Conformal Couplings in Induced Gravity *

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Abstract

It is found that the induced gravity with conformal couplings requires the conformal invariance in both classical and quantum levels for consistency. This is also true for the induced gravity with an extended conformal coupling interacting with torsion.

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I. INTRODUCTION

Far below the electro-weak scale, the weak interaction is well characterized by the dimensional Fermi’s coupling constant, \( G_F = (300\text{GeV})^{-2} \). However, from the success of Weinberg-Salam model, it turns out that the dimensional coupling constant is the low energy effective coupling which is determined by the dimensionless electro-weak coupling constants and the vacuum expectation value of Higgs scalar field through the spontaneous symmetry breaking. Indeed \( G_F \approx v_{\omega}^{-2} \), where \( v_{\omega} \approx 300 \text{ GeV} \) is the vacuum expectation value of Higgs field. The weakness of the weak interaction comes from the largeness of the vacuum expectation value of Higgs field. Thus, among the four fundamental interactions in nature, only gravitational interaction is characterized by the dimensional coupling constant, Newton’s constant \( G_N \approx (10^{19} \text{GeV})^{-2} \).

It’s well known that the interactions with dimensional coupling constants of inverse mass dimension are strongly diverse and nonrenormalizable. From the success of the Weinberg-Salam model, it might be considerable that gravity is also characterized by a dimensionless coupling constant \( \xi \), and that the weakness of gravity is associated with a symmetry breaking at high energy. Similarly to \( G_F \), \( G_N \) could be given by the inverse square of the vacuum expectation value of a scalar field, dilaton. It was independently proposed by Zee [2], Smolin [3], and Adler [4] that the Einstein-Hilbert action can be replaced by the induced gravity action

\[
S = \int d^4x \sqrt{g} \left( \frac{1}{2} \xi \phi^2 R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right),
\]

(1)

where the coupling constant \( \xi \) is dimensionless. The potential \( V(\phi) \) is assumed to attain its minimum value when \( \phi = \sigma \), then \( G_N = \frac{1}{8\pi \xi \sigma^2} \).

On the analogy of the \( SU(2) \times U(1) \) symmetry of the electro-weak interactions, we can consider a symmetry which is broken through spontaneous symmetry breaking in the gravitational interactions. Through the spontaneous symmetry breaking, the symmetric phase of the scalar field transits to an asymmetric phase of the scalar field. There have
been several attempts to apply some spontaneous symmetry breakings in induced gravity to inflationary models \[\text{[5–9]}\]. One of the most attractive symmetry in induced gravity is the conformal symmetry which rejects the Einstein-Hilbert action, but admits the induced gravity action Eq.(1) with the specific conformal coupling $\xi = \frac{1}{6}$.

In Riemann-Cartan space, the vector torsion plays the role of the conformal gauge field \[\text{[10,11]}\]. Without the vector torsion, the conformally invariant induced gravity action is unique with the specific conformal coupling. However, introducing the vector torsion field, a conformally invariant extension of induced gravity action can be considered \[\text{[12]}\].

Actually, we don’t know yet whether nature really shows conformal invariance at a sufficiently high energy scale. But there is some evidences for this conformal invariance from the renormalization group analysis of some induced gravity models. For some $SU(N)$ induced gravity models, it is found that, at high energy limit, the coupling $\xi$ approaches to the conformal coupling \[\text{[13–15]}\]. If all other interactions including scalar potential are conformally invariant in this limit, then the models show asymptotic conformal invariance. This may happen also for some Grand Unified Models with induced gravity action \[\text{[16]}\].

