Optimal additive Schwarz preconditioning for hypersingular integral equations on locally refined triangulations

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Abstract For the non-preconditioned Galerkin matrix of the hypersingular integral operator, the condition number grows with the number of elements as well as the quotient of the maximal and the minimal mesh-size. Therefore, reliable and effective numerical computations, in particular on adaptively refined meshes, require the development of appropriate preconditioners. We propose and analyze a local multilevel preconditioner which is optimal in the sense that the condition number of the corresponding preconditioned system is independent of the number of elements, the local mesh-size, and the number of refinement levels. The theory covers closed boundaries as well as open screens in 2D and 3D. Numerical experiments underline the analytical results and compare the proposed preconditioner to other multilevel schemes as well as techniques based on operator preconditioning.

Keywords Preconditioner · Multilevel additive Schwarz · Hypersingular integral equation

Mathematics Subject Classification 65N30 · 65F08 · 65N38
1 Introduction

Let \( \Omega_1 \subset \mathbb{R}^d \) be a bounded polygonal resp. polyhedral Lipschitz domain in \( \mathbb{R}^d \), \( d = 2, 3 \), with connected boundary \( \Gamma = \partial \Omega_1 \). For a given right-hand side \( f \) with zero integral mean, we consider the hypersingular integral equation

\[
Wu(x) := -\partial_n x \int_\Gamma \partial_n y G(x - y) u(y) \, ds_y = f(x) \quad \text{for } x \in \Gamma, \tag{1}
\]

which admits a unique solution \( u \) (up to additive constants). Here, \( \partial_n x \) is the normal derivative with respect to \( x \in \Gamma \), and \( G(z) \) denotes the fundamental solution of the Laplacian

\[
G(z) = \begin{cases} 
-\frac{1}{2\pi} \log |z| & \text{for } d = 2, \\
+\frac{1}{4\pi} \frac{1}{|z|} & \text{for } d = 3. 
\end{cases}
\tag{2}
\]

Details on the functional analytic setting of (1) are given in Sect. 2.1. The exact solution \( u \) of (1) cannot be computed analytically in general. For a given triangulation \( T_\ell \) of \( \Gamma \), one can, e.g., use the Galerkin boundary element method (BEM) to compute an approximation \( u_\ell \) of \( u \) instead. If a certain accuracy of the approximation \( u_\ell \approx u \) is required, adaptive mesh-refining algorithms of the type

\[
\text{SOLVE} \quad \rightarrow \quad \text{ESTIMATE} \quad \rightarrow \quad \text{MARK} \quad \rightarrow \quad \text{REFINE}
\]

are used, where, starting with a given initial triangulation \( T_0 \), a sequence of locally refined triangulations \( T_\ell \) and corresponding Galerkin solutions \( u_\ell \) are computed. The lowest-order BEM for (1) uses \( T_\ell \)-piecewise affine and globally continuous functions \( u_\ell \in S^1(T_\ell) \) to approximate \( u \), and the adaptive mesh-refinement leads to a nested sequence of spaces \( S^1(T_\ell) \subset S^1(T_{\ell+1}) \) for all \( \ell \geq 0 \).

In recent years, convergence and optimality of adaptive BEM has been proved [8,9,11,34]. Throughout, it is however assumed that the Galerkin solution \( u_\ell \) is computed exactly, i.e., the resulting linear system \( A_\ell x_\ell = b_\ell \) is solved exactly. As is well known, the accuracy of direct solvers as well as the effectivity of iterative solvers is usually spoiled by the conditioning of the matrix \( A_\ell \). For uniform triangulations \( T_\ell \) with \( N_\ell = \#T_\ell \) elements, it holds cond_2(\( A_\ell \)) \( \lesssim N_\ell^{1/(d-1)} \) for the \( \ell_2 \)-condition number. For adaptively refined triangulations \( T_\ell \) with maximal element diameter \( h_{\max,\ell} \) and minimal element diameter \( h_{\min,\ell} \) the situation is even worse [6], namely cond_2(\( A_\ell \)) \( \lesssim N_\ell (1 + |\log(N_\ell h_{\min,\ell})|) \) for \( d = 2 \) resp. cond_2(\( A_\ell \)) \( \lesssim N_\ell^{1/2}(h_{\max,\ell}/h_{\min,\ell})^2 \) for \( d = 3 \). Therefore, reliable and effective numerical computations require efficient preconditioners.

Prior work includes diagonal scaling of the BEM matrices which reduces the condition number for adaptive triangulations down to that of a uniform triangulation with the same number of elements [6,12]. Other multilevel preconditioners for the Galerkin BEM of hypersingular integral equations are proposed in [7,33,35] and the references therein, where mainly quasi-uniform triangulations are thoroughly analyzed. Another technique that also covers locally refined triangulations, is operator preconditioning.
based on the use of operators of opposite order, which is analyzed in [30] for closed boundaries. The authors show that this technique leads to bounded condition numbers independently of the triangulations. The case of open intervals and locally refined triangulations is covered in [23], where the bounds of the resulting condition numbers depend on $\log(h_{\text{min},\ell})^2$. A recent work based on operator preconditioning is [16], where the exact inverses of the boundary integral operators are used, which lead to mesh-independent condition numbers.

Our work focuses on additive Schwarz preconditioners for the Galerkin BEM of (1) with lowest-order polynomials $u_\ell \in S^1(T_\ell)$. For uniform triangulations, it is shown in [32] that this approach leads to bounded condition numbers for the preconditioned system, i.e., $\text{cond}((B^\ell)^{-1}A^\ell) \leq C < \infty$ with some $\ell$-independent constant $C > 0$. The same is proved for partially adapted triangulations in [5], where it is assumed that $T_\ell \cap T_{\ell+1} \subset T_{\ell+k}$ for all $\ell, k \in \mathbb{N}_0$, i.e., as soon as an element $T \in T_\ell$ is not refined, it remains non-refined in all succeeding triangulations. In our contribution, we remove such an assumption which is infeasible in practice, and only rely on nestedness $S^1(T_\ell) \subset S^1(T_{\ell+1})$ of the discrete ansatz spaces. To derive a stable subspace decomposition in $H^{1/2}$, we only use new nodes in $T_{\ell+1}\setminus T_\ell$ plus certain neighbours, see Fig. 2 below. Related work from the FEM literature with energy space $H^1$ includes [22,36,38].

While all constants and their dependencies are explicitly given in all statements, in proofs we use the symbol $\lesssim$ to abbreviate $\leq$ up to some multiplicative constant which is clear from the context. Moreover, we use $\simeq$ to abbreviate that both estimates $\lesssim$ and $\gtrsim$ hold.

The remainder of this work is organized as follows: Sect. 2 states the analytical main result of this work. We first recall the necessary notation to define the local multilevel preconditioner (LMLD) and then state that the proposed preconditioner leads to a uniformly bounded condition number even on adaptive meshes (Theorem 1). Furthermore, we analyze a global multilevel preconditioner (GMLD) and give a similar but weaker result (Theorem 2), which improves the result of [20]. For the ease of presentation, we first focus on closed boundaries $\Gamma = \partial \Omega$. In Sect. 3, numerical experiments on closed boundaries and slits in 2D and 3D underline the theoretical findings and prove that Theorems 1 and 2 are sharp. Additionally, we compare the preconditioner LMLD to a hierarchical preconditioner (HB) and a preconditioner based on operator preconditioning (OP). Moreover, we discuss and compare the computational complexity of these preconditioners in terms of arithmetic operations and storage requirements. In particular, the experiments from Sect. 3 underline the optimality of LMLD. In Sect. 4, we give the proof of Theorem 1. In Sect. 5, we give the proof of Theorem 2. The final Sect. 6 proves that Theorems 1 and 2 remain valid for open screens $\Gamma \subsetneq \partial \Omega$ in 2D and 3D.

2 Main result

2.1 Continuous setting

Let $\Gamma : = \partial \Omega$. By $H^s(\Gamma)$, $0 < s < 1$, we denote the usual Sobolev spaces. The space $H^{-s}(\Gamma) : = H^s(\Gamma)^*$ is the dual space of $H^s(\Gamma)$, where duality is understood with
with respect to the extended $L^2(\Gamma)$-scalar product $\langle \cdot, \cdot \rangle_{\Gamma}$. Finally, let $H_0^{\pm 1/2}(\Gamma) := \{ v \in H^{\pm 1/2}(\Gamma) : \langle v, 1 \rangle_{\Gamma} = 0 \}$.

It is known [17,21] that $\mathcal{W}$ induces a linear and bounded operator $\mathcal{W} : H^s(\Gamma) \rightarrow H^{s-1}(\Gamma)$, $0 \leq s \leq 1$, which is symmetric and positive semidefinite on $H^{1/2}(\Gamma)$. Moreover, it holds $\langle \mathcal{W} v, 1 \rangle_{\Gamma} = 0$. We thus suppose that the right-hand side $f$ in (1) satisfies $f \in H^{-1/2}_0(\Gamma)$.

As $\Gamma$ is connected, the kernel of $\mathcal{W}$ are precisely the constant functions, and thus $\mathcal{W} : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is linear, continuous, symmetric, and elliptic. Therefore, \[
\langle (v, w) \rangle := \langle \mathcal{W} v, w \rangle_{\Gamma} + \langle v, 1 \rangle_{\Gamma} \langle w, 1 \rangle_{\Gamma}
\] provides a scalar product on $H^{1/2}(\Gamma)$, and the induced norm $\| v \|^2 := \langle (v, v) \rangle$ is equivalent to the usual $H^{1/2}(\Gamma)$-norm. In particular, (1) is equivalently recast in the variational formulation \[
\langle (u, v) \rangle = \langle f, v \rangle_{\Gamma} \quad \text{for all } v \in H^{1/2}(\Gamma).
\]

According to the Lax–Milgram lemma, this formulation allows for a unique solution $u \in H^{1/2}(\Gamma)$. Due to $f \in H^{-1/2}_0(\Gamma)$, it follows $u \in H^{1/2}_0(\Gamma)$.

### 2.2 Triangulation and general notation

Let $T_\ell$ denote a regular triangulation of $\Gamma$ into compact affine line segments ($d = 2$) resp. compact plane surface triangles ($d = 3$). We define the local mesh-width function $h_\ell \in L^\infty(\Gamma)$ by

$$h_\ell |_{T} := h_\ell (T) := \text{diam}(T) \quad \text{for all } T \in T_\ell.$$  

We suppose that $T_\ell$ satisfies

$$\max \left\{ \frac{|T|}{|T'|} : T, T' \in T_\ell \text{ with } T \cap T' \neq \emptyset \right\} \leq \gamma \quad \text{for } d = 2,$$

$$\max \left\{ h_\ell (T)^2 / |T| : T \in T_\ell \right\} \leq \gamma \quad \text{for } d = 3.$$  

For $d = 2$ condition (6) is known as local quasi-uniformity (or: K-mesh property) and for $d = 3$ as uniform shape-regularity. Here and throughout, $|T|$ denotes the $(d-1)$-dimensional surface measure of $T \in T_\ell$ and hence $|T| = \text{diam}(T)$ for $d = 2$.

We consider lowest-order conforming boundary elements,

$$X_\ell := S^1(T_\ell) := \{ v \in C(\Gamma) : v|_T \text{ is affine for all } T \in T_\ell \} \subseteq H^{1/2}(\Gamma).$$  

Let $N_\ell$ denote the set of nodes of the mesh $T_\ell$. The natural basis of $X_\ell$ is given by the hat-functions. For each node $z \in N_\ell$, let $\eta_\ell^z \in S^1(T_\ell)$ be the hat-function characterized by

$$\eta_\ell^z (z) = 1 \quad \text{and} \quad \eta_\ell^z (z') = 0 \quad \text{for all } z' \in N_\ell \setminus \{z\}.$$  

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For any subset $\tau \subseteq \Gamma$, we define the patch $\omega^k_{\ell}(\tau) \subseteq \Gamma$ inductively by

$$\omega^1_{\ell}(\tau) := \omega_{\ell}(\tau) := \bigcup \{ T \in \mathcal{T}_{\ell} : T \cap \tau \neq \emptyset \}, \quad \omega^{k+1}_{\ell}(\tau) := \omega^k_{\ell}(\omega_{\ell}(\tau)) \text{ for } k \in \mathbb{N}. \quad (9)$$

For any node $z \in \mathcal{T}_{\ell}$, we abbreviate $\omega^k_{\ell}(z) := \omega^k_{\ell}(\{z\})$ and note that $\omega_{\ell}(z) = \operatorname{supp}(\eta^\ell_{z})$. We further define for every node $z \in \mathcal{N}_{\ell}$ the mesh-width

$$h_{\ell}(z) := \max_{T \in \mathcal{T}_{\ell}, T \subseteq \omega_{\ell}(z)} \operatorname{diam}(T).$$

It holds

$$h_{\ell}(T) \leq h_{\ell}(z) \lesssim h_{\ell}(T) \quad \text{for all } z \in \mathcal{N}_{\ell} \quad \text{and} \quad T \in \mathcal{T}_{\ell} \text{ with } z \in T, \quad (10)$$

where the hidden constant depends only on the $\gamma$-shape regularity of $\mathcal{T}_{\ell}$. It holds

$$0 \leq \eta^\ell_{z} \leq 1, \quad \| \nabla \eta^\ell_{z} \|_{L^\infty(\Gamma)} \lesssim h_{\ell}^{-1}(z), \quad \text{and} \quad \sum_{z \in \mathcal{N}_{\ell}} \eta^\ell_{z} = 1, \quad (11)$$

where the hidden constant depends only on the $\gamma$-shape regularity of $\mathcal{T}_{\ell}$.

