Abstract

A complete derivation, from first principles, of the reaction-rate formula for a generic process taking place in a heat bath of finite volume is given. It is shown that the formula involves no finite-volume correction. Through perturbative diagrammatic analysis of the resultant formula, the detailed-balance formula is derived. The zero-temperature limit of the formula is discussed. Thermal cutting rules, which are introduced in previous work, are compared with those introduced by other authors.

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1 Introduction

Ultrarelativistic heavy-ion-collision experiments at CERN and RHIC lead us to entertain a hope of reviving quark-gluon plasma (QGP) in the present day. As promising observables of the QGP formation, rates of various reactions taking place in a QGP (heat bath) have been computed by many authors. Almost all of them, however, concentrated on the analyses of particle production from a QGP or the decay rate of a particle in a QGP, whose computational method has long been known [1].

Since then, through analyses from first principles, a calculational scheme of the rate of a generic thermal reaction has been proposed [2, 3, 4, 5, 6, 7]. The resultant reaction-rate formula is written in terms of the Keldish variant of the real-time formalism (RTF) of thermal field theory [8]. However, complete analysis of classes of diagrams, which leads to diagrams in RTF including thermal propagators with \( n \geq 2 \) thermal self-energy insertion, is still lacking. Ref. [3] is the only work that discusses such classes of diagrams in scalar field theory. In the course of deduction [3] of such diagrams, there comes about an involved series, for which an identity is assumed. As for [4], where fermion fields are dealt with, the set of diagrams under consideration is not analyzed. This is also the case [3] for [7]. Incidentally, the thermal self-energy part in itself and the one thermal self-energy-inserted propagator are deduced in [2, 3, 4, 9].

The principal purpose of this paper is to present a complete derivation of the thermal reaction-rate formula (Secs. II - V).

There has been confusion regarding the issue of finite-volume corrections to the standard thermal perturbation theory. (Why and how has the confusion arisen is described historically in [10] with relevant references.) By employing a cubic system with periodic boundary condition, it has been shown in [10] that thermal expectation values of normal-ordered products of field operators can be chosen to be zero and there is no finite-volume correction on thermal amplitudes. It should be stressed that this statement is the statement within the RTF. The statement does not tell us whether or not the thermal reaction-rate formula deduced from first principles is free from finite-volume corrections. We shall derive in Secs. II - V the thermal reaction-rate

\footnote{In fact, in [7], an \( n \geq 2 \) thermal self-energy-inserted propagator is not deduced from the starting formula but is assumed at the start to have the correct form in RTF (cf. Eq. (17) in [7]).}
formula for the finite-volume system and explicitly see that there is no finite-volume correction.

It should be emphasized that the absence of finite-volume corrections here as well as in \[10\] is of rather academic since a cubic system with periodic boundary condition is taken. For physical finite-volume system, there are two sources of entering the finite-volume effects on the thermal perturbation theory constructed on the basis of (grand) canonical ensemble. The one comes from the physically sensible boundary condition on the single-particle wave function. The other comes from taking the physically sensible ensemble. For the case of nonequilibrium case, such as expanding QGP, the situation is of course much more involved.

In Sec. VI, through diagrammatic analysis for the reaction-rate formula, we derive the detailed-balance formula. In Sec. VII, we analyze the zero-temperature limit of the reaction-rate formula and reproduce a variant of the Cutkosky rules \[11\].

At zero temperature, the cutting (Cutkosky) rules \[11\] are the powerful device to investigate the imaginary or absorptive part of a scattering amplitude and a reaction rate like a scattering cross section. Then, it is natural to infer that a finite-temperature extensions of the cutting rules (thermal cutting rules) also plays an important role in thermal field theory.

Previously, several authors \[12, 2, 3, 4, 5, 6, 7, 9, 13, 14, 15, 16\] have discussed thermal cutting rules. However, because of the fact that the generalization of the notion of “cutting” in vacuum theory to the case of thermal field theory is not unique, the terms “cutting” and “(un)cuttable” are endowed with different meanings in \[12, 2, 3, 4, 5, 6, 7, 9, 13, 14, 15, 16\], which causes recent controversy. With this circumstances in mind, we pigeonhole different definitions of thermal cutting rules (Sec. VIII).

## 2 Preliminary

We consider a heat-bath system of temperature \(T\), composed of the fields \(\phi^{(\alpha)}\), with \(\alpha\) labeling collectively a field type and its internal degree of freedom. We assume \(T >> m\) and ignore \(m\) (hot plasma). The system is inside a cube with volume

\[^{\dagger}\text{Relationship between a thermal self-energy part (in imaginary-time formalism) and a rate of decay (production) of a particle in (from) a heat bath was clarified in \[1\], from which the cutting rules as applied to the self-energy part can be read off.}\]
Employing the periodic boundary conditions, we label the single-particle basis by its momentum \( p_k = \frac{2\pi k}{L}, \) \( k_j = 0, \pm 1, \pm 2, \ldots, \pm \infty (j = x, y, z). \)

Physically interesting thermal reactions are of the following generic type,

\[ \{A\} + \text{heat bath} \rightarrow \{B\} + \text{anything}. \]  

(2.1)

Here \( \{A\} \) and \( \{B\} \) designate group of particles, which are not thermalized, such as virtual photons and leptons. (Generalization to more general process, where among \( \{A\} \) and/or \( \{B\} \) are \( \phi^{(\alpha)} \)'s, is straightforward and will be dealt with in Sec. V.) The reaction rate \( R \) of the thermal process (2.1) is expressed \([2, 3, 4]\) as an statistical average of the transition probability \( W = S^\ast S \) (with \( S \) the \( S \)-matrix element) of the zero-temperature \((T = 0)\) process,

\[ \{A\} + \{n^{(\alpha)}_k\} \rightarrow \{B\} + \{n^{(\alpha)\prime}_k\}, \]  

(2.2)

where \( \{n^{(\alpha)}_k\} \) denotes the group of \( \phi^{(\alpha)} \)'s, which consists of the number \( n^{(\alpha)}_k \) of \( \phi^{(\alpha)}_k \) (\( \phi^{(\alpha)} \) in a mode \( k \) ):  

\[ R = \frac{N}{D}, \]  

(2.3)

\[ N \equiv \sum_{\{n^{(\alpha)}_k\}} \sum_{\{n^{(\alpha)\prime}_k\}} \rho \frac{W(\text{process (2.2)})}{2\pi \delta(0)}, \]  

(2.4)

\[ D \equiv \sum_{\{n^{(\alpha)}_k\}} \sum_{\{n^{(\alpha)\prime}_k\}} W_0(\{n^{(\alpha)}_k\} \rightarrow \{n^{(\alpha)\prime}_k\}), \]  

(2.5)

\[ \rho = N^{-1} \exp \left( -\beta \sum_{\alpha} \sum_k n^{(\alpha)}_k p_k \right). \]  

(2.6)

Here \( \beta = 1/T, \) \( p_k = |p_k|, \) and \( 2\pi \delta(0) = t_f - t_i (\sim \infty) \) is the time interval during which the interaction acts. \( W_0 = S_0^{\ast} S_0, \) the “thermal vacuum bubble,” is the \( T = 0 \) transition probability of the process indicated in Eq. (2.3), i.e., the reaction among the heat-bath particles \( \phi^{(\alpha)} \)'s alone. Note that the perturbation series for \( D \) starts from 1,

\[ D = 1 + \ldots. \]  

(2.7)

In Eq. (2.3), \( N \) is the normalization factor. In Eqs. (2.4) and (2.5), \( \sum \) stands for the summation with symmetry factors being respected, and, for a bosonic (fermionic) \( \phi^{(\alpha)}, n^{(\alpha)}_k \) runs over \( 0, 1, 2, \ldots, \infty \) (0 and 1). It is to be noted that \( \{A\} \) and \( \{B\} \) in \( S, \) which we write \( \{A, B\}_S, \) are not necessarily involved in one connected part of \( S. \) This
is also the case for $\{A, B\}_s$. We assume that, in $W = S^*S$, $\{A, B\}_s$ and $\{A, B\}_{s^*}$ are involved in one connected part, which we simply refer to as connected $W$. Then, a connected $W$ consists, in general, of two mutually disconnected parts, the one includes $\{A, B\}_s$ and $\{A, B\}_{s^*}$ and the other is a group of spectator particles. Generalization to other cases is straightforward. Examples of double-cut diagrams [17] for $S^*S$ are depicted in Fig. 1. It should be remarked on the form of $\rho$ in Eq. (2.6). Let us recall the following two facts. On the one hand, the statistical ensemble is defined by the density matrix at the very initial time $t_i (\sim -\infty)$. On the other hand, in constructing perturbative RTF, an adiabatic switching off of the interaction is required [18, 9, 8]. Then, the Hamiltonian $H$ in $\rho \equiv N^{-1}e^{-\beta H}$ should be the free Hamiltonian $H_0$, which leads to Eq. (2.6).

As will be seen below, diagrammatic analysis shows that $N$, Eq. (2.4), takes the form, $N = N_{con}D$, (2.8) where $N_{con}$ corresponds to a connected diagram and $D$ is as in Eq. (2.3). Then $R = N_{con}$.

The $T = 0$ $S$-matrix element is obtained through an application of the reduction formula. As an illustration, we take a heat-bath system of thermal neutral scalars $\phi$’s, and we take $\{A\}$ to be $\{\Phi(p_i); i = 1, \ldots, m\}$ and $\{B\}$ to be $\{\Phi(q_j); j = 1, \ldots, n\}$ with $\Phi$ a nonthermalized heavy neutral scalar. Assuming a $\Phi-\phi$ coupling to be of the form $\Phi \phi^l$, we have [2, 3]

$$
S = \prod_{j=1}^m (iK_{Pj, \phi_j}) \prod_{k=1}^n (iK_{Qk, \phi_k}) \prod_{k} \left[ \sum_{i_k=0}^{n_k} \sum_{i'_k=0}^{n'_k} \delta(n_k - i_k; n'_k - i'_k) \right]
\times N_{ik, i'k}^{n_k n'_k} \prod_{n'=1} \left( iK^*_{k,n'} \right) \prod_{n=1} \left( iK_{k,n} \right) \left\langle 0 | T \left[ \prod_{n'=1} \phi_{n'} \prod_{n=1} \phi_n \prod_{j=1}^m \prod_{k=1}^n \Phi_j \prod_{k=1}^n \Phi_k \right] | 0 \right\rangle,$$

(2.9)

where

$$
N_{ik, i'k}^{n_k n'_k} \equiv \left\{ \begin{array}{c}
n_k' \\
i_k'
\end{array} \right\} \left\{ \begin{array}{c}
n_k \\
i_k
\end{array} \right\} \frac{1}{i_k'! i_k!} \right\}^{1/2}. \quad (2.10)
$$

In Eq. (2.9) $\delta(\cdots; \cdots)$ denotes the Kronecker’s $\delta$-symbol,

$$
K_{k,n} \cdots \phi_n \equiv \frac{1}{\sqrt{2p_k V Z_\phi}} \int d^4x e^{-ip_kx \Box} \cdots \phi(x),
$$
\[ K_{p_j, \Phi_j} \cdots \Phi_j \equiv \frac{1}{\sqrt{2 E_j V Z_j}} \int d^4 x e^{-i p_j x} \times (\Box + M^2) \cdots \Phi_j(x), \quad (2.11) \]

where \( E_j = \sqrt{p_j^2 + M^2} \) with \( M \) the mass of \( \Phi \). \( Z_j \)'s in Eq. (2.11) are the wave-function renormalization constants. \( S_0 \) in \( W_0 = S_0^* S_0 \) is given by a similar expression to Eq. (2.9), where factors related to the \( \Phi \) fields are deleted. It is to be noted that, in Eq. (2.9), among \( n_k (n'_k) \) of \( \phi_k \)'s in the initial (final) state, \( i_k (i'_k) \) of \( \phi_k \)'s are absorbed in (emitted from) the \( i_k (i'_k) \) vertices in \( S \). Remaining \( n_k - i_k (= n'_k - i'_k) \) of \( \phi_k \)'s are merely spectators, which reflects only on the statistical factor in \( A \) in Eq. (3.14) below.