We have investigated the conformal couplings in induced gravity and found that induced gravity at conformal couplings should have conformal invariance for consistency. An extension of conformal coupling in induced gravity is also considered introducing the vector torsion.

\[\text{II. EXTENSION OF CONFORMAL COUPLING IN INDUCED GRAVITY}\]

In this section, we consider an extension of conformal coupling introducing the torsion in induced gravity action. The induced gravity action Eq.(1) is invariant under the conformal transformation,

$$g'_{\mu\nu}(x) = \exp(2\Lambda)g_{\mu\nu}(x), \quad \phi'(x) = \exp(-\Lambda)\phi(x),$$

at the conformal coupling $\xi = \frac{1}{6}$ for a conformally invariant scalar potential.
To consider an extension of conformal coupling with the torsion in induced gravity, we have to introduce Riemann-Cartan space-time first. However, it is found that the minimal extension to Riemann-Cartan space-time is sufficient. The conformal transformation of the affine connections $\Gamma^\gamma_{\beta\alpha}$ is determined from the invariance of the tetrad postulation,

$$D_\alpha e^i_\beta \equiv \partial_\alpha e^i_\beta + \omega^i_j e^j_\beta - \Gamma^\gamma_{\beta\alpha} e^i_\gamma = 0,$$

under the following tetrads $e^i_\alpha$ and the spin connections $\omega^i_j_\alpha$ transformations;

$$(e^i_\alpha)' = \exp(\Lambda)e^i_\alpha, \quad (\omega^i_j_\alpha)' = \omega^i_j_\alpha.$$  (4)

We have used Latin indices for the tangent space-time and Greek indices for the curved space-time. From the metric compatibility Eq.(3), the affine connections and the torsions which are the antisymmetric components of the affine connections transform as follows;

$$(\Gamma^\gamma_{\beta\alpha})' = \Gamma^\gamma_{\beta\alpha} + \delta^\gamma_\beta \partial_\alpha \Lambda, \quad (T^\gamma_{\beta\alpha})' = T^\gamma_{\beta\alpha} + \delta^\gamma_\beta \partial_\alpha \Lambda - \delta^\gamma_\alpha \partial_\beta \Lambda.$$  (5)

Therefore, the contracted vector torsion $T^\gamma_{\gamma\alpha}$ is effectively playing the role of a conformal gauge field. The torsion tensor can be decomposed into three irreducible components [17]. However, it is sufficient for the purpose of examination to decompose the torsion into the conformally invariant tracefree tensor part $A^\alpha_{\beta\gamma}$ and conformally non-invariant vector part $S_\alpha$ as follows;

$$T^\alpha_{\beta\gamma} = A^\alpha_{\beta\gamma} - \delta^\alpha_\gamma S_\beta + \delta^\alpha_\beta S_\gamma,$$  (6)

$$(S_\alpha)' = S_\alpha + \partial_\alpha \Lambda, \quad (A^\alpha_{\beta\gamma})' = A^\alpha_{\beta\gamma}.$$  (7)

This conformal transformations Eq.(4) and Eq.(7) are also considered in [3,18] to construct conformally invariant Ricci tensor.

Because minimal extension to Riemann-Cartan space-time is sufficient, we impose the conformally invariant torsionless condition

$$A^\alpha_{\beta\gamma} \equiv 0.$$  (8)
This condition is the conformally invariant extension of the torsionless condition in Riemann space-time $T^\alpha_{\beta\gamma} \equiv 0$. For this space, the affine connection can be written in terms of $g_{\mu\nu}$ and $S_\alpha$:

$$\Gamma^\alpha_{\beta\gamma} = \{^\alpha_\beta\gamma\} + S^\alpha g_{\beta\gamma} - S_\beta \delta^\alpha_\gamma.$$ (9)

Defining the conformally invariant connection $\Omega^\alpha_{\beta\gamma}$,

$$\Omega^\alpha_{\beta\gamma} \equiv \{^\alpha_\beta\gamma\} + S^\alpha g_{\beta\gamma} - S_\beta \delta^\alpha_\gamma,$$ (10)

the curvature tensors $R^\alpha_{\beta\mu\nu}(\Gamma)$ of the affine connections can be expressed in terms of the curvature tensors $R^\alpha_{\beta\mu\nu}(\Omega)$ of $\Omega^\alpha_{\beta\gamma}$ and $S_\alpha$:

