Levinson’s Theorem for Dirac Particles

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Abstract

Levinson’s theorem for Dirac particles constraints the sum of the phase shifts at threshold by the total number of bound states of the Dirac equation. Recently, a stronger version of Levinson’s theorem has been proven in which the value of the positive- and negative-energy phase shifts are separately constrained by the number of bound states of an appropriate set of Schrödinger-like equations. In this work we elaborate on these ideas and show that the stronger form of Levinson’s theorem relates the individual phase shifts directly to the number of bound states of the Dirac equation having an even or odd number of nodes. We use a mean-field approximation to Walecka’s scalar-vector model to illustrate this stronger form of Levinson’s theorem. We show that the assignment of bound states to a particular phase shift should be done, not on the basis of the sign of the bound-state energy, but rather, in terms of the nodal structure (even/odd number of nodes) of the bound state.

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I. INTRODUCTION

One of the most beautiful results in scattering theory was proven by Levinson in 1949 \cite{1}. Levinson’s theorem relates the number of bound states in a specific angular-momentum channel ($N_l$) to the value of the phase shift ($\delta_l$) at threshold

$$\delta_l(0) = N_l \pi . \quad (1)$$

In arriving at this form of Levinson’s theorem the modulo-$\pi$ ambiguity in the definition of the phase shift has been resolved by demanding that the phase shift approaches zero (and not only a multiple of $\pi$) in the high-energy limit. For simplicity, we have assumed, and will assume hereafter, the absence of zero-energy resonances.

Finding a generalization of Levinson’s theorem to Dirac particles, however, has proven to be a subtle task. Initially, it was believed that the generalization of Levinson’s theorem related the difference of positive- and negative-energy bound states to the corresponding difference of positive- and negative-energy phase shifts at threshold \cite{2}. It was also suggested that the modulo-$\pi$ ambiguity in the definition of the phase shift could be resolved, as in the nonrelativistic case, by allowing the phase shift to vanish in the high-energy ($E \to \pm \infty$) limit. These results, however, were later found to be incorrect and it was only until 1985 that a correct generalization of Levinson’s theorem to Dirac particles was found by Ma and Ni \cite{3}. They showed that the correct form of Levinson’s theorem for Dirac particles relates the sum (and not the difference) of the phase shifts at threshold to the total number of bound states \cite{3}

$$\delta_\kappa(E = +M) + \delta_\kappa(E = -M) = N_\kappa \pi . \quad (2)$$

In the above equation $\kappa$ is the generalized angular momentum channel ($j = |\kappa| - 1/2$) and $N_\kappa$ is the total number of (positive- plus negative-energy) bound states in that channel. In this form of Levinson’s theorem one implicitly assumes that the modulo-$\pi$ ambiguity in the definition of the phase shift has already been resolved. However, in contrast to the nonrelativistic case, where one is free to define the high-energy limit of the phase shift to be zero \cite{4,5}, the asymptotic value for the phase shift is given by \cite{6,7},

$$\lim_{E \to \pm \infty} \delta_\kappa(E) = \mp \int_0^\infty V(r) \, dr , \quad (3)$$

where $V(r)$ is the timelike component of a Lorentz-vector potential. In particular, this last relation implies

$$\delta_\kappa(E \to +\infty) + \delta_\kappa(E \to -\infty) = 0 , \quad (4)$$

and indicates that it is only the sum, but not the individual phase shifts, that vanishes in the high-energy limit. Yet, Eq. (3) establishes that the asymptotic behavior of the of the individual (positive- and negative-energy) phase shifts is known. In contrast, the behavior of the individual phase shifts at threshold is (as opposed to their sum) not constrained by Levinson’s theorem. Recently, however, Poliatzky was able to prove a stronger version of Levinson’s theorem by showing that the individual phase shifts at threshold are related to the number of bound states of an appropriate set of Schrödinger-like equations which
coincide with the Dirac equation in the limit of zero momentum [8]. In contrast to previous expectations [7], however, the phase shift at threshold with a particular sign of the energy does not coincide (in general) with the number of bound states of the Dirac equation having the same sign of the energy, i.e.,

\[ \delta_\kappa(E = \pm M) \neq N_\kappa^{(\pm)} \pi. \] (5)

As we will show later, this “naive” expectation is upset whenever the potential becomes sufficiently strong to cause the binding energy of certain states to exceed the rest mass of the particle.

