MINIMAL PATH AND ACYCLIC MODELS

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Abstract. In this paper, firstly, we will study the structure of the path complex \((\Omega_*(G)_Z, \partial)\) via the \(Z\)-generators of \(\Omega_*(G)_Z\), which is called the minimal path in [5]. In particular, we will study various examples of the minimal paths of length 3. Secondly, we will show that the supporting graph of minimal graph is acyclic in the path homology. Thirdly, we will consider the applications of the acyclic models.

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1. Introduction

In the past few years, there are several attempts to define the homotopy and (co)homology of (di)graphs, e.g. via cliques [2], or via Hochschild (co)homology [14]. The homotopy and (co)homology theory of digraphs considered in this paper were introduced in [7–11]. In particular, the (path) homology (see Definition 2.3) is defined via the path complex, which could be regarded as a generalization of the notion of a simplicial complex. Such a homology theory has many advantages in comparison with the previously studied notions of graph homologies:
(1) The chain complex associated with a path complex has a richer structure than simplicial chain complexes. It contains not only cliques but also binary hypercubes and many other subgraphs. (One can also see more examples in Subsection 3.4.)

(2) It satisfies the properties that are analogous to Eilenberg-Steenrod axioms.

(3) It is well linked to graph-theoretical operations. For example, the Künneth formula holds for both Cartesian product of two digraphs and join of two digraphs.

(4) There is a dual cohomology theory of digraph which coincides with the one developed by Dimakis and Müller-Hoissen in [3, 4] as a reduction of the “universal differential algebra”.

1.0.1. Motivations. There are rapidly increasing developments of such a digraph theory, such as its relation with simplicial complex and cubic complex [12] under some conditions, homotopy theory of digraph [10], the discrete Morse theory on digraph [15]. However, there still remain many basic and important questions. For example, they proved that the concatenation of paths induces a well defined cup product $\cup$ on cohomology. But they do not know whether the cup product is skew-symmetric, since one can not arbitrarily change the order of the path indices (due to the restriction of the directions of edges). To answer such a question becomes one of our motivations of this paper.

We checked the skew-symmetry holds in some simple examples by following the definitions, that is,

- Compute the cohomology of digraphs first;
- Choose representatives $\phi, \psi$, such that $[\phi] \in H^p(G)$ and $[\psi] \in H^q(G)$. Prove that $\phi \cup \psi$ and $(-1)^{pq} \psi \cup \phi$ are cohomologously equivalent.

We find that both steps are complicated (even in the case that $G$ is the Cartesian product of monotone cycles) and uninspiring. Then we switch to a traditional and but also more powerful method, via the acyclic model. Thus, a natural question arises: what is the acyclic model in path complex of digraph? Under the stronger regular condition, the answer is given by the minimal path (see, Definition 3.1) and its supporting digraph (see Definition 3.4).

The new question forces us to have a better understanding of the structure of the path complex of digraph.

1.0.2. Main theorems. The definition of minimal path here was introduced in Huang-Yau’s paper [5], where the minimal path and minimal relation played an important role to obtain an integral basis of a digraph. They proved that there exists an integral basis of $\Omega_*(G)_{\mathbb{Z}}$ consisting of minimal paths. With a choice of certain integral basis, they constructed a CW complex, whose integral singular cohomology is canonically isomorphic to the path cohomology of the digraph as introduced in [11]. In particular, they proved some structure of minimal path by using the Morse theory. Here we reprove and improve their result by deeply studying the definition of path complex. Our first main theorem is as follows.

**Theorem 1.1** (Theorem 3.11). Let $P \in \Omega_n(G)$ be a minimal path with the starting point $S$ and the ending point $E$, $\text{Supp}(P)$ be its supporting digraph and $d_S, d_E$ be the two distance functions on $P$. 

(1) Let \( S_1 = d^{-1}_S(1) \) and \( E_1 = d^{-1}_E(1) \). Then \( P \) is a linear combination of allowed elementary paths from \( S \) to \( E \), with coefficients being either 1 or -1. And
\[
\partial P = \sum_{\alpha \in E_1} P_{S,n-1,\alpha} + \sum_{\beta \in S_1} P_{\beta,n-1,E} + \sum_{k \in I_P} P^k_{S,n-1,E},
\]
where
- \( P_{S,n-1,\alpha}, P_{\beta,n-1,E}, P^k_{S,n-1,E} \in \Omega_n(\text{Supp}(P)) \), and moreover
  - \( P_{S,n-1,\alpha} \) is the minimal path of length \( n - 1 \) starting at \( S \) and ending at \( \alpha \);
  - \( P_{\beta,n-1,E} \) is the minimal path of length \( n - 1 \) starting at \( \beta \) and ending at \( E \);
  - \( P^k_{S,n-1,E} \) is the minimal path of length \( n - 1 \) starting at \( S \) and ending at \( E \).
- Such \( P_{S,n-1,\alpha}, P_{\beta,n-1,E} \) are unique (up to sign) in \( \text{Supp}(P) \) for each \( \alpha \in E_1, \beta \in S_1 \).
- The set \( I_P \) in the last summand depends on \( P \), and \( |I_P| \leq 1 \). That is, there exists at most one (up to sign) minimal path of length \( n - 1 \) in \( \text{Supp}(P) \) starting at \( S \) and ending at \( E \).

(2) For any \( v \in d^{-1}_E(k) \cap \text{Supp}(P) \), in \( \text{Supp}(P) \),
- There is only one minimal path (up to sign) of length \( n - k \) starting at \( S \) and ending at \( v \), denoted by \( P_{S,n-k,v} \), as well as only one minimal path (up to sign) of length \( k \) starting at \( v \) and ending at \( E \), denoted by \( P_{v,k,E} \).
- There is at most one minimal path (up to sign) of length \( k - 1 \) starting at \( v \) and ending at \( E \), denoted by \( P_{v,k-1,E} \).

(3) Each minimal 2-face with fixed starting and ending vertices in \( \text{Supp}(P) \) (\( n \geq 2 \)) is unique.

After studying several examples of minimal 3-paths, we learn that their supporting digraphs have acyclic path homologies. A general acyclic result holds for any minimal path, which is our second main theorem.

**Theorem 1.2** (Theorem 4.1). Let \( P \in \Omega_n(G) \) be a minimal path, and \( \text{Supp}(P) \) be its supporting digraph, then
\[
H_i(\text{Supp}(P); \mathbb{Z}) = 0, \ i > 0; \quad H_0(\text{Supp}(P); \mathbb{Z}) = \mathbb{Z}.
\]

Immediately, as an application of acyclic model, we obtain

**Theorem 1.3** (Theorem 5.8). For \( \varphi \in H^p(G), \ \psi \in H^q(G) \), we have
\[
\varphi \cup \psi = (-1)^{pq} \psi \cup \varphi.
\]

1.0.3. organizations. Our paper is organized as follows. In Section 2, we will briefly recall the definition of path complex of digraph, where we restricted to a stronger regular condition.

In Subsection 3.1, we introduce the definitions of minimal path and its supporting digraph. And then we make an intensive study of its structures step by step. In Subsections 3.2 and 3.3, we arrive at our structure theorem and give the proof. In Subsection 3.4, we will give various examples of minimal paths of length 3 and study their homotopic and homological results, where the homotopy theory of digraph will be briefly recalled in advance.
In Section 4, we will state and prove our second main theorem. In Subsection 4.1, we will recall the Mayer-Vietoris exact sequence in the path complex which is one of our main tools in the proof. The whole proof of the acyclic result will be shown in Subsection 4.2.

In Section 5, we first recall the definitions of cohomology of digraphs (Subsection 5.1). Then we reformulate the cup product through the diagonal approximation map and apply the acyclic result to prove the skew-symmetry property of cup product on path cohomology of digraph in Subsection 5.2.

The Appendix is devoted to giving an example of minimal path of length 4 and explaining the proof idea of our second main theorem via such an example.

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2. Path Complex of Digraph

Let $V$ be a finite set. For any $n \geq 0$, an elementary $n$-path is any ordered sequence $i_0, \ldots, i_n$ of $n + 1$ vertices of $V$, write it as $e_{i_0 \ldots i_n}$. Let $\Lambda_n(V)_\mathbb{Z}$ be the $\mathbb{Z}$-module generated by all such elementary $n$-paths. We also call $n$ the length of the path in $\Lambda_n(V)_\mathbb{Z}$.

The $\mathbb{Z}$-homomorphism $\partial : \Lambda_n(V)_\mathbb{Z} \to \Lambda_{n-1}(V)_\mathbb{Z}$ is defined via the generators:

$$\partial e_{i_0 \ldots i_n} = \sum_{j=0}^{n} (-1)^j e_{i_0 \ldots \hat{i}_j \ldots i_n}.$$

Clearly, $\partial (\Lambda_n(V)_\mathbb{Z}) \subset \Lambda_{n-1}(V)_\mathbb{Z}$, and $\partial^2 = 0$ ($\partial$ is a boundary operator).

**Definition 2.1** ([5]). Let $G = (V, E)$ be a simple\footnote{Here simple condition means that there are no multiple edges and self loops.} finite digraph, where $V$ is the set of vertices and $E$ is the set of directed edges.

1. An elementary path $e_{i_0 i_1 \ldots i_n}$ is called allowed, if $i_{k-1}i_k$ is a directed edge in $G$, $k = 1, 2, \ldots, n$.
2. An allowed elementary path $e_{i_0 i_1 \ldots i_n}$ is called regular, if $i_0, i_1, \ldots, i_n$ are all distinct.

**Remark 2.2.** (1) The regular condition here is more restrictive than the one in [7]. In [7], the set of regular $n$-paths is defined as a quotient space $R_n(V)$:

$$R_n(V) = \Lambda_n(V) / I_n(V),$$

where $I_n(V)$ is the sub-module generated by the irregular path $e_{i_0 i_1 \ldots i_n}$, where $i_{k-1} = i_k$ for some $k = 1, \ldots, n$. One can easily check that $(R_n(V), \partial)$ forms a quotient chain complex with the induced boundary operator, we still denote by $\partial$.

Moreover, combined with the allowed condition given by the directed edges, the monotone cyclic path will provide more allowed regular paths in the sense of [7]. In particular, the double arrows $a \leftrightarrow b$ will provide more regular allowed paths than ours, such as

$$e_{abab\ldots aba}, \ e_{abab\ldots aba}, \ e_{baba\ldots ba}, \ e_{baba\ldots bab},$$

and all of them are preserved by the boundary operator $\partial$.\footnote{Here simple condition means that there are no multiple edges and self loops.}
(2) As [5] writes, with this strong regular condition, the homology groups (see, Definition 2.3) are now obviously bounded above, and Lefschetz fixed point theorem holds, both of which are not true with the old regularity condition. The reader will also find that the strong regular condition is necessary for our following analysis for the structure of minimal path.

Let $\mathcal{A}_n(G)_\mathbb{Z}$ be the free $\mathbb{Z}$-module generated by all regular allowed elementary $n$-paths. Note that, $\mathcal{A}_n(G)_\mathbb{Z} = 0$, if $n > |V(G)|$.

If we take the boundary operator on the regular allowed elementary path $e_{i_0...i_n}$, in general, $\partial e_{i_0...i_n}$ may be not in $\mathcal{A}_{n-1}(G)_\mathbb{Z}$. Then furthermore, we consider the submodule $\Omega_n(G)_\mathbb{Z}$ of $\partial$-invariant regular allowed $n$-paths. That is

$$\Omega_n(G)_\mathbb{Z} = \{ p \in \mathcal{A}_n(G)_\mathbb{Z} | \partial p \in \mathcal{A}_{n-1}(G)_\mathbb{Z} \}.$$  

**Definition 2.3.** $(\Omega_*(G)_\mathbb{Z}, \partial)$ is called the path complex of digraph $G$ with coefficients in $\mathbb{Z}$. For each $n \geq 0$,

$$H_n(G, \mathbb{Z}) := \ker(\partial : \Omega_n(G)_\mathbb{Z} \to \Omega_{n-1}(G)_\mathbb{Z})/\text{im}(\partial : \Omega_{n+1}(G)_\mathbb{Z} \to \Omega_n(G)_\mathbb{Z}),$$

it is called the $n$-th path homology of $G$.

**Remark 2.4.** (1) Similarly, one can define the reduced path complex $(\tilde{\Omega}_*(G)_\mathbb{Z}, \partial)_{n \geq -1}$ by

- $\tilde{\Omega}_n(G)_\mathbb{Z} = \Omega_n(G)_\mathbb{Z}$, for $n \geq 0$, and $\Omega_{-1}(G)_\mathbb{Z} = \mathbb{Z}$,
- $\partial : \tilde{\Omega}_0(G)_\mathbb{Z} \to \tilde{\Omega}_{-1}(G)_\mathbb{Z}$, $\partial(\sum_v a_v v) = \sum_v a_v$, where $a_v \in \mathbb{Z}$ and $v \in V(G)$.

We denote $\tilde{H}_*(G; \mathbb{Z})$ for the reduced path homology of $G$. The relation between $\tilde{H}_*(G; \mathbb{Z})$ and $H_*(G; \mathbb{Z})$ is the same as the usual one.

(2) One can also consider the path homology over a more general coefficient. For example, Grigoryan-Jimenez-Y. Muranov [7], Grigoryan-Jimenez-Muranov-Yau [8] studied the homology theory of path complex over any abelian group $K$. Grigoryan-Muranov-Yau [13] studied the Künneth formula over the field $K$.

3. **Minimal Path**

In this section, we continue focusing on the $\mathbb{Z}$-coefficient and write $\Omega_*(G)_\mathbb{Z}$ (respectively, $\mathcal{A}_*(G)_\mathbb{Z}$) as $\Omega_*(G)$ (respectively, $\mathcal{A}_*(G)$) for short. We will recall the definition of minimal path in [5] and prove some structure theorems. And then, we will study the minimal paths of length 3 in details.

### 3.1. Definitions and basic properties

For any $P = \sum_{p=1}^m c_p e_p \in \Omega_n(G)$ with $e_p \in \mathcal{A}_n(G)$ being elementary, we define

$$w(P) = \sum_{p=1}^m |c_p|,$$

and call it the width of the path $P$.

**Definition 3.1.** The path $P \in \Omega_n(G)$ is called minimal, if there does not exist sequence of integers $(d_1, \ldots, d_m) \in \mathbb{Z}^m \setminus \{(0,0,\ldots,0)\}$, such that

1. $|c_p - d_p| \leq |c_p|$ and $|d_p| \leq |c_p|$ for each $p = 1, \ldots, m$;
(2) \( P' := \sum_{p=1}^{m} d_{p}e_{p} \in \Omega_{n}(G) \) and \( w(P') < w(P) \).

The following observations are immediate from this definition.

- If such a \( P' \) in Definition 3.1 exists, we say that \( P' \) is strictly smaller than \( P \), denote by \( P' < P \). And by definition, \( P - P' \in \Omega_{n}(G) \) is also strictly smaller than \( P \).
- If \( P \) is minimal, \( -P \) is also minimal.
- Any element in \( \Omega_{n}(G) \) is a linear combination of minimal elements.
- Minimal path of length 0 is just represented by a single point (up to sign); minimal path of length 1 is just represented by a directed edge \( e_{12} \) (up to sign).

**Example 3.2.** Let us look at the minimal path in the following digraphs \( G_{1} \) and \( G_{2} \).

(1) For the digraph \( G_{1} \), \( P_{1} = e_{013} - e_{023} \in \Omega_{2}(G_{1}) \). But \( P_{1} \) is not minimal, \( e_{013} \) is strictly smaller than \( P_{1} \), and so is \( -e_{023} = P_{1} - e_{013} \). All the minimal paths of length 2 in \( G_{1} \) are given by

\[ \pm e_{013}, \pm e_{023}. \]

(2) For the digraph \( G_{2} \), \( P_{2} = e_{014} - 2e_{024} + e_{034} \in \Omega_{2}(G_{2}) \). But \( P_{2} \) is not minimal, \( e_{014} - e_{024} \) is strictly smaller than \( P_{2} \), so is \( e_{024} - e_{034} = P_{2} - (e_{014} - e_{024}) \). All the minimal paths of length 2 in \( G_{2} \) are given by

\[ \pm (e_{014} - e_{024}), \pm (e_{014} - e_{034}), \pm (e_{024} - e_{034}). \]

More examples will be studied in Subsection 3.4 and Appendix.

The following result of minimal path is elementary but important for our later argument.

**Lemma 3.3.** Any minimal path is a linear combination of allowed elementary paths with the same starting and ending vertices.

