Virtual Reality Simulations of Curved Spaces

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Abstract

Previous virtual-reality simulations of curved space, which typically present honeycombs or other periodic structures, have proven effective in letting mathematicians experience curved space directly. By contrast, for students and other non-mathematicians, a game like Non-Euclidean Billiards is more effective because it gives students not just something to see, but also something to do in the curved space. However, such simulations encounter a geometrical problem: they must track the player’s hands as well as her head, and in curved space the effects of holonomy would quickly lead to violations of body coherence. This is, what the player sees with her eyes would disagree with what she feels with her hands. The present article presents a solution to the body coherence problem, as well as several other questions that arise in interactive VR simulations in curved space.
1 Introduction

Seeing curved spaces on a computer monitor is informative and fun, but being in a curved space is a far richer experience and far more informative. Indeed, for me personally, even having studied curved spaces for 45 years, when I first put on the virtual reality (VR) headset and “visited” the 3-sphere and hyperbolic 3-space, I found surprises. And after playing a few games of billiards in those spaces (Figure 1), I got an intuitive feel for them unlike any I had ever had before. The reason for that deeper gut-level understanding is that VR connects not only with our conscious minds, but it also completely hijacks our subconscious understanding of our environment: it feels real!

Figure 1: In hyperbolic 3-space, the billiards table is a regular pentagon with all 90\(^\circ\) angles (right). The table agrees locally with a square Euclidean table in the lab (left), which adds a tactile component to the simulation for greater realism.
1.1 Why billiards?

Non-Euclidean Billiards offers two advantages over earlier curved-space VR simulations:

- *Familiarity.* Until now, curved-space VR simulations have presented abstract mathematical structures such as honeycombs. Such structures are fine for mathematicians, who use the structure’s periodicity to help understand the nature of the space. But when non-mathematicians see an unfamiliar structure in an unfamiliar space, the novelty of the structure often hides the novelty of the space. The present project instead presents familiar objects—a billiards table, billiard balls, a cue stick, and a lamp—so that the familiarity of the objects may illuminate, not hide, the novel properties of the space.

- *Interactivity.* Previous curved-space VR simulations have been largely passive, which is fine for mathematicians who are already motivated to have a look around and think about what they’re seeing. Non-mathematicians, however, come to understand the properties of curved space more easily and more deeply when they’re given not only something to see, but also something to do. In the Non-Euclidean Billiards app, the player is invited to put on the VR headset and play billiards for as long as she wants. The unhurried nature of a billiards game gives the player plenty of time to reflect on what she sees. Most importantly, the need to line up a good shot forces the player to continually walk around the table and look at the ball positions and the table itself from many different viewpoints.

The author and one of his colleagues are currently making plans for an enrichment unit for 7th–12th grade students that will build on the familiarity and interactivity of Non-Euclidean Billiards to teach some fundamental concepts of curved space—namely angle sums, geodesic convergence/divergence, curvature (defined as angle deficit per unit area), and holonomy—in a more structured way.
1.2 Structure of this article

I have chosen to present the mathematics of Non-Euclidean Billiards in a pair of companion articles. The two articles share a common theme, but are written for very different audiences. The first article [Weeks 2020] appears in the proceedings of the conference

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Mathematics, Art, Music, Architecture, Education, Culture

Given the unusual breadth of that audience, I wrote that first article to present the basic concepts of curved space, along with its surprising optical effects, in the most visual and least technical way possible. All implementation details were omitted.

By contrast, the present article is an account of my lecture at the September 2019 workshop

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whose participants were my fellow illustrators of geometry. They, like the readers of Experimental Mathematics, were already well acquainted with the geometry of curved space, so I took the opportunity to dive “under the hood” and explain recent progress in making curved-space VR simulations more interactive, pedagogically effective, and internally efficient. The present article provides a written account of this progress, for the benefit of other mathematicians who write curved-space VR simulations, both now and in the future. The most substantive—and most surprising—discovery is the need for a visitor to curved space to use her muscles to provide internal resistance to the effects of holonomy, in order to maintain body coherence.

Sections 2 and 3 explain body coherence in detail, and propose a strategy for dealing with it in a VR simulation. Section 4 reviews the sequence of mappings used in curved-space graphics, and offers some small conceptual improvements that make working with them easier. Section 5 explains how stereoscopic vision would lead a Euclidean-born tourist to grossly misjudge distances in curved space. That same section then goes on to recommend that curved-space VR apps be written to simulate distances as each space’s native-born inhabitants perceive them, and shows how to modify the sequence of mappings from section 4 to achieve that goal. Finally, section 6 reviews the state of the art in simulating spaces that are homogeneous but anisotropic.

