Non-Markovian Dynamics of Spin Squeezing

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Abstract. We evaluate the spin squeezing dynamics of $N$ independent spin-1/2 particles with exchange symmetry. Each spin interacts with its own reservoir, and the reservoirs are independently and identical. The spin squeezing parameter is analytically calculated with different kinds of decoherence. The spin squeezing property vanishes with evolution time under the Markovian decoherence. Whereas coupled to the non-Markovian decoherence channels, the spin squeezing property collapses and revives with time. As spin squeezing can be regarded as a witness of multipartite entanglement, thus our scheme shows the collapse and revival of multipartite entanglement under non-Markovian decoherence.

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1. Introduction

Entanglement is a kind of quantum correlations that has been playing a central role in quantum information and communication theory. Quantum correlations have attracted much interest as fascinating demonstration of nonclassical phenomena and they have also found many promising applications such as achieving interferometric [1, 2, 3, 4] and enhancing the ratio-to-noise ratio in spectroscopy [5, 6] beyond the standard quantum noise limit. The spin squeezed state is one kind of quantum correlated states [7] with reduced fluctuations in one of the collective spin components, with possible applications in atomic interferometers and high-precision atomic clocks. Entanglement is based on the superposition principle combined with the Hilbert space structure, while spin squeezing is originated from another fundamental principle of quantum mechanics—the uncertainty principle. It is found that spin squeezing is closely related to and implies quantum entanglement [3, 5, 8, 9, 10]. As a multipartite entanglement measure spin squeezing is relative easy to be operated and measured.

However, realistic quantum systems are not closed and therefore it is of fundamental importance to study the quantum correlations when the system loses its coherence due to the information flows from the systems to the environment [11]. The unidirectional flow of information in which the decoherence and noise act consistently characterizes a Markovian process. The dynamics of entanglement in open systems was broadly studied [12]. A peculiar aspect of the entanglement dynamics is the well known “entanglement sudden death” phenomenon [13, 14, 15] and recently it is investigated the “sudden death” of spin squeezing under a Markovian process [16, 17]. However, there are some systems such as soft- and condensed-matter systems strongly coupling to the environment and the coupling leads to a different regime where information also flows back into the system from the surroundings, which characterizes a non-Markovian process. Memory effects caused by the information flowing back to the system during a non-Markovian process can temporarily interrupt the monotonic increase or decrease of distinguishability such as spin squeezing parameter. In this paper we will study the spin squeezing dynamics of $N$ independent spin-$1/2$ particles with exchange symmetry coupled to individual non-Markovian environments.

2. Spin Squeezing Definitions

We consider an ensemble of $N$ two-level particles with lower (upper) state $|\downarrow\rangle$ ($|\uparrow\rangle$). Adopting the nomenclature of spin-$1/2$ particles, we introduce the total angular momentum (i.e., Bloch vector)

$$\mathbf{J} = \sum_{j=1}^{N} \mathbf{S}_j,$$  \hspace{1cm} (1)
where $\vec{S}_j = \frac{1}{2} \hat{\sigma}_j^z = \frac{1}{2} \left( |\uparrow\rangle_j \langle \uparrow| - |\downarrow\rangle_j \langle \downarrow| \right)$. At this point, it is convenient to introduce the following spin squeezing parameter \[5, 18\]

$$\xi^2 = \frac{N(J_{\vec{r}})^2}{\langle \vec{J} \rangle^2},$$

(2)

Here, the minimization is over all directions denoted by $\vec{n}_\perp$, perpendicular to the mean spin direction $\vec{n} = \langle \vec{J} \rangle / |\langle \vec{J} \rangle|$. If $\xi^2 < 1$ is satisfied, the spin squeezing occurs and the $N$-qubit state is entangled.

### 3. One-Axis Twisted Spin Squeezed States

Now we introduce one kind of spin squeezed states—one axis twisted spin squeezed states. Consider an ensemble of $N$ spin-$1/2$ particles with exchange symmetry and assume that its dynamical properties can be described by collective operators $J_\alpha$, $\alpha = x, y, z$. The one-axis twisting Hamiltonian \[19, 20\] is an Ising-type Hamiltonian

$$\hat{H} = \sum_{j \neq k} \frac{1}{4} f(j, k) (I - \hat{\sigma}_j^z) \otimes (I - \hat{\sigma}_k^z),$$

(3)

which involves all pairwise interactions with coupling constant $f(j, k)$.

