General conditions for vanishing the current $J^z$ for Dirac field on boundaries of the domain between two planes

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Abstract

In connection with the Casimir effect for a spinor field in presence of external magnetic field, of special interest are solutions of the Dirac equation in the domains restricted by two planes, which have vanishing the third projection of the conserved current $J^z$ on two boundaries. General conditions for vanishing the current are formulated, they reduce to linear homogeneous algebraic system, for which solutions exist only when vanishing the determinant of the linear system, that is for the roots of a 4-th order algebraic equation with respect to the variable $e^{2ika}$, where $a$ is a half-distance between the planes, and $k$ stands for the third projection of the Dirac particle momentum. All solutions of the equation have been found explicitly, each of them provides us in principle with a special possibility to get the quantization rules for parameter $k$; the most of produced expression for the roots can be solved with respect to parameter $k$ only numerically. Generally, solutions $e^{2ika}$ depend on 4 arbitrary phase parameters which influence the appropriate wave functions with vanishing current: $J^z(z = -z, +a) = 0$.

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1 Introduction

The boundary conditions imposed on a quantum field leads to the modification of the vacuum energy levels and can be observed experimentally as a vacuum pressure. This effect (called Casimir effect) has been predicted by Casimir in his original work in 1948 [1], [2], and has been experimentally observed for electromagnetic field several years later. Until now, in many theoretical works Casimir energy has been computed for various types of boundary geometry and for fields other than electromagnetic one. It has received much attention since its discovery – see review [3], [4],

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and more recent [5]. The Casimir effect for charged relativistic fields can be also influenced by external electromagnetic fields.

In contrast to boson fields (for instance, scalar or electromagnetic), for the Dirac field one cannot impose directly restriction of vanishing the field on the boundary of the restricted domain. In 1975, Johnson [6] calculated for the first time the Casimir fermionic effect in the MIT-bag model. Johnson used as restriction the condition of vanishing the normal projection of the current of the Dirac field at the boundary surface of a domain (the bag).

The perturbation of Casimir energy by an external magnetic field was firstly considered by Elizalde et al. [7] and Cougo-Pinto et al. [8]. The influence on the Casimir energy of the Dirac field was firstly studied for an antiperiodic boundary condition by Farina – Tort – Cougo–Pinto [9]. These authors showed that the Casimir effect can be enhanced by a magnetic field.

More recently the influence of the external magnetic field on the complex scalar field as well as on the Casimir energy of the Dirac field was considered by Ostrowski [10] who calculates these by direct solving the field equations using the mode summation method. He has considered a complex scalar field confined between two infinite plates with Dirichlet boundary conditions with magnetic field in the direction perpendicular to the plates. Also he considered the Dirac field with antiperiodic boundary conditions. In scalar case the most simple and popular is the Dirichlet boundary conditions in the form:

\[ \Phi(x, y, z = -a) = \Phi(x, y, z = +a) \]

which implies that field is equal to zero on the two parallel planes (plates). The plates are perpendicular to the \( z \)-axis and the distance between them is equal to \( 2a \). In the fermionic case the most popular is antiperiodic boundary condition in the form:

\[ \Psi(x, y, z + 2a) = -\Psi(x, y, z) \]

Finally in the recent study of the Casimir effect for charged scalar field in presence of magnetic field [11], it is shown that in the case of a weak magnetic field and a small separation of the plates, the Casimir force is either attractive or repulsive, depending on the choice of a boundary condition. In the case of a strong magnetic field and a large separation of the plates, the Casimir force is repulsive, being independent of the choice of a boundary condition, as well as of the distance between the plates.

In the present paper we turn back to the case of Dirac field bounded by two parallel planes and submitted to external uniform magnetic field; the aim is to study in full generality the Johnson MIT-bag condition for the current \( J_z \) on the boundaries of the domain between two planes. As demonstrates the present analysis, the known in the literature associated restriction is (see for instance in [3]) only simplest and particular example, whereas many other possibilities exist here as well.

The outline of the paper is the following. In connection with the Casimir effect for the Dirac field in external magnetic field, of special interest are solutions of the Dirac equation in the domains between two planes, which have vanishing third projection of the conserved current \( J_z \) on the boundaries. Solutions with such properties are reachable when considering 4-dimensional linear space \( \{ \Psi \} \) based of four solutions for Dirac particle with opposite signs of the third projection of momentum \( +k \) and \( -k \) along the direction of the magnetic field. General conditions for vanishing such a current at the boundaries are formulated, they reduce to linear homogeneous algebraic system, for which equation \( \det S = 0 \) is a 4-th order algebraic equation, the roots \( K \) of which are
\[ K = e^{2ika}, \text{ where } a \text{ is a half-distance between the planes, and } k \text{ stands for the third projection of the Dirac particle momentum. Each root of this equation, if it represents a complex number of the unit length, will give a certain rule for quantization of the third projection of } k. \text{ The complete analysis and classification of all such solutions has been performed.} \]

2 Solutions of the Dirac equation in magnetic field

Let us consider the Dirac equation in Cartesian coordinates in presence of external uniform magnetic field \( \mathbf{A} = (-By, 0, 0) \)

\[
\begin{align*}
\left[ \gamma^0 \frac{\partial}{\partial t} + \gamma^1 (i \frac{\partial}{\partial x} - By) \\
+i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} - M \right] \Psi &= 0. 
\end{align*}
\]

Below we use the spinor basis for the Dirac matrices and the following substitution for the wave functions

\[
\Psi = e^{-i\epsilon t} e^{iax} e^{ikz} \begin{vmatrix} f_1(y) \\ f_2(y) \\ f_3(y) \\ f_4(y) \end{vmatrix};
\]

which results in equations for \( f_a(y) \)

\[
\begin{align*}
\epsilon f_3 + (a + By) f_4 - \frac{\partial}{\partial y} f_4 + kf_3 - M f_1 &= 0, \\
\epsilon f_4 + (a + By) f_3 + \frac{\partial}{\partial y} f_3 - kf_4 - M f_2 &= 0, \\
\epsilon f_1 - (a + By) f_2 + \frac{\partial}{\partial y} f_2 - kf_1 - M f_3 &= 0, \\
\epsilon f_2 - (a + By) f_1 - \frac{\partial}{\partial y} f_1 + kf_2 - M f_4 &= 0.
\end{align*}
\]

Let us impose linear restrictions (below we will prove their consistency)

\[ f_3 = Af_1, \quad f_4 = Af_2; \]

system (3) will take the form

\[
\begin{align*}
(\epsilon + k - \frac{M}{A}) f_1 + [(a + By) - \frac{\partial}{\partial y}] f_2 &= 0, \\
(-\epsilon + k + \frac{M}{A}) f_2 - [(a + By) + \frac{\partial}{\partial y}] f_1 &= 0, \\
(-\epsilon + k + MA) f_1 + [(a + By) - \frac{\partial}{\partial y}] f_2 &= 0, \\
(\epsilon + k - MA) f_2 - [(a + By) + \frac{\partial}{\partial y}] f_1 &= 0.
\end{align*}
\]
Equations 1 and 3, as well as 2 and 4, will coincide, if the condition below holds (we will use notation $+\sqrt{\epsilon^2 - M^2} = p$)

$$A = \frac{\epsilon \pm p}{M}. \quad (6)$$

Depending on values of $A$, we have two cases:

$A = \frac{\epsilon + p}{M} = \alpha,$

$$(k + p) f_1 + [(a + B y) - \frac{\partial}{\partial y}] f_2 = 0,$$

$$(k - p) f_2 - [(a + B y) + \frac{\partial}{\partial y}] f_1 = 0; \quad (7)$$

$A = \frac{\epsilon - p}{M} = \beta,$

$$(k - p) f_1 + [(a + B y) - \frac{\partial}{\partial y}] f_2 = 0,$$

$$(k + p) f_2 - [(a + B y) + \frac{\partial}{\partial y}] f_1 = 0. \quad (8)$$

Let us specify explicit form of an operator related to linear restriction (4). For a free Dirac particle, that is the helicity operator

$$\Sigma = \frac{1}{2} \left( \gamma^2 \gamma^3 \frac{\partial}{\partial x} + \gamma^3 \gamma^1 \frac{\partial}{\partial y} + \gamma^1 \gamma^2 \frac{\partial}{\partial z} \right).$$

In presence of the magnetic field, it becomes more complicated

$$\Sigma = \frac{1}{2} \left( \gamma^2 \gamma^3 (\frac{\partial}{\partial x} + iB y) + \gamma^3 \gamma^1 \frac{\partial}{\partial y} + \gamma^1 \gamma^2 \frac{\partial}{\partial z} \right).$$

The eigenvalue equation $\Sigma \Psi = p \Psi$ gives

$$(a + B y) f_2 - \frac{d}{dy} f_2 + k f_1 = pf_1,$$

$$(a + B y) f_1 + \frac{d}{dy} f_1 - k f_2 = pf_2,$$

$$(a + B y) f_4 - \frac{d}{dy} f_4 + k f_3 = pf_3,$$

$$(a + B y) f_3 + \frac{d}{dy} f_3 - k f_4 = pf_4.$$

Considering them jointly with (3), we arrive at the linear algebraic system

$$p f_3 + (\epsilon f_3 - M f_1) = 0, \quad p f_4 + (\epsilon f_4 - M f_2) = 0,$$

$$p f_1 - (\epsilon f_1 - M f_3) = 0, \quad p f_2 - (\epsilon f_2 - M f_4) = 0,$$

from whence it follows

$$p = +\sqrt{\epsilon^2 - M^2}, \quad f_3 = \frac{\epsilon \pm p}{M} f_1, \quad f_4 = \frac{\epsilon \pm p}{M} f_2;$$
these coincide with the above used restrictions.