**Proof.** Let \( P \) be a minimal \( n \)-path of \( G \), we write \( P \) as

\[ P = P_{1} + P_{2} + \cdots + P_{m} \in \Omega_{n}(G), \]

where each \( P_{i} \in A_{n}(G) \) with distinct starting points \( \alpha_{i} \) respectively. Let \( \delta_{0} \) be 0-th component of \( \partial \), that is

\[ \delta_{0}(e_{i_{0}i_{1}\ldots i_{n}}) = e_{i_{1}i_{2}\ldots i_{n}}. \]

It is of course a map from \( A_{s}(G) \) to \( A_{s-1}(G) \). Thus by definition,

\begin{equation}
\partial P - \delta_{0}P = \sum_{i=1}^{m} (\partial - \delta_{0})P_{i} \in A_{n-1}(G).
\end{equation}
By assumption, each \((\partial - \delta_0)P_i\) has the distinct starting points \(\alpha_i\), then they can not be cancelled by each other. Thus, (3.1) implies
\[
(\partial - \delta_0)P_i \in A_{n-1}(G), \quad i = 1, 2, \ldots, m.
\]
Furthermore,
\[
P_i \in \Omega_n(G), \quad i = 1, 2, \ldots, m.
\]
Again since \(P_i\) has distinct starting points, we have \(P_i \leq P\), and the equality holds if and only if the remaining \(P_j = 0\). Thus the starting vertex of \(P\) is unique. One can do the similar argument for the ending vertex. \(\square\)

Since the minimal path depends on the digraph, to study its structure further, we introduce the following definition of supporting digraphs.

**Definition 3.4.** For each minimal path \(P\) in the digraph \(G\), we define \(\text{Supp}(P)\) to be the minimal subgraph of \(G\) such that \(P \in \Omega_\ast(\text{Supp}(P))\).

**Example 3.5.** In the following 3-simplex digraph \(G\), \(P = e_{0123}\) is minimal, and its supporting subdigraph is given by

\[
\begin{array}{c}
G = 0 \\
1 \\
2 \\
3 \\
\end{array} \quad \text{Supp}(P) = 0 \\
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

For each allowed elementary path \(e_I = e_{i_0i_1\ldots i_n}\), we have the position function
\[
f_I : V(e_I) \to \mathbb{N}, \quad f_I(i_a) = a.
\]
Let \(P \in \Omega_n(G)\) be a minimal path with the starting vertex \(S\) and ending vertex \(E\). For the convenience of the following discussion, we introduce the following two “distance functions” on \(V(P)\):
\[
\begin{align*}
        d_S : V(P) & \to \mathbb{N}, \quad d_S(v) = \min \{f_I(v) | e_I \text{ is a component of } P\}; \\
        d_E : V(P) & \to \mathbb{N}, \quad d_E(v) = \min \{n - f_I(v) | e_I \text{ is a component of } P\}.
\end{align*}
\]
One can understand \(d_S\) and \(d_E\) as the distance from \(v\) to \(S\) and \(E\) in \(P\) (not in \(\text{Supp}(P)\)) respectively. Clearly,
\[
d_S(S) = 0, \quad d_S(E) = n, \quad d_E(S) = n, \quad d_E(E) = 0, \quad d_S(v) + d_E(v) \leq n.
\]

**Example 3.6.** For the following digraph \(G\), the path \(P = e_{SaeE} - e_{SaeE} + e_{SdeE} + e_{SbdE} - e_{SbeE}\) is minimal. We have
Let us decompose the differential $\partial = \partial_n : \Omega_n(G) \to \Omega_{n-1}(G)$ according to the positions of the vertices in the path

$$\partial_n = \delta_0 + \delta_1 + \cdots + \delta_n.$$ 

Immediately, we have

**Lemma 3.7.** Let $P \in \Omega_n(G)$ be a minimal path, then

$$\delta_i P \in A_{n-1}(\text{Supp}(P)) \subset A_{n-1}(G), \text{ for each } i = 0, \ldots, n.$$ 

Moreover, for a regular allowed $n$-path $A$, $A \in \Omega_n(G)$ if and only if

$$\delta_i A \in A_{n-1}(G), \text{ for each } i = 0, \ldots, n.$$ 

**Proof.** For $n = 1$, the result holds naturally. Let us assume that the result holds for any $k < n$, i.e. for any minimal path $P_k$ of length $k$ in $G$,

$$\delta_i P_k \in A_{k-1}(\text{Supp}(P_k)) \subset A_{k-1}(G), \text{ for each } i = 0, \ldots, k.$$ 

Now let us consider the case $k = n$, first since $\partial P \in \Omega_{n-1}(\text{Supp}(P))$, we can expand it in terms of minimal paths in $\Omega_{n-1}(\text{Supp}(P))$ and organize it according to the starting and ending points

$$\partial P = \sum_{\alpha \in E_1} \sum_{j=1, \ldots, i_{\alpha}} c_{\alpha}^j P_{S,n-1,\alpha}^j + \sum_{\beta \in S_1} A_{\beta,n-1,E} + A_{S,n-1,E},$$ 

where

- $E_1 = d_{E}^{-1}(1)$, $S_1 = d_{S}^{-1}(1)$;
- For each $\alpha \in E_1$, $j = 1, \ldots, i_{\alpha}$, $P_{S,n-1,\alpha}^j \in \Omega_{n-1}(\text{Supp}(P))$ is the minimal path of length $n-1$ starting at $S$ and ending at $\alpha$; and the corresponding coefficients $c_{\alpha}^j \in \mathbb{Z} - \{0\}$.
- $A_{\beta,n-1,E} \in \Omega_{n-1}(\text{Supp}(P))$ is of length $n-1$ starting at $\beta$ and ending at $E$;
- $A_{S,n-1,E} \in \Omega_{n-1}(\text{Supp}(P))$ is of length $n-1$ starting at $S$ and ending at $E$.

It is easy to obtain

$$\delta_n P = \sum_{\alpha \in E_1} \sum_{j=1, \ldots, i_{\alpha}} c_{\alpha}^j P_{S,n-1,\alpha}^j.$$ 

By Lemma 3.3, we have

$$P = (-1)^n \sum_{\alpha \in E_1} \sum_{j=1, \ldots, i_{\alpha}} c_{\alpha}^j P_{S,n-1,\alpha}^j E,$$
where \( P_{S,n-1,\alpha}^j \) means the concatenation of the path \( P_{S,n-1,\alpha} \) and \( e_{\alpha}E \). That is, if we write \( P_{S,n-1,\alpha}^j \) as \( P_{S,n-1,\alpha}^j = \sum d_m e_{S,v_1,...,v_{n-2},\alpha} \), then

\[
P_{S,n-1,\alpha} = \sum d_m e_{S,v_1,...,v_{n-2},\alpha} E.
\]

Now for \( i = 0, \ldots, n - 1 \), we can compute

\[
\delta_i P = (-1)^n \sum_{\alpha \in E_1} \sum_{j=1,...,i_\alpha} \left( c_1^j \delta_i P_{S,n-1,\alpha}^j \right) E.
\]

By induction hypothesis, for each \( \alpha \in E_1 \), \( j = 1, \ldots, i_\alpha \) and for each \( i = 0, \ldots, n - 1 \) we have

\[
\delta_i P_{S,n-1,\alpha}^j \in A_{n-2}(\text{Supp}(P_{S,n-1,\alpha}^j)) \subset A_{n-2}(\text{Supp}(P)) \subset A_{n-2}(G).
\]

In particular, all the non-zero components of \( \delta_0 P_{S,n-1,\alpha}^j, \delta_1 P_{S,n-1,\alpha}^j, \ldots, \delta_{n-2} P_{S,n-1,\alpha}^j \end{equation} \) end at \( \alpha \). which means for each \( k = 0, \ldots, n - 2 \),

\[
\delta_k P = (-1)^n \sum_{\alpha \in E_1} \sum_{j=1,...,i_\alpha} \left( c_1^j \delta_k P_{S,n-1,\alpha}^j \right) E \in A_{n-1}(\text{Supp}(P)) \subset A_{n-1}(G).
\]

It is obvious that \( \delta_n P \in A_{n-1}(\text{Supp}(P)) \subset A_{n-1}(G) \), thus

\[
\delta_{n-1} P = \partial P - \sum_{k=0}^{n-2} \delta_k P - \delta_n P \in A_{n-1}(\text{Supp}(P)) \subset A_{n-1}(G).
\]

For any \( A \in \Omega_n(G) \), we can write \( A = \sum a_P P \). Then we are done. \( \square \)

Moreover, let us decompose the minimal path \( P \) according to the \( k \)-th vertices in each elementary path component of \( P \). That is,

\[
P = \sum_{I} c_I e_I = \sum_{v \in \{f_I^{-1}(k)\}_I} A_{SvE},
\]

where \( A_{SvE} \in A_n(\text{Supp}(P)) \) is the sum of path components of \( P \) with the \( k \)-th vertex being \( v \). Then we have

**Lemma 3.8.** Let \( P \in \Omega_n(G) \) be a minimal path with the decomposition (3.4), then for each \( v \in \{f_I^{-1}(k)\}_I \),

\[
\delta_j A_{SvE} \in A_{n-1}(\text{Supp}(P)), \quad j = 0, 1, \ldots, k - 1, k + 1, \ldots, n.
\]

**Proof.** The proof is almost the same as above. For \( j \neq k \), by Lemma 3.7, we have

\[
\delta_j P = \sum_{v \in \{f_I^{-1}(k)\}_I} \delta_j A_{SvE} \in \Omega_{n-1}(\text{Supp}(P)) \subset A_{n-1}(\text{Supp}(P)).
\]

Since \( \delta_j A_{SvE} \) does not kill the \( k \)-th vertex \( v \) for \( j \neq k \), then any two different \( \delta_j A_{SvE} \) and \( \delta_j A_{Sv'E} \) can not cancel each other, by (3.5), it means,

\[
\delta_j A_{SvE} \in A_{n-1}(\text{Supp}(P)).
\]

The remaining statement is obvious. \( \square \)
According to our decomposition, we can write the above $A_{S \nu E}$ as,

$$A_{S \nu E} = A_{S,k,v} \bullet A_{v,n-k,E},$$

where $A_{S,k,v}$ and $A_{v,n-k,E}$ are the front $k$-path and back $(n-k)$-path of $A_{S \nu E}$ respectively, and $\bullet$ by connecting two paths $A_{S,k,v}$ and $A_{v,n-k,E}$ at the point $v$.

**Example 3.9.** Let us look at Example 3.6 again, we can expand the minimal path $P$ according to the 1-st and 2-nd positions:

$$P = e_{Sa} \bullet (e_{aeE} - e_{aeE}) - e_{Sb} \bullet (e_{bcE} - e_{bdE}) + e_{Sd} \bullet e_{deE};$$

$$P = (e_{Sa} - e_{Sbc}) \bullet e_{cE} + e_{Sbd} \bullet e_{de} - (e_{Sae} - e_{Sde}) \bullet e_{eE}.$$

Let us state another simple structure result for $\text{Supp}(P)$ before our main structure theorem, which will play a role in our proof of acyclic result in Section 4.

**Lemma 3.10.** Let $P \in \Omega_n(G)$ be a minimal path and $f_i$ be the position function for each path component $e_i$ of $P$. Then there is no directed edge from points of $f_i^{-1}(k)$ to points of $f_i^{-1}(l)$, if $|l-k| > 2$.

**Proof.** We prove the result by induction. For minimal paths of length 0,1,2, it is obvious. Assume the result holds for minimal path of length $k < n$. Now for minimal $n$-path $P$, we have seen that

$$P = (-1)^n \sum_{\alpha \in E_1} \sum_{j=1,...,i_\alpha} c^{j}_{\alpha} P^{j}_{S,n-1,\alpha} E.$$

Similarly, one can also write $P$ as

$$P = \sum_{\beta \in S_1} \sum_{k=1,...,j_{\beta}} d^{k}_{\beta} S P^{k}_{\beta,n-1,E}.$$

Applying the induction hypothesis on the two kinds of the expression for $P$, we are done. □

3.2. The structure theorem. Now, let us improve the decomposition result for $\partial P$ and explore the deep structure of the minimal path. Let us state our first main theorem as follows.

**Theorem 3.11** (Structure Theorem). Let $P \in \Omega_n(G)$ be a minimal path with the starting point $S$ and ending point $E$, $\text{Supp}(P)$ be its supporting directed graph and $d_S$, $d_E$ be the functions defined above.

(1) Let $S_1 = d_S^{-1}(1)$ and $E_1 = d_E^{-1}(1)$. Then $P$ is a linear combination of allowed elementary paths from $S$ to $E$, with coefficients being either 1 or -1. And

$$\partial P = \sum_{\alpha \in E_1} P_{S,n-1,\alpha} + \sum_{\beta \in S_1} P_{\beta,n-1,E} + \sum_{k \in I_\rho} P^{k}_{S,n-1,E},$$

where

- $P_{S,n-1,\alpha}, P_{\beta,n-1,E}, P^{k}_{S,n-1,E} \in \Omega_{n-1}(\text{Supp}(P))$, and moreover
  - $P_{S,n-1,\alpha}$ is the minimal path of length $n-1$ starting at $S$ and ending at $\alpha$;
  - $P_{\beta,n-1,E}$ is the minimal path of length $n-1$ starting at $\beta$ and ending at $E$;
  - $P^{k}_{S,n-1,E}$ is the minimal path of length $n-1$ starting at $S$ and ending at $E$.
- Such $P_{S,n-1,\alpha}, P_{\beta,n-1,E}$ are unique (up to sign) in $\text{Supp}(P)$ for each $\alpha \in E_1$, $\beta \in S_1$. 

The set $I_P$ in the last summand depends on $P$, and $|I_P| \leq 1$. That is, there exists at most one (up to sign) minimal path of length $n - 1$ in $\text{Supp}(P)$ starting at $S$ and ending at $E$.

(2) For any $v \in d_E^1(k) \cap \text{Supp}(P)$, in $\text{Supp}(P)$,

- There is only one minimal path (up to sign) of length $n - k$ starting at $S$ and ending at $v$, denoted by $P_{S,n-k,v}$, as well as only one minimal path (up to sign) of length $k$ starting at $v_k$ and ending at $E$, denoted by $P_{v,k,E}$.

- There is at most one minimal path (up to sign) of length $k - 1$ starting at $v$ and ending at $E$, denoted by $P_{v,k-1,E}$.

(3) Each minimal 2-face with fixed starting and ending vertices in $\text{Supp}(P)$ ($n \geq 2$) is unique.

We will prove the results (1) (2) (3) at the same time by induction. Before the general proof, let us look at the minimal path of length 2. There are only two kinds of minimal path of length 2: write the minimal path $P$ as

$$P = \sum_{a \in V(P) \setminus \{S,E\}} c_a e_{SaE}, \quad c_a \in \mathbb{Z}. $$

By Lemma 3.3, the requirement $\delta_1 P = -\sum_a c_a e_{SE} \in A_1(G)$ tells us

- If $S \rightarrow E$ in $G$, then for $a$ with $c_a \neq 0$, $\tilde{P} := \text{sign}(c_a)e_{SaE} \leq P$. By minimal condition for $P$, the equality holds if and only if $c_a = \pm 1$, and other $c_b = 0$ for $b \neq a$. The corresponding subdigraph $\text{Supp}(P)$ is the triangle

$$\text{Supp}(P) = \begin{array}{c}
\uparrow \\
S \rightarrow \\
\downarrow \rightarrow \ \\
E
\end{array}$$

- If $S \nrightarrow E$ in $G$, then

$$\delta_1 P = -\sum_a c_a e_{SE} = 0.$$ 

which means there exist $a, b$ such that $c_a, c_b \neq 0$, thus $\tilde{P} := \text{sign}(c_a)(e_{SaE} - e_{SbE}) \leq P$. By minimal condition for $P$, the equality holds if and only if $c_a = -c_b = \pm 1$, and other $c_i = 0$ for $i \neq a, b$. The corresponding subdigraph $\text{Supp}(P)$ is the square

$$\text{Supp}(P) = \begin{array}{c}
\uparrow \\
S \rightarrow \\
\uparrow \rightarrow \ \\
E
\end{array}$$

Now let us illustrate the proof idea of the theorem. First, by the rough decomposition (3.2), we can write

(3.6) 

$$P = (-1)^n \sum_{a \in E_{k}} \sum_{i_1, \ldots, i_a} c_{i_1}^{j_1} P_{S,n-1,\alpha}^j \alpha E, \quad c_{i_1}^{j_1} \in \mathbb{Z} - \{0\}.$$ 

each $P_{S,n-1,\alpha}^j \in \Omega_{n-1}(\text{Supp}(P))$ is the minimal path starting at $S$ and ending at $\alpha$. If $i_\alpha > 1$, or if there exists some $c_\alpha^j$, $|c_\alpha^j| \geq 2$, we will construct a path $\tilde{P} \in \Omega_{n}(\text{Supp}(P))$ which is strictly smaller than $P$, as we did in the length 2 case. To simplify the combinatorial discussion in the proof, we
generalize the partial order on $\Omega_n(G)$ in Definition 3.1 to the partial order on $A_n(G)$ and introduce the definition of $\partial$-invariant completion.

**Definition 3.12.** Let $u = \sum_{p=1}^{m} u_p e_p$, $v = \sum_{p=1}^{m} v_p e_p \in A_n(G)$, $u_p, v_p \in \mathbb{Z}$. We say $u \leq v$ if

1. $|u_p| \leq |v_p|$ and $|u_p - v_p| \leq |v_p|$, for each $p = 1, \ldots, m$;
2. $\sum_{p=1}^{m} |u_p| \leq \sum_{p=1}^{m} |v_p|$.

