Nonplanar Integrability

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Abstract: In this article we study operators with a dimension \(\Delta \sim O(N)\) and show that simple analytic expressions for the action of the dilatation operator can be found. The operators we consider are restricted Schur polynomials. There are two distinct classes of operators that we consider: operators labeled by Young diagrams with two long columns or two long rows. The main complication in working with restricted Schur polynomials is in building a projector from a given \(S_{n+m}\) irreducible representation to an \(S_n \times S_m\) irreducible representation (both specified by the labels of the restricted Schur polynomial). We give an explicit construction of these projectors by reducing it to the simple problem of addition of angular momentum in ordinary non-relativistic quantum mechanics. The diagonalization of the dilatation operator reduces to solving three term recursion relations. The fact that the recursion relations have only three terms is a direct consequence of the weak mixing at one loop of the restricted Schur polynomials. The recursion relations can be solved exactly in terms of symmetric Kravchuk polynomials or in terms of Clebsch-Gordan coefficients. This proves that the dilatation operator reduces to a decoupled set of harmonic oscillators and therefore it is integrable.

Keywords: Giant Gravitons, AdS/CFT correspondence, super Yang-Mills theory.
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1. Introduction

Ultraviolet violet divergences inevitably appear in perturbative calculations of various correlation functions in the $\mathcal{N} = 4$ super Yang-Mills theory. These divergences require renormalization, which induces a mixing among operators with the same bare conformal dimension. The eigenvalues of the new eigenstates under conformal rescalings are a sum of the classical scaling dimension and a loop correction, the anomalous dimension. By studying the diagrams which contribute to the renormalization of these operators it is possible to obtain an expression for the dilatation operator. The eigenvalues of this operator are the anomalous dimensions. A key discovery has been the understanding that the dilatation operator can be identified with the Hamiltonian of an integrable spin chain [2]. This integrability has been found in the planar limit of the theory.

Integrability has proved to be a key ingredient towards finding the exact spectrum of composite operators in $\mathcal{N} = 4$ super Yang-Mills theory. Further, direct perturbative calculations become very cumbersome at high loop orders but can be avoided, by assuming that the observed integrability persists to all loop orders. The dilatation operator can then be determined using general arguments. Clearly, unraveling the integrable structures in the gauge theory is an important problem. In the AdS/CFT correspondence [1], these planar results are directly relevant to the problem of determining the exact spectrum of strings in the $\text{AdS}_5 \times \text{S}^5$ background.

Summing the planar diagrams gives a valid description of the large $N$ limit of those operators in the theory with a bare dimension $\Delta$ such that $\Delta^2/N \ll 1$. To correctly construct the large $N$ limit for operators in the theory with an even larger dimension, one has to sum much more than just the planar diagrams. There are many good reasons to study operators with a large bare dimension. Giant gravitons [4] are dual to operators with a dimension of $O(N)$ [5, 3, 6] while new background geometries are dual to operators with a dimension of $O(N^2)$ [3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. New methods sparked by the pioneering works [3], employing group representation techniques, probe the theory beyond the planar limit. It is now known how to build a basis for these operators, with the very nice feature that the basis diagonalizes the two point function [3, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Techniques to compute correlation functions of these large operators have been developed [30, 12, 13, 14] and some investigation into loop corrections have been carried out [19, 20, 26, 31, 32, 33, 34, 35]. Formulas for the one loop dilatation operator have been found and these results show that mixing at one loop is highly constrained. Much more work is needed. In particular, although formulas for the one loop dilatation operator are known, they are difficult to evaluate explicitly. Ultimately we would like both explicit formulas for the action of the dilatation operator as well as its eigenvalues and eigenvectors.
Special cases of the problem in the $SU(2)$ sector have been studied numerically in [33, 35]. The computations are rather involved and the expressions for the dilatation operator are complicated. However, upon solving for the spectrum of the dilatation operator numerically, one finds a strikingly simple result: the dilatation operator is equivalent to a sum of harmonic oscillators. Given this simplicity, one might hope that a general analytic approach is possible. In this article we study operators with a dimension $\Delta \sim O(N)$ and show that simple analytic expressions for the action of the dilatation operator can be found. Using this construction of the dilatation operator for this class of operators, we can test whether or not it is integrable. We will argue that the dilatation operator reduces to a decoupled set of harmonic oscillators and hence that it is indeed integrable.

The operators we consider are restricted Schur polynomials built using $n$ complex matrices $Z$ and $m$ complex matrices $Y$. They are labeled by three Young diagrams. The first Young diagram specifies a representation of the symmetric group $S_{n+m}$ and the second two a representation of $S_n \times S_m$. There are two distinct classes of operators that we consider: operators labeled by Young diagrams with two long columns or two long rows. Previous experience with the half BPS case [3] suggests that these operators are dual to sphere giants (long columns) or AdS giants (long rows). When we say the Young diagrams have two rows (or columns) we mean that all three Young diagram labels have two rows (or columns). By saying the columns (or rows) are large we mean that both of the columns (or rows) of the first Young diagram label contain $O(N)$ boxes.

The main complication in working with restricted Schur polynomials is in building a projector from a given $S_{n+m}$ irreducible representation to an $S_n \times S_m$ irreducible representation (both specified by the labels of the restricted Schur polynomial). One of the main technical advances of this article is an explicit and simple construction of these projectors. We will now outline the logic of this construction. The projectors organize (partially) labeled Young diagrams into irreducible representations of $S_m$. Our first step entails showing that these partially labeled Young diagrams can be traded for a (reducible) polynomial representation of the symmetric group. The advantage of the polynomial representation is that it admits the action of an operator $d$ introduced by Dunkl in his study of intertwining functions [36]. This operator is then used to construct a Casimir $d^\dagger d$ whose eigenspaces of definite eigenvalue are precisely the irreducible representations we are after. Thus, we are able to substitute the problem of constructing projectors with the eigenproblem of $d^\dagger d$. We then define one more map, which maps our original partially labeled Young diagrams into states of a spin chain. When acting on the spin chain $d^\dagger d$ has a particularly simple form and its eigenproblem is easily solved explicitly. Indeed, we reduce the problem of computing projectors to the problem of
addition of angular momentum in ordinary non-relativistic quantum mechanics! This allows us to give rather explicit and simple formulas for the action of the dilatation operator. The diagonalization of the dilatation operator reduces to solving three term recursion relations. The fact that the recursion relations have only three terms is a direct consequence of the weak mixing at one loop of the restricted Schur polynomials. At this point we find that the dilatation operator is very closely related to certain discrete models for the harmonic oscillator [37] and as a consequence the recursion relations can be solved exactly in terms of symmetric Kravchuk polynomials or in terms of Clebsch-Gordan coefficients. If we consider the limit in which the recursion relation can be replaced by a differential equation, which is the large $N$ limit that we have taken, we find a direct connection to the usual harmonic oscillator. In this way we claim that when acting on the class of operators belonging to the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory and having a dimension $\Delta \sim O(N)$ the dilatation operator is integrable.

One of the lessons learned from the study of the half BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory is that sphere giants and AdS giants are not independent solutions, but instead are dual to each other. Indeed, the half BPS states are in one to one correspondence with Young diagrams of $SU(N)$. A given Young diagram may be regarded either as labeling a collection of as many sphere giant gravitons as there are columns or as labeling a collection of as many AdS giant gravitons as there are rows. This correspondence can also be transparently seen from the 1/2 BPS geometries with a white annulus [7]. For example, a single AdS giant of angular momentum $k$ may equally well be thought of as collection of $k$ sphere giant gravitons, each with unit angular momentum. A similar equivalence works for the full spectrum of $1/8$ BPS states: the partition function obtained by quantizing the sphere giant graviton is the same as the partition function obtained by quantizing the AdS giant graviton [41, 42], see also [46]. Our computation of the dilatation operator at one loop uncovers an incredibly simple relation between the action of the dilatation operator on sphere giants and its action on AdS giants. Our result suggests that the duality discovered in the BPS sector may be enlarged to the non-BPS sector of the theory.

In the next section we study the problem in the context of the two sphere giant system. The new method used to construct the symmetric group projectors is most easily developed for AdS giants. Setting the problem up and constructing the projectors is accomplished in sections 3 and 4 respectively. The explicit expression for the action of the dilatation operator acting on AdS giants is the last formula obtained in section 4. Section 5 explains how to build the construction for sphere giants and uncovers a very simple relation between the action of the dilatation operator on sphere giants and its action on AdS giants. In section 6 we consider the problem of diagonalizing the
dilatation operator and obtain the relevant discrete wave equations. Finally we briefly
discuss interesting features of our results in Section 7.

2. Two Sphere Giant Gravitons

In this section we first review the action of the dilatation operator on a system of two
sphere giant gravitons. The two sphere giant graviton system is described by restricted
Schur polynomials labeled by Young diagrams that have two long columns. The ex-
pression for the dilatation operator has been evaluated explicitly in some special cases;
a general expression is not known. This action simplifies considerably in a particular
limit that is described in detail. Given this simplicity, the general construction of the
dilatation operator directly in this limit, is considered. We argue that the simplicity
of this limit can be exploited when constructing the dilatation operator by employing
Young’s orthogonal representation for the symmetric group. We are able to obtain a
general expression for the dilatation operator acting on the two sphere giant system
because we manage to give a general explicit construction of the projection operators
appearing in the definition of the restricted Schur polynomials.

2.1 Operators dual to a Two Sphere Giant System

There are six scalar fields $\phi^i_{ab}$ taking values in the adjoint of $u(N)$ in $\mathcal{N} = 4$ super
Yang Mills theory. Assemble these scalars into the three complex com-
binations

$$Z = \phi_1 + i\phi_2, \quad Y = \phi_3 + i\phi_4, \quad X = \phi_5 + i\phi_6.$$ 

We will study operators built using $O(N)$ $Z$ and $O(N)$ $Y$ fields. We always use $n$
to denote the number of $Z$s, $m$ to denote the number of $Y$s and will often refer to
the $Y$ fields as “impurities”. These operators have a large $\mathcal{R}$-charge and belong to
the $SU(2)$ sector of the theory. As a consequence of this large $\mathcal{R}$-charge, non-planar
contributions to the correlation functions of these operators are not suppressed at large
$N$. The computation of the anomalous dimensions of these operators is then a problem
of considerable complexity. This problem has been effectively handled by new methods
which employ group representation theory, allowing one to sum all diagrams (planar
and non-planar) contributing. The new methods provide bases for the local operators
which diagonalize the free two point function and which have highly constrained mixing
at the quantum level. For the applications that we have in mind, these bases are clearly
far superior to the trace basis. Mixing between operators in the trace basis with this
large $\mathcal{R}$-charge is completely unconstrained even at the level of the free theory. The
particular basis we employ is provided by the restricted Schur polynomials

$$\chi_{R,(r,s)}(Z^\otimes n, Y^\otimes m) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_{(r,s)}(\Gamma_R(\sigma)) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} Y_{i_{\sigma(n+1)}}^{i_{n+1}} \cdots Y_{i_{\sigma(n+m)}}^{i_{n+m}}.$$ 

$R$ is a Young diagram with $n + m$ boxes or equivalently an irreducible representation of $S_{n+m}$. $r$ is a Young diagram with $n$ boxes or equivalently an irreducible representation of $S_n$ and $s$ is a Young diagram with $m$ boxes or equivalently an irreducible representation of $S_m$. The $S_n$ subgroup acts on $1, 2, ..., n$ and therefore permutes indices belonging to the $Z$s. The $S_m$ subgroup acts on $n + 1, n + 2, ..., n + m$ and hence permutes indices belonging to the $Y$s. Taken together $(r, s)$ specify an irreducible representation of $S_n \times S_m$. Tr$(r, s)$ is an instruction to trace over the subspace carrying the irreducible representation$^1$ $(r, s)$ of $S_n \times S_m$ inside the carrier space for irreducible representation $R$ of $S_{n+m}$. This trace is easily realized by including a projector $P_{R \to (r,s)}$ (from the carrier space of $R$ to the carrier space of $(r, s)$) and tracing over all of $R$, i.e. $\text{Tr}(P_{R \to (r,s)} \Gamma_R(\sigma))$.

We are interested in operators dual to sphere giant gravitons that wrap a 3-sphere in the $S^5$ of the $\text{AdS}_5 \times S^5$ background. The two sphere giant system is described by restricted Schur polynomials labeled by Young diagrams with at most two columns. Further, the number of $Z$s in the operator is $\alpha N$ where $0 < 2 - \alpha \equiv \zeta < 1$ and the number of $Y$s is fixed to be $O(N)$. We have chosen the number of $Z$s so that both columns of the Young diagram are long, that is, they both always have $O(N)$ boxes. We will explain our choice for the number of $Y$s below. It is reasonable to assume that these restricted Schur polynomials are all excitations (including BPS states and non-BPS states) of the two giant system.

The mixing of these operators with restricted Schur polynomials that have $n \neq 2$ columns (or of even more general shape) is suppressed at least by a factor of order $\frac{1}{\sqrt{N}}^2$. This factor arises from the normalization of the restricted Schur polynomials. For example, the three column restricted Schur polynomials (with one short column - mixing is greatest for this type of operator) have a two point function which is smaller than the two point function of the two column restricted Schur polynomials by a factor of order $\frac{1}{N}[33]$. Thus, at large $N$ the two column restricted Schur polynomials do not mix with other operators, which is a huge simplification. This is the analog of the statement that for operators with a dimension of $O(1)$, different trace structures do

$^1$In general, because $(r, s)$ can be subduced more than once, we should include a multiplicity index. We will not write or need this index in this article.

$^2$Here we are talking about mixing at the quantum level. There is no mixing in the free theory. This suppression factor is equal to 1 over the square root of the number of boxes in second column of the Young diagram, and it is for this reason that we must ensure that the second column in the Young diagram has $O(N)$ boxes.
not mix at large $N$. The fact that the two column restricted Schur polynomials are a decoupled sector at large $N$ is expected: these operators correspond to a well defined stable semi-classical object in spacetime (the two giant system). In this large $N$ limit, and $b_0 \sim O(N)$, we can decouple the two giant system from the three giant system.

$n$ column restricted Schur polynomials are also a decoupled sector at large $N$, for the same reason.

We will end this section with a few comments on notation. The $m$ impurity operators are built using $n$ Zs and $m$ Ys. There are three Young diagrams labeling the restricted Schur polynomial, $\chi_{R,(r,s)}$. The representation $r$ (which specifies an irreducible representation of $S_n$) is specified by stating the number of rows with two boxes ($= b_0$) and the number of rows with a single box ($= b_1$). The representation $s$ specifies an irreducible representation of $S_m$ that has at most two columns, while $R$, which specifies an irreducible representation of $S_{n+m}$, can be specified in terms of $r$ by stating which boxes in $R$ are to be removed to obtain $r$. Once $R$ has been given, there are a finite number of possible labels $R, (r, s)$ even in the limit $N \to \infty$. To illustrate the problem, it is useful to first study examples with small $m$ of order $O(1)$. In the next subsection we will be interested in the case $m = 4$. In this case there are nine operators that can be produced once $R$ is given. These operators are given below:

\[
\begin{align*}
\chi_A(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_B(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_C(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_D(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_E(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_F(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_G(b_0, b_1) &= \chi_{(Z,Y)} \\
\chi_H(b_0, b_1) &= \chi_{(Z,Y)}
\end{align*}
\]
For the operators we study the number of rows with two boxes in the first and second Young diagram labels scale as $N$. Thus, the length of the columns in the first two labels are taken to be very large at large $N$. The third label is exactly as shown for the $m = 4$ case even at large $N$. The number of boxes removed from each column of the first label to obtain the second label are exactly as shown, even at large $N$. In what follows we will study the case for which $m$ is of order $O(N)$. Thus, in general, the column lengths of all the three Young diagrams are of order $O(N)$ so that they are very long and thus are not shown in the figure.