This project as a whole takes its inspiration from, and builds upon, the pioneering work of [Hart et al. 2017a].
2 Headset tracking and body coherence: the problem

When simulating a curved space in VR, a fundamental question is how to map a pose of the user’s headset in the physical lab to a pose of the user’s virtual self in the curved space. The simplest algorithm would be to map local motions of the headset (as measured in its own local coordinate system) to the same local motions of the user’s virtual head (as measured in its own local coordinate system). In other words, if the physical headset moves 1 cm to the left in the lab, the user’s virtual head moves 1 cm to the left in the curved virtual space; if physical headset rotates 2°, the user’s virtual head rotates 2°; and so on.

That naive algorithm would work fine if we were tracking only the user’s head. But in practice we must track the user’s two hands as well, so that, for example, a player in the Non-Euclidean Billiards game can take a shot. Unfortunately holonomy can force the player’s head and hands out of alignment when using the naive tracking algorithm. To see how, consider what happens if the player parallel-translates her head a small distance $d$ forward, then the same distance to her left, the same distance backwards, and finally the same distance to right. In the physical Euclidean lab, her head returns to its original position and orientation (Figure 2). But in the 3-sphere, those forward, leftward, backward, and rightward motions are realized not by Euclidean translations, but by rotations of the 3-sphere. If we place the player at the north pole $(0, 0, 0, 1)$ and let $\theta = d/R$, where $R$ is the radius of the

![Figure 2: The player moves her head around a small square in the physical Euclidean lab, while leaving her body and hands still.](image)
3-sphere, the final placement of the player’s head is given by the matrix product

\[
\begin{pmatrix}
\cos \theta & 0 & 0 & -\sin \theta \\
0 & 1 & 0 & 0 \\
\sin \theta & 0 & 0 & \cos \theta \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & 0 & \cos \theta \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & -\sin \theta \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

This product assumes the left-to-right convention in which matrices act as

\((row \ vector)(first \ factor)(second \ factor)\ldots\)

but may also be sensibly interpreted using the right-to-left convention

\((second \ factor)(first \ factor)(column \ vector)\)

Either way, the matrices for the forward, leftward, backward, and rightward motions of the player’s head get applied in the opposite order from which the player makes those motions, which is somewhat counterintuitive but nevertheless correct. The distance \(d\) is typically small relative to the 3-sphere’s radius \(R\), in which case the matrix product multiplies out to approximately

\[
\begin{pmatrix}
1 & \theta^2 & 0 & -\frac{\theta^3}{2} \\
-\theta^2 & 1 & 0 & -\frac{\theta^3}{2} \\
0 & 0 & 1 & 0 \\
\frac{\theta^3}{2} & \frac{\theta^3}{2} & 0 & 1 \\
\end{pmatrix}
\]

The entries in the bottom row tell us that the player’s virtual head ends up offset from its original position by about \(\frac{1}{2}\theta^3\) radians in both the \(x\) and \(y\) directions, while the upper-left \(3 \times 3\) block tells us that the player’s head ends up rotated by an angle of approximately \(\theta^2\) radians. While the absolute offset is only a third-order effect, the offset as a fraction of the square’s width \(\theta\), namely \(\frac{\frac{1}{2}\theta^3}{\theta} = \frac{1}{2}\theta^2\) is a second-order effect, just like the rotation.

The preceding computation proves that when the player moves her head around a small square in the physical Euclidean lab, as shown in Figure 2, the naive head-tracking algorithm would move her virtual head around a path in the 3-sphere like the one shown in Figure 3(a). In the hyperbolic case, an analogous matrix product

\[
\begin{pmatrix}
\cosh \theta & 0 & 0 & \sinh \theta \\
0 & 1 & 0 & 0 \\
\sinh \theta & 0 & 0 & \cosh \theta \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cosh \theta & 0 & -\sinh \theta \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\sinh \theta & 0 & 0 & \cosh \theta \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cosh \theta & 0 & \sinh \theta \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

shows that the naive head-tracking algorithm would move the player’s virtual head around a path in hyperbolic 3-space like the one shown in Figure 3(b).
In both cases, when the player’s virtual head returns to the starting point in the physical Euclidean lab, her virtual head ends up slightly offset and slightly rotated relative to its original pose in the virtual curved space. In and of itself that’s not a problem, but if she keeps her hands still while moving her head, then she does have a problem: she sees her own hands sitting slightly offset and slightly rotated (Figure 3), even though she still feels her hands sitting straight in front of her (Figure 2).

