Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

Jianlong Wang, Xiaolei Leng*, Xianbin Liu and Ronghui Zheng

State Key Laboratory of Mechanics and Control of Mechanical Structure, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People’s Republic of China
E-mail: jlwang@muaa.edu.cn, lengxl@muaa.edu.cn, xbliu@muaa.edu.cn and rhzheng@muaa.edu.cn

Received 17 July 2021
Accepted for publication 25 September 2021
Published 26 October 2021

Online at stacks.iop.org/JSTAT/2021/103211
https://doi.org/10.1088/1742-5468/ac2a9f

Abstract. Sine-Wiener (SW) noise is increasingly adopted in realistic stochastic modeling for its bounded nature. However, many features of the SW noise are still unexplored. In this paper, firstly, the properties of the SW noise and its integral process are explored as the parameter $D$ in the SW noise tends to be infinite. It is found that although the distribution of the SW noise is quite different from Gaussian white noise, the integral process of the SW noise shows many similarities with the Wiener process. Inspired by the Wiener process, which uses the diffusion coefficient to denote the intensity of the Gaussian noise, a quantity is put forward to characterize the SW noise’s intensity. Then we apply the SW noise to a one-dimensional double-well potential system and the Maier–Stein system to investigate the escape behaviors. A more interesting result is observed that the mean first exit time also follows the well-known Arrhenius law as in the case of the Gaussian noise, and the quasi-potential and the exit location distributions are very close to the results of the Gaussian noise.

Keywords: diffusion, fluctuation theorem, large deviation, stochastic processes

*Author to whom any correspondence should be addressed.
Metastability is a generic feature of many natural systems [1]. Due to the fluctuations of the environmental variables, such as the temperature, humidity, and velocity field, when we study the dynamics of these systems we have to consider the influence of the random noise [2]. For the non-equilibrium systems, noise is indeed proved to play a constructive role, e.g. the picophytoplankton dynamics [2], multi-species ecological network [3], the kinetics of chemical reactions [4], and electron systems [5].

Usually, the random noise is modeled as Gaussian white noise. Since the supremum of the Gaussian noise is infinite with probability 1 as the time tends to be infinite, the system will eventually exhibit large fluctuations no matter how slight the noise is [6]. For the Gaussian noise-induced escape or transition, extensive work has been done [7–13], and some interesting results concerning the escape behavior have been obtained [6, 14–16]. For example, in the presence of noise, the mean first exit time (MFET) of the metastable state or unstable one is larger than the deterministic decay time, which means the stability of unstable states is enhanced by the noise [1, 17]. In the weak noise limit, the MFET from one attractor to one another is exponentially proportional to the inverse of the noise intensity [6, 18]. When the escape occurs, the realization of the transition moves along the most probable exit path (MPEP) with an overwhelming probability [19, 20], and the way that the MPEP approaches the exit boundary determines the shape of the exit location distribution (ELD), a Gaussian distribution, or a non-Gaussian distribution [16, 21–23].

However, the unboundedness of the Gaussian noise makes it contradict the very nature of real physical quantity [24, 25]. Hence, nowadays various bounded noises are introduced in the modeling of stochastic systems [26–31]. One of the most widely employed bounded noises is called sine-Wiener (SW) noise, which can be presented as a sinusoidal function with a constant amplitude and a random phase described by

https://doi.org/10.1088/1742-5468/ac2a9f
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

A Wiener process [32, 33]. It has been widely adopted in various dynamic systems as a random perturbation after the first time being treated as a turbulent fluctuation in the wind flow [30, 31, 34, 35]. Due to the complexity of the dynamical system perturbed by the SW noise, so far, little research has been done in studying the properties of the SW noise-induced escape.

Recently, when we study the properties of the SW noise, we found that the system under the SW noise with large $D$ is very diffusion-like. Inspired by this, we have detailedly studied the properties of the SW noise and its induced escape in the limit case $D \to \infty$, and some interesting results have been obtained. The paper is organized as follows. First, the properties of the SW noise and its integral process are analyzed in sections 2 and 3, respectively, and a quantity is put forward to characterize the noise’s intensity. Then in section 4, a fundamental requirement for the SW noise-induced escape is given and the escape from a one-dimensional double-well potential system is studied. Then, in section 5, we extended the research to a two-dimensional system, the Maier–Stein system, to study the MFETs and ELDs. At last, the conclusion is drawn in section 6.

2. The sine-Wiener noise

In this section, we give some essential properties of the SW noise, which are crucial for the escape problem of the SW noise. First, the mathematical model of the SW noise is expressed as [24]:

$$\xi(t) = A \sin(\sqrt{D}w_t).$$

where $A$ denotes the amplitude of the noise, $w_t$ is a standard Weiner process that initially starts from zero, and $D$ is the Weiner process’ diffusion coefficient. Using the well-known properties of the Weiner process, it is easy to verify that $\langle \xi(t) \rangle = 0$, and

$$\langle \xi^2(t) \rangle = \frac{A^2}{2} (1 - \exp(-2Dt)),$$

and

$$\langle \xi(t) \xi(t + \tau) \rangle = \frac{A^2}{2} \exp\left(\frac{-D\tau}{2}\right) (1 - \exp(-2Dt)), \quad \tau \geq 0. \quad (2)$$

Applying the Fourier-transform to the correlation function in equation (2), then the power spectrum of the SW noise can be estimated as:

$$\frac{A^2}{2\pi} (1 - \exp(-2Dt)) \frac{2D}{D^2 + 4\omega^2}. \quad (3)$$

When $D$ is small, the noise becomes a narrow-band random process, and its correlation time is so considerable that it can almost be viewed as a deterministic process in a short period of time. Especially when $D = 0$, this noise becomes a deterministic excitation. For the papers [24, 32, 36] where a small $D$ is employed, the observed MFETs are very short, and the induced transitions almost have no difference with those induced by a deterministic harmonic excitation.
When $D$ is large, this noise becomes a wide-band random process, the motion of the process changes direction frequently and becomes very unpredictable. The following work is mainly attempted to investigate the features of the noise as $D \to \infty$.