2.2 Simplified Action of the Dilatation Operator

The action of the one loop dilatation operator in the $SU(2)$ sector

$$D = -g^2_{YM} \text{Tr}[Y, Z][\partial_Y, \partial_Z]$$

on the restricted Schur polynomial has been studied in [33]. We will find it convenient to work with operators normalized to give a unit two point function. Towards this end, note that the two point function for restricted Schur polynomials has been computed in [24]

$$\langle \chi_{R,(r,s)}(Z,Y)\chi_{T,(t,u)}(Z,Y) \rangle = \delta_{R,(r,s)}\delta_{T,(t,u)}f_R \frac{\text{hooks}_R}{\text{hooks}_r \text{hooks}_s}.$$ 

In this expression $f_R$ is the product of the weights in Young diagram $R$ and hooks$_R$ is the product of the hook lengths of Young diagram $R$. Using this result, the normalized operators $O_{R,(r,s)}(Z,Y)$ can be obtained from

$$\chi_{R,(r,s)}(Z,Y) = \sqrt{f_R \frac{\text{hooks}_R}{\text{hooks}_r \text{hooks}_s}} O_{R,(r,s)}(Z,Y).$$

An important intermediate result is

$$D \chi_{R,(r,s)}(Z^\otimes n, Y^\otimes m) = \frac{g^2_{YM}}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m}} \text{Tr}_{(r,s)}(\Gamma_R((n,n+1)\psi - \psi(n,n+1))) \times$$

$$\times Z_{i_\psi(1)}^{i_\psi(n)} \cdots Z_{i_\psi(n-1)}^{i_\psi(n-1)} (YZ - ZY)^{i_\psi(n)} T^{i_{\psi(n+1)}} Y^{i_{\psi(n+2)}} \cdots Y^{i_{\psi(n+m)}}. \quad (2.1)$$

Notice that one of the indices associated to a $Z$ field and one of the indices associated to a $Y$ field participates. This is reflected in the formula below by the appearance of
the group element $\Gamma_R(n, n+1)$ which does not belong to the $S_m \times S_n$ subgroup. In terms of these normalized operators

$$DO_{R,(r,s)}(Z, Y) = \sum_{T,(t,u)} N_{R,(r,s); T,(t,u)} O_{T,(t,u)}(Z, Y)$$

$$N_{R,(r,s); T,(t,u)} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'}}{d_{R'} d_{T} d_{u} (n + m)} \sqrt{f_T Hooks_T Hooks_s Hooks_{R'} Hooks_{u}} \times$$

$$\times \text{Tr} \left( \left[ \Gamma_R((n, n+1)), P_{R\rightarrow (r,s)} \right] I_{R' T'} \left[ \Gamma_T((n, n+1)), P_{T\rightarrow (t,u)} \right] I_{T' R'} \right).$$

The $c_{RR'}$ is the weight of the corner box removed from Young diagram $R$ to obtain diagram $R'$, and similarly $T'$ is a Young diagram obtained from $T$ by removing a box. The interwiner $I_{AB}$ is a map from the carrier space of irreducible representation $A$ to the carrier space of irreducible representation $B$. Consequently, $A$ and $B$ must be Young diagrams of the same shape. The interwiner operator relevant for our study will be evaluated in sections 2.4, 3.3.

This last expression has been evaluated in [33] for the case of two impurities and in [35] for the case of three or four impurities. The results are rather complicated. However, in the limit that $N - b_0 = O(N)$, $b_0 = O(N)$ and $b_1 = O(N)$ the dynamics simplifies considerably. For $m = 4$ impurities, the action of the dilatation operator becomes

$$DO_A(b_0, b_1) = g_{YM}^2 \Delta O_A(b_0, b_1) \times O \left( \frac{1}{b_1} \right)$$

$$DO_B(b_0, b_1) = -\frac{3}{2} g_{YM}^2 \Delta O_B(b_0, b_1) + \frac{\sqrt{3}}{2} g_{YM}^2 \Delta O_C(b_0, b_1)$$

$$DO_C(b_0, b_1) = \sqrt{3} g_{YM}^2 \Delta O_B(b_0, b_1) - \frac{1}{2} g_{YM}^2 \Delta O_C(b_0, b_1)$$

$$DO_D(b_0, b_1) = -2 g_{YM}^2 \Delta O_D(b_0, b_1) + \frac{2}{\sqrt{3}} g_{YM}^2 \Delta O_E(b_0, b_1)$$

$$DO_E(b_0, b_1) = -2 g_{YM}^2 \Delta O_E(b_0, b_1) + \frac{2}{\sqrt{3}} \Delta O_D(b_0, b_1) + \frac{2\sqrt{6}}{3} \Delta O_F(b_0, b_1)$$

$$DO_F(b_0, b_1) = -2 \Delta O_F(b_0, b_1) + \frac{2\sqrt{6}}{3} \Delta O_E(b_0, b_1)$$

$$DO_G(b_0, b_1) = -\frac{3}{2} g_{YM}^2 \Delta O_G(b_0, b_1) + \frac{\sqrt{3}}{2} g_{YM}^2 \Delta O_H(b_0, b_1)$$

$$DO_H(b_0, b_1) = -\frac{1}{2} g_{YM}^2 \Delta O_H(b_0, b_1) + \frac{\sqrt{3}}{2} g_{YM}^2 \Delta O_G(b_0, b_1)$$
\[
DO_I(b_0, b_1) = g_{YM}^2 \Delta O_I(b_0, b_1) \times O \left( \frac{1}{b_1} \right) 
\] (2.11)

where

\[
\Delta O_X(b_0, b_1) = \sqrt{(N - b_0 - b_1)(N - b_0)}(O_X(b_0 + 1, b_1 - 2) + O_X(b_0 - 1, b_1 + 2)) 
- (2N - 2b_0 - b_1)O_X(b_0, b_1). 
\] (2.12)

with \( X = A, B, ..., I \). A slightly different limit was considered in [35] - there it was assumed that \( b_1 \sim O(\sqrt{N}) \). Since \( b_1 \) sets the difference in angular momentum of the two giants and the angular momentum of the two giants sets their radii [44], this limit sets the distance between the two giants to be string size. This is the natural limit to look for (open) stringy excitations of the giants. As we will see, the limit we consider here will allow us to reproduce spectra obtained by numerically diagonalizing the dilatation operator. We can easily identify combinations of operators that are annihilated by \( D \). Apart from \( O_A(b_0, b_1) \) and \( O_I(b_0, b_1) \) we have \( O_B(b_0, b_1) + \sqrt{3}O_C(b_0, b_1), O_D(b_0, b_1) + \sqrt{2}O_E(b_0, b_1) \) and \( O_F(b_0, b_1) + \sqrt{3}O_H(b_0, b_1) \). If we set \( \sqrt{3}O_B(b_0, b_1) - O_C(b_0, b_1) \equiv O_{B-C}(b_0, b_1), \sqrt{2}O_D(b_0, b_1) - O_E(b_0, b_1) \equiv O_{D-E}(b_0, b_1), \sqrt{3}O_G(b_0, b_1) - O_H(b_0, b_1) \equiv O_{G-H}(b_0, b_1) \) and \( \sqrt{3}O_F(b_0, b_1) \equiv O_{F}(b_0, b_1) \), we have

\[
DO_{B-C}(b_0, b_1) = -2g_{YM}^2 \Delta O_{B-C}(b_0, b_1) \\
DO_{D-E}(b_0, b_1) = -2g_{YM}^2 \Delta O_{D-E}(b_0, b_1) \\
DO_{D-F}(b_0, b_1) = -4g_{YM}^2 \Delta O_{D-F}(b_0, b_1) \\
DO_{G-H}(b_0, b_1) = -2g_{YM}^2 \Delta O_{G-H}(b_0, b_1) 
\] (2.13)

An exact diagonalization of the dilatation operator gives the spectrum of nine harmonic oscillators. Five of the oscillators have a level spacing \( \omega = 0 \), three have a level spacing \( \omega = 8g_{YM}^2 \) and one has a level spacing \( \omega = 16g_{YM}^2 \). These results can all be obtained from the simplified action of the dilatation operator given above. The generic form of the equation coming from the action of the dilatation operator is

\[
DO(b_0, b_1) = -\alpha g_{YM}^2 [\sqrt{(N - b_0)(N - b_0 - b_1)}(O(b_0 + 1, b_1 - 2) + O(b_0 - 1, b_1 + 2)) 
- (2N - 2b_0 - b_1)O(b_0, b_1)] 
\] (2.14)

which corresponds to an oscillator of level spacing \( 4\alpha g_{YM}^2 \). The derivation of this oscillator level spacing is in Appendix B. The goal of this article is to derive these simplified equations directly in the \( b_1 \sim O(N) \) limit and then to diagonalize them analytically. The main difficulty in deriving these equations is in the explicit construction of the projectors used in defining the restricted Schur polynomials.
2.3 Young’s Orthogonal Representation

There is a representation for the symmetric group, Young’s orthogonal representation, that we will make extensive use of. To define the representation, we will give the rule which determines matrix elements of the matrices representing adjacent permutations, that is, permutations of the form \((i, i+1)\), where \(i\) denotes the \(i\)th box. The matrix representing any other element of the group can easily be constructed as some product of the matrices representing the adjacent permutations. We will use the Young-Yamanouchi basis in which the boxes in the Young diagram are numbered. We choose our conventions so that if the boxes are removed according to the above numbering, removing box 1 first, box 2 second and so on, at each step one must always obtain a legal Young diagram. Thus, for the two Young diagrams shown below, the diagram on the right corresponds to a valid state; the one on the left does not.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 2 & 1 & 5 \\
5 & 4 & 3 & 2 \\
\end{array}
\]

The dimension of the \(S_5\) irreducible representation labeled by \(\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 2 & 1 & 5 \\
5 & 4 & 3 & 2 \\
\end{array}\) is 5. The reader can check that there are five possible labels each giving a state in the carrier space of this irreducible representation. Recall that to each labeled box we can associate a weight. Denote the weight of the box labeled \(i\) by \(c_i\). If box \(i\) is in row \(b\) and column \(a\), it has \(c_i = N + a - b\). We can now state the rule for the action of adjacent transpositions: If \((i, i+1)\) acts on a given state, it gives the same state back with coefficient \(\frac{1}{c_i-c_{i+1}}\) and it gives the state corresponding to the Young diagram with \(i\) and \(i+1\) swapped with coefficient \(\sqrt{1 - \frac{1}{(c_i-c_{i+1})^2}}\). Here are a few examples

\[
\begin{array}{cccc}
\text{State 1} & \text{State 2} & \text{State 3} \\
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}
\]

\[
\Gamma(\text{\(\{(12)\}\)}} \left| \begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}\right) = \frac{1}{2} \left| \begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}\right| + \frac{\sqrt{3}}{2} \left| \begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}\right|
\]

\[
\Gamma(\text{\(\{(12)\}\)}} \left| \begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}\right) = \frac{1}{2} \left| \begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}\right| + \frac{\sqrt{3}}{2} \left| \begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
\end{array}\right|
\]

\[\text{Footnote: Here to define the representation of } S_n \text{ we have used the weights } c_i \text{ which depend on } N. \text{ The } S_n \text{ group should know nothing about } N. \text{ Since only differences between weights appear, this is indeed the case.}\]
Young’s orthogonal representation is particularly useful because it simplifies dramatically when \( \frac{m}{m_0} \ll 1 \). Since the generic operator we consider has \( b_1 = O(N) \), we will consider \( m = \gamma N \) impurities with \( \gamma \sim O(N^0) \ll 1 \). In this simplification, we can consider the ratio \( \alpha/\gamma \sim O(N^0) \gg 1 \) as a fixed large number, which does not need to scale as \( N \) to some positive power. Indeed, if the boxes \( i \) and \( i + 1 \) are in the same column, \( i + 1 \) must sit above \( i \) so that

\[
\Gamma_R ((i, i + 1)) |\text{same column state}\rangle = - |\text{same column state}\rangle \tag{2.15}
\]

If \( i \) and \( i + 1 \) are in different columns, then since \( b_1 = O(N) \), even if we stack all of the impurities in the longer column \( c_i - c_{i+1} \) must itself be \( O(N) \). In this case, at large \( N \) replace \( \frac{1}{c_i - c_{i+1}} = O(b_1^{-1}) \) by 0 and \( \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1}) \) by 1 so that

\[
\Gamma_R ((i, i + 1)) |\text{different column state}\rangle = |\text{swapped different column state}\rangle \tag{2.16}
\]

The notation in this last equation is indicating two things: \( i \) and \( i + 1 \) are in different columns and the states on the two sides of the equation differ by swapping the \( i \) and \( i + 1 \) labels. All of the states which enter the trace in the restricted Schur polynomial (and hence in the dilatation operator) belong to a particular \( S_n \times S_m \) subspace. As we explain now, it is possible to directly extract part of this subspace and hence one need not work with the general state in \( R \). Start numbering our states using a Young-Yamanouchi basis. We can obtain \( r \) by removing boxes from \( R \). If we only number the first \( m \) boxes and further only boxes that are to be removed are numbered, then each partially labeled Young diagram stands for a collection of states, all belonging to the correct \( S_n \) subspace. Thus, all we need to do now is to take the correct combinations of these states (i.e. of the partially labeled Young diagrams) to get the required \( S_m \) subspace. The group element \( \Gamma(n, n + 1) \) appears in the expression of the dilatation operator, and it acts on one of the \( m \) boxes associated to the impurities and one box associated to the \( Z \)s. Taken together, these facts imply that all of the operators that we need to consider only have a non-trivial action on the first \( m + 1 \) boxes. Thus we will not label all of the boxes — it is good enough to label the first \( m + 1 \) boxes. Bear in mind that each partially labeled Young diagram corresponds to a collection of states. An example illustrating these rules: (assuming just \( m = 2 \) impurities so that we only

\[\text{Recall that } m \text{ is the number of Y's.}\]
label 3 boxes)

\[ \Gamma_R ((1,2)) |3 \rangle = |3 \rangle \quad \Gamma_R ((1,2)) |3 \rangle = - |3 \rangle \]

Thus, the representations of the symmetric group simplify dramatically in this limit.

