MULTIPLE LATTICE TILES AND RIESZ BASES OF EXPONENTIALS

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Abstract. Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded and measurable set and $\Lambda \subseteq \mathbb{R}^d$ is a lattice. Suppose also that $\Omega$ tiles multiply, at level $k$, when translated at the locations $\Lambda$. This means that the $\Lambda$-translates of $\Omega$ cover almost every point of $\mathbb{R}^d$ exactly $k$ times. We show here that there is a set of exponentials $\exp(2\pi it \cdot x)$, $t \in T$, where $T$ is some countable subset of $\mathbb{R}^d$, which forms a Riesz basis of $L^2(\Omega)$. This result was recently proved by Grepstad and Lev under the extra assumption that $\Omega$ has boundary of measure 0, using methods from the theory of quasicrystals. Our approach is rather more elementary and is based almost entirely on linear algebra. The set of frequencies $T$ turns out to be a finite union of shifted copies of the dual lattice $\Lambda^*$. It can be chosen knowing only $\Lambda$ and $k$ and is the same for all $\Omega$ that tile multiply with $\Lambda$.

Notation. We write $e(x) = e^{2\pi ix}$. If $E$ is a set, then $\chi_E$ is its indicator function. If $A$ is a non-singular $d \times d$ matrix and $\Lambda = AZ^d$ is a lattice in $\mathbb{R}^d$, then $\Lambda^* = A^{-}\top Z^d$ denotes the dual lattice.

1. Introduction

1.1. Riesz bases. In this paper we deal with the question of the existence of a Riesz (unconditional) basis of exponentials

$$e_t(x) := e(t \cdot x) = e^{2\pi it \cdot x}, \quad t \in L,$$

for the space $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is a domain of finite Lebesgue measure and $L \subseteq \mathbb{R}^d$ is a countable set of frequencies. By Riesz basis we mean that every $f \in L^2(\Omega)$ can be written uniquely in the form

$$f(x) = \sum_{t \in L} a_t \cdot e(t \cdot x)$$

with the coefficients $a_t \in \mathbb{C}$ satisfying

$$C_1 \|f\|_2^2 \leq \sum_{t \in L} |a_t|^2 \leq C_2 \|f\|_2^2,$$

for some positive and finite constants $C_1, C_2$. 

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1.2. Orthogonal bases. One very special example of a Riesz basis occurs when the exponentials $e(t \cdot x), t \in L$, can be chosen to be orthogonal and complete for $L^2(\Omega)$. One can then choose $a_t = |\Omega|^{-1/2} \langle f, e_t \rangle$ and $C_1 = C_2 = 1/|\Omega|$ for (2) to hold as an equality. For instance, if $\Omega = (0,1)^d$ is the unit cube in $\mathbb{R}^d$, then one can take $L = \mathbb{Z}^d$ and obtain such an orthogonal basis of exponentials. This case, where an orthogonal basis of exponentials exists, is a very rigid situation though and many "reasonable" domains do not have such a basis (a ball is one example [4][11], or any other smooth convex body or any non-symmetric convex body [7]).

The problem of which domains admit an orthogonal basis of exponentials has been studied intensively. The so-called Fuglede or Spectral Set Conjecture [4] (claiming that for $\Omega$ to have such a basis it is necessary and sufficient that it can tile space by translations) was eventually proved to be false in dimension at least 3 [2][3][9][12][20], in both directions. Yet the conjecture may still be true in several important special cases such as convex bodies [8], and it generated many interesting results even after the disproof of its general validity (a rather dated account may be found in [11]).

It is expected that the existence of a Riesz basis for a domain $\Omega$ is a much more general, and perhaps even generic, phenomenon, although proofs of existence of a Riesz basis for specific domains are still rather rare, especially in higher dimension [13][14][16]. Also no domain is known not to have a Riesz basis of exponentials [13].

