NORMAL FORMS FOR SINGULAR HOLOMORPHIC ENGEL SYSTEMS

M. CORRÊA JR AND LUIS. G. MAZA

ABSTRACT. We prove a normal form theorem for germs of holomorphic singular Engel systems with good conditions on its singular set. As an application, we prove that there exists an integral analytic curve passing through the singular points of the system. Also, we prove that a globally decomposable Engel system on a four dimensional projective space has singular set with atypical codimension.

1. INTRODUCTION

A germ of holomorphic Pfaff system of codimension $k$ on $(\mathbb{C}^n, 0)$ is a subsheaf $\mathcal{I}$ of the cotangent sheaf $\Omega^1_{\mathbb{C}^n}$ of $(\mathbb{C}^n, 0)$ spanned by $k$ germs of holomorphic differential 1-forms $\omega_1, \ldots, \omega_k$ assumed generically linearly independent near 0. We will write $\mathcal{I} = \langle \omega_1, \ldots, \omega_k \rangle$. This system can be represented by the holomorphic $k$-form $\omega_1 \wedge \ldots \wedge \omega_k$. The singular set of $\mathcal{I}$ is the analytic subset given by

$$\text{Sing}(\mathcal{I}) = \{ p \in (\mathbb{C}^n, 0); (\omega_1 \wedge \cdots \wedge \omega_k)(p) = 0 \}.$$ 

Therefore $\text{Sing}(\mathcal{I})$ is defined by $k \times k$ determinants of an $n \times k$ matrix. Therefore, each irreducible component has codimension at most $k + 1$. We say that the singular set of $\mathcal{I}$ has expected codimension if it has a component of codimension $k + 1$.

Let $\mathcal{C} = V(\mathfrak{a})$ be a germ of analytic subset in $(\mathbb{C}^n, 0)$ of codimension $\leq k$, with zeros ideal $\mathfrak{a}$. If $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle$, then we denote by $d\mathfrak{a}$ the Pfaff system spanned by $df_1, \ldots, df_r$.

We say that $\mathcal{C} = V(\mathfrak{a})$ is an integral variety of $\mathcal{I} = \langle \omega_1, \ldots, \omega_k \rangle$ if and only if

$$\omega_i \wedge d\mathfrak{a} \in \mathfrak{a} \otimes \Omega^{k+1}_{\mathbb{C}^n}, \text{ for each } i = 1, \ldots, k.$$ 

A Pfaff system $\mathcal{I}$ is integrable if and only if

$$d\mathcal{I} \equiv 0 \mod \mathcal{I}.$$ 

If $\mathcal{I}$ is integrable, by the classical Frobenius’s Theorem, for all points or all points $p \in (\mathbb{C}^n, 0) \setminus \text{Sing}(\mathcal{I})$ there exists an integral complex analytic manifold of codimension $k$ passing through $p$. In [11] B. Malgrange obtained a Frobenius’s Theorem for singular integrable systems with singular set of
codimension $\geq 3$, showing the existence of integral varieties passing through the singular points of the system.

For a germ of Pfaff system $\mathcal{I}$, we can define its derived flag $\mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \cdots$ by the relations $\mathcal{I}^{(0)} = \mathcal{I}$ and

$$\mathcal{I}^{(i+1)} = \{ \alpha \in \mathcal{I}^{(i)} : d\alpha \equiv 0 \mod \mathcal{I}^{(i)} \}.$$ 

Then, the derived flag of a Pfaff system $\mathcal{I}$ is defined inductively by the exact sequence

$$0 \longrightarrow \mathcal{I}^{(i+1)} \longrightarrow \mathcal{I}^{(i)} \longrightarrow d\mathcal{I}^{(i)}/\left(\mathcal{I}^{(i)}d\mathcal{I}^{(i)}\right) \longrightarrow 0.$$ 

Since the codimension of each Pfaff system $\mathcal{I}^{(i)}$ is generically constant then there will be an integer $N$ such that $\mathcal{I}^{(N)} = \mathcal{I}^{(N+1)}$. This integer $N$ is called the derived length of $\mathcal{I}$. The Pfaff system $\mathcal{I}^{(N)}$ is always integrable by definition since

$$d\mathcal{I}^{(N)} \equiv 0 \mod \mathcal{I}^{(N)}.$$ 

If $\mathcal{I}^{(N)} = 0$ we say that the system $\mathcal{I}$ is completely nonholonomic. See [3] for more details.

