avub)$ which suffices.

(b) Suppose $k > 0$ and $\delta = k + 1$, so that $l > 0$ while $(i - 1, j + k + 1)$ and $(i, j + k + 1)$ are both in $rpath \preceq(T,v)$. We cannot have $T_{i-1,j+k} = u_{j+k} - 1$, since then $(i - 1, j + k)$ would be in $rpath \preceq(T,v)$ and not $rpath \preceq(T,u)$, meaning that $(i - 1, j + k)$ would have to belong to $cpath \preceq(T,u,i)$. Therefore $(i - 1, j + k + 1)$ is also in $rpath \preceq(T,v)$. This means that terms $i$ and $i + 1$ of $\Delta_{n+1}^{bump}(avub)$ are

$$(j + k, j + k + 1, u_{j+k}, \theta) \quad \text{and} \quad (y, \tilde{y}, u_{j+k+1}, \theta)$$

for the cycle $\theta := \gamma_{i-1,j+k+1}(T, uvb) = \gamma_{i-1,j+k+1}(T, vub)$ and some columns $y \leq \tilde{y} \leq j + k + 1$. By Lemma 4.6, terms $i$ and $i + 1$ of $\Delta_{n+2}^{bump}(avub)$ are

$$(j + k + 1, j + k + 1, u_{j+k}, \eta) \quad \text{and} \quad (y, \tilde{y}, u_{j+k+1}, \eta)$$

\(^6\)Most of the boxes labeled by question marks in $T \overset{O}{\prec} u$ contain the same entries as the corresponding positions of $T$. Such an entry could be different if its position belongs to $rpath \preceq(T,u) \cap rpath \preceq(T,u)$. A given row has at most one such position, which must be strictly to the left of any terms of $rpath \preceq(T,v)$ in the same row.
for $\eta := \gamma_{i,j+k+1}(T,uvb) = \gamma_{i.j+k+1}(T,vub)$ and the same values of $\theta$, $y$, $\tilde{y}$. Thus $\Delta^{\text{bump}}_{n+1}(avub)$ and $\Delta^{\text{bump}}_{n+2}(avub)$ only differ in their $i$th terms, so their final terms coincide as needed.

c) Suppose $k = 0$ and $\delta = k + 1 = 1$, so that again $l > 0$. Then cases (1) and (2) would each lead to a contradiction of our assumption that $\text{rpath}^\leq(T,u) \cap \text{rpath}^\leq(T,v)$ is empty: case (1) would imply that this intersection contains $(i,i)$ while case (2) could imply that it contains $(i-1,i-1)$. Therefore we are in case (3) so position $(i,j)$ in $T$ contains an entry that is at most $u_j - 1$ while position $(i+1,j)$ in $T$ contains an entry that is at most $u_j$. It follows that terms $i$ and $i + 1$ of $\Delta^{\text{bump}}_{n+1}(avub)$ have the form

$$(j + 1, j + 1, u_j, \theta) \quad \text{and} \quad (j + 1, j + 1, u_j + 1, \eta)$$

while terms $i$ and $i + 1$ of $\Delta^{\text{bump}}_{n+2}(avub)$ have the form

$$(j + 1, j + 1, u_j, \eta) \quad \text{and} \quad (j + 1, j + 1, u_j + 1, \theta)$$

for $\theta := \gamma_{i-1,j+1}(T,uvb) = \gamma_{i-1,j+1}(T,vub)$ and $\eta := \gamma_{i,j+1}(T,uvb) = \gamma_{i,j+1}(T,vub)$. As in the previous paragraph, it follows that $\Delta^{\text{bump}}_{n+1}(avub)$ and $\Delta^{\text{bump}}_{n+2}(avub)$ do not differ outside these two terms, so either both sequences end in the same tuple in view of (4.7) or $\text{rpath}^\leq(T,v)$ and $\text{rpath}^\leq(T \leftarrow Q u,v)$ never reach the diagonal since $(i + 1, j + 1)$ is not a diagonal position. This is again sufficient to conclude that the lemma holds.

d) We claim that the case $\delta = k = 1$ cannot occur. In this event, it would follow in view of Proposition 4.2 that $(i - 1, j + k)$ and $(i, j + k)$ are both in $\text{rpath}^\leq(T,v)$ with $u_{j+k-1} \leq T_{i-1,j+k} < u_{j+k}$, which contradicts the fact that $T_{i-1,j+k} < u_{j+k-1}$ as $(i - 1, j + k) \notin \text{rpath}^\leq(T,u)$.

e) If $k > 0$ and $1 < \delta < k$, then it follows from Lemma 4.6 that $\Delta^{\text{bump}}_{n+1}(avub)$ and $\Delta^{\text{bump}}_{n+2}(avub)$ differ only in their $i$th term, where if this term of $\Delta^{\text{bump}}_{n+1}(avub)$ is $(y, \tilde{y}, d, \eta)$ then the corresponding term of $\Delta^{\text{bump}}_{n+2}(avub)$ is $(1 + y, 1 + \tilde{y}, d, \eta)$. Both sequences then have more than $i$ terms so they end with the same tuple as needed.

f) Next suppose $k > 0$ and $\delta = 1$. If $u_j < v_{i-1}$ then the argument in subcase (e) still applies.

Assume $v_{i-1} \leq u_j$. Then we cannot be in cases (1) or (2) without contradicting $\text{rpath}^\leq(T,u) \cap \text{rpath}^\leq(T,v) = \emptyset$, so $u_j$ appears in column $j$ of $T$ above row $i$ and position $(i+1,j)$ in $T$ contains an entry that is at most $u_j$. The entry in position $(i,j)$ of $T$ cannot be greater than $v_{i-1}$ since $(i,j+1) \in \text{rpath}^\leq(T,v)$, and this entry must also not be equal to $v_{i-1}$ since then we would have $u_{j+1} = v_{i-1} + 1$ which can only hold if $u_j = v_{i-1}$, in which case column $j$ of $T$ would have two equal entries, contradicting the fact that all columns of $T$ are strictly increasing. Thus position $(i,j)$ in $T$ contains an entry that is less than $v_{j-1}$.

It follows that $\Delta^{\text{bump}}_{n+1}(avub)$ and $\Delta^{\text{bump}}_{n+2}(avub)$ only differ in terms $i$ and $i + 1$: while these terms in $\Delta^{\text{bump}}_{n+1}(avub)$ must have the form $(j + 1, j + 1, v_{i-1}, \theta)$ and $(j + 1, j + 1, u_{j+1}, \eta)$ respectively for some cycles $\theta$ and $\eta$, the corresponding terms of $\Delta^{\text{bump}}_{n+2}(avub)$ are respectively

$$(j + 1, j + 2, v_{i-1}, \theta) \quad \text{and} \quad (j + 1, j + 1, u_{j+1}, \theta)$$

when $v_{i-1} = u_j$, or

$$(j + 1, j + 1, v_{i-1}, \theta) \quad \text{and} \quad (j + 1, j + 1, u_j, \phi)$$
when \( v_{i-1} < u_j \), where we may have \( \phi \neq \eta \). As in our earlier cases, we conclude that either both sequences end in the same tuple in view of (4.7), or we observe that \((i + 1, j + 1)\) is not a diagonal position so \( rpath(\leq) (T, v) \) and \( rpath(\leq) (T \overset{\circ}{,} u, v) \) never reach the diagonal.

