MODEL STRUCTURES FROM A MONAD ON PRESHEAVES

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ABSTRACT. In this note we describe conditions under which the algebras for
a monad on a presheaf category equipped with some additional structure are
fibrant objects in a model structure. We also prove that when these conditions
are satisfied the resulting model structure is, in a suitable sense, the smallest
model structure for which the units of the monad give a fibrant replacement.

1. Homotopy theory of presheaves

Throughout \( \mathcal{C} \) is a small category and \( \hat{\mathcal{C}} \) denotes the category of presheaves on
\( \mathcal{C} \). We recall some useful results about the homotopy theory of presheaves due to
Cisinski [1, 2].

1.1. Elementary homotopy data.

Definition 1.1. A cylinder for a presheaf \( X \) on \( \mathcal{C} \) consists of a tuple \( (X \otimes I, \partial^0, \partial^1, \sigma) \) as indicated
in the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\partial_0} & X \otimes I \\
\downarrow{1_X} & & \downarrow{\sigma} \\
X & & X
\end{array}
\]

such that \([\partial^0, \partial^1] : X + X \to X \otimes I\) is a monomorphism.

A morphism of cylinders \( X \otimes I \to Y \otimes I \) consists of maps \( \phi : X \to Y \) and
\( \psi : X \otimes I \to Y \otimes I \) such that

\[
\psi \circ \partial^e = \partial^e \circ \phi \\
\phi \circ \sigma = \sigma \circ \psi
\]

for \( e = 0, 1 \). This data determines a category \( \text{Cyl}(\mathcal{C}) \) of cylinders in \( \hat{\mathcal{C}} \).

A functorial cylinder is a section of the forgetful functor \( \text{Cyl}(\mathcal{C}) \to \mathcal{C} \).

Elementary homotopy data on \( \hat{\mathcal{C}} \) consists of a functorial cylinder \((\_ \otimes I)\) which
commutes with small colimits and preserves monomorphisms and for which each
square

\[
\begin{array}{ccc}
K & \xrightarrow{\partial^e} & L \\
\downarrow{j} & & \downarrow{\partial^e} \\
K \otimes I & \xrightarrow{j \otimes I} & L \otimes I
\end{array}
\]

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is a pullback when \( j : K \to L \) is a monomorphism, for \( e = 0, 1 \).

**Example 1.2.** Multiplication by the simplicial interval \( \Delta[1] \) provides the category \( \SSet \) of simplicial sets with elementary homotopy data. We consider later a different choice of elementary homotopy data on \( \SSet \).

Given elementary homotopy data \((- \otimes I)\) on \( \hat{C} \) there is an associated notion of homotopy obtained by defining a \((- \otimes I)\)-homotopy from \( f : X \to Y \) to \( g : X \to Y \) to consist of a map \( \theta : X \otimes I \to Y \) such that \( \theta \circ \partial^0 = f \) and \( \theta \circ \partial^1 = g \). We write \( \simeq \) for the homotopy relation and \( \simeq^\ast \) for the equivalence relation on hom-sets generated by \( \simeq \). \([X,Y]\) denotes the quotient of the hom-set \( \SSet(X,Y) \) by \( \simeq^\ast \).

Finally, note that these relations are congruences for composition.

### 1.2. Anodyne extensions.

Let \( T \) be any set of maps in \( \hat{C} \) and define

\[
\Lambda(T) := \{ K \otimes I \cup L \otimes \partial I \to L \otimes I \mid K \to L \in T \}.
\]

Then, given elementary homotopy data consisting of a cylinder \((- \otimes I)\) on \( \hat{C} \) and a set \( S \) of monomorphisms we define a collection of maps \( \Lambda_I(S) \) inductively as follows:

\[
\Lambda^n_I(S) := S \cup \{ K \otimes I \cup L \otimes \{e\} \to L \otimes I \mid K \to L \in M, \ e = 0, 1 \}
\]

and then:

\[
\Lambda^{n+1}_I(S) := \Lambda(\Lambda^n_I(S))
\]

\[
\Lambda_I(S) := \bigcup_{n \geq 0} \Lambda^n_I(S),
\]

where \( M \) is any set of monomorphisms such that the saturated class generated by \( M \) is the class of all monomorphisms (such a \( M \) will always exist in a presheaf topos). In this case we define the class \( \mathfrak{A}_I(S) \) of anodyne extensions generated by \( I \) and \( S \) by

\[
\mathfrak{A}_I(S) := \mathfrak{A}(\Lambda_I(S)\mathfrak{A}).
\]

