A Combinatorial formula for the nabla operator

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Abstract

We present an LLT-type formula for a general power of the nabla operator of [BG99] applied to the Cauchy product for the modified Macdonald polynomials, and use it to deduce a new proof of the generalized shuffle theorem describing $\nabla^k e_n$ [HHL+05a, CM18, Mel16], and the formula for $(\nabla^k p^a_1, e_n)$ from [EH16, GH17] as corollaries. We give a direct proof of the theorem by verifying that the LLT expansion satisfies the defining properties of $\nabla^k$, such as triangularity in the dominance order, as well as a geometric proof based on a method for counting bundles on $\mathbb{P}^1$ due to the second author [Mel17]. These formulas are related to an affine paving of the type A unramified affine Springer fiber studied by Goresky, Kottwitz, and MacPherson in [GKM04], and also to Stanley’s chromatic symmetric functions.

1 Introduction

There is a well-studied connection between the combinatorics of the nabla operator of [BG99, BGHT99], and the homology or cohomology of the affine Springer fibers $X_\gamma$ of the sort studied in [GKM03], see for instance [LS91, GKM04, GM13, GORS14, GMV14, Hik14, OY14, CO18, Kiv20]. In this picture, objects such as parking functions $(\pi, w)$ are seen to be in bijection with cells in an affine paving of $X_\gamma$, and combinatorial statistics such as $\text{dinv}(\pi, w)$ that appear in the shuffle theorem [Hag08, HHL+05a, CM18] and other nabla-type formulas are essentially the dimensions of the corresponding cells. In this way, combinatorial formulas may be interpreted as graded characters of the homology of some $X_\gamma$, with the $q$-degree representing half the homological degree, the $t$-degree being more subtle.
For example, consider the following power series

\[ H_{m,n}(q,t) = \frac{1}{(1-q)^{\gcd(n,m)}} \sum_{w \in W_n^+} t^{\text{area}(w)} q^{\text{dinv}_m(w)}. \]  

(1)

Here \( W_n^+ \) is a set of extended affine permutations in which \( w_i \geq 1 \) for \( 1 \leq i \leq n \), \( v \) is \( m \)-stable if \( v_{i+m} > v_i \) for all \( i \), and area and \( \text{dinv}_m \) are defined in Section 3.1. A slightly different version of this series was presented in [GMV14] in the case when \( n, m \) are relatively prime, which the authors showed describes the combinatorics of the rational version of the shuffle theorem [BGSLX16, Mel16]. The corresponding Springer fiber in this case is \( X_{n,m} \subset \mathcal{F}_n \), which is the one associated to the nil-elliptic operator \( \gamma = N^m \), where \( N(e_i) = e_{i+1} \) for \( i < n \), \( N(e_n) = te_1 \).

There is an extension of \( X_{n,m} \) for \( n, m \) not relatively prime, which in the case of \( m = kn \), becomes the unramified affine Springer fiber studied in [GKM04]. In this case, the equivariant homology \( H^*_T(X_{n,\mathbf{kn}}) \) for a standard torus action \( T \acts X_{n,\mathbf{kn}} \) is equipped with two commuting actions of the symmetric group, generally known as the “dot” and “star” actions which act on the left and right respectively, due to Knutson and Tymoczko [Knu03, Tym08]. The dot action comes from a space-level action on the affine flag variety, which permutes different fibers \( X_{n,\mathbf{kn}} \), whereas the right action comes from the Springer action. In this paper, we present an LLT-type expansion

\[ \Omega_k[X,Y] = \sum_{\mathbf{m} \in \mathbb{Z}^n} t^{[\mathbf{m}]} q^{\text{dinv}_k(\mathbf{m},a,b)} X_a Y_b, \]

for \( k \geq 1 \), where the quantities in the summand are defined in Section 2.2. We predict that \( \Omega_k[X,Y] \) corresponds to a Frobenius character extension of (1), namely \( \Omega_k[X,Y] = \mathcal{F}_{n,\mathbf{kn}}[X,Y; q,t] \), where the \( Y \)-variables represent the dot action, and the \( X \)-variables correspond to star. In particular, the coefficient of the monomial with all exponents equal to one in \( \Omega_k[X,Y] \) is shown to agree with \( H_{n,\mathbf{kn}}(q,t) \).

Our main theorem is that \( \Omega_k[X,Y] \) is computed by powers of the \( \nabla \)-operator applied to the Cauchy product for the modified Macdonald polynomials, shown in plethystic notation:

**Theorem A.** For \( k \geq 1 \), we have

\[ \nabla^k e_n \left[ \frac{XY}{(1-q)(1-t)} \right] = \Omega_k[X,Y]. \]

Notice that unlike most combinatorial formulas involving the nabla operator formulas, the one in Theorem A completely determines \( \nabla^k \), and could therefore be taken
as a definition. This is the key point to our first proof, which is done by verifying that \( \Omega_k[X, Y] \) satisfies the defining properties of \( \nabla^k \), similar to the approach taken in [HHL05b]. We give a second proof based on a method developed by the second author for counting bundles on \( \mathbb{P}^1 \) over a finite field, which we hope will lead to further connections with geometry and number theory.

We also deduce as corollaries some well-known formulas involving the \( \nabla \)-operator, namely the generalization of the shuffle theorem [HHL'05a, CO18] for arbitrary powers \( \nabla^k e_n \), and the Elias-Hogancamp expression for \( \nabla^k p^n_1 \) [EH16, GH17]. Our primary motivation for studying this formulas is part of an ongoing study of Tor groups of certainly polygraph-type modules, in connection with the nabla positivity conjecture of Bergeron, Garsia, Haiman, and Tesler [BGHT99], which predicts that the coefficients of the Schur expansion of \( \nabla^k s_\lambda \) are polynomials in \( c_{\lambda, \mu}(q, t) \) whose coefficients are entirely positive or entirely negative, which we will address in future papers.

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2 Premilinary definitions and notations

In this section we give general background on plethysm, affine permutations, and the combinatorial constructions that appear in our main theorem.

2.1 Macdonald polynomials

Given a symmetric function \( f \), we will adopt the usual plethysm notation of \( f[X] \) when \( X \) is an element of some \( \lambda \)-ring, so that \( f[x_1 + \cdots + x_N] \) is the substitution \( f(x_1, \ldots, x_N) \). If \( X = (x_1, x_2, \ldots) \) is some alphabet, we will use the same letter \( X \) to denote the sum in plethystic formulas. For details, we refer the reader to [Hai01b].

Let \( \tilde{H}_\lambda(X; q, t) \) denote the modified Macdonald polynomial [BGHT99], defined by

\[
\tilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda[X/(1 - t^{-1}); q, t^{-1}].
\]

Let \( \nabla \) be the Garsia-Haiman-Bergeron-Tesler operator

\[
\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda) t^{n(\lambda)}} \tilde{H}_\lambda(X; q, t),
\]

where

\[
n(\lambda) = \sum_i (i - 1) \lambda_i
\]
is the usual statistic from Macdonald’s book \cite{Mac95}. In this paper, $\nabla$ will always denote an operator applied to the $X$ variables.

## 2.2 Combinatorial definitions

Fix $n$ and define a label to be an $n$-tuple of positive integers $a = (a_1, ..., a_n)$ with $a_i \geq 1$. We will write $\text{labs}(n)$ for the set of all labels of length $n$, and will also call the individual $a_i$ labels. For any label $a$, we have a multiset $A = A(a) = (|A|, m_A)$ where $|A| = \{a_1, ..., a_n\}$ is the total set, and $m_A : |A| \rightarrow \mathbb{Z}_{\geq 1}$ is the multiplicity. We define a (strict) composition of $n$

$$\alpha(a) = (\alpha_1, ..., \alpha_l), \quad |A| = \{c_1 < \cdots < c_l\}, \quad \alpha_i = m_A(c_i).$$

In other words, $\alpha(a)$ is the result of sorting $a$ in increasing order, and reading of the sizes of the groups, for instance

$$\alpha((1, 1, 1, 4, 2, 1, 4)) = (4, 1, 3).$$

We may also define the corresponding partition $\mu(a) = \mu(\alpha(a))$ which is the result of sorting $\alpha(a)$ in decreasing order, so $\mu(a) = (4, 3, 1)$ in the above example. Given a multiset $A$, let $\text{labs}(A)$ denote the set of labels $a$ with $A(a) = A$, with similar definitions for $\text{labs}(\alpha)$ and $\text{labs}(\mu)$.

If $A, B, ...$ are totally ordered sets, we define the ordering on $A \times B \times \cdots$ as the corresponding lexicographic order, breaking ties from left to right. If $a \in A^n$, $b \in B^n$, ... are some elements, we define $[a, b, ...]$ to be the sorted representative of the simultaneous action of $S_n$ on all components. In other words, view $(a, b, ...)$ as a matrix, transpose the matrix, sort according to the order on $A \times B \times \cdots$, and transpose back. For instance, in the case $a \in A^n$, $b \in B^n$ for $A = B = \mathbb{Z}_{\geq 1}$, we have

$$[(1, 2, 1, 1, 2, 1), (3, 2, 3, 1, 1, 3)] = ((1, 1, 1, 1, 2, 2), (1, 3, 3, 3, 1, 2)).$$

We can then define $\alpha(a, b, ...)$ using the same rules as above, so in the above example $\alpha(a, b) = (1, 3, 1, 1)$. We make a similar definition for $\mu$, which also applies when the sets are unordered.

## 2.3 The dinv statistic

Let $a, b$ be labels, let $m \in \mathbb{Z}_{\geq 0}^n$ with the decreasing order on the $m_i$, so that a triple $[m, a, b]$ means one sorted as in the following way.
Definition 2.1. Let \( m \in \mathbb{Z}_{\geq 0}^n \) and let \( a, b \) be labels. We will say that \( (m, a, b) \) is sorted if for every \( i < j \) we have

1. \( m_i \geq m_j \), and
2. if \( m_i = m_j \) then \( a_i \leq a_j \), and
3. if \( m_i = m_j \) and \( a_i = a_j \) then \( b_i \leq b_j \).

For instance,

\[
[(1, 0, 1, 0), (2, 1, 1, 1), (1, 2, 2, 1)] = ((1, 1, 0, 0), (1, 2, 1, 1), (2, 1, 1, 2)).
\]

We will often write such lists as arrays, as in Example 2.5 below.

We now define a statistic \( \mathrm{dinv}_k(m, a, b) \) on triples which are sorted according to Definition 2.1.

Definition 2.2. Let \( m \in \mathbb{Z}_{\geq 0}^n \), let \( a, b \in \text{labs}(n) \) be labels, and suppose that \( (m, a) \) are sorted. We define

\[
\mathrm{dinv}_k(m, a, b) = \sum_{i < j} \mathrm{dinv}^{i,j}_k(m, a, b)
\]

where

\[
\mathrm{dinv}^{i,j}_k(m, a, b) = \max (m_j - m_i - 1 + k + \delta(a_i > a_j) + \delta(b_i > b_j), 0),
\]

and \( \delta(a_1 > a_2) \) is one if \( a_1 > a_2 \), zero otherwise.

