Note

Black hole formation by incoming electromagnetic radiation

José M M Senovilla

Física Teórica, Universidad del País Vasco, Apartado 644, E-48080 Bilbao, Spain

E-mail: josemm.senovilla@ehu.es

Received 19 August 2014
Accepted for publication 31 October 2014
Published 5 December 2014

Abstract
I revisit a known solution of the Einstein field equations to show that it describes the formation of non-spherical black holes by the collapse of pure electromagnetic monochromatic radiation. Both positive and negative masses are feasible without ever violating the dominant energy condition. The solution can also be used to model the destruction of naked singularities and the evaporation of white holes by emission or reception of light.

Keywords: black holes, Einstein–Maxwell, electromagnetic radiation

PACS numbers: 04.70.Bw, 04.40.Nr, 04.20.Jb

Today there are many models describing the formation, or the evaporation, of black holes in general relativity; some of them deal with the collapse of matter or fluids, others with out- or in-coming ‘incoherent radiation’. However, there is no identified case of the formation/evaporation of a black hole by means of electromagnetic radiation solely. The purpose of this note is to call attention to a family of solutions which describe the formation/destruction of black holes (also naked singularities) by reception or emission of pure monochromatic light. Actually, the solution was known long ago, and has been used to describe the collapse of null dust to non-spherical black holes with negative cosmological constant [13, 14], and to analyze the first example of a dynamical horizon with toroidal topology [5]. However, the fundamental point is that the radiation can be properly identified as monochromatic light—it should not be treated as ‘incoherent radiation’.

The line-element of the solution was first presented—among many other exact solutions—by Robinson and Trautman in their celebrated paper [22], and is given in local coordinates \(\{u, r, x, y\}\) by (see also [24], p 430)

\[
dx^2 = r^2 \left( dv^2 + a^2 \right) + 2\epsilon dv dr + \left( \frac{2m(v)}{r} + \frac{\Lambda r^2}{3} \right) dv^2,
\]

(1)
where \( r > 0, \epsilon = \pm 1 \) determines the retarded (\( \epsilon = -1 \)) or advanced (\( \epsilon = 1 \)) character of the null coordinate \( v \), \( k = -dv \) is a null one-form chosen to be future pointing, \( \Lambda \) is the cosmological constant (allowed to have any sign) and \( m(v) \) is a function of \( v \).

The metric (1) is a solution of the Einstein–Maxwell equations with \( \Lambda \) for a null electromagnetic field given by

\[
F = dv \wedge \left( h_x(v)dx + h_y(v)dy \right)
\]

so that \( k \) is the wave vector of this pure radiation field. Here, \( h_x(v) \) and \( h_y(v) \) are arbitrary functions. An appropriate observer with unit timelike tangent vector \( \vec{u} \) orthogonal to the \( \{ x, y \} \) surfaces and with \( u^\mu k_\mu = -1 \) will measure electric \( E \) and a magnetic \( B \) fields given by

\[
E = h_x(v)dx + h_y(v)dy, \quad B = h_x(v)dy - h_y(v)dx.
\]

Therefore, by choosing the functions \( h_x, h_y \) judiciously one can describe light that is linearly polarized (e.g. \( h_y = 0 \)), circularly polarized (e.g. \( h_x = A \cos v, h_y = A \sin v \)), or elliptically polarized.

The energy–momentum tensor of the electromagnetic radiation reads

\[
T_{\mu\nu} = \frac{\epsilon 4\pi G}{c^4} \int_{v_0}^{v} \left[ h_x^2(w) + h_y^2(w) \right] dw + M,
\]

from where one determines the function \( m(v) \) on the line-element (1):

\[
m(v) = \epsilon \frac{4\pi G}{c^4} \int_{v_0}^{v} \left[ h_x^2(w) + h_y^2(w) \right] dw + M.
\]

where \( v_0 \) is a fixed value of \( v \) and \( M = m(v_0) \) is an integration constant (\( v_0 \) can be \( -\infty \)). The energy–momentum tensor (3) always satisfies the dominant energy condition, and we have

\[
m(v) = \epsilon \frac{4\pi G}{c^4} \left[ h_x^2(v) + h_y^2(v) \right],
\]

where dots denote derivatives with respect to \( v \), so that the function \( m(v) \) is non-decreasing for \( \epsilon = 1 \) and non-increasing for \( \epsilon = -1 \). Observe that \( M \) can have any sign.