The class \( \mathfrak{A}_I(S) \) is, in fact, the smallest saturated class of monomorphisms which contains

\[
S \cup \{ K \otimes I \cup L \otimes \{e\} \to L \otimes I \mid K \to L \in M, \ e = 0, 1 \}
\]

and is closed under the operation \( \Lambda(-) \). We say that an arrow \( f : X \to Y \) is a **naive fibration** if it has the RLP with respect to \( \mathfrak{A}_I(S) \). A map \( f : X \to Y \) is then said to be a **weak equivalence** if the induced map

\[
[Y,A] \to [X,f^*A]
\]

is bijective for every (naively) fibrant object \( A \). Note that, given elementary homotopy data \((- \otimes I)\) on \( \hat{C} \), it is possible to describe the sets \([Y,A]\) as consisting of the set of maps \( Y \to A \) modulo the equivalence relation generated by \( I \)-homotopy. We note the following lemma used by Cisinski in the proof of Theorem 1.4 below.

**Lemma 1.3.** If \( A \) is (naively) fibrant, then the homotopy relation \( \simeq \) is an equivalence relation on \( \hat{C}(X,A) \), for any presheaf \( X \).

We will sometimes make free use of Lemma 1.3 later on without mention. We now arrive at one of Cisinski’s main results about the homotopy theory of presheaf categories.
Theorem 1.4 (Cisinski). If $I$ is elementary homotopy data on $\hat{C}$ and $S$ is a set of monomorphisms, then there is a model structure on $\hat{C}$ with the cofibrations the monomorphisms, the weak equivalences as above and the fibrations those maps having the RLP with respect to maps which are simultaneously cofibrations and weak equivalences.

When $S$ is a set of monomorphisms and $I$ is elementary homotopy data on $\hat{C}$ the model structure from Theorem 1.4 is referred to as the $(S,I)$-generated model structure on $\hat{C}$ or, where $I$ is understood, as the $S$-generated model structure on $\hat{C}$. Observe that in such a model structure the $I$-homotopy equivalences are always also weak equivalences.

1.3. Example: Kan complexes. When $\hat{C}$ is the category $\text{SSet}$ of simplicial sets, $I$ is the simplicial interval $\Delta[1]$ and the set $S$ from above is the emptyset, Theorem 1.4 gives the classical model structure on $\text{SSet}$ in which weak equivalences are weak homotopy equivalences and the fibrant objects are exactly the Kan complexes.

1.4. Example: Quasi-categories. We include here as an example the case of the quasi-category model structure on $\text{SSet}$ [4]. Recall that here $J^\infty$ is the infinite dimensional sphere: the nerve of the groupoid interval consisting of two object $\bot$ and $\top$ with one non-identity isomorphism $u : \bot \to \top$ with inverse $d : \top \to \bot$. In this instance $J^\infty$ is the appropriate interval for which we apply Theorem 1.4.

We write $\tau_1 : \text{SSet} \to \text{Cat}$ for the left-adjoint to the nerve functor. I.e., the left-Kan extension of the inclusion $\Delta \to \text{Cat}$ along the Yoneda embedding. We say that $\tau_1(X)$ is the fundamental category of $X$ and write $\tau_0(X)$ for the set of isomorphism classes of objects of $\tau_1(X)$. A map of $f : X \to Y$ is said to be a weak categorical equivalence if, for every quasi-category $A$, the induced map

$$\tau_0(Y, A) \xrightarrow{\tau_1(f, A)} \tau_0(X, A)$$

is bijective. Using a lifting property of $J^\infty$ established by Joyal [3] it is straightforward to establish that, when $A$ is a quasi-category, the set $\tau_0(X, A)$ has another description as the coequalizer

$$\text{SSet}(X \times J^\infty, A) \xrightarrow{X\bot} \text{SSet}(X, A) \xrightarrow{X\top} \tau_0(X, A).$$

Thus, the definition of weak categorical equivalences given here would correspond exactly with that given by Theorem 1.4 if it can be established that, for $J^\infty$ and an appropriate set $S$ of maps as above, the quasi-categories are precisely the naïvely fibrant objects.