We similarly define \( \mathrm{dinv}_k(m, a) \) as the result of removing \( \delta(b_i > b_j) \), which is the same as setting \( b = (1^n) \) by default.

Recall that a Dyck path is a path of North and East steps in the \( n \times n \) grid beginning at the origin \((0, 0)\), placed in the South-West, or lower left corner, and ending at \((n, n)\), which never goes below the diagonal. It is determined uniquely by the set

\[
D(\pi) = \{(i, j) : 1 \leq i < j \leq n \text{ is between the path and the diagonal}\}.
\]

Definition 2.3. Fix \( k \geq 0 \), suppose \( (m, a) \) is sorted, and let \( i < j \). We will say that \( i \) \( k \)-attacks \( j \) (or just attacks) if

\[
m_j - m_i - 1 + k + \delta(a_i > a_j) \geq 0.
\]
In other words, \( i \) \( k \)-attacks \( j \) if switching the order of \( b_i, b_j \) has an effect on \( \text{dinv}_k \).

For instance, for \( k = 1 \) we have that \( i \) attacks \( j \) if

1. \( m_i = m_j + 1 \) and \( a_i > a_j \), or
2. \( m_i = m_j \).

Note that in the second condition, we necessarily have \( a_i \leq a_j \).

**Definition 2.4.** Let \( \pi = \pi_k(m, a) \) denote the Dyck path such that the elements of \( D(\pi) \), are the pairs \( i < j \) for which \( i \) \( k \)-attacks \( j \).

We now have that

\[
dinv_k(m, a, b) = \text{dinv}_k(m, a) + \text{inv}_{\pi_k(m, a)}(b)
\]

(4)

where

\[
\text{inv}_\pi(b) = \#\{(i, j) \in D(\pi) : b_i > b_j\}.
\]

(5)

**Example 2.5.** Let \( m, a \) be given in array notation by

\[
\begin{pmatrix}
m & 3 & 3 & 3 & 2 & 0 & 0 \\
a & 1 & 1 & 5 & 4 & 2 & 5
\end{pmatrix}
\]

which is a sorted term for \( n = 6 \). Then we find that \( \pi_2(m, a) \) is the Dyck path given in Figure [ ] as the attacking pairs are the elements of \( D(\pi) \) listed in the caption.

### 2.4 Examples

A sum over all \( a \) will mean the infinite sum over all labels, unless some upper bound is specified, \( a_i \leq N \). We will adopt a convenient convention that a sum over \( [a, b, ...] \) means a sum over cosets, with the assumption that \( (a, b, ...) \) is the sorted representative in the summand. We will also allow for some summands in which only some of the summands are grouped, which means that just those terms are sorted. For instance, the symbol

\[
\sum_{[a, b, c, [d]]} \ldots
\]

indicates the sum over quadruples \( (a, b, c, d) \) so that for every \( i < j \) we have \( a_i \leq a_j \), \( b_i \leq b_j \) if \( a_i = a_j \), \( d_i \leq d_j \), and there are no constraints on \( c \). We also define automorphism factors for the cosets

\[
\text{aut}(a, b, ...) = \prod_i \mu_i!, \quad \text{aut}_q(a, b, ...) = \prod_i [\mu_i]_q!,
\]
Figure 1: A Dyck path of size $(6, 6)$ with area sequence $a(\pi) = (0, 1, 2, 3, 1, 1)$, and $D(\pi) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (4, 5), (5, 6)\}$.

where $\mu = \mu(a, b, ...)$, and

$$[k]_q = 1 + q + \cdots + q^{k-1}, \quad [k]_q^! = \prod_{j=1}^{k} [j]_q,$$

are the $q$-number and $q$-factorial.

We give some examples in symmetric functions. Let

$$X_a = x_{a_1} \cdots x_{a_n} = \prod_{a \in A} x_{a}^{m(a)}$$

be the associated monomial to $a$, where $(A, m)$ is the associated multiset.

**Example 2.6.** The complete and monomial symmetric functions are given by

$$h_n(x_1, x_2, ...) = \sum_{a} \frac{1}{\text{aut}(a)} X_a = \sum_{[a]} X_a = \sum_{\mu} m_\mu(x_1, x_2, ...),$$

$$m_\mu(x_1, x_2, ...) = \frac{1}{\mu_1! \cdots \mu_l!} \sum_{\mu(a) = \mu} X_a.$$  

We also have the quasi-symmetric monomials defined by

$$m_\mu = \sum_{\mu(a) = \mu} M_\alpha, \quad M_\alpha(x_1, x_2, ...) = \frac{1}{\alpha_1! \cdots \alpha_l!} \sum_{\alpha(a) = \alpha} X_a.$$
Example 2.7. We have
\[
e_n \left[ \frac{X}{1 - q} \right] = \sum_{[a]} \frac{q^{n(\mu(a)')}}{(1 - q)^n \text{aut}_q(a)} X_a,
\]
which follows from using the Cauchy product, and the well-known specializations for \( h_\mu(1, q, \ldots) \). Replacing \( e_n \) with \( h_n \) simply removes the \( q^{n(\mu(a)')} \) factor.

Example 2.8. We have the Cauchy product for the modified Macdonald polynomials from Section 2.1:
\[
e_n \left[ \frac{XY}{(1 - q)(1 - t)} \right] = \sum_{[m,a,b]} \frac{t^{|m|} q^{n(\mu(a)')}}{(1 - q)^n \text{aut}_q(m, a, b)} X_a Y_b.
\]
(6)
The order we chose for \( m \) does not affect the answer here, but it does make a difference in Theorem (1.4) which is about powers of the nabla operator applied to this expression. Again, replacing \( e_n \) with \( h_n \) simply removes the \( q^{n(\mu(a)')} \).

3 Motivation

We explain the underlying motivation behind Theorem A, which was discovered experimentally using conjectural relations between Haiman’s polygraph rings and the homology of the unramified affine Springer fiber \( H^T_*(X_{n,kn}) \). We connect the combinatorics of Section 2.2 to cells in \( X_{n,kn} \).

3.1 Affine permutations

We describe the connection between affine permutations and the combinatorics of the dinv statistic and rational slope parking functions, following [GMV14].

Define the set of positive affine permutations as
\[
W_n^+ = \{ w : \mathbb{Z} \rightarrow \mathbb{Z} : w(i + n) = w(i) + n, w_i \geq 1 \text{ for } i \geq 1 \}.
\]
Each one is determined by its values in window notation, \( w = (w_1, \ldots, w_n) \). Notice that these all have positive numbers as entries, and are not normalized in the usual way so that the sum of the entries is \( n(n + 1)/2 \), which is a condition for affine \( SL_n \) as opposed to \( GL_n \). Let
\[
W_{n,d}^+ = \{ w \in W_n^+ : w_1 + \cdots + w_n = dn + n(n + 1)/2 \}.
\]
We may still multiply any two such permutations, which results in adding the values of $d$. For each $d$ we have the Bruhat order $\leq_{bru}$ on $W_{n,d}^+$.

Following [GMV14], we have

**Definition 3.1.** An affine permutation is called $m$-stable if $w_{i+m} > w_i$ for all $i$, and is called $m$-restricted if $w^{-1}$ is $m$-stable.

These are the fixed points of the affine Springer fiber $X_{n,m} = X_\gamma$ of the type studied in [GKM03], in which $\gamma$ is the topologically nilpotent operator

$$
\gamma(e_i) = a_i e_{i+m}.
$$

Here $e_i \in \mathbb{C}^n((t))$ is the standard basis vector in $\mathbb{C}^n$ for $1 \leq i \leq n$, and otherwise $e_{i+n} = te_i$, and the $a_i$ are distinct nonzero complex numbers for for $1 \leq i \leq d$, and $a_{i+d} = a_i$ for $d = \gcd(n,m)$. For $m = kn$, we have $\gamma = \text{diag}(a_1 t^k, \ldots, a_n t^k)$, corresponding to the unramified case studied in [GKM03]. In this case the $n$-dimensional torus $T \subset GL_n(\mathbb{C})$ acts by multiplication on the left via, as well as the extended $(n+1)$-dimensional torus $\widehat{T}$, which includes loop rotation, both having discrete fixed points described by Definition 3.1.

For integers $a, b \in \mathbb{Z}$ which are not congruent modulo $n$, we have an affine transposition $t_{a,b}$ which switches the two. Given an $m$-restricted permutation, let

$$
\text{edges}_m(w) = \{ t_{a,b} : t_{a,b} w \leq_{bru} w, |a - b| < m \}.
$$

(7)

The statistic $|a - b|$ does not depend on the representatives $a, b$ or their order, and is called the height of the transposition. The set $\text{edges}_m(w)$ represent directed edges $w \to v$ with $v = t_{a,b} w$ in the GKM graph of $X_{n,m}$, corresponding to the one-dimensional orbits under $\widehat{T}$.

Recall that an $(n, m)$-rational slope Dyck path is one that begins at $(0, 0)$ and ends at $(m, n)$, never crossing the line of slope $n/m$. Again, we have the area and coarea sequences $\text{area}(\pi), \text{coarea}(\pi)$, and also $D(\pi)$. For any $m$-restricted permutation $w$, there is a rational $(n, m)$-Dyck path with coarea sequence

$$
b(\tilde{\pi}_m(w)) = \text{sort}(\mathbf{w}_m(w), <),
$$

where

$$
\mathbf{w}_m(w)_j = \# \{ t_{a,b} \in \text{edges}_m(w) : w^{-1} t_{a,b} w = t_{i,j} \text{ for some } i < j \}.
$$

This is the underlying Dyck path of sequence $\mathbf{w}_m(w) = \mathcal{PS}_{w^{-1}}$ of [GMV14], which is shown to define a bijection from the set of $m$-stable affine permutations in $W_n$ to rational parking functions for $(n, m)$ coprime.

We define
Definition 3.2. Let
\[ \text{dinv}_m(w) = \text{area}(\pi_{n,m}) - \# \text{edges}_m(w) = \text{area}(\tilde{\pi}_m(w)), \]
where \( \pi_{n,m} = (1^n0^m) \) is the \((n, m)\)-Dyck path of maximal area.

We now explain the connection with the statistic \( \text{dinv}_k(m, a, b) \) of Section 2.2. Recall the definition of Standardization from [HHL05b]:

Definition 3.3. The standardization of a label is the unique permutation \( \sigma = \text{Std}(a) \) such that \( a_{\sigma}^{-1} \) is weakly increasing, and the restriction of \( \sigma \) to \( a^{-1}(\{x\}) \) is increasing if \( x \) is positive, decreasing if \( x \) is negative.