$$R^\alpha_{\beta\mu\nu}(\Gamma) = R^\alpha_{\beta\mu\nu}(\Omega) + \delta^\alpha_\beta H_{\mu\nu}, \quad R_{\alpha\nu}(\Gamma) = R_{\alpha\nu}(\Omega) + H_{\alpha\nu},$$ (11)

where $H_{\mu\nu} \equiv \partial_\mu S_\nu - \partial_\nu S_\mu$ is the field strength of the vector torsion $S_\alpha$. With the help of Eq.(9) and Eq.(11), we obtain the identity,

$$\sqrt{g} R(\Omega) = \sqrt{g} R(\{} + 6 \sqrt{g} (\nabla_\alpha S^\alpha - S_\alpha S^\alpha),$$ (12)

where $\nabla_\alpha$ is the ordinary covariant derivative in Riemann space-time.

Introducing the conformally covariant derivative $D_\alpha$,

$$D_\alpha \phi \equiv \partial_\alpha \phi + S_\alpha \phi,$$ (13)

we have an extended conformal coupling in induced gravity up to total derivatives as follow;

$$S = \int d^4 x \sqrt{g} (\frac{\xi}{2} R(\{} \phi^2 + \frac{1}{2} D_\alpha \phi D^\alpha \phi - \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta} - V(\phi)),$$ (14)

where we have excluded the curvature square terms and conformally non-invariant torsion terms like $S^\mu S_\mu$. Now, the coupling $\xi$ is a dimensionless arbitrary constant. Using Eq.(12) we can rewrite this action in terms of Riemann curvature scalar $R(\{})$;

$$S = \int d^4 x \sqrt{g} (\frac{\xi}{2} R(\{} \phi^2 + \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta} + (1 - 6\xi) S^\alpha (\partial_\alpha \phi) \phi + \frac{1}{2} (1 - 6\xi) S_\alpha S^\alpha \phi^2 - V(\phi)).$$ (15)

In the limit of $\xi \to \frac{1}{6}$, this extended conformal coupling is reduced to the ordinary conformal coupling in Riemann space-time without torsion.
III. CONFORMAL INVARIANCE AT CONFORMAL COUPLINGS

We analyze the equations of motion for the action Eq. (15), in which the potential \( V(\phi) \) does not need to be classical, but can be an effective scalar potential \( V_{\text{eff}}(\phi) \) after integrating out all fluctuating quantum fields. Moreover, in general, the effective scalar potential may depend on the other background fields, the metric \( g_{\mu\nu} \) and the vector torsion \( S_\alpha \). Thus, we will consider the potential in the action Eq. (15) as an effective scalar potential \( V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha) \).

Varying the action, we obtain the three equations of motion;

\[
\Box \phi = \xi R\{\} \phi + (1 - 6\xi) \phi (S^\mu S_\mu - \nabla_\mu S^\mu) - \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial \phi}, \quad (16)
\]

\[
\partial_\mu (\sqrt{g} H^{\mu\nu}) = -(1 - 6\xi) \sqrt{g} \{ (\partial^\nu \phi) \phi + S^\nu \phi^2 \} + \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{S_\nu}, \quad (17)
\]

\[
\xi \phi^2 G_{\mu\nu} = (H_{\mu\alpha} H^{\alpha}_\nu - \frac{1}{4} g_{\mu\nu} H^{\alpha\beta} H_{\alpha\beta}) - (\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi) - (1 - 6\xi) \phi^2 (S^\alpha S_\alpha - \frac{1}{2} g_{\mu\nu} S_\alpha S^\alpha)
\]

\[-(1 - 6\xi) (S_\mu \phi \partial_\nu \phi + S_\nu \phi \partial_\mu \phi - g_{\mu\nu} S^\alpha \phi \partial_\alpha \phi) + \xi \{ \nabla_\mu (\phi \partial_\nu \phi) + \nabla_\nu (\phi \partial_\mu \phi) - g_{\mu\nu} \Box \phi \}
\]

\[-g_{\mu\nu} V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma}) + 2 \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial g^{\mu\nu}}. \quad (18)
\]

