Pseudo–Hermiticity and weak pseudo–Hermiticity: Equivalence of complementarity and the coordinate transformations in position–dependent mass

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Abstract

The complementarity between the twin concepts of pseudo–Hermiticity and weak pseudo–Hermiticity, established by Bagchi and Quesne [Phys. Lett. A 301 (2002) 173-176], can be understood in terms of coordinate transformations.

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1 Introduction

In recent years, the concept of pseudo–Hermiticity has attracted much attention on behalf of physicists [1-9]. The basic mathematical structure underlying the properties of pseudo-Hermiticity is revealed [3-5] and it has been found to be a more general concept then those of Hermiticity and \( \mathcal{PT} \)–symmetry [10-15]. By definition, a linear operator \( H \) (here a Hamiltonian) acting in a Hilbert space \( \mathcal{H} \) is called \( \eta \)–pseudo–Hermitian if it obeys to [3-5]

\[
\eta H = H^\dagger \eta, \tag{1}
\]

where \( \eta \) is a Hermitian linear invertible operator and a dagger stands for the adjoint of the corresponding operator. Then (non–Hermitian) Hamiltonian \( H \) has a real spectrum [3] if there is an invertible linear operator \( d : \mathcal{H} \rightarrow \mathcal{H} \) such that \( \eta = d^\dagger d \). As a consequence of this, the reality of the bound–state eigenvalues of \( H \) can be associated with \( \eta \)–pseudo–Hermiticity. Note that choosing \( \eta = 1 \) reduces the assumption (1) to the Hermiticity.

In a very interesting work [7], Bagchi and Quesne point out that the twin concepts of pseudo–Hermiticity and weak pseudo–Hermiticity are complementary to one another.
by admitting that it is possible to break up $\eta$ into two operators, i.e. $\eta_+$ and $\eta_-$, following combinations

$$\eta_+ H = H^\dagger \eta_+ \quad \text{and} \quad \eta_- H = H^\dagger \eta_-,$$

(2)

where $\eta_\pm = \eta \pm \eta^\dagger$. The first assumption corresponds to the pseudo–Hermiticity where $\eta_+$ is a second–order differential realization while the second is associated with weak pseudo–Hermiticity and $\eta_-$ is a first–order realization.

In the present paper, we take up the study of a complementarity between pseudo–Hermiticity and weak pseudo–Hermiticity under the concept of coordinate transformation and examine how the pseudo–Hermiticity should map to the weak pseudo–Hermiticity. In fact, our primary concern is to point out that the coordinate transformations can be looked upon as a toy model for understanding the complementarity. In this light, the complementarity acquires a mathematical meaning which, unfortunately, was not established in [7].

We end this section by defining a quite formalism used throughout the present work. In the case of a spatially varying mass [16-19] which will be denoted by $M (x) = m_0 m (x)$, the Hamiltonian proposed by von Roos [16] reads

$$H = \frac{1}{4} \left( m^\alpha (x) p m^\beta (x) p m^\gamma (x) + m^\gamma (x) p m^\beta (x) p m^\alpha (x) \right) + V (x),$$

(3)

where $\alpha, \beta$ and $\gamma$ are three parameters which obey the relation $\alpha + \beta + \gamma = -1$ in order to grant the classical limit and $V (x) = V_{Re} (x) + i V_{Im} (x) \in \mathbb{C}$. Here, $p = -i \frac{d}{dx}$ is a momentum with $\hbar = m_0 = 1$, and $m (x)$ is dimensionless–real valued mass.

Using the restricted Hamiltonian from the $\alpha = \gamma = 0$ and $\beta = -1$ constraints [17], the Hamiltonian (3) becomes

$$H = p U^2 (x) p + V (x),$$

(4)

with $U^2 (x) = \frac{1}{2m(x)}$ and $U (x) \in \mathbb{R}$. The shift on the momentum $p$ in the manner

$$p \rightarrow p' = p - \frac{A (x)}{U (x)},$$

(5)

where $A (x) = a (x) + ib (x) \in \mathbb{C}$ and $a (x), b (x)$ are real functions, allows to bring the Hamiltonian (4) in the form

$$H \rightarrow \mathcal{H} = \left( p - \frac{A (x)}{U (x)} \right) U^2 (x) \left( p - \frac{A (x)}{U (x)} \right) + V (x).$$

(6)

