Calculation of the dynamical critical exponent in the model $A$ of critical dynamics to order $\varepsilon^4$

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Abstract

A new method based on the $R'$-operation of the renormalization theory is proposed for the numerical calculation of the renormalization constants in the theory of critical behaviour. The problem of finding residues of the poles of the Green’s functions at $\varepsilon = 0$, where $\varepsilon = 4 - d$, is reduced to the evaluation of multiple UV-finite integrals, which can be performed by means of standard integration programs. The method is used to calculate the renormalization group functions of the model $A$ of critical dynamics in four-loop approximation. Dynamical exponent $z$ of the model $A$ is calculated in the fourth order of the $\varepsilon$-expansion.

Introduction

The method of the renormalization group (RG) nowadays is the main tool of calculation of critical exponents in the theory of critical phenomena. The basis of that method is the technique of the ultraviolet renormalization of a model. Critical exponents are found from the calculated renormalization constants by means of the standard RG rules. Thus calculation of the renormalization constants is the primary concern of the method. The most consistent scheme of applying the RG method is combining it with the $\varepsilon$-expansion (with $\varepsilon = 4 - d$, where $d$ is spatial dimension) and that is the way it is used throughout the paper.

Creation of subtle analytical methods of calculations allowed to attain 5-loop accuracy (fifth order of the $\varepsilon$-expansion) in the theory of static critical phenomena [1] and maximum 3-loop accuracy (third order of the $\varepsilon$-expansion) in the theory of critical dynamics [2]. These record results have held since 1991 and 1984 respectively. Attempts of analytical calculations in higher orders met fundamental difficulties [3].

For numerical calculation of the renormalization constants it is essential to be able to extract residues at $\varepsilon = 0$ from the graphs. We developed such technique based on well-known $R'$-operation of the theory of renormalizations. It reduces calculation of renormalization constants to the evaluation of multiple UV-convergent integrals which can be performed by standard programs for numerical integration. We used the technique for calculation of dynamical exponent of the model $A$ in 4-loop approximation (fourth order of the $\varepsilon$-expansion). Brief formulation of the model and the results obtained are given below.

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1 Model

The model A describes dynamical behavior of pure homogeneous Ising system like uniaxial magnetic materials. Particularly the critical dynamical exponent $z$ determines the growth of the relaxation time near the critical point. This paper presents the results of 4-loop calculation of the exponent $z$ with the help of the renormalization group (RG) in the $\varepsilon = 4 - d$ expansion framework.

The model A in the field-theoretic formulation (see e.g. [4]) can be described by the action
\[ S(\psi, \psi') = \lambda_0 \psi' \psi' + \psi' \left[ -\partial_t \psi + \lambda_0 (\partial^2 \psi - \tau_0 \psi - g_0 \psi^3 / 6) \right], \tag{1} \]
with the $n$-component order parameter field $\psi(x, t)$ and the auxiliary field $\psi'(x, t)$. At dimension $d = 4 - \varepsilon$ graphs of the perturbation theory are ultraviolet divergent that manifest itself as pole at $\varepsilon = 0$. These divergences are eliminated by introducing the renormalization constants $Z$ for which we use the minimal subtraction scheme (MS). Renormalized action has the form
\[ S_R(\psi, \psi') = Z_1 \lambda \psi' \psi' + \psi' \left[ -Z_2 \partial_t \psi + \lambda (Z_3 \partial^2 \psi - Z_4 \tau \psi - Z_5 g \mu^2 \psi^3 / 6) \right]. \tag{2} \]

It is obtained from (1) by multiplicative renormalization of the parameters $\lambda_0 = \lambda Z_\lambda$, $\tau_0 = \tau Z_\tau$, $g_0 = g \mu^2 Z_g$ and the fields $\psi \to Z_\psi \psi$, $\psi' \to Z_{\psi'} \psi'$, with the assumptions $Z_1 = Z_\lambda Z_{\psi}^2$, $Z_2 = Z_{\psi} Z_\psi$, $Z_3 = Z_\psi Z_\lambda Z_\psi$, $Z_4 = Z_{\psi} Z_\lambda Z_\tau Z_\psi$, $Z_5 = Z_{\psi} Z_{\lambda} Z_{\psi}^2 Z_\psi^3$.

