ASYMPTOTIC DIRICHLET PROBLEM FOR $\mathcal{A}$-HARMONIC AND
MINIMAL GRAPH EQUATIONS IN CARTAN-HADAMARD
MANIFOLDS

JEAN-BAPTISTE CASTERAS, ILKKA HOLOPAINEN, AND JAIME B. RIPOLL

Abstract. We study the asymptotic Dirichlet problem for $\mathcal{A}$-harmonic equations and for the minimal graph equation on a Cartan-Hadamard manifold $M$ whose sectional curvatures are bounded from below and above by certain func-
tions depending on the distance to a fixed point $o \in M$. We are, in particular,
interested in finding optimal (or close to optimal) curvature upper bounds.

1. Introduction

In this paper, we are interested in the asymptotic Dirichlet problem for $\mathcal{A}$-
harmonic functions and for the minimal graph equation on a Cartan-Hadamard
manifold $M$ of dimension $n \geq 2$. We first recall that a Cartan-Hadamard manifold
is a simply connected, complete Riemannian manifold having nonpositive sectional
curvature. It is well-known, since the exponential map $\exp_o : T_o M \to M$ is a diffeo-
morphism for every point $o \in M$, that $M$ is diffeomorphic to $\mathbb{R}^n$. One can define
an asymptotic boundary $\partial_\infty M$ of $M$ as the set of all equivalence classes of unit
speed geodesic rays on $M$ (see Section 2.1 for more details). The so-called geo-
metric compactification $\bar{M}$ of $M$ is then given by $\bar{M} = M \cup \partial_\infty M$ equipped with
the cone topology. We also notice that $\bar{M}$ is homeomorphic to a closed Euclidean
ball; see [19]. The asymptotic Dirichlet problem on $M$ for some operator $Q$ is then
the following: Given a continuous function $f$ on $\partial_\infty M$ does there exist a (unique)
function $u \in C(\bar{M})$ such that $Q[u] = 0$ on $M$ and $u|_{\partial_\infty M} = f$? We will consider
this problem for two kinds of operators: the minimal graph operator (or the mean
curvature operator) $M$ defined by

$$M[u] = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},$$

and the $\mathcal{A}$-harmonic operator (of type $p$)

$$Q[u] = -\text{div}\mathcal{A}_x(\nabla u),$$

where $\mathcal{A} : TM \to TM$ is subject to certain conditions; for instance $\langle \mathcal{A}(V), V \rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. The $p$-Laplacian
is an example of an $\mathcal{A}$-harmonic operator (see Section 2.3 for the precise definition).
We also note that $u$ satisfies $M[u] = 0$ if and only if $G := \{(x, u(x)) | x \in \Omega\}$ is a
minimal hypersurface in the product space $M \times \mathbb{R}$.

2000 Mathematics Subject Classification. 58J32, 53C21, 31C45.
Key words and phrases. minimal graph equation, Dirichlet problem, Hadamard manifold.
J.-B.C. supported by the CNPq (Brazil) project 501559/2012-4; I.H. supported by the Academy
of Finland, project 252293; J.R. supported by the CNPq (Brazil) project 302955/2011-9.
We will now give a brief overview of the results known for the asymptotic Dirichlet problem on Cartan-Hadamard manifolds. The first result for this problem in the case of the usual Laplace-Beltrami operator was obtained by Choi. In [11], he solved the asymptotic Dirichlet problem assuming that the sectional curvatures satisfy $K \leq -a^2 < 0$ and that $M$ satisfies a “convex conic neighborhood condition”, i.e. given $x \in \partial_{\infty} M$, for any $y \in \partial_{\infty} M$, $y \neq x$, there exist $V_x \subset M$, a neighborhood of $x$, and $V_y \subset M$, a neighborhood of $y$ such that $V_x$ and $V_y$ are disjoint open sets of $\bar{M}$ in terms of the cone topology and $V_x \cap M$ is convex with $C^2$ boundary.

Anderson [5] proved that the convex conic neighborhood condition is satisfied for manifolds of pinched sectional curvature $-b^2 \leq K \leq -a^2 < 0$ and therefore he was able to solve the asymptotic Dirichlet problem for the Laplace-Beltrami operator (see also [6] for a different approach). We point out that the asymptotic Dirichlet problem was solved independently by Sullivan [40] using probabilistic arguments. Ancona in a series of papers [1], [2], [3], and [4], was able to replace the curvature lower bound by a bounded geometry assumption that each ball up to a fixed radius is $L$-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$ for some fixed $L \geq 1$; see [1].

Finally, we give the following theorem by Hsu where the most general curvature bounds under which the asymptotic Dirichlet problem for the Laplacian is solvable are given. Here and throughout the paper $r(x)$ stands for the distance between $x \in M$ and a fixed point $o \in M$.

**Theorem 1.** [28, Theorems 1.1 and 1.2] Let $M$ be a Cartan-Hadamard manifold. Suppose that:
- there exist a positive constant $a$ and a positive and non-increasing function $h$ with $\int_0^{\infty} th(t) \, dt < \infty$ such that
  
  $$-h(r(x))^2 e^{2ar(x)} \leq \text{Ric}_x \quad \text{and} \quad K_x \leq -a^2,$$

or
- there exist positive constants $r_0$, $\alpha > 2$, and $\beta < \alpha - 2$ such that
  
  $$-r(x)^{2\beta} \leq \text{Ric}_x \quad \text{and} \quad K_x \leq \frac{\alpha(\alpha - 1)}{r(x)^2}$$

for all $x \in M$, with $r(x) \geq r_0$. Then the Dirichlet problem at infinity for the Laplacian is solvable.

The asymptotic Dirichlet problem has been studied for more general operators than the Laplacian. The first result in this direction has been obtained in [25] for the $p$-Laplacian under a pinched negative sectional curvature assumption by modifying the direct approach of Anderson and Schoen [6]. In [27] Holopainen and Väihäkangas have been able to relax the assumption on the curvature (see Theorem 3 for a more precise statement of these curvature assumptions). Of particular interest is the case of the minimal graph operator. In [12], Collin and Rosenberg were able to construct harmonic diffeomorphisms from the complex plane $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$ disproving this way a conjecture of Schoen and Yau [37]. This result has been generalized by Gálvez and Rosenberg [20] to any Hadamard surface $M$ whose curvature is bounded from above by a negative constant. A fundamental ingredient in their constructions is to solve the Dirichlet problem on unbounded ideal polygons with boundary values $\pm \infty$ on the sides of the ideal polygons. These unexpected results have raised interest in (entire) minimal hypersurfaces in the product space
Let $M \times \mathbb{R}$, where $M$ is a Cartan-Hadamard manifold (see for example, [15], [18], [30], [32], [35], [36], [39]). Very recently in [8], the authors generalized (most of) the solvability results to a larger class of operators $Q$ of the form

$$Q[u] = \text{div}(P(|\nabla u|^2)\nabla u),$$

with $P$ subject to the following growth conditions. Let $P: (0, \infty) \rightarrow [0, \infty)$ be a smooth function such that

$$P(t) \leq P_0 t^{(p-2)/2}$$

for all $t > 0$, with some constants $P_0 > 0$ and $p \geq 1$, and that $B := \mathcal{P}/P$ satisfies

$$-\frac{1}{2t} < B(t) \leq \frac{B_0}{t}$$

for all $t > 0$ with some constant $B_0 > -1/2$. Furthermore, assume that $tP(t^2) \rightarrow 0$ as $t \rightarrow 0+$ and define $P(|X|^2)X = 0$ whenever $X$ is a zero vector.

Following [8] we call a relatively compact open set $\Omega \Subset M$ Q-regular if for any continuous boundary data $h \in C(\partial \Omega)$ there exists a unique $u \in C(\Omega)$ which is $Q$-solution in $\Omega$ and $u|\partial \Omega = h$. In addition to the growth conditions on $P$, assume that

(A) there is an exhaustion of $M$ by an increasing sequence of $Q$-regular domains $\Omega_k$, and that

(B) locally uniformly bounded sequences of continuous $Q$-solutions are compact in relatively compact subsets of $M$.

It is well-known that the conditions above are satisfied by the minimal graph operator and the $p$-Laplacian (see [16], [22] and [39]).

The main theorem in [8] is a solvability result for the asymptotic Dirichlet problem for operators $Q$ that satisfy (1.4), (1.5), and conditions (A) and (B) under curvature assumption

$$-b(r(x))^2 \leq K(P) \leq -a(r(x))^2$$

on $M$, where $P \subset T_x M$ is a 2-plane and $a, b: [0, \infty) \rightarrow [0, \infty)$, $b \geq a$, are smooth functions satisfying suitable assumptions. Here, instead of giving the precise assumptions on functions $a$ and $b$, we state the following two solvability results as special cases of their main theorem (Theorem 1.6 in [8]).

**Theorem 2.** [8] Theorem 1.5] Let $M$ be a Cartan-Hadamard manifold of dimension $n \geq 2$. Fix $o \in M$ and set $r(\cdot) = d(o, \cdot)$, where $d$ is the Riemannian distance in $M$. Assume that

$$-r(x)^{2(\phi-2)-\varepsilon} \leq \text{Sect}_x(P) \leq -\frac{\phi(\phi-1)}{r(x)^2},$$

for some constants $\phi > 1$ and $\varepsilon > 0$, where $\text{Sect}_x(P)$ is the sectional curvature of a plane $P \subset T_x M$ and $x$ is any point in the complement of a ball $B(o, R_0)$. Then the asymptotic Dirichlet problem for the minimal graph equation (1.1) is uniquely solvable for any boundary data $f \in C(M(\infty))$.

**Theorem 3.** [8] Corollary 1.7] Let $M$ be a Cartan-Hadamard manifold of dimension $n \geq 2$. Fix $o \in M$ and set $r(\cdot) = d(o, \cdot)$, where $d$ is the Riemannian distance in $M$. Assume that

$$-r(x)^{2-\varepsilon}e^{2kr(x)} \leq \text{Sect}_x(P) \leq -k^2$$

for some constants $\varepsilon > 0$ and $k > 0$. The solvability conditions for the minimal graph equation (1.1) is then satisfied by any boundary data $f \in C(M(\infty))$. In addition to these assumptions, assume that $r(x)^2 \leq C_0$ for all $x \in M$, where $C_0$ is a constant that only depends on $M$. Then the minimal graph equation (1.1) is uniquely solvable for any boundary data $f \in C(M(\infty))$.
for some constants \( k > 0 \) and \( \varepsilon > 0 \) and for all \( x \in M \setminus B(o, R_0) \). Then the asymptotic Dirichlet problem for the operator \( Q \) (defined as in (1.3)) is uniquely solvable for any boundary data \( f \in C(M(\infty)) \).

The Dirichlet problem at infinity for \( A \)-harmonic function has been considered in [41] and [42]. In [42], Vähätängas was able to generalize the result obtained in [27] (for the \( p \)-Laplacian) to the \( A \)-harmonic case. In [41], by generalizing a method due to Cheng [10], he solved the asymptotic Dirichlet problem for \( A \)-harmonic equations of type \( p \) provided the radial sectional outside a compact set satisfy

\[
K(P) \leq -\frac{\phi(\phi - 1)}{r^2(x)}
\]

for some constant \( \phi > 1 \) with \( 1 < p < 1 + \phi(n - 1) \) and

\[
|K(P)| \leq C|K(P')|
\]

for some constant \( C \), where \( P \) and \( P' \) are any 2-dimensional subspaces of \( T_x M \) containing the (radial) vector \( \nabla r(x) \). It is worth observing that no curvature lower bounds are needed here.

The goal of this paper is threefold. First of all, we are looking for an optimal (or at least close to optimal) curvature upper bound under which asymptotic Dirichlet problems for equations (1.1) and (1.2) are solvable provided an appropriate curvature lower bound holds. Secondly, we are using PDE methods, like Caccioppoli-type inequalities (Lemma 18), Moser iteration scheme (Lemma 20), and Young complementary functions to study the minimal graph equation. As far as we know such methods are not frequently used in the context of the minimal graph equation. Last but not least, we want to publicize the results and methods of the still unpublished preprint [42] of Vähätängas. Our main results are the following two theorems.

**Theorem 4.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \). Assume that

\[
-(\log r(x))^2 \leq K(P) \leq -\frac{1 + \varepsilon}{r(x)^2 \log r(x)},
\]

for some constants \( \varepsilon > \bar{\varepsilon} > 0 \), where \( K(P) \) is the sectional curvature of any plane \( P \subset T_x M \) that contains the radial vector \( \nabla r(x) \) and \( x \) is any point in the complement of a ball \( B(o, R_0) \). Then the asymptotic Dirichlet problem for the \( A \)-harmonic equation (1.2) is uniquely solvable for any boundary data \( f \in C(\partial \infty M) \) provided that \( 1 \leq p < n\alpha/\beta \).

**Theorem 5.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 3 \) satisfying the curvature assumption (1.7) for all 2-planes \( P \subset T_x M \), with \( x \in M \setminus B(o, R_0) \). Then the asymptotic Dirichlet problem for the minimal graph equation (1.1) is uniquely solvable for any boundary data \( f \in C(\partial \infty M) \).