This discrepancy between what the player sees and what she feels, we call body incoherence. It’s a challenging problem that all interactive curved-space VR simulations, present and future, will face. Section 3 presents a solution that works well in the case of Non-Euclidean Billiards, along with some more general suggestions for other apps.

3 Headset tracking and body coherence: a solution

If you could truly visit hyperbolic 3-space or a 3-sphere, would you really experience body incoherence? No, of course not. What you’d see and what you’d feel would remain perfectly coherent. But as you parallel-translated your head around in a small circle, you’d feel a slight torque in your neck. You’d need to use your neck muscles to counter that torque, to keep your
head pointed straight relative to your body. This mysterious uninvited torque would feel roughly analogous to what you feel when you rotate a spinning gyroscope about one axis and must apply a mysterious uninvited torque to prevent the gyroscope from rotating about a perpendicular axis.

Curved-space simulations, such as the Non-Euclidean Billiards game, must pretend that the user’s neck is providing whatever torque is necessary to keep her head aligned with her body. Hence we must abandon the naive tracking algorithm and devise a new algorithm that’s guaranteed to place the user’s head and hands coherently in the curved virtual space.

3.1 Tracking algorithm for Non-Euclidean Billiards

The Non-Euclidean Billiards game makes use of a square physical table (Figure 1(left)), which locally agrees with the virtual pentagonal (in $\mathbb{H}^3$) or triangular (in $\mathbb{S}^3$) billiards table in the game (Figure 1(right)), and adds a tactile component to the simulation. But the presence of this square physical table means that we must also ensure coherence between where the player sees the virtual table with her eyes and where she feels the physical table with her hands. The simplest algorithm is to track each object—the player’s head and each of her two hands—relative to the nearest edge of the physical table. Although the algorithm applies equally well to each of the player’s hands, the following paragraphs will describe it only for her head, for brevity.

The algorithm expresses the player’s pose in the Euclidean lab relative to a coordinate system attached to a nearby point on the edge of the physical table, then transfers that pose to the simulated space by mapping into the tangent space at the corresponding point on the corresponding edge of the simulated table, and thence into the curved space itself via the exponential map. This ensures that the simulated table edge that the player sees with her eyes always agrees with the physical table edge that she feels with her hands.

At what point on the table edge should we base the aforementioned tangent space? One might be tempted to use the closest point on the closest edge, but this would introduce a small discontinuity: if the player stands near a corner and leans forward with her head over the table itself, then as she moves her head side to side, the “nearest edge point” may jump suddenly from one edge of the table to the next, without passing through the corner point. This jump in the basepoint for the tangent space would cause a small—but potentially perceptible—jump in the headset’s position in the simulated space. To avoid the discontinuous jump, one may instead base the tangent space not at the nearest point on the nearest table edge, but
rather at the table edge point whose azimuth agrees with the azimuth of the player’s head, as seen from the center of the table.

The preceding paragraphs define the mapping from the Euclidean physical space to the curved virtual space almost uniquely. The only free parameter is deciding which edge of the physical table corresponds to which edge of the virtual table. The correspondence between the edges of the square physical table and the edges of the virtual triangular, square or pentagonal table is not fixed, but is something that must be traced around in real time, as the player walks around the table. For example, if a player keeps walking around the physical table—always in the same direction, never turning back—she’ll see the following edges in the following order:

- square physical table: 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 ... 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 ...
- pentagonal virtual table: 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 ...

This matching of an edge of the physical table to an edge of the virtual table is the only thing that depends on the player’s history. Once that matching is known, the headset’s pose in the Euclidean physical space uniquely determines its pose in the curved virtual space, with no additional history dependence.

The same algorithm may be used to place each of the player’s hands in the curved virtual space. This guarantees the coherence of the player’s body, as well as its consistent placement relative to the table.