In the present work we show, using some of the ideas developed by Poliatzky [8], that the behavior of the phase shift at threshold with a particular sign of the energy is directly related to the number of bound states of the Dirac equation having a specific nodal structure. We will establish that the assignment of bound states to a particular phase shift should be done, not on the basis of the sign of the bound-state energy but, instead, by determining if the combined number of nodes in the upper and lower components of the bound-state wavefunction is even or odd. As we shall see, this criterion is inextricably linked to the orthogonality and boundary conditions satisfied by the eigenstates of the Dirac equation.

Our paper has been organized as follows. In Sec. II we review most of the formalism needed to prove the stronger form of Levinson’s theorem. Particular emphasis is placed on the boundary conditions satisfied by eigenstates of the Dirac equation which are ultimately responsible for the nodal structure of the bound state. In Sec. III we illustrate Levinson’s theorem using a mean-field approximation to the Walecka model as an example. In particular, we show that the “naive” relation between the phase shift at threshold and the corresponding number of bound states with the same sign of the energy is upset whenever the potential becomes strong enough to generate binding energies that exceed the rest mass of the particle. Finally, in Sec. IV we offer our conclusions and speculate on future applications of Levinson’s theorem.

II. FORMALISM

In the presence of a spherically symmetric potential the eigenstates of the Dirac equation can be classified according to a generalized angular momentum \( \kappa \) and can be written in a two component representation

\[ \Psi_{E\kappa m}(\mathbf{r}) = \begin{bmatrix} \frac{g_{E\kappa}(r)}{r} Y_{+\kappa m}(\hat{\mathbf{r}}) \\ i \frac{f_{E\kappa}(r)}{r} Y_{-\k\kappa}(\hat{\mathbf{r}}) \end{bmatrix} \],

with the upper and lower components expressed in terms of spin-spherical harmonics

\[ Y_{\kappa m}(\hat{\mathbf{r}}) \equiv \langle \hat{\mathbf{r}} | l \frac{\hat{z}}{2} j m \rangle, \quad j = |\kappa| - \frac{1}{2}, \quad l = \begin{cases} \kappa & \text{if } \kappa > 0; \\ -(1 + \kappa) & \text{if } \kappa < 0. \end{cases} \] (7)

For a particle moving in the presence of Lorentz scalar (\( S \)) and time-like vector (\( V \)) potentials the Dirac equation
\[ \alpha \cdot \mathbf{P} + \beta M + \beta S(r) + V(r) \Psi_{E_{km}}(r) = E \Psi_{E_{km}}(r), \]  
(8)

reduces to a set of first order, coupled differential equations

\[ \left( \frac{d}{dr} + \frac{\kappa}{r} \right) g_{E\kappa}(r) = \left[ M^*(r) + E^*(r) \right] f_{E\kappa}(r), \]  
(9a)

\[ \left( \frac{d}{dr} - \frac{\kappa}{r} \right) f_{E\kappa}(r) = \left[ M^*(r) - E^*(r) \right] g_{E\kappa}(r), \]  
(9b)

written in terms of a local mass and a local energy defined by

\[ M^*(r) \equiv M + S(r), \quad E^*(r) \equiv E - V(r). \]  
(10)

It is a well known fact that the above set of coupled equations can be reduced to a single second-order differential equation identical in form to a Schrödinger equation. For example, one can use Eq. (9a) to express the lower component of the Dirac wavefunction in terms of the upper component

\[ f_{E\kappa}(r) = \left[ \frac{1}{M^*(r) + E^*(r)} \right] \left( \frac{d}{dr} + \frac{\kappa}{r} \right) g_{E\kappa}(r), \]  
(11a)

\[ g_{E\kappa}(r) = [\xi(r)]^{1/2} u_{E\kappa}(r) = \left[ \frac{M^*(r) + E^*(r)}{M + E} \right]^{1/2} u_{E\kappa}(r), \]  
(11b)

which upon substitution into Eq. (9b) yields the following Schrödinger-like equation for the (transformed) upper component

\[ \left[ \frac{d^2}{dr^2} + p^2 - \frac{\kappa(\kappa + 1)}{r^2} - U_{\text{eff}}(E, \kappa; r) \right] u_{E\kappa}(r) = 0; \quad \left( p^2 \equiv E^2 - M^2 \right). \]  
(12)

