Approximation Algorithms for Clustering via Weighted Impurity Measures

Ferdinando Cicalese
University of Verona, Italy
ferdinano.cicalese@univr.it

Eduardo Laber
PUC-Rio, Brazil
laber@inf.puc-rio.br

Abstract

An impurity measures $I : \mathbb{R}^k \rightarrow \mathbb{R}^+$ is a function that assigns a $k$-dimensional vector $v$ to a non-negative value $I(v)$ so that the more homogeneous $v$, with respect to the values of its coordinates, the larger its impurity. Well known examples of impurity measures are the Entropy and the Gini impurities. We study the problem of clustering based on impurity measures: given a collection of $n$ many $k$-dimensional vectors $V \subset \mathbb{N}^k$ with non-negative integer coordinates and an impurity measure $I$, the goal is to find a partition $P$ of $V$ into $L$ groups $V_1, \ldots, V_L$ so as to minimize the sum of the impurities of the groups in $P$, i.e., $I(P) = \sum_{m=1}^{L} I\left(\sum_{v \in V_m} v\right)$.

Impurity minimization has been widely used as quality assessment measure in probability distribution clustering as well as in categorical clustering where it is not possible to rely on geometric properties of the data set. However, in contrast to the case of metric based clustering, the current knowledge of impurity measure based clustering in terms of approximation and inapproximability results is very limited.

Our research contributes to fill this gap. We first present a simple linear time algorithm that simultaneously achieves 3-approximation for the Gini impurity measure and $O(\log \sum_{v \in V} \|v\|_1)$-approximation for the Entropy impurity measure. Then, for the Entropy impurity measure—where we also show that finding the optimal clustering is strongly NP-hard—we are able to design a polynomial time $O\left(\log^2(\min\{k, L\})\right)$-approximation algorithm. Our algorithm relies on a nontrivial characterization of a class of clusterings that necessarily includes a partition achieving $O\left(\log^2(\min\{k, L\})\right)$-approximation of the impurity of the optimal partition. Remarkably, this is the first polynomial time algorithm with approximation guarantee independent of the number of points/vector and not relying on any restriction on the components of the vectors for producing clusterings with minimum entropy.
1 Introduction

Data clustering is a fundamental tool in the analysis of large datasets to reduce the computational resources required to handle them. For a recent comprehensive description of different clustering methods and their applications we refer to [14]. In general, clustering is the problem of partitioning a set of multidimensional points so that, in the output partition, similar points are grouped together and dissimilar points are separated. When data are numerical, the quality of a clustering is naturally based on the pairwise distance in some metric space where the data points lie. In many applications, however, data is categorical, i.e., data points are described by binary attributes or qualitative features, e.g., ethnicity, music preferences, place of residence, hair color, etc. In categorical clustering, rather than employing distance functions to measure the pairwise distance of data points, many clustering algorithms rely on so called impurity measures, that estimate the "purity" of a set of data points. A partition is then evaluated by considering the total impurity of the sets in which it splits the data set.

The design of clustering methods based on impurity measures is the central theme of this paper. More formally, an impurity measures \( I : \mathbf{v} \in \mathbb{R}^k \rightarrow I(\mathbf{v}) \in \mathbb{R}^+ \) is a function that assigns a vector \( \mathbf{v} \) to a non-negative value \( I(\mathbf{v}) \) so that the more homogeneous \( \mathbf{v} \), with respect to the values of its coordinates, the larger its impurity. Well known examples of impurity measures\(^1\) are the Entropy impurity \([7, 1]\) and the Gini impurity \([8]\):

\[
I_{\text{Ent}}(\mathbf{v}) = \|\mathbf{v}\|_1 \sum_{i=1}^{k} \frac{v_i}{\|\mathbf{v}\|_1} \log \frac{\|\mathbf{v}\|_1}{v_i} \quad \text{and} \quad I_{\text{Gini}}(\mathbf{v}) = \|\mathbf{v}\|_1 \sum_{i=1}^{k} \frac{v_i}{\|\mathbf{v}\|_1} \left(1 - \frac{v_i}{\|\mathbf{v}\|_1}\right)
\]

**Problem Description.** We are given a collection of \( n \) many \( k \)-dimensional vectors \( V \subset \mathbb{N}^k \) with non-negative integer coordinates and we are also given an impurity measure \( I \). The goal is to find a partition \( \mathcal{P} \) of \( V \) into \( L \) disjoint groups of vectors \( V_1, \ldots, V_L \) so as to minimize the sum of the impurities of the groups in \( \mathcal{P} \), i.e.,

\[
I(\mathcal{P}) = \sum_{m=1}^{L} I\left(\sum_{\mathbf{v} \in V_m} \mathbf{v}\right).
\]

We refer to this problem as the **Partition with Minimum Weighted Impurity Problem (PMWIP)**.

The complexity of clustering in metric spaces, e.g., in the case of k-median, k-center and k-means, is well understood from the perspective of approximation algorithms in the sense that the gap between the ratios achieved by the best known algorithms and the largest known inapproximability factors, assuming \( P \neq NP \), are somehow tight (see [6] and references therein). In contrast, despite its wide use in applications, the understanding of clusterings based on impurity measures is much more limited as we detail further. As an example, for information theoretic clustering \([13]\), which is closely related to clustering based on entropy impurity, no hardness result beyond the NP-Completeness proved in [3] is available and the best approximation ratio known, when no assumptions on the the data input are made and all probability distributions have the same weight, is \( O(\log n) \), where \( n \) is the number of points to be clustered \([10]\).

\(^1\)In the literature, also the form without the scaling factor \( \|\mathbf{v}\|_1 \) is found and the function used here is also referred to as scaled/weighted impurity. In the section Preliminaries we will give a more precise definition of a wider class of impurity measures.
In this paper we present algorithms and complexity results that contribute to the understanding of clustering based on impurity measures. In particular, for our formulation of the information theoretic clustering problem we manage to obtain an approximation factor that depends on the logarithmic of the number of clusters rather than on the number $n$ of data points.

Our Results and Techniques. First we present a simple linear time algorithm that simultaneously guarantees a $3$-approximation for the $I_{\text{Gini}}$ and an $O(\log \sum_{v \in V} \|v\|_1)$ approximation for $I_{\text{Ent}}$. In addition, for the relevant case where all vectors in $V$ have the same $\ell_1$ norm the algorithm provides an $O(\log n + \log k)$ approximation. Then, we present a second algorithm that provides an $O(\log^2(\min\{k, L\}))$-approximation for $I_{\text{Ent}}$ in polytime. We also prove that the problem considered here is strongly NP-Hard for $I_{\text{Ent}}$.

When $k > L$, both algorithms employ an extension of the technique introduced in [18] that allows to reduce the dimensionality of the vectors in $V$ from $k$ to $L$ with a controllable additive loss in the approximation ratio. In [18], where the case $L = 2$ is studied, after the reduction step, an optimal clustering algorithm is used. However, for arbitrary $L$, the same strategy cannot be applied since the problem is NP-Complete. Thus, it is crucial to devise novel procedures to handle the case where $k \leq L$.

The procedure employed by the first algorithm is quite simple: it assigns vectors to groups according to the dominant coordinate, that is, one with the largest value. The procedure of the second algorithm is more involved, it relies on the combination of the following results: (i) the existence of an optimal algorithm for $k = 2$ [17]; (ii) the existence of a mapping $\chi : \mathbb{R}^k \rightarrow \mathbb{R}^2$ such that for a set of vectors $B$ which is pure, i.e., a set of vectors with the same dominant component, $I_{\text{Ent}}(\sum_{v \in B} v) = O(\log k)I_{\text{Ent}}(\sum_{v \in B} \chi(v))$ and (iii) a structural theorem that states that there exists a partition whose impurity is at an $O(\log^2 k)$ factor from the optimal one and such that at most one of its groups is mixed, i.e., it is not pure, namely contains vectors with different dominant coordinate. The items (i) and (ii) would be sufficient to obtain a $\log k$ approximation had we had $I_{\text{Ent}}(\sum_{v \in S} v) = O(\log k) \cdot I_{\text{Ent}}(\sum_{v \in S} \chi(v))$ for all sets $S \subseteq V$. However, this property does not hold for arbitrary $S$ but it does for sets $S$ that are not mixed. This is the reason why item (iii) is important - it allows to only consider partitions in which at most one group is mixed. The search for a partition of this type with low impurity can be achieved in pseudo-polynomial time via Dynamic Programming. To obtain a polynomial time algorithm we then employ a filtering technique similar to that employed for obtaining a FPTAS for the subset sum problem.

Related Work. Partition optimization based on impurity measures as defined in the PMWIP is also employed in the construction of decision trees/random forests to assess the quality of nominal attributes in the attribute selection step (see, e.g., [8, 9, 11, 12, 18, 17], and references quoted therein). Kurkoski and Yagi [17] showed that for the entropy impurity measure the problem can be solved in polynomial time when $k = 2$. The correctness of this algorithm relies on a theorem, proved in [8], which is generalized for $k > 2$ and $L$ groups in [11, 9, 12]. Basically, these theorems state that there exists an optimal solution that can be separated by hyperplanes in $\mathbb{R}^k$. These results imply the existence of $O(n^k)$ optimal algorithm when $L = 2$. Recently, one of authors proved that the problem is NP-Complete for $I_{\text{Ent}}$, even when $L = 2$, and presented constant approximation algorithms for a class of impurity measures that include Entropy and Gini for $L = 2$ [18].

Kurkoski and Yagi [17] also observe that the (PMWIP) with the Entropy impurity measure corresponds to the problem of designing a quantizer for the output $Y$ of a discrete memoryless channel in order to maximize the mutual information between the channel’s input $X$ and the quantizer’s output $Z$. This problem (also motivated by the construction of polar codes) has recently
attracted large interest in the information theory community \cite{17,16,20,19}. The correspondence to PMWIP is obtained by taking the channel inputs \(X\) as the components of the vectors in \(V\), the channel’s output \(Y\) as the points/vectors in \(V\) and the quantizer’s outputs \(Z\) as the clusters. The impurity of the clustering coincides with the conditional entropy \(H(X \mid Z)\). The focus in the Information Theory community is proving bounds, as a function of \(|X|, |Y|\) and \(|Z|\), on the mutual information degradation due to quantization/clustering rather than designing approximation algorithms.

Another problem which is strictly related to PMWIP with the entropy impurity measure is the problem of clustering probability distributions based on the Kullback-Leibler (KL) divergence. In particular, the \(MTC_{KL}\) problem of \cite{10} asking for the clustering of a set of \(n\) probability distributions of dimension \(k\) into \(L\) clusters minimizing the total Kullback-Leibler (KL) divergence of the points from the centroids of the clusters, corresponds to the particular case of PMWIP where each vector has the same \(\ell_1\) norm. In \cite{10} an \(O(\log n)\) approximation for \(MTC_{KL}\) is given. Under the additional assumption that every element of every probability distribution is larger than a constant, Ackermann et. al. \cite{4,2,5} presents an \((1 + \epsilon)\)-approximation algorithm for \(MTC_{KL}\) that runs in \(O(nkL + k^2O(L/\epsilon)\log L+2(n))\) time. By using similar assumptions on the components of the input probability distributions, Jegelka et. al. \cite{15} show that Lloyds \(K\)-means algorithm—which is however also exponential time in the worst case \cite{22}—obtains an \(O(\log L)\) approximation for \(MTC_{KL}\).

It has to be noted that although the optimal solutions of \(MTC_{KL}\) and PMWIP are the same, the problems differ with regard to the approximation guarantee pursued. Let \(\text{OPT}(V)\) denote the minimum possible impurity of a partition of the input set of vectors \(V\) and \(I(\mathcal{P}(V))\) the impurity of partition \(\mathcal{P}\) of \(V\). The approximation goal in our study of PMWIP is to find a partition \(\mathcal{P}\) that minimizes the ratio \(I(\mathcal{P}(V))/\text{OPT}(V)\) while in the case of the above papers on \(MTC_{KL}\) the goal is to find \(\mathcal{P}\) that minimizes \((I(\mathcal{P}) - H)/\text{OPT}(V) - H)\), where \(H\) is the sum of the entropies of the vectors in \(V\). Among the algorithms mentioned for \(MTC_{KL}\), the one that allows a more direct comparison with ours is the one proposed in \cite{10} since it runs in polytime and does not rely on assumptions over the input data. An \(\alpha\)-approximation for the \(MTC_{KL}\) problem implies \(\alpha\)-approximation for the special case of PMWIP with vectors of the same \(\ell_1\) norm, so the approximation measure used in \cite{10} is more general. However, our results apply to a more general problem and nonetheless we are able to provide approximation guarantee depending on the logarithm of the number of clusters while the guarantee in \cite{10} depends on the logarithm of the number of input vectors.

## 2 Preliminaries

We start defining some notations employed throughout the paper. An instance of PMWIP is a triple \((V, L, I)\), where \(V\) is a collection of non-null vectors in \(\mathbb{R}^k\) with non-negative integer coordinates, \(L\) is an integer larger than 1 and \(I\) is a scaled impurity measure.

We assume that for each coordinate \(i = 1, \ldots, k\) there exists at least one vector \(v \in V\) whose \(i\)th coordinate is non-zero, i.e., the vector \(\sum_{v \in V} v\) has no zero coordinates—for otherwise we could consider an instance of PMWIP with the vectors lying in some dimension \(k' < k\). For a set of vectors \(S\), the impurity \(I(S)\) of \(S\) is given by \(I(\sum_{v \in S} v)\). The impurity of a partition \(\mathcal{P} = (V^{(1)}, \ldots, V^{(L)})\) of the set \(V\) is then \(I(\mathcal{P}) = \sum_{i=1}^{L} I(V^{(i)})\). We use \(\text{OPT}(V, I, L)\) to denote the minimum possible impurity for an \(L\)-partition of \(V\) and, whenever the context is clear, we simply
talk about instance \( V \) (instead of \((V, I, L)\)) and of the impurity of an optimal solution as \( \text{OPT}(V) \) (instead of \( \text{OPT}(V, I, L) \)). We say that a partition \((V^{(1)}, \ldots, V^{(L)})\) is optimal for input \((V, L, I)\) iff 
\[
\sum_{i=1}^{L} I(V^{(i)}) = \text{OPT}(V, I, L).
\]
For an algorithm \( \mathcal{A} \) and an instance \((V, I, L)\), we denote by \( \mathcal{A}(V, I, L) \) and \( I(\mathcal{A}(V, I, L)) \) the partition output by \( \mathcal{A} \) on instance \((V, I, L)\) and its impurity, respectively. Whenever it is clear from the context, we omit to specify the instance and write \( I(\mathcal{A}) \) for \( I(\mathcal{A}(V, I, L)) \).

We use bold face font to denote vectors, e.g., \( \mathbf{u}, \mathbf{v}, \ldots \). For a vector \( \mathbf{u} \) we use \( u_i \) to denote its \( i \)th component. Given two vectors \( \mathbf{u} = (u_1, \ldots, u_k) \) and \( \mathbf{v} = (v_1, \ldots, v_k) \) we use \( \mathbf{u} \cdot \mathbf{v} \) to denote their inner product and \( \mathbf{u} \circ \mathbf{v} = (u_1v_1, \ldots, u_kv_k) \) to denote their component-wise (Hadamard) product. We use \( \mathbf{0} \) and \( \mathbf{1} \) to denote the vectors in \( \mathbb{R}^k \) with all coordinates equal to 0 and 1, respectively. We use \([m]\) to denote the set of the first \( m \) positive integers. For \( i = 1, \ldots, k \) we denote by \( \mathbf{e}_i \) the vector in \( \mathbb{R}^k \) with the \( i \)th coordinate equal to 1 and all other coordinates equal to 0.

The following properties will be useful in our analysis.

**Proposition 1.** Let \( p \in [0, 0.5] \). Then, \( p \log(1/p) \geq (1 - p) \log[1/(1 - p)] \)

**Proof.** It is enough to show that \( g(p) = p \log(1/p) - (1 - p) \log[1/(1 - p)] \geq 0 \) in the interval \([0, 1/2]\). For this, simply observe that \( g(0) = g(1/2) = 0 \) and that \( g(p) \) is concave in the interval \([0, 1/2]\) (the second derivative is negative). \( \square \)

**Proposition 2.** Let \( A > 0 \). The function \( f(x) = x \log(A/x) \) is increasing in the interval \((0, A/e]\) and decreasing in the interval \([A/e, A]\) so that its maximum value in the interval \([0, A]\) is \((A \log e)/e\).

**Proof.** The result follows because \( f'(x) = (\ln(A) - \ln x - 1)/\ln 2 \), the derivative of \( f(x) \) is positive in the interval \((0, A/e]\) and negative in the interval \([A/e, A]\). \( \square \)

### 2.1 Frequency weighted impurity measures with subsystem property

The impurity measures we will focus on, namely Gini and Entropy, are special cases of a larger class of impurity measures, which we denote by \( \mathcal{C} \), that satisfy the following definition

\[
I(\mathbf{u}) = \|\mathbf{u}\|_1 \sum_{i=1}^{\dim(\mathbf{u})} f\left(\frac{u_i}{\|\mathbf{u}\|_1}\right), \tag{P0}
\]

where \( \dim(\mathbf{u}) \) is the dimension of vector \( \mathbf{u} \) and \( f : \mathbb{R} \to \mathbb{R} \) is a function satisfying the following conditions:

1. \( f(0) = f(1) = 0 \) \hspace{2cm} \tag{P1}
2. \( f \) is strictly concave in the interval \([0,1]\) \hspace{2cm} \tag{P2}
3. For all \( 0 < p \leq q \leq 1 \), it holds that \( f(p) \leq \frac{p}{q} f(q) + q \cdot f\left(\frac{q}{p}\right) \) \hspace{2cm} \tag{P3}

Impurity measures satisfying the conditions (P0)-(P2) are called **frequency-weighted impurity measures based on concave functions** [12]. A fundamental properties of such impurities measures is that they are **superadditive** as shown in [12]. We record this property in the following lemma.

