Upper bound for the number of privileged words

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Abstract

A non-empty word $w$ is a border of a word $u$ if $|w| < |u|$ and $w$ is both a prefix and a suffix of $u$. A word $u$ is privileged if $|u| \leq 1$ or if $u$ has a privileged border $w$ that appears exactly twice in $u$. Peltomäki (2016) presented the following open problem: “Give a nontrivial upper bound for $B(n)$”, where $B(n)$ denotes the number of privileged words of length $n$.

Let $\ln^{[0]}(n) = n$ and let $\ln^{[j]}(n) = \ln(\ln^{[j-1]}(n))$, where $j, n$ are positive integers. We show that if $q > 1$ is a size of the alphabet and $j \geq 3$ is an integer then there are constants $\alpha_j$ and $n_j$ such that

$$B(n) \leq \alpha_j q^n \sqrt{\frac{\ln n}{n}} \ln^{[j]}(n) \prod_{i=2}^{j-1} \sqrt{\ln^{[i]}(n)},$$

where $n \geq n_j$.

This result improves the upper bound of Rukavicka (2020).

1 Introduction

Let $u, w$ be non-empty words. We say that $w$ is a border of $u$ if $|w| < |u|$ and $w$ appears as both a prefix and a suffix of $u$. Let

$$\Theta(u) = \{ w \mid w \text{ is a border of } u \}.$$
We say that \( w \) is the *maximal border* of \( u \) if for every \( w \in \Theta(u) \) we have that \( |w| \leq |w| \). We say that \( u \) is *closed* if there is \( w \in \Theta(u) \) such that \( u \) contains exactly two occurrences of \( w \); realize that these two occurrences are a prefix and a suffix of \( u \).

We say that \( u \) is *privileged* if \( |u| \leq 1 \) or if there is \( w \in \Theta(u) \) such that \( u \) contains exactly two occurrences of \( w \); realize that these two occurrences are a prefix and a suffix of \( u \).

The closed and privileged words attracted some attention in recent years [2], [4], [7]. To find a lower and an upper bound for the number of privileged words are two topics that have been researched. Concerning the lower bound, it was shown that there are constants \( c \) and \( n_0 \) such that for all \( n > n_0 \), there are at least \( \frac{cn}{(\log n)^2} \) privileged words of length \( n \) [3], where \( q \) denote the size of the alphabet in question. This improves the lower bound for the number of privileged words from [1].

Let \( B(n) \) denote the number of privileged words of length \( n \). As for an upper bound for the number of privileged words, the following open problem can be found [5]: “Give a nontrivial upper bound for \( B(n) \”).

Let \( D(n) \) denote the number of closed words of length \( n \). In [6], it was shown that if \( q > 1 \) is a size of the alphabet then there is a positive real constant \( c \) such that

\[
D(n) \leq c \ln n \frac{q^n}{\sqrt{n}}, \text{ where } n > 1.
\]  

Since every privileged word \( u \) with \( |u| > 1 \) is also a closed word, we have that [11] is also an upper bound for the number of privileged words. Hence, the upper bound [11] gave a response to the open problem in [5].

**Definition 1.1.** Let \( \mathbb{N} \) denote the set of positive integers and let \( \mathbb{R} \) denote the set of real numbers. Let \( \ln^0(n) = n \) and let \( \ln^{[j]}(n) = \ln(\ln^{[j-1]}(n)) \), where \( j, n \in \mathbb{N} \).

Given \( j \in \mathbb{N} \), let \( \sigma^{[j]}, \rho^{[j]} : \mathbb{N} \to \mathbb{R} \) be functions defined as follows:

\[
\sigma^{[1]}(n) = \sqrt{\ln n}.
\]

\[
\sigma^{[2]}(n) = \ln^{[2]}(n).
\]

\[
\sigma^{[j]}(n) = \ln^{[j]}(n) \prod_{i=2}^{j-1} \sqrt{\ln^{[i]}(n)}, \text{ where } j \geq 3.
\]
\[ \rho^{[j]}(n) = \sigma^{[j]}(n) \sqrt{\frac{\ln n}{\sqrt{n}}}, \text{ where } j \in \mathbb{N}. \]

In the current article we improve the upper bound (1) for the number of privileged words. We prove the following theorem.