We will also define \( \text{Std}_< (a) \) and \( \text{Std}_> (a) \) with respect to the usual, and reverse order on \( \mathbb{Z}_{\geq 1} \), so that \( \text{Std} = \text{Std}_< \). For instance, if \( a = (3, 3, 3, 1, 2, 3, 1) \), then \( \text{Std}_< (a) = (4, 5, 6, 1, 3, 7, 2) \), \( \text{Std}_> (a) = (1, 2, 3, 6, 5, 4, 7) \).

In particular, \( \text{dinv}_k \) respects standardization, i.e. \( \text{dinv}_k(m, \text{Std}(a), \text{Std}(b)) = \text{dinv}_k(m, a, b) \).

Now given a tuple \((m, a, b)\) which is sorted, we define an affine permutation
\[ \text{aff}(m, a, b) = \text{Std}_>(\text{rev}(b))t(m)\text{Std}_<(a)^{-1}, \] (8)
where \( t(m) = (n + m_1n, ..., 1 + m_nn) \) is the maximal representative of its coset in \( S_n \backslash W_\alpha^+ / S_n \), and \( \text{rev}(b) \) is the result of writing \( b \) in the reverse order. We similarly define \( \text{aff}(m, a) \) as the left coset \( S_n \text{ aff}(m, a, b) \), which is independent of \( b \). The proof of the following proposition is tedious, and will be omitted.

Proposition 3.4. Fix multisets \( A, B \) of size \( n \) with \( |A|, |B| \in \mathbb{Z}_{\geq 1} \). Then \( \text{aff}(m, a, b) \) defines a bijection from the set of sorted triples
\[ (m, a, b) \in \mathbb{Z}_{\geq 0}^n \times \text{labs}(A) \times \text{labs}(B) \]
to the double coset \( S_{\text{rev}(\alpha(B))}W_\alpha^+ S_{\alpha(A)} \), where \( S_0 \) is the Young subgroup, and \( \text{aff}(m, a, b) \) is the unique representative of its double coset of maximal length. Moreover, we have
\[ \text{dinv}_k(m, a, b) = \text{dinv}_k(\text{aff}(m, a, b)), \]
and the Dyck path is determined by
\[ a(\tilde{\pi}_k(m, a)) = a(\tilde{\pi}_k(w_{\min})) - a(\tilde{\pi}_k(w_{\max})), \]
where \( w_{\min}, w_{\max} \) are the unique representatives of the coset \( \text{aff}(m, a) \) which are minimal and maximal in the Bruhat order.
Example 3.5. Take \((m, a, b) = ((2, 1, 0, 0), (2, 3, 1, 1), (1, 2, 1, 1))\), which is sorted. Then we have
\[ t(m) = (12, 7, 2, 1), \quad \text{Std}_<(a) = (3, 4, 1, 2), \quad \text{Std}_>(\text{rev}(b)) = (2, 3, 1, 4), \]
which gives \(w = \text{aff}(m, a, b) = (3, 2, 12, 5)\). This is the maximal length element in the double coset
\[ S_{(1,3)}wS_{(2,1,1)} = \{(2, 3, 12, 5), (2, 4, 11, 5), (3, 2, 12, 5), (3, 4, 10, 5), (4, 2, 11, 5), (4, 3, 10, 5)\}. \]

Now for \(k = 1\), we have
\[ \text{edges}_4(w) = \{(2, 3, 12, 5), (3, 2, 9, 8), (4, 2, 11, 5), (3, 4, 10, 5)\} \]
so that \(\text{dinv}_4(w) = 6 - 4 = 2\). On the other hand, \((m, a)\) has three attacking pairs, \{\((2, 3), (2, 4), (3, 4)\)\}. Since \((2, 3)\) and \((2, 4)\) are the pairs for which \(b_i > b_j\), we see that \(\text{dinv}_1(m, a, b) = 2\), in agreement with Proposition 3.4.

Example 3.6. Let us compute the Dyck path for the terms \(m, a\) from Example 2.5 using Proposition 3.4. Then we have that
\[ w_{\text{min}} = (19, 20, 5, 16, 21, 6), \quad w_{\text{max}} = (24, 23, 2, 15, 22, 1) \]
are the minimal and maximal representatives of the left coset of \(\text{aff}(m, a) \in S_n \backslash W_n^+\). Then
\[ a(w_{12}(w_{\text{min}})) = (0, 2, 4, 4, 1, 2), \quad a(w_{12}(w_{\text{max}})) = (0, 1, 2, 1, 0, 1), \]
and the unique \((n, n)\)-Dyck path whose area sequence is the difference \((0, 1, 2, 3, 1, 1)\) is the expected one from Figure 11.

### 3.2 Polygraphs and the Hilbert scheme

If \(M\) is a representation of \(S_n \times \cdots \times S_n\) with \(k\) factors, we will denote the Frobenius character by
\[ \mathcal{F}_{X_1, \ldots, X_k} M \in \mathbb{C}[x_{i,j}]^{S_n \times \cdots \times S_n}, \]
which is a function in \(k\) sets of variables, \(X_i = (x_{i,1}, x_{i,2}, \ldots)\), individually symmetric in each one. For doubly graded modules, the Frobenius character encodes the degrees with the \(q, t\) variables, namely
\[ \mathcal{F}M = \sum_{i,j} q^i t^j \mathcal{F}M^{(i,j)} \]
where $M^{(i,j)}$ is the homogeneous component of the bigrading.

In Haiman’s theory \cite{Hai01b}, the sum $\Omega_k[X,Y]$ is the equivariant index of a sheaf on the Hilbert scheme of points in the complex plane $\text{Hilb}_n \mathbb{C}^2$, with respect to the usual torus action $T = \{(q^{-1}, t^{-1})\} \circ \text{Hilb}_n \mathbb{C}^2$. Let $P$ be the Procesi bundle of rank $n!$ whose fibers carry an action of $S_n$ isomorphic to the regular representation. The modified Macdonald polynomial is the Frobenius character $\tilde{H}_\lambda = F^P_{\lambda}$ of the fibers of torus-fixed points, which are isomorphic to the Garsia-Haiman module. Then we have

\[
\nabla^k e_n \left[ \frac{XY}{(1-q)(1-t)} \right] = \sum_i (-1)^i \mathcal{F}_{Y,X} R^i \Gamma \left( P \otimes P^* \otimes \mathcal{L}^k \right). \tag{9}
\]

Now recall Haiman’s polygraph modules \cite{Hai01a}: Fix $n$ and let $x$ denote the set of variables $(x_1, ..., x_n)$, for some variable $x$. Let $\mathbb{C}[x,y] \cdot S_n$ denote the free left $\mathbb{C}[x,y]$-module with one free generator for each permutation $\tau \in S_n$. Consider the following variant of Haiman’s map from \cite{Hai01a} equation (152):

\[
\phi : \mathbb{C}[x,y,z,w] \to \mathbb{C}[x,y] \cdot S_n, \quad g(x,y,z,w) \mapsto \sum_{\tau \in S_n} g(x,y,\tau(x),\tau(y))\tau. \tag{10}
\]

We define a module $M$ as the image of $\phi$, as a $\mathbb{C}[x,y]$-module. We have the usual bigrading on $M$ compatible with the grading on the ring $\mathbb{C}[x,y,z,w]$, in which the degree of the $x, z$ variables are $(1, 0)$, and the $y, w$ variables have degree $(0, 1)$. Note that $x, y$ have nothing to do with the symmetric function variables $X, Y$.

There is an action of $S_n \times S_n$ on $M$, which may also be interpreted as a commuting left and right action by

\[
(\sigma_1, \sigma_2) \cdot f(x,y)\tau = \sigma_1 \cdot f(x,y)\tau \cdot \sigma_2^{-1} = f(\sigma_1(x), \sigma_1(y)) \cdot (\sigma_1 \tau \sigma_2^{-1}).
\]

Then $\phi$ intertwines this action with the one where the first factor simultaneously permutes $x, y$, and the second factor permutes $z, w$. Notice that the left $S_n$-action on $M$ is compatible with the action by permuting the variables, whereas the right $S_n$-action does not act on the variables. Another way to say this is that $M$ is a bigraded module over the smash product $\mathbb{C}[x,y] \rtimes S_n$, which is the noncommutative ring by adjoining a generator for each $\sigma \in S_n$ with the relation

\[
\sigma x_i = x_{\sigma_i}, \quad \sigma y_i = y_{\sigma_i},
\]

and that the right action of $S_n$ acts by automorphisms of $M$.

**Conjecture 3.7.** As a module over $\mathbb{C}[x,y] \rtimes S_n$, $M$ is the image of the Procesi bundle $P$ under the Haiman-Bridgeland-King-Reid isomorphism

\[
F \mapsto R\Gamma_{\text{Hilb}_n}(P \otimes F).
\]
The higher derived functors $R^i\Gamma(P \otimes P)$ vanish, and so $M \cong \Gamma_{\text{Hilb}}(P \otimes P)$. Moreover, we have that $M$ is free when regarded as a module over $\mathbb{C}[x]$, in other words forgetting the $\mathbb{C}[y]$-action.

**Remark 3.8.** Haiman identified the module in the case $F = B^{\otimes l}$ as the polygraph module $R(n, l)$ defined in [Hai01b], where $B$ is the tautological bundle, and he proved the second two statements for $R(n, l)$. To the best of our knowledge Conjecture 3.7 is not known. We point out that the vanishing statement is definitely false for three powers of the Procesi bundle $P^{\otimes 3}$, which may be seen by observing the Atiyah-Bott localization actually has negative terms.

**Remark 3.9.** The conjecture is motivated by the following geometric picture. Recall the commutative diagram ([Hai01a])

$$
\begin{array}{ccc}
X_n & \rightarrow & \mathbb{C}^{2n} \\
\downarrow \pi & & \downarrow \\
\text{Hilb}_n \mathbb{C}^2 & \rightarrow & \mathbb{C}^{2n}/S_n
\end{array}
$$

The diagram is a reduced cartesian product and the space $X_n$ is the *isospectral Hilbert scheme*. The map $\pi$ is finite and $P = \pi_*O_{X_n}$. Thus the ring $\Gamma_{\text{Hilb}}(P \otimes P)$ is the ring of functions on $X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n$, which is a closed subscheme of $X_n \times X_n$. On the other hand, $X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n$ is reduced, so it coincides with the reduced fiber product

$$X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n = \left(\text{Hilb}_n \times_{\mathbb{C}^{2n}/S_n} \left(\mathbb{C}^{2n} \times_{\mathbb{C}^{2n}/S_n} \mathbb{C}^{2n}\right)\right)_{\text{red}}.$$

The space $\mathbb{C}^{2n} \times_{\mathbb{C}^{2n}/S_n} \mathbb{C}^{2n}$ is covered by graphs of permutations viewed as maps $\mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$. This induces a covering of $X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n$ by $n!$ copies of $X_n$. Passing to the rings of functions we obtain ring homomorphisms:

$$\Gamma(O_{X_n}) \otimes \Gamma(O_{X_n}) \rightarrow \Gamma(O_{X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n}) \rightarrow \bigoplus_{\sigma \in S_n} \Gamma(O_{X_n}) ,$$

whose composition is the map $\phi$ of (10). The second map above is injective because the functor $\Gamma$ is left exact. If we knew that the first map is surjective, we would have

$$\Gamma_{\text{Hilb}} \mathbb{C}^2 (P \otimes P) = \Gamma(O_{X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n}) = \text{Im} \phi .$$

The conjecture is then reduced to the vanishing of the higher cohomologies of the ideal sheaf of $X_n \times_{\text{Hilb}} \mathbb{C}^2 X_n$ in $X_n \times X_n$. 