1.3. Lattice tiles. One general class of domains for which an orthogonal basis of exponentials is known to exist is the class of lattice tiles. A domain $\Omega \subseteq \mathbb{R}^d$ is said to tile space when translated at the locations of the lattice $L$ (a discrete additive subgroup of $\mathbb{R}^d$ containing $d$ linearly independent vectors) if

$$\sum_{t \in L} \chi_{\Omega}(x - t) = 1, \text{ for almost all } x \in \mathbb{R}^d. \quad (3)$$

Intuitively this condition means that one can cover $\mathbb{R}^d$ with the $L$-translates of $\Omega$, with no overlaps, except for a set of measure zero (usually the translates of $\partial \Omega$, for "nice" domains $\Omega$).

It is not hard to see that when $\Omega$ has finite and non-zero measure the set $L$ has density equal to $1/|\Omega|$. If $L$ is a lattice, then we call $\Omega$ an almost fundamental domain of $L$ and $|\Omega| = (\text{dens } L)^{-1}$. A fundamental domain of $L$ is any set which contains exactly one element of each coset mod $L$, for instance a fundamental parallelepiped. There are of course many others, as indicated in Figure 1.

It is easy to see [4][11] that every lattice tile by the lattice $L$ has an orthogonal basis of exponentials, namely those with frequencies $t \in L^*$, where $L^*$ is the dual lattice.

1.4. Multiple tiling by a lattice. We say that a domain tiles multiply when its translates cover space the same number of times, almost everywhere.

**Definition 1.1.** Let $\Omega \subseteq \mathbb{R}^d$ be measurable and $L \subseteq \mathbb{R}^d$ be a countable set. We say that $\Omega$ tiles $\mathbb{R}^d$ when translated by $L$ at level $k \in \mathbb{N}$ if

$$\sum_{t \in L} \chi_{\Omega}(x - t) = k, \quad (4)$$

for almost every $x \in \mathbb{R}^d$. If we do not specify $k$, then we mean $k = 1$. 

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Multiple tiles are a much wider class of domains than level-one tiles. For instance, any centrally symmetric convex polygon in the plane whose vertices have integer coordinates tiles multiply by the lattice $\mathbb{Z}^2$ at some level $k \in \mathbb{N}$. In contrast, only parallelograms or symmetric hexagons can tile at level one.

Another difference is the fact that if two disjoint domains $\Omega_1$ and $\Omega_2$ both tile multiply when translated at the locations $L$, then so does their union. In the case of multiple lattice tiling this operation gives essentially the totality of multiple tiles starting from level-one tiles, according to the following easy lemma.

**Lemma 1.** Suppose $\Omega \subseteq \mathbb{R}^d$ is a measurable set which tiles $\mathbb{R}^d$ at level $k$ when translated by the lattice $\Lambda \subseteq \mathbb{R}^d$. Then we can partition

\[ \Omega = \Omega_1 \cup \cdots \cup \Omega_k \cup E, \]

where $E$ has measure 0 and the $\Omega_j$ are measurable, mutually disjoint and each $\Omega_j$ is an almost fundamental domain of the lattice $\Lambda$.

**Proof.** Let $D \subseteq \mathbb{R}^d$ be a measurable fundamental domain of $\Lambda$, for instance one of its fundamental parallelepipeds. For almost every $x \in D$ (call the exceptional set $E \subseteq D$) it follows from our tiling assumption that $\Omega \cap (x + \Lambda)$ contains exactly $k$ points, which we denote by

\[ p_1(x) < p_2(x) < \cdots < p_k(x), \]

ordered according to the lexicographical ordering in $\mathbb{R}^d$. We also have that almost every point of $\Omega$ belongs to exactly one such list.