2 Pseudo–Hermiticity generating function

As $\eta_+ (= d^\dagger d)$ is pseudo–Hermitian and following the ordinary supersymmetric quantum mechanics, the operators $d$ and $d^\dagger$ are connecting to the first–order differential realization through [8,9]

$$d = U (x) \frac{d}{dx} + \Phi (x),$$

(7.a)

$$d^\dagger = -U' (x) - U (x) \frac{d}{dx} + \Phi^* (x),$$

(7.b)
where \( \Phi(x) = F(x) + iG(x) \in \mathbb{C} \) and \( F(x), G(x) \) are real functions. Here, the prime denotes derivative with respect to \( x \). It is obvious that Eqs.(7.a–b) become, under the transformation (5),

\[
d \to \mathcal{D} = U(x) \frac{d}{dx} - iA(x) + \Phi(x),
\]

\[
d^\dagger \to \mathcal{D}^\dagger = -U''(x) - U(x) \frac{d}{dx} + iA^*(x) + \Phi^*(x),
\]

and in terms of these, \( \eta_+ \) is transformed into \( \tilde{\eta}_+ \) (\( = \mathcal{D}^\dagger \mathcal{D} \)) such as

\[
\tilde{\eta}_+ = -U^2(x) \frac{d^2}{dx^2} - 2\mathcal{K}(x) \frac{d}{dx} + \mathcal{L}(x),
\]

where \( \mathcal{K}(x) \) and \( \mathcal{L}(x) \) are defined as

\[
\mathcal{K}(x) = U(x)U''(x) + iU'(x)(G(x) - a(x)),
\]

\[
\mathcal{L}(x) = \Phi^*(x)\Phi(x) + A^*(x)A(x) - [U(x)(iA(x) - \Phi(x))]'
- i\Phi^*(x)A(x) + i\Phi(x)A^*(x).
\]

Taking the adjoint of Eq.(9), one can easily check that \( \tilde{\eta}_+ \) is Hermitian; since it is written in the form \( \tilde{\eta}_+ = \mathcal{D}^\dagger \mathcal{D} \). On the other hand, the Hamiltonian (6) may be expressed as

\[
\mathcal{H} = -U^2(x) \frac{d^2}{dx^2} - 2\mathcal{M}_1(x) \frac{d}{dx} + \mathcal{N}_1(x) + V(x),
\]

where

\[
\mathcal{M}_1(x) = U(x)U''(x) - iU'(x)A(x),
\]

\[
\mathcal{N}_1(x) = i[U(x)A(x)]' + A^2(x).
\]

It should be noted that \( \mathcal{D} \) and \( \mathcal{D}^\dagger \) are two intertwining operators, and then the defining assumption (1) can be generalized into \( \tilde{\eta}_+ \mathcal{H} = \mathcal{H}^\dagger \tilde{\eta}_+ \). Using Eqs.(9), (11) and the adjoint of Eq.(11) on both sides of the last equation and comparing between their varying differential coefficients, we can recognized from the third–derivative that \( b(x) = 0 \), while the second–derivative connects the potential to its conjugate through

\[
V(x) = V^*(x) - 4iU(x)G'(x).
\]

However, the coefficients corresponding to the first–derivative give the shape of the potential, where after integration, we get

\[
V(x) = F^2(x) - G^2(x) - [U(x)F(x)]' - 2iU(x)G'(x) + \delta,
\]

where \( \delta \) is some constant of integration. The last remaining coefficient corresponds to the null–derivative and gives the pure–imaginary differential equation

\[
F^2(x) - [U(x)F(x)]' = \frac{G(x)}{G'(x)} \left( -F(x)F'(x) + \frac{1}{2}[U(x)F(x)]'' \right)
+ \frac{1}{4G'(x)} \left[ \frac{1}{4} U^2(x)G''(x) \right]' - \frac{G(x)}{4} \left[ U(x)U''(x) \right]'
+ \frac{U'(x)U(x)}{4} \left[ G(x) \right]' \left[ \frac{1}{U(x)} \right]''
+ \frac{U^2(x)U(x)}{2} \left[ \frac{G(x)}{U(x)} \right]'
- \frac{U''(x)U(x)}{4}.
\]
which is not easy to solve. However, the $\tilde{\eta}_+^\dagger$–orthogonality suggests that the eigenvector, here $\Psi (x)$, is related to $\mathcal{H}$ through

$$\tilde{\eta}_+ \Psi (x) = 0, \quad \text{or} \quad D \Psi (x) = 0,$$

leading, after integration, to the ground–state wave function

$$\Psi (x) = \Lambda (x) \psi (x),$$

$$= \exp \left[ i \int^x dy \frac{A(x)}{U(x)} \right] \psi (x)$$

$$= \mathcal{N}_0 \exp \left[ -i \int^x dy \frac{F(x)}{U(x)} - i \int^x dy \frac{G(x) - a(x)}{U(x)} \right],$$

