Space Mapping of Spline Spaces over Hierarchical T-meshes

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Abstract

In this paper, we construct a bijective mapping between a biquadratic spline space over the hierarchical T-mesh and the piecewise constant space over the corresponding crossing-vertex-relationship graph (CVR graph). We propose a novel structure, by which we offer an effective and easy operative method for constructing the basis functions of the biquadratic spline space. The mapping we construct is an isomorphism. The basis functions of the biquadratic spline space hold the properties such as linearly independent, completeness and the property of partition of unity, which are the same with the properties for the basis functions of piecewise constant space over the CVR graph. To demonstrate that the new basis functions are efficient, we apply the basis functions to fit some open surfaces.

Keywords: Spline spaces over T-meshes, Dimension, CVR graph, Space mapping, Basis functions

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1. Introduction

Splines are useful tools for representing functions and surface models. Non-uniform Rational B-Splines (NURBS), which are defined on tensor product meshes, are the most popular splines in the industry. Due to the tensor product structure, however, the local refinement of NURBS is impossible; furthermore, NURBS models generally contain a large number of superfluous control points. Therefore, many splines that are defined on T-meshes are developed and can be adaptively locally refined.

There are four main types of splines that can be defined over T-meshes. Hierarchical B-splines provides a classical approach to obtain local refinement in geometric modeling, the construction of the basis guarantees nested spaces and linear independence of the basis functions. The definition is improved as hierarchical B-splines with the partition of unity in \cite{2}. An increasing number of published papers \cite{3, 4, 5} discuss the completeness and partition of unity. T-splines \cite{6, 7} are defined over T-meshes, where T-junctions between axis aligned segments are allowed. T-splines have been used efficiently in CAD applications, being able to produce watertight and locally refined models. However, the use of the most general T-spline concept in IGA is limited by the risk of linear dependence of the resulting splines \cite{8}. Therefore, analysis-suitable T-splines are introduced in \cite{9}. Polynomial splines over hierarchical T-meshes (PHT-splines) \cite{10} are developed directly from the spline spaces. The basis functions of PHT-splines are linearly independent and form a
partition of unity. An adaptive extended IGA (XIGA) approach based PHT-splines for modeling crack propagation is presented in \[11\]. More works are done for IGA in \[12, 13\]. LR-splines \[14\] are also an important kind of the splines defined over T-meshes, and their definition is inspired by the knot insertion refinement process of tensor B-splines, they also proposed an efficient algorithm to seek and destroy linear dependence relations. In practice, linear dependence of LR B-splines can be controlled and much knowledge exists with respect to mesh configurations resulting in linear dependent LR B-splines. In \[15\], a first analysis on the necessary conditions for encountering a linear dependence relation has been presented. In \[16\], different properties of the LR-splines are analyzed: in particular the coefficients for polynomial representations and their relation with other properties such as linear independence and the number of B-splines covering each element.

To discuss the splines from the view of spline spaces, \[17\] proposed the spline space over a T-mesh \(S(m, n, \alpha, \beta, \mathcal{T})\) which is a bi-degree \((m, n)\) piecewise polynomial spline space over the T-mesh \(\mathcal{T}\), with the smoothness order \(\alpha\) and \(\beta\) in two directions. When \(m \geq 2\alpha + 1\) and \(n \geq 2\beta + 1\), a dimension formula is given in \[17\], and the basis functions are constructed in \[10\]. In 2011, \[18\] discovered that the dimension of the associated spline space has instability over particular T-meshes, i.e., the dimension is associated not only with the topological information of the T-mesh but also with the geometric information of the T-mesh. In addition, \[19\] gives two additional examples of \(S(5, 5, 3, 3, \mathcal{T})\) and \(S(4, 4, 2, 2, \mathcal{T})\) for the instability of dimensions. To overcome the instability of dimensions, weighted T-meshes \[20\], diagonalizable T-meshes \[21\], and T-meshes for a hierarchical B-spline \[22\], over which the dimensions are stable, are developed. \[23\] addresses hierarchical T-meshes, which have a nature tree structure and have existed in the finite element analysis community for a long time. For a hierarchical T-mesh, \[24\] derives a dimension formula for biquadratic \(C^1\) spline spaces, and \[22\] provides a dimension formula for \(S(d, d, d − 1, d − 1, \mathcal{T})\) over a very special hierarchical T-mesh using the homological algebra technique. Using tools from homological algebra, \[25\] discusses the dimension of polynomial splines of mixed smoothness on T-meshes. \[26\] provides combinatorial bounds on the dimension for polynomial spline spaces of non-uniform bi-degree on T-meshes. \[27\] gives a dimension formula of \(S(3, 3, 2, 2, \mathcal{T})\) over a T-mesh that is more general than that in \[22\] but also a special hierarchical T-mesh. Using a corresponding crossing-vertex-relationship graph (CVR graph), \[28\] constructed basis functions of \(S(2, 2, 1, 1, \mathcal{T})\) over hierarchical T-meshes and all basis functions are B-spline basis functions. However, the basis construction in \[28\] need to obey the limitation that the level differences of the hierarchical T-meshes are not more than one. In other words, the basis construction method is not considered on the general hierarchical T-meshes.

In this paper, we overcome the limitations in \[28\], and we discuss the dimensions and construct the basis functions from a space mapping standpoint. For the hierarchical T-mesh \(\mathcal{T}\), we denote the corresponding CVR graph as \(\mathcal{G}\), we do the works as follows:

1. Without any additional restrictions over \(\mathcal{T}\), we give a bijective mapping between \(\mathcal{S}(2, 2, 1, 1, \mathcal{T})\) and \(\mathcal{S}(0, 0, −1, −1, \mathcal{G})\). And \(\mathcal{S}(2, 2, 1, 1, \mathcal{T})\) is isomorphic to \(\mathcal{S}(0, 0, −1, −1, \mathcal{G})\).
2. By the tools which are called T-structures, we give a general method to construct each basis function for \(\mathcal{S}(2, 2, 1, 1, \mathcal{T})\) when there is no limitation for level difference of \(\mathcal{T}\).
3. By the isomorphic space, we prove that the basis functions of \(\mathcal{S}(2, 2, 1, 1, \mathcal{T})\) hold the properties of linearly independence, completeness and partition of unity.

This paper is organized as follows. In Section \[2\], we recall some notations about hierarchical T-meshes, spline spaces over hierarchical T-meshes and B-net method. In Section \[23\] we give a bijective mapping for univariate spline spaces. It illuminates us for considering the mapping between the spline space over a hierarchical T-mesh and the piecewise constant space over the corresponding CVR graph in Section \[11\]. To ensure the mapping is bijective, we introduce some conclusions about T-structures, by which we describe a general method to construct the basis functions of \(\mathcal{S}(2, 2, 1, 1, \mathcal{T})\) in Section \[5\]. We discuss the properties of the mapping and the properties of the basis functions in Section \[6\]. In Section \[7\] the basis functions are applied to fit some open surfaces. We end the paper with conclusions and future works in Section \[8\].
2. Hierarchical T-meshes, spline spaces and B-net method

In this section, we recall some notations about hierarchical T-meshes, spline spaces and B-net method.

2.1. Hierarchical T-meshes and some notations for hierarchical T-meshes

Instead of considering general T-meshes, we focus our attention on $2 \times 2$ division hierarchical T-meshes as follows:

**Definition 2.1.** [10] Given a tensor product mesh (level 0), at least one cell of level $k$ be subdivided into $2 \times 2$ equal subcells, which are cells at level $k + 1$. The resulting T-mesh is called a **hierarchical T-mesh of $2 \times 2$ division**. The maximal level number that appears is defined as the **level** of the hierarchical T-mesh, we denote the mesh of level $k$ as $\mathcal{T}^k$.

![Diagram of hierarchical T-mesh](image1)

Figure 1: A hierarchical T-mesh.

![Diagram of vertices, edges, and cells](image2)

Figure 2: The vertices, edges and cells.

Fig. 1 illustrates the process of generating a hierarchical T-mesh. In the hierarchical T-mesh $\mathcal{T}$, the definitions of vertex, edge and cell are the same as in [17], we recall the notations of $\mathcal{T}$ as follows:

In the hierarchical T-mesh $\mathcal{T}$, a grid point in $\mathcal{T}$ is also called a **vertex** of $\mathcal{T}$. If a vertex is on the boundary grid line of $\mathcal{T}$, then it is called a **boundary-vertex**. Otherwise, it is called an **interior-vertex**. There are two types of interior-vertices. An interior-vertex of valence four is called a **crossing-vertex**. An interior-vertex of valence three is called a **T-junction**.

The line segment connecting two adjacent vertices on a grid line is called an **edge** of $\mathcal{T}$. If an edge is on the boundary of $\mathcal{T}$, then it is called a **boundary-edge**; otherwise it is called an **interior-edge**. If an edge is the longest possible line segment whose two end points are either boundary vertices or T-junctions, we refer to the edge as a **l-edge**. If an l-edge is comprised of some boundary-edges, then it is called a **boundary-l-edge**; otherwise, it is called an **interior-l-edge**. If the two end points of an interior-l-edge are both T-junctions, the l-edge is called a **T-l-edge**. [27]
If an edge is the longest possible line segment whose inner-vertices are T-junctions, the edge is referred to as a c-edge.

Each rectangular grid element is referred to as a cell of $\mathcal{T}$. A cell is called an interior-cell if all its edges are interior edges; otherwise, it is called a boundary-cell.

In Fig. 2, $b_i$, $i = 0, ..., 5$ are boundary-vertices, while $v_i$, $i = 0, ..., 5$ are interior-vertices, $v_2$ is a crossing-vertex while $v_0$ is a T-junction. $b_1v_2$ is an interior-edge while $b_1b_2$ is a boundary-edge, $b_2b_3$ is a boundary-l-edge while $v_0v_1$ is an interior-l-edge, $v_0v_1$ is also a T-l-edge, and $v_2v_3$ is a c-edge. The blue cell is an interior-cell while the green cell is a boundary-cell.

2.2. Spline spaces

Given a T-mesh $\mathcal{T}$, we use $\mathcal{F}$ to denote all of the cells in $\mathcal{T}$ and $\Omega$ to denote the region occupied by the cells in $\mathcal{T}$. The spline spaces are defined as [17]:

$$S(m, n, \alpha, \beta, T) := \{ f(x, y) \in C^{\alpha, \beta}(\Omega) : f(x, y)|_\phi \in P_{mn}, \forall \phi \in \mathcal{F} \},$$  

(1)

where $P_{mn}$ is the space of the polynomials with bi-degree $(m, n)$ and $C^{\alpha, \beta}$ is the space consisting of all of the bivariate functions continuous in $\Omega$ with order $\alpha$ along the $x$-direction and $\beta$ along the $y$-direction. It is obvious that $S(m, n, \alpha, \beta, \mathcal{T})$ is a linear space.

For a T-mesh $\mathcal{T}$ of $S(m, n, \alpha, \beta, \mathcal{T})$, we can obtain an extended T-mesh in the following fashion. $m$ edges are added to the horizontal boundaries averagely, $n$ edges are added to the vertical boundaries averagely, and then connect the boundary-vertices of $\mathcal{T}$ to the outermost edges. The resulting mesh, which we denote as $\mathcal{T}^e$, is called the extended T-mesh of $\mathcal{T}$ associated with $S(m, n, \alpha, \beta, \mathcal{T})$. $\mathcal{T}^e$ is also called an extended T-mesh. Fig. 3 shows an example of the extended T-mesh.

The corresponding biquadratic spline spaces over $\mathcal{T}$ with homogeneous boundary conditions (HBC) were defined as follows [23]:

$$S(m, n, \alpha, \beta, \mathcal{T}) := \{ f(x, y) \in C^{\alpha, \beta}(\mathbb{R}^2) : f(x, y)|_\phi \in P_{22}, \forall \phi \in \mathcal{F}, \text{and } f|_{\mathbb{R}^2 \setminus \Omega} \equiv 0 \}. $$  

(2)

One important observation in [23] is that the two spline spaces $S(m, n, \alpha, \beta, \mathcal{T})$ and $\overline{S}(m, n, \alpha, \beta, \mathcal{T}^e)$ are closely related.