**Theorem 1.2.** If \( q > 1 \) is a size of the alphabet and \( j \in \mathbb{N} \) then there are constants \( \alpha_j \) and \( n_j \) such that

\[ B(n) \leq \alpha_j \rho^{[j]}(n)q^n, \text{ where } n \geq n_j. \]

**Remark 1.3.** Note in Theorem 1.2 that the constants \( \alpha_j \) and \( n_j \) depend on the size of the alphabet \( q \) and on the constant \( j \).

**Remark 1.4.** It is easy to verify that \( \lim_{n \to \infty} \frac{\rho^{[j]}(n)q^n}{\rho^{[j+1]}(n)q^n} = \infty \) for every positive integer \( j \). It means that the bigger \( j \) the better upper bound \( \alpha_j \rho^{[j]}(n)q^n \).

**Example 1.5.** We have that

- \( \rho^{[1]}(n)q^n = n^{-\frac{1}{2}}q^n \ln n \),
- \( \rho^{[2]}(n)q^n = n^{-\frac{1}{2}}q^n \sqrt{\ln n \ln \ln n} \),
- \( \rho^{[3]}(n)q^n = n^{-\frac{1}{2}}q^n \sqrt{\ln n (\ln \ln n) \sqrt{\ln \ln n}} \), and
- \( \rho^{[4]}(n)q^n = n^{-\frac{1}{2}}q^n \sqrt{\ln n (\ln \ln n \ln \ln n) \sqrt{\ln \ln n \ln \ln \ln n}} \).

To prove our result, we apply in principle the same ideas like in [6]. It means that we enumerate the privileged words depending on the length of the maximal border. We distinguish “short” and “long” borders. It turns out that the number of privileged words with a short border is bigger than the number of privileged words with a long border. When comparing the proof in the current article and the proof in [6], the essential difference is that we consider only privileged words instead of all words when enumerating the borders. Due to this difference Theorem 1.2 does not hold for closed words; recall that the upper bound (1) holds for closed words. To facilitate the comprehension we use mostly the same notation like in [6].

3
2 Preliminaries

Let $A$ be an alphabet with $q$ letters, where $q > 1$. Let $\varepsilon$ denote the empty word. Let $A^m$ denote the set of all words of length $m$, let $A^+ = \bigcup_{m \geq 1} A^m$, and let $A^* = A^+ \cup \{\varepsilon\}$. We have that $A^0 = \{\varepsilon\}$ and that $|A^m| = q^m$.

Let $A_w(n)$ denote the number of words of length $n$ that do not contain the factor $w \in A^*$. Let $\mu(n, m) = \max\{A_w(n) \mid w \in A^m\}$.

The function $\mu(n, m)$ represents the maximal value of $A_w(n)$ for all $w$ of length $m$.

Let $\hat{B}(n) \subseteq A^*$ denote the set of all privileged words of length $n$ and let $\hat{B}(n, m)$ denote the set of all privileged words of length $n$ having the maximal border of length $m$. Let $B(n) = |\hat{B}(n)|$ and $B(n, m) = |\hat{B}(n, m)|$.

Remark 2.1. Note that:

- $\hat{B}(n) = \bigcup_{m=1}^{n-1} \hat{B}(n, m)$.
- $\hat{B}(n, m) \cap \hat{B}(n, \overline{m}) = \emptyset$, where $m \neq \overline{m}$.

Let $\omega(n) = \frac{1}{\ln q} (\ln n - \ln \ln n) \in \mathbb{R}$, where $n \in \mathbb{N}$.

Let $\Pi$ denote the set of all functions $\pi(n) : \mathbb{N} \rightarrow \mathbb{N}$ such that $\pi(n) \in \Pi$ if and only if $1 \leq \pi(n) \leq \max\{1, \omega(n)\}$ and $\pi(n) \leq \pi(n + 1)$ for all $n \in \mathbb{N}$. We apply the function max, because $\omega(n) < 1$ for some small $n$.