Remembering on this, let us write down the substitutions for the wave functions of opposite polarization states:

\[
\Psi_{\epsilon ak,\alpha} = e^{-i\epsilon t} e^{iax} e^{ikz} \\
\begin{array}{c|c}
  f_1(y) & \alpha f_1(y) \\
  f_2(y) & \alpha f_2(y) \\
\end{array},
\]

\[
\Psi_{\epsilon ak,\beta} = e^{-i\epsilon t} e^{iax} e^{ikz} \\
\begin{array}{c|c}
  f_1(y) & \beta f_1(y) \\
  f_2(y) & \beta f_2(y) \\
\end{array};
\]

(9)

these waves should be solutions of respective systems: (note the notation: \( \lambda^2 = \epsilon^2 - m^2 - k^2 \))

\[
\text{type-}\alpha,
\]

\[
\frac{d^2 f_1}{dy^2} + [B - (a + By)^2 + \lambda^2] f_1 = 0,
\]

\[
f_2 = \frac{1}{k-p} \left( \frac{d}{dy} + (a + By) \right) f_1;
\]

\[
\text{type-}\beta \quad (\text{note the notation: } f_1, f_2 \rightarrow g_1, g_2,)
\]

\[
\frac{d^2 g_1}{dy^2} + [B - (a + By)^2 + \lambda^2] g_1 = 0,
\]

\[
g_2 = \frac{1}{k+p} \left( \frac{d}{dy} + (a + By) \right) g_1.
\]

Because \( f_1 \) and \( g_1 \) obey the same differential equation, one can set them equal to each other:

\[
(\alpha) \quad f_1(y) = g_1(y) = f(y),
\]

\[
(\alpha) \quad f_2 = \frac{1}{k-p} \left( \frac{d}{dy} + (a + By) \right) f \equiv \frac{1}{k-p} g,
\]

\[
(\beta) \quad g_2 = \frac{1}{k+p} \left( \frac{d}{dy} + (a + By) \right) f \equiv \frac{1}{k+p} g.
\]

Correspondingly, (9) will read

\[
\Psi_{\epsilon ak,\alpha} = e^{-i\epsilon t} e^{iax} e^{ikz} \\
\begin{array}{c|c}
  f(y) & (k-p)^{-1} g(y) \\
  \alpha f(y) & \alpha (k-p)^{-1} g(y) \\
\end{array},
\]

\[
\Psi_{\epsilon ak,\beta} = e^{-i\epsilon t} e^{iax} e^{ikz} \\
\begin{array}{c|c}
  f(y) & (k+p)^{-1} g(y) \\
  \beta f(y) & \beta (k+p)^{-1} g(y) \\
\end{array};
\]

(10)
3 Vanishing the current on the boundaries of the domain between two planes

Now let us derive condition for vanishing the current $J^z$ on the boundaries $z = -a, z = +a$ of the domain between two planes. We use the above solutions of the types $(\alpha, \beta)$ and similar to them with the change $k \mapsto -k$ (total for all four solutions factor $e^{-i\epsilon t} e^{i\omega x}$ will be omitted)

$$\Phi_1 = \Psi_{\epsilon, a, k, \alpha} = e^{ikz}$$

$$\Phi_2 = \Psi_{\epsilon, a, k, \beta} = e^{ikz}$$

$$\Phi_3 = \Psi_{\epsilon, a, -k, \alpha} = e^{-ikz}$$

$$\Phi_4 = \Psi_{\epsilon, a, -k, \beta} = e^{-ikz}$$

Let us make a linear combination of these:

$$\Phi = A_1\Phi_1 + A_2\Phi_2 + A_3\Phi_3 + A_4\Phi_4$$

the constituents of such a wave function are

$$\Phi_1 = \frac{e^{ikz}(A_1 + A_2) + e^{-ikz}(A_3 + A_4)}{f}$$

$$\Phi_2 = \frac{e^{ikz}(A_1 - A_2) - e^{-ikz}(A_3 - A_4)}{g}$$

$$\Phi_3 = \frac{e^{ikz}(A_1 + A_2) + e^{-ikz}(A_3 + A_4)}{f}$$

$$\Phi_4 = \frac{e^{ikz}(A_1 - A_2) - e^{-ikz}(A_3 - A_4)}{g}$$

note the structure of the current (in the spinor basis)

$$J^z = \Phi^+ \gamma^0 \gamma^3 \Phi = (\Phi_1^* \Phi_1 - \Phi_3^* \Phi_3) - (\Phi_2^* \Phi_2 - \Phi_4^* \Phi_4)$$

The pair $(\Phi_1, \Phi_3)$ contains one the same factor $f(y)$; in turn, the pair $(\Phi_2, \Phi_4)$ contains other factor $g(y)$. The current on the boundaries $z = -a, z = +a$ will vanish (with no influence of $y$-dependence), if the following restrictions are satisfied

$$z = -a, \quad \Phi_3 = e^{i\phi} \Phi_1, \quad \Phi_4 = e^{i\sigma} \Phi_2$$

$$e^{-ika}(A_1\alpha + A_2\beta) + e^{ika}(A_3\alpha + A_4\beta) = e^{i\phi}[e^{-ika}(A_1 + A_2) + e^{ika}(A_3 + A_4)],$$

$$[e^{-ika}(\frac{A_1\alpha}{k-p} + \frac{A_2\beta}{k+p}) - e^{ika}(\frac{A_3\alpha}{k+p} + \frac{A_4\beta}{k-p})] = e^{i\sigma}[e^{-ika}(\frac{A_1}{k-p} + \frac{A_2}{k+p}) - e^{ika}(\frac{A_3}{k+p} + \frac{A_4}{k-p})]$$
\[ z = +a, \quad \Phi_3 = e^{i\mu} \Phi_1, \quad \Phi_4 = e^{i\nu} \Phi_2 \implies \]
\[ e^{ika} (A_1 \alpha + A_2 \beta) + e^{-ika} (A_3 \alpha + A_4 \beta) = e^{i\mu} [e^{ika} (A_1 + A_2) + e^{-ika} (A_3 + A_4)], \]
\[ [e^{ika} (A_1 \alpha + A_2 \beta) + e^{-ika} (A_3 \alpha + A_4 \beta)] = e^{i\nu} [e^{ika} (A_1 + A_2) + e^{-ika} (A_3 + A_4)]. \]

These can be rewritten as homogeneous linear system with respect to \( A_1, A_2, A_3, A_4 \) (let \( K = e^{2iak} \))
\[ A_1(\alpha - e^{ip}) + A_2(\beta - e^{ip}) + A_3(\alpha - e^{ip})K + A_4(\beta - e^{ip})K = 0, \]
\[ A_1K(\alpha - e^{i\mu}) + A_2K(\beta - e^{i\mu}) + A_3(\alpha - e^{i\mu}) + A_4(\beta - e^{i\mu}) = 0, \]
\[ A_1(\alpha - e^{i\sigma})(k + p) + A_2(\beta - e^{i\sigma})(k - p) - A_3K(\alpha - e^{i\sigma})(k - p) - A_4K(\beta - e^{i\sigma})(k + p) = 0, \]
\[ A_1K(\alpha - e^{i\nu})(k + p) + A_2K(\beta - e^{i\nu})(k - p) - A_3(\alpha - e^{i\nu})(k - p) - A_4(\beta - e^{i\nu})(k + p) = 0. \]

Let us write down explicit form of the matrix for the system (13)
\[ S = \begin{vmatrix} (\alpha - e^{ip}) & (\beta - e^{ip}) & (\alpha - e^{ip})K & (\beta - e^{ip})K \\ (\alpha - e^{i\mu})K & (\beta - e^{i\mu})K & (\alpha - e^{ip}) & (\beta - e^{i\mu}) \\ (\alpha - e^{i\sigma})(k + p) & (\beta - e^{i\sigma})(k - p) & -(\alpha - e^{i\sigma})(k - p)K & -(\beta - e^{i\sigma})(k + p)K \\ (\alpha - e^{i\nu})(k + p)K & (\beta - e^{i\nu})(k - p)K & -(\alpha - e^{i\nu})(k - p) & -(\beta - e^{i\nu})(k + p) \end{vmatrix}. \]

\section{Variant with one independent phase}

There exist 6 variants with one independent phase:
\[ \rho = \Delta - \Delta \Delta \Delta \Delta \Delta \Delta \}
\[ \mu = \Delta - \Delta \Delta \Delta \Delta - \Delta \Delta \Delta \}
\[ \sigma = \Delta \Delta \Delta \Delta - \Delta \Delta \Delta - \Delta \}
\[ \nu = \Delta \Delta \Delta \Delta - \Delta - \Delta - \Delta \Delta \}

Let us consider the first variant in (15), the main equation takes the form
\[ \det S = \begin{vmatrix} (\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\ (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K & (\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) \\ (\alpha - e^{i\Delta})(k + p) & (\beta - e^{i\Delta})(k - p) & -(\alpha - e^{i\Delta})(k - p)K & -(\beta - e^{i\Delta})(k + p)K \\ (\alpha - e^{i\Delta})(k + p)K & (\beta - e^{i\Delta})(k - p)K & -(\alpha - e^{i\Delta})(k - p) & -(\beta - e^{i\Delta})(k + p) \end{vmatrix}. \]

