Noncommutative $N = 2 \, p - p'$ System

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Abstract

We analyse several open and mixed sector tree-level amplitudes in $N = 2 \, p - p'$ systems with a constant magnetic $B$ turned on. The 3-point function vanishes on-shell. The 4-point function, in the Seiberg-Witten (SW) low energy limit\cite{2}, is local, indicating the possible topological nature of the theory (in the SW low energy limit) and the possible relation between noncommutative $N = 2 \, p - p'$ system in two complex dimensions and in the SW limit, and (non)commutative $N = 2 \, p' - p'$ system in two real dimensions. We discuss three extreme noncommutativity limits (after having taken the Seiberg-Witten low energy limit) of the mixed 3-point function, and get two kinds of commutative non-associative generalized star products. We make some speculative remarks related to reproducing the above four-point tree level amplitude in the open sector, from a field theory.
1 Introduction

$N = 2$ strings have been studied in the past for a variety of reasons. They are extremely useful from the point of view of studying self-dual gravity and Yang-Mills theories \[3, 4\]. Also, they are thought to be intimately connected to M(atrix) and F theories \[5, 6, 7\]. Noncommutative $N = 2$ strings were first studied in \[8\] which showed several interesting features - (a) appearance of Moyal star product in open-string amplitudes and hence the topological nature of the purely open (and closed) sector(s), (b) construction of abelian noncommutative effective field theory in the purely open sector (equivalently, abelian noncommutative self-dual Yang-Mills in flat space), unlike its commutative $N = 2$ counterpart, (c) appearance of generalized star product in the mixed 3-point function, and (d) vanishing of the mixed 4-point function $A_{oooc}$ in the extreme noncommutativity limit.

There are the following three motivating reasons for studying noncommutative $N = 2$ $p - p'$ system. (i) The work of \[8\] had to do with noncommutative $N = 2$ $p' - p'$ system, i.e., the open strings starting and ending on the same brane (or branes of the same dimensionality with identical boundary conditions at the two ends of the open string) in the presence of external magnetic field. It is hence natural to extend this to the case where the two ends of the open string end on branes of different dimensionality (in the presence of a magnetic background), i.e., noncommutative
$N = 2 \ p - p'(> p)$ system. As the boundary conditions are different at the two ends of the open string, one would expect that the theory is not topological due to the shift in the vacuum energy. In this work, we discuss, among other things, whether it is possible, in any limit, to get a topological noncommutative $N = 2$ theory, at least in the purely open sector\footnote{There is no difference between noncommutative $N = 2 \ p - p'$ and $p' - p'$ theories in the purely closed sector. Further the noncommutative $N = 2 \ p' - p'$ theory is the same as the commutative $N = 2 \ p' - p'$ theory in the closed sector, which is known to be topological \footnote{Hence, the closed sector of noncommutative $N = 2 \ p - p'$ system is topological.}.} (ii) Secondly, the mixed-sector of noncommutative $N = 2 \ p' - p'$ system, involved a generalized star product in the mixed 3-point function and hence the field theory that would reproduce the string amplitude. It is hence natural to ask whether generalized star product(s) of the same or different type(s) appear in the mixed sector of noncommutative $N = 2 \ p - p'$ system. (iii) Finally, the field theory that reproduced the string amplitudes of noncommutative $N = 2 \ p' - p'$ system in \footnote{Hence, the closed sector of noncommutative $N = 2 \ p - p'$ system is topological.} consisted of the open-string metric and the Moyal star product in the purely open sector, and a generalized star product and two linear combinations of the open-string metric and the noncommutativity parameter in the mixed-string sector. It is of importance to construct the field theory that would reproduce the string amplitudes of noncommutative $N = 2 \ p - p'$ system.

We study amplitudes involving either two $p-p'$ open strings and one or two $p' - p'$
open strings, or two $p - p'$ open strings and one closed string, or two $p' - p'$ open strings and one closed string. We summarize our results vis-a-vis the abovementioned three motivating reasons. (i) Interestingly, we find that in the purely open sector of $N = 2 p - p'$ system, the 3-point function vanishes on-shell and the 4-point function, in the Seiberg-Witten (SW) low energy limit, is local. This strongly suggests that the purely open sector of noncommutative $N = 2 p - p'$ system, is topological in the SW low energy limit. (ii) The mixed sector is more non-trivial than in $[3]$. For finite noncommutativity, we get contact-term delta function divergences. For infinite noncommutativity (taking the infinite noncommutativity limit in a specific manner), after having taken the SW low energy limit, we show that the abovementioned divergence can be disregarded, and depending on whether one takes infinite noncommutativity along the common and/or the uncommon directions, one gets commutative non-associative generalized star products of two kinds in the mixed 3-point function. Also, in this limit, one gets an infinite series of local interactions for the mixed 3-point function. (iii) We show that it is not possible to construct a field theory that would reproduce the local 4-point function of the purely open sector, using the 3-point functions evaluated in this paper, in the purely open and mixed sectors. We speculate on possible resolution of this problem.

The paper is organised as follows. In section 2, we evaluate 3-point and 4-point string amplitudes in the purely open sector of the types mentioned above, by reading
off results from the corresponding expressions given for $N = 1$ $p-p'$ systems in the presence of magnetic $B$ in [1], after a suitable identification. We show that the 3-point function vanishes on-shell, and the 4-point function, in the SW low energy limit, is local. In section 3, we evaluate two mixed 3-point functions of the types mentioned above, for finite and infinite noncommutativity, in the SW low energy limit. We explicitly show the appearance of $(\delta(0))^{n=1,2}$-type divergences for finite noncommutativity, which can be taken care of by taking infinite noncommutativity limit suitably after having taken the SW low energy limit. We show the appearance of generalized star products of two kinds in this limit. Section 4 has a summary of and a discussion on results obtained in this work.

2 Purely open sector

In this section, we discuss 3- and 4-point string amplitudes involving vertex operators for two $p-p'$ open strings and one or two $p' - p'$ open strings, the latter in the SW low energy limit.

In the purely open sector, one can read off results from [1] after suitable identification of the polarization vector that figures in the vector vertex operator of [1]. This does not imply that a four-dimensional $N = 2$ theory can be mapped to a ten-dimensional $N = 1$ theory. What is implied and hence what gets used in the
calculations below, is that the open-string $N = 2$ vertex operators, and hence the open-string $N = 2$ open amplitudes, can be obtained from open-string $N = 1$ vertex operators and hence open-string $N = 1$ amplitudes, after the abovementioned identification; the closed string vertex operators that are constructed in this work, were not considered in [1].

The closed-([3]) and open-string ([4]) vertex operators, in the notations of [1] are given by:

$$V_c^{\text{int}} \sim \left( i k \cdot \partial \bar{x} - i k \partial x - \alpha' k \cdot \bar{\psi}_R k \cdot \psi_R \right) \left( i k \cdot \bar{\partial} \bar{x} - i k \bar{\partial} x - \alpha' \bar{k} \cdot \bar{\psi}_L \bar{k} \cdot \psi_L \right) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)} ,$$

$$V_o^{\text{int}} \sim \left( i k \cdot \partial \bar{x} - i \bar{k} \cdot \partial x - \alpha' k \cdot (\bar{\psi}_L + \bar{\psi}_R) - \bar{k} \cdot (\psi_L + \psi_R) \right) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)} ,$$

In $N = 1$ notations,

$$V_o^{\text{int}} = \int d\eta e^{i \left( \frac{1}{2} (k \cdot X + i k \cdot X) - \eta (D_L + D_R)(k \cdot X - \bar{k} \cdot X) \right)} |_{\theta_L = \theta_R} ,$$

which can be identified with

$$\int d\eta e^{i \left( \frac{1}{2} (k \cdot X + \bar{k} \cdot X) + \eta (D_L + D_R)(\zeta \cdot X + \bar{\zeta} \cdot X) \right)} |_{\theta_L = \theta_R}$$

of [1], $X$ being a chiral superfield and $\eta$ a Grassmanian parameter and $\zeta$ being the polarization vector, by setting $\zeta \equiv i k$. In the above, $D_L = \frac{\partial}{\partial \theta_L} + \theta_L \bar{\partial}$ and $D_R = \frac{\partial}{\partial \theta_R} + \theta_R \partial$.