### 3.2 General principles

Looking to the future, all designers of interactive curved-space VR simulations will face the body coherence problem. While a particular app’s ideal solution may depend heavily on its context, some general principles apply:

- **Ensure visual stability.** In a scene with no fixed reference points, it’s better to let the player’s virtual head move freely and tweak the player’s virtual hands to stay consistent with it, rather than to let the hands move freely and tweak the head to stay consistent. The reason for this is that artificially tweaking the player’s head would slightly shift her view of the whole scene. Hand motion alone must never affect how the player sees the scene, not even by the slightest amount. But if head motion causes a slight change in the position of the player’s hands, that would be undesirable but ultimately acceptable. This asymmetry is due to the primacy of the human visual system: a person’s sense of “where she is in the world” is tied most strongly to what she sees;
her hands and feet are then perceived as being 50–150 cm below that primary viewpoint.

- *Keep the player away from the walls.* If the player in the Non-Euclidean Billiards game weren’t anchored to the table, then if she parallel-translated herself around a small circle several times, she’d see the virtual table orbit around her, eventually passing behind her back. At that point, if she turned around to face the table and take a shot, she’d risk walking into a wall of the physical lab (or, more likely, she’d see the “chaperone” bounds that the VR system inserts into the scene to prevent players from walking into walls). To avoid such a scenario, all curved-space VR apps should, if possible, keep their primary virtual content centered in middle of the physical lab, and keep the player’s position coherent with that primary content.

The most challenging curved-space VR apps to design may be those whose game content is inherently large-scale, such as a potential VR version of HyperRogue [Kopczyński et al. 2019]. Even games whose content may be only a few meters across, such as the author’s 3D mazes in any of several closed 3-manifolds, nevertheless appear infinite to the player, because the player sees such a maze as an infinite periodic structure in the “universal cover”, beckoning her to wander beyond the bounds of the fundamental domain (http://www.geometrygames.org/TorusGames). How might we let the player travel longer distances than her real-world room would permit? One solution would be to give her the ability to press a button and fly through the scene, but that approach risks motion sickness, due to the player’s eyes seeing accelerations that her inner ear doesn’t feel. A safer solution—one used in some flat-space VR apps—would be teleportation: after selecting a destination and pressing a button, the player “fades out” from her current location, briefly sees total darkness, and then “fades back in” at her selected destination.

4 Computer graphics in curved space

Most 3D graphics, whether in flat space or curved space, whether traditional or VR (but excluding ray tracing and ray marching), relies on the following
Each object in the simulated world begins in its own local model space. The model matrix places the object into a shared world space, along with all the other objects. Conceptually one then places a camera into that same world space, to take a picture of the various objects; in practice, though, one takes the whole world and places it in front of the camera, in the camera’s own camera space. Of course the camera—like a real physical camera—sees not the whole space, but only the portion of it lying within a pyramid extending from the camera’s lens outward. That pyramid gets further limited by a near clipping plane and (optionally) a far clipping plane, leaving only objects within the resulting view frustum visible. The projection transformation maps the view frustum to a rectangular box in Euclidean 3-space.

An elementary exposition of computer graphics in $S^3$, $E^3$ and $H^3$ appears in [Weeks 2002]. The following subsections summarize the essentials of that article, but with a new approach to radians in the Euclidean case, a comment on units in VR, a new way to visualize the projection transformation, and an algorithm for drawing the “back hemisphere” in the spherical case.

4.1 Radians in $S^3$, $E^3$ and $H^3$

In the spherical case, the model, world, and camera spaces are all 3-spheres of some desired radius $\rho$ (Figure 4(left)). The 3-sphere is defined as a subset of Euclidean 4-space $E^4$, where its radius is measured using the Euclidean metric

$$|\left(x, y, z, w\right)|^2 = x^2 + y^2 + z^2 + w^2 = \rho^2.$$  

The model and view matrices are elements of the orthogonal group $O(4)$.

In the hyperbolic case, the model, world, and camera spaces are all hyperbolic 3-spaces of some desired radius $\rho$ (Figure 4(right)). Hyperbolic 3-space is defined as a subset of Minkowski space $E^{3,1}$, where its radius is measured using the Minkowski metric

$$|\left(x, y, z, w\right)|^2 = x^2 + y^2 + z^2 - w^2 = -\rho^2.$$  

The model and view matrices are elements of the Lorentz group $O(3,1)$.
Surprisingly, it’s the Euclidean case that requires the greatest care. A standard computer graphics trick is to take the coordinates \((x, y, z)\) and append a “1” as a fourth coordinate, so we can package up a rotation \(R\) and a translation \((\Delta x \ \Delta y \ \Delta z)\) as a single \(4 \times 4\) matrix

\[
\begin{pmatrix}
R_{00} & R_{01} & R_{02} & 0 \\
R_{10} & R_{11} & R_{12} & 0 \\
R_{20} & R_{21} & R_{22} & 0 \\
\Delta x & \Delta y & \Delta z & 1
\end{pmatrix}
\]

