SPECTRAL DISTRIBUTION OF THE FREE JACOBI PROCESS, REVISITED

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Abstract. We obtain a description for the spectral distribution of the free Jacobi process for any initial pair of projections. This result relies on a study of the unitary operator $RU_tSU_t^*$ where $R, S$ are two symmetries and $U_t$ a free unitary Brownian motion, freely independent from $\{R, S\}$. In particular, for non-null traces of $R$ and $S$, we prove that the spectral measure of $RU_tSU_t^*$ possesses two atoms at $\pm 1$ and an $L^\infty$-density on the unit circle $\mathbb{T}$, for every $t > 0$. Next, via a Szegő type transform of this law, we obtain a full description of the spectral distribution of $PU_tQU_t^*$ beyond the $\tau(P) = \tau(Q) = 1/2$ case. Finally, we give some specializations for which these measures are explicitly computed.

1. Introduction

Let $P, Q$ be two projections in a $W^*$-probability space $(\mathcal{A}, \tau)$ which are free with $\{U_t, U_t^*\}$. The present paper is a companion to the series of papers [5, 6, 7, 8, 9, 10] devoted to the study of the spectral distribution, hereafter $\mu_t$, of the self-adjoint-valued process $(X_t := PU_tQU_t^*P)_{t \geq 0}$. Viewed in the compressed algebra $(P \mathcal{A} P, \tau/\tau(P))$, $X_t$ coincides with the so-called free Jacobi process with parameter $(\tau(P)/\tau(Q), \tau(Q))$, introduced by Demni in [6] via free stochastic calculus, as solution to a free SDE there. Properties of its measure play important roles in free entropy and free information theory (see e.g. [15, 16, 17, 18, 24]). Furthermore, $\mu_t$ completely determines the structure of the von Neumann algebra generated by $P$ and $U_tQU_t^*$ (see e.g. [17, 22]) for any $t \geq 0$, yielding a continuous interpolation from the law of $PQP$ (when $t = 0$) to the free multiplicative convolution of the spectral measures of $P$ and $Q$ separately (when $t$ tends to infinity). Indeed, the pair $(P, U_tQU_t^*)$ tends towards $(P, UQU^*)$ as $t \to \infty$, where $U$ is a Haar unitary free from $\{P, Q\}$. The two projections $P$ and $UQU^*$ are therefore free (see [21]) and hence $\mu_{PUQU^*P} = \mu_P \boxtimes \mu_{UQU^*} = \mu_P \boxtimes \mu_Q$. This measure was explicitly computed in [11, Example 3.6.7]. Generally, the operators $P$ and $U_tQU_t^*$ are not free for finite $t$ and the process $t \mapsto (P, U_tQU_t^*)$ is known as the free liberation of the pair $(P, Q)$ (cf. [24]). When both projections coincide, the series of papers [7, 8, 9, 10] aim to determine $\mu_t$ for any $t > 0$. In particular, when $P = Q$ and $\tau(P) = 1/2$, Demni, Hmidi and myself proved in [9, Corollary 3.3] that the measure $\mu_t$ possesses a continuous density on $(0, 1)$ for $t > 0$ which fits that of the random variable $(I + U_{2t} + (I + U_{2t})^*)/4$. In [5], Collins and Kemp extended this result to the case of two projections $P, Q$ with traces $1/2$. Afterwards this result was partially extended by Izumi and Ueda to the arbitrary traces case. They proved the following.

$$\mu_t = (1 - \min\{\tau(P), \tau(Q)\})\delta_0 + \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_1 + \gamma_t$$

where $\gamma_t$ is a positive measure with no atom on $(0, 1)$ for every $t > 0$ (cf. [18, Proposition 3.1]). When $\tau(P) = \tau(Q) = 1/2$, this measure coincide with the Szegő transformation of the distribution of $UU_t$ where $U$ is a unitary random variable determined by the law of $PQP$.
(cf. [18, Proposition 3.3]). In [3] Lemma 3.2, Lemma 3.6, Collins and Kemp studied the support of the measure $\gamma_t$, for the general case of traces, and the way in which the edges of this support are propagated, but they were still not able to prove the continuity of $\gamma_t$.

