RATIONAL TANGLES AND THE MODULAR GROUP

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Abstract. There is a natural way to associate with a transformation of an isotopy class of rational tangles to another, an element of the modular group. The correspondence between the isotopy classes of rational tangles and rational numbers follows, as well as the relation with the braid group $B_3$.

Introduction

The main result on rational tangles [1] is the theorem stating that it is possible to associate with every rational tangle one and only one rational number, so that two rational tangles are isotopy equivalent iff they are represented by the same rational number.

We obtain this result associating with every rational tangle an element of $\text{PSL}(2, \mathbb{Z})$.

A tangle with four ends (here called simply tangle) is an embedding of two closed segments in a ball, such that their endpoints are four distinct points of the bounding sphere, and the image of the interior of the segments lie at the interior of the ball.

Let our sphere be centered at the origin of the three space with Cartesian coordinates $X$, $Y$ and $Z$, and the endpoints of the strands lie in the plane $Z = 0$.

Figure 1. Two tangles

We depict a tangle projected to the $XY$-plane, which contains the endpoints of the strands. Without lost of generality, we put these endpoints at the intersection of the sphere with the main diagonals of the plane (Fig. 2).

Definition. Two tangles are said isotopic if one can be deformed continuously to the other in the set of tangles with fixed endpoints.

1. Rational Tangles

A tangle is said rational if can be deformed continuously, in the set of tangles with non fixed endpoints, to a tangle consisting of two unlinked and unknotted segments.
**Example.** The tangles in Figure 1 are not rational, since the white strand of the left tangle is knotted, and the two strands of the tangle at right cannot be unlinked if we allow the endpoints of the strands moving on the bounding sphere.

For short we write *r-tangle* for rational tangle, and we denote by $\mathcal{R}$ the space of rational tangles.

![Isotopic r-tangles](image)

**Figure 2.** Isotopic r-tangles.

We denote by $\Gamma ||$ and $\Gamma = $ the simplest r-tangles (see Fig. 3) and by $1, 2, 3, 4$ the endpoints of the strands, starting with point $1 (X > 0, Y > 0)$ and going to 4 in the clockwise direction.

![Scheme of a tangle Γ and the basic tangles Γ || and Γ =](image)

**Figure 3.** Scheme of a tangle $\Gamma$ and the basic tangles $\Gamma ||$ and $\Gamma =$.

By the constructive definition of r-tangles in [1], an r-tangle is obtained from $\Gamma ||$ or $\Gamma = $ by a series of moves, consisting in twisting pairs of adjacent endpoints. We denote by $X_1^+$ and $X_1^-$ the positive and the negative twists, as shown in Fig. 4.

Given an r-tangle $\Gamma$, the r-tangle obtained by applying the move $X_1^\sigma$ $(\sigma = + \text{ or } -)$ to $\Gamma$ is denoted by $X_1^\sigma \Gamma$. Thus any expression
\begin{equation}
X_{i_1}^{\sigma_1}X_{i_2}^{\sigma_2}\ldots X_{i_n}^{\sigma_n}\Gamma_0,
\end{equation}
where $\Gamma_0 = \Gamma ||$ or $\Gamma = $, represents the r-tangle obtained from $\Gamma_0$ by applying the moves $X_1^\pm$ in the order from $X_{i_n}^{\sigma_n}$ to $X_{i_1}^{\sigma_1}$.

Any r-tangle, written in terms of the $X_1^\pm$, can be therefore represented such that each double point corresponds to one of the moves $X_1^\pm$ and its sign is *positive* or *negative* according to the scheme of Figure 5. We call such representations standard representations.

**Remark.** An r-tangle in a standard representation, where the double points are either all positive or all negative, is alternating, i.e., each strand passes alternatively up and down in the sequence of double points encountered along it.

Isotopic r-tangles may have different standard representations.

**Example.** The representations of isotopic r-tangles of Figure 2 at middle and at right, are standard, and they are respectively:
\[X_3^+X_2^+X_4^-X_4^+X_3^\pm\Gamma = , \quad X_3^+X_2^-X_3^\pm\Gamma = .\]
1.1. **Symmetries of the isotopy classes of r-tangles.** For the classification of the isotopy classes of the r-tangle the following observation is essential.