When you do the matrix multiplication, the “1” multiplies the \(\Delta x\), \(\Delta y\), and \(\Delta z\) and they get added in. The awkward question here is: What are the units on \(\Delta x\), \(\Delta y\), and \(\Delta z\)? If we’re measuring \(x\), \(y\), and \(z\) in meters, are \(\Delta x\), \(\Delta y\), and \(\Delta z\) also in meters? Yet the rotational components \(R_{ij}\) are dimensionless. Few mathematicians are comfortable having a transformation matrix with some dimensionless entries and other entries in meters. The situation becomes much clearer if instead of representing Euclidean 3-space as a hyperplane at height \(w = 1\), we represent it as a hyperplane at height \(w = \rho\), and call \(\rho\) the “radius” of this Euclidean space (Figure 4(center)). So now we can rewrite our matrix product as

\[
\begin{pmatrix}
R_{00} & R_{01} & R_{02} & 0 \\
R_{10} & R_{11} & R_{12} & 0 \\
R_{20} & R_{21} & R_{22} & 0 \\
\Delta x/\rho & \Delta y/\rho & \Delta z/\rho & 1
\end{pmatrix}
\]

and it’s abundantly clear that \(\rho\) is in meters, just like \(x\), \(y\), and \(z\), and \(\Delta x/\rho\), \(\Delta y/\rho\), and \(\Delta z/\rho\) are dimensionless, just like the \(R_{ij}\). The best way to think of this is that \(\Delta x/\rho\), \(\Delta y/\rho\), and \(\Delta z/\rho\) are translation distances in \textit{Euclidean radians}. 

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The big practical advantage to using Euclidean radians is that, by having a purely dimensionless transformation matrix acting on a position vector \((x, y, z, \rho)\) in pure meters, we make the Euclidean case consistent with the spherical and hyperbolic cases. So a game like Non-Euclidean Billiards can often use exactly the same computer code for all three geometries, with no need to split into separate cases.

4.2 VR needs explicit units

When writing traditional non-VR animations, most mathematicians (including the author) treated distances as dimensionless quantities, in effect measuring all distances in spherical, hyperbolic, or Euclidean radians. And that was fine: in a non-VR maze in a 3-torus or a non-VR flight through a Poincaré dodecahedral space, explicit units aren’t needed.

In VR, by contrast, units are essential. The player’s two eyes and two hands are immersed in the simulated world, so the distances the player sees with her eyes must agree with the distances she feels with her hands. The player perceives herself as part of the scene, not as an onlooker peering into a simulated world from the outside.

For an acceptable VR simulation, we must know (1) the space’s radius of curvature in meters and (2) the size of each object in meters. To see why, consider what happens when a player in a hyperbolic billiards game changes the radius of curvature of the space. If she increases the radius of curvature, the space as a whole will scale up, and any geometrical structures whose size is tied to the radius of curvature will also scale up. For example, the billiards table, which we’ve defined to be a right-angled regular pentagon, will get larger. But the player’s body, whose size in meters is fixed, will not get larger. Nor will the billiard balls, whose diameter is fixed at 57 mm, nor the cue stick, whose length is fixed at 1 m. The final result is that when the player increases the space’s radius of curvature, she’ll find herself playing billiards on a larger right-angled pentagon, but with the same familiar billiard balls and cue stick.

4.3 Visualizing the projection transformation

As part of the standard graphics pipeline, a vertex shader—sometimes more correctly called a vertex function—computes and reports each vertex’s position in homogeneous coordinates \((x, y, z, w)\). The Graphics Processing Unit (GPU) then divides through by the \(w\)-coordinate

\[
(x, y, z, w) \mapsto \left( \frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1 \right)
\]
to map the desired *view frustum* onto a rectangular box in the $w = 1$ hyperplane.