Our major result in these notes is a complete analysis of the spectral distribution of the unitary operator $RU_tSU^*_t$ (hereafter $\nu_t$) for any symmetries $R,S \in \mathcal{A}$ which are free with $\{U_t, U^*_t\}$. In particular, we prove that the measure

$$\nu_t - \frac{1}{2}|\tau(R) - \tau(S)| \delta_\pi - \frac{1}{2}|\tau(R) + \tau(S)| \delta_0$$

possesses an $L^\infty$-density $\kappa_t$ on $\mathbb{T} = (-\pi, \pi]$. Using the relationship between $\mu_t$ and $\nu_t$, when $\{P,Q\}$ and $\{R,S\}$ are associated (cf. [15, Theorem 4.3]), we deduce the regularity of $\mu_t$ for any initial projections. In particular, we prove that the measure $\gamma_t$ possesses a continuous density on $[0,1]$. Here is our result.

**Theorem 1.1.** Let $P,Q$ be orthogonal projections and $U_t$ a free unitary Brownian motion, freely independent from $P,Q$. For every $t > 0$, the spectral distribution $\mu_t$ of the self adjoint operator $PU_tQU^*_tP$ is given by

$$\mu_t = (1 - \min\{\tau(P), \tau(Q)\})\delta_0 + \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_1 + \frac{\kappa_t(2\arccos(\sqrt{x}))}{2\pi \sqrt{x(1-x)}} 1_{[0,1]}(x)dx.$$  

We conclude the paper with the following ‘unexpected’ identities for the measure $\nu_t$ when the initial operators are assumed to be freely, classically, boolean and monotone independent with law $\frac{\delta_1 + \delta_1 - 1}{2}$. We have $\nu_t$ is constant in $t$ in the first case, and its given by a dilation of the law of $U_t$ in the rest of cases. The result is as follows.

**Theorem 1.2.** Let $\lambda_t$ be the probability distribution of the free unitary Brownian motion $U_t$ and $\mu = \frac{\delta_1 + \delta_1 - 1}{2}$ (considered as a law on $\mathbb{T}$). We denote respectively by $\boxtimes, \star, \otimes$ and $\triangleright$ the free, classical, boolean and monotone multiplicative convolutions. Then, for all $t \geq 0$,

1. The measure $(\mu \boxtimes \mu) \boxtimes \lambda_t$ coincide with $\mu \boxtimes \mu$.
2. The push-forward of $(\mu \star \mu) \boxtimes \lambda_t$ by the map $z \mapsto z^2$ coincide with the law of $U_{2t}$.
3. The push-forward of $(\mu \otimes \mu) \boxtimes \lambda_t$ by the map $z \mapsto z^3$ coincide with the law of $U_{3t}$.
4. The push-forward of $(\mu \triangleright \mu) \boxtimes \lambda_t$ by the map $z \mapsto z^4$ coincide with the law of $U_{4t}$.

The paper is organized as follows. We start in Section 2 with some preliminaries which gathers useful information about the Herglotz transform of probability measures on the unit circle, and the spectral distribution of the free unitary Brownian motion. Section 3 fixes the basic ideas and notations for the rest of the work presented. Section 4 deals with regularity properties of the spectral measure $\nu_t$ and gives a proof of the Theorem 1.1. Section 5 consists in explicit computations of densities in certain special cases for initial operators.

### 2. Preliminaries

This section gives a concise review about some ideas we will use to prove our main results.

#### 2.1. The Herglotz transform

Let $\mathcal{M}_\mathbb{T}$ denotes the set of probability measures on the unit circle $\mathbb{T}$. The normalized Lebesgue measure on $\mathbb{T}$ will be denoted $m$. The Herglotz transform
$H_\mu$ of a measure $\mu \in \mathcal{M}_T$ is the analytic function in the unit disc $\mathbb{D}$ defined by the formula

$$H_\mu(z) = \int \frac{\zeta + z}{\zeta - z} d\mu(\zeta).$$

This function is related to the moments generating function of the measure $\mu$

$$\psi_\mu(z) = \int \frac{z}{\zeta - z} d\mu(\zeta), \quad z \in \mathbb{D}$$

by the simple formula $H_\mu(z) = 1 + 2\psi_\mu(z)$. Since any distribution on the unit circle is uniquely determined by its moments, we deduce that $H_\mu$ determines uniquely $\mu$. One of its major importance is due to the following result (see e.g. [4, Theorem 1.8.9]):

**Theorem 2.1** (Herglotz). The Herglotz transform sets up a bijection between the analytic functions $H$ on $\mathbb{D}$ with $\Re H \geq 0$ and $H(0) > 0$ and the non-zero measures $\mu \in \mathcal{M}_T$.