**Lemma 1.1.** Every r-tangle $\Gamma$ is isotopic to the tangle $\hat{\Gamma}$ obtained from $\Gamma$ by a rotation about the axis $X = 0$ and to the tangle $\Gamma^\dagger$ obtained from $\Gamma$ by a rotation about the axis $Y = 0$.

**Proof.** The basic tangles $\Gamma^\parallel$ and $\Gamma^\perp$ are invariant under these rotations. The tangles with only one double point, obtained as $X_4^+\Gamma^\parallel$, $X_3^+\Gamma^\parallel$ or as $X_4^+\Gamma^\parallel$, $X_4^+\Gamma^\parallel$, are also invariant under such rotations. Given a tangle $\Gamma$, we write its expression (1) in terms of $n$ moves. Therefore we have

$$\Gamma = X_i^\parallel \Phi,$$

where the r-tangle $\Phi$ is expressed in terms of $n - 1$ moves. Let us suppose that the isotopy class of $\Phi$ is invariant under the considered rotations, i.e.

$$\hat{\Phi} \sim \Phi \sim \Phi.$$

The following figure shows that $\Gamma$ is isotopy equivalent, by consequence, to $\Gamma^\dagger$ and to $\hat{\Gamma}$, when $\Gamma = X_4^+\Phi$.

Indeed:

$$\Gamma^\dagger = X_4^+\Phi \sim X_4^+\Phi = \Gamma,$$

and

$$\hat{\Gamma} = X_3^\parallel \Phi \sim X_3^\parallel \Phi \sim X_4^+\Phi = \Gamma.$$
Similarly, if \( \Gamma = X_3^\pm \Phi \):

\[
\Gamma \rangle = X_4^+ \Phi \sim X_2^+ \Phi = \Gamma,
\]

and

\[
\hat{\Gamma} = X_2^+ \Phi \sim X_2^+ \Phi = \Gamma.
\]

The cases with negative twists, as well as the cases with \( \Gamma = X_3^\pm \Phi \) and \( \Gamma = X_4^\pm \Phi \) are analogous. The lemma follows. \( \square \)

2. The group of moves of r-tangles

From the preceding section we deduce evidently also the following

**Corollary 2.1.** For every r-tangle \( \Gamma \in \mathcal{R} \):

\[
X_1^\pm \Gamma \sim X_3^\pm \Gamma, \quad X_2^\pm \Gamma \sim X_3^\pm \Gamma.
\]

Therefore, in order to classify the r-tangles up to isotopies, it is possible to consider only two basic moves, that change the isotopy class of a r-tangle, that we call A and B, and precisely:

\[
A := X_3^+, \quad B := X_2^+.
\]

Their inverses are denoted, respectively, \( A^{-1} \) and \( B^{-1} \) and satisfy \( A^{-1} = X_3^- \), \( B^{-1} = X_2^- \).

*Example.* The r-tangles of Figure 2 at middle and at right, have the following representation:

\[
A^1 B^1 B^{-2} A^2 \Gamma =, \quad A^1 B^{-1} A^2 \Gamma =. \]

Note that the moves A and B keep invariant the tangle endpoint lying in the first quadrant of the XY-plane.

There is another move keeping invariant this endpoint. It is the twist of the endpoints 2 and 4 by a rotation by \( \pi \) about the diagonal \( X = Y \), as shown in Fig. \( \square \). We denote this move by \( R \).

We now define a group of moves, generated by A, B, A\(^{-1}\) and B\(^{-1}\), and we denote it by \( \mathcal{T} \).

The identity \( E \) is the absence of any move.

Any word \( W = T_1 T_2 \cdots T_n \) where \( T_i \) are either A or B, or A\(^{-1}\), or B\(^{-1}\), represents an element of the group, and will be interpreted as the composition of the moves \( T_i \), which are applied to any r-tangle \( \Gamma \) starting from \( T_n \) and ending with \( T_1 \):

\[
T_1 T_2 \cdots T_n \Gamma = T_1 (T_2 (\cdots (T_n \Gamma))).
\]

Two words \( W \) and \( Q \) represents the same element iff, for every \( \Gamma \in \mathcal{R} \),

\[
Q \Gamma \sim W \Gamma.
\]
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Figure 6. The moves $A$, $B$ and $R$ obtained rotating by $\pi$ one half of the sphere containing the end-points of the r-tangle

Figure 7. Action of the moves $A$, $B$ and $R$ on a tangle $\Gamma$

For instance, the equation $AA^{-1} = E$ means that, for any r-tangle $\Gamma$, $AA^{-1}\Gamma \sim EI = \Gamma$. This equivalence is indeed obtained by the Reidemeister move which eliminates two double points.