**Lemma 1** (Lemma 1 in [12]). If \( I \) satisfies (P0)-(P2) then for every vectors \( \mathbf{u}_L \) and \( \mathbf{u}_R \) in \( \mathbb{R}^k_+ \), we have \( I(\mathbf{u}_L + \mathbf{u}_R) \geq I(\mathbf{u}_L) + I(\mathbf{u}_R) \).
The Entropy and the Gini impurity measure satisfy the definition (P0) by means of the functions \( f_{\text{Entr}}(x) = -x \log x \) and \( f_{\text{Gini}}(x) = x(1-x) \). In fact, for a vector \( u \in \mathbb{R}^k \) the Entropy impurity \( I_{\text{Ent}}(u) \) and the Gini impurity \( I_{\text{Gini}}(u) \) are defined by

\[
I_{\text{Ent}}(u) = \|u\|_1 \frac{k}{\|u\|_1} \sum_{i=1}^{k} f_{\text{Entr}} \left( \frac{u_i}{\|u\|_1} \right) \quad \text{and} \quad I_{\text{Gini}}(u) = \|u\|_1 \frac{k}{\|u\|_1} \sum_{i=1}^{k} f_{\text{Gini}} \left( \frac{u_i}{\|u\|_1} \right),
\]

(1)

It is also easy to see that \( I_{\text{Ent}}(u) = \|u\|_1 H \left( \frac{u_1}{\|u\|_1}, \frac{u_2}{\|u\|_1}, \ldots, \frac{u_k}{\|u\|_1} \right) \) where \( H(\cdot) \) denotes the Shannon entropy function.

The following fact states that both the Gini and Entropy impurity measures belong to the class \( C \).

**Fact 1.** Both \( f_{\text{Entr}} \) and \( f_{\text{Gini}} \) satisfy properties (P1)-(P3), and, in particular, we have that \( f_{\text{Entr}} \) satisfies (P3) with equality. Therefore both the Gini impurity measure \( I_{\text{Gini}} \) and the Entropy impurity measure \( I_{\text{Ent}} \) belong to \( C \).

We now show that the impurity measures of class \( C \) satisfy a special subsystem property which will be used in our analysis to relate the impurity of partitions for instances of dimension \( k \) with the impurity of partitions for instances of dimension \( L \).

**Lemma 2** (Subsystem Property). Let \( I \) be an impurity measure in \( C \). Then, for every \( u \in \mathbb{R}^k_+ \) and pairwise orthogonal vectors \( d^{(1)}, \ldots, d^{(L)} \in \{0,1\}^k \), such that \( \sum_{i=1}^{L} d^{(i)} = 1 \), we have

\[
I(u) \leq I \left( (u \cdot d^{(1)}, u \cdot d^{(2)}, \ldots, u \cdot d^{(L)}) \right) + \sum_{i=1}^{L} I(u \circ d^{(i)}).
\]

(2)

Moreover, for \( I = I_{\text{Ent}} \) we have that (2) holds with equality.

**Proof.** Let \( f \) be the concave function used by the frequency-weighted impurity measure \( I \).

For \( i = 1, \ldots, L \), let \( u^{(i)} = u \circ d^{(i)} \). We have

\[
I(u) = \|u\|_1 \sum_{j=1}^{k} f \left( \frac{u_j}{\|u\|_1} \right)
\]

(3)

\[
= \|u\|_1 \sum_{i=1}^{L} \sum_{j|d^{(i)}=1} f \left( \frac{u_j}{\|u\|_1} \right)
\]

(4)

\[
\leq \|u\|_1 \sum_{i=1}^{L} \sum_{j|d^{(i)}=1} \frac{u_j}{\|u\|_1} \|u^{(i)}\|_1 f \left( \frac{\|u^{(i)}\|_1}{\|u\|_1} \right) + \frac{\|u^{(i)}\|_1}{\|u\|_1} \sum_{j|d^{(i)}=1} \frac{u_j}{\|u^{(i)}\|_1} f \left( \frac{u_j}{\|u^{(i)}\|_1} \right)
\]

(5)

\[
= \|u\|_1 \sum_{i=1}^{L} \sum_{j|d^{(i)}=1} \frac{u_j}{\|u^{(i)}\|_1} f \left( \frac{\|u^{(i)}\|_1}{\|u\|_1} \right) + \sum_{i=1}^{L} \sum_{j|d^{(i)}=1} \frac{u_j}{\|u^{(i)}\|_1} f \left( \frac{u_j}{\|u^{(i)}\|_1} \right)
\]

(6)

\[
= \|u\|_1 \sum_{i=1}^{L} f \left( \frac{\|u^{(i)}\|_1}{\|u\|_1} \right) + \sum_{i=1}^{L} \|u^{(i)}\|_1 \sum_{j|d^{(i)}=1} f \left( \frac{u_j}{\|u^{(i)}\|_1} \right)
\]

(7)

\[
= I \left( (u \cdot d^{(1)}, u \cdot d^{(2)}, \ldots, u \cdot d^{(L)}) \right) + \sum_{i=1}^{L} I(u \circ d^{(i)})
\]

(8)
where (4) follows from (3) by splitting the second summation according to the partition of \(|k|\) induced by the non zero components of the vectors \(d^{(i)}\); (5) follows from (4) by applying property (P3) with \(p = \frac{u_j}{\|u_i\|}\) and \(q = \frac{\|u^{(i)}\|_1}{\|u_i\|}\); (6) follows from (5) by simple algebraic manipulations; (7) follows from (6) since by definition of \(u(i)\) we have \(\sum_{j\in d^{(i)}} u_j = \|u^{(i)}\|_1\); (7) follows from (6) since \(\|u^{(i)}\|_1 = u \cdot d^{(i)}\) and \(I(u \circ d^{(i)}) = \sum_{j\in d^{(i)}} \|u \circ d^{(i)}\|_1 f\left(\frac{u_j}{\|u \circ d^{(i)}\|_1}\right)\) and \(\|u \circ d^{(i)}\|_1 = \|u^{(i)}\|_1\).

The second statement of the lemma follows immediately by the fact that the concave function \(f_{Entr}\) satisfies property (P3) with equality (see Fact [3]). Hence, for \(I_{Entr}\) the inequality in (5) becomes an equality.

**Remark 1.** The Subsystem property in the previous lemma holds also under the stronger assumption that vectors \(d\)'s are from \([0,1]^k\) and not necessarily orthogonal.

### 3 Handling high dimensional vectors

In this section we present an approach to address instances \((V, I, L)\) with \(I \in C\) and \(k > L\). It consists of two steps: finding a 'good' projection of \(\mathbb{R}^k\) into \(\mathbb{R}^L\) and then solving PMWIP for the projected instance with \(k = L\). Thus, in the next sections we will be focusing on how to build this projection and how to solve instances with \(k \leq L\). The material of this section is a generalization for arbitrary \(L\) of the results introduced in [3] for \(L = 2\).

Let \(D\) be the family of all sequences \(D\) of \(L\) pairwise orthogonal directions in \(\{0,1\}^k\), such that \(\sum_{d \in D} d = 1\). For each \(D = (d^{(1)}, \ldots, d^{(L)}) \in D\) and any \(v \in \mathbb{R}^k\) we define the operation \(\text{collapse}_D : \mathbb{R}^k_+ \rightarrow \mathbb{R}^L_+\) by

\[
\text{collapse}_D(v) = (v \cdot d^{(1)}, \ldots, v \cdot d^{(L)}).
\]

We also naturally extend the operation to sets of vectors \(S\), by defining \(\text{collapse}_D(S)\) as the multisets of vectors obtained by applying \(\text{collapse}_D\) to each vector of \(S\).

Let \(A\) be an algorithm that on instance \((V, I, L)\) chooses a sequence of vectors \(D = \{d^{(1)}, \ldots, d^{(L)}\} \in D\) and returns a partition \((V^{(1)}, \ldots, V^{(L)})\) such that \((\text{collapse}_D(V^{(1)}), \ldots, \text{collapse}_D(V^{(L)}))\) is a 'good' partition for the \(L\)-dimensional instance \((\text{collapse}_D(V), I, L)\). In this section we quantify the relationship between the approximation attained by \((\text{collapse}_D(V^{(1)}), \ldots, \text{collapse}_D(V^{(L)}))\) for instance \((\text{collapse}_D(V), I, L)\) and the corresponding approximation attained by \((V^{(1)}, \ldots, V^{(L)})\) for instance \((V, I, L)\).

Let \(u^{(i)} = \sum_{v \in V^{(i)}} v\). From the subsystem property we have the following upper bound on the impurity of the partition returned by \(A\).

\[
I(A) = \sum_{i=1}^{L} I(u^{(i)}) \leq \sum_{i=1}^{L} I\left((u^{(i)} \cdot d^{(1)}, \ldots, u^{(i)} \cdot d^{(L)})\right) + \sum_{d \in D} I(u \circ d).
\]

Thus, by the superadditivity of \(I\) we have

\[
I(A) \leq \sum_{i=1}^{L} I\left((u^{(i)} \cdot d^{(1)}, \ldots, u^{(i)} \cdot d^{(L)})\right) + \sum_{d \in D} I(u \circ d).
\]
We now show two lower bounds on $\OPT(V, I, L)$. For the sake of simplifying the notation we will use $\OPT(V)$ for $\OPT(V, I, L)$.

**Lemma 3.** For any instance $(V, I, L)$ of PMWIP and any $D = \{d^{(1)}, \ldots, d^{(L)}\} \in \mathcal{D}$ we have $\OPT(V) \geq \OPT(\text{collapse}_D(V))$.

**Proof.** Let $V^{(1)}, \ldots, V^{(L)}$ be an optimal partition for $V$, i.e.,

$$\sum_{i=1}^{L} I(\sum_{v \in V^{(i)}} v) = \OPT(V). \tag{10}$$

We define the corresponding partition on the vectors $\tilde{v}$ in $\text{collapse}_D(V)$ by letting $\tilde{V}^{(i)} = \{\text{collapse}_D(v) \mid v \in V^{(i)}\}$. We have

$$\sum_{i=1}^{L} I(\sum_{\tilde{v} \in \tilde{V}^{(i)}} \tilde{v}) \geq \OPT(\text{collapse}_D(V)). \tag{11}$$

Let $u^{(i)} = \sum_{v \in V^{(i)}} v$. Moreover, by the subadditivity of $f$, we have that for each $i = 1, \ldots, L$, it holds that

$$I(\sum_{v \in V^{(i)}} v) = ||u^{(i)}||_1 \sum_{j=1}^{k} f\left(\frac{u^{(j)}_j}{||u^{(j)}||_1}\right) = ||u^{(i)}||_1 \sum_{j=1}^{L} \left(\sum_{\ell=1}^{L} f\left(\frac{u^{(i)}_\ell}{||u^{(i)}||_1}\right)\right) \geq

||u^{(i)}||_1 \sum_{i=1}^{L} f\left(\frac{\sum_{\ell=d^{(j)}_j=1}^{d^{(j)}_j} u^{(i)}_\ell}{||u^{(j)}||_1}\right) = I(\sum_{\tilde{v} \in \tilde{V}^{(i)}} \tilde{v})$$

which implies

$$\OPT(V) = \sum_{i=1}^{L} I(\sum_{v \in V^{(i)}} v) \geq \sum_{i=1}^{L} I(\sum_{\tilde{v} \in \tilde{V}^{(i)}} \tilde{v})$$

that combined with (11) gives the desired result. \hfill \square

The following result, proved in [9, 12], states that the groups in the optimal solution can be separated by hyperplanes in $\mathbb{R}^L$. We recall it here as it will be used to derive our second lower bound on $\OPT(V)$ contained in Lemma 5 below.

**Lemma 4** (Hyperplanes Lemma [9, 12]). Let $I$ be an impurity measure satisfying properties (P0)-(P2). If $(V_i)_{i=1,\ldots,L}$ is an optimal partition of a set of vectors $V$, then there are vectors $v^{(1)}, \ldots, v^{(L)} \in \mathbb{R}^k$ such that $v \in V_i$ if and only if $v \cdot v^{(i)} < v \cdot v^{(j)}$ for each $j \neq i$.

**Lemma 5.** Let $(V, I, L)$ be an instance of PMWIP. Let $u = \sum_{v \in V} v$. It holds that

$$\OPT(V) \geq \min_{D \in \mathcal{D}} \sum_{d' \in D} I(u \circ d'),$$

**Proof.** Let $W$ be the multiset of vectors built as follows: for each $v = (v_1, \ldots, v_k) \in V$ we add the vectors $v_1 e_1, \ldots, v_k e_k$ to $W$. Hence, $W$ has $nk$ vectors, all of them with only one non-zero coordinate.
It is not hard to see that for every partition $V^{(1)}, \ldots, V^{(L)}$ of $V$ there is a corresponding partition $W^{(1)}, \ldots, W^{(L)}$ such that $\sum_{i=1}^{L} I(\sum_{v \in V^{(i)}} v) = \sum_{i=1}^{L} I(\sum_{w \in W^{(i)}} w)$, hence,

$$\text{OPT}(V) \geq \text{OPT}(W).$$

Let us now employ Lemma \[4\] to analyze $\text{OPT}(W)$. Let $W^{(1)}, \ldots, W^{(L)}$, be a partition of $W$ with impurity $\text{OPT}(W)$. From Lemma \[4\] if two vectors $w, w' \in W$ are such that $w = we_i$ and $w' = w'e_i$ for some $i$ (i.e., they have the same non-zero component) then there is a $j$ such that both $w$ and $w'$ belong to $W^{(j)}$.

For $j = 1, \ldots, L$, let $d^{(j)}$ be the vector in $\{0, 1\}^k$ such that $d^{(j)}_i = 1$ if and only if the vectors of $W$ whose only non-zero coordinate is the $i$th one are in $W^{(j)}$. Then $\{d^{(1)}, \ldots, d^{(L)}\} \in D$ and we have

$$\text{OPT}(W) = \sum_{i=1}^{L} I(\sum_{w \in W} w \cdot d^{(i)}) = \sum_{i=1}^{L} I(\sum_{v \in V} v \cdot d^{(i)}) = \sum_{i=1}^{L} I(u \cdot d^{(i)}) \geq \min_{D \in \mathcal{D}} \sum_{d' \in D} I(u \cdot d').$$

Putting together \[9\] and Lemmas \[4\] we have

$$\frac{I(A)}{\text{OPT}(V)} \leq \frac{\sum_{i=1}^{L} I\left((u^{(i)} \cdot d^{(1)}, \ldots, u^{(i)} \cdot d^{(L)})\right) + \sum_{d \in D} I(u \cdot d)}{\max \{\text{OPT}(\text{collapse}_D(V)), \min_{D \in \mathcal{D}} \sum_{d' \in D} I(u \cdot d')\}} \leq \frac{\sum_{i=1}^{L} I\left((u^{(i)} \cdot d^{(1)}, \ldots, u^{(i)} \cdot d^{(L)})\right)}{\text{OPT}(\text{collapse}_D(V))} + \frac{\sum_{d \in D} I(u \cdot d)}{\min_{D \in \mathcal{D}} \sum_{d' \in D} I(u \cdot d')} \tag{12}$$

Since the first ratio in the last expression is the approximation attained by the partition $\text{collapse}_D(V^{(1)}), \ldots, \text{collapse}_D(V^{(L)})$ on the instance $(\text{collapse}_D(V), I, L)$, this inequality says that we can obtain a good approximation for instance $(V, I, L)$ of PMWIP (where the vectors have dimension $k > L$) by properly choosing: (i) a set $D$ of $L$ orthogonal directions in $\{0, 1\}^k$, and—given the choice of $D$—(ii) a good approximation for the instance $(\text{collapse}_D(V), I, L)$, where the vectors have dimension $L$.

4 The dominance algorithm

For a vector $v$ we say that $i$ is the dominant component for $v$ if $v_i \geq v_j$ for each $j \neq i$. In such a case we also say that $v$ is $i$-dominant. For a set of vectors $U$ we say that $i$ is the dominant component in $U$ if $i$ is the dominant component for $u = \sum_{v \in U} v$.

Given an instance $(V, I)$ let $u = \sum_{v \in V} v$ and let us assume that, up to reordering of the components, it holds that $u_i \geq u_{i-1}$, for $i = 1, \ldots, k - 1$.

Let $A^{\text{Dom}}$ be the algorithm that proceeds according to the following cases:

i. $k > L$. $A^{\text{Dom}}$ assigns each vector $v = (v_1, \ldots, v_k) \in V$ to group $i$ where $i$ is the dominant component of vector $v' = (v_1, \ldots, v_{L-1}, \sum_{j=L}^{k} v_j)$

ii. $k \leq L$. $A^{\text{Dom}}$ assigns each vector $v \in V$ to group $i$ where $i$ is the dominant component of $v$. 


The only difference between cases (i) and (ii) is the reduction of dimensionality employed in the former to aggregate the smallest components with respect to \( u \).

Let \( D = \{d^{(1)}, \ldots, d^{(L)}\} \in \mathcal{D} \) where \( d^{(i)} = e_i \) for \( i = 1, \ldots, L-1 \) and \( d^{(L)} = 1 - \sum_{i=1}^{L-1} d^{(i)} \).

We notice that that vector \( v' \) in case (i) is exactly \( \text{collapse}_D(v) \). Thus, if \( k > L \), we can rewrite (12) as

\[
\frac{I(A^{\text{Dom}}(V))}{\text{OPT}(V)} \leq \frac{I(A^{\text{Dom}}(\text{collapse}_D(V)))}{\text{OPT}(\text{collapse}_D(V))} + \frac{\sum_{d \in D} I(u \circ d)}{\min_{d \in D} \sum_{d' \in D} I(u \circ d')}
\] (13)

The next lemma is useful to prove an upper bound on the approximation of \( A^{\text{Dom}} \), when \( k \leq L \).

**Lemma 6.** Let \((V, I)\) be an instance of \((L, k)\)-PMWIP with \( I \in \mathcal{C} \) and \( k \leq L \). For a subset \( S \) of \( V \) let \( u^S = \sum_{v \in S} v \). If there exist positive numbers \( \alpha, \beta \) such that for each \( S \subseteq V \) we have

\[
\beta(\|u^S\|_1 - \|u^S\|_{\infty}) \leq I(u^S) \leq \alpha(\|u^S\|_1 - \|u^S\|_{\infty})
\]

then the algorithm \( A^{\text{Dom}} \) guarantees \( \alpha/\beta \) approximation, i.e.,

\[
\frac{I(A^{\text{Dom}}(V))}{\text{OPT}(V)} \leq \frac{\alpha}{\beta}
\]

**Proof.** Let \((V^{(1)}, \ldots, V^{(L)})\) be the partition of \( V \) returned by \( A^{\text{Dom}} \). Then, by the superadditivity of \( I \)

\[
\frac{I(A^{\text{Dom}})}{\text{OPT}(V)} = \frac{\sum_{i=1}^{k} I(\sum_{v \in V^{(i)}} v)}{\sum_{i=1}^{k} \sum_{v \in V^{(i)}} I(v)}
\]

Thus, it is enough to prove that for \( i = 1, \ldots, k \)

\[
\frac{I(\sum_{v \in V^{(i)}} v)}{\sum_{v \in V^{(i)}} I(v)} \leq \frac{\alpha}{\beta}
\]

Fix \( i \in [k] \) and let \( u = \sum_{v \in V^{(i)}} v \). By hypothesis, we have

\[
I(u) \leq \alpha(\|u\|_1 - \|u\|_{\infty}) \quad \text{and} \quad I(v) \geq \beta(\|v\|_1 - \|v\|_{\infty}) \quad \text{for every} \quad v \in V^{(i)}.
\]

Moreover, by construction, for every vector \( v \in V^{(i)} \) we have \( \|v\|_{\infty} = v_i \), so that \( \sum_{v \in V^{(i)}} \|v\|_{\infty} = \|u\|_{\infty} \). Putting everything together we have

\[
\frac{I(\sum_{v \in V^{(i)}} v)}{\sum_{v \in V^{(i)}} I(v)} \leq \frac{\alpha(\|u\|_1 - \|u\|_{\infty})}{\sum_{v \in V^{(i)}} \beta(\|v\|_1 - \|v\|_{\infty})} = \frac{\alpha}{\beta(\|u\|_1 - \|u\|_{\infty})} = \frac{\alpha}{\beta},
\]

as desired. \( \square \)

### 4.1 Analysis of \( A^{\text{Dom}} \) for the Gini impurity measure \( I_{\text{Gini}} \)

In this section we show that algorithm \( A^{\text{Dom}} \) achieves constant 3-approximation when the impurity measure is \( I_{\text{Gini}} \).