It follows, that the saturated class generated by the set $S$ of inner horns $\Lambda^k[n] \to \Delta[n]$ contains all maps of the form

$$(\partial \Delta[n] \times J^\infty) \cup (\Delta[n] \times \{e\}) \to \Delta[n] \times J^\infty.$$ 

Therefore, to see that sat($S$) = $\mathfrak{A}_{J^\infty}(S)$, it suffices to prove that sat($S$) is closed under $\Lambda^i(-)$, which follows from results in [4]. Thus, the model structure coming from Theorem 1.4 in this case is precisely the quasi-category model structure.
2. A model structure for $T$-algebras

In this section we study model structures on $\mathcal{C}$ derived from a monad $T$ on $\mathcal{C}$ possessing appropriate structure. I.e., we here describe conditions on a monad $T$ on $\mathcal{C}$ which yield a model structure on $\mathcal{C}$ in which all $T$-algebras are fibrant.

We henceforth assume given a fixed choice of elementary homotopy data $(- \otimes I)$ for $\mathcal{C}$.

2.1. $T$-weak equivalences. We arrive now at a modification of the notion of weak equivalence from above.

**Definition 2.1.** A map $f : X \to Y$ is a $T$-weak equivalence (relative to $(- \otimes I)$) if, for any $T$-algebra $A$, the induced map 

$$[Y, A] \xrightarrow{[f, A]} [X, A]$$

is a bijection.

It immediately follows from this definition that the $T$-weak equivalences satisfy the familiar three-for-two property and that the collection $W_T$ of $T$-weak equivalences contains all $I$-homotopy equivalences.

2.2. Axioms on $T$. In this first attempt to arrive at a model structure on $\mathcal{C}$ related to $T$ in the way described above we first assume that $T$ satisfies the following two axioms.

1. **(M1)**: $T$ preserves monomorphisms and the units $\eta_X : X \to T(X)$ are monomorphisms.
2. **(M2)**: There exists a set of presheaves $(G_i)_{i \in I}$ on $\mathcal{C}$ such that, for every presheaf $X$, the unit $\eta_X : X \to TX$ is in the saturated class generated by the maps

$$G^i \xrightarrow{\eta_{G^i}} T(G^i)$$

for $i \in I$.

Let $T$ be a monad on $\mathcal{C}$ which satisfies these axioms and write $\mathfrak{S}$ for the generating set of units $\{G^i \to TG^i\}$. Then, by Cisinski’s Theorem 1.4, there is a model structure on $\mathcal{C}$ in which the weak equivalences, fibrations and cofibrations are as described in Section 1 where now the set $S$ is $\mathfrak{S}$. We now investigate conditions under which the $T$-weak equivalences coincide with the weak equivalences obtained from the $\mathfrak{S}$-generated model structure.

**Lemma 2.2.** All units $\eta_X : X \to T(X)$ are acyclic cofibrations in the $\mathfrak{S}$-generated model structure on $\mathcal{C}$.

**Proof.** By the generation axiom, all units are in the saturated class $\text{sat}(\mathfrak{S})$ of monomorphisms generated by $\mathfrak{S}$. Moreover, the collection of anodyne extensions $\mathfrak{A}_I(\mathfrak{S})$ is saturated and contains $\mathfrak{S}$. Therefore, $\text{sat}(\mathfrak{S})$ is contained in $\mathfrak{A}_I(\mathfrak{S})$. 

**Lemma 2.3.** All $T$-weak equivalences are weak equivalences.
Proof. Assume $f : X \to Y$ is a $T$-weak equivalence and let a fibrant object $A$ be given. Then, by Lemma 2.2, there exists a retraction $\alpha : TA \to A$ as indicated in the following diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow{\eta_A} & & \downarrow{\alpha} \\
TA & & \\
\end{array}
$$

So, given a map $g : X \to A$ there exists, because $f$ is $T$-weak equivalence, a $k : Y \to TA$ such that $k \circ f$ is in the same homotopy class as $\eta_A \circ g$. Defining $\bar{g}$ to be $\alpha \circ k$, it follows that $\bar{g} \circ f$ is in the same homotopy class as $g$. Moreover, this construction yields the same homotopy class given any $g' : X \to A$ which is homotopic to $g$. Thus, $f$ is a weak equivalence. \qed

When $T$ satisfies $(M2)$ it makes sense to consider whether it also satisfies the following, rather strong, axiom:

$$(M3): \text{If } A \text{ is a } T\text{-algebra, then } A \to 1 \text{ has the RLP with respect to elements of } \mathbb{N}_I(\mathbb{G}).$$

Clearly $(M3)$ is equivalent to the statement that all $T$-algebras are fibrant and so, as might be expected, it is in general non-trivial to establish that it is satisfied by a given monad. Consequently, the following proposition is certainly in accord with the principle of “preservation of difficulty”.

