UNIQUENESS OF REGULAR SHRINKERS WITH 2 CLOSED REGIONS

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Abstract. Regular shrinkers describe blow-up limits of a finite-time singularity of the motion by curvature of planar network of curves. This follows from Huisken’s monotonicity formula. In this paper, we show that there is only one regular shrinker with 2 closed regions. This regular shrinker is the Cisgeminate eye. Moreover, we find some degenerate regular shrinkers with 2 closed regions.

1. Introduction

A regular network is an embedded network which satisfies the Herring condition: all multi-points are of degree 3 and the angles between curves are \(\frac{2\pi}{3}\). The reader can refer to [16] for detail. Given an initial regular network \(\Gamma_0\), a network flow is a family of networks with Herring condition and fixed boundary points that satisfy that \((\frac{\partial X_i}{\partial t})^\perp = \bar{k}_i\). Here \(X_i\) is the position vector and \(\bar{k}_i\) is the curvature vector of the curve \(\gamma_i\). Recently, many researchers have studied this flow in [1, 6, 12, 13, 14, 15, 16, 17, 18].

The short time existence of this flow of an initial regular \(C^2\) network with a triple junction is proved by L. Bronsard and F. Reitich in [7]. Recently, the short time existence of this flow of an initial regular \(C^2\) network with multiple junctions is proved by C. Mantegazza, M. Novaga, and A. Pluda in [15]. Using a parabolic rescaling procedure at the singular time and Huisken’s monotonicity formula [11], there is a subsequence which converges to a possibly degenerate regular network. This limit network shrinks self-similarly to the origin and it may be an open network. An open regular network is called a regular shrinker if it satisfies (1.1) \(\bar{k} + x^\perp = 0\) at any point, where \(\bar{k}\) is the curvature vector. A regular shrinker will move by homothety with respect to the origin under the network flow. Such a network describes the behavior of the flow at the singular time.

We are interested in the classification of regular shrinkers. If there are no triple junctions, the network flow is the curve shortening flow and the self-similarly shrinking solution of the flow is described in the work of U. Abresch and J. Langer [2]. They classify all immersed curves and show that the only embedded self-similarly shrinking curves are a line or a circle. A regular shrinker with exactly 1 triple junction must be a standard triod or a Brakke spoon, where the latter one is first described in the work of K. Brakke [3]. The Brakke spoon is shown to be the blow-up limit for all spoon-shaped network in the work of Pluda [18]. The classification of regular shrinker with 1 closed region is done by X. Chen and J. -S. Guo [9]. From the work of Mantegazza, Novaga, and Pluda [14], for an evolving network with at most two triple junctions, the multiplicity-one conjecture holds. P. Baldi, E. Haus, and Mantegazza [4, 5] exclude the \(\Theta\)-shaped network. Together with the work by Chen and

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Guo [9], all regular shrinkers with 2 triple junctions are completely characterized. There are only 2 such networks: the lens and the fish. This classification is used to study the general behavior of networks with 2 triple junctions in the work of Mantegazza, Novaga, Pluda [14]. The lens is shown to be the rescaling limit of any flow starting from a symmetric lens-shaped network in [1] and the work of G. Bellettini and Novaga [6]. The appendix of [16] contains a collection of all known regular shrinkers and some possible numerical results.

Apart from the cases described above, the classification of regular shrinkers remains open. In this paper, we complete the classification of all regular shrinkers with 2 closed regions. We establish the following result.

**Theorem 1.1.** The only regular shrinker with 2 closed regions is the Cisgeminate eye.

The paper is organized as follows. For any regular shrinker, it must be Abresch-Langer curves which intersect at triple junctions with angle $\frac{2\pi}{3}$. In section 2, we introduce the phase space to describe the behavior of Abresch-Langer curves. We also define some terminology which will be used throughout this article. In section 3, we focus on the possible topology of such networks and show that the topology must be a $\Theta$-shaped network with rays attached. Among the 2 closed regions, at least one of them does not contain the origin. In section 4, using the estimation of change of angle in [5], we are able to show the region which does not contain the origin must be a 4-cell. Therefore, the topology of the network must be a 4-cell attached to either a 2, 3, 4 or 5 cell. In section 5, we eliminate the possibility for the other cell to be either a 5-cell or a 2-cell. In section 6, we deal with the remaining case and establish the uniqueness of such a network. In section 7, we relax the condition to allow the regular shrinker to be degenerate and find some solutions for degenerate regular shrinkers. Some of the solutions may have curves with multiplicity greater than 1.

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2. Phase plane of Abresch-Langer curves

For a curve $\gamma(s)$ going around the origin in the counterclockwise direction, let $R$ be the distance to the origin and $\theta$ be the angle in polar coordinates. Let $\psi$ be the signed
angle from $\gamma$ to $\gamma_s$. We have $0 \leq \psi \leq \pi$. The following expression in terms of $R - \psi$ is derived in the work of Chen and Guo \cite{9}. For any curve, from the definition of $\psi$, we have $e \frac{dR}{ds} = \cos \psi$, $\frac{ds}{d\psi} = \frac{1}{R} \sin \psi$. For $\sin \psi \neq 0$, dividing the equations yields $\frac{dR}{d\psi} = R \cot \psi$. Let $\phi$ be the angle of the unit tangent vector. On a self-similarlyshrinking curve, we have $\frac{d\phi}{ds} = k = \langle \gamma, N \rangle = R \sin \psi$. Note that $\phi = \theta + \psi$. Therefore, $\frac{d\psi}{ds} = \frac{ds}{d\psi} - \frac{d\phi}{ds} = (R - \frac{1}{R}) \sin \psi$. Combining the equation involve $R$ and $\psi$, we have $(R - \frac{1}{R}) dR = \cot \psi d\psi$. Therefore, on a self-similarlyshrinking curve, we have

\[(2.1) \quad K(R) = c \sin \psi,\]

for some $c \geq 1$, where $K(R)$ is given by

\[(2.2) \quad K(R) = \frac{\exp(R^2 - 1)}{R}.\]

We define $c$ to be the energy of the curve. For the special case $\sin \psi = 0$, $\theta$ is a constant and the solution is a line through the origin. We define the energy for such curve to be infinite.

From now on, we call a curve which satisfies $\dot{k} + x = 0$ an AL-curve. Define $R - \psi$ plane as the phase plane and we will consider the trajectory $K(R) = c \sin \psi$ for some $c \geq 1$. The function $K(R)$ is strictly decreasing in $(0, 1)$, strictly increasing in $(1, \infty)$ and attains its absolute minimum 1 at $R = 1$. Therefore, $\psi$ attains the maximum $\pi - \sin^{-1}(\frac{1}{c})$ and the minimum $\sin^{-1}(\frac{1}{c})$ at $R = 1$.

Using the phase plane, we want to compute the change of angle $\theta$ when we move from one point to another point on the trajectory. If we use $R$ as the variable, it can be expressed as

\[(2.3) \quad \Delta \theta = \int_{R_1}^{R_2} \frac{d\theta}{dR} dR = \int_{R_1}^{R_2} \frac{\tan \psi}{R} dR = \int_{R_1}^{R_2} \frac{K(R)}{R \sqrt{c^2 - K(R)^2}} dR.\]

Note that if we fix $R_1$ and $R_2$, $\Delta \theta$ is monotonically decreasing with respect to $c$.

There are expressions of $\Delta \theta$ in terms of other variables. $\Delta \theta$ and $\Delta \phi$ are related by $\Delta \theta = \Delta \phi - \Delta \psi$, where $\Delta \psi$ can be determined by the starting and the ending point on the phase plane. Let $\eta = 1 + 2 \log c$. Taking log in both side of the equation (2.1), we obtain another expression of conservation law with respect to $\eta$.

\[(2.4) \quad R^2 - 2 \log k = \eta.\]

Consider the lower half of the trajectory where $0 < \psi < \frac{\pi}{2}$. Since $\frac{dk}{ds} = \frac{d}{ds} (R \sin \psi) = R^2 \cos \psi \sin \psi$, using the conservation law (2.4), it gives

\[(2.5) \quad \Delta \phi = \int_{k_1}^{k_2} \frac{d\phi}{dk} dk = \int_{k_1}^{k_2} \frac{1}{R \cos \psi} dk = \int_{k_1}^{k_2} \frac{1}{\sqrt{R^2 - k^2}} dk = \int_{k_1}^{k_2} \frac{1}{\sqrt{\eta - V(k)}} dk,

where $V(k) = k^2 - 2 \log k$ and the third equality comes from $R^2 \cos^2 \psi = R^2 - R^2 \sin^2 \psi = R^2 - k^2$. The potential $V(k)$ attains its minimum at $k = 1$. Also, for a fixed $\eta$, we define $k_{\text{min}}$ to be the unique $k < 1$ which satisfies $V(k) = \eta$. The variable $\eta$ can be regarded as the energy in terms of $k$. Note that this equation is derived in \cite{8}.

The following are expressions in terms of $\psi$. Since the trajectory is not symmetric with respect to the $R = 1$ line, we need to deal with $R < 1$ case and $R > 1$ case separately. Let $R = R^-(s)$ and $R = R^+(s)$ be the two inverses of $s = K(R)$. The domains of $R^-$, $R^+$ are
both \((1, \infty)\). The range of \(R^-\) and \(R^+\) are \((0, 1), (1, \infty)\) respectively. The change of angle, \(\Delta \theta\), is given by

\[
\Delta \theta = \int_{\psi_1}^{\psi_2} \frac{d\theta}{d\psi} d\psi = \int_{\psi_1}^{\psi_2} \frac{d\psi}{1 - [R-(c \sin \psi)]^2},
\]

\[
\Delta \theta = \int_{\psi_1}^{\psi_2} \frac{d\theta}{d\psi} d\psi = \int_{\psi_1}^{\psi_2} \frac{d\psi}{[R+(c \sin \psi)]^2 - 1},
\]

for \(R < 1\) and \(R > 1\), respectively.

**Lemma 2.1.** For any \(x > 1\), we have \(1 - [R^-(x)]^2 < [R^+(x)]^2 - 1\). Therefore, for \(\sin^{-1}(\frac{1}{c}) \leq \psi_1 < \psi_2 \leq \pi - \sin^{-1}(\frac{1}{c})\),

\[
\int_{\psi_1}^{\psi_2} \frac{d\psi}{1 - [R-(c \sin \psi)]^2} > \int_{\psi_1}^{\psi_2} \frac{d\psi}{[R+(c \sin \psi)]^2 - 1}.
\]

**Proof.** Let \(V(R) = R^2 - 2 \log R\) and \(\eta = 1 + 2 \log(x)\), we have \(V(R^+) = V(R^-) = \eta\). Let \(\alpha = 1 - R^-\). We have \(0 < \alpha < 1\) and define \(f(\alpha) = V(1+\alpha) - V(1-\alpha) = 4\alpha - 2 \log(1+\alpha) + 2 \log(1-\alpha)\). Since \(\frac{d}{d\alpha} f(\alpha) = \frac{-4\alpha^2}{1-\alpha^2} < 0\) for \(0 < \alpha < 1\), we have \(f(\alpha) < f(0) = 0\). Therefore, \(V(R^+) = V(R^-) = V(1-\alpha) = V(1+\alpha) - f(\alpha) > V(1+\alpha)\). This means \(R^+ > 1+\alpha\) and \(1 - R^- = \alpha < R^+ - 1\). We obtain

\[
1 - (R^-)^2 = (1 - R^-)(R^- + 1) < (R^+ - 1)(R^+ + 1) = (R^+)^2 - 1.
\]

The inequality (2.7) is an immediate consequence of the inequality (2.8). \(\square\)

Now, we consider the behavior of the network at a triple junction. On a trajectory satisfying \(K(R) = c \sin \psi\), the points where \(\psi = \frac{\pi}{3}, \frac{2\pi}{3}\) are important. Define \(A(c), B(c), C(c), D(c)\) be the points on the trajectory with coordinates \((R^+ (\frac{\sqrt{3}}{2} c), \frac{\pi}{3}), (R^+ (\frac{\sqrt{3}}{2} c), \frac{2\pi}{3}), (R^- (\frac{\sqrt{3}}{2} c), \frac{\pi}{3})\), \((R^- (\frac{\sqrt{3}}{2} c), \frac{2\pi}{3})\) respectively. Also, define \(M(c) = (1, \pi - \sin^{-1}(\frac{1}{c}))\), \(N(c) = (1, \sin^{-1}(\frac{1}{c}))\) to be the points with extreme \(\psi\) value.

**Figure 2.** The points A, B, C, D, M, N on the trajectory

**Lemma 2.2.** If one of the AL-curves into a triple junction in a regular shrinker is a ray or a line segment, the other 2 curves must have the same energy. Therefore, if we move in the counterclockwise direction, the corresponding point on the phase plane must switch from A to B or from D to C on the trajectory at such a triple junction.
Remark 2.3. When $c < c_\ast = \frac{2}{\sqrt{3}}$, the trajectory does not intersect the lines $\psi = \frac{\pi}{3}$, $\psi = \frac{2\pi}{3}$. In this case, the points $A(c)$, $B(c)$, $C(c)$, $D(c)$ is undefined. However, we can still define $M(c)$ and $N(c)$ where $R = 1$ and $\psi$ attains the extreme value.

