Estimates of the error of interval quadrature formulas on some classes of differentiable functions

In a symmetric class $H=2\pi$-periodic functions the interval quadrature formula

$$
\int_0^{2\pi} f(t) dt - \sum_{k=1}^{n} c_k \int_{x_k-h}^{x_k+h} f(t) dt = R_n(f; \vec{c}_n, \vec{x}_n; h), \quad (0.1)
$$

is considered. The coefficients $c_k = \{c_k\}_{k=1}^{n}$, $c_k \in R$ are the quadrature coefficients of the formula (0.1), $x_k = \{x_k\}_{k=1}^{n}$, $x_k \in [0, 2\pi)$ are the zeros, $h \in (0, \pi/n)$ and $R_n(f; \vec{c}_n, \vec{x}_n; h)$ is the error of the quadrature formula (0.1) for a function $f$. The right side

$$
R_n(H; \vec{c}_n, \vec{x}_n; h) := \sup_{f \in H} R_n(f; \vec{c}_n, \vec{x}_n; h)
$$

defines the error of the interval quadrature formula on the class $H$. In the symmetric class of functions the interval quadrature formulas were studied, e.g., by Khavinson A.L., Papadopoulos N.H., Babenko V.F., Motornyi V.P., Bondarev B.Y., and many other mathematicians. Under the condition (0.1) the error tends to zero as $h \to 0$, which is confirmed by numerical calculations.

Problem 1.2.2. To find the optimal interval quadrature formulas (0.1) is a problem of finding the error

$$
E_n(H; \vec{c}_0, \vec{x}_0; h) = \inf_{\vec{c}_n, \vec{x}_n} R_n(H; \vec{c}_n, \vec{x}_n; h), \quad (0.2)
$$

which is called the optimal interval quadrature formula on the class $H = \{f \mid f \in H\} \cup \{0\}$. The problem was solved, e.g., by Khavinson and Papadopoulos [8 - 11]. It is known that

$$
\int_0^{2\pi} f(t) dt - \frac{\pi}{n} \sum_{k=0}^{n-1} \int_{-h}^{h} f(t + \frac{2k\pi}{n}) dt = R_n(f; \vec{c}_0, \vec{x}_0; h), \quad (0.3)
$$

to the interval quadrature formula (0.1), where $c_0 = \{c_k\}_{k=1}^{n}$, $c_k = 2\pi/n$ are the coefficients of the quadrature formula (0.1), $x_0 = \{x_k\}_{k=1}^{n}$, $x_k = \frac{2\pi k}{n}$.
ESTIMATES OF THE ERROR OF INTERVAL QUADRATURE FORMULAS ON SOME CLASSES OF DIFFERENTIABLE FUNCTIONS

Let $H$ be a class of $2\pi$–periodic continuous functions such that together with the function $f(t)$ the class $H$ contains the functions $-f(t), f(t + a), f(t) + a$ for an arbitrary number $a$, we denote by $H_n$ subclass $2\pi/n$–of periodic functions. An interval quadrature formula is called the formula

$$
\int_0^{2\pi} f(t) dt - \frac{\pi}{nh} \sum_{k=0}^{n-1} \int_{x_k}^{x_k+h} f(t + \frac{2k\pi}{n}) dt = R_n(f; \vec{c}_0; \vec{x}_0; h),
$$

(1)

where $\vec{c}_n = \{c_k\}_{k=1}^n$, $c_k \in R$ are the coefficients of the quadrature (1), $\vec{x}_n = \{x_k\}_{k=1}^n$, $x_k \in [0, 2\pi]$—are the nodes, $h \in (0, \pi/n)$ and $R_n(f; \vec{c}_n; \vec{x}_n; h)$ is the error of the quadrature formula (1) for the function $f$. Then the quantity

$$
R_n(H; \vec{c}_n; \vec{x}_n; h) := \sup_{f \in H} R_n(f; \vec{c}_n; \vec{x}_n; h)
$$

is the error of the quadrature formula (1)(corresponding to the vectors $\vec{c}_n; \vec{x}_n; h$) on the class $H$.

