PARTIAL CLASSICALITY OF HILBERT MODULAR FORMS

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ABSTRACT. Let $F$ be a totally real field and $p$ a rational prime unramified in $F$. We prove a partial classicality theorem for overconvergent Hilbert modular forms: when the slope is small compared to a subset of weights, an overconvergent form is partially classical. We use the method of analytic continuation.

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1. INTRODUCTION

Coleman [Col96] proved that a $p$-adic overconvergent modular form of weight $k \in \mathbb{Z}$ must be classical if its slope, i.e., the $p$-adic valuation of the $U_p$-eigenvalue, is less than $k - 1$. His proof involves analyzing the rigid cohomology of modular curves. On the other hand, Buzzard [Buz03] and Kassaei [Kas06] developed the alternate method of analytic continuation to prove classicality theorems. The key is to understand the dynamic of the $U_p$ Hecke operator.

Let $F$ be a totally real field of degree $g$ over $\mathbb{Q}$. In the situation of Hilbert modular forms associated to $F$, many results about classicality are also known. Coleman’s cohomological method was developed by Tian–Xiao [TX16] to prove a classicality theorem, assuming $p$ is unramified in $F$. The method of analytic continuation was worked out first in the case when $p$ splits completely in $F$ by Sasaki [Sas10], then in the case when $p$ is unramified by Kassaei [Kas16] and Pilloni–Stroh [PS17], and finally when $p$ is allowed to be ramified by Bijakowski [Bij16].

Let $\Sigma$ be the set of archimedean embeddings of $F$, which we identify with the set of $p$-adic embeddings of $F$ through some fixed isomorphism $\mathbb{C} \cong \mathbb{C}_p$. For each prime $\mathfrak{p}$ of $F$ above $p$, denote by $\Sigma_{\mathfrak{p}} \subseteq \Sigma$ the subset of $p$-adic embeddings inducing $\mathfrak{p}$. Let $e_{\mathfrak{p}}$ be the ramification index, and $f_{\mathfrak{p}}$ the residue degree of $\mathfrak{p}$. Then the classicality theorem for overconvergent Hilbert modular forms proved by analytic continuation is as follows.

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**Theorem 1** (Bijakowski). Let $f$ be an overconvergent Hilbert modular form of weight $k \in \mathbb{Z}^\Sigma \cong \mathbb{Z}^g$. Assume that for all $p \mid p$, $U_p(f) = a_p f$ such that

$$\text{val}_p(a_p) < \frac{1}{e_p} \inf_{\tau \in \Sigma_p} \{k_\tau\} - f_p,$$

where $\text{val}_p$ is the $p$-adic valuation normalized so that $\text{val}_p(p) = 1$. Then $f$ is classical.

**Remark 1.1.** When $p$ is unramified in $F$, namely $e_p = 1$ for all $p \mid p$, Tian–Xiao proved the classicality theorem with weaker slope assumption: $\text{val}_p(a_p) < \inf_{\tau \in \Sigma_p} \{k_\tau\} - 1$. This slope bound is believed to be optimal [Bre10, Proposition 4.3].

In this paper, we prove some “partial” classicality theorems for overconvergent Hilbert modular forms. Let $I$ be a subset of $\Sigma$. Breuil defined the notion of $I$-classical overconvergent Hilbert modular forms (see [Bre10, p. 3] or Definition 2.4). When $I = \emptyset$, they are the usual overconvergent forms; when $I = \Sigma$, they are the classical forms.

**Theorem 2** (Theorem 3.1). Assume that $p$ is unramified in $F$. Let $f$ be an overconvergent Hilbert modular form of weight $k \in \mathbb{Z}^\Sigma$. Let $I \subseteq \Sigma$. Assume that for all $p \mid p$, $U_p(f) = a_p f$ such that

$$\text{val}_p(a_p) < \inf_{\tau \in I \cap \Sigma_p} \{k_\tau\} - f_p.$$

Then $f$ is $I$-classical.

We use the method of analytic continuation to prove Theorem 2. In the situation when $I = \Sigma$, this recovers the classicality theorem proven by Kassaei or Pilloni–Stroh, who assumed $p$ is unramified. Although when $I = \Sigma$, Bijakowski proved a classicality theorem not assuming $p$ is unramified, it is Kassaei’s approach that is more suitable for partial classicality. Indeed, when studying the dynamic of $U_p$-operators, it has been proven to be successful to use degree to parametrize regions on the Hilbert modular variety, and analyze how $U_p$-operators influence degrees. Kassaei made efforts to analyze how $U_p$-operators affect the more refined direction degrees, but only when $p$ is unramified. On the other hand, Bijakowski was able to use only the degree function to prove a classicality theorem allowing $p$ to be ramified. In the situation of partial classicality, the weight $k_\tau$ with $\tau \in \Sigma$ in the slope condition is independent of each other, while the $U_p$-operator intertwines all directional degrees inducing $p$. As a result, we cannot avoid analyzing the direction degrees like Bijakowski did.

We mention some related work on partial classicality theorems. Barrera Salazar and Williams [BSW21] took the perspective of overconvergent cohomology for a general quasi-split reductive group $G$ over $\mathbb{Q}$ with respect to a parabolic subgroup $Q$ of $G = G/\mathbb{Q}_p$. Applying their work to the situation of Hilbert modular forms (i.e., $G = \text{Res}_{F/\mathbb{Q}} \text{GL}_2$), we would recover Theorem 2 in the restrictive case of $I \subseteq \Sigma$ such that $I \cap \Sigma_p$ is either $\Sigma_p$ or $\emptyset$ for each $p \mid p$. In [Bre10, Proposition 4.3(i)] for the special case $S > 1 = S$, Breuil gave a conjecture about partial classicality: In the restrictive case when $I$ is either $\Sigma_p$ or $\emptyset$ for each $p \mid p$, if $f$ satisfies the weaker assumption $\text{val}_p(a_p) < \inf_{\tau \in \Sigma_p} \{k_\tau\} - 1$ for all $p \mid p$ such that $I \cap \Sigma_p \neq \emptyset$, then $f$ is $I$-classical. Yiwen Ding [Din17, Appendix A] studied partial classicality from the perspective of Galois representations. He also did not restrict to the case when $I \cap \Sigma_p$ is either $\Sigma_p$ or $\emptyset$. Namely, let $\rho_f$: $\text{Gal}_F \to \text{GL}_2(L)$ be the Galois representation associated to an overconvergent Hilbert Hecke eigenform $f$, where $L$ is a finite extension of $\mathbb{Q}_p$. If $\text{val}_p(a_p) < \inf_{\tau \in I \cap \Sigma_p} \{k_\tau\} - 1$, then $\rho_f|_{\text{Gal}_{F_p}}$ is $(I \cap \Sigma_p)$-de Rham.

There are many interesting questions related to $I$-classical overconvergent forms. In the direction of classicality, can we prove Theorem 2 not assuming $k_\tau$ is an integer for $\tau \notin I$? If $f$ is $I$-classical and $\text{val}_p(a_p) < \inf_{\tau \in I \cap \Sigma_p} \{k_\tau\} - f_p$, will $f$ be $I \cup J$-classical (see [Bre10, Conjecture 3.2 (ii)])? Relating to Galois representations, if $f$ is an $I$-classical Hilbert Hecke eigenform, does the Galois representation $\rho_f$ satisfy the condition that $\rho_f|_{\text{Gal}_{F_p}}$ is $(I \cap \Sigma_p)$-de Rham for all $p \mid p$? If this is true, one can further...
ask in the flavor of Kisin’s interpretation of Fontaine–Mazur conjecture: if \( f \) is overconvergent and \( \rho_f|_{{\text{Gal}}_{\mathbb{Q}_p}} \) is \((I \cap \Sigma_p)\)-de Rham for all \( p \mid p \), is \( f \) \( I \)-classical?

For the organization of this paper: In Section 2, we define the degree function and partially classical overconvergent forms. In Section 3, we prove Theorem 2.

**Notations.** Fix a totally real field \( F \) of degree \( g \) over \( \mathbb{Q} \). Let \( \Sigma \) denote the set of archimedean places of \( F \); in particular \( \# \Sigma = g \). Fix a rational prime \( p \) which is unramified in \( F \) and \((p) = p_1 \cdots p_r \) in \( F \).