After elementary transformation it gives
\[ \det S = (\alpha - e^{i\Delta})^2(\beta - e^{i\Delta})^2 \begin{vmatrix} 1 & K & K \\ K & 1 & (k + p)K \\ K & (k + p) & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ (k + p) & (k - p) & - (k + p) \end{vmatrix}. \]

We see two evident roots \( K = e^{2iak} = \pm 1 \). For instance, let \( K = +1 \), then
\[ \det S = (\alpha - e^{i\Delta})^2(\beta - e^{i\Delta})^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ (k + p) & (k - p) & - (k + p) & - (k + p) \end{vmatrix}. \]
They read as a linear system with respect to \( A_3, A_4 \); its solution is

\[
A_3 = -\frac{k+p}{p} A_1 - \frac{k}{p} \frac{(\beta - e^{i\Delta})}{(\alpha - e^{i\Delta})} A_2, \quad A_4 = \frac{k}{p} \frac{(\alpha - e^{i\Delta})}{(\beta - e^{i\Delta})} A_1 + \frac{k-p}{p} A_2. \tag{19}
\]

The corresponding wave function and quantization rule for \( k \) are

\[
J^2(z = \pm a) = 0, \quad K = e^{2ika} = +1, \quad k = \frac{\pi}{a} n, \ n = 0, \pm 1, \pm 2, \ldots;
\]

\[
\Phi = A_1 \Psi_1 + A_2 \Psi_2 + \left( -\frac{(k+p)}{p} A_1 - \frac{k}{p} \frac{(\beta - e^{i\Delta})}{(\alpha - e^{i\Delta})} A_2 \right) \Psi_3 + \left( \frac{k}{p} \frac{(\alpha - e^{i\Delta})}{(\beta - e^{i\Delta})} A_1 + \frac{k-p}{p} A_2 \right) \Psi_4. \tag{20}
\]

It should be emphasized that the phase \( \Delta \) does not determine quantization of the \( k \), instead it only influences on the coefficients of the linear combination in \( \Phi \). A simplest choice for the phase is \( \Delta = 0 \):

\[
\Phi = A_1 \Psi_1 + A_2 \Psi_2 + \left( -\frac{(k+p)}{p} A_1 + \alpha^{-1} \frac{k}{p} A_2 \right) \Psi_3 + \left( -\alpha \frac{k}{p} A_1 + \frac{k-p}{p} A_2 \right) \Psi_4. \tag{21}
\]

To examine two remaining solutions, let us turn back to the initial polynomial \( (17) \) and find all its roots. Explicitly, equation \( S = 0 \) reads

\[
(K^2 - 1)^2 = 0 \quad \implies \quad K = +1, -1, +1, -1. \tag{22}
\]

Let us consider the second variant in \( (15) \):

\[
S = \begin{vmatrix}
\alpha - e^{-i\Delta} & \beta - e^{-i\Delta} & (\alpha - e^{-i\Delta})K & (\beta - e^{-i\Delta})K \\
(\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
(\alpha - e^{i\Delta})(k+p) & (\beta - e^{i\Delta})(k-p) & -(\alpha - e^{i\Delta})(k-p)K & -(\beta - e^{i\Delta})(k+p)K \\
(\alpha - e^{i\Delta})(k+p)K & (\beta - e^{i\Delta})(k-p)K & -(\alpha - e^{i\Delta})(k-p)K & -(\beta - e^{i\Delta})(k+p)K
\end{vmatrix},
\]

or

\[
\det S = (\alpha - e^{i\Delta})^2 (\beta - e^{i\Delta})^2 
\begin{vmatrix}
\frac{\alpha - e^{-i\Delta}}{\alpha - e^{i\Delta}} & \frac{\beta - e^{-i\Delta}}{\beta - e^{i\Delta}} & \frac{\alpha - e^{-i\Delta}}{\alpha - e^{i\Delta}}K & \frac{\beta - e^{-i\Delta}}{\beta - e^{i\Delta}}K \\
K & 1 & 1 & 1 \\
(k+p) & (k-p) & -(k-p)K & -(k+p)K \\
(k+p)K & (k-p)K & -(k-p)K & -(k+p)K
\end{vmatrix}. \tag{23}
\]
In [23], the rows 3 and 4 are proportional if $K = +1, K = -1$, which means that the values $K = +1, -1$ are the roots of the polynomial. At this, the rank of the matrix equal to 3, because the left upper block $S_{3 \times 3}$ has a non-zero determinant:

$$
K = +1, \quad \text{det } S_{3 \times 3} = \frac{2k(1 - \alpha^2)}{x(\alpha x - 1)(x - \alpha)}(1 - x^2), \quad e^{i\Delta} = x;
$$

$$
K = -1, \quad \text{det } S_{3 \times 3} = \frac{2k(1 - \alpha^2)}{x(\alpha x - 1)(x - \alpha)}(1 - x^2), \quad e^{i\Delta} = x. \tag{24}
$$

However, this determinant vanishes if $x = +1, -1$ ($\Delta = 0, \pi$), and in this case the rank of the matrix equals to 2.

Let us detail in the case $\text{det } a_{3 \times 3} \neq 0$ the linear when $K = +1$; the fourth equation coincides with the third and we have only three different equations

$$
A_1(\alpha - e^{-i\Delta}) + A_2(\beta - e^{-i\Delta}) + A_3(\alpha - e^{-i\Delta}) = -A_4(\beta - e^{-i\Delta}),
$$

$$
A_1(\alpha - e^{i\Delta}) + A_2(\beta - e^{i\Delta}) + A_3(\alpha - e^{i\Delta}) = -A_4(\beta - e^{i\Delta}), \tag{25}
$$

$$
A_1(\alpha - e^{i\Delta})(k + p) + A_2(\beta - e^{i\Delta})(k - p) - A_3(\alpha - e^{i\Delta})(k - p) = A_4(\beta - e^{i\Delta})(k + p).
$$

With the use of notation $e^{i\Delta} = x$, solution of these equation looks (as should be expected, the formulas are meaningful when $x = \pm 1$)

$$
A_1 = \frac{2kx^3\alpha + (-k\alpha^2 + p\alpha^2 - 3k + p)x^2 + 2\alpha(k - 2p)x - k\alpha^2 + p\alpha^2 + k + p}{2\alpha(x - 1)(x + 1)(x - k)} A_4 = c_1 A_4,
$$

$$
A_2 = -\frac{(x - \alpha)x}{(x - 1)(x + 1)} A_4 = c_2 A_4, \tag{26}
$$

$$
A_3 = -\frac{2kx^3\alpha + (-k\alpha^2 + p\alpha^2 - k + p)x^2 - 2\alpha + (k + 2p)x + k\alpha^2 + p\alpha^2 + k + p}{2\alpha(x - 1)(x + 1)(x - k)} A_4 = c_3 A_4.
$$

The corresponding wave function and quantization rule for $k$ are

$$
\Psi = A_4 (c_1 \Psi_1 + c_2 \Psi_2 + c_3 \Psi_3 + \Psi_4); \tag{27}
$$

at this an arbitrary phase parameter $x = e^{i\Delta}$ influences $c_1, c_2, c_3$ in [27].

Turning back to general case [23], we see that det $S = 0$ is a bi-quadratic equation (let $\Lambda = K^2$)

$$
(kx^2\alpha^2 + px^2\alpha^2 - kx^2 - k\alpha^2 + px^2 - 4px\alpha + p\alpha^2 + k + p) \Lambda^2
$$

$$
+ (-2px^2\alpha^2 - 2px^2 + 8px\alpha - 2p\alpha^2 - 2p) \Lambda
$$

$$
- kx^2\alpha^2 + px^2\alpha^2 + kx^2 + k\alpha^2 + px^2 - 4px\alpha + p\alpha^2 - k + p = 0; \tag{28}
$$

its roots are

$$
\Lambda_1 = 1, \quad \Lambda_2 = -\frac{[(x^2 - 1)\alpha^2 - x^2 + 1]k + [(x^2 - 1)\alpha^2 + 4\alpha x - x^2 - 1]}{[(x^2 - 1)\alpha^2 - x^2 + 1]k + [(x^2 + 1)\alpha^2 - 4\alpha x + x^2 + 1]}p. \tag{29}
$$

In order to obtain a complex conjugate $\Lambda_2(x)$, it is enough in $\Lambda_2(x)$ to make the formal change $x \mapsto x^{-1}$, after that by direct calculation one proves identity $\Lambda_2\Lambda_2^* = \Lambda_2(x)\Lambda_2(x^{-1}) = +1$. This means that the root $\Lambda_2$ is of a phase type and it is appropriate to give quantization rule for $k$:

$$
e^{i\Delta k} = \Lambda_2(\epsilon, p, k, e^{i\Delta}). \tag{30}$$
It should be specially noted that eq. (30) contain an arbitrary phase \( x = e^{i\Delta} \), and it can be solved only numerically, which makes it of little significance in the context of theoretical analysis. However if \( x = +1, -1(\Delta = 0, \pi) \), the root \( \Lambda_2 = 1 \).