Interval quadrature formulas were considered, for example, in [1 - 8]. From the point of view of application, interval quadrature formulas are more natural than ordinary quadrature formulas, because the results of measuring a physical quantity, in some cases, due to the installation of measuring instruments, is the averaging of the function that characterizes the measured value. If we take a limit in (1) when $h \to 0$, then we obtain the usual quadrature formula. The problem of optimization of interval quadrature formulas (1) consists in finding the value

$$
E_n(H; \vec{c}_0; \vec{x}_0) = \inf_{x_k; x_k} R_n(H; \vec{c}_n; \vec{x}_n; h)
$$

(2)

which is called the error of the optimal quadrature formula in the class $H$, and the sets of coefficients $\vec{c}_0 = \{d_k\}_{k=1}^n$ and nodes $\vec{x}_0 = \{x_0\}_{k=1}^n$, for which the exact lower limit (2) is reached. The problem of optimization of interval quadrature formulas was considered,
for example, in [3 - 8]. It turned out that in many cases for classes of $2\pi$-periodic continuous functions the optimal interval quadrature formula is the formula

$$\int_0^{2\pi} f(t)dt - \frac{\pi}{nh} \sum_{k=0}^{n-1} \int_{-h}^{h} f(t + \frac{2k\pi}{n})dt = R_n(f; \vec{c}_0; \vec{x}_0; h),$$  \hspace{1cm} (3)

where $c_0 = \{c_k^0\}_{k=1}^n$, $c_k^0 = 2\pi/n$ are the coefficients of the quadrature (1), $\vec{x}_0 = \{x_k^0\}_{k=1}^n$, and $x_k^0 = \frac{2\pi(k-1)}{n}$ are the nodes, $h \in (0, \pi/n)$.

Quadrature formula (3) is exact for any constant, i.e. the integral over the segment $[0, 2\pi]$ is equal to the quadrature sum. So we may assume that the function $f$ in (3) is equal to zero in the average i.e.

$$-\frac{\pi}{nh} \sum_{k=0}^{n-1} \int_{-h}^{h} f(t + \frac{2k\pi}{n})dt = R_n(f; \vec{c}_0; \vec{x}_0; h),$$ \hspace{1cm} (4)

Let $f(t)$ be a $2\pi$-periodic function integrable on the period, let us denote by $f_h(t)$ the Steklov function, and let $S_h f$ be the Steklov operator, that is

$$S_h f = f_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} f(u)du.$$

The quadrature sum in (3) may be represented as

$$\frac{2\pi}{n} \sum_{k=0}^{n-1} f_h(\frac{2k\pi}{n})dt,$$

that is, the interval quadrature formula (3) coincides with the rectangles formula for the function $f_h(t)$.

We introduce the following classes:

$H^\omega$: continuous functions whose moduli of continuity satisfy $\omega(f; t) \leq \omega(t)$, where $\omega(t)$ is a given convex modulus of continuity.

$W^rH^\omega (r = 1, 2, ...)$ – the class of $2\pi$-periodic functions, for which $r$-th derivative $f^{(r)} \in H^\omega$, where $\omega(t)$ is a given convex modulus of continuity.

$W^r_1 (r = 1, 2, ...)$ – the class of all $2\pi$-periodic functions that have absolutely continuity $(r-1)$-st derivative and

$$\int_0^{2\pi} |f^{(r)}(x)|dx \leq 1$$

$W^r_\infty (r = 1, 2, ...)$ – the class of all $2\pi$-periodic functions that have a absolutely continuity derivative $f^{(r-1)}(x)$ and $|f^{(r)}(x)| \leq 1$ almost everywhere.

