Extended supersymmetry for the Bianchi-type cosmological models

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Abstract

In this paper we propose a superfield description for all Bianchi-type cosmological models. The action is invariant under world-line local $n = 4$ supersymmetry with $SU(2)_{\text{local}} \otimes SU(2)_{\text{global}}$ internal symmetry. Due to the invariance of the action we obtain the constraints, which form a closed superalgebra of the $n = 4$ supersymmetric quantum mechanics. This procedure provides the inclusion of supermatter in a systematic way.

I. INTRODUCTION

In the absence of a fundamental understanding of physics at very high energies and, in particular, in the absence of a consistent quantum theory of gravity, there is no hope, at present, to meet an understanding of the quantum origin of the Universe in a definite way. However, in order to come nearer to this presently unattainable goal it appears desirable to develop highly simplified, but consistent models, which contain as many as possible of those features which are believed to be present in a future complete theory. Spatially homogeneous minisuperspace models obtained by dimensional reduction from $(1+3)$ to $(1+0)$ dimensions have, therefore, played an important role in quantum cosmology.$^1$ On the other hand, there are several reasons for studying locally supersymmetric theories rather than non-supersymmetric ones. Four-dimensional model with local supersymmetry called supergravity (SUGRA) theory, leads to a constraint which can be thought of as square root of the Wheeler-DeWitt constraint, and it is related to it in the same way as the Dirac equation is related to the Klein-Gordon equation.$^2$ However, due to the technical complexities, the early papers on canonical supergravity $^3$ make no attempt at exploiting this idea, but content themselves with setting up the canonical formalism and discussing the classical constraint algebra in terms of Poisson (or Dirac) brackets.

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In the case of SUGRA theories one can find one-dimensional supersymmetric quantum mechanics (SQM) models by reducing four-dimensional $N = 1$ SUGRA coupled to supermatter.\textsuperscript{4} For this purpose it is necessary to consider the homogeneity of space, that is, the metric and the matter fields are independent of spatial coordinates, as a consequence one has a finite number of degrees of freedom. Thus, the study of the associated quantum model becomes analogous to a supersymmetric quantum mechanical problem. The hope we have in these models is that they could give us the notion of the full quantum theory of SUGRA. The supersymmetric quantum cosmological models have been intensively studied with the hope to get a consistent quantum theory for the cosmological models. However, not all the results obtained in supersymmetric quantum cosmology have their counterpart in the full theory of SUGRA. Some of these problems have already been mentioned in two extensive works.\textsuperscript{5}

More recently, we have proposed a new approach to investigating the supersymmetric quantum cosmology.\textsuperscript{6} In this approach we started with the action of the spatially homogeneous minisuperspace models and proceeded with supersymmetrization. Because the starting action preserves the invariance under local time reparametrization, then the supersymmetric action must be invariant under the extended local symmetry (supersymmetry). In order to have a local $n = 2$ supersymmetry in \textsuperscript{6}, the odd “time” parameter $\theta$ and its complex conjugate $\bar{\theta}$ were introduced. This involved introducing the superfield formulation, because superfields defined on superspace allow all the component fields in a supermultiplet to be manipulated simultaneously in a manner, that automatically preserves supersymmetry. This approach has the advantage of being simpler than the proposed models based on full SUGRA \textsuperscript{4,5,7}, and by means of this local symmetry procedure it gives the corresponding fermionic partners in a direct manner. Using the superfield procedure we have constructed the superfield action for all Bianchi-type models.\textsuperscript{8} The inclusion of the real scalar matter fields, as well as the parameter of spontaneous breaking of local supersymmetry were discussed in Ref. 9. Using the last results a normalizable wavefunction was obtained for the FRW model in Ref. 10. Although these models do not attempt to describe the real world, they keep many features occurring in four-dimensional space-time, which could really be studied in the quantum versions of simplified models.

The most physically interesting case is provided by $n = 4$ local supersymmetry, since it can be applied to the description of the systems resulting from the “realistic” $N = 1, D = 4$ SUGRA subject to an appropriate dimensional reduction down to $D = 1$.

