Sarben Sarkar

Implications of Space-Time foam for Entanglement Correlations of Neutral Kaons

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Abstract The role of $CPT$ invariance and consequences for bipartite entanglement of neutral (K) mesons are discussed. A relaxation of $CPT$ leads to a modification of the entanglement which is known as the $\omega$ effect. The relaxation of assumptions required to prove the $CPT$ theorem are examined within the context of models of space-time foam. It is shown that the evasion of the EPR type entanglement implied by $CPT$ (which is connected with spin statistics) is rather elusive. Relaxation of locality (through non-commutative geometry) or the introduction of decoherence by themselves do not lead to a destruction of the entanglement. So far we find only one model which is based on non-critical strings and D-particle capture and recoil that leads to a stochastic contribution to the space-time metric and consequent change in the neutral meson bipartite entanglement. The lack of an omega effect is demonstrated for a class of models based on thermal like baths which are generally considered as generic models of decoherence.

Keywords CPT · Decoherence · Entanglement · D-particles

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1 Introduction

It is established that the laws of Nature do not satisfy, $CP$, $PT$, and $CT$ symmetry where the operators $C$, $P$, and $T$ denote charge conjugation, parity and time reversal respectively [1]. Currently the successful theories are
based on local Lorentz invariant lagrangians, and it has been shown given the spin-statistics connection that \( CPT \) is a symmetry. The symmetry implies that the solution set of a theory is invariant under reversal of parity, time and interchange of particle and antiparticle. Consequently any violations of the consequences of \( CPT \) symmetry \[2\] would entail physics beyond the standard model of particle physics which is based on lagrangians. Typically the consequences of \( CPT \) that are considered are those of equal masses and lifetimes for particles and antiparticles. Recently it was noted \[3\] that when the \( CPT \) operator is not well defined there are implications for the symmetry structure of the initial entangled state of two neutral mesons in meson factories such as DAΦNE, the Frascati \( \phi \) factory. Indeed, if \( CPT \) can be defined as a quantum mechanical operator, then the decay of a (generic) meson with quantum numbers \( J^{PC} = 1^{--} \), leads to a pair state of neutral mesons \(|i\rangle\) having the form of the entangled state

\[
|i\rangle = \frac{1}{\sqrt{2}} \left( |M_0(k)\rangle |M_0(-k)\rangle - |M_0(-k)\rangle |M_0(k)\rangle \right) + \omega \sqrt{2} |\Delta(k)\rangle.
\]

This state has the Bose symmetry associated with particle-antiparticle indistinguishability \( CP = + \), where \( C \) is the charge conjugation and \( P \) is the permutation operation. If, however, \( CPT \) is not a good symmetry (i.e. ill-defined due to space-time foam), then \( M_0 \) and \( \bar{M}_0 \) may not be identified and the requirement of \( CP = + \) is relaxed \[3\]. Consequently, in a perturbative framework, the state of the meson pair can be parametrised to have the following form:

\[
|i\rangle = \frac{1}{\sqrt{2}} \left( |M_0(k)\rangle |M_0(-k)\rangle - |M_0(-k)\rangle |M_0(k)\rangle \right) + \omega \sqrt{2} |\Delta(k)\rangle
\]

where

\[
|\Delta(k)\rangle \equiv |M_0(k)\rangle |M_0(-k)\rangle + |M_0(-k)\rangle |M_0(k)\rangle
\]

and \( \omega = |\omega| e^{i\Omega} \) is a complex \( CPT \) violating (CPTV) parameter. For definiteness in what follows we shall term this quantum-gravity effect in the initial state as the “\( \omega \)-effect” \[3\]. There is actually another dynamical “\( \omega \)-effect” which is generated during the time evolution of the meson pair and this will be discussed further in passing \[5\].

Gravitation is one interaction which is not included in the standard model and we shall assume that it has a quantum manifestation. We will examine whether any such manifestation implies an \( \omega \)-effect. Although there is no satisfactory theory of quantum gravity there are features which will most likely survive a full resolution. One current attempt is based on string/brane theory \[6\] and will be for us a primary motivation. On more general grounds the existence of black hole solutions in general relativity and semi-classical arguments which imply thermal Hawking radiation emitted by a black hole have thrown into doubt that unitarity is preserved in quantum evolution \[7\]. Since the semi-classical calculation embodies the classical causality structure of horizons it is perhaps not surprising that unitarity is found not to hold. Clearly the relevance of an absence of unitarity is in the lack of inverse for \( T \) which will then have implications for \( CPT \). There has been a
continuing debate on the absence or otherwise of unitarity. The more recent contributions have suggested that the correlations between the Hilbert space of states within the horizon and the outgoing radiation would allow a recovery of the information in the black hole [8]. However considerations from quantum information theory based on estimates of mutual information have indicated that the evolution in the presence of even evaporating black holes is non-unitary [9,10]. In the presence of space-time foam which can be due to formation and destruction of microscopic black holes or other space-time singularities such as D-particles from the bulk in string/brane theory impinging on a 3 dimensional brane [11], we feel non-unitary behaviour is likely. Furthermore in non-critical string theory there are plausible arguments based on the renomalization group which could be interpreted in terms of non-unitary time-evolution. In section 2 we shall first discuss how the existence of the $\Theta \equiv \text{CPT}$ is incompatible with non-unitary evolution. We will elaborate on this with some arguments favouring the existence of non-unitary evolution both on general grounds and also within the context of non-critical strings. In section 3 we will give a model inspired by string induced decoherence [12] and show that the $\omega$-effect is generated. In section 4 we will evaluate plausible alternative models of space-time foam such as noncommutative geometry [13] and thermal baths and conclude that the omega effect is an extremely fragile phenomenon.

2 Master Equation and $\text{CPT}$

The implications of non-unitary evolution for $\Theta$ were first addressed by Wald [14]. Unitary time evolution is an implicit assumption in the proof of the CPT theorem and so his work serves as a conceptual cornerstone for the approach to the omega effect that we develop in this paper. A brief review of the discussion of Wald will now be given. Within the context of scattering theory let $\mathcal{H}_{in}$ denote the Hilbert space of in states and $\mathcal{H}_{out}$ the dual space. We can define the analogous entities for the out states. Let $S$ be a mapping from the set of in-states $\mathcal{G}_{in}$ to the set of out-states $\mathcal{G}_{out}$. In our framework $\mathcal{G}_{in}$ is isomorphic to $\mathcal{H}_{in} \otimes \mathcal{H}_{out}$ since the states are represented as density matrices $\rho_{in}^{A,B}$ where $A$ is a vector index associated with $\mathcal{H}_{in}$ and $B$ is a vector index associated with $\mathcal{H}_{out}$. The indexed form of $S$ is $S^{a}{}_{a,C}{}^{D}{}_{D}$ where the lower case indices refer to $\mathcal{H}_{out}$ and $\mathcal{H}_{out}$. If probability is conserved

$$\text{tr} \left( S \rho \right) = \langle \rho \rangle$$

which in index notation can be written as

$$S^{a}{}_{a,C}{}^{D}{}_{D} = \delta^{C}{}_{D}$$

where we have adopted the summation convention for repeated indices. Consider now operators $\Theta_{in}$ and $\Theta_{out}$ which implement the $\text{CPT}$ transformation on $\mathcal{G}_{in}$ and $\mathcal{G}_{out}$ respectively i.e.

$$\Theta_{in} : \mathcal{G}_{in} \rightarrow \mathcal{G}_{out}$$

$$\Theta_{out} : \mathcal{G}_{out} \rightarrow \mathcal{G}_{in}.$$
In particular under a CPT transformation let $\rho_{in} \in \mathcal{G}_{in}$ be mapped into $\rho_{out}^{'}$ and $\rho_{out} \in \mathcal{G}_{out}$ be mapped into $\rho_{out}^{'}$. If the theory (including quantum gravity) is assumed to be invariant under CPT then

$$
\rho_{out} = S\rho_{in},
$$  
(6)

$$
\rho_{out}^{'} = S\rho_{in}^{'}
$$  
(7)

$$
\Theta_{in}\rho_{in} = \rho_{out}^{'}
$$  
(8)

$$
\Theta_{out}\rho_{out} = \rho_{in}^{'}
$$  
(9)

and

$$
\Theta_{in}\Theta_{out} = I, \quad \Theta_{out}\Theta_{in} = I.
$$  
(10)

Since

$$
\Theta_{out} = \Theta_{in}^{-1}
$$  
(11)

it is convenient to drop the suffix $in$ in this last relation. Hence from (6), (7), (8) and (9) we can deduce that

$$
\Theta\rho_{in} = \rho_{out}^{'}
$$  
(12)

$$
= S\rho_{in}^{'}
$$  
(13)

$$
= S\Theta^{-1}\rho_{out}
$$  
(14)

$$
= S\Theta^{-1}S\rho_{in},
$$  
(15)

From (12) and (15) we can deduce the important result that $S$ has an inverse given by $\Theta^{-1}S\Theta^{-1}S$. Hence if $\Theta$ exists then time reversed evolution is permitted. Consequently for non-unitary evolution $\Theta$ cannot be defined.

