To the study of the dynamics of layered structures with plane inclusions

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Abstract. We consider the problem of vibration for a layered package with defects such as hard inclusions on interlayer interfaces. Using the factorization approach, we obtain recurrence relations for the construction of functional-matrix equation systems for problems of elastic vibrations of a multilayer structure with a finite number of planar defects. When studying the resonance properties of the structures under consideration, the obtained representations of matrix kernel symbols of the integral equation systems allow us to implement numerical algorithms and construct the root and polar sets of their elements and determinants for a wide range of problems. The approach presented in this work makes it possible to efficiently study vibrational regimes for multilayer structures taking into account non-ideal interlayer contact. The developed scheme can also be applied to the case of an arbitrary number and alternation of parallel cracks and inclusions. Theoretical results can find applications in engineering practice when studying the strength properties of designed materials, when assessing the operational characteristics of composite structural elements, etc.

1. Introduction
To solve various kinds of technical problems (wave monitoring, flaw detection, etc.), as well as to study the mechanisms of geophysical fields generation, it is necessary to develop mathematical models and methods for studying the deformation for materials of complex structure, in particular layered-block composites and geological materials, under the conditions of multifactor influence. The expansion of the material variety entails the necessity to construct their defining relationships and study the corresponding boundary value problems, since the physicomechanical properties of the structure can differ significantly from the properties of heterogeneous components in interaction.

The problem of vibration of a layered package with defects such as hard inclusions on interlayer interfaces is considered. Such mechanical objects, called vibration-strength «viruses» \cite{1,2}, are typical of layered-heterogeneous geological structures. In addition, modern composite materials that are currently used in high-tech industries are made of polymers, metal, carbon, fiberglass and other materials, which are layered structures with specified physical and mechanical characteristics of the layers. The problems of oscillation for bodies with a single defect or with a system of defects under the effect of loads in various settings are considered in the works of several authors \cite{1–6}. The choice of the number and type of layers, as well as the arrangement of inclusions determines the various strength properties of the material.
Currently, the majority of publications devoted to the study of problems for layered structures of various types contain the results obtained with numerical methods based on grid approximation (finite difference method (FDM), finite element method (FEM) [7–9], widely available as commercial software products. Paying tribute to their simplicity in implementation and versatility, it should be pointed out that there are a number of problems for which the direct application of FEM or FDM is ineffective, for example, in the study of dynamic problems for extended media as well as media containing defects in the presence of singularities of various kinds [10]. In the first case, the problems are associated with the necessity to store large amounts of data, in the second, with the need to downscale the grid near inhomogeneities, even if the quantity of the of the latter is small, since the stress-strain state of such mechanical systems depends on a large number of parameters. In the present work the factorization method [1, 11] we applied to solving problems for multilayered media with rigid inclusions.

2. The formulation of the problem
Let us consider the problem of harmonic vibrations for a multilayered structure consisting of \( N \) layers with plane-parallel boundaries with Lame coefficients \( \lambda_i, \mu_i \), density \( \rho_i \) and thickness \( h_k, \ k = 1, N \). Numbering of layers goes from bottom to top. As the latter, one can also consider a system of \( N \) layers on the surface of an elastic half-space, which is considered a layer of unlimited thickness. It is assumed that the oscillations are steady with frequency \( \omega \). In the interface planes at heights \( h_j, \ j = 2, N - 1 \), tiered rigid inclusions are located. Areas of inclusions \( \Omega_j \) are considered simply connected, their boundaries \( \Sigma_j \) are piecewise smooth.

The vectors of normal and tangential stresses in the planes \( x_3 \rightarrow h_n \pm 0 \) are denoted by \( \tau^+_n \exp(-i\omega t), \quad \tau^-_n = \{\tau^+_n1, \tau^+_n2, \tau^+_n3\} \), and the displacement vector is \( u^+_n \exp(-i\omega t) \), \( u^-_n = \{u^+_n1, u^+_n2, u^+_n3\} \).

Further, we consider the amplitude values of unknown and given functions, the time factor is omitted.

