Fourier-Mukai partners for general special cubic fourfolds

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Abstract

We exhibit explicit examples of general special cubic fourfolds with discriminant $d$ admitting an associated (twisted) K3 surface, which have non-isomorphic Fourier-Mukai partners. In particular, in the untwisted setting, we show that the number of Fourier-Mukai partners for a general cubic fourfold in the moduli space of special cubic fourfolds with discriminant $d$ and having an associated K3 surface, is equal to the number $m$ of Fourier-Mukai partners of its associated K3 surface, if $d = 2 \pmod{6}$; else, if $d = 0 \pmod{6}$, the cubic fourfold has $m$ Fourier-Mukai partners.

1 Introduction

A cubic fourfold $Y$ is a (smooth) hypersurface of degree 3 in $\mathbb{P}^5_C$. In [13], Kuznetsov studied the derived category $D^b(Y)$ of bounded complexes of coherent sheaves on $Y$ to address the problem of the (non)rationality of the cubic fourfold. More precisely, for $i = 0, 1, 2$, let $\mathcal{O}_Y(i)$ be the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^5}(i)$ with respect to the embedding of $Y$ in $\mathbb{P}^5_C$. The derived category $D^b(Y)$ admits a semiorthogonal decomposition of the form

$$D^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle,$$

where $\mathcal{A}_Y$ is the right orthogonal of the subcategory of $D^b(Y)$ generated by $\{\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\}$, i.e.

$$\mathcal{A}_Y := \{ F \in D^b(Y) : H^i(Y, F) = H^i(Y, F(-1)) = H^i(Y, F(-2)) = 0 \}.$$

Moreover, the triangulated subcategory $\mathcal{A}_Y$ has certain similarities with the bounded derived category of coherent sheaves on a K3 surface. Indeed, the Serre functor on $\mathcal{A}_Y$ is the shift by two and the Hochschild cohomology of $\mathcal{A}_Y$ is isomorphic to that of a K3 surface (see [12], Corollary 4.3 and [14], Proposition 4.1). The only non-trivial piece $\mathcal{A}_Y$ of the decomposition above should carry the information about the birational type of the cubic hypersurface. Infact, it has been conjectured that the cubic fourfold $Y$ is rational if and only if the category $\mathcal{A}_Y$ is equivalent to the derived category of coherent sheaves on a K3 surface (see [13], Conjecture 1.1). To support this guess, Kuznetsov proved in [13] that the cubic fourfolds which are known to be rational satisfy this condition. Furthermore, this conjecture descends from a more general one, concerning the Clemens-Griffiths components associated to a (maximal) semiorthogonal decomposition (see [14], Section 3).

On the level of the Hodge theory, the existence of an associated K3 surface as an indicator of rationality was conjectured before by Harris and Hassett (see [7], for a complete survey). Actually, Hassett’s and Kuznetsov’s conjectures are (generically) equivalent, by a result of Addington and Thomas (see [1], Theorem 1.1). Nevertheless, all the conjectures we mentioned have not been proved yet.

In [11], Huybrechts deeply studied the category $\mathcal{A}_Y$, in order to develop a theory for cubic fourfolds which parallels that of the derived category of a (twisted) K3 surface. In particular, he proved the analogous version for $\mathcal{A}_Y$ of some results concerning Fourier-Mukai partners of a K3 surface. In general,
we recall that a functor $\Xi : \mathcal{D}^b(Z) \to \mathcal{D}^b(Z')$, between the derived categories of two algebraic varieties $Z$ and $Z'$, is of Fourier-Mukai type if there exists an object $K$ in the derived category $\mathcal{D}^b(Z \times Z')$ of the product and an isomorphism of exact functors

$$
\Xi \cong \Phi_K(-) := Rp_{Z'}^* (K \otimes Lp_{Z'}^*(-)),
$$

where $p_Z : Z \times Z' \to Z$ and $p_{Z'} : Z \times Z' \to Z'$ are the natural projections (see [14], Section 1.5). A cubic fourfold $Y$ is a Fourier-Mukai partner of $Y$ if there exists an equivalence of categories

$$
A_Y \sim A_Y',
$$

which is of Fourier-Mukai type, i.e. such that the composition

$$
\mathcal{D}^b(Y) \xrightarrow{i^*} A_Y \sim A_Y' \hookrightarrow \mathcal{D}^b(Y')
$$

is a Fourier-Mukai functor; here, $i^*$ denotes the left adjoint functor of the full inclusion $i : A_Y \hookrightarrow \mathcal{D}^b(Y)$. Huybrechts showed that the number of (isomorphism classes of) Fourier-Mukai partners for a cubic fourfold $Y$ is finite (see [11], Theorem 1.1), as in the case of Fourier-Mukai partners for a K3 surface (see [3], Proposition 5.3). Moreover, he proved that the general cubic fourfold $Y$, i.e. such that $\text{rk}(H^{2,2}(Y, Z)) = 1$, has no non-trivial Fourier-Mukai partners (see [11], Corollary 3.7) and the same holds even if the rank of $H^{2,2}(Y, Z)$ is greater than 13 (see [11], Corollary 3.8).

It is natural to ask whether a special cubic fourfold $Y$, i.e. such that $\text{rk}(H^{2,2}(Y, Z)) \geq 2$ (see Section 2.1 and [6], Definition 3.1.1), admits Fourier-Mukai partners which are not isomorphic to $Y$. In particular, we may wonder if for special cubic fourfolds it is possible to prove a version of Theorem 1.7 and Corollary 1.8 of [18], which state that there are examples of K3 surfaces having a prescribed number of non-isomorphic Fourier-Mukai partners.

In the third section of this paper, we prove that the answer is positive in the case that the rank of $H^{2,2}(Y, Z)$ is exactly two and the cubic fourfold $Y$ admits an associated K3 surface $X$ with “enough” non-trivial Fourier-Mukai partners (see Proposition 5.7). More precisely, we give a counting formula for the number of Fourier-Mukai partners for general special cubic fourfolds admitting an associated K3 surface, as explained in the following theorem.

**Theorem 1.1.** Let $d$ be a positive integer such that $C_d$ is the moduli space of special cubic fourfolds with discriminant $d$, admitting an associated K3 surface, and $C_d$ is non empty. Let $Y$ be a general cubic fourfold in $C_d$ and let $m$ be the number of Fourier-Mukai partners of an associated K3 surface to $Y$. Then, the cubic fourfold $Y$ has exactly $m$ non-isomorphic Fourier-Mukai partners, when $d \equiv 2(\text{mod } 6)$; otherwise, if $d \equiv 0(\text{mod } 6)$, the number of non-isomorphic Fourier-Mukai partners of $Y$ is equal to $[m/2]$.

As a consequence of Theorem 1.1 we deduce that there exist cubic fourfolds admitting an arbitrary number of Fourier-Mukai partners, depending on the number of distinct odd primes in the prime factorization of the discriminant (see Corollary 5.11).

A weaker formulation of Theorem 1.1 holds for general cubic fourfolds $Y$ in $C_d$, admitting an associated twisted K3 surface $(X, \alpha)$ (see Section 2.4 and [11], Section 2.4 for the definition), if 9 does not divide the discriminant $d$. Indeed, in Section 4.1, we show that the number of twisted Fourier-Mukai partners of $(X, \alpha)$ with the Brauer class of the same order as $\alpha$, is a lower bound for the number of Fourier-Mukai partners of the cubic fourfold.

**Theorem 1.2.** Let $d$ be a positive integer such that $C_d$ is the moduli space of special cubic fourfolds with discriminant $d$, admitting an associated twisted K3 surface, and $C_d$ is non empty. Assume that 9 does not divide $d$. Let $Y$ be a general cubic fourfold in $C_d$ with associated twisted K3 surface $(X, \alpha)$, where $\alpha$ has order $\kappa$; let $m'$ be the number of Fourier-Mukai partners of $(X, \alpha)$ with Brauer class of

$$
\kappa.
$$
order $\kappa$. Then the cubic fourfold $Y$ admits at least $m'$ non-isomorphic Fourier-Mukai partners, when $d \equiv 2 \pmod{6}$; otherwise, if $d \equiv 0 \pmod{6}$, the number of non-isomorphic Fourier-Mukai partners of $Y$ is at least $[m'/2]$.

In particular, under the hypotheses of Theorem [1.2] we have that $m'$ is controlled by the number of distinct primes in the prime factorization of $d/2$ divided by the square of the order of the Brauer class $\alpha$ and by the Euler function evaluated in $\text{ord}(\alpha)$ (see Proposition 4.8).

Notice that our construction represents the first example of non-trivial Fourier-Mukai partners for a cubic fourfold. Actually, these results complete the expected analogy between cubic fourfolds and K3 surfaces, stated in [11]. Finally, we point out that this problem has already been studied in [2], in the case of cubic fourfolds containing a plane. In particular, they proved that the general cubic fourfold in $C_8$ has only one isomorphism class of Fourier-Mukai partners (see [2], Proposition 6.3).

Plan of the paper. The work is organized as follows. In Section 2 we recall some preliminary material on cubic fourfolds and K3 surfaces we will use in the next sections. In Section 2.1 and 2.2 we give an outline of the Hodge theory for special cubic fourfolds, following [6]; in particular, we state the conditions on the discriminant of the cubic fourfold to admit an associated K3 surface and we explain how this reflects on the level of moduli spaces of special cubic fourfolds and polarized K3 surfaces. In Section 2.3 as in [1], we describe the Mukai lattice of a cubic fourfold, which carries the connection between the derived category point of view and Hassett’s results. It is fundamental for our analysis about Fourier-Mukai partners, since two general special cubic fourfolds with isometric Mukai lattices are Fourier-Mukai partners, as showed in [11]. In Section 2.4 we see how these results generalize to the case of cubic fourfolds with an associated twisted K3 surface, as proved in [11]. Finally, in Section 2.5, we recall the counting formula in [18], which gives the precise number of untwisted Fourier-Mukai partners of a general K3 surface, and the construction in [16] to obtain an upper bound to the number of twisted Fourier-Mukai partners of a twisted K3 surface (modulo isomorphism).

Section 3 is devoted to the proof of Theorem 1.1 Let $Y$ be a general cubic fourfold of discriminant $d$ with associated K3 surface $X$: the isomorphism classes of Fourier-Mukai partners of $X$ correspond to general points in the period domain of special cubic fourfolds. Actually, they define distinct points in the period domain of marked cubic fourfolds. In Section 3.1 and 3.2 we show that the number of distinct points they define in the period domain of cubic fourfolds with discriminant $d$ is controlled by the number of distinct odd primes in the prime factorization of $d$. In Section 3.3 we complete the proof of the theorem.

In Section 4 we extend our result to the case of general cubic fourfolds of discriminant $d$ with associated twisted K3 surface $(X, \alpha)$ of order $\kappa$. The proof of Theorem 1.2 is contained in Section 4.1: we show that twisted non-isomorphic Fourier-Mukai partners of $(X, \alpha)$ of order $\kappa$ defines distinct points in the period domain of marked cubic fourfolds of discriminant $d$. In Section 4.2 we give a counting formula for twisted non-isomorphic Fourier-Mukai partners of $(X, \alpha)$ and in Section 4.3 we explicit it for Fourier-Mukai partners of order $\kappa$.

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2 Recollection of results and notation

In this section we recall some results about cubic fourfolds and K3 surfaces we will use in the next.
2.1 Special cubic fourfolds and associated K3 surface

Let $Y$ be a cubic fourfold; by classical results of Hodge theory and classification of quadratic forms, we have that the lattice given by the degree four integral cohomology group $H^4(Y, \mathbb{Z})$, endowed with the intersection form $(\cdot, \cdot)$, is isometric to

$$L := \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 21},$$

which is the unique odd, unimodular lattice of signature $(2, 21)$. If $H$ is the class in $H^2(Y, \mathbb{Z})$ corresponding to a hyperplane in $Y$, then the lattice $L$ contains the class $h := H^2$ such that $(h, h) = -3$. Moreover, the primitive lattice $H^4(Y, \mathbb{Z})_{\text{prim}}$ of $Y$, which is exactly the orthogonal complement of the class $h$ in $L$, is isometric to the lattice

$$L^0 := A_2(-1) \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2};$$

where $U$ denotes the hyperbolic plane, which is the free group $\mathbb{Z}^2$ with intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_8(-1)$ is the unique even, unimodular lattice of signature $(0, 8)$ and

$$A_2(-1) := \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

(see [6], Proposition 2.1.2). Let

$$Q := \{ y \in \mathbb{P}(L^0 \otimes \mathbb{C}) : (y, y) = 0, \langle y, \bar{y} \rangle > 0 \}$$

be the open set of the quadric defined by the intersection form; the choice of a connected component $D'$ of $Q$ determines the local period domain for cubic fourfolds. Let $\Gamma^+$ be the subgroup of the group of automorphism of $L$, preserving the class $h$ and the component $D'$. The global period domain of cubic fourfolds is the quotient

$$D := \Gamma^+ \backslash D'.$$

We denote by $C$ the moduli space of cubic fourfolds and let

$$\tau : C \to D$$

be the period map, which for every $Y$ in $C$ is defined by

$$\tau(Y) = y := [H^{1,1}(Y)].$$

Voisin proved that $\tau$ is an open immersion, i.e. Torelli Theorem holds for cubic fourfolds (see [6], Section 2.2 for more details).