2.1 Non-unitary evolution

Given the lack of a full theory of quantum gravity it may seem that we cannot say anything rigorous about the issue of non-unitarity when quantum effects and gravity are significant. This lack of a theory has permitted various suggestions which have typically had a certain speculative element. We will nonetheless examine some clues from the current understanding. One finding based on semi-classical analysis is thermal radiation produced by evaporating black holes which are predicted by general relativity. Thermal radiation carries no information about the initial state of the black hole and hence there is a loss of information. Loss of information is related to non-unitary evolution in the presence of black holes \[.\] This was the intriguing possibility suggested (quite sometime ago) by this analysis. Clearly these (semi-classical) considerations become more and more suspect when the black hole approaches the size of the Planck scale (based on heuristic arguments combining quantum mechanics and general relativity). More recently it was suggested that information is actually not lost but remains in a stable remnant of the size of the Planck scale. This is problematic because the entropy of a black hole (admittedly based on semi-classical analysis) is proportional to the square of its mass \[15][16]. Hence a Planck sized black hole has a much smaller mass $M_f$ compared to the original black hole of mass $M$ and so cannot contain
the original information. If there were many remnants (of the order of \( M^2 \) accommodating the \( M^2 \) bits of information) then these would also have had observable consequences for low energy physics \([10]\). A more recent suggestion considered the possibility that the information was actually locked in correlations between the emitted radiation and evaporating black hole. This is akin to quantum cryptography where the key is the small black hole. Although classically the bits required for the key would be similar to that of the original black hole, quantum mechanics allows keys that require far less bits. Such a mechanism would give the possibility of recovering the information of the black hole from correlations with the black hole remnant. However this possibility can be ruled out \([9]\) if the Hawking radiation picture persists beyond the semi-classical analysis as follows. We first note that, for a composite system made of two parts \( A \) and \( B \), any pure state \( |\Psi\rangle \) has the Schmidt decomposition

\[
|\Psi\rangle = \sum_s \eta_s |s_A\rangle |s_B\rangle
\]

where \( \eta_s \geq 0, \sum_s \eta_s^2 = 1 \) and \( |s_A\rangle \) and \( |s_B\rangle \) are orthonormal states for \( A \) and \( B \) respectively. The black hole evaporation process can be regarded as a mapping \( S \) from an initial state \( \rho_j \) in a space \( \mathcal{I} \) to a state \( \rho_f \) in a space \( \mathcal{O} \); \( \rho_f \) is independent of the states in \( \mathcal{I} \). This structure formalises the picture obtained for black hole evaporation and Hawking radiation from semi-classical analysis. The remainder of the final space can be denoted by \( \mathcal{F} \), and within a quantum information context is labelled the ancilla space. If \( \rho_j \) is taken to be \( |\Psi\rangle_{\mathcal{I}} \langle \Psi| \) then we can write \( S|\Psi\rangle_{\mathcal{I}} \), using the Schmidt decomposition, as

\[
S|\Psi\rangle_{\mathcal{I}} = \sum_k \sqrt{p_k} |k\rangle_{\mathcal{I}} \otimes |F_k(\Psi)\rangle_{\mathcal{F}}
\]

where \( \{ |F_k(\Psi)\rangle \} \) are orthonormal states in \( \mathcal{F} \), \( |k\rangle \) is an eigenvector (in \( \mathcal{O} \)) of the map \( F \) with a non-zero eigenvalue \( p_k \); there are \( K \) such eigenvalues. If \( \mathcal{I} \) has dimensionality \( d \) with a basis set \( |\psi_j\rangle \) then \( \mathcal{F} \) is spanned by \( \{ |F_k(\psi_j)\rangle \} \) where \( \{ |F_k(\psi_j)\rangle \} \equiv \{ |F_k(\psi_j)\rangle \} \}. Using a unitary transformation it is possible to map \( \{ |F_k(\psi_j)\rangle \} \) to \( \{ |q_k\rangle \otimes |\psi_j\rangle \} \) where \( \{ |q_k\rangle \} \) is an orthonormal set in \( \mathcal{O} \). Hence

\[
S|\Psi\rangle_{\mathcal{I}} = \sum_k \sqrt{p_k} |k\rangle_{\mathcal{I}} \otimes (|q_k\rangle \otimes |\psi_j\rangle)_{\mathcal{F}}.
\]

Hence information concerning \( |\Psi\rangle \) lies purely in \( \mathcal{F} \) and not in correlations between \( \mathcal{F} \) and \( \mathcal{O} \) \([9]\). Consequently the suggestion that correlations can be the key to the information problem does not seem to be viable, at least within this simple framework. It should be stated that other treatments based on extremality of black holes, anti-de Sitter (supersymmetric) space-times and the Euclidean formulation for summing over space-time geometries purport to support that information can be hidden (through entanglement) within correlations in a holographic way \([8]\). The recent arguments of Hawking and
other (string) theorists are somewhat special in their details and require constructions such as the extremality of black holes, anti-de Sitter (super-symmetric) space-times and the Euclidean formulation for summing over space-time geometries \[17\]. Although the special properties present in the solutions can be regarded as a drawback, nevertheless they indicate that the semi-classical argument on which the earlier argument was based on may be a weak point of the analysis. We are unable to comment further on these issues and will follow arguments that we find appealing.

2.2 Master equation from strings

Rather than just postulating non-unitary evolution based on semi-classical arguments already alluded to in general relativity we will give arguments in string theory that suggest non-unitary evolution in certain circumstances. Strings are fields over a two dimensional world sheet. The renormalization group for field theories in two dimensions has properties which will have an interesting reinterpretation for string theories \[18, 19\]. In theory space, i.e. in the infinite dimensional space of couplings \(\{g_i\}\) for all renormalizable two-dimensional unitary quantum field theories, there exists a function \(c(\{g_i\})\) which has the following properties:

- \(c(\{g_i\})\) is non-negative and non-increasing on renormalization group flows towards an infrared fixed point,
- the renormalization group fixed points are also critical points of \(c(\{g_i\})\),
- value of critical value of \(c(\{g_i\})\) is the conformal anomaly.

More precisely in terms of a renormalization group flow parameter \(t\)

\[
\frac{d}{dt}c(\{g^i\}) = \beta^i G_{ij} \beta^j \tag{19}
\]

where the renormalization group \(\beta\) function is defined by \(\beta^i = \frac{d}{dt}g^i(t)\) and \(G_{ij}\) is referred to as the Zamolodchikov metric. \(G_{ij}\) is negative definite and is the matrix of second derivatives of the free energy. In the world sheet conformal field theory approach to perturbative string theory it has long been thought that the time evolution of the string backgrounds and world sheet renormalization group flows are connected \[20, 21\]. In string theory for the backgrounds to be consistent with a classical space-time interpretation the central charge will need to have the critical values. In order to move away from such a classical background and allow in principle for fluctuating backgrounds it is necessary to deform the conformal points through vertex operators \(V_{g^i}\) associated with background fields \(g^i\). Hence the conformal invariant action \(S^\ast\) becomes deformed to \(S\) where

\[
S = S^\ast + g^i \int d^2z V_{g^i}(\Xi)
\]