In Theorem 4 above \( \alpha \) and \( \beta \) are so-called structural constants of the operator \( A \); see [2,3] for details. We notice that the Laplace-Beltrami operator corresponds to the case \( p = 2 \) and \( \alpha = \beta = 1 \), and therefore is covered by Theorem 4 in dimensions \( n > 2 \). Thus we obtain a generalization to higher dimensions of a recent result by Neel [31]. The curvature upper bound (1.7) appears also in a recent paper [34] where Ripoll and Telichevesky solved the asymptotic Dirichlet problem for the minimal graph equation on rotationally symmetric Hadamard surfaces. Notice
dimension $n = 2$ is excluded in Theorem 5. However, we believe that the result holds also in the 2-dimensional setting.

We point out that our curvature assumptions are in a sense optimal. Assume that

$$K(P_x) \geq -\frac{1}{r(x)^2 \log r(x)}$$

and let us consider first the case of an $A$-harmonic operator of type $p \geq n$. The standard Bishop-Gromov volume comparison theorem gives

$$\text{Vol}(\partial B_\rho) \leq C(\rho \log \rho)^{n-1}$$

for some constant $C$ and for all $\rho \geq r_0$ large enough. It is then easy to see that

$$\int_{r_0}^{\infty} \frac{d\rho}{(\text{Vol}(\partial B_\rho))^{1/(p-1)}} \geq C \int_{r_0}^{\infty} \frac{d\rho}{\rho \log \rho} = \infty$$

which implies that $M$ is so-called $p$-parabolic and hence every bounded $A$-harmonic function (with $A$ of type $p$) is constant; see e.g. [24] and [13]. On the other hand, in [33] Rigoli and Setti proved the following nonexistence theorem:

**Theorem 6.** Let $M$ be a complete manifold and $u \in C^1(M)$ be a solution of

$$\text{div} \frac{\varphi(|\nabla u|) \nabla u}{|\nabla u|} = 0,$$

where $\varphi \in C^1((0, \infty)) \cap C^0([0, \infty))$ satisfies the following conditions:

1. $\varphi(0) = 0$,
2. $\varphi(t) > 0$, for all $t \geq 0$,
3. $\varphi(t) \leq At^\delta$, for all $t \geq 0$,

for some positive constants $A$ and $\delta$. Assume that

$$(\text{Vol}(\partial B_\rho)^{1/\delta})^{-1} \notin L^1(\infty),$$

then $M$ is $\varphi$-parabolic i.e. $u$ is constant.

Using this theorem, we also see that the curvature upper bound would be sharp for the minimal graph equation in dimension $n = 2$. Notice that $\delta = 1$ for the minimal graph equation. We close this introduction with some comments on the necessity of curvature lower bounds. Indeed, Ancona’s and Borbély’s examples ([4], [7]) show that a (strictly) negative curvature upper bound alone is not sufficient for the solvability of the asymptotic Dirichlet problem for the Laplace equation. In [23], Holopainen generalized Borbély’s result to the $p$-Laplace equation, and very recently, Holopainen and Ripoll [26] extended these nonsolvability results to the operator $Q$ (as defined in (1.3)), in particular, to the minimal graph equation.

The plan of the paper is the following: Section 2 is devoted to preliminaries. We recall some well-known facts on Cartan-Hadamard manifolds, Jacobi equations, $A$-harmonic functions, the minimal graph equation, and Young functions. In Section 3 we prove Theorem 4. We adopt the same strategy as the one used in [42]. It is based on a Moser iteration procedure involving a weighted Poincaré inequality. Finally, in Section 4 we prove Theorem 5 adapting to the minimal graph equation the method used in Section 3 for $A$-harmonic functions. In this case since this equation does not satisfy (2.2), some extra difficulties appear.

**Acknowledgement.** We would like to thank Joel Spruck for his help to obtain the decay estimate for $|\nabla \log W|$ in Lemma 22.
2. Preliminaries

2.1. Cartan-Hadamard manifolds. We recall that Cartan-Hadamard manifolds are complete simply connected Riemannian n-manifolds, $n \geq 2$, with nonpositive sectional curvature. Let $M$ be a Cartan-Hadamard manifold, $\partial_\infty M$ the sphere at infinity, and $\bar{M} = M \cup \partial_\infty M$. Recall that the sphere at infinity is defined as the set of all equivalence classes of unit speed geodesic rays in $M$; two such rays $\gamma_1$ and $\gamma_2$ are equivalent if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$. For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^x:y: \mathbb{R} \rightarrow M$ such that $\gamma^x_0 = x$ and $\gamma^x_t = y$ for some $t \in (0, \infty]$. If $v \in T_x M \setminus \{0\}$, $\alpha > 0$, and $R > 0$, we define a cone

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\}: \angle(v, \dot{\gamma}^x_0) < \alpha\}$$

and a truncated cone

$$T(v, \alpha, R) = C(v, \alpha) \setminus \bar{B}(x, R),$$

where $\angle(v, \dot{\gamma}^x_0)$ is the angle between vectors $v$ and $\dot{\gamma}^x_0$ in $T_x M$. All cones and open balls in $\bar{M}$ form a basis for the cone topology on $\bar{M}$.

2.2. Jacobi equation. We use the curvature upper bound in order to prove a weighted Poincaré inequality and to estimate from above the norm of the gradient to a 1-dimensional Jacobi equation. If $k: [0, \infty) \rightarrow [0, \infty)$ is a smooth function, we denote by $f_k \in C^\infty([0, \infty))$ the solution to the initial value problem

$$\begin{cases}
    f_k(0) = 0, \\
    f'_k(0) = 1, \\
    f''_k = k^2 f_k.
\end{cases} \tag{2.1}$$

It follows that the solution $f_k$ is a nonnegative smooth function. Concerning the curvature upper bound in \cite{[11], Prop. 3.4} we have the following estimates:

**Proposition 7.** \cite{[11]} Prop. 3.4] Suppose that $f: [R_0, \infty) \rightarrow \mathbb{R}$, $R_0 > 0$, is a positive strictly increasing function satisfying the equation $f''(r) = a^2(r)f(r)$, where

$$a^2(r) \geq \frac{1 + \varepsilon}{r^2 \log r},$$

for some $\varepsilon > 0$ on $[R_0, \infty)$. Then, for any $0 < \bar{\varepsilon} < \varepsilon$, there exists $R_1 \geq R_0$ such that, for all $r \geq R_1$,

$$f(r) \geq r(\log r)^{1 + \bar{\varepsilon}}, \quad \frac{f''(r)}{f(r)} \geq \frac{1}{r} + \frac{1 + \bar{\varepsilon}}{r \log r}.$$  

2.3. $\mathcal{A}$-harmonic functions and Perron’s method. In this section we define $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions and record their basic properties that will be relevant in the sequel. We refer to \cite{[22]} for the proofs and for the nonlinear potential theory of $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions.

Let $\Omega$ be an open subset of a Riemannian manifold $M$. Suppose that for a.e. $x \in \Omega$ we are given a continuous map $\mathcal{A}_x: T_x M \rightarrow T_y M$ such that the map $x \mapsto \mathcal{A}_x(X_x)$ is a measurable vector field whenever $X$ is. We assume further that there
are constants $1 < p < \infty$ and $0 < \alpha \leq \beta < \infty$ such that for a.e. $x \in \Omega$, for all $v, w \in T_x M$, $v \neq w$, and for all $\lambda \in \mathbb{R} \setminus \{0\}$ we have

$$
\begin{align*}
\langle A_x(v), v \rangle &\geq \alpha |v|^p; \\
|A_x(v)| &\leq \beta |v|^{p-1}; \\
\langle A_x(v) - A_x(w), v - w \rangle &> 0; \\
A_x(Av) &= \lambda |\lambda|^{p-2}A_x(v).
\end{align*}
$$

We denote the set of such operators by $\mathcal{A}^p(\Omega)$ and we say that $A$ is of type $p$. The constants $\alpha$ and $\beta$ are called the structure constants of $A$.

A function $u \in W^{1,p}_0(\Omega)$ is called a (weak) solution of the equation

$$
- \text{div} A_x(\nabla u) = 0
$$

in $\Omega$ if

$$
\int_\Omega \langle A_x(\nabla u), \nabla \varphi \rangle = 0
$$

for all $\varphi \in C_0^\infty(\Omega)$. If $|\nabla u| \in L^p(\Omega)$, it is equivalent to require (2.4) for all $\varphi \in W^{1,p}_0(\Omega)$ by approximation. Continuous solutions of (2.3) are called $A$-harmonic functions (of type $p$). By the fundamental work of Serrin [38], every solution of (2.3) has a continuous representative. In the special case $A_\alpha(h) = |h|^{p-2}h$, $A$-harmonic functions are called $p$-harmonic and, in particular, if $p = 2$, we obtain the usual harmonic functions.

A function $u \in W^{1,p}_0(\Omega)$ is a subsolution of (2.3) in $\Omega$ if

$$
- \text{div} A_x(\nabla u) \leq 0
$$

weakly in $\Omega$, that is

$$
\int_\Omega \langle A_x(\nabla u), \nabla \varphi \rangle \leq 0
$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$. A function $v$ is called supersolution of (2.3) if $-v$ is a subsolution. Finally, a lower semicontinuous function $u: \Omega \to (-\infty, +\infty]$ that is not identically $+\infty$ in any component of $\Omega$ is called $A$-superharmonic if for every open $D \supseteq \Omega$ and for every $h \in C(\overline{D})$ that is $A$-harmonic in $D$, $h \leq u$ on $\partial D$ implies $h \leq u$ in $\overline{D}$.

A fundamental feature of (sub/super)solutions of (2.3) is the following well-known comparison principle: If $u \in W^{1,p}(\Omega)$ is a supersolution and $v \in W^{1,p}$ a subsolution of (2.3) in $\Omega$ such that $\max(v - u, 0) \in W^{1,p}_0(\Omega)$, then $u \geq v$ a.e. in $\Omega$. The existence of $A$-harmonic functions is given by the following result. Suppose that $\Omega \subseteq M$ is a relatively compact (nonempty) open set and that $\theta \in W^{1,p}(\Omega)$. Then there exists a unique $A$-harmonic function $u$ in $\Omega$, with $u - \theta \in W^{1,p}_0(\Omega)$.

Given a function $f \in C(\partial_\infty M)$ the Dirichlet problem at infinity for $A$-harmonic functions consists in finding a function $u \in C(M)$ such that $A(u) = 0$ in $M$ and $u|_{\partial_\infty M} = f$. In order to solve the Dirichlet problem for the $A$-harmonic functions, we will use Perron’s method. Let $A \in \mathcal{A}^p(M)$, with $p \in (1, \infty)$. We begin by recalling the definition of the upper class of a function $f \in \partial_\infty M$.

**Definition 8.** A function $u: M \to (-\infty, \infty]$ belongs to the upper class $\mathcal{U}_f$ of $f: \partial_\infty M \to [-\infty, \infty]$ if

1. $u$ is $A$-superharmonic in $M$,
2. $u$ is bounded from below and,
(3) \( \liminf_{x \to x_0} u(x) \geq f(x_0) \), for all \( x_0 \in \partial_\infty M \).

The function

\[
\mathcal{H}_f = \inf \{ u : u \in U_f \}
\]

is called the upper Perron solution and \( \mathcal{H}_f = -\mathcal{H}_{-f} \) the lower Perron solution.

**Theorem 9.** One of the following is true:

1. \( \mathcal{H}_f \) is \( A \)-harmonic in \( M \),
2. \( \mathcal{H}_f \equiv \infty \) in \( M \),
3. \( \mathcal{H}_f \equiv -\infty \) in \( M \).

Next we define \( A \)-regular points at infinity.

**Definition 10.** A point \( x_0 \in \partial_\infty M \) is called \( A \)-regular if

\[
\lim_{x \to x_0} \mathcal{H}_f(x) = f(x_0)
\]

for all \( f \in C(\partial_\infty M) \).

It is easy to see that the Dirichlet problem at infinity for \( A \)-harmonic functions is uniquely solvable if every point at infinity is \( A \)-regular.

### 2.4. Minimal graph equation.

Let \( \Omega \subset M \) be an open set. We say that a function \( u \in W^{1,1}_{\text{loc}}(\Omega) \) is a (weak) solution of the minimal graph equation \((1.1)\) if

\[
(2.5) \quad \int_{\Omega} \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}} = 0
\]

for every \( \varphi \in C_0^\infty(\Omega) \). Note that the integral above is well-defined since

\[
\sqrt{1 + |\nabla u|^2} \geq |\nabla u| \quad \text{a.e.,}
\]

and therefore

\[
\int_{\Omega} \left| \langle \nabla u, \nabla \varphi \rangle \right| \leq \int_{\Omega} \frac{|\nabla u| |\nabla \varphi|}{\sqrt{1 + |\nabla u|^2}} \leq \int_{\Omega} |\nabla \varphi| < \infty.
\]

In fact, it is equivalent to require \((2.5)\) for every \( \varphi \in \dot{W}^{1,1}_0(\Omega) \). Indeed, let \( \varphi \in \dot{W}^{1,1}_0(\Omega) \) and let \( (\varphi_j) \) be a sequence in \( C_0^\infty(\Omega) \) such that \( \nabla \varphi_j \to \nabla \varphi \) in \( L^1(\Omega) \).

