DISCRETE NAHM EQUATIONS FOR SU($N$) HYPERBOLIC MONOPOLES

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ABSTRACT. In a paper of Braam and Austin, SU(2) magnetic monopoles in hyperbolic space $H^3$ were shown to be the same as solutions to matrix-valued difference equations called the discrete Nahm equations. Here, I discover the $(N-1)$-interval discrete Nahm equations and show that their solutions are equivalent to SU($N$) hyperbolic monopoles. These discrete time evolution equations on an interval feature a jump in matrix dimensions at certain points in the evolution, which are given by the mass data of the corresponding monopole. I prove the correspondence with higher rank hyperbolic monopoles using localisation and Chern characters. I then prove that the monopole is determined up to gauge transformations by its “holographic image” of U(1) fields at the asymptotic boundary of $H^3$.

1. OUTLINE

The Nahm equations are the following system of ODE

$$\frac{d(σ + σ^*)}{dt} = [σ, σ^*] + [τ, τ^*]$$
$$\frac{dτ}{dt} = [σ, τ]$$

where $σ$ and $τ$ are complex-valued $k \times k$ matrices, $k \in \mathbb{N}$ and $t \in [-p, p]$, $p \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.

The solutions of the Nahm equations are in one-to-one correspondence with SU(2) magnetic monopoles in $\mathbb{R}^3$ of mass $p$ and charge $k$ \[1\].

| Euclidean $\mathbb{R}^3$ | Nahm equations | $\text{SU}(N)$ magnetic monopoles |
|--------------------------|----------------|---------------------------------|
| Hyperbolic $H^3$        | discrete Nahm equations | $(N-1)$-interval discrete Nahm equations |

Table 1. Monopoles and Nahm equations

Hurtubise and Murray \[2\] discovered what I call $(N-1)$-interval Nahm equations for SU($N$) magnetic monopoles in $\mathbb{R}^3$. The $(N-1)$-interval Nahm equations resemble the Nahm equations on intervals $[p_1, p_2], \ldots, [p_{N-1}, -p_N]$ where $p_1, \ldots, p_N \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.

Date: 29 June 2015.
Across each boundary $t = p_i$ for some $i \in \{1, \ldots, N - 1\}$, the matrices $\sigma, \tau$ change dimensions from $(k_1 + \ldots + k_{i-1}) \times (k_1 + \ldots + k_i)$ to $(k_1 + \ldots + k_i) \times (k_1 + \ldots + k_i)$. $\sigma$ and $\tau$ have a simple pole at each boundary and their residue at a pole is a representation of $\text{SU}(2)$.

Braam and Austin [3] then found the discrete Nahm equations

$$\left[ \beta_{i+\frac{1}{2}}, \beta_{i+\frac{1}{2}}^* \right] + \gamma_{i+1} \gamma_{i+1}^* - \gamma_i^* \gamma_i = 0$$

$$\beta_{i-\frac{1}{2}} \gamma_i - \gamma_i \beta_{i+\frac{1}{2}} = 0$$

where $\beta_i$ and $\gamma_i$ are complex-valued $k \times k$ matrices and $i \in \{-p, -p+1, \ldots, p-1, p\}$, $p \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$ (Notably, Braam and Austin only treat the half-integer case). The solutions to the discrete Nahm equations are in one to one correspondence with $\text{SU}(2)$ magnetic monopoles in hyperbolic 3-space $H^3$.

In this paper, I introduce the $(N-1)$-interval discrete Nahm equations whose solutions are in one-to-one correspondence with (framed) $\text{SU}(N)$ magnetic monopoles in hyperbolic space. As in the continuous case, the $(N-1)$-interval discrete Nahm equations resemble discrete Nahm equations on $(N-1)$ intervals and at each boundary between adjacent intervals, the matrices $\beta_i$ and $\gamma_i$ jump in dimensions. As far as I am aware, this is the first time that this change of dimensions behaviour has been found in a system of matrix difference equations.

Atiyah showed that hyperbolic magnetic monopoles are $S^1$-invariant instantons on $\mathbb{R}^4$ [4]. The $(N-1)$-interval discrete Nahm equations arise from the ADHM construction applied to $S^1$-invariant instantons. The matrices $\beta_i$ and $\gamma_i$ are found to be the block matrices within the ADHM matrices equivariant with respect to the induced $S^1$ action. The $(N-1)$-interval discrete Nahm equations are then the ADHM equations restricted to these equivariant blocks.

The $(N-1)$-interval discrete Nahm equations can be interpreted as the discrete evolution of block matrices within the ADHM matrices. The solution matrices at a boundary are to be thought of as boundary data for the evolution equations.

Atiyah also proved that there is an isomorphism between the moduli of monopoles and the moduli of rational maps [4, 5]. I produce explicit formulae for the rational map of an $\text{SU}(N)$ hyperbolic monopole in terms of the boundary data of a solution of the $(N-1)$-interval discrete Nahm equations.

Finally, Braam and Austin [3] showed that the boundary data of an $\text{SU}(2)$ hyperbolic monopole was equivalent with the boundary data in the sense of discrete Nahm equations and so determined the monopole (up to gauge equivalence). The proof of the
analogous theorem for the SU(N) case follows the same lines. However, it is notable
that the generalisation of the map
$$\mathbb{P}^1 \to \mathbb{P}^k$$
which appears in Braam and Austin’s theorem generalises to \((N-1)\) maps from \(\mathbb{P}^1\)
into the manifold of two term partial flags.

2. Monopoles and Instantons

An SU\((N)\) instanton on \(\mathbb{R}^4\) is a connection 1-form \(A_{\square}\) on the (trivial) principal SU\((N)\)
bundle \(P \to \mathbb{R}^4\) which satisfies the (anti-)self-duality equations

$$F_{\square} = \pm \star F_{\square}$$

where \(F_{\square}\) is the curvature form of \(A_{\square}\), and the asymptotic decay condition, that \(A_{\square}\) must extend to a connection on \(S^4\) (the conformal compactification of \(\mathbb{R}^4\)). We will restrict to the anti-self-dual instantons. For an instanton, the Yang-Mills Lagrangian

$$-\int_{\mathbb{R}^4} \text{Tr} F_{\square} \wedge \star F_{\square}$$

is an \(L^2\)-norm of the curvature and is equal to \(8\pi \kappa\) where \(\kappa\) is an integer. \(\kappa\) is a
topological invariant called the instanton charge. (See [6] for a complete treatment.)

A magnetic monopole \((A,\phi)\) on \(\mathbb{R}^3\) (euclidean) is a connection 1-form \(A\) on the principal SU\((N)\) bundle \(P \to \mathbb{R}^3\) and a section \(\phi\) of the adjoint bundle ad \(P\) which satisfies the Bogolomy equations

$$F_A = \star e D_A \phi$$

where the Hodge star dual \(\star e\) is defined by the euclidean metric, and a choice of asymptotic decay conditions. The moduli of euclidean monopoles is foliated by mass numbers \(p_1, \ldots, p_{N-1} \in \mathbb{R}\) and magnetic charge numbers \(k_1, \ldots, k_{N-1} \in \mathbb{Z}\).

A magnetic monopole \((A,\phi)\) in hyperbolic space \(H^3\) can be defined as an instanton on \(\mathbb{R}^4\) invariant under the following circle \(S^1\) action [4]. Choose coordinates \((x_1, x_2, x_3, x_4)\) for \(\mathbb{R}^4\) and rotate the \(x_3x_4\) plane with the \(x_1x_2\) plane as the axis of rotation. Then we may use new coordinates \((x_1, x_2, r, \theta)\) where \(e^{i\alpha} \in S^1\) acts by \(\theta \mapsto \alpha \theta\). The euclidean metric in these coordinates is

$$ds^2 = r^2 \left( \frac{dx_1^2 + dx_2^2 + dr^2}{r^2} + d\theta^2 \right).$$
Without the axis of rotation, \( \mathbb{R}^4 \) is foliated by upper half spaces and this metric induces the Poincaré hyperbolic metric on each. Conformally,

\[
\mathbb{R}^4 - \mathbb{R}^2 \simeq S^1 \times H^3.
\]

The instantons which are invariant under this circle action may be interpreted as a connection \( A \) on \( H^3 \) with all the right asymptotic decay conditions following from the original instanton.

A monopole connection \( A_\square \) in these coordinates is equivalent to a potential \( A = A_{x_1} dx_1 + A_{x_2} dx_2 + A_r dr \) and a Higgs field \( \phi \) (the \( d\theta \) part), a section of the adjoint bundle. The self-duality condition reduces to the hyperbolic Bogolmony equations

\[
F_A = \star D_A \phi
\]

where the Hodge star \( \star \) is defined by the above hyperbolic metric.

