Lectures on
Heavy Quarks in Quantum Chromodynamics

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Abstract

A pedagogical introduction to the heavy quark theory is given. It is explained that various expansions in the inverse heavy quark mass $1/m_Q$ present a version of the Wilson operator product expansion in QCD. A systematic approach is developed and many practically interesting problems are considered. I show how the $1/m_Q$ expansions can be built using the background field technique and how they work in particular applications. Interplay between perturbative and nonperturbative aspects of the heavy quark theory is discussed.

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1 Lecture 1. Heavy Quark Symmetry

The statement that Quantum Chromodynamics (QCD) is the theory of hadrons has become common place. It is a very strange theory, since many questions concerning dynamics of the quarks and gluons at large distances – however simple they might seem – remain unanswered or, at best, understood only at a qualitative level. Progress in the direction of the quantitative description of the hadronic properties is slow – every step bringing us closer to such a description is painfully difficult. At the same time new results, even modest, have a special weight for obvious reasons – QCD, unlike many other trendy theories in the modern high energy physics, definitely has a direct relation to Nature and will stay with us forever.

Every hadron in a sense is built from quarks and/or gluons. I say “in a sense” because these are no ordinary building blocks. The number of degrees of freedom fluctuates and is not fixed; this we know for sure. At large distances we have to deal with a genuine strongly coupled field theory, and, as usual, the strong coupling creates complicated structures which can not be treated by perturbative methods. Then we feel helpless and are ready to use every opportunity, no matter where it comes from, if only it gives the slightest hope of getting a solid quantitative approach based on QCD.

QCD has two faces, two components – hard and soft. The hard component is the realm of perturbative QCD. Not much will be said in these lectures about this aspect. Instead, we will concentrate on the soft component. Many years ago, at the dawn of the QCD era, it was noted [1] that heavy quarks are, probably, the best probe of the soft component of the gluon fields out of all probes we have at our disposal. The developments we witnessed in recent years confirm this conclusion.

The dynamics of soft degrees of freedom in QCD is the realm of non-perturbative phenomena. Having said this I hasten to add that there is an element of luck – transition from the perturbative regime to the non-perturbative one is very abrupt in QCD. In a sense the gauge coupling constant is abnormally small. I do not mean here the conventional logarithmic suppression of the running constant but, rather, the fact that $b$, the first coefficient in the Gell-Mann-Low function, is numerically large. This fact allows us to forget, in the first approximation, about perturbative effects and focus on non-perturbative ones in a wide range of problems. It is more exact to say that we will concentrate on studying the soft degrees of freedom, but due to the fortunate circumstance of “abnormal” smallness of $\alpha_s(\mu)/\pi$ for as low normalization point as $\mu \sim 1$ GeV, all effects due to the soft degrees of freedom are essentially non-perturbative. I will elucidate the precise meaning of this statement later.

It would be great if we could just switch off – by adjusting some parameter – all hard processes in QCD without changing its soft component. Then we would be left with the confining dynamics in a clean and uncontaminated form; formulation of the theory would be much easier. The only parameter which might do the job is $b$. If we could tend $b \rightarrow \infty$ with $\Lambda_{\text{QCD}}$ fixed, the hard gluons would be suppressed
by powers of $1/b$ while the soft component would presumably remain unaltered or almost unaltered. Unfortunately, nobody knows how to make the enhancement of $b$ parametric. (The limit of the large number of colors, $N_c \to \infty$, does not work since, although $b$ is definitely proportional to $N_c$ in this case, the perturbative expansion for all planar graphs goes in $N_c/b$, not in $1/b$ \cite{2}.) Therefore, we will have to rely on the numerical enhancement of $b$. In the first lectures I will merely assume that the hard gluon exchanges are non-existent. Later on, at the very end, we will return to this issue and will briefly discuss the impact of hard gluons.

The purpose of these lectures is mainly pedagogical – the coverage of the topic is neither chronological nor comprehensive. Technically sophisticated issues and calculations are avoided whenever possible; instead I discuss particularly illuminating problems, in a simplified setting. The readers interested in specific advanced applications (e.g. combining the $1/m_Q$ expansions with the chiral perturbation theory \cite{3}) are referred to the original publications and the review papers \cite{4} summarizing a wealth of results obtained in the heavy quark theory after 1990. The presentation of the heavy quark theory below as a rule does not follow the standard pattern and is, rather, complementary with respect to the more traditional reviews \cite{4}. We try to emphasize that the heavy quark theory and the heavy quark expansion is nothing else than a version of the Wilson operator product expansion (OPE) \cite{5}, an aspect which usually remains fogged.

### 1.1 Why heavy quarks?

The quark-gluon dynamics is governed by the QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} G_{\mu \nu}^{a} G_{\mu \nu}^{a} + \sum_{q} \bar{q} i \not{D} q + \sum_{Q} \bar{Q} (i \not{D} - m_{Q}) Q = \mathcal{L}_{\text{light}} + \sum_{Q} \bar{Q} (i \not{D} - m_{Q}) Q \quad (1.1)$$

where $G_{\mu \nu}^{a}$ is the gluon field strength tensor, the light quark fields ($u, d$ and $s$) are generically denoted by $q$ and are assumed, for simplicity, to be massless while the heavy quark fields are generically denoted by $Q$. To qualify as a heavy quark $Q$ the corresponding mass term $m_{Q}$ must be much larger than $\Lambda_{\text{QCD}}$. The charmed quark $c$ can be called heavy only with some reservations and, in discussing the heavy quark theory, it is more appropriate to keep in mind $b$ quarks. The hadrons to be considered are composed from one heavy quark $Q$, a light antiquark $\bar{q}$, or diquark $qq$, and a gluon cloud which can also contain light quark-antiquark pairs. The role of the cloud is, of course, to keep all these objects together, in a colorless bound state which will be generically denoted by $H_{Q}$.

Quite naturally in the heavy quark theory, the gamma matrices used are those of the standard representation,

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^{5} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.2)$$
With these definitions of the gamma matrices the left-handed spinor has the form
\[ \psi_L = (1 + \gamma_5)\psi. \]

The light component of \( H_Q \), its light cloud, has a complicated structure – the soft modes of the light fields are strongly coupled and strongly fluctuate. Basically, the only fact which we know for sure is that the light cloud is indeed light; typical frequencies are of order of \( \Lambda_{\text{QCD}} \). One can try to visualize the light cloud as a soft medium. The heavy quark \( Q \) is then submerged in this medium. If the hard gluon exchanges are discarded the momentum which the heavy quark can borrow from the light cloud is of order of \( \Lambda_{\text{QCD}} \), and the corresponding uncertainty in the energy of the heavy quark is of order \( \Lambda_{\text{QCD}}^2/m_Q \). Since these quantities are much smaller than \( m_Q \) this means, in particular, that the heavy quark-antiquark pairs cannot play a role. In other words, the field-theoretic (second-quantized) description of the heavy quark becomes redundant, and under the circumstances it is perfectly sufficient to treat one single heavy quark \( Q \) within quantum mechanics, which is infinitely simpler, of course, than any field theory. Moreover, one can systematically expand in \( 1/m_Q \). Thus, in the limit \( m_Q/\Lambda_{\text{QCD}} \to \infty \) the heavy quark component of \( H_Q \) becomes easily manageable allowing one to use the heavy quark as a probe of the light cloud dynamics. The special advantages of this limit in QCD were first emphasized by Shuryak.

1.2 Descending Down

In field theory one has to specify the normalization point \( \mu \) where all operators are defined; in particular, the gauge coupling constant \( g \) and the quark mass \( m_Q \) are functions of \( \mu \). The original QCD Lagrangian is formulated at very short distances, or, which is the same, at a high normalization point \( \mu = M_0 \) where \( M_0 \) is the mass of an ultraviolet regulator. In other words, the normalization point is assumed to be much higher than all mass scales in the theory, \( \mu \gg m_Q \). Constructing an effective theory intended for description of the low-energy properties of the heavy flavor hadrons we must evolve the Lagrangian from the original high scale \( M_0 \) down to a normalization point \( \mu \) lying below the heavy quark masses \( m_Q \). By evolving down I mean that we integrate out, step by step, all high-frequency modes in the theory thus calculating the Lagrangian \( L(\mu) \) describing dynamics of the soft modes, with characteristic frequencies less than \( \mu \). The hard (high-frequency) modes determine the coefficient functions in \( L(\mu) \) while the contribution of the soft modes is hidden in the matrix elements of (an infinite set) of operators appearing in \( L(\mu) \).

This approach, which in the context of QCD was put forward by K. Wilson long ago, has become common. It is widely recognized and exploited in countless applications – from the ancient problem of the \( K \) meson decays to fresh trends in the lattice

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\[2\] In some papers devoted to the subject the light cloud is referred to as ‘brown muck’. I think it is absolutely unfair with respect to the soft components of the quark and gluon fields to call them ‘brown muck’ only because we are not smart enough to fully understand the corresponding dynamics.
calculations \[6\]. The peculiarity of the heavy quark theory is due to the fact that the in and out states we deal with contain heavy quarks. Therefore, although we do integrate out the field fluctuations with the frequencies down to \(\mu\) the heavy quark fields themselves are not integrated out since we will be interested in physics in the sector with the \(Q\) charge \(\neq 0\). The effective Lagrangian \(L(\mu)\) acts in this sector.

If QCD was solved we could include in our explicit calculation of the effective Lagrangian all modes, descending down to \(\mu = 0\). The Lagrangian obtained in this way would be built in terms of the fields of physical mesons and baryons, not in terms of quarks and gluons, since the latter become irrelevant degrees of freedom in the infrared limit \(\mu \rightarrow 0\). This Lagrangian would give us the full set of all conceivable amplitudes and would, thus, represent the final answer for the theory. There would be no need for any further calculations – one would just pick up the amplitude of interest and compare it with experimental data.

This picture is quite utopian, of course. The real QCD is not solved in the closed form, and in doing explicit calculations of the coefficients in the effective Lagrangian one can not put \(\mu = 0\). The lower the value of \(\mu\) the larger part of dynamics is accounted for in the explicit calculation. Therefore, we would like to have \(\mu\) as low as possible; definitely \(\mu \ll m_Q\). The heavy quark can be treated as a non-relativistic object moving in the soft background field only provided the latter condition is met. On the other hand, to keep theoretical control over the explicit calculations of the coefficient functions we must stop at some \(\mu \gg \Lambda_{\text{QCD}}\), so that \(\alpha_s(\mu)/\pi\) is still a sufficiently small expansion parameter. In practice this means that the best choice (which we will always stick to) is \(\mu \sim \) several units times \(\Lambda_{\text{QCD}}\). All coefficients in the effective Lagrangian obtained in this way will be functions of \(\mu\).

Since \(\mu\) is an auxiliary parameter predictions for physical quantities must be \(\mu\) independent, of course. The \(\mu\) dependence of the coefficients must be canceled by that coming from the physical matrix elements of the operators in \(L(\mu)\). However, in calculating in the hard and soft domains (i.e. above \(\mu\) and below \(\mu\)) we make different approximations, so that the exact \(\mu\) independence of the physical quantities can be lost. Since the transition from the hard to soft physics is very steep one may hope that our predictions will be very insensitive to the precise choice of \(\mu\) provided that \(\mu \sim \) several units times \(\Lambda_{\text{QCD}}\). Below, if not stated to the contrary we will assume that the normalization point \(\mu\) is chosen in this way.

In descending from \(M_0\) down to \(\mu\) the form of the Lagrangian (1.1) changes, and a series of operators of higher dimension appears. It is important that all these operators are Lorentz scalars. For instance, the heavy quark part of the Lagrangian takes the form

\[
L_{\text{heavy}} = \sum_Q \left\{ \bar{Q}(i\not\!D - m_Q)Q + \frac{c_G}{2m_Q} \bar{Q}(i/2)\sigma_{\mu\nu}G_{\mu\nu}Q + \sum_{\Gamma, q} \frac{d_Q^{(\Gamma)}}{m_Q^2} \bar{Q}\Gamma Q\bar{q}\Gamma q \right\} + \mathcal{O}\left(\frac{1}{m_Q^3}\right)
\]

(1.3)

where \(c_G\) and \(d_Q^{(\Gamma)}\) are coefficient functions, \(G_{\mu\nu} \equiv gG^{a}_{\mu\nu}t^a\) and \(t^a\) is the color generator, \((\text{Tr} t^a t^b = \delta^{ab}/2)\); below we will often use the short-hand notation \(i\sigma G =
\[ i \sigma_{\mu \nu} G_{\mu \nu} = i \gamma_{\mu} \gamma_{\nu} G_{\mu \nu}. \]

The sum over the light quark flavors is shown explicitly as well as the sum over possible structures \( \Gamma \) of the four-fermion operators. All masses and couplings, as well as the coefficient functions \( c_G \) and \( d^{(\Gamma)} \), depend on the normalization point. For example, the coefficient \( c_G \) in the leading logarithmic approximation can be written as

\[
c_G(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(m_Q)} \right)^{-\frac{3}{b}} - 1, \quad b = 11 - \frac{2}{3} n_f, \tag{1.4}
\]

where \( n_f \) is the number of the light flavors. The power \(-3/b\) was first calculated in Ref. [8]. In Sect. 5.3 I will explain how to derive Eq. (1.4).

The operators of dimension five and higher in Eq. (1.3) are due to the contribution of hard gluons, with offshellness from \( \mu \) up to \( M_0 \). Since we agreed that in this lecture we will ignore the existence of such gluons, we will forget about these operators for the time being. Does this mean that what remains from the Lagrangian (1.3) contains no \( 1/m_Q \) terms?

The answer to this question is negative. The \( 1/m_Q \) expansion is generated by the first ("tree-level") term in the Lagrangian (1.3),

\[
\mathcal{L}^0_{\text{heavy}} = \bar{Q} (\mathcal{P} - m_Q) Q. \tag{1.5}
\]

Although the field \( Q \) in this Lagrangian is normalized at a low point \( \mu \) the field \( Q \) carries a hidden large parameter, \( m_Q \); isolating this parameter opens the way to the \( 1/m_Q \) expansion. Indeed, the interaction of the heavy quark with the light degrees of freedom enters through \( \mathcal{P}_\mu = i D_\mu \), where

\[
D_\mu = \partial_\mu - ig A^a_\mu t^a.
\]

The background gluon field \( A_\mu \) is weak if measured in the scale \( m_Q \), which means, of course, that there is a large "mechanical" part in the \( x \) dependence of \( Q(x) \), known from the very beginning [9],

\[
Q(x) = e^{-i m_Q t} \tilde{Q}(x) \tag{1.6}
\]

where \( \tilde{Q}(x) \) is a "rescaled" bispinor field which, in the leading approximation, carries no information about the heavy quark mass. It describes a residual motion of the heavy quark inside the heavy hadron [10] with typical momenta of order \( \Lambda_{\text{QCD}} \).

Equation (1.6) is written in the rest frame of \( H_Q \). In the arbitrary frame one singles out the factor \( \exp(-i m_Q v_\mu x_\mu) \) where \( v_\mu \) the four-velocity of the heavy hadron,

\[
v_\mu = p_\mu / M_H Q.
\]

The covariant momentum operator \( \mathcal{P}_\mu \) acting on the original filed \( Q \), when acting on the rescaled field \( \tilde{Q} \), is substituted by the operator \( m_Q v_\mu + \pi_\mu \),

\[
i D_\mu Q(x) = e^{-i m_Q v_\mu x_\mu} (m_Q v_\mu + i D_\mu) \tilde{Q}(x) \equiv e^{-i m_Q v_\mu x_\mu} (m_Q v_\mu + \pi_\mu) \tilde{Q}(x). \tag{1.7}
\]
Below we will consistently use different letters, \( P_\mu \) and \( \pi_\mu \) for the momentum operators \( iD_\mu \) acting on \( Q \) and \( \tilde{Q} \), respectively. If not stated to the contrary, we will use the rescaled field \( \tilde{Q} \), omitting the tilde in all expressions where there is no risk of confusion\(^3\). In the local colorless operators bilinear in the heavy quark fields it does not matter whether the original field \( Q \) or the rescaled one is used, since, say,

\[
\tilde{Q}Q = \tilde{Q}\tilde{Q}, \quad \tilde{Q}P_\mu Q = \tilde{Q}\pi_\mu \tilde{Q}, \ldots
\]

and so on. Using these distinct notations for the momentum operator is convenient since all expressions written in terms of \( \pi_\mu \) and \( \tilde{Q} \) do not contain implicitly the large parameter \( m_Q \).

I pause here to make a reservation. The rescaled field \( \tilde{Q} \) is a four-component Dirac bispinor, not a two component non-relativistic spinor which is usually introduced in the heavy quark effective theory (HQET)\(^4\). HQET is a formalism invented in the very beginning of the 90’s\(^5\) which is very often used in connection with the heavy quark physics\(^1\). It is convenient in a range of problems but can be quite misleading in some other problems. I prefer to discuss the heavy quark expansions directly and systematically in full QCD in the framework of the Wilson OPE. In many instances the careful reader will certainly recognize a significant overlap, but the Wilson language, being more general, seems to give a better understanding and command over the \( 1/m_Q \) expansions. Moreover, some issues can not be addressed in the framework of HQET at all.

The Dirac equation \( (\not P - m_Q)Q = 0 \) in terms of the rescaled field can be written as follows:

\[
\frac{1 - \gamma_0}{2} Q = \frac{\not\pi}{2m_Q} Q, \tag{1.8}
\]

and

\[
\pi_0 Q = -\frac{\pi^2 + (i/2)\sigma G}{2m_Q} Q. \tag{1.9}
\]

The last equation is actually the squared Dirac equation,

\[
\frac{1}{2m_Q}(\not P + m_Q)(\not P - m_Q) Q = \frac{1}{2m_Q} \left( \not P^2 + \frac{i}{2} \sigma G - m_Q^2 \right) Q = 0.
\]

In deriving Eq. (1.9) we used the fact that

\[
[\mathcal{P}_\mu, \mathcal{P}_\nu] = [\pi_\mu, \pi_\nu] = igC_{\mu\nu}^a t^a. \tag{1.10}
\]

Armed with this knowledge one can easily obtain the \( 1/m_Q \) expansion of \( \mathcal{L}_{\text{heavy}}^0 \), up to terms \( 1/m_Q^2 \):

\[
\mathcal{L}_{\text{heavy}}^0 = Q(i \not P - m_Q)Q = Q \left( \frac{1 + \gamma_0}{2} - \frac{(\tilde{\sigma} \tilde{\pi})^2}{8m_Q^2} \right) \left( \pi_0 - \frac{1}{2m_Q} (\tilde{\pi} \tilde{\sigma})^2 \right) - \]

\(^3\)Whenever one sees an expression containing \( \pi \)'s one may be sure that it refers to the rescaled fields \( \tilde{Q} \) even if the tildes are not written out explicitly.

\(^4\)HQET

\(^5\)90’s
\[- \frac{1}{8m_Q^2} \left( - \bar{D} \dot{E} + 2 \bar{\sigma} \cdot \dot{E} \times \vec{\pi} \right) \left( 1 + \frac{(\bar{\sigma} \vec{\pi})^2}{8m_Q^2} \right) \left( 1 + \frac{\gamma_0}{2} \right) Q + \mathcal{O} \left( \frac{1}{m_Q^3} \right), \tag{1.11}\]

where \(\bar{\sigma}\) denote the Pauli matrices and

\[(\bar{\pi} \bar{\sigma})^2 = \bar{\pi}^2 + \bar{\sigma} \vec{B},\]

\(\vec{E}\) and \(\vec{B}\) denote the background chromoelectric and chromomagnetic fields, respectively. The coupling constant \(g\) and the color matrix \(t^a\) are included in the definition of these fields. The derivation of this Lagrangian is a good home exercise. I encourage everyone to obtain Eq. (1.11) by using the commutation relation (1.10) and the properties of the gamma matrices. Those who will have problems with getting Eq. (1.11) should consult Chapter 4 of Bjorken and Drell [11] or Sect. 33 of the Landau-Lifshitz course [12] from where this Lagrangian follows immediately. It is worth noting that

\[\mathcal{L}^0_{\text{heavy}} \equiv \varphi^+(\pi_0 - \mathcal{H}_Q) \varphi \quad \text{(1.12)}\]

where

\[\varphi = \left( 1 + \frac{(\bar{\sigma} \vec{\pi})^2}{8m_Q^2} \right) \left( 1 + \frac{\gamma_0}{2} \right) Q \quad \text{(1.13)}\]

and \(\mathcal{H}_Q\) is a non-relativistic Hamiltonian, through second order in \(1/m_Q\),

\[\mathcal{H}_Q = \frac{1}{2m_Q}(\bar{\pi}^2 + \bar{\sigma} \vec{B}) + \frac{1}{8m_Q^2} \left( - (\bar{D} \dot{E}) + 2 \bar{\sigma} \cdot \dot{E} \times \vec{\pi} \right) \tag{1.14}\]

well-known (in the Abelian case) from the text-book expressions [11, 12]. Equation (1.13) is merely the Foldy-Wouthuysen transformation which is necessary to keep the term linear in \(\pi_0\) in its canonic form.

### 1.3 \(m_Q \to \infty\); The heavy quark symmetry

Let us first neglect all \(1/m_Q\) corrections altogether. In this limit \(m_Q\) drops out from \(\mathcal{L}^0_{\text{heavy}}\),

\[\mathcal{L}^0_{\text{heavy}} = \dot{Q} \frac{1 + \gamma_0}{2} \pi_0 Q. \quad \text{(1.15)}\]

This expression takes place in the rest frame of \(H_Q\); in the arbitrary frame \([10]\]

\[\mathcal{L}^0_{\text{heavy}} = \dot{Q} \frac{1 + \gamma^\prime}{2} \pi_\mu v_\mu Q. \quad \text{(1.16)}\]

In the limit \(m_Q \to \infty\) the masses of all \(Q\)-containing hadrons become equal to that of the heavy quark \(Q\),

\[M_{H_Q} = m_Q + \mathcal{O}(\Lambda_{\text{QCD}}). \]

The mass splittings between different hadrons are generically of order \(\Lambda_{\text{QCD}} \ll m_Q\). Soon, we will relate these mass splittings to the expectation values of certain operators.
The assertion that all \(Q\)-containing hadrons are degenerate to the zeroth order in \(m_Q\) is trivial. This "degeneracy" by no means implies that the internal structure of all \(Q\)-containing hadrons is the same. A little less trivial is the fact that there exist hadrons whose masses are degenerate to much better accuracy, \(O(m_Q^{-1})\), and whose internal structure is, indeed, identical in the limit \(m_Q \to \infty\).

Since all effects due to the heavy quark spin are, obviously, proportional to \(1/m_Q\), in this limit the heavy quark spin becomes irrelevant, see Eqs. (1.11), (1.16). Correspondingly, there emerges a symmetry between the states which differ only by the spin orientation of the heavy quark. The pseudoscalar and vector mesons of the type \(B\) and \(B^*\) (both are the ground state \(S\) wave mesons) present an example of such spin family. In the limit \(m_Q \to \infty\) their masses must be degenerate up to terms \(O(m_Q^{-1})\), and the light clouds of \(B\) and \(B^*\) coincide. If there is more than one heavy quark, say \(Q_1\) and \(Q_2\), the theory is symmetric with respect to the interchange \(Q_1 \leftrightarrow Q_2\) even if their masses are not close to each other (in physical applications we, of course, keep in mind \(b\) and \(c\)). Indeed, the heavy quark \(Q\) plays the role of the static force center inside \(H_Q\); the light cloud is flavor-blind and does not notice the substitution of \(Q_1\) by \(Q_2\) provided that the four-velocities of both quarks are the same. Notice that at this level the four-velocity of the heavy quark coincides with that of the heavy hadron. (Only when higher order corrections in \(1/m_Q\) are taken into account the difference between the four-velocities becomes important and the symmetry \(Q_1 \leftrightarrow Q_2\) is violated. At the level of \(1/m_Q\) also the spin symmetry is not valid any more.) If the hard gluon effects are neglected the interaction with the light cloud can not change the heavy quark four-velocity; therefore, this quantity is conserved in the strong interactions \([10]\). (This conservation is, of course, destroyed by the hard gluons which can easily carry away a finite fraction of the heavy quark momentum.)

The symmetry connecting \(Q_1\) and \(Q_2\) emerges in the limit \(m_{Q_{1,2}} \to \infty\) even if the masses of the heavy quarks are not close to each other. What is important is that both must be much larger than \(\Lambda_{\text{QCD}}\). We encounter here a situation which is conceptually close to the problem of the isotopic symmetry of the strong interactions. Everybody knows that the strong amplitudes are isotopically invariant with the accuracy up to a few percent, and, at the same time, the masses of the \(d\) and \(u\) quarks are not too close to each other, \(m_d/m_u \sim 2\). It is not the proximity of these masses which counts, but the fact that the both masses are much less than the QCD scale \(\Lambda_{\text{QCD}}\).

Usually the existence of an internal symmetry implies a degeneracy of the spectrum. For instance, the isotopic symmetry mentioned above, apart from certain relations between the scattering amplitudes, predicts that the proton and neutron masses are the same, up to small corrections due to the symmetry breaking effects. The heavy quark symmetry does not manifest itself as a degeneracy in the spectrum – the \(D\) and \(B\) masses are very far from each other. One has to subtract the mechanical part of the heavy quark mass in order to see that all dynamical parameters are insensitive to the substitution \(Q_1 \leftrightarrow Q_2\) in the limit \(m_{Q_{1,2}} \to \infty\) \([13]\). Perhaps,
this is the reason why it was discovered so late.

To elucidate the issue of the heavy quark symmetry let us consider a practical problem, semileptonic decay of the $B$ meson induced by the weak $b \to c$ transition. The initial $B$ meson decays into an electron-neutrino pair plus the $D$ meson. Since we do not now discuss the $1/m_Q$ corrections we may make no distinction between the four-velocities of the quark $Q$ and the hadron $H_Q$, and between their masses. Assume that the $B$ meson is at rest. Furthermore, let us assume that the four-momentum $q$ carried away by the lepton pair is maximal, $q^2 = (M_B - M_D)^2$. This means that the $D$ meson produced is also at rest – the hadronic system experiences no recoil. The corresponding regime is sometimes called the point of zero recoil.