3 Previous results

In this section we recall the results from [6] that we will need for the current article.

In [6], an upper bound for $\mu(n, m)$ was shown; it means an upper bound for the number of words of length $n$ that avoid some factor of length $m$.

Lemma 3.1. ([6, Lemma 2.1]) If $n, m \in \mathbb{N}$ then

$$\mu(n, m) \leq q^n \left(1 - \frac{1}{q^m}\right)^{\lfloor \frac{n}{m} \rfloor}.$$
Let $\beta = \frac{1}{\ln q} \in \mathbb{R}$. Using Lemma 3.1 it was shown that the number of words of length $n$ avoiding some factor of length shorter than $\pi(n) \in \Pi$ grows with $n$ approximately as the number of all words of length $n - \beta \ln n$.

**Theorem 3.2.** ([6, Theorem 2.3]) If $\pi(n) \in \Pi$ then there is a constant $c_1 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have that

$$\frac{\mu(n, \pi(n))}{q^{n - \beta \ln n}} \leq c_1.$$ 

Let $h(n) = \lfloor \beta \ln n \rfloor$. We present Theorem 3.2 in a more useful form for our next proofs.

**Corollary 3.3.** ([6, Corollary 2.4]) If $\pi(n) \in \Pi$ then there is a constant $c_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have that

$$\frac{\mu(n - 2\pi(n), \pi(n))}{q^{n - h(n)}} \leq c_2.$$ 

Let $\kappa > 1$ be a real constant and let

$$\overline{h}(n) = \max\{1, \frac{1}{\kappa} \omega(n)\}. \quad (2)$$

We apply the function $\max$ to assert that $\overline{h}(n) \geq 1$ for all $n$.

**Remark 3.4.** In our proofs, the privileged words will be enumerated depending on the length of the maximal border; recall the “short” and “long” borders mentioned in the introduction. We will distinguish the length of the maximal border for $m < \overline{h}(n)$ and for $m \geq \overline{h}(n)$.

The next technical lemma shows an upper bound for $q^{-h(n) + \overline{h}(n)}$.

**Lemma 3.5.** ([6, Lemma 2.7]) There is a constant $c_3 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have that

$$q^{-h(n) + \overline{h}(n)} \leq c_3 q^{\frac{\ln n}{\ln q} (\frac{1}{\kappa} - 1)}.$$ 

### 4 Upper bound for privileged words

We show that if $w$ is the maximal border of a privileged word $u$ then $w$ is privileged.
Lemma 4.1. If \( n, m \in \mathbb{N} \), \( u \in \hat{B}(n, m) \), \( w \in \Theta(u) \), and \( |w| = m \) then \( w \in \hat{B}(m) \).

Proof. Suppose that \( w \) is not privileged. Then since \( u \) is privileged there is a privileged border \( \overline{w} \in \Theta(u) \setminus \{w\} \) with exactly two occurrences in \( u \). We have that \( |\overline{w}| < |w| \), \( \overline{w} \) is a prefix of \( w \), and \( \overline{w} \) is a suffix of \( u \). Since \( w \) has at least two occurrences in \( u \), it follows that \( \overline{w} \) has at least three occurrences in \( w \). This is a contradiction. We conclude that \( w \) is privileged word. This ends the proof. \( \square \)

We show a recursive upper bound for the number of privileged words \( B(n, m) \). The following lemma is a variation of [6, Lemma 2.5] for privileged words.

Lemma 4.2. Suppose \( n, m \in \mathbb{N} \). We have that

- If \( 2m > n \) then \( B(n, m) \leq q^{\lceil \frac{n}{2} \rceil} \).
- If \( 2m \leq n \) then \( B(n, m) \leq B(m) \mu(n - 2m, m) \).

