Waves of spin-current in magnetized dielectrics

P. A. Andreev

Department of General Physics, Faculty of Physics, Moscow State University, Moscow, Russian Federation.

L. S. Kuz’menkov

Department of Theoretical Physics, Physics Faculty, Moscow State University, Moscow, Russian Federation.

(Dated: May 3, 2014)

Spin-current is an important physical quantity in present day spintronics and it might be very useful in the physics of quantum plasma of spinning particles. Thus it is important to have an equation of the spin-current evolution. This equation naturally appears as a part of the set of quantum hydrodynamics (QHD) equations. Consequently, we present the set of the QHD equations derived from the many-particle microscopic Schrodinger equation, which consists of the continuity equation, the Euler equation, the Bloch equation and equation of the spin-current evolution. We use these equations to study dispersion of the collective excitations in the three dimensional samples of the magnetized dielectrics. We show that dynamics of the spin-current leads to formation of new type of the collective excitations in the magnetized dielectrics, which we called spin-current waves.

I. INTRODUCTION

The spin-current is a very important characteristic for various physical systems. For instance it is useful for description of such processes as spin injection [1] and other process, where spin transport is involved. Most of them are accumulated in a grate field of physics: spintronics [2], [3]. There are different methods of the spin-current generation, such as spin pumping by sound waves [4], optical spin injection [5], and at using of junctions as Polarizers [6], spin flip in the result of electron interaction with electromagnetic wave [7]. Spin-polarized currents are used in spin lasers development [8] and in spin-diode structures [9]. It is interesting to admit that in some devises the spin-current exhibits the sine wave-like behavior [10]. New modification of spin-field effect transistors have been suggested (see for example [11], [12]), one of the spintronic devices utilize the electron’s spin properties in addition to its charge properties. Effects in junction play key role in spintronic devises. The spin-dependent Peltier effect was observed experimentally [13]. It is based on the ability of the spin-up and spin-down channels to transport heat independently. The use of graphene has been involved in this field either [14], [15]. Processes of spin transport and relaxation in graphene have also been considered [16]. It has been expected that silicon spintronics has potential to change information technology, this possibility discussed in Ref. [17].

We have described a number of examples there the spin-current plays important role. Thus, it is necessary to have analytical definition of the spin-current and equation of the spin-current evolution. In Refs. [18]- [22] authors discussed the definition of the spin-current in terms of the one-particle wave function describing the quantum state of the spinning particle. Knowledge of the wave function allows to calculate the spin-current and make conclusions on the system behavior. We keep the spin-current as an independent macroscopic variable defined via the many-particle wave function governs system behavior. We use the spin-current along with the particle concentration, the velocity field, and the magnetization. We consider the spin-current evolution due to interparticle interaction. For this purpose we derive the spin-current evolution equation as a part of the set of quantum hydrodynamic equations. Total angular momentum current was presented in Ref. [18] as a generalization of the spin current, which conserve even when the spin-current does not conserve. In our paper we consider the spin-current only, since the spin-current has appeared in the quantum hydrodynamic equations. It is not necessary to have a conserved quantity for the spin-current. We study the spin-current evolution due to interparticle interaction.

It can be expected that the spin-current evolution also gives influence on the spin waves, which are well-known in condensed matter physics. For example, quantum dots show interesting spin waves behavior [23]- [27]. Dynamics of the magnetic moments in the quantum plasma has also been studied. In Ref. [28] existence of the self-consistent spin waves in the magnetized plasma was shown. Thus new branches of the wave dispersion appear in the magnetized plasma due to the dynamics of magnetic moment of electrons and ions. It was demonstrated by means of the quantum hydrodynamics (QHD) method [29]- [32]. Later, the generalization of the Vlasov equation for the plasma of spinning charged particles was used to study the same problem [33]. Magnetic moment dynamics also leads to existence of the effect of resonances interaction of the neutron beam with the magnetized plasma, which gives new method of wave generation in the magnetized plasma [34], [35], [36]. Method of the QHD has become very popular and powerful method of studying influence of the magnetic
moment dynamics on various processes in the magnetized plasma \cite{34,37,41}. The method of the QHD can be used for systems of the neutral particles with the magnetic moment. Such systems are more preferable to demonstrate results given by the QHD method at description of the physical effects caused by evolution of the magnetic moments. The QHD description of the self-consistent spin waves in a system of neutral particles in the one-, two-, and three-dimensional dielectrics was described in Ref. \cite{42}. Usually the set of QHD equations for the spinning charged particles consists of the three equations for evolution of the material fields, these are the continuity equation for the particles concentration, the Euler equation for the velocity field, and the generalized Bloch equation for the magnetic moment evolution, and the Maxwell equations for the electromagnetic field description.

The spin-current and an equation of its evolution are in the center of attention of this paper. The spin-current for many-particle systems of charged spinning particles appears in the QHD equations \cite{34,43}. Usually we need to find approximate relation between the spin-current and other hydrodynamic variables to get closed set of the QHD equations. In this paper we go further, we derive an equation for the spin-current evolution by means of the QHD method. Recently, an analogous equation was derived for graphene electrons which have also been obtained by the QHD method \cite{44}. The spin-current definition and corresponding equations appears in the QHD in the semi-relativistic approximation. The spin-current have been universally defined according to the quantum electrodynamics in Ref. \cite{18}.

The first step in the development of the QHD was made after the Schrodinger equation had been suggested. E. Madelung represented the Schrodinger equation for the one charged particle in an external field as a set of two equations \cite{45}. These equations are the continuity equation and the Euler equation for the eddy-free motion. Later T. Takabayasi considered the QHD representation of the Pauli equation \cite{46}.

It is well-known that the one-particle Pauli equation can be represented as the set of the QHD equations. This set of equations is almost equivalent to the first three equations of the chain of the many-particle QHD equations taken in the self-consistent field approximation. The differences are the follows. One-particle equations do not contain the thermal pressure and the thermal contribution in the spin-current. Hence, the one-particle approximation may be used in semiconductors at low temperatures. However, the hydrodynamical equations should be coupled with the set of Maxwell equations.

If we derive the QHD equations from the Pauli equation for the one-particle in an external field we find three equations of matter field evolution mentioned above. However, if we have deal with many-particle system the set of the QHD equations contains some new functions, for example the kinetic pressure \(p^{\alpha\beta}\) caused by the thermal motion of particles, and the spin-current \(J^{\alpha\beta}\). Two-particle macroscopic functions also appear in the terms describing inter-particle interaction. In the self-consistent field approximation, containing no contribution of the exchange interaction, the terms describing interaction have same form as the terms caused by the external fields.