Let then $\Omega_j = \bigcup_{x \in D \setminus E} p_j(x)$, for $j = 1, 2, \ldots, k$. In other words, for (almost) each one of the classes mod $\Lambda$ we distribute its $k$ occurrences in $\Omega$ into the sets $\Omega_j$. It is easy to see that the $\Omega_j$ are disjoint and measurable and that they are almost fundamental domains of $\Lambda$. $\square$

1.5. **Multiple lattice tiles have Riesz bases of exponentials.** It is not true that domains that tile multiply by a lattice have an orthogonal basis of exponentials. For instance, it is known that the only convex polygons that have such a basis are parallelograms and symmetric hexagons, yet every symmetric convex polygon with integer vertices is a multiple tile, a much wider class.

It is however true that multiple tiles have a Riesz basis of exponentials. The main result of this paper is the following theorem.
Theorem 1. Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and tiles $\mathbb{R}^d$ multiply at level $k$ with the lattice $\Lambda$. Then there are vectors $a_1, \ldots, a_k \in \mathbb{R}^d$ such that the exponentials
\begin{equation}
    e((a_j + \lambda^*) \cdot x), \quad j = 1, 2, \ldots, k, \quad \lambda^* \in \Lambda^*
\end{equation}
form a Riesz basis for $L^2(\Omega)$.

The vectors $a_1, \ldots, a_k$ depend on $\Lambda$ and $k$ only, not on $\Omega$.

Theorem 1 was proved by Grepstad and Lev [6] with the additional topological assumption that the boundary $\partial \Omega$ has Lebesgue measure 0.

In [6] the result is proved following the method of [17,18] on quasicrystals. Our approach is more elementary and almost entirely based on linear algebra. The authors of [6] have pointed out that there are similarities in the method of this paper and the methods in [14–16]. The method essentially appears also in [19, § 3.2].

As an interesting corollary of Theorem 1 let us mention, as is done in [6], that, according to the recent result of [5], if $\Omega$ is a centrally symmetric polytope in $\mathbb{R}^d$, whose codimension 1 faces are also centrally symmetric and whose vertices all have rational coordinates, then $L^2(\Omega)$ has a Riesz basis of exponentials.

Open Problem 1. Is Theorem 1 still true if $\Omega$ is of finite measure but unbounded?

2. PROOF OF THE MAIN RESULT

The essence of the proof is contained in the following lemma.

Lemma 2. Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and tiles $\mathbb{R}^d$ multiply at level $k$ with the lattice $\Lambda$. Then there exist vectors $a_1, a_2, \ldots, a_k \in \mathbb{R}^d$ such that the following is true.

For any $f \in L^2(\Omega)$ there are unique measurable functions $f_j : \mathbb{R}^d \to \mathbb{C}$ such that
\begin{enumerate}
    \item The $f_j$ are $\Lambda$-periodic.
    \item The $f_j$ are in $L^2$ of any almost fundamental domain of $\Lambda$.
    \item We have the decomposition
        \begin{equation}
            f(x) = \sum_{j=1}^k e(a_j \cdot x)f_j(x), \quad \text{for a.e. } x \in \Omega.
        \end{equation}
\end{enumerate}

Finally we have
\begin{equation}
    C_1 \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^k \|f_j\|_{L^2(\Omega)}^2 \leq C_2 \|f\|_{L^2(\Omega)}^2,
\end{equation}
where $0 < C_1, C_2 < \infty$ do not depend on $f$.

Proof. Using Lemma 1 we can write $\Omega$ as the disjoint union
\[ \Omega = \Omega_1 \cup \cdots \cup \Omega_k, \]
where each $\Omega_k$ is a measurable almost fundamental domain of $\Lambda$. We can now define for $j = 1, 2, \ldots, k$ and for almost every $x \in \mathbb{R}^d$
\begin{enumerate}
    \item $\omega_j(x)$ as the unique point in $\Omega_j$ s.t. $x - \omega_j(x) \in \Lambda$, and
    \item $\lambda_j(x) = x - \omega_j(x)$.
\end{enumerate}
(The maps $\omega_j$ are clearly measurable and measure-preserving when restricted to a fundamental domain of $\Lambda$.) Since the sought-after $f_j$ are to be $\Lambda$-periodic it is
enough to define them on $\Omega_1$ and extend them to $\mathbb{R}^d$ by their $\Lambda$-periodicity. We may therefore rewrite our target decomposition (17) equivalently as follows.