Displacements in the separation planes of physicomechanical properties are continuous (in the inclusion areas they are considered given), stresses in the inclusion areas have discontinuity. For a packet of layers (thickness of the \( k \)-th layer \( h_k = h_{k+1} - h_k \)) with a free upper and pinched lower edges, the mathematical notation of the conditions formulated above has the form:

\[
\begin{align*}
x_3 = h_{N+1}: & \quad \tau_{N+1} = 0, \quad -\infty < x_1, x_2 < +\infty; \\
x_3 = h_k: & \quad \begin{cases} u_n = u^-_n = u^+_n, & (x_1, x_2) \in \Omega_n, \\
\tau_n = \tau^-_n = \tau^+_n = 0, & (x_1, x_2) \notin \Omega_n; \\
x_3 = h_1: & \quad \underline{u}_n = 0, \quad -\infty < x_1, x_2 < +\infty.
\end{cases}
\end{align*}
\]

Under the influence of a stamp on the upper edge in the area \( \Omega_{N+1} \), the first condition is written as:

\[
\begin{align*}
x_3 = h_{N+1}: & \quad \begin{cases} u_{N+1} = u^+_n, & (x_1, x_2) \in \Omega_{N+1}, \\
\tau_{N+1} = 0, & (x_1, x_2) \notin \Omega_{N+1}.
\end{cases}
\end{align*}
\]

Following the terminology of [1], the formulated problem describes the vibration-strength «virus».

3. Functional-matrix relations for a multilayer structure containing a system of plane rigid inclusions
In the general case, contact problems can be reduced to solving the systems of integral equations (SIE). The boundary-value problem of harmonic vibrations of a homogeneous isotropic layer described by the Lame equations, according to [12], can be reduced to a SIE of the following form:

\[
\int_0^\frac{1}{2} \int_0^\frac{1}{2} M \left( \int_0^\frac{1}{2} \int_0^\frac{1}{2} \lambda \sigma^2 \tau^m_n + 2\mu \sigma^k \sigma^l \right) u_{mm} (\xi_1, \xi_2, \xi_3) \exp(i(\sigma \xi)) \, d\xi = \int_0^\frac{1}{2} \left( \sum_{m=1}^2 \alpha_m \int_{\tilde{S}} \tilde{r}_{mm} (\xi_1, \xi_2, \xi_3) \exp(i(\sigma \xi)) \, d\xi \right),
\]

\[
\alpha_3 = \pm \sqrt{\gamma^2 - \alpha^2}, \quad \alpha^2 = \alpha_1^2 + \alpha_2^2,
\]
\[
\begin{align*}
\mu \sum_{i=1}^{3} \sum_{m=1}^{3} \iint_{S_m} \left[ (\alpha_i l^n_m - \alpha_i l^n_0) \alpha_m + \alpha_i l^n_m \left( \alpha_i \delta_{m0} - \alpha_i \delta_{m0} \right) \right] u_{m0}(\xi, \xi_1, \xi_2, \xi_3) \exp(i(\sigma \xi)) \, ds = \\
= \int_{S_m} \left[ \alpha_i \tau_{m0}(\xi, \xi_1, \xi_2) - \alpha_i \tau_{m0}(\xi, \xi_1, \xi_2) \right] \exp(i(\sigma \xi)) \, ds,
\end{align*}
\]

(2)

In the given ratios \(-\infty \leq \alpha_i, \alpha_i \leq \infty ; I^n = (l^n_1, l^n_2, l^n_3) - \) external normal to the boundary \( S_n \), \( l^n_i - \) functions of parameters \( \xi_i, \xi_i \); \( \delta_{m0} \)- Kronecker symbol.

For the inclusion, located at the interface of different module layers with \( \{l^n_i = l^n_0 = 0, l^n_i = \pm 1\} \), relations (1), (2) in Fourier transforms take the following form:

\[
E^\pm_{k,l-k} L^\pm_{k,l-k} U^\pm_{k,l-k} - E^\pm_{k,l-k} L^\pm_{k,l-k} U^\pm_{k,l-k} = E^\pm_{k,l-k} D^\pm_{k,l-k} T^\pm_{k,l-k} - E^\pm_{k,l-k} D^\pm_{k,l-k} T^\pm_{k,l-k}, \quad k = 2, N+1,
\]

(3)

In (3) the indices indicate the inclusion number and the layer number, respectively, \( T^\pm, U^\pm \)- two-dimensional Fourier transforms of vector function \( \mathbf{t}_k, \mathbf{u}_k \):

\[
U_k(\alpha_1, \alpha_2) = \int \int_{-\infty}^{\infty} \mathbf{u}_k(x_1, x_2, h) \exp(i(\alpha_1 x_1 + \alpha_2 x_2)) \, dx_1 \, dx_2,
\]

\[
T_k(\alpha_1, \alpha_2) = \int \int_{-\infty}^{\infty} \mathbf{t}_k(x_1, x_2, h) \exp(i(\alpha_1 x_1 + \alpha_2 x_2)) \, dx_1 \, dx_2.
\]

In this case, matrices with a \( \leftarrow \rightarrow \) sign correspond to waves propagating in the lower layer, with a \( \leftarrow \rightarrow \) sign – upper.