The transformation factor \([\xi(r)]^{1/2}\) relating the Dirac upper component \(g_{E\kappa}\) to the Schrödinger-like component \(u_{E\kappa}\) was introduced in order to remove all terms linear in the derivative of \(g_{E\kappa}\). The effective potential, which has now acquired an energy dependence due to the transformation,

\[ U_{\text{eff}}(E, \kappa; r) \equiv U_c(E; r) - \kappa U_{so}(E; r) + U_d(E; r), \]  
(13)

is defined in terms of central, spin-orbit, and Darwin contributions that are, respectively, given by

\[ U_c(E; r) = 2M \left[ \left( S(r) + \frac{E}{M} V(r) \right) + \left( \frac{S^2(r) - V^2(r)}{2M} \right) \right], \]  
(14a)

\[ U_{so}(E; r) = -\frac{1}{r} \left[ \frac{1}{\xi(r)} \frac{d\xi(r)}{dr} \right], \]  
(14b)

\[ U_d(E; r) = \frac{3}{4} \left[ \frac{1}{\xi(r)} \frac{d\xi(r)}{dr} \right]^2 - \frac{1}{2} \left[ \frac{1}{\xi(r)} \frac{d^2\xi(r)}{dr^2} \right]. \]  
(14c)
In particular, the energy dependence displayed by the (central) potential now makes evident the high-energy limit of the phase shifts [Eq. (3)] through the well known asymptotic behavior of the Schrödinger phase shift \[\lim_{E \to \pm \infty} \delta_\kappa(E) = -\frac{M_p}{p} \int_0^\infty V_{\text{eff}}(r) \, dr \to \pm \int_0^\infty V(r) \, dr .\] (15)

Thus, the asymptotic behavior of the phase shift can ultimately be traced to the Lorentz transformation properties of the vector potential. In addition, this relation indicates that the high-energy limit of the phase shift is unaffected by the presence of a Lorentz-scalar potential. Moreover, this finding contradicts the nonrelativistic notion that a distortion-free description of the projectile should be appropriate at large enough energies.

In reducing the original couple set of equations to an effective Schrödinger-like equation one could have, alternatively, chosen to eliminate the upper component of the Dirac wavefunction in favor of the lower component. This can be achieved, because of the structure of the equations [see Eq. (9)], by simply changing the sign of \(\kappa\) and of all timelike quantities (i.e., \(E \to -E\) and \(V \to -V\)). These modifications give rise to the following relations

\[
\begin{align*}
g_{E\kappa}(r) &= \left[\frac{1}{M^*(r) - E^*(r)}\right] \left(\frac{d}{dr} - \frac{\kappa}{r}\right) f_{E\kappa}(r) , \\
f_{E\kappa}(r) &= \left(\bar{\xi}(r)\right)^{1/2} v_{E\kappa}(r) = \left[\frac{M^*(r) - E^*(r)}{M - E}\right]^{1/2} v_{E\kappa}(r) ,
\end{align*}
\]

(16a, 16b)

and generate the corresponding Schrödinger-like equation for the (transformed) lower component

\[
\left[\frac{d^2}{dr^2} + p^2 - \bar{\kappa}(\bar{\kappa} + 1) \frac{1}{r^2} - \bar{U}_{\text{eff}}(E, \bar{\kappa}; r)\right] v_{E\kappa}(r) = 0 .
\]

(17)

In the above equation we have defined \(\bar{\kappa} \equiv -\kappa\) and introduced the effective potential \(\bar{U}_{\text{eff}}\) that is obtained from \(U_{\text{eff}}\) by changing the sign of all timelike quantities.

The system of equations Eqs. (11) and (12), or equivalently Eqs. (13) and (17), with the appropriate boundary conditions, are equivalent to the original set of coupled equations [Eq. (9)]. Unfortunately, the nonrelativistic form of Levinson’s theorem can not be applied to these equations. The difficulty stems from the fact that the theorem is not valid once the effective potential has acquired an energy dependence. Fortunately, Poliatzky has recently been able to show how to overcome this difficulty [8]. He has proven that for an equivalent set of Schrödinger-like equations,

\[
\begin{align*}
\left[\frac{d^2}{dr^2} + p^2 - \frac{\kappa(\kappa + 1)}{r^2} - U_{\text{eff}}(E = +M, \kappa; r)\right] \tilde{u}_{E\kappa}(r) &= 0 , \\
\left[\frac{d^2}{dr^2} + p^2 - \frac{\bar{\kappa}(\bar{\kappa} + 1)}{r^2} - \bar{U}_{\text{eff}}(E = -M, \bar{\kappa}; r)\right] \tilde{v}_{E\kappa}(r) &= 0 ,
\end{align*}
\]