The following lemma together with Lemma 6 will show that \( A^{\text{Dom}} \) guarantees 2-approximation on instances with \( k \leq L \).

**Lemma 7.** For a vector \( v \in \mathbb{R}^k_+ \) we have \( \|v\|_1 - \|v\|_{\infty} \leq I_{\text{Gini}}(v) \leq 2(\|v\|_1 - \|v\|_{\infty}) \).
Proof. First we prove the upper bound. We have that

\[ I_{\text{Gini}}(v) = \left\| v \right\|_1 \sum_{i=1}^{k} \frac{v_i}{\left\| v \right\|_1} \left( 1 - \frac{v_i}{\left\| v \right\|_1} \right) = \sum_{i=1}^{k} \sum_{j \neq i} \frac{v_i}{\left\| v \right\|_1} \frac{v_j}{\left\| v \right\|_1} \]  \quad (14)

\[ = \left\| v \right\|_\infty \left( \left\| v \right\|_1 - \left\| v \right\|_\infty \right) + \sum_{i} v_i \sum_{j \neq i} \frac{v_j}{\left\| v \right\|_\infty} \]  \quad (15)

\[ = 2 \left\| v \right\|_\infty \left( \left\| v \right\|_1 - \left\| v \right\|_\infty \right) + \sum_{i} v_i \sum_{j \neq i} \frac{v_j}{\left\| v \right\|_\infty} \]  \quad (16)

\[ \leq 2 \left\| v \right\|_\infty \left( \left\| v \right\|_1 - \left\| v \right\|_\infty \right) + \left( \left\| v \right\|_1 - \left\| v \right\|_\infty \right)^2 \]  \quad (17)

\[ = \frac{\left( \left\| v \right\|_1 - \left\| v \right\|_\infty \right) \left( \left\| v \right\|_1 + \left\| v \right\|_\infty \right)}{\left\| v \right\|_1} \]  \quad (18)

\[ \leq 2 \left( \left\| v \right\|_1 - \left\| v \right\|_\infty \right) \]  \quad (19)

For the lower bound we observe that \( \sum_{i=1}^{k} v_i^2 \leq \left\| v \right\|_\infty \left\| v \right\|_1 \). Therefore, we have

\[ I_{\text{Gini}}(v) = \left\| v \right\|_1 \sum_{i=1}^{k} \frac{v_i}{\left\| v \right\|_1} \left( 1 - \frac{v_i}{\left\| v \right\|_1} \right) = \left\| v \right\|_1 - \left( \sum_{i=1}^{k} v_i^2 \right) \geq \left\| v \right\|_1 - \frac{\left\| v \right\|_\infty \left\| v \right\|_1}{\left\| v \right\|_1} = \left\| v \right\|_1 - \left\| v \right\|_\infty. \]  \quad (20)

\[ \square \]

**Theorem 1.** Algorithm \( A^\text{Dom} \) is a 2-approximation algorithm for instances \((V, I)\) with \( I = I_{\text{Gini}} \) and \( k \leq L \).

**Proof.** Directly from Lemmas 6 and 7. \( \square \)

The following lemma will provide an upper bound (in fact an exact estimate) of the second ratio in (13).

**Lemma 8.** Fix a vector \( u \in \mathbb{R}^k \) such that \( u_i \geq u_{i+1} \) for each \( i = 1, \ldots, k-1 \) and \( D = \{d^{(1)}, \ldots, d^{(L)}\} \subseteq \mathcal{D} \) with \( d^{(i)} = e_i \) for \( i = 1, \ldots, L-1 \) and \( d^{(L)} = \sum_{j=L}^{k} e_j = 1 - \sum_{j=1}^{L-1} d^{(j)} \). It holds that

\[ \sum_{d \in D} I(u \circ d) = \min_{D' \subseteq D} \left\{ \sum_{d' \in D'} I(u \circ d') \right\} \]

\[ \sum_{d \in D} I(u \circ d^*) = \min_{D' \subseteq D} \left\{ \sum_{d' \in D'} I(u \circ d') \right\} \]  \quad (21)

and \( |D^* \cap D| \) is maximum among all \( D^* \) satisfying (21).

Let us assume for the sake of contradiction that \( D^* \neq D \). Let \( \hat{d} \in D^* \) such that \( \hat{d}_k = 1 \). We note that \( \hat{d} \neq d^{(L)} \) for otherwise we would have \( D^* = D \).

Let \( c \in D^* \setminus (D \cup \{ \hat{d} \}) \) such that for all other \( d \in D^* \setminus (D \cup \{ \hat{d} \}) \) we have \( \min \{ i \mid c_i = 1 \} < \min \{ i \mid d_i = 1 \} \), i.e., \( c \) is the vector in \( D^* \setminus (D \cup \{ \hat{d} \}) \) with the smallest non-zero component.
Let \( \mathbf{v} = \mathbf{c} + \hat{\mathbf{d}} \) and \( i^* \) be the minimum integer such that \( \mathbf{v}_{i^*} = 1 \). Note that \( i^* \leq L - 1 \), for otherwise we would have \( D^* \notin \mathcal{D} \). Let \( F \) be the set of vectors from \( \mathcal{D} \) defined by
\[
F = (D^* \setminus \{ \mathbf{d}, \mathbf{c} \}) \cup \{ \mathbf{d}^{(i^*)}, \mathbf{v} - \mathbf{d}^{(i^*)} \}.
\]

The following claim directly follows from [18, Lemma 4.1]. For the sake of self-containment we defer its proof to the appendix.

**Claim.** Fix \( \mathbf{u} \in \mathbb{R}^k \) such that \( u_i \geq u_{i+1} \) for each \( i = 1, \ldots, k - 1 \). Let \( \mathbf{z}^{(1)} \) and \( \mathbf{z}^{(2)} \) two orthogonal vectors from \( \{0,1\}^k \setminus \{0\} \). Let \( i^* = \min\{i \mid \max\{z_i^{(1)}, z_i^{(2)}\} = 1\} \) and \( \mathbf{v}^{(1)} = \mathbf{e}_{i^*} \) and \( \mathbf{v}^{(2)} = \mathbf{z}^{(1)} + \mathbf{z}^{(2)} - \mathbf{e}_{i^*} \). Then
\[
I(\mathbf{u} \circ \mathbf{v}^{(1)}) + I(\mathbf{u} \circ \mathbf{v}^{(2)}) \leq I(\mathbf{u} \circ \mathbf{z}^{(1)}) + I(\mathbf{u} \circ \mathbf{z}^{(1)}).
\]

By the Claim, we have that
\[
\sum_{\mathbf{d} \in F} I(\mathbf{u} \circ \mathbf{d}) = I(\mathbf{u} \circ \mathbf{d}^{(i^*)}) + I(\mathbf{u} \circ (\mathbf{v} - \mathbf{d}^{(i^*)})) + \sum_{\mathbf{d} \in F \cap D^*} I(\mathbf{u} \circ \mathbf{d})
\]
\[
\leq I(\mathbf{u} \circ \hat{\mathbf{d}}) + I(\mathbf{u} \circ \mathbf{c}) + \sum_{\mathbf{d} \in F \cap D^*} I(\mathbf{u} \circ \mathbf{d}) = \sum_{\mathbf{d} \in D^*} I(\mathbf{u} \circ \mathbf{d}),
\]
hence since \( D \) satisfies (21) we have that \( F \) also satisfies (21).

In addition we observe that \( |D \cap F| > |D \cap D^*| \) as by definition it shares with \( D \) all that was shared by \( D^* \) and also \( \mathbf{d}^{(i^*)} \). This would be in contradiction with the maximality of the intersection of \( D^* \). Therefore, we must have \( D^* = D \) which concludes the proof.

Putting together inequalities [13], Theorem 1, and Lemma 8 we get that

**Theorem 2.** Algorithm \( A^{\text{Dom}} \) is a linear time 3-approximation for the \( I_{\text{Gini}} \).

### 4.2 Analysis of \( A^{\text{Dom}} \) for the Entropy impurity measure \( I_{\text{Ent}} \)

The following lemma will be useful for applying Lemma [8] to the analysis of the performance of \( A^{\text{Dom}} \) with respect to the entropy impurity measure \( I_{\text{Ent}} \).

**Lemma 9.** For a vector \( \mathbf{v} \in \mathbb{R}_+^k \) we have
\[
(\|\mathbf{v}\|_1 - \|\mathbf{v}\|_{\infty}) \log \left( \frac{\|\mathbf{v}\|_1}{\min\{\|\mathbf{v}\|_1 - \|\mathbf{v}\|_{\infty}, \|\mathbf{v}\|_{\infty}\}} \right) \leq I_{\text{Ent}}(\mathbf{v}) \leq 2(\|\mathbf{v}\|_1 - \|\mathbf{v}\|_{\infty}) \log \left( \frac{k}{\|\mathbf{v}\|_1 - \|\mathbf{v}\|_{\infty}} \right).
\]

**Proof.** Let \( i^* \) be an index in \([k]\) such that \( v_{i^*} = \|\mathbf{v}\|_{\infty} \). We have that
\[
I_{\text{Ent}}(\mathbf{v}) = \|\mathbf{v}\|_{\infty} \log \frac{\|\mathbf{v}\|_1}{\|\mathbf{v}\|_{\infty}} + \sum_{i \neq i^*} v_i \log \frac{\|\mathbf{v}\|_1}{v_i} \tag{22}
\]
\[
= \|\mathbf{v}\|_{\infty} \log \frac{\|\mathbf{v}\|_1}{\|\mathbf{v}\|_{\infty}} + (\|\mathbf{v}\|_1 - \|\mathbf{v}\|_{\infty}) \log \|\mathbf{v}\|_1 - \sum_{i \neq i^*} v_i \log v_i. \tag{23}
\]
For the upper bound, we observe that the expression in (23) is maximum when \( v_i = (\|v\|_1 - \|v\|_\infty)/(k - 1) \) for \( i \neq i^* \). Thus,

\[
I_{\text{Ent}}(v) \leq \|v\|_\infty \log \frac{\|v\|_1}{\|v\|_\infty} + (\|v\|_1 - \|v\|_\infty) \log \frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty} + (\|v\|_1 - \|v\|_\infty) \log(k - 1). \tag{24}
\]

To show that this satisfies the desired upper bound, we split the analysis into two cases:

If \( \|v\|_\infty \geq \frac{\|v\|_1}{2} \), we have that

\[
I_{\text{Ent}}(v) \leq 2(\|v\|_1 - \|v\|_\infty) \log \frac{k\|v\|_1}{\|v\|_1 - \|v\|_\infty},
\]

where the second inequality follows from Proposition 1 using \( p = (\|v\|_1 - \|v\|_\infty)/\|v\|_1 \).

If \( \|v\|_\infty \leq \frac{\|v\|_1}{2} \), we have that

\[
I_{\text{Ent}}(v) \leq 2(\|v\|_1 - \|v\|_\infty) \log \frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty} + (\|v\|_1 - \|v\|_\infty) \log(k - 1)
\]

For the lower bound, consider the same two cases:

If \( \|v\|_\infty > \frac{\|v\|_1}{2} \), the expression in (23) is minimum when there is a unique index \( j \neq i^* \) such that \( v_j = \|v\|_1 - \|v\|_\infty \) and \( v_i = 0 \) for each \( i \in \{j, i^*\} \). Thus,

\[
I_{\text{Ent}}(v) \geq \|v\|_\infty \log \frac{\|v\|_1}{\|v\|_\infty} + (\|v\|_1 - \|v\|_\infty) \log \frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty} \geq (\|v\|_1 - \|v\|_\infty) \log \frac{\|v\|_1}{\min\{\|v\|_1 - \|v\|_\infty, \|v\|_\infty\}}
\]

If \( \|v\|_\infty < \frac{\|v\|_1}{2} \), the expression in (23) is minimum when there exists a set of indexes \( A \subseteq \{k\} \) with \( |A| = \lfloor \|v\|_1/\|v\|_\infty \rfloor - 1 \) such that \( v_i = \|v\|_\infty \) for each \( i \in A \) and (possibly) an index \( j \notin A \) such that \( v_j = \|v\|_1 - |A| \cdot \|v\|_\infty \). Thus,

\[
I_{\text{Ent}}(v) \geq (\|v\|_1 - \|v\|_\infty) \log \frac{\|v\|_1}{\|v\|_\infty} \geq (\|v\|_1 - \|v\|_\infty) \log \frac{\|v\|_1}{\min\{\|v\|_1 - \|v\|_\infty, \|v\|_\infty\}}
\]

From the bounds in the previous lemma and Lemma 6 we obtain our first guarantee on the approximation of algorithm \( A^{\text{Dom}} \) for the Entropy Impurity measure on instances with \( k \leq L \).

**Theorem 3.** Let \((V, I, L)\) be an instance of PMWIP with \( I = I_{\text{Ent}} \) and \( k \leq L \). Let \( p = \log k + \log(\sum_{v \in V} \|v\|_1) \). Then, \( A^{\text{Dom}} \) guarantees a \( 2p \)-approximation on instance \((V, I, L)\).
Proof. Let $S$ be a subset of $V$ and let $u_S = \sum_{v \in S} v$. Define $\alpha = 1$ and $\beta = 2p$.

If $\|u_S\|_1 = \|u_S\|_\infty$ then $I(u_S) = 0$ so that the conditions of Lemma 6 is satisfied. Otherwise, Lemma 9 guarantees that

$$\|u^S\|_1 - \|u^S\|_\infty \leq I(u^S) \leq 2(\|u^S\|_1 - \|u^S\|_\infty) \log(k\|u^S\|_1) \leq \|u^S\|_1 - \|u^S\|_\infty 2p.$$ 

Thus, it follows from Lemma 6 that we have a $2p$-approximation. \qed

Remark 2. Let $s$ be a large integer. The instance $\{(s, 0), (2, 1), (0, 1)\}$ and $L = 2$ shows that the analysis is tight up to constant factors. In fact, the impurity of $A_{Dom}$ is larger than $\log s$ while the impurity of the partition that leaves $(s, 0)$ alone is 4.

Theorem 4. Let Uniform-PMWIP (U-PMWIP) be the variant of PMWIP where all vectors have the same $\ell_1$ norm. We have that $A_{Dom}$ is an $O(\log n + \log k)$-approximation algorithm for U-PMWIP with $I = I_{Ent}$ and $k \leq L$.

Proof. Let $(V, I, L)$ be an instance of U-PMWIP with $I = I_{Ent}$ and vectors of dimension $k \leq L$. Let $(V^{(1)}, \ldots, V^{(L)})$ be the partition of $V$ returned by $A_{Dom}$. By the superadditivity of $I$ it holds that

$$\frac{I(A_{Dom})}{OPT(V)} = \frac{\sum_{i=1}^k I(\sum_{v \in V^{(i)}} v)}{\sum_{i=1}^k \sum_{v \in V^{(i)}} I(v)}$$

Thus, it is enough to prove that for $i = 1, \ldots, k$

$$\frac{I(\sum_{v \in V^{(i)}} v)}{\sum_{v \in V^{(i)}} I(v)} = O(\log n + \log k)$$

Let $s$ be the $\ell_1$ norm of all vectors in $V$, let $u = \sum_{v \in V^{(i)}} v$ and let $c = \|u\|_1 - \|u\|_\infty$. By Lemma 9 we have that

$$I(u) \leq c \log \frac{kns}{c}.$$ 

Moreover, we have

$$\sum_{v \in V^{(i)}} I(v) \geq \max \left\{ c, c \log \frac{s}{c} - \sum_{v \in V^{(i)}} (\|v\|_1 - \|v\|_\infty) \log(\|v\|_1 - \|v\|_\infty) \right\}$$

If $c \geq s/2$ then we have a $O(\log n + \log k)$ approximation using $c$ as a lower bound. If $c < s/2$ we get that

$$\sum_{v \in V^{(i)}} I(v) \geq c \log s - c \log c = c \log(s/c)$$

and the approximation is $O(\log n + \log k)$ as well. \qed

Remark 3. Let $s$ be a large integer. The instance with $n - 1$ vectors equal to $(s, 0)$, one vector equals to $(s, s/2)$ and $L = 2$ shows that the analysis is tight.

To obtain an approximation of $A_{Dom}$ for $I_{Ent}$ for general $L$ and $k$ we need an upper bound on the second fraction in Equation (12). This is given by the next lemma.
**Lemma 10.** Fix a vector $\mathbf{u} \in \mathbb{R}^k$ such that $u_i \geq u_{i+1}$ for each $i = 1, \ldots, k-1$ and $D = \{d^{(1)}, \ldots, d^{(L)}\} \in \mathcal{D}$ with $d^{(i)} = \mathbf{e}_i$ for $i = 1, \ldots, L-1$ and $d^{(L)} = \sum_{j=L}^{k-1} \mathbf{e}_j = 1 - \sum_{j=1}^{L-1} d^{(j)}$. It holds that

$$\sum_{\mathbf{d} \in \mathcal{D}} I_{\text{Ent}}(\mathbf{u} \circ \mathbf{d}) \leq O(\log L) \min_{D' \in \mathcal{D}} \left\{ \sum_{\mathbf{d}' \in D'} I_{\text{Ent}}(\mathbf{u} \circ \mathbf{d}') \right\}$$

**Proof.** Let $D^* = \{d_{*}^{(1)}, \ldots, d_{*}^{(L)}\} \in \mathcal{D}$ be such that

$$\sum_{\mathbf{d} \in \mathcal{D}^*} I_{\text{Ent}}(\mathbf{u} \circ \mathbf{d}) = \min_{D' \in \mathcal{D}} \left\{ \sum_{\mathbf{d}' \in D'} I_{\text{Ent}}(\mathbf{u} \circ \mathbf{d}') \right\}$$

and $|D \cap D^*|$ is maximum among all set of vectors in $\mathcal{D}$ satisfying [25]. Assume that $D \neq D^*$ for otherwise the claim holds trivially.

