ON KERNEL BUNDLES OVER REDUCIBLE CURVES WITH A NODE

SONIA BRIVIO AND FILIPPO F. FAVALE

Abstract. Given a vector bundle $E$ on a complex reduced curve $C$ and a subspace $V$ of $H^0(E)$ which generates $E$, one can consider the kernel of the evaluation map $ev_V : V \otimes \mathcal{O}_C \to E$, i.e. the kernel bundle $M_{E,V}$ associated to the pair $(E,V)$. Motivated by a well known conjecture of Butler about the semistability of $M_{E,V}$ and by the results obtained by several authors when the ambient space is a smooth curve, we investigate the case of a curve with one node. Unexpectedly, we are able to prove results which go in the opposite direction with respect to what is known in the smooth case. For example, $M_{E,H^0(E)}$ is actually quite never $w$-semistable. Conditions which gives the $w$-semistability of $M_{E,V}$ when $V \subset H^0(E)$ or when $E$ is a line bundle are then given.

Introduction

Let $C$ be a complex reduced projective curve. Let $(E,V)$ be a pair on $C$ given by a vector bundle $E$ of rank $r$ and a vector space $V \subseteq H^0(E)$ of dimension $k \geq r + 1$. We say that $(E,V)$ is a generated pair if global sections of $V$ generate $E$, i.e. the evaluation map $ev_V : V \otimes \mathcal{O}_C \to E$ is surjective. In this case, its kernel $M_{E,V}$ is a vector bundle of rank $k - r$ on $C$ which fits into the following exact sequence

$$(0.1) \quad 0 \longrightarrow M_{E,V} \longrightarrow V \otimes \mathcal{O}_C \xrightarrow{ev_V} E \longrightarrow 0$$

and it is called the kernel bundle (or the Lazarsfeld bundle) of the pair $(E,V)$. When $V = H^0(E)$, then it is denoted by $M_E$ and this case is said the complete case.

Generated pairs encode a lot of the geometry of the curve as well as a lot of interesting informations about it. For example, a generated pair $(E,V)$ defines a morphism

$$\varphi_{E,V} : C \to G(k - r, V), \quad x \mapsto \text{Ker}(ev_{V,x}),$$

where $G(k-r,V)$ denotes the Grassmannian variety of $(k-r)$-dimensional linear subspaces of $V$. Then, the exact sequence $0.1$ is actually the pull back by $\varphi_{E,V}$ of the following exact sequence on $G(k - r, V)$:

$$(0.2) \quad 0 \longrightarrow \mathcal{U} \longrightarrow V \otimes \mathcal{O}_{G(k-r,V)} \longrightarrow \mathcal{Q} \longrightarrow 0$$

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Both authors are members of GNSAGA-INdAM.
where $\mathcal{U}$ and $\mathcal{Q}$ are respectively the universal and quotient bundle on $G(k-r,V)$. These bundles have been studied by many authors when the curve $C$ is smooth and irreducible, because of their rich applications (see [Laz89] for an overview). In particular, their stability properties, which have been studied with respect to different point of view (see, for instance, [EL92,Mis08]) are closed related to higher rank Brill-Noether theory and moduli spaces of coherent systems (see [BGMN03,BBN08,BB12] for example). Finally, they have been useful in studying theta divisors and the geometry of moduli space of vector bundles on curves (see [Bea03,Pop07,Bri18,BF19a], for example).

In the complete case, Butler, in his seminal work [But94], proved that on a smooth irreducible curve of genus $g$ the kernel bundle $M_E$ is semistable for any semistable $E$ of degree $d \geq 2rg$ and it is actually stable if $E$ is stable of degree $d > 2gr$. In the case of line bundles, this result has been improved in several works by taking into consideration the Clifford index of the curve. For example, see [But97,BP08,Cam08,MS12].

For the general case, in [But97] Butler made the following conjecture:

**Conjecture.** For a general smooth curve of genus $g \geq 3$ and a general choice of a generated pair $(E,V)$, where $E$ is a semistable vector bundle, the kernel bundle $M_{E,V}$ is semistable.

Much works has been done in the direction of solving this conjecture. In particular it has been completely proved in [BBN15] in the case of line bundles. Moreover, many conditions for stability are given (see also [BBN08]).

It seems natural to ask whether similar results hold in the case of singular curves. The aim of this paper is to investigate stability properties for kernel bundles on a nodal reducible curve. More precisely, we will consider a complex reducible projective curve $C$ with two smooth irreducible components $C_i$, of genus $g_i \geq 2$, $i = 1, 2$, and a single node $p$. Some modifications to the environment are required, obviously. For example, the notion of semistability needs to be replaced with the notion of $w$-semistability for a given polarization $w$. The theory and the results that we will need for this are summarized in Section 1. We will say that $M_{E,V}$ is **strongly unstable** if it is $w$-unstable for any polarization on the curve.

Our first result is Theorem 2.4, which is proved in Section 2.