The expression for \( S^* \), the complex conjugate of \( S \), is obtained by taking the complex conjugate of Eq. (2.9), where we make the substitution,

\[ i_k \rightarrow j_k \quad i'_k \rightarrow j'_k. \]

This applies also to the expression for \( S_0^* \).

### 3 Derivation of the reaction-rate formula

In this section, we take self-interacting neutral scalar theory. Generalization to the complex-scalar theory is straightforward (cf. Sec. VIII). A comment on gauge theories is made at the end of this section. Fermion fields are dealt with in Sec. IV.

#### 3.1 Analysis of non mode-overlapping diagrams, \( i_k + i'_k + j_k + j'_k \leq 2 \)

In this subsection, for completeness, we briefly recapitulate the heart of the analysis of [2, 3]. Let us analyze \( N \) in Eq. (2.4) with \( S \) in Eq. (2.9).

(a) \( \{ i_k = i'_k = j_k = j'_k = 0 \} \).

Let us take a diagram for \( W = S^* S \). Let \( v_1 \) and \( v_2 \) be the vertices inside \( S \), which are connected by the propagator

\[ \frac{1}{V} \int_{-\infty}^{\infty} \frac{d p_0}{2\pi} \frac{i}{P^2 + i0^+}. \quad (3.1) \]
(b) \( \{ i_k = i'_k = 1, j_k = j'_k = 0 \} \) and \( \{ i_{-k} = i'_{-k} = 1, j_{-k} = j'_{-k} = 0 \} \).

We first deal with the case \( \{ i_k = i'_k = 1, j_k = j'_k = 0 \} \). We take out the diagram for \( W = S^*S \), which is obtained from \( W \) above as follows. Remove the propagator (3.1), connect \( \phi_{n=1; k} \), Eq. (2.9), to the vertex \( v_1 \) in \( S \), and connect \( \phi_{n'=1; k} \) to \( v_2 \).

Here \( \phi_{n=1; k} [\phi_{n'=1; k}] \) designates that, in Eq. (2.9), \( iK_{k,n=1} [iK_{k,n'=1}] \) operates on \( \phi_{n=1} [\phi_{n'=1}] \). We pick out from Eq. (2.9),

\[
N_{i,i'}^{n,n'} = N_{11}^{n,n} = n.
\]

Here and below, we suppress the suffix “\( k \)”, whenever no confusion arises. In \( S^* \), \( N_{j,j'}^{n,n'} = N_{00}^{nn} = 1 \). Inserting \( N_{j,j'}^{n,n'} N_{i,i'}^{n,n'} = n \) into Eq. (2.4) with Eq. (2.6), we obtain

\[
\langle n \rangle = \frac{1}{e^{\beta p} - 1} \equiv n_B(p).
\]

Here \( n_B(p) = 1/(e^{\beta p} - 1) \) is the Bose distribution function and the angular brackets denotes the statistical average,

\[
\langle \Omega_n \rangle \equiv \frac{\sum_{n=0}^{\infty} e^{-\beta n p} \Omega_n}{\sum_{n=0}^{\infty} e^{-\beta n p}}.
\]

Then, in \( \mathcal{N} \) in Eq. (2.4), the portion corresponding to Eq. (3.1) turns out to

\[
\frac{1}{2pV} n_B(p) = \frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \theta(p_0) 2\pi \delta(P^2) n_B(p),
\]

where \( 1/(2pV) \) has come from \( iK_{k,n'=1} [iK_{k,n=1}] \) in Eq. (2.9) with Eq. (2.11). It is to be noted that \( Z_\phi^{-1/2} \) in \( K \)'s, Eq. (2.11), may be dealt with just as in vacuum theory, so that we ignore \( Z_\phi^{-1/2} \) throughout this paper.

\( \{ i_{-k} = i'_{-k} = 1, j_{-k} = j'_{-k} = 0 \} \).

The relative diagram to the above diagram for \( W = S^*S \), same as above \( W \) except that \( \phi_{n=1; -k} [\phi_{n'=1; -k}] \) is connected to the vertex \( v_2 (v_1) \), yields, in place of Eq. (3.4),

\[
\frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \theta(-p_0) 2\pi \delta(P^2) n_B(p).
\]

Adding Eqs. (3.1), (3.4), and (3.5), we extract

\[
\frac{i}{P_k^2 + i0^+} + 2\pi n_B(p_k) \delta(P_k^2) \equiv iD_{11}(P_k) \equiv iD_{11}^{(0)}(P_k) + iD_{11}^{(T)}(P_k).
\]
Here $iD_{11}^{(0)}$ and $iD_{11}^{(T)}$ stand, respectively, for the $T$-independent part (the first term on the left-hand side (LHS)) and the $T$-dependent part (the second term) of $iD_{11}$.

(c) $\{i_k = j_k = 0, \ i'_k = j'_k = 1\}$ and $\{i_{-k} = j_{-k} = 1, \ i'_{-k} = j'_{-k} = 0\}$.

In order to extract the contribution of $\{i_k = j_k = 0, \ i'_k = j'_k = 1\}$, we take a diagram for $W = S^*S$ in $N$, Eq. (2.4), where $\phi_{n'=1; k}$ in $S$ is connected to the vertex $v_1$ in $S$ and $\phi_{n'=1; k}$ in $S^*$ is connected to the vertex $v_2$ in $S^*$.

We pick out from Eq. (2.9) and from the form of $S^*$,

$$N_{i'i'}^{nn'} N_{jj'}^{nn'} = N_{01}^{n,n+1} N_{01}^{n,n+1} = n + 1.$$ 

Inserting into Eq. (2.4) yields

$$1 + n \rightarrow 1 + n_B(p). \quad (3.7)$$

Then, in $N$ in Eq. (2.4), the portion under consideration takes the form

$$\frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \theta(p_0) 2\pi \{1 + n_B(p)\} \delta(P^2). \quad (3.8)$$

$$\{i_{-k} = j_{-k} = 1, \ i'_{-k} = j'_{-k} = 0\}.$$ 

We consider the relative diagram for $W = S^*S$, which is the same as above except that $\phi_{n=1; -k}$ in $S$ is connected to the vertex $v_1$ and $\phi_{n=1; -k}$ in $S^*$ is connected to the vertex $v_2$. In a similar manner as above, we have

$$\frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \theta(-p_0) 2\pi n_B(p) \delta(P^2). \quad (3.9)$$

Adding Eqs. (3.8) and (3.9), we extract

$$2\pi \left[ \theta(p_{k0}) + n_B(p_{k}) \right] \delta(P^2_{k}) \equiv iD_{21}(P_k). \quad (3.10)$$

(d) $\{i_k = i'_k = j_k = j'_k = 0\}$, $\{i_k = i'_k = 0, \ j_k = j'_k = 1\}$, and $\{i_{-k} = i'_{-k} = 0, \ j_{-k} = j'_{-k} = 1\}$.

In a similar manner as in (a) and (b) above, we extract

$$\frac{-i}{P^2_k - i0^+} + 2\pi n_B(p_{k}) \delta(P^2_{k}) \equiv iD_{22}(P_k) \quad (3.11)$$

$$\equiv iD_{22}^{(0)}(P_k) + iD_{22}^{(T)}(P_k)$$

$$= (iD_{11}(P_k))^*.$$
(e) \{i_k = j_k = 1, \ i'_k = j'_k = 0\} \text{ and } \{i_{-k} = j_{-k} = 0, \ i'_{-k} = j'_{-k} = 1\}.

In a similar manner as in (c) above, we extract

\[ 2\pi [\theta(-p_{k0}) + n_B(p_k)] \delta(P_k^2) \equiv iD_{12}(P_k) = iD_{21}(-P_k). \quad (3.12) \]

The forms of \( D_{ij}(P) \) \((i, j = 1, 2)\) defined above are nothing but the thermal propagators in the Keldish variant of RTF, which is defined on the time path \( C, -\infty \rightarrow +\infty \rightarrow -\infty \rightarrow -\infty - i\beta, \) in a complex time plane. The above derivation shows that the suffix “1” of \( D_{ij} \) stands for the vertex in \( S \) and the suffix “2” stands for the vertex in \( S^* \). On the other hand, in RTF, the suffix “1” stands for physical or type-1 field and “2” stands for thermal-ghost or type-2 field.

Let us turn to identify the vertex factors. We take the interaction Lagrangian density,

\[ \mathcal{L}_{\text{int}} = g \Phi^\ell / \ell! + \lambda \phi^{\ell'} / \ell'!. \quad (3.13) \]

Then, a \( \Phi^\ell (\phi^{\ell'}) \) vertex in \( S \) receives the factor \( ig \) \((i\lambda)\), and then a \( \Phi^\ell (\phi^{\ell'}) \) vertex in \( S^* \) receives the factor \(-ig\) \((-i\lambda)\). This again is in accord with RTF, where \( ig \) \((-ig)\) and \( i\lambda\) \((-i\lambda)\) are the factors which are associated with, in respective order, \( \Phi^\ell \)- and \( \phi^{\ell'} \)-vertices of type-1 \((\text{type-2})\) fields.

Repeating the above procedure, we finally obtain

\[ \frac{1}{V} \left( \prod_{j=1}^{n} 2q_j V \right) R = \left( \prod_{i=1}^{m} \frac{1}{2p_i V} \right) A(P^{(2)}_1, \ldots, P^{(2)}_m, Q^{(1)}_1, \ldots, Q^{(1)}_n, P^{(1)}_1, \ldots, P^{(1)}_m, Q^{(2)}_1, \ldots, Q^{(2)}_n). \quad (3.14) \]

Here \( A \) represents the thermal amplitude in the Keldish variant of RTF for the forward process,

\[ \sum_{i=1}^{m} \Phi_1(P_i) + \sum_{j=1}^{n} \Phi_2(Q_j) \rightarrow \sum_{i=1}^{m} \Phi_2(P_i) + \sum_{j=1}^{n} \Phi_1(Q_j), \]

where \( \Phi_1 \) \((\Phi_2)\) is a type-1 \((\text{type-2})\) field. The thermal amplitude \( A \) is diagrammed in Fig. 2. As we have assumed that \( W = S^*S \) represents the connected diagram \((\text{cf. above after Eq. (2.7)})\), the diagram for \( A \) is connected.
Each loop momentum $P$ in $A$ accompanies

$$\frac{1}{V} \sum_{p_k} \int \frac{dp_0}{2\pi}.$$  \hspace{1cm} (3.15)

In the large $V$ limit the LHS of Eq. (3.14) becomes

$$\frac{1}{V} \left( \prod_{j=1}^{n} 2q_j V \right) \mathcal{R} \rightarrow \frac{1}{V} \left( \prod_{j=1}^{n} 2E_j \frac{d}{d\mathbf{q}_j/(2\pi)^3} \right) \mathcal{R}$$

and Eq. (3.13) becomes

$$\frac{1}{V} \sum_{p_k} \int \frac{dp_0}{2\pi} \rightarrow \int \frac{d^4P}{(2\pi)^4}. \hspace{1cm} (3.16)$$

So far, $D$ in Eq. (2.5) does not participate; $D = 1$ (cf. Eq. (2.7)). The role of $D$ will be discussed below.

