Freiheitssatz and phase transition for the density model of random groups

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Abstract
Magnus’ Freiheitssatz [20] states that if a group is defined by a presentation with \( m \) generators and a single cyclically reduced relator, and this relator contains the last generating letter, then the first \( m - 1 \) letters freely generate a free subgroup. We study an analogue of this theorem in the Gromov density model of random groups [14], showing a phase transition phenomenon at density \( d_r = \min\{\frac{1}{2}, 1 - \log_2 m - 1\} \) with \( 1 \leq r \leq m - 1 \): we prove that for a random group with \( m \) generators at density \( d \), if \( d < d_r \) then the first \( r \) letters freely generate a free subgroup; whereas if \( d > d_r \) then the first \( r \) letters generate the whole group. This result partially answers a general problem proposed by Gromov in 2003 [15]: existence/nonexistence of non-free subgroups in a random group.

Keywords Random group · Freiheitssatz · Gromov density model · van Kampen diagram

Mathematics Subject Classification 20F05 · 20F06 · 60C05

Contents
1 Introduction ............................................... 2
   The density model of random groups ................................. 2
   Main results ............................................... 3
   Outline of the paper ........................................... 4
2 Preliminaries on group theory ...................................... 5
   2.1 Stallings graphs (\( X \)-labeled graphs generating subgroups) ........ 5
   2.2 Van Kampen diagrams ....................................... 6
   2.3 Distortion van Kampen diagrams .................................. 7
   2.4 Hyperbolic groups .......................................... 8
3 Random subsets and random groups ................................... 9
   3.1 Densable sequences of random subsets ....................... 9
   3.2 The intersection formula .................................. 10
   3.3 The density model of random groups ....................... 11
4 Abstract diagrams ............................................ 12
   4.1 Abstract van Kampen diagrams ................................ 12
1 Introduction

The Freiheitssatz (freedom theorem in German) is a fundamental theorem in combinatorial group theory. It was proposed by M. Dehn and proved by W. Magnus in his doctoral thesis [20] in 1930 (see [19] II.5). The theorem states that for a group presentation $G = \langle x_1, \ldots, x_m \mid r \rangle$ where the single relator $r$ is a cyclically reduced word, if $x_m$ appears in $r$, then $x_1, \ldots, x_{m-1}$ freely generate a free subgroup of $G$.

Random groups are groups obtained by a probabilistic construction. Its first mentions, in terms of “generic property” for finitely presented groups, appear in the works of Guba [16] and Gromov [13] §0.2 in the late 1980s. The simplest model of random groups is the few relator model ([24] Definition 1). A few relator random group is defined by a group presentation $G_\ell = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_k \rangle$ where the set of generators $X = \{x_1, \ldots, x_m\}$ is fixed, and the relators $r_1, \ldots, r_k$ are chosen uniformly at random among all reduced words of $X^\pm$ of length at most $\ell$. The first well-known result of random groups ([13] §0.2) is that asymptotically almost surely (denoted by a.a.s., which means with probability converges to 1 when $\ell$ goes to infinity), a few relator random group $G_\ell$ is non-elementary hyperbolic.

For detailed surveys on random groups, see (in chronological order) [12] by Ghy, [24] by Ollivier, [18] by Kapovich and Schupp and [6] by Bassino, Nicaud and Weil.

The density model of random groups

In 1993, Gromov introduced the density model of random groups in [14] 9.B. He considered a group presentation with a fixed set of $m$ generators $X = \{x_1, \ldots, x_m\}$ and $\lfloor (2m - 1)^d \ell \rfloor$ randomly chosen relators, among the $2m(2m - 1)^{\ell-1}$ reduced words of $X^\pm$ of length $\ell$. The parameter $d \in [0, 1]$ is called the density. Compare to the few relator model, the number of relators grows exponentially with the length $\ell$. The main result of [14] 9.B is the phase transition at density one half: if $d > \frac{1}{2}$, then a.a.s. the group is trivial; if $d < \frac{1}{2}$, then a.a.s. the group is non-elementary hyperbolic.

In a 1996 paper [5], Arzhantseva and Ol’shanskii proved a few relator random group version of the Freiheitssatz: a.a.s. every $(m - 1)$-generated subgroup of a few relator random group $G_\ell$ is free. Arzhantseva proved several free subgroup properties subsequently for the few relator model in [2–4]. Kapovich-Schupp [18] showed the existence of a small positive density $d(m)$ such that these results [2–5] can be generalized to a random group at any density $d < d(m)$. It was showed in [29] that the “every $(m - 1)$-generated subgroup is free” property [5] holds a.a.s. for a random group at any density $d < \frac{1}{120m^2 \ln(2m)}$.

In 2003, Gromov defined the general notion of random groups in [15] and proposed in Section 1.9 the following general problem: determining asymptotic invariants and phase transition phenomena for random groups. Since then, several variants of the phase transition phenomena have been discovered. For instance, Žuk [30] showed the freeness-property ($T$) phase transition for random triangular groups at density $1/3$ (see also [1] by Antoniuk-Łuczak-Świątkowski). Ollivier proved in 2004 [23] the hyperbolicity-triviality transition for
hyperbolic random groups, and in 2007 [25] the phase transition at density 1/5 for Dehn’s algorithm. In 2015 [8], D. Calegari and A. Walker showed that a random group at density \( d < 1/2 \) contains surface subgroups.

As we shall see, the main result of this paper is to highlight a new phase transition phenomenon, giving an analogue of the Freiheitssatz in the density model of random groups. In particular, it partially answers Gromov’s problem [15] 1.9 (iv): existence/nonexistence of non-free subgroups.

**Main results**

We say that a finite group presentation \( G = \langle X | R \rangle \) satisfies the *Magnus Freiheitssatz property* if every subset of \( X \) of cardinality \( |X| - 1 \) freely generates a free subgroup of \( G \). In particular, by Arzhantseva-Ol’shanskii’s result [5], a few-relator random group \( G_\ell \) has this property a.a.s.

We study the Magnus Freiheitssatz property in the density model of random groups.

Fix a set of \( m \geq 2 \) elements \( X = \{x_1, \ldots, x_m\} \) as generators of group presentations. Denote \( B_\ell \) as the set of cyclically reduced words of \( X^\pm = \{x_1^\pm, \ldots, x_m^\pm\} \) of length at most \( \ell \). A sequence of random groups \( (G_\ell(m, d))_{\ell \in \mathbb{N}} \) with \( m \geq 2 \) generators at density \( d \in [0, 1] \) is defined by group presentations \( G_\ell(m, d) := \langle X | R_\ell \rangle \) where \( R_\ell \) is a permutation invariant random subset of \( B_\ell \) with density \( d \). For example, it can be a uniform distribution on subsets of \( B_\ell \) of cardinality \( |B_\ell|^d \), or a Bernoulli sampling on \( B_\ell \) of parameter \( |B_\ell|^d - 1 \). We are interested in the asymptotic behavior of \( G_\ell(m, d) \) when \( \ell \to \infty \). See Section 3 or [29] for detailed definitions of random groups.

An *\( X \)-labeled graph* is a combinatorial graphs labeled by the generators \( x_1, \ldots, x_m \) and their inverses. The words read on the loops starting at a given vertex of the graph form a subgroup of the free group generated by \( X \). Details of \( X \)-labeled graphs are provided in Subsection 2.1.

The main result of this paper is a phase transition stated as follows.

**Theorem 1** (Theorem 5.1) *Let \( m, r \) be integers with \( m \geq 2 \) and \( 1 \leq r \leq m - 1 \). Let \( (G_\ell(m, d)) \) be a sequence of random groups with \( m \) generators at density \( d \in [0, 1] \). There is a phase transition at density\( d_r = \min \left\{ \frac{1}{2}, 1 - \log_2(m-1)(2r-1) \right\} \).

1. If \( d > d_r \), then a.a.s. \( x_1, \ldots, x_r \) generate the whole group \( G_\ell(m, d) \).
2. If \( d < d_r \), then a.a.s. every subgroup of \( G_\ell(m, d) \) generated by a reduced \( X \)-labeled graph \( \Gamma \) with \( b_1(\Gamma) \leq r \) and \( |\Gamma| \leq \frac{d_r - d}{2} \ell \) is a free group of rank \( r \).

In particular, when \( \Gamma \) is the wedge of \( r \) cycles of length 1 labeled by \( x_1, \ldots, x_r \) respectively, we have a.a.s. \( x_1, \ldots, x_r \) freely generate a free subgroup of \( G_\ell(m, d) \).

When \( r = 1 \), the consequence of the second assertion is actually a special case of a known result: for any \( m \geq 2 \), if \( 0 \leq d < 1/2 \), then a.a.s. the random group \( G_\ell(m, d) \) is torsion free. See Remark 5.2 and [24] V.d.

Let us focus on the particular case of the second assertion. By symmetry, the set \( \{x_1, \ldots, x_r\} \) can be replaced by any subset \( X_r \) of \( X \) of cardinality \( r \). In particular, if \( 0 \leq d < d_{m-1} \), then the group presentation \( G_\ell(m, d) = \langle X | R_\ell \rangle \) has the Magnus Freiheitssatz property. More precisely for the first assertion, we prove that if \( d > d_r \) then a.a.s. any generator \( x_i \) equals to a reduced word of \( X_r^\pm \) of length \( \ell - 1 \) in \( G_\ell(m, d) \). Therefore, any relator \( r_i \in R_\ell \) can be replaced by a reduced word \( r_i' \) of \( X_r^\pm \) of length at most \( \ell(\ell - 1) \). Construct \( R_\ell' \) by replacing every word of \( R_\ell \), we have the following result.
Corollary 2 Let $m \geq 2$ be an integer. Let $d_r$ be the number given in Theorem 1 for $1 \leq r \leq m - 1$. Let $(G_\ell (m, d))$ be a sequence of random groups with $m$ generators at density $d \in [0, 1]$.

For any integer $r$ with $2 \leq r \leq m - 1$, if the density $d$ satisfies $d_r < d < d_{r-1}$, then a.a.s. the random group $G_\ell (m, d) = \langle X | R_\ell \rangle$ admits a presentation with $r$ generators $\langle X_r | R'_\ell \rangle$ satisfying the Magnus Freiheitssatz property.

Remark 3 We emphasize that $R'_\ell$ contains relators of lengths varying from $\ell$ to $\ell^2$. Such a presentation cannot be studied using known methods in geometric or combinatorial group theory. Nevertheless, it gives us new examples of groups having the Magnus Freiheitssatz property.