**Theorem.** Let $C$ be a nodal curve as above. Let $(E,V)$ be a generated pair on the curve $C$ and let $E_i$ be the restriction of $E$ to the component $C_i$. If $E_i$ is semistable and
\[ \dim(V \cap H^0(E_i(-p))) \geq 1, \text{ then } M_{E,V} \text{ is strongly unstable and both its restrictions to the components are unstable.} \]

As a corollary of this result, we have that the kernel bundle \( M_E \) is strongly unstable for any globally generated vector bundle \( E \) whose restrictions \( E_i \) are semistable and not trivial (see Corollary 2.6). This is a somewhat unexpected result as in the smooth case, by the result of Butler, \( M_E \) is always semistable for any semistable \( E \) with degree sufficiently big. This impressive difference shows that it is worth going on this kind of problems.

It is natural to consider the restrictions of a generated pair \( (E, V) \) to the component \( C_i \). In this way, we show that we get again a generated pair \( (E_i, V_i) \). Nevertheless, it is not always true that the kernel bundle associated to \( (E_i, V_i) \) is the restriction of \( M_{E,V} \) to \( C_i \). Indeed in the case of Theorem 2.4, we have that \( M_{E_i,V_i} \) is a destabilizing quotient for the restriction of \( M_{E,V} \) to \( C_i \).

In section 3 we study the stability of \( M_{E,V} \) when \( M_{E,V|C_i} \simeq M_{E_i,V_i} \). We first show that this is equivalent to the following condition:

\[ (*) \quad V \cap H^0(E_1(-p)) = V \cap H^0(E_2(-p)) = \{0\}. \]

Then, we prove Theorem 3.2:

**Theorem.** Let \( C \) be a nodal curve as above. Let \( (E, V) \) be a generated pair on the curve \( C \) satisfying condition \((*)\). If both \( M_{E_1,V_1} \) and \( M_{E_2,V_2} \) are semistable then there exists a polarization \( w \) such that \( M_{E,V} \) is \( w \)-semistable.

We conclude Section 3 by applying the above theorem to special cases where we know semistability of the kernel bundles \( M_{E_i,V_i} \). The results are stated in Theorem 3.3 and 3.5.

### 1. Vector bundles on nodal reducible curves

Let \( C \) be a complex projective curve with two smooth irreducible components and one single node \( p \), i.e. \( p \) is an ordinary double point. Assume that

\[ \nu: C_1 \sqcup C_2 \to C \]

is a normalization map for \( C \), where \( C_i \) is a smooth irreducible projective curve of genus \( g_i \geq 2 \). Then \( \nu^{-1}(x) \) is a single point except when \( x \) is the node, in this case we have: \( \nu^{-1}(p) = \{q_1, q_2\} \) with \( q_i \in C_i \). Since for \( i = 1, 2 \) the restriction

\[ \nu_{|C_i}: C_i \to \nu(C_i) \]
is an isomorphism, we will call as well $C_1$ and $C_2$ the components of $C$. As $C$ is of compact type, i.e. every node of $C$ disconnects the curve, the pull back map $\nu^*$ induces an isomorphism

$$\text{Pic}(C) \simeq \text{Pic}(C_1 \sqcup C_2) \simeq \text{Pic}(C_1) \times \text{Pic}(C_2).$$

Moreover, the curve $C$ is contained in a smooth irreducible projective surface $X$. On it, $C, C_1$ and $C_2$ are effective divisors and we have: $C = C_1 + C_2$ with $C_1 \cdot C_2 = 1$. In particular $C$ is 1-connected, so the sheaf $\omega_C = \omega_X \otimes \mathcal{O}_C(C)$ is a dualizing sheaf for $C$ and we have $h^1(C, \omega_C) = 1$, see [FT14].

Moreover, for $i, j \in \{1, 2\}$ with $i \neq j$, we have an exact sequence

$$(1.1) \quad 0 \rightarrow \mathcal{O}_{C_i}(-C_j) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C_j} \rightarrow 0$$

from which we get

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_{C_i}(-C_j)) + \chi(\mathcal{O}_{C_j}) = \chi(\mathcal{O}_{C_i}(-p)) + \chi(\mathcal{O}_{C_j}).$$

Let $p_a(C) = 1 - \chi(\mathcal{O}_C)$ be the arithmetic genus of $C$, then we have:

$$p_a(C) = g_1 + g_2.$$
Let $E$ be a depth one sheaf on $C$, let $E_i = E|_{C_i}$ denote the restriction on the component $C_i$. We define the relative rank and the relative degree of $E$ with respect to the component $C_i$ as follows:

$$ r_i = \text{Rk}(E_i), \quad d_i = \text{deg}(E_i) = \chi(E_i) - r_i\chi(O_{C_i}), $$

where $\chi(E_i)$ is the Euler characteristic of $E_i$. We say that $E$ has multirank $(r_1, r_2)$ and multidegree $(d_1, d_2)$. For a vector bundle $E$, the restriction $E_i$ is a vector bundles too and we have $r_1 = r_2 = r$, we denote $r$ as the rank of $E$.