\(z\) is the usual holomorphic world sheet co-ordinate, \(\Xi\) are target space matter fields and \(V_{g^i}\) has conformal weight \((1, 1)\); however (for convenience now dropping the index \(i\)) \(V_g\) has a scaling dimension \(\alpha_g\) to \(O(g)\) with \(\alpha_g = \ldots\)
\(-gC_{ggg} + \ldots\) and \(C_{ggg}\) is the expansion coefficient in the operator product expansion of \(V_g\) with itself. In non-critical strings conformal invariance is restored by gravitational dressing with a Liouville field \(\varphi\) (which can be viewed as a local world sheet scale) so that

\[
\int d^2 z gV_g(\Xi) \to \int d^2 z g e^{\alpha \varphi} V_g(\Xi). \tag{20}
\]

In a small \(g\) expansion

\[
\int d^2 z gV_g(\Xi) \to \int d^2 z gV_g(\Xi) - \int d^2 z g^2 C_{ggg}.
\]

Scale invariance is restored by defining a renormalized coupling \(g_R\)

\[
g_R = g - C_{ggg} \varphi g^2 \tag{21}
\]

The local scale interpretation of \(\varphi\) is clearly consistent with the renormalisation group \(\beta\) function that we have noted earlier. The integration over world sheet metrics \(\gamma_{\alpha \beta}\) (in the Polyakov string action) implies an integration over \(\varphi\) on noting that \(\gamma_{\alpha \beta} = e^{\hat{\varphi}} \hat{\gamma}_{\alpha \beta}\) where \(\hat{\gamma}_{\alpha \beta}\) is a fiducial metric. In this way \(\varphi\) becomes a dynamical variable with a kinetic term. For matter fields with central charge \(c_m > 25\) the signature of this term is opposite to the kinetic terms for the fields \(\Xi\) and it has been suggested that in this case the zero mode of \(\varphi\) is a target time \(t\). The requirement of renormalizability of the world sheet \(\sigma\) model implies that for the density matrix \(\rho\) of a string state propagating in a background \(\{g_i\}\)

\[
\frac{d}{dt} \rho(g^i, p_i, t) = 0 \tag{22}
\]

where \(p_i\) is the conjugate momentum to \(g^i\) within the framework of a dynamical system with hamiltonian \(H\) and action the Zamolodchikov \(c\)-function \[22], i.e.

\[
c[g] = \int dt (p_i \dot{g}^i - H). \tag{23}
\]

From \[22\] we deduce that

\[
\frac{\partial \rho}{\partial t} + \dot{g}^i \frac{\partial \rho}{\partial g^i} + \dot{p}_i \frac{\partial \rho}{\partial p^i} = 0. \tag{24}
\]

The piece \(\dot{p}_i \frac{\partial \rho}{\partial p^i}\) in \[24\] can be written as \(G_{ij} \beta^j \frac{\partial \rho}{\partial p^i}\); using the canonical relationship of \(g^i\) and \(p^i\) this can be recast as \(-iG_{ij} \beta^j [\rho, g^i]\). As discussed in \[22\], such a term leads to a non-unitary evolution of \(\rho\). However this analysis cannot be regarded as settling the issue of non-unitarity even within the framework of non-critical string theory since there are issues relating to the time like signature of the Liouville field and the identification of the local renormalization group scale. Furthermore the role of D branes and in particular D particles have not been incorporated into the analysis.

An additional reason \[23\] for considering quantum decoherence models of quantum gravity, comes from recent astrophysical evidence on a current-era
acceleration of our Universe. Indeed, observations of distant supernovae [24], as well as WMAP data [25] on the thermal fluctuations of the cosmic microwave background (CMB), indicate that our Universe is at present in an accelerating phase, and that 73% of its energy-density budget consists of an unknown substance, termed Dark Energy. Best-fit models of such data include Einstein-Friedman-Robertson-Walker Universes with a non-zero cosmological constant. However, the data are also currently compatible with (cosmic) time-dependent vacuum-energy-density components, relaxing asymptotically to zero [26]. In colliding brane-world models, by treating the dark energy component of the Universe as a non-equilibrium energy density of the (observable) brane world [27, 11] this density is identified with the central charge surplus of the supercritical σ-models describing the (recoil) string excitations of the brane after the collision. The relaxation of the dark-energy density component can be a purely stringy feature of the logarithmic conformal field theory [28] describing the D-brane recoil [29] in a (perturbative) σ-model framework.

3 D-particle space-time foam

D-particles are D-branes with zero spatial dimension. Instead of considering microscopic black holes as agents for the foaminess of space-time we shall consider that D-particles can play that role. Typically open strings interact with D-particles and satisfy Dirichlet boundary conditions when attached to them. Closed strings may be cut by D-particles. D-particles are allowed in certain string theories such as bosonic, type IIA and type I and here we will here consider them to be present in string theories of phenomenological interest. Furthermore even when elementary D particles cannot exist consistently there can be effective D-particles formed by the compactification of higher dimensional D-branes. Moreover D particles are non-perturbative constructions since their masses are inversely proportional to the string coupling $g_s$. The study of D-brane dynamics has been made possible by Polchinski’s realisation that such solitonic string backgrounds can be described in a conformally invariant way in terms of world sheets with boundaries [30]. On these boundaries Dirichlet boundary conditions for the collective target-space coordinates of the soliton are imposed [31]. Although recently a particular supersymmetric version of D-particle foam has been suggested involving stacks of $D8$ branes and $O8$ orientifold planes enclosing a bulk space [11], it will suffice for us to consider for simplicity a model based on D-particles populating a bulk geometry between parallel D-brane worlds. The model is termed D-foam [32] (c.f. figure 2), since our world is modelled as a three-brane moving in the bulk geometry. Ordinary matter lives on the brane. As the brane moves through the bulk, stringy matter interacts with the D-particles. The spectrum of open strings attached to a Dp brane (with $p > 0$) contains a Maxwell field and the ends of the open string carry charge. The associated Maxwell field is confined to the world volume of the D-brane. Hence a conventionally charged string cannot end on a D-particle. We will thus restrict our consideration to neutral particles that are "captured" by D-particles. An important symmetry of this first quantised string theory is conformal.
invariance and the requirement of the latter does determine the space-time dimension and/or structure. This symmetry leads to a scaling of the metric and permits the representation of interactions through the construction of measures on inequivalent Riemann surfaces [33]. In and out states of stringy matter are represented by punctures at the boundaries. The D-particles as solitonic states [30] in string theory do fluctuate themselves, and this is described by stringy excitations, corresponding to open strings with their ends attached to the D-particles. In the first quantised (world-sheet) language, such fluctuations are also described by Riemann surfaces of higher topology with appropriate Dirichlet boundary conditions (c.f. fig. 1). The plethora of Feynman diagrams in higher order quantum field theory is replaced by a small set of world sheet diagrams classified by moduli which need to be summed or integrated over [6]. In order to understand possible consequences for CPT due to space-time foam we will have to characterise the latter. As a result, D-particles cross the brane world, and thereby appear as foamy flashing on and off structures for an observer on the brane. When low energy matter given by a closed string propagating in a $(d + 1)$-dimensional space-time collides with a very massive D-particle (0-brane) embedded in this space-time, the D-particle recoils as a result [36]. We shall consider the simple case of bosonic stringy matter coupling to D-particles and so we can only discuss matters of principle and ignore issues of stability. However we should note that an open string model needs to incorporate for completeness, higher dimensional D-branes such as the D3 brane. This is due to the vectorial charge

\[ \text{Fig. 1} \]

Upper picture: A Fluctuating D-particle is described by open strings attached to it. As a result of conservation of fluxes [7,30,34] that accompany the D-branes, an isolated D-particle cannot occur, but it has to be connected to a D-brane world through flux strings. Lower picture: World-sheet diagrams with annulus topologies, describing the fluctuations of D-particles as a result of the open string states ending on them. Conformal invariance implies that pinched surfaces, with infinitely long thin strips, have to be taken into account. In bosonic string theory, such surfaces can be resummed [35].
Logarithmic conformal field theory describes the impulse at stage (II)