Even though the GPU really does divide through by $w$, and that’s how computer graphicists think about it, I personally find it easier to think in terms of dividing through by $z$ instead. Dividing through by $z$ makes the geometrical meaning clearer. Moreover, essentially all vertex shaders apply a $90^\circ$ rotation in the $zw$-plane, which sends our conceptually clear division-by-$z$ to the GPU’s hardwired division-by-$w$. Combining this with our convention of letting $\rho$ be the radius of the space even in the Euclidean case (Section 4.1), the projection formula for all three geometries becomes

$$(x, y, z, \rho) \mapsto \left( \frac{x}{z}, \frac{y}{z}, 1, \frac{\rho}{z} \right)$$

In the Euclidean case, the division-by-$z$ takes the view frustum to a rectangular box, as desired (Figure 5). It’s perfectly acceptable to push the frustum’s far wall off to infinity, thus mapping an infinite portion of a pyramid onto a finite rectangular box. By contrast, if the frustum’s near wall gets too close to the observer, it will push the rectangular box’s top face upwards towards infinity, which quickly degrades the numerical precision of the results as the floating-point arithmetic’s finite precision gets spread out over an ever taller box.

![Figure 5: In Euclidean graphics, dividing by $z$ maps the view frustum to a rectangular box ($y$ coordinate suppressed).](image-url)
In the spherical case, that same division-by-$z$ takes not a frustum, but rather a solid I like to call the view banana (Figure 6), to a rectangular box. Like a real banana, the view banana has flat sides and tapers down at both ends.

In the hyperbolic case (Figure 7), we again map a frustum-like solid onto a rectangular box, and again it’s perfectly acceptable to push the frustum’s far wall off to infinity.

Figure 6: In spherical graphics, dividing by $z$ maps the view banana to a rectangular box ($y$ coordinate suppressed).

Figure 7: In hyperbolic graphics, dividing by $z$ maps the view frustum to a rectangular box ($y$ coordinate suppressed).
4.4 The 3-sphere

Rendering images in the 3-sphere is a two-part process. To see why, first recall that in Euclidean graphics, if we let the near clipping distance go to zero, the short edge of the red trapezoid in Figure 5 would approach the observer (the black dot), which would send the top edge of its projected image (the top edge of the red rectangle in the vertical plane) upwards to infinity. To avoid that problem, in Euclidean graphics we always set the near clipping distance to some \( \varepsilon > 0 \).

In spherical graphics, we face that same problem at both the top and the bottom of the red view banana on the sphere in Figure 6. To avoid trouble, we must set the near clipping distance to some \( \varepsilon > 0 \) to keep the top edge of its projected image (the top edge of the red rectangle in the vertical plane) from going upwards to infinity, and we must also set the far clipping distance to \( \pi - \varepsilon \) to keep the bottom edge of the projected image from going downwards to infinity.

One immediate consequence of this approach is that an \( \varepsilon \)-neighborhood of the observer’s antipodal point is always excluded from the rendered image. This is not surprising, given that a tiny fleck of dust sitting at the antipodal point would fill the observer’s entire sky! While ray-tracing methods would have no problem rendering the antipodal point, our mesh-based methods always omit the antipodal point and some tiny neighborhood surrounding it.

Our mesh-based methods can, however, easily see through the antipodal point and render content that sits on the “back side” of the 3-sphere. That is, we may render content whose line-of-sight distance from the observer falls in the range \( [\pi + \varepsilon, 2\pi - \varepsilon] \). Because the observer’s lines-of-sight all reconverge at the antipodal point, what the observer sees at distances \( [\pi + \varepsilon, 2\pi - \varepsilon] \) is precisely the same as what her antipodal twin would see at distances \( [\varepsilon, \pi - \varepsilon] \). In other words, to render the back hemisphere, we may simply apply our usual methods to render what the antipodal twin sees at distances \( [\varepsilon, \pi - \varepsilon] \). Technical detail: the most efficient way to implement this on a Tile-Based Deferred Render is to encode commands taking the 3-sphere’s back hemisphere into the back half of the clipping box \( (\frac{1}{2} \leq z \leq 1) \) and the front hemisphere into the front half of the clipping box \( (0 \leq z \leq \frac{1}{2}) \), and then let the renderer process of the whole scene at once. That way back-hemisphere objects that are eclipsed by front-hemisphere objects will never get rendered at all.

If the observer’s head were perfectly transparent, would we also have to render objects that she sees at distances in the range \( [2\pi + \varepsilon, 3\pi - \varepsilon] \)? No, this
isn’t necessary, because these would be the same objects that the observer sees at distances $[\varepsilon, \pi - \varepsilon]$, and they would fill exactly the same portions of observer’s sky.

