Equation of state for Entanglement in a Fermi gas

Christian Lunkes\textsuperscript{1} \textsuperscript{1},\textsuperscript{1} \textsuperscript{1} Caslav Brukner\textsuperscript{1} \textsuperscript{1},\textsuperscript{2},\textsuperscript{2} and Vlatko Vedral\textsuperscript{1} \textsuperscript{1},\textsuperscript{2} \textsuperscript{2}

\textsuperscript{1}QOLS, Blackett Laboratory, Imperial College London, London SW7 2BZ, England
\textsuperscript{2}Institut für Experimentalphysik, Universität Wien, Boltzmannasse 5, A-1090 Vienna, Austria
\textsuperscript{3}The Schrödinger Institute for Mathematical Physics, Boltzmannasse 9, Vienna, Austria
\textsuperscript{4}The School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, England

(Dated: April 1, 2022)

Entanglement distance is the maximal separation between two entangled electrons in a degenerate electron gas. Beyond that distance, all entanglement disappears. We relate entanglement distance to degeneracy pressure both for extreme relativistic and non-relativistic systems, and estimate the entanglement distance in a white dwarf. Treating entanglement as a thermodynamical quantity, we relate the entropy of formation and concurrence to relative electron distance, pressure, and temperature, to form a new equation of state for entanglement.

PACS numbers: 03.67.Mn, 03.65.Ud

The relation between thermodynamics and entanglement is of fundamental importance, and has been studied by many authors in the recent years [1, 2, 3, 4, 5, 6]. The second law of thermodynamics states that entropy $S$ can be regarded as a measure of disorder. It is a macroscopic variable and can be related to other state variables like pressure and temperature. In quantum mechanics, the entropy of formation is a measure of entanglement [7]. Entanglement is a purely quantum mechanical feature which the fermion gas behaves as an ideal gas. Since the particles in an ideal gas are not interacting, we can use Green’s function approach to find the subsystem of the two electrons, separated by a relative distance $r$. It was found that all entanglement vanishes if the relative distance $r$ between electrons is greater then the entanglement distance $r_e \propto 1/k_F$, where $k_F$ is the Fermi momentum. In this case, entanglement is purely due to particle statistics and not due to any physical interaction. At zero temperature, the electron gas is in its lowest energy configuration. All energy states are occupied up to the Fermi energy $\epsilon_F$. The only pressure the system exerts is the degeneracy pressure $P(n)$ (due to the Pauli exclusion principle), which solely depends on its density $n$. We now want to relate this pressure to entanglement. For the non-relativistic case, the expression for pressure remains valid if the temperature is low compared to the Fermi temperature $T_F = \epsilon_F/k_B$, where $k_B$ is the Boltzmann constant. This condition is met if the density is sufficiently high, at which the fermion gas behaves as an ideal gas. Since the particles in an ideal gas are not interacting, we can use the same treatment as in [1, 10], provided the density of the fermion gas is high enough. For the extreme relativistic case, the expression for pressure is valid for all temperatures.

We will first relate $r_e$ to the degeneracy pressure $P$. We then estimate $r_e$ in white dwarfs. Third, we express entropy of formation and concurrence, which are measures of entanglement, as functions of degeneracy pressure, relative distance between the electrons $r$, and temperature $T$. This represents our equation of state for entanglement.

Vedral [2], and then Oh and Kim [10], first described entanglement of noninteracting electron gases. They used Green’s function approach to find the subsystem of the two electrons, separated by a relative distance $r$. It is given by the two-spin density matrix [10]:

$$\rho_{12}(r) = \frac{n^2}{8} \left[ \delta_{\sigma_1,\sigma'_1} \delta_{\sigma_2,\sigma'_2} - f(r)^2 \delta_{\sigma_1,\sigma'_2} \delta_{\sigma'_1,\sigma_2} \right], \quad (1)$$

where $\sigma_1, \sigma_2, \sigma'_1, \sigma'_2$ are the spin variables of the two electrons under consideration. Following [2, 10], the state is entangled iff the Peres-Horodecki (partial transposition) condition [11] is satisfied:

