BRS Cohomology of a Bilocal Model

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Abstract

We present a model in which a gauge symmetry of a field theory is intrinsic in the geometry of an extended space time itself. A consequence is that the dimension of our space time is restricted through the BRS cohomology. If the Hilbert space is a dense subspace of the space of all square integrable $C^\infty$ functions, the BRS cohomology classes are nontrivial only when the dimension is two or four.

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1. Introduction

The problem why the dimension of our space time is four has not been answered so far. In the string models the dimensions $D = 26$ or $10$ play a special role, but any dynamical mechanisms by which the extra dimensions are uniquely compactified to make our four dimensional world have not been known. In this paper we point out another possibility which restricts, if not determines, the dimension of the space time. A basic idea is to convert the gauge principle of a field theory into an intrinsic geometry of the space time.

The Minkowski geometry is characterized as the invariant properties under the transformations which leave the interval, $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, invariant. The trajectories of a particle are the geodesics which minimize $-\int ds$. In the presence of an external electromagnetic field the trajectories of a particle with mass $m$ and electric charge $e'$ are modified from the geodesics to the ones which minimize the action

$$I_0 = -m\int ds + e'\int A_\mu(x)dx^\mu.$$  

\[ (1.1) \]

It is possible to reinterpret the trajectories as geodesics of an extended space spanned by $(x^\mu, a^\mu)$ where $a$’s are fictitious coordinates. By introducing the internal time $\tau$ and the einbein $V$ the integrated world length is defined as

$$I_0' = \int d\tau \left[ \frac{1}{2V(\tau)}(\dot{x}^\mu(\tau)\dot{x}_\mu(\tau)) - \frac{1}{2}m^2V(\tau) + ea_\mu(\tau)\dot{x}^\mu(\tau) \right], \quad (1.2)$$

where dots denote derivatives with respect to $\tau$. Then the trajectories of the charged particle are obtained by minimizing the world length $I_0'$ within the hypersurface defined by $ea_\mu = e'A_\mu(x)$. (In this interpretation the coupling constant $e$ has the dimension of mass square.)

The canonical theory is obtained by regarding $\tau$ as time. Introducing canonical variables $p_\mu$ and $\Pi$ conjugate to $x^\mu$ and $V$, respectively, we can write the Hamiltonian as

$$H' = \lambda L + \Lambda \Pi, \quad (1.3)$$

$$L = (p - ea)^2 - m^2, \quad (1.4)$$

where $\lambda = V$ and $\Lambda = \dot{V}$ are arbitrary functions of canonical variables. We get the primary constraint $\Pi \sim 0$ and the secondary constraint $L \sim 0$. The first quantization is achieved by replacing $p_\mu$ and $\Pi$ by $-i\frac{\partial}{\partial x^\mu}$ and $-i\frac{\partial}{\partial V}$, respectively. The canonical constraint is represented as the wave equation $L\Psi(x) = 0$, which coincides, on the hypersurface $ea_\mu = e'A_\mu(x)$, with the Klein-Gordon equation for a charged particle in the external electromagnetic field. Furthermore, the action of the field theory is given by

$$I' = \int d^Dx dv Q\Psi \bigg|_{ea=e'A(x)}, \quad (1.5)$$

\[ \dagger \text{We use the convention, } \eta_{\mu\nu} = \text{diag}(-+\ldots+). \]
where \( Q = cL \) is the BRS operator and \( c \) is the ghost variable corresponding to the reparametrization gauge of the action \( I_0' \).

We want to promote the above procedure into the more intrinsic geometrical construction. That is, we start neither with external field nor constrained hypersurface, and treat whole space time spanned by \((x^\mu, a^\mu)\) as the basic space time. The field theory may contain multiple of fields, hopefully of gauge fields, in an expansion of a basic field as \( \Psi(x, a) = \phi(x) + a^\mu A_\mu(x) + \ldots \), and an action like eq. (1.5), but without the restriction to a hypersurface, may describe the dynamics for the fields.

The world length defined by eq.(1.2) is clearly inadequate for the above purpose, since it has no information on the gauge properties of the electromagnetic field. For example the physical degrees of freedom should be the spatial coordinates, \( \vec{x} \), and the transverse polarizations, \( \vec{a}_T \), but the reparametrization invariance of \( I_0' \) can eliminate only one component among 2D coordinates.

Instead we start with the following world length, which permits just the desired degrees of freedom,

\[
I = \int d\tau \left[ \frac{1}{2V_1(\tau)} \dot{x}^\mu(\tau) \dot{x}_\mu(\tau) + \frac{1}{2V_2(\tau)} \dot{a}^\mu(\tau) \dot{a}_\mu(\tau) + 2ea_\mu(\tau) \dot{x}^\mu(\tau) \right].
\]

(1.6)

(the factor 2 in front of \( e \) is a convention). Apart from the above interpretation eq.(1.6) is formally regarded as an action for a bilocal particle each ingredient of which is described by the coordinates \( x^\mu \) (or \( a^\mu \)). This model was considered in previous papers [1, 2]. There it was shown that the action has a hidden local symmetry of \( SL(2, R) \), and it is sufficient to eliminate three of 2D coordinates. These facts are clearly seen in the canonical theory, but in the Lagrangian formalism the gauge symmetry is hidden and quite unexpected, since apparently there is only one reparametrization parameter if \( e \neq 0 \).