For $u \leq v$, we say $u < v$ if $u \neq v$.

Immediately, we have

**Proposition 3.13.** For each $n \in \mathbb{N}$, the above $\leq$ on $A_n(G)$ defines a partial order on $A_n(G)$. The subset of allowed elementary paths of length $n$ up to sign serve as the minimal set of $A_n(G)$. More explicitly, for $u = \sum_{n=1}^{m} u_n e_n \in A_n(G)$, with $u_n \in \mathbb{Z}$, then

$$\text{sign}(u_n) e_n \leq u, \quad n = 1, \ldots, m, \text{ if } u_n \neq 0.$$  

**Definition 3.14** ($\partial$-invariant completion). Let $u \in A_n(G)$. We say $u$ admits a $\partial$-invariant completion, if there exists a path $\tilde{u} \in \Omega_n(G)$ such that $u \leq \tilde{u}$. Furthermore, if $u$ admits $\partial$-invariant completions, let

$$C_{\text{min}}(G, u) = \{ \tilde{u} \in \Omega_n(G) | u \leq \tilde{u}, \text{there does not exist } v \in \Omega_n(G), \text{such that } v < \tilde{u} \}.$$  

be the subset of the minimal $\partial$-invariant completions of $u$.

It is obvious that if $u_{S \rightarrow E} \in A_n(G)$ is a regular allowed path with starting point $S$ and ending point $E$ and admits $\partial$-invariant completions, then any $\tilde{u} \in C_{\text{min}}(G, u_{S \rightarrow E})$ has the same starting point $S$ and the same ending point $E$.

**Example 3.15.** For the path $P = a e_{024} \in A_2(G)$, $a \in \mathbb{N}_+$ in the following digraph $G$, we know that

$$\tilde{P} := b e_{024} - (b + c) e_{014} + c e_{034}, \quad \text{for any } c \in \mathbb{Z}, b \in \mathbb{N}_+, b \geq a$$  

is a $\partial$-invariant completion for $P$. In particular, we have

$$C_{\text{min}}(G, a e_{024}) = \{ a e_{024} - a e_{014}, a e_{024} - a e_{034} \}.$$
3.3. Proof of Theorem 3.11. To prove the theorem, we consider one more claim as follows.

Claim 3.16. (1) For $2 \leq j \leq n$, each front elementary $j$-path component $e_{S_{i_1\ldots i_{j-1}u}}$ of $P$ admits a unique minimal $\partial$-invariant completion in $\text{Supp}(P)$, denote it by $P_{S,j,u}$.

(2) For $0 \leq k \leq n-2$, each behind elementary $(n-k)$-path component $e_{v_{k+1}\ldots v_{n-1}E}$ of $P$ admit a unique minimal $\partial$-invariant completion in $\text{Supp}(P)$, denote it by $P_{v,n-k,E}$.

We prove our result (including the claim) by double inductions where the first induction is in the increasing order while the second induction is in the decreasing order.

**Initial result 1:** It is clearly that the theorem holds for minimal paths of length $0, 1, 2$.

**Initial result 2 (for the claim):** For $j = n$, the minimal $\partial$-invariant completion of $e_{S_{i_1\ldots i_{n-1}E}}$ is exactly $P$, since $P$ is minimal. (The proof for the second part is similar.)

**Hypothesis 1:** Assume that the result holds for minimal path of length $k < n$. In particular, for each $P_{S,n-1,\alpha}^j$ in (3.6), we have

$$P_{S,n-1,\alpha}^j = (-1)^{n-1} \sum_{\alpha' \in E_2 \cap V(P_{S,n-1,\alpha}^j)} P_{S,n-2,\alpha'\alpha}^j,$$

where $E_2 = d_{\partial E}^{-1}(2)$, $P_{S,n-2,\alpha'}^j \in \Omega_{n-2}(\text{Supp}(P_{S,n-1,\alpha}^j))$ is the unique minimal path of length $n - 2$ starting at $S$ and ending at $\alpha' \in E_2 \cap V(P_{S,n-1,\alpha}^j)$.

**Hypothesis 2:** Assume that the claim holds for $k + 1 \leq j \leq n$.

Now let us look at the second induction first. For the case $j = k$, let $e_{S_{i_1\ldots i_{k-1}u}}$ be such a front $k$-path component of $P$, by Lemma 3.4, it admits $\partial$-invariant completion in $\text{Supp}(P)$. Now assume that $P_{S,k,u}^1$ and $P_{S,k,u}^2$ are two minimal $\partial$-invariant completions of $e_{S_{i_1\ldots i_{k-1}u}}$ in $\text{Supp}(P)$. By Hypothesis 1, both $P_{S,k,u}^1$ and $P_{S,k,u}^2$ are also minimal paths.

Now let us $e_{S_{i_1\ldots i_{k-1}u}}$ be the front $(k+1)$-path component of $P$. Then

- by Hypothesis 2, it admits a unique minimal $\partial$-invariant completion $P_{S,k+1,v}$ in $\text{Supp}(P)$.
- By Hypothesis 1, there is only one minimal path of length $k$ starting at $S$ and ending at $u$ in $\text{Supp}(P_{S,k+1,v})$.

The two uniqueness results imply that $P_{S,k,u}^1 = P_{S,k,u}^2$. Then we finish our proof of the claim.

Now let us turn to our first induction. For the minimal path $P$ of length $n$, recall we can write it as

$$P = (-1)^n \sum_{\alpha \in E_1} \sum_{j=1,\ldots,i_{\alpha}} c_{\alpha}^j P_{S,n-1,\alpha}^j E, \quad c_{\alpha}^j \in \mathbb{Z} - \{0\}.$$  

3.3.1. The case $|E_1| = 1$. Let us start with the simplest case that $|E_1| = 1$ with $E_1 = \{\alpha\}$. By (3.7) and (3.8), we have

$$P = (-1)^n \sum_{j=1,\ldots,i_{\alpha}} c_{\alpha}^j P_{S,n-1,\alpha}^j E = - \sum_{j=1,\ldots,i_{\alpha}} \sum_{\gamma \in E_2 \cap V(P_{S,n-1,\alpha}^j)} c_{\alpha}^j P_{S,n-2,\gamma\alpha}^j E.$$  

- If there exists $j \in \{1, \ldots, i_{\alpha}\}$ such that all vertices $\gamma \in E_2 \cap V(P_{S,n-1,\alpha}^j)$ connect to $E$ by a direct edge $\gamma \rightarrow E$, then

$$\widetilde{P} := P_{S,n-1,\alpha}^j E \in \Omega_n(\text{Supp}(P)).$$
Thus, \( \widetilde{P} \leq P \), and the equality holds if and only if \( i_\alpha = 1, \overline{c}_\alpha = \pm 1 \), which must hold by the minimal condition for \( P \).

- If for all \( j \in \{1, \ldots, i_\alpha\} \), there exists a vertex \( \gamma \in E_2 \cap V(P_{S,n-1,\alpha}^j) \) such that there is no direct edge from \( \gamma \) to \( E \), i.e. \( \gamma \not\leftrightarrow E \). Set

\[
I_j = \left\{ \gamma \in E_2 \cap V(P_{S,n-1,\alpha}^j) \mid \gamma \not\leftrightarrow E \right\},
\]

\[
I_j^c = E_2 \cap V(P_{S,n-1,\alpha}^j) - I_j.
\]

Then we consider \( \delta_{n-1}P \), by (3.9),

\[
\delta_{n-1}P = (-1)^n \sum_{j=1, \ldots, i_\alpha} \sum_{\gamma \in E_2 \cap V(P_{S,n-1,\alpha}^j)} \overline{c}_\alpha P_{S,n-2,\gamma}^j E
\]

\[
= (-1)^n \sum_{j=1, \ldots, i_\alpha} \sum_{\gamma \in I_j^c} \overline{c}_\alpha P_{S,n-2,\gamma}^j E + (-1)^n \sum_{j=1, \ldots, i_\alpha} \sum_{\gamma \in I_j} \overline{c}_\alpha P_{S,n-2,\gamma}^j E,
\]

where the first summands are allowed paths; while for each \( \gamma \in I_j \), each term \( P_{S,n-2,\gamma}^j E \) in the second summand is non-allowed.

Since \( \delta_{n-1}P \in A_{n-1}(\text{Supp}(P)) \), then we get

\[
\sum_{j=1, \ldots, i_\alpha} \sum_{\gamma \in I_j^c} \overline{c}_\alpha P_{S,n-2,\gamma}^j E = 0,
\]

which also means

\[
\sum_{j=1, \ldots, i_\alpha} \sum_{\gamma \in I_j} \overline{c}_\alpha P_{S,n-2,\gamma}^j E = 0.
\]

Then we have

\[
P = \sum_{j=1, \ldots, i_\alpha} \sum_{\gamma \in I_j^c} \overline{c}_\alpha P_{S,n-2,\gamma}^j E.
\]

Then we reduce to the first situation.

Let us finish the argument of the case \( |E_1| = 1 \). For \( P = P_{S,n-1,\alpha}E \), first by hypothesis 1, we have

\[
\partial P_{S,n-1,\alpha} = \delta_{n-1}P_{S,n-1,\alpha} + \delta_0P_{S,n-1,\alpha} + (\delta_1 + \cdots + \delta_{n-2})P_{S,n-1,\alpha}
\]

\[
= \sum_{\gamma \in d_{S}^{-1}(2) \cap V(P_{S,n-1,\alpha})} \pm P_{S,n-2,\gamma}E + \sum_{\beta \in d_{S}^{-1}(1)} P_{\beta,n-2,\alpha} + P_{S,n-2,\alpha},
\]

where in particular, either \( P_{S,n-2,\alpha} \) is a minimal path of length \( n - 1 \) or \( P_{S,n-2,\alpha} = 0 \). Similarly, we have

\[
\partial P = \partial P_{S,n-1,\alpha}E = \delta_0P_{S,n-1,\alpha}E + \delta_0P_{S,n-1,\alpha}E + (\delta_1 + \cdots + \delta_{n-1})P_{S,n-1,\alpha}E
\]

\[
= \pm P_{S,n-1,\alpha} + \sum_{\beta \in d_{S}^{-1}(1)} P_{\beta,n-2,\alpha}E \pm \left( P_{S,n-2,\alpha}E - \sum_{\gamma \in d_{S}^{-1}(2) \cap V(P_{S,n-1,\alpha})} P_{S,n-2,\gamma}E \right).
\]

The remaining statements are trivial except the following one:

\[
P_{S,n-1,\alpha} := P_{S,n-2,\alpha}E - \sum_{\gamma \in d_{S}^{-1}(2) \cap V(P_{S,n-1,\alpha})} P_{S,n-2,\gamma}E
\]
We want to construct a path from (1) naturally. By our construction, we can see that \( P \subset I \) where the path is a linear combination of minimal paths in \( \text{Supp}(P) \):

\[
P_{S,n-1,E} = P_{S,n-1,E}^1 + P_{S,n-1,E}^2 + \cdots + P_{S,n-1,E}^k, \quad k \geq 2.
\]

According to the decomposition at the \((n-2)\)-th vertices, then there exists some \( i = 1, \ldots, k \), and \( I \subset d_E^{-1}(2) \cap V(P_{S,n-1,a}) \) such that

\[
P_{S,n-1,E}^i = \sum_{\gamma \in I} P_{S,n-2,\gamma} E.
\]

By our construction, we can see that

\[
\tilde{P} := \sum_{\gamma \in I} P_{S,n-2,\gamma} \alpha E \in \Omega_\alpha(G), \quad \tilde{P} < P.
\]

Contradiction. Thus, \( P_{S,n-1,E} \) must be minimal. The statement (2) and (3) in this theorem follow from (1) naturally.

### 3.3.2. The case \(|E_1| = 2\)

Now let us consider the case \(|E_1| = 2\), i.e. \( E_1 = \{\alpha_1, \alpha_2\} \). We write \( P \) as

\[
P = (-1)^n \sum_{j=1,\ldots,i_{\alpha_1}} c_1^j P_{S,n-1,\alpha_1}^j E + (-1)^n \sum_{j=1,\ldots,i_{\alpha_2}} c_2^j P_{S,n-1,\alpha_2}^j E
\]

\[
= - \sum_{j=1,\ldots,i_{\alpha_1}} \sum_{\gamma_1 \in E \cap V(P_{S,n-1,\alpha_1}^j)} c_1^j P_{S,n-2,\gamma_1}^j \alpha_1 E
\]

\[
- \sum_{j=1,\ldots,i_{\alpha_2}} \sum_{\gamma_1 \in E \cap V(P_{S,n-1,\alpha_2}^j)} c_2^j P_{S,n-2,\gamma_2}^j \alpha_2 E.
\]

Now, without loss of generality, assume that \( i_{\alpha_1} \geq 1 \) or \( c_1^1 \) with \(|c_1^1| \geq 1\). Let us first consider the path

\[
P_0 = (-1)^n P_{S,n-1,\alpha_1} E \in A_n(\text{Supp}(P)).
\]

Then we have

- \( P_0 \) is a linear combination of allowed elementary path starting at \( S \) and ending at \( E \) with coefficients \( \pm 1 \) by Hypothesis 1;
- \( \delta_i P_0 \in A_{n-1}(\text{Supp}(P)), i = 0, 1 \ldots, n-2, n \) on \( P_{S,n-1,\alpha_1}^1 \);
- \( P_0 < P \).

We want to construct a \( \partial \)-invariant completion of \( P_0 \) that is smaller than \( P \).

By Hypothesis 1, we can write \( P_{S,n-1,\alpha_0}^1 \) as

\[
P_{S,n-1,\alpha_0}^1 = (-1)^{n-1} \sum_{\gamma_1 \in E \cap V(P_{S,n-1,\alpha_1}^j)} P_{S,n-2,\gamma_1}^j \alpha_1
\]

\[
= (-1)^{n-1} \sum_{\gamma_1 \in I_1} P_{S,n-2,\gamma_1}^1 \alpha_1 + (-1)^{n-1} \sum_{\gamma_1 \in I_1} P_{S,n-2,\gamma_1}^1 \alpha_1
\]

where as before,

\[
I_1 = \{ \gamma \in E \cap V(P_{S,n-1,\alpha_1}^1) \mid \gamma_1 \to E \}.
\]
\[ I_1^i = E_2 \cap V(P_{\delta,n-1,\alpha_i}) - I_1. \]

Then, we have
\[ \delta_{n-1}P_0 = (-1)^n \sum_{\gamma_1 \in I_1^i} P_{\delta,n-2,\gamma_1}^1 E + (-1)^n \sum_{\gamma_1 \in I_1} P_{\delta,n-2,\gamma_1}^1 E, \]
where the first summands are allowed paths; while for each \( \gamma_1 \in I_1 \), each \( P_{\delta,n-2,\gamma_1}^1 E \) is not allowed.

The following two observations help us modify \( P_0 \) to become \( \delta_{n-1} \) invariant.

- Since \( \delta_{n-1}P \in \mathcal{A}_{n-1}(\text{Supp}(P)) \), there must be terms in \( \delta_{n-1}P \) which cancel each \( P_{\delta,n-2,\gamma_1}^1 E \).
- By Claim 3.16, \( P_{\delta,n-2,\gamma_1}^1 E \) must be canceled by terms coming from \( \delta_{n-1}P_{\delta,n-1,\alpha_2}^j E \).

Now let us consider the allowed path
\[ P_1 = (-1)^{n-1} \sum_{\gamma_1 \in I_1} P_{\delta,n-2,\gamma_1}^1 \alpha_2 E, \]
and modify the path \( P_0 \) to be \( P_0 + P_1 \). Then by our construction, \( P_0 + P_1 \) satisfies

- \( P_0 + P_1 \) is a linear combination of allowed elementary path starting at \( S \) and ending at \( E \) with coefficients \( \pm 1 \) by Hypothesis 1;
- \( \delta_i(P_0 + P_1) \in \mathcal{A}_{n-1}(\text{Supp}(P)), \ i = 0, \ldots, n-3, n-1, n; \)
- \( P_0 + P_1 \leq P. \)

If \( \delta_{n-2}P_1 \in \mathcal{A}_{n-1}(\text{Supp}(P)) \), then we are done. Otherwise, we can consider the minimal \( \partial \)-invariant completion of \( P_1 \).