Let $r = r(m, d)$ be the maximal number such that a.a.s. $x_1, \ldots, x_r$ freely generate a free subgroup of $G_\ell (m, d)$. By the phase transition at density $\frac{1}{2}$ [14], if $d > \frac{1}{2}$, then $r(m, d) = 0$. If $d \leq \frac{1}{2}$, by Theorem 1,

$$\frac{(2m - 1)^{1-d} - 1}{2} \leq r(m, d) \leq \frac{(2m - 1)^{1-d} + 1}{2}.$$

As shown in Fig. 1, because $r(m, d)$ is an integer, there is only one choice when $d$ is not $1/2$ or one of the $d_r$. Note that the value of $r(m, d)$ is not clear when $d \in \{d_1, \ldots, d_{m-1}, 1/2\}$.

Our main theorem (Theorem 5.1) is a generalized version of Theorem 1. In the second assertion, we can replace the set $\{x_1, \ldots, x_r\}$ by any set of $r$ words of $X^{\pm}$ of lengths at most $\frac{d_r - d}{3r - \ell}$.

Outline of the paper

In Section 2, we first recall some essential tools in combinatorial group theory (Stallings graphs [28] and van Kampen diagrams [17]). We introduce distortion van Kampen diagrams to study the distortion of subgroups of a finitely presented group.

In order to give a concrete construction of random groups with density, we discuss probabilistic models of random subsets in Section 3 and recall the intersection formula by Gromov [14]. Technical details are treated in [29].

Section 4 is dedicated to abstract van Kampen diagrams defined by Ollivier [23]. We apply his idea to distortion diagrams and define abstract distortion diagrams. The main technical
lemma for our main theorem (Theorem 5.1) is to estimate the number of fillings of a given abstract distortion diagram (Lemma 4.13).

In the last section, we state a local result on distortion van Kampen diagrams (Lemma 5.3) and prove the main theorem by this lemma. The last subsection is then devoted to the proof of Lemma 5.3.

2 Preliminaries on group theory

In this section, we fix a finite group presentation \( G = \langle X | R \rangle \) where \( X \) is the set of generators and \( R \) is the set of relators. A word \( u \) in the alphabet \( X^\pm \) is called reduced if it has no sub-words of type \( xx^{-1} \) or \( x^{-1}x \) for any \( x \in X \). If \( u \) and \( v \) are words that represent the same element in \( G \), we denote \( u =_G v \).

We consider oriented combinatorial graphs and 2-complexes as defined in Chapter III.2. of Lyndon and Schupp [19]. Hence a graph is a pair \( \Gamma = (V, E) \) where \( V \) is the set of vertices (also called points) and \( E \) is the set of (oriented) edges. Every edge \( e \in E \) has a starting point \( \alpha(e) \in V \), an ending point \( \omega(e) \in V \) and an inverse edge \( e^{-1} \in E \), satisfying \( \alpha(e^{-1}) = \omega(e), \omega(e^{-1}) = \alpha(e) \) and \( (e^{-1})^{-1} = e \). The vertices \( \alpha(e) \) and \( \omega(e) \) are called the endpoints of \( e \). An undirected edge is a pair of inverse edges \( \{e, e^{-1}\} \).

A path on a graph \( \Gamma \) is a non-empty finite sequence of edges \( p = e_1 \ldots e_k \) such that \( \omega(e_i) = \alpha(e_{i+1}) \) for \( i \in \{1, \ldots k-1\} \). The starting point and the ending point of the path \( p \) are defined by \( \alpha(p) = \alpha(e_1) \) and \( \omega(p) = \omega(e_k) \). The inverse of \( p \) is the path \( p^{-1} = e_k^{-1} \ldots e_1^{-1} \). A path is called reduced if there is no subsequence of the form \( ee^{-1} \). A loop is a path whose starting point and ending point coincide. In this case \( \alpha(p) = \omega(p) \) is called the starting point of the loop. A loop \( p = e_1 \ldots e_k \) is cyclically reduced if it is a reduced path with \( e_k \neq e_1^{-1} \).

An arc of a graph \( \Gamma \) is a reduced path passing only by vertices of degree 2, except possibly for its endpoints. A maximal arc is an arc that can not be extended to another arc. Note that the endpoints of a maximal arc are not of degree 2.

Denote by \( b_1(\Gamma) \) the first Betti number of a graph \( \Gamma \), which is the rank of its fundamental free group. The following two elementary facts for finite connected graphs can be deduced by Euler’s characteristic.

Lemma 2.1 Let \( r \geq 1 \) be an integer. Let \( \Gamma \) be a finite connected graph with \( b_1(\Gamma) = r \) and with no vertices of degree 1.

1. The number of vertices of degree at least 3 is bounded by \( 2(r - 1) \).
2. The number of maximal arcs of \( \Gamma \) is bounded by \( 3(r - 1) \).

Lemma 2.2 Let \( r \geq 1 \) be an integer. The number of topological types of finite connected graphs \( \Gamma \) with \( b_1(\Gamma) \leq r \) with no vertices of degree 1 is bounded by \( (2r)^{6r} \).

Proof If \( r = 1 \) then the only topological type is a simple cycle. If \( r \geq 2 \), we may draw \( a \leq 3(r - 1) \) arcs on a set of \( v \leq 2(r - 1) \) vertices. There are at most \((v^2)^a \leq (2r)^{6r}\) ways.

2.1 Stallings graphs (\( X \)-labeled graphs generating subgroups)

The idea of representing subgroups of a free group by graph immersions is introduced by Stallings [28]. It is then interpreted as graphs labeled by the generators of the free group by
Margolis and Meakin [21]. The strategy is applied by Arzhantseva and Ol’shanskii [5] to study subgroups of a few relator random group, then in Arzhantseva’s subsequent articles [2, 3] and [4].

In this article, we follow the methods [5] to study subgroups of a random group in the Gromov density model.

A \textit{X-labeled graph} (Stallings graph) is a graph \( \Gamma = (V, E) \) with a labeling function on edges by generators \( \phi : E \to X^\pm \), satisfying \( \phi(e^{-1}) = \phi(e)^{-1} \). We denote briefly \( \Gamma = (V, E, \phi) \). The labeling function \( \phi \) extends naturally on the paths of \( \Gamma \). If \( p = e_1 \ldots e_k \) is a path of \( \Gamma \), then the word \( \phi(p) = \phi(e_1) \ldots \phi(e_k) \) is called the \textit{labeling word} of \( p \). We say that a word \( u \) is \textit{readable} on a \( X \)-labeled graph \( \Gamma \) if there exists a path \( p \) of \( \Gamma \) whose labeling word is \( u \).

Let \( \Gamma = (V, E, \phi) \) be a finite connected \( X \)-labeled graph. Labeling words of the loops starting at a vertex \( o \in V \) form a subgroup \( H \) of \( G = \langle X \mid R \rangle \), which is the image of the fundamental group \( \pi_1(\Gamma, o) \) by the group homomorphism induced by \( \phi \).

Since we are interested in the freeness of a subgroup, conjugacy preserving operations do not matter. Observe that the conjugacy class of the obtained subgroup is unchanged by the following three \textit{reduction operations} on the graph with a based vertex \((\Gamma, o)\):

- Change the base vertex \( o \in V \).
- Fold a pair of edges with the same label and the same starting point.
- Eliminate a vertex of degree 1 together with its only adjacent edge.

\textbf{Definition 2.3} (c.f. [5] §1) A \( X \)-labeled graph is called \textit{reduced} if it has no pair of edges with the same label and the same starting point, and, it has no vertices of degree 1.

\textbf{Definition 2.4} If a subgroup \( H \) is a \textit{conjugate} of \( \pi_1(\Gamma, o) \) in \( G = \langle X \mid R \rangle \) for some vertex \( o \) of \( \Gamma \), we say that \( H \) is \textit{generated} by the \( X \)-labeled graph \( \Gamma \).

Conversely, any finitely generated subgroup \( H \) of rank \( r \) can be generated by a \textit{reduced} \( X \)-labeled graph of first Betti number \( r \). One can choose a system of generators \( h_1, \ldots, h_r \) of \( H \), label them on the wedge of \( r \) simple cycles of lengths \( |h_1|, \ldots, |h_r| \), and apply the three reduction operations.

### 2.2 Van Kampen diagrams

We consider van Kampen diagrams defined by Lyndon and Schupp in [19] Chapter III.9. A \textit{2-complex} is a triplet \( W = (V, E, F) \), where \((V, E)\) is a graph and \( F \) is the set of (oriented) faces. Every face \( f \in F \) has a boundary \( \partial f \), which is a cyclically reduced loop of \((V, E)\), and an inverse face \( f^{-1} \in F \) satisfying \( \partial(f^{-1}) = (\partial f)^{-1} \) and \( (f^{-1})^{-1} = f \). An \textit{undirected face} is a pair of inverse faces \( \{f, f^{-1}\} \). The size \( |W| \) is the number of undirected faces.

Note that our definition is slightly more precise than [19]: Every face \( f \in F \) has a starting point and an orientation given by \( \partial f \). If \( \partial f = e_1 \ldots e_k \), we say that \( e_i \) is \textit{attached} to \( f \) and is the \( i \)-th boundary edge of \( f \) for \( 1 \leq i \leq k \). In this case, we say that \( \{e_i, e_i^{-1}\} \) is attached to \( \{f, f^{-1}\} \). An edge is called \textit{isolated} if it is not attached to any face.

A \textit{van Kampen diagram} (with respect to \( G = \langle X \mid R \rangle \)) is a finite, planar (embedded in \( \mathbb{R}^2 \)) and simply connected 2-complex \( D = (V, E, F) \) with two compatible labeling functions, on edges by generators \( \varphi_1 : E \to X^\pm \) and on faces by relators \( \varphi_2 : F \to R^\pm \). Compatible means that \((V, E, \varphi_1)\) is a \( X \)-labeled graph, \( \varphi_2(f^{-1}) = \varphi_2(f)^{-1} \) and \( \varphi_1(\partial f) = \varphi_2(f) \). Note that if a diagram \( D \) has no isolated edges (for example, a disk), then \( \varphi_1 \) is determined by \( \varphi_2 \). We denote briefly \( D = (V, E, F, \varphi_1, \varphi_2) \).
65

According to [10] p.159, a van Kampen diagram is either a disk or a concatenation of disks and segments. The boundary \( \partial D \) is the boundary of \( \mathbb{R}^2 \setminus D \), which is a sub-graph of its underlying graph \((V, E)\). A boundary path is a path on \( \partial D \) defined in a natural way in [19] p.150. A boundary word of \( D \) is then the labeling word of a boundary path, unique up to cyclic conjugations and inversions. The boundary length of \( D \) is the length of a boundary path, denoted \( |\partial D| \).

Let \( D = (V, E, F, \varphi_1, \varphi_2) \) be a van Kampen diagram. A pair of faces \( f, f' \in F \) is reducible if they have the same label and there is a common edge on their boundaries at the same position (see Fig. 2). A van Kampen diagram is called reduced if there is no reducible pair of faces.

