A KIND OF GENERALIZED TRANSVERSALITY THEOREM
FOR C r MAPPING WITH PARAMETER

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ABSTRACT. The author considers a generalized transversality theorem of the mappings with parameter in infinite dimensional Banach space. If the mapping is generalized transversal to a single point set, and in the sense of exterior parameters, the mapping is a Fredholm operator, then there exists a residual set of parameter, such that the Fredholm operator is generalized transversal to the single point set.

1. Introduction. Transversality is an important and basic notion in geometry and analysis. The celebrated transversality theorem of Thom is always used to characterize the stability, and has many important applications in nonlinear differential equations. Zeidler has pointed out that transversality is certainly one of the most important concepts in modern mathematics, which provides an answer to the question that when the preimage of a manifold is still a manifold.

We often meet an equation with a parameter

\[ F(x, s) = 0. \]

The following phenomenon has been observed: a branch of solutions \( x(s) \) depending on \( s \), is either disappeared or split into several branches, as \( s \) attains some critical values. This kind of phenomenon is called bifurcation.

It is well known that all eigenvalues of second-order ordinary differential equations on bounded intervals with Dirichlet data are simple, but it is not true for partial differential equations, for example, the Laplacian on a ball. In order to study the simplicity of eigenvalues of the Laplacian on bounded domains with Dirichlet data, Chang established a transversality theorem with one-parameter (see Theorem 1.3.18 in [1]). Let \( F(u, s) : X \times S \to Z \) be a \( C^r \) mapping, where \( X, Z \) are Banach spaces, \( S \) is a \( C^r \) Banach manifold. Assume that \( F(u, s) \nmid \{\theta\} \), \( f_s(u) = F(u, s) \) is a Fredholm mapping. Then there exists a residual set \( \Sigma \subset S \), such that \( f_s \nmid \{\theta\} \), for all \( s \in \Sigma \). This theorem showed that for most domains, all eigenvalues of second-order partial differential equations on bounded intervals with Dirichlet data are simple. If \( \{\theta\} \) is a critical value of \( F(u, s) \), does the transversality theorem hold?

In this paper, the author mainly considers the above transversality theorem if \( \{\theta\} \) is a regular value or some special critical value, and establishes two examples and a generalized transversality theorem. In the Example 3.1, the mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \)

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does not have any nontrivial regular value. The preimage of every critical value of \( f \) is a smooth regular submanifold of preimage space \( \mathbb{R}^2 \). This example shows that there are some critical values which have similar properties to those of regular values. This kind of critical values and regular values are called generalized regular values. On the other hand, the concept of transversality, \( f \cap P \text{ mod } N \), requires that \( \text{Im}(f'(x)) + TP_{f(x)} = TN_{f(x)} \) for any \( x \in f^{-1}(P) \). However, it is often the case that this equation does not hold.

In [4], Ma established the concept of generalized transversality and proved the generalized transversal preimage theorem by some continuity characteristics of generalized inverses of singular bounded linear operators in Banach spaces, and provided three examples of generalized transversality. The mappings \( f : X \to Z \) in his examples are generalized transversal to some submanifold of Banach space \( Z \). However, the mappings in his examples are not generalized transversal to \( \{ \theta \} \), where \( \theta \) is the origin of \( Z \). Here, the author provides the mapping \( f \) in Example 3.1 which is generalized transversal to the origin \( \{ \theta \} \), and the mapping \( f \) in Example 3.2 which is generalized transversal to a submanifold. The figures of these examples show the relative position of the submanifold \( P \) and the image of \( f \). After that, the author considers the transversality theorem for generalized regular values which contains some critical values. A generalized transversality theorem in infinite dimensional Banach spaces is established. The proofs are mainly based on the theory of generalized inverse and singularities.

The structure of this paper is as follows. In Section 2, some preliminaries needed in the sequel are presented. In Section 3, two examples for generalized regular value and generalized transversality are established. In Section 4, the generalized transversality theorem of Fredholm operators with parameter between infinite dimensional Banach spaces is established.

In this paper, all the derivatives are Fréchet derivatives, all symbols could be found in [5, 6].

2. Preliminaries. This section presents an introduction to generalized inverse, regular value and generalized transversality.

