CONCENTRATION PHENOMENON OF THE ENDEMIC EQUILIBRIUM OF A REACTION-DIFFUSION-ADVECTION SIS EPIDEMIC MODEL WITH SPONTANEOUS INFECTION

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Abstract. In this paper, we investigate the effect of spontaneous infection and advection for a susceptible-infected-susceptible epidemic reaction-diffusion-advection model in a heterogeneous environment. The existence of the endemic equilibrium is proved, and the asymptotic behaviors of the endemic equilibrium in three cases (large advection; small diffusion of the susceptible population; small diffusion of the infected population) are established. Our results suggest that the advection can cause the concentration of the susceptible and infected populations at the downstream, and the spontaneous infection can enhance the persistence of infectious disease in the entire habitat.

1. Introduction. Classical epidemic models assume that the entire population lives in one area and is well mixed. This assumption is not necessarily satisfied; for instance, the population may be living on isolated islands or in different countries while traveling from one location to another. This spatial heterogeneity affects the transmission of the disease. Spatially explicit models are also more effective in evaluating control strategies, particularly those applied to movement of individuals; one may refer to [32] for related discussion. To understand the impact of spatial heterogeneity on the dynamics of a disease, an increasing number of reaction-diffusion models have been proposed and studied in recent years. In particular, the SIS epidemic reaction-diffusion models have been developed to investigate the effects of the spatial heterogeneity and the movement of populations on the persistence or extinction of diseases, see, for instance, [1, 4, 10, 13, 15, 18, 22, 24, 25, 26, 35, 36, 37, 38, 39, 41, 43].

In the SIS epidemic models as mentioned above, infection is considered to be only transmitted by the contact between the infected and the susceptible populations. However, according to [16, 17], the social infection can also arise because of spontaneous factors other than transmission. For such a reason, the term whereby

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uninfected individuals turn into infected at the rate can be added to extend the primary SIS models, so that the effect of spontaneous infection can be incorporated. On the other hand, in some heterogeneous environments, the movement of populations may be subjected to some passive directions definitely as in \[5, 33\], which can be considered as external natural forces like the wind as \[9\] or streamflow as \[21, 27, 28, 29, 30\], etc. Therefore, it is important to explore how the advection and spontaneous infection affect the persistence and extinction of the infectious diseases modeled by an SIS system under consideration.

In this paper, we are concerned with the following reaction-diffusion-advection system with spontaneous infection over the interval habitat \([0, L]\) in a spatially heterogeneous environment:

\[
\begin{aligned}
S_t &= d_S S_{xx} + q S_x - \beta(x) \frac{SI}{S + I} - \eta(x) S + \gamma(x) I, \quad 0 < x < L, \quad t > 0, \\
I_t &= d_I I_{xx} + q I_x + \beta(x) \frac{SI}{S + I} + \eta(x) S - \gamma(x) I, \quad 0 < x < L, \quad t > 0, \\
d_S S_x + q S &= 0 = d_I I_x + q I, \quad x = 0, L, \quad t > 0, \\
(S(x, 0), I(x, 0)) &= (S_0(x), I_0(x)) \geq (0, 0), \quad I_0(x) \neq 0, \quad 0 < x < L.
\end{aligned}
\]

Here, the unknown functions \(S(x, t)\) and \(I(x, t)\) represent, respectively, the population densities at position \(x \in [0, L]\) and time \(t > 0\) of the susceptible individuals and infected individuals, and the positive constants \(d_S\) and \(d_I\) are their respective diffusion coefficients; the functions \(\beta(x)\) and \(\gamma(x)\) are Hölder continuous positive functions in \([0, L]\) and stand for the rates of epidemic transmission and recovery at position \(x\), respectively; the function \(\eta(x)\) is also positive and Hölder continuous while it stands for the spread of the disease according to the imported cases of the infectious disease. It is noteworthy that this kind of import can be considered as brief contacts with individuals who are not included in the population in \([17, 19, 34]\); meanwhile, one can refer to \([2, 16, 40, 44]\) regarding some other different explanations for \(\eta\). The positive constant \(q\) represents the advection speed of a stream or wind, which carries the susceptible and infected populations from the upstream \(x = L\) to the downstream \(x = 0\). This boundary value condition assumes the no-flux situation at the downstream and the upstream. Given initial data \(S_0, I_0\), thanks to the boundary value conditions, one can integrate the sum of the equations of (1) to yield the conservation law:

\[
\int_0^L (S(x, t) + I(x, t)) \, dx = \int_0^L (S_0(x) + I_0(x)) \, dx = N, \quad \forall t \geq 0.
\]

We would like to mention that the non-advection version \((q = 0)\) of (1) was proposed and studied by Tong and Lei \([42]\) and the non-spontaneous-infection version \((\eta = 0)\) was investigated in \([6, 20]\). More precisely, the systems in \([42]\) and \([6, 20]\) read, respectively, as the following

\[
\begin{aligned}
S_t &= d_S S_{xx} - \beta(x) \frac{SI}{S + I} - \eta(x) S + \gamma(x) I, \quad 0 < x < L, \quad t > 0, \\
I_t &= d_I I_{xx} + \beta(x) \frac{SI}{S + I} + \eta(x) S - \gamma(x) I, \quad 0 < x < L, \quad t > 0, \\
S_x &= I_x = 0, \quad x = 0, L, \quad t > 0, \\
S(x, 0) &= S_0(x) \geq 0, \quad I(x, 0) = I_0(x) \geq 0, \quad \neq 0.
\end{aligned}
\]
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and

\[
\begin{cases}
S_t = d_S S_{xx} + q S_x - \beta(x) \frac{SI}{S+I} + \gamma(x) I, & 0 < x < L, \ t > 0, \\
I_t = d_I I_{xx} + q I_x + \beta(x) \frac{SI}{S+I} - \gamma(x) I, & 0 < x < L, \ t > 0, \\
d_S S_x + q S = 0 = d_I I_x + q I, & x = 0, L, \ t > 0, \\
(S(x, 0), I(x, 0)) = (S_0(x), I_0(x)) \geq (0, 0), \ I_0(x) \neq 0, & 0 < x < L.
\end{cases}
\]  

(4)

The SIS reaction-diffusion-advection model with mass action infection mechanism was considered in [7].

In the present paper, our main goal is to understand the combined influence of the advection and spontaneous infection on the persistence, extinction, and spatial distribution of the infectious diseases modeled by system (1). To study the spatial distribution of the infectious diseases, we are then led to analyze the steady-state (i.e., stationary) solution problem corresponding to (1). The steady-state problem of (1) consists of a pair of nonlinear ordinary differential equations:

\[
\begin{align*}
&d_S S_{xx} + q S_x - \beta(x) \frac{SI}{S+I} - \eta(x) S + \gamma(x) I = 0, \ x \in (0, L), \\
&d_I I_{xx} + q I_x + \beta(x) \frac{SI}{S+I} + \eta(x) S - \gamma(x) I = 0, \ x \in (0, L),
\end{align*}
\]  

(5a)

constrained with the boundary value conditions

\[
d_S S_x + q S = 0 = d_I I_x + q I, \ x = 0, L,
\]  

(5c)

and the conservation law associated with (2)

\[
\int_0^L (S(x) + I(x)) \, dx = N.
\]  

(6)

From now on, we set \(N\) to be a given positive constant.

From the ecological point of view, we are only interested in nonnegative steady-state solutions. With regard to (5) and (6), a solution \((S, I)\) with \(I = 0\) is called a disease-free equilibrium (DFE), whereas there also is an endemic equilibrium (EE), which is the componentwise positive solution. However, it is easy to see that no DFE can exist in (5) and (6) due to the presence of spontaneous infection term in (1).

First of all, we will prove the existence of EE via the abstract theory dynamical system combined with a priori estimate of solutions to (1), which indeed shows the uniform persistence of system (1); see Theorem 2.1. Then, we study the spatial profile of EE in the following three cases: (i) \(q \to \infty\); (ii) \(d_S \to 0\); (iii) \(d_I \to 0\). Our first main result in this regard concerns the asymptotic behavior of EE as \(q \to \infty\), and our conclusion reads as follows.

**Theorem 1.1.** Fix \(d_S, d_I\) and denote by \((S(x, q), I(x, q))\) any EE of (5)-(6). Then, as \(q \to \infty\), we have

- \((S(x, q), I(x, q)) \to (0, 0)\) locally uniformly in \((0, L)\);
- at the downstream,

\[
\lim_{q \to \infty} \frac{1}{q} S\left(\frac{y}{q}, q\right) = C_S e^{-\frac{y}{S_0}}, \quad \lim_{q \to \infty} \frac{1}{q} I\left(\frac{y}{q}, q\right) = C_I e^{-\frac{y}{I_0}}
\]
uniformly for \( y \) in any compact subset of \([0, \infty)\), where \((C_S, C_I)\) is a positive solution of
\[
\begin{align*}
&d_SC_S + d_IC_I = N, \\
&\int_0^\infty \frac{\eta(0)C_Se^{-\frac{\beta y}{S}} + [\beta(0) + \eta(0)]C_Ie^{-\frac{\gamma y}{S}} - \frac{\eta y}{S} - d_I}{C_Se^{-\frac{\beta y}{S}} + C_Ie^{-\frac{\gamma y}{S}}} dy = \frac{\gamma(0)C_I}{C_S} d_I.
\end{align*}
\]  

Our second main result on the asymptotic behavior of EE as \( d_S \to 0 \) can be stated as follow.

