WARD IDENTITIES AND THE
VANISHING THEOREM FOR LOOP AMPLITUDES
OF THE CLOSED N=2 STRING

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Abstract

The existence of a ground ring of ghost number zero operators in the chiral BRST cohomology of the $N=2$ string is used to derive an infinite set of Ward identities for the closed-string scattering amplitudes at arbitrary genus. These identities are sufficient to rederive the well known vanishing theorem for loop amplitudes with more than three external legs.
1 Introduction

The explicit computation of loop amplitudes in string theory is notoriously difficult. Even for the technically most simple theory – the bosonic string – the level of mathematical complexity is impressing if one tries to go beyond one loop. Adding fermions and supersymmetry on the world-sheet does not improve the situation. On the contrary, the calculations become still more intricate and only a few explicit results exist. It seems that even the general formalism has not yet been fully worked out [1].

Fortunately, explicit computations can sometimes be replaced by more indirect methods, often related to symmetry arguments. It is thus not surprising that for the \( \mathcal{N}=2 \) string (i.e. the theory based on extended supersymmetry on the world-sheet; see [2] for a general review and [3] for a discussion of loop amplitudes) Berkovits and Vafa succeeded to avoid the evaluation of the path integral and obtained powerful results for loop amplitudes by embedding the theory into an \( \mathcal{N}=4 \) topological theory [4]. In fact, they found that all amplitudes with more than three external legs vanish to all orders in the loop expansion. The purpose of this letter is to give an alternative derivation of this result. Our approach has the advantage that conceptually it is very clear what is going on since the equations used to derive the vanishing of the amplitudes can nicely be interpreted as Ward identities of an infinite set of unbroken symmetries in target space. Another interesting point is that from a technical point of view our analysis rests on the picture dependence of the BRST cohomology of the \( \mathcal{N}=2 \) string at zero momentum and demonstrates what kind of information may be stored in the still somewhat obscure picture phenomenon [5]. Maybe, this lesson can also be useful in some way for the \( \mathcal{N}=1 \) string.

The letter is organised as follows: In the next section we recall some facts about the BRST cohomology of the \( \mathcal{N}=2 \) string. These results will be used in section three to derive an infinite set of target space Ward identities which will then be explicitly evaluated so that the vanishing of the loop amplitudes directly follows. We conclude with some further remarks and a brief discussion of the reliability of our arguments.

2 Symmetries and ground ring of the \( \mathcal{N}=2 \) string

One of the attractive features of the BRST approach to closed string theory is that it provides an efficient means to analyse symmetries in target space. More precisely, unbroken target space symmetries generally lead to the existence of ghost number one cohomology classes (in conventions where physical states have ghost number two). A detailed explanation of this fact is given in [7] (see also [8]) where in addition an elegant method to derive the corresponding Ward identities – briefly reviewed below – is described.

Due to the fact that the closed string Fock space factorises into right- and left-moving parts ghost number one cohomology classes can be further charac-

\footnote{There are exceptions, see section five of [6].}
terised: they are most conveniently constructed as a product of a holomorphic piece of ghost number zero and an antiholomorphic piece of ghost number one. The latter is usually the right-moving part of a physical vertex operator, taken at some discrete value of the momentum whereas the former very often is just the unit operator. If, however, the chiral (= left-moving) cohomology at ghost number zero contains further elements besides the unit operator more closed string operators of ghost number one can be constructed resulting in a much richer symmetry structure. An example is the bosonic string in two dimensions \[9\]. Moreover, interesting algebraic structures emerge. The BRST cohomology possesses a natural multiplication rule, additive in ghost number. The ghost number zero cohomology therefore forms a ring under this multiplication (the so-called ground ring). As has been emphasised in \[10\] the structure constants of this ground ring encode much information about the symmetry of the theory\[2\].