For each prime \( p \) of \( F \) above \( p \), let \( f_p \) be the residue degree of \( p \). Fix an isomorphism \( \iota_p: \mathbb{C} \cong \overline{\mathbb{Q}}_p \), and identify archimedean embeddings \( \tau: F \to \mathbb{C} \) with \( p \)-adic embeddings \( \iota_p \circ \tau: F \to \overline{\mathbb{Q}}_p \). For each prime \( p \) of \( F \) above \( p \), let \( \Sigma_p \subseteq \Sigma \) be the subset of \( p \)-adic embeddings inducing \( p \). Hence \( \# \Sigma_p = f_p \).

Let \( L \) be a finite extension of \( \mathbb{Q}_p \), containing the image of all \( p \)-adic embeddings \( \iota \circ \tau \) of \( F \). Since \( p \) is assumed to be unramified in \( F \), we may also assume that \( L \) is an unramified extension of \( \mathbb{Q}_p \).

Let \( k_L \) denote the residue field of \( L \). Let \( \delta_F \) be the different ideal of \( F \).

## 2. Partially classical overconvergent forms

### 2.1. Hilbert modular varieties.

Let \( N \geq 4 \) be an integer, and \( p \nmid N \). Let \( \mathfrak{c} \) be a fractional ideal of \( F \). Denote by \( \mathfrak{c}^+ \subseteq \mathfrak{c} \) the cone of totally positive elements, i.e., the elements in \( \mathfrak{c} \) which are positive under every embedding \( \tau: F \to \mathbb{R} \). Let \( Y_\mathfrak{c} \to \text{Spec} \mathcal{O}_L \) be the Hilbert modular scheme classifying \( (A, H) = (A/S, i, \lambda, \alpha, H) \) where

- \( A \) is an abelian scheme of relative dimension \( g \) over an \( \mathcal{O}_L \)-scheme \( S \),
- \( i: \mathcal{O}_F \to \text{End}_S(A) \) is a ring homomorphism. Here \( i \) is called a real multiplication on \( A \),
- \( \lambda: (\mathcal{P}_A; \mathcal{P}_A^+) \to (\mathfrak{c}, \mathfrak{c}^+) \) is an isomorphism of \( \mathcal{O}_F \)-modules identifying the positive elements, and inducing an isomorphism \( A \otimes \mathcal{O}_F \mathfrak{c} \cong A' \). Here \( \mathcal{P}_A = \text{Hom}_{\mathcal{O}_F}(A, A')^{\text{sym}} \) is the projective \( \mathcal{O}_F \)-module of rank \( 1 \) consisting of symmetric morphisms from \( A \) to its dual abelian scheme \( A' \), and \( \mathcal{P}_A^+ \subseteq \mathcal{P}_A \) is the cone of polarizations. Here \( \lambda \) is called a \( \mathfrak{c} \)-polarization of \( A \),
- \( \alpha: \mu_N \otimes \delta_F^{-1} \to A \) is a closed immersion of \( \mathcal{O}_F \)-group schemes. Here \( \alpha \) is called a \( \Gamma_1(N) \)-level structure, and
- \( H \subseteq A[p] \) is a finite flat \( \mathcal{O}_F \)-subgroup scheme of rank \( p^\beta \) which is isotropic with respect to the \( \mu \)-Weil pairing for some polarization \( \mu \in \mathcal{P}_A^+ \) of degree prime to \( p \).

Let \( \text{Cl}(F)^+ \) be the narrow class group of \( F \), namely the quotient of the abelian group of fractional ideals of \( F \) by the subgroup of principal ideals generated by totally positive elements. Let \( \{c_i\} \) be a set of representatives of \( \text{Cl}(F)^+ \). Define \( Y = \prod_i Y_{c_i} \), which is independent of the choice of the representatives \( \{c_i\} \). Denote by \( \mathcal{Y} \) the completion of \( Y \) along its special fiber, and by \( \mathcal{Y} \) the rigid generic fiber of the formal scheme \( \mathcal{Y} \). We also use this convention of letter styles for other schemes: when \( K/\mathbb{Q}_p \) is a finite extension and \( S \) is a scheme over \( \mathcal{O}_K \), we denote by \( \mathcal{S} \) the associated formal scheme and by \( \mathcal{S} \) the rigid generic fiber of \( \mathcal{S} \).

### 2.2. Directional degrees.

We first recall the definition of the degree for a commutative finite flat group scheme. See [Far10] for more detailed studies of the concept.

Let \( S \) be a scheme and \( G \) a commutative finite flat group scheme over \( S \). Let \( \omega_G \) be the sheaf of invariant differentials on \( G \). Define

\[
\delta_G := \text{Fitt}_0 \omega_G
\]

as the 0-th Fitting ideal of \( \omega_G \). This is an invertible ideal sheaf in \( \mathcal{O}_S \).

Now let \( K/\mathbb{Q}_p \) be a finite extension and \( S = \text{Spec} \mathcal{O}_K \). Then the degree of \( G \) is defined as [Far10, Définition 4] the rational number

\[
\deg G = \deg \omega_G := \text{val}_p(\delta_G).
\]

Writing \( \omega_G = \bigoplus_i \mathcal{O}_K/x_i \mathcal{O}_K \), then \( \deg G = \sum_i \text{val}_p(x_i) \). Equivalently, \( \deg G = \ell(\omega_G)/e_K \), where \( \ell(\omega_G) \) is the length of the \( \mathcal{O}_K \)-module \( \omega_G \), and \( e_K \) is the ramification index of \( K \). Recall that the
height $\text{ht} G$ of $G$ is such that $|G| = p^{\text{ht} G}$. Hence $G$ is étale if and only if $\deg G = 0$, and $G$ is multiplicative if and only if $\deg G = \text{ht} G$.

More generally, let $S$ be a scheme over $\mathcal{O}_K$. Each closed point $s$ in the rigid analytic space $S$ is defined over the ring of integer of a finite extension of $K$ [BLR95, Section 8.3, Lemma 6]. Hence we obtain the degree function

$$\deg: S \to [0, \infty) \cap \mathbb{Q} \quad s \mapsto \deg G_s.$$ 

The inverse image of a (open, closed, or half-open) interval in $[0, \infty)$ is an admissible open of $S$. Moreover, when the interval is closed and its end points $a \leq b$ are rational numbers, then the inverse image is quasi-compact.

We record some properties of $\deg$ which we will constantly use for computation.

**Lemma 2.1.** [Far10, lemme 4] Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of finite flat group schemes over $S$. Then $\deg G = \deg G' + \deg G''$.

**Lemma 2.2.** [Far10, p. 2] Let $\lambda: A \to B$ be an isogeny of $p$-power degree between abelian schemes over $S$. Let $G := \ker \lambda$. Let $\omega_{A/S}$ and $\omega_{B/S}$ be the sheaves of invariant differentials of $A$ and $B$, respectively. Let $\lambda^*: \omega_{B/S} \to \omega_{A/S}$ be the induced pullback map. Then

$$\deg G = \text{val}_p(\det \lambda^*).$$

In particular, if $A$ is of dimension $g$, then $\deg A[p] = g$.

When $G$ has an $\mathcal{O}_F$-module structure, we can define directional degree functions on $S$. Instead of a general exposition, we only explain this for $S = \mathcal{Y}$, the Hilbert modular variety over $L$. See also [PS17, Section 4.2] or [Kas16, Section 2.9]. Let $(\mathcal{A}^{\text{univ}}, H^{\text{univ}})$ be the universal abelian scheme over the Hilbert modular variety $\mathcal{Y}$. Let $\omega_{H^{\text{univ}}}$ be the sheaf of invariant differentials of $H^{\text{univ}}$, which is an $\mathcal{O}_F/p\mathcal{O}_F$-module. Since $p$ is unramified in $F$, $\Sigma$ is in bijection with the embeddings $\mathcal{O}_F/p\mathcal{O}_F \hookrightarrow k_L$.