Consider the variant 3 in [15]:

\[
| \begin{array}{cccc}
X & 1 & KX & K \\
-K & -KX & -1 & -X \\
X(k+p) & k-p & -KX(k-p) & -(k+p)K \\
X(k+p)K & K(k-p) & -X(k-p) & -k-p
\end{array} | = 0 , \quad x = e^{i\Delta}, \quad \frac{\alpha - x}{1 - \alpha x} = X ; \quad (31)
\]

it reduces to a bi-quadratic equation (let \( K^2 = \Lambda \))

\[
[(k+p)X^2 - k + p]\Lambda^2 - 2p(X^2 + 1)\Lambda - [(k-p)X^2 - k - p] = 0 ; \quad (32)
\]

its roots are

\[
\Lambda_1 = 1, \quad \Lambda_2 = \frac{(-k+p)X^2 + k + p}{(k+p)X^2 - k + p} , \quad \Lambda_2 \Lambda_2^* = \Lambda_2(X) \Lambda_2\left(\frac{1}{X}\right) = 1 . \quad (33)
\]

Consider the variant 4 in [15]:

\[
| \begin{array}{cccc}
X & 1 & KX & K \\
-KX & -X(k-p) & K(k-p) & X(k+p)K \\
X(k+p)K & K(k-p) & -X(k-p) & -k-p
\end{array} | = 0 , \quad x = e^{i\Delta}, \quad \frac{\alpha - x}{1 - \alpha x} = X . \quad (34)
\]

Further we get a bi-quadratic equation (let \( K^2 = \Lambda \))

\[
[(k+p)X^2 - k + p]\Lambda^2 - 2p(X^2 + 1)\Lambda - [(k-p)X^2 - k - p] = 0 ; \quad (35)
\]

its roots are

\[
\Lambda_1 = 1, \quad \Lambda_2 = \frac{(-k+p)X^2 + k + p}{(k+p)X^2 - k + p} ; \quad \Lambda_2 \Lambda_2^* = \Lambda_2(X) \Lambda_2\left(\frac{1}{X}\right) = 1 . \quad (36)
\]

Consider the variant 5 in [15]:

\[
| \begin{array}{cccc}
X & 1 & KX & K \\
-KX & K & X & 1 \\
X(k+p) & k-p & -KX(k-p) & -(k+p)K \\
-(k+p)K & -(k-p)K & k-p & k+p
\end{array} | = 0 ; \quad (37)
\]

we arrive at a bi-quadratic equation with respect to \( \Lambda = K^2 \)

\[
-[(k-p)X - k - p]\Lambda^2 - 2p(X-1)\Lambda + [(k+p)X - k + p] = 0 ; \quad (4.14b)
\]

its roots are

\[
\Lambda_1 = 1, \quad \Lambda_2 = \frac{-X(k+p) + k - p}{X(k-p) - k - p} , \quad \Lambda_2 \Lambda_2^* = 1 . \quad (38)
\]
Consider the variant 6 in (15):

\[
\begin{vmatrix}
X & 1 & KX & K \\
KX & \alpha & 1 & \alpha X \\
-k-p & -X(k-p) & (k-p)K & X(k+p)K \\
-(k+p)K & -(k-p)K & k-p & k+p
\end{vmatrix} = 0, \quad x = e^{i\Delta}, \quad \frac{\alpha - x}{1 - \alpha x} = X. \quad (39)
\]

Further we get a bi-quadratic equation (let \(K^2 = \Lambda\))

\[
-2[(k^2 - p^2)(X^2 + 1) - 2Xk^2](X + 1)\Lambda + [(k-p)X^2 - k-p][k+p]X - k + p = 0, \quad (40)
\]

with the roots

\[
\Lambda_1 = 1, \quad \Lambda_2 = \frac{[X(k+p) - k+p][(k-p)X^2 - k-p]}{[(k+p)X^2 - k+p][X(k+p) - k-p]}, \quad \Lambda_2 \Lambda_2 = 1. \quad (41)
\]

Consider the variant 7 in (15)

\[
\begin{vmatrix}
(\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) & K(\alpha - e^{i\Delta}) & K(\beta - e^{i\Delta}) \\
K(\alpha - e^{-i\Delta}) & K(\beta - e^{-i\Delta}) & (\alpha - e^{-i\Delta}) & (\beta - e^{-i\Delta}) \\
(\alpha - e^{i\Delta})(k+p) & (\beta - e^{i\Delta})(k-p) & -K(k-p)(\alpha - e^{i\Delta}) & -K(k+p)(\beta - e^{i\Delta}) \\
K(k+p)(\alpha - e^{-i\Delta}) & K(k-p)(\beta - e^{-i\Delta}) & -(k-p)(\alpha - e^{-i\Delta}) & -(k+p)(\beta - e^{-i\Delta})
\end{vmatrix} = 0. \quad (42)
\]

or with notation \(e^{i\Delta} = x\):

\[
\begin{vmatrix}
(\alpha - x) & (1 - \alpha x) & K(\alpha - x) & K(1 - \alpha x) \\
-K(1 - \alpha x) & -(\alpha - x) & K(\alpha - x) & -K(1 - \alpha x) \\
-(k+p)(\alpha - x) & (k-p)(1 - \alpha x) & -K(k-p)(\alpha - x) & -K(k+p)(1 - \alpha x) \\
-K(k+p)(1 - \alpha x) & -(k-p)(\alpha - x) & K(k-p)(\alpha - x) & K(k+p)(\alpha - x)
\end{vmatrix} = 0. \quad (42)
\]

Further with the use of notation \(\frac{\alpha - x}{1 - \alpha x} = X\), we arrive at

\[
\begin{vmatrix}
X & 1 & KX & K \\
-K & -KX & 1 & -X \\
X(k+p) & (k-p) & -KX(k-p) & -K(k+p) \\
-K(k+p) & -KX(k-p) & (k-p) & X(k+p)
\end{vmatrix} = 0. \quad (43)
\]

So we get a bi-quadratic equation (let it be \(K^2 = \Lambda\))

\[
\Lambda^2 - \Lambda[L(W - 2) + 2] + 1 = 0, \quad L = k^2/p^2, \quad W = X^2 + \frac{1}{X^2}. \quad (44)
\]

The roots are

\[
\Lambda_1 = 1 + \frac{L(W - 2)}{2} + \frac{\sqrt{L(W - 2)[4 + L(W - 2)]}}{2}, \quad \Lambda_2 = 1 + \frac{L(W - 2)}{2} - \frac{\sqrt{L(W - 2)[4 + L(W - 2)]}}{2}. \quad (45)
\]
Due to easily checked identities $W^* = W$ and
\[
\Lambda_1^* = \Lambda_1 = 1 + \frac{L(W - 2)}{2} + \frac{\sqrt{L(W - 2)[4 + L(W - 2)]}}{2},
\]
\[
\Lambda_2^* = \Lambda_2 = 1 + \frac{L(W - 2)}{2} - \frac{\sqrt{L(W - 2)[4 + L(W - 2)]}}{2}.
\]
we derive identities $\Lambda_1^* \Lambda_1 = 1$, $\Lambda_2^* \Lambda_2 = 1$. So, all the roots are complex number of the unit length.

Consider variant 8 in (15):
\[
\det S = \begin{vmatrix}
(\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
(\alpha - e^{-i\Delta})(k + p) & (\beta - e^{-i\Delta})(k - p) & -(\alpha - e^{-i\Delta})(k - p)K & -(\beta - e^{-i\Delta})(k + p)K \\
(\alpha - e^{i\Delta})(k + p) & (\beta - e^{i\Delta})(k - p)K & -(\alpha - e^{i\Delta})(k - p) & -(\beta - e^{i\Delta})(k + p)K \\
\end{vmatrix} = 0;
\]
with notation $e^{i\Delta} = x$, $\frac{\alpha - x}{1 - x} = X$ we rewrite this as
\[
\det S = \frac{(1 - \alpha x)^4}{x^2\alpha^2} \begin{vmatrix}
X & 1 & XK & K \\
x^{-1}K & -X & x^{-1} & -X \\
x^{-1}(k + p) & -X(x^{-1} - 1)K & x^{-1}(k - p)K & x^{-1}(k - p) - X^{-1}(k + p) \\
K & (k - p)K & -X(k - p) & -(k + p) \\
\end{vmatrix} = 0. \tag{46}
\]
Which results in a bi-quadratic equation
\[
[(X^2(k + p) - k + p)K^2 + X^2(k - p) - k - p]^2 = 0; \tag{47}
\]
its 2-multiple roots are given by (let $K^2 = \Lambda$)
\[
\Lambda = \frac{(-k + p)X^2 + k + p}{(k + p)X^2 - k + p}, \quad \Lambda^* = 1. \tag{48}
\]