**Theorem 2.2.** [23] Given a T-mesh $\mathcal{T}$, assume that $\mathcal{T}^e$ is its extension associated with $S(m, n, \alpha, \beta, \mathcal{T})$ and that $\Omega$ is the region occupied by the cells in $\mathcal{T}$. Then,

$$S(m, n, \alpha, \beta, \mathcal{T}) = \overline{S}(m, n, \alpha, \beta, \mathcal{T}^e)|_\Omega, $$

(3)

$$\dim S(m, n, \alpha, \beta, \mathcal{T}) = \dim \overline{S}(m, n, \alpha, \beta, \mathcal{T}^e). $$

(4)
2.3. B-net method

The B-net method is based on Bernstein-Bézier representation of polynomials. Refer to [17, 30] for details.

In Fig. 4 let \( f_1(x, y) \) and \( f_2(x, y) \) be two polynomials with bi-degree \((2,2)\) defined over two adjacent cells \( C_1 : [x_0, x_3] \times [y_0, y_1] \) and \( C_2 : [x_1, x_2] \times [y_0, y_1] \), respectively. They can be expressed in the Bernstein - Bézier forms:

\[
f_1(x, y) = \sum_{j=0}^{2} \sum_{k=0}^{2} b_{j,k}^1 B_j^2 \left( \frac{x-x_0}{x_3-x_0} \right) B_k^2 \left( \frac{y-y_0}{y_1-y_0} \right),
\]

\[
f_2(x, y) = \sum_{j=0}^{2} \sum_{k=0}^{2} b_{j,k}^2 B_j^2 \left( \frac{x-x_1}{x_2-x_1} \right) B_k^2 \left( \frac{y-y_1}{y_2-y_1} \right),
\]

where \( B_j^2(t) \) and \( B_k^2(t) \) are the Bernstein polynomials. \( b_{j,k}^1 \) and \( b_{j,k}^2 \) are referred to as the Bézier-ordinates (B-ordinates) of \( f_1(x, y) \) and \( f_2(x, y) \), respectively. \( b_{j,k}^1 \) corresponds to the point \( P_{j,k}^1 : \left( \frac{(2-j)x_0+jx_3}{2}, \frac{(2-k)y_0+ky_1}{2} \right) \), which is referred to as the domain-points \[31\] associated with \( C_1 \). \( b_{j,k}^2 \) corresponds to the point \( P_{j,k}^2 : \left( \frac{(2-j)x_1+jx_2}{2}, \frac{(2-k)y_0+ky_2}{2} \right) \), which is referred to as the domain-points associated with \( C_2 \). The domain-points of \( C_1 \) and \( C_2 \) are denoted by “•” and “◦” respectively.

As \( f_1(x, y) \) and \( f_2(x, y) \) are \( C^1 \) continuous across their common boundary, when \( 0 \leq j \leq 2, 1 \leq k \leq 2 \) are given, \( b_{j,k}^1 \), \( 0 \leq j \leq 2, 0 \leq k \leq 1 \) are determined. As shown in Fig. 4 if \( f_1(x, y) \) and \( f_2(x, y) \) are \( C^1 \) continuous across their common boundary, when the two rows of the B-ordinates in the green domain are given, the two rows of B-ordinates in the yellow domain are determined. When \( C_1 \) and \( C_2 \) are two vertical adjacent cells, we have similar conclusions. We call the B-ordinates that correspond to the domain points on the cell \( C \) as the B-ordinates on \( C \) for convenience.

By the preliminary knowledge above, we mainly discuss the spline space \( S(2,2,1,\mathcal{T}) \) in this paper. \[23\] gives the conclusion as

\[
\dim S(2,2,1,\mathcal{T}) = N_{\mathcal{T}},
\]

where \( N_{\mathcal{T}} \) is the number of cells in \( \mathcal{T} \). The piecewise constant space on \( \mathcal{T} \) is \( S(0,0,-1,-1,\mathcal{T}) \), and

\[
\dim S(0,0,-1,-1,\mathcal{T}) = N_{\mathcal{T}},
\]

we obtain

\[
\dim S(2,2,1,\mathcal{T}) = \dim S(0,0,-1,-1,\mathcal{T}). \tag{7}
\]

By Theorem \[2.2\] to consider the spline space over a T-mesh, we only need to consider the corresponding spline space with homogeneous boundary conditions over its extended T-mesh. The mapping for univariate spline spaces can enlighten us well.
3. Mapping for univariate spline spaces

In this section, we construct a bijective mapping between the univariate quadratic spline space and the corresponding univariate piecewise constant spline space. By the massage of basis function of the univariate piecewise constant spline space, a new method for constructing the basis functions of the quadratic spline space is given.

3.1. The univariate spline space and some notations

Given the knots \( T : t_0 < t_1 < ... < t_n \), we use \( \mathcal{F} \) to denote all the intervals in \( T \), \( \phi \) is referred to as an element of \( \mathcal{F} \), and \( \Omega \) to denote the range occupied by the intervals in \( T \). All of the interior knots \( t_1 < t_2 < ... < t_{n-1} \) are referred to as the C-knots of \( T \), which is denoted as \( \mathcal{G} \).

The quadratic spline spaces is defined as:

\[
S^{(2, 1)}(T) := \{ p(t) \in C^1(\Omega) : p(t)|_{\phi} \in \mathbb{P}_2, \forall \phi \in \mathcal{F} \}. \tag{8}
\]

For the knot sequence \( T \) of \( S^{(2, 1)}(T) \), we can obtain an extended knot sequences by inserting two knots at each end of \( T \). The extension of \( T \) is referred to as \( T^\varepsilon \):

\[
t^-2 < t^-1 < t^0 < t^1 < ... < t^n < t_{n+1} < t_{n+2}.
\]

The corresponding quadratic spline spaces over \( T \) with homogeneous boundary conditions (HBC) can be defined as follows:

\[
\overline{S}^{(2, 1)}(T) := \{ p(t) \in C^1(\mathbb{R}) : p(t)|_{\phi} \in \mathbb{P}_2, \forall \phi \in \mathcal{F}, p(t)|_{\mathbb{R}\setminus\Omega} \equiv 0 \}. \tag{9}
\]

The two spline spaces in Equations (8) and (9) are closely related as follows:

\[
S^{(2, 1)}(T) = \overline{S}^{(2, 1)}(T)|_{\Omega}. \tag{10}
\]

\[
dim S^{(2, 1)}(T) = dim \overline{S}^{(2, 1)}(T^\varepsilon). \tag{11}
\]

With Equation (10) and Equation (11), to consider the spline space over the knots, we need only to consider the corresponding spline space with homogeneous boundary conditions over its extended knots. We just need to construct the bijective mapping between \( S^{(2, 1)}(T) \) and \( \overline{S}^{(0, -1)}(0, -1, G) \), where \( G^\varepsilon \) is the C-knots of \( T^\varepsilon \).

3.2. The mapping between \( \overline{S}^{(2, 1)}(T) \) and \( \overline{S}^{(0, -1)}(0, -1, G) \)

In this subsection, we first define a mapping functional by the B-ordinates. And then, we use the mapping functional to define the mapping formulae between \( \overline{S}^{(2, 1)}(T) \) and \( \overline{S}^{(0, -1)}(0, -1, G) \). Let \( p(t) \) be a polynomial with degree 2 defined over the interval \( I : [t_0, t_1] \). It can be expressed in the Bernstein - Bézier form:

\[
p(t) = \sum_{j=0}^{2} b_j B^2_j(u), u = \frac{t - t_0}{t_1 - t_0}, \tag{12}
\]

where \( B^2_j(u) \) is the quadratic Bernstein polynomial and \( \sum_{j=0}^{2} B^2_j(u) = 1 \). \( b_j \) is referred to as the Bézier-ordinates(B-ordinates) corresponding to \( t_j : \frac{(2-j)t_0+jt_1}{2}, j = 0, 1, 2 \).

With Equation (12) \( (t - t_0)^2 \) is the factor of \( B_0^2(u) \) and \( (t - t_1)^2 \) is the factor of \( B_1^2(u) \), together with \( \sum_{j=0}^{2} B^2_j(u) = 1 \), the mapping functional is defined as follows:
Definition 3.1. Given the quadratic polynomial function \( p(t) \) and \( I = [t_0, t_1] \), let 
\[
u = \frac{t - t_0}{t_1 - t_0},
\]
we define the mapping functional \( \varphi \),
\[
\varphi(p(t) : I) = b_1|_I,
\]
where \( b_1|_I \) denotes a piecewise constant function whose value is \( b_1 \) on \( I \).

We can define the mapping formulae via the mapping functional in Definition 3.1 as follows:

Definition 3.2. Given the knots \( T : t_0 < t_1 < ... < t_n \), the C-knots of \( T \) are denoted as \( G : t_1 < t_2 < ... < t_{n-1} \). Each interior interval of \( T \) is denoted as \( I \), the corresponding interval of \( I \) on \( G \) is denoted as \( IG \). We give the mapping formulae between \( S(2, 1, T) \) and \( S(0, -1, G) \) as:
\[
\Phi : S(2, 1, T) \rightarrow S(0, -1, G),
\]
\[
\varphi(p(t)|_I : I) \rightarrow q(t)|_{IG},
\]
where \( p(t) \in S(2, 1, T) \), \( p(t)|_I \) denotes the expression of \( p(t) \) on \( I \), \( q(t) \in S(0, -1, G) \), and \( q(t)|_{IG} \) denotes the expression of \( q(t) \) on \( IG \).

Fig. 5(a) shows \( p(t) \in S(2, 1, T) \) on some interior intervals of \( T \), Fig. 5(c) shows the mapping result \( q(t) \in S(0, -1, G) \) on the corresponding intervals. In fact, the value of \( q(t) \) on each interval in Fig. 5(c) is the B-ordinate on the centre of each interval, which is shown in Fig. 5(b).

Lemma 3.3. The mapping defined in Definition 3.2 is injective.

Proof. Obviously, \( \Phi(p(t)) \equiv 0 \) implies \( p(t) \equiv 0 \) for \( p(t) \in S(2, 1, T) \). Thus, \( \Phi \) is injective.

From Lemma 3.3, the mapping we defined in Definition 3.2 is an injective mapping. To ensure the mapping is a bijective mapping, for each basis function of \( S(0, -1, G) \), we need to construct a basis function of \( S(2, 1, T) \).

3.3. Construction of the basis functions for \( S(2, 1, T) \)

In this subsection, we first initialize the B-ordinates of the quadratic basis function in \( S(2, 1, T) \) via the value of a basis function in \( S(0, -1, G) \), and then give an algorithm to calculate the B-ordinates of the quadratic basis function.

Given the basis function of \( S(0, -1, G) \) in Fig. 6(a):
\[
q(t) = \begin{cases} 
1, & [t_1, t_2] \\
0, & \text{other intervals}.
\end{cases}
\]
With Equation (14), we initialize the B-ordinates of \( p(t) \in S(2, 1, T) \) as:

\[
b^i_j = \begin{cases} 
1, & i = 1 \\
0, & i \neq 1
\end{cases}
\]

which are the B-ordinates on "•" in Fig. 7. Obviously, the support of \( p(t) \) is \([t_0, t_3]\). To calculate all of the B-ordinates \( b^i_j, i = 0, 1, 2; j = 0, 2 \) for the polynomial function \( p(t) \in S(2, 1, T) \) we give some conclusions as follows:

**Proposition 3.4.** We use Fig. 7 to illustrate some conclusions as follows:

1. \( p(t) \) is \( C^1 \) continuous on \( t_1 \) if and only if \( (t_1, b^0_2) \) occupies on the linear function that determined by \( (\frac{t_0 + t_1}{2}, b^0_1) \) and \( (\frac{t_1 + t_2}{2}, b^1_1) \).

2. \( p(t) \) is \( C^1 \) continuous on \( t_2 \) if and only if \( (t_2, b^2_1) \) occupies on the linear function that determined by \( (\frac{t_1 + t_2}{2}, b^1_1) \) and \( (\frac{t_2 + t_3}{2}, b^2_1) \).