Let \( M, N \) be \( C^k \)-Banach manifolds, \( k \geq 1 \). Then \( f : M \to N \) is called \( C^r \) \((r \leq k)\), which means that for each point \( x \in M \) and admissible charts \((U, \varphi), (V, \psi)\) of \( M, N \) respectively, with \( x \in U, f(x) \in V \), \( \overline{f} = \psi \circ f \circ \varphi^{-1} : X_\varphi \to Y_\psi \) is \( C^r \) at the point \( x_\varphi = \varphi(x) \), where \( X_\varphi \) is the chart space of \( M \), and \( Y_\psi \) is the chart space of \( N \). \( \overline{f} \) is called the representative of \( f \).

The linearization \( f'(x) \) of \( f : M \to N \) at the point \( x \) is the tangent map from \( TM_x \) to \( TN_{f(x)} \), where \( TM_x \) is the tangent space of Banach manifold \( M \) at the point \( x \), and \( TN_{f(x)} \) is the tangent space of Banach manifold \( N \) at the point \( f(x) \). There is no denying the fact that the tangent map is linear and continuous. If \( \overline{f} \) is the representative of \( f \) in the admissible charts \((U, \varphi)\) and \((V, \psi)\) with \( x \in U \) and \( f(x) \in V \). Then

\[
\overline{f}(x_\varphi) : X_\varphi \to Y_\psi
\]

is the representative of the tangent map

\[
f'(x) : TM_x \to TN_{f(x)},
\]

where \( \overline{f}(x_\varphi) \) is the Fréchet derivative of the representative \( \overline{f} \).

The basic ideas of the theory of generalized inverse are as follows. Let \( X, Y \) be Banach spaces, and \( B(X, Y) \) be the space of all bounded linear operators from the
Banach space $X$ to the Banach space $Y$. An operator $A^+ \in B(Y, X)$ is said to be a \textit{generalized inverse} of $A \in B(X, Y)$ provided $A^+ A A^+ = A^+$ and $AA^+ A = A$.

A bounded linear operator $T_x : X \rightarrow B(X, Y)$ is \textit{locally fine} at $x_0 \in X$, if $T_{x_0}$ has a generalized inverse $T_{x_0}^+$, and there exists a neighborhood $U_0$ at $x_0$ such that $\text{Im}(T_x) \cap \text{Ker}(T_{x_0}^+) = \{0\}$, where the symbols $\text{Im}(T_x)$ denotes the image of $T_x$, and $\text{Ker}(T_{x_0}^+)$ denotes the kernel of $T_{x_0}^+$. (See [2]) This concept induces the following two concepts of \textit{locally fine point}. One is defined in Banach spaces, the other in Banach manifolds. Let $X, Y$ be Banach spaces, $f : X \rightarrow Y$ be a $C^1$ mapping, the point $x \in X$ is called to be locally fine point of $f$ if $f'(x)$ is locally fine at the point $x$. Assume that $M, N$ are $C^k$—Banach manifolds ($k \geq 1$), $f : M \rightarrow N$ is a $C^1$ mapping. A point $x_0 \in M$ is locally fine point of $f$, if and only if $x_0 \in \varphi(x_0)$ is a locally fine point of the representative $\mathcal{T}$ of $f$, here $\mathcal{T} = \psi \circ f \circ \varphi^{-1}$ with charts $(U, \varphi)$ of $M$ at $x$ and $(V, \psi)$ of $N$ at $y = f(x)$. That is, there exists a bounded generalized inverse $(\mathcal{T}(x_0^\varphi))^+ \mathcal{T}(x_0^\varphi)$ such that

$$\text{Im}(\mathcal{T}(x_0^\varphi)) \cap \text{Ker}((\mathcal{T}(x_0^\varphi))^+) = \{0\}$$

at the point $x_0 \in \varphi(U)$ which is close to the point $x_0^\varphi$.

The concepts of locally fine point in different cases are quite useful in the theory of generalized inverse.

\textbf{Lemma 2.1.} \textit{Assume that $M, N$ are $C^r$ ($r \geq 1$) Banach manifolds, $f : M \rightarrow N$ is a $C^1$ mapping. $y \in N$ is a generalized regular value of $f$, which means that the set $f^{-1}(y)$ is empty or consists only of locally fine points of $f$. Then the pre-image $S = f^{-1}(y)$ is a submanifold of $M$ with the tangent space $TS_x = \text{Ker}(f'(x))$, for all $x \in S$. (See [3])}

Suppose that $f : U \subset E \rightarrow F$ is a $C^1$ map, where $U$ is open, $x_0$ is a regular point of $f$ if and only if $f'(x_0)$ is surjective and the kernel space $\text{Ker}(f'(x_0))$ splits $E$. According to the concept of locally fine point mentioned above, the regular point $x_0$ is a locally fine point of $f$, but the converse statement may not be true (see the Example [4] in this paper).