Fig. 2 Schematic representation of a D-foam. The figure indicates also the capture/recoil process of a string state by a D-particle defect for closed (left) and open (right) string states, in the presence of D-brane world. The presence of a D-brane is essential due to gauge flux conservation, since an isolated D-particle cannot exist. The intermediate composite state at $t = 0$, which has a life time within the stringy uncertainty time interval $\delta t$, of the order of the string length, and is described by world-sheet logarithmic conformal field theory, is responsible for the distortion of the surrounding space time during the scattering, and subsequently leads to induced metrics depending on both coordinates and momenta of the string state. This results on modified dispersion relations for the open string propagation in such a situation [32], leading to non-trivial “optics” for this space time.
has the form \( g_{ij} = \delta_{ij}, \quad g_{00} = -1, \quad g_{0i} = \varepsilon (\varepsilon y_i + u_i t) \Theta_\varepsilon (t), \quad i = 1, \ldots, d. \) (25)

where the suffix 0 denotes temporal (Liouville) components and

\[
\Theta_\varepsilon (t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dq \frac{e^{iqt}}{q - i\varepsilon},
\]

\[ u_i = (k_1 - k_2)_i, \]

with \( k_1 (k_2) \) the momentum of the propagating closed-string state before (after) the recoil; \( y_i \) are the spatial collective coordinates of the D particle and \( \varepsilon^{-2} \) is identified with the target Minkowski time \( t \) for \( t \gg 0 \) after the collision [29]. These relations have been calculated for non-relativistic branes where \( u_i \) is small. Now for large \( t \), to leading order,

\[ g_{0i} \sim \frac{u_i}{\varepsilon} \propto \frac{\Delta p_i}{M_P} \] (27)

where \( \Delta p_i \) is the momentum transfer during a collision and \( M_P \) is the Planck mass (actually, to be more precise \( M_P = M_s/g_s \), where \( g_s < 1 \) is the (weak) string coupling, and \( M_s \) is a string mass scale); so \( g_{0i} \) is constant in space-time but depends on the energy content of the low energy particle [38]. Such a feature does not arise in conventional approaches to space-time foam and will be important in our formulation of one of the microscopic models that we will consider. In a D-particle recoil the velocity \( u_i \) is random and in the absence of any particular prior knowledge it is natural to take a distribution which is gaussian with 0 mean and variance \( \sigma \). However there is a further uncertainty in \( u_i \), viz. quantum-fluctuation aspects of \( u^i \) about its classical trajectory.

Going to higher orders in perturbation theory of the quantum field theory at fixed genus does not qualitatively alter the situation in the sense that the equation for \( u_i \) remains deterministic. The effect of string perturbation theory where higher genus surfaces are considered (on taking into account the sub-leading leading infrared divergences) is to produce diffusive behaviour of \( u^i \). The resulting probability distribution \( p(u) \) for \( u \) (ignoring the vectorial index of \( u \)) is [39]

\[ p(u) = \frac{15\alpha'}{2\pi (g_s^2 \eta(t) + 15\alpha' \sigma^2)} \exp \left[ -\frac{15\alpha'}{2 (g_s^2 \eta(t) + 15\alpha' \sigma^2)} u^2 \right]. \] (28)

where

\[ \eta(t) = 2t_0^2 + 3(t + t_0)^2 \sqrt{1 + \frac{t}{t_0} - 5t_0^2(t + t_0)^2}. \] (29)

The interaction time includes both the time for capture and re-emission of the string by the D-particle, as well as the time interval until the next capture, during string propagation. In a generic situation, this time could be much larger than the capture time, especially in dilute gases of D-particles, which include less than one D-particle per string \( (\alpha^{3/2}) \) volume. Indeed, as discussed in detail in [40], using generic properties of strings consistent with the space-time uncertainties [41], the capture and re-emission time \( t_0 \), involves the
growth of a stretched string between the string state and the D-brane world (c.f. fig. 2) and is found proportional to the incident string energy $p^0$:

$$t_0 \sim \alpha' p^0 \ll \sqrt{\alpha'}.$$  

(30)

From the knowledge of $p(u)$ we can extract $\text{var}(u)$ the variance of $u$ which is an important input for our estimate of the omega effect:

$$\text{var}(u) = \eta(t) g_0^2 + 15\alpha'\sigma^2.$$

(31)

Further details and discussion on string perturbation theory leading to (31) can be found in [22].

4 Recoil model for $\omega$-effect

For an observer on the brane world the crossing D-particles will appear as twinkling space-time defects, i.e. microscopic space-time fluctuations. This will give the four-dimensional brane world a “D-foamy” structure. The target space-time metric state, which is close to being flat, can be represented schematically as a density matrix

$$\rho_{\text{grav}} = \int d^5 r \ f(r_\mu) |g(r_\mu)\rangle \langle g(r_\mu)|.$$  

(32)

The parameters $r_\mu$ $(\mu = 0, \ldots, 5)$ are stochastic with a gaussian distribution $f(r_\mu)$ characterised by the averages

$$\langle r_\mu \rangle = 0, \langle r_\mu r_\nu \rangle = \Delta_\mu \delta_{\mu\nu}.$$  

(33)

The fluctuations experienced by the two entangled neutral mesons will be assumed to be independent and $\Delta_\mu \sim O \left( \frac{E^2}{M^2} \right)$ i.e. very small. As matter moves through the space-time foam, assuming ergodicity, the effect of time averaging is assumed to be equivalent to an ensemble average. As far as our present discussion is concerned we will consider a semi-classical picture for the metric and so $|g(r_\mu)\rangle$ in (32) will be a coherent state. In order to address flavour oscillation phenomena [42] [43] the fluctuations of each component of the metric tensor $g^{\alpha\beta}$ will not be simply given by the simple recoil distortion (27), but instead will be taken to have a $2 \times 2$ (“flavour”) structure:

$$g^{00} = (-1 + r_4) 1$$
$$g^{01} = g^{10} = r_0 1 + r_1 \sigma_1 + r_2 \sigma_2 + r_3 \sigma_3$$
$$g^{11} = (1 + r_5) 1 $$

(34)

The above parametrisation has been taken for simplicity and we will also consider motion to be in the $x$- direction which is natural since the meson pair moves collinearly in the Center-of-Mass frame. A metric with this type of structure is compatible with the view that the D-particle defect is a “point-like” approximation for a compactified higher-dimensional brane black hole.
whose no hair theorems permit non-conservation of flavour. In the case of neutral mesons the concept of “flavour” refers to either particle/antiparticle species or the two mass eigenstates, by changing appropriately the relevant coefficients. Space-time deformations of the form \[ \text{(34)} \] and the associated Hamiltonians \[ \text{(47)} \] have been derived in the context of conformal field theory \[ \text{[32, 44, 39, 22]} \] and details on the relevant derivations will not be given here. We only mention that the variable \( r_1 \) in particular, expresses a momentum transfer during the interaction of the (string) matter state with the D-particle defect. In this sense, the off-diagonal metric component \( g_{01} \) can be represented as

\[
g_{01} \sim u_1
\]  

where \( u_1 = g_s \frac{\Delta k}{M_s} \) expresses the momentum transfer along the direction of motion of the matter string (taken here to be the x direction). In the above equation, \( M_s/g_s \) is the mass of the D-particle, which for weakly coupled strings with coupling \( g_s \) is larger than the string mass scale \( M_s \). In order to address oscillation phenomena, induced by D-particles, the fluctuations of each component of the metric tensor are taken in \[ \text{[5]} \] to have a 2 \( \times \) 2 (“flavour”) structure, as in \[ \text{(34)} \].