Proof. If \( 2m > n \) and \( w \in A^m \) then there is obviously at most one word \( u \) with \( |u| = n \) having a prefix and a suffix \( w \); realize that the prefix \( w \) and the suffix \( w \) would overlap with each other. If such \( u \) exists then the first half of \( u \) uniquely determines the second half of \( u \). If follows that

\[
B(n, m) \leq q^{\lceil \frac{n}{2} \rceil}.
\] (3)

Let \( F(v) \) denote the set of all factors of \( v \in A^* \). If \( n \geq 2m \) then let

\[
T(n, m) = \{uwv \mid u \in A^{n-2m} \text{ and } w \in \hat{B}(m) \text{ and } w \not\in F(u)\}.
\]

If \( n \geq 2m \) then Lemma 4.1 implies that

\[
\hat{B}(n, m) \subseteq T(n, m).
\] (4)

It is easy to see that

\[
|T(n, m)| \leq B(m) \mu(n - 2m, m).
\] (5)

The lemma follows from (3), (4), and (5). This ends the proof. \( \square \)

Definition 4.3. Let \( \text{upb} \) be a set of functions \( \rho : \mathbb{N} \to \mathbb{R} \) such that \( \rho \in \text{upb} \) if and only if there is \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \) we have that

6
1. \( q^n \rho(n) \geq B(n) \),

2. \( \rho(n) \geq \rho(n + 1) \), and

3. \( q^n \rho(n) \leq q^{n+1} \rho(n + 1) \).

Thus \( \text{upb}(X) \) is a set of non-increasing functions \( \rho \) such that \( q^n \rho(n) \) are upper bounds for the number of privileged words.

Using the functions of \( \text{upb} \), we can restate Lemma 4.2 as follows. We omit the proof, as it follows immediately from Lemma 4.2 and Definition 4.3.

**Lemma 4.4.** If \( \rho \in \text{upb} \) then there is \( n_0 \in \mathbb{N} \) such that for all \( n, m \in \mathbb{N} \) with \( n > n_0 \) we have that

- If \( 2m > n \) then \( B(n, m) \leq q^{\left\lceil \frac{n}{2} \right\rceil} \).
- If \( 2m \leq n \) then \( B(n, m) \leq q^m \rho(m) \mu(n - 2m, m) \).

We show an upper bound for the number of privileged words of length \( n \) having the maximal border shorter than \( h(n) \).

**Lemma 4.5.** If \( \rho \in \text{upb} \) then there are constants \( c_4 \in \mathbb{R} \) and \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \) we have that

\[
\Sigma_{m=1}^{h(n)-1} B(n, m) \leq c_4 \frac{\ln n}{\kappa} q^{n \frac{1}{\kappa} - 1} \rho(h(n)).
\]

**Proof.** From (2) it follows that \( h(n) < \frac{n}{2} \) for sufficiently large \( n \). Hence Lemma 4.2 implies that

\[
\Sigma_{m=1}^{h(n)-1} B(n, m) \leq \Sigma_{m=1}^{h(n)-1} B(m) \mu(n - 2m, m). \quad (6)
\]

From (2) it follows that

\[
\overline{h}(n) \leq \frac{\ln n}{\kappa \ln q} \text{ for sufficiently large } n. \quad (7)
\]
Corollary 3.3 implies that $\mu(n - 2m, m) \leq c_2 q^{n-h(n)}$, where $1 \leq m \leq \overline{h}(n) \leq h(n)$ and $c_2 \in \mathbb{R}$ is some constant. Then it follows from Lemma 3.5, Property 3 of Definition 4.3, and (7) that

$$
\begin{align*}
\overline{h}(n) - 1 \sum_{m=1}^{\overline{h}(n)} B(m) \mu(n - 2m, m) &\leq \sum_{m=1}^{\overline{h}(n)} q^m \rho(m) c_2 q^{n-h(n)} \\
&\leq \overline{h}(n) q^{\overline{h}(n)} \rho(\overline{h}(n)) c_2 q^{n-h(n)} \\
&\leq c_2 c_3 \overline{h}(n) q^{n + \frac{\ln n}{\ln q} (\frac{1}{\kappa} - 1)} \rho(\overline{h}(n)) \\
&= c_2 c_3 \frac{\ln n}{\kappa \ln q} q^{n + \frac{\ln n}{\ln q} (\frac{1}{\kappa} - 1)} \rho(\overline{h}(n)) \\
&= c_2 c_3 \frac{\ln n}{\kappa \ln q} q^n n^{\frac{1}{\kappa} - 1} \rho(\overline{h}(n)).
\end{align*}
$$