In 1933, E. van Kampen showed [17] that a word \( u \) of \( X^\pm \) is trivial in a finitely presented group \( G = \langle X | R \rangle \) if and only if it is a boundary word of a van Kampen diagram of \( G \). In [22] §11.6, A. Ol’shanskii improved this result to reduced diagrams.

**Lemma 2.5** (Van Kampen’s lemma, Ol’shanskii’s version) A word \( w \) of \( X^\pm \) is trivial in \( G = \langle X | R \rangle \) if and only if it is a boundary word of a reduced van Kampen diagram.

### 2.3 Distortion van Kampen diagrams

Let \( G = \langle X | R \rangle \) be a group presentation. For any word \( u \) of \( X^\pm \), we denote \( |u| \) its word length and \( \|u\|_G \) the distance between the endpoints of its image in the Cayley graph \( \text{Cay}(G, X) \).

Let \( \Gamma \) be a finite, connected and reduced \( X \)-labeled graph. Its universal covering \( \tilde{\Gamma} \) is an infinite, connected and reduced labeled tree, with a natural label-preserving graph morphism \( \tilde{\Gamma} \to \text{Cay}(G, X) \). If the map \( \tilde{\Gamma} \to \text{Cay}(G, X) \) is a \( \lambda \)-bi-Lipschitz embedding for some \( \lambda \geq 1 \), then every reduced word \( u \) readable on \( \Gamma \) satisfies \( \|u\|_G \geq \frac{1}{\lambda} |u| > 0 \), including those that form loops on \( \Gamma \). Hence the freeness of a subgroup generated by \( \Gamma \).

To solve this word problem, we introduce distortion van Kampen diagrams.

**Definition 2.6** (Distortion diagram) A distortion van Kampen diagram of \((G, \Gamma)\) is a pair \((D, p)\) where \( D \) is a van Kampen diagram of \( G \) and \( p \) is a cyclic sub-path of \( \partial D \) whose labeling word is readable on \( \Gamma \). (See Fig. 3.)
Lemma 2.7  Let $\lambda \geq 1$. If every disk-like and reduced distortion van Kampen diagram $(D, p)$ of $(G, \Gamma)$ satisfies

$$|p| \leq \frac{\lambda}{1+\lambda} |\partial D|,$$

then the map $\tilde{\Gamma} \to \text{Cay}(G, X)$ is a $\lambda$-bi-Lipschitz embedding.

In particular, any subgroup generated by $\Gamma$ is free.

Proof  Let $u$ be a reduced word that is readable on $\Gamma$. Let $v$ be one of the shortest word (whose image is a geodesic in $G$) such that $uv = G 1$. We shall check that $|u| \leq \lambda |v|$.

By van Kampen’s lemma (Lemma 2.5), there exists a reduced van Kampen diagram $D$ whose boundary word is $uv$. If $D$ is disk-like, then by the hypothesis $(\star)$ we have $|u| \leq \frac{1}{1+\lambda} (|u| + |v|)$, which gives $|u| \leq \lambda |v|$.

Otherwise, we decompose $D$ into disks and segments $D_1, \ldots, D_k$ (as in [10] p.159). The path of $v$ does not intersect itself because it is a geodesic in $G$. The path of $u$ on $D$ does not intersect itself. If it did, as $u$ is reduced, there would be a disk-like sub-diagram whose boundary word is readable on $\Gamma$, which is impossible because of $(\star)$.

Hence, for any $1 \leq i \leq k$, there are exactly two vertices on $\partial D_i$ separating $u$ and $v$, which are the only possible vertices of degree not equal to 2. The boundary word of $D_i$ is written as $u_i v_i$ where $u_i$ is a subword of $u$ and $v_i$ is a subword of $v$. If $D_i$ is a segment, then it is read once by $u$ and once by $v$ with opposite directions, so $|u_i| = |v_i| \leq \lambda |v_i|$. If $D_i$ is a disk, then $|u_i| \leq \lambda |v_i|$ by $(\star)$. We conclude that

$$|u| = \sum_{i=1}^k |u_i| \leq \sum_{i=1}^k \lambda |v_i| = \lambda |v|.$$  

$\square$

2.4 Hyperbolic groups

In this subsection, we recall several facts of hyperbolic groups defined by Gromov [13]. Let $G = \langle X | R \rangle$ be a finite group presentation. The Cayley graph $\text{Cay}(G, X)$ with the usual length metric is $\delta$-hyperbolic if each side of any geodesic triangle is $\delta$-close to the two other sides ([11] Chapter 1). In this case, $G$ is called a hyperbolic group.

We start by a criterion of hyperbolicity in [13] Chapter 2.3. See also [27] by Short and [11] Chapter 6. For a precise estimation of hyperbolicity constants, see [9] Lemma 3.11 by C. Champetier.

Theorem 2.8 (Isoperimetric inequality) Let $\ell$ be the longest relator length in $R$. The group $G = \langle X | R \rangle$ is hyperbolic if and only if there exists a real number $\beta > 0$ such that every reduced van Kampen diagram $D$ satisfies the following isoperimetric inequality:

$$|\partial D| \geq \beta \ell |D|.$$  

In this case, the Cayley graph $\text{Cay}(G, X)$ is $\delta$-hyperbolic with

$$\delta = \frac{4\ell}{\beta}.$$  

The local-global principle of hyperbolicity is due to M. Gromov in [13]. For other proofs, see [7] Chapter 8 by Bowditch or [26] by Papasoglu. We state here a sharpened version by Ollivier [25] Proposition 8.
Theorem 2.9 (Local-global principle of hyperbolicity) For any $\alpha > 0$ and $\varepsilon > 0$, there exists an integer $K = K(\alpha, \varepsilon)$ such that, if every reduced disk-like diagram $D$ with $|D| \leq K$ satisfies

$$|\partial D| \geq \alpha \ell |D|,$$

then every reduced diagram $D$ satisfies

$$|\partial D| \geq (\alpha - \varepsilon) \ell |D|.$$

Recall that a path $p$ in $\text{Cay}(X, R)$ is a $\lambda$-quasi-geodesic if every sub-path $u$ of $p$ satisfies $|u| \leq \lambda \|u\|_G$. It is a $L$-local $\lambda$-quasi geodesic if such an inequality is satisfied by every sub-path of length at most $L$. Here is the local-global principle for quasi-geodesics in hyperbolic groups, stated by Gromov [13] 7.2.A and 7.2.B. See [11] Chapter 3 for a proof.

Theorem 2.10 Let $G = \langle X | R \rangle$ be a group presentation such that $\text{Cay}(G, X)$ is $\delta$-hyperbolic. Let $\lambda \geq 1$.

1. Every $\lambda$-quasi-geodesic is $100\delta(1 + \log \lambda)$ close to any geodesic joining its endpoints.
2. Every $1000\delta$-local $\lambda$-quasi-geodesic is a (global) $2\lambda$-quasi-geodesic.

3 Random subsets and random groups

In this section, we recall the definition of random groups with density by Gromov [14]. Proofs of Proposition 3.2, Proposition 3.3, Theorem 3.4 and Theorem 3.5 are in [29].

3.1 Densable sequences of random subsets

A random subset $A$ of a finite set $E$ is a $\mathcal{P}(E)$-valued random variable, where $\mathcal{P}(E)$ is the set of subsets of $E$. We say that $A$ is permutation invariant if $\Pr(A = a) = \Pr(A = \sigma(a))$ for any permutation $\sigma$ of $E$ and any subset $a$ of $E$.

In this subsection, we consider a sequence of finite sets $E = (E_\ell)_{\ell \in \mathbb{N}}$ with $|E_\ell| \to \infty$. Let $(Q_\ell)$ be a sequence of events. We say that the event $Q_\ell$ holds asymptotically almost surely if $\Pr(Q_\ell) \to 1$. We denote briefly a.a.s. $Q_\ell$. Note that the intersection of a finite number of events that hold a.a.s. is an event that holds a.a.s. In addition, we have the following proposition.

Proposition 3.1 Let $Q = (Q_\ell)$, $R = (R_\ell)$ be sequences of events. If a.a.s. $Q_\ell$ and a.a.s. “$R_\ell$ under the condition $Q_\ell$”, then a.a.s. $R_\ell$.

Proof Denote by $\overline{Q}_\ell$ the complement of $Q_\ell$. By the two hypotheses, $\Pr(Q_\ell) \to 1$ and $\Pr(R_\ell | Q_\ell) \to 1$. Either $\overline{Q}_\ell$ is empty and $\Pr(R_\ell) = \Pr(R_\ell | Q_\ell) \to 1$, or by the formula of total probability

$$\Pr(R_\ell) = \Pr(Q_\ell) \Pr(R_\ell | Q_\ell) + \Pr(\overline{Q}_\ell) \Pr(R_\ell | \overline{Q}_\ell) \to 1.$$  

Let $d \in \{-\infty\} \cup [0, 1]$. A sequence of random subsets $A = (A_\ell)$ of $E = (E_\ell)$ is densable with density $d$ if the sequence of real-valued random variables

$$\log |E_\ell|(|A_\ell|)$$
converges in probability (or in distribution) to the constant $d$. We denote 
\[ \text{dens } A = d. \]

By definition, \( \text{dens } A = d \) if and only if 
\[ \forall \varepsilon > 0 \text{ a.a.s. } |E_\ell|^{d-\varepsilon} \leq |A_\ell| \leq |E_\ell|^{d+\varepsilon}. \]

In particular, \( \text{dens } A = -\infty \) if and only if a.a.s. \( A_\ell = \emptyset \); \( \text{dens } A = 0 \) if and only if a.a.s. \( A_\ell \neq \emptyset \) and \( |A_\ell| \) is sub-exponential.

Here is the main example of a densable sequence of permutation invariant random subsets. The proofs of Theorem 3.4 and Theorem 3.5 are much simpler in this model (see [29]).

**Proposition 3.2** (Bernoulli density model, [29] Proposition 1.12) Let \( 0 < d \leq 1 \). Let \((A_\ell)\) be a sequence of random subsets of \((E_\ell)\) such that every element \( e \in E_\ell \) is taken independently with probability \( p_\ell = |E_\ell|^{d-1} \). Then \( A = (A_\ell) \) is a densable sequence of permutation invariant random subsets with density \( d \).

Note that in the case \( d = 0 \), the Bernoulli model is not densable. If \( A_\ell \) is a Bernoulli sequence with density \( d > 0 \), then for any distinct elements \( e_1, \ldots, e_k \) in \( E_\ell \), we have 
\[ \Pr(e_1, \ldots, e_k \in A_\ell) = p_\ell^k = |E_\ell|^{k(d-1)} \]
by independence. This property is, in general, not true for an arbitrary densable sequence of permutation invariant random subsets. Nevertheless, it can be approached asymptotically.

