INVERSE EIGENVALUES PROBLEM OF NONNEGATIVE MATRICES VIA UNIT LOWER TRIANGULAR MATRICES

ALIMOHAMMAD NAZARI\textsuperscript{1} AND ATIYEH NEZAMI\textsuperscript{2}

Abstract. The main goal of this work is to use the unit lower triangular matrices for solving inverse eigenvalue problem of nonnegative matrices (NIEP) and present the easier method to solve this problem. We solve the problem for any given number of the real and complex eigenvalues. Finally, we present the necessary and sufficient conditions to solve this problem with this method.

1. Introduction And Preliminaries

A matrix is called unit lower triangular if it is lower triangular matrix and all entries on its main diagonal are one. The inverse of these matrices also is unit lower triangular. In Gaussian elimination method and LU factorization unit lower triangular matrices play a very important roll.

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of real or complex numbers in order that it be the spectrum of a nonnegative matrix $A$ with spectrum $\sigma$, we will say that $\sigma$ is realizable and that it is realization of $\sigma$.

Some necessary conditions on the list of real number $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.

(1.1)

(1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to $\sigma$ (Perron-Frobenius theorem).

(2) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$.

(3) $s_k^m \leq n^{m-1}s_{km}$ for $k, m = 1, 2, \ldots$ (JLL inequality)[1, 2].

Although many mathematicians have worked on the inverse eigenvalue problem of nonnegative matrices 3, 19, 5, 6, 7, 8, 9, 10, 11, 12, 13, our aim of this paper is solving the problem in the easier method via similarity of a matrix with upper triangular matrix.

2010 Mathematics Subject Classification. 15A18, 15A60, 15A09, 93B10.

Key words and phrases. Nonnegative matrices, Unit triangular matrices, Inverse eigenvalue problem.

Received: dd mmmm yyyy, Accepted: dd mmmm yyyy.

* Corresponding author.
Two matrix $A$ and $B$ is called similar if there exist an invertible matrix $P$ such that $PAP^{-1} = B$. We recall that two similar matrices have same eigenvalues and this Theorem plays a very important roll in this paper.

From now on in this paper $\lambda_1$ is the symbol of Perron eigenvalue, and assume that all of given list of real spectrum satisfy in necessary conditions (1), (2) and (3). Our method is the following, started by Guo in [13] and we continue Guo’s method in this paper.

The paper is organized as follows: At first for a given real set of eigenvalues that the number of its positive eigenvalues is less than or equal the number of negative eigenvalues, we solve the NIEP via unit lower triangular matrix. In continue for a given set of eigenvalues as $\sigma$ with nonnegative summation, in which the number of negative eigenvalues $\sigma$ less than the number of negative eigenvalues, we find a nonnegative matrices such that $\sigma$ is its spectrum. In the any case of problem at first we solve the problem for a finite number of eigenvalues $\sigma$ and then we provide the general solution. Finally we present a Theorem that shows the necessary and sufficient conditions for solving the problem in our way.

In section 3 we consider complex spectrum with Perron eigenvalue $\lambda_1$ and nonnegative summation and again solve the NIEP.

2. Real spectrum

Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a given spectrum such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$. We divide the problem into two parts, the first part is $k \leq \frac{n}{2}$ and the another part $k > \frac{n}{2}$ and solve the problem.

2.1. The spectrum with $k \leq \frac{n}{2}$. In this section we study NIEP with condition $k \leq \frac{n}{2}$.

2.1.1. Spectrum with one positive eigenvalues.

**Theorem 2.1.** [13]. Assume that given $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\sum_{i=1}^{n} \lambda_i \geq 0$, then there exist a set of nonnegative matrix that $\sigma$ is its spectrum.

**Proof.** Let $n = 2$, then consider the upper triangular matrix $A = \begin{bmatrix} \lambda_1 & \alpha_2 \\ 0 & \lambda_2 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, therefore the following matrix

$$C = LAL^{-1} = \begin{bmatrix} \lambda_1 - \alpha_2 & \alpha_2 \\ \lambda_1 - \alpha_2 - \alpha_2 + \lambda_2 & \lambda_1 - \alpha_2 - \alpha_2 + \lambda_2 \end{bmatrix},$$

has eigenvalues $\lambda_1$ and $\lambda_2$ and if $-\lambda_2 \leq \alpha_2 \leq \lambda_1$, then the matrix $C$ is nonnegative.

For $n = 3$ we consider $A = \begin{bmatrix} \lambda_1 & \alpha_2 & \alpha_3 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then the matrix

$$C = LAL^{-1} = \begin{bmatrix} \lambda_1 - \alpha_2 - \alpha_3 & \alpha_2 & \alpha_3 \\ \lambda_1 - \alpha_2 - \alpha_2 - \alpha_3 + \lambda_2 & \lambda_1 - \alpha_2 - \alpha_2 - \alpha_3 + \lambda_2 & \alpha_3 \\ \lambda_1 - \alpha_2 - \alpha_2 - \alpha_3 - \lambda_3 & \lambda_1 - \alpha_2 - \alpha_2 - \alpha_3 - \lambda_3 & \alpha_3 + \lambda_3 \end{bmatrix},$$

is similar to the matrix $A$ and if $-\lambda_2 \leq \alpha_2$, $-\lambda_3 \leq \alpha_3$ and $\alpha_2 + \alpha_3 \leq \lambda_1$ then the matrix $C$ is nonnegative.
In continue, we consider
\[ A = \begin{bmatrix}
\lambda_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\
0 & \lambda_2 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ddots & \lambda_n
\end{bmatrix}, \]
and
\[ L = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 0 & \ddots & \ddots & 0 \\
1 & 0 & 0 & \cdots & 1
\end{bmatrix}, \]
then the matrix
\[ C = LAL^{-1} = \begin{bmatrix}
\lambda_1 - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\
\lambda_1 - \lambda_2 - t & \alpha_2 + \lambda_2 & \alpha_3 & \cdots & \alpha_n \\
\lambda_1 - \lambda_{n-1} - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\
\lambda_1 - \lambda_n - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n + \lambda_n
\end{bmatrix}, \]
that \( t = \sum_{i=2}^{n} \alpha_i \) is similar to the matrix \( A \), and if
\[-\lambda_i \leq \alpha_i, \quad i = 2, 3, \cdots, n, \]
\[ t \leq \lambda_1. \]
then the matrix \( C \) is nonnegative and has eigenvalues \( \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \).

**Remark 2.2.** If given the spectrum \( \sigma \) with zero summation, then in the above Theorem it is necessary that we lie the value \( \lambda_i \) on the \((i, i)\) entry of main diagonal of the matrix \( A \) for \( i = 1, 2, \cdots, n \) and \(-\lambda_i\) in the entry \((1, i)\) of this matrix for \( i = 1, 2, \cdots, n \) and then the all elements on main diagonal of matrix \( C \) are zero.