The moduli space of hyperbolic monopoles \( (A, \phi) \) has components labelled by mass numbers \( p_1, \ldots, p_{N-1} \in \mathbb{Z} \) (or in \( \frac{1}{2} + \mathbb{Z} \) if \( N \) is even) which I order \( p_1 < p_2 < \ldots < p_{N-1} \) and corresponding charge numbers \( k_1, \ldots, k_{N-1} \in \mathbb{Z} \). Note the restriction (compared to the euclidean case) on the mass numbers which arise as the weights of the \( S^1 \)-action; this is a drawback of defining hyperbolic monopoles as \( S^1 \)-invariant instantons. For the rest of the paper, the mass numbers will be assumed to be distinct; this is the case of maximal symmetry breaking where the SU\((N)\) symmetry is reduced to the symmetry of a maximal torus U(1)\(^N\)-1 which preserves \( \phi \) at a point on the conformal sphere at infinity.

To employ the ADHM construction \([6, 7]\), we need to work in the twistor space \( \mathbb{P}^3 \) of \( \mathbb{R}^4 \subset S^4 \). Consider the fibration

\[
(2.1) \quad \mathbb{C} \mathbb{P}^3 \to \mathbb{H} \mathbb{P}^1 \simeq S^4
\]

\[
[x : y : z : w] \mapsto [x + y j : z + w j].
\]

The left multiplication by \( j \in \mathbb{H} \) leaves \( S^4 \) invariant but induces an involution on \( \mathbb{P}^3 \)

\[
J[x : y : z : w] = [\bar{y} : -\bar{x} : \bar{w} : -\bar{z}]
\]

acting as the antipodal map on the \( \mathbb{P}^1 \) fibres of the twistor fibration(2.1), commonly called a “real structure” on \( \mathbb{P}^3 \).

The Penrose-Ward transform is a correspondence between
(1) instantons on $S^4$ realised as vector bundles with unitary structure and an
connection with anti-self-dual curvature, and
(2) holomorphic vector bundles $E$ on $\mathbb{P}^3$ with a real form.

The circle action on $\mathbb{R}^4$ lifts to $\mathbb{P}^3$ along this fibration as the action
\[
[x : y : z : w] \mapsto [c^{-1/2}x : c^{1/2}y : c^{-1/2}z : c^{1/2}w]
\]

where $c \in S^1 \subset \mathbb{C}^\times$.

In $\mathbb{P}^3$, there are two fixed lines $\mathbb{P}_1^+ = \{[x : 0 : z : 0]\}$ and $\mathbb{P}_1^- = \{[0 : y : 0 : w]\}$ of the
$\mathbb{C}^\times$-action which cover the fixed $S^2_{\partial H} \subset S^4$. The $\mathbb{C}^\times$-action is free on $\mathbb{P}^3 - \mathbb{P}_1^+ \cup \mathbb{P}_1^-$ so we can decompose it into $\mathbb{C}^\times$-orbits. The boundary of each $\mathbb{C}^\times$-orbit is a pair of points,
one from each fixed line and each point in $\mathbb{P}_1^+ \times \mathbb{P}_1^-$ uniquely determines a $\mathbb{C}^\times$-orbit.

Thus the space of orbits
\[
Q = \frac{\mathbb{P}^3 - \mathbb{P}_1^+ \cup \mathbb{P}_1^-}{\mathbb{C}^\times}
\]
is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. $Q$ is known as the hyperbolic monopole mini-twistor space.

The projective plane $\mathbb{P}^2$ satisfying $w = 0$ contains the fixed line $\mathbb{P}_1^+$ and intersects $\mathbb{P}_1^-$ at a point $X_-$. This choice of $\mathbb{P}^2$ picks out a unique point $\{\infty\} \in \partial H^3$ covered by $\mathbb{P}_\infty^1 = \{[x : y : 0]\}$, the only fibre over a point of $\partial H^3$ contained in $\mathbb{P}^2$. Assume that
$z = -1$ by projectivity and then $\mathbb{P}^2 - \mathbb{P}_1^+$ is decomposed into a family of orbits $\{\mathbb{P}^1_{x_0}\}$ of the $\mathbb{C}^\times$-action, indexed by $x_0 \in \mathbb{P}_1^+$ where the orbits intersect $\mathbb{P}_1^+$. $\mathbb{P}^2 - \mathbb{P}_1^+$ also decomposes into a family of lines $\{\mathbb{P}^1_t\}_{t \in \mathbb{P}^1_{x_0}}$ (for some fixed choice of $x_0$) intersecting the point $[1 : 0 : 0]$ (the intersection of $\mathbb{P}_1^+$ and $\mathbb{P}_\infty^1$) which map to horospheres in $H^3$ at $\{\infty\}$.
Figure 2.2. The \( \mathbb{C}^\times \) orbits of \( \mathbb{P}^2 \) and the fibres of horospheres intersecting \( \{ \infty \} \in \partial H^3 \).

A framing of an instanton is an isomorphism \( P_\infty \to \text{SU}(N) \) for the fibre of \( P \) at the point at infinity of \( S^4 \). A framed \( \text{SU}(N) \) instanton is an instanton together with a framing.

The ADHM construction can be carried out over either \( \mathbb{P}^2 \) or \( \mathbb{P}^3 \). The \( \mathbb{P}^3 \) construction can always yield the \( \mathbb{P}^2 \) construction via geometric invariant theory but the converse is not true.

By a theorem of Donaldson [8], there is a natural correspondence between framed instantons and holomorphic bundles on \( \mathbb{P}^2 \subset \mathbb{P}^3 \) (with first Chern class \( c_1 = 0 \) since \( \text{SU}(N) \) has determinant 1) with a fixed holomorphic trivialisation at the fibre \( \mathbb{P}^1 \) of infinity via the twistor fibration (2.1).

Such a holomorphic bundle \( E \) on \( \mathbb{P}^2 \) can be constructed as the cohomology of monads [9]. A monad over \( \mathbb{P}^2 \) is the following pair of maps

\[
H \xrightarrow{A_X} K \xrightarrow{B_X} L
\]

where

1. \( H = H \otimes \mathcal{O}(-1), K = K \otimes \mathcal{O}, L = L \otimes \mathcal{O}(1) \);
2. \( H, K, L \) are \( \kappa, e + N, \kappa \) dimensional vector spaces over \( \mathbb{C} \) respectively;
3. \( \mathcal{O}(1) \) is the Hopf bundle over \( \mathbb{P}^2 \) and
4. \( A_X, B_X \) are linear maps for each \( [x : y : z] = X \in \mathbb{P}^2 \) and depend linearly on \( X \).

The map \( A_X \) needs to be injective, the map \( B_X \) needs to be surjective and \( B_X A_X \equiv 0_\kappa \).
Since the maps $A_X, B_X$ vary holomorphically with $X \in \mathbb{P}^2$, the holomorphic bundle $E$ can be defined fibre-wise by the cohomology
\[ E_X = \ker B_X / \im A_X \]
of the monad. For an instanton, this construction is unique up to an action of $\text{GL}_{HKL} = \text{GL}(H) \times \text{GL}(K) \times \text{GL}(L)$.

Following Donaldson, the conditions on $A_X$ and $B_X$ imply that a basis can be chosen such that
\[ A_X = \begin{bmatrix} x + z\alpha_1 \\ y + z\alpha_2 \\ za \end{bmatrix} \quad B_X = \begin{bmatrix} -y - z\alpha_2 & x + z\alpha_1 & zb \end{bmatrix} \]
where $\alpha_1$ and $\alpha_2$ are $\kappa \times \kappa$ matrices, $a$ is a $N \times \kappa$ matrix, $b$ is a $\kappa \times N$ matrix which we call ADHM matrices; they satisfy the complex ADHM equation
\begin{equation}
[\alpha_1, \alpha_2] + ba = 0. \tag{2.2}
\end{equation}

The action of $\text{GL}_{HKL}$ on the monad induces the following action of $\text{GL}(\kappa, \mathbb{C}) \times \text{GL}(N, \mathbb{C})$ on the data $\alpha_1, \alpha_2, a$ and $b$
\begin{align*}
\alpha_i & \mapsto g\alpha_ig^{-1} \\
a & \mapsto \lambda ag^{-1} \\
b & \mapsto gb\lambda^{-1}
\end{align*}
where $g \in \text{GL}(\kappa, \mathbb{C})$ and $\lambda \in \text{GL}(N, \mathbb{C})$. We call this a “gauge transformation” of the ADHM data.

For the fibre $\mathbb{P}^1_{\infty} = \{ [x : y : 0] \}$ over infinity,
\[ A_X = \begin{bmatrix} xI_\kappa \\ yI_\kappa \\ 0_{N \times \kappa} \end{bmatrix} \]
\[ B_X = \begin{bmatrix} -yI_\kappa & xI_\kappa & 0_{\kappa \times N} \end{bmatrix}. \]
Thus the trivialisation $\Psi : E|_{\mathbb{P}^1_{\infty}} \to \mathbb{C}^N$ fixes a basis (the “frame”) for the last $N$ entries of $K$.