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3.3 Relation with GKM theory

The space $X_{n,m}$ has an paving by affines by the results of [LS91] in the coprime case, and [GKM03] for the general case, including $X_{n,m}$ for general $(n, m)$ in type A. In the unramified case of $m = kn$ studied in [GKM04], the equivariant cohomology $H^*_T(X_{n,kn})$ is a submodule of the free $\mathbb{C}[x]$-module with basis $W_n$, and there is a basis in which the leading coefficient in the Bruhat order is

$$a_{w,k}(x) = \prod_{i<j} (x_i - x_j)^{\# \text{edges}^i_j(w)}.$$

Here $\text{edges}^i_j(w)$ is the set of transpositions $t_{a,b} \in \text{edges}_m(w)$ for which $\{a, b\} = \{i, j\}$, where the bar is the congruence class modulo $n$. The standard description of the corresponding homology is given as a subspace of the $\mathbb{C}(x)$-vector space with the same fixed point basis, which is different from the description of [GKM04]. For instance, in the case of $k = \infty$, we can compare the coefficient $a_{w,x}(x)$ with the leading terms in Kostant and Kumar’s nil Hecke ring [KK86, LLM+14], which encodes the equivariant homology of the affine flag variety $A_w$.

We expect that $M$ embeds as a submodule of the $GL_n$ version of $H^*_T(X_{n,kn})$, in which the fixed points only consist of positive permutations

$$M \subset \bigoplus_{\tau \in \mathbb{W}^+} \mathbb{C}[x]y^m \tau = \bigoplus_{w \in \mathbb{W}^+} e_w, \quad w = (\tau_1 + m_1, n, \ldots, \tau_n + m_n).$$

A construction of this type was used in [CO18] for instance, in which the authors exhibited an isomorphism $DR_n \cong H_*(X_{n,n+1})$ related to the ones studied in [OY14], and used it to study the diagonal coinvariant algebra $DR_n$ as a module over $\mathbb{C}[x]$. In another example, O. Kivinen showed that Haiman’s alternant ideal $J_n \subset \mathbb{C}[x,y]$ in general Lie type satisfies a suitable version of the GKM relations, and therefore injects into the equivariant Borel-Moore homology of the Grassmannian version of $X_{n,kn}$. In type A, when combined with Haiman’s results, it follows that the map is an isomorphism when the $y$-variables are inverted [Kiv20].

Now let

$$b_{w,k}(x) = \prod_{i<j} (x_i - x_j)^{k-\# \text{edges}^i_j(w)},$$

whose degree is $\text{dinv}_{kn}(w)$. The following conjecture illustrates the connection with Theorem A in the case $k = 1$:
Conjecture 3.10. There exist free generators $A_w = \sum_v c_{v,w}(x)e_v \in M$ as $v, w \in W^+_n$ satisfying the following properties:

1. The $A_w$ freely generate $M$ as a $C[x]$-module.
2. The coefficients satisfy $c_{v,w}(x) = 0$ unless $v \leq_{bru} w$, and the leading term is given by $c_{w,w}(x) = b_{w,1}(x)$.
3. For any compositions $\alpha, \beta$, if $w$ is the element of maximal length in $S_\alpha W^+_n / S_\beta$, then $A_w \in M^{\alpha, \beta}$, the invariant subspace with respect to the product of the corresponding Young subgroups.

In particular, there is the expected freeness of $M$ over $C[x]$.

To connect this with Theorem A, observe that combining (9) with Conjecture 3.7, we would have

$$\nabla e_n \left[ \frac{XY}{(1-q)(1-t)} \right] = F_{Y,X}M.$$ (11)

On the other hand, in light of Conjecture 3.10, we expect that $F_{Y,X}M = \Omega_1[X,Y]$ with respect to the dot and star actions mentioned in the introduction. To see this in the case of the Hilbert series, we take the contribution to $\Omega_1[X,Y]$ for which $a, b$ have all distinct entries. Then the automorphism factor is trivial, and we obtain the sum in (11), using Proposition 3.4 to relate the corresponding dinv statistics.

4 Main results

We can now state and prove our main Theorem, and some consequences.

4.1 Main theorem

Recall the conventions for summations of sorted representatives described in Section 2.4. We have our main theorem:

**Theorem 4.1.** For any $k \geq 1$, we have

$$\nabla^k e_n \left[ \frac{XY}{(1-q)(1-t)} \right] = \sum_{[m,a,b]} \frac{\ell[m]^d \text{dinv}_k(m,a,b)}{(1-q)^{\text{aut}_q(m,a,b)}} X_a Y_b.$$ (12)
Before proving Theorem 4.1 we state a few immediate consequences. Let
\[ \xi_\pi[Y; q] = \sum_b q^{\text{inv}_\pi(b)} Y_b, \] (13)
where \( \text{inv}_\pi(b) \) is defined in Section 2.2.

**Proposition 4.2.** The right hand side of (12) is given by
\[ \sum_{[m,a]} \frac{q^{\text{dinv}_k(m,a)}}{(1-q)^\text{naut}_q(m,a)} X_a \xi_{\pi_k(m,a)}[Y; q]. \]

**Proof.** Notice that whenever \( m_i = m_j, a_i = a_j, \) and \( b_i < b_j, \) switching the order of \( b_i \) and \( b_j \) always increases \( \text{dinv}_k(m,a,b) \) by one. Therefore we may remove the sorting condition in \( b, \) i.e. replace \([m,a,b]\) with \([m,a]\) in the sum, and remove it from the automorphism factor without changing the answer. Then we have
\[ \text{dinv}_k(m,a,b) = \text{dinv}_k(m,a) + \text{inv}_{\pi_k(m,a)}(b). \]

We have the following interpretation of \( \xi_\pi[Y; q]. \) Let
\[ X_\pi[Y; q] = \sum_{b : (i,j) \in D(\pi) \Rightarrow b_i \neq b_j} q^{\text{inv}_\pi(b)} Y_b \] (14)
be Stanley’s chromatic symmetric function.

**Proposition 4.3.** We have that
\[ \xi_\pi[Y; q] = (1 - q)^\omega X_\pi[Y(1-q)^{-1}; q]. \] (15)
In particular, it is a symmetric function.

**Proof.** We have that \( \xi_\pi[Y; q] \) is the same as the LLT polynomial \( \chi_\pi[Y; q] \) in [CM18], and the statement follows from Proposition 3.5 of that paper.

**Remark 4.4.** This can also be seen using a conjecture of Shareshian and Wachs [SW12], later proved in two different ways in [BC15, GP16], which would show that both sides of (15) are equal to the Frobenius character of the equivariant cohomology of the regular semisimple Hessenberg variety. This should have a geometric interpretation, and we expect that it corresponds to the paving of the affine Springer fiber by Hessenberg varieties from [GKM03].

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As a corollary, we have the expression for \((\nabla^k e_1^n, e_n)\) from [EH16]. It was later proved in [GH17], where it was shown that both sides equal the Poincaré polynomial for the Khovanov-Rozansky knot homology of \(k\)th power of the full twist, and also the Hilbert series for the \(k\)th power of Haiman’s alternant ideal \(J^k_n\).

**Corollary 4.5.** We have that

\[
(\nabla^k e_1^n, e_n) = \frac{1}{(1 - q)^n} \sum_m t^{\im(m)} q^{d_k(m)},
\]

(16)

where

\[
d_k(m) = \sum_{i<j} \max(k - m_i + m_j + 1, k - m_j + m_i).
\]

**Proof.** The left hand side of (16) is the result of taking the coefficient of \(X_a\) and \(Y_b\) for \(a = (1, \ldots, n)\) and \(b = (1, \ldots, 1)\) in (12). Since the entries of \(a\) are distinct, there is no automorphism factor. Now notice that the compositions \(m\) are in bijection with sorted pairs \([m', a]\) where \(a\) has distinct elements, meaning it is a permutation, and that \(d_k(m) = \text{dinv}_k(m', a)\).

4.2 Proof of Theorem A

We give our first proof of Theorem 4.1 by taking equation (12) as the definition of an operator on symmetric functions, and verifying that it satisfies the defining properties of \(\nabla^k\), similar to the approach in [HHL05b]. Let us define an operator \(\nabla^1_k\) on symmetric functions by

\[
\nabla^1_k e_n \left[ \frac{XY}{(1 - q)(1 - t)} \right] = \Omega_k[X, Y],
\]

(17)

where \(\Omega_k[X, Y]\) denotes the right hand side of (12). By Propositions 4.2 and 4.3, and the symmetry in \(X\) and \(Y\), the right side of (17) is symmetric in both sets of variables, and so this formula uniquely defines an operator acting on the \(X\)-variables.

We will prove the following proposition.

**Proposition 4.6.** We have the following properties of \(\Omega_k[X, Y]\):

1. It is symmetric in the two sets of variables, \(\Omega_k[X, Y] = \Omega_k[Y, X]\).

2. If \(\lambda, \mu\) are partitions, then the coefficient of \(X^\lambda Y^\mu\) in \(\Omega_k[X(t-1), Y(q-1)]\) is zero unless \(\lambda \leq \mu'\) in the dominance order.
3. The leading coefficient in front of $X^\lambda Y^{\lambda'}$ is $q^{k n(\lambda') + k n(\lambda)}$.

In particular, we have

**Corollary 4.7.** We have that $\nabla_{k}' = \nabla_k$, proving Theorem 4.7.

**Proof.** We have already shown that $\Omega_k[X, Y]$ is a symmetric function in the $Y$-variables in Proposition 4.3 and by the $X \leftrightarrow Y$ symmetry it is symmetric in both. The three properties in Proposition 4.6 then correspond to the defining properties of $\nabla_k$, namely self-adjointness in the modified Macdonald inner product, triangularity, and the correct leading term.

The $X \leftrightarrow Y$ symmetry item is clear from the symmetry in the definition of $\text{dinv}_k$. We now turn to the hard part which is showing triangularity. We first evaluate the plethystic substitution $Y \mapsto Y(q - 1)$ using (14) and (15) to obtain

\[ \Omega_k[X, Y(q - 1)] = (-1)^n \sum_{i \text{ k-attacks } j \neq b_i} t^{[m, a, b]} q^{\text{dinv}_k(m, a, b)} X_a Y_b, \tag{18} \]

noting that the automorphism factors all disappear because we never have a nonzero term with $(m_i, a_i, b_i) = (m_j, a_j, b_j)$ for $i \neq j$.