Let $c_4 = \frac{c_2 c_3}{\ln q}$. The lemma follows from (6) and (8). This ends the proof.

We show an upper bound for the number of privileged words of length $n$ with the maximal border longer than $\overline{h}(n)$ and shorter than $\frac{n}{2}$.

**Lemma 4.6.** There are constants $c_5 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$
\left\lceil \frac{n}{2} \right\rceil \sum_{m=\overline{h}(n)} B(n, m) \leq c_5 q^n (\ln n)^{\frac{1}{\kappa} - \frac{1}{2}}, \text{ where } n > n_0.
$$

**Proof.** From Lemma 4.2 and from $\mu(n - 2m, m) \leq q^{n-2m}$ we have for sufficiently large $n$ that

$$
\begin{align*}
\left\lceil \frac{n}{2} \right\rceil \sum_{m=\overline{h}(n)} B(n, m) &\leq \left\lceil \frac{n}{2} \right\rceil \sum_{m=\overline{h}(n)} B(m) \mu(n - 2m, m) \\
&\leq \left\lceil \frac{n}{2} \right\rceil \sum_{m=\overline{h}(n)} B(m) q^{n-2m}.
\end{align*}
$$

From (2) we have for sufficiently large $n$ that

$$
\begin{align*}
q^{-\overline{h}(n)} &\leq q^{-\frac{\ln n}{\kappa \ln q} (\ln n - \ln \ln n) + 1} \\
&= q(\ln n)^{\frac{1}{\kappa}} q^{-\frac{1}{\kappa \ln q} \ln n} \\
&= q(\ln n)^{\frac{1}{\kappa}} n^{-\frac{1}{2}}.
\end{align*}
$$
From (10) it follows that
\[
\sum_{m=\overline{h}(n)}^{\lfloor q/2 \rfloor} B(m)q^{n-2m} \leq \sum_{m=\overline{h}(n)}^{\lfloor q/2 \rfloor} q^m q^{n-2m}
\leq q^n \sum_{m=\overline{h}(n)}^{\lfloor q/2 \rfloor} q^{-m}
= q^n \frac{1 - q^{-(\lfloor q/2 \rfloor+1)}}{1 - q^{-1}} - q^n \frac{1 - q^{-\overline{h}(n)}}{1 - q^{-1}}
\leq q^{n-\overline{h}(n)}\frac{1}{1 - q^{-1}}
\leq q^{n+1}(\ln n)^{\frac{1}{\kappa}} n^{-\frac{1}{\kappa}}.
\] (11)

Let \(c_5 = \frac{q}{1-q^{-r}}\). Then the lemma follows from (9) and (11). This ends the proof. \(\square\)

We show an approximation for the function \(\frac{\sigma[j](\overline{h}(n))}{\sigma[j+1](n)}\) as \(n\) tends to infinity.

**Lemma 4.7.** If \(j \in \mathbb{N}\) then
\[
\lim_{n \to \infty} \frac{\sigma[j](\overline{h}(n))\sqrt{\ln(\overline{h}(n))}}{\sigma[j+1](n)} = 1.
\]

**Proof.** It follows from (2) that for sufficiently large \(n\) we have that \(\overline{h}(n) = \lfloor \frac{1}{\kappa \ln q} (\ln n - \ln \ln n) \rfloor\). Then it is easy to see that
\[
\lim_{n \to \infty} \frac{\ln[j](\overline{h}(n))}{\ln[j+1](n)} = \lim_{n \to \infty} \frac{\ln[j+1] \left( \lfloor \frac{1}{\kappa \ln q} (\ln n - \ln \ln n) \rfloor \right)}{\ln[j+1](n)} = 1.
\] (12)