The metric (1) is the unique Petrov type D solution of the Einstein–Maxwell equations with a null electromagnetic field [25]. It must be remarked that there exists no analogue solution in higher dimensions [21].

The space-time has three independent Killing vectors in general given by \( \partial_\theta, \partial_\phi, y \partial_r - x \partial_\phi \), the corresponding isometry group acts transitively on spacelike two-dimensional surfaces spanned by \( \{ \partial_\theta, \partial_\phi \} \). Thus, the third Killing vector is an isotropy. There is also a Kerr–Schild vector field [4] \( \xi = \partial_\phi \) satisfying

\[
\xi^\mu g_{\mu\nu} = \frac{2m}{r} k_\mu k_\nu \quad \xi^\mu k_\mu = 0.
\]

This vector field is analogous to the Kodama vector field in spherically symmetric spacetimes [1, 10]. The electromagnetic field inherits the symmetry \( \{ \partial_\theta, \partial_\phi \} \) but not the isotropy:

\[
\xi_{\gamma \theta \phi \phi} F = dv \wedge \left( h_x dx - h_y dy \right), \quad \xi^\phi F = dv \wedge \left( h_x dx + h_y dy \right).
\]

Observe that the Kerr–Schild vector field is a symmetry of the electromagnetic field whenever \( h_x, h_y \) are both constant, in which case the function \( m(v) \) is linear in \( v \).

The surfaces of transitivity, defined by constant values of \( v \) and \( r \), can describe (i) flat tori if one identifies \( x \leftrightarrow x + a \) and \( y \leftrightarrow y + b \), in which case they are compact with an area equal to \( abr^2 \); (ii) flat cylinders if only one of the previous identifications is performed; and
(iii) flat planes when \(-\infty < x, y < \infty\). The mean curvature one-form \([1, 11]\) for these surfaces is simply \(H = dr\) from where their two future null expansions can be easily extracted

\[
\theta_1 = -e, \quad \theta_2 = -e \left( \frac{2 m(v)}{r} + \frac{\Lambda}{3} r^2 \right).
\]

Hence, the surfaces of transitivity are trapped if and only if \(2 m/r + \Lambda r^2/3 > 0\). They are future- or past-trapped for \(\epsilon = 1\) or \(-1\), respectively. Notice that the transitivity surfaces are always trapped for large enough \(r\) if \(\Lambda > 0\) (de Sitter behavior at infinity), and for small enough \(r\) if \(m(v) > 0\) independently of the sign of \(\Lambda\). Actually, they are always trapped in the region \(m(v) > 0\) if \(\Lambda = 0\). For \(\Lambda < 0\), the transitivity surfaces are trapped only in the region with \(2 m(v) > -\Lambda r^2/3 > 0\) if this exists, and they can never be trapped for large enough \(r\) (anti-de Sitter (AdS) behavior).

The hypersurface defined by

\[
H: 2 m(v) + \frac{\Lambda}{3} r^3 = 0
\]

(if this is feasible) is a marginally trapped tube, that is, a hypersurface foliated by marginally trapped surfaces. Observe that \(H\) exists for \(\Lambda > 0\), \(= 0\) or \(<0\) only if \(m(v)\) is negative, zero or positive, respectively, somewhere. One can easily compute the causal character of \(H\): it is non-spacelike if \(\Lambda > 0\) and actually timelike or null whenever \(\dot{m} \neq 0\) or \(\ddot{m} = 0\) respectively; non-timelike if \(\Lambda < 0\) with null portions wherever \(\ddot{m} = 0\) and spacelike parts where \(\dot{m} \neq 0\), in the last case these parts are dynamical horizons \([5]\); if \(\Lambda = 0\) it is given by the null hypersurfaces \(v = \dot{v}\) such that \(m(\dot{v}) = 0\).

