Completely positive compact operators on non-commutative symmetric spaces

P. G. Dodds · B. de Pagter

Received: 16 February 2010 / Accepted: 19 June 2010 / Published online: 23 July 2010
© The Author(s) 2010. This article is published with open access at Springerlink.com

Abstract Under natural conditions, it is shown that a completely positive operator between two non-commutative symmetric spaces of \( \tau \)-measurable operators which is dominated in the sense of complete positivity by a completely positive compact operator is itself compact.

Keywords Completely positive operators · Compact operators · Non-commutative symmetric spaces

Mathematics Subject Classification (2000) Primary 46L52;
Secondary 46E30 · 47B60

1 Introduction

It was shown in [9] that each positive operator from the Banach lattice \( E \) to the Banach lattice \( F \), which is dominated by a positive compact operator, is itself compact, provided the norms on \( F \) and the Banach dual \( E^* \) are order continuous. Special cases of particular interest occur when \( E \) is an abstract \( M \)-space and \( F \) is an abstract \( L \)-space or the cases that \( E = L^p \), \( 1 < p \leq \infty \) and \( F = L^q \), \( 1 \leq q < \infty \). Full details of this theorem, and its subsequent development may be found in the monographs [1,13,21].

P. G. Dodds (✉)
School of CSEM, Flinders University, GPO Box 2100, Adelaide 5001, Australia
e-mail: peter@csem.flinders.edu.au

B. de Pagter
Department of Mathematics, Faculty of EEMCS, Delft University of Technology,
P.O. Box 5031, 2600 GA Delft, The Netherlands
e-mail: b.depagter@tudelft.nl
In the setting of non-commutative $L^p$-spaces, a non-commutative version of this (so-called) compact majorisation theorem was given by Neuhardt [15, 16]. In this setting, key technical difficulties arise from the fact that there are no non-commutative counterparts to the arguments of [9] based on properties of order-bounded operators between Banach lattices and technical lattice arguments characterising approximately order-bounded sets in Banach lattices. The key new idea introduced by Neuhardt was to consider domination in the sense of complete positivity, a notion which goes back to Stinespring [18], and to replace arguments in the Banach lattice case based on formulae for the infimum of positive linear operators between Banach lattices by representation theorems for $C^*$-algebras and linear functionals, using the assumption of complete positivity.

The purpose of the present paper is to place Neuhardt’s theorem within the more general framework of symmetric spaces of $\tau$-measurable operators affiliated with a semi-finite von Neumann algebra. The principal result of the paper (Theorem 5.5) is that if $E, F$ are strongly symmetric spaces on the positive half-line, if $0 \leq S, T : E(\tau) \to F(\sigma)$ are completely positive operators with $S$ dominated by $T$ in the sense of complete positivity, then $S$ is compact provided $T$ is compact and order continuous and the norms on $F$ and the Köthe dual $E^*$ are order continuous. This result uses Neuhardt’s theorem in the special case that $E = L^\infty$ and $F = L^1$ together with a characterisation of compact subsets of non-commutative spaces with order continuous norm (Proposition 4.6) in terms of sets of uniformly absolutely continuous norm.

2 Notation and preliminaries

Throughout this paper $\mathcal{M}$ will denote a von Neumann algebra on some Hilbert space $\mathcal{H}$. Unless otherwise stated, it will be assumed throughout that $\mathcal{M}$ is equipped with a fixed semifinite faithful normal trace $\tau$. For standard facts concerning von Neumann algebras, we refer to [19]. The identity in $\mathcal{M}$ is denoted by $1$ and we denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of all (self-adjoint) projections in $\mathcal{M}$. A linear operator $x : D(x) \to \mathcal{H}$, with domain $D(x) \subseteq \mathcal{H}$, is said to be affiliated with $\mathcal{M}$ if $ux = xu$ for all unitary $u$ in the commutant $\mathcal{M}'$ of $\mathcal{M}$. For any self-adjoint operator $x$ on $\mathcal{H}$, its spectral measure is denoted by $e^x$. A self-adjoint operator $x$ is affiliated with $\mathcal{M}$ if and only if $e^x(B) \in \mathcal{P}(\mathcal{M})$ for any Borel set $B \subseteq \mathbb{R}$. The closed and densely defined operator $x$, affiliated with $\mathcal{M}$, is called $\tau$-measurable if and only if there exists a number $s \geq 0$ such that $\tau(e^{[x]}(s, \infty)) < \infty$. The collection of all $\tau$-measurable operators is denoted by $S(\tau)$. With the sum and product defined as the respective closures of the algebraic sum and product, it is well known that $S(\tau)$ is a *-algebra. For $\epsilon, \delta > 0$, we denote by $V(\epsilon, \delta)$ the set of all $x \in S(\tau)$ for which there exists an orthogonal projection $p \in \mathcal{P}(\mathcal{M})$ such that $p(\mathcal{H}) \subseteq D(x)$, $\|x p\|_{\mathcal{B}(\mathcal{H})} \leq \epsilon$ and $\tau(1 - p) \leq \delta$. The sets $\{V(\epsilon, \delta) : \epsilon, \delta > 0\}$ form a base at 0 for a metrizable Hausdorff topology on $S(\tau)$, which is called the measure topology. Equipped with this topology, $S(\tau)$ is a complete topological *-algebra. These facts and their proofs can be found in the papers [14, 20].
For \( x \in S(\tau) \), the singular value function \( \mu(\cdot; x) = \mu(\cdot; |x|) \) is defined by

\[
\mu(t; x) = \inf \left\{ s \geq 0 : \tau \left( e^{s|x|} (s, \infty) \right) \leq t \right\}, \quad t \geq 0.
\]

It follows directly that the singular value function \( \mu(x) \) is a decreasing, right-continuous function on the positive half-line \([0, \infty)\). Moreover, \( \mu(uv) \leq \|u\| \|v\| \mu(x) \) for all \( u, v \in \mathcal{M} \) and \( x \in S(\tau) \) and \( \mu(f(x)) = f(\mu(x)) \) whenever \( 0 \leq x \in S(\tau) \) and \( f \) is an increasing continuous function on \([0, \infty)\) which satisfies \( f(0) = 0 \).

It should be observed that a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( S(\tau) \) converges to zero for the measure topology if and only if \( \mu(\cdot; x_n) \to 0 \) as \( n \to \infty \) for all \( t \geq 0 \).

If \( m \) denotes Lebesgue measure on the semiaxis \([0, \infty)\), and if we consider \( L^\infty(m) \) as an Abelian von Neumann algebra acting via multiplication on the Hilbert space \( \mathcal{H} = L^2(m) \), with the trace given by integration with respect to \( m \), then \( S(m) \) consists of all measurable functions on \([0, \infty)\) which are bounded except on a set of finite measure, and for \( f \in S(m) \), the generalized singular value function \( \mu(f) \) is precisely the classical decreasing rearrangement of the function \( |f| \), which is usually denoted by \( f^* \). In this setting, convergence for the measure topology coincides with the usual notion of convergence in measure. If \( \mathcal{M} = \mathcal{L}(\mathcal{H}) \) and \( \tau \) is the standard trace, then \( S(\tau) = \mathcal{M} \), the measure topology coincides with the operator norm topology.