In this regime the $B \to D$ transition form factor is exactly unity! More exactly,

$$\langle D|\bar{c}\gamma_0 b|B\rangle = (2M_B^2 M_D)^{1/2} \times \text{unity (at zero recoil)} \quad (1.17)$$

where the square root factors are due to the relativistic normalization of our amplitudes. By the same token

$$\langle D^*|\bar{c}\gamma_i \gamma_5 b|B\rangle = i(2M_B^2 M_D)^{1/2} D_i^* \times \text{unity (at zero recoil)} \quad (1.18)$$

where $D_i^*$ is the polarization vector of $D^*$. As well-known, the exact relations of this type always reflect an underlying symmetry. They can never emerge accidentally because only a symmetry can protect the form factors from renormalizations.

It is very easy to understand why Eqs. (1.17) and (1.18) take place. Indeed, the space-time picture is very transparent. The $b$ quark at rest is surrounded by its light cloud, the latter being the eigenstate of the problem of color interaction with a static force center. At time zero the weak current instantaneously substitutes the $b$ quark by $c$; the charmed quark is also at rest, and since the color interactions are flavor-blind the same light cloud continues to be the eigenstate, this time with the $c$ quark as the static center. If instead of the field-theoretic light cloud we had a quantum-mechanical problem one could say that the overlap integral for these identical wave functions is 1. The light cloud will feel the substitution $b \to c$ only to the extent the heavy quark momentum inside the heavy meson does not vanish exactly – this effect is, of course, suppressed by powers of $1/m_Q$. As we will see later corrections in the right-hand side of Eqs. (1.17) and (1.18) are actually of order $1/m_Q^2$; there are no linear corrections in $1/m_Q$. In the $B \to D^*$ transition generated by the axial-vector current the current, additionally, changes the orientation of the heavy quark spin. As was already mentioned, all effects related to the heavy quark spin are suppressed by $1/m_Q$; $D$ and $D^*$ are in the same multiplet, and the $B \to D^*$ transition is governed by the same symmetry. This symmetry allows one to rotate arbitrarily four states,

$$b \text{ spin up, } b \text{ spin down, } c \text{ spin up, } c \text{ spin down;}$$

therefore, we obviously deal here with an SU(4) invariance.
The symmetry relations (1.17) and (1.18) were first derived in Refs. [14, 15]. Shortly after it was realized [16] that the actual symmetry is much stronger – the SU(4) invariance takes place for any given value of \( v_\mu \), the four-velocity of the recoiling \( c \) quark, not necessarily at the point of zero recoil or close to it. Thus, many different form factors connecting \((B,B^*)\) and \((D,D^*)\) can be expressed in terms of one function depending only on the velocity of the recoiling hadron (in the rest frame of the decaying hadron). The universal form factor is called the Isgur-Wise function.

1.4 The Isgur-Wise function

Now we are finally ready to discuss a very elegant observation due to Isgur and Wise [16]. Let us consider now the amplitudes induced by the transition \( \bar{c} \Gamma b \) off the zero recoil point. Here \( \Gamma \) is any Lorentz matrix; of special interest are, of course, the vector and the axial-vector cases, \( \Gamma = \gamma_\mu \) or \( \gamma_\mu \gamma_5 \); the weak decays of the \( B \) meson are induced by the \( V-A \) currents. The physically measurable amplitudes are \( \langle D|\bar{c} \gamma_\mu b|B \rangle \) and \( \langle D^*|\bar{c} \gamma_\mu b|B \rangle \); for completeness one can also consider the amplitudes of the type \( \langle D|\bar{c} \Gamma b|B^* \rangle \), or \( \langle D|\bar{c} \Gamma b|B^* \rangle \), or \( \langle B|b \Gamma b|B \rangle \) – this adds nothing new. The four-velocity of the particle \( H_Q \) is defined as

\[
v_\mu = \frac{(p_{H_Q})_\mu}{M_{H_Q}}; \tag{1.19}\]

the four-velocities of the initial particles will be denoted by \( v \) while those of the final particles by \( v' \). It is obvious that \( v^2 = 1 \), and, additionally, in the rest frame \( v = \{1, 0, 0, 0\} \). In the most general case the amplitude \( \langle D|\bar{c} \gamma_\mu b|B \rangle \) can be expressed in terms of two form factors, the amplitude \( \langle D^*|\bar{c} \gamma_\mu \gamma_5 b|B \rangle \) in terms of three form factors and the amplitude \( \langle D^*|\bar{c} \gamma_\mu b|B \rangle \) in terms of one form factor. The heavy quark symmetry tells us that in the limit \( m_Q \to \infty \) these six functions, \textit{a priori} independent, reduce to one and the same function which depends only on the scalar product \( vv' \). Specifically,

\[
\langle D|\bar{c} \gamma_\mu b|B \rangle = \sqrt{M_B M_D} \left[(v + v')_\mu\right] \xi(y); \tag{1.20}
\]
\[
\langle D^*|\bar{c} \gamma_\mu \gamma_5 b|B \rangle = \sqrt{M_B M_D} i \left[ D_\mu^*(1 + vv') - (D^*_\alpha v_\alpha) v'_\mu \right] \xi(y); \tag{1.21}
\]
\[
\langle D^*|\bar{c} \gamma_\mu b|B \rangle = \sqrt{M_B M_D} \left[ -\epsilon_{\mu\nu\lambda\sigma} D^*_\mu v_\nu v_\lambda v_\sigma \right] \xi(y). \tag{1.22}
\]

where

\[
y = vv'
\]

and \( \xi(y) \) is the Isgur-Wise function, \( D_\mu^* \) is the polarization vector of \( D^* \). The Isgur-Wise function is independent of the heavy quark masses. The square root \( \sqrt{M_B M_D} \)
reflects the relativistic normalization of the states. The symmetry relations (1.17) and (1.18) imply that the normalization of the Isgur-Wise function at zero recoil is fixed,

$$\xi(y = 1) = 1. \quad (1.23)$$

Perhaps, it is worth noting that the phases in Eqs. (1.21) and (1.22) differ from what you might see in the literature. They are, of course, a matter of convention and reflect the definition of the states. The definition I follow is in accord with the standard relativistic convention, see Eq. (1.25) and Sect. 3.5.

The fact that a large set of form factors degenerate into a single function depending only on $y$ might seem a miracle; but after the assertion is made, with the knowledge you already have, it should be not difficult to understand why it happens. Indeed, let us turn again to the space-time picture described above. A $b$ quark at rest, surrounded by the light cloud, instantaneously converts into a $c$ quark. This time the four-momentum carried away by the lepton pair is not maximal; therefore, the $c$ quark is not at rest. This force center flies away with the velocity $\vec{v}'$. But the light cloud stays intact. So, the question is: “what is the amplitude for the flying $c$ quark and the cloud at rest to form a $D$ or $D^*$ meson?” We can look at this process in another way. After the $b \rightarrow c$ transition happened let us proceed to the rest frame of $c$. In this reference frame the $c$ quark produced is at rest, but the cloud, as a whole, moves away with the velocity $-\vec{v}'$. It is clear that this system – the static charmed force center plus a moving light cloud – has a projection on $D$ or $D^*$. The amplitude per se, with the kinematic structures excluded, can depend only on $|\vec{v}'|$ – there is no preferred orientation in the space, and the direction of $\vec{v}'$ is irrelevant. Using covariant notations one can say that the amplitude depends only on $vv'$ since in the $B$ rest frame $vv' = \sqrt{(1 + v'^2)}$. There is simply no place for the dependence on the heavy quark masses, apart from the overall normalization factors appearing because we stick to the relativistic normalization of states.

Since the heavy quark spin is irrelevant in the limit $m_Q \rightarrow \infty$ to warm up let us consider a toy model where the heavy quarks are deprived of their spins from the very beginning. In other words, I replace the genuine spin-1/2 heavy quarks of QCD by spin-0 color triplets with the same mass. We will turn to this toy model more than once below.

In QCD, $B$ and $B^*$ form a multiplet which includes 4 states: the total angular momentum of the light cloud (1/2) combines with the heavy quark spin (1/2) to produce either spin-0 state ($B$) or three spin-1 states ($B^*$). In our toy model the analog of this ground-state multiplet is obviously a baryon of spin 1/2; let us denote the corresponding field by $N^\alpha_Q$, where $\alpha$ is the spinorial index. The current generating the transition $N_b \rightarrow N_c$ has the form $c^\dagger b$, where the fields $b$ and $c$ are assumed to be scalar now. At first sight the amplitude $\langle N_c | c^\dagger b | N_b \rangle$ might contain four different

---

4Warning: an additional dependence on the heavy quark masses may emerge if we include the hard gluon exchanges neglected so far. For details see Ref. [11]
kinematic structures,
\[ \bar{N}_c N_b, \quad \bar{N}_c \not\! q N_b, \quad \bar{N}_c \not\! q^\prime N_b, \quad \bar{N}_c \sigma_{\mu\nu} N_b v_\mu v_\nu', \]
where I list only P-even structures, of course. On mass shell they all reduce to the first one, however; for instance, \( \bar{q} N_b = N_b \). Hence
\[ \langle N_c | c^\dagger b | N_b \rangle = \bar{N}_c N_b \xi(y). \]

Returning to real QCD what remains to be done is to work out the consequences of spin. The most concise general formula can be written in terms of the matrices
\[ \mathcal{M} = B(i\gamma_5) + B_\mu \gamma_\mu \] and \[ \mathcal{M}' = D^\ast(i\gamma_5) + D'^\ast_\mu \gamma_\mu, \] (1.24)
where \( B_\mu \) and \( D_\mu \) are the polarization vectors of \( B^\ast \) and \( D^\ast \), respectively. The (heavy quark) spin independence of the strong interactions at \( m_Q \to \infty \) manifests itself in the fact that the couplings of the ground state pseudoscalar to \( i\gamma_5 \) and the ground state vector to \( \gamma_\mu \) are the same, see Sect. 3.4. Now, the whole set of the transition amplitudes can be expressed by one compact cute formula \[ \frac{1}{\sqrt{2M_D 2M_B}} \langle H_c(\nu') | \bar{c} \tilde{\Gamma} b | H_b(\nu) \rangle = \frac{1}{2} \text{Tr} \{ \mathcal{M} \frac{1 + \not\! \nu'}{2} \Gamma \frac{1 + \not\! \nu}{2} \mathcal{M} \} \xi(y = \nu \nu'). \] (1.25)

Completing the trace we recover Eqs. (1.20) – (1.22).

Equation (1.25) can be derived in many different ways. Originally it was obtained in Ref. [17] (see also [18]). In Sect. 3.5 we will discuss one of the possible derivations – perhaps, not the simplest, but very instructive. Before we will be able to do that it is necessary to make a digression and study some elements of the background field technique.

Equations (1.20) – (1.22) are valid not only in the space-like domain (the form factor kinematics) but also in the time-like domain. The latter assertion calls for an immediate reservation, though. The heavy quark symmetry implies that \( M_B = M_{B^\ast} \). In the real world this equality is not exact: the heavy quark symmetry is violated by small \( 1/m_Q \) terms. This small violation can be strongly enhanced in the near threshold domain, \( E \approx 2M_B \), where the symmetry breaking parameter turns out to be of order one [19]. Indeed, let us consider a kinematical point above the threshold of the \( B\bar{B} \) production but below \( B\bar{B}^\ast \). In this domain all form factors describing three amplitudes
\[ \langle B\bar{B}| J_\mu|0 \rangle, \quad \langle B\bar{B}^\ast| J_\mu|0 \rangle, \quad \langle B^\ast\bar{B}^\ast| J_\mu|0 \rangle, \]
(here \( J_\mu \) is some heavy quark current, say, \( J_\mu = \bar{b} \gamma_\mu b \)) have imaginary parts associated with the normal thresholds due to the intermediate state \( B\bar{B} \). On the other hand, there is no contribution to the imaginary part from the intermediate state \( B\bar{B}^\ast \)
\footnote{Strictly speaking, for the outgoing particles one must use \( \mathcal{M}' = \gamma_0 (\mathcal{M}')^\dagger \gamma_0 \). With our conventions, however, \( \mathcal{M}' = \mathcal{M}' \).}
and $B^*\bar{B}^*$. In the pseudoscalar meson $B$ the spin of the heavy quark $Q$ is rigidly correlated with that of the light cloud. Hence, the spin independence of the heavy quark interaction is totally lost in the imaginary part in this point. In particular, in the amplitude $\langle B^*\bar{B}^*|J_\mu|0\rangle$ a kinematic structure forbidden by the Isgur-Wise formula appears. An even more pronounced effect of the heavy quark symmetry violation takes place in the anomalous thresholds generated by the pion exchange which can start parametrically much below the normal thresholds depending on the interplay between $M_{B^*}^2 - M_{B}^2$ and the pion mass [21].

1.5 The mass formula

To complete our first encounter with the basics of the heavy quark theory we will now derive a $1/m_Q$ expansion for the masses of the $Q$-containing hadrons.

It is intuitively clear that the heavy hadron mass can be expanded in terms of that of the heavy quark as follows

$$M_{H_Q} = m_Q + \bar{\Lambda} + O(m_Q^{-1})$$

(1.26)

where $\bar{\Lambda}$ is a constant, of order of $\Lambda_{\text{QCD}}$, which depends on the light quark content and the quantum numbers of $H_Q$ but is independent of $m_Q$. (It first appeared in Ref. [21]. Later on we will see that this expression is not as trivial as it might naively seem and requires thoughtful definitions of all parameters involved. In particular, since the quarks are never observed as isolated objects, one may ask what the quark mass $m_Q$ actually means. In due time we will return to this question, of course. For the time being we agreed to disregard hard gluon exchanges; then $m_Q$ is just the mass parameter in the Lagrangian (1.1).

Formally Eq. (1.26) can be most easily derived by analyzing the trace of the energy-momentum tensor in QCD,

$$\theta_{\mu\nu} = m_Q\bar{Q}Q + \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^a_{\mu\nu}$$

(1.27)

where $\beta(\alpha_s)$ is the Gell-Mann-Low function. For simplicity we assume the light quarks to be massless; introduction of the light quark masses changes only technical details at intermediate stages of our analysis. If the mass term of the light quarks is set equal to zero the light quark fields do not appear explicitly in the trace of the energy-momentum tensor. The expression (1.27) contains two terms: the first one is a mechanical part while the second term is the famous trace anomaly of QCD [22] (for a review see e.g. Ref. [23]).

Furthermore, as well-known, for any given one-particle state the expectation value of the trace of the energy-momentum tensor reduces to the mass of the state. Then, the hadron mass can be expressed in terms of two expectation values,

$$M_{H_Q} = \frac{1}{2M_{H_Q}} \langle H_Q|m_Q\bar{Q}Q|H_Q\rangle + \frac{1}{2M_{H_Q}} \langle H_Q|\frac{\beta(\alpha_s)}{4\alpha_s} G^2|H_Q\rangle$$

(1.28)
where the relativistic normalization of the states is implied

\[ \langle H_Q | H_Q \rangle = 2M_{H_Q} V \]

in the rest frame; \( V \) is the normalization volume. We will always use only the relativistic normalization of states which will routinely result in the factors \((2M_{H_Q})^{-1}\) in all expressions.

Let us discuss the expectation values of the operators in Eq. (1.28) in turn. The first one is explicitly proportional to \( m_Q \). To be more quantitative we must determine the matrix element of the heavy quark density \( \bar{Q}Q \). To this end it is convenient to use an argument suggested in Ref. [24] which will show us that the expectation value of \( \bar{Q}Q \) is very close to unity; as a matter of fact, with our present accuracy it is just equal to unity. The second expectation value reduces to \( \bar{A} \).

Indeed, in the rest frame of \( H_Q \) a typical momentum of \( Q \) is of order \( \Lambda_{QCD} \), i.e. the heavy quark is very slow. This means that the lower components of the bispinor field \( Q \) are small compared to the upper ones and, hence, the scalar density of the heavy quark is close to its vector charge, \( \bar{Q}Q \approx \bar{Q}\gamma_0 Q \). The difference is only due to the lower components. The vector charge, however, just measures the number of the heavy quarks inside \( H_Q \); therefore, its matrix element is exactly unity.

It is instructive to do the simple derivation outlined above in some detail. Combining the equations of motion, (1.8) and (1.9), it is easy to get that

\[ \frac{1 - \gamma_0}{2} Q = -\frac{1}{2m_Q} \bar{\pi} \gamma^1 + \frac{\gamma_0}{2} Q + \mathcal{O}(m_Q^{-2}), \] (1.29)

which implies, in turn,

\[ \bar{Q}Q = \bar{Q}\gamma_0 Q - \frac{1}{2m_Q^2} \bar{Q}(\bar{\pi}^2 + \bar{\sigma} \bar{B}) Q + \text{higher orders} \] (1.30)

where \( \bar{B} \) is the chromomagnetic field, \( B_i = \epsilon_{ijk} G_{jk} \). Equation (1.30) is the desired result demonstrating that

\[ \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q}Q | H_Q \rangle = \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q}\gamma_0 Q | H_Q \rangle + \mathcal{O}(m_Q^{-2}) = 1 + \mathcal{O}(m_Q^{-2}). \] (1.31)

The matrix element of the vector charge (appropriately normalized) is set equal to unity, as was discussed above.

This digression has been undertaken merely to familiarize the reader with the basics of the \( 1/m_Q \) expansion in QCD. As our understanding progresses the level of the explanatory remarks will be reduced so that in the subsequent lectures many derivations of a more technical nature will be suggested as an exercise.

Thus, we have established that the first expectation value in Eq. (1.28) produces \( m_Q \) in the expansion for the heavy hadron mass. The second expectation value
which also has the dimension of mass obviously does not scale with $m_Q$ in the limit $m_Q \to \infty$, so one can define \[ \bar{\Lambda} = \frac{1}{2M_{H_Q}} \langle H_Q| \frac{\beta(\alpha_s)}{4\alpha_s} G^2 |H_Q \rangle_{m_Q \to \infty}. \] (1.32)

Thus, the parameter $\bar{\Lambda}$ of the heavy quark theory is, in a sense, similar to the gluon condensate [27]. The latter is the expectation value of the same gluon operator over the vacuum state. In the case of $\bar{\Lambda}$ the gluon operator is averaged over the lowest state of the system with the given (unit) value of the heavy quark charge. The lowest state is, of course, the ground state pseudoscalar meson, $B$. Generally speaking, $H_Q$ can be any $\bar{Q}$-containing hadron. $B$ mesons are most interesting from the point of view of applications; of practical interest also are $Q$-containing baryons which are the lowest-lying states in the given channel with the baryon quantum numbers. Therefore, strictly speaking, unlike the gluon condensate, there exist many different parameters $\bar{\Lambda}$, one for every channel considered. Usually we will tacitly assume that $\bar{\Lambda}$ is defined with respect to the $B$ mesons.

Both expectation values,

\[
\frac{1}{2M_{H_Q}} \langle H_Q| Q \bar{Q} |H_Q \rangle \quad \text{and} \quad \frac{1}{2M_{H_Q}} \langle H_Q| \frac{\beta(\alpha_s)}{4\alpha_s} G^2 |H_Q \rangle,
\]

have $1/m_Q$ corrections which show up at the level $O(m_Q^{-1})$ in Eq. (1.26). Later on we will derive the expansion for $M_{H_Q}$ which takes into account these terms $O(m_Q^{-1})$.

The $1/m_Q$ corrections in the expectation value of the gluon anomaly are due to the fact that in our approach the states $|H_Q\rangle$ are physical heavy flavor states, rather than the asymptotic states corresponding to $m_Q = \infty$ which are usually considered within HQET. Instead of working with these fictitious states I prefer to explicitly keep track of all $1/m_Q$ corrections, both in the operators and in the definition of the states, appealing directly to the Wilsonian operator product expansion.

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6 This expression relating $\bar{\Lambda}$ to the expectation value of the gluon anomaly operator was obtained in Ref. [25]. Some subtleties left aside in the derivation presented here are discussed in detail in this paper. It is instructive to compare Eq. (1.32) with a similar expression for the nucleon mass, $M_N = \frac{1}{2M_N} \langle N| \frac{\beta(\alpha_s)}{4\alpha_s} G^2 |N \rangle$, known from ancient times [26].
Lecture 2. Basics of the Background Field Technique

The essence of our approach is separation of all momenta into two classes – hard and soft. For the time being we will continue to pretend that the role of the gluon degrees of freedom reduces to a soft gluon medium. This is an ideal situation where the gluons can be treated as a background field. A powerful method allowing one to put calculations in the background fields on an industrial basis was developed by Schwinger in electrodynamics many years ago. In the eighties it was adapted to QCD. We will be unable to submerge in all details of this technique, and will, rather, present some basic elements in particular examples. The review paper is recommended for further education. This lecture will be rather technical – its primary goal is to teach how the heavy quark mass expansions can be constructed in a systematic way in different problems.

The starting point of the method is decomposition of fields into two parts – the quantum part and the background one. The propagation of quanta is described by the correlation functions of the quantum part of the fields considered in the external field. Later on the external field is to be considered as a fluctuating field of the light cloud, but this stage need not concern us at the moment.

Let us start with a brief review of the Schwinger method, as it can be applied in QCD. We introduce the coordinate and momentum operators, $X_\mu$ and $p_\mu$, respectively, $[p_\mu, X_\nu] = i g_{\mu\nu}$, $[p_\mu, p_\nu] = [X_\mu, X_\nu] = 0$. Moreover, introduce a formal set of states $|x\rangle$ which are the eigenstates of the coordinate operator $X_\mu$,

$$X_\mu|x\rangle = x_\mu|x\rangle. \quad (2.1)$$

Please, note that $|x\rangle$ has nothing to do with the field-theoretic eigenstates, e.g. $|H_Q\rangle$. To emphasize this fact the use the regular bracket ) in the notation instead of the angle one, which is reserved for the field-theoretic eigenstates.

Then define the covariant momentum operator $P_\mu$ satisfying the following commutation relations

$$[P_\mu, X_\nu] = i g_{\mu\nu}, \quad [P_\mu, P_\nu] = i g t^a G^a_{\mu\nu}, \quad (2.2)$$

where $t^a$ are the generators of the color group, $G^a_{\mu\nu}$ is the external field.

The algebra (2.2) is the basic tool of the Schwinger formalism. We will expand the Green functions in the background field, and in each order of the expansion we will need to use only this algebra.

In the coordinate basis $P_\mu$ acts as a covariant derivative, namely

$$(y|P_\mu|x) = \left(i \frac{\partial}{\partial x_\mu} + g t^a A^a_\mu(x)\right) \delta(x - y) \quad (2.3)$$

if

$$(y|x) = \delta(x - y). \quad (2.4)$$
Now we can write formal expressions for the Green functions. For instance, for the quark Green function (mass $m_q$) describing propagation from the point 0 to the point $x$ we have

$$S(x, 0) = (x| \frac{1}{\slashed{P} - m_q}|0). \quad (2.5)$$

Eq. (2.5), rather obvious by itself, is readily verified by applying the Dirac operator to both sides of Eq. (2.5). Furthermore, it can be identically rewritten as follows

$$S(x, 0) = (x| (\slashed{P} + m_q) \frac{1}{\slashed{P}^2 - m_q^2 + (i/2)G_{\mu\nu}\sigma_{\mu\nu}}|0). \quad (2.6)$$

where

$$G_{\mu\nu} \equiv g \tau^a G^a_{\mu\nu}.$$ 

Please, note that the ordering is important here since $\slashed{P}$ does not commute with $\slashed{P}$ of $G_{\mu\nu}$.

If we are aimed at calculating the coefficient functions in the Born approximation we need nothing else – Eq. (2.6) is just systematically expanded in powers of the background field by using the commutation relations (2.2).

Observe that one can always shift $\slashed{P}$ by a c-number vector due to the fact that

$$e^{i\slashed{q}X} \slashed{P} e^{-i\slashed{q}X} = \slashed{P} + \slashed{q}.$$ \quad (2.7)

Hence, the Fourier transformed propagator reduces to

$$\int d^4x e^{i\slashed{q}X} S(x, 0) = \int d^4x (x| e^{i\slashed{q}X} \frac{1}{\slashed{P} - m_q} e^{-i\slashed{q}X} |0) = \int d^4x (x| \frac{1}{\slashed{P} + \slashed{q} - m_q} |0).$$

This simple trick allows one to readily develop the expansion sought for. Indeed, assume that $\slashed{q}$ is large (hard momentum) and $\slashed{P}$ represents soft modes and is small in this sense. Then we can expand in $\slashed{P}$,

$$\int d^4x e^{i\slashed{q}X} S(x, 0) \rightarrow \frac{1}{\slashed{q} - m_q} - \frac{1}{\slashed{q} - m_q} \slashed{P} \frac{1}{\slashed{q} - m_q} + ... \quad (2.8)$$

Next, we transpose $\slashed{P}$ to the right-most (left-most) position and act on the states using the equations of motion.

It may seem that so far we got almost nothing compared to the standard Feynman graph calculations. Let us demonstrate the efficiency of the background field technique in a few examples.

### 2.1 Inclusive decay of the heavy quark – toy model

One of the most important practical problems in the heavy quark theory is the description of the inclusive decays of heavy flavors. The semileptonic and radiative decays of the $B$ mesons $B \rightarrow X_c l \nu$ and $B \rightarrow X_s \gamma$ are particular examples. Both
are two key elements of the ongoing experimental efforts, in quest of new physics. Needless to say that a reliable QCD-based theory of such decays is badly needed. In this section we start discussing basics of such a theory.

Since this is our first exercise, for pedagogical reasons, it seems reasonable to “peel off” all inessential technicalities, like the quark spins, and resort to a simplified model. In this toy model we will consider the inclusive decay of a spinless heavy quark into a spinless lighter quark plus a photon. Of course, our photon is also a toy photon. We will assume it to be scalar and the corresponding field will be denoted by $\phi$.

The Lagrangian describing the transition of a heavy quark $Q$ into a lighter quark $q$ and a “photon” has the form

$$L_{\phi} = \mathcal{h}{\bar{Q}\phi q} + \text{h.c.}, \quad (2.9)$$

where $h$ is the coupling constant and $\bar{Q} = Q^\dagger$. The masses of the quarks $Q$ and $q$ are both large (and I remind that they are both spinless). Moreover, to further simplify the problem we will analyze a special limit (the so called small velocity or SV limit suggested in Ref. [15]) in which

$$\Lambda_{\text{QCD}} \ll m_Q - m_q \ll m_Q. \quad (2.10)$$

The field $\phi$ carries color charge zero; the reaction $Q \to q + \phi$ could be considered a toy model for the radiative decays of the type $B \to X_s \gamma$ where $X_s$ is an arbitrary inclusive hadronic state containing the $s$ quark produced in the $b$ quark decay.