**Definition 1.2.** A polarization $w$ of $C$ is given by a pair of rational weights $(w_1, w_2)$ such that $0 < w_i < 1$ and $w_1 + w_2 = 1$. For any sheaf $E$ of depth one on $C$ of multirank $(r_1, r_2)$ we define the polarized slope as

$$ \mu_w(E) = \frac{\chi(E)}{w_1r_1 + w_2r_2}. $$

**Definition 1.3.** A vector bundle $E$ on $C$ is said $w$-semistable if for any proper subsheaf $F \subset E$ we have $\mu_w(F) \leq \mu_w(E)$; it is said $w$-stable if $\mu_w(F) < \mu_w(E)$.

If $w$ and $w'$ are two polarizations, it can happen that a depth one sheaf $E$ is $w$-semistable and it is not $w'$-semistable. We will say that $E$ on $C$ is strongly unstable if it is $w$-unstable with respect to any polarization on $C$.

We will need the following result, see [Tei95, Tei11].

**Theorem 1.1.** Let $C$ be a nodal curve with two smooth irreducible components $C_i$ and a single node. Let $E$ be a vector bundle on $C$ of rank $r$ with restrictions $E_i$, $i = 1, 2$ and $w$ a polarization. Then we have the following properties:

1. if $E$ is $w$-semistable then:

\[
(1.2) \quad w_1\chi(E) \leq \chi(E_i) \leq w_1\chi(E) + r;
\]

2. if $E_1$ and $E_2$ are semistable $E$ satisfies the above condition, then $E$ is $w$-semistable. Moreover, if at least one of the restrictions is stable, then $E$ is $w$-stable too.

**2. Strongly unstable kernel bundles**

Let $C$ be a nodal reducible curve with two smooth irreducible components $C_i$ of genus $g_i \geq 2$ and a single node $p$ as in Section 1. Let $(E, V)$ be a generated pair on $C$, with $E$ a vector bundle of rank $r$ on $C$ and $V \subseteq H^0(E)$ of dimension $k \geq r + 1$. Consider the kernel bundle $M_{E,V}$ associated to the pair $(E, V)$. It is a vector bundle of rank $k - r \geq 1$ and Euler-characteristic

$$ \chi(M_{E,V}) = k(1 - p_a(C)) - \chi(E). $$
Let $E_i$ be the restriction of $E$ to the component $C_i$ and let $d_i$ be the degree of $E_i$. Let

(2.1) \[ \rho_i: H^0(E) \rightarrow H^0(E_i) \]

be the restriction map of global sections of $E$ to the component $C_i$ and let’s consider its image

$$ V_i = \rho_i(V). $$

We have a pair $(E_i, V_i)$ on the curve $C_i$. This pair is said to be of type $(r, d_i, k_i)$ as $r = \text{Rk}(E_i)$, $d_i$ is the degree of $E_i$ and $k_i = \dim V_i$. We will denote $(E_i, V_i)$ as the restriction of the pair $(E, V)$ to the curve $C_i$.

We have the following lemmas:

**Lemma 2.1.** Let $C$ be a nodal reducible curve as in Section 1. Let $(E, V)$ be a generated pair on $C$. Then

1. $(E_i, V_i)$ is a generated pair on the curve $C_i$ and $k_i \geq r$, $i = 1, 2$;
2. $h^0(E) = h^0(E_1) + h^0(E_2) - r$;
3. the restriction map $\rho_i: H^0(E) \rightarrow H^0(E_i)$ is surjective, for $i = 1, 2$.

*Proof.* (1) Consider the exact sequence 0.1 defining $M_{E,V}$. Since $\text{Tor}^1_{\mathcal{O}_C}(\mathcal{O}_{C_i}, E) = 0$ by tensoring the exact sequence with $\mathcal{O}_{C_i}$ we get the following exact sequence of locally free sheaves on $C_i$:

$$ 0 \rightarrow M_{E,V} \otimes \mathcal{O}_{C_i} \rightarrow V \otimes \mathcal{O}_{C_i}^{ev|C_i} \rightarrow E_i \rightarrow 0. $$

We have the following commutative diagram

(2.2) $$
\begin{array}{cccc}
0 & \rightarrow & M_{E,V} \otimes \mathcal{O}_{C_i} & \rightarrow & V \otimes \mathcal{O}_{C_i}^{ev|C_i} & \rightarrow & E_i & \rightarrow & 0 \\
& & \downarrow{\rho_i|_V} & & \downarrow{ev|_i} & & \end{array}
$$

From it we deduce that $ev|_i$ is surjective and thus, $(E_i, V_i)$ is a generated pair. In particular, $k_i = \dim(V_i) \geq r$.