The ADHM construction over $\mathbb{P}^3$ can be expressed in the same way but with a dependence on the coordinate $w$ and an isomorphism $J^*(E) \cong E^*$ that covers the real structure $J$ on $\mathbb{P}^3$ (See [3, 8] for details).
The maps $A_X$ and $B_X$ over $\mathbb{P}^3$ are

$$A_X = \begin{bmatrix}
x + z\alpha_1 - wa_2^* \\
y + z\alpha_2 + wa_1^* \\
z a + wb^*
\end{bmatrix}$$

$$B_X = \begin{bmatrix}
-x - z\alpha_2 - wa_1^* \\
y + z\alpha_1 - wa_2^* \\
z b - wa^*
\end{bmatrix}.$$  

They satisfy both the complex ADHM equation (2.2) and the real ADHM equation (2.3)

$$\mu = [\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a = 0$$

which is a moment map $\mu : \mathbb{C}^{2(\kappa+N)} \to u(\kappa)$ for the system. This equation is only preserved by the subgroup of $GL(\kappa, \mathbb{C})$ whose elements obey $g^{-1} = g^*$. Thus there is a reduction to an action of $U(\kappa) \times U(N)$ on the data $\alpha_1, \alpha_2, a$ and $b$.

The holomorphic vector bundle constructed on $\mathbb{P}^3$ agrees with the bundle constructed over $\mathbb{P}^2$ for the same ADHM data $(\alpha_1, \alpha_2, a, b)$ - we will call them both $E$.

Over the fixed line $\mathbb{P}^1_+$, the $\mathbb{C}^x$-action induces a representation on the fibres of the holomorphic vector bundle $E$. All the irreducible representations of $\mathbb{C}^x$ are 1-dimensional so up to conjugation, the circle action (for $SU(N)$) takes the form

$$c \mapsto \lambda(c) = \begin{bmatrix}
c^{p_1} \\
\cdots \\
c^{p_{N-1}} \\
c^{p_N}
\end{bmatrix}$$

where $p_1 < \ldots < p_{N-1}$ (since they are assumed to be distinct) are the weights of the $\mathbb{C}^x$-action and they are either all integers or if $N$ is even, they can also be all half-integers (this is well-defined since the $\mathbb{C}^x$ action comes from a double cover of $\mathbb{C}^x$). Since the structure group is $SU(N)$, $p_N = -p_1 - \ldots - p_{N-1}$.

To study hyperbolic monopoles via the ADHM construction, we examine what it means for a monad to be “circle invariant”. Work has been done in this direction by Norbury in his PhD thesis [10] for the $SU(2)$ case; however, his results apply equally to the $SU(N)$ case. Since this PhD thesis is not widely available, a proof will be supplied.

**Proposition 1** (Norbury). A monad over $\mathbb{P}^2$ whose cohomology is a holomorphic $\mathbb{C}^N$-vector bundle with trivialisation data corresponding to a framed instanton on $\mathbb{R}^4$ is $\mathbb{C}^x$-invariant if and only if there exists a homomorphism $P_c : \mathbb{C}^x \to GL(\kappa, \mathbb{C})$ such that
\begin{enumerate}
\item \( \alpha_1 = P_c \alpha_1 P_c^{-1} \)
\item \( \alpha_2 = c P_c \alpha_2 P_c^{-1} \)
\item \( a = \lambda a P_c^{-1} \)
\item \( b = c P_c b \lambda^{-1} \)
\end{enumerate}

**Proof.** For the monopole to be \( \mathbb{C}^* \)-invariant, the monad maps need to be \( \mathbb{C}^* \)-equivariant. There needs to be an element \( (\sigma, \rho, \sigma') \) of \( GL_{KH} \) for which the maps \( A_X \) and \( B_X \) satisfy 
\[ \rho(c) A_{(x,y,z)} = A_{(x, cy, z)} \sigma(c) \quad \text{and} \quad \sigma'(c) B_{(x, y, z)} = B_{(x, cy, z)} \rho(c). \]
We can ask that the choice of basis made for \( K \) be preserved which means that \( \rho(c) \) should split into blocks on the diagonal, \( \text{diag} (\rho_1, \rho_2, \rho_3) \in GL(\kappa, \mathbb{C}) \times GL(\kappa, \mathbb{C}) \times GL(\mathcal{N}, \mathbb{C}) \).

The condition \( A_{(x, cy, z)} = \rho(c) A_{(x, y, z)} \sigma^{-1}(c) \) in this basis is
\[
\begin{bmatrix}
x + z\alpha_1 \\
y + z\alpha_2 \\
z\alpha
\end{bmatrix}
\mapsto
\begin{bmatrix}
x + z\alpha_1 \\
cy + z\alpha_2 \\
z\alpha
\end{bmatrix} = \text{diag}(\rho_1, \rho_2, \rho_3)
\begin{bmatrix}
x + z\alpha_1 \\
y + z\alpha_2 \\
z\alpha
\end{bmatrix} \sigma^{-1}.
\]

Note that \( x = \rho_1 x \sigma^{-1} \) implies that \( \rho_1 = \sigma \) and \( cy = \rho_2 y \sigma^{-1} \) implies that \( \rho_2 = c \sigma \).

Likewise, \( B_{(x, cy, z)} = \sigma'(c) B_{(x, y, z)} \rho^{-1}(c) \) in the chosen basis reads as
\[
\begin{bmatrix}
-cy - z\alpha_2 & x + z\alpha_1 & zb
\end{bmatrix} = \sigma'
\begin{bmatrix}
-y - z\alpha_2 & x + z\alpha_1 & zb
\end{bmatrix} \text{diag}(\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1}).
\]

From the first two blocks, \( -cy = -\sigma' y \rho_1^{-1} \) implies that \( cp_1 = \sigma' \) and \( x = \sigma' x \rho_2^{-1} \) implies that \( \rho_2 = \sigma' \).

Together, this means \( \sigma = P_c = \rho_1 \) and \( \sigma' = c P_c = \rho_2 \) for some \( P_c \in GL(\kappa, \mathbb{C}) \). Recall that the last \( N \) basis elements of \( K \) provide the framing so \( \rho_3 \) needs to be the representation \( \lambda_c \). Thus, the conditions (1)-(4) of the theorem are exactly the conditions for the \( \mathbb{C}^* \)-equivariance of \( A_X \) and \( B_X \).

Thus we see that in the case of a circle invariant monopole, the \( \mathbb{C}^* \)-action on the monad’s bundles is multiplication by
\[
c \mapsto \text{diag} (P_c, \text{diag} (P_c, cP_c, \lambda_c), cP_c) \in GL(H) \times GL(K) \times GL(L).
\]
The homomorphism \( P_c \) is a representation of \( \mathbb{C}^* \) so we can diagonalise it. This means that \( H, K \) and \( L \) can be decomposed into weight spaces for the \( \mathbb{C}^* \)-action. The ADHM data \( \alpha_1, \alpha_2, a, b \) must then preserve these weight spaces.

Austin and Braam \cite{AB} found the weight space decomposition for the SU(2) case via the equivariant index theorem. In the next section, we will see a calculation of the weight spaces for any SU(\( N \)). It is enough to compute the \( \mathbb{C}^* \)-representation \( P_c \) over the fixed line \( \mathbb{P}^1 \) since this is enough to determine the ADHM data \( (\alpha_1, \alpha_2, a, b) \).
3. A Chern Characters Calculation

The starting point of the calculation is the following display (which can be found in [9]) for a monad

\( \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \ker B_X & E & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & H & K & \text{coker } A_X & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
B_X & L & L & 0 & 0 & 0
\end{array} \)  \hspace{1cm}

where the rows and columns are all exact.

The equivariant Chern character of \( \mathbb{P}^1 \) is a map \( K_{\mathbb{C}^\times}(\mathbb{P}^1) \to H_{\mathbb{C}^\times}(\mathbb{P}^1) \), from the equivariant K-theory to the equivariant cohomology of a space \( \mathbb{P}^1 \). By the additivity of the Chern character, the right vertical and bottom horizontal exact sequences of the display gives us the following

\[
\text{ch(}\text{coker } A_X\text{)} = \text{ch}(E) + \text{ch}(L)
\]

\[
\text{ch}(K) = \text{ch}(H) + \text{ch(}\text{coker } A_X\text{)}
\]

where \( \text{ch} \) denotes the \( \mathbb{C}^\times \text{-equivariant} \) Chern character. Putting them together yields

\[ \text{ch}(E) = \text{ch}(K) - \text{ch}(H) - \text{ch}(L). \]

The upshot is that if we know the equivariant Chern character of the holomorphic bundle \( E \), we can compute the equivariant Chern character of the monad vector spaces \( H, K \) and \( L \) over \( \mathbb{P}^1_+ \) and hence their \( \mathbb{C}^\times \) weight decomposition. Concretely, this data is encoded in the exponents of the matrix \( P_\epsilon \) and will induce a decomposition of the ADHM matrices.