(18a, 18b)

obtained from replacing the energy-dependence of the effective potential by its value at threshold, the nonrelativistic form of Levinson’s theorem can indeed be applied [8]. This
stronger form of Levinson’s theorem constrains, not only the sum of the phase shifts [Eq. (2),
but, in addition, the behavior of the individual phase shifts at threshold
\[ \delta_\kappa(E = +M) = \delta_l^{(+)}(p = 0) = n_l^{(+)} \pi, \]  
\[ \delta_\kappa(E = -M) = \delta_l^{(-)}(p = 0) = n_l^{(-)} \pi. \]  

In the above equations the orbital angular momentum \( l \) and \( \bar{l} \) are related, respectively, to \( \kappa \) and \( \bar{\kappa} \) through Eq. (7). In addition, \( n_l^{(+)} \) and \( n_l^{(-)} \) are the number of (nonrelativistic) bound states supported by the energy-independent potentials (\( U_{\text{eff}} \) and \( \bar{U}_{\text{eff}} \) respectively). The equality between phase shifts is a direct consequence of the equivalence of the Dirac equation to Eq. (18a) and Eq. (18b) at the corresponding thresholds. The relation to the number of bound states follows directly from applying the nonrelativistic form of Levinson’s theorem which is now valid because the potentials have become independent of energy.

We now show that the consistency of the above set of equations enables one to relate the value of the phase shifts at threshold to the number of bound states of the Dirac equation having, either, an even or odd number of nodes. Essential to the proof is the understanding of the asymptotic behavior (both as \( r \to 0 \) and as \( r \to \infty \)) of the ratio of lower to upper components of the Dirac wavefunction.

For small values of \( r \) the upper and lower components of the Dirac wavefunction can be, respectively, written as
\[ g_{E \kappa}(r) \simeq A \hat{j}_l(p_0^* r) ; \quad f_{E \kappa}(r) \simeq B \hat{j}_l(p_0^* r), \]  
where \( \hat{j}_l(z) \equiv z j_l(z) \) is the Ricatti-Bessel function (i.e., spherical Bessel function times its argument) [5]. The coefficients \( A, B \) and the wavenumber \( p_0^* \) are determined (after using the Dirac equation), by solving the secular equation
\[ \begin{vmatrix} p_0^* \text{sgn} \kappa & -M_0^* - E_0^* \\ -M_0^* + E_0^* & -p_0^* \text{sgn} \kappa \end{vmatrix} = 0, \]  
which results in
\[ p_0^* = \sqrt{E_0^{*2} - M_0^{*2}}, \]  

\[ \begin{pmatrix} B \\ A \end{pmatrix} = +\text{sgn} \kappa \left( \frac{p_0^*}{M_0^* + E_0^*} \right), \quad \text{or} \quad \begin{pmatrix} A \\ B \end{pmatrix} = -\text{sgn} \kappa \left( \frac{p_0^*}{M_0^* - E_0^*} \right), \]  

where we have defined \( \text{sgn} \kappa = \kappa/|\kappa|, M_0^* = M^*(r=0), \) and \( E_0^* = E^*(r=0). \)

The asymptotic (\( r \to \infty \)) behavior of the wavefunction can be obtained in a similar fashion. In this case, however, one must distinguish between scattering states (\( E > +M \) or \( E < -M \)) and bound states (\( -M < E < +M \)). For scattering states the asymptotic behavior is given by
\[ g_{E \kappa}(r) \simeq C \sin \left[ pr - \frac{l \pi}{2} + \delta_\kappa(E) \right] ; \quad f_{E \kappa}(r) \simeq D \sin \left[ pr - \frac{l \pi}{2} + \delta_\kappa(E) \right], \]  

\[ \begin{pmatrix} B \\ A \end{pmatrix} = +\text{sgn} \kappa \left( \frac{p_0^*}{M_0^* + E_0^*} \right), \quad \text{or} \quad \begin{pmatrix} A \\ B \end{pmatrix} = -\text{sgn} \kappa \left( \frac{p_0^*}{M_0^* - E_0^*} \right), \]  

where we have defined \( \text{sgn} \kappa = \kappa/|\kappa|, M_0^* = M^*(r=0), \) and \( E_0^* = E^*(r=0). \)
and yields the following relations,

\[ p = \sqrt{E^2 - M^2}, \tag{25} \]

\[ \left( \frac{D}{C} \right) = \text{sgn} \kappa \left( \frac{p}{E + M} \right), \quad \text{or} \quad \left( \frac{C}{D} \right) = \text{sgn} \kappa \left( \frac{p}{E - M} \right). \tag{26} \]

For bound states, on the other hand, the asymptotic behavior is given by

\[ g_{E \kappa}(r) \simeq C e^{-pr}; \quad f_{E \kappa}(r) \simeq D e^{-pr}, \tag{27} \]

which in turn implies

\[ p = \sqrt{M^2 - E^2}, \tag{28} \]

\[ \left( \frac{D}{C} \right) = - \left( \frac{p}{M + E} \right), \quad \text{or} \quad \left( \frac{C}{D} \right) = - \left( \frac{p}{M - E} \right). \tag{29} \]

In particular, this last relation establishes (because \(-M < E < +M\)) that the ratio of the two bound-state components is always negative irrespective of the value of \( \kappa \).

