A NOTE ON JET AND GEOMETRIC APPROACH TO HIGHER ORDER CONNECTIONS

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Abstract. We compare two ways of interpreting higher order connections. The geometric approach lies in the decomposition of higher order tangent space into the horizontal and vertical structures while the jet-like approach considers a higher order connection as the section of a jet prolongation of a fibered manifold. Particularly, we use the Ehresmann prolongation of a general connection and study the result from the point of view of geometric theory. We pay attention to linear connections, too.

1. Introduction

Several models of real objects are given as a smooth manifold and one or more linear connections, e.g. material elasticity, see [4]. To obtain a manifold with just one characterization, one has to consider a concept of a higher order connection. In this paper, we recall the basic concepts of higher order connections from both geometric and jet-like point of view, sections 2 and 4. Let us note that the original ideas are those of C. Ehresmann, i.e. the definition of a connection by means of a horizontal distribution in a tangent space, the double fibered manifolds and holonomic and nonholonomic jets of fibered mappings. The first idea can be found in [2], the second one in [3]. The second idea was used for the case of vector bundles by Pradines, [9]. Finally, the concept of holonomic and nonholonomic jets is widely studied in [6, 7, 12, 13, 14]. The first idea was extended in [10], where the main formula of higher order objects in multiple tangent spaces are derived, see also [11]. In this paper we compare the jet-like and geometric approach. We also recall a product of general connections which leads to the so called Ehresmann prolongation and show the reason why this operation is outstanding, especially concerning semiholonomic connections, section 2. We study Ehresmann prolongation of a connection from both points of view and show the analogues in both approaches.

2. Jet prolongation of a fibered manifold

Classical theory reads that r-th nonholonomic prolongation $\tilde{\mathcal{J}}^rY$ of $Y$ is defined by the following iteration:

(1) $\tilde{\mathcal{J}}^1Y = J^1Y$, i.e. $\tilde{\mathcal{J}}^1Y$ is a space of 1-jets of sections $M \to Y$ over the target space $Y$.

(2) $\tilde{\mathcal{J}}^rY = J^1(\tilde{\mathcal{J}}^{r-1}Y \to M)$.

Clearly, we have an inclusion $J^rY \subset \tilde{\mathcal{J}}^rY$ given by $j^r_\gamma \mapsto j^1_\gamma(\tilde{\mathcal{J}}^{r-1}_\gamma)$. Further, r-th semiholonomic prolongation $\bar{\mathcal{J}}^rY \subset \tilde{\mathcal{J}}^rY$ is defined by the following induction.

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It remains to describe coordinates on the semiholonomic prolongation. Let us denote by \( \pi \) the projection \( J^1Y \to Y \) and by \( \beta_r = \beta_{r-1} \) the projection \( \bar{J}^rY = J^1\bar{J}^{r-1}Y \to \bar{J}^{r-1}Y \), \( r = 2, 3, \ldots \). If we set \( J^1Y = J^1Y \) and assume we have \( \bar{J}^{-1}Y \subseteq \bar{J}^{-1}Y \) such that the restriction of the projection \( \beta_{r-1} : \bar{J}^{-1}Y \to \bar{J}^{-2}Y \) maps \( \bar{J}^{-1}Y \to \bar{J}^{-2}Y \), we can construct \( J^1\beta_{r-1} : J^1\bar{J}^{-1}Y \to J^1\bar{J}^{-2}Y \) and define
\[
\bar{J}^rY = \{ A \in J^1\bar{J}^{-1}Y ; \beta_r(A) = J^1\beta_{r-1}(A) \in \bar{J}^{-1}Y \}.
\]

If we denote by \( \mathcal{FM}_{m,n} \) the category with objects composed of fibered manifolds with \( m \)-dimensional bases and \( n \)-dimensional fibres and morphisms formed by locally invertible fiber-preserving mappings, then, obviously, \( J^r, \bar{J}^r \) and \( \bar{J}^{-r} \) are bundle functors on \( \mathcal{FM}_{m,n} \).