**Remark 2.3.** In Theorem 2.1 if \(-\sum_{i=2}^{n} \lambda_i \leq \lambda_1 \) and we set \( \alpha_i = -\lambda_i \) for \( i = 2, 3, \cdots, n \), then we can construct a nonnegative matrix that all elements of its main diagonal are zero except elements of the first row and the fist column. For instance if \( \sigma = \{10, -2, -2, -2, -1, -1\} \), then by (2.1) the following matrix has spectrum \( \sigma \),
\[ C = \begin{bmatrix}
2 & 2 & 2 & 2 & 1 & 1 \\
4 & 0 & 2 & 2 & 1 & 1 \\
4 & 2 & 0 & 2 & 1 & 1 \\
4 & 2 & 2 & 0 & 1 & 1 \\
3 & 2 & 2 & 2 & 0 & 1 \\
3 & 2 & 2 & 2 & 1 & 0
\end{bmatrix}, \]
that \( \alpha_i \) for \( i = 2, \cdots n \) satisfy in the conditions (2.2). In this case we can determine all elements that lie on the main diagonal of matrix \( C \) that summation of them is equal to the
summations of eigenvalues of the matrix $A$, for instance if we want the first four elements of main diagonal of matrix (2.1) are $\frac{1}{2}$, we must consider $\alpha_2 = 2.5, \alpha_3 = 2.5, \alpha_4 = 2.5, \alpha_5 = 1, \alpha_6 = 1$ then $t = 9.5$ that satisfy in conditions (2.4), and and the matrix $C$ is as below

$$
\begin{bmatrix}
0.5 & 2.5 & 2.5 & 1 & 1 \\
2.5 & 0.5 & 2.5 & 1 & 1 \\
2.5 & 2.5 & 0.5 & 1 & 1 \\
2.5 & 2.5 & 2.5 & 0 & 1 \\
1.5 & 2.5 & 2.5 & 2.5 & 0
\end{bmatrix}
$$

2.1.2. Spectrum with two positive eigenvalues. In this section, at first we consider $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ such that $\lambda_1 \geq \lambda_2 > 0 \geq \lambda_3 \geq \lambda_4$ and $\sum \lambda_i \geq 0$, $\lambda_1 \geq |\lambda_i|, i = 3, 4$ and in continue for a given set $\sigma$ with two positive eigenvalues and three negative eigenvalues solve the problem and finally for two positive eigenvalues and more than three negative eigenvalues with nonnegative summation again study the problem.

**Theorem 2.4.** Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ such that $\lambda_1 \geq \lambda_2 > 0 \geq \lambda_3 \geq \lambda_4$ and let $\sum \lambda_i \geq 0$, $\lambda_1 \geq |\lambda_i|, i = 3, 4$, then there exist a set of nonnegative matrices that $\sigma$ is its spectrum.

**Proof.** We start with the following matrices

$$
A = \begin{bmatrix}
\lambda_1 & \alpha_2 + \alpha_4 & \alpha_3 & 0 \\
0 & \lambda_2 & \alpha & \alpha_4 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}.
$$

Then the following matrix

$$
C = LAL^{-1} = \begin{bmatrix}
\lambda_1 - t & \alpha_2 + \alpha_4 & \alpha_3 & 0 \\
\lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 & \alpha_3 + \alpha & \alpha_4 \\
\lambda_1 - \lambda_3 - t & \alpha_2 + \alpha_4 & \alpha_3 + \lambda_3 & 0 \\
\lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 - \lambda_4 & \alpha_3 + \alpha & \alpha_4 + \lambda_4
\end{bmatrix},
$$

that $t = \sum_{i=2}^{4} \alpha_i$ is similar to the matrix $A$. If

$$
-\lambda_1 \leq \alpha_i, \quad i = 2, 3, 4, \\
t \leq \lambda_1
$$

(2.4)

$$
-\alpha_3 \leq \alpha \leq \lambda_1 - \lambda_2 - t,
$$

then the matrix $C$ is nonnegative.

**Remark 2.5.** In above Theorem one of the interesting solution is $\alpha_i := \lambda_i, i = 1, 2$ and $\alpha = \lambda_3$ that satisfies in condition (2.4) with zero summation and the all another elements of matrix $C$ are zero. For instance let $\sigma = \{7, 3, -5, -5\}$, then $\sigma$ is realizable by following nonnegative matrices

$$
C = \begin{bmatrix}
0 & 2 & 5 & 0 \\
2 & 0 & 0 & 5 \\
5 & 2 & 0 & 0 \\
2 & 5 & 0 & 0
\end{bmatrix},
$$
this spectrum is studied in [14] and we solve this problem easier.

Now we consider the set of \( \sigma \) with two positive eigenvalues and three negative eigenvalues with special conditions.

**Theorem 2.6.** Consider spectrum \( \sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} \) such that \( \lambda_1 \geq \lambda_2 > 0 > \lambda_3 \geq \lambda_4 \geq \lambda_5 \) with nonnegative summation. If \( t = \sum_{i=2}^{5} \alpha_i \) and there exist a nonnegative \( 5 \times 5 \) matrix that \( \sigma \) is its spectrum.

**Proof.** In this case we consider

\[
A = \begin{bmatrix}
\lambda_1 & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_3 & 0 & 0 \\
0 & \lambda_2 & \alpha & \alpha_4 & \alpha_5 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{bmatrix},
\]

where \( \alpha_i \geq -\lambda_i, i = 4, 5 \) and \( \alpha_5 < \lambda_2 \) and also \( \alpha_4 + \alpha_5 \geq \lambda_2 \) and

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1
\end{bmatrix},
\]

then

\[
C = LAL^{-1} = \begin{bmatrix}
\lambda_1 - t & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_3 & 0 & 0 \\
\lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 & \alpha_3 + \alpha & \alpha_4 & \alpha_5 \\
\lambda_1 - \lambda_3 - t & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_3 + \lambda_3 & 0 & 0 \\
\lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 - \lambda_4 & \alpha_3 + \alpha & \alpha_4 + \lambda_4 & \alpha_5 \\
\lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 - \lambda_5 & \alpha_3 + \alpha & \alpha_4 & \alpha_5 + \lambda_5
\end{bmatrix}.
\]

The matrix \( C \) is similar to the matrix \( A \) and if satisfy the conditions

\[
-\lambda_i \leq \alpha_i, \quad i = 2, 3, 4, 5
\]

\[
t \leq \lambda_1
\]

\[
-\alpha_3 \leq \alpha \leq \lambda_1 - \lambda_2 - t,
\]

then this matrix is nonnegative.

**Theorem 2.7.** Consider spectrum \( \sigma = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \) such that \( \lambda_1 \geq \lambda_2 > 0 \geq \lambda_3 \geq \cdots \geq \lambda_n \) with nonnegative summation. If \( t = \sum_{i=2}^{n} \alpha_i \) , then there exist a set of nonnegative \( n \times n \) matrix that \( \sigma \) is its spectrum.
**Proof.** The proof is provided as the proof of previous Theorem. In this case we consider the matrices $A$ and $L$ respectively as

$$A = \begin{bmatrix}
\lambda_1 & \alpha_2 + (\alpha_r + \cdots + \alpha_n) & \alpha_3 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\
0 & \lambda_2 & \beta_3 & \cdots & \beta_{r-1} & \alpha_r & \cdots & \alpha_n \\
0 & 0 & \lambda_3 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \lambda_{r-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_r & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_n \\
\end{bmatrix},$$

$$L = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
\end{bmatrix},$$

where the values of the second row of the matrix $A$, $\alpha_i, i = 3, \ldots, n$, must be at least equal to the values of the $-\lambda_i, i = 3, \ldots, n$ values in their columns respectively, and the value of $\alpha_r$ is the last value in the second row with starting of the last element on this row, such that $\sum_{i=r}^n \alpha_i \geq \lambda_2$.