The \(N=2\) string has been studied along these lines in \[12, 13, 14\]. Based on the fact that so many of its scattering amplitudes are known or conjectured to vanish \([4, 15, 16]\) and comparison with the field theory that reproduces tree-level scattering \[15, 17\] it seemed very plausible that in this theory a large symmetry group is realised. In fact, a ground ring of the \(N=2\) string has recently been found in \[14\] and will now briefly be reviewed. The construction looks somewhat unconventional because it does not restrict to operators of a single picture only, but takes into account the full picture degeneracy of the Fock space\[3\]. However, starting from this ground ring one may derive powerful Ward identities as has been shown for tree amplitudes in \[14\] and will be demonstrated in this letter for loop amplitudes.

At zero ghost number chiral cohomology classes occur only for vanishing momentum. For low-lying picture numbers and ghost number zero the cohomology problem is rather straightforward to solve:\[4\]:

- The cohomology is empty for pictures \((\pi^+, \pi^-) = \{(-1, -1), (-1, 0), (0, -1)\}\) and consists of the unit operator in the \((0, 0)\) picture.
- In the pictures \((-1, 1)\) and \((1, -1)\) the cohomology consists of the spectral flow operators
  \[A(z) = (1 - cb')J^- e^\phi^+ e^{-\phi^-}(z)\]
  \[A^{-1}(z) = (1 + cb')J^{++} e^{-\phi^+} e^{\phi^-}(z)\]

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2 There exist two further operations – the Gerstenhaber-bracket and the \(\Delta\) operation – which, together with the ring multiplication, give the BRST cohomology the structure of a BV-algebra.\[3\]

3 This construction is non-trivial due to the picture dependence of the BRST cohomology of the \(N=2\) string at zero momentum \[15\].

4 Poincaré-duality provides an isomorphism between the cohomologies for pictures \((\pi^+, \pi^-)\) and \((-\pi^- - 2, -\pi^+ - 2)\) \[4\]. Moreover, the cohomologies for pictures \((\pi^+ + \rho, \pi^- - \rho)\) with \(\rho \in \frac{1}{2}\mathbb{Z}\) coincide due to spectral flow. It is therefore sufficient to consider the case \(\pi^\pm \geq -1\) only.
with $J^{++} = \frac{1}{4} \epsilon_{a\bar{b}} \psi^a \psi^{+\bar{b}}$ and $J^{--} = -\frac{1}{4} \epsilon_{\bar{a}b} \bar{\psi}^{-\bar{a}} \bar{\psi}^{-b}$ (see [14] for conventions and a description of the $N=2$ string ghost system). One may check that $A$ and $A^{-1}$ are inverse to each other with respect to ring multiplication.

- In the $(1,0)$ picture the cohomology consists of the picture changing operator
  \[ X^+(z) = \{ Q, \xi^+(z) \} \]
  and the operator
  \[ A \cdot X^- \quad \text{with} \quad X^-(z) = \{ Q, \xi^-(z) \}. \]

It should be emphasized that $A \cdot X^-$ is BRST inequivalent to $X^+$. Analogously, the $(0,1)$ cohomology consists of the operators $X^-$ and $A^{-1} \cdot X^+$. We see that the size of the cohomology grows as the picture increases. To obtain cohomology classes with higher integral picture numbers one may simply consider polynomials of the operators $A$, $A^{-1}$ and $X^\pm$,

\[ (X^+)^k \cdot (X^-)^\ell \cdot A^n, \quad k, \ell \in \mathbb{N}, \quad n \in \mathbb{Z}. \]

Note that $k$ and $\ell$ must not be negative since, contrary to $N=1$ strings, there do not exist local inverse picture changing operators for the $N=2$ string [21] (the cohomology at vanishing momentum and ghost number is empty for picture numbers $(-1,0)$ and $(0,-1)$).