Alternatively, one can define the \( r \)-th order semiholonomic prolongation \( \bar{J}^rY \) by means of natural target projections of nonholonomic jets, see [14]. For \( r \geq q \geq 0 \) let us denote by \( \pi_q^r \) the target surjection \( \pi_q^r : \bar{J}^rY \to J^qY \) with \( \pi_q^r \) being the identity on \( \bar{J}^rY \). We note that the restriction of these projections to the subspace of semiholonomic jet prolongations will be denoted by the same symbol. By applying the functor \( J^k \) we have also the surjections \( J^k\pi_q^{r-k} : \bar{J}^rY \to J^qY \) and, consequently, the element \( X \in \bar{J}^rY \) is semiholonomic if and only if
\[
(J^k\pi_q^{r-k})(X) = \pi_q^r(X) \text{ for any integers } 1 \leq q \leq k \leq r.
\]

In [14], the proof of this property can be found and the author finds it useful when handling semiholonomic connections and their prolongations.

Now let us recall local coordinates on higher order jet prolongations of a fibered manifold, \( Y \to M \). Let us denote by \( x^i \), \( i = 1, \ldots, m \) the local coordinates on \( M \) and \( y^p \), \( p = 1, \ldots, n \) the fiber coordinates of \( Y \to M \). We recall that the induced coordinates on the holonomic prolongation \( J^rY \) are given by \((x^i, y^p_\alpha)\), where \( \alpha \) is a multiindex of range \( m \) satisfying \( |\alpha| \leq r \). Clearly, the coordinates \( y^p_\alpha \) on \( J^rY \) are characterized by the complete symmetry in the indices of \( \alpha \). Having the nonholonomic prolongation \( \bar{J}^rY \) constructed by the iteration, we define the local coordinates inductively as follows:

1) Suppose that the induced coordinates on \( \bar{J}^{-1}Y \) are of the form
\[
(x^i, y^p_{k_1 \ldots k_{r-1}}), \ k_1, \ldots, k_{r-1} = 0, 1, \ldots, m.
\]

2) We define the induced coordinates on \( \bar{J}^rY \) by
\[
(x^i, y^p_{k_1 \ldots k_{r-1}0} = y^p_{k_1 \ldots k_{r-1}1}, y^p_{k_1 \ldots k_{r-1}i} = \frac{\partial}{\partial x^j} y^p_{k_1 \ldots k_{r-1}}).
\]

It remains to describe coordinates on the semiholonomic prolongation \( \bar{J}^rY \). Let \( (k_1, \ldots, k_r), \ k_1, \ldots, k_r = 0, 1, \ldots, m \) be a sequence of indices and denote by \( \{k_1, \ldots, k_s\} \), \( s \leq r \) the sequence of non-zero indices in \( (k_1, \ldots, k_r) \) respecting the order. Then the definition of \( \bar{J}^rY \) reads that the point \((x^i, y^p_{k_1 \ldots k_r}) \in \bar{J}^rY \) belongs to \( \bar{J}^rY \) if and only if \( y^p_{k_1 \ldots k_r} = y^p_{l_1 \ldots l_r} \) whenever \( \{k_1, \ldots, k_r\} = \{l_1, \ldots, l_r\} \)

3. iterated tangents

Another concept, in this paper called geometric, of a connection rises from the theory of iterated tangent spaces. Let us recall that the bundle \( T^kM \to T^{k-1}M \) is
equipped with the structure of a $k$-fold vector bundle. Particularly, $T^kM$ admits $k$ different projections to $T^{k-1}M$,

$$\rho_s := T^{k-s}\pi_s : T^kM \to T^{k-1}M,$$

where $\pi_s$ is the natural projection $T^sM \to T^{s-1}M$, $s = 1, 2, \ldots, k$. Each projection defines a vector bundle with basis $T^{k-1}M$ and the total space is composed of $2^{k-1}n$-dimensional vector spaces as fibers. The local coordinates on the neighborhoods

$T^sU \subset T^sM$, where $T^{s-1}U = \pi_s(T^sU)$, $s = 1, 2, \ldots, k$,

are derived from coordinates, or coordinate mappings, $(u^i)$, which are given on the neighborhood $U \subset M$:

- $U$: $(u^i)$, $i = 1, 2, \ldots, n$,
- $TU$: $(u^i, u^i_1)$, where $u^i := u^i \circ \pi_1$, $u^i_1 := du^i$,
- $T^2U$: $(u^i, u^i_1, u^i_2, u^i_{12})$,

where $u^i := u^i \circ \pi_2$, $u^i_1 := du^i \circ \pi_2$, $u^i_2 := d(u^i \circ \pi_1)$, $u^i_{12} := d^2u^i$,

etc.