We briefly outline the main idea of [1] when one considers evaluation of amplitudes for $p - p'$ systems in the presence of nonzero $B$. As there is no space-time
supersymmetry and no Ramond-Ramond fields in $N = 2$ theory, the $p(p')$ branes are branes defined in the sense of open-string boundary conditions:

\[
\partial_\sigma X^{1,1} + (B \partial_\tau X)^{1,1} = 0, \quad \text{at } \sigma = 0,
\]

\[
\partial_\tau X^{2,2} = 0, \quad \text{at } \sigma = 0,
\]

\[
\partial_\sigma X^{1,1,2} + (B \partial_\tau X)^{1,1,2} = 0, \quad \text{at } \sigma = \pi,
\]

(4)

where we have complexified the space-time coordinates. The $p$-brane is a brane with $(2,0)$ signature on its world volume and the $p'$-brane is a brane with $(2,2)$ signature on its world volume. As there is no tachyon in the $p - p$ or $p' - p'$ open strings, and as shown below, there are no tachyons in the $p - p'$ open strings, hence, these nonsupersymmetric branes are stable. As the boundary conditions at the ends of the $p - p'$ open strings are different, this implies that the vacuum energy gets shifted relative to the (non)commutative $p' - p'$ theory. One thus has fields in addition to the massless scalar of the $p' - p'$ (non)commutative $N = 2$ strings. As explained in [1], one has to introduce “shift” $\sigma^\pm(\tau)$ and “twist” $\tau^\pm(\tau)$ fields that change the boundary conditions as one would go from $\sigma = 0$ to $\sigma = \pi$. As done in [1], we will fix $\tau_1, \tau_2, \tau_3$ at $0, -\infty, -1$ respectively. We now evaluate several tree-level amplitudes in the open and mixed sectors (in the next section).

(I) $A_{oo0'}$

We first evaluate the 3-point function $A_{oo0'}$ involving two $p - p'$ open strings
denoted by \( o \) each, and one \( p' - p' \) open string denoted by \( o' \). Now, \( A_{oo'o'} \) can be read off from equation (4.25) of [1] and is given by (having used the Jacobian for gauge-fixing the super-Möbius symmetry (See [4])):

\[
A_{oo'o'} = \frac{(\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1)}{(\tau_1 - \tau_2)^2} \int d\theta_3
\]

\[
\langle 0 | : \sigma^+(\tau_1) \tau^+(\tau_1) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)^{11}(\tau_1)} :: \sigma^-(\tau_2) \tau^-(\tau_2) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)^{11}(\tau_2)} :
\]

\[
\times \int d\eta_1 : e^{i(k \cdot \bar{x} + \bar{k} \cdot x)(\eta_3) + ik\eta_1(D_L + D_R)(k \cdot \bar{x} - \bar{k} \cdot x)(\eta_3)} : | 0 \rangle
\]

\[
\sim i\delta(\sum_{a=1}^{3} k_{a1})\delta(\sum_{b=1}^{3} k_{b1}) \sqrt{\frac{\alpha'}{2}} (k_2 - k_1)(\gamma) k_3 e^{C_3(\nu)} \gamma^{\frac{1}{2}} C_1 k_1 k_2
\]

(See (A2) for definition of \( C_3(\nu) \)). The vacuum is a tensor product of the vacuum corresponding to the usual SL(2,\( R \))-invariant vacuum for directions 1, \( \bar{1} \), and the vacuum corresponding to the uncommon directions 2, \( \bar{2} \) that is the analog of the “oscillator vacuum” of [1]. From (3), one sees that \( A_{oo'o'} \) vanishes on-shell. This is analogous to the similar result in [10]. Note that (3) for off-shell scalars, is written entirely in terms of the \( 1 - \bar{1} \) subspace of the target space.

(II) \( A_{oo'o'} \)

Next, we evaluate the 4-point function \( A_{oo'o'} \) involving two \( p - p' \) open string and two \( p' - p' \) open string vertex operators. The four-point function, defined as:

\[
A_{oo'o'} = \frac{(\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1)}{(\tau_1 - \tau_2)^2} \int d\tau_4 d\theta_3 d\theta_4
\]

\[
\langle 0 | : \sigma^+(\tau_1) \tau^+(\tau_1) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)^{11}(\tau_1)} :: \sigma^-(\tau_2) \tau^-(\tau_2) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)^{11}(\tau_2)} :
\]
\begin{align}
\times \int d\eta_1 : e^{i(k\cdot \bar{x} + \bar{k}\cdot x)(\tau_3) + ik\eta_1(D_L + D_R)(k\cdot \bar{x} - \bar{k}\cdot x)(\tau_3) :} \\
\times \int d\eta_2 : e^{i(k\cdot \bar{x} + \bar{k}\cdot x)(\tau_4) + ik\eta_2(D_L + D_R)(k\cdot \bar{x} - \bar{k}\cdot x)(\tau_4) :} |0\rangle,
\end{align}

(6)

can be read off from equation (4.27) of [1] after identification of the polarization vector in the vector vertex operator \( \zeta_\mu \) in [1] with \( ik_\mu \), and is given in (A1) in Appendix A. One can extract the pole structure of (A1), as in [1], by evaluating in the Seiberg-Witten (SW) low energy limit, the integral around \( x = 0 \) (\( \int_0^1 \)) corresponding to the \( t \)-channel process and around \( x = 1 \) (\( \int_1^1 \)) corresponding to the \( s \)-channel exchange. As (almost) massless particle-exchange will dominate the contributions of various states to the above four-point function, we have to find the almost massless poles from the above expression.

The following observations are useful. (a) Using \( L_0 = \alpha' G^{11} |k_1|^2 + \frac{\nu}{2} = 0 \) for \( N = 2 \) theories, one gets \( \alpha' m_\nu^2 = \frac{\nu}{2} (> 0) \sim O(1) \), (b) \( \alpha' \rightarrow \sqrt{\epsilon} \), \( b_{22} \rightarrow \frac{1}{\sqrt{\epsilon}} \), and \( \nu \equiv 1 + O(\sqrt{\epsilon}) \).

The integral \( \int_0^\delta \) gives terms of the type \( \frac{\alpha' O(1)}{\alpha' t + O(1)} \) and \( \frac{O(1)}{\alpha' t + O(1)} \). One sees there are no terms of the type \( \frac{1}{\alpha' t + O(\sqrt{\epsilon})} \), which would have corresponded to an almost massles pole. Hence, in the SW low energy limit, only \( \lim_{\alpha' \rightarrow 0} \frac{O(1)}{\alpha' t + O(1)} \equiv O(1) \) terms survive. These \( O(1) \) terms are local.

