Abstract

We apply the coadjoint orbit method to construct relativistic nonlinear sigma models (NLSM) on the target space of coadjoint orbits coupled with the Chern-Simons (CS) gauge field and study self-dual solitons. When the target space is given by Hermitian symmetric space (HSS), we find that the system admits self-dual solitons whose energy is Bogomol’nyi bounded from below by a topological charge. The Bogomol’nyi potential on the Hermitian symmetric space is obtained in the case when the maximal torus subgroup is gauged, and the self-dual equation in the $CP(N-1)$ case is explored. We also discuss the self-dual solitons in the non-compact $SU(1,1)$ case and present a detailed analysis for the rotationally symmetric solutions.

1. INTRODUCTION

Recently, a coadjoint orbit method to formulate the nonlinear sigma model defined on the target space of homogeneous space $G/H$ was proposed [1]. It was first applied to a non-relativistic spin system whose Poisson bracket between the dynamical variables defined on the coadjoint orbit satisfies the classical $G$ algebra. The Euler-Lagrange equation of motion yields the generalized continuous Heisenberg ferromagnet [2, 3]. When the target space of coadjoint orbit is given by HSS which is a symmetric space equipped with complex structure [4], the generalized ferromagnet system becomes completely integrable in 1+1 dimension [4]. Later, this method was exploited to produce a class of integrable extension of relativistic NLSM in 1+1 dimension [5]. It was also discovered that incorporation the CS gauge field in 2+1 dimension on the same target space produces a class of self-dual field theories which admit Bogomol’nyi self-dual equations saturating the energy functional [6]. A detailed numerical investigation in the compact $SU(2)$ case [6] showed a rich structure of self-dual solitons in the system.

In this paper, we apply the coadjoint orbit method to construct relativistic NLSM on the target space of coadjoint orbits coupled with the CS gauge field and study self-dual solitons. When the target space is HSS, the Hamiltonian is bounded from below by a topological charge, and the resulting self-dual CS solitons satisfy a vortex-type equation, thus producing a class of new self-dual theories on HSS. This construction
provides a unified framework for treating the previous gauged $O(3)$ model on $S^2$ and $CP(N-1)$ models which are well known examples of the coadjoint orbit $G/H$ with $S^2 = SO(3)/SO(2) \approx SU(2)/U(1)$ and $CP(N-1) = SU(N)/SU(N-1) \times U(1)$. We also study the self-dual solitons in the non-compact HSS with $SU(1,1)$ group in which the target space is given by the upper sheeted hyperboloid and find various topological and nontopological solitons.

We first give a brief summary of NLSM on the target space of coadjoint orbit for completeness. Consider a group $G$, Lie algebra $\mathfrak{g}$ and its dual $G^*$:

$$X \in G; \quad u \in G^*.$$  

The adjoint action of $G$ on the Lie algebra is defined by

$$\text{Ad}(g)X = gXg^{-1}, \quad g \in G. \quad (1.1)$$

Denoting inner product between $G$ and $G^*$ by $<u,X>$, the coadjoint action of the group on $G^*$ is defined in such a way to make the inner product invariant:

$$<\text{Ad}^*(g)u,X> = <u,\text{Ad}(g^{-1})X>. \quad (1.2)$$

The coadjoint orbit is given by the orbit of coadjoint action of the group $G$:

$$O_u = \{x|x = \text{Ad}^*(g)u, g \in G\}. \quad (1.3)$$

It can be shown that $O_u \approx G/H$, where $H$ is the stabilizer of the point $u$.