To continue the comparison of the many-particle QHD with the one-particle one we admit that in the one-particle case \(p^{\alpha\beta} = 0\) and \(J^{\alpha\beta} = M^{\alpha}v^{\beta}\), where \(M^{\alpha}\) is the density of magnetic moments, and \(v^{\beta}\) is the velocity field. We can also notice that in the one-particle case the kinetic energy field \(\varepsilon\) easily related with the particle concentration \(n\), magnetization \(M^{\alpha}\) and velocity field \(v^{\beta}\), so we have \(\varepsilon = mnv^2/2\). For the many-particle system we have that \(p^{\alpha\beta}, J^{\alpha\beta}\) and \(\varepsilon\) are independent material fields additional to \(n, v^{\alpha}\), and \(M^{\alpha}\). In the many-particle system \(J^{\alpha\beta}\) and \(\varepsilon\) are partly connected with \(n, v^{\alpha}\), and \(M^{\alpha}\). Let us make an example to describe the last statement. For the energy we have \(\varepsilon = mnv^2/2 + ne\), where the last term corresponds to internal energy caused by the thermal motion of particles \cite{29}. A set of QHD equations including the energy evolution equation and the non-zero thermal pressure in the Euler equation can be derived from a quantum kinetic equation, but derivation of the quantum kinetic equation needs some additional assumption to construct a distribution function \cite{47,48}. Thus, to make more detailed study of magnetic moment dynamics we are going to present the straightforward derivation of equation for the magnetic moment current (or the spin current) \(J^{\alpha\beta}\) evolution from the many-particle Schrodinger and study its influence on the spin wave dispersion. During several last decades a lot of different ways of the quantum kinetic equation derivation were suggested, but there are a lot of open questions in this field. Derivation of a kinetic equation via the Wigner distribution function is the most popular and actively used in recent publications. Moreover we keep developing the method of many-particle quantum hydrodynamics which is more direct way of derivation of equations for collective quantum dynamics \cite{34,49,50}.

Waves in systems of neutral and charged spinning particles have been considered by means of many-particle quantum hydrodynamics with no account of the spin-current equation. It has been assumed that the spin-current \(J^{\alpha\beta}\) appearing in the magnetic moment evolution equation can be approximately considered as \(J^{\alpha\beta} = M^{\alpha}v^{\beta}\). In this paper studying waves in the magnetized dielectrics we consider two kind of equilibrium states. One of them corresponds to the case when the equilibrium spin current \(J^{\alpha\beta}\) equals to zero, in second case we consider a non zero equilibrium spin current \(J^{\alpha\beta}\), but we suppose that the equilibrium velocity field equals to zero. Such structure might be realized by means two currents (flows of neutral particles) directed in opposite directions and having opposite equilibrium spin.

We also need to accent the fact that the many-particle QHD method has been used for the different physical sys-
tems, thus the sets of the QHD equations have been obtained for graphene [44], the neutral ultracold quantum gases [51], the Bose-Einstein condensate of excitons in graphene [52], along with the physics of plasma described above.

Presented here results are also important for the physics of magnetized ultracold quantum gases. Used where models are equivalent to the first three equations of the QHD, they are the continuity equation, the Euler equation, and the Bloch equation [53], [54].

This paper is organized as follows. In Sec. II we present and describe the set of the QHD equations derives in the paper. In Sec. III dispersion of the spin waves is considered, a contribution of the spin-current in the spin wave properties is studied. In Sec. IV brief summary of obtained results are presented.

II. THE MODEL

The many-particle QHD equations are derived from the microscopic many-particle Schrodinger equation

\[ i\hbar \partial_t \Psi(R, t) = \hat{H} \Psi(R, t), \]

where \( \Psi(R, t) \) is the wave function of \( N \) interacting particles. \( \Psi(R, t) \) depends on coordinate of all particles. We present it shortly by means of \( i \) th particle. The structure of the QHD equations depends on the explicit form of the Hamiltonian of considered system of particles. We do not described here the method of derivation of the QHD equations, a lot of paper are dedicated to this topic [29]-[32]. However, to be certain we present the Hamiltonian

\[
\hat{H} = \sum_p \left( \frac{1}{2m_p} \mathbf{p}_p^2 + e_p \mathbf{A}_{p,ext} - \gamma_p \sigma^\alpha_p \mathbf{D}^\alpha_p \right) + \frac{1}{2} \sum_{p,n\neq p} (e_p e_n G_{pn} - \gamma_p \gamma_n G_{pn}^{\alpha\beta} \sigma^\alpha_p \sigma^\beta_n) \tag{1}
\]

used for derivation of equations presented below, where \( \mathbf{D}^\alpha_p = -i\hbar \partial^\alpha_p - e_p \mathbf{A}_{p,ext}/c, \varphi_{p,ext}, A_{p,ext}^\alpha \) are the potentials of the external electromagnetic field, \( \partial^\alpha_p = \nabla^\alpha_p \) is the derivatives on space variables, and \( G_{pn} = 1/r_{pn} \) is the Green functions of the Coulomb interaction, \( C_{pn}^{\alpha\beta} = 4\pi \delta^{\alpha\beta} \delta(r_{pn}) + \partial^\beta_p \partial^\alpha_p (1/r_{pn}) \) is the Green function of spin-spin interaction, \( \gamma_p \) is the gyromagnetic ratio, \( \sigma^\alpha_p \) is the Pauli matrix, a commutation relations for them is

\[
[\sigma^\alpha_p, \sigma^\beta_q] = 2i \delta_{pq} \epsilon^{\alpha\beta\gamma} \sigma^\gamma_q,
\]

\( e_n, m_n \) are the charge and the mass of particle, \( \hbar \) is the Planck constant and \( c \) is the speed of light. For electrons \( \gamma_p \) reads \( \gamma_p = e_p \hbar/(2m_p c) \), \( e_p = -|e| \).

This Hamiltonian (1) consists of two parts. The first of them is presented by the first three terms, which describe motion of independent particle in an external electromagnetic field. The first term is the kinetic energy, which contains vector potential via covariant derivative \( \mathbf{D}_\mu \). Therefore, it contains action of an external magnetic field and a rotational electric field on particle charge. The second term in formula (1) describes interaction of particle charge with an external potential electric field. And the third term presents the action of an external magnetic field on the magnetic moments. The second group of terms consists of the two last terms. They describe the interparticle interaction. In this paper we consider the Coulomb and the spin-spin interactions presented by the fourth and fifth terms correspondingly.

Using explicit form of the Hamiltonian, we obtain the chain of equations, we truncate the chain of the QHD equations including the four equations only. These are the evolution equations for the particle concentration \( n \), the velocity field \( v^\mu \), the density of magnetic moment or spin \( M^\alpha \), and the spin-current \( j^{\alpha\beta} \).