From now on, for any 2 points $P$, $Q$ on the trajectory $K(R) = c \sin \psi$, use the notation $\Delta \theta_{PQ}$ to express the change of angle $\theta$ when we traverse the trajectory in the counterclockwise direction from $P$ to $Q$ without achieving a complete period. Define

\[ h_1(c) = \Delta \theta_{CD}(c) = \int_0^{\frac{2\pi}{3}} \frac{d\psi}{1 - (R^-(c \sin \psi))^2}, \]

\[ h_2(c) = \Delta \theta_{DA}(c) = \Delta \theta_{BC}(c), \]

\[ h_3(c) = \Delta \theta_{AB}(c) = \int_0^{\frac{2\pi}{3}} \frac{d\psi}{[R^+(c \sin \psi)]^2 - 1}, \]

for $c \geq c_\ast = \frac{2}{\sqrt{3}}$. Note that $c_\ast$ is the lowest energy if the curve connects to a line at a regular triple junction. We also define $\eta^\ast = 1 + 2 \log c^\ast = 1 + \log \frac{4}{3}$. Since $R^-$ is decreasing and $R^+$ is increasing, $h_1$ and $h_3$ are decreasing functions of $c$. From lemma 2.1, we have $h_1 > h_3$. Use $T(c)$ to denote the change of angle of a complete period. For $c \geq c_\ast$,

\[ T(c) = h_1(c) + 2h_2(c) + h_3(c). \]

Note from [2], $T(c)$ is decreasing and $\sqrt{2\pi} > T(c) > \pi$.

Lemma 2.4. The function $h_1$, $h_2$, $h_3$ is defined on $(c_\ast, \infty)$ with the following properties.

\[ \lim_{c \to \infty} h_1(c) = \lim_{c \to \infty} h_2(c) = \frac{\pi}{3}, \]

\[ \lim_{c \to \infty} h_3(c) = 0. \]

Proof. This lemma is established in [9]. We include the proof here for the completeness. Since $\lim_{s \to \infty} R^-(s) = 0$, $\lim_{s \to \infty} R^+(s) = \infty$,

\[ \lim_{c \to \infty} \int_0^{\frac{2\pi}{3}} \frac{d\psi}{1 - (R^-(c \sin \psi))^2} = \int_0^{\frac{2\pi}{3}} \lim_{c \to \infty} \frac{d\psi}{1 - (R^-(c \sin \psi))^2} = \int_0^{\frac{2\pi}{3}} d\psi = \frac{\pi}{3}, \]

\[ \lim_{c \to \infty} \int_0^{\frac{2\pi}{3}} \frac{d\psi}{[R^+(c \sin \psi)]^2 - 1} = \int_0^{\frac{2\pi}{3}} \lim_{c \to \infty} \frac{d\psi}{[R^+(c \sin \psi)]^2 - 1} = \int_0^{\frac{2\pi}{3}} 0 \cdot d\psi = 0. \]

Using the result from [2] about the change of angle of a complete period, we have $\lim_{c \to \infty} (h_1 + 2h_2 + h_3) = \pi$. We can deduce $\lim_{c \to \infty} h_2 = \frac{\pi}{3}$. \qed
We are now going to estimate the change of angle which corresponds to each part of the trajectories. The following estimation as a lower bound of the potential function $V(k)$ is needed.

**Lemma 2.5.** For $k_{\text{min}} \leq k_1 < k \leq 1$, $k_0 \leq k_1$, let $\bar{V} = (1 + \frac{1}{k_0})(k - 1)^2 + H$, where $H$ is chosen such that $\bar{V}(k_1) = V(k_1)$. We have $V(k) > \bar{V}(k)$ for all $k \in (k_1, 1)$. Therefore, for $k_1 < k_2 \leq 1$,

\[
(2.15) \quad \Delta \phi = \int_{k_1}^{k_2} \frac{dk}{\sqrt{\eta - V(k)}} \geq \frac{1}{\sqrt{1 + \frac{1}{k_0}}} \left( \sin^{-1} \left( \frac{1 - k_1}{\eta - H} \right) - \sin^{-1} \left( \frac{1 - k_2}{\eta - H} \right) \right).
\]

For the special case $k_1 = k_{\text{min}}$, $\sqrt{\eta - H} = 1 - k_{\text{min}}$.

**Proof.** For $k_{\text{min}} \leq k_1 < k \leq 1$, we have

\[
(2.16) \quad V'(k) = 2(k - \frac{1}{k}) = 2(1 + \frac{1}{k})(k - 1) > 2(1 + \frac{1}{k_0})(k - 1) = \bar{V}'(k).
\]

Use $\bar{V}(k_1) = V(k_1)$, we can deduce $V(k) > \bar{V}(k)$ for all $k \in (k_1, 1)$. We can obtain the estimation of the integral by direct computation. \(\square\)

We need a lower bound for $h_1 + 2h_2$. This quantity plays an important role when we are excluding some impossible cases.

**Theorem 2.6.** For every $\eta \geq \eta_*(i.e. c \geq c_*)$, we have $h_1(c) + 2h_2(c) > 0.7789\pi(> \frac{2\pi}{3})$. Furthermore, if $\eta \geq \frac{4}{3}$, we have $h_1 + 2h_2 > 0.9456\pi$. For $\eta \geq 1.38$, we have $h_1 + 2h_2 > \pi$.

**Proof.** We want to estimate $\Delta \phi$. Let $\hat{R}(c) = R^2(\frac{\sqrt{3}}{2}c) \geq 1$ be the $R$ value at point $A$. Note that $\eta$ is strictly increasing with respect to $\hat{R}$.

Case 1: $\eta \geq 1.38$.

\[
(2.17) \quad \Delta \phi = \int_{k_{\text{min}}}^{\frac{\sqrt{3}}{2}R} \frac{2dk}{\sqrt{\eta - V(k)}} = \int_{k_{\text{min}}}^{1} \frac{2dk}{\sqrt{\eta - V(k)}} + \int_{1}^{\frac{\sqrt{3}}{2}R} \frac{2dk}{\sqrt{\eta - V(k)}}.
\]

Define $L(\hat{R}) = \int_{k_{\text{min}}}^{1} \frac{2dk}{\sqrt{\eta - V(k)}}$ and $R(\hat{R}) = \int_{1}^{\frac{\sqrt{3}}{2}R} \frac{2dk}{\sqrt{\eta - V(k)}}$ to be the contribution of the left side and right side of the potential function to $\Delta \phi$.

For the left side, from lemma 2.5 we have

\[
(2.18) \quad L(\hat{R}) \geq \int_{k_{\text{min}}}^{1} \frac{2dk}{\sqrt{\eta - V(k)}} = \frac{\pi}{\sqrt{1 + \frac{1}{k_{\text{min}}}}}.
\]

For the right side, let $\kappa = \sqrt{2\log \hat{R} - 2\log \frac{2}{\sqrt{3}} + 1}$. Let $\bar{V}(k) = k^2 - 2\log \hat{R} + 2\log \frac{2}{\sqrt{3}}$ for $k \in (\kappa, \frac{\sqrt{3}}{2}R)$ and $\bar{V}(k) = 1$ for $k \in (1, \kappa)$. We have $\bar{V}(k) < V(k)$. The right side is bounded below by

\[
(2.19) \quad R(\hat{R}) \geq \int_{1}^{\frac{\sqrt{3}}{2}R} \frac{2dk}{\sqrt{\eta - V(k)}} = \frac{2\pi}{3} - 2\sin^{-1}(\frac{\kappa}{\hat{R}}) + 2\frac{\kappa - 1}{\sqrt{\eta - 1}}.
\]
Note that

\[
(2.20) \quad \frac{d}{dR} \left( \frac{\kappa - 1}{\sqrt{R^2 - \kappa^2}} - \sin^{-1} \left( \frac{\kappa}{R} \right) \right) = \frac{1}{\sqrt{R^2 - \kappa^2}} \left( \frac{R^2 - \kappa^3}{R} + \frac{\kappa - 1}{R} \right).
\]

Use \( \frac{d}{dR}(\hat{R}^2 - \kappa^2) = \frac{4}{3} \hat{R}^2 - 2\hat{R}^{-1} \), the minimum for \( \hat{R}^2 - \kappa^2 \) happens when \( \hat{R}^2 = \frac{3}{2} \). Therefore, \( \hat{R}^2 - \kappa^2 \geq \frac{3}{2} - \frac{3}{2} \log \frac{3}{2} + 2 \log \frac{3}{\sqrt{3}} - 1 > 0 \). We have \( \hat{R}^2 > \kappa^3 \) and \( \frac{d}{dR} \left( \frac{\kappa - 1}{\sqrt{R^2 - \kappa^2}} - \sin^{-1} \left( \frac{\kappa}{R} \right) \right) > 0 \).

\( R(\hat{R}) \) is bounded below by a function which is increasing when \( \hat{R} \) increases.

The following bound can be obtained by using a scientific calculator for elementary functions.

| \( \eta \) | \( k_{\min} > \) | Lower bound for \( L(\hat{R}) \) | \( R > \) | Lower bound for \( R(\hat{R}) \) |
|---|---|---|---|---|
| 1.38 | 0.60 | 0.6123\pi | 1.22 | 0.0585\pi |
| 1.4 | 0.59 | 0.6091\pi | 1.24 | 0.0748\pi |
| 1.45 | 0.56 | 0.5991\pi | 1.29 | 0.1123\pi |
| 1.5 | 0.52 | 0.5848\pi | 1.34 | 0.1453\pi |
| 2 | 0.39 | 0.5927\pi | 1.64 | 0.2761\pi |
| 3 | 0.32 | 0.4246\pi | 2.03 | 0.3631\pi |
| 4 | 0.13 | 0.3392\pi | 2.32 | 0.4035\pi |
| 5 | 0.08 | 0.2722\pi | 2.56 | 0.4287\pi |

When \( \eta \) increases, \( \hat{R} \) increases and \( k_{\min} \) decreases. Therefore, the lower bound for the right side increases and the lower bound for the left side decreases. We have

\[
\text{Range for eta } \eta \quad \text{Lower bound for } \Delta \phi
\]

| \( [1.38, 1.4] \) | \( [1.4, 1.45] \) | \( [1.45, 1.5] \) | \( [1.5, 2] \) | \( [2, 3] \) | \( [3, 4] \) | \( [4, 5] \) |
|---|---|---|---|---|---|---|
| 0.0585\pi \leq 0.6091\pi > \frac{3\pi}{4} | 0.0748\pi \leq 0.5991\pi > \frac{3\pi}{4} | 0.1123\pi \leq 0.5848\pi > \frac{3\pi}{4} | 0.1453\pi \leq 0.5296\pi > \frac{3\pi}{4} | 0.2792\pi \leq 0.4246\pi > \frac{3\pi}{4} | 0.3631\pi \leq 0.3392\pi > \frac{3\pi}{4} | 0.4035\pi \leq 0.2722\pi > \frac{3\pi}{4} |

For all \( \hat{R} > \frac{5}{2} \), first we compare \( k_{\min} \) with \( \frac{1}{2R^2} \). Note that \( V(k_{\min}) = \eta \). Let

\[
(2.21) \quad U(\hat{R}) = \eta - V \left( \frac{1}{2R^2} \right) = \hat{R}^2 - 6 \log \hat{R} + 2 \log \frac{1}{\sqrt{3}} - \frac{1}{4\hat{R}^4}.
\]

we have \( \frac{d}{dR} U = \frac{1}{R^3} (2\hat{R}^6 - 6\hat{R}^4 + 1) \). Note that \( 2\hat{R}^6 - 6\hat{R}^4 + 1 \) is increasing for \( \hat{R} > \sqrt{2} \), \( \frac{d}{dR} U(2) > 0 \), \( U \) is increasing for \( \hat{R} > 2 \). Together with \( U(\frac{5}{2}) > 0 \), we can deduce for \( \hat{R} > \frac{5}{2} \), \( U > 0 \) and therefore \( k_{\min} < \frac{1}{2R^2} \).

Use \( L(\hat{R}) = \int_{k_{\min}}^{\frac{2\kappa}{\sqrt{\eta - V(k)}}} > \frac{2(1-k_{\min})}{\sqrt{\eta - 1}} \),

\[
(2.22) \quad \Delta \phi \geq \frac{2\pi}{3} - 2 \sin^{-1} \left( \frac{\kappa}{\hat{R}} \right) + \frac{2(\kappa - k_{\min})}{\sqrt{R^2 - \kappa^2}} \geq 2 \left( \frac{\pi}{3} - \sin^{-1} \left( \frac{\kappa}{\hat{R}} \right) + \frac{\kappa - \frac{1}{2R^2}}{\sqrt{R^2 - \kappa^2}} \right).
\]
Consider $\kappa$ as function of $\hat{R}$. Let $F(\hat{R}) = -\sin^{-1}(\frac{\kappa}{\hat{R}}) + \frac{\kappa - \frac{3\kappa}{\sqrt{2}}}{\sqrt{R^2 - \kappa^2}}$. We have

\begin{equation}
F'(\hat{R}) = \frac{1}{\hat{R}\sqrt{R^2 - \kappa^2}} \left( \kappa(1 - \kappa^2) + \frac{1}{2}(3 - \frac{1}{R^2} - \frac{2\kappa^2}{R^2}) \right)
\end{equation}

For $\hat{R} > \frac{\sqrt[3]{4}}{2} > \frac{2\sqrt{3}}{\sqrt[3]{3}}$, we can deduce that $F'(\hat{R}) < 0$. Since $\lim_{\hat{R} \to \infty} F = 0$, we get $\Delta \phi > \frac{2\pi}{3}$. In each case, we have $\Delta \theta = \Delta \phi - \Delta \psi > \pi$.