For the neutral Kaon system, the case of interest, \( K_0 - \bar{K}_0 \), is produced by a \( \phi \)-meson at rest, i.e. \( K_0 - \bar{K}_0 \) in their C.M. frame. The CP eigenstates (on choosing a suitable phase convention for the states \( |K_0\rangle \) and \( |\bar{K}_0\rangle \) ) are, in standard notation, \( |K_{\pm}\rangle \) with

\[
|K_{\pm}\rangle = \frac{1}{\sqrt{2}} (|K_0\rangle \pm |\bar{K}_0\rangle).
\]  

The mass eigenstates \( |K_S\rangle \) and \( |K_L\rangle \) are written in terms of \( |K_{\pm}\rangle \) as

\[
|K_L\rangle = \frac{1}{\sqrt{1 + |\varepsilon_2|^2}} [ |K_-\rangle + \varepsilon_2 |K_+\rangle] \tag{37}
\]

and

\[
|K_S\rangle = \frac{1}{\sqrt{1 + |\varepsilon_1|^2}} [ |K_+\rangle + \varepsilon_1 |K_-\rangle]. \tag{38}
\]

In terms of the mass eigenstates

\[
|i\rangle = C \left\{ \left| K_L \left( \frac{k}{\sqrt{2}} \right) \right\rangle \left| K_S \left( -\frac{k}{\sqrt{2}} \right) \right\rangle - \left| K_S \left( \frac{k}{\sqrt{2}} \right) \right\rangle \left| K_L \left( -\frac{k}{\sqrt{2}} \right) \right\rangle + \omega \right\}
\]

\[
= C \left\{ \left| K_L \left( \frac{k}{\sqrt{2}} \right) \right\rangle \left| K_S \left( -\frac{k}{\sqrt{2}} \right) \right\rangle - \left| K_S \left( \frac{k}{\sqrt{2}} \right) \right\rangle \left| K_L \left( -\frac{k}{\sqrt{2}} \right) \right\rangle \right\}
\]

where \( C = \sqrt{\frac{1 + |\varepsilon_1|^2}{1 + |\varepsilon_2|^2}} \). In the notation of two level systems (on suppressing the \( \frac{k}{\sqrt{2}} \) label) we write

\[
|K_L\rangle = |\uparrow\rangle \tag{40}
\]

\[
|K_S\rangle = |\downarrow\rangle.
\]
These will be our “flavours” and represent the two physical eigenstates, with masses $m_1 \equiv m_L$, $m_2 \equiv m_S$, with

$$\Delta m = m_L - m_S \sim 3.48 \times 10^{-15} \text{ GeV}. \quad (41)$$

The stochastic variables $r_\mu$ \((33)\) in \((34)\), are linked with the fluctuations of the D-particle recoil velocity, by representing the latter as:

$$u_1 \sim r g_s \frac{k}{M_s} \quad (42)$$

upon the above-mentioned technicality of considering flavour changes in addition to the momentum transfer. Hence, the detailed discussion in the previous session on the stochastic fluctuations of the recoil velocity about a zero average value, translates into rewriting \((33)\) with variances (c.f. \((31)\))

$$\Delta \mu \sim g_0^2 t_0^2 \alpha' + O(\sigma^2) \sim g_0^2 \left(\frac{p^0}{M_s}\right)^2 + O(\sigma^2), \quad \mu = 1, 2. \quad (43)$$

Here we considered the capture time $t_0 \sim \alpha' p^0$, with $p^0$ the energy of the probe, as spanning the essential interaction time for the initial entangled meson state \([39]\). In this way we extrapolate the result \((28)\) to times smaller than $\sqrt{\alpha'}$. This is acceptable, as long as such times are finite. From a conformal field theory point of view, this means that we consider the world-sheet scaling parameter $1/z^2 \sim \ln(L/a)^2 \sim t_0 \ll \sqrt{\alpha'}$ for probe energies $p^0 \ll M_s$.

The Klein-Gordon equation for a spinless neutral meson field $\Phi = \left(\phi_1, \phi_2\right)$ with mass matrix $m = \frac{1}{2} (m_1 + m_2) 1 + \frac{1}{2} (m_1 - m_2) \sigma_3$ in a gravitational background is

$$(g^{\alpha\beta} D_\alpha D_\beta - m^2) \Phi = 0 \quad (44)$$

where $D_\alpha$ is the covariant derivative. Since the Christoffel symbols vanish for $a_t$ independent of space time the $D_\alpha$ coincide with $\partial_\alpha$. Hence

$$(g^{00} \partial_0^2 + 2g^{01} \partial_0 \partial_1 + g^{11} \partial_1^2) \Phi - m^2 \Phi = 0. \quad (45)$$

It is useful at this stage to rewrite the state $|i\rangle$ in terms of the mass eigenstates.

The unnormalised state $|i\rangle$ will then be an example of an initial state

$$|\psi\rangle = |k, \uparrow\rangle^{(1)}|\tilde{k} , \downarrow\rangle^{(2)} - |k, \downarrow\rangle^{(1)}|\tilde{k}, \uparrow\rangle^{(2)} + |\Delta\rangle \quad (46)$$

with

$$|\Delta\rangle = \xi |k, \uparrow\rangle^{(1)}|\tilde{k}, \downarrow\rangle^{(2)} + \xi' |k, \downarrow\rangle^{(1)}|\tilde{k}, \uparrow\rangle^{(2)}$$

where $|K_L (\tilde{k})\rangle = |k, \uparrow\rangle$ and we have taken $\tilde{k}$ to have only a non-zero component $k$ in the $x$-direction; superscripts label the two separated detectors of the collinear meson pair, $\xi$ and $\xi'$ are complex constants and we have left
the state $|\psi\rangle$ unnormalised. The evolution of this state is governed by a hamiltonian $\hat{H}$

$$\hat{H} = g^{01} (g^{00})^{-1} \hat{k} - (g^{00})^{-1} \sqrt{(g^{01})^2 k^2 - g^{00} (g^{11} k^2 + m^2)} \tag{47}$$

which is the natural generalisation of the standard Klein-Gordon hamiltonian in a one particle situation. Moreover $\hat{k} \{\pm k, \uparrow\} = \pm k |k, \uparrow\rangle$ together with the corresponding relation for $\downarrow$. We next note that the Hamiltonian interaction terms

$$\hat{H}_I = -(r_1 \sigma_1 + r_2 \sigma_2) \hat{k} \tag{48}$$

are the leading order contribution in the small parameters $r_\mu$ in the Hamiltonian $\hat{H}$ \cite{17}, since the corresponding variances $\sqrt{\Delta_\mu}$ are small. The term \cite{48}, has been used in \cite{5} as a perturbation in the framework of non-degenerate perturbation theory, in order to derive the “gravitationally-dressed” initial entangled meson states, immediately after the $\phi$ decay. The result is:

$$|k, \uparrow\rangle_{QG}^{(1)} | -k, \downarrow\rangle_{QG}^{(2)} - |k, \downarrow\rangle_{QG}^{(1)} | -k, \uparrow\rangle_{QG}^{(2)} = |\Sigma\rangle + |\bar{\Delta}\rangle \tag{49}$$

where

$$|\Sigma\rangle = |k, \uparrow\rangle^{(1)}_{QG} | -k, \downarrow\rangle^{(2)}_{QG} - |k, \downarrow\rangle^{(1)}_{QG} | -k, \uparrow\rangle^{(2)}_{QG},$$

$$|\bar{\Delta}\rangle = |k, \downarrow\rangle^{(1)} | -k, \uparrow\rangle^{(2)} (\beta^{(1)} - \beta^{(2)}) + |k, \uparrow\rangle^{(1)} | -k, \downarrow\rangle^{(2)} (\alpha^{(2)} - \alpha^{(1)}) + \beta^{(1)} \alpha^{(2)} |k, \downarrow\rangle^{(1)} | -k, \uparrow\rangle^{(2)} - \alpha^{(1)} \beta^{(2)} |k, \uparrow\rangle^{(1)} | -k, \downarrow\rangle^{(2)}$$

and

$$\alpha^{(i)} = \frac{(i) \langle \uparrow, k | \hat{H}_I | k^{(i)} \rangle^{(i)}}{E_2 - E_1}, \quad \beta^{(i)} = \frac{(i) \langle \uparrow, k | \hat{H}_I | k^{(i)} \rangle^{(i)}}{E_1 - E_2}, \quad i = 1, 2 \tag{50}$$

where the index $(i)$ runs over meson species (“flavours”) $(1 \to K_L, 2 \to K_S)$. The reader should notice that the terms proportional to $(\alpha^{(2)} - \alpha^{(1)})$ and $(\beta^{(1)} - \beta^{(2)})$ in \cite{41} generate $\omega$-like effects. We concentrate here for brevity and concreteness in the strangeness conserving case of the $\omega$-effect in the initial decay of the $\phi$ meson \cite{15}, which corresponds to $r_1 \propto \delta_2$. We should mention, however, that in general quantum gravity does not have to conserve this quantum number, and in fact strangeness-violating $\omega$-like terms are generated in this problem through time evolution \cite{5}.

