Disordered loops in the two-dimensional antiferromagnetic spin-fermion model

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Abstract

The spin-fermion model has long been used to describe the quantum-critical behavior of 2d electron systems near an antiferromagnetic (AFM) instability. Recently, the standard procedure to integrate out the fermions to obtain an effective action for spin waves has been questioned in the clean case. We show that in the presence of disorder, the single fermion loops display two crossover scales: upon lowering the energy, the singularities of the clean fermionic loops are first cut off, but below a second scale new singularities arise that lead again to marginal scaling. In addition, impurity lines between different fermion loops generate new relevant couplings which dominate at low energies. We outline a non-linear $\sigma$ model formulation of the single-loop problem, which allows to control the higher singularities and provides an effective model in terms of low-energy diffusive as well as spin modes.

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1 Introduction

The spin-fermion model is a low-energy effective model describing the interaction of conductance electrons (fermions) with spin waves (bosons). It has been used, e.g., to describe the quantum critical behavior of an electron system near an antiferromagnetic instability [123]. An important example where this might be realized experimentally is in itinerant heavy-fermion materials [4]. By integrating out the fermions completely, a purely bosonic effective action for spin waves is obtained. This action is written in terms of a bare spin propagator and bare bosonic vertices with any even number of spin lines. The value of each bosonic vertex is given by a fermionic loop with spin-vertex insertions: in general, these are complicated functions of all external bosonic frequencies.
and momenta. Hertz [1] and Millis [2] considered only the static limit of these vertices, i.e., setting all frequencies to zero at finite momenta. In this limit the 4-point vertex and all higher vertices vanish for a linear dispersion relation, while they are constants proportional to a power of the inverse bandwidth if the band curvature is taken into account. For the AFM the dynamic critical exponent is $z = 2$ due to Landau damping of spin modes by particle-hole pairs. The scaling of these vertices in $d = 2$ under an RG flow toward low energy scales is marginal for the 4-point vertex while all higher vertices are irrelevant ($d + z = 4$ is the upper-critical dimension). Thus, a well-defined bosonic action with only quadratic and quartic parts in the spin field is obtained.

Recently, Lercher and Wheatley [5] as well as Abanov et al. [6] considered for the 2d case not only the static limit of the 4-point vertex but the full functional dependence on frequencies and momenta. Surprisingly, they found that in the dynamic limit, setting the momenta to zero at finite frequency, the 4-point vertex is strongly divergent as the external frequencies tend to zero, implying an effective spin interaction nonlocal in time. The higher bosonic vertices display an even stronger singularity [7].

To assess the relevance of the singular bosonic vertices, Ref. [7] considered the scaling limit $\omega \sim q^2$ with $z = 2$. In this limit, the bosonic vertices are less singular than in the dynamic limit but the related couplings are still marginal, i.e., they cannot be neglected in the effective bosonic action, in apparent contradiction to Hertz and Millis. Employing an expansion in a large number of hot spots $N$ or fermion flavors, Ref. [6] argues that vertex corrections are resummed to yield a spin propagator with an anomalous dimension $\eta = 2/N = 1/4$ (for $N = 8$). At the same time, $z = 2$ remains unchanged up to two-loop order, i.e., the frequency dimension is given by $x_\omega = 2(1 - \eta/2)$.

The infinite number of marginal vertices renders the purely bosonic theory difficult to use for perturbative calculations. A relevant question is whether the above difficulty persists upon the inclusion of a weak static disorder potential present in real materials. To address this issue we insert disorder corrections into single fermionic loops and find two different crossover scales: at frequencies $\omega \gg 1/\tau$ (i.e., much larger than the impurity scattering rate) and momenta $q \gg 1/\ell$ (with mean free path $\ell = v_F \tau$, where $v_F$ is the Fermi velocity), the fermionic loops resemble the clean case, while below this scale the singularity is cut off by self-energy corrections and the loops saturate. However, at yet lower frequencies, a second crossover scale $\omega \sim 1/(\tau k_F \ell)$, $q \sim 1/(\ell \sqrt{k_F \ell})$ appears where the loops acquire a diffusive form due to impurity ladder corrections and the related couplings again scale marginally, as in the clean case. Therefore, in an intermediate energy range disorder regularizes the singular vertices and appears to restore Hertz and Millis theory, while ultimately at the lowest scales the disordered loops are as singular as the clean ones, albeit with a different functional form: the linear dispersion of the electrons is replaced by a diffusive
form. We outline a non-linear $\sigma$ model formulation of the disordered single-loop problem which allows us to identify all disorder corrections which exhibit the maximum singularity, and provides an action for spin modes coupled to low-energy diffusive electronic modes, instead of the original electrons.

Finally, while all disorder corrections to a single fermion loop lead to couplings which scale at most marginally, impurity lines connecting different fermion loops are a relevant perturbation in $d = 2$. We find that these diagrams may dominate the single-loop contributions below $\omega \simeq 1/\tau$, depending on the typical values of the bosonic momenta.

We proceed as follows: in the remaining part of this section, we introduce the model and the scaling arguments for the clean case. We then insert disorder corrections into a single fermion loop and discuss a class of most singular diagrams in section 2. Their scaling behavior and the emergence of two crossover scales is the subject of section 3. The multi-loop diagrams are discussed in section 4. Appendix A contains the non-linear $\sigma$ model for the disordered single-loop case.

1.1 The spin-fermion model in 2d

The 2d spin-fermion model is defined by the action

$$S[\psi, \bar{\psi}, \phi] = (\bar{\psi}, G_0^{-1} \psi) + (\phi, \chi_0^{-1} \phi) + g \bar{\psi} \psi \phi$$

for a fermionic field $\psi, \bar{\psi}$ and a bosonic spin field $\phi$. The inverse fermionic propagator is $G_0^{-1}(i\epsilon, \mathbf{p}) = i\epsilon - \xi_p$ in terms of the Matsubara frequency $i\epsilon$ and a dispersion relation $\xi_p$ with a roughly circular Fermi surface (FS), which we approximate by a quadratic dispersion $\xi_p = \frac{\mathbf{p}^2}{2m_e} - \mu$ with electron mass $m_e$, chemical potential $\mu$, Fermi momentum $k_F = \sqrt{2m_e\mu}$, and constant density of states $2\pi \rho_0 = \epsilon_F/v_F^2$. $\chi_0(q)$ is the bare spin propagator.

We assume that the above model describes an AFM quantum critical point at finite $q = q_c$. The Fermi surface has so-called hot regions connected by exchange of $q_c$, and cold regions where scattering off spin waves is weak. Here we shall assume an underlying lattice and a commensurate $q_c = (\pi, \pi)$, which is equivalent (up to a reciprocal lattice vector) to $-q_c$.