In this work we extend the transformations of time reparametrization to the $n = 4$ local supersymmetry with $SU(2)_{\text{local}} \otimes SU(2)_{\text{global}}$ internal symmetry for all Bianchi-type-A cosmological models, and we give a procedure for including other matter fields in a systematic way. This paper generalizes the $n = 4$ construction described in Ref. 11. The extension presented is desirable for two reasons: 1) supersymmetric minisuperspace models are related to full $N = 1, D = 4$ dimensional SUGRA by dimensional reduction to $(1 + 0)$ dimensions.\textsuperscript{8} By such reduction $N = 1, D = 4$ SUGRA goes over to an $n = 4, D = 1$ supersymmetric model; 2) the gravitational field should be coupled to a supersymmetric matter field, like a complex scalar field.
II. $N = 4$ SUPERCONFORMAL TRANSFORMATIONS AND THE ACTION

The Bianchi models are the most general homogeneous cosmologies with a 3-dimensional group of isometries. These groups are in a one-to-one correspondence with 3-dimensional Lie algebras, which were classified long time ago by Bianchi. There are nine distinct 3-dimensional Lie algebras, and consequently nine types of Bianchi cosmologies. The 3-metric for each of these models can be written in the generalized coordinates

$$ds^2 = G_{\mu\nu}(q^\lambda) dq^\mu dq^\nu,$$

where the generalized coordinates $q^\lambda(\alpha, \beta_+, \beta_-)$ with $\nu = 0, 1, 2$ span the minisuperspace with the metric $G_{\mu\nu}$, which we may choose as flat, making use of the fact that the metric in minisuperspace is fixed only up to an arbitrary conformal factor, written as $\exp[2\omega(q)]$. All Bianchi-types models are conformally flat, i.e. its metric takes the form

$$G_{\mu\nu}(q) = e^{2\omega(q)} G_{\mu\nu}^{(0)},$$

with $G_{\mu\nu}^{(0)} = diag(-1, 1, 1)$. The inverse of this conformal factor appears in the potential of each Bianchi model,

$$U(q) = e^{-2\omega(q)} U_0(q),$$

with the potential $U_0 = -(3)^g(3)R$, where $(3)^g$ is the determinant and $(3)^R$ is the scalar curvature of the 3-metric. The action for the Bianchi type models may be written as

$$S = \frac{1}{2} \int \left\{ \frac{1}{N} G_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu + NU(q) \right\} dt.$$

The lapse function $N(t)$ and the coordinates $q^\mu(t)$ depend on the time parameter $t$ only. The action (4) is invariant under reparametrization of $t' \rightarrow t + a(t)$, if the transformations of $q^\mu$ and $N(t)$ are defined as

$$\delta q^\mu(t) = a(t) \dot{q}^\mu, \quad \delta N(t) = (aN).$$

That is, $q^\mu(t)$ transforms as a scalar and $N(t)$ as a one-dimensional vector and its dimensionality is inverse to that of $a(t)$. It is easy to see, that the action (4) gives a simple one-dimensional model for the somehow interacting homogeneous “matter” field $q^\mu$ and gravity field $N(t)$. Using the superfield formalism the $n = 2$ local SQM for cosmological models was constructed in Ref. 6,8.

In the action (4), $U(q)$ corresponds to the potential of each Bianchi-type models. This potential may be written as

$$U(q) = \frac{1}{2} G^{\mu\nu} \frac{\partial \phi}{\partial q^\mu} \frac{\partial \phi}{\partial q^\nu} = \frac{1}{2} G_{\mu\nu} m^\mu m^\nu.$$

Thanks to this relation, the hidden symmetry of the cosmological models was found (see Ref. 7). This allows to construct a corresponding SQM. However, in this case the supersymmetry is global. It is natural to demand, that any cosmological action is invariant under local transformations. For this reason the more extended symmetry must be local.
To construct the superfield action in the world-line superspace \((t, \theta^a, \bar{\theta}_a)\) [with \(t\) being a time parameter, and \(\theta^a\) and \(\bar{\theta}_a = (\theta^a)^*\), where \(a = 1, 2\) is an \(SU(2)\) index, being two complex Grassmann coordinates] one introduces a real “matter” superfield \(Q^\mu(t, \theta^a, \bar{\theta}_a)\) and a world-line supereinbein \(N(t, \theta^a, \bar{\theta}_a)\) which has the following properties with respect to the \(SU(2)\) \(n = 4\) superconformal transformations of the world-line superspace \(\Sigma\):

\[
\begin{align*}
\delta t &= \Lambda - \frac{1}{2} \theta^a D_a \Lambda - \frac{1}{2} \bar{\theta}_a \bar{D}_a \Lambda, \\
\delta \theta^a &= i \bar{D}^a \Lambda, \\
\delta \bar{\theta}_a &= i D_a \Lambda, \\
\delta \bar{\theta}^a &= i \bar{D}_a \Lambda, \\
\delta \theta \bar{\theta} &= i \bar{\theta} \theta - (\theta \bar{\theta})^* \Lambda,
\end{align*}
\]