It is very easy to calculate the total width for the free quark decay $Q \to q + \phi$,

$$\Gamma_{\text{free quark}}(Q \to q\phi) = \frac{h^2 E_0}{8\pi m_Q^2} \equiv \Gamma_0 \quad (2.11)$$

where

$$E_0 = \frac{m_Q^2 - m_q^2}{2m_Q}. \quad (2.12)$$

This free quark expression is valid for the total inclusive probability in the asymptotic limit when $m_Q \to \infty$. We are interested, however, in the preasymptotic corrections proportional to powers of $1/m_Q$.

First of all we must formulate what object we must deal with in order to be able to calculate these corrections systematically. Upon reflection one concludes that it can not be the decay amplitude $Q \to q\phi$ itself. Instead we must consider the $Q \to Q$ forward “scattering” amplitude depicted on Fig. 1. By scattering I mean that $Q$ scatters off the $\phi$ quantum and off the background gluon field which is not shown on Fig. 1 explicitly but is implied. It is implied that all quark lines, $Q$ and $q$, are submerged into this soft-gluon background field. Through the optical theorem the imaginary part of the amplitude of Fig. 1 is related to the inclusive probability of the $Q \to q\phi$ transition. More specifically, if we introduce the transition operator

$$\hat{T} = i \int d^4x e^{-iQx}T\{\bar{Q}(x)q(x), \bar{q}(0)Q(0)\}. \quad (2.13)$$
then the energy spectrum of the φ particle in the inclusive decay is obtained from \( \hat{T} \) in the following way:

\[
\frac{d\Gamma}{dE} = \frac{\hbar^2 E}{4\pi^2 M_{HQ}} \text{Im} \langle H_Q|\hat{T}|H_Q \rangle.
\] (2.14)

Here as usual \( H_Q \) denotes a hadron built from the heavy quark \( Q \) and the light cloud (including the light antiquark), \( q \) in the exponent is the four-momentum carried away by \( \phi \) and \( E \) is the energy of the \( \phi \) quantum.

Equation (2.14) immediately translates the \( 1/m_Q \) expansion for the transition operator in the \( 1/m_Q \) expansion for the inclusive decay rate. The fact that the transition operator must be the primary object of the analysis in all problems of this type was realized in Refs. [9, 29, 30].

Now we use what we have already learnt about the background field technique to write the transition operator in the form

\[
\hat{T} = -\int d^4x e^{-iqx} (x|\bar{Q} \frac{1}{P^2 - m^2_q} Q|0) = \\
-\int d^4x (x|\bar{Q} \frac{1}{(P_0 - q + \pi)^2 - m^2_q} Q|0),
\] (2.15)

where

\[
(P_0)_\mu = m_Q v_\mu.
\]

The Green function of the quark \( q \) differs from the Green function given in Eq. (2.6) in an obvious way since we assume for the time being that our quarks \( Q \) and \( q \) have spin zero, and, correspondingly, instead of Eq. (2.6) referring to the spinor quarks we have

\[
S(x, 0) = (x|\frac{1}{P^2 - m^2_q}|0).
\] (2.16)

In the second line of Eq. (2.15) we proceeded to the rescaled fields \( \tilde{Q} \) which singles out the mechanical part of the momentum operator.

One more thing which will be needed is the equation of motion for the scalar field \( Q \) substituting the Dirac equation. Starting from \((P^2 - m^2_Q)Q = 0\) we obviously get

\[
\pi_0 \tilde{Q} = -\frac{\pi^2}{2m_Q} \tilde{Q}.
\] (2.17)

Finally we are ready to begin constructing the \( 1/m_Q \) expansion. Since \( \pi \) is of order \( \Lambda_{QCD} \) while \( P_0 - q \) scales as \( m_Q \), in the leading approximation \( \pi \) in the denominator of Eq. (2.15) can be neglected altogether. Then, obviously,

\[
\hat{T}^{(0)} = \bar{Q}Q \frac{1}{m^2_q - k^2}
\] (2.18)
where

\[ k = P_0 - q. \]

We see that the leading operator appearing in the expansion is \( \bar{Q}Q \) and it has dimension 2 (let us recall that the scalar \( Q \) field has dimension 1 in contrast to the real quark fields of dimension 3/2, which leads in particular to different normalization factors in the matrix elements). Taking the imaginary part we conclude that in the leading approximation

\[
\frac{1}{\pi} \text{Im} \langle H_Q | \hat{T}^{(0)} | H_Q \rangle = \frac{\langle \bar{Q}Q \rangle}{2m_Q} \delta(E - E_0). \tag{2.19}
\]

Here and below I will use a very convenient short-hand notation,

\[ \langle \bar{Q}Q \rangle \equiv \langle H_Q | \bar{Q}Q | H_Q \rangle. \]

The delta function in the imaginary part is characteristic of a two-body decay. As a matter of fact, combining Eq. (2.19) with the general expression (2.14) and approximating \( \langle \bar{Q}Q \rangle \) by unity – which can and must be done in the leading order in \( 1/m_Q \) – we get the delta-function spectrum of the the free quark decay. Integrating over the energy we then arrive at the free quark decay width (2.11).

Although this little achievement is quite gratifying and shows that we are on the right track the real \( 1/m_Q \) expansion begins when the preasymptotic terms switch on. To this end the terms with \( \pi \) in the denominator of Eq. (2.15) must be kept, and then the expansion in \( (k\pi + \pi^2) \) must be carried out. The general term of this expansion is

\[
\hat{T} = \frac{1}{m_q^2 - k^2} \sum_{n=0}^{\infty} \bar{Q} \left( \frac{2m_Q\pi_0 + \pi^2 - 2q\pi}{m_q^2 - k^2} \right)^n Q. \tag{2.20}
\]

In Lecture 4 where the theory of the end point spectrum will be presented we will need the whole sum. At the moment our purpose is more limited – we are aimed at getting the first correction in the total decay width. This task does not require the infinite sum; only two terms, with \( n = 1 \) and \( n = 2 \), are relevant. Both terms are especially simple.

Indeed, if \( n = 1 \) the combination \( 2m_Q\pi_0 + \pi^2 \) in the numerator acting on \( Q \) is nothing else than the equation of motion, and can be dropped, see Eq. (2.17). We can further discard the \( \vec{q}\vec{\pi} \) part – since the \( H_Q \) spin is assumed to be zero there is no preferred orientation and, hence, \( \langle \bar{Q}\vec{\pi}Q \rangle = 0 \). In this way we arrive at

\[
\langle H_Q | \hat{T}^{(1)} | H_Q \rangle = -\frac{q_0}{m_Q} \frac{\langle \bar{\pi}^2 \rangle}{(m_q^2 - k^2)^2} \tag{2.21}
\]

plus terms of higher order in \( 1/m_Q \). Here the same equation of motion (2.17) was applied to eliminate \( \pi_0 \) in favor of \( \pi^2 \) which, in the given order in \( 1/m_Q \), coincides with \(-\bar{\pi}^2\). The notation is even more concise than previously, namely \( \langle \bar{\pi}^2 \rangle \) stands for \( \langle H_Q | \bar{Q}\vec{\pi}^2Q | H_Q \rangle \). You will often see similar short-hand below.
I pause here to make a side remark. The physical meaning of the matrix element \( \langle \vec{\pi}^2 \rangle \) is quite transparent – it merely represents the average value of the square of the momentum of the heavy quark \( Q \) inside the heavy hadron \( H_Q \). This quantity is of order \( \Lambda_{\text{QCD}}^2 \). This is one of the most important parameters of the heavy quark theory, along with \( \bar{\Lambda} \). Note the gap in dimensions of the operators appearing in the expansion. The dimension-2 operator \( \bar{Q}Q \) is followed by dimension-4 operator \( \bar{Q}\vec{\pi}^2Q \). No relevant operator of dimension 3 exists. Due to this reason the contribution of \( \hat{T}^{(1)} \) in the total width is “unnaturally” suppressed by two powers of the inverse heavy quark mass, not one power as one would expect \textit{apriori}. The observation of the dimension gap was first made in Ref. [30] in the context of HQET; it is crucial in phenomenological applications.

Let us return now to the construction of the \( 1/m_Q \) expansion, and consider the term with \( n = 2 \) in the sum (2.20). One of two factors \( (2m_Q\pi_0 + \pi^2) \) can be applied to the right, the other one to the left. The difference between \( \pi \) applied to the right and to the left is a total derivative which vanishes anyway in the forward matrix element \( \langle H_Q|...|H_Q \rangle \). This simple observation implies that the combination \( (2m_Q\pi_0 + \pi^2) \) in the numerator again vanishes by virtue of the equation of motion and we are left with

\[
\langle H_Q|\hat{T}^{(2)}|H_Q \rangle = \frac{4}{3} \frac{\pi^2}{q^2 - k^2} \frac{1}{(m_Q^2 - k^2)^3} \langle \vec{\pi}^2 \rangle ,
\]

where I have singled out and retained only the spin-0 part of the operator \( \bar{Q}\pi_i\pi_jQ \rightarrow (1/3)\delta_{ij}\bar{Q}\vec{\pi}^2Q \) for the reasons explained above.

We are almost done. The imaginary parts of \( \hat{T}^{(1)} \) and \( \hat{T}^{(2)} \) are expressible in terms of the first and second derivatives of the delta function, and after some simple algebra it is not difficult to get

\[
\frac{1}{\pi} \text{Im} \langle \hat{T} \rangle = \left( \frac{\langle \bar{Q}Q \rangle}{2m_Q} - \frac{\langle \bar{Q}\vec{\pi}^2Q \rangle}{12m_Q^3} \right) \delta(E - E_0) - \frac{E_0\langle \bar{Q}\vec{\pi}^2Q \rangle}{12m_Q^3} \delta'(E - E_0) + \frac{E_0^2\langle \bar{Q}\vec{\pi}^2Q \rangle}{12m_Q^3} \delta''(E - E_0) + ...
\]

where operators of higher dimension are ignored; I have taken into account that \( q_0 = E \) and \( q^2 = E^2 \) and played a little with the delta functions.

The expansion of \( \text{Im} \hat{T} \) into local operators generates more and more singular terms at the point where the \( \phi \) spectrum would be concentrated in the free quark approximation. You should not be surprised by this circumstance which will be elucidated in every detail in due time. What is important is that the physical spectrum is a smooth function of \( E \). One could derive a smooth spectrum by summing up the infinite set of operators in Eq. (2.20) – this will be the subject of Lecture 4. There is no need to carry out this summation now, however, since we are interested only in the integral characteristics of the type of the total probability.
As far as such integral characteristics are concerned, the expansion in Eq. (2.23) is perfectly legitimate.

At first, we calculate the total width by substituting eq. (2.23) into eq. (2.14) and integrating over $E$,

$$\Gamma = \int dE \frac{d\Gamma}{dE} = \Gamma_0 \frac{m_Q}{M_{H_Q}} \langle \bar{Q}Q \rangle$$

(2.24)

where the integration runs from 0 to the physical boundary $E_{0,\text{phys}}$, expressed in terms of the hadron masses

$$E_{0,\text{phys}} = \frac{M_{H_Q}^2 - M_{H_q}^2}{2M_{H_Q}}.$$  

(2.25)

The power correction proportional to $\langle \bar{Q} \pi^2 Q \rangle / m_Q$ which might have appeared cancels in the total width! Is this cancellation unexpected? No, we could have anticipated it on general grounds. Indeed, the total width $\Gamma$ is a Lorentz scalar, and, quite naturally, the $1/m_Q$ expansion for this quantity must run over the Lorentz scalar operators; $\bar{Q}Q$ is Lorentz scalar while $\bar{Q} \pi^2 Q$ is not. The fact that there are no explicit $1/m_Q$ corrections in Eq. (2.24) does not mean that they are absent in $\Gamma$ at all. They could appear through $\langle \bar{Q}Q \rangle m_Q / M_{H_Q}$. Hence, our next task is to find the expansion for $\langle \bar{Q}Q \rangle$ in the toy model at hand. To solve the problem we will use the very same idea as in Sect. 1.5; the only difference is the form of the heavy quark current. For the scalar quarks the current whose diagonal matrix element counts the number of quarks is $\bar{Q} i \gamma_\mu Q$. Hence, in the rest frame of $H_Q$ we have:

$$\frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} i \gamma_\mu Q | H_Q \rangle = 1.$$  

(2.26)

Passing to the rescaled fields we arrive at the relation

$$1 = \frac{1}{M_{H_Q}} \langle H_Q | m_Q \bar{Q}Q + \bar{Q} \pi_0 Q | H_Q \rangle =$$

$$\frac{m_Q}{M_{H_Q}} \langle H_Q | \bar{Q}Q | H_Q \rangle + \frac{1}{2M_{H_Q} m_Q} \langle H_Q | \bar{Q} \pi^2 Q | H_Q \rangle,$$

(2.27)

where the second line is due to the equation of motion. Equation (2.27) leads us to the result sought for,

$$\frac{m_Q}{M_{H_Q}} \langle H_Q | \bar{Q}Q | H_Q \rangle = \left( 1 - \frac{\mu^2_\pi}{2m_Q^2} + \ldots \right);$$

(2.28)

I have introduced here the standard notation for the expectation value of $\pi^2$,

$$\left( \mu^2_\pi \right)_{\text{toy model}} = \frac{1}{2M_{H_Q}} \langle H_Q | 2m_Q \bar{Q} \pi^2 Q | H_Q \rangle.$$  

(2.29)
As was already mentioned, $\mu_2^2$ is a crucial parameter of the heavy quark theory. Its definition in QCD will be slightly different from that in our toy model, but the physical meaning will be the same.

Plugging in Eq. (2.28) in (2.24) we finally arrive at the desired expression,

$$\Gamma = \Gamma_0 \left( 1 - \frac{\mu_2^2}{2m_Q^2} \right). \quad (2.30)$$

The inclusive width coincides with the parton-model result up to terms of order $1/m_Q^2$. There is no correction of order $1/m_Q$. Moreover, the term $1/m_Q^2$ is calculable and its physical meaning is quite transparent: it reflects the time dilation for the moving quark inside the heavy hadron at rest (the Doppler effect). The coefficient $(-1/2)$ in front of $\langle \bar{\pi}^2 \rangle/m_Q^2$ could, therefore, have been guessed from the very beginning, without explicit calculations, were we a little bit smarter.

This situation is quite general, it takes place not only in the toy model at hand but in real QCD as well. The absence of the correction of order $1/m_Q$ in the total inclusive widths (say, the semileptonic width of the $B$ mesons, or $\Gamma(B \to X_s \gamma)$, and so on) is called the CGG/BUV theorem [30, 24].

I hasten to add, though, that the absence of the $1/m_Q$ correction is not merely a consequence of the dimension gap in the set of the relevant operators, as it is sometimes stated in the literature. Indeed, let me give a counterexample. Let us calculate the average energy of the $\phi$ particle, or, more exactly, the first moment of the spectrum,

$$I_1 = \int_{E_0}^{E_0^{phys}} dE \left( E_0^{phys} - E \right) \frac{1}{\Gamma_0} \frac{d\Gamma}{dE}. \quad (2.31)$$

In the parton model the $\phi$ spectrum is a pure delta function and, consequently, $I_1$ vanishes. The heavy quark expansion does generate a non-vanishing result, a $1/m_Q$ effect. To see that this is indeed the case we integrate the theoretical spectrum (2.23) which yields us

$$I_1 = \Delta - \frac{\mu_2^2 E_0^{phys}}{2m_Q^2}, \quad (2.32)$$

where $\Delta$ is defined as follows

$$\Delta = E_0^{phys} - E_0, \quad (2.33)$$

and the parameters $E_0$ and $E_0^{phys}$ are given in Eqs. (2.12) and (2.25). Now, invoking what we have already learnt in Sect. 1.5 about the heavy hadron masses we find that

$$\Delta = \frac{1}{2} v_0^2 \bar{\Lambda} + O(\Lambda_{QCD}^2) \quad (2.34)$$

where

$$v_0 = \frac{M_{HQ} - M_{Hq}}{M_{HQ}}.$$
In the SV limit \( v_0 \) is small and coincides with the velocity of the final hadron produced in the transition \( Q \to q\phi \).

With all these definitions

\[
I_1 = \frac{1}{2} v_0^2 \bar{\Lambda} + \mathcal{O}(\Lambda_{QCD}^2) \tag{2.35}
\]

i.e. the preasymptotic correction in the first moment is of the order of \( \Lambda_{QCD} \), not \( \Lambda_{QCD}^2 \). The reason for the occurrence of a “wrong” power of the QCD parameter is that the leading correction term in the \( 1/m_Q \) expansion in this particular quantity is unrelated to any local operator. As we will see later on such a situation is not rare in the heavy quark theory. The sum rule (2.35) is just a version of Voloshin’s optical sum rule \[31\], while that of Eq. (2.30) can be interpreted in terms of the Bjorken sum rule \[32\]. We will dwell on the both sum rules in the real QCD in Sect. 3.6.

I apologize for this little waterfall of new letters and definitions and hope that a simple picture behind our results is not overshadowed. Notice that in the SV limit \( E_{0\text{phys}} - E \) reduces to the excitation energy of the final hadron produced in the decay. The factor \( E_{0\text{phys}} - E \) in the integrand eliminates the “elastic” peak, so that the integral is saturated only by the inelastic contributions. Say, in the \( b \to c\phi \) transition the contribution of \( B \to D\phi \) is eliminated, only the excited \( D \) mesons survive in the first moment. Since the excitation energies are of order of \( \Lambda_{QCD} \), or \( \bar{\Lambda} \), the prediction (2.35) means that the probabilities of the inelastic transitions \( B \to \text{excited} \) \( D \)'s are all proportional to \( v^2 \). This is in full agreement with the theorem \[15\] discussed in Sect. 1.3 – that in the point of zero recoil the only transition that can occur is the elastic \( B \to D \) transition, with the unit probability. Away from the point of the zero recoil (but in the SV limit) the inelastic transitions are generated. However, Eq. (2.30) shows, that up to small corrections \( \mathcal{O}(\Lambda_{QCD}^2/m_Q^2) \) which can be neglected if we are interested only in the linear in \( \Lambda_{QCD} \) effects, the total probability remains unity. In other words, the total probability is just reshuffled: a small \( v^2 \) part is taken away from the elastic transition and is given to the inelastic transitions. The QCD analog of this assertion is the essence of the Bjorken sum rule \[32\].

It is quite evident that the series of such sum rules can readily be continued further. For the next moment, for instance, we get

\[
I_2 = \int_0^{E_{0\text{phys}}} dE \left( E_{0\text{phys}} - E \right)^2 \frac{1}{\Gamma_0} \frac{d\Gamma}{dE} = \Delta^2 + \frac{\mu_x E_0^2}{3m_Q^2}. \tag{2.36}
\]

Analyzing this sum rule in the SV limit one obtains, in principle, additional information, not included in the results of Refs. \[13, 31, 32\]. It is worth emphasizing that in Eqs. (2.24), (2.32) and (2.30) we have collected all terms through order \( \Lambda_{QCD}^2 \), whereas those of order \( \Lambda_{QCD}^3 \) are systematically omitted. Predictions for higher moments would require calculating terms \( \mathcal{O}(\Lambda_{QCD}^3) \) and higher.

Concluding this part let me suggest to you an exercise which will show whether the technology introduced above is well understood by you. Try to repeat in real
QCD, with the quark spins switched on, everything we have done in the toy model.
Of particular interest to us will be the transition operator
\[ \hat{T} = i \int d^4x e^{-iqx} T\{ Q(x) \Gamma_\mu q(x) , \bar{q}(0) \Gamma_\nu Q(0) \}. \] (2.37)
where \( \Gamma_\mu \) is either \( \gamma_\mu \) or \( \gamma_\mu \gamma_5 \). This transition operator is relevant for the semileptonic \( b \) to \( c \) decays. To facilitate the task consider special kinematics: (i) zero recoil (the vanishing spatial momentum of the lepton pair, \( \vec{q} = 0 \); (ii) small velocity limit \( \vec{q}^2 \ll m_Q^2 \). To further facilitate the task limit yourself to the spatial components of \( \Gamma_\mu \). If you still have problems go over this lecture again and consult the original works [33, 34, 35]. The full answer for the transition operator (2.37) is given, for instance, in Appendix of Ref. [33].

### 2.2 The Fock-Schwinger gauge

In some situations (especially when one deals with massless quarks) a variant of the background field technique based on the so called Fock-Schwinger gauge for the external filed turns out to very efficient (for a review and extensive list of references see [28]). The gauge condition on the background gluon field has the form

\[ x_\mu A_\mu(x) = 0. \] (2.38)

What is remarkable in this condition is that in this gauge the gauge four-potential can be represented as an expansion which runs only over the gauge covariant quantities, the gluon field strength tensor and its covariant derivatives,

\[ A_\mu(x) = \frac{1}{2} x_\rho G_{\rho\mu}(0) + \frac{1}{3!} x_\alpha x_\rho D_\alpha G_{\rho\mu}(0) + \ldots . \] (2.39)

This expression implies, in particular, that \( A(0) = 0 \). It is worth noting that the gauge condition (2.38) singles out the origin and, hence, breaks the translational invariance. The latter is restored only in the final answer for the gauge invariant amplitudes.

It is rather easy to show (see [28]) that the massless quark Green function in the coordinate space is

\[ S(x, 0) = \frac{1}{2\pi^2} \frac{x'}{x^4} - \frac{1}{8\pi^2} \frac{x_\alpha}{x^2} \tilde{G}_{\alpha\phi\gamma_5} \gamma_5 + \ldots , \quad \tilde{G}_{\alpha\phi} = \frac{1}{2} \epsilon_{\alpha\phi\mu\nu} G_{\mu\nu}. \] (2.40)

One can also construct a similar expansion for the Green function in the momentum space \( S(q) \).

If the quark is not massless, \( m_q \neq 0 \), the expansion of the Green function in the background field becomes much more cumbersome. Although we will hardly need it in full I quote it here for the sake of completeness,

\[ S(x, 0) = \frac{1}{2\pi^2} \frac{x'}{x^4} \left\{ \frac{1}{2} m^2 x^2 K_2(m \sqrt{-x^2}) \right\} - \frac{1}{8\pi^2} \frac{x_\alpha}{x^2} \tilde{G}_{\alpha\phi\gamma_5} \gamma_5 \left\{ \frac{-x^2 m K_1(m \sqrt{-x^2})}{\sqrt{-x^2}} \right\} \]
\[-\frac{im^2}{4\pi^2}K_1(m\sqrt{-x^2})\frac{m}{\sqrt{-x^2}}G_{\rho\lambda}\sigma^{\rho\lambda}K_0(m\sqrt{-x^2}) + \ldots \]  

(2.41)

Here \( K \) is the McDonald function. This result was obtained in Ref. [36]. Further education on the Fock-Schwinger gauge technique can be obtained from Ref. [28]. The best way to master those aspects which are most common in the heavy quark theory is merely to play with this tool kit. Let us see how it works in the calculation of the \( 1/m_Q^2 \) correction in the total probability of the semileptonic decay of the heavy quark in the real QCD. Unlike Sect. 2.1 we will address directly the total width bypassing the stage of the spectrum. The first calculation of the power correction in \( \Gamma(B \rightarrow X_c l \nu) \) along these lines was carried out in Ref. [24] (see also [37]).

2.3 The \( 1/m_Q \) corrections to the semileptonic inclusive width in QCD

In this section I will describe probably the most elegant application of the ideas developed above – calculation of the leading correction in the total semileptonic widths. In the toy model considered in the previous section it was established that the \( 1/m_Q \) correction was absent, and the first non-trivial correction \( 1/m_Q^2 \) was associated with the matrix element of \( \langle \bar{Q}Q \rangle \). As a matter of fact, through the heavy quark expansion, we managed to express it in terms of the matrix element \( \langle \bar{Q}\pi Q \rangle \).

When we take the quark spins into account a new dimension-5 Lorentz scalar operator appears, \( \langle \bar{Q}(i/2)\sigma GQ \rangle \). On general grounds one may expect that the main lesson abstracted from the toy model – the absence of the \( 1/m_Q \) term – persists but the \( 1/m_Q^2 \) correction will receive a contribution from the operator \( \bar{Q}(i/2)\sigma GQ \). The conclusion will be confirmed by the analysis presented below. You will see how efficiently the Fock-Schwinger technique is in this case.

Thus, let us proceed to calculation of semileptonic widths. The final quark mass \( m_q \) is arbitrary – we do not assume the SV limit now, nor is any other constraint imposed on \( m_q \). The weak Lagrangian responsible for the semileptonic decays has the following generic form

\[
\mathcal{L} = \frac{G_F}{\sqrt{2}}V_{Qq}(\bar{q}\Gamma_\mu Q)(\bar{l}\Gamma_\nu \nu), \quad \Gamma_\mu = \gamma_\mu (1 + \gamma_5),
\]  

(2.42)

where \( l \) is a charged lepton, electron for definiteness. The mass of the charged lepton will be neglected. Moreover, \( G_F \) and \( V_{Qq} \) are constants irrelevant for our purposes.

As usual, at the first stage we construct the transition operator \( \hat{T}(Q \rightarrow X \rightarrow Q) \),

\[
\hat{T} = i \int d^4x T\{\mathcal{L}(x)\mathcal{L}(0)\} = \sum_i C_i \mathcal{O}_i \]  

(2.43)

describing a diagonal amplitude with the heavy quark \( Q \) in the initial and final state (with identical momenta). The lowest-dimension operator in the expansion of \( \hat{T}(Q \rightarrow X \rightarrow Q) \) is \( \bar{Q}Q \), and the complete perturbative prediction – the spectator
model – corresponds to the perturbative calculation of the coefficient of this operator. For the time being we are not interested in perturbative calculations. Our task is the analysis of the influence of the soft modes in the gluon field manifesting themselves as a series of higher-dimension operators in $\hat{T}$.

At the second stage we average $\hat{T}$ over the hadronic state of interest, say, $B$ mesons. At this stage the non-perturbative large distance dynamics enters through matrix elements of the operators of dimension 5 and higher.