**Proposition 1.** Coordinate mappings given on the neighborhood $T^{s-1}U$ induce coordinate mappings on the neighborhood $T^sU$ with respect to the projection $\pi_s$ by adding the differentials of these mappings.

Local coordinates are obtained by the following principle: to the coordinates of a point of a manifold we attach the coordinates of the vector tangent to the manifold at that point. We use the following notation: the coordinates of a neighborhood $T^kU$ consist of two copies of local coordinates on $T^{k-1}U$ where the second copy is equipped with an additional subscript $k$. This principle is suitable in the sense that the coordinates with index $s$ are recognized as the fiber coordinates for projections $\rho_s$, $s = 1, 2, \ldots, k$, i.e. the coordinates with index $s$ disappear after the application of projection $\rho_s$.

The coordinate form of the three projections $\rho_s : T^3U \to T^2U$, $s = 1, 2, 3$, is given by the following diagram:

$$(u^i, u^i_1, u^i_2, u^i_{12}, u^i_3, u^i_{13}, u^i_{23}, u^i_{123})$$

$$\rho_1 \downarrow \rho_2 \downarrow \rho_3$$

$$(u^i, u^i_1, u^i_3, u^i_{23})$$  $$(u^i, u^i_1, u^i_{13})$$  $$(u^i, u^i_1, u^i_{12}).$$

**Remark.** Let us note that the semiholonomity condition is connected to the notion of the osculating bundle, see [11], and can be defined as the equalizer of all possible projections, which corresponds to (1).

4. **Connections**

We start with the jet-like approach to connections. This rather structural description is quite suitable for determining natural operators on connections, for details see [6].
Definition 4.1. A general connection on the fibered manifold $Y \rightarrow M$ is a section $\Gamma : Y \rightarrow J^1Y$ of the first jet prolongation $J^1Y \rightarrow Y$.

Further generalization of this idea leads us to the definition of $r$-th order connection, which is a section of $r$-th order jet prolongation of a fibered manifold. According to the character of the target space we distinguish holonomic, semiholonomic and nonholonomic general connections. The coordinate form of a second order nonholonomic connection $\Delta : Y \rightarrow \tilde{J}^2Y$ is given by

$$y^p_i = F^p_i(x, y), \quad y^p_{0i} = G^p_i(x, y), \quad y^p_{ij} = H^p_{ij}(x, y),$$

where $F, G, H$ are arbitrary smooth functions. In case of linear connections all functions are linear in fiber coordinates.

Let us now recall the geometric concept of a connection and its extension to higher order connections. The following section is based on the paper [11].

Definition 4.2. A connection on bundle $\pi : M_1 \rightarrow M$ is defined by the structure $\triangle_h \oplus \triangle_v$ on a manifold $M_1$ where $\triangle_v = \ker T\pi$ is vertical distribution tangent to the fibers and $\triangle_h$ is horizontal distribution complementary to the distribution $\triangle_v$. The transport of the fibers along the path $\gamma \subset M$ is realized by the horizontal lifts given by the distribution $\triangle_h$ on the surface $\pi^{-1}(\gamma)$. If the bundle is a vector one and the transport of fibers along an arbitrary path is linear, then the connection is called linear.

We will assume that the base manifold $M$ is of dimension $n$ and the fibers are of dimension $r$. Then

$$\dim \triangle_h = n, \quad \dim \triangle_v = r.$$  

On the neighborhood $U \subset M_1$, let us consider local base and fiber coordinates:

$$(u^i, u^\alpha), \quad i = 1, 2, \ldots, n; \quad \alpha = n + 1, \ldots, n + r.$$  

Base coordinates $(u^i)$ are determined by the projection $\pi$ and the coordinates $(\bar{u}^i)$ on a neighborhood $\bar{U} = \pi(U), \ u^i = \bar{u}^i \circ \pi$.