**Theorem 1.2.** Fix \( q, d_I \) and denote by \((S(x, d_S), I(x, d_S))\) any EE of (5)-(6). Then, as \( d_S \to 0 \), we have
- \( d_SS(dy, d_S) \to C_Se^{-qy}(0 < C_S < Nq) \) uniformly for \( y \) in any compact subset of \([0, \infty)\);
- \((S(\cdot, d_S), I(\cdot, d_S)) \to (\tilde{S}(\cdot) + \tilde{N}\delta_0, \tilde{I}(\cdot))\) weakly in \( L^1((0, L))\), where \( \delta_0 \) is the Dirac measure centered at 0. Here \((\tilde{S}, \tilde{I})\) is a positive solution of the following system:

\[
\begin{align*}
q\tilde{S}_x - \beta(x)\frac{\tilde{S}I}{S + I} - \eta(x)\tilde{S} + \gamma(x)\tilde{I} = 0, & \quad x \in (0, L), \\
d_I\tilde{I}_x + q\tilde{I}_x + \beta(x)\frac{\tilde{S}I}{S + I} + \eta(x)\tilde{S} - \gamma(x)\tilde{I} = 0, & \quad x \in (0, L),
\end{align*}
\]

with \( \int_0^L [\tilde{S}(x) + \tilde{I}(x)] dx + \tilde{N} = N \) and \( 0 < \tilde{N} < N \).

Our last main result focuses on the asymptotic behavior of EE as \( d_I \to 0 \). Indeed, we can state the following.

**Theorem 1.3.** Fix \( q, d_S \) and denote by \((S(x, d_I), I(x, d_I))\) any EE of (5)-(6). Then, as \( d_I \to 0 \), we have
- \( d_I(d_Iy, d_I) \to C_Ie^{-qy}(0 < C_I < Nq) \) uniformly for \( y \) in any compact subset of \([0, \infty)\);
- \((S(\cdot, d_I), I(\cdot, d_I)) \to (\bar{S}(\cdot), \bar{I}(\cdot) + \bar{N}\delta_0)\) weakly in \( L^1((0, L))\), where \( \delta_0 \) is the Dirac measure centered at 0. And \((\bar{S}, \bar{I})\) is a positive solution of the following system:

\[
\begin{align*}
&d_S\bar{S}_{xx} + q\bar{S}_x - \beta(x)\frac{\bar{S}I}{S + I} - \eta(x)\bar{S} + \gamma(x)\bar{I} = 0, & \quad x \in (0, L), \\
&q\bar{I}_x + \beta(x)\frac{\bar{S}I}{S + I} + \eta(x)\bar{S} - \gamma(x)\bar{I} = 0, & \quad x \in (0, L), \\
&d_S\bar{S}_x(0) + q\bar{S}(0) + q\bar{I}(0) = 0, \quad d_S\bar{S}_x(L) + q\bar{S}(L) = 0, \\
&q\bar{I}(0) = \left[ \frac{\gamma(0) - \beta(0)}{\frac{\bar{S}(0)}{S(0) + I(0)}} \right] \bar{N}, \quad \bar{I}(L) = 0,
\end{align*}
\]

with \( \int_0^L [\bar{S}(x) + \bar{I}(x)] dx + \bar{N} = N \), \( 0 < \bar{N} < N \) and \( \gamma(0) > \frac{\beta(0)}{\frac{\bar{S}(0)}{S(0) + I(0)}} \).
The implication of Theorems 1.1-1.3 in the control strategy of infectious disease will be discussed in the conclusion section.

Indeed, compared to [6, 20], the presence of the spontaneous infection causes several mathematical difficulties in the establishment of fine lower and upper positive bounds of the two components $S$ and $I$ with respect to large advection or small diffusion rates, as well as in the analysis of the asymptotic behaviors of $S$ and $I$. We have to introduce some new techniques to overcome those difficulties. More interestingly, the incorporation of the spontaneous infection can produce some very different spatial distribution behaviors of the disease, as discussed in the final section.

The paper is organized as follows. In section 2, we will prove the existence of EE. Section 3 is devoted to the analysis of the asymptotic profiles of the EE, and Theorems 1.1-1.3 are proved there. Section 4 ends the paper with a discussion.

2. The existence of EE. In this section, we establish the uniform persistence of solution to system (1) for any given parameters $d_S$, $d_I$, $\beta$, $\gamma$, $\eta$ and $q$, which in turn yields the existence of EE of (5)-(6).

**Theorem 2.1.** There exists a positive constant $\epsilon_0$ independent of the initial data $(S_0(x), I_0(x))$ with $\int_0^L [S_0(x) + I_0(x)] \, dx = N$, such that for any solution $(S, I)$ of (1), it holds that

$$\liminf_{t \to \infty} S(x, t) \geq \epsilon_0, \quad \liminf_{t \to \infty} I(x, t) \geq \epsilon_0$$

uniformly for $x \in [0, L]$, which means that the disease persists uniformly. Furthermore, (5)-(6) admits at least one EE.

**Proof.** For any positive solution $(S, I)$ of (1), we make the transformations $u(x, t) = e^{\frac{S}{S+I}} S(x, t)$, $v(x, t) = e^{\frac{S}{S+I}} I(x, t)$. Then system (1) becomes equivalent to the following one:

$$\begin{cases}
  u_t = d_S u_{xx} - q u_x - \beta \frac{e^{\frac{S}{S+I}} u v}{e^{\frac{S}{S+I}} u + e^{\frac{S}{S+I}} v} - \eta u + \gamma e^{\frac{S}{S+I}} v, & 0 < x < L, \quad t > 0, \\
  v_t = d_I v_{xx} - q v_x + \beta \frac{e^{\frac{S}{S+I}} u v}{e^{\frac{S}{S+I}} u + e^{\frac{S}{S+I}} v} + \eta e^{\frac{S}{S+I}} u - \gamma v, & 0 < x < L, \quad t > 0, \\
  u_x = v_x = 0, & x = 0, L, \quad t > 0, \\
  u(x, 0) = e^{\frac{S}{S+I}} S(x, 0), \quad v(x, 0) = e^{\frac{S}{S+I}} I(x, 0), & 0 < x < L,
\end{cases} \tag{9}$$

and $(u, v)$ satisfies

$$\int_0^L \left[ e^{\frac{S}{S+I}} u(x, t) + e^{\frac{S}{S+I}} v(x, t) \right] \, dx = N, \quad t \geq 0. \tag{10}$$

By (10), for any solution $(u(x, t), v(x, t))$, it is clear that $\|u(\cdot, t)\|_{L^1((0, L))}$ + $\|v(\cdot, t)\|_{L^1((0, L))}$ is bounded, independent of $t \in [0, \infty)$. Now, in view of [11, Lemma 2.1] with $p_0 = 1$, $q = 2$ and $q = 0$, we can deduce the bound of $\|u(\cdot, t)\|_{L^\infty((0, L))}$ and $\|v(\cdot, t)\|_{L^\infty((0, L))}$, which is independent of $t \in [0, \infty)$. We want to point out that [11, Lemma 2.1] holds in the presence of the advection terms; this is a straightforward consequence of [12, Theorem 1 and Corollary 1]. That is, there is a positive constant $C$ which is independent of the initial data $(u(x, 0), u(x, 0))$ with

$$\int_0^L \left[ e^{\frac{S}{S+I}} u(x, 0) + e^{\frac{S}{S+I}} v(x, 0) \right] \, dx = N,$$
such that

$$\|u(\cdot,t)\|_{L^\infty((0,L))}, \|v(\cdot,t)\|_{L^\infty((0,L))} \leq C, \quad \forall t \geq 0.$$  

Based on such a priori estimates, one can easily follow the abstract theory developed in [31, 45] to assert the uniform persistence property and the existence of EE for system (9); see, for example, [23, Theorem 3.1, Theorem 5.1]. The details are omitted here. Equivalently, we have obtained the desired conclusion of the theorem due to the transformation

$$u(x,t) = e^{\frac{x}{\beta S(x,t)}}, \quad v(x,t) = e^{\frac{x}{\gamma I(x,t)}}.$$  

3. Asymptotic profiles of EE: proof of Theorems 1.1-1.3. In this section, we first present a priori estimates for any EE of (5)-(6) and then prove the three main results: Theorems 1.1-1.3.

3.1. A priori estimates. This subsection is devoted to a priori estimates for any EE of (5)-(6). We start with the estimates of the lower and upper bounds of the integral of the $S$-component and $I$-component for any EE $(S,I)$.

**Lemma 3.1.** Let $(S,I)$ be any EE of (5)-(6). Then we have

$$0 < \int_0^L S(x) \, dx \leq \frac{N \max_{x \in [0,L]} \gamma(x)}{\max_{x \in [0,L]} \beta(x) + \eta(x) + \min_{x \in [0,L]} \gamma(x)} \quad (11)$$

and

$$0 < \int_0^L I(x) \, dx \leq \frac{N \max_{x \in [0,L]} \eta(x)}{\max_{x \in [0,L]} \beta(x) + \eta(x) + \min_{x \in [0,L]} \gamma(x)}.$$

**Proof.** According to the boundary condition (5c), we integrate (5a) or (5b) over $(0,L)$ and find that

$$\int_0^L \left[ \beta(x) \frac{S(x)I(x)}{S(x) + I(x)} + \eta(x)S(x) - \gamma(x)I(x) \right] \, dx = 0.$$  

Thus, we have

$$\int_0^L \eta(x)S \, dx \leq \int_0^L \gamma(x)I \, dx \quad (12)$$

and

$$\int_0^L \gamma(x)I \, dx = \int_0^L \left[ \beta(x) \frac{SI}{S + I} + \eta(x)S \right] \, dx \leq \int_0^L \left[ \beta(x) + \eta(x) \right] S \, dx. \quad (13)$$

Substituting

$$\int_0^L I \, dx = N - \int_0^L S \, dx$$

into (12) and (13), we obtain

$$\int_0^L S \, dx \leq N - \int_0^L \eta(x)S \, dx \quad (14)$$

and

$$\int_0^L I \, dx \leq \int_0^L \gamma(x) \, dx \quad (15).$$  

The details of the proof are omitted for brevity.
Similarly, we get
\[
\min_{x \in [0, L]} \eta(x) \int_0^L S \, dx \leq \max_{x \in [0, L]} \gamma(x) \left( N - \int_0^L S \, dx \right),
\]
which in turn gives
\[
\int_0^L S(x) \, dx \leq \frac{N \max_{x \in [0, L]} \gamma(x)}{\max_{x \in [0, L]} \gamma(x) + \min_{x \in [0, L]} \eta(x)}.
\]
Similarly, we get
\[
\min_{x \in [0, L]} \eta(x) \left( N - \int_0^L I \, dx \right) \leq \max_{x \in [0, L]} \gamma(x) \int_0^L I \, dx,
\]
from which it follows that
\[
\int_0^L I(x) \, dx \geq \frac{N \min_{x \in [0, L]} \eta(x)}{\max_{x \in [0, L]} \gamma(x) + \min_{x \in [0, L]} \eta(x)}.
\]
Using (13), we can apply a similar analysis as above to assert that
\[
\int_0^L S(x) \, dx \geq \frac{N \min_{x \in [0, L]} \eta(x)}{\max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) + \min_{x \in [0, L]} \gamma(x)},
\]
\[
\int_0^L I(x) \, dx \leq \frac{N \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right)}{\max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) + \min_{x \in [0, L]} \gamma(x)}.
\]
The proof is complete. \[\square\]

Now we will establish the pointwise estimate for the $S$-component of any EE ($S, I$) of (5)-(6).