As the \( s \)-channel corresponds to exchange of \( p' - p' \) open string that has only a massless scalar in its spectrum, the result can be read off from equation (5.11) of
\[ A_4 \sim \frac{1}{2s} \left\{ (k_2 - k_1)\,(\not p)k_3 - ik_4\,(\not p'\not p)Jk_3 \right\} k_3\,(\not p')k_4 \]

\[ -k_4\,(\not p')k_3 \left\{ (k_2 - k_1)\,(\not p)k_4 - ik_3\,(\not p'\not p')Jk_4 \right\} \]

\[ -2\nu \left( k_3 \times k_4 \right) \frac{1}{2} k_3\,(\not p')k_4 - 2k_3\,(\not p')k_4\nu \left( k_3 \times k_4 \right) \frac{1}{2} \]

\[ + \left\{ -t + m_0^2 + k_3\,(\not p'\not p')k_4 - (1 - 2\nu) \left( k_3 \times k_4 \right) \right\} \frac{1}{2} k_3\,(\not p')k_4 \]

\[ \times \exp \left[ 2\alpha' \left( k_3 \times k_4 \right) \frac{1}{2} \left\{ \gamma + \frac{1}{2} (\psi(\nu) + \psi(1 - \nu)) \right\} \right] \]

\[ + (k_3 \leftrightarrow k_4), \]  \hspace{1cm} (7)

which we see is local. In (7)

\[ s \equiv -(k_3 + k_4)\,(\not p')(k_3 + k_4) = -2k_3\,(\not p')k_4; \]

\[ J \equiv (J_\mu^\rho) \equiv \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \]  \hspace{1cm} (8)

To see if one is able to generate the expression for \( A_{oo'oc'} \) from two 3-point functions, given that \( A_{oo} \) vanishes on-shell, one will have to evaluate mixed 3-point functions corresponding to scattering of a graviton from a \( p - p' \) open string - \( A_{oooc} \), as well as scattering of a graviton from a \( p' - p' \) open string - \( A_{o'oc'c} \). We discuss this in the next section.
3 The mixed sector

We now consider the mixed 3-point functions involving two \( p - p' \) or \( p' - p' \) open strings and a closed string, \( A_{ooc} \) and \( A_{o'o'c} \), respectively. Even though an exact answer can be obtained, we work in the Seiberg-Witten low energy limit followed by infinite noncommutativity limit eventually, as only then can we get an answer that has no contact-term divergences.

(I) \( A_{ooc} \)

One can show that the following vertex operator written in term of \( N = 1 \) notations, reproduces the \( N = 2 \) vertex operator for closed strings.

\[
V_c^{\text{int}} = \int \int d\eta_1 d\eta_2 \int \int d\theta_L d\theta_R \exp \left[ E_I X^I + \bar{E}_I \bar{X}^I \right],
\]

where

\[
E_I \equiv i \sqrt{\frac{\alpha'}{2}} l_I + il_I (\eta_1 D_L + \eta_2 D_R);
\]

\[
\bar{E}_I \equiv i \sqrt{\frac{\alpha'}{2}} l_I \bar{l}_I - il_I (\eta_1 D_L + \eta_2 D_R).
\]

For calculating self-contractions for the third closed vertex operator, that needs to be done to go from the \( SL(2,R) \)-normal ordering \( :; \) to oscillator-normal ordering \( :: \);, one has to evaluate: \( \exp \left( E_I \bar{E}_I G^{\text{sub} \ II} \right) \), where \( G^{\text{sub}} \) is the subtracted 2-point function of \( \mathbb{1} \) (also defined in appendix B). This calculation of self-contraction is done in appendix B. As the closed string metric \( g^{II} \), the open-string metric \( G^{II} \) and
the noncommutativity parameter $\frac{\Theta^{IJ}}{\pi \alpha'}$ are given by $\frac{\delta^{IJ}}{\epsilon}$, $\frac{2\delta^{IJ}}{\epsilon(1+\beta^I)}$ and $\frac{\delta^{IJ}}{\epsilon(1+\beta^I)}$, one sees the explicit appearance of all three, in particular the closed-string metric, in (B1).

In the superspace formalism, for calculating $A_{oooc}$, one has to evaluate:

$$\int d(Rez_3) \int d\theta_L d\theta_R \int d\theta_1 d\theta_2 \int d\eta_1 d\eta_2$$

$$(0|: \sigma^+ (\tau_1) \tau^+ (\tau_1) \theta_1 E \exp[i k_1 X^1 + i \bar{k}_1 \bar{X}^1] (\tau_1, \theta_1) :: \sigma^- (\tau_2) \tau^- (\tau_2) \theta_2 E \exp[i q_1 X^1 + i \bar{q}_1 \bar{X}^1] (\tau_2, \theta_2) :$$

$$\times : E \exp[E_1 X^1 + \bar{E}_1 \bar{X}^1] (z_3, \bar{z}_3; \theta_L, \theta_R) : |0\rangle.$$  \hspace{1cm} (11)

Then, using $G_{\text{sub}}$ to go from the SL(2,R)-invariant vacuum $|0\rangle$ to the oscillator vacuum $|\sigma, s\rangle$, in the $2, \bar{2}$-subspace for the closed-string vertex operator, the above gives:

$$\int d(Rez_3) \int d\theta_L d\theta_R \int d\theta_1 d\theta_2 \int d\eta_1 d\eta_2$$

$$(0|: \theta_1 E \exp[i k_1 X^1 + i \bar{k}_1 \bar{X}^1] (\tau_1, \theta_1) :: \theta_2 E \exp[i q_1 X^1 + i \bar{q}_1 \bar{X}^1] (\tau_2, \theta_2) :$$

$$\times : E \exp[E_1 X^1 + \bar{E}_1 \bar{X}^1] (z_3, \bar{z}_3; \theta_L, \theta_R) : |0\rangle$$

$$\times E \exp[E_2 X^2 + \bar{E}_2 \bar{X}^2] (z_3, \bar{z}_3; \theta_L, \theta_R) \times \langle \sigma, s | : E \exp[E_2 X^2 + \bar{E}_2 \bar{X}^2] (z_3, \bar{z}_3; \theta_L, \theta_R) :: |\sigma, s\rangle,$$  \hspace{1cm} (12)

which is the same as:

$$\int d(Rez_3) \int d\theta_L d\theta_R \int d\theta_1 d\theta_2 \int d\eta_1 d\eta_2$$

$$(0|: \theta_1 E \exp[i k_1 X^1 + i \bar{k}_1 \bar{X}^1] (\tau_1, \theta_1) :: \theta_2 E \exp[i q_1 X^1 + i \bar{q}_1 \bar{X}^1] (\tau_2, \theta_2) :$$

$$\times E \exp[E_2 X^2 + \bar{E}_2 \bar{X}^2] (z_3, \bar{z}_3; \theta_L, \theta_R) \times |\sigma, s\rangle.$$  \hspace{1cm} (13)
\[ \times : \text{Exp}\left[E_1 X^1 + \bar{E}_1 \bar{X}^1\right](z_3, \bar{z}_3; \theta_L, \theta_R) : |0\rangle \]

\[ \times \text{Exp}\left[E_I \bar{E}_I G_{\text{sub}}^{II}\right](z_3, \bar{z}_3; \theta_L, \theta_R). \quad (13) \]

Equation (13) of the form

\[
\int d(Rez_3) \int d\theta_L d\theta_R \int d\theta_1 d\theta_2 \int d\eta_1 d\eta_2 \rho e^{\eta_1 \theta_L \alpha_1 + \eta_2 \theta_R \beta_1} \text{Exp}\left[E_I \bar{E}_I G_{\text{sub}}^{II}\right] = \int d(Rez_3) \rho e^{\gamma_1 (\delta_2 + \gamma_2 \delta_1 - (\alpha_1 \frac{\Theta^{11}_{11}}{\pi \alpha'} + \alpha_2 \frac{\delta(0)}{\epsilon} + \alpha_3 \frac{\Theta^{22}_{22}}{\pi \alpha'}) + \alpha_4 (\beta_1 \frac{\Theta^{11}_{11}}{\pi \alpha'} + \beta_2 \frac{\delta(0)}{\epsilon} + \beta_3 \frac{\Theta^{22}_{22}}{\pi \alpha'}) + \beta_4 (\omega_1 \frac{\Theta^{22}_{22}}{\pi \alpha'} + \omega_2)}),
\]