A sequence of $k$ consecutive $A$ ($B$) moves will be denoted by $A^k$ ($B^k$) and its inverse by $A^{-k}$ ($B^{-k}$).

Any word $W = T_1T_2 \cdots T_n$ will be thus written as

$W = A^{a_1}B^{a_2}A^{a_3}B^{a_4} \cdots$ or $W = B^{a_4}A^{a_2}B^{a_3}A^{a_4} \cdots$

where $a_i$ are non-zero integers. The word $W$ has the inverse

$W^{-1} = \cdots B^{-a_4}A^{-a_3}B^{-a_2}A^{-a_1} \ \text{or} \ \ W = \cdots A^{-a_4}B^{-a_3}A^{-a_2}B^{-a_1}$

satisfying evidently $Q^{-1}Q = QQ^{-1} = E$.

**Theorem 2.2.** The group $\mathcal{T}$ is isomorphic to $\text{PSL}(2, \mathbb{Z})$.

**Proof.** We define a map $\mu : \mathcal{T} \mapsto \text{PSL}(2, \mathbb{Z})$, sending $A$ and $B$ to the following generators $A, B$ of $\text{PSL}(2, \mathbb{Z})$, which are defined up to multiplications by $-1$:

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$

so that $\mu(A^{-1}) = A^{-1}$, $\mu(B^{-1}) = B^{-1}$, and $\mu(E) = E$, $E$ being the identity $(2 \times 2)$-matrix, up to multiplication by $-1$.

The map $\mu$ sends any sequence of moves $A$ and $B$ and their inverses to the operator of $\text{PSL}(2, \mathbb{Z})$ obtained as the corresponding product of $A$, $B$ and their inverses.

To prove that $\mu$ is an isomorphism, we have to prove that

$\mu(W) = \mu(W') \Leftrightarrow \forall \Gamma \in \mathcal{R} \ W\Gamma \sim W'\Gamma.$
Proof of (4)⇒. To every element of PSL(2, Z) there correspond different words in the generators \( A, B, \ A^{-1} \) and \( B^{-1} \), since they are not independent. We have therefore to prove that whenever \( \mu(W) = \mu(W') \), where \( W \) and \( W' \) are different words in \( T \), then \( W\Gamma \sim W'\Gamma \) for every \( r \)-tangle \( \Gamma \). This is true if the following relation among the considered generators of PSL(2, Z) (which is the unique non trivial relation among them):

\[
(5) \quad AB^{-1}A^{-1}B^{-1} = E
\]

is the image by \( \mu \) of the analog relation holding in \( T \):

\[
(6) \quad AB^{-1}A^{-1}B^{-1} = E.
\]

Let \( S := A^{-1}BA^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), up to multiplication by \(-1\). Observe that \( S^{-1} = S \) in PSL(2, Z). Therefore Eq. (5) is equivalent to the equations

\[
(7) \quad B^{-1}AB^{-1} = S, \quad \text{and} \quad S^2 = E.
\]

We define \( S := AB^{-1}A \in T \), and to prove Eq. (6) we prove:

(i) \( B^{-1}AB^{-1} = S \);
(ii) \( S^2 = E \).

Proof of (i). The next figure shows that for any tangle \( \Gamma \),

\[
AB^{-1}A\sim R\Gamma, \quad B^{-1}AB^{-1}\Gamma \sim R\Gamma.
\]

Hence item (i) is satisfied by \( S = R \).

Proof of (ii).
Both \( R\Gamma \) and \( R^{-1}\Gamma \) are isotopic to the tangle obtained by a rotation of the whole ball containing \( \Gamma \) about the diagonal \( Y = X \). Hence \( R^2\Gamma \sim \Gamma \) for any tangle \( \Gamma \), and we conclude \( R^2 = E \).

**Proof of (4)$$.$$ To conclude the proof of the theorem, we have to exclude that there exist some words \( W \) and \( W' \) in \( \mathcal{T} \) such that, for every \( r\)-tangle \( \Gamma \), the isotopy classes of \( W\Gamma \) and \( W'\Gamma \) coincide, but \( \mu(W) \neq \mu(W') \). In this case there should be a move \( Q \), namely \( Q = W^{-1}W' \), which is isotopy equivalent to the identity, and satisfies \( \mu(Q) \neq E \).