Supposing that \((2.5)\) holds for all such \( \varphi_j \), we get

\[
\left| \int_{\Omega} \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}} \right| = \int_{\Omega} \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}} - \int_{\Omega} \frac{\langle \nabla u, \nabla \varphi_j \rangle}{\sqrt{1 + |\nabla u|^2}} \leq \int_{\Omega} \frac{\langle \nabla u, \nabla \varphi - \nabla \varphi_j \rangle}{\sqrt{1 + |\nabla u|^2}} \leq \int_{\Omega} \frac{\sqrt{1 + |\nabla u|^2}}{\sqrt{1 + |\nabla u|^2}} \leq \int_{\Omega} |\nabla \varphi - \nabla \varphi_j| \to 0
\]

as \( j \to 0 \). The following lemma guarantees the existence of (strong) solutions of \((1.1)\) with given boundary values.

**Lemma 11.** Suppose that \( \Omega \Subset M \) is a smooth relatively compact open set whose boundary has nonnegative mean curvature with respect to inwards pointing unit normal field. Then for each \( f \in C^{2,\alpha}(\Omega) \) there exists a unique \( u \in C^\infty(\Omega) \cap C^{2,\alpha}(\bar{\Omega}) \) that solves the minimal graph equation \((1.1)\) in \( \Omega \) with boundary values \( u|\partial\Omega = f|\partial\Omega \).
Proof. This lemma follows from well-known techniques used in the continuity method of elliptic PDE theory and therefore we just sketch the argument. Set

\[ V = \{ t \in [0,1] : \exists u \in C^{2,\alpha}(\bar{\Omega}) \text{ such that } \mathcal{M}[u] = 0 \text{ in } \Omega \text{ and } u|\partial\Omega = tf|\partial\Omega \}. \]

We have \( V \neq \emptyset \) since \( 0 \in V \). Moreover, by the implicit function theorem, \( V \) is open in \( [0,1] \). Given \( t \in V \), let \( u \) be a solution of (1.1) such that \( u|\partial\Omega = tf|\partial\Omega \). Since constant functions are solutions of (1.1), we have \( \sup_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |f| \) by the comparison principle (see e.g. [8, Lemma 1]. Also, since \( \partial\Omega \) has nonnegative mean curvature with respect to inwards pointing unit normal field, we may use classical logarithmic type barriers to prove that \( \max_{\partial\Omega} |\nabla u| \leq C \) where \( C \) is a constant that depends only on \( f \) and \( \Omega \) (see e.g. [15, Section 4] for details). By [36, Lemma 3.1] we have \( \max_{\bar{\Omega}} |\nabla u| \leq C \) for some constant independent of \( u \) and \( t \). Hölder estimates and theory of linear elliptic PDEs imply that the \( C^{2,\beta} \) norm of \( u \) is bounded by a constant depending only on \( f \) and \( \Omega \) for some \( 0 < \beta < 1 \).

Then, if \( t_n \in V \) converges to \( t \in [0,1] \) and \( u_n \) is a solution of (1.1) such that \( u_n|\partial\Omega = t_nf|\partial\Omega \), then \( (u_n) \) contains a subsequence converging in the \( C^{2} \) norm on \( \bar{\Omega} \) to a solution \( u \in C^{2}(\Omega) \) of (1.1) in \( \Omega \) such that \( u|\partial\Omega = tf|\partial\Omega \). Regularity theory implies that \( u \in C^{\infty}(\Omega) \cap C^{2,\alpha}(\bar{\Omega}) \). It follows that \( t \in V \), so that \( V \) is closed and hence \( V = [0,1] \). □

From now on we will mainly consider solutions of (1.1) that are at least \( C^{2} \)-smooth.

2.5. Young functions. Let \( \phi : [0,\infty) \to [0,\infty) \) be a homeomorphism and let \( \psi = \phi^{-1} \). Define Young functions \( \Phi \) and \( \Psi \) by setting, for each \( t \in [0,\infty) \),

\[ \Phi(t) = \int_{0}^{t} \phi(s)ds \]

and

\[ \Psi(t) = \int_{0}^{t} \psi(s)ds. \]

Then we have the Young inequality

\[ ab \leq \Phi(a) + \Psi(b) \]

for all \( a, b \in [0,\infty) \). The functions \( \Phi \) and \( \Psi \) are said to form a complementary Young pair. Furthermore, \( \Phi \) (and similarly \( \Psi \)) is a continuous, strictly increasing, and convex function satifying

\[ \lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0 \]

and

\[ \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty. \]

Such Young functions are usually called \( N \)-functions (nice Young functions) in the literature; see e.g. [29] for a more general definition of Young functions.

Following [42] we consider complementary Young pairs of a special type. Suppose that a homeomorphism \( G : [0,\infty) \to [0,\infty) \) is a Young function that is a diffeomorphism on \( (0,\infty) \) and satisfies

\[ \int_{0}^{1} \frac{1}{G^{-1}(t)} dt < \infty, \]

where \( G^{-1} \) is the inverse of \( G \). Then, for each \( t \in [0,\infty) \),

\[ G^{-1}(t) = \{(s, r) \in \mathbb{R}^2 : 0 < s < 1, 0 < r < 1, G(s)r = t \}. \]

Let \( \bar{G}^{-1}(t) = \{ (s, r) \in \mathbb{R}^2 : 0 < s < 1, 0 < r < 1, G(s)r = t \} \).

Then \( \bar{G}^{-1}(t) \) is a Jordan curve with center \( (0, \frac{1}{2}) \), and the area of \( \bar{G}^{-1}(t) \) is \( 2G^{-1}(t) \) for each \( t \in [0,\infty) \).
We collect the properties of \( G \) and then setting \( G \) and a homeomorphism \( H \) sketch the construction. The function \( F \) is obtained by first choosing \( \lambda \in (1, 1 + \varepsilon_0) \) and a homeomorphism \( H : [0, \infty) \to (0, \infty) \) that is diffeomorphic on \((0, \infty)\) and satisfies

\[
H(t) = \begin{cases} (\log \frac{1}{t})^{-1} (\log \frac{1}{t})^{-\lambda} & \text{if } t \text{ is small enough;} \\ \frac{p}{\varepsilon_0} & \text{if } t \text{ is large enough,} \end{cases}
\]

and then setting \( G(t) = \int_0^t H(s)ds \). Then \( G \) and \( \tilde{G} \), \( \tilde{G}(t) = G(t^{1/p}) \), are Young functions. Let \( \tilde{F} \) be the complementary Young function to \( \tilde{G} \) and, finally, define \( F \) by setting \( F(t) = c\tilde{F}(t^p) \) for a suitable positive constant \( c \).

Since \( G \) is convex, we have \( G(t) \geq ct \) for all \( t \geq 1 \). Therefore \( G^{-1}(t) \leq ct \) for all \( t \) large enough and, consequently, \( \int_0^\infty 1/G^{-1} = \infty \). Taking into account \( 2.6 \) we conclude that the function \( \gamma \), defined by

\[
\gamma(t) = \int_0^t \frac{1}{G^{-1}(s)} ds,
\]

is a homeomorphism \([0, \infty) \to [0, \infty)\) that is a diffeomorphism on \((0, \infty)\). Hence the same is true for its inverse

\[
\varphi = \gamma^{-1} : [0, \infty) \to [0, \infty).
\]

We collect the properties of \( \varphi \) to the following lemma.

**Lemma 13.** \([42]\) Lemma 4.5] The function \( \varphi : [0, \infty) \to [0, \infty) \) is a homeomorphism that is smooth on \((0, \infty)\) and satisfies

\[
G \circ \varphi' = \varphi,
\]

and

\[
\lim_{t \to 0^+} \frac{\varphi''(t)\varphi(t)}{\varphi'(t)^3} = 1.
\]

From now on, \( \varphi \) will be the function defined in \( 2.9 \) such that the corresponding \( F \in \mathcal{F}_p \) satisfies \( 2.8 \). We define an auxiliary function \( \psi = (\varphi')^{p-1} \varphi \). It is easy to see that \( \psi : [0, \infty) \to [0, \infty) \) is a homeomorphism that is smooth on \((0, \infty)\). It follows from \( 2.11 \) that

\[
\lim_{t \to 0^+} \frac{\psi'(t)}{\varphi'(t)^p} = p.
\]
Consequently, for every $\delta > 0$, there exists $t_\delta > 0$ such that
\begin{equation}
\frac{\psi'(t)}{2p} \leq \varphi'(t)^p \leq \frac{(1 + \delta)^p \psi'(t)}{p}
\end{equation}

and
\begin{equation}
\frac{\psi(t)^p}{\psi'(t)^{p-1}} \leq \frac{(1 + \delta)^p \varphi(t)^p}{p^{p-1}}
\end{equation}
whenever $t \in (0, t_\delta]$.

\section{Dirichlet problem at infinity for $A$-harmonic functions}

This section is devoted to the proof of Theorem \ref{thm:dirichlet}. We assume that $A \in A_p(M)$, with $1 < p < \infty$. Throughout the section the function $F \in F_p$ satisfies (2.8) (see Proposition \ref{prop:fi}) and the function $\varphi$ is related to $G$ and $F$ by (2.10) as explained in 2.5. Furthermore, $r$ stands for the distance function $r(x) = d(x, o)$.

We start with stating a Caccioppoli-type inequality that will be crucial in the sequel.

\begin{lemma}
\cite[Lemma 2.15]{42}
Suppose that $\Psi: [0, \infty) \to [0, \infty)$ is a homeomorphism that is smooth on $(0, \infty)$. Let $U \Subset M$ be an open, relatively compact set and let $\eta \geq 0$ be a Lipschitz function in $U$. Suppose that $\theta, u \in L^\infty(U) \cap W^{1,p}(U)$ are continuous functions and that $u$ is $A$-harmonic in $U$. Denote $h = |u - \theta|$ and suppose that $\eta^p \Psi(h) \in W^{1,p}_0(U)$. Then
\begin{equation}
\left( \int_U \eta^p \Psi'(h) |\nabla u|^p \right)^{1/p} \leq \frac{\beta}{\alpha} \left( \int_U \eta^p \Psi'(h) |\nabla \theta|^p \right)^{1/p} + \frac{p \beta}{\alpha} \left( \int_U \frac{\Psi^p}{\Psi'_{p-1}}(h) |\nabla \eta|^p \right)^{1/p}.
\end{equation}
\end{lemma}

The proof is a straightforward application of the $A$-harmonic equation (2.4) for $u$ with the test function $f = \eta^p \Psi((u - \theta)^+) - \eta^p \Psi((u - \theta)^-)$. We omit the details and refer to \cite{42} for the proof. In Section 4 we prove a Caccioppoli inequality for solutions of the minimal graph equation.

Combining the Caccioppoli inequality (3.1) with a local Sobolev inequality (see (3.2) below) and running a Moser-type iteration we obtain pointwise estimates for the difference of an $A$-harmonic function and its boundary data in sufficiently small balls in terms of certain integral quantities in bigger balls. Recall that a local Sobolev inequality holds on any Cartan-Hadamard manifold. More precisely, there exist two constants $r_S > 0$ and $C_S < \infty$ such that
\begin{equation}
\left( \int_B |\eta|^{n/(n-1)} \right)^{(n-1)/n} \leq C_S \int_B |\nabla \eta|
\end{equation}
for every ball $B = B(x, r_S) \subset M$ and every function $\eta \in C_0^\infty(B)$. Such an inequality can be obtained e.g. from Croke’s estimate of the isoperimetric constant; see \cite{13} and \cite{9}. The following lemma is proved in \cite{42} Lemma 2.20. Below $\Omega \subset M$ is a nonempty open set.

\begin{lemma}
\cite[Lemma 2.20]{42}
Suppose that $\|\theta\|_{L^\infty} \leq 1$. Suppose that $s \in (0, r_S)$ is a constant and $x \in M$. Denote $B = B(x, s)$. Suppose that $u \in W^{1,p}_{loc}(M)$ is

\end{lemma}
a function that is \( A \)-harmonic in the open set \( \Omega \cap B \), satisfies \( u - \theta \in W^{1,p}_0(\Omega) \), inf \( _M \theta \leq u \leq sup^M \theta \), and \( u = \theta \) a.e. in \( M \setminus \Omega \). Then

\[
\text{ess sup}_{B(x,n/2)} \varphi (|u - \theta|)^{(n+1)/p} \leq c \int_B \varphi (|u - \theta|)^p,
\]

where the constant \( c \) is independent of \( x \).

In Section 4 we will state and prove a similar estimate for solutions of the minimal graph equation.

Next we show that the integral appearing in Lemma 15 can be estimated from above by another integral that will be uniformly bounded provided sectional curvatures of \( M \) are bounded as in Theorem 4.

**Lemma 16.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \). Suppose that

\[
K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)},
\]

for some constant \( \varepsilon > 0 \), where \( K(P) \) is the sectional curvature of any plane \( P \subset T_x M \) that contains the radial vector \( \nabla r(x) \) and \( x \) is any point in \( M \setminus B(o,R_0) \).