Since the bundle \( E \) is trivial over \( \mathbb{P}^1_+ \), we have a representation of \( \mathbb{C}^\times \) on the fibres which allows us to compute the equivariant Chern character of \( E|_{\mathbb{P}^1_+} \). Over any \( \mathbb{P}^1 \), all holomorphic vector bundles split into line bundles by the Birkoff-Grothendieck splitting.
principle [9]. The strategy is to localise to \( \mathbb{P}^1 \), split all the relevant bundles and compute the exponents of \( P_c \). Since the ADHM matrices are constant, any conditions on them over any line will hold globally.

3.1. The bundle \( E \).

For \( SU(2) \), Atiyah showed that over \( \mathbb{P}^1 \), \( E = \mathcal{O}(k) \otimes \mathcal{L}^{-p} \oplus \mathcal{O}(-k) \otimes \mathcal{L}^p \) where \( \mathcal{L} \) is the trivial line bundle with the \( e^p \) representation of \( \mathbb{C}^\times \) [4]. This follows from a result of equivariant K-theory that over a fixed point set \( M \),

\[
K_{\mathbb{C}^\times}(M) = K(M) \otimes R(\mathbb{C}^\times)
\]

where \( R(\mathbb{C}^\times) = \mathbb{Z}[u] \) is the ring of characters of the representations of \( \mathbb{C}^\times \) [11].

The \( \mathbb{C}^\times \)-representation on \( E \) over \( \mathbb{P}^1 \)

\[
c \mapsto \lambda(c) = \text{diag} \left( e^{p_1} \ldots e^{p_N} \right)
\]

ordered \( p_1 < p_2 < \ldots < p_N \) splits \( E \) into a sum of line bundles. Since these line bundles are algebraic, we invoke Birkhoff-Grothendieck [Okonek-Schneider-Spindler 1980] to see the unique splitting

\[
E = \mathcal{O}(k_1) \otimes \mathcal{L}^{p_1} \oplus \ldots \oplus \mathcal{O}(k_{N-1}) \otimes \mathcal{L}^{p_{N-1}} \oplus \mathcal{O}(k_N) \mathcal{L}^{p_N}
\]

where \( k_N = -(k_1 + \ldots + k_{N-1}) \) and \( p_N = -(p_1 + \ldots + p_{N-1}) \).

Using results in [4, 12], we calculate the equivariant first Chern class and the total Chern class of \( E \). The equivariant first Chern class of a line bundle of the form \( \mathcal{O}(k) \otimes \mathcal{L}^p \) is

\[
c_1^e = kx + pu
\]

where \( x \) is the second degree generator of the usual \( H^2(\mathbb{P}^1) \) and \( u \) is the first degree generator of \( R(\mathbb{C}^\times) \).

This is enough to calculate the equivariant Chern character

\[
\text{ch}(E) = e^{k_1 x + p_1 u} \ldots + e^{k_N x + p_N u}
\]

and since \( H^*(\mathbb{P}^1) = \mathbb{Z}[x]/\langle x^2 \rangle \), the following series expansion with respect to \( x \) is exact

\[
\text{ch}(E) = e^{p_1 u} \ldots + e^{p_N u}
\]

\[
+ x (k_1 e^{p_1 u} + \ldots + k_N e^{p_N u}) .
\]

The equivariant total Chern class of \( E \) is given by

\[
\prod_{i=1}^{N} (1 + k_i x + p_i u) \mod x^2.
\]
The localisation formula from Atiyah and Bott \[12\] tells us that the second Chern class \(c_2\) (remember that \(c_1(E) = 0\)) can be found by looking at the coefficient of \(x\) and dividing it by \(u\). This is positive integer

\[
(3.4) \quad c_2(E) = - \left[ 2 \sum_{i=1}^{N-1} k_ip_i + \sum_{i=1}^{N-2} (k_i p_j + k_j p_i) \right]
\]

which reduces to \(2kp\) as expected for the SU(2) case \(p_1 = -p\) which is known.

### 3.2. The main calculation.

Since the \(x\)-terms in the Chern character of \(E\) only has terms up to \(e^{p_1u} \text{ and } e^{p_Nu}\), the lowest weight of \(P_c\) and highest weight of \(cP_c\) are \(c^{p_1}\) and \(c^{p_N}\) respectively. This is required because for the \(x\)-terms, the lowest weight term of \(H\) and the highest weight term of \(L\) do not cancel with any other terms on the right side of (3.2) and therefore must exactly match \(x\)-terms of \(\text{ch}(E)\).

The homomorphism \(P_c\) has the form

\[
\text{diag} \quad \left( e^{p_1} \ldots e^{p_1} e^{p_1+1} \ldots e^{p_1+1} \ldots e^{p_N-1} \ldots e^{p_N-1} \right)
\]

\[\leftarrow \chi_{p_1} \rightarrow \leftarrow \chi_{p_1+1} \rightarrow \ldots \leftarrow \chi_{p_N-1} \rightarrow \]

and the \(p_N - p_1\) numbers \(\chi_{p_1}, \ldots, \chi_{p_N-1}\) are what we need to calculate.

The vector bundles \(H, K\) and \(L\) decompose as follows:

\[
H = \bigoplus_{i=p_1}^{p_N-1} (\mathcal{O}(-1) \otimes \mathcal{L}^i)^{\otimes \chi_i}
\]

\[
K = \bigoplus_{i=p_1}^{p_N-1} (\mathcal{L}^i)^{\otimes \chi_i} \oplus \bigoplus_{i=p_1}^{p_N-1} (\mathcal{L}^{i+1})^{\otimes \chi_{i+1}} \oplus (\mathcal{L}^{p_1} \oplus \ldots \oplus \mathcal{L}^{p_N})
\]

\[
L = \bigoplus_{i=p_1}^{p_N-1} (\mathcal{O}(1) \otimes \mathcal{L}^{i+1})^{\otimes \chi_{i+1}}.
\]

Note that \(K\) has been arranged into the parts on which the \(\mathbb{C}^\times\)-action is via \(P_c\), \(cP_c\) and \(\lambda\) respectively.
The corresponding equivariant Chern characters are:

\[
\text{ch}(H) = \sum_{i=p_1}^{p_N-1} \chi_i e^{-iu} = \sum_{i=p_1}^{p_N-1} \chi_i e^{iu} - x \left( \sum_{i=p_1}^{p_N-1} \chi_i e^{iu} \right)
\]

\[
\text{ch}(K) = \sum_{i=p_1}^{p_N-1} \chi_i e^{iu} + \sum_{i=p_1}^{p_N-1} \chi_i e^{(i+1)u} + (e^{p_1u} + \ldots + e^{p_Nu})
\]

\[
= \chi_{p_1} e^{p_1u} + \sum_{i=p_1+1}^{p_N-1} (\chi_{i-1} + \chi_i) e^{iu} + \chi_{p_N-1} e^{p_Nu} + (e^{p_1u} + \ldots + e^{p_Nu})
\]

\[
\text{ch}(L) = \sum_{i=p_1}^{p_N-1} \chi_i e^{x+(i+1)u}
\]

\[
= \sum_{i=p_1}^{p_N-1} \chi_i e^{(i+1)u} + x \left( \sum_{i=p_1}^{p_N-1} \chi_i e^{(i+1)u} \right).
\]

We proceed by comparing coefficients. The \(x\)-terms are enough to determine the unknowns \(\chi_{p_1}, \ldots, \chi_{p_N-1}\).

\[xe^{p_1u} : k_1 = \chi_{p_1}\]

\[xe^{p_Nu} : k_N = -\chi_{p_N}\]

\[xe^{p_iu}, \text{ for } 1 < i \leq N - 1 : k_i = \chi_{p_i} - \chi_{p_i-1}\]

and all the other \(x\)-terms require that \(\chi_j = \chi_{j-1}\) when \(j \neq p_i\) for any of the \(1 \leq i \leq N\).

The interesting 1-terms are the ones of the form \(e^{p_iu}\). The rightmost terms of (3.5) supply the 1-terms of \(\text{ch}(E)\). We expected to see this because in the monad, \(K\) carries the trivialisation/framing data of \(E\) in its last \(N\) basis elements. The rest of the 1-terms \(\text{ch}(K)\) cancel with the 1-terms of \(\text{ch}(H)\) and \(\text{ch}(L)\) to show that they are consistent with the constraints set by the \(x\)-terms.

In the case of \(\text{SU}(3)\), the weights run from \(p_1\) to \(p_2\) with coefficients \(\chi_i = k_1\) and then from \(p_2\) to \(-p_1 - p_2\) with coefficients \(\chi_i = k_1 + k_2\). At \(p_2\), the coefficient jumps from \(\chi_{p_2-1} = k_1\) to \(\chi_{p_2} = k_1 + k_2\). This is illustrated by the following diagram (which should be viewed as an interval - the domain of an evolution equation)

\[
\begin{array}{ccccccc}
\bullet & p_2-p_1 & \bullet & -2p_2-p_1 & \bullet & p_1 & p_2 & k_1 & k_1+k_2 & p_3 = -p_1-p_2
\end{array}
\]
where the quantity above the line is the number of distinct weights with corresponding coefficient being the quantity under the line. The dimensions of $P_c$ (as a square matrix) are given by

$$(p_2 - p_1)k_1 - (2p_2 + p_1)(k_1 + k_2) = -(2p_1k_1 + 2p_2k_2 + p_1k_2 + p_2k_1)$$

which is exactly the formula for the second Chern class $c_2(E)$ from the previous subsection.