The world length defined by eq.(1.6) is the basic quantity of our model, and we will often call it as action in the context of the canonical theory which was described in ref. [2] as a model for a bilocal particle. The bilocal particle interpretation, however, cannot be extended to the curved space time, since the coordinates \( a^\mu \) would not be vector under general coordinate transformations thus \( I \) cannot be invariant. But the arguments in the subsequent sections are all in the flat space time, and the results are independent of which interpretation one chooses.

A dynamical theory on the space time who’s geometry is determined by \( I \) should be restricted by the BRS cohomology associated with the \( SL(2, R) \) symmetry. There is, however, an arbitrariness of the function space which we choose as the basic Hilbert space. We choose a dense subspace of the space, \( \tilde{H} \), of all square integrable \( C^\infty \) functions, which is defined in section 3. Our main result is that the BRS cohomology classes are nontrivial only when \( D = 2 \) or 4, i.e., the theory should be empty otherwise. When \( D = 2 \) the nontrivial physical states have only spin one, while when \( D = 4 \) they have only spin zero.

Therefore, a field theory appropriate to the BRS structure of the space time may contain physical gauge fields only when \( D = 2 \), a rather disappointing result. For \( D \neq 2 \) the gauge symmetries of a field theory are not physical and the gauge fields are all pure gauge. This
means that we must make some extension of the basic space time or the Hilbert space in order to get a realistic gauge field theory. Possible extensions are briefly discussed in the final section.

In section 2 we give a brief review of the canonical theory in the previous paper [2], correcting some minor errors of signatures contained in it. And we define a Hilbert space which is the base of the argument on the BRS cohomology. In section 3 the BRS cohomology is obtained. In section 4 a free field theory is defined which is based on the BRS structure of the geometry. Section 5 is devoted to outlooks. In Appendix, two facts, i.e., the linear independence of the basis we use, and that the basic function space is dense in \( \tilde{H} \), are proved.

2. Quantization

Let us briefly review the result of ref. [2]. We can derive the canonical theory by regarding \( \tau \) as time. It is convenient to add a total derivative term, \(-e \frac{d}{d\tau}(ax)\), in \( I \) to symmetrize \( x \)'s and \( a \)'s.

The canonical momenta of \( x \)'s, \( a \)'s and \( V \)'s are denoted by \( p \)'s, \( b \)'s and \( \Pi \)'s, respectively. Then the Hamiltonian is given by

\[
\mathcal{H} = \lambda_1 L_1 + \lambda_{-1} L_{-1} + \sum_{a=1,2} \Lambda_a \Pi_a,
\]

(2.1)

where \( \lambda \)'s and \( \Lambda \)'s are arbitrary functions of the canonical variables, and

\[
L_1 = \frac{i}{4e}(p - ea)^2, \quad L_{-1} = \frac{i}{4e}(b + e\tau)^2,
\]

(2.2)

(the factor \( i \) is a convention, and we omit the space time suffices here and hereafter). The primary constraints are \( \Pi_a \sim 0 \), the stability condition of them along \( \tau \) development leads to the secondary constraints \( L_{\pm 1} \sim 0 \), and finally the stability of the latter requires the tertiary constraint:

\[
L_0 = \frac{i}{4e}(p - ea)(b + e\tau) \sim 0.
\]

(2.3)

The constraints \( L \)'s form a first class algebra and according to the Dirac conjecture [3] they may generate a gauge symmetry of the action (1.6). In fact \( I \) is invariant under transformations with two local parameters \( \epsilon_0, \epsilon_1 \) and their \( \tau \) derivatives up to 2 ranks, \( \dot{\epsilon}_1, \dot{\epsilon}_0, \dot{\epsilon}_0 \) [2]. Thus the independent Cauchy data in a time like surface are reduced by five, two of which correspond to the einbeins and there remains \( 2D - 3 \) physical coordinates as expected.
Proceeding to quantum theory, we replace $p$ and $b$ by $-i\frac{\partial}{\partial x}$ and $-i\frac{\partial}{\partial a}$, respectively. The commutators of $L$’s form $SL(2, R)$ algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + 2(\alpha - \frac{D}{4})\delta_{n+m}, \quad (n, m = 0, \pm 1), \quad (2.4)$$

where the central term is caused by the ordering ambiguity in $L_0$ defined by $L_0 = \frac{i}{4e}(p - ea)(b + ex) - \alpha$. The BRS operator is

$$Q = \sum_{n=0,\pm1} c_n L_n - \frac{1}{2} \sum_{n,m=0,\pm1} (n - m) c_n c_m \frac{\partial}{\partial c_{n+m}}, \quad (2.5)$$

where $c_n(n = 0, \pm 1)$ are the ghost variables. The nilpotency of $Q$ fixes the ordering ambiguity in $L_0$. In fact we see

$$Q^2 = 2 \left( \alpha - \frac{D}{4} \right) c_1 c_{-1}, \quad (2.6)$$

which requires $\alpha = \frac{D}{4}$, so we have

$$L_0 = \frac{i}{4e}(-i\frac{\partial}{\partial x} - ea)(-i\frac{\partial}{\partial a} + ex) - \frac{D}{4}. \quad (2.7)$$

(Note $L_0 = \frac{i}{8e}(p - ea, b + ex)$.)