### 2.3 Galerkin discretization

The Galerkin approximation $u_{\ell} \in X^\ell$ to $u$ solves

$$\langle \langle u_{\ell}, v_{\ell} \rangle \rangle = \langle f, v_{\ell} \rangle_{\Gamma} \quad \text{for all } v_{\ell} \in X^\ell. \quad (12)$$

Fixing a numbering of the nodes $\mathcal{N}_{\ell} = \{z_1, \ldots, z_N\}$, the discrete solution $u_{\ell}$ from (12) is obtained by solving a linear system of equations $A^\ell x^\ell = b^\ell$ in $\mathbb{R}^N$, where

$$A^\ell_{jk} = \langle \langle \eta^\ell_{z_k}, \eta^\ell_{z_j} \rangle \rangle, \quad b^\ell_j = \langle f, \eta^\ell_{z_j} \rangle_{\Gamma}, \quad \text{and} \quad u_{\ell} = \sum_{k=1}^N x^\ell_k \eta^\ell_{z_k}. \quad (13)$$

### 2.4 Mesh-refinement and hierarchical structure

We assume that $\mathcal{T}_{\ell}$ is obtained from an initial triangulation $\mathcal{T}_0$ by use of bisection. For $d = 2$, we assume a bisection algorithm that guarantees $\ell$-independent $\gamma$-shape regularity (6), see [2] for such an algorithm. For $d = 3$, we use 2D newest vertex bisection; see Fig. 1 or, e.g., [18] and the references therein. We note that $\ell$-independent $\gamma$-shape regularity (6) is guaranteed. We suppose that $\mathcal{T}_{\ell+1} = \text{refine}(\mathcal{T}_{\ell}; \mathcal{M}_{\ell})$ for all $\ell \in \mathbb{N}_0$, where $\text{refine}(\cdot)$ abbreviates the mesh-refinement strategies mentioned and $\mathcal{M}_{\ell} \subseteq \mathcal{T}_{\ell}$ is an arbitrary set of marked elements. The mesh $\mathcal{T}_{\ell+1}$ is then the
For each surface triangle $T \in \mathcal{T}_\ell$ for $d = 3$, there is one fixed reference edge, indicated by the double line (left, top). Refinement of $T$ is done by bisecting the reference edge, where its midpoint becomes a new node. The reference edges of the son triangles are opposite to this newest vertex (left, bottom). To avoid hanging nodes, iterated newest vertex bisection leads to 2, 3, or 4 son triangles if more than one edge, but at least the reference edge, is marked for refinement (right).

The left figure shows a mesh $\mathcal{T}_{\ell-1}$, where the two elements in the lower left corner are marked for refinement (green). Bisection of these two elements provides the mesh $\mathcal{T}_\ell$ (right), where two new nodes are created. The set $\mathcal{N}_\ell$ consists of these new nodes plus those immediate neighbours, where $\text{supp}(\eta^{\ell}_{z}) \subseteq \text{supp}(\eta^{\ell-1}_{z})$ (red). The union of the support of basis functions in $\tilde{X}^\ell = \text{span}\{\eta^{\ell}_{z} : z \in \mathcal{N}_\ell\}$ is given by the light- and dark-green areas in the right figure. The coarsest regular triangulation of $\Gamma$ such that all marked elements $T \in \mathcal{M}_\ell$ have been bisected.

Clearly, $\mathcal{N}_\ell \subseteq \mathcal{N}_{\ell+1}$. To provide an optimal additive Schwarz scheme on locally refined meshes, we define

$$\tilde{\mathcal{N}}_0 = \mathcal{N}_0 \quad \text{and} \quad \tilde{\mathcal{N}}_\ell : = \mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1} \cup \{z \in \mathcal{N}_\ell \cap \mathcal{N}_{\ell-1} : \omega_\ell(z) \subseteq \omega_{\ell-1}(z)\} \quad \text{for } \ell \geq 1,$$

i.e., $\tilde{\mathcal{N}}_\ell$ contains all new nodes plus certain neighbours, see also Fig. 2. We stress that smoothing on all nodes $z \in \mathcal{N}_\ell$ or on new nodes $\mathcal{N}_{\ell+1} \setminus \mathcal{N}_\ell$ only will lead to suboptimal schemes, whereas smoothing on the nodes $z \in \tilde{\mathcal{N}}_\ell$ will prove to be optimal. For $\ell \geq 0$ and $z \in \mathcal{N}_\ell$, we define the subspaces

$$\tilde{X}^\ell : = \text{span}\{\eta^{\ell}_{z} : z \in \tilde{\mathcal{N}}_\ell\} \quad \text{and} \quad X^\ell_{z} : = \text{span}\{\eta^{\ell}_{z}\}.$$

**2.5 Local multilevel diagonal preconditioner (LMLD)**

For any $L \in \mathbb{N}_0$, we aim to derive a preconditioner $(\tilde{\mathbf{B}}^L)^{-1}$ for the Galerkin matrix $\mathbf{A}^L$ from (13) with respect to the space $X^L$ and the basis $\{\eta^{L}_{z} : z \in \mathcal{N}_L\}$. 
For all $0 \leq \ell \leq L$, let $A^\ell$ be the Galerkin matrix with respect to $X^\ell$ and the associated basis $\{\eta^\ell_z : z \in N^\ell\}$. Let $\widetilde{D}^\ell$ be a diagonal matrix, where the entries are given by the diagonal elements of $A^\ell$ for all degrees of freedom that correspond to the nodes in $\tilde{N}^\ell$ and are zero otherwise. Define $N^\ell = \#N^\ell$. We consider the embedding $I^\ell : X^\ell \to X^L$, i.e., the formal identity. Let $I^\ell \in \mathbb{R}^{N^L \times N^\ell}$ be the matrix representation of the operator $I^\ell$ with respect to the bases of $X^\ell$ resp. $X^L$. With this notation, we consider the matrix

$$
\left(\widetilde{B}^L\right)^{-1} = \sum_{\ell=0}^L I^\ell \left(\widetilde{D}^\ell\right)^{-1} \left(I^\ell\right)^T. \tag{16}
$$

Instead of solving $A^L x^L = b^L$, we now consider the preconditioned linear system

$$
\left(\widetilde{B}^L\right)^{-1} A^L x^L = \left(\widetilde{B}^L\right)^{-1} b^L. \tag{17}
$$

As is shown in Sect. 4.1, $\left(\widetilde{B}^L\right)^{-1}$ corresponds to a diagonal scaling on each local subspace $\tilde{X}^\ell$. Therefore, this type of preconditioner is called local multilevel diagonal scaling.

For a symmetric and positive definite matrix $C \in \mathbb{R}^{N_L \times N_L}$, we denote by $\langle \cdot, \cdot \rangle_C = \langle C \cdot, \cdot \rangle_2$ the induced scalar product on $\mathbb{R}^{N_L}$, and by $\| \cdot \|_C$ the corresponding norm resp. induced matrix norm. Here $\langle \cdot, \cdot \rangle_2$ denotes the Euclidean inner product on $\mathbb{R}^{N_L}$.

We define the condition number $\text{cond}_C$ of a matrix $A \in \mathbb{R}^{N_L \times N_L}$ as

$$
\text{cond}_C(A) := \|A\|_C \|A^{-1}\|_C. \tag{18}
$$

**Theorem 1** The matrix $\left(\widetilde{B}^L\right)^{-1}$ is symmetric and positive definite with respect to $\langle \cdot, \cdot \rangle_2$, and $\overrightarrow{P}^L_{AS} := \left(\widetilde{B}^L\right)^{-1} A^L$ is symmetric and positive definite with respect to $\langle \cdot, \cdot \rangle_{\overrightarrow{B}^L}$. Moreover, the minimal and maximal eigenvalues of the matrix $\overrightarrow{P}^L_{AS}$ satisfy

$$
c \leq \lambda_{\min} \left(\overrightarrow{P}^L_{AS}\right) \quad \text{and} \quad \lambda_{\max} \left(\overrightarrow{P}^L_{AS}\right) \leq C, \tag{19}
$$

where the constants $c$, $C > 0$ depend only on $\Gamma$ and the initial triangulation $\mathcal{T}_0$. In particular, the condition number of the additive Schwarz matrix $\overrightarrow{P}^L_{AS}$ is $L$-independently bounded by

$$
\text{cond}_{\overrightarrow{B}^L} \left(\overrightarrow{P}^L_{AS}\right) \leq C/c. \tag{20}
$$

The proof of Theorem 1 is given in Sect. 4 below, and we focus on the relevant application first: Consider an iterative solution method, such as CG, see e.g. [25], to solve (17), where the relative reduction of the $j$-th residual depends only on the condition number $\text{cond}_{\overrightarrow{B}^L} \left(\overrightarrow{P}^L_{AS}\right)$. Then, Theorem 1 proves that the iterative scheme, together with the preconditioner $\overrightarrow{B}^L$ is optimal in the sense that the number of iterations to reduce the relative residual under the tolerance $\varepsilon$ is bounded by a constant that
depends only on $\Gamma$, and the initial triangulation $T_0$, but is independent of the current triangulation $T_L$.

### 2.6 Global multilevel diagonal preconditioner (GMLD)

In addition to the new local multilevel diagonal preconditioner from Sect. 2.5, we consider a global multilevel diagonal preconditioner, where we use all nodes $z \in N_\ell$ of the triangulation $T_\ell$ instead of only $\tilde{N}_\ell$ to construct the preconditioner. For 2D hypersingular integral equations with graded meshes on an open curve, it is proved [20] that the maximal eigenvalue of the associated additive Schwarz operator is $O(L^2)$. For $\gamma$-shape regular meshes, the following theorem proves $O(L)$ growth, and numerical experiments in Sect. 3.3 underline that this improved result is sharp.

Let $D^\ell$ denote the diagonal matrix of $A^\ell$, i.e. $D^\ell_{jk} := \delta_{jk} A^\ell_{jj}$, with Kronecker’s delta $\delta_{jk}$. We define the global multilevel preconditioner $(B_L)^{-1}$ by

$$\left( B_L \right)^{-1} := \sum_{\ell=0}^{L} I^\ell \left( D^\ell \right)^{-1} (I^\ell)^T. \tag{21}$$

**Theorem 2** The matrix $(B_L)^{-1}$ is symmetric and positive definite with respect to $\langle \cdot, \cdot \rangle_2$, and $P_{\text{AS}}^L := (B_L)^{-1} A^L$ is symmetric and positive definite with respect to $\langle \cdot, \cdot \rangle_{B_L}$. Moreover, the minimal and maximal eigenvalues of the matrix $P_{\text{AS}}^L$ satisfy

$$c \leq \lambda_{\min} \left( P_{\text{AS}}^L \right) \quad \text{and} \quad \lambda_{\max} \left( P_{\text{AS}}^L \right) \leq C(L + 1), \tag{22}$$

where the constants $c, C > 0$ depend only on $\Gamma$ and the initial triangulation $T_0$, but are independent of the level $L$. In particular, the condition number of the additive Schwarz matrix $P_{\text{AS}}^L$ is bounded by

$$\text{cond}_{B_L} \left( P_{\text{AS}}^L \right) \leq (L + 1) C/c. \tag{23}$$

### 3 Numerical experiments

In this section, different experiments in 2D and 3D show the optimality of the proposed local multilevel diagonal preconditioner $\tilde{B}_L$ (LMLD) from (16) numerically.

In all examples, the exact solution is known, and uniform mesh-refinement will lead to suboptimal convergence rates for the energy error. Since adaptive refinement regains the optimal order of convergence, the use of adaptive methods is preferable. To steer the mesh-refinement, we employ some ZZ-type error estimator [4,10]. The resulting linear systems are solved by CG. Overall, we compare the following five preconditioners with respect to the condition number of the resulting system, the overall storage requirements, and the arithmetic complexity for its applications:
• **LMLD** local multilevel diagonal preconditioner from (16); cf. Theorem 1;

• **GMLD** global multilevel diagonal preconditioner from (21); cf. Theorem 2;

• **HB** hierarchical basis preconditioner, where only new nodes are considered for preconditioning [33]: Define $\mathcal{N}_\ell := \mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}$ as the set of new nodes and define $D^\ell$ as the diagonal matrix, where the entries are given by the diagonal elements of $A^\ell$ for those indices corresponding to the set $\mathcal{N}_\ell$ and are zero otherwise. The hierarchical basis preconditioner is then given by

$$\left(B^L_{HB}\right)^{-1} = \sum_{\ell=0}^L I^\ell \left(D^\ell\right)^{-1} \left(I^\ell\right)^T.$$  \hfill (24)

The preconditioned matrix reads $P^L_{HB} := \left(B^L_{HB}\right)^{-1}A^L$.