$$f(r)^2 > \frac{1}{2}, \quad (2)$$

where $f$ at zero temperature and non zero temperature is given by:

$$f(r, 0) = \frac{3}{r k_F^2} \int_0^{k_F} k \sin kr \, dk \quad (3)$$

and

$$f(r, T) = \frac{3}{r k_F^2} \int_0^{\infty} k n_k \sin kr \, dk, \quad (4)$$

for the relativistic case, the expression for pressure remains valid if the temperature is low compared to the Fermi temperature $T_F = \epsilon_F/k_B$, where $k_B$ is the Boltzmann constant. This condition is met if the density is sufficiently high, at which the fermion gas behaves as an ideal gas. Since the particles in an ideal gas are not interacting, we can use the same treatment as in [2, 10], provided the density of the fermion gas is high enough. For the extreme relativistic case, the expression for pressure is valid for all temperatures.
where
\[ n_k = \frac{1}{\exp(\frac{\epsilon_k - \mu}{k_B T}) + 1} \] (5)
is the Fermi-Dirac distribution, \( \mu \) is the chemical potential, and \( \epsilon_k \) are the energy levels. The region of entanglement is:

\[ 0 \leq r < r_e, \] (6)

where \( r \) is the relative distance between the two electrons, and the entanglement distance \( r_e \) is found to be inversely proportional to the Fermi momentum \( k_F \):

\[ r_e = \frac{\zeta}{k_F}. \] (7)
The proportionality constant \( \zeta \) can be found numerically by solving \( f^2 = \frac{1}{3} \). The integral in Eq.(4) can be approximated for different temperatures \( T \). In general, \( \zeta \) is a function of temperature, but for \( T \) sufficiently close to zero, \( \zeta \) and the entanglement distance \( r_e \) only change slightly from the ones at zero temperature. However, the functional dependence of the relation \( k_F = \frac{\zeta}{r} \) remains the same for a gas with constant density. A typical value for small temperatures is \( \zeta \approx 1.8 \). We note that there will be higher order entanglement present (3, 4 etc. electrons), but we will not consider it here.

Eq.(6) is the upper bound for the distance of two entangled electrons. The entanglement distance \( r_e \) is of the order of the Fermi wavelength, which corresponds to the spread of the momentum wavefunction of the electrons that are close to the Fermi sphere. In general, only these electrons are of physical significance. Eq.(6) is the most restrictive bound for entanglement distance and is valid for all electrons in the gas. Any randomly chosen electrons that are closer then \( r_e \) are definitely entangled. However, due to the spread in their wavefunction, two electrons whose momenta lie far below the Fermi surface can also be entangled even if they are further apart then \( r_e \). This however implies the knowledge of their momenta \( k \), which is not accessible in this case. Eq.(6) was derived by making no preselection of momenta. The function \( f(r) \) in Eq.(11) is computed by summing over all possible momenta, from \( k = 0 \) to \( k = k_F \), for \( T = 0 \). In order to have entanglement, \( f(r) \) must then satisfy the Peres-Horodecki criterion. In [10], the entanglement was already discussed as a function of the relative distance between the electrons. It was found that the entanglement is maximal if the relative distance between the electrons is zero. In this case, the spatial wavefunctions fully overlap, and because of the antisymmetric nature of the two electron wavefunction, the particles must have opposite spin. No spin configuration \( (|\uparrow\downarrow>, |\downarrow\uparrow>) \) is discriminated and therefore the two electrons are in the maximally entangled Bell state, \( |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \).