Let
\[
y(n) = \frac{\sigma[j](\overline{h}(n))\sqrt{\ln(\overline{h}(n))}}{\sigma[j+1](n)}
\]
and let
\[
\overline{y}(n) = \frac{\sqrt{\ln(\overline{h}(n))) \prod_{i=2}^{j-1} \sqrt{\ln[i](\overline{h}(n))}}}{\prod_{i=2}^{j} \sqrt{\ln[i](n)}}.
\]
From (12) it follows that
\[
\lim_{n \to \infty} y(n) = \lim_{n \to \infty} \frac{\sqrt{\ln (\overline{h}(n)) \prod_{i=2}^{j-1} \sqrt{\ln^i(\overline{h}(n))}}}{\prod_{i=2}^{j} \sqrt{\ln^i(n)}}
\]
\[
= \lim_{n \to \infty} \frac{\sqrt{\ln (\overline{h}(n)) \ln^2(\overline{h}(n)) \ln^3(\overline{h}(n)) \ldots \ln^{j-1}(\overline{h}(n)) \ln^j(n)}}{\sqrt{\ln^2(n) \ln^3(n) \ldots \ln^{j-1}(n) \ln^j(n)}}
\]
\[
= 1.
\] (13)

From (12) and (13) it follows that
\[
\lim_{n \to \infty} y(n) = \lim_{n \to \infty} \frac{\ln^2(\overline{h}(n)) \left( \prod_{i=2}^{j-1} \sqrt{\ln^i(\overline{h}(n))} \right) \sqrt{\ln (\overline{h}(n))}}{\ln^{j+1}(n) \prod_{i=2}^{j} \sqrt{\ln^i(n)}} = 1.
\] (14)

From (12) it follows that
\[
\lim_{n \to \infty} \sigma^{[1]}(\overline{h}(n)) \sqrt{\ln (\overline{h}(n))} = \lim_{n \to \infty} \frac{\sqrt{\ln (\overline{h}(n)) \ln (\overline{h}(n))}}{\ln^2(n)} = 1
\] (15)
and
\[
\lim_{n \to \infty} \sigma^{[2]}(\overline{h}(n)) \sqrt{\ln (\overline{h}(n))} = \lim_{n \to \infty} \frac{\ln^2(\overline{h}(n)) \sqrt{\ln (\overline{h}(n))}}{\ln^3(n) \sqrt{\ln^2(n)}} = 1.
\] (16)

The lemma follows from (14), (15), and (16). This ends the proof.

The next technical lemma will be used in the proof of Theorem 4.9.

**Lemma 4.8.** If $j \in \mathbb{N}$ then there are constants $c_6$ and $n_0$ such that for all $n \in \mathbb{N}$ with $n > n_0$ we have that
\[
c_6 \sigma^{[j+1]}(n) \sqrt{\ln n q^n n^{-\frac{1}{2}}} \geq \max \left\{ \ln (n) q^n n^{-\frac{1}{2}} \rho^{[j]}(\overline{h}(n)), \sqrt{\ln n q^n n^{-\frac{1}{2}}} \right\}.
\]
Proof. Let

\[ y(n) = \frac{\ln (n) q^n - \frac{1}{2} \rho[j](h(n))}{\sigma^{[j]}(n) \sqrt{\ln n q^n n^{-\frac{1}{2}}} + \sigma^{[j]}(n) \sqrt{\ln (h(n))}} \]

Lemma 4.7 and (2) imply that

\[ \lim_{n \to \infty} y(n) = \lim_{n \to \infty} \frac{\sqrt{\ln n}}{\sqrt{h(n)}} \]

\[ = \lim_{n \to \infty} \frac{\sqrt{\ln n}}{\sqrt{\frac{1}{\kappa \ln q} (\ln n - \ln \ln n)}} \]

\[ = \sqrt{\kappa \ln q}. \] (17)

Obviously \( \sigma^{[j]}(n) \sqrt{\ln n q^n n^{-\frac{1}{2}}} \geq \sqrt{\ln n q^n n^{-\frac{1}{2}}} \) for sufficiently large \( n \). Thus the lemma follows from (17). This ends the proof.