By Lemma 2 we have that for every $\mathcal{D} = \{\hat{d}^{(1)}, \ldots, \hat{d}^{(L)}\} \in \mathcal{D}$

$$\sum_{i=1}^{L} I_{\text{Ent}}(\mathbf{u} \circ \hat{d}^{(i)}) = I_{\text{Ent}}(\mathbf{u}) - I_{\text{Ent}}(\mathbf{u} \cdot \hat{d}^{(1)}, \ldots, \mathbf{u} \cdot \hat{d}^{(L)})$$

$$= \|\mathbf{u}\|_1 \left( H\left(\frac{u_1}{\|\mathbf{u}\|_1}, \ldots, \frac{u_k}{\|\mathbf{u}\|_1}\right) - H\left(\frac{\mathbf{u} \cdot \hat{d}^{(1)}}{\|\mathbf{u}\|_1}, \ldots, \frac{\mathbf{u} \cdot \hat{d}^{(L)}}{\|\mathbf{u}\|_1}\right) \right)$$

where $H()$ denotes the Entropy function. Let us define $H(\hat{D}) = H\left(\frac{\mathbf{u} \cdot \hat{d}^{(1)}}{\|\mathbf{u}\|_1}, \ldots, \frac{\mathbf{u} \cdot \hat{d}^{(L)}}{\|\mathbf{u}\|_1}\right)$.

Then $\hat{D}$ is a set of vectors that minimizes $\sum_{i=1}^{L} I_{\text{Ent}}(\mathbf{u} \circ \hat{d}^{(i)})$ iff it maximizes $H(\hat{D})$.

We can think of the vectors in $\hat{D}$ as buckets containing components of $\mathbf{u}$, and we say that $u_j$ is in bucket $i$ if $\hat{d}_{j}^{(i)} = 1$. From the above formula and the concavity property of the Entropy function we have that the following claim holds.

**Claim 1.** Assume that there exists a subset $A \subseteq \{j \mid \hat{d}_{j}^{(i)} = 1\}$ of bucket $i$ and a subset $B \subseteq \{j' \mid \hat{d}_{j'}^{(i')} = 1\}$ of bucket $i'$ such that

$$\left| \left( \hat{d}^{(i)} \cdot \mathbf{u} - \sum_{j \in A} u_j + \sum_{j' \in B} u_{j'} \right) - \left( \hat{d}^{(i')} \cdot \mathbf{u} - \sum_{j' \in B} u_{j'} + \sum_{j \in A} u_j \right) \right| \leq \|\hat{d}^{(i)} \cdot \mathbf{u} - \hat{d}^{(i')} \cdot \mathbf{u}\|_1$$

i.e., swapping bucket for elements in $A$ and $B$ does not increase the absolute difference between the sum of elements in buckets $i$ and $i'$. Then, for the set of vectors $\hat{D} = \{\hat{d}^{(1)}, \ldots, \hat{d}^{(L)}\} \in \mathcal{D}$ defined by

$$\hat{d}^{(\ell)} = \begin{cases} \hat{d}^{(i)} & \ell \notin \{i, i'\} \\ \hat{d}^{(i)} - \sum_{j \in A} \mathbf{e}_j + \sum_{j' \in B} \mathbf{e}_{j'} & \ell = i \\ \hat{d}^{(i')} - \sum_{j' \in B} \mathbf{e}_{j'} + \sum_{j \in A} \mathbf{e}_j & \ell = i' \\ \end{cases}$$

i.e., for the set of vectors corresponding to the new buckets, it holds that $H\left(\frac{\mathbf{u} \cdot \hat{d}^{(1)}}{\|\mathbf{u}\|_1}, \ldots, \frac{\mathbf{u} \cdot \hat{d}^{(L)}}{\|\mathbf{u}\|_1}\right) \leq H\left(\frac{\mathbf{u} \cdot \hat{d}^{(1)}}{\|\mathbf{u}\|_1}, \ldots, \frac{\mathbf{u} \cdot \hat{d}^{(L)}}{\|\mathbf{u}\|_1}\right)$, with the equality holding iff inequality [26] is tight.
Because of Claim 1, we have that $D^*$ satisfying \((\ref{25})\) is a set of vectors that coincides with buckets that distribute the components of $u$ in the most balanced way, i.e., $H(D^*)$ is maximum among all $D \in \mathcal{D}$.

From these observations, we can characterize the structure of buckets of $D^*$. For the sake of a simpler notation, let us denote with $S^{(i)}$ the sum of components in bucket $d^{(i)}_i$, i.e., $S^{(i)} = u \cdot d^{(i)}_i$.

We have the following

**Claim 2.** The set $D^*$ satisfies the following properties:

(i) there is no bucket $i$ that consists of a single element $u_j$ with $j \geq L$;

(ii) if $u_j$ is not alone in bucket $i$ then for each $i' \neq i$ it holds that $S^{(i')} \geq u_j$;

(iii) if $u_j$ is not alone in bucket $i$ then for each $i' \neq i$ it holds that $S^{(i')} \geq S^{(i)} - u_j$.

For (i), assume, by contradiction that such $i$ and $j$ exists. Then, since $D^* \neq D$, there exists a bucket $i' \neq i$ that contains at least two elements, with one of them being $u_{j'}$ for some $j' < L$. Then, by Claim 1, swapping the buckets for $u_j$ and $u_{j'}$ produces a new set of vectors with entropy not smaller than $H(D^*)$ and intersection with $D$ larger than that of $D^*$, which is a contradiction.

For (ii), we observe that if there exists a bucket $i'$ such that $S^{(i')} < u_j$ by moving every element of bucket $i'$ into bucket $i$ and moving only $u_j$ from bucket $i$ into bucket $i'$, by Claim 1, we get a new set of vectors with entropy larger than $H(D^*)$, which is a contradiction.

For (iii), we observe that if there exists a bucket $i'$ such that $S^{(i')} < S^{(i)} - u_j$ swapping all the elements of bucket $i'$ with all the elements of bucket $i$ except for $u_j$, by Claim 1, we get a new set of vectors with entropy larger than $H(D^*)$, which is a contradiction.

We are now ready to prove the statement of the lemma. From the definition of $D$, since for $i = 1, \ldots, L - 1$ the bucket $i$ contains only one element, we have $I(u \circ d^{(i)}) = 0$. Let $S = \sum_{j \geq L} u_j$, and define $i(j)$ to be the bucket of $D^*$ that contains $u_j$, for each $j = 1, \ldots, k$. We have

$$\sum_{i=1}^{L} I_{\text{Ent}}(u \circ d^{(i)}) = I_{\text{Ent}}(u \circ d^{(L)}) = \sum_{j \geq L} u_j \log \frac{S}{u_j} = \sum_{j \geq L} u_j \log \frac{S}{u_j} \sum_{j \leq \frac{S^{(i)}}{2}} u_j \log \frac{S}{u_j} \sum_{j > \frac{S^{(i)}}{2}} u_j \log \frac{S}{u_j} \quad (27)$$

where in the last expression we split the summands according to whether $u_j \geq \frac{S^{(i)}}{2}$ or $u_j < \frac{S^{(i)}}{2}$. We will argue that

$$\sum_{j \geq L} u_j \log \frac{S}{u_j} \sum_{j \leq \frac{S^{(i)}}{2}} u_j \log \frac{S}{u_j} = O(\log L) \sum_{i=1}^{L} I_{\text{Ent}}(u \circ d^{(i)}) \quad (28)$$

$$\sum_{j \geq L} u_j \log \frac{S}{u_j} \sum_{j > \frac{S^{(i)}}{2}} u_j \log \frac{S}{u_j} = O(\log L) \sum_{i=1}^{L} I_{\text{Ent}}(u \circ d^{(i)}) \quad (29)$$

from which the statement of the lemma follows.

**Proof of Inequality \((\ref{25})\).**
Since
\[
\sum_{i=1}^{L} I_{\text{Ent}}(u \circ d^{(i)}) \geq \sum_{j > L} u_j \log \frac{S(i(j))}{u_j},
\]
(30)

it is enough to show that for each \(j \geq L\), with \(u_j \leq S(i(j))/2\), we have
\[
\frac{u_j \log \frac{S(i(j))}{u_j}}{u_j \log \frac{S(i(j))}{u_j}} \leq \log(4L).
\]
(31)

The above inequality can be established by showing that \(S \leq 2L \cdot S(i(j))\) and, then, using the bound \(\frac{\log a}{\log b} \leq \log(2a/b)\), which holds whenever \(b \geq 2\) and \(a \geq b\).

To see that \(S \leq 2L \cdot S(i(j))\), let \(\ell\) be a bucket in \(D^*\) containing some \(u_j\) for \(j' \geq L\). By Claim 2 (i) we have that bucket \(\ell\) contains at least two elements. Let \(e(\ell)\) be the element in bucket \(\ell\) of minimum value. Then, by Claim 2 (iii), we have
\[
S(i(j)) \geq S(\ell) - u_{e(\ell)} \geq S(\ell)/2,
\]
(32)
where the last inequality follows from the fact that bucket \(\ell\) has at least two elements. Let \(B = \{\ell \mid \text{bucket } \ell \text{ has at least one element } u_{j'} \text{ with } j' \geq L\}\). Then, we have \(LS(i(j)) = \sum_{\ell \in B} S(\ell)/2 \geq S/2\), that gives \(S/S(i(j)) \leq 2L\), as desired.

Proof of Inequality (24).

First we argue that we can assume that there exists at most one \(j\), with \(j \geq L\), with \(u_j > S(i(j))/2\). In fact, if there exist \(j \neq j'\) such that \(u_j > S(i(j))/2\) and \(u_{j'} > S(i(j'))/2\) then \(i(j') \neq i(j)\) and no element \(u_r\), with \(r < L\), is either in bucket \(i(j)\) or in \(i(j')\). Hence, by the pigeonhole principle, there must exist elements \(u_r\) and \(u_s\), with \(r, s < L\) that are both in some bucket \(i' \notin \{i(j), i(j')\}\). Thus, by Claim 1, swapping buckets for \(u_r\) and \(u_j\) we get a new set of vectors \(D'\) whose buckets are at least as balanced as those of \(D^* (H(D') \geq H(D^*))\) and \(|D' \cap D| \geq |D^* \cap D|\). However, in \(D'\) there is one less index \(j\) with \(j \geq L\) and \(u_j > S(i(j))/2\). Thus, by repeating this argument, we eventually obtain a \(D'\) satisfying \(H(D^*) = H(D')\) (maximum) and there is at most one \(j\) satisfying \(u_j > S(i(j))/2\).

We also have that \(u_j = u_L\). For otherwise, if \(u_L > u_j\), by the previous observation we have that \(S(i(L)) \geq 2u_L\) hence swapping \(u_L\) and \(u_j\) we obtain a more balanced set of vectors \(D'\) with \(H(D') > H(D^*)\), against the hypothesis that \(H(D^*)\) is maximum. Therefore, we can assume, w.l.o.g., that \(j = L\) and \(i(L) = L\).

Finally, for each \(\ell, \ell' < L\) we can assume that \(u_\ell\) and \(u_{\ell'}\) are in different buckets. For otherwise, swapping buckets for \(u_\ell\) and \(u_{\ell'}\) we get a new set \(D'\) with \(H(D') \geq H(D^*)\) and for all \(j \geq L\), \(u_j \leq S(i(j))/2\). Then, the desired result would follow because we already proved that inequality (24) holds.

Because of the previous observation we can assume that in \(D^*\), up to renaming the buckets, for each \(m \in [L]\) the element \(u_m\) is in bucket \(m\). Let \(X_m = S^{(m)} - u_m\). Note that \(u_L + \sum_{m=1}^{L} X_m = S\).
Then, we have the following lower bound on the impurity of the buckets of $D^*$:

$$
\sum_{i=1}^{L} I_{\text{Ent}}(u \circ d_s^{(i)}) \geq \sum_{m=1}^{L} u_m \log \frac{S^{(m)}}{u_m} \geq u_L \left( \sum_{m=1}^{L} \log \frac{S^{(m)}}{u_m} \right) \tag{33}
$$

$$
= u_L \log \left( \prod_{m=1}^{L} \frac{(u_m + X_m)}{u_m} \right) = u_L \log \left( \frac{(u_L + X_L) \prod_{m=1}^{L-1} (u_m + X_m)}{u_L \prod_{m=1}^{L-1} u_m} \right) \tag{34}
$$

On the other hand, because of the standing assumption $j = L$ we can write as upper bound on the only summand in the left hand side of (29)

$$
u_j \log \frac{S}{u_j} = u_L \log \left( \frac{(u_L + X_L) + \sum_{m=1}^{L-1} X_m}{u_L} \right).
$$

Therefore, to prove the bound in (29) it is enough to show

$$(u_L + X_L) + \sum_{m=1}^{L-1} X_m \leq \left( \frac{(u_L + X_L) \prod_{m=1}^{L-1} (X_m + u_m)}{\prod_{m=1}^{L-1} u_m} \right).
$$

We can now show that this inequality holds by using Claim 2 (ii), which gives $(u_L + X_L) \geq u_s$ for each $s < L$ such that $X_s \neq 0$. Therefore, we have

$$
\left( (X_L + u_L) + \sum_{s=1}^{L-1} X_s \right) \prod_{m=1}^{L-1} u_m = (X_L + u_L) \prod_{m=1}^{L-1} u_m + \sum_{s=1}^{L-1} X_s \prod_{m=1}^{L-1} u_m
$$

$$
\leq (X_L + u_L) \prod_{m=1}^{L-1} u_m + \sum_{s=1}^{L-1} \frac{u_L + X_L}{u_s} X_s \prod_{m=1}^{L-1} u_m
$$

$$
= (X_L + u_L) \prod_{m=1}^{L-1} u_m + (u_L + X_L) \sum_{s=1}^{L-1} X_s \prod_{m \in [L-1] \setminus s} u_m
$$

$$
= (u_L + X_L) \left( \prod_{m=1}^{L-1} u_m + \sum_{s=1}^{L-1} X_s \prod_{m \in [L-1] \setminus s} u_m \right)
$$

$$
\leq (u_L + X_L) \prod_{m=1}^{L-1} (u_m + X_m),
$$

which concludes the proof of (29).

The proof of the lemma is complete. $\square$

By (13), combining the results in the previous lemma with Theorems 3, 4 and the fact that $L \leq n$, we have the following results that apply regardless the relation between $k$ and $L$.

**Theorem 5.** Let $(V, I_{\text{Ent}}, L)$ be an instance of PMWIP and let $p = \min \{ \log L, \log k \} + \log (\sum_{v \in V} \|v\|_1)$. Then, $A^{\text{Dom}}$ on instance $(V, I_{\text{Ent}}, L)$ guarantees $2p$-approximation.

**Theorem 6.** Let Uniform-PMWIP (U-PMWIP) be the variant of PMWIP where all vectors have the same $\ell_1$ norm. We have that $A^{\text{Dom}}$ is an $O(\log k + \log n)$-approximation algorithm for U-PMWIP with $I = I_{\text{Ent}}$. 

18
5 An $O(\log^2(\min\{k, L\}))$-approximation for PMWIP with $I_{\text{Ent}}$

In this section we present our main result on the entropy measure. Under the assumption $k \leq L$, we will show the existence of an $O(\log^2 k)$-approximation polynomial time algorithm. Note that in the light of Lemma 10 and the approach of Section 3 (see, in particular equation (12)), this implies an $O(\log^2(\min\{k, L\}))$-approximation algorithm for any $k$ and $L$.

Recall that a vector $v$ is called $i$-dominant if $i$ is the largest component in $v$, i.e., $v_i = \|v\|_\infty$. Accordingly, we say that a set of vectors $B$ (often, in this section, referred to as a bucket) is $i$-dominant if $i$ is the largest component in the bucket, i.e., $\|\sum_{v \in B} v\|_\infty = \sum_{v \in B} v_i$. We use $\text{dom}(v)$ and $\text{dom}(B)$, respectively, to denote the index of the dominant component of vectors $v$ and $\sum_{v \in B} v$.

We will say that a bucket $B$ is $i$-pure if each vector in $B$ is $i$-dominant. A bucket which is not $i$-pure for any $i$ will be called a mixed bucket. Following the bound on the impurity of a vector $v$ given by Lemma 9, we define the ratio of a vector $v$ as

$$\text{ratio}(v) = \frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty}.$$ 

and, accordingly, the ratio of bucket $B$ as

$$\text{ratio}(B) = \frac{\|\sum_{v \in B} v\|_1}{\|\sum_{v \in B} v\|_1 - \|\sum_{v \in B} v\|_\infty}.$$

Abusing notation, for a set of vectors $B$ we will use $\|B\|_1$ to denote $\|\sum_{v \in B} v\|_1$ and $\|B\|_\infty$ to denote $\|\sum_{v \in B} v\|_\infty$. Moreover, we use $B(j)$ to denote the set of the $j$ vectors in $B$ of minimum ratio. Since in this section we are only focusing on the entropy impurity measure, we will use $I$ to denote $I_{\text{Ent}}$.

We will find it useful to employ the following corollary of Lemma 9.

**Corollary 1.** For a vector $v \in \mathbb{R}^k_+$ and $i \in [k]$ we have

$$\left(\|v\|_1 - \|v\|_\infty\right) \max\left\{1, \log\left(\frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty}\right)\right\} \leq I_{\text{Ent}}(v) \leq 2(\|v\|_1 - v_i) \log\left(\frac{2k\|v\|_1}{\|v\|_1 - v_i}\right) \quad (35)$$

**Proof.** The second inequality follows from Lemma 9 and Proposition 2 using $A = 2k\|v\|_1$. 

5.1 Our Tools

In this section we discuss the main tools employed to design our algorithms.