The real vector space \( S_h(\tau) = \{ x \in S(\tau) : x = x^* \} \) is a partially ordered vector space with the ordering defined by \( x \geq 0 \) if and only if \( \langle x \xi, \xi \rangle \geq 0 \) for all \( \xi \in \mathcal{D}(x) \). The positive cone in \( S_h(\tau) \) will be denoted by \( S(\tau)_+ \). If \( 0 \leq x_\alpha \uparrow a \leq x \) holds in \( S(\tau)_+ \), then \( \sup_{\alpha} x_\alpha \) exists in \( S(\tau)_+ \). The trace \( \tau \) extends to \( S(\tau)_+ \) as a non-negative extended real-valued functional which is positively homogeneous, additive, unitarily invariant and normal. This extension is given by \( \tau(x) = \int_0^\infty \mu(t; x) \, dt, \quad x \in S(\tau)_+ \), and satisfies \( \tau(x^*x) = \tau(xx^*) \) for all \( x \in S(\tau) \). It should be observed that if \( f \) is an increasing continuous function on \([0, \infty)\) satisfying \( f(0) = 0 \), then

\[
\tau(f(|x|)) = \int_0^\infty \mu(t; f(|x|)) \, dt = \int_0^\infty f(\mu(t; x)) \, dt \quad (2.1)
\]

for all \( x \in S(\tau) \).

If \( 1 \leq p < \infty \), we set \( L^p(\tau) = \{ x \in S(\tau) : \tau(|x|^p) < \infty \} \). Note that it follows from (2.1) that \( L^p(\tau) \) is also given by \( L^p(\tau) = \{ x \in S(\tau) : \mu(x) \in L^p(m) \} \), where \( m \) denotes Lebesgue measure on \([0, \infty)\). The space \( L^p(\tau) \) is a linear subspace of \( S(\tau) \) and the functional \( x \mapsto \|x\|_{L^p(\tau)} = \tau(|x|^p)^{1/p}, \quad x \in L^p(\tau) \), is a norm. It should be observed that \( \|x\|_{L^p(\tau)} = \|\mu(x)\|_{L^p(m)} \) for all \( x \in L^p(\tau) \). Equipped with this norm, \( L^p(\tau) \) is a Banach space. In this setting, we also have that \( L^\infty(\tau) = \mathcal{M} \).

In the commutative setting, the spaces \( L^p(\tau) \) are the familiar Lebesgue spaces. In the special case that \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) equipped with standard trace, the corresponding \( L^p \)-spaces are the Schatten classes \( \mathcal{S}_p \). As is well known, the space \( L^1(\tau) \) may be identified with the von Neumann algebra predual of \( \mathcal{M} \) with respect to trace duality. If \( x \in S(\tau) \), then the projection onto the closure of the range of \( |x| \) is called the support of \( x \) and is denoted by \( s(x) \). We set \( \mathcal{F}(\tau) = \{ x \in \mathcal{M} : \tau(s(x)) < \infty \} \).
If \((\mathcal{N}, \sigma)\) is a semifinite von Neumann algebra, possibly on some different Hilbert space, if \(x \in S(\tau)\) and \(y \in S(\sigma)\), then \(x\) is said to be submajorised by \(y\) (in the sense of Hardy, Littlewood and Polya) if and only if \(\int_0^t \mu(s; x)ds \leq \int_0^t \mu(s; y)ds\) for all \(t \geq 0\). We write \(x \ll y\), or equivalently, \(\mu(x) \ll \mu(y)\) (with respect to Lebesgue measure on \((0, \infty)\)).

We shall need, in particular, the submajorization inequality \(\mu(xy) \ll \mu(x)\mu(y)\) whenever \(x, y \in S(\tau)\). A proof of this inequality may be found in [3] in the case that \(M\) is non-atomic, and this latter assumption may be removed by a standard argument (see [10]). If this inequality is combined with an inequality of Hardy (see [12, Chapter II.2.18]), then it is easily seen that if \(x, y, z \in S(\tau)\), then \(\mu(xyz) \ll \mu(x)\mu(y)\mu(z)\).

For further details and proofs, we refer the reader to [4, 6, 10].

A linear subspace \(E \subseteq S(\tau)\) is called an \(\mathcal{M}\)-bimodule if \(uxv \in E\) whenever \(x \in E\) and \(u, v \in \mathcal{M}\). If the \(\mathcal{M}\)-bimodule \(E\) is equipped with a norm \(\|\cdot\|_E\) which satisfies \(\|uxv\|_E \leq \|u\|_{B(H)} \|v\|_{B(H)} \|x\|_E\), \(x \in E, u, v \in \mathcal{M}\), then \(E\) is called a normed \(\mathcal{M}\)-bimodule (of \(\tau\)-measurable operators). If \(E \subseteq S(\tau)\) is an \(\mathcal{M}\)-bimodule, and if \(x \in S(\tau)\), then \(x \in E \iff |x| \in E \iff x^* \in E\); and if \(y \in E\) is such that \(|x| \leq |y|\), then \(x \in E\). Further, if \(E\) is a normed \(\mathcal{M}\)-bimodule, then \(\|x\|_E = \|x\|_E\) and \(\|x^*\|_E = \|x\|_E\) for all \(x \in E\) and \(\|x\|_E \leq \|y\|_E\) whenever \(x, y \in E\) satisfy \(|x| \leq |y|\). A normed \(\mathcal{M}\)-bimodule which is a Banach space is called a Banach \(\mathcal{M}\)-bimodule. It is easily seen that \(\mathcal{F}(\tau)\) is an \(\mathcal{M}\)-bimodule and that each of the Banach spaces \(\mathcal{M}, L^1(\tau), L^1(\tau) \cap \mathcal{M}, L^1(\tau) + \mathcal{M}\) are Banach \(\mathcal{M}\)-bimodules. If \(E \subseteq S(\tau)\) is a normed \(\mathcal{M}\)-bimodule, then \(E\) will be called symmetrically normed if \(x \in E, y \in S(\tau)\) and \(\mu(y) \leq \mu(x)\) imply that \(y \in E\) and \(\|y\|_E \leq \|x\|_E\); strongly symmetrically normed if \(E\) is symmetrically normed and its norm has the additional property that \(\|y\|_E \leq \|x\|_E\) whenever \(x, y \in E\) satisfy \(y \ll x\); fully symmetric if \(E\) is symmetrically normed and if, whenever \(x \in E\) and \(y \in S(\tau)\) satisfy \(y \ll x\) then \(x \in E\) and \(\|y\|_E \leq \|x\|_E\).

If a strongly symmetrically normed space is Banach, then it will be simply called a strongly symmetric space. It may be shown that any strongly symmetrically normed space, with the property that \(\sqrt{\chi_{E} s(x)} = 1\) (which will always be assumed) satisfies \(\mathcal{F}(\tau) \subseteq E \subseteq L^1(\tau) + \mathcal{M}\), with continuous inclusions (where \(\mathcal{F}(\tau)\) is equipped with the \(L^1 \cap L^\infty\)-norm). If, in addition, \(E\) is a Banach space, then \(L^1(\tau) \cap \mathcal{M} \subseteq E\), with continuous embedding. If \(E \subseteq S(\tau)\) is a strongly symmetrically normed \(\mathcal{M}\)-bimodule, then the embedding of \(E\) into \(S(\tau)\) is continuous from the norm topology of \(E\) to the measure topology on \(S(\tau)\). A wide class of strongly symmetrically normed \(\mathcal{M}\)-bimodules may be constructed as follows. If \(E \subseteq S(m)\) is a strongly symmetrically normed space, set \(E(\tau) = \{x \in S(\tau) : \mu \in \mathcal{M}, \mu(x) \in E\}\), \(\|x\|_{E(\tau)} := \|\mu(x)\|_E\). It may be shown as in [4] (see [5]) that \((E(\tau), \|\cdot\|_{E(\tau)})\) is a strongly symmetrically normed \(\mathcal{M}\)-bimodule and is a Banach \(\mathcal{M}\)-bimodule if \(E\) is a Banach space.