We show how to improve the upper bound for the number of privileged words on condition that there is already an upper bound of the form \( \alpha_j \rho[j](n) q^n \), where \( \alpha_j \) is a constant.

Theorem 4.9. If there is \( j \in \mathbb{N} \) and a constant \( \alpha_j \) such that \( \alpha_j \rho[j] \in \text{upb} \) then there is a constant \( \alpha_{j+1} \) such that \( \alpha_{j+1} \rho[j+1] \in \text{upb} \).

Proof. Obviously we have that

\[ B(n) = \sum_{m=1}^{n-1} B(n, m) = \sum_{m=1}^{\lceil \frac{n}{2} \rceil} B(n, m) + \sum_{m=\lceil \frac{n}{2} \rceil + 1}^{n-1} B(n, m). \] (18)

From Lemma 4.4 we get that

\[ \sum_{m=\lceil \frac{n}{2} \rceil + 1}^{n-1} B(n, m) \leq \sum_{m=\lceil \frac{n}{2} \rceil + 1}^{n-1} q^{\lceil \frac{n}{2} \rceil} \leq \frac{n}{2} q^{\lceil \frac{n}{2} \rceil}. \] (19)
From Lemma 4.5 and Lemma 4.6, it follows that there are constants $c_4, c_5 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have that
\[
\left\lfloor \frac{n}{2} \right\rfloor \sum_{m=1}^{\left\lfloor \frac{n}{2} \right\rfloor} B(n, m) = \sum_{m=1}^{\left\lfloor \frac{n}{2} \right\rfloor} B(n, m) + \sum_{m=\overline{n}}^{\left\lfloor \frac{n}{2} \right\rfloor} B(n, m) \leq c_4 \frac{\ln n}{\kappa} q^n n^{1/2 - \alpha_j \rho[j]}(\overline{n}(n)) + c_5 (\ln n)^{1/2} n^{1/2 - \frac{1}{\kappa}}. \tag{20}
\]

Let $\kappa = 2$. From Lemma 4.8 and (20), it follows that there are constants $c_6 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have that
\[
\left\lfloor \frac{n}{2} \right\rfloor \sum_{m=1}^{\left\lfloor \frac{n}{2} \right\rfloor} B(n, m) \leq 2c_6 \sigma^{[j+1]}(n) \sqrt{\ln n} q^n n^{-\frac{1}{2}}. \tag{21}
\]

From (18), (19), and (21) it follows that
\[
B(n) \leq 2c_6 \sigma^{[j+1]}(n) \sqrt{\ln n} q^n n^{-\frac{1}{2}} + \frac{n}{2} q^{\left\lfloor \frac{n}{2} \right\rfloor}. \tag{22}
\]

We have that
\[
\lim_{n \to \infty} \frac{n q^{\frac{n}{2}}}{q^n n^{-\frac{1}{2}}} = 0. \tag{23}
\]

From (22) and (23) it follows that there is a constant $\alpha_{j+1}$ such that
\[
B(n) \leq \alpha_{i+1} \sigma^{[j+1]}(n) \sqrt{\frac{\ln n}{n}} q^n = \alpha_{i+1} \rho^{[j+1]}(n) q^n.
\]

This completes the proof. \hfill \square

Now we can step to the proof of the main theorem of the article.

**Proof.** (Proof of Theorem 1.2) From (1) it follows that there is a constant $\alpha_1$ such that $\alpha_1 \rho^{[1]}(n) \in \text{upb}$. Hence Theorem 4.9 implies that for every $j \in \mathbb{N}$ there is a constant $\alpha_j$ such that $\alpha_j \rho^{[j]}(n) \in \text{upb}$. This ends the proof. \hfill \square

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