Chiral Dynamics in Few–Nucleon Systems

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We employ the chiral nucleon–nucleon potential derived using the method of unitary transformation up to next–to–next–to–leading order (NNLO) to study bound and scattering states in the two–nucleon system. The predicted partial wave phase shifts and mixing parameters for higher energies and higher angular momenta beyond the ones which are fitted are mostly well described for energies below 300 MeV. Various deuteron properties are discussed. We also apply the next–to–leading order (NLO) potential to 3N and 4N systems. The resulting 3N and 4N binding energies are in the same range what is found using standard NN potentials. Experimental low–energy 3N scattering observables are also very well reproduced like for standard NN forces. Surprisingly the long standing $A_\pi$–puzzle is resolved at NLO. The cut–off dependence of the scattering observables is rather mild.

1. INTRODUCTION

Over the past decade effective field theory methods have been successfully applied to many different physical processes and became, in particular, a standard tool in calculating low–energy properties of the purely pionic and pion–nucleon systems. In this method one starts from the most general Lagrangian for pions and nucleons. Apart from the usual symmetry constraints, spontaneously and explicitly broken chiral symmetry of QCD is taken into account. As a consequence, the couplings of pion fields, which play the role of Nambu-Goldstone bosons within this formalism, are of derivative type. As has been shown by Weinberg more than 20 years ago, one can use such an effective Lagrangian to derive the expansion of low energy S-matrix elements in powers of some scale $Q$ related to low external momenta of pions and nucleons \[1\] (the 4-momenta of the external pions and the 3-momenta of the external nucleons).

Motivated by successful applications of CHPT in the $\pi\pi$ and $\pi N$ sectors, Weinberg proposed in 1990 to extend the formalism to the NN interaction \[2\]. The crucial difference is, however, that the nucleon–nucleon interaction is non–perturbative at low energies: it is strong enough to bind two nucleons in the deuteron. Thus, direct application of CHPT
to the NN amplitude will necessarily fail. The way out of this problem, proposed by Weinberg, is to apply CHPT not to the amplitude but to a kernel of the corresponding integral Lippmann–Schwinger (LS) equation using time–ordered perturbation theory. Such a kernel or effective NN potential is then defined as a sum of all diagrams with two incoming and outgoing nucleons that do not contain purely nucleonic intermediate states (reducible diagrams). A systematic power counting for the potential can be derived in a similar manner as for the $\pi\pi$ and $\pi N$ systems. These ideas have been extensively studied by the Texas–Seattle group \cite{3}. They have obtained an energy dependent two–nucleon potential\cite{2} at NNLO. With altogether 26 free parameters\cite{3} they were able to achieve a qualitative agreement with the NN data.

Alternatively, one can also do the expansion directly on the level of the scattering amplitude, as it has been proposed by Kaplan et al. \cite{4}. In their power counting scheme the leading order non–perturbative contribution in the S–channels is given by resummation of the iterated non–derivative NN contact interaction. All corrections to this result including those from various pion exchanges are treated as small perturbations \cite{4}.

In what follows we will concentrate on the potential approach to the few–nucleon scattering. In ref. \cite{6} (referred to as I from here on) we constructed the NN and 3N potential based on the most general chiral effective pion–nucleon Lagrangian using the method of unitary transformations. For that, we developed a power counting scheme consistent with this projection formalism. In contrast to previous results obtained in old–fashioned time–ordered perturbation theory, the method employed leads to energy–independent potentials. To LO, the potential consists of the undisputed OPE and two NN contact interactions. Corrections at NLO stem from the leading chiral TPE and additional contact terms with two derivatives. At NNLO one has to include the subleading TPE which contains pion–nucleon interactions with two derivatives (pion mass insertions). In contrast to ref. \cite{3}, we take the novel LECs from systematic studies of pion–nucleon scattering in CHPT \cite{7}. The resulting effective potential is renormalized and applied to two–body bound state and scattering problems \cite{8} (referred to as II).

We also consider 3N and 4N systems within the chiral effective field theory \cite{9} (referred to as III). To the best of our knowledge this is the first time that $\chi$PT has been practically applied to nuclear systems beyond $A = 2$ within the Hamiltonian approach. In this first application we restrict ourselves to the NLO NN potential. At this order no three–nucleon forces (3NFs) occur\cite{10}. The NLO results presented here are therefore parameter free and can serve as a good testing ground for the usefulness of the approach. Of course, some aspects of the 3N system have already been studied in nuclear EFT \cite{10,11}, but not as direct extensions of the NN system as done here.