There is a curvature singularity at \(r \to 0\) unless \(m(v) = 0\) (in which case there is no electromagnetic radiation). This particular case with \(m(v) = 0\) has constant curvature \(\Lambda/3\), so that the metric represents a region of de Sitter, flat, or AdS space-time depending on the sign of \(\Lambda\). In these cases \(r = 0\) is actually a horizon through which the metric is extendible. Black holes in AdS space-time obtained by identification along one symmetry generator were deeply analyzed in \([6]\). Other interesting particular cases arise if the electromagnetic radiation is absent (\(h_i = h_i = 0\)) but we retain a non-vanishing constant \(m(v) = M \neq 0\). These are metrics describing planar, cylindrical or toroidal black holes when \(\Lambda < 0\), and have been largely studied in the literature \([3, 7, 9, 12, 15, 17, 26]\). These cases, as follows from \((5)\), have \(\xi^\nu\) as another Killing vector, so that they are stationary outside \(H\) which is a Killing horizon in this situation. One can check \([3, 7, 9, 12, 15, 17, 26]\) that then \(m(v) = M\) is proportional to the mass in the toroidal case, to mass per unit length in the cylindrical case, and to mass per unit area in the planar case. Therefore, negative values of \(m(v)\) can be interpreted, at least when \(\Lambda < 0\), as negative values of the mass. Black holes with negative mass were discussed in \([16]\), but the remarkable thing about solution \((1)\) is that the dominant energy condition holds everywhere. This may be related to recent discussions on similar situations in de Sitter backgrounds \([2, 19]\). When \(\Lambda = 0\) but keeping \(m(v) = M\) the metric can be seen to be isometric to the Kasner space-time \([24]\) with exponents \(p_1 = p_2 = 2/3, p_3 = -1/3\) if \(M > 0\), and isometric to the plane-symmetric Taub solution \([24]\) if \(M < 0\).

The collapse to form non-spherical stationary black holes with a constant \(m(v) = M\) and \(\Lambda < 0\) has been studied in several papers, such as for instance in \([23]\) where the toroidal case treated herein was not explicitly considered but was later carried out in \([20]\). In \([20]\), the collapse of perfect fluids describing anisotropic, as well as spatially inhomogeneous, interiors was fully described by matching these interiors to an exterior \((1)\) with constant \(m(v) = M\). A collapse by matching was also considered in \([13]\), where the dynamical metric \((1)\) was studied.
without realizing that the incoming flow of radiation is actually a null electromagnetic field, but this matching is incorrect\(^1\).

The point I want to make in this note is that the metric (1) describes, appropriately, the generation of black holes by collapse of fully identified matter content: pure electromagnetic monochromatic waves with a well defined polarization. And there is no need for a matching procedure. Actually, this is just one situation of interest among a rich family of different behaviors that can be properly represented by (1). Some outstanding cases are enumerated and briefly analyzed next.

(1) *Formation of a toroidal, cylindrical or planar black hole by sending light into an AdS background.* By choosing \(\Lambda < 0, \epsilon = 1\) and letting \(h_0(v) = h_\rho(v) = 0\) for all \(v < v_0\) the function \(m(v) = M\) is constant for all \(v < v_0\). When \(M\) is set to zero then the space-time is AdS in this entire region. If light is then sent into the space-time by letting \(h_\rho(v), h_\tau(v)\) to be non-zero in the interval \(v_0 \leq v \leq v_1\), a black hole enclosing a future singularity censored by an event horizon forms with a final constant \(m(v) = M_f > 0\) given by

\[ r = 0 \text{ singularity} \]

\[ \text{AdS} \]

\[ \text{EH} \]

\[ \text{H} \]

\[ \text{AdS} \]

