Abstract

The influence of node mobility on the convergence time of averaging gossip algorithms in networks is studied. It is shown that a small number of fully mobile nodes can yield a significant decrease in convergence time. A method is developed for deriving lower bounds on the convergence time by merging nodes according to their mobility pattern. This method is used to show that if the agents have one-dimensional mobility in the same direction the convergence time is improved by at most a constant. Upper bounds are obtained on the convergence time using techniques from the theory of Markov chains and show that simple models of mobility can dramatically accelerate gossip as long as the mobility paths significantly overlap. Simulations show that these bounds are still valid for more general mobility models that seem analytically intractable, and further illustrate that different mobility patterns can have significantly different effects on the convergence of distributed algorithms.

1 Introduction

Gossip algorithms are distributed message passing schemes that are used to disseminate and process information in networks. Average consensus [1–3] and averaging gossip algorithms [4, 5] form an important special case of schemes that can compute linear functions of the data in a robust and distributed way. Such schemes have found numerous uses for distributed estimation, localization and optimization [6–8] and also for compressive sensing of sensor measurements and field estimation [9]. In this paper we study gossip algorithms that compute linear functions and will not discuss related problems like information dissemination (see e.g. [10, 11] and references therein).

Gossip algorithms are a natural fit for wireless ad-hoc and sensor network applications because of their distributed and robust nature. Recently the broadcast nature of wireless communication has been exploited to improve convergence [12]. Another key feature of some wireless networks is node mobility; to the best of our knowledge, the impact of mobility on gossip algorithms has not been significantly investigated. In this paper we attempt to analyze how mobility can (or cannot) help the convergence of gossip algorithms. For fixed nodes in a random geometric graph or grid (both popular model topologies for large wireless ad-hoc and sensor networks), standard gossip is extremely wasteful in terms of communication requirements; even optimized standard gossip algorithms on a grid converge very slowly, requiring $\Theta(n^2 \log \epsilon^{-1})$ messages [5, 13] to compute the average within accuracy $\epsilon$. Observe that this is of the same order as requiring every node to flood its estimate to all other nodes. The obvious solution of averaging numbers on a spanning tree and flooding back the average to all the nodes requires only $\Theta(n)$ messages. Clearly, constructing and maintaining a spanning tree in dynamic and ad-hoc networks introduces significant overhead and complexity, but a quadratic number of messages is a high price to pay for fault tolerance. In this context, what kind of mobility patterns are beneficial and how many mobile agents are needed to boost the convergence speed? Our results suggest that certain kinds of mobility can, in some cases, significantly accelerate convergence.

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Our results are a first step to understanding how mobility can impact the convergence of iterative message-passing schemes, at least for the special case of pairwise averaging where the convergence behavior is better understood.

**Main Results:** Our first result is that if \( m \) nodes have full mobility and the others are fixed, the convergence time drops to \( \Theta(n^2/m \log \epsilon^{-1}) \). Therefore, even a vanishingly small fraction of mobile nodes can change the order of messages required for convergence. In particular, if any constant fraction of nodes have full mobility, the convergence time drops to \( \Theta(n \log \epsilon^{-1}) \), the same order as a fully connected graph.

Our second result is that some mobility patterns might not be beneficial. We show that even if all the nodes of the network have one dimensional mobility in the same direction (e.g. horizontal), this yields no benefit in the convergence time, up to constants. Intuitively, this is because the information must still diffuse across the other direction (e.g. vertical). Finally we show that one dimensional mobility with a randomly selected direction is as good as full mobility.

In order to obtain these results, we develop a novel method for deriving lower bounds on the convergence time of gossip algorithms with mobile nodes by merging nodes with similar mobility regions. This method is based on a characterization of the convergence time of Markov chains in terms of a functional called the Dirichlet form [14]. Our upper bounds are derived using the so-called Poincaré inequality [15] and the related canonical path method [16]; a version of this technique has also been previously used to study gossip algorithms [17].

## 2 Network model and preliminaries

### 2.1 Time model

We use the asynchronous time model [5, 18], which is well-matched to the distributed nature of wireless networks. In particular, we assume that each sensor has an independent clock whose “ticks” are distributed as a rate \( \lambda \) Poisson process. Our analysis is based on measuring time in terms of the number of ticks of an equivalent single virtual global clock ticking according to a rate \( n \lambda \) Poisson process. An exact analysis of the time model can be found in [5]. We will refer to the time between two consecutive clock ticks as one timeslot.

Throughout this paper we will be analyzing the number of required messages without worrying about delay. We can therefore adjust the length of the timeslots relative to the communication time so that only one packet exists in the network at each timeslot with high probability. Note that this assumption is made only for analytical convenience; in a practical implementation, several packets might co-exist in the network, but the associated issues are beyond the scope of this work.

### 2.2 Network and mobility model

Suppose we have a collection of \( n \) agents \( \mathcal{A} \). At the first timeslot, each agent \( i \) starts at some initial location with a scalar \( x_i(0) \). We will denote the vector of their initial values by \( \mathbf{x}(0) \). The objective of our algorithm is for every agent to estimate the average

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i(0) .
\]

In order to accomplish this goal, the agents pass messages between each other to communicate their estimates. We assume that this communication always succeeds\(^1\). We also assume that the messages involve real numbers; the effects of message quantization in gossip and consensus algorithms is an active area of research [19, 20].

The \( n \) agents can move in an area \( \mathcal{G} \). For example, we may take \( \mathcal{G} \) to be a graph with vertex set \( \mathcal{V} \) and edge set \( \mathcal{E} \). Agents at locations \( v \) and \( v' \) can communicate if either \( v = v' \) or \( (v, v') \in \mathcal{E} \). Another example

\(^1\)Note however that gossip algorithms remain robust to communication and agent failures.
is taking $G$ to be the unit square and allowing agents at $v$ and $v'$ to communicate if the distance $d(v, v')$ is less than some radius $r(n)$.

In this paper we will use two main examples. The first is the $\sqrt{n} \times \sqrt{n}$ discrete lattice on the torus. We will assume that the agents start at time 0 at different sites on the torus. The second example is the random geometric graph (RGG) model on the unit torus. In this model, the agent’s initial locations are generated by selecting $n$ uniformly chosen locations on the unit square. Agents can communicate with each other if the distance between them is less than $r(n) = \sqrt{5c \log n} n$, where $c \geq 10$ ensures some useful regularity properties [17] discussed subsequently.

Under agent-based mobility, at each time step agent $i$ moves to a new location in $G$ chosen according to a fixed probability distribution $\mu_i$. Therefore the sequence of agent locations $l_i(1), l_i(2), \ldots, l_i(t)$ is independent and identically distributed (iid) according to the distribution $\mu_i$. We call the collection of distributions $\{\mu_i : i \in A\}$ an agent-based mobility pattern. Our theoretical results in this paper are for agent-based mobility. Some examples of agent-based mobility are:

1. A simple example of agent-based mobility is full uniform mobility on the whole graph. In this model, $\mu_i$ is the uniform distribution on $G$ for each $i \in A$. This corresponds to the case where each agent is equiprobably at any location in the graph at time $t$. This is similar to the model proposed by Grossglauser and Tse [21]. We will also consider a static network with $m$ fully mobile agents added to the network.