**Proposition 3.3** (Similar to [29] Lemma 3.10) Let \( A = (A_\ell) \) be a densable sequence of permutation invariant random subsets of \( E = (E_\ell) \) with density \( d \). Let \( \varepsilon > 0 \). Denote \( Q_\ell \) the event \( |E_\ell|^{d-\varepsilon} \leq |A_\ell| \leq |E_\ell|^{d+\varepsilon} \) (we have a.a.s. \( Q_\ell \) by definition). Let \( e_1, \ldots, e_k \) be distinct elements in \( E_\ell \). For \( \ell \) large enough,
\[ |E_\ell|^{k(d-1-2\varepsilon)} \leq \Pr(e_1, \ldots, e_k \in A_\ell | Q_\ell) \leq |E_\ell|^{k(d-1+2\varepsilon)}. \]

### 3.2 The intersection formula

We recall here the intersection formula for random subsets. See [14] for the original version by M. Gromov, and [29] Section 2 for a proof.

**Theorem 3.4** (The intersection formula) Let \( A = (A_\ell) \), \( B = (B_\ell) \) be independent densable sequences of permutation invariant random subsets.

1. If \( \text{dens } A + \text{dens } B < 1 \), then a.a.s. \( A_\ell \cap B_\ell = \emptyset \).
2. If \( \text{dens } A + \text{dens } B > 1 \), then \( A \cap B := (A_\ell \cap B_\ell) \) is a densable sequence of permutation invariant random subset. In addition,
\[ \text{dens}(A \cap B) = \text{dens } A + \text{dens } B - 1. \]
In particular, a.a.s. \( A_\ell \cap B_\ell \neq \emptyset \).

A fixed subset can be regarded as a constant random subset. The density of a sequence of fixed subsets can be defined by the same way. Note that a sequence of subsets \( F = (F_\ell) \) of \( E = (E_\ell) \) is densable with density \( d \) if and only if 
\[ |F_\ell| = |E_\ell|^{d+o(1)}. \]

We consider also the intersection between a sequence of random subsets and a sequence of fixed subsets. See [29] Section 3 for a proof.
Theorem 3.5 ([29] Theorem 3.7) Let $A = (A_\ell)$ be a densable sequence of permutation invariant random subsets of $E$. Let $F = (F_\ell)$ be a densable sequence of fixed subsets.

1. If $\text{dens } A + \text{dens } F < 1$, then a.a.s. $A_\ell \cap F_\ell = \emptyset$.
2. If $\text{dens } A + \text{dens } F > 1$, then the sequence $A \cap F$ is densable in $E$, with density

$$\text{dens } A + \text{dens } F - 1.$$ 

In addition, $A \cap F$ is densable and permutation invariant in $F$, with density

$$\frac{\text{dens } A + \text{dens } F - 1}{\text{dens } F}.$$

$\square$

3.3 The density model of random groups

Fix an alphabet $X = \{x_1, \ldots, x_m\}$ as generators of group presentations. Let $B_\ell$ be the set of cyclically reduced words on $X^\pm = \{x_1^\pm, \ldots, x_m^\pm\}$ of lengths at most $\ell$. Note that $|B_\ell| = (2m - 1)\ell + o(\ell)$.

We consider a sequence of random groups $G(m, d) = (G_\ell(m, d))$ defined by random presentations $G_\ell(m, d) := \langle X | R_\ell \rangle$ where $R = (R_\ell)$ is a densable sequence of permutation invariant random subsets of $B = (B_\ell)$ with density $d$. Such a sequence is called a sequence of random groups at density $d$.

The number of relators $|R_\ell|$ is a real-valued random variable and is concentrated to $(2m - 1)^d\ell$. More precisely, for any $\varepsilon > 0$ a.a.s.

$$(2m - 1)^{d\ell - \varepsilon \ell} \leq |R_\ell| \leq (2m - 1)^{d\ell + \varepsilon \ell}.$$ 

We are interested in asymptotic behaviors of a sequence of random groups. In his book [14], Gromov observed that there is a phase transition at density $1/2$.

Theorem 3.6 (Phase transition at density $1/2$) Let $G(m, d) = (G_\ell(m, d)) = (\langle X | R_\ell \rangle)$ be a sequence of random groups at density $d$.

1. If $d > 1/2$, then a.a.s. $G_\ell(m, d)$ is a trivial group.
2. If $d < 1/2$, then a.a.s. $G_\ell(m, d)$ is a hyperbolic group, and the Cayley graph $\text{Cay}(G_\ell, X)$ is $\delta$-hyperbolic with $\delta = \frac{4\ell}{1 - 2d}$.

In addition, for any $s > 0$, a.a.s. every reduced van Kampen diagram $D$ of $G_\ell(m, d)$ satisfies the isoperimetric inequality

$$|\partial D| \geq (1 - 2d - s)\ell |D|.$$ 

The proof of our main theorem (Theorem 5.1) is very similar to the strategy of proving this theorem: For the first point we apply the intersection formula to show that some type of relation exists; for the second assertion we convert the problem into a diagram problem and apply some local-global argument in hyperbolic groups.

For Theorem 3.6, we give here a proof for the first assertion and an idea of proof for the second assertion.

Proof of Theorem 3.6.1 Let $S_\ell$ be the set of cyclically reduced words of length exactly $\ell$. The sequence $(S_{\ell-1})$ is a fixed sequence of subsets of $B = (B_\ell)$ of density 1. By the
intersection formula (Theorem 3.5), the two sequences \( (x_1 R_\ell \cap x_1 S_{\ell-1}) \) and \( (R_\ell \cap x_1 S_{\ell-1}) \) are both sequences of random subsets of \( (x_1 S_{\ell-1}) \) with density \( d \). By the intersection formula between random subsets (Theorem 3.4), their intersection is a sequence of random subsets with density \( (2d - 1) > 0 \), which is a.a.s. not empty. Thus, a.a.s. there exists a word \( w \in S_{\ell-1} \) such that \( w \in R_\ell \) and \( x_1 w \in R_\ell \), so a.a.s. \( x_1 = 1 \) in \( G_\ell \) by canceling \( w \).

The argument works for every generator \( x_i \in X \). By intersecting a finite number of a.a.s. events, a.a.s. \( G_\ell \) is isomorphic to the trivial group. \( \square \)

By Theorems 2.8 and 2.9, to prove Theorem 3.6.2, it is sufficient to find a local isoperimetric inequality. See [24] for a proof by Ollivier.

**Lemma 3.7 (Local isoperimetric inequality)** Let \( s > 0 \). If \( d < 1/2 \), then for \( K = K (1 - 2d - \frac{s}{2}, \frac{s}{2}) \) provided by Theorem 2.9, a.a.s. any reduced disc-like diagram \( D \) of \( G_\ell(m, d) \) with at most \( K \) faces satisfies the isoperimetric inequality

\[
|\partial D| \geq \left(1 - 2d - \frac{s}{2}\right) \ell |D|.
\]

\( \square \)

### 4 Abstract diagrams

According to Proposition 3.3, the probability that a van Kampen diagram appears in a given random group at density is determined by the number of relators used in this diagram. Two van Kampen diagrams having the same underlying 2-complex may not use the same number of relators, and should be treated separately.

For example, to check that if a group satisfies the \( C'(\lambda) \) small cancellation condition (see [29] Theorem 4.3), we consider van Kampen diagrams whose underlying 2-complex consists of two faces \( f_1, f_2 \) sharing a common path of length \( \lambda \min\{|\partial f_1|, |\partial f_2|\} \). We then need to consider the two types of diagrams in Fig. 4, one using two distinct relators and the other one using one relator.

When the area of the common underlying 2-complex is larger, the number of types of diagrams increases and can be very sophisticated. In 2004, Ollivier introduced abstract van Kampen diagrams ([23] p.10) to surround this problem.

#### 4.1 Abstract van Kampen diagrams

**Definition 4.1 (Abstract diagram, Ollivier [24])** An abstract van Kampen diagram \( \tilde{D} \) is a finite, planar and simply-connected 2-complex \( (V, E, F) \) with a labeling function on faces by integer numbers \( \tilde{\varphi}_2 : F \to \{1, 1^-, 2, 2^-, \ldots, k, k^-, \ldots, k\} \) satisfying \( \tilde{\varphi}_2(f^{-1}) = \tilde{\varphi}_2(f)^- \). We denote \( D = (V, E, F, \tilde{\varphi}_2) \),
By convention, \((i^-)^- = i\) for any \(1 \leq i \leq k\). The numbers \(\{1, \ldots, k\}\) are called abstract relators of \(\widetilde{D}\).

Similarly to a van Kampen diagram, a pair of faces \(f, f' \in F\) is reducible if they have the same label, and they share an edge at the same position of their boundaries. An abstract diagram is called reduced if there is no reducible pair of faces.

Let \(D = (V, E, F, \varphi_1, \varphi_2)\) be a van Kampen diagram of a group presentation \(G = \langle X \mid R \rangle\). Let \(\{r_1, \ldots, r_k\} \subset R\) be the set of relators used in \(D\). Define \(\widetilde{\varphi}_2 : F \rightarrow \{1, 1^-, \ldots, k, k^-\}\) by \(\widetilde{\varphi}_2(f) = i\) if \(\varphi_2(f) = r_i\). We obtain an abstract diagram \(\widetilde{D} = (V, E, F, \widetilde{\varphi}_2)\) with \(k\) abstract relators, called an underlying abstract diagram of \(D\).

An abstract diagram \(\widetilde{D}\) is fillable by a group presentation \(G = \langle X \mid R \rangle\) (or by a set of relators \(R\)) if there exists a van Kampen diagram \(D\) of \(G\), called a filled diagram of \(\widetilde{D}\), whose underlying abstract diagram is \(\widetilde{D}\). That is to say, there exists \(k\) different relators \(r_1, \ldots, r_k \in R\) such that the construction \(\varphi_2(f) := r_{\widetilde{\varphi}_2(f)}\) gives a diagram \(D = (V, E, F, \varphi_1, \varphi_2)\) of \(G\). In Fig. 5, the abstract diagram has two abstract relators 1, 2 and is filled by the relators \(r_1, r_2\). The \(k\)-tuple \((r_1, \ldots, r_k)\) is called a filling of \(\widetilde{D}\). As we picked different relators, \(\widetilde{D}\) is reduced if and only if a filled diagram \(D\) is reduced.

We assume that faces with the same label of \(\widetilde{D}\) have the same boundary length, otherwise \(\widetilde{D}\) would never be fillable. Denote \(\ell_i\) the length of the abstract relator \(i\) for \(1 \leq i \leq k\). Let \(\ell = \max\{\ell_1, \ldots, \ell_k\}\) be the maximal boundary length of faces of \(\widetilde{D}\).

**Notation** The pairs of integers \((i, 1), \ldots, (i, \ell_i)\) are called abstract letters of \(i\).

The set of abstract letters of \(\widetilde{D}\), denoted \(\widetilde{X}\), is then a subset of \(\{1, \ldots, k\} \times \{1, \ldots, \ell\}\), endowed with the lexicographic order.