If the cubic fourfold $Y$ is special, then there exists a rank-two, negative definite, saturated sublattice $(K, (\cdot, \cdot))$ of $L \cap H^{2,2}(Y)$, containing the class $h$. This lattice $K$ is a labelling for $Y$ and the discriminant of the pair $(Y, K)$ is the determinant of the intersection matrix of $K$. We will write $K_d$ to underline the fact that the labelling has discriminant $d$. By Hassett’s work, we know that special cubic fourfolds are contained in a countable union of irreducible divisors $C_d$ in the moduli space $C$, where the union runs over the discriminants of every possible labelling $K_d$. We recall the construction of the divisors $C_d$, following [6], Section 3. Let $K_d^0$ be the primitive part of the lattice $K_d$; the set

$$D'_{K_d} := \{ y \in \mathbb{P}(L^0 \otimes \mathbb{C}) : y^\perp \cap K_d^0 \neq 0 \}$$

is the local period domain which parametrizes special Hodge structures with labelling $K_d$. We denote by $\Gamma_{K_d}^+$ the subset of $\Gamma^+$ containing the automorphisms $g$ such that

$$g(K_d) \subseteq K_d.$$
Hassett proved that the quotient
\[ \Gamma_{K_d}^+ \backslash \mathcal{D}_K \rightarrow \mathcal{D} \]
defines an irreducible divisor in the period domain \( \mathcal{D} \). Notice that two labellings \( K \) and \( K' \) defines the same divisor in \( \mathcal{D} \) if and only if there exists an automorphism \( g \) in \( \Gamma^+ \) such that \( K' = g(K) \); moreover, he observed that this condition is equivalent to ask that \( K \) and \( K' \) have the same discriminant. Hence, we write \( \mathcal{D}_d \) to denote the divisor in \( \mathcal{D} \) parametrizing special Hodge structures, having a labelling of discriminant \( d \), modulo the action of \( \Gamma^+ \); finally, we define \( \mathcal{C}_d \) to be the intersection of \( \mathcal{C} \) with \( \mathcal{D}_d \) (through the period map). Moreover, the divisor \( \mathcal{C}_d \) is non empty if and only if
\[ d > 6 \quad \text{and} \quad d \equiv 0, 2 \pmod{6} \]
(see [6], Theorem 1.0.1). We refer to the elements in \( \mathcal{C}_d \) as special cubic fourfolds with discriminant \( d \).

He also gave numerical conditions for the discriminant \( d \) which ensure the existence of an associated K3 surface for special cubic fourfolds.

**Theorem 2.1** ([6], Theorem 1.0.2). Let \( Y \) be a cubic fourfold in \( \mathcal{C}_d \) with labelling \( K_d \). There exist a K3 surface \( X \) with polarization class of degree \( d \) and an isometry of Hodge structures
\[ K_d^+ \cong H^2(X, \mathbb{Z})_{\text{prim}} \]
between the orthogonal sublattice to the lattice \( K_d \) in \( H^4(Y, \mathbb{Z}) \) and the degree two primitive cohomology of the K3 surface, if and only if the discriminant \( d \) of \( K_d \) satisfies the following conditions:
\[ 4 \nmid d, \quad 9 \nmid d, \quad p \nmid d \quad \text{for any odd prime } p \quad \text{such that } p \equiv -1 \pmod{3}. \] (2)

We recall that for any primitive class \( v_Y \) in \( K_d \) for \( d \) such that \( \mathcal{C}_d \) is non empty, the self-intersection of \( v_Y \) is equal to \(-d/3\) if \( d \equiv 0 \pmod{6} \), or to \(-3d\) if \( d \equiv 2 \pmod{6} \), and the primitive embedding of \( v_Y \) in \( H^4(Y, \mathbb{Z})_{\text{prim}} \) is completely determined up to isometry of \( H^4(Y, \mathbb{Z})_{\text{prim}} \) (see [6], Proposition 3.2.2 and Proposition 3.2.5). More precisely, he studied all primitive embeddings of \( v_Y \) in \( L^0 \) and he showed that, up to isometries of \( L^0 \), the class \( v_Y \) has the form
\[ e_1 - (d/6)f_1 \quad \text{if } d \equiv 0 \pmod{6} \]
or
\[ 3(e_1 - ((d - 2)/6)f_1) + \mu_1 - \mu_2 \quad \text{if } d \equiv 2 \pmod{6}, \]
where \( e_1, f_1 \) is the standard basis of the hyperbolic plane and \( \mu_1, \mu_2 \) span the lattice \( A_2(-1) \) (see also [11], Section 2.2). We also point out the following property, which will be useful in the next sections, where it is computed the discriminant group \( d(K_d^+) \) of \( K_d^+ \), endowed with the discriminant form \( q_{K_d^+} \) induced by the intersection form.

**Proposition 2.2** ([6], Proposition 3.2.6). If \( d \equiv 0 \pmod{6} \), then \( d(K_d^+) \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \), which is cyclic unless nine divides \( d \). Furthermore, we may choose this isomorphism so that
\[ q_{K_d^+}((1, 0)) \equiv \frac{2}{3} \pmod{2\mathbb{Z}} \quad \text{and} \quad q_{K_d^+}((1, 0)) \equiv \frac{3}{d} \pmod{2\mathbb{Z}}. \]

If \( d \equiv 2 \pmod{6} \), then \( d(K_d^+) \cong \mathbb{Z}/d\mathbb{Z} \). Furthermore, we may choose a generator \( g \) so that
\[ q_{K_d^+}(g) \equiv \frac{2d - 1}{3d} \pmod{2\mathbb{Z}}. \]
2.2 Immersion into the moduli spaces of K3 surfaces

In [6], Section 5.3, Hassett proved that the existence of an isometry of Hodge structures as in Theorem 2.1 allows an identification between the moduli space of marked special cubic fourfolds of discriminant \( d \) and the moduli space of degree \( d \) polarized K3 surfaces. Let us explain this observation. We fix a labelling \( K_d \) and we write \( \Gamma_d^+ \) to denote the subgroup of the group of automorphisms of \( L \) fixing the class \( h \) and preserving the labelling \( K_d \). Let \( D_d^{\text{lab}} \) be the domain which parametrizes Hodge structures in \( \mathcal{D}' \) with labelling \( K_d \), modulo the action of \( \Gamma_d^+ \). Then, the domain \( D_d^{\text{lab}} \) is birational to \( D_d \) via the map \( D_d^{\text{lab}} \to D \) (see [6], Section 3.1); actually, a general point in \( D_d \) has a unique labelling. We say that \( D_d^{\text{lab}} \) is the domain of labelled special Hodge structures with discriminant \( d \).

Let, now, \( G_d^+ \) be the subgroup of \( \Gamma_d^+ \) of automorphisms acting trivially on \( K_d \). Then, he defines the global period domain of marked special Hodge structures of discriminant \( d \) as the quotient

\[
D_d^{\text{mar}} := G_d^+ \backslash D_d.
\]

In this new space, two cubic fourfolds having the same labelling \( K_d \) which comes from different primitive embeddings in \( H^{2,2}(-) \cap L \) are not identified. The relation between \( D_d^{\text{mar}} \) and \( D_d^{\text{lab}} \) is explained in the following proposition.

**Proposition 2.3** ([6], Proposition 5.3.1). The group \( G_d^+ \) is equal to \( \Gamma_d^+ \) (resp. the group \( G_d^+ \) is an index-two subgroup of \( \Gamma_d^+ \)), if \( d \equiv 2 \pmod{6} \) (resp. if \( d \equiv 0 \pmod{6} \)).

The forgetful map \( D_d^{\text{mar}} \to D_d^{\text{lab}} \) is an isomorphism (resp. a double cover), if \( d \equiv 2 \pmod{6} \) (resp. if \( d \equiv 0 \pmod{6} \)).

**Remark 2.4.** Consider the situation with \( d \equiv 0 \pmod{6} \). Let \( y \) be a point in \( D_d^{\text{lab}} \); by Proposition 2.3 we have that the fiber on \( y \) of the map \( D_d^{\text{mar}} \to D_d^{\text{lab}} \) contains two elements, which we denote by \( y_1 \) and \( y_2 \). Then, the Hodge structures on \( L^0 \) represented by \( y_1 \) and \( y_2 \) are related by the automorphism \( \gamma \) of \( L^0 \), acting as the multiplication by \(-1\) on the two hyperbolic planes and as the identity elsewhere. Indeed, the automorphism \( \gamma \) is an element of \( \Gamma^+ \), but it does not belong to \( G_d^+ \) for these values of the discriminant (see [6], the proof of Proposition 5.3.1).

On the other hand, let us recall the construction of the global period domain for degree \( d \) polarized K3 surfaces. The degree two cohomology group of a K3 surface with the usual intersection form \((,\,\,\,)\) is isometric to the lattice \( \Lambda := E_8(-1)^{\otimes 2} \oplus U^{\otimes 3} \). We denote by \( \Sigma_d \) the group of automorphisms of \( \Lambda \), preserving the element \( l := e_1 + (d/2)f_1 \in U \). Let \( \Lambda_d^0 \) be the orthogonal complement of \( l \) in \( \Lambda \). Then, the quadric

\[
Q_{K3} := \{ x \in \mathbb{P}((\Lambda_d^0 \otimes \mathbb{C}) : (x, x) = 0, (x, \bar{x}) > 0 \}
\]

has two connected components \( \mathcal{N}_d \) and \( \bar{\mathcal{N}}_d \). Let \( \Sigma_d^+ \) be the subgroup of \( \Sigma_d \) preserving the component \( \mathcal{N}_d \); the quotient

\[
\mathcal{N}_d := \Sigma_d^+ \backslash \mathcal{N}_d
\]

is the global period domain for K3 surfaces with polarization class of degree \( d \).

Assume that \( d \) is a positive integer such that the divisor \( C_d \) is non empty and conditions (2) of Theorem 2.1 are satisfied. Then, Hassett proved that an isometry between \( K_d^+ \) and \( \Lambda_d^0 \) induces an isomorphism \( D_d^{\text{mar}} \to \mathcal{N}_d \). Indeed, the choice of such an isometry of lattices induces an isomorphism of the corresponding discriminant groups \( d(K_d^+) \) and \( d(\Lambda_d^0) \), which preserves the \( \mathbb{Q} / 2\mathbb{Z} \) valued discriminant forms on these groups. We denote by \( \text{Iso}(d(K_d^+), d(\Lambda_d^0)) / \{ \pm 1 \} \) the quotient of the set of all such isomorphisms of discriminant groups by the action of the group \( \{ n \in \mathbb{Z} / d\mathbb{Z} : n^2 = 1 \} \). Using [17], Theorem 1.14.2, Hassett verified that every element of this set is induced by an isometry \( K_d^+ \cong \Lambda_d^0 \). In conclusion, we have the following result.
Theorem 2.5 ([3], Theorem 5.3.2, 5.3.3). Let $d$ be a positive integer satisfying conditions (2) of Theorem 2.1 and such that the divisor $\mathcal{C}_d$ is non empty. Then, there exists an isomorphism

$$j_d : D_d^{mar} \to \mathcal{N}_d$$

between the period domain of marked special Hodge structures of discriminant $d$ and the global period domain of degree $d$ polarized K3 surfaces, which is unique up to the choice of an element in

$$\text{Iso}(d(K_d^0), d(\Lambda_d^0))/(\pm 1).$$

2.3 Mukai lattice for $\mathcal{A}_Y$

Let us now consider the categorical framework. We have already observed in Section 1 that the subcategory $\mathcal{A}_Y$ of a cubic fourfold $Y$ behaves in a certain way as the derived category of a K3 surface. In [13], Kuznetsov proved that for certain special cubic fourfolds $Y$, there exist a K3 surface $X$ and an equivalence of categories $\mathcal{A}_Y \xrightarrow{\sim} \mathcal{D}^b(X)$. In general, if this condition is satisfied, we say that $\mathcal{A}_Y$ is geometric. In [1], Addington and Thomas explained the relation between Kuznetsov’s K3 surface and Hassett’s Hodge theoretic associated K3 surface. Indeed, let

$$\tilde{\Lambda} : \mathcal{A}_Y \xrightarrow{\sim} \mathcal{D}^b(X)$$

be the isomorphism which associates to every element in the rational topological K-theory of $Y$ the corresponding Mukai vector in the rational cohomology ring of $Y$ endowed with the Mukai pairing $\langle \cdot, \cdot \rangle$. Using $\tilde{\Lambda}$, they defined a weight two Hodge structure on $\mathcal{M}_Y$, which is unique up to the choice of an element in $\mathcal{A}_Y$. Then, there exist two elements $\lambda_1, \lambda_2$ in $N(\mathcal{A}_Y)$ spanning a rank two sublattice with intersection matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$
Using the previous result, they proved that a cubic fourfold \( Y \) belongs to \( C_d \) with \( d \) satisfying conditions (2) of Theorem 2.1 if and only if the Mukai lattice contains a copy of the hyperbolic plane; in particular, this observation implies that if \( \mathcal{A}_Y \) is geometric, then \( Y \) belongs to some \( C_d \) with \( d \) satisfying (2) (see [1], Theorem 3.1). By deformation theory, they also proved that the converse holds on a Zariski open subset of \( C_d \).

**Theorem 2.7** ([1], Theorem 1.1). If \( \mathcal{A}_Y \) is geometric, then \( Y \) belongs to \( C_d \) for some \( d \) satisfying conditions (2) of Theorem 2.1. Conversely, for each \( d \) satisfying (2), the set of cubic fourfolds \( Y \) in \( C_d \) for which \( \mathcal{A}_Y \) is geometric forms a Zariski open dense subset.

**Remark 2.8.** On the lattice level, we can fix the embedding of \( A_2 \) in \( \tilde{\Lambda} \) given by

\[
A_2 \hookrightarrow U \oplus U \hookrightarrow \tilde{\Lambda}, \quad \lambda_1 \mapsto e_2 + f_2, \lambda_2 \mapsto e_1 + f_1 - e_2,
\]

where \( e_1, f_1 \) (resp. \( e_2, f_2 \)) is the standard base of the first (resp. the second) copy of the hyperbolic plane. As observed in [11], Section 2.1, by [17], Theorem 1.1, we have that this embedding is unique, up to isometries of \( \tilde{\Lambda} \). Moreover, the orthogonal complement of \( A_2 \) with respect to this embedding is

\[
A_2^\perp \cong E_8(1) \oplus U \oplus A_2(-1).
\]

**Remark 2.9.** Since, by definition, the lattice \( A_2 \) is contained in \( N(\mathcal{A}_Y) \), the orthogonality condition implies that \( T(\mathcal{A}_Y) \) is in \( A_2^\perp \). In particular, as observed in [11], Section 3.3, the orthogonal complement to the transcendental lattice in \( A_2^\perp \) is \( N(\mathcal{A}_Y) \cap A_2^\perp \).