The first step in derivation of the many-particle QHD equations is the definition of particle concentration. Which is the first collective quantum mechanical observable in our model. The particle concentration is the quantum mechanical average of the microscopic concentration

\[
n(r, t) = \int \Psi^+(R, t) \sum_p \delta(r - r_p) \Psi(R, t) dR, \tag{2}
\]

where we integrate over the 3N dimensional configurational space, and \( dR = \prod_{p=1}^N dr_p \). The formula (2) is more than the first collective quantum mechanical observable. Using of the particle concentration operator \( \hat{n} = \sum_p \delta(r - r_p) \) gives the projector of the 3N dimensional configurational space in the three dimensional physical space. Waves propagation, the charge-current flow, the spin-current flow happen in the three dimensional physical space. Consequently it is worthwhile to have a model, which explicitly describes the dynamic of quantum many-particle system in the physical space. The QHD is an example of such model.

We now differentiate the particle concentration with respect to time and find the continuity equation, where the particles current \( \mathbf{j} = n \mathbf{v} \) emerges.

Hence, the first equation of the QHD set of equations is the continuity equation

\[
\partial_t n + \nabla (n \mathbf{v}) = 0. \tag{3}
\]

At derivation of the continuity equation (3) the explicit form of the particles current appears as

\[
\mathbf{j} = \int \sum_p \delta(r - r_p) \left( \frac{1}{2m_p} \left( \Psi^+(R, t) \mathbf{D}_\mu \Psi(R, t) \right) + h.c. \right) dR, \tag{4}
\]

where h.c. means the hermitian conjugation.

Differentiating the function of current with respect to time, we obtain the momentum balance equations, this equation is an analog of the Euler equation

\[
\frac{m}{n} (\partial_t + \mathbf{v} \nabla) v^\alpha + \partial_\beta p^{\alpha\beta}
\]
\[ -\frac{\hbar^2}{4m} \partial^\alpha \Delta n + \frac{\hbar^2}{4m} \partial^\beta \left( \frac{\partial^\alpha n \cdot \partial^\beta n}{n} \right) = enE^\alpha + \frac{e}{c} \epsilon^\alpha_\beta_\gamma n \nu^\beta B^\gamma + M^\beta \nabla^\alpha B^\beta, \]  

(5)

where \( E \) and \( B \) are the electric and magnetic fields, \( M \) is the density of magnetic moments, \( \epsilon^\alpha_\beta_\gamma \) is the antisymmetric symbol (the Levi-Civita symbol), \( p^\alpha_\beta \) is the kinetic pressure tensor. The momentum balance equation (5) has usual form, we see that evolution of the velocity field caused by momentum current on thermal velocities \( p^\alpha_\beta \), the quantum Bohm potential specific for quantum kinematics (two terms in the left-hand side of equation (5), which proportional to \( \hbar^2 \)), and interaction, which is presented in the right-hand side of the momentum balance equation (5). We derive this equation for charged spinning particles, thus the force field contains the density of the Lorentz force describing action of electromagnetic field on charges presented by two first terms and force acting on the magnetic moment density from the magnetic field presented by the last term. We present the force field in the self-consistent field approximation. General form of the force field appearing in the Euler equation and introducing of the self-consistent field approximation are presented in Appendix A.

The second-fifth terms in the left-hand side of equation (5) appear due to representation of the momentum flux \( \Pi^\alpha_\beta \). At derivation of the Euler equation the momentum flux \( \Pi^\alpha_\beta \) emerges in the following explicit form

\[ \Pi^\alpha_\beta = \int \sum_p \delta(\mathbf{r} - \mathbf{r}_p) \frac{1}{2m_p} \times \]

\[ \times \left( \Psi^\dagger(R, t) \hat{D}_p^\beta \hat{D}_p^\alpha \Psi(R, t) + h.c. \right) dR. \]  

(6)

To represent the momentum flux via the hydrodynamic variables we need to introduce the velocity of a quantum particle \( \nu^\alpha_i(R, t) \). It appears via the phase \( S(R, t) \) of the many-particle wave function \( \Psi(R, t) = a(R, t) e^{iS(R, t)} \).

The velocity of i th particle is \( \nu^\alpha_i(R, t) = \frac{\hbar}{m} \nabla_i S(R, t) \). We can also introduce the thermal velocity of i th particle \( \nu^\alpha_i = \nu^\alpha_i(R, t) - \nu^\alpha(R, t) \) as difference of the velocity of i th particle and the thermal center of mass velocity (the velocity field) \( \mathbf{v}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t)/n(\mathbf{r}, t) \) (for details see Refs. [29], [32]).

The definition of magnetization

\[ M^\alpha(\mathbf{r}, t) = \int \sum_p \delta(\mathbf{r} - \mathbf{r}_p) \Psi^\dagger(R, t) \hat{D}_p^\alpha \Psi(R, t) dR, \]  

(7)

appears at derivation of the Euler equation (5).

The equation of evolution of the magnetic moments

\[ \partial_t M^\alpha + \nabla^\beta J^{\alpha\beta} = \frac{2\gamma}{\hbar} \epsilon^{\alpha\beta\gamma} M^\beta B^\gamma \]  

(8)

is derived at differentiating of the magnetization with respect to time and using of the Schrodinger equation for the time derivatives of the wave function. This equation is a generalization of the Bloch equation. From equation (8) we see that evolution of magnetic moment density caused by both the spin current \( J^{\alpha\beta} \) and interaction of the magnetic moments with the magnetic field. Charge of particles gives no interference in dynamics of the magnetic moment density.

The explicit form of spin-current in the many-particle system is

\[ J^{\alpha\beta} = \int \sum_p \delta(\mathbf{r} - \mathbf{r}_p) \frac{1}{2m_p} \times \]

\[ \times \left( \Psi^\dagger(R, t) \hat{D}_p^\beta \hat{D}_p^\alpha \Psi(R, t) + h.c. \right) dR. \]  

(9)

Next equation is the equation of spin-current evolution. The spin-current appears in the Bloch equation, and if we include the spin-orbit and the spin-current interaction we get that the spin-current gives contribution in the force field in the Euler equation (5), for example see Ref. [34]. The spin-current evolution equation for system of neutral particles is

\[ \partial_t J^{\alpha\beta} + \partial^\gamma (J^{\alpha\beta} \gamma) \]

\[ = \frac{\gamma^2}{m} n \partial^\beta B^\alpha - \frac{2\gamma}{\hbar} \epsilon^{\alpha\beta\gamma} B^\gamma J^{\delta\gamma}. \]  

(10)

for flux of the spin-current \( \delta^{\alpha\beta\gamma} \) which emerges in the second term in the left-hand side of equation (10). Its explicit form is

\[ J^{\alpha\beta\gamma} = \int \sum_p \delta(\mathbf{r} - \mathbf{r}_p) \frac{1}{4m^2_p} \times \]

\[ \times \left( \Psi^\dagger(R, t) \hat{D}_p^\gamma \hat{D}_p^\alpha \hat{D}_p^\beta \Psi(R, t) + h.c. \right) dR. \]  

(11)