This completes our argument if all positions in \( cpath(\leq) (T, u, i) \cup cpath(\leq) (T, u, i) \) are occupied in \( T \).

When this does not occur, we must have \( l = 0 \) and \((i, j + k) \notin T\). In this case row \( i \) of \( T \) is

\[
\begin{array}{ccccccc}
T_{i1} & T_{i2} & \cdots & T_{ij} & u_{j+1} & u_{j+2} & \cdots & u_{j+k-1}
\end{array}
\]

while row \( i \) of \( T \overset{\circ}{,} u \) is

\[
\begin{array}{ccccccc}
T_{i1} & T_{i2} & \cdots & T_{ij} & u_j & u_{j+1} & \cdots & u_{j+k-2} & u_{j+k-1}
\end{array}
\]

Here, cases (1) or (3) from above must apply. We cannot have \((i, j + k) \in rpath(\leq) (T, v) \setminus rpath(\leq) (T, v) \) if \((i, j + k) \notin T\), so again \((i, j + \delta) \in rpath(\leq) (T, v) \cap cpath(\leq) (T, u, i) \) for some \( \delta \in [k] \). By the minimality of \( i \), this position is the unique element in both \( rpath(\leq) (T, v) \) and \( cpath(\leq) (T, u, i) \), since \( rpath(\leq) (T, v) \) contains at most one position in each row, and since all positions of \( rpath(\leq) (T, v) \) above row \( i \) contain entries greater than \( u_{j+\delta} \) while all positions of \( rpath(\leq) (T, v) \cup rpath(\leq) (T, v) \) above row \( i \) contain entries that are at most \( u_j \). We are left with two further subcases:

(g) If \( u_j < v_{i-1} \), then it follows from Lemma 4.6 as in subcase (e) that \( \Delta^{bump}_{n+1}(avub) \) and \( \Delta^{bump}_{n+2}(avub) \) differ only in their \( i \)th term, where if this term of \( \Delta^{bump}_{n+1}(avub) \) is \((y, \tilde{y}, d, \eta)\) then the corresponding term of \( \Delta^{bump}_{n+2}(avub) \) is \((1 + y, 1 + \tilde{y}, d, \eta)\). In this event, both sequences have more than \( i \) terms unless \( y = \tilde{y} = j + k \). Since \((j + j + k)\) is not a diagonal position, we conclude that the lemma holds holds either way.

(h) Assume \( v_{i-1} \leq u_j \). Then we cannot be in case (1) without contradicting \( rpath(\leq) (T, u) \cap rpath(\leq) (T, v) = \emptyset \), so \( i < j \) and \( u_j \) appears in column \( j \) of \( T \) above row \( i \). If \( \delta < k \) then we can repeat the argument given in subcase (f) to deduce our result. If \( \delta = k \) then we must have \( k = 1 \) and \( v_{i-1} < u_j \). In this situation, \( \Delta^{bump}_{n+1}(avub) \) has only \( i \) terms and ends with a term of the form \((j + 1, j + 1, v_{i-1}, \theta)\) for some cycle \( \theta \), and it is easy to see that \( \Delta^{bump}_{n+2}(avub) \) is formed from \( \Delta^{bump}_{n+1}(avub) \) by appending the tuple \((j + 1, j + 1, u_j, \phi)\) for some cycle \( \phi \). Since neither \((i, j + 1)\) nor \((i + 1, j + 1)\) is a diagonal position, this shows that \( rpath(\leq) (T, v) \) and \( rpath(\leq) (T \overset{\circ}{,} u, v) \) never reach the diagonal so the lemma again holds.

This completes our proof of the lemma.

\[\square\]

### 4.4 Final arguments

Combining the preceding results, we can now explain how to derive Theorem 3.16. There are three main steps in our argument, handling each of cases in which we can have \( \text{o}ck_i(a) = a \).

**Lemma 4.13.** Suppose \( a = a_1a_2\cdots a_n \) is a primed involution word. Write \( \square_i \) for \( i \in [n] \) to denote the box of \( Q_{\text{EG}}^O(a) \) containing \( i \) or \( i' \). Assume that \( i \in \{-1, 0\} \), or that \( i \in [n - 2] \) and \( \square_i \) and \( \square_{i+2} \) are both on the main diagonal. Then \( P_{\text{EG}}^O(\text{o}ck_i(a)) = P_{\text{EG}}^O(a) \) and \( Q_{\text{EG}}^O(\text{o}ck_i(a)) = \delta_i(Q_{\text{EG}}^O(a)) \).

41
Proof. If $i \in \{−1, 0\}$, then the desired identities follows easily from the definitions. Assume $i \in [n−2]$ and $\square_i$ and $\square_{i+2}$ are both on the main diagonal. Write $\square_i = (q−1, q−1)$ and $Q = Q_{\text{EG}}^O(a)$.

Then we must have $\square_{i+1} = (q−1, q)$ and $\square_{i+2} = (q, q)$, and consequently $\delta_i(Q) = s_i(Q) = s_{i+1}(Q)$ is formed from $Q$ by swapping $i + 1$ and $i' + 1$, and then reversing the primes on the entries in the diagonal boxes $(q−1, q−1)$ and $(q, q)$ if these entries are not both primed or both unprimed.

After possibly invoking Proposition 3.15 to interchange $Q$ with $\delta_i(Q)$, we may assume that the entry in position $(q−1, q)$ of $Q$ is $i + 1$ rather than $i' + 1$. Then it is evident from Lemma 4.7 that $\tau_i(a) = \tau_{i+1}(a) = \tau_{i+2}(a) = 1$. Write $b := \text{ock}_i(a)$. Since we know from Lemma 4.1 that $Q_{\text{EG}}^O(\text{unprime}(b))$ is formed by applying $\delta_i$ to $Q_{\text{EG}}^O(\text{unprime}(a)) = \text{unprime}_{\text{diag}}(Q)$, which adds a prime to position $(q−1, q)$, it is also clear from Lemma 4.7 that $\tau_i(b) = 1$.

To compute $\tau_{i+1}(b)$, we consider the weak bumping paths $\text{path}_{i+1}^<(b)$, $\text{path}_{i+1}^\leq(b)$, and $\text{path}_{i+2}^<(b)$ that result from inserting $b_i, b_{i+1},$ and $b_{i+2}$ successively into $P_{\text{EG}}^O(a_1a_2\cdots a_{i−1}) = P_{\text{EG}}^O(b_1b_2\cdots b_{i−1})$. In view of Proposition 4.2, the first path must terminate at position $(q−1, q−1)$, the last two positions of the second path must be $(q−1, q−1)$ followed by $(q−1, q)$, and the last two positions of the third path must be $(q−1, q)$ followed by $(q, q)$.