2. In the horizontal mobility model, each agent selects a new horizontal location uniformly at each time. For the torus, the agent selects a new column uniformly. For the RGG, it selects a new horizontal coordinate uniformly from $[0,1]$.

3. In the bidirectional model each agent selects equiprobably whether it will move horizontally or vertically for all time. At each time step, the horizontal agents select a new horizontal coordinate uniformly, and the vertical agents select a new vertical coordinate uniformly.

4. In a local model for the torus, an agent that starts initially at location $(i, j)$ chooses a new location uniformly in the square of size $(2m+1)^2$ centered at $(i, j)$. That is, the horizontal coordinate is uniformly distributed in $\{i - m, \ldots, i + m\} \mod \sqrt{n}$ and the vertical coordinate is chosen uniformly in $\{j - m, \ldots, j + m\} \mod \sqrt{n}$.

The key assumption in all our mobility models is that in each gossip timeslot, the positions of the mobile agents are selected independently from some distribution supported on a sub-region of the space, similarly to Grossglauser and Tse [21]. Popular mobility models like the random walk model [22, 23], random waypoint model [24], and random direction model [25] have time dependencies. If however the duration of one gossip timeslot is comparable or larger than the mixing time of the mobility model, the positions of the agents will be approximately independent (see also [26]). If delay is not an issue, we can always set the duration of the gossip timeslot to have that property, and in simulations we show that if we do not allow the mobility model to mix, mobility is not as helpful. Therefore, our preliminary experimental evaluation suggests that our analytic results could be used to bound these more realistic mobility models.

3 Algorithm and main results

3.1 The algorithm

The gossip algorithm that we will consider is a simple extension of the standard nearest-neighbor gossip of Boyd et al. [5] that includes the mobility model in a natural way. At each time step, the agents move independently to new locations. One agent is selected at random, chooses one of its neighbors according to the graph $G$, and performs a pairwise average with that neighbor. More precisely, at each time $t = 1, 2, \ldots$ the following events occur:

\footnote{The unit torus is formed from the unit square by “glueing” opposite edges together.}
1. Each agent $i \in A$ chooses a new location $l_i(t)$ according to the mobility distribution $\mu_i$.

2. A agent $i$ is selected at random and selects a neighbor $j$ uniformly from the set

$$N_i(t) = \{ k \in A : (l_i(t), l_k(t)) \in V \}.$$  

3. The agents $i$ and $j$ exchange values and update their estimates:

$$x_k(t) = \begin{cases} \frac{1}{2}(x_i(t-1) + x_j(t-1)) & k = i, j \\ x_k(t-1) & k \neq i, j \end{cases}$$  

Since the algorithm is randomized, we are interested in providing probabilistic bounds on its running time. Given $\epsilon > 0$, the $\epsilon$-averaging time [5] is the earliest time at which the vector $x(t)$ is $\epsilon$ close to the normalized true average with probability greater than $1 - \epsilon$:

$$T_{\text{ave}}(n, \epsilon) = \sup_{x(0)} \inf_{t=0,1,2,\ldots} \left\{ \mathbb{P} \left( \frac{\|x(t) - \bar{x}\|}{\|x(0)\|} \geq \epsilon \right) \leq \epsilon \right\},$$  

where $\|\cdot\|$ denotes the $\ell_2$ norm. Note that this is essentially measuring a rate of convergence in probability. The analysis of Denantes et al. [27] shows that bounds on the spectral gap yield an asymptotic deterministic rate of vanishing error. Our bounds can be used to bound both the rate of convergence in probability and to show that the averaging error decays exponentially asymptotically almost surely.

3.2 Main results

Our main results characterize the benefit (or lack thereof) of mobility in speeding up the convergence of gossip algorithms:

- For horizontal mobility on the random geometric graph and the torus, the averaging time improves by at best a constant factor over the case where the agents are not mobile at all:

$$T_{\text{ave}}^{(\text{torus, horiz})}(n, \epsilon) = \Omega(n^2 \log \epsilon^{-1})$$  

$$T_{\text{ave}}^{(\text{RGG, horiz})}(n, \epsilon) = \Omega\left( \frac{n^2 \log \epsilon^{-1}}{\log n} \right)$$  

- For bidirectional mobility where each agent initially selects whether to move vertically or horizontally, the convergence time is within a constant factor of full mobility:

$$T_{\text{ave}}^{(\text{torus, bi})}(n, \epsilon) = O(n \log \epsilon^{-1})$$  

$$T_{\text{ave}}^{(\text{RGG, bi})}(n, \epsilon) = O(n \log \epsilon^{-1})$$  

- For $n$ non-mobile agents on a $\sqrt{n} \times \sqrt{n}$ torus with $m \leq n$ agents having full mobility, the convergence time is

$$T_{\text{ave}}^{(\text{torus plus m, 2D})}(n, \epsilon) = \Theta\left( \frac{n^2}{m} \log \epsilon^{-1} \right).$$  

- For the local mobility model with each agent moving in a square of size $(2m + 1)^2$,

$$T_{\text{ave}}^{(\text{torus, local})}(n, \epsilon) = O\left( \frac{n^2 \log m}{m^2 \log \epsilon^{-1}} \right)$$
4 Upper and lower bounds on convergence time

4.1 Convergence analysis

At each step of the algorithm, the agents update their estimates of the average $\bar{x}$. Let $x(t)$ denote the average estimates at time $t$. For agents $i$ and $j$ define the matrix $W^{(i,j)}$

$$W^{(i,j)} = I - \frac{1}{2}(e_i - e_j)(e_i - e_j)^T,$$

where $e_i$ is the $i$-th elementary vector. The new vector of averages is given by

$$x(t) = W^{(i,j)}x(t-1).$$

The randomness in the mobility and in the agent selection induces a probability distribution on the matrices \{ $W^{(i,j)} : i, j \in A$ \}. Since the mobility and selection are iid across time, we can write the update as

$$x(t) = \left( \prod_{s=1}^{t} W(s) \right) x(0),$$

where \{ $W(s)$ \} are iid random matrices. Denote the expected value of this random matrix by $\bar{W} = E[W(s)]$. It is not hard to see that $\bar{W}$ is a (symmetric) stochastic matrix and therefore corresponds to a Markov chain. Let $P_{ij}$ be the probability that agent $i$ is selected in step 2 of the algorithm and it selects agent $j$ in its neighbor set. Then it is clear that $P(W(s) = W^{(i,j)}) = P_{ij} + P_{ji}$, and that