### 3.2 Analysis of mode-overlapping diagrams, $i_k + i'_k + j_k + j'_k \geq 4$

Above derivation of the thermal-reaction-rate formula is not complete in that we have only considered the cases where $i_k + i'_k + j_k + j'_k \leq 2$. When generalized self-energy parts are involved in $W = S^*S$, $i_k + i'_k + j_k + j'_k \geq 4$. [We call the diagram with $i_k + i'_k + j_k + j'_k \geq 4$ the mode-overlapping diagram.] As mentioned in Sec. I, a complete analysis of the classes of diagrams that leads to RTF diagrams including thermal propagators with $n \geq 2$ thermal self-energy insertions is still lacking. In this subsection, dealing with mode-overlapping diagrams, we shall complete the derivation of the thermal-reaction-rate formula. We shall show at the same time that there is no finite-volume correction to the formula.

For illustration of the procedure, we start with analyzing the diagram (for $W = S^*S$) with $\{i_k = j_k = i'_k = j'_k = 1\}$. Let us focus our attention on $\phi$ with mode $k$. Both in $S$ and in $S^*$, there are one “absorber vertex” ($v'_1$ and $v_2$ in Fig. 3 below) and one “emitter vertex” ($v_1$ and $v'_2$ in Fig. 3).

From $S^*S$, pick out the factor,

$$N_{11} N_{11}' = n^2,$$

where and below, the suffix “$k$” is dropped whenever no confusion arises. In $\mathcal{N}$ in Eq. (2.4), we have, in place of Eq. (3.3),

$$\langle n^2 \rangle = 2n_B^2 + n_B$$

10
\[ n_B(1 + n_B) + n_B^2, \quad (3.17) \]

where \( n_B \equiv n_B(p) \).

The first term on the right-hand side (RHS) of Eq. (3.17) goes to

\[
\{2\pi\theta(p_0)n_B(p)\delta(P^2)\} \{2\pi\theta(p_0)(1 + n_B(p))\delta(P^2)\} = iD_{12}^{(+)}(P) iD_{21}^{(+)}(P), \quad (3.18)
\]

where \( D_{12/21}^{(\pm)}(P) \equiv \theta(\pm p_0) D_{12/21}(P) \). The (part of) thermal propagator \( iD_{12}^{(+)}(P) [iD_{21}^{(+)}(P)] \) is diagrammed in the double-cut diagram for \( W = S^*S \), Fig. 3 (a), as the line that connects the emitter vertex \( v_2 \) \([v_1]\) with the absorber vertex \( v_1' \) \([v_2]\).

The second term of Eq. (3.17) goes to

\[
\left[2\pi\theta(p_0) n_B(E) \delta(P^2)\right]^2 = iD_{11}^{(T)(+)}(P) iD_{22}^{(T)(+)}(P). \quad (3.19)
\]

\( iD_{11}^{(T)(+)}(P) [iD_{22}^{(T)(+)}(P)] \) is diagrammed in the double-cut diagram, Fig. 3 (b), as the line that connects the emitter vertex \( v_1 \) \([v_2]\) with the absorber vertex \( v_1' \) \([v_2]\). Thus, with obvious notation, \( n_B(1 + n_B) \) part in Eq. (3.17) “supplies” \((++)\) portion of \( iD_{12}(P) iD_{21}(P) \), Fig. 3 (a), and \( n_B^2 \) part “supplies” \((++)\) portion of \( iD_{11}^{(T)}(P) iD_{22}^{(T)}(P) \) in Fig. 3 (b).

The \((-\cdot)\) portion of \( iD_{12}(P) iD_{21}(P) \) emerges from \( W = S^*S \), which is the same as Fig. 3 except that \( \{i_{-k} = j_{-k} = i'_{-k} = j'_{-k} = 1\}. \) Now, \( v_1 \) and \( v_2 \) \([v_1' \) and \( v_2]\) are absorber \([\text{emitter}]\) vertices. The \((+, -)\) portion comes from \( W = S^*S \) with \( \{i_k = i_{-k} = j_k = j_{-k} = 1\}. \) This time, \( v_1 \) and \( v_1' \) \([v_2 \) and \( v_2]\) are absorber \([\text{emitter}]\) vertices. The \((-+)\) portion comes from \( W = S^*S \) with \( \{i'_{-k} = j'_{-k} = j_k = j'_{-k} = 1\}, \) where the absorber \([\text{emitter}]\) vertices are \( v_2 \) and \( v_2' \) \([v_1 \) and \( v_1]\). Adding all these contributions to the contribution (3.18), we obtain Eq. (3.18) with complete \( iD_{12}(P) iD_{21}(P) \). In a similar manner, we can find a set of relative diagrams, which, together with Eq. (3.19), yield the complete \( iD_{11}(P) iD_{22}(P) \).

All the vertices “\( v_1 \)”, “\( v_1' \)”, “\( v_2 \)”, and “\( v_2' \)” (cf. Fig. 3) are not necessarily within one connected diagram. There is a diagram as depicted, e.g., in Fig. 4. Figure 4 (a) [(b)] contains the factor \( iD_{12}^{(+)} iD_{21}^{(+)} [iD_{11}^{(T)(+)} iD_{22}^{(T)(+)}] \) in Eq. (3.18) [Eq. (3.19)]. Let us inspect Fig. 4 (a). As stated above after Eq. (2.7), we are considering the case where \( \{A, B\}_S \) and \( \{A, B\}_S^* \) \( \{\{A\} = \{\Phi(P)\}\} \) and \( \{B\} = \{\Phi(Q_j)\}\) are involved in
one connected part of $W = S^* S$. Then, all $\Phi$’s are in, e.g., the bottom subdiagram in Fig. 4 (a) (and then also in Fig. 4 (b)) and, in the middle subdiagram, only constituent particles $\phi$’s of the heat bath participate. $iD_{21}^{(+)}(P)$ is involved in the middle subdiagram, which goes to $D$, while $iD_{12}^{(+)}(P)$ is involved in the bottom subdiagram, which goes to $N_{\text{con}}$. Thus, Fig. 4 (a) is in $N_{\text{con}} D$ with $D \neq 1$ in Eq. (2.8) with Eq. (2.7). As a matter of fact, $N_{\text{con}}$ here is obtained from $W = S^* S$ with $\{i_k = j_k = 1, i'_k = j'_k = 0\}$ and $D$ is obtained from $W_0 = S_0^* S_0$ (cf. Eq. (2.3)) with $\{i_k = j_k = 0, i'_k = j'_k = 1\}$. Thus, Fig. 4 (a) does contribute to $R$ in Eq. (2.3) as $R = N_{\text{con}}$, which already appears at lower order of perturbation series. As above, it is straightforward to find a set of relative diagrams, which, together with Fig. 4 (a), yields the complete $iD_{12}(P) iD_{21}(P)$. Similarly one can find a set of relative diagrams, which, together with Fig. 4 (b), yields the complete $iD_{11}(P) iD_{22}(P)$.

The relevant part of Fig. 4 (b) and its “relatives” sits in $A$, Eq. (3.14), as a (1, 2) component of a thermal self-energy-inserted propagator. Thus, $W = S^* S$ with $\{i_k = j_k = i'_k = j'_k = 1\}$ together with its “relatives” has turned out to take the proper seat in $A$ in Eq. (3.14).

It is straightforward to generalize the above argument to a generic diagram for $W = S^* S$. Let us focus our attention on a mode $k$. We analyze $N$ in Eq. (2.4). Let $\phi_k$ be $\phi$ in the mode $k$. In $S$ in Eq. (2.4), $i_k \phi_k$’s in the initial state and $i'_k \phi_k$’s in the final state participate directly in the reaction. In $S^*$, $j_k (j'_k) \phi_k$’s in the initial (final) state participate directly; $i_k - i'_k = j_k - j'_k = n_k - n'_k$. In $S$, there are $i_k (i'_k)$ “absorber vertices” (“emitter vertices”) and, in $S^*$, there are $j_k (j'_k)$ “emitter vertices” (“absorber vertices”). [Recall that, in the case of Figs. 3 and 4, $v_1'$ and $v_2$ are absorber vertices and $v_1$ and $v_2'$ are emitter vertices.]

We pick out from $W = S^* S$,

$$ N_{n n'}^{ij} N_{i' n'}^{i' j'} = \frac{n! n'}{(n-i)! (n-j')!} \frac{1}{i! i'! j! j'!} \prod_{k=0}^{j'} (n + i' - i - k) \prod_{k=0}^{j-1} (n - k) \quad (3.20) $$

where and below the suffix “$k$” has been dropped. From the form for $S$, Eq. (2.3), we see that the permutation of $\phi_{n'} (n' = 1, \ldots, i'_k)$ and the permutation of $\phi_n (n = 1, \ldots, i_k)$ give the same diagram, and then $i_k! i'_k! j_k! j'_k!$ same diagrams emerge for $W = S^* S$, which eliminates the first factor on the RHS of
Eq. (3.20). In $N$ in Eq. (2.4), we have, in place of Eq. (3.3),

$$
\langle i' - 1 \prod_{k=0}^{j-1} (n + i' - i - k) \prod_{k=0}^{j-1} (n - k) \rangle \equiv H_{j,j'}^{i,i'}.
$$

Here it is convenient to introduce a generating function of $H_{j,j'}^{i,i'}$,

$$
f(y, z) \equiv \sum_{n=0}^{\infty} y^n z^n e^{-x^n} = \beta p = \beta p_k.
$$

(3.21)

In fact, from Eq. (3.21), we obtain

$$
H_{j,j'}^{i,i'} = \frac{\partial^2 f}{\partial y^i \partial x^j} \bigg|_{y=z=1}.
$$

(3.22)

From Eq. (3.22) with Eq. (3.21), it can be shown that

$$
H_{j,j'}^{i,i'} = \min(i', j') \sum_{k=0}^{\frac{i'}{j' - k}} \left( \begin{array}{c} i' \\ k \end{array} \right) \frac{j'! (j + i' - k)!}{(j' - k)!} (n_B(x))_{j + i' - k}.
$$

(3.23)

Since $i - i' = j - j'$, we can readily see that $H_{j,j'}^{i,i'}$, Eq. (3.23), is symmetric under $(i, i') \leftrightarrow (j, j')$. Then, without loss of generality, we assume $i \geq j$.