If \(E \subseteq S(\tau)\) is a strongly symmetrically normed \(\mathcal{M}\)-bimodule, set

\[E^\times = \{y \in S(\tau) : \sup \{\tau(|xy|) : x \in E, \|x\|_E \leq 1\} < \infty\}\]

and

\[\|y\|_{E^\times} = \sup \{\tau(|xy|) : x \in E, \|x\|_E \leq 1\}, \quad y \in E^\times.\]
If \( y \in S(\tau) \), then

\[
y \in E^\times \iff \sup \left\{ \int_{(0,\infty)} \mu(x)\mu(y)dm : x \in E, \|x\|_E \leq 1 \right\} < \infty.
\]

in which case, the latter quantity is equal to \( \|y\|_{E^\times} \). The space \( (E^\times, \| \cdot \|_{E^\times}) \) is a normed Banach \( \mathcal{M} \)-bimodule. If \( y \in E^\times \), define \( \phi_y : E \to \mathbb{C} \) by \( \phi_y(x) = \tau(xy), x \in E \). The Banach \( \mathcal{M} \)-bimodule \( E^\times \) has the following properties (see [6]): (i) \( \phi_y \in E^* \) and the map \( y \to \phi_y \in E^* \) is an isometry; (ii) \( E^\times \) has the Fatou property, that is, \( 0 \leq y_\alpha \uparrow \alpha \subseteq E^\times, \sup_\alpha \| y_\alpha \|_{E^\times} < \infty \implies y = \sup_\alpha y_\alpha \) exists in \( E^\times \) and \( \|y\|_{E^\times} = \sup_\alpha \| y_\alpha \|_{E^\times} \); (iii) \( E^\times \) is fully symmetric; (iv) If \( E \subseteq S(m) \) is a strongly symmetrically normed space, then \( E^\times(\tau) = E(\tau)^\times \).

3 Completely positive mappings

Let \( A \) be a \( C^* \)-algebra of operators acting in some Hilbert space \( \mathcal{H} \). It is assumed that the identity operator \( 1 \) is an element of \( A \). Denote by \( M_n(A) \) the set of all \( n \times n \)-matrices \( a = [a_{ij}]_{i,j=1}^n \) with entries from \( A \). With the obvious definitions of addition, scalar multiplication and matrix multiplication, together with the \( * \)-operation defined by setting \( (a^*)_i = a_i^* \), the set \( M_n(A) \) is an involutive algebra. If \( a = [a_{ij}]_{i,j=1}^n \in M_n(A) \), then \( a \) induces, in the obvious manner, a bounded linear operator on the \( n \)-fold direct product Hilbert space \( \mathcal{H}^n = \bigoplus_{i=1}^n \mathcal{H} \). Equipped with the corresponding operator norm, \( M_n(A) \) is then a \( C^* \)-algebra with identity. The element \( a \in M_n(A) \) is positive if and only if there exists \( b \in M_n(A) \) such that \( a = b^*b \). We denote by \( e_{i,j}, 1 \leq i, j \leq n \), the usual matrix basis for \( M_n(\mathbb{C}) \). Note that if \( a = [a_{ij}]_{i,j=1}^n \in M_n(A) \), then \( a = \sum_{i,j=1}^n a_{ij} \otimes e_{ij} \). It is shown in [19] that an element \( a = [a_{ij}]_{i,j=1}^n \) of \( M_n(A) \) is positive if and only if it is a sum of matrices of the form \( [a_{ij}^*a_{ij}]_{i,j=1}^n \) with \( a_1, \ldots, a_n \in A \).

We suppose that \( (\mathcal{M}, \tau) \) is a semifinite von Neumann algebra acting in some Hilbert space \( \mathcal{H} \). For each \( n \in \mathbb{N} \), we let \( M_n = \mathcal{M} \overline{\otimes} M_n(\mathbb{C}) \) be the von Neumann algebra tensor product acting in the tensor product Hilbert space \( \mathcal{H} \otimes \mathbb{C}^n \) and with the tensor product trace \( \tau_n := \tau \otimes \text{tr}_n \). Here \( \text{tr}_n \) denotes the standard matrix trace. As is well known (see, for example [19, Chapter IV, Proposition 1.6]), the von Neumann algebra tensor product \( M_n \) coincides with the algebraic tensor product \( \mathcal{M} \otimes M_n(\mathbb{C}) \) and may be identified with the space \( M_n(\mathcal{M}) \) of all \( n \times n \)-matrices \( [x_{ij}]_{i,j=1}^n \) with values in \( \mathcal{M} \).

Given \( x \in S(\tau) \) and \( y \in M_n(\mathbb{C}) \), the linear operator \( x \otimes y : \mathcal{D}(x) \otimes \mathbb{C}^n \to \mathcal{H} \otimes \mathbb{C}^n \) is defined by setting \( (x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta \) for all \( \xi \in \mathcal{D}(x) \) and \( \eta \in \mathbb{C}^n \). The operator \( x \otimes y \) is pre-closed, affiliated with \( \mathcal{M} \otimes M_n(\mathbb{C}) \) and the domain \( \mathcal{D}(x) \otimes \mathbb{C}^n \) is \( \tau_n \)-dense. Consequently, its closure, denoted by \( x \otimes y \), is \( \tau_n \)-measurable.

Observing that \( (x_1 + x_2) \otimes y_1 = x_1 \otimes y_1 + x_2 \otimes y_1, x_1 \otimes (y_1 + y_2) = x_1 \otimes y_1 + x_1 \otimes y_2 \) and \( (x_1 \otimes x_2)(y_1 \otimes y_2) = x_1x_2 \otimes y_1y_2 \) for all \( x_1, x_2 \in S(\tau) \) and \( y_1, y_2 \in M_n(\mathbb{C}) \), it is easily verified that the linear subspace of \( S(\tau_n) \) generated by the operators \( x \otimes y, x \in S(\tau) \) and \( y \in M_n(\mathbb{C}) \), is a subalgebra of \( S(\tau_n) \) which may be identified.
with the tensor product \( S(\tau) \otimes M_n(\mathbb{C}^n) \cong M_n(S(\tau)) \). We claim that \( S(\tau_n) = S(\tau) \otimes M_n(\mathbb{C}^n) \). Indeed, defining \( p_i = 1 \otimes e_{ij}, 1 \leq i \leq n \), it is clear that the map \( x \mapsto x \otimes e_{i1}, x \in M \), is a trace preserving \(*\)-isomorphism from \( M \) onto the reduced von Neumann algebra \( p_1Mnp_1 \), which has a unique extension to a trace preserving \(*\)-isomorphism from \( S(\tau) \) onto \( S(p_1Mnp_1, \tau_n) \), given by \( x \mapsto x \otimes e_{i1}, x \in S(\tau) \).