For $0 < p < \infty$, let $H^p(\mathbb{D})$ be the space of analytic functions $f$ on $\mathbb{D}$ such that

$$\sup_{0 < r < 1} \int |f(r\zeta)|^p d\zeta < \infty.$$ 

For $p = \infty$, let $H^\infty(\mathbb{D})$ denote the Hardy space consisting of all bounded analytic functions on $\mathbb{D}$ with the sup-norm. Let $L^p(\mathbb{T})$ denote the Lebesgue spaces on the circle $\mathbb{T}$ with respect to the normalized Lebesgue measure. The following result proves the existence of a boundary function for all $f \in H^p(\mathbb{D})$ (see [4, Theorem 1.9.4]).

**Theorem 2.2** ([4]). Let $0 < p \leq \infty$ and $f \in H^p(\mathbb{D})$, the boundary function $\tilde{f}(\zeta)$ exists for $m$-almost all $\zeta$ in $\mathbb{T}$ and belongs to $L^p(\mathbb{T})$. Furthermore, the norms of $f$ in $H^p(\mathbb{D})$ and of $\tilde{f}(\zeta)$ in $L^p(\mathbb{T})$ coincide.

We know (see e.g. [4, Lemma 2.1.11]) that $H_\mu \in H^p(\mathbb{D})$ for all $0 < p < 1$, then $\tilde{H}_\mu(\zeta)$ exists for $m$-almost all $\zeta$ in $\mathbb{T}$. The density of $\mu$ can be recovered then from the boundary values of $\Re H_\mu$ by Fatou’s theorem ([4, Theorem 1.8.6]) since $\Re H_\mu = d\mu/dm$ $m$-a.e. Note that the atoms of $\mu \in \mathcal{M}_T$ can also be recovered from $H_\mu$ by Lebesgues dominated convergence theorem, via

$$\lim_{r \to 1^-} (1 - r)H_\mu(r\zeta) = 2\mu\{\zeta\} \quad \text{for all } \zeta \in \mathbb{T}.$$ 

### 2.2. Spectral distribution of the free unitary Brownian motion.

For $\mu \in \mathcal{M}_T$, let $\psi_\mu$ denote its moments generating function and $\chi_\mu$ the function $\frac{\psi_\mu}{1 + \psi_\mu}$. If $\mu$ has nonzero mean, we denote by $\chi_\mu^{-1}$ the inverse function of $\chi_\mu$ in some neighborhood of zero. In this case the $\Sigma$-transform of $\mu$ is defined by $\Sigma_\mu(z) = \frac{1}{z} \chi_\mu^{-1}(z)$. The distribution $\lambda_t$ of the free unitary Brownian motion was introduced by Biane in [2] as the unique probability measure on $\mathbb{T}$ such that its $\Sigma$-transform is given by

$$\Sigma_{\lambda_t}(z) = \exp\left(\frac{t}{2} \frac{1 + z}{1 - z}\right).$$

This measure $\lambda_t$ is known as the multiplicative analogues of semicircular distributions. Its moments follow from the large-size asymptotic of observables of the free Brownian motion.
(of dimension $d$) $(U^t_{d})_{t \geq 0}$ on the unitary group $U(d)$ as follows.

$$\lim_{d \to \infty} \frac{1}{d} \mathbb{E} \left( \text{tr}[U^t_{d}]^{k} \right) = \int_{\mathbb{T}} \zeta^k d\lambda_t(\zeta), \quad k \geq 0.$$ 

This result was proved independently by Biane and Rains in [2, 23] where these moments are explicitly calculated:

$$\tau(U^t_k) = e^{-kt/2} \sum_{j=0}^{k-1} \frac{(-t)^j}{j!} \binom{k}{j+1} k^{j-1}, \quad k \geq 0. \quad (2.1)$$

The equality (2.1) can be transformed into the PDE

$$\partial_t H + zH \partial_z H = 0, \quad (2.2)$$

with the initial condition $H(0, z) = (1 + z)/(1 - z)$ for the Herglotz transform $H_{\lambda_t}(z)$ (see e.g. the proof of [18, Proposition 3.3]). The measure $\lambda_t$ is described in [3] from the boundary behaviour of the inverse function of $H_{\lambda_t}(z)$ as follows.