First, we need to work out the stationary density of the SW noise process. From equation (1), one have

$$\xi(t) = A \sin(\sqrt{D}w(t)) = A \sin(\text{mod}(\sqrt{D}w(t), 2\pi)).$$

The phased process $\sqrt{D}w(t)$ can be viewed as a motion on a circle $S(\text{mod}2\pi)$. It is easy to verify that the stationary probability distribution on this circle is a uniform distribution $U(0,2\pi)$. Therefore, according to equation (4), the SW noise process has a stationary probability density function (PDF), which can be given as

$$\rho(x) = \begin{cases} \frac{1}{\pi \sqrt{A^2 - x^2}} & -A \leq x \leq A \\ 0 & \text{others} \end{cases}.$$ (5)

As shown in figure 1, equation (5) is verified by the numerical simulation. The second moment of the stationary SW noise process equals precisely to the value $A^2/2$ derived from equation (2) as $t \to \infty$. From the form of equation (5), we can see the stationary PDF of the SW noise only depends on $A$ but not $D$. While, from equation (2), it can be observed that the $D$ controls the convergence speed of the SW noise. The larger $D$ is chosen, the faster the noise converges to the stationary state. Hence, in the limit case, as $D \to \infty$, $\xi(t_1)$ and $\xi(t_2)$ become independent of each other for any $t_1 \neq t_2$; in addition, it can be seen that the mean and the autocorrelation function of the SW noise process are independent of time $t$, $\langle \xi(t) \rangle = 0$, $\langle \xi(t) \xi(t + \tau) \rangle = 0$. Hence, as $D \to \infty$, the SW noise becomes a weak stationary process.

Based on the above analysis, we summarize the following properties of the SW noise in the limit case $D \to \infty$:

\begin{itemize}
  \item Stationary PDF of the SW noise with $A = 1$.
  \item The stationary PDF of the SW noise only depends on $A$ but not $D$.
  \item The larger $D$ is chosen, the faster the noise converges to the stationary state.
  \item As $D \to \infty$, the SW noise becomes a weak stationary process.
\end{itemize}
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

(a) $E[\xi(t)] = 0$ for $t \geq 0$.
(b) $t_1 \neq t_2 \Rightarrow \xi(t_1)$ and $\xi(t_2)$ are independent of each other.
(c) $\xi(t)$ is a stationary process.

As is seen, in the limit case, the SW noise has a lot in common with the Gaussian white noise, while the bounded nature of the SW noise makes it more suitable than the Gaussian noise to model the real physical perturbation. For some further properties of the SW noise, the readers are referred to the recent paper [37], where some properties and relationships of the most commonly employed bounded noises are investigated within a solid mathematical ground.

3. The integral process of the sine-Wiener noise

As we all know, the movement of a particle in a liquid or gas, caused by being hit by molecules of that liquid or gas, forms a Brownian motion. The higher the temperature is, the greater the impact force on the particle is, and the more violent the Brown motion is. In math, the impact force of the molecules is modeled by the Gaussian white noise, and the Brown motion is hence depicted by the integral process of that noise. Furthermore, the diffusion capability of the perturbed particles is weighed by the second increment moment of that integral process per unite time, called the diffusion coefficient. In the following, by theoretical and numerical analysis, we show that the integral process of the SW noise in the limit case $D \to \infty$ also forms a Brown motion. Hence we use the diffusion coefficient to denote the intensity of the SW noise as in the case of Gaussian noise.

First of all, we define the integral process of the SW noise as

$$W(t) \equiv \int_0^t \xi(s) \, ds, \quad t > 0.$$  \hfill (6)

Since the SW noise is continuous and bounded, $-A \leq \xi(s) \leq A$, the integral process should be differentiable with respect to time $t$. In the limit $D \to \infty$, the SW noise $\xi(t)$ is almost white and obeys the identical distribution, as shown in equation (5). Therefore, the integral equation in equation (6) can be viewed as a generalization of the summation of independent random variables and should be asymptotic to the Gaussian distribution. If the first and second moments of this integral are known, then the specific distribution of the integral equation (6) can be determined.

The first moment of the integral process is derived as

$$E(W(t)) = E\left(\int_0^t \xi(s) \, ds\right) = \int_0^t E(\xi(s)) \, ds = 0, \quad \text{as} \ D \to \infty.$$  \hfill (7)

The calculation of the second moment is a bit more complicated. According to Ito’s formula,

$$d \left( \sin \left( \sqrt{D} w_t \right) \right) = \sqrt{D} \cos \left( \sqrt{D} w_t \right) \, dw_t - \frac{D}{2} \sin \left( \sqrt{D} w_t \right) \, dt.$$  \hfill (8)
the integral equation (8) can be rewritten as:

\[ W(t) = \int_0^t A \sin \left( \sqrt{D} w_s \right) \, ds \]

\[ = \frac{2A}{D} \left[ \int_0^t \sqrt{D} \cos \left( \sqrt{D} w_s \right) \, dw_s - \sin \left( \sqrt{D} w_t \right) + \sin \left( \sqrt{D} w_0 \right) \right]. \tag{9} \]