Next let us define the generators of the kinematic symmetry:

$$\tilde{p} = p + ea, \quad \tilde{b} = b - ex, \quad (2.8)$$

$$M_{\mu\nu} = x_{[\mu}p_{\nu]} + a_{[\mu}b_{\nu]}, \quad (2.9)$$

These generators satisfy the ordinary commutation relations except that $[\tilde{p}_\mu, \tilde{b}_\nu] = 2ei\eta_{\mu\nu}$ which is interpreted as an uncertainty relation. It is important to note that the kinematic generators all commute with $L_n, (n = 0, \pm 1)$:

$$[\tilde{p}_\mu, L_{0,\pm1}] = [\tilde{b}_\mu, L_{0,\pm1}] = [M_{\mu\nu}, L_{0,\pm1}] = 0. \quad (2.10)$$

There is the unique ground state, $|0\rangle$, which is annihilated by $L_{-1}$ and Lorentz invariant and has vanishing momentum:\[1]

$$|0\rangle = e^{-ieax}, \quad (2.11)$$

$$M_{\mu\nu}|0\rangle = \tilde{p}_\mu|0\rangle = L_{-1}|0\rangle = 0. \quad (2.12)$$

We define the state with momentum $k$ by

$$|k\rangle = e^{-\frac{i}{\hbar}\tilde{b}k}|0\rangle = e^{ikx}|0\rangle. \quad (2.13)$$

\[1\] The state $|0\rangle$ here is different from that defined in ref.[2]. The latter is an eigenstate of total momentum, $\tilde{p} + \tilde{b}$. 
According to the Dirac prescription the first quantization would be achieved by requiring
the wave equations, \[ L_{0,\pm 1} \Psi(x, a) = 0. \] Here \( \Psi \) would be any function which is square
integrable and differentiable to an arbitrary order. Let us denote the set of all such functions
by \( \tilde{H} \). We assume, however, that the Hilbert space is not the whole space \( \tilde{H} \) but a dense
subspace of \( \tilde{H} \). One can prove the existence of the subspace, \( H_1 \), which contains an arbitrary
momentum and spin states, and a function in \( H_1 \) is general enough as \( H_1 \) is dense in the
whole function space \( \tilde{H} \). We find that the following functions span the dense subspace \( H_1 \)
in \( \tilde{H} \):

\[
u_{nJj}(k) = L_1^n e^{-\frac{\pi ik}{2}} F_{Jj}(a) |0\rangle \quad (n, J, j = 0, 1, \ldots),
\]
(2.14)

where \( F_{Jj}(a) \) are the harmonic polynomials of \( a \) with homogeneous order \( J \), which satisfy
\[ \Box_a F_{Jj} = 0, \] and \( j \) varies from 1 to \( (2J + D - 2)(J + D - 3)!/J!(D - 2)! \). We denote by
\( H_1 \) the set of all functions which can be written as linear combinations (and integrations
over \( k \)) of \( u_{nJj}(k) \), with vanishing coefficients except a finite number of ones.

In Appendix, we show that \( H_1 \) is dense in \( \tilde{H} \), and that a finite number of element in
\( \{u_{nJj}(k)\} \) are linearly independent. Thus the function space \( H_1 \) satisfies all the requirements
mentioned before, and we choose it as the basic Hilbert space. A merit of our basis (2.14)
is that they belong to a representation of the \( SL(2, \mathbb{R}) \) as expressed in eqs.(A.5), (A.6) and
(A.7) in Appendix, and a calculation to obtain the BRS cohomology becomes algebraic.

3. BRS cohomology

Let us obtain the BRS cohomology classes of the first quantized system. Our task is to
obtain all classes of functions \( \Psi \)'s in \( H_1 \), which satisfy the Kugo-Ojima(KO) condition [4],
\[ Q|\Psi\rangle = 0, \]
(3.1)
and are not written as \( |\Psi\rangle = Q|\chi\rangle \) for some \( |\chi\rangle \) in \( H_1 \). Since the ghost variables are
Grassmann odd we can divide \( H_1 \) according to the ghost numbers, \( N_g \), varying from 0 to
3. We can search such functions separately in each sector because \( Q \) has ghost number 1
and the KO condition does not mix sectors with different ghost numbers.

