Relaxation schemes for mathematical programmes with switching constraints

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ABSTRACT
Switching-constrained optimization problems form a difficult class of mathematical programmes since their feasible set is almost disconnected while standard constraint qualifications are likely to fail at several feasible points. That is why the application of standard methods from nonlinear programming does not seem to be promising in order to solve such problems. In this paper, we adapt several relaxation methods which are well known from the numerical treatment of mathematical programmes with complementarity constraints to the setting of switching-constrained optimization. A detailed convergence analysis is provided for the adapted relaxation schemes of Scholtes as well as Kanzow and Schwartz. While Scholtes’ method and the relaxation scheme of Steffensen and Ulbrich only find weakly stationary points in general, it is shown that the adapted relaxation scheme of Kanzow and Schwartz is capable of identifying Mordukhovich-stationary points of switching-constrained programmes under suitable assumptions. Some computational experiments and a numerical comparison of the proposed methods based on examples from logical programming, switching control, and portfolio optimization close the paper.

1. Introduction
This paper is dedicated to so-called mathematical programmes with switching constraints, MPSCs for short. These are optimization problems of the form

\[ \begin{align*}
    f(x) & \rightarrow \min \\
    g_i(x) & \leq 0, \quad i \in M, \\
    h_j(x) & = 0, \quad j \in P, \\
    G_l(x)H_l(x) & = 0, \quad l \in Q,
\end{align*} \]

(MPSC)

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where $\mathcal{M} := \{1, \ldots, m\}$, $\mathcal{P} := \{1, \ldots, p\}$, $\mathcal{Q} := \{1, \ldots, q\}$ are index sets and the functions $f, g_i, h_j, G_l, H_l: \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable for all $i \in \mathcal{M}$, $j \in \mathcal{P}$, and $l \in \mathcal{Q}$. For brevity, $g: \mathbb{R}^n \to \mathbb{R}^m$, $h: \mathbb{R}^n \to \mathbb{R}^p$, $G: \mathbb{R}^n \to \mathbb{R}^q$, and $H: \mathbb{R}^n \to \mathbb{R}^q$ are the mappings which possess the component functions $g_i$ ($i \in \mathcal{M}$), $h_j$ ($j \in \mathcal{P}$), $G_l$ ($l \in \mathcal{Q}$), and $H_l$ ($l \in \mathcal{Q}$), respectively. The last $q$ constraints in (MPSC) force $G_l(x)$ or $H_l(x)$ to be zero for all $l \in \mathcal{Q}$, which gives rise to the terminology 'switching constraints'.

Switching structures appear frequently in the context of optimal control, see [8,13,14,21,29,32,34], and references therein, or as a reformulation of so-called *either-or constraints*, see [23, Section 7]. As we will show in Section 6.2.3, portfolio optimization problems with so-called semi-continuous variables, see e.g. [11], can be modelled as switching-constrained programmes. Naturally, (MPSC) is related to other problem classes from disjunctive programming such as *mathematical programmes with complementarity constraints*, MPCCs for short, see [22,24], or *mathematical programmes with vanishing constraints*, MPVCs for short, see e.g. [1,15]. Indeed, similarly to MPCCs and MPVCs, standard constraint qualifications are likely to be violated at the feasible points of (MPSC). Recently, stationarity conditions and constraint qualifications for (MPSC) were introduced in [23].

Here, we focus on the computational treatment of (MPSC). Clearly, standard methods from nonlinear programming may run into difficulties when applied to (MPSC) due to two reasons: first, the feasible set of (MPSC) is likely to be disconnected or at least *almost* disconnected. Secondly, standard regularity conditions like the Mangasarian–Fromovitz constraint qualification, MFCQ for short, are likely to fail at the feasible points of (MPSC) under mild assumptions, see [23, Lemma 4.1]. Similar issues appear in the context of MPCCs and MPVCs where several different relaxation schemes were introduced to overcome these shortcomings, see [16,18] and references therein for an overview. Basically, the idea is to relax the *irregular* constraints using a relaxation parameter such that the resulting surrogate problems are (regular) standard nonlinear problems which can be tackled by common methods. The relaxation parameter is then iteratively reduced to zero and, in each iteration, a Karush–Kuhn–Tucker (KKT) point of the surrogate problem is computed. Ideally, the resulting sequence possesses a limit point and, under some problem-tailored constraint qualification, this point satisfies a suitable MPSC-tailored stationarity condition. Furthermore, it is desirable that the relaxed problems satisfy standard constraint qualifications in a neighbourhood of the limit point under reasonable assumptions.

Noting that (MPSC) can be reformulated by means of

$$
\begin{align*}
 f(x) & \to \min_{x,y,z} \\
 g_i(x) & \leq 0, \quad i \in \mathcal{M}, \\
 h_j(x) & = 0, \quad j \in \mathcal{P}, \\
 -G_l(x) & \leq y_l, \quad G_l(x) \leq y_l, \quad l \in \mathcal{Q}, \\
 -H_l(x) & \leq z_l, \quad H_l(x) \leq z_l, \quad l \in \mathcal{Q}, \\
 y_l, z_l & \geq 0, \quad l \in \mathcal{Q}, \\
 y_lz_l & = 0, \quad l \in \mathcal{Q},
\end{align*}
$$

(1.1)
which is an MPCC, a first idea on how to proceed would be to tackle (1.1) with the relaxation methods discussed in [18]. However, there are some essential difficulties connected to this approach. First, for the handling of each switching constraint in (MPSC), one needs to introduce two slack variables, four (possibly) nonlinear inequality constraints, and one complementarity constraint on the respective slack variables, i.e. programme (1.1) comes at a high price. Secondly, one can easily check that in situations where a feasible point \((\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q\) induces a biactive complementarity constraint (i.e. when \(G_l(\bar{x}) = H_l(\bar{x}) = \bar{y}_l = \bar{z}_l = 0\) holds for some \(l \in Q\)), the MPCC-tailored version of the regularity conditions MFCQ is generally violated at \((\bar{x}, \bar{y}, \bar{z})\). Thus, many results obtained in [18] and the underlying literature are not applicable. A third minor point for this approach seems to be the observation that due to the introduction of slack variables, (1.1) is likely to possess far more local minimizers than the original problem (MPSC), see [23, Section 7] where related issues are discussed.

In this paper, we show that several relaxation schemes which are well known from the theory of MPCCs can be adapted in order to allow a direct computational treatment of the model (MPSC). More precisely, we will focus on

- the global relaxation scheme of Scholtes, see [28],
- the global relaxation scheme of Kanzow and Schwartz from [20],
- the local relaxation method of Steffensen and Ulbrich introduced in [30], and
- the nonsmooth relaxation method by Kadrani, Dussault, and Benchakroun suggested in [19].

In Section 3, a theoretical analysis of the adapted scheme of Scholtes reveals that this method possesses uncomfortably weak convergence properties. Namely, it generally computes only weakly stationary points of (MPSC). This motivates the study of an adapted version of the relaxation scheme due to Kanzow and Schwartz which is known to produce Mordukhovich-stationary points of MPCCs under reasonable assumptions. As we will show in Section 4, this result can be carried over to the setting of switching-constrained programming. A brief discussion of the relaxation schemes of Steffensen and Ulbrich as well as Kadrani et al. is provided in Section 5. We study the numerical capability of the first three relaxation schemes by means of three representative examples from switching-constrained programming.

The remaining part of the paper is structured as follows. In Section 2, we describe the general notation used throughout the paper and recall some fundamental theory on nonlinear as well as switching-constrained programming. An adapted version of Scholtes’ global relaxation method is investigated in Section 3. Section 4 is dedicated to the study of the relaxation approach due to Kanzow and Schwartz. In Section 5, we briefly discuss the adapted relaxation schemes of Steffensen and Ulbrich as well as Kadrani, Dussault, and Benchakroun without stating theoretical details. Section 6 contains a numerical comparison of the relaxation methods based on three different applications of switching-constrained programming, namely either-or constrained programming, switching-constrained optimal control, and semi-continuous portfolio optimization. We conclude the paper with some final remarks in Section 7.
2. Notation and preliminaries

2.1. Basic notation

The subsequently introduced tools of variational analysis can be found in [26].

For any vectors \( x, y \in \mathbb{R}^n \), let \( x \cdot y \) be their Euclidean inner product. Furthermore, we use \( \|x\|_2 := \sqrt{x \cdot x} \) in order to represent the Euclidean norm of \( x \in \mathbb{R}^n \). For a nonempty set \( A \subset \mathbb{R}^n \), we call

\[
A^\circ := \{ y \in \mathbb{R}^n \mid \forall x \in A : x \cdot y \leq 0 \}
\]
the polar cone of \( A \). It is well known that \( A^\circ \) is a nonempty, closed, convex cone. For any two sets \( B_1, B_2 \subset \mathbb{R}^n \), the polarization rule \( (B_1 \cup B_2)^\circ = B_1^\circ \cap B_2^\circ \) holds by definition. The polar of a polyhedral cone can be characterized by means of, e.g., Motzkin’s theorem of alternatives. Note that we interpret the relations \( \leq \) and \( \geq \) for vectors componentwise.

**Lemma 2.1:** For matrices \( C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{p \times n} \), let \( K \subset \mathbb{R}^n \) be the polyhedral cone

\[
K := \{ d \in \mathbb{R}^n | Cd \leq 0, Dd = 0 \}.
\]

Then \( K^\circ = \{ C^T \lambda + D^T \rho | \lambda \in \mathbb{R}^m, \lambda \geq 0, \rho \in \mathbb{R}^p \} \).

Let \( A \subset \mathbb{R}^n \) be a nonempty set and \( \bar{x} \in A \). Then the closed cone

\[
T_A(\bar{x}) := \{ d \in \mathbb{R}^n | \exists \{x_k\}_{k \in \mathbb{N}} \subset A \exists \{\tau_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+: x_k \rightarrow \bar{x}, \tau_k \downarrow 0, (x_k - \bar{x})/\tau_k \rightarrow d \}
\]
is called tangent or Bouligand cone to \( A \) at \( \bar{x} \). Here, \( \mathbb{R}_+ := \{ r \in \mathbb{R} | r > 0 \} \) denotes the set of all positive reals.

The union \( \{v^1, \ldots, v^r\} \cup \{w^1, \ldots, w^s\} \) of sets \( \{v^1, \ldots, v^r\}, \{w^1, \ldots, w^s\} \subset \mathbb{R}^n \) is called positive-linearly dependent if there exist vectors \( \alpha \in \mathbb{R}^r, \alpha \geq 0, \) and \( \beta \in \mathbb{R}^s \) which do not vanish at the same time such that

\[
0 = \sum_{i=1}^r \alpha_i v^i + \sum_{j=1}^s \beta_j w^j.
\]

Otherwise, \( \{v^1, \ldots, v^r\} \cup \{w^1, \ldots, w^s\} \) is called positive-linearly independent. Clearly, if the set \( \{v^1, \ldots, v^r\} \) is empty, then the above definitions reduce to linear dependence and independence, respectively. The following lemma will be useful in this paper; its proof is similar to that of [25, Proposition 2.2] and therefore omitted.

**Lemma 2.2:** Let \( \{v^1, \ldots, v^r\}, \{w^1, \ldots, w^s\} \subset \mathbb{R}^n \) be given sets whose union \( \{v^1, \ldots, v^r\} \cup \{w^1, \ldots, w^s\} \) is positive-linearly independent. Then there exists \( \varepsilon > 0 \) such that, for all vectors \( \tilde{v}^1, \ldots, \tilde{v}^r, \tilde{w}^1, \ldots, \tilde{w}^s \in \{ z \in \mathbb{R}^n | \|z\|_2 \leq \varepsilon \} \), the union \( \{v^1 + \tilde{v}^1, \ldots, v^r + \tilde{v}^r\} \cup \{w^1 + \tilde{w}^1, \ldots, w^s + \tilde{w}^s\} \) is positive-linearly independent.

For some vector \( z \in \mathbb{R}^n \) and an index set \( I \subset \{1, \ldots, n\} \), \( z_I = \mathbb{R}^{|I|} \) denotes the vector which results from \( z \) by deleting all \( z_i \) with \( i \in \{1, \ldots, n\} \setminus I \). Finally, let us mention that \( \text{supp} z := \{ i \in \{1, \ldots, n\} | z_i \neq 0 \} \) is called the support of the vector \( z \in \mathbb{R}^n \).
2.2. Standard nonlinear programmes

Here, we recall some fundamental constraint qualifications from standard nonlinear programming, see, e.g. [3]. Therefore, we consider the nonlinear programme

\[
\begin{align*}
    f(x) & \to \min \\
    g_i(x) & \leq 0, \quad i \in \mathcal{M}, \\
    h_j(x) & = 0, \quad j \in \mathcal{P},
\end{align*}
\]

i.e. we forget about the switching constraints in (MPSC) for a moment.

Let \( \tilde{X} \subset \mathbb{R}^n \) denote the feasible set of (NLP) and fix some point \( \bar{x} \in \tilde{X} \). Then

\[ I^\mathcal{L}(\bar{x}) := \{ i \in \mathcal{M} | g_i(\bar{x}) = 0 \} \]

is called the index set of active inequality constraints at \( \bar{x} \). Furthermore, the set

\[
L_{\tilde{X}}(\bar{x}) := \left\{ d \in \mathbb{R}^n \middle| \begin{array}{c}
\nabla g_i(\bar{x}) \cdot d \leq 0 \quad i \in I^\mathcal{L}(\bar{x}) \\
\nabla h_j(\bar{x}) \cdot d = 0 \quad j \in \mathcal{P}
\end{array} \right\}
\]

is called the linearization cone to \( \tilde{X} \) at \( \bar{x} \). Obviously, \( L_{\tilde{X}}(\bar{x}) \) is a polyhedral cone, and thus closed and convex. It is well known that \( T_{\tilde{X}}(\bar{x}) \subset L_{\tilde{X}}(\bar{x}) \) is always satisfied. The converse inclusion generally only holds under some constraint qualification.