Square both sides of the equation (9), and take \( w_0 = 0 \), it yields

\[ [W(t)]^2 = \frac{4A^2}{D^2} \left[ \int_0^t \sqrt{D} \cos \left( \sqrt{D} w_s \right) \, dw_s - \sin \left( \sqrt{D} w_t \right) \right]^2. \tag{10} \]

Taking the expectation on both sides of the last equation, then the second moment of the integral equation (6) is obtained as

\[ E[W(t)]^2 = \frac{4A^2}{D^2} E \left\{ \left[ \int_0^t \sqrt{D} \cos \left( \sqrt{D} w_s \right) \, dw_s \right]^2 \right\}
\[ - 2 \left[ \int_0^t \sqrt{D} \cos \left( \sqrt{D} w_s \right) \, dw_s \right] \left[ \sin \left( \sqrt{D} w_t \right) \right] \}
\[ = \frac{4A^2}{D^2} E \left[ \int_0^t D \cos^2 \left( \sqrt{D} w_s \right) \, ds \right] + \frac{4A^2}{D^2} E \left[ \sin \left( \sqrt{D} w_t \right) \right]^2 \]
\[ - \frac{8A^2}{D^2} E \left[ \int_0^t \sqrt{D} \cos \left( \sqrt{D} w_s \right) \sin \left( \sqrt{D} w_t \right) \, dw_s \right] \]
\[ = \frac{4A^2}{D} \int_0^t E \left[ \cos^2 \left( \sqrt{D} w_s \right) \right] \, ds + \frac{2A^2}{D^2} \left( 1 - \exp \left( -2Dt \right) \right) \]
\[ = \frac{4A^2}{D} \int_0^t \left\{ 1 - \frac{1}{2} \left[ 1 - \exp \left( -2Ds \right) \right] \right\} \, ds + \frac{2A^2}{D^2} \left( 1 - \exp \left( -2Dt \right) \right) \]
\[ = \frac{2A^2}{D} t + \frac{3A^2}{D^2} - \frac{3A^2}{D^2} \exp \left( -2Dt \right). \tag{11} \]

Therefore, as \( D \to \infty \), the distribution of the integral process \( W(t) \) asymptotically obeys a normal distribution with \( N \left( 0, \frac{2A^2}{D} t \right) \). Furthermore, as is seen, the second moment of the integral process is proportional to time \( t \), hence it is really a diffusion process. Similarly, it can be proved that the increment of the process \( W(t) - W(s) \) should also obey a normal distribution with \( N \left( 0, \frac{2A^2}{D} (t-s) \right) \) for any \( t > s > 0 \), which means the increment of the integral process \( W(t) \) is stationary.

Next, we need to prove the increments of the integral process \( W(t) \) are independent of each other, i.e.

\[ W(t_1), W(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1}) \text{ are independent for all } 0 \leq t_1 < t_2 \cdots < t_k. \tag{12} \]
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

**Figure 2.** Probability distribution of the integral process at the time $t = 100$ s, 200 s, 500 s: the lines are theoretical prediction: a normal distribution with $N (0, 2A^2t/D)$; circles are Monte Carlo simulation.

### Table 1. The probability of $\sup_{0 < t < T} W_t > a$, with $T = 500$ s.

| $a$   | $P \left( \sup_{0 < t < T} W_t > a \right)$ | $2P (W_T > a)$ |
|-------|---------------------------------|----------------|
| 2.5   | 0.4170                          | 0.4296         |
| 5     | 0.1006                          | 0.1142         |
| 8     | 0.0098                          | 0.0114         |

For the fact that normal random variables are independent, if they are uncorrelated, it is enough to prove that

$$E[(W(t_i) - W(t_{i-1})) (W(t_j) - W(t_{j-1}))] = 0 \text{ when } t_i \leq t_{j-1}. \quad (13)$$

Replacing $W(t)$ by equation (9) into the left side of the above function (13), the equation’s correctness is easily verified. Here, we do not give proof because the procedure is very similar to equation (11).

Now, we use numerical simulation to validate those theoretical results. 2000 sample trajectories of the integral process (6) are conducted by the Monte Carlo simulation, with $D = 100$ and $A = 1$. The distributions of these trajectories at different times are shown in figure 2. The comparison shows that the theoretical and numerical results are in good agreement. In addition, the probabilities of the process surpassing the value $a$ within time $T = 500$ s and at the time $T = 500$ s are tallied respectively by these sample trajectories. As is shown in table 1, the following useful identity is also verified:

$$P \left( \sup_{0 < t < T} W_t > a \right) = 2P (W_T > a). \quad (14)$$

Therefore, we conclude that the integral process $W(t)$ of the SW noise is almost a Wiener process in the limit $D \to \infty$. For a Wiener process generated by the Gaussian white noise, its diffusion coefficient could denote the diffusion capability of the particles under the noise. Hence, it is reasonable for us to use the quantity $2A^2/D$ to denote

https://doi.org/10.1088/1742-5468/ac2a9f
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case \( D \rightarrow \infty \)

the diffusion capability of the particles under the SW noise or denote the intensity for the SW noise. Compared with the classical construction of the Wiener process which is almost nowhere differentiable, the integral process of the SW noise has a good nature that it is almost everywhere differentiable.