Having established the boundary conditions satisfied by the two components of the Dirac equation, we now turn to the set of equations that have determined the threshold behavior of the phase shifts. We start by inspecting Eq. (11b) which defined the relation between \( g_{E \kappa} \) and \( u_{E \kappa} \) and ultimately determined the value of \( \delta \kappa(E = +M) \). The fact that both \( g_{E \kappa} \) and \( u_{E \kappa} \) satisfy real differential equations with real boundary conditions \[ \text{[8]}, \]

demands that the transformation factor

\[ \xi(r) = \left[ M^*(r) + E^*(r) \right]/[M + E] \]

be positive for all values of \( r \) and in particular positive for \( r = 0 \). This implies that bound states of the Dirac equation satisfying this condition, and therefore being “counted” by \( \delta \kappa(E = +M) \), have the sign of the ratio of the two Dirac components at the origin determined solely by the sign of \( \kappa \), i.e., from Eq. (23) we obtain

\[ \text{sgn} \left[ \frac{f_{E \kappa}(r)}{g_{E \kappa}(r)} \right]_{r \to 0} = \text{sgn} \left( \frac{B}{A} \right) = \text{sgn} \kappa. \]

Since, in addition, the ratio of the two Dirac components is always negative at large separations \[ \text{[see Eq. (29)]}, \]

these relations constrain the combined number of nodes in the bound-state wavefunction (i.e., the number of nodes in \( g_{E \kappa} \) plus the number of nodes in \( f_{E \kappa} \)) to be even for \( \kappa < 0 \) and odd for \( \kappa > 0 \). In the case of \( \kappa > 0 \), for example, the ratio is positive near the origin and the lower component must change sign an odd number of times more than the upper component in order for the ratio to become negative at large separations.

Relating the behavior of \( \delta \kappa(E = -M) \) to the number of bound states with the complimentary nodal structure now follows from Eq. (16b). Again, the fact that both \( f_{E \kappa} \) and \( u_{E \kappa} \) satisfy real differential equations with real boundary conditions forces the transformation factor

\[ \bar{\xi}(r) = \left[ M^*(r) - E^*(r) \right]/[M - E] \]

to be positive for all values of \( r \). Consequently, the behavior of the ratio of Dirac wavefunctions at the origin is now determined, not by the sign of \( \kappa \) but, instead, by the sign of \( \bar{\kappa} = -\kappa \), i.e.,
Consequently, bound states of the Dirac equation satisfying the above relation, and hence being counted by $\delta_\kappa(E = -M)$, must have a combined number of nodes that is now even for $\kappa > 0$ and odd for $\kappa < 0$. Thus, the boundary conditions have proved essential in establishing the connection between the threshold behavior of the phase shifts and the number of bound states of the Dirac equation with a specific nodal structure.

We can now formulate the stronger version of Levinson's theorem for a Dirac particle as follows. If the modulo-$\pi$ ambiguity in the definition of the phase shifts is resolved by demanding the fulfillment of the high-energy relations

$$\lim_{E \to \pm \infty} \delta_\kappa(E) = \mp \int_0^\infty V(r) \, dr ,$$

and if $N^{(e)}_\kappa(N^{(o)}_\kappa)$ represents the number of bound states of the Dirac equation having an even(odd) number of nodes, then the phase shifts at threshold are given by

$$\delta_\kappa(E = \pm M) = \begin{cases} N^{(e)}_\kappa \pi & \text{if } \kappa < 0; \\ N^{(o)}_\kappa \pi & \text{if } \kappa > 0. \end{cases}$$

(31)

So far we have ignored the possible existence of zero-energy resonances. The presence of zero-energy resonances can only occur in channels having, either, $\kappa = -1$ or $\kappa = +1$ and their existence modify the above relations by the addition of a factor of $\pi/2$ in the value of $\delta_\kappa(E = +M)$ or $\delta_\kappa(E = -M)$ respectively [2,3,8].