$$\bar{W}_{ij} = \frac{1}{2}(P_{ij} + P_{ji}).$$

The pioneering work of Boyd et al. [5] showed that the convergence time of a randomized gossip algorithm is dictated by the mixing time of the Markov chain associated to $\bar{W}$. Mathematically, our problem is how to analyze the mixing time of the new graph induced by the new feature (in this case mobility) and then compare it to the old graph without mobility. For a Markov chain $M$ with transition matrix $\bar{W}$, the convergence rate to the stationary distribution is given by $\lambda_2(\bar{W})$, the second largest eigenvalue of $\bar{W}$. Note that the largest eigenvalue $\lambda_1(\bar{W})$ is 1. Define the relaxation time $T_{\text{relax}}$ to be the reciprocal of the spectral gap:

$$T_{\text{relax}}(\bar{W}) = \frac{1}{1 - \lambda_2(\bar{W})}.$$ 

**Theorem 1** (Convergence with $T_{\text{relax}}$ [5]). If $P = (P_{ij})$ is symmetric and $n$ is sufficiently large, then $T_{\text{ave}}(n, \epsilon)$ is bounded by

$$T_{\text{ave}}(n, \epsilon) = \Theta \left( T_{\text{relax}}(\bar{W}) \log \epsilon^{-1} \right)$$

4.2 Lower bounds

In this section we turn to a general method for constructing lower bounds on the convergence time for pairwise gossip algorithms under agent-based mobility. The main intuition is to partition the set of vertices in the graph and merge all agents whose mobility is supported in the same element of the partition. This induces a transformation on the Markov chain associated to the gossip algorithm. By using an extremal characterization of the relaxation time for Markov chains we can lower bound the $T_{\text{relax}}(\bar{W})$ in the original gossip algorithm by that for the induced Markov chain. The only remaining issue is to choose a partition that yields a tight lower bound. At the moment, this must be done by inspection, but we can use this technique to show that horizontal mobility cannot improve the convergence of gossip for the torus or the RGG.
**Theorem 2.** Let \( \{ \mathcal{U}_r \} \) be any partition of the set of locations \( \mathcal{G} \), and let \( \hat{\mathcal{W}} \) be the transition matrix of the chain induced by merging all agents whose mobility is restricted to a single set in the partition. Then

\[
T_{\text{ave}}(n, \epsilon) = \Omega(T_{\text{relax}}(\hat{W}) \log \epsilon^{-1}).
\]

**Proof.** We begin with the set \( \mathcal{G} \) on which the agents in \( \mathcal{A} \) can move. Let \( \{ \mathcal{U}_r : r = 1, 2, \ldots, M \} \) be a partition of \( \mathcal{G} \). Given an agent-based mobility pattern \( \{ \mu_i \} \), let

\[
\mathcal{C}_r = \{ v \in \mathcal{A} : \mu_v(\mathcal{U}_r) = 1 \},
\]

be the set of agents whose mobility is restricted to \( \mathcal{U}_r \). We can create a map \( F \) on the state set \( \mathcal{A} \) of the Markov chain corresponding to the gossip algorithm:

\[
F(a) = \begin{cases} 
  r & \text{if } a \in \mathcal{C}_r \\
  a & \text{otherwise}
\end{cases}
\]

The map \( F \) merges agents whose mobility is restricted to \( \mathcal{U}_r \) and leaves the other agents invariant. Let \( B \) denote the image of \( F \). For a Markov chain on \( \mathcal{A} \) with transition probabilities \( W_{ij} \) and stationary distribution \( \pi(\cdot) \), we can define a new Markov chain on \( B \) with transitions \( \hat{W}_{kl} \):

\[
\hat{W}_{kl} = \frac{1}{\sum_{i:F(i)=k} \pi_i} \sum_{i:F(i)=k} \sum_{j:F(j)=l} \pi_i W_{ij}.
\]

This is the induced chain from the function \( F \) [14, Chapter 4, p.37]. The stationary distribution of this chain is \( \hat{\pi}_k = \sum_{i:F(i)=k} \pi_i \).

We can express the relaxation time of a Markov chain in terms of the Dirichlet form [14]. Given a real-valued function \( g \) on the state space of the Markov chain with transition matrix \( \bar{W} \) and stationary distribution \( \pi(\cdot) \), the Dirichlet form is given by

\[
D(g, g) = \frac{1}{2} \sum_{k,l} \pi(k) \bar{W}_{kl} (g(k) - g(l))^2.
\]

The relaxation time is then given by

\[
T_{\text{relax}}(\bar{W}) = \sup_g \left\{ \frac{\sum_k \pi(k) g(k)^2}{D(g, g)} : \sum_k \pi(k) g(k) = 0 \right\}.
\]

The following contraction principle shows that \( T_{\text{relax}} \) for an induced chain is at most that of the original chain. The validity of this lemma is mentioned in [14, Chapter 4, p.37] and here we present a proof which easily follows from similar arguments from [14]:

**Lemma 1.** Let \( \mathcal{M} \) be a Markov chain on a finite state space \( \mathcal{A} \) with transition matrix \( W \) and let \( F : \mathcal{A} \rightarrow \mathcal{B} \) be an arbitrary mapping. Then the relaxation time of the chain \( \mathcal{M} \) on \( \mathcal{B} \) with transition matrix \( \hat{W} \) given by (20) induced by \( F \) lower bounds the relaxation time of the original chain:

\[
T_{\text{relax}}(\hat{W}) \leq T_{\text{relax}}(W).
\]

**Proof.** We use the extremal property of the relaxation time in (22). Let \( \hat{g} \) achieve the supremum in (22) for the induced chain given by \( \hat{W} \). We can create a function \( g \) from \( \hat{g} \) to lower bound \( T_{\text{relax}}(\mathcal{M}) \). Simply set \( g(i) = \hat{g}(k) \) for \( \{ i : F(i) = k \} \). Then

\[
\sum_{i \in \mathcal{A}} \pi(i) g(i)^2 = \sum_{k \in \mathcal{B}} \hat{\pi}(k) \hat{g}(k)^2.
\]

Furthermore, from (20) we can see that the Dirichlet form \( D(g, g) \) is also unchanged. Therefore the supremum of (22) for the original chain is at least as large as that for the induced chain.
Note that while the mixing time of a Markov chain decreases when states are merged, as argued, the same is not true for other quantities like the expected time to go from one state to another. The preceding lemma and Theorem 1 gives a lower bound on the benefit on the convergence speed of gossip in a network of mobile nodes.

In theory we could optimize the lower bound over all partitions \( \{U_r\} \), but for our examples there is an “obvious” partition that yields a meaningful lower bound. We turn first to the \( \sqrt{n} \times \sqrt{n} \) torus.