For a layered elastic structure, the application of the differential factorization method [13] leads to the same relations (3). The value of the approach used in this work lies in the ability to obtain recurrence relations when constructing systems of functional-matrix equations for an arbitrary number of layers and inclusions.

For the problem under consideration, the Fourier images of the desired stress jumps on the edges of the inclusions can be expressed in terms of known values: given displacements in the lower interface (rigid fastening).

Inclusions and displacement under the stamp on the upper interface, the absence of displacements on the inclusions can be expressed in terms of known values: given displacements in the upper interface.

The value of the approach used in this work lies in the ability to obtain recurrence relations when constructing systems of functional-matrix equations for an arbitrary number of layers and inclusions.
\[
R_{NN} = \mathbf{K}_N^{-1} \cdot R_{Nj} = (-1)^{N+1-j} \mathbf{K}_N^{-1} \prod_{i=1}^{j-1} F_i N_i^{-1} \cdot R_{jN} = \left( \prod_{i=1}^{N} N_i^{-1} F_i^* \right) \mathbf{K}_N^3, \quad j = 1, N-1;
\]
\[
R_{N,j} = (-1)^{N+1-j} \mathbf{K}_N^{-1} \prod_{i=j+1}^{j-1} F_i N_i^{-1} - N_i^{-1} \mathbf{K}_N, \quad j < N-1;
\]
\[
R_{N=1,j} = (-1)^{N+1-j} \mathbf{K}_N^{-1} \prod_{i=j+1}^{j-1} F_i N_i^{-1} \cdot R_{j,1} = \left( \prod_{i=1}^{N} N_i^{-1} F_i^* \right) R_{1,j} \cdot, \quad j = 1, N-1;
\]
\[
R_{j,j} = -N_{j+1}^{-1} F_{j+1} F_{j+1} N_{j+1}^{-1} - N_{j+1}^{-1}, \quad j < N-1;
\]
\[
R_{k,j} = -R_{k,j+1} N_{j+1}^{-1} \quad \text{for } j < k < N-1, \quad R_{k,j} = N_{k+1}^{-1} F_{k+1} R_{k+1,j} \quad \text{for } k < j < N-1.
\]

The notations here
\[
F_j^* = \left( \left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{J}_j^* \mathbf{D}_j, -\left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{J}_j^* \mathbf{D}_{j,j} \right)^{-1} \left( \left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{L}_{j,j} - \left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{L}_{j,j} \right), \quad J_j = \mathbf{E}_{j,j} \mathbf{F}_{j,j}^*;
\]
\[
N_j = \mathbf{Z}_j - \mathbf{Z}_j^2, \quad N_{j+1} = \mathbf{Z}_j^2 + \mathbf{F}_j \mathbf{N}_j \mathbf{F}_j^*, \quad j = 2, N-1;
\]
\[
Z_j^* = \left( \left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{J}_j^* \mathbf{D}_j, -\left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{J}_j^* \mathbf{D}_{j,j} \right)^{-1} \left( \left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{L}_{j,j} - \left( \mathbf{D}_{j,j} \right)^{-1} \mathbf{L}_{j,j} \right).
\]

Matrix function \( \mathbf{K}_N \) have representations:
\[
\mathbf{K}_N = \mathbf{Z}_N, \quad \mathbf{K}_j = \left[ \mathbf{Z}_j^2 + \mathbf{F}_j \mathbf{N}_j \mathbf{F}_j^* \right], \quad j = 2, N.
\]

Stresses \( \mathbf{T} = \left\{ T_1, ..., T_N \right\} \) in the interface planes are subsequently determined through stress jumps in the inclusion areas:
\[
\mathbf{T} = \mathbf{W}^T, \quad (5)
\]
where the elements of the block matrix \( \mathbf{W} = \left\{ \mathbf{W}_{kj} \right\}_{k,j=1}^{N} \) have representations:
\[
\mathbf{W}_{kj} = \mathbf{K}_k \mathbf{R}_{kj} \quad \text{for } k \leq j, \quad \mathbf{W}_{kj} = (-1)^{j-k} \prod_{i=k}^{j-1} F_i N_i^{-1} + \mathbf{K}_k \mathbf{R}_{kj} \quad \text{for } j < k < N.
\]