The example of Remark 2, apart from establishing the tightness of $A^{\text{Dom}}$ for $I_{\text{Ent}}$, also shows that we cannot obtain a very good partition by just considering those containing only pure buckets. However, perhaps surprisingly, the situation is different if we allow at most one mixed bucket. This is formalized in Theorem 7, our first and main tool to obtain good approximate solutions for instances of PMWIP. This structural theorem will be used by our algorithms to restrict the space where a partition with low impurity is searched. Its proof, presented in the next section, is reasonably involved: it consists of starting with an optimal partition and then showing how to exchange vectors from its buckets so that a new partition $P'$ satisfying the desired properties is obtained.
Theorem 7. There exists a partition \( \mathcal{P}' \) with the following properties: (i) it has at most one mixed bucket; (ii) if \( \mathbf{v} \) is an \( i \)-dominant vector in the mixed bucket and \( \mathbf{v}' \) is an \( i \)-dominant vector of an \( i \)-pure bucket, then \( \text{ratio}(\mathbf{v}) \leq \text{ratio}(\mathbf{v}') \); (iii) the impurity of \( \mathcal{P}' \) is at an \( O(\log^2 k) \) factor from the minimum possible impurity.

Our second tool is a transformation \( \chi^{2C} \) that maps vectors in \( \mathbb{R}^k \) into vectors in \( \mathbb{R}^2 \). The nice property of this transformation is that it preserves the entropy of a set of \( i \)-pure vectors up to an \( O(\log k) \) distortion as formalized by Proposition 3. Thus, in the light of Theorem 7, instead of searching for low-impurity partitions of \( k \)-dimensional vectors with at least \( L \)-1 pure buckets, we can search for those in a 2-dimensional space.

The transformation \( \chi^{2C} \) is defined as follows

\[
\chi^{2C}(\mathbf{v}) = \begin{cases} 
(\|\mathbf{v}\|_\infty, \|\mathbf{v}\|_1 - \|\mathbf{v}\|_\infty) & \text{if } \|\mathbf{v}\|_\infty \geq \frac{1}{2}\|\mathbf{v}\|_1 \\
(\|\mathbf{v}\|_1/2, \|\mathbf{v}\|_1/2) & \text{if } \|\mathbf{v}\|_\infty < \frac{1}{2}\|\mathbf{v}\|_1.
\end{cases}
\]

Let \( I_2(B) \) to denote the 2-impurity of the set \( B \), that is, the impurity of the set of 2-dimensional vectors obtained by applying \( \chi^{2C} \) to each vector in \( B \). We have that

Proposition 3. Fix \( i \in [k] \) and let \( B \) be an \( i \)-pure bucket. It holds that

\[
(1/2)I_2(B) \leq I(B) \leq 2I_2(B) + 4(\log k) \sum_{\mathbf{w} \in B} I(\mathbf{w}).
\]

Finally, our last tool is the following result from [17], here stated following our notation, that shows that PMWIP can be optimally solved when \( k = 2 \).

Theorem 8 ([17]). Let \( V \) be a set of 2-dimensional vectors and let \( L \) be an integer larger than 1. There exists a polynomial time algorithm to build a partition of \( V \) into \( L \) buckets with optimal impurity.

In addition, the partition computed by the algorithm satisfies the following property: if \( B \) is a bucket in the partition and if \( \mathbf{v} \in V \setminus B \) then either \( \text{ratio}(\mathbf{v}) \geq \max_{\mathbf{v}' \in B} \{\text{ratio}(\mathbf{v}')\} \) or \( \text{ratio}(\mathbf{v}) \leq \min_{\mathbf{v}' \in B} \{\text{ratio}(\mathbf{v}')\} \).

Motivated by the previous results we define \( \mathcal{A}^{2C} \) as the algorithm that takes as input a set of vectors \( B \) and an integer \( b \) and produces a partition of \( B \) into \( b \) buckets by executing the following steps: (i) every vector \( \mathbf{v} \in B \) is mapped to \( \chi^{2C}(\mathbf{v}) \); (ii) the algorithm given by Theorem 3 is applied over the transformed set of vectors to distribute them into \( b \) buckets; (iii) the partition of \( B \) corresponding to the partition produced in step (ii) is returned.

Algorithm \( \mathcal{A}^{2C} \) is employed as a subroutine of the algorithms presented in the next section. The following property holds for \( \mathcal{A}^{2C} \).

Proposition 4. Let \( B \) be an \( i \)-pure set of vectors. The impurity of the partition \( \mathcal{P} \) constructed by the algorithm \( \mathcal{A}^{2C} \) on input \((B,b)\) is at most an \( O(\log k) \) factor from the minimum possible impurity for a partition of \( B \) into \( b \) buckets.

Proof. Let \( \mathcal{P}^* \) be the partition of \( B \) into \( b \) buckets with minimum impurity. We have that

\[
I(\mathcal{P}) \leq 2I_2(\mathcal{P}) + 4\log k \sum_{\mathbf{w} \in B} I(\mathbf{w}) \leq 2I_2(\mathcal{P}^*) + 4\log k \sum_{\mathbf{w} \in B} I(\mathbf{w}) \\
\leq 4I(\mathcal{P}^*) + 4(\log k) \sum_{\mathbf{w} \in B} I(\mathbf{w}) = O(\log k)I(\mathcal{P}^*),
\]

20
where the first inequality follows from Proposition 8 (applied to each bucket of \( \mathcal{P} \)), the second one from the optimality of \( \mathcal{P} \), the third one by Proposition 3 (applied to each bucket of \( \mathcal{P}^* \)), and the last one by observing that by superadditivity of \( I \) we have \( I(\mathcal{P}^*) \geq \sum_{w \in B} I(w) \).

5.2 Proof of Theorem 7

The proof proceeds in steps. Lemma 11 shows that there exist a partition with at most one mixed bucket whose impurity is \( O(\log k) \) factor from \( \text{OPT}(V) \). Next, we explain how to modify this partition in order to obtain a new partition \( \mathcal{P} \) with at most one mixed bucket, impurity limited by \( O(\log k)\text{OPT}(V) \) and such that the vectors in its \( i \)-pure buckets are ordered according to their ratios. Finally, we show how to modify \( \mathcal{P} \) so that we obtain a partition \( \mathcal{P}' \) that satisfies the properties of Theorem 7.

**Lemma 11.** There exists a partition with at most one mixed bucket that satisfies: (i) the impurity of the mixed bucket is at a \( O(\log k) \) factor from the optimal impurity and (ii) the sum of the impurities of the pure buckets is at most the optimal impurity.

**Proof.** Let \( \mathcal{P}^* \) be an optimal partition. If \( \mathcal{P}^* \) has at most one mixed bucket we are done. Otherwise, let \( B_1, \ldots, B_j \), with \( j \geq 2 \), be the mixed buckets in \( \mathcal{P}^* \). We assume w.l.o.g. that \( B_1 \) is the bucket with the smallest ratio among the mixed buckets.

For \( i = 2, \ldots, j \), let \( S_i = \{ v \mid v \in B_i \text{ and } \text{dom}(v) \neq \text{dom}(B_i) \} \). Let \( \mathcal{P} \) be a new partition obtained from \( \mathcal{P}^* \) by replacing \( B_1 \) with \( B'_1 = B_1 \cup S_2 \cup \ldots \cup S_j \) and \( B_i \) with \( B'_i = B_i \setminus S_i \), for \( i \geq 2 \). It is clear that \( B'_1 \) is the unique mixed bucket in \( \mathcal{P} \).

It follows from subadditivity that \( I(B'_1) \leq I(B_i) \) for \( i > 1 \), which establishes (ii). Thus, in order to complete the proof it is enough to establish an upper bound on \( I(B'_1) \).

For \( i = 2, \ldots, j \), let \( u^{(i)} = \sum_{v \in B_i} v \) and \( w^{(i)} = \sum_{v \in S_i} v \). Moreover, let \( s_i = \| w^{(i)} \|_1 \). Thus,

\[
\| u^{(i)} \|_1 - \| w^{(i)} \|_\infty = \| u^{(i)} \|_1 - \sum_{v \in B_i} v_{\text{dom}(B_i)} \geq \| w^{(i)} \|_1 - \sum_{v \in S_i} v_{\text{dom}(B_i)} \geq \| w^{(i)} \|_1 / 2 = \frac{s_i}{2},
\]

where the leftmost inequality holds because for each \( v \in S_i \) we have \( \text{dom}(v) \neq \text{dom}(B_i) \), so that \( \| v \|_1 / 2 \geq v_{\text{dom}(B_i)} \).

Therefore, it follows from Corollary 1 that

\[
I(B_i) \geq (\| u^{(i)} \|_1 - \| u^{(i)} \|_\infty) \max\{1, \log(\text{ratio}(B_i))\} \geq \frac{s_i}{2} \max\{1, \log(\text{ratio}(B_i))\},
\]

for each \( i > 1 \).

We assume w.l.o.g. that \( B_1 \) is 1-dominant. Let \( u^{(1)} = \sum_{v \in B_1} v \) and let \( s_1 = \| u^{(1)} \|_1 \) and \( c_1 = \| u^{(1)} \|_1 - \| u^{(1)} \|_\infty \). Again, from Corollary 1 we have

\[
I(B_1) \geq c_1 \max\{1, \log(\text{ratio}(B_1))\}
\]

(37)

For \( i = 2, \ldots, j \) let \( c_i = \| w^{(i)} \|_1 - w^{(i)}_1 \). Let \( u = \sum_{v \in B'_1} v \). Then, \( u = u^{(1)} + \sum_{i=2}^j w^{(i)} \), hence \( \| u \|_1 = \sum_{i=1}^j s_i \) and \( \| u \|_1 - u_1 = \sum_{i=1}^j c_i \).
By Corollary \textbf{9} (with \(i = 1\)) we have that

\[
I(B_1) \leq 2 \left( \sum_{i=1}^{j} c_i \right) \cdot \log \left( 2 k \frac{\sum_{i=1}^{j} s_i}{\sum_{i=1}^{j} c_i} \right) \\
\leq 2 \left( c_1 + \sum_{i=2}^{j} s_i \right) \log \left( 2 k \frac{s_1 + \sum_{i=2}^{j} s_i}{c_1 + \sum_{i=2}^{j} s_i} \right) \\
\leq 2 \left( c_1 + \sum_{i=2}^{j} s_i \right) \log \left( 2 k \frac{s_1}{c_1} \right) = 2 \left( c_1 + \sum_{i=2}^{j} s_i \right) \log(2k \cdot \text{ratio}(B_1)), \tag{38}
\]

where the second inequality follows from Proposition \textbf{2}.

Since \(\text{ratio}(B_1) \leq \text{ratio}(B_t)\) for \(t > 1\) we can conclude, by using the lower bounds (36) and (37) that \(I(B_1) = O(\log k) \sum_{i=1}^{j} I(B_i)\).

\[\square\]

Using the mapping \(\chi^{2C}\) and Proposition \textbf{4} we can derive the following result.

\textbf{Lemma 12.} There exists a partition with the following properties: (i) it has at most one mixed bucket; (ii) if \(B_i\) is a \(i\)-pure bucket and \(v\) is a \(i\)-dominant vector that belongs to an \(i\)-pure bucket different from \(B_i\) then either \(\text{ratio}(v) \geq \max_{v' \in B_i \setminus \{v\}} \{\text{ratio}(v')\}\) or \(\text{ratio}(v) \leq \min_{v' \in B_i \setminus \{v\}} \{\text{ratio}(v')\}\) and (iii) its impurity is at a \(O(\log k)\) factor from the minimum possible impurity.

\textbf{Proof.} Let \(\mathcal{P}\) be a partition that satisfies Lemma \textbf{11}. Let \(V_i\) be the set of \(i\)-dominant vectors that are not in the mixed bucket. If \(V_i \neq \emptyset\) let \(B_1^i, \ldots, B_n^i\) be the \(i\)-pure buckets where they lie. We replace these \(t(i)\) buckets by the \(t(i)\) buckets obtained by running algorithm \(A^{2C}\) for input \((V_i, t(i))\). This replacement is applied for every \(i\). It follows from Proposition \textbf{4} that the total impurity of the pure buckets in the new partition is at most at a \(O(\log k)\) factor from the total impurity of the pure buckets in \(\mathcal{P}\).

The property (ii) is assured by the structure of the partition constructed by Algorithm \(A^{2C}\). In order to guarantee that the ties are broken correctly we present the \(i\)-dominant vector for algorithm \(A^{2C}\) in the order of their ratios.

\[\square\]

Now, we conclude the proof of Theorem \textbf{7}. Our starting point is the partition \(\mathcal{P}\) that satisfies items (i)-(iii) of Lemma \textbf{12}. We show how to obtain a partition \(\mathcal{P}'\) from \(\mathcal{P}\) that satisfies the properties of Theorem \textbf{7}.

Let \(B_{\text{mix}}\) be the mixed bucket in \(\mathcal{P}\). We assume w.l.o.g that \(\text{dom}(B_{\text{mix}}) = 1\). Moreover, let \(B_i\) be the \(i\)-pure bucket that contains the \(i\)-dominant vectors with the smallest ratios. In what follows we assume that the vectors in \(B_i\) are sorted by increasing order of their ratios so that by the \(j\)th first vector in \(B_i\) we mean the one with the \(j\)th smallest ratio.

Let \(s_{i,p} = \|B_i\|_1\) (\(p\) indicates a pure bucket, \(i\) indicates the dominance, and \(s\) indicates that we are considering the total sum of the components of the vectors). Let \(V_{i,\text{mix}}\) be the set of \(i\)-dominant vectors in \(B_{\text{mix}}\), i.e., \(V_{i,\text{mix}} = \{v \in B_{\text{mix}} \mid \text{dom}(v) = i\}\).

Let \(s_{i,\text{mix}} = \|V_{i,\text{mix}}\|_1\), i.e., \(s_{i,\text{mix}}\) denotes the total sum of the components of the \(i\)-dominant vectors from bucket \(B_{\text{mix}}\).

In order to explain the construction of \(\mathcal{P}'\) we need to define \(2k\) set of vectors \(X_1, Y_1, \ldots, X_k, Y_k\) that will be moved among the buckets of \(\mathcal{P}\) to obtain \(\mathcal{P}'\). Those are defined according to the following cases:
impurity is an efficient construct a good partition for the case where the partition better understanding of the strategy at the basis of our algorithm, let us first discuss how one can its pure buckets.

The key idea is to look among the partitions that satisfy the properties of Theorem 7 for one that then we show how to convert it into a polynomial time algorithm with the same approximation.

Proof. See the Appendix.

We first present a pseudo-polynomial time algorithm that provides an approximation, has no mixed buckets.

5.3 The approximation algorithm

We first present a pseudo-polynomial time algorithm that provides an approximation, has no mixed buckets.

A special case: no mixed bucket. Theorem 7 establishes the existence of a partition \( \mathcal{P}^* \) whose impurity is an \( O(\log^2 k) \) approximation of the optimum and has at most one mixed bucket. For a better understanding of the strategy at the basis of our algorithm, let us first discuss how one can efficiently construct a good partition for the case where the partition \( \mathcal{P}^* \), achieving the \( O(\log^2 k) \) approximation, has no mixed buckets. In this case, we can employ algorithm \( \mathcal{A}^{2C} \) to obtain a partition with minimum 2-impurity among those that only have pure buckets. By Proposition 3 it follows that the impurity of a partition made only of pure buckets, is upper bounded by its 2-impurity plus \( O(\log k) \) times a lower bound on the optimal impurity. Proceeding like in the proof of Proposition 4 then we can show that the impurity of the partition of minimum 2-impurity is upper bounded by the same upper bound on the impurity of \( \mathcal{P}^* \).

\[ \text{case 1. } s_{i,p} < s_{i,mix}. \]

subcase 1.1 \( i > 1 \) (i is not the dominant component of \( B_{mix} \)). Let \( r_i \) be the largest ratio among the ratios of the vectors from \( B_i \). In addition, let \( Y_i = B_i \) and let \( X_i \) be the set of \( i \)-dominant vectors from \( B_{mix} \) whose ratios are larger than \( r_i \).

subcase 1.2 \( i = 1 \). Let \( m \) be such that \( \|V_{1,mix}(m-1)\|_1 \leq s_{1,p} \) and \( \|V_{1,mix}(m)\|_1 > s_{1,p} \).

Moreover, let \( r_1 \) be the ratio of the \( m \)-th first \( 1 \)-dominant vector of \( V_{1,mix} \). Let \( X_1 = V_{1,mix} \setminus V_{1,mix}(m-1) \) (the set containing all the \( 1 \)-dominant vector of \( B_{mix} \) but the first \( m - 1 \) ones) and let \( Y_1 = \{ v \in B_1 \mid \text{ratio}(v) < r_1 \} \) (the set containing every vector in \( B_1 \) with ratio smaller than \( r_1 \)).

\[ \text{case 2. } s_{i,p} \geq s_{i,mix}. \]

In this case, let \( m \) be such that \( \|B_i(m-1)\|_1 < s_{i,mix} \) and \( \|B_i(m)\|_1 \geq s_{i,mix} \). Moreover, let \( r_i \) be the ratio of the \( m \)-th vector of \( B_i \). We define \( Y_i = B_i(m-1) \) (the set containing the \( m - 1 \) first vectors of \( B_i \)) and \( X_i = \{ v \in V_{1,mix} \mid \text{ratio}(v) > r_1 \} \) (the set containing every \( i \)-dominant vector in \( B_{mix} \) with ratio larger than \( r_1 \)).

Let \( X = \bigcup_{i=1}^{k} X_i \) and \( Y = \bigcup_{i=1}^{k} Y_i \). The partition \( \mathcal{P}' \) is obtained from \( \mathcal{P} \) by replacing the bucket \( B_{mix} \) with the bucket \( B_{mix}' \) and the bucket \( B_i \), for every \( i \), with \( B_i' = (B_i \cup X_i) \setminus Y_i \).

Lemma 13. The partition \( \mathcal{P}' \) satisfies item (i) and (ii) from Theorem 7.

Proof. By construction every \( i \)-dominant vector in \( B_{mix}' \) has ratio at most \( r_i \) and every \( i \)-dominant vector in \( V \setminus B_{mix}' \) has ratio at least \( r_i \). \( \square \)

Lemma 14. The impurity of the partition \( \mathcal{P}' \) is at most \( O(\log k) \) times larger than that of \( \mathcal{P} \).

Proof. See the Appendix. \( \square \)
The partition with minimum 2-impurity made only of pure buckets can be obtained by means of dynamic programming.