Furthermore, it should be observed that \( x \in S(\tau_n) \) and \( 1 \leq i, j \leq n \), we have \((1 \otimes e_{1i})x\), \((1 \otimes e_{1j})p_1 \in S(\tau_n)p_1 \) and so, there exists \( x_{ij} \in S(\tau) \) such that \((1 \otimes e_{ii})x\), \((1 \otimes e_{1j}) = x_{ij} \otimes p_1 \). Consequently,

\[
p_ixp_j = (1 \otimes e_{i1})(x_{ij} \otimes p_1)(1 \otimes e_{1j}) = x_{ij} \otimes e_{ij}
\]

and hence,

\[
x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in S(\tau) \otimes M_n(\mathbb{C}^n).
\]

This proves the claim. Identifying \( S(\tau) \otimes M_n(\mathbb{C}^n) \) with \( M_n(S(\tau)) \), the elements \( x_{ij} \) in (3.1) are the matrix elements of \( x \); we also write \( x = [x_{ij}]_{i,j=1}^{n} \). The proposition which now follows may be established using standard arguments, and the details of proof will be omitted.

**Proposition 3.1** Suppose that \( x = [x_{ij}]_{i,j=1}^{n} \in M_n(S(\tau)) \).

(i) \( \mu(x_{ij}) = \mu(x_{ij} \otimes e_{ij}) = \mu(p_1xp_j) \leq \mu(x) \) for all \( 1 \leq i, j \leq n \).

(ii) \( x \in L^1(\tau_n) \) if and only if \( x_{ij} \in L^1(\tau) \) for all \( 1 \leq i, j \leq n \), in which case \( \tau_n(x) = \sum_{i=1}^{n} \tau(x_{ii}) \).

If \( E \subseteq S(m) \) is any strongly symmetric space on \([0, \infty)\), and if \( x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in S(\tau_n) \), then \( x \in E(\tau_n) \) if and only if \( x_{ij} \in E(\tau) \) for all \( 1 \leq i, j \leq n \). We may therefore identify \( E(\tau_n) \) with the space of all \( n \times n \) matrices with entries in \( E(\tau) \) and will write \( E(\tau_n) = M_n(E(\tau)) \). With this identification, observe that if \( x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in E(\tau_n) \) and if \( y = \sum_{i,j=1}^{n} y_{ij} \otimes e_{ij} \in S(\tau_n) \) then \( y \in E^\times(\tau_n) = E(\tau_n)^\times \) if and only if \( y_{ij} \in E(\tau)^\times, 1 \leq i,j \leq n \), in which case

\[
xy = \sum_{i,j=1}^{n} \sum_{k=1}^{n} x_{ik}y_{kj} \otimes e_{ij} \in L^1(\tau_n) \text{ and } \tau_n(xy) = \sum_{i=1}^{n} \sum_{k=1}^{n} \tau(x_{ik}y_{ki}).
\]

Observe that if \( x \in S(\tau) \), then \( x \geq 0 \) if and only if there exists \( z \in S(\tau) \) such that \( x = z^*z \). We shall need the following simple result, which is proved exactly as in [19, Lemma IV.3.1].

**Lemma 3.2** If \( x \in S(\tau_n) \), then \( x \geq 0 \) if and only if \( x \) is a sum of elements of the form \( \sum_{i,j=1}^{n} x_i^*x_j \otimes e_{ij} \) with \( x_i \in S(\tau), 1 \leq i \leq n \).

The preceding lemma now yields the following criterion for positivity in spaces \( E(\tau_n) \) in terms of the trace \( \tau \).
Proposition 3.3 Suppose that $E \subseteq S(m)$ is a strongly symmetric space on $[0, \infty)$ and suppose that $x \in E(\tau_n)$. If $x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in M_n(E(\tau))$, then $x \geq 0$ if and only if $\sum_{i,j=1}^{n} \tau(y_{ij} x_{ij} y_{ij}^{*}) \geq 0$ for all choices $y_1, y_2, \ldots, y_n \in S(\tau)$ such that $y_{ij} y_{ij}^{*} \in E^{\times}(\tau), 1 \leq i, j \leq n$.

Proof It will suffice to show that $\tau_n(xy) \geq 0$ for all $0 \leq y \in E^{\times}(\tau_n)$. If $0 \leq y \in E^{\times}(\tau_n)$, then it may be assumed by Lemma 3.2 that there exist $y_1, y_2, \ldots, y_n \in S(\tau)$ such that $y = \sum_{i,j=1}^{n} y_{ij} \otimes e_{ij}$. It follows, in particular, that $\mu(y_{ij}^{*} y_{ij}) = \mu(y_{ij} y_{ij}^{*}) \in E^{\times}, 1 \leq i \leq n$, and so also, for all $1 \leq i, j \leq n$,

$$\mu(y_{ij}) \mu(y_{ij}^{*}) \leq \frac{1}{2} \left( \mu(y_{ij})^{2} + \mu(y_{ij}^{*})^{2} \right) = \frac{1}{2} \left( \mu(y_{ij}^{*} y_{ij}) + \mu(y_{ij} y_{ij}^{*}) \right) \in E^{\times}.$$ 

From the submajorisation

$$\mu(y_{ij} x_{ij} y_{ij}^{*}) \ll \mu(x_{ij}) \mu(y_{ij}) \mu(y_{ij}^{*}), \quad 1 \leq i, j \leq n,$$

and the fact that $\mu(x_{ij}) \in E$, it follows that $\mu(y_{ij} x_{ij} y_{ij}^{*}) \in L^{1}(m)$ so that $y_{ij} x_{ij} y_{ij}^{*} \in L^{1}(\tau), 1 \leq i, j \leq n$. Since it is clear that $x_{ij} y_{ij}^{*} y_{ij} \in L^{1}(\tau)$, it follows from the first assertion of [6] Proposition 3.4 that $\tau(y_{ij} x_{ij} y_{ij}^{*}) = \tau(x_{ij} y_{ij}^{*} y_{ij})$ for all $1 \leq i, j \leq n$. By the remarks preceding Lemma 3.2, it now follows that

$$\tau_n(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} \tau(x_{ij} y_{ij}^{*} y_{ij}) = \sum_{i,j=1}^{n} \tau(y_{ij} x_{ij} y_{ij}^{*})$$

and this suffices to complete the proof. \hfill \Box

Suppose now that $(\mathcal{N}, \sigma)$ is a semifinite von Neumann algebra, that $E \subseteq S(\tau), F \subseteq S(\sigma)$ are strongly symmetric spaces and that $T : E \rightarrow F$ is a linear mapping. For each $n \in \mathbb{N}$, we let $T_n : M_n(E) \rightarrow M_n(F)$ be defined by setting

$$T_n \left( [x_{ij}]_{i,j=1}^{n} \right) = [T(x_{ij})]_{i,j=1}^{n}.$$

for all $[x_{ij}]_{i,j=1}^{n} \in M_n(E)$. The mapping $T$ is said to be completely positive if and only if $T_n \geq 0$ for every $n \in \mathbb{N}$, that is, $T_n$ maps $M_n(E) \cap S(\tau_n)^{+}$ into $M_n(F) \cap S(\sigma_n)^{+}$, for each $n \in \mathbb{N}$. Denote by $CP(E, F)$ the collection of all completely positive maps $T : E \rightarrow F$. If $T \in CP(E, F)$, then we will write $0 \leq_{cp} T : E \rightarrow F$. If $S, T \in CP(E, F)$, then we write $0 \leq_{cp} S \leq_{cp} T$ if and only if $0 \leq_{cp} T - S$. Note that, if $E, F \subseteq S(m)$ are strongly symmetric spaces on $[0, \infty)$, then via the identifications $E(\tau_n) = M_n(E(\tau)), F(\sigma_n) = M_n(F(\sigma))$, each of the mappings $T_n$ defined above induces a linear mapping from $E(\tau_n)$ to $F(\sigma_n)$. Without risk of confusion, we continue to denote these mappings by $T_n$. In this setting, the linear mapping $T : E(\tau) \rightarrow F(\sigma)$ is completely positive if and only if each of the mappings $T_n : E(\tau_n) \rightarrow F(\sigma_n)$ are positive in the usual sense, that is, $0 \leq T_n(x) \in F(\sigma_n)$ whenever $0 \leq x \in E(\tau_n)$.