5 Cases with two independent phases

There exist only 4 substantially different possibilities to fix four phases with the use of 2 parameters:
\[
\begin{array}{cccccc}
\rho & = & \Delta & \Delta & \Delta & \Delta \\
\mu & = & \Delta & \Delta & -\Delta & -\Delta \\
\sigma & = & W & W & W & W \\
\nu & = & W & -W & W & -W \\
\end{array} \tag{49}
\]
Let us the variant 1 in (49):
\[
\det S = \begin{vmatrix}
(\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
(\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K & (\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) \\
(\alpha - e^{iW})(k + p) & (\beta - e^{iW})(k - p) & -(\alpha - e^{iW})(k - p)K & -(\beta - e^{iW})(k + p)K \\
(\alpha - e^{iW})(k + p)K & (\beta - e^{iW})(k - p)K & -(\alpha - e^{iW})(k - p) & -(\beta - e^{iW})(k + p)K \\
\end{vmatrix}.
\]
With notation $e^{i\Delta} = x$, $e^{iW} = y$, it reads

$$\det S = \begin{vmatrix}
\alpha - x & (1 - \alpha x)^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
\alpha - x & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
(\alpha - y)(k + p) & (1 - \alpha y)\alpha^{-1}(k - p) & -(\alpha - y)(k - p)K & -(1 - \alpha y)\alpha^{-1}(k + p)K \\
(\alpha - y)(k + p)K & (1 - \alpha y)\alpha^{-1}(k - p)K & -(\alpha - y)(k - p) & -(1 - \alpha y)\alpha^{-1}(k + p)
\end{vmatrix};$$

introducing notation $\frac{\alpha - x}{1 - \alpha x} = X$, $\frac{\alpha - y}{1 - \alpha x} = Y$; after simple manipulation we arrive at

$$\det S = \frac{(1 - \alpha x)^2(1 - \alpha y)^2}{\alpha^2} \begin{vmatrix}
X & 1 & XK & K \\
XK & K & X & 1 \\
Y(k + p) & (k - p) & -Y(k - p)K & -(k + p)K \\
Y(k + p)K & (k - p)K & -Y(k - p) & -(k + p)
\end{vmatrix}. \quad (50)$$

Equation $\det S = 0$ turns to have the form

$$[X(k + p) - Y(k - p)] [X(k - p) - Y(k + p)] (K - 1)^2(K + 1)^2 = 0; \quad (51)$$

its roots are $K = 1$, $K = 1$, $K = -1$, $K = -1$.

Let us consider the variant 2 in [49].

$$\det S =$$

$$\begin{vmatrix}
\alpha - e^{i\Delta} & \beta - e^{i\Delta} & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
\alpha - e^{i\Delta}K & \beta - e^{i\Delta} & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
(\alpha - e^{iW})(k + p) & (\beta - e^{iW})(k - p) & -(\alpha - e^{iW})(k - p)K & -(\beta - e^{iW})(k + p)K \\
(\alpha - e^{-iW})(k + p)K & (\beta - e^{-iW})(k - p)K & -(\alpha - e^{-iW})(k - p) & -(\beta - e^{-iW})(k + p)
\end{vmatrix}. \quad (52)$$

With notation $e^{i\Delta} = x$, $e^{iW} = y$, it reads

$$\det S =$$

$$\begin{vmatrix}
\alpha - x & (1 - \alpha x)^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
\alpha - x & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
(\alpha - y)(k + p) & (1 - \alpha y)\alpha^{-1}(k - p) & -(\alpha - y)(k - p)K & -(1 - \alpha y)\alpha^{-1}(k + p)K \\
-(1 - \alpha y)(k + p)K & -(\alpha - y)\alpha^{-1}(k - p)K & -(k - p)(1 - \alpha y)\alpha^{-1} & -(\alpha - y)\alpha^{-1}(k + p)
\end{vmatrix},$$

from whence we arrive at

$$\det \begin{vmatrix}
X & 1 & XK & K \\
XK & K & X & 1 \\
Y(k + p) & (k - p) & -Y(k - p)K & -(k + p)K \\
-(k + p)K & -Y(k - p)K & (k - p) & Y(k + p)
\end{vmatrix} = 0. \quad (53)$$

In explicit form, equation $\det S = 0$ looks as

$$-(K^2 - 1) \left\{ [(k^2 - p^2)Y(X^2 + 1) - (Y^2(k - p)^2 + (k + p)^2)X]K^2 \right. \left. - (k^2 - p^2)Y(X^2 + 1) + [Y^2(k + p)^2 + (k - p)^2]X \right\} = 0; \quad (54)$$
its roots are

\[ K_{1,2} = \pm 1, \quad K_{3,4} = \pm \sqrt{\frac{YX(k + p) - k + p}{YX(k - p) - k + p} \frac{X(k - p) - Y(k + p)}{X(k + p) - Y(k - p)}}. \] (55)

We easily derive identities \( K_3 K^*_4 = 1, \quad K_4 K^*_3 = 1 \).

Let us consider the variant 3 in [49]:

\[
\begin{vmatrix}
(\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
(\alpha - e^{-i\Delta})K & (\beta - e^{-i\Delta})K & (\alpha - e^{-i\Delta}) & (\beta - e^{-i\Delta}) \\
(\alpha - e^{iW})(k + p) & (\beta - e^{iW})(k - p) & -(\alpha - e^{iW})(k - p)K & -(\beta - e^{iW})(k + p)K \\
(\alpha - e^{iW})(k + p)K & (\beta - e^{iW})(k - p)K & -(\alpha - e^{iW})(k - p) & -(\beta - e^{iW})(k + p)K
\end{vmatrix} = 0;
\]

with notation \( e^{i\Delta} = x, \quad e^{iW} = y \) it reads

\[
\begin{vmatrix}
(\alpha - x) & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
-(1 - \alpha x)x^{-1}K & -(\alpha - x)\alpha^{-1}x^{-1}K & -(1 - \alpha x)x^{-1} & -(\alpha - x)\alpha^{-1}x^{-1} \\
(\alpha - y)(k + p) & (1 - \alpha y)\alpha^{-1}(k - p) & -(\alpha - y)(k - p)K & -(1 - \alpha y)(k + p)K \\
(\alpha - y)(k + p)K & (1 - \alpha y)\alpha^{-1}(k - p)K & -(\alpha - y)(k - p) & -(1 - \alpha y)\alpha^{-1}(k + p)
\end{vmatrix} = 0.
\]

From this we derive

\[
\begin{vmatrix}
X & 1 & XK & K \\
-K & -XK & -1 & -X \\
Y(k + p) & (k - p) & -Y(k - p)K & -(k + p)K \\
Y(k + p)K & (k - p)K & -Y(k - p) & -(k + p)
\end{vmatrix} = 0. \tag{56}
\]

In explicit form, equation \( \det S = 0 \) is

\[
(K^2 - 1)\{(X^2 + (k - p)^2)Y - (k^2 - p^2)(Y^2 + 1)X\}K^2 - \\
-[(k - p)^2X^2 + (k + p)^2]Y + (k^2 - p^2)(Y^2 + 1)X = 0; \tag{57}
\]

its roots are

\[
K_{1,2} = \pm 1, \quad K_{3,4} = \pm \sqrt{\frac{YX(k - p) - k - p}{YX(k + p) - k + p} \frac{X(k + p) - Y(k - p)}{X(k + p) - Y(k - p)}}; \tag{5.6b}
\]

it is readily verified that \( K_3 K^*_4 = 1, \quad K_4 K^*_3 = 1 \).

Consider the variant 4 in [49]:

\[
\begin{vmatrix}
(\alpha - e^{i\Delta}) & (\beta - e^{i\Delta}) & (\alpha - e^{i\Delta})K & (\beta - e^{i\Delta})K \\
(\alpha - e^{-i\Delta})K & (\beta - e^{-i\Delta})K & (\alpha - e^{-i\Delta}) & (\beta - e^{-i\Delta}) \\
(\alpha - e^{iW})(k + p) & (\beta - e^{iW})(k - p) & -(\alpha - e^{iW})(k - p)K & -(\beta - e^{iW})(k + p)K \\
(\alpha - e^{iW})(k + p)K & (\beta - e^{iW})(k - p)K & -(\alpha - e^{iW})(k - p) & -(\beta - e^{iW})(k + p)
\end{vmatrix};
\]

which can be translated to to form

\[
\begin{vmatrix}
X & 1 & XK & K \\
-K & -XK & -1 & -X \\
Y(k + p) & (k - p) & -Y(k - p)K & -(k + p)K \\
-(k + p)K & -Y(k - p)K & (k - p) & Y(k + p)
\end{vmatrix} = 0. \tag{58}
\]
It reads explicitly as (let it be $K^2 = \Lambda$

$$[(k + p)X - (k - p)Y]^2 \Lambda^2 - [(4k^2Y^2 - 2k^2 + 2p^2)X^2
+(-4k^2 + 4p^2)YX + (-2k^2 + 2p^2)Y^2 + 4k^2]\Lambda + [X(k - p) - (k + p)Y]^2 = 0; \hspace{1cm} (59)$$

the roots are

$$\Lambda_1 = \frac{2k\sqrt{(YX - 1)^2(((Y^2 - 1)X^2 - Y^2 + 1)k^2 + p^2(X + Y)^2) +
+ (2k^2Y^2 + p^2 - k^2)X^2 + Y(p^2 - k^2)(2X + Y) + 2k^2}}{((k + p)X - (k - p)Y)^2},$$

$$\Lambda_2 = \frac{-2k\sqrt{(YX - 1)^2(((Y^2 - 1)X^2 - Y^2 + 1)k^2 + p^2(X + Y)^2) +
+ (2k^2Y^2 + p^2 - k^2)X^2 + Y(p^2 - k^2)(2X + Y) + 2k^2}}{((k + p)X - (k - p)Y)^2}. \hspace{1cm} (60)$$