To see this, for each \( j = 1, \ldots, k \) let
\[
V_j = \{ v | \text{dom}(v) = j \} \quad \text{and} \quad S_j = \{ v | \text{dom}(v) = j' \text{ for some } j' \leq j \} \tag{40}
\]
Moreover, for each \( b = 1, \ldots, L \) let \( Q^*(S_j, b) \) be a partition of the vectors of \( S_j \) into \( b \) pure buckets such that its 2-impurity, denoted by \( \text{OPT}_2(j, b) \), is minimum. It is not hard to see that the following recurrence holds:
\[
\text{OPT}_2(j, b) = \begin{cases} 
I_2(A^{2C}(V_j, b)) & \text{if } j = 1 \\
\min_{1 \leq b' < b - j} \{ I_2(A^{2C}(V_j, b')) + \text{OPT}_2(j - 1, b - b') \} & \text{if } j > 1
\end{cases} \tag{41}
\]
where \( A^{2C}(V_j, b) \) is the partition of \( V_j \) into \( b \) buckets obtained by the Algorithm \( A^{2C} \) discussed in the previous section.

Thus, if there exists a partition \( P^* \), without mixed buckets, for which \( I(P^*) = O(\log^2 k)\text{OPT}(V) \), then the impurity of the partition \( Q^*(S_k, L) \) constructed by a DP algorithm based on the equation \( \text{(41)} \) satisfies
\[
I(Q^*(S_k, L)) \leq I_2(Q^*(S_k, L)) + O(\log k) \sum_{v \in V} I(v) \leq I_2(P^*) + O(\log k)\text{OPT}(V) \leq O(\log^2 k)\text{OPT}(V),
\]
where the first inequality in the first line follows from Proposition \( \text{[3]} \) the first inequality in the second line is due to the minimality of the 2-impurity of \( Q^*(S_k, L) \) and the superadditivity of \( I \) implying that \( \sum_{v \in V} I(v) \) is a lower bound on \( \text{OPT}(V) \).

A pseudopolynomial time algorithm for the general case. Now, we turn to the case where there exists at most one mixed bucket in the partition given by Theorem \( \text{[7]} \). Given an instance \((V, I, L)\) of PMWIP, let \( C = \sum_{v \in V} |v| \|v\|_1 \) and for each \( i = 1, \ldots, k \), let \( V_i \) and \( S_i \) be as in \( \text{(40)} \). For fixed \( w, i \in [k], \ell \in [\|V\|], c \in [C], b \in [L] \) let us denote by \( Q^*(w, \ell, S_i, b, c) \) a partition of \( S_i \) into \( b \) buckets that satisfies the following properties:

a it has one bucket, denoted by \( B^{Q^*} \), that contains exactly \( \ell \) vectors that are \( w \)-dominant;

b it contains at most one mixed bucket. This mixed bucket, if it exists, is the bucket \( B^{Q^*} \).

c For every \( i \), if \( v \) and \( v' \) are, respectively, \( i \)-dominant vectors in \( B^{Q^*} \) and \( V \setminus B^{Q^*} \); then ratio\((v) \leq \text{ratio}(v')\);

d the total sum of all but the \( w \)-component of vectors in \( B^{Q^*} \) is equal to \( c \), i.e., \( c = \|B^{Q^*}\|_1 - (\sum_{v \in B^{Q^*}} v_w) \);

e the sum of the 2-impurities of the buckets in \( Q^*(w, \ell, S_j, b, c) \setminus B^{Q^*} \) is minimum among the partitions for \( S_j \) into \( b \) buckets that satisfy the previous items.

The algorithm builds partitions \( Q^* = Q^*(w, \ell, S_k, L, c) \) for all possible combinations of \( w, \ell \) and \( c \) and, then, returns the one with minimum impurity.
This approach is motivated by the following: Let $\mathcal{P}^*$ be a partition that contains one mixed bucket, denoted by $B_{mix}^*$, and satisfies the properties of Theorem 4. For such a partition, let $w^* = \text{dom}(B_{mix}^*)$, $\ell^*$ be the number of $w^*$-dominant vectors in $B_{mix}^*$ and $c^* = \|B_{mix}^*\|_1 - \sum_{v \in B_{mix}^*} v_{w^*}$ (the sum of all but the $w^*$ component of the vectors in $B_{mix}^*$). Then, it is possible to prove that the impurity of a partition $Q^* = Q^*(w^*, \ell^*, S_k, L, c^*)$ is at an $O(\log k)$ factor from that of $\mathcal{P}^*$ (see the proof of Theorem 4 below). The key observations are: (i) the impurity of the bucket $BQ^*$ of $Q^*$ is at an $O(\log k)$ factor from that of $B_{mix}^*$ since $\|BQ^*\|_1$ is at most twice $\|B_{mix}^*\|_1$ and $\|BQ^*\|_1 - \sum_{v \in BQ^*} v_{w^*} = \|B_{mix}^*\|_1 - \sum_{v \in B_{mix}^*} v_{w^*} = c^*$; (ii) the sum of the 2-impurity of the buckets in $Q^* \setminus BQ^*$ is at most the sum of the 2-impurity of the buckets $\mathcal{P}^* \setminus B_{mix}^*$ so that their standard impurities differ by not more than a logarithmic factor.

**Building the partitions $Q^*(w, \ell, S_i, b, c)$**. To simplify our discussion let us assume w.l.o.g. that $w = 1$.

Let $Q^* = Q^*(w, \ell, S_i, b, c)$ be a partition that satisfies properties (a)-(e) above and let $I_{2\text{pure}}(Q^*) = I_2(Q^* \setminus BQ^*)$ be the total 2-impurity of the buckets of $Q$ which are surely pure. Moreover, let $V_i(j)$ be the set of the $j$ vectors of $V_i$ of smallest ratio, and let $c_i(j) = \|V_i(j)\|_1 - \sum_{v \in V_i(j)} v_1$, i.e., the total sum of all components but the first of the vectors in $V_i(j)$.

For $i = 1$ we have

$$I_{2\text{pure}}(Q^*(1, \ell, S_i, b, c)) = \begin{cases} I_2(A^{2C}(V_1 \setminus V_1(\ell), b - 1)), & \text{if } c = c_1(\ell) \\ \infty, & \text{otherwise} \end{cases} \tag{42}$$

For $i > 1$ we have

$$I_{2\text{pure}}(Q^*(1, \ell, S_i, b, c)) = \min_{0 \leq j \leq |V_i| \atop 0 \leq j' < b} \{I_2(A^{2C}(V_i \setminus V_i(j), b')) + I_{2\text{pure}}(Q^*(1, \ell, S_{i-1}, b - b', c - c_i(j)))\} \tag{43}$$

Algorithm 1 relies on equations (42) and (43). First, at line 1 it preprocesses the partitions generated by algorithm $A^{2C}$ that are used by these equations. Next, it runs over the possible combinations $(w, \ell)$ and, for each of them, the procedure $\mathcal{M}$ is called to search for a partition with impurity smaller than those found so far.

For a fixed pair $(w, \ell)$, procedure $\mathcal{M}$ constructs partitions $Q^*(w, \ell, S_i, b, c)$ for all the possible combinations of $i$ and $b$ and all the possible corresponding $c$. Thus, to simplify we use $Q^*(S_i, b, c)$ to refer to $Q^*(w, \ell, S_i, b, c)$. The first step of procedure $\mathcal{M}$, where component $w$ is relabeled to 1 is only meant to keep a direct correspondence with the assumption $w = 1$ in equations (42) and (43). Equation (42) is implemented in lines 8-10 to build the list $U_1$ that contains all the partitions $Q^*(1, b, c)$ for which $I_{2\text{pure}}(Q^*(1, b, c)) \neq \infty$. The loop of lines 11-12 calls procedure GenerateNewList, that employs Equation (43), to build a list $U_i$, from list $U_{i-1}$, containing all partitions $Q^*(i, b, c)$ with $I_{2\text{pure}}(i) \neq \infty$. We note that at line 20 the special bucket $B'$ of the new partition under construction, is obtained as an extension of the bucket $BQ^*$ of the partition $Q^*(i - 1, b_i)$ in $U_{i-1}$, which includes the vectors in $V_1(\ell)$.

At the end of the procedure $\mathcal{M}$ the partition of minimum impurity in $U_k$ is returned. This is the partition of minimum impurity among the partition $Q^*(w, \ell, S_k, b, c)$ stored in list $U_k$ for some $b$ and $c$. Hence, for $w = w^*$ and $\ell = \ell^*$, in particular, it is a partition that has impurity not larger than the partition $Q^*(w^*, \ell^*, S_k, L, c^*)$ which we already observed to be an $O(\log k)$ approximation of the minimum impurity partition satisfying Theorem 7.
Since \( c \leq C = \sum_{v \in V} \|v\|_1 \) and the lists \( U_i \) cannot grow larger than \( kLC \) it is easy to see that the proposed algorithm runs in polynomial time on \( n = |V| \) and \( C = \sum_{v \in V} \|v\|_1 \).

The following theorem gives a formal proof of the approximation guarantee for the solution returned by Algorithm \[\text{Algorithm 1}\]

\begin{algorithm}[H]
\begin{algorithmic}
\State Preprocess \( A^{2^k}(V_j \setminus V_j(j'), b) \) for \( j = 1, \ldots, k, j' = 1, \ldots, |V_j| \) and \( b = 1, \ldots, L \)
\State \( Q_{\text{Best}} \leftarrow \) arbitrarily chosen partition of \( S_k \) into \( L \) buckets
\For{\( w = 1, \ldots, k \) and \( \ell = 1, \ldots, |V_w| \)}
\If{\( I(M(w, \ell)) < I(Q_{\text{Best}}) \)}
\State Update \( Q_{\text{Best}} \) to \( M(w, \ell) \)
\EndIf
\EndFor
\Function{GenerateNewList}{\( U_i \), \( i \)}
\For{every partition \( Q \) in the list \( U \)}
\State \( i, b, c \) be the values s.t. \( Q = Q(i, b, c) \)
\If{\( b < L \)}
\For{\( b' = 1, \ldots, L - b \)}
\For{\( j = 0, \ldots, |V_i| \)}
\State \( b' \leftarrow b' \cup V_i(j) \)
\State \( Q' \leftarrow Q' \cup (Q \setminus B') \cup A^{2^k}(V_i \setminus V_i(j), b'). \)
\State Add \( Q' \) to \( U \)
\State \( c' \leftarrow \|b'\|_1 - \sum_{v \in B'} v \)
\If{\( U \) contains another \( Q'' \) with parameters \( (i, b + b', c') \)}
\If{\( I_2^{\text{opt}}(Q'') > I_2^{\text{opt}}(Q') \)}
\State remove \( Q'' \) from \( U \)
\Else
\State remove \( Q' \) from \( U \)
\EndIf
\EndIf
\EndFor
\EndFor
\EndFunction
\State Return \( U \)
\end{algorithmic}
\end{algorithm}

**Theorem 9.** For instances with vectors of dimension \( k \leq L \), there exists a pseudo-polynomial time \( O(\log^2 k) \)-approximation algorithm for PMWIP.

**Proof.** Let \( Q \) be the partition with smallest impurity between the one returned by Algorithm \[\text{Algorithm 1}\] and the one returned by the DP based algorithm that implements Equation \[\text{(41)}\]. In addition, let \( P^* \) be a partition that satisfies the conditions of Theorem \[\text{7}\]. In particular, we have \( I(P^*) \leq O(\log^2 k) \text{OPT}(V) \).

To show that \( I(Q) \) is \( O(\log^2 k) \text{OPT}(V) \) we compare \( I(Q) \) with \( I(P^*) \). We argue according to whether \( P^* \) has a mixed bucket or not.

**Case 1.** \( P^* \) has a mixed bucket. We can assume that \( P^* \) coincides with the partition \( P' \) of Lemma \[\text{14}\].
Let $B'_{\text{mix}}$ be the mixed bucket of $\mathcal{P}'$ and assume w.l.o.g. that $w'$ is the dominant component in $B'_{\text{mix}}$. Let $s' = \|B'_{\text{mix}}\|_1$, $c' = s' - \|B'_{\text{mix}}\|_\infty$ and $c = s' - \sum_{v \in B'_{\text{mix}}} v_1$ (recall that in the proof of Lemma 14 component 1 is the dominant component of the bucket $B_{\text{mix}}$ from the partition $\mathcal{P}$ that is used as a basis to obtain $\mathcal{P}'$; note that it is possible to have $1 \neq w'$). From Proposition 2, since $c' \leq c$, and the proof of Lemma 14 we have that

$$2c' \log \frac{2ks'}{c'} \leq 2c' \log \frac{2ks'}{c} \leq O(\log^2 k) \text{OPT}(V).$$

In particular, the second inequality in (44) is proved in Appendix D, **Bounds on the mixed buckets $B'_{\text{mix}}$**—note that with our present definition of $s'$ and $c$ the middle term of (44) coincides with the right hand side of (69).

Let $\ell'$ be the number of $w'$-dominant vectors from $\mathcal{P}'$ that lie in $B'_{\text{mix}}$. We know that the impurity of the output partition $Q$ is not larger than that of $Q^*(w', \ell', S_k, L, c')$, one of the partitions built by Algorithm 1. Thus, it is enough to show that the impurity of $Q^*(w', \ell', S_k, L, c')$ is at a $O(\log^2 k)$ factor from the optimum. For this we will show that $I(Q^*)$ is $O(I(\mathcal{P}') + \text{OPT}(V) \log k)$. In what follows we use $Q^*$ to refer to $Q^*(w', \ell', S_k, L, c')$, and as before, $BQ^*$ denotes the special bucket in $Q^*$.

Let

$$s_1 = \left\| \sum_{v \in B'_{\text{mix}}} v \right\|_1 \quad \text{and} \quad c_1 = s_1 - \sum_{v \in B'_{\text{mix}}} v_{w'},$$

By Corollary 1 with $i = w'$, we have that

$$I(BQ^*) \leq 2c' \log \left( \frac{2k}{c'} (2(c' - c_1) + s_1) \right) \leq 2c' \log \frac{4ks'}{c'} \leq O(\log^2 k) \text{OPT}(V) \quad (45)$$

where

- for the first inequality, we are also using the fact that $\|BQ^*\|_1 \leq 2(c' - c_1) + s_1$. To see that the last relation holds we note that

$$\left\| \sum_{v \in B_{Q^*}} v \right\|_1 - \sum_{v \in B_{Q^*}} v_{w'} = c' - c_1,$$

hence $2(c' - c_1)$ is an upper bound on the total mass of the vectors in $B_{Q^*}$ which are not $w'$-dominant. Therefore, we have the upper bound $2(c' - c_1) + s_1$ used in the first inequality for $\|BQ^*\|_1$.

- for the second inequality we are using $s' \geq c' - c_1 + s_1$.

- the last inequality follows from (44).

We now focus on the buckets of $Q^*$ different from $BQ^*$—which are surely pure. From the proof of Lemma 14 (Appendix D, **Bounds on the $i$-pure buckets**) we have that the total impurity of the buckets in $\mathcal{P}'$ different from $B'_{\text{mix}}$ satisfies

$$\sum_{B \in \mathcal{P}' \setminus B'_{\text{mix}}} I(B) = O(\log^2 k) \text{OPT}(V). \quad (46)$$

27
In addition, we have

\[
\sum_{B \in Q^* \setminus B^Q} I(B) \leq \sum_{B \in Q^* \setminus B^Q} (2I_2(B) + 4(\log k) \sum_{w \in B} I(w)) \tag{47}
\]

\[
= 2 \sum_{B \in Q^* \setminus B^Q} I_2(B) + 4(\log k) \sum_{w \in V} I(w) \tag{48}
\]

\[
\leq 2 \sum_{B \in P' \setminus B'_{mix}} I_2(B) + 4(\log k) \sum_{w \in V} I(w) \tag{49}
\]

\[
\leq 4 \sum_{B \in P' \setminus B'_{mix}} I(B) + 4(\log k) \sum_{w \in V} I(w) \tag{50}
\]

\[
\leq O(\log^2)OPT(V) + OPT(V) \log k, \tag{51}
\]

where the inequality in (47) follows from Proposition 3; (49) follows from (48) by the property (e); (50) follows from (49) by Proposition 3 and, finally, to obtain (51) we use (46) for the left term and superadditivity for the right term;

From (45) and (47)-(50) we have

\[
I(Q) \leq I(Q^*) = I(B^Q) + \sum_{B \in Q^* \setminus B^Q} I(B) = O(\log^2 k)OPT(V)
\]

and the proof for Case 1 is complete.

Case 2. \( P^* \) does not have a mixed bucket. In this case, let \( Q' \) be the partition built according to the recurrence in (41). It was argued right after this inequality that \( I(Q') \) is \( O(\log^2 k)OPT(V) \). Thus, \( I(Q) \) is also \( O(\log^2 k)OPT(V) \).

The polynomial time algorithm. Let \( P^* \) be a partition that satisfies the conditions of Theorem 7. If \( P^* \) does not have a mixed bucket then the DP based on Equation (41) is a polynomial time algorithm that builds a partition whose impurity is at most \( O(\log k) \) times larger than that of \( P^* \). Thus, we just need to focus in the case where \( P^* \) has a mixed bucket.

Let \( \text{Algo-Prune} \) be the variant of Algorithm 1 that together with the instance takes as input an extra integer parameter \( t \) and uses the following additional conditions regarding the way the lists \( U_i \)'s are handled: (i) only partitions for which the fifth parameter \( c \) is at most \( t \) are added to \( U_i \); (ii) after creating the list \( U_i \) in line 12 and before proceeding to list \( U_{i+1} \) the following pruning is performed: the interval \([0, \ell]\) is split into 4k subintervals of length \( \ell / 4k \) and while there exist two partitions \( Q(w, \ell, S_i, b, c) \) and \( Q'(w, \ell, S_i, b, c') \) in \( U_i \) with both \( c' \) and \( c \) lying in the same subinterval, the one for which the \( I_{p}^{2\text{pure}}(\cdot) \) is larger is removed. This step guarantees that a polynomial number of partitions are kept in \( U_i \).

Let us consider the algorithm \( A_{poly} \) that executes \( \text{Algo-Prune} \) \( e = \lceil \log(\sum_{v \in V} \|v\|_1) \rceil \) times. In the \( j \)th execution \( \text{Algo-Prune} \) is called with \( t = 2^j \). After execution \( j \) the partition with the minimum impurity found in \( U_k \) is kept as \( Q^{(j)} \). After all the \( e \) executions have been performed, the partition with minimum impurity in \( \{Q^{(1)}, \ldots, Q^{(e)}\} \) is returned.

From the above observation that in each call of \( \text{Algo-Prune} \) the number of partitions kept in the lists is polynomial in size of the instance and the fact that the number of calls to \( \text{Algo-Prune} \)...
is also polynomial in the size of the input, we have that $A_{\text{poly}}$ is a polynomial time algorithm for our problem.