We decorate undirected edges of \(\widetilde{D}\) by abstract letters and directions. Let \(f \in F\) labeled by \(i\) and let \(e \in E\) at the \(j\)-th position of \(\partial f\). The edge \(\{e, e^{-1}\}\) is decorated, on the side of \(\{f, f^{-1}\}\), by an arrow indicating the direction of \(e\) and the abstract letter \((i, j)\). This decoration on \(\{e, e^{-1}\}\) is called the decoration from \(f\) at the position \(j\). The number of decorations on an edge \(\{e, e^{-1}\}\) is the number of its adjacent faces \(\{f, f^{-1}\}\) with multiplicity \((0, 1\ or\ 2\ when\ \widetilde{D}\ is\ planar)\).

For any filling \((r_1, \ldots, r_k)\) of \(\widetilde{D}\), we construct the canonical function \(\phi : \widetilde{X} \rightarrow X^\pm\) such that \(r_i = \phi(i, 1) \ldots \phi(i, \ell_i)\) for any \(1 \leq i \leq k\). If an edge \(\{e, e^{-1}\}\) is decorated by two abstract letters \((i, j), (i', j')\), then \(\phi(i', j') = \phi(i, j)\) if they have the same direction, or \(\phi(i', j') = (\phi(i, j))^{-1}\) if they have opposite directions. For example, in the diagram of Fig. 6, there is an edge decorated by two abstract letters \((1, 4)\) and \((2, 3)\) with opposite directions, so we have \(\phi(1, 4) = \phi(2, 3)^{-1}\).

Note that if \(\widetilde{D}\) is reduced, then by definition an abstract letter can not be decorated twice on an edge with the same direction (Fig. 7 left-hand side). If \(\widetilde{D}\) is fillable (by the set of all relators), then an abstract letter \((i, j)\) can not be decorated twice on an undirected edge with opposite directions (Fig. 7 right-hand side), otherwise we have \(\phi(i, j) = (\phi(i, j))^{-1}\) in the set of generators \(X\).
In the following, we assume that $\tilde{D}$ is fillable and reduced, so that the abstract letters decorated on an edge $\{e, e^{-1}\}$ are all different, and the two types of sub-diagrams in Fig. 7 can not appear in $\tilde{D}$. In particular, for any edge $\{e, e^{-1}\}$, there exists a unique face $\{f, f^{-1}\}$ (at a unique position) from which the decoration is (lexicographically) minimal. Whence the following two definitions.

**Definition 4.2** (Preferred face of an edge) Let $\{e, e^{-1}\}$ be an edge of $\tilde{D}$. Let $\{f, f^{-1}\}$ be the adjacent face of $\{e, e^{-1}\}$ from which the decoration is minimal. Then $\{f, f^{-1}\}$ is called the preferred face of $\{e, e^{-1}\}$.

**Definition 4.3** (free-to-fill) An abstract letter $(i, j)$ of $\tilde{D}$ is free-to-fill if, for any edge $\{e, e^{-1}\}$ decorated by $(i, j)$, it is the minimal decoration on $\{e, e^{-1}\}$.

Note that an abstract letter $(i, j)$ is free-to-fill if and only if every face $f$ labeled by $i$ is the preferred face of its $j$-th boundary edge. In other words, if $(i, j)$ is not free-to-fill, then there exists an edge $\{e, e^{-1}\}$ decorated by $(i, j)$ that has another decoration $(i', j') < (i, j)$.

For example, in the abstract diagram of Fig. 8, (1, 4), (2, 1) and (2, 2) are not free-to-fill. The other abstract letters are free-to-fill.

Denote $F^+ = \{ f \in F \mid \tilde{\varphi}_2(f) \in \{1, \ldots, k\} \}$. It gives a preferred orientation for each undirected face of $\tilde{D} = (V, E, F, \tilde{\varphi}_2)$. Let $\overline{E}$ be the set of undirected edges of $\tilde{D}$.

**Lemma 4.4** Let $\tilde{D}$ be a reduced fillable abstract diagram without isolated edges. For every face $f \in F^+$, let $\overline{E}_f$ be the set of edges $\{e, e^{-1}\}$ on the boundary of $\{f, f^{-1}\}$ such that $\{f, f^{-1}\}$ is the preferred face of $\{e, e^{-1}\}$. Then

$$\overline{E} = \bigsqcup_{f \in F^+} \overline{E}_f.$$
Proof For every edge \{e, e^{-1}\} there exists a unique face \( f \in F^+ \) such that \{e, e^{-1}\} \in \overline{E}_f. Hence, the sets \( \overline{E}_f \) with \( f \in F^+ \) are pairwise disjoint. Their reunion is the set of edges because every edge is adjacent to at least one face.

\[\square\]

4.2 Abstract distortion van Kampen diagrams

We generalize the idea of abstract diagrams to distortion van Kampen diagrams.

Definition 4.5 (Abstract distortion diagram) An abstract distortion van Kampen diagram is a pair \((\tilde{D}, p)\) where \(\tilde{D}\) is an abstract diagram and \(p\) is a path on \(\partial \tilde{D}\).

Let \( G = \langle X | R \rangle \) be a group presentation and let \( \Gamma \) be a \(X\)-labeled graph. An abstract distortion diagram \((\tilde{D}, p)\) is fillable by the pair \((G, \Gamma)\) (or by the pair \((R, \Gamma)\)) if there exists a filled diagram \(D\) of \(\tilde{D}\) such that \((D, p)\) is a distortion diagram of \((G, \Gamma)\). The distortion diagram \((\tilde{D}, p)\) is called a filled distortion diagram of \((\tilde{D}, p)\).

In the following, an abstract distortion diagram \((\tilde{D}, p)\) is reduced, fillable and without isolated edges. Recall that \(\tilde{X} \subset \{1, \ldots, k\} \times \{1, \ldots, \ell\}\) is the set of abstract letters. Let \(\overline{p}\) be the set of undirected edges given by \(p\). In an abstract distortion diagram, we distinguish between two types of free-to-fill abstract letters: those that decorate an edge of \(\overline{p}\) and those that do not.

Definition 4.6 Let \((i, j)\) be an abstract letter of \((\tilde{D}, p)\).

(i) \((i, j)\) is free-to-fill if it is free-to-fill for the abstract diagram \(\tilde{D}\) and it does not decorate any edge of \(\overline{p}\).

(ii) \((i, j)\) is semi-free-to-fill if it is free-to-fill for the abstract diagram \(\tilde{D}\) and it decorates an edge of \(\overline{p}\).

(iii) Otherwise, \((i, j)\) is not free-to-fill.

Notation Let \(i\) be an abstract relator of \(\tilde{D}\). We denote \(\alpha_i\) the number of faces labeled by \(i\), \(\eta_i\) the number of free-to-fill abstract letters of \(i\), and \(\eta'_i\) the number of semi-free-to-fill abstract letters of \(i\).

Note that \(\ell_i - \eta_i - \eta'_i\) is the number of non free-to-fill edges.

Lemma 4.7 Recall that \(\overline{E}_f\) is the set of edges on the boundary of \(f\) that prefers \(\{f, f^{-1}\}\). Let \(i\) be an abstract relator. For any face \( f \in F \) with \(\tilde{\varphi}_2(f) = i\), we have

\[\eta'_i \leq |\overline{E}_f \cap \overline{p}| \quad \text{and} \quad \eta_i \leq |\overline{E}_f| - |\overline{E}_f \cap \overline{p}|.\]

Proof Let \(\{e, e^{-1}\}\) be the edge at the \(j\)-th position of \(\partial f\). It is decorated by \((i, j)\). If \(\{f, f^{-1}\}\) is not preferred by \(\{e, e^{-1}\}\), then \((i, j)\) is not free-to-fill because there is a smaller decoration on \(\{e, e^{-1}\}\).

Thus, if \(\{e, e^{-1}\} \in \overline{E}_f \cap \overline{p}\) then \((i, j)\) is semi-free-to-fill, which gives the first inequality. Similarly, if \(\{e, e^{-1}\} \in \overline{E}_f \setminus \overline{p}\) then \((i, j)\) is free-to-fill, so we have the second inequality.

\[\square\]

Lemma 4.8 Recall that \(\overline{E}\) is the set of undirected edges. The following two inequalities hold.

\[\sum_{i=1}^{k} \alpha_i \eta'_i \leq |\overline{p}|, \quad \sum_{i=1}^{k} \alpha_i \eta_i \leq |\overline{E}| - |\overline{p}|.\]
Proof By Lemma 4.7, for every $1 \leq i \leq k$
\begin{equation*}
\alpha_i \eta_i' \leq \sum_{f \in F, \widetilde{\varphi}_2(f) = i} |E_f \cap |p||.
\end{equation*}
Apply Lemma 4.4,
\begin{equation*}
\sum_{i=1}^{k} \alpha_i \eta_i' \leq \sum_{f \in F^+} |E_f \cap |p|| \leq |p|.
\end{equation*}
We get the second inequality by replacing $\eta_i'$ by $\eta_i$ and $|p|$ by $|E \setminus |p||$. □

4.3 The number of fillings of an abstract distortion diagram

Recall that $X = \{x_1, \ldots, x_m\}$ is a fixed set of $m \geq 2$ generators and that $B_\ell$ is the set of cyclically reduced words of $X^\pm = \{x_1^\pm, \ldots, x_m^\pm\}$ of length at most $\ell$. For a graph $\Gamma$, denote by $|\Gamma|$ the number of its undirected edges and $b_1(\Gamma)$ its first Betti number, which is the rank of its fundamental free group.

Throughout this subsection, we fix an integer $1 \leq r \leq m - 1$, a $X$-labeled graph $\Gamma$ with $b_1(\Gamma) = r$, and an abstract distortion diagram $(\widetilde{D}, p)$ with $k \geq 1$ abstract relators. Assume that $\widetilde{D}$ is reduced, fillable and has no isolated edges, and that $\ell$ is the longest boundary length of faces of $\widetilde{D}$.

Denote by $N_\ell(\widetilde{D}, p, \Gamma)$ the set of fillings $(r_1, \ldots, r_k)$ of $(\widetilde{D}, p)$ by $(B_\ell, \Gamma)$. In Lemma 4.13, we give an upper bound of the number of fillings $|N_\ell(\widetilde{D}, p, \Gamma)|$.

Lemma 4.9 The number of reduced words $u$ of length $L$ that is readable on $\Gamma$ is at most $2|\Gamma|(2r - 1)^L$.