Let us assume that the inner product is given by the trace: $<u,X> = \text{Tr}(Xu)$. Then, $G$ and $G^*$ are isomorphic and the coadjoint orbit can be parameterized by

$$Q = gKg^{-1} = Q\, t^B \eta_{AB}; \quad t^A, K \in G \quad (A = 1, \cdots, \text{dim} \, G), \quad (1.4)$$

where $\eta_{AB}$ is the $G$-invariant metric given by $\text{Tr}(t^At^B) = -\frac{1}{2} \eta^{AB}$ with $t^A$'s being the generator of $G$. The action for the NLSM on the target space of coadjoint orbit can be constructed as

$$S(g) = \epsilon \text{Tr} \int d^3x \partial_\mu Q \partial^\mu Q. \quad (1.5)$$

$\epsilon = +1$ for the compact case $-1$ for the non-compact case. Let us first choose the element $K$ to be the central element of the Cartan subalgebra of $G$ whose centralizer in $G$ is $H$. Then, for the HSS, we have $J = \text{Ad}(K)$ acting on the coset is a linear map satisfying the complex structure condition $J^2 = -1$, which gives the useful identity:

$$[Q,[Q,\partial_\mu Q]] = -\partial_\mu Q. \quad (1.6)$$

This paper is organized as follows: In Section 2, starting from a CS gauged action of (1.5) on arbitrary HSS, we derive self-dual equations and Bogomol’nyi potential. We give explicit expressions in $CP(N-1)$ case. In Section 3, we deal with non-compact minimal $SU(1,1)$ model and discuss rotationally symmetric solutions in detail. In Section 4, we give the conclusion.

2. COMPACT MODEL

Let us consider the following CS gauged action of (1.5):

$$S_G = \int d^3x \left[ -\frac{1}{2} \left( D_\mu Q^A D^\mu Q^B \eta_{AB} \right) - W_G(Q^A) - \kappa \epsilon \mu \nu \rho \text{Tr} \left( \partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \right]. \quad (2.1)$$
where the covariant derivative is defined by
\[ D_\mu Q = \partial_\mu Q + [A_\mu, Q], \quad A_\mu = A^A_{\mu B} \eta_{AB}. \] (2.2)

We assume that the potential is given by
\[ W_G(Q^A) = \frac{1}{2} I^{AB} Q^A Q^B, \] (2.3)
where \( I^{AB} \) is the symmetric tensor and its content will be determined by the self-duality condition. The equations of motion are given by
\[ D_\mu [Q, D^\mu Q] + [\bar{Q}, Q] = 0, \quad (\bar{Q} = I^{AB} Q^A t^B). \] (2.4)

We first treat the compact case with \( \eta_{AB} = \delta_{AB} \). To study self-dual solitons, we bring the energy functional into Bogomol’nyi expression:
\[ E_G = \int d^2 x \left[ \frac{1}{2} (D_0 Q^A)^2 + \frac{1}{4} (D_i Q^A)^2 + W(Q^A) \right] \]
\[ = \int d^2 x \left[ \frac{1}{2} (D_0 Q^A)^2 + \frac{1}{4} (D_i Q^A + \epsilon_{ij} [Q, D_j Q]^A)^2 \right] \]
\[ + W(Q^A) \pm \frac{1}{2} \epsilon_{ij} F_{ij}^A Q^A \right] \pm 4\pi T_G, \] (2.5)
where the topological charge \( T_G \) is given by
\[ T_G = \frac{1}{8\pi} \int d^2 x [\epsilon_{ij} Q^A [\partial_i Q, \partial_j Q]^A - 2\epsilon_{ij} \partial_i (Q^A A_j^A)]. \] (2.6)

In deriving (2.5), we used the gauged version of (1.3) where \( \partial_\mu \) is replaced by the covariant derivative \( D_\mu \). Thus, the Hamiltonian is bounded from below by the topological charge \( T_G \) when the potential \( W_G \) is chosen such that
\[ W_G \pm \frac{1}{2} \epsilon_{ij} F_{ij}^A Q^A = 0. \] (2.7)

Here, \( F_{ij}^A \) is determined in terms of \( Q^A \) by the Gauss’s law which is the time-component of (2.4).

The minimum energy arises when the self-duality equation is satisfied:
\[ D_i Q = \mp \epsilon_{ij} [Q, D_j Q]. \] (2.8)
Consistency with the static equations of motion (2.4) forces
\[ F_{ij} = 0, \quad A_0 = \pm \frac{1}{\kappa} Q, \] (2.9)
which in turn puts the potential \( W_G = 0 \) and \( I^{AB} = 0 \). Note that the gauge field can be chosen as a pure gauge in this case and the contents of the Bogomol’nyi solitons are precisely the two dimensional instantons which were completely classified on each HSS [10].