We decompose $\omega_{H^{\text{univ}}}$ according to the embeddings $\mathcal{O}_F/p\mathcal{O}_F \hookrightarrow k_L$ to obtain

$$\omega_{H^{\text{univ}}} = \bigoplus_{\tau \in \Sigma} \omega_{H^{\text{univ}}, \tau}.$$ 

For each $\tau \in \Sigma$, define $\delta_\tau := \text{Fit}_{\delta}(\omega_{H^{\text{univ}, \tau}})$, which is an invertible ideal sheaf in $\mathcal{O}_Y$.

Let $y = (\mathcal{A}, H)$ be a closed point of $\mathcal{Y}$. Let $K$ be the finite extension of $L$ over which $y$ is defined. Then we have the rational number $\deg_{\omega_{H, \tau}}$. In addition, $\deg_{\omega_{H, \tau}} \in [0, 1]$. Indeed, for each $p | p_\tau$, the subgroup scheme $H[p]$ of $H$ is a Raynaud group scheme over $\text{Spec} \mathcal{O}_K$, namely a $k_p = \mathcal{O}_F/p\mathcal{O}_F$-vector space scheme of dimension 1. For each tuple $(d_{\tau})_{\tau \in \Sigma_p}$ of elements of $\mathcal{O}_K$ with $\text{val}_p(d_{\tau}) \leq 1$, Raynaud associates a $k_p$-vector space scheme of dimension 1

$$H_{(d_{\tau})} := \text{Spec} \mathcal{O}_K[X_{\tau}, \tau \in \Sigma_p]/(X_{\sigma-1 \tau}^p - d_{\tau} X_{\tau}),$$

where $\sigma$ is the Frobenius automorphism of $L$ over $\mathbb{Q}_p$ lifting $x \mapsto x^p$ modulo $p$, and the $k_p$-action on $X_{\tau}$ is given by the character $k_p^* \to \mathcal{O}_K^*$ induced by $\tau: F \to L$. Moreover, each $k_p$-vector space scheme of dimension 1 over $\mathcal{O}_K$ is isomorphic to some $H_{(d_{\tau})}$ [Ray74, THÉORÈME 1.4.1]. Since $\omega_{H_{(d_{\tau})}, \tau} = \mathcal{O}_K/d_{\tau} \mathcal{O}_K$, we have $\deg_{\omega_{H, \tau}} = \deg_{\omega_{H_{(d_{\tau})}, \tau}} = \text{val}_p(d_{\tau}) \in [0, 1]$.

Hence for each $\tau \in \Sigma$, we can define the directional degree function

$$\deg_{\tau}: \mathcal{Y} \to [0, 1] \cap \mathbb{Q}, \quad y = (\mathcal{A}, H) \mapsto \deg_{\omega_{H, \tau}},$$

as well as

$$\deg: \mathcal{Y} \to [(0, 1] \cap \mathbb{Q})^\Sigma, \quad y \mapsto (\deg_{\tau} y).$$

As before, the inverse image of $\deg_{\tau}$ (resp. $\deg$) of a subset of $[0, 1]$ (resp. $[0, 1])^\Sigma$ defined by a finite number of affine inequalities is an admissible open of $\mathcal{Y}$. Moreover, when the inequalities are all non-strict and the coefficients are all rational numbers, then the inverse image is quasi-compact.
Given $I \subseteq \Sigma$, we define
\[
\mathcal{F}_I := \prod_{\tau \in \Sigma} \mathcal{F}_{I, \tau}, \quad \text{where } \mathcal{F}_{I, \tau} = \begin{cases}
[0, 1], & \tau \in I \\
[1, 1], & \tau \notin I.
\end{cases}
\]
Then $\mathcal{F}_I$ is a closed $|I|$-dimensional hypercube in $([0, 1] \cap \mathbb{Q})^\Sigma = \mathcal{F}_\Sigma$. We also define $x_I \in [0, 1]^\Sigma$ to be the vertex
\[
x_{I, \tau} = \begin{cases}
0, & \tau \in I \\
1, & \tau \notin I.
\end{cases}
\]
Hence the vertices of $\mathcal{F}_I$ are exactly the $x_J$’s with $J \subseteq I$. Denote by $\mathcal{Y}_I$ the quasi-compact admissible open $\deg^{-1}\mathcal{F}_I$ of $\mathcal{Y}$.

**Definition 2.3.** Let $p \mid p$ be a prime of $F$. For $\tau \in \Sigma_p$, define the *twisted directional degree* 
\[
\tilde{\deg}_\tau : \mathcal{Y} \to [0, \frac{p^f - 1}{p - 1}] \cap \mathbb{Q}
\]
by
\[
\tilde{\deg}_\tau := \sum_{j=0}^{f_p - 1} p^{f_p - 1 - j} \deg_{\sigma_j \circ \tau} = p^{f_p - 1} \deg_\tau + p^{f_p - 2} \deg_{\sigma_0 \circ \tau} + \cdots + \deg_{\sigma_{f_p - 1} \circ \tau}.
\]
Here $\sigma$ is the Frobenius automorphism of the unramified extension $L$ over $\mathbb{Q}_p$, lifting $x \mapsto x^p$ mod $p$. We also define
\[
\tilde{\deg} : \mathcal{Y} \to ([0, \frac{p^f - 1}{p - 1}] \cap \mathbb{Q})^\Sigma \quad y \mapsto (\tilde{\deg}_\tau y)_{\tau}.
\]
We use the overhead tilde notation ($\tilde{\cdot}$) to denote the image under the linear transformation
\[
\mathbb{R}^\Sigma \to \mathbb{R}^\Sigma \quad (x_\tau)_{\tau} \mapsto (\tilde{x}_\tau)_{\tau}, \quad \text{where } \tilde{x}_\tau = \sum_{j=0}^{f_p - 1} p^{f_p - 1 - j} x_{\sigma_j \circ \tau} \text{ for } \tau \in \Sigma_p.
\]
In particular, if $(x_\tau)_{\tau} = \deg y$ for some $y \in \mathcal{Y}$, then $(\tilde{x}_\tau)_{\tau} = \tilde{\deg} y$. For example, $\tilde{x}_I$ is the vertex of $\tilde{\mathcal{F}}_\Sigma$ given by
\[
\tilde{x}_I, \tau = \sum_{j=0}^{f_p - 1} p^{f_p - 1 - j} x_{I, \sigma_j \circ \tau} \text{ for } \tau \in \Sigma_p.
\]
See Figures 1 and 2 for an example of $\mathcal{F}_\Sigma$ and $\tilde{\mathcal{F}}_\Sigma$.
2.3. Hilbert modular forms. Let $\omega = \omega_{\text{Am}}$ be the sheaf of relative differentials of the universal abelian scheme over $Y$. The sheaf $\omega$ is an $\mathcal{O}_F \otimes_\mathbb{Z} \mathcal{O}_Y$-module, locally free of rank 1. The $\mathcal{O}_F$-module structure on $\omega$ provides the decomposition with respect to embeddings $\tau : F \to L$

$$\omega = \bigoplus_{\tau \in \Sigma} \omega_\tau,$$

where each $\omega_\tau$ is an $\mathcal{O}_Y$-module, locally free of rank 1. Given $k = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$, we define an invertible sheaf on $Y$

$$\omega^k = \bigotimes_{\tau \in \Sigma} \omega_\tau^{k_\tau}.$$

We use the same notation $\omega^k$ for the invertible sheaf on $\mathcal{Y}$ coming from analytifying $\omega^k$.

The space of *Hilbert modular forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $k$* is defined to be $H^0(Y, \omega^k)$. By GAGA and Koecher principle, it is the same as $H^0(\mathcal{Y}, \omega^k)$ [PS17, Proposition 5.1.2].

**Definition 2.4.** Let $I \subseteq \Sigma$. The space of *I-classical overconvergent* Hilbert modular forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $k$ is

$$H^0(I, \omega^k) := \varprojlim_{\mathcal{V}} H^0(\mathcal{V}, \omega^k),$$

where $\mathcal{V}$ runs through strict neighborhoods of $\mathcal{Y}_{\mathcal{F}_I}$ in $\mathcal{Y}$.