5 Native-inhabitant view vs. tourist view

To focus on a distant object in hyperbolic 3-space, an observer must look slightly cross-eyed, to compensate for the space’s inherent geodesic divergence (Figure 8). For native inhabitants of hyperbolic space, this is fine, because when they were babies they learned that a certain positive vergence angle means that the object they’re looking at is far away. But a Euclidean-born tourist who visits hyperbolic space would misinterpret that positive vergence angle as meaning that the object is close. In fact the tourist would see all of hyperbolic space crammed into a small ball, of radius only a meter or two in the case of the hyperbolic Billiards game.

Figure 8: In hyperbolic space, an observer must look slightly cross-eyed to focus on a distant object.
In a 3-sphere, by contrast, an observer must look less cross-eyed than in Euclidean space, to compensate for the space’s inherent geodesic convergence (Figure 9). When an object is exactly 90° away, like blue (#2) billiard ball in Figure 9, the observer doesn’t need to look crosseyed at all. For those of us who grew up in Euclidean space, our binocular vision makes that blue (#2) ball seem infinitely far away. If a ball is more than 90° away, like the red (#3) ball in Figure 9, the observer must look slightly “walleyed”—with a negative vergence angle—to focus on it.

When designing a virtual reality simulation like the non-Euclidean billiards game, the developer must decide whether to show the space the way the native sees it, to show it the way the Euclidean-born tourist sees it, or to offer the user the choice. For me, the tourist’s view is just too weird: as you walk around hyperbolic 3-space, the entire contents of the universe seem to move along with you, all trapped inside that finite ball. Spherical space is even worse, because you have to look walleyed to focus on distant objects. The human visual system can do this to some extent, but I personally find it uncomfortable and vaguely distressing. To avoid such discomfort, Non-Euclidean Billiards offers the native-inhabitant view only.
To simulate a rigorously correct native-inhabitant view, we must trick the user’s eyes and brain into perceiving each object’s true hyperbolic or spherical distance. This is most easily accomplished by inserting an extra step into the graphics pipeline:

\[
\begin{align*}
\text{Model Space} & \rightarrow_{\text{model matrix}} \text{World Space} \\
& \rightarrow_{\text{view matrix}} \text{Camera Space} \\
& \rightarrow_{\text{azimuthal equidistant map}} \text{Euclidean Tangent Space} \\
& \rightarrow_{\text{projection transformation}} \text{Projection Space}
\end{align*}
\]

The extra step maps the simulation’s contents from the hyperbolic or spherical camera space onto the observer’s Euclidean tangent space in such a way that all distances from the observer’s head are preserved. In cartography this is called an azimuthal equidistant map. In differential geometry it could be called a logarithmic map because it’s the inverse of the exponential map, but given the more common meanings of logarithmic and exponential, those names feel awkward here. Note that the azimuthal equidistant map gets computed relative to the observer’s head (more precisely, relative to the bridge of her nose), not relative to each eye separately.

Unlike the preceding steps in the above pipeline, the azimuthal equidistant map cannot be realized as a matrix multiplication. Fortunately, though, it can be realized with a simple trigonometric computation, which puts only a small extra burden on the vertex shader. For example, in the spherical case, say an observer at \( P = (0, 0, 0, \rho) \) is viewing some point of interest \( Q = (x, y, z, w) \) on a 3-sphere of radius \( \rho \) meters. Let \( \theta \) be the distance in radians from \( P \) to \( Q \), and let \( d \) be that same distance in meters (measured along the 3-sphere itself, not in the 4-dimensional ball that it bounds). We want to find a point \( Q' \) in the tangent space that sits in the same direction from \( P \) as \( Q \) does, and also sits \( d \) meters away (but in the tangent space, not on the 3-sphere). Geometrically it’s easy to see that

\[
Q' = \left( \frac{\theta}{\sin \theta} x, \frac{\theta}{\sin \theta} y, \frac{\theta}{\sin \theta} z, \rho \right).
\]

To compute \( \theta \), note that \( w = \rho \cos \theta \), where \( w \) and \( \rho \) are both already known,
so our shader code becomes

```plaintext
CosineTheta = clamp(w/rho, -1, +1)
Theta = acos(CosineTheta)
SineTheta = sin(Theta)
if SineTheta > 0.0001
    Factor = Theta / SinTheta
else
    Factor = 1 // correct near north pole,
              // but not near south pole
Qprime = (Factor * x, Factor * y, Factor * z, rho)
```

The shader code for the hyperbolic case is essentially the same, but with \( \text{acosh} \) and \( \text{sinh} \) instead of \( \text{acos} \) and \( \text{sin} \), and different clamping bounds.