The one-axis twisted spin squeezed state \[21, 22, 23\] is obtained by the evolution of the above Hamiltonian

$$|\psi_t\rangle = \exp \left( -i\hat{H}t \right) |+\rangle^\otimes N = \prod_{j \neq k} \exp \left[ -\frac{i}{4} f(j, k) t \hat{\sigma}_j^z \hat{\sigma}_k^z \right] |+\rangle^\otimes N,$$

(4)

where $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle) / \sqrt{2}$. If we choose the evolution time to satisfy $f(j, k) t = m\pi$ with $m$ an integer, the state $|\psi_t\rangle$ is a product state. If $f(j, k) t = (2m + 1) \pi/2$, $|\psi_t\rangle$ becomes a graph state. And for $0 < f(j, k) t < \pi/2$, $|\psi_t\rangle$ is a one-axis twisted spin squeezed state with spin squeezing parameter $\xi$.

The spin squeezing parameter of the one-axis twisted spin squeezed state with all coupling coefficients satisfying $f(j, k) t = \alpha$ takes this form

$$\xi^2 = \frac{1 - (N-1) \left[ \sqrt{A^2 + B^2} - A \right]}{\cos^{2N-2} \alpha},$$

(5)

where

$$A = 1 - \cos^{N-2} (2\alpha), B = 4 \sin \alpha \cos^{N-2} \alpha.$$  

(6)

The mean spin direction for the one-axis twisted spin squeezed state is

$$\vec{n} = (\cos (N\alpha), \sin (-N\alpha), 0),$$

(7)

and the orthogonal direction is then

$$\vec{n}_\perp = (- \cos \phi \sin (-N\alpha), \cos \phi \cos (N\alpha), \sin \phi).$$

(8)

The minimum spin squeezing parameter with respect to $\alpha$ is obtained $\xi \propto 1/N^{1/3}$ shown in Fig. 1.
4. Evolution of Spin Squeezing in The Presence of Decoherence

We consider a single qubit coupled to an environment which is described by a thermal reservoir. The evolution of this qubit is governed by a general master equation of Lindblad form

$$\frac{d}{dt}\chi = i \left[ \hat{H}_r, \chi \right] + \mathcal{L}\chi, \quad (9)$$

where the reference system is

$$\hat{H}_r = \frac{\Delta}{2} \sum_{j=1}^{N} \hat{\sigma}_z^j \quad (10)$$

with $\Delta$ is the strength of the external field proved that the optimal spin squeezed states that maximize the sensitivity of the Ramsey spectroscopy are eigensolutions of the Hamiltonian $[24]$. Whereas, the incoherent processes are described by the superoperator $\mathcal{L}$:

$$\mathcal{L}\chi = -\frac{b}{2} (1-s) \left[ \hat{\sigma}_x \hat{\sigma}_- \chi + \chi \hat{\sigma}_+ \hat{\sigma}_- - 2 \hat{\sigma}_- \chi \hat{\sigma}_+ \right]$$

$$-\frac{b}{2} s \left[ \hat{\sigma}_- \hat{\sigma}_+ \chi + \chi \hat{\sigma}_- \hat{\sigma}_+ - 2 \hat{\sigma}_+ \chi \hat{\sigma}_- \right]$$

$$-\frac{2c-b}{8} \left[ 2\chi - 2\hat{\sigma}_z \chi \hat{\sigma}_z \right], \quad (11)$$

with $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y) / 2$. For an arbitrary $s$, $b = 0$ and $c = \gamma$, the generator Eq. (11) describes the coupling between the qubit and a decoherence channel equivalent to a dephasing channel. For $s = 1/2$ and $b = c = \gamma$, the qubit is coupled to a depolarizing
Non-Markovian Dynamics of Spin Squeezing

Figure 2. The time evolution of spin squeezing under non-Markovian and Markovian dephasing. (a) The spin squeezing parameter $\xi$ for one-axis twisted spin squeezed states coupled to non-Markovian dephasing channels v.s. the time $t$ optimized with respect to coefficient $\alpha$ and fixed $N = 10$, $\gamma = 0.01$, and $\eta_0 = 1$. (b) Under the Markovian dephasing, the spin squeezing parameter $\xi$ v.s. the time $t$ optimized with respect to coefficient $\alpha$ and $\kappa(t) = e^{-0.005t}$.