**Corollary 1** (Torus with horizontal mobility). Let \( G = (V,E) \) be the \( \sqrt{n} \times \sqrt{n} \) torus and suppose that the set of agents \( A = V \). Let the mobility pattern for the \((i,j)\)-th agent be uniformly distributed on the set \( U_i \{ (i,k) : k \leq \sqrt{n} \} \), which corresponds to mobility only in the horizontal direction. Then

\[
T_{\text{ave}}(n, \epsilon) = \Omega \left( n^2 \log \frac{1}{\epsilon} \right). \tag{25}
\]

**Proof.** Consider the partition \( \{U_i\} \) of \( V \), where \( U_i \) is the \( i \)-th row of the torus. Consider two agents, one starting at \((i,j)\) and the other at \((k,l)\), where \( k = i \pm 1 \). Then the probability in the algorithm that \((i,j)\) and \((k,l)\) average with each other is the chance that \((i,j)\) is selected times the probability (over the mobility) that \((i,j)\) and \((k,l)\) are adjacent to each other times the chance that \((i,j)\) selects \((k,l)\) out of its neighbors. We can upper bound this probability:

\[
W_{ij} = O \left( \frac{1}{n} \times \frac{1}{\sqrt{n}} \right). \tag{26}
\]

The chain induced from this partition is a cycle with \( \sqrt{n} \) states, where each state corresponds to a row in the original Markov chain. The transitions from row to row are given by \( \hat{W}_{kl} \):

\[
\hat{W}_{kl} = \frac{1}{\sum_{i:F(i)=k} \pi_i} \sum_{i:F(i)=k} \sum_{j:F(j)=l} \pi_j W_{ij} \tag{27}
\]

\[
= \sqrt{n} \cdot \sqrt{n} \cdot \frac{1}{n} \cdot O \left( \frac{1}{n} \times \frac{1}{\sqrt{n}} \right) \tag{28}
\]

\[
= O \left( \frac{1}{n} \right). \tag{29}
\]

Therefore the self-transition for each state is \( 1 - O(1/n) \). Let \( \alpha = \hat{W}_{kl} \), the transitions from row to row. The matrix \( \hat{W} \) is circulant and generated by the vector \((\alpha, 1-2\alpha, \alpha, 0, \ldots, 0)\). The eigenvalues are given by the discrete Fourier transform of the vector (c.f. [13]):

\[
\lambda_k(\hat{W}) = 1 - 2\alpha + 2\alpha \cos \left( \frac{2\pi(k-1)}{\sqrt{n}} \right). \tag{30}
\]

In particular, the second-largest eigenvalue can be bounded using the Taylor expansion of the cosine:

\[
\lambda_2(\hat{W}) \geq 1 - 2\alpha + 2\alpha \left( 1 - \frac{1}{2} \frac{4\pi^2}{n} \right) = 1 - O \left( \frac{1}{n^2} \right). \tag{31}
\]

Therefore the relaxation time is

\[
T_{\text{relax}} = \Omega(n^2), \tag{31}
\]

and the averaging time is bounded by Theorem 1. 

The preceding theorem shows that allowing nodes to move in only one direction gives the same order convergence time as the the torus without any node mobility. That is, *sometimes mobility can yield no significant benefits in terms of convergence.* In the case where we add a single agent moving in the vertical direction we still do not gain anything. The proof follows from the same arguments as Corollary 1.
**Corollary 2** (A single vertical mover doesn’t help). Let $G = (\mathcal{V}, \mathcal{E})$ be the $\sqrt{n} \times \sqrt{n}$ torus and suppose that the set of agents $\mathcal{A} = \mathcal{V} \cup \{e\}$. Let the mobility pattern for the $(i, j)$-th agent in $\mathcal{V}$ be uniformly distributed on the set $\{(i, k) : k \leq \sqrt{n}\}$, which corresponds to mobility only in the horizontal direction. Let the mobility pattern for $e$ be uniform on $\{(i, 1) : i \in \sqrt{n}\}$. Then for this gossip algorithm,

$$T_{\text{ave}}(n, \epsilon) = \Omega \left( n^2 \log \epsilon^{-1} \right).$$

(32)

We could prove in a similar way that adding a constant number of agents in the vertical direction does not speed up the convergence appreciably. Our final result in this section shows that 1D unidirectional mobility cannot help speed up the convergence time of gossip on random geometric graphs as well. Boyd et al. [5] have shown that the averaging time for standard pairwise gossip on the RGG is $\Theta(nr^{-2} \log \epsilon^{-1})$, which for $r(n) = \Theta(\sqrt{n^{-1}} \log n)$ is $\Theta((n^2 / \log n) \log \epsilon^{-1})$.

### 4.3 Upper bounds

**Canonical Path method** [16]: For any ergodic and reversible Markov chain on a state space $\Omega$, for each pair $i, j$ of states define the *capacity* of a directed edge $e_{i,j}$ to be

$$C(e) = \pi(i) \overline{W}_{ij}. \quad (33)$$

For each pair of states we define a *demand* $D(i, j) = \pi(i) \pi(j)$. A flow is any way of routing $D(i, j)$ units of liquid from $i$ to $j$ for all pairs $i, j$ simultaneously. Formally, a flow $F : \mathcal{P} \rightarrow \mathbb{R}^+$ is a function on the set $\mathcal{P}$ of all simple paths on the transition graph of the Markov chain that satisfies the demand:

$$\sum_{p \in \mathcal{P}_{ij}} F(p) = D(i, j), \quad (34)$$

where $\mathcal{P}_{ij}$ denotes all the paths from $i$ to $j$.

For a flow $F$ we can define the *load* on an edge $e$ to be total flow routed across that edge:

$$f(e) = \sum_{p \in \mathcal{P}_{ij} : e \in p} F(p) \quad (35)$$

The *cost* of a flow $F$ is the maximum overload of any edge:

$$\rho(F) = \max_e \frac{f(e)}{C(e)}, \quad (36)$$

Finally, define the *length* of a flow $l(F)$ to be longest flow-carrying path, i.e. the longest $p$ for which $F(p) \neq 0$.

Using these definitions, we can use the following Poincaré inequality [16] to yield an upper bound on the inverse spectral gap (relaxation time) of the Markov chain:

$$\frac{1}{1 - \lambda_2(W)} \leq \rho(F) l(F). \quad (37)$$

Intuitively, if there are no 'bottlenecks’ on the transitions for every pair of states, the relaxation time of the chain will be very small. Any flow $F$ gives an upper bound that depends on the cost $\rho(F)$ of its most congested edge.

**Corollary 3** (Full mobility is optimal). Let $G = (\mathcal{V}, \mathcal{E})$ be the $\sqrt{n} \times \sqrt{n}$ torus and suppose that the set of agents $\mathcal{A} = \mathcal{V}$. Let the mobility pattern every agent in $\mathcal{V}$ be uniformly distributed on all of $\mathcal{V}$, which corresponds to full mobility. Then for this gossip algorithm,

$$T_{\text{ave}}(n, \epsilon) = \Omega \left( n \log \epsilon^{-1} \right).$$

(38)
Proof. The stationary distribution is uniform, so \( \pi_i = 1/n \) for all \( i \) and the demand \( D(i, j) = 1/n^2 \) for all pairs \((i, j)\). Furthermore, the probability of \( i \) and \( j \) averaging is \( \Omega(1/n^2) \), so the state diagram of the Markov chain is the complete graph with edge capacities \( \Omega(1/n^3) \). The simplest flow is to route directly the demand \( 1/n^2 \) on the edge from \( i \) to \( j \), which gives a cost of \( O(n) \) with a flow of length 1, so the relaxation time is \( O(n) \).