We would now like to evaluate the substitution $X \mapsto X(t - 1)$. To do this, we must write equation (18) as a quasi-symmetric function in the $X$-variables. We first sort the triples in a different order, so that $[a, m, b]'$ is a triple in which the $a_i$ are in descending order, the $m_i$ are in increasing order to break ties, and the $b_i$ are in descending order to breaking ties. This is the reverse of the usual order, modified so that $a$ has priority over $m$. Define two conditions $W$ (wrong) and $NW$ (not wrong) on pairs $m, b$:

\[
W_i : \text{ if } m_i > m_{i+1} \text{ or } m_i = m_{i+1}, \ b_i < b_{i+1}, \\
NW_i : \text{ if } m_i < m_{i+1} \text{ or } m_i = m_{i+1}, \ b_i > b_{i+1}.
\]

We can reconstruct the condition of when a nonzero term in (18) must have the inequality $a_i > a_{i+1}$ based on the ordering $m, b$, to produce a quasi-symmetric expansion.

For $(a, m, b) = [a, m, b]'$ reverse sorted, we have that

\[
d_{k}([m, a, b]) = d_{k}(m, b) = \sum_{i < j} \max(d_{k}^{i, j}(m, b), 0)
\]

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for every nonzero term in (18), where
\[
d_{k}^{i,j}(\mathbf{m}, \mathbf{b}) = \begin{cases} 
    k + m_j - m_i + \delta(b_i > b_j) & m_i > m_j \\
    k - 1 + m_i - m_j + \delta(b_i < b_j) & m_i \leq m_j
\end{cases}
\] (19)
and \( \delta \) is the delta function, 1 for true, 0 for false. For any \( \mathbf{m} \) we will say that \( i \) \( k \)-attacks \( j \) if
\[m_i \notin \{m_j - k + 1, \ldots, m_j + k\},\]
which is the same as \( i' \) and \( j' \) attacking each other in \([\mathbf{m}, \mathbf{a}]\) if \( \mathbf{m} \) is the representative in which \([\mathbf{a}, \mathbf{m}]\) are sorted.

We can now write
\[
\Omega_k[X, (q - 1)Y] = (-1)^n \sum_{\mathbf{m}, \mathbf{b}: \ i \ k\text{-attacks } j \mapsto b_i \neq b_j} t^{[\mathbf{m}]} q^{d_k(\mathbf{m}, \mathbf{b})} Y_b \sum_{a_1 \geq \cdots \geq a_n: \ W_i \Rightarrow a_i > a_{i+1}} X_a,
\] (20)
which is a quasi-symmetric expansion. We may apply the operator of the substitution \( F[X] \mapsto F[(t - 1)X] \) using the standardization approach from [HHL05b] to obtain
\[
\Omega_k[(t - 1)X, (q - 1)Y] = \sum_{\mathbf{m}, \mathbf{b}: \ i \ k\text{-attacks } j \mapsto b_i \neq b_j} t^{[\mathbf{m}]} q^{d_k(\mathbf{m}, \mathbf{b})} Y_b \times
\]
\[
\sum_{l=0}^{n} (-t)^l \sum_{a_1 \leq \cdots \leq a_n-l: \ W_i \Rightarrow a_i < a_{i+1}} X_a = \sum_{\mathbf{m}, \mathbf{b}: \ i \ k\text{-attacks } j \mapsto b_i \neq b_j} t^{[\mathbf{m}]} q^{\text{dinv}_k(\mathbf{m}, \mathbf{b})} Y_b \sum_{l=0}^{n} (-t)^l \sum_{a_1 \leq \cdots \leq a_n-l: \ W_i \Rightarrow a_i > a_{i+1}} X_a.
\] (21)

Before proving the vanishing, it will be helpful to write equation (21) in a more convenient form by collecting powers of \( t \). Define the rotation operator \( \rho \) on pairs \((\mathbf{m}, \mathbf{b})\) by \( \rho(\mathbf{m}, \mathbf{b}) = (\mathbf{m}', \mathbf{b}') \), where
\[
m_i' = m_{i-1}, \quad b_i' = b_{i-1}, \quad m_1' = m_n + 1, \quad b_1' = b_n, \]
which satisfies
\[
d_k(\rho(\mathbf{m}, \mathbf{b})) = d_k(\mathbf{m}, \mathbf{b}), \quad \text{area}(\rho(\mathbf{m}, \mathbf{b})) = \text{area}(\mathbf{m}, \mathbf{b}) + 1,
\]
where area\((\mathbf{m}, \mathbf{b})\) = |\(\mathbf{m}\)|. Moreover, for \(1 \leq i < n - 1\), \(W_i\) for \((\mathbf{m}, \mathbf{b})\) is equivalent to \(W_{i+1}\) for \(\rho(\mathbf{m}, \mathbf{b})\). The triples in (21) are then bijectively mapped via \(\rho\) to triples satisfying \(m_1, \ldots, m_l \geq 1\), so the right hand side of (21) becomes
\[
\Omega_k[X(t - 1), Y(q - 1)] = \sum_{(l, a, m, b) \in A(n, k)} (-1)^{l|\mathbf{m}|} q^{d_i(\mathbf{m}, \mathbf{b})} X_a Y_b,
\]
and the summation set is given in the following definition:

**Definition 4.8.** Let \(A(n, k)\) be the set of all quadruples \((l, a, \mathbf{m}, \mathbf{b})\) satisfying

1. The terms \((a_i, m_i, b_i)\) are sorted for \(l + 1 \leq i \leq n\), and in reverse order for \(1 \leq i \leq l\).
2. \(m_i > 0\) for \(1 \leq i \leq l\).
3. If \(i\) \(k\)-attacks \(j\), i.e. \(m_i \in \{m_j - k + 1, \ldots, m_j + k\}\) for \(i < j\), then \(b_i \neq b_j\).

**Example 4.9.** For instance, we would have
\[
A = \begin{pmatrix}
\mathbf{a} & 3 & 2 & 1 & 1 & 2 & 4 \\
\mathbf{m} & 3 & 1 & 0 & 0 & 0 & 0 \\
\mathbf{b} & 1 & 3 & 2 & 4 & 1 & 5
\end{pmatrix} \in A(6, 2),
\]
where we are drawing a dividing line to indicate that \(l = 2\). Below is the table of the contributions (before taking the max with zero) to \(d_2(\mathbf{m}, \mathbf{b})\):
\[
(d_2^{i,j}(\mathbf{m}, \mathbf{b})) = \begin{pmatrix}
1 & 0 & -1 & -1 & -1 & -1 \\
-1 & 1 & 2 & 1 & 2 & 1 \\
-2 & 1 & 1 & 2 & 1 & 2 \\
-2 & 0 & 1 & 1 & 2 & 1 \\
-2 & 1 & 2 & 2 & 1 & 2 \\
-2 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
We see that \(d_2(\mathbf{m}, \mathbf{b}) = 16\), by adding up the positive entries above the diagonal.

We now demonstrate the triangularity of equation (23) by finding an involution \(\iota_k : A(n, k) \rightarrow A(n, k)\) which sends a quadruple \((l, a, \mathbf{m}, \mathbf{b})\) to itself, or sends it to one which cancels it in (23). We then show that the set of fixed points are empty unless the dominance order property is satisfied.
Definition 4.10. For any $i$ and any quadruple $(l, a, m, b)$, we define

$$\text{move}_i(l, a, m, b) = (l', a', m', b')$$

where $l' = l - 1$ if $i \leq l$ or $l + 1$ if $i > l$, and $(a', m', b')$ is the result of inserting $(a_i, m_i, b_i)$ in the unique position on the opposite side of the dividing line $l$ so that $(l', a', m', b')$ is sorted as in Condition 1 of Definition 4.8.

Notice that for any element of $A(n, k)$, we never have $(a_i, m_i, b_i) = (a_j, m_j, b_j)$ unless $i = j$, because of condition 3. We therefore have a unique permutation $\sigma$ so that $(a_\sigma, m_\sigma, b_\sigma)$ is overall sorted, not in reverse order for $i \leq l$.

Definition 4.11. Given $A = (l, a, m, b) \in A(n, k)$, we will say that $i$ is $k$-movable if $\text{move}_i(A) \in A(n, k)$, and for any $j$ with $\sigma_j < \sigma_i$, we have $d_{k,j}^i(m, b), d_{k,i}^j(m, b) \leq 0$. Let $\iota_k : A(n, k) \rightarrow A(n, k)$ be the involution defined by setting

$$\iota_k(A) = \begin{cases} A & \text{no element } i \text{ is } k\text{-movable,} \\ \text{move}_i(A) & i \text{ the movable element with smallest value of } \sigma_i. \end{cases}$$

Then if $A' = \iota_k(A)$ is not a fixed point, the criteria for being $k$-movable ensures that

$$d_k(m', b') = d_k(m, b), \quad |m'| = |m|, \quad (-1)^l = -(-1)^{l'},$$

which turns out to be enough to cancel terms in (23).

Example 4.12. Let us compute the involution on the term $A \in A(6, 2)$ from Example 4.9. We have that $\sigma = (5, 3, 1, 2, 4, 6)$. The smallest element is therefore $i = 3$, which is not moveable because $m_3 = 0$, so we cannot move it to the left of the dividing line $l = 2$ without violating condition 2 of the definition of $A(n, k)$. The next smallest values of $i = 4, 2, 5$ cannot be moved because we have $d_{2,j}^i(m, b) > 0$ or $d_{2,i}^j(m, b) > 0$ for some $j$ earlier in the list. However, $i = 1$ is moveable, and we end up with

$$\iota_k(A) = \begin{pmatrix} a & 2 & 1 & 1 & 2 & 3 & 4 \\ m & 1 & 0 & 0 & 3 & 0 \\ b & 3 & 2 & 4 & 1 & 1 & 5 \end{pmatrix}.$$

Let $A'(n, k)$ denote the fixed points of $\iota_k$.

Proposition 4.13. If $\lambda, \mu$ are partitions, then the set of all $(l, a, m, b) \in A'(n, k)$ for which $\alpha(a), \alpha(b) = \lambda, \mu$ is empty unless $\lambda \lessdot \mu'$ in the dominance order. If $\lambda = \mu'$, then it contains a unique element.
We will prove it through some lemmas. Let \( T(A) = T(a, m, b) \) be the result of filling the \( r \)th row of the composition \( \alpha = \alpha(a) \) with the pairs \((m_i, b_i)\) for which \( a_i = \alpha_r \), in such a way that the order is compatible with the reading order. For instance,

\[
\begin{array}{c|cccccc}
\text{a} & 2 & 2 & 1 & 2 & 1 & 3 \\
\text{m} & 1 & 0 & 0 & 1 & 2 & 1 \\
\text{b} & 1 & 1 & 3 & 2 & 3 & 1 \\
\end{array} = \begin{array}{ccc}
23 & 01 & 03 \\
11 & 12 & 01 \\
11 & & \\
\end{array}
\tag{25}
\]

Notice that the diagram does not depend on the ordering or on \( l \).