(14)

where \( \rho, \gamma_i, \alpha_i, \beta_i, \omega_i \) are defined in (B5). We will now work in the infinite noncommutativity limit in which one can drop the \( \delta(0) \)-dependent terms that appear as additive terms in (14), relative to the \( \alpha_i \frac{\Theta^{11}_{11}}{\pi \alpha'} \) terms in (14). This can be seen more explicitly by choosing the representation \( \delta(x) = \lim_{\epsilon' \to 0} \frac{\epsilon'}{x^2 + \epsilon'^2} \). Thus, for \( \epsilon' \sim \epsilon, \frac{\delta(0)}{\epsilon} \sim \frac{1}{\epsilon} \). Hence, in the infinite noncommutativity limit, if one assumes that \( \Theta^{11} \) and/or \( \Theta^{22} \to \infty \) as \( \frac{1}{\epsilon} \), this justifies dropping the \( (\delta(0))^n, n = 1, 2 \) terms relative to the \( \alpha_i \frac{\Theta^{11}_{11}}{\pi \alpha'} \) terms. The terms proportional to \( \delta'(0) \) can be dropped as \( \delta'(0) \equiv \int_{-\infty}^{\infty} \delta'(x)\delta(x) = 0 \) as \( \delta(x) \) is an even function and \( \delta'(x) \) is an odd function. Also, as the step function \( \Theta(x) \) is related to the sign function \( \epsilon(x) \) by the relation \( \frac{1}{2} \epsilon(x) = \Theta(x) - \frac{1}{2} \), hence using the integral representation for \( \epsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha (\frac{\epsilon \delta_{\alpha}}{\alpha - i\epsilon} - \frac{\epsilon \delta_{-\alpha}}{\alpha + i\epsilon}) \), we see that \( \lim_{x \to 0} \epsilon(x) = 0 \).
implying \( \lim_{x \to 0} \Theta(x) = \frac{1}{2} \). Now, in (14) and (B5), we have assumed that one has fixed \( \text{Im} z_3 \), and hence one requires to integrate only over \( \text{Re} z_3 \).

The sum in equation (B5) for the expression for \( \gamma_1 \) should be evaluated as follows. Consider the \( \cos \) -dependent and independent terms separately. The \( \cos \)-independent terms can be written as \(-2\gamma - (\Psi(1 - \nu) + \Psi(\nu))\), where \( \gamma \equiv \text{Euler number} \), and \( \Psi \equiv \Gamma'/\Gamma \). Then use identity (A.5) of Itoyama’s appendix, and one sees that at \( \nu = 1 \), the above is proportional to \( \alpha'b_{22} \to \beta \). \( \cos \)-dependent terms can be evaluated by writing \( 1 - \nu = \sqrt{\epsilon} \). One will get \( 1/\sqrt{\epsilon} - \ln \sin^2 \epsilon_1 \). We demand that \( \epsilon, \epsilon_1 \to 0 \) in such a way that:

\[
\ln \sin^2 \epsilon_1 + \frac{1}{\sqrt{\epsilon}} = 0, 
\tag{15}
\]

which is reasonable as \( \ln \sin^2 \epsilon_1 \) approaches \( -\infty \), and \( 1/\sqrt{\epsilon} \) approaches \( +\infty \). Hence, the \( \alpha' \ln [\sin^2 (\epsilon_1)] \)-type singularity is replaced by \( \alpha'/\sqrt{\epsilon} \sim \beta \). Now comes the point of evaluating the sum in \( \gamma_1 \). Take the Seiberg-Witten low energy limit and hence take \( 1 - \nu = \sqrt{\epsilon} \). The sum becomes:

\[
2 \sum_{m=1}^{\infty} \frac{\cos[m\sigma_3]}{m} = -\ln(\sin^2 \frac{\sigma_3}{2}). \tag{16}
\]

For the evaluation of the integral, it is more convenient to evaluate the integral using \( \text{Re} z_3 \). One has to evaluate:

\[
\int_{-\infty}^{\infty} d\text{Re}z_3 (\text{Re}z_3 + i\text{Im}z_3)^{\lambda_4} (\text{Re}z_3 - i\text{Im}z_3)^{\lambda_5} e^{\lambda_1 \ln(1+\cos \sigma_3)} (1, \lambda_3 e^{-2\tau_3}). \tag{17}
\]
The above integral is evaluated in Appendix B. We now consider three cases for infinite noncommutativity.

(a) $\Theta^{1,1} \rightarrow \infty$, $\Theta^{22} \equiv \text{finite}$

From (B5) and (C3), one sees that in the Seiberg-Witten low energy limit, $\lambda_1 \sim O(1/\sqrt{\epsilon})$, $\lambda_4 - \lambda_5 \sim 2\lambda_4 \sim -2\lambda_5 \equiv O(\Theta^{11}/\sqrt{\epsilon}) \equiv O(1/\epsilon) \gg \lambda_1$. Then, using

\[
\begin{align*}
(a) \text{Appell} F_1(a, b_1, b_2; a; x, y) &= \text{Appell} F_1(1, b_1, b_2; 1; x, y); \\
(b) F_1(a, b; a; x) &= F_1(1, b; 1; x); \\
(c) \text{Appell} F_1(1, a, b; 1; -i, i) &= (1 - i)^{-b}(1 + i)^{-a}; \\
(d) F_1(1, a; 1; -1) &= 2^{-a},
\end{align*}
\]

one sees that one gets:

\[
\lambda_3 \Gamma(2\lambda_1) e^{\frac{-i\pi(\lambda_4 - \lambda_5)}{2\lambda_4} + i\pi\lambda_5} = \frac{1}{-4\lambda_4} + (k \leftrightarrow q) .
\]

(19)

Hence, one gets a factor of $e^{\frac{-i(\lambda_4 - \lambda_5)\pi}{2\lambda_4} + i\pi\lambda_5} \Theta^{11} \rightarrow \infty \frac{e^{-i\pi(\lambda_4 - \lambda_5)}}{(\lambda_4 - \lambda_5)} = e^{i\Theta^{11}(k_1\bar{q}_1 - \bar{k}_1q_1)}$, which gives a generalized star product after Bose symmetrization in the noncommutative $p' - p'$ $A_{oooc}$ amplitude.

In the $\Theta^{11} \rightarrow \infty$ limit, and setting $(Imz_3)_0 = 1$, (19) simplifies to give:

\[
A_{oooc}(\epsilon \rightarrow 0, \Theta^{11} \rightarrow \infty) \sim
\frac{2}{\alpha'} e^{-\frac{2\alpha|q_1|^2}{\epsilon}} \left( \Theta^{11}(k_1\bar{q}_1 - \bar{k}_1q_1) \right)^2 \frac{2\sin[\Theta^{11}(k_1\bar{q}_1 - \bar{k}_1q_1)]}{\Theta^{11}(k_1\bar{q}_1 - \bar{k}_1q_1)} \Gamma\left( \frac{2|q_1|^2}{\epsilon} \right).
\]

(20)
The above result consists of product of four factors - a gaussian damping factor analogous to the open sector but here it depends on the closed string metric, the factor “$c_L$” where (in the extreme noncommutativity limit), “$c_L = c_R = \Theta_{\bar{1}1}(k_1q_{\bar{1}} - \bar{k}_1q_1)$ (See [8]), a generalized star product, and a kinematic factor $\Gamma(\frac{2|l_2|^2\alpha'}{\epsilon})$. Hence, $A_{oooc}$ also involves the closed-string metric in addition to the open-string metric. The expression (20) corresponds to an infinite series of local interactions.