Then we have

$$C = LAL^{-1}$$

$$= \begin{bmatrix}
c_{11} & c_{12} & \alpha_3 & \alpha_4 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\
c_{21} & \alpha_2 + \lambda_2 & \alpha_3 + \beta_3 & \alpha_4 + \beta_4 & \cdots & \alpha_{r-1} + \beta_{r-1} & \alpha_r & \cdots & \alpha_n \\
c_{31} & \alpha_3 + \lambda_3 & \alpha_4 + \beta_4 & \cdots & \alpha_{r-1} + \beta_{r-1} & 0 & \cdots & 0 \\
c_{4,1} & c_{42} & \alpha_3 + \beta_3 & \lambda_4 + \alpha_4 & \cdots & \alpha_{r-1} + \beta_{r-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{r-1,1} & c_{r-1,2} & \alpha_3 + \beta_3 & \alpha_4 + \beta_4 & \cdots & \lambda_{r-1} + \alpha_{r-1} & 0 & \cdots & 0 \\
c_{r1} & c_{r2} & \alpha_3 + \beta_3 & \alpha_4 + \beta_4 & \cdots & \alpha_{r-1} + \beta_{r-1} & \lambda_r + \alpha_r & \cdots & \alpha_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \alpha_3 + \beta_3 & \alpha_4 + \beta_4 & \cdots & \alpha_{r-1} + \beta_{r-1} & 0 & \cdots & \lambda_n + \alpha_n \\
\end{bmatrix}.$$
where
\[
\begin{align*}
c_{11} &= \lambda_1 - t, \\
c_{31} &= c_{41} = \cdots = c_{r-1,1} = \lambda_1 - \lambda_2 - t, \\
c_{21} &= c_{r+1} = \cdots = c_{n1} = \lambda_1 - \lambda_2 - t - (\beta_3 + \cdots + \beta_{r-1}), \\
c_{12} &= c_{32} = \alpha_2 + \alpha_r + \cdots + \alpha_n, \\
c_{22} &= \alpha_2 + \lambda_2 - \lambda_i, \quad i = 4 \cdots n,
\end{align*}
\]
that \( t = \sum_{i=2}^{n} \alpha_i \). The matrix \( C \) is nonnegative matrix if hold the following conditions
\[
\begin{align*}
-\lambda_i &\leq \alpha_i, \quad i = 2, \cdots, n \\
t &\leq \lambda_1 \\
-\alpha_i &\leq \beta_i \leq \lambda_1 - \lambda_2 - t, \quad i = 3, \cdots, r-1,
\end{align*}
\]
and then \( \sigma \) is its spectrum.

**Example 2.8.** Let \( \sigma = \{19, 1, -5, -5, -3, -3, -2, -2\} \). This spectrum is chosen form [16] and by our method we again solve this problem. We select two matrices \( A \) and \( L \) as:
\[
A = \begin{bmatrix}
19 & 1 & 5 & 5 & 3 & 3 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & a_{27} & 2 \\
0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix},
\]
\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]
Then the matrix $C = LAL^{-1}$ is computed as follows:

$$
C = \begin{bmatrix}
0 & 1 & 5 & 5 & 3 & 3 & 2 & 0 \\
-1 - a_{27} & 0 & 5 & 5 & 3 & 3 & 2 + a_{27} & 2 \\
5 & 1 & 0 & 5 & 3 & 3 & 2 & 0 \\
5 & 1 & 5 & 0 & 3 & 3 & 2 & 0 \\
3 & 1 & 5 & 5 & 0 & 3 & 2 & 0 \\
3 & 1 & 5 & 5 & 3 & 0 & 2 & 0 \\
2 & 1 & 5 & 5 & 3 & 3 & 0 & 0 \\
-1 - a_{27} & 2 & 5 & 5 & 3 & 3 & 2 + a_{27} & 0
\end{bmatrix}.
$$

If $-2 \leq a_{27} \leq -1$, then the matrix $C$ is nonnegative and has spectrum $\sigma$.

2.1.3. **NIEP for spectrum three positive eigenvalues.** In this section we study the NIEP for three positive eigenvalues and more than or equal three negative eigenvalues.

**Theorem 2.9.** Consider spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5 \geq \lambda_6$ with nonnegative summation. Then there exist a set of nonnegative $6 \times 6$ matrices that $\sigma$ is its spectrum.

**Proof.** We consider the matrix $A$ as follow

$$
A = \begin{bmatrix}
\lambda_1 & \alpha_2 + \alpha_5 + \alpha_6 + \alpha_3 & 0 & \alpha_4 & 0 & 0 \\
0 & \lambda_2 & \alpha_6 + \alpha_3 & a_{24} & \alpha_5 & 0 \\
0 & 0 & \lambda_3 & a_{34} & a_{35} & \alpha_6 \\
0 & 0 & 0 & \lambda_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_6
\end{bmatrix}.
$$

The unit lower triangular matrix that solve the problem, we can find as

$$
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
$$

In this case, the following matrix will be obtained with similarity transformations of the unit lower triangular matrix

$$
C = LAL^{-1}
$$

(2.6)
Consider spectrum diagonal located the elements of $\sigma$. The spectrum with and in general in the next subsection, we describe this process.

and by similarity $\sigma$ is its spectrum.

**Theorem 2.10.** Consider spectrum $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \cdots \geq \lambda_n$ with nonnegative summation. Then there exist a set of nonnegative $n \times n$ matrix that $\sigma$ is its spectrum.

**Proof.** The method of proof this Theorem is continue of process of previous Theorem and in general in the next subsection, we describe this process.

2.1.4. The spectrum with $k$ positive eigenvalues. In this case we study NIEP with $k$ positive eigenvalues and more than or equal $k$ non-positive eigenvalues. One of the most important points that we have to consider is where non-zero values elements lie in the unit lower triangular matrix and another important point is the detection and the amount and locations of nonzero value of the upper triangular matrix $A$ which in the main its diagonal located the elements of $\sigma$.

**Theorem 2.11.** Consider spectrum $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_n\}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$ with nonnegative summation. Then there exist a set of nonnegative $n \times n$ matrix that $\sigma$ is its spectrum.

**Proof.** For construction of matrix $C$ in which is nonnegative and has eigenvalues $\sigma$, we explain that how to construct the upper triangular matrix $A$ and unit lower triangular matrix $L$.

For construction of the matrix $A$ we put $\lambda_1, \lambda_2, \ldots, \lambda_n$ on main diagonal of this matrix where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are positive and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n$ are nonpositive eigenvalues and $k \leq \frac{n}{2}$.