Definition of the flux of spin-current contains two operators of the long derivative \( \hat{D}_p^\alpha \) when the spin-current contains one operator of the long derivative. Since \( J^{\alpha\beta\gamma} \) is the flux of the spin-current we use an approximate formula \( J^{\alpha\beta\gamma} = J^{\alpha\beta} \gamma \). In general case \( J^{\alpha\beta\gamma} \) has more complex structure and contains additional contribution of both the thermal motion and quantum kinematics as the quantum Bohm potential. The last one shows in the form of a term analogous to the quantum Bohm potential. We do not consider these contributions and pay attention to the spin current evolution caused by inter-particle interaction. We should pay special attention to equation (10) because all this paper is dedicated to consideration of the influence of the spin-current evolution on dynamics of particles system. Equation (10) is presented for the chargeless spinning particles, and this form will be used in the paper. However, we now present it for the charged spinning particles

\[ \partial_t J^{\alpha\beta} + \partial^\gamma (J^{\alpha\beta} \gamma) = \frac{e}{m} M^\alpha E^\beta \]
we have no dependency on wave vector, consequently the field equals to zero. We pose that an equilibrium spin-current equals to zero, and in two different equilibrium states. In the first case we consider small amplitude excitations of the quantum Bohm potential giving dispersion of the de Broglie wave. Moreover the evolution of spin-current leads to the Fermi pressure. The second term is the contribution of the quantum magnetic susceptibility $\kappa_a$ and the magnetic permeability $\nu_a = 1 + 4\pi\kappa_a$. In the case $\kappa_a \ll 1$ we have $\nu_a \simeq \kappa_a$.

### III. DISPERSION EQUATION

We consider the collective eigen-waves in a system of the neutral spinning particles being in an external uniform magnetic field. Hence, we have deal with the paramagnetic and diamagnetic dielectrics. There are the two fundamental collective excitations in such physical systems, they are the sound waves and the spin waves. Following the QHD description of the three dimensional magnetized dielectrics one can show that there is one type of the spin waves in such systems. These waves have a constant eigen-frequency $\omega = 2\gamma B_0/\hbar$, which is the cyclotron frequency, where $B_0$ is an external magnetic field. Here we have no dependency on wave vector, consequently the group velocity $\partial\omega/\partial k$ of these waves equal to zero.

We are interested in an interference of the spin-current evolution on the dispersion properties of the medium. To find the dispersion dependence of eigen-waves in the described systems we consider small amplitude excitations around an equilibrium state of the medium. We consider two different equilibrium states. In the first case we suppose that an equilibrium spin-current equals to zero, and in the second case we consider a medium with an equilibrium spin-current under condition that an equilibrium velocity field equals to zero.

For getting of solution we consider hydrodynamic variables as the sum of an equilibrium part and a small perturbation

$$n = n_0 + \delta n, \quad E = 0 + E,$$

$$B = B_0 e_z + \delta B, \quad v = 0 + \delta v,$$

$$M^\alpha = M_0^\alpha + \delta M^\alpha, \quad M_0^\alpha = \chi B_0^\alpha, \quad J^{\alpha\beta} = J_0^{\alpha\beta} + \delta J^{\alpha\beta},$$

$$P^{\alpha\beta} = p^{\alpha\beta} + \delta p = m v^2 \delta n_a,$$  \hspace{1cm} (14)

where $\delta^{\alpha\beta}$ is the Kronecker symbol, $v_F$ is the Fermi velocity, $\chi_a = \kappa_a/\nu_a$ is the ratio between the equilibrium magnetic susceptibility $\kappa_a$ and the magnetic permeability $\nu_a = 1 + 4\pi\kappa_a$. Substituting relations (13) in the set of equations (3), 5, 8 and (13) and neglecting by the nonlinear terms, we obtain a system of linear homogeneous equations in the partial derivatives with constant coefficients. Passing to the following representation for the small perturbations $\delta f$

$$\delta f = f(\omega, k)e^{ikx}$$

yields a homogeneous system of algebraic equations.

Even when we consider an equilibrium spin-current the linear set of QHD equations splits on four independent sets. One of them contains $\delta v_x, \delta M_z, J_{xx}$, the second one contains $\delta M_z, \delta M_y, \delta J_{xx}, \delta J_{yy}$, the third set includes $\delta J_{xy}, \delta J_{yy}$, and fourth includes $\delta J_{xx}, \delta J_{yz}$. Two last sets have the same solution, which is the cyclotron frequency $\omega = 2\gamma B_0/\hbar$. Hence, we have to consider solutions of the first and second sets.

#### A. The first group of dispersion branches

The first set give the following dispersion equation

$$\omega^4 + \omega^2 \left(4\pi\gamma^2 \frac{n_0}{m} - \nu^2\right)k^2$$

$$+ \frac{4\pi\hbar^3 M_0}{mm_0} J_0^{xx} \omega - \frac{4\pi\gamma n_0^2}{m} \nu^2 k^2 = 0, \quad (15)$$

where

$$\nu^2 = v_F^2 + \frac{\hbar^2 k^2}{4m^2}, \quad (16)$$

and

$$v_F = (3\pi^2 n_0)^{1/3} \frac{\hbar}{m}. \quad (17)$$

We can see that a non-zero equilibrium spin-current leads to existence of the additional term in the dispersion equation.

In the absence of an equilibrium spin-current we have two solutions of the equation (15)

$$\omega^2 = -4\pi\gamma n_0 k^2/m, \quad (18)$$

and

$$\omega^2 = v^2 k^2, \quad (19)$$

where solution (19) is the well-known sound wave. Dispersion of the sound wave consists of two parts (16). The first of them is the usual linear term, which appears due to the Fermi pressure. The second term is the contribution of the quantum Bohm potential giving dispersion of the de Broglie wave. Moreover the evolution of spin-current leads
to existence of the second solution \( \text{(18)} \). This solution can be rewritten as

\[
\omega = \pm i \sqrt{\frac{4 \pi n_0}{m}} \gamma k. \quad \text{(20)}
\]

It shows faster damping and gives no wave behavior.

The sound wave solution \( \text{(19)} \) and spin wave \( \omega = 2 \gamma B_0 / \hbar \) appear in the simpler model without account of the spin-current evolution. In this case a system of neutral spinning particles can be described by the continuity, Euler, and magnetic moment evolution equations, where we can put \( J^{\alpha \beta} = M^{\alpha \beta} \) to close the set of equations. It has been mostly used for systems of charged spinning particles \[28\], \[34\], \[37\].