Figure 3. The time evolution of spin squeezing under non-Markovian and Markovian depolarizing. (a) The spin squeezing parameter $\xi$ for one-axis twisted spin squeezed states coupled to non-Markovian depolarizing channels v.s. the time $t$ optimized with respect to coefficient $\alpha$ and fixed $N = 10$, $\gamma = 0.01$ and $\eta_0 = 1$. (b) Under the Markovian depolarizing, the spin squeezing parameter $\xi$ v.s. the time $t$ optimized with respect to coefficient $\alpha$ and $\kappa(t) = e^{-0.005t}$.

channel. Whereas, for $s = 1$ and $b = 2c = \gamma$, that is coupled to a decay channel (pure damping).

Equivalently, one can use the resulting completely positive map $\mathcal{E}$ with $\chi = \mathcal{E}\chi$ as follows:

$$\mathcal{E}\chi = \sum_{j=0}^{3} p_j \hat{\sigma}_j \chi \hat{\sigma}_j,$$

(12)

with $\chi$ a density matrix for a single-qubit state and $\sum_{j=0}^{3} p_j = 1$. These noise channels are of particles interest in quantum information theory. This class contains for example: (i) for $p_0 = (1 + 3\kappa^2) / 4$ and $p_1 = p_2 = p_3 = (1 - \kappa^2) / 4$ with $\kappa = e^{-\gamma t}$ the above depolarizing channel; (ii) for $p_0 = (1 + \kappa^2) / 2$, $p_1 = p_2 = 0$ and $p_3 = (1 - \kappa^2) / 2$ the above dephasing channel. Finally, the decay channel is obtained as

$$\mathcal{E}\chi = E_0 \chi E_0^\dagger + E_1 \chi E_2^\dagger,$$

(13)
with the Kraus operators $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}$ and $E_1 = \begin{pmatrix} 0 & \sqrt{1 - \kappa^2} \\ 0 & 0 \end{pmatrix}$.

The quantum master equations with the time-local structures are also very useful for the description of non-Markovian processes. Suppose we have a time-local master equation of the form

$$\frac{d}{dt} \chi = i \left[ \hat{H}_r, \chi \right] + K(t) \chi, \tag{14}$$

where $K(t)$ is a time-dependent generator and takes the similar form in Eq. (11) with time-dependent parameter $\gamma(t)$.

If the relaxation rate $\gamma(t)$ is positive function the generator $K(t)$ is in the Lindblad form for each fixed $t \geq 0$. In the Markovian regime, information of the qubit state leaks to its surroundings and $|\kappa(t)|$ is a monotonically decreasing function of times. In the non-Markovian environment, in contrast, information also flows back into the system of qubit state and a revival of distinguishability (here the spin squeezing parameter) can be observed in the time evolution. With the definition of non-Markovianity [26], we see that an increase of $|\kappa(t)|$ leads to a negative rate $\gamma(t)$ in the generator $K(t)$. As an example, we consider the case of a Lorentzian reservoir spectral density which is on the resonance with the spin qubit transition frequency and leads to an exponential two point correlation function

$$f(t) = \frac{1}{2} \eta_0 \gamma e^{-\gamma t}. \tag{15}$$

The rate $\kappa(t)$ is defined as the solution of the integrodifferential equation

$$\frac{d}{dt} \kappa(t) = - \int_0^t dt' f(t - t') \kappa(t') \tag{16}$$

corresponding to an initial condition $\kappa(0) = 1$. The parameter $\gamma$, defining the spectral width of the coupling between the qubit and reservoir, is connected to the reservoir correlation time $\tau \approx \gamma^{-1}$. For a weak coupling regime $\eta_0 < \gamma/2$, the relaxation...
Non-Markovian Dynamics of Spin Squeezing

7

time is greater than the reservoir correlation time and the behavior of \( \kappa(t) \) is a Markovian exponential decay. In the strong coupling regime \( \eta_0 > \gamma/2 \), the reservoir correlation time is greater than the relaxation time and non-Markovian effects become relevant \[12, 26, 27, 28, 29, 30, 31, 32, 33, 34\]. Thus we obtain

\[
\kappa(t) = e^{\frac{d}{2}} \left[ \cos \left( \frac{dt}{2} \right) + \frac{\gamma}{d} \sin \left( \frac{dt}{2} \right) \right],
\]

(17)

where \( d = \sqrt{2\eta_0 \gamma - \gamma^2} \).