**Theorem 2.3** ([3]). For every $t > 0$, $\lambda_t$ has a continuous density $\rho_t$ with respect to the normalized Lebesgue measure on $\mathbb{T}$. Its support is the connected arc $\{e^{i\theta} : |\theta| \leq g(t)\}$ with

$$g(t) := \frac{1}{2} \sqrt{t(4-t)} + \arccos \left(1 - \frac{t}{2}\right)$$

for $t \in [0, 4]$, and the whole circle for $t > 4$. The density $\rho_t$ is determined by $\Re h_t(e^{i\theta})$ where $z = h_t(e^{i\theta})$ is the unique solution (with positive real part) to

$$\frac{z-1}{z+1} e^{2z} = e^{i\theta}.$$ 

3. Reminder and notations

We use here the same symbols as in [15, 16]. To a given pair of projections $P, Q$ in $\mathcal{A}$ that are independent of $(U^t)_{t \geq 0}$ we associate the symmetries $R = 2P - I$ and $S = 2Q - I$. Denote by $\alpha = \tau(R)$ and $\beta = \tau(S)$. We sometimes use the notations $a = |\alpha - \beta|/2$ and $b = |\alpha + \beta|/2$ for simplicity. Keep the symbols $\mu_t$ and $\nu_t$ above. The unit circle is identified with $(-\pi, \pi)$ by $e^{i\theta}$. According to [15, Theorem 4.3], the measure $\nu_t$ is connected to $\mu_t$ by the following formula

$$\nu_t = 2\tilde{\mu}_t - \frac{2 - \alpha - \beta}{2} \delta_{-\pi} - \frac{\alpha + \beta}{2} \delta_0,$$

where

$$\tilde{\mu}_t := \frac{1}{2} \left( \mu_t + (\mu_t|_{(0, \pi)}) \circ j^{-1} \right) \quad (3.1)$$

is the symmetrization on $(-\pi, \pi)$, with the mapping $j : \theta \in (0, \pi) \mapsto -\theta \in (-\pi, 0)$, of the positive measure $\mu_t(d\theta)$ on $[0, \pi]$ obtained from $\mu_t(dx)$ via the variable change $x = \cos^2(\theta/2)$. Equivalently, we obtain the following relationship between the Herglotz transforms $H_{\mu_t}$ and $H_{\nu_t}$ (see [15, Corollary 4.2]).

$$H_{\nu_t}(z) = \frac{z-1}{z+1} H_{\mu_t} \left( \frac{4z}{(1+z)^2} \right) - 2(\alpha + \beta) \frac{z}{z^2 - 1}. \quad (3.2)$$
The function \( H_\nu(z) \), which we shall denote by \( H(t, z) \), is analytic in both variables \( z \in \mathbb{D} \) and \( t > 0 \) (see [5, Theorem 1.4]) and solves the PDE (see [15, Proposition 2.3])

\[
\partial_t H + z H \partial_z H = \frac{2z(\alpha z^2 + 2\beta z + \alpha)(\beta z^2 + 2\alpha z + \beta)}{(1 - z^2)^3}.
\] (3.3)

Let

\[
K(t, z) := \sqrt{H(t, z)^2 - \left(a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z}\right)^2}.
\] (3.4)

The PDE (3.3) is then transformed into

\[
\partial_t K + z H(t, z) \partial_z K = 0.
\]

Note that steady state solution \( K(\infty, z) \) is the constant \( \sqrt{1 - (a + b)^2} \) (see [15, Remark 3.3]). The ordinary differential equations (ODEs for short) of characteristic curve associated with this PDE are as follows.

\[
\begin{cases}
\partial_t \phi_t(z) = \phi_t(z) H(t, \phi_t(z)), & \phi_0(z) = z, \\
\partial_t [K(t, \phi_t(z))] = 0
\end{cases}
\] (3.5)

The second ODE of (3.5) implies that \( K(t, \phi_t(z)) = K(0, z) \), while the first one is nothing else but the radial Loewner ODE (see [20, Theorem 4.14]) which defines a unique family of conformal transformations \( \phi_t \) from some region \( \Omega_t \subset \mathbb{D} \) onto \( \mathbb{D} \) with \( \phi_t(0) = 0 \) and \( \partial_z \phi_t(0) = e^t \). Moreover, from [20, Remark 4.15], \( \phi_t \) is invertible from \( \Omega_t \) onto \( \mathbb{D} \) and it has a continuous extension to \( T \cap \Omega_t \) by [16, Proposition 2.1]. Integrating the first ODE in (3.5), we get

\[
\phi_t(z) = z \exp \left( \int_0^t H(s, \phi_s(z)) ds \right).
\]