(2) Following [Ses82], the vector bundle $E$ on the curve $C$ is obtained by gluing the vector bundles $E_1$ and $E_2$ along the fibers at the node. More precisely $E$ is determined by the triple $(E_1, E_2, \sigma)$, where $\sigma \in \text{GL}(E_{1,p}, E_{2,p})$ is an isomorphism. In particular, $\sigma$ induces
the following commutative diagram, (see [BF19b] for more details)

\[
\begin{array}{cccccc}
0 & \rightarrow & E & \rightarrow & E_1 \oplus E_2 & \rightarrow & E_{2,p} & \rightarrow & 0 \\
& \downarrow{\rho_{1,p} \oplus \rho_{2,p}} & \downarrow{=} & \downarrow{=} & \downarrow{=} & \downarrow{=} & \downarrow{=} & \downarrow{=} & \downarrow{=}
\end{array}
\]

\[
E_{1,p} \oplus E_{2,p} \rightarrow \delta \rightarrow E_{2,p} \rightarrow 0
\]

where \( \rho_{i,p} \) is the restriction map to the fiber at \( p \) and \( \delta(u, v) = \sigma(u) - v \). Since by (1) \( E_i \) is globally generated, passing to cohomology we obtain the exact sequence

\[
0 \rightarrow H^0(E) \rightarrow H^0(E_i) \oplus H^0(E_2) \rightarrow E_{2,p} \rightarrow 0,
\]

which proves (2).

(3) Let \( i, j \in \{1, 2\} \) with \( i \neq j \). If we tensor the exact sequence 1.1 with \( E \), we obtain

\[
0 \rightarrow E_j(-p) \rightarrow E \rightarrow E_i \rightarrow 0.
\]

Passing to cohomology we have:

\[
0 \rightarrow H^0(E_j(-p)) \rightarrow H^0(E) \rightarrow H^0(E_i) \rightarrow \cdots
\]

from which we deduce that \( \text{Ker} \rho_i \simeq H^0(E_j(-p)) \). Since \( E_j \) is globally generated we have:

\[
\text{Rk} \rho_i = h^0(E) - h^0(E_j(-p)) = h^0(E) - h^0(E_j) + r = h^0(E_i).
\]

Lemma 2.2. Let \( C \) be a nodal reducible curve as in Section 1. Let \( (E, V) \) be a generated pair on \( C \). If \( E_i \) is semistable and \( \dim(V \cap H^0(E_i(-p))) \geq 1 \), \( i = 1, 2 \), then we have:

1. \( (E_i, V_i) \) is of type \( (r, d_i, k_i) \), with \( d_i \geq r \) and \( k_i \leq k - 1 \);
2. the kernel bundle \( M_{E_i,V_i} \) is a non trivial quotient of the restriction \( M_{E,V} \otimes O_{C_i} \);
3. the restriction \( M_{E,V} \otimes O_{C_i} \) is an unstable vector bundle on \( C_i \).

Proof. (1)-(2) First of all note that the assumption \( \dim(V \cap H^0(E_i(-p))) \geq 1 \) implies \( d_i \geq r \). Indeed, as \( E_i(-p) \) has a non zero global section \( s_i \), the zero locus of \( s_i \) is an effective divisor \( Z_0(s_i) \) on \( C_i \) of degree at least 1 and we have an inclusion of sheaves \( O_{C_i}(Z_0(s_i)) \subset E_i \). Since \( E_i \) is semistable, we must have

\[
1 \leq \deg(Z_0(s_i)) = \mu(O_{C_i}(Z_0(s_i))) \leq \mu(E_i) = d_i/r,
\]

which gives \( d_i \geq r \).

Let

\[
S_j = V \cap H^0(E_j(-p)),
\]
since by lemma 2.1 \( \text{Ker} \rho_i \simeq H^0(E_j(-p)) \) then we have the following exact sequence:

\[
0 \longrightarrow S_j \longrightarrow V \xrightarrow{\rho_i|V} V_i \longrightarrow 0,
\]

where \( \dim S_j \geq 1 \) and \( \dim V_i \leq k - 1 \).

We can then complete diagram 2.2 obtaining the following one:

(2.4) \[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & S_j \otimes \mathcal{O}_{C_i} & S_j \otimes \mathcal{O}_{C_i} & 0 \\
0 & M_{E_i, V} \otimes \mathcal{O}_{C_i} & V \otimes \mathcal{O}_{C_i} & E_i \longrightarrow 0 \\
0 & M_{E_i, V_i} & V_i \otimes \mathcal{O}_{C_i} & E_i \longrightarrow 0 \\
0 & 0 & 0 & 0
\end{array}
\]

By definition the kernel of \( ev_{V_i} \) is the kernel bundle \( M_{E_i, V_i} \), which turns out to be a non trivial quotient of \( M_{E_i, V} \otimes \mathcal{O}_{C_i} \).