More interesting cases in which the system offers other solitons arise when we gauge the subgroup \( H \). We consider gauging the maximal torus subgroup of \( G \):
\[ S_H = \int d^3 x \left[ -\frac{1}{2} \left( D_\mu Q^A D^\mu Q^A \right) - W_H(Q^A) + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\mu A^a_\nu A^a_\rho \right]. \] (2.10)
Here, the index \( a = 1, \ldots, \) rank \( G \) denotes the maximal Abelian subgroup. Again, the content of the potential \( W_H \) will be determined from the self-duality condition.

Using the Gauss’s law given by

\[
\frac{\kappa}{2} \epsilon_{ij} F^a_{ij} = -[Q, D_0 Q]^a;
\]  

(2.11)

we find that the energy functional satisfies

\[
E_H = \frac{1}{2} \int d^2 x \left[ \left( D_0 Q^A \pm \frac{1}{\kappa} [Q, Q_H]^A \right)^2 + \frac{1}{2} (D_i Q^A \pm \epsilon_{ij} [Q, D_j Q]^A)^2 \right] \pm 4\pi T_H,
\]

\[
T_H = \frac{1}{8\pi} \int d^2 x \epsilon_{ij} Q^A [\partial_i Q, \partial_j Q]^A - 2 \epsilon_{ij} \partial_i (Q_H^A A^a_j),
\]

(2.12)

when the Bogomol’nyi potential \( W_H \) is chosen as

\[
W_H = \frac{1}{2\kappa^2} ([Q_H, Q]^A)^2, \quad Q_H \equiv Q_H^a t^a = (Q^a - V^a)t^a.
\]

(2.13)

Note that \( V^a \)'s are free parameters associated with the vacuum symmetry breaking \( \Box \).

When the self-duality equations

\[
D_i Q^A = \mp \epsilon_{ij} [Q, D_j Q]^A, \quad D_0 Q^A = \mp \frac{1}{\kappa} [Q, Q_H]^A,
\]

(2.14)

are satisfied, we see that the energy is saturated by the topological charge:

\[
E_H = 4\pi |T_H|.
\]

(2.15)

The first order equation \( (2.14) \) in the static case fixes \( A_0^a \) to be

\[
A_0^a = \pm \frac{1}{\kappa} Q_H^a,
\]

(2.16)

which automatically solves the Euler-Lagrange equations of motion of the action \( (2.10) \) with the potential given by \( (2.13) \).

Let us examine \( (2.11) \) and \( (2.14) \) more closely in \( CP(N-1) \) case. We use the expression of \( Q \) \( [1] \),

\[
Q = i \Psi \Psi^\dagger - i \frac{I}{N}
\]

(2.17)

where the column vector \( \Psi \) can be expressed by the Fubini-Study coordinate \( \psi_a \)'s \( (a = 1, 2, \ldots, N-1) \):

\[
\Psi = \frac{1}{\sqrt{1 + |\psi|^2}} \begin{pmatrix} 1 \\ \psi_1 \\ \vdots \\ \psi_{N-1} \end{pmatrix},
\]

(2.18)

with \( |\psi|^2 = |\psi_1|^2 + \cdots + |\psi_{N-1}|^2 \). Using the complex notation; \( z = x + iy, \bar{z} = x - iy, \)

\( A_z = \frac{1}{2}(A_1 - i A_2), A_{\bar{z}} = \frac{1}{2}(A_1 + i A_2), \) and \( D_z = \frac{1}{2}(D_1 - i D_2), D_{\bar{z}} = \frac{1}{2}(D_1 + i D_2), \) we obtain an alternative expression of the self-duality equation,

\[
D_z Q = \mp i [Q, D_z Q].
\]

(2.19)

With the above parameterization of \( Q \), the self-duality equation \( (2.19) \) for the plus sign becomes a set of \( N - 1 \) equations \( [6] \):

\[
D^a_z \equiv \partial_z + i \frac{1}{2} (A_2 + \frac{1}{\sqrt{3}} A_2^2 + \cdots + \frac{2}{a(a-1)} A_{a-1}^a + \frac{2(a+1)}{a} A_a^a),
\]