**Proof.** 1. As \( p(t) \) is \( C^1 \) continuous on \( t_1 \), we obtain \( b^0_1 = b^2_0 \) and

\[
\frac{b^0_2 - b^0_1}{t_1 - t_0} = \frac{b^1_1 - b^0_2}{t_2 - t_1}.
\]

with Equation (16), we obtain

\[
\frac{b^0_2 - b^0_1}{b^1_1 - b^0_1} = \frac{t_1 - \frac{t_1 + t_0}{2}}{t_2 - t_1},
\]

with Equation (17), the point \((t_1, b^0_0)\) is on the linear function that is determined by the points \( (\frac{t_0 + t_1}{2}, b^0_0) \) and \( (\frac{t_1 + t_2}{2}, b^1_1) \).

The reverse proving process can be derived naturally.

2. Similar to 1, the proposition is correct.
Algorithm 1: Calculate the B-ordinates of \( p(t) \in \mathbb{S}(2,1,T) \)

**Input:** The support \([t_0,t_1,t_2,t_3]\); \( b_0^i, b_1^i, b_2^i \).

**Output:** All of the B-ordinates of \( p(t) \) on \([t_0,t_3]\) in Fig. 7.

1. Using the two points \((\frac{t_1+t_2}{2},0)\) and \((\frac{t_2+t_3}{2},1)\) to calculate the linear function on \((t_1,b_2^0)\);
2. Calculate the B-ordinate on \( t_1 \) as \( b_2^0 = \frac{t_2-t_1}{t_2-t_0} \);
3. Using the two points \((\frac{t_1+t_2}{2},1)\) and \((\frac{t_2+t_3}{2},0)\) to calculate the linear function on \((t_2,b_2^0)\);
4. Calculate the B-ordinate on \( t_2 \) as \( b_0^2 = \frac{t_2-t_1}{t_1-t_3} + 1 \);
5. Obtain the B-ordinates on \([t_1,t_2]\) as \( \{b_2^1,1,b_0^0\} \);
6. Using the \( C^1 \) continuous condition to calculate that the B-ordinates on \( t_0 \) and \( \frac{t_1+t_2}{2} \) are 0;
7. Obtain the B-ordinates on \([t_0,t_1]\) as \( \{0,0,b_2^0\} \);
8. Using the \( C^1 \) continuous condition to calculate that the B-ordinates on \( \frac{t_2+t_3}{2} \) and \( t_3 \) are 0;
9. Obtain the B-ordinates on \([t_2,t_3]\) as \( \{b_2^0,0,0\} \);

Then, given the B-ordinates by Equation (11), we can calculate the B-ordinates of \( p(t) \in \mathbb{S}(2,1,T) \) via Algorithm 1. We show \( p(t) \) in Fig. 6(b), \( p(t) \) is \( C^1 \) continuous, and \( p(t) = \Phi^{-1}(q(t)) \).

### 3.4. The isomorphic univariate spaces and properties

In this subsection, we prove that the mapping is bijective, \( \mathbb{S}(2,1,T) \) is isomorphic to \( \mathbb{S}(0,-1,G) \), the basis functions of \( \mathbb{S}(2,1,T) \) hold the properties of are linearly independence, partition of unity and completeness.

**Theorem 3.5.** The mapping defined in definition 3.2 holds the property of bijectivity.

**Proof.** For each basis function of \( \mathbb{S}(0,-1,G) \), we can obtain a quadratic function of \( \mathbb{S}(2,1,T) \) via Algorithm 1; the mapping is surjective. As the mapping is injective, the mapping holds the property of bijectivity.

As the mapping between \( \mathbb{S}(2,1,T) \) and \( \mathbb{S}(0,-1,G) \) is bijective. We obtain the following corollary naturally.

**Corollary 3.6.** \( \mathbb{S}(2,1,T) \) is isomorphic to \( \mathbb{S}(0,-1,G) \).

**Theorem 3.7.** The basis functions of \( \mathbb{S}(2,1,T) \), which are constructed in 3.3, hold the properties of linearly independence, partition of unity and completeness on \([t_0,t_n]\).

**Proof.** Assume that the basis functions of \( \mathbb{S}(0,-1,G^c) \) are \( q_i(t) \), the basis functions of \( \mathbb{S}(2,1,T^c) \) are \( p_i(t), \), \( i = 1,...,N \), where \( G^c \) is the C-knots of \( T^c \). We obtain that the mapping between \( \mathbb{S}(2,1,T^c) \) and \( \mathbb{S}(0,-1,G^c) \) is bijective, and \( \mathbb{S}(2,1,T^c) \) is isomorphic to \( \mathbb{S}(0,-1,G^c) \).

As the spaces are linear spaces and \( \Phi^{-1}(q_i(t)) = p_i(t) \), we obtain
\[
\sum_{i=1}^{N} p_i(t) = \sum_{i=1}^{N} \Phi^{-1}(q_i(t)) = \Phi^{-1}(\sum_{i=1}^{N} q_i(t)).
\]

As \( \sum_{i=1}^{N} q_i(t) = 1, t \in [t_0,t_{n+1}] \), the B-ordinate on the centre position of each interior interval on \( T^c \) are 1. By Algorithm 1, the B-ordinates on each interval of \( T \) is 1. With Equation (10) and Equation (11), the basis functions of \( \mathbb{S}(2,1,T) \) have partition of unity on \([t_0,t_n]\).

As the basis functions of \( \mathbb{S}(0,-1,G^c) \) are linearly independent and complete on \([t_0,t_n]\), and \( \mathbb{S}(2,1,T^c) \) is isomorphic to \( \mathbb{S}(0,-1,G^c) \), the polynomial functions of \( \mathbb{S}(2,1,T^c) \) are linearly independent and complete on \([t_0,t_n]\). With Equation (10) and Equation (11), the basis functions of \( \mathbb{S}(2,1,T) \), which are constructed in 3.3 are linearly independent and complete on \([t_0,t_n]\). The theorem is proved.
T-cell

An interior-cell that at least one of four corner vertices is a T-junction.

One-neighbour-cell

The level of the T-connection

The level of the one-neighbour-cells corresponds to the T-connection.

Table 1: Some notations for hierarchical T-meshes

Till now, we construct a bijective mapping between $S(2, 1, T)$ and $S(0, -1, G)$, the two spaces are isomorphic to each other, some important properties are the same for the basis functions of the two spaces. For $S(2, 2, 1, 1, T)$, we want to obtain similar conclusions. We denote $S(d, d, d - 1, d - 1, T)$ as $S^d(T)$ for convenience. By Equation 7 we first discuss the spline space $S^2(T)$ with homogeneous boundary conditions, which is denoted by $S^2(T)$.

4. The mapping between $S^2(T)$ and $S^0(G)$

In this section, we will introduce the mapping between $S^2(T)$ and $S^0(G)$.

4.1. Some notations and CVR graphs

Before we give the mapping, we introduce some notations for a hierarchical T-mesh in Table 1, we also give some abbreviations in brackets for convenience.

We use Fig. 8(a) to introduce the notations in Table 1. In Fig. 8(a), cell 0 and cell 5 are P-cells, while cell 1, cell 2 and cell 3 are T-cells. Cell 1 and cell 2 are T-connected, cell 2 and cell 3 are T-connected. The T-connection, which can be denoted as $TC_0$, is the union that consists of cell 1, cell 2 and cell 3. As cell 5 is a P-cell, the gray domain is a P-domain. The green domain is the T-connection-domain of $TC_0$, the domain inside the red square is the T-rectangle-domain of $TC_0$, the domain-centre of the T-rectangle-domain is denoted as “•” in Fig. 8(a). Cell 4 is the one-neighbour-cell of $TC_0$, the level of cell 4 is the level of $TC_0$.

In [23], Definition 4.1 is introduced to propose a topological explanation to the dimension formula of $S(2, 2, 1, 1, T)$.
Given a hierarchical T-mesh $\mathcal{T}$, we can construct a graph $\mathcal{G}$ by retaining the crossing-vertices and the line segments with two end points that are crossing-vertices and removing the other vertices and the edges in $\mathcal{T}$. $\mathcal{G}$ is called the crossing-vertex-relationship graph (CVR graph for short) of $\mathcal{T}$.

We introduce some notations of CVR graph for the mapping in Table 2, we also give some abbreviations in brackets for convenience.

We also use Fig. 8(a) to illustrate the notations in Table 2. Fig. 8(b) shows the CVR graph of the hierarchical T-mesh $\mathcal{T}$ for short) of $\mathcal{T}$ in Fig. 8(a). The P-cell 0 in 8(a) corresponds to the P-g-cell 0 in 8(b). In Fig. 8(a), for the T-connection $\mathcal{T}G$, the T-connection-domain of $\mathcal{T}G$ corresponds to the T-g-cell 1 in Fig. 8(b). From the relationship between the cells of a hierarchical T-mesh and its CVR graph, we consider the mapping between $\mathcal{S}^2(\mathcal{T})$ and $\mathcal{S}^0(\mathcal{G})$.

### Table 2: Cells for CVR graphs

| Notations | Definitions |
|-----------|-------------|
| g-cell ($GC$) | A grid element in CVR graph. |
| P-g-cell ($P[GC]$) | A g-cell corresponds to a P-cell in $\mathcal{T}$. |
| T-g-cell ($T[GC]$) | A g-cell corresponds to a T-connection-domain in $\mathcal{T}$. |

#### 4.2. The mapping formulae between $\mathcal{S}^2(\mathcal{T})$ and $\mathcal{S}^0(\mathcal{G})$

By the notations in Table 1 and Table 2, we use the B-ordinates to define a functional, and then use the functional to construct the mapping between $\mathcal{S}^2(\mathcal{T})$ and $\mathcal{S}^0(\mathcal{G})$.

For the Bernstein polynomials $B^2_j(u)B^2_k(v), j, k = 0, 1, 2$ on $[x_0, x_1] \times [y_0, y_1]$, where $u = \left(\frac{x-x_0}{x_1-x_0}\right), v = \left(\frac{y-y_0}{y_1-y_0}\right)$, we obtain

$$B^2_j(u)B^2_k(v) = 1 - \sum_{j=0}^{2} B^2_j(u)B^2_0(v) - \sum_{j=0, j \neq 1}^{2} B^2_j(u)B^2_1(v) - \sum_{j=0}^{2} B^2_j(u)B^2_2(v). \quad (18)$$

As $B^2_j(u), j = 0, 2$ possess the factors $(x-x_0)^2$ or $(x-x_1)^2$, and $B^2_k(v), k = 0, 2$ possess the factors $(y-y_0)^2$ or $(y-y_1)^2$. We can give the functional as follows:

#### Definition 4.4. Given $f(x, y) \in \mathcal{S}^2(\mathcal{T})$, and $\mathcal{D} := [x_0, x_1] \times [y_0, y_1]$ is a rectangular domain. Let $u = \frac{x-x_0}{x_1-x_0}, v = \frac{y-y_0}{y_1-y_0}, f(x, y)$ can be expressed as:

$$f(x, y) = \sum_{j=0}^{2} \sum_{k=0}^{2} b_{j,k}B^2_j(u)B^2_k(v),$$

we obtain

$$f(x, y) = b_{1,1} + \sum_{j=0}^{2} (b_{j,0} - b_{1,1})B^2_j(u)B^2_0(v) + \sum_{j=0, j \neq 1}^{2} (b_{j,1} - b_{1,1})B^2_j(u)B^2_1(v) + \sum_{j=0}^{2} (b_{j,2} - b_{1,1})B^2_j(u)B^2_2(v).$$

We define a mapping functional $\varphi$,

$$\varphi(f(x, y) : \mathcal{D}) = b_{1,1}|_{\mathcal{D}},$$

where $b_{1,1}|_{\mathcal{D}}$ denotes a piecewise constant function whose value is $b_{1,1}$ on $\mathcal{D}$.
4.3. The injectivity property of the mapping

In Fig. 9(a), \( C_0 \) and \( C_1 \) are two aligned P-cells in \( \mathcal{T} \). \( P_{11}^1 \) is the center domain point of \( C_i, i = 0, 1 \). Let \( f_i(x, y) \) be the polynomial with bi-degree \((2, 2)\) defined over \( C_i \), the B-ordinate on \( P_{11}^1 \) is denoted as \( b_{11}^1 \), the domain that covers \( C_i \) is denoted as \( PD_i, i = 0, 1 \). Applying the functional \( \varphi \) in Definition 4.2, \( \varphi(f_i(x, y): PD_i) = b_{11}^1|PD_i, i = 0, 1 \).