In [11], Proposition 3.5, Huybrechts proved that, given two cubic fourfolds \( Y \) and \( Y' \), the existence of a Fourier-Mukai equivalence \( \mathcal{A}_Y \sim \mathcal{A}_{Y'} \) implies the existence of a Hodge isometry of the corresponding Mukai lattices. The surprising fact is that, for general cubic fourfolds in \( C_d \), the category \( \mathcal{A}_Y \) is completely determined by the Hodge structure on \( H(\mathcal{A}_Y, \mathbb{Z}) \).

**Theorem 2.10** ([11], Theorem 1.5, (iii)). For arbitrary \( d \) and general \( Y \) in \( C_d \), there exists a Fourier-Mukai equivalence \( \mathcal{A}_Y \sim \mathcal{A}_{Y'} \) if and only if there exists a Hodge isometry \( \tilde{H}(\mathcal{A}_Y, \mathbb{Z}) \cong \tilde{H}(\mathcal{A}_{Y'}, \mathbb{Z}) \).

### 2.4 Associated twisted K3 surface

In [11], Huybrechts generalized Theorem 2.11 and Theorem 2.12 to the case of cubic fourfolds admitting an associated **twisted** K3 surface. We recall that a twisted K3 surface is the datum of a K3 surface \( X \) and a class in the Brauer group \( H^2(X, O_X^\text{tors}) \) of \( X \). Following [9], Section 2, let \( B \) be a rational class of \( H^2(X, \mathbb{Q}) \), which is sent to \( \alpha \) through the composition

\[
H^2(X, \mathbb{Q}) \to H^2(X, O_X) \xrightarrow{\exp} H^2(X, O_X^\text{tors}).
\]

We say that \( B \) is a B-field lift of \( \alpha \). We denote by \( \check{H}(X, \alpha, \mathbb{Z}) \) the cohomology ring \( H^*(X, \mathbb{Z}) \) with the Mukai pairing and the weight two Hodge structure defined by

\[
\check{H}^{2,0}(X, \alpha) := \exp(B)H^{2,0}(X)
\]

and

\[
\check{H}^{1,1}(X, \alpha) := \exp(B)H^{1,1}(X).
\]

We see that \( \check{H}(X, \alpha, \mathbb{Z}) \) is isomorphic as a lattice to \( \tilde{\Lambda} \) and we call it the Mukai lattice of \( (X, \alpha) \). Notice that, for a different choice of the class \( B \), we obtain isomorphic Hodge structures on the Mukai lattice. We point out that the Hodge structure we have defined is determined by the form
Mukai partners of a twisted K3 surface will be useful in Section 4.

Theorem 2.11 ([11], Theorem 1.3). Let $Y$ be a cubic fourfold. There exist a twisted K3 surface $(X, \alpha)$ and a Hodge isometry $\tilde{H}(A_Y, \mathbb{Z}) \cong \tilde{H}(X, \alpha, \mathbb{Z})$ if and only if $Y$ belongs to $C_d$ for $d$ such that

$$d \equiv 0, 2(\text{mod } 6), d > 6, n_i \equiv 0(\text{mod } 2) \text{ for all primes } p_i \equiv 2(\text{mod } 3) \text{ in } 2d = \prod p_i^{n_i}. \quad (3)$$

On the other hand, if there exists a twisted K3 surface $(X, \alpha)$ such that the category $A_Y$ is equivalent to the derived category $\mathcal{D}^b(X, \alpha)$ of bounded complexes of $\alpha$-twisted coherent sheaves on $X$, then the cubic fourfold $Y$ belongs to $C_d$ for $d$ satisfying conditions (3) of Theorem 2.11; moreover, the converse holds on a Zariski open subset of the divisor $C_d$ (see [11], Theorem 1.4).

2.5 Counting formulas for Fourier-Mukai partners of a K3 surface

The aim of this subsection is to recollect some known formulas which count the number of (twisted) Fourier-Mukai partners of a given (twisted) K3 surface. We recall that a twisted Fourier-Mukai partner of a K3 surface $X$ (resp. a twisted K3 surface $(X, \alpha)$) is a twisted K3 surface $(X', \alpha')$ such that there exists an equivalence of categories $\mathcal{D}^b(X) \cong \mathcal{D}^b(X', \alpha')$ (resp. $\mathcal{D}^b(X, \alpha) \cong \mathcal{D}^b(X', \alpha')$); if the Brauer class $\alpha'$ is trivial, we say that the Fourier-Mukai partner is untwisted.

The first result concerns the number of isomorphism classes of untwisted Fourier-Mukai partners of a general K3 surface, which is determined by the number of distinct primes in the factorization of the degree of the polarization class.

Theorem 2.12 ([18], Proposition 1.10). Let $X$ be a K3 surface with Néron-Severi lattice $\text{NS}(X)$ of rank one generated by a polarization class $l_X$ such that $l_X^2 = 2n$. Let $m$ be the number of (isomorphism classes of) Fourier-Mukai partners of $X$; then we have:

- $m = 1$, if $l_X^2 = 2$ or $l_X^2 = 2^a$,
- $m = 2^{h-1}$, if $l_X^2 = 2^{e_1} \cdots 2^{e_h}$,
- $m = 2^h$, if $l_X^2 = 2^a 2^{e_1} \cdots 2^{e_h}$,

where $a, h$ and the $e_i$'s are natural numbers with $a \geq 2$, the $p_i$'s are different primes such that $p_i \geq 3$.

More generally, Ma proved in [16] a counting formula for isomorphism classes of twisted Fourier-Mukai partners of a twisted K3 surface $(X, \alpha)$ which admits an untwisted Fourier-Mukai partner (see [16], Theorem 1.1). Moreover, relaxing this hypothesis, he obtained an upper bound to the number of Fourier-Mukai partners of $(X, \alpha)$. We conclude this paragraph by resuming Ma’s construction, which will be useful in Section 4.
Let \((X, \alpha)\) be a twisted K3 surface with \(\text{ord}(\alpha) = \kappa\). We recall that a twisted K3 surface \((X', \alpha')\) is isomorphic to \((X, \alpha)\) if there exists an isomorphism \(F : X \cong X'\) such that \(F^* \alpha' = \alpha\). We denote by \(\text{FM}^r(X, \alpha)\) the set of isomorphism classes of Fourier-Mukai partners \((X', \alpha')\) of \((X, \alpha)\), having \(\alpha'\) of order \(r\). We say that \((X_1, \alpha_1)\) and \((X_2, \alpha_2)\) in \(\text{FM}^r(X, \alpha)\) are \(\sim\)-equivalent if there exists a Hodge isometry \(g : T_{X_1} \cong T_{X_2}\) such that \(g^* \alpha_2 = \alpha_1\). We define the quotient

\[
\text{FM}^r(X, \alpha) := \text{FM}^r(X, \alpha) / \sim
\]

and we denote by \(\pi : \text{FM}^r(X, \alpha) \to \text{FM}^r(X, \alpha)\) the quotient map. Let \(I'(d(T(X, \alpha)))\) be the set of all isotropic subgroups of order \(r\) of the discriminant group \((d(T(X, \alpha), q_{T(X, \alpha)})\) of \(T(X, \alpha)\), i.e.

\[
I'(d(T(X, \alpha))) := \{x \in d(T(X, \alpha)) : q_{T(X, \alpha)}(x) = 0 \in \mathbb{Q}/2\mathbb{Z}, \text{ord}(x) = r\}.
\]

We define the map

\[
\mu : \text{FM}^r(X, \alpha) \to \text{O}_{\text{Hdg}}(T(X, \alpha))/I'(d(T(X, \alpha))),
\]

where \(\text{O}_{\text{Hdg}}(T(X, \alpha))\) is the group of Hodge isometries of the generalized transcendental lattice, in the following way. For every \((X_1, \alpha_1)\) in \(\text{FM}^r(X, \alpha)\), there exists a Hodge isometry \(g_1 : T(X_1, \alpha_1) \cong T(X, \alpha)\). Then

\[
g_1^*(T_{X_1}) \cong \frac{T_{X_1}}{T(X_1, \alpha_1)} \cong \frac{\mathbb{Z}}{r\mathbb{Z}},
\]

is an isotropic, cyclic subgroup of \(d(T(X, \alpha))\) of order \(r\). Thus, for every class \([(X_1, \alpha_1)]\) in \(\text{FM}^r(X, \alpha)\), we set

\[
\mu([(X_1, \alpha_1)]) = x := [g_1(\alpha_1^{-1}(1))] \in \text{O}_{\text{Hdg}}(T(X, \alpha))/I'(d(T(X, \alpha))).
\]

We have that:

1. The map \(\mu\) is well-defined and injective (see \([10]\), Lemma 3.2);

2. The image of \(\mu\) is contained in

\[
\text{O}_{\text{Hdg}}(T(X, \alpha))/J'(d(T(X, \alpha))),
\]

where

\[
J'(d(T(X, \alpha))) = \{x \in I'(d(T(X, \alpha))) : \text{there exists a primitive embedding } U \hookrightarrow \langle N(X, \alpha), \lambda(x) \rangle, \text{ for } \lambda : d(T(X, \alpha)) \cong d(N(X, \alpha)) \text{ (see } [10], \text{ Proposition 3.4)}\).

On the other hand, for every \((X_1, \alpha_1)\) in \(\text{FM}^r(X, \alpha)\), we can define a map

\[
\nu : \pi^{-1}(\pi(X_1, \alpha_1)) \to \Gamma(X_1, \alpha_1)^+ \setminus \text{Emb}(U, N(X_1)),
\]

where \(\text{Emb}(U, N(X_1))\) is the set of primitive embeddings of \(U\) in

\[
N(X_1) = H^0(X_1, \mathbb{Z}) \oplus \text{NS}(X_1) \oplus H^2(X_1, \mathbb{Z})
\]

and \(\Gamma(X_1, \alpha_1)^+\) is the set of orientation-preserving isometries of \(N(X_1)\), which come from isometries of \(T_{X_1}\) fixing \(\alpha_1\). Indeed, for every \((X_2, \alpha_2)\) in the fiber of \(\pi\) over \((X_1, \alpha_1)\), there exists a Hodge isometry \(g : T_{X_2} \cong T_{X_1}\) such that \(g^* \alpha_2 = \alpha_2\). By \([17]\), Theorem 1.14.4, the isometry \(g\) extends to an isometry

\[
\phi : \tilde{H}(X_2, \mathbb{Z}) \cong \tilde{H}(X_1, \mathbb{Z}).
\]

Since the restriction \(\phi|_{N(X_2)}\) is orientation-preserving, we obtain the primitive embedding

\[
\varphi : U \cong H^0(X_2, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z}) \xrightarrow{\partial} N(X_1).
\]

Thus, we set \(\nu(X_2, \alpha_2) = [\varphi]\). We have that:
1. The map $\nu$ is injective (see [16], Lemma 3.5);

2. The map $\nu$ is surjective if and only if the Căldărașu’s Conjecture holds (see [16], Remark 3.7).

We recall the statement of Căldărașu’s Conjecture, which was proposed for the first time in [4], Conjecture 5.5.5.

**Conjecture 2.13** ([16], Question 3.8). Let $(X, \alpha)$ be a twisted K3 surface. For each untwisted Fourier-Mukai partner $X'$ of $X$ and each Hodge isometry $g : T_{X'} \cong T_X$, the twisted K3 surface $(X', g^*\alpha)$ is a Fourier-Mukai partner of $(X, \alpha)$.

**Remark 2.14.** We point out that Conjecture 2.13 is related to another conjecture due to Căldărașu, which asks whether two twisted K3 surfaces having Hodge isometric twisted transcendental lattices are Fourier-Mukai partners. This conjecture is known to be false in general by [9], Example 4.11.

To state Ma’s formula, we need to introduce some notations. For every $x$ in $\Gamma(d(T(X, \alpha)))$, we define the overlattice $T_x := \langle x, T(X, \alpha) \rangle$ of $T(X, \alpha)$ and the morphism

$$\alpha_x : T_x \twoheadrightarrow \frac{T_x}{T(X, \alpha)} \cong \langle x \rangle \cong \frac{\mathbb{Z}}{r\mathbb{Z}}.$$  

For a pair $(x, M)$ such that $\langle \lambda(x), N(X, \alpha) \rangle \cong U \oplus M$, we define the number

$$\tau(x, M) := \#(O_{\text{Hdg}}(T_x, \alpha_x) \backslash O(d(M))/O(M)),$$

where $O_{\text{Hdg}}(T_x, \alpha_x)$ is the set of Hodge isometries $g$ of $T_x$, such that $g^*\alpha_x = \alpha_x$. For a natural number $r$, we define

$$\varepsilon(r) = \begin{cases} 1, \text{ if } r = 1, 2 \\ 2, \text{ if } r \leq 3. \end{cases}$$

Finally, if $G(L)$ is the genus of a lattice $L$, $O(L)_0$ is the kernel of the map $r_L : O(L) \rightarrow O(d(L))$ and $O(L)_0^+$ is the subgroup of $O(L)_0$ of orientation-preserving isometries, we define the subsets

$$G_1(L) := \{L' \in G(L) : O(L')_0^+ \neq O(L'_0)\},$$

$$G_2(L) := \{L' \in G(L) : O(L')_0^+ = O(L'_0)\}.$$

Using the previous observations, Ma proved that the following inequality holds.

**Theorem 2.15** ([16], Proposition 4.3). We have the inequality

$$\# \text{FM}^r(X, \alpha) \leq \sum_x \left\{ \sum_M \tau(x, M) + \varepsilon(r) \sum_{M'} \tau(x, M') \right\}. \tag{6}$$

**Here:**

- $x$ runs over the set $O_{\text{Hdg}}(T(X, \alpha)) \backslash J^r(d(T(X, \alpha)))$;
- the lattices $M$ and $M'$ run over the sets $G_1(M_\varphi)$, $G_2(M_\varphi)$ respectively, where $M_\varphi$ is a lattice satisfying $\langle \lambda(x), N(X, \alpha) \rangle \cong U \oplus M_\varphi$. 

11
3 Construction of the examples (untwisted case)

The aim of this section is to prove Theorem 1.1. In the first two paragraphs we exhibit some preliminary computations on the level of period domain of (marked) cubic fourfolds, while in the last section we provide the proof of the theorem.