A slightly less simple example is a cycle with one fully mobile agent. The cycle has averaging time \( \Theta(n^3 \log \epsilon^{-1}) \) (see [13]). With one mobile agent the averaging time drops to \( O(n^2 \log \epsilon^{-1}) \).

**Corollary 4** (Cycle with one fully mobile agent). Let \( G = (V, E) \) be a cycle of \( n \) locations and suppose the agents are \( A = V \cup \{v'\} \). Suppose no agent in \( V \) can move, but \( v' \) has mobility uniformly distributed on \( V \), which corresponds to full mobility. Then for this gossip algorithm,

\[
T_{\text{ave}}(n, \epsilon) = \Omega \left( n^2 \log \epsilon^{-1} \right).
\]

Proof. The stationary distribution for this chain is uniform, so \( \pi_i = 1/(n+1) \) for all \( i \) in \( A \). The probability that \( i \) and \( j \) average for \( i, j \in A \) is 0 unless \( i \) and \( j \) are neighbors. Otherwise, with probability \( 3/n \) the mobile node \( v' \) is a neighbor of \( i \), so:

\[
P_{ij} = \frac{1}{n} \left( \frac{1 - 3/n}{2} + \frac{3/n \cdot 1/3}{n} \right) = \frac{1}{2n} \left( 1 - \frac{1}{n} \right).
\]

For \( i \in A \) and \( j = v' \) we have

\[
P_{iv'} = \frac{1}{n} \cdot \frac{3/n \cdot 1/3}{1} = \frac{1}{n^2}.
\]

Thus the capacity is

\[
C(i, j) = \begin{cases} 
\frac{1}{2n(n+1)} \left( 1 - \frac{1}{n} \right) & j \in V \\
\frac{1}{n^2(n+1)} & j = v'
\end{cases}
\]

The demand is just \( D(i, j) = 1/(n+1)^2 \) between each pair of nodes.

To construct a flow \( F \), we just route all flow through the mobile agent \( v' \). An edge \((i, v')\) for \( i \in V \) carries \( n \) flows to all agents \( j \neq i \), each of size \( 1/(n+1)^2 \) for a total of \( f(i, v') = n/(n+1)^2 \). Similarly, an edge \((v', i)\) the same total flow. All flows are of length 2, so \( l(F) = 2 \). The overload is

\[
\rho(F) = \frac{n/(n+1)^2}{1/(n^2(n+1))} = \frac{n^3}{(n+1)}.
\]

And thus for large \( n \) we get an upper bound of \( O(n^2) \) for the relaxation time of the chain. The averaging time then follows from Theorem 1.

5 Examples revisited

We now turn to our examples of mobility and derive scaling results for gossip with mobility. For the torus we will show that local mobility in a square of area \( m^2 \) cuts the convergence time by \( m^2 \) and adding \( m \) fully mobile agents cuts the convergence time by \( m \). For the random geometric graph we will prove the same result for bidirectional mobility and a lower bound for unidirectional mobility.
5.1 The torus

5.1.1 Local mobility

An important step in bridging the mobility model here with more reasonable mobility models is to consider local mobility, in which an agent moves uniformly in a square of side length \((2m + 1)\) centered at its initial location.

**Theorem 3.** Consider gossip with \(n\) agents on the \(\sqrt{n} \times \sqrt{n}\) torus \(G\). Each agent moves uniformly in a square of side-length \(2m + 1\) centered at itself. Then the averaging time is given by

\[
T_{\text{ave}}(n, \epsilon) = O\left(\frac{n^2 \log m}{m^2} \log \epsilon^{-1}\right).
\]

**Proof.** Divide the grid into squares of side length \(m\). Then each square contains \(m^2\) agents and mobility of each agent spans at least its own square and each square adjacent to it. For each pair of agents we must route \(D(i, j) = 1/n^2\) units of flow. We will do this by routing flows in L-shaped paths, as shown in Figure 1. Agent \(i\) will spread its \(1/n^2\) units of flow evenly to the \(m^2\) nodes in the adjacent square – these will send the flow split among the \(m^2\) agents in the next square, and so on. Thus flow is routed only along inter-square edges. Each left-to-right edge carries flow from the \(O(\sqrt{n}/m)\) squares to the left of it and to the \(O(n/m^2)\) squares to the right and above it. Each square has \(m^2\) agents so there are \(O(n^{3/2}/m)\) flows carrying \(1/n^2m^2\) per flow, so the load on the edge is

\[
f(i, j) = O\left(\frac{1}{\sqrt{nm^3}}\right).
\]

The same bound holds for down-to-up edges.

To find the capacity of these edges, we calculate the probability that agents \(i\) and \(k\) in adjacent squares average with each other. The probability is \(1/n\) to select agent \(i\) and the overlap in agent \(i\) and \(k\)’s mobility area is \(\Omega(m^2)\), so the chance \(i\) and \(k\) are adjacent after moving is \(\Omega(1/m^2)\). With high probability there will
be no more than \(O(\log m)\) nodes for \(i\) to choose from, so the chance of selecting \(k\) is at worst \(\Omega(1/\log m)\). Thus:

\[
C(i, k) = \Omega\left(\frac{1}{n^2 m^2 \log m}\right).
\]

The maximum length of any flow is \(O(\sqrt{n}/m)\), so the Poincaré inequality gives

\[
\frac{1}{1 - \lambda_2(W)} = O\left(\frac{n^2 \log m}{m^2}\right).
\]

\[\square\]

5.1.2 Adding mobile agents

The question motivating this work is this: how much can agent mobility improve the convergence speed of gossip or consensus algorithms? Put another way, how much mobility is needed to gain a certain factor improvement in the convergence? A simple model for which we can answer this question is the following: consider \(n\) static agents in the \(\sqrt{n} \times \sqrt{n}\) torus together with \(m\) mobile agents whose mobility \(\mu_i\) is uniform on the torus. We use our techniques from earlier sections below to show that the averaging time of gossip in this model is \(\Theta(n^2/m \log \epsilon^{-1})\), which for \(m = n^\alpha\) is \(\Theta(n^{2-\alpha})\). For example, adding \(\sqrt{n}\) mobile nodes can speed convergence by a factor of \(\sqrt{n}\).