Fix \( \bar{\varepsilon} \in (0,\varepsilon) \) and let \( R_1 \geq R_0 \) be given by Proposition 7. Suppose that \( U \subset M \) is an open, relatively compact set and that \( u \) is an \( A \)-harmonic function in \( U \), with \( u - \theta \in W^{1,p}_0(U) \), where \( A \in \mathcal{A}(M) \), with

\[
p < \frac{n\alpha}{\beta},
\]

and \( \theta \in W^{1,\infty}(M) \) is a continuous function, with \( ||\theta||_\infty \leq 1 \). Then there exist a bounded \( C^1 \)-function \( C: [0,\infty) \to [0,\infty) \) and a constant \( c_0 \geq 1 \) that is independent of \( \theta, U, \) and \( u \) such that

\[
\int_U \varphi (|u - \theta|/c_0)^p (\log(1+r) + C(r)) \leq c_0 + c_0 \int_U \left( \frac{c_0 |\nabla \theta| r \log(1+r)}{\log(1+r) + C(r)} \right) (\log(1+r) + C(r)).
\]

**Proof.** We begin by proving a weighted Poincaré-type inequality. First of all, we have

\[
\Delta r \geq \frac{n-1}{r}
\]

in \( M \setminus \{o\} \) since \( M \) is a Cartan-Hadamard manifold. Moreover, by applying the standard Laplace comparison theorem and Proposition 7, we find that

\[
\Delta r(x) \geq (n-1) \left( \frac{1}{r(x)} + \frac{1+\bar{\varepsilon}}{r(x) \log r(x)} \right)
\]

whenever \( r(x) \geq R_1 \). Therefore

\[
r \log(1+r) \Delta r \geq (n-1) (\log(1+r) + \mathcal{E}(r))
\]

in \( M \), where \( \mathcal{E}: [0,\infty) \to [0,\infty) \) is a bounded \( C^1 \)-function satisfying

\[
\mathcal{E}(r) = \begin{cases} 0, & \text{if } 0 \leq r \leq R_1; \\ \frac{(1+\bar{\varepsilon}) \log(1+r)}{\log r}, & \text{if } r \geq 2R_1. \end{cases}
\]
By the assumption (3.3), we can choose $\delta > 0$ such that
\begin{equation}
(3.7) \quad p < \frac{n\alpha}{(1 + \delta)^2\beta}.
\end{equation}
Denote $h = |u - \theta|/c_0$, where the constant $c_0 > 0$ will be specified in due course. Since $-1 \leq \inf_U \theta \leq u \leq \sup_U \theta \leq 1$ in $U$, we may assume that $c_0$ is so large that $\|h\|_{\infty} \leq t_\delta$, where $t_\delta > 0$ is a constant such that (2.13) and (2.14) hold for all $t \in (0, t_\delta]$.

Using (3.5) and integration by parts, we obtain
\begin{equation}
(n - 1) \int_U \varphi(h)^p \left( \log(1 + r) + \mathcal{E}(r) \right) \leq \int_U \varphi(h)^p r \log(1 + r) \Delta r
\end{equation}
\begin{equation}
= - \int_U \left\langle \nabla \left( \varphi(h)^p r \log(1 + r) \right), \nabla r \right\rangle
\end{equation}
\begin{equation}
= - \int_U \varphi(h)^p \left( \frac{r}{1 + r} + \log(1 + r) \right) - p \int_U r \log(1 + r) \varphi(h)^{p-1} \varphi'(h) \langle \nabla h, \nabla r \rangle.
\end{equation}
This, together with Hölder’s inequality, gives rise to
\begin{equation}
\int_U \varphi(h)^p \left( \log(1 + r) + \mathcal{C}(r) \right) \leq p \int_U r \log(1 + r) \varphi(h)^{p-1} \varphi'(h) |\nabla h|
\end{equation}
\begin{equation}
\leq p \left( \int_U |\nabla h|^p \varphi'(h)^p \left( \frac{r \log(1 + r)}{(\log(1 + r) + \mathcal{C}(r)_{p-1}} \right)^p \right)^{1/p}
\end{equation}
\begin{equation}
\times \left( \int_U \varphi(h)^p \left( \log(1 + r) + \mathcal{C}(r) \right) \right)^{(p-1)/p},
\end{equation}
where
\begin{equation}
(3.8) \quad \mathcal{C}(r) = \frac{r}{n(1 + r)} + \frac{(n - 1)\mathcal{E}(r)}{n}.
\end{equation}
To simplify notation, we set
\begin{equation}
(3.9) \quad L(r) = \log(1 + r) + \mathcal{C}(r)
\end{equation}
and
\begin{equation}
(3.10) \quad w = \frac{r \log(1 + r)}{(\log(1 + r) + \mathcal{C}(r))^{(p-1)/p}}.
\end{equation}
Hence
\begin{equation}
(3.11) \quad n \left( \int_U \varphi(h)^p L(r) \right)^{1/p} \leq p \left( \int_U |\nabla h|^p \varphi'(h)^p w^p \right)^{1/p}
\end{equation}
The gradient of $w$ is given by
\begin{equation}
(3.12) \quad \nabla w = L(r)^{1/p} \left( \frac{\log(1 + r) + \frac{r}{1 + r}}{L(r)} + \left( \frac{1}{p} - 1 \right) \frac{r \log(1 + r) \left( \frac{1}{1 + r} + \mathcal{C}'(r) \right)}{L(r)^2} \right) \nabla r.
\end{equation}
We claim that
\begin{equation}
(3.13) \quad |\nabla w| \leq L(r)^{1/p}
\end{equation}
for all $r$ large enough, say $r \geq R_2$, and
\begin{equation}
(3.14) \quad |\nabla w| \leq c
\end{equation}
in $B(o, R_2)$. To prove (3.13), we first note that $C'(r) r \to 0$ as $r \to \infty$, and therefore

$$\log(1 + r) + \frac{r}{1 + r} - \frac{r \log(1 + r)(\frac{1}{1+r} + C'(r))}{L(r)} \geq 0$$

whenever $r$ is large enough. We have, for $r \geq R_2 \geq R_1$,

$$0 \leq \frac{\log(1 + r) + \frac{r}{1+r} + \frac{1}{p} - 1}{L(r)} \frac{r \log(1 + r)(\frac{1}{1+r} + C'(r))}{L(r)^2} \leq \frac{\log(1 + r) + \frac{r}{1+r} + \frac{1}{p} - 1 \frac{1}{n} \frac{1}{1+r} + \frac{1}{n} \frac{1}{1+r} \frac{1}{n} \frac{1}{1+r} \log(1 + r)}{L(r)} \leq 1$$

since

$$\frac{\frac{1}{n} - 1}{1 + r} \frac{1}{n} \frac{1}{1+r} \log(1 + r) = \frac{1}{n} \frac{1}{n} \frac{1}{1+r} \log(1 + r) - \frac{r}{1 + r} > 0.$$  

Hence (3.13) follows. The estimate (3.14) holds since $w$ is smooth in $M \setminus \{o\}$ and $w(r)/r \to 0$ as $r \to 0$.

Using the estimate $|\nabla h| \leq (|\nabla u| + |\nabla \theta|)/c_0$, Minkowski's inequality, and (2.13), we obtain

$$\left( \int_U |\nabla h|^p \varphi'(h)^p w^p \right)^{1/p} \leq c_0^{-1} \left( \int_U (\varphi'(h) w |\nabla u| + \varphi(h) w |\nabla \theta|)^p \right)^{1/p} \leq c_0^{-1} \left( \int_U \varphi(h)^p |\nabla u|^p w^p \right)^{1/p} + c_0^{-1} \left( \int_U \varphi(h)^p |\nabla \theta|^p w^p \right)^{1/p} \leq \frac{1 + \delta}{c_0^{1/p}} \left[ \left( \int_U \psi'(h)|\nabla u|^p w^p \right)^{1/p} + \left( \int_U \psi'(h)|\nabla \theta|^p w^p \right)^{1/p} \right].$$  

(3.15)

Applying the Caccioppoli inequality (3.14) with $u$ and $\theta$ replaced by $u/c_0$ and $\theta/c_0$, respectively, to the first term on the right-hand together with (2.14), we obtain

$$\left( \int_U w^p \psi'(h)|\nabla u|^p \right)^{1/p} \leq \frac{\beta}{\alpha} \left( \int_U w^p \psi'(h)|\nabla u|^p \right)^{1/p} + \frac{p\beta c_0}{\alpha} \left( \int_U \psi(h)^{p-1}(h)|\nabla u|^p \right)^{1/p} \leq \frac{\beta}{\alpha} \left( \int_U w^p \psi'(h)|\nabla \theta|^p \right)^{1/p} + \frac{p^{1/p} \beta c_0 (1 + \delta)}{\alpha} \left( \int_U \varphi(h)^p |\nabla u|^p \right)^{1/p}.$$  

(3.16)
Now combining (3.11), (3.15), and (3.16), we find
\[ n \left( \int_{U} \varphi(h)^{p} L(r) \right)^{1/p} \leq p \left( \int_{U} |\nabla h|^{p} \varphi(h)^{p} w^{p} \right)^{1/p} \]
\[ \leq (1 + \delta)p^{-1} c_{0}^{-1} \left[ \left( \int_{U} \psi'(h)|\nabla u|^{p} w^{p} \right)^{1/p} + \left( \int_{U} \psi'(h)|\nabla \theta|^{p} w^{p} \right)^{1/p} \right] \]
\[ \leq (1 + \delta)p^{1-1/p} c_{0}^{-1} \left[ (1 + \frac{\beta}{\alpha}) \left( \int_{U} \psi'(h)|\nabla \theta|^{p} w^{p} \right)^{1/p} \right] \]
\[ + \frac{p^{1/p} \beta c_{0}(1 + \delta)}{\alpha} \left( \int_{U} \varphi(h)^{p} \nabla u|^{p} \right)^{1/p} \]
\[ \leq (1 + \delta)p^{1-1/p} c_{0}^{-1} (1 + \frac{\beta}{\alpha}) \left( \int_{U} \psi'(h)|\nabla \theta|^{p} w^{p} \right)^{1/p} \]
\[ + \frac{p^{1/p} \beta c_{0}(1 + \delta)^{2}}{\alpha} \left( \int_{U} \varphi(h)^{p} L(r) \right)^{1/p} + C, \]

where in the last step we used (3.13) and (3.14) to estimate
\[ \int_{U} \varphi(h)^{p} \nabla u|^{p} = \int_{U \cap B(\Omega, R_{2})} \varphi(h)^{p} \nabla u|^{p} + \int_{U \setminus B(\Omega, R_{2})} \varphi(h)^{p} \nabla u|^{p} \]
\[ \leq \bar{C} + \int_{U} \varphi(h)^{p} L(r). \]

Since
\[ p < \frac{n\alpha}{(1 + \delta)^{2} \beta}, \]
it follows that there exists a constant \( C \) depending on \( p, n, \alpha, \beta \) such that
\[ (3.17) \int_{U} \varphi(h)^{p} L(r) \leq C \int_{U} \varphi'(h)^{p} |\nabla \theta|^{p} w^{p} + C_{0}. \]

Next, recalling that \( F(-1/p) \) and \( G(-1/p)^{p} \) are complementary Young functions, we have, for all \( x, y \geq 0 \) and \( k > 0 \),
\[ (3.18) \ xy = kx(y/k) \leq k \left( G(x^{1/p})^{p} + F(k^{-1/p} y^{1/p}) \right) = kG(x^{1/p})^{p} + kF(k^{-1/p} y^{1/p}). \]

The definition of \( w \), previous inequalities (3.17), (3.13), and (2.10) yield
\[ \int_{U} \varphi(h)^{p} L(r) \leq C \int_{U} \varphi'(h)^{p} L(r) \left( \frac{|\nabla \theta|^{r} \log(1 + r)}{L(r)} \right)^{p} + C_{0} \]
\[ \leq Ck \int_{U} G(\varphi'(h))^{p} L(r) + Ck \int_{U} F \left( \frac{k^{-1/p} |\nabla \theta|^{r} \log(1 + r)}{L(r)} \right) L(r) + C_{0} \]
\[ \leq Ck \int_{U} \varphi(h)^{p} L(r) + Ck \int_{U} F \left( \frac{k^{-1/p} |\nabla \theta|^{r} \log(1 + r)}{L(r)} \right) L(r) + C_{0}. \]

Taking \( k > 0 \) small enough, we finally obtain
\[ \int_{U} \varphi(h)^{p} L(r) \leq \frac{Ck}{1 - Ck} \int_{U} F \left( \frac{k^{-1/p} |\nabla \theta|^{r} \log(1 + r)}{L(r)} \right) L(r) + \frac{C_{0}}{1 - Ck}. \]
\[ \square \]
We are now in position to prove Theorem \[4\] In fact, we prove the following localized version concerning the \(A\)-regularity of a point \(x_0 \in \partial_\infty M\) which then implies Theorem \[4\] since the uniqueness statement follows from the comparison principle.

**Theorem 17.** Let \(M\) be a Cartan-Hadamard manifold of dimension \(n \geq 2\). Suppose that

\[
(3.19) \quad - \frac{(\log r(x))^2}{r(x)^2} \leq K(P) \leq - \frac{1 + \varepsilon}{r(x)^2 \log r(x)},
\]

for some constants \(\varepsilon > \bar{\varepsilon} > 0\), where \(K(P)\) is the sectional curvature of any plane \(P \subset T_x M\) that contains the radial vector \(\nabla r(x)\) and \(x\) is any point in a cone neighborhood \(U\) of \(x_0 \in \partial_\infty M\). Then \(x_0\) is \(A\)-regular for every \(A \in \mathcal{A}_p(M)\), with \(1 \leq p < n\alpha/\beta\).