Proof We estimate the number of paths $p$ on $\Gamma$ whose labeling word can be reduced. Take an oriented edge of $\Gamma$ as the first edge of $p$, there are $2|\Gamma|$ choices. Every vertex is of degree at most $2r$ because $b_1(\Gamma) = r$. As $p$ is reduced, every time we take the next edge, there are at most $(2r - 1)$ choices. Hence, there are at most $2|\Gamma|(2r - 1)^L$ paths. □

Remark 4.10 Note that when $r = 1$, a reduced $X$-labeled graph $\Gamma$ with $b_1(\Gamma) = 1$ is a simple cycle (since there is no isolated edges). The number of reduced words $u$ of length $L$ that is readable on $\Gamma$ is exactly $2|\Gamma|$, as the choice of a starting point and an orientation on $\Gamma$ decides such a word.

A vertex of $(\widetilde{D}, p)$ is called distinguished if it is either of degree at least 3, or the starting point of a face, or an endpoint of $p$. Let $i$ be an abstract letter of $(\widetilde{D}, p)$. It can be regarded as a 2-complex (see Fig. 9) with two inverse faces $\{i, i^-\}$ and $2\ell_i$ edges $(i, 1), \ldots, (i, \ell_i)$ with their inverses, such that $\partial i = (i, 1) \ldots (i, \ell_i)$. 

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A vertex of \( \partial i \) is marked if there exists a face \( f \) of \( \tilde{D} \) labeled by \( i \) such that the corresponding vertex is distinguished. Note that the starting point of \( \partial i \) is marked. Marked vertices divide the loop \( \partial i \) into segments, called elementary segments.

Consequently, an elementary segment is a sequence of abstract letters \( (i, j)(i, j + 1) \ldots (i, j + t) \) such that, if a path \( e_j \ldots e_{j+t} \) on \( \tilde{D} \) is decorated by \( (i, j) \ldots (i, j + t) \), then it passes by no distinguished points except for its endpoints.

**Lemma 4.11** Let \( (i, j) \ldots (i, j + t) \) be an elementary segment of \( (\tilde{D}, p) \). The abstract letters \( (i, j), \ldots, (i, j + t) \) are either all free-to-fill, or all semi-free-to-fill, or all not free-to-fill.

**Proof** We shall check that if the vertex between two consecutive abstract letters \( (i, j) \) and \( (i, j + 1) \) is not marked, then they are of the same type.

Recall that if an edge \( \{e_1, e_1^{-1}\} \) is decorated by \( (i, j) \) from the face \( \{f, f^{-1}\} \), then there is an edge \( \{e_2, e_2^{-1}\} \) next to \( \{e_1, e_1^{-1}\} \), decorated by \( (i, j + 1) \) from the same face \( \{f, f^{-1}\} \). Assume that the vertex between \( (i, j) \) and \( (i, j + 1) \) is not marked so that the vertex between \( \{e_1, e_1^{-1}\} \) and \( \{e_2, e_2^{-1}\} \) is not distinguished.

We suppose by contradiction that \( (i, j) \) and \( (i, j + 1) \) are not of the same type. There are \( 3^2 - 3 = 6 \) cases, grouped into three cases.

**case 1.** \((i, j)\) is semi-free-to-fill and \((i, j + 1)\) is free-to-fill, or inversely:
Recall that if \((i, j)\) is semi-free-to-fill in the abstract distortion diagram \((\tilde{D}, p)\), then it decorates an undirected edge \(\{e_1, e_1^{-1}\}\) on \(p\). As \((i, j + 1)\) is free-to-fill, the edge \(\{e_2, e_2^{-1}\}\) decorated by \((i, j + 1)\) from the same face is not on \(p\) (see Fig. 10). So the vertex between \(\{e_1, e_1^{-1}\}\) and \(\{e_2, e_2^{-1}\}\) is distinguished, contradiction.

**case 2.** \((i, j)\) is not free-to-fill, and \((i, j + 1)\) is free-to-fill or semi-free-to-fill:
By definition, there is an edge \(\{e_1, e_1^{-1}\}\) decorated by \((i, j)\) having a smaller decoration \((i', j') < (i, j)\) (see Fig. 11). Let \(\{f, f^{-1}\}, \{f', f'^{-1}\}\) be the faces attached by \(\{e_1, e_1^{-1}\}\) such that \(f\) is labeled by \(i\) and \(f'\) is labeled by \(i'\).
Let \(\{e_2, e_2^{-1}\}\) be the edge next to \(\{e_1, e_1^{-1}\}\), decorated by \((i, j + 1)\) from the face \(\{f, f^{-1}\}\). As the vertex between \(e_1\) and \(e_2\) is not distinguished, \(\{e_2, e_2^{-1}\}\) is attached to the face \(\{f', f'^{-1}\}\). It is then decorated by \((i', j' + 1)\) or \((i', j' - 1)\) from \(\{f', f'^{-1}\}\). Because \((i, j + 1)\) is free-to-fill, we have \((i, j + 1) < (i', j' + 1)\) or \((i, j + 1) < (i', j' - 1)\). Both are impossible because \((i, j) > (i', j')\).

**case 3.** \((i, j)\) is free-to-fill or semi-free-to-fill, and \((i, j + 1)\) is not free-to-fill:
There is an edge \(\{e_1, e_1^{-1}\}\) decorated by \((i, j + 1)\) having a smaller decoration \((i', j') < (i, j + 1)\) (Fig. 12). By the same argument of case 2, \((i, j) < (i', j' + 1)\) or \((i, j) < (i', j' - 1)\). The second one is obviously impossible. If the first one held, then \((i', j') < (i, j + 1) < (i', j' + 2)\), so \((i', j') = (i, j)\), and there was an edge decorated by \((i, j)\) and \((i, j + 1)\) with opposite directions. The canonical function \(\phi : \tilde{X} \rightarrow X\) gives \(\phi(i, j + 1) = \phi(i, j)^{-1}\), which is impossible because \(r_i = \phi(i, 1) \ldots \phi(i, \ell_i)\) should be a reduced word.
Lemma 4.12 Let \((\widetilde{D}, p)\) be an abstract distortion diagram with no isolated edges.

(i) The number of distinguished vertices of \((\widetilde{D}, p)\) is at most \(3|\widetilde{D}|\).

(ii) The number of elementary segments of an abstract letter \(i\) is at most \(3|\widetilde{D}|^2\).

Proof The underlying 1-complex of \(\widetilde{D}\) is a graph of first Betti number \(|\widetilde{D}|\) without isolated edges. By Lemma 2.1 there are at most \(2(|\widetilde{D}|−1)\) vertices of degree \(≥3\). We add \(k ≤|\widetilde{D}|\) starting points and 2 endpoints of \(p\), there are at most \(3|\widetilde{D}|\) distinguished vertices on \((\widetilde{D}, p)\).

The number of faces of \(\widetilde{D}\) labeled by \(i\) is at most \(|\widetilde{D}|\). Every face brings at most \(3|\widetilde{D}|\) marked vertices to \(\partial i\), so there are at most \(3|\widetilde{D}|^2\) marked vertices on \(\partial i\).

Recall that for any abstract relator \(i\) of an abstract diagram, we denote \(\eta_i\) as the number of free-to-fill abstract letters of \(i\) and \(\eta'_i\) as the number of semi-free-to-fill abstract letters of \(i\).

Lemma 4.13 Let \(X\) be a set of \(m ≥ 2\) generators. Let \(1 ≤ r ≤ m − 1\) be an integer. Let \(\Gamma\) be a \(X\)-labeled graph with \(b_1(\Gamma) = r\). Let \((\widetilde{D}, p)\) be a reduced abstract distortion diagram with no isolated edges and with \(k\) abstract relators.

\[
|N_\ell(\widetilde{D}, p, \Gamma)| \leq \left(\frac{2m}{2m - 1}\right)^k (2|\Gamma|)^{3|\widetilde{D}|^2} (2m − 1)^k \sum_{i=1}^k \eta_i (2r − 1)^{\eta'_i}
\]

Proof We fill the abstract letters of \(\widetilde{D}\) in lexicographic order. We shall prove that if the abstract relators \(1, \ldots, i − 1\) are filled, then there are at most \(\left(\frac{2m}{2m - 1}\right)^k (2|\Gamma|)^{3|\widetilde{D}|^2} (2m − 1)^{\eta_i} (2r − 1)^{\eta'_i}\) ways to fill the \(i\)-th abstract relator.

By Lemma 4.11, we fill elementary segments of \(i\) in order. Let \(u\) be an elementary segment of \(i\). If \(u\) is free-to-fill, then there are at most \((2m − 1)^{|u|}\) ways to fill \(u\) (or at most \(2m(2m − 1)^{|u|−1}\) ways if \(u\) is the first segment of \(i\)). If \(u\) is semi-free-to-fill, then there are at most \(2|\Gamma|(2r − 1)^{|u|}\) ways to fill \(u\) by Lemma 4.9. If \(u\) is not free-to-fill, there is only one choice.

The sum of the lengths of free-to-fill segments is \(\eta_i\), and the sum of the lengths of semi-free-to-fill segments is \(\eta'_i\). As the number of semi-free-to-fill segments is at most \(3|\widetilde{D}|^2\) (Lemma 4.12), there are at most \(2m(2m − 1)^{\eta_i−1}(2|\Gamma|)^{3|\widetilde{D}|^2} (2r − 1)^{\eta'_i}\) ways to fill the abstract relator \(i\). □
5 Freiheitssatz for random groups

Recall that $B_\ell$ is the set of cyclically reduced words of $X^\pm = \{x_1^\pm, \ldots, x_m^\pm\}$ of length at most $\ell$, and that $|B_\ell| = (2m - 1)^{\ell + o(\ell)}$. For $2 \leq r \leq m$, the set of cyclically reduced words on $X^\pm_r = \{x_1^\pm_r, \ldots, x_r^\pm_r\}$ of length at most $\ell$ is of cardinality $(2r - 1)^{\ell + o(\ell)}$, so its density in $B_\ell$ is $\log_{2m-1}(2r - 1)$. For the case $r = 1$, the set of cyclically reduced words on $\{x_1^\pm\}$ is of cardinality $2\ell$, hence with density 0 in $B_\ell$.

**Notation** Let $m \geq 2$ and $1 \leq r \leq m - 1$ be integers. In this section, we denote

$$c_r = \log_{2m-1}(2r - 1)$$

and

$$d_r = \min\left\{\frac{1}{2}, 1 - c_r\right\}.$$

5.1 The phase transition

For any $1 \leq r \leq m - 1$, we exhibit a phase transition at density $d_r$, characterizing the freeness of certain $r$-generated subgroups in random groups with $m$ generators.

**Theorem 5.1** Let $m \geq 2$ and $1 \leq r \leq m - 1$ be integers. Let $(G_\ell(m, d))$ be a sequence of random groups at density $d \in [0, 1]$.