We now relate this entanglement to \( P \). First we discuss the domain of validity of our approach. Let us start by considering an electron gas at \( T = 0 \). All the electrons fill up the quantum states in accordance with the Pauli exclusion principle so that the total energy has its smallest value. The electrons have momenta from \( p = 0 \) to \( p_F = (\frac{3\pi^2}{2})^{\frac{3}{2}}k_B T \). The radius of the Fermi sphere in momentum space is the Fermi momentum \( k_F \), and \( n = N/V \) is the density. The number of states in a volume element of width \( dp \) is proportional to \( p^2 \). The total energy can then be obtained by multiplying the number of states by \( \frac{p^2}{2m} \) and integrating over all momenta. This leads to the general equation of state of a Fermi gas, from which the degeneracy pressure can be deduced. We distinguish between the non-relativistic and the extreme relativistic case [12]:

\[ P = \begin{cases} \frac{(3\pi^2)^{\frac{3}{2}}k^2_F}{5m}n \frac{\text{K}}{\text{mol}} & \text{non-rel.} \\ \frac{(3\pi^2)^{\frac{3}{2}}k_F}{4\pi^3}n^\frac{4}{3} & \text{rel.} \end{cases} \] (8)

where \( m \) is the mass of the electron. The non-relativistic expression is valid for low temperatures, provided that \( T \ll T_F \), which is the the Fermi temperature. Rewriting this gives \( T \ll \frac{(3\pi^2)^{\frac{3}{2}}k^2_F}{2mk}n^\frac{4}{3} \), which is satisfied if the gas density is sufficiently high. A degenerate electron gas has the peculiar property that it increasingly approaches the ideal gas as the density increases [12]. This can be seen from the following argument. Following Landau and Lifshitz [12], we consider a plasma of electrons and positively charged nuclei. When the electrons move independently as in an ideal gas, the mean kinetic energy (which in order of magnitude is equal to the Fermi energy) is much higher than the Coulomb interaction. For a nucleus of charge \( Z e \) and with the mean distance between the electrons and the nuclei being \( a \):

\[ \frac{E_{\text{Coulomb}}}{E_{\text{kinetic}}} = \frac{Ze^2}{a} \ll \frac{p_F^2}{2m} \]

\[ \frac{Ze^2}{a} \ll \frac{h^2n^{\frac{4}{3}}}{2m} \quad \text{with} \quad a \sim \left(\frac{Z}{n}\right)^{\frac{1}{6}} \]

\[ n \gg \left(\frac{e^2m}{\hbar^2}\right)^3 Z^2 \]

There is no electron interaction if the gas density is sufficiently high.

Let us now relate \( r_e \) to the degeneracy pressure. The non-relativistic expression for the degeneracy pressure is given by Eq.(5), which is valid at low temperatures if the density is high. High density means no electron interaction, so we use the same treatment as in [8] [10]. The extreme relativistic expression for
the degeneracy pressure is valid at all temperatures. The Fermi momentum can be expressed in terms of the density \( n \), and Eq. (7) is rewritten as:

\[
\rho_e = \frac{\zeta}{(3\pi^2)^{\frac{1}{3}} n^{\frac{4}{3}}}.
\]

(9)

The entanglement distance is approximately equal to the average distance between the electrons in the gas. Rewriting the degeneracy pressure in terms of \( \rho_e \), we obtain for the non-relativistic and extreme relativistic case:

\[
P = \begin{cases} 
\frac{\zeta^2 k^2}{15\pi^2 m} r_e^{-5} & \text{non-rel.} \\
\frac{\zeta^4 \hbar c}{12\pi^2} r_e^{-4} & \text{rel.} 
\end{cases}
\]

(10)

This expression is valid if \( T \ll T_F \). The constants \( \zeta \) and \( \zeta' \) are different because in the extreme relativistic limit, the density is much higher than in the non-relativistic case. When the degeneracy pressure of a given gas is high, \( \rho_e \) is small and the Fermi momentum is high. This means that the electrons are moving at high velocities and are localized in space with no overlap of their wavefunctions, hence no entanglement is present in this limit.