**Lemma 2.3.** Every operator \( Q \in PSL(2, \mathbb{Z}) \) can be written as \( VT \) where \( T \) is either a word in sole \( A, B \) or a word in sole \( A^{-1}, B^{-1} \), and \( V = E \) or \( V = S \).

**Proof of the Lemma.** It follows from the relations (7) and from the derived relations holding in \( PSL(2, \mathbb{Z}) \):

\[
SA = B^{-1}S, \quad SB = A^{-1}S, \quad SA^{-1} = BS, \quad SB^{-1} = AS.
\]

We then write the operator \( Q := \mu(Q) \) in the form \( Q = VT \). \( T \) is a word either in \( A, B \) or in \( A^{-1}, B^{-1} \). Suppose that the word \( T \) ends with \( A^a \), for some non zero \( a \in \mathbb{Z} \). Consider the \( r\)-tangle \( Q'T'=Q'' \), where \( Q' = VT \), \( T \) being the word in the moves \( A \) and \( B \) obtained translating \( A \) to \( B \) and \( B \) to \( A \) from the word \( T \), and \( V \) is equal to \( R \) or to \( E \), according to \( V \). The double points of \( TT' \) are either all positive or all negative, and, since the move \( R \) is equivalent to a rotation changing the sign of all double points, the \( r\)-tangle \( Q'T' \) is alternating. It is therefore evident that \( Q'T'=Q'' \). But we have \( Q'T'=Q'' \), since \( \mu(Q') = \mu(Q) \). Therefore \( Q'I'=Q'' \). This contradicts the hypothesis that the move \( Q \) is isotopy equivalent to the identity. If \( T \) ends with \( B^b \) for some non zero \( b \in \mathbb{Z} \), then we consider the analogous move \( Q' \) and the \( r\)-tangle \( Q'I'' \), getting the relations \( Q'I'' \sim I'' \) and \( Q'I'' \sim Q'I'' \), from which \( Q'I'' \sim Q'I'' \), contradicting the hypothesis. This concludes the proof of the theorem.

Let us denote \( \Gamma_0 \) the \( r\)-tangle \( \Gamma_\rightarrow \). Observe that

\[
\Gamma_\rightarrow \sim B^{-1}A\Gamma_0, \quad \text{or} \quad \Gamma_\rightarrow \sim A^{-1}B\Gamma_0
\]

We obtain the following corollary:

**Corollary 2.4.** The space of the isotopy classes of rational tangles coincides with the orbit of \( \Gamma_0 \) under the action of the group \( \mathcal{T} \).

**Proof.** Every \( r\)-tangle is isotopy equivalent to an \( r\)-tangle reached from \( \Gamma_0 \) or \( \Gamma_\rightarrow \) by a series of twists \( X_i \). By the corollary (2.1) to generate all the isotopy classes in \( \mathcal{R} \) the moves \( A, B \) and their inverses are sufficient. The same \( \Gamma_\rightarrow \) can be obtained from \( \Gamma_0 \) by \( A, B \) and their inverses, by Eq. (1). Since every word in \( A, B \) and their inverses is an element of \( \mathcal{T} \), and all classes are representable starting from \( \Gamma_0 \), the corollary follows.

3. **Rational tangles and rational numbers**

We denote by \( \bar{\Gamma} \) the isotopy class of the \( r\)-tangle \( \Gamma \), and by \( \bar{\mathcal{R}} \) the space of the isotopy classes of \( r\)-tangles.
In this section we prove that every class of r-tangles different from that of $\Gamma^{||}$ is uniquely represented by a rational number, and, vice versa, that every rational number represents uniquely a class of rational tangles.

Let

$$\tilde{Q} := \{(p, q) \in \mathbb{Z}^2 \setminus (0, 0) \mod (p, q) \sim (rp, rq), \forall r \in \mathbb{Z} \setminus 0\}.$$ 

Let $v_0 = (0, 1)$ and $v_\infty = (1, 0)$. Observe that $(0, r)$, for all nonzero $r \in \mathbb{Z}$, represent the same element as $v_0$ in $\tilde{Q}$, and $(r, 0)$, for all nonzero $r \in \mathbb{Z}$, represent the same element as $v_\infty$.