When $I = \varnothing$, $I$-classical simply means overconvergent, and when $I = \Sigma$, $I$-classical means classical. Whenever $J \subseteq I$, we have a map

$$H^0(I, \omega^k) \to H^0(J, \omega^k)$$

given by restriction. This is an injective map.

2.4. *$U_p$-operators.* Let $p \mid p$ be a prime of $F$ above $p$ and $f_p$ the residue degree of $p$.

Let $Y(p) \to \text{Spec } L$ be the moduli space whose $S$-points consist of $(A, H, H_1)$, where $(A, H) \in Y(S)$ and $H_1 \subseteq A[p]$ is a finite flat isotropic $\mathcal{O}_F$-subgroup scheme of rank $p^{f_p}$ and $H_1 \neq H[p]$. We have the $U_p$-correspondence of $Y \otimes_{\mathcal{O}_L} L$:

$$\begin{array}{ccc}
Y(p) & \xleftarrow{p_1} & Y \otimes_{\mathcal{O}_L} L \\
Y \otimes_{\mathcal{O}_L} L & \xrightarrow{p_2} & Y \otimes_{\mathcal{O}_L} L
\end{array}$$
Here the projections are
\[ p_1: (\overline{A}, H, H_1) \mapsto (A, H), \]
and
\[ p_2: (\overline{A}, H, H_1) \mapsto (A/H_1, \tilde{H}), \]
where \( \tilde{H} \) is the image of \( H \) under \( A \to A/H_1 \).

Let \( Y(p)^{an} \) be the rigid analytification of \( Y(p) \) \cite[Section 5.4, Corollary 5]{BLR95}, which is a rigid analytic space over \( L \). We have the induced \( U_p \)-correspondence, \( p_1 \) and \( p_2 \) over \( (Y \otimes L)^{an} \). Note that \( (Y \otimes L)^{an} \) contains \( Y \). Let \( \mathcal{Y}(p) := Y(p)^{an} \times_{(Y \otimes L)^{an}, p_1} Y \). We then have the \( U_p \)-correspondence, \( p_1 \) and \( p_2 \) over \( \mathcal{Y} \).

Given a subset \( U \) of \( \mathcal{Y} \), we then obtain a subset of \( \mathcal{Y} \)
\[ U_p(U) := p_2 p_1^{-1}(U). \]
Given two admissible opens \( U, V \subseteq \mathcal{Y} \) such that \( U_p(V) \subseteq U \), we have \( U_p: \omega^{\tilde{L}}(U) \to \omega^{\tilde{L}}(V) \) defined by
\[ (U_p f)(A, H) = \frac{1}{p_1(L)} \sum_{(A/H_1, H) \in U_p(A, H)} \text{pr}^* f(A/H_1, \tilde{H}), \]
where \( \text{pr}: A \to A/H_1 \) is the natural projection.

We record the dynamic of \( U_p \) with respect to the (twisted) directional degrees. See \cite[Proposition 5.1.4, 5.1.14]{Kas16} or \cite[Proposition 4.4.1, 4.4.2]{PS17}.

**Proposition 2.5.** Let \( y = (\overline{A}, H) \in \mathcal{Y} \). Let \( p \mid p \) be a prime of \( F \) above \( p \), and \( y' = (A/H_1, \tilde{H}) \in U_p(y) \). Then
1. \( \deg_{\tau}(y') \geq \deg_{\tau}(y) \) for all \( \tau \in \Sigma_p \), and
2. if
   \[ \sum_{\tau \in \Sigma_p} \deg_{\tau} y' = \sum_{\tau \in \Sigma_p} \deg_{\tau} y, \]
equivalently, \( \sum_{\tau \in \Sigma_p} \deg_{\tau} y' = \sum_{\tau \in \Sigma_p} \deg_{\tau} y \), then \( \deg_{\tau} y \in \{0, 1\} \) for all \( \tau \in \Sigma_p \).

3. **Partial classicality**

The content of this section is to prove the following partial classicality theorem.

**Theorem 3.1.** Let \( f \) be an overconvergent Hilbert modular form of weight \( k \). Let \( I \subseteq \Sigma \). Assume that for all \( p \mid p \), \( U_p(f) = a_p f \) such that
\[ \text{val}_p(a_p) < \inf_{\tau \in I \cap \Sigma_p} \{k_{\tau}\} - f_p. \]
Then \( f \) is \( I \)-classical.

**Remark 3.2.** In the case of \( I = \Sigma \), this is a theorem of Kassaei \cite{Kas16} or Pilloni–Stroh \cite{PS17}. Although when \( I = \Sigma \), Bijakowski \cite{Bij16} proved a classicality theorem not assuming \( p \) is unramified, it is Kassaei’s approach that is more suitable for partial classicality. Both use the idea of analytic continuation. Kassaei made efforts to analyze how \( U_p \)-operators affect \( \deg_{\tau} \) for all \( \tau \in \Sigma_p \), but only when \( p \) is unramified. On the other hand, Bijakowski was able to use only \( \deg H[p] \) to prove the classicality even when \( p \) is ramified. In the situation of partial classicality, the weight \( k_{\tau} \) with \( \tau \in \Sigma \) in the slope condition is independent of each other, while the \( U_p \)-operator intertwines all directional degrees inducing \( p \), so we do need to understand the directional degrees.
Throughout the section, we will assume that \( p \) is inert in \( F \). To prove Theorem 3.1 for a general unramified \( p \), we can apply the same argument to each prime \( p \mid p \). For example, see [Sas10] and [PS17, Lemma 7.4.2].

Now we begin to prove Theorem 3.1 assuming \( p \) is inert in \( F \); in particular, \( f_p = g \). We will show that if \( U_p(f) = a_p f \) such that \( \text{val}_p(a_p) < \inf_{\tau \in I} k_{\tau} - g \), then \( f \) is \( J \)-classical for all \( J \subseteq I \), and hence \( f \) is \( I \)-classical. We do this by induction on \( |J| \).

3.1. Automatic analytic continuation. In the subsection, with the assumption that the slope of \( f \) is finite (but not necessarily small), we can already show that \( f \) can be analytically continued to a large region in \( \mathcal{Y} \).

Let \( I \subseteq \Sigma \) and \( \epsilon > 0 \). Define

\[
U_I(\epsilon) = \{ y \in \mathcal{Y} : \sum_{\tau \in I} \hat{\deg}_\tau y \geq \sum_{\tau \in I} \hat{x}_{I,\tau} + \epsilon, \hat{\deg}_\tau y \geq p^{g-2} + \cdots + 1 + \epsilon, \forall \tau \notin I \}. 
\]

See Figures 3 and 4 for examples of the image of \( U_I(\epsilon) \) under \( \deg \), and Figures 5 and 6 for examples of the image of \( U_I(\epsilon) \) under \( \tilde{\deg} \).

Because \( U_I(\epsilon) \) is defined by a finite number of affine inequalities with \( \hat{\deg}_\tau \) (equivalently, with \( \deg \)), we know that \( U_I(\epsilon) \) is an admissible open of \( \mathcal{Y} \). Note that whenever \( \epsilon' < \epsilon \), we have \( U_I(\epsilon') \supseteq U_I(\epsilon) \).

Let \( f \) be an overconvergent Hilbert modular form of weight \( k \). Assume that \( U_p(f) = a_p f \) with \( \text{val}_p(a_p) < \infty \).

**Lemma 3.3.** Let \( I \subseteq \Sigma \). Suppose that \( f \) is defined on a strict neighborhood of \( \deg^{-1} x_J = \tilde{\deg}^{-1} \tilde{x}_J \) for all \( J \subseteq I \). Then \( f \) can be extended to \( U_I(\epsilon) \) for any rational number \( \epsilon > 0 \).