Figure 1. Schematic diagram of the formation of a toroidal (cylindrical or planar) black hole by sending light into an anti-de Sitter (AdS) background. As usual, null radial lines are drawn at 45° and the future direction is upwards. The spacetime is AdS until monochromatic light enters from \(J\) (an infinity that is timelike) at \(v = v_0\) and flows along null hypersurfaces until it ceases at \(v = v_f\). Thus, the shaded region contains a non-vanishing energy–momentum content due to the electromagnetic radiation solely. In the AdS part, the null hypersurfaces labeled as \(r = 0\) and \(v = -\infty\) are horizons through which the space-time can be regularly extended. The spacetime has a final mass proportional to \(M_f\) in (7). When the first photon reaches the \(r = 0\)-horizon a dynamical horizon \(H\) develops that eventually (at \(v = v_f\)) merges with the event horizon \(EH\) of the black hole, which encloses a future spacelike singularity \(r = 0\). Observe that \(EH\) starts developing in the AdS region.

\(^1\) In [13] the metric (1) is claimed to be matched to an interior Robertson–Walker metric for dust. However, this is impossible, as the Israel conditions [8, 18] for a matching would require that the normal components of the energy–momentum tensor be continuous at the matching hypersurface, and this cannot happen with a dust on one side and (3) on the other side.
A Penrose-like diagram of this example is given in figure 1.

(2) Transformation of a naked singularity into a black hole enclosing a clothed singularity by sending light into the former. When $M$ is not set to zero but rather is a negative constant in the previous situation, the spacetime has a naked timelike singularity at $r = 0$ for all $v \leq v_0$. Sending light again as before, and assuming that

$$M_f = \frac{4\pi G}{c^4} \int_{v_0}^{v_1} \left[ h_+^2(v) + h_\times^2(v) \right] dv + M$$

(8)

is strictly positive, the singularity at $r = 0$ transmutes into a spacelike one clothed by an event horizon which merges with the dynamical horizon $\mathcal{H}$. An illustrative diagram is presented in figure 2. Similar situations arise for $\Lambda > 0$ and $\Lambda = 0$, but now without the formation of event horizons; in the former case there is a marginally trapped tube $\mathcal{H}$ which is partly null and partly timelike but not in the latter. Moreover, in the former case there is a past infinity $\mathcal{J}^-$ which is spacelike while in the latter it is null.

(3) Annihilation of a naked singularity by sending a fine-tuned amount of light. Under the same assumptions as in the previous case, if $M < 0$ and $h_+(v)$, $h_\times(v)$ are fine tuned together with $v_1 - v_0$ such that the final $M_f$ in (8) vanishes, the original timelike naked singularity
simply disappears and the final outcome is a portion of AdS, de Sitter, or flat space-time for negative, positive or vanishing $\Lambda$, respectively. The case with $\Lambda > 0$ is the only one containing future trapped surfaces and a marginally trapped tube $\mathcal{H}$—which in this case is non-spacelike everywhere. Corresponding diagrams are given in figure 3.

(4) Time reversals. By setting $\epsilon = -1$ in (1) one describes situations where the electromagnetic radiation is emitted outwardly towards the future; observe that now $m \lesssim 0$. Then, for instance, models for the evaporation of a white hole by pure emission of light leading to AdS space-time arise: this is simply the time reversal of (1) and the corresponding diagram is the same as in figure 1 but turned upside down. Similarly, one can model the appearance of a naked timelike singularity in vacuum (with arbitrary $\Lambda$) by spontaneous emission of photons. Again, these are the time reversals of (3) and the corresponding diagrams are the same as in figure 3, interchanging future and past.