(2.20)
Similarly, for the minus sign, we have
\[ D^a_a \equiv \partial_z - \frac{i}{2} A_z^1 + \frac{1}{\sqrt{3}} A_z^2 + \cdots + \sqrt{\frac{2}{a(a-1)}} A_z^{a-1} + \sqrt{\frac{2(a+1)}{a}} A_z^a, \]  
\[ D^a_+ \psi_a = 0. \]  

(2.23)

We concentrate on the plus sign from here on. With \( \bar{\psi}_a = w_a \exp(i \phi_a) \), we find that (2.11), (2.14) and (2.21) produce the following new vortex-type equation:
\[ \nabla^2 \log w_a + \epsilon_{ij} \partial_i \partial_j \phi_a = \sum_{i=1}^{a-1} \sqrt{\frac{1}{2i(i+1)}} \Gamma^i + \sqrt{\frac{a+1}{2a}} \Gamma^a, \]  
where \( \Gamma^a \) is given by
\[ \Gamma^a = \frac{V^b - Q^b}{\kappa^2} \left[ \frac{2}{N} \delta^{ab} + d^{abc} Q^c - Q^a Q^b \right]. \]  

(2.24)

We used the normalization: \( \{ \lambda^A, \lambda^B \} = \left( \frac{4}{N} \right) \delta^{AB} I + 2d^{ABC} \lambda^C \). Also the Bogomol’nyi potential (2.13) can be expressed by
\[ W_H = \frac{1}{2\kappa^2} (V^a - Q^a)(V^b - Q^b) \left[ \frac{2}{N} \delta^{ab} + d^{abc} Q^c - Q^a Q^b \right]. \]  

(2.25)

(2.26)

Let us give an example in the case of \( CP(1) \). With \( w_1 = w, \phi_1 = \phi, V^1 = V \), the above potential becomes
\[ W_H = \frac{1}{2\kappa^2} (V - Q^3)^2 (1 - (Q^3)^2), \]  
which is exactly the same as the potential in the \( O(3) \) model \[ \[\]. \]  
Next, we find that (2.24) becomes
\[ \nabla^2 \log w + \epsilon_{ij} \partial_i \partial_j \phi = \frac{1}{\kappa^2} \left[ V - \frac{1 - w^2}{1 + w^2} \right] \left[ 1 - \left( \frac{1 - w^2}{1 + w^2} \right)^2 \right] \]  

(2.28)

A detailed numerical study of the above equation showed that the equation has various kinds of rotationally symmetric solitons solutions connected with symmetric and broken phases, and they are anyons carrying fractional angular momentum \[ \[\]. \]  
Similar results are expected in the more complicated higher \( CP(N) \) case, but a detailed study will be addressed elsewhere.

3. NON-COMPACT \( SU(1,1) \) SOLITON

In this section, we consider a non-compact HSS with \( \epsilon = -1 \). We restrict to the \( SU(1,1) \) group with \( \eta_{AB} = (-, -, +) \). The target space is given by the two-sheeted hyperboloid \( H = SU(1,1)/U(1) \). Using the expression for the group element \( g \) of (1.4) given by
\[ g = \frac{1}{\sqrt{1 - |\psi|^2}} \begin{pmatrix} 1 & \psi^* \\ \psi & 1 \end{pmatrix}, \]  

(3.1)

which satisfies \( gMg^\dagger = M \) with \( M = \text{diag}(1, -1) \), we have (with \( K = i\sigma^3/2 \))
\[ Q = \frac{i}{2(1 - |\psi|^2)} \begin{pmatrix} 1 + |\psi|^2 & -2\psi^* \\ 2\psi & -(1 + |\psi|^2) \end{pmatrix}. \]  

(3.2)
We restrict to $|\psi| < 1$, which corresponds to the upper sheet of $\mathcal{M} = SU(1,1)/U(1)$. A couple of remarks at this point concerning the ungauged case are in order. First, some soliton solutions associated with non-compact NLSM were discussed in connection with the Ernst equation [12], which are not self-dual. Secondly, using the above expression, one can check that there actually exist self-dual soliton solutions which are analytic or anti-analytic as in the compact case [13], but the energy and topological charge diverge at the boundary $|\psi| = 1$. Coupling with CS gauge field greatly improves the situation, because the gauge field effectively provides a potential barrier to the boundary (see (3.7)) and prevents the system from diverging.