Suppose the first row of $cseq_{i+2}(a)$ is $[γ_1 \cdots γ_q]$. Since $\text{cseq}_{i+2}(a) = \text{cseq}_{i+2}(b)$ by Lemma 4.1, we deduce from Lemma 4.6 that the first rows of $\text{cseq}_{i−1}(a) = \text{cseq}_{i−1}(b), \text{cseq}_i(b)$, and $\text{cseq}_{i+1}(b)$ are $[γ_1 \cdots γ_q−2], [γ_1 \cdots γ_q−2 γ_q],$ and $[γ_1 \cdots γ_q−2 γ_{q−1}],$ respectively. Thus $\tau_{i+1}(b)$ is the permutation of $\text{cyc}(z)$ that swaps $γ_{q−1}$ and $γ_q$. Multiplying $\tau_1(a)\tau_2(a)\cdots \tau_{i+2}(a)$ on the right by this permutation gives $\tau_1(b)\tau_2(b)\cdots \tau_{i+2}(b)$ and vice versa. By Lemmas 4.9 and 4.1 this means that $P_{\text{EG}}^O(a_1a_2\cdots a_{i+2}) = P_{\text{EG}}^O(b_1b_2\cdots b_{i+2})$ and $Q_{\text{EG}}^O(b_1b_2\cdots b_{i+2}) = \delta_i(Q_{\text{EG}}^O(a_1a_2\cdots a_{i+2})).$ □

Our next lemma also has a relatively short proof.

**Lemma 4.14.** Suppose $a = a_1a_2\cdots a_n \in R^+(z)$ for some $z \in I_E$. Assume $i \in [n−2]$ and $[a_{i+1}] < [a_i] < [a_{i+2}]$. Then $\tau(\text{ock}_i(a)) = \tau(a)$ and $P_{\text{EG}}^O(\text{ock}_i(a)) = P_{\text{EG}}^O(a)$.

**Proof.** Let $b := \text{ock}_i(a) = a_1\cdots a_i a_{i+2} a_{i+1}\cdots a_n$. Since Lemma 4.1 implies that $P_{\text{EG}}^O(\text{unprime}(a)) = P_{\text{EG}}^O(\text{unprime}(b))$, it suffices by Lemma 4.9 to show that $\tau(\text{unprime}(a)) = \tau(\text{unprime}(b))$. We may therefore assume that $a = \text{unprime}(a)$ has no primed letters and prove that $\tau(a) = \tau(b)$.

To this end, write $\square_j$ for $j \in [n]$ to denote the box of $Q_{\text{EG}}^O(a)$ containing $j$ or $j'$. We first check that $\square_i$ and $\square_{i+2}$ are not both on the main diagonal. Arguing by contradiction, we observe that these positions could only both be on the diagonal if the weak bumping paths $\text{path}_{i−1}^<(a)$, $\text{path}_{i−1}^\leq(a)$, and $\text{path}_{i+2}^<(a)$ that result from inserting $a_i, a_{i+1},$ and $a_{i+2}$ successively into $P_{\text{EG}}^O(a_1a_2\cdots a_{i−1})$ respectively terminate at $(q−1, q−1), (q−1, q),$ and $(q, q)$ for some $q > 0$. Assume this is the case, so that we have $\text{path}_{i−1}^<(a) = \text{rpath}_{i−1}^<(a)$ and $\text{path}_{i+1}^<(a) = \text{rpath}_{i+1}^<(a)$.

Since $a_i > a_{i+1}$, Proposition 4.3 implies that the positions in $\text{rpath}_{i+1}^<(a)$ are all weakly to the left of the corresponding positions in $\text{rpath}_{i−1}^<(a)$. The second to last position in $\text{path}_{i−1}^<(a)$ must therefore be $(q−1, q−1)$, so the entry in position $(q−1, q−1)$ of $P_{\text{EG}}^O(a_1a_2\cdots a_{i−1})$ is the same as the entry in position $(q−1, q−1)$ of $P_{\text{EG}}^O(a_1a_2\cdots a_i)$. Since $a_{i+1} < a_i < a_{i+2}$, it is easy to check that the first $q−1$ positions in $\text{path}_{i+2}^<(a)$ are strictly to the right of the corresponding positions in $\text{path}_{i}^<(a)$, and that if $\text{path}_{i+2}^<(a)$ reaches row $q$ then its position in that row must be strictly to the right of $(q−1, q)$.

But this makes it impossible for $\text{path}_{i+2}^<(a)$ to terminate at $(q, q)$.

Thus $\square_i$ and $\square_{i+2}$ are not both on the diagonal. By Lemma 4.1 we have $P_{\text{EG}}^O(a_1a_2\cdots a_j) = P_{\text{EG}}^O(b_1b_2\cdots b_j)$ for all $j \in [n] \setminus \{i+1\}$ along with $Q_{\text{EG}}^O(b) = \delta_i(Q_{\text{EG}}^O(a))$, so $\tau_j(a) = \tau_j(b)$ for all $j \in [n] \setminus \{i, i+1, i+2\}$. It remains to show that $\tau_{i+1}(a)\tau_{i+2}(a) = \tau_{i+1}(b)\tau_{i+2}(b)$. Evidently
cseq_i(a) = cseq_i(b) and cseq_{i+2}(a) = cseq_{i+2}(b) and a_{i+1} < a_{i+2}. Since $Q^O_{\text{EG}}(b) = \delta_i(Q^O_{\text{EG}}(a))$ and $\square_i$ and $\square_{i+2}$ are not both on the diagonal, it follows from Proposition 3.15 that conditions (b) and (c) in Lemma 4.10 also hold, so that result implies that $\tau_{i+1}(a)\tau_{i+2}(a) = \tau_{i+1}(b)\tau_{i+2}(b)$. 

The next lemma has a significantly longer proof, though much of this is routine case analysis.

**Lemma 4.15.** Let $a = a_1a_2 \cdots a_n \in R^+_{\text{inv}}(z)$ for some $z \in \mathbb{Z}$. Write $\square_i$ for $j \in [n]$ to denote the box of $Q^O_{\text{EG}}(a)$ containing $j$ or $j'$. Suppose $i \in [n-2]$ is such that $[a_i] < [a_{i+2}] < [a_{i+1}]$, but $\square_i$ and $\square_{i+1}$ are not both on the main diagonal. Then $\tau(\text{ock}_i(a)) = \tau(a)$ and $P^O_{\text{EG}}(\text{ock}_i(a)) = P^O_{\text{EG}}(a)$.

**Proof.** Define $b = \text{ock}_i(a)$ and assume $a = \text{unprime}(a)$ has no primed letters. As in the proof of Lemma 4.14, it suffices by Lemmas 4.9 and 4.1 to check that $\tau(a) = \tau(b)$. We have either $a_i < a_{i+2} < a_{i+1}$ and $b = a_1 \cdots a_{i+2}a_{i+3} \cdots a_n$, or $a_i = a_{i+2} < a_{i+1}$ and $b = a_1 \cdots a_{i+1}a_{i+2} \cdots a_n$. In either case, Lemma 4.1 implies that $P^O_{\text{EG}}(a_{i+2}b_{i+3} \cdots b_j) = P^O_{\text{EG}}(b_1b_2 \cdots b_j)$ for $i \in [n] \setminus \{i, i+1\}$ so we have $\text{cseq}_i(a) = \text{cseq}_i(b)$ for $j \in [n] \setminus \{i, i+1\}$. Thus $\tau_j(a) = \tau_j(b)$ for $j \in [n] \setminus \{i, i+1, i+2\}$ and it is enough to show that $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b)$.