In the definition below, we recall several standard constraint qualifications which are applicable to (NLP).

**Definition 2.3:** Let \( \bar{x} \in \mathbb{R}^n \) be a feasible point of (NLP). Then \( \bar{x} \) is said to satisfy the

(a) **linear independence constraint qualification (LICQ)** if the following vectors are linearly independent:

\[ \{ \nabla g_i(\bar{x}) | i \in I^\mathcal{L}(\bar{x}) \} \cup \{ \nabla h_j(\bar{x}) | j \in \mathcal{P} \}. \]

(b) **Mangasarian–Fromovitz constraint qualification (MFCQ)** if the vectors in (2.1) are positive-linearly independent.

(c) **constant positive linear dependence condition (CPLD)** if, for any sets \( I \subset I^\mathcal{L}(\bar{x}) \) and \( J \subset \mathcal{P} \) such that the gradients

\[ \{ \nabla g_i(\bar{x}) | i \in I \} \cup \{ \nabla h_j(\bar{x}) | j \in J \} \]

are positive-linearly dependent, there exists a neighbourhood \( U \subset \mathbb{R}^n \) of \( \bar{x} \) such that the gradients

\[ \{ \nabla g_i(x) | i \in I \} \cup \{ \nabla h_j(x) | j \in J \} \]

are linearly dependent for all \( x \in U \).

(d) **Abadie constraint qualification (ACQ)** if \( T_{\tilde{X}}(\bar{x}) = L_{\tilde{X}}(\bar{x}) \).

(e) **Guignard constraint qualification (GCQ)** if \( T_{\tilde{X}}(\bar{x})^\circ = L_{\tilde{X}}(\bar{x})^\circ \).
Note that the following relations hold between the constraint qualifications from Definition 2.3:

\[ \text{LICQ} \implies \text{MFCQ} \implies \text{CPLD} \implies \text{ACQ} \implies \text{GCQ}, \]

see [18, Section 2.1] for some additional information.

It is well known that the validity of GCQ at some local minimizer \( \bar{x} \in \mathbb{R}^n \) of \((\text{NLP})\) implies that the KKT conditions

\[
0 = \nabla f(\bar{x}) + \sum_{i \in I^g(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in P} \rho_j \nabla h_j(\bar{x}), \quad \forall i \in I^g(\bar{x}) : \lambda_i \geq 0
\]

provide a necessary optimality condition. Thus, the same holds for the stronger constraint qualifications ACQ, CPLD, MFCQ, and LICQ.

### 2.3. Mathematical Programmes with Switching Constraints

The statements of this section are mostly taken from [23]. Let \( X \subset \mathbb{R}^n \) denote the feasible set of \((\text{MPSC})\) and fix a point \( \bar{x} \in X \). Then the index sets

\[
I^G(\bar{x}) := \{ l \in Q | G_l(\bar{x}) = 0, H_l(\bar{x}) \neq 0 \},
\]

\[
I^H(\bar{x}) := \{ l \in Q | G_l(\bar{x}) \neq 0, H_l(\bar{x}) = 0 \},
\]

\[
I^{GH}(\bar{x}) := \{ l \in Q | G_l(\bar{x}) = 0, H_l(\bar{x}) = 0 \}
\]

form a disjoint partition of \( Q \). It is easily seen that MFCQ (and thus LICQ) cannot hold for \((\text{MPSC})\) at \( \bar{x} \) if \( I^{GH}(\bar{x}) \neq \emptyset \). Taking a look at the associated linearization cone

\[
L_X(\bar{x}) = \left\{ d \in \mathbb{R}^n \left| \begin{array}{c} \nabla g_i(\bar{x}) \cdot d \leq 0 \quad i \in I^g(\bar{x}) \\
\nabla h_j(\bar{x}) \cdot d = 0 \quad j \in P \\
\n\nabla G_l(\bar{x}) \cdot d = 0 \quad l \in I^G(\bar{x}) \\
\n\n\nabla H_l(\bar{x}) \cdot d = 0 \quad l \in I^H(\bar{x}) \end{array} \right. \right\},
\]

which is always convex, one can imagine that ACQ is likely to fail as well if \( I^{GH}(\bar{x}) \neq \emptyset \) since, in the latter situation, \( T_X(\bar{x}) \) might be nonconvex. Note that GCQ may hold for \((\text{MPSC})\) even in the aforementioned context.

Due to the inherent lack of regularity, stationarity conditions for \((\text{MPSC})\) which are weaker than the associated KKT conditions were introduced.

**Definition 2.4:** A feasible point \( \bar{x} \in X \) of \((\text{MPSC})\) is called

(a) **weakly stationary** (W-stationary) if there exist multipliers \( \lambda_i \ (i \in I^g(\bar{x})) \), \( \rho_j \ (j \in P) \), \( \mu_l \ (l \in Q) \), and \( \nu_l \ (l \in Q) \) which solve the following system:

\[
0 = \nabla f(\bar{x}) + \sum_{i \in I^g(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in P} \rho_j \nabla h_j(\bar{x}) + \sum_{l \in Q} \left[ \mu_l \nabla G_l(\bar{x}) + \nu_l \nabla H_l(\bar{x}) \right].
\]
\[ \forall i \in I^G(\bar{x}) : \lambda_i \geq 0, \]
\[ \forall l \in I^H(\bar{x}) : \mu_l = 0, \]
\[ \forall l \in I^G(\bar{x}) : \nu_l = 0. \]

(b) **Mordukhovich-stationary** (M-stationary) if it is W-stationary and the associated multipliers additionally satisfy
\[ \forall l \in I^{GH}(\bar{x}) : \mu_l \nu_l = 0. \]

(c) **strongly stationary** (S-stationary) if it is W-stationary while the associated multipliers additionally satisfy
\[ \forall l \in I^{GH}(\bar{x}) : \mu_l = 0 \land \nu_l = 0. \]

Clearly, the following implications hold:

\[ \text{S-stationarity} \implies \text{M-stationarity} \implies \text{W-stationarity}. \]

Moreover, the KKT conditions of (MPSC) are equivalent to the S-stationarity conditions from Definition 2.4. One may check Figure 1 for a geometric interpretation of the Lagrange multipliers associated with the switching conditions from \( I^{GH}(\bar{x}) \).

In the literature on MPCCs, the concept of so-called Clarke stationarity (C-stationarity) is famous as well. It naturally appears as a stationarity condition if the complementarity conditions are reformulated by nonsmooth equations (using the min-operator or the projection onto the nonnegative orthant) while the resulting surrogate problem is tackled with Clarke's tools of generalized differentiation, see [7] and [27, Lemma 1]. C-stationarity is weaker than Mordukhovich- but stronger than weak stationarity in general. It needs to be noted that the C-stationarity system for MPCCs cannot be obtained using Clarke's normal cone to the complementarity set. Now, the question arises whether there exists a reasonable system of C-stationarity which addresses the model programme (MPSC). Due to the above arguments, this issue is directly related to the existence of a nonsmooth reformulation of the switching condition \( G_l(x) H_l(x) = 0, l \in Q \). It is, however, not clear how such a reasonable nonsmooth reformulation, which does not only induce W-stationarity, should

**Figure 1.** Geometric illustrations of W-, M-, and S-stationarity for an index \( l \in I^{GH}(\bar{x}) \).
look like. Furthermore, the symmetry of the switching condition should enforce a symmetric behaviour of the multipliers in the appearing stationarity systems, and by means of Figure 1, there does not seem to be a reasonable symmetric set of multipliers such that the resulting stationarity condition is stronger than weak but weaker than M-stationarity. Summarizing these arguments, it seems like there does not exist a reasonable concept of C-stationarity which addresses (MPSC).

In order to ensure that one of the stationarity notions from Definition 2.4 plays the role of a necessary optimality condition for (MPSC), suitable problem-tailored constraint qualifications need to be valid. For the definition of such conditions, the following so-called tightened nonlinear problem is of interest:

\[
\begin{align*}
    f(x) &\rightarrow \min \\
    g_i(x) &\leq 0, \quad i \in \mathcal{M}, \\
    h_j(x) &= 0, \quad j \in \mathcal{P}, \\
    G_l(x) &= 0, \quad l \in I^G(\bar{x}) \cup I^{GH}(\bar{x}), \\
    H_l(x) &= 0, \quad l \in I^H(\bar{x}) \cup I^{GH}(\bar{x}).
\end{align*}
\]

(TNLP)

Note that (TNLP) is a standard nonlinear programme.

**Definition 2.5:** Let \(\bar{x} \in X\) be a feasible point of (MPSC). Then MPSC-LICQ (MPSC-MFCQ) is said to hold for (MPSC) at \(\bar{x}\) if LICQ (MFCQ) holds for (TNLP) at \(\bar{x}\), i.e. if the vectors

\[
\{\nabla g_i(\bar{x})|i \in I^E(\bar{x})\} \cup \{\nabla h_j(\bar{x})|j \in \mathcal{P}\} \\
\cup \{\nabla G_l(\bar{x})|l \in I^G(\bar{x}) \cup I^{GH}(\bar{x})\} \\
\cup \{\nabla H_l(\bar{x})|l \in I^H(\bar{x}) \cup I^{GH}(\bar{x})\}
\]

are linearly independent (positive-linearly independent).

It is obvious that MPSC-LICQ is stronger than MPSC-MFCQ. Furthermore, MPSC-LICQ implies standard GCQ for (MPSC) at the reference point.

In this paper, we will use another MPSC-tailored constraint qualification called MPSC-NNAMCQ, where NNAMCQ stands for the No Nonzero Abnormal Multiplier Constraint Qualification which has been introduced to investigate optimization problems whose feasible sets are preimages of closed but not necessarily convex sets under continuously differentiable mappings, see [26, Section 6.D] and [33]. Clearly, (MPSC) belongs to this problem class as well if one reformulates the switching constraints as

\[(G_l(x), H_l(x)) \in C, \quad l \in \mathcal{Q},\]

with \(C := \{(a, b) \in \mathbb{R}^2 | ab = 0\} \).
**Definition 2.6:** Let $\bar{x} \in X$ be a feasible point of (MPSC). Then MPSC-NNAMCQ is said to hold for (MPSC) at $\bar{x}$ if the following condition is valid:

$$
0 = \sum_{i \in I^g(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in P} \rho_j \nabla h_j(\bar{x}) + \sum_{l \in Q} \left[ \mu_l \nabla G_l(\bar{x}) + \nu_l \nabla H_l(\bar{x}) \right] \quad \forall i \in I^g(\bar{x}): \lambda_i \geq 0,
\forall l \in I^H(\bar{x}): \mu_l = 0,
\forall l \in I^G(\bar{x}): \nu_l = 0,
\forall l \in I^{GH}(\bar{x}): \mu_l \nu_l = 0
\implies \lambda = 0, \quad \rho = 0, \quad \mu = 0, \quad \nu = 0.
$$

It is easy to check that MPSC-NNAMCQ is implied by MPSC-MFCQ. Note that a local minimizer of (MPSC) where MPSC-LICQ holds is an S-stationary point. Furthermore, one can easily check that the associated multipliers which solve the system of S-stationarity are uniquely determined. Under validity of MPSC-NNAMCQ (and, thus, under validity of MPSC-MFCQ), a local minimizer of (MPSC) is, in general, only M-stationary. Finally, there exist several problem-tailored constraint qualifications for (MPSC) which are weaker than MPSC-NNAMCQ but also imply that local solutions are M-stationary, see [23].

**3. The relaxation scheme of Scholtes**

For some parameter $t \geq 0$, let us consider the surrogate problem

$$
f(x) \rightarrow \min \quad \begin{align*}
g_i(x) & \leq 0, \quad i \in \mathcal{M}, \\
h_j(x) & = 0, \quad j \in \mathcal{P}, \\
-t & \leq G_l(x) H_l(x) \leq t, \quad l \in \mathcal{Q}.
\end{align*}
$$

(P$_S$(t))

This idea is inspired by Scholtes’ global relaxation method which was designed for the computational treatment of MPCCs, see [28] and [18, Section 3.1], but is also used in related settings of disjunctive programming, see e.g. [2,4,5]. Here, each of the original switching constraints is replaced by two nonlinear inequality constraints which relax the original switching requirement. The feasible set of (P$_S$(t)) is denoted by $X_S(t)$ and visualized in Figure 2.

The upcoming lemma justifies that (P$_S$(t)) is indeed a relaxation of (MPSC). Its proof follows by standard arguments and is, therefore, omitted.

**Lemma 3.1:** The following statements hold:

(P1) $X_S(0) = X$,
(P2) $0 \leq t_1 \leq t_2 \implies X_S(t_1) \subset X_S(t_2)$,
(P3) $\bigcap_{t>0} X_S(t) = X$. 
Now, we are in position to characterize a conceptual method for the numerical solution of (MPSC): First, a sequence \( \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+ \) of positive relaxation parameters is chosen which converges to zero. Next, one solves the surrogate problem \((PS(t_k))\) via standard methods. If one computes a (local) minimizer of one of these surrogate problems which is feasible to (MPSC), then it is already a (local) minimizer of (MPSC). In general, it will be only possible to compute KKT points of the surrogate problem. However, if such a sequence of KKT points converges to some point \( \bar{x} \in \mathbb{R}^n \), then this point must be feasible to (MPSC) by construction. Below, we investigate under which conditions the point \( \bar{x} \) is stationary for (MPSC).

It is well known from [18, Theorem 3.1] that Scholtes’ relaxation approach finds C-stationary points of MPCCs under an MPCC-tailored version of MFCQ. Here, we want to generalize this result to problem (MPSC).