When computing fermionic loops with only spin-vertex insertions, the momentum integration can be reduced to the region around two hot spots separated by $q_c$, which we shall denote by $\alpha$ and $\bar{\alpha}$ (Fig. 1). The fermionic dispersion

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1 The explicit spin structure of the spin-fermion vertex is not relevant for the fermionic loops and needs to be specified only when spin lines are contracted.
Fig. 1. The Fermi surface with the hot spots separated by the wave vector $q_c$.

The relation $\xi_p$ is linearized around any hot spot $\alpha$ at momentum $p^\alpha_{hs}$ as

$$\xi_p = v_F (p - p^\alpha_{hs}) = v^\alpha_x \bar{p}_x + v^\alpha_y \bar{p}_y \equiv \xi^\alpha_p$$

where $\bar{p}$ denotes the distance from the hot spot. The components $v_x$ ($v_y$) of the Fermi velocity $v_F$ parallel (perpendicular) to $q_c$ at a given hot spot $\alpha$ are related by $v^2_F = v^2_x + v^2_y$ and $v_x/v_y = \tan(\phi_0/2)$, with $\phi_0$ the angle between hot spots $\alpha$ and $\bar{\alpha}$ as seen from the center of the circular Fermi surface. The case $\phi_0 = \pi$ ($v_y = 0$) corresponds to perfect nesting, but here we consider a generic $\phi_0$ without nesting. For a pair of hot spots, the momentum integration can be written as

$$\int \frac{d^2p}{(2\pi)^2} = J \sum \int d\xi^\alpha d\xi^{\bar{\alpha}}$$

where $\xi^\alpha$ and $\xi^{\bar{\alpha}}$ are two independent momentum directions at hot spot $\alpha$ ($\xi^{\bar{\alpha}}$ coincides with the radial direction at hot spot $\bar{\alpha}$), $J^{-1} = 4\pi^2 v^2_F \sin \phi_0$ is the corresponding Jacobian which depends on the shape of the Fermi surface and the filling, and one still has to perform the summation over all $N = 8$ hot spots.

1.2 Clean fermionic $2n$-loops

The fermionic loops with $2n$ spin-vertex insertions—i.e., the $2n$-point functions—are in general complicated functions of the external frequencies and momenta. The loop with two spin insertions contributes to the self-energy for the spin propagator and has the well-known Landau damping form for small frequencies $\omega$ and momenta near $q_c$ [8],

$$\Sigma(i\omega, q \approx q_c) = -\gamma |\omega|, $$
with the dimensionless strength of the spin fluctuations [6]

\[
\gamma \equiv 2\pi JNg^2 = \frac{g^2N}{2\pi v_F^2 \sin \phi_0} = \frac{g^2N}{4\pi v_xv_y}.
\]

The inverse spin propagator resummed in the random-phase approximation has dynamical exponent \( z = 2 \),

\[
\chi^{-1}(i\omega, \mathbf{q}) = m + \gamma |\omega| + \nu |\mathbf{q} - \mathbf{q}_c|^2,
\]

where the mass term \( m \) measures the distance from the quantum critical point and \( \nu \approx g^2/\epsilon_F \). Near criticality \( m \approx 0 \), the momentum exchanged by scattering off a spin wave is peaked near \( \mathbf{q}_c \). From here on, we shall denote by \( \mathbf{q} \) the deviation from \( \mathbf{q}_c \). In the commensurate case there are logarithmic singularities in the clean bosonic self-energy (from contracting two spin lines diagonally in Fig. 2) that may lead to an anomalous dimension of the spin propagator [6].

![Diagram of clean fermionic four-loop](image)

Fig. 2. The clean fermionic four-loop \( b_4 \), indicating the notation for the external frequencies and momenta.

The 4-point function is given by the fermion loop with four spin insertions, which provides the bare two-spin interaction:

\[
b_4 = -\pi Jg^4 \sum_\alpha \left[ |\omega_1 + \omega| + |\omega_1 - \omega| - |\omega_2 + \omega| - |\omega_2 - \omega| \right] \left[ i(\omega_1 + \omega_2) - \xi_{\mathbf{q}_1 + \mathbf{q}_2}^\alpha \right] \left[ i(\omega_1 - \omega_2) - \xi_{\mathbf{q}_1 - \mathbf{q}_2}^\alpha \right],
\]

where we have labeled the three independent external frequencies and momenta as shown in Fig. 2. This is a nonanalytic function whose value for \( \omega \to 0, \mathbf{q} \to 0 \) depends on the order of the limits: it vanishes in the static limit (\( \omega \to 0 \) first) while it diverges as \( 1/\omega \) in the dynamic limit (\( \mathbf{q} \to 0 \) first). There is an important difference from the forward-scattering loop with small external momenta [9]: here, symmetrization of the external lines does not lead to loop cancellation, i.e., a reduction of the leading singularity. Instead, symmetrization only modifies the prefactors but does not change the scaling dimension.
1.3 Scaling of the clean fermionic loops

We recall the scaling behavior of the $2n$-point functions [7]. Since no cancellation of the leading singularity occurs, we only need to consider the power of external frequency and momentum and not the particular linear combinations of frequencies and momenta involved. Introducing a symbolic notation where $\omega$ denotes a positive linear combination of external frequencies and $q$ a linear combination of momenta, the $2n$-point functions have the scaling form

$$b_{2n} \sim g^{2n} \frac{\omega}{v_F^2 (\omega + iv_F q)^{2(n-1)}} ,$$

where an average over hot spots is understood, which leads to a real positive function of the frequencies and momenta represented by $\omega$ and $q$. For the purpose of scaling, we have substituted $J \sim 1/v_F^2$ and $\gamma \sim g^2/v_F^2$.

To estimate the relevance of the vertices in the scaling limit $\omega^{2/z} \sim q^2 \to 0$, where $q$ dominates $\omega$ in the denominator for $z > 1$, consider the $\phi^{2n}$ term in the effective action [7]:

$$g_{2n} \int (d^2 q d\omega)^{2n-1} \frac{\omega}{(v_F q)^{2(n-1)}} \phi^{2n},$$

where $g_{2n}$ is the coupling strength related to the vertex function $b_{2n}$. Using the scaling dimension of the field $[\phi^2] = -(d+z+2)$ (in frequency and momentum space) in two dimensions,

$$[g_{2n}] = -(2n-1)(2+z) - [z - 2(n-1)] - n(-4-z) = (2-z)n.$$

The scaling dimension of all $2n$-point functions is zero for $z = 2$, i.e., all bosonic vertices are marginal in the scaling limit, and it is not clear how to perform calculations with such an action. Our aim is to see if and how the disorder present in real systems changes the scaling dimension of the fermionic loops.