We next remark that on averaging the density matrix over the random variables $r_i$, which are treated as independent variables between the two meson particles of the initial state \cite{41}, we observe that only terms of order $|\omega|^2$ will survive, with the order of $|\omega|^2$ being

$$|\omega|^2 = \bar{\Delta}_{(1),(2)} \left( \mathcal{O} \left( \frac{1}{(E_1 - E_2)^2} \langle \downarrow, k | \hat{H}_I | k, \uparrow \rangle^2 \right) \right),$$

$$= \bar{\Delta}_{(1),(2)} \left( \mathcal{O} \left( \frac{\Delta_2 k^2}{(E_1 - E_2)^2} \right) \right) \sim \bar{\Delta}_{(1),(2)} \left( \frac{\Delta_2 k^2}{(m_1 - m_2)^2} \right) \tag{51}$$

for the physically interesting case of non-relativistic Kaons in $\phi$ factories, in which the momenta are of order of the rest energies. The notation $\bar{\Delta}_{(1),(2)} (\ldots)$
above indicates that one considers the differences of the variances \( \Delta_2 \) between the two mesons 1 - 2, in that order.

The variances in our model of D-foam, which are due to quantum fluctuations of the recoil velocity variables about the zero average (dictated by the imposed requirement on Lorentz invariance of the string vacuum) are given by \( (43) \), with \( p_0 \sim m_i \) the energy of the corresponding individual (non-relativistic) meson state \( (i) \) in the initial entangled state \( (4) \). It is important to notice that, on taking the difference of the variance \( \Delta_2 \) between the mesons (1) and (2), the terms proportional to the dispersion \( \sigma^2 \) in the initial recoil velocity \( u_0 \) Gaussian distribution cancel out, since \( \sigma^2 \) is assumed universal. In this sense, one is left with the contributions from the first term of the right-hand-side of \( (43) \), and thus we obtain the following estimate \[39\] for the square of the amplitude of the (complex) \( \omega \)-parameter:

\[
|\omega|^2 \sim g_0 \frac{(m_1^2 - m_2^2)}{M_s^2} \frac{k^2}{(m_1 - m_2)^2} = \frac{m_1 + m_2}{m_1 - m_2} \frac{k^2}{(M_s^2/g_0^2)}, \tag{52}
\]

where \( M_P \equiv M_s/g_0 \) represents the (average) quantum gravity scale, which may be taken to be the four-dimensional Planck scale. In general, \( M_s/g_0 \) is the (average) D-particle mass, as already mentioned. In the modern version of string theory, \( M_s \) is arbitrary and can be as low as a few TeV, but in order to have phenomenologically correct string models with large extra dimensions one also has to have in such cases very weak string couplings \( g_0 \), such that even in such cases of low \( M_s \), the D-particle mass \( M_s/g_0 \) is always close to the Planck scale \( 10^{19} \text{ GeV} \). But of course one has to keep an open mind about ways out of this pattern, especially in view of the string landscape.

The result \[52\], implies, for neutral Kaons in a \( \phi \) factory, for which \[41\] is valid, a value of: \( |\omega| \sim 10^{-11} \), which in the sensitive \( \eta^+ \) bi-pion decay channel, is enhanced by three orders of magnitude, as a result of the fact that the \( |\omega| \) effect always appears in the corresponding observables \[45\] in the form \( |\omega|/|\eta^+| \), and the CP-violating parameter \( |\eta^+| \sim 10^{-3} \). Unfortunately, this value is still some two orders of magnitude away from current bounds of the \( \omega \)-effect at, or the projected sensitivity of upgrades of, the DAΦNE detector \[46\].

The estimate does not change much if one considers relativistic meson states. Although in such a case the non-relativistic quantum mechanics formalism leading to \[52\] in the Kaon systems should strictly be replaced by an appropriate relativistic treatment, nevertheless, one may still use the expression \[52\] in that case, in order to have a rough idea of the order of magnitude of the effect in such relativistic cases. The major difference in this case is the form of the Hamiltonian, which stems from the expansion of the Dirac Hamiltonian for momenta \( k \gg m_i, m_i \) the masses. The quantities \( E_i \sim k + \frac{m_i^2}{2k}, \ i = 1, 2 \) (due to momentum conservation, assumed on average), and the capture times \( t_c \sim \alpha E_i \). In such a case then, it is straightforward to estimate \( |\omega|^2 \) as:

\[
|\omega|^2 \sim \Delta_{(1),(2)} \left( \frac{k^2}{(m_1 - m_2)^2} \right) \sim \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \frac{g_0^2}{M_s^2} \frac{k^2}{M_s^2}, \tag{53}
\]
which does not lead to significant changes in the order of magnitude for probes of energies up to 10 GeV. This completes our discussion on the estimates of the $\omega$-effect in the initial entangled state of two mesons in a meson factory. As discussed in [5], $\omega$-like terms can also be generated due to the time evolution but we will not discuss this here.

5 Alternative models of space-time foam

At this point the omega effect has been successfully generated through the string type decoherence model. Clearly the ability of the environment, in this case the D-particles, to change the flavour is important. For example models of fuzzy space-time foam embodied in non-commutative geometry capture effectively the non-commutative space-time commutation relations due to the presence (generally) of many D-branes [6] in string theory and introduce non-locality. The resulting formalism, purely from the non-commutative structure, does not lead to change in flavours; for our model it was important that D-particle capture, recoil and emission of strings does not conserve flavour. A simple calculation, within the framework of non-commutative geometry [13], involving twisted tensor products implies a change of the EPR-type correlation of the CPT invariant form during time evolution (cf [69] with $\omega = 0$) in the form of a small admixture of a term of the form $\langle k, \uparrow \rangle^{(1)} \langle -k, \downarrow \rangle^{(2)} + \langle k, \downarrow \rangle^{(1)} \langle -k, \uparrow \rangle^{(2)}$. This term is multiplied by the antisymmetric tensor $\theta^{\mu \nu}$ where $[x^\mu, x^\nu] = i\theta^{\mu \nu}$. However this is not the correlation for the $\omega$-effect. Hence within a string inspired framework it seems to be necessary to consider a space-time foam with a flavour changing mechanism. We will now consider a class of models of decoherence due to space-time foam which satisfies this criterion and is not directly related to string theory. It is based on a heuristic non-local effective theory approach to gravitational fluctuations. The non-locality arises because the fluctuation scale is taken to be intermediate between the Planck scale and the low energy scale. Furthermore the dominant non-locality is assumed to be bilocal, and, from similarities to quantum Brownian motion [47],[48], plausible arguments can be given for a thermal bath model for space-time foam [49]. The robustness or otherwise of the $\omega$-effect can be gauged from the predictions of this class of models.

Garay [49] has argued that the effect of non-trivial topologies related to space-time foam and a non-zero minimum length can be modelled by a field theory with non-local interactions on a flat background in terms of a complete set of local functions $\{h_j(\phi, x)\}$ of the fields $\phi$ at a space-time point $x$. His argument is based on general arguments related to problems of measurement [48]. Also, by considering the infinite redshifting near the horizon for an observer far away from the horizon of a black hole, Padmanabhan [50] has argued that a foam consisting of virtual black holes would magnify Planck scale physics for observers asymptotically far from the horizon and thus an effective non-local field theory description would be appropriate. The non-
local action $I$ can be written in terms of a sum of non-local terms $I_n$ where

$$I_n = \frac{1}{n!} \int dx_1 \ldots \int dx_n \, f^{i_1 \ldots i_n} (x_1, \ldots, x_n) h_{i_1} (x_1) h_{i_2} (x_2) \ldots h_{i_n} (x_n)$$  \hspace{1cm} (54)

(where a summation convention for the indices has been assumed; the $f^{i_1 \ldots i_n}$ depend on relative co-ordinates such as $x_1 - x_2$ and are expected to fall off for large separations, the scale of fluctuations being $l_* > \frac{M_P}{M}$. Assuming a form of weak coupling approximation it was further argued [49] that the retention of only the $n = 2$ term would be a reasonable approximation. Formally (i.e. ignoring the validity of Euclidean to Minkowski Wick rotations) the resulting non-locality can be written in terms of an auxiliary field $\phi$ through the functional identity

$$\exp \left( i \int dx_1 \int dx_2 \, f^{i_1 i_2} (x_1 - x_2) h_{i_1} (x_1) h_{i_2} (x_2) \right) = \wp [\phi]$$

where

$$\wp [\phi] = \int d\phi \exp \left( - i \int dx_1 dx_2 \, k_{i_1 i_2} (x_1 - x_2) \phi^{i_1} (x_1) \phi^{i_2} (x_2) + i \int dx \, \phi (x) h_j (x) \right)$$

and $k_{i_1 i_2}$ is the inverse of $f^{i_1 i_2}$. The bilocality is now represented as a local field theory for $\phi$ subjected to a stochastic field $\phi$. As shown in [51] and [52] this stochastic behaviour results [49] in a master equation of the type found in quantum Brownian motion, i.e. unitary evolution supplemented by diffusion. There is no dissipation which might be expected from the fluctuation diffusion theorem because the noise is classical. Since the energy scales for typical experiments are much smaller than those associated with gravitational quantum fluctuations Garay considered $f^{i_1 i_2} (x_1 - x_2)$ to be proportional to a Dirac delta function and argued that the master equation was that for a thermal bath.