(3) We claim that \( S_j \otimes \mathcal{O}_{C_i} \) is a non trivial destabilizing subsheaf of \( M_{E_i, V} \otimes \mathcal{O}_{C_i} \). In fact we have:

\[
0 = \mu(S_j \otimes \mathcal{O}_{C_i}) > \mu(M_{E_i, V} \otimes \mathcal{O}_{C_i}) = \frac{-d_i}{k - r} < 0,
\]

since as we have seen \( d_i \geq r \).

\[\square\]

**Remark 2.2.1.** Note that, under the assumption of lemma 2.2, \( M_{E_i, V_i} \) is actually a destabilizing quotient for \( M_{E_i, V} \otimes \mathcal{O}_{C_i} \).

**Remark 2.2.2.** Let \( (E, V) \) be a generated pair with \( E_i \) semistable and not trivial. Then

(1) \( h^0(E_i(-p)) \geq 1 \).

Indeed, the restriction \( (E_i, V_i) \) of \( (E, V) \) is a generated pair and \( \dim(V_i) \geq r \) by Lemma 2.2. Equality holds if and only if \( E_i \simeq V_i \otimes \mathcal{O}_{C_i} \) which is impossible by assumption. Hence \( h^0(E_i) \geq \dim(V_i) \geq r + 1 \) and then \( h^0(E_i(-p)) \geq 1 \).

(2) If, moreover, \( \dim(V) > h^0(E_i) \) then \( V \cap H^0(E_j(-p)) \neq \{0\} \).
It is enough to notice that the restriction \( \rho_i|_V : V \to H^0(E_i) \) cannot be injective in this case. Hence \( V \cap \ker(\rho_i) = V \cap H^0(E_j(-p)) \neq \{0\} \).

In the sequel we will need the following result of Xiao which generalizes Clifford’s Theorem for semistable vector bundles of rank \( r \geq 2 \) on smooth curves.

**Theorem 2.3.** Let \( E \) be a semistable vector bundle of rank \( r \) on a smooth irreducible complex projective curve \( C \) of genus \( g \geq 2 \). If we assume that \( 0 \leq \mu(E) \leq 2g - 2 \), then

\[
h^0(E) \leq \deg(E)/2 + r.
\]

For the proof see [BGN97].

Our first result is the following condition for strongly unstable kernel bundles:

**Theorem 2.4.** Let \( C \) be a reducible nodal curve as in Section 1. Let \( (E, V) \) be a generated pair on \( C \). If \( E_i \) is semistable and \( \dim(V \cap H^0(E_i(-p))) \geq 1, i = 1, 2 \), then the kernel bundle \( M_{E,V} \) is strongly unstable.

**Proof.** We have to prove that \( M_{E,V} \) is \( w \)-unstable with respect to any polarization \( w \) on \( C \). Assume on the contrary, that there exists a polarization \( w = (w_1, w_2) \) such that \( M_{E,V} \) is \( w \)-semistable. As in the proof of Lemma 2.2, let \( S_j = V \cap H^0(E_j(-p)) \) and \( s_j = \dim S_j \geq 1, j = 1, 2 \). As we have proved in Lemma 2.2, for \( i \neq j \), the sheaf \( S_j \otimes \mathcal{O}_C \) is a subsheaf of \( M_{E,V} \otimes \mathcal{O}_C \). Hence we also have the following inclusion of locally free sheaves on \( C \):

\[
S_j \otimes \mathcal{O}_C(-p) \hookrightarrow M_{E,V} \otimes \mathcal{O}_C(-p).
\]

If we tensor with \( M_{E,V} \) the exact sequence 1.1 we obtain

\[
0 \to M_{E,V} \otimes \mathcal{O}_C(-p) \to M_{E,V} \to M_{E,V} \otimes \mathcal{O}_C \to 0,
\]

from which we deduce the following inclusion of sheaves of depth one on \( C \):

\[
S_j \otimes \mathcal{O}_C(-p) \hookrightarrow M_{E,V}.
\]

Since we are assuming that \( M_{E,V} \) is \( w \)-semistable, we have

\[
(2.5) \quad \mu_w(S_j \otimes \mathcal{O}_C(-p)) \leq \mu_w(M_{E,V}).
\]

As

\[
\mu_w(S_j \otimes \mathcal{O}_C(-p)) = \frac{\chi(S_j \otimes \mathcal{O}_C(-p))}{w_is_j} = \frac{-g_i}{w_i},
\]

and

\[
\mu_w(M_{E,V}) = \frac{\chi(M_{E,V})}{k-r} = \frac{-(d_1 + d_2) + (k-r)(p_a(C)-1)}{k-r},
\]

where

\[
p_a(C) = \frac{\deg(C) - 2g + 2}{2g - 2}.
\]
from the inequality 2.5 we obtain
\[(2.6)\]
\[w_i \leq \frac{g_i(k - r)}{d_1 + d_2 + (k - r)(p_a(C) - 1)}, \quad i = 1, 2.\]

Now recall that \((w_1, w_2)\) is a polarization so \(w_1 + w_2 = 1\). Hence,
\[(2.7)\]
\[1 = w_1 + w_2 \leq \frac{p_a(C)(k - r)}{d_1 + d_2 + (k - r)(p_a(C) - 1)} = \frac{p_a(C)(k - r)}{p_a(C)(k - r) + (d_1 + d_2 - k + r)}.\]

Claim: \(k < d_1 + d_2 + r\).