By the fact \( \delta_{n-2}P \in \mathcal{A}_{n-2}(\text{Supp}(P)) \), Claim 3.16 and Hypothesis 1, there exists a modified term \( P_1 \) such that

- \( P_1 + P_1' \) is a sum of some terms in \( \{ \pm P_{\delta,n-1,\alpha_2}^j \}_{j=1}^{\alpha_2} \).
- \( P_1 + P_1' < P. \)

Then we look at the path \( P_0 + P_1 + P_1' \), it satisfies

- \( P_0 + P_1 + P_1' \) is a linear combination of allowed elementary path starting at \( S \) and ending at \( E \) with coefficients \( \pm 1 \) by Hypothesis 1;
- \( \delta_i(P_0 + P_1 + P_1') \in \mathcal{A}_{n-1}(\text{Supp}(P)), \ i = 0, \ldots, n-3, n-2, n; \)
- \( P_0 + P_1 + P_1' \leq P. \)

If \( \delta_{n-1}(P_0 + P_1 + P_1') \in \mathcal{A}_{n-1}(\text{Supp}(P)) \), we are done. Otherwise, we continue modifying our path, to get \( P_2, P_2', P_3, P_3', \ldots \), there are two points in the construction:

**Point 1:** By Claim 3.16, the term cancel
\[ \delta_{n-1}P_{\delta,n-2,\gamma_1}^1 \alpha_1 E, \ \gamma_1 \rightarrow E \quad \text{(respectively, } \delta_{n-1}P_{\delta,n-2,\gamma_2}^1 \alpha_2 E, \ \gamma_2 \rightarrow E) \]
must come from
\[ \delta_{n-1}P_{\delta,n-2,\gamma_1}^1 \alpha_2 E, \ \gamma_1 \rightarrow E \quad \text{(respectively, } \delta_{n-1}P_{\delta,n-2,\gamma_2}^1 \alpha_1 E, \ \gamma_2 \rightarrow E). \]

At the \( k \)-th time, we get the path \( P_k \).

**Point 2:** As we construct the \( P_1' \), we get \( P_k' \). If \( P_k' \neq 0 \), then new vertices in \( E_2 \) will come in.
Since we work with a finite graph, it will stop at finite steps, which means, for some positive integer $l$, we obtain

$$\bar{P} = P_0 + P_1 + P'_1 + \cdots + P_l + P'_l \in \Omega_n(Supp(P)), \quad (P'_l \text{ may be } 0),$$

is a linear combination of allowed elementary path starting at $S$ and ending at $E$ with coefficients $\pm 1$; in particular, $\bar{P} \leq P$.

Moreover, let us look at our construction more carefully and see that the modification will stop before $P_2$. That is,

$$P_0 + P_1 + P'_1 \in \Omega_n(Supp(P)), \quad (P'_1 \text{ may be } 0).$$

Let $P_{ S,n - 3, \epsilon} \gamma_1 \alpha_1 E$, $\gamma_1 \in I_1$ be a component of $P_0$, by our construction,

- first, we need to add a term $-P_{ S,n - 3, \epsilon} \gamma_1 \alpha_2 E$ to do the modification.
  - If $\epsilon \to \alpha_2$, we do not need to modify this term again, done;
  - If $\epsilon \Rightarrow \alpha_2$, we need to add the term of the form $P_{ S,n - 3, \epsilon} \gamma_2 \alpha_2 E$ (the existence is obvious).
    * If $\gamma_2 \to E$, we do not need to modify this term again, done;
    * If $\gamma_2 \to E$, we need to add the term of the form $P_{ S,n - 3, \epsilon} \gamma_2 \alpha_1$ (the existence is obvious).

Then we will obtain a subgraph of the form

![Diagram](image-url)

which gives us a minimal $\partial$-invariant completion of $e_{\epsilon \gamma_1 \alpha_1} E$, that is

$$P_{\epsilon, 3, E} = e_{\epsilon \gamma_1 \alpha_1} E - e_{\epsilon \gamma_2 \alpha_1} E + e_{\epsilon \gamma_2 \alpha_2} E - e_{\epsilon \gamma_1 \alpha_2} E.$$  

Since $\gamma_1 \Rightarrow E$, $P_{\epsilon, 3, E}$ is a minimal path. Apply Hypothesis 1 (3) for $P_{\epsilon, 3, E}$, there is no directed edge from $\epsilon \to \alpha_1$, otherwise $e_{\epsilon, \gamma_1, \alpha_1}$, $e_{\epsilon, \gamma_2, \alpha_1}$ become two different minimal 2-faces with the same starting and ending vertices in $\text{Supp}(P_{\epsilon, 3, E})$. Since $P_{ S,n - 3, \epsilon} \gamma_1 \alpha_1 < P_{ S,n - 3, \epsilon} \gamma_1 \alpha_1$, there must exist terms $-P_{ S,n - 3, \epsilon} \gamma_1 \alpha_1 < P_{ S,n - 3, \epsilon} \gamma_1 \alpha_1$. Then $e_{\epsilon \gamma_1 \alpha_1} E - e_{\epsilon \gamma_1 \alpha_1} E$ has a minimal $\partial$-invariant completion different from $P_{\epsilon, 3, E}$, contradiction.

Thus, we get

$$\tilde{P} = P_0 + P_1 + P'_1 = \pm P_{ S,n - 1, \alpha_1} E \pm \sum_j P_{ S,n - 1, \alpha_2} E \in \Omega_n(Supp(P)), \quad \tilde{P} \leq P.$$  

Furthermore, we can consider the $\partial$-invariant completion of $P_{ S,n - 1, \alpha_2} E$ in $\text{Supp}(\tilde{P}) \subset \text{Supp}(P)$, then we a smaller path

$$\hat{P} = \pm P_{ S,n - 1, \alpha_2} E \pm P_{ S,n - 1, \alpha_1} E \in \Omega_n(Supp(\tilde{P})), \quad \hat{P} \leq \tilde{P} \leq P.$$  

Then after a similar analysis of $\partial P$ as we do for the case $|E_1| = 1$, using the structure result for $P_{ S,n - 1, \alpha_1}$ and $P_{ S,n - 1, \alpha_2}$, we finish the case of $|E_1| = 2$.  

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**MINIMAL PATH AND ACYCLIC MODELS**  

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3.3.3. Induction on $|E_1|$. Now we consider one more induction on $|E_1|$.

**Initial result 3:** We have proved the result when $P$ is of length $n$ and $|E_1| = 1, 2$ under the hypothesis 1.

**Hypothesis 3:** Assume that the result holds for $P$ is of length $n$ and $|E_1| \leq m$.

For $|E_1| = m + 1$, $E_1 = \{ \alpha_1, \ldots, \alpha_{m+1} \}$, first, we can prove as for the case $|E_1| = 2$ that starting from $P_0 = \pm P_{S,n-1, \alpha_1} E$, we obtain

$$\tilde{P} = P_0 + P_1 + P_1' + \cdots + P_l + P_l' = \sum \in \Omega_n(\text{Supp}(P)),$$

where $P_0 = P_{S,n-1, \alpha_1} E$.

Similarly let us consider the component $P_{S,n-3, \epsilon \gamma_1 \alpha_1} E$ of $P_0$, where $\gamma_1 \not\rightarrow E$. Then we repeat the argument as the case $|E_1| = 2$: if the modification does not stop before $P_2$. Then $e_{\epsilon \gamma_1 \alpha_1} E$ admits two different minimal $\partial$-invariant completions, contradiction. Then we are done.

For the convenience of the discussion in the next section, we study the intersections of some subdigraphs in $\text{Supp}(P)$. The result follows from Theorem 3.11 directly.

**Corollary 3.17.** For any $v_k \in d^{-1}_E(n-k)$, $v_l \in d^{-1}_E(n-l)$, $v_k \neq v_l$, let $P_{S,k,v_k} \in \Omega_k(\text{Supp}(P))$ and $P_{S,l,v_l} \in \Omega_l(\text{Supp}(P))$ be the corresponding minimal paths, then their intersection contains the starting point $S$ and

$$\text{Supp}(P_{S,k,v_k}) \cap \text{Supp}(P_{S,l,v_l})$$

is a union of minimal paths of length less than $\min\{k,l\}$. More precisely, if $v_i \in \text{Supp}(P_{S,k,v_k}) \cap \text{Supp}(P_{S,l,v_l}) \cap d^{-1}_E(n-i)$, then

$$\text{Supp}(P_{S,i,v_i}) \subset \text{Supp}(P_{S,k,v_k}) \cap \text{Supp}(P_{S,l,v_l}).$$

We will see the above structure results in the following subsection of examples of minimal path of length 3.

3.4. **Minimal path of length 3 and some properties.** In this subsection, we will study the minimal paths of length 3 as well as their basic homotopic and homological properties. We will see that unlike the length 2 case, there are infinitely many types of minimal paths of length 3. We will see that some of the minimal paths are contractible, while some of them are not.

3.4.1. **Homotopy theory of digraphs.** Before the discussion of the examples, let us briefly recall some basic homotopy theory of digraph developed by Grigoryan-Lin-Muranov-Yau [10].

**Definition 3.18** (Digraph map). A morphism from a digraph $G = (V(G), E(G))$ to a digraph $H = (V(H), E(H))$ is a map $f : V(G) \rightarrow V(H)$ such any edge $v \rightarrow w$ on $G$, we have

$$f(v) \rightarrow f(w) \quad \text{or} \quad f(v) = f(w) \quad \text{in} \quad H.$$

The requirement is also denoted by $f(v) \equiv f(w)$. We refer to such morphisms also as digraphs maps, and also denote by $f : G \rightarrow H$. 

Definition 3.19 (Cartesian product). For two digraphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, we define the Cartesian product $G \boxtimes H$ as a digraph with the set of vertices $V(G) \times V(H)$ and with the set of edges as follows: for $v, v' \in V(G)$ and $w, w' \in V(H)$, we have $(v, w) \rightarrow (v', w')$ in $G \boxtimes H$ if and only if

either $v = v'$ and $w \rightarrow w'$, or $v \rightarrow v'$ and $w = w'$.

Fix $n \geq 0$. Denote by $I_n$ any digraph, with $V(I_n) = \{0, 1, \ldots, n\}$ and $E(I_n)$ containing exactly one of the edges $i \rightarrow i + 1$, $i + 1 \rightarrow i$ for any $i = 0, 1, \ldots, n - 1$. We call it the line digraph.

Definition 3.20 (Homotopy). Let $G$, $H$ be two digraphs. Two digraph maps $f, g : G \rightarrow H$ are called homotopic if there exists a line digraph $I_n$ for some $n \geq 1$ and a digraph map $F : G \boxtimes I_n \rightarrow H$, such that

$$F|_{G \boxtimes \{0\}} = f \quad \text{and} \quad F|_{G \boxtimes \{n\}} = g.$$ 

We shall write $f \simeq g$ and call $F$ a $n$-step homotopy between $f$ and $g$.

Definition 3.21 (Homotopy equivalent). Two digraphs $G$ and $H$ are called homotopy equivalent if there exist digraph maps

$$f : G \rightarrow H, \quad g : H \rightarrow G$$

such that

$$f \circ g \simeq \text{Id}_H, \quad g \circ f \simeq \text{Id}_G.$$ 

A digraph $G$ is called contractible if $G \simeq \{\ast\}$ where $\{\ast\}$ is a single vertex digraph.

Similarly, two homotopy equivalent digraphs have the same homology groups, see in [10]. In particular, a contractible digraph has acyclic homologies.

Definition 3.22 (Deformation Retraction). Let $G$ be a digraph and $H$ be its sub-digraph.

1. A retraction of $G$ onto $H$ is a digraph map $r : G \rightarrow H$ such that $r|_H = \text{Id}_H$.

2. A retraction $r : G \rightarrow H$ is called a deformation retraction if $i \circ r \simeq \text{Id}_G$, where $i : H \rightarrow G$ is the natural inclusion digraph map.

Proposition 3.23 ([10]). Let $r : G \rightarrow H$ be a retraction of a digraph $G$ onto a sub-digraph $H$. Assume that there exists a finite sequence $\{f_k\}_{k=0}^n$ of digraph maps $f_k : G \rightarrow G$ with the following properties

1. $f_0 = \text{Id}_G$, $f_n = i \circ r$;

2. for any $k = 1, \ldots, n$ either $f_{k-1}(x) \rightarrow f_k(x)$, any $x \in V(G)$, or $f_{k-1}(x) \leftarrow f_k(x)$, any $x \in V(G)$.

Then $r$ is a deformation retraction, the digraphs $G$ and $H$ are homotopy equivalent.

Corollary 3.24 ([10]). Let $r : G \rightarrow H$ be a retraction of a digraph $G$ onto a sub-digraph $H$ and

$$r(x) \rightarrow x \text{ for all } x \in V(G), \quad \text{or} \quad x \leftarrow r(x) \text{ for all } x \in V(G).$$

Then $r$ is a deformation retraction, the digraphs $G$ and $H$ are homotopy equivalent.
3.4.2. Examples: minimal paths of length 3.

**Example 3.25.** If an elementary path \( P = e_{0123} \) is \( \partial \)-invariant, the corresponding supporting digraph is as follows.

![Diagram](image)

Then we can compute \( \partial e_{0123} \) and present it in terms of the minimal path of length 2 in \( \text{Supp}(P) \),

\[
\partial e_{0123} = -e_{012} + (e_{013} - e_{023}) + e_{123}.
\]

Let \( r \) be the digraph map

![Diagram](image)

given by \( r(3) = 2 \) and \( r|_H = \text{Id}_H \). By Corollary 3.24, we get

\[
i \circ r \simeq \text{Id}_{\text{Supp}(P)}
\]

by a one-step homotopy. Furthermore, \( H \) is obviously contractible, so is \( \text{Supp}(P) \). In particular, \( \text{Supp}(P) \) is acyclic.

**Example 3.26.** The minimal digraph which supports \( P = e_{0134} - e_{0234} \) is as follows.

![Diagram](image)

Similarly, we have

\[
\partial(e_{0134} - e_{0234}) = -(e_{013} - e_{023}) + e_{134} - e_{234} + (e_{014} - e_{024}).
\]

and the deformation retraction

![Diagram](image)
is given by $r(4) = 3$ and $r|_H = \text{Id}_H$. Thus, Supp($P$) is contractible.

**Example 3.27.** The minimal digraph which supports $P = e_{0135} - e_{0235} + e_{0245}$ is as follows.

![Diagram](image)

Similarly, we have

$$\partial(e_{0135} - e_{0235} + e_{0245}) = -(e_{013} - e_{023}) - e_{024} + e_{135} + (e_{245} - e_{235}) + (e_{015} - e_{045})$$

There are several deformation retractions to its subgraphs as follows:

$r_1$: 

![Diagram](image)

(1) $r_1(5) = 3$, $r_1(4) = 2$ and $r_1|_{H_1} = \text{Id}_{H_1}$.

$r_2$: 

![Diagram](image)

(2) $r_2(5) = 4$, $r_2(3) = 2$, $r_2(1) = 0$ and $r_2|_{H_2} = \text{Id}_{H_2}$.

Thus, Supp($P$) is contractible.

**Example 3.28.** The minimal digraph which supports $P = e_{0136} - e_{0146} - e_{0236} + e_{0256}$ is as follows.

![Diagram](image)

Similarly, we have

$$\partial(e_{0136} - e_{0146} - e_{0236} + e_{0256})$$

$$= (e_{023} - e_{013}) + e_{014} - e_{025} + (e_{136} - e_{146}) + (e_{256} - e_{236}) + (e_{046} - e_{056})$$

There are several deformation retractions to its subdigraphs as follows:
(1) $\mathbf{r}_1(6) = 3$, $\mathbf{r}_1(5) = 2$, $\mathbf{r}_1(4) = 1$ and $\mathbf{r}_1|_{H^1} = \text{Id}_{H^1}$.

(2) $\mathbf{r}_2(6) = 4$, $\mathbf{r}_2(3) = 1$, $\mathbf{r}_2(5) = \mathbf{r}_2(2) = 0$ and $\mathbf{r}_2|_{H^2} = \text{Id}_{H^2}$.

(3) $\mathbf{r}_3(1) = 4$, $\mathbf{r}_3(2) = 3$, $\mathbf{r}_3(3) = 6$ and $\mathbf{r}_3|_{H^3} = \text{Id}_{H^3}$.

(4) $\mathbf{r}_4(0) = 1$, $\mathbf{r}_4(2) = 3$, $\mathbf{r}_4(5) = 3$ and $\mathbf{r}_4|_{H^4} = \text{Id}_{H^4}$, where the homotopy

\[ F : \text{Supp}(P) \sqcup I_3 \to \text{Supp}(P), \quad \text{with} \quad I_3 = 0 \leftarrow 1 \to 2 \leftarrow 3 \]

connecting $\text{Id}_{\text{Supp}(P)}$ and $i_4 \circ \mathbf{r}_4$ is given by

- $F(v, 0) = \text{Id}_{\text{Supp}(P)}$;
- $F(0, 1) = 0, F(1, 1) = 1, F(2, 1) = 2, F(3, 1) = 3, F(4, 1) = 1, F(5, 1) = 2, F(6, 1) = 3$. \quad (\text{We have } F(x, 0) \leftarrow F(x, 1).)\]
- $F(0, 2) = 1, F(1, 2) = 4, F(2, 2) = 3, F(3, 2) = 6, F(4, 2) = 4, F(5, 2) = 3, F(6, 2) = 6$. \quad (\text{We have } F(x, 1) \leftarrow F(x, 2).)$
\( F(v, 3) = i_4 \circ r_4, \) that is
\[
F(0, 3) = 1, F(1, 3) = 1, F(2, 3) = 3, F(3, 3) = 3,
\]
\[
F(4, 3) = 4, F(5, 3) = 3, F(6, 3) = 6. \quad \text{(We have } F(x, 2) \xleftarrow{\sim} F(x, 3).)\]

Thus \( \mathrm{Supp}(P) \) is contractible via alternative homotopies.