Let us define

\[
h_t(r, \theta) = 1 - \int_0^t \frac{1 - |\phi_s(re^{i\theta})|^2}{-\ln r} \int_\mathbb{T} \frac{1}{|\xi - \phi_s(re^{i\theta})|^2} \nu_s(\xi) ds,
\]

so that

\[
\ln |\phi_t(re^{i\theta})| = \ln r + \Re \int_0^t H(s, \phi_s(re^{i\theta})) ds = (\ln r) h_t(r, \theta).
\] (3.6)

Define \( R_t : [-\pi, \pi] \to [0, 1] \) as follows

\[
R_t(\theta) = \sup \{ r \in (0, 1) : h_t(r, \theta) > 0 \},
\]

and let

\[
I_t = \{ \theta \in [-\pi, \pi] : h_t(\theta) < 0 \}
\]

where \( h_t(\theta) = \lim_{r \to 1^+} h_t(r, \theta) \in \mathbb{R} \cup \{-\infty\} \) (see the fact exposed under Lemma 3.2 in [16]).

The next result, giving a description of \( \Omega_t \) and its boundary, was proved in [16, Proposition 3.3].

**Proposition 3.1** ([16]). For any \( t > 0 \), we have

1. \( \Omega_t = \{ re^{i\theta} : h_t(r, e^{i\theta}) > 0 \} \)
2. \( \partial \Omega_t \cap \mathbb{D} = \{ re^{i\theta} : h_t(r, e^{i\theta}) = 0 \text{ and } \theta \in I_t \} \).
(3) \( \partial \Omega_t \cap \mathbb{T} = \{ e^{i\theta} : h_t(r, e^{i\theta}) = 0 \text{ and } \theta \in [-\pi, \pi] \setminus I_t \} \).

In closing, we recall the following result which will be of use later on (see the proof of Theorem 1.1 in [16]).

**Lemma 3.2** ([16]). *For every* \( t > 0 \), *the function* \( K(t, \cdot) \) *has a continuous extension to the unit circle* \( \mathbb{T} \).

### 4. Analysis of spectral distributions of \( RU_t SU_t^* \)

In this section, we shall prove the Theorem 1.1. To this end, we start by giving a description of the spectral measure \( \nu_t \) of \( RU_t SU_t^* \) for any \( t > 0 \), and deriving a formula for its density. We notice that from the asymptotic freeness of \( R \) and \( U_t SU_t^* \), the measure \( \nu_t \) converges weakly as \( t \to \infty \) (see [15, Proposition 2.6]) to

\[

\nu_\infty = a\delta_\pi + b\delta_0 + \frac{\sqrt{-\cos \theta - r_+ \cos \theta - r_-}}{2\pi \lvert \sin \theta \rvert} 1_{(\theta_-, \theta_+), \cup(-\theta_+, -\theta_-)} d\theta \quad (4.1)

\]

with \( r_\pm = -\alpha \beta \pm \sqrt{(1 - \alpha^2)(1 - \beta^2)} \) and \( \theta_\pm = \arccos r_\pm \). The following theorem asserts that an analogous result holds for finite \( t \).

**Theorem 4.1.** *For every* \( t > 0 \), *\( \nu_t - a\delta_\pi - b\delta_0 \) is absolutely continuous with respect to the normalized Lebesgue measure on \( \mathbb{T} = (-\pi, \pi] \). Moreover, its density \( \kappa_t \) *at the point* \( e^{i\theta} \) *is equal to the real part of* \( \sqrt{K(t, e^{i\theta})^2 + (a + b)^2 - 1 - \frac{(\cos \theta - r_+)(\cos \theta - r_-)}{\sin^2 \theta}} \).

**Proof.** Define the function

\[

L(t, z) = \int e^{i\theta} (\nu_t - a\delta_\pi - b\delta_0)(d\theta) = H(t, z) - a \frac{1 - z}{1 + z} - b \frac{1 + z}{1 - z}.

\]

The real part of this function is nothing else but the Poisson integral of the measure \( \nu_t - a\delta_\pi - b\delta_0 \). Using (3.1) and multiplying by the conjugate, we get

\[

L(t, z) = \frac{K(t, z)^2}{\sqrt{K(t, z)^2 + (a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z})^2 + a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z}}} = \frac{(1 - z^2)K(t, z)^2}{\sqrt{[(1 - z^2)K(t, z)]^2 + [a(1 - z)^2 + b(1 + z)^2]^2 + a(1 - z)^2 + b(1 + z)^2}}.