(1) \( N_g = 0 \): Let us write a function \( \Psi^{(0)} \in H_1 \), as
\[
\Psi^{(0)} = \sum_{n=0}^{\infty} \sum_{J=0}^{\infty} \sum_j \int_k \alpha_{nJj}(k) u_{nJj}(k).
\]
(3.2)
In this sector the KO condition (3.1) amounts to \( L_n \Psi = 0 \) for \( n = 0, \pm 1 \). Although
the subscript of the summation in eq.(3.2) extend to infinity, only a finite number of the
coefficients are nonvanishing. In Appendix we show an arbitrary finite subset of \( \{u_{nJj}(k)\} \)
is linearly independent. Therefore, using only \( L_1 \Psi = 0 \), we see, without no subtleties on
divergence or infinite summations, that all \( \alpha \)'s vanishes. Thus in this sector of \( H_1 \) there is
no physical state.
(2) $N_g = 1$: In this sector we can write an arbitrary state as
\[ \Psi^{(1)} = \sum_{n=0}^{\infty} \sum_{J=0}^{\infty} (\alpha_{nJ} c_0 + \beta_{nJ} c_1 + \gamma_{nJ} c_{-1}) u_{nJ}, \] (3.3)
where $\alpha$'s, $\beta$'s and $\gamma$'s are numerical coefficients. Integrations over $k$ are implicitly assumed and we suppress the $k$-dependence in the expressions, since all $L$'s commute with $\tilde{b}$ and they play no role in the present argument ($j$ dependence is also suppressed).

The KO condition requires the following equations to the coefficients, for $n \geq 1$:
\[
\begin{pmatrix}
2 & (n + 1)(n + \frac{D}{2} + J) & (n + \frac{D}{2} + J)
\end{pmatrix}
\begin{pmatrix}
\alpha_{nJ} \\
\beta_{n+1J} \\
\gamma_{n-1J}
\end{pmatrix}
= 0,
\] (3.4)

and
\[
\begin{pmatrix}
2 & \frac{D}{4} + J \\
1 & \frac{D}{4} + J
\end{pmatrix}
\begin{pmatrix}
\alpha_{0J} \\
\beta_{1J}
\end{pmatrix}
= 0,
\] (3.5)

\[
(1 - \frac{D}{4} - \frac{J}{2}) \beta_{0J} = 0.
\] (3.6)

These equations are a consequence of the (finite) linear independence of our basis (2.14). Since the matrices appearing in eqs. (3.4) and (3.5) have vanishing determinants, we get the following solution
\[
\Psi^{(1)} = \sum_{n=0}^{\infty} \sum_{J=0}^{\infty} \left[ \beta_{0J} c_{1u0J} + \beta_{1J} (-\left(\frac{D}{4} + \frac{J}{2}\right)c_0 u_{0J} + c_1 u_{1J})
\right.
\]
\[
+ \sum_{n=1}^{\infty} \left. \frac{\alpha_{nJ}}{n + \frac{D}{4} + \frac{J}{2}} \right] \left( (n + \frac{D}{4} + \frac{J}{2}) c_0 u_{nJ} - c_1 u_{n+1J} - n(n - 1 + \frac{D}{2} + J)c_{-1} u_{-1J} \right).
\] (3.7)

A part of expression in r.h.s of eq.(3.7) may be written as a BRS trivial form. Since the BRS trivial quantities in the present sector have the ghost number 1, only candidates are of the form $Qu_{nJ}$:
\[
Qu_{nJ} = -(n + \frac{D}{4} + \frac{J}{2}) c_0 u_{nJ} + c_1 u_{n+1J} + n(n - 1 + \frac{D}{2} + J)c_{-1} u_{-1J} \quad (n \geq 1),
\] (3.8)

\[
Qu_{0J} = -(\frac{D}{4} + \frac{J}{2}) c_0 u_{0J} + c_1 u_{1J}.
\] (3.9)

Comparing eq.(3.4) with eqs.(3.8, 3.9) we find
\[
\Psi^{(1)} = \sum_{J=0}^{\infty} \beta_{0J} c_{1u0J} - Q \left( \sum_{J=0}^{\infty} \left( \beta_{1J} u_{0J} + \sum_{n=0}^{\infty} \frac{\alpha_{nJ} u_n}{n + \frac{D}{4} + \frac{J}{2}} \right) \right).
\] (3.10)
Furthermore, by eq. (3.6) we find that if \((D, J) \neq (2, 1), (4, 0)\) then \(\beta_0 = 0\). Thus if \(D \neq 2, 4\), the solution to the KO condition is BRS trivial, while if \(D = 2(J = 1)\) or \(D = 4(J = 0)\) we get the BRS nontrivial physical states, \(c_1 u_{0J}\).

(3) \(N_g = 2\): In this sector we have the expansion

\[
\Psi^{(2)} = \sum_{n=0}^{\infty} \sum_{J=0}^{\infty} (\alpha'_{nJ} c_0 c_{-1} + \beta'_{nJ} c_0 c_1 + \gamma'_{nJ} c_1 c_{-1}) u_{nJ}.
\]

The solution to the KO condition is similarly obtained as before:

\[
\Psi^{(2)} = \sum_{J=0}^{\infty} \left[ \beta'_{0J} c_0 c_1 u_{0J} + \beta'_{1J} (c_0 c_1 u_{1J} + 2 c_1 c_{-1} u_{0J}) + \sum_{n=1}^{\infty} \gamma'_{nJ} (c_1 c_{-1} u_{nJ} - (n + \frac{D}{4} + \frac{J}{2}) c_0 c_{-1} u_{n-1J}) + \sum_{n=2}^{\infty} \beta'_{nJ} (c_0 c_1 u_{nJ} + n(n - 1 + \frac{D}{2} + J) c_0 c_{-1} u_{n-2J}) \right],
\]

where the coefficients in eq. (3.12) are all arbitrary. The list of the BRS trivial states in the present sector is as follows:

\[
\begin{pmatrix}
Q c_0 u_{nJ} \\
Q c_1 u_{n+1J} \\
Q c_{-1} u_{n-1J}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{2} (n + 1)(n + \frac{D}{2} + J) & 0 & n(n - 1 + \frac{D}{2} + J) \\
(n + \frac{D}{4} + \frac{J}{2}) & 0 & 0 \\
-1 & n + \frac{D}{4} + \frac{J}{2} & n + \frac{D}{4} + \frac{J}{2}
\end{pmatrix}
\begin{pmatrix}
c_{-1} c_1 u_{nJ} \\
c_1 c_0 u_{n+1J} \\
c_{-1} c_0 u_{n-1J}
\end{pmatrix}
\]