We now give an estimate for \( \rho_e \) in white dwarfs. White dwarfs are examples of highly degenerate matter. They are objects of extremely high density. We express the electron density as:

\[
n = \frac{Z \rho}{4m n_A H},
\]

(11)

where \( Z \) and \( A \) are the number of protons and nucleons and \( m_H \) is the mass of the Hydrogen atom. If now assume that the density is constant, we can deduce entanglement distance for a white dwarf of mass \( M \) and radius \( R \). The density is:

\[
\rho = \frac{M}{4/3\pi R^3}.
\]

(12)

Using Eq. (11), we write:

\[
r_e \propto R M^{-\frac{4}{3}}.
\]

(13)

We estimate the entanglement distance of SIRIUS B, which is a carbon-oxygen white dwarf, of mass \( 1M_\odot \), radius \( R = 0.008 R_\odot \), and a temperature of \( T = 27000K \). Assuming non-relativistic electrons, and calculating the Fermi temperature gives \( T/T_F \sim 10^{-6} \), which justifies our low temperature approximation. From Eq. (13), the entanglement distance is found to be \( r_e \simeq 6 \times 10^{13} m \). This is two orders of magnitude bigger then the size of a typical nucleus. Note that here entanglement length is much smaller than the size \( R \) of the system. This should be contrasted with the limit of entanglement length being of the order of \( R \), i.e \( r_e \simeq R \), for which the mass of the system is \( m \simeq 2.7 \times 10^{-27} kg \), which is about the mass of a nucleus.

Let us finally express the entropy of formation as a function of the degeneracy pressure, relative distance between the electrons \( r \), and temperature \( T \). Relating the Fermi momentum to the degeneracy pressure for the non-relativistic and extreme relativistic case:

\[
k_F = \begin{cases} 
\left(\frac{15\pi^2 m}{\hbar^2}\right)^{\frac{1}{3}} P^{\frac{1}{3}} & \text{non-rel.} \\
\left(\frac{12\pi^2}{\hbar c}\right)^{\frac{1}{3}} P^{\frac{1}{3}} & \text{rel.}
\end{cases}
\]

(14)

so Eq. (11) becomes

\[
f_{r,P,T} = \begin{cases} 
\frac{\gamma}{rP_T} \int_0^\infty k n_k \sin kr dk & \text{non-rel.} \\
\gamma' \int_0^\infty k n_k \sin kr dk & \text{rel.}
\end{cases}
\]

(15)

with

\[
\gamma = \left(\frac{\hbar^2 3^\frac{4}{3}}{15\pi^2 m}\right)^\frac{1}{3} \text{ and } \gamma' = \left(\frac{\hbar c^2 3^\frac{4}{3}}{12\pi^2}\right)^\frac{1}{3}.
\]

(16)

We now describe the system as it is compressed. If the density \( n \) changes, so does the Fermi momentum \( k_F = (3\pi^2 n)^{\frac{1}{3}} \) and the degeneracy pressure \( P \). This corresponds to a white dwarf, where the particles are forced into a smaller and smaller volume by the gravitational pressure. This process can continue until the extreme relativistic limit is reached where the energy of the electrons becomes large compared to \( mc^2 \).

The integrals in Eq. (15) can be solved numerically for a given temperature. At low temperatures, we take the chemical potential to be \( \mu \simeq \epsilon_F \). All the information needed to compute entanglement is given in the two particle density matrix Eq. (11). From this density matrix, any entanglement measure \( E \) can be computed as a function of \( f_{r,P,T} \). We express the entropy of formation in terms of electron distance, degeneracy pressure and temperature for an electron gas. The equation of state for entanglement becomes:

\[
E_F(r,P,T) = \hbar \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{(f_{r,P,T})^2 - 1}{2}} \right],
\]

(17)

with \( \hbar(y) = -y \log_2 y - (1-y) \log_2 (1-y) \), and \( f_{r,P,T} \) is either the non-relativistic or extreme relativistic expression. We can also use the concurrence \( C \) to express entanglement:

\[
C(r,P,T) = \max \left\{ \frac{2f_{r,P,T}^2 - 1}{2 - f_{r,P,T}^2}, 0 \right\}.
\]

(18)
We find that the average amount of entanglement for electron pairs is given by integrating from a microscopic variable to a macroscopic one. For convenience, first define the variable $x$ as a function of non-relativistic degeneracy pressure $P$, for a gas with initial entanglement distance $r_e = 10^{-8} m$.