**Proof.** Let \(f: \partial_\infty M \to \mathbb{R}\) be a continuous function. To prove that \(x_0\) is \(A\)-regular, we need to show that

\[
\lim_{x \to x_0} \overline{f}(x) = f(x_0).
\]

Fix an arbitrary \(\varepsilon' > 0\). Let \(v_0 = \xi_o^{x_0}\) be the initial vector of the geodesic ray from \(o\) to \(x_0\). Furthermore, let \(\delta \in (0, \pi)\) and \(R_0 > 0\) be such that \(T(v_0, \delta, R_0) \subset U\) and that \(|f(x_1) - f(x_0)| < \varepsilon'\) for all \(x_1 \in C(v_0, \delta) \cap M(\infty)\); see \[2.4\] for the notation. Next we fix \(\delta > \varepsilon > 0\) are the constants in the curvature assumption \([3.19]\). Let \(r_1 > \max(2, R_1)\), where \(R_1 \geq R_0\) is given by Proposition \(7\). We denote \(\Omega = T(v_0, \delta, r_1) \cap M\) and define \(\theta \in C(M)\) by setting

\[
\theta(x) = \min \left(1, \max \left(r_1 + 1 - r(x), \delta^{-1} \varphi_o(x_0, x)\right)\right).
\]

Note that \(\theta = 1\) on \(\partial \Omega\). Let \(\Omega_j = \Omega \cap B(o, j)\) for integers \(j > r_1\) and let \(u_j\) the unique \(A\)-harmonic function in \(\Omega_j\) with \(u_j - \theta \in W_{0}^{1,p}(\Omega_j)\). It is clear that each \(y \in \partial \Omega_j\) is \(A\)-regular and hence \(u_j\) can be continuously extended to \(\partial \Omega_j\) by setting \(u_j = \theta\) on \(\partial \Omega_j\). Since \(0 \leq u_j \leq 1\), the sequence \((u_j)\) is equicontinuous, and therefore by the Ascoli-Arzelà theorem, there exists a subsequence, still denoted by \((u_j)\), that converges locally uniformly to a continuous function \(u: \Omega: \to [0, 1]\). It follows that \(u\) is \(A\)-harmonic in \(\Omega\); see e.g. [22, Chapter 6] for these boundary regularity and convergence results. Next we prove that

\[
(3.20) \quad \lim_{x \to x_0} u(x) = 0.
\]

Denote \(\bar{\theta} = \theta/c_0\), \(\bar{u}_j = u_j/c_0\), and \(\bar{u} = u/c_0\), where \(c_0\) is given by Lemma \(16\). Fatou’s lemma and Lemma \(16\) applied to \(U = \Omega_j\) imply that

\[
\begin{align*}
\int_{\Omega} \varphi(|\bar{u} - \bar{\theta}|) &\leq \int_{\Omega} \varphi(|u - \theta|/c_0) \leq \liminf_{j \to \infty} \int_{\Omega_j} \varphi(|u_j - \theta|/c_0)^p L(r) \leq \liminf_{j \to \infty} \int_{\Omega_j} \varphi(|u_j - \theta|/c_0)^p L(r) \\
&\leq c_0 + c_0 \int_{\Omega} F \left( \frac{c_0 |\nabla \theta| r \log(1 + r)}{L(r)} \right) L(r).
\end{align*}
\]

We will show at the end of the proof that the right-side in \(3.21\) is finite. Meanwhile we extend each \(u_j\) to a function \(u_j \in W_{0}^{1,p}(M) \cap C(M)\) by setting \(u_j(y) = \theta(y)\) for
We deduce that
\[ u \]

Note that we may replace \( \text{ess sup} \) by \( \text{sup} \) because \( u_j - \theta \) is continuous in \( M \). The dominated convergence theorem implies that
\[
\sup_{B(x,s/2)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^p \leq c \int_{B(x,s)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^p.
\]

One can prove in a similar way that
\[
\lim_{k \to \infty} \sup_{B(x,s/2)} \varphi(|\tilde{u}_j - \tilde{\theta}|)^p = c \int_{B(x,s)} \varphi(|\tilde{u} - \tilde{\theta}|)^p.
\]

Let \( (x_k) \) be a sequence of points in \( \Omega \) so that \( x_k \to x_0 \) as \( k \to \infty \). Applying the estimate (3.22) above with \( x = x_k \) and a fixed \( s \in (0, r_s) \) and assuming that the right-side of (3.21) is finite we obtain
\[
\lim_{k \to \infty} \sup_{B(x,s/2)} \varphi(|\tilde{u} - \tilde{\theta}|)^p = 0.
\]

Hence
\[
\lim_{k \to \infty} |\tilde{u}(x_k) - \tilde{\theta}(x_k)| = 0
\]

and, consequently, (3.20) holds. Next we define \( w: M \to \mathbb{R} \) by
\[
w(x) = \begin{cases} \min(1, 2\tilde{u}(x)), & \text{if } x \in \Omega; \\ 1, & \text{if } x \in M \setminus \Omega. \end{cases}
\]

Then \( w \) is \( A \)-superharmonic in \( M \) (see [22, Lemma 7.2]) and hence, by the definition of \( \overline{\mathcal{H}}_f \), we have
\[
\overline{\mathcal{H}}_f \leq f(x_0) + \varepsilon' + 2(\sup |f|)w.
\]

Hence, by (3.20)
\[
\limsup_{x \to x_0} \overline{\mathcal{H}}_f(x) \leq f(x_0) + \varepsilon'.
\]

One can prove in a similar way that
\[
\liminf_{x \to x_0} \underline{\mathcal{H}}_f(x) \geq f(x_0) - \varepsilon'.
\]

We deduce that
\[
\lim_{x \to x_0} \overline{\mathcal{H}}_f(x) = f(x_0),
\]

and therefore \( x_0 \) is \( A \)-regular.

To conclude the proof, it remains to show that
\[
\int_{\Omega} F \left( \frac{c_0 |\nabla \theta| r \log(1 + r)}{L(r)} \right) L(r) < \infty.
\]

Recall that above \( \Omega = T(r_0, \delta, r_1) \cap M \), with \( v_0 = \tilde{\gamma}_{0,x_0} \). The integral (3.23) will be estimated from above by using geodesic polar coordinates \((r, v)\) for points \( x \in \Omega \). Here \( r = r(x) \in [r_1, \infty) \) and \( v = \tilde{\gamma}_{0,x} \). Let \( \lambda(r, v) \) be the Jacobian for these polar coordinates. We need to estimate \( \lambda \) and the function \( F \) from above. To this end,
let \( a, b : [0, \infty) \to [0, \infty) \) be smooth functions such that they are constant in some neighborhood of 0,
\[
-b^2(r(x)) \leq K(P) \leq -a^2(r(x))
\]
for all \( x \in C(v_0, \delta) \) and for all 2-planes \( P \subset T_xM \) containing the radial vector \( \nabla r \), and that
\[
a^2(t) = \frac{1 + \varepsilon}{t^2 \log t},
b^2(t) = \frac{(\log t)^{2\varepsilon}}{t^2}
\]
for \( t \geq R_0 \). For \( x \in \Omega \), we denote by \( J(x) \) the supremum and by \( j(x) \) the infimum of \( |V(r(x))| \) over all Jacobi fields \( V \) along \( \gamma^{o.x} \) that satisfy \( V_0 = 0 \), \( |V'_0| = 1 \), and \( V'_0 + \dot{\gamma}^{o.x}_0 \). By applying the Rauch comparison theorem we get the estimates
\[
(3.24)\quad j(x) \geq f_a(r(x));
(3.25)\quad J(x) \leq f_b(r(x)),
\]
where \( f_a \) and \( f_b \) are the solutions to corresponding Jacobi equations \((2.1)\); see e.g. \[27\] Proposition 2.5. Thus we have
\[
(3.26)\quad \lambda(r, v) \leq f_b(r)^{n-1}
\]
for all points \( x = (r, v) \in \Omega \). We also recall from \[41\] Lemma 2 that
\[
(3.27)\quad |\nabla \theta(x)| \leq \frac{c}{j(x)} \leq \frac{c}{f_a(r(x))}
\]
in \( \Omega \). It follows that there exists a constant \( c_1 \) such that
\[
(3.28)\quad \frac{c_0 |\nabla \theta| r \log(1 + r)}{L(r)} = \frac{c_0 |\nabla \theta| r \log(1 + r)}{r} \leq \frac{c}{c_1 f_a(r)}
\]
for all \( r \) large enough. Since the functions \( \varphi \) and \( F \in F_p \) were fixed so that \( F \) satisfies \((2.8)\) we have in particular, \( F \leq \tilde{F} \), where
\[
\tilde{F}(s) = \exp \left( -\frac{1}{s} \left( \frac{1}{s} \right)^{-1-\varepsilon_0} \right)
\]
for all \( s \) small enough and \( \varepsilon_0 \in (0, 1) \). In what follows, we assume that \( t_0 \geq R_1 \) is a sufficiently large constant. For \( t \geq t_0 \), we define
\[
\Phi(t) = \left( \frac{t}{c_1 f_a(t)} \right)^{\frac{1}{n}}
= t^{-\frac{2}{n-1}} \exp \left( \frac{1}{n-1} \frac{c_1 f_a(t)}{t} \left( \log \frac{c_1 f_a(t)}{t} \right)^{-1-\varepsilon_0} \right),
\]
and thus
\[
\frac{\Phi'(t)}{\Phi(t)} = \frac{-2t + c_1 \left( 1 - (1 + \varepsilon_0) \log \frac{c_1 f_a(t)}{t} \right) \left( tf_a'(t) - f_a(t) \right) \left( \log \frac{c_1 f_a(t)}{t} \right)^{-1-\varepsilon_0}}{(n-1)t^2}.
\]
Straightforward computations, using Proposition 4 yield to
\[
\left( \frac{tf_a'(t)}{f_a(t)} - 1 \right) \frac{f_a(t)}{t} \geq (1 + \varepsilon)(\log t)^\varepsilon
\]
for all \( t \geq R_1 \). It follows that
\[
\frac{\Phi'(t)}{\Phi(t)} \geq \frac{2(\log t)^{\bar{\epsilon}}}{t} = 2b(t)
\]
for all \( t \geq t_0 \). Since \( b'(t)/b(t)^2 \to 0 \) as \( t \to \infty \), we obtain
\[
\lim_{t \to \infty} \frac{f'_b(t)}{b(t)f_b(t)} = 1
\]
by [27, Lemma 2.3]. Therefore we have
\[
\frac{\Phi'(t)}{\Phi(t)} \geq 2b(t) \geq f'_b(t) f_b(t)
\]
for \( t \geq t_0 \). It follows that \( \Phi(t) \geq c f_b(t) \), for all \( t \geq t_0 \). Thus we have
\[
F \left( \frac{c_0 |\nabla \theta(r, v)| r \log(1 + r)}{L(r)} \right) L(r) \lambda(r, v)
= F \left( \frac{c_0 |\nabla \theta(r, v)| r \log(1 + r)}{\log(1 + r) + C(r)} \right) (\log(1 + r) + C(r)) \lambda(r, v)
\leq c F \left( \frac{r}{c_1 f_b(r)} \right) (\log(1 + r) + C(r)) \Phi(r)^{n-1}
= c(\log(1 + r) + C(r)) r^{-2}
\]
for all \( x = (r, v) \in U \cap M \) outside a compact set. Since \( C \) is a bounded function, this shows that [3223] holds and therefore concludes the proof of Theorem 17. □

4. Dirichlet Problem at Infinity for the Minimal Graph Equation

In this section we will prove Theorem 5. We will use a slightly different approach than the one adopted in the proof of Theorem 4 but the main ingredients will be the same. However, to solve the Dirichlet problem at infinity for the minimal graph equation, some extra difficulties appear. The first one is the fact that the minimal graph operator does not satisfy (2.2). Therefore, we need to adapt the previous Caccioppoli inequality proved in Lemma 14. The second difficulty is linked to the fact that it may not be possible, in general, to solve the minimal graph equation on the sets \( \Omega_j \) as defined in the proof of Theorem 4.

4.1. Caccioppoli Inequality and Some Consequences. We begin this section with the following Caccioppoli-type inequality. In what follows we use the customary notation \( W(x) = \sqrt{1 + |\nabla u(x)|^2} \) for a smooth solution \( u \) of the minimal graph equation.