**Case 2:** $\frac{4}{3} \leq \eta(\hat{R}) < 1.38$. From the proof of the theorem above, we have $\Delta \phi > L(\hat{R}) > 0.6123\pi$. Therefore, $\Delta \theta > 0.9456\pi$.

**Case 3:** $\eta_* \leq \eta < \frac{4}{3}$. In this case,

\begin{equation}
1 + 2\log\frac{2}{\sqrt{3}} \leq \eta(\hat{R}) = \hat{R}^2 - 2\log \hat{R} + 2\log\frac{2}{\sqrt{3}} \leq \frac{4}{3}.
\end{equation}

Note that $V(\frac{1}{\sqrt{3}}) = \frac{1}{3} + \log 3 > \frac{4}{3}$. On the other hand, $V(\frac{2}{3}) = \frac{4}{9} + 2\log\frac{9}{4} < 1 + 2\log\frac{2}{\sqrt{3}}$, we can deduce $\frac{1}{\sqrt{3}} < k_{\min} < \frac{2}{3} < \frac{5}{6} < \frac{\sqrt{3}}{2} \leq \frac{\sqrt{3}}{2}. \hat{R}$. Therefore, from lemma 2.5

\begin{equation}
\Delta \phi > \frac{1}{\sqrt{1 + \frac{2}{\sqrt{3}}}}(\pi - 2\sin^{-1}\frac{1 - \frac{\sqrt{3}}{2} \hat{R}}{1 - k_{\min}}) > 0.4456\pi(> \frac{\pi}{3}).
\end{equation}

Therefore, $\Delta \phi > 0.4456\pi$, $h_1(c) + 2h_2(c) = \Delta \psi + 0.4456\pi > 0.7789\pi(> \frac{2\pi}{3})$. □

**Corollary 2.7.** The Cisgeminate 3-ray star proposed in the appendix of [16] does not exist.

**Proof.** By symmetry, the change of angle is $\frac{2\pi}{3}$ for each 5-cell. On the other hand, the change of angle should be $h_1 + 4h_2$ for the corresponding energy, which is impossible since $h_1 + 2h_2 > \frac{2\pi}{3}$. □

**Proposition 2.8.** For any $c > 1$, $\Delta \theta_{MN}(c) < \pi$.

**Proof.** Let $\psi$ be the $\phi$ value at $N(c)$. We have $c = \frac{1}{\sin(\psi)}$ and $\eta = 1 + 2\log c = 1 - 2\log \sin \psi$. The curvature corresponds to $(1, \psi_0)$ is $k_0 = \sin \psi_0$.

\begin{equation}
\Delta \theta = \Delta \phi - \Delta \psi = \Delta \phi + \pi - 2\psi_0.
\end{equation}

We need to estimate $\Delta \phi$. Let $\hat{V}$ be the linear function passing through $(k_{\min}, \eta) = (k_{\min}, 1 - 2\log \sin \psi_0)$ and $(k_0, \hat{V}(k_0)) = (\sin \psi_0, \sin^2 \psi_0 - 2\log \sin \psi_0)$. Since $\hat{V}'' = 0$, $\hat{V}'' > 0$, we have $\hat{V} > V$ for all $k \in (k_{\min}, k_0)$.

\begin{equation}
\Delta \phi = \int_{k_{\min}}^{k_0} \frac{2dk}{\sqrt{\eta - \hat{V}(k)}} \leq \int_{k_{\min}}^{k_0} \frac{2dk}{\sqrt{\eta - \hat{V}(k)}} = 4\frac{k_0 - k_{\min}}{\cos \psi_0}.
\end{equation}

Note that

\begin{equation}
V(\sin \frac{\psi_0}{\sqrt{e}}) = \sin^2 \frac{\psi_0}{e} - 2\log(\sin \frac{\psi_0}{\sqrt{e}}) = \frac{\sin^2 \psi_0}{e} - 2\log \sin \psi_0 + 1 > \eta.
\end{equation}
We have \( \frac{\sin \psi}{\sqrt{\pi}} < k_{\text{min}} \). Therefore,

\[
(2.29) \quad \Delta \phi \leq 4 \frac{k_0 - k_{\text{min}}}{\cos \psi_0} < 2 \frac{2(1 - \frac{1}{\sqrt{\pi}})}{\cos \psi_0} \sin \psi_0.
\]

For \( \psi_0 < 0.21\pi < \cos^{-1}(2(1 - \frac{1}{\sqrt{\pi}})) \), we have \( \Delta \phi \leq 2 \sin \psi_0 \leq 2\psi_0 \).

For \( 0.21\pi \leq \psi_0 < \frac{\pi}{2} \), we can improve the lower bound for \( k_{\text{min}} \). Since

\[
(2.30) \quad V\left(\frac{\sin \psi_0}{\sqrt{2}}\right) = \frac{\sin^2 \psi_0}{2} - 2 \log(\frac{\sin \psi_0}{\sqrt{2}}) = \frac{\sin^2(\frac{\pi}{2})}{2} - 2 \log \sin \psi_0 + \log 2 > \eta,
\]

\( k_{\text{min}} > \frac{\sin \psi_0}{\sqrt{2}} \) in this case. Let \( \hat{V}(k) = 2(k - 1)^2 + L \) where \( L \) is chosen such that \( \hat{V}(k_{\text{min}}) = \eta \).

We have \( \hat{V} > V \) in our interval of integration since \( \hat{V}'' < V'' \), \( \hat{V}(k_{\text{min}}) = V(k_{\text{min}}) \) and \( \hat{V}'(1) = V'(1) = 0 \).

\[
(2.31) \quad \Delta \phi \leq \int_{k_{\text{min}}}^{\sin \psi_0} \frac{2dk}{\sqrt{\eta - L - 2(k - 1)^2}} = \sqrt{2} \cos^{-1}\left(\frac{1 - \sin \psi_0}{1 - k_{\text{min}}} \right) < \sqrt{2} \cos^{-1}\left(1 - \frac{\sin \psi_0}{1 - k_{\text{min}}} \right).
\]

Note that we use the substitution \( -\sqrt{\frac{\eta - L}{2}} \cos \alpha = k - 1 \).

Let \( f(\psi_0) = \sin \psi_0 + \cos \sqrt{2} \psi_0(1 - \frac{\sin \psi_0}{\sqrt{2}}) \), we want to show that \( f < 1 \) for \( \frac{\pi}{5} \leq k < \frac{\pi}{2} \). We have \( f < 1 \) for \( \psi_0 \geq \frac{\pi}{2\sqrt{2}} \).

| Range for \( \psi_0 \) | Upper bound for \( f(\psi_0) \) |
|------------------------|--------------------------------|
| \([0.3\pi, \frac{\pi}{2\sqrt{2}}]\) | \( \frac{\pi}{2\sqrt{2}} + \cos(\sqrt{2} \cdot 0.3\pi)(1 - \frac{\sin 0.3\pi}{\sqrt{2}}) < 0.9969 \) |
| \([0.27\pi, 0.3\pi]\) | \( 0.3\pi + \cos(\sqrt{2} \cdot 0.27\pi)(1 - \frac{\sin 0.27\pi}{\sqrt{2}}) < 0.9794 \) |
| \([0.24\pi, 0.27\pi]\) | \( 0.27\pi + \cos(\sqrt{2} \cdot 0.24\pi)(1 - \frac{\sin 0.24\pi}{\sqrt{2}}) < 0.9990 \) |
| \([0.22\pi, 0.24\pi]\) | \( 0.24\pi + \cos(\sqrt{2} \cdot 0.22\pi)(1 - \frac{\sin 0.22\pi}{\sqrt{2}}) < 0.9917 \) |
| \([0.21\pi, 0.22\pi]\) | \( 0.22\pi + \cos(\sqrt{2} \cdot 0.21\pi)(1 - \frac{\sin 0.21\pi}{\sqrt{2}}) < 0.9748 \) |

This is equivalent to \( \Delta \phi < 2\psi_0 \).

\( \square \)

If we set \( \psi_0 = \frac{\pi}{5} \), \( A(c_*) = N(c_*) = D(c_*) \) and \( B(c_*) = M(c_*) = D(c_*) \) in the proposition, we have the following corollary.

**Corollary 2.9.** The upper bound of \( h_1 \) is given by

\[
(2.32) \quad h_1(c_*) < \pi.
\]

3. **The Possible Topology of a Regular Shrinker with 2 Closed Regions**

Now, we turn our attention to the topology of a regular shrinker with possibly more than 2 closed regions. Remove all the rays from such a regular shrinker and consider it as a graph \( G \) with \( E \) edges and \( V \) vertices.

**Lemma 3.1.** For any regular shrinker with at least one closed region, let \( F_i \) be the closed regions enclosed by the network. Then \( \bigcup_i \text{cl}(F_i) \) is star-shaped with respect to the origin \( O \).
Proof. From the graph $G$ defined above, define $\rho: I \subset S^1 \to \mathbb{R}$ by

$$\rho(t) = \max_{\{x \in \mathbb{R}^+ | x \in G\}} x,$$

where $I$ is the maximal subset of $\mathbb{S}^2$ such that $\rho$ can be defined. Since $G$ is a compact set, if the set $\{x \in \mathbb{R}^+ | x \in G\}$ is not empty, we can get the maximum value.

For any $t \in I$, $\rho(t)t \in G$. If $\rho(t)t$ is a vertex of $G$, since the edges intersection at $\rho(t)t$ and the angle between the curves are $\frac{2\pi}{3}$, there should be at least 1 curve going clockwise and 1 curve going counterclockwise from $\rho(t)t$. Therefore, there should be a neighborhood of $t$ which is contained in $I$. If $\rho(t)t$ lies on an edge of $G$, this edge cannot be a line segment since there exist one endpoint of the line segment corresponds to the same $t \in S^1$ with larger distance from the origin. Therefore, it must lies on a segment of a nondegenerate AL-curves and there should be a neighborhood of $t$ which is contained in $I$. $I$ is open in $S^1$.

For any sequence $t_i \in I$, $t_i \to t$, $\rho(t_i)t_i$ is a sequence in $G$. Since $G$ is compact, there must be a limit point $x$ of $\rho(t_i)t_i$ in $G$. Since there are only finitely many nondegenerate AL-curves in $G$ and each $\rho(t_i)t_i$ lies on either a segment of a nondegenerate AL-curve or an endpoint of a segment of a nondegenerate AL-curve. $\rho(t_i)$ is bounded away from 0. Therefore $\|x\| > 0$, $x = \|x\|t$ and $t \in I$. $I$ is closed in $S^1$. Since $I$ is nonempty, we have $I = S^1$.

Note that $\rho$ is upper semi-continuous, $\limsup_{t \to t_0} \rho(t) \leq \rho(t_0)$. For any $t_0 \in S^1$, $\rho(t_0)t_0$ is a vertex or belongs to a non-degenerate AL-curve. Again, there exists a neighborhood of $\rho(t_0)t_0$ in $G$. We can find a sequence $P_i \in G$ in the neighborhood such that it converges to $\rho(t_0)t_0$ and $t_i = \frac{P_i}{\|P_i\|} \neq t_0$, $t_i$ converges to $t_0$. We obtain

$$\rho(t_0) = \lim_{P_i \to \rho(t_0)t_0} \|P_i\| \leq \liminf_{t_i \to t_0} \rho(t) \leq \limsup_{t_i \to t_0} \rho(t) \leq \rho(t_0).$$

Therefore, $\rho(t_0) = \lim_{t \to t_0} \rho(t)$ and $\rho$ is continuous on $S^1$. Let $\Gamma(t) = \rho(t)t$ and $F$ be the finite region enclosed by the curve $\Gamma(t)$. Note that the origin $O$ belongs to $F$ because $I = S^1$. Since $\Gamma \subset G_0$ and $G_0 \subset \text{cl}(F)$, we obtain $\cup_i \text{cl}(F_i) = \text{cl}(F)$ is star-shaped with respect to the origin. $\square$

Now, we turn our attention to the topology of regular shrinker with 2 closed regions.

**Theorem 3.2.** The topology of a regular shrinker with 2 closed regions must be a Θ-shaped network with possibly multiple rays attached to the outer curves.

**Proof.** Use lemma 3.1 the 2 closed regions share at least an edge. If they share more than 1 edge, we obtain either there are more than 2 closed regions or some multipoints are not triple junctions. It is impossible.

We need to exclude the case that one of the regions is a 1-cell surrounded by another region with 4-cell or 5-cell. Let $\gamma_1$ be the boundary of the 1-cell and $\gamma_2$ be the piecewise smooth curve which is the boundary of $\text{cl}(F_1) \cup \text{cl}(F_2)$, where $F_1$ are the closed regions of the network. There can be at most one ray attached to $\gamma_2$. Using lemma 2.2 the energies of all smooth AL-curves of $\gamma_2$ are the same. Let $c_1$ be the energy of $\gamma_1$. Since $T(c) < \sqrt{2}\pi$ and the change of angle on $\gamma_1$ is $2\pi$, the curve $\gamma_1$ consists more than a complete period. The $R$ value of $\gamma_1$ must achieve the maximum $R^+(c_1)$ and the minimum $R^-(c_1)$. Since $\gamma_1$ is included in the region enclosed by $\gamma_2$, there exists a value $R$ on $\gamma_2$ with $R > R^+(c_1)$. Therefore, we have $c_2 > c_1$. 
Proof. For a regular triple junction, \( \psi \) is the starting point or the ending point of Proposition 3.3. Let \( \psi \) go from the starting point to the ending point, depending on whether they go in the counterclockwise direction or the clockwise direction. Assume \( \gamma \) is an AL-curve which goes from the starting point to the ending point. We call them \( \gamma \) AL-curves which correspond to the \( \Theta \) network on the right as the starting point and the other triple junction of the original \( \Theta \) network as the ending point. We call the inner curve of the \( \Theta \) network \( \gamma \). Aside from \( \gamma \), there are 2 piecewise smooth curves consisting of AL-curves which goes from the starting point to the ending point. We call them \( \gamma_{\text{up}}, \gamma_{\text{down}} \) depending on whether they go in the counterclockwise direction or the clockwise direction from the starting point to the ending point.