Important extra information about renormalization constants could be obtained from consideration of the static (simultaneous) Green’s functions of the theory (1). It could be shown that such functions coincide with renormalized Green’s functions of $\phi^4$ theory with static action
\[ S_R^{\text{stat}}(\Phi) = Z_3 \partial^2 \psi / 2 - Z_4 \tau \psi^2 / 2 - Z_5 g \mu^2 \psi^4 / 24, \tag{3} \]
where static renormalization constants $Z_3$, $Z_4$ and $Z_5$ in the MS scheme coincide with the respective dynamical ones in (2) so the renormalization constants of the field $\psi$ and parameters $g$ and $\tau$ are equal to the static ones [4]. Moreover the requirement that the static and the dynamical theories agree leads to relation
\[ Z_{\psi} Z_\lambda = Z_\psi, \tag{4} \]
from which it follows
\[ Z_1 = Z_2 = Z_{\lambda} Z_{\psi}^2, \quad Z_3 = Z_{\psi}^2, \quad Z_4 = Z_\tau Z_{\psi}^2, \quad Z_5 = Z_g Z_{\psi}^4. \tag{5} \]

The only new renormalization constant in addition to the static theory is $Z_\lambda$ that can be found from the 1-irreducible functions $\langle \psi' \psi' \rangle$ or $\langle \psi' \partial_t \psi \rangle$. At present the static renormalization constants are known up to 5-loop accuracy (fifth order of the perturbation theory). So in the MS scheme one can use coordinate of the fixed point $g^*$ (root of $\beta$-function $\beta(g^*) = 0$) calculated in the static model [3]. In the following in place of $g$ we will use for convenience the charge $u = g S_d / (2\pi)^d$, where $S_d = 2\pi^{d/2} / \Gamma(d/2)$ is the area of the unit sphere in $d$ dimensions. The value of the charge at the fixed point $u^*$ is given in [4]. With the accuracy required for the following calculation we have
\[ u^* = \frac{2}{3k_1} \varepsilon + \frac{34}{81} k_0 \frac{1}{k_1^3} \varepsilon^2 + \frac{2}{27} k_1^2 \left( -\frac{33n^3 + 110n^2 + 1760n + 4544}{5832} - 4\zeta(3)k_1 k_4 \right) \varepsilon^3 + O(\varepsilon^4). \tag{6} \]

From now on we use the notation:
\[ k_0 = (3n + 14) / 17, \quad k_1 = (n + 8) / 9, \quad k_2 = (n^2 + 6n + 20) / 27, \quad k_3 = (n + 2) / 3, \quad k_4 = (5n + 22) / 27, \tag{7} \]
all $k_i$ are normalized so that $k_i = 1$ at $n = 1$. 2
2 Calculation of the dynamical critical exponent

The dynamical critical exponent \( z \) is expressed by the relation \( z = 2 + \gamma^* - \eta \) by means of the Fisher critical exponent \( \eta \) and the value \( \gamma^* \equiv \gamma(u^*) \) of RG-function \( \gamma(u) = \beta(u) \partial_u \ln Z_2 \) at the fixed point \( u^* \) \([4]\). Traditionally \( z \) is written in the form \( z = 2 + R \eta \) with \( R \equiv \gamma^*/\eta - 1 \). The quantity \( R \) is convenient because the first two terms of its \( \varepsilon \)-expansion being independent of \( n \) (the number of field components). Indeed the leading terms of the \( \varepsilon \)-expansion of the Fisher exponent are determined by the series \([4]\)

\[
\eta = k_3 u^2 \left[ 1 + a_1 k_1 u^* + (a_2 k_2 + a_3 k_3 + a_4 k_4) u^* \right] + O(u^{*5}),
\]

where

\[
a_1 = -3/8, \quad a_2 = -15/64, \quad a_3 = -5/32, \quad a_4 = 45/32,
\]

while the analogous expansion for the \( \gamma^* \) is given by

\[
\gamma^* = k_3 u \left[ 1 + c_1 k_1 u^* + (c_2 k_2 + c_3 k_3 + c_4 k_4) u^* \right] + O(u^{*5}),
\]

with some constants \( c_i, h \). The quantities \([8]\) and \([10]\) depend on \( n \) only via \( k_i \). Calculating the quotient \( \gamma^*/\eta \) from these relations and using \([6]\) one finds

\[
\frac{\gamma^*}{\eta} = h \left\{ 1 + \frac{2}{3} (c_1 - a_1) \varepsilon + \left[ \frac{4}{9} a_1 (a_1 - c_1) + \frac{4}{3} (c_2 - a_2) + \frac{nb_1 + b_0}{(n + 8)^2} \right] \varepsilon^2 + O(\varepsilon^3) \right\},
\]

where

\[
b_0 = 28 (c_1 - a_1) - \frac{176}{3} (c_2 - a_2) + 24 (c_3 - a_3) + \frac{88}{3} (c_4 - a_4),
b_1 = 6 (c_1 - a_1) - \frac{40}{3} (c_2 - a_2) + 12 (c_3 - a_3) + \frac{20}{3} (c_4 - a_4).
\]