The results may seem quite trivial: according to the central limit theorem, the sum of amounts of independent identical distribution random variables guarantees the Gaussianity, hence if we assume that the random variables of the noise at different moments are i.i.d. the integral process of this noise will be naturally a Wiener process. So can we construct a Wiener process as a limit in distribution of any kind of i.i.d. bounded random variables?

Now, assume \( \tilde{\xi}(s) \) obeys the same distribution as \( \xi(s) \), and \( \langle \tilde{\xi}(s) \tilde{\xi}(t) \rangle = \delta(s - t) \):

\[
\rho(\tilde{\xi}(s) = x) = \begin{cases} \frac{1}{\pi \sqrt{A^2 - x^2}} & -A \leq x \leq A \\ 0 & \text{others} \end{cases}
\]

Again, we define the integral process \( \tilde{W}(t) \) of \( \tilde{\xi}(s) \) as

\[
\tilde{W}(t) = \int_0^t \tilde{\xi}(s) \, ds, \quad t > 0.
\]

According to the central limit theorem, the \( \tilde{W}(t) \) follows a normal distribution, and due to each moment \( \langle \tilde{\xi}(s) \tilde{\xi}(t) \rangle = \delta(s - t) \), then the \( \tilde{W}(t) \) is a Wiener process. But what is the distribution of the \( \tilde{W}(t) \) or the diffusion coefficient of the \( \tilde{W}(t) \)? According to our definition, can we construct \( \tilde{W}(t) \) as follows?

\[
\tilde{W}(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\xi}(k \Delta t), \quad \Delta t = t/n.
\]

The first moment of the above integral process \( E(\tilde{W}(t)) = 0 \), and the second moment \( \left[ \tilde{W}(t) \right]^2 = \frac{4t^2}{n} \), with \( n \rightarrow \infty \). The second moment of \( \tilde{W}(t) \) varies as the \( n \) changes and is not proportional to time \( t \). Obviously, \( \tilde{W}(t) \) is not a Wiener process.

The problem lines in that, for each time interval \( \Delta t \), the increment \( \tilde{W}(k \Delta t + \Delta t) - \tilde{W}(k \Delta t) \) is a random variable, and cannot be expressed as \( \tilde{\xi}(k \Delta t) \Delta t \). But what is the distribution of the increment \( \tilde{W}(k \Delta t + \Delta t) - \tilde{W}(k \Delta t) \)? It is a vicious circle.

Or in another way, as the classical construction of the Wiener process, we present the bounded noise-induced Wiener process as i.i.d. bounded random variables \( \Delta \tilde{W} \) taken at very short time intervals of non-random length \( t/n \), with \( E(\Delta \tilde{W}) = 0, E(\Delta \tilde{W})^2 = \frac{t}{n} \). The sum \( \tilde{W}(t) \) is asymptotic to the normal distribution \( N(0, t) \), and since the increments are independent, \( \tilde{W}(t) \) is a Wiener process. Therefore, the boundary of the noise can be derived as \( \sup_{0 < s < t} \left( \tilde{\xi}(s) \right) = \sup_{0 < s < t/n} \left( \tilde{\xi}(s) \right) = \frac{\max(\Delta \tilde{W})}{t/n} \equiv B \). Now, another serious
problem comes up. If we represent the same Wiener process through a more fined divi-
sion, $t/mn$, then the bounded random variables $\Delta \tilde{W}$ should obey $E\left(\Delta \tilde{W}\right)^2 = \frac{t}{nm}$, resulting in $\sup_{0<s<t/mn} \left(\tilde{\xi}(s)\right) = B/\sqrt{m}$. The boundary of the noise varies as the time division varies. Furthermore, as $m \to \infty$, the noise’s boundary tends to zero, $\sup_{0<s<t} \left(\tilde{\xi}(s)\right) \to 0$.

When such a noise acted on a dynamic system, the perturbed system will exhibit the
dynamic behavior no more than a deterministic system.

As is seen, it is hard to construct a Wiener process if we only rely on the central
limit theorem. Up to now, there is no mathematical model of the Wiener process that
is differentiable with respect to time $t$. This means that the velocity of the Brownian
motion can not be measured. While Li has successfully measured the instantaneous
velocity of a Brownian particle in 2010 [38], which contradicts the classical mathematic
model of the Brownian motion. Hence, there is an urgent need for a mathematic model
of the Wiener process which is differentiable, and the integral process $W(t)$ of the SW
noise might offer an inspiring perspective.

Now, note that, if the amplitude $A$ is fixed, and as $D \to \infty$, the high-frequency
switching direction together with the symmetrical probability distribution of the noise
will make the action of the SW noise very weak, $2A^2/D \to 0$. For a system perturbed
by Gaussian white noise, the features of the escape behavior in weak noise limit have
been studied extensively. For example: the MFET from one attractor to one another
is exponentially proportional to the inverse of the noise intensity; When the escape
occurs, the realization of the transition moves along the MPEP with an overwhelming
probability. So what about the escape behavior of the SW noise when the noise intensity
$2A^2/D \to 0$? That is what we want to investigate in the following section.

4. The escape from one-dimensional well potential system

In the $\mathbb{R}^r$ space, we consider the following type of differential equation:

$$\dot{x}_t = b(x_t) + \xi_t. \quad (15)$$

Here, $b(x) = (b^1(x), \ldots, b^r(x))$ is an $r$-dimensional vector, $\xi_t$ is an $r$-dimensional random
process with each component independent from each other.