In Appendix A, we show that

$$
H_{j,j'}^{i,i'} = \sum_{k=0}^{\min(j', j')} \frac{i!}{(i - j + k)! (j' - k)!} \left( \begin{array}{c} i \\ k \end{array} \right) (n_B)^{i+k-1} (1 + n_B)^{j'-k}
$$

(3.24)

$$
= \sum_{k=0}^{\min(j, j')} \left\{ C_{i,j}^{k} (n_B)^{j-k} \right\} \left\{ C_{i' - j + k}^{0} (n_B)^{i-j+k} \right\} \times \left\{ C_{j', k}^{0} (n_B)^{k} \right\} \left\{ C_{j' - k, j' - k}^{0} (1 + n_B)^{j' - k} \right\}.
$$

(3.25)

Here $n_B \equiv n_B(p)$ and

$$
C_{i,j}^{k} \equiv \frac{i!}{(i - j + k)!} \left( \begin{array}{c} i \\ k \end{array} \right).
$$

In Eq. (3.25), the factor $C_{i,j}^{k}$ may be identified to the number of ways of connecting $j - k$ (out of $j$) emitter vertices in $S^*$ to $i$ absorber vertices in $S$, the factor $C_{i' - j + k}^{0}$ to the number of ways of connecting $i - j + k$ absorber vertices in $S$ to $i'$ emitter vertices in $S$, the factor $C_{j', k}^{0}$ to the number of ways of connecting $k$ emitter vertices in $S^*$ to $j'$ absorber vertices in $S^*$, and the factor $C_{j' - k, j' - k}^{0}$ to the number of ways of
connecting \( j' - k \) absorber vertices in \( S^* \) to \( i' - (i - j + k) = j' - k \) emitter vertices in \( S \). Then, in \( \mathcal{R} \) in Eqs. (2.3), we have, in place of Eqs. (3.18) and (3.19),
\[
\begin{align*}
\min(j, j') & \sum_{k=0}^{\min(j, j')} \left[ C_{i, j}^k \{ iD_{12}^{(+)}(p) \}^{j-k} \right] \left[ C_{i', i-j+k}^0 \{ iD_{11}^{(T)(+)}(p) \}^{i-j+k} \right] \\
& \times \left[ C_{j', k}^0 \{ iD_{22}^{(T)(+)}(p) \}^{k} \right] \left[ C_{j' - k, j' - k}^0 \{ iD_{21}^{(+)}(p) \}^{j' - k} \right].
\end{align*}
\]
(3.26)

This is just a portion of “right” thermal amplitude in RTF. Just as in the simple case, \( \{ i_k = j_k = i'_k = j'_k = 1 \} \), analyzed above, we can find a set of relative diagrams for \( W = S^* S \), which, together with Eq. (3.26), leads to Eq. (3.26) with complete \( D \)’s.

Among the diagrams that accompany Eq. (3.26) with complete \( D \)’s, are disconnected ones like Fig. 4 (a). Such diagrams belong to \( \mathcal{N} = \mathcal{N}_{con} D \) with \( D \neq 1 \) (cf. Eq. (2.8)), and then do contribute to \( \mathcal{R} \) in Eq. (2.3) as \( \mathcal{R} = \mathcal{N}_{con} \). Connected diagrams that accompany Eq. (3.26) with complete \( D \)’s take the proper seat in \( A \) in Eq. (3.14).

Conversely, for any diagram for \( A \) in Eq. (3.14), through the analysis running in the opposite direction, one can identify a set of diagrams for \( W = S^* S \). The analysis made above is so general that no additional comment is necessary on the diagrams that leads to \( A \), Eq. (3.14), which includes thermal propagator(s) with \( n \) (\( \geq 2 \)) thermal self-energy insertion.

This completes the derivation of the formula (3.14) for the rate of a generic thermal reaction taking place in a heat bath of finite volume. Keeping in mind a suitable normalization for incident fluxes of \( \Phi \)’s, the formula (3.14) “smoothly” goes to the formula for the infinite-volume (\( V = \infty \)) system (cf. Eq. (3.16)) in the sense that there do not exist extra contributions in Eq. (3.14) with \( V < \infty \), which disappear in the limit \( V \to \infty \). Thus, there is no finite-volume correction to the thermal reaction-rate formula (3.14).

Here we make a comment on gauge theories. Choosing a physical gauge like Coulomb gauge, the gauge boson may be dealt with in a similar manner to the above scalar-field case. When we adopt a covariant gauge, Faddeev-Popov (FP) ghost field comes on the stage. The first summations in Eqs. (2.4) and (2.5) are carried out over the modes of physical degrees of freedom. This can be implemented by inserting the projection operator \( \mathcal{P} \) onto the physical space on the left side of \( \rho \) in Eqs. (2.4) and (2.5) and sum is taken over \( \{ n_k^{(\alpha)} \} \) for all, unphysical as well as physical, modes \( \alpha \)’s. As far as the ensemble average of physical quantities like reaction rate are concerned,
all the role of $\mathcal{P}$ is to make the antiperiodic boundary condition for FP-ghost field the periodic one, $\phi_{FP}(t - i\beta, x) = \phi_{FP}(t, x)$, so that the bare FP-ghost propagator is the same in form to the scalar propagator. Keeping this fact in mind, we can deduce Eq. (3.14), where $A$ is evaluated using standard gauge-field and FP-ghost thermal propagators in the covariant gauge.

4 The Dirac fermion

We study the case of Dirac fermion. The expression for $S$ in Eq. (2.9) with Eqs. (2.10) and (2.11) is changed accordingly. Let $n_k^{(\sigma)}$ [shift $\bar{n}_k^{(\sigma)}$] ($\sigma = \pm$) be the number of mode-$k$ fermion [anti fermion] with helicity $\sigma$. The combinatorial factor $N_{n_k n_k'}$ in Eq. (2.9) is changed to

$$N_f = \prod_{\sigma=\pm} \left( \frac{n_k^{(\sigma)} n_k'^{(\sigma)}}{i_k^{(\sigma)} i_k'^{(\sigma)}} \frac{n_k'^{(\sigma)} n_k^{(\sigma)}}{i_k'^{(\sigma)} i_k^{(\sigma)}} \right)$$

$$= \prod_{\sigma=\pm} \left( \begin{array}{c} n_k^{(\sigma)} \\ i_k^{(\sigma)} \end{array} \right) \left( \begin{array}{c} n_k'^{(\sigma)} \\ i_k'^{(\sigma)} \end{array} \right) \left( \begin{array}{c} \bar{n}_k^{(\sigma)} \\ \bar{i}_k^{(\sigma)} \end{array} \right) \left( \begin{array}{c} \bar{n}_k'^{(\sigma)} \\ \bar{i}_k'^{(\sigma)} \end{array} \right),$$

where $n_k^{(\sigma)} - i_k^{(\sigma)} = n_k'^{(\sigma)} - i_k'^{(\sigma)}$ and $\bar{n}_k^{(\sigma)} - \bar{i}_k^{(\sigma)} = \bar{n}_k'^{(\sigma)} - \bar{i}_k'^{(\sigma)}$. In Eqs. (2.4) and (2.5), the summations on $n_k^{(\sigma)}$, $i_k^{(\sigma)}$, $\bar{n}_k^{(\sigma)}$, and $\bar{i}_k^{(\sigma)}$ are taken over 0 and 1. We assume that the interaction Lagrangian is bilinear in fermion fields, which include fermion fields constituting the heat bath and possibly nonthermalized heavy fermion fields, the counterpart of $\Phi$’s in Eq. (2.9).

4.1 Analysis of non mode-overlapping diagrams

We proceed as in Sec. III A using the same notation.

(a) $\{i_k^{(\sigma)} = i_k'^{(\sigma)} = j_k^{(\sigma)} = j_k'^{(\sigma)} = \bar{n}_k^{(\sigma)} = \bar{n}_k'^{(\sigma)} = \bar{i}_k^{(\sigma)} = \bar{i}_k'^{(\sigma)} = 0\}$ ($\sigma = \pm$).

In place of Eq. (3.1), we have

$$\frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{iP}{P^2 + i0^+},$$

which comes from the following contraction in $S$ (cf. Eq. (2.9)),

$$\langle 0| T \left[ \cdots \bar{\psi}(x_1) \psi(x_1) \cdots \bar{\psi}(x_2) \psi(x_2) \cdots \right] |0\rangle = iS_F(x_1 - x_2) \langle 0| T \left[ \cdots \bar{\psi}(x_1) \cdots \psi(x_2) \cdots \right] |0\rangle.$$
Here $\bar{\psi}\psi$'s in Eq. (4.3) come from the interaction Lagrangian $\mathcal{L}_{int}$.

(b) Fermion mode with $\{i_k^{(\sigma)} = i_k^{(\sigma)'}, j_k^{(\sigma)} = j_k^{(\sigma)'}, k = 1, j = \pm\}$ and its relative.

We consider the positive-helicity ($\sigma = +$) fermion mode with $\{i_k^{(+)} = i_k^{(+)'}, j_k^{(+)} = j_k^{(+)'}, k = 1, j = 0\}$. In place of Eqs. (3.2) and (3.3), we have, in respective order,

$$N_f = n^2.$$ and

$$\langle n^2 \rangle = \frac{1}{e^{\beta p} + 1} \equiv n_F(p),$$

where

$$n_F(p) = \frac{1}{e^{\beta p} + 1}$$

is the Fermi-distribution function and $\langle \Omega_n \rangle \equiv \sum_{n=0}^{1} e^{-\beta np} \Omega_n / \sum_{n=0}^{1} e^{-\beta np}$. We note that the contribution corresponding to Eq. (4.3) above is (cf. Eq. (2.9))

$$\langle 0 | T \left[ \cdots \psi_{n=1}(y) \bar{\psi}(x_1) \psi(x_1) \cdots \bar{\psi}(x_2) \psi(x_2) \bar{\psi}_{n=1}(z) \cdots \right] | 0 \rangle = -iS_F(y - x_2) iS_F(x_1 - z) \langle 0 | T \left[ \cdots \bar{\psi}(x_1) \cdots \psi(x_2) \cdots \right] | 0 \rangle.$$

Then, the LHS of Eq. (3.4) is replaced by

$$-\frac{1}{2pV} n_F(p) u^{(+)}(P) \bar{u}^{(+)}(P).$$

Adding the contribution from the negative-helicity fermion mode with $\{i_k^{(-)} = i_k^{(-)'}, j_k^{(-)} = j_k^{(-)'}, k = 1, j = 0\}$, we have

$$-\frac{1}{2pV} n_F(p) \sum_{\sigma = \pm} u^{(\sigma)}(P) \bar{u}^{(\sigma)}(P)$$

$$= -\frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \theta(p_0) 2\pi \delta(P^2) n_F(p) \, \mathbf{p}.$$ (4.4)

Adding further the contribution from the antifermion modes with $\{i_{-k}^{(\sigma)} = i_{-k}^{(\sigma)'}, j_{-k}^{(\sigma)} = j_{-k}^{(\sigma)'}, k = 1, j = \pm\}$ to Eqs. (4.2) and (4.4), we extract

$$\left[ \frac{i}{P_k^2 + i0^+} - 2\pi n_F(p_k) \delta(P_k^2) \right] \mathbf{p}_k$$

$$\equiv iS_{11}(P_k) = iS_{11}^{(0)}(P_k) + iS_{11}^{(T)}(P_k).$$ (4.5)

(c) Fermion mode with $\{i_k^{(\sigma)} = j_k^{(\sigma)} = 0, i_k^{(\sigma)'}, j_k^{(\sigma)'} = 1\}$ ($\sigma = \pm$) and its relative.
In place of Eq. (3.7), we have

$$1 - n_F(p).$$

Then, Eq. (3.8) is replaced by

$$\frac{1}{V} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \theta(p_0) 2\pi \{1 - n_F(p)\} \delta(P^2) \hat{P}.$$ 

Adding the contribution from the antifermion mode with \(\{i^{(\sigma)}_k = j^{(\sigma)}_k = 1, i^{(\sigma)}'_{-k} = j^{(\sigma)}'_{-k} = 0\}\) (\(\sigma = \pm\)), we extract

$$2\pi [\theta(p_0) - n_F(p_k)] \delta(P^2_k) \hat{P}_k \equiv iS_{21}(P_k).$$

(d) Interchanging the roles of \(S\) and \(S^*\) in (a) and (b) above, we obtain, in place of Eq. (3.11),

$$\left[ \frac{-i}{P_k^2 - i0^+} - 2\pi n_F(p_k) \delta(P^2_k) \right] \hat{P}_k$$

$$\equiv iS_{22}(P_k) = iS_{22}^{(0)}(P_k) + iS_{22}^{(T)}(P_k).$$

(e) Fermion mode with \(\{i^{(\sigma)}_k = j^{(\sigma)}_k = 1, i^{(\sigma)}'_{-k} = j^{(\sigma)}'_{-k} = 0\}\) (\(\sigma = \pm\)) and its relative.