(b) $\Theta_{1\bar{1}}, \Theta_{2\bar{2}} \to \infty$

In this limit, one has to evaluate:

$$\int_{-\infty}^{\infty} dRez_3 e^{\lambda_1 \ln(1+\cos\sigma_3)}(Rez_3 + iImz_3)^{\lambda_4}(Rez_3 - iImz_3)^{\lambda_5} \left( \alpha_1 \beta_1 + \alpha_1 \beta_3 + \beta_1 \alpha_3 + \alpha_3 \beta_3 \right).$$

(21)

Thus, one gets:

$$A_{oooc}(\epsilon \to 0, \Theta_{1\bar{1}}, \Theta_{2\bar{2}} \to \infty) \sim e^{-2\epsilon |l_2|^2 \alpha'} \frac{\Gamma(\frac{2|l_2|^2\alpha'}{\epsilon}) \sin[\Theta_{1\bar{1}}(k_1q_{\bar{1}} - \bar{k}_1q_1)]}{\Theta_{1\bar{1}}(k_1q_{\bar{1}} - \bar{k}_1q_1)} \left[ -2\alpha' \left( \Theta_{1\bar{1}}(k_1q_{\bar{1}} - \bar{k}_1q_1) \right)^2 + \left( \frac{\alpha' 4i|l_2|^2 \Theta_{2\bar{2}}}{2 Imz_3 \pi \alpha'} \right)^2 \right].$$

(22)

Again, one sees the appearance of a generalized product in the amplitude. As in (20), (22) corresponds to an infinite series of local terms.

(c) $\Theta_{2\bar{2}} \to \infty, \Theta_{1\bar{1}} \equiv \text{finite}$
One has to evaluate in this limit:
\[
\int_{-\infty}^{\infty} dRez_3 \alpha_3 \beta_3 e^{\lambda_1 \ln(1 + \cos \sigma_3)} (Rez_3 + iImz_3)^{\lambda_4} (Rez_3 - iImz_3)^{\lambda_5}.
\] (23)

Hence, one gets:
\[
A(\epsilon \to 0, \Theta^{22} \to \infty) \sim 
\frac{\left(\sqrt{\frac{\alpha'}{2}} \frac{|l_2|\alpha'}{|m_2| \pi \alpha'}\right)^2 e^{-2\pi |l_2|^2 \alpha'}}{\left(\frac{|l_2|^2 \alpha'}{e}\right)} 2^{\alpha' G^{11}(k_1 \bar{l}_1 + \bar{k}_1 l_1)} \left(\Gamma(-1 - \alpha' G^{11}(k_1 \bar{l}_1 + \bar{k}_1 l_1))\cos\frac{3\Theta^{11}(k_1 \bar{q}_1 - \bar{k}_1 q_1)}{2}\right)
+ \frac{\Gamma(2|l_2|^2 \alpha')}{\Gamma(\frac{|l_2|^2 \alpha'}{e})} 2F1\left(-\frac{|l_2|^2 \alpha'}{e}, 2 + \alpha' G^{11}(k_1 \bar{l}_1 + \bar{k}_1 l_1), -1\right) \cos[\Theta^{11}(k_1 \bar{q}_1 - \bar{k}_1 q_1)]
\] (24)

One gets another generalized star product different from the one that appears in cases (a) and (b). By writing \(\epsilon = \epsilon_1 + i\epsilon_2\), and then using Stirling’s asymptotic expression for the gamma function, and also using \(\lim_{x \to \infty} 2F1(x, a; -x; -1) = 2F1(1, a; 1; 1)\), one sees that (24) also corresponds to an infinite series of local terms.

To calculate \(A_{oo'o'}\), one needs to calculate \(A_{o'o'c}\) in addition to \(A_{ooc}\).

(II) \(A_{o'o'c}\)

As the “tachyon” vertex operators for the \(p' - p'\) open strings are \(e^{i(k \cdot x + \bar{k} \cdot x)}\), where all four space-time coordinates are included, hence, the vacuum relevant to this amplitude is only the SL(2,R)-invariant vacuum. Hence, this amplitude is the
same as the one calculated in [8]. The result is:

\[ A_{o'o'c} = c_{LR} \frac{\sin(kq)}{(kq)}, \quad (25) \]

where \( c_{L,R} \) are as defined in [8].

It does not seem plausible to be able to obtain the local \( t \)- and \( s \)-channel results of \( A_{oo'o'} \) for finite noncommutativity or in the extreme noncommutativity limit using \( A_{ooc} \) of (21) or (22) or (24) and \( A_{o'o'c} \) of (25). Some of the possible field theory graphs in the \( t \) and \( s \) channels are drawn in Fig.1. As \( A_{ooc} \) vanishes, graphs (a) and (e) in Fig.1 vanish. One requires to evaluate a four-point function with an internal \( p - p' \) open-string exchange. The non-local pieces of all allowed field theory graphs should cancel and the graphs should possibly give (though not necessary) the local part of \( A_{oo'o'} \) as obtained from string theory. One has to remember to include the contribution from \( \int_{1-\delta}^1 \) to get the complete form of the local expression. Alternatively, it is possible that loop graphs in the field theory are required to be evaluated for reproducing a tree-level string amplitude.

4 Summary and discussion

To summarize, we have evaluated (by mapping the \( N = 2 \) vertex operators and hence the amplitudes to their \( N = 1 \) counterparts) the 3-point and 4-point amplitudes involving two \( p - p' \) open strings and one or two \( p' - p' \) open strings. While
the former was found to vanish on-shell (suggesting the possibility that the noncommutative $N = 2 \ p - p'$ system in two complex dimensions in SW low energy limit$^3$ is related to a (non)commutative $N = 2 \ p' - p'$ system in two real dimensions), the latter in the Seiberg-Witten low-energy limit, gave a local result for the $t$-channel and $s$-channel processes indicating the possible topological nature (in the SW low energy limit) of the theory. We also evaluate the mixed 3-point function involving a closed string and two $p - p'$ or $p' - p'$ open strings. While for finite noncommutativity, for the former, one obtains $\delta(0)$- and $(\delta(0))^2$-type singularities (tree-level amplitudes in light cone $N = 1$ string field theory are known to be singular which hence require local divergent contact interactions as counter terms (to give finite results)$^12$ whose existence was argued earlier from the super-Poincare algebra$^13$) while evaluating self-contractions for the closed-string vertex operator, these singularities can be avoided by taking the infinite noncommutativity limit in a suitable way. We consider three infinite noncommutativity limits (alongwith the Seiberg-Witten low energy limit): (a) $\Theta^{11} \to \infty$ and $\Theta^{23} \equiv$ finite, (b) $\Theta^{11}, \Theta^{22} \to \infty$, and (c) $\Theta^{22} \to \infty$ and $\Theta^{11} \equiv$ finite. Cases (a) (intriguing similar to the $A_{oo}$ result of$^8$) and (b) give the commutative non-associative generalized star products involving $\frac{\sin(\theta \Theta_0)}{(\theta \Theta_0)}$.

$^3$The SW limit needs to be taken as even though the 3-point function vanishes without having to take this limit, the 4-point function is local only after having taken the SW low energy limit. The 4-point function of (non)commutative $N = 2 \ p' - p'$ in two real dimensions, is local$^10$. 

19
and case (c) gives another commutative non-associative generalized star product involving $\cos(\partial \Theta \partial)$. All three cases involve a gaussian damping factor, similar to the $N = 1$ calculations in [1, 11]. The mixed 3-point function involving two $p' - p'$ open strings is identical to the corresponding 3-point function calculated for noncommutative $p' - p'$ system. It will be interesting to work on the field theory that would reproduce the local 4-point function $A_{ooo' o'}$.