It has been shown in [14] that all these operators are BRST inequivalent! For a given picture $(\pi^+, \pi^-)$ we thus have constructed $\pi^+ + \pi^- + 1$ operators,

\[ O_{\pi^+, \pi^-, n} = (X^+)^{\pi^+ + n} \cdot (X^-)^{\pi^- - n} \cdot A^n, \quad n = -\pi^+, ..., \pi^- . \tag{2} \]

To obtain ghost number one cohomology classes of the closed string connected to the symmetries of the theory the operators in (2) have to be combined with right-moving cohomology classes of zero momentum and ghost number one. These operators can be found in a similar way: In [21] it has been shown that the relevant cohomology in the $(0,0)$ picture is spanned by the four elements

\[ -iP^a = c \partial Z^a - 2\gamma^- \psi^+ a, \quad -i\bar{P}^{\bar{a}} = c \partial \bar{Z}^{\bar{a}} - 2\gamma^+ \bar{\psi}^{-\bar{a}}. \tag{3} \]

Here the target space Lorentz indices $a$ and $\bar{a}$ range from 0 to 1. Multiplication with $O_{\pi^+, \pi^-, n}$ gives similar operators in higher pictures:

\[ P^a_{\pi^+, \pi^-, n} = O_{\pi^+, \pi^-, n} \cdot P^a, \quad \bar{P}^{\bar{a}}_{\pi^+, \pi^-, n} = O_{\pi^+, \pi^-, n} \cdot \bar{P}^{\bar{a}}. \tag{4} \]

\[ ^5\text{Multiplication of two operators, denoted by a dot in the following, means to take the regular term in their operator product expansion [10].} \]
We are now ready to write down the sought for closed string cohomology classes of ghost number one:

\[ \Sigma^{a}_{\pi^{+}, \pi^{-}, m, n} = O^{\pi^{+}, \pi^{-}, m}_{\pi^{+}, \pi^{-}, n}(z) \tilde{P}^{a}_{\pi^{+}, \pi^{-}, m}(z), \quad m, n = -\pi^{+}, ..., \pi^{-}. \quad (5) \]

To save space the analogous operators \( \Sigma_{\bar{a}} \) will not be explicitly mentioned in the following. Using the descent equations one may now construct an infinite set of symmetry charges and work out the transformation laws of the physical state. This has been done in [14].

We conclude this section with one further remark. So far, we have only considered the relative cohomology of states that are annihilated by the zero modes of all fermionic antighosts. It would, however, be more appropriate also to take into account states that are not annihilated by \( b_{0} + \bar{b}_{0} \) which defines the so-called semi-relative cohomology (one way to see that this is the right space to consider is to write down a kinetic term in a string field formalism). Allowing for more states generally changes the cohomology. But fortunately, one can show that the operators (5) are still non-trivial in the semi-relative cohomology. One may also wonder whether new cohomology classes turn up, as happens for the bosonic string in two dimensions [7]. We do not know the general answer to this question, but explicit calculations for low-lying pictures indicate that this is not the case.

3 Ward identities

We will now use the results from the previous section to derive Ward identities for \( N=2 \) string amplitudes at arbitrary genus. Actually, an \( N=2 \) string scattering amplitude is further characterised by a Chern number classifying \( U(1) \) bundles over the world-sheet Riemann surface. It is, however, sufficient to focus on vanishing Chern number in the following. This will be justified in section four. For reasons of space the general formalism will not be reviewed in detail here. Instead, we refer to [22, 23] for more extensive explanations.

The basic object involved in the computation of scattering amplitudes is the vertex operator of the single degree of freedom in the theory. As usual, it splits into holomorphic and antiholomorphic parts:

\[ V(z, \bar{z}, k) = V^{left}(z, k) \tilde{V}^{right}(\bar{z}, k) \]

The left-moving operator is

\[ V^{left}_{(-1,-1)}(z, k) = c \bar{c} e^{-\phi^{+}} e^{-\phi^{-}} e^{ikZ^{left}} \]

in the \((-1,-1)\) picture and

\[ V^{left}_{(\pi^{+}, \pi^{-})}(z, k) = (X^{+})^{\pi^{+} + 1} \cdot (X^{-})^{\pi^{-} + 1} \cdot V^{left}_{(-1,-1)}(z, k) \]

in higher pictures (the right-moving piece \( \tilde{V}^{right} \) looks similar). Counting both metric and \( U(1) \) but not supersymmetry ghost number vertex operators in