Finally, denote by $W^rH^\omega_n$ the subclass of $2\pi/n$-periodic functions from $W^rH^\omega$. In [5] the optimality of the interval quadrature formula (3) on the class $W^r_1$ is proved, and in the paper [6] the optimality of the interval quadrature formula (3) on the class $W^r_\infty$ is proved.
In the present paper, we obtain the exact estimate of the error of the interval quadrature formula (3) on the class $W^r H_\omega$.

**Theorem 1.** The equality

$$R_n(H; c_0; x_0; h) = R_n(H_n; c_0; x_0; h),$$

holds, where $H$ and $H_n$ are the classes of functions defined in above.

**Proof.** Since $H_n$ is a subset of the set $H$, the left part does not exceed the right one. On the other hand, due to the properties of the class $H$ (symmetry, shift invariance), for any function $f \in H$, the function

$$\psi_f(t) = \frac{1}{n} \sum_{i=0}^{n-1} f(t + \frac{2\pi i}{n})$$

has the following properties

$$\int_0^{2\pi} \psi_f(t) dt = \int_0^{2\pi} f(t) dt,$$

and the quadrature sums for the functions $f$ and $\psi_f$ coincide. Indeed,

$$\frac{\pi}{nh} \sum_{k=0}^{n-1} \int_{-h}^h \psi_f(t + \frac{2k\pi}{n}) dt =$$

$$= \frac{\pi}{n^2h} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \int_{-h}^h f(t + \frac{2\pi(k + i)}{n}) dt =$$

$$= \frac{\pi}{n^2h} \sum_{j=0}^{n-1} \sum_{k+i=j (mod n)} \int_{-h}^h f(t + \frac{2\pi(k + i)}{n}) dt =$$

$$= \frac{\pi}{nh} \sum_{j=0}^{n-1} \int_{-h}^h f(t + \frac{2\pi j}{n}).$$

Since the integrals and quadratic sums for the functions $f(t)$ and $\psi_f(t)$ coincide, then the errors of quadrature formula (2) for functions $f(t)$ and $\psi_f(t)$ as well, and this implies equality (4).

**Corollary 1.** Due to equations (3 - 5) we have

$$R_n(W^r H_\omega; c_0; x_0; h) = R_n(W^r H_n^\omega; c_0; x_0; h) =$$

$$= \sup_{f \in W^r H_\omega} \frac{\pi}{nh} \sum_{k=0}^{n-1} \int_{-h}^h f(t + \frac{2\pi}{n}) dt|$$

Indeed, since the function $f(t)$ is $2\pi/n-$ periodic, the terms in the right-hand side of equation (6) coincide, therefore
\[
\sup_{f \in W^{r+1}_n} \frac{\pi}{nh} \left| \sum_{k=0}^{n-1} \int_{-h}^{h} f(t + \frac{2k\pi}{n}) dt \right| = \sup_{f \in W^{r+1}_n} \frac{\pi}{h} \left| \int_{-h}^{h} f(t) dt \right|.
\]

(7)

Denote by \( I_r \) (natural number) the integration operator, i.e.

\[
I_r f(x) = \frac{1}{\pi} \int_{0}^{2\pi} D_r(x-t) f(t) dt, \quad \int_{0}^{2\pi} f(t) dt = 0,
\]

where \( D_r(t) \) - Bernoulli kernel:

\[
D_r(t) = \sum_{k=1}^{\infty} \frac{\cos(kt - \frac{r\pi}{2})}{k^r}, \quad r = 1, 2, \ldots
\]

The right-hand side of equation (7) may be represented by the integration operator:

\[
\sup_{f \in W^{r+1}_n} \frac{\pi}{h} \left| \int_{-h}^{h} f(t) dt \right| = \sup_{f \in W^{r+1}_n} \frac{\pi}{h} |I_1 f(h) - I_1 f(-h)| = \sup_{f \in H^\infty} \frac{\pi}{h} |I_{r+1} f(h) - I_{r+1} f(-h)|
\]

(9)

Thus it is necessary to estimate the modulus of difference \( I_{r+1} f(h) - I_{r+1} f(-h) \) of functions \( f \) from class \( W^{r+1}_n \) with step \( 2h \), i.e. estimate the modulus of continuity of functions \( f \) from the class \( W^{r+1}_n \).