\]

Note that \( K(t, z) \) extends continuously to \( \mathbb{T} \) by Lemma 3.2. The denominator of the above expression does not vanish on the closed unit disc and

\[

z \mapsto (1 - z^2)^2 K(t, z)^2 + [a(1 - z)^2 + b(1 + z)^2]^2 = (1 - z^2)H(t, z)^2
\]

does not take negative values. These together imply that \( L(t, z) \) has a continuous extension on the boundary \( \mathbb{T} \). Hence, by uniqueness of Herglotz representation (see Theorem 2.1), the
measure $\nu_t - a\delta_\pi - b\delta_0$ is absolutely continuous with respect to the Haar measure in $\mathbb{T}$ and its density is given by:

$$\Re \left[ H(t, e^{i\theta}) - a \frac{1 - e^{i\theta}}{1 + e^{i\theta}} - b \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right] = \Re \sqrt{\left[ K(t, e^{i\theta}) \right]^2 + \left[ a \frac{1 - e^{i\theta}}{1 + e^{i\theta}} - b \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right]^2}$$

To complete the proof, we need only show that

$$[a \tan(\theta/2) - b \cot(\theta/2)]^2 = 1 - (a + b)^2 + \frac{(\cos \theta - r_+)(\cos \theta - r_-)}{\sin^2 \theta}$$
or equivalently that

$$(1 - a^2 - b^2) \sin^2 \theta - a^2 \sin^2 \theta \tan^2(\theta/2) - b^2 \sin^2 \theta \cot^2(\theta/2) = -(\cos \theta - r_+)(\cos \theta - r_-).$$

Working from the left-hand side and using the identities

$$\sin^2 \theta = 1 - \cos^2 \theta, \quad \sin^2 \theta \tan^2(\theta/2) = (1 - \cos \theta)^2, \quad \sin^2 \theta \cot^2(\theta/2) = (1 + \cos \theta)^2,$$

we get

$$(1 - a^2 - b^2)(1 - \cos^2 \theta) - a^2(1 - \cos \theta)^2 - b^2(1 + \cos \theta)^2.$$

Rearranging these terms, we obtain

$$-\cos^2 \theta + 2(a^2 - b^2) \cos \theta - 2(a^2 + b^2) + 1.$$

So, by substituting the equalities $\alpha \beta = b^2 - a^2$ and $\alpha^2 + \beta^2 = 2(a^2 + b^2)$, we obtain the required formula:

$$-\cos^2 \theta - 2\alpha \beta \cos \theta + 1 - a^2 - \beta^2 = -(\cos \theta - r_+)(\cos \theta - r_-).$$

\[\Box\]

**Proposition 4.2.** The support of $\nu_t$ is a subset of $\{\phi_t (R_t(\theta) e^{i\theta}) : \theta \in I_t\}$.

**Proof.** By (3.6), we have

$$\int_0^t \Re H(s, \phi_s (R_t(\theta) e^{i\theta})) \, ds = -\ln R_t(\theta)$$

where we used the fact that $\ln |\phi_t (R_t(\theta) e^{i\theta})| = 0$ due to the equality $|\phi_t (R_t(\theta) e^{i\theta})| = 1$. Then, by continuity of $s \mapsto \Re H(s, \phi_s (R_t(\theta) e^{i\theta}))$ on $[0, t]$, we deduce that the assertion $\Re H(t, \phi_t (R_t(\theta) e^{i\theta})) > 0$ yields $R_t(\theta) \neq 1$. Finally, by definition of $R_t(\theta)$ and $I_t$, we have

$$\{\theta : R_t(\theta) \neq 1\} = \{\theta : \exists r_0 \in (0, 1), \ h_t(r_0, e^{i\theta}) = 0\} = \{\theta : h_t(\theta) < 0\} = I_t.$$

\[\Box\]

**Proposition 4.3.** The density $\kappa_t$ of $\nu_t - a\delta_\pi - b\delta_0$ belongs to $L^\infty(\mathbb{T})$. 
Proof. By (3.4), we have
\[
K(t, z)^2 = H(t, z)^2 - \left( a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z} \right)^2.
\]
Then
\[
\Re L(t, z) \leq K(t, z)^2.
\]
Since the function \( K(t, z) \) is analytic in \( D \) and extends continuously to \( T \), it becomes then of Hardy class \( H_\infty(D) \), and hence the density of \( \nu_t = a \delta_\pi + b \delta_0 \) belongs to \( L_\infty(T) \) by [19, Theorem p. 15]. □