First of all, by Lemma 2.1 we have:
\[(2.8)\]
\[k \leq h^0(E) = h^0(E_1) + h^0(E_2) - r.\]

In order to prove the claim, we can consider the following cases:

**Case 1:** \(\mu(E_i) > 2g_i - 2\) for \(i = 1, 2\). Since \(E_i\) are semistable this implies that \(h^1(E_i) = 0\). So, by Riemann-Roch theorem,
\[k \leq h^0(E) = h^0(E_1) + h^0(E_2) - r = d_1 + d_2 + r(1 - g_1 - g_2) < d_1 + d_2 + r\]
as \(g_1 + g_2 \geq 2\).

**Case 2:** \(\mu(E_i) > 2g_i - 2\) and \(\mu(E_j) \leq 2g_j - 2\). We have \(h^1(E_i) = 0\) so by Riemann-Roch we can compute \(h^0(E_i)\). On the other hand, we can use Clifford’s inequality in order to give a bound on \(h^0(E_j)\). Then we get
\[k \leq h^0(E) = h^0(E_i) + h^0(E_j) - r \leq d_i + r(1 - g_i) + \frac{d_j}{2} + r - r < d_1 + d_2 + r\]
as \(g_i \geq 2\) and \(d_j > 0\).

**Case 3:** \(\mu(E_i) \leq 2g_i - 2\) for \(i = 1, 2\). We use Clifford’s inequality for a bound on both vector bundles:
\[k \leq h^0(E) = h^0(E_1) + h^0(E_2) - r \leq \frac{d_1}{2} + r + \frac{d_2}{2} + r - r < d_1 + d_2 + r.\]

From the claim and inequality 2.7 we obtain \(1 = w_1 + w_2 < 1\) which is a contradiction. Hence \(M_{E,V}\) is actually \(w\)-unstable for all polarizations \(w\).

Let \(E\) be a globally generated vector bundle of rank \(r\) on the curve \(C\). Then we can apply Lemma 2.2, Remark 2.2.2 and Theorem 2.4 to the generated pair \((E, H^0(E))\), in order to obtain the following result on its kernel bundle \(M_E\):

**Corollary 2.5.** Let \(C\) be a reducible nodal curve as in Section 1. Let \(E\) be a globally generated vector bundle on \(C\). If \(E_1\) and \(E_2\) are semistable and not trivial, then the restriction of \(M_E\) to each component \(C_i\) is unstable and \(M_E\) is strongly unstable.

In particular, for line bundles we obtain the following:
Corollary 2.6. Let $C$ be a reducible nodal curve as in Section 1. Let $L$ be a globally generated line bundle on $C$ with non trivial restrictions. Then the restriction of $M_L$ to each component $C_i$ is unstable and $M_L$ is strongly unstable.

3. $w$-semistable kernel bundles

Let $C$ be a nodal reducible curve with two smooth irreducible components $C_i$ of genus $g_i \geq 2$ and a single node $p$ as in Section 1. Let $(E, V)$ be a generated pair on $C$, where $E$ is a vector bundle of rank $r$ on $C$ and $V \subset H^0(E)$ of dimension $k \geq r + 1$. Let $E_i$ be the restriction of $E$ to the component $C_i$ and assume that $E_1$ and $E_2$ are both semistable. As we have seen in the previous section, when $V \cap H^0(E_j(-p)) \neq \{0\}$ for $j = 1, 2$, the kernel bundle of the pair $(E, V)$ is strongly unstable and both restrictions are unstable. In this section we would like to study generated pairs $(E, V)$ defining $w$-semistable kernel bundles with semistable restrictions. Hence it is natural to assume the following condition:

\[(*) \quad V \cap H^0(E_1(-p)) = V \cap H^0(E_2(-p)) = \{0\}.\]

Let $M_{E,V}$ be the kernel bundle of the pair $(E, V)$ and $(E_i, V_i)$ the restriction of the pair $(E, V)$ to the component $C_i$. Then we have the following lemma:

Lemma 3.1. Let $C$ be a reducible nodal curve as in Section 1. Let $(E, V)$ be a generated pair, then we have the following properties:

1. $(E, V)$ satisfies $(*)$ if and only if $M_{E,V} \otimes O_{C_i} \simeq M_{E_i, V_i}$, $i = 1, 2$, where $M_{E_i, V_i}$ is the kernel bundle of $(E_i, V_i)$ on $C_i$;