(11)
For each $x \in \Omega_1$ and $r = 1, 2, \ldots, k$: $f(\omega_r(x)) = \sum_{j=1}^{k} e (a_j \cdot (x - \lambda_r(x)))f_j(x)$.

We view (11) as a $k \times k$ linear system

(12) $M \tilde{F} = F$

whose right-hand side is the column vector

$F = (f(\omega_1(x)), f(\omega_2(x)), \ldots, f(\omega_k(x)))^\top$

and the unknowns form the column vector

$\tilde{F} = (f_1(x), f_2(x), \ldots, f_k(x))^\top$.

We have a different linear system for each $x \in \Omega_1$ and its matrix is $M = M(x) \in \mathbb{C}^{k \times k}$ with

(13) $M_{r,j} = M_{r,j}(x) = e (a_j \cdot (x - \lambda_r(x))), \ r, j = 1, 2, \ldots, k$.

Factoring we can write this matrix as

(14) $M(x) = N(x) \operatorname{diag}(e (a_1 \cdot x), e (a_2 \cdot x), \ldots, e (a_k \cdot x))$,

with the matrix $N = N(x)$ given by

$N_{r,j} = N_{r,j}(x) = e (-a_j \cdot \lambda_r(x)), \ r, j = 1, 2, \ldots, k$.

The key observation here is that when varying $x \in \Omega_1$ the number of different $N(x)$ matrices that arise (the $a_j$ are fixed) is finite and bounded by a quantity that depends on $\Omega$ and $\Lambda$ only. The reason for this is that the vectors $\lambda_r(x)$ are among the $\Lambda$ vectors in the bounded set $\Omega - \Omega$, hence they take values in a finite set. (This is the only place where the boundedness of $\Omega$ is used.)

Let us now see that the vectors $a_1, \ldots, a_k$ can be chosen so that all the (finitely many) possible matrices $N$ are invertible. We have

(15) $\det N(x) = \sum_{\pi \in S_k} \operatorname{sgn}(\pi) e \left(-\sum_{j=1}^{k} a_j \cdot \lambda_{\pi_j}(x)\right)$,

where $S_k$ denotes the permutation group on $\{1, 2, \ldots, k\}$. By the definition of the vectors $\lambda_r(x)$ and the disjointness of the sets $\Omega_r$ it follows that for each $x$ no two $\lambda_r(x)$ can be the same. View now the expression (15) as a function of the vector $a = (a_1, \ldots, a_k) \in \mathbb{R}^{dk}$. Clearly it is a trigonometric polynomial and it is not identically zero as all the frequencies (for $\pi$ in the symmetric group $S_k$)

(16) $\lambda_{\pi}(x) = (\lambda_{\pi_1}(x), \ldots, \lambda_{\pi_k}(x)) \in \mathbb{R}^{dk}$,

are distinct precisely because all the $\lambda_r(x)$ are distinct. Since the zero-set of any trigonometric polynomial (that is not identically zero) is a set of co-dimension at least 1 it follows that the vectors $a_1, \ldots, a_k$ can be chosen so that all the $N(x)$ matrices that arise are invertible.

Now let $x \in \Omega_1$ and consider the solution of the linear system (12) at $x$ that now takes the form

(17) $\tilde{F}(x) = \operatorname{diag}(e (-a_1 \cdot x), e (-a_2 \cdot x), \ldots, e (-a_k \cdot x)) N(x)^{-1} F(x)$.
Since $N(x)$ runs through a finite number of invertible matrices and the diagonal matrix in (17) is an isometry it follows that there are finite constants $A_1, A_2 > 0$, independent of $f$, such that for any $x \in \Omega_1$ we have

$$A_1 \|F(x)\|_{L^2}^2 \leq \|\tilde{F}(x)\|_{L^2}^2 \leq A_2 \|F(x)\|_{L^2}^2.$$  

Integrating (18) over $\Omega_1$ we obtain

$$A_1 \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^k \|f_j\|_{L^2(\Omega_1)}^2 \leq A_2 \|f\|_{L^2(\Omega)}^2.$$  

This implies (8) with $C_j = k \cdot A_j$, $j = 1, 2$. To show the uniqueness of the decomposition (7) observe that any such decomposition must satisfy the linear system (17), whose non-singularity has been ensured by our choice of the $a_j$. □

We can now complete the proof of our main result.