Deeply rooted in the proof of Levinson's theorem is the completeness of eigenstates of a Hermitian Hamiltonian. The completeness property establishes that the appearance of a bound state must always be accompanied by the corresponding disappearance of a state in the continuum. Alternatively, the existence of an attractive potential does not alter the (infinite) number of states but simply "pulls down" some scattering states into the bound-state region [2,3]. The phase shift at threshold may, thus, be regarded as a bound-state "counter" that triggers every time a scattering state is pulled down into the bound-state region.

Examined against this background, and with the added constraint imposed by orthogonality, it may not be surprising that the phase shifts at the two different thresholds ($E = \pm M$) are related to the bound states of the Dirac equation with a complimentary nodal structure. As the potential becomes strong enough to support the existence of bound states, the phase shifts at threshold will "trigger" every time a scattering state is pull down into the bound-state region. States moving into the bound-state region from different ends of the continuum, however, must necessarily have a different nodal structure. This is a consequence of the orthogonality of states with different energies. For example, consider the first state (i.e., the most deeply bound) that has been pulled down from the negative-energy continuum and the corresponding one from the positive-energy continuum. These are the bound states that are expected to have the fewest number of nodes [see, e.g., Fig. 2]. These two states, however, can not be simultaneously free of nodes; otherwise they could not be orthogonal and still satisfy the asymptotic condition [Eq. (29)]. Therefore, all states that have been pulled down from the negative-energy continuum must have a complimentary
nodal structure to those states that have originated in the positive-energy continuum. Furthermore, this nodal structure is preserved even when the potential is strong enough to cause the binding energy of some of the states to exceed the rest mass of the particle. Therefore, it is the nodal structure (even or odd number of nodes) and not the energy (positive or negative) that determines the behavior of the phase shift at threshold.

Implicit in the above derivation of the stronger form of Levinson’s theorem is the assumption that the wavenumber in the interior ($\vec{p}_0^*$) is real. This fact conforms to the intuitive notion that a bound-state wavefunction oscillates in the interior and falls off exponentially in the exterior. In particular, this implies that $(M_0^* + E_0^*)$ and $(M_0^* - E_0^*)$ must have opposite signs; a fact that follows directly from Eq. (22), i.e.,

$$-\vec{p}_0^{*2} = (M_0^{*2} - E_0^{*2}) = (M_0^* + E_0^*)(M_0^* - E_0^*) < 0.$$  \hspace{1cm} (32)

Consequently, all bound states can be classified according to, either, $(M_0^* + E_0^*) > 0$ [those related to $\delta_\kappa(+M)$] or according to $(M_0^* - E_0^*) > 0$ [those related to $\delta_\kappa(-M)$].

There are, however, some very interesting bound states for which the interior wavenumber, as defined by Eq. (22), is purely imaginary. These bound states, which appear in the context of strong-coupling field theories, display exponential behavior, not only at large distances, but, in addition, in the (interior) region where one customarily observes oscillating behavior \[9,10\]. These shell-like states are driven by a strong scalar potential that generates a strong binding energy and, thus, a rapid variation of the wavefunction. The existence of nodes (which leads to an increase in the kinetic energy), of an angular momentum barrier, or of an additional vector potential are all factors that may destroy the shell-like nature of these states \[9\]. These node-free states are, thus, believed to exist only in channels with $\kappa = \pm 1$ and, as we show later, do not seem to violate the stronger form of Levinson’s theorem.

We now proceed to illustrate some of the ideas developed so far by means of some examples.

### III. RESULTS

To illustrate Levinson’s theorem we use a mean-field approximation to the Walecka model \[11,12\]. The Walecka model is a relativistic quantum field theory of nucleons interacting via the exchange of scalar and vector mesons. The ground-state of the system is obtained from solving self-consistently a set of mean-field (Hartree) equations \[13\]. The outcome of the calculation is a set of single-particle Dirac orbitals together with scalar and (timelike) vector mean fields. In Fig. 1 we show the self-consistently determined mean fields for $^{40}$Ca. The potentials are strong ($\sim M/2$) and opposite in sign in order to reproduce the weak-binding energy but strong spin-orbit splitting characteristic of single-particle nucleon (i.e., positive-energy) states.

