On group averaging for non–compact groups

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Abstract

We review some aspects of the use of a technique known as group averaging, which provides a tool for the study of constrained systems. We focus our attention on the case where the gauge group is non–compact, and a ‘renormalized’ group averaging method must be introduced. We discuss the connection between superselection sectors and the rate of divergence of the group averaging integral.

1 Introduction

What follows is primarily based in the article [1] written in collaboration with Donald Marolf. A short review of some previous work has been added in order to make it self–contained. The focus has also been changed, stressing the physical issues, and referring the reader to the original literature for details on the mathematical technology.

Quantization of constrained systems was first studied by Dirac [2](a complete treatment of the subject can be found in Ref. [3]). His method involves introducing the constraints as operators on some space. The physical Hilbert space is then defined to contain only those states annihilated by all the constraints. This last step is the one in which we will concentrate our attention. In particular, we will discuss the conditions under which a physical inner product can be constructed explicitly. There are many ways of implementing the Dirac method. Here we are interested in what is called refined algebraic quantization (RAQ) [4, 5, 6, 7, 8], which is particularly suitable for dealing with canonical quantization within a generally applicable mathematical framework.

RAQ becomes much more powerful when a technique known as group averaging can be applied. Group averaging uses the integral

\[ I = \int_G \langle \phi_1 | U(g) | \phi_2 \rangle \, dg \]  

over the gauge group \( G \) to define the physical Hilbert space. Here \( dg \) is a Haar measure on \( G \). Once a space of states has been found for which this procedure converges, group averaging gives an algorithm for the implementation of RAQ. When group averaging converges sufficiently strongly this algorithm gives the unique implementation of RAQ [9]. However, it will often happen that group averaging

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fails to converge on some interesting domain. As described in [3], the fact that convergent group averaging ensures a unique representation (compatible with RAQ) of the algebra of observables shows that group averaging must in fact diverge in the presence of any superselection rules. However, as was described in [4], one can sometimes construct a renormalized group averaging operation, even when group averaging does not properly converge. Our goal here is to study the possibility of implementing a general method for constructing renormalized group averaging integrals. We will discuss in parallel the particular case fully treated in [1], where the method has been successfully applied to the $SO(n,1)$ group acting on Minkowski space.

We will start in section 2 with a short and simplified review of RAQ. In section 3 we will study the convergency properties of the integrals of the form (1). In section 4 we will discuss the regularization of the divergent group averaging integrals. In section 5 we will show the emergence of superselected sectors, and we will finish in section 6 with conclusions and final comments.

## 2 Refined Algebraic Quantization

The starting point of Refined Algebraic Quantization is an auxiliary Hilbert space $\mathcal{H}_{aux}$ with an inner product $\langle , \rangle$. We also have a set of constraints $C^A = 0$ generating a Lie Group $G$, a unitary representation of it, $U(g)$, acting on $\mathcal{H}_{aux}$ and an algebra of selfadjoint observables $\mathcal{A}$ commuting with all the constraints. The core of the method stands in finding the so called rigging map $\eta$,

$$\eta : \mathcal{H}_{aux} \rightarrow \mathcal{H}_{aux}^{\ast}$$

such that

$$\langle \phi | C^A = 0 .$$

Here $\mathcal{H}_{aux}^{\ast}$ is the dual of $\mathcal{H}_{aux}$. The rigging map $\eta$ defines both the physical Hilbert space $\mathcal{H}_{phy}$ and its inner product. $\mathcal{H}_{phy}$ is simply the image of the rigging map, and

$$\langle \phi_1 | \phi_2 \rangle_{phy} = \eta(\langle \phi_1 | \phi_2 \rangle) .$$

The rigging map must satisfy two requirements: It must be such that (4) is a definite positive inner product and must commute with all the observables.

It has been shown[4] that if the integral $I$ in (1) converges for all $|\phi_1\rangle$ and $|\phi_2\rangle$ then the rigging map is uniquely determined. The inner product (1) is then given precisely by the integral (1). This procedure for implementing the rigging map is known as group averaging.