Then we start with the last row and the last column of the matrix $A$ entry $\lambda_n$. We add the value $\alpha_n \geq -\lambda_n$ on the row $k$ and column $n$ of matrix $A$ and if $\lambda_k - \alpha_n \geq 0$, we add $\alpha_{n-1} \geq -\lambda_{n-1}$ on the row $k$ and column $n-1$ and if again $(\lambda_k - \alpha_n - \alpha_{n-1}) \geq 0$ we add the value $\alpha_{n-2} \geq \lambda_{n-2}$ on the row $k$ and column $n-2$ and continue this way until the first time $\lambda_k - \sum_{i=1}^{n} \alpha_i < 0$, then we add the values $-(\lambda_k - \sum_{i=r}^{n} \alpha_i)$ on the row $k$ and column $k-1$ and we add in row $k$ from column $k+1$ until $r-1$ the values $\beta_{k,j}$ for $j = k+1, \ldots, r-1$ respectively. The values $\alpha_i$ help to us that the effect of the negative eigenvalues in the matrix solution to be eliminated. We select the values $\beta_{k,j}$ to obtain at last nonnegative matrix. After this step we use the second positive eigenvalue $\lambda_{k-1}$. We add the value $\alpha_{r-1}$ to the row $k-1$ and column $r-1$ and if $\lambda_{k-1} - \alpha_{r-1} > 0$ we add the value $\alpha_{r-2} \geq \lambda_{r-2}$ on the row $k-1$ and column $r-2$ and similar the previous step continue. Assume that the index $m$ is the value such that for the first time $\lambda_{k-1} - \sum_{i=k+1}^{m} \alpha_i < 0$. In this step again the values $\beta_{k-1,j}$ started from the column $k+1$ and finished $m-1$. We continue
this method from $\lambda_n$ to $\lambda_{k-1}$. Then we have

$$A = \begin{pmatrix} 
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
\lambda_s & a_{s,s+1} & \lambda_{s+1} & 0 & \alpha_{k+1} \\
& \ddots & \ddots & \ddots & \ddots \\
\lambda_{m-2} & a_{k-2,k-1} & \lambda_{k-1} & a_{k-1,k} & \ddots & \ddots & \alpha_{m-1} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\lambda_{m-1} & \alpha_{n-1} & \alpha_n \\
\end{pmatrix},$$

where

$$a_{s,s+1} = -(\lambda_{s-1} - \sum_{i=k+1}^{n} \alpha_i),$$

$$a_{k-2,k-1} = -(\lambda_{k-1} - \sum_{i=m}^{r} \alpha_i),$$

$$a_{k-1,k} = -(\lambda_k - \sum_{i=r}^{n} \alpha_i).$$

For construction matrix $L$, we act the following procedure. Whereas $L$ is unit lower triangular matrix we set on the main diagonal of this matrix the number 1. For all $\alpha_j, j = n, \cdots, k + 1$ that lies on entry $(i, j)$ of matrix $A$, we set the entry $(j, i)$ of the matrix $L$ again 1 and in this row the previous entries will be 1 or appropriate number (that help us for getting nonnegative matrix $C$ in product of three matrices $LAL^{-1}$) until the column of the last positive $\lambda_i, i = k, k - 1, \cdots$ that in the construction the matrix $A$ is used. For all entries of matrix $A$ in which get positive value for $\lambda_k - \sum_{i=r}^{n} \alpha_i < 0$, we set 1 on the transpose entry of these elements and again in its row the previous elements of these entries will be 1 or appropriate number until the column of the last positive that in the construction the matrix $A$ is used. For simplicity, we display these entries of matrix
Let $L$ be a matrix with $L_{st} = 1$ when $s = t$. Then the matrix $L$ can be written as:

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
$$

with $s^{th}$ row $\rightarrow$ 1
$k^{th}$ row $\rightarrow$ 1, 1
$m^{th}$ row $\rightarrow$ 1, 1, 1
$p^{th}$ row $\rightarrow$ 1, 1, 1, 1
$n^{th}$ row $\rightarrow$ 1, 1, 1, 1

$\uparrow s^{th} \text{col}$ $\uparrow k^{th} \text{col}$
where the empty entries have value zero. By a simple induction we have

\[
L^{-1} = \begin{bmatrix}
1 & & & \\
1 & & & \\
1 & & & \\
\vdots & & & \\
1 & 1 & \ldots & 1 \\
0 & 0 & \ldots & -1 & 1 \\
0 & 0 & \ldots & 0 & -1 & 1 \\
1 & 1 & \ldots & & & \\
\vdots & \vdots & \vdots & & & \\
1 & 1 & \ldots & 1 & 0 & \ldots & 1 \\
1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 1 \\
1 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

**Remark 2.12.** It is important to note that a given list of eigenvalues \(\sigma\) is realizable with our method, when the value of Perron eigenvalue or the last positive eigenvalue that is used for difference between positive eigenvalues \(\lambda_s\) and \(\sum_\alpha\) at least equal the minimum of negative eigenvalues. For instance the list \(\sigma = \{6, 1, 1, 1, -4, -4\}\) is solvable and the list \(\sigma = \{6, 1, 1, 1, 1, -4, -4\}\) is not solvable by our method, although we add a positive amount in this list, since in the solving process we can not use the Perron eigenvalue for the last eigenvalue -4. When we put the elements of main diagonal of upper triangular matrix \(A\) from up to down by decreasing order, but if we interchange the elements of main diagonal of matrix \(A\) as the list \(\sigma = \{6, 1, 1, 1, -4, -4, 1\}\) is solvable and we do not consider the last column in our algorithm. So far for the convenience, we assume that the values of main diagonal of matrix \(A\) is decreasing and in general it is not necessary.

2.2. **The spectrum with** \(k > \frac{n}{2}\). From the beginning discussion until now we study NIEP with the number of negative eigenvalues more than or equal the number of positive eigenvalues. In this section we study in which the number of positive eigenvalues in the given spectrum \(\sigma\) more than the number of negative eigenvalues.

The spectrum with one negative eigenvalues is very simply solved. In continue we study the cases that the spectrum \(\sigma\) has more than or equal two negative eigenvalues. So there are at least 5 members in \(\sigma\). Unfortunately in this section there exist some cases that the lower triangular matrix must have some non zero-one values.

2.2.1. **The spectrum with two negative eigenvalues.** In this subsection we solve NIEP with three positive and two negative eigenvalues that satisfies in condition (1) and then we
solve the extension of problem of NIEP with three positive eigenvalues and more than three negative eigenvalues.

**Theorem 2.13.** Let \( \sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} \) such that \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 > \lambda_4 \geq \lambda_5 \) and satisfies in (1.1), then there exist a set of nonnegative eigenvalues in which \( \sigma \) is its spectrum.