**Theorem 3.1.** The map $\rho : \tilde{R} \to \tilde{Q}$ defined in this way:

$$\rho(\tilde{\Gamma}_0) = v_0;$$

if $\Gamma = Q\Gamma_0$, then

$$\rho(\tilde{\Gamma}) = \mu(Q)v_0,$$

associates with every isotopy class of r-tangles one and only one element of $\tilde{Q}$.

Before proving this theorem, we make some observations.

**Remark 3.2.** The r-tangles $\Gamma_0$ and $\Gamma^{||}$ possess the following invariances: for every $m \in \mathbb{Z}$

$$B^m\Gamma_0 \sim \Gamma_0, \quad A^m\Gamma^{||} \sim \Gamma^{||}.$$ 

No other element of $\mathcal{I}$ keeps invariant $\Gamma_0$ nor $\Gamma^{||}$.

Observe that, from Eq. (10) and Remark 3.2, it follows also that

$$R\Gamma_0 \sim \Gamma^{||}, \quad R\Gamma^{||} \sim \Gamma_0.$$

**Lemma 3.3.** There is no element of $\mathcal{I}$ keeping invariant a class of $\tilde{R}$ different from $\tilde{\Gamma}_0$ or $\tilde{\Gamma}^{||}$.

**Proof.** Suppose the equation $W\Gamma \sim \Gamma$ be fulfilled by some r-tangle $\Gamma$ and some word $W \neq E$. Let $\Gamma = Q\Gamma_0$ for some $Q \in \mathcal{I}$. By hypothesis we have $WQ\Gamma_0 \sim \Gamma_0$, and, by consequence, $Q^{-1}WQ\Gamma_0 \sim \Gamma_0$. By Remark 3.2, we obtain that $Q^{-1}WQ = B^{k}$, for some integer $k$. The last equation is fulfilled either if both $W$ and $Q$ are powers of $B$, and hence $\Gamma \sim \Gamma_0$; or if $Q = A^2R$, and hence, by Eq. (10), $\Gamma \sim \Gamma^{||}$, and $W = A^{-k}$, being $RA^{-j}A^{-k}A^{2}R = B^k$.

**Lemma 3.4.** Every class in $\tilde{R}$ is uniquely represented by an element of $\mathcal{I}$, modulo multiplication at right by $B^m$, $m \in \mathbb{Z}$.

**Proof.** Every class of r-tangles can be represented as $Q\Gamma_0$, for some $Q \in \mathcal{I}$, by Corollary 2.4. With the r-tangle $\Gamma = Q\Gamma_0$ we associate the element $Q \in \mathcal{I}$. By Remark 3.2, $\Gamma = QB^m\Gamma_0$, therefore if we associate $Q$ to $\Gamma$, we may as well associate $QB^m$, for all $m \in \mathbb{Z}$. On the other hand, if $Q' \neq QB^m$ for some $m$, then $Q'\Gamma_0 \sim \Gamma_0$, by the same remark.

**Proof of Theorem 3.1.** We observe firstly that the map $\rho$ respects the invariances of Remark 3.2.

Indeed, $\rho(B^m\tilde{\Gamma}_0) = B^m v_0 = (\frac{1}{m} \ 0)(\tilde{v}_0) = (0_m) = m v_0 \sim v_0 = \rho(\tilde{\Gamma}_0)$. Moreover, using Eq. (10),

$$\rho(\tilde{\Gamma}^{||}) = B^{-1} \rho(\tilde{\Gamma}) v_0 = (\frac{1}{-1} \ 1)(\tilde{v}_0) = (\tilde{v}_0) = v_\infty.$$ 

Therefore $\rho(A^n\tilde{\Gamma}^{||}) = A^n B^{-1} \rho(\tilde{\Gamma}) v_0 = A^n v_\infty = (\frac{1}{1} \ m)(\tilde{v}_0) = (\tilde{v}_0) = m v_\infty \sim v_\infty$, i.e., $\rho(A^n\tilde{\Gamma}^{||}) = \rho(\tilde{\Gamma}^{||})$.