In Fig. 9(b), \( C_0 \) is the one-neighbor-cell of \( TC \) in \( \mathcal{T} \), the T-rectangle-domain of \( TC \) is denoted as \( TRD \). Let \( f_i(x, y) \) be the polynomial with bi-degree \((2, 2)\) defined over \( C_i, i = 0, ..., k \),

\[
\begin{align*}
  f_1(x, y) &= f_0(x, y) + (x - x_1)^2v(y), \\
  f_2(x, y) &= f_0(x, y) + (x - x_1)^2v(y) + c_1(x - x_1)^2(y - y_1)^2, \\
  &\vdots \\
  f_k(x, y) &= f_0(x, y) + (x - x_1)^2v(y) + c_1(x - x_1)^2(y - y_1)^2 + ... + c_{k-1}(x - x_1)^2(y - y_k)^2, v(y) \in \mathbb{P}_2(y), c_i \in \mathbb{R}.
\end{align*}
\]

As \( f_i(x, y), i = 1, ..., k \) possess the cofactor \((x - x_1)^2\), \( \varphi(f_i(x, y): TRD) = \varphi(f_0(x, y): TRD) \).

Given a hierarchical T-mesh \( \mathcal{T} \), \( \mathcal{G} \) denotes the CVR graph of \( \mathcal{T} \). For \( f(x, y) \in \mathbb{S}^2(\mathcal{T}) \), we denote the support of \( f(x, y) \) as \( \text{Sup}(f) \). From Table 1 and Table 2, \( PC \) is denoted as a P-cell of \( \text{Sup}(f) \), the P-domain of \( PC \) is denoted as \( PD \), the P-g-cell corresponds to \( PC \) in \( \mathcal{G} \) is denoted as \( PGc \); \( TC \) is denoted as a T-connection of \( \text{Sup}(f) \), the T-rectangle-domain of \( TC \) is denoted as \( TRD \). And the T-g-cell corresponds to \( TC \) in \( \mathcal{G} \) is denoted as \( TGc \). We can define the mapping between \( \mathbb{S}^2(\mathcal{T}) \) and \( \mathbb{S}^0(\mathcal{G}) \) as follows:

**Definition 4.3.** The mapping formulae between \( \mathbb{S}^2(\mathcal{T}) \) and \( \mathbb{S}^0(\mathcal{G}) \) is defined as:

\[
\Phi: \mathbb{S}^2(\mathcal{T}) \to \mathbb{S}^0(\mathcal{G}),
\]

\[
\begin{cases}
  \varphi(f(x, y)|_{PC}: PD) \to g(x, y)|_{PGc} \\
  \varphi(f(x, y)|_{TC}: TRD) \to g(x, y)|_{TGc}.
\end{cases}
\]

Where \( f(x, y) \in \mathbb{S}^2(\mathcal{T}) \), \( f(x, y)|_{PC} \) denotes the expression of \( f(x, y) \) on \( PC \), \( g(x, y) \in \mathbb{S}^0(\mathcal{G}) \), and \( g(x, y)|_{PGc} \) denotes the expression of \( g(x, y) \) on \( PGc \), which corresponds to \( PC \); \( f(x, y)|_{TC} \) denotes the expression of \( f(x, y) \) on the one-neighbor-cell of \( TC \), and \( g(x, y)|_{TGc} \) denotes the expression of \( g(x, y) \) on \( TGc \), which corresponds to \( TC \).

4.3. The injectivity property of the mapping

**Lemma 4.4.** Given a hierarchical T-mesh \( \mathcal{T} \), for each T-connection \( TC \in \mathcal{T} \), the T-rectangle-domain of \( TC \) is denoted as \( TRD \). At least one one-neighbor-cell of \( TC \) exists, and for all the one-neighbor-cells of \( TC \), apply the functional \( \varphi \) with the polynomial of each one-neighbor-cell on \( TRD \), the results are the same.
Proof. Without loss of generality, we use Fig. 10 to illustrate the lemma.

1. At least one one-neighbour-cell of $\mathcal{T}C$ exists.

In Fig. 10(a), we denote the level of $C_i$ as $l_i$, where $i = 0, 1, 2, 3$. As $v_0$ is a T-junction, we get $l_2 \leq l_0$ and $l_2 \leq l_1$. As $v_1$ is a T-junction, we get $l_3 \leq l_2$. Then, $C_3$ is the cell with the lowest level of $C_0, C_1, C_2, C_3$. In a similar manner, one-neighbour-cell must exist.

2. If at least two one-neighbour-cells of $\mathcal{T}C$ exist, they are connected to $\mathcal{T}C$ and have the same level.

According to Fig. 10 (b), the neighbour cells of $\mathcal{T}C$ are $C_1, C_2, C_3$ and $C_4$. The maximum level of the cells in $\mathcal{T}C$ is $l_0$, and the level of $C_1$ is $l_i$, where $i = 1, 2, 3, 4$. As the mesh is a hierarchical T-mesh, we can assume $l_1 = l_3 = l$, and by (1), $l < l_0$.

(1) Assume $l_2 > l$, then $l_4 < l$.

We use proofs by contradiction to prove $l_4 < l$. If $l_4 > l$, by the assumption, $l_2 > l$, then $\mathcal{T}C$ will be divided, and we obtain a contradiction. Thus, $l_4 < l$ and $C_4$ is the only one-neighbour-cell.

(2) Assume $l_2 < l$ and $l_4 < l$.

If $l_2 < l_4$, $C_2$ is the only one-neighbour-cell. If $l_2 > l_4$, $C_4$ is the only one-neighbour-cell. If $l_2 = l_4$, $C_2$ and $C_4$ are connected to $\mathcal{T}C$.

Thus, if $\mathcal{T}C$ has at least two one-neighbour-cells, the levels of the one-neighbour-cells are same.

3. The one-neighbour-cell is aligned with $\mathcal{T}C$.

According to Fig. 10 (c), $v_0$ must be cross-vertexes; otherwise, the one-neighbour-cell of $\mathcal{T}C$ is not the lowest level cell.

From the above, the mapping is well defined and the mapping result is unique. The lemma is proved.

Theorem 4.5. The mapping defined in Equation (19) is injective.

Proof. By Lemma 4.4, the mapping result on each T-connection is unique, and $\Phi(f(x, y)) \equiv 0$ implies $f(x, y) \equiv 0$ for $f(x, y) \in S_2^T(\mathcal{T})$. Thus, $\Phi$ is injective.

The mapping we defined in Definition 4.3 is injective. To verify that the mapping is a bijective mapping, we need to confirm the mapping is surjective. In other words, given a basis function $q(x, y) \in S_0^G(\mathcal{F})$, we need to construct the corresponding basis function $p(x, y) \in S_2^T(\mathcal{T})$, and $p(x, y)$ is the inverse image of $q(x, y)$.
5. Construction of the basis functions for $S^2(\mathcal{T})$

In this section, we give a general method to construct the basis functions of $S^2(\mathcal{T})$ when there is no limitation for level difference on $\mathcal{T}$. Each basis function corresponds to a basis function of $S_0(\mathcal{G})$. First, we propose a new structure and introduce how to use T-structures for calculating the B-ordinates. Second, for each basis function of $S^2(\mathcal{T})$, we initialize the weights on each domain of $\mathcal{T}$ via the basis function of $S_0(\mathcal{G})$, we use the T-structures to calculate the B-ordinates for the basis function of $S^2(\mathcal{T})$. Finally, we propose that our computation can be reduced by simplifying $\mathcal{T}$.

5.1. T-structures and some conclusions for T-structures

T-structures will play an important role in calculating the B-ordinates for each basis function of $S^2(\mathcal{T})$. In this subsection, we introduce how to use T-structures for calculating the B-ordinates of a polynomial function on the T-structure.

5.1.1. Some notations for T-structures

![T-structure](image.png)

**Figure 11:** The T-structure $\mathcal{T}$.

**Definition 5.1.** In the hierarchical T-mesh $\mathcal{T}$, a c-edge and all of the cells that have at least one common vertex with the c-edge constitute a **T-structure** of $\mathcal{T}$, we denote the T-structure as $\mathcal{T}$ for convenience. The c-edge is referred to as the **mid-edge** of $\mathcal{T}$. All of the vertices on the c-edge are referred to as the **interior-vertices** of $\mathcal{T}$. The end-points of the c-edge are referred to as the **end-points** of $\mathcal{T}$. The lowest level cell that has the common edge with the c-edge is referred to as the **mother-cell** of $\mathcal{T}$. The cells adjacent to the mid-edge except the mother-cell are referred to as the **sub-cells** of $\mathcal{T}$. The level of the mother-cell is denoted as the **level** of $\mathcal{T}$. If the mid-edge is horizontal(vertical), the T-structure is referred to as a **horizontal(vertical) T-structure**.

In Fig. 11, the T-structure $\mathcal{T}$ consists of the c-edge $v_0 v_m$ and the cells $C_0, ..., C_n$. $v_0 v_m$ is the mid-edge of $\mathcal{T}$. $v_0, ..., v_m$ are the interior-vertices of $\mathcal{T}$. $v_0$ and $v_m$ are the end-points of $\mathcal{T}$. $C_0$ is the mother-cell of $\mathcal{T}$. $C_5, ..., C_n$ are the sub-cells of $\mathcal{T}$. The level of the mother-cell $C_0$ is the level of the T-structure $\mathcal{T}$. $\mathcal{T}$ is a horizontal T-structure.

**Definition 5.2.** One horizontal T-structure and one vertical T-structure are **connected** if they have one common interior-vertex. The union of all connected T-structures is referred to as a **T-structure-branch**, which is denoted as $\mathcal{TSB}$. The minimal level of the T-structures in a T-structure-branch is denoted as the **level** of the T-structure-branch.

We show an example in Fig. 12: $\mathcal{T}_0$ in Fig. 12(b) consists of the c-edge $v_0 v_2$ and the surrounding cells 0, 1, ..., 7, which are shown in Fig. 12(a); $\mathcal{T}_1$ in Fig. 12(b) consists of the c-edge $v_1 v_3$ and the surrounding cells 0, 4, 5, 8, 9, 10, which are shown in Fig. 12(a); and $\mathcal{T}_2$ in Fig. 12(b) consists of the c-edge $v_3 v_4$ and the surrounding cells 5, 8, 9, ..., 13, which are shown in Fig. 12(a). $\mathcal{T}_0$ and
Lemma 5.3. Given the T-connection $\mathcal{T}C \in \mathcal{T}$, $\mathcal{T}C$ will be covered by a T-structure-branch.

Proof. Assume $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n$ are the T-structures that cover the T-cells $\{C_1, C_2, ..., C_m\}$.

(1) When $m = 1$, obviously, only one T-structure $\mathcal{T}_1$ covers $\mathcal{T}C$. The conclusion is right.

(2) When $m > 1$, we prove it by reduction to absurdity.

Without loss of generality, assume $\mathcal{T}_1$ is not connected with any T-structure of $\mathcal{T}_2, \mathcal{T}_3, ..., \mathcal{T}_n$.

Assume that the sub-cells of $\mathcal{T}_0$ are $I = \{C_{i_1}, ..., C_{i_k}\}$, and $\{i_1, ..., i_k\}$ is a sub set of $\{1, ..., m\}$. We denote $\{C_1, C_2, ..., C_m\} \setminus I$ as the sub T-cell set of $\{C_1, C_2, ..., C_m\}$ except $\{C_{i_1}, ..., C_{i_k}\}$. T-cells in $\{C_1, C_2, ..., C_m\} \setminus I$ are not T-connected to the T-cells in $I$, and $\mathcal{T}C$ will be divided into two, it is a contradiction of the assumption. Thus, T-structures $\mathcal{T}_0, \mathcal{T}_1, ..., \mathcal{T}_n$ comprise a T-structure-branch.

The lemma is proved.

We also show an example by Fig. 12. In Fig. 12(a), the T-connection $\mathcal{T}C$ consists of cell 3, cell 4, cell 5, cell 8, cell 11; in Fig. 12(b), the T-structure Branch $\mathcal{T}SB$ consists of $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$; obviously, the cells of $\mathcal{T}C$ are covered by the cells belonging to $\mathcal{T}SB$.