Presence of the equilibrium spin-current in the equation \( \text{(15)} \) makes it rather complicate. So, we are going to solve it numerically. It seems reasonable to choose \( \Omega \equiv \omega / \lambda = \sqrt{m / (\pi n_0)} \omega / (2 \gamma k) \) as dimensionless frequency, where we introduced

\[
\lambda^2 = \frac{4 \pi \gamma^2 n_0 k^2}{m}. \quad \text{(21)}
\]

In this case we get equation \( \text{(15)} \) in the form of

\[
\Omega^4 + (1 - \alpha) \Omega^2 + \beta \Omega - \alpha = 0, \quad \text{(22)}
\]

where \( \alpha = m v^2 / (4 \pi n_0 \gamma^2) \) is the parameter describing contribution of the quantum Bohm potential, \( \beta = k M_0 J^{xz}_0 / (\gamma^2 n_0^2) \) is the parameter describing contribution of the equilibrium spin-current. On figures we present a limit case: de-Broglie regime when \( \alpha \simeq \alpha \hbar = \hbar^2 k^2 / (16 \pi n_0 \gamma^2) \). The both parameters \( \alpha \) and \( \beta \) depend on module of the wave vector \( k \). Consequently equation \( \text{(22)} \) allows to get \( \Omega(k) \) dependence. Equation \( \text{(22)} \) gives two solutions. One of them is stable solution, which dispersion presented on Fig. \( \text{(1)} \). The second solution of equation \( \text{(22)} \) shows an instability, which exists due to the existence of the equilibrium spin-current \( J^{xz}_0 \). This solution is presented on Fig. \( \text{(2)} \).

Solutions \( \text{(18)} \) and \( \text{(19)} \) can be briefly written in terms of reduced frequency \( \Omega \) and \( \alpha \), so we have \( \Omega_1^2 = -1 \) and \( \Omega_2^2 = \alpha \).

Let us consider approximate solutions of equation \( \text{(22)} \) at nonzero equilibrium spin-current under assumption that \( \beta \) gives small contribution in dispersion of waves \( \text{(19)} \) and \( \text{(20)} \). Thus solutions of equation \( \text{(22)} \) emerge as

\[
\omega = v k + \zeta_1 \quad \text{(23)}
\]

and

\[
\omega = \pm i \sqrt{\frac{4 \pi n_0}{m}} \gamma k + \zeta_2, \quad \text{(24)}
\]

where

\[
\zeta_i = \frac{\beta \omega_i}{2 \omega_i (2 (\omega_i / \lambda)^2 + 1 - \alpha) + \lambda \beta} \approx \frac{\beta}{2 (2 (\omega_i / \lambda)^2 + 1 - \alpha)} \quad \text{(25)}
\]

\[
\omega_i \quad \text{is used for short representation of solutions \( \text{(19)} \) and \( \text{(20)} \). Including } \alpha \lambda^2 = v k \text{ we come to the following dispersion relation for the sound wave}
\]

\[
\omega = v k + \frac{\beta \lambda^2}{2 (v^2 k^2 + \lambda^2)} \equiv U_1^* k, \quad \text{(26)}
\]

where we introduced a modified sound velocity \( U_1^* \), which slightly depends on the wave vector \( k \) via the quantum Bohm potential.

\[
U_1^* = v + \frac{2 \pi \gamma^2 n_0 (\beta / k)}{mv^2 + 4 \pi \gamma^2 n_0}. \quad \text{(27)}
\]

Since \( \beta > 0 \) and \( \beta \sim k \) we see that \( U_1^* > v \).
Contribution of the spin-current in unstable branch \((24)\) in considering limit
\[
\omega = \pm i \lambda - \frac{\beta \lambda^2}{2(\nu^2 k^2 + \lambda^2)}
\]
\[
= \pm i \sqrt{\frac{4\pi n_0}{m} \gamma k - \frac{2\nu^2 n_0 \beta}{m\nu^2 + 4\pi^2 n_0}} \tag{28}
\]
appears to be real and negative. We see that real part of solution \((28)\) is almost a linear function of the wave vector \(k\), but it contains small additional dependence on the wave vector via \(\nu\) containing contribution of the quantum Bohm potential. This results differs from previously considered case depicted on Fig.2. Formula \((28)\) is obtained in the limit of small contribution of the equilibrium spin-current. When Fig.2 is obtained for a finite value of the equilibrium spin-current, so it reveals an interesting dependence of real and imaginary parts of the frequency on the wave vector discussed above.

B. Wave dispersion for the second group of variables

The second set of equations containing evolution of \(\delta M_x\), \(\delta M_y\), \(\delta J_{xx}\), \(\delta J_{yx}\) gives the following dispersion equation
\[
\omega^2 + 8\pi \frac{\Omega_\gamma}{\hbar \omega} \gamma k J_{xx}^z
+ 4\pi k \omega^2 (1 + \frac{\Omega_\gamma}{\hbar \omega}) \rho_{k n_0} \gamma h + \Omega_\gamma \gamma k \rho_{J_{xx}^z}
+ 2\Omega_\gamma \gamma k \rho_{J_{xx}^z}
- \frac{\Omega_\gamma^2}{\hbar \omega} \rho_{\Omega_\gamma^2} (1 - 4\pi \xi) = 0.
\]
\[
\Omega_\gamma = \frac{2\nu B_0}{k} \tag{29}
\]
where \(\Omega_\gamma = \frac{2\nu B_0}{k}\) and \(\xi = M_0/B_0\). For the ferromagnetic samples \(\xi\) is larger than one \(\xi \gg 1\), while \(\xi\) is rather small \(\xi \ll 1\) for the para- and the dia-magnetics.

In the absence of the equilibrium spin-current this equation is simplified to
\[
\omega^4 + \left(\lambda^2 - 2\Omega_\gamma^2 (1 - 2\pi \xi)\right) \omega^2 + \Omega_\gamma^2 \left(\lambda^2 + 2\Omega_\gamma^2 (1 - 4\pi \xi)\right) = 0.
\]
\[
\omega^4 + \left(\lambda^2 - 2\Omega_\gamma^2 (1 - 2\pi \xi)\right) \omega^2 + \Omega_\gamma^2 \left(\lambda^2 + 2\Omega_\gamma^2 (1 - 4\pi \xi)\right) = 0.
\]
\[
\omega^2 = \Omega_\gamma^2 \left(1 - 2\pi \xi\right) - \frac{1}{2} \lambda^2 \tag{30}
\]
Solving this equation we get
\[
\omega^2 = \Omega_\gamma^2 \left(1 - 2\pi \xi\right) - \frac{1}{2} \lambda^2
\]
\[
\pm \sqrt{\frac{1}{4} \lambda^4 - 2\lambda^2 \Omega_\gamma^2 (1 - \pi \xi) + 16\pi^2 \xi^2 \Omega_\gamma^4}, \tag{31}
\]
These are new solutions. Their occurrence is caused by the account of the spin-current evolution, so we called them spin-current waves.