Definition 4.3. On a neighborhood $U \subset M_1$ we define a local (adapted) basis of the structure $\triangle_h \oplus \triangle_v$,

$$(X_i, X_\alpha) = \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^\alpha}, \delta^i_\beta, 0 \right), \quad \left( \omega^i, \omega^\alpha \right) = \left( \delta^i_\alpha - \Gamma^i_j \delta^\alpha_\beta, 0 \right) \cdot \left( du^j, du^\beta \right).$$

The horizontal distribution $\triangle_h$ is the linear span of the vector fields $(X_i)$ and the annihilator of the forms $(\omega^\alpha)$,

$$X_i = \partial_i + \Gamma^\beta_i \partial_\beta, \quad \omega^\alpha = du^\alpha - \Gamma^\alpha_i du^i.$$

Definition 4.4. A classical affine connection on manifold $M$ is seen as a linear connection on the bundle $\pi_1 : TM \rightarrow M$. On the tangent bundle $TM \rightarrow M$ one can define the structure $\triangle_h \oplus \triangle_v$. The indices in the formulas are denoted by Latin letters all of them ranging from 1 to $n$. The functions $\Gamma^\alpha_i, X_i, \omega^\alpha$ are of the form...
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(in $\Gamma_i^\alpha$ the sign is changed to comply with the classical theory):

$$
\Gamma_i^\alpha \mapsto -\Gamma_{jk}^i u_1^k, \\
X_i = \partial_i + \Gamma_i^\alpha \partial_\alpha \mapsto X_i = \partial_i - \Gamma_{kj}^i u_1^k \partial_k, \\
\omega^\alpha = du^\alpha - \Gamma_i^\alpha du_i \mapsto U^i_{12} = u_1^i + \Gamma_{jk}^i u_1^k u_2^j.
$$

**Definition 4.5.**
Higher order connections are defined as follows: on tangent bundle $TM$ the structure $\Delta \oplus \Delta_1$ is defined where $\ker T\rho_1 = \Delta_1$, on $T(TM)$ the structure $\Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12}$ is defined where $\ker T\rho_s = \Delta_s \oplus \Delta_{12}$, $s = 1, 2$, etc.

5. **Connections on two-fold fibered manifolds**

More generally, one can define a second order connection by means of a two-fold fibered manifold. Note that the Definition 4.5 is a special case of the following. A two-fold fibered manifold is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\rho_2} & \mathcal{M}_1 \\
\rho_1 & \downarrow & \downarrow \\
\mathcal{M}_2 & \xrightarrow{\pi_2} & \mathcal{M}
\end{array}
$$

where $\rho_1, \rho_2$ and $\pi_1, \pi_2$ — four fibered manifolds

$$
\dim \mathcal{M} = n, \quad \dim \mathcal{M}_1 = n + r_1, \quad \dim \mathcal{M}_2 = n + r_2, \quad \dim \mathcal{M} = n + r_1 + r_2 + r_{12}.
$$

The double projection

$$
\pi = \pi_1 \circ \rho_2 = \pi_2 \circ \rho_1 : \mathcal{M} \to \mathcal{M}
$$

divides a manifold $\mathcal{M}$ to $n$-parameter family of fibers of dimensions $(r_1 + r_2 + r_{12})$. Each fiber carries structure of another two fibers of dimensions $r_1 + r_{12}$ and $r_2 + r_{12}$ and these two fibers have the common intersection of dimension $r_{12}$.

A two-fold fibered manifold is called a vector bundle if both fibrations $\pi_1, \pi_2, \rho_1, \rho_2$ — form vector bundles.

An example of a two-fold fibered manifold is the second order tangent bundle $T^2M$ of a manifold $M$. In this case $n = r_1 = r_2 = r_{12}$.

**Definition 5.1.** A connection on a two-fold fibered manifold is defined by a structure on a manifold $\mathcal{M}$:

$$
\Delta \otimes \Delta_1 \otimes \Delta_2 \otimes \Delta_{12}, \quad \dim \Delta = n, \quad \dim \Delta_1 = r_1, \quad \dim \Delta_2 = r_2, \quad \dim \Delta_{12} = r_{12},
$$

$$
\ker T\rho_2 = \Delta_2 \oplus \Delta_{12}, \quad \ker T\rho_1 = \Delta_1 \oplus \Delta_{12}
$$

$$
T\rho_2(\Delta \oplus \Delta_1) = TM_1, \quad T\rho_1(\Delta \oplus \Delta_2) = TM_2, \quad T\pi \Delta = TM.
$$

**Remark.** A connection on a two-fold vector fibered manifold is called linear if the structure $\mathcal{M}$ induces on the manifolds $\pi_1, \pi_2, \rho_1$ and $\rho_2$ linear connections.