We can prove additional properties of these roots:

$$a\Lambda^2 + b\Lambda + c = 0, \hspace{0.5cm} a(\Lambda - \Lambda_1)(\Lambda - \Lambda_2) = 0, \hspace{0.5cm} \Lambda_1\Lambda_2 = \frac{c}{a}, \hspace{1cm} (5.8c)$$

first checking $\Lambda_1\Lambda_1^* = 1$, and then with the help of Vieta’s theorem we derive a needed identity for second root

$$(\Lambda_1\Lambda_1^*)(\Lambda_2\Lambda_2^*) = \frac{c}{a} \frac{c^*}{a^*}; \hspace{1cm} \Lambda_2\Lambda_2^* = \frac{(X(k - p) - (k + p)Y)^2(\frac{k - p}{X} - \frac{k + p}{X})}{((k + p)X - (k - p)Y)^2(\frac{k - p}{X} - \frac{k + p}{X})^2} \implies \Lambda_2\Lambda_2^* = 1.$$

6 The cases of three independent phases

There exist 6 possibilities to determine phases with the help of 3 parameters:

$$\rho = \begin{array}{cccccc}
F & G & F & F & F & F \\
F & H & G & G & G & G
\end{array} \hspace{1cm} (61)$$

Let us consider the variant 1 in (61):

$$\text{det } S = \begin{vmatrix}
(\alpha - e^{iF}) & (\beta - e^{iF}) & (\alpha - e^{iF})K & (\beta - e^{iF})K \\
(\alpha - e^{iG})K & (\beta - e^{iG})K & -(\alpha - e^{iG})(k - p)K & -(\beta - e^{iG})(k + p)K \\
(\alpha - e^{iH})(k + p) & (\beta - e^{iH})(k - p) & -(\alpha - e^{iH})(k - p) & -(\beta - e^{iH})(k + p) \\
(\alpha - x) & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K
\end{vmatrix}. \hspace{1cm} (62)$$

With notation $e^{iF} = x, \hspace{0.1cm} e^{iG} = y, \hspace{0.1cm} e^{iH} = z$ ; it reads

$$\text{det } S = \begin{vmatrix}
(\alpha - x) & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
(\alpha - y)(k + p) & (1 - \alpha y)\alpha^{-1}(k - p) & -(\alpha - y)(k - p)K & -(1 - \alpha y)\alpha^{-1}(k + p)K \\
(\alpha - z)(k + p)K & (1 - \alpha z)\alpha^{-1}(k - p)K & -(\alpha - z)(k - p) & -(1 - \alpha z)\alpha^{-1}(k + p)
\end{vmatrix}.$$
in notation e

which results in a bi-quadratic equation (let Λ = K²)

its roots are

Consider the variant 2 in (61):

in notation \( e^F = x, e^G = y, e^H = z \), it reads

\[
\text{det } S = \begin{vmatrix}
(\alpha - y) & (1 - \alpha y)\alpha^{-1} & (\alpha - y)K & (1 - \alpha y)\alpha^{-1}K \\
(\alpha - z)K & (1 - \alpha z)\alpha^{-1}K & (\alpha - z) & (1 - \alpha z)\alpha^{-1} \\
(\alpha - x)(k + p) & (1 - \alpha x)\alpha^{-1}(k - p) & -(\alpha - x)(k - p)K & -(1 - \alpha x)\alpha^{-1}(k + p)K \\
(\alpha - x)(k + p)K & (1 - \alpha x)\alpha^{-1}(k - p)K & -(\alpha - x)(k - p) & -(1 - \alpha x)\alpha^{-1}(k + p) \\
\end{vmatrix}
\]

\[
= (1 - \alpha y)\alpha^{-1}(1 - \alpha z)\alpha^{-1}(1 - \alpha x)^2\alpha^{-2} \alpha^2
\]

\[
\times \begin{vmatrix}
Y & 1 & YK & K \\
ZK & K & Z & 1 \\
X(k + p) & (k - p) & -X(k - p)K & -(k + p)K \\
X(k + p)K & (k - p)K & -X(k - p) & -(k + p) \\
\end{vmatrix};
\]
thus we arrive at
\[
\det \begin{vmatrix}
Y & 1 & YK & K \\
ZK & K & Z & 1 \\
X(k+p) & (k−p) & −X(k−p)K & −(k+p)K \\
X(k+p)K & (k−p)K & −X(k−p) & −(k+p)
\end{vmatrix} = 0 .
\] (67)

Explicitly, we have a bi-quadratic equation (let \( \Lambda = K^2 \))
\[
[(k+p)X - Z(k-p)] [(k-p)X - Y(k+p)] \Lambda^2 + 
+ 2[(k^2 + p^2)(Y + Z)X - (k^2 - p^2)(X^2 + YZ)] \Lambda + 
+ [(k+p)X - Y(k-p)] [(k-p)X - Z(k+p)] = 0;
\] (68)
its roots are
\[
\Lambda_1 = 1, \quad \Lambda_2 = \frac{[(k+p)X - (k+p)Z] [(k+p)X - (k-p)Y]}{[(k+p)X - (k-p)Z] [(k-p)X - (k+p)Y]}, \quad \Lambda_2 \Lambda_2^* = 1 .
\] (69)

Consider the variant 3 in (61):
\[
\det S = \begin{vmatrix}
(\alpha - e^{iF}) & (\beta - e^{iF}) & (\alpha - e^{iF})K & (\beta - e^{iF})K \\
(\alpha - e^{iG})K & (\beta - e^{iG})K & (\alpha - e^{iG}) & (\beta - e^{iG}) \\
(\alpha - e^{iF})(k+p) & (\beta - e^{iF})(k-p) & -(\alpha - e^{iF})(k-p)K & -(\beta - e^{iF})(k+p)K \\
(\alpha - e^{iH})(k+p)K & (\beta - e^{iH})(k-p)K & -(\alpha - e^{iH})(k-p) & -(\beta - e^{iH})(k+p)
\end{vmatrix};
\]
\[
= \begin{vmatrix}
(\alpha - x) & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
(\alpha - y)K & (1 - \alpha y)\alpha^{-1}K & (\alpha - y) & (1 - \alpha y)\alpha^{-1} \\
(\alpha - x)(k+p) & (1 - \alpha x)\alpha^{-1}(k-p) & -(\alpha - x)(k-p)K & -(1 - \alpha x)\alpha^{-1}(k+p)K \\
(\alpha - z)(k+p)K & (1 - \alpha z)\alpha^{-1}(k-p)K & -(\alpha - z)(k-p) & -(1 - \alpha z)\alpha^{-1}(k+p)
\end{vmatrix}
\]
\[
= (1 - \alpha x)\alpha^{-1}(1 - \alpha y)\alpha^{-1}(1 - \alpha x)\alpha^{-1}(1 - \alpha z)\alpha^{-1}\alpha^2
\]
\[
\times \begin{vmatrix}
X & 1 & XK & K \\
YK & K & Y & 1 \\
X(k+p) & (k−p) & −X(k−p)K & −(k+p)K \\
Z(k+p)K & (k−p)K & −Z(k−p) & −(k+p)
\end{vmatrix} = 0,
\] (70)

which gives a bi-quadratic equation (let \( \Lambda = K^2 \))
\[
pX[k(Y - Z) - (Y + Z)p] \Lambda^2 + 2[(X^2 + YZ)k^2 - (k^2 - p^2)(Y + Z)X] \Lambda - pX[k(Y - Z) + (Y + Z)p] = 0 ;
\] (71)
its roots are
\[
\begin{align*}
\Lambda_1 &= \frac{\sqrt{k^2(X-Z)^2(X-Y)^2k^2 + 2p^2((Y+Z)X-2YZ)X(-1/2Z-1/2Y+X)}}{pX((Y-Z)k-(Y+Z)p)} - \frac{k^2(X^2+YZ)-(Y+Z)(k^2-p^2)X}{pX((Y-Z)k-(Y+Z)p)}, \\
\Lambda_2 &= -\frac{\sqrt{k^2(X-Z)^2(X-Y)^2k^2 + 2p^2((Y+Z)X-2YZ)X(-1/2Z-1/2Y+X)}}{pX((Y-Z)k-(Y+Z)p)} - \frac{k^2(X^2+YZ)-(Y+Z)(k^2-p^2)X}{pX((Y-Z)k-(Y+Z)p)}. 
\end{align*}
\] (72)

By direct calculation we get \(\Lambda_1\Lambda_1^* = 1\); then with the help of Vieta’s theorem we prove the same for second root
\[
(\Lambda_1\Lambda_1^*)(\Lambda_2\Lambda_2^*) = \frac{c e^s}{a a^*},
\]
\[
\Lambda_2\Lambda_2^* = \frac{(k(Y-Z)+(Y+Z)p)(k(Y^{-1}-Z^{-1})+(Y^{-1}+Z^{-1}))}{(k(Y-Z)-(Y+Z)p)(k(Y^{-1}-Z^{-1})-(Y^{-1}+Z^{-1}))} \implies \Lambda_2\Lambda_2^* = 1.
\]