In the small \(k\) limit we come to
\[
\omega^2 = \Omega_\gamma^2 \left(1 + 2\pi \xi\right) + \lambda^2 \left(\frac{1}{4} - \frac{1}{4} \xi + \frac{1}{4} \xi\right) \tag{32}
\]
In the paramagnetic limit formula \((30)\) gives
\[
\omega^2 = \Omega_\gamma^2 - \frac{1}{2} \lambda^2 \pm \lambda \sqrt{\frac{1}{4} \lambda^2 - 2\Omega_\gamma^2} \tag{33}
\]
At large \(k\) at small magnetic field \((\lambda^2 \gg \Omega_\gamma^2)\) only one solution exists, which corresponds to the sign plus in front of the square root in formula \((31)\). This solution appears as
\[
\omega_{\lambda, +} = \Omega_\gamma \sqrt{16\pi^2 \xi^2 \Omega_\gamma^2 - 1}, \tag{34}
\]
where \(\frac{\Omega_\gamma^2}{\lambda^2} \gg 1\) we have that this solution exists at \(\xi > \frac{\lambda^2}{4\pi \Omega_\gamma^2}\) \(\gg 1\). It corresponds to large magnetization \(M_0\).

Equation \((29)\) can be rewritten as an equation of fifth degree. One of its solution is \(\omega = 0\). So we have equation of fourth degree
\[
\omega^4 + \left(\lambda^2 - 2\Omega_\gamma^2 (1 - 2\pi \xi)\right) \omega^2 + \vartheta \omega + \Omega_\gamma^2 \left(2\pi \xi + \lambda^2\right) = 0, \tag{35}
\]
where
\[
\vartheta = 16\pi^2 \gamma \Omega_\gamma J_{xx}^z k / h. \tag{36}
\]
This equation \((35)\) differs from \((30)\) by one term only. It is the third term \(\vartheta \omega\). Thus we have changing of solutions \((31)\) caused by the equilibrium spin-current \(\vartheta \sim J_{xx}^z\). Solutions of equation \((35)\) we calculated approximately under assumption that the equilibrium spin-current gives small contribution in dispersion dependence. Designating solutions of equation \((30)\) presented by formula \((31)\) as \(\varpi\). Then solutions of equation \((35)\) appear as
\[
\omega = \varpi + \frac{\vartheta \varpi}{2 \varpi^2 + \lambda^2 - 2\Omega_\gamma^2 (1 - 2\pi \xi) + \vartheta}
\]
\[
\simeq \varpi + \frac{\vartheta}{2 (2 \varpi^2 + \lambda^2 - 2\Omega_\gamma^2 (1 - 2\pi \xi))}. \tag{37}
\]
Using explicit form of \(\varpi\) in the second term of formula \((37)\) we come to the following formula
\[
\omega = \varpi \pm \frac{\vartheta}{2 \sqrt{\lambda^4 - 8\lambda^2 \Omega_\gamma^2 (1 - \pi \xi) + 64\pi^2 \xi^2 \Omega_\gamma^4}}. \tag{38}
\]
Signs plus and minus in front of the second term of formula \((38)\) correspond to signs in formula \((31)\).
C. Discussion

Formulas (18), (19), (31) and the cyclotron frequency \( \omega = 2\gamma B_0\sqrt{\nu_0} \) appear as solutions of a dispersion equation obtained at account of the spin-current evolution in the absence of the equilibrium spin-current.

Solution (18) reveals two branches (20). One of them has an increasing amplitude. Another one has a decreasing amplitude. Since we have no source of the energy in the system we have no mechanism for the amplitude increasing. So, we conclude that the decreasing branch takes place in considering case. This decreasing solution gives no contribution in the spectrum as it reveals monotonous decreasing of the amplitude (non oscillating solution). Thus we have got left the two unchanged solutions: the sound wave (19) and the spin wave \( \omega = 2\gamma B_0/h \), which can be found with no account of the spin-current evolution. We have also found solutions (31), which present the two spin-current waves reaching the wave spectrum of magnetized dielectrics.

At this step we can conclude that the account of the spin-current evolution, without changing of conditions system being at, we obtained additional information about processes happen in the system. In considering case we have got the two additional wave branches (the spin-current waves) described by formula (31). Formula (32) reveals that at small \( k \) and large enough external magnetic field \( B_0 \) the cyclotron frequency \( \Omega_c = 2\gamma B_0/h \) gives main contribution in the dispersion of the spin-current waves.

Consideration of the spin-current as an independent physical variable gives us possibility to consider some conditions, which cannot be included in more simple model. One of these conditions is the existence of an equilibrium spin-current with the zero equilibrium velocity field \( \nu_0 = 0 \). Presence of an equilibrium spin-current leads to changes of solutions (18) and (19), and to complication of dispersion equation (30) (degree of this equation increases, so we get equation (29)). However it gives no change in the dispersion of spin waves with the cyclotron frequency \( \omega = 2\gamma B_0/h \).

Presence of an equilibrium spin-current \( J_0^{sx} \) leads to change of the solutions (19) and (20). Real part of solution (20) appears due to \( J_0^{zx} \). In the de-Brolie regime it is pictured by lowest curve on Fig. (2). It reveals as a curve with two linear areas, one at small \( k \), and the second one at \( k \geq 4 \cdot 10^5 \text{ cm}^{-1} \). They connect smoothly around \( k = 3 \cdot 10^5 \text{ cm}^{-1} \). Thus, we can assume \( \Omega = \nu k \), with different \( \nu_i \) for each area. Consequently, we have \( \text{Re} \omega = \sqrt{m\nu} 2\gamma \nu_i k^2 \). Imaginary part of solution \( \text{Im} \omega \) is pictured by upper curve on Fig.(2). Its form is similar to the form of the real part of the frequency \( \text{Re} \omega \), but it has larger value. Limit of small contribution of the equilibrium spin-current allows to obtain some analytical solution (28). In this approximation the equilibrium spin-current \( J_0^{sx} \) gives rise to appearance of the negative real part of the frequency. It gives no changes in the imaginary part of the solution.

Let us discuss the sound wave. In the absence of an equilibrium spin-current it is described by formula (19). In the presence of an equilibrium spin-current we numerically consider a limit case: the de-Broglie regime. In the de-Broglie regime we find that the equilibrium spin-current does not change form of the dispersion curve (formula (19) gives us \( \omega = \frac{\hbar k^2}{2m} \) in the de-Broglie regime). Thus we see that \( \Omega \) linearly depends on the wave vector \( k \), and \( \omega \sim k^2 \).

The approximation of small equilibrium spin-current lets to trace contribution of the equilibrium spin-current on the sound wave analytically. In the de-Broglie regime an addition to spectrum of the free quantum particles as

\[
\Delta \omega_s = \frac{8\pi mn_0\gamma^2 \beta}{\hbar^2 k^2 + 16\pi mn_0\gamma^2} \sim \frac{k}{k^2 + \chi^2},
\]

where \( \chi \) does not depend on the wave vector \( k \). One can see that the frequency shift is positive and decreases at the increasing of the wave vector. In the Fermi regime we find that the sound velocity increases on a constant value defined by formula (27).