Finally, the imaginary part of $\langle H_Q|T|H_Q \rangle$ presents the $H_Q$ semileptonic width sought for,

$$\Gamma = \frac{1}{M_{H_Q}} \text{Im}\langle H_Q|T|H_Q \rangle. \quad (2.44)$$

The diagram determining the transition operator is depicted on Fig. 2. The lepton propagators are, of course, free – they do not feel the background gluon field. Thick lines refer to the initial quark $Q$. Although the gluon field is not shown one should understand that the lines corresponding to $Q$ and $q$ are submerged into a soft gluon background.

In the Fock-Schwinger gauge the line corresponding to the final quark $q$ (Fig. 2) remains free, and the only source of the dimension-5 operators is the external line corresponding to $Q$ (or $\bar{Q}$). Let us elaborate this point in more detail.

If we do not target corrections higher than $1/m_Q^2$ it is sufficient to use the expression for the quark Green function given in Eq. (2.40) or (2.41). The particular form is absolutely inessential; the only important point is the chiral structure of the vertices in the weak Lagrangian and the fact that the leptons are massless.

The currents in the weak Lagrangian (2.42) are left-handed. Therefore, the Green function of the quark $q$ is sandwiched between $\Gamma_\mu$ and $\Gamma_\nu$. This means that $1 + \gamma_5$ projectors annihilate the part of the Green function with the even number of the $\gamma$ matrices. Then the only potential contribution is associated with the first line in eq. (2.41).

The non-perturbative term is the one containing $\tilde{G}_{\alpha\phi}$. This term vanishes, however, after convoluting it with the lepton part. Indeed, the lepton loop (with the massless leptons) has the form

$$L_{\mu\nu} = -\frac{2}{\pi^4} \frac{1}{x^8} (2x_\mu x_\nu - x^2 g_{\mu\nu}). \quad (2.45)$$

Here I take the product of two massless fermion propagators in the coordinate space, with the appropriate $\gamma$ matrices inserted, and do the trace. Actually we need to know only the last bracket. Now, convoluting it with the $\tilde{G}$ term from the quark Green function we get

$$(\Gamma_\mu x_\alpha \tilde{G}_{\alpha\phi} \gamma_\phi \Gamma_\nu)(2x_\mu x_\nu - x^2 g_{\mu\nu}) \equiv 0,$$

q.e.d. [24, 38].

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Thus, if one uses the Fock-Schwinger gauge the only source for the $1/m_Q$ correction in the total semileptonic widths (at the level up to $1/m_Q^2$) is through the equations of motion for $Q$. Here is how it works.

The expression for the amplitude corresponding to the diagram of Fig. 2 can be generically written as follows:

$$A = \int d^4 x \bar{Q}(0) F(x) Q(x),$$  \hfill (2.46)

where a function $F(x)$ incorporates the lepton loop and the $q$ quark Green function. It may include Lorentz and color matrices, etc. Now, let us single out the large, mechanical part of the motion of the heavy quark,

$$Q(x) = e^{-iP_0 x} \bar{Q}(x),$$

Then

$$A = \int d^4 x \left\{ \bar{Q}(0) F(x) e^{-iP_0 x} [\bar{Q}(0) + x_\mu \partial_\mu \bar{Q}(0) + \frac{1}{2} x_\mu x_\nu \partial_\mu \partial_\nu \bar{Q}(0) + ...] \right\} =$$

$$\left\{ \bar{Q}(0) [\tilde{F}(P_0) \bar{Q}(0) + i \frac{\partial}{\partial P_{0\mu}} \tilde{F}(P_0) \partial_\mu \bar{Q}(0)$$

$$+ i^2 \frac{1}{2} \partial_{P_{0\mu}} \partial_{P_{0\nu}} \tilde{F}(P_0) \partial_\mu \partial_\nu \bar{Q}(0) + ...] \right\} =$$

$$\bar{Q}(0) \tilde{F}(P_0 + i \partial) \bar{Q}(0) = Q(0) \tilde{F}(i \partial) Q(0),$$  \hfill (2.47)

where $\tilde{F}$ is the Fourier transform of $F(x)$.

Our next goal is to convert $i \partial$ in the covariant derivative and then use the equation of motion, $i D_\mu Q = m_Q Q$. More exactly, we start from the expressions of the form

$$Q(0) p(p^2) Q(0), \quad p_\mu = i \partial_\mu,$$  \hfill (2.48)

rewrite $p_\mu$ in terms of $P_\mu = i D_\mu$ plus terms with the gluon field strength tensor (in the Fock-Schwinger gauge) and then substitute $P$ acting on $Q$ by $m_Q$. Expressions (2.48) appear in Im$\tilde{T}$. If the final $q$ quark is massless, $m_q = 0$, the only relevant power is $k = 2$. Switching on the quark mass, $m_q \neq 0$, brings in other values of $k$ as well. (Warning: in the procedure sketched above all operators $p$ in Eq. (2.48) should be considered as acting either only to the right or only to the left. I will assume they act to the right. We can not make some of them act to the right and others to the left and neglect full derivatives. Question: do you understand why?)

Since we focus now on $Q \sigma GQ$ it is sufficient to keep only the terms linear in the gluon field strength tensor; the terms with derivatives of $G_{\mu \nu}$ are to be neglected as well. In this approximation in the Fock-Schwinger gauge $A_\mu = (1/2) x_\mu G_{\rho \alpha}$.

Furthermore,

$$p^2 = P^2 - 2 A p$$  \hfill (2.49)
where we neglected the terms quadratic in $A$ and used the fact that $[p_\mu, A_\mu] = 0$. Equation (2.49) should be substituted in Eq. (2.48), and then we start transposing $Ap$ trying to put it in the left-most position, next to $\bar{Q}(0)$. If $Ap$ appears in this position the result is zero since $A(0) = 0$. Notice that

$$[p^2, Ap] \propto G_{\alpha\beta}p_\alpha p_\beta = 0,$$

so, one can freely transpose $Ap$ through $p^2$. In this way we arrive at

$$\bar{Q}(0) p(p^2)^k Q(0) = \bar{Q}(0) \left[ \mathcal{P}(p^2)^k - 2k \mathcal{P}(AP)(p^2)^{k-1} \right] Q(0). \tag{2.50}$$

Moreover, with our accuracy the last term reduces to

$$\gamma_\alpha [P_\alpha, A_\mu] \mathcal{P}_\mu (p^2)^{k-1} = \gamma_\alpha \frac{i}{2} G_{\alpha\mu} \mathcal{P}_\mu (p^2)^{k-1}$$

$$= -\frac{i}{8} (\mathcal{P} \sigma G - \sigma G \mathcal{P}) p^{2k-2} \tag{2.51}$$

and, hence, using the equations of motion we conclude that

$$\bar{Q}(0) \mathcal{P}(AP)(p^2)^{k-1} Q(0) = 0. \tag{2.52}$$

As a result, the $\sigma G$ terms emerge only due to the fact that

$$p^2 = p^2 - \frac{i}{2} \sigma G,$$

and the final expression is as follows:

$$\bar{Q}(0) p(p^2)^k Q(0) \rightarrow m_Q^{2k+1} \bar{Q}(0) Q(0) - \frac{ik}{2} \bar{Q}(0) \sigma G Q(0) m_Q^{2k-1}. \tag{2.53}$$

After these explanatory remarks the procedure of calculating the leading $1/m_Q^2$ correction in the total semileptonic width should be perfectly clear. Let us summarize it in the form of a prescription.

(i) Calculate the semileptonic width in the parton model. The result has the form

$$\frac{G_F^2 |V_{Qq}|^2}{192 \pi^3} m_Q^5 F_0 \left( \frac{m_q^2}{m_Q^2} \right) \tag{2.54}$$

where $F_0(m_Q^2/m_Q^2)$ is the phase space factor, a function of the ratio $m_q^2/m_Q^2$ well-known in the literature (see Eq. (2.57) below).

(ii) Then construct the expression for $\Gamma$ including the $O(m_Q^{-2})$ corrections in the following way [24]:

$$\Gamma = \frac{G_F^2 |V_{Qq}|^2}{192 \pi^3} m_Q^5 \left\{ \frac{1}{2 M_{H_Q}} \langle H_Q | \bar{Q} Q | H_Q \rangle F_0(\rho) \right\}.$$
\[ + \frac{\mu_G^2}{m_Q^2} \left( \rho \frac{d}{d\rho} - 2 \right) F_0(\rho) \]  

(2.55)

where

\[ \mu_G^2 = \frac{1}{2M_{HQ}} \langle H_Q | \bar{Q} (i/2) \sigma G Q | H_Q \rangle \]  

(2.56)

and

\[ \rho = \frac{m_Q^2}{m_{\pi}^2} . \]

A few comments are in order concerning this beautiful expression for the total semileptonic width. The expansion contains two Lorentz scalar operators, $\bar{Q}Q$ and $\bar{Q}(i/2)\sigma G Q$, of dimension 3 and 5, respectively. The fact that only the Lorentz scalars contribute is obvious since $\Gamma$ is a Lorentz invariant quantity. We observe here the very same gap in dimensions mentioned previously in the context of the toy model – there is no operator of dimension 4 \[30]. The only element still needed to complete the derivation is the matrix element $\langle \bar{Q}Q \rangle$. Fortunately, the corresponding heavy quark expansion has been already built, see Eq. \[1.30\].

Borrowing the explicit expression for $F_0(\rho)$ from textbooks (it is singled out below in the braces) and assembling all other pieces together we finally get

\[ \Gamma = \frac{G_F^2 |V_{qQ}|^2}{192 \pi^2} m_Q^5 \times \]

\[ \left[ \left( 1 + \frac{\mu_G^2 - \mu_\pi^2}{2m_Q^2} \right) \left\{ 1 - 8\rho + 8\rho^3 - \rho^4 - 12 \rho^2 \ln \rho \right\} - 2 \frac{\mu_G^2}{m_Q^2} (1 - \rho)^4 \right] , \]  

(2.57)

where $\mu_\pi^2$ is defined in QCD as follows

\[ \mu_\pi^2 = \frac{1}{2M_{HQ}} \langle H_Q | \bar{Q} \pi^2 Q | H_Q \rangle . \]  

(2.58)

This result is due to Bigi et al. \[24\]. The absence of the $1/m_Q$ correction is a manifestation of the CGG/BUV theorem.

If the mass of the final charged lepton is non-negligible, the property of no soft gluon emission from the quark line (in the Fock-Schwinger gauge) is lost. The expansion for $\Gamma(B \to X_c \tau \nu_\tau)$ in this case is much more cumbersome; it was constructed in Ref. \[39\].

Dimension-5 operators are responsible for the leading non-perturbative corrections in the total semileptonic widths. To assess the convergence of the expansion it may be instructive to have an idea of the higher order terms in the expansion. The $1/m_Q^3$ terms were estimated in Ref. \[40\]. This is a rather messy and time-consuming analysis, and it is hardly in order to comment on it in this lecture. Surprising though it is, from what we already know it is practically trivial to find the coefficient of the dimension-7 operator

\[ \bar{Q} G_{\mu\nu} G_{\mu\nu} Q , \]  

(2.59)
with two gluon field strength tensors fully contacted over the Lorentz indices for massless final quarks (say, \( b \to u \) transition). Of course, this is a purely academic exercise, for many reasons. In particular, because it is only one of a rather large number of dimension-7 operators. Since we do not know their matrix elements anyhow, it seems to be meaningless to carry out full classification and calculate all coefficients. The operator (2.59) is chosen since one can at least use factorization for a rough estimate of the corresponding matrix element and since we get its coefficient essentially for free.

The point is that the massless quark propagator (in the Fock-Schwinger gauge) does not contain \( G_{\mu \nu}G_{\mu \nu} \) term at all (see ref. [41]). This fact implies that the only source of the operator (2.59) is the same as in the case of \( \bar{Q}\sigma GQ \). It is not difficult to get that

\[
\bar{Q}(0) \gamma^\mu \gamma^\nu Q(0) = \bar{Q}(0) \gamma^5 Q(0) + \frac{1}{2} \bar{Q}(0) G_{\mu \nu} G_{\mu \nu} Q(0) = \\
\bar{Q}(0) \gamma^5 Q(0) + \bar{Q}(0) G_{\mu \nu} G_{\mu \nu} Q(0),
\]

(2.60)

where we omitted structures of the type \( [G_{\mu \nu} G_{\mu \alpha} - (1/4) g_{\mu \alpha} G_{\mu \rho} G_{\mu \rho}] \).

### 2.4 Digression

This section is intended for curious readers – those who are anxious to find out where and how else, beyond the theory of the \( H_Q \) states, the background field technique can be used to obtain interesting predictions. Here I will discuss an estimate of the mass splittings between the levels of the highly excited quarkonium states. This part can be safely omitted in first reading since it is unrelated to the remainder of these lectures.

The quarkonium states to be considered below consist of one quark \( Q \), one antiquark \( \bar{Q} \) and the soft gluon cloud connecting them together. To begin with we will assume that \( m_Q \) is large, \( m_Q \gg \Lambda_{\text{QCD}} \) (later on this assumption will be relaxed). The \( Q\bar{Q} \) mesons can have different quantum numbers. We will analyze the excited \( S \) and \( P \) wave states with the quantum numbers \( 0^- \) and \( 0^+ \), respectively. The naive quark model language is used to name the states; this does not mean, of course, that we accept any of the dynamical assumptions of the naive quark model. It would be more accurate to say that the mesons of interest are produced from the vacuum by the currents

\[
J_P = \bar{Q} i \gamma_5 Q \quad \text{and} \quad J_S = \bar{Q} Q.
\]

(2.61)

The central object of our analysis is the difference between the two-point functions in the pseudoscalar and scalar channels. In terms of the Green functions in the background field this difference takes the form (see Fig. 3)

\[
\Pi_P - \Pi_S = i \int dx e^{iqx} \left( \langle \text{vac} | T \{ J_P(x) J_P(0) \} | \text{vac} \rangle - \langle \text{vac} | T \{ J_S(x) J_S(0) \} | \text{vac} \rangle \right) =
\]
\[ i \text{Tr} \left\{ i \gamma_5 \frac{1}{\mathcal{P} + (q/2) - m_Q} i \gamma_5 \frac{1}{\mathcal{P} - (q/2) - m_Q} - \frac{1}{\mathcal{P} + (q/2) - m_Q} \frac{1}{\mathcal{P} - (q/2) - m_Q} \right\} . \quad (2.62) \]

A comment is in order here concerning the trace operation in this expression. It implies not only the trace over the Lorentz and color indices, as usual, but also the trace in the momentum space substituting \( \int \frac{d^4 p}{(2\pi)^4} \) in the conventional Feynman integral. With the help of Eq. (2.6) the difference \( \Pi_P - \Pi_S \) can be identically rewritten as

\[ \Pi_P - \Pi_S = -2m_Q^2 i \text{Tr} \left\{ \frac{1}{D_+} \frac{1}{D_-} \right\} \quad (2.63) \]

where

\[ D_\pm = [\mathcal{P} \pm (q/2)]^2 - m_Q^2 + (i/2)\sigma G . \]

We continue to ignore hard gluons assuming that the only role of the gluon field is to provide a soft cementing background. This is certainly an idealization, but let us see how far one can go within the framework of this simplified picture. Neglecting hard gluons means, in particular, that we will be unable to analyze the low-lying levels of heavy quarkonium where an essential role is played by the short-distance Coulomb interaction. Each \( \sigma G \) insertion in the denominator is of order of \( \Lambda_{\text{QCD}}^2 \), i.e. does not scale with the external momentum \( q \) when \( q \) is large. Let us expand Eq. (2.63) in \( \sigma G \) and take the trace over the Lorentz indices. Then the first order term drops out; the first surviving term is bilinear in \( G \),

\[ \Pi_P - \Pi_S = -8m_Q^2 i \text{Tr} \left\{ \frac{1}{D_+} \frac{1}{D_-} + \frac{1}{2} \frac{1}{D_+} G_{\alpha\beta} \frac{1}{D_+} G_{\alpha\beta} \frac{1}{D_-} \frac{1}{D_-} + \frac{1}{2} \frac{1}{D_-} G_{\alpha\beta} \frac{1}{D_-} G_{\alpha\beta} \frac{1}{D_+} \frac{1}{D_+} \right\} + \cdots . \quad (2.64) \]

A closer look at this expression reveals some peculiar features. First of all, one can interpret each term as a certain correlation function in the theory where the quark \( Q \) is scalar, not spinor. Take, for instance, this first line. It is nothing else than the two-point function of the \( L = 0 \) quarkonium in the scalar QCD (i.e. QCD with the scalar quarks; \( L \) is the total angular momentum of the meson). The current producing the scalar quarkonium from the vacuum in the scalar QCD is \( Q^\dagger Q \) (Fig. 4). The second and the third line, together, represent the four-point function of the type depicted on Fig. 5. The current denoted by the dashed line on this figure is \( Q^\dagger G_{\alpha\beta} Q \); the momentum flowing through this line vanishes. Two insertions of \( G \) imply that this four-point function is explicitly proportional to \( \Lambda_{\text{QCD}}^4 \).

Now, let us examine Eq. (2.64) in the complex \( q^2 \) plane. At some positive values of \( q^2 \) the two-point function of Fig. 4 has simple poles corresponding to positions of
the $L = 0$ quarkonium levels in the scalar QCD. The four-point function of Fig. 5 has double and single poles at the very same values of $q^2$ and single poles at some other values of $q^2$ corresponding to the production of $L = 1$ states in the scalar QCD. The latter are produced from the $L = 0$ states by applying to them the current $Q^\dagger G_{\alpha\beta} Q$.

On the other hand, the original difference $\Pi_P - \Pi_S$ in real QCD has only single poles at the positions of the $S$ and $P$ wave states. These positions are shifted compared to the levels in the scalar QCD. Expanding in the shift one generates double poles. The $L = 1$ pole – the partner to the $P$ wave meson states in $\Pi_P - \Pi_S$ appears only in the four-point function of Fig. 5. From this figure it is quite clear that the residue of the $L = 1$ pole scales as $\Lambda_{QCD}^4/(\Delta(M^2))^2$ where $\Delta(M^2)$ is a characteristic splitting between the $L = 0$ and $L = 1$ states. On the other hand, the residue of the $P$ wave meson in $\Pi_P - \Pi_S$, on general grounds, scales as $\vec{p}^2/M^2$ where $\vec{p}$ is a characteristic quark momentum, and I assume that $\Lambda_{QCD} \ll |\vec{p}| \ll M$. Equating these two estimates we find that

$$\Delta M \sim \Lambda_{QCD}^2/|\vec{p}|.$$ 

One may observe, with satisfaction, that this is exactly the characteristic level splitting (between radial or orbital excitations) for two heavy quarks interacting through a string (“linear potential”). What is remarkable is that in no place our estimate invokes any reference to the linear potential or other models. It was based only on some general features of QCD. For me this is a strong evidence that a string-like picture should take place in QCD, at least, approximately, for high excitations.

What changes if the quarks are light or even massless, $m_Q \to \infty$? The only difference is that now the residues of the $P$ wave mesons in $\Pi_P - \Pi_S$ are of the same order as those of the $S$ wave mesons for highly excited states, which implies that

$$\Lambda_{QCD}^4/(\Delta(M^2))^2 \sim 1,$$

or

$$\Delta(M^2) \sim \Lambda_{QCD}^2.$$

In other words, we got the linear Regge trajectories, at least for highly excited states. Moreover, this analysis makes clear a potentially important point – the empirical observation that even the lowest states in every channel lie on the linear Regge trajectories looks like a numerical coincidence and can not be exact.
3 Lecture 3. Classic Problems with Heavy Quarks

The number of problems successfully solved within the heavy quark expansion is quite large. Even a brief review of the main applications is beyond the scope of these lectures. Some issues, however, are quite general and are important in a variety of applications. Everybody, not only the heavy quark practitioners, should know them. In this lecture we will discuss several such topics – the scaling of the heavy meson coupling constants, some properties of the Isgur-Wise function, and, finally, analysis of corrections violating the heavy quark symmetry at the point of zero recoil. We begin, however, from a systematic classification of all local operators which appear in the heavy quark expansion up to the level $O(m_Q^{-3})$. Some terms of order $1/m_Q^3$ in particular heavy quark expansions are actually not expressible in terms of the local operators and are, rather, related to non-local correlation functions. A full classification of such correction also exists [42, 25], but we will not go into details only marginally mentioning them here and there. The interested reader is referred to the original publications [42, 25].

3.1 Catalogue of relevant operators

The local operators in the heavy quark expansion are bilinear in the heavy quark field. They are certainly gauge invariant, and in many instances, when the expansion is built for scalar quantities, the operators must be Lorentz scalars. As in any operator product expansion in QCD they can be ordered according to their dimension. We will limit ourselves here to dimension 6 and lower. This leaves us with quite a few possibilities listed below. We start with the Lorentz scalar operators. The only appropriate operators are

$$\bar{Q}Q, \quad i\frac{\sigma_{\mu\nu}G_{\mu\nu}}{2}Q, \quad \text{and} \quad \bar{Q}\Gamma Q \Gamma q$$

(3.1)

where $\Gamma$ stands here for a combination of $\gamma$ and color matrices. All other structures that might come to one's mind reduce to those listed above and full derivatives by virtue of the equations of motion.

(Exercise: prove that this is the case, for instance, for the operators $\bar{Q}D^2Q$ and $\bar{Q}G_{\mu\nu}\gamma_\mu D_\nu Q$.)

(i) The only operator of dimension 3 is $\bar{Q}Q$. This operator is related to the heavy quark current $\bar{Q}\gamma_0 Q$ plus terms suppressed by powers of $1/m_Q$. The leading term of this expansion has been already discussed, see Eq. (1.30). Actually it is not difficult to continue this expansion one step further. The following relation is exact:

$$\bar{Q}Q = \bar{Q}\gamma_0 Q + 2\bar{Q}\left(\frac{1 - \gamma_0}{2}\right)^2 Q = \bar{Q}\gamma_0 Q - 2\bar{Q}\frac{\vec{\not{p}}}{2m_Q} \cdot \frac{\vec{\not{p}}}{2m_Q} Q = \bar{Q}\gamma_0 Q + \bar{Q}\frac{\vec{p}^2}{2m_Q^2} Q + \text{a total derivative} ;$$

(3.2)
in the first relation above the operators $\pi_\mu$ act on the $\bar{Q}$ field. Keeping in mind that we always consider only the forward matrix elements, with the zero momentum transfer, we can drop all terms with total derivatives. Applying now the equations of motion (1.8) and (1.9) generates the $1/m_Q$ expansion for the scalar density,

$$
\bar{Q}Q = \bar{Q}\gamma_0 Q + \frac{1}{2m_Q^2}\bar{Q} (\pi^2 + \frac{i}{2}\sigma G) Q = \bar{Q}\gamma_0 Q - \frac{1}{2m_Q^2}\bar{Q} (\pi\bar{\pi})^2 Q - \frac{1}{4m_Q^3}\bar{Q} \left(-\langle \bar{D}E \rangle + 2\bar{\pi} \cdot E \times \bar{\pi} \right) Q + \mathcal{O}\left(\frac{1}{m_Q^4}\right). \tag{3.3}
$$

Here $E_i = G_{i0}$ is the chromoelectric field, and its covariant derivative is defined as $D_j E_k = -i[\pi_j, E_k]$; we have omitted the term $\bar{Q}([\pi_k, [\pi_0, \pi_i]] - [\pi_i, [\pi_0, \pi_k]])Q$ using the Jacobi identity. Moreover,

$$
\bar{D}E = g^2 t^a J_0^a
$$

by virtue of the QCD equation of motion (here $J_\mu^a = \sum_q \bar{q}\gamma_\mu t^aq$ is the color quark current). Therefore the first of the $1/m_Q^3$ terms can be rewritten as a four-fermion operator.

(ii) As has been already mentioned, no operators of dimension 4 exist.

(iii) Dimension five. There is only one such operator,

$$
\mathcal{O}_G = \bar{Q}\frac{i}{2}\sigma_{\mu\nu} G_{\mu\nu} Q, \tag{3.4}
$$

where $\sigma_{\mu\nu} = (1/2)[\gamma_\mu, \gamma_\nu]$. Again, it can be expanded in the powers of $1/m_Q$,

$$
\mathcal{O}_G = -\bar{Q}\bar{\sigma}\bar{B}Q - \frac{1}{2m_Q^2}\bar{Q} \left(-\langle \bar{D}E \rangle + 2\bar{\pi} \cdot E \times \bar{\pi} \right) Q + \mathcal{O}\left(\frac{1}{m_Q^4}\right). \tag{3.5}
$$

where $\bar{B}$ is the chromomagnetic field, $\bar{B} = \nabla \times \bar{A} = -(1/2)\epsilon_{ijk} G_{jk}$.

(iv) Dimension 6 four-quark operators $\sum_i \bar{Q}\Gamma_i Q\bar{q}q$. Generally speaking, the matrix $\Gamma_i$ can be any Lorentz matrix $(1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5$ or $\sigma_{\mu\nu})$ or any of the above multiplied by $t^a$. Of course, in specific problems only a subset of these matrices may appear. The four-quark operators differ by the chiral properties of the light quark field $q$. Some of them carry non-zero chirality (they are non-singlet with respect to $SU(N_f)_L \times SU(N_f)_R$). Hence, they do not show up in the transitions associated with the weak currents of the $V - A$ type.

Further remarks will concern operators that are spatial scalars but not Lorentz scalars. They appear in the low-energy effective Lagrangian (1.11) and in the expansions of the type (3.3) and (3.5). The most important operator from the class is

$$
\mathcal{O}_\pi = \bar{Q}\bar{\pi}\pi^2 Q, \tag{3.6}
$$

which we have already encountered more than once.