Let $s(a)$ be the number of diagonal entries in $Q^O_{\text{EG}}(a)$ equal to $i$, $i+1$, or $i+2$. We must have $s(a) \in \{0, 1\}$ since $i$ and $i+2$ are not both on the diagonal. Let $r(a) \in \{0, 1, 2\}$ be the number of (off-diagonal) entries in $Q^O_{\text{EG}}(a)$ equal to $i'$, $i'+1$, or $i'+2$. Since $Q^O_{\text{EG}}(b) = \delta_i(Q^O_{\text{EG}}(a))$ by Lemma 4.1, we deduce from Proposition 3.15 that $s(a) = s(b)$ and $r(a) = r(b)$.

Suppose the intersection of $rpath_i^<(a)$ and $rpath_i^<(b)$ includes a position off the diagonal. Then Lemma 4.11 implies that $\text{cseq}_i(a) = \text{cseq}_i(b)$ so $\tau_i(a) = \tau_i(b)$. Because $s(a) = s(b)$ and $r(a) = r(b)$, we can then apply Lemma 4.10 to deduce that $\tau_{i+1}(a)\tau_{i+2}(a) = \tau_{i+1}(b)\tau_{i+2}(b)$.

Instead suppose $a_i < a_{i+2} < a_{i+1}$ and $rpath_i^<(a)$ and $rpath_i^<(b)$ are disjoint. Then Lemma 4.12 implies that $\text{cseq}_i(a) = \text{cseq}_i(b)$ so $\tau_{i+2}(a) = \tau_{i+2}(b)$. Because $s(a) = s(b)$ and $r(a) = r(b)$, we can again apply Lemma 4.10 to deduce that $\tau_i(a)\tau_{i+1}(a) = \tau_i(b)\tau_{i+1}(b)$.

Thus, we may assume that $rpath_i^<(a)$ and $rpath_i^<(b)$ intersect in at most one position, which is on the main diagonal, and that if $a_i < a_{i+2} < a_{i+1}$ then $rpath_i^<(a)$ and $rpath_i^<(b)$ intersect in at least one position.

Assume that if $a_i < a_{i+2} < a_{i+1}$ then the first position in the intersection of $rpath_i^<(a)$ and $rpath_i^<(b)$ is off the diagonal in row $j > 0$. This position cannot belong to $rpath_i^<(a) \cap rpath_i^<(b)$, so it must be occupied by some entry $u$ in $T$. If instead $a_i = a_{i+2} < a_{i+1}$, then we set $j := 0$ and $u := a_i$. Assume the last position in $rpath_i^<(a)$ is in row $k$. Then $j < k$ and the following claims are easy to deduce from our assumption that $rpath_i^<(a)$ and $rpath_i^<(b)$ do not intersect off the diagonal:

(A1) Suppose $t \in \{1, 2, \ldots, k - j - 1\}$ or $t = 0 < j$. Then row $j + t$ of $T$ contains both $u + t$ and $u + t + 1$, and the positions of $u + t$ and $u + t + 1$ in row $j + t$ of $T$ are in $rpath_i^<(a) \cap rpath_i^<(b)$ and $rpath_i^<(b)$, respectively. Moreover, if row $j + t$ of $T$ contains $u + t - 1$ then its position is in $rpath_i^<(a)$, and otherwise the position of $u + t$ in row $j + t$ of $T$ is in $rpath_i^<(a)$. It follows that if $j > 0$ then row $j$ of $T$ does not contain $u - 1$, since $rpath_i^<(a)$ and $rpath_i^<(b)$ share a position in this row.

(A2) The position $(k, k)$ is in $rpath_i^<(a)$, since otherwise the last position in $rpath_i^<(a)$ would be an off-diagonal element of $rpath_i^<(a) \cap rpath_i^<(b)$. If occupied, the entry of position $(k, k)$ in $T$ must be at least $u + k - j - 1$.

Suppose $x, y \in \mathbb{Z}$ are such that $row(T)xy$ is an involution word. The tableau $T \overset{\circ}{\to} x$ differs from $T$ only in the positions that belong to $\text{path}^<(T, x)$, which contain successively increasing entries until the last position which is not in $T$. 43
If we know only the first \(k-1\) positions of \(\text{path}^< (T, x)\) and \(\text{path}^< (T, x)\), but we know that the entry of \(T\) in the \(k\)th term of \(\text{path}^< (T, x)\) is bounded below by some number \(N\) when this position is present in \(T\), then we can compute the subtableau of \(T \preceq x\) formed by omitting all entries greater than \(N\). In this event, we can then also compute the initial subsequences of \(\text{path}^<= (T \preceq x, y)\) and \(\text{path}^<= (T \preceq x, y)\) that consist of positions with entries of \(T \preceq x\) that are bounded above by \(N\). These observations let us deduce the following additional properties:

(A3) The first \(k-1\) terms of \(\text{path}^<_i (a)\) and \(\text{path}^<_i+1 (b)\) coincide, as do the first \(k-1\) terms of \(\text{path}^<_1 (b)\) and \(\text{path}^<_i+1 (a)\), as do the first \(k-1\) terms of \(\text{path}^<_i (a)\) and \(\text{path}^<_i+1 (b)\).

(A4) The first \(k-1\) terms of \(\text{path}^<_i (b)\) and \(\text{path}^<_i (a)\) are the same except in the rows \(j+t\) where \(T\) does not contain \(u+t-1\), for \(t \in \{1, 2, \ldots, k-j-1\}\) or \(t = 0 < j\). In these rows, \(\text{path}^<_i+1 (a)\) contains the position of \(u+t+1\) in \(T\), rather than the position of \(u+t\) which is in \(\text{path}^<_i (b)\).

(A5) The first \(j\) terms of \(\text{path}^<_i+2 (a)\) and \(\text{path}^<_i+2 (b)\) coincide, as do the first \(j\) terms of \(\text{path}^<_i+2 (a)\) and \(\text{path}^<_i (a)\) and \(\text{path}^<_i (b)\). If \(j > 0\) then term \(j\) of all four paths is the position of \(u+1\) in row \(j\) of \(T\).

(A6) If \(t \in [k-j-1]\), then the \((j+t)\)th terms of \(\text{path}^<_i+2 (a)\), \(\text{path}^<_i+2 (a)\), \(\text{path}^<_i+2 (b)\), and \(\text{path}^<_i+2 (b)\) are either the respective positions in row \(j+t\) of \(T\) of \(u+t-1\), \(u+t\), \(u+t\), and \(u+t+1\) when row \(j+t\) of \(T\) contains the entry \(u+t-1\), or the respective positions of \(u+t\), \(u+t+1\), \(u+t+1\), and \(u+t+1\) when the same row does not contain \(u+t-1\).

Combining the preceding observations, we arrive at the following key claim:

(A7) Let \(v = u+k-j-1\) and assume \(k > 1\). Then the entries of the shifted tableaux \(T, T \preceq a_i, a_{i+1}\) in the \(k-1\)th positions of \(\text{path}^<_i (a)\), \(\text{path}^<_i+1 (a)\), and \(\text{path}^<_i+2 (a)\) are \(v, v+1, \) and \(v\), respectively. Likewise, the entries of the shifted tableaux \(T, T \preceq b_i, a_{i+1}\) in the \(k-1\)th positions of \(\text{path}^<_i (b)\), \(\text{path}^<_i+1 (b)\), and \(\text{path}^<_i+2 (b)\) are \(v+1, v, \) and \(v+1\), respectively.