5.1.2. B-net method on T-structures

In order to connect T-structures with B-ordinates, we introduce the B-net method on T-structures as follows.

Let $p(x, y) \in \mathbb{S}^2(\mathcal{T})$ be the polynomial defined over the cells of the T-structure $\mathcal{T} \in \mathcal{T}$. We assume $\mathcal{T}$ as a horizontal T-structure in Fig. 13(a), we refer to the B-ordinates on “•”, “○” and
“φ” as the corresponding B-ordinates of \( p(x, y) \) on \( \mathcal{T} \). For the T-structure in Fig. 13(b), the corresponding B-ordinates can be defined similarly. For the vertical T-structures, we can also define the notations similarly.

**Lemma 5.4.** For a T-structure \( T \in \mathcal{T} \), let \( p(x, y) \in \mathbb{S}^2(\mathcal{T}) \) be the polynomial defined over the cells of \( \mathcal{T} \). Then, when the two rows (columns) B-ordinates on the mother-cell that near the mid-edge are given, the two rows (columns) B-ordinates on sub-cells that near the mid-edge are determined.

**Proof.** We prove the lemma for horizontal T-structures. In Fig. 13 when the B-ordinates on “φ” of the mother-cell are given, the B-ordinates on “φ” of the sub-cells can be calculated via B-net method. If it is a vertical T-structure, the lemma can be similarly proved.

By Lemma 5.4, if we want to obtain the two rows (column) B-ordinates of sub-cells that are near the mid-edge, we need to obtain the six B-ordinates of the mother-cell that are near the mid-edge.

5.1.3. The corresponding B-ordinates on a crossing-vertex

In this subsection, we introduce the B-ordinates associated with a crossing-vertex, which can help us to obtain the six B-ordinates of the mother-cell that are near the mid-edge.

Let \( p(x, y) \in \mathbb{S}^2(\mathcal{T}) \) be the polynomial defined over the cells around a crossing-vertex \( v^+ \in \mathcal{T} \), which are shown in Fig. 14(a). We define the sixteen B-ordinates on the green domain points in Fig. 14(a) as the corresponding B-ordinates of \( p(x, y) \) on \( v^+ \). In Fig. 14, the crossing-vertex \( v^+ \) is denoted as “•”, the cells around \( v^+ \) are denoted as \( C_i \), where \( i = 1, 2, 3, 4 \), and \( C_1 \) is denoted as the cell with the top level. We use Fig. 14 to give the lemma that describes the relationship between the corresponding B-ordinates on a crossing-vertex and the bilinear function as follows:

![Figure 14: Corresponding B-ordinates around a crossing-vertex.](image)

**Lemma 5.5.** For each crossing-vertex \( v^+ \in \mathcal{T} \), the corresponding B-ordinates of \( p(x, y) \in \mathbb{S}^2(\mathcal{T}) \) on \( v^+ \) are on a bilinear function.

**Proof.** We also use Fig. 14 to illustrate the proving process. In Fig. 14(a), the domain points of \( C_1 \) around \( v^+ \) are \( P_{j,k} = (s_{j,k}^1, t_{j,k}^1) \), where \( j, k = 1, 2 \), which are denoted as “•” on \( C_1 \). The bilinear function \( f_1(s, t) = a_{1}s + b_{1}s + c_{1}t + d_{1} \), where \( j, k = 1, 2 \), which satisfies \( f_1(s_{j,k}^1, t_{j,k}^1) = b_{j,k}^1 \), exists. Similarly, the domain points of \( C_i \) around \( v^+ \) are denoted as “•” on \( C_i \), where \( i = 2, 3, 4 \), respectively, and the corresponding bilinear functions are denoted as \( f_i(s, t) = a_is + b_is + c_it + d_i \), where \( i = 2, 3, 4 \), respectively.

We use Fig. 14(b) to illustrate \( f_1 = f_2 = f_3 = f_4 \). Apply the \( C^2 \) continuous conditions to \( C_i \), where \( i = 2, 3, 4 \). The B-ordinates on “φ” and the B-ordinates on “φ” belong to \( C_i \).
Theorem 5.6. Assume that the two end-points of the T-structure $B$-ordinates and the adaptive nodes on each end-point of $T$ we need to determine the bilinear function.

5.1.4. The adaptive nodes for a bilinear function

Given the corresponding $f$ function; thus $f$ can be calculated by Lemma 5.4, and the corresponding B-ordinates $f$ on a common bilinear function $f(s,t) = ast + bs + ct + d$ of $v^\ast$. For $f(s,t)$, to determine $a, b, c$ and $d$, we recall the notation of the adaptive nodes $\mathbf{bs}$ for a bilinear function.

By Lemma 5.5, if we want to calculate the corresponding B-ordinates for a crossing-vertex $v^\ast$, we need to determine the bilinear function $f(s,t) = ast + bs + ct + d$ of $v^\ast$. For $f(s,t)$, to determine $a, b, c$ and $d$, we recall the notation of the adaptive nodes $\mathbf{bs}$ for a bilinear function.

5.1.4. The adaptive nodes for a bilinear function

We use Fig. 15 to illustrate how to obtain a group of adaptive nodes for a bilinear function.

Step 1 Choose the point $v_1(s_1, t_1) \in \mathbb{R}^2$.
Step 2 Draw the cross $X_1 \in \mathbb{R}^2$, and $v_1 \notin X_1$. Choose two points $v_2(s_2, s_2)$ and $v_3(s_3, t_3)$ on one edge of $X_1$, and choose another point $v_4(s_4, t_4)$ on the other edge of $X_1$.

$v_1, v_2, v_3$ and $v_4$ are referred to as adaptive nodes of the bilinear function $f(s,t) = ast + bs + ct + d$. Given the corresponding $f(s_i, t_i), i = 1, ..., 4$, we can use the linear equations $f(s_i, t_i) = as_i + bs_i + ct_i + d, i = 1, ..., 4$ to calculate the coefficients $a, b, c$ and $d$.

And then, we give the following lemma to illustrate the relationship between the corresponding B-ordinates and the adaptive nodes on each end-point of $T$:

**Theorem 5.6.** Assume that the two end-points of the T-structure $T \in \mathcal{T}$ are crossing-vertices. Let $p(x,y) \in \mathcal{S}(\mathcal{T})$ be the polynomial defined over the cells of $\mathcal{T}$. Given the values on a group of adaptive nodes for each end-point of $T$, the corresponding B-ordinates on $T$ can be calculated by Lemma 5.5.

**Proof.** By Lemma 5.5, the corresponding B-ordinates on each end-point are on a bilinear function. For each end-point, as the value on each adaptive node is given, the coefficients of each bilinear function are calculated by four equations, the corresponding B-ordinates on each end-points can be calculated by the corresponding bilinear functions, the two rows (column) B-ordinates of the mother-cell that near the mid-edge are obtained, and the corresponding B-ordinates on $T$ can be calculated by Lemma 5.5, and the corresponding B-ordinates satisfy the $C^1$ continuous conditions.

Till now, we obtain the conclusion that if we want to obtain the corresponding B-ordinates on a T-structure, we need to obtain a group of adaptive nodes and the corresponding values on the nodes. We use Fig. 16 to give the following proposition to discuss the values on the adaptive nodes for the end-points of a T-structure.

In Fig. 16(a), $C_0$ and $C_1$ are two aligned P-cells in $\mathcal{T}$. $P_{01}^0(x_1, \frac{m+1}{2}, \frac{n+1}{2})$ and $P_{11}^0(\frac{x_1+m}{2}, \frac{n+1}{2})$ are two domain-points of $C_0$. $P_{11}^1(x_1, \frac{m+1}{2}, \frac{n+1}{2})$ and $P_{11}^2(x_1, \frac{m+1}{2})$ are two domain-points of $C_1$. 
Let \( p_i(x, y) \in \mathbb{S}^2(\mathcal{S}) \) be the polynomial defined over \( C_i, i = 0, 1 \). The B-ordinate of \( p_0(x, y) \) on \( P^0_{11} \) is denoted as \( b^0_{11}, i = 0, 1 \). The B-ordinate of \( p_1(x, y) \) on \( P^0_{11} \) is denoted as \( b^1_{11}, i = 1, 2 \).

In Fig. 16(b), \( C_0 \) is the one-neighbor-cell of \( \mathcal{T}C \) in \( \mathcal{S} \), the T-connection-domain of \( \mathcal{T}C \) is denoted as \( \mathcal{T}CD \), the T-rectangle-domain of \( \mathcal{T}C \) is denoted as \( \mathcal{T}RD \). Let \( p_i(x, y) \in \mathbb{S}^2(\mathcal{S}) \) be the polynomial defined over \( C_i, i = 0, \ldots, k, \varphi(p_i(x, y) : \mathcal{T}RD) = \varphi(p_0(x, y) : \mathcal{T}RD) = \omega_{\mathcal{TCP}} \). The center position of \( \mathcal{T}RD \) is denoted as \( P\left(\frac{x_1 + x_2}{2}, \frac{y_0 + y_2}{2}\right) \). \( P^0_{10}(x_1, \frac{y_0 + y_2}{2}) \) and \( P^1_{11}(x_1 + x_2, \frac{y_0 + y_2}{2}) \) are denoted as two domain-points of \( C_0 \), the B-ordinate on \( P^0_{11} \) is denoted as \( b^0_{11}, i = 0, 1 \).

**Proposition 5.7.** From the illustrations of Fig. 16, we give the conclusions as follows:

1. In Fig. 16(a), \( p(x, y) \) is \( C^1 \) continuous on \( v_1v_2 \) if and only if \( (x_1, \frac{y_0 + y_2}{2}, b^1_{11}) \) is on the linear function that is determined by \( \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_2}{2}, b^1_{01}\right) \) and \( \left(\frac{x_1 + x_2}{2}, \frac{y_0 + y_2}{2}, b^1_{11}\right) \).

2. In Fig. 16(b), \( p(x, y) \) is \( C^1 \) continuous on \( v_1v_2 \) if and only if \( (x_1, \frac{y_0 + y_2}{2}, b^0_{10}) \) is on the linear function that is determined by \( \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_2}{2}, \omega\right) \) and \( \left(\frac{x_1 + x_2}{2}, \frac{y_0 + y_2}{2}, b^0_{11}\right) \).

**Proof.**

1. Similar to Proposition 3.4, the proposition is true.

2. In Fig. 16(a), we denote \( p(x, y) \) on \( C_i \) as \( p_i(x, y), i = 0, 1 \). As \( p_1(x, y) = p_0(x, y) + (x - x_1)^2v(y), b^1_{11} \) is the mapping result of \( \varphi(p_0(x, y) : [x_0, x_1] \times [y_0, y_2]) \). In Fig. 16(b), \( \omega \) is also the mapping result of \( \varphi(p_0(x, y) : [x_0, x_1] \times [y_0, y_2]) \). Thus, we have a similar conclusion as that \( (x_1, \frac{y_0 + y_2}{2}, b^0_{10}) \) is on the linear function that is determined by \( \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_2}{2}, \omega\right) \) and \( \left(\frac{x_1 + x_2}{2}, \frac{y_0 + y_2}{2}, b^0_{11}\right) \). The reverse proving process can be derived naturally. The proposition is proved.