### 2.4 Computation of the Dilatation Operator

The dilatation operator includes the coefficient

\[
-g^2_Y c_{RR'} d_T d_{R^{m+1}} n m \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_s \text{hooks}_u}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}}. 
\]

In the above formula \( d_{R^{m+1}} \) labels an irreducible representation of \( S_{n-1} \) and is obtained from \( R \) (which labels an irreducible representation of \( S_{n+m} \)) by removing \( m + 1 \) boxes. The dimension factor \( d_{R^{m+1}} \) is included in the above coefficient for convenience - it naturally appears when we evaluate the trace

\[
\text{Tr} \left( \left[ \Gamma_R ((n, n+1)), P_{R \rightarrow (r,s)} \right] I_{R'}^{T'} \left[ \Gamma_T ((n, n+1)), P_{T \rightarrow (t,u)} \right] I_T^{R'} \right). \tag{2.17}
\]

There are different choices for the specific \( R^{m+1} \)s but, in the limit that we consider, they all give the same contribution. In the limit we consider

\[
\frac{n}{n + m} = \frac{\alpha}{\alpha + \gamma} \rightarrow 1 \quad \frac{d_{R^{m+1}}}{d_{R'}} = 2^{-m} \quad d_u = \frac{m!}{\text{hooks}_u}
\]

so that

\[
-g^2_Y c_{RR'} d_T d_{R^{m+1}} n m \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_s \text{hooks}_u}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}} = -g^2_Y \sqrt{c_{RR'} c_{TT'}} \sqrt{\text{hooks}_s \text{hooks}_u} \frac{1}{(m - 1)!}. \tag{2.18}
\]
All that remains in the evaluation of $D$ is the computation of the trace (2.17). Recall that, because we have only labeled $m + 1$ boxes, when we talk about a “state” we actually mean a collection of states. All of the operators we specify below have the same action on the collection of states being considered. Since we are talking about collections of states the reader should bear in mind that we have often suppressed a factor of the identity matrix acting on the collection of states. We do this in all formulas that follow. It is straightforward to verify that for the case of $m$ impurities, the intertwiners are $2^{m+1} \times 2^{m+1}$ matrices, that there are four possible intertwiners and that their non zero elements are \(^5\)

$$(I_1)_{ij} = 1, \quad i = j = 1, 2, 3, ..., 2^m$$

(this intertwiner is used when evaluating terms in $N_{R,(r,s);T,(t,u)}$ for which $R = T$ and the box is removed from the first column of $R$ to obtain $R'$)

$$(I_2)_{ij} = 1, \quad i = j = 2^m + 1, 2^m + 2, 2^m + 3, ..., 2^{m+1}$$

(again $R = T$ but now the box is removed from the second column of $R$ to obtain $R'$) for the terms with no changes in $R$ and

$$(I_3)_{ij} = 1, \quad j = i + 2^m = 2^m + 1, 2^m + 2, 2^m + 3, ..., 2^{m+1}$$

(now $R \neq T$ and the box is removed from the first column in $R$ to obtain $R'$ and from the second column in $T$ to obtain $T'$)

$$I_4 = (I_3)^T$$

(again $R \neq T$ and the box is removed from first column in $T$ to obtain $T'$ and from the second column in $R$ to obtain $R'$). The non zero elements of $\Gamma ((n, n + 1))$ are

$$\Gamma ((n, n + 1))_{ij} = -1, \quad i = j = 1 + 2p \quad p = 0, 1, ..., 2^{m-1} - 1$$

$$\Gamma ((n, n + 1))_{ij} = -1, \quad i = j = 2^m + 2 + 2p \quad p = 0, 1, ..., 2^{m-1} - 1$$

$$\Gamma ((n, n + 1))_{ij} = (-1)^{m-1}, \quad i = 2 + 2p, \quad j = 2^m + 1 + 2p \quad p = 0, 1, 2^m - 1$$

$$\Gamma ((n, n + 1))_{ij} = (-1)^{m-1}, \quad j = 2 + 2p, \quad i = 2^m + 1 + 2p \quad p = 0, 1, 2^m - 1$$

Next one needs to compute the projector and then the trace. Without a few new ideas it is not possible to find formulas for general $m$ for the projectors used in constructing restricted Schur polynomials labeled by Young diagrams that have two long columns. These new ideas are most easily developed by studying restricted Schur polynomials labeled by Young diagrams with two long rows. This is the case that we study next.

\(^5\)This next formula nicely illustrates our conventions. The indices $i$ and $j$ do not select a unique state - they select a collection of states. Thus, there is an identity matrix acting on the collection of states which is suppressed on the right hand side of this equation.
3. Two AdS Giant Gravitons

In this section we first review the action of the dilatation operator on a system of two AdS giant gravitons. This two giant graviton system is described by restricted Schur polynomials labeled by Young diagrams that have two long rows. We again argue that the action of the dilatation operator simplifies considerably in a particular limit. The general construction of the dilatation operator directly in this limit, is considered. We argue that Young’s orthogonal representation for the symmetric group in this setting reduces to a representation that is known and has been studied. Using these results we introduce an operator $d^\dagger d$ whose eigenspaces are irreducible representations of the symmetric group. Thus, a general explicit construction of the projection operators appearing in the definition of the restricted Schur polynomials is reduced to solving the eigenvalue problem of $d^\dagger d$. We will use these results to give an explicit formula for the dilatation operator acting on a general system of two AdS giant gravitons in the next section.

3.1 Operators dual to a Two AdS Giant System

We have focused on operators dual to sphere giant gravitons that wrap a 3-sphere in the $S^5$ of the AdS$_5 \times S^5$ background. In this section we would like to focus on giant gravitons that are large in the AdS directions but continue to carry angular momentum on the $S^5$. The two AdS giant system is described by restricted Schur polynomials labeled by Young diagrams with at most two rows. We will continue to assume that the number of Zs in the operator is $\alpha N$ where $\alpha \sim O(N^0) > 1$ and the number of Ys is fixed to be $\gamma N$ with $\gamma \sim O(N^0) \ll 1$. The mixing of these operators with restricted Schur polynomials that have $n \neq 2$ rows (or of even more general shape) is again suppressed at least by a factor of order $\frac{1}{N^6}$. The fact that the two row restricted Schur polynomials are a decoupled sector at large $N$ is again expected: these operators correspond to a well defined stable semi-classical object in spacetime (the two AdS giant system).

We only need to make very minor changes in our notation. There are still three Young diagrams labeling the restricted Schur polynomial, $\chi_{R,(r,s)}$. The representation $r$ (which specifies an irreducible representation of $S_n$) is specified by stating the number of columns with two boxes ($= b_0$) and the number of columns with a single box ($= b_1$). The representation $s$ specifies an irreducible representation of $S_m$ that has at most two rows, while $R$, which specifies an irreducible representation of $S_{n+m}$, can be specified in terms of $r$ by stating which boxes in $R$ are to be removed to obtain $r$. Recall that

---

Here we are again talking about mixing at the quantum level. There is again no mixing in the free theory.
in the sphere giant system, once $r$ has been given there are a finite number of possible labels $R, (r, s)$ even in the limit $N \to \infty$. This is also the case for the AdS giant system.

The two AdS giant system and two sphere giant system have an important difference. For the two sphere giant system the parameter $b_1$ was bounded from above and the parameter $b_0$ was bounded from below. Indeed, the largest possible value for $b_1$ and the smallest possible value for $b_0$ were obtained when the first column contains $N$ boxes. Of course these bounds on $b_0, b_1$ are a consequence of the fact that the number of boxes in any given column is bounded by $N$. For the AdS giant system the number of boxes in the first row is not bounded, so that $b_0$ can range all the way down to 0. For the sphere giant case the fact that $b_0$ was bounded from below, and the fact that this bound was $O(N)$, implied a clean decoupling of the two sphere giant system. For the AdS giants, when $b_0 \ll N$ we are transitioning to the state of one giant graviton plus strings, or plus Kaluza-Klein gravitons, or both. The two AdS giant system is still effectively decoupled because it would take a very long time\(^7\) before we decay from the two giant sector. This point does however need to be considered when we study the dilatation operator numerically. In our numerical studies we will only consider Young diagrams $R$ with two rows. This is strictly speaking not always justified because when the second row contains $O(1)$ boxes mixing with Young diagrams with one long row and more than one short row is not suppressed. We will discuss this point further below (see section 7) and argue that it does not affect the accuracy of our results.

3.2 Dilatation Operator

For the two sphere giant system we have seen that when $b_1 \gg 1$ the action of the dilatation operator simplifies dramatically. Is this also the case for the action of the dilatation operator on the two AdS giant system? To compute the action of the dilatation operator we need to again evaluate (2.2), but now for the case that $R$ and $T$ have two rows. The result of this computation for the case of two impurities is given in Appendix A. There are a number of points worth noting. First, there is a simple relation between the sphere giant and AdS giant results. To obtain the action of the dilatation operator on the AdS giant system from the action of the dilatation operator on the sphere giant system one simply replaces the sphere weights $N-a$ to $N+a$. This is completely explicit if the reader compares the first two expressions in Appendix A to the expressions appearing in [33]. In the last two expressions of Appendix A, this is not manifest because we have combined terms. Second, the numerical spectrum for the AdS giant system is surprisingly similar to the spectrum obtained for the sphere giant

\(^7\)What we mean by this is that we need to apply the Hamiltonian $O(N)$ times before we leave the two giant sector.
system. 3/4 of the states are massless while 1/4 of the states match those of an oscillator with an energy spacing of $8g^2_{YM}$. This looks identical to what was obtained for the sphere giants. One important difference however, is the fact that for the sphere giant system the levels were of the form $8g^2_{YM}n$ with $n = 1, 2, 3, ...$ where as for the AdS giant system the levels are of the form $4g^2_{YM} + 8g^2_{YM}n$ with $n = 1, 2, 3, ...$. Particularly for the low levels, it seems that these results can be trusted, because the operator with good scale dimension receives no contribution from restricted Schur polynomials with one very long column and one short column. Finally, perhaps the most important result we find is that in the limit that $\frac{b_0}{m} \gg 1$ the action of the dilatation operator reduces to a collection of equations of the form

$$DO(b_0, b_1) = -\alpha g^2_{YM} \left[ \sqrt{(N + b_0)(N + b_0 + b_1)}(O(b_0 + 1, b_1 - 2) + O(b_0 - 1, b_1 + 2)) ight. - (2N + 2b_0 + b_1)O(b_0, b_1) \right].$$

(3.1)

This corresponds to an oscillator of level spacing $4\alpha g^2_{YM}$. More detailed derivations are in Appendix B. This result is again related to our previous result for the sphere giant system, by replacing for example $N - b_0 \rightarrow N + b_0$.

### 3.3 Young’s Orthogonal Representation

In the case of AdS giants, Young’s orthogonal representation reduces to a representation which has already been studied in the mathematics literature [38]. The reader may wish to consult Appendix C where we review a little of the relevant background. The idea is to define a map from a labeled Young diagram to a monomial. We will consider the case of $m$ $Y$ fields. In this case, consider a labeled Young diagram that has $m$ boxes labeled; the labels are distributed arbitrarily between the upper and lower rows. Ignore the boxes that appear in the lower row. For boxes labeled $i$ in the upper row include a factor of $x_i$ in the monomial. If none of the boxes in the first row are labeled, the Young diagram maps to 1. Thus, for example, when $m = 4$

\[
\begin{array}{cccc}
\tiny{1} & \tiny{2} & \tiny{3} & \tiny{4} \\
\tiny{1} & \tiny{2} & \tiny{3} & \tiny{4} \\
\end{array}
\leftrightarrow x_3
\quad
\begin{array}{cccc}
\tiny{1} & \tiny{2} & \tiny{3} & \tiny{4} \\
\tiny{1} & \tiny{2} & \tiny{3} & \tiny{4} \\
\end{array}
\leftrightarrow x_1x_2x_3
\]

The symmetric group acts by permuting the labels on the factors in the monomial. Thus, for example, $(12)x_1x_3 = x_2x_3$. This defines a reducible representation of the symmetric group, $S_m$. It is clear that the operator

$$d = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}$$

(3.2)

\footnote{The contribution is zero to the accuracy of our numerical diagonalization.}

\footnote{Up to now we have labeled $m + 1$ boxes when considering $m$ impurities. For the present discussion where we want to understand how to decompose into irreducible $S_m$ representations it is more convenient to label only $m$ boxes.}
commutes with the action of the symmetric group. This operator was introduced by Dunkl in the study of intertwining functions [36]. It acts on the monomials by producing the sum of terms that can be produced by dropping one factor at a time. For example

$$d(x_1 x_2 x_3) = x_2 x_3 + x_1 x_3 + x_1 x_2.$$  

The adjoint\(^{10}\) \(d^\dagger\) produces the sum of monomials that can be obtained by appending a factor, without repeating any of the \(x_i\)s (this is written for \(m = 4\) impurities but the generalization to any \(m\) is obvious)

$$d^\dagger(x_1 x_2) = x_1 x_2 x_3 + x_1 x_2 x_4.$$  

The fact that \(d\) commutes with all elements of the symmetric group, implies that \(d^\dagger\) will too. Indeed, take the dagger of

$$[d, \Gamma(\sigma)] = 0$$

to obtain (use the fact that we are working in an orthogonal representation in the next line)

$$[d^\dagger, \Gamma(\sigma)] = -([d, \Gamma(\sigma^{-1})])^\dagger = 0.$$  

Thus, \(d^\dagger d\) will also commute with all the elements of the symmetric group and consequently its eigenspaces will furnish representations of the symmetric group. These eigenspaces are irreducible representations - consult [38] for further details and results. This last fact implies that the problem of computing the projectors needed to define the restricted Schur polynomials can be replaced by the problem of constructing projectors onto the eigenspaces of \(d^\dagger d\). This amounts to solving for the eigenvectors and eigenvalues of \(d^\dagger d\). In the next section we will argue that this is a surprisingly simple problem.

For the remainder of this section we switch back to our previous convention and again label \(m + 1\) boxes. See section 2.4 for a careful description of our notation. The intertwiners are again \(2^{m+1} \times 2^{m+1}\) matrices, and again there are four possible intertwiners. Their non zero elements are

\(^{(I_1)}_{ij} = 1, \quad i = j = 1, 2, 3, \ldots, 2^n\)

\((R = T \text{ and the box is removed from first column})\)

\(^{(I_2)}_{ij} = 1, \quad i = j = 2^m + 1, 2^m + 2, 2^m + 3, \ldots, 2^{m+1}\)

\(^{10}\)Consult Appendix C for details on the inner product on the space of monomials.
(R = T and the box is removed from second column) for the terms with no changes in R and

$$(I_3)_{ij} = 1, \quad j = i + 2^m = 2^m + 1, 2^m + 2, 2^m + 3, \ldots, 2^{m+1}$$

(the box is removed from first column in R and from the second column in T)

$$I_4 = (I_3)^T$$

(the box is removed from first column in T and from second column in R). The non zero elements of $\Gamma((n, n + 1))$ are

$$\Gamma((n, n + 1))_{ij} = 1, \quad i = j = 1 + 2p \quad p = 0, 1, \ldots, 2^{m-1} - 1$$

$$\Gamma((n, n + 1))_{ij} = 1, \quad i = j = 2^m + 2 + 2p \quad p = 0, 1, \ldots, 2^{m-1} - 1$$

$$\Gamma((n, n + 1))_{ij} = 1, \quad i = 2 + 2p, \quad j = 2^m + 1 + 2p \quad p = 0, 1, 2^m - 1$$

$$\Gamma((n, n + 1))_{ij} = 1, \quad j = 2 + 2p, \quad i = 2^m + 1 + 2p \quad p = 0, 1, 2^m - 1$$

### 3.4 Computation of the Dilatation Operator

To get to the AdS giant case, we have “flipped” the Young diagrams swapping rows and columns, and mapped the weights $N - b \rightarrow N + b$. It is easy to check that the dimension of any Young diagram R is the same as the dimension of the flipped diagram, and (this is the same fact that) the product of hooks is unchanged. The weights $c_{RR'}$ are now for example $N + b_0$ instead of $N - b_0$.

Thus, the coefficient needed to compute the dilatation operator is

$$-g_2^2 \frac{c_{RR'} d_T d_{R^{m+1}} n m}{d_R d_i d_u(n + m)} \sqrt{\frac{f_R^{\text{hooks}} s_R^{\text{hooks}} t_R^{\text{hooks}} u_R^{\text{hooks}}}{f_T^{\text{hooks}} s_T^{\text{hooks}} t_T^{\text{hooks}} u_T^{\text{hooks}}}} = -g_2^2 \frac{\sqrt{c_{RR'} c_{TT'} \sqrt{\text{hooks}}}}{(m - 1)!}$$

$$d_{R^{m+1}}$$ again labels an irreducible representation of $S_{n-1}$ and is again obtained from $R$ (which labels an irreducible representation of $S_{n+m}$) by removing $m + 1$ boxes. The dimension factor $d_{R^{m+1}}$ is again included in the above coefficient for convenience - it naturally appears when we evaluate the trace (2.17). All that we need to evaluate now are the traces (2.17). We will show how to evaluate these for general $m$ in the next section.