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**Appendix A**

We give below the result for the 4-point function $A_{ooo' o'}$ obtained after the identification of $\zeta = ik$ in the corresponding result in [1].

\[
A_{ooo' o'} \sim -\delta(\sum_{a=1}^{4} k_{a1}) \delta(\sum_{b=1}^{4} k_{b1}) \int_{0}^{1} dx x^{-\alpha' t + \alpha' m^{2}} (1 - x)^{2\alpha' k_{3} \hat{p}} k_{4} \exp \left( C_{3}(\nu) + C_{4}(\nu) + (NC) \right) \times \exp \left[ -\alpha' \left\{ \left( k_{3} \odot_{(p,p')} k_{4} + k_{3} \times_{(p,p')} k_{4} \right) \mathcal{H} \left( \nu; \frac{1}{x} \right) + \left( k_{3} \odot_{(p,p')} k_{4} - k_{3} \times_{(p,p')} k_{4} \right) \mathcal{H} \left( \nu; x \right) \right\} \right] \times \left[ \frac{1}{(1 - x)^{2}} k_{3} \hat{p} k_{4} \left( 1 - 2\alpha' k_{3} \hat{p} k_{4} \right) \right]
\]
\[
\frac{\alpha'}{2} \frac{1}{x} \left\{ \left[ (k_2 - k_1) \right]_{(p)k_3} - k_4 \right\} \left\{ \left[ (k_2 - k_1) \right]_{(p)k_4} + k_3 \right\} \\
+ \frac{\alpha'}{1 - x} \left\{ \left[ (k_2 - k_1) \right]_{(p)k_3} k_3 \right\} \left[ k_4 \right]_{(p)k_3} - k_4 \left[ (k_2 - k_1) \right]_{(p)k_4} \right\} \\
+ \frac{x^{-\nu}}{(1 - x)^2} \left\{ -\alpha' \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
+ \left( \frac{1 - \nu}{2} - \alpha' k_3 \right) \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
+ \frac{x^\nu}{(1 - x)^2} \left\{ -\alpha' \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
+ \left( \frac{1 - \nu}{2} - \alpha' k_3 \right) \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
+ \frac{\nu}{2} \frac{x^{\nu+1}}{(1 - x)^2} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
- \frac{\alpha'}{2} \frac{x^{-2\nu}}{(1 - x)^2} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
- \frac{x^\nu}{2} \frac{1}{(1 - x)^2} \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
- \frac{\alpha'}{2} \frac{x^{-2\nu}}{1 - x} \left( k_4 \right)_{(p,p')k_3} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
- \frac{\alpha'}{2} \frac{x^{2\nu-1}}{1 - x} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
+ \frac{\alpha'}{2} \frac{x^{\nu-1}}{1 - x} \left\{ \left( (k_2 - k_1) \right)_{(p)k_3} - k_4 \right\} \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
- \left( \left( (k_2 - k_1) \right)_{(p)k_4} + k_3 \right) \left( k_4 \right)_{(p,p')k_3} + k_4 \times k_3 \right\} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
+ \frac{\alpha'}{2} \frac{x^{-\nu}}{1 - x} \left\{ \left( (k_2 - k_1) \right)_{(p)k_3} + k_4 \right\} \left( k_3 \right)_{(p,p')k_4} + k_3 \times k_4 \right\} \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \\
- \left( \left( (k_2 - k_1) \right)_{(p)k_4} - k_3 \right) \left( k_4 \right)_{(p,p')k_3} + k_4 \times k_3 \right\} \left( k_3 \right)_{(p,p')k_4} - k_3 \times k_4 \right\} \left( k_3 \right)_{(p)k_4} \right) \right] . \quad (A1)
where

\[(NC) \equiv \sum_{1 \leq a < a' \leq N} i^2 (x_a - x_{a'}) \sum_{i,j=1}^p \theta^{ij} k_{ai} k_{a'i};\]

\[k_{(p,p')} \equiv 2G^{\mathbb{R}} k_{2\mathbb{R}};\]

\[\left( k \circ (p,p') \right)_2 \equiv G^{\mathbb{R}} (k_{2\mathbb{R}} + k_{2\mathbb{R}'}) , \quad \left( k \times (p,p') \right)_2 \equiv G^{\mathbb{R}} (k_{2\mathbb{R}} - k_{2\mathbb{R}'}) ,\]

\[C_\alpha (\nu) \equiv 2\alpha' G^{22} |k|^2 \{ \gamma + \frac{1}{2} (\psi (\nu) + \psi (1 - \nu)) \};\]

\[H(\nu; z) = \begin{cases} \mathcal{F} \left( 1 - \nu_1; \frac{1}{z} \right) - \frac{\pi}{2} b_I = \sum_{n=0}^{\infty} \frac{z^{-n-1+\nu}}{n+1-\nu} - \frac{\pi}{2} b_I & \text{for } |z| > 1 \\ \mathcal{F} (\nu_1; z) + \frac{\pi}{2} b_I = \sum_{n=0}^{\infty} \frac{z^{n+\nu}}{n+\nu} + \frac{\pi}{2} b_I & \text{for } |z| < 1 \end{cases} \] (A2)

\[\mathcal{F} \text{ defined in (B3) again.}\]

### Appendix B

In this appendix, we discuss the self-contraction calculation for the closed-string vertex operator. For this purpose, one starts with:

\[\text{Exp} \left( E_I \bar{E}_I G^{\text{sub II}} \right) = \lim_{3 \to 3'} \text{Exp} \left[ \left( -\frac{\alpha'}{2} l_I l_I - \sqrt{\frac{\alpha'}{2}} l_I l_I \left[ \eta_1 (D_L - D'_L) + \eta_2 (D_R - D'_R) \right] + l_I l_I \eta_1 \eta_2 (D_L D'_R + D_R D'_L) \right) \right] \]

\[= \text{Exp} \left( \gamma_1 + \theta_L \theta_R \gamma_2 + \eta_1 \theta_L (\alpha_2 - \frac{\delta(0)}{\epsilon} + \alpha_3 \Theta_{\alpha}^{22} + \alpha_4) + \eta_2 \theta_R \alpha_5 + \eta_2 \theta_R (\omega_1 \Theta_{\alpha}^{22} + \omega_2) + \eta_2 \theta_R (\beta_2 - \frac{\delta(0)}{\epsilon} + \beta_3 \Theta_{\alpha}^{22} + \beta_4) + \eta_1 \eta_2 \delta_1 + \eta_1 \eta_2 \theta_L \theta_R \delta_2 \right), \] (B1)
where $\rho, \alpha_i, \beta_j, \omega_k, \gamma_l, \delta_m$ are defined in (B3), and the $G^{\text{sub}}$ is the subtracted Green’s function of [1].

Now, $G^{\text{sub}} \equiv G - G$, and using notations of [1], the following results (of Itoya et al) are used in arriving at (B5):

\[
G^{I\bar{J}}(z_1, \bar{z}_1 | z_2, \bar{z}_2) \equiv \langle \sigma, s | R^{I}(z_1, \bar{z}_1)X^{\bar{J}}(z_2, \bar{z}_2) | \sigma, s \rangle
\]

\[
= \Theta(|z_1| - |z_2|) \frac{2 \delta^{I\bar{J}}}{\varepsilon} \left[ F \left( 1 - \nu_I; \frac{z_2 + \theta_1 \theta_2}{z_1} \right) + F \left( 1 - \nu_I; \frac{\bar{z}_2 + \bar{\theta}_1 \bar{\theta}_2}{z_1} \right) - F \left( 1 - \nu_I; \frac{\bar{z}_2 + \bar{\theta}_1 \bar{\theta}_2}{z_1} \right) - F \left( 1 - \nu_I; \frac{z_2 + \theta_1 \theta_2}{z_1} \right) \right] + \Theta(|z_2| - |z_1|) \frac{2 \delta^{I\bar{J}}}{\varepsilon} \left[ F \left( \nu_I; \frac{z_1}{z_2 + \theta_1 \theta_2} \right) + F \left( \nu_I; \frac{\bar{z}_1}{\bar{z}_2 + \bar{\theta}_1 \bar{\theta}_2} \right) - F \left( \nu_I; \frac{\bar{z}_1}{\bar{z}_2 + \bar{\theta}_1 \bar{\theta}_2} \right) - F \left( \nu_I; \frac{z_1}{z_2 + \theta_1 \theta_2} \right) \right],
\]

where $\Theta(x)$ is the step function, $F(\nu; z)$ is defined as

\[
F(\nu; z) = \frac{z^{\nu}}{\nu} {}_2F_1(1, \nu; 1 + \nu; z) = \sum_{n=0}^{\infty} \frac{1}{n + \nu} z^{n+\nu}, \tag{B3}
\]

and ${}_2F_1(a, b; c; z)$ is the hypergeometric function, and

\[
G^{IJ}(z_1, \bar{z}_1 | z_2, \bar{z}_2) \equiv \langle 0 | R^{I}(z_1, \bar{z}_1)X^{\bar{J}}(z_2, \bar{z}_2) | 0 \rangle
\]

\[
= -g^{IJ} \ln(z_1 - z_2 - \theta_1 \theta_2)(\bar{z}_1 - \bar{z}_2 - \bar{\theta}_1 \bar{\theta}_2) + (g^{IJ} - 2G^{IJ}) \ln(z_1 - z_2 - \theta_1 \theta_2)(\bar{z}_1 - \bar{z}_2 - \bar{\theta}_1 \bar{\theta}_2)
\]

\[
- 2 \frac{g^{IJ}}{2\pi \alpha^I} \ln \left( \frac{z_1 - \bar{z}_2 - \theta_1 \theta_2}{\bar{z}_1 - z_2 - \bar{\theta}_1 \theta_2} \right). \tag{B4}
\]