**Theorem 4.** Consider the gossip with \(n + m\) agents on the \(\sqrt{n} \times \sqrt{n}\) torus \(G\). The \(n\) static agents \(S\) are positioned on the \(n\) nodes of the torus and the \(m\) mobile agents \(M\) have mobility that is uniform on \(G\), where \(m \leq n\). Then the averaging time is given by

\[
T_{\text{ave}}(n, \epsilon) = \Theta\left(\frac{n^2}{m \log \epsilon^{-1}}\right).
\]

**Proof.** We first show that for \(i \in S\) and \(j \in M\), the probability \(P_{ij}\) that agent \(i\) contacts agent \(j\) and averages is \(\Theta(1/n(m+n))\). Agent \(i\) is selected with probability \(1/(m+n)\) and agent \(j\) is in the neighborhood of agent \(i\) with probability \(5/n\). Therefore:

\[
P_{ij} = \frac{5}{n(m+n)} \sum_{l=0}^{m-1} \frac{1}{5+l} \mathbb{P}(L = l),
\]

where \(L\) is the number of agents in \(M\) that land in the neighborhood of \(i\). The summation is just

\[
\sum_{l=0}^{m-1} \frac{1}{5+l} \mathbb{P}(L = l) = \mathbb{E}[1/(5 + L)],
\]

which is clearly upper bounded by 1, so

\[
P_{ij} = O\left(\frac{1}{n(m+n)}\right).
\]

Since \(1/(5 + L)\) is convex, Jensen’s inequality can be used to lower bound

\[
\mathbb{E}[1/(5 + L)] \geq 1/\mathbb{E}[5 + L] = 1/(5 + m/n).
\]

Therefore \(P_{ij} = \Omega(1/n(m+n))\). By symmetry, we have the same bound on \(P_{ji}\).
To get the lower bound, consider the function $G : \mathcal{S} \cup \mathcal{M} \to \mathcal{S} \cup \{M\}$ that is the identity on $\mathcal{S}$ and merges $\mathcal{M}$ into a single state $M$. We can bound the transition probabilities of the new chain using (20):

$$
\hat{W}_{Mi} = \frac{1}{\sum_{j \in \mathcal{M}} \pi(j)} \sum_{j \in \mathcal{M}} \pi(j) \frac{P_{ij} + P_{ji}}{2} = \Theta \left( \frac{1}{n(m+n)} \right)
$$

(50)

$$
\hat{W}_{iM} = \frac{1}{\pi(i)} \sum_{j \in \mathcal{M}} \pi(i) \frac{P_{ij} + P_{ji}}{2} = \Theta \left( \frac{m}{n(m+n)} \right).
$$

(51)

For $i, k \in \mathcal{S}$ we have $\hat{W}_{ik} = \hat{W}_{ki}$.

The new chain is a torus plus an additional central node $M$. The probability of transitioning from the torus to the central node is $\Theta((m/n)/(m+n))$ and for transitioning back it is $\Theta((1/n)/(m+n))$. It can be seen (see the Appendix) that the relaxation time for this chain is $\Omega(n^2/m)$ via the extremal characterization in (22). Thus $T_{\text{ave}}(n, \epsilon) = \Omega \left( \frac{n^2}{m \log \epsilon^{-1}} \right) \cdot$

We now turn to the upper bound. As before, we construct a flow on the chain. The demand between any two agents $(i, j)$ is $1/(n + m)^2$. Since $P_{ij} = \Theta(1/(n + m))$, the capacity

$$
C(e) = \Theta(1/(n + m)^2),
$$

for $e = (i, j)$. We must now construct a flow that will yield an upper bound on the relaxation time of $n^2/m$. For a pair of states $i \in \mathcal{S}$ and $j \in \mathcal{M}$ we assign $1/(n + m)^2$ to the direct path $(i, j)$. For a pair $i \in \mathcal{S}$ and $j \in \mathcal{S}$ we split $1/(n + m)^2$ equally into the $m$ paths $(i, k, j)$ for $k \in \mathcal{M}$. Finally, for $i \in \mathcal{M}$ and $j \in \mathcal{M} \cup \mathcal{S}$ we again route $1/(n + m)^2$ directly on $(i, j)$. Then

$$
f((i, j)) = \begin{cases} 
\frac{1}{(m+n)^2} & i,j \in \mathcal{M} \\
\frac{1}{(m+n)^2} + \frac{n}{m(m+n)} & i \in \mathcal{S}, j \in \mathcal{M} \\
0 & i,j \in \mathcal{S} 
\end{cases}
$$

Therefore $\rho(F) = \Theta(n^2/m)$. Since all paths are of the same length, the Poincaré inequality implies that $T_{\text{relax}}(\hat{W}) = O \left( \frac{n^2}{m} \right)$, so Theorem 4 gives $T_{\text{ave}}(n, \epsilon) = O \left( \frac{n^2}{m \log \epsilon^{-1}} \right) \cdot$

5.2 Random geometric graphs

5.2.1 Bidirectional mobility

We now turn to the case where some agents move horizontally and some vertically. We will prove our results for the random geometric graph model, where $n$ nodes are initially placed uniformly in the unit square $G$. In the bidirectional mobility model, before the gossip algorithm starts, each node flips a fair coin, is assigned to move horizontally or vertically, and moves like this throughout the process. Note that this is a one-dimensional mobility model since each node is moving only horizontally or vertically throughout the execution of the gossip algorithm, never changing direction. Our result is that this mobility model is as good as complete node connectivity.

Theorem 5. Consider the gossip algorithm with $n$ agents under the random geometric graph model and bidirectional mobility. We can choose a connectivity radius $r(n) = \Theta \left( \sqrt{\frac{\log n}{n}} \right)$ such that the gossip averaging time is

$$
T_{\text{ave}}(n, \epsilon) = \Theta(n \log \epsilon^{-1}).
$$

(52)
Let $B_i$ denote the number of agents whose initial position was in square $i$.

It is well known [17, 28–30] that a combination of a Chernoff and a union bound, yields uniform bounds on the maximum and minimum load of all the squares:

$$\Pr \left( \frac{c_1}{2} \log n \leq B_i \leq 2c_1 \log n \ \forall i \right) \geq 1 - n^{1-c_1/8} \frac{2}{c_1 \log n}.$$ 

Therefore, selecting $c_1 \geq 10$ we can show that all the squares have $\Theta(\log n)$ agents with probability at least $1 - \frac{1}{n \log n}$. This guarantees balanced square loads even if the experiment is repeated $n^2$ times. We set the transmission radius to $r(n) = \sqrt{\frac{\log n}{c_1 \log n}}$ to guarantee that a agent in a square can always communicate with agents in the four adjacent squares.

Recall that initially each agent is assigned to be a horizontally moving or vertically moving node by flipping a coin and keeps this directionality throughout the process. Denote by $H_i$ the set of nodes that move horizontally and whose initial position was in the $i$-th row of squares. These agents always stay in the $i$-th row. Similarly, let $V_i$ be the set of agents who move vertically in the $i$-th column of squares.