We have used these spherically-symmetric potentials to generate all bound states of the Dirac equation having $\kappa = -1$. In Fig. 2 we display the single-particle spectrum generated from these mean-field potentials by scaling, both scalar and vector fields, by factors of 0.6, 1.0 (unscaled) and 1.4 respectively. The two numbers enclosed in brackets correspond, respectively, to the number of nodes in the upper $(g_{E\kappa})$ and lower $(f_{E\kappa})$ components of the bound-state wavefunction. The fact that the vector field, but not the scalar, changes sign
under charge conjugation leads to spectroscopy of negative-energy states that is driven, in contrast to the positive-energy states, by a very strong central attraction and weak spin-orbit splitting. In particular, for $\lambda = 1.4$, the state with the least number of odd nodes (and, thus, “originally” in the negative-energy continuum) has crossed into the positive-energy region. Figures 3, 4, and 5 display the energy dependence of the positive- and negative-energy phase shifts and illustrate the stronger form of Levinson’s theorem (see also Table I). These figures show how the phase shifts at threshold act as bound-state counters once the modulo-$\pi$ ambiguity in the definition of the phase shift has been resolved according to Eq. (3). For the first two cases ($\lambda=0.6, 1.0$) the potential (although quite strong) is not strong enough to bind any Dirac orbital by more than the rest mass of the nucleon. Hence, the classification of bound states can be done according to their nodal structure or, equivalently, according to the sign of the energy, i.e.,

\begin{align}
\delta_\kappa(E=+M) &= N^{(e)}_\kappa = N^{(+)}_\kappa, \\
\delta_\kappa(E=-M) &= N^{(o)}_\kappa = N^{(-)}_\kappa.
\end{align}

For $\lambda = 1.4$, however, the binding energy of one bound-state orbital (the one with one single node) exceeds the rest mass of the nucleon and it becomes inappropriate to count bound states according to the sign of the energy (see Table I), i.e.,

$$\delta_\kappa(E=-M) = N^{(o)}_\kappa \neq N^{(-)}_\kappa.$$ 

So far we have only studied bound-state orbitals that display (the traditional) oscillating behavior inside the region of the potential and exponential falloff in the exterior. We now turn to the very interesting case of shell-like bound states. For simplicity, we consider a relativistic Lorentz-scalar square well. The width ($c$) of the potential is fixed at $cM = 20$. The strength of the potential, on the other hand, is varied from $0 < |S_0|/M < 2$. Fig. 6 shows the $\kappa=-1$ single-particle spectrum along with the threshold behavior of the positive-energy phase shift as a function of the scalar strength. Bound states may be obtained (graphically) by finding the intersection of a circle of radius $R^2 = (M^2 - M'^2)c^2$ with tangent-like lines. In particular, this expression reduces, in the weak-coupling limit ($|S_0|/M << 1$), to the well-know nonrelativistic relation

$$R^2 = \left[M^2 - M'^2\right]c^2 \rightarrow 2M|S_0|c^2.$$ 

Hence the behavior of the spectrum is, at least for $|S_0|/M < 1$ easily understood. As the strength of the potential increases the radius $R$ increases accordingly and gives rise to the formation of bound states. All these states are characterized by oscillating behavior inside the region of the well and the appearance of any bound state is accompanied by a corresponding increase in the value of the phase shift at threshold. As $|S_0|/M = 1$ (and $M'^* = 0$) the radius attains its maximum value and leads to the maximum number of bound states that can be supported by this ($cM = 20$) potential. As the strength of the potential increases even further, however, the radius $R$ decreases leading to a weaker binding and ultimately to the disappearance of all but one bound state. This interesting dynamics is nicely reflected in the threshold behavior of the positive-energy phase shift. In fact, since for pure scalar potentials the binding energy never exceeds the rest mass of the particle, even the naive expectation for Levinson’s theorem [Eq. (33)] is satisfied.
The sole remaining bound state is characterized by having a negative effective mass inside the region of the potential and satisfies $0 < E < |M^\ast| = -M^\ast$. In particular, the effective wavenumber $p_0^\ast$, as defined in Eq. (22), is purely imaginary and leads, as shown in Fig. 4, to the formation of the shell-like state. Since the nodal structure of this state remains unchanged during the oscillating-like to exponential-like transition, the existence of this shell-like state is still properly accounted for by the value of the phase shift at threshold.

Notice that for this shell-like state the transformation factor $\xi(r)$, relating the upper component of the Dirac wavefunction to the corresponding Schrödinger-like wavefunction [Eq. (11b)], is purely imaginary in the interior and purely real in the exterior. This suggests that, although it should be possible to obtain this shell-like solution from solving the equivalent Schrödinger-like equation, one should exercise care in enforcing these very peculiar set of boundary conditions.