\footnote{Note that in our notations $\bar{D} = -\partial/\partial x - i\bar{A}$, therefore $\langle \bar{D}E \rangle = -\text{div}\, E$ in the Abelian case.}
Dimension 4 operators are all reducible to those of dimension 5 and higher. For instance,

\[
\bar{Q} \gamma \vec{\pi} \gamma_{0} Q = O(m_{Q}^{-2}),
\]

\[
Q_{\pi_{0}} Q = \frac{1}{2m_{Q}} Q(\vec{\pi}^{2} - \frac{i}{2} \sigma G)Q + O(m_{Q}^{-2}).
\]

At the level of dimension 6 only one additional operator emerges (apart from the four-fermion operators), namely,

\[
\bar{Q} \vec{\sigma} \cdot \vec{E} \times \vec{\pi} Q.
\]

(3.7)

At first sight it might seem that one could build extra operators of dimension 6, from the gluon fields, e.g.

\[
\bar{Q} \pi_{i} E_{i} Q \text{ or } \bar{Q} \sigma_{i} \epsilon_{ijk} (D_{j} E_{k}) Q.
\]

Actually they are reducible to operators of higher dimension via the equations of motion. Indeed, using the fact that

\[
E_{i} = -i[\pi_{0}\pi_{i}]
\]

one can rewrite

\[
\bar{Q} \pi_{i} E_{i} Q = -i \bar{Q} \pi_{i} [\pi_{0}\pi_{i}] Q = -i \bar{Q} (\pi_{i} \pi_{0} \pi_{i} - \pi_{i}^{2} \pi_{0}) Q =
\]

\[
- i \bar{Q} ([\pi_{i} \pi_{0}] \pi_{i} + \pi_{0} \pi_{i}^{2} - \pi_{i}^{2} \pi_{0}) Q = - i \bar{Q} ([\pi_{i} \pi_{0}] \pi_{i} + \frac{1}{2m_{Q}} (\vec{\sigma} \vec{\pi})^{2} \pi_{i}^{2} - \frac{1}{2m_{Q}} \pi_{i}^{2} (\vec{\sigma} \vec{\pi})^{2}) Q
\]

\[
- \frac{i}{2} \bar{Q} [\pi_{i} [\pi_{0} \pi_{i}]] Q \text{ + dimension seven } =
\]

\[
= -i \bar{Q} (\text{div} \vec{E}) Q \text{ + dimension seven.}
\]

(3.8)

This is a four-fermion operator. By the same token,

\[
\bar{Q} \sigma_{i} \epsilon_{ijk} (D_{j} E_{k}) Q = \bar{Q} \sigma_{i} D_{0} B_{i} Q =
\]

\[
- i \bar{Q} \sigma_{i} [\pi_{0} B_{i}] Q = - \frac{i}{2m_{Q}} \bar{Q} ([\vec{\sigma} \vec{\pi}]^{2}, \vec{\sigma} \vec{B}] Q
\]

(3.9)

which is obviously of the next order in $1/m_{Q}$ (a dimension-seven operator).
3.2 Extracting/determining the matrix elements

Construction of the operator product expansion is only the first step in any theoretical analysis. The heavy quark expansion must be converted into predictions for the physical quantities. To this end it is necessary to take the matrix elements of the operators involved in the expansion. The latter carry all information about the large distance dynamics responsible for the hadronic structure, in all its peculiarity. These matrix elements in our QCD-based approach play the same role as the wave functions in the non-relativistic quark models.

In this section we will summarize what is known about the matrix elements of the operators from the list presented above.

(i) The most favorable situation takes place at the level of dimension three. Indeed, the only Lorentz scalar operator of dimension 3 is $\bar{Q}Q$ which has a nice expansion (3.3). The operator $\bar{Q}\gamma^0 Q$ is the time component of the conserved current, measuring the number of the quarks $Q$ in $H_Q$. Therefore, both for mesons and baryons

$$\frac{1}{2M_H} \langle H_Q | \bar{Q}\gamma^0 Q | H_Q \rangle = 1.$$ \hspace{1cm} (3.10)

(As usual, we stick to the rest frame of $H_Q$; in the case of baryons averaging over the baryon spin is implied).

(ii) The status of two operators of dimension 5 is different. Let us consider first $O_G$ whose matrix elements are expressible in terms of experimentally measurable quantities.

To leading order in $1/m_Q$ the parameter $\mu_G^2$ defined in Eq. (2.56) reduces to

$$\mu_G^2 = \frac{1}{2M_{H_Q}} \langle H_Q | O_G | H_Q \rangle = -\frac{1}{2M_{H_Q}} \langle H_Q | Q \bar{\sigma} \vec{B} Q | H_Q \rangle.$$ \hspace{1cm} (3.11)

For pseudoscalar mesons this quantity can be related to the measured hyperfine mass splittings. Indeed, $Q \bar{\sigma} \vec{B} Q$ is the leading spin-dependent operator in the heavy quark Hamiltonian (1.14). Hence

$$\mu_G^2(B_Q) = \frac{3}{4} (M_{B^*}^2 - M_B^2)$$ \hspace{1cm} (3.12)

where $B^*$ and $B$ are generic notations for the vector and pseudoscalar mesons, respectively, and the limit $m_Q \to \infty$ is implied. Assuming that the $b$ quark already belongs to this asymptotic limit one estimates $\mu_G^2$ from the measured B meson masses,

$$\mu_G^2 \approx 0.35 \text{ GeV}^2.$$ 

Furthermore, in the baryon family four baryons are expected to decay weakly and are, thus, long-living states: $\Lambda_Q$, $\Sigma_Q$, $\Xi_Q$ and $\Omega_Q$. In the first three of them

\footnote{Some authors prefer a different nomenclature \[\text{[13]}\]. The expectation values of the chromomagnetic and kinetic energy operators are sometimes called $\lambda_2$ and $\lambda_1$.}
the total angular momentum of the light cloud is zero; hence the chromomagnetic field has no vector to be aligned with, and
\[ \mu_G^2(\Lambda_Q) = \mu_G^2(\Sigma_Q) = \mu_G^2(\Xi_Q) = 0. \]  
(3.13)

In the case of \( \Omega_Q \) the total angular momentum of the light cloud is 1. Hence,
\[ \mu_G^2(\Omega_Q) = \frac{2}{3} \left( \frac{M_{\Omega_Q}^2}{\Omega_Q^{3/2}} - \frac{M_{\Omega_Q}^2}{\Omega_Q^{1/2}} \right) \]  
(3.14)

where the superscripts \( 3/2 \) and \( 1/2 \) mark the spin of the baryon. Although the mass splitting on the right-hand side of Eq. (3.14) is in principle measurable, it is not known at present, and if one wants to get an estimate one has to resort to quark models or lattice calculations. Both approaches are not mature enough at the moment to give reliable predictions for this quantity and I suggest we wait until experimental measurements appear.

Let us proceed now to the discussion of the matrix element of the operator \( O_\pi \). The physical meaning of this matrix element is the average kinetic energy (more exactly, the spatial momentum squared) of the heavy quark \( Q \) inside \( H_Q \). This operator is spin-independent, and it is much harder to extract \( \mu_\pi^2 \) (the parameter \( \mu_\pi^2 \) is defined in Eq. (2.58)) from phenomenology, although such an extraction is possible, in principle (see Ref. [25] and Lectures 4 and 5 for details). Since the phenomenological analysis has not been carried out yet one has to rely on theoretical estimates. Several calculations of \( \mu_\pi^2 \) within the QCD sum rules yield [44, 45]

\[ \mu_\pi^2(B) = \frac{1}{2M_B} \langle B | O_\pi | B \rangle = 0.5 \pm 0.1 \ \text{GeV}^2. \]

A remarkable model-independent lower bound on \( \mu_\pi^2(B) \) exists in the literature,
\[ \mu_\pi^2(B) > \mu_G^2(\Lambda) \approx 0.35 \ \text{GeV}^2 \]  
(3.15)

The quantum-mechanical derivation of this inequality due to Voloshin (see Ref. [4]) is straightforward. Indeed, start from the square of the Hermitian operator \( (\vec{\sigma}\vec{\pi})^2 \) and average it over the \( B \) meson state. It is obvious then that \( \langle (\vec{\sigma}\vec{\pi})^2 \rangle > 0 \). Using the fact that \( (\vec{\sigma}\vec{\pi})^2 = \vec{\pi}^2 + \vec{\sigma}\vec{B} \) we immediately arrive at Eq. (3.15). A field-theoretic derivation of the same result can be found in Ref. [25]. It is remarkable that the inequality (3.15) almost saturates the QCD sum rule estimate quoted above. Another lower bound on \( \mu_\pi^2(B) \), obtained from a totally different line of reasoning, is discussed in Sect. 3.6. It turns out to be close to Eq. (3.15) numerically.

It is plausible that \( \mu_\pi^2 \) for mesons and baryons is different – there is no reason why they should coincide. The task of estimating \( \mu_\pi^2 \) for baryons remains open.

These parameters, \( \mu_\pi^2 \) and \( \mu_G^2 \), along with \( \Lambda \), are most important in applications. In most applications one deals with the expectation values over the \( B \) meson state. Therefore, let us agree that \( \mu_\pi^2, \mu_G^2, \) and \( \Lambda \), with no subscripts or arguments, are
defined with respect to the $B$ mesons. This is a standard convention. In a few cases
when these quantities are defined with respect to other heavy flavor hadrons we will
mark them by the corresponding subscripts or indicate with parentheses.

(iii) Operators of dimension 6 are studied to a much lesser extent than those
of dimension 5. Perhaps, the least favorable is the situation with the operator $O_E$
given in Eq. (3.7). Let us parametrize its matrix element as follows:

$$
\frac{1}{2M_H} \langle H_Q | \bar{Q} \vec{\sigma} \times \vec{E} Q | H_Q \rangle = \mu^3_E. \tag{3.16}
$$

This operator in the heavy quark Hamiltonian is responsible for the spin-orbit inter-
actions and consequently generates the spin-orbit splittings between the masses
of the ground states and the orbital excitations. Hence, in the non-relativistic limit
(non-relativistic with respect to the spectator light quark) $\mu^3_E$ vanishes for the $S$
wave states. Of course, the non-relativistic approximation with respect to the light
quark is very bad. The estimate of $\mu^3_E$ existing in the literature [40] is so rough that
it, probably, does not deserve to be discussed here.

As for the four-quark operators the only method of estimating their matrix ele-
ments which does not rely heavily on the most primitive (and hence totally unreli-
able) quark models is the old idea of factorization applicable only in mesons but –
alas – not in baryons.

First of all let us observe that each of the four-quark operators exists in two
variants differing by the color flow. One can always rearrange the operators, using
the Fierz identities, in the form

$$
O_{4q} = \bar{Q} \gamma^\mu \gamma^5 q \Gamma Q \tag{3.17}
$$

and

$$
\tilde{O}_{4q} = \bar{Q} t^a \gamma^\mu q t^a Q \tag{3.18}
$$

Take for definiteness $\Gamma = \gamma^\mu \gamma^5$. (Other $\gamma$ matrices can also appear, of course.) In the
first operator color is transferred from the initial heavy to the final light quark and
from the initial light to the final heavy quark. The second operator is essentially
color-exchange. Now, if we are interested in the matrix elements over the meson
states we can simply factorize the currents appearing in Eqs. (3.17) and (3.18) (i.e.
saturate by the vacuum intermediate state),

$$
\frac{1}{2M_B} \langle B_Q | O_{4q} | B_Q \rangle = \frac{1}{2} M_B f_B^2, \quad \frac{1}{2M_B} \langle B_Q | \tilde{O}_{4q} | B_Q \rangle = 0 \tag{3.19}
$$

where $f_B$ is the pseudoscalar decay constant,

$$
\langle 0 | \bar{Q} \gamma^\mu \gamma^5 q | B_Q \rangle = i f_B p_\mu \tag{3.20}
$$

As we will discuss shortly, in the limit $m_Q \to \infty$ the combination $M_B f_B^2$, scales as
$m_Q^0$ (modulo logarithmic corrections) so that the right-hand side of Eq. (3.19) is the
cube of a typical hadronic mass, as it should be.
Factorization in Eq. (3.19) is justified by \(1/N_c\) arguments. Indeed, corrections to Eq. (3.19) are of the order of \(N_c^{-1}\).

Thus, from the whole set of the four-quark operators we can say something about the meson expectation values of those operators which are reducible to

\[
(\bar{Q} \Gamma q)(q \Gamma Q)
\]  

(3.21)

where \(\Gamma\) stands for a Lorentz matrix but not for the color one, and the color indices are contracted within each of two brackets separately. Up to terms \(O(N_c^{-1})\) two brackets can be factorized.

To get an idea about the numerical value of \((1/2)f_B^2M_B\) we should substitute a numerical value for \(f_B\) which is not measured so far. Theoretical ideas about this fundamental constant will be discussed later. Now let me say only that \((1/2)f_B^2M_B \sim 0.1 \text{ GeV}^3\), with a significant uncertainty. Those matrix elements which are due to nonfactorizable contributions (see Eq. (3.18)) are essentially undetermined, although they are expected to be suppressed compared to the factorizable matrix elements \((1/2)f_B^2M_B\).

As for the baryon matrix elements of the four-quark operators next-to-nothing is known about them at the moment. Some very crude estimates within the naive quark model are available [46] but they are very unreliable.

In conclusion of this section a remark is in order concerning numerical estimates of the key parameter of the heavy quark theory, \(\bar{\Lambda}\). I postponed discussing the issue because its value continues to be controversial. QCD sum rules indicate [48, 45, 49] that \(\bar{\Lambda} \sim 0.5 \text{ GeV}\). This number is in full agreement with the lower bound stemming from Voloshin’s sum rule, see Sect. 3.6. However, some lattice calculations yield a factor of 2 lower estimate. I am inclined to think that there is something wrong in the lattice results. Perhaps, the lattice definition of \(\bar{\Lambda}\) does not fully correspond to that of the continuum theory. It is inconceivable that such a low value of \(\bar{\Lambda}\) as 0.2 or even 0.3 GeV could be reconciled with the lower bound implied by Voloshin’s sum rule.

In the discussion above we have totally disregarded logarithmic dependence of the operators and their matrix elements due to anomalous dimensions – i.e. the issue of the normalization point (including the normalization point of \(\bar{\Lambda}\)). This is in line with that so far we pretend that hard gluons do not exist. A brief excursion into this topic will be undertaken later; here it is only worth mentioning that all numerical estimates presented above refer to a low normalization point, of order of several units times \(\Lambda_{QCD}\).

We are ready now to review classic problems of the heavy quark theory. We will gradually move from simpler to more sophisticated problems.

### 3.3 Mass formula revisited

In Sect. 1.5 we have found the first subleading term in the mass formula for the heavy flavor hadrons. The parameter \(\bar{\Lambda}\) was related to the expectation value of
the gluon anomaly, see Eq. (1.32). It is very easy to continue the expansion one
step further and find the next subleading term, of order $1/m_Q$. One could have
extended the derivation along the lines suggested in Sect. 1.5. This was done in Ref.
[25]. This is not the fastest route, however. Instead, let us observe that the $1/m_Q$
term in the Hamiltonian (1.14) can be considered as the first order perturbation;
the corresponding correction to the mass is merely the expectation value of this
perturbation,

$$M_{H_Q} = m_Q + \bar{\Lambda} + \frac{1}{2m_Q}(2M_{H_Q})^{-1}\langle H_Q|\pi^2 + \vec{\sigma}\vec{B}|H_Q\rangle + ... =$$

$$m_Q + \bar{\Lambda} + \frac{(\mu^2 - \mu_G^2)\rho_Q}{2m_Q} + ...$$

(3.22)

The terms of order $1/m_Q^2$ are neglected. If we keep only the terms up to $1/m_Q$
it does not matter whether the state $H_Q$ we average over is an asymptotic state
(corresponding to $m_Q = \infty$) or the real physical heavy flavor state. I remind that,
unlike HQET, we work with the physical states. The difference becomes noticeable
only at the level $1/m_Q^2$. In this order the mass formula does not reduce any more to
the expectation values of local operators. A part of the $1/m_Q^2$ correction is due to
non-local correlation functions, see Ref. [25] for further details. Eq. (3.22) was first
presented in Ref. [43].

### 3.4 The scaling law of the pseudoscalar and vector coupling
constants

The pseudoscalar and vector meson constants $f_P$ and $f_V$ are defined as

$$\langle 0|\bar{Q}\gamma_\mu\gamma_5 q|B_Q\rangle = if_P p_\mu \quad \langle 0|\bar{Q}\gamma_\mu q|B_Q^*\rangle = if_V M_V \epsilon_\mu .$$

(3.23)

An alternative definition of the pseudoscalar constant can be given in terms of the
pseudoscalar current,

$$\langle 0|\bar{Q}i\gamma_5 q|B_Q\rangle = f'_P M_B .$$

(3.24)

The constant $f_B$ is one of the key parameters of the heavy quark physics, just in
the same way the constant $f_\pi$ is a key parameter of the soft pion physics. Below
we will show that in the limit $m_Q \to \infty$ all three constants, $f_P, f_V$ and $f'_P$, coincide
with each other and scale as $m_Q^{-1/2}$ modulo a weak logarithmic dependence on $m_Q$.
(Needless to say, that both masses, $M_P$ and $M_V$ also coincide in this limit.) For
definiteness let us consider $f'_P$. Two other constants can be treated in a similar
manner. The subscripts will be omitted in the remainder of this section to avoid
overloaded expressions.

Start from the two-point function

$$\mathcal{A}(k) = i \int e^{ikx} d^4x \langle T\{\bar{Q}(x)i\gamma_5 q(x) \bar{q}(0)i\gamma_5 Q(0)\}\rangle ,$$

(3.25)
where \( k \) is the external momentum. The two-point function \( A(k) \) develops a pole at \( k^2 = M^2 \), the position of the ground state pseudoscalar,

\[
A(k) = \frac{f^2 M^2}{k^2 - M^2} + \text{excitations},
\]

(3.26)

Of course, the currents produce from the vacuum not only the ground state mesons but also all excitations in the given channel. It is clear that to isolate the lowest-lying pole we should keep \( k^2 \) close to \( M^2 \). Keeping in mind Eq. (1.26) it is natural to represent \( k \) as

\[
k_\mu = \{ m_Q + \epsilon, 0, 0, 0 \}
\]

where \( \epsilon \) scales like \( \Lambda_{\text{QCD}} \) while \( m_Q \to \infty \). With this parametrization of \( k_\mu \) we merely separate the mechanical (uninteresting) part of the momentum. The pole is achieved at \( \epsilon = \bar{\Lambda} \); near the pole

\[
A(\epsilon) \approx \frac{f^2 M}{2(\epsilon - \bar{\Lambda})}.
\]

(3.27)

The value of the coupling constant is obtained by amputating the pole,

\[
f^2 = \lim_{\epsilon \to \bar{\Lambda}} \left\{ \frac{2(\epsilon - \bar{\Lambda})}{M} A(\epsilon) \right\}.
\]

(3.28)

Let us now examine the theoretical expression for the same two-point function. In the background field technique (which is already pretty familiar, right?) we write

\[
A(k) = i \text{Tr} \left\{ i\gamma_5 \frac{1}{\mathcal{P} - m_q} i\gamma_5 \frac{1}{\mathcal{P} - m_Q} \right\}.
\]

(3.29)

Superficially this expression looks the same as if the quarks were treated as free; they are not, however; the coupling to the background field is reflected in the fact that \( \mathcal{P}_\mu \) is the momentum operator, not just a \( c \)-number four-vector.

Now we will take advantage of the fact that \( m_Q \to \infty \). As usual, we close our eyes on any possible hard contributions, assuming that \( \mathcal{P} \), the momentum operator of the light quark, is soft, i.e. does not scale with \( m_Q \) in the large mass limit but, rather \( \mathcal{P} \sim \Lambda_{\text{QCD}} \). (This is the reason, by the way, why the large external momentum \( k \) was directed through the heavy quark line in Eq. (3.29).) Intuitively it is clear that the hard components of \( \mathcal{P} \) should be irrelevant for the lowest-lying state whose “excitation energy” measured from \( m_Q \) is of order \( \Lambda_{\text{QCD}} \).

If \( \mathcal{P} \) is soft and \( m_Q \to \infty \) the heavy quark Green function in the leading approximation takes the form

\[
\frac{1}{k' + \mathcal{P} - m_Q} = (k' + \mathcal{P} + m_Q) \frac{1}{(k + \mathcal{P})^2 - m_Q^2 + (i/2)\sigma G} = \frac{\gamma_0 + 1}{2} \frac{1}{\epsilon + \mathcal{P}_0},
\]

(3.30)
where in the second line all $1/m_Q$ terms are omitted. No explicit $m_Q$ dependence is left! This means that $A(\epsilon)$ scales as $m_Q^0$. Equation (3.28) immediately implies then that $f$ scales as

$$f \sim m_Q^{-1/2}. \quad (3.31)$$

Equation (3.30) for the heavy quark Green function in the limit $m_Q \to \infty$ is in one-to-one correspondence with the leading term $\bar{Q}\pi_0(1 + \gamma_0)/2Q$ in the low-energy Lagrangian (1.11). The analysis of the scaling law of the coupling constants presented above is a simplified version of that carried out many years ago by Shuryak [6]. Later it was established that the power dependence on $m_Q$ in Eq. (3.31) is supplemented by a logarithmic dependence appearing due to the hard gluon exchanges [50].

A few words about the numerical value of the coupling constants. Although in principle $f_D$ and $f_B$ are measurable experimentally, practically it is a very hard measurement, especially for $B$. No experimental number for $f_B$ exists so far. Its value was estimated in the QCD sum rules and on lattices more than once. Leaving aside a dramatic evolution of the issue I will say only that the recent and most reliable results cluster around 160 MeV both, in the sum rules [47, 48] and in the lattice calculations [51]. It is curious to note that in $(m_b)^{1/2} f_B$ the preasymptotic $1/m_Q$ correction turned out to be unexpectedly large and negative [47, 45, 52, 51]; at the same time in $(m_b)^{1/2} f'_B$ the preasymptotic $1/m_Q$ correction is much more modest [47, 48].

### 3.5 Proof of the Isgur-Wise formula

I return to my promise to prove the Isgur-Wise formula (1.25). Consider the three-point function depicted on Fig. 6. The sides of the triangle are the Green functions of the quarks in the background gluon field. The reduction theorems tell us that in order to get the transition amplitudes $\langle H_c|\bar{c}\Gamma b|H_b\rangle$ from this three-point function we must “amputate” it: multiply by $(p^2 - M_B^2)$ and $(p'^2 - M_D^2)$, tending $p^2$ to $M_B^2$ and $p'^2$ to $M_D^2$. This singles out the meson states we want to pick up. For the vector mesons we must also multiply the three-point functions by its polarization vector $\epsilon_\mu$. The last step necessary for amputation is dividing by the coupling constants (residues) connecting the currents $\bar{b}i\gamma_5 c$ and $\bar{b}\gamma_\mu c$ to the respective mesons. If the currents are normalized appropriately – and we will always do that – the corresponding coupling constants in the pseudoscalar and vector channels are the same, $fM$ (see Sect. 3.4). It is convenient to combine the pseudoscalar and vector channels together by introducing the currents

$$J = B\bar{b}i\gamma_5 q + B_\mu \bar{b}\gamma_\mu q \quad \text{and} \quad J' = D^*\bar{q}i\gamma_5 c + D'_\mu \bar{q}\gamma_\mu c \quad (3.32)$$

where $B$ and $D$ are external constants marking the annihilation and creation of the initial and final $B$’s and $D$’s ($B_\mu$ and $D'_\mu$ denote the polarization vectors of $B^*$ and $D^*$, respectively).
The expression for the three-point function of Fig. 6 takes the form

\[ i \text{Tr} \left\{ \mathcal{M}' \frac{1}{\gamma' + \gamma - m_c} \Gamma \frac{1}{\gamma' + \gamma - m_b} \mathcal{M} \frac{1}{\gamma - m_q} \right\} \]  

(3.33)

where the matrices \( \mathcal{M}' \) and \( \mathcal{M} \) are introduced in Eq. (1.24). In Sect. 3.4 the \( m_Q \rightarrow \infty \) limit of the quark propagator was obtained in the rest frame. Here we have two heavy flavor states, initial and final, and both can not be at rest simultaneously. Therefore, we need the very same propagator in the arbitrary frame. Let \( p_\mu = m_Q v_\mu + \epsilon_\mu \). Then a trivial generalization of Eq. (3.30) is

\[
\frac{1}{\gamma' + \gamma - m_Q} = \frac{(\gamma' + \gamma + m_Q)}{(p + \gamma)^2 - m_Q^2 + (i/2)\sigma G} \rightarrow \frac{\gamma' + 1}{(\epsilon + \gamma)v} \frac{1}{2} \frac{1}{\epsilon + \gamma} v 
\]  

(3.34)

Using this propagator in Eq. (3.34) we rewrite the three-point function of Fig. 6 as follows

\[
\left( \mathcal{M}' \frac{1}{2} \Gamma \frac{1}{2} \mathcal{M} \right)_{\alpha\beta} \times \left\{ \frac{1}{(\epsilon' + \gamma)\epsilon'} \frac{1}{(\epsilon + \gamma)v} \left( \frac{1}{\gamma - m_q} \right)_{\beta\alpha} \right\}. 
\]  

(3.35)

The expression in the braces is independent of the heavy quark masses; moreover, it is proportional to the three-point function in the theory with the scalar heavy quarks considered in Sect. 1.4. As was explained there, in this theory in the limit \( m_Q \rightarrow \infty \) only one form factor survives.

One subtle point deserves discussing here. When I speak about the heavy flavor mesons I keep in mind particles built from the heavy quark and a light antiquark, which is not always in line with the accepted nomenclature. Say, they call \( B \) meson a particle with the \( b \) antiquark, not quark. Since this distinction plays no role in my lectures I will continue to ignore this linguistic nuance referring to the \( b\bar{q} \) states as \( B \) mesons. All equations presented above assume that the \( b \) quark in the initial state annihilates to produce the \( c \) quark in the final state. Simultaneously a light antiquark in the initial meson is annihilated and the same light antiquark reappears in the final meson.

Return now to the model with spinless heavy quarks. The heavy flavor hadrons we now deal with are spin-1/2 baryons. More exactly, we have \( \text{anti}b \text{aryon} \) in the initial state and \( \text{anti}b \text{aryon} \) in the final state. This means that near the mass shell the expression in the braces in Eq. (3.35) takes the form

\[
\left( \frac{-\gamma' + 1}{2} - \frac{\gamma' + 1}{2} \right)_{\beta\alpha} \sqrt{M_B M_D f_B f_D} \frac{1}{\nu' \epsilon'} - \frac{1}{\nu \epsilon - \Lambda} \left( \xi(y) \right); 
\]  

(3.36)
the minus sign between the unit term and the \( v \) term in the density matrices is due to the fact that we deal with the antibaryons. Amputating the legs and combining Eqs. (3.36) and (3.35) we get the Isgur-Wise formula (1.25) since \( M(-\varepsilon + 1) = (\varepsilon + 1)M \), and so on.