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6Application of spectral flow only leads to vertex operators proportional to those above.
closed $N=2$ string theory therefore have ghost number four (in our conventions picture changing operators have ghost number zero, see [4]). Moreover, they are not annihilated by the zero modes $b_0$ and $\tilde{b}_0$ of the $U(1)$ antighosts. On the other hand the ghost number one operators constructed in the previous section are all elements of the relative cohomology, i.e. they are all killed by the zero modes of all fermionic antighosts. It is, however, not too difficult to relate relative cohomology classes to operators of higher ghost number, essentially by multiplying with the relevant ghosts. In this way we can construct from the ghost number one operators in equation (5) new cohomology classes of ghost number three:

$$\Sigma^{a}_{\pi^{+},\pi^{-},m,n} \rightarrow \Omega^{a}_{\pi^{+},\pi^{-},m,n} \equiv c^{'}c' \Sigma^{a}_{\pi^{+},\pi^{-},m,n} + \ldots$$

Here the dots refer to further terms that might be necessary to achieve BRST invariance but are unimportant otherwise.

We are now ready to derive a Ward identity involving a genus $g$ scattering amplitude of $N$ external states with momenta $k_1,\ldots,k_N$ (denoted $A^g_N(k_1,\ldots,k_N)$ in the following). One starts with the correlator

$$\langle \Omega^{a}_{\pi^{+},\pi^{-},m,n}(z,\bar{z}) \prod_{i=1}^{N} V^{cl}_{\pi^{+}_{i},\pi^{-}_{i}}(z_{i},\bar{z}_{i},k_{i}) \prod_{l}(\mu_{l},B)(\tilde{\mu}_{l},\tilde{B}) \rangle_{g}$$

where $(\mu_{l},B)$ and $(\tilde{\mu}_{l},\tilde{B})$ are the the appropriate Beltrami differentials integrated with the corresponding antighosts and the index $g$ indicates that the correlator is meant to be evaluated with respect to the conformal field theory living on a Riemann surface of genus $g$. The antighosts can be applied to the vertex operators and the integrations can be pulled out of the brackets. Let us denote the remaining integrand by $\Theta$. If the operator $\Omega$ in (6) were replaced by an ordinary physical vertex operator $V$ one could integrate $\Theta$ over the moduli space of a genus $g$ surface with $N+1$ punctures. From counting dimensions and ghost numbers it follows, however, that $\Theta$ as defined by (6) can be integrated only over the boundary of moduli space. In fact, it can be considered as a differential form on moduli space of codimension one. Since $\Theta$ can also be shown to be a closed form $\Theta$ Stokes’ theorem leads to the desired Ward identity

$$\int_{\partial M_{g,N+1}} \Theta = \int_{M_{g,N+1}} d\Theta = 0. \quad (7)$$

The next step is to have a closer look at the $N=2$ string moduli space $M_{g,N+1}$, i.e. the moduli space of $N=2$ super Riemann surfaces with genus $g$ and $N + 1$ punctures (and vanishing Chern number in our case). In addition to the usual metric and super moduli, we also have to consider the so-called $U(1)$ moduli describing a continuum of possible monodromy phases for the worldsheet fermions arising from their transport along non-trivial homology cycles.

\footnote{For simplicity we only consider closed string operators whose left- and right-moving picture numbers coincide.}
However, the $U(1)$ moduli space is compact (it always has the topology of a torus) and therefore does not contribute to the boundary of moduli space. As a result, in our Ward identity (7) only the familiar boundary components of the metric moduli space appear.

The metric moduli are of two different types. One corresponds to the shape of the underlying Riemann surface whereas the other describes punctures, i.e. the locations of the vertex operators. If we move to the boundary of moduli space the Riemann surface degenerates in some way. In the following it is convenient to distinguish four different cases: First of all, the underlying surface may pinch either along a trivial or a non-trivial homology cycle. If a genus $g$ surface pinches along a non-trivial cycle it becomes a surface of genus $g - 1$ with two points coinciding. If it pinches along a trivial cycle the result is a connected pair of Riemann surfaces with genera $g_1$ and $g_2$ such that $g_1 + g_2 = g$. For a $g=2$ surface with four punctures these two cases are illustrated in the top row of the figure below. It may also happen that a number of punctures approach each other. This is conformally equivalent to a situation where a sphere containing the relevant punctures splits off of the rest of the surface. This is illustrated in the bottom line of the figure, where we also distinguished whether two vertex operators $V$ approach each other or one $V$ approaches the ghost number three operator $Ω$.