**Theorem 2.4.** Let $T$ be a monad on $\mathcal{C}$ satisfying $(M1)$–$(M3)$, then there exists a model structure on $\mathcal{C}$ for which the weak equivalences are the $T$-weak equivalences, the cofibrations are the monomorphisms, all $T$-algebras are fibrant and the units $\eta_X : X \to T(X)$ present $T(X)$ as a fibrant replacement of $X$. Moreover, this model structure is precisely the one obtained by applying Theorem 1.4 to the set $\mathbb{G}$ of generating units.

**Proof.** It suffices, by the foregoing lemmata, to prove that all weak equivalences are also $T$-weak equivalences. That this is so follows immediately from $(M3)$. \qed

When a monad $T$ on $\mathcal{C}$ with elementary homotopy data $- \otimes I$ satisfies the hypotheses of Theorem 2.4 we will refer to the resulting model structure as the $T$-minimal model structure on $\mathcal{C}$ (with respect to $- \otimes I$). We now mention some consequences which justify the definition of $T$-weak equivalences and which explain this choice of terminology.

2.3. **Further consequences of the axioms.** Throughout this section we assume that $T$ satisfies axioms $(M1)$–$(M3)$ unless otherwise stated.

**Lemma 2.5.** Given maps $f, g : X \to Y$, $f \simeq g$ implies $Tf \simeq Tg$.

**Proof.** We first prove that $f \simeq g$ implies $Tf \simeq Tg$, as this clearly implies the case for $\simeq$. Suppose $\theta : X \otimes I \to Y$ is a homotopy from $f$ to $g$. Then there exists a map

$$(X \otimes I) \cup (TX \otimes \partial I) \xrightarrow{\theta'} TY$$
induced by the commutativity of the following square:

\[
\begin{array}{ccc}
X \otimes \partial I & \xrightarrow{\eta_X \otimes 1_{\partial I}} & TX \otimes \partial I \\
\downarrow & & \downarrow [Tf,Tg] \\
X \otimes I & \xrightarrow{\eta_Y \circ \theta} & TY.
\end{array}
\]

Thus, because \( TY \) is fibrant, there exists a lift \( \tilde{\theta} \) as indicated in the following diagram

\[
\begin{array}{ccc}
(X \otimes I) \cup (TX \otimes \partial I) & \xrightarrow{\theta'} & TY \\
\downarrow & \nearrow \tilde{\theta} & \\
TX \otimes I & & \\
\end{array}
\]

This \( \tilde{\theta} \) is the required homotopy. \( \square \)

**Corollary 2.6.** Given a \( T \)-algebra \( A \) and maps \( f,g : X \to A \), if \( f \simeq g \), then \( f' \simeq g' \) where \( f' \) is the induced \( T \)-algebra homomorphism \( TX \to A \) and similarly for \( g' \).

One important consequence of Lemma 2.5 is the following alternative characterization of the weak equivalences.

**Lemma 2.7.** A map \( f : X \to Y \) is a weak equivalence if and only if

\[
TX \xrightarrow{T(f)} TY
\]

is a homotopy equivalence with its homotopy inverse \( \tilde{f} : TY \to TX \) a \( T \)-algebra homomorphism.

**Proof.** Because, as remarked earlier, homotopy equivalences are always weak equivalences it suffices to prove the “only if” direction.