**Proof.** First of all, note that \( U_I(\epsilon) \) is \( U_p \)-stable because \( U_p \) increases twisted directional degrees (Proposition 2.5(1)).
By Proposition 2.5(2), $U_p$ strictly increases $\sum_{\tau \in \Sigma} \tilde{d}_{\tau}$ except at points $y \in \mathcal{Y}$ such that $\deg y \in \{0, 1\}^g$, i.e., $\deg y = x_J$ for some $J \subseteq \Sigma$. Suppose that $y \in \mathcal{U}_i(\varepsilon)$ satisfies $\deg y = x_J$. We claim that $J \subseteq I$. Indeed, for $\tau \in J$, $\deg_{\tau} y \leq p^{g-2} + \cdots + 1$. Hence the second condition of $\mathcal{U}_i(\varepsilon)$

$$\deg_{\tau} y \geq p^{g-2} + \cdots + 1 + \varepsilon, \forall \tau \notin I$$

says that $\tau \notin I$ implies $\tau \notin J$, i.e., $J \subseteq I$. The first condition of $\mathcal{U}_i(\varepsilon)$

$$\sum_{\tau \in I} \tilde{d}_{\tau} y \geq \sum_{\tau \in I} x_{I, \tau} + \varepsilon$$

then says that $J \neq I$.

For each $J \subseteq I$, let $\mathcal{V}_J$ be a strict neighborhood of $\deg^{-1} x_J$ on which $f$ is defined. Moreover we can choose $\mathcal{V}_J$ in the form

$$\mathcal{V}_J = \{ y \in \mathcal{Y} : \deg_{\tau} y \leq \varepsilon_{\tau} \text{ if } \tau \in J, \deg_{\tau} y \geq 1 - \varepsilon_{\tau}, \text{ if } \tau \notin J \},$$

for some rational $\varepsilon_{\tau} > 0$. On the other hand, let $\varepsilon_{\tau}' < \varepsilon_{\tau}$ be a rational number, and define

$$\mathcal{V} = \left\{ y \in \mathcal{Y} : \begin{array}{l}
\deg_{\tau} y \geq \varepsilon_{\tau}' \text{ if } \tau \in I, \deg_{\tau} y \leq 1 - \varepsilon_{\tau}', \text{ if } \tau \notin I; \\
\sum_{\tau \in I} \deg_{\tau} y \geq \sum_{\tau \in I} x_{I, \tau} + \varepsilon, \deg_{\tau} y \geq p^{g-2} + \cdots + 1 + \varepsilon, \forall \tau \notin I
\end{array} \right\}.$$

Because $\mathcal{V}_J$’s and $\mathcal{V}$ are defined by a finite number of affine non-strict inequalities with rational coefficients, they are quasi-compact admissible opens of $\mathcal{U}_i(\varepsilon)$. We hence have an admissible cover $\mathcal{U}_i(\varepsilon) = \bigcup_{J \subseteq I} \mathcal{V}_J \cup \mathcal{V}$.

Since $\mathcal{V}$ is disjoint from $\deg^{-1} x_J$ for any $J \subseteq I$ from its definition, $U_p$ strictly increases $\sum_{\tau \in \Sigma} \tilde{d}_{\tau}$ on $\mathcal{V}$. Using the Maximum Modulus Principle, the quasi-compactness of $\mathcal{V}$ implies that there is a positive lower bound for the increase of $\sum_{\tau \in \Sigma} \deg_{\tau}$ under $U_p$ on $\mathcal{V}$. Because $\mathcal{U}_i(\varepsilon)$ is $U_p$-stable, there exists $M > 0$ such that $U_p^M \mathcal{V} \subseteq \bigcup_{J \subseteq I} \mathcal{V}_J$. Since $f$ is defined on $\bigcup_{J \subseteq I} \mathcal{V}_J$, we may define $f$ on $\mathcal{V}$ by $(U_p^{\varepsilon})^M f$. On the intersection $(\bigcup_{J \subseteq I} \mathcal{V}_J) \cap \mathcal{V}$, the definitions of $f$ coincide since $a_p$ is the $U_p$-eigenvalue of $f$. We can then define $f$ on the whole $\mathcal{U}_i(\varepsilon)$ through the admissible cover $\mathcal{U}_i(\varepsilon) = \bigcup_{J \subseteq I} \mathcal{V}_J \cup \mathcal{V}$.  

### 3.2. Analytic continuation near vertices

In this subsection, we will make use of the small slope assumption (1) to extend $f$ to a strict neighborhood of $\deg^{-1} x_J$.

Let’s first give an outline of the strategy. By (1), for any small enough $\varepsilon > 0$ we have

$$\val_p(a_p) \leq \inf_{\tau \in I} k_{\tau} - g - \varepsilon \sum_{\tau \in I} k_{\tau}.$$

Possibly making it smaller, we will first fix such a rational number $\varepsilon$. Then we will choose a rational number $\delta > 0$ based on $\varepsilon$, and define a sequence of strict neighborhoods

$$S_{I,0}(\delta) \supseteq S_{I,1}(\delta) \supseteq \cdots$$

of $\deg^{-1} x_J$. When $\delta' < \delta$ we will show that $S_{I,0}(\delta') \subseteq S_{I,0}(\delta)$. We have extended $f$ to $\mathcal{U}_i(\delta)$ by Lemma 3.3. Further applying some power of $U_p^{\varepsilon}$, we will be able to extend $f$ to $S_{I,0}(\delta) \setminus S_{I,0}(\delta')$, named $f_m$. We will also define $F_m$ on $S_{I,0}(\delta)$. With the help of the estimates in Section 3.3, we will show that when $m \to \infty$, $f_m$ and $F_m$ glue to define an extension of $f$ on $S_{I,0}(\delta)$.

To begin, we prove the following lemma regarding the twisted directional degrees of points in the set $U_p(y)$, when $y \in \mathcal{Y}$ satisfies $\deg y = x_J$. The lemma will be used to decompose the $U_p$-correspondence $\mathcal{Y}(p)$ over $S_{I,0}(\delta)$ into the special part $\mathcal{Y}(p)^{sp}$ and the non-special part $\mathcal{Y}(p)^{nsp}$, and so the $U_p$-operator becomes $U_p^{sp} + U_p^{nsp}$.

**Lemma 3.4.** Let $y = (A/H) \in \mathcal{Y}$. Let $y_1 = (A/H_1, \bar{H} = A[p]/H_1)$ and $y_2 = (A/H_2, \bar{H} = A[p]/H_2)$ be in $U_p(y)$ and $y_1 \neq y_2$. 


ii. There exists arbitrarily small positive rational number \( \epsilon \) so that if \( |\text{deg}_\tau(y) - \bar{x}_{1,\tau}| \leq \epsilon \) and \( |\text{deg}_\tau(y_1) - \bar{x}_{1,\tau}| \leq \epsilon \) for some \( I \subseteq \Sigma \), then

\[
\text{deg}_\tau H_2 = \inf(\text{deg}_\tau H, \text{deg}_\tau H_1), \quad \text{for all } \tau \in \Sigma.
\]

In particular, \( y_2 \in U_\phi(\epsilon) \).

Proof. For the proof of \( i. \), see \cite[Lemma 5.1.5 2(a)]{Kas16}. The first statement of \( ii. \) follows from \cite[Lemma 5.1.5 2(a)]{Kas16}.

The only statement remained to be proved is the one after “In particular”. By assumption,

\[
\text{deg}_\tau H_2 = \inf(\text{deg}_\tau H, \text{deg}_\tau H_1) = \begin{cases} 
\text{deg}_\tau H & \text{if } \tau \in I \\
\text{deg}_\tau H_1 & \text{if } \tau \not\in I
\end{cases}
\]

and

\[
\text{deg}_\tau y_2 = (p^{g-1} + \cdots + 1) - \text{deg}_\tau H_2 = \begin{cases} 
(p^{g-1} + \cdots + 1) - \text{deg}_\tau H & \tau \in I \\
(p^{g-1} + \cdots + 1) - \text{deg}_\tau H_1 & \tau \not\in I.
\end{cases}
\]

If we further require that \( \epsilon < \frac{1}{2}(p^{g-1} - p^{g-2} - \cdots - 1) \), then \( \text{deg}_\tau y_2 \geq p^{g-2} + \cdots + 1 + \epsilon \), i.e., \( y_2 \in U_\phi(\epsilon) \).

\[ \square \]

**Corollary 3.5.** Let \( I \subseteq \Sigma \) and \( I \neq \emptyset \). Let \( \epsilon \) be a rational number as in Lemma 3.4 \( ii. \) such that \( \epsilon < \frac{1}{2}(p^{g-1} - p^{g-2} - \cdots - 1) \). Let \( y \in \mathcal{Y} \) be such that \( |\text{deg}_\tau(y) - \bar{x}_{1,\tau}| \leq \epsilon \) for all \( \tau \in \Sigma \). Then there exists at most one point \( y_1 \in U_p(y) \) such that \( |\text{deg}_\tau(y_1) - \bar{x}_{1,\tau}| \leq \epsilon \) for all \( \tau \in \Sigma \).