To see how a pinch (denoted by $P$ in the figure) can properly be included in the computation let us recall that it can equivalently be described by an infinitely long cylinder. This cylinder can be taken into account by inserting a complete set of physical states. In this formulation the twist angle of the cylinder is one of the moduli leading to an insertion of the metric antighost combination $b(z) - \tilde{b}(\bar{z})$. So the pinch can be represented by the sum

$$\sum_i |\hat{O}_i\rangle\langle O_i|$$

where $i$ labels a basis of the absolute BRST cohomology and

$$\langle O_j|\hat{O}_i\rangle = \delta_{ji}, \quad |\hat{O}_i\rangle = (b_0 - \tilde{b}_0)|O_i\rangle.$$  

What about the fermionic and $U(1)$ moduli? The former are correctly taken into account by obeying the right selection rules for picture numbers [24]. Moreover, a pinch contributes one complex $U(1)$ modulus. This corresponds to the fact that the complete set of states (8) carries two units of $U(1)$ ghost number – just enough to compensate the antighost insertion due to the $U(1)$ modulus of the pinch.

Let us now become more explicit: We assume $N \geq 3$, i.e. the presence of at least three vertex operators, and genus $g > 0$ since tree-level amplitudes have been discussed in [14]. It will also be sufficient and technically simpler to consider only operators $Σ$ (and the corresponding $Ω$) of the form

$$Σ_n^a(z, \bar{z}) := Σ_{-n,n,n,n}^a(z, \bar{z}) = A^n(z) \hat{A}^a \cdot \hat{P}^a(\bar{z}),$$

which have picture numbers ($-n, n$). The four cases mentioned above will now be discussed in turn.
3.1 Case 1: A non-trivial homology cycle pinches

Besides the $N$ physical vertex operators already present the pinching leads to an insertion of two further vertex operators $O_i$ and $\hat{O}_i$, as explained above. So we have to evaluate the expression

$$\sum_i \langle \langle \Omega^{n_1}_{a_1} V_1 \ldots V_N \hat{O}_i O_i \rangle \rangle_{g-1}.$$  \hspace{1cm} (10)

Here, the notation for the vertex operators has been simplified in a hopefully obvious way. The double bracket as usual denotes evaluation of the full amplitude including integration over moduli space.

To further evaluate the expression (10) let us note that it contains at least six operators (since we assumed $N \geq 3$ in the beginning). Regardless of the value of $g$ integration over moduli space leads for this number of operators to insertions of metric antighosts that transform cohomology classes into integrated vertex operators. Since this effect will be crucial in the following, we briefly review some details:

Assume the operator $B(z)\hat{B}(\bar{z})$ represents a closed string cohomology class. From the explicit form of the BRST operator it follows that

$$B^{(1)}(z) = \oint_z \frac{dw}{2\pi i} b(w) B(z)$$

satisfies the relation

$$[Q, B^{(1)}] = \partial B,$$

$Q$ being the left-moving part of the BRST operator. Since this argument goes
through for the right-moving half, as well, a $b$-ghost insertion leads to the integrated operator
\[ \int d^2 \bar{z} B^{(1)}(\bar{z}) \tilde{B}^{(1)}(\bar{z}) \]
which is BRST invariant since the integrand transforms into a total derivative. In practice, going over from a cohomology class to an integrated vertex operator simply amounts to getting rid of the undifferentiated $c$- and $\tilde{c}$-ghosts. If some cohomology class does not contain both these ghost fields (as for example the unit operator) its integrated form is zero.

We are always free to choose where to locate the $b$-ghost insertions\[8\] i.e. which cohomology class to convert into an integrated operator. In the present case we can pick $\Omega$. From the explicit form of $A$ in equation (1) one sees that stripping off a $c$-ghost necessarily leads to the presence of a $b'$-ghost, for example $A^{(1)} = -b' J - e^\phi - e^{-\phi}$. However, there is no corresponding $c'$-ghost in sight to compensate $b'$ in a correlation function. So we learn from simple $U(1)$ ghost number counting that the amplitude (10) vanishes! In other words, the kind of degeneration considered in this subsection does not contribute to the Ward identity.