**Lemma 3.2.** Let ($S, I$) be any EE of (5)-(6). Then the following assertions hold.

(i) For any $q$, $d_S$, $d_I > 0$, it holds that
\[
S(0)e^{-\frac{q}{d_S}s} \left[ 1 + \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) \frac{d_S}{q} \right] x \leq S(x), \quad \forall x \in [0, L].
\]

(ii) For any $q$, $d_S$, $d_I > 0$, it holds that
\[
S(x) \leq S(0)e^{-\frac{q}{d_S}s} + \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) \frac{N}{q} \left( 1 - e^{-\frac{q}{d_S}s} \right), \quad \forall x \in [0, L].
\]

**Proof.** We first verify (i). We apply a similar technique introduced by Cui and Lou [8, Lemma 2.5]. Let
\[
S(x) = e^{-\frac{q}{d_S}s}A_x w(x),
\]
where $A$ is a positive constant which will be determined later. Hence,
\[
S_x = e^{-\frac{q}{d_S}s}A_x \left( w_x - \frac{q}{d_S} A w \right),
\]
\[
S_{xx} = e^{-\frac{q}{d_S}s}A_x \left( w_{xx} - \frac{2q}{d_S} A w_x + \frac{q^2}{d_S^2} A^2 w \right).
\]

Thus (5a) reduces to the following equation
\[ d_S w_{xx} + q(1 - 2A)w_x + \left[ \frac{q^2}{d_S} A(A - 1) - \beta(x) \frac{I}{S + I} - \eta(x) + \gamma(x) \frac{I}{S} \right] w = 0, \quad (16) \]
for \( x \in (0, L) \). By (14) and (15), the boundary condition (5c) becomes
\[ d_S w_x = q(A - 1)w, \quad x = 0, L. \quad (17) \]

In (14), we choose
\[ A = 1 + \frac{Cd_S}{q^2}, \]
where \( C \) is a positive constant to be determined below. Then by (17), we see that
\[ d_S w_x = \frac{Cd_S}{q} w > 0, \quad x = 0, L. \]

Let \( w(x_*) = \min_{x \in [0, L]} w(x) \). We prove \( x_* = 0 \) when \( C \) is suitably chosen. Clearly, the boundary condition (17) implies \( x_* \neq L \). To produce a contradiction, we suppose that \( 0 < x_* < L \). Since \( w_{xx}(x_*) \geq 0 \) and \( w_x(x_*) = 0 \), one deduces from (16) that
\[ C \left( 1 + \frac{Cd_S}{q^2} \right) - \beta(x_*) \frac{I(x_*)}{S(x_*) + I(x_*)} - \eta(x_*) + \gamma(x_*) \frac{I(x_*)}{S(x_*)} \leq 0. \]

This infers that
\[ C \left( 1 + \frac{Cd_S}{q^2} \right) < \beta(x_*) + \eta(x_*). \]

Therefore, a contradiction occurs if we take \( C = \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \). This contradiction enables us to conclude that \( x_* = 0 \), that is,
\[ \min_{x \in [0, L]} w(x) = w(0) \leq w(x), \quad \forall x \in [0, L]. \]

As a result, by (14), we derive the desired estimate:
\[ S(0) = w(0) \leq w(x) = e^{ \mu \gamma \left[ 1 + \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) \frac{d_S}{q^2} \right] x} S(x), \quad \forall x \in [0, L]. \]

We now verify (ii). Integrating (5a) over \((0, x)\), we obtain
\[ d_S S_x(x) + qS(x) = \int_0^x \left[ \beta(y) \frac{S(y)I(y)}{S(y) + I(y)} + \eta(y)S(y) - \gamma(y)I(y) \right] dy \]
\[ \leq \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) \int_0^L S(x) \, dx. \]

By (6), we should also notice that
\[ d_S S_x(x) + qS(x) \leq \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) N. \]

That is, we have
\[ S_x(x) + \frac{q}{d_S} S \leq \frac{N}{d_S} \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) =: M, \quad \forall x \in [0, L]. \]

Then the function
\[ S_*(x) := S(x) - \frac{d_S}{q} M \]
satisfies
\[ \frac{d}{dx} \left( e^{\frac{d_S}{q} x} S_* \right) \leq 0, \quad \forall x \in [0, L]. \]
Integrating the above inequality over \((0, x)\), we obtain
\[
e^{\frac{dS}{q} x} S(x) - S(x) \leq 0,
\]
which means,
\[
S(x) - \frac{dS}{q} x M \leq \left[ S(0) - \frac{dS}{q} x M \right] e^{-\frac{x}{S} r}, \quad \forall x \in [0, L].
\]

In light of the expression of \(M\), the desired estimate is proved. \(\Box\)

Using a similar analysis to that of Lemma 3.2, we can establish the pointwise estimate for the I-component of any EE \((S, I)\) of (5)-(6). More precisely, we have

**Lemma 3.3.** Let \((S, I)\) be any EE of (5)-(6). Then the following assertions hold.

(i) For any \(q, d_s, d_I > 0\), it holds
\[
I(0) e^{-\frac{dS}{q} \left(1 + \max_{x \in [0,L]} \frac{\gamma(x) d_I}{q} \right) x} \leq I(x), \quad \forall x \in [0, L].
\]

(ii) For any \(q, d_s, d_I > 0\), it holds
\[
I(x) \leq I(0) e^{-\frac{dS}{q} x} + \frac{\max_{x \in [0,L]} \gamma(x) N}{q} \left(1 - e^{-\frac{dI}{q} x}\right), \quad \forall x \in [0, L].
\]

With the help of Lemmas 3.2 and 3.3, we can obtain the upper estimates for any EE \((S, I)\) of (5)-(6). Precisely, we can conclude that

**Lemma 3.4.** Let \((S, I)\) be any EE of (5)-(6). Then the following assertions hold.

(i) For any \(q, d_s, d_I > 0\), we have
\[
S(x) \leq \frac{qN \left(1 + \left(\max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \eta(x)\right) \frac{dS}{q} \right)}{d_s \left[1 - e^{-\frac{dS}{q} \left(1 + \max_{x \in [0,L]} \frac{\beta(x) + \max_{x \in [0,L]} \eta(x) d_I}{q} \right) x}\right]} e^{-\frac{dS}{q} x}
\]
\[
\quad + \frac{\max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \eta(x) N}{q} \left(1 - e^{-\frac{dI}{q} x}\right), \quad \forall x \in [0, L].
\]

(ii) For any \(q, d_s, d_I > 0\), we have
\[
I(x) \leq \frac{qN \left(1 + \max_{x \in [0,L]} \gamma(x) \frac{dI}{q} \right)}{d_I \left[1 - e^{-\frac{dS}{q} \left(1 + \max_{x \in [0,L]} \frac{\gamma(x) d_I}{q} \right) x}\right]} e^{-\frac{dI}{q} x}
\]
\[
\quad + \max_{x \in [0,L]} \frac{\gamma(x) N}{q} \left(1 - e^{-\frac{dI}{q} x}\right), \quad \forall x \in [0, L].
\]

**Proof.** According to (i) of Lemma 3.2, we obtain
\[
S(0) e^{-\frac{dS}{q} \left(1 + \left(\max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \eta(x)\right) \frac{dS}{q} \right) x} \leq S(x), \quad \forall x \in [0, L].
\]

Integrating this inequality over \((0, L)\), we then get
\[
S(0) \int_{0}^{L} e^{-\frac{dS}{q} \left(1 + \left(\max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \eta(x)\right) \frac{dS}{q} \right) x} dx \leq \int_{0}^{L} S(x) dx.
\]
Applying the same procedure to the inequality (i) of Lemma 3.3 leads to
\[ I(0) \int_0^L e^{-\frac{\gamma}{q} \left[ 1 + \max_{x \in [0, L]} \beta(x) \frac{d}{dx} \right]} dx \leq \int_0^L I(x) dx. \]

Due to the conservation law (6), we can easily find
\[ S(0) I(0) \int_0^L e^{-\frac{\gamma}{q} \left[ 1 + \max_{x \in [0, L]} \beta(x) \frac{d}{dx} \right]} dx \leq N \]
and
\[ I(0) \int_0^L e^{-\frac{\gamma}{q} \left[ 1 + \max_{x \in [0, L]} \beta(x) \frac{d}{dx} \right]} dx \leq N. \]

Hence, for any \( d_I, d_S, q > 0 \), we obtain
\[ S(0) \leq \frac{qN \left[ 1 + \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \frac{d}{dx} \right) \right]}{d_S \left[ 1 - e^{-\frac{\gamma}{q} \left[ 1 + \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \frac{d}{dx} \right]} L \right]} \]
and
\[ I(0) \leq \frac{qN \left[ 1 + \max_{x \in [0, L]} \gamma(x) \frac{d}{dx} \right]}{d_I \left[ 1 - e^{-\frac{\gamma}{q} \left[ 1 + \max_{x \in [0, L]} \gamma(x) \frac{d}{dx} \right]} L \right]}. \]

Now, substituting (18) and (19) into the inequalities in Lemma 3.2(ii) and Lemma 3.3(ii) respectively, we deduce the desired results.