**Lemma 4.14.** If \( A \in \mathcal{A}'(n, k) \), then every \( m \)-number in row \( r \) of \( T(A) \) is at most \((r - 1)k\).

**Proof.** Since the rows of the diagram are decreasing, it suffices to check the inequality for the first element of each row. The first element of the first row must be zero, otherwise the lowest element would be movable by simply switching the position of the dividing line, which can only violate Condition 2 of Definition 4.8. If some element is greater than \((r - 1)k\), then there must be a rows whose first entry exceeds all previous entries by more than \( k \), in which case it is movable according to (19).

**Lemma 4.15.** If \( A \in \mathcal{A}'(n, k) \), then the same \( b \)-number can appear in the first \( r \) rows of \( T(A) \) at most \( r \) times. If it appears the maximum \( r \) times, then they all occur in different rows, and all occurrences \( b_i \) are to the right of the dividing line, \( i \geq l + 1 \).

**Proof.** Suppose the number \( b \) appears \( r + 1 \) times in rows 1 through \( r \). Let \( x_1, ..., x_r \) denote the set of the corresponding values of \( m_i \) in the order they appear in \( m \), for instance \((1, 0, 1, 0)\) for the \( b \)-value of 1 in (25). Then we must have that

\[
x_s \leq x_t - k \quad \text{or} \quad x_s \geq x_t + k + 1 \tag{26}
\]

for \( 1 \leq s < t \leq r + 1 \) by condition 3 of Definition 4.8. Now let \( 0 \leq y_1 \leq \cdots \leq y_{r+1} \) denote the same set of numbers as the \( x_s \) but in sorted order. By (26) we have that \( y_{s+1} \geq y_s + k \), so that \( y_{r+1} \geq rk \), which contradicts Lemma 4.14.

To prove the second statement, define \( 0 \leq y_1 \leq \cdots \leq y_r \) as above. Then by the same reasoning we have \( y_r \geq (r - 1)k \) and also \( y_r \leq (r - 1)k \) by the same lemma, so we must have \( y_s = sk \). Then only the first case is possible in (26), and so all the \( x_s \) are same order \( x_s = y_s \). Since \( x_1 = 0 \), it must be to the right of the dividing line because of condition 2 and so the rest are as well. Then if two \( b \)-values appear in the same row, there will be increasing \( m \)-values for the same \( a \)-value, so in the wrong order for the entries to the right of the dividing line. \( \square \)
We now prove Proposition 4.13.

Proof. The shape of \( T(p, \lambda) \) is just \( \lambda \), and the first statement of Lemma 4.15 easily shows that \( \lambda \equiv \mu \).

For the second statement, if \( \lambda = \mu \), then the \( i \)-th lowest \( b \)-number appears \( \lambda_i \) times. By the second statement of Lemma 4.15, it appears once in every row up to \( \lambda_i \), to the right of the dividing line, and (by the proof) with corresponding \( m \)-values \( 0, k, 2k, \ldots \). There is a unique term with these properties, namely \( l = 0 \) since all elements are to the right, and

\[
\mathbf{a} = (1^{\lambda_1}, \ldots, l^{\lambda_r}), \quad \mathbf{m} = (0^{\lambda_1}, \ldots, (rk)^{\lambda_r}), \quad \mathbf{b} = (1, \ldots, \lambda_1, \ldots, 1, \ldots, \lambda_r).
\]

Finally, we can prove Proposition 4.6 and therefore Theorem 4.1.

Proof. The first statement is clear from the symmetry of \( \text{dinv}_k \) in \( \mathbf{a} \) and \( \mathbf{b} \). The second statement follows from Proposition 4.13, since all terms in (23) corresponding to \( A \in \mathcal{A}(n, k) - \mathcal{A}'(n, k) \) cancel with \( t_k(A) \) by (24). Finally, the leading term from the proof is easily seen to be \( q^{kn(\lambda)} t^{kn(\lambda)} \).

4.3 A new proof of the shuffle theorem

We now show how to recover the shuffle theorem from Theorem 4.1.

Notice that if we have \( b_i = b_j \) for any \( i \neq j \) in (18), then we cannot have \( m_i = m_j \), or \( m_i = m_j + 1 \) and \( a_i > a_j \). By the first condition, we can uniquely sort the orbits so that the \( b_i \) are sorted in reverse order, \( b_1 \geq \ldots \geq b_n \), and if \( b_i = b_{i+1} \) then \( m_i < m_{i+1} \). Then the second condition says:

\[
b_i = b_{i+1} \Rightarrow m_{i+1} > m_i + 1 \text{ or } m_{i+1} = m_i + 1 \text{ and } a_{i+1} \leq a_i.
\]

For a pair of sequences \( (\mathbf{m}, \mathbf{a}) \) and a position \( i \) we will define two conditions, “parking function at \( i \),” and “not parking function at \( i \),” noting that one is the negation of the other:

\[
\text{PF}_{k,i} : \text{ if } m_{i+1} \leq m_i + k - 1 \text{ or } m_{i+1} = m_i + k, \text{ and } a_{i+1} > a_i,
\]

\[
\text{NPF}_{k,i} : \text{ if } m_{i+1} > m_i + k \text{ or } m_{i+1} = m_i + k, \text{ and } a_{i+1} \leq a_i.
\]

We can now write

\[
\nabla^k h_n \left[ \frac{XY}{1-t} \right] = \sum_{\mathbf{m}, \mathbf{a}} X_{\mathbf{a} t^{\mathbf{m}}} q^{d_k(\mathbf{m}, \mathbf{a})} \sum_{b_1 > \ldots > b_n : \text{PF}_{k,i} \Rightarrow b_i > b_{i+1}} Y_{\mathbf{b}}, \tag{27}
\]

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where \( d_k(m, a) \) is the same one defined in (19). We would like to evaluate the substitution \( Y \rightarrow (1-t)Y \). To do this, notice that (27) is a sum of quasi-symmetric functions in \( Y \). We can therefore compute the substitution using the standardization approach from [HHL05b]. The result is

\[
\nabla^k h_n[XY] = \sum_{m,a} \sum_{l=0}^n (-t)^l X_a \sum_{b_1 \geq \cdots \geq b_n \text{ s.t. } b_i \geq b_{i+1}} \sum_{b_1, b_2, \ldots, b_n} Y_b. \tag{28}
\]

Finally, let us make the evaluation \( Y = -1 \) in (28) adding an extra sign \((-1)^n\), which amounts to counting only the terms in which the quasi-symmetric functions have strict inequalities. We obtain

\[
\nabla^k e_n[X] = \sum_{l=0}^n (-t)^l \sum_{m,a: \text{PF}_{k,i} \text{ for } 1 \leq i \leq n-l-1, \text{NPF}_{k,i} \text{ for } n-l+1 \leq i \leq n-1} X_a t^{[m]} q^{d_k(m,a)}. \tag{29}
\]

We would like to cancel certain terms in the right hand side of (29), this time using the rotation operator \( \rho \) defined in (22). Let

\[
\mathcal{A}(n,k) = \{(l, m, a) : \text{PF}_{k,i} \text{ for } 1 \leq i \leq n-l-1, \text{NPF}_{k,i} \text{ for } n-l+1 \leq i \leq n-1\},
\]

and notice that for \( 1 \leq i < n-1 \), we have that \( \text{PF}_{k,i} \) for \((m, a)\) is equivalent to \( \text{PF}_{k,i+1} \) for \( \rho(m, a) \). Consider those triples \((l, m, a)\) satisfying

(1A) \( l > 0 \),

(2A) \( \text{PF}_{k,1} \) for \( \rho(m, a) \) if \( l < n \).

The image of \( \rho \) on these triples is the set of triples \((l, m, a)\) satisfying

(1B) \( l < n \),

(2B) \( \text{NPF}_{k,n-1} \) for \( \rho^{-1}(m, a) \) if \( l > 0 \),

(3B) \( m_1 > 0 \).

It is clear that these are the sets of all \( A \in \mathcal{A}(n,k) \) so that \( \rho(A) \in \mathcal{A}(n,k) \) for the first set, or \( \rho^{-1}(A) \in \mathcal{A}(n,k) \) for the second.

We can now check the following proposition, which implies that the two sets have no elements in common, and so the terms coming from the two sets cancel each other out in (29).
Proposition 4.16. The set of triples satisfying (1A), (2A), (1B), (2B), (3B) is empty.

We let \( A'(n, k) \subset A(n, k) \) denote the subset of contributing terms, which are not in either set. We can now give a new proof of the shuffle theorem [HHL05b, CM18], noting that the conditions of the summation in (30) mean that \( m \) is the area sequence of a Dyck path, and \( a \) is a word parking function, see [Hag08].

Theorem 4.17.

\[
\nabla^k e_n[X] = \sum_{m, a: PF_{k,i} \text{ for all } i, m_1=0} X_{\rho(l,m)|} q^{d_i(m,a)}. \quad (30)
\]

Proof. Using Proposition 4.16 to cancel terms in (29), the terms that remain are the ones that fail to satisfy at least one out of (1A) and (2A), and also fail at least one of (1B), (2B), (3B). If a term does not satisfy (2A), it means \( l < n \) and NPF\(_{k,1}\) holds for \( \rho(m,a) \). In particular, we have \( 0 < m_n + k \leq m_1 \), so the only property that can fail among (1B), (2B), (3B) is (2B). Therefore, \( l > 0 \) and PF\(_{k,n-1}\) holds for \( \rho^{-1}(m,a) \), which is equivalent to PF\(_{k,1}\) for \( \rho(m,a) \), a contradiction. Then among (1A), (2A), the property (1A) is the one that fails, so we have \( l = 0 \). The only property among (1B), (2B), (3B) that can fail in the case \( l = 0 \) is (3B), so we have \( m_1 = 0 \). \( \square \)

5 Parabolic bundles

In this Section we deduce Theorem 4.1 by counting parabolic bundles in two different ways.

5.1 Counting formula

On the first side, we will need a result from [Mel17] for counting bundles on \( \mathbb{P}^1 \) over a finite field. Let \( q \) be a prime power, and let \( k \) be the finite field with \( |k| = q \) elements. Let \( S = \{s_1, \ldots, s_k\} \subset \mathbb{P}^1(k) \) be a collection of rational points. Let \( N \) be a big integer (this will correspond to the number of variables in each alphabet). We need \( k \) alphabets \( X_1, \ldots, X_k \). The variables in alphabet \( X_i \) are denoted \( x_{i,j} \) (\( 1 \leq i \leq k, 1 \leq j \leq N \)).