**Remark.** Similarly, one can define a connection on a $k$–fold fibered manifold. In such case the commutative diagram would be represented by a $k$–dimensional cube. These manifolds would correspond to the $k$–th tangent bundle $T^kM$ of a manifold $M$. 

On the neighborhoods
\[ \mathcal{U} \subset \mathcal{M}, \quad \mathcal{U}_1 = \rho_2(\mathcal{U}) \subset \mathcal{M}_1, \quad \mathcal{U}_2 = \rho_1(\mathcal{U}) \subset \mathcal{M}_2, \quad U = \pi(\mathcal{U}) \subset M \]
we have the coordinate systems \((u^i, u^{\alpha_1}, u^{\alpha_2}, u^{\alpha_{12}}), (u^i, u^{\alpha_1}), (u^i, u^{\alpha_1}), (u^i)\).

The transformation of coordinates on the neighborhoods \(\mathcal{U}\),
\[ (u^i, u^{\alpha_1}, u^{\alpha_2}, u^{\alpha_{12}}) \leadsto (\tilde{u}^i, \tilde{u}^{\alpha_1}, \tilde{u}^{\alpha_2}, \tilde{u}^{\alpha_{12}}) = (a^i, a^{\alpha_1}, a^{\alpha_2}, a^{\alpha_{12}}), \]
gives a Jacobi matrix:
\[ \begin{pmatrix} \alpha^i_j & 0 & 0 & 0 \\ \alpha^j_1 & \alpha^j_{1i} & 0 & 0 \\ \alpha^j_2 & 0 & a^{\alpha_2}_j & 0 \\ a^{\alpha_{12}}_j & a^{\alpha_{12}}_{1j} & a^{\alpha_{12}}_{2j} & a^{\alpha_{12}}_{12} \end{pmatrix}. \]

See [1], [10]. Let us mention that the local (adapted) basis of such decomposition is represented by a matrix of the form
\[ \begin{pmatrix} \delta^i_j & 0 & 0 & 0 \\ \Gamma^j_{i1} & \delta^j_{i2} & 0 & 0 \\ \Gamma^j_{i2} & 0 & \delta^j_{i2} & 0 \\ \Gamma^j_{i12} & \Gamma^j_{i121} & \Gamma^j_{i122} & \Gamma^j_{i1212} \end{pmatrix}. \]
The dual reper is given by the system of 1–forms:
\[
\begin{align*}
\omega^i &= du^i, \\
\omega^{\alpha_1} &= du^{\alpha_1} - \Gamma^j_{i1} du^i, \\
\omega^{\alpha_2} &= du^{\alpha_2} - \Gamma^j_{i2} du^i, \\
\omega^{\alpha_{12}} &= du^{\alpha_{12}} - \Gamma^j_{i12} du^{\alpha_1} - \Gamma^j_{i12} du^{\alpha_2} - \Gamma^j_{i12} du^i,
\end{align*}
\]
where \(\Gamma^j_{i1} - \Gamma^j_{i12} = \Gamma^j_{i121} \Gamma^j_{i122} + \Gamma^j_{i121} \Gamma^j_{i122}\).

In case of linear connection the elements of the matrix \([5]\) are of the form
\[
\begin{align*}
\Gamma^j_{i1} &= \Gamma^j_{j\beta_1} u^{\beta_1}, \\
\Gamma^j_{i2} &= \Gamma^j_{j\beta_2} u^{\beta_2}, \\
\Gamma^j_{i12} &= \Gamma^j_{j\beta_1\beta_2} u^{\beta_1} + \Gamma^j_{j\beta_12} u^{\beta_2}, \\
\Gamma^j_{j\beta_12} &= \Gamma^j_{j\beta_1\beta_2} u^{\beta_1} + \Gamma^j_{j\beta_12} u^{\beta_2} + \Gamma^j_{j\beta_1} u^{\beta_2}, \\
\Gamma^j_{j\beta_1\beta_2} &= \Gamma^j_{j\beta_1\beta_2} u^{\beta_1} + \Gamma^j_{j\beta_1\beta_2} u^{\beta_2},
\end{align*}
\]
where the coefficients depend on the base coordinates \(u^i\) only.