3.1 Some remarks about general special cubic fourfolds

Let us consider the case of a general cubic fourfold $Y$ in $C_d$, i.e. such that the rank of $H^{2,2}(Y, \mathbb{Z})$ is equal to two. We denote by $K_d$ the labelling with discriminant $d$ of $Y$, containing the class $h$ with self-intersection $-3$, and let $v_Y$ be a primitive class in $K_d$.

First of all, we observe that the orthogonal complement of the generalized transcendental lattice $T(A_Y)$ in $A^1_d$ is the rank one sublattice generated by the element $v_Y := v^{-1}_M(v_Y)$. Indeed, as observed in Remark 2.9, it is equal to the intersection $N(A_Y) \cap A^1_d$. Then, the isometry of Hodge structures

$$v_M : A^1_d \cong H^4(Y, \mathbb{Z})_{\text{prim}}$$

of Proposition 2.6 induced by the Mukai vector, yields the isometries

$$N(A_Y) \cap A^1_d \cong H^{2,2}(Y, \mathbb{Z}) \cap H^4(Y, \mathbb{Z})_{\text{prim}} \cong \mathbb{Z} v_Y.$$

Moreover, the primitive embedding of $v_Y$ in $A^1_d$ is uniquely determined up to isometry of $A^1_d$, since it holds for $v_Y$ in $H^4(Y, \mathbb{Z})_{\text{prim}}$ (see Section 2.1).

As a consequence, still by Proposition 2.6 we have an isometry of Hodge structures

$$T(A_Y) = \langle \lambda_1, \lambda_2, v_Y \rangle^\perp \cong \langle h, v_Y \rangle^\perp \cong K_d^\perp,$$  \hspace{1cm} (7)

induced by $v_M$, between the generalized transcendental lattice of $Y$ and the orthogonal complement of $K_d$ in $H^4(Y, \mathbb{Z})_{\text{prim}}$.

In addition, let us assume that $Y$ verifies one of these two conditions:

(a) the cubic fourfold $Y$ is contained in $C_d$, where $d$ is a positive integer verifying conditions (2) of Theorem 2.1 which guarantee the existence of an associated K3 surface $X$ of degree $d$;

(a') the cubic fourfold $Y$ belongs to $C_d$ for $d$ satisfying conditions (3) of Theorem 2.1, which guarantee the existence of an associated twisted K3 surface $(X, \alpha)$ with $\kappa := \text{ord}(\alpha)$.

As observed in 11, Section 2.5, the fact that $Y$ is general in $C_d$ implies that also the K3 surface $X$ determines a general point in the period domain of polarized K3 surfaces. Indeed, if (a') holds (the argument is the same in the untwisted case), we have that $\tilde{H}^{1,1}(A_Y, \mathbb{Z}) \cong \tilde{H}^{1,1}(X, \alpha, \mathbb{Z})$ has rank three. Since it has the same rank of $\tilde{H}^{1,1}(X, \mathbb{Z}) \cong H^{1,1}(X, \mathbb{Z}) \oplus U$, we deduce that the Néron-Severi group $\text{NS}(X)$ of $X$ has rank one. Thus, if $l_X$ denotes a polarization class for $X$, then $\text{NS}(X) = \mathbb{Z} l_X$.

Notice that if $Y$ satisfies (a), then the class $l_X$ has self-intersection $d$ and the transcendental lattice of $X$, which we denote by $T_X$, is exactly the degree two primitive lattice $H^2(X, \mathbb{Z})_{\text{prim}}$. Moreover, composing the isomorphism (7) with the isometry of Hodge structures

$$\psi : K_d^\perp \cong H^2(X, \mathbb{Z})_{\text{prim}}$$  \hspace{1cm} (8)

given by Theorem 2.1 we get the isometry of Hodge structures

$$\varphi : T(A_Y) \cong H^2(X, \mathbb{Z})_{\text{prim}} = T_X.$$  \hspace{1cm} (9)
On the other hand, we can make a similar argument when \((a')\) holds. Indeed, a Hodge isometry
\[
\phi : \tilde{H}(A_Y, \mathbb{Z}) \cong \tilde{H}(X, \alpha, \mathbb{Z}),
\]
which exists under our hypotheses, restricts to the Hodge isometry
\[
\phi_T : T(A_Y) \cong T(X, \alpha),
\]
(10)
between the generalized transcendental lattices. Moreover, if \(c := 2n\) is the self-intersection of \(l_X\), then the K3 surface \(X\) defines a general point \(x\) in the local period domain \(N^*_c\) of degree \(c\) polarized K3 Hodge structures. Finally, by definition, we have
\[
|d(T(X, \alpha))| = \text{ord}(\alpha)^2 \cdot |d(T_X)|,
\]
i.e. \(d = \kappa^2 \cdot c\). 

Now, assume that condition \((a)\) holds and let \(m\) be the number of isomorphism classes of untwisted Fourier-Mukai partners of \(X\). We fix a representative for each class of isomorphism and we denote them by \(X_1, \ldots, X_m\), choosing \(X_1 := X\). By [19], Theorem 3.3, this is equivalent to ask that, for every index \(2 \leq k \leq m\), there exists a Hodge isometry
\[
g_k : T_X \cong T_{X_k}
\]
among the transcendental lattices of the K3 surfaces. Notice that, by the counting formula of Theorem 2.12, we have that \(m\) depends on the number of distinct odd primes in the prime factorization of \(d\).

Moreover, since the K3 surface \(X\) has Néron-Severi lattice of rank one, the same property has to hold for its Fourier-Mukai partners. Thus, for every \(k = 2, \ldots, m\), there exists a class \(l_k\) in the Néron-Severi lattice of \(X_k\) such that \(\text{NS}(X_k) = \mathbb{Z} l_k\) and the self-intersection of \(l_k\) is still equal to \(d\). We denote by \(x_k\) the point in the local period domain \(N^*_d\), which is determined by the Hodge structure on the transcendental lattices of the K3 surfaces \(X_k\). These points also descend to different points in the global period domain \(N^*_d\), since they come from non-isomorphic K3 surfaces.

### 3.2 Some preliminary computations

Let \(Y\) be a general special cubic fourfold satisfying condition \((a)\). The strategy to produce our examples is to use the untwisted Fourier-Mukai partners \(X_k\)'s of \(X\) to determine different points in the period domain \(D_d\) of special cubic fourfolds with discriminant \(d\), which are good candidates to be Fourier-Mukai partners for \(Y\). Hence, we need to be under the hypothesis that \(X\) admits non-trivial Fourier-Mukai partners. By Theorem 2.12 it is equivalent to ask that \(d\) satisfies the following condition:

\((b)\) there exist an integer \(h > 1\) and some different prime numbers \(p_1, \ldots, p_h \geq 3\) such that \(d = 2p_1^{e_1} \cdots p_h^{e_h}\), for some positive integers \(e_1, \ldots, e_h\).

**Example 3.1.** Notice that the case \(d = 2^a p_1^{e_1} \cdots p_h^{e_h}\), for an integer \(a > 1\), cannot happen, because, for such a value of the discriminant, the cubic fourfold \(Y\) does not admit an associated K3 surface. We also observe that there exist positive integers satisfying conditions \((a)\) and \((b)\): for example, we can choose \(d = 42\), which satisfies \(d \equiv 0\mod 6\), or \(d = 182\), if we want \(d \equiv 2\mod 6\): in both cases, the associated K3 surface \(X\) admits one Fourier-Mukai partner, which is not isomorphic to \(X\).

We observe that the isometry \(\varphi\) of [19], between the generalized transcendental lattice of \(Y\) and the transcendental lattice of the associated K3 surface \(X\), induces an isomorphism
\[
j' : D'_d \rightarrow N^*_d
\]
between the local period domains: as verified in the proof of Theorem 5.3.2 of [6], the isomorphism $j'$ descends to an isomorphism
\[ j : D_d^{mar} \to \mathcal{N}_d \]
among the period domains of marked special Hodge structure of discriminant $d$ and degree $d$ polarized K3 surfaces, respectively. Actually, the isomorphism
\[ \varphi : d(T(A_Y)) \to d(T_X) \]
defined by $\varphi$ over the discriminant groups, respecting the $\mathbb{Q}/2\mathbb{Z}$-valued discriminant quadratic forms, determines an element of the set $\text{Iso}(d(T(A_Y)), d(T_X))/\langle \pm 1 \rangle$, which, by Theorem 2.5, corresponds to the datum of an isomorphism of period domains as above.

For every $1 \leq k \leq m$, we denote by $y_k$ the preimage of $x_k$, which is the point in $\mathcal{N}_d'$ corresponding to $X_k$, with respect to $j'$: by definition, the point $y_k$ parametrizes a special Hodge structure with labelling of discriminant $d$ on $A_2^k$. Explicitly, the point $y_k$ correspond to the following data:

1. A sublattice $T_k$ of $A_2^k$ equipped with a weight two Hodge structure, denoted by
   \[ T_k \otimes \mathbb{C} = V^{3,1} \oplus W \oplus V^{1,3}; \]

2. An isometry $\varphi_k : T_{X_k} \cong T_k$, induced by the isomorphism of discriminant groups $\varphi$, such that
   \[ V^{3,1} := (\varphi_k)_{\mathbb{C}}(H^{2,0}(X_k)) \]
   and
   \[ W := (\varphi_k)_{\mathbb{C}}(H^{1,1}(X_k) \cap (T_{X_k} \otimes \mathbb{C})). \]

Indeed, the transcendental lattice $T_{X_k}$ is isometric to a sublattice of $A_2^k$, via an isometry induced by the isomorphism $\varphi$ of discriminant groups. So, for every $1 \leq k \leq m$, there exists a class $v_k$ in $A_2^k$ with $(v_k, v_k) = (v_Y, v_Y)$ and such that its orthogonal complement in $A_2^k$ is $T_k$. We denote by $K_2^d$ the rank three negative definite sublattice of $A$ given by (the saturation of)
\[ A_2 \oplus \mathbb{Z}v_k. \]

Then, a point $y_k$ corresponds to the datum of the Hodge structure on $\tilde{A}$ with $V^{3,1}$ as $(2,0)$ part and
\[ V^{2,2} := (K_2^d \otimes \mathbb{C}) \oplus W, \]
as $(1,1)$ part.

In conclusion, we have fixed $m$ points $y_1, \ldots, y_m$ in the local period domain $D_d'$, which parametrize general special Hodge structures on $A_2^k$ with labelling of discriminant $d$.

**Remark 3.2.** Notice that the point $y_1$ represents the special Hodge structure $y$ on $A_2^k$ of the cubic fourfold $Y$ from which we started. Indeed, by definition of $j'$, we have that
\[ j'(y) = x = j'(y_1), \]
since $x_1 := x$ is the Hodge structure on $H^2(X, \mathbb{Z})$.

**Remark 3.3.** We observe that, by definition as we recalled in Section 2.1, points in $D_d'$ represents special Hodge structures on $L^0$ with labelling of discriminant $d$. Thus, we have that, for every $1 \leq k \leq m$, the point $y_k$ determines the Hodge structure on $L = \mathbb{Z}^{B_2} \oplus \mathbb{Z}(-1)^{B_21}$ given by
\[ V^{3,1} = (\varphi_k)_{\mathbb{C}}(H^{2,0}(X_k)). \]
and
\[ V^{2,2} := \mathbb{C} h_k \oplus \mathbb{C} v_k \oplus (\varphi_k)_{\mathbb{C}} (H^{1,1}(X_k) \cap (T_{X_k} \otimes \mathbb{C})); \]

here, \( h_k \) is a class in \( L \) of self-intersection \(-3\) and \( v_k \) is an element in \( L \) with the same self-intersection as the primitive vector \( v_Y \) of \( Y \). In particular, the point \( \gamma_k \) parametrizes the special Hodge structure on \( L^0 \) with respect to the labelling \( K_2 = \mathbb{Z} h_k \oplus \mathbb{Z} v_k \). In this way, the map \( j' \) is determined by the isomorphism induced by the map \( \psi : K_2 \cong H^2(X, \mathbb{Z})_{\text{prim}} \) over the discriminant groups. The point is that this construction is equivalent to the previous one on \( A_2^+ \). Indeed, by composing with the isometry \( v_M \), sending every element in the topological K-theory of \( Y \) to the corresponding Mukai vector, we get the isometry \( \varphi \).

Now, we observe that the \( m \) points \( y_1, \ldots, y_m \), determines distinct points in the period domain \( \mathcal{D}_d^{\text{mar}} \). Indeed, the map \( j : \mathcal{D}_d^{\text{mar}} \rightarrow \mathcal{N}_d \) is an isomorphism such that
\[ j(y_k) = x_k. \]

The points \( x_1, \ldots, x_m \) are distinct K3 Hodge structures in \( \mathcal{N}_d \); hence, the points \( y_1, \ldots, y_m \) cannot coincide, because \( j \) is an isomorphism.

Let us see the points \( y_1, \ldots, y_m \) in the global period domain \( \mathcal{D}_d^{\text{lab}} \), defined as the quotient of \( \mathcal{D}'_d \) by the action of the group \( \Gamma_d^2 \) of the automorphisms of \( \Lambda \), preserving \( A_2 \), the connected component \( \mathcal{D}_d' \) and the labelling \( K_2 \). These points could be identified in this domain. By the way, we observe that, if some of them are not identified in \( \mathcal{D}_d^{\text{lab}} \), then they correspond to distinct points in the global period domain \( \mathcal{D}_d \). Indeed, the map sending \( \mathcal{D}_d^{\text{lab}} \) in \( \mathcal{D}_d \), which forgets the labelling, is an isomorphism on general points of \( \mathcal{D}_d \). This is a consequence of the fact that, for general points in \( \mathcal{D}_d \), the integral \((2,2)\) part of the Hodge structure has rank two and, then, the labelling is unique (see Section 2.2).

In particular, it is enough to study the behavior of the forgetful map \( \rho : \mathcal{D}_d^{\text{mar}} \rightarrow \mathcal{D}_d^{\text{lab}} \) over the \( y_1, \ldots, y_m \), to understand how many of them define different general special Hodge structures of discriminant \( d \). According to Proposition 2.26 we have to distinguish two cases depending on the value of the discriminant.