Proof. By the proof of Lemma 3.4 \( ii. \), if \( y_2 \in U_p(y) \) and \( y_2 \neq y_1 \), then \( \text{deg}_\tau(y_2) \geq p^{g-1} - \epsilon \) for all \( \tau \in \Sigma \). Since \( I \neq \emptyset \), we pick an arbitrary \( \tau_0 \in I \). Then

\[
\text{deg}_{\tau_0}(y_2) - \bar{x}_{1,\tau_0} \geq (p^{g-1} - \epsilon) - (p^{g-2} + \cdots + 1) > \epsilon.
\]

\[ \square \]

For any rational number \( \delta > 0 \), consider the strict neighborhood of \( \text{deg}_\tau^{-1}x_I \):

\[
S_{I,0}(\delta) := \left\{ y \in \mathcal{Y} : \sum_{\tau \in I} \text{deg}_\tau y \leq \sum_{\tau \in I} \bar{x}_{1,\tau} + \delta, \text{deg}_\tau y \geq \bar{x}_{1,\tau} - \delta, \forall \tau \not\in I \right\},
\]

which is a quasi-compact admissible open. Recall from Section 2.4 that the \( U_p \)-correspondence is given by \( p_1 : \mathcal{Y}(p) \to \mathcal{Y}, (A, H, H_1) \mapsto (A, H) \) and \( p_2 : \mathcal{Y}(p) \to \mathcal{Y}, (A, H, H_1) \mapsto (A/H_1, \bar{H}) \). Define

\[
S_{I,1}(\delta) := p_1(p_1^{-1}S_{I,0}(\delta) \cap p_2^{-1}S_{I,0}(\delta)),
\]

which is a quasi-compact admissible open of \( \mathcal{Y} \) because it is the pushforward of a quasi-compact admissible open by the finite étale morphism \( p_1 \). Note that

\[
S_{I,1}(\delta) = \{ y \in S_{I,0}(\delta) : \exists y_1 \in U_p(y) \text{ also in } S_{I,0}(\delta) \},
\]

so \( S_{I,1}(\delta) \) is called the special locus of order 1 in \( S_{I,0}(\delta) \).
Let $\epsilon$ be a small enough rational number as in Lemma 3.4 ii. such that $\epsilon < \frac{1}{2}(p^{g-1} - p^{g-2} - \cdots - 1)$, and that the small slope condition (2) is satisfied. Note that the $S_{I,0}(\delta)$’s contain a fundamental system of strict neighborhoods of $\deg^{-1} \omega x_1$. Hence we choose a rational number $\delta > 0$ so that $S_{I,0}(\delta) \subseteq \{ y \in \mathcal{Y}: |\deg_{\tau} y - \tilde{x}_{I,\tau}| < \epsilon \}$ and $S_{I,0}(\delta) \subseteq \{ y \in \mathcal{Y}: |\deg_{\tau} y - x_{I,\tau}| < \epsilon \}$. With this choice of $\delta$, we see by Corollary 3.5 that the $y_{1}$ in the definition of $S_{I,1}(\delta)$ is unique.

Hence we have a correspondence $\mathcal{Y}(p)^{sp} := p_{1}^{-1}S_{I,0}(\delta) \cap p_{2}^{-1}S_{I,0}(\delta) \subseteq \mathcal{Y}(p)$

\[ \begin{array}{c}
\mathcal{Y}(p)^{sp} \\
S_{I,1}(\delta) \\
p_{1}^{sp} \\
p_{2}^{sp}
\end{array} \]

where $p_{i}^{sp}$ is the restriction of $p_{i}$ to $\mathcal{Y}(p)^{sp}$, and $p_{1}^{sp}$ is an isomorphism. Then as before in Section 2.4, for any subset $\mathcal{U} \subseteq S_{I,1}(\delta)$, let $U_{p}^{sp}(\mathcal{U}) := p_{2}^{sp}(p_{1}^{sp})^{-1}(\mathcal{U})$. If $\mathcal{U} \subseteq S_{I,0}(\delta)$ is an admissible open, then $(U_{p}^{sp})^{-1} \mathcal{U} = p_{1}^{sp}(p_{2}^{sp})^{-1} \mathcal{U}$ is also an admissible open because $p_{1}^{sp}$ is finite étale (indeed an isomorphism). For $f \in \omega^{k}(\mathcal{U})$, let $U_{p}^{sp} f \in \omega^{k}(U_{p}^{sp})^{-1} \mathcal{U}$ be $(U_{p}^{sp} f)(A, H) := \frac{1}{p_{1}^{sp}} p_{1}^{sp} f(A/H_{1}, H)$, where $H_{1}$ is such that $p_{1}^{sp}(A, H, H_{1}) = (A, H)$.

We also define $\mathcal{Y}(p)^{nsp} := (\mathcal{Y}(p) \times_{\mathcal{Y}, p_{1}} S_{I,1}(\delta)) \setminus \mathcal{Y}(p)^{sp}$. By Lemma 3.4 ii., we have $p_{2}(\mathcal{Y}(p)^{nsp}) \subseteq U_{\varnothing}(\epsilon)$. Hence

\[ \begin{array}{c}
\mathcal{Y}(p)^{nsp} \\
S_{I,1}(\delta) \\
p_{1}^{nsp} \\
p_{2}^{nsp}
\end{array} \]

where $p_{i}^{nsp}$ is again the restriction of $p_{i}$. We similarly define $U_{p}^{nsp}$ on subsets $\mathcal{U} \subseteq S_{I,1}(\delta)$ and on $f \in \omega^{k}(\mathcal{U})$ when $\mathcal{U} \subseteq S_{I,0}(\delta)$ is an admissible open.

Define the quasi-compact admissible open

\[ \mathcal{V}_{I}(\delta) = \{ y \in \mathcal{Y}: \sum_{\tau \in I} \deg_{\tau} y \geq \sum_{\tau \in I} \tilde{x}_{I,\tau} + \delta, \deg_{\tau} y \geq \tilde{x}_{I,\tau} - \delta, \forall \tau \notin I \} \]

Then $S_{I,0}(\delta) \cup \mathcal{V}_{I}(\delta)$ is $U_{p}$-stable because $U_{p}$ increases twisted directional degrees (Proposition 2.5(1)). Hence we have

\[ U_{p}(S_{I,0}(\delta) \setminus S_{I,1}(\delta)) \subseteq \mathcal{V}_{I}(\delta) \]

Note that $\mathcal{V}_{I}(\delta) \subseteq \mathcal{U}_{I}(\delta)$, and the latter was defined in Section 3.1.

**Lemma 3.6.** Let $\delta' < \delta$ be two positive rational numbers. Then $S_{I,1}(\delta)$ is a strict neighborhood of $S_{I,1}(\delta')$.

**Proof.** Because $S_{I,0}(\delta)$ is defined by inequalities of twisted directional degrees, when $\delta' < \delta$ are two positive rational numbers, then $S_{I,0}(\delta)$ is a strict neighborhood of $S_{I,0}(\delta')$. By definition, $S_{I,1}(\delta) = p_{1}^{-1}S_{I,0}(\delta) \cap p_{2}^{-1}S_{I,0}(\delta))$. Since $p_{1}$ is finite étale, pushforward by $p_{1}$ preserves quasi-compact admissible opens, and hence $S_{I,1}(\delta)$ is a strict neighborhood of $S_{I,1}(\delta')$. \qed

As explained above, for any admissible open $\mathcal{U} \subseteq S_{I,0}(\delta)$, $(U_{p}^{sp})^{-1} \mathcal{U}$ is also an admissible open. Define the admissible open

\[ S_{I,m}(\delta) = (U_{p}^{sp})^{-m}S_{I,0}(\delta), \]

which is quasi-compact because $S_{I,0}(\delta)$ is. Lemma 3.6 says that if $\delta < \epsilon$ are two positive rational numbers, then $S_{I,1}(\delta)$ and $S_{I,0}(\delta) \setminus S_{I,1}(\delta')$ form an admissible covering of $S_{I,0}(\delta)$. Then $S_{I,m}(\delta)$ and $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$ also form an admissible covering of $S_{I,0}(\delta)$.