## 3 On convergence of the group averaging integrals

Let us now focus our attention in the convergence of the integral $I$ in (1).

$$I \leq \int_G |\langle \phi_1 | U(g) | \phi_2 \rangle| \ dg = \int_G dg = V_G ,$$

\[1\]This is an simplification of the actual definition, but is enough for our purpose here. Rigorously speaking, the rigging map goes from a dense subspace $\Phi$ of $\mathcal{H}_{aux}$ to its dual $\mathcal{H}_{aux}^{\ast}$ (see for example Ref. [4]).
and therefore the integral is bounded by the volume of $G$ and it is guaranteed to converge for any compact group. The integral will diverge, in the worst case, as the volume of the group. This will happen for example if we plug in Eq. (1) $|\phi_2\rangle = |\phi_1\rangle = |\psi\rangle$ for some $G$–invariant state $|\psi\rangle$. For non–compact groups there are some particular cases where the group averaging procedure converges (see for example [10]), but generically, convergency will fail. Our goal is to find some regularization procedure in order to define a finite, (renormalized) generalization of $I$.

4 Regularization of the group averaging integrals

The possibility of regularizing the integral $I$ lays in our ability to find a parameter, $\lambda \in [0, \infty]$, such that:

- $G(\lambda)$ is a finite domain on $G$ such that $G(\lambda) \rightarrow G$ when $\lambda \rightarrow \infty$.
- $\langle \phi_1|\phi_2\rangle_{\lambda} = \int_{G(\lambda)} \langle \phi_1|U(g)|\phi_2\rangle \, dg \rightarrow f^i(\phi_1, \phi_2)\Lambda^i(\lambda)$

for big values of $\lambda$,

where $f^i(\phi_1, \phi_2)$ are finite functions of the states and $\Lambda^i(\lambda)$ diverges for $\lambda \rightarrow \infty$. $\Lambda^i(\lambda)$ may depend on the states only through the integer $i$, which labels different degrees of divergency that might occur.

If we can find such parameter we will define:

$$\langle \phi_1|\phi_2\rangle_{phys} = f^i(\phi_1, \phi_2) . \tag{6}$$

We do not know a general method for obtaining $G(\lambda)$. A particular example has been given in [1] for the Lorentz group $SO(n, 1)$. In that case, however, the function $G(\lambda)$ was also a function of the states $|\phi_i\rangle$. This adds more difficulty to our problem, because one has to check explicitly the consistency of the inner product with the requirements of Refined Algebraic Quantization which are automatically satisfied in the present case.

5 The normalized group averaging integral and superselected sectors

Let us now study the consequences of the above definition of the rigging map. In general, $U(g)$ does not have to be an irreducible representation. For instance, if we take $SO(n, 1)$ acting on an auxiliary Hilbert space defined by the set of functions with compact support on Minkowski space $\mathbb{R}^{2+1}$, any hyperboloid $x_1^2 + \cdots x_n^2 - t^2 = \text{constant}$ is an invariant subspace where the action of the group is irreducible. The auxiliary Hilbert space can be written as a direct sum:

$$\mathcal{H}_{aux} = \bigoplus_a \mathcal{H}_a \tag{7}$$

Now we will show that for two states $|\chi_1\rangle$ and $|\chi_2\rangle$ belonging to the same irreducible subspace, the degree of divergency of $\langle \chi_1|\chi_1\rangle_{\lambda}$ and $\langle \chi_2|\chi_2\rangle_{\lambda}$ is the same. In fact, as there exist some $g_0$ such that $|\chi_2\rangle = U(g_0)|\chi_1\rangle$,

$$\langle \chi_2|\chi_2\rangle_{\lambda} = \int_{G_\lambda} \langle \chi_1|U(g_0)^{\dagger}U(g)U(g_0)|\chi_1\rangle \, dg ,$$
and because we can go continuously from the identity to $g_0$ for any $\lambda$, the function $\Lambda(\lambda)$ cannot have a discontinuity from some $\Lambda^i(\lambda)$ to another function $\Lambda^j(\lambda)$ with different asymptotic behavior. By a similar argument it is straightforward to note that given any two states in some $H_a$, their inner product $\langle \chi_i|\chi_j \rangle_\lambda$ is either zero or shows the same degree of divergency. Now we rewrite the auxiliary Hilbert space as a direct sum

$$H_{\text{aux}} = \bigoplus_i H_i,$$

where we have just grouped together the $H_a$ showing the same degree of divergency in $\lambda$. For the $SO(n,1)$ case, the Hilbert space is divided in two sectors, defined by functions supported inside or outside the light cone $[1]$. The group average integrals converge inside the light cone. Outside, the integrals diverge as $\lambda^{n-2}$ for $n > 2$ and as $\log \lambda$ for $n = 2$. For $\lambda = 1$ they converge. The physical inner product is given by

$$\langle x|y \rangle_{\text{phy}} = \frac{1}{(x^2)^{(n-1)/2}} \delta(x^2 - y^2),$$

where $x^2 = x_1^2 + \cdots + x_n^2 - t^2$, and the states $|x\rangle$ form a distributional basis of the auxiliary Hilbert space (localized delta functions), with $\langle x|y \rangle = \delta(x - y)$.