Before moving on, it should be noted that the SW noise’s value is bounded by its
amplitude; therefore, a particle excited by this noise can never transcend the domain
where the field intensity is stronger than the amplitude. Here, we give a fundamental
requirement for the SW noise-induced escape. In math, it is expressed as follows: if a
particle wants to fluctuate from point $x_1$ to point $x_2$, only if there exists a differentiable
path $\varphi_t$, $0 \leq t \leq T$, connecting $x_1$ and $x_2$ such that $\sup_{0 \leq t \leq T} |\varphi_t - b'(\varphi_t)| < A$, $\forall i \in 1, 2, \ldots, r$, can the fluctuation be realized. While, for the Gaussian white noise, no matter
how small the noise is, the supremum of the noise is infinite as $t \to \infty$. Hence the particle
excited by the Gaussian noise always has the opportunity to overcome the field and
fluctuates to anywhere it wants to go [6].

https://doi.org/10.1088/1742-5468/ac2a9f
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

Figure 3. The potential well of $V(x) = -0.25x^2 + 0.125x^4$. (Inset) A particle propelled by a constant force $F = \sqrt{3}/9$ stagnated at the position $x = -\sqrt{3}/3$.

Now we consider the escape of a particle from a double-well potential field [36]. The potential function is given by $V(x) = -0.25x^2 + 0.125x^4$ and the vector field $b(x) = -V'(x)$. For this system, two stable states are located on $x = \pm 1$, respectively, and an unstable steady state on $x = 0$, see figure 3. Due to the symmetry of the system, we need only to consider the escape from $x = -1$ to $x = 1$. If a particle wants to escape from one stable position to another, it must overcome the height of potential well from the stable state $x = -1$ to the unstable state 0.

In the first case, we assume the random perturbation $\xi_t$ is an SW noise. Through a simple analysis, it can be found that the steepest slope from $x = -1$ to $x = 0$ lies in the position $x = -\sqrt{3}/3$ with $\dot{x}_{\text{min}} = -\sqrt{3}/9$. Hence the escape only happens when the amplitude of the SW noise $A > \sqrt{3}/9$. Otherwise, even a constant excitation $F = \sqrt{3}/9$ cannot bring the particle from $x = -1$ to $x = 1$, see figure 3. Now assuming the noise's amplitude is large enough to escape, three groups of numerical simulations are conducted. In each group, the parameter $D$ is fixed, and various noise amplitudes $A$ are chosen. The MFETs of these groups are shown in table 2. Plotting the MFETs versus the quantity $2A^2/D$, which is obtained to denote the diffusion capability under the SW noise in section 3, we can see the MFETs in each group in figure 4 obey the following asymptotic law

$$\langle \tau_D \rangle \sim \exp \left( \frac{\Delta \phi}{2A^2/D} \right). \quad (16)$$

This is identical to the Arrhenius law, which is obtained by Kramers [22]. The quasi potential $\Delta \phi$ found by fitting these data are 0.466, 0.283, and 0.269 for $D = 5$, $D = 50$, and $D = 2000$, respectively.

For comparison, the perturbation under the white Gaussian noise with noise intensities $\tilde{D}$ is also considered. Because it is a one-dimensional system, according to the large deviation theory, the optimal fluctuation path should move along with the inverse of the vector field, and the quasi-potential of the system can be directly given as

https://doi.org/10.1088/1742-5468/ac2a9f
### Table 2. The MFET of the SW noise.

| $D$  | $A^2$ | MFET(s) | $A^2$ | MFET(s) | $A^2$ | MFET(s) | $A^2$ | MFET(s) | $A^2$ | MFET(s) |
|------|-------|---------|-------|---------|-------|---------|-------|---------|-------|---------|
| 5    | 0.36  | 0.25    | 0.16  |        | 0.144 | 0.12    |       |         |       |         |
| 60   | 188   | 2180    | 5400  | 42498  |       |         |       |         |       |         |
| 50   | 2.5   | 1.69    | 1.44  | 1.25   | 1.0   |         |       |         |       |         |
| 77   | 318   | 588     | 1289  | 5487   |       |         |       |         |       |         |
| 2000 | 100   | 67.6    | 57.6  | 50     | 40    |         |       |         |       |         |
| 80   | 298   | 591     | 1206  | 4535   |       |         |       |         |       |         |

### Table 3. The MFET of the Gaussian noise.

| $\tilde{D}$ | MFET(s) |
|-------------|---------|
| 0.100       | 64      |
| 0.0636      | 206     |
| 0.0576      | 384     |
| 0.0500      | 729     |
| 0.0400      | 2452    |

Through numerical simulation, the MFETs for different noise intensities $\tilde{D}$ are shown in table 3. Plotting the MFET versus the noise intensity $\tilde{D}$ shows the coefficient $\Delta \phi = 0.243$ derived by fitting the data according to equation (16) is very close to the theoretical value $\Delta \phi = 0.25$. Compared with the $\Delta \phi$ of Gaussian noise and that of the SW noise where $D$ is large, we can see that they are in good agreement. For the Gaussian white noise, Freidlin’s large deviation theory has established the action functional and hence the quasi-potential equals $\phi(x) = 0.25$ for this example, while it is interesting that the SW noise, of which the distribution is totally different from the Gaussian noise, shows its quasi-potentials very close to that of the Gaussian noise.