1. If $d > d_r$, then a.a.s., $G_\ell(m, d)$ is generated by $x_1, \ldots, x_r$ (or by any subset of $X$ of cardinality $r$).
2. If $d < d_r$, then a.a.s. for every reduced $X$-labeled graph $\Gamma$ with $b_1(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r - d}{5} \ell$, the map $\widehat{\Gamma} \to \text{Cay}(G_\ell, X)$ is a $\frac{10}{d_r - d}$-bi-Lipschitz embedding.
   In addition, a.a.s. every subgroup of $G_\ell(m, d)$ generated by a reduced $X$-labeled graph $\Gamma$ with $b_1(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r - d}{5} \ell$ is a free group of rank $r$.
   In particular, a.a.s. $x_1, \ldots, x_r$ freely generate a free subgroup of $G_\ell(m, d)$.

**Remark 5.2** Let us discuss the special case that $r = 1$ of Theorem 5.1.

(a) Since $c_1 = \log_{2m-1}(2 \cdot 1 - 1) = 0$, we always have $d_1 = \min\{1/2, 1 - c_1\} = 1/2$, independent of the number $m$. The critical density coincides to that of Gromov’s triviality-hyperbolicity phase transition (Theorem 3.6). The first assertion would be obvious as a random group at density $d$ is a.a.s. trivial when $d > d_1 = 1/2$.

(b) The consequence of the second assertion gives that when $d < d_1 = 1/2$, a.a.s. every cyclically reduced word of $X^\pm$ of length at most $\frac{1 - 2d}{10} \ell$ is a torsion free element in $G_\ell(m, d)$.
   It is actually a special case of another known result: for any $m \geq 2$, if $0 \leq d < 1/2$ then a.a.s. $G_\ell(m, d)$ is torsion free. See [24] V.d for an argument by Ollivier.

Let us prove the first assertion of Theorem 5.1, which requires only the intersection formulae (Theorem 3.4 and Theorem 3.5).

**Proof of Theorem 5.1.1** We separate the two cases $d_r = 1/2 < d$ and $d_r < d < 1/2$.

(a) If $d_r = 1/2 < d$, then a.a.s. the random group $G_\ell(m, d)$ is trivial by Theorem 3.6. We admit the convention that the trivial element of a trivial group generates itself, so the set of elements $\{x_1, \ldots, x_r\}$ generates the whole group. Note that it is always the case when $r = 1$ according to Remark 5.2 (a).
(b) Assume that $d_r < d < 1/2$. Recall the notations $X = \{x_1, \ldots, x_m\}$ and $X_r = \{x_1, \ldots, x_r\}$.

Let $A_\ell$ be the set of words of type $x_{r+1}w$ where $w$ is a cyclically reduced word of $X_r^\pm$ of length $\ell - 1$. The density of $(A_\ell)$ in $(B_\ell)$ is $c_\ell$. Indeed, if $2 \leq r \leq m$, then the cardinality of $A_\ell (2r - 1)\ell + o(\ell)$, so $\log_{|B_\ell|} |A_\ell| \to 2r - 1 = c_\ell$; if $r = 1$ (although it would never be concerned as $d_1 = 1/2$ by Remark 5.2 (a)), then the cardinality of $A_\ell$ is 2, so $\log_{|B_\ell|} |A_\ell| \to 0 = c_1$.

Since $d_r < 1/2$, we have $d_r = 1 - c_r$. Together with the hypothesis $d_r < d$, we obtain $c_r + d > 1$. Apply the intersection formula (Theorem 3.2.4), a.a.s. the intersection $R_\ell \cap A_\ell$ is not empty. Hence, a.a.s. there exists a cyclically reduced word $w_{r+1}$ of $X_r^\pm$ such that $x_{r+1}w_{r+1} \in R_\ell$, which implies $x_{r+1} = G_\ell$ $w_{r+1}$.

Apply the same argument to the other generators $x_{r+2}, \ldots, x_m$. A.a.s. there are cyclically words $w_{r+1}, \ldots, w_m$ of $X_r^\pm$ such that $x_i = G_\ell w_i$ for any $r + 1 \leq i \leq m$. Hence, a.a.s. every word of $X_r^\pm$ equals to a word of $X_r^\pm$ in $G_\ell$.

\[ \square \]

To prove the second assertion of Theorem 5.1, we state first a local result, showing that the boundary of local diagrams of a random group at density $d$ satisfy some inequality.

**Lemma 5.3** Let $m \geq 2$ and $1 \leq r \leq m - 1$ be integers. Let $(G_{\ell}) = (G_{\ell}(m, d))$ be a sequence of random groups with density $d < d_r$. For any $K > 0$, a.a.s. for every reduced $X$-labeled graph $\Gamma$ with $b_1(\Gamma) \leq r$ and $|\Gamma| \leq \frac{d_r - d}{5} \ell$, every disc-like reduced distortion van Kampen diagram $(D, \ell)$ of $(G_{\ell}, \Gamma)$ with $|D| \leq K$ satisfies

\[
|p| \leq \left(1 - \frac{d_r - d}{5}\right) |\partial D|. \tag{\star}
\]

The proof of this lemma is in the next subsection.

**Proof of Theorem 5.1.2 by Lemma 5.3** Apply Lemma 3.1 on the results of Theorem 3.6.2, let us work under the conditions that every diagram $D$ of $G_{\ell}(m, d)$ satisfies $|\partial D| \geq (1 - 2d)/\ell |D|$ and that $G_{\ell}(m, d)$ is $\delta$-hyperbolic with $\delta = \frac{4\ell}{1 - 2d}$.

Let $\Gamma$ be a reduced $X$-labeled graph with $|\Gamma| \leq \frac{d_r - d}{5} \ell$ and $b_1(\Gamma) \leq r$. Let $\lambda = \frac{5}{d_r - d}$. By the local-global principle of quasi-geodesics in a hyperbolic space (Theorem 2.10), in order to prove that the image of any geodesic of $\Gamma$ in Cay($X, G_\ell$) is a (global) $2\alpha$-quasi geodesic, we shall prove that every reduced word $u$ readable on $\Gamma$ is a $1000\lambda\delta$-local $\lambda$-quasi-geodesic.

Let $u$ be a reduced word that is readable on $\Gamma$ with $|u| \leq 1000\lambda\delta$. Let $v$ be a geodesic in $G_\ell$ joining endpoints of the image of $u$ in $G_\ell$. We shall prove that $|u| \leq \lambda |v|$. By van Kampen’s lemma (Lemma 2.5) there exists a diagram $D$ of $G_\ell$ whose boundary word is $uv$. By the isoperimetric inequality (Theorem 3.6.2) and the fact that $|\partial D| = |u| + |v| \leq 2|u| \leq 2000\lambda\delta$, we have

\[
|D| \leq \frac{2|\partial D|}{(1 - 2d)\ell} \leq \frac{40000}{(1 - 2d)^2(d_r - d)}. \]

Apply Lemma 5.3 with $K = \frac{40000}{(1 - 2d)^2(d_r - d)}$. If $D$ is disk-like, then by (\star), we have

\[
|u| \leq \left(1 - \frac{d_r - d}{5}\right) (|u| + |v|) \leq \frac{\lambda}{1 + \lambda} (|u| + |v|),
\]

which implies $|u| \leq \lambda |v|$.
Otherwise, we decompose $D$ into discs and segments. By the same argument of Lemma 2.7, because every disc-like sub-diagram is a distortion diagram satisfying (⋆), we still have $|u| \leq \lambda |v|$. Hence, the word $u$ is a $1000\lambda\delta$-local $\lambda$-quasi-geodesic with $\lambda = \frac{5}{d_r - d}$. We conclude that the map $\widetilde{\Gamma} \to \text{Cay}(G_\ell, X)$ is a $\frac{10}{d_r - d}$-bi-Lipschitz embedding.

Let $H$ be a subgroup generated by $\Gamma$. If $H$ were not free, then there would be a reduced loop $p$ of $\Gamma$ whose labeling word is a trivial word in the random group $G_\ell(m, d)$. But this word should be a quasi-geodesic by the previous result, which gives a contradiction. To prove that $x_1, \ldots, x_r$ generate freely a free subgroup, one can take $\Gamma'$ as the wedge of $r$ cycles of length one labeled by $x_1, \ldots, x_r$. \hfill $\square$

5.2 Local distortion diagrams of a random group

We will prove Lemma 5.3 in this subsection. The proof is similar to Ollivier’s proof of the local isoperimetric inequality for random groups (Lemma 3.7) in [24] p. 86.

Let $(G_\ell) = (G_\ell(m, d)) = ((X| R_\ell))$ be a sequence of random groups with density $d < d_r$. We work first on the fillability of an abstract distortion diagram. Denote

$$\varepsilon_d = \frac{d_r - d}{5}.$$ 

Since the sequence of random subsets $(R_\ell)$ is densable with density $d$, the probability event

$$Q_\ell := \left\{ (2m - 1)^{(d - \varepsilon_d)\ell} \leq |R_\ell| \leq (2m - 1)^{(d + \varepsilon_d)\ell} \right\}$$

is a.a.s. true.

**Lemma 5.4** (Fillability of an abstract distortion diagram) Let $K > 0$. Let $\Gamma$ be a reduced $X$-labeled graph with $b_1(\Gamma) \leq r$ and $|\Gamma| \leq \varepsilon_d \ell$. Let $(\widetilde{D}, p)$ be a disc-like abstract distortion diagram with $|\widetilde{D}| \leq K$ that satisfies

$$|p| > (1 - \varepsilon_d) |\partial \widetilde{D}|.$$

Then for $\ell$ large enough,

$$P = \text{Pr} \left( (\widetilde{D}, p) \text{ is fillable by } (G_\ell, \Gamma) \mid Q_\ell \right) \leq \ell^{10 K^3 (2m - 1)^{-2\varepsilon_d \ell}}.$$

**Proof** We shall prove the lemma in four steps. We omit “for $\ell$ large enough” in every step. Recall that $a_i$ is the number of faces labeled by $i$, $\eta_i$ the number of free-to-fill abstract letters of $i$, and $\eta'_i$ the number of semi-free-to-fill abstract letters of $i$.

**Step 1:** $\log_{2m - 1} P \leq \sum_{i=1}^{k} (\eta_i + c_r \eta'_i + (d - 1 + 2\varepsilon_d)\ell) + 10 K^3 \log_{2m - 1} \ell$. \hfill (1)

According to Proposition 3.3, if $(r_1, \ldots, r_k)$ is a filling of $\widetilde{D}$ by $B_\ell$, then for $\ell$ large enough $\text{Pr} \left( r_1, \ldots, r_k \in R_\ell \mid Q_\ell \right) \leq (2m - 1)^{k(d - 1 + 2\varepsilon_d)\ell}$. Recall that $N_\ell(\widetilde{D}, p, \Gamma)$ is the set of fillings of $(\widetilde{D}, p)$ by $(B_\ell, \Gamma)$.