**Case** \( d \equiv 2(\text{mod} \ 6) \): by Theorem 2.25 we have that the map \( \rho \) sending \( \mathcal{D}_d^{\text{mar}} \) in \( \mathcal{D}_d^{\text{lab}} \) is an isomorphism. Hence, the \( y_1, \ldots, y_m \) are not identified by the action of \( \Gamma_d^2 \) and they determine \( m \) distinct general special Hodge structures of discriminant \( d \).

**Case** \( d \equiv 0(\text{mod} \ 6) \): by Theorem 2.25 the map \( \rho \) from the period domain of marked special Hodge structures of discriminant \( d \) is a double cover for the domain of labelled ones. Thus, it is possible that there exist two indexes \( 1 \leq k_1 \neq k_2 \leq m \) such that \( y_{k_1} \) and \( y_{k_2} \) belong to the fiber of a point \( y_{k_1,2} \) in \( \mathcal{D}_d^{\text{lab}} \) with respect to \( \rho \). As recalled in Remark 2.24 this is equivalent to ask that the diagram
\[ \begin{array}{ccc}
T_{k_1} & \longrightarrow & A_2^+ \\
\equiv & \downarrow \gamma & \\
T_{k_2} & \longrightarrow & A_2^+ 
\end{array} \]

(11)

commutes, i.e. the isometry of Hodge structures between \( T_{k_1} \) and \( T_{k_2} \) extends to the automorphism \( \gamma \) of \( A_2^+ \). Moreover, since, by definition, the lattice \( T_{k_i} \) is Hodge isometric to the trascendental lattice \( T_{X_{k_i}} \) of the K3 surface \( X_{k_i} \) for every \( i = 1, 2 \), we have that \( \gamma \) induces an isometry of Hodge structures between \( T_{X_{k_1}} \) and \( T_{X_{k_2}} \), which we denote by \( \gamma' \). The isometry \( \gamma' \) does not extend to an automorphism of the whole \( \Lambda \), as we prove in the next lemma.

**Lemma 3.4.** Let us suppose that there exist two indexes \( 1 \leq k_1 \neq k_2 \leq m \) such that the points \( y_{k_1} \) and \( y_{k_2} \) belong to the same fiber with respect to the forgetful map \( \mathcal{D}_d^{\text{mar}} \rightarrow \mathcal{D}_d^{\text{lab}} \). Then, the K3 surfaces
$X_{k_1}$ and $X_{k_2}$, corresponding, respectively, to the points $x_{k_1} = j(y_{k_1})$ and $x_{k_2} = j(y_{k_2})$ in $\mathcal{N}_d$, are not isomorphic.

Proof. Keeping the notations used above, let $T_{k_i}$ be the sublattice of $A_2^1$ carrying the Hodge structure represented by the point $y_{k_i}$, for every $i = 1, 2$; by hypothesis, the lattices $T_{k_1}$, $T_{k_2}$ sit in the diagram [11].

First of all, we prove that the isometry $\gamma'$ does not extend to an automorphism of $\Lambda$. Indeed, let $\tilde{\gamma}'$ be the isomorphism over the discriminant groups induced by $\gamma'$, which respects the discriminant quadratic forms. By construction, we have

$$\gamma' := (\varphi_{k_i})^{-1} \circ \gamma \circ \varphi_{k_i};$$

thus, passing to the discriminant groups, we have the following commutative diagram:

$$\begin{array}{ccc}
d(T_{X_{k_1}}) & \xrightarrow{\varphi_{k_1}} & d(T_{k_1}) \\
\downarrow{\gamma'} & & \downarrow{\tilde{\gamma}} \\
d(T_{X_{k_2}}) & \xrightarrow{\varphi_{k_2}} & d(T_{k_2})
\end{array} \tag{12}$$

where $\varphi_{k_i}$ denotes the isomorphism of discriminant groups (respecting the discriminant quadratic forms) induced by the isometry $\varphi_{k_i}$ for every $i = 1, 2$.

Since the lattice $\Lambda$ is unimodular, we have that $d(T_{X_{k_i}})$ is isomorphic to the discriminant group of the Néron-Severi lattice $d(\text{NS}(X_{k_i}))$ for every $i = 1, 2$ (see [17], Proposition 1.6.1). As a consequence, there exists an induced isomorphism

$$\tilde{\gamma}'_N : d(\text{NS}(X_{k_1})) \to d(\text{NS}(X_{k_2}))$$

between the discriminant groups of the Néron-Severi lattices of the K3 surfaces. Now, by [17], Proposition 1.5.2, we have that the isometry $\gamma'$ extends to the whole $\Lambda$ if and only if the isomorphism $\tilde{\gamma}'_N$ comes from an isometry of the form $\text{NS}(X_{k_1}) \cong \text{NS}(X_{k_2})$. But, we recall that

$$\text{NS}(X_{k_i}) = \mathbb{Z} l_{k_i}$$

for every $i = 1, 2$. Hence, there exist only two isometries between $\text{NS}(X_{k_1})$ and $\text{NS}(X_{k_2})$, defined by sending the polarization class $l_{k_1}$ to $l_{k_2}$ (resp. to $-l_{k_2}$).

Let us suppose that the isomorphism of discriminant groups $\tilde{\gamma}'_N$ comes from one of these two isometries. Then, it has to act as the multiplication by $1$ or $-1$ on the generators of the discriminant groups. Consequently, we have that also $\tilde{\gamma}'$ acts as the multiplication by $\pm 1$ on the discriminant groups of the transcendental lattices of the K3 surfaces $X_{k_i}$’s. Furthermore, by diagram [12], we deduce that the same property holds for the isomorphism $\tilde{\gamma} : d(T_{k_1}) \cong d(T_{k_2})$.

Now, we recall that, for every $i = 1, 2$, the lattice $T_{k_i}$ is isometric to the orthogonal complement in $L$ of a labelling $K_{d_i}^{K_d}$ (see [8]). Thus, the induced isomorphism between the discriminant groups $d(K_{d_i}^{K_d})$’s acts as the multiplication by $\pm 1$ on the generators. Composing with the canonical isomorphism of $d(K_{d_i}^{K_d})$ with $d(K_{d_i}^{K_d})$, we obtain that the isomorphism of discriminant groups induced by $\gamma$ between $d(K_{d_i}^{K_d})$ and $d(K_{d_i}^{K_d})$ acts as the multiplication by $\pm 1$ on the generators. However, this observation yields a contradiction with the definition of $\gamma$ (actually, the isometry $\gamma$ sends $h_{k_1}$ to $h_{k_2}$ and $v_{k_1}$ to $-v_{k_2}$, see Remark 2.4). Thus, we deduce that the isomorphism $\tilde{\gamma}'_N$ does not arise from an isometry $\text{NS}(X_{k_1}) \cong \text{NS}(X_{k_2})$ and, hence, the isometry $\gamma'$ does not extend to an isometry of $\Lambda$, as we stated.

Finally, we observe that it cannot exist an isometry between the cohomology groups $H^2(X_{k_1}, \mathbb{Z})$ and $H^2(X_{k_2}, \mathbb{Z})$, because it should be an extension of $\gamma'$. Hence, by Torelli Theorem for K3 surfaces, we deduce that the K3 surfaces $X_{k_1}$ and $X_{k_2}$ are not isomorphic, as we wanted.
Anyway, the fibers of the map $\rho$ contain at most two points. Hence, we deduce that our points $y_1, \ldots, y_m$ descend to at least $m/2$ different Hodge structures in $\mathcal{D}_d$ (actually, the integer $m > 1$ is even by the counting formula of Theorem 2.1.

On the other hand, we observe that if $T$ is a sublattice of $\Lambda$ which is Hodge isometric to $T_X$, then the lattice $\gamma(T)$, with the Hodge structure induced by that one on $T$ through $\gamma_C$, still satisfies the same property. As a consequence, we obtain the following fact.

**Lemma 3.5.** If $X_T$ is a K3 surface with trascendental lattice $T$, which is a Fourier-Mukai partner of $X$, then the K3 surface $X_{\gamma(T)}$ with trascendental lattice $\gamma(T)$ is a Fourier-Mukai partner of $X$.

Let us denote by $y_T$ (resp. $y_{\gamma(T)}$) the point in $\mathcal{D}_d^{\text{mar}}$ such that $j(y_T) = x_T$ (resp. $j(y_{\gamma(T)}) = x_{\gamma(T)}$), where $x_T$ (resp. $x_{\gamma(T)}$) denotes the point in the period domain $\mathcal{N}_d$ corresponding to $X_T$ (resp. to $X_{\gamma(T)}$). By definition, the points $y_T$ and $y_{\gamma(T)}$ belong to the same fiber of the forgetful map $\rho$. By Lemma 3.4, we have that $x_T$ and $x_{\gamma(T)}$ are different points in $\mathcal{N}_d$; since $j$ is an isomorphism, we get that also the points $y_T$ and $y_{\gamma(T)}$ are distinct in $\mathcal{D}_d^{\text{mar}}$. Moreover, by Lemma 3.5 there exists an index $1 \leq k \leq m$ such that $X_{\gamma(T)} \cong X_k$, where $X_k$ is a Fourier-Mukai partner of $X$. Thus, we have that the points $x_k$ and $x_{\gamma(T)}$ coincide in $\mathcal{N}_d$: we obtain that $y_k$ is identified to the point $y_{\gamma(T)}$, because they are preimage of the same point by the isomorphism $j$. In other words, we have proved that, if $y_T$ is one of the $m$ points we found as preimage by $j$ of the Fourier-Mukai partners of $X$, then the point belonging to the same fiber of $y_T$ with respect to the double cover $\rho$, is distinct from $y_T$ and it appears among the $m$ points $y_1, \ldots, y_m$.

In conclusion, we have shown that the $m$ points $y_1, \ldots, y_m$ determine $m/2$ different special Hodge structures of discriminant $d$, because, passing to the period domain $\mathcal{D}_d^{\text{lab}}$, for every index $1 \leq k_1 \leq m$, the point $y_{k_1}$ is identified with a unique point $y_{k}$, corresponding to an index $1 \leq k_2 \leq m$ such that $k_2 \neq k_1$ and representing the marked Hodge structure in the same fiber of $y_{k_1}$. Moreover, in order to have $m/2 > 0$, we need to assume that the factorization of $d/2$ is given by at least three different prime numbers; hence, we add this condition to hypothesis (b) if $d \equiv 0 (\text{mod} \ 6)$:

(b') there exist an integer $h > 2$ and some different prime numbers $p_1, \ldots, p_h \geq 3$ such that $d = 2p_1^{e_1} \ldots p_h^{e_h}$, for some positive integers $e_1, \ldots, e_h$.

We will use condition (b') if $d \equiv 0 (\text{mod} \ 6)$; instead, we will assume condition (b), if $d \equiv 2 (\text{mod} \ 6)$.

**Remark 3.6.** Notice that there exist values for the discriminant $d$ satisfying condition (a) and (b'), when $d \equiv 0 (\text{mod} \ 6)$. For example, we can take $d = 546$; in this case, the associated K3 surface admits $m = 2^3 - 1 = 4$ Fourier-Mukai partners. In particular, they induce two special Hodge structures with discriminant $d$: one of them represents the isomorphism class of the cubic fourfold $Y$ and the other determines a special Hodge structure, which is not isomorphic to that of $Y$.

3.3 Proof of Theorem 1.1

Let $Y$ be a general cubic fourfold in $\mathcal{C}_d$ satisfying condition (a). Let us assume that:

- the discriminant $d$ verifies condition (b), if $d \equiv 2 (\text{mod} \ 6)$;
- the discriminant $d$ verifies condition (b'), if $d \equiv 0 (\text{mod} \ 6)$.

We denote by $p$ the number of distinct points in $\mathcal{D}_d$ which are image via $\rho$ of the points $y_1, \ldots, y_m$ we found in Section 3.2. By the previous argument, if $d \equiv 2 (\text{mod} \ 6)$, then $p$ is equal to $m$; else, if $d \equiv 0 (\text{mod} \ 6)$, then $p$ is equal to $m/2$.

Now, we would like to understand if for every $k = 1, \ldots, p$, there exists a cubic fourfold $Y_k \in \mathcal{C}_d$ such that $T_k$ is the generalized trascendental lattice $T(\mathcal{A}_{Y_k})$ naturally associated to $Y_k$. Equivalently, we
Section 3, this condition is satisfied if and only if \( y_k \) does not define a Hodge structure of discriminant 2 or 6, i.e., the point \( y_k \) is not contained neither in \( D_2 \) nor in \( D_6 \). But, by construction, the integral (2, 2) part of the Hodge structure defined by \( y_k \) is given by the lattice \( K_d^\text{prim} \), which has discriminant \( d \).

Since the divisor \( C_d \) is not empty (it contains the cubic fourfold \( Y \) we chose since the very beginning), the discriminant \( d \) is not equal to 2 and to 6. We conclude that there exist \( p \) general special cubic fourfolds \( Y_1, \ldots, Y_p \in C_d \) such that

\[
T(\mathcal{A}_{Y_k}) = T_k \quad \text{for every } k = 1, \ldots, p.
\]

We observe that for the index \( k = 1 \), the cubic fourfold \( Y_1 \) is isomorphic to \( Y \); indeed, by Remark 3.2 they satisfy

\[
T(\mathcal{A}_Y) = T_1 = T(\mathcal{A}_{Y_k}),
\]

which means that \( y_1 = y \) in \( D_d \). Since the period map \( \tau \) is an open immersion (by Torelli Theorem for cubic fourfolds, see Section 2.1), we must have \( Y \cong Y_1 \). For the same reason, the cubic fourfolds \( Y_1, \ldots, Y_p \) are not isomorphic to each other.