2 Disorder corrections to a single fermion loop

We consider static impurities modeled by a random local potential with mean squared amplitude $u^2$ [10]. As long as no spin-vertex insertions appear between impurity scatterings, the disorder corrections in the Born approximation have the standard form. In the fermionic propagator $G(i\epsilon, \mathbf{p}) = (i\epsilon - \xi_{\mathbf{p}})^{-1}$ the Matsubara frequency $\epsilon$ is cut off as $\bar{\epsilon} \equiv \epsilon + \text{sgn}(\epsilon)/(2\tau)$ at the scale of the impurity scattering rate $1/\tau = 2\pi \rho_0 u^2$. Although the particle-hole bubble $B(i\epsilon + i\omega, i\epsilon, \mathbf{q})$ with small momentum transfer $\mathbf{q}$ is cut off by disorder, the
direct ladder resummation has the diffusive form \( L(i\epsilon + i\omega, i\epsilon, q) \approx u^2/(|\omega|\tau + Dq^2\tau) \) if both frequencies lie on different sides of the branch cut on the real line. The diffusion constant is \( D = v_F^2\tau/2 = v_F\ell/2 \) in \( d = 2 \). Throughout this work we assume that impurity scattering is weak, \( 1/(k_F\ell) = 1/(\epsilon_F\tau) \ll 1 \).

2.1 2-point function

Due to the linearized dispersion around the hot spots, the bosonic self-energy is unchanged by disorder corrections to the fermionic propagators:

\[
\Sigma_{\text{dirty}}^{(0)}(i\omega, q_c + q) = \begin{array}{c}
i\omega, q_c + q \\
\hline
i\epsilon, p
\end{array}
\]

\[
= -\sum_{\text{spin}} g^2 \int \frac{d\epsilon}{2\pi} \frac{d^2p}{(2\pi)^2} \frac{1}{\xi_p} \frac{1}{i(\epsilon + \omega) - \xi_{p + q_c + q}}
\]

\[
= -\gamma |\omega| .
\]

As the direct impurity ladder with large momentum transfer \( q_c \) is not singular, the leading vertex correction is given by a single impurity line across the bubble:

\[
\Sigma_{\text{dirty}}^{(1)}(i\omega, q_c + q) = \begin{array}{c}
i\omega, q_c + q \\
\hline
i\epsilon, p
\end{array}
\]

\[
= -2g^2 \int \frac{d\epsilon}{2\pi} \frac{d^2p}{(2\pi)^2} \frac{1}{\xi_p} \frac{1}{i\epsilon - \xi_p - \xi_{p + q_c}} \times u^2 \int \frac{d^2p'}{(2\pi)^2} \frac{1}{\xi_{p'}} \frac{1}{i\epsilon - \xi_{p'} + q_c}
\]

\[
\approx -\gamma \frac{1}{\tau} .
\]  \( \text{(2)} \)

where the cutoff scale \( \epsilon_F \) is used for the divergent frequency integral. The mass correction shifts the position of the critical point as a function of the control parameter by a finite amount. We assume that the system can be fine-tuned again to the critical point by adjusting the control parameter.

The correction to the bosonic self-energy from the maximally crossed ladder (cf. equation (B.1)) is

\[
\Sigma_{\text{dirty}}^{\text{crossed}}(i\omega, q_c + q) \sim \gamma |\omega| \frac{\ln(|\omega|\tau)}{k_F\ell} .
\]

The logarithm does not change the bare scaling dimension, however the absence of a corresponding term in \( q \) could tend to increase \( z \) and possibly make
the clean vertices irrelevant starting from \( z = 2 \). We will not further discuss this possibility and eliminate the crossed ladders by a small magnetic field because, as we shall see, the vertices in the presence of impurities will acquire new singularities due to diffusive ladders which will not be affected by the value of \( z \).

### 2.2 4-point function

Including only self-energy disorder corrections cuts off the frequencies in the denominator by \( \tilde{\omega} = \omega + \text{sgn}(\omega)/\tau \) and yields the scaling form (where \( \omega \) and \( q \) represent the same linear combinations of external frequencies and momenta as in equation (1))

\[
b^{(0)}_{4,\text{dirty}} = -\pi J g^4 \sum_\alpha \frac{\omega}{(i\tilde{\omega} - \xi_\alpha^2)} \sim \frac{g^4}{v_F(\omega + 1/\tau + iv_F q)^2}.
\]

For \( \omega, v_F q \ll 1/\tau \), this contribution vanishes linearly in \( \omega \) and is, therefore, irrelevant in the scaling limit. Instead, a singular contribution is obtained by including diffusive ladders into the loop. Since direct ladders between two propagators with momenta separated by \( q_c \) are not diffusive, we insert direct ladders only between propagators with almost equal momenta—i.e., near the same hot spot (see Fig. 3).

![Fig. 3. The four-point vertex with one direct ladder insertion factorizes into pairs of spin vertices with three propagators between them (R vertices) and the direct ladder (L).](image)

The \( R_{\alpha\beta\alpha} \) subdiagram is constructed from two propagators near one hot spot \( \alpha \) and one propagator near the associated hot spot \( \bar{\alpha} \), with two spin vertices in between:

\[
R_{\alpha\beta\alpha}(q, q') = g^2 \int \frac{d^2 p}{(2\pi)^2} \frac{1}{i(\epsilon + \omega) - \xi_{p+q}} \frac{1}{i(\epsilon + \omega') - \xi_{p+q_c+q'}} \frac{1}{i\tilde{\epsilon} - \xi_p} \\
= -2\pi i J g^2 \sum_\alpha \frac{\Theta(-\epsilon[\epsilon + \omega])}{|\tilde{\omega}| + i\tilde{\xi}_q} \text{sgn}(\epsilon + \omega'),
\]

which saturates to a constant for small \( \omega, q \). Combining the parts, we obtain for the 4-point vertex with one direct ladder (by the superscript we denote
the number of ladders

\[ b_{4,\text{dirty}}^{(1)} = - \sum_{\text{spin}} \int \frac{d\epsilon}{2\pi} R(q_1 + q_2, q + q_1) L(q_1 + q_2) R(q_1 + q_2, q + q_2) \]

\[ \sim g^4 \frac{\omega}{v_F^2 (\bar{\omega} + iv_F q)^2} \frac{1/(k_F \ell)}{\omega \tau + D q^2 \tau} \]

as the scaling form for typical external \( \omega, q \). We have checked explicitly that symmetrization of the spin insertions does not change the leading singularity. Note that the last factor in the ladder contribution (4) becomes larger than unity only for \( \omega \tau, D q^2 \tau < 1/(k_F \ell) \), while \( b_{4,\text{dirty}}^{(0)} \) is cut off for \( \omega \tau, D q^2 \tau < 1 \). This appearance of two crossover scales is a central observation of our work.