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. Using Theorem 4.1 and in [15], we have
\[
\nu_t = a \delta_\pi + b \delta_0 = 2 |\hat{\mu}_t - (1 - \min \{\tau(P), \tau(Q)\}) \delta_\pi - \max \{\tau(P) + \tau(Q) - 1, 0\} \delta_0|.
\]
From Theorem 4.3, this measure is absolutely continuous with respect to the normalized Lebesgue measure \( d\theta/2\pi \) on \( T = (-\pi, \pi] \) with density the function \( \kappa_t \). Hence, by (3.1), we have
\[
(\hat{\mu}_t - (1 - \min \{\tau(P), \tau(Q)\}) \delta_\pi - \max \{\tau(P) + \tau(Q) - 1, 0\} \delta_0)(d\theta) = \kappa_t(\theta) \frac{d\theta}{2\pi}, \quad \theta \in [0, \pi]
\]
and so the desired assertion holds via the variable change \( \theta = 2 \arccos(\sqrt{t}) \). □

5. Special cases

We present here some specializations for which the measure \( \nu_t \) (and hence \( \mu_t \)) is explicitly determined.

5.1. Centered initial operators. i.e. \( \tau(R) = \tau(S) = 0 \) or \( a = b = 0 \). In this case, the PDE (3.3) rewrites
\[
\partial_t H + z H \partial_z H = 0,
\]
and the measure \( \nu_t \) becomes identical to the probability distribution of \( UU_2t \) where \( U \) is a free unitary whose distribution is \( \nu_0 \) (see [18, Proposition 3.3] or [15, Remark 4.7]). Hence, the measure \( \nu_t \) is given by the multiplicative free convolution \( \nu_0 \boxtimes \lambda_{2t} \) studied by Zhong in [25]. The density of this measure and its support are explicitly computed in [25, Theorem 3.8 and Corollary 3.9]. In particular, when \( \nu_0 \) is a Dirac mass at 1 (on the unit circle), the Herglotz transforms \( H(t, z) \) of \( \nu_t \) satisfy the PDE
\[
\partial_t H + z H \partial_z H = 0, \quad H(0, z) = \frac{1 + z}{1 - z}.
\]
Then it follows from uniqueness of solution of (2.2) that \( H(t, z) = H_{\lambda_{2t}}(z) \), and by uniqueness of Herglotz representation, \( \nu_t \) coincide with the law \( \lambda_{2t} \) of \( U_{2t} \). Hence, by Theorem 2.3 the density of \( \nu_t \) is given by the formula \( \kappa_t(\omega) = \rho_{2t}(\omega) \) and the support is the full unit circle for \( t > 2 \) and the set \( \{e^{i\theta} : |\theta| < g(2t)\} \) for \( t \in [0, 2] \).
In the rest of the paper, we illustrate how the family of measure \((\nu_t)_{t \geq 0}\) provides a continuous interpolation between freeness and different type of independence.

5.2. **Free initial operators.** If \(R\) and \(S\) are free, then Proposition 2.5 in \([15]\), implies that

\[
H(0, z) = \sqrt{1 + 4z \left( \frac{b^2}{(1 - z)^2} - \frac{a^2}{(1 + z)^2} \right)}.
\]

Then it follows from (3.4) that

\[
K(0, z) = \sqrt{H(0, z)^2 - \left( a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z} \right)^2} = \sqrt{1 - (a + b)^2}.
\]

But the facts exposed (under the ODEs (3.3)) in section 3 show that \(K(t, z) = K(0, \phi_t^{-1}(z))\) holds for every \(z \in \mathbb{D}\). This implies that \(K(t, z) = \sqrt{1 - (a + b)^2}\) for any \(t \geq 0\), and therefore \(\nu_t\) coincides with the measure \(\nu_\infty\).

5.3. **Classically independent initial operators.** In this case, the measure \(\nu_t\) is considered as a \(t\)-free convolution which interpolates between classical independence and free independence (see \([1]\)). Let \(R, S\) two independent symmetries, from the facts exposed above Lemma 5.4 in \([15]\), we have

\[
H(0, z) = 1 + 2 \sum_{n \geq 1} \tau(R^n)\tau(S^n)z^n = \frac{1 + z^2 + 2z\tau(R)\tau(S)}{1 - z^2}.
\]

In particular, when \(\tau(R) = \tau(S) = 0\), the function \(H(t, z)\) satisfies the PDE

\[
\partial_t H + z\partial_z H = 0, \quad H(0, z) = \frac{1 + z^2}{1 - z^2}
\]

and hence, by (2.2), it coincide with \(H_\chi_t(z^2)\). We retrieve then the result obtained in \([1]\) Theorem 3.6: for any \(t \geq 0\), the push-forward of \(\nu_t\) by the map \(z \mapsto z^2\) coincide with the law of \(U_{4t}\). In particular, the density of \(\nu_t\) is given by \(\kappa_t(\omega) = \rho_{4t}(\omega^2)\) for any \(\omega\) in the unit circle and the support is the full unit circle for \(t > 1\) and the set \(\{e^{i\theta} : |\theta| < g(4t)/2\}\) for \(t \in [0, 1]\).