By Proposition 5.7, the **weight** on the domain-centre of \( \mathcal{T}C \) and the B-ordinate on the center-domain-point of the one-neighbor-cell \( C \in \mathcal{T}C \) can be used as values in Theorem 5.6 to calculate the coefficients of the bilinear functions for \( v_1 \) and \( v_2 \) in Fig. 16(b). We obtain the conclusion that we can use the weights on the domain-centres and the B-ordinates on the domain-points to calculate the corresponding B-ordinates of the polynomial \( p(x, y) \in \mathbb{S}^2(\mathcal{S}) \). We introduce the method for using T-structures to calculate the B-ordinates for the basis functions of \( \mathbb{S}^2(\mathcal{S}) \) as follows:

### 5.2. Evaluate the B-ordinates of the basis functions of \( \mathbb{S}^2(\mathcal{S}) \)

In this subsection, we will evaluate the B-ordinates for the basis functions of \( \mathbb{S}^2(\mathcal{S}) \). First, we use each basis function of \( \mathbb{S}^2(\mathcal{S}) \) to initialize the weights on each domain-center of \( \mathcal{S} \), and we obtain a domain \( \mathbb{E} \) that covers \( sup(p(x, y)) \), \( p(x, y) \) is denoted as the basis function in \( \mathbb{S}^2(\mathcal{S}) \). Second, we give an order for the T-structure-branches that corresponds to the T-connections in \( \mathbb{E} \). Third, we use the lowest level T-structure-branch to calculate the B-ordinates on each T-cell that belongs to the lowest level T-connection. Finally, in a similar way to the lowest level T-connection, we calculate the B-ordinates on the T-cells of the rest T-connections.
5.2.1. Initialize the weights on the domain-centres

In this subsection, to make use of the CVR graph \( G \), we initialize the weights for the basis function \( p(x, y) \in S^2(\mathcal{T}) \) by a basis function \( q(x, y) \in S^0(\mathcal{G}) \).

Given the basis function \( q(x, y) \in S^0(\mathcal{G}) \) as:

\[
q(x, y) = \begin{cases} 
1, & \text{GC} \\
0, & \text{other g-cells}
\end{cases}
\]  

(21)

With Equation (21), we give the weight on each domain-centre as:

\[
\omega = \begin{cases} 
1, & \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\
0, & \text{other domain-centres}
\end{cases}
\]  

(22)

We use the weights in Equation (22) to construct the basis function \( p(x, y) \in S^2(\mathcal{T}) \) and \( p(x, y) \) is a piecewise quadratic polynomial on some cells of \( \mathcal{T} \). The weight on the domain-centre of \( \mathcal{D} \) is 1, while the weights on the other domain-centres are 0, we give the domain \( \mathcal{E} \in \mathcal{G} \) that covers \( Sup(p(x, y)) \). The sketch of \( \mathcal{E} \) is shown in Fig. 17: the basis-cell is the P-cell or the T-connection that corresponds to \( \mathcal{D} \). \( S_1 \) consists of the center-cell, the P-cells and T-connections whose domains that cover, are covered, or adjacent to \( \mathcal{D} \). \( S_2 \) consists of the P-cells and T-connections whose domains that cover, are covered, or adjacent to \( S_1 \). Delete each repeat element in \( S_1 \cup S_2 \), we obtain the domain \( \mathcal{E} \) that covered by a series of T-connections and P-cells.

![Figure 17: The sketch of \( \mathcal{E} \)](image)

5.2.2. The order of T-structure-branches in \( \mathcal{E} \)

By Lemma 5.3, each T-connection is covered by a T-structure-branch. Assume that the T-connections in \( \mathcal{E} \) are \( TC_0, TC_1, \ldots, TC_n \), we denote \( TSB_i \) as the corresponding T-structure-branch of \( TC_i, i = 0, \ldots, n \). For each \( TSB_i \), we give the Algorithm 2 to order the T-structures in \( TSB_i, i = \)
0, ..., n. And then, sort \( TSB \), \( i = 0, ..., n \) in descending level order.

**Algorithm 2:** The algorithm to sort the T-structures for each T-structure-Branch

Input: \( TSB : \{ T_0, ..., T_m \} \)

Output: The order sorted T-structure-branch

1. Give an empty T-structure-branch \( TSB_0 \);
2. \( TSB_0 \leftarrow \) the T-structure (T-structures) with the lowest level in \( TSB \);
3. Remove each T-structure of \( TSB_0 \) from \( TSB \);
4. while \( |TSB| > 0 \) do
   5. for \( i = 0, i < |TSB_0|, i++ \) do
      6. Give an empty T-structure vector \( V \);
      7. for \( j = 0; j < |TSB|; j++ \) do
         8. if \( TSB_0[i] \) is connected to \( TSB[j] \) then
            9. \( V \leftarrow TSB[j] \);
      10. Remove each T-structure of \( V \) from \( TSB \);
      11. Sort T-structures of \( V \) in descending level order;
      12. \( TSB_0 \leftarrow \) each T-structure of \( V \);
   5. Output \( TSB_0 \);

5.2.3. The B-ordinates of \( p(x, y) \) on \( TC_0 \)

In this subsection, we use T-structures to calculate the B-ordinates of \( p(x, y) \) on \( TC_0 \). We denote the T-rectangle-domain of \( TC_0 \) as \( TRD_0 \). By Lemma 5.3, \( TC_0 \) is covered by \( TSB_0 \). \( TSB_0 \) is the lowest level T-structure-branch in \( E \). \( TC_0 \) is also the lowest level T-connection in \( E \). The B-ordinates on the T-cells that belong to \( TC_0 \) can be calculated by T-structures in \( TSB_0 : T_0, ..., T_m \). \( T_0, ..., T_m \) are already sorted by Algorithm 2 and \( T_0 \) is the T-structure with the lowest level.

As \( TC_0 \) is also the lowest level T-connection in \( E \), we give the initialization as follows:

**Initialization 1.** As \( TC_0 \) is also the lowest level T-connection in \( E \), we give the initialization as follows:

As \( T_0 \) is the T-structure with the lowest level in \( TSB_0 \), we give the lemma as follows:

**Lemma 5.8.** The two end-points of \( T_0 \) are crossing-vertices.

**Proof.** We prove the lemma by reduction to absurdity. If one of the end-points is a T-junction, then a lower level T-structure \( T_0' \), which connects to \( T_0 \) at a T-junction, exists. Then, the level of \( T_0' \) is lower than the level of \( T_0 \) and \( T_0' \) belongs to \( TSB_0 \). It contradicts the order of the T-structures in \( TSB_0 \). Thus, the two end-points of \( T_0 \) are crossing-vertices. The lemma is proved.

By Lemma 5.8, the two end-points of \( T_0 \) are crossing-vertices. Without loss of generality, we use the vertical T-structure in Fig. 18(a) to illustrate \( T_0 \). We denote the two end-points of \( T_0 \) as \( v_1 \) and \( v_2 \) respectively, denote the mother-cell of \( T_0 \) as \( C_1 \). \( C_1 \) is also the one-neighbor-cell of \( TC_0 \). We can calculate the corresponding B-ordinates of \( T_0 \) via the bilinear function associated with each end-point of \( T_0 \). From Theorem 5.3, if the weights or B-ordinates on the adaptive nodes of the bilinear function on each end-point of \( T_0 \) are given, we can use the bilinear function to calculate the corresponding B-ordinates for each end-point. We give the theorem to obtain the corresponding B-ordinates on \( T_0 \) as follows:

**Theorem 5.9.** For each end-point of \( T_0 \), there exists a group of adaptive nodes that the weight or B-ordinate on each node is given, the corresponding B-ordinates of \( p(x, y) \) on \( T_0 \) can be calculated by Theorem 5.6.
that is parallel to C two of the four adaptive nodes. As analysis of the type of C the nine B-ordinates of pconnection corresponding to TSB belongs to the T-structure-branch C of the T-structure T

If denoted as l for E in Fig 18(b) are given by the initialization, we can choose 0 1.

In the next discussion, we need to obtain the whole group of adaptive nodes for C If is a T-cell, without loss of generality, we assume the C belongs to TC, the level of TC is denoted as l(TCi), i = 1, 2, 3 for convenience. By Definition 5.2, the level of TC is l(Ci).

If C1 is a T-cell, we use Fig 18(b) to illustrate the adaptive nodes. Assume that C1 is a sub-cell of the T-structure T'. As C1 is the mother-cell of TC, the level of T' is lower than l(C1). If T' belongs to the T-structure-branch TSB', the level of TSB' is lower than l(C1). We denote the T-connection corresponding to TSB' as TC', the level of TC' is lower than l(Ci). By Initialization 11 the nine B-ordinates of p(x, y) on each T-cell belongs to TC are 0. The four B-ordinates on "•" of C0 in Fig 18(b) are given by the initialization, we can choose P1 (s1, t1), P1 (s1, t1), P0 (s0, t0) and P1 (s1, t1) as the adaptive nodes of v1.

If C1 is a P-cell, we use Fig. 18(c) to illustrate the two given adaptive nodes: the weights on "▲" and "•" are given by Equation 22. As C1 is a P-cell, the weight on "•" is also the B-ordinate on the centre domain point of C1. By Proposition 3.7, the points on "▲" and "•" can be used as two of the four adaptive nodes. As C1 is the one-neighbor-cell of TC0, "▲" and "•" are on the line that is parallel to E1, which is shown as the green dashed line in Fig. 18(c).

In the next discussion, we need to obtain the whole group of adaptive nodes for v1. We use Fig. 18 to illustrate the adaptive nodes, the notations are the same as that in Fig. 18(c). A careful analysis of the type of C1 and l(Ci), i = 1, 2, 3 will help us to obtain the four adaptive nodes. We have the following two situations by discuss the type of C1, i = 2, 3:

1. C2 is a P-cell. If C2 is a P-cell, the vertices of C1 and C2 on E are the same, and it is a crossing-vertex, then l(C2) = l(C1). As the B-ordinate of p(x, y) on the centre domain-point of C2 is given by Equation 22, the centre domain-point can be used as one node of the four adaptive nodes. And then, we discuss the type of C3 to obtain the adaptive nodes.

2. C2 is T-cell. If C2 is a T-cell, we assume that C2 belongs to the T-connection TC2. As C1 is a P-cell that is adjacent to TC2, the level of the one-neighbor-cell of TC2 is less than or equal to l(C1), we obtain that l(TC2) ≤ l(C1). If l(TC2) = l(C1), C1 is the one-neighbor-cell of TC2; otherwise, C1 is not the one-neighbor-cell of TC2. And then, we discuss the type of C3 to obtain the adaptive nodes.
Consider case 1. If \( C_2 \) is a P-cell, the B-ordinate of \( p(x, y) \) on the centre domain-point of \( C_2 \) is given by Equation 22. We use Fig. 19(a) and (b) to discuss the adaptive nodes. We denote the domain-point of \( C_2 \) as \( P_{11}^3(s_{11}^3, t_{11}^3) \) in Fig. 19(a) and (b). By Proposition 5.7, \( P_{11}^3 \) can be used as one of the four adaptive nodes. \( P_{11}^3 \) is on a line that is perpendicular to the green dashed line, the two lines construct a cross, which can be denoted as \( X_0 \). We discuss the whole group of adaptive nodes as follows:

(1) \( C_3 \) is a P-cell. If \( C_3 \) is a P-cell, the B-ordinate of \( p(x, y) \) on the centre domain-point of \( C_3 \) is given by Equation 22. We denote the domain-point of \( C_3 \) as \( P_{11}^3(s_{11}^3, t_{11}^3) \) in Fig. 19(a). By Proposition 5.7, \( P_{11}^3 \) can be used as one of the four adaptive nodes. As \( P_{11}^3 \) is not on the cross \( X_0 \), we can choose the four nodes as \( P_{11}^3(s_{11}^3, t_{11}^3), P_{11}^2(s_{11}^2, t_{11}^2), P_{11}^1(s_{11}^1, t_{11}^1) \) and \( P_0^0(s_0^0, t_0^0) \) in Fig. 19(a).

(2) \( C_3 \) is a T-cell. If \( C_3 \) is a T-cell, we assume \( C_3 \) as a T-cell belongs to the T-connection \( \mathcal{T}_C \), as \( C_3 \) is a P-cell adjacent to \( \mathcal{T}_C \), the level of the one-neighbor-cell of \( \mathcal{T}_C \) is not higher than \( l(C_2) \), we obtain that \( l(\mathcal{T}_C) \leq l(C_2) \).

\( a \) \( l(\mathcal{T}_C) \leq l(C_2) \). If \( l(\mathcal{T}_C) \leq l(C_2) \), as \( l(C_2) = l(C_1) \), we obtain \( l(\mathcal{T}_C) < l(C_1) \). By initialization, the nine B-ordinates of \( p(x, y) \) on each T-cell of \( \mathcal{T}_C \) are given, the B-ordinate of \( p(x, y) \) on the centre domain-point of \( C_3 \) is given. Similar to (1) in case 1, we can choose the four nodes as \( P_{11}^3(s_{11}^3, t_{11}^3) \), \( P_{11}^2(s_{11}^2, t_{11}^2) \), \( P_{11}^1(s_{11}^1, t_{11}^1) \) and \( P_0^0(s_0^0, t_0^0) \) in Fig. 19(a).