If $y \in F$ is a regular value of $f$, which means that $f^{-1}(y)$ is empty or consists of only regular points of $f$, then $y$ must be the generalized regular value of $f$. We call the complement of generalized regular value as \textit{sharp—critical value}. The following Sard-Smale Lemma holds in the sense of sharp critical value.

\textbf{Lemma 2.2} (Sard-Smale Lemma). \textit{Suppose that $X$ is a separable Banach space and $Y$ is a Banach space. Let $f \in C^r(U, Y)$ be a Fredholm mapping, where $U \subset X$ is an open set. If $r > \max\{0, \text{ind}(f)\}$, then the set of critical values is of the first category.}

Transversality is a promotion of regular value. Assume that $M, N$ are $C^r$ ($r \geq 1$) Banach manifolds, $P$ is a submanifold of $N$, $f : M \rightarrow N$ is a $C^r$ mapping. Then $f$ is called transversal to $P$ in $N$ if and only if

$$\text{Im}(f'(x)) + TP_{f(x)} = TN_{f(x)},$$

for any $y \in f(M) \cap P$, the preimage $(f'(x))^{-1}(TP_{f(x)})$ splits $TM_x$ for all $x$ with $f(x) = y$. We write as $f \cap P \mod N$.

Thom’s famous result, the transversality theorem, which provides an answer to the question that when the preimage of a manifold is still a manifold. Let $M, N$ be $C^r$ ($r \geq 1$) Banach manifolds, $P$ be a submanifold of $N$. Suppose that $f : M \rightarrow N$
is a $C^r$ mapping. If $f \cap P \mod N$, then the preimage $S = f^{-1}(P)$ is a submanifold of $M$ with the tangent space $TS_x = (f'(x))^{-1}(TP_{f(x)})$, for any $x \in S$.

However, in our daily studying, we often need transversality theorem in case that $\text{Im}(f'(x)) + TP_{f(x)} \neq TN_{f(x)}$. So, the concept of generalized transversality and generalized transversality theorem are necessary.

**Definition 2.3.** Let $f : M \to N$ be a $C^r$ mapping and $P$ be a submanifold of $N$. Then $f$ is generalized transversal to $P$, and write as $f \cap_G P \mod N$, if for each $x_0 \in f^{-1}(P)$, the following two conditions are satisfied.

(i) For each $x \in f^{-1}(P)$, $\text{Im}(f'(x)) + TP_{f(x)}$, $(f'(x))^{-1}(TP_{f(x)})$ and $TP_{f(x)}$ split $TN_{f(x)}$, $TM_x$ and $\text{Im}(f'(x)) + TP_{f(x)}$, respectively.

(ii) For any $x_0 \in f^{-1}(P)$, there exists a neighborhood $U_0$ at $x_0$ and a subbundle $\bigcup m_x$ of $T_{U_0}M$ (the restriction of $TM$ to $U_0$) such that $m_{x_0}$ is a topological complement of $(f'(x_0))^{-1}(TP_{f(x_0)})$, and $(f'(x))^{-1}(TP_{f(x)}) + m_x = TM_x$ for any $x \in U_0$. (See [4])

**Lemma 2.4.** Let $M, N$ be $C^k$ Banach manifolds, $f : M \to N$ be a $C^r$ mapping, and $P$ be a submanifold of $N$. If $f \cap_G P \mod N$, then the preimage $S = f^{-1}(P)$ is a submanifold of $M$ with the tangent space

$$TS_x = (f'(x))^{-1}(TP_{f(x)})$$

for any $x \in S$. If $P$ only consists of a single point $y \in N$, then $f$ is generalized transversal to $P$ if and only if $y$ is a generalized regular value of $f$. (See [4])

3. Examples. There are two examples in this section. The difference between regular value and generalized value is showed in the first example. The preimage of every critical value of $f$ in this example is a smooth regular submanifold of the preimage space. The second one is an example of generalized transversality. By the way, the mapping in the first example is generalized transversal to the origin $\{\theta\}$, the proof of this generalized transversality is presented in the beginning of the next section.