Note that in all the above examples of minimal 3-paths, there is exactly one minimal 2-path of type \( P_{S, 2, E} \). The following 3 examples tells us there may be no such a term.

**Example 3.29.** The minimal digraph which supports \( P = e_{0135} - e_{0145} + e_{0245} - e_{0235} \) is as follows.

\[
\begin{array}{c}
\text{Supp}(P) = 0 \ar 1 \ar 3 \ar 5 \\
2 \ar 4
\end{array}
\]

Similarly, we have
\[
\partial(e_{0135} - e_{0145} + e_{0245} - e_{0235})
\]
\[
= (e_{023} - e_{013}) + (e_{014} - e_{024}) + (e_{135} - e_{145}) + (e_{245} - e_{235}).
\]

There is a deformation retraction as follows:

\[
\begin{array}{c}
\text{r : } 0 \ar 1 \ar 3 \ar 5 \ar 0 \\
2 \ar 4
\end{array}
\]

\( r(5) = r(4) = 3, \ r\big|_H = \text{Id}_H, \) where the homotopy
\[
F : \text{Supp}(P) \boxtimes I_3 \to \text{Supp}(P), \quad \text{with} \quad I_3 = 0 \leftrightarrow 1 \leftrightarrow 2 \to 3
\]

connecting \( \text{Id}_{\text{Supp}(P)} \) and \( i \circ r \) is given by

- \( F(v, 0) = \text{Id}_{\text{Supp}(P)}; \)
- \( F(0, 1) = 0, F(1, 1) = 1, F(2, 1) = 0, \)
  \( F(3, 1) = 1, F(4, 1) = 1, F(5, 1) = 3. \quad F(x, 0) \xleftarrow{\sim} F(x, 1). \)
- \( F(0, 2) = 0, F(1, 2) = 1, F(2, 2) = 0, \)
  \( F(3, 2) = 1, F(4, 2) = 1, F(5, 2) = 1. \quad F(x, 1) \xleftarrow{\sim} F(x, 2). \)
- \( F(v, 3) = i \circ r, \) that is
  \( F(0, 3) = 1, F(1, 3) = 1, F(2, 3) = 2, \)
  \( F(3, 3) = 3, F(4, 3) = 3, F(5, 3) = 3. \quad F(x, 2) \xleftarrow{\sim} F(x, 3). \)

Note that one can also define \( F(v, 3) = 0 \) for all \( v \) and then we obtain a deformation retraction to the single point 0.
Example 3.30. The minimal digraph which supports
\[ P = e_{0136} - e_{0156} + e_{0456} + e_{0246} - e_{0236} \]
is as follows.

Similarly, we have
\[
\partial(e_{0136} - e_{0156} + e_{0456} + e_{0246} - e_{0236})
= (e_{023} - e_{013}) - e_{024} + (e_{015} - e_{045}) + (e_{136} - e_{156}) + e_{456}.
\]

There are several deformation retractions to its subdigraphs as follows:

(1) \( r_1(6) = 5, r_1(3) = 1, r_1(2) = 0 \) and \( r_1\big|_H = \text{Id}_H \).

(2) \( r_2(6) = 4, r_2(5) = 4, r_2(3) = 2, r_2(1) = 0 \) and \( r_2\big|_H = \text{Id}_H \).

Example 3.31 (3-Cube). The minimal digraph which supports
\[ P = e_{0137} - e_{0237} + e_{0267} - e_{0467} + e_{0457} - e_{0157} \]
is as follows.
Similarly, we have
\[ \partial(e_{0137} - e_{0237} + e_{0267} - e_{0467} + e_{0457} - e_{0157}) \]
\[ = (e_{023} - e_{013}) + (e_{015} - e_{045}) + (e_{046} - e_{026}) \]
\[ + (e_{137} - e_{157}) - (e_{237} - e_{267}) + (e_{457} - e_{467}). \]

It is known as the 3-cube digraph, which is isomorphic (there exists a bijection of digraph map) to \( I_1^{\Box 3} \). Thus it is contractible. Also note that there is no minimal path of length 2 of type \( P_{S,2,E} \) in this example as well as in Examples 3.29, 3.30.

**Remark 3.32.** We learn from the above examples that all of them can deform retract to some (actually all, one can construct one by one) minimal faces. Thus, they have acyclic homologies. In fact, one can also construct the digraph map \( h : I_1^{\Box 3} \rightarrow \text{Supp}(P) \) such that the above \( \text{Supp}(P) \) is exactly the image of \( h \). One can regard such a map as the analogue of the characteristic map in CW complex. We will do the further discussion in a separation paper.

Now let us continue looking at two different minimal 3-paths.

**Example 3.33.** The minimal digraph which supports
\[ P = e_{0137} - e_{0237} + e_{0267} - e_{0467} + e_{0457} \]
is as follows.

```
  1 -- 2 -- 3
 /   /   /  \
\(\text{Supp}(P)\) = 0 -- 4 -- 5 -- 6 -- 7
```

We can compute \( \partial P \) and obtain
\[ \partial(e_{0137} - e_{0237} + e_{0267} - e_{0467} + e_{0457}) \]
\[ = (e_{023} - e_{013}) + (e_{015} - e_{045}) - e_{026} \]
\[ + (e_{137} - e_{157}) - (e_{237} - e_{267}) + e_{457} + (e_{047} - e_{067}). \]

One can compute directly that \( \tilde{H}_*(\text{Supp}(P)) = 0 \).

**Example 3.34** (Exotic cube). The minimal digraph which supports
\[ P = e_{0258} - e_{0158} - e_{0268} + e_{0368} - e_{0378} + e_{0478} \]
is as follows.
We can compute $\partial P$ and obtain

$$
\partial P = (e_{015} - e_{025}) + (e_{026} - e_{036}) + (e_{037} - e_{047}) \quad \text{(starting at 0 and ending in } E_1) \\
+ e_{478} - e_{158} + (e_{258} - e_{268}) + (e_{368} - e_{378}) \quad \text{(starting in } S_1 \text{ and ending at 8)} \\
- (e_{018} - e_{048}) \quad \text{(starting at 0 and ending at 8)}
$$

One can compute directly that $\tilde{H}_4(\text{Supp}(P)) = 0.$

**Remark 3.35.** Unlike the above other examples, there seem to be no deformation retractions from $\text{Supp}(P)$ in Examples 3.33, 3.34 to its minimal face. One can check one by one that such $\text{Supp}(P)$ can not deform retract to any of its sub-digraph, which means that it is not contractible.

From another point of view, one can see that there are no digraph maps from the 3-cube $I_1^{33}$ onto the above 2 examples by a simple analysis:

- For Example 3.33, the $\text{Supp}(P)$ has one more directed edge than 3-cube, then it can not be the image of $I_1^{33}$;
- For Example 3.34, the exotic cube has one more vertex than 3-cube, then it can also not be the image of $I_1^{33}$.

Such an interpretation could be understood via the singular cubic homology of digraph, which is introduced in [7] by Grigoryan-Jimenez-Muranov. They proved that such a singular cubic homology is a homotopic invariant (Corollary 4.6). However, the exotic cube has a non-trivial singular cubic homology group $H_2^{sc}$, which also means that the exotic cube is not contractible.

If we increase the number of the vertex and edge in an appropriate way, there are more examples of minimal 3-paths.

**Example 3.36.** Let us consider the path

$$
P = e_{S05E} - e_{S15E} + e_{S17E} - e_{S37E} + e_{S39E} - e_{S49E} \\
+ e_{S48E} - e_{S28E} + e_{S26E} - e_{S06E}.
$$

Its supporting digraph is given by
Note that one can think of this digraph as two cubes pasting together along

\[ S \to 2 \to 7 \to E \]

with the edge 2 → 7 removed, see, as the following picture shows.

Note that the edge 2 → 7 is necessary for the two paths

\[
P_1 = e_{S05E} - e_{S15E} + e_{S17E} - e_{S27E} + e_{S26E} - e_{S06E}
\]

\[
P_2 = e_{S27E} - e_{S37E} + e_{S39E} - e_{S49E} + e_{S48E} - e_{S28E},
\]

to be \(\partial\)-invariant individually.

We can continue doing such a pasting and removing the necessary edge, and then we get infinitely many digraphs which give us the corresponding minimal path.

**Example 3.37.** Let us consider the path

\[
P = e_{S06E} - e_{S16E} + e_{S17E} - e_{S27E} + e_{S28E} - e_{S38E}
\]

\[+ e_{S39E} - e_{S49E} + e_{S4(10)E} - e_{S5(10)E}.
\]

Its supporting digraph is given by
One can also think of this digraph as two smaller digraphs pasting together with some edge removed, see, for examples

One can also compute directly that the supporting digraphs in Examples 3.36, 3.37 have trivial reduced path homologies.

4. ACYCLIC MODELS

We have learnt that all the examples of supporting digraphs in the last subsection are acyclic. In this section, we will prove such an observation in the general case, which is our second main theorem.

Theorem 4.1 (Acyclic Models). Let $P$ be a minimal path of $G$, and $\text{Supp}(P)$ be its supporting subgraph. Then $\text{Supp}(P)$ is acyclic, that is,

$$H_i(\text{Supp}(P); \mathbb{Z}) = 0, \quad i > 0; \quad H_0(\text{Supp}(P); \mathbb{Z}) = \mathbb{Z}.$$  

Or equivalently, $\tilde{H}_*(\text{Supp}(P); \mathbb{Z}) = 0$.

4.1. Mayer-Vietoris exact sequences in path complex. We will use the technique of Mayer-Vietoris exact sequence in the path complex in [8]. Let us state the corresponding result first. It is worth to mention that such an exact sequence result holds for any abelian group as the coefficient. Here we still use it with integer coefficient $\mathbb{Z}$.

Let $Y_1$ and $Y_2$ be two digraphs, and

$$X = Y_1 \cup Y_2, \quad Z = Y_1 \cap Y_2.$$
Let \(i_1 : Z \to Y_1, i_2 : Z \to Y_2\) and \(j_1 : Y_1 \to X, j_2 : Y_2 \to X\) be the corresponding inclusion digraph maps. Suppose that any allowed elementary path on \(X\) lies in \(Y_1\) or \(Y_2\). Then for any \(p \geq -1\), there exists a short exact sequence of abelian groups:

\[
(4.1) \quad 0 \longrightarrow \mathcal{A}_p(Z) \xrightarrow{\delta} \mathcal{A}_p(Y_1) \oplus \mathcal{A}_p(Y_2) \xrightarrow{d} \mathcal{A}_p(X) \longrightarrow 0,
\]

where \(\delta = (j^1_*, j^2_*)\), \(d(a, b) = j^1_*(a) - j^2_*(b)\).

**Lemma 4.2** ([8] Mayer-Vietoris exact sequence). Let \(Y_1, Y_2, X, Z\) be as above. If the homomorphism

\[
\tilde{\Omega}_p(Y_1) \oplus \tilde{\Omega}_p(Y_2) \to \tilde{\Omega}_p(X)
\]

that is induced by \(d\) is an epimorphism for any \(p \geq -1\). Then there is a short exact sequence of chain complexes

\[
0 \longrightarrow \tilde{\Omega}_p(Z) \xrightarrow{\delta} \tilde{\Omega}_p(Y_1) \oplus \tilde{\Omega}_p(Y_2) \xrightarrow{d} \tilde{\Omega}_p(X) \longrightarrow 0.
\]

And moreover we have the long exact sequence

\[
\cdots \to \tilde{H}_p(Z) \xrightarrow{\delta} \tilde{H}_p(Y_1) \oplus \tilde{H}_p(Y_2) \xrightarrow{d} \tilde{H}_p(X) \xrightarrow{\partial} \tilde{H}_{p-1}(Z) \xrightarrow{\delta} \tilde{H}_{p-1}(Y_1) \oplus \tilde{H}_{p-1}(Y_2) \to \cdots
\]

4.2. **Proof of Theorem 4.1.** This subsection is devoted to proving Theorem 4.1. The proof is done by several inductions on the length first and then on the number of the points in the same position of the elementary path component of \(P\).

The first induction is on the length of \(P\). Below are the initial result (1) and the hypothesis (1).

**Initial result (1).** If \(P\) is a minimal path of length 0 or 1 or 2, then \(\tilde{H}_s(Supp(P)) = 0\).

**Hypothesis (1).** We assume the acyclic result holds for any minimal path of length \(k < n\).

To prove the result for the case \(k = n\), we further do the second induction on the number \(|E_1|\).

**Initial result (2)** Let us look at the simplest case, \(|E_1| = 1\). By Theorem 3.11, if \(|E_1| = 1\), the minimal path \(P\) is of the form

\[
P = P_{S,n-1,a}E, \quad E_1 = d_{E}^{-1}(1) = \{\alpha\},
\]

and the corresponding supporting digraph \(Supp(P)\) owns the property that each point \(\beta \in d_{E}^{-1}(2)\), we have the directed edge \(\beta \to E\). Then it is clear that \(Supp(P_{S,n-1,a})\) is a deformation retract of \(Supp(P)\), where the deformation retraction \(r\) is given by

\[
r : Supp(P) \to Supp(P_{S,n-1,a}),
\]

\[
r(E) = \alpha, \quad r|_{Supp(P_{S,n-1,a})} = Id_{Supp(P_{S,n-1,a})}.
\]

Thus

\[
H_*(Supp(P)) \cong H_*(Supp(P_{S,n-1,a})).
\]

By Hypothesis (1), we get our second initial result.

Next, let us continue discussing the case \(|E_1| = 2\). It is the key point to apply the induction method. We state the idea into the following three steps.

**Step 1:** We embed \(Supp(P)\) into a larger digraph \(\hat{Supp}(P)\) by adding some new edges from points in \(E_2 = d_{E}^{-1}(2)\) to \(E\).
Step 2: Show that $\widehat{\text{Supp}}(P)$ is acyclic by using the Mayer-Vietoris method, where we need the third induction for the intersection of the MV pair.

Step 3: Prove that the embedding $\text{Supp}(P) \hookrightarrow \widehat{\text{Supp}}(P)$ induces the isomorphism on the homology groups.

Now assume that $E_1 = \{\alpha_1, \alpha_2\}$. We embed $\text{Supp}(P)$ to $\widehat{\text{Supp}}(P)$ which is obtained by adding the new edges $\beta \rightarrow E$, whenever $\beta \in E_2 = d^{-1}_E(2), \beta \not\rightarrow E$.

For the following example, we add $2 \rightarrow 5$ (the blue edge) to get $\widehat{\text{Supp}}(P)$.

The following two observations for the new digraph $\widehat{\text{Supp}}(P)$ are important for us.

- The two paths
  $$P_{S,n-1,\alpha_1} E, P_{S,n-1,\alpha_2} E \in \Omega_n(\widehat{\text{Supp}}(P))$$
  are two minimal paths. For simplicity, we denote them by $P_{\alpha_1}, P_{\alpha_2}$ respectively.

- We have the following relations among $\widehat{\text{Supp}}(P), \text{Supp}(P_{\alpha_1})$ and $\text{Supp}(P_{\alpha_2})$
  $$\widehat{\text{Supp}}(P) = \text{Supp}(P_{\alpha_1}) \cup \text{Supp}(P_{\alpha_2})$$
  $$\text{Supp}(P_{\alpha_1}) \cap \text{Supp}(P_{\alpha_2}) = \bigcup_{\beta \in d^{-1}_E(2), \beta \not\rightarrow E \text{ in } \text{Supp}(P)} \text{Supp}(P_{S,n-2,\beta}) E.$$

Lemma 4.3. Let $X = \widehat{\text{Supp}}(P), Y_1 = \text{Supp}(P_{\alpha_1})$ and $Y_2 = \text{Supp}(P_{\alpha_2})$. Then the digraph pair $(Y_1, Y_2)$ forms a Mayer-Vietoris pair of the new digraph $X$.

Proof. According to the above observations, it suffices to prove that the homomorphism in Lemma 4.2
  $$\tilde{\Omega}_p(Y_1) \oplus \tilde{\Omega}_p(Y_2) \rightarrow \tilde{\Omega}_p(X)$$
  is an epimorphism for any $p \geq -1$.

It is obvious for $p = -1, 0, 1, 2$. By the construction of $X, Y_1, Y_2$, we only need to consider the minimal path $u \in \Omega_p(X)$ starting at some vertex $U$ and ending at $E$. We can write $u$ as
  $$u = \sum_{\gamma_1 \in V'_1} P_{U,p-1,\gamma_1} E + \sum_{\gamma_2 \in V'_2} P_{U,p-1,\gamma_2} E + \sum_{\gamma_12 \in V'_{12}} P_{U,p-1,\gamma_12} E,$$
  where
  - $V'_1 \subset V_1 = \{\gamma_1 \in V(Y_1) | \gamma_1 \rightarrow E \text{ in } \text{Supp}(P), \gamma_1 \not\in V(Y_2)\}$;
  - $V'_2 \subset V_2 = \{\gamma_2 \in V(Y_1) | \gamma_2 \rightarrow E \text{ in } \text{Supp}(P), \gamma_2 \not\in V(Y_1)\}$;
  - $V'_{12} \subset V_{12} = \{\gamma_{12} \in V(Y_1) \cap V(Y_2) | \gamma_{12} \rightarrow E \text{ in } X\}$.