We would be interested in the effect of decoherence on the spin squeezing properties of a system including \( N \) two-level particles. A decoherence model individual coupling of each of the qubits to a thermal bath is considered in this paper, where the evolution of the \( k \)th qubit is described by the map \( \mathcal{E}_k \) with Pauli operators \( \hat{\sigma}_j \) \((j = 0, 1, 2, 3)\) acting on qubit \( k \). We are interested in the dynamical evolution of a given one-axis twisted spin squeezed state \( \psi \) of \( N \) qubit in the presence of decoherence. The initial state \( \psi \) evolves in time to a mixed state \( \rho(t) \) given by

\[
\rho(t) = \mathcal{E}_1 \mathcal{E}_2 \ldots \mathcal{E}_N |\psi_t\rangle \langle \psi_t|.
\]

(18)

For one-axis twisted spin squeezed states, we consider the three kinds of decoherent channels. The modified mean spin direction and the orthogonal direction are calculated as

\[
\vec{n}' = (\cos (\Delta t - N\alpha) , \sin (\Delta t - N\alpha) , 0),
\]

\[
\vec{n}'_\perp = (-\cos \phi \sin (\Delta t - N\alpha), \cos \phi \cos (\Delta t - N\alpha) , \sin \phi).
\]

(19)

Coupling to the individual dephasing channel which is the main type of decoherence for a spin ensemble, the spin squeezing parameter of the one-axis twisted spin squeezed state evolves to

\[
\xi^2_{\text{deph}}(t) = \frac{\zeta}{\cos^{2N-2} \alpha},
\]

(20)

where

\[
\zeta = 1 + \frac{1}{4} \kappa^2(t) (N - 1) \left( A - \frac{A^2}{\sqrt{A^2 + B^2}} \right) - \frac{1}{4} \kappa(t) (N - 1) \frac{B^2}{\sqrt{A^2 + B^2}}.
\]

(21)

Then the spin squeezing parameter of the one-axis twisted spin squeezed state evolves to

\[
\xi^2_{\text{depol}}(t) = \frac{\zeta}{\kappa^2(t) \cos^{2N-2} \alpha}.
\]

(22)

under the individual depolarizing. Whereas, the spin squeezing parameter of the one-axis twisted spin squeezed state evolves to

\[
\xi^2_{\text{damp}}(t) = \frac{\zeta}{\left\{ \kappa(t) \cos^{N-1} \alpha - [1 - \kappa(t)] \right\}^2}.
\]

(23)

in the individual damping channel.

Figs. 2-4 show the time evolution of spin squeezing of 10-particle prepared initially in one-axis twisted spin squeezed state coupled to individual dephasing, depolarizing
and damping channels. We compare the evolution of spin squeezing under Markovian and non-Markovian decoherence channels. Here we consider a Lorentzian reservoir and thus the decohherence function $\kappa(t)$ can be written as exponential decay $e^{-\gamma t/2}$ modified by a periodical time-dependent function $\cos(dt/2) + \gamma/d \sin(dt/2)$. In short time regime, the spin squeezing property collapses and revives under either of three kinds of non-Markovian decoherence. With time increasing, the part of the exponential decay becomes more important and the spin squeezing property vanishes finally as that under the Markovian decoherence does.

5. Conclusion

In summary, we study the dynamics processes of the spin squeezing in a spin ensemble in which each spin is coupled to independent identical decoherence channel. We analytically calculate the dynamics of the spin squeezing parameters under three different types of decoherence. As we know the Heisenberg scaling $1/N$ in the decoherence-free case can be achieved. In the presence of Markovian decoherence the spin squeezing property of one-axis twisted states vanishes with the evolution time. Whereas, in the presence of non-Markovian decoherence and in the short time limit, the spin squeezing property collapses and revives with the evolution time due to short-time memory effect during non-Markovian processing. With time increasing, the spin squeezing vanishes finally even under the non-Markovian decoherence. As spin squeezing can be regarded as a measure/witness of multiqubit entanglement, thus our scheme for the first time shows the collapse and revival of multiqubit entanglement under non-Markovian decoherence.

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Non-Markovian Dynamics of Spin Squeezing

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