Let \( R_{\text{start}}, R_{\text{end}} \) be the \( R \) value for the starting point and the ending point respectively.

**Proposition 3.3.** Let \( \psi_{\text{start,up}}, \psi_{\text{end,up}}, \psi_{\text{start,in}}, \psi_{\text{end,in}} \) be the corresponding \( \psi \) at the starting point or the ending point of \( \gamma_{\text{up}}, \gamma_{\text{in}} \) respectively. Then \( \psi_{\text{start,up}} + \psi_{\text{end,up}} = \pi \), \( \psi_{\text{start,in}} + \psi_{\text{end,in}} = \pi \), and \( K(R_{\text{start}}) = K(R_{\text{end}}) \).

**Proof.** For a regular triple junction, \( \psi_{\text{start,up}} + \frac{2\pi}{3} = \psi_{\text{start,in}} \). Similarly, \( \psi_{\text{end,up}} - \frac{2\pi}{3} = \psi_{\text{end,in}} \).

Compute the energy of \( \gamma_{\text{up}} \) and \( \gamma_{\text{in}} \) gives

\[
\begin{align*}
c_{\text{up}} &= \frac{K(R_{\text{start}})}{\sin(\psi_{\text{start,up}})} = \frac{K(R_{\text{end}})}{\sin(\psi_{\text{end,up}})}, \\
c_{\text{in}} &= \frac{K(R_{\text{start}})}{\sin(\psi_{\text{start,in}})} = \frac{K(R_{\text{end}})}{\sin(\psi_{\text{end,in}})}.
\end{align*}
\]

We obtain

\[
\frac{\sin(\psi_{\text{start,up}})}{\sin(\psi_{\text{start,up}} + \frac{2\pi}{3})} = \frac{\sin(\psi_{\text{end,up}})}{\sin(\psi_{\text{end,up}} - \frac{2\pi}{3})}.
\]

We omit the subscript ”up” in the next equation. The equation is equivalent to

\[
\sin(\psi_{\text{start}})[\frac{1}{2}\sin(\psi_{\text{end}}) - \frac{\sqrt{3}}{2}\cos(\psi_{\text{end}})] = \sin(\psi_{\text{end}})[\frac{1}{2}\sin(\psi_{\text{start}}) + \frac{\sqrt{3}}{2}\cos(\psi_{\text{start}})]
\]

After combining some terms, we have

\[
0 = \frac{\sqrt{3}}{2}[\sin(\psi_{\text{start}})\cos(\psi_{\text{end}}) + \sin(\psi_{\text{end}})\cos(\psi_{\text{start}})] = \frac{\sqrt{3}}{2}\sin(\psi_{\text{start}} + \psi_{\text{end}}).
\]

Therefore, \( \psi_{\text{start}} + \psi_{\text{end}} = \pi \). \( \square \)

If we move along \( \gamma_{\text{down}} \) from the starting point to the ending point, we are moving counterclockwise. In order to use the setting for counterclockwise oriented AL-curve in section 2. We
use clockwise direction as the positive direction for the $\theta$, $\phi$, $\psi$ value related to $\gamma_{\text{down}}$. In this setting, we have $\psi_{\text{start, down}} = \frac{2\pi}{7} - \psi_{\text{start, up}}$.

Define $\theta_{\text{up}}$, $\theta_{\text{in}}$, $\theta_{\text{down}}$ be the total change of angle for the curves respectively. Note that $\theta_{\text{down}}$ is measured clockwise. We have

\begin{equation}
\theta_{\text{up}} = \theta_{\text{in}} = 2\pi - \theta_{\text{down}}.
\end{equation}

From the symmetry, the $\psi$ value at the starting point gives suffice information for the $\psi$ value at the ending point. From now on, use $\psi_{\text{up}}$, $\psi_{\text{in}}$, $\psi_{\text{down}}$ to describe the $\psi$ value for each curve at the starting point for simplicity. $\psi_{\text{in}} = \psi_{\text{up}} + \frac{2\pi}{3}$, $\psi_{\text{down}} = \frac{2\pi}{3} - \psi_{\text{up}}$.

4. The cell which does not contain the origin

Since there are 2 closed regions, at least one of them does not contain the origin in the interior. We can follow the argument in [5] to show it must be a 4-cell. The following theorem concerning 2-cell is established in [5].

**Theorem 4.1 ([5]).** In a self-similarly shrinking network moving by curvature, there are no 2-cells without the origin inside.

If the closed region which does not contain the origin is a 3-cell, a 4-cell, or a 5-cell, we have the following lemma.

**Lemma 4.2.** We have the following result concerning $\gamma_{\text{up}}$ and $\gamma_{\text{in}}$.

(1) If $\gamma_{\text{up}}$ passes through the point corresponding to $(R^-(c_{\text{up}}), \frac{\pi}{2})$, we have $c_{\text{in}} > c_{\text{up}}$.

(2) It is impossible for $\gamma_{\text{up}}$ has a complete period on its trajectory.

**Proof.** We need part 1 to establish part 2.

(1) If $\gamma_{\text{up}}$ pass through the point $(R^-(c_{\text{up}}), \frac{\pi}{2})$ on the trajectory, since $\gamma_{\text{in}}$ lies inside, if we connect the origin with the point corresponding to $(R^-(c_{\text{up}}), \frac{\pi}{2})$ with a line segment, the line segment must intersect $\gamma_{\text{in}}$. From this, we have $R^-(c_{\text{in}}) < R^-(c_{\text{up}})$, this is equivalent to $c_{\text{in}} > c_{\text{up}}$.

(2) If $\gamma_{\text{in}}$ is nondegenrate, since $\psi_{\text{in}} > \frac{2\pi}{3}$, the starting point of $\gamma_{\text{in}}$ lies on the $BC$ arc of the trajectory, the ending point of $\gamma_{\text{in}}$ lies on the $DA$ arc of the trajectory. Assume $\gamma_{\text{in}}$ passes through the point corresponding to $(R^+(c_{\text{in}}), \frac{\pi}{2})$, the point with largest $R$ on the trajectory, since $c_{\text{in}} > c_{\text{up}}$, $R^+(c_{\text{in}}) > R^+(c_{\text{up}})$, $\gamma_{\text{in}}$ and $\gamma_{\text{up}}$ must intersect and we get a contradiction. Therefore, on the phase plane, $\gamma_{\text{in}}$ only achieve the part from point $B$ to point $A$ on its trajectory.

\begin{equation}
\theta_{\text{in}} \leq (h_1 + 2h_2)(c_{\text{in}}) < T(c_{\text{in}}) < T(c_{\text{up}}).
\end{equation}

If $\gamma_{\text{up}}$ has a complete period on its trajectory, $\theta_{\text{up}} > T(c_{\text{up}})$ and this contradict $\theta_{\text{up}} = \theta_{\text{in}}$. If $\gamma_{\text{in}}$ is degenerate, we have $\theta_{\text{in}} = \pi < T(c_{\text{up}}) < \theta_{\text{up}}$ and we get a contradiction.

□

To eliminate the possibility that this cell is a 3-cell, we need the following lemma from [5].

**Lemma 4.3 ([5]).** Let $\gamma$ be a shrinking curve, parametrized counterclockwise by arc length, with positive curvature and let $(s_0, s_1)$ be an interval where $R(s)$ is increasing. If $R_s(s_0) \geq \frac{1}{2}$, namely, $\psi(s_0) \leq \frac{\pi}{3}$, then

\begin{equation}
\int_{s_0}^{s_1} \frac{d\theta}{ds} ds < \frac{\pi}{2}.
\end{equation}
Similarly, if \( R(s) \) is decreasing on \((s_0, s_1)\) and \( \frac{dR}{ds}(s_1) \leq -\frac{1}{2}, \) namely, \( \psi(s_1) \geq \frac{2\pi}{3}, \) then the same conclusion holds. This is equivalent to \( 2h_2 + h_3 \leq \pi. \)

**Theorem 4.4.** The upper cell cannot be a 3-cell.

**Proof.** For the 3-cell we are studying, label the triple junction connected to the ray as \( P \). The curve \( \gamma_{in} \) goes from \( S \) to \( E \). The piecewise smooth curve \( \gamma_{up} \) goes from \( S \) to \( P \) and then from \( P \) to \( E \). We name the part from \( S \) to \( P \) as \( \gamma_{up}^1 \) and the second part from \( P \) to \( E \) as \( \gamma_{up}^2 \).

If \( \gamma_{up} \) passes through the point \( (R^+(c_{up}), \frac{\pi}{2}) \) on the trajectory, without loss of generality, assume it happens on \( \gamma_{up}^1 \). On the phase plane, \( \gamma_{up}^1 \) starts at a point on the \( DA \) arc, passes through \( (R^+(c_{up}), \frac{\pi}{2}) \) on the trajectory and at the ending point \( P \), the corresponding point must be either \( D \) or \( A \). If it ends at point \( A \) on the phase plane, we have a complete period of \( (R, \psi) \) when traversing \( \gamma_1 \). If it ends at point \( D \), consider the curve \( \gamma_{up}^2 \), the starting point on the phase plane is \( C \) and it ends somewhere between \( B \) and \( C \). Therefore, \( \gamma_{up} \) covers a complete period of the trajectory \( (R, \psi) \). This is impossible from lemma 4.2.

From the previous part, \( \gamma_{up}^1 \) and \( \gamma_{up}^2 \) do not pass through \( (R^+(c_{up}), \frac{\pi}{2}) \) on the phase plane. \( R \) is strictly increasing on \( \gamma_{up}^1 \) and is strictly decreasing on \( \gamma_{up}^2 \). Now, we separate into 2 cases. From the previous section, we choose the coordinate such that the line passes through point \( S \) and point \( E \) is parallel to the \( x \)-axis. Let \( y = m \) be the equation for this line.

**Case 1:** \( m > 0 \). Let \( s_1^* \) be the arc length parameter at the start of \( \gamma_{up}^1 \). Since \( \gamma_{up}^1, \gamma_{in} \) are above \( L \), the angle between \( \frac{d\gamma_{up}^1}{ds}(s_1^*) \) and \( (1,0) \) is less than or equal to \( \frac{\pi}{3} \). Using the ODE describing the self-shrinking curve, we can extend \( \gamma_{up}^1 \) to \( s < s_1^* \). This curve must intersect positive \( x \)-axis at some \( s_1 \). Since the curvature is positive, the angle between \( \gamma_{up}^1(s_1) \) and \( (1,0) \) is less than or equal to \( \frac{\pi}{3} \). Similarly, let \( s_2^* \) be the arc length parameter at the end of \( \gamma_{up}^2 \). We can extend \( \gamma_{up}^2 \) beyond \( s_2^* \) to intersect negative \( x \)-axis at \( \gamma_{up}^2(s_2) \) and the angle between \( \frac{d\gamma_{up}^2}{ds}(s_2) \) and \( (1,0) \) is less than or equal to \( \frac{\pi}{3} \). The change of angle on the extended curve from \( \gamma_{up}^1(s_1) \) to \( \gamma_{up}^2(s_2) \) is exactly \( \pi \). There should be at least one extended curve with change of angle greater then or equal to \( \frac{\pi}{2} \). Without loss of generality, assume extended \( \gamma_{up}^1 \) has this property. Since from the starting point to the end point of extended \( \gamma_{up}^1 \), \( R \) is monotonically increasing, we obtain a contradiction by lemma 4.3.

**Case 2:** \( m \leq 0 \). Either the change of angle \( \psi_{up} \) or \( \gamma_{up}^2 \) is greater than or equal to \( \frac{\pi}{2} \) since their summation must exceed \( \pi \). Without loss of generality, we can assume \( \gamma_{up}^1 \) satisfies condition. Note that \( \psi_{up} \leq \frac{\pi}{3} \) at the start of \( \gamma_{up}^1 \). Since from the starting point to the end point of \( \gamma_{up}^1 \), \( R \) is monotonically increasing, we obtain a contradiction by lemma 4.3.

**Theorem 4.5.** If the upper cell is a 4-cell, the curve \( \gamma_{up} \) on the phase plane must be \( SA \rightarrow BA \rightarrow BE \). We also have \( c_{in} > c_{up} \) and \( \frac{\pi}{6} < \psi_{up} \leq \frac{\pi}{3} \).

**Proof.** On \( \gamma_{up} \), the starting point lies on the \( DA \) arc and the ending point lies on the \( BC \) arc. The only possibility for \( \gamma_{up} \) does not have a complete period on the trajectory is that all the triple junction goes from \( A \) to \( B \). Note that the curve from a triple junction to another triple junction must pass through \( (R^+(c_{up}), \frac{\pi}{2}) \), we have \( c_{in} > c_{up} \). Use \( c_{in} = \frac{K(R_{start})}{\psi_{in}} \), \( c_{up} = \frac{K(R_{start})}{\psi_{up}} \) and \( \psi_{in} = \psi_{up} + \frac{2\pi}{3} \), we have \( \frac{\pi}{6} < \psi_{up} \leq \frac{\pi}{3} \).
Theorem 4.6. The upper cell cannot be a 5-cell.