The relevant statistical factor is \(n_F(p)\). Let us show that the part under consideration turns out to \(iS_{12}(P_k)\). In place of \(p_0 > 0\) portion of Eq. (3.12), we have

$$2\pi n_F(p_k) \delta(P^2_k) \hat{P}_k$$

which seems to be the \(p_0 > 0\) portion of \(iS_{12}(P_k)\). However this is not the case. Within the resultant reaction-rate formula, which is an amplitude in RTF, the above factor \(2\pi n_F(p_k) \delta(p^2_k) \hat{P}_k\) necessarily appears in association with a thermal fermion loop (see below for detail). The thermal fermion loop carries an extra minus sign, so that we have, for the portion under consideration,

$$iS_{12}^{(+)}(P_k) = 2\pi [-n_F(p_k)] \delta(P^2_k) \hat{P}_k.$$ 

Adding the contribution from the antifermion mode with \(\{i^{(\sigma)}_{-k} = j^{(\sigma)}_{-k} = 0, i^{(\sigma)}'_{-k} = j^{(\sigma)}'_{-k} = 1\}\) (\(\sigma = \pm\)), we extract

$$2\pi [\theta(-p_0) - n_F(p_k)] \hat{P}_k \delta(P^2_k) \equiv iS_{12}(P_k).$$ (4.6)

In the process of deduction, \(iS_{jl} (j, l = 1, 2)\) appears in succession. At the final stage, sets of \(\langle W \rangle = \langle S^*S \rangle\) turn out to be thermal amplitudes \(A\)'s (cf. Eq (3.14)),

17
which includes thermal loops of the fermion $\psi$. Out of $A$'s, we take a “standard” $A$: Each fermion loop contains at most one $iS_{12}$. (Note that the number of $iS_{21}$ in a fermion loop is equal to the number of $iS_{12}$.) From $A$, we take a “standard” $A$: Each fermion loop contains at most one $iS_{12}$. (Note that the number of $iS_{21}$ in a fermion loop is equal to the number of $iS_{12}$.) From $A$, we take two fermion loops $L_1$ and $L_2$ and let $iS_{21}(P) \in L_1$ and $iS_{21}(Q) \in L_2$. $iS_{21}(P)iS_{21}(Q)$ comes, with obvious notation, from $S^*S = S^*(p, q, ...) S(p, q, ...) \equiv W_s$, where $S$ is the $S$-matrix element obtained using Feynman rules (in vacuum theory). The $S$-matrix element which is related to $S(p, q, ...)$ through exchange $p \leftrightarrow q$ is $-S(q, p, ...)$, where $S(q, p, ...) \equiv W_s$. Then, we have

$$W_s \rightarrow W = -S^*(p, q, ...) S(q, p, ...) ,$$

which brings in an extra minus sign into the corresponding thermal amplitude $A$. Observe here that, through the above replacement of $S$, $L_1$ and $L_2$ in $A_s$ turns out to be an one thermal fermion loop $L$ in $A$. A thermal fermion loop carries a minus sign. Then $L_1$ and $L_2$ in $A_s$ carries $+ = (-)^2$ while $L$ in $A$ carries $-$. In reducing $\langle W \rangle$ to $A$, the extra minus sign in Eq. (4.7) eliminates one $-$, being present in $A_s$, and is left with one $-$, which is interpreted as the minus sign associated with $L$ in $A$. What we have shown is that $A$ is a “right thermal amplitude.”

Repeating the above procedure for “parent” $A$’s and “children” $A$’s, as “constructed” above, we can exhaust all $A$’s that contributes to the reaction-rate formula, and see that they are “right” thermal amplitudes.

### 4.2 Analysis of mode-overlapping diagrams

Let us turn to analyze the mode-overlapping diagrams. Noting that $n^{(\sigma)}_i$ etc. and then also $i^{(\sigma)}_k$ etc. take two values 0 and 1, we shall exhaust all the mode-overlapping configurations.

(a) $\{i^{(\sigma)}_k = j^{(\sigma)}_k = j^{(\sigma)}_k = 1\} (\sigma = \pm)$ and its relatives.

From Eq. (4.1), $N_f = \langle (n^{(\sigma)})^4 \rangle (\sigma = \pm)$, which leads to $\langle (n^{(\sigma)})^4 \rangle = n_F$. Through by now familiar manner, we extract

$$n_F \sum_{\sigma = \pm} \left\{ 2\pi \theta(p_0) u^{(\sigma)}_j(P) \bar{u}^{(\sigma)}_j(P) \right\} \times \left\{ 2\pi \theta(p_0) u^{(\sigma)}_i(P) \bar{u}^{(\sigma)}_i(P) \right\} .$$

(4.8)

$u^{(\sigma)}_i$ and $\bar{u}^{(\sigma)}_j$ in Eq. (4.8) are attached to the vertices in $S [S^*]$. 

18
Adding Eqs. (4.10) and (4.11), we have
\[ \frac{n_F^2}{\pi} \sum_{\sigma = \pm} \left\{ 2\pi \theta(p_0) u_j^{(-\sigma)}(P) \overline{u}_{j'}^{(-\sigma)}(P) \right\} \times \left\{ 2\pi \theta(p_0) u_i^{(\sigma)}(P) \overline{u}_{i'}^{(\sigma)}(P) \right\} , \quad (4.9) \]
and the latter yields
\[ n_F(1 - n_F) \sum_{\sigma = \pm} \left\{ 2\pi \theta(p_0) u_j^{(-\sigma)}(P) \overline{u}_{j'}^{(-\sigma)}(P) \right\} \times \left\{ 2\pi \theta(p_0) u_i^{(\sigma)}(P) \overline{u}_{i'}^{(-\sigma)}(P) \right\} . \quad (4.10) \]
Adding Eqs. (4.8) and (4.9), we obtain
\[ \left( iS_{22}^{(T)(+)}(P) \right)_{jj'} \left( iS_{11}^{(T)(+)}(P) \right)_{i'i'} + n_F(1 - n_F) \sum_{\sigma = \pm} \left\{ 2\pi \theta(p_0) u_j^{(-\sigma)}(P) \overline{u}_{j'}^{(-\sigma)}(P) \right\} \times \left\{ 2\pi \theta(p_0) u_i^{(\sigma)}(P) \overline{u}_{i'}^{(-\sigma)}(P) \right\} . \quad (4.11) \]
Adding Eqs. (4.10) and (4.11), we have
\[ \left( iS_{22}^{(T)(+)}(P_k) \right)_{jj'} \left( iS_{11}^{(T)(+)}(P_k) \right)_{i'i'} - \left( iS_{12}^{(+)}(P_k) \right)_{ij'} \left( iS_{21}^{(+)}(P_k) \right)_{j'i} . \quad (4.12) \]
Recalling the fact that \( i \) and \( i' \) [\( j \) and \( j' \)] attach to the vertices in \( S[S^*] \), we see that Eq. (4.12) is just a portion of “right” thermal amplitude in RTF. Adding an appropriate sets of relative diagrams, we can extract Eq. (4.12) with complete \( S \)'s. (b) \( \{ i^{(+)}_k = i^{(-)}_k = j^{(+)}_k = j^{(-)}_k = 1 \} \) and its relatives.

Taking care of the anticommutativity of fermion fields, we extract
\[ n_F^2 \left[ 2\pi \theta(p_0) \left\{ u_i^{(+)}(P) u_{i2}^{(-)}(P) - u_i^{(+)}(P) u_{i1}^{(-)}(P) \right\} \right] \times \left[ 2\pi \theta(p_0) \left\{ \overline{u}_{j1}^{(+)}(P) \overline{u}_{j2}^{(-)}(P) - \overline{u}_{j2}^{(+)}(P) \overline{u}_{j1}^{(-)}(P) \right\} \right] . \quad (4.13) \]
Here \( u_i \)'s [\( \overline{u}_j \)'s] are attached to the vertices in \( S[S^*] \). Simple manipulation yields
\[ \text{Eq. (4.13)} = \left( iS_{12}^{(+)}(P_k) \right)_{i1j2} \left( iS_{12}^{(+)}(P_k) \right)_{i2j2} - \left( iS_{12}^{(+)}(P_k) \right)_{i1j2} \left( iS_{12}^{(+)}(P_k) \right)_{i2j1} . \quad (4.14) \]
Adding appropriate relative diagrams, we can extract Eq. (4.14) with complete $S$’s, which sits on the “right seat” in thermal amplitude in RTF (cf. Eq. (3.14)).

We extract

\[
\hat{\rho}_{j1j2j3}^{i_1i_2i_3}(P) = \hat{\rho}_{j1j2j3}^{i_1i_2i_3}(P) - \hat{\rho}_{j1j2j3}^{i_1i_2i_3}(P)
\]

where the spinors with suffices $i_1$, $i_2$, and $i_3$ [$j_1$, $j_2$, and $j_3$] are attached to the vertices in $S [S^*]$.

We shall show that

\[
\text{Eq. (4.15)} = \hat{\rho}_{j1j2j3}^{i_1i_2i_3}(P) - \hat{\rho}_{j1j2j3}^{i_1i_2i_3}(P)
\]

where

\[
\hat{\rho}_{j1j2j3}^{i_1i_2i_3}(P) \equiv \left( iS_{12}^{(+)}(P) \right)_{i_1j_1} \left( iS_{12}^{(+)}(P) \right)_{i_2j_2} \left( iS_{21}^{(+)}(P) \right)_{j_3i_3}
\]

\[
+ \left( iS_{12}^{(T)(+)}(P) \right)_{i_1i_3} \left( iS_{12}^{(+)}(P) \right)_{i_2j_2} \left( iS_{21}^{(T)(+)}(P) \right)_{j_3j_1}
\]

\[
+ \left( iS_{12}^{(+)}(P) \right)_{i_1j_1} \left( iS_{12}^{(T)(+)}(P) \right)_{i_2i_3} \left( iS_{21}^{(T)(+)}(P) \right)_{j_3j_2}.
\]

We shall prove this by running in the opposite direction, i.e., starting from Eq. (4.16), we derive Eq. (4.13). The first term on the RHS of Eq. (4.17) consists of two terms, the one is proportional to $n_F^2$ and the one is proportional to $n_F^3$. The second and third terms are proportional to $n_F^3$. \(iS_{12}^{(+)}(P)\) may be written as (cf. Eq. (4.4))

\[
\left( iS_{12}^{(+)}(P) \right)_{i_1j_1} = -2\pi n_F(p) \delta(p^2) \sum_{\sigma=\pm} u_{i_1}^{(\sigma)}(p) P_{j_1}^{(\sigma)}(p).
\]