If we consider an \( N \)-layer packet on an elastic half-space, then to the relations (2) we should add the functional-matrix equation for a structure element of unlimited thickness with index «0», whose properties are characterized by \( \hat{\lambda}_0, \mu_0, \rho_0 \):
\[
\mathbf{L}_{0j} \mathbf{U}_j = \mathbf{D}_{0j} \mathbf{T}_j.
\]

The system of functional-matrix equations that connect the Fourier transforms of the displacements at the interfaces between the layers and the stress jumps in the inclusion areas will contain an \( N+1 \) equation and can be written as:
\[
\mathbf{V} \mathbf{T} = \mathbf{U},
\]
where \( \mathbf{T} = \left\{ T_1, ..., T_N \right\}, \quad \mathbf{U} = \left\{ U_1, ..., U_N, U_{N+1} \right\} \), the blocks of matrix function \( \mathbf{V} = \left\{ \mathbf{V}_{kj} \right\}_{k,j=1}^{N} \) have the following representation:
\[
\mathbf{V}_{N+1,1} = \left( \mathbf{D}_{0k} \right)^{-1} \mathbf{L}_{0k}, \quad \mathbf{V}_{N+1,j} = (-1)^{j-1} \left( \mathbf{D}_{0k} \right)^{-1} \prod_{i=1}^{j} F_i N_i^{-1} \cdot \mathbf{V}_{j+1} = \left( \prod_{i=1}^{N} N_i^{-1} F_i^* \right) \mathbf{K}_N^{-1}, \quad j = 1, N-1;
\]
\[
\mathbf{V}_{N,j} = \left( \mathbf{D}_{0k} \right)^{-1} \mathbf{L}_{0k} \cdot \mathbf{V}_{j,1} = \left( \prod_{i=1}^{N} N_i^{-1} F_i^* \right) \mathbf{V}_{j,N}, \quad j = 1, N-1;
\]
\[
\mathbf{V}_{j,j} = -\mathbf{V}_{j+1,k} F_{j+1} N_{j+1}^{-1} \mathbf{F}_{j+1}^* \quad \text{for } j < k < N, \quad \mathbf{V}_{j,j} = \left[ \mathbf{Z}_j^2 + \mathbf{F}_j \mathbf{N}_j \mathbf{F}_j^* \right], \quad j = 1, N-1;
\]
\[
\mathbf{K}_{0j} = \left( \mathbf{D}_{0k} \right)^{-1} \mathbf{L}_{0k}, \quad \mathbf{K}_{j,j} = \left[ \mathbf{Z}_j^2 + \mathbf{F}_j \mathbf{N}_j \mathbf{F}_j^* \right], \quad j = 1, N-1.
\]
If the upper interface of the considered layered structure is free of stresses \( \mathbf{T}_{\nu+1} = 0 \) we need to remove the last row and column from matrices \( \mathbf{R} \) (4), \( \mathbf{V} \) (6). In the event that on some interfaces \( (x_j = h_j, j = j_1, j_m) \) we have the ideal \( (\mathbf{T}_{j} = 0) \), rows and columns with corresponding numbers \( j = j_1, j_m \) are removed from the system matrix. Also, from the vector of unknowns we remove \( \mathbf{T}_{j} \) and from the right side \( -\mathbf{U}_j \).

4. Systems of integral equations for the problems under consideration

The system of functional-matrix equations (4), (6) lead to the SIE with respect to contact stresses \( \tau_{\nu+1} = \tau_{\nu+1}^* (x_1, x_2) \) under the stamp and stress jumps \( \tau_{\nu}^* = \tau_{\nu}^* (x_1, x_2) - \tau_{\nu}^* (x_1, x_2) \) on the edges of inclusions occupying flat areas \( \Omega_n \) \((n = 2, N)\):

\[
\sum_{j=1}^{N-1} K_{jj} (\Omega_{\nu+1}) \tau_{\nu+1}^* + \sum_{j=1}^{N-1} K_{\nu,j} (\Omega_{\nu+1}) \tau_{\nu+1} = u_{\nu+1}, \quad x_j = h_{\nu+1}, \quad (x_1, x_2) \in \Omega_{\nu+1}, \quad n = 1, N-1;
\]