(3.13)

\[
\begin{pmatrix}
Q c_0 u_{0J} \\
Q c_1 u_{1J}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & 1 \\
D/2 + J & D/4 + J
\end{pmatrix}
\begin{pmatrix}
c_{-1} c_1 u_{0J} \\
c_1 c_0 u_{1J}
\end{pmatrix},
\]

(3.14)

\[
Q c_1 u_{0J} = \left(1 - \frac{D}{4} - \frac{J}{2}\right) c_0 c_1 u_{0J}.
\]

(3.15)

Note the matrices appeared in eqs. (3.13, 3.14) are transposition of those in eqs. (3.4, 3.5), hence have vanishing determinants and are not invertible. Thus only specific combinations of \(c_0 c_1 u_{nJ}\) are BRS trivial. We find all terms, except the first one, in r.h.s. of eq. (3.12) is BRS trivial,

\[
\Psi^{(2)} = \sum_{J=0}^{\infty} \beta'_{0J} c_0 c_1 u_{0J}
\]

\[
+ Q \left( \sum_{J=0}^{\infty} \left\{ - \beta'_{0J} c_0 u_{0J} + \sum_{n=1}^{\infty} \gamma'_{nJ} c_{-1} u_{n-1J} - \sum_{n=2}^{\infty} \frac{\beta'_{nJ}}{n - 1 + \frac{D}{4} + \frac{J}{2}} (c_1 u_{n+1J} + (n + 1)(n + \frac{D}{2} + J) c_{-1} u_{n-1J}) \right\} \right). (3.16)
\]
Furthermore we find that if $D \neq 2,4$ then the factor $(1 - \frac{D}{4} - \frac{J}{2})$ in eq.(3.15) is invertible, and the first term in r.h.s. of eq.(3.16) is also BRS trivial. Thus if $D \neq 2,4$, the solution to the KO condition is BRS trivial, while if $D = 2(J = 1)$ or $D = 4(J = 0)$ we get the BRS nontrivial physical states, $c_0 c_1 u_0 J$.

(4) $N_g = 3$: In this sector there exists no BRS nontrivial physical states, since

$$Q c_{1c-1} u_{nJ} = - \left( n + \frac{D}{4} + \frac{J}{2} \right) c_0 c_1 c_{-1} u_{nJ}. \quad (3.17)$$

This is the final possibility.

We conclude from the above arguments for the four cases that if $D \neq 2,4$ then the BRS cohomology of our system is trivial, while if $D = 2(J = 1)$ or $D = 4(J = 0)$ then it contains two classes which are BRS equivalent to $c_1 u_{0J}(k)$ or $c_0 c_1 u_{0J}(k)$. In the framework of the first quantization, the mass spectrum of the physical states of the bilocal system is continuous. This is because we have no interactions, in the ordinary sense, between the two particles, but the two particles are correlated through the internal $SL(2, R)$ local symmetry generalizing the reparametrization of the world lines. A discrete mass spectrum, if exists, might be obtained after introducing a possible interaction between the bilocal particles in the framework of the second quantization. In the free field theory defined in the next section we see such an indication of the discrete mass spectrum.

4. Field theory

Let us examine how the inherent BRS structure of the basic space time $M^{2D}$, spanned by $(x^\mu, a^\mu)$, is connected with a gauge symmetry of a field theory on $M^D$. As an example we present here the free field theory.

A field $\Psi$ is a function of $x$’s, $a$’s and $c$’s, and we assume $\Psi$ is Grassmann odd. The action is defined by

$$I = \int d^Dx d^D a d c_0 d c_1 d c_{-1} \Psi^* Q \Psi, \quad (4.1)$$

which is invariant under the following gauge transformation,

$$\delta \Psi = Q \Lambda, \quad (4.2)$$

where $\Lambda$ is an arbitrary function of $x$’s, $a$’s and $c$’s, which is Grassmann even. Since the BRS operator has the ghost number one, the only component fields of $\Psi$ with a non-zero ghost number are subjected to the gauge transformations. The component fields of $N_g = 1$ sector have kinetic terms in $I$ among themselves, and the fields of $N_g = 0$ or 2 have mixed kinetic terms.

In general, however, the gauge symmetry is not a physical one. Since vector fields appear as component of $a^\mu$ in $\Psi$ and belong to $J = 1$ sector (and scalars belong to $J = 0$ sector),...
there exists physical vector fields only when $D = 2$ as was shown in the previous section. For $D \neq 2$ the component fields of $J = 1$ sector are expressed as $Q\chi$ for some $\chi$. In other words the gauge fields are pure gauge, in the sense of eq. (4.2), in all cases except $D = 2$.