In general, we have

$$p_1k_1 - p_2k_2 - \cdots - p_Nk_N = (p_1k_1 + \cdots + p_Nk_N)$$

and this gives us the dimensions of $P_c$

$$(3.6) \quad \kappa = \sum_{i=1}^{N-1} \left( (p_{i+1} - p_i) \sum_{j=1}^i k_j \right).$$

In [10], Norbury proved the SU(2) case of the following proposition by a different method.

**Proposition 2.** The dimensions $\kappa \times \kappa$ of $P_c$ are given by $\kappa = c_2(E)$ for all $G = SU(N)$, $N \in \mathbb{N}_{\geq 3}$.

**Proof.** We proceed by induction. The SU(3) case above is our base step. (For the SU(2) case, it is compatible too; $c_2(E) = 2kp = \kappa$.)

For the inductive step, we assume that the proposition holds for SU($N - 1$). The difference in (3.4) between the $N$ and $N - 1$ cases is

$$(p_{N-2} - p_{N-1})(k_1 + \ldots + k_{N-2}) - (2p_{N-1} + p_{N-2} + \ldots + p_1)(k_1 + \ldots + k_{N-1})$$

$$+ (2p_{N-2} + p_{N-3} + \ldots + p_1)(k_1 + \ldots + k_{N-2})$$

$$= -p_{N-1}(k_1 + \ldots + k_{N-2}) - (2p_{N-1} + p_{N-2} + \ldots + p_1)k_{N-1}$$

which is exactly the extra terms of $c_2(E)$ in (3.6) in going from $N - 1$ to $N$.

3.3. **Discrete Nahm equations.**

The preceding section proves that

**Proposition 3.** Let $E$ be a $\mathbb{C}^\times$-equivariant holomorphic vector bundle on $\mathbb{P}^3$ ($\mathbb{C}^\times$-action $[x : y : z : w] \mapsto [c^{-1/2}x : c^{1/2}y : c^{-1/2}z : c^{1/2}w]$) corresponding to a monopole with mass numbers $p_1, \ldots, p_{N-1} \in \mathbb{Z}$ (or $\frac{1}{2} + \mathbb{Z}$ if $N$ is even) ordered $p_1 < \ldots < p_{N-1}$, and charge numbers $k_1, \ldots, k_{N-1} \in \mathbb{Z}$.
Then the $\mathbb{C}^\times$ weight space decomposition of the monad

$$H \xrightarrow{A} K \xrightarrow{B} L$$

restricted to $\mathbb{P}^1_+$ is

$$H = \mathbb{C}^{k_1}_{p_1} \oplus \ldots \oplus \mathbb{C}^{k_1}_{p_{-1}} \oplus \mathbb{C}^{k_1+k_2}_{p_{-2}} \oplus \mathbb{C}^{k_1+k_2}_{p_{-3}} \oplus \ldots \oplus \mathbb{C}^{-k_N}_{p_{-N-1}}$$

$$K = \mathbb{C}^{k_1}_{p_1} \oplus \mathbb{C}^{2k_1}_{p_{-1}+1} \oplus \ldots \oplus \mathbb{C}^{2k_1}_{p_{-2}} \oplus \mathbb{C}^{2(k_1+k_2)+1}_{p_{-3}+1} \oplus \mathbb{C}^{2(k_1+k_2)}_{p_{-4}+1} \oplus \ldots \oplus \mathbb{C}^{2(k_1+\ldots+k_{N-1})+1}_{p_{-N}+1} \oplus \mathbb{C}^{-k_N+1}_{p_{-N}}$$

$$L = \mathbb{C}^{k_1}_{p_1} \oplus \ldots \oplus \mathbb{C}^{k_1}_{p_2} \oplus \mathbb{C}^{k_1+k_2}_{p_{-2}+1} \oplus \mathbb{C}^{k_1+k_2}_{p_{-3}+2} \oplus \ldots \oplus \mathbb{C}^{-k_N}_{p_{-N}}$$

where the subscript denotes the weight of the $\mathbb{C}^\times$ representation on that component. The final mass and charge numbers are defined $p_N = -\sum_{i=1}^{N-1} p_i$ and $k_N = -\sum_{i=1}^{N-1} k_i$ respectively.

Note that anti-self-dual instantons have instanton charge $\kappa < 0$ which constrains the allowed mass and charge numbers of a hyperbolic monopole.

The conditions of Proposition 1 imply that the ADHM data $(a_1, a_2, a, b)$ for a magnetic monopole only map between components of the same weight. Now I will describe the form of the ADHM data $(a_1, a_2, a, b)$ which preserve the above weight decomposition.

The matrix $\alpha_1$ is a sparse matrix with square blocks $\{\beta_{i+1/2}\}, p_1 \leq i \leq p_N - 1$ running down the diagonal of the indicated size. The matrix dimensions increase from $(k_1 + \ldots + k_{j-1}) \times (k_1 + \ldots + k_j)$ to $(k_1 + \ldots + k_j) \times (k_1 + \ldots + k_j)$ at each $i = p_j$, $2 \leq j \leq N - 1$. The subscripts of $\beta_{i+1/2}$, $\gamma_i$, $a_i$ and $b_i$ indicate that they map between spaces of weight $i$ of the $\mathbb{C}^\times$-action (between $i$ and $i+1$ for the $\beta$s).

The sparse matrix $\alpha_2$ has (square except at transitions) blocks $\{\gamma_i\}, p_1 + 1 \leq i \leq p_N - 1$ along the super-diagonal. At $i = p_j$, $2 \leq j \leq N - 1$, the diagonal block of zeros increases in dimensions from $(k_1 + \ldots + k_{j-1}) \times (k_1 + \ldots + k_{j-1})$ to $(k_1 + \ldots + k_j) \times (k_1 + \ldots + k_j)$. The matrix $\gamma_{p_j}$ sitting in the transition is a rectangular matrix of dimensions $(k_1 + \ldots + k_{j-1}) \times (k_1 + \ldots + k_j)$.

The $N \times \kappa$ matrix $a$ is divided by $P_e$ into columns labelled by weight space. The non-zero entries are row vectors $\{a_1, \ldots, a_{N-1}\}$ in the columns with weight $p_i$, $1 \leq i \leq N - 1$ and $i$-th rows of length $k_1 + \ldots + k_i$. The last weight space of the domain of $a$ corresponding to the last $-k_N$ columns has weight $p_N - 1$.

The $\kappa \times N$ matrix $b$ is divided into rows labelled by weight space. The non-zero entries are column vectors $\{b_2, \ldots, b_N\}$ in the rows with weight $p_i$, $2 \leq i \leq N - 1$ and
Figure 3.1. The weight decomposition of the monad of an SU(3) hyperbolic monopole with $p_1 = -3$ and $p_2 = -1$ (hence $\kappa = 7k_1 + 5k_2$).

$p_N$, and $i$-th columns of length $k_1 + \ldots + k_{i-1}$. Note that the first weight space of the image of $b$ corresponding to the first $k_1$ rows has weight $p_1 + 1$.

The complex equation (2.2) is now a series of equations in terms of the blocks \[
\{ \beta_{i+1/2} \}_{p_1 \leq i \leq p_N - 1} \quad \text{and} \quad \{ \gamma_j \}_{p_1 + 1 \leq j \leq p_N - 1},
\]

\[
(3.7) \begin{cases}
\beta_{i+\frac{1}{2}} \gamma_{i+1} - \gamma_{i+1} \beta_{i+\frac{3}{2}} + b_{i+1} a_{i+1} = 0 & \text{for } i + 1 = p_j, \ 2 \leq j \leq N - 1 \\
\beta_{i+\frac{1}{2}} \gamma_{i+1} - \gamma_{i+1} \beta_{i+\frac{3}{2}} = 0 & \text{otherwise}
\end{cases}
\]

which we call the complex discrete Nahm equations.