Adding the contributions with zero and one ladder,

\[ b_{4,\text{dirty}} \sim g^4 \frac{\omega}{v_F^2 (\bar{\omega} + iv_F q)^2} \frac{1/(k_F \ell)}{\omega \tau + D q^2 \tau} \times \left\{ \begin{array}{ll} \omega \tau \text{ and/or } q \ell \gg 1 \\ \omega \tau^2 \left[ 1 + \frac{1/(k_F \ell)}{\omega \tau + D q^2 \tau} \right] \end{array} \right\} \quad (\omega \tau \text{ and } q \ell \ll 1) . \]

There are, of course, many more ways to insert ladders into the 4-point vertex. Since we are interested in the most singular contribution that determines the scaling, we have already excluded direct ladders between different hot spots on the basis that they are not diffusive. However, one could also add a second ladder in Fig. 3 between the \( \bar{\alpha} \) and \( \bar{\beta} \) lines (with \( \bar{\alpha} = \bar{\beta} \)). This does not change the singularity, but gives a much smaller prefactor \( 1/(k_F \ell)^2 \). In general, all such crossings of direct ladders yield internal integrations over the ladder momenta (as in the case of the cooperon ladder) and give at most logarithmic corrections but not a higher singularity. Logarithmic corrections do not change the scaling dimension, and because of the higher order in \( 1/(k_F \ell) \), all diagrams with crossings of direct ladders will be neglected. Note that the numerical values of the coefficients of course depend on these diagrams. We argue further that the insertion of crossed ladders leads at most to the same singularity as those with direct ladders, up to logarithmic terms.

In conclusion, this leaves us with a much smaller set of diagrams displaying the leading singularity: \textit{all possible insertions of direct ladders between propagators of the same spin (at the same hot spot), which do not cross each other.}

For the 4-point vertex, the ladder diagram in Fig. 3 (with summation over hot spots and symmetrization of external lines understood) is the only one meeting these conditions and is therefore sufficient to obtain the leading singularity. In the scaling limit \( \omega \sim q^2 \), the ladder correction has the same scaling behavior (constant) as the clean vertex \( g_4 \), hence the 4-point coupling remains asymp-

\[ \text{Alternatively, as discussed above in connection with } \Sigma^{\text{crossed}}_{\text{dirty}} \text{ one could apply a small magnetic field, which does not cut off the singularity in the clean case and with direct ladders, but suppresses the contribution from crossed ladders.} \]
totically marginal even when disorder is included. This leads to the question how the higher $2n$-point vertices behave.

2.3 Higher bosonic vertices

Fig. 4. The disordered $2n$ loop (shown here for $n = 3$) with $n - 1$ ladder insertions arranged in a chain. This can be extended for larger $n$ by repeating the ($XL$) block.

Following the above discussion, we consider a particular insertion of ladders into the bare $2n$-point vertex which meets the above condition for a maximally singular contribution. As shown in Fig. 4, we group two adjacent spin vertices together (the $R$ part as in Fig. 3),

\[
R \sim i \frac{g^2}{v_F^2} \tau \\
RL \sim i \frac{g^2}{k_F \ell} \frac{1}{\omega + Dq^2} \sim \frac{g^2/\tau}{(\omega + Dq^2)(\epsilon_F)} .
\]

This is connected via a direct ladder $L$ to an $X$ vertex made from four fermion propagators with two spin and two ladder insertions,

\[
X \sim \frac{g^2}{v_F^2} \tau^2 \quad \text{(for small external $\omega$, $q$)} \\
XL \sim \frac{g^2}{k_F \ell} \frac{\tau}{\omega + Dq^2} \sim \frac{g^2}{(\omega + Dq^2)(\epsilon_F)} .
\]

There is another contribution to $X$ not depicted in Fig. 4 with two spin insertions on the same fermion line: this term is of the same order of magnitude but generically does not cancel the one shown.

Repeating the ($XL$) part $n - 2$ times and finishing with another $R$ vertex, we obtain a chain-like diagram with $n - 1$ ladders which, for small $\omega$ and $q$, can
be estimated by the scaling form

\[ b^{(n-1)}_{2n, \text{dirty}} = - \sum_{\text{spin}} \int d\epsilon RL( XL)^{n-2} R \]

\[ \sim \frac{g^2}{\tau} \left( \frac{g^2}{(\omega + Dq^2)(\epsilon_F)} \right)^{n-2} \frac{g^2}{v_F^2} \tau \omega \]

\[ \sim \frac{g^{2n}}{v_F^2} \frac{\omega}{(\omega + Dq^2)^{n-1}(\epsilon_F)^{n-1}}. \]

Diagrams with fewer than \( n - 1 \) ladders have a weaker singularity and give an irrelevant contribution to the coupling in the scaling limit.

Fig. 5. The disordered 2n loop for even \( n \) (shown here for \( n = 4 \)) with \( n \) ladder insertions and a Hikami vertex \( H_n \) in the middle.

There are other diagrams with \( n \) ladders, which at first appear to be even more divergent but upon closer inspection turn out to have the same singularity. As shown in Fig. 5 for even \( n \) one can connect \( n \) \( R \) parts via \( n \) ladders to an \( n \)-point Hikami vertex \[ H_n \sim \frac{\epsilon_F}{v_F^2} \tau^n (\omega + Dq^2). \]

The additional factor of an inverse diffusion propagator in \( H_n \) is due to the insertion of further single impurity lines which cancel the constant term and leave only terms linear in \( \omega \) and \( q^2 \) for small \( \omega \) and \( q \); this effectively cancels one of the \( n \) diffusive ladders. Connecting the \( RL \) parts to the Hikami vertex and adding the frequency integration along this one large fermion loop,

\[ b^{(n-1)}_{2n, \text{dirty}} = - \sum_{\text{spin}} \int d\epsilon H_n( RL)^{n} \sim \frac{g^{2n}}{v_F^2} \frac{\omega}{(\omega + Dq^2)^{n-1}(\epsilon_F)^{n-1}}. \]
has the same singularity as the chain-type diagram.