By our construction, note that
• Each \( P_{U,p-1,\gamma_1} \in \Omega_{p-1}(\text{Supp}(P)) \) is the unique minimal \((p-1)\)-path starting at \( U \) and ending at \( \gamma_1 \), which also lives in \( Y_1 \) or \( Y_2 \) depending on \( \gamma \in V(Y_1) \) or \( V(Y_2) \), respectively;

• \( V_{12} = \{ \beta \in d^{-1}_E(2) | \beta \to E \text{ in } \text{Supp}(P) \} \), i.e.

\[
P_{U,p-1,\gamma_{12}} E \in A_p(X) \setminus A_p(\text{Supp}(P)).
\]

Without loss of generality, assume that \( u \notin \Omega_p(Y_1) \) and \( u \notin \Omega_p(Y_2) \), which corresponds to \( V_1', V_2' \neq \emptyset \). Now for \( i = 1, 2 \), we consider the minimal \( \partial \)-invariant completion of \( \sum_{\gamma_1 \in V_i'} P_{U,p-1,\gamma_1} \) in \( Y_i \), we denote it as \( P_{U,p,E}^i \).

**Claim 4.4.** \( P_{U,p,E}^1 - \sum_{\gamma_1 \in V_1'} P_{U,p-1,\gamma_1} E \) or \( P_{U,p,E}^2 - \sum_{\gamma_2 \in V_2'} P_{U,p-1,\gamma_2} E \) must be of the form

\[
\sum_{\gamma_i' \in V_i' \subset V_{12}} P_{U,p-1,\gamma_i'} E \in A_{p-1}(Y_1) \cap A_{p-1}(Y_2).
\]

**Proof of Claim 4.4.** As the minimal \( \partial \)-invariant completion, we can decompose \( P_{U,p,E}^1 \in \Omega_p(Y_1) \) and \( P_{U,p,E}^2 \in \Omega_p(Y_2) \) as

\[
P_{U,p,E}^1 = \sum_{\gamma_1 \in V_1'} P_{U,p-1,\gamma_1} E + \sum_{\gamma_1' \in V_1''} P_{U,p-1,\gamma_1'} E + \sum_{\gamma_1'' \in V_{12} \subset V_{12}} P_{U,p-1,\gamma_{12}} E,
\]

\[
P_{U,p,E}^2 = \sum_{\gamma_2 \in V_2'} P_{U,p-1,\gamma_2} E + \sum_{\gamma_2' \in V_2''} P_{U,p-1,\gamma_2'} E + \sum_{\gamma_2'' \in V_{12} \subset V_{12}} P_{U,p-1,\gamma_{12}} E,
\]

where \( V_i'' \subset V_i \) and \( V_i' \cap V_i'' = \emptyset \), \( i = 1, 2 \).

We want to prove that \( V_1'' = \emptyset \) or \( V_2'' = \emptyset \). If \( V_1'', V_2'' \neq \emptyset \), it means that there exists \( \gamma_i' \in V_i'' \setminus V_i' \) (since \( V_i' \cap V_i'' = \emptyset \)), such that \( P_{U,p-1,\gamma_i'} E \in A_p(\text{Supp}(P)) \), and it will cancel some terms in \( P_{U,p-1,\gamma_i} E \) for some \( \gamma_i \in V_i' \).

If we can not decompose \( u \) into two \( \partial \)-invariant paths in \( Y_1 \) and \( Y_2 \) respectively, it means that there must exist a term \( P_{U,p-1,\gamma_1} E \) for some \( \gamma_1 \in V_1' \), which cancels some terms in \( P_{U,p-1,\gamma_2} E \) for some \( \gamma_2 \in V_2' \).

Then in \( \text{Supp}(P) \), there will exist a point \( \beta \), such that

\[
e_{\beta \gamma_1} E - e_{\beta \gamma_2} E \text{ and } e_{\beta \gamma_1} E - e_{\beta \gamma_2} E \in \Omega_2(\text{Supp}(P)).
\]

It is contradiction with Theorem 3.11 (3), since \( P \) is a minimal path. \( \square \)

It follows from the claim that

\[
u = P_{U,p,E}^1 - \sum_{\gamma_i' \in V_i' \subset V_{12}} P_{U,p-1,\gamma_i'} E + \sum_{\gamma_i'' \in V_{12} \subset V_{12}} P_{U,p-1,\gamma_{12}} E
\]

\[
(\text{ or } P_{U,p,E}^2 - \sum_{\gamma_i'' \in V_{12} \subset V_{12}} P_{U,p-1,\gamma_{12}} E)
\]

\[
\in \Omega_p(Y_1) \oplus \Omega_p(Y_2).
\]

Then we are done. \( \square \)

Let \( Z = Y_1 \cap Y_2 \), it follows from Lemma 4.3 that we obtain the long exact sequence

\[
\cdots \to \tilde{H}_p(Z) \xrightarrow{\delta} \tilde{H}_p(Y_1) \oplus \tilde{H}_p(Y_2) \xrightarrow{d} \tilde{H}_p(X) \xrightarrow{\partial} \tilde{H}_{p-1}(Z) \xrightarrow{\delta} \tilde{H}_{p-1}(Y_1) \oplus \tilde{H}_{p-1}(Y_2) \to \cdots
\]
By the argument of initial result (2), we know that $\widetilde{H}(Y_1) = \widetilde{H}(Y_2) = 0$, thus

(4.2) \[ \widetilde{H}_*(\text{Supp}(P)) \cong \widetilde{H}_*(Z). \]

We want to show that $\text{Supp}(P)$ is acyclic through $Z$. We will understand the structure of $Z$ as follows.

(a): $Z$ could be regarded as a proper subgraph of $\text{Supp}(P_{S,n-1,\alpha_1})$ or $\text{Supp}(P_{S,n-1,\alpha_2})$ (it can embed to the two digraphs by mapping $E$ to $\alpha_1$ or $\alpha_2$ with other points fixed). For convenience, we call $n-1$ the length of the digraph $Z$.

(b): If we remove the point $E$, the reduced subgraph of $Z$ is a union of supporting digraphs of minimal path of length $n-2$ with the same starting point $S$.

We will prove the acyclic result of $Z$ via the third induction on the length of $Z$, that is $n-1$.

**Initial result (3).** Note that

- For $n = 2$, $Z = \{S \to E\}$. It is clearly acyclic.
- For $n = 3$, $Z$ is a digraph of the form as follows.

\[ Z = S \hspace{1cm} \longrightarrow \hspace{1cm} E \]

\[ \begin{array}{c}
1 \\
\vdots \\
k-1 \\
k
\end{array} \]

It is easy to check that $Z$ is acyclic in this case.

**Hypothesis (3).** We assume that $Z$ of the above form (satisfying (a) (b)) with $k < n-1$ is acyclic. Now let us prove the case $k = n-1$. We repeat the above analysis and do the induction on the number of points in $E_2 \cap V(Z) = d^{-1}_E(2) \cap V(Z)$.

**Initial result (2’)** If $|E_2 \cap V(Z)| = 1$, the argument is the same as the proof of the initial result (2).

If $|E_2 \cap V(Z)| = 2$ and $E_2 \cap V(Z) = \{\beta_1, \beta_2\}$. We embed $Z$ into a larger digraph $\hat{Z}$ by adding new edges:

\[ \gamma \to E, \quad \gamma \in d^{-1}_E(3) \cap V(Z). \]

Similarly, we have the MV pair $(Z_1, Z_2)$ for $\hat{Z}$, where $Z_1$ and $Z_2$ are supporting digraphs of the minimal paths $P_{S,n-2,\beta_1}E$ and $P_{S,n-2,\beta_2}E$ respectively. By repeating using the Mayer-Vietoris sequence, and applying Hypothesis (3) to $Z_1 \cap Z_2$, we obtain

\[ \widetilde{H}_*(\hat{Z}) = 0. \]

**Lemma 4.5.** Let $Z$ and $\hat{Z}$ be two digraphs as above corresponding to the case $E_2 \cap V(Z) = \{\beta_1, \beta_2\}$. Then

\[ \widetilde{H}_*(Z) = \widetilde{H}_*(\hat{Z}) = 0. \]
Proof. Let \( u \in \Omega_p(Z) \), and \( \partial u = 0 \). Since \( \tilde{H}_*(\tilde{Z}) = 0 \), there exists \( v \in \Omega_{p+1}(\tilde{Z}) \), such that

\[
\partial v = u.
\]

Now we want to replace \( v \) by \( v' \in \Omega_{p+1}(G) \), such that \( \partial v' = u \).

By construction, \( \Omega_*(\tilde{Z}) \) only has more paths ending at \( E \) than \( \Omega_*(Z) \). Thus, we only need to consider the path component ending with \( E \) in \( v \). Meanwhile, since \((Z_1, Z_2)\) is the Mayer-Vietoris pair for \( \tilde{Z} \). Thus, we can represent \( v \in \Omega_{p+1}(\tilde{Z}) \) as

\[
v = \left( \sum_{\gamma_1} p_{\gamma_1} + \sum_{\epsilon_1} p_{\epsilon_1} \right) \beta_1 E + \sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} \gamma_1 E \quad (:= v_1 \in \Omega_{p+1}(Z_1))
\]

\[
\left( \sum_{\gamma_2} p_{\gamma_2} + \sum_{\epsilon_2} p_{\epsilon_2} \right) \beta_2 E + \sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} \gamma_2 E \quad (:= v_2 \in \Omega_{p+1}(Z_2)),
\]

where the summands are explained as follows:

- \( \left( \sum_{\gamma_1} p_{\gamma_1} + \sum_{\epsilon_1} p_{\epsilon_1} \right) \beta_1 \in \Omega_p(Z) \) ends at \( \beta_1, p_{\gamma_1} \) ends at \( \gamma_1, p_{\epsilon_1} \) ends at \( \epsilon_1 \), where \( \gamma_1 \in d_E^{-1}(3) \), \( \delta_1 \in d_E^{-1}(4) \).
- \( \left( \sum_{\gamma_2} p_{\gamma_2} + \sum_{\epsilon_2} p_{\epsilon_2} \right) \beta_2 \in \Omega_p(Z) \) ends at \( \beta_2, p_{\gamma_2} \) ends at \( \gamma_2, p_{\epsilon_2} \) ends at \( \epsilon_2 \), where \( \gamma_2 \in d_E^{-1}(3) \), \( \delta_2 \in d_E^{-1}(4) \).
- \( \sum_{\gamma_i} \sum_{\delta_i} p_{\delta_i} \gamma_i \in \Omega_p(Z) \), \( \sum_{\delta_i} \) depends on \( \gamma_i, i = 1, 2 \). While \( \sum_{\gamma_i} \sum_{\delta_i} p_{\delta_i} \gamma_i E \in A_{p+1}(\tilde{Z}) \setminus A_{p+1}(Z) \).

Now let us analyze the condition \( \partial v = u \in \Omega_p(Z) \) as follows.

\[
\partial v = \left[ \partial \left( \sum_{\gamma_1} p_{\gamma_1} + \sum_{\epsilon_1} p_{\epsilon_1} \right) \right] \beta_1 E \pm \left( \sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} \gamma_1 E \right) + \left( \sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} \gamma_1 E \right)
\]

\[
+ \left[ \partial \left( \sum_{\gamma_2} p_{\gamma_2} + \sum_{\epsilon_2} p_{\epsilon_2} \right) \right] \beta_2 E \pm \left( \sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} \gamma_2 E \right) + \left( \sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} \gamma_2 E \right)
\]

First, \( v_1 \in \Omega_{p+1}(Z_1) \) and \( v_2 \in \Omega_{p+1}(Z_2) \) imply that

\[
\sum_{\epsilon_1} p_{\epsilon_1} E + \sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} E = 0, \quad \text{since they are non-allowed in } \partial v_1;
\]

\[
(4.3) \quad \sum_{\epsilon_2} p_{\epsilon_2} E + \sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} E = 0, \quad \text{since they are non-allowed in } \partial v_2.
\]
Second, $\partial v = u \in \Omega_p(Z)$ and the terms with underlines are non-allowed in $Z$. Combined with (4.3), we get

\begin{equation}
\sum_{\gamma_1} p_{\gamma_1} + \sum_{\gamma_2} p_{\gamma_2} + \sum_{\gamma_1} \left( \partial \sum_{\delta_1} p_{\delta_1} \right) + \sum_{\gamma_2} \left( \partial \sum_{\delta_2} p_{\delta_2} \right) = 0.
\end{equation}

We want to find $w \in \Omega_{p+2}(Z)$ such that $v + \partial w \in \Omega_{p+1}(Z)$. To do this, let us study the above equations (4.3), (4.4) carefully.

Let us look at Equations (4.3). To cancel the terms $\sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} \gamma_1 E$ and $\sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} \gamma_2 E$ in $v$, we consider the following allowed path in $Z$:

\[ w_1 = \sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} \gamma_1 \beta_1 E + \sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} \gamma_2 \beta_2 E \in A_{p+2}(Z). \]

Then we can compute $v + (-1)^p \partial w_1$ and obtain the corresponding non-allowed terms come from

\begin{equation}
\left[ \sum_{\gamma_1} p_{\gamma_1} + (-1)^p \partial \left( \sum_{\gamma_1} \sum_{\delta_1} p_{\delta_1} \right) \gamma_1 \right] \beta_1 E + \left[ \sum_{\gamma_2} p_{\gamma_2} + (-1)^p \partial \left( \sum_{\gamma_2} \sum_{\delta_2} p_{\delta_2} \right) \gamma_2 \right] \beta_2 E.
\end{equation}

To cancel the non-allowed path, let us read (4.4) by dividing $\gamma_1, \gamma_2 \in d_{E}^{-1}(3) \cap V(Z)$ into three disjoint parts

- $\gamma_1': \gamma_1' \rightarrow \beta_1$, but $\gamma_1' \not\rightarrow \beta_2$;
- $\gamma_2': \gamma_2' \rightarrow \beta_2$, but $\gamma_2' \not\rightarrow \beta_1$;
- $\gamma_{12}': \gamma_{12}' \rightarrow \beta_1$, and $\gamma_{12}' \not\rightarrow \beta_2$.

Then (4.4) implies, according to the endpoints,

\[ p_{\gamma_1'} + \partial \left( \sum_{\delta_1} p_{\delta_1} \right) \gamma_1' = 0, \quad p_{\gamma_2'} + \partial \left( \sum_{\delta_2} p_{\delta_2} \right) \gamma_2' = 0, \]

\[ p_{\gamma_{12}'}^Z + p_{\gamma_{12}'}^Z + \partial \left( \sum_{\delta_{12}} p_{\delta_{12}} \right) \gamma_{12}' = 0. \]

The last equation allows us to decompose the left hand side as follows:

\[ p_{\gamma_{12}'}^Z + p_{\gamma_{12}'}^Z + \partial \left( \sum_{\delta_{12}} p_{\delta_{12}} \right) \gamma_{12}' = \left[ p_{\gamma_{12}'}^Z + \partial \left( \sum_{\delta_{12}} p'_{\delta_{12}} \right) \gamma_{12}' \right] + \left[ p_{\gamma_{12}'}^Z + \partial \left( \sum_{\delta_{12}} (p_{\delta_{12}} - p'_{\delta_{12}}) \right) \gamma_{12}' \right]. \]

such that

\[ p_{\gamma_{12}'}^Z + \partial \left( \sum_{\delta_{12}} p'_{\delta_{12}} \right) \gamma_{12}' = p_{\gamma_{12}'}^Z + \partial \left( \sum_{\delta_{12}} (p_{\delta_{12}} - p'_{\delta_{12}}) \right) \gamma_{12}' = 0. \]
Then, we consider another allowed \((p + 2)\)-path in \(Z\):

\[
w_2 = \sum_{\gamma_1} \left( \sum_{\delta_1} (p\delta_{12} - p'_\delta_{12}) \right) \gamma_{12}^\prime (\beta_1 - \beta_2)E.
\]

Then we can compute \(\partial w_2\) and find that the corresponding non-allowed term cancels the one in (4.5). Thus we get

\[
w_1 - w_2 \in \Omega_{p+1}(\hat{Z})
\]

and furthermore,

\[
v' := v + (-1)^p \partial w_1 - (-1)^p \partial w_2 \in \Omega_{p+1}(Z).
\]

Then we are done. \(\square\)

Now let us return to the proof \(\tilde{H}_s(Z) = 0\) for \(|E_2 \cap V(Z)| > 2\).