A contact system on $(\mathbb{C}^3, 0)$ is a completely nonholonomic system. The classical Darboux-Pfaff theorem gives a normal form for non-singular contact system. That is, if $\mathcal{I}$ is a non-singular contact system, then there exist a germ of coordinate system on $(\mathbb{C}^3, 0)$ such that

$$\mathcal{I} = \langle dz_3 - z_2dz_1 \rangle.$$ 

In [5] D. Cerveau provided a singular version of the Pfaff-Darboux Theorem. More precisely, Let $\beta$ be a germ of holomorphic 1-form on $(\mathbb{C}^n, 0)$. We define the class of $\beta$ to be the integer $r$ for which

$$\beta \wedge (d\beta)^r \neq 0, \quad \beta \wedge (d\beta)^{r+1} \equiv 0.$$ 

**Theorem 1.1.** [5] Darboux-Pfaff-Cerveau] Let $\beta$ be a germ of holomorphic 1-form on $(\mathbb{C}^n, 0)$ of class $r$ and $\text{cod}(\text{Sing}(d\beta)) \geq 3$. Then there exist $f_1, \ldots, f_{r+1}, g_1, \ldots, g_r \in \mathcal{O}_0^n$ such that

$$\beta = \sum_{i=1}^r f_idg_i + df_{r+1}.$$ 

In this work we are interested in completely nonholonomic system on $(\mathbb{C}^4, 0)$ of codimension 2 and derived length equal to 2.

**Definition 1.1.** A germ of Engel system in $(\mathbb{C}^4, 0)$ is a Pfaff system $\mathcal{I} = \langle \alpha, \beta \rangle$ of codimension 2 in $(\mathbb{C}^4, 0)$ satisfying the following conditions:

(i) $\alpha \wedge \beta \wedge d\alpha \neq 0$
(ii) $\alpha \wedge \beta \wedge d\beta \equiv 0$
(iii) $\beta \wedge d\beta \neq 0,$
We can see that a germ of Engel system in $(\mathbb{C}^4,0)$ is a system of codimension 2 such that, for $0 \leq i \leq 2$, the elements of its derived flag satisfy $\text{cod}(\mathcal{I}^{(i)}) = 2 - i$. In fact, $\mathcal{I}^{(0)} = \langle \alpha, \beta \rangle$, $\mathcal{I}^{(1)} = \langle \beta \rangle$ and $\mathcal{I}^{(2)} = 0$. Thus, an Engel system has derived length equal to 2.

In the real non-singular case, these Pfaff systems were introduced by E. von Weber in 1898 and studied by several authors [4][7][12]. F. Engel [6] shows that a non-singular Engel system is locally isomorphic, at a generic point, to the canonical system

$$\mathcal{I}_0 = \langle dz_4 - z_3dz_1, dz_3 - z_2dz_1 \rangle.$$  

That is, F. Engel provides a kind of Pfaff-Darboux type theorem for non-singular Pfaff systems of codimension 2 and derived length equal to 2. M. Zhitomirskii in [15] obtained normal forms for real non-singular Engel along non generic points.

The canonical system appears naturally as a system called canonical contact system on the space $J^2(\mathbb{C}, \mathbb{C})$ of 2-jets of holomorphic maps of $\mathbb{C}$, see [12]. Nonsingular global holomorphic Engel systems have been studied by L. Solá Conde and F. Presa in [13].

We prove the following result for germs of holomorphic Engels system in $(\mathbb{C}^4,0)$

**Theorem 1.2.** Let $\mathcal{I}$ be a germ of holomorphic Engel system on $(\mathbb{C}^4,0)$ with $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$. Then there exist $f_1, \ldots, f_4 \in \mathcal{O}^4_0$ such that

$$\mathcal{I} = \langle df_4 - f_3df_1, df_3 - f_2df_1 \rangle.$$  

More precisely, there exists a germ of holomorphic map $f := (f_1, f_2, f_3, f_4) : (\mathbb{C}^4,0) \to \mathbb{C}^4$ which is a biholomorphism outside $\text{Sing}(\mathcal{I}) \cup \text{Sing}(d\mathcal{I}^{(1)})$ such that $f^*\mathcal{I}_0 = \mathcal{I}$.