**Example 3.1.** Let $U = \{p = (x, y) \mid x^2 + y^2 \neq 0\}$. Define $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ as follows: $f(p) = (f_1(p), f_2(p))$, $f_1(p) = e^{x^2+y^2} - x$, $f_2(p) = x^2 + y^2 - 1$, then

$$f'(P) = \begin{pmatrix} 2xe^{x^2+y^2} & 2ye^{x^2+y^2} \\ 2x & 2y \end{pmatrix}.$$  

Notice that $f'(p)$ is not surjective, for all $p \in U$, $f$ has not regular points and nontrivial regular values. However, for any $p \in U$, $p$ is a locally fine point for $f$, then $f$ has nontrivial generalized regular values. All $q \in f(U)$ are critical values. In fact, for $z > -1$, $f^{-1}(e^{z+1} - e, z) = \{p = (x, y) \mid x^2 + y^2 - 1 = z\}$ consists of only locally fine points of $f$. That is to say, $(e^{z+1} - e, z)$ is a generalized regular value of $f$. In particular, we obtain that the critical value $\theta = (0, 0)$ of $f$ is a generalized regular value of $f$.

**Example 3.2.** An example of generalized transversality. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $(x, y, z) = f(s, t) = (s, s^3, t)$, and let $P$ be the $z$-axis in $\mathbb{R}^3$. Then, $f^{-1}(P)$ is the $t$-axis in $\mathbb{R}^3$, and $t$-axis is a submanifold of $\mathbb{R}^2$. $f$ is not transversal to $P \mod \mathbb{R}^3$. Indeed, $f \cap_G P \mod \mathbb{R}^3$ (see Figure 1).
Figure 1: \( f(s,t) = (s, s^3, t) \) is generalized transversal to \( P = \{(0,0, z) \mid z \in \mathbb{R} \} \mod \mathbb{R}^3 \).

Proof. For any \( (s,t) \in \mathbb{R}^2 \),

\[
f'(s,t) = \begin{pmatrix} 1 & 0 \\ 3x^2 & 0 \\ 0 & 1 \end{pmatrix}.
\]

\( f^{-1}(P) \) is the \(-\)axis in \( \mathbb{R}^2 \), and \( \text{Im}(Tf(s,t)) = \{(a_1, 3s^2a_1, a_2) \mid \forall (a_1, a_2) \in \mathbb{R}^2 \} \).

So for each \( (s,t) \in f^{-1}(P) \), we have \( \text{Im}(Tf(s,t)) + TP(0,0,t) \not\subseteq \mathbb{R}^3 \), that is, \( f \) is not transversal to \( P \mod \mathbb{R}^3 \). However, \( f \nabla G \mod \mathbb{R}^3 \). In fact, \( (Tf(s,t))^{-1}(TP(s,s^3,t)) \) is the \(-\)axis in \( \mathbb{R}^2 \), let \( E_0 \) be the \( s \)-axis in \( \mathbb{R}^2 \), then for any \( (s,t) \in \mathbb{R}^2 \),

\[
(Tf(s,t))^{-1}(TP(s,s^3,t)) + E_0 = \mathbb{R}^2.
\]

This proves that \( f \nabla G \mod \mathbb{R}^3 \). $\square$

4. Main results. Consider the mapping \( F(u,s) : \mathbb{R}^2 \to \mathbb{R}^2 \),

\[
F(u,s) = (e^{u^2+s^2} - e, u^2+s^2 - 1),
\]
then \( F'(u,s) \) is a \( C^r \) \((r \geq 1) \) Fredholm operator with index \( \text{ind}F'(u,s) = 0 < r \), where

\[
F'(u,s) = \begin{pmatrix} 2ue^{u^2+s^2} & 2se^{u^2+s^2} \\ 2u & 2s \end{pmatrix}.
\]

Let \( \theta = (0,0) \in \mathbb{R}^2 \), \( P = \{\theta\} \), \( F^{-1}(P) = \{(u,s) \mid u^2 + s^2 = 1\} \). For any \( (u,s) \in F^{-1}(P) \), \( \dim \text{Im}F'(u,s) = 1 \), \( \theta = (0,0) \) is a critical value, \( TP_F(u,s) = TP(0,0) = 0 \), then \( \dim \text{Im}F(u,s) + TP_F(u,s) \not\subseteq \mathbb{R}^2 \). That is, \( F(u,s) \) is not transversal to \( P = \{\theta\} \mod \mathbb{R}^2 \).