**Hypothesis (2)** Assume when \(|E_2 \cap V(Z)| = k < m\), \(Z\) is acyclic.

For \(|E_2 \cap V(Z)| = m\), assume that \(E_2 \cap V(Z) = \{\beta_1, \ldots, \beta_m\}\). Let

\[
Z_1 = \text{Supp}(P_{S,n-2,\beta_1})E, \quad Z_2 = \cup_{\beta_1, i=2, \ldots, m} \text{Supp}(P_{S,n-2,\beta_i})E.
\]

Assume that

\[
Z_1 \cap Z_2 \cap f^{-1}(n - 3) = \{\gamma_1, \ldots, \gamma_l\},
\]

then consider the new digraphs \(\hat{Z}\) by adding the edges \(\gamma_i \to E\), \(i = 1, \ldots, l\). Then \(\hat{Z}_1 := Z_1 \cup \cup_{i=1}^{l} \{\gamma_i \to E\}\) and \(\hat{Z}_2 := Z_2 \cup \cup_{i=1}^{l} \{\gamma_i \to E\}\) form a Mayer-Vietoris pair for \(\hat{Z}\), more explicitly

- \(\hat{Z}_1 \cup \hat{Z}_2 = \hat{Z}\)
- \(\hat{Z}_1 \cap \hat{Z}_2 = \cup j \text{Supp}(P_{S,n-3,\gamma_j})E\), it is of the form in our second induction.

Also, we have the deformation retraction by mapping \(E\) to \(\beta_1\) and fixing other points. By Hypothesis (3), we known that

\[
\tilde{H}_s(\hat{Z}_1) = 0.
\]

Thus, again by Hypothesis (3) for \(\hat{Z}_1 \cap \hat{Z}_2\) and by Mayer-Vietoris exact sequence, we have

\[
\tilde{H}_s(\hat{Z}) \cong \tilde{H}_s(\hat{Z}_2).
\]

Moreover, by Hypothesis (2) for \(|E_2 \cap V(\hat{Z}_2)| = m - 1\), we have

\[
\tilde{H}_s(\hat{Z}) \cong \tilde{H}_s(\hat{Z}_2) = 0.
\]

Let us repeat the argument for the case \(|E_2 \cap V(Z)| = 2\), (note that the classification of \(\gamma_i\) in the \(|E_2| = 2\) case still work for the general case by Theorem 3.11 (3)), thus we have

\[
\tilde{H}_s(Z) = \tilde{H}_s(\hat{Z}) = 0.
\]

Then, we finish the proof of acyclic property for \(Z\) of length \(n - 1\).

Thus, by (4.2), we obtain

\[
\tilde{H}_s(\text{Supp}(P)) = 0, \quad \text{when} \ |E_1| = 2.
\]

Then, repeating the relative argument for the pair \((\text{Supp}(P), \text{Supp}(P))\) as Lemma 4.5, we obtain,

\[
\tilde{H}_s(\text{Supp}(P)) = \tilde{H}_s(\text{Supp}(P)) = 0, \quad \text{when} \ |E_1| = 2.
\]
Hypothesis (2). Assume when $|E_1| = k < m$, $\text{Supp}(P)$ is acyclic. For $|E_1| = m$, assume that $E_1 = \{\alpha_1, \ldots, \alpha_m\}$. To do the induction, we have

Claim 4.6. There exists $\alpha_i \in E_1$, such that if we add the edges

$$\gamma \to E, \quad \gamma \in d_{E}^{-1}(2) \cap V(P_{S, n-1, \alpha_i})$$

then the two paths $P_1 := P_{S, n-1, \alpha_i} E$ and $P_2 := P - P_1$ become two minimal paths, with $|E_1 \cap V(P_2)| = m - 1$.

Proof of Claim 4.6. For any $\alpha_i$, if we add such edges, we get the minimal path $P_1$. Meanwhile $P_2$ becomes $\partial$-invariant, but may not be minimal. Then we can write $P_2$ as

$$P_2 = P_2^1 + P_2^2 + \cdots + P_2^l, \text{ each } P_2^j \text{ is minimal.}$$

We keep $P_2^1$ minimal and remove the redundant new edges. Then the claim is proved. An example will be given in the appendix. \qed

It follows from this claim we can repeat the above argument for the proof for $Z$, then we are done.

4.2.1. Proof idea via an example. Let us use the idea in the proof to compute the path homology of exotic cube. A more complicated example will be given in the appendix.

In this example, $E_1 = \{5, 6, 7\}$, $|E_1| = 3$. We add the direct edges $2 \to 8$, and get the following larger digraph $\widehat{\text{Supp}(P)}$

Then the digraphs $Y_1$ and $Y_2$ below form a Mayer-Vietoris pair for $\widehat{\text{Supp}(P)}$. 

$$\widehat{\text{Supp}(P)} = \begin{array}{c}
\text{In this example, } E_1 = \{5, 6, 7\}, |E_1| = 3. \text{ We add the direct edges } 2 \to 8, \text{ and get the following larger digraph } \widehat{\text{Supp}(P)} \\
\end{array}$$
Here $Z = Y_1 \cap Y_2 = \{0 \to 2 \to 8\}$. Now we have,

(a): $Y_1$ can deform contract to a minimal path of length 2, by Hypothesis (1), $\tilde{H}(Y_1) = 0$.

(b): $Y_2$ is a minimal path with $\{6, 7\} = E_1 \cap V(Y_2)$, i.e. $|E_1 \cap V(Y_2)| = 2 < 3 = |E_1|$. By Hypothesis (2), $\tilde{H}(Y_2) = 0$.

By Mayer-Vietoris exact sequence, we get

$$\tilde{H}^*(\widehat{\text{Supp}(P)}) \cong \tilde{H}^*(Z) = 0.$$  

It remains to prove that

$$\tilde{H}^*(\widehat{\text{Supp}(P)}) \cong \tilde{H}^*(\widehat{\text{Supp}(P)}).$$

Note that

$$\Omega_2(\widehat{\text{Supp}(P)}) = \Omega_2(\widehat{\text{Supp}(P)}) \oplus \text{Span}_\mathbb{Z}\{e_{018} - e_{028}, e_{258}\}.$$  

If $u \in \Omega_1(\widehat{\text{Supp}(P)})$, and $\partial u = 0$. Then there exists $v \in \Omega_2(\widehat{\text{Supp}(P)})$, such that $\partial v = u$. We write $v$ as

$$v = v_1 + a(e_{018} - e_{028}) + be_{258}, \quad a, b \in \mathbb{Z}$$

where $v_1 \in \Omega_2(\widehat{\text{Supp}(P)})$.

We can compute

$$\partial v = \partial v_1 + a(e_{01} - e_{02} + e_{18} - e_{28}) + b(e_{25} - e_{28} + e_{58})$$

$$= \partial v_1 + a(e_{01} - e_{02} + e_{18}) + b(e_{25} + e_{58}) - (a + b)e_{28}.$$  

Then $\partial v = u \in \Omega_1(\widehat{\text{Supp}(P)})$ implies that $a + b = 0$.

Let $w = ae_{0158} - ae_{0258} \in \Omega_3(\widehat{\text{Supp}(P)})$, we have

$$\partial w = a(e_{158} - e_{258} + e_{018} - e_{028} - e_{015} + e_{025}).$$

Furthermore,

$$v' := v + \partial w \in A_3(\widehat{\text{Supp}(P)}).$$

Since $\partial v' = \partial(v + \partial w) = \partial v = u \in \Omega_2(\widehat{\text{Supp}(P)})$, we have $v' \in \Omega_3(\widehat{\text{Supp}(P)})$. Thus $H_1(\widehat{\text{Supp}(P)}) = 0$.

Similarly, one check that $H_2(\widehat{\text{Supp}(P)}) = H_3(\widehat{\text{Supp}(P)}) = 0$.

Remark 4.7. Note that one can also embed $\text{Supp}(P)$ to two obvious acyclic digraphs.

1. $\text{Supp}(P)$, whose vertex and edge sets are given by
   - $V(\text{Supp}(P)) = V(\widehat{\text{Supp}(P)})$;
   - $E(\text{Supp}(P)) = \{i_a \to i_b \mid \text{if } i_a \text{ and } i_b \text{ are two distinct vertices in } e_{i_0 \cdots i_{a-1} \cdots i_b \cdots i_n} \in A_n(\widehat{\text{Supp}(P)})\}$. 

(2) $\hat{\text{Supp}}(P)$, by adding (if there are not such edges in $\text{Supp}(P)$) all the edges

$$v \to E, \quad \text{for any } v \in d^{-1}_E(k), \text{ and any } k = 0, 1, \ldots, n - 1.$$ 

The two kinds of bigger digraphs of exotic cube are given as follows.

Both digraphs $\overline{\text{Supp}(P)}$ and $\hat{\text{Supp}}(P)$ can deform retract to its end point $E$. In particular, the first digraph is known as transitive closure [1]. Lin-Wang-Yau [15] developed the Morse theory on the digraph and studied the homology relation between a digraph $G$ and its transitive closure $\overline{G}$ under some condition on discrete gradient vector field.

One can also consider the Morse theory on the digraph through the minimal path or the singular cubic chain. We will study such a project in a separation paper.

**Remark 4.8.** In fact, we prove a more general acyclic result of the digraphs of the form

$$G_1 = \bigcup_i \text{Supp}(P_{S,n-1,\alpha_i})E, \quad G_2 = \bigcup_j \text{Supp}(P^j_{S,n,E}),$$

where $P_{S,n-1,\alpha}$ is the minimal $(n-1)$-paths and $P^j_{S,n,E}$’s are all minimal $n$-paths with the same starting and ending vertices.

We obtain such a result by observing that the intersection digraph $Z$ and $\hat{Z}$ are of such two forms. But we can learn from our examples in this papers that we could only add one edge to split $\text{Supp}(P)$ into two small supporting sub-digraphs, which means that $Z$ may have a simple look such as $\text{Supp}(P_{S,n-1,\alpha})E$, then the proof of acyclic result will become much more simpler. To do that, we need to improve the structure theorem of minimal path $P$ and $\text{Supp}(P)$. We leave such a question to the readers who are interested in it.

**Remark 4.9 (About the regular condition).** In fact, one can loose the regular condition in Definition 2.1 to be the one in Remark 2.2, and then define the minimal path in the similar way. It is easy to check that many properties of minimal paths in this paper still hold. A corresponding structure theorem and vanishing result may be obtained by a more careful analysis.
5. Applications

In this section, we will briefly recall the path cohomology of digraph [11] and its cup product structure, here we change to the field coefficient \( \mathbb{K} \). Next, we will use the method of acyclic models to construct the chain homotopy in the proof of the skew-symmetry of cup product.

5.1. The cohomology of digraphs. Let \( V \) be a finite set. For any \( n \geq 0 \) define a \( n \)-form on \( V \) as any linear functional \( \omega : \Lambda_n(V) \rightarrow \mathbb{K} \). The linear space of all \( n \)-forms is denoted by \( \Lambda^n(V) \). That is, \( \Lambda^n(V) \) is the dual space of \( \Lambda_n(V) \).

For any elementary \( n \)-path \( e_{i_0\cdots i_n} \), there is a dual elementary \( n \)-form \( e^{i_0\cdots i_n} \) such that
\[
(e^{i_0\cdots i_n}, e_{j_0\cdots j_n}) = \delta^{i_0\cdots i_n}_{j_0\cdots j_n}.
\]

There is a linear operator \( d : \Lambda^n(V) \rightarrow \Lambda^{n+1}(V) \), which defines on the basis \( \{e^{i_0\cdots i_n}\} \) by
\[
d e^{i_0\cdots i_n} = \sum_{k \in V} \sum_{p=0}^{n+1} (-1)^p e^{i_0\cdots i_{p-1} k i_p \cdots i_n}.
\]

**Proposition 5.1** ([11]). The pair \((\Lambda^*(V), d)\) is the dual complex of \((\Lambda_*(V), \partial)\). More explicitly,

1. \( d^2 = 0 \);
2. \( (d\omega, u) = (\omega, \partial u) \), for any \( \omega \in \Lambda^n(V), u \in \Lambda_{n+1}(V) \).

Let \( \mathcal{R}^n(V) \) be the subspace of \( \Lambda^n(V) \) spanned by \( \{e^{i_0\cdots i_n} : i_0 \cdots i_n \text{ is regular}\} \). Moreover, \((\mathcal{R}^*(V), d)\) is the dual complex of \((\mathcal{R}_*(V), \partial)\) as Remark 2.2 says.

Let \( G = (V, E) \) be a digraph. For any \( n \geq 0 \), consider the following subspaces of \( \mathcal{R}^n(V) \):

- \( \mathcal{A}^n(G) = \text{Span}\{e^{i_0\cdots i_n} : i_0 \cdots i_p \text{ is allowed}\} \)
- \( \mathcal{N}^n(G) = \text{Span}\{e^{i_0\cdots i_n} : i_0 \cdots i_p \text{ is not allowed but regular}\} \)

such that
\[
\mathcal{R}^n(V) = \mathcal{A}^n(G) \oplus \mathcal{N}^n(G).
\]

Similarly, \( \mathcal{A}^*(G) \) is not necessary preserved by the differential operator \( d \). To obtain a cohomology theory of digraph, set \( \mathcal{J}^n(G) = \mathcal{N}^n(G) + d\mathcal{N}^{n-1}(G) \subset \mathcal{R}^n(V) \) and
\[
\Omega^n(G) = \mathcal{R}^n(V)/\mathcal{J}^n(G) = \mathcal{A}^n(G)/(\mathcal{J}^n(G) \cap \mathcal{A}^n(G)).
\]

**Proposition 5.2** ([11]). The pair \((\Omega^*(G), d)\) forms a well defined quotient complex of \((\mathcal{R}^*(V), d)\), which is the dual complex of \((\Lambda_*(G), \partial)\). The corresponding cohomology space is denoted by \( H^*(G) \), and it is the dual space of \( H_*(G) \).

5.2. Cup product. The cup product \( \cup : \Omega^p(G) \otimes \Omega^q(G) \rightarrow \Omega^{p+q}(G) \) is induced by the concatenation on the space \( \Lambda^*(V) \):
\[
e^{i_0\cdots i_p} e^{j_0\cdots j_q} = \delta_{i_p, j_0} e^{i_0\cdots i_p j_1\cdots j_q}.
\]

One can also formulate the cup product as follows.

For \( u \in \Omega_n(G) \), we write it more explicitly as
\[
u = \sum_{I, |I| = n} c_I e_I, \quad e_I = e_{i_0 i_1 \cdots i_n} \in \mathcal{A}_n(G).
\]
And for each allowed path \( e_I = e_{i_0i_1...i_n} \), denote
\[
e_I|_{0...p} = e_{i_0...i_p}, \quad e_I|_{p...n} = e_{i_p...i_n}.
\]

For \( \alpha \in \Omega^p(G) \), \( \beta \in \Omega^q(G) \), we define \( \alpha \cup \beta \) on \( \Omega^{p+q}(G) \) as
\[
(\alpha \cup \beta, u) = \sum_{I, |I| = p+q} c_I(\alpha, e_I|_{0...p})(\beta, e_I|_{p...p+q}),
\]
where \( u = \sum_{I, |I| = p+q} c_I e_I \in \Omega_{p+q}(G) \).

**Proposition 5.3.** (1) \( \alpha \cup \beta \) in (5.1) is well defined for \( \alpha, \beta \in \Omega^*(G) \). We call it the cup product of \( \alpha \) and \( \beta \).

(2) The cup product is well defined on \( H^*(G) \).

(3) The cup product coincides with concatenation on \( \Omega^*(G) \).

**Proof.** (1) It suffices to check that for any \( n^p \in \mathcal{N}^p(G) \), \( n^{p-1} \in \mathcal{N}^{p-1}(G) \), the following identity holds
\[
\sum_{I, |I| = p+q} c_I(n^p + dn^{p-1}, e_I|_{0...p})(\beta, e_I|_{p...p+q}) = 0.
\]

Note that \( e_I|_{0...p} \) is still allowed path in \( G \), then
\[
(n^p, e_I|_{0...p}) = 0, \quad n^p \in \mathcal{N}^p(G).
\]
then
\[
\sum_{I, |I| = p+q} c_I(n^p + dn^{p-1}, e_I|_{0...p})(\beta, e_I|_{p...p+q})
= \sum_{I, |I| = p+q} c_I(n^{p-1}, \partial e_I|_{0...p})(\beta, e_I|_{p...p+q}).
\]

- If \( \delta e_I|_{0...p} = (-1)^i e_I|_{0...\hat{i}...p} \in \mathcal{A}_{p-1}(G) \), the corresponding terms in the above identity vanish.
- If \( \delta e_I|_{0...p} = (-1)^i e_I|_{0...\hat{i}...p} \) is not allowed, it means \( \delta e_I \) is not allowed. Since
\[
u = \sum_{I, |I| = p+q} c_I e_I \in \Omega_{p+q}(G),
\]

it means that there exists another summand in \( u \), and its corresponding term cancels the non-allowed path. So the corresponding summation will be zero.

(2) (3) It’s obvious by definition. \( \square \)

To study the skew-symmetry property of the cup product, we express the cup product in terms of the star product and diagonal map. Here we briefly introduce the definitions and basic properties we need, and refer to [11] for more details.

**Definition 5.4** (cross product and star product, [6]). Let \( X \) and \( Y \) be two digraphs.
Lemma/Definition 5.7. Let $u \in \mathbb{R}$, we will not use the explicit expression of the cross product, but the following important properties.