Other $S$’s in Eq. (4.17) may be expressed similarly. Straightforward but tedious manipulation shows that the “$n_F^3$ part” of $S_{j1j2j3}^{i_1i_2i_3} - S_{j1j2j3}^{i_2i_1i_3}$ vanishes. Then, in Eq. (4.16), we are left with “$n_F^2$ part”, which turns out to be Eq. (4.13).
We extract

\[ n_F^2 \left[ 2\pi\theta(p_0) \left\{ u_{i_1}^{(+)}(P) u_{i_2}^{(-)}(P) - u_{i_1}^{(-)}(P) u_{i_2}^{(+)}(P) \right\} \right. \]
\[ \times \left[ 2\pi\theta(p_0) \left\{ \pi_{j_1}^{(+)}(P) \overline{\pi}_{j_2}^{(-)}(P) - \pi_{j_1}^{(-)}(P) \overline{\pi}_{j_2}^{(+)}(P) \right\} \right. \]
\[ \times \left[ 2\pi\theta(p_0) \left\{ \pi_{i_3}^{(+)}(P) \overline{\pi}_{i_4}^{(-)}(P) - \pi_{i_3}^{(-)}(P) \overline{\pi}_{i_4}^{(+)}(P) \right\} \right. \]
\[ \times \left[ 2\pi\theta(p_0) \left\{ u_{j_3}^{(+)}(P) u_{j_4}^{(-)}(P) - u_{j_3}^{(-)}(P) u_{j_4}^{(+)}(P) \right\} \right] . \] (4.19)

As in the above case (c), through straightforward but tedious calculation, we obtain

\[
\text{Eq. (4.19) } = S_{j_1j_2j_3j_4}^{i_1i_2i_3i_4}(P) - S_{j_1j_2j_3j_4}^{i_1i_2i_3i_4}(P) 
- S_{j_1j_2j_3j_4}^{i_1i_2i_3i_4}(P) + S_{j_1j_2j_3j_4}^{i_1i_2i_3i_4}(P) ,
\] (4.20)

where

\[
S_{j_1j_2j_3j_4}^{i_1i_2i_3i_4}(P) \equiv (iS_{12}^{(+)}(P))_{i_1j_1} (iS_{12}^{(+)}(P))_{i_2j_2} (iS_{21}^{(+)}(P))_{j_3i_3} (iS_{21}^{(+)}(P))_{j_4i_4} 
+ (iS_{11}^{(T)(+)}(P))_{i_1j_1} (iS_{12}^{(+)}(P))_{i_2j_2} (iS_{22}^{(T)(+)}(P))_{j_3j_4} (iS_{21}^{(+)}(P))_{j_4i_4} 
+ (iS_{12}^{(+)}(P))_{i_1j_1} (iS_{11}^{(T)(+)}(P))_{i_2j_2} (iS_{22}^{(T)(+)}(P))_{j_3j_4} (iS_{21}^{(+)}(P))_{j_4i_4} 
+ (iS_{11}^{(T)(+)}(P))_{i_1j_1} (iS_{12}^{(+)}(P))_{i_2j_2} (iS_{22}^{(T)(+)}(P))_{j_3j_4} (iS_{21}^{(+)}(P))_{j_4i_4} 
+ (iS_{11}^{(T)(+)}(P))_{i_1j_1} (iS_{21}^{(+)}(P))_{i_2j_2} (iS_{22}^{(T)(+)}(P))_{j_3j_4} (iS_{21}^{(+)}(P))_{j_4i_4} 
\times (iS_{22}^{(T)(+)}(P))_{j_3j_4} (iS_{22}^{(T)(+)}(P))_{j_3j_4} .
\]

The same comment as above after Eq. (4.14) applies here.

There remains following two configurations to be analyzed; (e) \( \{ i_{k_1}^{(+)} = i_{k_1}^{(-)} = j_{k_1}^{(+)} = j_{k_1}^{(-)} = 1 \} \) and its relatives and (f) \( \{ i_{k_1}^{(+)} = i_{k_1}^{(-)} = j_{k_1}^{(+)} = j_{k_1}^{(-)} = 1, i_{k_1}^{(\sigma)} = j_{k_1}^{(\sigma)} = 1 \} \) (\( \sigma = \pm \)) and its relatives. The case (e) [(f)] may be analyzed in a similar manner as (b) [(c)] above and the “right combination” of thermal propagators is extracted.

As in the scalar-field case, Sec. III B, there appear disconnected \( N \)'s; \( N = N_{con}D \) with \( D \neq 1 \). Such cases are treated in a same manner as in the scalar-field case.

This completes the analysis of all mode-overlapping configurations.

Conversely, we take a diagram for \( A \) in the reaction-rate formula (cf. Eq. (3.14)). The amplitude \( A \) contains “vanishing contributions,” which should vanish. By this
we mean the contributions coming from the configurations, in which at least one of $i_k^{(h)}$, $i_k^{(h)'}$, $j_k^{(h)}$, $j_k^{(h)'}$, $\bar{i}_k^{(h)}$, $\bar{i}_k^{(h)'}$, $\bar{j}_k^{(h)}$, $\bar{j}_k^{(h)'}$ ($h = \pm$) is equal to or greater than 2. Let us show that such contributions really vanish. Suppose that $A$ contains

$$
\prod_{k=1}^{3} (iS_{12}(R_k))_{i_kj_k},
$$

(4.21)

where $R_k$ ($k = 1, 2, 3$) is the loop momentum (cf. Eq. (3.13)) and the suffix ‘$i_kj_k$’ stands for the $(i_k,j_k)$ element of $iS_{12}$ in the $4 \times 4$ Dirac-matrix space. In the loop-momentum space, there are “points,” where $R_1 = R_2 = R_3 \equiv R = (r_0, r)$. Adding the contributions from the five relative diagrams, we have, in place of Eq. (4.21),

$$
\sum_{\text{perm}} \sigma_{l_1l_2l_3}^{j_1j_2j_3} \prod_{k=1}^{3} (iS_{12}(R))_{i_kl_k},
$$

(4.22)

where summation is taken over all permutations of $(j_1j_2j_3)$. $\sigma_{l_1l_2l_3}^{j_1j_2j_3} = +/−$ when $(l_1l_2l_3)$ is an even/odd permutation of $(j_1j_2j_3)$, which is a reflection of the anticommutativity of fermion fields. We take the case $r_0 > 0$. The “type-1 side” of Eq. (4.22) comes from $i_k^{(+)} + i_k^{(-)} = 3$, and then $i_k^{(+)} \geq 2$ or $i_k^{(-)} \geq 2$. Then the contribution under consideration should vanish. In order to see that this is really the case, using the expression (4.18), we further extract from Eq. (4.22)

$$
\sum_{\text{perm}} \sigma_{l_1l_2l_3}^{j_1j_2j_3} \prod_{k=1}^{3} \left[ \sum_{\sigma_k=\pm} u_{i_k}^{(\sigma_k)}(R) \mathbf{\pi}_{i_k}^{(\sigma_k)}(R) \right].
$$

(4.23)

Again straightforward but tedious manipulation shows that Eq. (4.23) is in fact vanishes. In a similar manner, we can show that Eq. (4.22) with $r_0 < 0$ also vanishes.

We can also see that $\left( iS_{11}^{(T)}(R) \right)_{i_1j_1} \prod_{k=2}^{3} (iS_{12}(R))_{i_kj_k}$ and its relatives add up to vanish. When product of $n$ ($\geq 4$) $iS_{12}(R)$ and/or $iS_{11}^{(T)}(R)$ appears in $A$, pick out three of them and apply the above argument to show that the contribution vanishes.

The above analysis applies to all other “vanishing contributions,” which include $\prod_{k=1}^{3} (iS_{21}(R_k))$ with its relatives etc. This completes the proof of absence of “vanishing contributions.”
5 The rate of reactions between the constituent particles of the heat bath

In the heat bath composed of scalar fields $\phi$'s, taking place is the reaction,

$$\phi(p_1) + ... + \phi(p_m) + \text{heat bath} \rightarrow \phi(q_1) + ... + \phi(q_n) + \text{anything},$$ \hspace{1cm} (5.1)

where $\phi$'s are the constituent particle of the heat bath. One can easily show that the reaction rate takes the form,

$$\frac{1}{V} \left( \prod_{j=1}^{n} 2q_j V \right) R = \left( \prod_{i=1}^{m} \frac{1}{2p_i V} \right) \left( \prod_{i=1}^{m} n_B(p_i) \right) \left( \prod_{j=1}^{n} \{1 + n_B(q_j)\} \right) \times A(P_1^{(2)}, ..., P_m^{(2)}, Q_1^{(1)}, ..., Q_n^{(1)}; P_1^{(1)}, ..., P_m^{(1)}, Q_1^{(2)}, ..., Q_n^{(2)}),$$ \hspace{1cm} (5.2)

where $A$ is the RTF amplitude for the forward process,

$$\phi_1(P_1) + ... + \phi_1(P_m) + \phi_2(Q_1) + ... + \phi_2(Q_n) \rightarrow \phi_2(P_1) + ... + \phi_2(P_m) + \phi_1(Q_1) + ... + \phi_1(Q_n).$$ \hspace{1cm} (5.3)

It is worth noting that Eq. (5.2) may be rewritten as

$$\frac{1}{V} R = \left[ \prod_{i=1}^{m} \frac{1}{V} \int \frac{dp_{i0}}{2\pi} \theta(p_{i0}) iD_{12}(P_i) \right] \times \left[ \prod_{j=1}^{n} \frac{1}{V} \int \frac{dq_{j0}}{2\pi} \theta(q_{j0}) iD_{21}(Q_j) \right] A \equiv \tilde{A}_{\text{bubble}}.$$

(5.4)

The RHS, $\tilde{A}_{\text{bubble}}$, is a no-leg thermal amplitude, in which no summation is taken over $p_i$ $(i = 1, ..., m)$ and $q_j$ $(j = 1, ..., n)$.

Generalization of the above result to the theories with gauge bosons and/or fermions is straightforward.
6 Detailed balance

In this section, on the basis of the generalized reaction-rate formula, Eq. (5.2), we derive the detailed-balance formula through diagrammatic analysis.

The purpose of this section is to show that the rate (5.2) for the process (5.1) is equal to the rate for the inverse process to (5.1). [For the case of theories with gauge bosons and/or fermions, the same result is obtained.] This is well known for the cases of decay- and production-processes, which corresponds to \( m = 1, n = 0 \) and \( m = 0, n = 1 \), respectively, in Eq. (5.2).

Take a diagram for \( A \), Eq. (5.2), and let \( N_1 \) and \( N_2 \) be the number of \( iD_{21} \)'s and \( iD_{12} \)'s, respectively, which is involved in \( A \),

\[
\prod_{j=1}^{N_1} iD_{21}(R_j) \prod_{k=1}^{N_2} iD_{12}(R_{N_1+k}). \tag{6.1}
\]

By cutting all the lines \( iD_{12} \)'s and \( iD_{21} \)', we divide \( A \) into one or several “type-1 islands” and one or several “type-2 islands”. Here, the type-1 (type-2) island is a “maximal” amputated subdiagram of \( A \), which consists of only type-1 (type-2) vertices and of the propagators \( iD_{11} \)'s (\( iD_{22} \)'s) connecting them. Then, a type-1 (type-2) island includes no type-2 (type-1) vertex. A type-1 (type-2) island is connected by \( iD_{21} \)'s and/or \( iD_{12} \)'s to type-2 (type-1) island(s).