The cubic fourfolds \( Y_1, \ldots, Y_p \) are good candidates to be Fourier-Mukai partners of \( Y \). Indeed, for every \( 2 \leq k \leq p \), we have an isometry of Hodge structures

\[
T(\mathcal{A}_Y) \cong T_X \cong T_{X_k} \cong T(\mathcal{A}_{Y_k}). \tag{13}
\]

Moreover, these lattices have signature (2, 19) and cyclic discriminant groups for the considered values of the discriminant by Proposition 2.22. By [17], Theorem 1.14.4, a lattice satisfying these conditions has a unique primitive embedding in the Mukai lattice \( \Lambda \). As a consequence, we deduce that, for every \( 2 \leq k \leq p \), the isometry (13) extends to an isometry

\[
\tilde{H}(\mathcal{A}_Y, \mathbb{Z}) \cong \tilde{H}(\mathcal{A}_{Y_k}, \mathbb{Z})
\]

which preserves the Hodge structures. By Theorem 2.10, the existence of such an isometry of Hodge structures implies the existence of a Fourier-Mukai equivalence between \( \mathcal{A}_Y \) and \( \mathcal{A}_{Y_k} \). In conclusion, we have proved the following property.

**Proposition 3.7.** Let \( d \) be a positive integer such that \( d > 6, d \equiv 0, 2 \pmod{6} \) and satisfying conditions (2) of Theorem 2.1. Let us assume that \( d \) satisfies condition (b) (resp. condition (b')) in the case that \( d \equiv 2 \pmod{6} \) (resp. in the case that \( d \equiv 0 \pmod{6} \)). Then, for every general \( Y \) in \( C_d \), the cubic fourfolds \( Y_2, \ldots, Y_p \), constructed above, are non-isomorphic Fourier-Mukai partners of \( Y \), with \( p > 1 \).

We conclude this section, by showing that, for a general special cubic fourfold \( Y \) in \( C_d \) with associated K3 surface \( X \), the number of Fourier-Mukai partners of \( X \) is an upper bound for the number of Fourier-Mukai partners for \( Y \). Together with Proposition 3.7 this result gives the proof of Theorem 1.1. On the other hand, it represents an alternative way to prove the finiteness result of [11], Corollary 3.6, under the previous hypotheses.

**Proposition 3.8.** Let \( Y \) be a general cubic fourfold in \( C_d \) satisfying conditions (2) of Theorem 2.1. If the associated K3 surface \( X \) admits \( m \) (possibly equal to one) Fourier-Mukai partners, then, the cubic fourfold \( Y \) cannot have more than \( m \) (resp. \( \lfloor m/2 \rfloor \)) Fourier-Mukai partners if \( d \equiv 2 \pmod{6} \) (resp. if \( d \equiv 0 \pmod{6} \)).

**Proof.** Let \( Y' \) be a Fourier-Mukai partner of \( Y \); we denote by \( y' \) a point in \( D_d^\text{mar} \) identifying the weight two Hodge structure on the primitive cohomology of \( Y \). We denote by \( x' \) the image of \( y' \) via the map \( j : D_d^\text{mar} \to \mathcal{N}_d \), which defines a general K3 surface \( X' \) with polarization of degree \( d \). By definition of \( j \), we have that there exists an isometry of Hodge structures

\[
K_d^{\text{prim}} \cong H^2(X', \mathbb{Z})_{\text{prim}}.
\]

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where $K'_d$ denotes the labelling of discriminant $d$ of $Y'$.

Now, we observe that by [11], Proposition 3.5, an equivalence $\mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_{Y'}$ of Fourier-Mukai type, induces an isometry of Hodge structures
\[
\tilde{H}(\mathcal{A}_Y, \mathbb{Z}) \cong \tilde{H}(\mathcal{A}_{Y'}, \mathbb{Z})
\]
between the Mukai lattices of $Y$ and $Y'$, respectively. Hence, there exists an induced Hodge isometry on the transcendental lattices
\[
T(\mathcal{A}_Y) \cong T(\mathcal{A}_{Y'}).
\]
Moreover, the same argument applied at the beginning of this section shows the existence of a Hodge isometry $T(\mathcal{A}_{Y'}) \cong T_X$ among the generalized transcendental lattice and the transcendental lattice of $X'$. As a consequence, we have the isometry of Hodge structures $T_X \cong T_{X'}$. This means that $X'$ is a Fourier-Mukai partner of $X$, thanks to [19], Theorem 3.3.

Keeping the notations of the previous part of this section, let $X := X_1, \ldots, X_m$ be all the Fourier-Mukai partners of $X$, up to isomorphism. Hence, there exists an index $k \in \{1, \ldots, m\}$ such that $X'$ is (isomorphic to) $X_k$. Passing to the period domain $\mathcal{N}_d$, we have that the points $x'$ and $x_k$, which correspond respectively to $X'$ and $X_k$, are identified.

Let $Y_k$ be the special cubic fourfold with discriminant $d$ represented in $\mathcal{D}_d^{\text{mar}}$ by the point $y_k$ such that
\[
j(y_k) = x_k,
\]
which exists since $j$ is an isomorphism. Since, by construction, the preimage of $x' = x_k$ is $y'$, the fact that $j$ is an isomorphism implies that
\[
y_k = y' \in \mathcal{D}_d^{\text{mar}}.
\]
In particular, they represent the same point in $\mathcal{D}_d$: by the Torelli Theorem for cubic fourfolds, we conclude that $Y'$ is isomorphic to $Y_k$.

In conclusion, we have proved that, fixing an associated K3 surface for $Y$, every Fourier-Mukai partner of $Y$ is a cubic fourfold with associated K3 surface given by a Fourier-Mukai partner of $X$, and that Fourier-Mukai partners of $Y$ having the same associated K3 surface, are isomorphic as marked cubic fourfolds. Therefore, the cubic fourfold $Y$ can have at most $m$ Fourier-Mukai partners if $d \equiv 2(\text{mod } 6)$. Otherwise, if $d \equiv 0(\text{mod } 6)$, we have at most $m$ Fourier-Mukai partner of $Y$, which are non-isomorphic as marked cubic fourfolds. Hence, by Lemmas 3.4 and 3.5 and Proposition 2.3, we conclude that the number of Fourier-Mukai partners of $Y$ is at most $[m/2]$, as we stated.

**Remark 3.9.** Notice that, to prove these results, we have fixed an associated K3 surface to $Y$ and, consequently, an isomorphism between the period domains $\mathcal{D}_d^{\text{mar}}$ and $\mathcal{N}_d$. Actually, we could choose a Fourier-Mukai partner of $X$ as fixed associated K3 surface to $Y$: this would have given a different isomorphism $\tilde{j}$ on the level of period domains and a different identification of Fourier-Mukai partners of $Y$ with Fourier-Mukai partners of $X$. Anyway, the considerations about the number of Fourier-Mukai partners hold in the same way (see [7], Remark 27).

Using the counting formula in Theorem 2.12 we rewrite the statement of Theorem 1.1 in a more explicit way.

**Corollary 3.10.** Let $d$ be a positive integer such that $d > 6$, $d \equiv 0, 2(\text{mod } 6)$ and satisfying conditions 2 of Theorem 2.1. Let us suppose that the prime factorization of $d$ has $h > 1$ (resp. $h > 2$) distinct odd primes if $d \equiv 2(\text{mod } 6)$ (resp. if $d \equiv 0(\text{mod } 6)$). Then, the general cubic fourfold in $\mathcal{C}_d$ admits $2^{h-1}$ (resp. $2^{h-2}$) Fourier-Mukai partners.
4 Construction of the examples (twisted case)

This section is devoted to the proof of Theorem 1.2. In particular, in Section 4.2 and 4.3 we describe the lower bound to the number of Fourier-Mukai partners of a cubic fourfold $Y$ as in Theorem 1.2, in terms of the number of primes in the prime factorization of the discriminant of $Y$ and the Euler function evaluated in the order of the associated twisted K3 surface.

4.1 Proof of Theorem 1.2

Keeping the notations of Section 3.1, let $Y$ be a general special cubic fourfold satisfying condition $(a')$. Let us make the following hypothesis:

(c) $9$ does not divide the discriminant $d$.

We recall that $T(A_Y)$ is Hodge isometric to the orthogonal complement $K_d$ of the labelling of $Y$ (see (7)); in particular, they have isometric discriminant groups. Hence, by Proposition 2.2 and hypothesis (c), we deduce that the discriminant group $d(T(A_Y))$ is cyclic. Notice that the same property holds for the discriminant group of $T(X, \alpha)$, because they are Hodge isometric under $\phi_T$ (see (10)).

The strategy to prove the result is quite similar to that one used for the untwisted case. Assume that $\alpha$ has order $\kappa$. Let $m_1$ be the cardinality of the set $\text{FM}^{\kappa}(X, \alpha)$. We consider the representatives $p_X, \alpha_1, \ldots, p_X, \alpha_{m_1}$ of the $m_1$ isomorphism classes of twisted Fourier-Mukai partners of order $\kappa$ of $(X, \alpha)$. These $m_1$ twisted K3 surfaces actually define general points in the local period domain of discriminant $d$ cubic fourfolds. Then, we prove that these points are distinct in the period domain of marked Hodge structures of discriminant $d$, thanks to condition (c).

In order to see this, let us fix an index $2 \leq k \leq m'$ and we consider the twisted K3 surface $(X_k, \alpha_k) = (X', \alpha')$. We denote by $\varphi'$ the point in the quadric

$$\tilde{Q} := \{ y \in \mathbb{P}(\tilde{\Lambda} \otimes \mathbb{C}) : (y, y) = 0, (y, \bar{y}) > 0 \},$$

defined by the Hodge structure $\tilde{H}(X', \alpha', \mathbb{Z})$. By [9], Proposition 4.3, the isometry

$$\phi_H : \tilde{H}(X, \alpha, \mathbb{Z}) \cong \tilde{H}(X', \alpha', \mathbb{Z})$$

induced by the equivalence of categories preserves the Hodge structures. In particular, the restriction

$$T(X, \alpha) \cong T(X', \alpha')$$

of $\phi_H$ is an isometry of Hodge structures, which can be used to construct the primitive embedding

$$T(X', \alpha') \cong T(X, \alpha) \cong T(A_Y) \hookrightarrow A_2^1.$$

In particular, we deduce that the point in the quadric

$$\tilde{Q} := \{ y \in \mathbb{P}(\tilde{\Lambda} \otimes \mathbb{C}) : (y, y) = 0, (y, \bar{y}) > 0 \},$$

defined by the Hodge structure $\tilde{H}(X', \alpha', \mathbb{Z})$, belongs to the local period domain $D_d^\prime$. By [15], we conclude that there exists a cubic fourfold $Y'$ in $C_d^\prime$ such that $\tilde{H}(A_Y, \mathbb{Z}) \cong \tilde{H}(A_{Y'}, \mathbb{Z})$. By construction, the cubic fourfolds $Y$ and $Y'$ have Hodge isometric transcendental lattices. We are ready to prove the following result.

**Lemma 4.1.** The cubic fourfolds $Y$ and $Y'$ are not isomorphic as marked cubic fourfolds.
Proof. We will actually prove that if $Y$ and $Y'$ are isomorphic as marked cubic fourfolds, then the twisted K3 surfaces $(X, \alpha)$ and $(X', \alpha')$ are isomorphic, in contradiction with our assumption.

We denote by $y$ and $y'$ the points in the local period domain $D^\alpha_d$ representing the Hodge structures associated to $Y$ and $Y'$, respectively. If $y$ and $y'$ are the same point in the period domain $D^\alpha_d^\text{mar}$, then there exists an isometry of Hodge structures

$$ \eta : T(X, \alpha) \cong T(X', \alpha'), $$

such that the induced isomorphism $\bar{\eta}$ between the discriminant groups $d(T(X, \alpha))$ and $d(T(X', \alpha'))$ is trivial. More precisely, there exists a lattice $T$, which is Hodge isometric to $T(X, \alpha)$ and $T(X', \alpha')$, such that the map $\eta_T$, which sits in the diagram

$$ T(X, \alpha) \xrightarrow{\eta} T(X', \alpha') $$

acts as the identity on the discriminant group $d(T)$.

First of all, we prove that the Hodge isometry $\eta$ extends to a Hodge isometry of the transcendental lattices $T_X$ and $T_{X'}$. Indeed, we set

$$ H = \frac{T_X}{T} \quad \text{and} \quad H' = \frac{T_{X'}}{T}, $$

which are cyclic subgroups of $d(T)$ of order $\kappa$. We recall that a cyclic group of order $d$ admits a unique cyclic subgroup of order $o$ for every natural number $o$ dividing $d$. Thus, we have that $H$ and $H'$ are the same subgroup, because they have the same order. Moreover, if $\bar{\eta}_T$ denotes the automorphism of $d(T)$ induced by $\eta_T$, then

$$ \bar{\eta}_T(H) = \text{id}_{d(T)}(H) = H. $$

By [17], Proposition 1.4.2, we conclude that the isometry $\eta_T$ extends to an isometry $g : T_X \cong T_{X'}$. By construction, the isometry $g$ preserves the Hodge structures on $T_X$ and $T_{X'}$. If we define the embeddings $i$ and $i'$ as the compositions

$$ i : T \cong T(X, \alpha) \to T_X \cong S $$

and

$$ i' : T \cong T(X', \alpha') \to T_{X'} \cong S, $$

then these maps sit in the commutative diagram

$$ T \xrightarrow{i} S \xrightarrow{g_S} S $$

$$ T \xrightarrow{i'} S \xrightarrow{g_{S'}} S $$

where $g_S$ is the isometry induced by $g$ via the identification of $T_X$ and $T_{X'}$ with a lattice $S$.