**Proof.** In this case we construct the members \( A \) and \( L \) as follows.

\[
A = \begin{bmatrix}
\lambda_1 & -\lambda_2 - \lambda_4 - \lambda_3 - \lambda_5 & 0 & 0 & 0 \\
0 & \lambda_2 & -\lambda_3 - \lambda_5 & -\lambda_4 & 0 \\
0 & 0 & \lambda_3 & a_{34} & -\lambda_5 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{bmatrix}

\]

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
l_{31} & 1 & 1 & 0 & 0 \\
l_{51} & 1 & 1 & 1 & 0 \\
l_{51} & 1 & 1 & 1 & 1
\end{bmatrix},
\]

so for simplicity, we use the following symbols to represent the matrix \( C \):

\[
C = LAL^{-1} = \begin{bmatrix}
s_1 & \lambda_1 - s_1 & 0 & 0 & 0 \\
c_{21} & 0 & - (\lambda_3 + \lambda_5) & -\lambda_4 & 0 \\
c_{31} & c_{32} & 0 & -\lambda_4 + a_{34} & -\lambda_5 \\
c_{41} & -\lambda_4 & - (\lambda_3 + \lambda_5) & 0 & 0 \\
c_{51} & c_{52} & -\lambda_5 & -\lambda_4 + a_{34} & 0
\end{bmatrix},
\]

where

\[
c_{21} = c_{41} = s_1 - \lambda_2 - (\lambda_3 + \lambda_5)(1 - l_{31}),
\]

\[
c_{31} = l_{31}s_1 - \lambda_2 - \lambda_5(1 - l_{51}),
\]

\[
c_{51} = l_{51}s_1 - \lambda_2 - \lambda_5(1 - l_{31}),
\]

\[
c_{32} = l_{31}(\lambda_1 - s_1) + s_1 - \lambda_1 - \lambda_3 - a_{34},
\]

\[
c_{52} = l_{51}(\lambda_1 - s_1) + s_1 - \lambda_1 - \lambda_3 - a_{34}.
\]

The matrix \( C \) is nonnegative if all entries of this matrix will be nonnegative and then we have

\[
1 - \frac{\lambda_3 + a_{34}}{s_1 - \lambda_1} \leq l_{31} \leq 1 - \frac{s_1 - \lambda_2}{\lambda_3 + \lambda_5},
\]

\[
1 - \frac{\lambda_3 + a_{34}}{s_1 - \lambda_1} \leq l_{51} \leq 1 - \frac{l_{31}s_1 - \lambda_2}{\lambda_5},
\]

\[
\lambda_4 \leq a_{34} \leq \frac{(s_1 - \lambda_1)(s_1 - \lambda_2)}{\lambda_3 + \lambda_5} - \lambda_3.
\]

\( \Box \)

**Remark 2.14.** If the number of positive eigenvalues more than three, we can continue the above method for construction of matrices \( A \) and \( L \) and get nonnegative matrix \( C \), for instance if for \( n = 6 \) in spectrum \( \sigma \) we have \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \geq 0 > \lambda_{n-1} \geq \lambda_n \),
In general case we can construct the matrices $A$ and $L$ as

\[
A = \begin{bmatrix}
\lambda_1 & -\lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_2 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & -\lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & -\lambda_4 - \lambda_6 & -\lambda_5 & 0 \\
0 & 0 & 0 & \lambda_4 & \alpha_{45} & -\lambda_6 \\
0 & 0 & 0 & 0 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_6
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
l_{41} & l_{42} & 1 & 1 & 0 & 0 \\
l_{61} & l_{62} & 1 & 1 & 0 & 1 \\
l_{61} & l_{62} & 1 & 1 & 0 & 1
\end{bmatrix},
\]

therefore the nonnegative matrix $C$ obtain as follows

\[
C = \begin{bmatrix}
s_1 & \lambda_1 - s_1 & 0 & 0 & 0 & 0 \\
s_1 - \lambda_2 & 0 & \lambda_1 + \lambda_2 - s_1 & 0 & 0 & 0 \\
c_{31} & c_{32} & 0 & -\lambda_4 - \lambda_6 & -\lambda_5 & 0 \\
c_{41} & c_{42} & c_{43} & 0 & -\lambda_5 + \alpha_{45} & -\lambda_6 \\
c_{51} & c_{52} & -\lambda_5 & -\lambda_4 - \lambda_6 & 0 & 0 \\
c_{61} & c_{62} & c_{62} & -\lambda_6 & -\lambda_5 + \alpha_{45} & 0
\end{bmatrix},
\]

where

\[
\begin{align*}
c_{31} &= s_1 - \lambda_2 + (-\lambda_4 - \lambda_6)(-l_{41} + l_{42}), \\
c_{41} &= l_{41}s_1 - l_{42}\lambda_2 - \lambda_6 (-l_{61} + l_{62}), \\
c_{51} &= s_1 - \lambda_2 + (-\lambda_4 - \lambda_6)(-l_{41} + l_{42}), \\
c_{61} &= l_{61}s_1 - l_{62}\lambda_2 - \lambda_6 (-l_{41} + l_{42}), \\
c_{32} &= -\lambda_3 + (-\lambda_4 - \lambda_6)(1 - l_{42}), \\
c_{42} &= (\lambda_1 - s_1)(l_{41} - l_{42}) - \lambda_3 - \lambda_6 (1 - l_{62}), \\
c_{52} &= -\lambda_3 + (-\lambda_4 - \lambda_6)(1 - l_{42}), \\
c_{62} &= l_{62}(\lambda_1 + \lambda_2 - s_1) + \lambda_3 + \lambda_6 + \lambda_5 - \alpha_{45}, \\
c_{43} &= l_{42}(\lambda_1 + \lambda_2 - s_1) + \lambda_3 + \lambda_6 + \lambda_5 - \alpha_{45}, \\
c_{63} &= (\lambda_1 - s_1)(l_{61} - l_{62}) - \lambda_3 - \lambda_6 (1 - l_{42}).
\end{align*}
\]

In general case we can construct the matrices $A, L$ and $C$ as above method and the non zero-one entries of matrix $L$ is located in the row number of the last positive eigenvalues and row number of the last negative eigenvalues of matrix $A$. 
Example 2.15. Let \( \sigma = \{6, 1, 1, -4, -4\} \). This spectrum is solved in [7] and in [14] is discussed about its C-realizability. We can find the nonnegative matrix \( C \) that has spectrum \( \sigma \) by our method and simpler. For this we consider matrices \( A \) and \( L \) as:

\[
\begin{bmatrix}
6 & 6 & 0 & 0 & 0 \\
0 & 1 & 3 & 4 & 0 \\
0 & 0 & 1 & -4 & 4 \\
0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & -4
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
l_{21} & 1 & 1 & 0 & 0 \\
l_{51} & 1 & 1 & 0 & 1
\end{bmatrix}
\]

then the matrix \( C \) is computed by relation \( C = LAL^{-1} \) i.e.

\[
\begin{bmatrix}
0 & 6 & 0 & 0 & 0 \\
2 - 3l_{21} & 0 & 3 & 4 & 0 \\
3 - 4l_{51} & -3 + 6l_{21} & 0 & 0 & 4 \\
2 - 3l_{21} & 4 & 3 & 0 & 0 \\
3 - 4l_{21} & -3 + 6l_{51} & 4 & 0 & 0
\end{bmatrix}.
\]

The interval that of \( l_{21} \) and \( l_{51} \) that by them them matrix \( C \) is nonnegative very simple is computed.