**Proof of Theorem 1.** Let $f \in L^2(\Omega)$. By Lemma 2 we can write $f$ as in (7). Since the $f_j$ are $\Lambda$-periodic and are in $L^2$ of any almost fundamental domain $D$ of $\Lambda$ it follows that we can expand each $f_j$ in the frequencies of $\Lambda^*$ (the dual lattice of $\Lambda$)

$$f_j(x) = \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e(\lambda^* \cdot x), \; j = 1, 2, \ldots, k,$$

with

$$\|f_j\|_{L^2(D)}^2 = \sum_{\lambda^* \in \Lambda^*} |f_{j,\lambda^*}|^2,$$

since the exponentials $e(\lambda^* \cdot x)$, $\lambda^* \in \Lambda^*$, form an orthogonal basis of $L^2(D)$ (we assume without loss of generality that $|D| = 1$).

The completeness of (6) follows from (7):

$$f(x) = \sum_{j=1}^k \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e((a_j + \lambda^*) \cdot x).$$

The fact that (6) is a Riesz sequence follows from (8):

$$\frac{k}{C_2} \sum_{j,\lambda^*} |f_{j,\lambda^*}|^2 \leq \left( \sum_{j,\lambda^*} f_{j,\lambda^*} e((a_j + \lambda^*) \cdot x) \right)_{L^2(\Omega)}^2 \leq \frac{k}{C_1} \sum_{j,\lambda^*} |f_{j,\lambda^*}|^2.$$  

As is clear from the proof above, the $k$-tuples of vectors $a_1, \ldots, a_k$ that appear in Theorem 1 are a generic choice: almost all $k$-tuples will do. The exceptional set in $\mathbb{R}^{dk}$ is a set of lower dimension.

With a little more care one can see that one can choose the vectors $a_1, \ldots, a_k$ to depend on $\Lambda$ and $k$ only and not on $\Omega$. In the proof of Lemma 2 the $a_j$ were chosen to ensure that the trigonometric polynomials (15) are all non-zero. Fix $\Lambda$ and $k$ and form the set of all polynomials of the form (15) which are not identically zero. This set of polynomials is countable and each such polynomial vanishes on a set of codimension at least 1 in $\mathbb{R}^{dk}$. It follows that the union of their zero-sets cannot possibly exhaust $\mathbb{R}^{dk}$ and we only have to choose the $a_j$ to avoid that union.

Thus there is a choice of $a_j$ that works for all $\Omega$ of the same lattice. This proof does not give uniform values for the constants $C_1$ and $C_2$ in (8) though. □
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REFERENCES

[1] U. Bolle, On multiple tiles in \( E^2 \), Intuitive geometry (Szeged, 1991), Colloq. Math. Soc. János Bolyai, vol. 63, North-Holland, Amsterdam, 1994, pp. 39–43. MR1383609 (97e:52032)

[2] Bálint Farkas, Máte Matolcsi, and Péter Móra, On Fuglede’s conjecture and the existence of \( \sigma \)-spectra, J. Fourier Anal. Appl. 12 (2006), no. 5, 483–494, DOI 10.1007/s00041-005-5069-7. MR2267631 (2007h:52022)

[3] Bálint Farkas and Szilárd Gy. Révész, Tiles with no spectra in dimension 4, Math. Scand. 98 (2006), no. 1, 44–52. MR2221543 (2007g:52016)

[4] Bent Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Functional Analysis 16 (1974), 101–121. MR0470754 (57 #10500)

[5] Nick Gravin, Sinai Robins, and Dmitry Shiryaev, Translational tilings by a polytope, with multiplicity, Combinatorica 32 (2012), no. 6, 629–649, DOI 10.1007/s00493-012-2860-3. MR3063154

[6] Sigrid Grepstad and Nir Lev, Multi-tiling and Riesz bases, arXiv preprint arXiv:1212.4679 (2012).