1. For any elementary $p$-path $e_x$ in $X$ and $q$-path $e_y$ in $Y$, the cross product $e_x \times e_y$ is defined as a $(p + q)$-path in $X \square Y$ by

$$e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)}e_z,$$

where $\Sigma_{x,y}$ is the set of all stair-like paths $z$ on $X \square Y$ whose projections on $X$ and $Y$ are respectively $x$ and $y$, $L(z)$ is the number of cells in $\mathbb{N}_0^2$ below the staircase $S(z)$. And it extend by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(X \square Y)$.

2. For a $p$-form $\alpha$ on $X$ and a $q$-form $\beta$ on $Y$, define their star product $\alpha \star \beta$ as a $(p + q)$-form on $Z = X \square Y$ as follows: for elementary forms, set

$$e^{ia_1 \ldots a_p}e^{ja_1 \ldots j_q} = e^{(ia_1)(ja_1) \ldots (ia_p)(ja_q)}.$$

**Example 5.5.** Let $a, b \in V(X), 1, 2 \in V(Y)$, we have

1. $e_a \times e_1 = e_{(a1)(a2)}, e_{ab} \times e_1 = e_{(a1)(b1)}$;
2. $e_{ab} \times e_2 = e_{(a1)(b1)(b2)} - e_{(a1)(a2)(b1)}$.

We will not use the explicit expression of the cross product, but the following important properties.

**Lemma 5.6 ([6]).** (1) Let $\alpha \in \mathcal{R}^p(G), \beta \in \mathcal{R}^q(G), u \in \mathcal{R}_p(G)$ and $v \in \mathcal{R}_q(G)$, then

$$(\alpha \star \beta, u \times v) = (\alpha, u)(\beta, v).$$

(2) If $a \in \mathcal{A}_p(G), b \in \mathcal{A}_q(G)$, then $a \times b \in \mathcal{A}_{p+q}(G)$. Moreover, if $u \in \Omega_p(G)$, and $v \in \Omega_q(G)$, then $u \times v \in \Omega_{p+q}(G)$. In particular,

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v).$$

(3) The star product is well defined for all $\alpha \in \Omega^p(G)$ and $\beta \in \Omega^q(G)$, and $\alpha \star \beta \in \Omega^{p+q}(G \square G)$. In particular,

$$d(\alpha \star \beta) = (d\alpha) \star \beta + (-1)^p \alpha \star (d\beta).$$

Now for any $p, q \geq 0$, any $\alpha \in \Omega^p(G), \beta \in \Omega^q(G), u \in \Omega_{p+q}(G)$,

$$(\alpha \cup \beta, u) = \sum_{I, |I| = p+q} c_I(\alpha, e_I|_{0 \ldots p})(\beta, e_I|_{p \ldots p+q})$$

$$= \sum_{I, |I| = p+q} c_I(\alpha \star \beta, e_I|_{0 \ldots p} \times e_I|_{p \ldots p+q})$$

$$= (\alpha \star \beta, \sum_{I, |I| = p+q} c_I e_I|_{0 \ldots p} \times e_I|_{p \ldots p+q}).$$

**Lemma/Definition 5.7.** Let $G = (V, E)$ be a digraph, we define the following two linear operators.

- $\tilde{\Delta}_2 : \mathcal{R}_n(G) \rightarrow \mathcal{R}_n(G \square G)$: for elementary regular path, set

$$\tilde{\Delta}_2 e_I = \sum_{i=0}^{|I|} e_I|_{0 \ldots i} \times e_I|_{i \ldots n}.$$
the transposition operator $t_\sharp$:

$$t_\sharp(e_{i_0i_1\ldots i_p} \times e_{j_0j_1\ldots j_q}) = (-1)^{pq}e_{j_0j_1\ldots j_q} \times e_{i_0i_1\ldots i_p}. $$

1. If $u = \sum_{I, |I| = n} c_I e_I \in \Omega_n(G)$, then

$$\Delta_t(u) = \sum_{i=0}^n I, |I| = n c_I e_I \big|_{0\ldots i} \times e_I \big|_{i\ldots n} \in \Omega_n(G \square G).$$

$$t_\sharp \circ \Delta_t(u) = \sum_{i=0}^n (-1)^{(n-i)} \sum_{I, |I| = n} c_I e_I \big|_{i\ldots n} \times e_I \big|_{0\ldots i} \in \Omega_n(G \square G).$$

2. Both $\Delta_t$ and $t_\sharp \circ \Delta_t$ commute with the boundary operator $\partial$.

We call $\Delta_t$ and $t_\sharp \circ \Delta_t$ the diagonal approximations.

Proof. (1) By Lemma 5.6 (2), we have

$$\partial(\Delta_t(u)) = \sum_{i=0}^n \left( \sum_{I, |I| = n} c_I \partial e_I \big|_{0\ldots i} \right) \times e_I \big|_{i\ldots n} + \sum_{i=0}^n (-1)^{(n-i)} \sum_{I, |I| = n} c_I e_I \big|_{i\ldots n} \times (\partial e_I \big|_{i\ldots n})$$

Since $u = \sum_I c_I e_I \in \Omega_n(G)$, by Lemma 3.7, we have

$$\delta_j u = \sum_{I} c_I \delta_j e_I \in \mathcal{A}_{n-1}(G), \quad j = 0, 1, \ldots, n.$$  

It implies that, with a shift of index,

$$\sum_{I} c_I \delta_j \left(e_I \big|_{0\ldots i}\right) \in \mathcal{A}_{i-1}(G), \quad j = 0, 1, \ldots, i;$$

$$\sum_{I} c_I \delta_j \left(e_I \big|_{i\ldots n}\right) \in \mathcal{A}_{n-i-1}(G), \quad j = 0, \ldots, n-i.$$

Thus, we have $\partial(\Delta_t(u)) \in \mathcal{A}_{n-1}(G)$, i.e. $\Delta_t(u) \in \Omega_n(G \square G)$. The similar argument implies that $t_\sharp \circ \Delta_t(u) \in \Omega_n(G \square G)$.

(2) By definition, one can check directly, for $u \in \Omega_n(G)$,

$$\Delta_t(\partial u) = \partial(\Delta_t(u)), \quad t_\sharp \circ \Delta_t(\partial u) = \partial(t_\sharp \circ \Delta_t(u)).$$

In particular, we learn from the proof that for each fixed $i$,

$$\sum_{I, |I| = n} c_I e_I \big|_{0\ldots i} \times e_I \big|_{i\ldots n} \in \Omega_n(G \square G).$$

Dually we have, for $\alpha \in \Omega^p(G), \beta \in \Omega^q(G)$,

$$\alpha \cup \beta = \Delta_t(\alpha \star \beta), \quad \beta \star \alpha = (-1)^{pq}t_\sharp(\alpha \star \beta).$$
which induce, for \( \varphi \in H^p(G) \), \( \psi \in H^q(G) \),

\[
\varphi \cup \psi = \tilde{\Delta}^*(\varphi \star \psi), \quad (-1)^{pq} \psi \cup \varphi = t^* \tilde{\Delta}^*(\varphi \star \psi).
\]

**Theorem 5.8.** The two chain maps \( \tilde{\Delta}_t \) and \( t_2 \circ \tilde{\Delta}_t \) are chain homotopic. Furthermore, for \( \varphi \in H^p(G) \), \( \psi \in H^q(G) \), we have

\[
\varphi \cup \psi = (-1)^{pq} \psi \cup \varphi.
\]

**Proof.** We will construct the chain homotopy \( F : \Omega_* (G) \to \Omega_{*+1}(G \sqcup G) \) one by one. For simplicity, assume that \( G \) is connected.

First, for the case \( k = 0 \), \( \Omega_0(G) = \text{Span}\{e_i\}_{i \in V(G)} \), we have

\[
t_2 \circ \tilde{\Delta}_t(e_i) - \tilde{\Delta}_t(e_i) = e_i \times e_i - e_i \times e_i = 0.
\]

Thus, \( F(e_i) \) is a cycle in \( \Omega_1(G \sqcup G) \). For simplicity, we fix \( e_{ab} \in \Omega_1(G) \), and set

\[
F(e_i) = e_a \times e_{ab} - e_b \times e_{ab} + e_{ab} \times e_b - e_{ab} \times e_a \quad \text{for any } e_i \in \Omega_0(G).
\]

Clearly, \( \partial F(e_i) = 0 \) and \( t_2 \circ \tilde{\Delta}_t(e_i) - \tilde{\Delta}_t(e_i) = \partial F(e_i) + F(\partial e_i) \) holds naturally.

By our choice of \( F \) on \( \Omega_0(G) \), furthermore, we can obtain

\[
t_2 \circ \tilde{\Delta}_t(e_{ij}) - \tilde{\Delta}_t(e_{ij}) - F(\partial e_{ij}) = e_{ij} \times e_i + e_j \times e_{ij} - e_i \times e_{ij} - e_{ij} \times e_j = \partial(e_{ij} \times e_{ij}).
\]

So we can define \( F(e_{ij}) = e_{ij} \times e_{ij} \in \Omega_2(G \sqcup G) \).

Now to construct \( F : \Omega_n(G) \to \Omega_{n+1}(G \sqcup G) \), we only need to define \( F(u) \) for a minimal path \( u \in \Omega_n(G) \). Note that

\[
t_2 \circ \tilde{\Delta}_t(u) - \tilde{\Delta}_t(u) - F(\partial u)
\]

is a cycle in \( \Omega_n \left( \text{Supp}(u) \sqcup \text{Supp}(u) \right) \subset \Omega_n(G \sqcup G) \), because by Lemma 5.7,

\[
\partial \left( t_2 \circ \tilde{\Delta}_t(u) \right) - \partial \tilde{\Delta}_t(u) - \partial F(\partial u)
= t_2 \circ \tilde{\Delta}_t(\partial u) - \tilde{\Delta}_t(\partial u) - \left( -F(\partial^2 u) + t_2 \circ \tilde{\Delta}_t(\partial u) - \tilde{\Delta}_t(\partial u) \right) = 0.
\]

Since \( H_{i>0} \left( \text{Supp}(u) \right) = 0 \), then by Kunneth formula\(^2\) in [13],

\[
H_{i>0} \left( \text{Supp}(u) \sqcup \text{Supp}(u) \right) = 0.
\]

Then there exists \( v \in \Omega_{n+1} \left( \text{Supp}(u) \sqcup \text{Supp}(u) \right) \subset \Omega_{n+1}(G \sqcup G) \) such that

\[
\partial v = t_2 \circ \tilde{\Delta}_t(u) - \tilde{\Delta}_t(u) - F(\partial u),
\]

then we can define

\[
F(u) = v.
\]

\(^2\)One can also use the acyclic result to prove the Kunneth formula, see the last subsection.
By $\mathbb{K}$-linear extension, we have, for any $p_n \in \Omega_n(G)$,

$$t_\sharp \circ \tilde{\Delta}_\sharp(p_n) - \tilde{\Delta}_\sharp(p_n) - F(\partial p_n) = \partial F(p_n).$$

After descending to homology, we have

$$t_\ast \circ \tilde{\Delta}_\ast(q_n) = \tilde{\Delta}_\ast(q_n)$$

for $q_n \in H_n(G)$.

Moreover, for $\varphi \in H^p, \psi \in H^q(G)$, with $p + q = n$ we have

$$(\psi \cup \varphi, q_n) = (\tilde{\Delta}_\ast(\psi \star \varphi), q_n) = (\psi \star \varphi, \tilde{\Delta}_\ast(q_n))$$

$$= ((-1)^{pq} t_\ast(\psi \star \varphi), \tilde{\Delta}_\ast(q_n))$$

$$= (-1)^{pq} (\varphi \star \psi, t_\ast \tilde{\Delta}_\ast(q_n))$$

$$= (-1)^{pq} (\varphi \star \psi, \tilde{\Delta}_\ast(q_n))$$

$$= (-1)^{pq} (\tilde{\Delta}_\ast(\varphi \star \psi), q_n) = (-1)^{pq} (\varphi \cup \psi, q_n).$$

\[\square\]

5.3. Other applications. The acyclic model of a homology theory has a wide application, such as Künnett formula, the universal coefficient theorem and so on. In [13], they show the Kunneth formula by building the isomorphism between

$$\Omega_*(X \boxtimes Y) \quad \text{and} \quad \Omega_*(X) \otimes \Omega_*(Y).$$

But if we consider the strong product $X \# Y$, which is defined as follows:

- $V(X \# Y) = V(X) \times V(Y)$;
- the arrows are defined as follows:
  - horizontal edge: $(x, y) \rightarrow (x', y)$, where $x \rightarrow x'$,
  - vertical edge: $(x, y) \rightarrow (x, y')$, where $y \rightarrow y'$,
  - diagonal edge: $(x, y) \rightarrow (x', y')$, where $x \rightarrow x'$ and $y \rightarrow y'$.

In this case, the chains $\Omega_*(X \# Y)$ and $\Omega_*(X) \otimes \Omega_*(Y)$ are not isomorphic, but there exist a chain map and its homotopic inverse, where the existence of the chain homotopy can be shown by our acyclic result.

**Appendix**

In this section, we will explain our proof of acyclic model theorem 4.1 via a minimal path of length 4.

Let us consider the following path

$$P = e_{S159}E - e_{S169}E + e_{S269}E$$

$$+ e_{S16(10)}E - e_{S26(10)}E + e_{S27(10)}E - e_{S37(10)}E$$

$$- e_{S27(11)}E + e_{S37(11)}E - e_{S28(11)}E + e_{S48(11)}E.$$

It could be a minimal path of length 4 with the following supporting digraph.
Now we add the new directed edges $6 \rightarrow E$ which make the paths

$$P_1 = e_{S159E} - e_{S169E} + e_{S269E}$$

and

$$P_2 = e_{S16(10)E} - e_{S26(10)E} + e_{S27(10)E} - e_{S37(10)E} - e_{S27(11)E} + e_{S37(11)E} - e_{S28(11)E} + e_{S48(11)E}$$

are $\partial$-invariant in the new larger digraph $\hat{\text{Supp}(P)}$. In particular, both $P_1$ and $P_2$ are minimal as Claim 4.6 says. Denote $X_i = \text{Supp}(P_i)$, $i = 1, 2$. We have $(X_1, X_2)$ form a Mayer-Vietoris pair for $\hat{\text{Supp}(P)}$. See the following picture for each digraph.
As we do for the exotic cube, first, by applying the Mayer-Vietoris exact sequence for the pair \((X_1, X_2)\) and hypothesis, we have

\[
\tilde{H}_*(\text{Supp}(P)) \cong \tilde{H}_*(Z) \cong 0.
\]

Second, one can show that the embedding

\[
(\Omega_*(\text{Supp}(P)), \partial) \hookrightarrow (\Omega_*(\text{Supp}(P)), \partial)
\]

induce the isomorphisms on homologies. Then we are done.

**Remark 5.9.** Actually, one can further add new edge \(7 \to E\) to embed \(X_2\) into a larger graph \(\hat{X}_2\) such that,

\[
P_{21} = e_{S16(10)}E - e_{S26(10)}E + e_{S27(10)}E - e_{S37(10)}E,
\]

\[
P_{22} = -e_{S27(11)}E + e_{S37(11)}E - e_{S28(11)}E + e_{S48(11)}E
\]

are \(\partial\)-invariant in \(\hat{X}_2\). Similarly, it is obvious that \((\text{Supp}(P_{21}), \text{Supp}(P_{22}))\) form a Mayer-Vietoris pair for \(\hat{X}_2\). By the same reason, we have

\[
\tilde{H}_*(\hat{X}_2) = 0.
\]

Then one can show that in this case the two embeddings

\[
(\Omega_*(X_2), \partial) \hookrightarrow (\Omega_*(\hat{X}_2), \partial)
\]

induce the isomorphisms on homologies.

Thus, the essential idea in the proof of acyclic is to find some resolution of the original supporting digraph.

**Remark 5.10** (Related to Claim 4.6). Alternatively, one can first add two edges \(6 \to E\) and \(7 \to E\) to get \(\text{Supp}(P)\), which makes

\[
P_1' = e_{S16(10)}E - e_{S26(10)}E + e_{S27(10)}E - e_{S37(10)}E,
\]

\[
P_2' = P - P_1
\]

are \(\partial\)-invariant in \(\text{Supp}(P)\). One can also consider the other Mayer-Vietoris pair \((X'_1, X'_2)\) related to \(P_1', P_2'\) as follows:
However, in this case, \( P'_2 \) is not a minimal path but the summand of the two minimal path\(^3\), then we cannot apply our induction assumption to say that \( \bar{H}_s(X'_2) = 0 \). And in this case, \( Z \) also becomes more complicated than the previous case. We have two choices to deal with such a situation.

1. Remove the redundant new edges \( 6 \to E \) or \( 7 \to E \), then we return to our original situation.
2. Add more edges \( 1 \to E \), \( 2 \to E \) and \( 3 \to E \) and get the larger digraph \( \hat{X}'_2 \). Then compute its homology via the \( MV \) pair and then compare the homologies of \( X'_2 \) and \( \hat{X}'_2 \).

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\(^3\)In particular, note that there are more than one minimal 2-paths starting at 2 and ending at \( E \) in \( X'_2 \): \( e_{26E} - e_{27E} \), \( e_{26E} - e_{29E} \) and \( e_{27E} - e_{29E} \). They exist a linear relation among the three paths.
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