• **DIAG** diagonal scaling of the Galerkin matrix [6, 12]. The preconditioned matrix reads $P^L_{\text{diag}} := \left(D^L\right)^{-1}A^L$. Note that in 2D $\langle \eta_z^L, \eta_z^L \rangle \simeq C$ for some constant $C > 0$ and all $z \in \mathcal{N}_L$. Thus, a diagonal preconditioner in 2D will not affect the condition number significantly.

• **OP** operator preconditioning [30]. For simplicity we consider the 2D case and denote by $T^*_L$ the dual mesh of $T_L$, see, e.g., [29, Section 2.2]. Let $V^L$ be the Galerkin matrix of the single-layer operator

$$\mathcal{V}\phi(x) := \int_G G(x - y)\phi(y) \, ds_y$$

with respect to the space $P^0(T^*_L)$ of piecewise constants on the dual mesh. Let $M^L$ denote the mass matrix with respect to the spaces $P^0(T^*_L)$ and $S^1(T_L)$. Then, the preconditioner reads

$$\left(B^L_{OP}\right)^{-1} := \left(M^L\right)^{-1} V^L \left(M^L\right)^{-T}$$

and the preconditioned system is given by $P^L_{OP} := \left(B^L_{OP}\right)^{-1}A^L$.

The preconditioners GMLD and HB are formally similar to LMLD, but the condition number depends on the level $L$ resp. on the mesh-width function $h_L$. Since the diagonal elements of the matrix $A^L$ in 2D are essentially constant, the simple diagonal preconditioner has no significant effect on the condition numbers; see also [6].

All computations were performed with MATLAB (2012b) on an x86_64 GNU/Linux system. For the preconditioned CG algorithm, we use the MATLAB function `pcg.m`. The assembly of the boundary integral operators in 2D was done with help of the MATLAB BEM-library HILBERT [1]. In 3D, we used the library BEM++ [27].

Before we come to the experiments, we comment on computational complexity and storage requirements of the preconditioners LMLD, GMLD, HB, and OP in the following section.
3.1 Storage requirements and computational complexity

The storage requirements for the Galerkin matrix $A_L$ resp. $V_L$ are in general $N_L^2$, where $N_L = \#N_L$ is the number of degrees of freedom. The cost of assembly of these matrices is also of the same order. For iterative solvers, the complexity of matrix–vector multiplications is of order $O(N_L^2)$. Note that matrix compression techniques like FMM [14] or $\mathcal{H}$-matrices [15] reduce memory consumption, assembly, and matrix–vector multiplication down to $O(N_L \log \beta N_L)$, where $\beta \geq 0$ depends on the method used.

**Operator preconditioning** In 2D, the mass matrix $M_L$ is sparse and diagonally dominant and can hence be stored resp. inverted in $O(N_L)$. Thus, memory consumption, assembly, and matrix–vector multiplication for OP is $O(N_L \log \beta N_L)$.

**LMLD** For the application of this preconditioner on a vector, we need to know how to realize $I^\ell$. To that end, let $I^\ell_{\ell+1}$ denote the matrix form of the formal identity $\chi^\ell_{\ell+1} \to \chi^\ell$, hence,

$$I^\ell = I^\ell_{\ell+1} \ldots I^\ell_{L}. $$

Altogether, we can write

$$ (\tilde{B}^L)^{-1} = I^L_{L-1} \ldots I^0_1 (\tilde{D}^0)^{-1} (I^0_T)_T \ldots (I^L_T)_T + \ldots + I^L_{L-1} (\tilde{D}^{L-1})^{-1} (I^{L-1})_T + (\tilde{D}^L)^{-1}. $$

Since $\tilde{N}^\ell$ consist of newly created nodes and some neighbours, we have

$$ \tilde{N}^\ell = \#\tilde{N}^\ell \leq C (N^\ell - N^{\ell-1}) = C \#(\tilde{N}^\ell \setminus N^{\ell-1}), $$

where for $d = 2$ there holds $C = 3$ and for $d = 3$ the constant $C$ depends only on shape-regularity (6). Therefore, the overall storage requirements of all matrices $(\tilde{D}^\ell)^{-1}, \ell = 0, \ldots, L$, are (with $N_{-1} := 0$)

$$ \mathcal{O} \left( \sum_{\ell=0}^{L} (N^{\ell} - N^{\ell-1}) \right) = \mathcal{O} (N_L). $$

For the application of $(\tilde{B}^L)^{-1}$ we need the evaluation of $I^\ell_{\ell+1}$ on a vector $x$. This can be done in an efficient way: The assignment

$$ x \leftarrow I^\ell_{\ell+1} x $$

can be done in $\mathcal{O}(N_{\ell+1} - N_{\ell})$ operations, since all values of the vector $x$ with indices corresponding to nodes in $N_{\ell}$ remain unchanged under the operation above. We only have to take care of newly created nodes $N_{\ell+1} \setminus N_{\ell}$. Also the assignment involving the transposed matrix
\[ x \leftarrow \left( I_{\ell+1} \right)^T x \]

needs only \( O(N_{\ell+1} - N_\ell) \) operations. In particular, the storage requirements are \( O(N_{\ell+1} - N_\ell) \) as we only have to store information corresponding to newly created nodes. Hence, for all matrices \( I_{\ell+1} \) the memory consumption is \( O(N_L) \). This is well-known in the context of multilevel algorithms, see [39, Section 4]. The considerations above allow us to analyze the complexity of the following algorithm.

**Algorithm 1** Evaluation of \( y = (\tilde{B}_L)^{-1} x \)

| Input: \( x \), matrices \( \{ I_{\ell+1} \}_{\ell=0}^{L-1} \), \( \{ (\tilde{D}_\ell)^{-1} \}_{\ell=0}^{L-1} \) |
|---|
| for \( k = L, \ldots, 1 \) do |
| \( x^k \leftarrow (\tilde{D}^k)^{-1} x \) |
| \( x \leftarrow (I_{k+1}^k)^T x \) |
| end for |
| x \leftarrow (D^0)^{-1} x |
| for \( k = 0, \ldots, L - 1 \) do |
| \( x \leftarrow I_{k+1}^k x \) |
| \( x \leftarrow x + x^{k+1} \) |
| end for |

| Output: \( y = x \) |

Note that we only need the additional storage for the vectors \( x^k \), which is \( O(N_k - N_{k-1}) \) due to the fact that \( \tilde{D}^k \) only has \( O(N_k - N_{k-1}) \) non-zero entries, hence, for all \( x^k \) we need \( O(N_L) \) memory units. The number of arithmetic operations can be obtained by summing all operations in Algorithm 1. This gives us the optimal linear complexity \( O(N_L) \).

**GMLD and HB** With the same arguments as for LMLD, we can analyze the complexity of the other multilevel preconditioners GMLD and HB. For HB, we obtain the same (asymptotic) complexity/storage requirements, whereas for GMLD the application and storage cost of \( (D^\ell)^{-1} \) is \( O(N_\ell) \) leading to an overall cost of \( O(\sum_{\ell=0}^{L} N_\ell) \).

Table 1 concludes the comparison of the different precondition strategies for closed boundaries in 2D. However, the results extend to 3D and open boundaries, where the condition number estimate of HB is even worse [24]. For OP on open boundaries, one has to consider a different operator instead of the single-layer operator to obtain uniformly bounded condition numbers, see [16] for the 2D case. Moreover, it is clear that a comparison of OP and the multilevel preconditioners is not fair in terms of computational complexity, so that in Sect. 3.2 we restrict ourselves to compare OP with the other preconditioners only with respect to the condition number and number of iterations used. For 3D problems that are solved with BEM++ using adaptive cross approximation compression techniques, experiments with operator preconditioning are presented in the extended preprint [26], where the authors summarize their results by [26, Page A:34]:

[…] The table indicates that calculations with mass-matrix or opposite-order preconditioners are significantly slower than without preconditioning. The reason is that we need to define the space of piecewise constant functions on the
Table 1 Comparison of the different preconditioners in terms of computational complexity (assembly, memory, evaluation) and bounds for the condition numbers in the case of 2D and adaptivity as discussed in Sect. 3.1

| Prec. | Assembly | Memory consumption | Evaluation | Cond. number |
|-------|----------|---------------------|------------|--------------|
| LMLD  | $\mathcal{O}(N_L)$ | $\mathcal{O}(N_L)$ | $\mathcal{O}(N_L)$ | $\mathcal{O}(1)$ |
| GMLD  | $\mathcal{O}(N_L)$ | $\mathcal{O}(\sum_{\ell=0}^{L} N_\ell)$ | $\mathcal{O}(\sum_{\ell=0}^{L} N_\ell)$ | $\mathcal{O}(L)$ |
| HB    | $\mathcal{O}(N_L)$ | $\mathcal{O}(N_L)$ | $\mathcal{O}(N_L)$ | $\mathcal{O}(\log h_{\min,L})^2)$ |
| DIAG  | —        | —                   | $\mathcal{O}(N_L)$ | $\mathcal{O}(N_L^{1/(d-1)})$ |
| OP    | $\mathcal{O}(N_L (\log N_L)^{\beta_1})$ | $\mathcal{O}(N_L (\log N_L)^{\beta_2})$ | $\mathcal{O}(N_L (\log N_L)^{\beta_3})$ | $\mathcal{O}(1)$ |

Here, the assembly costs of the multilevel preconditioners LMLD, GMLD, and HB are given by the assembly costs to realize the matrices $I_{l+1}$. The results transfer to 3D (where the bound on the condition number for HB gets worse) as well as to open boundaries. The constants $\beta_i$ for OP depend on the chosen matrix compression technique. Without matrix compression we have $\mathcal{O}(N_L^2)$. The condition number estimate for DIAG holds in 2D up to a logarithmic factor [6]

dual grid, which is realized by a barycentric refinement of the original mesh, leading to about six times as many elements. This is an unavoidable cost if we need to work with stable dual pairings. […]

Note that these results clearly depend on the underlying compression technique, see Table 1. A thorough comparison study between different kinds of preconditioners for adaptive BEM in 2D and 3D is left for future work.

3.2 Adaptive BEM for hypersingular integral equation for 2D Neumann problem on L-shaped domain

We consider the boundary $\Gamma = \partial \Omega$ of the L-shaped domain $\Omega$, sketched in Fig. 3. With the 2D polar coordinates $(r, \varphi)$ of $x \in \mathbb{R}^2 \setminus \{0\}$, the function

![L-shaped domain and initial triangulation](image)

**Fig. 3** L-shaped domain $\Omega$ and initial triangulation $T_0$ with $\# T_0 = 8$
satisfies $\Delta w = 0$ and has a generic singularity at the reentrant corner. With the adjoint double-layer integral operator $K'$, we define

$$f := (1/2 - K') \partial_n w.$$ 

The exact solution $u$ of (1) is, up to some additive constant, the trace $u = w|_{\Gamma}$ of the potential $w$. While uniform mesh-refinement leads to a reduced convergence order $O(N^{-2/3})$, the adaptive BEM of [10] generates a sequence of meshes $T_\ell$ for $\ell = 0, 1, 2, \ldots$, which regains the optimal rate $O(N^{-3/2})$ (not displayed).

In Fig. 4, we compare the condition numbers of the different preconditioned matrices and the Galerkin matrix $A^L$, i.e., the ratio between maximal and minimal eigenvalue. As predicted by Theorem 1 resp. [30], the condition number of LMLD resp. OP stays bounded, whereas the preconditioned systems for GMLD resp. HB, are suboptimal. This is also reflected by the number of CG iterations, see Fig. 4. On the one hand, we observe that OP needs less than half the iterations of LMLD. On the other hand, the overall solution time for the finest mesh with 18,294 elements is more than 3 times higher for OP (14.32 s) than for LMLD (4.57 s).

### 3.3 Artificial mesh-adaptation for hypersingular integral equation for 2D Neumann problem on L-shaped domain

We consider the problem from Sect. 3.2. However, we use a stronger mesh-adaptation towards the reentrant corner: We start with the initial mesh $T_0$ as given in Fig. 3, and obtain $T_\ell$ from $T_{\ell-1}$ by marking and bisecting only the two elements closest to the origin $(0,0)$. Simple calculations show $h_{\min,\ell} = 2^{-\ell} h_0$ and $h_{\max,\ell} = h_0$, where $h_0 \in \mathbb{R}$ denotes the constant mesh-width of the initial mesh. We compare the condition numbers of the unpreconditioned and preconditioned systems in Fig. 5. The...
results from Fig. 5 show that the condition number of LMLD and OP are uniformly bounded, while the condition number of GMLD and HB grow with $O(L)$ and $O(L^2)$, respectively. In particular, this proves that Theorems 1 and 2 are sharp. The condition number of $P_{LB}$ grows even worse. It is proved in [33, Corollary 1] that the condition number of HB is bounded by $|\log(h_{\min,L})|^2 \simeq L^2$.