[7] Alex Iosevich, Nets Hawk Katz, and Terry Tao, Convex bodies with a point of curvature do not have Fourier bases, Amer. J. Math. 123 (2001), no. 1, 115–120. MR1827279 (2002g:52011)

[8] Alex Iosevich, Nets Katz, and Terence Tao, The Fuglede spectral conjecture holds for convex planar domains, Math. Res. Lett. 10 (2003), no. 5-6, 559–569, DOI 10.4310/MRL.2003.v10.n5.a1. MR2024715 (2004i:42020)

[9] Mihail N. Kolountzakis and Máte Matolcsi, Complex Hadamard matrices and the spectral set conjecture, Collect. Math. Vol. Extra (2006), 281–291. MR2264214 (2007h:52023)

[10] M. N. Kolountzakis, On the structure of multiple translational tilings by polygonal regions, Discrete Comput. Geom. 23 (2000), no. 4, 537–553, DOI 10.1007/s004540010014. MR1753701 (2001c:52025)

[11] Mihail N. Kolountzakis, The study of translational tiling with Fourier analysis, Fourier analysis and convexity, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004, pp. 131–187. MR2087242 (2005e:42071)

[12] Mihail N. Kolountzakis and Máte Matolcsi, Tiles with no spectra, Forum Math. 18 (2006), no. 3, 519–528, DOI 10.1515/FORUM.2006.026. MR2237932 (2007d:20088)

[13] Gady Kozma and Shahaf Nitzan, Combining Riesz bases, arXiv preprint arXiv:1210.6383 (2012).

[14] Yuri I. Lyubarskii and Alexander Rashkovskii, Complete interpolating sequences for Fourier transforms supported by convex symmetric polygons, Ark. Mat. 38 (2000), no. 1, 139–170, DOI 10.1007/BF02384495. MR1749363 (2001m:32013)

[15] Yuri I. Lyubarskii and Kristian Seip, Sampling and interpolating sequences for multiband-limited functions and exponential bases on disconnected sets, J. Fourier Anal. Appl. 3 (1997), no. 5, 597–615, DOI 10.1007/BF02648887. Dedicated to the memory of Richard J. Duffin. MR1491937 (99f:30007)

[16] Jordi Marzo, Riesz basis of exponentials for a union of cubes in \( \mathbb{R}^d \), arXiv preprint math/0601288 (2006).

[17] Basarab Matei and Yves Meyer, Quasicrystals are sets of stable sampling (English, with English and French summaries), C. R. Math. Acad. Sci. Paris 346 (2008), no. 23-24, 1235–1238, DOI 10.1016/j.crma.2008.10.006. MR2473299 (2010g:94050)

[18] Basarab Matei and Yves Meyer, Simple quasicrystals are sets of stable sampling, Complex Var. Elliptic Equ. 55 (2010), no. 8-10, 947–964, DOI 10.1080/174769390903394689. MR2674875 (2011k:42019)

[19] Alexander Olevskii and Alexander Ulanovskii, On multi-dimensional sampling and interpolation, Anal. Math. Phys. 2 (2012), no. 2, 149–170, DOI 10.1007/s13324-012-0027-4. MR2917231

[20] Terence Tao, Fuglede’s conjecture is false in 5 and higher dimensions, Math. Res. Lett. 11 (2004), no. 2-3, 251–258, DOI 10.4310/MRL.2004.v11.n2.a8. MR2067470 (2005i:42037)

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