Each square contains in expectation $c_1 \log n$ nodes and there are $\sqrt{\frac{n}{c_1 \log n}}$ squares in each row and column. Since each node flips a fair coin and is assigned in a vertically or horizontally moving class, the expected cardinalities will be:

$$\mathbb{E}|H_i| = \mathbb{E}|V_i| = \frac{1}{2} c_1 \log n \sqrt{\frac{n}{c_1 \log n}} = \Theta(\sqrt{n \log n}) .$$ 

Using standard Chernoff bounds we can show that the cardinalities of $|H_i|, |V_i|$ are sharply concentrated near their expectation.

Theorem 1 shows that the averaging time of the gossip algorithm is bounded by the inverse spectral gap (relaxation time) of the average matrix $\overline{W}$, where the expected matrix $\overline{W} = \mathbb{E}W(s)$ is computed over mobility of the nodes and random selection of which nodes are gossiping.

We now proceed to bound the spectral gap using a canonical flow and we need to select paths for every pair of states for the Markov chain defined by $\overline{W}$. The state space is the set of $n$ agents and $\pi(i) = 1/n$ for each agent $i$ since $\overline{W}$ is doubly stochastic. The capacities of the edges will be proportional to the entries of $\overline{W}$ (see 14), where $\overline{W}_{ij}$ is the average of the probabilities $P_{ij}$ and $P_{ji}$, measuring how often agents $i$ and $j$ are
pairwise averaged. For each pair of agents \((i, j)\) we must specify how to satisfy the demand \(D(i, j) = n^{-2}\) by assigning flows to some (appropriately chosen) paths in \(P_{ij}\).

Our flow construction uses four different cases depending on whether \(i\) and \(j\) move horizontally or vertically:

Case 1 Suppose \(i \in H_k\) and \(j \in H_l\). To satisfy the demand \(n^{-2}\) node \(i\) assigns \(\Theta(n^{-3})\) units to each path \((i, v, j)\), where \(v \in V_r\) for some \(r\). There are \(\Theta(n)\) agents who move vertically, so the total flow that reaches \(j\) can be made equal to \(n^{-2}\). See Figure 3.

Case 2 Suppose \(i \in V_k\) and \(j \in V_l\). This is the same as the previous case, except that \(\Theta(n^{-3})\) units are assigned to each path \((i, h, j)\) for \(h \in H_r\).

Case 3 Suppose \(i \in H_k\) and \(j \in V_l\). To satisfy the demand \(n^{-2}\) assign \(n^{-2}\) to the direct path \((i, j)\).

Case 4 Suppose \(i \in V_k\) and \(j \in H_l\). We again assign \(n^{-2}\) to the direct path \((i, j)\).

Our construction therefore only uses the edges in the graph between \(H\) sets and \(V\) sets. In other words it is only the averaging between nodes that move vertically with nodes that move horizontally that allows information to spread fast in the network. The averaging between two nodes in \(H\) or \(V\) could be omitted and still the bound would not change in order. The total load on an edge \(e = (h, v)\) between a horizontal moving agent and a vertical moving agent is the sum of the direct flow \((h, v)\), the the sum of the flows \((h, v, j)\) for all horizontal moving \(i\) and \((i, h, v)\) for all vertical moving \(i\).

\[
f(e) = \frac{1}{n^2} + \Theta \left( \frac{1}{n^3} \right) \sum |V_r| + \Theta \left( \frac{1}{n^3} \right) \sum |H_r| \\
= \Theta \left( \frac{1}{n^2} \right).
\]

The same bound holds for \(e = (v, h)\).
Finally, we calculate the capacity for the edges \((v, h)\). It is sufficient to calculate a lower bound on the probability that agents \(v \in V_k\) and \(h \in H_l\) average. Agent \(v\) is selected with probability \(1/n\). Based on our assumptions on the communication radius, \(v\) can communicate with \(\Theta(\log n)\) neighbors. The probability that \(v\) lands in a row within \(r(n)\) of row \(l\) is \(\Theta(\sqrt{n - 1} \log n)\) and the probability that \(h\) lands within \(r(n)\) of row \(k\) is also \(\Theta(\sqrt{n - 1} \log n)\). Therefore we have

\[
P_{vh} = \Omega\left(\frac{1}{n} \sqrt{\frac{\log n}{n}} \sqrt{\frac{\log n}{n}} \frac{1}{\log n}\right) = \Omega\left(\frac{1}{n^2}\right).
\]

The capacity of each edge \((v, h)\) is then \(C(e) = \Omega(n^{-3})\). By symmetry, the same formulae hold for \((h, v)\).

We can now calculate the overload for this flow on any edge \(e = (v, h)\):

\[
f(e) = \frac{\Theta(n^{-2})}{\Omega(n^{-3})} = O(n).
\]

Since this holds for all edges we have \(\rho(F) = O(n)\). The maximum length of any path used in the flow is 2, so by the Poincaré inequality we have

\[
T_{\text{relax}}(W) = \frac{1}{1 - \lambda_2(W)} = \rho(F) l(F) = O(n).
\]

Theorem 1 gives the result.

One intuition for this result is that bidirectional mobility enables the construction of “short” routes between all pairs of agents. We can derive the identical result for the torus using the same arguments. Under bidirectional mobility the averaging time for the torus is \(O(n \log \epsilon^{-1})\), which is the same as full mobility.

### 5.2.2 Unidirectional mobility

We now prove an analogous lower bound to the unidirectional mobility model for the torus that shows unidirectional mobility does not improve the scaling performance for random geometric graphs.

**Corollary 5** (Random geometric graph with 1D mobility). Consider gossip on the random geometric graph with \(n\) agents with the 1D unidirectional mobility model. Then for this gossip algorithm,

\[
T_{\text{ave}}(n, \epsilon) = \Omega\left(\frac{n^2 \log \epsilon^{-1}}{\log n}\right).
\]

**Proof.** We first divide the unit square into sub-squares of side length \(c_1 \sqrt{\log n/n}\) for some constant \(c_1\). This creates a \(\Theta\left(\sqrt{\log n/n}\right) \times \Theta\left(\sqrt{\log n/n}\right)\) torus on which the mobility can be defined. We must first characterize the Markov chain corresponding to the gossip algorithm under the 1D unidirectional mobility model. If we set the communication radius to \(c_2 \sqrt{\log n/n}\) then an agent in the \(i\)-th row of sub-squares can communicate with agents in rows \(\{i - c_3, \ldots, i + c_3\}\), where \(c_3\) is again a constant. Moreover, each sub-square will have \(\Theta(\log n)\) agents with high probability. Therefore we can upper bound the probability that an agent in row \(i\) will average with an agent in one of the rows \(\{i - c_3, \ldots, i - 1, i + 1, \ldots, i + c_3\}\):

\[
\beta_{ij} = O\left(\frac{1}{n} \times \sqrt{\frac{\log n}{n}} \times \frac{1}{\log n}\right).
\]

Thus the chance a given agent averages with someone not in their row of sub-squares is \(O(1/\sqrt{n^3 \log n})\).
As in the torus, we apply the induced chain method using the partition that merges each row of sub-squares. This creates a new Markov chain with $\sqrt{n}/\log n$ states that is a kind of cycle where there are positive transition probabilities from state $k$ (corresponding to the $k$-th row) to states $l \in \{i - c_3, \ldots, i + c_3\}$.