### 4. Construction of AdS Giant Projectors and the Dilatation Operator

In this section we define a map from labeled Young diagrams to spin chain states. The operator $d^d d$ takes a particularly simple form and its eigenvalue value problem
is solved explicitly. These results then allow a general construction of the projection operators used to define the restricted Schur polynomials and then ultimately of the dilatation operator itself. Our final formula for the dilatation operator is given as the last equation of this section.

We can map the labeled Young diagrams into states of a spin chain, for general $m$. The spin at site $i$ can be in state spin up (1) or state spin down (0). Initially in this section it is again convenient to label only $m$ boxes. In this case, the spin chain has $m$ sites and the box $i$ tells us the state of site $i$. If box $i$ appears in the first row, site $i$ is in state 1; if it appears in the second row site $i$ is in state 0. For example,

$$
\begin{array}{c|c|c|c|c|c|c|c}
\hline
& & & & & & & \\
\hline
& & & & & & & \\
\hline
\end{array} \leftrightarrow |1100\rangle
$$

$d^dd$ has a very simple action on this spin chain as we now explain: Introduce the states

$$
|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

for the two spins and the operators

$$
a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad a^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

which act on these states

$$
a^\dagger |0\rangle = |1\rangle, \quad a^\dagger |1\rangle = 0
$$

$$
a |1\rangle = |0\rangle, \quad a |0\rangle = 0.
$$

Here is an example of a state of the spin chain

$$
|001011\rangle = |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle
$$

for a system with 6 lattice sites. Label the sites starting from the left, as site 1, then site 2 and so on till we get to the last site, which is site 6. The operator $a$ acting at the third site (for example) is

$$
a_3 = 1 \otimes 1 \otimes a \otimes 1 \otimes 1 \otimes 1.
$$

We can then write $d^dd$ as the following operator

$$
d^dd = \sum_p \sum_n a^\dagger_n a_p. \quad (4.1)
$$
This is a long ranged spin chain. In terms of the Pauli matrices
\[ \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
we can write
\[ a = \frac{1}{2}(\sigma^1 + i\sigma^2), \quad a^\dagger = \frac{1}{2}(\sigma^1 - i\sigma^2), \quad a^\dagger a = \frac{1}{2}(1 - \sigma^3), \]
\[ d^\dagger d = \frac{1}{4} \sum_p \sum_n (\sigma^1 - i\sigma^2)_n (\sigma^1 + i\sigma^2)_p = -\sum_p \frac{1}{2} \sigma^3_p + \frac{1}{4} \sum_p \sum_n (\sigma^1_n \sigma^1_p + \sigma^2_n \sigma^2_p). \quad (4.2) \]

The total spins of the system are
\[ J^3 = \sum_p \frac{1}{2} \sigma^3_p, \quad J^1 = \sum_p \frac{1}{2} \sigma^1_p, \quad J^2 = \sum_p \frac{1}{2} \sigma^2_p, \]
\[ J^2 = J^3 J^3 + J^1 J^1 + J^2 J^2. \]

We use capital letters for operators and little letters for eigenvalues. In terms of these total spins we have
\[ d^\dagger d = -J^3 + (J^2 - (J^3)^2) = J^2 - J^3 (J^3 + 1). \]

Thus, eigenspaces of \( d^\dagger d \) can be labeled by the eigenvalues of \( J^2 \) and eigenvalues of \( J^3 \), and hence the labels \( R, (r, s) \) of the restricted Schur polynomial can be traded for these eigenvalues. We will illustrate the connection using a specific example and then state the general rule. Consider the case of 8 spins. The impurities can be organized into any irreducible representation corresponding to a Young diagram with 8 blocks and two rows. The possible irreducible representations are

\[
\begin{array}{cccccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

and they have dimensions 14, 28, 20, 7 and 1 respectively. Coupling 8 spin-\( \frac{1}{2} \) particles using the usual rule for the addition of angular momentum, we have
\[ \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 14 \oplus 28 \oplus 20 \oplus 7 \oplus 3 \oplus 4. \]

Notice that the number of angular momentum multiplets matches the number of possible Young diagrams for the impurities and the degeneracy of each multiplet matches the dimension of the \( S_8 \) irreducible representation associated to the Young diagram.
These connections are a consequence of the Schur-Weyl duality between the symmetric groups and the unitary groups.

The general rule is now clear: consider the restricted Schur polynomial $\chi_{R,(r,s)}$. The $J^2 = j(j+1)$ quantum tells you the shape of the Young diagram $s$ that organizes the impurities. If there are $N_1$ boxes in the first row of $s$ and $N_2$ boxes in the second, then $2j = N_1 - N_2$. The $J^3$ eigenvalue of the state is always a good quantum number, both in the basis we start in where each spin has a sharp angular momentum or in the basis where the states have a sharp total angular momentum. The $j^3$ quantum number tells you how many impurities sit in the first and second rows of $R$, that is, it tells you how many boxes must be removed from each row of $R$ to obtain $r$. Denote the number of boxes removed from the first row by $n_1$ and the number of impurities in the second row by $n_2$. We have $2j^3 = n_1 - n_2$. Here are some examples of the projection operators appearing in the restricted Schur polynomials, written in terms of the $j,j^3$ states:

\[
P_R, (r,s) = \sum_{i=1}^{d} |j = 0, j^3 = 0, i\rangle \langle j = 0, j^3 = 0, i| \]

\[
P_R, (r,s) = \sum_{i=1}^{d} |j = 3, j^3 = 0, i\rangle \langle j = 3, j^3 = 0, i| \]

\[
P_R, (r,s) = \sum_{i=1}^{d} |j = 3, j^3 = -1, i\rangle \langle j = 3, j^3 = -1, i| \]

\[
P_R, (r,s) = \sum_{i=1}^{d} |j = 3, j^3 = 1, i\rangle \langle j = 3, j^3 = 1, i| \]

In the above, $i$ labels all the states with the displayed $(j,j^3)$ quantum numbers; it runs from 1 to the dimension of the irreducible representation organizing the impurities. These results are a general construction of the projection operators used to define the restricted Schur polynomials dual to the two AdS giant system.

When we evaluate the traces (2.17) we need to consider the action of $\Gamma((n,n+1))$. Towards this end, we again need to switch to our previous convention of labeling $m+1$ boxes in the Young diagram. Equivalently, we need add another spin site to our chain. $\Gamma((n,n+1))$ can then be taken to act on the first and $(m+1)$th site of the spin chain\(^{11}\). Including this extra site our projectors become

\[
P_{R,(r,s)} = \sum_i |j, j^3, i\rangle \langle j, j^3, i| \otimes 1
\]

\(^{11}\)We could allow $\Gamma((n,n+1))$ to act on any of the first $m$ sites and the $(m+1)$th site of the spin chain without changing the final result.
\[
= \sum_i |j, j^3, i\rangle\langle j, j^3, i| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)
\]
where the identity above sits in the \((m+1)\)th slot. We will also use the notation
\[
p_{R,(r,s)} = \sum_i |j, j^3, i\rangle\langle j, j^3, i|
\]
when the \((m+1)\)th slot is not included.

In terms of the spin chain language, the intertwiners can be written as
\[
I_1 = \left( \frac{1}{2} + \frac{1}{2} \sigma_1^3 \right), \quad I_2 = \left( \frac{1}{2} - \frac{1}{2} \sigma_1^3 \right), \quad I_3 = a_1, \quad I_4 = a_1^\dagger.
\]

When \(\Gamma ((n, n+1))\) acts it does so by swapping the first and \((m+1)\)th spins, so that
\[
\Gamma ((n, n+1)) I_1 = |0\rangle\langle 0| \otimes 1 \otimes \cdots \otimes 1 \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes 1 \otimes \cdots \otimes 1 \otimes |0\rangle\langle 1|
= (\frac{1}{2} + \frac{1}{2} \sigma_1) (\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) + a_1^\dagger a_{m+1}.
\]

In exactly the same way
\[
I_1 \Gamma ((n, n+1)) = (\frac{1}{2} + \frac{1}{2} \sigma_1) (\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) + a_1^\dagger a_{m+1}.
\]

One of the terms contributing to the dilatation operator is
\[
\text{Tr}(I_1[P_{R,(r,s)}, \Gamma((n, n+1))]I_1^T[P_{R,(r,s)}, \Gamma((n, n+1))]).
\]

This is a term that does not change the shape of the Young diagram labels of the restricted Schur polynomial. Using the facts that \(I_1 = I_1^T, \Gamma((n, n+1))^T = \Gamma((n, n+1))\) and \(P_{R,(r,s)}^T = P_{R,(r,s)}\) we can rewrite this term as
\[
2 \left( \text{Tr}(I_1 P_{R,(r,s)} \Gamma((n+1, n)) I_1 P_{R,(r,s)} \Gamma((n+1, n))) - \text{Tr}(I_1 P_{R,(r,s)} \Gamma((n+1, n)) I_1 \Gamma((n+1, n)) P_{R,(r,s)})) \right)
= 2 \text{Tr}(P_{R,(r,s)} [(\frac{1}{2} + \frac{1}{2} \sigma_1^3)(\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) + a_1^\dagger a_{m+1}] P_{R,(r,s)} [(\frac{1}{2} + \frac{1}{2} \sigma_1^3)(\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) + a_1^\dagger a_{m+1}])
- 2 \text{Tr}((\frac{1}{2} + \frac{1}{2} \sigma_1^3) P_{R,(r,s)} (\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) P_{R,(r,s)})
\]

Tracing over the \(m\)th slot we easily find
\[
\text{Tr} \left( P_{R,(r,s)} [(\frac{1}{2} + \frac{1}{2} \sigma_1^3)(\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) P_{R,(r,s)} a_1^\dagger a_{m+1}] \right) = \text{Tr} \left( p_{R,(r,s)} [(\frac{1}{2} + \frac{1}{2} \sigma_1^3) p_{R,(r,s)} a_1^\dagger] \right) \langle 0 | 1 \rangle = 0
\]
and

\[ \text{Tr} \left( R_{(r,s)} a_1^t a_{m+1} R_{(r,s)} a_1^1 a_{m+1} \right) = \text{Tr} \left( p_{R_{(r,s)}} a_1^1 p_{R_{(r,s)}} a_1^1 \right) \text{Tr} \left( |0 \rangle \langle 1 | \right) = 0. \]

The expression for the term in the dilatation operator that we are considering becomes

\[ \text{Tr} ( I_1 [R_{(r,s)}, \Gamma ((n, n + 1))] J^T [R_{(r,s)}, \Gamma ((n, n + 1))] ) \]

\[ = 2 \text{Tr} \left( R_{(r,s)} (\frac{1}{2} + \frac{1}{2} \sigma_1^3) (\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) R_{(r,s)} (\frac{1}{2} + \frac{1}{2} \sigma_1^3) (\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) \right) \]

\[ - 2 \text{Tr} (\frac{1}{2} + \frac{1}{2} \sigma_1^3) R_{(r,s)} (\frac{1}{2} + \frac{1}{2} \sigma_{m+1}^3) R_{(r,s)} \right) . \]

Expanding these expressions out and making use of the identities

\[ \text{Tr} (P_{R_{(r,s)}} P_{R_{(r,s)}}) = \text{Tr} (P_{R_{(r,s)}} \sigma_{m+1}^3 P_{R_{(r,s)}} \sigma_{m+1}^3) \]

\[ \text{Tr} (P_{R_{(r,s)}} P_{R_{(r,s)}} \sigma_1^3) = \text{Tr} (P_{R_{(r,s)}} \sigma_{m+1}^3 P_{R_{(r,s)}} \sigma_1^3 \sigma_{m+1}^3) \]

we obtain

\[ \text{Tr} (I_1 [R_{(r,s)}, \Gamma ((n + 1, n))]) \]

\[ = \frac{1}{4} \text{Tr} (P_{R_{(r,s)}} \sigma_1^3 P_{R_{(r,s)}} \sigma_1^3) - \frac{1}{4} \text{Tr} (P_{R_{(r,s)})}) \]

\[ = \text{Tr} (P_{R_{(r,s)}} I_1 P_{R_{(r,s)}} I_1) - \text{Tr} (P_{R_{(r,s)}} I_1) . \]

Using exactly the same types of arguments we find

\[ \text{Tr} (I_2 [R_{(r,s)}, \Gamma ((n+1, n))]) \]

\[ = \text{Tr} (P_{R_{(r,s)}} I_2 P_{S_{(t,u)}} I_2) - \text{Tr} (P_{R_{(r,s)}} I_2) \delta_{R_{(r,s)}} S_{(t,u)} \]

\[ \text{Tr} (I_3 [R_{(r,s)}, \Gamma ((n+1, n))]) \]

\[ = \text{Tr} (P_{R_{(r,s)}} I_3 P_{S_{(t,u)}} I_3) - \text{Tr} (P_{R_{(r,s)}} I_3) \delta_{R_{(r,s)}} S_{(t,u)} \]

\[ \text{Tr} (I_4 [R_{(r,s)}, \Gamma ((n+1, n))]) \]

\[ = \text{Tr} (P_{R_{(r,s)}} I_4 P_{S_{(t,u)}} I_4) . \]
To evaluate these traces write each projector as \( \langle i \rangle \) is a multiplicity label that runs from 1 to the number of times the irreducible \( SU(2) \) representation \( j_1 \) appears

\[
p_1 = \sum_i |j_1, j_1^3, i \rangle \langle j_1, j_1^3, i |.
\]

Recall that we use a little letter \( p \) for the projectors when the \((m + 1)\)th site is not included. In this last expression the subscript 1 on the projector stands for \( R, (r, s) \) and is a notation which simplifies the equations dramatically. Introduce the following states (the first state on the RHS is the ket describing particle 1 in the lattice; the second ket is a good total angular momentum state obtained by coupling the states of the remaining \( m - 1 \) particles):

\[
|\phi_1, i \rangle = |\frac{1}{2}, \frac{1}{2}; j_1 - \frac{1}{2}, j_1^3 - \frac{1}{2}, i \rangle \langle \frac{1}{2}, \frac{1}{2}; j_1 - \frac{1}{2}, j_1^3 - \frac{1}{2}, i |j_1, j_1^3 \rangle
\]

\[
|\phi_2, i \rangle = |\frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i \rangle \langle \frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i |j_1, j_1^3 \rangle
\]

\[
|\phi_3, i \rangle = |\frac{1}{2}, -\frac{1}{2}; j_1 - \frac{1}{2}, j_1^3 - \frac{1}{2}, i \rangle \langle \frac{1}{2}, -\frac{1}{2}; j_1 - \frac{1}{2}, j_1^3 - \frac{1}{2}, i |j_1, j_1^3 \rangle
\]

\[
|\phi_4, i \rangle = |\frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i \rangle \langle \frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i |j_1, j_1^3 \rangle.
\]

Using these states the projector can be written as

\[
p_1 = \sum_i (|\phi_1, i \rangle + |\phi_3, i \rangle)(\langle \phi_1, i | + \langle \phi_3, i |) + \sum_i (|\phi_2, i \rangle + |\phi_4, i \rangle)(\langle \phi_2, i | + \langle \phi_4, i |).
\]