Consider a non-relativistic gas with high density at zero temperature, which undergoes compression. This is shown in Fig. 1. If we fix the relative distance $r$ of the two electrons to be smaller than the entanglement distance $r_e$, then the two electrons are certainly entangled. The smaller the distance $r$ is, the more entangled are the electrons. If we now compress the system, the density $n$ increases, and $r_e$ in Eq.(13) decreases. The entanglement distance $r_e$ approaches the relative electron distance $r$ from above, and the entanglement between the two electrons decreases. If the system is compressed further, $r_e$ becomes smaller than $r$, and entanglement vanishes. This means that the electrons are forced into a smaller and smaller volume, so their momenta increase and their wavefunctions overlap less, until there is no overlap and therefore no entanglement.

Eq.(17) or Eq.(18) are not equations of state in the "classical sense" of that word. We call them like that in order to explore considerations of entanglement distance to degeneracy pressure. We then estimated entanglement distance for the simplest model of a white dwarf. For the first time, the entropy of formation and the concurrence was related to temperature, pressure and relative distance between the electrons. It is valid for temperatures which are sufficiently close to zero and high density for the non-relativistic case, and for all temperatures for the extreme relativistic case. In this work entanglement is purely due to particle statistics (Pauli exclusion principle) and not to fermion interaction. Future work would consist of investigating entanglement for fermions that physically interact. It would also be interesting to extend the analysis to multipartite entanglement and to see if one could indeed treat entanglement as a thermodynamical variable, and write an equation for entanglement similar to the well-known fundamental thermodynamic equation for energy $U$, $dU = TdS - PdV$.

In conclusion, we have considered a degenerate electron gas with high density at low temperatures, and an extremely relativistic electron gas. We first related entanglement distance to degeneracy pressure. We then estimated entanglement distance for the simplest model of a white dwarf. For the first time, the entropy of formation and the concurrence was related to temperature, pressure and relative distance between the electrons. It is valid for temperatures which are sufficiently close to zero and high density for the non-relativistic case, and for all temperatures for the extreme relativistic case. In this work entanglement is purely due to particle statistics (Pauli exclusion principle) and not to fermion interaction. Future work would consist of investigating entanglement for fermions that physically interact. It would also be interesting to extend the analysis to multipartite entanglement and to see if one could indeed treat entanglement as a thermodynamical variable, and write an equation for entanglement similar to the fundamental thermodynamic equation for energy $U$, $dU = TdS - PdV$.

We are grateful to Marcelo França Santos for helpful discussions.

* Electronic address: christian.lunkes@ic.ac.uk
‡ Electronic address: c.brukner@ic.ac.uk
† Electronic address: v.vedral@ic.ac.uk
[1] V. Vedral and E. Kashefi, Phys. Rev. Lett. 89, 037903, (2002).
[2] M. B. Plenio and V. Vedral, Contemp. Phys. 39, 431, (1998).
[3] V. Vedral, New J. Phys. 6, 22, (2004).
[4] S. Popescu and D. Rohrlich, Phys. Rev. A 56, R3319 (1997).
[5] M. C. Arnesen, S. Bose, V. Vedral, Phys. Rev. Lett. 87, 017901 (2001).
[6] S. Gosh, T. F. Rosenbaum, G. Aeppli, S. N. Copper smith, Nature 425, 48 (2003).
[7] C. H. Bennett et al., Phys. Rev. A 54, 3824 (1996).
[8] C. Brukner and V. Vedral, e-print quant-ph/0406040 (2004).
[9] V. Vedral, CEJP 2, 289 (2003).
[10] S. Oh and J. Kim, Phys. Rev. A 69, 054305 (2004).
[11] A. Peres, Phys. Lett. A 202, 16 (1995).
[12] Landau and Lifshitz, Statistical Physics 1, (Butterworth-Heinemann, 1980).
[13] B. W. Carroll and D. A. Ostlie, Modern Astrophysics, (Addison-Wesley,1996).
[14] V. Coffman, J. Kundu, W. Wootters, Phys. Rev. A 61, 052306 (2000).