Here, two sample trajectories of the SW noise-perturbed system are shown in figure 5. The amplitudes $A$ of both noises are the same, while the parameters $D$ are different.
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

As is seen in figure 5(a), the sample trajectory with a small $D$ is relatively smooth, and the noise mainly manifests as a positive excitation when the transition happens. While for the trajectory with large $D$, as shown in figure 5(b), the transition trajectory is zigzag and erratic, and the direction of the noise changes frequently when the transition happens. The transition mechanism is totally different for these two noises. For the noise with a small parameter $D$, due to the long-term correlation of the noise, the noise behaviors more like a deterministic excitation, and the continuous positive excitation pushes the particle to transfer quickly. That is the reason that the observed MFETs are very short in the papers [24, 32, 36]. In contrast, for the noise with a large parameter $D$, due to the frequent change of the noise’s direction, the particle wanders aimless, and the transition happens very occasionally. The larger the diffusion coefficient $2A^2/D$ is, the more violently the particle diffuses, the more likely the transition happens.

Now we consider another important exit phenomenon, the noise enhanced stability (NES): at the presence of noise, the stability of metastable or unstable states will be enhanced and the MFET of the metastable state will be larger than the deterministic decay time $\tau_{\text{det}}$.

Consider the vector field $b(x) = -V'(x)$, where the potential function is given by $V(x) = 0.3x^2 - 0.2x^3$. The potential file is shown in figure 6. The system has a local

https://doi.org/10.1088/1742-5468/ac2a9f
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

Figure 6. Potential profile of $V(x) = 0.3x^2 - 0.2x^3$.

stable point located at $x = 0$, an unstable equilibrium at $x = 1$, and the potential well intersects the $x$-axis at $x = 1.5$.

Under the assumption that the perturbation is a Gaussian white noise, and with noise intensity $\tilde{D}$, i.e. $\langle \xi(t) \xi(s) \rangle = \tilde{D} \delta(t - s)$, the MFET of the system has a closed analytical expression \cite{39}:

$$\langle \tau(x_0, x_F) \rangle = \frac{2}{\tilde{D}} \int_{x_0}^{x_F} e^{2V(z)/\tilde{D}} \int_{-\infty}^{z} e^{-2V(y)/\tilde{D}} \, dy \, dz.$$  \hspace{1cm} (17)

Here, $x_0$ is the initial point, and $x_F$ is the exit point.

We choose $x_F = 2.2$ as the paper \cite{1} does, and the MFETs from the different initial points $x_0$ are shown in figure 7. As is seen, the MFET exhibits two different patterns. For $1 \leq x_0 < 1.5$, the MFET diverges as the noise intensity tends to zero, due to the small possibility that the Brownian particle transits into the potential well and is trapped in the well in the weak noise limit. And for $x_0 \geq 1.5$, the MFET shows a nonmonotonic behavior as a function of the noise intensity.

Now let us check whether the system under the SW noise with large $D$ will exhibit the same NES phenomenon. First, we fix $D = 200$, and vary the amplitude $A$. The MFET obtained by simulation is also shown in figure 7. Each MFET is derived from a total of 1000000 sample trajectories. As is seen, MFETs of the SW noise agree quite well with the theoretical MFETs corresponding to the equivalent intensity Gaussian white noise. For lower values of $\tilde{D}$, due to the need for much more realizations, we can not obtain the simulated MFETs as the theoretical curves. A different result from the Gaussian noise can be expected that when the amplitude $A$ decreases lower than the steepest slope of the potential profile, $A \leq \max_{1 \leq x \leq x_0} |V'(x)|$, the Brownian particle propelled by the SW noise can never transit into the well bottom, resulting in the fact that the MFET tends to the deterministic escape time $\tau_{\text{det}}$. 

https://doi.org/10.1088/1742-5468/ac2a9f
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

Figure 7. MFET from the different initial positions $x_0$: the lines are the theoretical prediction for the Gaussian white noise, and the symbols are the numerical results for the SW noise.

5. The escape from the Maier–Stein system

Now we extend our investigation to a two-dimensional system, the Maier–Stein system. The system is given as follows [40]:

$$
\dot{x} = x - x^3 - \alpha xy^2, \quad \dot{y} = -\mu y(1 + x^2) .
$$

(18)

Here, $\alpha$ and $\mu$ are parameters. The vector field of the system is symmetric about both the $x$-axis and the $y$-axis. Two stable points are located at $(\pm 1, 0)$, and a saddle point at $(0, 0)$. The basins of attraction are separated by the $y$-axis, which are the stable manifolds of the saddle point. Assuming the system is perturbed by random noise, the motion equation of the particle becomes

$$
\dot{x} = x - x^3 - \alpha xy^2 + \xi_1 \quad \dot{y} = -\mu y(1 + x^2) + \xi_2 .
$$

(19)

Here, $\langle \xi_i (t) \xi_j (s) \rangle = 0$. For the symmetry about the $y$-axis, we need only to study the exit from the left half-plane.

If the perturbation is Gaussian white noise, and $\langle \xi_i (t) \xi_j (s) \rangle = \tilde{D} \delta_{ij} (t - s)$, a steady PDF is supposed to assume the form [18]

$$
p(x) \approx C(x) \exp(-\phi(x)/\tilde{D}), \quad x = (x, y) .
$$

(20)

asymptotically for small $\tilde{D}$. Here, $\phi(x)$ is the quasi-potential of the system.