Our manuscript is organized as follows. In Section 2 we will introduce the effective NN potential up to NNLO and give explicit expressions in momentum space. In Section 3 we apply it to the NN bound state and scattering problems. In the next section 4 we discuss NLO results for 3N and 4N systems. Our findings are summarized in Section 5.

\footnote{Such explicit energy dependence of the potential complicates its application to three– and more–nucleon problems.}
\footnote{Some of them are redundant and can be eliminated from the Lagrangian via Fierz transformation.}
\footnote{The 3NF appears first at NNLO and contains 5 undetermined parameters.}
2. THE NN POTENTIAL

In I, we derived the two–nucleon potential at next–to–leading order. To leading order, it consists of the one–pion–exchange potential (OPEP) and two S–wave (non–derivative) contact interactions. The latter are parametrized by two LECs $C_S$ and $C_T$ [2],

\[ V_{NN}^{LO} = -\left(\frac{g_A}{2f_\pi}\right)^2 \vec{r}_1 \cdot \vec{r}_2 \frac{\vec{q} \cdot \vec{q}_1 \cdot \vec{q}_2 \cdot \vec{q}}{q^2 + M^2_\pi} + C_S + C_T \vec{\sigma}_1 \cdot \vec{\sigma}_2, \tag{1} \]

where $f_\pi = 92.4$ MeV is the pion decay constant, $M_\pi = 138.03$ MeV the pion mass and $g_A = 1.26$ the axial–vector coupling. Further, $\vec{q} = \vec{p}' - \vec{p}$, where $\vec{p}$ and $\vec{p}'$ stand for initial and final nucleon momenta, respectively. Clearly, the LO potential can only serve as a very crude approximation of the NN force. At NLO order, we have three distinct contributions. The first one represents the one–loop self–energy and vertex corrections to the OPEP and to LO contact interactions. As discussed in II and [12], such contributions do not lead to form factors and only renormalize the appropriate couplings. Secondly, one has seven different NN contact interactions with two derivatives:

\[ V_{NN,cont} = C_1 \vec{q}^2 + C_2 \vec{k}^2 + (C_3 \vec{q}^2 + C_4 \vec{k}^2)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + iC_5 \frac{1}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{k}) + C_6 (\vec{q} \cdot \vec{\sigma}_1)(\vec{q} \cdot \vec{\sigma}_2) + C_7 (\vec{k} \cdot \vec{\sigma}_1)(\vec{k} \cdot \vec{\sigma}_2). \tag{2} \]

Thirdly, there is the genuine TPEP. After performing the renormalization it reads:

\[ V_{TPEP}^{NLO} = -\frac{\vec{r}_1 \cdot \vec{r}_2}{384\pi^2 f_\pi^4} L(q) \left\{ 4M^2_\pi (5g_A^4 - 4g_A^2 - 1) + q^2 (23g_A^4 - 10g_A^2 - 1) + \frac{48g_A^4 M^4_\pi}{4M^2_\pi + q^2} \right\} \]

\[ -\frac{3g_A^4}{64\pi^2 f_\pi^4} L(q) \left\{ \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} - q^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right\} + P(\vec{k}, \vec{q}), \tag{3} \]

with

\[ L(q) = \frac{1}{q} \sqrt{4M^2_\pi + q^2} \ln \frac{\sqrt{4M^2_\pi + q^2} + q}{2M_\pi}. \tag{4} \]

Here $P(\vec{k}, \vec{q})$ represents the polynomial part.
At NNLO we have the additional TPEP with one $\pi\pi NN$ vertex with two derivatives as shown in Figure 1. Three different interactions of this type enter the effective Lagrangian. For the corresponding LECs $c_{1,3,4}$ we take the values obtained from fitting the invariant amplitudes inside the Mandelstam triangle, i.e. in the unphysical region [7]. Various NNLO self–energy and vertex correction diagrams lead again only to renormalization of the corresponding couplings. Note further that the contact interactions are renormalized by an infinite amount in order to absorb all UV divergences of the NNLO TPEP. The explicit expressions for the renormalized NNLO TPEP can be found in II and in [12].