### 3.2 Case 2: A trivial homology cycle pinches

The contribution to the Ward identity of this component of the boundary is
\[ \sum_{i, \alpha} \langle \langle V_{u_1} \ldots V_{u_p} \Omega_n^a \hat{O} \rangle \rangle_{g_1} \langle \langle O_i V_{u_{p+1}} \ldots V_{u_N} \rangle \rangle_{g_2} \]  
(11)
with $g_1 + g_2 = g$ and $g_1, g_2 > 0$. The sum over $\alpha$ runs over all possible ways to divide the set of $N$ physical vertex operators into a subset $\{V_{u_1} \ldots V_{u_p}\}$ on the genus $g_1$ surface and the remainder $\{V_{u_{p+1}} \ldots V_{u_N}\}$ located on the other surface.

Since $g_1$ is strictly positive and the correlation function involving $\Omega$ contains at least one further operator the expression (11) can again be evaluated by transforming $\Omega$ to its integrated form. As in the previous subsection the vanishing of (11) then follows from $U(1)$ ghost number counting.

### 3.3 Case 3: A sphere not including $\Omega$ splits off

In this case we have to evaluate the expression
\[ \sum_{i, \alpha} \langle \langle V_{u_1} \ldots V_{u_p} \Omega_n^a \hat{O} \rangle \rangle_{g} \langle \langle O_i V_{u_{p+1}} \ldots V_{u_N} \rangle \rangle_{g=0}. \]  
(12)
Since $g > 0$ by assumption the correlator involving $\Omega$ vanishes by the same argument as above.

\[8\]Since we are dealing with vertex operators of non-standard ghost number, this is not completely obvious in the path integral formulation. In the operator formalism, however, one may explicitly check that the location of the $b$-ghost insertion is immaterial.
3.4 Case 4: A sphere including $\Omega$ splits off

In this final case the contribution to the Ward identity reads

$$
\sum_{i,\alpha} \langle \langle V_{u_1} \ldots V_{u_p} \Omega^a_n \hat{O}^i \rangle \rangle_{g=0} \langle \langle O_i V_{u_{p+1}} \ldots V_{u_N} \rangle \rangle_{g} = 0.
$$  \hspace{1cm} (13)

The ghost number three operator $\Omega$ now appears in a tree-level amplitude whose evaluation involves metric antighost insertions as soon as more than three operators are present. Correspondingly, terms in the $\alpha$-sum vanish by the standard argument whenever the $g=0$ correlator involves more than one operator $V$ besides $\Omega$ and $\hat{O}^i$. What remains are those degenerations where $\Omega$ splits off with precisely one vertex operator $V_i$. These are the only contributions to the Ward identity:

$$
N \sum_{u=1}^{u} \langle \langle \hat{O}_i V_{u_{i-1}} V_{u_{i+1}} \ldots V_{u_N} \rangle \rangle_{g=0} = 0.
$$  \hspace{1cm} (14)

Obviously, the only non-vanishing term in the above sum over $i$ occurs when $O_i$ coincides with the vertex operator $V_u$. In each term of the $u$-sum the second correlator therefore is just the genus $g$ amplitude of $N$ physical states $A^g_N$. Reinserting the momenta $k_u$ allows us to rewrite the Ward identity as

$$
A^g_N (k_1, \ldots, k_N) \cdot \sum_{u=1}^{N} \langle \langle V(k_u) \Omega^a_n \hat{V}(-k_u) \rangle \rangle_{g=0} = 0.
$$  \hspace{1cm} (15)

These identities have already been derived in [14] for tree amplitudes. Equations (15) tell us that they do not get modified for higher genera. The remaining correlator can be evaluated as

$$
\langle \langle V(k) \Omega^a_n \hat{V}(-k) \rangle \rangle_{g=0} = \left( \frac{\vec{k}^0}{k^4} \right)^n n^a \equiv h(k)^n k^a.
$$  \hspace{1cm} (16)