2. under the above assumption, there exists a polarization $w = (w_1, w_2)$ such that

\[w_i \chi(M_{E,V}) \leq \chi(M_{E_i, V_i}) \leq w_i \chi(M_{E,V}) + \text{Rk}(M_{E,V}), \quad i = 1, 2.\]

Proof. (1) As we have seen in the proof of Lemma 2.2, the kernel of $\rho_i : H^0(E) \to H^0(E_i)$ is $H^0(E_j(-p))$. So, $\rho_{i,V} : V \to V_i$ is an isomorphism if and only if $H^0(E_j(-p)) \cap V = \{0\}$. From commutative diagramm 2.4 of Lemma 2.2, $M_{E,V} \otimes O_{C_i} \simeq M_{E_i, V_i}$, if and only if

\[S_j = V \cap H^0(E_j(-p)) = \{0\}.\]

(2) We want to prove the existence of $w = (w_1, w_2) \in \mathbb{Q}^2$ with $0 < w_i < 1$ and $w_1 + w_2 = 1$ satisfying the conditions:

\[w_i \chi(M_{E,V}) \leq \chi(M_{E_i, V_i}) \leq w_i \chi(M_{E,V}) + \text{Rk}(M_{E,V}), \quad i = 1, 2.\]

Since $\chi(M_{E,V}) = \chi(M_{E_1, V_1}) + \chi(M_{E_2, V_2}) - \text{Rk}(M_{E,V})$, it is easy to verify that whenever $w_1$ satisfies the above condition for $i = 1$, then $w_2 = 1 - w_1$ satisfies the condition for $i = 2$. 

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So it is enough to find \( w_1 \in (0, 1) \cap \mathbb{Q} \) satisfying the following system of inequalities:

\[
\begin{align*}
\chi(M_{E_1,V_1}) &\leq w_1 \chi(M_{E,V}) + \text{Rk}(M_{E,V}) \\
\chi(M_{E_1,V_1}) &\geq w_1 \chi(M_{E,V}).
\end{align*}
\]

From the exact sequences defining \( M_{E,V} \) and \( M_{E_1,V_1} \) one can get

\[
\chi(M_{E,V}) = (k - r)(1 - p_a(C)) - (d_1 + d_2) \quad \chi(M_{E_1,V_1}) = (k - r)(1 - g_1) - d_1
\]

\[\text{Rk}(M_{E,V}) = k - r.\]

By substituting in the above system we obtain:

\[
\frac{(k - r)(g_1 - 1) + d_1}{(k - r)(p_a(C) - 1) + (d_1 + d_2)} \leq w_1 \leq \frac{(k - r)g_1 + d_1}{(k - r)(p_a(C) - 1) + (d_1 + d_2)}.
\]

By denoting

\[
a_1 = \frac{(k - r)(g_1 - 1) + d_1}{(k - r)(p_a(C) - 1) + (d_1 + d_2)} \quad b_1 = \frac{(k - r)g_1 + d_1}{(k - r)(p_a(C) - 1) + (d_1 + d_2)}
\]

it is immediate to see that

\[0 < a_1 < b_1 < 1\]

under the hypothesis of the lemma. Hence, for any \( w_1 \in (a_1, b_1) \cap \mathbb{Q} \), we have that \((w_1, w_2)\) satisfies both the conditions of the lemma. \(\square\)

Using Lemma 3.1 we get the first result of this section.

**Theorem 3.2.** Let \( C \) be a nodal reducible curve as in Section 1. Let \((E, V)\) be a generated pair satisfying condition \((\ast)\). If both \(M_{E_1,V_1}\) and \(M_{E_2,V_2}\) are semistable then there exists a polarization \(w\) such that \(M_{E,V}\) is \(w\)-semistable.

**Proof.** We have to prove that \(M_{E,V}\) is \(w\)-semistable for a suitable polarization. By Theorem 1.1, it is enough to verify that both the restrictions of \(M_{E,V}\) to \(C_1\) and \(C_2\) are semistable and to find a polarization \(w\) satisfying the conditions 1.2:

\[
(3.1) \quad w_i \chi(M_{E,V}) \leq \chi(M_{E,V} \otimes \mathcal{O}_{C_i}) \leq w_i \chi(M_{E,V}) + \text{Rk}(M_{E,V}), \quad i = 1, 2.
\]

By Lemma 3.1 we have that the restrictions of \(M_{E,V}\) are \(M_{E_i,V_i}\) which are semistable by assumption. Finally conditions 3.1 are exactly the ones given in lemma 3.1(2). \(\square\)