Consider the variant 4 in (61):
\[
\det S = \begin{vmatrix} 
(\alpha-e^{iF}) & (\beta-e^{iF}) & (\alpha-e^{iF})K & (\beta-e^{iF})K \\
(\alpha-e^{iG})K & (\beta-e^{iG})K & (\alpha-e^{iG}) & (\beta-e^{iG}) \\
(\alpha-e^{-iF})(k+p) & (\beta-e^{-iF})(k-p) & -(\alpha-e^{-iF})(k-p)K & -(\beta-e^{-iF})(k+p)K \\
(\alpha-e^{iH})(k+p)K & (\beta-e^{iH})(k-p)K & -(\alpha-e^{iH})(k-p) & -(\beta-e^{iH})(k+p) 
\end{vmatrix}.
\] (73)

from whence with the use of notation \(e^{iF} = x, e^G = y, e^H = z\); we get
\[
\det S = \begin{vmatrix} 
(\alpha-x) & (1-\alpha x)\alpha^{-1} & (\alpha-x)K & (1-\alpha x)\alpha^{-1}K \\
(\alpha-y)K & (1-\alpha y)\alpha^{-1}K & (\alpha-y) & (1-\alpha y)\alpha^{-1} \\
-(1-\alpha x)x^{-1}(k+p) & -(\alpha-x)\alpha^{-1}x^{-1}(k-p) & (1-\alpha x)(k-p)x^{-1}K & (\alpha-x)\alpha^{-1}x^{-1}(k+p)K \\
(\alpha-z)(k+p)K & (1-\alpha z)\alpha^{-1}(k-p)K & -(\alpha-z)(k-p)K & -(1-\alpha z)\alpha^{-1}(k+p) 
\end{vmatrix}
= (1-\alpha x)\alpha^{-1}(1-\alpha y)\alpha^{-1}(1-\alpha x)\alpha^{-1}x^{-1}(1-\alpha z)\alpha^{-1}x^{-1}(1-\alpha z)\alpha^{-1}x^{-1}(1-\alpha z)\alpha^{-1}x^{-1}
\]
\[
\times \begin{vmatrix} 
X & 1 & XK & K \\
YK & K & Y & 1 \\
-(k+p) & -X(k-p) & (k-p)K & X(k+p)K \\
Z(k+p)K & (k-p)K & -Z(k-p) & -(k+p) 
\end{vmatrix};
\] (74)
equation \(\det S = 0\) is a bi-quadratic one (let \(\Lambda = K^2\))
\[
[(X^2-1)k + p(X^2+1)][(Y-Z)k - p(Y+Z)]\Lambda^2
- \left\{2(Y+Z)X^2 - 4(Y+Z+1)X + 2(Y+Z)\right\}k^2 - 2p^2(X^2+1)(Y+Z) \right\} \Lambda
+[(X^2-1)k - (X^2+1)p] [(Y-Z)k + p(Y+Z)] = 0.
\] (75)

its roots are
\[
\Lambda_1
\]
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\[
\begin{align*}
\det S &= 2k\sqrt{(XZ-1)(X-Z)(XY-1)(X-Y)k^2 - [YZ(X^2+1) - X(Y+Z)][1 + X^2 - X(Y+Z)]p^2} \\
&= \frac{[(X^2-1)k + p(X^2+1)][(Y-Z)k - p(Y+Z)]}{[(X^2-1)k + p(X^2+1)][(Y-Z)k - p(Y+Z)]} \\
&+ \frac{[(Y+Z)X^2 - 2(YZ+1)X + Y + Z]k^2 - p^2(Y + Z)(X^2 + 1)}{[(X^2-1)k + p(X^2+1)][(Y-Z)k - p(Y+Z)]}.
\end{align*}
\]

The first root is a phase \( \Lambda_1 \Lambda^*_1 = 1 \), with the use of Vieta’s theorem we prove the same for second root:

\[
(\Lambda_1 \Lambda^*_1)(\Lambda_2 \Lambda^*_2) = \frac{c}{a^*}, \quad \Lambda_2 \Lambda^*_2
\]

\[
\frac{((X^2-1)k - p(X^2+1))(k(Y-Z) + (Y+Z)p)((\frac{1}{X^2} - 1)k - p((\frac{1}{X^2} + 1))(k(\frac{1}{X^2} - \frac{1}{Z}) + (\frac{1}{X^2} + \frac{1}{Z})p)}{((X^2-1)k + p(X^2+1))(k(Y-Z) - (Y+Z)p)((\frac{1}{X^2} - 1)k + p((\frac{1}{X^2} + 1))(k(\frac{1}{X^2} - \frac{1}{Z}) - (\frac{1}{X^2} + \frac{1}{Z})p)} = 1.
\]

Consider the variant 5 in \( (76) \):

\[
\det S = \begin{vmatrix}
(\alpha - e^{iF}) & (\beta - e^{iF}) & (\alpha - e^{iF})K & (\beta - e^{iF})K \\
(\alpha - e^{iG}) & (\beta - e^{iG})K & (\alpha - e^{iG}) & (\beta - e^{iG}) \\
(\alpha - e^{iH})(k + p) & (\beta - e^{iH})(k - p) & -(\alpha - e^{iH})(k - p) & -(\beta - e^{iH})(k + p) \\
(\alpha - e^{iF})(k + p)K & (\beta - e^{iF})(k - p)K & -(\alpha - e^{iF})(k - p) & -(\beta - e^{iF})(k + p) \\
\end{vmatrix}
\]

with the notation \( e^{iF} = x, \ e^{iG} = y, \ e^{iH} = z \), it reads

\[
\det S = \begin{vmatrix}
(\alpha - x) & (1 - \alpha x)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
(\alpha - y)K & (1 - \alpha y)\alpha^{-1} & (\alpha - y) & (1 - \alpha y)\alpha^{-1} \\
(\alpha - z)(k + p) & (1 - \alpha z)\alpha^{-1}(k - p) & -(\alpha - z)(k - p)K & -(1 - \alpha z)\alpha^{-1}(k + p)K \\
(\alpha - x)(k + p)K & (1 - \alpha x)\alpha^{-1}(k - p)K & -(\alpha - x)(k - p) & -(1 - \alpha x)\alpha^{-1}(k + p) \\
\end{vmatrix}
\]

\[
= (1 - \alpha x)\alpha^{-1}(1 - \alpha y)\alpha^{-1}(1 - \alpha z)\alpha^{-1}(1 - \alpha x)\alpha^{-1}\alpha^{*}X
\]

\[
\times \begin{vmatrix}
X & 1 & XK & K \\
YK & K & Y & 1 \\
Z(k + p) & (k - p) & -Z(k - p)K & -(k + p)K \\
X(k + p)K & (k - p)K & -X(k - p) & -(k + p) \\
\end{vmatrix}; \quad (77)
\]

The explicit form of \( \det S = 0 \) is (let \( \Lambda = K^2 \))

\[
\{[(k + p)X - Y(k - p)]K^2 + X(k - p) - Y(k + p)\} \times \{[(k + p)X - Z(k - p)]K^2 + X(k - p) - Z(k + p)\} = 0 \quad (78)
\]
the roots are

$$\Lambda_1 = -\frac{X(k-p) - Z(k+p)}{(k+p)X - Z(k-p)}; \quad \Lambda_2 = -\frac{X(k-p) - Z(k+p)}{(k+p)X - Z(k-p)};$$

identities hold $\Lambda_1 \Lambda_1^* = 1, \quad \Lambda_2 \Lambda_2^* = 1$.

Consider the variant 6 in (81):

$$\det S = \begin{vmatrix}
(\alpha - x) & (\beta - e^{iF}) & (\alpha - e^{iF})K & (\beta - e^{iF})K \\
(\alpha - e^{iG})K & (\alpha - e^{iG}) & (\alpha - e^{iG}) & (\alpha - e^{iG}) \\
(\alpha - e^{iH})(k+p) & (\beta - e^{iH})(k+p) & (\beta - e^{iH})(k+p) & (\beta - e^{iH})(k+p) \\
(\alpha - e^{-iF})(k+p)K & (\beta - e^{-iF})(k+p)K & (\beta - e^{-iF})(k+p)K & (\beta - e^{-iF})(k+p)K
\end{vmatrix}$$

(80)

with notation $e^{iF} = x, e^{iG} = y, e^{iH} = z$ it can be presented as

$$\det S = \begin{vmatrix}
(\alpha - x) & (1 - \alpha y)\alpha^{-1} & (\alpha - x)K & (1 - \alpha x)\alpha^{-1}K \\
(\alpha - e^{iF})K & (1 - \alpha y)\alpha^{-1}K & (\alpha - y)K & (1 - \alpha y)\alpha^{-1}K \\
(\alpha - e^{iH})(k+p) & (1 - \alpha z)(k-p) & -(\alpha - z)(k-p)K & (1 - \alpha z)(k-p)K \\
-(1 - \alpha x)x^{-1}(k+p)K & -(\alpha - x)\alpha^{-1}x^{-1}(k+p)K & (1 - \alpha x)x^{-1}(k+p)K & (\alpha - x)\alpha^{-1}x^{-1}(k+p)K
\end{vmatrix}$$

(81)

$$= (1 - \alpha \alpha^{-1})(1 - \alpha y)\alpha^{-1}(1 - \alpha z)\alpha^{-1}(1 - \alpha x)x^{-1}\alpha^{-1}\alpha^2$$

$$\times\begin{vmatrix}
X & 1 & XK & K \\
YK & K & Y & 1 \\
Z(k+p) & (k-p) & -Z(k-p)K & -(k+p)K \\
-(k+p)K & -(X(k-p)K & -(k-p) & X(k+p)
\end{vmatrix};$$