Definition 5.1. A parabolic bundle is a pair \((E,F)\), where \( E \) is a vector bundle on \( \mathbb{P}^1 \) over \( k \), and \( F = (F_{i,j})_{1 \leq i \leq k, 0 \leq j \leq N} \) is a collection of vector spaces so that for each \( i \) we have

\[
0 = F_{i,0} \subseteq F_{i,1} \subseteq \cdots \subseteq F_{i,N-1} \subseteq F_{i,N} = E(s_i).
\]
An endomorphism of \((\mathcal{E}, \mathbf{F})\) is an endomorphism of \(\mathcal{E}\) preserving each \(F_{i,j}\). An endomorphism \(\theta\) is nilpotent if \(\theta^n = 0\) for some \(n\).

Here \(\mathcal{E}(s_i)\) is the fiber of \(\mathcal{E}\) over \(s_i\). If \(\mathcal{E}\) had rank \(n\), then \(\mathcal{E}(s_i)\) is an \(n\)-dimensional vector space.

Parabolic bundles have the following discrete invariants:

- \(\text{rank}(\mathcal{E}) = \text{rank of } \mathcal{E}\),
- \(\deg(\mathcal{E}) = \text{degree of } \mathcal{E}\),
- \(r_{i,j} = \dim(F_{i,j}/F_{i,j-1})\).

Note that \(r_{i,\bullet}\) is a composition of \(n\) for each \(i = 1, \ldots, k\) (of length \(N\) with zeros allowed). These invariants are packaged in the following weight:

\[
\text{weight}(\mathcal{E}, \mathbf{F}) = t^{\deg} \prod_{i=1}^{k} \prod_{j=1}^{N} x_{i,j}^{r_{i,j}}.
\]

It is well-known that over \(\mathbb{P}^1\) every vector bundle is a sum of line bundles, so we can write \(\mathcal{E} = O(m_1) \oplus \cdots \oplus O(m_n)\). We write \(\mathcal{E} \geq 0\) if all \(m_i \geq 0\). The following formula has been proved in [Mel17]:

\[
\Omega = \sum_{(\mathcal{E}, \mathbf{F}) : \mathcal{E} \geq 0} \frac{\text{weight}(\mathcal{E}, \mathbf{F}) \cdot \left| \Nilp(\mathcal{E}, \mathbf{F}) \right|}{\left| \Aut(\mathcal{E}, \mathbf{F}) \right|} = \sum_{n=0}^{\infty} \left( \sum_{\lambda \vdash n} \frac{\prod_{i=1}^{k} \tilde{H}_{\lambda}(X_i ; q, t)}{\prod_{a,d}(q^a+1-t^d)(q^a-t^d+1)} \right).
\]  

The summation goes over the isomorphism classes of pairs \((\mathcal{E}, \mathbf{F})\). Here \(\Nilp(\mathcal{E}, \mathbf{F})\) denotes the set of all nilpotent endomorphisms of \(\mathcal{E}\) which preserve \(\mathbf{F}\), and similarly \(\Aut(\mathcal{E}, \mathbf{F})\) is the set of automorphisms. Equivalently, one can sum over the isomorphism classes of triples \((\mathcal{E}, \mathbf{F}, \theta)\) where \(\theta \in \Nilp(\mathcal{E}, \mathbf{F})\), and each summand is

\[
\text{weight}(\mathcal{E}, \mathbf{F}) \cdot \left| \Nilp(\mathcal{E}, \mathbf{F}, \theta) \right| / \left| \Aut(\mathcal{E}, \mathbf{F}, \theta) \right|.
\]

We transform the summation as follows. Assume \(s_1 = 0\) and \(s_2 = \infty\). Let us run the summation over \(\mathcal{E}, \mathbf{F}^0 := \mathbf{F}_1\), \(\mathbf{F}^\infty := \mathbf{F}_2\) and \(\theta\) first, and then go over the possible flags \(\mathbf{F}_3, \ldots, \mathbf{F}_k\). We obtain

\[
\Omega = \sum_{(\mathcal{E}, \mathbf{F}^0, \mathbf{F}^\infty)} \frac{t^{\deg} \prod_{i=1}^{k} x_{1,i,j}^{r_{1,i,j}} x_{2,i,j}^{r_{2,i,j}}}{\left| \Aut(\mathcal{E}, \mathbf{F}^0, \mathbf{F}^\infty) \right|} \sum_{\mathbf{F}_3, \ldots, \mathbf{F}_k} \prod_{i=3}^{k} \prod_{j=1}^{N} x_{i,j}^{r_{i,j}}.
\]
We can interpret triple $\mathcal{E}, \mathbf{F}^0, \mathbf{F}^\infty$ as a parabolic bundle with only two marked points $0, \infty$, and $\theta$ as its endomorphism. For fixed $\mathcal{E}, \theta$ the flag $\mathbf{F}_{i,\bullet}$ runs over all flags in $\mathcal{E}(s_i)$ preserved by the restriction $\theta(s_i)$ of the endomorphism $\theta$ to $\mathcal{E}(s_i)$. In particular, the summation and the product can be interchanged: $\sum_{\mathbf{F}_{i,\bullet}, \ldots, \mathbf{F}_{k,\bullet}} \prod_{i=3}^k = \prod_{i=3}^k \sum_{\mathbf{F}_{i,\bullet}}$. The contribution $\sum_{\mathbf{F}_{i,\bullet}} \prod_{j=1}^N x_{i,j}^{r_{i,j}}$ clearly depends only on the Jordan form of $\theta(s_i)$ and is given by the corresponding Hall-Littlewood polynomial $\tilde{H}_{\text{type} \theta(s_i)}[X_i; q, 0]$ (see [Mel17] for details). The notation $\text{type} \theta(s_i)$ stands for the partition whose conjugate specifies the sizes of the Jordan blocks of $\theta(s_i)$. We obtain the following

**Corollary 5.2.** The generating function $\Omega$ can be written as follows:

$$\Omega = \sum_{n=0}^\infty \sum_{\lambda \vdash n} \prod_{i=1}^k \tilde{H}_\lambda[X_i; q, t] \prod_{a,l}(q^{a+1} - t^l)(q^a - t^{l+1})$$

$$= \sum_{(\mathcal{E}, \mathbf{F}^0, \mathbf{F}^\infty)} \frac{t^{\deg} \prod_{j=1}^N x_{1,j}^{r_{1,j}} x_{2,j}^{r_{2,j}}}{|\text{Aut}(\mathcal{E}, \mathbf{F}^0, \mathbf{F}^\infty)|} \sum_{\theta \in \text{Nilp}(\mathcal{E}, \mathbf{F}^0, \mathbf{F}^\infty)} \prod_{i=3}^k \tilde{H}_{\text{type} \theta(s_i)}[X_i; q, 0],$$

where the first summation on the right hand side runs over the isomorphism classes of parabolic bundles with marked points $0, \infty$.

Here we are interested in expressions of the form

$$\nabla^k h_n \left[ -\frac{XY}{(q-1)(t-1)} \right] = \sum_{\lambda \vdash n} \left( q^{n(\lambda)} t^n(\lambda) \right)^k \frac{H_\lambda[X; q, t] H_\lambda[Y; q, t]}{\prod_{a,l}(q^{a+1} - t^l)(q^a - t^{l+1})}.$$  

Recall that

$$(H_\lambda[X; q, t], s_1^n) = q^{n(\lambda)} t^n(\lambda),$$

and by setting $t = 0$

$$(H_\lambda[X; q, 0], s_1^n) = \begin{cases} q^{\binom{n}{2}} & \lambda = (n), \\ 0 & \text{otherwise.} \end{cases}$$

Applying $(-, s_1^n)$ in the alphabets $X_3, \ldots, X_k$ to both sides of Corollary 5.2 and replacing $k$ by $k+2$ and relabeling $s_i$, we obtain
Corollary 5.3. Let $k \geq 0$, and let $\{s_1, \ldots, s_k\}$ be an arbitrary set of points on $\mathbb{P}(k)$ disjoint from 0 and $\infty$. We have

$$q^k \binom{n}{2} \sum_{\substack{\ell \in \mathbb{N}^N \\
\text{rank}(\mathcal{E}) = n}} |\text{Aut}(\mathcal{E}, \mathcal{F}^0, \mathcal{F}^\infty)| |\text{Nilp}_k(\mathcal{E}, \mathcal{F}^0, \mathcal{F}^\infty)| = \nabla^{k} h_n \left[ -\frac{XY}{(q-1)(t-1)} \right],$$

where $\text{Nilp}_k(\mathcal{E}, \mathcal{F}^0, \mathcal{F}^\infty)$ denotes the set of nilpotent endomorphisms $\theta$ satisfying $\theta(s_i) = 0$ for $i = 1, \ldots, k$.

5.2 Parabolic bundles with two marked points

Next we will use an explicit classification of triples $(\mathcal{E}, \mathcal{F}^0, \mathcal{F}^\infty)$ to give an alternative formula for the generating function in Corollary 5.3. The building blocks of the classification will be parabolic bundles of rank 1, i.e. parabolic line bundles:

Example 5.4. Consider Definition 5.1 in the case $\text{rank}(\mathcal{E}) = 1$. Then $\mathcal{E}(s_i)$ is a vector space of dimension 1, so the sequence of vector spaces $0 = F_{i,0} \subseteq \cdots \subseteq F_{i,N} = \mathcal{E}(s_i)$ is determined by an integer $j_i$ such that $F_{i,j_i-1} = 0$, $F_{i,j_i} \neq 0$. Since we are on $\mathbb{P}^1$, the line bundle $\mathcal{E}$ is uniquely determined by its degree $m$. So a parabolic line bundle is uniquely determined by an integer $m$ and a tuple $(j_1, j_2, \ldots, j_k)$, $1 \leq j_i \leq N$. In the case $k = 2$, we will denote $a = j_1$, $b = j_2$. The corresponding parabolic line bundle is denoted by $O(m; a, b)$.

Proposition 5.5. Let $(\mathcal{E}, \mathcal{F}^0, \mathcal{F}^\infty)$ be a parabolic vector bundle of rank $n$ on $\mathbb{P}^1$ with two marked points. There exists a unique multiset of triples $(m_1, a_1, b_1), \ldots, (m_n, a_n, b_n)$ such that

$$(\mathcal{E}, \mathcal{F}^0, \mathcal{F}^\infty) = \bigoplus_{i=1}^n O(m_i; a_i, b_i).$$

This can be thought of as a generalization of the classical Bruhat decomposition for $GL_n$. There is a tedious direct proof based on several applications of the standard Bruhat decomposition, but we will use homological algebra instead. The proof will occupy the rest of the section.

Of course, the category of parabolic bundles is not an abelian category, but it can be embedded as a full subcategory into the abelian category of parabolic coherent sheaves, which has global dimension 1, so all $\text{Ext}^i$ vanish for $i > 1$. The Euler form is given by (see [Mel17])

$$\dim \text{Hom}(\mathcal{E}, \bar{\mathcal{E}}) - \dim \text{Ext}(\mathcal{E}, \bar{\mathcal{E}}) = \text{rank } \mathcal{E} \text{ rank } \mathcal{E}' + \text{rank } \mathcal{E} \deg \mathcal{E}' - \text{rank } \mathcal{E}' \deg \mathcal{E}$$

(32)
We denote by $\overline{E}$ the pair $(\mathcal{E}, F)$. In the case $k = 2$ we write $(\mathcal{E}, F^0, F^\infty)$.