Now we are ready to prove analytic continuation near vertices.
Proposition 3.7. Let \( f \) be an overconvergent Hilbert modular form of weight \( k \). Let \( I \subseteq \Sigma \). Suppose that \( f \) is defined on a strict neighborhood of \( \deg^{-1}x_J \) for all \( J \subseteq I \). Let \( \epsilon \) be a small enough rational number as in Lemma 3.4 ii. such that \( \epsilon < \frac{1}{2}(p^{q-1} - p^{q-2} - \cdots - 1) \), and that
\[
\text{val}_p(a_p) \leq \inf_{\tau \in I} k_\tau - g - \epsilon \sum_{\tau \in I} k_\tau.
\]

Let \( \delta > 0 \) be a rational number so that \( S_{I,0}(\delta) \subseteq \{ y \in Y : |\deg_\tau y - \bar{x}_{I,\tau}| < \epsilon \} \) and \( S_{I,0}(\delta) \subseteq \{ y \in Y : |\deg_\tau y - x_{I,\tau}| < \epsilon \} \). Then \( f \) can be extended to \( S_{I,0}(\delta) \), which is a strict neighborhood of \( \deg^{-1}x_I \).

Proof. By definition, \( S_{I,m-1}(\delta) \supseteq S_{I,m}(\delta) \). In addition, \( U_p^m(S_{I,0}(\delta) \setminus S_{I,m}(\delta)) \subseteq V_I(\delta) \). By Lemma 3.3, we can extend \( f \) to \( V_I(\delta) \subseteq U_I(\delta) \). Then we can further extend \( f \) by \( (\frac{U_I}{a_p})^m f \) to \( (U_p)^{m}V_I(\delta) \supseteq S_{I,m}(\delta) \). Similarly, for any other rational number \( \delta' < \delta \), we can extend \( f \) by \( (\frac{U_I}{a_p})^m f \) to \( (U_p)^{m}V_I(\delta') \supseteq S_{I,0}(\delta') \setminus S_{I,m}(\delta') \). Because \( S_{I,0}(\delta) \setminus S_{I,m}(\delta) \) and \( S_{I,0}(\delta') \setminus S_{I,m}(\delta') \) form an admissible covering of \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \), we can actually extend \( f \) to \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \).

We denote by \( f_m \) the extension of \( f \) to \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \).

On the other hand, by Lemma 3.3, we can extend \( f \) to \( U_\omega(\epsilon) \). Then
\[
F_m := \sum_{j=0}^{m-1} (\frac{U_I}{a_p})^{j+1}(U_p)^{j} f
\]
can be defined on \( (U_p)^{-m}U_\omega(\epsilon) \supseteq S_{I,m}(\delta) \).

Assume the norm estimates in Proposition 3.8 in the next subsection. By (2), we can choose a subsequence so that \( F_m \mod p^m \) and \( f_m \mod p^m \) glue as \( h_m \) (only defined modulo \( p^m \)) under the admissible covering \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \) and \( S_{I,m}(\delta) \) of \( S_{I,0}(\delta) \). We have \( h_m \equiv f \mod p^m \) on \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \). By (3), we can further choose a subsequence so that \( h_{m+1} \mod p^m \) agrees with \( h_m \mod p^m \) on \( S_{I,m+1}(\delta) \). Hence \( h = \lim_{m \to \infty} h_m \) is defined on \( S_{I,0}(\delta) \), and \( h = f \) on \( S_{I,0}(\delta) \setminus \bigcap_m S_{I,m}(\delta') \). Hence \( h \) is the desired extension of \( f \) to \( S_{I,0}(\delta) \). \( \Box \)

3.3. Norm estimates. Assume that \( \text{val}_p(a_p) \leq \inf_{\tau \in I} k_\tau - g - \epsilon \sum_{\tau \in I} k_\tau \). Choose a rational number \( \delta > 0 \) so that \( S_{I,0}(\delta) \subseteq \{ y \in Y : |\deg_\tau y - \bar{x}_{I,\tau}| < \epsilon \} \) and \( S_{I,0}(\delta) \subseteq \{ y \in Y : |\deg_\tau y - x_{I,\tau}| < \epsilon \} \). Also let \( \delta' < \delta \) be another positive rational number.

Let \( f_m \) defined on \( S_{I,0}(\delta) \) and \( F_m \) defined on \( S_{I,0}(\delta) \setminus S_{I,m}(\delta') \) as in the previous section. The following proposition records the norm estimates used to glue \( f_m \) and \( F_m \) in the previous section.

Proposition 3.8.

(1) \( |F_m|_{S_{I,m}(\delta)} \) and \( |f_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')} \) are bounded.

(2) \( |F_m - f_m|_{S_{I,m}(\delta) \setminus S_{I,m}(\delta')} \to 0 \).

(3) \( |F_{m+1} - F_m|_{S_{I,m+1}(\delta)} \to 0 \).

We need the following two lemmas to prove Proposition 3.8.

Lemma 3.9. Let \( V \subseteq S_{I,1}(\delta) \) and \( h \in \omega V(U_p^p(V)) \). Then
\[
|U_p^p(h)|_V \leq p^g - \sum_{\tau \in I} k_\tau (1 - \epsilon)|h|_{U_p^p(V)}.
\]
In particular, if \( \text{val}_p(a_p) < \inf_{\tau \in I} k_\tau - g - \epsilon \sum_{\tau \in I} k_\tau \), then
\[
\frac{U_p^p(h)}{a_p}|_V \leq p^{-\mu}|h|_{U_p^p(V)}
\]
for some small enough \( \mu > 0 \).
Lemma 3.10. For $1 \leq j \leq m$, $f_m - \frac{U_p^{sp}}{a_p} \delta f_m = F_j$ on $S_{I,j}(\delta) \setminus S_{I,m}(\delta')$.

Proof. Recall that we have fixed $\delta' < \delta$, and $f_m$ is defined on $S_{I,0}(\delta) \setminus S_{I,m}(\delta')$. In particular, $(\frac{U_p^{sp}}{a_p})^j f_m$ is defined on $(U_p^{sp})^{-j} S_{I,0}(\delta) \setminus S_{I,m}(\delta') = S_{I,j}(\delta) \setminus S_{I,j+m}(\delta')$. By definition, $F_j = \sum_{\ell=0}^{j-1} (\frac{1}{a_p})^{\ell+1} U_p^{nsp} (U_p^{sp})^\ell f$ on $S_{I,j}(\delta)$. Hence $F_j + (\frac{U_p^{sp}}{a_p})^j f_m$ is defined on $S_{I,j}(\delta) \setminus S_{I,m}(\delta')$. A simple calculation using the fact that $U_p = U_p^{sp} + U_p^{sp}$ yields the claimed equality $F_j + (\frac{U_p^{sp}}{a_p})^j f_m = f_m$. □

Proof of Proposition 3.8.

(1) Because $f$ is defined on the quasi-compact open $V_I(\delta)$, $|f|_{V_I(\delta)}$ is bounded. Since $U_p$ is a compact operator, $|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta)} \leq \frac{U_p}{a_p} |f|_{V_I(\delta)}$ is also bounded. Similarly, $|f_1|_{S_{I,0}(\delta') \setminus S_{I,1}(\delta')}$ is bounded, and hence $|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}$ is bounded.

We will show that $|f_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)})$ for all $m \geq 1$. Because $f_m$'s are compatible, it suffices to show that $|f_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)})$ for all $m \geq 1$. We do this by induction on $m$. By Lemma 3.10, $f_m - \frac{U_p^{sp}}{a_p} f_m = F_1$ on $S_{I,1}(\delta) \setminus S_{I,m}(\delta')$. Then it suffices to show that $\frac{U_p^{sp}}{a_p} f_m|_{S_{I,0}(\delta) \setminus S_{I,m}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)})$.