**Lemma 18.** Suppose that \( \Psi : [0, \infty) \to [0, \infty) \) is a homeomorphism that is smooth on \( (0, \infty) \). Let \( U \subseteq M \) be an open and relatively compact set. Suppose that \( \eta \geq 0 \) is a locally Lipschitz function on \( U \setminus \{0\} \). Suppose that \( \theta, u \in L^\infty(U) \cap W^{1,2}(U) \) are continuous functions and that \( u \in C^2(U) \) is a solution of the minimal graph equation in \( U \). Denote
\[
h = \frac{|u - \theta|}{\nu},
\]
where \( \nu > 0 \) is a constant, and suppose that
\[
\eta^2 \Psi(h) W \in W^{1,2}_0(U).
\]
We estimate the terms on the right-side as

\[ (4.1) \quad \int_U \eta^2 \Psi'(h) |\nabla u|^2 \leq 4 \int_U \eta^2 \Psi'(h) |\nabla \theta|^2 + 8 \nu^2 \int_U \frac{\Psi^2}{\Psi'}(h) |\nabla \eta|^2 + 4 \nu^2 \int_U \eta^2 \frac{\Psi^2}{\Psi'}(h) |\nabla \log W|^2. \]

**Proof.** We begin by defining

\[ f = \nu \eta^2 \Psi \left( (u - \theta)^+ / \nu \right) W - \nu \eta^2 \Psi \left( (u - \theta)^- / \nu \right) W. \]

It is easy to see that \( f \in W^{1,2}_0(U) \) and its gradient is given by

\[ \nabla f = \eta^2 \Psi'(h) W (\nabla u - \nabla \theta) + 2 \nu \eta \operatorname{sgn}(u - \theta) \Psi(h) W \nabla \eta + \nu \eta^2 \operatorname{sgn}(u - \theta) \Psi(h) \nabla W. \]

Using \( f \) as a test function in the minimal graph equation, we obtain that

\[ \int_U \eta^2 \Psi'(h) |\nabla u|^2 = \int_U \eta^2 \Psi'(h) |\nabla u, \nabla \theta| - 2 \nu \int_U \operatorname{sgn}(u - \theta) \eta \Psi(h) |\nabla u, \nabla \eta| \]

\[ \quad - \nu \int_U \operatorname{sgn}(u - \theta) \eta^2 \Psi(h) |\nabla \log W, \nabla u| \]

\[ \leq \int_U \eta^2 \Psi'(h) |\nabla u| |\nabla \theta| + 2 \nu \int_U \eta \Psi(h) |\nabla u||\nabla \eta| \]

\[ + \nu \int_U \eta^2 \Psi(h) |\nabla u||\nabla \log W|. \]

We estimate the terms on the right-side as

\[ \int_U \eta^2 \Psi'(h) |\nabla u||\nabla \theta| \leq \epsilon/2 \int_U \eta^2 \Psi'(h) |\nabla u|^2 + 1/(2 \epsilon) \int_U \eta^2 \Psi'(h) |\nabla \theta|^2, \]

\[ 2 \nu \int_U \eta \Psi(h) |\nabla u||\nabla \eta| \leq \epsilon \int_U \eta^2 \Psi'(h) |\nabla u|^2 + \nu^2 / \epsilon \int_U \frac{\Psi^2}{\Psi'}(h) |\nabla \eta|^2, \]

and

\[ \nu \int_U \eta^2 \Psi(h) |\nabla u||\nabla \log W| \leq \epsilon/2 \int_U \eta^2 \Psi'(h) |\nabla u|^2 + \nu^2 / (2 \epsilon) \int_U \frac{\Psi^2}{\Psi'}(h) |\nabla \log W|^2. \]

Choosing \( \epsilon = 1/4 \) above proves the claim. \( \square \)

**Remark 19.** As can be seen later in the proof of Lemma 21, the second term

\[ 8 \nu^2 \int_U \frac{\Psi^2}{\Psi'}(h) |\nabla \eta|^2 \]

on the right-side of (4.1) is the only term that affects the dimension restriction \( n \geq 3 \) in Theorem 5. One could improve the factor \( 8 \nu^2 \) to \( (4 + \epsilon) \nu^2 \) for any \( \epsilon > 0 \) but, nevertheless, the dimension bound \( n \geq 3 \) still remains.

Before we state and prove a counterpart of Lemma 15 for the minimal graph equation, we recall from 2.5 that \( \varphi : [0, \infty) \to [0, \infty) \) is a homeomorphism, smooth on \( (0, \infty) \), and satisfies (2.10), i.e.

\[ G \circ \varphi' = \varphi, \]
where the homeomorphic Young function

\[ G : [0, \infty) \rightarrow [0, \infty) \]

is, in particular, convex. Hence there exist positive constants \( t_1 \) and \( c_2 \) such that

\[ \varphi(t) \leq 1, \quad \varphi'(t) \leq 1, \quad \text{and} \quad \varphi(t) \leq c_2 \varphi'(t) \]

for all \( t \in (0, t_1] \).

**Lemma 20.** Let \( \Omega = B(o, R) \) and suppose that \( \theta \in C^1(\Omega) \) with \( \| \theta \|_{C^1(\Omega)} \leq C_1 \). Let \( u \in C^2(\Omega) \) be a solution of the minimal graph equation in \( \Omega \) such that \( \inf_{\Omega} \theta \leq u \leq \sup_{\Omega} \theta \) and \( \| \nabla \log W \| \leq C_1 \). Fix \( s \in (0, r_S) \), where \( r_S \) is the radius in the Sobolev inequality (3.2), and suppose that \( B = B(x, s) \subset \Omega \). Then there exists a positive constant \( \nu_0 = \nu_0(\varphi, C_1) \) such that for all \( \nu \geq \nu_0 \)

\[ \sup_{B(x,s/2)} \varphi |u - \theta|^{2(n+1)} \leq c_3 \int_B \varphi |u - \theta|^2, \]

where \( c_3 \) is a positive constant depending only on \( n, s, C_S, C_1 \) and \( \varphi \).

**Proof.** We denote \( \kappa = n/(n-1) \), \( B/2 = B(x, s/2) \), and \( h = |u - \theta|/\nu \), where \( \nu \geq \nu_0 > 0 \) will be fixed in due course. For each \( j \in \mathbb{N} \) we denote \( s_j = s(1 + \kappa^{-j})/2 \) and \( B_j = B(x, s_j) \). Furthermore, let \( \eta_j \) be a Lipschitz function such that \( 0 \leq \eta_j \leq 1 \), \( \eta_j \{ B_j \} = 0 \), and that

\[ |\nabla \eta_j| \leq \frac{1}{s_j - s_{j+1}} = 2n\kappa^j/s. \]

For \( \Phi = \varphi^2 \) and \( m \geq 1 \) we have

\[ |\nabla (\eta_j^2 \Phi(h)^m)| \leq 2\eta_j \Phi(h)^m |\nabla \eta_j| + m \eta_j^2 \Phi'(h) \Phi(h)^{m-1} |\nabla h|. \]

We claim that

\[ \left( \int_{B_{j+1}} \Phi(h)^m \right)^{1/\kappa} \leq c(\kappa^j + m + \kappa^j/m) \int_{B_j} \Phi(h)^{m-1}. \]

For every \( m, j \geq 1 \), \( \eta_j^2 \Phi(h)^m \) is a Lipschitz function supported in \( B_j \). By the Sobolev inequality (3.2) we first have

\[ \left( \int_{B_{j+1}} \Phi(h)^m \right)^{1/\kappa} \leq \left( \int_{B_j} (\eta_j^2 \Phi(h)^m)^{1/\kappa} \right) \leq C_S \int_{B_j} |\nabla (\eta_j^2 \Phi(h)^m)| \]

\[ \leq 2C_S \int_{B_j} \eta_j \Phi(h)^m |\nabla \eta_j| + C_S \int_{B_j} \eta_j^2 (\Phi^m)'(h)|\nabla h| \]

\[ \leq c\kappa^j \int_{B_j} \Phi(h)^m + C_S/\nu \int_{B_j} (\Phi^m)'(h)|\nabla \theta| \]

\[ + C_S/\nu \int_{B_j} \eta_j^2 (\Phi^m)'(h)|\nabla u|. \]

Next we use the assumption

\[ -C_1 \leq \inf_{\Omega} \theta \leq u \leq \sup_{\Omega} \theta \leq C_1 \]

to observe that \( |u - \theta| \leq 2C_1 \). Hence, by (4.2), we can choose \( \nu_0 \) large enough so that

\[ \varphi(h) \leq 1, \quad \varphi'(h) \leq 1, \quad \text{and} \quad \varphi(h) \leq c_2 \varphi'(h) \]
for $\nu \geq \nu_0$. Consequently,

\begin{equation}
\Phi(h) \leq 1, \quad \Phi'(h) \leq 2, \quad \text{and } \Phi(h) \leq \frac{c}{2}\Phi'(h).
\end{equation}

We obtain estimates

\begin{equation}
\int_{B_j} \Phi(h)^m \leq \int_{B_j} \Phi(h)^{m-1}
\end{equation}

and

\begin{equation}
\int_{B_j} (\Phi^m)'(h) |\nabla \theta| = m \int_{B_j} \Phi(h)^{m-1} \Phi'(h) |\nabla \theta| \leq 2mC_1 \int_{B_j} \Phi(h)^{m-1}.
\end{equation}

We estimate the third term on the right-side of (4.4) first as

\begin{equation}
\int_{B_j} \eta_j^2 (\Phi^m)'(h) |\nabla u|^2 \leq 4 \int_{B_j} \eta_j^2 (\Phi^m)'(h)(1 + |\nabla u|^2)
\end{equation}

\begin{equation}
\leq 2m \int_{B_j} \Phi(h)^{m-1} + \int_{B_j} \eta_j^2 (\Phi^m)'(h) |\nabla u|^2.
\end{equation}

Next we notice that $\eta_j^2 \Phi(h)^m W \in W^{1,2}_0(B_j)$ since $\text{ supp } \eta_j \subset \bar{B}_j$. Thus we may apply the Caccioppoli-type inequality (4.1) with $\Psi = \Phi^m$ to obtain

\begin{equation}
\int_{B_j} \eta_j^2 (\Phi^m)'(h) |\nabla \log W|^2 \leq c(m + \kappa^2/m + 1/m) \int_{B_j} \Phi(h)^{m-1}.
\end{equation}

Now the estimate (4.3) follows by inserting estimates (4.6)-(4.9) into (4.4). We apply (4.3) with $m = m_j + 1$, where $m_j = (n+1)\kappa^j - n$. Since $m_{j+1} = \kappa(m_j + 1)$, (4.3) takes the form

\begin{equation}
\left( \int_{B_{j+1}} \Phi(h)^{m_{j+1}} \right)^{1/\kappa} \leq C \kappa^j \int_{B_j} \Phi(h)^{m_j}.
\end{equation}

By denoting

\begin{equation}
I_j = \left( \int_{B_j} \Phi(h)^{m_j} \right)^{1/\kappa^j},
\end{equation}

we can write the previous inequality as

\begin{equation}
I_{j+1} \leq C^{1/\kappa^j} \kappa^{j/\kappa^j} I_j.
\end{equation}

Since

\begin{equation}
\limsup_{j \to \infty} I_j \geq \lim_{j \to \infty} \left( \int_{B_{j+1}} \Phi(h)^{m_{j+1}} \right)^{(n+1)/m_j} = \sup_{B/2} \Phi(h)^{n+1},
\end{equation}

we finally get

\begin{equation}
\text{ess sup } \Phi(h)^{n+1} \leq \limsup_{j \to \infty} I_j \leq C^m \kappa^S I_0 \leq c \int_B \Phi(h),
\end{equation}
where
\[ S = \sum_{j=0}^{\infty} j \kappa^{-j} < \infty. \]

Next we will prove the counterpart of Lemma 16. We point out that some extra difficulties will appear due to the presence of \(|\nabla \log W|\) in the right-side of the Caccioppoli inequality (4.1). Moreover, we have to assume that the dimension of \(M\) is at least 3. Let us recall the definitions of the bounded \(C^1\)-function \(C\): \([0, \infty) \to [0, \infty)\) from (3.8) and (3.9) and functions
\[ L(r) = \log(1 + r) + C(r) \]
and
\[ w = \frac{r \log(1 + r)}{\sqrt{L(r)}}. \]
from (3.9) and (3.10) (with \(p = 2\)), respectively.

**Lemma 21.** Let \(M\) be a Cartan-Hadamard manifold of dimension \(n \geq 3\). Suppose that
\[ K(P) \leq -\frac{1 + \varepsilon}{r(x)^2 \log r(x)}, \]
for some constant \(\varepsilon > 0\), where \(K(P)\) is the sectional curvature of any plane \(P \subset T_xM\) that contains the radial vector \(\nabla r(x)\) and \(x\) is any point in \(M \setminus B(o, R_0)\). Fix \(\varepsilon \in (0, \varepsilon)\) and let \(R_1 \geq R_0\) be given by Proposition 7. Let \(U = B(o, R)\), with \(R > R_1\), and suppose that \(u \in C^2(U)\) is the unique solution of the minimal graph equation (1.1) in \(U\), with \(u|\partial U = \theta|\partial U\), where \(\theta \in C^\infty(M)\), with \(\|\theta\|_\infty \leq C\). Furthermore, suppose that \(|\nabla \log W(x)| \leq W(r(x))\), where \(W: [0, \infty) \to [0, \infty)\) is a continuous function that is independent of \(u\), and \(W(r) = o(1/r)\) as \(r \to \infty\). Then there exists a constant \(c_4 \geq 1\) that is independent of \(u\) such that
\[ \int_U \varphi(|u - \theta|/c_4)^2 L(r) \leq c_4 + c_4 \int_U F \left( \frac{c_4 |\nabla \theta| r \log(1 + r)}{L(r)} \right) L(r). \]