**IV. CONCLUSIONS**

Levinson’s theorem for Dirac particles relates the sum of the positive- and negative-energy phase shifts at threshold to the total number of bound states of the Dirac equation. Recently, Poliatzky was able to prove a stronger version of Levinson’s theorem in which the value of each individual phase shift at threshold is related to the number of bound states of a pair of Schrödinger-like equations which coincide with the Dirac equation at zero momentum. In this paper we have elaborated on some of these ideas and have shown that the stronger version of Levinson’s theorem relates the value of the Dirac phase shift at threshold to the number of bound states of the Dirac equation with a specific nodal structure (i.e., even or odd number of nodes).

We have shown that the simple picture of Levinson’s theorem developed in the nonrelativistic context and based on the completeness relation is preserved in the relativistic case. This picture suggests that the existence of an attractive potential does not alter the number of states but simply pulls down some scattering states into the bound-state region. The merit of Levinson’s theorem is to identify the phase shift at threshold as the bound-state counter.

We have illustrated the stronger form of Levinson’s theorem using a mean-field approximation to the Walecka model. We have explicitly shown that the “naive” nonrelativistic generalization of Levinson’s theorem, namely, $\delta_\kappa(E = \pm M) = N^{(\pm)}_\kappa \pi$, is violated as soon as the binding energy exceeds the rest mass of the particle. The nodal structure of the bound states, however, is robust and remains unchanged even when the qualitative structure of the bound state is modified (e.g., shell-like solutions).

One essential feature that must be addressed in the study of Levinson’s theorem for Dirac particles is the asymptotic behavior of the phase shift [Eq. (3)]. The fact that the phase shift remains finite even at very large energies is a relativistic effect that might lead to interesting consequences. For example, many coincidence $(e, e'p)$ experiments have already been proposed at CEBAF in the hope of identifying novel behavior in the propagation of nucleons through the nuclear medium (color transparency). To date, most relativistic analyses of the $(e, e'p)$ reaction employ an ejectile wavefunction obtained from solving a Dirac equation having Lorentz scalar and (timelike) vector potentials. Because most of the analyses conducted so far have been restricted to low-energy ejectiles, the consequences
of the high-energy behavior of the phase shift have not yet been fully explored. As the energy of the ejectile increases, as seems to be required for the onset of color transparency, it will become essential to understand the relativistic effects associated with the high-energy propagation of the ejectile.

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FIGURES

FIG. 1. Self-consistent scalar and vector potentials for \(^{40}\)Ca obtained from a mean-field approximation to the Walecka model.

FIG. 2. Single-particle spectrum for the \(\kappa = -1\) channel in \(^{40}\)Ca for different values of the scaling parameter \(\lambda\).

FIG. 3. Positive- and negative-energy \((\kappa = -1)\) phase shifts as a function of energy for \(\lambda = 0.6\). The dashes represent the asymptotic values of the phase shift (i.e., ± the integral of the vector potential).

FIG. 4. Positive- and negative-energy \((\kappa = -1)\) phase shifts as a function of energy for \(\lambda = 1.0\). The dashes represent the asymptotic values of the phase shift (i.e., ± the integral of the vector potential).

FIG. 5. Positive- and negative-energy \((\kappa = -1)\) phase shifts as a function of energy for \(\lambda = 1.4\). The dashes represent the asymptotic values of the phase shift (i.e., ± the integral of the vector potential).

FIG. 6. Single-particle spectrum and threshold behavior of the positive-energy phase shift as a function of the strength of the scalar square-well potential \((\kappa = -1)\).

FIG. 7. Upper \((g)\) and lower \((f)\) components of the lowest energy \(\kappa = -1\) bound state for three different values of the strength of the scalar square-well potential.
TABLE I. Illustration of Levinson’s theorem for $\kappa = -1$ using a mean-field approximation to the Walecka model for $^{40}$Ca.

| $\lambda$ | $\delta^\kappa_\kappa (+M)/\pi$ | $N^{(e)}_\kappa$ | $N^{(+)}_\kappa$ | $\delta^\kappa_\kappa (-M)/\pi$ | $N^{(o)}_\kappa$ | $N^{(-)}_\kappa$ |
|----------|---------------------------------|------------------|------------------|-------------------------------|-----------------|-----------------|
| 0.20     | 1                               | 1                | 1                | 3                             | 3               | 3               |
| 0.60     | 2                               | 2                | 2                | 6                             | 6               | 6               |
| 1.00     | 2                               | 2                | 2                | 8                             | 8               | 8               |
| 1.40     | 2                               | 2                | 3                | 10                            | 10              | 9               |