### 3.4 Adaptive BEM for 2D slit problem

We consider the hypersingular equation (1) on the slit $\Gamma = (-1, 1) \times \{0\}$ with right-hand side $f = 1$ and exact solution $u(x, 0) = 2\sqrt{1-x^2}$. For this example, the correct energy space is $\tilde{H}^{1/2}(\Gamma)$, and the exact solution belongs to $u \in (\tilde{H}^{1/2}(\Gamma) \cap H^{1-\varepsilon}(\Gamma)) \setminus H^1(\Gamma)$ for all $\varepsilon > 0$. In particular, uniform mesh-refinement thus leads to the reduced order of convergence $O(N^{-1/2})$, while the adaptive mesh-refinement of [10] regains the optimal order $O(N^{-3/2})$.

As is shown in Sect. 6 below, Theorems 1 and 2 also hold in this setting. We compare LMLD, GMLD, and HB with respect to condition numbers and number of CG iterations in Fig. 6.

### 3.5 Adaptive BEM for hypersingular integral equation for 3D Neumann problem on L-shaped domain

We consider a similar problem as in Sect. 3.2 on the boundary of the L-shaped domain from Fig. 7. We set

\[ w(x, y, z) = r^{2/3} \cos(2/3\varphi), \]

where $(r, \varphi, z)$ denote the cylindrical coordinates of $(x, y, z) \in \mathbb{R}^3$. Note that the Neumann data $\phi$ has a singularity at the reentrant edge. In Fig. 8, we compare the
condition numbers of the different preconditioned matrices and the Galerkin matrix \( A^L \). As predicted by Theorem 1, the condition number of LMLD stays bounded, whereas the other preconditioned systems are suboptimal. This is also reflected by the number of CG iterations; see Fig. 8.

3.6 Artificial mesh-adaptation for hypersingular integral equation for 3D Neumann problem on L-shaped domain

We consider the problem from Sect. 3.5 together with a strong refinement towards the node \((0, 0, 0)\), i.e., in each step we mark only the elements which share the node
(0, 0, 0). Iterated NVB leads to a mesh with $h_{\text{min},L} \simeq 2^{-L} h_0$ and $h_{\text{max},L} = h_0$, where $h_0 \in \mathbb{R}$ is the constant mesh-width of the initial mesh. The results from Fig. 9 show, that the condition number for GMLD behaves as $O(L)$, whereas the condition number for LMLD is optimal. Moreover, we also see that the simple diagonal scaling leads to suboptimal results. This is also reflected by the number of CG iterations; see Fig. 9.

4 Proof of Theorem 1

4.1 Abstract analysis of additive Schwarz operators

In this subsection, we show that the multilevel diagonal scaling from Sect. 2 is a multilevel additive Schwarz method. Recall the notation from Sect. 2.4 and Sect. 2.5.
For each subspace \( \mathcal{X}_z^\ell = \text{span}\{\eta_z^\ell\} \), we define the orthogonal projection \( P_z^\ell : H^{1/2}(\Gamma) \rightarrow \mathcal{X}_z^\ell \) by
\[
\langle\langle P_z^\ell v, w_z^\ell \rangle\rangle = \langle\langle v, w_z^\ell \rangle\rangle \quad \text{for all } w_z^\ell \in \mathcal{X}_z^\ell
\] (25)
with the explicit representation
\[
P_z^\ell v = \frac{\langle\langle v, \eta_z^\ell \rangle\rangle}{\|\eta_z^\ell\|^2} \eta_z^\ell.
\] (26)

Define \( \tilde{P}_{AS}^L : = \sum_{\ell=0}^L \sum_{z \in \tilde{N}_\ell} P_z^\ell : H^{1/2}(\Gamma) \rightarrow \mathcal{X}_L \). (27)

Then, \( \tilde{P}_{AS}^L : = (\tilde{B}^L)^{-1} A^L \) satisfies \( \langle\langle \tilde{P}_{AS}^L v, w \rangle\rangle = \langle\langle \tilde{P}_{AS}^L x, y \rangle\rangle_{A^L} \) for all \( v = \sum_{j=1}^{N_L} x_j \eta_z^L \), \( w = \sum_{k=0}^{N_L} y_k \eta_z^L \), i.e., the additive Schwarz operator \( \tilde{P}_{AS}^L \) (restricted to \( \mathcal{X}_L \)) generates the preconditioner from Theorem 1.

In our concrete setting, the additive Schwarz operator \( \tilde{P}_{AS}^L \) satisfies the following spectral equivalence estimate (29) which is proved in Sect. 4.5 (lower bound) resp. Sect. 4.6 (upper bound). Linearity, boundedness and symmetry of additive Schwarz operators are well-known, see e.g. [13, Lemma 2].

**Proposition 3** The operator \( \tilde{P}_{AS}^L \) is linear and bounded as well as symmetric
\[
\langle\langle \tilde{P}_{AS}^L v, w \rangle\rangle = \langle\langle v, \tilde{P}_{AS}^L w \rangle\rangle \quad \text{for all } v, w \in H^{1/2}(\Gamma)
\] (28)
and satisfies
\[
c \|v\|^2 \leq \langle\langle \tilde{P}_{AS}^L v, v \rangle\rangle \leq C \|v\|^2 \quad \text{for all } v \in \mathcal{X}_L.
\] (29)
The constants \( c, C > 0 \) depend only on \( \Gamma \) and the initial triangulation \( T_0 \).

The relation between \( \tilde{P}_{AS}^L \) and the symmetric matrix \( \tilde{P}_{AS}^L \) yields the eigenvalue estimates from Theorem 1.

**Proof of Theorem 1** Symmetry of \( (\tilde{B}^L)^{-1} \) follows from the definition (16). The other properties are obtained using the identity
\[
\langle\langle \tilde{P}_{AS}^L v, w \rangle\rangle = \langle\langle \tilde{P}_{AS}^L x, y \rangle\rangle_{A^L} \quad \text{for all } v = \sum_{j=1}^{N_L} x_j \eta_z^L, w = \sum_{j=1}^{N_L} y_j \eta_z^L.
\]

With Proposition 3, we see that \( \tilde{P}_{AS}^L \) is a symmetric and positive definite matrix with
\[
c \|x\|^2_{A^L} \leq \langle\langle \tilde{P}_{AS}^L x, x \rangle\rangle_{A^L} \leq C \|x\|^2_{A^L} \quad \text{for all } x \in \mathbb{R}^{N_L}.
\]
Thus, a bound for the minimal resp. maximal eigenvalue of $\tilde{P}_{AS}$ is given by

$$\lambda_{\min}(\tilde{P}_{AS}) \geq c, \quad \lambda_{\max}(\tilde{P}_{AS}) \leq C.$$ 

In the next step, we prove positive definiteness of $(\tilde{B}^L)^{-1}$. Note that

$$0 < \|v\|^2 \lesssim \langle \tilde{P}_{AS}^L v, v \rangle = \langle (\tilde{B}^L)^{-1} A^L x A^L x \rangle_2.$$

Since $A^{L}$ is regular, we obtain $\langle (\tilde{B}^L)^{-1} y, y \rangle > 0$ for all $y \in \mathbb{R}^{N_L}$. In particular the inverse $\tilde{B}^L$ of $(\tilde{B}^L)^{-1}$ is well-defined, symmetric, and positive definite. The identity

$$\langle \tilde{P}_{AS}^L x, y \rangle_{\tilde{B}^L} = \langle A^L x, y \rangle_2 \quad \text{for all } x, y \in \mathbb{R}^{N_L},$$

and the symmetry of $A^L$ prove that $\tilde{P}_{AS}$ is symmetric with respect to $\langle \cdot, \cdot \rangle_{\tilde{B}^L}$. Finally, we stress that the condition number $\text{cond}_C(A)$ of a matrix $A$ is given by $\text{cond}_C(A) = \lambda_{\max}(A) / \lambda_{\min}(A)$, if $C$ is symmetric and positive definite and if $A$ is symmetric with respect to $\langle \cdot, \cdot \rangle_C$. Therefore,

$$\text{cond}_{\tilde{B}^L}(\tilde{P}_{AS}^L) = \text{cond}_{A^L}(\tilde{P}_{AS}^L) = \frac{\lambda_{\max}(\tilde{P}_{AS}^L)}{\lambda_{\min}(\tilde{P}_{AS}^L)} \leq \frac{C}{c},$$

which also concludes the proof. \qed

The remainder of this section is concerned with the proof of Proposition 3. This requires some preparations and auxiliary results (Sect. 4.2–4.4), before we face the lower bound (Sect. 4.5) and the upper bound (Sect. 4.6) of (29).

### 4.2 Level function and uniform mesh-refinement

For an element $T \in \mathcal{T}_0$, let $T_0 \in \mathcal{T}_0$ be the unique ancestor element with $T \subseteq T_0$. The generation of $T$ is defined as

$$\text{gen}(T) := \frac{\log(|T|/|T_0|)}{\log(1/2)} \in \mathbb{N}_0.$$

Similarly to [36], we assign to each node $z \in \mathcal{N}_\ell$ the level

$$\text{level}_\ell(z) := \left\lceil \max \{\text{gen}(T)/(d-1) : T \in \mathcal{T}_\ell \text{ with } T \subseteq \omega_\ell(z) \} \right\rceil,$$

where $\lceil \cdot \rceil$ denotes the Gaussian ceil function, i.e., $\lceil x \rceil = \min \{n \in \mathbb{N}_0 : x \leq n\}$ for $x \geq 0$. Note that (30) slightly differs from the level function in [36]. However, both definitions of the level function are equivalent.

Besides the sequence of locally refined triangulations $\mathcal{T}_\ell$, we consider a second unrelated sequence $\mathcal{H}_m$ of uniform triangulations: Let $\mathcal{H}_0 := \mathcal{T}_0$ and let $\mathcal{H}_{m+1}$ be
obtained from \( \hat{T}_m \) by uniform refinement, i.e., all elements of \( \hat{T}_m \) are bisected into son elements with half diameter. For \( d = 2 \), this corresponds to one bisection per element, while three bisections are used for \( d = 3 \), cf. Fig. 1. Let \( \hat{\mathcal{N}}_m \) denote the set of all nodes of \( \hat{T}_m \). Define \( \hat{h}_0 := \max_{T \in T_0} h_0(T) \) as well as the constant
\[
\hat{h}_m := 2^{-m} \hat{h}_0 \quad \text{for each } m \geq 1.
\]
(31)
Note that \( \hat{h}_m \) is equivalent to the usual local mesh-size function on \( \hat{T}_m \), i.e., \( \hat{h}_m \simeq \text{diam}(T) \) for all \( T \in \hat{T}_m \) and all \( m \geq 0 \).

**Lemma 4** Let \( z \in \mathcal{N}_\ell \) and \( m := \text{level}_\ell(z) \). Then, it holds \( z \in \hat{\mathcal{N}}_m \) and \( \eta^\ell_z \in \hat{\mathcal{X}}^m \) as well as
\[
C_1 \hat{h}_m \leq h_\ell(z) \leq C_2 \hat{h}_m.
\]
(32)
The constants \( C_1, C_2 > 0 \) depend only on the initial triangulation \( T_0 \).

**Proof** By definition (30), it follows that
\[
\text{gen}(\hat{T}) = m(d - 1) \geq \text{gen}(T) \quad \text{for all } T \in T_\ell \text{ with } T \subseteq \omega_\ell(z) \text{ and } \hat{T} \in \hat{T}_m.
\]
(33)
Let \( z' \in \omega_\ell(z) \cap \mathcal{N}_\ell \). For \( T \subseteq T_\ell \) with \( z \in T_\ell \), let \( T_0 \in T_0 \) be its unique ancestor. Due to (33), there exists some \( \hat{T} \in \hat{T}_m \) with \( \hat{T} \subseteq T \subseteq T_0 \) and \( z' \in \hat{\mathcal{N}}_m \cap \hat{T} \). This proves that all nodes of \( \omega_\ell(z) \cap \mathcal{N}_\ell \) belong to \( \hat{\mathcal{N}}_m \) and hence \( \eta^\ell_z \in \hat{\mathcal{X}}^m \). To see (32), we note that the definition (30) of \( m = \text{level}_\ell(z) \) guarantees the existence of \( T' \in T_\ell \) with \( \text{gen}(T') + 1 > m(d - 1) \geq \text{gen}(T') \) and \( T' \in \omega_\ell(z) \). Therefore, \( \text{diam}(\hat{T}) \simeq \text{diam}(T') \simeq \text{diam}(T) \) for all \( \hat{T} \in \hat{T}_m \) and \( T \subseteq \omega_\ell(z) \).