### 3.6 The Bjorken sum rule and all that

The Isgur-Wise function \( \xi(y) \) carries information about the structure of the light cloud. Needless to say that the heavy quark expansion \emph{per se} does not help to calculate this function. One has to rely on methods applicable in the strong coupling regime which are outside the scope of my lectures (QCD sum rules, lattices, ...). Still, some interesting and important relations emerge. Here we will discuss a sum rule for the slope of the Isgur-Wise function and related topics.

We are already familiar with the sum rule technology in the heavy quark theory. In Sect 2.1 we dwelled on a simplified problem: inclusive decays of a spinless heavy quark \( Q \) into a lighter spinless quark \( q \) and a fictitious spin-zero photon \( \phi \). The “photon” was assumed to be on mass shell, \( q^2 = 0 \). The predictions obtained referred to the moments of the “photon” energy. Now you are mature enough to face actual problems from real life. We will concentrate on the decays of a \( b \) containing hadron into a \( c \) containing hadron plus the lepton pair \( l\nu \). The four-momentum of the lepton pair is a free parameter, in particular, \( q^2 \neq 0 \). We can and will choose the value of \( q \) to our advantage.

Consider a transition \( H_b \to H_c \) induced by some particular current, say, axial-vector. At zero recoil \( \xi = 1 \). In the SV limit where the velocity of the recoiling hadron is small

\[
\xi(y) = 1 - \rho^2(y - 1) + ... = 1 - \rho^2 \frac{\vec{v}^2}{2} + ...
\]  

(3.37)

where \( y = vv' \), \( \vec{v} \) is the spatial velocity of \( H_c \) in the \( H_b \) rest frame, and the slope parameter \( \rho^2 \) was introduced in Ref. [32]. It plays the same role as, say, the charge radius of pions.

To get relations involving \( \rho^2 \) we start from consideration of the transition operator similar to that of Eq. (2.13). The expectation value of the transition operator over the \( B \) meson state yields the hadronic amplitude whose imaginary part is proportional to the probability of the inclusive decay \( B \to X_c l\nu \) with the fixed value of \( q \), the momentum carried away by the lepton pair \( l\nu \). (Here \( X_c \) denotes an inclusive hadronic state containing one \( c \) quark.) A new element compared to the toy model of Sect. 2.1 is the heavy quark spin. Another distinction is the fact that, to achieve the SV limit, we do not need now to assume that \( m_c \) is close to \( m_b \). In the semileptonic decay \( B \to X_c l\nu \) one can fine-tune the lepton pair momentum in such a way that \( q^2 \) is close to its maximal value, \( q_{\text{max}}^2 = (M_B - M_D)^2 \); then the \( c \) containing hadronic state produced is almost at rest, and we are in the SV limit even though the charmed quark is significantly lighter than the \( b \) quark. In other words, for such values of \( q^2 \) the \( c \) quark is always slow.
This transition operator describes the forward scattering of $B$ to $B$ via intermediate states $D^*$ and “excitations”. (We focus for definiteness on the axial-vector current, $\bar{c}\gamma_\mu\gamma_5b$. The vector current can be treated in a similar way.) The excitations can include, for instance, $D^*\pi\pi$, and so on. In general, all intermediate states except the lowest-lying $D^*$ will be referred to as excitations. The transition operator

$$\hat{T}_{\mu\nu} = i \int d^4 x e^{-iqx} T\{\bar{b}(x)\gamma_\mu\gamma_5c(x), \bar{c}\gamma_\mu\gamma_5b\}$$

in the Born approximation is given by the diagram of Fig. 1. The hadronic amplitude obtained by averaging $\hat{T}_{\mu\nu}$ over the $B$ meson state,

$$h_{\mu\nu} = \frac{1}{2M_B} \langle B|\hat{T}_{\mu\nu}|B \rangle$$

contains various kinematical factors. In the general case the hadronic tensor $h_{\mu\nu}$ consists of five different covariant structures [30, 33]:

$$h_{\mu\nu} = -h_1g_{\mu\nu} + h_2v_\mu v_\nu - ih_3\epsilon_{\mu\nu\alpha\beta}v_\alpha q_\beta + h_4q_\mu q_\nu + h_5(q_\mu v_\nu + q_\nu v_\mu).$$

Moreover, the invariant hadronic functions $h_1$ to $h_5$ depend on two variables, $q_0$ and $\vec{q}^2$, or $q_0$ and $|\vec{q}|$. For $\vec{q} = 0$ only one variable survives, and only two of five tensor structures in $h_{\mu\nu}$ are independent.

Each of these hadronic invariant functions satisfies a dispersion relation in $q_0$,

$$h_i(q_0) = \frac{1}{2\pi} \int \frac{w_i(\tilde{q}_0)d\tilde{q}_0}{\tilde{q}_0 - q_0} + \text{polynomial}$$

where $w_i$ are observable structure functions,

$$w_i = 2 \text{Im } h_i.$$

For our purposes it is quite sufficient to analyze only one function, namely, $h_1$. Moreover, for the time being we will disregard all non-perturbative corrections $O(\Lambda^2_{\text{QCD}})$ which means that operators in the expansion of $\hat{T}_{\mu\nu}$ other than $\bar{b}b$ can be neglected, and the $B$ meson expectation value of $\bar{b}b$ can be replaced by unity. Calculating $h_1$ in this approximation is a trivial problem (it was a part of the exercise suggested in Sect. 2.1). Specifically,

$$- h_1^{AA} = (m_b + m_c - q_0)\frac{1}{z} + O(\Lambda_{\text{QCD}}^2)$$

where

$$z = (m_b - q_0 - E_c)(m_b - q_0 + E_c), \quad E_c^2 = m_c^2 + \vec{q}^2.$$
\[ q_{\text{max}} = M_B - E_{D^*}, \quad E_{D^*} = M_{D^*} + \frac{q^2}{2M_{D^*}}. \]  

(3.45)

When \( \epsilon \) is real and positive we are on the physical cut where the actual intermediate states (e.g. \( D^* \)) are produced. Here the imaginary part of \( h_1 \) is given by the \textquotedblleft elastic\textquotedblright\ contribution of \( D^* \) plus inelastic excitations. For negative \( \epsilon \) we are below the cut. The result for \( h_1 \) above can be trusted if \( -\epsilon \gg \Lambda_{\text{QCD}} \) since the expansion actually runs in \( \Lambda_{\text{QCD}}/\epsilon \). The expansion in the inverse heavy quark mass also requires, of course, that \( |\epsilon| \ll m_{c,b} \). A bridge between the physical domain of positive \( \epsilon \) and the Euclidean domain of negative \( \epsilon \) where the calculation is done is provided by the dispersion relations.

At the next stage the amplitude \( h_1 \) is expanded in powers of \( \Lambda_{\text{QCD}}/\epsilon \) and \( \epsilon/m_{b,c} \). Polynomials in \( \epsilon \) can be discarded since they have no imaginary part. We are interested only in negative powers of \( \epsilon \). The coefficients in front of \( 1/\epsilon^n \) are related, through dispersion relations, to the integrals over the imaginary part of \( h_1 \) with the weight functions proportional to the excitation energy to the power \( n-1 \). Indeed,

\[ -h_1(\epsilon, q^2) = \frac{1}{2\pi} \int d\bar{\epsilon} \frac{w_1(\bar{\epsilon}, q^2)}{\epsilon - \bar{\epsilon}} = \]

\[ \frac{1}{\epsilon} \frac{1}{2\pi} \int d\bar{\epsilon} w_1(\bar{\epsilon}, q^2) + \frac{1}{\epsilon^2} \frac{1}{2\pi} \int d\bar{\epsilon} \bar{\epsilon} w_1(\bar{\epsilon}, q^2) + \frac{1}{\epsilon^3} \frac{1}{2\pi} \int d\bar{\epsilon} \bar{\epsilon}^2 w_1(\bar{\epsilon}, q^2) + ... \]  

(3.46)

Thus, our immediate task is to built the \( 1/\epsilon \) expansion from the amplitude (3.42).

The theoretical expression for the amplitude \( h_1 \) above knows nothing, of course, about the meson masses; it contains only the quark masses. Correspondingly, it is very convenient to build first the expansion of \( h_1 \) in an auxiliary quantity,

\[ \epsilon_q = m_b - E_c - q_0, \quad E_c = m_c + \frac{q^2}{2m_c}. \]  

(3.47)

Then, if necessary, we reexpress the expansion obtained in this way in terms of \( \epsilon \),

\[ \frac{1}{\epsilon_q} = \frac{1}{\epsilon} + \frac{(\epsilon - \epsilon_q)}{\epsilon^2} + ... \]  

(3.48)

The difference between \( \epsilon_q \) and \( \epsilon \) is \( O(\Lambda_{\text{QCD}} \cdot q^2/m_{b,c}^2) \) and \( O(\Lambda_{\text{QCD}}^2/m_{b,c}) \).

After these introductory remarks, assembling all information in our disposal, we get

\[ -h_1 = \left( 1 - \frac{\bar{q}^2}{4} \right) \frac{1}{\epsilon} + \bar{\Lambda} \frac{\bar{q}^2}{2} \frac{1}{\epsilon^2} + ... \]  

(3.49)

plus terms of higher order in \( \bar{q}^2 \) or \( \Lambda_{\text{QCD}} \). In deriving Eq. (3.49) I used the fact that \( \epsilon - \epsilon_q = \bar{\Lambda} \bar{q}^2/2 \) plus terms of higher order in \( \bar{q}^2 \) or \( \Lambda_{\text{QCD}} \).

This completes the theoretical aspect of the calculation. The coefficients in front of \( 1/\epsilon \) and \( 1/\epsilon^2 \) in \( h_1 \) are known; Eq. (3.48) tells us that these coefficients are equal to
integrals over $w_1$, the spectral density. So, what remains to be done is to express the spectral density in terms of the contribution coming from the physical intermediate states. Let us assume for simplicity that the spectrum of the intermediate states is discrete. Denote the mass of the $i$-th state by $M_i$ and the energy by $E_i$; the lowest lying meson, $D^*$, corresponds to $i = 0$. Then the propagator of the $i$-th meson

$$\frac{1}{(M_B - q_0 - E_i)(M_B - q_0 + E_i)}$$

at positive $\epsilon$ has the imaginary part

$$(2E_i)^{-1}\pi\delta(\epsilon - \delta_i)$$

where $\delta_i$ is the excitation energy (including the corresponding kinetic energy),

$$\delta_i = E_i - E_{D^*}.$$ 

For the “elastic” $B \rightarrow D^*$ transition $\delta_0$ vanishes, of course.

Now it is rather obvious that the structure function $w_1$ reduces to

$$w_1(\epsilon) = \sum_{i=0}^{\infty} \frac{|f_{B\rightarrow i}|^2}{2E_i} 2\pi\delta(\epsilon - \delta_i),$$

(3.50)

where the sum runs over all possible final hadronic states, the term with $i = 0$ corresponds to the “elastic” transition $B \rightarrow D^*$ while $i = 1, 2, \ldots$ represent excited states with the energies $E_i = M_i + \vec{q}\vec{v}/(2M_i)$; furthermore, $|f_{B\rightarrow i}|^2$ looks like the square of a form factor. Strictly speaking $|f_{B\rightarrow i}|^2$ is not exactly the square of a form factor; rather this is the (appropriately normalized) contribution to the given structure function coming from the multiplet of the degenerate states which includes summation over spin states as well. By appropriate normalization I mean that we routinely insert the normalization factor $(2M_B)^{-1}$. In the particular example considered (the axial-vector current) $D$ is not produced in the elastic transition, so that in the elastic part one needs to sum only over polarizations of $D^*$. Say, at zero recoil $f_{B\rightarrow D^*} = \sqrt{2M_{D^*}}F_{B\rightarrow D^*}$ where $F_{B\rightarrow D^*}$ is the $B \rightarrow D^*$ form factor at zero recoil, see Eq. (3.58) below.

Let us examine in more detail the elastic contribution, $i = 0$. The form factor of the $B \rightarrow D^*$ transition generated by the axial-vector current is given in Eq. (1.21). Using this expression we readily obtain

$$(2E_{D^*})^{-1}|f_{B\rightarrow D^*}|^2 = \frac{M_{D^*}}{E_{D^*}} \left( \frac{1 + \vec{v}\vec{v}'}{2} \right)^2 |\xi(\vec{v}\vec{v}')|^2 \approx 1 - \rho^2 \vec{v}\vec{v}'^2.$$ 

(3.51)

Comparing the $1/\epsilon$ coefficient in the dispersion representation (3.46) with that of Eq. (3.49) we conclude that

$$\frac{1}{2\pi} \int d\epsilon w_1(\epsilon) = 1 - \rho^2 \vec{v}\vec{v}'^2 + \sum_{i=1}^{\infty} \frac{|f_{B\rightarrow i}|^2}{2E_i} = 1 - \frac{\vec{v}\vec{v}'^2}{4}.$$ 

(3.52)
which implies, in turn

\[ \rho^2 = \frac{1}{4} + \sum_{i=1}^{\infty} \frac{|f_{B\rightarrow i}|^2}{2M_i \bar{v}^2}. \]  

(3.53)

In Sect. 1.3 we learnt that at zero recoil (i.e. \( \bar{v} = 0 \)) only the elastic transition survives. As a consequence of the heavy quark symmetry for all non-elastic transitions \( |F_{B\rightarrow i}|^2 \sim \bar{v}^2 \). The ratio \( |f_{B\rightarrow i}|^2/\bar{v}^2 \) stays finite in the limit of small \( \bar{v} \). Eq. (3.53) is the Bjorken sum rule proper. Since the contribution of the excited states on the right-hand side is obviously positive it tells us, in particular, that \( \rho^2 > 1/4 \). This inequality is not very informative, though, since both, the QCD sum rule \([48, 53]\) and lattice calculations indicate that \( \rho^2 \) is only slightly less than unity, perhaps, close to 0.8.

Leaving technicalities aside let me summarize the physical meaning of the result obtained. The coefficient in front of \( 1/\epsilon^2 \) in Eq. (3.49) does not contain \( \Lambda_{\text{QCD}} \). This means that we calculate the probability of the decay \( b \rightarrow c^{ll\nu} \) with given value of \( \bar{v}^2 \) merely in the parton model; this probability is equal to that of the physical decay \( B \rightarrow X_c^{ll\nu} \); the latter is comprised of the elastic transition \( B \rightarrow D^*^{ll\nu} \) and the transition of \( B \) into excitations. (The quotation marks are used to emphasize the fact that the decays that are measured are induced by both, the axial-vector and vector, currents, while we focus now only on the transitions induced by the axial-vector current.) All probabilities of production of the excited states are proportional to \( \bar{v}^2 \) (at small \( \bar{v}^2 \)), and so is the part of the elastic transition containing \( \rho^2 \). The sum of these two contributions must coincide with the \( \bar{v}^2 \) term obtained in the parton model. The very same analysis, by the way, presents a proof of the fact that \( \xi(y = 1) = 1 \). (Do you see this?)

Now, we make the next step, proceeding to the \( 1/\epsilon^2 \) terms. The \( 1/\epsilon^2 \) term in Eq. (3.49) is proportional to \( \Lambda \), hence the result we are about to get evidently goes beyond the parton model. Combining Eq. (3.49) with the dispersion representation (3.46) we find

\[ \frac{\Lambda \bar{v}^2}{2} = \sum_{i=1}^{\infty} \frac{|f_{B\rightarrow i}|^2}{2E_i} \delta_i = \sum_{i=1}^{\infty} \frac{|f_{B\rightarrow i}|^2}{2M_i} (M_i - M_{D^*}). \]  

(3.54)

The sum runs not from zero to infinity but from 1 to infinity since \( \delta_0 = 0 \). Moreover, since all \( |F_{B\rightarrow i}|^2 \) for \( i = 1, 2, ... \) are proportional to \( \bar{v}^2 \), and we are interested only in the \( \bar{v}^2 \) term, it is legitimate to substitute, as I did, \( E_i \) by \( M_i \) and \( \delta_i \) by \( M_i - M_{D^*} \).

Eq. (3.54) is the optical (or Voloshin’s) sum rule; superficially it looks the same as in the toy model of Sect. 2.1. Please, remember this sum rule – it gives a unique opportunity to measure \( \Lambda \), one of the key parameters of the heavy quark theory. To this end one has to measure the inelastic transition probabilities in the semileptonic decays \( B \rightarrow X_c^{ll\nu} \) in the SV limit. This is a difficult measurement, but not impossible, at least in principle. Before venturing into this noble task – extraction of \( \Lambda \) from experimental data – I must warn you that acceptable accuracy can be achieved only provided that the perturbative corrections (hard gluons) as
well as nonperturbative ones, of the next order in $\Lambda_{\text{QCD}}$, are included in the sum rules. We will briefly discuss the impact of the perturbative corrections in Lecture 5.

Those who are anxious to get something practical from the optical sum rule in the absence of the necessary measurements should not be discouraged. We can still get a lower bound on $\bar{\Lambda}$. Indeed, let us rewrite Eq. (3.54) as follows:

$$\bar{\Lambda} \frac{\vec{v}^2}{2} = \sum_{i=1}^{\infty} \frac{|f_{B \rightarrow i}|^2}{2M_i} (M_1 - M_{D^*}) + \sum_{i=2}^{\infty} \frac{|f_{B \rightarrow i}|^2}{2M_i} (M_i - M_1) =$$

$$= (M_1 - M_{D^*}) \left( \rho^2 - \frac{1}{4} \right) \vec{v}^2 + \sum_{i=2}^{\infty} \frac{|f_{B \rightarrow i}|^2}{2M_i} (M_i - M_1), \quad (3.55)$$

where Eq. (3.53) is substituted. Since the second term on the right-hand side is obviously positive we conclude that

$$\bar{\Lambda} > 2(M_1 - M_{D^*}) \left( \rho^2 - \frac{1}{4} \right) \sim 500 \text{ MeV} \quad (3.56)$$

where $M_1$ is the mass of the first excited resonance with the quantum numbers of $D^*$ ($M_1 - M_{D^*} \sim 0.5 \text{ GeV}$).

Following the same line of reasoning one can derive the “third” sum rule relating $\mu_\pi^2$ to an appropriately weighted sum over excitations [54]. The corresponding inequality analogous to Eq. (3.56) takes the form

$$\mu_\pi^2 > 3(M_1 - M_{D^*})^2 \left( \rho^2 - \frac{1}{4} \right) \sim 0.45 \text{ GeV}^2. \quad (3.57)$$

### 3.7 Deviations of the $B \rightarrow D^*$ form factor from unity at zero recoil

The heavy quark theory began from the observation that the $B \rightarrow D^*$ axial-vector form factor at zero recoil is exactly unity in the limit $m_b \rightarrow \infty$, $m_c \rightarrow \infty$, $m_b/m_c$ arbitrary, see Sect. 1.3. This is purely a symmetry statement, as usual, dynamics resides in the corrections. In this section we will discuss deviations from unity.

When the heavy quark mass is infinite it is nailed at the origin, both in the initial $B$ meson and in the final $D^*$. The light cloud then does not notice the replacement of one quark by another, the overlap is unity. If we make the quark masses finite they start jiggling inside the mesons, and this motion is different in $B$ and $D^*$ since the heavy quark velocities are different. On top of this the difference in the relative spin orientations of the heavy quarks and light clouds shows up. These two effects lead to deviations from unity. At a heuristic level there is no doubt that the deviations are of order of (i) the square of the characteristic heavy quark momentum ($\vec{p}$ itself can not enter since there is no preferred orientation) or (ii) chromomagnetic correlation $\vec{\sigma} \vec{B}$. In both cases dimensional arguments prompt us that the deviation from unity
at zero recoil is proportional to \(1/m_{c,b}^2\); linear effects in \(1/m_{c,b}\) are absent. The assertion was first formulated in Ref. \[15\] and was cast in the form of a theorem (Luke’s theorem) in Ref. \[21\]. The proof presented below is abstracted from the recent work \[55\].

Let us define the \(B \to D^*\) form factor at zero recoil as follows

\[
\langle D^*|\bar{c}\gamma_\mu\gamma_5 b|B\rangle = i\sqrt{4M_B M_{D^*}} F_{B \to D^*} D^*\mu,
\]

(3.58)

to be compared with Eq. (1.18). Conceptually our present derivation is very close to that leading to the Bjorken and Voloshin sum rules (Sect. 3.6). We will again consider the transition operator induced by the axial-vector current limiting ourselves to the spatial components of the current. Technically it is simultaneously simpler and more involved. Simpler – because at the point of zero recoil one must put \(\vec{q} = 0\), so that kinematics is trivial. In particular, from the very beginning only one structure \((h_1)\) survives in the general decomposition (3.40). The calculation is more complicated on the other hand since now one has to keep track of terms of order \(\Lambda_{\text{QCD}}^3\). Those of order \(\Lambda_{\text{QCD}}\) are simply absent!

The quantity \(\epsilon\) is defined now as

\[
\epsilon = M_B - M_{D^*} - q_0 \tag{3.59}
\]

and we continue to assume that \(\Lambda_{\text{QCD}} \ll |\epsilon| \ll m_{c,b}\) and continue to examine our old acquaintance, \(h_1\), expanded in powers of \(1/\epsilon\) and \(\epsilon/m_{c,b}\). The result of a relatively simple calculation (which the reader is encouraged to do) is

\[
-h_1 = \{1 - \Delta\} \frac{1}{\epsilon} + \mathcal{O}(\Lambda_{\text{QCD}}^3) \left(\frac{1}{\epsilon^2}\right) + ...
\]

(3.60)

where

\[
\Delta = \frac{\mu_G^2}{3m_c^2} + \frac{\mu_{\pi}^2}{4} \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b}\right).
\]

An explanatory remark is in order here concerning the \(1/\epsilon^2\) term in \(h_1\). The theoretical expression for \(h_1\), as it naturally emerges from the computations, depends on \(\epsilon_q\), not on \(\epsilon\) where \(\epsilon_q\) is the energy measured from the “quark” threshold, see Eq. (3.47). In the kinematics at hand, when we are at the point of zero recoil,

\[
\epsilon - \epsilon_q = M_B - M_{D^*} - (m_b - m_c) = \epsilon - \epsilon_q = M_B - M_{D^*} - (m_b - m_c) = \epsilon - \epsilon_q = M_B - M_{D^*} - (m_b - m_c) = \epsilon - \epsilon_q = M_B - M_{D^*} - (m_b - m_c) = \epsilon - \epsilon_q = M_B - M_{D^*} - (m_b - m_c) = \]

(3.61)

where I invoked Eq. (3.22). (Can you figure out why the coefficients in this expression and in Eq. (3.61) are different? Hint: The parameters \(\mu_G^2\) and \(\mu_{\pi}^2\) in Eq. (3.61) are defined as the expectation values of the corresponding operators over the pseudoscalar meson. This is not the case in Eq. (3.22).)
We first expand $-h_1$ in $1/\epsilon_q$ and then pass to the physical variable $\epsilon$ and rearrange the expansion. In the $1/\epsilon_q$ expansion the corrections of order $O(\Lambda_{\text{QCD}})$ are absent from the very beginning – an obvious fact hardly requiring further comments – while the term $O(\Lambda_{\text{QCD}}^2/\epsilon_q^2)$ does appear explicitly. This term, however, is killed in passing from $1/\epsilon_q$ to $1/\epsilon$, see Eq. (3.61).

That is why I assert that the coefficient in front of $1/\epsilon^2$ is $O(\Lambda_{\text{QCD}}^3)$. Moreover, as we will see shortly this coefficient in Eq. (3.60) is positive. In principle, it is calculable (more exactly, expressible in terms of several new phenomenological parameters) but this will be of no concern to us in this lecture.

Repeating, step by step, the derivation of Sect. 3.6 we conclude that

$$|F_{B \to D^*}|^2 + \sum_{i=1,2,...} |F_{B \to \text{excit}}|^2 = 1 - \Delta \quad (3.62)$$

and

$$\sum_{i=1,2,...} |F_{B \to \text{excit}}|^2 (M_i - M_{D^*}) = O(\Lambda_{\text{QCD}}^3). \quad (3.63)$$

The latter sum rule, by the way, is the reason why we know that the coefficient $O(\Lambda_{\text{QCD}}^3)$ in front of $1/\epsilon^2$ is positive – the left-hand side of the sum rule is obviously positive-definite. These two relations, taken together, plus positivity of $|F_{B \to i}|^2$, imply that $|F_{B \to D^*}|^2$ is limited from below and from above,

$$1 - \Delta - \frac{O(\Lambda_{\text{QCD}}^3)}{M_1 - M_{D^*}} < |F_{B \to D^*}|^2 < 1 - \Delta \quad (3.64)$$

where $M_1$ is the mass of the first excited state produced by the axial-vector current, $M_1 - M_{D^*} = O(\Lambda_{\text{QCD}})$.

Not only is it seen that the deviation of $|F_{B \to D^*}|^2$ from unity starts from terms scaling like $1/m_{c,b}^2$, with no $1/m_{c,b}$ corrections, but we understand now the reasons lying behind this remarkable fact. Moreover, we have an idea of how large the actual deviations are since Eq. (3.64) establishes a lower limit for these deviations in terms of the parameter $\Delta$ which is determined numerically rather well. In this aspect the derivation I present here goes beyond the more conventional analysis of Ref. [43, 42]. The reader is nevertheless advised to consult the latter works to get a broader perspective of the heavy quark theory – the more approaches you master the better for you.