3.2. Profile of EE as \( q \to \infty \): proof of Theorem 1.1. In order to explore the profile of any EE \( (S(x, q), I(x, q)) \) of (5)-(6) as \( q \to \infty \), we make use of the following transformations:
\[ u(y, q) := \frac{1}{q} S \left( \frac{y}{q}, q \right), \quad v(y, q) := \frac{1}{q} I \left( \frac{y}{q}, q \right), \quad 0 \leq y \leq qL. \]

By (5)-(6), we then obtain the following nonlinear ordinary differential equations which \((u, v)\) satisfies over \((0, qL)\):
\[ \begin{align*}
  &d_S u_{yy} + u_y - v(y, q) \left[ \beta \frac{y}{q} \frac{u(y, q)}{u(y, q) + v(y, q)} \right] - \gamma \frac{y}{q} \frac{u(y, q)}{q^2} = 0, \quad \text{(20a)} \\
  &d_I v_{yy} + v_y + \frac{v(y, q)}{q^2} \left[ \beta \frac{y}{q} \frac{u(y, q)}{u(y, q) + v(y, q)} \right] - \gamma \frac{y}{q} \frac{u(y, q)}{q^2} = 0, \quad \text{(20b)}
\end{align*} \]

with the boundary conditions
\[ \begin{align*}
  &d_s u_y + u = 0 = d_I v_y + v, \quad y = 0, qL. \quad \text{(20c)}
\end{align*} \]

Meanwhile, the conservation law (6) becomes
\[ \int_0^{qL} (u(y, q) + v(y, q)) \, dy = N, \quad \forall q > 0. \]

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all, by (i) and (ii) of Lemma 3.4, it is easily seen that \( S(x, q) \) and \( I(x, q) \) converge to zero locally uniformly for \( x \in (0, L) \) as \( q \to \infty \), respectively. It remains to explore the limiting behaviors of \( S(x, q) \) and \( I(x, q) \) at the downstream \( x = 0 \). Our proof consists of four steps as follows.
Step 1: Convergence of \( \{(u(y, q), v(y, q))\} \). It follows from (i), (ii) of Lemma 3.4 that for any fixed \( K > 0 \), when \( q \) satisfies \( q > K/L \),

\[
0 \leq u(y, q) < \frac{2N}{d_S} e^{-\frac{\pi}{d_S}} + \frac{N}{q} \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) (1 - e^{-\frac{\pi}{d_S}}), \quad \forall y \in [0, qL],
\]

\[
0 \leq v(y, q) < \frac{2N}{d_I} e^{-\frac{\pi}{d_I}} + \frac{N}{q^2} \max_{x \in [0, L]} \gamma(x) (1 - e^{-\frac{\pi}{d_I}}), \quad \forall y \in [0, qL].
\]

Therefore, for any fixed \( K > 0 \), both \( \{u(y, q)\} \) and \( \{v(y, q)\} \) are uniformly bounded in \([0, K]\) with respect to \( q \) satisfying \( q > K/L \). Thus, for a given constant \( K > 0 \), the standard \( L^p\)-estimates (up to the boundary \( y = 0 \)) for elliptic equations (see [14]) ensure that

\[
\|u(y, q)\|_{W^{2, p}((0, K))} \leq M, \quad \|v(y, q)\|_{W^{2, p}((0, K))} \leq M
\]

for some positive constant \( M = M(K, p) \) and any \( p > 1 \). Moreover, the constant \( M \) does not depend on all \( q > K/L \) and is allowed to vary when necessary. Due to the embedding theorem (by choosing \( p \) to be properly large), we have

\[
\|u(y, q)\|_{C^{1,\alpha}([0, K])} \leq M, \quad \|v(y, q)\|_{C^{1,\alpha}([0, K])} \leq M
\]

for some \( \alpha \in (0, 1) \).

Because of the above estimates, we can apply the Ascoli-Arzelà theorem, combined with a diagonal argument, to find a negative sequence \( \{q_n\} \) with \( \lim_{n \to \infty} q_n = \infty \) and a pair of nonnegative functions \((u^*, v^*)\) such that \( u_n(y) := u(y, q_n) \) and \( v_n(y) := v(y, q_n) \) satisfy

\[
\lim_{n \to \infty} (u_n, v_n) = (u^*, v^*) \quad \text{in} \quad C_{\text{loc}}^1([0, \infty)) \times C_{\text{loc}}^1([0, \infty)). \tag{22}
\]

As \((u_n, v_n)\) solves (20)-(21), we let \( n \to \infty \) to find that \((u^*, v^*)\) satisfies the limiting system (in the weak sense and then in the classical sense)

\[
d_S u_{yy}^* + u_y^* = 0, \quad y \in (0, \infty), \tag{23a}
\]

\[
d_I v_{yy}^* + v_y^* = 0, \quad y \in (0, \infty) \tag{23b}
\]

with the boundary conditions

\[
d_S u_y^*(0) + u^*(0) = 0 = d_I v_y^*(0) + v^*(0). \tag{23c}
\]

By solving the whole limiting system (23), one has

\[
u^*(y) = C_S e^{-\frac{\pi}{d_S}}, \quad v^*(y) = C_I e^{-\frac{\pi}{d_I}}
\]

for some constants \( C_S, C_I \geq 0 \). That is, it follows from (22) that

\[
\lim_{n \to \infty} (u_n, v_n) = (C_S e^{-\frac{\pi}{d_S}}, C_I e^{-\frac{\pi}{d_I}}) \quad \text{in} \quad C_{\text{loc}}^1([0, \infty)) \times C_{\text{loc}}^1([0, \infty)). \tag{24}
\]

Step 2: \( C_S, C_I > 0 \). Employing the change of variable \( x = \frac{y}{q_n} \) in the second inequality of (11), we obtain

\[
\int_0^{q_nL} u_n(y) \, dy \geq \frac{N \min_{x \in [0, L]} \gamma(x)}{\max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) + \min_{x \in [0, L]} \gamma(x)} > 0. \tag{25}
\]

In view of

\[
\lim_{n \to \infty} u_n(y) = C_S e^{-\frac{y}{q_n}} \quad \text{in} \quad C_{\text{loc}}^1([0, \infty))
\]
proved in Step 1, we let \( n \to \infty \) in the left-hand side of (25) to conclude that
\[
\int_{0}^{q_{n}L} u_{n}(y) \, dy \to \int_{0}^{\infty} C_{S} e^{-\frac{q}{\bar{y}}} \, dy,
\]
which then infers that
\[
d_{S}C_{S} = C_{S} \int_{0}^{\infty} e^{-\frac{q}{\bar{y}}} \, dy \geq \frac{N \min_{x \in [0,L]} \gamma(x)}{\max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \eta(x) + \min_{x \in [0,L]} \gamma(x)}.
\]
This shows
\[
C_{S} \geq \frac{N \min_{x \in [0,L]} \gamma(x)}{d_{S} \left( \max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \eta(x) + \min_{x \in [0,L]} \gamma(x) \right)} > 0.
\]
Similarly, we can prove \( C_{I} > 0 \) using the equations satisfied by \( v_{n} \).

**Step 3:** \((C_{S}, C_{I})\) satisfies (7). By (21), we have
\[
\int_{0}^{q_{n}L} (u_{n}(y) + v_{n}(y)) \, dy = N.
\]
Then, as \( n \to \infty \), using (24) we see that
\[
\int_{0}^{\infty} \left( C_{S} e^{-\frac{q}{\bar{y}}} + C_{I} e^{-\frac{q_{n}}{\bar{y}}} \right) \, dy = N, \quad \text{that is,} \quad C_{S} d_{S} + C_{I} d_{I} = N.
\]
We then integrate (20a) or (20b) over \((0, q_{n}L)\) to assert
\[
\int_{0}^{q_{n}L} \left\{ v_{n}(y) \left[ \beta \left( \frac{y}{q_{n}} \right) u_{n}(y) + v_{n}(y) \right] - \gamma \left( \frac{y}{q_{n}} \right) \right\} \, dy = 0, \quad \forall n \geq 1.
\]
Letting \( n \to \infty \) in the above equality allows us to know that
\[
\int_{0}^{\infty} \left\{ C_{I} e^{-\frac{q_{n}}{\bar{y}}} \left[ \beta(0) e^{-\frac{q_{n}}{\bar{y}}} + C_{S} e^{-\frac{q}{\bar{y}}} \right] - \gamma(0) \right\} \, dy = 0.
\]
Thanks to \( C_{S}, C_{I} > 0 \), this is equal to
\[
\int_{0}^{\infty} \frac{\eta(0) C_{S} e^{-\frac{q}{\bar{y}}} + \left[ \beta(0) + \eta(0) \right] C_{I} e^{-\frac{q_{n}}{\bar{y}}} - \gamma(0) C_{I}}{C_{S} e^{-\frac{q}{\bar{y}}} + C_{I} e^{-\frac{q_{n}}{\bar{y}}}} \, dy = \frac{\gamma(0) C_{I}}{C_{S} d_{I}}.
\]
Thus, \((C_{S}, C_{I})\) is a positive solution of (7).