\( b \) \( l(\mathcal{T}_C) = l(C_2) \). If \( l(\mathcal{T}_C) = l(C_2) \), \( C_2 \) is the one-neighbor-cell of \( \mathcal{T}_C \). We denote the domain-centre of \( \mathcal{T}_C \) as \( P_3^3(s_3^3, t_3^3) \) in Fig. 19(b). As the weight on \( P_3^3 \) is given by Equation 22. By Proposition 5.7, \( P_3^3 \) can be used as one of the four adaptive nodes. As \( P_{11}^3 \) is not on the cross \( X_0 \), we can choose the four nodes as \( P_3^3(s_3^3, t_3^3), P_{11}^1(s_{11}^1, t_{11}^1) \), \( P_{11}^2(s_{11}^2, t_{11}^2) \) and \( P_0^0(s_0^0, t_0^0) \) in Fig. 19(b).

Consider case 2. If \( C_2 \) is a T-cell, \( C_2 \) is assumed as a cell belongs to \( \mathcal{T}_C \). As \( l(\mathcal{T}_C) \leq l(C_1) \), we discuss the adaptive nodes as follows:

(1) \( l(\mathcal{T}_C) = l(C_1) \). If \( l(\mathcal{T}_C) = l(C_1) \), \( C_1 \) is the one-neighbor-cell of \( \mathcal{T}_C \). We use Fig. 19(c) and Fig. 19(d) to discuss the adaptive nodes. We denote the domain-centre of \( \mathcal{T}_C \) as \( P_2^2(s_2^2, t_2^2) \) in Fig. 19(c) and Fig. 19(d). The weight on the domain-centre is given by Equation 22. By Proposition 5.7, \( P_2^2 \) can be used as one of the four adaptive node, and \( P_2^2 \) is on a line that is perpendicular to the green dashed line, the two lines construct a cross, which can be denoted as \( X_1 \).

\( a \) \( C_3 \) is a P-cell. If \( C_3 \) is a P-cell, the B-ordinate on the centre domain-point of \( C_3 \) is given by Equation 22, we denote the centre domain-point of \( C_3 \) as \( P_{11}^3(s_{11}^3, t_{11}^3) \) in Fig. 19(c). By Proposition 5.7, \( P_{11}^3 \) can be used as one of the four nodes. As \( P_{11}^3 \) is not on the cross \( X_1 \), we can choose the four nodes as \( P_{11}^3(s_{11}^3, t_{11}^3), P_{11}^1(s_{11}^1, t_{11}^1), P_{11}^2(s_{11}^2, t_{11}^2) \) and \( P_0^0(s_0^0, t_0^0) \) in Fig. 19(c).

\( b \) \( C_3 \) is a T-cell. If \( C_3 \) is a T-cell, we assume \( C_3 \) as a T-cell of the T-connection \( \mathcal{T}_C \). Then, \( l(\mathcal{T}_C) \) is not equal to \( l(C_1) \). Otherwise, if \( l(\mathcal{T}_C) = l(C_1) \), as the one-neighbor-cell of \( l(\mathcal{T}_C) \) is neither on \( E_2 \) nor on \( E_3 \). There exist some cross vertices of \( \mathcal{T}_C \) on \( E_2 \) or \( E_3 \). \( \mathcal{T}_C \) is split into several parts, it is contrary to the assumption. Thus, \( l(\mathcal{T}_C) < l(C_1) \) or \( l(\mathcal{T}_C) > l(C_1) \).

i. if \( l(\mathcal{T}_C) < l(C_1) \), by initialization, the nine B-ordinates of \( p(x, y) \) on each T-cell in \( \mathcal{T}_C \) are given, the B-ordinate on the center domain-point of \( C_3 \) is given. Similar to (1)(a) in case 2, we can choose the four nodes as \( P_{11}^3(s_{11}^3, t_{11}^3), P_{11}^1(s_{11}^1, t_{11}^1), P_{11}^2(s_{11}^2, t_{11}^2) \) and \( P_0^0(s_0^0, t_0^0) \) in Fig. 19(c).

ii. if \( l(\mathcal{T}_C) > l(C_1) \), if \( l(\mathcal{T}_C) > l(C_1) \), as \( C_3 \) is a cell divided by one cell of \( \mathcal{T}_C \), \( k = l_1 \), \( \nu_3 \) is a crossing-vertex, and the one-neighbor-cell is either on \( E_2 \) or on \( E_3 \). As the weight on the domain-centre of \( \mathcal{T}_C \) is given by Equation 22 by Proposition 5.7, the domain-centre of \( \mathcal{T}_C \) is denoted as \( P_3^3(s_3^3, t_3^3) \) in Fig. 19(d), can be used.
as one of the four adaptive nodes. As \( P^3 \) is not on the cross \( X_1 \), we can choose the four nodes as \( P^3(s^3, t^3) \), \( P^1_1(s^1_1, t^1_1) \), \( P^2(s^2, t^2) \) and \( P^0(s^0, t^0) \) in Fig. 11(d).

(2) \( l(TC_2) < l(C_1) \). If \( l(TC_2) < l(C_1) \). By Initialization \[ \text{Initialization} \] the B-ordinates of \( p(x, y) \) on each T-cell of \( \mathcal{T}C_2 \) are given. By Proposition \[ \text{Proposition} \] the two domain-points, which are denoted as \( P^2_{01}(s^2_{01}, t^2_{01}) \) and \( P^2_{11}(s^2_{11}, t^2_{11}) \) in Fig. 11(e), can be used as two of the four adaptive nodes. \( P^2_{01} \) is on a line that is perpendicular to the green dashed line, the two lines construct a cross, which can be denoted as \( X_2 \). As \( P^2_{01} \) is not on \( X_2 \), we can choose one group of adaptive nodes as \( P^2_{01}(s^2_{01}, t^2_{01}) \), \( P^2_{11}(s^2_{11}, t^2_{11}) \), \( P^0(s^0, t^0) \) and \( P^1_1(s^1_1, t^1_1) \) in Fig. 11(e).

From the discussion above, each group of nodes is adaptive. As the weight or B-ordinate on each node is given by the discussion, we can calculate the coefficients of the bilinear function on \( v_1 \). Similar to the situation of \( v_1 \), we can calculate the coefficients of the bilinear function on \( v_2 \). And then, using Theorem \[ \text{Theorem} \] we obtain the corresponding B-ordinates that satisfy the \( C^1 \) continuous condition on each edge of \( \mathcal{T}_0 \), we obtain the corresponding B-ordinates of \( p(x, y) \) on \( \mathcal{T}_0 \). The theorem is proved.

![Diagrams](image_url)

**Figure 20:** Other T-structures.

**Corollary 5.10.** The corresponding B-ordinates of \( p(x, y) \) on \( \mathcal{T}_i \), \( i = 1, \ldots, m \) can be calculated by Lemma \[ \text{Lemma} \] 5.4.

**Proof.** We can obtain all of the corresponding B-ordinates of \( \mathcal{T}_0 \) via the Theorem 5.9. As the T-structures in \( \mathcal{T}SB_0 \) are sorted as \( \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_m \) via Algorithm 2, \( \mathcal{T}_1 \) is connected to \( \mathcal{T}_0 \). Without loss of generality, we assume that \( \mathcal{T}_1 \) is connected to \( \mathcal{T}_0 \) at \( v_2 \), which is shown in Fig. 20. \( v_2 \) is a crossing-vertex in Fig. 20(a) or T-junction in Fig. 20(b), the corresponding B-ordinates of \( v_2 \) are obtained. In Fig. 20 \( C_0 \) is denoted as the mother-cell of \( \mathcal{T}_1 \), and the B-ordinates on \( P^0_{01} \) and \( P^0_{11} \) are given. Replace \( P^0 \) in Fig. 13 with \( P^0_{01} \) in Fig. 20. We can obtain at least one group adaptive nodes for \( v_1 \) in Fig. 20. The corresponding bilinear function for \( v_1 \) can be obtained naturally. And then, we can calculate the corresponding B-ordinates of \( \mathcal{T}_1 \) via Lemma 5.4. By Algorithm 2, \( \mathcal{T}_i \) is connected to one of \( \mathcal{T}_0, \ldots, \mathcal{T}_{i-1}, i = 1, \ldots, m \). In a similar way to \( \mathcal{T}_1 \), we can obtain the corresponding B-ordinates of \( p(x, y) \) on \( \mathcal{T}_i \), \( i = 2, \ldots, m \). The corollary is proved.

We give algorithm \[ \text{Algorithm} \] 5 to illustrate the process for calculating the corresponding B-ordinates on
each T-structure in $\mathcal{TSB}_0$.

**Algorithm 3: Calculate the B-ordinates for each T-cell in $\mathcal{TC}_0$**

**Input:** $\mathcal{TSB}_0 : \{T_0, \ldots, T_m\}$

**Output:** The B-ordinates of each T-cell in $\mathcal{TC}_0$

1. Calculate the corresponding B-ordinates of $\mathcal{TSB}_0[0]$;
2. $V \leftarrow \mathcal{TSB}_0[0]$;
3. Remove $\mathcal{TSB}_0[0]$ from $\mathcal{TSB}_0$;
4. while $|\mathcal{TSB}_0| > 0$ do
5. for $i = 1, i < |\mathcal{TSB}_0|, i + +$ do
6.   Bool Connected=false;
7.   for $j = 0; j < |V|; j + +$ do
8.     if $\mathcal{TSB}_0[i]$ is connected to $V[j]$ at one end point of $\mathcal{TSB}_0[i]$ then
9.       Connected=true;
10.      if Connected then
11.         Calculate the corresponding B-ordinates on the other end-point of $\mathcal{TSB}_0[i]$;
12.         Using Lemma 5.4 to calculate the corresponding B-ordinates on $\mathcal{TSB}_0[i]$;
13.         Remove $\mathcal{TSB}_0[i]$ from $\mathcal{TSB}_0$;
14.         $V \leftarrow \mathcal{TSB}_0[i]$;
15.     Using the $C^1$ continuous conditions on the rest edges that the B-ordinates are not obtained;

**Theorem 5.11.** The B-ordinates of $p(x, y)$ on each T-cell belongs to $\mathcal{TC}_0$ can be calculated by Algorithm 3.

**Proof.** By Lemma 5.3, the T-cells in $\mathcal{TC}_0$ are covered by $\mathcal{TSB}_0$. The B-ordinate on the centre domain-point of each T-cell in $\mathcal{TC}_0$ is obtained. As the corresponding B-ordinates on each T-structure in $\mathcal{TSB}_0$ are calculated, we can calculate the rest B-ordinates on each T-cell of $\mathcal{TC}_0$ via $C^1$ continuous conditions. The theorem is proved.

5.2.4. The B-ordinates of $p(x, y)$ on $\mathcal{TC}_i, i = 1, \ldots, n$

The B-ordinates on the T-cells of $\mathcal{TC}_0$ are calculated in Section 5.2.3. The B-ordinates on the T-cells of the rest T-connections $\mathcal{TC}_i, i = 1, \ldots, n$ can be calculated similarly.

**Theorem 5.12.** For each T-connection in $\mathcal{E}$, we can obtain the B-ordinates of $p(x, y)$ on the T-cells belong to the T-connection.

**Proof.** As $\mathcal{TC}_i, i = 0, \ldots, n$ are in the same descending level order as $\mathcal{TSB}_i, i = 0, \ldots, n$. We can calculate all of the B-ordinates on $\mathcal{TC}_0$ via algorithm 3. And then, $\mathcal{TC}_1$ is the T-connection whose level is the lowest. We can calculate B-ordinates on $\mathcal{TC}_1$ in the same way as $\mathcal{TC}_0$. By this analogy, we can calculate the B-ordinates on each T-connection $\mathcal{TC}_i, i = 2, \ldots, n$ by Algorithm 3. The theorem is proved.