For the fixed parameter \( t > 0 \) and a feasible point \( x \in X_S(t) \) of \((PS(t))\), we introduce the index sets

\[
I^+_t(x) := \{ l \in \mathcal{Q} | G_l(x) H_l(x) = t \},
\]

\[
I^-_t(x) := \{ l \in \mathcal{Q} | G_l(x) H_l(x) = -t \}.
\]

In the upcoming theorem, we provide a convergence result of Scholtes’ relaxation scheme for problem (MPSC).

**Theorem 3.2:** Let \( \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+ \) be a sequence of positive relaxation parameters converging to zero. For each \( k \in \mathbb{N} \), let \( x_k \in X_S(t_k) \) be a KKT point of \((PS(t_k))\). Assume that the sequence \( \{x_k\}_{k \in \mathbb{N}} \) converges to a point \( \bar{x} \in X \) where MPSC-MFCQ holds. Then \( \bar{x} \) is a W-stationary point of (MPSC).

**Proof:** Noting that \( x_k \) is a KKT point of \((PS(t_k))\), we find multipliers \( \lambda^k \in \mathbb{R}^m \), \( \rho^k \in \mathbb{R}^p \), and \( \xi^k \in \mathbb{R}^q \) which satisfy the following conditions:
\[ 0 = \nabla f(x_k) + \sum_{i \in P(x_k)} \lambda_i^k \nabla g_i(x_k) + \sum_{j \in P} \rho_j^k \nabla h_j(x_k) + \sum_{l \in Q} \xi_l^k \left[ H_l(x_k) \nabla G_l(x_k) + G_l(x_k) \nabla H_l(x_k) \right], \]
\[ \forall i \in I^g(x_k) : \lambda_i^k \geq 0, \quad \forall i \in \mathcal{M} \setminus I^g(x_k) : \lambda_i^k = 0, \]
\[ \forall l \in I^+_I(x_k) : \xi_l^k \geq 0, \]
\[ \forall l \in I^-_I(x_k) : \xi_l^k \leq 0, \]
\[ \forall l \in Q \setminus (I^+_I(x_k) \cup I^-_I(x_k)) : \xi_l^k = 0. \]

For each \( k \in \mathbb{N} \) and \( l \in Q \), let us define artificial multipliers \( \mu_l^k, v_l^k \in \mathbb{R} \) as stated below:
\[ \mu_l^k := \begin{cases} \xi_l^k H_l(x_k) & l \in I^G(\tilde{x}) \cup I^{GH}(\tilde{x}), \\ 0 & l \in I^H(\tilde{x}), \end{cases} \quad v_l^k := \begin{cases} \xi_l^k G_l(x_k) & l \in I^H(\tilde{x}) \cup I^{GH}(\tilde{x}), \\ 0 & l \in I^G(\tilde{x}). \end{cases} \]

Thus, we obtain
\[ 0 = \nabla f(x_k) + \sum_{i \in P(x_k)} \lambda_i^k \nabla g_i(x_k) + \sum_{j \in P} \rho_j^k \nabla h_j(x_k) + \sum_{l \in Q} \left[ \mu_l^k \nabla G_l(x_k) + v_l^k \nabla H_l(x_k) \right] + \sum_{l \in I^H(\tilde{x})} \xi_l^k H_l(x_k) \nabla G_l(x_k) + \sum_{l \in I^{GH}(\tilde{x})} \xi_l^k G_l(x_k) \nabla H_l(x_k). \tag{3.1} \]

Next, we are going to show that the sequence \( \{(\lambda^k, \rho^k, \mu^k, v^k, \xi_l^k)\}_{k \in \mathbb{N}} \) is bounded where we used \( I := I^G(\tilde{x}) \cup I^H(\tilde{x}) \) for brevity. We assume on the contrary that this is not the case and define
\[ \forall k \in \mathbb{N} : \quad (\tilde{\lambda}^k, \tilde{\rho}^k, \tilde{\mu}^k, \tilde{v}^k, \tilde{\xi}_l^k) := \frac{(\lambda^k, \rho^k, \mu^k, v^k, \xi_l^k)}{\| (\lambda^k, \rho^k, \mu^k, v^k, \xi_l^k) \|_2}. \]

Clearly, \( \{(\tilde{\lambda}^k, \tilde{\rho}^k, \tilde{\mu}^k, \tilde{v}^k, \tilde{\xi}_l^k)\}_{k \in \mathbb{N}} \) is bounded and, thus, converges w.l.o.g. to a nonvanishing vector \( (\tilde{\lambda}, \tilde{\rho}, \tilde{\mu}, \tilde{v}, \tilde{\xi}_l) \) (otherwise, a suitable subsequence is chosen). The continuity of \( g \) ensures that \( I^g(x_k) \subset I^g(\tilde{x}) \) is valid for sufficiently large \( k \in \mathbb{N} \). Dividing (3.1) by \( \| (\lambda^k, \rho^k, \mu^k, v^k, \xi_l^k) \|_2 \) and taking the limit \( k \to \infty \) while respecting the continuous differentiability of all involved functions, we come up with
\[ 0 = \sum_{i \in I^g(\tilde{x})} \tilde{\lambda}_i \nabla g_i(\tilde{x}) + \sum_{j \in P} \tilde{\rho}_j \nabla h_j(\tilde{x}) + \sum_{l \in Q} \left[ \tilde{\mu}_l \nabla G_l(\tilde{x}) + \tilde{v}_l \nabla H_l(\tilde{x}) \right], \]
\[ \forall i \in I^g(\tilde{x}) : \tilde{\lambda}_i \geq 0, \quad \forall i \in \mathcal{M} \setminus I^g(\tilde{x}) : \tilde{\lambda}_i = 0, \]
\[ \forall l \in I^H(\tilde{x}) : \tilde{\mu}_l = 0, \]
\[ \forall l \in I^{GH}(\tilde{x}) : \tilde{v}_l = 0. \]

Now, the validity of MPSC-MFCQ yields \( \tilde{\lambda} = 0, \tilde{\rho} = 0, \tilde{\mu} = 0, \) and \( \tilde{v} = 0. \) Hence, \( \tilde{\xi}_{l_0} \neq 0 \) holds for at least one index \( l_0 \in I \). Let us assume \( l_0 \in I^H(\tilde{x}) \). Then we have \( v_{l_0}^k = \xi_{l_0}^k G_{l_0}(x_k), \)
which leads to
\[
\hat{\nu}_0 = \lim_{k \to \infty} \frac{v^k_{l_0}}{\| (\lambda^k, \rho^k, \mu^k, v^k, \xi^k_l) \|_2} = \lim_{k \to \infty} \frac{\xi^k_{l_0} G_{l_0}(x^k)}{\| (\lambda^k, \rho^k, \mu^k, v^k, \xi^k_l) \|_2} = \hat{\nu}_0 G_{l_0}(\bar{x}) \neq 0.
\]

This, however, is a contradiction since \( \hat{\nu} \) vanishes due to the above arguments. Similarly, the case \( l_0 \in I^G(\bar{x}) \) leads to a contradiction. As a consequence, the sequence \( \{ (\lambda^k, \rho^k, \mu^k, v^k, \xi^k_l) \}_{k \in \mathbb{N}} \) is bounded.

Thus, we may assume w.l.o.g. that this sequence converges to \( (\lambda, \rho, \mu, v, \xi_I) \). Again, we take the limit in (3.1) and obtain
\[
0 = \nabla f(\bar{x}) + \sum_{i \in I^g(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in P} \rho_j \nabla h_j(\bar{x}) + \sum_{l \in Q} \left[ \mu_l \nabla G_l(\bar{x}) + v_l \nabla H_l(\bar{x}) \right],
\]
\[
\forall i \in I^g(\bar{x}): \lambda_i \geq 0, \quad \forall i \in M \setminus I^g(\bar{x}): \lambda_i = 0,
\]
\[
\forall l \in I^H(\bar{x}): \mu_l = 0,
\]
\[
\forall l \in I^G(\bar{x}): v_l = 0
\]
which shows that \( \bar{x} \) is a W-stationary point of (MPSC).

As already mentioned in Section 2, there does not seem to be a suitable definition of C-stationarity which applies to (MPSC) (in particular, a reasonable stationarity concept which is stronger than W- but weaker than M-stationarity). Thus, the statement from Theorem 3.2 does not seem to be too surprising at all. The following example shows that we cannot expect any stronger results in general. Thus, the qualitative properties of Scholtes’ relaxation method are uncomfortably weak in the setting of switching-constrained programming. On the other hand, one has to admit that local minimizers of (MPSC) which are only W-stationary can be found by Scholtes’ relaxation scheme which puts this result in some positive light.

**Example 3.3:** Let us consider the switching-constrained optimization problem
\[
\frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 1)^2 \rightarrow \min \quad x_1, x_2 \geq 0.
\]
The globally optimal solutions of this programme are given by \((1, 0)\) as well as \((0, 1)\), and these points are S-stationary. Furthermore, there exists a W-stationary point at \( \bar{x} = (0, 0) \) which is no local minimizer.

One can easily check that the associated problem \( (P_S(t)) \) possesses a KKT point at \((\sqrt{t}, \sqrt{t})\) for each \( t \in (0, 1] \). Taking the limit \( t \downarrow 0 \), this point tends to \( \bar{x} \) which is, as we already mentioned above, only W-stationary for the switching-constrained problem of interest. Clearly, MPSC-LICQ is valid at \( \bar{x} \).

Although the theoretical properties of Scholtes’ relaxation approach do not seem to be promising in light of (MPSC), we check the applicability of the approach since it has been reported in [18] that this relaxation scheme is the most robust one for the numerical handling of complementarity-constrained programmes when computational practice
is investigated. Thus, we analyze the restrictiveness of the assumption that a sequence of KKT points associated with \((P_S(t))\) can be chosen.

In order to guarantee that locally optimal solutions of the nonlinear relaxed surrogate problems \((P_S(t))\), which are located closely to the limit point from Theorem 3.2, are KKT points, a constraint qualification needs to be satisfied. Adapting [18, Theorem 3.2] to the switching-constrained situation, it is possible to show that whenever MPSC-MFCQ is valid at a feasible point \(\bar{x} \in X\) of \((MPSC)\), then standard MFCQ is valid for \((P_S(t))\) in a neighbourhood of \(\bar{x}\).

**Theorem 3.4:** Let \(\bar{x} \in X\) be a feasible point of \((MPSC)\) where MPSC-MFCQ is satisfied. Then there exists a neighbourhood \(U \subset \mathbb{R}^n\) of \(\bar{x}\) such that MFCQ holds for \((P_S(t))\) at all points from \(X_S(t) \cap U\) for all \(t > 0\).

**Proof:** Due to the validity of MPSC-MFCQ at \(\bar{x}\), the union

\[
\{\nabla g_i(\bar{x})| i \in I^S(\bar{x})\} \cup \left[\{\nabla h_j(\bar{x})| j \in \mathcal{P}\} \cup \{\nabla G_l(\bar{x})| l \in I^G(\bar{x}) \cup I^{GH}(\bar{x})\} \right. \\
\left. \cup \{\nabla H_l(\bar{x})| l \in I^H(\bar{x}) \cup I^{GH}(\bar{x})\right]\]

is positive-linearly independent. Invoking Lemma 2.2, there is a neighbourhood \(U\) of \(\bar{x}\) such that the vectors

\[
\{\nabla g_i(x)| i \in I^S(\bar{x})\} \cup \left[\{\nabla h_j(x)| j \in \mathcal{P}\} \cup \{\nabla G_l(x)| l \in I^G(\bar{x}) \cup I^{GH}(\bar{x})\} \right. \\
\left. \cup \{\nabla H_l(x)| l \in I^H(\bar{x}) \cup I^{GH}(\bar{x})\right]\]

are positive-linearly independent for any choice of \(x \in U\).

Now, fix \(t > 0\) as well as \(x \in X_S(t) \cap U\) and set \(I^0_l(x) := I^+_l(x) \cup I^-_l(x)\). Note that \(t > 0\) guarantees \(I^+_l(x) \cap I^-_l(x) = \emptyset\). Clearly, we have

\[
\forall l \in I^H(\bar{x}): \quad G_l(x) \neq 0 \quad H_l(x) \approx 0, \quad \forall l \in I^G(\bar{x}): \quad G_l(x) \approx 0 \quad H_l(x) \neq 0
\]

if \(U\) is sufficiently small. Exploiting Lemma 2.2 once more while recalling that \(G\) and \(H\) are continuously differentiable, we obtain that the vectors

\[
\{\nabla g_i(x)| i \in I^S(\bar{x})\} \cup \left[\{\nabla h_j(x)| j \in \mathcal{P}\} \cup \{H_l(x)\nabla G_l(x) + G_l(x)\nabla H_l(x)| l \in I^G(\bar{x}) \cap I^+_l(x)\} \cup \{H_l(x)\nabla G_l(x) + G_l(x)\nabla H_l(x)| l \in I^H(\bar{x}) \cap I^-_l(x)\} \right. \\
\left. \cup \{\nabla G_l(x)| l \in I^{GH}(\bar{x}) \cap I^+_l(x)\} \cup \{\nabla H_l(x)| l \in I^{GH}(\bar{x}) \cap I^-_l(x)\}\right]
\]

are positive-linearly independent if the neighbourhood \(U\) is chosen small enough.
Suppose that there are vectors \( \lambda \in \mathbb{R}^m \), \( \rho \in \mathbb{R}^p \), and \( \xi \in \mathbb{R}^q \) which satisfy
\[
0 = \sum_{i \in I^G(x)} \lambda_i \nabla g_i(x) + \sum_{j \in P} \rho_j \nabla h_j(x) + \sum_{l \in Q} \xi_l \left[ H_l(x) \nabla G_l(x) + G_l(x) \nabla H_l(x) \right],
\]
\[
\forall i \in I^G(x) : \lambda_i \geq 0, \quad \forall i \in M \setminus I^G(x) : \lambda_i = 0,
\]
\[
\forall l \in I^+_t(x) : \xi_l \geq 0,
\]
\[
\forall l \in I^-_t(x) : \xi_l \leq 0,
\]
\[
\forall l \in Q \setminus I^+_t(x) : \xi_l = 0.
\]