We note that the only place the dimension assumption is used in the proof is to guarantee that $\beta \wedge (d\beta)^2 \equiv 0$. Therefore, we obtain the following result which will be useful in application to Engel systems on four dimensional projective spaces.

**Corollary 1.1.** Let $\mathcal{I} = \langle \alpha, \beta \rangle$ be a Pfaff system of codimension 2 on $(\mathbb{C}^n,0)$ satisfying the following conditions:

- (i) $\alpha \wedge \beta \wedge d\alpha \neq 0$
- (ii) $\alpha \wedge \beta \wedge d\beta \equiv 0$
- (iii) $\beta \wedge d\beta \neq 0$ and $\beta \wedge (d\beta)^2 \equiv 0$

If $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$, then there exist $f_1, \ldots, f_4 \in \mathcal{O}^n_0$ such that

$$\mathcal{I} = \langle df_4 - f_3df_1, df_3 - f_2df_1 \rangle.$$  

Normal forms allows us to prove the existence of integral submanifolds. An interesting consequence of Theorem 1.2 is the existence of an integral analytic curve passing through the singular points of the system.
**Theorem 1.3.** Let $\mathcal{I}$ be a germ of holomorphic Engel system on $(\mathbb{C}^4,0)$ with $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$ and $\text{cod}(\text{Sing}(d\mathcal{I}^{(1)})) \geq 3$, then there exists a germ of an analytic curve passing through $\text{Sing}(\mathcal{I})$ which is a solution of $\mathcal{I}$.

**Proof.** In fact, it follows from Theorem 1.2 that the analytic curve \{ $f_1 = f_3 = f_4 = 0$ \} is a solution of $\mathcal{I} = \langle df_4 - f_3 df_1, df_3 - f_2 df_1 \rangle$. \hfill $\square$

Finally, we give another application of Theorem 1.2 for globally decomposable Engel system on four dimensional projective space.

It is well known that all codimension one integrable systems in $\mathbb{P}^n$ have in its singular set an irreducible component of codimension two.

**Theorem 1.4.** Let $\mathcal{I}$ be a codimension one integrable system on $\mathbb{P}^n$, $n \geq 3$, such that $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$. Then $\text{Sing}(\mathcal{I})$ has an irreducible component of codimension two.

Let $\mathcal{I}$ be a codimension one integrable system on a Fano manifold such that $\text{Sing}(\mathcal{I}) \neq \emptyset$. In Corollary 4.7 F. Loray, J. V. Pereira and F Touzet show that if the canonical class of $\mathcal{I}$ is numerically trivial then its singular set has a component of codimension two.

A similar situation appears in the study of singularities of Poisson structures on Fano manifolds motivated by Bondal’s conjecture Conjecture 4. A. Polishchuk in showed that the rank of a nondegenerate Poisson structure on a Fano variety of odd dimension drops along a subset of codimension two.

As an application of Theorem 1.2 we prove that a globally decomposable Engel system on four dimensional projective space has a singular set with atypical codimension. In fact, the expected codimension of the singular set of a Pfaff system of codimension 2 should be 3. But, the following Theorem shows that the singular set of these systems has codimension $\leq 2$.

**Theorem 1.5.** Let $\mathcal{I}$ be a globally decomposable holomorphic Engel system on $\mathbb{P}^4$. Then, either $\text{Sing}(d\mathcal{I}^{(1)})$ has a component of codimension two, or $\text{Sing}(\mathcal{I})$ has a component of codimension one. Moreover, if $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$, then $\text{Sing}(\mathcal{I})$ has a component of codimension two.