Take a type-1 island and we write its contribution (to \( A \))

\[
\mathcal{I}_1(Q_{j_1}, ..., Q_{j_{\ell'}}; P_{i_1}, ..., P_{i_\ell}). \tag{6.2}
\]

Here \( \{P_{i_k}, 1 \leq k \leq \ell \} \) is a subset of \( \{P_i, 1 \leq i \leq m\} \) on the LHS of Eq. (5.3) and \( \{Q_{j_k}, 1 \leq k \leq \ell' \} \) is a subset of \( \{Q_j, 1 \leq j \leq n\} \) on the RHS of Eq. (5.3), where \( \ell, \ell' \geq 0 \). This type-1 island is connected by \( s_1(\geq 0) \) propagators \( iD_{21} \)'s and \( s_2(\geq 0) \) propagators \( iD_{12} \)'s to one or several type-2 islands. With the help of the identity,

\[
D_{21}(R) = e^{\beta r_0} D_{12}(R), \tag{6.3}
\]

and the momentum-conservation condition, we obtain, for \( iD \)'s that are attached to \( \mathcal{I}_1 \),

\[
\prod_{j=1}^{s_1} iD_{21}(R_j) \prod_{k=1}^{s_2} iD_{12}(R_{s_1+k})
= \exp \left( \beta \left[ \sum_{k=1}^{\ell} P_{i_k} - \sum_{k=1}^{\ell'} Q_{j_k} \right] \right) \prod_{j=1}^{s_1} iD_{12}(R_{j}) \prod_{k=1}^{s_2} iD_{21}(R_{s_1+k}). \tag{6.4}
\]
We now take a type-2 island, whose contribution is written as

$$\mathcal{I}_2(P_{i_1}, ..., P_{i_{\ell'}}, Q_{j_1}, ..., Q_{j_{\ell'}}),$$

(6.5)

where \(\{Q_{j_k}, 1 \leq k \leq \ell'\}\) is a subset of \(\{Q_j, 1 \leq j \leq n\}\) on the LHS of Eq. (5.3) and \(\{P_{i_k}, 1 \leq k \leq \ell\}\) is a subset of \(\{P_i, 1 \leq i \leq m\}\) on the RHS of Eq. (5.3). \(\ell (\ell')\) here is not necessarily equal to \(\ell (\ell')\) in Eq. (6.2). In a similar manner as above, in place of Eq. (6.4), we have, with obvious notation,

$$s_1' \prod_{j=1}^{s_1'} iD_{12}(R_j) \prod_{k=1}^{s_2'} iD_{21}(R_{s_1'+k}) = \exp \left( \beta \left[ \sum_{k=1}^{\ell'} p_{i_k} - \sum_{k=1}^{\ell} q_{j_k} \right] \right)$$

$$\times \prod_{j=1}^{s_1'} iD_{12}(R_j) \prod_{k=1}^{s_2'} iD_{21}(R_{s_1'+k}).$$

(6.6)

For all the islands, we make the above replacements, i.e., the LHS of Eqs. (6.4) and (5.6) are replaced with respective RHS. Through this procedure, each \(iD_{21}\) and each \(iD_{12}\) in Eq. (6.1) is “used” twice. Then we obtain

$$\text{Eq. (6.1)} = \exp \left( \beta \left[ \sum_{j=1}^{m} p_j - \sum_{j=1}^{n} q_j \right] \right)$$

$$\times \prod_{j=1}^{N_1} iD_{12}(R_j) \prod_{k=1}^{N_2} iD_{21}(R_{N_1+k}).$$

Now we note that the propagators in \(\mathcal{I}_1\)’s (\(\mathcal{I}_2\)’s) are \(iD_{11}\)’s (\(iD_{22}\)’s), and vertices in \(\mathcal{I}_1\)’s (\(\mathcal{I}_2\)’s) are \(i\lambda (-i\lambda)\) [cf. above after Eq. (3.13)]. Then, using the relation (3.11),

$$[iD_{11}(R)]^* = iD_{22}(R),$$

and \([i\lambda]^* = -i\lambda\), we easily see that

$$\left[ \mathcal{I}_1(Q_{j_1}, ..., Q_{j_{\ell'}}; P_{i_1}, ..., P_{i_{\ell'}}) \right]^* = \mathcal{I}_2(Q_{j_1}, ..., Q_{j_{\ell'}}; P_{i_1}, ..., P_{i_{\ell'}}).$$

(6.7)

\(^\dagger\)A comment on QCD (QED) is in order. As to the 4-gluon vertex, when compared to the scalar theory, no new feature arises. Let \(\mathcal{V}_i (i = 1, 2)\) be the factor that is associated to a trigluon vertex in a type-\(i\) island. \(\mathcal{V}_1\) is real and \(\mathcal{V}_2 = -\mathcal{V}_1\). Then, in place of Eq. (6.7), we have \(\mathcal{I}_1^* = (-)^N \tau_2\) with \(N\) the number of trigluon vertices in \(\mathcal{I}_1\). Since \(\mathcal{A}\) in Eq. (5.2) contains even number of trigluon vertices, Eq. (6.8) holds unchanged. Let us turn to analyze the quark-gluon vertex. In a standard notation, the factor associated to a quark-gluon vertex in a type-1/2 island is \(\pm ig\gamma^\mu T^a\). Taking trace, in \(\mathcal{A}\) in Eq. (5.2), of the products of \(\gamma\)-matrices and of color matrices yield a real function of \(P\)’s and \(Q\)’s. Then, \((ig)^* = -ig\) leads to Eq. (6.8). To sum up, Eq. (6.8) holds for QCD (QED).
Here we note that, from the first-principle derivation above, it is obvious that, to any order of perturbation series, the amplitude $A$ in Eq. (5.2) is real, provided that all the contributing diagrams are added. This fact, together with Eq. (5.7), shows that

$$A(P_1^{(2)}, ..., P_m^{(2)}, Q_1^{(1)}, ..., Q_n^{(1)}; P_1^{(1)}, ..., P_m^{(1)}, Q_1^{(2)}, ..., Q_n^{(2)})$$

$$= \exp \left( \beta \left[ \sum_{i=1}^{m} p_i - \sum_{j=1}^{n} q_j \right] \right)$$

$$\times A(Q_1^{(2)}, ..., Q_n^{(2)}, P_1^{(1)}, ..., P_m^{(1)}, Q_1^{(1)}, ..., Q_n^{(1)}; P_1^{(2)}, ..., P_m^{(2)}) \right). \quad (6.8)$$

Using Eq. (6.3), we obtain

$$e^{\beta p_i} n_B(p_i) = 1 + n_B(p_i),$$

$$e^{-\beta q_j} \left\{ 1 + n_B(q_j) \right\} = n_B(q_j). \quad (6.9)$$

Substituting Eq. (6.8) into Eq. (5.2) and using Eq. (6.9), we finally obtain

$$\frac{1}{V} \left( \prod_{j=1}^{n} 2q_j V \right) R \frac{1}{V} \left( \prod_{i=1}^{m} 2p_i V \right) R'. \quad (6.10)$$

Here, the LHS is the rate of the thermal reaction (1.1) while the RHS is the rate of its inverse process

$$\phi(q_1) + ... + \phi(q_n) + \text{heat bath}$$

$$\rightarrow \phi(p_1) + ... + \phi(p_m) + \text{anything}.$$ 

Equation (6.10) is the desired detailed-balance formula.

## 7 $T \rightarrow 0$ limit and Cutkosky rules

In this section, we show that, in the limit, $T \rightarrow 0$, the reaction-rate formula (3.14) reduces to the formula that is obtained using the Cutkosky rules. Then, in the case of $m = 2$ and $n = 0$, Eq. (3.14) goes to the optical theorem and, for $m = 2$ and $n = 1$, Eq. (3.14) goes to the Mueller formula [20] for inclusive reactions.

In the previous section, for a given diagram for $A$ in Eq. (3.14), we have defined a set of “islands”. The islands in the set may be classified in two groups. The first
group consists of the islands, which contains at least one external vertex. Here the external vertex is the vertex, in which or from which the external momentum flows. The second group consists of the isolated islands, which have no external vertex.

Let us take the scalar field theory and investigate zero-temperature limit \((T \to 0)\) of the reaction-rate formula, Eq. (3.14). [Again, generalization to other theories is straightforward.] In this limit, \(iD_{21}(P) \to 2\pi\theta(p_0)\delta(P^2)\) and \(iD_{12}(P) \to 2\pi\theta(-p_0)\delta(P^2)\). It can readily be seen that, due to momentum conservation, \(I_1\) and \(I_2\), Eqs. (6.2) and (6.5), corresponding to the isolated islands vanish. Then, the nonvanishing amplitude \(A\) contains only the islands belonging to the first group. Thus, we obtain

\[
A = \prod_{j=1}^{s} \left[ 2\pi\theta(r_{j0})\delta(R_j^2) \right]^{N_1} \prod_{i=1}^{N_1} I_1(\{P\}_i; \{Q\}_i) \\
\times \prod_{j=1}^{N_2} I_2(\{Q\}_j; \{P\}_j),
\]

where \(\{P\}_i\) etc. denotes the subset of \(P_1, ..., P_m\), which flow in the \(i\)th “type-1 island” etc. \(\{P\}_i \cup \{Q\}_i\) and \(\{Q\}_j \cup \{P\}_j\) are not empty. In Eq. (7.1), the direction of all the \(s\) momenta, \(R\)'s, each of which connects a “type-1 island” and a “type-2 island,” is taken to flow from the “type-1 island” to the “type-2 island”. As noted before, the diagram representing \(A\) in Eq. (7.1) is connected.

The RHS of Eq. (7.1) is just the quantity, which is obtained by applying the Cutkosky rules \(\prod\) (in vacuum theory) to the present case. As a special case, consider Eq. (7.1) with \(m = 2\) and \(n = 0\). Since the particle represented by \(\phi\) is stable at \(T = 0\), in Eq. (7.1), \(N_1 = N_2 = 1\) and \(\{P\}_{i=1} = \{P\}_{j=1} = \{P_1, P_2\}\). Thus Eq. (7.1) is the optical theorem in vacuum theory. Similarly, for \(m = 2\) and \(n \geq 1\), Eq. (7.1) is just the (generalized) Mueller formula [20] for the inclusive process,

\[
\Phi(p_1) + \Phi(p_2) \to \Phi(q_1) + ... + \Phi(q_n) + \text{anything}.
\]

8 Thermal cutting rules

In view of controversy mentioned in Sec. I, we survey in this section the discussions made in the past for the thermal Cutkosky formula and thermal cutting rules.
Although no new result is involved here, it is worth pigeonholing the issue. The Cutkosky formula in vacuum theory is the formula that relates the imaginary or absorptive part of an amplitude $A$ to the sum of cut amplitudes $\sum_{\text{cuts}} B^{(\text{cut})}$. For simplicity, in this section, we take a self-interacting complex scalar field theory. Generalization to other theories are straightforward. $B^{(\text{cut})}$s are constructed from $A$ by so cutting the propagators $iD$’s in $A$ that $A$ is divided into $A_S$ and $A_{S^*}$, which are amputated. Here $A_S$ is a part(s) of $A$ and $A_{S^*}$ is the complex conjugate of the amplitude that is obtained from $A$ by removing $A_S$ and $iD$’s. Cutting the propagator $iD(P)$ makes $iD(P)$

$$2\pi\theta(\pm p_0) \delta(P^2 - m^2),$$  

(8.1)

where the upper (lower) sign is taken when $P$ flows from a vertex in $A_S$ ($A_{S^*}$) to a vertex in $A_{S^*}$ ($A_S$). When the Cutkosky formula is applied to a forward amplitude $A$, we see that $\text{Im} A$ is proportional to the corresponding reaction rate, where cutted propagators represent the (on-shell) particles in the final state.