**Proof.** As in the proof of Lemma 16 we denote \(h = |u - \theta|/\nu\), where \(\nu \geq \nu_0\) will be fixed later. Recall from (5.11) with \(p = 2\) that
\[ n \left( \int_U \varphi(h)^2 L(r) \right)^{1/2} \leq 2 \left( \int_U |\nabla h|^2 \varphi'(h)^2 w^2 \right)^{1/2}. \]
We estimate the right-side as
\[ 2 \left( \int_U |\nabla h|^2 \varphi'(h)^2 w^2 \right)^{1/2} \leq 2/\nu \left( \int_U (\varphi'(h)|\nabla u| w + \varphi'(h)|\nabla \theta| w)^2 \right)^{1/2}, \]
\[ \leq 2/\nu \left( \int_U \varphi'(h)^2 |\nabla u|^2 w^2 \right)^{1/2} + 2/\nu \left( \int_U \varphi'(h)^2 |\nabla \theta|^2 w^2 \right)^{1/2}. \]
Let \(\delta \in (0, 1/1000)\) and suppose that \(\nu\) is so large that \(\|h\|_\infty \leq t_\delta\), where \(t_\delta > 0\) is a constant such that (2.13) and (2.14) hold for all \(t \in (0, t_\delta]\) with \(p = 2\). Then by
the Caccioppoli inequality \( \text{[1.1]}, \text{[2.13]}, \text{and [2.14]} \), the first term on the right-side of (4.11) can be estimated from above as

\[
\begin{align*}
2/\nu \left( \int_U \varphi'(h)^2 |\nabla u|^2 w^2 \right)^{1/2} &\leq \sqrt{2}(1+\delta)/\nu \left( \int_U \varphi'(h)|\nabla u|^2 w^2 \right)^{1/2} \\
&\leq \sqrt{2}(1+\delta)/\nu \left( \int_U \varphi'(h)|\nabla \theta|^2 w^2 + 8\nu \int_U |\nabla \log W|^2 w^2 \right)^{1/2} \\
&\leq \sqrt{2}(1+\delta)/\nu \left( \int_U \varphi'(h)|\nabla \theta|^2 w^2 + 4\nu^2(1+\delta)^2 \int_U \varphi(h)^2 |\nabla w|^2 \right)^{1/2} \\
&\leq \sqrt{2}(1+\delta)/\nu \left( 16 \int_U \varphi'(h)^2 |\nabla \theta|^2 w^2 + 4\nu^2(1+\delta)^2 \int_U \varphi(h)^2 |\nabla w|^2 \right)^{1/2} \\
&\leq \sqrt{2}(1+\delta)/\nu \left( \int_U \varphi'(h)^2 |\nabla \theta|^2 w^2 \right)^{1/2} + \sqrt{8}(1+\delta)^2 \left( \int_U \varphi(h)^2 |\nabla w|^2 \right)^{1/2} \\
&\leq 4\sqrt{2}(1+\delta)/\nu \left( \int_U \varphi'(h)^2 |\nabla \theta|^2 w^2 \right)^{1/2} + \sqrt{8}(1+\delta)^2 \left( \int_U \varphi(h)^2 |\nabla w|^2 \right)^{1/2} + 2(1+\delta)^2 \left( \int_U \varphi(h)^2 |\nabla \log W|^2 w^2 \right)^{1/2} + C.
\end{align*}
\]

Taking into account the upper bounds (3.13) and (3.14) for \(|\nabla w|\) we obtain

\[
\int_U \varphi(h)^2 |\nabla w|^2 \leq c + \int_U \varphi(h)^2 L(r),
\]

and therefore

\[
(n - \sqrt{8}(1+\delta)^2) \left( \int_U \varphi(h)^2 L(r) \right)^{1/2} \leq \frac{4\sqrt{2}(1+\delta) + 2}{\nu} \left( \int_U \varphi'(h)^2 |\nabla \theta|^2 w^2 \right)^{1/2} + 2(1+\delta)^2 \left( \int_U \varphi(h)^2 |\nabla \log W|^2 w^2 \right)^{1/2} + C.
\] (4.12)

Next we apply the complementary Young functions \( F(\sqrt{\cdot}) \) and \( G(\sqrt{\cdot})^2 \) as in the proof of Lemma 16 to estimate the first term on the right-side of (4.12)

\[
\begin{align*}
\int_U \varphi'(h)^2 |\nabla \theta|^2 w^2 &= \int_U \varphi'(h)^2 L(r) \left( \frac{|\nabla \theta|r \log(1+r)}{L(r)} \right)^2 \\
&\leq k \int_U G(\varphi'(h))^2 L(r) + k \int_U F \left( \frac{|\nabla \theta|r \log(1+r)}{\sqrt{k}L(r)} \right) L(r) \\
&= k \int_U \varphi(h)^2 L(r) + k \int_U F \left( \frac{|\nabla \theta|r \log(1+r)}{\sqrt{k}L(r)} \right) L(r)
\end{align*}
\]
for all \( k > 0 \). By the assumption \( |\nabla \log W| = o(1/r) \) we may estimate the second term on the right-side of (4.12) as
\[
\int_{U} \varphi(h)^2 |\nabla \log W|^2 w^2 = \int_{U} \varphi(h)^2 L(r) \left( \frac{|\nabla \log W| r \log(1 + r)}{\log(1 + r) + C(r)} \right)^2 
\leq \delta \int_{U} \varphi(h)^2 L(r) + C_6.
\]
Choosing \( k > 0 \) small enough and \( c_4 = \nu \) large enough we finally obtain (4.10).

4.2. Solving the asymptotic Dirichlet problem with Lipschitz boundary values. Since the asymptotic boundary \( \partial_\infty M \) is homeomorphic to the unit sphere \( S^{n-1} \subset T_o M \), we may interpret the given boundary value function \( f \in C(\partial_\infty M) \) as a continuous function on \( S^{n-1} \). In this section we solve the asymptotic Dirichlet problem for (1.1) with Lipschitz continuous boundary values \( f \in C(S^{n-1}) \). First we construct an extension of \( f \) as in [25]. We assume that, for all \( x \in M \) and for all 2-planes \( P \subset T_x M \),
\[
- b^2(r(x)) \leq K(P) \leq - a^2(r(x)),
\]
where \( a, b \colon [0, \infty) \to [0, \infty) \) are smooth functions that are constant in some neighborhood of 0 and
\[
a^2(t) = \frac{1 + \varepsilon}{t^2 \log t}, \\
b^2(t) = \frac{(\log t)^{2\varepsilon}}{t^2}
\]
for \( t \geq R_0 \). We identify \( \partial_\infty M \) with the unit sphere \( S^{n-1} \subset T_o M \) and assume that \( f \colon S^{n-1} \to \mathbb{R} \) is \( L \)-Lipschitz. We extend \( f \) radially to a continuous function \( \tilde{\theta} \) on \( M \setminus \{o\} \). The Lipschitz continuity of \( f \) and the curvature upper bound imply that
\[
\text{osc}(\tilde{\theta}, B(x, 3)) \leq \frac{eL}{f_a(r(x))}
\]
for \( x \in M \setminus \{o\} \), where \( f_a \) is the solution to the Jacobi equation [24]. Next we will define a smooth function \( \theta \) on \( M \) such that
\[
\lim_{x \to \xi} \theta(x) = f(\xi),
\]
for every \( \xi \in \partial_\infty M \) and that first and second order derivatives of \( \theta \) are controlled. In order to construct \( \theta \), we first fix a maximal 1-separated set \( Q = \{q_1, q_2, \ldots\} \subset M \setminus \{o\} \). For each \( x \in M \), we write \( Q_x = Q \cap B(x, 3) \). The curvature lower bound implies that
\[
\text{card} Q_x \leq c
\]
for some constant \( c \) independent of \( x \). We then define \( \theta \) as
\[
\theta(x) = \sum_{q_i \in Q} \tilde{\theta}(q_i) \varphi_i(x),
\]
where \( \{\varphi_i\} \) is a partition of the unity subordinate to \( \{B(q_i, 3)\} \) defined as follows. First fix a \( C^\infty \)-function \( \zeta \colon [0, \infty) \to [0, 1] \) such that \( \zeta[0, 1] = 1 \), \( \zeta[2, \infty] = 0 \), and
\[
\max\{|\zeta'(t)|, |\zeta''(t)|\} \leq c\chi_{[1,2]}(t).
\]
For \( q_i \in Q \) and \( x \in M \), let \( \eta_i(x) = \zeta(d(x, q_i)) \) and finally define

\[
\varphi_i(x) = \frac{\eta_i(x)}{\sum_j \eta_j(x)}.
\]

Following [25], one can easily check that \( \theta \) satisfies all the required properties. Moreover, the gradient of \( \theta \) satisfies

\[
(4.16) \quad |\nabla \theta|(x) \leq \frac{cL}{f_\alpha(r(x))}.
\]

for all \( r(x) \geq 1 \).

The next lemma is devoted to prove the decay assumption on \( \nabla \log W(x) \) used above in Lemma 21. We will use ideas from Ding, Jost, and Xin (see [18, Section 4]). We are grateful to J. Spruck for his help to obtain the decay estimate.

**Lemma 22.** Let \( M \) be a Cartan-Hadamard manifold satisfying the curvature assumption (4.13) for all 2-planes \( P \subset T_x M \). Suppose that \( \theta \in C(\bar{M}) \cap C^\infty(M) \) is an extension of a Lipschitz function \( f \in C(\partial_\infty M) \) as in (4.15). Let \( \Omega = B(o,S) \) and let \( u \in C^\infty(\Omega) \cap C(\bar{\Omega}) \) be the unique solution of (1.1) in \( \Omega \) with \( u|\partial \Omega = \theta|\partial \Omega \).

Then there exists a continuous function \( W : \mathbb{R}_+ \to \mathbb{R}_+ \) that is independent of \( S \) such that \( W(r) = o(1/r) \) as \( r \to \infty \) and

\[
(4.17) \quad |\nabla \log W(x)| \leq W(r(x))
\]

for \( x \in \Omega \).

**Proof.** Since sectional curvatures are bounded from below by a negative constant and \( |u| \leq \max_{\partial_\infty M} |f| \), we have

\[
\max_{\bar{\Omega}} |\nabla u| \leq C,
\]

with \( C \) independent of the radius \( S \). This estimate is obtained by using classical logarithmic type barriers to obtain boundary gradient estimates and then applying [36, Lemma 3.1]. In local coordinates \( x = (x^1, \ldots, x^n) \) the minimal graph equation can be written as

\[
\partial_j \left( \sqrt{\sigma} \frac{\sigma^{ij} u_i}{\sqrt{1 + |\nabla u|^2}} \right) = 0,
\]

where \( \{\partial_j\} \) is the associated coordinate frame, \( \sigma_{ij}dx^i dx^j \) is the Riemannian metric, \( \sigma = \det(\sigma_{ij}) \), and \( (\sigma^{ij}) = (\sigma_{ij})^{-1} \). Differentiating the equation in the direction \( \partial_k \) and setting \( w = \partial_k u \), we see that \( w \) satisfies

\[
L(w) + \partial_j f^j_k = 0,
\]

where \( L \) is defined by

\[
L(w) = \partial_j \left( \frac{\sigma}{\sqrt{1 + |\nabla u|^2}} \left( \sigma^{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} \right) w_i \right),
\]

with \( u^i = \sigma^{ij}u_j = \sigma^{ij}\partial_j u \), and

\[
f^j_k = \frac{u_i}{\sqrt{1 + |\nabla u|^2}} \partial_k (\sqrt{\sigma}\sigma^{ij}) - \frac{1}{2} \sqrt{\sigma \sigma^{ij}} \frac{u_i u_p u_q}{(1 + |\nabla u|^2)^2} \partial_k \sigma^{pq}.
\]

Fix \( p \in M \) and denote \( R = d(o, p) \) and

\[
\rho = \rho(R) = \left( \frac{R}{(\log R)^2} \right)^{2/3}.
\]
so that
\[ \tilde{\rho} := \frac{R}{(\log R)^\alpha \rho} \to \infty \]
as \( R \to \infty \). We claim that there are positive constants \( \alpha', \theta_1 \in (0, 1) \) and \( C \) such that there exist harmonic coordinates \((x_1, \ldots, x^n)\) on \( B(p, \theta_1 \rho) \) satisfying
\[
(4.18) \quad \sigma_{ij} + \frac{R}{(\log R)^\alpha \rho} |\nabla \sigma_{ij}| + \left( \frac{(\log R)^\alpha \rho}{R} \right)^{-1 - \alpha'} [\nabla \sigma_{ij}]_{C(p, \theta_1 \rho)} \leq C,
\]
where
\[ [\varphi]_{\alpha', C(p, \theta_1 \rho)} = \sup_{x, y \in B(p, \theta_1 \rho)} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^{\alpha'}}. \]
Since we are interested in the asymptotic behavior of \( \nabla \log W \) we may assume without loss of generality that \( R \) is so large that \( R - \rho \geq R/2 \geq R_0 \). Hence we have
\[ |\text{Riem}| \leq \frac{c (\log R)^\beta R}{R^2} \]
for all sectional curvatures in \( B(p, \rho) \). By the standard volume comparison theorem we obtain
\[ \text{Vol}(B(p, \rho)) \leq C \rho^n e^{-\frac{c (n-2)(\log R)^\beta}{R}}. \]
It follows that
\[ \|\text{Riem}\|_{L^{n/2}(B(p, \rho))} \leq C \left( \frac{(\log R)^\beta \rho}{R} \right)^2, \]
and, for \( q > n \),
\[ \rho^{2 - 2n/q} \|\text{Ric}\|_{L^q/2(B(p, \rho))} \leq C \left( \frac{(\log R)^\beta \rho}{R} \right)^2. \]
Then, using these last two estimates, [43, Theorem 7.1] applies and gives the existence of the harmonic coordinates described above. Using this system of coordinates, we will prove that \( \nabla u \) is uniformly Hölder.