In order to show the validity of MFCQ for \((P_\delta(t))\) at \(x, \lambda = 0, \rho = 0\), and \(\xi = 0\) have to be deduced. We get
\[
0 = \sum_{i \in I^G(x)} \lambda_i \nabla g_i(x) + \sum_{j \in P} \rho_j \nabla h_j(x)
+ \sum_{l \in I^G(\bar{x}) \cup I^H(\bar{x}) \cap I^+_t(x)} \xi_l \left[ H_l(x) \nabla G_l(x) + G_l(x) \nabla H_l(x) \right],
\]
\[
+ \sum_{l \in I^{GH}(\bar{x}) \cap I^+_t(x)} \xi_l G_l(x) \nabla H_l(x) + \sum_{l \in I^{GH}(\bar{x}) \cap I^+_t(x)} \xi_l H_l(x) \nabla G_l(x).
\]

Noting that \(\lambda_i \geq 0\) holds for all \(i \in I^G(x)\) while \(I^G(x) \subset I^G(\bar{x})\) holds whenever \(U\) is chosen sufficiently small, we obtain \(\lambda = 0, \rho = 0, \xi_l = 0 (l \in (I^G(\bar{x}) \cup I^H(\bar{x}) \cap I^+_t(x)), \xi_l G_l(x) = 0 (l \in I^{GH}(\bar{x}) \cap I^+_t(x)), \) and \(\xi_l H_l(x) = 0 (l \in I^{GH}(\bar{x}) \cap I^+_t(x))\) from the positive-linear independence of the vectors in (3.2). Since we have \(G_l(x) \neq 0\) and \(H_l(x) \neq 0\) for all \(l \in I^+_t(x)\) from \(t > 0\), \(\xi_l = 0\) follows for all \(l \in I^+_t(x)\) since \(I^G(\bar{x}) \cup I^H(\bar{x}) \cup I^{GH}(\bar{x}) = Q\) is valid. This yields \(\xi = 0\). Thus, MFCQ holds for \((P_\delta(t))\) at \(x\).

4. The relaxation scheme of Kanzow and Schwartz

Recalling that there does not seem to exist an appropriate C-stationarity notion for (MPSC), one has to suspect that any relaxation scheme for MPCCs which computes C-stationary points yields only W-stationary points in the setting of switching-constrained programming. This has been visualized for Scholtes' global relaxation method in Section 3.

In order to deal with this issue, we now adapt the relaxation scheme from [20], which computes M-stationary points of MPCCs under suitable assumptions, to (MPSC). Clearly, we hope that the adapted method can be used to identify M-stationary points of (MPSC) as well. The papers [6,17], where the Kanzow–Schwartz relaxation scheme is adapted to cardinality- and vanishing-constrained optimization problems, respectively, suggest that this is a reasonable expectation.

4.1. On the relaxation scheme

For our relaxation approach, we will make use of the function \(\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}\) defined below:
\[
\forall (a, b) \in \mathbb{R}^2 : \quad \varphi(a, b) := \begin{cases} 
abla \varphi(a, b) = \begin{cases} 1 & \text{if } a + b \geq 0, \\
-\frac{1}{2} & \text{if } a + b < 0.
\end{cases}
\end{cases}
\]
The function $\varphi$ was introduced in [20] to study a relaxation method for the numerical solution of MPCCs. In the following lemma, which parallels [20, Lemma 3.1], some properties of $\varphi$ are summarized.

**Lemma 4.1:**

(a) The function $\varphi$ is an NCP-function, i.e. it satisfies
\[
\forall (a, b) \in \mathbb{R}^2: \quad \varphi(a, b) = 0 \iff a \geq 0 \land b \geq 0 \land ab = 0.
\]

(b) The function $\varphi$ is continuously differentiable and satisfies
\[
\forall (a, b) \in \mathbb{R}^2: \quad \nabla \varphi(a, b) = \begin{cases} 
\begin{pmatrix} b \\ a \end{pmatrix} & \text{if } a + b \geq 0, \\
\begin{pmatrix} -a \\ -b \end{pmatrix} & \text{if } a + b < 0.
\end{cases}
\]

For some parameter $t \geq 0$ as well as indices $s \in \mathcal{S} := \{1, 2, 3, 4\}$ and $l \in \mathcal{Q}$, we define functions $\Phi_i^j(\cdot; t): \mathbb{R}^n \to \mathbb{R}$ via
\[
\Phi_1^1(x; t) := \varphi(G_l(x) - t, H_l(x) - t), \quad \Phi_1^2(x; t) := \varphi(-G_l(x) - t, H_l(x) - t), \\
\Phi_1^3(x; t) := \varphi(-G_l(x) - t, -H_l(x) - t), \quad \Phi_1^4(x; t) := \varphi(G_l(x) - t, -H_l(x) - t)
\]
for each $x \in \mathbb{R}^n$. Now, we are in position to introduce the surrogate problem of our interest:
\[
\begin{align*}
\text{min} & \quad f(x) \\
g_i(x) & \leq 0, \quad i \in \mathcal{M}, \\
h_j(x) & = 0, \quad j \in \mathcal{P}, \\
\Phi_i^j(x; t) & \leq 0, \quad s \in \mathcal{S}, l \in \mathcal{Q}.
\end{align*}
\]

(P_{KS}(t))

The feasible set of (P_{KS}(t)) will be denoted by $X_{KS}(t)$. We observe that one switching constraint is replaced by four nonlinear inequality constraints in (P_{KS}(t)). Figure 3 provides an illustration of $X_{KS}(t)$. Note that the family $\{X_{KS}(t)\}_{t \geq 0}$ possesses the same properties as the family $\{X_S(t)\}_{t \geq 0}$ described in Lemma 3.1, see [20, Lemma 3.2] as well. Thus, the adapted Kanzow–Schwartz method is reasonable for switching-constrained problems as well.

Let $x \in X_{KS}(t)$ be a feasible point of $P_{KS}(t)$ for some fixed parameter $t > 0$. Later on, it will be beneficial to work with the index sets defined below:
\[
\begin{align*}
I_{0,0}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) = t, H_l(x) = t\}, & \quad I_{1,0}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) = -t, H_l(x) = t\}, \\
I_{1,1}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) = t, H_l(x) > t\}, & \quad I_{1,2}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) = -t, H_l(x) > t\}, \\
I_{1,3}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) > t, H_l(x) = t\}, & \quad I_{1,4}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) < -t, H_l(x) = t\}, \\
I_{1,3}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) = -t, H_l(x) < t\}, & \quad I_{1,4}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) = t, H_l(x) < t\}, \\
I_{1,3}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) < -t, H_l(x) = t\}, & \quad I_{1,4}^{00}(x) := \{l \in \mathcal{Q} | G_l(x) > t, H_l(x) = -t\}.
\end{align*}
\]

Note that all these sets are pairwise disjoint. The index sets $I_{1,1}^{00}(x)$, $I_{1,2}^{00}(x)$, and $I_{1,4}^{00}(x)$ subsume the three possible cases where the constraints $\Phi_1^1(x; t) \leq 0$ ($l \in \mathcal{Q}$) are active.
Similarly, the other index sets cover those indices where the constraints $\Phi^s_l(x; t) \leq 0$ ($l \in Q, s \in \{2, 3, 4\}$) are active. It follows that an index $l \in Q$ which does not belong to any of the above sets is inactive for $(P_{KS}(t))$ and can therefore be disregarded (locally).

In order to address any of the four quadrants separately, we will exploit

$$I^0_{t,1}(x) := t^0_{t,1}(x) \cup t^0_{t,2}(x) \cup I^+_{t,1}(x), \quad I^0_{t,2}(x) := t^0_{t,2}(x) \cup t^0_{t,3}(x) \cup I^-_{t,2}(x),$$

$$I^0_{t,3}(x) := t^0_{t,3}(x) \cup t^0_{t,4}(x) \cup I^+_{t,3}(x), \quad I^0_{t,4}(x) := t^0_{t,4}(x) \cup t^0_{t,4}(x) \cup I^-_{t,4}(x),$$

i.e. for fixed $s \in S$, $I^0_{t,s}(x)$ collects all indices $l \in Q$ where the constraint $\Phi^s_l(x; t) \leq 0$ is active.

For brevity, we set

$$I^0_t(x) := \bigcup_{s \in S} I^0_{t,s}(x),$$

$$I^0\pm_t(x) := I^0_{t,1}(x) \cup I^0_{t,2}(x) \cup I^0_{t,3}(x) \cup I^0_{t,4}(x),$$

$$I^0\mp_t(x) := I^+_{t,1}(x) \cup I^-_{t,2}(x) \cup I^-_{t,3}(x) \cup I^+_{t,4}(x).$$

Thus, we collect all indices in $I^0\pm_t(x)$ where $G_l(x) \in [-t, t]$ holds while $|H_l(x)| > t$ is valid. Similarly, $I^0\mp_t(x)$ comprises all indices where $H_l(x) \in [-t, t]$ and $|G_l(x)| > t$ hold true. The set $I^0_t(x)$ contains all those indices where $G_l(x), H_l(x) \in [-t, t]$ is valid.

If $\bar{x} \in X$ is feasible to (MPSC), $x$ lies in a sufficiently small neighbourhood of $\bar{x}$, and $t \geq 0$ is sufficiently small, then the following inclusions follow from the continuity of all appearing functions:

$$I^0_t(x) \cup I^0\pm_t(x) \subset I^G(\bar{x}) \cup I^{GH}(\bar{x}),$$

$$I^0_t(x) \cup I^0\pm_t(x) \subset I^H(\bar{x}) \cup I^{GH}(\bar{x}).$$

In the lemma below, we present explicit formulas for the gradients of $\Phi^s_l(\cdot; t)$ with $l \in Q$ and $s \in S$. They can be derived exploiting Lemma 4.1 as well as the chain rule.
Lemma 4.2: For $x \in \mathbb{R}^n$, $t > 0$, and $l \in Q$, the following formulas are valid:

$$
\nabla_x \Phi_l^1(x; t) = \begin{cases} 
(H_l(x) - t) \nabla G_l(x) + (G_l(x) - t) \nabla H_l(x) & \text{if } G_l(x) + H_l(x) \geq 2t, \\
-(G_l(x) - t) \nabla G_l(x) - (H_l(x) - t) \nabla H_l(x) & \text{if } G_l(x) + H_l(x) < 2t,
\end{cases}
$$

$$
\nabla_x \Phi_l^2(x; t) = \begin{cases} 
(t - H_l(x)) \nabla G_l(x) - (G_l(x) + t) \nabla H_l(x) & \text{if } -G_l(x) + H_l(x) \geq 2t, \\
-(G_l(x) + t) \nabla G_l(x) - (H_l(x) - t) \nabla H_l(x) & \text{if } -G_l(x) + H_l(x) < 2t,
\end{cases}
$$

$$
\nabla_x \Phi_l^3(x; t) = \begin{cases} 
(H_l(x) + t) \nabla G_l(x) + (G_l(x) + t) \nabla H_l(x) & \text{if } -G_l(x) - H_l(x) \geq 2t, \\
-(G_l(x) + t) \nabla G_l(x) - (H_l(x) + t) \nabla H_l(x) & \text{if } -G_l(x) - H_l(x) < 2t,
\end{cases}
$$

$$
\nabla_x \Phi_l^4(x; t) = \begin{cases} 
-(H_l(x) + t) \nabla G_l(x) + (t - G_l(x)) \nabla H_l(x) & \text{if } G_l(x) - H_l(x) \geq 2t, \\
-(G_l(x) - t) \nabla G_l(x) - (H_l(x) + t) \nabla H_l(x) & \text{if } G_l(x) - H_l(x) < 2t.
\end{cases}
$$

Particularly, we have

$$
\forall l \in I_{t,1}^0(x) : \nabla_x \Phi_l^1(x; t) = \begin{cases} 
(G_l(x) - t) \nabla H_l(x) & \text{if } l \in I_{t,1}^0(x), \\
(H_l(x) - t) \nabla G_l(x) & \text{if } l \in I_{t,1}^{+1}(x), \\
0 & \text{if } l \in I_{t,1}^{00}(x),
\end{cases}
$$

$$
\forall l \in I_{t,2}^0(x) : \nabla_x \Phi_l^2(x; t) = \begin{cases} 
-(G_l(x) + t) \nabla H_l(x) & \text{if } l \in I_{t,2}^{-0}(x), \\
(t - H_l(x)) \nabla G_l(x) & \text{if } l \in I_{t,2}^{+0}(x), \\
0 & \text{if } l \in I_{t,2}^{00}(x),
\end{cases}
$$

$$
\forall l \in I_{t,3}^0(x) : \nabla_x \Phi_l^3(x; t) = \begin{cases} 
(G_l(x) + t) \nabla H_l(x) & \text{if } l \in I_{t,3}^{-0}(x), \\
(H_l(x) + t) \nabla G_l(x) & \text{if } l \in I_{t,3}^{+0}(x), \\
0 & \text{if } l \in I_{t,3}^{00}(x),
\end{cases}
$$

$$
\forall l \in I_{t,4}^0(x) : \nabla_x \Phi_l^4(x; t) = \begin{cases} 
(t - G_l(x)) \nabla H_l(x) & \text{if } l \in I_{t,4}^{+0}(x), \\
-(H_l(x) + t) \nabla G_l(x) & \text{if } l \in I_{t,4}^{00}(x), \\
0 & \text{if } l \in I_{t,4}^{00}(x).
\end{cases}
$$

As a corollary of the above lemma, we obtain an explicit formula for the linearization cone associated with $(P_{KS}(t))$.