Finally let us write the action $\mathcal{I}$ as an integration over $x$’s of ordinary fields on $M^D$. For that purpose let us expand $\Psi$ as

$$
\Psi = N_\epsilon^{-1/2} e^{-\frac{1}{2} \epsilon a^2} \sum_{\alpha=1,\pm 1} c_\alpha (\phi_\alpha (x) + A_{\mu \alpha} (x) a^\mu + \ldots) |0\rangle + \ldots, \quad (4.3)
$$

$$
N_\epsilon = \int d^D x e^{-\epsilon a^2} = \text{const.} e^{-\frac{D}{4}}, \quad (4.4)
$$

where the factor $e^{-\frac{1}{2} \epsilon a^2}$ is introduced for making integrals convergent. The explicit form of the action (in the sector with ghost number one) is expressed as

$$
\mathcal{I} = \int d^D x \left\{ \frac{i}{4\epsilon} \left[ \phi_0^* \left( \Box - \frac{2e^2 D}{\epsilon} \right) \phi_0 - \phi_1^* \left( \Box - \frac{2e^2 D}{\epsilon} \right) \phi_1 \right] + \frac{i}{8\epsilon \epsilon} \left[ A_{\mu}^{\mu*} \left( \Box - \frac{2(D + 2)e^2}{\epsilon} \right) A_0 - A_0^{\mu*} \left( \Box - \frac{2(D + 2)e^2}{\epsilon} \right) A_{-1\mu} \right] + \frac{1}{2\epsilon} \left[ \phi_0^* \partial_\mu A_0 - \phi_0^* \partial_\mu A_0^{\mu*} + \phi_1^* \partial_\mu A_1 - \phi_1^* \partial_\mu A_1^{\mu*} \right] + \frac{i}{32} D - \frac{1}{2\epsilon} \left( \phi_0^* \phi_0 - \phi_1^* \phi_1 \right) + \frac{i(D + 10)}{64\epsilon} \left( A_{\mu*} A_0^{\mu*} - A_0^{\mu*} A_{\mu*} \right) \right\}. \quad (4.5)
$$

Writing the gauge parameter as

$$
\Lambda (x, a) = N_\epsilon^{-1/2} e^{-\frac{1}{2} \epsilon a^2} (\Lambda (x) + \Lambda_\mu (x) a^\mu + \ldots) |0\rangle + \ldots, \quad (4.6)
$$

the gauge transformation is expressed as

$$
\delta \phi_1 = -\frac{i}{4\epsilon} \Box \Lambda, \quad \delta \phi_0 = -\frac{D}{4\epsilon} \Lambda - \frac{i}{4\epsilon} \partial_\mu \Lambda^\mu, \quad \delta \phi_{-1} = 0, \quad (4.7)
$$

$$
\delta A_{\mu 1} = -\partial_\mu \Lambda - \frac{i}{4\epsilon} \Box \Lambda_\mu, \quad \delta A_{0\mu} = -\frac{i}{4\epsilon} \epsilon \partial_\mu \Lambda - \frac{D + 10}{4\epsilon} \Lambda_\mu, \quad \delta A_{-1\mu} = \frac{i}{4\epsilon} \epsilon \Lambda_\mu. \quad (4.8)
$$

In the above formulas, the divergent factor $\frac{1}{\epsilon}$ can be absorbed into the coupling constant $\epsilon$ and renormalization factors of fields.
5. Outlooks

We have shown that the BRS cohomology classes in the Hilbert space $H_1$ are nontrivial only when $D = 2$ or 4. In the case $D = 2$ the nontrivial physical states are $c_1u_{01j}(k)$ and $c_0c_1u_{01j}(k)$, i.e., momentum eigenstates with spin one, while in the case $D = 4$ they are $c_1u_{00}(k)$ and $c_0c_1u_{00}(k)$ which are spin zero states. In particular the ground state with the vacuum quantum number is physical only when $D = 4$ (see [3]).

Our original hope was that the BRS structure of the basic space time would be translated into a gauge symmetry of the corresponding field theory. But it turned out that a possible gauge field belong to a BRS trivial sector and is pure gauge except $D = 2$. Hence a realistic model would be obtained by some modifications or extensions of the present one.

First possibility is to extend the fictitious dimensions, $a^\mu$, to a multiple of ones, $a^\mu_i, (i = 1, \ldots, N)$. A preliminary investigation shows that the symmetry of the world length is enlarged to $Sp(N + 1, 2)$ for odd $N$, and the physical degrees of freedom of the coordinates reduces by $\frac{1}{2}(N + 1)(N + 2)$. In order to maintain at least one physical component of the coordinates we have the inequality, $\frac{1}{2}(N + 2) < D$, which means $N \leq 5$ for $D = 4$. This may provide us the more abundant structure, though it is not certain whether one of them includes a realistic theory. A second possibility is to supersymmetrize the model, which may introduce fermionic fields. It is tempting to seek for a supersymmetric world length.