Let \( \hat{\mathcal{X}}^m := S^1(\hat{T}_m) \) and denote by \( \hat{\Pi}_m \) the \( L^2 \)-orthogonal projection onto \( \hat{\mathcal{X}}^m \).

**Lemma 5** For all \( v \in H^{1/2}(\Gamma) \) holds
\[
\sum_{m=0}^{\infty} \hat{h}_m^{-1} \| v - \hat{\Pi}_m v \|^2_{L^2(\Gamma)} \leq C_{\text{norm}} \| v \|^2_{H^{1/2}(\Gamma)}.
\]
(34)
The constant \( C_{\text{norm}} > 0 \) depends only on \( \Gamma \) and the initial triangulation \( T_0 \).

**Proof** We note that \( \hat{\mathcal{X}}^k \subseteq \hat{\mathcal{X}}^{k+1} \) and \( \lim_{k \to \infty} \| v - \hat{\Pi}_k v \|_{L^2(\Gamma)} = 0 \) for all \( v \in L^2(\Gamma) \). Thus,
\[
\| v - \hat{\Pi}_m v \|^2_{L^2(\Gamma)} = \sum_{k=m+1}^{\infty} \| (\hat{\Pi}_k - \hat{\Pi}_{k-1}) v \|^2_{L^2(\Gamma)}.
\]
(35)
Plugging (35) into the left-hand side of (34) and changing the order of summation, we see
\[
\sum_{m=0}^{\infty} \hat{h}_m^{-1} \| v - \hat{\Pi}_m v \|_{L^2(\Gamma)}^2 = \sum_{m=0}^{\infty} \hat{h}_m^{-1} \sum_{k=m+1}^{\infty} \| (\hat{\Pi}_k - \hat{\Pi}_{k-1}) v \|_{L^2(\Gamma)}^2
\]
\[= \sum_{k=1}^{\infty} \left( \sum_{m=0}^{k-1} \hat{h}_m^{-1} \right) \| (\hat{\Pi}_k - \hat{\Pi}_{k-1}) v \|_{L^2(\Gamma)}^2. \tag{36}\]

With the definition (31) of $\hat{h}_m$ and the geometric series we infer
\[
\sum_{m=0}^{k-1} \hat{h}_m^{-1} = \hat{h}_0^{-1} \sum_{m=0}^{k-1} 2^m < \hat{h}_0^{-1} 2^k = \hat{h}_1^{-1}. \tag{37}\]

[5, Theorem 5] states that for $s \in [0, 1]$ and $v \in H^s(\Gamma)$ it holds
\[
\| v \|_{H^s(\Gamma)}^2 \simeq \| \hat{\Pi}_0 v \|_{H^s(\Gamma)}^2 + \sum_{k=1}^{\infty} \hat{h}_k^{-2s} \| (\hat{\Pi}_k - \hat{\Pi}_{k-1}) v \|_{L^2(\Gamma)}^2. \tag{38}\]

The hidden constants in (38) depend only on $\Gamma$, the initial triangulation $T_0$, and on $s$. Using equations (36)–(37) and norm equivalence (38) for $s = 1/2$, we conclude the proof of (34).

\[\square\]

### 4.3 Scott–Zhang projection

We require a variant of the Scott–Zhang quasi-interpolation operator [31], see also [4] for higher-order polynomials $S^p(T_\ell)$ for $p \geq 1$ and $H^s(\Gamma)$ resp. $\tilde{H}^s(\Gamma)$ with $\Gamma \subseteq \partial \Omega$:

For $z \in \mathcal{N}_\ell$, let $T^\ell_z \in T_\ell$ be an element with $z \in T^\ell_z$. Let $\psi^\ell_z$ denote the $L^2$-dual basis function with
\[
\int_{T^\ell_z} \psi^\ell_z(x) \eta^\ell_{z'}(x) \, ds_x = \delta_{zz'} \quad \text{for all } z' \in \mathcal{N}_\ell. \tag{39}\]

Then, the linear operator $J^\ell : L^2(\Gamma) \to S^1(T_\ell)$ defined by
\[
J^\ell v = \sum_{z \in \mathcal{N}_\ell} \eta^\ell_z \int_{T^\ell_z} \psi^\ell_z(x) v(x) \, ds_x \quad \text{for all } v \in L^2(\Gamma), \tag{40}\]

is an $H^1$-stable projection onto $S^1(T_\ell)$, i.e.,
\[
J^\ell v_\ell = v_\ell \quad \text{and} \quad \| J^\ell v \|_{H^1(\Gamma)} \leq C_4 \| v \|_{H^1(\Gamma)} \quad \text{for all } v_\ell \in S^1(T_\ell) \quad \text{and} \quad v \in H^s(\Gamma), \tag{41}\]
and satisfies for all \( v \in H^1(\Gamma) \)

\[
\| \nabla J_\ell v \|_{L^2(T)} \leq C_3 \| \nabla v \|_{L^2(\omega_{\ell}(T))} \quad \text{and} \\
\| v - J_\ell v \|_{L^2(T)} \leq C_3 h_\ell(T) \| \nabla v \|_{L^2(\omega_{\ell}(T))}.
\]

(42)

The constant \( C_3 > 0 \) depends only on \( \gamma \)-shape regularity of \( T_\ell \), while \( C_4 > 0 \) additionally depends on \( \Gamma \). Moreover, if \( v \) is linear on \( T_\ell^z \) it holds

\[
J_\ell v(z) = v(z).
\]

(43)

Note that the choice of \( T_\ell^z \) is arbitrary, but for \( z \in \mathcal{N}_\ell \setminus \mathcal{N}_\ell \subseteq \mathcal{N}_{\ell-1} \) we require that \( T_\ell^z = T_\ell^z = T_\ell \cap T_\ell-1 \). For \( z \in \mathcal{N}_\ell \setminus \mathcal{N}_\ell \), it thus follows \( \eta_\ell^z = \eta_{\ell-1}^z \) as well as \( T_\ell^z = T_{\ell-1}^z \) and consequently \( \psi_\ell^z = \psi_{\ell-1}^z \). This yields

\[
(J_\ell - J_{\ell-1}) v(z) = 0 \quad \text{for all } z \in \mathcal{N}_\ell \setminus \mathcal{N}_\ell.
\]

(44)

In particular, we have

\[
(J_\ell - J_{\ell-1}) v \in \text{span}\{ \eta_\ell^z : z \in \mathcal{N}_\ell \} = \mathcal{X}_\ell.
\]

(45)

Finally and with the second-order patch from (9), we have the following pointwise estimate.

**Lemma 6** For all \( v \in L^2(\Gamma) \), it holds

\[
| (J_\ell - J_{\ell-1}) v(z) | \leq |J_\ell v(z)| + |J_{\ell-1} v(z)| \\
\leq C_5 h_\ell(z)^{-(d-1)/2} \| v \|_{L^2(\omega_{\ell-1}^2(z))} \quad \text{for all } z \in \mathcal{N}_\ell.
\]

(46)

The constant \( C_5 > 0 \) depends only on \( \gamma \)-shape regularity of \( T_\ell \).

**Proof** According to [31, Lemma 3.1], it holds \( \| \psi_\ell^z \|_{L^\infty(T_\ell^z)} \lesssim |T_\ell^z|^{-1} \). For any node \( z \in \mathcal{N}_\ell \), we have \( T_\ell^z \subseteq \omega_\ell(z) \subseteq \omega_{\ell-1}^2(z) \) and thus

\[
|J_\ell v(z)| \leq \int_{T_\ell^z} |\psi_\ell^z(x) v(x)| \, ds_x \leq \| \psi_\ell^z \|_{L^\infty(T_\ell^z)} |T_\ell^z|^{1/2} \| v \|_{L^2(T_\ell^z)} \\
\lesssim |T_\ell^z|^{-1/2} \| v \|_{L^2(\omega_{\ell-1}^2(z))} \lesssim h_\ell(z)^{-(d-1)/2} \| v \|_{L^2(\omega_{\ell-1}^2(z))}.
\]

(47)

For \( z \in \mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1} \), there exist two nodes \( z_1, z_2 \in \mathcal{N}_{\ell-1} \) such that

\[
J_{\ell-1} v(z) = \eta_{z_1}^{\ell-1}(z) \int_{T_{z_1}^{\ell-1}} \psi_{z_1}^{\ell-1}(x) v(x) \, ds_x + \eta_{z_2}^{\ell-1}(z) \int_{T_{z_2}^{\ell-1}} \psi_{z_2}^{\ell-1}(x) v(x) \, ds_x.
\]
For \( z \in \hat{\mathcal{N}}_\ell \cap \mathcal{N}_{\ell-1} \), this equality is understood with \( z_1 = z \) and \( n_{z_2}^{\ell-1} = 0 \). In either case, we note that \(|T_{z_2}^{\ell-1}| \simeq h_{z_2}^{d-1}(z)\) as well as \( T_{z_1}^{\ell-1} \subseteq \omega_{\ell-1}^1(z_1) \subseteq \omega_{\ell-1}^2(z)\). Analogously to (47), we derive

\[
|J_{\ell-1}v(z)| \lesssim |T_{z_1}^{\ell-1}|^{-1/2}\|v\|_{L^2(T_{z_1}^{\ell-1})} + |T_{z_2}^{\ell-1}|^{-1/2}\|v\|_{L^2(T_{z_2}^{\ell-1})} \lesssim h_\ell(z)^{-(d-1)/2}\|v\|_{L^2(\omega_{\ell-1}^2(z))}.
\]  

(48)

Combining (47)–(48), we prove (46).

\[\square\]

4.4 Further auxiliary results

The proof of Proposition 3 requires some additional definitions and technical results. For a given node \( z \in \hat{\mathcal{N}}_\ell \), it may hold \( z \in \hat{\mathcal{N}}_{\ell+m} \) even with the same level \( \text{level}_\ell(z) = \text{level}_{\ell+m}(z) \). We count how often a node \( z \in \mathcal{N}_L \) with a fixed level \( k \in \mathbb{N}_0 \) shows up in the sets \( \mathcal{N}_L \). For \( z \in \mathcal{N}_L \) and \( k \in \mathbb{N}_0 \), we therefore define

\[\tilde{K}_k(z) := \{\ell \in \{0, 1, \ldots, L\} : z \in \hat{\mathcal{N}}_\ell \text{ and } \text{level}_\ell(z) = k\}.
\]  

(49)

The following lemma from [36, Lemma 3.1] proves that the cardinality of the set \( \tilde{K}_k(z) \) is uniformly bounded.

**Lemma 7** For all \( z \in \mathcal{N}_L \) and \( k \in \mathbb{N}_0 \), it holds \#\(\tilde{K}_k(z) \leq C_6\), and the constant \( C_6 > 0 \) depends only on the initial triangulation \( T_0 \), but is independent of \( L, k, \) and \( z \).

For each \( z \in \mathcal{N}_L \), we further define the quantities

\[
r_\ell(z) := \min \{\text{gen}(T) : T \in T_{\ell-1} \text{ with } T \subseteq \omega_{\ell-1}^2(z)\}
\]

and \( R_\ell(z) := [r_\ell(z)/(d - 1)] \),

where \([x] := \max \{n \in \mathbb{N}_0 : x \geq n\}\) is the Gaussian floor function for \( x \geq 0\). Analogously to the patch \( \omega_k^1(z) \) from (9), we define the patch \( \hat{\omega}_m^k(z) \) corresponding to the uniformly refined triangulation \( \hat{T}_m \). The proof of (i)–(ii) of the following lemma is found in [36, Proof of Lemma 3.3] for \( d = 3 \) and the level function defined in [36]. With our definition (30) of the level function, the proof basically follows the same lines. Details are left to the reader.

**Lemma 8** (i) For all \( z \in \mathcal{N}_L \) holds \( \text{level}_\ell(z) \leq R_\ell(z) + C_7 \), and the constant \( C_7 > 0 \) depends only on the initial triangulation \( T_0 \).

(ii) For all \( z \in \mathcal{N}_L \) and \( T \in T_{\ell-1} \) with \( T \subseteq \omega_{\ell-1}^2(z) \), there exists an element \( \hat{T} \in \hat{T}_{R_\ell(z)} \) such that \( T \subseteq \hat{T} \).