Apply Lemma 4.13 with $|\Gamma| \leq \varepsilon_d \ell$ and $|\widetilde{D}| \leq K$,

$$\text{Pr} \left( (\widetilde{D}, p) \text{ is fillable by } (G_\ell, \Gamma) \mid Q_\ell \right) \leq \sum_{(r_1, \ldots, r_k) \in N_\ell(\widetilde{D}, p, \Gamma)} \text{Pr} \left( r_1, \ldots, r_k \in R_\ell \mid Q_\ell \right).$$
Hence the inequality (1) by applying $\log_{2m-1}$.

**Step 2:** $|\tilde{D}| (\log_{2m-1} P - 10K^3 \log_{2m-1} \ell) \leq \sum_{i=1}^{k} \alpha_i (\eta_i + c_r \eta'_i + (d - 1 + 2\varepsilon_d)\ell)$. (2)

Let $\tilde{D}_i$ be the sub-diagram of $\tilde{D}$ consisting of the faces labeled by the first $i$ abstract relators $1^\pm, \ldots, i^\pm$ and the edges attached to them. Apply (1) to $\tilde{D}_i$, and denote $P_i$ the probability obtained. We have

$$\log_{2m-1} P \leq \log_{2m-1} P_i \leq \sum_{s=1}^{i} (\eta_s + c_r \eta'_s + (d - 1 + 2\varepsilon_d)\ell) + 10K^3 \log_{2m-1} \ell.$$ 

Without loss of generality, we assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Note that $\log_{2m-1} P$ is negative and that $\alpha_1 \leq |\tilde{D}|$. By Abel’s summation formula, with convention $\alpha_{k+1} = 0$,

$$\sum_{i=1}^{k} \alpha_i (c_r \eta'_i + \eta_i + (d - 1 + 2\varepsilon_d)\ell)$$

$$= \sum_{i=1}^{k} \left[ (\alpha_i - \alpha_{i+1}) \sum_{s=1}^{i} (c_r \eta'_s + \eta_s + (d - 1 + 2\varepsilon_d)\ell) \right]$$

$$\geq \sum_{i=1}^{k} \left[ (\alpha_i - \alpha_{i+1}) (\log_{2m-1} P - 10K^3 \log_{2m-1} \ell) \right]$$

$$\geq \alpha_1 (\log_{2m-1} P - 10K^3 \log_{2m-1} \ell)$$

$$\geq |\tilde{D}| (\log_{2m-1} P - 10K^3 \log_{2m-1} \ell).$$

**Step 3:** $\log_{2m-1} P \leq \left( d - \frac{1}{2} + 2\varepsilon_d \right) \ell + \left( c_r - \frac{1}{2} + \varepsilon_d \right) |\partial \tilde{D}| + 10K^3 \log_{2m-1} \ell$. (3)

Let $\varepsilon'_d > 0$ such that $|\overline{p}| = (1 - \varepsilon'_d) |\partial \tilde{D}|$. By hypothesis $\varepsilon'_d < \varepsilon_d$. Because $\tilde{D}$ is disc-like and the boundary length of every face is $\leq \ell$, the number of undirected edges $|E|$ is less than $|\tilde{D}|(\varepsilon'_d - |\partial \tilde{D}|)/2 + |\partial \tilde{D}|$. Apply Lemma 4.8, we get

$$\sum_{i=1}^{k} \alpha_i \eta'_i \leq |\overline{p}| = (1 - \varepsilon'_d) |\partial \tilde{D}|,$$

$$\sum_{i=1}^{k} \alpha_i \eta \leq |E| - |\overline{p}| \leq \frac{|\tilde{D}| \ell}{2} + \left( \varepsilon'_d - \frac{1}{2} \right) |\partial \tilde{D}|.$$ 

Note that $\sum_{i=1}^{k} \alpha_i = |\tilde{D}|$. So we have

$$\sum_{i=1}^{k} \alpha_i (c_r \eta'_i + \eta_i + (d - 1 + 2\varepsilon_d)\ell)$$

$$\leq c_r (1 - \varepsilon'_d) |\partial \tilde{D}| + \frac{|\tilde{D}| \ell}{2} + \left( \varepsilon'_d - \frac{1}{2} \right) |\partial \tilde{D}| + (d - 1 + 2\varepsilon_d) |\tilde{D}| \ell.$$
\[ \leq \left( d - \frac{1}{2} + 2\varepsilon_d \right) |\tilde{D}| \ell + \left( c_r - \frac{1}{2} + \varepsilon_d \right) |\partial \tilde{D}|. \]

Combine this inequality with (2), we get (3)

**Step 4:** \( \left( d - \frac{1}{2} + 2\varepsilon_d \right) \ell + \left( c_r - \frac{1}{2} + \varepsilon_d \right) \frac{|\partial \tilde{D}|}{|D|} \leq -2\varepsilon_d \ell. \) (4)

Recall that \( c_r = \log_{2m - 1}(2r - 1) \). Note that \(|\partial \tilde{D}| \leq \ell |\tilde{D}|\) and that \( d = d_r - 5\varepsilon_d \).

There are two cases:

(a) If \( c_r \geq \frac{1}{2} \), then \( d = 1 - c_r - 5\varepsilon_d \) and \( c_r - \frac{1}{2} + \varepsilon_d \geq 0 \), so

\[
\left( d - \frac{1}{2} + 2\varepsilon_d \right) |\tilde{D}| \ell + \left( c_r - \frac{1}{2} + \varepsilon_d \right) |\partial \tilde{D}|
\leq (d + c_r - 1 + 3\varepsilon_d) |\tilde{D}| \ell
\leq -2\varepsilon_d |\tilde{D}| \ell.
\]

(b) If \( c_r < \frac{1}{2} \), then \( d = \frac{1}{2} - 5\varepsilon_d \) and \( c_r - \frac{1}{2} + \varepsilon_d \leq \varepsilon_d \), so

\[
\left( d - \frac{1}{2} + 2\varepsilon_d \right) |\tilde{D}| \ell + \left( c_r - \frac{1}{2} + \varepsilon_d \right) |\partial \tilde{D}|
\leq \left( d - \frac{1}{2} + 3\varepsilon_d \right) |\tilde{D}| \ell
\leq -2\varepsilon_d |\tilde{D}| \ell.
\]

By (3) and (4), for \( \ell \) large enough \( \log_{2m - 1}(p) \leq -2\varepsilon_d \ell + 10K^3 \log_{2m - 1} \ell. \) □

By Lemmas 2.1 and Lemma 2.2, we have the following two results.

**Lemma 5.5** For \( \ell \) large enough, the number of reduced labeled connected graphs \( \Gamma \) (with respect to \( X \)) with \(|\Gamma| \leq \varepsilon_d \ell \) and \( b_1(\Gamma) \leq r \) is bounded by

\[(2m - 1)^{\varepsilon_d \ell} \ell^{3r}.\]

□

**Lemma 5.6** For \( \ell \) large enough, the number of disc-like abstract distortion diagrams \((\tilde{D}, p)\) with \(|\tilde{D}| \leq K \) and \(|\partial f| \leq \ell \) for all faces \( f \in F \) is bounded by

\[\ell^{5K}.\]

□

**Proof of Lemma 5.3** Recall that \( \varepsilon_d = \frac{d_r - d}{5} \).

We shall prove that a.a.s. for every reduced \( X \)-labeled graph \( \Gamma \) with \( b_1(\Gamma) \leq r \) and \(|\Gamma| \leq \varepsilon_d \ell \), every reduced distortion diagram \((D, p)\) of \((G_\ell, \Gamma)\) with \(|D| \leq K\) satisfies \(|p| \leq (1 - \varepsilon_d) |\partial D|\).

Apply Lemmas 5.4, 5.5 and 5.6. The probability that there exists a reduced \( X \)-labeled graph \( \Gamma \) with \( b_1(\Gamma) \leq r \), \(|\Gamma| \leq \varepsilon_d \ell \) and there exists a disc-like reduced abstract distortion diagram \((\tilde{D}, p)\) with \(|\tilde{D}| \leq K \), \(|p| > (1 - \varepsilon_d) |\partial \tilde{D}|\) such that \((\tilde{D}, p)\) is fillable by \((G_\ell, \Gamma)\) is bounded by

\[(2m - 1)^{\varepsilon_d \ell} \ell^{3r} \times \varepsilon^{5K} \times \ell^{10K^3} (2m - 1)^{-2\varepsilon_d \ell} = (2m - 1)^{-\varepsilon_d \ell + O(\log \ell)}.\]
So the probability that there exists a reduced $X$-labeled graph $\Gamma$ with $b_1(\Gamma) \leq r$, $|\Gamma| \leq \epsilon d \ell$ and there exists a disc-like reduced distortion diagram $(D, p)$ of $(G_\ell, \Gamma)$ with $|D| \leq K$ that satisfies $|p| > (1 - \epsilon_d) |\partial \tilde{D}|$ is bounded by
\[(2m - 1)^{-\epsilon_d \ell + O(\log \ell)},\]
which goes to 0 when $\ell$ goes to infinity. \(\square\)

This completes the proof of Theorem 5.1.

**Remark 5.7** In this proof, the number of edges of graphs gives an exponential factor $\sim (2m - 1)^{|\Gamma|}$ by Lemma 5.5, which is the main reason that by our current method we need the condition $|\Gamma| \leq \frac{d_r - d}{5} \ell$ in Theorem 5.1.2.

Note that in Step 4 of Lemma 5.4, the numbers “$+2\epsilon_d$” and “$+\epsilon_d$” in the left hand side can be replaced by arbitrary small positive numbers, so the estimation “$-2\epsilon_d$” in the right hand side can be arbitrary close to $-5\epsilon_d = -(d_r - d)$. Hence, the condition $|\Gamma| \leq \frac{d_r - d}{5} \ell$ in Theorem 5.1.2 can be pushed up to $|\Gamma| \leq (d_r - d - s) \ell$ for any small $s > 0$ in our current method.

**Remark 5.8** While our method for proving Theorem 5.1.2 does not work for $X$-labeled graphs with $|\Gamma| \geq (d_r - d) \ell$, it does not mean that the conclusion fails for these graphs.

For instance, since a.a.s. the random group $G_\ell(m, d)$ is $4\ell \frac{4d}{1 - 2d}$ hyperbolic for any $d < 1/2$, according to [13] 5.3.A, a.a.s. there exists a large enough $K = K(m, d)$ such that, if $\Gamma$ is a $X$-labeled graph with $b_1(\Gamma) = r$ whose maximal arcs are longer than $K \ell$, then a subgroup generated by $\Gamma$ is free and quasi-convex.

Note that $K \ell$ is much larger than the hyperbolicity constant $\delta = \frac{4}{1 - 2d} \ell > \ell$, which is still much larger than the bound $(d_r - d) \ell < \ell$ of Theorem 5.1.2.

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