Again, with the parameterization $\psi = w \exp(i\phi)$, we find the Bogomol’nyi potential (2.13) and the self-dual equation (2.24) are produced as follows:

$$W_H = \frac{1}{2\kappa^2} \left[ V - \left( \frac{1 + w^2}{1 - w^2} \right)^2 \left( \frac{1 + w^2}{1 - w^2} \right) - 1 \right],$$

$$\nabla^2 \log w + \epsilon_{ij} \partial_i \partial_j \phi = \frac{1}{\kappa^2} \left[ V - \frac{1 + w^2}{1 - w^2} \right] \left( \frac{1 + w^2}{1 - w^2} \right) - 1.$$ (3.3)

$$\nabla^2 \log w + \epsilon_{ij} \partial_i \partial_j \phi = \frac{1}{\kappa^2} \left[ V - \frac{1 + w^2}{1 - w^2} \right] \left( \frac{1 + w^2}{1 - w^2} \right) - 1.$$ (3.4)

Let us look for the rotationally symmetric solutions with the ansatz in the cylindrical coordinate $(r, \theta)$ given by

$$w = \tanh \frac{f(r)}{2}, \quad \phi = n\theta, \quad A_i = \frac{\epsilon_{ij} x_j}{r^2} a(r).$$ (3.5)

Then, the Gauss’s law and self-dual equation become ($' = d/dr$)

$$a'(r) = \left( \frac{r}{k^2} \right)(-V + \cosh f(r))(1 - \cosh^2 f(r)), \quad \rho f'(r) = (a(r) - n) \sinh f(r).$$ (3.6)

Now, the combined equation of motion in (3.4) becomes an analogue of the one dimensional Newton’s equation for $r > 0$, if we regard $r$ as “time” and $u(r) \equiv \log \tanh \frac{f(r)}{2}$ as the “position” of the hypothetical particle with unit mass under a time-dependent friction, $(1/r)u'$, and an effective potential $V_{eff}$:

$$V_{eff}(u) = \frac{1}{2\kappa^2} \coth^2 u + \frac{V}{\kappa^2} \coth u.$$ (3.7)

The exerting force also includes an impact term at $r = 0$ due to $\epsilon^{ij} \partial_i \partial_j \phi = \frac{2}{r} \delta(r)$ in (3.7).

The inspection of the effective potential suggests that solitons are basically of two types; the non-topological vortices with $n \neq 0$ (negative integer) and the non-topological solitons with $n = 0$. In the former case, the “particle” starts from $u = -\infty$, reaches a turning point where it stops, changes the direction, and finally rolls down to $u = -\infty$. In the latter, the “particle” starting at some finite position, either rolls down to $u = -\infty$ directly, or moves to a turning point, changes the direction, and rolls down to $u = -\infty$. Let us look at the solutions more closely. Near $r = 0$, the condition for $A_i$ to be non-singular forces $a(0) = 0$. First, when $n \neq 0$, we must have $f(0) = 0$. When $n = 0$, $\alpha \equiv f(0)$ can be arbitrary. The behavior of the solution near $r = \infty$ can be also read off from the conditions $f'(\infty) = a'(\infty) = 0, \beta \equiv f(\infty) = 0$ for arbitrary $\gamma \equiv a(\infty)$ and $V$. Putting $f(r) = f_\infty r^l, \quad a(r) = \gamma + a_\infty r^s (l, s < 0)$ near $r = \infty$, we find $l = \gamma - n, s = 2\gamma + 2 - 2n$ for $V \neq 1$. Since $l, s < 0$, we have consistency condition $\gamma < n - 1$. When $V = 1$, we have $l = \gamma - n, s = 4\gamma + 2 - 4n$ and $\gamma < n - \frac{1}{2}$. When
\[ \beta = \cosh^{-1} V \] for \( V > 1 \), \( \gamma \) must be equal to \( n (\neq 0) \). This solution which will show oscillatory behavior before it comes to rest does not exist. Near \( r = \infty \), we assume an exponential approach \( f(r) = \beta + f_{\infty} e^{-ar}, \quad a(r) = \gamma + \alpha_{\infty} r^s e^{-br} \) \( (a, b > 0) \). Then substitution leads to a contradictory output, \( l = s \) and \( l = s + 1 \). Power law approach with \( a, b = 0 \) and \( l, s < 0 \) also leads to a contradiction. In view of the Bogomol'nyi potential \( (3.3) \), this excludes any solitons in the broken vacuum with \( V = \cosh f(\infty) \), and all the solitons are in the symmetric phases.