These states are particularly convenient because the action of the intertwiners \( I_1 \) and \( I_2 \) on these states is very simple

\[
I_1 |\phi_1, i \rangle = |\phi_1, i \rangle, \quad I_1 |\phi_2, i \rangle = |\phi_2, i \rangle
\]

\[
I_1 |\phi_3, i \rangle = 0, \quad I_1 |\phi_4, i \rangle = 0,
\]

\[
I_2 |\phi_1, i \rangle = 0, \quad I_2 |\phi_2, i \rangle = 0,
\]

\[
I_2 |\phi_3, i \rangle = |\phi_3, i \rangle, \quad I_2 |\phi_4, i \rangle = |\phi_4, i \rangle.
\]

We can now compute the traces we need

\[
\text{Tr}(P_1 I_1) = \sum_i \langle \phi_1, i |\phi_1, i \rangle + \sum_i \langle \phi_2, i |\phi_2, i \rangle.
\]
\[ s'_1 \text{ is obtained by dropping a box from the first row of } s \text{ and } d_{s'_{1}} \text{ is the dimension of this irreducible representation. } s'_{2} \text{ is obtained by dropping a box from the second row of } s \text{ and } d_{s'_{2}} \text{ is the dimension of this irreducible representation. We have summed over } i \text{ to obtain the factors of } d_{s'_{1}} \text{ and } d_{s'_{2}}. \text{ The overlaps } \langle \phi_{1}, i | \phi_{1}, i \rangle \text{ and } \langle \phi_{2}, i | \phi_{2}, i \rangle \text{ are independent of } i \text{ so that in the last line above we could sum over } i \text{ and there is no further need for this index. Recall the general expression for the Clebsch-Gordan coefficient}

\[
\langle j_{1}, j_{1}'; j_{2}, j_{2}' | j, j' \rangle = \delta_{j_{1} + j_{1}' + j_{2}' - j_{2}} \sqrt{\frac{(2j + 1)(j + j_{1} - j_{2})!(j - j_{1} + j_{2})!(j_{1} + j_{2} - j)!}{(j_{1} + j_{2} + j + 1)!}} \times \\
\times \sqrt{(j + j_{1} - j_{1}')!(j - j_{1}')!(j_{1} + j_{2})!(j_{1} + j_{2} - j_{2})!(j_{2} + j_{2}')!} \times \\
\times \left( \sum_{k} k!(j_{1} + j_{2} - j - k)!(j_{1} - j_{1}' - k)!(j_{2} + j_{2}' - k)!(j - j_{1} + j_{2}' + k)!(j_{1} - j_{2} + j_{2} + k)! \right)^{2}
\]

where the sum runs over all values of } k \text{ for which the argument of each factorial is non-negative. It is now straightforward to find}

\[
\langle \frac{1}{2}, \frac{1}{2}; j - \frac{1}{2}, j' - \frac{1}{2} | j, j' \rangle = \sqrt{\frac{j + j' + 1}{2(j + 1)}}, \quad \langle \frac{1}{2}, \frac{1}{2}; j + \frac{1}{2}, j' + \frac{1}{2} | j, j' \rangle = \sqrt{\frac{j + j'}{2j}},
\]

so that

\[
\text{Tr}(P_{1} I_{1}) = 2d_{s'} d_{R^{m+1}} \frac{j + j' + 1}{2(j + 1)} + 2d_{s'}' d_{R^{m+1}} \frac{j + j'}{2j},
\]

where } R^{m+1} \text{ is a Young diagram of } S_{n-1}, \text{ obtained from } R \text{ by removing } m + 1 \text{ boxes, and the factor of 2 comes from tracing over the } (m + 1)\text{th slot. Now, since } I_{1} + I_{2} \text{ is the identity we find}

\[
2d_{s'} d_{R^{m+1}} = \text{Tr}(P_{1}) = \text{Tr}(P_{1} I_{1}) + \text{Tr}(P_{1} I_{2}) = d_{s'} d_{R^{m+1}} + \text{Tr}(P_{1} I_{2})
\]

which implies that

\[
\text{Tr}(P_{1} I_{2}) = d_{s'} d_{R^{m+1}}.
\]

Now consider the second type of term

\[
\text{Tr}(P_{1} I_{1} P_{1} I_{1}) = \text{Tr}(I_{1} P_{1} I_{1} P_{1} I_{1})
\]
where we used the fact that the trace is cyclic and $I_1$ is a projector so that $(I_1)^2 = I_1$.

This is a useful observation because

$$I_1P_1I_1P_1I_1 = \sum_i (|\phi_1, i\rangle \langle \phi_1, i| + |\phi_2, i\rangle \langle \phi_2, i|) \sum_j (|\phi_1, j\rangle \langle \phi_1, j| + |\phi_2, j\rangle \langle \phi_2, j|)$$

It immediately follows that

$$\text{Tr}(P_1I_1P_1I_1) = 2 \left( d_{s_1} \left( \frac{j + j^3 + 1}{2(j + 1)} \right)^2 + d_{s_2} \left( \frac{j + j^3}{2j} \right)^2 \right) d_{R_{m+1}} \text{.}$$

Again using the fact that $I_1$ and $I_2$ sum to the identity, it is clear that

$$\text{Tr}(P_1I_2P_1I_1) = \text{Tr}(P_1I_1) - \text{Tr}(P_1I_1P_1I_1) \nonumber$$

$$= 2d_{R_{m+1}} \left( d_{s_1} \left( \frac{j + j^3 + 1}{2(j + 1)} \right) + d_{s_2} \left( \frac{j + j^3}{2j} \right) - d_{s_1} \left( \frac{j + j^3 + 1}{2(j + 1)} \right) - d_{s_2} \left( \frac{j + j^3}{2j} \right) \right) \text{.}$$

Finally,

$$d_s d_{R_{m+1}} = \text{Tr}(P_1) = \text{Tr}(P_1P_1) = \text{Tr}(P_1(I_1 + I_2)P_1(I_1 + I_2)) \nonumber$$

$$= \text{Tr}(P_1I_1P_1I_1) + 2\text{Tr}(P_1I_2P_1I_1) + \text{Tr}(P_1I_2P_1I_2) \text{.}$$

We can solve this last equation for $\text{Tr}(P_1I_2P_1I_2)$. This complete the evaluation of the terms we were considering.

We now need to consider the case that the two projectors appearing in the trace have different labels. There is only one term to compute because

$$\text{Tr}(P_1I_1P_2I_1) = \text{Tr}(P_1I_1P_2(I_1 + I_2)) - \text{Tr}(P_1I_1P_2I_2) = -\text{Tr}(P_1I_1P_2I_2) \nonumber$$

$$= -\text{Tr}(P_1(I_1 + I_2)P_2I_2) + \text{Tr}(P_1I_2P_2I_2) = \text{Tr}(P_1I_2P_2I_2) \text{.}$$

The only time we get a non-zero result for this trace is when $j_2 = j_1 \pm 1$. Without any loss of generality, consider the case that $j_2 = j_1 + 1$. The result for $j_2 = j_1 - 1$ is obtained by swapping $1 \leftrightarrow 2$. We can write

$$p_2 = \sum_i (|\psi_1, i\rangle + |\psi_3, i\rangle)(\langle \psi_1, i| + \langle \psi_3, i|) + \sum_i (|\psi_2, i\rangle + |\psi_4, i\rangle)(\langle \psi_2, i| + \langle \psi_4, i|) \text{.}$$

where

$$|\psi_1, i\rangle = |\frac{1}{2}, \frac{1}{2}; j_1 + 1, j^3 - \frac{1}{2}, \frac{1}{2}; j_1 + 1, j^3 - \frac{1}{2}, i|j_1 + 1, j^3\rangle \nonumber$$

$$|\psi_2, i\rangle = |\frac{1}{2}, \frac{1}{2}; j_1 + \frac{3}{2}, j^3 - \frac{1}{2}, \frac{1}{2}; j_1 + \frac{3}{2}, j^3 - \frac{1}{2}, i|j_1 + 1, j^3\rangle \nonumber$$

$$|\psi_3, i\rangle = |\frac{1}{2}, \frac{1}{2}; j_1 + 1, j^3 - \frac{1}{2}, \frac{1}{2}; j_1 + 1, j^3 - \frac{1}{2}, i|j_1 + 1, j^3\rangle \nonumber$$

$$|\psi_4, i\rangle = |\frac{1}{2}, \frac{1}{2}; j_1 + \frac{3}{2}, j^3 - \frac{1}{2}, i|j_1 + 1, j^3\rangle \nonumber$$
\[|\psi_3, i\rangle = \left[\frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i\right]\langle\frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i| j_1 + 1, j_1^3\rangle\]

\[|\psi_4, i\rangle = \left[\frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i\right]\langle\frac{1}{2}, -\frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 + \frac{1}{2}, i| j_1 + 1, j_1^3\rangle.\]

We now find

\[\text{Tr}(P_1 I_1 P_2 I_1) = 2d_{R^m+1}\text{Tr} \left[ \sum_i (|\psi_1, i\rangle \langle \psi_1, i| + |\psi_2, i\rangle \langle \psi_2, i|) \sum_j (|\phi_1, j\rangle \langle \phi_1, j| + |\phi_2, j\rangle \langle \phi_2, j|) \right] \]

\[= 2d_{R^m+1} \sum_{i,j} \langle \phi_2, j| \psi_1, i\rangle \langle \psi_1, i| \phi_2, j\rangle \]

\[= 2d_{R^m+1} d_{s_c} \left( \left[\frac{1}{2}, \frac{1}{2}, j_1 + 1, j_1^3 + \frac{1}{2}, i| j_1, j_1^3\right] \right)^2 \left[\frac{1}{2}, \frac{1}{2}, j_1 + 1, j_1^3 + \frac{1}{2}, i| j_1 + 1, j_1^3\right] \]

\[= 2d_{R^m+1} d_{s_c} \left( \frac{j_1 + j_1^3 + 1}{2(j_1 + 1)} \right) \left( \frac{j_1 - j_1^3 + 1}{2(j_1 + 1)} \right). \]

If \(s_1\) is the third label for \(p_1\) and \(s_2\) is the third label of \(p_2\), then by removing a box from \(s_1\) we can get the same Young diagram as when we remove a box from \(s_2\) - in the last line above we have called this Young diagram which can be reached from either \(s_1, s_2\) by removing a single box \(s'_c\).

Putting things together we find (when \(j = 0\) the term \((m+2)(j^3)^2\) in round braces on the first line below and the last term in the equation must be omitted)

\[DO_{j,j^3}(b_0, b_1) = g_{YM}^2 \left[ -\frac{1}{2} \left( m - \frac{(m+2)(j^3)^2}{j(j+1)} \right) \Delta O_{j,j^3}(b_0, b_1) + \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}} \frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} \Delta O_{j+1,j^3}(b_0, b_1) \right. \]

\[\left. + \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}} \frac{(j+j^3)(j-j^3)}{2j} \Delta O_{j-1,j^3}(b_0, b_1) \right] \quad (4.3) \]

where

\[\Delta O(b_0, b_1) = \sqrt{(N+b_0)(N+b_0+b_1)} \{ O(b_0+1, b_1-2) + O(b_0-1, b_1+2) \}

\[-(2N+2b_0+b_1)O(b_0, b_1). \quad (4.4)\]

This completes our evaluation of the dilatation operator for \(m\) impurities.
5. Construction of Sphere Giant Projectors and the Dilatation Operator

The starting point of our AdS giant analysis was a polynomial representation which was isomorphic to the Young diagrams. There is a polynomial representation isomorphic to the sphere giants too. For $m$ impurities, introduce the $2m$ variables $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$. The $x$’s and $y$’s are Grassman numbers and all these variables anticommute. The $x$’s and $y$’s are associated to the two columns respectively. Consider three impurities for illustration; in this case the isomorphism is defined by

\[
\begin{align*}
3 & \leftrightarrow y_1 y_2 y_3 & 2 & \leftrightarrow x_1 y_2 y_3 & 1 & \leftrightarrow x_2 y_1 y_3 & 1 & \leftrightarrow x_3 y_1 y_2 \\
2 & \leftrightarrow x_1 y_2 y_3 & 3 & \leftrightarrow x_1 y_2 y_3 & 1 & \leftrightarrow x_2 y_1 y_3 & 1 & \leftrightarrow x_3 y_1 y_2 \\
1 & \leftrightarrow x_1 x_2 y_3 & 2 & \leftrightarrow x_1 x_3 y_2 & 1 & \leftrightarrow x_2 x_3 y_1 & 1 & \leftrightarrow x_1 x_2 y_3.
\end{align*}
\]

The generalization to any $m$ is obvious. The polynomials are ordered with (i) $x$’s to the left of the $y$’s and (ii) so that their subscripts (within the $x$’s and $y$’s separately) increase. The action of $S_n$ is to act on the subscripts without changing the order of the variables.

The natural definition for $d$ is

\[
d = \sum_{i=1}^{m} y_i \frac{\partial}{\partial x_i}. \quad (5.1)
\]

It is easy to check that $d$ does indeed commute with the symmetric group. Using the inner product: $\langle y_1 y_2 y_3, y_1 y_2 y_3 \rangle = 1$ (so the inner product of same polynomials is 1 and of different polynomials is 0) we find

\[
d^\dagger = \sum_{i=1}^{m} x_i \frac{\partial}{\partial y_i}. \quad (5.2)
\]
The Casimir we want is
\[ d^\dagger d = \sum_i x_i \frac{\partial}{\partial x_i} - \sum_{i,j} x_i y_j \frac{\partial}{\partial y_i} \frac{\partial}{\partial x_j}. \]

This has a nice expression in terms of the following spin model
\[ d^\dagger d = \sum_{n,p} (-1)^{n-p} a_n^\dagger a_p = \frac{1}{4} \sum_{p,n} (-1)^{n+p} (\sigma_n^1 \sigma_p^1 + \sigma_n^2 \sigma_p^2) - \frac{1}{2} \sum_p \sigma_p^3. \quad (5.3) \]

If we further define
\[ \tilde{\sigma}_n^i = (\sigma_n^3)^i (\sigma_n^3)^n \]
we map this into the AdS giant problem
\[ d^\dagger d = \sum_{n,p} (-1)^{n-p} a_n^\dagger a_p = \frac{1}{4} \sum_{p,n} (\tilde{\sigma}_n^1 \tilde{\sigma}_p^1 + \tilde{\sigma}_n^2 \tilde{\sigma}_p^2) - \frac{1}{2} \sum_p \tilde{\sigma}_p^3 = \tilde{J}^2 - \tilde{J}^3 (\tilde{J}^3 + 1). \]