The proof of Theorem 1.1 is complete. \(\square\)

3.3. **Profile of EE as** \(d_{S} \to 0\): **proof of Theorem 1.2.** In what follows, we will present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** For any EE \((S(x, d_{S}), I(x, d_{S}))\) of (5)-(6), we make the transformations:
\[
u(y, d_{S}) := d_{S} S(d_{S} y, d_{S}), \quad v(y, d_{S}) := I(d_{S} y, d_{S}), \quad 0 \leq y \leq d_{S}^{-1} L.
\]
Then, for \( y \in (0, d_{S}^{-1} L) \), (5)-(6) will become the following
\[
u_{yy} + q u_{y} - d_{S}^{2} v \left( \frac{u}{u + d_{S} v} - \gamma \right) - d_{S} \eta u = 0, \quad (27a)
\]
\[
I v_{yy} + q d_{S} v_{y} + d_{S}^{2} v \left( \frac{u}{u + d_{S} v} - \gamma \right) + d_{S} \eta u = 0 \quad (27b)
\]
with the boundary conditions
\[ u_y + qu = 0 = d_I v_y + q d_S v, \quad y = 0, \quad d_S^{-1} L, \] (27c)
and the conservation law
\[ \int_0^{d_S^{-1} L} (u + d_S v) \, dy = N. \] (28)

For our purpose, we also make the following transformations:
\[ S^*(x, d_S) = e^{\frac{\theta}{d_S}} S(x, d_S), \quad I^*(x, d_S) = e^{\frac{\theta}{d_S}} I(x, d_S). \]

Thus the original system (5) is reduced to the equivalent one:
\[
\begin{cases}
  d_S S_{xx}^* - q S_x^* - \beta(x) \frac{e^{-\frac{\theta}{d_S}} S^* I^*}{e^{-\frac{\theta}{d_S}} S^* + e^{-\frac{\theta}{d_S}} I^*} \\
  - \eta(x) S^* + \gamma(x) e^{\left(\frac{\theta}{d_S} - \frac{\theta}{d_I}\right) x} I^* = 0, \quad 0 < x < L, \\
  d_I I_{xx}^* - q I_x^* + \beta(x) \frac{e^{-\frac{\theta}{d_S}} S^* I^*}{e^{-\frac{\theta}{d_S}} S^* + e^{-\frac{\theta}{d_S}} I^*} \\
  + \eta(x) e^{\left(\frac{\theta}{d_I} - \frac{\theta}{d_S}\right) x} S^* - \gamma(x) I^* = 0, \quad 0 < x < L, \\
  S^*_x(0) = S^*_x(L) = 0, \quad I^*_x(0) = I^*_x(L) = 0, \\
  \int_0^L \left[ e^{-\frac{\theta}{d_S}} S^* + e^{-\frac{\theta}{d_S}} I^* \right] \, dx = N.
\end{cases}
\] (29)

The remaining argument consists of five steps.

**Step 1: Estimates of \( u \) and \( v \) for small \( d_S \).** Firstly, from (i) of Lemma 3.4, we can immediately notice that \( d_S S(\cdot, d_S) \) is uniformly bounded on \([0, L]\) with respect to \( d_S \) for \( 0 < d_S \leq 1 \). Thus, according to the transformations (26), for any fixed constant \( K > 0 \), one can find a positive constant \( C \) such that
\[ u(y, d_S) \leq C, \quad \forall y \in [0, K]. \] (30)

Hereafter, the positive constant \( C \) may be different from line to line but is independent of all \( 0 < d_S < L/K \).

Fix \( K > 0 \), we need to estimate \( \|v(\cdot, d_S)\|_{C([0, K])} \). Using system (29), we know that
\[
\begin{cases}
  -d_I I_{xx}^* + q I_x^* + \gamma(x) I^* = \beta(x) \frac{e^{-\frac{\theta}{d_S}} S^* I^*}{e^{-\frac{\theta}{d_S}} S^* + e^{-\frac{\theta}{d_S}} I^*} \\
  + \eta(x) e^{\left(\frac{\theta}{d_I} - \frac{\theta}{d_S}\right) x} S^* \geq 0, \quad 0 < x < L, \\
  I^*_x(0) = I^*_x(L) = 0.
\end{cases}
\]

Appealing to [14, Theorem 8.18], we can see that
\[ \|I^*(\cdot, d_S)\|_{L^1([0, L])} \leq C \min_{x \in [0, L]} I^*(x, d_S), \]
which in turn gives
\[ \|I(\cdot, d_S)\|_{L^1([0, L])} \leq C \min_{x \in [0, L]} I(x, d_S) \]
due to \( I^*(x, d_S) = e^{\frac{\theta}{d_S}} I(x, d_S) \). Thus, it holds that
\[ \min_{x \in [0, L]} I(x, d_S) \geq C. \] (31)
On the other hand, from (ii) of Lemma 3.4, we notice that
\[ \max_{x \in [0,L]} I(x,d_S) \leq C, \quad \forall 0 < d_S \leq 1, \]
and so there holds
\[ \max_{y \in [0,d_S^2L]} v(y,d_S) \leq C, \quad \forall 0 < d_S \leq 1. \]

Hence, in view of (30) and (32), one can apply the standard regularity of elliptic equations to (27a), (27b) and (27c) to conclude that, for any fixed constant \( K > 0 \),
\[ \|u(\cdot, d_S)\|_{C^{1,\alpha}([0,K])} \leq C, \quad \|v(\cdot, d_S)\|_{C^{1,\alpha}([0,K])} \leq C, \]
for some constant \( \alpha \in (0,1) \).

**Step 2: Convergence of \( \{ (u(y,d_S), v(y,d_S)) \} \).** Thanks to (33), there exists a sequence of \( d_S \), denoted by itself, and the corresponding positive solution \( (u(y,d_S), v(y,d_S)) \) such that
\[ \lim_{d_S \to 0} (u(\cdot, d_S), v(\cdot, d_S)) = (\hat{u}(\cdot), \hat{v}(\cdot)) \text{ in } C^1_{\text{loc}}([0,\infty)) \times C^1_{\text{loc}}([0,\infty)). \]
Since \( (u(x,d_S), v(x,d_S)) \) solves (27)-(28), it is easily checked that \( (\hat{u}, \hat{v}) \) satisfies
\[
\begin{align*}
\hat{u}_{yy} + q\hat{u} &= 0, \quad y \in (0, \infty), \quad (35a) \\
\hat{v}_{yy} &= 0, \quad y \in (0, \infty) \quad (35b)
\end{align*}
\]
with the boundary conditions
\[ \hat{u}_y(0) + q\hat{u}(0) = 0 = \hat{v}_y(0). \] (35c)

By solving (35a), (35b) and (35c), we find
\[ \hat{u}(y) = \hat{C}_S e^{-qy}, \quad \hat{v}(y) = \hat{C}_I, \quad y \geq 0 \]
with some nonnegative constants \( \hat{C}_S, \hat{C}_I \). Arguing as in Step 2 in the proof of Theorem 1.1, one can show \( \hat{C}_S > 0, \hat{C}_I > 0 \). In particular, it follows that \( I(0,d_S) \to \hat{C}_I > 0 \) as \( d_S \to 0 \) from (26) and (34).

Now we set \( S(\hat{x}_{d_S}, d_S) = \min_{x \in [0,L]} S(x,d_S) \). And then \( \hat{x}_{d_S} \in [0,L] \). Without loss of generality, we can assume that \( \hat{x}_{d_S} \to \hat{x} \in [0,L] \) as \( d_S \to 0 \).

In light of \( I^* = e^{\frac{q}{4\pi}I} \), the equation satisfied by \( I^*(x,d_S) \) reads as
\[
\begin{align*}
-\frac{d}{dx}I^*_{xx} + qI^*_x &= \frac{SI^*}{S + e^{-\frac{q}{4\pi}I^*}} + \eta(x)e^{\frac{q}{4\pi}I^*}S - \gamma(x)I^*, \quad 0 < x < L, \\
I_x^*(0) &= I_x^*(L) = 0.
\end{align*}
\]
Clearly, it holds that
\[
\begin{align*}
\int_0^L \left| \frac{SI^*}{S + e^{-\frac{q}{4\pi}I^*}} + \eta(x)e^{\frac{q}{4\pi}I^*}S - \gamma(x)I^* \right| dx \\
&\leq \left( \max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \gamma(x) \right) \int_0^L I^* \, dx + \max_{x \in [0,L]} \eta(x) e^{\frac{q}{4\pi}L} \int_0^L S \, dx \\
&\leq \left( \max_{x \in [0,L]} \beta(x) + \max_{x \in [0,L]} \gamma(x) + \max_{x \in [0,L]} \eta(x) \right) e^{\frac{q}{4\pi}N}.
\end{align*}
\]
Thus, using the $L^1$-estimate theory for elliptic equations (see, for instance, [3]), for any given $p \in [1, \infty)$, we can assert that $\|I^*(\cdot, d_S)\|_{W^{1,p}(0,L)} \leq C$, and so
\[ \|I(\cdot, d_S)\|_{W^{1,p}(0,L)} \leq C. \]

Owing to the fact that for any fixed $p > 1$, $W^{1,p}(0,L)$ is compactly embedded into $C([0,L])$ (see [14]), passing to a subsequence of $d_S$, denoted still by itself for convenience, we can obtain
\[ I(\cdot, d_S) \to \hat{I}(\cdot) \text{ in } C([0,L]), \quad \text{as } d_S \to 0, \] (36)
where $\hat{I} \in C([0,L])$ is positive on $[0,L]$ due to (31).