The effective potential discussed above is only meaningful for momenta below a certain scale and therefore needs regularization. That is done in the following way:

$$V(\vec{p},\vec{p}') \to f_\Lambda(\vec{p}) V(\vec{p},\vec{p}') f_\Lambda(\vec{p}')$$

where $f_\Lambda(\vec{p})$ is a regulator function. Here we use two different regulator functions:

$$f_\Lambda^\text{sharp}(\vec{p}) = \theta(\Lambda^2 - p^2)$$

$$f_\Lambda^\text{expon}(\vec{p}) = \exp(-p^4/\Lambda^4)$$

3. RESULTS FOR THE NN SYSTEM

Having introduced the effective potential up to NNLO we will now apply it to the NN system.

3.1. Phase shifts

In order to apply the effective potential to bound state and scattering problems, one first needs to determine the unknown couplings related to the NN contact interactions. At LO one has two such constants ($C_S, C_T$), whereas nine constants ($C_S, C_T, C_1 \ldots C_7$) occur at NLO and NNLO, see eqs. (1), (2). To pin down this couplings we do not perform global fits as done in ref. [3]. Rather we introduce independent new parameters by projections on to appropriate partial waves. To leading order, the two S–waves are depending on one parameter each. At NLO and NNLO, we have one additional parameter for $^1S_0$ and $^3S_1$ as well as one parameter in each of the four P–waves and in $\epsilon_1$. We thus can fit each partial wave separately, which makes the fitting procedure not only extremely simple but also unique. All fits have been performed to the phase shifts of the Nijmegen phase shift analysis (NPSA) [3] for laboratory energies smaller than (50) 100 MeV at (NLO) NNLO as described in detail in II. Having determined the parameters, we can now predict the S– and P–waves for energies above 100 MeV. All other partial waves are parameter free predictions for all energies considered. In Figure 2 we show the results using the sharp regulator function with $\Lambda = 500$ and 875 MeV for LO (NLO) and NNLO, as discussed in II. At NNLO both S–waves agree well with the data. In the case of $^1S_0$ the OPEP plus non–derivative contact terms is insufficient to describe the phase shift (as it is well–known from effective range theory and previous studies in EFT approaches). This also holds true for some P–waves. It is known that TPEP alone is too strong in some of the D– and F–waves. To the order we are working, there are no contact interactions so that apart from varying the cut–off we have no freedom here. Still, for our best global fit, the D–waves are quite reasonably described. The result for the $^3D_3$ phase is rather astonishing, since e.g. in the Bonn potential [3] the correlated two–pion exchange is very important in the description of $^3D_3$. Such $\pi\pi$ correlations are only implicitly present at NNLO. They are
Figure 2. Predictions for various partial wave phase shifts (in degrees) for nucleon laboratory energies $E_{lab}$ below 300 MeV (0.3 GeV). The dotted, dashed and solid curves represent LO, NLO and NNLO results, in order. The squares depict the Nijmegen PSA results.

hidden in the strengths of the LECs $c_1$ and $c_3$\cite{15}. From the higher partial waves the $^3G_5$ is the most interesting one. As in the case of $^3D_3$, the OPEP alone is not sufficient to describe the phase shifts at energies above 100 MeV, as it could be expected for such a high value of angular momentum. The leading TPEP makes the situation even worse. Only inclusion of the subleading TPEP at NNLO allows for a very accurate and parameter free\cite{4} description of the phase shift.

Table 1
Scattering lengths, effective range and shape parameters for the $^3S_1$ wave at NLO and NNLO compared to the NPSA.

|       | $a$ [fm] | $r$ [fm] | $v_2$ [fm$^3$] | $v_3$ [fm$^5$] | $v_4$ [fm$^7$] |
|-------|----------|----------|---------------|---------------|---------------|
| $^3S_1$ NLO   | 5.434    | 1.711    | 0.075         | 0.77          | −4.2          |
| $^3S_1$ NNLO  | 5.424    | 1.741    | 0.046         | 0.67          | −3.9          |
| $^3S_1$ NPSA  | 5.420    | 1.753    | 0.040         | 0.67          | −4.0          |

We have also calculated the values of various effective range and shape parameters for

\footnote{The cut–off dependence in this channel is rather weak.}
In Table 1 we compare our NLO and NNLO predictions for these quantities with the results from the Nijmegen PSA. As expected, in all cases the NNLO results go closer to the data than the NLO ones. Note further that in both cases one has the same number of free parameters. Therefore we consider the improvement as an indication of a good convergence of our expansion.