(7)

\[
\begin{align*}
\delta Q^\mu &= -\Lambda \dot{Q}^\mu + \dot{\Lambda} Q^\mu - i (D_a \Lambda) (\bar{D}^a Q^\mu) - i (\bar{D}_a \Lambda) (D^a Q^\mu), \\
\delta N &= -\Lambda \dot{N} - \dot{\Lambda} N - i (D_a \Lambda) (\bar{D}^a N) - i (\bar{D}_a \Lambda) (D^a N),
\end{align*}
\]

(8)

where the overdot denotes the time derivative \(d/dt\). The transformation law \(\delta N\) for the superfield \(Q^\mu\) shows that this superfield is a vector superfield in the one-dimensional \(n = 4\) superspace, while the superfield \(N Q^\mu\) is a scalar.\(^{12}\)

The superfield \(Q^\mu\) obeys the quadratic constraints

\[
[D_a, \bar{D}^a] Q^\mu = -4m^\mu,
\]

\[
D^a D_a Q^\mu = 0, \\
\bar{D}_a \bar{D}^a Q^\mu = 0,
\]

(10)

and it is irreducible representation of \(n = 4\) supersymmetry.\(^{12}\) The vector \(m^\mu\) depends on the concrete cosmological model in consideration. For example, in the case of the Friedmann-Robertson-Walker model \(m^\mu\) has the form \(m = \sqrt{R}/2\), where \(k\) takes the value \(1, 0, -1\), and the metric \(G_{\mu\nu}\) has one component \(G_{00} = -R\) (see Ref. 11).

The superfield \(N\) obeys the constraints

\[
[D_a, \bar{D}^a] \frac{1}{N} = 0,
\]

\[
D^a D_a \frac{1}{N} = 0, \\
\bar{D}_a \bar{D}^a \frac{1}{N} = 0,
\]

(11)

which are imposed in order to have a one-to-one correspondence between the number of transformation parameters and that of fields, and

\[
D_a = \frac{\partial}{\partial \theta^a} - \frac{i}{2} \bar{\theta}_a \frac{\partial}{\partial \theta}, \\
\bar{D}^a = \frac{\partial}{\partial \bar{\theta}^a} - \frac{i}{2} \theta^a \frac{\partial}{\partial \bar{\theta}},
\]

are the supercovariant derivatives. The infinitesimal superfield \(\Lambda\), which appears in (7-9)

\[
\Lambda(t, \theta, \bar{\theta}) = a(t) + \theta^a \bar{\alpha}_a(t) - \bar{\theta}_a \alpha^a(t) + \theta^a (\sigma^b)^a \bar{\theta}_b b_i(t) \\
+ \frac{i}{4} (\theta \bar{\theta}) \bar{\theta}_a \alpha^a(t) - \frac{i}{4} (\theta \bar{\theta})(\theta \bar{\theta}) \theta^a \bar{\theta}_a(t) + \frac{1}{16} (\theta \theta)(\bar{\theta} \bar{\theta}) \bar{a}(t),
\]

(12)

contains the parameters of the local time reparametrizations \(a(t), \bar{\alpha}(t), b_i(t)\) being a local \(SU(2)\) parameter of the world-line superspace.