Corollary 4.3: Fix $t > 0$ and a feasible point $x \in X_{KS}(t)$ of $(P_{KS}(t))$. Then the following formula is valid:

$$
\mathcal{L}_{X_{KS}(t)}(x) = \left\{ d \in \mathbb{R}^n : \begin{array}{c}
\nabla g_i(x) \cdot d \leq 0 \quad i \in I^8(x) \\
\nabla h_j(x) \cdot d = 0 \quad j \in P \\
\nabla G_l(x) \cdot d \leq 0 \quad l \in I_{t,1}^{0+}(x) \cup I_{t,4}^{0-}(x) \\
\nabla G_l(x) \cdot d \geq 0 \quad l \in I_{t,2}^{0+}(x) \cup I_{t,3}^{0-}(x) \\
\nabla H_l(x) \cdot d \leq 0 \quad l \in I_{t,1}^{+0}(x) \cup I_{t,2}^{-0}(x) \\
\nabla H_l(x) \cdot d \geq 0 \quad l \in I_{t,3}^{0-}(x) \cup I_{t,4}^{+0}(x) \end{array} \right\}.
$$
4.2. Convergence properties

In this section, we analyze the theoretical properties of the adapted Kanzow–Schwartz relaxation scheme. The underlying idea of the numerical method is the same as already stated in Section 3: For a sequence of positive relaxation parameters converging to zero, we compute a sequence of KKT points associated with \((PKS(t))\) and (in case of existence) study the properties of its limit point. First, we will show that whenever MPSC-NNAMCQ is valid at \(\bar{x}\), then it is an M-stationary point of \((MPSC)\). Second, it will be demonstrated that validity of MPSC-LICQ at \(\bar{x}\) implies that GCQ holds for the surrogate problem \((PKS(t))\) at all feasible points from a sufficiently small neighbourhood of \(\bar{x}\) and sufficiently small \(t > 0\). This property ensures that local minima of the surrogate problem \((PKS(t))\) which are located near \(\bar{x}\) are in fact KKT points. This way, it is shown that the underlying assumptions of the method are reasonable.

**Theorem 4.4:** Let \(\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+\) be a sequence of positive relaxation parameters converging to zero. For each \(k \in \mathbb{N}\), let \(x_k \in X_{KS}(t_k)\) be a KKT point of \((PKS(t_k))\). We assume that \(\{x_k\}_{k \in \mathbb{N}}\) converges to \(\bar{x} \in X\) where MPSC-NNAMCQ holds. Then \(\bar{x}\) is an M-stationary point of \((MPSC)\).

**Proof:** Since \(x_k\) is a KKT point of \((PKS(t_k))\), there exist multipliers \(\lambda^k \in \mathbb{R}^m\), \(\rho^k \in \mathbb{R}^p\), and \(\alpha^k, \beta^k, \gamma^k, \delta^k \in \mathbb{R}^q\) which solve the following system:

\[
0 = \nabla f(x_k) + \sum_{i \in I^S(x_k)} \lambda_i^k \nabla g_i(x_k) + \sum_{j \in P} \rho_j^k \nabla h_j(x_k)
\]

\[
+ \sum_{l \in Q} \left[ \alpha_l^k \nabla x \Phi_1^l(x_k; t_k) + \beta_l^k \nabla x \Phi_2^l(x_k; t_k) + \gamma_l^k \nabla x \Phi_3^l(x_k; t_k) + \delta_l^k \nabla x \Phi_4^l(x_k; t_k) \right],
\]

\(\forall i \in I^S(x_k): \lambda_i^k \geq 0, \forall i \in \mathcal{M} \setminus I^S(x_k): \lambda_i^k = 0,\)

\(\forall l \in I^p_{k,1}(x_k): \alpha_l^k \geq 0, \forall l \in \mathcal{Q} \setminus I^p_{k,1}(x_k): \alpha_l^k = 0,\)

\(\forall l \in I^p_{k,2}(x_k): \beta_l^k \geq 0, \forall l \in \mathcal{Q} \setminus I^p_{k,2}(x_k): \beta_l^k = 0,\)

\(\forall l \in I^p_{k,3}(x_k): \gamma_l^k \geq 0, \forall l \in \mathcal{Q} \setminus I^p_{k,3}(x_k): \gamma_l^k = 0,\)

\(\forall l \in I^p_{k,4}(x_k): \delta_l^k \geq 0, \forall l \in \mathcal{Q} \setminus I^p_{k,4}(x_k): \delta_l^k = 0.\)

Next, let us define new multipliers \(\alpha^k_G, \alpha^k_H, \beta^k_G, \beta^k_H, \gamma^k_G, \gamma^k_H, \delta^k_G, \delta^k_H \in \mathbb{R}^q\) as stated below for all \(l \in \mathcal{Q}:\)

\[
\alpha^k_{G,l} := \begin{cases} 
\alpha^k_l (H_l(x_k) - t_k) & l \in I^p_{k,1}(x_k), \\
0 & \text{otherwise},
\end{cases}
\]

\[
\alpha^k_{H,l} := \begin{cases} 
\alpha^k_l (G_l(x_k) - t_k) & l \in I^p_{k,1}(x_k), \\
0 & \text{otherwise},
\end{cases}
\]

\[
\beta^k_{G,l} := \begin{cases} 
\beta^k_l (t_k - H_l(x_k)) & l \in I^p_{k,2}(x_k), \\
0 & \text{otherwise},
\end{cases}
\]

\[
\beta^k_{H,l} := \begin{cases} 
\beta^k_l (-G_l(x_k) - t_k) & l \in I^p_{k,2}(x_k), \\
0 & \text{otherwise},
\end{cases}
\]
Furthermore, we set $\mu^k := \alpha^k_G + \beta^k_G + \gamma^k_G + \delta^k_G$ and $\nu^k := \alpha^k_H + \beta^k_H + \gamma^k_H + \delta^k_H$. By definition, we have supp $\mu^k \subset I_{tk}^{0\pm}(x_k)$ as well as supp $\nu^k \subset I_{tk}^{\pm0}(x_k)$. Thus, for sufficiently large $k \in \mathbb{N}$, (4.1) yields

\[
\text{supp } \mu^k \subset I^G(\bar{x}) \cup I^{GH}(\bar{x}), \quad \text{supp } \nu^k \subset I^H(\bar{x}) \cup I^{GH}(\bar{x}).
\]

Using Lemma 4.2,

\[
0 = \nabla f(x_k) + \sum_{i \in I^S(x_k)} \lambda_i^k \nabla g_i(x_k) + \sum_{j \in P} \rho_j^k \nabla h_j(x_k) + \sum_{l \in Q} \left[ \mu_l^k \nabla G_l(x_k) + \nu_l^k \nabla H_l(x_k) \right]
\]

is obtained.

Next, it will be shown that the sequence $\{(\lambda^k, \rho^k, \mu^k, \nu^k)\}_{k \in \mathbb{N}}$ is bounded. Assuming the contrary, we define

\[
\forall k \in \mathbb{N}: \quad (\tilde{\lambda}^k, \tilde{\rho}^k, \tilde{\mu}^k, \tilde{\nu}^k) := \frac{(\lambda^k, \rho^k, \mu^k, \nu^k)}{\|\lambda^k, \rho^k, \mu^k, \nu^k\|_2}.
\]

Clearly, $\{(\tilde{\lambda}^k, \tilde{\rho}^k, \tilde{\mu}^k, \tilde{\nu}^k)\}_{k \in \mathbb{N}}$ is bounded and, thus, possesses a converging subsequence (without relabelling) with nonvanishing limit $(\bar{\lambda}, \bar{\rho}, \bar{\mu}, \bar{\nu})$. The continuity of $g$ yields supp $\bar{\lambda} \subset I^S(\bar{x})$. The above considerations yield

\[
\text{supp } \bar{\mu} \subset I^G(\bar{x}) \cup I^{GH}(\bar{x}), \quad \text{supp } \bar{\nu} \subset I^H(\bar{x}) \cup I^{GH}(\bar{x}).
\]

Choose $l \in I^{GH}(\bar{x})$ arbitrarily. If $\bar{\mu}_l \neq 0$ holds true, then $\bar{\mu}_l^k \neq 0$ must be valid for sufficiently large $k \in \mathbb{N}$. By definition of $\mu^k_l$, $l \in I_{tk}^{0\pm}(x_k)$ follows. Since $I_{tk}^{0\pm}(x_k)$ and $I_{tk}^{\pm0}(x_k)$ are disjoint, $\nu_l^k = 0$ holds for sufficiently large $k \in \mathbb{N}$. This yields $\bar{\nu}_l^k = 0$ for sufficiently large $k \in \mathbb{N}$, i.e. $\bar{\nu}_l = 0$ is obtained. This shows

\[
\forall l \in I^{GH}(\bar{x}): \quad \bar{\mu}_l \bar{\nu}_l = 0.
\]

Dividing (4.2) by $\|\lambda^k, \rho^k, \mu^k, \nu^k\|_2$, taking the limit $k \to \infty$, respecting the continuous differentiability of $f, g, h, G$, as well as $H$, and invoking the above arguments, we obtain

\[
0 = \sum_{i \in I^S(\bar{x})} \tilde{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j \in P} \tilde{\rho}_j \nabla h_j(\bar{x}) + \sum_{l \in Q} \left[ \tilde{\mu}_l \nabla G_l(\bar{x}) + \tilde{\nu}_l \nabla H_l(\bar{x}) \right],
\]

\[
\forall i \in I^S(\bar{x}): \quad \tilde{\lambda}_i \geq 0, \quad \forall i \in \mathcal{M} \setminus I^S(\bar{x}): \quad \tilde{\lambda}_i = 0,
\]

\[
\forall l \in I^H(\bar{x}): \quad \tilde{\mu}_l = 0,
\]

\[
\forall l \in I^G(\bar{x}): \quad \tilde{\nu}_l = 0,
\]

\[
\forall l \in I^{GH}(\bar{x}): \quad \tilde{\mu}_l \tilde{\nu}_l = 0.
\]
Due to the fact that \((\tilde{x}, \tilde{\rho}, \tilde{\mu}, \tilde{v})\) does not vanish, this is a contradiction to the validity of MPSC-NNAMCQ. Thus, \(\{ (\lambda^k, \rho^k, \mu^k, v^k) \}_{k \in \mathbb{N}} \) is bounded.

We assume w.l.o.g. that \(\{ (\lambda^k, \rho^k, \mu^k, v^k) \}_{k \in \mathbb{N}} \) converges to \((\lambda, \rho, \mu, v)\) (otherwise, we choose an appropriate subsequence). Reprising the above arguments, we have

\[
\forall i \in I^S(\tilde{x}): \lambda_i \geq 0, \quad \forall i \in M \setminus I^S(\tilde{x}): \lambda_i = 0,
\]
\[
\forall l \in I^H(\tilde{x}): \mu_l = 0,
\]
\[
\forall l \in I^G(\tilde{x}): v_l = 0,
\]
\[
\forall l \in I^{GH}(\tilde{x}): \mu_l v_l = 0.
\]

Taking the limit in (4.2) and respecting the continuous differentiability of all appearing mappings, we obtain

\[
0 = \nabla f(\tilde{x}) + \sum_{i \in I^S(\tilde{x})} \lambda_i \nabla g_i(\tilde{x}) + \sum_{j \in P} \rho_j \nabla h_j(\tilde{x}) + \sum_{l \in Q} \left[ \mu_l \nabla G_l(\tilde{x}) + v_l \nabla H_l(\tilde{x}) \right].
\]

This shows that \(\tilde{x}\) is an M-stationary point of (MPSC).

In order to verify that the validity of MPSC-LICQ at the limit point \(\tilde{x}\) ensures that GCQ holds at all feasible points of \((P_{KS}(t))\) sufficiently close to \(\tilde{x}\) where \(t\) is sufficiently small, we need to study the variational geometry of the sets \(X_{KS}(t)\) in some more detail.

Fix \(t > 0\) and some point \(\tilde{x} \in X_{KS}(t)\). For an arbitrary index set \(I \subset I_t^{00}(\tilde{x})\), we consider the subsequent programme:

\[
\begin{align*}
 f(x) & \to \min \\
g_i(x) & \leq 0, \quad i \in M, \\
h_j(x) & = 0, \quad j \in P, \\
-\tau & \leq G_l(x) \leq \tau, \quad l \in I_t^{00}(\tilde{x}) \cup I, \\
-\tau & \leq H_l(x) \leq \tau, \quad l \in I_t^{00}(\tilde{x}) \cup (I_t^{00}(\tilde{x}) \setminus I), \\
\Phi_l^s(x; t) & \leq 0, \quad l \in Q \setminus I_t^{00}(\tilde{x}), s \in S.
\end{align*}
\]

The feasible set of \((P_{KS}(t, \tilde{x}, I))\) will be denoted by \(X_{KS}(t, \tilde{x}, I)\). Clearly, \(\tilde{x}\) is a feasible point of \((P_{KS}(t, \tilde{x}, I))\) for arbitrary \(I \subset I_t^{00}(\tilde{x})\). Furthermore, \(X_{KS}(t, \tilde{x}, I) \subset X_{KS}(t)\) is valid for any choice of \(I \subset I_t^{00}(\tilde{x})\).