We give below the definitions of $\alpha_i, \beta_j, \omega_k, \gamma_l, \delta_m$ that figure in the self-contractions
in \( A_{ooc} \).

\[
\begin{align*}
\gamma_1 & \equiv -\frac{l_2 \delta_{22} \bar{l}_2 \alpha'}{2 \epsilon} \left[ 2 \sum_{m=0}^{\infty} \left( \frac{1 - \cos(m+1-\nu)\sigma_3}{(m+1-\nu)} + \frac{1 - \cos(m+\nu)\sigma_3}{(m+\nu)} \right) \right] \\
-2 \ln \sin^2 \sigma_3 + 2 \ln \sin^2 \epsilon_1 + \frac{4}{(1 + B_{22}^2)} \ln(4 \text{Im} z_3^2) - \frac{4 B_{22} \pi}{(1 + B_{22}^2)} \\
\gamma_2 & \equiv -\frac{l_2 \delta_{22} \bar{l}_2}{\epsilon} \left( \frac{2i}{\text{Im} z_3} \left[ \cos(2 \nu \sigma_3) - \cos(2(1 - \nu)\sigma_3) - \frac{(1 - B_{22}^2)}{(1 + B_{22}^2)} \right] \right); \\
\delta_1 & \equiv \frac{4 l_2 \delta_{22} \bar{l}_2 \sin(2 \nu \phi) - \sin[2(1 - \nu)\phi] + \sin(2 \phi \nu)}{\text{Im} z_3}; \\
\rho & \equiv (-\text{Re} z_3 - i \text{Im} z_3) \left[ \sqrt{\alpha'} (k_1 \bar{l}_1 (G^{11} - \frac{\Theta_{11}^{11}}{\pi \alpha'}) + \bar{k}_1 l_1 (G^{11} + \frac{\Theta_{11}^{11}}{\pi \alpha'}) \right] (-\text{Re} z_3 + i \text{Im} z_3) \left[ \sqrt{\alpha'} (k_1 \bar{l}_1 (G^{11} + \frac{\Theta_{11}^{11}}{\pi \alpha'}) + \bar{k}_1 l_1 (G^{11} - \frac{\Theta_{11}^{11}}{\pi \alpha'}) \right]; \\
\alpha_1 & \equiv -\sqrt{2 \alpha'} \left( -G^{11} + \frac{\Theta_{11}^{11}}{\pi \alpha'} \right) k_1 \bar{l}_1 - (G^{11} + \frac{\Theta_{11}^{11}}{\pi \alpha'}) \bar{k}_1 l_1 \frac{1}{-\text{Re} z_3 - i \text{Im} z_3}; \\
\beta_1 & \equiv -\sqrt{2 \alpha'} \left( -G^{11} - \frac{\Theta_{11}^{11}}{\pi \alpha'} \right) k_1 \bar{l}_1 + (G^{11} + \frac{\Theta_{11}^{11}}{\pi \alpha'}) \bar{k}_1 l_1 \frac{1}{-\text{Re} z_3 + i \text{Im} z_3}. \\
\alpha_3 & \equiv \sqrt{\frac{\alpha'}{2 \text{Im} z_3 \pi \alpha'}} \Theta_{22}^2; \\
\alpha_4 & \equiv \sqrt{\frac{\alpha'}{2 (\text{Im} z_3)_{0}}} (e^{2i(\nu - 1)\phi} - e^{-2i \nu \phi}); \\
\alpha_5 & \equiv -\sqrt{\frac{\alpha'}{2 \text{Im} z_3}} G^{22}; \\
\omega_1 & \equiv \frac{\alpha'}{2 \text{Im} z_3} \frac{4 i l_2^2}{\pi \alpha'} \Theta_{22}^2; \\
\omega_2 & \equiv \sqrt{\frac{\alpha'}{2 \text{Im} z_3}} \frac{g^{22}}{\pi \alpha'} \sin(2 \nu \phi); \\
\beta_3 & \equiv \sqrt{\frac{\alpha'}{2 \text{Im} z_3 \pi \alpha'}} \Theta_{22}^2; \\
\beta_4 & \equiv -\sqrt{\frac{\alpha'}{2 \text{Im} z_3}} \frac{g^{22}}{\pi \alpha'} (e^{2i(1 - \nu)\phi} - \cos(2 \nu \phi)) \quad (B5)
\end{align*}
\]
Appendix C

We discuss below the evaluation of the integral (17) in $A_{soc}$.

\[
\int_{-\infty}^{\infty} dRez_3 (Rez_3 + iImz_3)^{\lambda_4} (Rez_3 - iImz_3)^{\lambda_5} e^{\lambda_1 \ln(1 + \cos \sigma_3)} (1, \lambda_3 e^{-2\tau_3})
\]

\[
= \int_{-\infty}^{\infty} d(Rez_3) \left[ 1 + \frac{Rez_3}{\sqrt{(Rez_3)^2 + (Imz_3)^2}} \right]^{\lambda_1} \left( \frac{\lambda_3}{(Rez_3)^2 + (Imz_3)^2} \right) \times (Rez_3 + i(Imz_3))^{\lambda_4} (Rez_3 - i(Imz_3))^{\lambda_5}
\]

\[
= (Imz_3)^{1+\lambda_4+\lambda_5} \int_{-\pi/2}^{\pi/2} d\theta \sec^2 \theta (1 + \sin \theta)^{\lambda_1} \left( e^{-i[\lambda_4-\lambda_5] \theta} \cos -2 - \lambda_4 - \lambda_5 \theta, \frac{\lambda_3}{(Imz_3)^2} e^{-i[\lambda_4-\lambda_5] \theta} \cos -[\lambda_4+\lambda_5] \theta \right)
\]

\[
= 2^{\lambda_1 - [\lambda_4 + \lambda_5]} e^{\frac{i\pi(\lambda_4 - \lambda_5)}{2}} (Imz_3)^{1+\lambda_4+\lambda_5} \int_{-\pi/2}^{\pi/2} d\theta \left( \frac{e^{-i\pi(\lambda_4 - \lambda_5)}}{4} e^{i[\lambda_4-\lambda_5] \theta} \cos 2\lambda_1 - (\lambda_4 + \lambda_5) \theta \frac{\theta}{2} \sin -2 - (\lambda_4 + \lambda_5) \theta \frac{\theta}{2} \right.
\]

\[
+ \frac{\lambda_3}{(Imz_3)^2} e^{\frac{i\pi(\lambda_4 - \lambda_5)}{2}} e^{i(\lambda_4 - \lambda_5) \theta} \cos 2\lambda_1 - (\lambda_4 + \lambda_5) \theta \frac{\theta}{2} \sin -2 - (\lambda_4 + \lambda_5) \theta \frac{\theta}{2}, \right.
\]

\[
+ \frac{\lambda_3}{(Imz_3)^2} e^{\frac{i\pi(\lambda_4 - \lambda_5)}{2}} e^{-i(\lambda_4 - \lambda_5) \theta} \cos (\lambda_4 + \lambda_5) \theta \frac{\theta}{2} \sin 2\lambda_1 - (\lambda_4 + \lambda_5) \theta \frac{\theta}{2}) \quad \text{(C1)}
\]