It is seen from these relations that first two terms of the \( \varepsilon \)-expansion of \( R \) indeed do not depend on \( n \). Evaluation of the renormalization constant \( Z_2 \) in 4-loop approximation gives the following values of the coefficients in \([10]\):

\[
h = 1.72609, \quad c_1 = -0.4939(4), \quad c_2 = -0.2512(0), \quad c_3 = -0.1699(4), \quad c_4 = 1.806(3).
\]

The first calculation of \( h \) was performed in \([5]\) (one 2-loop graph):

\[
h = 6 \ln(4/3).
\]

The constant \( c_1 \) was calculated in \([6]\) and corrected in \([2]\) (three 3-loop graphs):

\[
c_1 = \pi^2/8 - F(1/4) - \frac{3}{4} + \frac{13}{8} \ln 4 - \frac{21}{8} \ln 3,
\]

where \( F(x) = \int_x^1 dt \ln t/(t - 1) \) is dilogarithm (Spence’s function). The values of the constants \( h \) and \( c_1 \) obtained in \([13]\) agree with the analytic expressions from \([14]\) and \([15]\).
In regard to the constants we pioneer in calculating of, namely, $c_2$, $c_3$ and $c_4$ (twenty five 4-loop graphs). It turns out that one can verify the value of $c_2$ using the asymptotic expression for the constant $R$ in the limit $n \to \infty$ [5]:

$$R = \frac{4}{\varepsilon} \left\{ \frac{(4 - \varepsilon)\Gamma^2(1 - \varepsilon/2)}{8\Gamma(2 - \varepsilon) \int_0^{1/2} dx[x(2 - x)]^{-\varepsilon/2}} - 1 \right\} + O(1/n). \quad (16)$$

Calculating of the $\varepsilon$-expansion in the rhs of (16) gives

$$R = [6 \ln(4/3) - 1](1 - 0.188483 \varepsilon - 0.0999529 \varepsilon^2) + O(\varepsilon^3), \quad n \to \infty. \quad (17)$$

On the other hand from (11) we obtain

$$R = \frac{\gamma^*}{\eta} - 1 = h \left\{ 1 + \frac{2}{3} (c_1 - a_1) \varepsilon + \frac{4}{9} [a_1(a_1 - c_1) + 3(c_2 - a_2)] \varepsilon^2 \right\} - 1 + O(\varepsilon^3), \quad n \to \infty. \quad (18)$$

After the substitution of the values from (9) and (13) into (18) the agreement between the coefficients of the $\varepsilon$-expansions in (17) and (18) is established.

The final expression of the constant $R$ is obtained by the substitution of the quantities (13) to (2) and (11):

$$R = [6 \ln(4/3) - 1] \left( 1 - 0.18849(9) \varepsilon + \left[ 4.78(6) n + 21.5(4) \right] \varepsilon^2 \right) + O(\varepsilon^3). \quad (19)$$

The final expression for the dynamical critical exponent $z = 2 + R\eta$ for $n = 1$ results from (11), (8) and (19):

$$z|_{n=1} = 2 + 0.0134462 \varepsilon^2 + 0.011036(2) \varepsilon^3 - 0.00558(5) \varepsilon^4 + O(\varepsilon^5). \quad (20)$$

3 Discussion

The method that we used for extracting the residues of graphs at $\varepsilon = 0$ and for the numerical calculating of the renormalization constants, turned out to be highly effective for the RG-analysis of the dynamical model $A$ in the 4-loop approximation. Correctness of the obtained results was confirmed by the cancellation of the poles at $\varepsilon = 0$ in the RG function $\gamma(u) = \beta(u)\partial_u \ln Z_2$ and by comparison with the results known in two and three loops and with the leading order of the $1/n$-expansion. The main difficulty in the approach we used was achieving of high precision in numerical calculations of multiple integrals (in our case maximum multiplicity equals to nine and the achieved precision is of order 0.1%). The proposed method can be used for multiloop calculations of the renormalization constants for both dynamical and statical problems. In particular, we are planning 6-loop calculation in the static model [3].

It is obvious from (20) that the 4-loop correction to the critical exponent $z$ is relatively small for the real value of $\varepsilon = 1$ (i.e. $d = 3$) we obtain $z = 2.0188(9)$.

Acknowledgements

The authors thank N.V. Antonov for discussions. The work was supported by Russian Foundation for Basic Research (grant No 08-02-00125a).
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