It remains to show that $A_{\text{poly}}$ is also an $O(\log^3 k)$-approximation algorithm. For this, let us consider again the partitions $P^*$ and $Q^*(1, \ell^*, S_k, L, c)$ defined in the case 2 of the proof of Theorem 9. We can show that there is a partition $Q$ among those constructed by $A_{\text{poly}}$ such that $I_2^{\text{pure}}(Q) \leq I_2^{\text{pure}}(Q^*(1, \ell^*, S_k, L, c))$ and such that the special bucket $B^Q$ of $Q$ has $\ell^*$ vectors that are $1$-dominant and satisfies $\|B^Q\|_1 - \sum_{v \in B^Q} v_1 \leq 2(\|B_{\text{mix}}\|_1 - \|B_{\text{mix}}\|_\infty) = 2c$.

Note that these properties are enough to obtain our claim since, with them, proceeding as in the proof of Theorem 9 one can show that the impurity of $Q$ is at most an $O(\log^3 k)$ factor larger than the optimal impurity.

For the definition of $Q$ we need some additional notation. As in Theorem 9 let us denote with $Q^*$ the partition $Q^*(1, \ell^*, S_k, L, c)$. Then $B^{Q^*}$ denotes the special bucket of this partition.

For $i = 1, \ldots, k$ let $b_i$ be the number of $i$-pure buckets in $Q^*$ and let $n_i$ be the number of $i$-dominant vectors that lie in the bucket $B^{Q^*}$. Moreover, let $c_i = \|V_i(n_i)\|_1 - \sum_{v \in V_i(n_i)} v_1$. With this, we have that $\sum_{i=1}^k c_i = c = \|B_{\text{mix}}\|_1 - \|B_{\text{mix}}\|_\infty$.

The partition $Q$ is defined as the last partition of the sequence $Q_1, \ldots, Q_k$, where

- $Q_1$ is the partition $Q^*(1, \ell^*, S_1, b_1, c_1)$ constructed in the $\lceil \log c \rceil$-th execution of ALGO-PRUNE, i.e., with $t = 2^{\lceil \log c \rceil} > c$.

- For $i > 1$, let $Q_i'$ be the partition obtained by extending $Q_{i-1}$ with the $b_i$ buckets from the partition $A^{2C}(V_i \setminus V_i(n_i), b_i)$ and replacing the bucket $B^{Q_{i-1}}$, from $Q_{i-1}$, with $B^{Q_{i-1}} \cup V_i(n_i)$. Note that such a partition is added to $U_i$ before the pruning step (ii) is executed. Then, $Q_i$ is defined as the partition that survives (after the pruning step (ii)) in the subinterval where $Q_i'$ lies.

Let $c_i' = \|B^{Q_i}\|_1 - \sum_{v \in B^{Q_i}} v_1$ (the total mass of vectors in the special bucket $B^{Q_i}$ of $Q_i$, minus the mass of such vectors in the component 1).

We can prove by induction that

$$\left| c_i' - \sum_{j=1}^i c_j \right| \leq \frac{i \cdot t}{4k}.$$ 

For $i = 1$ the result holds since $c_1 = c_1'$. It follows from the induction that

$$c_{i-1}' = \sum_{j=1}^{i-1} c_j \leq \frac{(i-1) \cdot t}{4k}.$$ 

The result for $i$ is established by observing that the pruning step (ii) above, ensures that

$$\left| c_i' - (c_{i-1}' + c_i) \right| \leq \frac{t}{4k}.$$ 

Let $Q_i^*$ be the subpartition of $Q^*$ that contains bucket $B^{Q^*}$ and all $i'$-pure bucket for each $i' \leq i$. We can also prove by induction that $I_2^{\text{pure}}(Q_i) \leq I_2^{\text{pure}}(Q_i^*)$. For $i = 1$ the result holds since $Q_1 = Q_1^*$. For a general $i$ we have that

$$I_2^{\text{pure}}(Q_i) \leq I_2^{\text{pure}}(Q_i-1) + A^{2C}(V_i \setminus V(n_i), b_i) \leq I_2^{\text{pure}}(Q_i-1) + A^{2C}(V_i \setminus V(n_i), b_i) = I_2^{\text{pure}}(Q_i^*).$$

29
Thus, by using the same arguments employed in the proof of Theorem 9 on can show that the impurity of \( Q \) is at an \( O(\log^2 k) \) factor from the optimal one. We can now state the main theorem of the paper.

**Theorem 10.** There is a polynomial time \( O(\log^2(\min\{k, L\})) \) approximation algorithm for PMWIP.

**Proof.** By the above argument we have that Algorithm \( A_{poly} \) is a polynomial time \( O(\log^2 k) \) approximation algorithm for PMWIP with \( k \leq L \). For \( k > L \), applying Lemma 10 and the approach of Section 3 (see, in particular equation (12)), we have an \( O(\log^2 L) \)-approximation algorithm. Putting together the two cases we have the claim. \( \square \)

6 Strong Hardness of PMWIP for the Entropy Impurity measure

In this section we show that PMWIP is strongly NP-hard when \( I \) is the Entropy measure. This rules out an FPTAS for the problem under the standard complexity assumptions. For this we show a reduction from 3-PARTITION.

**Theorem 11.** The PMWIP for the Entropy impurity measure is strongly NP-Hard.

**Proof.** Consider an instance \( \mathcal{T}^{3-Par} \) of 3-PARTITION given by a multiset \( U = \{u_1, \ldots, u_k\} \) of \( k = 3L \) integers such that for each \( i = 1, \ldots k \) it holds that \( T/4 < u_i < T/3 \) where \( T = (\sum_{i=1}^{k} u_1)/L \).

The 3-PARTITION problem consists of deciding whether there exists a partition of \( U \) into \( L \) parts \( A_1, \ldots, A_L \) such that the sum of the elements in each part is equal to \( T \).

From \( \mathcal{T}^{3-Par} \) we can create in polynomial time an instance \( (V, I_{Ent}, L) \) of PMWIP as follows: for each number \( u_i \in U \) add the scaled canonical vector \( \mathbf{v}_i = u_i \mathbf{e}_i \) to \( V \).

Let \( (D^1, D^2, \ldots, D^L) \) be a partition of \( V \) and let \( \mathbf{u} = \sum_{\mathbf{v} \in V} \mathbf{v} = (u_1, \ldots, u_k) \). Let \( \mathbf{d}^{(i)} \in \{0, 1\}^k \) be defined by \( d_j^{(i)} = 1 \) iff \( v_j \in D^i \) then the impurity of \( (D^1, D^2, \ldots, D^L) \) is given by

\[
\sum_{i=1}^{L} I_{Ent}(\mathbf{u} \cdot \mathbf{d}^{(i)}).
\]

By the Subsystem Property—which holds with equality for \( I_{Ent} \) (see Lemma 2)—we have

\[
\sum_{i=1}^{L} I_{Ent}(\mathbf{u} \cdot \mathbf{d}^{(i)}) = I(\mathbf{u}) - I\left((\mathbf{u} \cdot \mathbf{d}^{(1)}, \mathbf{u} \cdot \mathbf{d}^{(2)}, \ldots, \mathbf{u} \cdot \mathbf{d}^{(L)})\right),
\]

hence, the right hand side is minimized when \( I((\mathbf{u} \cdot \mathbf{d}^{(1)}, \mathbf{u} \cdot \mathbf{d}^{(2)}, \ldots, \mathbf{u} \cdot \mathbf{d}^{(L)}) \) is maximum, i.e., the vector \( (\mathbf{u} \cdot \mathbf{d}^{(1)}, \mathbf{u} \cdot \mathbf{d}^{(2)}, \ldots, \mathbf{u} \cdot \mathbf{d}^{(L)}) \) is as balanced as possible.

By the well known properties of the entropy function, we have that deciding in polynomial time whether there is a partition of \( V \) such that the resulting impurity is at most \( I(\mathbf{u}) - \|\mathbf{u}\| \log L \) is equivalent to decide whether there exists a partition of \( V \) into sets \( D^1, \ldots, D^L \) such that \( \mathbf{u} \cdot \mathbf{d}^{(1)} = \mathbf{u} \cdot \mathbf{d}^{(2)} = \cdots = \mathbf{u} \cdot \mathbf{d}^{(L)} = \|\mathbf{u}\|_1/L \), which is the same as deciding whether there is a partition \( A_1, \ldots, A_L \) of \( U \) such that \( \sum_{u \in A^i} u = (\sum_{i=1}^{k} u_1)/L \), i.e., solving the instance \( \mathcal{T}^{3-Par} \) of 3-PARTITION. This concludes the reduction.

Thus, the strong hardness of 3-PARTITION implies the strong hardness of PMWIP for the Entropy impurity measure. \( \square \)
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A The proof of Fact 1

Fact 1 The Gini impurity measure defined by \( I_{\text{Gini}}(u) = ||u||_1 \sum_{i=1}^{k} \frac{u_i}{||u||_1} (1 - \frac{u_i}{||u||_1}) \) and the Entropy impurity measure defined by \( I_{\text{Ent}} = ||u||_1 \sum_{i=1}^{k} \frac{u_i}{||u||_1} \log \left( \frac{||u||_1}{u_i} \right) \) belong to \( C \). For \( f_{\text{Entr}} \), a simple inspection shows that (P3) holds at equality.

Proof. The measure \( I_{\text{Gini}} \) is obtained using the function \( f_{\text{Gini}}(x) = x(1-x) \), and \( I_{\text{Ent}} \) is obtained using the function \( f_{\text{Ent}}(x) = x \log \frac{x}{q} \). Clearly both functions satisfy property (P1), and it is known they also satisfy (P2) \[Q\]. So it remains to be shown that they satisfy property (P3).

For \( f_{\text{Gini}} \), (P3) becomes

\[
p(1-p) \leq p(1-q) + p \left( 1 - \frac{p}{q} \right) \quad \forall q \in [p, 1]
\]

which after canceling the \( p \)'s out and rearranging, is equivalent to \( p \geq q + \frac{q}{p} - 1 \) for all \( q \in [p, 1] \), or \( p \geq \max_{q \in [p, 1]} (q + \frac{q}{p} - 1) \). But the function in the max is convex in \( q \), and hence its maximum is attained at one of the endpoints \( q = p \) and \( q = 1 \); for these endpoints the inequality holds at equality, which then proves the desired property.

For the function \( f_{\text{Ent}}(x) = -x \log x \) we have that for any \( 0 < x \leq y < 1 \) it holds that

\[
-x \frac{\log(y)}{y} - y \frac{\log(x)}{y} = -x \log(y) - x \log(x) + x \log(y) = -x \log(x),
\]

showing that \( f_{\text{Ent}}(x) = -x \log x \) satisfies (P3) with equality. \( \square \)

B The proof of the Claim in Lemma 8

Claim. Fix \( u \in \mathbb{R}^k \) such that \( u_i \geq u_{i+1} \) for each \( i = 1, \ldots, k-1 \). Let \( z^{(1)} \) and \( z^{(2)} \) two orthogonal vectors from \( \{0, 1\}^k \setminus \{0\} \). Let \( i^* = \min \{ i \mid \max \{z_i^{(1)}, z_i^{(2)}\} = 1 \} \) and \( v^{(1)} = e_{i^*} \) and \( v^{(2)} = z^{(1)} + z^{(2)} - e_{i^*} \). Then

\[
I(u \circ v^{(1)}) + I(u \circ v^{(2)}) \leq I(u \circ z^{(1)}) + I(u \circ z^{(1)}).
\]

Proof. For the sake of simplifying the notation, let us assume that \( i^* = 1 \). Since \( v^{(1)} + v^{(2)} = z^{(1)} + z^{(2)} \), and the only significant components are the non-zero components of \( z^{(1)} + z^{(2)} \), for the analysis, we assume without loss of generality that \( z^{(2)} = 1 - z^{(1)} \). Setting \( d = z^{(1)} \), we have to prove that

\[
I_{\text{Gini}}(u \circ e_1) + I_{\text{Gini}}(u \circ (1-e_1)) \leq I_{\text{Gini}}(u \circ d) + I_{\text{Gini}}(u \circ (1-d)),
\]

for every \( d \in \{0, 1\}^k \setminus \{0\} \).

It follows from the definition of \( I_{\text{Gini}}(\cdot) \) that

\[
I_{\text{Gini}}(u \circ d) + I_{\text{Gini}}(u \circ (1-d)) = (u \cdot d) \left( \frac{(u \cdot d)^2 - \sum_{i|d_i=1}(u_i)^2}{(u \cdot d)^2} \right) + (u \cdot (1-d)) \left( \frac{(u \cdot (1-d))^2 - \sum_{i|d_i=0}(u_i)^2}{(u \cdot (1-d))^2} \right) =
\]

\[
||u||_1 \left( \frac{\sum_{i|d_i=1}(u_i)^2}{u \cdot d} \right) - \left( \frac{\sum_{i|d_i=0}(u_i)^2}{u \cdot (1-d)} \right)
\]

Define \( g(d) \) as the sum of two last terms of the above expression, that is,

\[
g(d) = \left( \frac{\sum_{i|d_i=1}(u_i)^2}{u \cdot d} \right) + \left( \frac{\sum_{i|d_i=0}(u_i)^2}{u \cdot (1-d)} \right)
\]

\]

33
It is enough to prove that \( g(e_1) \geq g(d) \) for an arbitrary \( d \). For that, we assume w.l.o.g. that \( d_1 = 1 \) due to the symmetry of \( g(d) \) with respect to \( d \).

Let
\[
\alpha = \frac{\sum_{i > 1 \mid d_i = 1} (u_i)^2}{\sum_{i > 1 \mid d_i = 1} u_i^2} \quad \text{and} \quad \beta = \frac{\sum_{i \mid d_i = 0} (u_i)^2}{\sum_{i \mid d_i = 0} u_i^2}
\]

Thus,
\[
g(d) = \frac{(u_1)^2 + \alpha (u \cdot d - u_1)}{u_1 + (u \cdot d - u_1)} + \beta
\]

Moreover, we can write \( g(e_1) \) as a function of \( d \)
\[
g(e_1) = u_1 + \frac{\alpha (u \cdot d - u_1) + \beta (u(1 - d))}{(u \cdot d - u_1) + u(1 - d)}
\]
The following inequalities will be useful: \( \alpha, \beta \leq u_1 \) since \( u_1 \geq u_i \) for all \( i \), \( (u \cdot d - u_1) \geq \alpha \) and \( u(1 - d) \geq \beta \).

We need to prove that
\[
g(e_1) = u_1 + \frac{\alpha (u \cdot d - u_1) + \beta (u(1 - d))}{(u \cdot d - u_1) + u(1 - d)} \geq \frac{(u_1)^2 + \alpha (u \cdot d - u_1)}{u_1 + (u \cdot d - u_1)} + \beta = g(d),
\]
or equivalently,
\[
\frac{u_1 (u \cdot d - u_1)}{u_1 + (u \cdot d - u_1)} - \frac{\alpha (u \cdot d - u_1)}{u_1 + (u \cdot d - u_1)} \geq \frac{\beta (u \cdot d - u_1)}{u(1 - d) + (u \cdot d - u_1)} - \frac{\alpha (u \cdot d - u_1)}{u(1 - d) + (u \cdot d - u_1)}
\]

Simplifying the terms we need to prove
\[
(\beta - \alpha) [u \cdot d - u_1 + u_1] \leq (u_1 - \alpha) [(u \cdot d - u_1) + u \cdot (1 - d)]
\]
which is equivalent to
\[
\beta u_1 - \alpha u_1 \leq (u_1 - \beta)(u \cdot d - u_1) + (u_1 - \alpha)u \cdot (1 - d), \tag{52}
\]
However, because \( \alpha, \beta \leq u_1 \), \( (u \cdot d - u_1) \geq \alpha \) and \( u \cdot (1 - d) \geq \beta \), we have
\[
(u_1 - \beta)\alpha + (u_1 - \alpha)\beta \leq (u_1 - \beta)(u \cdot d - u_1) + (u_1 - \alpha)u \cdot (1 - d).
\]
Thus, to establish inequality \( \tag{52} \), it is enough to prove that
\[
\beta u_1 - \alpha u_1 \leq (u_1 - \beta)\alpha + (u_1 - \alpha)\beta,
\]
or, equivalently,
\[
\alpha \beta \leq \alpha u_1.
\]
The last inequality holds because \( u_1 \geq \beta \).

\[\square\]

C The proof of Proposition 3

Proposition 3. Fix \( i \in [k] \) and let \( B \) be a set of vector in \( \mathbb{R}^k \) such that for each \( v \in B \), it holds that \( \|v\|_\infty = v_i \), i.e., \( B \) is \( i \)-pure. It holds that
\[
\frac{1}{2} I_2(B) \leq I(B) \leq 2 I_2(B) + 4(\log k) \sum_{w \in B} I(w)
\]
Proof. Let us assume w.l.o.g. that $B$ is 1-pure. Let $v$ be the vector corresponding to $B$, that is, $v = \sum_{v' \in B} v'$. Moreover, let

$$u = \sum_{v' \in B} \chi^{2C}(v')$$

$$u^L = \sum_{v' \in B : \|v'\|_\infty < \|v\|_1/2} \chi^{2C}(v')$$

and

$$u^H = \sum_{v' \in B : \|v'\|_\infty \geq \|v\|_1/2} \chi^{2C}(v')$$

Note that $u^L$ corresponds to the set of vectors for which the dominant component is affected by transformation $\chi^{2C}$. It shall be clear that $\|v\|_1 = \|u\|_1$ and $\|v\|_\infty \leq \|u^L\|_\infty + \|u^H\|_\infty = \|u^L\|_1/2 + \|u^H\|_\infty = \|u\|_\infty$

From Lemma 9 and Corollary 1 we have that

$$\left(\|v\|_1 - \|v\|_\infty\right) \max\left\{1, \log\left(\frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty}\right)\right\} \leq I(v) \leq 2(\|v\|_1 - \|v\|_\infty) \log\left(\frac{2k\|v\|_1}{\|v\|_1 - \|v\|_\infty}\right)$$

(53)

Let $\alpha = \|u\|_1 - \|u^H\|_\infty - \|u^L\|_1/2$. Then, we have

$$I(u) = \alpha \log\left(\frac{\|u\|_1}{\alpha}\right) + (\|u\|_1 - \alpha) \log\left(\frac{\|u\|_1}{\|u\|_1 - \alpha}\right);$$

(54)

Since $\|u^H\|_\infty + \|u^L\|_1/2 \geq \|u\|_1/2$, then $\alpha \leq \|u\|_1/2$, from Proposition 1 we have $\frac{\|u\|_1 - \alpha}{\|u\|_1} \log\frac{\|u\|_1}{\|u\|_1 - \alpha} \leq \frac{\alpha}{\|u\|_1} \log\frac{\|u\|_1}{\alpha}$. This, together with (54) implies that

$$\alpha \log\frac{\|u\|_1}{\alpha} \leq I(u) \leq 2\alpha \log\frac{\|u\|_1}{\alpha}.$$  

(55)

Now we note that $\|u^H\|_\infty + \|u^L\|_1/2 \geq \|v\|_\infty$, hence

$$\alpha = \|u\|_1 - \|u^H\|_\infty - \|u^L\|_1/2 \leq \|v\|_1 - \|v\|_\infty.$$ 

(56)

We first focus on the proof of the left bound $\frac{1}{2}I_2(B) \leq I(B)$. We split the analysis into two cases

Case 1. $\|v\|_1 - \|v\|_\infty \leq \frac{\|v\|_1}{e}$. Then

$$I(u) \leq 2\alpha \log\frac{\|v\|_1}{\alpha} \leq 2(\|v\|_1 - \|v\|_\infty) \log\frac{\|v\|_1}{\|v\|_1 - \|v\|_\infty} \leq 2I(v)$$

(57)

where the first inequality is from (55), the second inequality is from Proposition 2 and the last inequality is from (53) (using the hypothesis at the basis of this case).
Case 2. $\|v\|_1 - \|v\|_\infty > \frac{\|v\|_1}{e}$. Then,

$$I(v) \geq (\|v\|_1 - \|v\|_\infty) \log e \geq \|v\|_1 = \|u\|_1 \geq I(u)$$

(58)

where the first inequality is from (58) and the last inequality is because by definition of $I = I_{Ent}$, for a 2 dimensional vector $u$ we have $I(u) \leq \|u\|_1$.