Let us discuss now contribution of the equilibrium spin-current in the dispersion of the spin-current waves. Formula (33) shows that the spin-current wave having plus (minus) in front of the square root in formula (31) gets positive (negative) contribution of the equilibrium spin-current in the dispersion dependence. So we find an increasing (a decreasing) of the frequency. Including the fact that \( \vartheta \sim k \) and \( \lambda \sim k \) we have the following dependence of the last term in formula (33) (a frequency shift caused by the equilibrium spin-current) on the wave vector appears as

\[
\Delta \omega_s = \frac{8\sqrt{\pi} m \Omega_c, J_0^{sx}}{\hbar \sqrt{n_0}} \cdot \frac{\sqrt{k^4 + \tilde{a} k^2 + \bar{b}}}{\sqrt{k^4 - \tilde{a} k^2 + \bar{b}}},
\]

where \( \tilde{a} = 2m(1 - \pi \xi) \Omega_c^2/(n_0 \gamma^2) \) and \( \bar{b} = 4m^2 \xi^2 \Omega_c^4/(n_0^2 \gamma^4) \) are positive constants. At \( k \rightarrow 0 \) we obtain \( \Delta \omega_s \rightarrow 0 \). \( \Delta \omega_s \) increases with increasing of the wave vector. However \( \Delta \omega_s \) reaches its maximum at an intermediate wave vector \( k_0 = \frac{\sqrt{\pi mn_0 \Omega_c}}{\gamma \sqrt{n_0}} \). This maximum value of the frequency shift is

\[
\Delta \omega_s(k_0) = \frac{8\sqrt{\pi} m \Omega_c, J_0^{sx} \sqrt{n_0}}{\hbar \sqrt{n_0}} \frac{1}{\sqrt{2\sqrt{q - \tilde{a}}}}.
\]

At large \( k \) the frequency shift \( \Delta \omega_s \) decreases as \( 1/k \).

IV. CONCLUSION

To get influence of the spin-current on dynamics of magnetized dielectrics we have derived equation of the spin-current evolution as a part of the set of the QHD equations. In the result we have set of four equations: the continuity equation (particle number evolution equation), the Euler
equation (momentum balance equation), the Bloch equation (magnetic moment balance equation), and equation of the spin-current evolution. These equations are used in the self-consistent field approximation.

With no account of the spin-current evolution equation we find two wave solutions: the sound wave and one spin wave solution. Including the spin-current evolution equation leads to account of new solutions. We found three new wave solutions. Two of them have frequencies near the cyclotron frequency. These solutions make spectrum of spin waves richer. It appears as a splitting of one spin wave branch on three branches. The third solution has negative square of the frequency and reveal monotonic damping of perturbation amplitude. These solutions emerge when the equilibrium spin-current equals to zero. Account of an equilibrium spin-current does not give new solution, but a change of wave dispersion was obtained.

**APPENDIX A: GENERAL FORM OF THE FORCE FIELD AND THE SELF-CONSISTENT FIELD APPROXIMATION**

In the Euler equation (5) the force field

\[ F^\alpha = e n E^\alpha + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} n v^\beta B^\gamma + M^\beta \nabla^\alpha B^\beta \]  \tag{39}

is presented in the self-consistent field approximation. Here we are going to present general form of this force field and explain how we got formula (39) from the general formula.

Force field consists of two parts

\[ F^\alpha = F^\alpha_{ext} + F^\alpha_{int}. \]  \tag{40}

The first of them is the force of the particle interaction with an external field

\[ F^\alpha_{ext} = e n E^\alpha_{ext} + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} n v^\beta B^\gamma_{ext} + M^\beta \nabla^\alpha B^\beta_{ext}, \]  \tag{41}

and the second part is the inter-particle interactions

\[ F^\alpha_{int} = -e^2 \int (\nabla^\alpha G(\mathbf{r} - \mathbf{r}')) n_2(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}', \]
\[ + \int (\nabla^\alpha G^{\beta\gamma}(\mathbf{r} - \mathbf{r}')) M_2^{\beta\gamma}(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}', \]  \tag{42}

where

\[ n_2(\mathbf{r}, \mathbf{r}', t) = \sum_{p,n\neq p} \delta(\mathbf{r} - \mathbf{r}_p) \delta(\mathbf{r} - \mathbf{r}'_n) \Psi^*(R, t) \Psi(R, t) dR \]

\[ \times \mu_B^2 \Psi^*(R, t) \sigma_p^\alpha \sigma_n^\beta \Psi(R, t) dR \]  \tag{43}

is the two-particle concentration, and

\[ M_2^{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = \int \sum_{p,n\neq p} \delta(\mathbf{r} - \mathbf{r}_p) \delta(\mathbf{r} - \mathbf{r}'_n) \times \mu_B^2 \Psi^*(R, t) \sigma_p^\alpha \sigma_n^\beta \Psi(R, t) dR \]  \tag{44}

is the two-particle magnetization.

It has been shown that a two-particle function \( f_2(\mathbf{r}, \mathbf{r}', t) \) (see formulas (43) and (44)) appears as a sum of two terms

\[ f_2(\mathbf{r}, \mathbf{r}', t) = f(\mathbf{r}, t) f(\mathbf{r}', t) + g_2(\mathbf{r}, \mathbf{r}', t). \]  

The first of the two terms is the product of corresponding one-particle functions. It corresponds to the self-consistent field approach suitable for the long-range interaction. The second term gives contribution of quantum correlations, particularly the exchange correlation.

Thus, in the self-consistent field approximation we have that the two-particle concentration represents as the product of the concentrations in points \( \mathbf{r} \) and \( \mathbf{r}' \):

\[ n_2(\mathbf{r}, \mathbf{r}', t) = n(\mathbf{r}, t) n(\mathbf{r}', t); \]

and the two-particle magnetization gives us the product of the one-particle magnetization \( M_2^{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = M^\alpha(\mathbf{r}, t) M^\beta(\mathbf{r}', t) \). Putting these representations in formula (42) we can introduce electric field caused by the electric charges and the magnetic field caused by magnetic moments. These fields emerge as

\[ E^\alpha = -e \nabla^\alpha \int G(\mathbf{r} - \mathbf{r}') n(\mathbf{r}', t) d\mathbf{r}', \]  \tag{45}

and

\[ B^\alpha = \int G^{\alpha\beta}(\mathbf{r} - \mathbf{r}') M^\beta(\mathbf{r}', t) d\mathbf{r}'. \]  \tag{46}

They satisfy the Maxwell equations (13). Using these fields we can get to the force field (39).