Secondly, we prove that, if the isomorphism $\eta_T$, induced by $\eta_T$ on $d(T)$, acts as the identity, then also $g_S$, induced by $g_S$, is the identity on $d(S)$. Indeed, let us denote by $g_S^\vee$ (resp. $\eta_T^\vee$) the extension of $g_S$ to $S^\vee$ (resp. of $\eta_T$) to $T^\vee$. We recall that $g_S^\vee$ and $\eta_T^\vee$ are defined by the precomposition with $g_S$ and $\eta_T$, respectively, and they make the diagram

$$ T \xrightarrow{i} S \xrightarrow{g_S} S^\vee \xrightarrow{g_S^\vee} T^\vee $$

$$ T \xrightarrow{i'} S \xrightarrow{g_{S'}} S^\vee \xrightarrow{\eta_T^\vee} T^\vee $$

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to commute. Next, we observe that we have the isomorphisms of groups

$$r: \frac{S^\vee}{i(T)}_H \cong \frac{S^\vee \circ i(T)}{S} \quad \text{and} \quad r': \frac{S^\vee / i'(T)}{H'} \cong \frac{S^\vee}{S},$$

where \( H = S/i(T) \) and \( H' = S/i'(T) \). We claim that the isomorphism

$$\tilde{g}: \frac{S^\vee / i(T)}{H} \cong \frac{S^\vee / i'(T)}{H'},$$

induced by \( g\xi \), is identified with \( \tilde{g} \) via the isomorphisms \( r \) and \( r' \). Indeed, we have that \( g\xi = \eta_T|_{S^\vee} \) induces the isomorphism

$$\tilde{g}: \frac{S^\vee}{i(T)} \to \frac{S^\vee}{i'(T)},$$

which is actually the restriction of \( \eta_T \) to \( S^\vee / T \). Now, we denote by \( \pi \) and \( \pi' \) the quotient maps

$$\pi: \frac{S^\vee}{i(T)} \to \frac{S^\vee / i(T)}{H}, \quad \pi': \frac{S^\vee}{i'(T)} \to \frac{S^\vee / i'(T)}{H'}.$$

We recall that \( \eta_T \) sends the subgroup \( H \) to \( H' \), because \( \eta_T \) extends to \( gS \) (see [17], Proposition 1.4.2). Thus, the isomorphism \( \tilde{g} \), defined by \( \tilde{g} \) passing to the quotient, is well defined, because

$$\pi'(\tilde{g}(H)) = \pi'(\eta_T(H)) = \pi'(H') = 0.$$

We have that the diagram

$$\begin{array}{ccc}
\frac{S^\vee / i(T)}{H} & \xrightarrow{\tilde{g}} & \frac{S^\vee / i'(T)}{H'} \\
\downarrow r & & \downarrow r' \\
\frac{S^\vee}{S} & \xrightarrow{\tilde{g}S} & \frac{S^\vee}{S} \\
\end{array}$$

commutes, because the maps \( \tilde{g} \) and \( \tilde{g}S \) are induced by the same map \( g\xi \). Now, we observe that \( \tilde{g} \) acts as the identity, since it is induced by \( \eta_T|_{(S^\vee / T)} \) which is the identity map by our hypothesis. Since the diagram (4.1) commutes, we conclude that also \( \tilde{g}S \) acts as the identity map, as we stated.

Finally, we denote by \( \mathbb{Z}l \) rank one lattice which is the orthogonal complement of \( S \) in \( \Lambda \). Since \( d(S) \cong d(\mathbb{Z}l) \) by [17], Proposition 1.6.1, we have that \( \tilde{g}S \) acts trivially also on the discriminant group \( d(\mathbb{Z}l) \). Thus, by [17], Proposition 1.5.2, we conclude that the isometry \( gS \) extends to an isometry \( f_\Lambda \) of \( \Lambda \) and, therefore, the isometry \( g \) extends to \( f: H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z}) \). Furthermore, the restriction of \( f_\Lambda \) to \( \mathbb{Z}l \) is the identity, because by construction it induces the identity on the discriminant group of \( \mathbb{Z}l \). In particular, we deduce that the isometry \( f \) preserves the ample cones of \( X \) and \( X' \). By Torelli Theorem, we have that there exists an isomorphism \( F \) between the K3 surfaces \( X' \) and \( X \) such that \( F^* = f \). Since, by definition, the isometry \( f \) sends the class \( \alpha \) to \( \alpha' \), we conclude that \((X, \alpha)\) and \((X', \alpha')\) are isomorphic as twisted K3 surfaces, in contradiction with our assumption. Therefore, we conclude that \( y \) and \( y' \) are not the same point in \( D_d^\text{mar} \), as we wanted.

As a consequence, we have that the twisted K3 surfaces \((X_1, \alpha_1), \ldots, (X_{m'}, \alpha_{m'})\) determine \( m' \) distinct marked cubic forfolds \( Y_1 = Y, \ldots, Y_{m'} \) in \( C^\text{mar}_d \), corresponding to \( m' \) distinct Hodge structures in \( D_d^\text{mar} \). Indeed, the argument in the proof of Lemma 4.1 holds for every couple of cubic fourfolds in the previous list.

As in the untwisted case, the proof of Theorem 1.2 follows from Proposition 2.3 and Theorem 2.10.
Remark 4.2. Notice that it is necessary to assume that $\text{ord}(\alpha) = \text{ord}(\alpha')$, in order to extend the isometry $\eta$ to the transcendental lattices. Indeed, if this condition is not satisfied, then the discriminant groups of the transcendental lattices of $X$ and $X'$ could not be isomorphic. Actually, we can prove that Lemma 4.1 does not hold in general without this assumption, by giving a counterexample.

We set $d = 2 \cdot 13^2$, which is congruent to 2 modulo 6 and let $Y$ be a general cubic fourfold in $C_d$. Since $d$ satisfies conditions (2) of Theorem 2.1 there exists a K3 surface $X$, which is associated to $Y$. We denote by $j$ the isomorphism $D^n_d \cong \mathcal{N}_d$ induced by this association on the period domains. We observe that, by [10], Proposition 5.1, there exist $\varphi(13) \cdot 2^{-1} = 6$ isomorphism classes of Fourier-Mukai partners of order 13 of $X$. Hence, we denote by $(X_1, \alpha_1)$ one of this Fourier-Mukai partner, we know to exist. We observe that the point $y$ defined by the Hodge structure on $\hat{H}(X_1, \alpha_1, \mathbb{Z})$, belongs to the quadric $Q$ of [1], because this holds for $\hat{H}(X, \mathbb{Z})$ and there exists an isometry of Hodge structures $H(X, \mathbb{Z}) \cong \hat{H}(X_1, \alpha_1, \mathbb{Z})$. In particular, it defines a point $\varphi_1 =: y_1$ in $D^n_d$. Using [15], there exists a cubic fourfold $Y_1$ in $C_d$, which corresponds to this point via the period map. Now, we point out that the point $y_1$ have to be sent by $j$ in a point $x'$ of $\mathcal{N}_d$, corresponding to an untwisted Fourier-Mukai partner $X'$ of $X$. But, by the counting formula of Theorem 2.12 the K3 surface $X$ admits $2^0 = 1$ isomorphism class of Fourier-Mukai partners. Thus, we have that $X$ is isomorphic to $X'$. Since $j$ is an isomorphism and the period map for cubic fourfolds is injective, we conclude that $Y$ and $Y_1$ are isomorphic as marked cubics. On the other hand, the K3 surfaces $X$ and $(X_1, \alpha_1)$ cannot clearly be isomorphic.

As a consequence, we deduce that, given two cubic fourfolds with associated twisted K3 surface, which are isomorphic as marked cubics, is it not true in general that the associated twisted K3 surfaces are isomorphic. This prevent us to have a well-defined map between $D^n_d$ and the period domain of generalized Calabi-Yau structures of hyperkähler type (see [8] for the definition) and to generalize Theorem 5.3.2 and 5.3.3 of [4] to the twisted case.

4.2 Ma’s formula in our setting

In Section 2.5, we recalled that formula (6) gives an upper bound to the number of isomorphism classes of twisted Fourier-Mukai partners with Brauer class of a fixed order $r$ for a twisted K3 surface $(X, \alpha)$. The aim of this paragraph is to prove that if we consider a general cubic fourfold $Y$ in $C_d$ satisfying condition (a') and (c), then formula (6) gives precisely the number of elements in the set $FM'(X, \alpha)$. The key point of the proof is the fact that the Caldararu’s Conjecture 2.13 holds in this particular case.

Proposition 4.3. Let $(X, \alpha)$ be a twisted K3 surface such that there exist a special cubic fourfold $Y$ of discriminant $d$ and a Hodge isometry

$$\hat{H}(X, \alpha, \mathbb{Z}) \cong \hat{H}(\mathcal{A}_Y, \mathbb{Z})$$

between the Mukai lattices of $(X, \alpha)$ and $\mathcal{A}_Y$. If $X$ is generic and $9$ does not divide the discriminant $d$, then the number of (isomorphism classes of) Fourier-Mukai partners of $(X, \alpha)$ of order $r$ is given by formula (6).

Proof. Firstly, we observe that the Caldararu’s Conjecture 2.13 holds under our assumptions for every Fourier-Mukai partner $(X_1, \alpha_1)$ of $(X, \alpha)$. More precisely, we prove that if a K3 surface $X'_1$ has the transcendental lattice $T_{X'_1}$, Hodge isometric to $T_{X_1}$ via $g_1$, then the twisted K3 surface $(X'_1, \alpha'_1 := g_1^* \alpha_1)$ is a Fourier-Mukai partner of $(X_1, \alpha_1)$. Indeed, the isometry $g_1$ restricts to the isometry of Hodge structures

$$f := (g_1)|_{T(X_1, \alpha_1)} : T(X'_1, \alpha'_1) \cong T(X_1, \alpha_1).$$

Notice that there exists a Hodge isometry $T(X, \alpha) \cong T(X_1, \alpha_1)$, because the twisted K3 surface $(X_1, \alpha_1)$ belongs to $FM'(X, \alpha)$, where $r := \text{ord}(\alpha_1)$; therefore, the discriminant group $d(T(X_1, \alpha_1))$ is
still cyclic. Thus, by [17], Theorem 1.14.1, the isometry $f$ extends to an isometry of Hodge structures

$$\phi : \tilde{H}(X_1', \alpha_1', \mathbb{Z}) \cong \tilde{H}(X_1, \alpha_1, \mathbb{Z}).$$

Finally, since we have the primitive embedding

$$T(X_1, \alpha_1) \cong T(X, \alpha) \to \mathbb{A}_2^\perp,$$

the Hodge structure on $\tilde{H}(X_1, \alpha_1, \mathbb{Z})$ defines a point in the quadric $Q$ defined in (1). By [11], Lemma 2.3, we know that every Hodge structure on $\Lambda$ determined by a point in $Q$ admits a Hodge isometry that reverses any given orientation of the four positive directions. As a consequence, up to composing with this isometry, we can assume that $\phi_1$ is orientation-preserving: by [10], Theorem 0.1, we conclude that there exists an equivalence of categories $D^b(X_1', \alpha_1') \cong D^b(X_1, \alpha_1)$. In particular, we obtain that the map $\nu$ of (3) is bijective.

To conclude the proof, we show that the map $\mu$ of (1) is surjective on $O_{\text{Hdg}}(T(X, \alpha)) \setminus J^r(d(T(X, \alpha)))$; in particular, this implies that we have an equality in formula (5).

Let $x$ be in $J^r(d(T(X, \alpha)))$; by definition, $x$ is an element of $I^r(d(T(X, \alpha)))$ such that there exists a primitive embedding

$$\varphi : U \to \tilde{M}_x,$$

where

$$\tilde{M}_x := \langle \lambda(x), N(X, \alpha) \rangle \subset N(X, \alpha)$$

is an overlattice of $N(X, \alpha)$. By [17], Proposition 1.4.1, we have that

$$d(\tilde{M}_x) \cong d(\langle \lambda(x) \rangle^\perp / \langle \lambda(x) \rangle) \cong d(x^\perp / \langle x \rangle) \cong d(T_x).$$

Thus, by [17], Proposition 1.6.1, we have an embedding $\tilde{M}_x \oplus T_x \hookrightarrow \tilde{\Lambda}$, with $\tilde{M}_x$ and $T_x$ both embedded primitively. We define the lattice

$$\Lambda_\varphi := \varphi(U)^\perp \cap \tilde{\Lambda},$$

which is isometric to the K3 lattice $\Lambda$, with the Hodge structure induced from $T_x$. By the surjectivity of the period map, there exist a K3 surface $X_\varphi$ and a Hodge isometry

$$h : H^2(X, \mathbb{Z}) \cong \Lambda_\varphi.$$ 

We denote by $\alpha_\varphi$ the composition $\alpha_x \circ (h|_{T_{X_\varphi}})^{-1}$; then, we obtain a twisted K3 surface $(X_\varphi, \alpha_\varphi)$.

Now, we observe that the map $h$ induces the isometry

$$f : T(X_\varphi, \alpha_\varphi) = \ker \alpha_\varphi \cong \ker \alpha_x = T(X, \alpha).$$

Moreover, since $d(T(X, \alpha))$ is a cyclic group, we conclude that $f$ extends to a Hodge isometry

$$\tilde{f} : \tilde{H}(X_\varphi, \alpha_\varphi, \mathbb{Z}) \cong \tilde{H}(X, \alpha, \mathbb{Z}).$$

By [11], Lemma 2.3, there exists an isometry of $\tilde{\Lambda}$ that reverses the given orientation. Thus, we can assume that $\tilde{f}$ is orientation-preserving. By [10], Theorem 0.1, we conclude that $(X_\varphi, \alpha_\varphi)$ belongs to $\text{FM}^r(X, \alpha)$. By construction, we have that $\mu((X_\varphi, \alpha_\varphi)) = [x]$.