Kobes and Semenoff (KS) were the first who generalized the Cutkosky formula to the case of RTF. Namely they obtained the formula that relates the imaginary part of a thermal amplitude to the sum of “circled amplitudes,” each of which corresponds to the “circled” diagram that includes the so-called circled and uncircled vertices. The first paper of discusses general thermal amplitudes and the second one discusses physical amplitudes, i.e., amplitudes with all external vertices being of type 1. In the sequel, unless otherwise stated, we shall restrict our concern to the physical amplitudes. The thermal Cutkosky formula deduced in may be written in terms of thermal amplitudes in RTF:

$$\text{Im} \left[iG(P^{(1)}_1, \ldots, P^{(1)}_n)\right] = -\frac{1}{2} \sum'_{i_1, \ldots, i_n=1} 2 G(P^{(i_1)}_1, \ldots, P^{(i_n)}_n).$$

(8.2)

Here $G(P^{(ij)}_1, \ldots, P^{(ij)}_n)$ stands for the (amputated) thermal amplitude with type-$i_j$ ($j = 1, \ldots, n$) external vertex in which or from which $P_j$ flows. In Eq. (8.2), the sum $\sum'$ stands for taking summation excluding $i_1 = \ldots = i_n = 1$ and $i_1 = \ldots = i_n = 2$. Note that, as a matter of course, in $G$, sum is taken over the types (1 and 2) for all internal vertices.
KS then generalized the notion of cuttings. Comparison of $iD_{21}(P)$, Eq. (3.10), and $iD_{12}(P) = iD_{21}(-P)$, Eq. (3.12), with Eq. (8.1) leads them to regard $iD_{12}$ and $iD_{21}$ in $G$’s on the RHS of Eq. (8.2) as cutted propagators. Through cuttings, each $G$ is divided into several pieces. KS then introduced a notion of cuttable and uncuttable diagrams. The former diagram is the diagram that does not include isolated island(s) (cf. Sec. VII) while the latter diagram includes at least one isolated island. Note that, in the case of vacuum theory, all the diagrams are cuttable ones, which motivates KS to introduce the above definition. Thus, the terminology “uncuttable” sounded quite natural at the time of its introduction. In spite of the fact that this is a matter of definition, existence of uncuttable diagrams has aroused controversy.

Kobes analyzed [13] retarded Green functions in terms of circled diagrams. As to the usage of “cuttings”, “cuttable”, and “uncuttable,” he followed [12].

Jeon analyzed [14] two-point functions in imaginary-time formalism. Continuing to the real energies, he discussed thermal cutting rules. His definition of cutting is the same as in [12], i.e., the propagators $iD_{12}$ and $iD_{21}$ are regarded as cutted propagators. No mention was made on the cuttable and uncuttable diagrams, but no doubt that he supposed all diagrams to be cuttable.

Bedeque, Das, and Naik analyzed [15] the imaginary part of thermal amplitudes (physical and “unphysical) from the same starting formula as in [12], but with different route. Recall that the propagator $iD_{jk}$ ($j, k = 1, 2$) connects a type-$j$ vertex with a type-$k$ vertex. $iD_{jk}$ is defined to be a cutted propagator if and only if one of the type-$j$ and type-$k$ vertices is of circled and another is of uncircled (cf. the first paper of [12]). They then showed that the imaginary part of a thermal amplitude is written as the sum of cuttable diagrams, in the sense of KS stated above. In each cuttable diagram, connected subdiagram(s) at one side of the cut line contains only uncircled vertices (external and internal) while connected subdiagram(s) at the other side of the cut line contains only circled vertices. As was pointed out in [16], however, each connected part contains in general propagators that are proportional to the on-shell factor $\delta(P^2 - m^2)$. Of course, in the zero-temperature limit, their formula as well as KS’s one reduce to the Cutkosky formula.

Gelis extensively analyzed [16] thermal cutting rules for various formulations of real-time thermal field theory. As to the usage of “cuttings”, “cuttable”, and “un-
cuttable,” he followed [12].

Cutting rules for thermal reaction-rate formula are discussed in [2, 3, 4, 5, 6, 7]. Note that, as mentioned above, in vacuum theory, the cutted propagator, Eq. (8.1), corresponds to the (on-shell) final-state particle. The thermal cutting rules introduced in [2, 3, 4, 5, 6, 7] is a generalization of this fact. As we have seen above, $iG_{12}$ (which collectively denotes $iD_{12}$ and $iS_{12}$) $[iG_{21}]$ consists of two parts, the one comes from the particle [antiparticle] in the initial state and another comes from the antiparticle [particle] in the final state. While $iG^{(T)}_{11}$ and $iG^{(T)}_{22}$, the $T$-dependent parts of $iG_{11}$ and $iG_{22}$, come from the interplay of initial-state (anti)particle and the final-state (anti)particle. We recall that each of the thermal propagators $iG_{11}$ and $iG_{22}$ consists of two parts, the $T = 0$ part $iG^{(0)}$ and the $T$-dependent part $iG^{(T)}$. Then, $A$ in Eq. (3.14) or (5.2) is divided into $2^N$ contributions, where $N$ is the number of $iG_{11}$’s and $iG_{22}$’s. Above observation leads us to regard $iG_{12}$, $iG_{21}$, $iG^{(T)}_{11}$, and $iG^{(T)}_{22}$ as the cutted propagators.

Through the applications of the above cutting rules, $A$ is divided into several subparts. Each subpart contains only type-1 vertices or only type-2 vertices. The former (latter) belongs to $S$ ($S^*$) in $\langle S^* S \rangle$. The cuttings work as follows. The line that cut $iG_{12}(P)$ with $p_0 > 0$ ($p_0 < 0$) is the initial-state particle (final-state antiparticle) cut line. The line that cut $iG_{21}(P)$ with $p_0 > 0$ ($p_0 < 0$) is the final-state particle (initial-state antiparticle) cut line. The line that cut $iG^{(T)}_{11}(P)$ $[iG^{(T)}_{22}(P)]$ is the initial-state cut line and the final-state cut line in $S$ $[S^*]$ and, in $S^* [S]$, an one extra spectator particle with $P$ is. For the line that cut $iG^{(T)}_{11}(P)$ with $p_0 > 0$ ($p_0 < 0$) is the initial-state particle (antiparticle) cut line and the final-state particle (antiparticle) cut line. For the cut line on $iG^{(T)}_{22}(P)$, similar statement holds.

It is quite obvious that the “cutting rules” introduced above for thermal reaction rates may be used for general thermal amplitudes evaluated in the Keldish variant of RTF.

Finally, it is worth mentioning that it can easily be seen from Eqs. (3.14) and (8.2) that the RHS of Eq. (8.2), which represents the imaginary part of a physical amplitude, is a sum of various reaction rates times corresponding kinematical factors.
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Appendix A Proof of Equation (3.24)

Here we prove the identity Eq. (3.24). We expand the RHS of Eq. (3.24) in powers of \( n_B(x)(\equiv \xi) \) to obtain

\[
\begin{align*}
\sum_{k=0}^{\min(j, j')} \frac{i!}{(i-j+k)! (j'-k)!} \xi^k (1+\xi)^{j'-k} \\
= \sum_{k=0}^{\min(j, j')} \sum_{\ell=0}^{j'-k} \frac{i!}{(i-j+k)! \ell! (j'-k-\ell)!} \frac{j! j'!}{k! (j-\ell)!(j'-k-\ell)!} \xi^{i+j'-\ell} \\
= \sum_{k=0}^{j'} \sum_{\ell=0}^{\min(j'-k, j)} \frac{i!}{(i-j+\ell)! \ell! (j'-k-\ell)!} \frac{j!}{k! (j'-k-\ell)!} \xi^{i+j'-k},
\end{align*}
\]

(A.1)

where \( i \geq j \). Comparing Eq. (A.1) with Eq. (3.23), we see that it is sufficient to show that

\[
k \mathcal{F}_{j, j'}^{i, i'} = k \mathcal{G}_{j, j'}^{i, i'},
\]

(A.2)

where

\[
k \mathcal{F}_{j, j'}^{i, i'} \equiv \sum_{\ell=0}^{\min(j, j'-k)} \frac{i! j!}{\ell! (i-j+\ell)! (j'-k-\ell)! (j-\ell)!},
\]

(A.3)

\[
k \mathcal{G}_{j, j'}^{i, i'} \equiv \frac{(j+i'-k)!}{(i+k)! (j'-k)!}.
\]

(A.4)

Here we define two functions,

\[
F_{j, j'}^{i, i'}(x) \equiv \sum_{k=0}^{j'} x^{j'-k} k \mathcal{F}_{j, j'}^{i, i'},
\]

(A.5)

\[
G_{j, j'}^{i, i'}(x) \equiv \sum_{k=0}^{j'} x^{j'-k} k \mathcal{G}_{j, j'}^{i, i'}.
\]

(A.6)

It can easily be shown that \( F \)'s and \( G \)'s satisfy the same differential equation,

\[
\begin{align*}
\frac{d}{dx} F_{j, j'}^{i, i'}(x) &= F_{j, j'-1}^{i, i'-1}(x) + j F_{j-1, j'-1}^{i, i'}(x), \\
\frac{d}{dx} G_{j, j'}^{i, i'}(x) &= G_{j, j'-1}^{i, i'-1}(x) + j G_{j-1, j'-1}^{i, i'}(x).
\end{align*}
\]

(A.7)

(A.8)
From Eqs. (A.5), (A.6) with Eqs. (A.3) and (A.4), we obtain

\[ F_{i, i'}(0) = G_{i, i'}(0) = \frac{i!}{(i - j)!} \quad (A.9) \]

\[ F_{j, 0}(x) = G_{j, 0}(x) = \frac{i!}{(i - j)!} \quad (A.10) \]

We see from Eq. (A.7) [Eq. (A.8)] that \( F_{i, i'}(x) \) [\( G_{i, i'}(x) \)] may be obtained from \( F_{j, 0}(x) \) [\( G_{j, 0}(x) \)] in Eq. (A.10) with \( \hat{i}' \leq i' \), \( \hat{j} \leq j \), and \( F_{\hat{i}, \hat{j}}(0) \) [\( G_{\hat{i}, \hat{j}}(0) \)] in Eq. (A.9) with \( \hat{i}' \leq i' \), \( \hat{j} \leq j \), \( \hat{j}' \leq j' \). Since \( F \)'s and \( G \)'s subject to the same set of equations (A.7) - (A.10), we conclude that

\[ F_{j, j'}(x) = G_{j, j'}(x) \]

which proves Eq. (A.2).

Q.E.D.

32
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FIG. 1. Two examples of double-cut diagrams for the transition probability $W = S^*S$ in vacuum theory. Dashed lines are the final-state cut lines while the dotted lines are the initial-state cut lines. The left side of the cut lines represents the $S$-matrix element, $S$, while the right side does $S^*$. The line that is cutted by the final-state (initial-state) cut line represents a particle in the final (initial) state in $S$. The lines cutted by the initial-state [final-state] cut line include those corresponding to $\{A\} [\{B\}]$ in Eq. (2.2). The group of lines on top of diagrams stands for spectator particles. (a) Both $S$ and $S^*$ are connected. In addition to the spectator particles mentioned above, additional spectator particles are in $S^*$. (b) $S$ is connected while $S^*$ is disconnected. Note, however, that $S^*S$ is connected.

Fig. 2 Diagrammatic representation of the thermal amplitude $A$ in Eq. (3.14).

Fig. 3 Double-cut diagrams for $W = S^*S$, which yields (a) $iD_{12}^{(+)}(P) iD_{21}^{(+)}(P)$ and (b) $iD_{11}^{(T)(+)}(P) iD_{22}^{(T)(+)}(P)$. Here $P = (p, p)$.

Fig. 4 Double-cut diagrams for $W = S^*S$, which yields (a) $iD_{12}^{(+)}(P) iD_{21}^{(+)}(P)$ and (b) $iD_{11}^{(T)(+)}(P) iD_{22}^{(T)(+)}(P)$. Here $P = (p, p)$. 

35
FIG. 3
FIG. 4