The real ADHM equation becomes the real discrete Nahm equations
\[ \alpha_1 = \begin{array}{cccccccc} 
\beta_{p_1 + \frac{1}{2}} & & & & & & & k_1 \\
\beta_{p_1 + \frac{3}{2}} & & & & & & k_1 \\
& \ddots & & & & & & \\
\beta_{p_i - \frac{1}{2}} & \beta_{p_i + \frac{1}{2}} & & & & & & k_i - 1 \\
& & \beta_{p_i + \frac{3}{2}} & & & & & k_i \\
& & & \ddots & & & & \\
\beta_{p_N - \frac{1}{2}} & & & & & & \beta_{p_N + \frac{1}{2}} & -k_N \\
0 & \gamma_{p_1 + 1} & & & & & & k_1 \\
0 & 0 & \gamma_{p_1 - 1} & & & & & k_i - 1 \\
0 & 0 & 0 & \gamma_{p_i} & & & & k_i \\
0 & 0 & 0 & 0 & \gamma_{p_i + 1} & & & k_i \\
0 & 0 & 0 & 0 & 0 & \gamma_{p_i + 2} & & \\
& & & & & & \ddots & \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma_{p_N} & -k_N \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_N \\
\end{array} \]
\[
\begin{array}{cccccc}
& a_{p_1} & \cdots & a_{p_i} & \cdots & a_{p_{N-1}} \\
\hline
k_1 & k_1 + k_2 & k_1 + \ldots + k_i & -k_N \\
\end{array}
\]

\begin{align*}
\beta_{i+\frac{1}{2}}^+ \beta_{i+\frac{1}{2}}^- + \gamma_{i+1}^+ \gamma_{i+1}^- - \gamma_i^+ \gamma_i^- - a_i^* a_i &= 0 & \text{when } i = p_j, 1 \leq j \leq N - 1 \\
\beta_{i+\frac{1}{2}}^+ \beta_{i+\frac{1}{2}}^- + \gamma_{i+1}^+ \gamma_{i+1}^- - \gamma_i^+ \gamma_i^- + b_{i+1}^* b_{i+1}^- &= 0 & \text{when } i + 1 = p_j, 2 \leq j \leq N \\
\beta_{i+\frac{1}{2}}^+ \beta_{i+\frac{1}{2}}^- + \gamma_{i+1}^+ \gamma_{i+1}^- - \gamma_i^+ \gamma_i^- &= 0 & \text{otherwise}
\end{align*}

where \( \gamma_{p_1} = 0 = \gamma_{p_N} \) so the first real equation is

\[
\left[ \beta_{p_1+\frac{1}{2}}^+ \beta_{p_1+\frac{1}{2}}^- \right] + \gamma_{p_1+1}^+ \gamma_{p_1+1}^- - a_{p_1}^* a_{p_1} = 0
\]

and the last one is

\[
\left[ \beta_{p_N-\frac{1}{2}}^+ \beta_{p_N-\frac{1}{2}}^- \right] + b_{p_N+\frac{1}{2}}^* b_{p_N+\frac{1}{2}}^- - \gamma_{p_{N-1}}^* \gamma_{p_{N-1}}^- = 0.
\]
Definition 4. A solution of the \((N - 1)\)-interval discrete Nahm equations of type 
\((p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1}) \in \mathbb{Z}^{2(N-1)}\) is an equivalence class of matrices 
\([\{\beta_j\}, \{\gamma_j\}, \{a_{p_i}\}, \{b_{p_i}\}]\) 
labeled by half-integer points on an interval \(j \in [p_1, p_N]\) as shown

\[
\begin{array}{cccccccccc}
a & \beta & \gamma & \beta & \gamma & \beta & b, \gamma & a & \beta & \gamma & \beta & b \\
p_1 & p_1+1 & \ldots & p_{i-1} & p_i & p_{i+1} & \ldots & p_{N-1} & p_N
\end{array}
\]

with dimensions \((k_1 + \ldots + k_i) \times (k_1 + \ldots + k_i)\) at half integer points on an interval 
\((p_i, p_{i+1})\) and at a boundary point \(p_i\) between intervals, the matrices \(a_{p_i}\), \(\gamma_{p_i}\) and \(b_{p_i}\) 
have dimensions \(1 \times (k_1 + \ldots + k_i)\), \((k_1 + \ldots + k_{i-1}) \times (k_1 + \ldots + k_i)\) and 
\((k_1 + \ldots + k_{i-1}) \times 1\) respectively. The matrices must satisfy the \((N - 1)\)-interval 
discrete Nahm equations and satisfy the equivalence relation (\textit{“gauge transformations”})

\[
\begin{align*}
\beta_j & \sim g_j \beta_j g_j^{-1} \\
\gamma_j & \sim g_j^{-\frac{1}{2}} \gamma_j g_j^{\frac{1}{2}} \\
a_{p_i} & \mapsto \lambda_{p_i} a_{p_i} g_{p_i+\frac{1}{2}}^{-1} \\
b_{p_i} & \mapsto g_{p_i-\frac{1}{2}} b_{p_i} \lambda_{p_i}^{-1}
\end{align*}
\]

where \(g_j \in U(k_1 + \ldots + k_i)\) when \(j \in (p_i, p_{i+1})\).

Thus is our first main theorem proven:

Theorem 5. There is an equivalence between

(1) framed \(SU(N)\) monopoles \((A, \phi)\) on hyperbolic space \(H^3\) of mass \((p_1, \ldots, p_{N-1}) \in \mathbb{Z}^{N-1}\) (or \((\frac{1}{2} + \mathbb{Z})^{N-1}\) for \(N\) even) and charge \((k_1, \ldots, k_{N-1}) \in \mathbb{Z}^{N-1}\), and

(2) solutions of the \((N - 1)\)-interval discrete Nahm equations of type 
\((p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})\).
4. The rational map

Atiyah [5] showed that:

**Theorem 6 (Atiyah).** For a compact classical group $G$, the moduli space of circle-invariant instantons or equivalently, hyperbolic monopoles of charge $k = (k_1, \ldots, k_N)$ is isomorphic to the space of degree $k$ “rational maps”

$$f : \mathbb{P}^1 \to G/T$$

where $T$ is a maximal torus.

When $G = SU(N)$, $G/T = Fl_{\text{full}}(N) = \{0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^N\}$, the manifold of full flags in $N$-dimensional space. For magnetic monopoles, we have the following corollary.

**Corollary 7.** There is an isomorphism between the moduli of framed $SU(N)$ magnetic monopoles on $H^3$ and the moduli of degree $(k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_{N-1})$ rational maps such that $f(\infty) = 0$,

$$f : \mathbb{P}^1 \to Fl_{\text{full}}(N).$$

Along the lines of Braam and Austin [3], I will derive an explicit formula for the rational map of a hyperbolic monopole in terms of its discrete Nahm boundary data. To do this, restrict the bundle to the projective plane $\mathbb{P}^2 = \{[x : y : z : 0] \in \mathbb{P}^3\}$. Over this $\mathbb{P}^2$, the solutions of the discrete Nahm equations have a $\text{GL}(k, \mathbb{C})$ freedom. We first require two lemmas of Braam and Austin whose conditions are satisfied in our case.

**Lemma 8 (Braam-Austin 4.2).** If $(\{\gamma_i\}, \{\beta_i\}, \{a_{pi}\}, \{b_{pi+1}\})$ lies in a stable orbit then the $\gamma_i$ are all injective.

By the injectivity of the $\gamma_i$ and using the $\text{GL}(k, \mathbb{C})$ action,

$$g_{i-\frac{1}{2}} \gamma_i g_{i+\frac{1}{2}}^{-1} = I$$

we set all the interval $\gamma_i$ to the identity matrix. Then in each interval, the $\beta_i$ are all equal to constant matrix $\beta_{[pi]}$ with subscript labelling the boundary point before the interval. Square brackets in the subscript indicate that this is the matrix after the $\text{GL}(k, \mathbb{C})$ action has been applied.

**Lemma 9 (Braam-Austin 4.3).** The data $(\{\beta_{[pi]}\}, \{\gamma_{[pi]}\}, \{a_{[pi]}\}, \{b_{[pi+1]}\})$ defines a monad satisfying the ADHM equations if and only if $\{\beta_{l[p_i]} a_{[pi]}\}$ for $l = 0, \ldots, k_1 + \ldots + k_i$ span $\mathbb{C}^{k_1+\ldots+k_i}$. 
The procedure is as follows. Choose a “horosphere line” \( \mathbb{P}^1_h \) in \( \mathbb{P}^2 \) with coordinates say \( x \mapsto [x : h : -1] \). The trivialisation of \( E \) over \( \mathbb{P}^1_\infty \) is also a trivialisation of the monad in the sense that over \( \mathbb{P}^1_\infty \), \( (0,0,r) \in K \), \( r \in \mathbb{C}^N \) are representatives of the global sections of \( E|_{\mathbb{P}^1_\infty} \). Extended to \( \mathbb{P}^1_h \), this trivialisation is

\[
\begin{pmatrix}
-(h - \alpha_2)^{-1} b \\
0_\kappa \\
I_N
\end{pmatrix} r + \begin{pmatrix}
(h - \alpha_2)^{-1} (x - \alpha_1) \\
I_\kappa \\
0_N
\end{pmatrix} Y \in K
\]

where \( Y \in \mathbb{C}^\kappa \).