The final identities for the genus $g$ amplitude thus read

$$
A^g_N (k_1, \ldots, k_N) \cdot \sum_{i=1}^{N} h(k_i)^n k^a_i = 0 \quad \text{for any } n \in \mathbb{Z}
$$  \hspace{1cm} (17)

and imply the vanishing of all amplitudes with $N \geq 4$ [14]. The three point function, however, is generally non-zero. One may for example check that the tree-level amplitude

$$
A^{g=0}_{N=3}(k_1, k_2, k_3) = (\vec{k}_1 \cdot k_2 - \vec{k}_2 \cdot k_1)^2
$$

satisfies all identities without being zero. On dimensional grounds it seems very plausible that for higher genus the three point function is just a power of the tree-level result:

$$
A^g_{N=3}(k_1, k_2, k_3) = \alpha_g (\vec{k}_1 \cdot k_2 - \vec{k}_2 \cdot k_1)^{4g+2}
$$

Here the pre-factor $\alpha_g$ depends on the genus but not on the momenta. Explicit computations at one loop show that $\alpha_{g=1}$ is divergent [25, 26]. This concludes our discussion of the scattering amplitudes of the $N=2$ string.
4 Some remarks

So far we have ignored the possibility of non-vanishing Chern number $c$, corresponding to topologically non-trivial configurations of the $U(1)$ gauge field on the world-sheet. A careful evaluation of the path integral shows that a non-zero Chern number can be simulated by inserting (a power of) the spectral flow operator $A$ into the $c = 0$ correlation function and simultaneously adjusting the picture numbers of the vertex operators $[3]$. Since the derivative of the spectral flow operator is BRST trivial each $A$ (or $A^{-1}$) can be moved towards one of the vertex operators and simply pulls out a momentum factor $h(k)$ (or its inverse, see eq. (16) for a definition of $h$). Therefore, amplitudes with different Chern number are proportional to one another. Hence, it is sufficient to prove the vanishing of a scattering amplitude for one fixed value of $c$. Secondly we have ignored that, as a Riemann surface with $c = 0$ degenerates and splits into two, the resulting surfaces may have non-vanishing Chern numbers $c$ and $-c$. So we actually should include in our Ward identity a summation over all such splittings. However, we have just explained that this only leads to additional factors $h(k)^c$ and $h(k)^{-c}$ which cancel each other ($k$ is the momentum flowing through the pinch). This justifies our treatment where we completely neglected sectors with non-zero Chern number.

A further point that deserves to be mentioned is the question of non-linear contributions to the symmetries. One of the remarkable features of the $N=2$ string that make it such an interesting toy model is the fact that we know a simple field theory that reproduces the tree-level amplitudes to all orders in $\alpha'$. This field theory is well known to possess a highly non-linear symmetry structure. In $[4]$ the linearised version of the unbroken symmetries on the field theory side was compared to the transformation rules of the $N=2$ string vertex operators under the symmetries that lead to the above Ward identities. They were found to coincide. In fact, the Hilbert space in our formulation of the theory consists only of single string states. So it seems at first sight correct to restrict a comparison between symmetries in field theory and string theory to the linear level. However, it has been explained in $[6]$ (section 6) that non-linear symmetry structures can make their appearance in a first quantised string theory at the level of Ward identities. More precisely, a non-linear contribution to a Ward identity corresponds to a situation where the ghost number one (three for $N=2$ strings) operator $\Omega$ splits off with more than one further vertex operator. In this case only the overlap between the charge acting on a single vertex operator with a multi-string state is sent through the pinch. In other words, a symmetry is realized non-linearly precisely when the tree-level amplitude

$$\langle \langle \Omega V(k_1) \ldots V(k_{n-1}) V(k_n) \rangle \rangle_{g=0}$$

is non-vanishing for $n \geq 3$. A model where this indeed happens is the bosonic string in two dimensions. Yet it has been argued in section three that in our case

$^9$This sum is finite since supersymmetry ghost zero modes kill correlators when $|c|$ exceeds a certain value.
of the $N=2$ string the relevant correlation functions vanish. As a consequence
the Ward identity (17) is linear. This indicates a clear discrepancy to the field
theory and suggests that the behaviour of the $N=2$ string is not fully captured
by its tree-level effective field theory.