Let us focus on the vicinity of \( r = 0 \). (a) \( V \leq 1 \), in which the effective potential \( (3.7) \) is a monotonically decreasing function. (a-i) \( n \neq 0 \); Trying power solutions of the form \( f(r) = f_0 \alpha^p, \quad a(r) = a_0 \alpha^q \) \( (p, q > 0) \), we find \( p = -n, \quad q = 2 - 2n \) for \( V < 1 \). Hence \( n \) must be a negative integer. When \( V = 1 \), we have \( p = -n, \quad q = 2 - 4n \). (a-ii) \( n = 0, \alpha \neq 0 \); Let us try \( f(r) = \alpha + f_0 \alpha^p, \quad a(r) = a_0 \alpha^q \) \( (p, q > 0) \). We find \( p = q = 2 \). We note that both \( a_0 \) and \( f_0 \) turns out to be negative, so that the solution rolls down to \( r = \infty \). Climbing up at first and then rolling down the hill solution does not exist. When (b) \( V > 1 \), the effective potential \( (3.7) \) develops a pool with a local minimum at \( f_m = \cosh^{-1} V \). (b-i) \( n \neq 0 \); The behavior is similar to (a-i) except the fact \( a(r) \) passes the minimum at \( r_m \) twice in the process of climbing up, passing the turning point, and rolling down the hill to its original position. (b-ii) \( n = 0, \alpha \neq 0 \); There are two cases. When \( \alpha < \cosh^{-1} V \), the solution, if exists, will behave similarly with (b-i) except that it starts at some finite point \( \alpha \). However, it cannot exist for the following reason; The initial “velocity” of the particle is given by \( u'(0) \propto f'(0) = 0 (f'(r) \propto r \text{ from (a-ii)}) \). Hence the particle does not carry enough kinetic energy to return to its starting point in this dissipative system with conservative potential. Note that when \( n \neq 0 \), even though the initial velocity is in general equal to 0 except \( n = -1 \), the solutions are possible because of the impact term at \( r = 0 \). In the opposite case \( \alpha > \cosh^{-1} V \), it is similar to (a-ii) and only rolling down the hill is permitted. A detailed numerical study given in Figs. 1 and 2 indeed confirms the existence of these solitons.

Note that there does not exist any topological lump solutions, because \( \pi_2(\mathcal{M}) = 0 \). And the topological vortices does not exist, because there is no bump in the effective potential where the particle can stop at the top. In the solutions, the magnetic flux is given by \( \Phi = 2\pi \gamma \), and the energy is saturated by the topological charge; \( E = 4\pi |T| = 2\pi |\gamma(1 - V)| \). The system also carry non-vanishing angular momentum. Let us define

\[ J = \int d^2x \epsilon_{ij} x_i D_0 Q^A D_j Q^B \eta_{AB}. \]  

A simple calculation using the Gauss's Law \( (2.11) \) (with plus sign in the right hand side due to the \( \epsilon \) factor), and self-dual equations \( (2.14) \) and \( (2.6) \), we find \( J = \pi \kappa (|\gamma - n|^2 - n^2) \). Thus the solitons in general carry a fractional angular momentum, representing anyons. For the non-topological solitons, it is simply \( J = \pi \kappa \gamma^2 \).