Till now, we can calculate the B-ordinates of $p(x, y)$ on the the T-cells belong to $\mathcal{TC}_i, i = 0, \ldots, n$. And we give the following theorem to obtain the B-ordinates of $p(x, y)$:

**Theorem 5.13.** For the polynomial function $p(x, y) \in \mathbb{S}^2(\mathcal{F})$, the weights are denoted by Equation (22), $\mathcal{E}$ denotes the domain that covers sup($p(x, y)$). Then, the B-ordinates on each cell in $\mathcal{E}$ can be calculated by Theorem 5.12 and $C^1$ continuous conditions.
Proof. As the B-ordinate on the center domain-point of each T-cell in $E$ is calculated by theorem 5.12, and the B-ordinate on the center domain-point of each P-cell in $E$ is given by Equation (22). The B-ordinates on each cell in $E$ can be calculated via $C^1$ continuous conditions. The theorem is proved.

For each cell $C \in E$, if at least one B-ordinate of $C$ is nonzero, we save $C$ to the support of $p(x, y)$. We can use Bernstein-Bézier to express $p(x, y)$, which is $C^1$ continuous on the support. Then, we obtained a polynomial function with local support of $\overline{S}^2(\mathcal{T})$.

5.3. Simplify the hierarchical T-mesh

In particular, if the T-l-edge $E$ only contains $V_0$ vertices, $\dim W[E] = (V_0 - d - 1)_+ := \max(0, V_0 - d - 1)$ holds for $S^d(\mathcal{T})$, $E$ is defined as a trivial l-edge \footnote{24} \footnote{27} if $(V_0 - d - 1)_+ = 0$. For the T-l-edge $E$ with one interior crossing-vertex, we say $E$ is a trivial l-edge. As $\dim W[E] = 0$, we can remove $E$ from $\mathcal{T}$, and the polynomial functions of the spline spaces will not change.

Fig. 21 shows the simplification of a hierarchical T-mesh. As the green lines are trivial l-edges in $\mathcal{T}_0$, remove them, we obtain $\mathcal{T}_1$. As the blue line is a trivial l-edges in $\mathcal{T}_1$, remove it, we obtain $\mathcal{T}_2$. $\mathcal{T}_2$ is denoted as the simplification of $\mathcal{T}_0$.

![Figure 21: Simplification of a T-mesh.](image-url)

It is natural that we can remove meshlines that do not contribute to the dimension of the spline space \footnote{14}. The simplification can reduce the number of T-structures, decrease the amount of calculation, and remove some overhanging edges from CVR graph. We give Algorithm 4 to construct a biquadratic polynomial function $p(x, y) \in \overline{S}^2(\mathcal{T})$, where $\mathcal{T}$ is the simplified T-mesh.

**Algorithm 4:** Calculate the B-ordinates for $p(x, y)$

- **Input:** $q(x, y) \in \overline{S}^d(\mathcal{T})$
- **Output:** The B-ordinates of $p(x, y) \in \overline{S}^2(\mathcal{T})$

1. Give the weights on each domain of $\mathcal{T}$ by Equation (22).
2. Obtain $E : \{TC_0, ..., TC_n\} \cup \{PC_0, ..., PC_l\}$;
3. Obtain each T-structure-branch $\mathcal{T}SB_i$ corresponding to $TC_i, i = 0, ..., n$;
4. Sort the T-structures in $\mathcal{T}SB_i$ via algorithm \footnote{2} \footnote{3} $i = 0, ..., n$ in descending level order;
5. Calculate the corresponding B-ordinates of $p(x, y)$ on $\mathcal{T}SB_i$ via Algorithm \footnote{3};
6. Calculate the B-ordinates on the cells in $E$ via $C^1$ continuous conditions;
7. Save the support of $p(x, y)$ on $\mathcal{T}$.

By Algorithm 4, we obtain the basis function of $\overline{S}^2(\mathcal{T})$ corresponding to the basis function of $\overline{S}^d(\mathcal{T})$, and we give the theorem as:

**Theorem 5.14.** Each basis function of $\overline{S}^d(\mathcal{T})$ corresponds to a biquadratic polynomial function of $\overline{S}^2(\mathcal{T})$. 
Proof. Given the basis function \( q(x, y) \in S^0(\mathcal{T}) \), denote the domain weights for \( p(x, y) \in S^2(\mathcal{T}) \) by Equation (22), we can calculate the B-ordinates for \( p(x, y) \) by algorithm 4. The theorem is proved.

So far, we construct the biquadratic polynomial function of \( S^2(\mathcal{T}) \) via a piecewise constant basis function of \( S^0(\mathcal{T}) \).

6. The isomorphic bivariate spaces and properties

In this section, we discuss the bijective property of the mapping that constructed in Section 4. Some properties of the basis functions we constructed in Section 5 are also discussed.

6.1. The bijective property of the mapping

First, by Section 4 and Section 5, we give a theorem about the mapping as follows:

**Theorem 6.1.** For the hierarchical T-mesh \( \mathcal{T} \), \( \mathcal{G} \) denotes the CVR graph of \( \mathcal{T} \). The mapping between \( S^2(\mathcal{T}) \) and \( S^0(\mathcal{G}) \) is bijective, and \( S^2(\mathcal{T}) \) is isomorphic to \( S^0(\mathcal{G}) \).

**Proof.** By Theorem 4.5, the mapping from \( S^2(\mathcal{T}) \) to \( S^0(\mathcal{G}) \) is an injective mapping. By Theorem 5.14, each basis function of \( S^0(\mathcal{G}) \) corresponds to a polynomial function of \( S^2(\mathcal{T}) \), the mapping is surjective. Thus, the mapping is bijective, and \( S^2(\mathcal{T}) \) is isomorphic to \( S^0(\mathcal{G}) \).

As the properties of the piecewise constant basis functions over the CVR graph are simple and clear, we use them to discuss the properties of the basis function belongs to biquadratic spline space over the hierarchical T-mesh as follows:

6.2. Properties

In this subsection, we denote the extension of \( \mathcal{T} \) as \( \mathcal{T}^* \), and we denote the CVR graph of \( \mathcal{T}^* \) as \( \mathcal{G}^* \). We discuss the properties of basis functions of \( S^2(\mathcal{G}^*) \) as follows:

**Theorem 6.2.** The basis functions of \( S^2(\mathcal{G}) \) hold the properties of linearly independence, completeness.

**Proof.** Apply the mapping to \( S^2(\mathcal{T}^*) \) and \( S^2(\mathcal{G}^*) \). By Theorem 6.1, the mapping between \( S^2(\mathcal{T}^*) \) and \( S^0(\mathcal{G}^*) \) is bijective, and \( S^2(\mathcal{T}^*) \) is isomorphic to \( S^0(\mathcal{G}^*) \). As the basis functions of \( S^0(\mathcal{G}^*) \) are linearly independent and complete, the basis functions of \( S^2(\mathcal{T}^*) \) are also linearly independent and complete. By Theorem 2.2 and the basis functions of \( S^2(\mathcal{T}) \) are linearly independent and complete on \( \mathcal{T} \).

**Theorem 6.3.** All the basis functions of \( S^2(\mathcal{T}) \) have the property of unit partition.

**Proof.** Assume that the basis functions of \( S^2(\mathcal{T}^*) \) are \( p_i(x, y) \), the basis functions of \( S^0(\mathcal{G}^*) \) are \( q_i(x, y) \), and \( \Phi^{-1}(q_i) = p_i \) where \( i = 1, 2, ..., N_{gs} \). As the space is a linear space, we obtain

\[
\sum_{i=1}^{N_{gs}} p_i = \sum_{i=1}^{N_{gs}} \Phi^{-1}(q_i) = \Phi^{-1}\left( \sum_{i=1}^{N_{gs}} (q_i) \right).
\]

As \( \sum_{i=1}^{N_{gs}} (q_i) = 1 \) is true for each g-cell of \( \mathcal{G}^* \), the weight on each interior domain of \( \mathcal{T}^* \) is 1. By Algorithm 4, the B-ordinates on each cell of \( \mathcal{G}^* \) are 1.

Thus, \( \sum_{i=1}^{N_{gs}} p_i = 1 \) is true on \( \mathcal{T} \), by Theorem 22 the theorem is proved.
| model            | n  | dim | max error  | t(s) |
|------------------|----|-----|------------|------|
| A surface patch  | 4  | 33  | 2.4 × 10^{-3} | 0.293 |
| × 10^{-5}       | 0.293 |
| Nefertiti face   | 4  | 913 | 2.2 × 10^{-3} | 5.67 |
| Gargoyle         | 11 | 4908| 7.8 × 10^{-3} | 101.25 |
| Female head      | 13 | 4256| 6.0 × 10^{-3} | 150.38 |

Table 3: Experiment data.

Thus, the mapping is an isomorphism and the basis functions that we construct for $S^2(\mathcal{T})$ hold the properties of linearly independence, completeness and partition of unity.

7. Surface fitting

Given an open surface triangulation with vertices $V_i, i = 0, \ldots, N$ in 3D space, the corresponding parameter values $(x_i, y_i), i = 0, \ldots, N$ are obtained by the parametrization in [32], we denote the triangle on the triangulation mesh as $\Delta$. The parameter mesh is a triangle mesh and the parameter domain is $[0, 1] \times [0, 1]$.

To construct a spline to fit the given surface, we need to compute all the basis functions $b_j(x, y), j = 1, \ldots, m$ and their corresponding control points $P_j, j = 1, \ldots, m$. We denote the fitting spline $S(x, y) = \sum_{j=1}^{m} P_j b_j(x, y)$. To find the control points, we just need to solve a linear system

$$S(x_k, y_k) = V'_k, k = 1, 2, \ldots, m,$$

where $(x_k, y_k) = (\frac{x_i + x_{i+1}}{2}, \frac{y_i + y_{i+1}}{2})$ is the domain-centre of the domain $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$. $(x_k, y_k) \in \Delta_k : ((x_{k1}, y_{k1}), (x_{k2}, y_{k2}), (x_{k3}, y_{k3})), (x_k, y_k) = w_{k1}(x_{k1}, y_{k1}) + w_{k2}(x_{k2}, y_{k2}) + w_{k3}(x_{k3}, y_{k3})$, and $V'_k = w_{k1} V_{k1} + w_{k2} V_{k2} + w_{k3} V_{k3}$.

The surface fitting scheme repeats the following two steps until the fitting error in each cell is less than the given tolerance $\varepsilon$.

1. Compute all the control points for all the basis functions.
2. Find all the cells whose errors are greater than the given error tolerance $\varepsilon$, then subdivide these cells into four subcells to form a new mesh, simplify the new mesh, and construct basis functions for the new mesh. The fitting error on cell $C$ is $\max_{(x,y) \in C} \| V(x, y) - S(x, y) \|$. Four examples are provided to illustrate the above surface fitting scheme in Fig. 22. The iteration number (n), the dimensions of the spline spaces, the max error and the CPU time on 64 bit operating system are shown in Table 3.

8. Conclusions and future works

We give bijective mapping between the biquadratic spline space over a hierarchical T-mesh and the piecewise constant spline space over the corresponding CVR graph. And we obtain the conclusion that the biquadratic spline space is isomorphic to the piecewise constant spline space. By the bijective mapping, we proposed a novel method to discuss the dimensions of the biquadratic spline spaces over hierarchical T-meshes. We construct the basis functions of the biquadratic spline spaces via a novel structure, which is called a T-structure. Our method is general when the level difference of the hierarchical T-meshes is more than one. We overcome the limitations in [28], and we need not subdivide the extra cells to maintain the level different is less or equal to 1. To reduce the computation, we give the simplifications of the hierarchical T-meshes. Our method is easy operative, and some numerical experiments are given to show our method is effective. By the bijective mapping, it is easy to prove that the basis functions hold the properties of linearly independence, completeness and partition of unity.
Figure 22: Original meshes (left), result surfaces (middle), surfaces with T-meshes (right)
As the mapping we construct is an isomorphism, we will apply our basis functions to IGA and models with high genus in the future. This mapping provides a new idea for us to study high-order spline spaces with low-order spline spaces. We are also working to extend our work to high order spline spaces. The 3-variate case is also a considerable question. As some edges do not contribute to our dimension, improving our subdivision rules is also a considerable idea. We will also consider improving our basis construction method to reduce the computational overload in the future.

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