This last property still makes sense when \(j = 0\) and \(k = 1\) if we define the entries in the “0th position” of \(\text{path}^<_m (a)\) and \(\text{path}^<_m (b)\) to be \(a_m\) and \(b_m\), respectively.

We need just one other observation. Let \(U\) be the shifted tableau formed from \(T\) by omitting the first \(k-1\) rows. Using Proposition 3.14 and property (A7), it is easy to check that \(a\) is equivalent under \(\preceq\) to a word that begins with \(\text{row} (U) v (v+1) v\). If \(U\) were empty or if all entries in \(U\) were greater than \(v+2\) then this word is an involution equivalent under \(\preceq\) to \(v (v+1) v \text{row} (U)\) which is impossible by Proposition 2.4. Thus:

(A8) The entry of \(T\) in position \((k,k)\) is occupied by \(v, v+1, \) or \(v+2\).

We can now reason precisely about the possibilities for \(\tau_i (a), \tau_i (b), \tau_i (a), \tau_i+2 (a), \tau_i+1 (b), \) and \(\tau_i+2 (b)\) under our assumptions that \(\text{path}^<_i (a)\) and \(\text{path}^<_i (b)\) do not intersect off the diagonal and that if \(a_i < a_{i+2} < a_{i+1}\) then \(\text{path}^<_i (a)\) and \(\text{path}^<_i (b)\) do intersect off the diagonal. Below, we will refer to the entries of the shifted tableaux arranged in the diagram:

\[
P_{EG} (a_1 \cdots a_{i-1}) \xrightarrow{a_i} P_{EG} (a_1 \cdots a_i) \xrightarrow{a_{i+1}} P_{EG} (a_1 \cdots a_{i+1}) \xrightarrow{a_{i+2}} P_{EG} (a_1 \cdots a_{i+2})
\]

\[
P_{EG} (b_1 \cdots b_i) \xrightarrow{b_i} P_{EG} (b_1 \cdots b_i) \xrightarrow{b_{i+1}} P_{EG} (b_1 \cdots b_{i+1})
\]

\[
\begin{align*}
(4.13)
\end{align*}
\]
where in this picture, an arrow $\rightarrow$ connects two tableaux if inserting $u$ into the first tableau according to Definition 3.1 gives the second. Write
\[
cseq_{i-1}(a) = cseq_{i-1}(b) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_q \\ c_1 & c_2 & \cdots & c_q \end{bmatrix}.
\] (4.14)

As noted above, there are three possibilities for the entry of $T$ in position $(k, k)$.

First suppose the entry of $T$ in position $(k, k)$ is $v$. Then, in view of Remark 3.5, the entries of $T$ in positions $\{k, k+1, k+2\} \times \{k, k+1, k+2\}$ must be $T_{k+i, k+j} = v+i+j$ for all $0 \leq i \leq j \leq 2$. Using Lemma 4.7 and property (A7), it is easy to see that the entries in these positions are the same for all six tableaux in (4.13), and that $\tau_i(a) = (\gamma_k, \gamma_{k+1}) = \tau_{i+2}(b)$ and $\tau_i(a) = (\gamma_k, \gamma_{k+2}) = \tau_{i+1}(b)$ and $\tau_{i+2}(a) = (\gamma_{k+1}, \gamma_{k+2}) = \tau_i(b)$. Thus $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \gamma_{k+2})$.

Suppose next that the entry of $T$ in position $(k, k)$ is $v+1$. Then, again in view of Remark 3.5, the entries of $T$ in positions $\{k, k+1\} \times \{k, k+1\}$ must be $T_{k+i, k+j} = v+i+j$ for all $0 \leq i \leq j \leq 1$. Assume $k > 1$. Then row $k-1$ of $T$ contains $v$ and $v+1$ in off-diagonal positions, so the entry in position $(k-1, k+1)$ of $T$ is at most $v+1$. If equality holds, then the entries of the six tableaux in (4.13) in positions $\{k-1, k, k+1\} \times \{k, k+1\}$ must be

![Diagram showing six tableaux with entries in positions as described.](image)

On the other hand, if the entry in position $(k-1, k+1)$ of $T$ is less than $v+1$ then position $(k-1, k+2)$ of $T$ must have an entry less than $v+2$. When this happens or when $k = 1$, the entries in the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1, k+2\}$ must instead be

![Diagram showing six tableaux with entries in positions as described.](image)

where \(?\) denotes a position that may be unoccupied. In both cases, it follows using Lemmas 4.6
that the values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

$$
\begin{array}{c}
\gamma_{k+1} \\
\gamma_k
\end{array} \xrightarrow{a_i} 
\begin{array}{c}
\gamma_{k+1} \\
\gamma_k
\end{array} \xrightarrow{a_{i+1}} 
\begin{array}{c}
\gamma_{k+1} \\
\gamma_k
\end{array} \xrightarrow{a_{i+2}} 
\begin{array}{c}
\gamma_k \\
\gamma_{k+1}
\end{array}
\begin{array}{c}
\gamma_k \\
\gamma_{k+1}
\end{array}
\begin{array}{c}
\gamma_k \\
\gamma_{k+1}
\end{array} \xrightarrow{b_i} 
\begin{array}{c}
\gamma_k \\
\gamma_{k+1}
\end{array} \xrightarrow{b_{i+1}} 
\begin{array}{c}
\gamma_k \\
\gamma_{k+1}
\end{array} 
\end{array}
$$

Thus, it follows by Lemma 4.7 that $\tau_i(a) = \tau_{i+1}(a) = \tau_{i+2}(b) = 1$ and $\tau_{i+2}(a) = \tau_i(b) = (\gamma_k, \gamma_{k+1})$, so $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \gamma_{k+1})$ as needed.

Finally, suppose the entry of $T$ in position $(k, k)$ is $v + 2$. If $k > 1$ then row $k - 1$ of $T$ contains $v$ and $v + 1$ off the diagonal, so the entry in position $(k - 1, k)$ of $T$ must be less than $v + 2$. There are two subcases depending on the entry in position $(k - 1, k + 2)$ of $T$. If $k > 1$ and this position contains a number less than $v + 2$, or if $k = 1$, then the entries in the six tableaux in (4.13) in positions $\{k, k + 1\} \times \{k, k + 1, k + 2\}$ are

$$
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & ? & ? \\
\end{array} \xrightarrow{a_i} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & ? & ? \\
\end{array} \xrightarrow{a_{i+1}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 2 & ? \\
\end{array} \xrightarrow{a_{i+2}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 2 & ? \\
\end{array} \xrightarrow{b_i} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 2 & ? \\
\end{array} \xrightarrow{b_{i+1}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 2 & ? \\
\end{array} \xrightarrow{b_{i+2}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 2 & ? \\
\end{array}
\end{array}
$$