The above integrals can be evaluated using mathematica. One needs the following:

\[
\int_0^{\pi/2} \cos^a(\frac{x}{2}) \sin^b(\frac{x}{2}) e^{i\alpha \theta} d\theta
\]

\[
= \frac{i2^{1-a}}{a + b - 2c} \left[ e^{(b-a)\pi/2} e^{ic\pi/2} \text{AppellF1}\left[-\frac{a}{2}, -\frac{b}{2} + c, -a, b; 1 - \frac{a}{2}, -\frac{b}{2} + c; -i, i \right] \right.
\]

\[
- \left. \frac{\Gamma[1 + b] \Gamma[1 - \frac{a}{2} - \frac{b}{2} + c]}{\Gamma[1 - \frac{a}{2} + \frac{b}{2} + c]} \text{F1}\left[-\frac{a}{2}, -\frac{b}{2} + c, -a; 1 - \frac{a}{2} + \frac{b}{2} + c; -1 \right] \right). \quad \text{(C2)}
\]

25
One gets the following:

\[
2^{\lambda_3 - [\lambda_4 + \lambda_5]} e^{\frac{i\pi(\lambda_4 - \lambda_5)}{2}} (Im z_3)_0^{1+\lambda_4 + \lambda_5} [\frac{\Gamma(1 + \lambda_4 + \lambda_5)}{\Gamma(1 + 2\lambda_1 - 2\lambda_3 + 2\lambda_4 + 2\lambda_5 - (\lambda_4 - \lambda_5); -i, i)}] \\
\text{AppellF1} \left( \frac{2 + \lambda_4 + \lambda_5 - 2\lambda_1}{2}, \frac{2 + \lambda_4 + \lambda_5}{2} - \lambda_4 - \lambda_5, \lambda_4 + \lambda_5 + 2, \frac{2 + \lambda_4 + \lambda_5 - 2\lambda_1}{2}, \frac{2 + \lambda_4 + \lambda_5}{2} + \lambda_4 - \lambda_5; -1 \right) \\
\frac{2 + \lambda_4 + \lambda_5 - 2\lambda_1}{2} + \frac{2 + \lambda_4 + \lambda_5}{2} - \lambda_4 - \lambda_5, \lambda_4 + \lambda_5 + 2, \frac{2 + \lambda_4 + \lambda_5 - 2\lambda_1}{2}, \frac{2 + \lambda_4 + \lambda_5}{2} + \lambda_4 - \lambda_5; -1 \right) \\
\frac{i e^{-i\pi(\lambda_4 - \lambda_5)}}{4[2\lambda_1 - 4 - 4\lambda_4 - 2\lambda_2]} \left( \frac{\Gamma(1 + \lambda_4 + \lambda_5)}{\Gamma(1 + 2\lambda_1 - 2\lambda_3 + 2\lambda_4 + 2\lambda_5 - (\lambda_4 - \lambda_5); -i, i)} \right) \\
\x_\text{AppellF1} \left( -\lambda_1 + 2\lambda_4, -2\lambda_1 + (\lambda_4 + \lambda_5); (\lambda_4 + \lambda_5); -\lambda_1 + 1 + 2\lambda_4; -i, i \right) \\
\frac{\Gamma(1 - \lambda_4 - \lambda_5) \Gamma(1 - \lambda_1 + 2\lambda_4)}{\Gamma(1 + \lambda_4 - \lambda_5 - \lambda_1)} \right)
\]
\begin{align}
\frac{\lambda_3}{(Im \lambda_3^2)^2} \frac{ie^{i \pi (\lambda_4 - \lambda_5)}}{4[2 \lambda_1 - 4 \lambda_4]} 2^{1+\lambda_1+\lambda_5-2 \lambda_1} e^{-i \pi (\lambda_4 - \lambda_5)}
\times \text{AppellF}_1 \left( -\lambda_1 + 2 \lambda_5, (\lambda_4 + \lambda_5), -2 \lambda_1 + (\lambda_4 + \lambda_5); 1 - 2 \lambda_1 + 2 \lambda_5; -i, i \right)
\frac{\Gamma(1 - \lambda_4 - \lambda_5) \Gamma(1 - \lambda_1 + 2 \lambda_4) 2 \text{F}_1 \left( -\lambda_1 + 2 \lambda_4, -2 \lambda_1 + \lambda_4 + \lambda_5; 1 + \lambda_4 - \lambda_5 - \lambda_1; -1 \right)}{\Gamma(1 - \lambda_4 + \lambda_5 - \lambda_1)}
\end{align}

(C3)

References

[1] B.Chen, H. Itoyama, T. Matsuo and K. Murakami, *Worldsheet and Spacetime Properties of p-p’ System with B Field and Noncommutative Geometry*, Nucl. Phys. B593, 505, 2001 [hep-th/0005283].

[2] N.Seiberg, E.Witten, *String Theory and Noncommutative Geometry*, JHEP 9909:032, (1999), [hep-th/9908142].

[3] H. Ooguri and C. Vafa, *Selfduality And N=2 String Magic*, Mod. Phys. Lett. A 5 (1990) 1389, *Geometry of N=2 strings*, Nucl. Phys. B 361, 469 (1991).

[4] N. Marcus, *A Tour through N=2 strings*, [hep-th/9211050]. The N=2 open string, Nucl. Phys. B387, 263 (1992) [hep-th/9207024].

[5] E.Martinec, *M-theory and N = 2 strings*, [hep-th/9710122].
[6] W. Siegel, The N=4 string is the same as the N=2 string, Phys. Rev. Lett. 69, 1493 (1992) [hep-th/9204005]; N=2, N=4 string theory is selfdual N=4 Yang-Mills theory, Phys. Rev. D 46 (1992) 3235; ibid D 47 (1993) 2504; 2512.

[7] S. Ketov, Do the critical (2,2) strings know about a supergravity in 2+10 dimensions?, [hep-th/9710086]; Self-duality and F theory [hep-th/9612171]; From N = 2 strings to F & M theory, Nucl. Phys. Proc. Suppl. 52A, 335 (1997) [hep-th/9606142].

[8] A. Kumar, A. Misra and K. L. Panigrahi, Noncommutative N = 2 Strings, JHEP 0102 (2001) 037 [hep-th/0011206].

[9] H. ’Liu and J. Michelson, *-TREK: The one loop N = 4 noncommutative SYM action, [hep-th/0008205]; T. Mehen and M. B. Wise, Generalized *-products, Wilson lines and the solution of the Seiberg-Witten equations, JHEP 0012, 008 (2000) [hep-th/0010204]; M. R. Garousi, Nucl. Phys. B579 (2000) 209, [hep-th/9909214]; Nucl. Phys. B602 (2001) 527-540 [hep-th/0011147].

[10] M. Ademollo et al., Dual String With U(1) Color Symmetry, Nucl. Phys. B111 (1976) 77.

[11] B. Chen, H. Itoyama, T. Matsuo, K. Murakami, Correspondence between Noncommutative Soliton and Open String/D-brane System via Gaussian Damping
Factor, hep-th/0010066.

[12] J. Greensite and F. R. Klinkhamer, *Superstring Amplitudes And Contact Interactions*, Nucl. Phys. B 304, 108 (1988).

[13] J. Greensite and F. R. Klinkhamer, *New Interactions For Superstrings*, Nucl. Phys. B 281, 269 (1987); *Contact Interactions In Closed Superstring Field Theory*, Nucl. Phys. B 291 (1987) 557.
Figure 1: Some field theory graphs in the t and s channels; thin line is $p - p'$ open-string scalar $o$, thick line is $p' - p'$ open-string scalar $o'$, and dashed line is closed-string scalar $c$. 