2.2.2. The spectrum with three negative eigenvalues.

Theorem 2.16. Assume that \( \sigma = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-3} \geq 0 > \lambda_{n-2} > \lambda_{n-1} > \lambda_n \) with \( \sum_{i=1}^{n} \lambda_i \geq 0 \), then there exist a set of nonnegative matrix \( C \) such that \( \sigma \) is its spectrum.

Proof. The proof process is the same as before, and we only need to give an example to illustrate this process. \( \square \)

Example 2.17. Let \( \sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\} = \{8, 2, 2, 1, -5, -5, -5\} \) that \( \sigma \) satisfies in conditions [1.1] special \( \sum \lambda_i = \sum \lambda_i^3 = 0 \), and \( \sum_{k=4}^{n} \lambda_k^6 > 0 \). By our method we consider

\[
A = \begin{bmatrix}
\lambda_1 & -s_1 + \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & -s_1 + \lambda_1 + \lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -s_1 + \lambda_1 + \lambda_2 & 0 & 0 & 0 & 0 & 0 \\
\lambda_3 & -\lambda_8 - \lambda_5 - \lambda_4 - \lambda_7 & 0 & -\lambda_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & -\lambda_8 - \lambda_5 & a_{46} & -\lambda_7 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 & a_{56} & a_{57} & -\lambda_8 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8
\end{bmatrix}.
\]
where $s_1 = \sum_{i=1}^{n} \lambda_i$ and

$$L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
l_{31} & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
l_{41} & l_{42} & 1 & 1 & 0 & 0 & 0 & 0 \\
l_{51} & l_{52} & l_{53} & 1 & 1 & 0 & 0 & 0 \\
l_{61} & l_{62} & 1 & 0 & 0 & 1 & 0 & 0 \\
l_{71} & l_{72} & l_{73} & 1 & 0 & 0 & 1 & 0 \\
l_{81} & l_{82} & l_{83} & l_{84} & 1 & 0 & 0 & 1
\end{bmatrix}. $$

If we select $a_{46} = -5, a_{56} = \frac{20}{7}, a_{57} = -5$ and also select $l_{31} = \frac{4}{7}, l_{41} = \frac{1}{7}, l_{42} = \frac{23}{70}, l_{51} = \frac{1}{37}, l_{61} = \frac{23}{70}, l_{62} = \frac{1}{7}, l_{71} = l_{41}, l_{72} = l_{42}, l_{81} = l_{51}, l_{82} = \frac{9}{70}, l_{83} = \frac{3}{7}$ the matrix $C$ is solution of problem is computed as follows:

$$C = \begin{bmatrix}
0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\
\frac{57}{140} & \frac{13}{7} & 0 & 7 & 0 & 5 & 0 & 0 \\
\frac{67}{140} & \frac{11}{7} & 0 & 0 & 4 & 0 & 5 & 0 \\
\frac{19}{140} & \frac{1}{70} & 0 & 0 & 0 & 0 & 0 & 5 \\
\frac{11}{30} & \frac{23}{70} & 0 & 7 & 0 & 0 & 0 & 0 \\
\frac{67}{140} & \frac{1}{70} & 0 & 5 & 4 & 0 & 0 & 0 \\
\frac{19}{140} & \frac{1}{70} & 0 & 0 & 5 & 0 & 0 & 0
\end{bmatrix}. $$

Now we present another solution for this problem. If we consider the matrix

$$A = \begin{bmatrix}
\lambda_1 & -\lambda_7 - \lambda_3 & -\lambda_2 & 0 & -\lambda_8 & -\lambda_5 & -\lambda_4 & 0 & -\lambda_6 & 0 & 0 \\
0 & \lambda_2 & -\lambda_7 & \lambda_3 & a_{24} & a_{25} & a_{26} & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & a_{34} & a_{35} & a_{36} & -\lambda_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & a_{46} & a_{47} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 & a_{56} & a_{57} & -\lambda_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8 & 0 & 0 & 0 \\
\end{bmatrix}. $$
and the matrix

\[ L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
l_{31} & 1 & 1 & 0 & 0 & 0 & 0 \\
l_{51} & l_{52} & l_{53} & 1 & 1 & 0 & 0 \\
l_{71} & l_{72} & 1 & 0 & 0 & 0 & 1 \\
l_{81} & l_{82} & l_{83} & l_{84} & 1 & 0 & 0 \\
\end{bmatrix}, \]

again if we select \( a_{24} = -2, a_{34} = a_{25} = a_{35} = a_{47} = a_{57} = 0, a_{46} = -5, a_{56} = -2, a_{26} = a_{26} = -5 \) and \( l_{31} = l_{51} = 2, l_{52} = \frac{4}{7}, l_{53} = 0, l_{71} = \frac{4}{7}, l_{72} = \frac{-2}{3}, l_{81} = 2, l_{82} = l_{83} = 0, l_{84} = 1, \)

then the nonnegative matrix

\[ C = \begin{bmatrix}
0 & 1 & 0 & 2 & 0 & 5 & 0 & 0 \\
2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{22}{3} & 0 & 2 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
1/2 & 7/4 & 3/4 & 1/2 & 0 & 7/4 & 0 & 0 \\
5 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
5/3 & 0 & 0 & 4 & 0 & 5 & 0 & 0 \\
5/4 & 3/4 & 0 & 1 & 5 & 3 & 0 & 0 \\
\end{bmatrix}, \]

is solution of problem.

2.2.3. The spectrum with \( k \) negative eigenvalues. In this subsection we consider the extended of problem.

**Theorem 2.18.** Consider \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-k} \geq 0 > \lambda_{n-k+1} \geq \cdots \geq \lambda_k \), then there exist a set of nonnegative matrices that \( \sigma \) is its spectrum.

**Proof.** For construction of matrix \( A \) and \( C \) we will do as follows.

In main diagonal elements of upper triangular matrix \( A \), the all elements of \( \sigma \) in order \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), we put the entries \( (n-k,n), (n-k-1,n-1), \ldots, (n-2k,n-k) \) the real numbers \( \alpha_i \geq -\lambda_i \) that \( i = n-k, \ldots, n-2k \) respectively. In addition the entries \( (n-j-1,n-j) \) for \( j = k, \ldots, 2k-1 \) we set \( -(\alpha_{n-j} - \lambda_{n-j}) \) respectively, and another element between rows \( 1 \) and \( n-k \) and between \( \alpha_i \) and \( \lambda_i \) we put appreciate real numbers whereas the matrix \( C = LAL^{-1} \) be nonnegative.