If $k > 1$ and position $(k - 1, k + 2)$ of $T$ is unoccupied or contains a number greater than or equal to $v + 2$, then positions $(k - 1, k)$ and $(k - 1, k + 1)$ of $T$ must contain the numbers $v$ and $v + 1$. In this case the entries in the six tableaux in (4.13) in positions $\{k - 1, k, k + 1\} \times \{k, k + 1\}$ are

$$
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & ? & ? \\
\end{array} \xrightarrow{a_i} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & ? & ? \\
\end{array} \xrightarrow{a_{i+1}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 1 & ? \\
\end{array} \xrightarrow{a_{i+2}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 1 & ? \\
\end{array} \xrightarrow{b_i} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 1 & ? \\
\end{array} \xrightarrow{b_{i+1}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 1 & ? \\
\end{array} \xrightarrow{b_{i+2}} 
\begin{array}{c|c|c}
? & ? & ? \\
v + 2 & v + 1 & ? \\
\end{array}
\end{array}
$$

Write $\eta_k$ and $\eta_{k+1}$ for the entries in the first row of $cseq_{i+2}(a)$ in columns $k$ and $k + 1$. The following assertions apply equally to both of the cases above. First, since $cseq_{i-1}(a) = cseq_{i-1}(b)$
and $cseq_{i+2}(a) = cseq_{i+2}(b)$, one can check using Lemmas 4.6 and 4.7 that $\gamma_k = \eta_k$. If $cseq_{i-1}(a)$ has only $k$ columns, then it follows similarly that the values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

\[
\begin{array}{c c c c c c}
\gamma_k & ? & \eta_k+1 & \gamma_k & \eta_k+1 & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
gamma_k & ? & \eta_k+1 & \gamma_k & \eta_k+1 & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & ? & \eta_k+1 & \gamma_k & \eta_k+1 & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \beta \\
\gamma_k & \emptyset & \eta_k+1 & \gamma_k & \eta_k+1 & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \emptyset & \eta_k+1 & \gamma_k & \eta_k+1 & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \emptyset & \eta_k+1 & \gamma_k & \eta_k+1 & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\end{array}
\]

where we set $\beta := \emptyset$ in the first subcase above and $\beta := \eta_{k+1}$ in the second. Thus $\tau_i(a) = \tau_{i+2}(a) = (\gamma_k, \eta_k+1)$ and $\tau_{i+1}(a) = \tau_{i+1}(b) = \tau_{i+2}(b) = 1$, giving $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = 1$ as desired. If $cseq_{i-1}(a)$ has at least $k+1$ columns, then it follows likewise that the values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

\[
\begin{array}{c c c c c c}
\gamma_k & \eta_{k+1} & ? & \gamma_k & \eta_{k+1} & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \eta_{k+1} & ? & \gamma_k & \eta_{k+1} & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \eta_{k+1} & ? & \gamma_k & \eta_{k+1} & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \eta_{k+1} & ? & \gamma_k & \eta_{k+1} & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \eta_{k+1} & ? & \gamma_k & \eta_{k+1} & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\gamma_k & \eta_{k+1} & ? & \gamma_k & \eta_{k+1} & \gamma_k \\
& a_i & & a_{i+1} & & a_{i+2} \\
& b_i & & & b_{i+1} & \\
\end{array}
\]

where the entry $\beta$ has the same definition as before. Thus Lemma 4.7 gives $\tau_i(a) = \tau_{i+2}(a) = (\gamma_k, \eta_k+1)$ and $\tau_{i+1}(a) = (\gamma_k, \gamma_k+1)$ while $\tau_i(b) = \tau_{i+1}(b) = 1$ and $\tau_{i+2}(b) = (\gamma_k+1, \eta_k+1)$, so $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k+1, \eta_k+1)$ as needed.

This completes the proof of the lemma under our first set of hypotheses. It remains to consider the case when $a_i < a_{i+2} < a_{i+1}$ and $rpath_i^\leq(a)$ and $rpath_i^\leq(b)$ do not intersect off the diagonal, but $rpath_i^\leq(a)$ and $rpath_i^\leq(b)$ intersect in a unique position which is on the diagonal. Suppose this position is $(k, k)$. This position must be occupied in $T$, since otherwise it is straightforward to check using Remark 3.5 that both $i$ and $i + 2$ would be on the diagonal of $Q^T_{E_{GQ}}(a)$. The reasoning we used to justify (A3) lets us similarly derive the following claims:

(B1) The first $k - 1$ terms of $rpath_i^\leq(a)$ and $rpath_{i+1}^\leq(b)$ coincide, as do the first $k - 1$ terms of $rpath_{i+1}^\leq(a)$ and $rpath_i^\leq(b)$. Each of the first $k - 1$ terms of the first two paths is strictly to the right of the main diagonal and strictly to the left of the corresponding term in the second two paths. The same statements hold for the corresponding weak bumping paths.

(B2) The first $k - 1$ terms of $rpath_{i+2}^\leq(a)$ and $rpath_{i+2}^\leq(b)$ coincide. Each of the first $k - 1$ terms of these paths is strictly to the right of the corresponding term in $rpath_i^\leq(a)$ or $rpath_{i+1}^\leq(b)$, and weakly to the left of corresponding term in $rpath_{i+1}^\leq(a)$ or $rpath_i^\leq(b)$. The same statements hold for the corresponding weak bumping paths.

47
If $k = 1$ then let $u := a_i = b_i + 1 < v := a_i + 2 = b_i + 2 < w := a_i + 1 = b_i$. If $k > 1$ then define $u$, $v$, and $w$ to be the entries of $T$, $T \Leftarrow a_i \Leftarrow a_i + 1$, and $T \Leftarrow a_i$, respectively, in position $k - 1$ of $\text{path}_i^T$, $\text{path}_{i+2}^T$, and $\text{path}_{i+1}^T$, respectively. It follows from (B1) and (B2) that:

(B3) Assume $k > 1$. Then $w$ is also the entry of $T \Leftarrow b_i$ in position $k - 1$ of $\text{path}_{i+1}^T$. Likewise, $v$ is also the entry of $T \Leftarrow b_i \Leftarrow b_i + 1$ in position $k - 1$ of $\text{path}_{i+2}^T$. In turn, $w$ is also the entry of $T$ in position $k - 1$ of $\text{path}_i^T$, and $u < v < w$.

(B4) The entry of $T$ in position $(k, k)$ is at least $w$ since $(k, k) \in \text{rpath}_i^T$.

This leaves us with three cases for the possible values of $\tau_i(a), \tau_{i+1}(a), \tau_{i+2}(a), \tau_i(b), \tau_{i+1}(b)$, and $\tau_{i+2}(b)$, which we will discuss below. Denote $\text{cseq}_{i-1}(a) = \text{cseq}_{i-1}(b)$ as in (4.14) above. Again write $\eta_k$ and $\eta_{k+1}$ for the entries in the first row of $\text{cseq}_{i+2}(a)$ in columns $k$ and $k + 1$.