We have to prove that for every pair of classes $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ in $\tilde{R}$

$$\tilde{\Gamma} \sim \tilde{\Gamma}' \iff \rho(\tilde{\Gamma}) = \rho(\tilde{\Gamma}').$$

**Proof of $\Rightarrow$.** By Lemma 3.4 if $\tilde{\Gamma} = \tilde{\Gamma}'$ then $\Gamma = Q\Gamma_0$ and $\Gamma' = Q'\Gamma_0$ with $Q' = QB^m$. If $\rho(Q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we obtain $\rho(\tilde{\Gamma}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} a + mb & b \\ c + md & d \end{pmatrix}$. Therefore $\rho(\tilde{\Gamma}) = \rho(\tilde{\Gamma}') = \mu(Q)v_0 = \begin{pmatrix} b \\ d \end{pmatrix}$ and $\rho(\tilde{\Gamma}') = \mu(Q')v_0 = \begin{pmatrix} b \\ d \end{pmatrix}$. 


Proof. Let \( \Gamma \sim Q\Gamma_0 \), \( \Gamma' \sim Q'\Gamma_0 \), \( \mu(Q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), \( \mu(Q') = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \). We suppose that 
\[ \rho(\Gamma) = \rho(\Gamma'), \] i.e., \((b, d) \sim (b', d') \in \mathbb{Q}.\] This equality implies \( b = b' \) and \( d = d' \), because \( b, d \) as well as \( b', d' \) are relatively prime, constituting a column of a discriminant one matrix. The matrices \( Q := \mu(Q) \) and \( Q' = \mu(Q') \) may differ for the first column, and they have the same unit discriminant. We obtain therefore \( a' = a + bk \) and \( c' = c + dk \) for some integer \( k \), and this implies \( Q' = QB^k \). We thus obtain \( \Gamma' \sim Q'\Gamma_0 \sim QB^k\Gamma_0 \sim Q\Gamma_0 \), and hence \( \Gamma' \sim \Gamma. \) \( \square \)

Consider the map \( t \) sending every element of \( \mathbb{Q} \) with \( q \neq 0 \) to a rational number: \( t : (p, q) \mapsto p/q \). Every rational number is the image by \( t \) of one and only one element of \( \mathbb{Q} \). We will write \( p/q = t(v) \), where \( v = (p, q) \in \mathbb{Q} \). Evidently \(mv = (mp, mq) \sim v \) in \( \mathbb{Q} \).

**Corollary 3.5.** The map \( \rho \circ t : \tilde{\mathbb{R}} \to \mathbb{Q} \) associates with every isotopy class of \( r \)-tangles different from \( \tilde{\Gamma} \) one and only one rational number.

**Proof.** By Theorem 3.1, the map \( \rho \) associates with every isotopy class of \( r \)-tangles an element of \( \tilde{\mathbb{Q}} \). The map \( t \) associates with every element of \( \mathbb{Q} \), \( q \neq 0 \), a rational number. The element \((r, 0) \in \tilde{\mathbb{Q}} \) is the image by \( \rho \) of the class of \( \Gamma_{||} \), by Eq. (10). The corollary follows.

**Corollary 3.6.** The map \((\rho \circ t)^{-1} : \mathbb{Q} \to \tilde{\mathbb{R}} \) associates with every rational number one and only one isotopy class of \( r \)-tangles.

**Proof.** With the rational number \( p/q \), \( t^{-1} \) associates \( v = (p, q) \in \mathbb{Q} \). Since \( q \neq 0 \), and \( p \) and \( q \) are coprime, the pair \((p, q)\) defines a matrix \( Q \in \text{PSL}(2, \mathbb{Z}) \) such that \( Qv_0 = v \), up to a right factor equal to \( B^m \), for some \( m \in \mathbb{Z} \) as we have seen in the proof of Eq. (12). Therefore also the element \( Q \in \mathcal{I} \) such that \( \mu(Q) = Q \) is defined up to a right factor equal to \( B^m \), for some \( m \in \mathbb{Z} \). But all elements \( QB^m \) define the same class \(\mathbb{Q}\Gamma_0 \), therefore the image of \( v \) by \( \rho^{-1} \) is well defined and unique by Remark 5.2. \( \square \)

We obtain as corollary that every \( r \)-tangle is alternating.

**Corollary 3.7.** A rational tangle different from \( \Gamma_{||} \) is equivalent to a tangle obtained by \( \Gamma_0 \) either by sole positive twists or sole negative twists.