(iii) There exists \( n \in \mathbb{N}_0 \), which depends only on the initial triangulation \( T_0 \), such that for all \( z \in \mathcal{N}_L \) holds \( \omega_\ell(z) \subseteq \omega_{\ell-1}^2(z) \subseteq \hat{\omega}_m^k(\text{level}_\ell(z)) \).
Proof We prove only (iii). Obviously, \( \omega_\ell(z) \subseteq \omega_{\ell-1}(z) \subseteq \omega_{\ell-1}^2(z) \) by definition of these patches. It remains to prove the second inclusion \( \omega_{\ell-1}^2(z) \subseteq \hat{\omega}_{\ell(z)}^n(z) \): Lemma 8 states that for \( T \in T_{\ell-1} \) with \( T \subseteq \omega_{\ell-1}^2(z) \), there exists \( \hat{T} \in \hat{T}_{R_\ell(z)} \) with \( T \subseteq \hat{T} \subseteq \hat{\omega}_{R_\ell(z)}^n(z) \). Each element \( \hat{T} \in \hat{T}_{R_\ell(z)} \) with \( \hat{T} \subseteq \hat{\omega}_{R_\ell(z)}^n(z) \) is bisected into \( 2(d-1)C_7 \) elements \( \hat{T}_j \in \hat{T}_{R_\ell(z)+C_7} \) such that

\[
\hat{T} = \bigcup_{j=1}^{2(d-1)C_7} \hat{T}_j.
\]

In particular, there exists a constant \( n \in \mathbb{N} \) with \( n \leq 4(d-1)C_7 \) such that \( \hat{T} \subseteq \hat{\omega}_{R_\ell(z)+C_7}^n(z) \). Lemma 8 (i) states that level \( \ell(z) \leq R_\ell(z) + C_7 \). Hence, \( \hat{\omega}_{R_\ell(z)+C_7}^n(z) \subseteq \hat{\omega}_{\ell(z)}^n(z) \) by the definition of the patches. \( \square \)

### 4.5 Proof of Proposition 3, lower bound in (29)

Let \( v \in \mathcal{X}^L \) and set \( J_{-1} := 0 \). With the property (45) of the Scott–Zhang projection \( J_\ell \), we define

\[
\tilde{v}^\ell := (J_\ell - J_{\ell-1})v \in \tilde{\mathcal{X}}^\ell \quad \text{for all} \quad 0 \leq \ell \leq L.
\]

The projection property of \( J_L \) and the telescoping series prove

\[
v = J_L v = (J_L - J_{-1})v = \sum_{\ell=0}^{L} \tilde{v}^\ell.
\]

We further decompose \( v \) into

\[
v = \sum_{\ell=0}^{L} \sum_{z \in \hat{\mathcal{N}}_\ell} \tilde{v}^\ell(z) \eta_z^\ell =: \sum_{\ell=0}^{L} \sum_{z \in \hat{\mathcal{N}}_\ell} v_z^\ell \quad \text{with} \quad v_z^\ell \in \mathcal{X}_z^\ell.
\]

Let \( z \in \hat{\mathcal{N}}_\ell \). According to the properties (11) of the hat-functions, standard interpolation techniques yield

\[
\| \eta_z^\ell \|_{H^1/2}^2 \lesssim \| \eta_z^\ell \|_{L^2}^2 \lesssim \| \eta_z^\ell \|_{H^1} \| \eta_z^\ell \|_{H^1} \| \eta_z^\ell \|_{H^1} \lesssim |\omega_\ell(z)| h_\ell(z)^{-1} \lesssim h_\ell(z)^{d-2}.
\]

This implies \( \| v_z^\ell \|_{H^1/2}^2 \lesssim h_\ell(z)^{d-2} |(J_\ell - J_{\ell-1})v(z)|^2 \). We set \( \hat{\Pi}_m := \hat{\Pi}_0 \) for \( m < 0 \). Lemma 8 yields that \( \hat{\Pi}_{1(z) - C_7} v \in \hat{\mathcal{X}}_{R_\ell(z)} \) and that \( (\hat{\Pi}_{1(z) - C_7} v)|T \) is linear for all \( T \in T_{\ell-1} \) with \( T \subseteq \omega_{\ell-1}^2(z) \). The Scott–Zhang projection preserves linearity on all elements \( T \in T_{\ell-1} \) resp. \( T \in T_{R_\ell(z)} \) with \( z \in T \subseteq \omega_{\ell-1}^2(z) \). This and Lemma 6 yield

\[
|(J_\ell - J_{\ell-1})v(z)|^2 = |(J_\ell - J_{\ell-1}) \left( v - \hat{\Pi}_{1(z) - C_7} v \right)(z)|^2
\]
\[ \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell} \| v^\ell_z \|^2 \lesssim \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell} h_\ell(z)^{-(d-1)} \| v - \widehat{\Pi}_{\ell}(z) - C_7 v \|^2_{L^2(\omega_{\ell-1}^2(z))}. \]

Altogether, we obtain \[ \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell} \| v^\ell_z \|^2 \lesssim h_\ell(z)^{-1} \| v - \widehat{\Pi}_{\ell}(z) - C_7 v \|^2_{L^2(\omega_{\ell-1}^2(z))}. \] Using the equivalence \( h_\ell(z) \simeq \widehat{\Pi}_{\ell}(z) \) from Lemma 4, we get

\[ \sum_{m=0}^{\infty} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell, \text{level}(z)=m} \widehat{h}^{-1}_m \| v - \widehat{\Pi}_{m-C_7} v \|^2_{L^2(\omega_m^2(z))} \lesssim \sum_{m=0}^{\infty} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell, \text{level}(z)=m} \widehat{h}^{-1}_m \| v - \widehat{\Pi}_{m-C_7} v \|^2_{L^2(\omega_m^2(z))}. \]

With Lemma 8 (iii) and the definition (49) of \( \widehat{K}_m(z) \), we see

\[ \sum_{m=0}^{\infty} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell, \text{level}(z)=m} \widehat{h}^{-1}_m \| v - \widehat{\Pi}_{m-C_7} v \|^2_{L^2(\omega_m^2(z))} \lesssim \sum_{m=0}^{\infty} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{N}_\ell, \text{level}(z)=m} \widehat{h}^{-1}_m \| v - \widehat{\Pi}_{m-C_7} v \|^2_{L^2(\omega_m^2(z))}. \]

For \( z \in \mathcal{N}_\ell \) with \( \text{level}(z) = m \), Lemma 4 states \( z \in \mathcal{N}_m \). This and \( \# \mathcal{K}_m(z) \leq C_6 \) from Lemma 7 give

\[ \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_L, \ell \in \mathcal{K}_m(z)} \widehat{h}^{-1}_m \| v - \widehat{\Pi}_{m-C_7} v \|^2_{L^2(\omega_m^2(z))} \leq \sum_{m=0}^{\infty} \sum_{z \in \mathcal{N}_L \cap \mathcal{N}_m} \sum_{\ell \in \mathcal{K}_m(z)} \widehat{h}^{-1}_m \| v - \widehat{\Pi}_{m-C_7} v \|^2_{L^2(\omega_m^2(z))}. \]
Uniform $\gamma$-shape regularity of $\tilde{T}_m$ and the definition $\tilde{\Pi}_m = \tilde{\Pi}_0$ for $m < 0$ yield
\[
\sum_{m=0}^{\infty} \sum_{z \in \tilde{N}_m} \tilde{h}_m^{-1} \| v - \tilde{\Pi}_m - C_7 v \|^2_{L^2(\tilde{\omega}_m^0(z))} \lesssim \sum_{m=0}^{\infty} \tilde{h}_m^{-1} \| v - \tilde{\Pi}_m - C_7 v \|^2_{L^2(\Gamma)}
\]
\[
\lesssim \sum_{m=0}^{\infty} \tilde{h}_m^{-1} \| v - \tilde{\Pi}_m v \|^2_{L^2(\Gamma)}
\]
We combine the last four estimates with Lemma 5 and norm equivalence on $H^{1/2}(\Gamma)$ to see
\[
\sum_{\ell=0}^{L} \sum_{z \in \tilde{N}_\ell} ||| \tilde{v}_\ell^z |||_2 \lesssim \sum_{m=0}^{\infty} \tilde{h}_m^{-1} \| v - \tilde{\Pi}_m v \|^2_{L^2(\Gamma)} \lesssim \sum_{m=0}^{\infty} \tilde{h}_m^{-1} \| v - \tilde{\Pi}_m v \|^2_{H^{1/2}(\Gamma)} \lesssim \| v \|^2_{L^2(\Gamma)}
\]
Standard results, also known as Lions’ lemma [19,37], show that the combination of (53) and (55) proves ellipticity of $\tilde{\mathcal{P}}_L^{AS}$
\[
\| v \|^2 \lesssim \langle \langle \tilde{\mathcal{P}}_L^{AS} v, v \rangle \rangle \quad \text{for all } v \in \mathcal{X}^L.
\]
This shows the lower bound in (29).

4.6 Proof of Proposition 3, upper bound of (29)

Let $M := \max_{z \in \tilde{N}_L} \text{level}_L(z)$ denote the maximal level of all nodes $z \in \tilde{N}_L$ and note that Lemma 4 yields $\tilde{N}_L \subseteq \tilde{N}_M$ and hence $\mathcal{X}^L \subseteq \tilde{\mathcal{X}}^M$. We rewrite the additive Schwarz operator $\tilde{\mathcal{P}}_L^{AS}$ as
\[
\tilde{\mathcal{P}}_L^{AS} = \sum_{\ell=0}^{L} \sum_{z \in \tilde{N}_\ell} P_z^\ell = \sum_{m=0}^{M} \tilde{\mathcal{Q}}_m^L \quad \text{with} \quad \tilde{\mathcal{Q}}_m^L := \sum_{\ell=0}^{L} \sum_{z \in \tilde{N}_\ell \text{ level}_L(z) = m} P_z^\ell.
\]
There holds the following strengthened Cauchy-Schwarz inequality.

**Lemma 9** For all $0 \leq m \leq M$, $k \leq m$
\[
0 \leq \langle \langle \tilde{\mathcal{Q}}_m^L \tilde{v}^k, \tilde{v}^k \rangle \rangle \leq C_8 2^{-(m-k)} \| \tilde{v}^k \|_2^2 \quad \text{for all } \tilde{v}^k \in \tilde{\mathcal{X}}^k.
\]
The constant $C_8 > 0$ depends only on $\Gamma$ and the initial triangulation $T_0$.

**Proof** By definition of $\tilde{\mathcal{Q}}_m$, it holds
\[
\langle \langle \tilde{\mathcal{Q}}_m^L \tilde{v}^k, \tilde{v}^k \rangle \rangle = \sum_{\ell=0}^{L} \sum_{z \in \tilde{N}_\ell \text{ level}_L(z) = m} \langle \langle P_z^\ell \tilde{v}^k, \tilde{v}^k \rangle \rangle = \sum_{\ell=0}^{L} \sum_{z \in \tilde{N}_\ell \text{ level}_L(z) = m} \| P_z^\ell \tilde{v}^k \|_2^2 \geq 0.
\]
Let \( z \in \tilde{\mathcal{N}}_k \) with \( \text{level}_\ell(z) = m \). Lemma 4 states \( h_{\ell}(z) \simeq \hat{h}_m \) and \( z \in \tilde{\mathcal{N}}_k \cap \tilde{\mathcal{N}}_m \). From the representation (26) of \( \mathcal{P}_z^k \), we get

\[
\langle \mathcal{P}_z^k \hat{v}^k, \hat{v}^k \rangle = \frac{\langle \langle \hat{v}^k, \eta_z^k \rangle \rangle^2}{\| \eta_z^k \|^2_2} \leq \frac{\langle \langle \mathcal{W} \hat{v}^k, \eta_z^k \rangle \rangle^2 + \langle \langle \hat{v}^k, 1 \rangle \rangle_\Gamma^2 \| \eta_z^k \|^2_2}{\| \eta_z^k \|^2_2}.
\] (59)

According to, e.g., [6, Theorem 4.8], it holds \( h_{\ell}(z)^s \| \eta_z^k \|^2_{H^s(\Gamma)} \simeq \| \eta_z^k \|^2_{L^2(\Gamma)} \), where the hidden constants depends on \( \Gamma \), \( 0 \leq s \leq 1 \), and \( \gamma \)-shape regularity of \( \mathcal{T}_q \). With \( h_{\ell}(z) \| \eta_z^k \|^2 \simeq h_{\ell}(z) \| \eta_z^k \|^2_{H^{1/2}(\Gamma)} \simeq \| \eta_z^k \|^2_{L^2(\Gamma)} \), the Cauchy-Schwarz inequality thus gives

\[
\frac{\langle \langle \mathcal{W} \hat{v}^k, \eta_z^k \rangle \rangle^2}{\| \eta_z^k \|^2_2} \leq h_{\ell}(z) \| \mathcal{W} \hat{v}^k \|^2_{L^2(\omega_k(z))} \leq \frac{\hat{h}_m \| \hat{h}_k^{1/2} \mathcal{W} \hat{v}^k \|^2_{L^2(\omega_k(z))}}{\hat{h}_k} = 2^{-(m-k)} \| \hat{h}_k^{1/2} \mathcal{W} \hat{v}^k \|^2_{L^2(\omega_k(z))}.
\]