Proof. Consider the curve from a triple junction to another triple junction. It starts at either \( B \) or \( C \) and it end at either \( D \) or \( A \) on the trajectory. It must pass through \((R - (c_{\text{up}}), \frac{\pi}{2})\), we have \( c_{\text{in}} > c_{\text{up}} \). Again, on \( \gamma_{\text{up}} \), the starting point lies on the \( DA \) arc and the ending point lies on the \( BC \) arc. The only possibility for \( \gamma_{\text{up}} \) does not have a complete period on the trajectory is \( SA \rightarrow BD \rightarrow CA \rightarrow BE \) or \( SA \rightarrow BA \rightarrow BA \rightarrow BE \). Therefore, \( \theta_{\text{up}} \geq (2h_1 + 2h_2)(c_{\text{up}}) \). Using \( h_1 > h_3 \), the change of angle is greater than \( T(c_{\text{up}}) \). Use the argument as in the proof of lemma 4.2, we can conclude that there does not exist such 5-cell. \( \square \)

Remark 4.7. The theorems about the upper cells are not restrict to a \( \Theta \)-shaped network. They can be applied to any closed region in a regular shrinker with only 1 edge connected to another closed region and without the origin inside.

Remark 4.8. From the theorem above, we can conclude the regular shrinker with the topology of Cisgeminate 4-ray star proposed in the appendix of [16] does not exist.

5. The structure of the lower curve

For a regular shrinker, any closed region has at most 5 edges. Furthermore, for a \( \Theta \)-shaped network with lines, there is at least one closed region which does not enclose the origin. From the previous section, such closed region must be a 4-cell. Now, there are 4 topology type remain possible: a 4-cell together with either a 5-cell, a 4-cell, a 3-cell, a 2-cell. From now on, we use \( S, E \) to denote the starting point and the ending point on the trajectory respectively.

Proposition 5.1. For the energy of the 3 curves, we have \( c_{\text{in}} > c_{\text{up}} \geq c_{\text{down}} \).

Proof. We have \( c_{\text{in}} > c_{\text{up}} \) and \( \frac{\pi}{6} < \psi_{\text{up}} \leq \frac{\pi}{5} \) from the previous section. Therefore, \( \frac{\pi}{3} \leq \psi_{\text{down}} < \frac{\pi}{5} \). From \( \psi_{\text{up}} \leq \frac{\pi}{3} \leq \psi_{\text{down}} \), we obtain \( c_{\text{up}} \geq c_{\text{down}} \). \( \square \)

Proposition 5.2. If \( R_{\text{start}} < 1 \) or \( R_{\text{end}} < 1 \), the change of angle \( \theta_{\text{in}} \leq \pi \).

Proof. For the special case \( c_{\text{in}} = \infty \), we have \( \theta_{\text{in}} = \pi \). Otherwise, when \( R_{\text{start}} = R_{\text{end}} < 1 \) since at the start of \( \gamma_{\text{in}} \), \( \frac{\pi}{6} \psi < \psi_{\text{in}} \leq \pi \) and at the end of \( \gamma_{\text{in}} \), \( \psi = \pi - \psi_{\text{in}} \) and it cannot contain a complete loop of the trajectory, the part of the trajectory is less than the change of angle going from \( M \) to \( N \) counterclockwisely on the trajectory. Therefore, it is less than \( \Delta \theta_{MN} \).

If either \( R_{\text{start}} > 1 \) or \( R_{\text{end}} > 1 \), without loss of generality, assume \( R_{\text{start}} < 1 < R_{\text{end}} \). We want to compare the change of angle from \( M \) to \( N \) and the change of angle from \( S \) to \( E \). Using lemma 2.1 we have

\[
\Delta \theta_{MS}(c) = \int_{\psi_{\text{in}}}^{\psi_{\text{max}}} \frac{d\psi}{1 - [R - (c \sin \psi)]^2} \geq \int_{\psi_{\text{in}}}^{\psi_{\text{max}}} \frac{d\psi}{[R + (c \sin \psi)]^2 - 1} = \Delta \theta_{NE}(c).
\]

Therefore, \( \Delta \theta_{SE}(c) < \Delta \theta_{MN}(c) \). From theorem 2.8, \( \Delta \theta_{SE}(c) < \Delta \theta_{MN}(c) < \pi \). \( \square \)

Lemma 5.3. It is impossible for \( \gamma_{\text{down}} \) to have a complete period of the trajectory.
The period is bounded below by
$$k > 1$$ is bounded below by $$2(x - 1)^2 + H$$, where $$H$$ is chosen such that this parabola pass through $$(k_{\text{max}}, V(k_{\text{max}}))$$. We have the lower bound $$\pi/\sqrt{2}$$.

The period is bounded below by
$$T(c_{\text{down}}) > 0.6123\pi + \frac{\pi}{\sqrt{2}} > 1.3194\pi.$$ 

We obtain $$\theta_{\text{down}} > T(c_{\text{down}}) > 1.3194\pi$$. On the other hand, from theorem 2.6 $$\theta_{\text{up}} > (h_1 + 2h_2)(c_{\text{up}}) > 0.7789\pi$$.

If $$\eta_{\text{up}} \geq 1.38$$, from theorem 2.6 $$\theta_{\text{up}} > (h_1 + 2h_2)(c_{\text{up}}) > \pi$$. From the result of Abresch and Langer [2], $$\theta_{\text{down}} > T(c_{\text{down}}) > \pi$$. In both case, it is impossible since $$\theta_{\text{up}} + \theta_{\text{down}} = 2\pi$$. \qed

We deal with the case that the bottom cell is a 2-cell first. This is quite different from the 3-cell, 4-cell, 5-cell cases.

**Theorem 5.4.** It is impossible for the bottom cell to be a 2-cell.

**Proof.** In this case, the trajectory for $$\gamma_{\text{down}}$$ in the phase plane may not touch $$\psi = \frac{\pi}{2}$$ and $$\psi = \frac{2\pi}{3}$$. Therefore, the point A, B, C, D may be undefined on the trajectory and the method of expressing angles in terms of $$h_1, h_2,$$ and $$h_3$$ may not be applicable. For this case, we only use the point $$M, N$$. Since there are no triple junctions on $$\gamma_{\text{down}}$$, we have $$\theta_{\text{down}} = \Delta\theta_{SE}$$. We separate into 3 cases.

When $$R_{\text{start}} = R_{\text{end}} < 1$$, if $$\eta_{\text{up}} \geq 1.38$$, using theorem 2.6 and proposition 5.2 we have $$\pi \geq \theta_{\text{in}} = \theta_{\text{up}} > (h_1 + 2h_2)(c_{\text{up}}) > \pi$$. It is impossible. If $$\eta_{\text{up}} < 1.38$$, we have $$c_{\text{up}} = e^{-\frac{c_{\text{up}}}{e^{1.9}}} < e^{0.19}$$. Since $$\sin(\psi_{\text{up}}) = \frac{K(R_{\text{start}})}{c_{\text{up}}} \geq e^{-0.19}$$, we have $$\psi_{\text{up}} \geq 0.3099\pi$$ and $$\psi_{\text{down}} = \frac{2\pi}{3} - \psi_{\text{up}} \leq 0.3568\pi$$.

$$\theta_{\text{down}} + \theta_{\text{in}} \leq T(c_{\text{down}}) - \int_{\psi_{\text{down}}}^{\psi_{\text{up}}} \frac{1}{1 - R^{-}(c_{\text{down}}) \sin(\psi)} d\psi + \pi$$

$$\leq \sqrt{2}\pi - \frac{\pi - 2\psi_{\text{down}}}{1 - R^{-}(c_{\text{down}})^2} + \pi \leq \sqrt{2}\pi - \frac{\pi - 2 \times 0.3568\pi}{1 - 0.6^2} + \pi < 2\pi,$$

the last inequality comes from $$R^{-}(c_{\text{down}}) \geq R^{-}(c_{\text{up}}) \geq R^{-}(e^{0.19}) > 0.6$$. Therefore, there does not exist a bottom 2-cell in this case.

When $$R_{\text{start}} < 1 < R_{\text{end}}$$, using the symmetry of the trajectory with respect to $$\psi = \frac{\pi}{2}$$ and lemma 2.1 we have

$$\theta_{\text{down}} = \Delta\theta_{SE}(c_{\text{down}}) = (\Delta\theta_{SN} + \Delta\theta_{NE})(c_{\text{down}}) \leq (\Delta\theta_{SN} + \Delta\theta_{MS})(c_{\text{down}}) = \Delta\theta_{MN}(c_{\text{down}}) < \pi,$$

where the last inequality is given by proposition 2.8. Combine $$\theta_{\text{in}} < \pi$$ from proposition 5.2. This contradicts $$\theta_{\text{down}} + \theta_{\text{in}} = 2\pi$$.

When $$R_{\text{start}} = R_{\text{end}} \geq 1$$, use the equation (2.3) and the monotonicity with respect to $$c$$ for fixed range of $$R$$, from $$c_{\text{down}} < c_{\text{in}}$$, we have $$\Delta\theta_{SM}(c_{\text{in}}) = \Delta\theta_{NE}(c_{\text{in}}) \leq \Delta\theta_{EM}(c_{\text{down}}) =$$
\[ \Delta \theta_{NS}(c_{\text{down}}). \] Note that if \( R_{\text{start}} = R_{\text{end}} = 1, \Delta \theta_{SM}(c_{\text{in}}) = \Delta \theta_{NE}(c_{\text{in}}) = \Delta \theta_{EM}(c_{\text{down}}) = \Delta \theta_{NS}(c_{\text{down}}) = 0. \] Therefore, 

\[ \theta_{\text{in}} + \theta_{\text{down}} = (\Delta \theta_{SM} + \Delta \theta_{MN} + \Delta \theta_{NE})(c_{\text{in}}) + \Delta \theta_{SE}(c_{\text{down}}) \leq \Delta \theta_{MN}(c_{\text{in}}) + (\Delta \theta_{NS} + \Delta \theta_{SE} + \Delta \theta_{EM})(c_{\text{down}}) \leq \Delta \theta_{MN}(c_{\text{in}}) + \frac{1}{2} T(c_{\text{down}}) \leq \pi + \frac{\pi}{\sqrt{2}} < 2\pi. \]

Note that we use \( \Delta \theta_{MN}(c) > \Delta \theta_{NM}(c) \) from lemma 2.1 and \( \Delta \theta_{MN}(c) + \Delta \theta_{NM}(c) = T(c). \) □

If the bottom cell is a 3-cell, a 4-cell or a 5-cell, \( c_{\text{down}} \geq c_*. \) We can describe the change of angle in terms of \( h_1, h_2, \) and \( h_3. \) Here we list all the possible cases. The arrow indicates a triple junction with a ray. The point on the phase plane will either jump from \( D \) to \( C \) or jump from \( A \) to \( B. \)

| Cell | Path on the trajectory | \( R_{\text{start}} = R_{\text{end}} < 1 \) | \( R_{\text{start}} < 1 < R_{\text{end}} \) | \( R_{\text{start}} = R_{\text{end}} > 1 \) |
|------|------------------------|---------------------------------|---------------------------------|---------------------------------|
| 5-cell | SD→CD→CD→CE | \( 2h_1^0 + 2h_1 \) | \( h_1^0 + 3h_1 + h_2 + h_3^0 \) | \( 4h_1 + 2h_2 + 2h_3^0 \) |
|      | SA→BD→CD→CE          |                                |                                |                                 |
|      | SD→CA→BD→CE          |                                |                                |                                 |
|      | SD→CD→CA→BE          | \( 2h_1^0 + 2h_1 + 2h_2 \) | \( h_1^0 + 3h_1 + 3h_2 + h_3^0 \) | \( 4h_1 + 4h_2 + 2h_3^0 \) |
|      | SA→BA→BD→CE          |                                |                                |                                 |
|      | SA→BD→CA→BE          |                                |                                |                                 |
|      | SD→CA→BA→BE          | \( 2h_1^0 + 2h_1 + 4h_2 \) | \( h_1^0 + 3h_1 + 5h_2 + h_3^0 \) | \( 4h_1 + 6h_2 + 2h_3^0 \) |
|      | SA→BA→BA→BE          | \( 2h_1^0 + 2h_1 + 6h_2 \) | \( h_1^0 + 3h_1 + 7h_2 + h_3^0 \) | \( 4h_1 + 8h_2 + 2h_3^0 \) |
| 4-cell | SD→CD→CE             | \( 2h_1^0 + h_1 \) | \( h_1^0 + 2h_1 + h_2 + h_3^0 \) | \( 3h_1 + 2h_2 + 2h_3^0 \) |
|      | SA→BD→CE             |                                |                                |                                 |
|      | SD→CA→BE             | \( 2h_1^0 + h_1 + 2h_2 \) | \( h_1^0 + 2h_1 + 3h_2 + h_3^0 \) | \( 3h_1 + 4h_2 + 2h_3^0 \) |
|      | SA→BA→BE             | \( 2h_1^0 + h_1 + 4h_2 \) | \( h_1^0 + 2h_1 + 5h_2 + h_3^0 \) | \( 3h_1 + 6h_2 + 2h_3^0 \) |
| 3-cell | SD→CE                | \( 2h_1^0 \) | \( h_1^0 + h_2 + h_3^0 \) | \( 2h_1 + 2h_2 + 2h_3^0 \) |
|      | SA→BE                | \( 2h_1^0 + 2h_2 \) | \( h_1^0 + h_1 + 3h_2 + h_3^0 \) | \( 2h_1 + 4h_2 + 2h_3^0 \) |

**Figure 3.** Possible places for S and E

Since \( \frac{\pi}{4} \leq \psi_{\text{down}} < \frac{\pi}{2}, \ S, E \) lie on either \( CD \) or \( AB \) arc of the trajectory corresponds to \( c_{\text{down}}. \) Use \( h_1^0 = \Delta \theta_{SD} = \Delta \theta_{CE} \) when \( S(c_{\text{down}}) \) or \( E(c_{\text{down}}) \) lie on the \( CD \) arc. \( h_3^0 = \Delta \theta_{SB} = \Delta \theta_{AE} \) when \( S(c_{\text{down}}) \) or \( E(c_{\text{down}}) \) lie on the \( AB \) arc. Note that we can eliminate the case
\( R_{\text{start}} = R_{\text{end}} = 1 \), we have \( c_{\text{down}} = c_s \), \( A = D = S \) and \( B = C = E \). Since when we go from the starting point to the triple junction, we either form a complete loop or the curve will be degenerate.