**Lemma 4.5:** For fixed \(t > 0\) and \(\tilde{x} \in X_{KS}(t)\), we have

\[
T_{X_{KS}(t)}(\tilde{x}) = \bigcup_{I \subset I_t^{00}(\tilde{x})} T_{X_{KS}(t, \tilde{x}, I)}(\tilde{x}).
\]

**Proof:** We show both inclusions separately.

‘\(\subset\)’ Fix an arbitrary direction \(d \in T_{X_{KS}(t)}(\tilde{x})\). Then we find sequences \(\{y_k\}_{k \in \mathbb{N}} \subset X_{KS}(t)\) and \(\{\tau_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+\) such that \(y_k \to \tilde{x}, \tau_k \downarrow 0,\) and \((y_k - \tilde{x})/\tau_k \to d\) as \(k \to \infty\). It is sufficient to verify the existence of an index set \(\tilde{I} \subset I_t^{00}(\tilde{x})\) such that \(\{y_k\}_{k \in \mathbb{N}} \cap X_{KS}(t, \tilde{x}, \tilde{I})\) possesses infinite cardinality since this already gives us \(d \in T_{X_{KS}(t, \tilde{x}, \tilde{I})}(\tilde{x})\).
Fix $k \in \mathbb{N}$ sufficiently large and $l \in I_t^{0\pm}(\bar{x})$. Then, due to continuity of $G_l$, $H_l$, as well as $\varphi$ and feasibility of $y_k$ to $(P_{KS}(t))$, we either have $l \in I_t^{0\pm}(y_k)$ and, thus, $G_l(y_k) \in \{-t, t\}$, or $-t < G_l(y_k) < t$. Similarly, we obtain $-t \leq H_l(y_k) \leq t$ for all $l \in I_t^{0\pm}(\bar{x})$. Due to feasibility of $y_k$ to $(P_{KS}(t))$, we have $-t \leq G_l(y_k) \leq t$ or $-t \leq H_l(y_k) \leq t$. Thus, setting

$$I_k := I_t^{00}(\bar{x}) \cap \{l \in \mathcal{Q} | -t \leq G_l(y_k) \leq t\},$$

$y_k \in X_{KS}(t, \bar{x}, I_k)$ is valid. Since there are only finitely many subsets of $I_t^{00}(\bar{x})$ while $\{y_k\}_{k \in \mathbb{N}}$ is infinite, there must exist $\bar{I} \subset I_t^{00}(\bar{x})$ such that $\{y_k\}_{k \in \mathbb{N}} \cap X_{KS}(t, \bar{x}, \bar{I})$ is of infinite cardinality.

By definition of the tangent cone, we easily obtain $T_{X_{KS}(t, \bar{x}, \bar{I})}(\bar{x}) \subset T_{X_{KS}(t)}(\bar{x})$ for each $I \subset I_t^{00}(\bar{x})$. Taking the union over all subsets of $I_t^{00}(\bar{x})$ yields the desired inclusion.

**Theorem 4.6:** Let $\bar{x} \in X$ be a feasible point of (MPSC) where MPSC-LICQ is satisfied. Then there exist $\bar{t} > 0$ and a neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ such that GCQ holds for $(P_{KS}(t))$ at all points from $X_{KS}(t) \cap U$ for all $t \in (0, \bar{t}]$.

**Proof:** Due to the validity of MPSC-LICQ at $\bar{x}$ and the continuous differentiability of $g$, $h$, $G$, and $H$, the gradients

$$\{\nabla g_i(x) | i \in I^g(\bar{x})\} \cup \{\nabla h_j(x) | j \in \mathcal{P}\} \cup \{\nabla G_l(x) | l \in I^G(\bar{x}) \cup I^{GH}(\bar{x})\} \cup \{\nabla H_l(x) | l \in I^H(\bar{x}) \cup I^{GH}(\bar{x})\}$$

are linearly independent for all $x$ which are chosen from a sufficiently small neighbourhood $V$ of $\bar{x}$, see Lemma 2.2. Invoking (4.1), we can choose a neighbourhood $U \subset V$ of $\bar{x}$ and $\bar{t} > 0$ such that for any $\tilde{x} \in X_{KS}(t) \cap U$, where $t \in (0, \bar{t}]$ holds, we have

$$I^g(\bar{x}) \subset I^g(\bar{x}),$$

$$I_t^{00}(\bar{x}) \cup I_t^{0\pm}(\bar{x}) \subset I^G(\bar{x}) \cup I^{GH}(\bar{x}),$$

$$I_t^{00}(\bar{x}) \cup I_t^{0\pm}(\bar{x}) \subset I^H(\bar{x}) \cup I^{GH}(\bar{x}).$$

Particularly, for any such $\tilde{x} \in \mathbb{R}^n$ and $I \subset I_t^{00}(\bar{x})$, the gradients

$$\{\nabla g_i(\tilde{x}) | i \in I^g(\tilde{x})\} \cup \{\nabla h_j(\tilde{x}) | j \in \mathcal{P}\} \cup \{\nabla G_l(\tilde{x}) | l \in I_l^{0\pm}(\tilde{x}) \cup I\} \cup \{\nabla H_l(\tilde{x}) | l \in I_l^{0\pm}(\tilde{x}) \cup (I_t^{00}(\bar{x}) \setminus I)\}$$

are linearly independent, i.e. standard LICQ is valid for $(P_{KS}(t, \tilde{x}, I))$ at $\tilde{x}$ for each set $I \subset I_t^{00}(\bar{x})$. This implies $T_{X_{KS}(t, \tilde{x}, I)}(\tilde{x}) = L_{X_{KS}(t, \tilde{x}, I)}(\tilde{x})$ for each $I \subset I_t^{00}(\bar{x})$. Exploiting
Lemma 4.5, we obtain
\[ T_{X_{KS}(t)}(\bar{x}) = \bigcup_{I \subseteq I^0_{t}(\bar{x})} \mathcal{L}_{X_{KS}(t,\bar{x},I)}(\bar{x}). \]

Computing the polar cone on both sides yields
\[ T_{X_{KS}(t)}(\bar{x})^\circ = \bigcap_{I \subseteq I^0_{t}(\bar{x})} \mathcal{L}_{X_{KS}(t,\bar{x},I)}(\bar{x})^\circ. \quad (4.4) \]

Define
\[
\begin{align*}
T_G^+(t, \bar{x}, I) & := I_{t,1}^0(\bar{x}) \cup I_{t,4}^0(\bar{x}) \cup \left[ I \cap \left( I^0_{t,1}(\bar{x}) \cup I^0_{t,4}(\bar{x}) \right) \right], \\
T_G^-(t, \bar{x}, I) & := I_{t,2}^0(\bar{x}) \cup I_{t,3}^0(\bar{x}) \cup \left[ I \cap \left( I^0_{t,2}(\bar{x}) \cup I^0_{t,3}(\bar{x}) \right) \right], \\
T_H^+(t, \bar{x}, I) & := I_{t,1}^0(\bar{x}) \cup I_{t,2}^0(\bar{x}) \cup \left[ \left( I^0_t(\bar{x}) \setminus I \right) \cap \left( I^0_{t,1}(\bar{x}) \cup I^0_{t,2}(\bar{x}) \right) \right], \\
T_H^-(t, \bar{x}, I) & := I_{t,3}^0(\bar{x}) \cup I_{t,4}^0(\bar{x}) \cup \left[ \left( I^0_t(\bar{x}) \setminus I \right) \cap \left( I^0_{t,3}(\bar{x}) \cup I^0_{t,4}(\bar{x}) \right) \right]
\end{align*}
\]
and observe that these sets characterize the indices \( l \in \mathcal{Q} \) where the constraints \( G_l(\bar{x}) \leq t, G_l(\bar{x}) \geq -t, H_l(\bar{x}) \leq t, \) and \( H_l(\bar{x}) \geq -t, \) respectively, are active in \( (P_{KS}(t, \bar{x}, I)) \). We therefore obtain

\[
\mathcal{L}_{X_{KS}(t,\bar{x},I)}(\bar{x}) = \left\{ d \in \mathbb{R}^n : \begin{array}{l}
\nabla g_l(\bar{x}) \cdot d \leq 0 \quad l \in I^k(\bar{x}) \\
\nabla h_j(\bar{x}) \cdot d = 0 \quad j \in \mathcal{P} \\
\n\nabla G_l(\bar{x}) \cdot d \leq 0 \quad l \in T_G^+(t, \bar{x}, I) \\
\n\nabla G_l(\bar{x}) \cdot d \geq 0 \quad l \in T_G^-(t, \bar{x}, I) \\
\n\nabla H_l(\bar{x}) \cdot d \leq 0 \quad l \in T_H^+(t, \bar{x}, I) \\
\n\nabla H_l(\bar{x}) \cdot d \geq 0 \quad l \in T_H^-(t, \bar{x}, I)
\end{array} \right\}.
\]

Exploiting Lemma 2.1, the polar of this cone is easily computed:
\[
\mathcal{L}_{X_{KS}(t,\bar{x},I)}(\bar{x})^\circ = \left\{ \eta \in \mathbb{R}^n : \begin{array}{l}
\eta = \sum_{i \in I^k(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in \mathcal{P}} \rho_j \nabla h_j(\bar{x}) \\
\quad + \sum_{l \in \mathcal{Q}} \left[ \mu_l \nabla G_l(\bar{x}) + \nu_l \nabla H_l(\bar{x}) \right] \\
\quad \forall i \in I^k(\bar{x}) : \lambda_i \geq 0 \\
\quad \forall l \in \mathcal{Q} : \mu_l \leq 0 \quad \text{if } l \in T_G^+(t, \bar{x}, I), \\
\quad \mu_l = 0 \quad \text{otherwise}, \\
\quad \forall l \in \mathcal{Q} : \nu_l \geq 0 \quad \text{if } l \in T_H^+(t, \bar{x}, I), \\
\quad \nu_l = 0 \quad \text{otherwise}
\end{array} \right\}. \quad (4.5)
\]

Observing that \( (P_{KS}(t)) \) is a standard nonlinear problem, \( T_{X_{KS}(t)}(\bar{x}) \subseteq \mathcal{L}_{X_{KS}(t)}(\bar{x}) \) and, thus, \( \mathcal{L}_{X_{KS}(t)}(\bar{x})^\circ \subseteq T_{X_{KS}(t)}(\bar{x})^\circ \) are inherent. It remains to show the inclusion \( T_{X_{KS}(t)}(\bar{x})^\circ \subseteq \mathcal{L}_{X_{KS}(t)}(\bar{x})^\circ \).
Thus, choose \( \eta \in T_{\text{KS}}(\tilde{x})^\circ \) arbitrarily. Then, in particular, (4.4) yields
\[
\eta \in \mathcal{L}_{\text{KS}}(\tilde{x}, \varphi)(\tilde{x})^\circ \subseteq \mathcal{L}_{\text{KS}}(\tilde{x}, \tilde{u}^*_t(\tilde{x}))(\tilde{x})^\circ.
\]

Exploiting the representation (4.5), we find \( \lambda_i, \lambda'_i \geq 0 \) \((i \in I^g(\tilde{x}))\), \( \rho, \rho' \in \mathbb{R}^p \), \( \mu, \mu' \in \mathbb{R}^q \), as well as \( \nu, \nu' \in \mathbb{R}^q \) which satisfy
\[
\eta = \sum_{i \in I^p(\tilde{x})} \lambda_i \nabla g_i(\tilde{x}) + \sum_{j \in \mathcal{P}} \rho_j \nabla h_j(\tilde{x}) + \sum_{l \in \mathcal{Q}} \left[ \mu_l \nabla G_l(\tilde{x}) + \nu_l \nabla H_l(\tilde{x}) \right]
\]
and
\[
\mu_l \begin{cases} 
\geq 0 & \text{if } l \in l_{1,1}^{0+}(\tilde{x}) \cup l_{1,4}^{0-}(\tilde{x}), \\
\leq 0 & \text{if } l \in l_{1,3}^{0+}(\tilde{x}) \cup l_{1,2}^{0-}(\tilde{x}), \\
= 0 & \text{otherwise,}
\end{cases}
\]
\[
\mu'_l \begin{cases} 
\geq 0 & \text{if } l \in l_{1,1}^{00}(\tilde{x}) \cup l_{1,2}^{00}(\tilde{x}) \cup l_{1,3}^{00}(\tilde{x}) \cup l_{1,4}^{00}(\tilde{x}), \\
\leq 0 & \text{if } l \in l_{1,2}^{00}(\tilde{x}) \cup l_{1,3}^{00}(\tilde{x}) \cup l_{1,4}^{00}(\tilde{x}), \\
= 0 & \text{otherwise,}
\end{cases}
\]
\[
\nu_l \begin{cases} 
\geq 0 & \text{if } l \in l_{1,1}^{10}(\tilde{x}) \cup l_{1,2}^{10}(\tilde{x}), \\
\leq 0 & \text{if } l \in l_{1,3}^{10}(\tilde{x}) \cup l_{1,4}^{10}(\tilde{x}), \\
= 0 & \text{otherwise,}
\end{cases}
\]
\[
\nu'_l \begin{cases} 
\geq 0 & \text{if } l \in l_{1,1}^{-0}(\tilde{x}) \cup l_{1,2}^{-0}(\tilde{x}), \\
\leq 0 & \text{if } l \in l_{1,3}^{-0}(\tilde{x}) \cup l_{1,4}^{-0}(\tilde{x}), \\
= 0 & \text{otherwise,}
\end{cases}
\]
for all \( l \in \mathcal{Q} \). Thus, we obtain
\[
0 = \sum_{i \in I^g(\tilde{x})} (\lambda_i - \lambda'_i) \nabla g_i(\tilde{x}) + \sum_{j \in \mathcal{P}} (\rho_j - \rho'_j) \nabla h_j(\tilde{x})
\]
\[
+ \sum_{l \in \mathcal{Q}} \left[ (\mu_l - \mu'_l) \nabla G_l(\tilde{x}) + (\nu_l - \nu'_l) \nabla H_l(\tilde{x}) \right].
\]
Observing \( \text{supp}(\mu - \mu') \subseteq I_{t_i}^{00}(\tilde{x}) \cup I_{t_i}^{0\pm}(\tilde{x}) \) as well as \( \text{supp}(\nu - \nu') \subseteq I_{t_i}^{00}(\tilde{x}) \cup I_{t_i}^{\pm0}(\tilde{x}) \) and using (4.3), we obtain \( \lambda_i = \lambda'_i \) \((i \in I^g(\tilde{x}))\), \( \rho = \rho' \), \( \mu = \mu' \), as well as \( \nu = \nu' \). Particularly,
\[
\eta = \sum_{i \in I^g(\tilde{x})} \lambda_i \nabla g_i(\tilde{x}) + \sum_{j \in \mathcal{P}} \rho_j \nabla h_j(\tilde{x}) + \sum_{l \in \mathcal{Q}} \left[ \mu_l \nabla G_l(\tilde{x}) + \nu_l \nabla H_l(\tilde{x}) \right]
\]
is obtained where \( \text{supp } \mu \subseteq I_{t_i}^{0\pm}(\tilde{x}) \) and \( \text{supp } \nu' \subseteq I_{t_i}^{\pm0}(\tilde{x}) \) hold true. Finally, we exploit Corollary 4.3 in order to see \( \eta \in \mathcal{L}_{\text{KS}}(\tilde{x})^\circ \). This shows the validity of the inclusion \( T_{\text{KS}}(\tilde{x})^\circ \subseteq \mathcal{L}_{\text{KS}}(\tilde{x})^\circ \) and, thereby, GCQ holds true for \((\text{P}_{\text{KS}}(t))\) at \( \tilde{x} \). Since \( t \in (0, \tilde{t}] \) and \( \tilde{x} \in X_{\text{KS}}(t) \cap U \) were arbitrarily chosen, the proof is completed. \( \blacksquare \)
5. Remarks on other possible relaxation schemes

In this section, we briefly discuss two more relaxation approaches for the numerical treatment of (MPSC) which are inspired by the rich theory on MPCCs. Particularly, the relaxation schemes provided in [30] as well as [19] are adapted to the setting of switching-constrained optimization.