Qualitatively it is quite clear why the deviation of $|F_{B \to D^*}|^2$ from unity starts from $\Lambda_{\text{QCD}}^2/m_{c,b}^2$. Indeed, let us return to the Bjorken formula (3.37). In this formula it is assumed $\vec{v} \gg \Lambda_{\text{QCD}}/m_Q$ so that actually we do not distinguish between the velocity of the recoiling final heavy hadron and that of the final quark. At zero recoil the heavy hadron is nailed, but not the heavy quark. The latter experiences a primordial motion inside the nailed hadron, with the velocity $\vec{v}^2 \sim \Lambda_{\text{QCD}}^2/m_Q^2$. So, a reasonable guess would be to extrapolate Eq. (3.37) down to $\vec{v}^2 \sim \Lambda_{\text{QCD}}^2/m_Q^2$. As we see, this guess works.
It is worth emphasizing that our analysis need not be confined to the transitions induced by the spatial components of the axial-vector current. We could consider the temporal components, or vector currents, or something else. Each time we get additional information. For instance, from the transition operator induced by the vector currents we get a sum rule proving the inequality $\mu_\pi^2 > \mu_G^2$ obtained in Sect. 3.2 from a quantum-mechanical argument.
4 Lecture 4. Theory of the Line Shape

In this lecture I will discuss one of the most interesting and practically important applications of the heavy quark theory, the spectra in the end point domain in the inclusive decays. Inclusive weak decays of heavy flavors, in particular, semileptonic decays, are close relatives of famous deep inelastic scattering – the processes where a highly virtual photon scatters off nucleons to produce an inclusive multiparticle state. The latter are related to the former via channel crossing. Deep inelastic lepton-nucleon scattering was in the focus of theoretical activity in the late sixties and the beginning of seventies and was instrumental in discovering and developing QCD [56]. It is thus quite surprising that for a long time there were hardly any attempts to treat the beauty decays in QCD proper along essentially the same lines as it was done in deep inelastic scattering. Realization of the idea that the $1/m_Q$ expansion in the theory of the line shape can play the same role as the twist $1/Q^2$ expansion in DIS came with the 20 years delay [57, 58, 59] – I see absolutely no reasons why the corresponding theory was worked out only recently and not 20 years ago.

The theory of the line shape in QCD resembles that of the Mössbauer effect. To explain what I mean it is convenient to consider, for definiteness, the transition $B \to X_s \gamma$ where $X_s$ denotes the inclusive hadronic state with the $s$ quark. This decay has been recently observed experimentally. (Description of $B \to X_q l\nu$ is conceptually similar but is more technically involved).

Again, to avoid inessential technicalities I will neglect the quark and photon spins. So we will consider the transition $Q \to q \phi$ where all fields $Q$, $q$ and $\phi$ are spinless. Thus, to begin with, we will limit ourselves to the toy model described in Sect. 2.1, see Eq. (2.9). The mass of the final quark $m_q$ will be treated as a free parameter which can vary from zero almost up to $m_Q$. For our approach to be valid we still need that $\Delta m \equiv m_Q - m_q \gg \Lambda_{QCD}$ although the mass difference may be small compared to the quark masses.

To warm up we will put the final quark mass to zero. At the level of the free quark decay the photon energy is then fixed by the two-body kinematics of the decay $Q \to q \phi$, namely, $E_\phi = m_Q/2$. In other words, in the rest frame of the decaying $Q$ quark the photon energy spectrum is a monochromatic line at $E_\phi = m_Q/2$ (Fig. 7). On the other hand, in the actual hadronic decays $H_Q \to X_q \phi$ the kinematical boundary of the spectrum lies at $M_{H_Q}/2$. Moreover, due to multiparticle final states (which are, of course, present at the level of the hadronic decays) the “photon” line will be smeared. In particular, the window – a gap between $m_Q/2$ and $M_{H_Q}/2$ – will be closed (Fig. 7). There are two mechanisms smearing the monochromatic line of the free-quark decay. The first is purely perturbative: the final quark $q$ can shake off a hard gluon, thus leading to the three-body kinematics. This mechanism tends to diminish the photon energy and may be important at $E < m_Q/2$. We will defer its discussion till later times. The second mechanism is due to the “primordial” motion
of the heavy quark $Q$ inside $H_Q$ and is non-perturbative. Even if the decaying $B$ is nailed at the origin so that its velocity vanishes, the $b$ quark moves inside the light cloud, its momentum being of order $\Lambda_{\text{QCD}}$. This is the QCD analog of the Fermi motion of the nucleons in the nuclei. It is quite clear that this motion affects the decay spectra. Say, if the “primordial” heavy quark momentum is parallel to that of the photon, the photon produced gets more energy, and vice versa, for the antiparallel momenta it gets less. It is quite clear that this effect is preasymptotic (suppressed by inverse powers of $m_b$): while typical energies of the decay products are of order $m_b$ a shift due to the heavy quark motion is of order $\Lambda_{\text{QCD}}$.

Only the second mechanism will be of interest for us in this lecture. The window (i.e. the domain kinematically inaccessible for free quarks) plus the adjacent domain below the window, of width several units $\times \Lambda_{\text{QCD}}$, taken together, form what is called the end point domain. Below I will outline the main elements of the theory allowing one to translate an intuitive picture of the $Q$ quark primordial motion inside $H_Q$ in QCD-based predictions for the spectrum in the end point domain\(^9\). The spectrum below the end point domain is the realm of the perturbative physics (hard gluon emission).

### 4.1 Formalism

Let us return back to Sect. 2.1 and consider the transition operator defined there. Since we are interested in the energy spectrum the “photon” momentum $q$ must be fixed. Let us assemble Eqs. (2.14) and (2.23) together. For convenience I will reproduce the result here again, taking into account the fact that now $m_q$ is assumed to vanish,

$$
\frac{d\Gamma}{dE} = \frac{m_Q}{M_{H_Q}} \Gamma_0 \left\{ \langle \bar{Q}Q \rangle \delta \left( E - \frac{m_Q}{2} \right) - \frac{(\vec{\pi}^2)}{4m_Q} \delta' \left( E - \frac{m_Q}{2} \right) + \frac{(\vec{\pi}^2)}{24} \delta'' \left( E - \frac{m_Q}{2} \right) + \ldots \right\}
$$

If in the leading approximation the spectrum is just a delta function, the corrections are more and more singular! The higher the correction the stronger the singularity. Nonsense? No, this was to be expected: the width of the $\phi$ line in the transition $H_Q \to X_q\phi$ is of order $\Lambda_{\text{QCD}}$. We expand in the powers of $\Lambda_{\text{QCD}}/m_Q$; hence we must expect the enhancement of the singularities in each successive order. Equation (4.1) gives all terms up to $\Lambda_{\text{QCD}}^2$. It is clear that to describe the shape of the line one needs to sum up the infinite number of terms in this expansion.

Then in the approximation of Fig. 1 (no hard gluon exchanges) the transition operator is given by Eq. (2.15) with $m_q$ set equal to zero. To construct the operator product expansion to all orders we observe that the momentum operator $\pi$ corresponding to the residual motion of the heavy quark is $\sim \Lambda_{\text{QCD}}$ and the expansion in $\pi/k$ is possible. Unlike the problem of the total widths, however, in the end

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\(^9\)Basics of the theory of the line shape were worked out in Refs. [57, 58, 59]. Further crucial steps were undertaken in Refs. [60, 61, 62, 63]. In my presentation I follow mainly Bigi et al. [59].
point domain $k^2$ is anomalously small, the expansion parameter is of order unity, and there exists an infinite set of terms where all terms are of the same order of magnitude.

To elucidate this statement let us examine different terms in the denominator of the propagator,

$$k^2 + 2\pi k + \pi^2.$$ 

In the end point domain

$$|E - (1/2)m_Q| \sim \bar{\Lambda}$$ (4.2)

It is quite trivial to find that in this domain

$$k_0 \sim |\vec{k}| \sim m_Q/2, \quad k^2 \sim m_Q\bar{\Lambda};$$

in particular, at the kinematical boundary (for the maximal value of the “photon” energy) $k_0 = M_Q/2$ and $k^2 = -m_Q\bar{\Lambda}$. Hence, in the end point domain

$$k^2 \sim k\pi \gg \pi^2.$$ 

In other words, when one expands the propagator of the final quark in the transition operator,

$$\bar{Q}_{k^2 + 2\pi k + \pi^2} Q,$$ (4.3)

in $\pi$, in the leading approximation all terms $(2k\pi/k^2)^n$ must be taken into account while terms containing $\pi^2$ can be omitted. The first subleading correction would contain one $\pi^2$ and arbitrary number of $2k\pi$’s, etc.

Thus, in this problem it is twist of the operators (≡ dimension - Lorentz spin) in the operator product expansion, not their dimension, that counts. For connoisseurs I will add that this aspect makes the theory of the line shape in the end point domain akin to that of deep inelastic scattering (DIS). Keeping only those terms in the expansion that do not vanish in the limit $m_Q \to \infty$ (analogs of the twist-2 operators in DIS) we get the following series for the transition operator

$$\hat{T} = -\frac{1}{k^2} \sum_{n=0}^{\infty} \left( -\frac{2}{k^2} \right)^n k^{\mu_1} \ldots k^{\mu_n} (\bar{Q}_{\pi_{\mu_1} \ldots \pi_{\mu_n}} Q - \text{traces}).$$ (4.4)

Traces are subtracted by hand since they are irrelevant anyway; their contribution is suppressed as $k^2\pi^2/(k\pi)^2 \sim \Lambda_{QCD}/m_Q$ to a positive power. Another way to make the same statement is to say that in Eq. (4.4) the four-vector $k$ can be considered as light-like, $k^2 = 0$.

### 4.2 The light cone distribution function

After the transition operator is built the next step is averaging of $\hat{T}$ over the hadronic state $H_Q$. Using only the general arguments of the Lorentz covariance one can write

$$\langle H_Q | \bar{Q}_{\pi_{\mu_1} \ldots \pi_{\mu_n}} Q - \text{traces} | H_Q \rangle = a_n \bar{\Lambda}^n (v_{\mu_1} \ldots v_{\mu_n} - \text{traces})$$ (4.5)
where $a_n$ are constants parametrizing the matrix elements. Their physical meaning will become clear momentarily. Right now it is worth noting that the term with $n = 1$ drops out ($a_1 = 0$). Indeed, $\langle H_Q|\bar{Q}\pi Q|H_Q\rangle$ is obviously zero for spinless $H_Q$ while $\pi_0$ through the equation of motion reduces to $\bar{\pi}^2/(2m_Q)$ and is of the next order in $1/m_Q$. Disappearance of $\bar{Q}\pi_\mu Q$ means that there is a gap in dimensions of the relevant operators.

Let us write $a_n$'s as moments of some function $F(x)$,

$$a_n = \int dx x^n F(x). \quad (4.6)$$

Then, $F(x)$ is nothing else than the primordial line-shape function! (That is to say, $F(x)$ determines the shape of the line before it is deformed by hard gluon radiation; this latter deformation is controllable by perturbative QCD). The variable $x$ is related to the photon energy,

$$x = \frac{2}{\Lambda} \left( E - \frac{m_Q}{2} \right).$$

If this interpretation is accepted – and I will prove that it is correct – it immediately implies that (i) $F(x) > 0$, (ii) the upper limit of integration in Eq. (4.6) is 1, (iii) $F(x)$ exponentially falls off at negative values of $x$ so that practically the integration domain in Eq. (4.6) is limited from below at $-x_0$ where $x_0$ is a positive number of order unity.

To see that the above statement is indeed valid we substitute Eqs. (4.5), (4.6) in $\hat{T}$,

$$\langle H_Q|\hat{T}|H_Q\rangle = -\frac{1}{k^2} \sum_n \int dy F(y) y^n \left( -\frac{2\bar{\Lambda}kv}{k^2 + i\delta} \right)^n, \quad (4.7)$$

and sum up the series. The $i\delta$ regularization will prompt us how to take the imaginary part at the very end. In this way we arrive at

$$\frac{d\Gamma}{dE} = -\frac{4}{\pi} \frac{\Gamma_0 m_Q E}{M_{H_Q}} \text{Im} \int dy F(y) \frac{1}{k^2 + 2y\Lambda kv + i\delta} = (2/\Lambda)\Gamma_0 F(x), \quad (4.8)$$

where the variable $x = k^2/(2\bar{\Lambda}kv)$ was written out above in terms of the “photon” energy, and $\Gamma_0$ is the total decay width in the parton approximation. Corrections to Eq. (4.8) are of order $\bar{\Lambda}/m_Q$.

Thus, we succeeded in getting the desired smearing: the monochromatic line of the parton approximation is replaced by a finite size line whose width is of order $\bar{\Lambda}$. The pre-asymptotic effect we deal with is linear in $\Lambda_{\text{QCD}}/m_Q$.

At this point you might ask me how this could possibly happen. We have already learnt that there is a gap in dimensions of the operators in the expansion – no operators of dimension 4 exist – and the correction to $QQ$ is also quadratic in $1/m_Q$ (the CGG/BUV theorem). There are no miracles – the occurrence of the effect linear in $\Lambda_{\text{QCD}}/m_Q$ became possible due to the summation of the infinite series in Eq. (4.4); no individual term in this series gives rise to $\Lambda_{\text{QCD}}/m_Q$. 

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To avoid misunderstanding it is worth explicitly stating that the primordial distribution function $F(x)$ is *not* calculated; rather $F(x)$ is related to the light-cone distribution function of the heavy quark inside $H_Q$, namely $\langle \bar{Q}(n\pi)^a Q \rangle$, $n^2 = 0$, or more explicitly

$$F(x) \propto \int dt e^{ixt} \langle H_Q | \bar{Q}(x = 0) e^{i\int_0^t nA(n\tau) d\tau} Q(x_\mu = n_\mu t) | H_Q \rangle,$$

where $n$ is a light-like vector\(^\text{10}\)

$$n_\mu = (1, 0, 0, 1).$$

Unfortunately, this primordial function is not the one that will be eventually measured from $d\Gamma/dE$; the actual measured line shape will be essentially deformed by radiation of hard gluons. I will say a few words about this in Lecture 5.

The primordial distribution function $F(x)$ which we defined here can be called the *light-cone* distribution function. This is clear from the expression (4.9) which has a very transparent physical meaning. The quark $q$ produced is massless and, therefore, propagates along the light cone from the point of emission to the point of absorption in the transition operator defining the distribution function.

If we looked at the physical line shape sketched on Fig. 7 more attentively, through a microscope, we would notice that a smooth curve is obtained as a result of adding up many channels, specific decay modes. A typical interval in $E$ that contains already enough channels to yield a smooth curve after summation is $\sim \Lambda^2_{\text{QCD}}/m_Q$. Roughly one can say that the spectrum of Fig. 7 covers altogether $m_Q/\Lambda_{\text{QCD}}$ resonance states produced in the $H_Q$ decays and composed of $q$ plus the spectator (I keep in mind here that the final hadronic state is produced through decays of highly excited resonances, as in the multicolor QCD). These states span the window between $m_Q/2$ and $M_{H_Q}$ and the adjacent domain to the left of the maximum at $E = m_Q/2$.

### 4.3 Varying the mass of the final quark

So far I was discussing the transition into a massless final quark. It is very interesting to trace what happens with the line shape and the primordial distribution function as the final quark mass $m_q$ increases.

Inspection of the transition operator shows that as long as $m_q^2 \ll m_Q\Lambda$ nothing changes in our formulae at all in the leading-twist approximation. Since the characteristic values of $k^2$ in the end point domain are of order $m_Q\Lambda$ and $m_q^2 \ll m_Q\Lambda$ one can merely neglect the final quark mass altogether.

A more interesting regime is $m_q \sim (m_Q\Lambda)^{1/2}$. In this regime one can not neglect $m_q^2$ in the denominator. It is not difficult to see, however [62], that, as in the massless

\(^{10}\) Similar light-cone distributions for light quarks are well known [64] in the theory of deep inelastic scattering, see also [65].
case, all traces can be neglected since \( k^2 \pi^2 / (k \pi)^2 \sim \Lambda_{\text{QCD}}/m_Q \). This means that the same light-cone distribution function \( F(x) \) that emerged in the massless case describes the line shape if \( m_q \sim m_Q \Lambda \) as well. The only change that occurs is a shift of the end point spectrum, as a whole, to the left. Indeed, if previously the variable \( x \) was defined as \((2/\Lambda)[E - (1/2)m_Q]\), now when \( m_q \neq 0 \)

\[
x = \frac{2}{\Lambda} (E - E_0)
\]

where \( E_0 = (2m_Q)^{-1}(m_Q^2 - m_q^2) \). The maximum of the distribution, in particular, shifts from \( m_Q/2 \) to \( E_0 = m_q/2 - \mathcal{O}(\Lambda) \).

What happens if one continues to increase \( m_q \)? Increasing the quark mass further results in more drastic changes. The trace terms can not be omitted any more, and the light-cone function gives place to other distribution functions. This is obvious already from a simple kinematical argument. Indeed, with \( m_q \) increasing the window shrinks. When we eventually come to the SV limit

\[
\Lambda_{\text{QCD}} \ll \Delta m = m_Q - m_q \ll m_{Q,q}
\]

it shrinks to zero. In this limit the photon energy in the two-body quark decay, \( \Delta m(1 + \Delta m(2m_q)^{-1}) \) differs from the maximal photon energy in the hadronic decay, \( \Delta M(1 + \Delta M(2M_q)^{-1}) \), only by a tiny amount inversely proportional to \( m_Q \) (\( \Delta m \) and \( \Delta M \) stand for the quark and meson mass differences, respectively).

Thus, the kinematical consideration prompts us that the line shape must essentially change. Anticipating the results of the calculation let me describe the situation pictorially. Simultaneously with the shrinkage of the window the peak becomes more asymmetric and develops a two-component structure (Fig. 8). The dominant component of the peak, on its right-hand side, becomes narrower and eventually collapses into a delta function when \( m_q \) becomes a finite fraction of \( m_Q \). A shoulder develops on the left-hand side; the number of the hadronic states populating the end point domain becomes smaller – instead of \( m_Q/\Lambda_{\text{QCD}} \) states at \( m_q = 0 \) we are speaking of just several states at \( m_q = \) a finite fraction of \( m_Q \). When we approach the SV limit the height of the shoulder corresponding to the production of the excited states becomes very small, proportional to \( \vec{v}^2 \ll 1 \) (Fig. 9). This is the end of the evolution – starting from the light-cone distribution function at \( m_q = 0 \) we continuously pass to the temporal distribution function in the SV limit. It is the temporal distribution function that shapes the inelastic shoulder on Fig. 9.

This rather sophisticated picture, hardly reproducible in naive quark models, emerges from the operator product expansion (in the leading approximation) if one follows along the same lines as previously. The transition operator \( \hat{T} \) for \( m_q \neq 0 \) is given in Eq. (2.20); I reproduce it here again for convenience,

\[
\hat{T} = \frac{1}{m_q^2 - k^2} \sum_{n=0}^{\infty} \hat{Q} \left( \frac{2m_Q \pi_0 + \pi^2 - 2q \pi}{m_q^2 - k^2} \right)^n Q, \quad \text{(4.10)}
\]
Notice that $2m_Q \pi_0 + \pi^2$ acting on $Q$ yields zero (the equation of motion) and in the SV limit $q$ must be treated as a small parameter,

$$q_0/m_Q \equiv E/m_Q = v \ll 1;$$

$v$ is the spatial velocity of the heavy quark produced. Although $v$ is small the inclusive description is still valid provided that $\Delta m \gg \Lambda_{QCD}$.

In the zeroth order in $q$ the only term surviving in the sum (4.10) is that with $n = 0$, and we are left with the single pole, the elastic contribution depicted on Fig. 9. This is the extreme realization of the quark-hadron duality. The inclusive width is fully saturated by a single elastic peak. We have already discussed this phenomenon in Lecture 2. What might seem to be a miracle at first sight has a symmetry explanation – the phenomenon is explained by the heavy quark symmetry. The fact that the parton-model monochromatic line is a survivor of hadronization is akin to the Mössbauer effect.

If terms $O(v^2)$ are switched on the transition operator acquires an additional part,

$$\hat{T}_{v^2} = \frac{4}{3} \hat{q}^2 \frac{1}{(m^2 - k^2)^3} \sum_{n=0}^{\infty} \left( \frac{2m_Q}{m^2 - k^2} \right)^n \bar{Q} \pi_i \pi_i^0 \pi_i Q. \quad (4.11)$$

From this expression it is obvious that the shape of the $v^2$ shoulder is given by the temporal distribution function $G(x)$ whose moments are introduced through the matrix elements

$$\langle H_Q|\bar{Q} \pi_i \pi_i^0 \pi_i Q|H_Q \rangle = \bar{\Lambda}^{n+2} \int dx x^n G(x). \quad (4.12)$$

Alternatively, $G(x)$ can be written as a Wilson line along the time direction,

$$G(x) \propto \int dt e^{ixt\bar{\Lambda}} \langle H_Q|\bar{Q}(t = 0, \vec{x} = 0) \pi_i e^{-i \int_0^t A_0(\tau) d\tau} \pi_i Q(t, \vec{x} = 0)|H_Q \rangle. \quad (4.13)$$

Intuitively it is quite clear why the light-cone distribution function gives place to the temporal one in the SV limit. Indeed, if the massless final quark propagates along the light-cone, for $\Delta m \ll m$ the quark $q$ is at rest in the rest frame of $Q$, i.e. propagates only in time.

In terms of $G(x)$ our prediction for the line shape following from Eq. (4.11) takes the form

$$\frac{d\Gamma}{dE} \propto \left[ 1 - \frac{v^2}{3} \int \left( \frac{1}{y^2 + \bar{\Lambda}/E_{max}} \right) G(y) dy \right] \delta(x) + \frac{v^2}{3} \left( \frac{1}{x^2 + \bar{\Lambda}/E_{max}} \right) G(x), \quad (4.14)$$

where $x = (E - E_{max})/\bar{\Lambda}$. The $v^2$ corrections affect both, the elastic peak (they reduce the height of the peak) and the shoulder (they create the shoulder). The total decay rate stays intact, however: the suppression of the elastic peak is compensated by the integral over the inelastic contributions in the shoulder. This is the Bjorken
sum rule thoroughly considered in Lecture 3. It is important that we do not have to
guess or make ad hoc assumptions – a situation typical for model-building – QCD
itself tells us what distribution function enters in this or that case and in what
particular way.

4.4 Real QCD: Inclusive semileptonic decays

From the analysis presented above the following remarkable fact should be clear.
The very same primordial distribution functions that determine the line shape in
the radiative transitions appear in the problem of the spectra in the semileptonic
decays. In particular, in $b \to u \ell \nu$ we deal with $F(x)$.

Of course, kinematical conditions are different. Now the hadronic part of the
process, $B \to X_u \ell \nu$ inclusive decay, depends on two variables, for instance, $q_0$ and
$q^2$, or $q_0$ and $|\vec{q}|$. The probability of the decay, in the free quark approximation, is
proportional to $\delta(m_Q - q_0 - |\vec{q}|)$ [66]. In other words, in this approximation only a
line on the $q_0, |\vec{q}|$ plane is populated (Fig. 10). (I assume that we are not interested
in the individual momenta of $\ell$ and $\nu$ and measure just the total momentum of the
lepton pair. This is quite a fantastic formulation of the problem since experimentally
the neutrino energy and momentum are not measured, of course; only the electron
energy is usually measured. Nevermind, let us keep in mind a gedanken experiment.)
The end point domain is defined now as a band whose width is several units $\times \Lambda_{QCD}$
adjacent to the above quark line (Fig. 10). Needless to say that in the physical
decay the whole large triangle is populated; the inner part of the triangle, to the
left of the end point band, is due to the hard gluon emission. The smearing of the
delta-like spectrum in the band is due to the primordial motion of $b$ inside $B$, and
is described by the light cone distribution.

A trivial modification compared to Sect. 4.3 is the occurrence of several structure
functions. All five structure functions are expressible, however, in terms of the
same light cone primordial distribution function $F(x)$ where, as previously, $x =
-\Lambda^{-1}k^2/2k_0$. Since $q_0$ and $\vec{q}$ are independent variables in the case at hand

$$x = -\Lambda^{-1}k^2/2k_0 = -\Lambda^{-1}k^2/(k_0 + |\vec{k}|) = -\Lambda^{-1}(m_Q - q_0 - |\vec{q}|) \quad \text{(4.15)}$$

where in the denominator the difference between $k_0$ and $|\vec{k}|$ is neglected which is
perfectly legitimate in the end point band. In this band $k_0 - |\vec{k}| = \mathcal{O}(\Lambda_{QCD})$, and the
difference between $k_0$ and $|\vec{k}|$ becomes important only at the level of the subleading
twists which are not included anyway.

Thus, we observe a scaling behavior: the structure functions that generally speaking
could depend on two variables, $q_0$ and $|\vec{q}|$, actually depend only on the single
light-cone combination (4.15). This is the analog of the Bjorken scaling in deep
inelastic scattering! In the rest of the phase space, outside the end point band, the
approximate equality $k_0 \approx |\vec{k}|$ is not valid, of course, and the above scaling is not
going to take place. The primordial distribution falls off – presumably exponentially
outside the end point band. The hard gluon emissions will populate the phase space outside this domain creating long logarithmic tails. The primordial part is buried under these tails. Therefore, outside the band one can not expect that the structure functions depend on the single combination $q_0 + |\vec{q}|$ anyway.

Guesses about a scaling behavior in the inclusive semileptonic decays are known in the literature \cite{66}. Now we are finally able to say for sure what sort of scaling takes place, where it is expected to hold and where and how it will be violated.

I will not go into further details which are certainly important if one addresses the problem of extraction of $V_{ub}$ from experimental data. Some of them are discussed in the literature, others still have to be worked out. Applications of the theory to data analysis is a separate topic going beyond the scope of this lecture.

What can be said about the light cone distribution function $F(x)$? This function depends on the structure of the light cloud of the $B$ meson and, thus, belongs to the realm of the soft physics. The moments of this function are related to the expectation values of the operators $\bar{Q} \pi_{\mu_1} ... \pi_{\mu_n} Q$ (see Eq. (4.5)); in real QCD the properly normalized matrix elements on the left-hand side include the factor $(2M_B)^{-1})$. The knowledge of the infinite set of these expectation values would be equivalent to the knowledge of the structure of the light cloud. Needless to say that this is beyond our abilities at present. Still, we know a few first moments of $F(x)$ and have a general idea of the shape of this function. It must be positive everywhere in the physical domain, vanish at $x = 1$ and have exponential fall-off at large negative $x$. The latter property ensures the existence of all moments. Moreover,

$$a_0 = \int dx F(x) = 1,$$

$$a_1 = \int dx x F(x) = 0,$$

and

$$a_2 = \int dx x^2 F(x) = \frac{\mu^2}{3\Lambda^2}.$$

Estimates of the third moment also exist in the literature \cite{59, 42}. I can not dwell on this issue now and will only mention that $a_3$ is constrained by exact inequalities, i.e. \cite{53}

$$a_2 < \frac{1}{4} + \sqrt{\frac{1}{4} - a_3}, \quad a_3 < \frac{1}{4} - \left(a_2 - \frac{1}{2}\right)^2. \quad (4.16)$$

To derive these inequalities one merely observes that for any $t$ the integral from $-\infty$ to 1 over $x$ over the function $(1 - x)(x - t)^2F(x)$ is positive; on the other hand, this integral is a second order polynomial in $t$ and, hence, its discriminant must be negative.