We choose the matrix \( L \) a lower unit triangular matrix and in the last row of \( L \) we consider \( 0 \leq l_{11}, l_{22}, \ldots, l_{kk} \) that \( \sum_{i=1}^{k} l_{ii} = k \) and element of row \( n-1 \) all be 1 for \( j = 1, 2, \ldots, k-1, \) and row \( n-2 \) all element from \( j = 1, \) to \( n-2 \) be 1, and the row \( n-k \) we have \( n-k+1 \) element 1 and another element be zero. We repeat above method with less that one element for row \( n-k-1, \) i.e. in row \( n-k-1 \) we set \( 0 \leq l_{n-k-1,1}, l_{n-k-1,2}, \ldots, l_{n-k-1,n-k-1}, \)
with $\sum_{j=1}^{k-1} l_{n-k-1,j} = k - 1$ with choosing above method then the matrix $C = LAL^{-1}$ is nonnegative.

\[ \square \]

### 3. Complex Spectrum

In this section we assume that $\sigma \in \mathbb{C}$. We recall that $\sigma = \overline{\sigma}$ is necessary condition for solvable problem. At first we consider the case $n = 3$, this case has two complex conjugate eigenvalues and a Perron eigenvalue $\lambda_1 \geq |\lambda_2 \pm i\mu_2|$ and for $n = 4$ we have two real eigenvalues and a pair complex conjugate eigenvalue. For $n = 5$ or more then we have several case that in continue we study all of them.

In this case we introduce the complex unit lower triangular matrix as follows:

The elements of main diagonal unit lower triangular matrix $L$ in this case are $i$ or $1$.

#### 3.1. The case $n = 3$

We consider this case in two parts. At first we assume that $\lambda_2 > 0$ in $\sigma = \{\lambda_1, \lambda_2 \pm i\mu_2\}$ and in another case $\lambda_2 < 0$.

**Case 1.** $\lambda_2 > 0$: Now we bring an important Theorem from [19] and by helping unit lower triangular matrix in complex case, we find a nonnegative matrix for this Theorem.

**Theorem 3.1.** Let $\sigma = \{\lambda_1, -\lambda_2 \pm i\mu_2\}$, with $\lambda_2 > 0$, $\mu_2 > 0$ and $\lambda_1 \geq \sqrt{\lambda_2^2 + \mu_2^2}$. Then, $\sigma$ is realizable by a nonnegative matrix $A$ if and only if

$$\lambda_1 \geq 2\lambda_2 + 3\max\{0, \frac{\mu_2}{\sqrt{3}} - \lambda_2\}.$$

Now we present solution of this Theorem by our method. We consider the matrix $A$ and $L$ as follows:

$$A = \begin{bmatrix} \lambda_1 & i\mu_2 & 0 \\ 0 & \lambda_2 - i\mu_2 & -i\mu_2 \\ 0 & 0 & \lambda_2 + i\mu_2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & i & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

then we have

$$C = \begin{bmatrix} -l_{21}\mu_2 + \lambda_1 & \mu_2 & 0 \\ -\mu_2(l_{21}^2 + 1) + l_{21}(\lambda_1 - \lambda_2) & l_{21}\mu_2 + \lambda_2 & \mu_2 \\ -\lambda_2 + \lambda_1 & 0 & \lambda_2 \end{bmatrix},$$

and if $\frac{(\lambda_1-\lambda_2)-\sqrt{(\lambda_1-\lambda_2)^2-4\mu_2^2}}{2\mu_2} \leq l_{21} \leq \frac{(\lambda_1-\lambda_2)+\sqrt{(\lambda_1-\lambda_2)^2-4\mu_2^2}}{2\mu_2}$ then $C$ is real nonnegative matrix.

**Case 2.** $\lambda_2 < 0$: In this case the matrices $A$ and $L$ present as

$$A = \begin{bmatrix} \lambda_1 & i\mu_2 & 0 \\ 0 & \lambda_2 - i\mu_2 & -i\mu_2 \\ 0 & 0 & \lambda_2 + i\mu_2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & i & 0 \\ \frac{\lambda_2^2}{\mu_2^2} + 1 & -\frac{\lambda_2 i}{\mu_2} + 1 & 1 \end{bmatrix},$$

then the matrix $C = LAL^{-1}$ find as

$$\begin{bmatrix} -l_{21}\mu_2 + \lambda_1 & \mu_2 & 0 \\ -\mu_2(l_{21}^2 + 1) - l_{21}\lambda_1\mu_2 + 2\mu_2 l_{21}\lambda_2 + \mu_2^2 + \lambda_2^2 & l_{21}\mu_2 + 2\lambda_2 & \mu_2 \\ \frac{(\mu_2^2 + \lambda_2^2)\lambda_1}{\mu_2^2} & 0 & 0 \end{bmatrix}.$$
If \( l_{21} \) lies in the interval
\[
\frac{\lambda_1 - 2\lambda_2 - \sqrt{-4\mu_2^2 + \lambda_1^2 - 4\lambda_1\lambda_2}}{2\mu_2} \leq l_{21} \leq \frac{\lambda_1 - 2\lambda_2 + \sqrt{-4\mu_2^2 + \lambda_1^2 - 4\lambda_1\lambda_2}}{2\mu_2},
\]
then the matrix \( C \) is nonnegative.

### 3.2. The case \( n=4 \)

The case \( n = 4 \) very similar to the previous case. Let \( \sigma = \{\lambda_1, \lambda_2, \lambda_3 \pm \mu_3i\} \) with nonnegative summation. If \( \lambda_2 > 0 \) and \( \lambda_1 \) is the Perron eigenvalue, then by the cases 2 of subsection 6.1, we construct the \( 3 \times 3 \) nonnegative matrix \( C_1 \) and we add the value of \( \lambda_2 \) on the main diagonal and find \( 4 \times 4 \) matrix \( C = \begin{pmatrix} C_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \).

The matrix \( C \) is solution of problem and has spectrum \( \sigma \).

If \( \lambda_2 < 0 \), we construction the matrix \( C \), by combination of subsection 6.1 and some of the related real above sections i.e. we add the value \(-\lambda_2\) in the first row of matrix \( A \) in the same column that the value \( \lambda_2 \) lies on the main diagonal of matrix \( A \), and the another entries of matrix \( A \) and the matrix \( L \) are the combination of above real sections and above complex subsection is constructed.

**Example 3.2.** Consider \( \sigma = \{6, -2, -2 - i, -2 + i\} \). Since the sum of \( \lambda_i \) equal zero, then by our method we find a nonnegative matrix with zero trace. For this we consider two matrices \( A \) and \( L \) as follows:
\[
A = \begin{bmatrix} 6 & 2 & i & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 - i & -i \\ 0 & 0 & 0 & -2 + i \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ l_{31} & 0 & i & 0 \\ 5 & 0 & 1 + 2i & 1 \end{bmatrix}.
\]

then the matrix \( C \) equals
\[
C = LAL^{-1} = \begin{bmatrix} 4 - l_{31} & 2 & 1 & 0 \\ 6 - l_{31} & 0 & 1 & 0 \\ 2l_{31} - l_{31}^2 - 5 & 2l_{31} & l_{31} - 4 & 1 \\ 20 & 10 & 0 & 0 \end{bmatrix},
\]
if \( 4 - \sqrt{11} \leq l_{31} \leq 4 \), then the matrix \( C \) is solution of problem.