We now claim that Eq. (6) also defines a direct product with the physical, normalized, inner product. Moreover, we will show that each $H_i$ defines a superselected sector of the theory, that is, for any two states $|\phi_i\rangle$, $|\phi_j\rangle$ belonging to different sectors $H_i$ and $H_j$ respectively, and for any observable $\hat{O}$ in $A$,

$$\langle \phi_i|\hat{O}|\phi_j \rangle_{\text{phy}} = 0.$$

In fact,

$$\lim_{\lambda \to \infty} \int_{G_{\lambda}} \langle \phi_i|\hat{O}U(g)|\phi_j \rangle \, dg = \langle \phi_i|\hat{O}|\phi_j \rangle_{\text{phy}} \Lambda^k(\lambda),$$

for some $k$. Now let us decompose

$$\hat{O}|\phi_i\rangle = \alpha|\phi'_i\rangle + \beta|\phi'_j\rangle + |\psi\rangle$$

$$\hat{O}|\phi_j\rangle = a|\tilde{\phi}_i\rangle + b|\tilde{\phi}_j\rangle + |\tilde{\psi}\rangle,$$

where the indices $i,j$ indicate where those states live, $(a,b,\alpha,\beta)$ are complex numbers and $|\psi\rangle$, $|\tilde{\psi}\rangle$ live in the orthogonal complement of $H_i \oplus H_j$. As $\hat{O}$ is selfadjoint in $H_{\text{aux}}$ we have

$$\langle \phi_i|\hat{O}|\phi_j \rangle_\lambda = \beta^*\langle \phi'_j|\phi_j \rangle_\lambda$$

$$= a \langle \phi_i|\phi_i \rangle_\lambda.$$

Therefore we see that the right hand side expressions must vanish in the $\lambda \to \infty$ limit. Otherwise, we will get a contradiction, for by definition they have different degree of divergency.

Finally, note that the inner product defined in (6) is not unique if there are superselected sectors. There is always an undetermined constant factor when one drops out the infinite factors $\Lambda^i$. When only one sector is present, this factor is given by the normalization of the states, but otherwise, we can always define different weights in different sectors, giving us a freedom in the election of the inner product.
6 Conclusions

We have seen that in certain circumstances it is possible to generalize the group averaging method in order to extend its applicability to non-compact groups. This is done by regularizing the integrals and dropping out the infinite factors that might occur. When the auxiliary Hilbert space contains different sectors for which the integrals have different degrees of divergency, the physical Hilbert space will be a direct sum of these sectors:

$$\mathcal{H}_{\text{phy}} = \bigoplus_i \mathcal{H}^i_{\text{phy}},$$

with the inner product for each sector defined in (6). Each $\mathcal{H}^i_{\text{phy}}$ is superselected (no observable has matrix element between their elements). Because of this, the inner product is defined inside each sector independently and we can multiply it by different weights, giving us a freedom in the definition of the inner product for the whole physical Hilbert space.

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References

[1] A. Gomberoff and D. Marolf, Int. J. Mod. Phys. D8, 519 (1999), [gr-qc/9902069].

[2] Dirac, P A M Lectures on Quantum Mechanics (New York: Belfer Graduate School of Science, Yeshiva University, 1964).

[3] M. Henneaux and C. Teitelboim Quantization of Gauge Systems (Princeton University Press, Princeton, 1992).

[4] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, J. Math. Phys. 36, 6456 (1995), [gr-qc/9504018].

[5] D. Giulini and D. Marolf, Class. Quant. Grav. 16, 2479 (1999), [gr-qc/9812024].

[6] N. Landsman, J. Geom. Phys. 15 (1995) 285-319, [hep-th/9305088].

[7] A. Higuchi, Class. Quant. Grav. 8 (1991) 1983.

[8] D. Marolf, Class. Quant. Grav. 12 (1995) 1199, [gr-qc/9404053].

[9] D. Giulini and D. Marolf, Class. Quant. Grav. 16 (1999) 2489, [gr-qc/9902045].

[10] D. Marolf, in Symplectic Singularities and Geometry of Gauge Fields, (Banach Center Publications, Polish Academy of Sciences, Institute of Mathematics Warsaw, 1997); [gr-qc/9508015].