where $\mathcal{N}_0$ is a constant of normalization. The wave function $\Psi (x)$ is then subjected to a gauge transformation in a manner of $\psi (x) \rightarrow \Psi (x) = \Lambda (x) \psi (x)$, where $\Lambda (x) = \sqrt{\tilde{\eta}_+ (x)}$ [1,7]. Now, using the Schrödinger equation $\mathcal{H} \Psi (x) = \mathcal{E} \Psi (x)$ where $\mathcal{E} = \mathcal{E}_{\text{Re}} + i \mathcal{E}_{\text{Im}}$, one obtain the differential equation

$$2F(x)G(x) + U(x)G'(x) - U'(x)G(x) = -\mathcal{E}_{\text{Im}} + i (\mathcal{E}_{\text{Re}} - \delta),$$

where $\delta$ is a constant introduced in Eq.(14). In order to solve suitably Eq.(18), we assume that both sides of Eq.(18) are equal to zero; which requires that $\mathcal{E}_{\text{Re}} = \delta$ and $\mathcal{E}_{\text{Im}} = 0$. Therefore, the energy eigenvalues $\mathcal{E}$ are real. In these settings, we end up by relating $F(x)$ to $G(x)$ and $U(x)$ through the differential equation

$$F(x) = \frac{G(x)}{2} \left[ \frac{U(x)}{G(x)} \right]',$$

and which proves to be the solution of Eq.(15). Hence, it becomes clear that $F(x)$ (i.e. $G(x)$) is a generating function leading to identify the potential $V(x)$.

### 3 Weak pseudo–Hermiticity generating function

For the first–order differential realization, $\eta_-$ may be anti–Hermitian and $\mathcal{H}$ can be relaxed to be weak pseudo–Hermitian. Then $\eta_-$ can be expressed as

$$\eta_- = U(x) \frac{d}{dx} \varphi (x),$$

where $\varphi (x) = f(x) + ig(x) \in \mathbb{C}$ and $f(x), g(x)$ are real functions. Using Eq.(5), $\eta_-$ and $\eta_{-}^\dagger$ become

$$\eta_- \rightarrow \tilde{\eta}_- = U(x) \frac{d}{dx} - iA(x) + \varphi (x),$$

$$\eta_{-}^\dagger \rightarrow \tilde{\eta}_{-}^\dagger = -U'(x) - U(x) \frac{d}{dx} + iA^*(x) + \varphi^*(x).$$

As now $\tilde{\eta}_-$ points to weak pseudo–Hermiticity, this amounts to writing

$$\tilde{\eta}_{-}^\dagger = -\tilde{\eta}_-,$$
which brings to the relation

\[ U'(x) = 2f(x) + 2b(x). \] (23)

Letting both sides of \( \tilde{\eta}_- \mathcal{H} = \mathcal{H} \tilde{\eta}_- \) act on every function and comparing their varying differential coefficients, one deduced from the second–derivative that \( b(x) = 0 \), therefore the generating function \( f(x) \) in Eq.(23) becomes

\[ f(x) = \frac{U'(x)}{2}, \] (24)

while the first–derivative gives the imaginary part of the potential

\[ V_{\text{Im}}(x) = iU(x)f'(x) - U(x)g'(x) - \frac{i}{2}U(x)U''(x). \] (25)

The last coefficient corresponds to the null–derivative which gives, after a double integration by parts, the real part of the potential

\[ V_{\text{Re}}(x) = -g^2(x) - \frac{1}{2}U(x)U''(x) - \frac{1}{4}U'^2(x) + \varepsilon, \] (26)

where \( \varepsilon \) is some constant of integration. In consequence, using Eqs.(22–24), we obtain the potential

\[ V(x) = -g^2(x) - iU(x)g'(x) - \frac{1}{2}U(x)U''(x) - \frac{1}{4}U'^2(x) + \varepsilon. \] (27)

4 Equivalence of Complementarity–Coordinate transformation

In this section, we bring to the notion of the complementarity a mathematical meaning by examining the way in which pseudo–Hermiticity should map into weak pseudo–Hermiticity through the generating functions \( F(x) \) and \( f(x) \). In fact, it is well known from Eqs.(19) and (24) that both generating functions belong to the same ordinary space representation \( \{X\} \), then there must be a transformation connecting them. For this reason, we assume that the required transformations are concerned with coordinate transformations (or point canonical transformations.)