5.1. The relaxation scheme of Steffensen and Ulbrich

Here, we adapt the relaxation scheme from [30] for the numerical treatment of (MPSC). For any $t > 0$, let us introduce $\phi(\cdot; t) : \mathbb{R} \to \mathbb{R}$ by means of

$$\forall z \in \mathbb{R} : \phi(z; t) := \begin{cases} |z| & \text{if } |z| \geq t, \\ t \theta(z/t) & \text{if } |z| < t, \end{cases}$$

where $\theta : [-1, 1] \to \mathbb{R}$ is a twice continuously differentiable function with the following properties:

(a) $\theta(1) = \theta(-1) = 1$, \hspace{1cm} (b) $\theta'(-1) = -1$ and $\theta'(1) = 1$,

(c) $\theta''(-1) = \theta''(1) = 0$, \hspace{1cm} (d) $\theta''(z) > 0$ for all $z \in (-1, 1)$.

A typical example for a function $\theta$ with the above properties is given by

$$\forall z \in [-1, 1] : \theta(z) := \frac{2}{\pi} \sin \left( \frac{\pi}{2} z + \frac{3\pi}{2} \right) + 1, \quad (5.1)$$

see [30, Section 3]. Noting that the function $\phi(\cdot; t)$ is smooth for each choice of $t > 0$, it can be used to regularize the feasible set of (MPSC). A suitable surrogate problem is given by

$$f(x) \to \min \quad g_i(x) \leq 0, \quad i \in \mathcal{M},$$

$$h_j(x) = 0, \quad j \in \mathcal{P},$$

$$G_l(x) + H_l(x) - \phi(G_l(x) - H_l(x); t) \leq 0, \quad l \in \mathcal{Q}, \quad (PSU(t))$$

$$G_l(x) - H_l(x) - \phi(G_l(x) + H_l(x); t) \leq 0, \quad l \in \mathcal{Q},$$

$$-G_l(x) + H_l(x) - \phi(-G_l(x) - H_l(x); t) \leq 0, \quad l \in \mathcal{Q},$$

$$-G_l(x) - H_l(x) - \phi(-G_l(x) + H_l(x); t) \leq 0, \quad l \in \mathcal{Q}.$$
introduce four inequality constraints to replace one of the original switching constraints. In contrast to the relaxation schemes presented in Sections 3 and 4 where the feasible set of (MPSC) has been changed globally, programme (PSU(t)) provides only a local relaxation of the switching structure.

It has been mentioned in [18] that the relaxation scheme of Steffensen and Ulbrich computes C-stationary points of MPCCs under an MPCC-tailored version of CPLD, see [18, Section 3.4] as well. Recalling some arguments from Sections 2.3 and 3, the adapted method may only find W-stationary points of (MPSC) in general. The upcoming example confirms this conjecture.

**Example 5.1:** Let us consider the switching-constrained optimization problem

\[
\begin{align*}
x_1 x_2 - x_1 - x_2 & \rightarrow \min \\
x_1^2 + x_2^2 - 1 & \leq 0, \\
x_1 x_2 & = 0.
\end{align*}
\]

(5.2)

Obviously, the globally optimal solutions of this problem are given by (1, 0) as well as (0, 1), and these points are S-stationary. Furthermore, there is a W-stationary point at \(\bar{x} = (0, 0)\) which is no local minimizer. The global maximizers \((-1, 0)\) and \((0, -1)\) do not satisfy any of the introduced stationarity concepts.

Let us consider the associated family of nonlinear problems (PSU(t)) for \(t \in (0, 1]\) where \(\theta\) is chosen as in (5.1). It can be checked that \(x(t) := (\frac{t}{2}(1 - \frac{2}{\pi}), \frac{t}{2}(1 - \frac{2}{\pi}))\) is a KKT point of (PSU(t)). Note that \(x(t) \rightarrow \bar{x}\) as \(t \downarrow 0\), and that MPSC-LICQ holds at \(\bar{x}\). However, \(\bar{x}\) is only a W-stationary point of the switching-constrained problem (5.2).
5.2. The relaxation scheme of Kadrani, Dussault, and Benchakroun

Finally, we want to take a brief look at the relaxation approach which was suggested in [19] for the treatment of MPCCs. For any $t \geq 0$, let us consider the optimization problem

$$
\begin{align*}
    f(x) &\rightarrow \min \\
    g_i(x) &\leq 0, \quad i \in \mathcal{M}, \\
    h_j(x) &= 0, \quad j \in \mathcal{P}, \\
    (G_l(x) - t)(H_l(x) - t) &\leq 0, \quad l \in \mathcal{Q}, \\
    (-G_l(x) - t)(H_l(x) - t) &\leq 0, \quad l \in \mathcal{Q}, \\
    (G_l(x) + t)(H_l(x) + t) &\leq 0, \quad l \in \mathcal{Q}, \\
    (G_l(x) - t)(-H_l(x) - t) &\leq 0, \quad l \in \mathcal{Q},
\end{align*}
$$

(P_{KDB}(t))

whose feasible set will be denoted by $X_{KDB}(t)$. The family $\{X_{KDB}(t)\}_{t \geq 0}$ only satisfies property (P1) from Lemma 3.1 while (P2) and (P3) are violated in general. Thus, the surrogate problem (P_{KDB}(t)) does not induce a relaxation technique for (MPSC) in the narrower sense. Figure 5 depicts that $X_{KBD}(t)$ is almost disconnected, i.e. it is close to crumbling into disjoint sets for any $t > 0$. In particular, (P_{KDB}(t)) still possesses combinatorial structure. This may cause serious problems when standard techniques are used to solve the associated surrogate problem (P_{KDB}(t)). Moreover, four inequality constraints are necessary to replace one switching constraint from (MPSC) in (P_{KDB}(t)).

On the other hand, it is clear from [18, Section 3.3] that the regularization approach of Kadrani, Dussault, and Benchakroun computes M-stationary points of MPCCs under an MPCC-tailored version of CPLD at the limit point. Furthermore, if an MPCC-tailored LICQ holds at the limit point, then standard GCQ holds for the surrogate problems in a neighbourhood of the point for sufficiently small relaxation parameters. These results are closely related to those for the relaxation approach from [20] which we generalized to (MPSC) in Section 4.

![Figure 5. Geometric interpretation of the relaxed feasible set $X_{KDB}(t)$.](image)
Although we abstain from a detailed analysis of the regularization method which is induced by the surrogate problem \((P_{\text{KDB}}(t))\) due to the aforementioned shortcomings, the above arguments motivate the formulation of the following two conjectures.

**Conjecture 5.2:** Let \(\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+\) be a sequence of positive regularization parameters converging to zero. For each \(k \in \mathbb{N}\), let \(x_k \in X_{\text{KDB}}(t_k)\) be a KKT point of \((P_{\text{KDB}}(t_k))\). Assume that the sequence \(\{x_k\}_{k \in \mathbb{N}}\) converges to a point \(\bar{x} \in X\) where \(\text{MPSC-MFCQ}\) holds. Then \(\bar{x}\) is an M-stationary point of \((\text{MPSC})\).

**Conjecture 5.3:** Let \(\bar{x} \in X\) be a feasible point of \((\text{MPSC})\) where \(\text{MPSC-LICQ}\) is satisfied. Then there exist \(\tilde{t} > 0\) and a neighbourhood \(U \subset \mathbb{R}^n\) of \(\bar{x}\) such that \(\text{GCQ}\) holds for \((P_{\text{KDB}}(t))\) at all points from \(X_{\text{KDB}}(t) \cap U\) for all \(t \in (0, \tilde{t})\).

### 6. Numerical results

This section is dedicated to a detailed analysis and comparison of four numerical methods for \((\text{MPSC})\). We consider the adapted relaxation schemes of Scholtes from Section 3, Kanzow and Schwartz from Section 4, as well as Steffensen and Ulbrich from Section 5.1. Additionally, we carry out the experiments using a standard solver from nonlinear programming and solve \((\text{MPSC})\) directly, i.e. the switching constraints are interpreted as standard equality constraints. All four methods are applied to multiple classes of switching-constrained programmes which possess significant practical relevance. The particular examples we analyze are:

- an either-or constrained problem with known local and global solutions,
- a switching-constrained optimal control problem involving the non-stationary heat equation in two dimensions, and
- optimization problems involving semi-continuous variables, in particular special instances of portfolio optimization.

For each of the examples, we first provide an overview of the corresponding problem structure, and then give some numerical results. To facilitate a quantitative comparison of the used algorithms, we use performance profiles, see [9], based on the computed function values. Note that we do not perform computational experiments w.r.t. the relaxation scheme of Kadrani, Dussault, and Benchakroun due to two reasons: First, the results from [18] suggest that the theoretical and numerical behaviour of this method is closely related to the one of the relaxation scheme by Kanzow and Schwartz. Secondly, this approach does not provide a classical relaxation method since it violates \((P2)\) and \((P3)\) from Lemma 3.1, and the variational geometry of the associated feasible set is of combinatorial structure.

### 6.1. Implementation

The numerical experiments in this section were all done in MATLAB R2018a. The particular algorithms we use for our computations are the following:

- **Schol:** the adapted Scholtes relaxation scheme from Section 3,
- **KanSch:** the adapted Kanzow–Schwartz relaxation scheme from Section 4,
SteUlb: the adapted Steffensen–Ulbrich relaxation scheme from Section 5.1, and SNOPT: the SNOPT nonlinear programming solver from [12], called through the TOMLAB programming environment.

The overall implementation is done in MATLAB, and each algorithm is called with user-supplied gradients of the objective functions and constraints. The stopping tolerance for all algorithms is set to $10^{-4}$ (although it should be noted that the methods use different stopping criteria, i.e. they impose the accuracy in different ways). For the relaxation algorithms, the relaxation parameters are chosen as $t_k := 0.01^k$, and these methods are also terminated as soon as $t_k$ drops below $10^{-8}$. For the implementation of the method SteUlb, we use the function $\theta$ from (5.1). Finally, to solve the relaxed nonlinear subproblems in the relaxation method, we employ the SNOPT solver with an accuracy of $10^{-6}$.

To accurately measure the performance of the solvers, it is important to note that switching-constrained programmes can, in general, admit a substantial amount of local minimizers. Therefore, the robustness is best measured by comparing the obtained function values (using different methods and starting points) with the globally optimal function value—if the latter is known; otherwise, a suitable approximate is used. To avoid placing too much emphasis on the accuracy of the final output (which does not make sense since the algorithms use different stopping criteria), we use the quantity

$$Q_\delta(x_p^a) := \begin{cases} f(x_p^a) - f_{\min} + \delta & \text{if } x_p^a \text{ is feasible within tolerance,} \\ +\infty & \text{otherwise} \end{cases}$$

as the base metric for the performance profiles, where $x_p^a$ is the final iterate of the given algorithmic run of algorithm $a \in A := \{\text{Schol, KanSch, SteUlb, SNOPT}\}$ for problem $p$, $f_{\min}$ is the (approximate) global minimal value of the underlying problem, and $\delta \geq 0$ is an additional parameter which reduces the sensitivity of the values to numerical accuracy. We have found that an appropriate choice of $\delta$ can eminently improve the meaningfulness of the results. In order to obtain significant results, different problem settings from a set $P$ are considered. In Sections 6.2.1 and 6.2.2, the problems from $P$ only differ w.r.t. given starting points for our algorithmic runs, while in Section 6.2.3, the instances from $P$ are different realizations of the same underlying model from portfolio optimization. The resulting performance ratio is given by

$$\forall p \in P \forall a \in A: \quad r_{p,a} := \frac{Q_\delta(x_p^a)}{\min\{Q_\delta(x_p^\alpha) | \alpha \in A\}}.$$ 

In the performance profiles, we plot the illustrative parts of the curves $\rho_a : [1, \infty) \rightarrow [0, 1]$ given by

$$\forall \tau \in [1, \infty): \quad \rho_a(\tau) := \frac{|\{p \in P | r_{p,a} \leq \tau\}|}{|P|}$$

for each algorithm $a \in A$. Here, $|\cdot|$ denotes the cardinality of a set.

6.2. Numerical examples

The following pages contain three examples of MPSCs. In Section 6.2.1, we deal with an either-or constrained problem, which can be reformulated as an MPSC, see [23, Section 7].
Section 6.2.2 is dedicated to a (discretized) switching-constrained optimal control problem based on the framework from [8]. Finally, in Section 6.2.3, we deal with a class of optimization problems with semi-continuous variables, which can again be reformulated as MPSCs. This section contains a particular example from portfolio optimization which originates from [11].