6. Ehresmann prolongation

We show that Ehresmann prolongation plays an important role in determining all natural operators transforming first order connections into higher order connections. Let us note that also natural transformations of semiholonomic jet prolongation functors \(\mathcal{T}\) are involved. To find the details about this topic we refer to [6], [7], [12]. For our purposes, it is enough to consider \(r = 2\). We use the notation of \(6\), where the map \(e : \mathcal{T}^2Y \rightarrow \mathcal{T}^2Y\) is obtained from the natural exchange map \(e_A : J^1J^1Y \rightarrow J^1J^1Y\) as a restriction to the subbundle \(\mathcal{T}^2Y \subset J^1J^1Y\). Note that while \(e_A\) depends on the linear connection \(\Lambda\) on \(M\), its restriction \(e\) is independent of
any auxiliary connections. We remark, that originally the map $e_{\Lambda}$ was introduced by M. Modugno. We also recall that J. Pradines introduced a natural map $J^2Y \to J^2Y$ with the same coordinate expression.

Now we are ready to recall the following assertion, see [12] for the proof.

**Proposition 2.** All natural operators transforming first order connection $\Gamma : Y \to J^1Y$ into second order semiholonomic connection $Y \to J^2Y$ form a one parameter family

$$\Gamma \mapsto k \cdot (\Gamma \ast \Gamma) + (1 - k) \cdot e(\Gamma \ast \Gamma), \quad k \in \mathbb{R}.$$ 

This shows the importance of Ehresmann prolongation in the theory of prolongations of connections. Let us now recall a more general concept of a product of two connections:

Given two higher order connections $\Gamma : Y \to \tilde{J}^rY$ and $\Gamma : Y \to \tilde{J}^sY$, the product of $\Gamma$ and $\Gamma$ is the $(r + s)$-th order connection $\Gamma \ast \Gamma : Y \to \tilde{J}^{r+s}Y$ defined by

$$\Gamma \ast \Gamma = \tilde{J}^s \Gamma \circ \Gamma.$$ 

Particularly, if both $\Gamma$ and $\Gamma$ are of the first order, then $\Gamma \ast \Gamma : Y \to \tilde{J}^2Y$ is semiholonomic if and only if $\Gamma = \Gamma$ and $\Gamma \ast \Gamma$ is holonomic if and only if $\Gamma$ is curvature-free, [5], [14].

As an example we show the coordinate expression of an arbitrary nonholonomic second order connection and of the product of two first order connections. The coordinate form of $\Delta : Y \to \tilde{J}^2Y$ is

$$y^p_i = F^p_i(x, y), \quad y^p_0 = G^p_i(x, y), \quad y^p_{ij} = H^p_{ij}(x, y),$$

where $F, G, H$ are arbitrary smooth functions. Further, if the coordinate expressions of two first order connections $\Gamma, \Gamma : Y \to J^1Y$ are

(7) $$\Gamma : \quad y^p_i = F^p_i(x, y), \quad \Gamma : \quad y^p_i = G^p_i(x, y),$$

then the second order connection $\Gamma \ast \Gamma : Y \to \tilde{J}^2Y$ has equations

(8) $$y^p_i = F^p_i, \quad y^p_0 = G^p_i, \quad y^p_{ij} = \frac{\partial F^p_i}{\partial x^j} + \frac{\partial F^p_i}{\partial y^q} G^q_j.$$ 

For linear connections, the coordinate form would be obtained by substitution

$$F^p_i = F^p_{iq} y^q, \quad G^p_i = G^p_{iq} y^q$$

in the equations (7), where $F^p_{iq}$ and $G^p_{iq}$ are functions of the base manifold coordinates $x_i$. For order three see [13].

### 7. Tangent Funktor and Ehresmann Prolongation

If we apply the tangent functor $T$ two times on a projection $\pi : E \to M$ and a section $\sigma : M \to E$ we obtain

$$T\pi : TE \to TM, \quad T^2\pi : T^2E \to T^2M,$$

$$T\sigma : TM \to TE, \quad T^2\sigma : T^2M \to T^2E,$$

respectively. The mappings $\sigma, T\sigma$ and $T^2\sigma$ define the sections of fibered manifolds $\pi, T\pi$ and $T^2\pi$. 