In appendix A, we propose a non-linear $\sigma$ model for the spin modes coupled to low-energy diffusion modes (instead of the original electrons) with only one local (constant) coupling. This allows us to control all single-loop diagrams exhibiting the leading singularity, thereby supporting the perturbative calculations in this work. Elimination of the diffusive modes would lead again to an effective action for the spin modes with infinitely many marginal couplings.

### 3 Scaling of the disordered single loops

#### 3.1 Smallness of ladder corrections and second crossover scale

As for $b_{4,\text{dirty}}$, the disordered loops (beyond the 2-point function) feature two crossover scales, one where disorder corrections in the self-energy of the fermion lines cut off the vertices, and another where ladder corrections lead again to marginal scaling. The existence of these two scales can be traced back to the presence of hot spots.

For comparison, consider the case of forward-scattering bosonic vertices. Self-energy disorder corrections become important and cut off the fermionic propagators at $\omega \tau \approx 1$ and $q\ell \approx 1$. Adding one disorder ladder implies adding also two fermionic propagators with nearby momenta and performing one momentum integration. This additional contribution can be estimated as

$$LG^2 = \frac{u^2 \sqrt{(1 + \omega \tau)^2 + (q\ell)^2}}{\sqrt{(1 + \omega \tau)^2 + (q\ell)^2} - 1} \int \frac{d^2p}{(2\pi)^2} G^2 = \frac{\sqrt{(1 + \omega \tau)^2 + (q\ell)^2}}{\sqrt{(1 + \omega \tau)^2 + (q\ell)^2} - 1}.$$  

The ladder correction becomes dominant exactly at the same scale $\omega \tau \approx q\ell \approx 1$ where the self-energy corrections appear. Hence, there is only a single crossover scale between clean and dirty behavior.

Also in the case of backscattering bosonic vertices, the self-energy corrections set in at $\omega \tau \approx q\ell \approx 1$. However, the ladder insertions are modified due to the presence of hot spots for the two additional fermionic propagators:

$$LG^2 = L J N \int d\xi \alpha d\xi^\alpha G^2 = \frac{N/(2 \sin \phi_0)}{k_F \ell} \frac{\sqrt{(1 + \omega \tau)^2 + (q\ell)^2}}{\sqrt{(1 + \omega \tau)^2 + (q\ell)^2} - 1}.$$  

In comparison with the forward-scattering case above, there is an additional
ξ integration which effectively replaces one factor τ by 1/ε_F, such that the
diffusive ladders become dominant only at a second, lower crossover scale
ωτ, (qℓ)^2 ≈ 1/(k_Fℓ). This means that the impurity ladder scattering of electron-
hole pairs with nearby momenta is less effective by a factor of 1/(k_Fℓ) if the
particles are forced by spin-vertex insertions to be near hot spots between
impurity ladders.

3.1.1 Scaling regimes

According to the above discussion, we can identify three different scaling
regimes for the disordered 2n-loops:

\[ b_{2n,\, dirty} \approx \begin{cases} 
   b_{2n,\, clean} \sim \frac{\omega}{(\omega + i\tau^q)\xi^2(n-1)} & (\omega, (q\ell)^2 \gg 1) \\
   b_{2n,\, dirty}^{(0)} \sim \frac{\omega}{(1/\tau^2)^{n-1}} & (1 \gg \omega, (q\ell)^2 \gg \frac{1}{k_F\ell}) \\
   b_{2n,\, dirty}^{(n-1)} \sim \frac{\omega}{(\omega + Dq^2)^{(n-1)}(\xi)^n} & (\frac{1}{k_F\ell} \gg \omega, (q\ell)^2) 
\end{cases} \]

Note that each pair of fermion-like propagators in the clean case is replaced
by one diffusive propagator and an additional factor of ε_F in the disordered
case, leaving g_{2n} marginal in the scaling limit. The contributions to the vertices
which become dominant in the different scaling regimes are visualized in Fig. 8
below, and in this figure it is made explicit how the constraints on ωτ and qℓ
are to be understood.

There are two different diffusive regimes in the model: a fast one for charge
modes with diffusion constant D, and a much slower one for spin modes with
ν/γ ≃ νq^2/(k_Fℓ). We can, therefore, look at two variants of the scaling limit
ω ∼ q^2. For charge diffusion we have ω ∼ Dq^2 (see the red/dashed scaling line
in Fig. 8), and

\[ b_{2n,\, dirty} \approx \begin{cases} 
   b_{2n,\, clean} \sim \frac{\omega}{(\omega^F)^{n-1}} & (\omega \gg 1) \\
   b_{2n,\, cutoff} \sim \frac{\omega}{(1/\tau^2)^{n-1}} & (1 \gg \omega \gg \frac{1}{k_F\ell}) \\
   b_{2n,\, ladder} \sim \frac{\omega}{(\omega F)^{n-1}} & (\frac{1}{k_F\ell} \gg \omega, \omega \gg \frac{1}{k_F\ell}) 
\end{cases} \]  (5)

shows a non-monotonous behavior. On the other hand, if the typical values of
ω and q are given by the spin propagators connected to the external legs of
the fermion loops, we put γω ≃ νq^2 (see the blue/solid scaling line in Fig. 8), and

\[ b_{2n,\, dirty} \approx \begin{cases} 
   b_{2n,\, clean} \sim \frac{\omega}{(\omega F)^{n-1}} & (\omega \gg \frac{1}{k_F\ell}) \\
   b_{2n,\, cutoff} \sim \frac{\omega}{(1/\tau^2)^{n-1}} & (\frac{1}{k_F\ell} \gg \omega, \omega \gg \frac{1}{(k_F\ell)^2}) \\
   b_{2n,\, ladder} \sim \frac{\omega}{(\omega F)^{n-1}(k_F\ell)^{n-1}} & (\frac{1}{k_F\ell} \gg \omega) 
\end{cases} \]  (6)

This scaling analysis suggests that even though the ladder corrections scale
marginally for asymptotically low frequencies, there is a range of frequencies
and momenta where the singular clean vertices are already cut off and the ladder corrections are still small, such that the Hertz-Millis theory might apply in this zone. As we shall see in the following section, such a regime may be hidden by further contributions from multiple loops.

4 Disorder corrections to multiple fermion loops

The multi-loop disorder corrections arise from impurity lines connecting different fermionic loops. In the simplest case, one takes $n$ static particle-hole bubbles with two spin insertions each (mass terms) and connects them with single impurity lines. As the impurity lines do not carry frequency, there are only $n$ independent frequencies in this $2n$-point spin vertex. The corresponding coupling in the action has therefore a different scaling dimension than the single-loop contribution (with $2n-1$ independent frequencies) and is generally more relevant. In fact, the missing frequency integrations lead to a scaling as in the classical field theory and the Harris criterion [12] applied to the bare model implies that such contributions are relevant in $d < 4$.