From (57) and (57) it immediately follows that $\frac{1}{2} I_2(B) = \frac{1}{2} I(u) \leq I(v) = I(B)$.

We now focus on the right inequality and show that $I(v) \leq 2 I(u) + 4(\|v\|_1 - \|v\|_\infty) \log k$, from which also the last inequality in the statement of the proposition immediately follows.

First, we observe that

$$2 \left( \|u\|_1 - \|u^H\|_\infty - \frac{\|u^L\|_1}{2} \right) = \|u^L\|_1 + 2(\|u^H\|_1 - \|u^H\|_\infty) \geq (\|u^L\|_1 + (\|u^H\|_1 - \|u^H\|_\infty))$$

(59)

and that we assumed $s_{i,mix} = \sum_{w\in B} \|w\|_1$ and $\|v\|_1 = \|u\|_1$ (for the second term).

Since $B$ is $i$-pure, we have that $\|v\|_1 = \sum_{w\in B} \|w\|_1$ and $\|v\|_\infty = \sum_{w\in B} \|w\|_\infty$. Then, we have

$$I(v) \leq 2 I(u) + 4(\|v\|_1 - \|v\|_\infty) \log k$$

(60)

where the last inequality follows from the left hand side of (54) together with the definition of $\alpha$ (for the first term) and from $\|u^H\|_\infty - \frac{\|u^L\|_1}{2} \geq \|v\|_\infty$ and $\|v\|_1 = \|u\|_1$ (for the second term).

We have then shown the right inequalities of the statement. The proof of the Proposition is complete.

\[\square\]

D Proof of Lemma 14

Proof. Recall that $s_{i,mix}$ denotes the total sum of the components of the $i$-dominant vectors from bucket $B_{mix}$ and that we assumed $\text{dom}(B_{mix}) = 1$. We let $c_{i,mix} = s_{i,mix} - \sum_{v\in V_{i,mix}} v_1$, i.e., the total sum of all but the first component of the $i$-dominant vectors in $B_{mix}$. Moreover, let
Mix = \sum_{i=1}^{k} s_{i,mix} \text{ and } c_{mix} = \sum_{i=1}^{k} c_{i,mix}. \text{ Furthermore, let } c_{i,p} = s_{i,p} - \sum_{\nu \in B_i} v_i, \text{ i.e., the total sum of the non-}i\text{ components of vectors in } B_i.

It follows from Corollary \ref{corollary1} that

\[ I(B_{mix}) \geq c_{mix} \max \left\{ 1, \log \left( \frac{s_{mix}}{c_{mix}} \right) \right\} \]  

(65)

Note that for \( i > 1 \), we have \( c_{i,mix} \geq (s_{i,mix})/2 \), for otherwise \( i \) would not be the dominant component in \( V_{i,mix} \). Thus, we also have that

\[ I(B_{mix}) \geq c_{mix} \geq c_{1,mix} + \sum_{i=2}^{k} s_{i,mix}/2 \]  

(66)

Moreover, if \( c_{mix} < s_{mix}/e \), from (65) we have that

\[ I(B_{mix}) \geq c_{mix} \geq c_{1,mix} + \frac{k}{2} \sum_{i=1}^{k} s_{i,mix}/2 \]  

(67)

where the last inequality follows from (66) and Proposition \ref{proposition2}.

From Corollary \ref{corollary1} we have that

\[ I(B_i) \geq c_{i,p} \max \left\{ 1, \log \left( \frac{s_{i,p}}{c_{i,p}} \right) \right\} \]  

(68)

for every \( i \)-pure bucket \( B_i \).

Now we derive upper bounds on \( B'_{mix}, B'_1, \ldots, B'_k \) and compare them with the lower bounds given by the previous equations.

**Bound on the mixed bucket \( B'_{mix} \).**

Let \( s_{i,mix} = \|V_{i,mix} \cap B'_{mix}\|_1 \), this is the total sum of the components of the \( i \)-dominant vectors in \( B_{mix} \cap B'_{mix} \).

Moreover, let

\[ c_{i,mix}^L = s_{i,mix} - \sum_{\nu \in V_{i,mix} \cap B'_{mix}}^\nu v_1, \]

i.e., the total sum of all but the first components in the \( i \)-dominant vectors in \( B_{mix} \cap B'_{mix} \).

Recall that \( Y_i \) is the set of \( i \)-dominant vectors moved from \( B_i \) to \( B_{mix} \) in order to obtain partition \( \mathcal{P}' \). Let \( s_{i,p}^L = \|Y_i\|_1 \) and let

\[ t_{i,p}^L = s_{i,p}^L - \sum_{\nu \in Y_i} v_1, \]

i.e., the total sum of all but the first components of vectors in \( Y_i \).

In these notations, the superscript \( L \) is used to remind the reader that these quantities refer to vectors with 'low' ratio.

From Corollary \ref{corollary1} (with \( i = 1 \)) we have that

\[ I(B'_{mix}) \leq 2 \left( \sum_{i=1}^{k} c_{i,mix}^L + t_{i,p}^L \right) \log \left( \frac{2k \left( \sum_{i=1}^{k} s_{i,mix}^L + s_{i,p}^L \right)}{\sum_{i=1}^{k} c_{i,mix}^L + t_{i,p}^L} \right) \]  

(69)
Moreover, we have
\[
\sum_{i=1}^{k} (c_{i,mix}^{L} + t_{i,p}^{L}) \leq c_{1,mix}^{L} + t_{1,p}^{L} + \sum_{i=2}^{k} (s_{i,mix}^{L} + s_{i,p}^{L}) \tag{70}
\]

Thus, we have that
\[
I(B_{mix}') \leq 2 \left( c_{1,mix}^{L} + \sum_{i=2}^{k} (s_{i,mix}^{L} + s_{i,p}^{L}) \right) \log \left( \frac{2k \left( \sum_{i=1}^{k} s_{i,mix}^{L} + s_{i,p}^{L} \right)}{c_{1,mix}^{L} + t_{1,p}^{L} + \sum_{i=2}^{k} (s_{i,mix}^{L} + s_{i,p}^{L})} \right) + 2t_{1,p}^{L} \log \left( \frac{2k \left( \sum_{i=1}^{k} s_{i,mix}^{L} + s_{i,p}^{L} \right)}{t_{1,p}^{L} + \sum_{i=2}^{k} (s_{i,mix}^{L} + s_{i,p}^{L})} \right) \tag{71}
\]
\[
\leq 2 \left( c_{1,mix}^{L} + 2 \sum_{i=2}^{k} s_{i,mix} \right) \log \left( \frac{2k s_{mix}^{L}}{c_{1,mix}^{L} + 2 \sum_{i=2}^{k} s_{i,mix}} \right) + 2t_{1,p}^{L} \log \left( \frac{2k s_{mix}^{L}}{t_{1,p}^{L} + \sum_{i=2}^{k} (s_{i,mix}^{L} + s_{i,p}^{L})} \right) \tag{72}
\]

where the first inequality follows from inequality (69), Proposition 2 and inequality (70); the second inequality follows from \(s_{i,p}^{L} \leq s_{i,mix}^{L}\) and the third inequality from \(s_{i,p}^{L} \leq s_{i,mix}^{L}\) together with Proposition 2.

We prove that the expression in (71)-(72) is at most an \(O(\log k)\) factor of \(I(B_{mix}') + I(B_1)\). First, we consider the term in (71) that we denote by \(\alpha\).

If \(c_{mix} \geq s_{mix}/e\) then
\[
\alpha = \left( c_{1,mix}^{L} + 2 \sum_{i=2}^{k} s_{i,mix} \right) \log \left( \frac{2k s_{mix}^{L}}{c_{1,mix}^{L} + 2 \sum_{i=2}^{k} s_{i,mix}} \right) \leq \left( 2c_{1,mix} + 2 \sum_{i=2}^{k} s_{i,mix} \right) \log \left( \frac{2k s_{mix}^{L}}{2c_{1,mix} + 2 \sum_{i=2}^{k} s_{i,mix}} \right) \leq \left( 2c_{1,mix} + 2 \sum_{i=2}^{k} s_{i,mix} \right) \log \left( \frac{2k s_{mix}^{L}}{2c_{mix}^{L}} \right) = \left( 2c_{1,mix} + 2 \sum_{i=2}^{k} s_{i,mix} \right) \log(ke) \leq 4 \left( c_{1,mix} + \sum_{i=2}^{k} \frac{s_{i,mix}}{2} \right) \log(ke) \tag{73}
\]

which is at a \(O(\log k)\) factor from the lower bound on \(I(B_{mix}')\) given by inequality (66).
On the other hand, if \( c_{\text{mix}} < s_{\text{mix}} / e \), then the first term of (72) is at \( O(\log k) \) factor from lower bound given by inequality (67), in fact we have

\[
\alpha = \left( c_{1, \text{mix}}^L + 2 \sum_{i=2}^k s_{i, \text{mix}} \right) \log \left( \frac{2k s_{\text{mix}}}{c_{1, \text{mix}}^L + 2 \sum_{i=2}^k s_{i, \text{mix}}} \right) \\
\leq \left( c_{1, \text{mix}} + 2 \sum_{i=2}^k s_{i, \text{mix}} \right) \log \left( \frac{2k s_{\text{mix}}}{c_{1, \text{mix}} + 2 \sum_{i=2}^k s_{i, \text{mix}}} \right) \\
\leq 4 \left( c_{1, \text{mix}} + \sum_{i=2}^k \frac{s_{i, \text{mix}}}{2} \right) \log \left( \frac{2k s_{\text{mix}}}{c_{1, \text{mix}} + \sum_{i=2}^k \frac{s_{i, \text{mix}}}{2}} \right) ,
\]

where the first inequality follows from Proposition 2.

Now, we turn to the second term of (72), which we will denote here by \( \beta \). We have that

\[
\beta = t_{1, p}^L \log \left( \frac{2k (s_{1, \text{mix}}^L + s_{1, p}^L) + 2k (\sum_{i=2}^k s_{i, \text{mix}}^L + s_{i, p}^L)}{t_{1, p}^L + \sum_{i=2}^k (s_{i, \text{mix}}^L + s_{i, p}^L)} \right) \\
\leq t_{1, p}^L \log \left( \max \left\{ \frac{2k (s_{1, \text{mix}}^L + s_{1, p}^L)}{t_{1, p}^L}, 2k \right\} \right) \\
\leq t_{1, p}^L \log \left( \max \left\{ \frac{4k \cdot s_{1, p}}{c_{1, p}}, 2k \right\} \right) \\
\leq c_{1, p} \log \left( \max \left\{ \frac{4k \cdot s_{1, p}}{c_{1, p}}, 2k \right\} \right) ,
\]

where the second inequality holds because \( s_{1, \text{mix}}^L \leq s_{1, p} \). Moreover, since \( t_{1, p}^L \leq c_{1, p} \) the last inequality holds due to Proposition 2. It is now easy to see that the quantity in the righthand side of the last inequality is at a \( O(\log k) \) factor from the lower bound on the impurity of \( B_1 \) given by inequality (68).

We have completed the proof that \( I(B_{\text{mix}}') = O(\log k)(I(B_{\text{mix}}) + I(B_1)) \) as desired.

**Bound on \( i \)-pure buckets.** Recall that \( X_i \) is the set of vectors moved from \( B_{\text{mix}} \) to \( B_i \) in order to obtain partition \( P' \). Let \( s_{i, \text{mix}}^H \) be the total sum of the components of all vectors in \( X_i \), i.e., \( s_{i, \text{mix}}^H = \| \sum_{v \in X_i} v \|_1 \). Let \( d_{i, \text{mix}}^H \) be the total sum of all but the \( i \)th components over all vectors in \( X_i \), i.e., \( d_{i, \text{mix}}^H = s_{i, \text{mix}}^H - \sum_{v \in X_i} v_i \). In addition, let \( s_{i, p}^H \) be the total sum of the components of the vectors in the set \( B_i \setminus Y_i \), i.e., \( s_{i, p}^H = \| \sum_{v \in B_i \setminus Y_i} v \|_1 \). Finally, let \( c_{i, p}^H \) be the total sum of all but the \( i \)th component over all vectors in \( B_i \setminus Y_i \) that is \( c_{i, p}^H = s_{i, p}^H - \sum_{v \in B_i \setminus Y_i} v_i \) and let \( c_{i, p}^L \) be the total sum of all but the \( i \)th component over all the vectors in \( Y_i \) that is \( c_{i, p}^L = s_{i, p}^L - \| Y_i \|_\infty \).

**Case 1.)** \( s_{i, p} \geq s_{i, \text{mix}} \). From Lemma 9 we have that

\[
I(B_i') \leq 2(c_{i, p}^H + d_{i, \text{mix}}^H) \log \left( \frac{k \cdot (s_{i, \text{mix}}^H + s_{i, p}^H)}{c_{i, p}^H + d_{i, \text{mix}}^H} \right) \leq 2(c_{i, p}^H + d_{i, \text{mix}}^H) \log \left( \frac{2k \cdot s_{i, p}}{c_{i, p}^H + d_{i, \text{mix}}^H} \right) \ (74)
\]

39
Let \( v \) be the first vector of bucket \( B_i \) that is not moved to \( B_{mix} \) and let \( s = \|v\|_1 \). In particular, \( v \) is the vector with the smallest ratio in \( B_i \) among those that are not moved to \( B_{mix} \). Let \( c = s - v \).

Recall definition of \( r_i \) in the construction of \( P' \). We have that

\[
\frac{s_{H, mix}^H}{d_{i, mix}^H} > r_i \geq \frac{s_{L, p}^L + s}{c_{i, p}^L + c},
\]

(75)

and

\[
s_{i, mix}^H \leq s_{i, mix} \leq s_{i, p}^L + s,
\]

(76)

where the second inequality follows by the definition of \( X_i \) and \( Y_i \) under the standing assumption \( s_{i, p} \geq s_{i, mix} \).

Thus, from (75) and (76) we conclude that

\[
d_{i, mix}^H \leq c_{i, p}^L + c \leq c_{i, p}^L + d_{i, mix}^H \leq 2c_{i, p}.
\]

Therefore, from (74) and Proposition 2 we have

\[
I(B_i') \leq 2(c_{i, p}^H + d_{i, mix}^H) \log \left( \frac{2k \cdot s_{i, p}^L + s_{i, mix}^H}{c_{i, p}^L + d_{i, mix}^H} \right) \leq 4c_{i, p} \log \left( \frac{k \cdot s_{i, p}^L}{c_{i, p}} \right).
\]

(77)

Case 2.) \( s_{i, p} < s_{i, mix} \).

\textbf{subcase 2.1) } \( i = 1 \). From Lemma 9 we have

\[
I(B_1') \leq 2(c_1^H + d_{1, mix}^H) \log \left( \frac{k(s_1^L + s_{1, mix}^H)}{c_1^L + d_{1, mix}^H} \right) \leq 2(c_1^H + d_{1, mix}^H) \log \left( \frac{2k \cdot s_{mix}^L}{c_1^L + d_{1, mix}^H} \right)
\]

Let \( v \) be the first vector of \( B_{mix} \) that is moved to \( B_1 \). Let \( s = \|v\|_1 \) and let \( c = s - v_1 \). By construction we have that \( s_{1, p}^H \leq s_{1, p} \leq s_{1, mix}^L + c \).

Moreover,

\[
\frac{s_{1, p}^H}{c_{1, p}^L} \geq r_1 \geq \frac{s_{1, mix}^L + s}{c_{1, mix}^L + c},
\]

hence \( c_{1, p}^H \leq c_{1, mix}^L + c \).

Therefore, \( c_{1, p}^H + c_{1, mix} \leq 2c_{1, mix} \). Thus, by the above inequality on \( I(B_1') \) and Proposition 2 we have

\[
I(B_1') \leq 4c_{1, mix} \log \left( \frac{2k \cdot s_{mix}^L}{c_{1, mix}} \right) \leq 4c_{mix} \log \left( \frac{2k \cdot s_{mix}^L}{c_{mix}} \right).
\]

(78)

\textbf{subcase 2.2) } \( i > 1 \).

In this case the bucket \( B_i' \) is exactly the set \( X_i \). Thus, it follows from the subadditivity of \( I \) that

\[
I(B_i') = I(X_i) \leq I(V_{i, mix})
\]

(79)

Thus, by aggregating the upper bounds given by Equations (77), (78) and (79), we get that

\[
\sum_{i=1}^{k} I(B_i') \leq \sum_{i=1}^{k} 4c_{i, p} \log \left( \frac{k \cdot s_{i, p}}{c_{i, p}} \right) + 4c_{mix} \log \left( \frac{2k \cdot s_{mix}^L}{c_{mix}} \right) + \sum_{i=2}^{k} I(V_{i, mix})
\]

40
The first term is at most $O(\log k) \sum_{i=1}^{k} I(B_i)$ due to the lower bound in (68). The second term is $O(\log k) I(B_{\text{mix}})$ due to the lower bounds in (66) and (67). Finally, the last term is at most $I(B_{\text{mix}})$ due to the subadditivity of $I$. \qed