Estimation of values of functions

\[
\max_x |f(x)|, \quad f \in K
\]

and also their differences

\[
\max_x |f(x) - f(x + h)|, \quad f \in K, \quad h > 0
\]

led to the problem of finding the exact upper bound of the functional

\[
\int_a^b \psi(t) f(t) dt
\]

(10)

on different classes of functions. In particular for the classes \( W^{r+1}_n \), this problem was solved by N.P. Korneichuk and he proposed a new method for estimating the functionalities of the form (10).

**Theorem 2.** Let \( \psi(t) \) – be an integrable function on the segment \([a, b]\), such that the function \( \Psi(x) = \int_a^x \psi(t) dt \) is strictly monotone on segments \([a, c]\) and \([c, b]\) and \( \Psi(b) = 0 \). If \( \omega \) is a given convex modulus of continuity, then the equality

\[
\sup_{f \in H^\infty} \int_a^b f(t) \psi(t) dt = \int_a^c |\psi(t)| \omega(\rho(t) - t) dt =
\]
ESTIMATES OF THE ERROR OF INTERVAL QUADRATURE FORMULAS ON SOME CLASSES OF DIFFERENTIABLE FUNCTIONS

\[ = \int_c^b |\psi(t)| \omega(t - \rho^{-1}(t)) dt, \]
holds, where the function \( \rho(x) \) is defined by the equation

\[ \Psi(x) = \Psi(\rho(x)), \; x \in [a, c], \rho(x) \in [c, b] \]
and \( \rho^{-1}(x) \) is a function inverse to \( \rho(x) \).

**Corollary 2.** If the function \( \psi(t) \) satisfies the conditions of Theorem 2 and

\[ \psi(a + u) = -\psi(b - u), \]  
then \( c = (a + b)/2 \) and \( \rho(t) = a + b - t \), and

\[ \sup_{f \in H^r} \int_a^b f(t) \psi(t) dt = \int_a^{(a+b)/2} |\psi(t)| \omega[a + b - 2t] dt = \]

\[ = \int_{(a+b)/2}^b |\psi(t)| \omega[2t - a - b] dt. \]

Note that the conditions of Theorem 2 for the function \( \psi(t) \) hold for Bernoulli functions \( D_r(t) \).

Really, if \( r = 2\nu \) is even, then on segments \([0, \pi]\) and \([\pi, 2\pi]\) the conditions of the theorem satisfy for the functions \( D_{2\nu}(t) \) and if \( r = 2\nu + 1 \) is odd, then for the functions \( D_{2\nu+1}(t) \) on the segments \([-\pi/2, \pi/2]\) and \([\pi/2, 3\pi/2]\), in addition to the conditions of the Theorem, (11) holds. It is obvious that the conditions of Theorem 2 and condition (11) hold for the corresponding segments and for the functions \( D_r(nt) \).

In the paper [9] the functions

\[ R_r(h, t) = D_r(h - t) - D_r(-h - t) \]
were considered as well, and it was observed that functions \( R_{2\nu}(h, t) \) have the same properties as functions \( D_{2\nu+1}(h, t) \), and functions \( R_{2\nu+1}(h, t) \) behaves similarly to functions \( D_{2\nu}(h, t) \). In particular, \( R_{2\nu}(h, t) \) is an odd function, and \( R_{2\nu+1}(h, t) \) is even, and Theorem 2 holds for them. It is easy to verify that similar properties have functions

\[ K_{\nu,n}(h, t) = D_{2\nu}(n(h - t)) - D_{2\nu}(n(-h - t)). \]