We first define the $n$-loop vertices $\Delta_{2n}$ in the disordered spin-fermion model and then discuss the energy scales where these additional vertices become quantitatively more important than the single-loop contributions. Let $\Delta[V]$ denote the particle-hole bubble at arbitrary momentum transfer $q$ (not just near $q_c$) and zero external frequency $\omega = 0$ in the presence of a particular configuration of the disorder potential $V$,

$$
\Delta[V] \equiv \Delta_{2n}
$$

$$
= - \int \frac{d\epsilon}{2\pi} \operatorname{Tr}\left( \frac{1}{i\epsilon - \xi - V} \Gamma \frac{1}{i\epsilon - \xi - V^*} \right)
$$

where $\xi$ is the hopping matrix, the spin-fermion vertex $\Gamma = \exp(-iqx)$ is a diagonal matrix in real space (with spin indices suppressed), and the trace runs over spatial indices. We diagonalize $\xi + V = \sum_k |k\rangle \xi_k \langle k|$ and perform the $\epsilon$ integration,

$$
\Delta[V] = - \int \frac{d\epsilon}{2\pi} \sum_{kl} \frac{1}{i\epsilon - \xi_k} \langle k|\Gamma|l\rangle \frac{1}{i\epsilon - \xi_l} \langle l|\Gamma^*|k\rangle
$$

$$
= - \sum_{kl} |\langle k|\Gamma|l\rangle|^2 \frac{\Theta(-\xi_k \xi_l)}{|\xi_k - \xi_l|}.
$$

We now specialize to the case $q_c = q_c$ and define the static connected $2n$-point
vertex $\Delta_{2n}$ as

$$\Delta_{2n} \equiv \left\langle \left( \begin{array}{c} \text{impurity lines} \\ \text{disorder average, connected part} \end{array} \right) \right\rangle^{n}$$

We assume that for a generic large momentum transfer near $q_c$ and generic band dispersion, all $\Delta_{2n}$ have a finite and nonzero limit as $\omega_i, |q_i - q_c| \to 0$. The corresponding terms in the action

$$\delta_{2n} \int (d\omega)^n (d^d q)^{2n-1} \left( \phi(\omega_i)\phi(-\omega_i) \right)^n_{\{q_i\}}$$

have only $n$ independent frequency integrations but $2n - 1$ momentum integrals. $\delta_{2n}$ stands for the running coupling with bare value $\Delta_{2n}$. The constraint on the frequency integration leads to a scaling dimension

$$[\delta_{2n}] = -nz - (2n - 1)d - n(-d - z - 2) = d - n(d - 2) = 2 \quad (z = 2) \quad (d = 2).$$

Thus, infinitely many couplings $\delta_{2n}$ are all equally relevant. The singularity found in two dimensions is so strong that one expects that even in three dimensions $\Delta_4$ remains a relevant perturbation and destabilizes a Gaussian fixed point in the disordered model, which may be related to the difficulty encountered in reconciling experiments with the Hertz-Millis theory for $d = 3, z = 2$ \cite{13,14}.

Fig. 6. Chain-type contribution to the $\Delta_{2n}$ vertex (shown here for $n = 3$) with zero momentum transfer on the impurity lines.

In order to estimate the quantitative importance of the $\Delta_{2n}$ vertices with respect to the $b_{2n}$ vertices, one needs to find the contributions to $\Delta_{2n}$ at lowest order in $1/(k_F\ell)$. For $n = 1$, the leading term is the mass correction in equation (2). For $n \geq 2$, leading contributions to $\Delta_{2n}$ are given by chains of $n$ $\Delta[V]$'s connected by $n - 1$ single impurity lines with zero momentum transfer (see Fig. 6). The terminal bubbles of the chain (with one impurity line) are roughly $g^2/v_F^2$, while the intermediate bubbles with two impurity lines are
approximately \( g^2/(v_F^2 \epsilon_F) \). The complete chain can therefore be estimated as

\[
\Delta_{2n} \approx g^2 v_F^2 u^2 \left[ \frac{g^2}{v_F^2 \epsilon_F} \right]^{n-2} \frac{g^2}{v_F^2} \\
\approx \frac{g^{2n}}{v_F^2} \frac{1}{\epsilon_F^{n-2}} \frac{1}{(k_F \ell)^n-1} \quad (n \geq 2).
\]

In the scaling limit dominated by spin diffusion, we can replace \((q/k_F)^2\) by \(\omega/\epsilon_F\), and the single-loop vertex functions \(b_{2n}\) scale as given in equation (6). Below a certain frequency scale \(\omega\), the relevant couplings \(\delta_{2n}\) will necessarily become larger than the marginal couplings \(g_{2n}\); in order to find this scale one has to compare \(b_{2n,\text{clean}}\) with \(\Delta_{2n}/\omega^{n-1}\), to account for the missing \(n-1\) frequency integrations in the couplings \(\delta_{2n}\) with respect to \(g_{2n}\):

\[
b_{2n,\text{clean}} \approx \frac{\Delta_{2n}}{\omega^{n-1}} \frac{\omega}{(\omega \epsilon_F)^{n-1}} \approx \frac{1}{\omega^{n-1} \epsilon_F^{n-2} (k_F \ell)^n-1} \frac{1}{1 (k_F \ell)^n-2}.
\]

Hence, in the spin scaling limit the higher vertices \(\Delta_{2n}\) start to dominate the clean single-loop vertices \(b_{2n,\text{clean}}\) at successively lower frequency scales. Likewise, the cutoff vertex \(b_{2n,\text{cutoff}}\) is of the same magnitude as \(\Delta_{2n}/\omega^{n-1}\) for \(\omega \tau \approx 1/(k_F \ell)^{(2n-3)/n}\). Extending these arguments beyond the scaling limit to the \(\omega-q^2\) plane, we obtain the crossover lines between single- and multiloop contributions indicated in Fig. 8. In summary, \(\Delta_4\) dominates the clean vertex \(b_4\) below \(\omega \lesssim v_F q/\sqrt{k_F \ell}\) for \(v_F q \gtrsim 1/\tau\), and the cutoff vertex \(b_4\) below \(\omega \lesssim 1/(\tau \sqrt{k_F \ell})\) for \(v_F q \lesssim 1/\tau\). The higher vertices \(\Delta_{2n>4}\) become dominant for \(v_F q \lesssim 1/\tau\) only below \(\omega \approx 1/(\tau k_F \ell)\).