Without loss of generality, we may assume that \( S \), the radius of \( \Omega \), is greater than \( 2R \). Let \( s \leq \theta_1 \rho/4 \) and recall that
\[ \tilde{\rho} = \frac{R}{(\log R)^\alpha \rho}. \]
We define \( M_4(s) = \sup_{B(p, 4s)} w, m_4(s) = \inf_{B(p, 4s)} w, M_1(s) = \sup_{B(p, s)} w, \) and \( m_1(s) = \inf_{B(p, s)} w \). Using (4.18) it is easily seen that
\[ |f_j^i| \leq \frac{C}{\tilde{\rho}} \]
on \( B(p, \theta_1 \rho) \). Next applying the weak Harnack inequality [21, Theorem 8.18], we have
\[
(4.19) \quad \frac{1}{s^n} \int_{B(p, 2s)} (M_4(s) - w) \leq C (M_4(s) - M_1(s) + s/\tilde{\rho}),
\]
and
\[
(4.20) \quad \frac{1}{s^n} \int_{B(p, 2s)} (w - m_4(s)) \leq C (m_1(s) - m_4(s) + s/\tilde{\rho}).
\]
Denote \( w(s) = M_1(s) - m_1(s) \). Since \( \text{Vol}(B(p, 2s)) \geq C_1 s^n \), for some constant \( C_1 \), using (4.19) and (4.20), we have
\[
C_1 w(4s) \leq \frac{\text{Vol}(B(p, 2s))}{s^n} w(4s) \leq C(w(4s) - w(s) + 2s/\tilde{\rho}).
\]
This implies that there exists \( \gamma \in (0, 1) \) such that, for all \( s \in [0, \theta_1 \rho/4] \),
\[
w(s) \leq \gamma w(4s) + 2s/\tilde{\rho}.
\]
Using Lemma 8.23 (notice that \( \tilde{\rho} \to \infty \) as \( R \to \infty \), we get that there exist \( \alpha \in (0, 1) \) and a positive constant \( C \) such that
\[
\|\nabla u\|_{C^\alpha(B(p, \theta_1 \rho))} \leq C \tilde{\rho}^{-\alpha}.
\]
Then the scaling invariant Schauder estimates implies that there exists a constant \( C \) depending on \( \alpha \) such that we have
\[
\sup_{B(p, \theta_1 \rho/2)} |D^i u| \leq C \rho^{-i} \sup_{B(p, \theta_1 \rho)} |u|, \quad \text{for } i = 1, 2.
\]
Since \( \sup_{B(p, \theta_1 \rho)} |u| \leq \max_{\partial_{\infty} M} |f| \) and
\[
|\nabla \log W| = \frac{|\nabla (\nabla u, \nabla u)|}{2\sqrt{1 + |\nabla u|^2}} \leq |\nabla (\nabla u, \nabla u)|,
\]
the claim (4.17) follows immediately from (4.21) by our choice of
\[
\rho = \left( \frac{R}{(\log R)^{2/3}} \right). \tag{4.24}
\]
\( \square \)

We are now ready to solve the asymptotic Dirichlet problem with Lipschitz boundary values.

**Lemma 23.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 3 \) satisfying the curvature assumption (1.17) for all 2-planes \( P \subset T_x M \), with \( x \in M \setminus B(o, R_0) \). Suppose that \( f \in C(\partial_{\infty} M) \) is \( L \)-Lipschitz when interpreted as a function on \( \mathbb{S}^{n-1} \subset T_o M \). Then the asymptotic Dirichlet problem for the minimal graph equation (1.1) is uniquely solvable with boundary values \( f \).

**Proof.** Let \( \theta \in C(\bar{M}) \cap C^\infty(M) \) be the extension of the given boundary data \( f \in C(\partial_{\infty} M) \) defined as above. We exhaust \( M \) by an increasing sequence of geodesic balls \( B_k = B(o, k), k \in \mathbb{N} \). Hence there exist smooth solutions \( u_k \in C(\bar{B_k}) \) of the minimal graph equation

\[
\begin{aligned}
\text{div} \left( \frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}} \right) &= 0, & \text{in } B_k, \\
\theta|_{\partial B_k} &= \theta.
\end{aligned}
\]

Then
\[
-\max_{x \in M} |\theta(x)| \leq u_k \leq \max_{x \in M} |\theta(x)|
\]
in \( B_k \) by the comparison principle. Standard arguments involving interior gradient estimates [39] Theorem 1.1] and (regularity) theory of elliptic PDEs imply that there exists a subsequence, still denoted by \( u_k \), that converges in \( C^2_{\text{loc}}(M) \) to a solution \( u \in C^\infty(M) \) of the minimal graph equation. Therefore the proof reduces to prove that \( u \) extends continuously to \( \partial_{\infty} M \) and satisfies \( u|_{\partial_{\infty} M} = f \). For each
By Lemma 20, we then get

\[
\int_M \varphi(|u - \theta|/\nu)^2 \leq \liminf_{k \to \infty} \int_{B(\nu, k)} \varphi(|u - \theta|/\nu)^2 \\
\leq \nu + \nu \int_M F\left(\frac{r \log(1 + r)}{L(r)}\right) L(r) < \infty.
\]

By Lemma 20 we then get

\[
\lim_{x \to \xi} \sup_{B(\overline{x}, s/2)} \varphi(|u - \theta|/\nu)^{2(n+1)} = 0
\]

for every \( \xi \in \partial_{\infty} M \). Hence \( u \) extends continuously to \( \partial_{\infty} M \) and satisfies \( u|\partial_{\infty} M = f \).

4.3. Solving the Dirichlet problem with continuous boundary values.

Proof of Theorem 5. Let \( f \in C(\partial_{\infty} M) \). Again we identify \( \partial_{\infty} M \) with the unit sphere \( S^{n-1} \subset T_{\infty} M \). Let \( (f_i) \) be a sequence of Lipschitz functions on \( S^{n-1} \) such that \( f_i \to f \) uniformly on \( S^{n-1} \). By the previous Lemma 20 there exist solutions \( u_i \in C(M) \cap C^\infty(M) \) of the minimal graph equation \((1.1)\) with \( u_i = f_i \) on \( \partial_{\infty} M \).

By the maximum principle,

\[
\sup_M |u_i - u_j| = \max_{\partial_{\infty} M} |f_i - f_j|,
\]

and applying the interior gradient estimate [39, Theorem 1.1], we conclude that the sequence \((u_i)\) converges in \( C(M) \cap C^2_{\text{loc}}(M) \) to a function \( u \in C(M) \) that is also a solution to \((1.1)\) in \( M \) and \( u = f \) on \( \partial_{\infty} M \). By regularity theory \( u \in C^\infty(M) \). To prove the uniqueness, suppose that \( u \) and \( v \) are both solutions of \((1.1)\), continuous in \( M \), with \( u = v \) on \( \partial_{\infty} M \), and \( u(y) > v(y) \) for some \( y \in M \). Let \( \delta = (u(y) - v(y))/2 \) and let \( U \) be the \( y \)-component of the set \( \{ x \in M : u(x) > v(x) + \delta \} \). Then \( U \) is a relatively compact domain and \( u = v + \delta \) on \( \partial U \). It follows that \( u = v + \delta \) in \( U \) which leads to a contradiction since \( y \in U \).
9. Cheeger, J., Gromov, M., and Taylor, M. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.* 17, 1 (1982), 15–53.

10. Cheng, S. Y. The Dirichlet problem at infinity for non-positively curved manifolds. *Comm. Anal. Geom.* 1, 1 (1993), 101–112.

11. Choi, H. I. Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds. *Trans. Amer. Math. Soc.* 287, 2 (1984), 691–716.

12. Collin, P., and Rosenberg, H. Construction of harmonic diffeomorphisms and minimal graphs. *Ann. of Math.* (2) 172, 3 (2010), 1879–1906.

13. Coulhon, T., Holopainen, I., and Saloff-Coste, L. Harnack inequality and hyperbolicity for subelliptic $p$-Laplacians with applications to Picard type theorems. *Geom. Funct. Anal.* 11, 6 (2001), 1139–1191.

14. Croke, C. B. Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. École Norm. Sup.* (4) 13, 4 (1980), 419–435.

15. Dajczer, M., Hinojosa, P. A., and de Lira, J. H. Killing graphs with prescribed mean curvature. *Calc. Var. Partial Differential Equations* 33, 2 (2008), 231–248.

16. Dajczer, M., Lira, J. H., and Ripoll, J. An interior gradient estimate for the mean curvature equation of Killing graphs. arXiv preprint arXiv:1206.2900 (2012).

17. Ding, Q., Jost, J., and Xin, Y. Minimal graphic functions on manifolds of non-negative Ricci curvature. Preprint arXiv:1310.2048v2, 2013.

18. do Espírito-Santo, N., and Ripoll, J. Some existence results on the exterior Dirichlet problem for the minimal hypersurface equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28, 3 (2011), 385–393.

19. Eberlein, P., and O’Neill, B. Visibility manifolds. *Pacific J. Math.* 46 (1973), 45–109.

20. Galvez, J. A., and Rosenberg, H. Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces. *Amer. J. Math.* 132, 5 (2010), 1240–1273.

21. Gilbarg, D., and Trudinger, N. S. Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

22. Heinonen, J., Kilpeläinen, T., and Martio, O. *Nonlinear potential theory of degenerate elliptic equations*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

23. Holopainen, I. Nonsolvability of the asymptotic Dirichlet problem for the $p$-Laplacian on Cartan-Hadamard manifolds. *Adv. Calc. Var.*. To appear.

24. Holopainen, I. Volume growth, Green’s functions, and parabolicity of ends. *Duke Math. J.* 97, 2 (1999), 319–346.

25. Holopainen, I. Asymptotic Dirichlet problem for the $p$-Laplacian on Cartan-Hadamard manifolds. *Proc. Amer. Math. Soc.* 130, 11 (2002), 3393–3400 (electronic).

26. Holopainen, I., and Ripoll, J. Nonsolvability of the asymptotic Dirichlet problem for some quasilinear elliptic PDEs on Hadamard manifolds. *Rev. Mat. Iberoamericana*. To appear.

27. Holopainen, I., and Väihäkangas, A. Asymptotic Dirichlet problem on negatively curved spaces. *J. Anal.* 15 (2007), 63–110.

28. Hsu, E. P. Brownian motion and Dirichlet problems at infinity. *Ann. Probab.* 31, 3 (2003), 1305–1319.

29. Kufner, A., John, O., and Fučík, S. *Function spaces*. Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

30. Meeks, W. H., and Rosenberg, H. The theory of minimal surfaces in $M \times \mathbb{R}$. *Comment. Math. Helv.* 80, 4 (2005), 811–858.

31. Ni, W. M. Brownian motion and the Dirichlet problem at infinity on two-dimensional Cartan-Hadamard manifolds. *Potential Anal.* 41, 2 (2014), 443–462.

32. Nelli, B., and Rosenberg, H. Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *Bull. Braz. Math. Soc. (N.S.)* 33, 2 (2002), 263–292.

33. Rigoli, M., and Setti, A. G. Liouville type theorems for $\phi$-subharmonic functions. *Rev. Mat. Iberoamericana* 17, 3 (2001), 471–520.

34. Ripoll, J., and Telichevesky, M. Complete minimal graphs with prescribed asymptotic boundary on rotationally symmetric Hadamard surfaces. *Geom. Dedicata* 161 (2012), 277–283.
[35] Ripoll, J., and Telichevesky, M. Regularity at infinity of Hadamard manifolds with respect to some elliptic operators and applications to asymptotic Dirichlet problems. Trans. Amer. Math. Soc. 367, 3 (2015), 1523–1541.

[36] Rosenberg, H., Schulze, F., and Spruck, J. The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$. J. Differential Geom. 95, 2 (2013), 321–336.

[37] Schoen, R., and Yau, S. T. Lectures on harmonic maps. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[38] Serre, J. Local behavior of solutions of quasi-linear equations. Acta Math. 111 (1964), 247–302.

[39] Spruck, J. Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbb{R}$. Pure Appl. Math. Q. 3, 3, Special Issue: In honor of Leon Simon. Part 2 (2007), 785–800.

[40] Sullivan, D. The Dirichlet problem at infinity for a negatively curved manifold. J. Differential Geom. 18, 4 (1983), 723–732 (1984).

[41] Vähäkangas, A. Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions. Potential Anal. 27, 1 (2007), 27–44.

[42] Vähäkangas, A. Dirichlet problem on unbounded domains and at infinity. Reports in Mathematics, Preprint 499, Department of Mathematics and Statistics, University of Helsinki, 2009.

[43] Yang, D. Convergence of Riemannian manifolds with integral bounds on curvature. II. Ann. Sci. École Norm. Sup. (4) 25, 2 (1992), 179–199.

UFRGS, INSTITUTO DE MATEMÁTICA, AV. BENTO GONÇALVES 9500, 91540-000 PORTO ALEGRE-RS, BRASIL.
E-mail address: jeanbaptiste.casteras@gmail.com

DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68, 00014 UNIVERSITY OF HELSINKI, FINLAND.
E-mail address: ilkka.holopainen@helsinki.fi

UFRGS, INSTITUTO DE MATEMÁTICA, AV. BENTO GONÇALVES 9500, 91540-000 PORTO ALEGRE-RS, BRASIL.
E-mail address: jaime.ripoll@ufrgs.br