Consider the splitting of \( E \) over \( \mathbb{P}^1_+ \),

\[
E = \mathcal{O}(k_1) \otimes \mathcal{L}^{p_1} \oplus \ldots \oplus \mathcal{O}(k_r) \otimes \mathcal{L}^{p_r} \oplus \ldots \oplus \mathcal{O}(k_N) \mathcal{L}^{p_N}.
\]

Atiyah showed that in the SU(2) case, the last factor extends by flowing along the \( \mathbb{C}^\times \)-action to a sub-line-bundle over \( \mathbb{P}^3 - \mathbb{P}^1_- \). The sum of the last two factors extend to a sub-plane-bundle and the sum of the last three extend to a rank 3 sub-bundle of \( E \), etc.

**Lemma 10.** On \( \mathbb{P}^2 - \mathbb{P}^1_- \), there exists unique holomorphic sub-bundles \( L_1^+ \subset L_2^+ \subset \ldots \subset L_{N-1}^+ \) of \( E \) which is preserved by the \( \mathbb{C}^\times \)-action and each \( L_i^+ \) restricted to \( \mathbb{P}^1_+ \) coincides with the last \( i \)-th factors.

**Proof.** The bundle \( E \) restricted to a \( \mathbb{C}^\times \)-orbit \( \mathbb{P}^1 - \{ \text{pt of } \mathbb{P}^1_- \} \) has the following \( \mathbb{C}^\times \)-action:

\[
c \cdot (z; u_1, \ldots, u_N) = (cz; c^{p_1}u_1, \ldots, c^{p_N}u_N).
\]

In the limit \( c \to 0 \), the global holomorphic sections of the form \( (0,0,\ldots,0,u_N(z)) \) are preserved by the \( \mathbb{C}^\times \)-action since multiplication by \( c \in \mathbb{C}^\times \) cannot change zero into a non-zero number. Since the space of such sections is one dimensional, they give us a sub-line bundle \( L_1^+ \) of \( E \). The sections have weight \(-p_N\) and so must coincide with the first factor in the splitting of \( E \) over \( \mathbb{P}^1_+ \).

Similarly for \( 1 < i < N \), in the \( c \to 0 \) limit, the global holomorphic sections

\[
(0,\ldots,0,u_i(z),u_{i+1}(z),\ldots,u_N(z))
\]

are preserved by the \( \mathbb{C}^\times \)-action and have weights \((p_i,\ldots,p_N)\). The set of them is \((N-i+1)\)-dimensional so they define a rank \((N-i+1)\) sub-bundle \( L_{N-i+1}^+ \) of \( E \).
By induction, a section of the form \((0, \ldots, 0, u_i(z), \ldots, u_N(z))\) is also a section of the sub-bundle given by sections of the form \((0, \ldots, u_{i-1}(z), \ldots, u_N(z))\) so \(L^+_{N-i+1} \subset L^+_N\) and thus the sub-bundles are a chain ordered by subset.

These are the only sections preserved by the \(\mathbb{C}^\times\)-action since the \(\mathbb{C}^\times\)-action is transitive on the non-zero entries of sections. Hence the holomorphic sub-bundles \(L^+_1 \subset \ldots \subset L^+_N\) preserved by the \(\mathbb{C}^\times\)-action thus defined are unique. \(\square\)

The rational map \(f\) is defined by sending each point \(x\) of \(\mathbb{P}^1_+\) to the fibre of the restriction of \(L^+_1 \subset \ldots \subset L^+_N \subset E\) to the orbit of \(\mathbb{C}^\times\) whose limit is \(x\). The chain of sub-bundles over the \(\mathbb{C}^\times\)-orbit is trivialised by taking the intersection of the \(\mathbb{C}^\times\)-orbit with the chosen horosphereline \(\mathbb{P}^1_h\) as the unit point and then the rest of the isomorphism is constructed by flowing along the \(\mathbb{C}^\times\)-orbit using the \(\mathbb{C}^\times\)-action. Canonically,

\[
(L^+_1, \ldots, L^+_N)|_{\mathbb{C}^\times} \cong (\mathbb{C}^1, \ldots, \mathbb{C}^{N-1}) \times \mathbb{C}^\times
\]

so that \(f(z)\) is an element of the manifold of full flags \(\text{Fl}_{\text{full}}(N)\).

Since \(E\) has a canonical trivialisation over \(\mathbb{P}^1_+\), we can find equations for the rational map. On the level of the monad, the rank \(i\) sub-bundle is produced exactly when the \(p_1, \ldots, p_{N-i}\) weight spaces are in the kernel of \(A_X\). This happens when the expression for each \(p_i\) weight space in the monad trivialisation is equal to the negative of some element of the image of \(A_X\).

Using Lemma \(8\) to linearly transform \(\{\gamma_{[j]}\}_{j \neq p_i}\) into identity matrices, we can invert \((h - \alpha_2)\). Writing \(r = (r_1, \ldots, r_N)\), we define the algebraic equations of a flag of subspaces by recursion. The condition that the \(p_i\) weight space be in the kernel of \(A_X\) is equivalent to solving the equations

\[
(-h)^{p_{N-1}}b_{[p_{N}]}r_n + (x - \beta_{[p_{N-1} + \frac{1}{2}]})w_{p_{N-1}} = 0
\]

\[
r_{N-1} + a_{[p_{N-1}]}w_{p_{N-1}} = 0.
\]

Solving for \(r_{N-1}\) in terms of \(r_N\), this is

\[
r_{N-1} = (-h)^{p_{N-1}}b_{[p_{N}]}(x - \beta_{[p_{N-1}]})^{-1}a_{[p_{N-1}]}r_N
\]

which defines a line in a plane for any \(x \in \mathbb{P}^1_+\).

Proceeding in the same way for the other weight spaces, we have:

**Proposition 11.** Let \((\{\gamma_i\}, \{\beta_i\}, \{a_{p_i}\}, \{b_{p_{i+1}}\})\) be a solution of the \((N-1)\)-interval discrete Nahm equations of type \((p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})\). Then the solution can be put into the form \((\{\beta_{[p_i]}\}, \{\gamma_{[p_i]}\}, \{a_{[p_i]}\}, \{b_{[p_{i+1}]}\})\) and the rational map,
\[ f : \mathbb{P}^1 \to Fl_{\text{full}}(N) \]
\[ x \mapsto (V_1, \ldots, V_{N-1}), \quad \dim V_i = i, \]

into the manifold of full flags in \( \mathbb{C}^N \) can be written as the maps \((r_1(x), \ldots, r_{N-1}(x))\),

\[ r_{N-1}(x) = (-h)^{p_{N-1} - p_N}a[p_{N-1}](x - \beta[p_{N-1}])^{-1}b[p_N]r_N(x) \]

\[ : \]

\[ r_j(x) = \sum_{i=j+1}^{N} (-h)^{p_{j} - p_{i}}a[p_{i}](x - \beta[p_{j}])^{-1}b^{k_1+\ldots+k_j}r_i(x) \]

\[ : \]

\[ r_1(x) = \sum_{i=2}^{N} (-h)^{p_{1} - p_{i}}a[p_{i}](x - \beta[p_{1}])^{-1}b^{k_1+k_2+\ldots+k_i}r_i(x) \]

where for each \( x \in \mathbb{P}^1 \), \( r_{N-1}(x) \) specifies an \((N-1)\)-dimensional linear subspace in \( \mathbb{C}^N \) and each successive \( r_i(x) \) specifies an \( i \)-dimensional linear subspace inside the \((i+1)\)-dimensional linear subspace specified by \( r_{i+1}(x) \). The superscript \( k_1 + \ldots + k_j \) indicates that only the first \( k_1 + \ldots + k_j \) entries of the vector are involved.

Note that when \( N = 2 \), the equation of the rational map is of the form

\[ r(x) = \frac{r_2(x)}{r_1(x)} = (-h)^{p_{1}}v(x - \beta)^{-1}v^t \]

which is the rational map found by Atiyah for SU(2) hyperbolic monopoles [3, 4].

5. The Boundary Value of a Monopole

On the conformal sphere at infinity, \( S^2_{\infty} \), the holomorphic vector bundle \( E \) splits into holomorphic line bundles \( \mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_{N-1}) \) and the gauge field \( A \) restricted to \( S^2_{\infty} \), induces a a U(1) connection \( A_i \) on each factor \( \mathcal{O}(k_i) \). We define the \((N-1)\)-tuple \((A_1, \ldots, A_{N-1})\) to be the boundary value or connections at infinity.

We shall prove the following generalisation of Braam-Austin’s theorem [3] regarding the boundary values of SU(2) hyperbolic monopoles.

**Theorem 12.** Let \((A, \Phi)\) be a framed SU(2) hyperbolic monopole. Then
(1) the \((N - 1)\) tuple of \(U(1)\) connections \((A_1, \ldots, A_{N-1})\) on \(S^2_\infty\) determines the connection \(A\) (up to gauge transformations);
(2) there exists for \(i = 1, \ldots, N - 1\), holomorphic maps
\[
F_i : \mathbb{P}^1 \rightarrow Fl(k_1 + \ldots + k_i, k_1 + \ldots + k_i + 1, 2k_1 + \ldots + 2k_{i-1} + k_i + 1)
\]
into the manifold of two term partial flags for which each \(A_i\) is the pullback of the unitary invariant connection on the “hyperplane bundle” \(O(1, -1)\) of the \(i\)-th flag manifold; and
(3) the map \(A \mapsto (A_1, \ldots, A_{N-1})\) is an immersion of the moduli space of \(SU(N)\) framed hyperbolic monopoles in the moduli of \((N - 1)\) tuples of \(U(1)\) connections on \(S^2\).