### 3.3. The case \( n \geq 5 \)

For \( n \geq 5 \) we combine the above complex case and above real cases and find solution. We will give some examples that already are solved and again solve them by our method.

**Example 3.3.** Let \( \sigma = \{12, i\sqrt{3}, -i\sqrt{3}, 4 + 3i, 4 - 3i\} \), with nonnegative real part of complex eigenvalues, then there exist a nonnegative matrix that \( \sigma \) is spectrum.
For solving this problem we consider two m matrices $A$ and $L$ as follows:

$$A = \begin{bmatrix}
12 & i\sqrt{3} & 0 & 3i & 0 \\
0 & -i\sqrt{3} & -i\sqrt{3} & 0 & 0 \\
0 & 0 & i\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 4 - 3i & -3i \\
0 & 0 & 0 & 4 + 3i
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
l_{21} & i & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
l_{41} & 0 & 0 & i & 0 \\
1 & 0 & 0 & 1 & 1
\end{bmatrix},$$

then the following matrix $C$,

$$C = \begin{bmatrix}
12 - l_{21}\sqrt{3} - 3l_{41} & \sqrt{3} & 0 & 3 & 0 \\
-l_{21}^2\sqrt{3} + (12 - 3l_{41})l_{21} - \sqrt{3} & l_{21}\sqrt{3} & \sqrt{3} & 3l_{21} & 0 \\
-12 - 3l_{41} & 0 & 0 & 3 & 0 \\
8l_{41} - l_{41}\sqrt{3}l_{21} - 3 - 3l_{41}^2 & l_{41}\sqrt{3} & 0 & 3l_{41} + 4 & 3 \\
8 - l_{21}\sqrt{3} & \sqrt{3} & 0 & 0 & 4
\end{bmatrix},$$

is nonnegative and has spectrum $\sigma$, if satisfies the conditions

$$0 \leq l_{41} \leq 4 - \frac{2}{3}\sqrt{3},$$

$$-1/2\sqrt{3}l_{41} + 2\sqrt{3} - 1/2\sqrt{3l_{41}^2 - 24l_{41} + 44} \leq l_{21},$$

$$-1/2\sqrt{3}l_{41} + 2\sqrt{3} + 1/2\sqrt{3l_{41}^2 - 24l_{41} + 44} \geq l_{21}.$$

**Example 3.4.** Let $\sigma = \{6, -2 - 3i, -2 + 3i, -1 - i, -1 + i\}$, with negative real part of complex eigenvalues, then there exist a nonnegative matrix that $\sigma$ is spectrum.

For this problem the matrices $A$ and $L$ will be as:

$$A = \begin{bmatrix}
6 & 3i & 0 & i & 0 \\
0 & -2 - 3i & -3i & 0 & 0 \\
0 & 0 & -2 + 3i & i & 0 \\
0 & 0 & 0 & -1 - i & -i \\
0 & 0 & 0 & 0 & -1 + i
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
l_{21} & i & 0 & 0 & 0 \\
l_{41} & 0 & 0 & i & 0 \\
2 & 0 & 0 & 1 + i & 1
\end{bmatrix},$$

then the matrix $C$ is computed as:

$$C = LAL^{-1} = \begin{bmatrix}
6 - 3l_{21} - l_{41} & 3 & 0 & 1 & 0 \\
10l_{21} - 3l_{21}^2 - 13/3 - l_{21}l_{41} & 3l_{21} - 4 & 3 & l_{21} & 0 \\
26/3 - 13/9l_{41} & 0 & 0 & 13/9 & 0 \\
8l_{41} - 3l_{21}l_{41} - 2 - l_{41}^2 & 3l_{41} & 0 & l_{41} - 2 & 1 \\
12 - 6l_{21} & 6 & 0 & 0 & 0
\end{bmatrix}.$$
REFERENCES

1. C.R. Johnson, Row stochastic matrices similar to doubly stochastic matrices, Linear and Multilinear Algebra 10 (2) (1981) 113-130.
2. R. Lowey, D. London, A note on an inverse problem for nonnegative matrices, Linear and Multilinear Algebra 6 (1978) 83–90.
3. A.M. Nazari, F. Sherafat, On the inverse eigenvalue problem for nonnegative matrices of order two to five, Linear Algebra and its Applications, 436, (2012) 1771-1790.
4. O. Rojo, R. L. Soto, Existence and construction of nonnegative matrices with complex spectrum, Linear Algebra and its Applications 368 (2003) 53-69.
5. M. Fiedler, Eigenvalues of nonnegative symmetric matrices, Linear Algebra Appl. 9 (1974) 119-142.
6. Helena Smigoc, The inverse eigenvalue problem for nonnegative matrices, Linear Algebra and its Applications 393(2004)365–374.
7. R. Reams, An inequality for nonnegative matrices and the inverse eigenvalue problem, Linear and Multilinear Algebra 41(1996)367-375.
8. T. J. Laffey, E. Meehan, A characterization of trace zero nonnegative $5 \times 5$ matrices, Linear Algebra Appl. 302-303(1999)295–302.
9. N. Radwan, An inverse eigenvalue problem for symmetric and normal matrices, Linear Algebra Appl. 248 (1996) 101-109.
10. Helena Smigoc, Construction of nonnegative matrices and the inverse eigenvalue problem, Linear and Multilinear Algebra, Vol. 53, No. 2, March 2005, 85-96.
11. J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcia-Estevez, C. Marijuan, M. Pisonero, The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs, Linear Algebra and its Applications 426 (2007) 729-773.
12. Eleanor Meehan, Some Results On Matrix Spectra, PhD thesis, University College Dublin, 1998.
13. W. Guo, Eigenvalues of nonnegative matrices, Linear Algebra Appl. 266 (1997) 261-270.
14. Alberto Borobia, Julio Moro, Ricardo L. Soto, A unified view on compensation criteria in the real nonnegative inverse eigenvalue problem, Linear Algebra and its Applications 428 (2008) 2574-2584.
15. Thomas J. Laffey , Eleanor Meehan, A characterization of trace zero nonnegative $5 \times 5$ matrices, Linear Algebra and its Applications 302-303 (1999) 295-302.
16. Collao, Charles R.Johnson, Ricardo L.Soto, Universal realizability of spectra with two positive eigenvalues, Linear Algebra and its Applications 545 (2018) 226-239.
17. Cristina Manzaneda, Enide Andrade, Mara Robbiano, Realizable lists via the spectra of structured matrices, Linear Algebra and its Applications 534 (2017) 51-72.
18. K. R. Suleimanova, Stochastic matrices with real characteristic values, Dokl. Akad. Nauk. SSSR, 66(1949), 343–345.
19. O. Rojo, R. L. Soto, Existence and construction of nonnegative matrices with complex spectrum, Linear Algebra and its Applications 368 (2003) 53-69.

1Department of Mathematics, Arak University, Arak, Iran. P. O. Box 38156-8943.
E-mail address: a-nazari@araku.ac.ir

2Department of Mathematics, Arak University, Arak, Iran. P. O. Box 38156-8943..
E-mail address: a-nezami@arshad.araku.ac.ir