**Lemma 1.** The functions \( K_{\nu,n}(h, t) \), if \( h \in [0, \pi/2n] \), have the following properties:

1. \( K_{\nu,n}(h, t) \) are equal to zero on the average.
2. \( K_{2\nu,n}(h, t) \) is odd, \( K_{2\nu+1,n}(h, t) \) is even. In particular, the functions \( K_{2\nu,n}(h, t) \) are equal to zero at the points \( k\pi/n, k \in Z \).
3. \( \frac{d}{dt} K_{\nu,n}(h, t) = -K_{\nu-1,n}(h, t) \).
4. \( \int_0^{\pi/n} K_{2\nu+1,n}(h, t) dt = 0, \int_0^{2\pi/n} K_{2\nu+1,n}(h, t) dt = 0. \)
5. The functions \( K_{\nu,n}(h, t) \) change the sign twice on the period \([0, 2\pi/n]\).
6. The functions \( K_{2\nu+1,n}(h, t) \) are strictly monotone on the segments \([0, \pi/n], \pi/n, 2\pi/n]\).
Proposition 1 follows from the definition of the functions $K_{r,n}(h, t)$.

We prove the oddness of the function $K_{2\nu,n}(h, t)$. To this end we use the periodicity and evenness of the function $D_{2\nu}(t)$.

$$K_{2\nu,n}(h, t) = D_{2\nu}(n(h - t)) - D_{2\nu}(n(-h - t)) =$$

$$= D_{2\nu}(n(-h + t)) - D_{2\nu}(n(h + t)) = -K_{2\nu,n}(h, -t)$$

Similarly we prove, the evenness of the function $K_{2\nu+1,n}(h, t)$.

Proposition 3 follows from the definition of the functions $K_{r,n}(h, t)$.

Proposition 4 follows from the Newton-Leibnitz formula and Propositions 3 and 2. To prove 5 we have to check 5 for the functions $K_{1,n}(h, t)$, and then to assume that for some positive integer $r > 1$ the Statement 5 does not hold and to get a contradiction.

Proposition 6 follows from the previous one.

Lemma is proved.

**Lemma 2.** If $h \in [0, \pi/2n]$, then for the functions $K_{2\nu,n}(h, t)$ the conditions of Theorem 2 and condition (11) are satisfied on segments $[-\pi/2n, \pi/2n]$ and $[\pi/2n, 3\pi/2n]$.

**Proof.** The fulfillment of the conditions of Theorem 2 follows from Lemma 1. Condition (11) for functions $K_{2\nu,n}(h, t)$ on segments $[-\pi/2n, \pi/2n]$ means the oddness of these functions and it is proved.

Consider the function $K_{2\nu,n}(h, t)$ on the segment $[\pi/2n, 3\pi/2n]$.

$$K_{2\nu,n}(h, \pi/n - t) = D_{2\nu}(n(h - \pi/n + t)) - D_{2\nu}(n(-x - \pi/n + t)) =$$

$$= D_{2\nu}(n(-h + \pi/n - t)) - D_{2\nu}(n(h + \pi/n - t)) =$$

$$= D_{2\nu}(n(-h - \pi/n - t)) - D_{2\nu}(n(h - \pi/n - t)) = -K_{2\nu,n}(h, \pi/n + t).$$

Lemma is proved.

Consider another integral representation of differential functions. It selected so as to apply N.P. Korneichuk method of the estimation of functionalities of form (10). Let $B_r(t) = \frac{1}{n^\nu - 1} D_r(nt)$.

**Theorem 3.** Any $2\pi/n$ periodic function, equal to zero on the average, whose $(r - 1)$-st derivative is absolutely continuous, may be represented in the form