**Proof.** From Lemma, we have a decomposition of the monad \(H \rightarrow K \rightarrow L\) restricted to \(\mathbb{P}^1_+\) (which by abuse of notation, I conflate with \(S^2_\infty\) since any connections on \(\mathbb{P}^1_+\) descend to connections on \(S^1_\infty\) along the twistor transform) into weight spaces. By considering the maps \(A_x\) and \(B_x\) restricted to a weight subspace, we get what is called a small monad. By dimensional considerations, the cohomology of a generic small monad \((p_i < j < p_{i+1})\)

\[
\begin{array}{ccc}
\mathbb{C}^{k_1 + \ldots + k_i} & \xrightarrow{\gamma_j} & \mathbb{C}^{k_1 + \ldots + k_{i-1}} \\
\beta_j + \frac{1}{2} & & \beta_j - \frac{1}{2} \\
\mathbb{C}^{k_1 + \ldots + k_i} & \xrightarrow{\gamma_j} & \mathbb{C}^{k_1 + \ldots + k_{i-1}}
\end{array}
\]

is trivial except for the weight spaces \(p_1, \ldots, p_N\) which take the form

\[
\begin{array}{ccc}
\mathbb{C}^{k_1 + \ldots + k_i} & \xrightarrow{\gamma_{p_i}} & \mathbb{C}^{k_1 + \ldots + k_{i-1}} \\
\beta_{p_i} + \frac{1}{2} & & \beta_{p_i} - \frac{1}{2} \\
\mathbb{C}^{k_1 + \ldots + k_i} & \xrightarrow{\gamma_{p_i}} & \mathbb{C}^{k_1 + \ldots + k_{i-1}}
\end{array}
\]

The cohomology of these small monads are holomorphic line bundles defined fibre-wise

\[
L_{p_i}(x) = \ker(C^{2k_1 + \ldots + 2k_{i-1} + k_i + 1} \rightarrow C^{k_1 + \ldots + k_{i-1}}) / A_x(C^{k_1 + \ldots + k_i})
\]
which are exactly the line bundles in the splitting of $\mathcal{E}$.

Furthermore, there is a natural interpretation of the maps $A_x$ and $B_x$, restricted to each weight space of weight $p_i$ as a pair of maps,

\[ B^l_x : \mathbb{C}^{k_1+\ldots+k_i-1} \to \mathbb{C}^{2k_1+\ldots+2k_{i-1}+k_i+1} \]

\[ A_x : \mathbb{C}^{k_1+\ldots+k_i} \to B^l_x(\mathbb{C}^{k_1+\ldots+k_i-1}) \perp \cong \mathbb{C}^{k_1+\ldots+k_i+1} \subset \mathbb{C}^{2k_1+\ldots+2k_{i-1}+k_i+1} \]

defining a map $F_i = (A_x(H_{p_i}), B_x(L_{p_i})^\perp)$ into the two term partial flag manifold $\mathrm{Fl}(k_1 + \ldots + k_i, k_1 + \ldots + k_i + 1, 2k_1 + \ldots + 2k_{i-1} + k_i + 1)$. Then each line bundle $L_{p_i}$ and its $U(1)$ connection is the pullback of the invariant line bundle and connection over the two term partial flag manifold. This proves (2) of the theorem.

The map $F_i$ thus defined is an embedding of $\mathbb{P}^1$ into the partial flag manifold, for the ADHM equations guarantee that the monad is non-degenerate \cite{8}, and so $\text{im} F_i$ has no self-intersections and its derivative is non-zero. Compose $F_i$ with the Plücker embedding and then the Segre embedding to get

\[ F_{i}^\mathbb{P} : \mathbb{P}^1 \hookrightarrow \mathbb{P}(i) \]

where

\[ \mathfrak{P}(i) = \left( \begin{array}{c} 2k_1 + \ldots + 2k_{i-1} + k_i+1 \\ k_1 + \ldots + k_i \end{array} \right) \left( \begin{array}{c} 2k_1 + \ldots + 2k_{i-1} + k_i+1 \\ k_1 + \ldots + k_i+1 \end{array} \right) - 1. \]

The pullback of the $U(\mathfrak{P}(i) + 1)$ invariant connection $A_i$ by the embedding $F_{i}^\mathbb{P}$ induces a Kähler form $F_{A_i}$ (the curvature form of $A_i$) on $\mathbb{P}^1$. The work of Calabi \cite{13} tells us that any such embedding $F_{i}^\mathbb{P}$ is locally rigid, that is, the embedding is determined by the Kähler form up to the isometry group of the target space.

Hence the boundary values $(A_1, \ldots, A_{N-1})$ descend by the twistor transform to $U(1)$ connections on $S^1$ and determine the small monad for the weight spaces corresponding to the weights $p_1, \ldots, p_{N-1}$. These small monads provide boundary values for the $(N-1)$-interval discrete Nahm equations and their propagation uniquely specifies a complete solution up to gauge transformations. Thus the boundary values on $S^1_\infty$ or equivalently $\mathbb{P}^1_+$ uniquely determine the monopole.

On the moduli space of $\text{SU}(N)$ framed hyperbolic monopoles, the boundary values $(A_1, \ldots, A_{N-1})$ are local coordinates. Thus $A \mapsto (A_1, \ldots, A_{N-1})$ is a local immersion of the moduli of monopoles into the moduli of $(N-1)$-tuples of $U(1)$ connections on $S^1$. \hfill \Box
6. Final Remarks

I have shown in this paper that

(1) There is an equivalence between framed SU(N) hyperbolic monopoles
\((A, \phi)\) of charge \((p_1, \ldots, p_{N-1})\) and charge \((k_1, \ldots, k_{N-1})\), and solutions
\((\{\beta_i\}, \{\gamma_i\}, \{a_{p_i}\}, \{b_{p_i}\})\) of the \((N-1)\)-interval discrete Nahm equations of type
\((p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})\);

(2) The rational map \(\mathbb{P}^1 \to SU(N)/U(1)^{N-1}\) of a hyperbolic monopole can be written explicitly from a solution
\((\{\beta_j\}, \{\gamma_j\}, \{a_{p_i}\}, \{b_{p_i}\})\) of the discrete Nahm equations; and

(3) An SU(N) hyperbolic monopole \((A, \phi)\) is determined by its boundary value
\((N-1)\)-tuple of U(1) connections \((A_1, \ldots, A_{N-1})\) on the conformal boundary
sphere \(\mathbb{P}^1_\infty\) of \(H^3\).

Note that the \((N - 1)\)-interval discrete Nahm equations are are essentially \((N - 1)\)
copies of the (SU(2)) discrete Nahm equations linked by an equation of the form
\[\beta_{p_i-\frac{1}{2}} \gamma_{p_i} - \gamma_{p_i} \beta_{p_i+\frac{1}{2}} + b_{p_i} a_{p_i} = 0.\]

It is interesting to interpret the \((N - 1)\)-interval discrete Nahm equations as a rep-resentation of the type \(A\) quiver. The \(A_{N-1}\) Dynkin quiver diagram is the directed
graph
\[
\circ \xrightarrow{\beta} \circ \xrightarrow{\gamma} \circ \xrightarrow{\beta} \cdots \circ \xrightarrow{\gamma} \circ
\]
with \((N - 1)\) vertices. Associate a vector space \(V_i \simeq \mathbb{C}^{k_1 + \ldots + k_i}\) to the \(i\)-th vertex,
the operator \(\beta_{[p_i]} : V_i \to V_i\) (from Section 4) to each curved arrow and the operator
\(b_{[p_i]} a_{[p_i]} : V_i \to V_{i+1}\) to each edge between distinct vertices. This is one way that
the \((N - 1)\)-interval discrete Nahm equations could be treated as a representation
of the \(A_{N-1}\) quiver. It would be interesting to study such representations and their
relationship with such quiver representations appearing in the study of supersymmetric
models whose Coulomb branches involve BPS monopoles [14].

Another interesting avenue of research would be to study the spectral curve associated
to SU(N) hyperbolic monopoles in terms of the \((N - 1)\)-interval discrete Nahm
equations. This is being studied in on-going work with M.K. Murray. It is known that
spectral data does determine the monopole for a (apparently) different set of decay
conditions [15].
Acknowledgements. I would like to thank my PhD supervisor, Paul Norbury for suggesting the project and for his guidance. I acknowledge the Australian Department of Education and Training for providing PhD funding via the Australian Postgraduate Award.

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