2. **Proof of the Theorem 1.2**

**Proof.** Since $d\beta \wedge \beta \neq 0$ and $(d\beta)^2 \wedge \beta \equiv 0$, we have that $\beta$ has class 1. By Theorem 1.1 there exist $f_1, f_3, f_4 \in \mathcal{O}_0^4$ such that

$$\beta = df_4 - f_3 df_1.$$ 

In particular, $d\beta = df_1 \wedge df_3$. Now, since $d\beta \wedge \alpha \wedge \beta = 0$ we get

$$0 = d\beta \wedge \alpha \wedge \beta = df_1 \wedge df_3 \wedge \alpha \wedge \beta.$$ 

This implies that there exist germs of holomorphic functions $\tilde{a}, \tilde{b}$ and $\tilde{\lambda}$ on $U = (\mathbb{C}^4,0) - \text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$ such that

$$\alpha|_U = \tilde{a} df_1 + \tilde{b} df_3 + \tilde{\lambda} \beta.$$
Since the codimension of $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$ is bigger than 2, by Hartogs’ extension Theorem we have the identity

$$\alpha - \lambda \beta = \text{ad} f_1 + \text{bd} f_3,$$

where $a, b, \lambda \in O^4_0$ on $(\mathbb{C}^4, 0)$. Now if either $a = 0$ or $b = 0$, then $\alpha \wedge \beta \wedge d\alpha \equiv 0$, a contradiction to Engel’s conditions. Thus $a \neq 0$ and $b \neq 0$, and therefore

$$\frac{1}{b} \alpha - \frac{\lambda}{b} \beta = \frac{a}{b} df_1 + df_3$$

and if we set $f_2 = -\frac{a}{b}$ then

$$\frac{1}{b} \alpha - \frac{\lambda}{b} \beta = df_3 - f_2 df_1.$$

Thus,

$$I = \langle \alpha, \beta \rangle = \left\langle \alpha, \frac{1}{b} \alpha - \frac{\lambda}{b} \beta \right\rangle = \langle df_4 - f_3 df_1, df_3 - f_2 df_1 \rangle.$$  

Now, we will prove that the map $f : (\mathbb{C}^4, 0) \to \text{defined by } f_1, f_2, f_3, f_4$ is a biholomorphism outside $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$. That is, we prove that

$$df_1 \wedge df_2 \wedge df_4 \wedge df_3$$

never vanishes outside $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$. Differentiating the identity

$$\frac{1}{b} \alpha = \frac{a}{b} df_1 + df_3 + \frac{\lambda}{b} \beta.$$

we get

$$d \left( \frac{1}{b} \right) \wedge \alpha + \frac{1}{b} d\alpha = d \left( \frac{a}{b} \right) \wedge df_1 + d \left( \frac{\lambda}{b} \right) \wedge \beta + \frac{\lambda}{b} d\beta.$$

Multiplying this identity by $\beta \wedge \alpha$ we obtain

$$\left[ d \left( \frac{a}{b} \right) \wedge df_1 + d \left( \frac{\lambda}{b} \right) \wedge \beta + \frac{\lambda}{b} d\beta \right] \wedge \beta \wedge \alpha = d \left( \frac{a}{b} \right) \wedge df_1 \wedge \beta \wedge \alpha$$

since $d\beta \wedge \beta \wedge \alpha \equiv 0$. Thus

$$d \left( \frac{a}{b} \right) \wedge df_1 \wedge \beta \wedge \alpha = \left[ d \left( \frac{1}{b} \right) \wedge \alpha + \frac{1}{b} d\alpha \right] \wedge \beta \wedge \alpha.$$  

Therefore

$$d \left( \frac{a}{b} \right) \wedge df_1 \wedge \beta \wedge \alpha = \frac{1}{b} d\alpha \wedge \beta \wedge \alpha \neq 0.$$

Using that $\alpha = adf_1 + bd f_3 + \lambda \beta$ and $\beta = df_4 - f_3 df_1$ and substituting in $d \left( \frac{a}{b} \right) \wedge df_1 \wedge \beta \wedge \alpha$ we conclude that

$$0 \neq d \left( \frac{a}{b} \right) \wedge df_1 \wedge \beta \wedge \alpha = bd \left( \frac{a}{b} \right) \wedge df_1 \wedge df_4 \wedge df_3 = bd f_2 \wedge df_1 \wedge df_4 \wedge df_3$$

is nowhere vanishing outside $\text{Sing}(\alpha \wedge \beta) \cup \text{Sing}(d\beta)$.