Since $S$ is locally uniformly bounded in $(0,L)$ due to Lemma 3.4, for any fixed small $\varepsilon > 0$, after passing to a subsequence of $d_S \to 0$, we may assume that
\[ S(\cdot, d_S) \to \hat{S}(\cdot) \text{ weakly in } L^2((\varepsilon,L)), \]
where $\hat{S} \in L^2((\varepsilon,L))$ and $\hat{S} \geq 0$ a.e. in $(\varepsilon,L)$. As $\int_0^L S(x, d_S) \, dx < N$ for all $d_S > 0$, by a diagonal argument, up to a further subsequence of $d_S \to 0$, we obtain
\[ \int_0^L \hat{S}(x) \, dx = \lim_{\varepsilon \to 0} \int_0^L S(x) \, dx = \lim_{\varepsilon \to 0} \lim_{d_S \to 0} \int_0^L S(x, d_S) \, dx \leq N. \] (37)
Thus, $\hat{S} \in L^1((0,L))$ and $\hat{S} \geq 0$ a.e. in $(0,L)$.

**Step 3:** $\lim_{d_S \to 0} S(\hat{x}_{d_S}, \hat{d}_S) = 0$ and $\hat{x}_{d_S} = \hat{x} = L$ for all small $d_S$. Arguing indirectly, we suppose that $S(x, d_S) \geq \delta > 0$, $\forall x \in [0,L]$, for all small $d_S > 0$, where the positive constant $\delta$ is independent of $d_S > 0$.

Since we now have the positive lower bound $\delta$ of $S(\cdot, d_S)$, we will use the similar technique as in the proof of Lemmas 3.2 and 3.3 to establish the improved upper bound of $S(\cdot, d_S)$ when $d_S > 0$ is small. By taking
\[ S(x, d_S) = e^{-\int_{L}^{x} \frac{2C_{d_S}}{q^2} \eta(x) \, dx} w(x, d_S) \] (38)
where the positive constant $C$ to be chosen later, we notice that $w$ satisfies
\[ d_sw_{xx} - q \left( 1 - \frac{2C_{d_S}}{q^2} \right) w_x + \left[ \frac{C_{d_S}}{q^2} - 1 \right] - \beta(x) \frac{I}{S + I} - \eta(x) + \gamma(x) \frac{I}{S} \right] w = 0, \] (39)
for $x \in (0,L)$ with the boundary conditions
\[ w_x = -\frac{C}{q} w < 0, \quad x = 0, L. \]

Take $w(x^*, d_S) = \max_{x \in [0,L]} w(x, d_S)$. It is easily seen that $x^* \neq L$. Suppose $0 < x^* < L$. Because of $w_{xx}(x^*, d_S) \leq 0$ and $w_x(x^*, d_S) = 0$, the equation (39) gives
\[ \frac{C}{q^2} \left( \frac{C_{d_S}}{q^2} - 1 \right) - \beta(x^*) \frac{I(x^*, d_S)}{S(x^*, d_S) + I(x^*, d_S)} - \eta(x^*) + \gamma(x^*) \frac{I(x^*, d_S)}{S(x^*, d_S)} \geq 0. \]
Remember that $\max_{x \in [0,L]} I(x, d_S) \leq C$ still holds and $S(x, d_S) \geq \delta > 0$, $\forall x \in [0,L]$, for all small $d_S > 0$. Hence, we have
\[ \frac{C}{q^2} \left( \frac{C_{d_S}}{q^2} - 1 \right) + \max_{x \in [0,L]} \gamma(x) \frac{C}{\delta} \geq 0. \] (40)
By choosing
\[ \frac{C}{q^2} = 2 \max_{x \in [0,L]} \gamma(x) \frac{C}{\delta} \] (41)
and requiring
\[ 0 < \delta < \frac{q^2}{2C}, \tag{42} \]
it can lead to a contradiction against (40). This implies that \( x^* = 0 \) for any \( \delta > 0 \) satisfying (42) with \( C \) being defined in (41). In conclusion, for all small \( \delta \), we have \( u(0, \delta) = \max_{x \in [0, L]} u(x, \delta) \). This and (38) lead to
\[
S(0, \delta) = w(0, \delta) \geq w(x, \delta) = e^{\frac{\delta}{qN}(1-\frac{q}{qN})x}S(x, \delta), \quad \forall x \in [0, L].
\]
Finally, making use of (18), we derive
\[
S(x, \delta) \leq \frac{qN\left[1 + \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) \frac{\delta}{\pi} \right]}{\delta \left[1 - e^{-\frac{x}{\delta} \left(1 + \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) \frac{\delta}{\pi} \right)} \right]} e^{-\frac{x}{\delta} \left(1 + \frac{\delta}{\pi} \right)n}, \tag{43}
\]
for all small \( \delta \) and \( x \in [0, L] \). Therefore, (43) implies that \( S(x, \delta) \) converges to zero locally uniformly in \((0, L)\) as \( \delta \to 0 \), which contradicts our previous hypothesis that \( S(x, \delta) \geq \delta > 0, \forall x \in [0, L], \) for all small \( \delta > 0 \). So it is necessary that \( \min_{x \in [0, L]} S(x, \delta) \to 0 \) as \( \delta \to 0 \).

We next prove \( \hat{x} = L \). Since in Step 2 we have already proved that
\[
\lim_{\delta \to 0} (d_\delta S(0, \delta)) = \hat{C} > 0,
\]
it is easy to see \( \hat{x} > 0 \). Suppose \( \hat{x} \in (0, L) \). Then \( \hat{x}_d \in (0, L) \) for all small \( \delta \). Due to the definition of \( \hat{x}_d, S_{xx}(\hat{x}_d, \delta) \geq 0 \) and \( S_x(\hat{x}_d, \delta) = 0 \), we are able to deduce from (5a) that
\[
-\beta(\hat{x}_d) \frac{S(\hat{x}_d, \delta)I(\hat{x}_d, \delta)}{S(\hat{x}_d, \delta) + I(\hat{x}_d, \delta)} - \eta(\hat{x}_d)S(\hat{x}_d, \delta) + \gamma(\hat{x}_d)I(\hat{x}_d, \delta) \leq 0,
\]
from which it follows that
\[
\min_{x \in [0, L]} \gamma(x)I(\hat{x}_d, \delta) \leq \left( \max_{x \in [0, L]} \beta(x) + \max_{x \in [0, L]} \eta(x) \right) S(\hat{x}_d, \delta) \to 0, \tag{44}
\]
as \( \delta \to 0 \). This is impossible because of (31). Thus, \( \hat{x} = L \).

In fact, \( \hat{x}_d \in (0, L) \) for all small \( \delta \). We apply the inequality (44) to derive a contradiction. Hence, it must hold that \( \hat{x}_d = L \). This implies that \( \hat{S}(L) = 0 \).

**Step 4:** (\( \hat{S}, \hat{I} \) is a classical solution of (8). To find the equations satisfied by \( \hat{S}, \hat{I} \), we first fix any small \( \varepsilon > 0 \). Take any \( \varphi \in C^2([\varepsilon, L]) \) such that \( \varphi = 0 \) near \( \varepsilon \) and \( L \). Multiplying the \( S \)-component equation by \( \varphi \) and integrating by parts, we can deduce
\[
d_{\delta} \int_{\varepsilon}^{L} S \varphi_{xx} \, dx - q \int_{\varepsilon}^{L} S \varphi_x \, dx - \int_{\varepsilon}^{L} \left[ \beta(x) \frac{\hat{S} \hat{I}}{\hat{S} + \hat{I}} - \eta(x) \hat{S} + \gamma(x) \hat{I} \right] \varphi \, dx = 0.
\]
Sending \( d_{\delta} \to 0 \) and according to the arbitrariness of \( \varepsilon \), we find that \( \hat{S} \) is a nonnegative, nontrivial solution (in the weak sense and then in the classical sense) of
\[
\begin{aligned}
q \hat{S}_x - \beta(x) \frac{\hat{S} \hat{I}}{\hat{S} + \hat{I}} - \eta(x) \hat{S} + \gamma(x) \hat{I} = 0, & \quad x \in (0, L), \\
\hat{S}(L) = 0.
\end{aligned} \tag{45}
\]
Here, we used the fact that \( \hat{I} \in C([0, L]), \hat{I} > 0 \) on \([0, L]\), and the claim proved in Step 3.

Similarly, one can show that \( \hat{I} \) is the classical solution of

\[
\begin{align*}
    d_t \hat{I}_x + q \hat{I}_x + \beta(x) \frac{\hat{S} \hat{I}}{\hat{S} + \hat{I}} + \eta(x) \hat{S} - \gamma(x) \hat{I} &= 0, \quad x \in (0, L), \\
    d_t \hat{I}_x(L) + q \hat{I}(L) &= 0.
\end{align*}
\]

(46)

Next, we will determine the boundary condition of \( \hat{I} \) at 0. To this end, we denote

\[ \hat{N} = N - \int_0^L (\hat{S} + \hat{I}) \, dx. \]

Obviously, \( 0 \leq \hat{N} < N \) due to \( \hat{I} > 0 \) on \([0, L]\). Notice that

\[
\begin{align*}
    \lim_{\varepsilon \to 0} \lim_{d_s \to 0} \int_0^\varepsilon S(x, d_s) \, dx &= N - \lim_{d_s \to 0} \int_0^L I(x, d_s) \, dx - \int_0^L \hat{I}(x) \, dx, \\
    \lim_{\varepsilon \to 0} \lim_{d_s \to 0} \int_\varepsilon^L S(x, d_s) \, dx &= N - \int_0^L \hat{I}(x) \, dx - \int_0^L \hat{S}(x) \, dx = \hat{N}.
\end{align*}
\]

(47)

Here we used (37). For any fixed small \( \varepsilon \in (0, L) \), we integrate the equation of \( I \)-component over \([0, \varepsilon]\) to deduce

\[
    d_t I_\varepsilon(x, d_s) + q I_\varepsilon(x, d_s) + \int_0^\varepsilon \left[ \beta(x) \frac{S I}{S + I} + \eta(x) S - \gamma(x) I \right] \, dx = 0. \quad (48)
\]

It follows from the mean value theorem that

\[
\begin{align*}
    \int_0^\varepsilon \left[ \beta(x) \frac{S I}{S + I} + \eta(x) S - \gamma(x) I \right] \, dx &= \beta(x_\varepsilon) \frac{I(x_\varepsilon)}{S(x_\varepsilon) + I(x_\varepsilon)} \int_0^\varepsilon S \, dx \\
    &\quad + \eta(x_\varepsilon) \int_0^\varepsilon S \, dx - \gamma(x_\varepsilon) I(x_\varepsilon) \cdot \varepsilon
\end{align*}
\]

(49)

for some \( x_\varepsilon \in (0, \varepsilon) \). Here, we see from (48) and (49) that

\[
    d_t I_\varepsilon(0) + q I_\varepsilon(0) + \beta(0) \frac{\hat{I}(0)}{\hat{S}(0) + \hat{I}(0)} \hat{N} + \eta(0) \hat{N} = 0, \quad (50)
\]

by first sending \( d_S \to 0 \) and then \( \varepsilon \to 0 \).