Substituting equation (20) into the Fokker–Plank equation and denoting the vector field equation (18) by $b(x)$, result in

$$
\left( b + \frac{1}{2} \nabla \phi \right) \cdot \nabla \phi - \tilde{D} \left[ \nabla \cdot b + \frac{1}{2} \nabla \cdot \nabla \phi \right] = 0 .
$$

(21)

https://doi.org/10.1088/1742-5468/ac2a9f
Obviously, the Freidlin Hamilton–Jacobi equation for \( \phi \)

\[
\left( b + \frac{1}{2} \nabla \phi \right) \cdot \nabla \phi = 0 \tag{22}
\]
yields the weak noise asymptotics. To solve (22) is to consider the Hamiltonian

\[
H = b \cdot p + \frac{1}{2} p \cdot p \quad \text{with the momenta } p \equiv \frac{\partial \phi}{\partial x},
\]

and to integrate

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p} = b(x) + p,
\]

\[
\frac{dp}{dt} = - \frac{\partial H}{\partial x} = - \left[ \frac{\partial b(x)}{\partial x} \right]^T p, \tag{23}
\]
as well as \( \dot{\phi} = p \cdot \dot{x} \).

More specifically, equation (23) can be rewritten as

\[
\dot{x} = x - x^3 - \alpha xy^2 + p_x
\]
\[
\dot{y} = -\mu y(1 + x^2) + p_y
\]
\[
\dot{p}_x = (1 - 3x^2 - \alpha y^2) p_x + 2\mu xyp_y
\]
\[
\dot{p}_y = 2\alpha xy p_x + \mu (1 + x^2) p_y. \tag{24}
\]

Then, solving the Hamilton–Jacobi equation (24), the quasi-potential of the system can be derived.

Maier and Stein [16, 40] have shown that when the MPEP approaches the saddle point, the parameters \( \alpha \) and the ration \( \mu = |\lambda_s|/\lambda_u \) (\( \lambda_s \) and \( \lambda_u \) are the stable and unstable eigenvalues of the linearized deterministic dynamics at the saddle point, respectively.) determine the angle of the MPEP to the attraction boundary. If the MPEP is perpendicular to the separatrix, then the ELD will be a Gaussian centered on the saddle. Otherwise, the ELD will be skewed, asymptotic to a non-Gaussian distribution.

By applying the ordered upwind method [41] to the Hamilton–Jocabi function (24), the system’s quasi-potential and the MPTPs are derived, shown in figure 8. The quasi-potential at the saddle point \((0, 0)\) are 0.500 and 0.456 for \( \alpha = 0.8 \) and \( \alpha = 1.6 \), respectively. For \( \alpha = 0.8 \), from the shape of the quasi-potential, it is easy to see that the MPTPs from point \((-1, 0)\) to the section \( x = a, \ -1 \leq a \leq 0 \) always lies on the \( x \)-axis, so the ELDs on these sections are Gaussian distributions centered on \( y = 0 \), as shown in figures 9(a) and (b). For another parameter \( \alpha = 1.6 \), we can see the MPTPs from point \((-1, 0)\) to the sections \( x = -0.3 \) and \( x = 0 \) move along near the red curves, resulting in the ELDs on those sections having double peaks, shown in figures 9(c) and (d).

For the SW noise, there is no such theoretical work can be done for the scarce of corresponding theory. Hence, we can only use the numerical simulation method to investigate the escape behavior. We choose the parameters of the SW noise such that \( 2A^2/D = \tilde{D} \), which means the diffusion ability under the SW noise and the Gaussian noise are equal. Simulating the system with such an SW noise, the ELDs on these sections are obtained as shown in figure 9, and the MFETs to the right half-plane are shown.
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

Figure 8. Quasi-potential of the system under white Gaussian noise, the red lines are the MPTPs: (a) $\alpha = 0.8$ and $\mu = 0.4$. (b) $\alpha = 1.6$ and $\mu = 0.4$.

Figure 9. ELD. Lines: white Gaussian noise with intensity 0.04; circles: SW noise with $D = 200$ and $A = 2$; (a) $\alpha = 0.8$, $\mu = 0.4$, $x = -0.5$. (b) $\alpha = 0.8$, $\mu = 0.4$, $x = 0$. (c) $\alpha = 1.6$, $\mu = 0.4$, $x = -0.3$. (d) $\alpha = 1.6$, $\mu = 0.4$, $x = 0$.

in figure 10. We can see that the ELDs on these sections agreed quite well with that of Gaussian noise. Based on fitting the MFETs, according to equation (16), the quasi-potential under the SW noise are 0.497 and 0.440 for $\alpha = 0.8$ and $\alpha = 1.6$, respectively. And the quasi-potential under the Gaussian noise 0.470 are and 0.433. Both the results are very close to the theoretical prediction derived from the ordered upwind method. Furthermore, by the good agreement of the ELDs on these sections, we can even judge that the MPEP under the SW noise should also follow along with the MPEP of white Gaussian noise. For the SW noise with a small parameter $D$, even assuming that the

https://doi.org/10.1088/1742-5468/ac2a9f
In this paper, the properties of the SW noise and its integral process are explored in the limit case $D \to \infty$. It is shown that although the probability distribution of the SW noise is totally different from the Gaussian noise, the integral process of the SW noise has many similarities with the Wiener process. Inspired by the Wiener process, which uses the diffusion coefficient to weigh the diffusion capability under the noise or the intensity of the noise, a quantity is put forward to characterize the intensity of the SW noise. By investigating noise-induced escape in the one-dimension potential systems and the Maier–Stein system, we found that when the amplitude of the SW noise is large enough for the escape, the MFET of the SW noise with large $D$ also follows the Arrhenius law with respect to the noise intensity and the quasi-potential is very close to that of Gaussian noise. Furthermore, when both noise intensities are equal, the ELDs also show good agreement, and we judge that the MPTP of the SW noise should also follow that of the Gaussian noise. Therefore, the excellent agreement between these two noises provides us a new window to consider the SW noise perturbed system. For the system excited by the SW noise, it is usually very difficult to analyze its dynamic behavior. According to the results of this paper, if an equivalent Gaussian white noise replaces the SW noise, then the study of the system can be greatly simplified while many statistical properties
are preserved. On the other hand, the good agreement between these two noises might explain why using the Gaussian noise to model the system could still get some useful statistical results while many random excitations in real physics are bounded.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant No. 11772149), the Research Fund of State Key Laboratory of Mechanics and Control of Mechanical Structures (Nanjing University of Aeronautics and Astronautics) (Grant No. 0113G01), and the Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD).