5.4. **Boolean independent initial operators.** To a given probability measure \(\mu\) on the unit circle, we keep the same notations \(\psi, H, \mu, \chi\) as in section 2. Let \(\mu_1, \mu_2 \in \mathcal{M}_\tau\) and set \(F_\mu(z) = \frac{1}{2} \chi_\mu(z)\). Then the multiplicative boolean convolution \(\mu = \mu_1 \uplus \mu_2\) is uniquely determined by (see \([14]\) or \([13]\) for more details)

\[
F_\mu(z) = F_{\mu_1}(z)F_{\mu_2}(z).
\]

Then, for boolean independent symmetries \(R, S\) with law \(\mu = \delta_t + \delta_{-t}\), we have

\[
\psi_t(z) = \frac{z^2}{1 - z^2}, \quad \mu(z) = z^2, \quad \chi_t(z) = z^2, \quad F_{\mu}(z) = z
\]

and therefore \(F_{\mu \uplus \chi}(z) = F_{\mu}(z)^2 = z^2\). It follows that

\[
\psi_{\mu \uplus \chi}(z) = \frac{z^3}{1 - z^3} \quad \text{and} \quad H_{\mu \uplus \chi}(z) = \frac{1 + z^3}{1 - z^3}.
\]
Hence, by (2.2) the Herglotz transform $H(t, z)$ of $\nu_t$ and $H_{\lambda t}(z^2)$ solve the same PDE with the initial condition $H(0, z) = (1 + z^3)/(1 - z^3)$. By uniqueness, it follows that the push-forward of $\nu_t$ by the map $z \mapsto z^3$ coincide with the law of $U_{\lambda t}$, for any $t \geq 0$. In particular, we have $\kappa_t(\omega) = \rho_{\lambda t}(\omega^3)$ for any $\omega$ in the unit circle and $\nu_t$ is supported in the full unit circle for $t > 2/3$ and the set $\{e^{i\theta} : |\theta| < g(\lambda t)/3\}$ for $t \in [0, 2/3]$.

### 5.5. Monotone independent initial operators.

For $\mu_1, \mu_2 \in M_T$, the multiplicative monotone convolution $\mu = \mu_1 \triangleright \triangleright \mu_2$ is uniquely determined by (see [14] or [12] for more details)

$$\chi_\mu(z) = \chi_{\mu_1}(\chi_{\mu_2}(z)).$$

Here, we shall compute the measure $\nu_t$ for monotone independent symmetries $R, S$ with law $\mu = \frac{\delta_1 + \delta_1}{2}$. As usual, we have

$$\psi_\mu(z) = \frac{z^2}{1 - z^2}, \quad \chi_\mu(z) = z^2,$$

and then $\chi_{\mu \triangleright \triangleright \mu}(z) = \chi_\mu(\chi_\mu(z)) = z^4$. Hence,

$$\psi_{\mu \triangleright \triangleright \mu}(z) = \frac{z^4}{1 - z^4} \quad \text{and} \quad H_{\mu \triangleright \triangleright \mu}(z) = \frac{1 + z^4}{1 - z^4}.$$

It follows that $H(t, z) = H_{\lambda t}(z^4)$ by uniqueness. Thus, the push-forward of $\nu_t$ by the map $z \mapsto z^4$ coincide with the law of $U_{\lambda t}$, for any $t \geq 0$. In particular, we have $\kappa_t(\omega) = \rho_{\lambda t}(\omega^4)$ for any $\omega$ in the unit circle and $\nu_t$ is supported in the full unit circle for $t > 1/2$ and the set $\{e^{i\theta} : |\theta| < g(\lambda t)/4\}$ for $t \in [0, 1/2]$.

Finally, we remind (see the section 5.1) that $\nu_t = \nu_0 \boxplus \lambda_{2t}$ for centered initial operators $R, S$ (i.e. $\tau(R) = \tau(S) = 0$). Hence, the discussions so far can be summarized in the Theorem 1.2.

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