According to the Example 3.1 for every \( z > -1 \), \( (e^{z+1} - e, z) \) is a generalized regular value of \( F(u,s) \),

\[
F^{-1}(e^{z+1} - e, z) = \{(u,s) \mid u^2 + s^2 - 1 = z\}
\]
is a smooth regular submanifold of \( \mathbb{R}^2 \). In particular, the critical value \( \theta = (0,0) \) is a generalized regular value of \( F(u,s) \). Then \( F(u,s) \nabla G \{\theta\} \mod \mathbb{R}^2 \) by Lemma 2.4 (see Figure 2).
Figure 2: $F(u, s) = (e^{u^2 + s^2} - e, u^2 + s^2 - 1)$ is generalized transversal to $P = \{\theta\}$ mod $\mathbb{R}^2$.

In the following, the generalized transversality theorem in case that $\theta = (0, 0)$ is a generalized regular value of $F(u, s)$ is established. All Banach spaces in the theorem are infinite dimensional.

**Theorem 4.1.** Suppose that $X, Z$ are Banach spaces, where $X$ is separable, $S$ is a $C^{r}$ Banach manifold, $\theta$ is a point in $Z$. Assume that $F \in C^{r}(X \times S, Z)$ satisfies

1. $F \cap_{G} \{\theta\}$, and
2. for all $s \in S, f_{s}(u) = F(u, s)$ is a Fredholm mapping with index satisfying $\max\{0, \text{ind}(f_{s})\} < r$.

Then there exists a residual set $\Sigma \subset S$, which is the countable intersection of open dense sets, such that for all $s \in \Sigma, f_{s} \cap_{G} \{\theta\}$.

**Proof.** Let $V = F^{-1}(\theta)$, by Lemma 2.4 we can draw the conclusion that $V$ is a submanifold of $X$ since $F \cap_{G} \theta$.

Define the injection: $i : V \to X \times S$, and the projection $\pi : X \times S \to S$, and $p : X \times S \to X$. In the following, all of them are restricted on $V$.

We claim that $\pi \circ i$ is a Fredholm mapping with $\text{ind}(\pi \circ i) = \text{ind}(f_{s})$.

Indeed, by Lemma 2.4 we know that $\theta$ is a generalized regular value of $F$ since $F \cap_{G} \{\theta\}$. Then, according to Lemma 2.1, $\text{Ker}(F'((v)) = T_{v}(V))$,

$$\text{Ker}(F'(v)) = T_{v}(V) = \text{Im}((p \circ i)'(v)) \oplus \text{Im}((\pi \circ i)'(v)),$$

for any $v = (x, s) \in T_{v}(V)$.

Since $F \cap_{G} \{\theta\}$, we have a direct sum decomposition

$$X \times T_{s}(S) = (F'(v))^{-1}(\theta) \oplus Y$$

$$= \text{Ker}(F'(v)) \oplus Y$$

$$= T_{v}V \oplus Y,$$

where $F'(v) : Y \to Z$ is an isomorphism. By the assumption that $f'_{s}(x)$ is a Fredholm mapping, we have direct sum decompositions

$$X = \text{Im}((p \circ i)') \oplus Y_{1}, \quad Z = Z_{1} \oplus Z_{2},$$

such that $f'_{s} : Y_{1} \to Z_{1}$. Thus $Y = Y_{1} \oplus Y_{2}$, where $T_{s}(S) = \text{Im}((\pi \circ i)'(v) \oplus Y_{2}$.