First, assume \( f \) is a weak equivalence. Then \( [f,TX] : [Y,TX] \to [X,TX] \) is a bijection and there exists a homotopy class

\[
[Y \xrightarrow{h} TX]
\]

of maps \( Y \to TX \) such that \( [h \circ f] = [\eta_X] \). Choose a representative \( h : Y \to TX \) and then note that, because \( TX \) is a \( T \)-algebra there exists a unique \( T \)-algebra homomorphism \( \tilde{f} : TY \to TX \) such that

\[
TY \xrightarrow{\tilde{f}} TX
\]

commutes. By choice of \( h \) we have that \( hf \simeq \eta_X \) and therefore, by Corollary 2.6,

\[
\tilde{f} \circ T(f) \simeq 1_X. \tag{1}
\]
Also,
\[
T(f) \circ \bar{f} \circ \eta_Y \circ f = T(f) \circ \bar{f} \circ T(f) \circ \eta_X
\]
\[
\simeq T(f) \circ \eta_X
\]
\[
= \eta_Y \circ f,
\]
where the homotopy is by (1). But then
\[
[f, TX][T(f) \circ \bar{f} \circ \eta_Y] = [f, TX][\eta_Y]
\]
and since \([f, TX]\) is injective \(T(f) \circ \bar{f} \circ \eta_Y \simeq \eta_Y\). Therefore, applying Corollary 2.6,
\[
T(f) \circ \bar{f} \simeq 1_Y.
\]
\[\square\]

The following proposition together with its corollary show that the model structure on \(\hat{C}\) established in Proposition 2.4 above is, in a suitable sense, the least model structure on \(\hat{C}\) for which the units \(\eta_X : X \to TX\) are fibrant replacements.

**Proposition 2.8.** If \(\mathcal{W}'\) is any collection of maps in \(\hat{C}\) such that
\begin{enumerate}
  \item \(\mathcal{W}'\) has the 3-for-2 property;
  \item \(\mathcal{W}'\) contains all homotopy equivalences; and
  \item \(\mathcal{W}'\) contains all units \(\eta_X : X \to T(X)\),
\end{enumerate}
then the collection \(\mathcal{W}\) of weak equivalences is contained in \(\mathcal{W}'\).

**Proof.** Let a map \(f : X \to Y\) in \(\mathcal{W}\) be given. Then, by Lemma 2.7, there exists a homotopy inverse \(\bar{f} : TY \to TX\) of \(Tf\) which is also a \(T\)-algebra homomorphism. Then \(Tf\) is in \(\mathcal{W}'\) by assumption and, because
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta_X & & \downarrow \eta_Y \\
TX & \xrightarrow{Tf} & TY \\
\end{array}
\]
commutes, the 3-for-2 axiom for \(\mathcal{W}'\) implies that \(f\) is also in \(\mathcal{W}'\). \[\square\]

**3. Examples**

We now give two simple examples to illustrate that there do in fact exist monads and elementary homotopy data for which the conditions \((M1)\) through \((M3)\) are satisfied.

**3.1. Monoids.** Consider the free-monoid monad \(T\) on the category \(Set\) of sets. We will prove that this monad satisfies conditions \((M1)\) to \((M3)\). First, observe that \((M1)\) is trivially satisfied. Next, we take as our set of generating presheaves the set \(N\) of natural numbers (coded as, say, Zermelo ordinals). We will now prove that the units of the natural numbers give rise to all units. Let \(\mathcal{E}\) be the set of units of natural numbers.

**Lemma 3.1.** With this \(\mathcal{E}\), \((M2)\) is satisfied.
Proof. Let a set $X$ be given and denote by $\mathcal{P}_f(X)$ the set of all pairs $(S, \sigma)$ such that $S$ is a finite subset of $X$ and $\sigma$ is an isomorphism $|S| \cong S$ of sets, where $|S|$ denotes the cardinal number of $S$. The map

$$\coprod_{(S, \sigma) \in \mathcal{P}_f(X)} |S| \xrightarrow{\prod_{(S, \sigma) \in \mathcal{P}_f(X)} \eta_{|S|}} \prod_{(S, \sigma) \in \mathcal{P}_f(X)} T|S|$$

is in the saturated class generated by $\mathcal{G}$, where $|S|$ denotes the cardinality of $S$. We claim that there exist maps making

$$
\begin{array}{c}
X \xrightarrow{\eta_X} \prod_{(S, \sigma) \in \mathcal{P}_f(X)} |S| \xrightarrow{\eta_{|S|}} \prod_{(S, \sigma) \in \mathcal{P}_f(X)} T|S| \xrightarrow{\eta_X} TX
\end{array}
$$

a retract diagram. Given $x \in X$, let $s(x) := ((\{x\}, !), 0)$ and let $r$ be the canonical map determined by the maps

$$
|S| \xrightarrow{\sigma} S \xrightarrow{r} X
$$

for $(S, \sigma)$ in $\mathcal{P}_f(X)$, where the unnamed map is the inclusion.