\(^{\S}\)Our conventions for spinors are as follows: \(\theta_a = \theta^b \epsilon_{ba}, \theta^a = \epsilon^{ab} \theta_b, \bar{\theta}_a = \bar{\theta}^b \epsilon_{ba}, \bar{\theta}^a = \epsilon^{ab} \bar{\theta}_b, \bar{\theta}_a = (\theta^a)^*, \bar{\theta}^a = -(\theta_a)^*, \theta \bar{\theta} \equiv \theta^a \theta_a = -2 \theta^1 \theta^2, \bar{\theta} \theta \equiv \bar{\theta}_a \bar{\theta}^a = (\bar{\theta} \theta)^*, \theta \bar{\theta} \equiv \bar{\theta}_a \theta^a, \epsilon^{12} = -\epsilon^{21} = 1, \epsilon_{12} = 1.\)
The constraints (10) can be explicitly solved, the solution is described by the superfield
\[ Q^\mu(t, \theta, \bar{\theta}) = q^\mu(t) + \theta^a \bar{\chi}_a^\mu(t) - \bar{\theta}_a \lambda^a \omega_\mu^a(t) + \theta^a(\sigma_i)_a \bar{\theta}_b F^{ij\mu}(t) + m^\mu(\theta \bar{\theta}) \] (13)
This superfield contains one bosonic field \( q^\mu \) and the Grassmann-odd fermionic fields (they are four). \( \lambda^a(t) \) and \( \bar{\chi}_a^\mu(t) \) are their superpartner spin degrees of freedom, and \( F^{ij\mu}_a = (\sigma^i)_a^b F_i \) are three auxiliary fields, where \( (\sigma^i)_a^b \) \( (i = 1, 2, 3) \) are the ordinary Pauli matrices.

The constraints (11) are described by the superfield
\[
\frac{1}{\mathcal{N}}(t, \theta, \bar{\theta}) = \frac{1}{\mathcal{N}(t)} + \theta^a \psi_a(t) - \bar{\theta}_a \psi'^a(t) + \theta^a (\sigma^i)_a \bar{\theta}_b V^i(t) \]
\[ + \frac{i}{4} (\theta \bar{\theta}) \frac{\bar{\theta}_a \psi'_a(t) - \theta^a \psi^a(t)}{4} + \frac{1}{16} (\theta \bar{\theta}) (\theta \bar{\theta}) d^2 \frac{1}{\mathcal{N}(t)} \]
\]
The superfield \( \mathcal{N} \) describes an \( n = 4 \) world-line supergravity multiplet consisting of the einbein “graviton” \( \mathcal{N}(t) \), two complex “gravitinos” \( \psi^a(t) \) and \( \bar{\psi}_a(t) \), and the \( SU(2) \) gauge field \( V^i(t) \). The components of \( \mathcal{N} \) play the role of Lagrange multipliers. Their presence mean that the dynamics of the model is subject to constraints.

The \( n = 4 \) superfield action for the Bianchi-type cosmological models invariant under \( n = 4 \) superconformal symmetry has the form \(^{11,12}\)
\[ S = -\frac{8}{\kappa^2} \int dt d^2 \theta d^2 \bar{\theta} \frac{\mathcal{N}^{-1}}{\mathcal{N}} A(\mathcal{N} Q^\mu), \] (15)
where \( \kappa^2 = 8\pi G_N, G_N \) is the Newtonian constant of gravity. The action (14) is the most general superfield action, which can be constructed with respect to the \( n = 4 \) conformal supersymmetry. \( A(\mathcal{N} Q^\mu) \) is an arbitrary function of the superfields \( \mathcal{N} Q^\mu \) called superpotential. Note, that in the case of \( n = 4 \) local supersymmetry it is sufficient to construct one invariant action possessing a minimal number of time derivatives, unlike of two invariants, a kinetic part and the potential one, as in the case of \( n = 2 \) local supersymmetry.\(^6\)