Last but not least we should give our opinion on the reliability of our argu-
ments. In fact, we must admit that the analysis of the boundary of the $N=2$
string moduli space has been somewhat heuristic. It is mainly based on count-
ing of dimensions and ghost numbers. Hidden subtleties might be detected by a
more careful investigation. For example, it is conceivable that the $U(1)$ moduli
space behaves in some discontinuous way as the Riemann surface degenerates.
Whether or not this is the case can only be answered by studying the relevant
index theorem. Other potential difficulties are related to the fermionic moduli
that we have treated in a rather straightforward way, ignoring possible ambi-
guities due to the location of picture changing operators. In any case it would
be helpful to have an explicit computation of the one-loop four point function.
If that turns out to be non-vanishing it will be extremely interesting to see by
which mechanism the derivation of the Ward identities must be modified.

References

[1] J. Polchinski, String Theory, volume 2, section 12.5, Cambridge University
Press, 1998.

[2] N. Marcus, A tour through $N=2$ strings, hep-th/9211059.

[3] J. Bischoff and O. Lechtenfeld, Int. J. Mod. Phys. A 27 (1997) 4933,
hep-th/9612218.

[4] N. Berkovits and C. Vafa, Nucl. Phys. B 433 (1995) 123, hep-th/9407190.

[5] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B 271 (1986) 93.

[6] T. Banks and L. Dixon, Nucl. Phys. B 381 (1988) 93.

[7] E. Witten and B. Zwiebach, Nucl. Phys. B 377 (1992) 55, hep-th/9201050.
[8] G. Barnich, F. Brandt and M. Henneaux, Comm. Math. Phys. **174** (1995) 57, hep-th/9405109.

[9] P. Ginsparg and G. Moore, *Lectures on 2–D Gravity and 2–D String Theory*, hep-th/9304011.

[10] E. Witten, Nucl. Phys. **B 373** (1992) 187, hep-th/9108004.

[11] B.H. Lian and G.J. Zuckerman, Comm. Math. Phys. **154** (1993) 613, hep-th/9211072.

[12] A. Giveon and A. Shapere, Nucl. Phys. **B 386** (1992) 43, hep-th/9203008.

[13] M. Li, Nucl. Phys. **B 395** (1993) 129, hep-th/9204027.

[14] K. Jünemann, O. Lechtenfeld and A. Popov, Nucl. Phys. **B 548** (1999) 449, hep-th/9901164.

[15] H. Ooguri and C. Vafa, Nucl. Phys. **B 361** (1991) 469.

[16] R. Hippmann, Diploma-Thesis (in German), http://www.itp.uni-hannover.de/~lechtenf/Theses/hippmann.ps.

[17] A.D. Popov, M. Bordemann and H. Römer, Phys. Lett. **B 385** (1996) 63, hep-th/9606077.

[18] K. Jünemann and O. Lechtenfeld, Comm. Math. Phys. **203** (1999) 53, hep-th/9712182.

[19] B.H. Lian and G.J. Zuckerman, Comm. Math. Phys. **125** (1989) 301.

[20] J. Bischoff, S. V. Ketov and O. Lechtenfeld, Nucl. Phys. **B 438** (1995) 37, hep-th/9406101.

[21] J. Bie{ń}kowska, Phys. Lett. **B 281** (1992) 59, hep-th/9111047.

[22] E. Verlinde, Nucl. Phys. **B 381** (1992) 141, hep-th/9202021.

[23] I. Klebanov and A. Pasquinucci, Nucl. Phys. **B 393** (1993) 261, hep-th/9204052.

[24] O. Lechtenfeld, Nucl. Phys. **B 338** (1990) 403.

[25] M.Bonini, E. Gava and R. Iengo, Mod. Phys. Lett **A6** (1991) 795.

[26] B. Niemeyer, private communication.