Apart from the above directions it is conceivable to enlarge the Hilbert space. For example we may add base functions which are created from a vacuum annihilated by $L_1$. But it turns out, by a similar argument as in section 3, that the result on the dimensionality would not be altered from that obtained in the present analysis.

The model presented here can be regarded as a particular mode of the string models. If one put, e.g.,

$$X(\tau, \sigma) = x(\tau) + (2 - 9\sigma + 10\sigma^3)a(\tau),$$
\hspace{1cm} (5.1)

$$g_{nm} = \begin{pmatrix} 0 & e \frac{e^{-7 + 12\sigma}}{10V_1(\tau)} + \frac{14(1 - \sigma)}{15V_2(\tau)} \\ e \frac{e^{-7 + 12\sigma}}{10V_1(\tau)} + \frac{14(1 - \sigma)}{15V_2(\tau)} & 0 \end{pmatrix},$$
\hspace{1cm} (5.2)

the world length (1.6) is written as the world area $I = \int d\tau \int_0^1 d\sigma \sqrt{-g} g^{nm} \partial_n X \partial_m X$. The correspondence is not so beautiful and we may not expect any intrinsic connections between the two models. But it is impressive that both models require critical dimensions, though in quite different mechanisms.
Appendix

In this Appendix we show the linear independence of the basis \( \{u_{nJj}(k)\} \) and that the space \( H_1 \) is dense in \( \tilde{H} \), the set of all square integrable \( C^\infty \) functions. Since an element in \( H_1 \) is a linear combination of \( u_{nJj}(k) \) with a finite number of nonvanishing coefficients, we show that arbitrary finite subset of \( \{u_{nJj}(k)\} \) is linearly independent.

Before proceeding to the proof let us recapitulate useful relations. From the commutators of \( L_a \) (\( a = 0, \pm 1 \)) we get

\[
\begin{align*}
\{L_0, L_{n\pm 1}^n\} &= \mp nL_{n\pm 1}^n, \\
\{L_{n\mp 1}, L_{n\pm 1}^n\} &= L_{n\pm 1}^{n-1}n(n-1 \mp 2L_0).
\end{align*}
\]

(A.1)  (A.2)

For arbitrary polynomials \( G(a) \) of \( a \), we have

\[
L_{-1}G(a)|0\rangle = -\frac{i}{4e}\Box a G(a)|0\rangle,
\]

(A.3)

and for arbitrary homogeneous polynomials \( G_J(a) \) of order \( J \), we have

\[
L_0 G_J(a)|0\rangle = -\left(\frac{D}{4} + \frac{J}{2}\right) G_J(a)|0\rangle.
\]

(A.4)

Applying these relations to the harmonic polynomials, \( F_{Jj} \), satisfying \( \Box a F_{Jj} = 0 \), we get

\[
\begin{align*}
L_1 u_{nJj}(k) &= u_{n+1Jj}(k), \\
L_{-1} u_{nJj}(k) &= n\left( n - 1 + \frac{D}{2} + J \right) u_{n-1Jj}(k), \\
L_0 u_{nJj}(k) &= -\left( n + \frac{D}{4} + \frac{J}{2} \right) u_{nJj}(k).
\end{align*}
\]

(A.5)  (A.6)  (A.7)

Now let us give the proof of the (finite) linear independence of \( \{u_{nJj}(k)\} \).

1. Linear independence.

We prove that a finite subset of \( \{u_{nJj}(k)\} \) is linearly independent. Suppose that

\[
\sum_{n=0}^{n_0} \sum_{Jj} \int_k \alpha_{nJj}(k)u_{nJj}(k) = 0,
\]

(A.8)

(\( \int_k \equiv \int \frac{d^Dk}{(2\pi)^D} \)). Multiplying \( L_{-1}^{n_0} \) to eq.(A.8) and using eqs.(A.2) and (A.6) we see all terms except \( n = n_0 \) vanish and get

\[
\sum_{Jj} \int_k \alpha_{n_0Jj}(k)u_{0Jj}(k) = 0.
\]

(A.9)

Thus we get

\[
\sum_{Jj} \hat{\alpha}_{n_0Jj}(x)F_{Jj}(a) = 0,
\]

(A.10)
By the linear independence of the harmonic polynomials we see all \( \tilde{\alpha}_n \)’s vanish, hence also all \( \alpha_n \)’s vanish. Then multiplying \( L_{n-1}^{n-1} \) to eq. (A.8) we get \( \alpha_{n-1}j_j(k) = 0 \) in the same manner, and so on. Thus all \( \alpha \)’s vanish, which completes the proof for the finite linear independence of \( \{u_{n,j_j}\} \).

(2) \( H_1 = \tilde{H} \).

We show that \( H_1 \) is dense in \( \tilde{H} \), i.e., for an arbitrary element \( f \) in \( \tilde{H} \) there exists a sequence \( \{f_N\} \) in \( H_1 \), which converges to \( f \) in the limit \( N \to \infty \).