**Theorem 5.5.** For the case the bottom cell is either a 3-cell, 4-cell or 5-cell, it is impossible than \( R_{\text{start}} = R_{\text{end}} > 1 \).

**Proof.** In this case,

\[ 2h_3^2(c_{\text{down}}) = (\Delta \theta_{SB} + \Delta \theta_{AE})(c_{\text{down}}) > \Delta \theta_{AB}(c_{\text{down}}) = h_3(c_{\text{down}}). \]

We have

\[ (5.8) \quad \theta_{\text{down}} \geq (2h_1 + 2h_2 + 2h_3^2)(c_{\text{down}}) > (2h_1 + 2h_2 + h_3)(c_{\text{down}}) = (h_1 + T)(c_{\text{down}}) > \frac{4\pi}{3}. \]

This is impossible since \( \theta_{\text{up}} > \frac{2\pi}{3} \) and \( \theta_{\text{down}} + \theta_{\text{up}} = 2\pi \).

Therefore, for a regular shrinker, we have either \( R_{\text{start}} < 1 \) or \( R_{\text{end}} < 1 \).

**Proposition 5.6.** If \( R_{\text{start}} < 1 \) or \( R_{\text{end}} < 1 \), for a \( \Theta \)-shaped regular shrinker, we have \( \theta_{\text{up}} > (h_1 + 2h_2 + 2\Delta \theta_{NA})(c_{\text{up}}) \). Moreover, for \( c \geq \tilde{c} = e^{\frac{1.0065}{2}} \), \( (h_1 + 2h_2 + 2\Delta \theta_{NA})(c) > \pi \).

Therefore, we have \( c_{\text{up}} \in I_A = (c_s, \tilde{c}) \). In this case, we have \( \theta_{\text{up}} = \theta_{\text{in}} \in (0.9947\pi, \pi] \).

**Proof.** Without loss of generality, assume \( R_{\text{start}} < 1 \). Recall that

\[ (5.9) \quad \theta_{\text{up}} = (\Delta \theta_{SA} + \Delta \theta_{BA} + \Delta \theta_{BE})(c_{\text{up}}) = (h_1 + 2h_2)(c_{\text{up}}) + \Delta \theta_{SA}(c_{\text{up}}) + \Delta \theta_{BE}(c_{\text{up}}). \]

If \( R_{\text{end}} < 1 \), we have

\[ (5.10) \quad \Delta \theta_{SA}(c_{\text{up}}) + \Delta \theta_{BE}(c_{\text{up}}) \geq \Delta \theta_{NA}(c_{\text{up}}) + \Delta \theta_{BM}(c_{\text{up}}) = 2\Delta \theta_{NA}(c_{\text{up}}). \]

If \( R_{\text{end}} > 1 \), we have

\[ (5.11) \quad \Delta \theta_{SA}(c_{\text{up}}) + \Delta \theta_{BE}(c_{\text{up}}) = \Delta \theta_{SN}(c_{\text{up}}) + \Delta \theta_{NA}(c_{\text{up}}) + \Delta \theta_{BM}(c_{\text{up}}) - \Delta \theta_{EM}(c_{\text{up}}). \]

We want to compare \( \Delta \theta_{SN} \) and \( \Delta \theta_{EM} \). Using the symmetry of the trajectory with respect to \( \psi = \frac{\pi}{2} \) and lemma 2.1 \( \Delta \theta_{SN}(c_{\text{up}}) \geq \Delta \theta_{EM}(c_{\text{up}}) \). Therefore, we also have \( \Delta \theta_{SA}(c_{\text{up}}) + \Delta \theta_{BE}(c_{\text{up}}) \geq 2\Delta \theta_{NA}(c_{\text{up}}) \) and \( \theta_{\text{up}} \geq h_1(c_{\text{up}}) + 2h_2(c_{\text{up}}) + 2\Delta \theta_{NA}(c_{\text{up}}) \). From proposition 5.2 we have \( (h_1 + 2h_2 + 2\Delta \theta_{NA})(c_{\text{up}}) \leq \theta_{\text{up}} = \theta_{\text{in}} \leq \pi \).

For \( c \geq e^{0.19} \), we have \( \eta \geq 1.38 \), by using theorem 2.6 \( h_1 + 2h_2 + 2\theta_{NA} \geq (h_1 + 2h_2) > \pi \).

For \( e^{\frac{1}{19}} \leq c < e^{0.19} \), we have \( \frac{3}{c} \leq \eta < 1.38 \) and \( k_2 \geq 1 \). Recall that

\[ (5.12) \quad \Delta \theta_{NA}(c) = \Delta \phi_{NA}(c) + \Delta \psi_{NA}(c) = \int_{k_1}^{k_2} \frac{dk}{\sqrt{\eta - V(k)}} + \left( \sin^{-1}(\frac{1}{c}) - \frac{\pi}{3} \right), \]

where \( V(k) = k^2 - 2 \log k, k_1 = \frac{1}{c}, k_2 = \frac{\sqrt{3}}{2} R^+(\frac{\sqrt{3}}{2} c) \) is the curvature at \( A \), and \( \eta = 1 + 2 \log c \). Using lemma 2.5 with \( k_0 = \frac{1}{c} \) and \( H = V(\frac{1}{c}) - (1 + c)(\frac{1}{c} - 1)^2 \), we have \( \frac{\eta - H}{1 + k_0} = \frac{c-1}{c} \) and

\[ \Delta \theta_{NA}(c) \geq \int_{k_1}^{1} \frac{dk}{\sqrt{\eta - V(k)}} + \left( \sin^{-1}(\frac{1}{c}) - \frac{\pi}{3} \right) \geq \frac{1}{\sqrt{1 + c}} \sin^{-1}(\sqrt{\frac{c-1}{c}}) + \left( \sin^{-1}(\frac{1}{c}) - \frac{\pi}{3} \right). \]

When \( c \) increases, \( \frac{c-1}{c} \) increases. Since \( k_2 \geq 1 \) and \( e^{\frac{1}{19}} \leq c < e^{0.19} \), using theorem 2.6 we have

\[ 2\Delta \theta_{NA}(c) \geq \frac{2}{\sqrt{1 + e^{0.19}}} \sin^{-1} \frac{e^{\frac{1}{19}} - 1}{e^{\frac{1}{19}}} + 2 \left( \sin^{-1}(\frac{1}{e^{0.19}}) - \frac{\pi}{3} \right) \geq 0.1252\pi, \]
and \((h_1 + 2h_2 + 2\Delta \theta_N) > 0.9456\pi + 0.1252\pi > \pi\) for \(e^{\frac{1}{6}} < c < e^{\frac{1}{5}}\).

For \(e^{\frac{1.31 - 1}{2}} < c < e^{\frac{1}{5}}\), we have \(0.6235 \leq k_{\min} = R^{-}(c) \leq 0.6358\) and \(0.9590 \leq k_2 = \sqrt{3}\frac{\pi}{2}\). Combining Lemma 2.5 and the inequality (5.12),

\[
(h_1 + 2h_2)(c) > \frac{2}{\sqrt{1 + e^{\frac{1}{6}}}} \left( \pi - \sin^{-1}\left( \frac{1 - k_2}{1 - k_{\min}} \right) + \frac{\pi}{3} \right)
\]

and

\[
2\Delta \theta_N(c) \geq \frac{2}{\sqrt{1 + e^{\frac{1}{6}}}} \left[ \sin^{-1}\left( \sqrt{\frac{e^{\frac{1.31 - 1}{2}} - 1}{e^{\frac{1.31 - 1}{2}}} - 1} \left(1 - 0.9590\right) \right) \right]
\]

\[
+ 2 \left( \sin^{-1}\left( \frac{1}{e^{\frac{1.31 - 1}{2}}} \right) - \frac{\pi}{3} \right) \geq 0.0966\pi.
\]

Therefore, \((h_1 + 2h_2 + 2\Delta \theta_N)(c) > 0.9084\pi + 0.0966\pi > \pi\) for \(e^{\frac{1.31 - 1}{2}} \leq c < e^{\frac{1}{5}}\).

For \(e^{\frac{1.31 - 1}{2}} = \bar{c} \leq c < e^{\frac{1.31 - 1}{2}}\), we have \(0.6356 \leq k_{\min} = R^{-}(c) \leq 0.6377\) and \(0.9513 \leq k_2 = \sqrt{3}\frac{\pi}{2}\). Using Lemma 2.5 and the inequality (5.12), we have

\[
(h_1 + 2h_2)(c) > \frac{2}{\sqrt{1 + e^{\frac{1}{6}}}} \left( \pi - \sin^{-1}\left( \frac{1 - k_2}{1 - k_{\min}} \right) + \frac{\pi}{3} \right)
\]

and

\[
2\Delta \theta_N(c) \geq \frac{2}{\sqrt{1 + e^{\frac{1}{6}}}} \left[ \sin^{-1}\left( \sqrt{\frac{\bar{c} - 1}{\bar{c}} - 1} \left(1 - 0.9513\right) \right) \right]
\]

\[
+ 2 \left( \sin^{-1}\left( \frac{1}{e^{\frac{1.31 - 1}{2}}} \right) - \frac{\pi}{3} \right) \geq 0.0988\pi.
\]

Therefore, \((h_1 + 2h_2 + 2\Delta \theta_N)(c) > 0.9031\pi + 0.0988\pi > \pi\) for \(\bar{c} \leq c < e^{\frac{1.31 - 1}{2}}\).

Combining above estimates, we obtain an contradiction for \(c_{up} > \bar{c}\). Therefore, \(c_{up} \in I_A\).

Since the network is regular, \(\psi_{down} = \psi_{up} + \frac{\pi}{3}\). Using the conservation law (2.1),

\[
\frac{\sin(\psi_{up} + \frac{\pi}{3})}{\sin(\psi_{up})} = \frac{\sin(\psi_{down})}{\sin(\psi_{up})} = \frac{c_{up}}{c_{down}} \leq \frac{\bar{c}}{c_e}.
\]

Since \(\frac{\sin(\psi_{up} + \frac{\pi}{3})}{\sin(\psi_{up})}\) decreases on the interval \((0, \frac{\pi}{3})\), we have \(\psi_{up} > 0.3307\pi\). Note that on \(\gamma_{in}\), we have \(\Delta \phi > 0\). Hence,

\[
\theta_{up} = \theta_{in} = \Delta \phi - \Delta \psi \geq 0 + (\psi_{in} - (\pi - \psi_{in})) = \frac{\pi}{3} + 2\psi_{up} > 0.9947\pi.
\]

Combining Proposition 5.2, \(\theta_{up} = \theta_{in} \in (0.9947\pi, \pi]\).
Theorem 5.7. For the case that the bottom cell is either a 3-cell, a 4-cell, or a 5-cell, it is impossible either \( R_{\text{start}} > 1 \) or \( R_{\text{end}} > 1 \).