For the stabilization term, the same arguments together with \( \langle \eta_z^k, 1 \rangle_\Gamma \leq \| \eta_z^k \|^2_{L^2(\Gamma)} \) \( |\omega_{\ell}(z)|^{1/2} \) yield

\[
\frac{\langle \langle \hat{v}^k, 1 \rangle \rangle_\Gamma^2 \langle \eta_z^k, 1 \rangle_\Gamma^2}{\| \eta_z^k \|^2_2} \leq h_{\ell}(z) |\omega_{\ell}(z)| \| \hat{v}^k \|^2_{H^{1/2}(\Gamma)} \leq 2^{-(m-k)} |\omega_{\ell}(z)| \| \hat{v}^k \|^2_{H^{1/2}(\Gamma)},
\]

where the last estimate follows from \( h_{\ell}(z) \simeq \hat{h}_m \simeq \hat{h}_m / \hat{h}_k = 2^{-(m-k)} \). Combining these three estimates, we obtain

\[
\langle \langle \mathcal{P}_z^k \hat{v}^k, \hat{v}^k \rangle \rangle \leq 2^{-(m-k)} \left( \| \hat{h}_k^{1/2} \mathcal{W} \hat{v}^k \|^2_{L^2(\omega_k(z))} + |\omega_{\ell}(z)| \| \hat{v}^k \|^2_{H^{1/2}(\Gamma)} \right).
\]

By Lemma 8 (iii), we have \( \omega_{\ell}(z) \subseteq \hat{\omega}_m^n(z) \). The representation of \( \tilde{\mathcal{Q}}_m^L \) from (58) thus gives

\[
\langle \langle \tilde{\mathcal{Q}}_m^L \hat{v}^k, \hat{v}^k \rangle \rangle \leq 2^{-(m-k)} \sum_{\ell=0}^{L} \sum_{z \in \tilde{\mathcal{N}}_k, \text{level}_\ell(z) = m} \left( \| \hat{h}_k^{1/2} \mathcal{W} \hat{v}^k \|^2_{L^2(\hat{\omega}_m^n(z))} + |\hat{\omega}_m^n(z)| \| \hat{v}^k \|^2_{H^{1/2}(\Gamma)} \right).
\]

By definition (49) of \( \hat{\mathcal{K}}_m(z) \) and Lemma 7, the double sum can be rewritten and further estimated by

\[
\sum_{\ell=0}^{L} \sum_{z \in \tilde{\mathcal{N}}_k, \text{level}_\ell(z) = m} \left( \| \hat{h}_k^{1/2} \mathcal{W} \hat{v}^k \|^2_{L^2(\hat{\omega}_m^n(z))} + |\hat{\omega}_m^n(z)| \| \hat{v}^k \|^2_{H^{1/2}(\Gamma)} \right)
\]

\[
= \sum_{z \in \tilde{\mathcal{N}}_m \cap \mathcal{N}_L, \ell \in \hat{\mathcal{K}}_m(z)} \left( \| \hat{h}_k^{1/2} \mathcal{W} \hat{v}^k \|^2_{L^2(\hat{\omega}_m^n(z))} + |\hat{\omega}_m^n(z)| \| \hat{v}^k \|^2_{H^{1/2}(\Gamma)} \right).
\]
By \( \gamma \)-shape regularity of \( \widehat{T}_m \), it holds

\[
\sum_{z \in \hat{N}_m} \left( \| h_k^{1/2} \mathcal{W} \partial_k^\mathcal{G} \|^2_{L^2(\tilde{\Omega}_m(z))} + | \hat{\omega}_m^n(z) | \| \hat{v}_k \|^2_{H^{1/2}(\Gamma)} \right) \lesssim \| h_k^{1/2} \mathcal{W} \partial_k^\mathcal{G} \|^2_{L^2(\Gamma)} + \| \hat{v}_k \|^2_{H^{1/2}(\Gamma)}.
\]

Recall stability \( \mathcal{W} : H^1(\Gamma) \rightarrow L^2(\Gamma) \). Together with an inverse estimate between \( H^1(\Gamma) \) and \( H^{1/2}(\Gamma) \) for piecewise polynomials, see e.g. [4, Proposition 5], we obtain

\[
\| h_k^{1/2} \mathcal{W} \partial_k^\mathcal{G} \|^2_{L^2(\Gamma)} = h_k \| \mathcal{W} \partial_k^\mathcal{G} \|^2_{L^2(\Gamma)} \lesssim h_k \| \hat{v}_k \|^2_{H^1(\Gamma)} \lesssim \| \hat{v}_k \|^2_{H^{1/2}(\Gamma)}.
\]

We note that the latter estimate does not only hold for uniform triangulations \( \widehat{T}_k \), but also for shape-regular triangulations and higher-order polynomials [3, Corollary 2]. Combining the last four estimates with norm equivalence \( \| \hat{v}_k \|_{H^{1/2}(\Gamma)} \simeq \| \hat{v}_k \| \), we conclude the proof.

The rest of the proof follows along the lines of the proof of [32, Lemma 2.8] and is given for completeness. We note that Lemma 9 implies, in particular, that \( \langle \langle \hat{\mathcal{Q}}_M^k, v, w \rangle \rangle \) defines a positive semi-definite and symmetric bilinear form on \( \hat{X}^k \) for \( k \leq m \) and hence satisfies a Cauchy-Schwarz inequality.

**Proof of upper bound in (29)** Let \( \hat{G}_m : H^{1/2}(\Gamma) \rightarrow \hat{X}^m \) denote the Galerkin projection onto \( \hat{X}^m \) with respect to the scalar product \( \langle \langle \cdot, \cdot \rangle \rangle \), i.e.,

\[
\langle \langle \hat{G}_m v, \hat{w}^m \rangle \rangle = \langle \langle v, \hat{w}^m \rangle \rangle \quad \text{for all } \hat{w}^m \in \hat{X}^m. \tag{62}
\]

Note that \( \hat{G}_m \) is the orthogonal projection onto \( \hat{X}^m \) with respect to the energy norm \( \| \cdot \| \). We set \( \hat{G}_{-1} = 0 \). For any \( v \in \mathcal{X}^L \subseteq \hat{X}^M \), it holds \( \hat{G}_m v = \sum_{k=0}^m (\hat{G}_k - \hat{G}_{k-1}) v \) as well as \( \hat{G}_M v = v \). Lemma 4 yields \( \hat{Q}_m^L v \in \hat{X}^m \). The symmetry of the orthogonal projection \( \hat{G}_m \) hence shows
\[
\langle \widetilde{Q}_m^L v , v \rangle = \sum_{k=0}^{m} \langle \widetilde{Q}_m^L v , (\widehat{G}_k - \widehat{G}_{k-1})v \rangle \\
\leq \sum_{k=0}^{m} \langle \widetilde{Q}_m^L v , v \rangle^{1/2} \langle \widetilde{Q}_m^L (\widehat{G}_k - \widehat{G}_{k-1})v , (\widehat{G}_k - \widehat{G}_{k-1})v \rangle^{1/2},
\]

where we have used the Cauchy-Schwarz inequality for \(\langle \widetilde{Q}_m^L v , w \rangle\) with \(w = (\widehat{G}_k - \widehat{G}_{k-1})v \in \mathring{X}_k\). For the second scalar product, we apply Lemma 9 and obtain

\[
\langle \widetilde{Q}_m^L (\widehat{G}_k - \widehat{G}_{k-1})v , (\widehat{G}_k - \widehat{G}_{k-1})v \rangle \lesssim 2^{-(m-k)} \| (\widehat{G}_k - \widehat{G}_{k-1})v \|^2 = 2^{-(m-k)} \| (\widehat{G}_k - \widehat{G}_{k-1})v , v \|
\]

With the representation (56) of \(\widetilde{P}_AS^L\) and the Young inequality, we infer

\[
\langle \widetilde{P}_AS^L v , v \rangle = \sum_{m=0}^{M} \langle \widetilde{Q}_m^L v , v \rangle \\
\leq \frac{\delta}{2} \sum_{m=0}^{M} \sum_{k=0}^{m} 2^{-(m-k)/2} \langle \widetilde{Q}_m^L v , v \rangle \\
+ \frac{\delta^{-1}}{2} \sum_{m=0}^{M} \sum_{k=0}^{m} 2^{-(m-k)/2} \langle (\widehat{G}_k - \widehat{G}_{k-1})v , v \rangle,
\]

for all \(\delta > 0\). There holds \(\sum_{k=0}^{m} 2^{-(m-k)/2} \leq \sum_{k=0}^{\infty} 2^{-k/2} =: K < \infty\). Changing the summation indices in the second sum, we see

\[
\langle \widetilde{P}_AS^L v , v \rangle \leq K \frac{\delta}{2} \sum_{m=0}^{M} \langle \widetilde{Q}_m^L v , v \rangle + \frac{\delta^{-1}}{2} \sum_{k=0}^{M} \sum_{m=k}^{M} 2^{-(m-k)/2} \langle (\widehat{G}_k - \widehat{G}_{k-1})v , v \rangle \\
\leq K \frac{\delta}{2} \sum_{m=0}^{M} \langle \widetilde{Q}_m^L v , v \rangle + K \frac{\delta^{-1}}{2} \sum_{k=0}^{M} \langle (\widehat{G}_k - \widehat{G}_{k-1})v , v \rangle \\
= K \frac{\delta}{2} \langle \widetilde{P}_AS^L v , v \rangle + K \frac{\delta^{-1}}{2} \langle v , v \rangle,
\]

where the final equality follows from the telescoping series and \(\widehat{G}_M v = v\). Choosing \(\delta > 0\) sufficiently small and absorbing the first-term on the right-hand side on the left, we conclude the upper bound in (29).

5 Proof of Theorem 2

Clearly, the abstract analytical setting for additive Schwarz operators from Sect. 4.1 also applies for the operator

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\[ P_{\text{AS}}^L = \sum_{\ell=0}^{L} \sum_{z \in N_\ell} P_\ell^z \]  

associated to the preconditioner defined in Sect. 2.6. We stress that the properties of \((B^L)^{-1}\) and \(P_{\text{AS}}\) follow in the same way as for Theorem 1. It thus remains to provide a lower and upper bound for the operator \(P_{\text{AS}}^L\) in analogy to Proposition 3 for \(P_{\text{AS}}^L\).

**Proposition 10** The operator \(P_{\text{AS}}^L\) satisfies
\[ c \|v\|^2 \leq \langle P_{\text{AS}}^L v, v \rangle \leq C (L + 1) \|v\|^2 \quad \text{for all } v \in \mathcal{X}^L. \]  

The constants \(c, C > 0\) depend only on \(\Gamma\) and the initial triangulation \(T_0\).

In principle, the proof follows the same lines as in Sect. 4.5–4.6. We sketch the most important modifications only. Details are left to the reader.

### 5.1 Proof of lower bound in (64)

By virtue of Lions’ lemma (see the proof of lower bound of (29) in Sect. 4.5), we need to construct a decomposition \(v = \sum_{\ell=0}^{L} \sum_{z \in N_\ell} v_\ell^z\) with \(v_\ell^z \in \mathcal{X}^\ell_z\) and \(\sum_{\ell=0}^{L} \sum_{z \in N_\ell} \|v_\ell^z\|^2 \leq c^{-1} \|v\|^2\), for all \(v \in \mathcal{X}^L\). Since \(\tilde{N}_\ell \subseteq N_\ell\), we may rely on the same decomposition as in Sect. 4.5. This concludes the proof with the same constant \(c > 0\) for Proposition 3 and Proposition 10.

### 5.2 Proof of upper bound in (64)

We define the set
\[ K_k(z) := \{ \ell \in \{0, 1, \ldots, L\} : z \in N_\ell \text{ and level}_\ell(z) = k \}. \]  

**Lemma 11** For all \(z \in N_L\) and \(k \in \mathbb{N}_0\) there holds \(#K_k(z) \leq L + 1\).

**Proof** Obviously, there are at most \(L + 1\) indices in the set \(K_k(z)\). \(\square\)

We proceed as in Sect. 4.6 and provide a similar result as in Lemma 9, where, however, Lemma 11 plays an important role in the proof. To that end, we consider
\[ P_{\text{AS}}^L = \sum_{m=0}^{M} Q_m^L \quad \text{with} \quad Q_m^L := \sum_{\ell=0}^{L} \sum_{z \in N_\ell \text{ and level}_\ell(z) = m} P_\ell^z. \]

**Lemma 12** For all \(0 \leq m \leq M, k \leq m\)
\[ 0 \leq \langle Q_m^L \hat{v}^k, \hat{v}^k \rangle \leq C_9 (L + 1) 2^{-(m-k)} \|\hat{v}^k\|^2 \quad \text{for all } \hat{v}^k \in \hat{\mathcal{X}}^k. \]  

The constant \(C_9 > 0\) depends only on \(\Gamma\) and the initial triangulation \(T_0\).
Proof The proof follows the same lines as the proof of Lemma 9. The important modifications consist in replacing $\tilde{N}_\ell$ by $N_\ell$, $\tilde{K}_m(z)$ by $K_m(z)$, and $\tilde{Q}_L^m$ by $Q_L^m$. However, in estimate (60) a bound for the cardinality of the set $\tilde{K}_m(z)$ enters. Clearly, we have to replace this bound by the bound for $#K_m(z)$ from Lemma 11. Therefore, the factor $L + 1$ comes into play. □

The rest of the proof is a simple adaptation of Sect. 4.6. It is therefore left to the reader.