To get the sphere giant dilatation operator we will simply need to rewrite the formulas for the intertwiners and \( \Gamma ((n,n+1)) \) in terms of the \( \tilde{\sigma}^i \) and then trace. The computations in this case are parallel to those in section 4. \( I_1 \) and \( I_2 \) are unchanged; \( I_3 \) and \( I_4 \) each pick up a minus sign. Since they always appear together, this change does not affect the answer for the dilatation operator at all. As far as \( \Gamma ((n,n+1)) \), it continues to swap the first and last slots. If \( m \) is even, \( \Gamma ((n,n+1)) \) just picks up a minus sign and hence, since \( \Gamma ((n,n+1)) \) always appears twice, the dilatation operator is unaffected. When \( m \) is odd \( \Gamma ((n,n+1)) \) always appears twice, the dilatation operator is unaffected. When \( m \) is odd \( \Gamma ((n,n+1)) \) which are on the diagonal change sign. The intertwiners \( I_1, I_2 \) only pick up on the diagonal elements and the intertwiners \( I_3, I_4 \) only pick up off the diagonal elements. Thus, we only ever get products of off diagonal elements of \( \Gamma ((n,n+1)) \) with off diagonal elements of \( \Gamma ((n,n+1)) \) or products of on the diagonal elements of \( \Gamma ((n,n+1)) \) with on the diagonal elements of \( \Gamma ((n,n+1)) \). Thus, even for \( m \) odd the dilatation operator is unaffected. This proves that
\[ \text{Tr} \left( \left[ \Gamma_R((n,n+1)), P_{R \rightarrow (r,s)} \right] I_{R' T'} \left[ \Gamma_T((n,n+1)), P_{T \rightarrow (t,u)} \right] I_{T' R'} \right) \]
is the same for the AdS and sphere giant cases and proves our previous observation that to get sphere from AdS we just replace factors like \( N + b_0 + b_1 \rightarrow N - b_0 - b_1 \). We can thus use the exactly the same method as for AdS case. Consider the restricted Schur polynomial \( \chi_{R,(r,s)} \) where all Young diagrams has at most 2 columns. The \( \tilde{J}^2 = j(j+1) \) quantum number again tells you the shape of the Young diagram \( s \) that organizes the impurities. if there are \( N_1 \) boxes in the first column of \( s \) and \( N_2 \) boxes in the second,
Then \(2j = N_1 - N_2\). The \(j^3\) quantum number again tells you how many boxes are removed from the first and second columns of \(R\) to produce \(r\). Denote the number of boxes removed from the first column by \(n_1\) and the number of boxes removed from the second column by \(n_2\). We have \(2j^3 = n_1 - n_2\). The dilatation operator is thus

\[
DO_{j,j^3}(b_0, b_1) = g_{YM}^2 \left[ \frac{1}{2} \left( m - \frac{(m + 2)(j^3)^2}{j(j + 1)} \right) \Delta O_{j,j^3}(b_0, b_1) + \sqrt{\frac{(m + 2j + 4)(m - 2j)(j + j^3 + 1)(j - j^3 + 1)}{2(j + 1)}} \Delta O_{j+1,j^3}(b_0, b_1) + \sqrt{\frac{(m + 2j + 2)(m - 2j + 2)(j + j^3)(j - j^3)}{2j}} \Delta O_{j-1,j^3}(b_0, b_1) \right]
\]

(5.4)

where

\[
\Delta O(b_0, b_1) = \sqrt{(N - b_0 - b_1)(N - b_0)}(O(b_0 + 1, b_1 - 2) + O(b_0 - 1, b_1 + 2)) - (N - b_0)O(b_0, b_1) - (N - b_0 - b_1)O(b_0, b_1).
\]

(5.5)

6. Diagonalization of the Dilatation Operator

In this section we introduce a transformation that reduces the dilatation operator to a set of decoupled oscillators. The transformation is constructed by solving a three term recursion relation. This three term recursion relation is nothing but the recursion relation of certain Clebsch-Gordan coefficients. This allows the construction of operators with a good scaling dimension in terms of Hahn polynomials. In the limit that the number of impurities is very large, these wave functions become the wave functions of the two dimensional radial oscillator.

6.1 \(j^3 = 0\) case and the corresponding discrete wave equation

For this case the action of the dilatation operator is

\[
DO_{j,0}(b_0, b_1) = g_{YM}^2 \left[ \frac{1}{2} m \Delta O_{j,0}(b_0, b_1) + \sqrt{\frac{(m + 2j + 4)(m - 2j)(j + 1)}{2(j + 1)(2j + 3)}} \Delta O_{j+1,0}(b_0, b_1) + \sqrt{\frac{(m + 2j + 2)(m - 2j + 2)(j + j^3)(j - j^3)}{2j}} \Delta O_{j-1,0}(b_0, b_1) \right]
\]

(6.1)

where

\[
\Delta O(b_0, b_1) = \sqrt{(N + b_0)(N + b_0 + b_1)}(O(b_0 + 1, b_1 - 2) + O(b_0 - 1, b_1 + 2)) - (2N + 2b_0 + b_1)O(b_0, b_1).
\]

(6.2)
Make the following ansatz for the operators of good scaling dimension

\[
\sum_{b_1} f(b_0, b_1) O_p(b_0, b_1) = \sum_{j, b_1} C_p(j) f(b_0, b_1) O_{j,0}(b_0, b_1)
\]  

(6.3)

and require that

\[
-\alpha_p C_p(j) = -\frac{1}{2} m C_p(j) + \sqrt{\frac{(m + 2j + 4)(m - 2j)}{(2j + 1)(2j + 3)}} \frac{j + 1}{2} C_p(j + 1)
\]

\[
+ \sqrt{\frac{(m + 2j + 2)(m - 2j + 2)}{(2j + 1)(2j - 1)}} \frac{j}{2} C_p(j - 1).
\]

(6.4)

The dilatation operator does not change the number of Zs \(n\) or the number of Ys \(m\). Since \(n = 2b_0 + b_1\) we do not sum over \(b_0\) and \(b_1\) independently in (6.3). Further, \(b_1\) only takes odd or even values. The action of the dilatation operator then reduces to

\[
DO_p(b_0, b_1) = -\alpha_p g^2_{YM} \left[ \sqrt{(N + b_0)(N + b_0 + b_1)} (O_p(b_0 + 1, b_1 - 2) + O_p(b_0 - 1, b_1 + 2)) - (2N + 2b_0 + b_1) O_p(b_0, b_1) \right].
\]

(6.5)

Reducing the dilatation operator to a set of decoupled oscillators amounts to determining the coefficients \(C_n(j)\) and the values of \(\alpha_n\), by solving the recursion relation (6.4).

Now, introduce the Clebsch-Gordan coefficients

\[
C^{j_1, j_2, j_T}_{m_1, m_2, m_T} = \langle j_T, m_T | j_1, m_1, j_2, m_2 \rangle
\]

which couples the state with two angular momenta \(j_1\) and \(j_2\) to a state with good total angular momentum \(j_T\). The eigenvalue of the 3-component of angular momentum is denoted in the above using an \(m\). The recursion relation for the Clebsch-Gordan coefficients is

\[
(2n - m) C_{n-m-j}^{m, m, j-n, 0} = \sqrt{\frac{(m - 2j)(m + 2j + 4)}{(2j + 1)(2j + 3)}} \frac{j + 1}{2} C_{n-m-j+1}^{m, m, j+1,0} + \sqrt{\frac{(m - 2j + 2)(m + 2j + 2)}{(2j + 1)(2j - 1)}} \frac{j}{2} C_{n-m-j-1}^{m, m, j-1,0}.
\]

It is now clear that the solution to our recursion relation is \(C_p(j) = (-1)^{-j} C_{p-m, m, p, 0}\) with \(m\) the number of impurities and

\[-\alpha_p = -2p = 0, -2, -4, ..., -m.
\]

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From the rules for coupling angular momentum, for the \( j^3 = 0 \) case, the range of \( p \) is \( 0 \leq p \leq \frac{m}{2} \). The energy spacing of the oscillators in the 1-2 directions are \( 4 \alpha_p g_Y^2 M = 8 p g_Y^2 M \). Thus, in this \( j^3 = 0 \) case we obtain \( \frac{m}{2} + 1 \) oscillators with frequencies \( 0, 8 g_Y^2 M, 16 g_Y^2 M, \ldots, 4 m g_Y^2 M \).

Our “eigenfunctions”, the \( C_p(j) \), are given by Clebsch-Gordan coefficients, or equivalently by (dual) Hahn polynomials, reviewed for example in [37]. They can be written in terms of the \( _3F_2 \) hypergeometric function as

\[
C_p(j) = (-1)^{\frac{m}{2} - p} \left( \frac{m}{2} \right)! \sqrt{\frac{2}{m}} \frac{(2j + 1)}{(\frac{m}{2} - j)! (\frac{m}{2} + j + 1)!} \ _3F_2 \left( -j, j + 1, -p; 1 \right)
\]

where the range of \( j \) and \( p \) are \( 0 \leq j \leq \frac{m}{2}, 0 \leq p \leq \frac{m}{2} \). The lowest energy eigenfunction corresponds to the BPS states. We have fixed the number of \( Y \)s to be \( m = \gamma N \) with \( \gamma \ll 1 \) and \( b_1 = O(N) \) so that

\[
\frac{m}{b_1} \sim \gamma \ll 1.
\]

This is the condition needed to ensure the dramatic simplifications of Young’s orthogonal representation. We can think in terms of a double scaling limit \( m \to \infty \) and \( b_1 \sim N \to \infty \) keeping \( \gamma \) fixed and very small. In this limit \( _3F_2 \left( -j, j + 1, -p; 1 \right) \to L_p \left( \frac{2j^2}{m} \right) \) with \( L_p(\cdot) \) the Laguerre polynomial, so that

\[
C_p(j) \to (-1)^{\frac{m}{2} - p} \sqrt{\frac{2}{m}} \sqrt{2j + 1} e^{-\frac{j^2}{m}} L_p \left( \frac{2j^2}{m} \right), \quad 0 \leq j \leq \frac{m}{2}.
\]

These coefficients become the wave function of the 2d radial Harmonic oscillator for the \( s \)-wave, i.e. without a centripetal force

\[
\frac{1}{2} \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + r^2 \right] C_p(r) = (2p + 1) C_p(r)
\]

where \( r = \sqrt{\frac{2}{m} j} \) ranges over \( 0 \leq r \leq \sqrt{\frac{m}{2}} \) and \( C_p(r) = C_p(j)/\sqrt{r} \) in the continuum limit. This is a radial oscillator in 2d (the 3-4 directions) with the energy levels labeled by integer \( p \). What is the dual interpretation of these wave functions? To answer this we need to know how \( r \) maps into the spacetime description of the dual physics. Since \( r \) is a scaled version of \( j \), different values of the \( r \) coordinate correspond to different Young diagrams organizing the impurities. These Young diagrams all have two rows. Based on our experience with the half BPS case, it seems natural to associate each one of the rows with each one of the giant gravitons. We will simply assume that this is the correct interpretation; this point certainly deserves further study. Recalling that \( Y = \phi_3 + i \phi_4 \) we know that the number of \( Y \)s in each operator tells us the angular
momentum of the operator in the 3-4 plane. Denote the two angular momenta of the two giant gravitons $J^Y_1$ and $J^Y_2$. The relations between $j, m, J^Y_1$ and $J^Y_2$ are

$$J^Y_1 + J^Y_2 = m, \quad J^Y_1 - J^Y_2 = 2j,$$

$$J^Y_1 = \frac{m}{2} + j, \quad J^Y_2 = \frac{m}{2} - j.$$

$j$ (and hence $r$) is directly proportional to the difference in angular momenta of the two states. Giving an angular momentum to the gravitons will cause them to expand as a consequence of [44]. The separation between the two gravitons in the 3-4 plane will (for separations small compared to the radii of the giants) be directly proportional to the difference in angular momenta of the two giants. Consequently, it is natural to interpret $r$ as a coordinate for the radial separation between the two giants in the 3-4 plane. How large is this separation? Recall that the radius of a giant with angular momentum $J$, in units of the AdS scale $R$ is $\sqrt{\frac{J}{N}}R$. The maximum value of the difference between the angular of the giants, $j$, is bounded by $m$ so that the length between them is always very small $\sim \sqrt{\gamma R}$. The separation between the giants in 3-4 direction is thus seen to be small in AdS scale units for these states, due to the $\gamma$ factor, but it can be large in string scale units. In units of the string length the separation between the two giants is

$$\sqrt{\frac{2}{m} \frac{j}{N^2} (g_{YM}^2)} l_s$$

where in the above we used the approximation $(\frac{m}{2} + j)^{\frac{1}{2}} - (\frac{m}{2} - j)^{\frac{1}{2}} \to \sqrt{\frac{2}{m}j}$. This separation can be an $O(1)$ or very large length. Thus, we propose that $r$ is a coordinate for the radial separation between the two giants in the 3-4 plane, and the separation naturally ranges from the string length up to small (but non-zero) distances in units of the AdS scale. One still needs to solve the eigenvalue problem of (6.5) that will determine a “wave function” for the 1-2 plane and we see that $Z = \phi_1 + i\phi_2$ plane. Thus, our operators are described by a wave function in four dimensional space. It is rather natural to interpret this space as the 4d Kähler base appearing in the construction of the 1/4 BPS geometries in [40, 17]. We see very concretely the emergence of local physics on the 4d space from the system of Young diagrams labeling the restricted Schur polynomial. This is strongly reminiscent of the 1/2 BPS case where the Schur polynomials provide wave functions for fermions in a harmonic oscillator and further, these wave functions very naturally reproduce features of the geometries and the phase space [7] (for a review see for example [45]).

For the two matrix model we are studying here it is not true that the two matrices $Z, Y$ commute. For this reason, we can’t simultaneously diagonalize them and there is
no analog of the eigenvalue basis that is so useful for the large $N$ dynamics of single matrix models. For the subsystem describing the BPS states of the $Z,Y$ system however [46] has deduced that we can indeed assume that $Z,Y$ commute in the interacting theory and hence there should be a description in terms of eigenvalues. Further, the eigenvalue dynamics is again supposed to be dynamics in an oscillator potential with repulsions preventing the collision of eigenvalues. We have described a part of the BPS sector (as well as non-BPS operators) among the operators we have studied. In the case of a single matrix it is possible to associate the rows of the Young diagram labeling a Schur polynomial with the eigenvalues of the matrix. This provides a connection between the eigenvalue description and the Schur polynomial description for single matrix models.

Are the oscillators we find here a signal of simple underlying eigenvalue dynamics? Is there a connection between the Young diagram labels and eigenvalues?

We will now study in detail the state with $p = 0$ so that $\alpha_0 = 0$, i.e. we consider a BPS state. This is the ground state of the wave equation (6.7). In this case, since

$$\ _3F_2\left(-\frac{j}{2},1,0;1\right) = 1,$$

$$C_0(j) = (-1)^{\frac{m}{2}} \left(\frac{m}{2}\right)! \sqrt{\frac{(2j+1)}{(\frac{m}{2} - j)!((\frac{m}{2} + j + 1)!}}, \quad 0 \leq j \leq \frac{m}{2}. $$

If we now consider the large $m$ regime,

$$C_0(j) \rightarrow (\frac{m}{2})^{-\frac{j}{2}} (-1)^{\frac{j}{2}} \sqrt{2j + 1} e^{-\frac{j^2}{m}}, \quad \text{for } m \gg 1, \ 0 \leq j \ll m \quad (6.8)$$

This formula agrees beautifully with our numerically obtained eigenfunctions - see figure 1 where we make the comparison.

Another interesting eigenfunction to consider is

$$C_\frac{m}{2}(j) = \left(\frac{m}{2}\right)! \sqrt{\frac{(2j+1)}{(\frac{m}{2} - j)!((\frac{m}{2} + j + 1)!)} \ _2F_1\left(-\frac{j}{2},1;1\right)$$

Using the identity

$$\ _2F_1\left(-\frac{j}{2},1;\frac{1}{2}(1-z)\right) = P_j(z)$$

where $P_j(z)$ is Legendre polynomial and $P_j(-1) = (-1)^j$, we find

$$C_\frac{m}{2}(j) = \left(\frac{m}{2}\right)!(-1)^j \sqrt{\frac{(2j+1)}{(\frac{m}{2} - j)!((\frac{m}{2} + j + 1)!}}, \quad 0 \leq j \leq \frac{m}{2}. $$

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If we now consider the large \( m \) regime,

\[
C_{\frac{m}{2}}(j) \to \left( \frac{m}{2} \right)^{-\frac{1}{2}} (-1)^j \sqrt{2j + 1} e^{-\frac{j^2}{m}}, \quad \text{for } m \gg 1, \ 0 \leq j \ll m. \quad (6.9)
\]

This eigenfunction produces an oscillator with a level spacing of \( 4mg_{YM}^2 \). It is interesting to compare (6.8) and (6.9): There are alternating minus signs, i.e. \((-1)^j\) in (6.9) for superposing \( j \)-states, while there are same signs in (6.8) for superposing \( j \)-states; these different phases produce vastly different eigenvalues for the two states. This again agrees very well with the numerically computed eigenfunctions. We have compared the numerically computed \( C_0(j) \) and \( C_{\frac{m}{2}}(j) \) in figure 2.