A standard approach to quantum Brownian motion of a particle is to have the particle interact with a bath of quantum harmonic oscillators [47]. A thermal field represents a bath about which there is minimal information since only the mean energy of the bath is known, a situation which may be valid also for space-time foam. In applications of quantum information it has been shown that a system of two qubits (or two-level systems) initially in a separable state (and interacting with a thermal bath) can actually be entangled by such a single mode bath [53]. As the system evolves the degree of entanglement is sensitive to the initial state. The close analogy between two-level systems and neutral meson systems, together with the modelling by a phenomenological thermal bath of space-time foam, makes the study of thermal master equations an intriguing one for the generation of $\omega$-terms.

The hamiltonian $H$ representing the interaction of two such two-level 'atoms' with a single mode thermal field [54] is

$$H = \nu a^\dagger a + \frac{1}{2} \Omega \sigma_3^{(1)} + \frac{1}{2} \Omega \sigma_3^{(2)} + \gamma \sum_{i=1}^{2} \left( a \sigma_3^{(i)} + a^\dagger \sigma_-^{(i)} \right)$$  \hspace{1cm} (55)
where $a$ is the annihilation operator for the mode of the thermal field and the $\sigma$s are again the Pauli matrices for the 2-level systems (using the standard conventions). The superscripts label the particle. The harmonic oscillator operators $a$ and $a^\dagger$ satisfy
\[ [a, a^\dagger] = 1, \quad [a^\dagger, a^\dagger] = [a, a] = 0. \quad (56) \]

The $H$ here (commonly known as the Jaynes-Cummings hamiltonian [54]) is quite different from the $H$ of the D-particle foam of the last section. It explicitly incorporates the back reaction or entanglement between system and bath. There are no classical stochastic terms at this level of description and also $H$ is not separable. In the D-particle foam model the lack of separability came solely from the entangled nature of the initial unperturbed state. This is also in contrast with the Lindblad model. The thermal master equation comes from tracing over the oscillator degrees of freedom. We will however consider the dynamics before tracing over the bath because, although the thermal bath idea has a certain intuitive appeal, it cannot claim to be rigorous and so for attempts to find models for the $\omega$-effect it behoves us to entertain also deviations from the thermal bath state of the reservoir. An important feature of (55) is the block structure of subspaces that are left invariant by $H$.

It is straightforward to show that the family of invariant irreducible spaces $E_n$ may be defined by \( \{ |e_i^{(n)}\rangle, i = 1, \ldots, 4 \} \) where (in obvious notation, with $n$ denoting the number of oscillator quanta)
\[
|e_1^{(n)}\rangle \equiv |\uparrow^{(1)}, \uparrow^{(2)}, n\rangle, \quad |e_2^{(n)}\rangle \equiv |\uparrow^{(1)}, \downarrow^{(2)}, n + 1\rangle, \quad |e_3^{(n)}\rangle \equiv |\downarrow^{(1)}, \uparrow^{(2)}, n + 1\rangle, \\
|e_4^{(n)}\rangle \equiv |\downarrow^{(1)}, \downarrow^{(2)}, n + 2\rangle. \quad (57)
\]

The total space of states is a direct sum of the $E_n$ for different $n$. We will write $H$ as $H_0 + H_1$ where
\[
H_0 = \nu a^\dagger a + \frac{\Omega}{2} (\sigma_3^{(1)} + \sigma_3^{(2)}) \quad (58)
\]
and
\[
H_1 = \gamma^2 \sum_{i=1}^{2} \left( a\sigma_+^{(i)} + a^\dagger \sigma_-^{(i)} \right). \quad (59)
\]
n is a quantum number and gives the effect of the random environment. In our era the strength $\gamma$ of the coupling with the bath is weak. We expect heavy gravitational degrees of freedom and so $\Omega \gg \nu$. It is possible to associate both thermal and highly non-classical density matrices with the bath state. Nonetheless because of the block structure it can be shown completely non-perturbatively that the $\omega$ contribution to the density matrix is absent [55].

We will calculate the stationary states in $E_n$, using degenerate perturbation theory, where appropriate. We will be primarily interested in the dressing of the degenerate states $|e_2^{(n)}\rangle$ and $|e_3^{(n)}\rangle$ because it is these which contain
the neutral meson entangled state. In 2nd order perturbation theory the dressed states are
\[ |\psi_2^{(n)}\rangle = |\xi_2^{(n)}\rangle - |\xi_3^{(n)}\rangle + O(\gamma^3) \]  
(60)
with energy \( E_2^{(n)} = (n + 1)\nu + O(\gamma^3) \) and
\[ |\psi_3^{(n)}\rangle = |\xi_2^{(n)}\rangle + |\xi_3^{(n)}\rangle + O(\gamma^3) \]  
(61)
with energy \( E_3^{(n)} = (n + 1)\nu + \frac{2\gamma^2}{n^2}\nu + O(\gamma^3) \). It is natural to expect \( |\psi_2^{(n)}\rangle \) which can in principle give the \( \omega \)-effect. More precisely we would construct the state \( Tr \left( \rho_B \right) \) where \( \rho_B \) is the bath density matrix (and has the form \( \rho_B = \sum_{n,m} p_{nm} |n\rangle \langle n'| \) for suitable choices of \( p_{nm} \)). To this order of approximation \( |\psi_2^{(n)}\rangle \) cannot generate the \( \omega \)-effect since there is no admixture of \( |\xi_1^{(n)}\rangle \) and \( |\xi_4^{(n)}\rangle \). However it is a priori possible that this may change when higher orders in \( \gamma \). We can show that
\[ |\psi_2^{(n)}\rangle = |\xi_2^{(n)}\rangle - |\xi_3^{(n)}\rangle \]  
(62)
to all orders in \( \gamma \) by directly considering the hamiltonian matrix \( \mathcal{H}^{(n)} \) for \( \mathcal{H} \) within \( \mathcal{E}_n \); it is given by
\[
\mathcal{H}^{(n)} = \begin{pmatrix}
\Omega + n\nu & \gamma\sqrt{n+1} & \gamma\sqrt{n+1} & 0 \\
\gamma\sqrt{n+1} & (n+1)\nu & 0 & \gamma\sqrt{n+2} \\
\gamma\sqrt{n+1} & 0 & (n+1)\nu & \gamma\sqrt{n+2} \\
0 & \gamma\sqrt{n+2} & 0 & (n+2)\nu - \Omega
\end{pmatrix}.
\]
(62)
We immediately notice that
\[
\mathcal{H}^{(n)} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = (n+1)\nu \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}
\]
(63)
and so, to all orders in \( \gamma \), the environment does not dress the state of interest to give the \( \omega \)-effect; clearly this is independent of any choice of \( \rho_B \).