**Remark 3.2.1.** Let \( C \) be nodal reducible curve as in Section 1. Let \( E \) be a globally generated vector bundle of rank \( r \) on the curve \( C \). Assume that there exist \( k \) global sections generating \( E \), with \( r + 1 \leq k \leq \min(h^0(E_1), h^0(E_2)) \). Then a general pair \((E, V)\) with \(\dim V = k\) is generated and satisfies condition \((\ast)\).
Proof. For any $k \leq \min(h^0(E_1, h^0(E_2))$, let $G(k, H^0(E))$ denote the Grassmannian variety parametrizing $k$-dimensional linear subspaces of $H^0(E)$. Under our hypothesis, the set of linear subspaces $V$ such that $(E, V)$ is generated is a non empty open subset. So it is enough to show that a general pair $(E, V)$ satisfies $(\star)$. Let’s consider the Schubert cycle

$$\sigma_j = \{ V \in G(k, H^0(E)) \mid V \cap H^0(E_j(-p)) \geq 1 \},$$

since $k + h^0(E_j(-p)) \leq h^0(E)$, then it is a proper closed subvariety of $G(k, H^0(E))$. This concludes the proof. □

The first application of this Theorem deals with the case of line bundles. For each smooth irreducible component $C_i$ of the nodal curve $C$, we denote by $G_{d_i}^{k-1}(C_i)$ the variety parametrizing linear series of degree $d_i$ and dimension $k-1$ on the curve $C_i$. An element of $G_{d_i}^{k-1}(C_i)$ is given by a pair $(L_i, V_i)$ of type $(1, d_i, k)$ on $C_i$. We recall that, if $C_i$ is a general curve then $G_{d_i}^{k-1}(C_i)$ is non empty if and only if the Brill-Noether number

$$\rho_i = g_i - k(g_i - d_i + k - 1)$$

is non negative, hence if and only if $d_i \geq g_i + k - 1 - \frac{a_i}{k}$. For details one can see [ACGH85].

**Theorem 3.3.** Let $C$ be a reducible nodal curve as in Section 1. Let $(L, V)$ be a generated pair on $C$, where $L$ is a line bundle and $\dim V = k$. Let $(L_i, V_i)$ be its restriction to $C_i$. If $C_i$ is general and $(L_i, V_i)$ is general in $G_{d_i}^{k-1}(C_i)$, then there exists a polarization $w$ such that $M_{L,V}$ is $w$-semistable.

Proof. By Theorem 3.2 it is enough to show that $(L, V)$ satisfies condition $(\star)$ and that $M_{L_i, V_i}$ is semistable for $i = 1, 2$.

Since $C$ is general the same holds for its components. Note that $G_{d_i}^{k-1}(C_i)$ is non empty, since we are assuming $(L_i, V_i) \in G_{d_i}^{k-1}(C_i)$. Moreover, we have that

$$\dim(V_i) = k = \dim(V),$$

which implies that the restriction map $\rho_i$ induces an isomorphism of $V$ into $V_i$. In particular, this gives us that $V \cap H^0(L_j(-p)) = \{0\}$, i.e. condition $(\star)$ holds.

It remains to show that, under the assumptions of the Theorem, the vector bundle $M_{L_i, V_i}$ is semistable. This follows from [BBN15, Theorem 5.1] since $C_i$ is general and $(L_i, V_i)$ is general in $G_{d_i}^{k-1}(C_i)$. □

**Corollary 3.4.** Under the hypothesis of Theorem 3.3, if we require also that a component $C_i$ is Petri-general, $k \geq 6$ and $g_i \geq 2k - 6$, then $M_{L,V}$ is $w$-stable.

Proof. By theorem 3.3 there exists a polarization $w$ such that $M_{L,V}$ is $w$-semistable. According to Theorem 1.1, it is enough to show that one of the restrictions of $M_{L,V}$ is
stable. Since $C_i$ is Petri, $k \geq 6$ and $g_i \geq 2k - 6$, then $M_{L_i,V_i}$ is stable, from [BBN15, Theorem 6.1], so we can conclude that $M_{L,V}$ is $w$-stable too.

The second application of Theorem 3.2 deals with generated pairs $(E, V)$ whose restrictions to each irreducible component $C_i$ is the complete pair $(E_i, H^0(E_i))$. This occurs when $\rho_i|_V : V \to H^0(E_i)$ is an isomorphism.

**Theorem 3.5.** Let $C$ be a reducible nodal curve as in Section 1. Let $(E, V)$ be a generated pair on $C$, satisfying conditions (⋆), such that for $i = 1, 2$ the restriction to $C_i$ is $(E_i, H^0(E_i))$. If $d_i \geq 2rg_i$, then there exists a polarization $w$ such that $M_{E,V}$ is $w$-semistable. If $d_i > 2rg_i$, then $M_{E,V}$ is $w$-stable.

**Proof.** By Theorem 3.2 it is enough to prove that both $M_{E_i,V_i}$ are semistable. By hypothesis, we have $M_{E_i,V_i} = M_{E_i}$. Hence, the result follows from [But94, Theorem 1.2]: in fact $M_{E_i}$ is semistable when $d_i \geq 2rg_i$ and it is stable if $d_i > 2rg_i$.

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Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi, 55, I-20125 Milano, Italy

E-mail address: sonia.brivio@unimib.it

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi, 55, I-20125 Milano, Italy

E-mail address: filippo.favale@unimib.it