So, integrating (13) over the Grassmann coordinates \( \theta, \bar{\theta} \) and making the following redefinition of the component fields
\[ \psi = N^{3/2} \psi', \quad V^i = 2N(V^i + N(\psi^i \sigma_i \bar{\psi})), \quad \lambda^\mu = \sqrt{N}(\lambda^\mu - q^\mu \psi'), \]
\[ F^\mu_i = 2\sqrt{N}\{F^{\mu i}_a - q^a V^i + \frac{\sqrt{N}}{2}(\psi^i \sigma_i \bar{\psi}) + \frac{\sqrt{N}}{2}(\lambda^\mu \sigma_i \bar{\psi})\}, \quad q^\mu = Nq^\mu, \]
one obtains the component action
\[ S = \int \left\{ \frac{1}{2N} G_{\mu \nu} Dq^\mu Dq^\nu + i G_{\mu \nu}(\bar{\lambda}^\nu \tilde{D} \lambda^\mu + \bar{\lambda}^\mu \tilde{D} \lambda^\nu) + \frac{1}{2} G_{\mu \nu} F^\mu_i F^i\nu - 2\sqrt{N} \Gamma_{\mu \nu \rho} \lambda^\sigma (\sigma_i) \bar{\lambda}^\rho \lambda^\sigma \right\} dt, \]
(17)
where \( Dq^\mu = \dot{q}^\mu - i(\bar{\psi} \lambda^\mu + \bar{\lambda}^\mu \psi) \) is the supercovariant derivative, \( \tilde{D} \lambda^\mu = D\lambda^\mu + \Gamma^\mu_{\rho \sigma} \dot{q}^\rho \lambda^\sigma \), where \( D\lambda^\mu = \lambda^\mu - \frac{1}{2} V^\mu \) is the \( SU(2) \) covariant derivative. In order to give a geometrical form we have introduced in the action (17) the special metric
in this case the Christoffel connection takes the form
\[ \Gamma_{\mu\nu\rho}(q) = \frac{1}{2} \frac{\partial^3 A(q)}{\partial q^\mu \partial q^\nu \partial q^\rho}, \]
and the Riemann curvature tensor
\[ R_{\mu\nu,\rho\sigma} = \Gamma_{\eta\mu\rho}^n \Gamma_{\eta\nu\sigma}^n - \Gamma_{\eta\mu\nu}^n \Gamma_{\eta\rho\sigma}^n. \]

In the action (17) the components \( F_i \) of the superfield \( Q^\mu \) appear without derivatives and, therefore, they are non-dynamical variables. We can eliminate \( F_i \) by means of their equation of motion. Solving the equation of motion of the auxiliary fields \( F_i \) and substituting the solution back into Eq.(17) we obtain the component action. From the component action we derive the first-class constraints varying it with respect to \( N(t), \psi(t), \bar{\psi}(t) \) and \( V_i(t) \), respectively

\[ H_0 = \frac{\kappa^2}{2} G^{\mu\nu} P_\mu P_\nu + 2 G_{\mu\nu} m^\mu m^\nu + 4 D_\mu m^\nu \lambda^\nu \lambda^\mu - R_{\mu\nu,\rho\sigma} \lambda^\sigma \lambda^\rho \lambda^\nu \lambda^\mu - D_\mu \Gamma_{\nu\rho\sigma} (\lambda^\sigma \lambda^\nu) (\lambda^\rho \lambda^\mu), \]

\[ Q_a = \lambda^a \mu P_\mu - 2 i G_{\mu\nu} \lambda^a \mu m^\nu + i \Gamma_{\mu\nu\rho} \lambda^a \mu \lambda^\rho, \]

\[ Q^b = \lambda^b \mu P_\mu + 2 i G_{\mu\nu} \lambda^b \mu m^\nu + i \Gamma_{\mu\nu\rho} \lambda^b \mu \lambda^\rho, \]

and

\[ \mathcal{F}_i = G_{\mu\nu} \lambda^i (\sigma_i)_{ab} \lambda^a \lambda^b, \]

where \( H_0 \) is the Hamiltonian of the system, \( Q^a \) and \( \mathcal{Q}_a \) are the supercharges, and \( \mathcal{F}_i \) is the generator of \( SU(2) \) rotations.

So, following the standard procedure of quantization of the system with bosonic and fermionic degrees of freedom, we introduce the canonical Poisson brackets

\[ \{ q^\mu, P_\nu \} = \delta^\mu_\nu, \quad \{ \lambda^{a\mu}, \pi_{(\lambda)\nu} \} = -\delta^a_\nu \delta^\mu_\nu, \quad \{ \lambda^a \mu, \pi^{b \nu}_0 \} = -\delta^{a}_\mu \delta^{b}_\nu, \]

\[ \{ q^\mu, P_\nu \} = \delta^\mu_\nu, \quad \{ \lambda^{a\mu}, \pi_{(\lambda)\nu} \} = -\delta^a_\nu \delta^\mu_\nu, \quad \{ \lambda^a \mu, \pi^{b \nu}_0 \} = -\delta^{a}_\mu \delta^{b}_\nu, \]

where \( P_\mu, \pi_{(\lambda)am} \) and \( \pi^{a \nu}_{(\lambda)\mu} \) are the momenta conjugated to \( q^\mu, \lambda^\mu \) and \( \lambda^\nu \) respectively. From the explicit form of the momenta