Proof. Assume the contrary, without loss of generality, let \( R_{\text{start}} < 1 < R_{\text{end}} \). For the case that the bottom cell is either a 3-cell, 4-cell, or 5-cell, from lemma 2.1, we have \( \Delta \theta_{SD} \geq \Delta \theta_{EB} \) and

\[
(5.16) \quad (h_1^c + h_3^c)(c_{\text{down}}) = (\Delta \theta_{SC} + \Delta \theta_{AE})(c_{\text{down}}) > (\Delta \theta_{EB} + \Delta \theta_{AE})(c_{\text{down}}) = h_3(c_{\text{down}}).
\]

Therefore,

\[
(5.17) \quad \theta_{\text{down}} \geq (h_1^c + h_1 + h_2 + h_3^c)(c_{\text{down}}) \geq h_1(c_{\text{down}}) + h_3(c_{\text{down}}) \geq h_1(\bar{c}) + h_3(\bar{c}),
\]

where the last inequality comes from that \( h_1(c) \) and \( h_3(c) \) decrease as \( c \) increases. Since \( K(R^+(\bar{c}\sin \psi)) = \bar{c} \sin \psi \leq \bar{c} < K(\sqrt{2}) = \frac{\sqrt{5}}{\sqrt{2}} \), we have \( \sqrt{2} > R^+(\bar{c}\sin \psi) \) and

\[
(5.18) \quad h_3(\bar{c}) = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{d\psi}{[R^+(\bar{c}\sin \psi)]^2} - 1 \geq \frac{\pi}{3} \frac{1}{(\sqrt{2})^2} - 1 = \frac{\pi}{3}.
\]

On the other hand, using lemma 2.5

\[
(5.19) \quad h_1(\bar{c}) = \int_{k_{\text{min}}}^{k_2} \frac{2dk}{\sqrt{\bar{n} - V(k)}} + \frac{\pi}{3} \geq \frac{2}{\sqrt{1 + \frac{1}{k_{\text{min}}}}} \left( \frac{\pi}{2} - \sin^{-1} \left( \frac{1 - k_2}{1 - k_{\text{min}}} \right) \right) + \frac{\pi}{3},
\]

where \( k_{\text{min}} \) is the global minimum for curvature of \( \gamma_\bar{c} \), and \( k_2 \) is the curvature of \( \gamma_\bar{c} \) at the point \( D(\bar{c}) \), and \( \bar{n} = 1.3065 \). By calculating, \( k_{\text{min}} \in (0.6376, 0.6377) \) and \( k_2 \in (0.7834, 0.7835) \), and \( h_1(\bar{c}) \geq 0.7027\pi \). Therefore,

\[
(5.20) \quad \theta_{\text{in}} + \theta_{\text{down}} \geq 0.9947\pi + (0.7027\pi + \frac{\pi}{3}) > 2\pi.
\]

It is impossible. \( \square \)

From now on, for the case that the bottom cell is either a 3-cell, a 4-cell or a 5-cell, we have \( R_{\text{start}} = R_{\text{end}} < 1 \).

Theorem 5.8. There does not exist solution with bottom cell being a 5-cell.

Proof. The smallest possible angle for \( \theta_{\text{down}} \) is that the triple junctions are all of the \( D \to C \) type. Since \( R_{\text{start}} = R_{\text{end}} < 1 \),

\[
(5.21) \quad \theta_{\text{down}} \geq (2h_1^c + 2h_1)(c_{\text{down}}) \geq 2h_1(c_{\text{up}}).
\]

From theorem 2.6, for every \( c \geq c_* \), we have \( h_1(c) + 2h_2(c) > 0.7789\pi (> \frac{2\pi}{3}) \). This equation is a lower bound of \( \theta_{\text{up}} \). From the proof of theorem 5.7, we have \( h_1(\bar{c}) > 0.7027\pi \). Therefore,

\[
(5.22) \quad \theta_{\text{up}} + \theta_{\text{down}} > (h_1 + 2h_2)(c_{\text{up}}) + 2h_1(c_{\text{up}}) > 0.7789\pi + 2 \times 0.7027\pi > 2\pi
\]

and we get a contradiction. \( \square \)
6. 2 4-COMPLEMENT OR A 4-CELL AND A 3-CELL

First, we consider the case which $\gamma_{\text{down}}$ is $SA \rightarrow BA \rightarrow BE$. In this case, there is a symmetry between $\gamma_{\text{down}}$ and $\gamma_{\text{up}}$. Precisely speaking, for any $R_0$, on the trajectory of energy $c$, define $P(R_0) = (R_0, \sin^{-1} \frac{K(R_0)}{c})$. We have $S = P(R_{\text{start}})$ for $\gamma_{\text{up}}$ and $\gamma_{\text{down}}$. The change of angle can be expressed as

\[
\begin{align*}
\theta_{\text{up}} &= (h_1 + 2h_2 + 2\Delta \theta_{P(R_{\text{start}})A}(c_{\text{up}})), \\
\theta_{\text{down}} &= (h_1 + 4h_2 + 2h^2_1)(c_{\text{down}}) = (h_1 + 2h_2 + 2\Delta \theta_{P(R_{\text{start}})A}(c_{\text{down}})).
\end{align*}
\]

Note that

\[
\Delta \theta_{P(R_{\text{start}})A}(c) = \int_{R_{\text{start}}}^{R^+(\sqrt[3]{2}c)} \frac{K(R)dR}{R\sqrt{c^2 - K(R)^2}}.
\]

To obtain uniqueness and existence of the regular shrinker in this case, we need the following lemmas.

**Lemma 6.1.** Given a number $0.7 \leq R_0 \leq 1$, $\Delta \theta_{P(R_0)A}(c)$ strictly increases on the admissible interval $I_A = [c_*, c]$.

**Proof.** As $c \in I_A$, using $K(R^+(\sqrt[3]{2}c)) = \sqrt[3]{2}c$, we have $\frac{dR^+}{dc} = \frac{1}{c(R^+ - \frac{1}{R^+})}$ and

\[
\begin{align*}
\frac{d\Delta \theta_{P(R_0)A}(c)}{dc} &= \frac{\sqrt[3]{2}c}{R^+ \sqrt{c^2 - (\sqrt[3]{2}c)^2}} \frac{1}{c(R^+ - \frac{1}{R^+})} - \int_{R_0}^{R^+(\sqrt[3]{2}c)} \frac{cK(R)dR}{R \left(\sqrt{c^2 - K^2(R)}\right)^3}.
\end{align*}
\]

Let $W(R, c) = \frac{K(R)/c}{(\sqrt[3]{1 - (K(R)/c)^2})}$. Note that, fixed $R$, $W(R, c)$ is a decreasing function of $c$ and $W(R, c_*) > 0$ because of $K(R) < c_*$ for $R \in [0.7, 1.1]$. Therefore,

\[
\begin{align*}
\frac{d\Delta \theta_{P(R_0)A}(c)}{dc} &\geq \frac{1}{c} \left( \frac{\sqrt[3]{2}}{R^+ (\sqrt[3]{2}c)^2} - 1 \right) - \int_{R_0}^{R^+(\sqrt[3]{2}c)} \frac{W(R, c_*)dR}{R} \\
&\geq \frac{1}{c} \left( \frac{\sqrt[3]{2}}{1.1^2} - 1 \right) - \int_{0.7}^{1.1} \frac{W(R, c_*)dR}{R} \\
&\geq \frac{1}{c} \left( \frac{\sqrt[3]{2}}{0.21} - \sum_{i=1}^{4} \max_{R \in J_i} \frac{|J_i|}{R} \left( \max_{R \in J_i} W(R, c_*) \right) \right),
\end{align*}
\]

where $J_1 = [0.7, 0.73]$, $J_2 = [0.73, 0.8]$, $J_3 = [0.8, 0.9]$, $J_4 = [0.9, 1.1]$, and the second inequality comes from $\max_{c \in I_A} R^+(\sqrt[3]{2}c) < 1.1$ because of $K(1.1) > \frac{\sqrt[3]{2}}{2}c$. Since $W(R, c_*)$ decreases for
$R \in [0.7, 1]$ and increases for $R \in [1, 1.1]$, we have

\[
\left( \max_{R \in I_1} \frac{|J_1|}{R} \right) \left( \max_{R \in I_1} W(R, c_*) \right) = 0.03 \quad \frac{0.7}{\sqrt{3}} W(0.7, \frac{2}{\sqrt{3}}) < 1.81
\]

\[
\left( \max_{R \in I_2} \frac{|J_2|}{R} \right) \left( \max_{R \in I_2} W(R, c_*) \right) = 0.07 \quad \frac{0.73}{\sqrt{3}} W(0.73, \frac{2}{\sqrt{3}}) < 2.27
\]

\[
\left( \max_{R \in I_3} \frac{|J_3|}{R} \right) \left( \max_{R \in I_3} W(R, c_*) \right) = 0.1 \quad \frac{0.8}{\sqrt{3}} W(0.8, \frac{2}{\sqrt{3}}) < 1.47
\]

\[
\left( \max_{R \in I_4} \frac{|J_4|}{R} \right) \left( \max_{R \in I_4} W(R, c_*) \right) = 0.2 \quad \frac{0.9}{\sqrt{3}} \max \left\{ W(0.9, \frac{2}{\sqrt{3}}), W(1.1, \frac{2}{\sqrt{3}}) \right\} < 1.74.
\]

Therefore,

\[
d\Delta \theta_{P(R_0)A}(c) = \frac{1}{c} \left( \frac{\sqrt{3}}{0.21} - (1.81 + 2.27 + 1.47 + 1.74) \right) > 0.
\]

That is, $\Delta \theta_{P(R_0)A}(c)$ increases on the admissible interval $I_A$. \hfill \Box

**Lemma 6.2.** $(h_1 + 2h_2)(c)$ and $h_2(c)$ are increasing on the admissible interval $I_A$.

**Proof.** For any $c \in I_A$, let $Q^+$ and $Q^-$ be $(0.7, \pi - \sin^{-1} \frac{K(0.7)}{c})$ and $(0.7, \sin^{-1} \frac{K(0.7)}{c})$ on the $R - \psi$ plane.

\[
h_1(c) + 2h_2(c) = \Delta \theta_{Q^+(c)} + 2\Delta \theta_{Q^-(c)} = f(c) + 2\Delta \theta_{P(0.7)A}(c),
\]

where

\[
f(c) = \int_{\sin^{-1}(\frac{K(0.7)}{c})}^{\pi - \sin^{-1}(\frac{K(0.7)}{c})} \frac{d\psi}{1 - R^{-2}(c \sin \psi)^2}.
\]

As $c \in I_A = [c_*, \bar{c}]$,

\[
f'(c) = 2 \left( \frac{K(0.7)}{0.51} \right) \frac{1}{\sqrt{c^2 - K^2(0.7)}} - \int_{\sin^{-1}(\frac{K(0.7)}{c})}^{\pi - \sin^{-1}(\frac{K(0.7)}{c})} \frac{(R^-)^2}{1 - (R^-)^2} d\psi
\]

\[
\geq 2 \left( \frac{K(0.7)}{0.51} \right) \frac{1}{\sqrt{(\bar{c})^2 - K^2(0.7)}} - \frac{0.7^2}{1 - 0.7^2} \left( \pi - 2 \sin^{-1} \frac{K(0.7)}{c} \right) > 0,
\]

where the inequality holds since $\frac{2(R^-)^2}{1 - (R^-)^2}$ is a decreasing function of $\psi$. We have $f(c)$ increases strictly on $I_A$. Combining the equation (6.7) and using Lemma 6.1, $h_1(c) + 2h_2(c)$ increases on $I_A$. Since $h_1(c)$ is decreasing, the function $h_2(c)$ is increasing on $I_A$. \hfill \Box

**Theorem 6.3.** For the case that the bottom cell is a 4-cell with $(R, \psi)$ being $SA \rightarrow BA \rightarrow BE$, there exists a unique solution. The curve $\gamma_{in}$ is a line segment through the origin and the network is symmetric with respect to $\gamma_{in}$.

**Proof.** For this case, we have $R_{\text{start}} = R_{\text{end}} < 1$, therefore,

\[
\theta_{\text{down}} = (h_1 + 2h_2 + 2\Delta \theta_{P(R_{\text{start}})A})(c_{\text{down}}).
\]
Since \( c_{up} \in I_A \), we have

\[
R_{start} > R^\ast (-\frac{\sqrt{3}}{2} c_{up}) \geq R^\ast (-\frac{\sqrt{3}}{2} \tilde{c}) > 0.7.
\]

By lemma 6.1 and lemma 6.2, since \( c_s \leq c_{down} \leq c_{up} \leq \tilde{c} \), we have \( \theta_{up} \geq \theta_{down} \). On the other hand, using proposition 5.6, \( \theta_{down} = 2\pi - \theta_{up} \geq 2\pi - \pi = \pi \geq \theta_{up} \). Therefore, \( \theta_{up} = \theta_{down} = \pi \), \( c_{down} = c_{up} \), and \( \psi_{up} = \psi_{down} = \frac{\pi}{3} \). Moreover, \( \theta_{down} = (h_1 + 4h_2)(c_{down}) \).

Use corollary 2.9 and proposition 5.6

\[
(h_1 + 4h_2)(c_s) = h_1(c_s) < \pi
\]

and

\[
(h_1 + 4h_2)(\tilde{c}) > (h_1 + 2h_2)(\tilde{c}) + 2\Delta \theta_{N,A}(\tilde{c}) > \pi.
\]

By the continuity and the monotonicity of \( h_k \), there exists a unique \( c_0 \in I_A \) such that \( c_{up} = c_{down} = c_0 \) and \( \theta_{up} = \theta_{down} = h_1 + 4h_2 = \pi \).

**Proposition 6.4.** For the case that the bottom cell is a 3-cell or a 4-cell which is not the previous case. There is no solution.

**Proof.** For the case that the bottom cell \( \gamma_{down} \) is either a 3-cell, or 4-cell which is not the special case, we have

\[
\theta_{down} < (h_1 + 2h_2 + 2\Delta \theta_{P(R_{start})A})(c_{down}).
\]

Again, we have \( R_{start} \in [0.7, 1] \). Since \( c_s \leq c_{down} \leq c_{up} \leq \tilde{c} \), we have \( \theta_{up} \geq (h_1 + 2h_2 + 2\Delta \theta_{P(R_{start})A})(c_{down}) > \theta_{down} \). On the other hand, using proposition 5.6, \( \theta_{down} = 2\pi - \theta_{up} \geq 2\pi - \pi = \pi \geq \theta_{up} \). We obtain a contradiction.