□
3. Application to Engel systems on projective spaces

A Pfaff system $\mathcal{I}$ of codimension $k$ on a complex projective space $\mathbb{P}^n$ is a locally decomposable section

$$\omega_I \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n} \otimes L).$$

This means that for all $p \in \mathbb{P}^n$ there exists a neighborhood $U$ of $p$ and 1-forms $\omega_1, \ldots, \omega_k \in \Omega^1_U$, such that $\omega_I|_U = \omega_1 \wedge \cdots \wedge \omega_k$.

If $i : \mathbb{P}^k \to \mathbb{P}^n$ is a generic linear immersion then $i^*\omega_I \in H^0(\mathbb{P}^k, \Omega^k_{\mathbb{P}^k} \otimes L)$ is a section of a line bundle, and its divisor of zeros reflects the tangencies between $\mathcal{I}$ and $i(\mathbb{P}^k)$. The degree of $\mathcal{I}$ is, by definition, the degree of such a tangency divisor. Set $d := \text{deg}(\mathcal{I})$. Since $\Omega^k_{\mathbb{P}^k} \otimes L = \mathcal{O}_{\mathbb{P}^n}(\text{deg}(L) - k - 1)$, one concludes that $L = \mathcal{O}_{\mathbb{P}^n}(d + k + 1)$.

We say that $\mathcal{I}$ is globally decomposable if $\omega_I = \omega_1 \wedge \cdots \wedge \omega_k$ for suitable $\omega_i \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n} \otimes L_i)$.

Besides, the Euler sequence implies that a section $\omega$ of $\Omega^k_{\mathbb{P}^n}(d + k + 1)$ can be thought of as a polynomial $k$-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d + 1$, which we will still denote by $\omega$, satisfying

$$i_R \omega = 0$$

where

$$R = x_0 \frac{\partial}{\partial x_0} + \cdots + x_n \frac{\partial}{\partial x_n}$$

is the radial vector field. Thus the study of distributions of degree $d$ on $\mathbb{P}^n$ reduces to the study of locally decomposable homogeneous $k$-forms of degree $d + 1$ on $\mathbb{C}^{n+1}$ satisfying the relation (2).

We will use the following Jouanolou’s Lemma.

**Lemma 3.1.** [8, Lemme 1.2, pp. 3] If $\eta$ is a homogeneous $q$-form of degree $s$, then

$$i_R d\eta + d(i_R \eta) = (q + s)\eta$$

where $R$ is the radial vector field and $i_R$ denotes the interior product or contraction with $R$.

As an application of Theorem 1.2 we prove that a globally decomposable Engel system on four dimensional projective space has a singular set with atypical codimension. In fact, the expected codimension of the singular set of a codimension 2 should be 3. But, the following Theorem shows that the singular set of these systems has codimension $\leq 2$.

**Theorem 3.1.** Let $\mathcal{I}$ be a globally decomposable holomorphic Engel system on $\mathbb{P}^4$. Then, either $\text{Sing}(dI^{(1)})$ has a component of codimension two, or $\text{Sing}(\mathcal{I})$ has a component of codimension one. Moreover, if $\text{cod}(\text{Sing}(\mathcal{I})) \geq 2$, then $\text{Sing}(\mathcal{I})$ has a component of codimension two.

**Proof.** Firstly we observe that on $\mathbb{P}^4$ all holomorphic 1-forms $\beta \in H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(s))$ satisfy $\beta \wedge (d\beta)^2 = 0$. In fact, $\beta \wedge (d\beta)^2$ is a 5-form on $\mathbb{P}^4$. 

Suppose that \( \text{Cod}(\text{Sing}(I)) \geq 2 \) and \( \text{Cod}(\text{Sing}(dI^1)) \geq 3 \). Then, it follows from Corollary \[\text{Example 2.}\] that there exist homogeneous polynomials \( f_1, f_2, f_3, f_4 \) on \( \mathbb{C}^5 \) such that
\[
\alpha = df_4 - f_3 df_1, \quad \beta = df_3 - f_2 df_1.
\]
In particular, we have that \( \beta \wedge df = -df_4 \wedge df_3 \wedge df_1 \) and \( \alpha \wedge \beta \wedge d\alpha = df_1 \wedge df_2 \wedge df_3 \wedge df_4 \).