\[
\sum_{j=1}^{N-1} K_{\nu,j} (\Omega_{\nu+1}) \tau_{\nu+1}^* + \sum_{j=1}^{N-1} K_{\nu,N} (\Omega_{\nu+1}) \tau_{\nu+1} = u_{\nu+1}, \quad x_j = h_{\nu+1}, \quad (x_1, x_2) \in \Omega_{\nu+1},
\]

where the integral operators have the form:

\[
K_{jj} (\Omega_{\nu+1}) \tau_{\nu+1}^* = \int_{\Gamma_{k}} k_{jj} (x_1 - \xi_1, x_2 - \xi_2) \tau_{\nu+1}^* (\xi_1, \xi_2) d\xi_1 d\xi_2,
\]

\[
k_{jj} (x_1, x_2) = \frac{1}{4\pi^2} \int_{\Gamma_{k}} K_{jj} (x_1, x_2) \exp(-i(\alpha x_1 + \alpha x_2)) d\alpha_1 d\alpha_2,
\]

the kernel symbols (7), (8) for the packet of layers are described by relations \( K_{jj} = K_{jj} \left( k, j = 1, N \right) \), also the principle of limiting absorption is used in the work to determine the position of the contours \( \Gamma_k \) \((k = 1, 2)\) in the complex plane. For an \( N \)-layer packet on half-space, the dimension of the SIE increases. Kernel matrix-symbols are defined as \( K_{jj} = V_{jj} \left( k, j = 1, N + 1 \right) \). If some, for example, the \( l \)-th, interlayer interface contains several inclusions \( (M_l) \), then \( \Omega_{l} = \bigcup_{m=1}^{M_l} \Omega_{m} \) should be put in (7).

The resulting systems of integral equations can be solved using semi-analytical factorization methods or numerical methods.

The defects in the interlayer interfaces can have a significant effect on the waveguide and strength properties and, ultimately, on the reliability of the structure as a whole. The conditions of localization of the wave process by the vibration-strength «viruses», which are relations connecting the parameters of the problem under consideration (geometric and mechanical characteristics of the system, external influences, etc.) \([14]\), require us to find the real zeros of the determinant for the matrix \( \mathbf{R} \) of the system of obtained matrix-functional equations (4). Due to the block structure of the matrix in the system, the latter can be represented in the form of products of the auxiliary matrix determinants, their properties allow using well-known algorithms for the computation of determinants. In work \([2]\), the results of an analytical study of the root and polar sets for the determinants of the matrix-symbol kernels of certain problems for layered structures with defects are presented. The representations of matrix blocks of kernel symbols for the systems of integral equations obtained in this work allow us to implement numerical algorithms for constructing the root and polar sets of their elements and determinants for a wide range of problems.
5. Conclusion

The development of technology initiates the creation of new materials and, despite the current rich experience in studying problems for various layered composites and bodies with coatings, the solution of a number of problems remains relevant. Studying them attracts the attention of researchers with its complexity from an experimental and mathematical point of view and requires the improvement of methods for solving them. Multilayer composites made of carbon fiber, polymers, metals are widely used in construction materials. Layered structures with defects are widely used as models in vibrational geophysics and active seismology.

Dynamic and resonance effects in multilayer structures with multiple defects in the interface planes are currently under scrutiny. The presence of multiple inhomogeneities greatly complicates the description of the structural properties of the material containing them. But despite the mathematical difficulties of such problems they are of great interest. At the same time, direct numerical methods are sometimes completely inapplicable to their solution in extended regions with an increasing oscillation frequency; moreover, with a sharp contrast in the physical properties of the materials in the structure, the accuracy of the solution decreases. The approach presented in the work allows one to efficiently study the vibrational regimes for multilayer structures taking into account non-ideal interlayer contact (the presence of interface defects such as hard inclusions), providing an algorithm for construction of kernel symbol matrix-functions of SIE for the described class of problems with various combinations of physicomechanical properties of structurally inhomogeneous media. The developed scheme can also be applied to the case of an arbitrary number and alternation of parallel cracks and inclusions. The presented models and theoretical results can find applications in engineering practice when studying the strength properties of the materials being designed, when evaluating the operational characteristics of composite structural elements, etc.

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