From the analysis of the torus we can see that from row $k$ to $l$:

$$
\hat{W}_{kl} = \frac{1}{\sum_{i:F(i) = k} \pi_i} \sum_{i:F(i) = k} \sum_{j:F(j) = l} \pi_i W_{ij} \tag{60}
$$

$$
= \sqrt{\frac{n}{\log n}} \cdot n \log n \cdot O\left(\frac{1}{n^{3/2} \sqrt{\log n}}\right) \tag{61}
$$

$$
= O\left(\frac{1}{n}\right) \tag{62}
$$

Let $\beta$ denote this transition probability. The matrix of this new chain is still circulant and generated by the vector

$$
(\beta, \ldots, \beta, 1 - 2c_3\beta, \beta, \ldots, \beta, 0 \ldots 0) \tag{63}
$$

The DFT and Taylor expansion again gives the bound on the second-largest eigenvalue:

$$
\lambda_2(\hat{W}) = 1 - \beta \cdot O\left(\frac{\log n}{n}\right) \tag{64}
$$

$$
= 1 - O\left(\frac{\log n}{n^2}\right) \tag{65}
$$

Therefore $T_{relax}(\hat{W}) = \Omega(n^2/\log n)$. 

6 Experiments and simulations

We can gain some intuition about the benefits of mobility via simulations. All simulations are for a torus with a linearly varying field. Our first main result was a lower bound that stated that horizontal mobility was as bad as no mobility in terms of convergence. This is illustrated in Figure 4 where we can see that for a range of network sizes the error under horizontal mobility is close to that of the torus with no mobility. Indeed, as the network size gets larger, the gap vanishes, which suggests that our analysis is tight for this example. Our second major result was a positive one; the bidirectional mobility model was nearly as good as full mobility. This is illustrated in Figure 5. Although there is a gap between the error decay under the two mobility models, for a fixed error the number of iterations needed for to achieve that error is at most a constant factor more for the bidirectional mobility model.

Our final result was that adding $m$ mobile agents to a static grid with $n$ agents gives a convergence time of $\Theta(n^2/m \log \epsilon^{-1})$. Figure 6 shows how adding only a few additional mobile agents can dramatically improve the speed of convergence. As we add more nodes, $\log \epsilon$ decreases linearly, which corresponds to an exponential decay in the average error. This suggests that even in large networks investing in a small number of mobile agents can yield a major benefit in convergence time.

Finally, we can simulate gossip with a different random walk mobility model [22, 23] that will slow the convergence time. At each time, we assume the agents move according to a random walk in their row. Figure 7 shows how the error behaves as a function of the number of steps between each gossip iteration. The dashed line indicates the error under the horizontal mobility model, which corresponds to the nodes moving according to the stationary distribution of the random walk. Although the random walk seems difficult to handle analytically, these simulations indicate that our bounds may hold for these models as well.
Figure 4: Log average error versus number of iterations of the gossip algorithm for the torus with no mobility and with horizontal mobility. As the graph size increases, the gap between the two algorithms vanishes.

Figure 5: Log average error versus number of iterations of the gossip algorithm for the torus with full mobility and with bidirectional mobility. As the graph size increases, the gap between the two algorithms shrinks.
Figure 6: Adding a few mobile nodes to a static grid can exponentially decrease the estimation error for a fixed number of iterations (20000).

Figure 7: Error in the random walk model is lower bounded by the fast-mixing horizontal mobility model.
7 Discussion and future directions

In this work we investigated how agent mobility impacts the convergence speed of distributed averaging algorithms by developing new analytical tools derived from the theory of Markov chains. Using these tools we could show that different mobility patterns can have dramatically different effects depending on the overlap of the mobility paths. Perhaps surprisingly, even a sublinear number of mobile nodes can change the order of gossip messages required for convergence. We note that “mobility” in our model is a kind of time-varying network topology and in practical implementations need not come from the physical mobility of the agents.

The class of mobility models which are amenable to our analysis makes a strong assumption on the speed of the mobility or delay-tolerance of the gossip algorithm. One interesting direction for future research involves understanding the benefits of mobility for more realistic mobility models. It is possible that general mobility models with memory are tractable to analysis if the mobility is driven by a Markov chain since this would integrate naturally with the Markov structure of the averaging process. We conjecture that random walk models with slower mixing times will yield smaller benefits, and that our independent (fast mixing) model is always an upper bound. For these models, modifying the pairwise gossip paradigm (c.f. [17]) may yield a greater benefit than relying on mobility alone. The impact of node mobility on distributed optimization and general message-passing algorithms on probabilistic graphical models would also be a very interesting research direction.

Another interesting direction is understanding the impact of mobility for more general message-passing algorithms for example for optimizing convex functions. The analysis of [31] obtains a convergence theorem similar to the spectral gap and it would be interesting to investigate the scaling behavior of the number of required iterations for the min-sum algorithm to optimize a convex function under some node mobility model.

1.1 Relaxation time for the torus plus central node

We will construct a $g$ in (22) to show that the mixing time of a torus plus an additional central node $M$ with transition probabilities $\Theta((m/n)/(m+n))$ to $M$ and $\Theta((1/n)/(m+n))$ away from $M$ along with transitions $\Theta(1/n)$ between neighbors in the torus has relaxation time $\Omega(n^2/m)$. The stationary distribution for this chain has probability $\Theta(1/(m+n))$ on the nodes of the torus and $\Theta(m/(m+n))$ on $M$. Let $g(M) = 0$ and $g$ be constant on each column of the torus with the values on the columns being $\{-\alpha, -\alpha+1, \ldots, 0, 1, 2, \ldots, \alpha, \alpha, \alpha, \ldots, -\alpha+1, -\alpha\}$, where $\alpha = \Theta(\sqrt{n})$. Then clearly $\sum \pi_i g(i) = 0$. We can calculate the numerator and denominator in (22):

$$\sum_k \pi(k) g(k)^2 = \frac{1}{m+n} \sqrt{n^4} \sum_{i=0}^{\alpha} i^2 = \Theta \left( \frac{n^2}{m+n} \right)$$

$$\mathcal{D}(g,g) = \frac{m/n}{(m+n)^2} \sqrt{n^4} \sum_{0=1}^{\alpha} i^2 + \frac{1/n}{m+n} \sqrt{n^4} \sum_{i=1}^{\alpha} 1 = \Theta \left( 1 + \frac{mn}{m+n} \right).$$

Dividing gives the result.
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