(81)

equation $\det S = 0$ is a bi-quadratic one (let $\Lambda = K^2$):

$$[-(k^2 - p^2)(XY^2 + Z) + ((k-p)^2)ZY + (k+p)^2)X] \Lambda^2$$

$$+[(2(k^2 - p^2)Y - 4Zk^2)X^2 + (2(k^2 - p^2)(Z^2 + 1)X - 4k^2Y + 2(k^2 - p^2)Z] \Lambda$$

$$-(k-p)^2(YX^2 + Z) + ((k+p)^2)ZY + (k-p)^2)X = 0;$$

(82)

the roots are

$$\Lambda_1 = \frac{2k\sqrt{-(Y(Z)(XZ^2 - 1)(XZ - 1)(XY)k^2 - (Z^2 + 1)X(2Z)(X^2 + (Z - Y)X - 1)p^2})}{(Y(k-p)X - k-p)(X(k+p) - Z(k-p))}$$

$$\frac{((Y - 2Z)k^2 - Yp^2)X^2 + (Y + 1)k^2 - p^2)X + (Z^2 - 2Y)k^2 - Zp^2}{(Y(k-p)X - k-p)(X(k+p) - Z(k-p))},$$

$$\Lambda_2 = \frac{2k\sqrt{-(Y(Z)(XZ^2 - 1)(XZ - 1)(X - Y)k^2 - (Y^2 - 2Y)X(2Z)(X^2 + (Z - Y)X - 1)p^2})}{(Y(k-p)X - k-p)(X(k+p) - Z(k-p))}$$

$$\frac{((Y - 2Z)k^2 - Yp^2)X^2 + (Y + 1)k^2 - p^2)X + (Z^2 - 2Y)k^2 - Zp^2}{(Y(k-p)X - k-p)(X(k+p) - Z(k-p))}. $$

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The root $\Lambda_1$ is a phase, $\Lambda_1 \Lambda_1^* = 1$; then with the help of Vieta’s theorem we prove the same for the second root

$$
(\Lambda_1 \Lambda_1^*)(\Lambda_2 \Lambda_2^*) = \frac{c c^*}{a a^*},
$$

$$
\Lambda_2 \Lambda_2^* = \frac{(X(k - p) - Z(k + p))(Y(k + p) - k + p)(\frac{k + p}{X} - k)(\frac{k + p}{Z} - k)}{(Y(k - p)X - k - p)(X(k + p) - Z(k - p))(\frac{k + p}{X} - k - p)(\frac{k + p}{Z} - k - p)} = 0.
$$

7 Analysis of the general case of four phases

Let us turn to equations (14) in general case of all four independent phases:

Below, the shortening notation will be used:

$$
e^{i\rho} = x, \quad (\alpha - e^{i\mu}) = \alpha - x, \quad (\beta - e^{i\nu}) = (1 - \alpha x)/\alpha, \quad X = \frac{\alpha - x}{1 - \alpha x}, \quad X^* = \frac{1}{X},
$$

$$
e^{i\mu} = y, \quad (\alpha - e^{i\mu}) = \alpha - y, \quad (\beta - e^{i\nu}) = (1 - \alpha y)/\alpha, \quad Y = \frac{\alpha - y}{1 - \alpha y}, \quad Y^* = \frac{1}{Y},
$$

$$
e^{i\nu} = v, \quad (\alpha - e^{i\nu}) = \alpha - v, \quad (\beta - e^{i\nu}) = (1 - \alpha v)/\alpha, \quad V = \frac{\alpha - v}{1 - \alpha v}, \quad V^* = \frac{1}{V},
$$

$$
e^{i\nu} = w, \quad (\alpha - e^{i\nu}) = \alpha - w, \quad (\beta - e^{i\nu}) = (1 - \alpha w)/\alpha, \quad W = \frac{\alpha - w}{1 - \alpha w}, \quad W^* = \frac{1}{W}.
$$

Equation $\det S = 0$ can be presented as follows:

$$
\det S = \frac{(1 - \alpha x)(1 - \alpha y)(1 - \alpha v)(1 - \alpha w)}{\alpha \alpha \alpha \alpha} \alpha^2
\times
\begin{vmatrix}
X & 1 & \alpha X K & K \\
Y K & K & \alpha Y & 1 \\
(k + p) V & (k - p) (k - p) V K & -(k + p) K & -(k + p) K \\
(k + p) W K & (k - p) K & -(k - p) W & -(k + p)
\end{vmatrix},
$$

which results in a bi-quadratic equation (let $\Lambda = K^2$)

$$
[(V - X)k - p(V + X)][(W - Y)k + p(W + Y)] \Lambda^2
+ 2\{(2X - W - Y)V - (X - 2Y)W - XYk^2 + p^2(W + Y)(V + X)\} \Lambda
+ [(W - Y)k - p(W + Y)][(V - X)k + p(V + X)] = 0.
$$

The roots are

$$
\Lambda_1 =
$$

$$
2k \sqrt{[(X - Y)W + XY][V - WXY][V + X - Y - W][p^2 - (X - Y)(W - X)(V - Y)(V - W)]^2
+ [(W - 2X + Y)V + (X - 2Y)W + XYk^2 - (W + Y)(V + X)p^2][W - Y)k + (W + Y)p][(V - X)k - (V + X)p]}.
$$
\[ \Lambda_2 = \]
\[ -2k \sqrt{ [((X-Y)W+XY)V-WXY](V+X-Y-W)p^2 - (X-Y)(W-X)(V-Y)(V-W)k^2} \]
\[ \frac{([W-Y]k + (W+Y)p)[(V-X)k - (V+X)p]}{([W-2X+Y)V + (X-2Y)W + XY]k^2 - (W+Y)(V+X)p^2} + \frac{[W-Y]k + (W+Y)p][[(V-X)k - (V+X)p]}{([W-2X+Y)V + (X-2Y)W + XY]k^2 - (W+Y)(V+X)p^2} \].

We readily prove that both complex roots have a unit length:
\[ \Lambda_1 \Lambda_1^* = 1, \quad (\Lambda_1 \Lambda_1^*)(\Lambda_2 \Lambda_2^*) = \frac{c}{a} \frac{c^*}{a^*} \],
\[ \Lambda_2 \Lambda_2^* = \frac{([W-Y]k - p(W+Y)] [(V-X)k + p(V+X)]}{[(V-X)k - p(V+X)] [(W-Y)k + p(W+Y)]} \times \]
\[ \frac{[\frac{1}{(V-X)}k - p(\frac{1}{V} + \frac{1}{X})] [\frac{1}{(V-X)}k + p(\frac{1}{V} + \frac{1}{X})]}{[(V-X)k - p(\frac{1}{V} + \frac{1}{X})] [(W-Y)k + p(W+Y)]} = 1. \quad (87) \]

8 Conclusion

Thus, all solutions of the 4-th order polynomial have been found in explicit form. All of them are complex numbers of unit length, and thereby they are appropriate to produce a quantization rule for \( k \). Some of these rules are simple, and straightforwardly give needed quantization rule of the form \( 2ka = \pi n, \pi + pn, n = 0 \pm 1, \pm 2, \ldots \). The most of produced expression for the roots are complex enough and can be solved with respect to parameter \( k \) only numerically.

Till now, only simplest variants of quantization \( 2ka = \pi n, \pi + pn, n = 0 \pm 1, \pm 2, \ldots \) were examined in the literature [4] in the context of treating the Casimir effect for Dirac particle in external magnetic field.

A last remark should be added: the final algebraic problem in the form of polynomial and corresponding quantization for \( k \) preserve its form when considering the problem of electron in magnetic field on the base of cylindric coordinates as well. Moreover, the same polynomial arises in considering a free electron in the domain between two planes.

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References

[1] H.B.G. Casimir. The Attraction Between Two Perfectly Conducting Plates. Proc. Kon. Ned. Akad. Wetenschap. B. 51, 793–795 (1948).
[2] H.B.G. Casimir. On the Theory of Electromagnetic Waves in Resonant Cavities. Phillips Res. Rep. 6, 162–182 (1951).

[3] I. V.M. Mostenanenko, N.Ya. Trunov. Casimir Effect and its Applications. Uspekhi Fizicheskix nauk. - 1988. 156, 385–426.

[4] V.M. Mostepanenko, N.N. Trunov. The Casimir Effect and its Applications. Clarendon Press, Oxford 1997.

[5] M. Bordag, G.L. Klimchitskaya, U. Mohideen and V.M. Mostepanenko, Advances in the Casimir Effect. Oxford University Press, Oxford, 2009.

[6] K. Johnson. The MIT Bag Model. Acta Phys. Pol. B. 6, 865–892 (1975).

[7] E. Elizalde, A. Romeo. One-dimensional Casimir Effect Perturbed by an External Field. J. Phys. A. 30, 5393–5403 (1997).

[8] M.V. Cougo-Pinto, C. Farina M.R. Negrao, A.C. Tort. Bosonic Casimir Effect in External Magnetic Field. J. Phys. A. 32, 4457–4462 (1999).

[9] C. Farina, A.C. Tort, M.V. Cougo-Pinto. Fermionic Casimir Effect in an External Magnetic Field in: Work Shop on Quantum Field Theory under the Influence of External Conditions, Leipzig, 1998. The Casimir Effect 50 Years Later, World Scientific, Singapore 1999, p. 235–239.

[10] M. Ostrowski. Casimir Effect in External Magnetic Field. Acta Phys. Pol. B. 37, 1753–1767 (2006).

[11] Yu. A. Sitenko and S. A. Yushchenko. The Casimir Effect with Quantized Charged Scalar Matter in Background Magnetic Field. arXiv:1401.6950v1 [hep-th] 27 Jan 2014