**Proof.** We write \( \Gamma \sim Q\Gamma_0 \). Hence we consider \( Q = \mu(Q) \). \( Q \) is different from \( A^mS \) since \( \Gamma \sim \Gamma_{||} \), according to Eq. (9), Remark 5.2, and Eq. (10). By Lemma 2.3 and Theorem 2.2, \( Q \) can be written as \( VT \), where \( T \) is a word either in \( A \) and \( B \) or in \( A^{-1} \) and \( B^{-1} \), and \( V = R \) or \( V = E \). If \( V = E \), the proof is finished, since the tangle \( T\Gamma \) is alternating. If \( V = R \), then we apply to every generator in the word \( T \) the equations obtained by the relations in \( \mathcal{I} \) corresponding to the relations (8) in \( \text{PSL}(2, \mathbb{Z}) \). Therefore we obtain \( RT = TR \), where \( T \) is obtained by \( T \) exchanging \( B \) with \( A^{-1} \) and \( A \) with \( B^{-1} \). Hence \( \Gamma \sim T\Gamma_0 \sim \Gamma_{||} \). Since \( \Gamma_{||} \sim A^m\Gamma_{||} \), we can write \( \Gamma \sim T\Gamma_{||} \), where \( T' \) is a word whose last element is \( B^k \), with \( k \neq 0 \). If \( k \leq 1 \), and hence \( T' \) is a word in \( A^{-1} \) and \( B^{-1} \), we use the relation, from Eq. (9), \( B^{-1}\Gamma_{||} \sim A^{-1}\Gamma_0 \), to obtain the relation \( T\Gamma_{||} = T'\Gamma_0 \), where \( T'' \) is obtained substituting the last \( B^{-1} \) with \( A^{-1} \). In this way \( \Gamma \sim T''\Gamma_0 \) is obtained from \( \Gamma_0 \) by sole negative twists. If \( k \geq 1 \), and hence \( T' \) is a word in \( A \) and \( B \), we use the relation, from Eq. (10), \( B\Gamma_{||} \sim A\Gamma_0 \), to obtain the relation \( T\Gamma_{||} \sim T''\Gamma_0 \), where \( T'' \) is obtained substituting the last \( B \) with \( A \). Therefore \( \Gamma \sim T''\Gamma_0 \) is obtained from \( \Gamma_0 \) by sole positive twists. \( \square \)

### 3.1. Continued fraction procedure.

We prove now that the continued fraction procedure allows to associate with every rational number one and only one alternating rational tangle.

Let \( p, q \) and \( a_i \) \((i = 1 \ldots n)\) represent natural numbers. A positive rational number has a unique representation by a continued fraction with positive elements \( a_i \):

\[
\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}.
\]
As a consequence, a negative rational number has a unique representation by a continued fraction with negative elements $-a_i$:

$$\frac{-p}{q} = -a_1 - \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots + \frac{1}{a_n}}}} = (-a_1) + (-a_2) + (-a_3) + \ldots + (-a_n).$$

A continued fraction with positive elements is denoted by

$$\frac{p}{q} = [a_1; a_2, \ldots, a_n].$$

Note that $a_1$ may be zero, but all $a_i$, for $i > 1$, are positive, and $a_n$ is bigger or equal to 2. Therefore, we can always write an equivalent expression for $\frac{p}{q}$:

$$\frac{p}{q} = [a_1, a_2, a_3, \ldots, a_{n-1}, 1].$$

**Example:** $7/3 = [2, 3] = [2, 2, 1]$, since

$$2 + \frac{1}{3} = 2 + \frac{1}{2 + \frac{1}{3}}.$$

If the fraction is is negative, we will write evidently $\frac{-p}{q} = [-a_1, -a_2, -a_3, \ldots, -a_n + 1, -1]$.

In this way we may always suppose $n$ be odd, because if $n$ is even, then we decrease $a_n$ by one and we add the $(n + 1)$-th element equal to 1 (or $-1$). We call such a continued fraction odd continued fraction.