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] Fiasconaro A, Spagnolo B and Boccaletti S 2005 Signatures of noise-enhanced stability in metastable states Phys. Rev. E 72 61110
[2] Denaro G et al 2013 Spatio-temporal behaviour of the deep chlorophyll maximum in Mediterranean Sea: development of a stochastic model for picophytoplankton dynamics Ecol. Complex. 13 21–34
[3] Cuchi S, de Pasquale F and Spagnolo B 1996 Self-regulation mechanism of an ecosystem in a non-Gaussian fluctuation regime Phys. Rev. E 54 706–16
[4] Kuang Z B 2015 Energy and entropy equations in coupled nonequilibrium thermal mechanical diffusive chemical heterogeneous system Sci. Bull. 60 952–7
[5] Lin P V, Shi X, Jaroszynski J and Popović D 2012 Conductance noise in an out-of-equilibrium two-dimensional electron system Phys. Rev. B 86 3209–19
[6] Freidlin M and Wentzell A D 1988 Random Perturbations of Dynamical Systems (New York: Springer)
[7] Smelyanskiy V N, Dykman M I and Mcclintock P V E 2016 Optimal paths and the prehistory problem for large fluctuations in noise-driven systems Phys. Rev. Lett. 68 2718–21
[8] Dykman M I, Millonas M M and Smelyanskiy V N 1994 Observable and hidden singular features of large fluctuations in nonequilibrium systems Phys. Lett. A 195 53–8
[9] Chen Z 2017 Subtle escaping modes and subset of patterns from a nonhyperbolic chaotic attractor Phys. Rev. E 95 012208
[10] Schuss Z and Matkowsky B J 1979 The exit problem: a new approach to diffusion across potential barriers SIAM J. Appl. Math. 36 604–23
[11] Pizzolato N, Fiasconaro A, Adorno D P and Spagnolo B 2010 Resonant activation in polymer translocation: new insights into the escape dynamics of molecules driven by an oscillating field Phys. Biol. 7 34001
[12] Dubkov A A and Spagnolo B 2005 Acceleration of diffusion in randomly switching potential with supersymmetry Phys. Rev. E 72 41104

https://doi.org/10.1088/1742-5468/ac2a9f
Investigation on the properties of sine-Wiener noise and its induced escape in the particular limit case $D \to \infty$

[13] Mantegna R N and Spagnolo B 1998 Probability distribution of the residence times in periodically fluctuating metastable systems *Int. J. Bifurcation Chaos* **08** 783–90

[14] Bobrovsky B Z and Schuss Z 1982 A singular perturbation method for the computation of the mean first passage time in a nonlinear filter *SIAM J. Appl. Math.* **42** 174–87

[15] Schuss B J and Schuss Z 2018 The exit problem for randomly perturbed dynamical systems *SIAM J. Appl. Math.* **33** 365–82

[16] Maier R S and Stein D L 1997 Limiting exit location distributions in the stochastic exit problem *SIAM J. Appl. Math.* **57** 752–90

[17] Agudov N V and Spagnolo B 2001 Noise-enhanced stability of periodically driven metastable states *Phys. Rev. E* **64** 35102

[18] Chen Z and Liu X 2017 Singularities of fluctuational paths for an overdamped two-well system driven by white noise *Physica A* **469** 206–15

[19] Einchcomb S J B and McKane A J 1995 Use of Hamiltonian mechanics in systems driven by colored noise *Phys. Rev. E* **51** 2974–81

[20] Beale P 1989 Noise-induced escape from attractors *J. Phys. A. Math. Gen.* **22** 3283

[21] Roy R V and Nauman E 1995 Noise-induced effects on a non-linear oscillator *J. Sound Vib.* **183** 269–95

[22] Kramers H A 1940 Brownian motion in a field of force and the diffusion model of chemical reactions *Physica* **7** 284–304

[23] Lupu D G, Maier R S, Mannella R, McClintock P V E and Stein D L 1999 Observation of saddle-point avoidance in noise-induced escape *Phys. Rev. Lett.* **82** 1806–9

[24] Bobrovsky B Z and Schuss Z 1982 A singular perturbation method for the computation of the mean first passage time in a nonlinear filter *SIAM J. Appl. Math.* **42** 174–87

[25] Schuss B J and Schuss Z 2018 The exit problem for randomly perturbed dynamical systems *SIAM J. Appl. Math.* **33** 365–82

[26] Maier R S and Stein D L 2000 How an anomalous cusp bifurcates in a weak-noise system *Phys. Rev. Lett.* **85** 1358–61

[27] Cameron M K 2012 Finding the quasipotential for nongradient SDEs *Physica D* **241** 1532–50

https://doi.org/10.1088/1742-5468/ac2a9f