Taking the divergence of Eq. (17), we obtain

\[
(1 - 6\xi) \nabla_\mu (S^\mu \phi^2) = -\frac{1}{2} (1 - 6\xi) \Box \phi^2 + \nabla_\nu \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial S_\nu}. \quad (19)
\]

The trace of Einstein Eq. (18) is

\[
\xi R\{\} \phi^2 = -\partial_\alpha \phi \partial^\alpha \phi - (1 - 6\xi) (S^\alpha \partial_\alpha \phi^2 + S_\alpha S^\alpha \phi^2) + 3\xi \Box \phi^2
\]

\[+ 4 V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma}) - 2 \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial g^{\mu\nu}} g^{\mu\nu}. \quad (20)
\]

From Eq. (16) and Eq. (20), we have
Using Eq. (19), we have a ξ independent equation for a general effective potential from Eq. (21) as follows:

\[
4V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma}) - \phi \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial \phi} - 2 \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial g^{\mu\nu}} g^{\mu\nu} g_{\mu\nu}.
\]

(21)

Therefore the metric and vector torsion dependencies of an effective potential are directly related to the deviation of the effective potential from the quartic form.

Let’s consider the conformal transformations Eq. (2) and Eq. (4) of the action Eq. (15), in which the scalar potential is replaced by the effective potential \(V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)\). Because the kinetic terms are conformally invariant for the conformal couplings, only the scalar potential term contributes to the conformal variation:

\[
\delta S = \int d^4x \sqrt{g}(-\frac{1}{2}V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)g_{\mu\nu} \delta g^{\mu\nu} + \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial \phi} \delta \phi

+ \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial S_\alpha} \delta S_\alpha).
\]

(23)

Using the infinitesimal forms of the conformal transformations Eq. (2) and Eq. (4),

\[
\delta g^{\mu\nu} = -2\Lambda g^{\mu\nu}, \quad \delta \phi = -\Lambda \phi, \quad \delta S_\alpha = \partial_\alpha \Lambda,
\]

(24)

the conformal variation of the action can be written as

\[
\delta S = \int d^4x \sqrt{g}\Lambda(4V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha) - \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial \phi} \phi - 2 \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial g^{\mu\nu}} g^{\mu\nu}

- \nabla_\alpha \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial S_\alpha}) - \int d^4x \partial_\alpha(\sqrt{g}\Lambda \frac{\partial V_{\text{eff}}(\phi; g_{\mu\nu}, S_\alpha)}{\partial S_\alpha}).
\]

(25)

Because the last total derivative term can be eliminated if we consider a conformal transformation which have a vanishing \(\Lambda(x)\) at space-time infinity, the Eq. (22) we have obtained
from the equations of the motion analysis is the condition for the the conformal invariance of the induced gravity action, $\delta S \equiv 0$. The relation Eq.(22) also appears in case of the special conformal coupling $\xi = \frac{1}{6}$, where the vector torsion is decoupled from the scalar field. Therefore, we can say that the conformal couplings in induced gravity generally requires the conformal invariance of the induced gravity action for consistency.

IV. CONCLUSION

Without introducing the vector torsion, the conformal coupling in induced gravity is unique with $\xi = \frac{1}{6}$. However, in Riemann-Cartan space-time the vector torsions play the role of the conformal gauge fields, which make an extended conformal coupling possible

For some $SU(N)$ induced gravity models, it is found that the coupling $\xi$ approaches to the conformal coupling $\frac{1}{6}$ at high energy limit [13, 15]. If all other interactions are conformally invariant in this limit, then the models have asymptotic conformal invariance. This may happen also for some Grand Unified Models with induced gravity action [16].

We have investigated the conformal couplings in induced gravity and found that the induced gravity models at conformal couplings should have conformal invariance for consistency at classical and quantum levels.

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