By Lemma 3.9,

$$\frac{U_p^{sp}}{a_p} f_m|_{S_{I,0}(\delta) \setminus S_{I,m+1}(\delta')} \leq |f_m|_{S_{I,m-1}(\delta) \setminus S_{I,m}(\delta')} = |f_{m-1}|_{S_{I,m-1}(\delta) \setminus S_{I,m}(\delta')}$$

Hence

$$\frac{U_p^{sp}}{a_p} f_m|_{S_{I,0}(\delta) \setminus S_{I,m+1}(\delta')} \leq \sup(|f_1|_{S_{I,0}(\delta) \setminus S_{I,1}(\delta')}, |F_1|_{S_{I,m}(\delta)})$$

by induction hypothesis.
As for \( |F_m|_{S_{I,m}(\delta)} \), by Lemma 3.9,
\[
|F_m|_{S_{I,m}(\delta)} \leq \sup_{0 \leq j \leq m-1} \left| \left( \frac{1}{a_p} \right)^{j+1} U_p^{nsp} (U_p^{sp})^j f \right|_{S_{I,m}(\delta)}
\]
\[
= \sup_{0 \leq j \leq m-1} \left| \left( \frac{U_p^{sp}}{a_p} \right)^j F_1 \right|_{S_{I,m}(\delta)}
\]
\[
\leq \sup_{0 \leq j \leq m-1} |F_1|_{S_{I,m-j}(\delta)}
\]
\[
= |F_1|_{S_{I,1}(\delta)}.
\]

(2) By Lemma 3.10 and Lemma 3.9,
\[
|F_m - f_m|_{S_{I,m}(\delta) \setminus S_{I,m}(\delta')} = \left| \left( \frac{U_p^{sp}}{a_p} \right)^m f_m \right|_{S_{I,m}(\delta) \setminus S_{I,m}(\delta')}
\]
\[
\leq p^{- \mu} |f_m|_{S_{I,0}(\delta) \setminus S_{I,0}(\delta')}
\]
\[
= p^{- \mu} |f_0|_{S_{I,0}(\delta) \setminus S_{I,0}(\delta')}
\]
\[
\to 0 \text{ as } m \to \infty.
\]

(3) By Lemma 3.9,
\[
|F_{m+1} - F_m|_{S_{I,m+1}(\delta)} = \left| \left( \frac{1}{a_p} \right)^{m+1} U_p^{nsp} (U_p^{sp})^m f \right|_{S_{I,m+1}(\delta)}
\]
\[
= \left| \left( \frac{U_p^{sp}}{a_p} \right)^m F_1 \right|_{S_{I,m+1}(\delta)}
\]
\[
\leq p^{- \mu} |F_1|_{S_{I,1}(\delta)}
\]
\[
\to 0 \text{ as } m \to \infty.
\]

\[ \square \]

3.4. Finishing the proof of Theorem 3.1. Following the paragraph just before Section 3.1, we assume that the overconvergent form \( f \) is defined on a strict neighborhood of \( \overline{\deg^{-1} x_J} \) for all \( J \subseteq I \). We also assume that \( f \) satisfies the small slope condition (1). Let \( \varepsilon \) be a small enough rational number as in Lemma 3.4 ii. such that \( \varepsilon < \frac{1}{2} (p^{d-1} - p^{d-2} - \cdots - 1) \), and that
\[
\text{val}_p(a_p) \leq \inf_{\tau \in \mathcal{I}} k_\tau - g - \varepsilon \sum_{\tau \in \mathcal{I}} k_\tau.
\]
By Proposition 3.7 we can extend \( f \) to a strict neighborhood \( S_{I,0}(\delta) \) of \( \overline{\deg^{-1} x_I} \) for any small enough rational number \( \delta > 0 \).

Note that the vertices in \( \mathcal{F}_I \) are exactly the \( x_J \)'s with \( J \subseteq I \), so we have extended \( f \) to a strict neighborhood of the inverse image of \( \text{deg} \) of all the vertices of \( \mathcal{F}_I \). We will show that \( f \) can be extended to a strict neighborhood \( \mathcal{U} \) of \( \overline{\mathcal{Y} \mathcal{F}_I} \), again using the argument in Lemma 3.3 that \( U_p \) strictly increases the sum of twisted directional degrees when the \( \text{deg} \) is not one of the vertices of \( [0,1]^g \).

Define a quasi-compact admissible open
\[
\mathcal{U} = \{ y \in \mathcal{Y} : \deg_\tau y \geq p^{g-2} + \cdots + 1 + \varepsilon, \forall \tau \not\in I \}.
\]
Recall that \( \mathcal{Y} \mathcal{F}_I = \{ y \in \mathcal{Y} : \deg_\tau y = 1, \forall \tau \not\in I \} \). If \( y \in \mathcal{Y} \mathcal{F}_I \), then for \( \tau \not\in I \),
\[
\deg_\tau y \geq p^{g-1} > p^{g-2} + \cdots + 1 + \varepsilon.
\]
Hence $\mathcal{U}$ is a strict neighborhood of $\mathcal{V}F_I$. We have shown in the proof of Lemma 3.3 that the condition of $\mathcal{U}$ implies that if $y \in \mathcal{U}$ is such that $\deg(y) = x_J$ for some $J \subseteq \Sigma$, then $J \subseteq I$. Moreover, $\mathcal{U}$ is $U_p$-stable because $U_p$ increases twisted directional degrees (Proposition 2.5(1)).

For each $J \subseteq I$, let $\mathcal{V}_J$ be a strict neighborhood of $\deg^{-1}x_J$ on which $f$ is defined, and we explicitly choose $\mathcal{V}_J$ in the form

$$\mathcal{V}_J = \{ y \in \mathcal{V} : \deg_{\tau} y \leq \epsilon_\tau \text{ if } \tau \in J, \deg_{\tau} y \geq 1 - \epsilon_\tau \text{, if } \tau \not\in J \},$$

for some rational $\epsilon_\tau > 0$. Let $\epsilon'_\tau < \epsilon_\tau$ be a rational number, and define the quasi-compact admissible open

$$\mathcal{V} = \left\{ y \in \mathcal{V} : \begin{array}{l}
\deg_{\tau} y \geq \epsilon'_\tau \text{ if } \tau \in I, \\
\deg_{\tau} y \leq 1 - \epsilon'_\tau \text{, if } \tau \not\in I; \\
\deg_{\tau} y \geq p^{\delta-2} + \cdots + 1 + \epsilon, \forall \tau \not\in I
\end{array} \right\}.$$

We have an admissible cover $\mathcal{U} = \bigcup_{J \subseteq I} \mathcal{V}_J \cup \mathcal{V}$.

Since $\mathcal{V}$ is disjoint from $\deg^{-1}x_J$ for any $J \subseteq I$ from its definition, $U_p$ strictly increases $\sum_{\tau \in \Sigma} \deg_{\tau}$ on $\mathcal{V}$ by Proposition 2.5(2). Using the Maximum Modulus Principle, the quasi-compactness of $\mathcal{V}$ implies that there is a positive lower bound for the increase of $\sum_{\tau \in \Sigma} \deg_{\tau}$ under $U_p$ on $\mathcal{V}$. Because $\mathcal{U}$ is $U_p$-stable, there exists $M > 0$ such that $U_p^M \mathcal{V} \subseteq \bigcup_{J \subseteq I} \mathcal{V}_J$. Since $f$ is defined on $\bigcup_{J \subseteq I} \mathcal{V}_J$, we may define $f$ on $\mathcal{V}$ by $(\frac{L}{a_\tau})^M f$. On the intersection $(\bigcup_{J \subseteq I} \mathcal{V}_J) \cap \mathcal{V}$, the definitions of $f$ coincide since $a_\tau$ is the $U_p$-eigenvalue of $f$. We can then define $\tilde{f}$ on the whole $\mathcal{U}$ through the admissible cover $\mathcal{U} = \bigcup_{J \subseteq I} \mathcal{V}_J \cup \mathcal{V}$.

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