3.2. Deuteron

We now turn to the bound state properties. At NNLO (NLO), we consider an exponential regulator with \( \Lambda = 1.05 \) (0.60) GeV, which reproduces the deuteron binding energy \( E_d = -2.224575(9) \) MeV within an accuracy of about one third of a permille (2.5 percent)\(^5\): \( E_d = -2.2238 \) (-2.1650) MeV. The asymptotic \( D/S \) ratio, called \( \eta \), and the strength of the asymptotic wave function, \( A_S \), as well as the root–mean–square matter radius \( r_d \) are well described at NNLO (NLO): \( \eta = 0.0245 \) (0.0248), \( A_S = 0.884 \) (0.866) fm\(^{-1/2} \) and \( r_d = 1.967 \) (1.975) fm compared to experimentally measured values \( \eta = 0.0256(4) \), \( A_S = 0.8846(16) \) fm\(^{-1/2} \) and \( r_d = 1.9671(6) \) fm, respectively. The D–state probability, which is not an observable, seems to be most sensitive to small variations in the cut–off. We observe a small deviation in the values for the quadrupole moment, \( Q_d = 0.262 \) (0.266) fm\(^2 \) compared to the experimental value \( Q_d = 0.2859(3) \) fm\(^2 \), which is also typical for conventional potentials and possibly arises from the missing exchange current operators.

4. RESULTS FOR THE 3N AND 4N SYSTEMS

4.1. Bound states

In this section we present the binding energies for 3N and 4N systems calculated from our NLO potential. As already pointed out above no 3NF appear at this order. The NLO results shown here are therefore parameter free predictions and can serve as a good testing ground for the usefulness of the approach. In order to investigate the cut–off dependence of 3N and 4N observables we have generated several NN potentials corresponding to different exponential cut-offs between \( \Lambda \) =540 and 600 MeV. They were all fitted to the \( ^1S_0, ^3S_1-^3D_1, ^1P_1 \) and \( ^3P_{0,1,2} \) NN phase shifts up to \( E_{\text{lab}} = 100 \) MeV, see III for more details. We find for the fully converged solutions of the corresponding Faddeev-Yakubovsky equations

\(^5\)Note that the deuteron binding energy is not used to fit the free parameters in the potential.
the 3N and 4N binding energies as given in Table 2. The ranges are compatible with what is found using realistic potentials. Note that the NN forces were included up to the total NN angular momentum of $j_{\text{max}}=6$.

4.2. $nd$ scattering

Also, 3N scattering can be solved rigorously nowadays [16] and we show in Figs. 3 some elastic observables as a particular example, namely the angular distributions, the nucleon analyzing power $A_y$ and the tensor analyzing power $T_{20}$. Except for $A_y$ there are no $nd$ data for those energies. The discrepancies between data and theory for $T_{2k}$ and for the differential cross section can be traced back to the $pp$ Coulomb force [19]. Thus except for $A_y$ the agreement of CD-Bonn with the data is rather good, which is a well known fact and is just given for the sake of orientation. Our parameter free predictions for $d\sigma$ and $T_{20}$ agree rather well with the CD-Bonn result and thus with the data. We consider this to be an important result, demonstrating that the chiral NN forces are very well suited to also describe low-energy 3N scattering observables rather quantitatively. On top of that, surprisingly for us, the chiral force predictions are now significantly higher in the maxima of $A_y$ than for CD-Bonn and break the long standing situation, that all standard realistic NN forces up to now underpredict the maxima by about 30%. That used to be called the $A_y$-puzzle [16]. We are now in fact rather close to the experimental $nd$ values. Note that on a 2N level the chiral potential predictions for the analyzing power agree well with the predictions based on the Nijmegen PSA [3]. Our predictions for various break up observables agree also very well with the ones obtained from CD-Bonn, as it is

![Figure 3](image-url)
demonstrated in [9].

5. CONCLUSIONS

Our findings show that the scheme originally proposed by Weinberg works quantitatively and much better than it was expected. It extends the successful applications of effective field theory (chiral perturbation theory) in the pion and pion–nucleon sectors to systems with more than one nucleon. Most of the np partial waves are well described at NNLO. The NNLO TPEP turns out to be too strong in the triplet F-waves. This is expected to be cured at N^3LO.

The very first results using chiral NN forces in 3N and 4N systems are very promising and show that these effective chiral forces are very well suited to describe also low energy properties of nuclear systems beyond \( A = 2 \). They agree rather well with standard nuclear force predictions as exemplified with CD-Bonn and most importantly they break the stagnation in the \( A_y \)-puzzle. It will be very interesting to perform the next step and use the NNLO NN forces, which would also require inclusion of the leading 3NF.

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