The dimension of Hom between two parabolic line bundles is given by

$$\dim \text{Hom}(\mathcal{O}(m; j_1, \ldots, j_k), \mathcal{O}(m'; j'_1, \ldots, j'_k)) = \max(1 + m' - m - \#\{i : j_i < j'_i\}, 0).$$

By the formula for the Euler form we obtain

$$\dim \text{Ext}(\mathcal{O}(m; j_1, \ldots, j_k), \mathcal{O}(m'; j'_1, \ldots, j'_k)) = \max(m - m' - 1 + \#\{i : j_i < j'_i\}, 0).$$

Introduce a total order on parabolic line bundles in such a way that $\mathcal{O}(m; j_1, \ldots, j_k) \prec \mathcal{O}(m'; j'_1, \ldots, j'_k)$ lexicographically.

This order clearly satisfies

**Proposition 5.6.** For two parabolic line bundles $L, L'$ if $\text{Hom}(L, L') \neq 0$, then $L \preceq L'$.

**Proof of Proposition 5.6.** Let $k = 2$. We prove the existence first. The proof goes by induction on the rank $n$. The case $n = 1$ is clear. Assume $n > 1$ and suppose $\overline{E}$ is a parabolic bundle of rank $n$. Let $L \subset \overline{E}$ be the maximal in our order parabolic line subbundle. From the theory of parabolic coherent sheaves, we have a short exact sequence

$$0 \to L \to \overline{E} \to \overline{E}' \to 0$$

where $\overline{E}'$ is a parabolic bundle of rank $n - 1$. By the induction assumption, $\overline{E}' \cong \bigoplus_{l=1}^{n-1} L_l$. Suppose the short exact sequence does not split. Then there exist $l$ such that for $L' = L_l$ we have

$$\text{Ext}(L', L) \neq 0 \implies m' - m - 1 + \#\{i : j'_i < j_i\} \geq 1.$$

Note that since $k = 2$ this implies $m' \geq m$. Our plan is to construct a bundle $L''$ such that

1. $\text{Hom}(L'', L') \neq 0$,
2. $\text{Ext}(L'', L) = 0$,
3. $L'' > L$. 

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By the exact sequence
\[ \text{Hom}(L'', \mathcal{E}) \to \text{Hom}(L'', \mathcal{E}') \to \text{Ext}(L'', L) \]
these conditions would guarantee that any non-zero homomorphism \( h \in \text{Hom}(L'', L') \subset \text{Hom}(L'', \mathcal{E}') \) can be lifted to a non-zero homomorphism \( L'' \to \mathcal{E}' \), and we would obtain a contradiction with the maximality of \( L \).

If \( m' \geq m + 1 \), we pick \( L'' = O(m + 1; N, N) \). This guarantees that \( \dim \text{Hom}(L'', L') = m' - m > 0 \), \( \dim \text{Ext}(L'', L) = 0 \) and \( L'' > L \), so the required conditions are satisfied.

Otherwise, we must have \( m' = m \), \( j'_1 < j_1 \) and \( j'_2 < j_2 \). Picking \( L'' = O(m; j'_1, j'_2) \) (or \( O(m; j_1, j'_2) \)) satisfies \( \dim \text{Hom}(L'', L') = 1 \), \( \dim \text{Ext}(L'', L) = 0 \) and \( L'' > L \). So the existence have been proven.

Note that we have in particular demonstrated that the the maximal line subbundle is a direct summand. By Proposition 5.6 it must be present in any direct sum decomposition, and by successively splitting away the maximal subbundle we deduce the uniqueness.

\[ \text{Remark 5.7}. \text{ For } k > 2 \text{ the statement does not hold. For a counter-example for } k = 3, \text{ pick trivial bundle of rank 2 and three lines in general position over the marked points.} \]

5.3 Computation

We are ready to identify all the ingredients in the left hand side of Corollary 5.3. By Proposition 5.5 the summation runs over the set of sorted triples \([m, a, b]\) (see Section 2.2 for combinatorial notations). Each sorted triple corresponds to a direct sum of line bundles \( L_i = O(m_i; a_i, b_i) \), which satisfy \( L_1 \geq \cdots \geq L_n \). Denote
\[
O(m; a, b) = \bigoplus_{i=1}^n O(m_i; a_i, b_i).
\]

**Proposition 5.8.** Suppose \((m, a, b)\) is sorted. The number of automorphisms of \( O(m; a, b) \) is given by
\[
|\text{Aut}(O(m; a, b))| = (q - 1)^n \text{aut}_q(m, a, b) q^{\sum_{i<j} \max(1+m_i-m_j-\delta_{a_i}, a_i-\delta_{b_j}, b_i)}.
\]

**Proof.** By Proposition 5.6 automorphisms are given by block-upper-triangular matrices with block sizes equal to the multiplicities of triples \((m_i, a_i, b_i)\). A block-upper-triangular matrix is invertible precisely if the blocks are. So we obtain that the number of automorphisms is given by
\[
|\text{Aut}(O(m; a, b))| = \prod_{i<j: L_i \neq L_j} q^{\dim \text{Hom}(L_j, L_i)} \times \prod_i |GL_{m_i}(k)|,
\]
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where $\mu_1, \mu_2, \ldots$ denote the multiplicities, $\sum \mu_i = n$. The number of elements of $GL_r(k)$ is given by

$$|GL_r(k)| = q^{\binom{r}{2}} (q - 1)^r [r]_q!.$$  

Since $\dim \text{Hom}(L_i, L_i) = 1$, we have

$$|\text{Aut}(O(m; a, b))| = \prod_{i < j} q^{\dim \text{Hom}(L_j, L_i)} (q - 1)^n \text{aut}_q(m, a, b),$$

and (32) gives the formula.

The following statement is useful

**Proposition 5.9.** Suppose $\theta$ is an endomorphism of $E$ which vanishes in some point $s \in \mathbb{P}(k) \setminus \{0, \infty\}$. Then $\theta$ is nilpotent.

**Proof.** For every $r > 0$, $\text{Tr} \theta^r$ is a global function on $\mathbb{P}^1$, therefore it is constant. Since it is zero at $s$, it must be zero everywhere. So $\text{Tr} \theta^r = 0$ for all $r > 0$, hence $\theta$ is nilpotent.

Below we include the case $k = 0$ for completeness. We have

**Proposition 5.10.** Suppose $(m, a, b)$ is sorted and $k \geq 0$. We have

$$|\text{Nilp}_k(O(m; a, b))| = q^{\sum_{i < j} \max(1 - k + m_i - m_j - \delta_{a_j < a_i} - \delta_{b_i < b_j}, 0)} \times \begin{cases} 1 & (k > 0) \\ q^{\sum_{i < j} \binom{r}{2}} & (k = 0) \end{cases},$$

where $\mu = (\mu_1, \ldots, \mu_l)$ are the multiplicities of the triples $(m_i, a_i, b_i)$.

**Proof.** As in the proof of Proposition 5.8 the endomorphisms are given by block-upper-triangular matrices. In the case $k > 0$ the blocks are automatically zero. The space of off-diagonal entries in position $(i, j)$ is given by $\text{Hom}(L_j, L_i)$ ($i < j$). This is the space of polynomials of bounded degree. When polynomials are forced to have zeroes at $k$ further points, the dimension drops down by $k$. This completely describes the case $k > 0$. For the case $k = 0$ we need to count the number of nilpotent matrices in each block. This is given by $q^{r^2 - r}$ for a block of size $r \times r$. For each block, the factor $q^{\sum_{i < j}}$ contains $q^{\binom{r}{2}}$, so extra factor $q^{\binom{r}{2}}$ has to be added.

The remaining pieces of the left hand side of Corollary 5.3 are identified as follows:

$$t^{\deg} = t^{\lfloor m \rfloor}, \quad \prod_{j=1}^N x_j^{r_{1,j}} = X_a, \quad \prod_{j=1}^N y_j^{r_{2,j}} = Y_b.$$
Example 5.11. Let $k = 0$. The left hand side of Corollary 5.3 becomes

$$\sum_{[m,a,b]} \frac{t^{\binom{m}{2}} X_a Y_b q^{\sum \binom{a}{2}}}{(q - 1)^n \text{aut}_q(m, a, b)}.$$

So in each summand each triple $(m, a, b)$ with multiplicity $\mu$ contributes a factor of

$$\frac{\mu^m x_a^\mu y_b^\mu q^{\binom{\mu}{2}}}{(q - 1)^{\mu [\mu]_q}}.$$

Summing over all $n$ we obtain every possible triple with every multiplicity, so the result can be written as an infinite product

$$\prod_{m=0}^\infty \prod_{a,b=1}^N \prod_{\mu=0}^\infty \frac{t^{\mu^m x_a^\mu y_b^\mu q^{\binom{\mu}{2}}}}{(q - 1)^{\mu [\mu]_q}} = \prod_{m=0}^\infty \prod_{a,b=1}^N \prod_{r=0}^\infty (1 - x_a y_b t^{m} q^r) = \text{Exp} \left[ -\frac{X Y}{(1 - t)(1 - q)} \right],$$

which matches the right hand side of Corollary 5.3.

Our main conclusion is

Theorem 5.12. For $k \geq 1$ we have

$$\nabla^k h_n \left[ -\frac{X Y}{(q - 1)(t - 1)} \right] = \sum_{[m,a,b]} \frac{t^{\binom{m}{2}} q^{\text{dinv}_k(m,a,b)} X_a Y_b}{(q - 1)^n \text{aut}_q(m, a, b)}.$$

Proof. In view of the above computations and Corollary 5.3 it remains to match the $q$-degree in each summand with $\text{dinv}_k$. The $q$-degree is given by

$$k \binom{n}{2} + \sum_{i < j} \max(1-k+m_i-m_j-\delta_{a_i < a_i}-\delta_{b_j < b_i}, 0) - \sum_{i < j} \max(1+m_i-m_j-\delta_{a_j < a_i}-\delta_{b_j < b_i}, 0).$$

For each pair $i < j$ let $c_{i,j} = m_i - m_j - \delta_{a_j < a_i} - \delta_{b_j < b_i}$ and note that $c_{i,j} \geq -1$. Then the above sum can be written as

$$\sum_{i < j} (k + \max(1 - k + c_{i,j}, 0) - (1 + c_{i,j})) = \sum_{i < j} \max(k - 1 - c_{i,j}, 0).$$

Each summand matches the corresponding summand in Definition 2.2

$$\max(k - 1 - c_{i,j}, 0) = \text{dinv}_k^{i,j}(m, a, b),$$

and therefore the $q$-degree equals $\text{dinv}_k(m, a, b)$. 

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