**7. Degenerate regular shrinkers**

We can find some degenerate regular shrinkers by allowing some edges to be degenerate, which is a curve with zero length. The definition of degenerate regular shrinker can be found in [16]. The theorems concerning the topology of the network in section 3 are still applicable. Note that the curve with both ends attached to rays cannot be degenerate, since the two rays cannot form a \( \frac{\pi}{3} \) angle when they are not intersecting at the origin. Therefore, the degenerate curves can only be the curves attached to the starting or the ending point. Without loss of generality, assume the first AL-curve on \( \gamma_{up} \) which goes out from the starting point is degenerate. The angle \( \psi_{up} = \frac{\pi}{3} \) and the starting point must be either D or A. For the other two curves, we have \( \psi_{in} = \pi \) and \( \gamma_{in} \) must be a line segment through the origin. Furthermore, we have \( \theta_{up} = \theta_{down} = \pi \) and \( c_{up} = c_{down} \). From proposition 3.3, the ending point must be either the point B or C on the trajectory.

The upper cell cannot be a 2-cell, otherwise, the starting point and the ending point will be the same point and both \( \gamma_{up} \) and \( \gamma_{in} \) will be degenerate. It is impossible. Therefore, in the degenerate case, the upper cell must be a 3-cell, a 4-cell or a 5-cell. Here, we use \( p \)-cell to denote a cell with \( p \) edges which are possibly degenerate. From now on, we use the first curve, the second curve, etc. to describe the smooth AL-curves when we traverse \( \gamma_{up} \) from the starting point to the ending point. Since \( T(c) > \pi \) for any \( c \), the curve cannot have a complete period on the trajectory. Note that if we find a solution for the upper cell, since \( \gamma_{in} \) is a line segment on \( x \)-axis, we can get a solution by letting \( \gamma_{down} \) be the reflection of
\(\gamma_{\text{up}}\) with respect to \(x\)-axis. Furthermore, if \(R_{\text{start}} = R_{\text{end}}\), we can get the solution by letting \(\gamma_{\text{up}}\) and \(\gamma_{\text{down}}\) be symmetric with respect to the origin.

We need an estimation of angle to exclude some cases.

**Lemma 7.1.** \((2h_1 + h_2)(c) > \pi\) for any \(c \geq c_*\).

**Proof.** For the case \(c \geq \hat{c} = e^{0.19}\), we have \(\eta = 1 + 2 \log c \geq 1.38\). Using theorem 2.6, we obtain

\[
(2h_1 + h_2)(c) > \frac{3}{2} h_1(c) + \frac{1}{2} (h_1 + 2h_2)(c) > \frac{3}{2} \times \frac{\pi}{3} + 1 = \pi.
\]

On the other hand, using lemma 2.5,

\[
(7.1) \quad h_1(\hat{c}) = \int_{k_{\text{min}}}^{k_2} \frac{2dk}{\sqrt{1.38 - V(k)}} + \frac{\pi}{3} = \frac{2}{\sqrt{1 + \frac{1}{k_{\text{min}}}}} \left( \frac{\pi}{2} - \sin^{-1} \left( \frac{1 - k_2}{1 - k_{\text{min}}} \right) \right) + \frac{\pi}{3},
\]

where \(k_2\) is the curvature of \(\gamma_{\hat{c}}\) at the point \(D(\hat{c})\). By using a scientific calculator, \(k_{\text{min}} \in (0.6007, 0.6008)\), \(k_2 \in (0.6871, 0.6872)\), and \(h_1(\hat{c}) \geq 0.5945\pi\). For the case \(c < \hat{c}\), we have \((2h_1 + h_2)(c) \geq 2h_1(c) > 2h_1(\hat{c}) > \pi\). \(\square\)

**Theorem 7.2.** If the upper cell is a 3-cell, the type of \(\gamma_{\text{up}}\) is \(DD \rightarrow CB\) on the trajectory and \(\theta_{\text{up}} = h_1 + h_2 + h_3\).

**Proof.** If the first curve is \(DD\), which is degenerate, the second curve starts from \(C\). Since the second curve is neither degenerate nor contain a complete period, the end point most be the point \(B\). The second curve is the \(CB\) arc on the phase plane. In this case, \(\theta_{\text{up}} = h_1 + h_2 + h_3\). Since \(\lim_{c \to \infty} (h_1 + h_2 + h_3)(c) = \frac{2\pi}{3}\) and \((h_1 + h_2 + h_3)(c_*) = T(c_*) > \pi\), using intermediate value theorem, there exist a solution.

If the first curve is \(AA\), the second curve starts from \(B\). If the ending point is \(C\), \(\theta_{\text{up}} = h_2 < \frac{T(c_{\text{up}})}{2} < \pi\) is too small. Otherwise, it will form a complete loop. \(\square\)

![Figure 4. Degenerate case: heart. The ray on x-axis has multiplicity 2.](image)

**Theorem 7.3.** If the upper cell is a 4-cell

1. If the first curve is \(DD\), the type of \(\gamma_{\text{up}}\) should be \(DD \rightarrow CA \rightarrow BC\) and \(\theta_{\text{up}} = h_1 + 2h_2\).
2. If the first curve is \(AA\), the types of \(\gamma_{\text{up}}\) are the following two cases:
   - \(AA \rightarrow BA \rightarrow BB\) and \(\theta_{\text{up}} = h_1 + 2h_2\).
   - \(AA \rightarrow BA \rightarrow BC\) and \(\theta_{\text{up}} = h_1 + 3h_2\). In this case, the energy belongs to \(I_A\).

**Proof.** (1) If the first curve is \(DD\), which is degenerate, the second curve starts at \(C\).
• If the second curve ends at \( D \), the third curve starts at \( C \). If it immediately end at \( C \), \( \theta_{\text{up}} = h_1 < \pi \) from corollary 2.9. If the ending point is \( B \), we have \( \theta_{\text{up}} = 2h_1 + h_2 + h_3 > 2h_1 + h_2 > \pi \) from lemma 7.1. Both cases are impossible.

• If the second curve ends at \( A \), the third curve starts at \( B \). If it immediately ends at \( B \), using lemma 2.1 and proposition 2.8, \( \theta_{\text{up}} = h_1 + h_2 < \Delta \theta_{MN} < \pi \). If it ends at \( C \), \( \theta_{\text{up}} = h_1 + 2h_2 \). Since \( 2h_1 + 4h_2 = 2\pi \) has a unique solution which corresponds to the lens in the classification of regular shrinker with 1 closed region in [9], we have existence and uniqueness for this case.

(2) If the first curve is \( AA \), the second curve starts at \( B \).

• If the second curve ends at \( D \), the third curve starts at \( C \) and ends at either \( D \) or \( A \). If it ends at \( D \), suppose the fourth curve is not degenerate, \( \theta_{\text{up}} \geq 3h_1 > \pi \). Therefore, the fourth curve is degenerate and \( \theta_{\text{up}} = 2h_1 \). Note that \( h_1(c_*) > \frac{\pi}{2} \), \( \lim_{c \to \infty} h_1(c) = \frac{\pi}{3} \) and \( h_1 \) is decreasing, we have existence and uniqueness in this case. If the third curve ends at \( A \), \( \theta_{\text{up}} \geq 2h_1 + 2h_2 > \pi \) from lemma 7.1. This is impossible.

• If the second curve ends at \( A \), the third curve must start from \( B \) and end at either \( D \) or \( A \) and \( \theta_{\text{up}} \geq 2h_1 + 2h_2 > \pi \). This is impossible.

\begin{align}
(h_1 + 3h_2)(c_*) &= h_1(c) < \pi, \\
(h_1 + 3h_2)(\bar{c}) &> (h_1 + 2h_2)(\bar{c}) + 2\Delta \theta_{NA}(\bar{c}) > \pi.
\end{align}

Therefore, there exists a unique number \( c_{\text{up}} \) such that \((h_1 + 3h_2)(c_{\text{up}}) = \pi\).

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Degenerate cases: broken lens, cat, half lens, fox.}
\end{figure}

**Remark 7.4.** The broken lens is the only degenerate regular shrinker with multiplicity 1.

**Theorem 7.5.** If the upper cell is a 5-cell, the type of \( \gamma_{\text{up}} \) is \( DD \to CD \to CD \to CC \) and \( \theta_{\text{up}} = 2h_1 \).

**Proof.** If the first curve is \( DD \), which is degenerate, the second curve starts from \( C \).

• If the second curve ends at \( D \), the third curve starts at \( C \) and ends at either \( D \) or \( A \). If it ends at \( D \), suppose the fourth curve is not degenerate, \( \theta_{\text{up}} \geq 3h_1 > \pi \). Therefore, the fourth curve is degenerate and \( \theta_{\text{up}} = 2h_1 \). Note that \( h_1(c_*) > \frac{\pi}{2} \), \( \lim_{c \to \infty} h_1(c) = \frac{\pi}{3} \) and \( h_1 \) is decreasing, we have existence and uniqueness in this case. If the third curve ends at \( A \), \( \theta_{\text{up}} \geq 2h_1 + h_2 > \pi \) from lemma 7.1. This is impossible.

• If the second curve ends at \( A \), the third curve must start from \( B \) and end at either \( D \) or \( A \) and \( \theta_{\text{up}} \geq 2h_1 + 2h_2 > \pi \). This is impossible.
If the first curve is $AA$, the second curve starts from $B$.
- If the second curve ends at $D$, the third curve starts at $C$ and ends at $D$. We have $\theta_{up} \geq 2h_1 + h_2 > \pi$ from lemma $7.1$. This is impossible.
- If the second curve ends at $A$, the third curve start at $B$ and ends at $D$ or $A$ and need to cross $CD$ arc. Therefore $\theta_{up} \geq 2h_1 + 2h_2 > \pi$ from lemma $7.1$, which is impossible.

\[ \square \]

**Figure 6.** Degenerate case: half 4-ray star

**Lemma 7.6.** The degenerate regular shrinkers are either symmetric with respect to $x$-axis or symmetric with respect to the origin.

**Proof.** From theorem $6.3, 7.2, 7.3, 7.5$, the possible curves $\gamma_{up}$ in the (degenerate) regular shrinker with 2 closed regions are the following six cases:

1. $DD \to CA \to BC$ with $h_1 + 2h_2 = \pi$ and the energy $c_1$.
2. $DD \to CD \to CD \to DD$ with $2h_1 = \pi$ and the energy $c_2$.
3. $DA \to BA \to BC$ with $h_1 + 4h_2 = \pi$ and the energy $c_3$.
4. $DD \to CB$ with $h_1 + h_2 + h_3 = \pi$ and the energy $c_4$.
5. $AA \to BA \to BC$ with $h_1 + 3h_2 = \pi$ and the energy $c_5$.
6. $AA \to BA \to BB$ with $h_1 + 2h_2 = \pi$ and the energy $c_6$.

In the cases (1), (2), and (3), $R_{start} = R_{end} < 1$. In the case (4) and (5), we have either $R_{start} < 1 < R_{end}$ or $R_{start} > 1 > R_{end}$. In the case (6), we have $R_{start} = R_{end} > 1$. If there exists a degenerate regular shrinker with 2 closed regions, which $\gamma_{up}$ and $\gamma_{down}$ are of different types, it should be one of the following cases: $c_1 = c_2$, $c_1 = c_3$, $c_2 = c_3$, $c_4 = c_5$.

If $c_1 = c_2$ and $\left( h_1 + 2h_2 \right)(c_1) = 2h_1(c_2) = \pi$, from theorem $2.6$, we have $c_1 < e^{0.19}$. Since $h_1$ is decreasing, from the inequality $\left( 7.1 \right)$, $\frac{\pi}{2} = h_1(c_2) = h_1(c_1) > h_1(e^{0.19}) > 0.5945\pi$. This is a contradiction.

If $c_1 = c_3$ and $\left( h_1 + 2h_2 \right)(c_1) = \left( h_1 + 4h_2 \right)(c_3) = \pi$, $h_2(c_1) = h_2(c_3) = 0$, and $h_1(c_1) = \pi$. We obtain a contradiction from corollary $2.9$.

If $c_2 = c_3$ and $2h_1(c_2) = \left( h_1 + 4h_2 \right)(c_3) = \pi$, we obtain $h_1(c_2) = \frac{\pi}{2}$ and $h_2(c_2) = \frac{\pi}{8}$. From theorem $2.6$, it gives $0.7789\pi < \left( h_1 + 2h_2 \right)(c_2) = 0.75\pi$. This is a contradiction.

If $c_4 = c_5$ and $\left( h_1 + h_2 + h_3 \right)(c_4) = \left( h_1 + 3h_2 \right)(c_5) = \pi$, from theorem $7.3$, $c_5 \in I_4$. Using lemma $6.2$, the function $h_2$ is increasing on $I_4$ and the function $\left( h_1 + h_2 + h_3 \right)(c) = T(c) - h_2(c)$ is decreasing on $I_4$ from that $T$ is decreasing. Therefore, using the inequalities $\left( 5.18 \right)$ and $\left( 5.19 \right)$, we obtain $\pi = \left( h_1 + h_2 + h_3 \right)(c_5) \geq \left( h_1 + h_2 + h_3 \right)(\hat{c}) > 0.7027\pi + 0 + \frac{\pi}{5} > \pi$. This is a contradiction. \[ \square \]
By combining theorems 7.2, 7.3, 7.5, and lemma 7.6, we find some degenerate regular shrinkers with 2 closed regions, which are the heart, the broken lens, the cat, the half lens, the fox, the half 4-ray star.

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