As we noted this block structure of the Hilbert space is a consequence of the structure of \( \mathcal{H}_1 \) which is commonly called the hamiltonian in the ‘rotating wave’ approximation[56]. (This nomenclature arises within the context of quantum optics where this model is used extensively.) It is natural to expect that, once the rotating wave approximation is abandoned, our conclusion would be modified. We shall examine whether this expectation materialises by adding to \( \mathcal{H} \) a ‘non-rotating wave’ piece \( \mathcal{H}_2 \)
\[
\mathcal{H}_2 = \gamma \sum_{i=1}^{2} \left( a\sigma_+^{(i)} + a^\dagger \sigma_-^{(i)} \right).
\]
(64)
\( \mathcal{H}_2 \) does not map \( \mathcal{E}_n \) into \( \mathcal{E}_n \) as can be seen from
\[
\mathcal{H}_2 |\xi_2^{(n)}\rangle = \gamma' \left( \sqrt{n+2} |\xi_1^{(n+2)}\rangle + \sqrt{n+1} |\xi_4^{(n-2)}\rangle \right).
\]
We have noted that $|e^{(n)}_3\rangle - |e^{(n)}_2\rangle$ is an eigenstate of $H_0 + H_1$. However the perturbation $H_2$ annihilates $|e^{(n)}_3\rangle - |e^{(n)}_2\rangle$ and so this eigenstate does not get dressed by $H_2$. Hence the $\omega$ effect cannot be rescued by moving away from the rotating wave scenario.

We should comment that a variant of the bath model based on non-bosonic operators $a$ can be considered. This departs somewhat from the original philosophy of Garay but may have some legitimacy from analysis in M-theory. From an infinite momentum frame study it was suggested that D-particles may satisfy infinite statistics\([5\, 8]\). Of course these D-particles would be light contrary to the heavy D-particles that we have been considering and so we will consider a more general deformation of the usual statistics. If flavour changes can arise due to D-particle interactions then it may be pertinent to reinterpret the hamiltonian in this way. Generalised statistics is characterised conventionally by a c-number deformation\([5\, 8]\) parameter $q$ with $q = 1$ giving bosons. Although we have as yet not made a general analysis of such deformed algebras, we will consider the following particular deformation of the usual harmonic oscillator algebra:

$$a_q a_q^\dagger - q^{-1} a_q^\dagger a_q = q^{N_q}$$

where the $q$-operators $a_q, a_q^\dagger$ and $N_q$ act on a Hilbert space with a denumerable orthonormal basis $\{ |n\rangle_q, n = 0, 1, 2, \ldots \}$ as follows:

$$a_q^\dagger |n\rangle_q = |n + 1\rangle_i^{1/2} |n + 1\rangle_q, a_q |n\rangle_q = |n\rangle_i^{-1/2} |n - 1\rangle_q, N_q |n\rangle_q = n |n\rangle_q$$

and

$$[x] \equiv q^x - q^{-x}. \quad (67)$$

Clearly when $q = 1$ the boson case is recovered. We can consider the $q$ generalisation of $H$. The resulting $q$ operator is

$$H_q = h\nu a_q^\dagger a_q + \frac{1}{2} \hbar \Omega \sigma_3^{(1)} + \frac{1}{2} \hbar \Omega \sigma_3^{(2)} + \hbar \gamma \sum_{i=1}^2 (a_q \sigma_3^{(i)} + a_q^\dagger \sigma_3^{(i)})$$

and there is still the natural $q$ analogue $E^{(q)}_n$ of the subspace $E_n$ which is invariant under $H_q$ and spanned by the basis

$$|e^{(n)}_1\rangle_q \equiv |(1), (2)\rangle |n\rangle_q, |e^{(n)}_2\rangle_q \equiv |(1), (1)\rangle |n + 1\rangle_q, |e^{(n)}_3\rangle_q \equiv |(1), (2)\rangle |n + 1\rangle_q, \quad (69)$$

and

$$|e^{(n)}_4\rangle_q \equiv |(1), (1)\rangle |n + 2\rangle_q. \quad (70)$$

A very similar calculation to the bosonic case yields

$$H_q^{(n)} = \begin{pmatrix} \Omega + |n\rangle \nu & \gamma \sqrt{|n + 1\rangle} & 0 \\ \gamma \sqrt{|n + 1\rangle} & [n + 1\rangle \nu & \gamma \sqrt{|n + 2\rangle} \\ 0 & \gamma \sqrt{|n + 2\rangle} & [n + 2\rangle \nu - \Omega \end{pmatrix}. \quad (71)$$
Again the vector $|e^{(n)}_2\rangle_q - |e^{(n)}_3\rangle_q$ is an eigenvector with eigenvalue $[n + 1] \nu$. Hence this model using generalised statistics based on the Jaynes-Cummings framework also does not lead to the $\omega$-effect.

One cannot gainsay that other more complicated models of ‘thermal’ baths may display the $\omega$-effect but clearly a rather standard model rejects quite emphatically the possibility of such an effect. This by itself is very interesting. It shows that the $\omega$-effect is far from a generic possibility for space-time foams. Just as it is remarkable that various versions of the paradigmatic ‘thermal’ bath as well as entanglement associated with non-commutative geometry cannot accommodate the $\omega$-effect, it is also remarkable that the D-particle foam model manages to do so very simply.

6 Conclusions

In this work we have discussed in detail primarily two classes of space-time foam models, which may characterise realistic situations of the (still elusive) theory of quantum gravity. They both involve non-unitary evolution. Relevant to this is, as mentioned earlier, standard models of non-commutative geometry, even though they incorporate fuzziness of space-time, are unable to reproduce the omega effect. In one of our models, inspired by non-critical string theory, string matter on a brane world interacts with D particles in the bulk. Recoil of the heavy D particles owing to interactions with the stringy matter produces a gravitational distortion which has a backreaction on the stringy matter. This distortion, consistent with a logarithmic conformal field theory algebra, depends on the recoil velocity of the D-particle. It is modelled by a stochastic metric and consequently can affect the different mass eigenstates of neutral mesons. There are two contributions to the stochasticity. One is due to the stochastic aspects of the recoil velocity found in low order loop amplitudes of open or closed strings (performed explicitly in the bosonic string theory model) while the other is the leading behaviour of an infinite resummation of the subleading order infrared divergences in loop perturbation theory. The former is assumed to fluctuate randomly, with a dispersion which is viewed as a phenomenological parameter. The latter for long enough capture times of the D-particle with stringy matter gives a calculable dispersion which dominates the phenomenological dispersion for the usual expectations of the quantum gravity scale. This gives a prediction for the order of magnitude of the CPTV $\omega$-like effect in the initial entangled state of two neutral mesons in a meson factory based on stationary (non-degenerate) perturbation theory for the gravitational dressing of the correlated meson state. The Klein-Gordon Hamiltonian in the induced stochastic gravitational field was used for the calculation of the dressing. The order of magnitude of $\omega$ may not be far from the sensitivity of immediate future experimental facilities, such as a possible upgrade of the DAΦNE detector or a B-meson factory.

This causes a CPTV $\omega$-like effect in the initial entangled state of two neutral mesons in a meson factory, of the type conjectured in [3]. Using stationary perturbation theory it was possible to give an order of magnitude
estimate of the effect: the latter is momentum dependent, and of an order which may not be far from the sensitivity of immediate future experimental facilities, such as a possible upgrade of the DAΦNE detector or a B-meson factory. A similar effect, but with a sinusoidal time dependence, and hence experimentally disentanglable from the initial-state effect, is also generated in this model of foam by the evolution of the system.

A second model of space-time foam, that of a thermal bath of gravitational degrees of freedom, is also considered in our work, which, however, does not lead to the generation of an \( \omega \)-effect. Compared to the initial treatment of thermal baths further significant generalisations have been considered here. In the initial treatment (using a model derived in the rotating wave approximation) the structure of the Hamiltonian matrix was block diagonal, the blocks being invariant 4-dimensional subspaces. The real significance of the block diagonal structure was unclear for the absence of the omega effect in this model and has remained a matter for debate. We have analyzed the model (now without the rotating wave approximation) and to our surprise the omega effect is still absent. Furthermore we have also considered q oscillators for the heat bath modes since from Matrix theory we know that energetic D-particles satisfy q statistics. For the model of q statistics that we have adopted there is again no \( \omega \)-effect in the q version of the thermal bath.

It is interesting to continue the search for other (and more) realistic models of quantum gravity, either in the context of string theory or in alternative approaches, such as loop quantum gravity, exhibiting intrinsic CPT Violating effects in sensitive matter probes. Detailed analyses of global data in relation to CPTV, including very sensitive probes such as high energy neutrinos, is a good way forward.

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