Fig. 7. (a) Ring-type contribution to the \(\Delta_{2n}\) vertex (shown here for \(n = 4\)) which is peaked when the impurity lines carry a momentum near \(2k_F\). (b) Points on Fermi surface connected in the singular \(q_c-2k_F\) bubbles, with a commensurate \(q_c\) and incommensurate \(2k_F\).
In addition to the chain-type diagrams above, there are diagrams with \( n \) bubbles arranged in a ring and connected by single impurity lines. In this case, one has to integrate over the momentum carried by the impurity lines, involving also momenta near \( 2k_F \) where the bubble with two static \( q_c \) spin insertions and two static \( 2k_F \) charge insertions becomes singular in the clean case (see Fig. 7). This singularity is related to the well-known \( 2k_F \) singularity of the particle-hole bubble \cite{15}; while in the clean case only a small region of the Fermi surface around the hot spots is visited, impurity scattering visits the whole Fermi surface, including the parts separated by \( 2k_F \). The disorder correction to the fermionic self-energy provides a cutoff for this singularity, thus changing the estimated power of \( 1/(k_F \ell) \) in the expression for \( \Delta_{2n} \). We estimate that all such diagrams are at least of order \( 1/(k_F \ell)^2 \), which implies that they become dominant at the same scale as \( \Delta_0 \), i.e., only below \( \omega \approx 1/(\tau k_F \ell) \).

Therefore, in the spin scaling limit we still obtain clean anomalous behavior above \( \omega \approx 1/\tau \), which gets modified by a single relevant vertex \( \Delta_4 \) for \( 1/\tau \geq \omega \geq 1/(\tau k_F \ell) \). For \( \omega < 1/(\tau k_F \ell) \), ever higher multi-loop vertices \( \Delta_{2n} \) add to the single-loop vertices.

![Fig. 8. (Color online) Regions in the \( \omega-q^2 \) plane where different contributions to the spin vertices become dominant. The lines separating the different regions are meant as a guide only, as they are determined only up to prefactors of order \( O(1) \).](image)

The problem of a disordered AFM was considered in Ref. \cite{16}, where the authors analyzed an AFM \( \phi^4 \) model with a local \( u\phi^4 \) interaction and a random mass term \( m\phi^2 \) within an \( \epsilon = 4 - d \) expansion. Averaging over random mass
configurations yields a $\Delta(\phi^2)\omega(\phi^2)\omega'$ term from the static mass-mass correlation $\langle mm \rangle$, which depends on three independent momenta but only two frequencies. Therefore, $\Delta$ is a relevant coupling in $d < 4$ and the clean model is unstable against arbitrarily small disorder. Recently, this model has also been studied using a strong-disorder RG [14].

However, according to the above discussion the clean AFM $\phi^4$ model is not applicable in $d = 2$ because $u$ is nonlocal and there are infinitely many additional marginal couplings which even change the direction of the RG flow for $u$ [7]. Furthermore, we have pointed out that for a generic fermionic band dispersion the average over a random fermionic potential implies a fluctuating mass term which has non-vanishing higher cumulants $\langle m^n \rangle$. While these higher cumulants are irrelevant in $d = 4$, in two dimensions all couplings $\Delta_{2n}$ are equally relevant and may possibly change the direction of the RG flow also for $\Delta = \Delta_4$. Thus, it is not obvious which of the results of [16] hold in $d = 2$.

5 Discussion and conclusions

We have shown that the fermionic loops with large momentum transfer, which are relevant to describe an AFM transition, exhibit two crossover energy scales if disorder corrections are added. In the scaling limit set by charge diffusion $\omega \simeq Dq^2$, the singularities of the clean (marginal) vertices $g_{2n}$ are cut off at scale $\omega \approx 1/\tau$ and the 4-point vertex and beyond become irrelevant. Below $\omega \approx 1/(\tau k_F \ell)$, however, diffusive ladders lead again to marginal scaling, albeit with a diffusive functional form of the vertex functions different from the clean case. On the other hand, in the scaling limit set by the slower spin diffusion $\omega \simeq \xi q^2 \approx Dq^2/(k_F \ell)$, these crossovers occur at the same momentum scales but at frequency scales smaller by a factor of $1/(k_F \ell)$. A summary of crossover scales and boundaries is reported in Fig. 8.

In addition to the single fermion loops, there are diagrams made of multiple fermion loops connected only by impurity lines, which were not present in the clean case. In accordance with the Harris criterion applied to the bare model, these are relevant perturbations in $d = 2$ which make the clean model unstable against disorder. Indeed, in contrast to $d = 4$, in two dimensions the bosonic model contains infinitely many equally relevant vertices $\Delta_{2n}$ due to disorder corrections to multiple fermion loops, which we estimate to become important below $\omega \simeq 1/(\tau \sqrt{k_F \ell})$ (charge diffusion) or $\omega \simeq 1/\tau$ (spin diffusion), respectively.

Combining these results, we can distinguish two cases: if scaling is dominated by charge diffusion, the anomalous clean behavior [6] above $\omega \approx 1/\tau$ is cut off below to make place for an essentially non-interacting behavior (Hertz-
Millis theory with only irrelevant couplings) until \( \omega \approx 1/(\tau \sqrt{k_F \ell}) \), where the relevant disorder vertex \( \Delta_4 \) starts to dominate. Below \( \omega \approx 1/(\tau k_F \ell) \), infinitely many more disorder vertices \( \Delta_{2n} \) dominate the respective single-loop vertices.

In the case of scaling determined by spin diffusion, the clean anomalous behavior is modified below \( \omega \approx 1/\tau \) by a single disorder vertex \( \Delta_4 \), while the higher single-loop vertices \( g_{2n>4} \) are not yet strongly modified by disorder. Only below \( \omega \approx 1/(\tau k_F \ell) \) the single loops are cut off, but at the same time more disorder vertices \( \Delta_{2n>4} \) appear. In consequence, there is a direct crossover from clean anomalous to strongly disordered behavior which implies that the theory of Hertz and Millis may not be applicable for the two-dimensional antiferromagnet.