4. CONCLUSION

We showed that the coadjoint orbit approach for the relativistic NLSM coupled with CS gauge field leads to a class of new self-dual field theories on the target space of HSS which contain the previous \( O(3) \) and \( CP(N - 1) \) models, and a new non-compact \( SU(1, 1) \) model. We also found an explicit expression of the Bogomol'nyi potential when the maximal torus subgroup is gauged, and showed that the non-compact NLSM admits self-dual soliton solutions which are saturated by the Bogomol'nyi bound, and gave a complete description of the rotationally symmetric solutions.

There remains several further issues to be discussed. Firstly, note that the identity \( (1.6) \) and its gauged version on HSS is essential for the existence of self-duality. In
this respect, it would be an intriguing problem to extend the above formalism to other non HSS coadjoint orbits, and also to higher non-compact group. Quantization of the model is another problem to be addressed. Secondly, it would be interesting to see whether there exists a well-defined procedure in which the non-relativistic NLSM of the generalized CS Heisenberg ferromagnet system defined on the coadjoint orbits \cite{oh87a} could emerge as a non-relativistic limit of the present relativistic NLSM. This will require in the course a revelation of the connection between the symplectic structure of HSS \cite{oh87a} for the non-relativistic NLSM and phase space structure of relativistic NLSM.

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References

[1] P. Oh and Q-H. Park, Phys. Lett. B 383, 333 (1996).
[2] M. Lakshmanan, Phys. Lett. A 61, 53 (1977); L. A. Takhtajan, Phys. Lett. A 64, 235 (1977); J. Tjon and J. Wright, Phys. Rev. B 15, 3470 (1977).
[3] L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer-Verlag, Berlin, 1987).
[4] A. P. Fordy and P. P. Kulish, Commun. Math. Phys. 89, 427 (1983).
[5] P. Oh, J. Phys A: Math. Gen. 31, L325 (1998).
[6] P. Oh and Q-H. Park, Phys. Lett. B 400, 157 (1997); (E) 416, 452 (1998).
[7] Y. Kim, P. Oh, and C. Rim, Mod. Phys. Lett. A 12, 3169 (1997).
[8] G. Nardelli, Phys. Rev. Lett. 73, 2524 (1994); B. J. Schroers, Phys. Lett. B 356, 291 (1995); J. Gladikowski, B. M. A. G. Piete and B. J. Schroers, Phys. Rev. D 53, 844 (1996); K. Kimm, K. Lee and T. Lee, Phys. Rev. D 53, 4436 (1996); K. Arthur, D. H. Tchrakian and Y. Yang, Phys. Rev. D 54, 5245 (1996); P. K. Ghosh and S. K. Ghosh, Phys. Lett. B 366, 199 (1996); Y. M. Cho and K. Kimm, Phys. Rev. D 52, 7325 (1995); K. Kimm, K. Lee and T. Lee, Phys. Lett. B 380, 303 (1996); P. K. Ghosh, Phys. Lett. B 384, 185 (1996).
[9] We will only consider the simplest non-compact case $SU(1,1)$. Here, $\epsilon = -1$ is necessary to render the energy to be non-negative in the representation in which the element $K$ is anti-Hermitian. Higher orbits of non-compact group needs more careful treatment.
[10] A. M. Perelomov, Phys. Rep. 146, 135 (1987).
[11] K. Kimm, K. Lee and T. Lee, Phys. Lett. B 380, 303 (1996).
[12] S. Takeno, Progr. Theor. Phys. 66, 1250 (1981); J. Gruszczak, J. Phys. A: Math. Gen. 14, 3247 (1981).
[13] A. A. Belavin and A. M. Polyakov, JETP Lett. 22, 245 (1975).
Figure 1: $a(r)$ as a function of $r$ for $\kappa = 1$ with various $(n,V)$: $(0,0)$ for solid line, $(-1,0)$ for dotted line, and $(-1,1.5)$ for dashed line.
Figure 2: $f(r)$ as a function of $r$ for $\kappa = 1$ with various $(n, V)$: $(0, 0)$ for solid line, $(-1, 0)$ for dotted line, and $(-1, 1.5)$ for dashed line.