Now, given \( \varphi \in C([0, L]) \), for any small \( \varepsilon > 0 \), it is clear that

\[
\begin{align*}
    \int_0^L S(x, d_s) \varphi(x) \, dx &= \int_0^\varepsilon S(x, d_s) \varphi(x) \, dx + \int_\varepsilon^L S(x, d_s) \varphi(x) \, dx \\
    &= \varphi(x_\varepsilon) \int_0^\varepsilon S(x, d_s) \, dx + \int_\varepsilon^L S(x, d_s) \varphi(x) \, dx,
\end{align*}
\]

where \( x_\varepsilon \in [0, \varepsilon] \). Then, letting \( d_S \to 0 \) first and then \( \varepsilon \to 0 \), and using (47), we find

\[
\begin{align*}
    \int_0^L S(x, d_s) \varphi(x) \, dx &\to \int_0^L \hat{S}(x) \varphi(x) \, dx + \hat{N} \varphi(0).
\end{align*}
\]

That is, \( S(\cdot, d_s) \to \hat{S}(\cdot) + \hat{N} \delta_0 \) weakly in \( L^1([0, L]) \), where \( \delta_0 \) is the Dirac measure centered at 0.

Furthermore, we use the fact that \( \hat{I} \) is positive on \([0, L]\) and (45) to conclude that \( \hat{S} > 0 \) on \([0, L]\).
Step 5: \( \hat{C}_S < Nq \) and \( \hat{N} > 0 \). By virtue of (28) and (34), one can observe that

\[
N = \lim_{d_S \to 0} \int_0^L (S(x, d_S) + I(x, d_S)) \, dx > \lim_{d_s \to 0} \int_0^L S(x, d_S) \, dx
\]

By (28) and (34), one can observe that

\[
N = \lim_{d_S \to 0} \int_0^{d_S^{-1} L} u(y, d_S) \, dy = \int_0^{\infty} \hat{u}(y) \, dy = \frac{\hat{C}_S}{q}.
\]

Hence, \( \hat{C}_S < Nq \).

Moreover, adding up the first two equations in (45) and (46) and integrating over \([x, L]\), together with the conditions of \( \hat{S}, \hat{I} \) at \( x = L \), we have

\[
d_I \hat{I}_x(x) + q \hat{I}(x) + q \hat{S}(x) = d_I \hat{I}_x(L) + q \hat{I}(L) + q \hat{S}(L) = 0. \quad (51)
\]

By (50) and (51), it holds that

\[
d_I \hat{I}_x(0) + q \hat{I}(0) + q \hat{S}(0) = q \hat{S}(0) - \beta(0) \frac{\hat{I}(0)}{S(0) + \hat{I}(0)} \hat{N} - \eta(0) \hat{N} = 0,
\]

which yields that

\[
q \hat{S}(0) = \left[ \beta(0) \frac{\hat{I}(0)}{S(0) + \hat{I}(0)} + \eta(0) \right] \hat{N}.
\]

One can see from this equality that \( \hat{N} > 0 \) due to \( \hat{S}(0), \hat{I}(0) > 0 \).

The proof of Theorem 1.2 is now complete.

3.4. Profile of EE as \( d_I \to 0 \): proof of Theorem 1.3. In this section, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. For any EE \((S(x, d_S), I(x, d_S))\) of (5)-(6), we take the transformations:

\[
u(y, d_I) := S(d_I y, d_I), \quad v(y, d_S) := d_I I(d_I y, d_I), \quad 0 \leq y \leq d_I^{-1} L. \quad (52)
\]

For \( y \in (0, d_I^{-1} L) \), (5)-6 will become the following

\[
d_S u_{yy} + q d_I u_y - d_I^2 u \left( v \frac{d}{d_I} u + v \right) + d_I \gamma v = 0, \quad (53a)
\]

\[
v_{yy} + qv_y + d_I^2 u \left( \frac{v}{d_I u + v} + \eta \right) - d_I \gamma v = 0 \quad (53b)
\]

with the boundary conditions

\[
d_S u_y + q d_I u = 0 = v_y + qv, \quad y = 0, \quad d_I^{-1} L, \quad (53c)
\]

and the conservation law

\[
\int_0^{d_I^{-1} L} (d_I u + v) \, dy = N. \quad (54)
\]

By (ii) of Lemma 3.4, \( d_I I(\cdot, d_I) \) is uniformly bounded on \([0, L]\) concerning with \( d_I \) for \( 0 < d_I \leq 1 \). Thus, for any fixed constant \( K > 0 \), there is a positive constant \( C \) such that

\[
v(y, d_I) \leq C, \quad \forall y \in [0, K]. \quad (55)
\]

Henceforth, the positive constant \( C \) may be different from line to line but is independent of all \( 0 < d_I < L/K \).
Fix $K > 0$. From system (29) it follows that

$$
\begin{align*}
-d_S x S'_{xx} + q S'_{x} + \left[ \eta(x) + \beta(x) e^{-\frac{q}{q^*}} I^* \right] S' + e^{-\frac{q}{q^*}} S^* + e^{-\frac{q}{q^*}} I^* \\
= \gamma(x) e^{\left( \frac{q}{q^*} - \frac{q}{q^*} \right) I^*} \geq 0, \\
0 < x < L,
\end{align*}
$$

and so $S^*(0) = S^*(L) = 0$. Thanks to [14, Theorem 8.18], it holds that

$$
\|S^*(\cdot, d_I)\|_{L^1([0,L])} \leq C \min_{x \in [0,L]} S^*(x, d_I),
$$

and so

$$
\|S(\cdot, d_I)\|_{L^1([0,L])} \leq C \min_{x \in [0,L]} S(x, d_I).
$$

Moreover, Lemma 3.1 gives

$$
\min_{x \in [0,L]} S(x, d_I) \geq C, \quad \forall 0 < d_I \leq 1,
$$

and (i) of Lemma 3.4 implies

$$
\max_{x \in [0,L]} S(x, d_I) \leq C, \quad \text{and so} \quad \max_{y \in [0,d_I^{-1} L]} u(y, d_I) \leq C, \quad \forall 0 < d_I \leq 1.
$$

Hence, by (55) and (57), one can apply the standard regularity of elliptic equations to (53a),(53b) and (53c) to deduce that, for any fixed constant $K > 0$,

$$
\|u(\cdot, d_I)\|_{C^{1,\alpha}([0,K])} \leq C, \quad \|v(\cdot, d_I)\|_{C^{1,\alpha}([0,K])} \leq C
$$

for some constant $\alpha \in (0,1)$. In view of (58), there exists a sequence of $d_I$, denoted by itself, and the corresponding positive solution $(u_{d_I}(y), v_{d_I}(y)) := (u(y, d_I), v(y, d_I))$ such that

$$
\lim_{d_I \to 0} (u_{d_I}(\cdot), v_{d_I}(\cdot)) = (\tilde{u}(\cdot), \tilde{v}(\cdot)) \quad \text{in} \quad C^1_{\text{loc}}([0,\infty)) \times C^1_{\text{loc}}([0,\infty)).
$$

Since $(u_{d_I}, v_{d_I})$ solves (53)-(54), we can easily find that $(\tilde{u}, \tilde{v})$ satisfies

$$
\begin{align*}
\tilde{u}_{yy} &= 0, & y \in (0,\infty) , \\
\tilde{v}_{yy} + q \tilde{v}_y &= 0, & y \in (0,\infty)
\end{align*}
$$

with the boundary conditions

$$
\tilde{u}_y(0) = 0 = \tilde{v}_y(0) + q \tilde{v}(0).
$$

Solving (60a), (60b) and (60c), we deduce

$$
\tilde{u}(y) = \tilde{C}_S e^{-qy}, \quad \tilde{v}(y) = \tilde{C}_I e^{-qy}, \quad y \geq 0
$$

for some nonnegative constants $\tilde{C}_S$, $\tilde{C}_I$. Arguing as in Step 2 in the proof of Theorem 1.1, one can see that $\tilde{C}_S > 0$, $\tilde{C}_I > 0$. Especially, $S(0, d_I) \to \tilde{C}_S > 0$ as $d_I \to 0$ because of (52) and (59).

Using the equation of $S^*$ and the $L^1$-estimate theory for elliptic equations (see, for instance, [3]), we can obtain that, for any given $p \in [1,\infty)$, $\|S^*(\cdot, d_I)\|_{W^{1,p}([0,L])} \leq C$, and then $\|S(\cdot, d_I)\|_{W^{1,p}([0,L])} \leq C$. Thus, passing to a subsequence of $d_I$, denoted still by itself for convenience, we have

$$
S(\cdot, d_I) \to \tilde{S}(\cdot) \quad \text{in} \quad C([0,L]), \quad \text{as} \quad d_I \to 0,
$$

where $\tilde{S} \in C([0,L])$ is positive on $[0,L]$ due to (56).
Next we set $I(\tilde{x}_{d_t}, d_t) = \min_{x \in [0, L]} \{I(x, d_t)\}$. And $\tilde{x}_{d_t} \in [0, L]$. One may assume that $\tilde{x}_{d_t} \to \tilde{x} \in [0, L]$ as $d_t \to 0$. As in Step 3 of the proof of Theorem 1.2, we can claim that $\lim_{d_t \to 0} I(\tilde{x}_{d_t}, d_t) = 0$ and $\tilde{x}_{d_t} = \tilde{x} = L$ for all small $d_t$.