First suppose the entry in position $(k, k)$ of $T$ is $w$. Then, in view of Remark 3.5, the entries of $T$ in positions $\{k, k + 1\} \times \{k, k + 1\}$ must be $T_{k+i, k+j} = w + i + j$ for all $0 \leq i \leq j \leq 1$. If $k > 1$, then row $k - 1$ of $T$ contains both $u$ and $w$ in positions off the main diagonal, so the entry in position $(k - 1, k + 1)$ of $T$ is at most $w$. If $k > 1$ and this entry is equal to $w$, then the entries of the six tableaux in (4.13) in positions $\{k - 1, k, k + 1\} \times \{k, k + 1\}$ are

Alternatively, if $k > 1$ and the entry in position $(k - 1, k + 1)$ of $T$ is less than $w$, then the entry of $T$ in position $(k - 1, k + 2)$ must be occupied by a number less than $w + 1$. In this case, or if $k = 1$, the entries of the six tableaux in (4.13) in positions $\{k, k + 1\} \times \{k, k + 1, k + 2\}$ are
In both situations, it follows by Lemma 4.6 that the values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

so by Lemma 4.7 we have $\tau_i(a) = (\gamma_k, \eta_k)$ and $\tau_{i+1}(a) = \tau_{i+2}(a) = 1$ while $\tau_i(b) = \tau_{i+2}(b) = (\gamma_k, \gamma_{k+1})$ and $\tau_{i+1}(b) = (\eta_k, \gamma_{k+1})$, so $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \eta_k)$ as desired.

Suppose next that the entry in position $(k, k)$ of $T$ is $w + 1$. If $k > 1$ then the entry in position $(k-1, k+1)$ of $T$ is at most $w$, so the entries of the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

First assume the array $\text{cseq}_{i-1}(a)$ has only $k$ columns. Then it follows by Lemma 4.6 that the values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

so by Lemma 4.7 we have $\tau_i(a) = \tau_{i+1}(b) = (\gamma_k, \eta_k)$ and $\tau_{i+1}(a) = \tau_{i+2}(a) = \tau_i(b) = \tau_{i+2}(b) = 1$, so $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \eta_k)$ as needed. If $\text{cseq}_{i-1}(a)$ has at least $k + 1$ columns, then it follows likewise that the values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in
positions \(\{k, k + 1\} \times \{k, k + 1\}\) are

\[
\begin{array}{ccc}
\gamma_k & \gamma_{k+1} & a_i \\
\eta_k & \gamma_k & a_{i+1} \\
\gamma_{k+1} & \eta_k & a_{i+2} \\
\end{array}
\]

so by Lemma 4.7 we have \(\tau_i(a) = \tau_{i+1}(b) = (\gamma_k, \eta_k)\) and \(\tau_{i+1}(a) = \tau_{i+2}(b) = (\gamma_k, \eta_k)\) and \(\tau_{i+2}(a) = \tau_i(b) = 1\), so \(\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \eta_k)\) as desired.

Finally suppose that the entry in position \((k, k)\) of \(T\) is \(x > w + 1\). If \(k > 1\) then the entry in position \((k-1, k+1)\) of \(T\) is at most \(w\), so the entries of the six tableaux in (4.13) in positions \(\{k, k + 1\} \times \{k, k + 1\}\) are

If the array \(cseq_{i-1}(a)\) has only \(k\) columns, the values of \(\gamma_{xy}\) applied to the six tableaux in (4.13) in positions \(\{k, k + 1\} \times \{k, k + 1\}\) are

so by Lemma 4.7 we have \(\tau_i(a) = (\gamma_k, \eta_k)\) and \(\tau_{i+1}(a) = 1\) and \(\tau_{i+2}(a) = (\gamma_k, \eta_k)\) while \(\tau_i(b) = (\eta_k, \eta_k)\) and \(\tau_{i+1}(b) = (\eta_k, \eta_k)\) and \(\tau_{i+2}(b) = 1\), so we have \(\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \eta_k)\) as needed. If \(cseq_{i-1}(a)\) has at least \(k + 1\) columns, then the
values of $\gamma_{xy}$ applied to the six tableaux in (4.13) in positions $\{k, k+1\} \times \{k, k+1\}$ are

$$
\begin{array}{c}
\gamma_k & \gamma_{k+1} \\
\eta_k & \gamma_{k+1} \\
\end{array}
\quad
\begin{array}{c}
\gamma_k & \gamma_{k+1} \\
\eta_k & \gamma_{k+1} \\
\end{array}
\quad
\begin{array}{c}
\gamma_k & \gamma_{k+1} \\
\eta_k & \gamma_{k+1} \\
\end{array}
$$

so by Lemma 4.7 we have $\tau_i(a) = (\gamma_k, \eta_k)$ and $\tau_{i+1}(a) = (\gamma_k, \gamma_{k+1})$ and $\tau_{i+2}(a) = (\gamma_k, \eta_{k+1})$ while $\tau_i(b) = (\gamma_k, \eta_{k+1})$ and $\tau_{i+1}(b) = (\eta_k, \eta_{k+1})$ and $\tau_{i+2}(b) = (\gamma_{k+1}, \eta_{k+1})$, so $\tau_i(a)\tau_{i+1}(a)\tau_{i+2}(a) = \tau_i(b)\tau_{i+1}(b)\tau_{i+2}(b) = (\gamma_k, \eta_{k+1}, \gamma_{k+1}, \eta_k)$ as desired. This completes our proof of the lemma. \qed

Finally, we arrive at the proof of Theorem 3.16.

Proof of Theorem 3.16. Suppose $i \in \mathbb{Z}$ and $a$ is a primed involution word. Let $b = \text{o ck}_i(a)$. Combining Lemmas 4.13, 4.14, and 4.15 shows that $P^O_{EG}(a) = P^O_{EG}(b)$.

We have $Q^O_{EG}(b) = \delta_i(Q^O_{EG}(a))$ for $i \in \{-1, 0\}$ by Lemma 4.13. Assume $i \in [\ell(a) - 2]$. Lemma 3.13 implies that the operations $\text{unprime}$ and $\text{o ck}_i$ commute while Proposition 3.15 implies that $\text{unprime}_{\text{diag}}$ and $\delta_i$ commute. It follows by (3.1) and Lemma 4.1 that

$$
\text{unprime}_{\text{diag}}(\delta_i(Q^O_{EG}(a))) = \delta_i(\text{unprime}_{\text{diag}}(Q^O_{EG}(a))) = \delta_i(Q^O_{EG}(\text{unprime}(a))) = Q^O_{EG}(\text{o ck}_i(\text{unprime}(a))) = Q^O_{EG}(\text{unprime}(b)) = \text{unprime}_{\text{diag}}(Q^O_{EG}(b)).
$$

Thus $\delta_i(Q^O_{EG}(a))$ only differs from $Q^O_{EG}(b)$ in which of its diagonal entries are primed.

It is clear from the definitions of $\delta_i$ and $Q^O_{EG}$ that $\delta_i(Q^O_{EG}(a))$ and $Q^O_{EG}(b)$ each only differ from $Q^O_{EG}(a)$ in their entries in positions $\Box_i$, $\Box_{i+1}$, and $\Box_{i+2}$, where $\Box_j$ is the box of $Q^O_{EG}(a)$ containing $j$ or $j'$. Hence, if $\Box_i$, $\Box_{i+1}$, and $\Box_{i+2}$ are all off the diagonal then necessarily $\delta_i(Q^O_{EG}(a)) = Q^O_{EG}(b)$. If exactly two of these positions are on the diagonal then the same conclusion holds by Lemma 4.13. We cannot have all three of $\Box_i$, $\Box_{i+1}$, and $\Box_{i+2}$ on the diagonal, and if exactly one of these positions is on the diagonal then it suffices by Proposition 3.15 to show that its entry is primed in $\delta_i(Q^O_{EG}(a))$ if and only if it is also primed in $Q^O_{EG}(b)$. This holds since Proposition 3.15 asserts that $\text{primes}_{\text{diag}}(Q^O_{EG}(a)) = \text{primes}_{\text{diag}}(\delta_i(Q^O_{EG}(a)))$, and by definition we have $\text{primes}(P^O_{EG}(a)) + \text{primes}_{\text{diag}}(Q^O_{EG}(a)) = \text{primes}(a) = \text{primes}(b) = \text{primes}(P^O_{EG}(b)) + \text{primes}_{\text{diag}}(Q^O_{EG}(b))$.