---

[1] S. Takahashi, S. Maekawa, J. Magnetism and Magnetic Materials 272, 1423 (2004).
[2] I. Zutic, J. Fabian, S. Das Sarma, Rev. Mod. Phys. 76, 323 (2004).
[3] J. Sinova and I. Zutic, Nature Materials 11, 368 (2012).
[4] K. Uchida, H. Adachi, T. An, H. Nakayama, M. Toda, B. Hillebrands, S. Maekawa, and E. Saitoh, J. Appl. Phys. 111, 053903 (2012).
[5] F. Pezzoli, F. Bottegoni, D. Trivedi, F. Cicciacci, A. Giorgioni, P. Li, S. Cecchi, E. Grilli, Y. Song, M. Guzzi, H. Dery, and G. Isella, Phys. Rev. Lett. 108, 156603 (2012).
[6] Z. Rashidian, F. Kheirandish, Int. J. Theor. Phys. 51, 1989 (2012).
[7] R. T. Hammond, Appl. Phys. Lett. 100, 121112 (2012).
[8] J. Lee, R. Oszwaldowski, C. Gothigen, and I. Zutic, Phys. Rev. B 85, 045314 (2012).
[9] Y. Li, M. B. A. Jalil, and S. Ghee Tan, J. Appl. Phys. 111, 093717 (2012).
[10] Rong Zhang, Long Bai, Chen-Long Duan, Physica B 407, 2372 (2012).
[11] M. J. Ma, M. B. A. Jalil, and Z. B. Siu, J. Appl. Phys. 111, 07C326 (2012).
[12] N. T. Bagraev, O. N. Guimbitskaya, L. E. Klyachkin, A. A. Kudryavtsev, A. M. Malyarenko, V. V. Romanov, A. I. Ryskin, I. A. Shelykh, A. S. Shcheulin, Physica C 893, 470 (2010).
[13] J. Flipse, F. L. Bakker, A. Slachter, F. K. Dejene and B. J. van Wees, Nature Nanotechnology 7, 166 (2012).
[14] D. Pesin and A. H. MacDonald, Nature Materials 11, 409 (2012).
[15] I. J. Vera-Marun, V. Ranjan and B. J. van Wees, Nature Physics 8, 313 (2012).
[16] Wei Han, K. M. Mc Creary, K. Pi, W. H. Wang, Yan Li, H. Wen, J. R. Chen, R. K. Kawakami, J. Magnetism and Magnetic Materials 324, 369 (2012).
[17] R. Jansen, Nature Materials 11, 400 (2012).
[18] Z. An, F. Q. Liu, Y. Lin, C. Liu, Sci. Rep. 2, 388 (2012).
[19] Q. F. Sun, X. C. Xie, Phys. Rev. B 72, 245305 (2005).
[20] Q. F. Sun, X. C. Xie, J. Wang, Phys. Rev. B 77, 035327 (2008).
[21] P. Jin, Y. Li, F. Zhang, J. Phys. A 39, 7115 (2006).
[22] J. Shi, P. Zhang, D. Xiao, Q. Niu, Phys. Rev. Lett. 96, 076604 (2006).
[23] Farkhad G. Aliev, Juan F. Sierra, Ahmad A. Awad, Gleb N. Kakazei, Dong-Soo Han, Sang-Koog Kim, Vitali Metlushko, Bojan Ilic, and Konstantin Y. Guslienko, Phys. Rev. B 79, 174433 (2009).
[24] A. A. Awad, K. Y. Guslienko, J. F. Sierra, G. N. Kakazei, V. Metlushko, and F. G. Aliev, Appl. Phys. Lett. 96, 012503 (2010).
[25] G. N. Kakazei, P. E. Wigen, K. Yu. Guslienko, V. Novosad, A. N. Slavin, V. O. Golub, N. A. Lesniki, and Y. Otani, Appl. Phys. Lett. 85, 443 (2004).
[26] Ki-Suk Lee, Konstantin Y. Guslienko, Jun-Young Lee, and Sang-Koog Kim, Phys. Rev. B 76, 174410 (2007).
[27] K. Y. Guslienko, R. H. Heredero, O. Chubykalo-Fesenko, Phys. Rev. B 79, 014402 (2010).
[28] P. A. Andreev, L.S. Kuz’menkov, Moscow University Physics Bulletin 62, N.5, 271 (2007).
[29] L. S. Kuz’menkov and S. G. Maksimov, Teor. i Mat. Fiz., 118 287 (1999) [Theoretical and Mathematical Physics 118 227 (1999)].
[30] P. A. Andreev, L. S. Kuz’menkov, Phys. Rev. A 78, 053624 (2008).
[31] M. I. Trukhanova, Int. J. Mod. Phys. B, 26, 1250004 (2012).
[32] P. A. Andreev, L. S. Kuz’menkov, M. I. Trukhanova, Phys. Rev. B 84, 245401 (2011).
[33] G. Brodin, M. Marklund, J. Zamanian, B. Ericsson and P. L. Mana, Phys. Rev. Lett. 101, 245002 (2008).
[34] P. A. Andreev, L. S. Kuz’menkov, Int. J. Mod. Phys. B 26 1250186 (2012).
[35] P. A. Andreev, L. S. Kuz’menkov, Physics of Atomic Nuclei 71, N.10, 1724 (2008).
[36] P. A. Andreev and L. S. Kuz’menkov, PIERS Proceedings, p.1047, March 20-23, Marrakesh, MOROCCO 2011.
[37] L. S. Kuz’menkov, S. G. Maksimov, and V. V. Fedoseev, Vestn. Mosk. Univ., Ser. 3: Fiz., Astron., No. 5, 3 (2000) (Moscow Univ. Phys. Bull., No. 5, 1 (2000)).
[38] M. Marklund and G. Brodin, Phys. Rev. Lett. 98, 025001 (2007).
[39] S. M. Mahajan and F. A. Asenjo, Phys. Rev. Lett. 107, 195003 (2011).
[40] F. Haas, G. Manfredi, M. Feix, Phys. Rev. E 62, 2763 (2000).
[41] P. K. Shukla, B. Eliasson, Rev. Mod. Phys. 83, 885 (2011).
[42] P. A. Andreev and L. S. Kuz’menkov, PIERS Proceedings, p. 1055, August 19-23, Moscow, Russia 2012.
[43] L. S. Kuz’menkov, S. G. Maksimov, and V. V. Fedoseev, Theor. Math. Fiz. 126 136 (2001) [Theoretical and Mathematical Physics, 126 110 (2001)].
[44] P. A. Andreev, arXiv: 1201.0779.
[45] E. Madelung, Z. Phys., 40, 332 (1926).
[46] T. Takabayasi, Progr. Theor. Phys., 14, 283 (1955).
[47] R. Balescu, Equilibrium and nonequilibrium statistical mechanics, (Wiley, New York, 1975).
[48] M. Hillery, R.F. O’Connell, M.O. Scully, E.P. Wigner, Physics Reports, 106, 121 (1984).
[49] P. A. Andreev, arXiv: 1212.0099.
[50] P. A. Andreev, F. A. Asenjo, and S. M. Mahajan, arXiv: 1304.5780.
[51] P. A. Andreev and L. S. Kuz’menkov, arXiv:1208.1000.
[52] P. A. Andreev, arXiv: 1201.6553.
[53] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. 67, 1822 (1998).
[54] T. L. Ho, Phys. Rev. Lett. 81, 742 (1998).