Since \( i_R \alpha = i_R \beta = 0 \), we conclude that \( k_4 f_4 - k_1 f_3 f_1 = k_3 f_3 - k_1 f_2 f_1 = 0 \), where \( k_i = \deg(f_i), \ i = 1, 3, 4 \). These relations imply that
\[
\beta \wedge df = \alpha \wedge \beta \wedge d\alpha = 0.
\]
This is a contradiction. On the other hand, suppose that \( \text{cod}(\text{Sing}(I)) \geq 2 \). By lemma \[\text{Example 3.}\] we have
\[
i_R(df) = (s + 1)\beta
\]
since \( i_R \beta = 0 \). Using this relation we have that
\[
\text{Sing}(df) \subset \text{Sing}(i_R(df) \wedge \alpha) = \text{Sing}((s + 1)\beta \wedge \alpha) = \text{Sing}(\beta \wedge \alpha).
\]
We conclude that \( \text{Sing}(I) \) has a component of codimension two. \( \square \)

**Example 1.** Consider the differential system induced by the 1-forms
\[
\alpha = z_0^2 dz_4 - z_0 z_3 dz_1 + (z_1 z_3 - z_0 z_4) dz_0
\]
and
\[
\beta = z_0^3 dz_3 - z_0 z_2 dz_1 + (z_1 z_2 - z_0 z_3) dz_0.
\]

A calculation shows that the pair of 1-forms \((\alpha, \beta)\) satisfy the conditions \(i), ii) and iii)\) of definition \[\text{Example 4.}\] and \( i_R \alpha = i_R \beta = 0 \). Therefore, the differential system \( I = \langle \alpha, \beta \rangle \) induces a decomposable Engel system on \( \mathbb{P}^4 \).

We have that
\[
\alpha \wedge \beta = z_0^4 dz_4 \wedge dz_3 - z_0^3 z_2 dz_4 \wedge dz_1 + z_0^2 (z_1 z_2 - z_0 z_3) dz_4 \wedge dz_0 - z_0^3 z_3 dz_1 \wedge dz_3 - z_0 z_3 (z_1 z_2 - z_0 z_3) dz_1 \wedge dz_0 + z_0^2 (z_1 z_3 - z_0 z_4) dz_0 \wedge dz_3 - z_0 z_2 (z_1 z_3 - z_0 z_4) dz_0 \wedge dz_1.
\]

Therefore \( \text{Sing}(\alpha \wedge \beta) = \{z_0 = 0\} \) has codimension one. Moreover,
\[
\text{Sing}(df) = \{z_0 = z_1 = z_2 = 0\}
\]
has codimension 3.

**Example 2.** Consider the differential system induced by the 1-forms
\[
\alpha = z_0^3 dz_1 + z_0^2 z_0 dz_4 - (z_0^2 z_1 - z_0^2 z_4) dz_0
\]
and
\[
\beta = z_0^3 dz_2 + z_0 z_1 z_0 dz_1 - (z_0^2 z_2 + z_0 z_4) dz_0.
\]

A calculation shows that the pair of 1-forms \((\alpha, \beta)\) satisfies the conditions \(i), ii) and iii)\) of definition \[\text{Example 5.}\] and \( i_R \alpha = i_R \beta = 0 \). Therefore, the differential system \( I = \langle \alpha, \beta \rangle \) induces a decomposable Engel system on \( \mathbb{P}^4 \). We can see that
\[
\text{Sing}(\alpha \wedge \beta) = \{z_0 = 0\} \quad \text{and} \quad \text{Sing}(df) = \{z_0 = z_3 = z_4 = 0\}.
\]
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M. CORRÉA JR, ICEX - UFMG, DEPARTAMENTO DE MATEMÁTICA, AV. ANTÔNIO CARLOS 6627, 30123-970 BELO HORIZONTE MG, BRAZIL,
E-mail address: mauricio@mat.ufmg.br

LUIS. G. MAZA, UNIVERSIDADE FEDERAL DE ALAGOAS, INSTITUTO DE MATEMÁTICA, AV. LOURIVAL MELO MOTA, S/N, TABULEIRO DOS MARTINS 57072-970 - MACEIO, AL- BRASIL,
E-mail address: lmaza@im.ufal.br