6 Extension to screen problems

6.1 Continuous setting

Let $\Gamma \subseteq \partial \Omega$ denote an open screen. By $\tilde{H}^{1/2}(\Gamma)$, we denote the space of $H^{1/2}(\partial \Omega)$ functions which vanish outside of $\Gamma$. It is known [28] that the hypersingular integral operator $\mathcal{W} : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ from (1) is a linear, bounded, symmetric, and elliptic operator. The definition

$$\langle \langle v, w \rangle \rangle := \langle \mathcal{W}u, w \rangle_{\Gamma} \quad \text{for all } v, w \in \tilde{H}^{1/2}(\Gamma)$$

provides a scalar product on $\tilde{H}^{1/2}(\Gamma)$, and the induced norm $\|v\|_{\Gamma}^2 := \langle \langle v, v \rangle \rangle$ is an equivalent norm on $\tilde{H}^{1/2}(\Gamma)$. Instead of (4), we consider the variational formulation of the hypersingular integral equation (1)

$$\langle \langle u, v \rangle \rangle = \langle f, v \rangle_{\Gamma} \quad \text{for all } v \in \tilde{H}^{1/2}(\Gamma)$$

with right-hand side $f \in H^{-1/2}(\Gamma)$. The lemma of Lax–Milgram proves that this formulation admits a unique solution $u \in \tilde{H}^{1/2}(\Gamma)$.

6.2 Notations

We use the same notations as in Sect. 2.2–2.4. The discrete space $\mathcal{X}_\ell$ from (7) is replaced by the definition $\mathcal{X}_\ell := S_0^1(T_\ell) : = S^1(T_\ell) \cap \tilde{H}^{1/2}(\Gamma)$, i.e., the space of piecewise linear and globally continuous functions, which vanish outside the open boundary part $\Gamma$. Moreover, the set $\tilde{N}_\ell$ now does not consist of all nodes of the triangulation $T_\ell$, but only of the nodes which lie inside $\Gamma$, i.e.,

$$\tilde{N}_\ell := \left\{ \text{z is a node of } T_\ell : \eta_\ell z \in \tilde{H}^{1/2}(\Gamma) \right\}.$$

6.3 Multilevel diagonal preconditioner

We stick with the settings and notations as in Sect. 2.5–2.6.

Theorem 13 Theorem 1 and Theorem 2 hold for screen problems.
Proof First, we extend Theorem 1 to problems on open boundaries. Note that the abstract analysis of additive Schwarz operators from Sect. 4 holds also for $\Gamma \subseteq \partial \Omega$. In particular, we only need to prove the lower and upper bound from Proposition 3.

We stress that Lemma 4–8 hold accordingly if $H^{1/2}$ is replaced by $\tilde{H}^{1/2}$:

- Lemmas 4, 7–8 hold for this problem, since they are only related to the triangulations and the mesh-refinement procedure.
- Lemma 5 remains valid if $H^{1/2}$ is replaced by $\tilde{H}^{1/2}$, since the equivalence (38) also holds for $\Gamma \subseteq \partial \Omega$ and $H^{1/2}$ replaced by $\tilde{H}^{1/2}$, see [5, Theorem 5].
- Lemma 6 involves a variant of the Scott–Zhang operator, which has to be constructed appropriately, see Sect. 4.3 and the references therein. To this end, one may proceed as in [4] with the restriction on the choice of the elements $T_{\ell}^z$ required here, so that Lemma 6 remains valid.

Altogether, the proof of the lower bound follows the same lines as in Sect. 4.5. Clearly, the Sobolev space $H^{1/2}(\Gamma)$ has to be replaced by $\tilde{H}^{1/2}(\Gamma)$. Note that our re-definition (69) of $\mathcal{N}_{\ell}$ for $\Gamma \subseteq \partial \Omega$ ensures that the hat-function $\eta^\ell_z$ vanishes outside of $\Gamma$ for all $z \in \mathcal{N}_{\ell}$. Thus,

$$\|\eta^\ell_z\|_{\tilde{H}^{1/2}(\Gamma)} = \|\eta^\ell_z\|_{H^{1/2}(\partial \Omega)}. \quad (70)$$

Therefore, estimate (54) holds, since

$$\|\eta^\ell_z\|_{\tilde{H}^{1/2}(\Gamma)}^2 \simeq \|\eta^\ell_z\|_{H^{1/2}(\partial \Omega)}^2 \leq \|\eta^\ell_z\|_{L^2(\partial \Omega)} \|\eta^\ell_z\|_{H^1(\partial \Omega)} \lesssim h_\ell(\varepsilon)^{d-2}. \quad (71)$$

The rest of the proof holds verbatim with the notational adaptations mentioned above.

Finally, we stress that for the proof of the upper bound in Proposition 3, one has to verify Lemma 9 only. Due to (70) and [6, Theorem 4.8], we get

$$h_\ell(z)^{1/2}\|\eta^\ell_z\|_{\tilde{H}^{1/2}(\Gamma)} = h_\ell(z)^{1/2}\|\eta^\ell_z\|_{H^{1/2}(\partial \Omega)} \simeq \|\eta^\ell_z\|_{L^2(\partial \Omega)} = \|\eta^\ell_z\|_{L^2(\Gamma)}. \quad (72)$$

The inverse-type estimate (61) for the hypersingular integral operator $W$ still holds true for open boundary parts $\Gamma$ and $H^{1/2}(\Gamma)$ replaced by $\tilde{H}^{1/2}(\Gamma)$. Then, the same proof as for Lemma 9 can be used, if the stabilization terms from equation (59) and the following equations are omitted.

The extension of Theorem 2 to problems on open boundaries can be obtained by the modifications from above and Sect. 5. Details are left to the reader. \(\Box\)

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References

1. Aurada, M., Ebner, M., Feischl, M., Furraz-Leite, S., Führer, T., Goldenits, P., Karkulik, M., Mayr, M., Praetorius, D.: HILBERT—a MATLAB implementation of adaptive 2D-BEM. Numer. Algorit. 67(1), 1–32 (2014)
2. Aurada, M., Feischl, M., Führer, T., Karkulik, M., Praetorius, D.: Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element methods. Comput. Methods Appl. Math. 13(2013), 305–332 (2013)

3. Aurada, M., Feischl, M., Führer, T., Karkulik, M., Praetorius, D.: Local inverse estimates for non-local boundary integral operators. ASC Report, 12/2015, Vienna University of Technology (2015)

4. Aurada, M., Feischl, M., Führer, T., Karkulik, M., Praetorius, D.: Efficiency and optimality of some weighted-residual error estimator for adaptive BEM for hypersingular integral equations. Comput. Methods Appl. Math. 13(2013), 305–332 (2013)

5. Ainsworth, M., McLean, W.: Multilevel diagonal scaling preconditioners for boundary element equations on locally refined meshes. Numer. Math. 93(3), 387–413 (2003)

6. Ainsworth, M., McLean, W., Tran, T.: The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling. SIAM J. Numer. Anal. 36(6), 1901–1932 (1999)

7. Cao, T.: Adaptive-additive multilevel methods for hypersingular integral equation. Appl. Anal. 81(3), 539–564 (2002)

8. Feischl, M., Führer, T., Karkulik, M., Melenk, J.M., Praetorius, D.: ZZ-Type a posteriori error estimators for adaptive boundary element methods on a curve. Eng. Anal. Bound. Elem. 38, 49–60 (2014)

9. Feischl, M., Führer, T., Karkulik, M., Melenk, J.M., Praetorius, D.: Quasi-optimal convergence rates for adaptive boundary element methods with data approximation. Part I: weakly-singular integral equation. Calcolo 51(4), 531–562 (2014)

10. Feischl, M., Führer, T., Karkulik, M., Melenk, J.M., Praetorius, D.: Local inverse estimates for non-local boundary integral operators. ASC Report, 12/2015, Vienna University of Technology (2015)

11. Feischl, M., Führer, T., Karkulik, M., Praetorius, D.: Quasi-optimal convergence rates for adaptive boundary element methods with data approximation. Part II. Electron. Trans. Numer. Anal. 44, 153–176 (2015)

12. Feischl, M., Führer, T., Karkulik, M., Praetorius, D.: ZZ-Type a posteriori error estimators for adaptive boundary element methods on a curve. Eng. Anal. Bound. Elem. 38, 49–60 (2014)

13. Feischl, M., Karkulik, M., Melenk, J.M., Praetorius, D.: Quasi-optimal convergence rate for an adaptive boundary element method. SIAM J. Numer. Anal. 51(2), 1327–1348 (2013)

14. Griebel, M., Oswald, P.: On additive Schwarz preconditioners for sparse grid discretizations. Numer. Math. 66(4), 449–463 (1994)

15. Hackbusch, W.: A sparse matrix arithmetic based on $\mathcal{H}$-matrices. I. Introduction to $\mathcal{H}$-matrices. Computing 62(2), 89–108 (1999)

16. Hiptmair, R., Jerez-Hanckes, C., Urzúa-Torres, C.: Mesh-independent operator preconditioning for boundary elements on open curves. SIAM J. Numer. Anal. 52(5), 2295–2314 (2014)

17. Hsiao, G.C., Wendland, W.L.: Boundary integral equations In: Applied Mathematical Sciences, vol. 164. Springer, Berlin (2008)

18. Karkulik, M., Pavlicek, D., Praetorius, D.: On 2D newest vertex bisection: optimality of mesh-closure and $H^1$-stability of $L_2$-projection. Constr. Approx. 38(2), 213–234 (2013)

19. Lions, P.L.: On the Schwarz alternating method. I. In: First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987). SIAM, Philadelphia, pp. 1–42 (1988)

20. Maischak, M.: A multilevel additive Schwarz method for a hypersingular integral equation on an open curve with graded meshes. Appl. Numer. Math. 59(9), 2195–2202 (2009)

21. McLean, William: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)

22. Mitchell, W.F.: Optimal multilevel iterative methods for adaptive grids. SIAM J. Sci. Stat. Comput. 13(1), 146–167 (1992)

23. McLean, W., Steinbach, O.: Boundary element preconditioners for a hypersingular integral equation on an interval. Adv. Comput. Math. 11(4), 271–286 (1999)

24. Saad, Y.: Iterative Methods for Sparse Linear Systems, 2nd edn. Society for Industrial and Applied Mathematics, Philadelphia (2003)

25. Šmíšek, M., Betcke, T., Arridge, S., Phillips, J., Schweiger, M.: Solving boundary integral problems with BEM++. 2013. Extended and Revised Preprint. http://www.bempp.org/files/bempp-toms-preprint.pdf
27. Šmigaj, W., Betcke, T., Arridge, S., Phillips, J., Schweiger, M.: Solving boundary integral problems with BEM++. ACM Trans. Math. Softw. 41(2):Art. 6, 40 (2015)
28. Stephan, E.P.: Boundary integral equations for screen problems in $\mathbb{R}^3$. Integral Equ. Oper. Theory 10(2), 236–257 (1987)
29. Steinbach, O.: Stability estimates for hybrid coupled domain decomposition methods. In: Lecture Notes in Mathematics, vol. 1809. Springer, Berlin (2003)
30. Steinbach, Olaf, Wendland, Wolfgang L.: The construction of some efficient preconditioners in the boundary element method. Numerical treatment of boundary integral equations. Adv. Comput. Math. 9(1–2), 191–216 (1998)
31. Scott, L.R., Zhang, S.: Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comput. 54(190), 483–493 (1990)
32. Tran, T., Stephan, E.P.: Additive Schwarz methods for the $h$-version boundary element method. Appl. Anal. 60(1–2), 63–84 (1996)
33. Tran, T., Stephan, E.P., Mund, P.: Hierarchical basis preconditioners for first kind integral equations. Appl. Anal. 65(3–4), 353–372 (1997)
34. Tsogtorel, G.: Adaptive boundary element methods with convergence rates. Numer. Math. 124, 471–516 (2013)
35. Tran, T., Stephan, E.P., Zaprianov, S.: Wavelet-based preconditioners for boundary integral equations. Numerical treatment of boundary integral equations. Adv. Comput. Math. 9(1–2), 233–249 (1998)
36. Haijun, W., Chen, Z.: Uniform convergence of multigrid V-cycle on adaptively refined finite element meshes for second order elliptic problems. Sci. China Ser. A 49(10), 1405–1429 (2006)
37. Widlund, O.B.: Optimal iterative refinement methods. In: Domain Decomposition Methods. pp. 114–125 (SIAM, Philadelphia, 1989)
38. Xuejun, X., Chen, H., Hoppe, R.H.W.: Optimality of local multilevel methods on adaptively refined meshes for elliptic boundary value problems. J. Numer. Math. 18(1), 59–90 (2010)
39. Yserentant, H.: On the multilevel splitting of finite element spaces. Numer. Math. 49(4), 379–412 (1986)