As $I$ is locally uniformly bounded in $(0, L]$ due to Lemma 3.4, for any fixed small $\varepsilon > 0$, after passing to a subsequence of $d_t \to 0$, one may assume that

$$I(\cdot, d_t) \rightharpoonup \tilde{I}(\cdot) \text{ weakly in } L^2((\varepsilon, L]),$$

where $\tilde{I} \in L^2((\varepsilon, L])$ and $\tilde{I} \geq 0$ a.e. in $(\varepsilon, L)$. Since $\int_0^L I(x, d_t) dx < N$ for all $d_t > 0$, by a diagonal argument, up to a further subsequence of $d_t \to 0$, we derive

$$\int_0^L \tilde{I}(x) dx = \lim_{\varepsilon \to 0} \int_0^L \tilde{I}(x) dx = \lim_{\varepsilon \to 0} \int_0^L I(x, d_t) dx \leq N. \tag{61}$$

Thus, $\tilde{I} \in L^1((0, L])$ and $\tilde{I} \geq 0$ a.e. in $(0, L)$.

Similarly as in the proof of Theorem 1.2, we can show that $\tilde{I}$ is a classical solution of

$$\begin{cases} q\tilde{L} + \beta(x) \frac{S\tilde{I}}{S + I} + \eta(x)\tilde{S} - \gamma(x)\tilde{I} = 0, & x \in (0, L), \\ \tilde{I}(L) = 0, \end{cases} \tag{62}$$

and $\tilde{S}$ is the classical solution of

$$\begin{cases} d_S\tilde{S}_x + q\tilde{S} - \beta(x) \frac{S\tilde{I}}{S + I} - \eta(x)\tilde{S} + \gamma(x)\tilde{I} = 0, & x \in (0, L), \\ d_S\tilde{S}(L) + q\tilde{S}(L) = 0. \end{cases} \tag{63}$$

Let us denote

$$\tilde{N} = N - \int_0^L (\tilde{S} + \tilde{I}) dx.$$

Obviously, $0 \leq \tilde{N} < N$. Notice from (61) that

$$\lim_{\varepsilon \to 0} \lim_{d_t \to 0} \int_0^\varepsilon I(x, d_t) dx = N - \lim_{d_t \to 0} \int_0^L S(x, d_t) dx$$

$$- \lim_{\varepsilon \to 0} \lim_{d_t \to 0} \int_0^\varepsilon I(x, d_t) dx = N - \int_0^L \tilde{S}(x) dx - \int_0^L \tilde{I}(x) dx = \tilde{N}. \tag{64}$$

For any fixed small $\varepsilon \in (0, L)$, we integrate the equation of $S$-component over $[0, \varepsilon]$ to deduce

$$d_S S_x(\varepsilon, d_t) + q S(\varepsilon, d_t) - \int_0^\varepsilon \left[ \beta(x) \frac{S\tilde{I}}{S + I} + \eta(x) S - \gamma(x) I \right] dx = 0. \tag{65}$$

It follows from the mean value theorem that

$$\int_0^\varepsilon \left[ \beta(x) \frac{S\tilde{I}}{S + I} + \eta(x) S - \gamma(x) I \right] dx = \beta(x_v) \frac{S(x_v)}{S(x_v) + I(x_v)} \int_0^\varepsilon I dx$$

$$+ \eta(x_v) S(x_v) \cdot \varepsilon - \gamma(x_v) \int_0^\varepsilon I dx \tag{66}$$

for some $x_v \in (0, \varepsilon)$. Owing to the uniform convergence above, (65) and (66) imply that

$$d_S \tilde{S}_x(0) + q \tilde{S}(0) - \left[ \beta(0) \frac{\tilde{S}(0)}{S(0) + I(0)} \tilde{N} - \gamma(0) \tilde{N} \right] = 0,$$
by first sending $d_I \to 0$ and then $\varepsilon \to 0$.

Now, given $\varphi \in C([0, L])$, for any small $\varepsilon > 0$, it is clear that
\[
\int_0^L I(x, d_I)\varphi(x) \, dx = \int_0^\varepsilon I(x, d_I)\varphi(x) \, dx + \int_\varepsilon^L I(x, d_I)\varphi(x) \, dx
\]
\[= \varphi(x_\varepsilon) \int_0^\varepsilon I(x, d_I) \, dx + \int_\varepsilon^L I(x, d_I) \varphi(x) \, dx,
\]
where $x_\varepsilon \in [0, \varepsilon]$. Letting $d_I \to 0$. Then $\varepsilon \to 0$, and using (64), we derive
\[
\int_0^L I(x, d_I)\varphi(x) \, dx \to \int_0^L \bar{I}(x)\varphi(x) \, dx + N \varphi(0).
\]
This means that $I(\cdot, d_I) \to \bar{I}(\cdot) + \hat{N}\delta_0$ weakly in $L^1((0, L))$, where $\delta_0$ is the Dirac measure centered at 0. Moreover, by the fact that $\bar{S}$ is positive on $[0, L]$, one sees that $\bar{I} > 0$ in $[0, L]$ using (62).

In light of (54) and (59), one can further obtain that
\[
N = \lim_{d_I \to 0} \int_0^L (S(x, d_I) + I(x, d_I)) \, dx > \lim_{d_I \to 0} \int_0^L I(x, d_I) \, dx
\]
\[= \lim_{d_I \to 0} \int_0^{d_I^{-1}L} v(y, d_I) \, dy = \int_0^\infty \hat{v}(y) \, dy = \frac{C_I}{q}.
\]
This yields $\bar{C}_I < Nq$.

Finally, adding up the first two equations in (62) and (63) and integrating over $[x, L]$, together with the conditions of $\bar{S}, \bar{I}$ at $x = L$, we have
\[
d_S\bar{S}_x(x) + q\bar{S}(x) + q\bar{I}(x) \equiv d_S\bar{S}_x(L) + q\bar{S}(L) + q\bar{I}(L) = 0.
\]

Particularly, this gives
\[
d_S\bar{S}_x(0) + q\bar{S}(0) + q\bar{I}(0) = q\bar{I}(0) + \beta(0)\frac{\bar{S}(0)}{\bar{S}(0) + \bar{I}(0)} \bar{N} - \gamma(0)\bar{N} = 0,
\]
and so
\[
q\bar{I}(0) = \left[ \gamma(0) - \beta(0)\frac{\bar{S}(0)}{\bar{S}(0) + \bar{I}(0)} \right] \bar{N}.
\]

By the positivity of $\bar{I}(0)$, we have $\bar{N} > 0$ and $\gamma(0) > \beta(0)\frac{\bar{S}(0)}{\bar{S}(0) + \bar{I}(0)}$. So far, we have proved all assertions stated in Theorem 1.3.

4. Conclusion. In this paper, we study an SIS epidemic reaction-diffusion model (1) with spontaneous infection and advection in one-dimensional space. We first prove the uniform persistence property of the solution using the abstract theory of dynamical systems, which then implies the existence of endemic equilibrium (EE). We focus on the asymptotic behavior of EE in three cases: large advection; small diffusion of the susceptible population; small diffusion of the infected population. Our results suggest that the asymptotic profiles of the susceptible and infected populations behave differently from that of the corresponding system without advection and that of the system without spontaneous infection. In what follows, we would like to compare the results of those models and reveal the effect of advection and spontaneous infection of either the susceptible or infected population on the spatial distribution of disease.
(i) **Case of large advection.** When the advection coefficient $q$ becomes sufficiently large, Theorem 1.1 shows that the densities of both the susceptible and infected populations only concentrate at the downstream $x = 0$ behaving like delta functions with some masses at the downstream. These masses are determined by the conservation law and the integral equality in (7). Similar results hold for (4); see [20, Theorem 3.1] and [6, Theorem 1.2]. This shows that increasing the advection rate can not help to extinguish the infectious disease.

(ii) **Case of small diffusion of the susceptible.** When the mobility of the susceptible population is controlled to be small, we know from Theorem 1.2 that for model (1), the density of the susceptible population is positive on the domain except at the upstream, where it can reach its minimum value zero, and also concentrates at the downstream. But the density of the infected population tends to be positive on the entire habitat. Comparing to [42, Theorem 4.1] for (3), we see that the existence of advection will trigger the concentration phenomenon of the susceptible at the downstream while leading to its extinction at the upstream. In addition, the infected population for models (1) and (3) persist in the entire habitat.

On the other hand, when comparing to [20, Theorem 3.2] for model (4), one will notice that if the spontaneous infection is incorporated, the infected population persists in the entire habitat if the mobility of the susceptible population is controlled to be small. Thus, the presence of the spontaneous infection enhances the persistence of the disease.

(iii) **Case of small diffusion of the infected.** When the diffusion rate of the infected is restricted to be small, the density of the infected population modeled by system (1) is positive everywhere except at the upstream and concentrate at the downstream while the density of the susceptible population will stay positive on the whole domain. As far as system (3) is concerned, [42, Theorem 4.2] shows that the densities of both susceptible and infected populations remain positive everywhere. Hence, whether there is advection or not, the disease can not be eradicated if the diffusion of the infected is controlled to be small for both models (1) and (3). However, for (4), the disease will become extinct once we restrict the mobility of the infected, as shown by [20, Theorem 3.3]. In other words, the appearance of spontaneous infection can enhance the persistence of infectious disease.

In view of the above discussion, we may conclude that both spontaneous infection and advection play important roles in the spatial distribution of both the susceptible and infected populations: the advection can cause the concentration of the susceptible and infected populations at the downstream, and the spontaneous infection can enhance the persistence of infectious disease.

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