Let us consider for a function $f$ defined on a manifold $M$, its following differentials on $T^2M$ in local coordinate form:

$$f_1 \equiv f, \quad f_2 \equiv f, \quad f_12 \equiv f_1 x_1 x_2 + f_2 x_1 x_2,$$

where $f_i = \frac{\partial f}{\partial x^i}$, $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$.

Furthermore, $f_1 = df \circ \rho_1$, $f_2 = df \circ \rho_2$, $f_12 = d^2 f$. We use these notations in the formulae below.

If the section $\sigma$ is defined by local functions $\Gamma^p$, then the sections $T \sigma$ and $T^2 \sigma$ are defined by its differentials $\Gamma^{1}_1$, $\Gamma^{2}_2$, and $\Gamma^{12}$,

$$\sigma : x^i \rightsquigarrow y^p = \Gamma^p,$$

$$T \sigma : (x^i, x^i_1) \rightsquigarrow (y^p, y^p_1) = (\Gamma^p, \Gamma^p_1),$$

$$T^2 \sigma : (x^i, x^i_1, x^i_2) \rightsquigarrow (y^p, y^p_1, y^p_2, y^p_{12}) = (\Gamma^p, \Gamma^p_1, \Gamma^p_2, \Gamma^p_{12}),$$

(9) where $\Gamma^p_1 = \Gamma^p x^i_1$, $\Gamma^p_2 = \Gamma^p x^i_2$, $\Gamma^p_{12} = \Gamma^p x^i_1 x^i_2 + \Gamma^p x^i_2$.

The case when the coefficients $\Gamma^p_i$, $\Gamma^p_{ij}$ in (9) are arbitrary functions, corresponds to a nonholonomic connection on the fibered manifold $\pi$.

The case when $\Gamma^p_{ij} = \frac{\partial \Gamma^p_i}{\partial x^j}$, where $\Gamma^p_i$ are arbitrary functions corresponds to a semiholonomic connection on the fibered manifold $\pi$.

The case when $\Gamma^p_1 = df \circ \rho_1$, $\Gamma^p_2 = df \circ \rho_2$, $\Gamma^p_{12} = d^2 f$, corresponds to a holonomic connection on the fibered manifold $\pi$.

The functions $\Gamma^p_i, \Gamma^p_{ij}$ define nonholonomic, semiholonomic or holonomic Ehresmann prolongation of a connection, respectively.

**Remark.** Nonholonomic prolongation induces a connection on a double fibered manifold

$$J \rightarrow E \rightarrow M : \gamma^p_i \rightsquigarrow y^p \rightsquigarrow x^i.$$

On the fibered manifold $E \rightarrow M$ the fiber transformations are given by the Pfaff system

$$\omega^p \equiv dy^p - \Gamma^p_i dx^i = 0,$$

more precisely, along a curve $x^i(t)$ - by the system of first order ODEs

$$\ddot{y}^p = \Gamma^p_i \dot{x}^i.$$

In case $(\Gamma^p_{i2}, x^i_1, x^i_2, x^i_{12}) \rightsquigarrow (\dot{y}^p, \dot{x}^i, \ddot{x}^i, \dddot{x}^i)$ we obtain the system of second order ODEs:

$$\Gamma^p_{12} = \Gamma^p_i x^i_1 x^i_2 + \Gamma^p_i x^i_{12} \Rightarrow \dddot{y}^p = \Gamma^p_{ij} \dot{x}^i \dot{x}^j + \Gamma^p \dddot{x}^i.$$

Considering the system (10), we obtain for fiber coordinates $y^p, y^p_i$ system of first order ODEs

(11)$$\begin{cases}
\ddot{y}^p = \Gamma^p_i \dot{x}^i, \\
\dot{y}^p_i = \Gamma^p_{ij} \dot{x}^j.
\end{cases}$$

The sections of fibers along a curve $x^i(t)$ are given.

The horizontal distribution $\Delta_h$ is $n$-dimensional and described by the vector field

$$X_i = \partial_i + \Gamma^p_i \partial_p + \Gamma^p_{ij} \partial_p^j, \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x^i}, \quad \partial_p = \frac{\partial}{\partial y^p}, \quad \partial_p^j = \frac{\partial}{\partial y^p_j}.$$
A NOTE ON JET AND GEOMETRIC APPROACH TO HIGHER ORDER CONNECTIONS

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