The key result on which the main results of this paper are based is the following majorisation theorem, due to Neuhardt [15,16].
Lemma 3.6

If $E, T : \mathcal{M} \to L^1(\sigma)$ are linear maps, if $0 \leq_{cp} S \leq_{cp} T$ and if $T$ is compact, then $S$ is compact.

It was shown by Stinespring [18, Theorem 3], that a positive linear mapping from a commutative $C^*$-algebra $A$ to another $C^*$-algebra $B$ is necessarily completely positive. As noted in [18], the converse is false, even in the case that $A = B = M_2(\mathbb{C})$. See also [19] Proposition 3.9 of Chapter IV. Using Lemma 3.2 and Proposition 3.3, a modification of the proof given in [19] Proposition IV.3.9 yields the following variant. The details are omitted.

Proposition 3.5

Let $(\mathcal{M}, \tau)$ be a commutative von Neumann algebra and let $(\mathcal{N}, \sigma)$ be any semi-finite von Neumann algebra. Suppose that $E, F \subseteq S(m)$ are strongly symmetric spaces. If $0 \leq T : E(\tau) \to F(\sigma)$, then $T$ is completely positive.

It is worth remarking that the preceding proposition shows that the commutative specialisation of Neuhardt’s theorem actually coincides with an important special case of one of the main results of [9].

It should be observed that, if $E \subseteq S(\tau)$ is a strongly symmetric space, and if $\varphi \in E^\times$ then $\varphi \in E^\times$ if and only if, whenever $x_\alpha \downarrow_{\sigma} 0$ in $E$, it follows that $\varphi(x_\alpha) \to_{\alpha} 0$. See [6, Theorem 5.11]. It will now be convenient to make the following definition. Suppose that $E \subseteq S(\tau), F \subseteq S(\sigma)$ are strongly symmetric spaces. The continuous linear mapping $T : E \to F$ will be called order continuous if $\sigma(T(x_\alpha)z) \to_{\alpha} 0$ whenever $x_\alpha \downarrow_{\sigma} 0$ in $E$ and $0 \leq z \in F^\times$. Using the above remark, it follows that if $T : E \to F$ is order continuous, and if $T^* : E^* \to F^*$ denotes the Banach adjoint mapping, then $T^*z \in E^\times$ whenever $z \in F^\times$. In this case, the restriction of the adjoint $T^*$ to the Köthe dual $F^\times$ will be denoted by $T^\times$ so that $T^\times : F^\times \to E^\times$. It is clear that, if $T \geq 0$, then $T^\times \geq 0$. If $0 \leq T : E \to F$, then $T$ is order continuous if and only if $x_\alpha \downarrow_{\sigma} 0$ in $E$ implies $Tx_\alpha \downarrow_{\sigma} 0$ in $F$. It follows, in particular, that if $0 \leq S, T : E \to F$ are positive linear mappings with $0 \leq S \leq T$, then $S$ is order continuous if $T$ is order continuous.

Lemma 3.6

If $E, F \subseteq S(m)$ are strongly symmetric spaces and if $0 \leq T : E(\tau) \to F(\sigma)$ is order continuous, then $\sigma_n(zT_n x) = \tau_n(x(T^\times)_n z)$ for all $n \in \mathbb{N}, x \in E(\tau_n)$ and $z \in F^\times(\sigma_n) = F(\sigma_n)^\times$,

Proof If $x = [x_{ij}]_{i,j=1}^n \in E(\tau_n) = M_n(E(\tau))$ and if $z \in F(\sigma_n)^\times = F^\times(\sigma_n) = M_n(F(\sigma)^\times)$, then, using the remarks concerning the trace preceding Lemmas 3.2, it follows that

$$\sigma_n(zT_n x) = \sum_{i,j=1}^n \sigma(z_{ij} T x_{ji}) = \sum_{i,j=1}^n \tau(x_{ij} T^\times z_{ji}) = \tau_n(x(T^\times)_n z).$$

□

Proposition 3.7

Suppose that $E, F \subseteq S(m)$ are strongly symmetric spaces. If $T : E(\tau) \to F(\sigma)$ is order continuous, and if $n \in \mathbb{N}$, then
(i) $T_n : E(\tau_n) \to F(\sigma_n)$ is order continuous;

(ii) $(T_n)^\times = (T^\times)_n$;

(iii) $T$ is completely positive if and only if $T^\times$ is completely positive.

**Proof** (i) Suppose that $x_\alpha \downarrow_\alpha 0$ in $E(\sigma_n)$ and that $z \in F(\sigma_n)^\times = F^\times(\sigma_n)$. If $z = [z_{ij}]_{i,j=1}^n \in F(\sigma_n)^\times = F^\times(\sigma_n) = M_n(F^\times)$, then, using the fact that $T$ is order continuous so that $T^\times : F(\sigma) \to E(\sigma)^\times$, it follows that $(T^\times)^n_z = [T^\times z_{ij}]_{i,j=1}^n \in M_n(E(\sigma)^\times) = M_n(E^\times(\sigma)) = E^\times(\sigma_n) = E(\sigma_n)^\times$. Consequently, $\tau_n(x_\alpha(T^\times_n z)) \to_\alpha 0$. From Lemma 3.6, it now follows that $\sigma_n(zT_nx_\alpha) \to_\alpha 0$, and from this it follows that $T_n$ is order continuous.

(ii) This is now a straightforward reformulation of Lemma 3.6.

(iii) Observe that $T \geq_{cp} 0 \iff T_n \geq 0 \iff (T_n)^\times \geq 0$. Since $(T_n)^\times = (T^\times)_n$, as follows from (ii), the assertion of (iii) now follows readily.

$\square$

### 4 Characterisations of compactness

Throughout this section, $(\mathcal{M}, \tau)$ will denote a semifinite von Neumann algebra.

**Lemma 4.1** Suppose that $E \subseteq S(\tau)$ is a strongly symmetric space. If $x \in E$ and if $0 \leq h \in E^\times$, then $h^{1/2}xh^{1/2} \in L^1(\tau)$ and $\|h^{1/2}xh^{1/2}\|_1 \leq 4\|x\|_E\|h\|_{E^\times}$.

**Proof** That $h^{1/2}xh^{1/2} \in L^1(\tau)$ follows by adapting the argument used to establish the second assertion of [6, Proposition 3.4]. To establish the norm estimate, suppose first that $0 \leq x \in E$. Observe that, if $0 \leq z \in \mathcal{M}$ satisfies $0 \leq z \leq 1$, then, using the first assertion of [6, Proposition 3.4],

$$
\tau(h^{1/2}xh^{1/2}z) = \tau(xh^{1/2}zh^{1/2}) \leq \|x\|_E\|h^{1/2}zh^{1/2}\|_{E^\times} \leq \|x\|_E\|h\|_{E^\times}
$$

since $h^{1/2}zh^{1/2} \leq h$. This suffices to establish the estimate in the case that $x \geq 0$. The general case that $x \in E$ follows by writing $x = Re(x)^+ - Re(x)^- + i(Im(x)^+ - Im(x)^-)$ and noting that $Re(x) \leq |Re(x)|$, $\|Im(x)\| \leq |Im(x)|$. The estimate then follows readily from the inequalities $\|Re(x)\|_E$, $\|Im(x)\|_E \leq \|x\|_E$. $\square$

Let $E \subseteq S(\tau)$ be a strongly symmetric space. A bounded subset $\mathcal{K} \subseteq E$ is said to have uniformly absolutely continuous norm if and only if $\sup\{\|e_nx_n\|_{E(\tau)} : x_n \in \mathcal{K}\} \to 0$ as $n \to \infty$ for all sequences $\{e_n\}_{n=1}^\infty \subseteq P(\mathcal{M})$ for which $e_n \downarrow_n 0$. See, for example, [8,17]. If $\mathcal{K} \subseteq E$ is of uniformly absolutely continuous norm, then $\mathcal{K}$ is contained in the set $E^{oc}$ of elements of order continuous norm. See [8]. Here

$$
E^{oc} = \{x \in E : |x| \geq x_\alpha \downarrow_\alpha 0 \implies \|x_\alpha\|_E \downarrow_\alpha 0\}.
$$

We shall need the following convergence criterion.