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A Non-linear sigma model

In this appendix we outline the steps to formulate the problem of disordered single loops in terms of the non-linear \( \sigma \) model for interacting disordered electrons [17,18,19,20] in order to support the perturbative calculations in this work and control all contributions with the leading singularity. Assuming some familiarity with the non-linear \( \sigma \) model itself, we introduce only the modifications necessary to accommodate the spin-vertex insertions. We start with noninteracting electrons in the presence of disorder, add the spin vertices as couplings to an external field, perform the disorder average using the replica method, integrate over the fermionic degrees of freedom and obtain an effective action for the \( Q \) matrices (in standard notation),

\[
S[Q, \tilde{\phi}] \simeq \int dr \left\{ \frac{\alpha_0}{8\pi} \text{Tr} \, Q^2 - \frac{1}{2} \text{Tr} \ln \left( G_0^{-1} + \frac{i}{2\pi} Q - \tilde{\phi} \right) \right\}
\]

where \( Q \) are matrices in frequency and replica space with a weak real-space dependence representing the electron-hole pairs while \( \phi(r) = \tilde{\phi}(r) \exp(i r q_c) \) is the staggered external field, with the matrix \( \tilde{\phi} \) slowly varying in space.\(^3\) The \( Q \) matrices can be expressed as a rotation \( Q = T^{-1} Q_{sp} T \) of the saddle-point solution \( Q_{sp} = \text{sgn}(\epsilon) \) of the classical action for \( \phi = 0 \). In the vicinity of the saddle point, one obtains

\[
S[Q, \tilde{\phi}] \simeq \int dr \left\{ D \text{Tr}(\nabla Q)^2 - 4 \text{Tr}(\epsilon Q) - \text{terms in } \tilde{\phi} \text{ and } Q \right\},
\]

\(^3\) For simplicity, we do not write explicitly the frequency dependence of \( \tilde{\phi} \).
where the first two terms are the standard non-linear $\sigma$ model for disordered electrons and additional terms are obtained by expanding the logarithm in $\phi$. One thus obtains vertices $(Q\tilde{\phi})^k$ which couple the diffusons to the external spin field.

We parametrize the $Q$ matrices as $Q = e^{W/2}Q_{sp}e^{-W/2}$, where the diffuson propagator $\langle W W \rangle$ is represented by the direct ladder $L$. There is a term $Q_{sp}\tilde{\phi}W\tilde{\phi}$ in the action which corresponds to the $R$ vertex in equation (3), and terms $W\tilde{\phi}W\tilde{\phi}$ and $W^2\tilde{\phi}^2$ (corresponding to the $X$ vertex) with two diffusons connected to two spin-vertex insertions. Higher vertices $(Q\tilde{\phi})^k$ with more than two spin insertions generate sub-leading contributions by the scaling arguments presented below.

For the leading singularity it suffices to consider all possible ways to connect $X$ and $R$ vertices via diffusons, with the possible inclusion of the Hikami vertices $H_k$ [11], which represent the interaction of the diffusons in the absence of $\tilde{\phi}$. Each vertex contributes a $\delta$ function of all momenta, while each ladder implies an integration over its momentum. Hence, the power counting depends only on the number $n_\delta$ of additional $\delta$ functions beyond the overall $\delta$ function of external momenta, which is the number of vertices minus one, $n_\delta = n_V - 1 = n_R + n_X - 1$. The Hikami vertices do not contribute to $n_\delta$ since they scale as an inverse ladder. The number $2n$ of spin insertions determines $n_R + n_X = n$, such that $[g_{2n}] = z - 2n_\delta = z - 2(n - 1)$ in accordance with our previous calculation. This maximal scaling dimension is valid for a large class of diagrams, two representatives of which are the chain-type diagram in Fig. 4 and the star-shaped diagram in Fig. 5.

The class of diagrams with leading singularity contains, however, contributions with a different relative importance measured in powers of $1/(k_F\ell)$. The dominant terms of $O(1/(k_F\ell)^{n-1})$ correspond to a single fermion line and no crossings of direct ladders, otherwise an additional factor $1/(k_F\ell)^l$ is generated according to the formula

$$l = \sum_{k>1} n_{H_{2k}} (k - 1) - \frac{n_R}{2} + 1$$

which is valid for any connected diagram with $n_{H_{2k}} \geq 0$ Hikami vertices $H_{2k}$ and $0 \leq n_R \leq n$ vertices $R$ as well as $n_X = n - n_R$ vertices $X$. For instance, Fig. 4 corresponds to $RLR$ and has $l = 0$ (with $n_{H_{2k}} = 0$ and $n_R = 2$), while an additional ladder crossing the first one corresponds to $Tr(XLXL)$ with a single closed fermion loop which has $l = 1$ (with $n_R = 0$).

If we include the spin propagator in the action and integrate over the diffusons, the purely spin-wave action is recovered with its infinite number of marginal $\tilde{\phi}$ vertices, while integrating over $\tilde{\phi}$ will likely lead to an action with an infinite number of marginal diffuson vertices. It appears that the action
written in terms of both $\tilde{\phi}$ and diffusons provides the simplest formulation of the interaction of the low-energy spin and diffusive modes, with only one (constant) coupling associated to the coupling term $\int dr \, \text{Tr}[Q \tilde{\phi} Q \tilde{\phi}]$.

\section*{B Crossed impurity diagrams}

In this appendix we consider the disorder corrections to the bosonic self-energy beyond the Born approximation due to maximally crossed ladders $L_c$. In contrast to the direct ladders they can have a diffusive contribution also between two propagators separated by an incommensurate $q_c$ (e.g., in the bosonic self-energy), and as the ladder momentum is integrated over, this typically results in a logarithm:

$$
\Sigma_{\text{crossed}}(i\omega, q_c + q) = -2g^2 \int \frac{d\epsilon}{(2\pi)^2} \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2p'}{(2\pi)^2} \frac{G^{\alpha}_{p+q}(\epsilon + \omega) G^{\alpha}_{p}(\epsilon)}{L_c(\omega, p + p') G^{\alpha}_{p' + q}(\epsilon + \omega) G^{\alpha}_{p'}(\epsilon)}
$$

$$(\omega \tau \ll 1) \quad \rightarrow -\frac{g^2}{v_F^2} \frac{1}{k_F \ell} |\omega| \left( \frac{4}{N\pi} - \ln(|\omega| \tau) \right). \quad (B.1)$$

While the above expression $\propto |\omega| \ln(|\omega| \tau)$ vanishes for $|\omega| \rightarrow 0$, it is logarithmically larger than the Landau damping term $\propto |\omega|$ and has the same sign. Thus, $\Sigma_{\text{crossed}}$ enhances the frequency-dependent part of the bosonic propagator while leaving the momentum-dependent part unchanged for small frequencies and momenta. Therefore, there may be a tendency to increase $z$ beyond $z = 2$. On the other hand, for $N = 8$ the term in parentheses becomes of $O(1)$, i.e., comparable with the direct ladder contribution, only if $|\omega| \lesssim 10^{-3} \tau$. As previously mentioned, in this work we discarded these contributions assuming that a small magnetic field would cut off this logarithmic singularity.

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