**Acknowledgments**

Thanks to I Bengtsson for bringing [6] to my attention. Supported by grants FIS2010-15492 (MICINN), GIU12/15 (Gobierno Vasco), P09-FQM-4496 (J Andalucía—FEDER) and UFI 11/55 (UPV/EHU).
References

[1] Bengtsson I and Senovilla J M M 2011 Region with trapped surfaces in spherical symmetry, its core, and their boundaries Phys. Rev. D 83 044012
[2] Belletête J and Paranjape M B 2013 On negative mass Int. J. Mod. Phys. D 22 1341017
[3] Cai R-G and Zhang Y-Z 1996 Black plane solutions in four-dimensional spacetimes Phys. Rev. D 54 4891
[4] Coll B, Hildebrandt S R and Senovilla J M M 2001 Kerr–Schild symmetries Gen. Relativ. Gravit. 33 649
[5] Dadras P, Firouzjaee J T and Mansouri R 2012 A concrete anti-de Sitter black hole with dynamical horizon having toroidal cross-sections and its characteristics Europhys. Lett. 100 39001
[6] Holst S and Peldán P 1997 Black holes and causal structure in anti-de Sitter isometric spacetimes Class. Quantum Grav. 14 3433
[7] C-g Huang C-b Liang 1995 A torus-like black hole Phys. Lett. A 201 27
[8] Israel W 1966 Singular hypersurfaces and thin shells in general relativity Nuovo Cimento 44 1
Israel W 1967 Singular hypersurfaces and thin shells in general relativity Nuovo Cimento 48 463 (erratum)
[9] Klemm D, Moretti V and Vanzo L 1998 Rotating topological black holes Phys. Rev. D 57 6127
[10] Kodama H 1980 Conserved energy flux for the spherically symmetric system and the backreaction problem in the black hole evaporation Prog. Theor. Phys. 63 1217
[11] Kriele M 1999 Spacetime (Berlin, Germany: Springer)
[12] Lemos J P S 1995 Three dimensional black holes and cylindrical general relativity Phys. Lett. B 353 46
[13] Lemos J P S 1998 Gravitational collapse to toroidal, cylindrical, and planar black holes Phys. Rev. D 57 4600
[14] Lemos J P S 1999 Collapsing shells of radiation in anti-de Sitter spacetimes and the hoop and cosmic censorship conjectures Phys. Rev. D 59 044020
[15] Mann R B 1997 Pair production of topological anti-de Sitter black holes Class. Quantum Grav. 14 L109
[16] Mann R B 1997 Black holes of negative mass Class. Quantum Grav. 14 2927
[17] Mann R B 1997 Topological black holes: outside looking in Internal structure of black holes and space-time singularities (Proc. workshop in Haifa, Annals Israel Phys. Soc. vol 13) 2nd edn ed L M Burko and A Ori pp 311–42
[18] Mars M and Senovilla J M M 1993 Geometry of general hypersurfaces in spacetime: Junction conditions Class. Quantum Grav. 10 1865
[19] Mbarek S and Paranjape M B 2014 Negative mass bubbles in de Sitter space-time Phys. Rev. D 90 101502
[20] Mena F C, Natário J and Tod P 2008 Gravitational collapse to toroidal and higher genus asymptotically AdS black holes Adv. Theor. Math. Phys. 12 1163
[21] Ortaggio M, Podolský J and Žofka M 2008 Robinson–Trautman spacetimes with an electromagnetic field in higher dimensions Class. Quantum Grav. 25 025006
[22] Robinson I and Trautman A 1962 Some spherical gravitational waves in general relativity Proc. R. Soc. A 265 463
[23] Smith W L and Mann R B 1997 Formation of topological black holes from gravitational collapse Phys. Rev. D 56 4942
[24] Stephani H, Kramer D, MacCallum M A H, Hoenesaers C and Herlt E 2003 Exact Solutions to Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[25] van den Bergh N 1989 Einstein–Maxwell null fields of Petrov type D Class. Quantum Grav. 6 1871
[26] Vanzo L 1997 Black holes with unusual topology Phys. Rev. D 56 6475