The above algorithm lets the observer use her Euclidean binocular vision to see every point in the space at its correct hyperbolic or spherical distance. This eliminates the illusion that all of hyperbolic space sits in a small finite ball, and also eliminates the need for the observer to look walleyed in spherical space. However, even though the observer perceives all objects at their true hyperbolic or spherical distances, there’s still the rich and interesting question of how the observer’s brain integrates those distances into a mental model of the space. For a full discussion, see [Weeks 2020].

Note #1: In the spherical case, when rendering the back hemisphere we must set Factor = \( (\pi + \theta)/\sin \theta \) in the shader code above, so that back-hemisphere objects get drawn in the azimuthal equidistant map at distances in the desired range \([\pi + \varepsilon, 2\pi - \varepsilon]\).

Note #2: One of the referees has pointed out that the Hypernom VR game [Hart et al. 2015] uses a technique similar to the one described here, but with stereographic projection instead of the azimuthal equidistant map. Stereographic projection works well in Hypernom, which is a game for experiencing the 3-sphere as a double-cover of the rotation group \( \text{SO}(3) \), rather than a simulation of the 3-sphere for its own sake. However, because it grossly distorts radial distances, stereographic projection would work poorly in a game like Non-Euclidean Billiards, which strives to let players experience the 3-sphere and hyperbolic 3-space as closely as possible to the way that those spaces’ native inhabitants might experience them.
6 Thurston’s eight geometries

Thurston’s revolutionary Geometrization Theorem says that every closed 3-manifold may be cut into pieces in a natural way, and each piece will admit one of eight homogeneous geometries. Of the eight, the three isotropic geometries $S^3$, $E^3$ and $H^3$ are the easiest to simulate. The product geometries $S^2 \times E$, $E^2 \times E$ ($= E^3$) and $H^2 \times E$ aren’t too much harder, with simulations having been made first in traditional graphics [Weeks 2006] and more recently in VR [Hart et al. 2017b]. Adding some “vertical shear” to each product space gives a twisted product: twisted $S^2 \times E$ (grouped with $S^3$ in the Geometrization Theorem, but with an extra free parameter to set the amount of twist, which typically makes the geometry anisotropic), twisted $E^2 \times E$ (commonly known as Nil geometry) and twisted $H^2 \times E$ (commonly known as $\tilde{SL}(2,\mathbb{R})$ geometry). The eighth geometry is called Sol geometry, and is the strangest of all. Explicit geodesic computations have recently been applied with great success to simulate the twisted product geometries and even Sol [Berger 2015, 2017; Kopczyński et al. 2019, 2020; Novello et al. 2020a, 2020b; Coulon et al. 2020a, 2020b, 2020c].

One could easily port the Non-Euclidean Billiards game to the product geometries, and in fact I have already written a prototype of such an app (Figure 10). However, I recommend against using the product

Figure 10: A pentagonal billiards table in $H^2 \times E$. Even though its five edges are perfectly straight, the player sees them as curved because of $H^2 \times E$’s surprising optics. The more distant billiard balls appear slightly prolate for the same reason.
geometries with general audiences, for the simple reason that straight lines don’t look straight. The billiards game’s raison d’être is to introduce non-mathematicians to curved space. In the case of hyperbolic geometry, the traditional approach to this task was to show people a picture of the Poincaré disk model (Figure 11) and say that

the circular arcs are really straight lines in hyperbolic geometry,

but I’ve found this approach to be ineffective because non-mathematicians think you’re saying that they should

pretend that the circular arcs are straight lines
(even though obviously they’re not).

When a hyperbolic billiard table is embedded in $H^2 \times E$, the table’s edges and the balls’ trajectories, while intrinsically straight, nevertheless appear curved to the player (Figure 10), because of the surprising optics of $H^2 \times E$ itself. This would make it very difficult to convince people that those edges and trajectories really are straight, and that curved space is just as valid as flat space. That’s why I wrote the Non-Euclidean Billiards game in $H^3$ instead, so players can see and feel the straightness of hyperbolic lines directly (Figure 1). The same reasoning applies to favor $S^3$ over $S^2 \times E$ with general audiences.

By contrast, with an audience of geometers and topologists, simulations of all eight geometries are well received and provide much insight and food for thought.
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