$$f(x) = \frac{1}{\pi} \int_0^{2\pi/n} B_r(x - t) f^{(r)}(t) dt, \quad (12)$$

**Proof.** First conside the case $r = 1$. Apply the formula of integration by parts

$$\frac{1}{\pi} \int_0^{2\pi/n} B_r(t) f^{(r)}(x - t) dt = -\frac{1}{\pi} B_r(t) f(x - t) |_{0}^{2\pi/n} - \frac{n}{2\pi} \int_0^{2\pi/n} f(x - t) dt.$$
natural number \( r + 1 \). Using the equation \( \frac{d}{dt} B_{r+1}(t) = B_r(t) \) and integrating the parts we obtain
\[
\frac{1}{\pi} \int_0^{2\pi/n} B_{r+1}(x-t)f^{(r+1)}(t)dt = \frac{1}{\pi} B_{r+1}(x-t)f^{(r)}(t)_{0}^{2\pi/n} + \frac{1}{\pi} \int_0^{2\pi/n} B_r(x-t)f^{(r)}(t)dt = f(x).
\]

**Theorem 4.** If \( r \) is an even natural number, then for any \( h \in [0, \pi/2n] \) the equality
\[
\sup_{f \in W^r H^{\omega n}} \max_x |f(x + h) - f(x - h)| = \frac{4}{\pi n^{2\nu-1}} \int_0^{\pi/n} \left| \sum_{i=1}^{\infty} \frac{\sin(2i+1)nh\sin(2i+1)nt}{(2i+1)^{2\nu}} \omega(2t)dt \right|.
\]

holds.

**Proof.** Due to the periodicity of the function \( f(x) \), one may assume that
\[
\max_x |f(x + h) - f(x - h)| = |f(h) - f(-h)|
\]

We use images (12)
\[
|f(h) - f(-h)| = \frac{1}{\pi} \left| \int_0^{2\pi/n} [B_r(h-t) - B_r(-h-t)]f^{(r)}(t)dt \right| = \frac{1}{\pi n^{2\nu-1}} \left| \int_0^{2\pi/n} [D_{2\nu}(n(h-t)) - D_{2\nu}(n(-h-t))]f^{(2\nu)}(t)dt \right|
\]
So
\[
|f(h) - f(-h)| = \frac{1}{\pi n^{2\nu-1}} \int_0^{2\pi/n} K_{\nu,n}(h,t)f^{(2\nu)}(t)dt,
\]
where
\[
K_{\nu,n}(h,t) = D_{2\nu}(n(h-t)) - D_{2\nu}(n(-h-t)) = \left( -i \right)^\nu \frac{\cos nk(h-t) - \cos nk(-h-t)}{k^{2\nu}} = \frac{(-i)^\nu}{\pi} \sum_{k=1}^{\infty} \frac{\sin nkh\sin nk t}{k^{2\nu}}.
\]

By Lemma 2, the conditions of Theorem 1 hold for the function \( K_{\nu,n}(h,t) \), as well as condition (11) on the segments \([-\pi/2n, \pi/2n]\) and \([\pi/2n, 3\pi/2n]\). Therefore by Theorem 1 we have
\[
|f(h) - f(-h)| = \frac{1}{\pi n^{2\nu-1}} \left| \int_0^{2\pi/n} K_{\nu,n}(h,t)f^{(2\nu)}(t)dt \right| \leq
\]
\[ \begin{align*}
\leq & \frac{1}{\pi n^{2\nu - 1}} \left\{ \left| \int_{-\pi/2n}^{\pi/2n} K_{\nu,n}(h,t) f^{(2\nu)}(t) dt \right| + \\
& + \left| \int_{\pi/2n}^{3\pi/2n} K_{\nu,n}(h,t) f^{(2\nu)}(t) dt \right| \right\} \leq \\
\leq & \frac{1}{\pi n^{2\nu - 1}} \sup_{f \in H^\nu} \left| \int_{-\pi/2n}^{\pi/2n} K_{\nu,n}(h,t) f(t) dt \right| + \\
& + \frac{1}{\pi n^{2\nu - 1}} \sup_{f \in H^\nu} \left| \int_{\pi/2n}^{3\pi/2n} K_{\nu,n}(h,t) f(t) dt \right| = \\
= & \frac{1}{\pi n^{2\nu - 1}} \int_{0}^{\pi/2n} |K_{\nu,n}(h,t)| \omega(2t) dt + \\
& + \frac{1}{\pi n^{2\nu - 1}} \int_{\pi/2n}^{\pi/n} |K_{\nu,n}(h,t)| \omega(2\pi/n - 2t) dt. (15)
\end{align*} \]