Every word $w = x_1 \cdots x_n$ of length $n$ in $TX$ determines a finite subset $S = \{x_1, \ldots, x_n\}$ of $X$ together the obvious isomorphism $\sigma : |S| \to S$ and we let $u(w) := ((S, \sigma), (\sigma(x_1) \cdots \sigma(x_n)))$. Finally, $v$ is induced by the map $T|S| \to TX$, for $(S, \sigma)$ in $\mathcal{P}_f(X)$, obtained by sending a word $(a_1 \cdots a_n)$ to the word $(\sigma(a_1) \cdots \sigma(a_n))$. This data clearly determines a retract diagram as claimed and therefore $\eta_X$ is in the saturated class generated by $\mathcal{G}$. □

Next, note that Set possesses a natural choice of elementary homotopy data given by cartesian product with the subobject classifier $2 = \{0, 1\}$ (using the subobject classifier in this way is typical and arises in [1] as well). This is the elementary homotopy data which we will employ. With this choice we have

Lemma 3.2. With these choices, (M3) is satisfied.

Proof. Let a monoid $M$ be given. Since $M$ has the RLP with respect to all elements of $\mathcal{G}$, it suffices to show that there exists a map $\bar{f}$ making

$$
\begin{array}{c}
K \times 2 \cup L \times \{e\} \xrightarrow{f} M
\end{array}
$$

commute, for any such $\bar{f}$ ($e = 0, 1$) and any monomorphism $K \to L$. The existence of $\bar{f}$ is trivial in this case (and does not even require $M$ to be a monoid). □
3.2. Categories. Let \textbf{Cat} denote the category of small categories. Let \textbf{L} denote the free category on the graph consisting of two distinct objects 0 and 1 and two distinct edges \( u : 0 \to 1 \) and \( d : 1 \to 0 \).

Let \textbf{Graph} be the category of directed graphs and let \( T \) be the free category monad on \textbf{Graph}. As in the case of monoids, \( T \) trivially satisfies condition (M1). In what follows, we denote by \([n]\) the finite ordinal \( \{0, 1, \ldots, n\} \) regarded as a category. We will prove that when \( \mathcal{G} \) is the set of units of finite ordinals \([n]\) for \( n \geq 0 \), the generation axiom is satisfied.

\textbf{Lemma 3.3.} The free category monad \( T \) on \textbf{Graph} satisfies condition (M2).

\textbf{Proof.} Let \( \mathcal{G} \) be as just described and assume that \( G \) is an arbitrary graph. We first form a new graph \( \tilde{G}(0) \) as the pushout

\[
\begin{array}{ccc}
\coprod_{[0] \to G[0]} & \longrightarrow & G \\
\coprod_{\eta[0]} & \quad \downarrow h_0 & \\
\coprod_{[0] \to G T[0]} & \longrightarrow & \tilde{G}(0)
\end{array}
\]

and we observe that there is a canonical induced map \( k_0 : \tilde{G}(0) \to TG \) such that \( k_0 \circ h_0 = \eta_G \) and \( \coprod T[0] \to \tilde{G}(0) \to TG \) is the induced map \( \coprod T[0] \to TG \). In general, assuming we have constructed \( \tilde{G}(n) \) we form the pushout

\[
\begin{array}{ccc}
\coprod_{[n+1] \to \tilde{G}(n)[n+1]} & \longrightarrow & \tilde{G}(n) \\
\coprod_{\eta[n+1]} & \quad \downarrow h_{n+1} & \\
\coprod_{[n+1] \to \tilde{G}(n) T[n+1]} & \longrightarrow & \tilde{G}(n+1)
\end{array}
\]

and we obtain \( k_{n+1} : \tilde{G}(n+1) \to TG \) with \( h_{n+1} \circ k_{n+1} = k_n \) and also commuting with the map \( \coprod T[n+1] \to \tilde{G}(n+1) \) in the required way. This gives us a countable sequence \( G \to \tilde{G}(0) \to \tilde{G}(1) \to \ldots \) and we take the colimit \( \tilde{G} = \lim_{\longrightarrow} \tilde{G}(n) \). Since each of the maps \( h_n \) is in the saturated class generated by \( \mathcal{G} \) it follows that the resulting map \( h : G \to \tilde{G} \) is also in this class. There is also a canonical map \( k : \tilde{G} \to TG \) such that \( k \circ h = \eta_G \).