**Proposition 3.8.** Let the odd continued fraction of $\frac{p}{q}$ be

$$[a_1; a_2, a_3, \ldots, a_n],$$

where the $a_i$ are either all positive or all negative but $a_1$ that may be zero. Then $\frac{p}{q} = t(\rho(\widetilde{Q_{\Gamma_0}}))$, where $Q_{\Gamma_0}$ is the alternating r-tangle obtained applying to $\Gamma_0$ the sequence of moves

$$A^{a_1}B^{a_2}A^{a_3} \ldots B^{a_{n-1}}A^{a_n}.$$

**Proof.** We define the map $\tau_Q : Q \to Q$ in the following way: for every $z = \frac{p}{q} \in \mathbb{Q}$

$$\tau_Q(z) = t(\rho(\widetilde{Q_{\Gamma_0}})).$$

Evidently, $t(\rho(z)) = t(\rho(\widetilde{\gamma}))$. For every $\alpha \in \mathbb{Z}$ we have:

$$\tau_{A^\alpha}(z) = z + \alpha, \quad \tau_{B^\alpha}(z) = \frac{1}{\alpha + \frac{1}{z}}.$$

If $\alpha = 0$, we obtain $\tau_E(z) = z$. Moreover, if $Q'' = QQ'$, then $\tau_{Q''}(z) = \tau_Q(\tau_{Q'}(z))$. Therefore, if

$$Q = A^{a_1}B^{a_2}A^{a_3} \ldots B^{a_{n-1}}A^{a_n},$$

then

$$\tau_Q(0) = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots + \frac{1}{a_n}}}}.$$

We thus associate with the odd continued fraction [13] of $\frac{p}{q}$ the sequence of moves

$$Q = A^{a_1}B^{a_2}A^{a_3} \ldots B^{a_{n-1}}A^{a_n}$$

and the alternating tangle $Q_{\Gamma_0}$. By construction, $\frac{p}{q} = t(\rho(\widetilde{Q_{\Gamma_0}})).$ □
4. RATIONAL TANGLES AND THE BRAID GROUP $B_3$

It is well known that the modular group is isomorphic to $\tilde{B}_3 := B_3/\langle \omega \rangle$ where $B_3$ is the group of braids with 3 strands, generated by $\sigma_1$ and $\sigma_2$:

and $\langle \omega \rangle$ is the relation

$$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 = 1.$$

We describe here explicitly the isomorphism between $\tilde{B}_3$ and $\mathcal{T}$, giving its topological meaning.

**Theorem 4.1.** The following construction associates with every element of $\mathcal{T}$ an element of $\tilde{B}_3$ and vice-versa.

**Proof.** The construction can be described as follows: the endpoints 4, 3 and 2 of the r-tangle $\Gamma$ become, respectively, the upper endpoints of the strands of the braid, and the endpoints 4, 3 and 2 of the r-tangle $Q\Gamma$ become, respectively, the lower endpoints of the strands of the braid.

The topological resolution of each double point remains unchanged in the transformation, so that we obtain:

$$A \leftrightarrow \sigma_1 \quad B \leftrightarrow \sigma_2^{-1}.$$
To any word of $\mathcal{T}$ in $\mathcal{A}$ and $\mathcal{B}$ and their inverses there corresponds therefore the word in $B_3$ where the generators are translated according to (15) and are put in the inverse order, if the order from left to right of the generators in a word of $B_3$ is (as usually) interpreted as their order from top to bottom in the braid.

The relation $\langle \omega \rangle$ is therefore the translation of relation: $B^{-1}AB^{-1}AB^{-1}A = E$, which is equivalent to (16).

\[
\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sim \rightarrow
\]

\textbf{Remark.} Remember that in $\mathcal{R}$ the topological meaning of relation (6) is that $R^2 = E$, i.e., rotating by $2\pi$ an r-tangle about the diagonal containing the endpoints 1 and 3, its isotopy class does not change.

For the braids the element $\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$ is as well equivalent to a rotation by $2\pi$ about the central vertical axis of the segment containing the lower endpoints (or of that containing the upper endpoints) of the strands. This rotation changes the isotopy class of the braid. Therefore $\tilde{B}^3$ can be interpreted as the group of braids such that the segments containing the endpoints are allowed to rotate independently by $2\pi$.

\textbf{Remark.} The construction of Theorem 4.1 is similar to that showed in [2], but we haven’t any ambiguity problem, since we deal with $\mathcal{T}$, and not with rational tangles.

\textbf{References}

[1] Goldman J.R., Kauffman L.H. \textit{Rational Tangles}, Advances in Applied Mathematics, 18 (1997), 300-332

[2] Kauffman L.H., Lambropoulou S. On the classification of rational tangles. arXiv:math/0311499