**Proposition 4.2** If $\{x_n\}_{n=1}^\infty$ is a sequence in $E^{oc}$, then the following statements are equivalent.

The preceding proposition is proved in [8, Theorem 6.11]. In the case that the von Neumann algebra $\mathcal{M}$ does not contain any minimal projections and $E = E(\tau)$, with $E \subset S(m)$, has order continuous norm, this proposition may also be found in [3]. See also [17].

Before proceeding, some additional preparation is needed. Let $E \subseteq S(\tau)$ be a symmetrically normed $\mathcal{M}$-bimodule. It follows from the theory of ordered Banach spaces that $E^*$ is a (complex) ordered Banach space with a generating cone, and that there exists a constant $K_E$ such that, whenever $x \in E$,

$$
\|x\|_E \leq K_E \sup\{\psi(|x|) : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}
$$

See, for example, Ando [2]. We remark that, in the present setting, the constant $K_E$ may be taken to be 4 (see [8]), and we will use this remark in what follows.

**Lemma 4.3** Suppose that $E \subseteq S(\tau)$ is a normed $\mathcal{M}$-bimodule. If $0 \leq \psi \in E^*$, if $0 \leq x \in E$ and $0 \leq y \in \mathcal{M}$ and if $xy = u|x|y$ is the polar decomposition, then

$$
\psi(|xy|) \leq \psi(u^*xu)^{1/2}\psi(yxy)^{1/2}.
$$

**Proof** Using the fact that $\psi \geq 0$, a standard Cauchy-Schwartz type argument readily implies that

$$
\psi(|xy|) = \psi(u^*x^{1/2}x^{1/2}y) \leq \psi(u^*xu)^{1/2}\psi(yxy)^{1/2}.
$$

$\square$

**Lemma 4.4** Suppose that $E \subseteq S(\tau)$ is a symmetrically normed $\mathcal{M}$-bimodule. If $0 \leq x \in E$ and $0 \leq y \in \mathcal{M}$, then

$$
\|xy\|_E \leq 4\|x\|_E^{1/2}\|yxy\|_E^{1/2}
$$

and

$$
\|xy\|_E \leq 4\|x\|_E^{1/2}\|x^{1/2}y^2x^{1/2}\|_E^{1/2}.
$$

**Proof** Let $xy = u^*|xy|$ be the polar decomposition. Applying Lemma 4.3, it follows that

$$
\|xy\|_E \leq 4\sup\{\psi(|xy|) : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}
$$

$$
\leq 4\sup\{\psi(u^*xu)^{1/2}\psi(yxy)^{1/2} : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}
$$

$$
\leq 4\|x\|_E^{1/2}\sup\{\psi(yxy)^{1/2} : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}
$$

$$
\leq 4\|x\|_E^{1/2}\|yxy\|_E^{1/2}.
$$
The second assertion follows directly from the first by observing that

$$\mu(yxy) = \mu((x^{1/2}y^*)x^{1/2}y) = \mu(x^{1/2}y(x^{1/2}y^*)) = \mu(x^{1/2}y^2x^{1/2}).$$

Since $E$ is symmetrically normed, it follows that $\|yxy\|_E = \|x^{1/2}y^2x^{1/2}\|_E$ and the assertion follows. \hfill \Box

We note that a related inequality is given in [3] in the case that $M$ is non-atomic and the norm on $E$ is order continuous. We may now prove the following characterisation of sets of absolutely continuous norm contained in the positive cone of $E$ which will be needed in what follows.

**Proposition 4.5** Suppose that $E \subseteq S(\tau)$ is a strongly symmetric space. If $E$ has order continuous norm and if $K \subseteq E_+$ is bounded, then the following statements are equivalent.

(i) $K$ is of uniformly absolutely continuous norm.

(ii) For all $e_n \downarrow 0 \subseteq P(\mathcal{M})$, $\sup\{\|exe_n\|_E : x \in K\} \to 0$.

(iii) For all $e_n \downarrow 0 \subseteq P(\mathcal{M})$, $\sup\{\|e_nx\|_E : x \in K\} \to 0$.

**Proof** Since $\|exe_n\|_E = \|(exe_n)^*\|_E = \|exe_n\|_E$ for all $x \in K \subseteq E_+$ and all $n$, the equivalence $(ii) \iff (iii)$ is clear. Further, since $\|exe_n\|_E \leq \|e_nx\|_E$ for all $x \in E$ and all $n$, the implication $(i) \implies (ii)$ is also clear. The implication $(i) \implies (ii)$ now follows by observing that

$$\|exe_n\|_E \leq 4 \left( \sup_{x \in K} \|x\|_E^{1/2} \right) \|e_nxe_n\|_E^{1/2}$$

for all $x \in K$ and for all $n$, as follows from Lemma 4.4. \hfill \Box

We note that the above proposition fails if the assumption that $K$ lies in the positive cone of $E$ is omitted. The proposition which follows characterises norm compactness in spaces of order continuous norm in terms of sets of uniformly absolutely continuous norm.

**Proposition 4.6** Suppose that $E \subseteq S(\tau)$ is strongly symmetric. Suppose that $K \subseteq E$ is bounded, that the norm on $E$ is order continuous and consider the following statements.

(i) $K$ is relatively compact.

(ii) $K$ has uniformly absolutely continuous norm and $K$ is relatively compact for the measure topology.

(iii) $K$ has uniformly absolutely continuous norm and $h^{1/2}Kh^{1/2}$ is relatively compact in $L^1(\tau)$ for all $0 \leq h \in E^\times$.

The implications $(i) \iff (ii) \implies (iii)$ are always valid. If, in addition $K \subseteq E_+$, then all three statements are equivalent.
Proof The equivalence (i) \iff (ii) follows from Proposition 4.2.

(i) \implies (iii). Let \( 0 \leq h \in E^\times \). It follows from Lemma 4.1 that the map

\[ x \mapsto h^{1/2}xh^{1/2} : E \to L^1(\tau) \]

is continuous. Consequently, the set \( h^{1/2}K h^{1/2} \subset L^1(\tau) \) is relatively compact in \( L^1(\tau) \) whenever \( K \) is relatively compact in \( E \).