In the last integral we will make the replacement \( \pi/n - t = u \). Then from a chain of inequalities we will get

\[ |f(h) - f(-h)| \leq \frac{1}{\pi n^{2\nu - 1}} \int_{0}^{\pi/2n} |K_{\nu,n}(h,t)| + \\
+ K_{\nu,n}(h, \pi/n - t) \omega(2t) dt. \] (16)

With this, the sign of equality in (16) holds for an odd \( 2\pi/n \) periodic function, which \( r- \) th derivative is equal to \( \frac{1}{2} \omega(2t) \) on \( [0, \pi/2n] \) and \( \omega(2\pi/n - 2t) \) on \( x[\pi/2n, \pi/n] \). We denote the extreme function by \( f_{n,r}(t) \). Substituting instead of \( K_{\nu,n}(h,t) \) the sum (14) in (16), we obtain

\[ \sup_{f \in W^{2\nu} H^\nu} \omega(f,t) = \frac{4}{\pi n^{2\nu - 1}} \int_{0}^{\pi/2n} \sum_{j=0}^{\infty} \frac{\sin n(j+1)h \sin n(j+1)t}{(2j+1)^{2\nu}}. \]

So

\[ R_n(W^{\nu} H^\nu; \tilde{c}_0; \tilde{x}_0; h) = \frac{4}{h n^{2\nu - 1}} \int_{0}^{\pi/2n} \sum_{j=0}^{\infty} \frac{\sin n(j+1)h \sin n(j+1)t}{(2j+1)^{2\nu}}. \]

References

1. Pitinauer Fr., Peimer M. Interpolation mit Intervalfunktionalen// Math.Z. - 1976. - Bd.146, № 1.- S. 7-15.
2. Omladic M., Pohor S., Suhe dol A. On a new type of quadrature formulas// Numer. Math.-1976.-25, № 4.- P.421-426.
3. Кузьмина А.Л. Интерполяционные квадратурные формулы с кратными узловыми интервалами// Изв. вузов Математика-1980.-№ 7.- С. 39-44.
ESTIMATES OF THE ERROR OF INTERVAL QUADRATURE FORMULAS ON SOME
CLASSES OF DIFFERENTIABLE FUNCTIONS

4. Шарипов Р.Н. Наилучшие интервальные квадратурные формулы для классов Липшица// Конструктивная теория функций и функциональный анализ. Изд-во Казанского ун-та- 1983. выпуск 4. С. 124-132.

5. Бабенко В.Ф. Об одной задаче оптимального интегрирования. Исследования по современным проблемам суммирования и приближения функций и их приложениям. Сборник научных трудов.- Днепропетровск. ДГУ.- 1984.-С. 3-13.

6. Motornyi V.P. On the best interval quadrature formula in the class of functions with bounded $r^{th}$ derivative//EAST JORNAL OF APPROXIMATIONS. V. 4, Number 4 (1998). P.- 459-478.

7. Бородачев С.В. Оптимизация "интервальных"квадратурных формул для классов $H_{\omega^+}$-$\omega^-$//Вісн. Дніпропетров ун-ту Математика.-1998.- Вип.3.-С.19 -26.

8. Бородачев С.В. Об оптимизации "интервальных" квадратурных формул на некоторых несимметричных классах периодических функций //Вісн. Дніпропетров ун-ту. Математика.-1999.- Вип.4.-С.19 -24.

9. Корнійчук М.П. Про екстремальні властивості періодичних функцій//ДАН УССР. №8 (1962), С. 993-998.

10. Корнійчук Н.П. Экстремальные задачи теории приближения//М., Физматгиз, 1976. -320 с.

Received: 12.04.2020. Accepted: 02.05.2020