### 6.2 Arbitrary \( j^3 \) and the corresponding discrete wave equation

For arbitrary \( j^3 \), we make the following ansatz for the operators of good scaling dimension

\[
\sum_{b_1} f(b_0, b_1) O_{p,j^3}(b_0, b_1) = \sum_{j,b_1} C_{p,j^3}(j) f(b_0, b_1) O_{j,j^3}(b_0, b_1)
\]

Repeating the arguments in the last subsection 6.1, we find that the \( O_{p,j^3}(b_0, b_1) \)'s satisfy the recursion equation (6.5) with the prefactor given by \(-\alpha_{p,j^3}\). We also obtain the following recursion relation

\[
-\alpha_{p,j^3} C_{p,j^3}(j) = \sqrt{\frac{(m + 2j + 4)(m - 2j)(j + j^3 + 1)(j - j^3 + 1)}{2(j + 1)(2j + 3)}} C_{p,j^3}(j + 1)
\]
Figure 2: A comparison between the numerically computed $C_0(j)$ (plotted in blue) and $C_m^2(j)$ (plotted in red) for $m = 600$. The two solutions are clearly identical up to the alternating signs in $C_0(j)$, exactly as predicted by the analytic expressions for the eigenfunctions.

$$
\sqrt{(m+2j+2)(m-2j+2)(j+j^3)(j-j^3)} \frac{C_{p,j^3}(j-1)}{2j} - \frac{1}{2} \left( m - \frac{(m+2)(j^3)^2}{j(j+1)} \right) C_{p,j^3}(j).
$$

(6.10)

This equation has a $j^3 \to -j^3$ symmetry, and we will solve for both the $j^3 \geq 0$ case and the $j^3 \leq 0$ case. The range for $j$ is

$$0 \leq |j^3| \leq j \leq \frac{m}{2}.$$

The eigenfunctions $C_{p,j^3}(j)$ which solve the recursion relation are easily expressed in terms of the hypergeometric functions $3F_2$,

$$C_{p,j^3}(j) = (-1)^{\frac{m}{2} - p} \left( \frac{m}{2} \right)! \sqrt{\frac{(2j+1)}{(m/2-j)! (m/2+j+1)!}} 3F_2 \left( \frac{j^3-j,j+|j^3|+1,-p}{|j^3|-m/2,1} \right)$$

(6.11)

where the range of $j$ and $p$ are $|j^3| \leq j \leq \frac{m}{2}$, $0 \leq p \leq \frac{m}{2} - |j^3|$, and the associated eigenvalues are

$$-\alpha_{p,j^3} = -2p = 0, -2, -4, ..., -(m - 2|j^3|).$$

This implies an energy spacing for the oscillators in 1-2 direction of $4\alpha_p g^2_{YM} = 8pg^2_{YM}$. Thus, for a given value of $j^3$ we obtain $\frac{m}{2} - |j^3| + 1$ oscillators with frequencies $0, 8g^2_{YM}, 16g^2_{YM}, ..., (4m - 8|j^3|)g^2_{YM}$. The lowest energy eigenfunction corresponds to
the BPS states. The eigenfunctions (6.11) are related to the Hahn polynomials which are defined by

\[ Q_n(x; \alpha, \beta, N) = _3F_2 \left( \begin{array}{l} -n, n+\alpha, \beta+1-x \\ \alpha+1, -N \end{array}; 1 \right), \quad n = 0, 1, 2, \ldots, N. \]

It is a well known fact that the Hahn polynomials are closely related to the Clebsch-Gordan coefficients of \( SU(2) \) [39].

One can take a double scaling limit, so that \( m \to \infty, b_1 \to \infty, \frac{m}{m} \sim \gamma \ll 1 \), and \( \sqrt{\frac{2}{m}} j \) becomes a continuous variable. In this limit, \( _3F_2 \left( \begin{array}{l} |j|^2, |j|^2+1-p, j+1 \\ |j|^2, -\frac{m}{2} \end{array}; 1 \right) \to L_p(\frac{2(j^2-|j|^2)}{m}) \) with \( L_p(\cdot) \) the Laguerre polynomial. If we define the continuous variables \( u_0 = \sqrt{\frac{2}{m}} j^3, r = \sqrt{\frac{2}{m}} j, r^2 = u^2 + u_0^2, \) and \( j \geq |j^3| \), then we obtain the following differential equation as the continuous approximation of the difference equation

\[ \frac{1}{2} \left[ -\frac{1}{u} \frac{\partial}{\partial u} \left( u \frac{\partial}{\partial u} \right) + u^2 \right] C_{p, u_0}(u) = (2p + 1) C_{p, u_0}(u) \cdot \]

In this equation \( C_{p, u_0} = C_{p, j^3}(j)/\sqrt{R} \). Note that one can always introduce an overall (normalization) factor in \( C_{p, u_0} \). This limit captures the continuous limit that we took when replacing the hypergeometric function \( _3F_2 \) with the Laguerre polynomial \( L_p(\frac{2(j^2-|j|^2)}{m}) \),

\[ C_{p, j^3}(j) \to (-1)^{\frac{m}{2}-p} \sqrt{\frac{2}{m}} \sqrt{2j+1} e^{-\frac{(j^2-|j|^2)^2}{m}} L_p(\frac{2(j^2-|j|^2)}{m}) \]

where we have introduced an overall normalization factor.

For fixed \( m, j^3 \) ranges from \( 0 \leq |j^3| \leq \frac{m}{2} \). For the extreme case \( j^3 = \pm \frac{m}{2} \), there is only one oscillator with zero frequency. All states of the corresponding oscillator are BPS states. Another extreme case is \( j^3 = 0 \). The eigenfunctions we have obtained in this section nicely recover the results of subsection 6.1 and we see that both the \( j^3 = 0 \) and \( j^3 \neq 0 \) cases are captured by the eigenfunctions described in this section.

For a general \( j^3 \), the lowest energy state is given by the \( p = 0 \) eigenfunction

\[ C_{p=0, j^3}(j) = \left( \frac{m}{2} \right)!(-1)^{\frac{m}{2}} \sqrt{\frac{(2j+1)}{(\frac{m}{2} - j)! (\frac{m}{2} + j + 1)!}}. \] (6.12)

The lowest energy eigenfunction corresponds to the BPS states. The highest energy state is

\[ C_{p=\frac{m}{2}-|j^3|, j^3}(j) = \left( \frac{m}{2} \right)!(-1)^j \sqrt{\frac{(2j+1)}{(\frac{m}{2} - j)! (\frac{m}{2} + j + 1)!}}. \] (6.13)
where to obtain this simple result we have used the formulas relating the hypergeometric function and the Jacobi polynomial \( P_{j-|j^3|}^{0,2j^3}(z) \):

\[
\begin{align*}
2F_1(\frac{|j^3|-j,j+|j^3|+1}{2}; 1-z) &= P_{j-|j^3|}^{0,2j^3}(z), \\
P_{j-|j^3|}^{0,2j^3}(-1) &= (-1)^{|j^3|} P_{j-|j^3|}^{0,2j^3} \left( j+|j^3| \right).
\end{align*}
\]

This eigenfunction (6.13) corresponds to an oscillator in the 1-2 directions with the largest energy spacing of \((4m-8|j^3|)g_{YM}^2\) in this specific \( j^3 \) sector.

It is interesting to compare (6.12) and (6.13). There are two main differences. One is the extra alternating signs \((-1)^j\) in (6.13) for superposing \( j \)-states. This is familiar from our \( j^3 = 0 \) results. Another difference is the extra factor \( \frac{(j+|j^3|)}{(j-|j^3|)} \) which tends to 1 when \( j \to |j^3| \) and tends to \( \frac{(2|j^3|)!}{(2|j^3|)!} \) when \( j \gg |j^3| \). Of course, in the special case \( j^3 = 0 \) we recover the eigenfunctions discussed in subsection 6.1.

One can also look at the large \( m \) regime of the eigenfunction (6.13) which behaves as

\[
C_{p=\frac{m}{2}-|j^3|,j^3}(j) \to \left( \frac{m}{2} \right)^{-\frac{j}{2}} (-1)^j \frac{\sqrt{2j+1}}{\Gamma(2|j^3|+1)} \frac{\Gamma(j+|j^3|+1)}{\Gamma(j-|j^3|+1)} e^{-\frac{j^2}{m}}, \quad (6.14)
\]

for \( m \gg 1, |j^3| \leq j \ll \frac{m}{2} \), and where \( \Gamma(\cdot) \) is the gamma function. Notice that in both the limit \( j \to |j^3| \) and the limit \( j \gg |j^3| \) the eigenfunction behaves nicely. The numerically generated eigenfunctions are compared with the exact eigenfunctions in figure 3 below.

**Figure 3:** A comparison between the numerically computed \( C_{p=\frac{m}{2}-|j^3|,j^3}(j) \) (plotted in blue) and the analytic formula (6.14) (plotted in red) for \( m = 120 \) and \( j^3 = 4 \). The agreement is clearly excellent.
7. Discussion

In summary we have found that: If the number of impurities is even $= 2n$ we obtain a set of oscillators with frequency $\omega_i$ and degeneracy $d_i$ given by

$$\omega_i = 8ig_{YM}^2, \quad d_i = 2(n - i) + 1, \quad i = 0, 1, ..., n.$$ 

If the number of impurities is odd $= 2n + 1$ we obtain a set of oscillators with frequency $\omega_i$ and degeneracy $d_i$ given by

$$\omega_i = 8ig_{YM}^2, \quad d_i = 2(n - i + 1), \quad i = 0, 1, ..., n.$$ 

This is exactly the spectrum that was conjectured in [35]. In the present paper we have analytically explained the degeneracies of the oscillators.

It is useful to review the salient features of our results. The operators we consider, restricted Schur polynomials, are composed of order $O(N)$ $Y$’s and $O(N)$ $Z$’s. Consequently, our large $N$ spectra are obtained by summing both planar and nonplanar diagrams. The nonplanar diagrams can not be neglected. Recall that the restricted Schur polynomial has three labels $\chi_{R,(r,s)}$ with $R$ a Young diagram containing $n + m$ boxes, $r$ a Young diagram containing $n$ boxes and $s$ a Young diagram containing $m$ boxes. We have traded these labels for the integers $j$ (which specifies $s$), $b_0, b_1$ (which specify $r$) and $j^3$ (which specifies how $R$ and $r, s$ are related). By focusing on the family of operators labeled using two column/row Young diagrams and with fixed numbers of $Z$'s and $Y$'s we have simplified the action of the dilation operator to two recursion relations, one in the $b_0, b_1$ variables and one in the $j$ variables. A crucial ingredient in our construction is the construction of the projector $P$ used in defining the restricted Schur polynomial. This construction is achieved by mapping each Young diagram $s$ onto a unique state of a spin chain with $m$ spin variables. The computation of the projector is then reduced to the problem of coupling the individual spins in the spin model to obtain a good total spin.

The operators we have studied include both BPS and non-BPS states. The zero eigenvalue states are the BPS states. We find a discrete wave equation in the 3-4 directions (corresponding to $Y$) arising from the recursion relation involving $j$. The associated “energy levels” are labeled by an integer $p$ and set the parameter $\alpha_p$ which determines the frequencies of the oscillators described by a second discrete wave equation in the 1-2 directions (corresponding to $Z$). This second wave equation arises from the recursion relation involving $b_0, b_1$. The associated energy levels are described in detail at the start of this section. We have interpreted these states as the oscillations of the relative positions of the two giant graviton branes. Our results give a gauge theory description of the giant graviton brane worldvolume physics.
Thanks to the cut off on the number of rows in the Young diagram, the sphere giants form a cleanly decoupled sector. The same is not true of the AdS giants. The fact that the AdS giants do not decouple at large $N$ as cleanly as the sphere giants did, has not caused any problems. Indeed, we have checked that all of the low lying energy levels correspond to combinations of operators with all of their support on operators labeled by Young diagrams with two long rows.

The spectra for the AdS giants and sphere giants are closely related. This similarity between the spectra goes even further: we have proved that the action of the dilatation operator on sphere giants is related to its action on AdS giants upon making the substitution $N + b \rightarrow N - b$. It would be nice to study this property in more detail.

There are a number of new features of our results that deserve comment. The spectrum we obtain is that of a set of oscillators with frequency some multiple of $8g_{YM}^2$. It is somewhat unusual to find a dependence on $g_{YM}^2$ itself at large $N$ - we are used to the combination which defines the 't Hooft coupling $\lambda = g_{YM}^2 N$. Our one loop results suggest that we can hold $g_{YM}^2$ small but fixed. If we take the usual 't Hooft limit, $g_{YM}^2 \rightarrow 0$ and we obtain a continuous spectrum. Our discrete spectra are obtained in a limit that is not the usual 't Hooft limit. It would be very interesting to study the dilatation operator to two loops and see if this dependence on $g_{YM}^2$, with no dressing by factors of $N$ to some power, persists. Another possibility is that we have an $SU(2)$ gauge symmetry for the sector of the operators with two columns or two rows, and the associated 't Hooft coupling is $2g_{YM}^2$. The factor 2 may be understood as the number that multiplies the effective tension of the two coincident brane system.

The group $SU(2)$ has played a central role. It appears in at least three (apparently) unrelated ways

- The projectors $P_{R \rightarrow (r,s)}$ were written in terms of $SU(2)$ Clebsch-Gordan coefficients. There is a natural action of $SU(2)$ defined by our map to the spin chain (see sections 4, 5).

- The eigenvalue problem for the $Z$ oscillator has been solved in terms of symmetric Kravchuk polynomials. The symmetric Kravchuk polynomials satisfy a difference equation that follows from the raising and lowering relations between $SU(2)$ states e.g. [37].

- The eigenvalue problem for the $Y$ oscillator has been solved in terms of Hahn polynomials. Hahn polynomials are closely related to the Clebsch-Gordan coefficients of $SU(2)$. 
Since we deal with a two giant graviton system, the worldvolume theory will have an $SU(2)$ gauge symmetry. Could this be related to the $SU(2)$ groups we found above? It would be interesting to study their relation.

The $SU(2)$s found above are not related to the global $SU(2)$ subgroup of the $R$-symmetry, which rotates $Z$ and $Y$ into each other. Indeed, the global $SU(2)$ rotates $Z$ and $Y$ into each other so that their action mixes operators with different values of $m$ and $n$. The $SU(2)$s found above all have an action within a given $m,n$. It would be very interesting to understand this further. This may well shed further light on the integrability we have uncovered in this article.