Since the saturated class generated by \( \mathcal{G} \) is closed under retracts it suffices to define a section \( s : TG \to \tilde{G} \) of the map \( k \). On vertices \( s \) is just the identity. An edge \( a \to b \) in \( TG \) is a sequence \( (f_1, f_2, \ldots, f_n) \) of composable edges from \( G \). (Note that such a sequence is allowed to be empty (\( 0 \) when \( a = b \).) This gives graph homomorphisms \([n] \to G \) and, composing with the maps \( h_0, h_1, \ldots, h_{n-1}, [n] \to \tilde{G}(n-1) \). Such a sequence possesses a composite \( (f_n \circ \cdots \circ f_1) \) in \( \tilde{G}(n) \) and therefore determines an edge \([f_n \circ \cdots \circ f_1] : a \to b \) in \( \tilde{G} \) and we take \( s(f_1, \ldots, f_n) \) to be this edge (here the square brackets indicate that this edge is actually itself an equivalence class). With this definition, \( s \) is clearly a section of \( k \). \( \square \)

In \textbf{Graph} we have elementary homotopy data given by \((- \times I)\) where \( I \) is the graph with two distinct objects \( 0 \) and \( 1 \) and two distinct edges \( u : 0 \to 1 \) and \( d : 1 \to 0 \).

\textbf{Lemma 3.4.} Condition (M3) is satisfied for this choice of elementary homotopy data.
Proof. Given

\[
\begin{array}{c}
K \times I \cup L \times \{0\} \\
\downarrow \\
L \times I
\end{array}
\xrightarrow{f} C
\]

with \(C\) a category, we define \(\bar{f} : L \times I \to C\) on vertices by setting

\[
\bar{f}(x,t) := \begin{cases} f(x,0) & \text{when } x \notin K \\ f(x,t) & \text{otherwise.} \end{cases}
\]

For edges, we will consider each of the different cases one at a time. Assume given an edge \((e,j) : (a,s) \to (b,t)\) in \(L \times I\). For such \((e,j)\) in the domain of \(f\) we simply define \(\bar{f}(e,j) := f(e,j)\). As such, we now focus on those cases where \((e,j)\) is not in the domain \(\text{dom}(f)\) of \(f\).

- \(a \notin \text{dom}(f)\) and \(b \notin \text{dom}(f)\): If \(e = a = b\), then \(\bar{f}(e,j) := 1_{f(e,0)}\). Otherwise, \(\bar{f}(e,j) := f(e,0)\).
- \(a \in \text{dom}(f)\) and \(b \notin \text{dom}(f)\): If \(j = 1\), then \(\bar{f}(e,j) := f(e,0) \circ f(a,d)\). If \(j = d\), then \(\bar{f}(e,j) := f(e,0) \circ f(a,d)\). If \(j = u\), then \(\bar{f}(e,j) := f(e,0)\).
- \(a \notin \text{dom}(f)\) and \(b \in \text{dom}(f)\): Dual to the previous case.
- \(a \in \text{dom}(f)\) and \(b \in \text{dom}(f)\): If \(j = 1\), then \(a, b \in K\) and \(\bar{f}(e,j) := f(b,u) \circ f(e,0) \circ f(a,d)\). If \(j = d\), then \(\bar{f}(e,j) := f(e,0) \circ f(a,d)\). If \(j = u\), then \(\bar{f}(e,j) := f(b,u) \circ f(e,0)\).

Thus, we have defined \(\bar{f}\). Similarly, a category \(C\) has the RLP with respect to those maps \(K \times I \cup L \times \{1\} \to L \times I\) for \(K \to L\) a monomorphism. \(\square\)

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References

[1] Denis-Charles Cisinski, Théories homotopiques dans les topos, *J. Pure Appl. Algebra* **174**, 2002, 43–82.
[2] _, Les préfaisceaux comme modèles des types d’homotopie, Astérisque, vol. 308, Soc. Math. France, 2006.
[3] André Joyal, Quasi-categories and Kan complexes, *J. Pure Appl. Algebra* **175**, 2002, 207–222.
[4] _, The theory of quasi-categories and its applications. In *Advanced Course on Simplicial Methods in Higher Categories*, vol. 2, pages 149–496, Centre de Recerca Matemàtica, Barcelona, 2008.

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