Finally, we observe that if $x$ and $x'$ in $J^r(d(T(X, \alpha)))$ are in the same orbit for the action of $O_{\text{Hdg}}(T(X, \alpha))$, then the twisted K3 surfaces $(X_\varphi, \alpha_\varphi)$ and $(X'_\varphi, \alpha'_\varphi)$ such that $\mu((X_\varphi, \alpha_\varphi)) = [x]$ and $\mu((X'_\varphi, \alpha'_\varphi)) = [x']$ are $\sim$-equivalent. Indeed, by hypothesis, there exists a Hodge isometry $\eta$ of $T(X, \alpha)$ which induces an isomorphism $\tilde{\eta}$ on $d(T(X, \alpha))$ such that $\tilde{\eta}(x) = x'$. Then, by [17], Proposition 1.4.2, the overlattices $\langle x, T(X, \alpha) \rangle \cong T_{X_\varphi}$ and $\langle x', T(X, \alpha) \rangle \cong T_{X'_\varphi}$ are isomorphic. Moreover, this isomorphism sends $\alpha_\varphi$ to $\alpha'_\varphi$, because it is an extension of $\eta$; this observation completes the proof of the Proposition.
Remark 4.4. We point out that the result of Proposition 4.3 still holds without the assumptions that the K3 surface, or equivalently the cubic fourfold is generic, and that 9 does not divide $d$. Indeed, by Proposition 2.6, we still have the isometry of Hodge structures

$$T(X, \alpha) \cong T(A_Y) \cong K_d^\perp.$$  

In particular, the discriminant of $T(X, \alpha)$ is computed in Proposition 2.2 as a consequence, we have that

$$\text{rk}(\tilde{H}(X, \alpha, \mathbb{Z})) - \text{rk}(T(X, \alpha)) \geq 4 \geq l(d(T(A_Y))) + 2.$$  

Thus, by [17], Theorem 1.14.4, we deduce that any isometry of the form $T(X, \alpha) \cong T(X_1, \alpha_1)$ extends to the whole Mukai lattice. In particular, the Căldăraru’s Conjecture holds under our assumptions. Following the same steps of the proof of Proposition 4.3, we obtain the result. Actually, it is not useful for the construction of non-isomorphic Fourier-Mukai partners for cubic fourfolds, even in the case that 9 does not divide $d$. Indeed, we cannot control the passage from $C_d^\mathrm{ab}$ to $C_d$; finally, the counting formulas for (twisted) Fourier-Mukai partners of K3 surfaces are not explicit in the non-generic case.

4.3 Application of Proposition 4.3

Let $Y$ be a general special cubic fourfold of discriminant $d$ satisfying conditions (a’) and (c). By Proposition 4.3, we have that the number of isomorphism classes of Fourier-Mukai partners of order $\kappa$ of $(X, \alpha)$ is

$$m' = \sum_{x} \left\{ \sum_{M} \tau(x, M) + \varepsilon(x) \sum_{M'} \tau(x, M') \right\}.$$  

Here $x$ runs over the set $O_{\text{Hdg}}(T(X, \alpha)) \setminus J^\kappa(d(T(X, \alpha)))$ and the lattices $M$ and $M'$ run over the sets $\mathcal{G}_1(M_\varphi)$, $\mathcal{G}_2(M_\varphi)$ respectively, where $M_\varphi$ is a lattice satisfying $\langle \lambda(x), N(X, \alpha) \rangle \cong U \oplus M_\varphi$ (see formula (a) and Section 2.5). Let us write $m'$ in a more explicit way, in order to find numerical conditions on $d$ and $\kappa$, which guarantee the existence of non isomorphic Fourier-Mukai partners for $Y$. We consider only the case $\kappa \geq 2$, because we have already treated the untwisted case in Section 3.

Lemma 4.5. Let $g$ be a generator of the cyclic group $d(T(X, \alpha))$ of order $d$. Then

$$I^\kappa(d(T(X, \alpha))) = \{ (a\kappa c)g : a \in (\mathbb{Z}/\kappa\mathbb{Z})^\times \}.$$  

Proof. We observe that every element of the form $x = (a\kappa c)g$ with $a \in (\mathbb{Z}/\kappa\mathbb{Z})^\times$ belongs to $I^\kappa(d(T(X, \alpha)))$. Indeed, by Proposition 2.2, we can choose the generator $g$ such that

$$q_{T(X, \alpha)}(g) = \begin{cases} \frac{2d-1}{6} \pmod{2\mathbb{Z}} & \text{if } d \equiv 2 \pmod{6}, \\ \frac{2d-9}{3d} \pmod{2\mathbb{Z}} & \text{if } d \equiv 0 \pmod{6}. \end{cases}$$  

In particular, we have that

$$q_{T(X, \alpha)}((a\kappa c)g) = \begin{cases} a^2 \kappa^2 c^2 \frac{2d-1}{3d} = a^2 \kappa^2 c^2 \frac{2d-1}{3d} \in 2\mathbb{Z} & \text{if } d \equiv 2 \pmod{6}, \\ a^2 \kappa^2 c^2 \frac{2d-9}{3d} = a^2 \kappa^2 c^2 \frac{2d-9}{3d} \in 2\mathbb{Z} & \text{if } d \equiv 0 \pmod{6}. \end{cases}$$  

Finally, it is easy to verify that $(a\kappa c)g$ has order $\kappa$. On the other hand, if $x$ belongs to $I^\kappa(d(T(X, \alpha)))$, then $x = (n)g$ for $n$ in $(\mathbb{Z}/\kappa\mathbb{Z})^\times$. Indeed, the elements of $I^\kappa(d(T(X, \alpha)))$ are all the possible generators of the unique subgroup of order $\kappa$ of $d(T(X, \alpha)) \cong \mathbb{Z}/d\mathbb{Z}$.  

For every $x = (a\kappa c)g$ in $I^\kappa(d(T(X, \alpha)))$, we set

$$\tilde{M}_x := \langle \lambda(x), N(X, \alpha) \rangle.$$
and

$$H_x := \frac{\tilde{M}_x}{N(X, \alpha)}.$$  

We point out that

$$J^\kappa(d(T(X, \alpha))) = \{ x \in I^\kappa(d(T(X, \alpha))) : \tilde{M}_x \cong U \oplus \mathbb{Z} l \text{ with } l^2 = c \}.$$  

Indeed, let \((X_\alpha, \alpha_x)\) be the twisted K3 surface such that \(\mu([(X_\alpha, \alpha_x)]) = [x]\). Then, by definition, we have that

$$N(X_\alpha) \cong \langle \lambda(x), N(X, \alpha) \rangle \text{ and } T_{X_\alpha} \cong \langle x, T(X, \alpha) \rangle.$$  

Since \(T(X_\alpha, \alpha_x) \cong T(X, \alpha)\), we have that

$$d = |d(T(X_\alpha, \alpha_x))| = \text{ord}(\alpha_x^2|d(T_{X_\alpha})) = \kappa^2|d(T_{X_\alpha})|,$$

which implies that

$$d(\tilde{M}_x) \cong d(T_{X_\alpha}) \cong \mathbb{Z}/c\mathbb{Z}.$$  

On the other hand, the opposite inclusion follows from the definition of \(J^\kappa(d(T(X, \alpha)))\).

**Lemma 4.6.** Every element \(x \in I^\kappa(d(T(X, \alpha)))\) belongs to \(J^\kappa(d(T(X, \alpha)))\).

**Proof.** Let \(\bar{x} = (\bar{a}c \kappa)g\) be the image via \(\mu\) of the isomorphism class of the K3 surface \((X, \alpha)\), with \(\bar{a}\) in \((\mathbb{Z}/\kappa\mathbb{Z})^\times\). By definition, we have that

$$U \oplus \mathbb{Z} l \cong N(X) \cong \langle \lambda(\bar{x}), N(X, \alpha) \rangle,$$

with \(l^2 = c\); in particular, the lattice \(U \oplus \mathbb{Z} l\) is an overlattice of \(N(X, \alpha)\). Let \(x = (ac \kappa)g\) be an element in \(I^\kappa(d(T(X, \alpha)))\). Since the groups \(H_{x\bar{}}\) and \(H_x\) are cyclic subgroups of \(d(N(X, \alpha))\) of the same order, they are the same subgroup. Since by \([17]\), Theorem 1.4.1, there is a bijection between overlattices of \(N(X, \alpha)\) and isotropic subgroups of \(d(T(X, \alpha))\), we conclude that the overlattices \(U \oplus \mathbb{Z} l\) and \(\tilde{M}_x\) are isomorphic. In particular, the element \(x\) is in \(J^\kappa(d(T(X, \alpha)))\). \(\square\)

In the next lemma, we show that the cardinality of the set of Hodge isometries of \(T(X, \alpha)\) is two. We point out that this result holds for all generic twisted K3 surfaces and the proof is completely analogous to that of \([13]\), Lemma 4.1.

**Lemma 4.7.** We have that

$$\text{O}_{\text{Hdg}}(T(X, \alpha)) = \{ \pm \text{id} \}.$$  

**Proposition 4.8.** We have that

$$m' := \text{FM}^\kappa(X, \alpha) = \begin{cases} \varphi(\kappa)^{2h-2} & \text{if } \kappa > 2 \text{ and } c = 2 \\ \varphi(\kappa)^{2h-1} & \text{if } \kappa = 2 \text{ or } c > 2 \end{cases},$$

where \(h\) is the number of distinct prime factors in the prime factorization of \(c/2\) if \(c > 2\) and \(h = 1\) if \(c = 2\).

**Proof.** We observe that the three lemmas of this subsection imply that

$$\#(\text{O}_{\text{Hdg}}(T(X, \alpha)) \setminus J^\kappa(d(T(X, \alpha)))) = \begin{cases} 1 & \text{if } \kappa = 2, \\ \frac{1}{2} \varphi(\kappa) & \text{if } \kappa > 2 \end{cases}.$$  

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where \( \varphi \) denotes the Euler function. On the other hand, as we have already observed, the only lattice \( M_{\psi} \) such that \( \tilde{M}_{\psi} \cong U \oplus M_{\psi} \) is \( \mathbb{Z} l \) with \( l^2 = c \). Thus, our computation is actually the same used in \cite{16}, to prove Proposition 5.1. Indeed, using the notations introduced in Section 2.5, we have that

\[
G(\mathbb{Z}l) = \{ \mathbb{Z}l \} = \begin{cases} 
G_1(\mathbb{Z}l) & \text{if } c = 2, \\
G_2(\mathbb{Z}l) & \text{if } c > 2.
\end{cases}
\]

Moreover, we notice that

\[
O(\mathbb{Z}l) = \{ \pm \text{id} \}
\]

and

\[
O(d(\mathbb{Z}l)) = \begin{cases} 
\{ \text{id} \} & \text{if } c = 2, \\
\left( \frac{Z}{2Z} \right)^h & \text{if } c > 2.
\end{cases}
\]

In particular, the order of the set \( O(d(\mathbb{Z}l)) \) is \( 2^h \) if \( c > 2 \). Finally, we observe that

\[
O_{\text{Hdg}}(T_x, \alpha_x) = \begin{cases} 
\{ \pm \text{id} \} & \text{if } \kappa = 2, \\
\{ \text{id} \} & \text{if } \kappa > 2.
\end{cases}
\]

So, if \( \kappa > 2 \), then

\[
m' = \begin{cases} 
\frac{1}{2} \varphi(\kappa) & \text{if } c = 2, \\
\frac{1}{2} \varphi(\kappa) 2^h = \varphi(\kappa) 2^{h-1} & \text{if } c > 2.
\end{cases}
\]

Otherwise, if \( \kappa = 2 \), then

\[
m' = \begin{cases} 
1 & \text{if } c = 2, \\
2^{h-1} & \text{if } c > 2,
\end{cases}
\]

as we claimed.

By Proposition 4.3 and Proposition 4.8, we have that the lower bound given by Theorem 1.2 is explicitly determined. In particular, it is easy to construct examples of general twisted K3 surfaces and, consequently, of general cubic fourfolds with an arbitrary big number of non-isomorphic Fourier-Mukai partners.

**Example 4.9.** Let us take \( d = 50 \), which satisfies conditions (3) of Theorem 2.11 and (c). Therefore, a cubic fourfold in \( C_{50} \) has a twisted associated K3 surface with Brauer class of order \( \kappa = 5 \). By our result, we have that a general cubic fourfold with discriminant 50 admits at least \( \varphi(5)/2 = 4/2 = 2 \) (isomorphism classes of) Fourier-Mukai partners.

### References

[1] N. Addington, R. P. Thomas, *Hodge theory and derived categories of cubic fourfolds*, Duke Math. J. 163, no. 10 (2014), 1885-1927.

[2] M. Bernardara, E. Macrì, S. Mehrotra, P. Stellari, *A categorical invariant for cubic threefolds*, Adv. Math. 229 (2012), 770-803.

[3] T. Bridgeland, A. Maciocia, *Complex surfaces with equivalent derived categories*, Math. Z. 236 (2001), 677–697

[4] A. Căldăru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, PhD. Thesis (2000).

[5] B. Hassett, *Some rational cubic fourfolds*, Journal of Algebraic Geometry 8 (1999), 103-114.
[6] B. Hassett, *Special cubic fourfolds*, Compositio Mathematica 120 (2000), no. 1, 1-23.

[7] B. Hassett, *Cubic fourfolds, K3 surfaces, and rationality questions*, Lecture notes for a 2015 CIME-CIRM summer school, 36 pages.

[8] D. Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces, and B-fields*, Int. J. Math. 16 (2005), 13-36.

[9] D. Huybrechts, P. Stellari, *Equivalences of twisted K3 surfaces*, Math. Ann. 332 (2005), 901-936.

[10] D. Huybrechts, P. Stellari, *Proof of Caldararu’s conjecture. An appendix to a paper by K. Yoshioka*, In: *The 13th MSJ Inter. Research Inst. - Moduli Spaces and Arithmetic Geometry*, Adv. Stud. Pure Math. 45 (2006), 31-42.

[11] D. Huybrechts, *The K3 category of a cubic fourfold*, 39 pages, 10 June 2015, [arXiv:1505.01775v2], [math.AG].

[12] A. Kuznetsov, *Derived categories of cubic and $V_{14}$ threefolds*, Proc. Steklov Inst. Math. 3 246 (2004), 171–194.

[13] A. Kuznetsov, *Derived categories of cubic fourfolds*, Cohomological and geometric approaches to rationality problems, Progr. Math. 282 (2010), 219–243.

[14] A. Kuznetsov, *Derived categories view on rationality problems*, Lecture notes for the CIME-CIRM summer school, Levico Terme, June 22-27 (2015), 26 pages.

[15] E. Looijenga, *The period map for cubic fourfolds*, Inventiones mathematicae 177, Issue 1 (2009), 213-233.

[16] S. Ma, *Twisted Fourier-Mukai number of a K3 surface*, Trans. Amer. Math. Soc. 362 (2010), no. 1, 537–552.

[17] V.V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izvestija 14 (1980), 103–167.

[18] K. Oguiso, *K3 surfaces via almost-primes*, Math. Research Letters 9 (2002), 47-63.

[19] D. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York) 84 (1997), no. 5, 1361-1381.

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