From $F'(v) = f'_{s}(x) \oplus \partial_{s}F(v)$, we have $\partial_{s}F(v) : Y_{2} \to Z_{2}$ is an isomorphism. Thus

$$\text{Ker}f'_{s}(x) = \text{Im}(p \circ i)' = \text{Ker}(\pi \circ i)'(v),$$
\[ \text{Coker} f'_s(x) = Z_2 \cong Y_2 = \text{Coker}(\pi \circ i)'(v). \]

Therefore, \( \text{ind}(f_s) = \text{ind}(\pi \circ i). \)

According to the Sard-Smale Lemma, the set of sharp critical values of \( \pi \circ i \) is of the first category. That is to say, it is a countable union of closed nowhere dense sets. We note these sets of sharp critical values of \( \pi \circ i \) as \( \Pi_\alpha, \alpha \in \Lambda(\Lambda \text{ is an index set}) \). Then for any \( \alpha \in \Lambda, Z \setminus \Pi_\alpha \) is an open dense set. So the set of generalized regular values, which is the intersection of \( Z \setminus \Pi_\alpha \), is a residual set. That is to say, \( \Sigma \subset S \), which is the set of generalized regular value of \( \pi \circ i \), is a residual set.

In the following, we only need to show that for a generalized regular value \( s \in \Sigma, f_s \cap G \{ \theta \} \). This means that \( \theta \) is a generalized regular value of \( f_s \), that is, \( f_s^{-1}(\theta) \) is empty or only consists of generalized regular points of \( f_s \).

If \( f_s^{-1}(\theta) = \phi \), then the theorem will be true automatically. Now, we consider the case that \( f_s^{-1}(\theta) \neq \phi \). All we need to show is that for all \( x_0 \in f_s^{-1}(\theta) \), \( x_0 \) is a locally fine point of \( f_s \).

For all \( s \in \Sigma \subset S \), we have \( v_0 = (x_0, s) \in f_s^{-1}(\theta) \times \Sigma \subset X \times S \). Since \( F \cap G \{ \theta \} \)
\( v_0 \) is a locally fine point of \( F(v) \). There exists an operator \( F'(v_0)^+ \) which is a generalized inverse of \( F'(v_0) \), and a neighborhood \( U \) of \( v_0 \), such that
\[ \text{Im}(F'(v)) \cap \text{Ker}(F'(v_0)^+) = \{0\}, \]

hence
\[ \text{Im}(F'(v)) = \text{Im}(f'_s(x) \oplus \partial_s F(v)) = \text{Im}(f'_s(x)) \oplus \text{Im}(\partial_s F(v)), \]

and
\[ \text{Im}(f'_s(x)) \subset \text{Im}(F'(v)), \]

according to (1) and (2),
\[ \text{Im}(f'_s(x)) \cap \text{Ker}(F'(v_0)^+) = \{0\}. \]

On the other hand, we know that
\[ \text{Ker}(F'(v_0)^+) = \text{Ker}((f'_s(x_0) \oplus \partial_s F(v_0))'). \]

Now, let us consider the structure of \( \text{Ker}(F'(v_0)^+) \).
If \( f'_s(x_0) = 0 \), then
\[ \text{Ker}((f'_s(x_0) \oplus \partial_s F(v_0))') = \text{Ker}((\partial_s F(v_0))^+). \]

If \( \partial_s F(v_0) = 0 \), then
\[ \text{Ker}((f'_s(x_0) \oplus \partial_s F(v_0))') = \text{Ker}((f'_s(x_0))^+). \]

If \( f'_s(x_0) \neq 0 \), and \( \partial_s F(v_0) \neq 0 \), then we note \( \text{Ker}(F'(v_0)^+) \) as \( \text{Ker}_{\text{span}} \), that is
\[ \text{Ker}((f'_s(x_0) \oplus \partial_s F(v_0))') = \text{Ker}_{\text{span}}. \]

In summary, the null space \( \text{Ker}(F'(v_0)^+) \) is spanned by
\[ \{\text{Ker}((\partial_s F(v_0))^+), \text{Ker}((f'_s(x_0))^+)\}, \text{Ker}_{\text{span}}, \}

so
\[ \text{Ker}(F'(v_0)^+) = \text{span}\{\text{Ker}((\partial_s F(v_0))^+), \text{Ker}((f'_s(x_0))^+)\}, \text{Ker}_{\text{span}}, \}

that is to say
\[ \text{Ker}(f'_s(x_0)^+) \subset \text{Ker}(F'(v_0)^+), \]
by (3) and (4), it follows that
\[ \text{Im}(f'_s(x)) \cap \text{Ker}(f'_s(x_0))^+ = \{0\}, \]
which means that \( x_0 \) is a locally fine point of \( f_s \).

According to Lemma 2, the conclusion that \( f_s \in \{\theta\} \) is hold for any generalized regular value \( s \in \Sigma \).

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