We now assume, in addition, that \( K \subseteq E_+ \). It will suffice to show the implication (iii) \implies (ii). Assume then that \( K \) is of uniformly absolutely continuous norm and that \( h^{1/2}K h^{1/2} \) is relatively compact in \( L^1(\tau) \) for all \( 0 \leq h \in E^\times \). Since the measure topology is metrizable, it will suffice to show that any sequence \( \{x_n\}_{n=1}^\infty \subseteq K \) has a convergent subsequence. To this end, we set \( e := \sup_n s(x_n) \). Using the order continuity of the norm on \( E \), it follows from [8], Lemma 6.9 and Lemma 6.10, that \( e \) is a \( \sigma \)-finite projection. We let \( \{f_n\}_{n=1}^\infty \subseteq P(\mathcal{M}) \) be any sequence of mutually disjoint projections such that \( \tau(f_n) < \infty \) for all \( n \in \mathbb{N} \) and \( \sum_{n=1}^\infty f_n = e \). By considering the von Neumann algebra \( e\mathcal{M}e \), we may assume that \( e = 1 \).

We now set \( h := \sum_{n=1}^\infty 2^{-n} \| f_n \|^{-1} f_n \in E^\times \) and note that \( s(h) = 1 = s(h^{1/2}) \). We may suppose that the sequence \( \{h^{1/2}x_nh^{1/2}\}_{n=1}^\infty \) is Cauchy in measure. For each \( k \in \mathbb{N} \), we set \( e_k := e^{h^{1/2}}(1/k, \infty) \) and note that \( 1 - e_k \downarrow_k 0 \). We observe that, for all \( k, m, n \in \mathbb{N} \),

\[ x_n - x_m = (1 - e_k)(x_n - x_m) + e_k(x_n - x_m)(1 - e_k) + e_k(x_n - x_m)e_k \]

and let \( V(\epsilon, \delta) \) be any neighbourhood of 0 for the measure topology. Using the fact that the sequence \( \{x_n\}_{n=1}^\infty \subseteq E_+ \) is of uniformly absolutely absolutely continuous norm, it follows from Proposition 4.5 together with the continuity of the embedding of \( E \) into \( S(\tau) \) that there exists \( k_0 \in \mathbb{N} \) such that

\[ (1 - e_k)(x_n - x_m) \in V(\epsilon/3, \delta/3), \quad (x_n - x_m)(1 - e_k) \in V(\epsilon/3, \delta/3) \]

for all \( n, m \in \mathbb{N} \) and all \( k \geq k_0 \). It now suffices to show, for each fixed \( k \in \mathbb{N} \), that \( e_k(x_n - x_m)e_k \in V((\epsilon/3, \delta/3) \) for all sufficiently large \( n, m \in \mathbb{N} \). Since \( e_k h^{1/2} = h^{1/2}e_k \geq k^{-1}e_k \), there exists \( z_k \in e_k \mathcal{M} e_k \) such that \( z_k h^{1/2}e_k = e_k h^{1/2}z_k = e_k \).

It follows that

\[ e_k(x_n - x_m)e_k = z_k h^{1/2}e_k(x_n - x_m)e_k h^{1/2}z_k = z_k e_k h^{1/2}(x_n - x_m)h^{1/2}e_k z_k. \]

Since the sequence \( \{h^{1/2}x_nh^{1/2}\}_{n=1}^\infty \) is Cauchy for the measure topology, it follows from the fact that multiplication is continuous for the measure topology that \( e_k(x_n - x_m)e_k \in V(\epsilon/3, \delta/3) \) for all sufficiently large \( n, m \) and this suffices to conclude the proof of the proposition. \( \square \)

It should be noted that several of the characterisations of compactness in this section go back to [11] in the case of (commutative) \( L^p \)-spaces and may be found in [16] in the case of the Haagerup \( L^p \)-spaces.
5 Domination by completely positive compact operators

We begin with several immediate consequences of the characterisations given in the preceding section. Throughout this section, \((\mathcal{M}, \tau)\) and \((\mathcal{N}, \sigma)\) will denote semifinite von Neumann algebras, acting in (possibly different) Hilbert spaces. Suppose first that \(E \subseteq S(\tau), F \subseteq S(\sigma)\) are strongly symmetric spaces. If \(T : E \to F\) is a linear mapping, then, for all \(0 \leq h \in F^\times\), the mapping \(M(h)T\) is defined by setting

\[
M(h)T(x) := h^{1/2}(Tx)h^{1/2}, \quad x \in E
\]

The following result now follows directly from Proposition 4.6.

**Proposition 5.1** Suppose that the norm on \(F\) is order continuous. If \(0 \leq T : E \to F\) is a positive linear mapping, then the following statements are equivalent.

(i) \(T\) is compact.
(ii) \(T\) maps the unit ball of \(E\) into a set of uniformly absolutely continuous norm in \(F\) and the mapping \(M(h)T : E \to L^1(\sigma)\) is compact, for all \(0 \leq h \in F^\times\).

**Corollary 5.2** Suppose that the norm on \(F\) is order continuous. If \(0 \leq T : \mathcal{M} \to F\) is a positive linear mapping, then the following statements are equivalent.

(i) \(T\) is compact.
(ii) The mapping \(M(h)T : \mathcal{M} \to L^1(\sigma)\) is compact for all \(0 \leq h \in F^\times\).

**Proof** It needs only be observed that the image under \(T\) of the positive part of the unit ball of \(\mathcal{M}\) is contained in the order interval

\[
[0, T(1)] = \{z \in F(\sigma) : 0 \leq z \leq T(1)\}.
\]

Since the norm on \(F\) is order continuous, this clearly implies that the order interval \([0, T(1)]\) is of uniformly absolutely continuous norm. The assertion of the corollary now follows from Proposition 5.1. \(\square\)

**Lemma 5.3** If \(0 \leq T : E \to F\) is completely positive, and if \(0 \leq h \in F^\times\), then \(0 \leq M(h)T : E \to L^1(\sigma)\) is completely positive.

**Proof** It needs only be observed that, if \(n \in \mathbb{N}\) and if \(0 \leq [ai_{ij}]_{i,j=1}^n \in M_n(S(\sigma)) = S(\sigma_n)\), then \(0 \leq [h^{1/2}ai_{ij}h^{1/2}]_{i,j=1}^n \in M_n(S(\sigma)) = S(\sigma_n)\). If \(ai_{ij} = a_i^*a_j\) with \(a_1, a_2, \ldots, a_n \in S(\sigma)\), then \(h^{1/2}ai_{ij}h^{1/2} = (ai_i h^{1/2})a_j h^{1/2}\), and the assertion now follows from Lemma 3.2. \(\square\)

**Proposition 5.4** Suppose that the norm on \(F\) is order continuous, and that \(S, T : \mathcal{M} \to F\) are linear mappings. If \(0 \leq_{cp} S \leq_{cp} T\) and if \(T\) is compact then \(S\) is compact.

**Proof** By Corollary 5.2, it suffices to show that the map \(M(h)S : \mathcal{M} \to L^1(\sigma)\) is compact for all \(0 \leq h \in F^\times\). Suppose then that \(0 \leq h \in F^\times\). As follows from Lemma 5.3, \(0 \leq_{cp} M(h)S \leq_{cp} M(h)T : \mathcal{M} \to L^1(\sigma)\). That \(M(h)S\) is compact now follows from Theorem 3.4. \(\square\)
For the remainder of this section, it will be assumed that $E, F \subseteq S(m)$ are strongly symmetric spaces. Recall (see [6]) that if $E \subseteq S(m)$ has order continuous norm then the norm on $E(\tau)$ is order continuous and the Banach dual $E(\tau)^*$ coincides with the Köthe dual $E(\tau) = E^*(\tau)$. Note that, if $F$ has order continuous norm and if $T : E(\tau) \to F(\sigma)$ is an order continuous linear mapping, then the Banach adjoint $T^*$ coincides with the mapping $T^X$. We may now state the principal result of this paper.