Suppose an arbitrary function \( f \in \tilde{H} \) is given. Since \( f(x,a)e^{ie(x-a)a} \in \tilde{H} \) it is Fourier expandable:

\[
    f(x,a)e^{ie(x-a)a} = \int_k \int_{k'} e^{ikx + ik'a} \tilde{f}(k, k').
\]

If we define

\[
    f_N(x,a) = \sum_0^N \int_k \int_{k'} \tilde{f}(k, -k') e^{i(k-k')x} \left(-\frac{L_1}{n!} e^{-\frac{i}{\pi^2} (\frac{1}{4}k' + \frac{1}{8}k^2)}\right) 0,
\]

then, using eqs. (A.12) and

\[
    L_1|0\rangle = -i e^{a^2}|0\rangle, \quad e^{-\frac{i}{\pi^2} (\frac{1}{8}k' + \frac{1}{4}k^2)} e^{i e^a^2}|0\rangle = e^{i k(x-a)} e^{i e^a^2}|0\rangle,
\]

we see that

\[
    f(x,a) = \lim_{N \to \infty} f_N(x,a).
\]

Thus, if \( f_N \in H_1 \) we see \( \tilde{H} = \tilde{H} \).

By moving \( e^{i(k-k')x} \) in eq. (A.13) to the far right until to hit \( |0\rangle \) we get a polynomial of \( k, k' \) and \( p' \), where \( p' \equiv p - ea \). Using \( p'|0\rangle = -2ea|0\rangle \) we get a polynomial of \( a \) multiplied by \( e^{i(k-k')x} \). Using eq. (2.13) the latter factor is absorbed into \( e^{-\frac{i}{\pi^2} \frac{1}{8} k'} \). Thus \( f_N \) are written as (finite) linear combinations (and integrations over \( k \) and \( k' \)) of \( L_1^n e^{-\frac{i}{\pi^2} \frac{1}{8} k'} G(a, k) \), where \( G(a, k) \) is a polynomial of \( a \). The explicit expression is

\[
    f_N(x,a) = \sum_{n=0}^N \sum_{m=0}^n \sum_{J=0}^{n-m} \int_k \int_{k'} \tilde{f}_{nm, J}(k, k') L_1^n e^{-\frac{i}{\pi^2} \frac{1}{8} k'} ((k' - k)a)^J |0\rangle,
\]

\[
    \tilde{f}_{nm, J}(k, k') = e^{-\frac{i}{\pi^2} \frac{1}{8} k'^2} \frac{(\frac{1}{4}k'(k' - k)^2)^{n-m-J}}{(n-m-J)! m!} \tilde{f}(k, -k').
\]

Finally let us prove that the function \( L_1^n e^{-\frac{i}{\pi^2} \frac{1}{8} k'} G_j(a) |0\rangle \) in r.h.s. of eq. (A.16), where \( G_j \) is an arbitrary polynomial of order \( J \), belongs to \( H_1 \), i.e., it is written as linear combinations
(and integrations over $k$) of $u_{nJj}(k)$. Then, from eqs. (A.13) and (A.16), we see $f \in \tilde{H}$, which is the desired result.

Since a polynomial $G(a)$, satisfying $\Box_a G(a) = 0$, is written as a linear combination of the harmonic polynomials we see, from eq. (A.3), that,

$$L_{-1}G_J(a)|0\rangle = 0 \implies G_J(a)|0\rangle \in H_1.$$  \hspace{1cm} (A.18)

Let us prove the fact that there exist coefficients $a_{nJj'}(k)$ for any $J$, satisfying

$$G_J(a)|0\rangle = \left[ J \right] \sum_{n=0}^{\infty} \sum_{J'=0}^{J} \int_k a_{nJj'}(k)u_{nJj'}(k),$$  \hspace{1cm} (A.19)

i.e., $G_J|0\rangle \in H_1$. The proof is given by induction on $J$. For $J = 1$, $G_J$ is a linear function of $a$ and $\Box_a G_J = 0$, hence $L_{-1}G_J|0\rangle = 0$. Hence by (A.18) we see $G_J \in H_1$. Next assume $G_J$ satisfy eq. (A.19) for $J \leq J_0$. By eq. (A.3) we see

$$L_{-1}G_{J_0+1}|0\rangle = -\frac{i}{4\epsilon} \Box_a G_{J_0+1}|0\rangle.$$  \hspace{1cm} (A.20)

Since $\Box_a G_{J_0+1}$ is of order $J_0 - 1$ we can use the assumption of the induction. Then write

$$\Box_a G_{J_0+1}|0\rangle = \sum_{nJj} \int_k a_{nJj}(k)u_{nJj}(k).$$  \hspace{1cm} (A.21)

From eqs. (A.3) and (A.6) we see

$$L_{-1}L_1 u_{nJj}(k) = (n + 1)(n + \frac{D}{2} + J)u_{nJj}(k).$$  \hspace{1cm} (A.22)

Then from eqs. (A.20), (A.21) and (A.22) we see

$$L_{-1} \left( G_{J_0+1} + \frac{i}{4\epsilon} \sum_{nJj} \int_k \frac{a_{nJj}(k)}{(n + 1)(n + \frac{D}{2} + J)}u_{nJj}(k) \right) |0\rangle = 0.$$  \hspace{1cm} (A.23)

And finally from the fact of (A.18) we find the function inside the parenthesis of eq. (A.23) multiplied by $|0\rangle$, and hence $G_{J_0+1}|0\rangle$, belong to $H_1$, which completes the inductive proof.
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