\[ P_\mu = \frac{1}{\kappa^2} G_{\mu\nu} \{ q^\nu - i \kappa (\bar{\psi}_a \lambda^{a\nu} - \lambda^a \psi^a) \}, \]

\[ \pi_{(\lambda)am} = -i G_{\mu\nu} \lambda^{a \nu}, \quad \pi^{a \nu}_{(\lambda)\mu} = -i G_{\mu\nu} \lambda^{a \nu}, \]

\[ \pi^{a \nu}_{(\lambda)\mu} = -i G_{\mu\nu} \lambda^{a \nu}, \]

\[ \pi^{a \nu}_{(\lambda)\mu} = -i G_{\mu\nu} \lambda^{a \nu}, \]

\[ \pi^{a \nu}_{(\lambda)\mu} = -i G_{\mu\nu} \lambda^{a \nu}, \]
one can conclude, that the system possesses the second-class fermionic constraints

$$
\Pi^{(\lambda)\alpha}_{\mu} = \pi^{(\lambda)\alpha}_{\mu} + iG_{\mu\nu}\lambda^\nu, \quad \Pi^{(\lambda)b}_{\mu} = \pi^{(\lambda)b}_{\mu} + iG_{\mu\nu}\lambda^\nu,
$$

(27)

since

$$
\{\Pi^{(\lambda)a}_{\mu}, \Pi^{(\lambda)b}_{\nu}\} = -2iG_{\mu\nu}\delta^a_b.
$$

(28)

Therefore, the quantization has to be done using the Dirac brackets, defined by any of two functions \( F \) and \( G \) as

$$
\{F, G\}^* = \{F, G\} - \{F, \Pi_a\} \frac{1}{\{\Pi_a, \Pi_b\}} \{\Pi_b, G\}.
$$

As a result, we obtain the following Dirac brackets for the canonical variables

$$
\{q^\mu, \pi^\nu\}^* = \delta^\mu_\nu, \quad \{\lambda^{a\mu}, \pi^\nu\}^* = -\frac{i}{2}\delta^a_b G^{\mu\nu},
$$

$$
\{\lambda^{a\mu}, \pi^\nu\}^* = -\lambda^{a\rho}\Gamma^\mu_{\rho\nu}, \quad \{\pi^\mu, \gamma^\nu\}^* = -\pi^\mu_{\gamma\nu},
$$

(29)

$$
\{P^\mu, \lambda^{a\nu}\}^* = 2iR^{\mu\nu,\rho\sigma}\lambda^\rho\lambda^\sigma.
$$

The supercharges and the Hamiltonian form the following \( n = 4 \) SUSY QM algebra with respect to the introduced Dirac brackets

$$
\{Q_a^b, Q_i^a\}^* = -i\delta^b_a H_0, \quad \{F_j^i, F_k^i\}^* = \epsilon_{jkl} F_l,
$$

$$
\{F_i^a, Q_a^b\}^* = \frac{i}{2}(\sigma_i)^a_c Q_c, \quad \{F_i^a, Q^a\}^* = -\frac{i}{2}(\sigma_i)^a_c Q^c.
$$

(30)

On the quantum level we replace the Dirac brackets by (anti)commutators using the rule

$$
i\{,\}^* = \{,\}.
$$

one obtains the non-zero commutation relations

$$
[q^\mu, P^\nu] = i\delta^\mu_\nu, \quad \{\lambda^{a\mu}, \pi^\nu\} = \frac{1}{2}\delta^a_b G^{\mu\nu},
$$

(31)

$$
[P^\mu, \lambda^{a\nu}] = i\Gamma^{\mu\rho} \lambda^{a\rho}, \quad [P^\mu, \pi^\nu] = i\Gamma^{\mu\rho} \pi^{\rho\nu},
$$

$$
[P^\mu, P^\nu] = -2R^{\mu\nu,\rho\sigma}\lambda^\rho\lambda^\sigma.
$$

We observe that \( P^\mu \) has properties of covariant momenta when acting on fermionic variables \( \lambda^{a\mu} \) and \( \pi^{a\nu} \). The superalgebra of the constraints generates the \( SU(2)_{\text{local}} \otimes SU(2)_{\text{global}} \) \( n = 4 \) superconformal transformations of the components of the superfields \( Q^\mu \). In the quantum theory the first-class constraints (24) associated with the invariance of the action (15,17) become conditions on the wave function \( \Psi \) of the Universe. Therefore, any physically allowed states must obey the quantum constraints

$$
H_0 \Psi = 0, \quad Q^a \Psi = 0, \quad \overline{Q}_a \Psi = 0, \quad F_i \Psi = 0.
$$