**Theorem 5.5** Let $0 \leq S, T : E(\tau) \to F(\sigma)$ be linear mappings and suppose that $0 \leq_{cp} S \leq_{cp} T$. If $T$ is order continuous, if the norms on $E^X$ and $F$ are order continuous, and if $T$ is compact, then $S$ is compact.

*Proof* Since $T$ is compact, it follows from Corollary 5.1 that $T(B(E_+)) \subseteq F(\sigma)$ is of uniformly absolutely continuous norm. Since $0 \leq S \leq T$ it follows also that $S(B(E_+))$, and hence also $S(B(E))$, is of uniformly absolutely continuous norm. To show that $S$ is compact, it follows again from Corollary 5.1 that it suffices to show that $M(h)S : E \to L^1(\sigma)$ is compact for all $0 \leq h \in F(\sigma)^X$. Now observe that the map $M(h)T : E \to L^1(\sigma)$ is order continuous. Indeed, if $0 \leq h \in F(\sigma)^X$, and if $x_\alpha \downarrow 0 \subseteq E(\tau)$ then $Tx_\alpha \downarrow 0 \subseteq F(\sigma)$ and this implies that $h^{1/2}(T_x)h^{1/2} \downarrow 0$ holds in $L^1(\sigma)$. Since $0 \leq S \leq T$, it follows that $0 \leq M(h)S \leq M(h)T$ and so also $M(h)S : E \to L^1(\sigma)$ is order continuous. Consequently

$$0 \leq (M(h)S)^*, (M(h)T)^* : M \to E^X$$

and so $(M(h)S)^* = (M(h)S)^X$ and $(M(h)T)^* = (M(h)T)^X$. It follows from Proposition 3.7 (iii) that

$$0 \leq_{cp} (M(h)S)^* = (M(h)S)^X \leq_{cp} (M(h)T)^X = (M(h)T)^*.$$

Further, by Schauder’s theorem, $(M(h)T)^* : M \to E^X$ is compact. Since the norm on $E^X$ is order continuous, it follows from Proposition 5.4 that $(M(h)S)^* : M \to E^X$ is compact. Again using Schauder’s theorem, it follows that $M(h)S : E \to L^1(\sigma)$ is compact, and this completes the proof of the Theorem.

**Corollary 5.6** Suppose that $0 \leq S, T : E(\tau) \to E(\tau)$ are linear mappings which satisfy $0 \leq_{cp} S \leq_{cp} T$. If the norms on $E, E^X$ are order continuous and if $T$ is compact, then $S$ is compact.

*Proof* To apply the preceding Theorem, we need only note that, since the norm on $E(\tau)$ is order-continuous, then each positive linear map on $E(\tau)$ is necessarily order-continuous.

Finally, suppose that $\tau(1) < \infty$, and that $0 \leq S, T : E(\tau) \to E(\tau)$ are linear mappings with $T$ compact which satisfy $0 \leq_{cp} S \leq_{cp} T$. It can be shown that if the norm on $E$ is order continuous or if $E$ has the Fatou property and the norm on $E^X$ is order continuous, then $S^2$ is compact. This is a non-commutative counterpart to a well known theorem of Aliprantis and Burkinshaw [1], Meyer-Nieberg [13], and Zaanen [21]. Further, it may be shown that a completely positive operator on the predual of a finite von Neumann algebra which is dominated in the sense of complete positivity by a Dunford–Pettis operator, is itself a Dunford–Pettis operator.
In the case of abstract $L$-spaces this was first proved in [9]. In fact, this result continues to hold for the non-commutative counterparts of separable Lorentz spaces and certain Orlicz spaces. In addition, a completely positive mapping from a semifinite von Neumann algebra $\mathcal{M}$ to any non-commutative space $F(\sigma)$ with order continuous norm can be expressed uniquely as the sum of a completely positive compact operator and a completely positive operator which dominates no non-zero compact mapping, in the sense of complete positivity. This is shown in [16] in the case that $F$ is an $L^p$-space, $1 \leq p < \infty$ and in the Banach lattice setting again goes back to [9]. A similar decomposition holds for Dunford–Pettis operators in the case of finite von Neumann algebras. The details will appear elsewhere.

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

1. Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Academic Press, New York (1985)
2. Ando, T.: On fundamental properties of a Banach space with a cone. Pacific J. Math. 12, 1163–1169 (1962)
3. Chilin, V.I., Sukochev, F.A.: Weak convergence in non-commutative symmetric spaces. J. Oper. Theory 31, 35–65 (1994)
4. Dodds, P.G., Dodds, T.K., de Pagter, B.: Non-commutative Banach function spaces. Math. Z. 201, 583–597 (1989)
5. Dodds, P.G., Dodds, T.K., de Pagter, B.: A general markus inequality. Proc. Centre Math. A.N.U. 24, 47–57 (1990)
6. Dodds, P.G., Dodds, T.K., de Pagter, B.: Non-commutative Köthe duality. Trans. Am. Math. Soc. 339, 717–750 (1993)
7. Dodds, P.G., de Pagter, B.: Non-commutative Yosida-Hewitt theorems and singular functionals in symmetric spaces of $\tau$-measurable operators. In: Curbera, G.P., et al. (eds.) Vector Measures, Integration and Related Topics. Proceedings of Conference on “Vector Measures, Integration and Related topics” Eichstätt, Sept. 2008. Operator Theory: Advances and Applications, vol. 201, pp. 183–198. Birkhäuser Verlag, Basel (2010)
8. Dodds, P.G., de Pagter, B.: The non-commutative Yosida-Hewitt decomposition revisited (submitted)
9. Dodds, P.G., Fremlin, D.H.: Compact operators in Banach lattices. Israel J. Math. 34, 287–320 (1979)
10. Fack, T., Kosaki, H.: Generalized $s$-numbers of $\tau$-measurable operators. Pacific J. Math. 123, 269–300 (1986)
11. Krasnoselskii, M.A., Zabreiko, P.P., Pustylnik, E.I., Sobolevskii, P.E.: Integral Operators in Spaces of Summable Functions. Noordhoff, Leiden (1976)
12. Krein, S.G., Petunin, J.I., Semenov, E.M.: Interpolation of linear operators. In: Translations of Mathematical Monographs, vol. 54. AMS (1982)
13. Meyer-Nieberg, P.: Banach Lattices. Universitext. Springer-Verlag (1991)
14. Nelson, E.: Notes on non-commutative integration. J. Funct. Anal. 15, 103–116 (1974)
15. Neuhardt, E.: Zur theorie der Kernoperatoren auf $L^p$-Räumen von semifiniten von Neumann Algebren. PhD thesis, Saarbrücken (1987)
16. Neuhardt, E.: Order properties of compact maps on $L^p$-spaces associated with von Neumann algebras. Math. Scand. 66, 110–116 (1990)
17. Randrianantoanina, N.: Sequences in non-commutative $L_p$-spaces. J. Oper. Theory 48, 235–247 (2002)
18. Stinespring, W.F.: Positive functions on $C^*$-algebras. Proc. Am. Math. Soc. 6, 211–216 (1955)
19. Takesaki, M.: Theory of Operator Algebras I. Springer-Verlag, New York (1979)
20. Terp, M.: $L^p$-Spaces Associated with von Neumann Algebras. Notes, Copenhagen University (1981)
21. Zaanen, A.C.: Riesz Spaces II. North-Holland, Mathematical Library Amsterdam-New York-Oxford (1983)