In mathematical terms, a coordinate transformation \( x \equiv x(\xi) \) changes \( F(x) \) into \( f(\xi) \) in the following way

\[ F(x) = \frac{G(x)}{2} \left[ U(x) \right]' \xrightarrow{x \equiv x(\xi)} f(\xi) = \frac{U'(\xi)}{2}, \] (28)

where \( U(x) \equiv U[x(\xi)] = \mathcal{U}(\xi) \).

An interesting way to solve this problem, that can be described within coordinate transformation, is to build a differential equation from Eq.(19) and assume that it is maintained invariant if one applies a coordinate transformation. In fact, Eq.(19) can be expressed as

\[ U(x) \frac{dZ(x)}{dx} = 2F(x)Z(x), \] (29)
where $Z(x) = \frac{U(x)}{G(x)}$. It is then obvious that whenever Eq.(29) holds for the set of functions (i.e. $U(x)$, $F(x)$ and $Z(x)$), similar differential equation will holds for the transformed functions too (i.e. $\overline{U}(\xi)$, $\overline{F}(\xi)$ and $\overline{Z}(\xi)$) such as

$$\overline{U}(\xi) \frac{d\overline{Z}(\xi)}{d\xi} = 2\overline{F}(\xi)\overline{Z}(\xi),$$

(30)

where $F(x) \equiv F[x(\xi)] = \overline{F}(\xi)$; idem. for $Z(x)$. Therefore, from Eq.(30), the mass function $U(x)$ is changed in the following way

$$U(x) \rightarrow \overline{U}(\xi) = U[x(\xi)] \frac{d\xi(x)}{dx}.$$ 

(31)

Let us introduce two new functions $\mathcal{R}(\xi)$ and $\mathcal{S}(\xi)$ related, respectively, to $Z(x)$ and $F(x)$ by

$$Z(x) \rightarrow \overline{Z}(\xi) = Z[x(\xi)] \mathcal{R}(\xi),$$

(32.a)

$$F(x) \rightarrow \overline{F}(\xi) = F[x(\xi)] \mathcal{S}(\xi).$$

(32.b)

Substituting Eqs.(32.a–b) into Eq.(30) taking into account (31), we get

$$U(x) \frac{dZ(x)}{dx} = 2 \left[ \mathcal{S}(x) F(x) - U(x) \frac{d}{dx} \ln \sqrt{\mathcal{R}(x)} \right] Z(x),$$

(33)

and by identifying it to Eq.(29), one obtain

$$\mathcal{S}(x) F(x) = F(x) + U(x) \frac{d}{dx} \ln \sqrt{\mathcal{R}(x)},$$

(34)

which can be interpreted as a similarity transformation relating $F(x)$ to $f(x)$; i.e.

$$F(x) \rightarrow f(x) \equiv \mathcal{S}(x) F(x) = F(x) + U(x) \frac{d}{dx} \ln \sqrt{\mathcal{R}(x)}. $$

(35)

In this light, let us redefine the coordinate transformation on $F(x)$ following

$$F(x) \rightarrow \overline{F}(\xi) = F[x(\xi)] \mathcal{S}(\xi) = F[x(\xi)] \frac{d\xi(x)}{dx},$$

(36)

and from Eqs.(35) and (19), we get the identity

$$f(\xi) \equiv F[x(\xi)] \mathcal{S}(\xi) = \frac{G[x(\xi)] \mathcal{S}(\xi)}{2} \left[ \frac{U[x(\xi)]}{G[x(\xi)]} \right]' .$$

(37)

Now in order to recover our result, we assume that the condition $G[x(\xi)] \mathcal{S}(\xi) = 1$ holds, and by defining the generating function $G(x)$ as

$$G[x(\xi)] \equiv \mathcal{S}^{-1}(\xi) = \frac{dx(\xi)}{d\xi},$$

(38)

therefore Eq.(37) can be amply simplified, taking into consideration Eq.(30), to

$$f(\xi) \equiv \frac{1}{2} \left[ U[x(\xi)] \frac{d\xi(x)}{dx} \right]'$$

$$= \frac{\overline{U}(\xi)}{2} .$$

(39)

This completes the proof and leads to the identity (28).
5 Conclusion

In this paper, we have proposed to give a mathematical meaning for the notion of complementarity between the twin concepts of pseudo–Hermiticity and weak pseudo–Hermiticity within the framework of coordinate transformations, and as a consequence this has opened the way towards understanding the complementarity. Our primary concern in our work implies that all generating functions, whose the associated potentials are related to the pseudo–Hermiticity and weak pseudo–Hermiticity, can be connected into some generalized coordinate transformations.

As a concluding remark, we would like to point out the equivalence between the complementarity and coordinate transformations is concerned by a particular choice which the generating function $G(x)$ (i.e. $F(x)$) can take.

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