But $P^O_{EG}(a) = P^O_{EG}(b)$, so $\text{primes}_{\text{diag}}(Q^O_{EG}(b)) = \text{primes}_{\text{diag}}(Q^O_{EG}(a)) = \text{primes}_{\text{diag}}(\delta_i(Q^O_{EG}(a)))$. \qed

References

[1] S. Assaf. “Shifted dual equivalence and Schur P-positivity”. In: J. Combin. 9.2 (2018), pp. 279–308.

[2] A. Björner and F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics 231 (2005), Springer, New York.
[3] M. Brion. “On orbit closures of spherical subgroups in flag varieties”. In: *Comment. Math. Helv.* 76 (2001), no. 2, pp. 263–299.

[4] A. S. Buch and V. Ravikumar. “Pieri rules for the K-theory of cominuscule Grassmannians”. In: *J. Reine Angew. Math.* 668 (2012), pp. 109–132.

[5] M. B. Can, M. Joyce, and B. Wyser. “Chains in Weak Order Posets Associated to Involutions”. In: *J. Combin. Theory Ser. A* 137 (2016), pp. 207–225.

[6] E. Clifford, H. Thomas, and A. Yong. “K-theoretic Schubert calculus for OG(n, 2n + 1) and jeu de taquin for shifted increasing tableaux”. In: *J. Reine Angew. Math.* 690 (2014), pp. 51–63.

[7] P. Edelman and C. Greene. “Balanced tableaux”. In: *Adv. Math.* 63 (1987), pp. 42–99.

[8] M. Gillespie, J. Levinson, and K. Purbhoo. “A crystal-like structure on shifted tableaux”. In: *Algebraic Combinatorics* 3 (2020), pp. 693–725.

[9] M. D. Haiman. “On Mixed Insertion, Symmetry, and Shifted Young Tableaux”. In: *J. Combin. Theory Ser. A* 50 (1989), pp. 196–225.

[10] Z. Hamaker and E. Marberg. “Atoms for signed permutations”. In: *European J. Combin.* 94 (2021), 103288.

[11] Z. Hamaker, E. Marberg, and B. Pawlowski. “Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures”. In: *J. Combin. Theory Ser. A* 160 (2018), pp. 217–260.

[12] Z. Hamaker, E. Marberg, and B. Pawlowski. “Schur P-positivity and involution Stanley symmetric functions”. In: *IMRN*, Volume 2019, Issue 17, pp. 5389–5440.

[13] Z. Hamaker, E. Marberg, and B. Pawlowski. “Fixed-point-free involutions and Schur P-positivity”. In: *J. Combin.* 11.1 (2020), pp. 65–110.

[14] Z. Hamaker, A. Keilthy, R. Patrias, L. Webster, Y. Zhang, and S. Zhou, “Shifted Hecke insertion and K-theory of OG(n, 2n + 1)”. In: *J. Combin. Theory Ser. A* 151 (2017), pp. 207–240.

[15] M. Hansson and A. Hultman. “A word property for twisted involutions in Coxeter groups”. In: *J. Combin. Theory Ser. A* 161 (2019), pp. 220–235.

[16] T. Hiroshima. “Queer Supercrystal Structure for Increasing Factorizations of Fixed-Point-Free Involution Words”. Preprint (2019), arXiv:1907.10836.

[17] J. Hu and J. Zhang. “On involutions in symmetric groups and a conjecture of Lusztig”. In: *Adv. Math.* 287 (2016), pp. 1–30.

[18] J. E. Humphreys. Reflection groups and Coxeter groups. Cambridge University Press, 1990.

[19] T. Ikeda and H. Naruse. “K-theoretic analogues of factorial Schur P- and Q-functions”. In: *Adv. Math.* 243 (2013), pp. 22–66.
[20] R. La Scala, V. Nardozza and D. Senato. “Super RSK-algorithms and super plactic monoid”. In: *Internat. J. Algebra Comput.* 16 (2006), no. 2, pp. 377–396.

[21] J. B. Lewis and E. Marberg. “Enriched set-valued P-partitions and shifted stable Grothendieck polynomials”. In: *Math. Z.* (2021). [https://doi.org/10.1007/s00209-021-02751-5](https://doi.org/10.1007/s00209-021-02751-5).

[22] F. Maas-Gariépy and R. Patrias. “Set-valued domino tableaux and shifted set-valued domino tableaux”. Preprint (2020), arXiv:2011.12493.

[23] E. Marberg. “A symplectic refinement of shifted Hecke insertion”. In: *J. Combin. Theory Ser. A* 173 (2020), 105216.

[24] E. Marberg. “Bumping operators and insertion algorithms for queer supercrystals”. Preprint (2019), arXiv:1910.02261.

[25] E. Marberg. “Extending a word property for twisted Coxeter systems”. Preprint (2021), arXiv:2108.12020.

[26] E. Marberg and B. Pawlowski. “K-theory formulas for orthogonal and symplectic orbit closures”. In: *Adv. Math.* 372 (2020), 107299.

[27] E. Marberg and B. Pawlowski. “On some properties of symplectic Grothendieck polynomials”. In: *J. Pure Appl. Algebra* 225.1 (2021), 106463.

[28] R. Muth. “Super RSK Correspondence with Symmetry”. In: *Electron. J. Combin.* 26 (2019), no. 2, P2.27.

[29] R. Patrias and P. Pylyavskyy. “Dual filtered graphs”. In: *Algebraic Combinatorics* 1 (2018), pp. 441–500.

[30] R. W. Richardson and T. A. Springer. “The Bruhat order on symmetric varieties”. In: *Geom. Dedicata* 35 (1990), pp. 389–436.

[31] Bruce E. Sagan. “Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley”. In: *J. Combin. Theory Ser. A* 45.1 (1987), pp. 62–103.

[32] M. Shimozono and D. E. White. “A Color-to-Spin Domino Schensted Algorithm”. In: *Electron. J. Combin.* 8 (2001), 50 pp.

[33] J. R. Stembridge. “Shifted tableaux and the projective representations of symmetric groups”. In: *Adv. Math.* 74 (1989), pp. 87–134.

[34] D. R. Worley. “A theory of shifted Young tableaux”. PhD thesis, Massachusetts Institute of Technology, 1984.

[35] B. J. Wyser. “Symmetric subgroup orbit closures on flag varieties: Their equivariant geometry, combinatorics, and connections with degeneracy loci”. PhD thesis, University of Georgia, 2012, arXiv:1201.4397.

[36] B. J. Wyser and A. Yong. “Polynomials for symmetric orbit closures in the flag variety”. In: *Transform. Groups* 22.1 (2017), pp. 267–290.