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### A. Dilatation Operator for AdS Giants with Two Impurities

We are interested in the two AdS giant system for the case that our operators have two impurities. In this case there are four operators that can be produced once $r$ is given. These operators are

\[
\chi_A(b_0, b_1) = \chi_{\text{first label}}(Z, Y) \quad \chi_B(b_0, b_1) = \chi_{\text{second label}}(Z, Y)
\]

\[
\chi_C(b_0, b_1) = \chi_{\text{third label}}(Z, Y) \quad \chi_D(b_0, b_1) = \chi_{\text{fourth label}}(Z, Y)
\]

The length of the rows in the first two labels are taken to infinity at large $N$. The third label is exactly as shown even at large $N$. The number of boxes removed from each row of the first label to obtain the second label are exactly as shown even at large $N$. The dilatation operator is

\[
DO_A(b_0, b_1) = 2g_Y^2(N + b_0 + b_1 + 1) \left[ \frac{1}{(b_1 + 2)^2} O_A(b_0, b_1) \right]
\]
\[ + \frac{1}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 1}{b_1 + 3}} O_C(b_0 - 1, b_1 + 2) - \frac{b_1 + 4}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 1}{b_1 + 3}} O_B(b_0 - 1, b_1 + 2) \]

\[ + 2g^2_{YM} \sqrt{(N + b_0 + b_1 + 1)(N + b_0 - 1)} \left[ \frac{b_1}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} O_B(b_0, b_1) \right] \]

\[ - \frac{1}{b_1 + 2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} O_C(b_0, b_1) + 2 \frac{1}{(b_1 + 2)^2} O_D(b_0 - 1, b_1 + 2) \]

\[ DO_D(b_0, b_1) = 2g^2_{YM}(N + b_0) \left[ \frac{b_1 - 2}{b_1^2} \sqrt{\frac{b_1 + 1}{b_1 - 1}} O_B(b_0 + 1, b_1 - 2) \right] \]

\[ - \frac{1}{b_1} \sqrt{\frac{b_1 + 1}{b_1 - 1}} O_C(b_0 + 1, b_1 - 2) + 2 \frac{1}{b_1^2} O_D(b_0, b_1) \]

\[ + 2g^2_{YM} \sqrt{(N + b_0)(N + b_0 + b_1)} \left[ \frac{2}{b_1} O_A(b_0 + 1, b_1 - 2) \right] \]

\[ + \frac{1}{b_1} \sqrt{\frac{b_1 - 1}{b_1 + 1}} O_C(b_0, b_1) - \frac{b_1 + 2}{b_1^2} \sqrt{\frac{b_1 - 1}{b_1 + 1}} O_B(b_0, b_1) \]

\[ DO_C(b_0, b_1) = g^2_{YM} \sqrt{(N + b_0 + 1)(N + b_0 + b_1 + 1)} \left[ - \frac{2}{b_1 + 2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} O_A(b_0, b_1) \right] \]

\[ - O_C(b_0 - 1, b_1 + 2) + \frac{b_1 + 4}{b_1 + 2} O_B(b_0 - 1, b_1 + 2) \]

\[ + g^2_{YM} \sqrt{(N + b_0 + b_1)(N + b_0)} \left[ \frac{2}{b_1} \sqrt{\frac{b_1 - 1}{b_1 + 1}} O_D(b_0, b_1) \right] \]

\[ - O_C(b_0 + 1, b_1 - 2) + \frac{b_1 - 2}{b_1} O_B(b_0 + 1, b_1 - 2) \]

\[ + g^2_{YM} (2N + 2b_0 + b_1 - 3) O_C(b_0, b_1) \]

\[ + g^2_{YM} \frac{(N + b_0)(4 - 4b_1 - 2b_1^2) - b_1^3 - b_1^2 + 4b_1}{b_1(b_1 + 2)} O_B(b_0, b_1) \]

\[ - 2g^2_{YM} \frac{(N + b_0 - 1)}{b_1 + 2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} O_D(b_0 - 1, b_1 + 2) \]
These take the form of superpositions. We look for operators with good scaling dimension that diagonalize the equation (3.1).

\[ DO_B(b_0, b_1) = g^2_Y \sqrt{(N + b_0 - 1)(N + b_0 + b_1 + 1)} \left[ 2 \frac{b_1}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} O_A(b_0, b_1) \right. \\
+ \left. \frac{b_1}{b_1 + 2} O_C(b_0 - 1, b_1 + 2) - \frac{b_1(b_1 + 4)}{(b_1 + 2)^2} O_B(b_0 - 1, b_1 + 2) \right] \\
+ g^2_Y \sqrt{(N + b_0 + b_1)(N + b_0)} \left[ -2 \frac{b_1 + 2}{b_1^2} \sqrt{\frac{b_1 - 1}{b_1 + 1}} O_D(b_0, b_1) \right. \\
+ \left. \frac{b_1 + 2}{b_1} O_C(b_0 + 1, b_1 - 2) - \frac{(b_1 + 2)(b_1 - 2)}{b_1^2} O_B(b_0 + 1, b_1 - 2) \right] \\
+ g^2_Y \frac{(N + b_0)(4 - 2b_1^2 - 4b_1) - b_1^2 - b_1^2 + 4b_1}{b_1(b_1 + 2)} O_C(b_0, b_1) \\
+ g^2_Y \frac{2(N + b_0)(b_1^4 + 4b_1^2 + 4b_1^2 - 8) + b_1^5 + 5b_1^4 + 8b_1^3 - 16b_1}{b_1^2(b_1 + 2)^2} O_B(b_0, b_1) \\
+ \left. \frac{2g^2_Y (N + b_0 - 1)b_1}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} O_D(b_0 - 1, b_1 + 2) \right] \\
- \frac{2g^2_Y (N + b_0 + b_1)(b_1 + 2)}{b_1^2} \sqrt{\frac{b_1 - 1}{b_1 + 1}} O_A(b_0 + 1, b_1 - 2) \]

B. Oscillators

We look for operators with good scaling dimension that diagonalize the equation (3.1). These take the form of superpositions

\[ \sum_{b_1} f(b_0, b_1) O(b_0, b_1) \]

There are not independent sums over \( b_0 \) and \( b_1 \) because \( n = 2b_0 + b_1 \) is fixed. Since

\[ \sum_{b_1} f(b_0, b_1) DO(b_0, b_1) = -\alpha g^2_Y \left( \sum_{b_1} O(b_0, b_1) \sqrt{(N + b_0)(N + b_0 + b_1)} (f(b_0 - 1, b_1 + 2) \\
+ f(b_0 + 1, b_1 - 2)) \right) - \sum_{b_1} O(b_0, b_1)(2N + 2b_0 + b_1)f(b_0, b_1) = \sum_{b_1} \kappa f(b_0, b_1) O(b_0, b_1) \]
where we have assumed \( N + b_0, b_1 \gg 1 \), the \( f(b_0, b_1) \) satisfy the recursion relation

\[
-\alpha g_{YM}^2 \frac{1}{\sqrt{(N + b_0)(N + b_0 + b_1)}} \left[ \sqrt{(N + b_0)(N + b_0 + b_1)} \right] (f(b_0 - 1, b_1 + 2) + f(b_0 + 1, b_1 - 2)) - (2N + 2b_0 + b_1) f(b_0, b_1)] = \kappa f(b_0, b_1)
\]

In the large \( N + b_0, b_1 \) regime it is accurate to take a continuum limit of this recursion relation. This gives a particularly simple description of the coefficients \( f(b_0, b_1) \). Towards this end, introduce the continuous variable \( \rho = \frac{2b_1}{\sqrt{N + b_0}} \), replace \( f(b_0, b_1) \) with \( f(\rho) \) and expand

\[
\sqrt{(N + b_0 + b_1)(N + b_0)} = (N + b_0) \left( 1 - \frac{1}{2} \frac{b_1}{N + b_0} - \frac{1}{8} \frac{b_1^2}{(N + b_0)^2} + \ldots \right)
\]

\[
f \left( \rho - \frac{1}{\sqrt{N + b_0}} \right) = f(\rho) - \frac{1}{\sqrt{N + b_0}} \frac{\partial f}{\partial \rho} + \frac{1}{2(N + b_0)} \frac{\partial^2 f}{\partial \rho^2} + \ldots
\]

These expansions are only valid if \( b_1 \ll N + b_0 \), which is certainly not always the case. However, for eigenfunctions with all of their support in the small \( \rho \) region we do expect the continuum limit of the recursion relation to give accurate answers. We find the recursion relation becomes

\[
(2\alpha g_{YM}^2)^{\frac{1}{2}} \left[ -\frac{\partial^2}{\partial \rho^2} + \rho^2 \right] f(\rho) = \kappa f(\rho)
\]

which is a half of the harmonic oscillator with frequency \( 2\alpha g_{YM}^2 \). It is only half the oscillator because the lengths of the rows (or columns) of the Young diagram are non-increasing. This implies that \( \rho \geq 0 \), so that only half of the wavefunctions are selected (those that vanish at \( \rho = 0 \)) and the energy spacing of the remaining oscillator states is \( 4\alpha g_{YM}^2 \). Clearly the description of the coefficients \( f(b_0, b_1) \) obtained by solving (B.2) will be accurate for the operators corresponding to the low lying oscillator eigenstates.

It is also possible to solve the recursion relation (B.1) directly. Since we work in the large \( N + b_0, b_1 \) regime, we can replace (B.1) by

\[
kappa f(b_0, b_1) = -\alpha g_{YM}^2 \frac{1}{\sqrt{(N + b_0)(N + b_0 + b_1 + 1)}} f(b_0 - 1, b_1 + 2)
\]

\[
+ \sqrt{(N + b_0 + 1)(N + b_0 + b_1)} f(b_0 + 1, b_1 - 2) - (2N + 2b_0 + b_1) f(b_1, b_1)]
\]

This recursion relation is solved by [37]

\[
f(b_0, b_1) = (-1)^n \left( \frac{1}{2} \right)^{\frac{N + b_0 + b_1}{2}} \sqrt{\frac{(2N + 2b_0 + b_1)}{N + b_0 + b_1}} \left( \frac{(2N + 2b_0 + b_1)}{n} \right) F_1 \left( -n, -(N + b_0 + b_1), 2 \right)
\]

(B.3)

where the hypergeometric function \( F_1 \) which appears defines the symmetric Kravchuk polynomial \( K_n(x, 1/q, p) \)

\[
F_1 \left( -n, -x, q \right) = K_n(x, 1/q, p)
\]
The corresponding eigenvalue is \( \kappa = 2n\alpha g_{YM}^2 \). As for the continuous solutions, because \( b_1 \geq 0 \) only half of the wavefunctions are selected and the energy level spacing is again \( 4\alpha g_{YM}^2 \). The solutions (B.3) are accurate for all levels of the oscillator. We have checked that the low lying solutions of (B.3) are in excellent agreement with the harmonic oscillator wave functions that vanish at the origin. Note that for any finite value of \( N \) the spectrum is bounded. It is only in the \( N \to \infty \) limit that the tower of levels is infinite. In this limit, the harmonic oscillator wave functions are an excellent description for any arbitrarily high but finite energy level.

The analysis for the equation (2.14) is similar once we make the change \( N + b_0 \to N - b_0 \). We have a similar recursion relation for \( f(b_0, b_1) \) in this second case. By taking a continuum limit, and using a continuous variable \( \tilde{\rho} = \frac{2b_1}{\sqrt{N-b_0}} \), we have

\[
(2\alpha g_{YM}^2)^{1/2} \left[ -\frac{\partial^2}{\partial \tilde{\rho}^2} + \tilde{\rho}^2 \right] f(\tilde{\rho}) = \kappa f(\tilde{\rho})
\]

which is again an oscillator with frequency \( 2\alpha g_{YM}^2 \), but since \( \tilde{\rho} \geq 0 \), again, only half of the wavefunctions are selected and the energy spacing is \( 4\alpha g_{YM}^2 \). Finally, the original recursion relation can again be solved using symmetric Kravchuk polynomials.

C. The Space \( L(\Omega_p) \)

In this Appendix we discuss the representation relevant for the problem of multiple AdS giants. We highly recommend the article [38] for background material. Consider \( S_m \) the symmetric group on \( m \)-objects. Then

\[
\Omega_p = S_m / S_{m-p} \times S_p
\]

is the space of all \( p \) subsets of \( \{1, 2, ..., m\} \). If \( m = 4 \) then \( \Omega_1 = \{ \{1\}, \{2\}, \{3\}, \{4\} \} \) and \( \Omega_2 = \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \} \) etc. You can identify a \( p \) subset with a monomial. For example, we’d identify \( \{1, 3\} \) with \( x_1x_3 \) and \( \{1, 2, 4\} \) with \( x_1x_2x_4 \). Thus, we can consider \( \Omega_p \) to be the space of distinct monomials with \( p \) factors and no factor repeats. Ordering of the factors is not important so that \( x_1x_2x_4 \) and \( x_4x_1x_2 \) are exactly the same element of \( \Omega_3 \). Our main interest is in \( L(\Omega_p) \) which is the space of complex valued functions on \( \Omega_p \). The symmetric group has a very natural action on \( L(\Omega_p) \): we can define this action by defining it on each monomial. The symmetric group acts by permuting the labels on the factors in the monomial. Thus, for example,

\[
(12)x_1x_2x_3 = x_1x_2x_3 \quad (24)x_1x_2x_3 = x_1x_4x_3.
\]
There is a natural inner product under which the monomials are orthonormal, so that, for example
\[ \langle x_1 x_2 x_3, x_1 x_2 x_3 \rangle = 1, \quad \langle x_1 x_2 x_3, x_1 x_2 x_4 \rangle = 0 = \langle x_1 x_2 x_3, x_1 x_3 x_4 \rangle. \]

\( L(\Omega_p) \) furnishes a reducible representation of the symmetric group \( S_m \). The relevance of \( L(\Omega_p) \) for us here is that the projectors acting in \( L(\Omega_p) \) projecting onto an irreducible representation of \( S_m \) are precisely the projectors we need to define the restricted Schur polynomials. Consider the operator
\[ d = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}. \tag{C.1} \]
It maps from \( L(\Omega_p) \) to \( L(\Omega_{p-1}) \). Further, it commutes with the action of \( S_m \). Because of this, elements of the kernel of \( d \) form an invariant \( S_m \) subspace. The intersection of the kernel of \( d \) and \( L(\Omega_p) \) is called \( S^{m-p,p} \) in [38] and it is proved that \( S^{m-p,p} \) is an irreducible representation of \( S_m \).

An example will help to make this discussion concrete. Consider \( S_3^{1,1} \) which is spanned by the polynomials (this basis was found by writing the obvious polynomials linear in the \( x_i \)s that are annihilated by \( d \) and then using the Gram-Schmidt algorithm to get an orthonormal basis)
\[ \phi_1 = \frac{x_1 - x_2}{\sqrt{2}}, \quad \phi_2 = \frac{x_3 - x_4}{\sqrt{2}}, \quad \phi_3 = \frac{x_1 + x_2 - x_3 - x_4}{2}. \]

It is easy to check that
\[ (12) \phi_1 = -\phi_1, \quad (12) \phi_2 = \phi_2, \quad (12) \phi_3 = \phi_3, \]
\[ (23) \phi_1 = \frac{1}{2} \phi_1 - \frac{1}{2} \phi_2 + \frac{1}{\sqrt{2}} \phi_3, \quad (23) \phi_2 = -\frac{1}{2} \phi_1 + \frac{1}{2} \phi_2 + \frac{1}{\sqrt{2}} \phi_3, \quad (23) \phi_3 = \frac{1}{\sqrt{2}} \phi_1 + \frac{1}{\sqrt{2}} \phi_2, \]
\[ (34) \phi_1 = \phi_1, \quad (34) \phi_2 = -\phi_2, \quad (34) \phi_3 = \phi_3. \]

Thus, we have the following group elements
\[ \Gamma((12)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma((23)) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \Gamma((34)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Using these matrices it is possible to compute all elements of the group now, and then to compute characters. In this way, it is a simple matter to identify this as the irreducible representation.
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