### 6.2.1. An either-or constrained example

Let us consider the optimization problem

\[(x_1 - 8)^2 + (x_2 + 3)^2 \rightarrow \min\]
\[x_1 - 2x_2 + 4 \leq 0 \lor x_1 - 2 \leq 0,\]
\[x_1^2 - 4x_2 \leq 0 \lor (x_1 - 3)^2 + (x_2 - 1)^2 - 10 \leq 0.\]  

(6.2)

Here, \(\lor\) denotes the logical ‘or’. The feasible set of this programme is visualized in Figure 6. It is easily seen that (6.2) possesses the unique global minimizer \(\bar{x} = (2, -2)\) and another local minimizer \(\tilde{x} = (4, 4)\). Arguing as in [23, Section 7], we can transform (6.2) into a switching-constrained optimization problem by introducing additional variables:

\[(x_1 - 8)^2 + (x_2 + 3)^2 \rightarrow \min\]
\[x, z, z_1, z_2, z_3, z_4 \leq 0,\]
\[(x_1 - 2x_2 + 4 - z_1)(x_1 - 2 - z_2) = 0,\]
\[(x_1^2 - 4x_2 - z_3)((x_1 - 3)^2 + (x_2 - 1)^2 - 10 - z_4) = 0.\]  

(6.3)

Note that the local minimizers of (6.2) can be found among the local minimizers of (6.3) choosing suitable values for the variable \(z\), see [23, Section 7.1].

The algorithms in question are each tested with the starting points in the set \(\{0, 1\}^6\), which makes for a total of 64 starting points. The resulting performance profile can also be found in Figure 6; it is based on the metric (6.1) with \(\delta := 1\). Clearly, the method Schol excels the other three, finding the best function values in all the test runs, followed by KanSch.

### 6.2.2. Switching-constrained optimal control

Here, we intend to solve a switching-constrained optimal control problem with the proposed relaxation method. The underlying example is taken from [8, Section 5.2].

Let \(I := (0, T)\), with \(T := 10\) the final time, \(\Omega := (-1, 1)^2\), and let \(\Gamma\) be the boundary of \(\Omega\). Furthermore, we define \(\Omega_u := (-1, 0] \times (-1, 1)\) as well as \(\Omega_v := (0, 1) \times (-1, 1)\). Let us consider the optimal control of the non-stationary heat equation with zero initial and Neumann boundary conditions given below:

\[
\frac{\partial y(t, \omega)}{\partial t} - \Delta y(t, \omega) - \frac{1}{10} \chi_{\Omega_u}(\omega) u(t) - \frac{1}{10} \chi_{\Omega_v}(\omega) v(t) = 0, \quad \text{a.e. on } I \times \Omega,
\]
\[
\mathbf{n}(\omega) \cdot \nabla y(t, \omega) = 0, \quad \text{a.e. on } I \times \Gamma,
\]
\[
y(0, \omega) = 0, \quad \text{a.e. on } \Omega.
\]

(6.4)

Here, \(\chi_A : \Omega \to \mathbb{R}\) denotes the characteristic function of a measurable set \(A \subset \Omega\) which equals 1 on \(A\) and vanishes otherwise. Let \(y_d \in L^2(I; H^1(\Omega))\) be the solution of the state
equation associated with the desirable control functions $u_d, v_d \in L^2(I)$ given by
\[
\forall t \in I: \quad u_d(t) := 20 \sin^4(2\pi t/T), \quad v_d(t) := 10 \cos^4(1.4\pi t/T).
\]
All feasible controls $u, v \in H^1(I)$ shall satisfy the switching requirement
\[
u(t) = 0 \quad \text{a.e. on } I.
\]
Note that $u_d$ and $v_d$ violate this switching condition. We aim to find the minimum of the objective function defined by
\[
J(y, u, v) := \frac{1}{2} \| y - y_d \|_{L^2(I; L^2(\Omega_1))}^2 + \frac{\alpha}{2} \left( \| u \|_{L^2(I)}^2 + \| v \|_{L^2(I)}^2 \right) + \frac{\beta}{2} \left( \| \partial_t u \|_{L^2(I)}^2 + \| \partial_t v \|_{L^2(I)}^2 \right)
\]
with respect to $(y, u, v) \in L^2(I; H^1(\Omega_1)) \times H^1(I) \times H^1(I)$ such that $(u, v)$ satisfy the switching requirement (6.5) while $y$ solves the associated state equation (6.4). We chose $\alpha := 10^{-6}$ and $\beta := 10^{-5}$ for our experiments.

For the numerical solution of the problem, the domain $\Omega$ is tessellated using the function `generateMesh` from the MATLAB PDE toolbox and a discretization tolerance of $h := 10^{-1}$. The time interval $I$ is subdivided into equidistant intervals of size $\tau := 10^{-1}$. Both the spatial and temporal discretizations use standard piecewise linear (continuous) finite elements, which leads to a conforming approximation of the $H^1$-norm in (6.6).

After discretization, the problem turns into a finite-dimensional MPSC comprising the variables $\vec{u}, \vec{v} \in \mathbb{R}^{101}$, a quadratic objective function, and the switching constraints $\vec{u}_i \vec{v}_i = 0$ for all $i = 1, \ldots, 101$. These correspond to the simple constraint mappings $G(\vec{u}, \vec{v}) := \vec{u}$ and $H(\vec{u}, \vec{v}) := \vec{v}$. Note that the feasible set can be seen as the union of $2^{101}$ convex 'branches' (obtained by setting either $\vec{u}_i = 0$ or $\vec{v}_i = 0$ for each $i = 1, \ldots, 101$). Hence, the problem can be expected to admit a substantial amount of local minimizers, and it is unrealistic to expect algorithmic implementations to find the global minimizer, even when
Figure 7. A (possible) global minimizer for the switching-constrained optimal control problem from Section 6.2.2 (left) and the resulting performance profile (right).

tested with a large number of initial points. To facilitate a quantitative comparison of our numerical algorithms (as in the previous section), we use the following heuristic to obtain an upper estimate of the optimal value: using a coarser time discretization (with \( \tau = 0.4 \)), we compute the exact global minimizer of the resulting problem by minimizing the objective over each of the branches induced by the switching constraints. The corresponding global minimizer is then lifted to the finer time grid (with \( \tau = 0.1 \)) by linear interpolation, and the resulting point is used as an initial guess for all the used algorithms. The resulting estimate of the optimal value is 0.2997, and the associated controls are depicted in Figure 7.

For the numerical tests, we generated 1000 starting points with coordinates chosen randomly in the interval \([0, 10]\). The performance profile was constructed by using the metric (6.1) with \( \delta := 0 \), and it too can be found in Figure 7.

As in the previous example, the relaxation method Schol turns out to be the most robust of the tested algorithms, finding the smallest function values in more than 95% of the test runs. Next are the algorithms due to SteUlb and KanSch which do not significantly surpass SNOPT. However, it has to be noted that the method SteUlb does not find a feasible point in 30 of the test runs. A possible reason for that might be that the switching constraints are relaxed only locally around the switching points. Furthermore, the method SteUlb induces nonlinearities (more precisely, 404 nonlinear inequality constraints result from 101 simple switching constraints) while the original problem data is linear (in the sense that the functions modelling (MPSC) are linear).

When analyzing the results in more detail, it turns out that, as expected, the algorithms found an exorbitant amount of distinct points (possibly local minimizers). Interestingly, however, the associated function values actually lie quite close to each other. This explains the \( \tau \)-axis scaling in the performance profile plot.

6.2.3. Semi-continuous variables

In many optimization scenarios, it is desirable that a nonnegative decision variable is either exactly zero or contained in some interval whose lower bound is strictly positive. This is the case, for instance, in production planning, portfolio optimization, compressed sensing in signal processing, and subset selection in regression. More details can be found in \([6,31]\), and references therein.
Given a decision variable \( x \in \mathbb{R}^n \) and bounds \( \ell, u \in \mathbb{R}^n, \ell \leq u \), a requirement of the above form can be reformulated as the either-or type constraints

\[
x_i = 0 \lor x_i \in [\ell_i, u_i], \quad i = 1, \ldots, n.
\]

In this context, the variable \( x \) is often called semi-continuous since it is required to lie in some continuous interval, except for the outlier case when it is equal to zero. (One might also be inclined to call \( x \) a semi-discrete variable, but we have not seen this terminology elsewhere in the literature.)

Constraint systems of the form (6.7) can be reformulated as switching constraints by using slack variables. Indeed, there are two ways of doing so: On the one hand, we could introduce two nonnegative slack variables to transform the box constraints in (6.7) into equality constraints; this procedure eventually yields an MPSC with \( 2n \) switching constraints. On the other hand, assuming that \( u_i \geq 0 \) holds for all \( i = 1, \ldots, n \) (which is the case in nearly all relevant applications), we can simply treat the requirement \( x_i \leq u_i \) as a standard inequality constraint which should be fulfilled at all times. Clearly, if \( x_i = 0 \) is valid, then the inequality \( x_i \leq u_i \) holds automatically, so that we can rewrite (6.7) as

\[
x_i \leq u_i, \quad i = 1, \ldots, n,
\]

\[
x_i = 0 \lor x_i \geq \ell_i, \quad i = 1, \ldots, n.
\]

Using a single slack vector \( y \in \mathbb{R}^n \), we can now rewrite this system as

\[
x \leq u,
\]

\[
y \geq 0,
\]

\[
x_i(x_i - \ell_i - y_i) = 0, \quad i = 1, \ldots, n.
\]

In the notation of our general framework (MPSC), this corresponds to the switching mappings

\[
G(x, y) := x \quad \text{and} \quad H(x, y) := x - \ell - y.
\]

The inequality constraints \( x \leq u \) and \( y \geq 0 \) can be implemented as components of the mapping \( g \) (which may contain other constraints depending on the particular problem). Note that the above reformulation approach only results in \( n \) switching constraints.

We now present a concrete example of portfolio optimization based on the test examples in [11]. The problems in this reference have the form

\[
x^\top Q x \rightarrow \min
\]

\[
e^\top x = 1,
\]

\[
\mu^\top x \geq \rho,
\]

\[
x_i = 0 \lor x_i \in [\ell_i, u_i], \quad i = 1, \ldots, n,
\]

with randomly generated \( Q \in \mathbb{R}^{n \times n}, \mu, \ell, u \in \mathbb{R}^n, \) and \( \rho \in \mathbb{R} \). Here, \( e \in \mathbb{R}^n \) represents the all-ones vector. More details and a link to the underlying database can be found in [11] and [10, Section 4.3]. The particular examples we chose are the 30 instances with size 200.
Figure 8. Performance profile for the portfolio optimization problem from Section 6.2.3.

The corresponding problems (6.8) are reformulated as MPSCs by means of the aforementioned procedure, and the resulting problems are then attacked by the four test algorithms in question. Figure 8 depicts the resulting performance profile based on the metric (6.1) with $\delta := 0$.

For this particular problem class, it surprisingly turns out that the performance advantage of the relaxation method $\text{KanSch}$ is particularly large when compared to the other three methods. In 28 out of 30 examples, the $\text{KanSch}$ algorithm finds the best function value among the tested methods. Next is the algorithm $\text{Schol}$. Additionally, the methods $\text{KanSch}$ and $\text{Schol}$ seem to find feasible points much more reliably than the other two algorithms. To be precise, the algorithm $\text{SteUlb}$ always terminates at the origin which is not feasible, and this observation is independent from the particular choice of the starting point. This explains why the performance curve associated with the method $\text{SteUlb}$ does not appear in Figure 8. Again, the above phenomenon might be a consequence of the local nonlinear relaxation properties of this method.

### 6.3. A brief summary

The results of our computational experiments indicate that for the numerical solution of switching-constrained optimization problems, the use of global relaxation methods is reasonable. As we have seen in all examples, the methods $\text{Schol}$ and $\text{KanSch}$ reliably produce feasible points of (MPSC) which generally possess better function values than the output of the local relaxation method $\text{SteUlb}$ and the NLP solver $\text{SNOPT}$. To be exact, the local relaxation method $\text{SteUlb}$ struggles for the output of feasible solutions in many situations. We guess that this observation results from the fact that one original switching constraint is
replaced by four heavily nonlinear inequality constraints while the relaxation only enlarges the feasible set in a neighbourhood of the actual switching points.

Surprisingly, we observed that the precise choice of the global relaxation scheme one should use in order to solve (MPSC) depends on the structural properties of the underlying problem. While the algorithm Schol excelled the method KanSch in the setting of switching control, the situation is completely the other way around when portfolio optimization problems with semi-continuous variables are under consideration. This might be caused by the fact that the rectangular structure of the Kanzow–Schwartz regularization fits better to the geometrical properties of the feasible set which is induced by semi-continuity conditions. Without stating precise data, we need to admit that due to the simple structure of the relaxation technique Schol, where each switching constraint is replaced by two simple inequality constraints, this method consumes less computation time than the other relaxation schemes in order to terminate.

7. Final remarks

In this paper, we presented relaxation methods for the computational solution of mathematical programmes with switching constraints (MPSCs) which were inspired by related solution methods for mathematical programmes with complementarity constraints (MPCCs). Our analysis shows that an adapted version of Scholtes’ famous global relaxation scheme possesses poor theoretical convergence properties. On the other hand, adjusting the Kanzow–Schwartz relaxation method to MPSCs, we have proved that the resulting algorithm computes M-stationary points of MPSCs under a problem-tailored constraint qualification (MPSC-NNAMCQ) which is weaker than MPSC-MFCQ. The numerical experiments include a wide array of practically relevant problems and demonstrate the computational efficiency of these global relaxation methods in comparison to local relaxation schemes or standard solvers from nonlinear programming.

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