(32)

The quantum generators \( H_0, Q^a, \overline{Q}_a \) and \( F_i \) form a closed superalgebra of the \( n = 4 \) supersymmetric quantum mechanics
\[ \{ Q_a, Q^b \} = H_0 \delta_a^b, \quad [F_i, F_j] = i \epsilon_{ijk} F_k, \quad [F_i, \overline{Q}_a] = -\frac{1}{2} (\sigma_i)^b_a \overline{Q}_b, \quad (33) \]

\[ [F_i, Q^a] = \frac{1}{2} (\sigma_i)^a_b Q^b. \]

In order to obtain the quantum expression for the Hamiltonian \( H_0 \) and for the supercharges \( Q^a \) and \( \overline{Q}_a \) we may solve the operator ordering ambiguity, for example following the works.\(^{11,13} \)

### III. CONCLUSIONS

On the basis of the local \( n = 4 \) supersymmetry the superfield action for the Bianchi-types cosmological models is formulated. It is shown, that the action (33) has the form of the localized version of \( n = 4 \) supersymmetric quantum mechanics. Due to the quantum supersymmetric algebra (33), the Wheeler-DeWitt equation, which is of the second-order, can be replaced by the four first-order supercharge operator equations constituting its supersymmetric “square root”.

It would be very interesting to consider the interaction with matter fields and analyze the spontaneous breaking of \( n = 4 \) local supersymmetry. We hope, that for those more general supersymmetric cosmological models than in Ref. 10, we can find a normalizable wavefunction. The details of this study will be given elsewhere.

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1. C.W. Misner, in Magic Without Magic, edited by J.R. Klauder (Freeman, San Francisco, 1972).
2. C. Teitelboim, Phys. Rev. Lett. **38**, 1106 (1977).
3. S. Deser, J.H. Kay and K.S. Stelle, Phys. Rev. **D16**, 2448 (1977); E.S. Fradkin and M.A. Vasiliev, Phys. Lett. **B72**, 70 (1977); M. Pilati, Nucl. Phys. **B132**, 138 (1978); T. Jacobson, Class. Quantum Grav. **5**, 923 (1988).
4. A. Macías, O. Obregón and M.P. Ryan Jr, Class. Quantum Grav. **4**, 1477 (1987).
5. P.D. D’Eath, *Supersymmetric Quantum Cosmology*, (Cambridge: Cambridge University Press, 1996); P.V. Moniz, *Supersymmetric Quantum Cosmology*, Int. J. Mod. Phys. **A11**, 4321-4382 (1996).
6. O. Obregón, J.J. Rosales and V.I. Tkach, Phys. Rev. **D53**, 1750 (1996).
7. R. Graham, Phys. Rev. Letters 67, 1381 (1991); E.E. Donets, M. N. Tentyukov and M. M. Tsulaia, Phys. Rev.D59, 023515 (1999).

8. V.I. Tkach, J.J. Rosales and O. Obregón, Class. Quantum Grav. 13, 2349 (1996); V. I. Tkach, J. J. Rosales and J. Socorro, Class. Quantum Grav. 16, 797 (1999).

9. V.I. Tkach, O. Obregón and J.J. Rosales, Class. Quantum Grav. 14, 339 (1997).

10. O. Obregón, J.J. Rosales, J. Socorro and V.I. Tkach, Class. Quantum Grav. 16, 2861 (1999).

11. A. Pashnev, J.J. Rosales, V.I. Tkach and M. Tsulaia, Phys. Rev.D64, 087502, (2001).

12. E. A. Ivanov, S.O. Krivonos and A. I. Pashnev, Class. Quantum Grav. 8 19 (1991); E.E. Donets, A. Pashnev, V.O. Rivelles, D. Sorokin and M. Tsulaia, Phys. Lett. B 484, 337 (2000).

13. V. de Alfaro, S. Fubini, S. Furlan and M. Roncadelli, Nucl. Phys.B 296,402 (1998); E.E. Donets, A. Pashnev, J.J. Rosales and M.M. Tsulaia, Phys. Rev.D 61, 043512-1, (2000).