A sketch of a function satisfying all these requirements is given on Fig. 11.

A natural desire to extend the formalism described above to the semileptonic inclusive transitions $b \rightarrow c l \nu$ encounters serious technical difficulties. The essence of the problem is as follows. The final quark $c$ can be treated as heavy, although
at the same time, \( m_c^2 \ll m_b^2 \). The ratio \( m_c^2/m_b^2 \approx 0.07 \) is a small parameter while \( m_c^2/(\Lambda m_b) \sim 1 \). Under the circumstances the type of the distribution function describing the primordial motion of \( b \) inside \( B \) and determining the measurable structure functions \( w_i \) to \( w_5 \) will depend on the value of \( |\vec{q}| \), and the scaling property – dependence on one particular combination of variables – is lost.

The \( q_0, |\vec{q}| \) plane is shown on Fig. 12. In the free quark approximation the transition probability is proportional to \( \delta(m_b - q_0 - E_c) \) where \( E_q = (m_c^2 + \vec{q}^2)^{1/2} \), and all events are concentrated along the line indicated on Fig. 12. At the hadronic level the phase space consists of the full triangle, with one side curved. The end point band is also curved.

The fact that one side of the triangle is distorted compared to \( b \rightarrow ul\nu \) is not crucial. What is important is the change of dynamics as we move from the upper left corner to the lower right one. In the case of \( b \rightarrow ul\nu \) moving along the end point band in this direction does not affect the measured structure functions (apart from the extreme domain of soft \( u \) – the exclusive resonance domain – where our description fails altogether). The situation is different in the \( b \rightarrow cl\nu \) transition.

If \( |\vec{q}|^2 \gg m_c^2 \) one recovers \[62\] the same light-cone function \( F(x) \) as in the transition \( b \rightarrow ul\nu \) or \( b \rightarrow s\gamma \). Modifications are marginal. First, some extra terms explicitly proportional to \( m_c/m_b \) are generated in the structure functions due to the fact that \( \mathcal{P} + m_c \) replaces \( \mathcal{P} \) in the numerator of the quark Green function. Moreover, if in the \( b \rightarrow ul\nu \) transitions the scaling variable in the end-point domain is

\[
x = \bar{\Lambda}^{-1}(q_0 + |\vec{q}| - m_b),
\]

in the \( b \rightarrow cl\nu \) transition it is shifted by a constant term of order 1,

\[
x = \bar{\Lambda}^{-1}(q_0 + |\vec{q}| - m_b) + \frac{m_c^2}{\bar{\Lambda}m_b}.
\]

To see how this shift occurs \[62\] and to reveal limitations of the approximation let us start from the parton model variable \( m_b - q_0 - E_c \). In the limit \( |\vec{q}|^2/m_c^2 \rightarrow \infty \) inside the end point band formally one may substitute \( E_c \) by

\[
E_c \rightarrow |\vec{q}| + \frac{m_c^2}{2|\vec{q}|} \rightarrow |\vec{q}| + \frac{m_c^2}{m_b} \left(1 + \mathcal{O}(m_c^2/m_b^2)\right).
\]

If \( m_c^2 = \mathcal{O}((\bar{\Lambda}m_b) \) formally one may discard the \( \mathcal{O}(m_c^2/m_b^2) \) correction. In this way we arrive at the scaling variable (4.18).

It is worth emphasizing that the occurrence of the light-cone distribution function in this regime, the same as in the \( b \rightarrow u \) transition, is a remarkable fact. Indeed, if we could examine the measured structure functions “in the microscope” in these two cases we would see that their microstructure is quite different. As was already mentioned, in the \( b \rightarrow u \) transition the end point band is saturated by the production of \( \sim m_b/\sqrt{\bar{\Lambda}} \) states, with the spacing between the individual states of order \( \bar{\Lambda}^2/m_b \). In the \( b \rightarrow c \) transition, even if we are in the upper left corner of the phase space
where the light-cone distribution is relevant, the number of the states produced is \( \sim m_b/m_c \sim m_c/\bar{\Lambda} \) and the spacing is of order \( \bar{\Lambda}^2/m_c \). We deal with a much coarser structure in the latter case, and still all resonance contributions being summed up must add up to produce the light-cone distribution, formally the same one that is created by a much larger number of resonances in the \( b \to u \) transition. (Purely theoretically we can not predict fine grain versus coarse grain composition of the structure functions if we limit ourselves to the leading twist. Only analysis of all twists could resolve these details, would this analysis be possible).

The word “formally” is used above three times, not accidentally. Practically in the \( b \to cl\nu \) transition \( |\vec{q}| \) can never be much larger than \( m_c \). Indeed, the maximal value of \( |\vec{q}| \), corresponding to \( q^2 = 0 \) is \( (m_b^2 - m_c^2)/2m_b \sim 2 \) GeV, that is only \( \sim 1.5m_c \). Therefore, only by stretching a point and only in a narrow domain near \( q^2 = 0 \), one can expect that the light cone function of the variable (4.18) is, perhaps, more or less relevant.

As \( |\vec{q}| \) decreases and becomes less than \( m_c \) (this regime takes place in a large part of the phase space) the light-cone distribution function becomes irrelevant. The measurable structure functions are determined by a different distribution – the light-like vector \( n_\mu \) in Eq. (4.9) is replaced by \( w_\mu = \left(1, \frac{\vec{q}}{E_c}\right) \), and it becomes clear that the would be scaling variable \( x = \bar{\Lambda}^{-1}(q_0 + E_c - m_b) \) fails to represent all dependence of the structure functions on \( q_0 \) and \( |\vec{q}| \). When \( q^2 \) approaches its maximal value,

\[
q^2_{\text{max}} = (M_B - M_D)^2,
\]

\( |\vec{q}| \) tends to zero and we eventually approach the SV regime which I have already discussed, with fascination, in the toy example above. In the SV limit the velocity of \( H_c \) produced is small, and the structure functions probe the primordial motion described by the temporal distribution function \( G(x) \) where now

\[
x \approx \bar{\Lambda}^{-1}(q_0 - \Delta m),
\]

\( \Delta m \) is the quark mass difference coinciding, to the leading order with \( M_B - M_D \).

Thus, changing \( q^2 \) from zero to \( q^2_{\text{max}} \) results in an evolution of the distribution function appearing in theoretical formulae for \( d\Gamma(B \to X_c l\nu) \), from light-cone to temporal, through a series of intermediate distributions. The physical reason for this evolution is quite clear – what distribution function is actually measured depends on the parton-model velocity of the quark produced in the \( b \) decay. In the limiting cases of very large recoil and very small recoil the problem is solved in the sense that the structure functions are expressed in terms of the light-cone and temporal distribution functions, respectively. The intermediate case \( |\vec{q}| \sim m_c \) is not worked out in detail so far. It is beyond any doubt, however, that the parton-model type scaling will not take place.
5 Lecture 5. Including Hard Gluons. Generalities of the Operator Product Expansion.

Finally the time comes when I can not ignore any more the existence of hard gluons. Hard gluons are mere nuisance from the point of view of the theory of hadrons since they play no, or very little, role in the structure of the low-lying hadronic states. Yet, if we want to go beyond purely academic exercises, however beautiful they might look, and descend down into a messy world of real hadronic physics, hard gluons can not be forgotten about since they “contaminate” nearly every experimentally measurable quantity. To make contact with the real world we have to consider interplay between the soft and hard physics.

The hard gluons manifest themselves in many ways. They contribute to the coefficient functions in the effective Lagrangian (1.3) obtained by integrating out all degrees of freedom with the characteristic frequencies down to \( \mu \). They show up in the calculations of the total decay rates and spectra discussed in Lectures 3 and 4 resulting in perturbative corrections which, in some instances, change the answer quite drastically. They result in the fact that all basic parameters of the heavy quark physics – the heavy quark mass, \( \Lambda \), \( \mu^2 \pi \) and so on – generally speaking, become \( \mu \) dependent and can not be treated as universal constants. Here we will address some of these issues in brief.

5.1 Calculation of the effective Lagrangian

I have already started discussing this topic in Sect. 1.2. The original QCD Lagrangian (1.1) is formulated at very short distances. In principle, it codes all information necessary for calculation of all observable amplitudes. We just do the functional integral and ... Alas, there are very few functional integrals that can be calculated analytically; numerical evaluation on lattices may take years, and I even dare to assert that some amplitudes will never be calculated that way. So, we take the original Lagrangian and start evolving it down, integrating out all fluctuations with the frequencies \( \mu < \omega < M_0 \) where \( M_0 \) is the original normalization point, and \( \mu \) will be treated, for the time being, as a current parameter. In this way we get the Lagrangian which has the form

\[
\mathcal{L} = \sum_n C_n(M_0; \mu) O_n(\mu) .
\]  

The coefficient functions \( C_n \) represent the contribution of virtual momenta from \( \mu \) to \( M_0 \). The operators \( O_n \) enjoy full rights of the Heisenberg operators with respect to all field fluctuations with frequencies less than \( \mu \). The sum in Eq. (5.1) is infinite – it runs over all possible Lorentz singlet gauge invariant operators with the appropriate quantum numbers; for instance, if \( CP \) is conserved, only \( CP \)-even operators will appear in (5.1). If, say, the electromagnetic processes are included, the operators in the Lagrangian (5.1) may contain the photon and electron fields, and so on. All
operators can be ordered according to their dimension; moreover, we can use the equations of motion stemming from the original QCD Lagrangian to get rid of some of the operators in the sum. Those operators that are reducible to full derivatives give vanishing contributions to the physical (on mass shell) matrix elements and can thus be discarded as well.

If one just abstractly writes the expression (5.1) one is free to take any value of $\mu$; in particular, $\mu = 0$ would mean that everything is calculated and we have the full $S$ matrix, all conceivable amplitudes, at our disposal. Nothing is left to be done. In this case Eq. (5.1) is just a sum of all possible amplitudes. This sum then must be written in terms of the physical hadronic states, of course, not in terms of the quark and gluon operators since the latter degrees of freedom are simply non-existent at large distances.

This is day-dreaming, of course. Needless to say that in our explicit calculation of the coefficient functions we have to stop somewhere, at such virtualities that the quark and gluon degrees of freedom are still relevant, and the coefficient functions $C_n(M_0, \mu)$ are still explicitly calculable. On the other hand, for obvious reasons it is highly desirable to have $\mu$ as low as possible. In the heavy quark theory there is an additional requirement that $\mu$ must be much less than $m_Q$. The process of calculating the coefficients $C_n(M_0, \mu)$ is called matching in the more standard presentation of HQET. Actually we see that this procedure is nothing else than a generalization of Wilson’s idea of the renormalization group and the (Wilsonian) operator product expansion. Using the standard OPE language has an evident advantage: all well-studied elements of the latter approach can be immediately adapted in the environment of the heavy quark expansions. In particular all parameters one can read off from the Lagrangian (5.1) depend on $\mu$ (including, say, the heavy quark mass). Let us assume that $\mu$ is large enough so that $\alpha_s(\mu)/\pi \ll 1$, on the one hand, and small enough so that there is no large gap between $\Lambda_{\text{QCD}}$ and $\mu$. The possibility to make such a choice of $\mu$ could not be anticipated apriori and is an extremely fortunate feature of QCD, a gift from the Gods. Quarks and gluons with the offshellness larger than $\mu$ chosen that way are called hard.

Needless to say that the parameter $\mu$ is in our minds, not in Nature. All observable amplitudes must be $\mu$ independent. The $\mu$ dependence of the coefficient functions $C_n$ must conspire with that of the matrix elements of the operators $O_n$ in such a way as to ensure this $\mu$ independence of the physical amplitudes.

What can be said about the calculation of the coefficients $C_n$? Since $\mu$ is sufficiently large, see above, the main contribution comes from perturbation theory. We just draw all relevant Feynman graphs and calculate them, generating an expansion in $\alpha_s(\mu)$ which for brevity I will denote by $\alpha_s$, with the argument omitted,

$$C_n = \sum_l a_l \alpha_s^l.$$  

Sometimes some graphs will contain not only powers of $\alpha_s(\mu)$ but powers of $\alpha_s \ln(m_Q/\mu)$. This happens if the anomalous dimension of the operator $O_n$ is nonvanishing – quite
a typical situation – or if a part of a contribution to $C_n$ comes from characteristic momenta of order $m_Q$ and is, thus, expressible in terms of $\alpha_s(m_Q)$, and we rewrite it in terms of $\alpha_s(\mu)$. Nevermind, this is a trivial technicality. You are supposed to know how to sum up these logarithms.

As a matter of fact the expression (5.1) is not quite accurate theoretically. One should not forget that, in doing the loop integrations, in $C_n$ we must discard the domain of virtual momenta below $\mu$, by definition of $C_n(\mu)$. Subtracting this domain from the perturbative loop integrals we introduce in $C_n$ power corrections of the type $(\mu/m_Q)^n$ by hand. In principle, one should recognize the existence of such corrections and try to learn how to deal with them. The fact that they are there was realized long ago (see e.g. V. Novikov et al, Ref. [5]) and then largely ignored. If it is possible to choose $\mu$ sufficiently small these corrections may be insignificant numerically and can be omitted. This is what is actually done in practice. This is one of the elements of a simplification of the Wilsonian operator product expansion. The simplified version is called the practical version of OPE, see below. Certainly, at the modern stage of the theoretical development it is desirable to return to the issue to engineer a better procedure than just discarding these $\mu/m_Q$ terms in the coefficient functions. Attempts in this direction are under way [67].

Even if perturbation theory dominates in the coefficient functions they still contain also nonperturbative terms coming from short distances. Sometimes they are referred to as noncondensate nonperturbative terms. An example is provided by the so called direct instantons with the sizes of order $m_Q^{-1}$. These contributions fall off as high powers of $\Lambda_{QCD}/m_Q$ and are very poorly controllable theoretically. Since the fall off of the noncondensate nonperturbative corrections is extremely steep, basically the only thing we need to know is a critical value of $m_Q$. For lower values of $m_Q$ no reliable theoretical predictions are possible at present. For higher values of $m_Q$ one can ignore the noncondensate nonperturbative contributions. There are good reasons to believe that the $b$ quark, fortunately, lies above the critical point. Again, I must add that the noncondensate nonperturbative contributions are neglected in the practical version of OPE.

(Do we see seeds of the nonperturbative contribution in Eq. (5.1)? Yes, we do. At any finite order the perturbative contribution is well-defined. At the same time, if the coefficients in the series (5.1) grow factorially with $l$ – and this is actually the case – the tail of the series, $l > 1/\alpha_s$, must be regularized which may bring in terms of order

$$\exp \left(-\frac{C}{\alpha_s(m_Q)}\right) \sim \left(\frac{\Lambda_{QCD}}{m_Q}\right)^\gamma$$

where $C$ is some positive constant and the exponent $\gamma$ need not be integer. In a sense, one may say that contributions to $C_n$ of this type are vaguely related to diagrams with $1/\alpha_s$ hard gluon loops.)

Thus, two sources of nonperturbative corrections in the physical amplitudes are indicated. Those due to nonperturbative terms in the coefficient functions are systematically ignored (and, perhaps, rightly so, as I tried to convince you) in these
lectures and in all works based on the practical version of OPE which constitute the overwhelming majority of all works devoted to the $1/m_Q$ expansions. The second source is operators of higher dimensions in the Lagrangian (5.1), the so-called condensate corrections. The latter were in the center of our attention; they generate the $1/m_Q$ expansions discussed above. One new element which I would like to add here, is that the series of $1/m_Q$ terms generated by higher-dimensional operators is also asymptotic and divergent in high orders $[68]$. Of course, we always calculate only one, at best two, first $1/m_Q$ corrections, truncating the series. If, however, one would ask what the impact of the high-order tail of the power series is, the answer would be: this tail is reflected in exponentially small terms $\sim \exp(-m_Q)$. This type of contribution is certainly not seen in OPE truncated at any finite order. A transparent example is again provided by instantons. This time one has to fix the size of the instanton $\rho$ by hand, $\rho_0 \sim \Lambda^{-1}$. Then their contribution to physical amplitudes is $\mathcal{O}(\exp(-m_Q \rho_0))$. The relation between $\exp(-m_Q \rho_0)$ piece and the high-order terms of the power series is conceptually akin to the connection between $\exp(-1/\alpha_s)$ terms and $l \sim 1/\alpha_s$ orders in the perturbative expansion.

Summarizing, Wilsonian OPE (5.1) leads to expansions in different parameters. Purely logarithmic terms $(\ln m_Q)^{-l}$ are due to ordinary perturbation theory. Terms of the type $(m_Q^2)^{-k}(\ln m_Q)^{-\gamma}$ reflect higher-dimension operators and direct instantons. In the former case the values of $k$ are integer, the latter case may produce non-integer values of $k$. In the practical version of OPE we calculate the coefficient functions perturbatively. All non-perturbative terms come from condensates within this approximation. The condensate power series is truncated: only those operators whose dimension is smaller than some number are retained.

The practical version of OPE was heavily used in connection with the QCD sum rule method. It was checked $[27]$ that in the majority of channels this is a valid approximation allowing one to calculate in the Euclidean domain down to $\mu$ as low as 0.6 or 0.7 GeV. The validity of this approximation is an element of luck; it relies, among other things, on the fact that $\Lambda_{QCD}$ is significantly smaller than 1 GeV, and $\alpha_s(1\text{GeV})/\pi$ is already a small parameter.

I hasten to add that some exceptional channels where the practical version of OPE fails at much larger values of $\mu$ were detected in the analysis of glueballs $[69]$. It would be interesting to explore the issue in the context of the heavy quark theory. The existing theory gives no clues for establishing the domain of validity of the practical version of OPE from first principles, neither does it tell us about when the exponential terms, not visible by standard methods, become negligibly small. At this point we have to rely on indirect methods and phenomenological information.

5.2 Untangling hard gluons from soft ones

The coefficient functions $C_n$ in Eq. (5.1) contain, generally speaking, an infinite number of perturbative terms, and non-perturbative contributions of different types. Practically we often calculate them to the first nontrivial order. For instance, in
Lecture 3 we treated the transition operator in the Born approximation; thus, all coefficients in OPE were found to order $\alpha_s^0$. For a number of purposes (although not always, of course) such a calculation, ignoring the hard gluon exchanges altogether, is quite sufficient. Let me remind that by hard gluons I mean those with offshellness from $\mu$ up to $m_Q$. Let us ask a question – can one find a theoretical parameter which would justify the approximation of no hard gluon exchanges? In other words, does a parameter exist that would allow one to switch the hard gluons on/off?

Each extra hard loop contains the running gauge coupling $\alpha_s(\mu)$,

$$\frac{\alpha_s(\mu)}{\pi} = \frac{2}{b} \left[ \ln \left( \frac{\mu}{\Lambda_{\text{QCD}}} \right) \right]^{-1} \tag{5.3}$$

where $b$ is the first coefficient of the Gell-Mann-Low function,

$$b = \frac{11}{3} N_c - \frac{2}{3} n_f.$$ 

If we could make $b$ very large the running law of $\alpha_s$ would be very steep effectively switching off all hard gluons. Indeed, once $\mu$ is bigger than, say, $2\Lambda_{\text{QCD}}$ and $b \to \infty$ the gauge coupling constant $\alpha_s(\mu) \to 0$. The first idea which immediately comes to one’s mind is to make $b$ large by tending the number of colors $N_c$ to infinity. Alas, this idea does not work. It is known from the early days of QCD that the expansion parameter in all planar diagrams is $N_c \alpha_s$, not $\alpha_s$ itself [2]. Thus, the diagram of Fig. 13 is of the same order in $N_c$ as the Born graph of Fig. 1. So, we have to rely on numerical smallness of $1/b$. For instance, in the theory with three light flavors and three colors $b = 9$, quite a large number. This is not the first time in physics we have to deal with numerical enhancements. It is true that it is always better to have an adjustable parameter, which could be sent to infinity at will, than to deal with just a large fixed number. It is quite unfortunate that we do not have such a parameter at our disposal in the real world QCD. If one still wants to have $b$ as an adjustable parameter one could try a trick. Let us assume that, apart from quarks and gluons, our theory contain quark ghost fields. These ghost fields are perfectly the same as the quark fields, with a single exception – each ghost loop has an extra minus sign. The quark ghost fields may or may not have a mass term. Let us say that they do have a mass term $m_{gh}$ equal to $\Lambda_{\text{QCD}}$. Then they would automatically decouple in the soft contributions. The action of such a crazy theory has the form

$$iS = iS_{\text{QCD}} + \sum_q \bar{q}_{gh}(i\not\partial - m_{gh})q_{gh} =$$

$$iS_{\text{QCD}} - N_{gh} \ln \det \{i\not\partial - m_{gh}\} \tag{5.4}$$

where $S_{\text{QCD}}$ is the action of quantum chromodynamics, see Eq. (1.1), and $N_{gh}$ is the number of the quark ghosts, a free parameter assumed to be large. Notice the ghostly minus sign in front of the logarithm of the determinant. After some thinking one may conclude that, perhaps, this theory is not so crazy. Let us postulate that
the initial particles we consider belong to our world – \( B , D \) and so on – i.e. they do not carry these quark ghosts. Of course, if \( B \) decays the quark ghosts do appear in the final state, and the probability of their emission is negative. This does not mean, however, that the total amplitude is not unitary, as one could suspect from the fact that we introduced the fields with a wrong metric. Indeed, it is obvious that the only role of the quark ghosts is to switch off all hard gluons in the limit \( N_{gh} \to \infty \) since in this limit

\[
    b = 11 - \frac{2}{3} N_f + \frac{2}{3} N_{gh} \to \infty
\]

and \( \alpha_s(\mu) \to 0 \), according to Eq. \((5.3)\). In particular, the diagram of Fig. 13 where the gluon line is dressed with the bubble insertions vanishes. All soft contributions with \( \mu \lesssim \Lambda_{QCD} \) remain intact, however, and the positivity of the forward scattering amplitudes is not violated.

If there exists a stringy representation of QCD it should refer to the fake “QCD”, Eq. \((5.2)\), rather than to the real one since in the string amplitudes there is no place for hard gluons.

The idea of treating \( b \) as a numerically large parameter is not new in QCD. In the purely perturbative calculations it constitutes the basis of the so called BLM approach \([70]\)\footnote{In the limit \( b \to \infty \) the coefficient given in this expression does indeed vanish, in full accord with the argument of the previous section.}. Originally the BLM approach was engineered as a scale-setting procedure intended as a substitute for full computations of \( \mathcal{O}(\alpha_s^2) \) corrections. Assume that \( \mathcal{O}(\alpha_s) \) corrections in some amplitude are known exactly. In order \( \alpha_s^2 \) typically one has to deal with a large number of graphs. The idea is to pick up only those which contain a “large parameter”, \( b \alpha_s^2 \), presuming that the graphs without \( b \) are numerically suppressed. Typically there are very few graphs producing \( b \alpha_s^2 \). By doing so we can approximately determine the scale \( \mu \) in the \( \mathcal{O}(\alpha_s) \) term without labor and time-consuming calculation of a large number of all \( \alpha_s^2 \) contributions. Later, it was suggested \([71, 72]\) to extend the prescription of the “\( b \) graph dominance” to even higher orders, a more extremist and dangerous approach. In both cases the limit of large \( b \) is used to get some information about perturbation theory. I use this limit in order to switch off the perturbative hard gluons in the first place pushing the theory to the mode where only the soft gluons survive, hopefully providing a more transparent picture of the infrared dynamics determining the regularities of the hadronic world.

### 5.3 Impact of hard gluons

Having said all that let us return to the real world where \( b \) is fixed, not infinity, and examine several examples of corrections due to hard gluons. An instructive example to begin with is the calculation of the coefficient in front of the chromomagnetic operator \( \mathcal{O}_G \) in the effective Lagrangian \( \mathcal{L}_{\text{heavy}}(\mu) \), see Eq. \((1.3)\) \footnote{In the limit \( b \to \infty \) the coefficient given in this expression does indeed vanish, in full accord with the argument of the previous section.}. This coefficient
takes into account virtual gluons with offshellness from \( \mu \) to \( m_Q \).

The line of reasoning is as follows. Our starting point is \( \mu = m_Q \). At this
normalization point the Lagrangian we deal with is the QCD Lagrangian (1.1) with
the coupling constant and heavy quark mass normalized at \( m_Q \). We then descend
a little further, down to \( \mu \) equal to a finite fraction of \( m_Q \), say, \( m_Q/5 \). This is
sufficient to make the \( Q \) quark nonrelativistic and make all nonrelativistic expansions
work. Being interested only in logarithms of \( m_Q \) we ignore any nonlogarithmic \( \alpha_s \)
corrections that may appear at this stage. The nonrelativistic expansion of the
Lagrangian \( \bar{Q}(\not{i}D - m_Q)Q \) implies that the operator \( \bar{Q}(i/2)\sigma GQ \) appears with the
coefficient \( C_0 = 1/(2m_Q) \). Further evolution down to \( \mu = \) several units \( \times \Lambda_{\text{QCD}} \) will
change \( C \); in particular, at one loop

\[
C_0 \to C(\mu) = C_0 \left( 1 + \gamma \frac{\alpha_s}{4\pi} \ln \frac{m_Q^2}{\mu^2} + \text{non-log terms} \right)
\]

(5.5)

where \( \gamma \) is a number. Our goal is to find \( \gamma \) and then sum up all leading logarithms.
This is not the end of the story, however, if one wants to represent the result in the
form (1.3), where the sum over the operators includes only the Lorentz invariant
ones. The leading operator is \( \mathcal{L}_{\text{heavy}}^0(\mu) \). The coefficient \( C(\mu) \) should be represented
as

\[
C(\mu) = C_0 + (C(\mu) - C_0);
\]

then \( C_0 \) is swallowed back in the definition of \( \mathcal{L}_{\text{heavy}}^0(\mu) \) while the expression in the
brackets represents \( c_G \) in Eq. (1.3).

The relevant one-loop graphs are depicted on Fig. 14. At first sight the number
of diagrams is rather large, and the computation might seem rather cumbersome.
My task is to reduce it to a back-of-the-envelope calculation by using several smart
observations and the background field technique.

First of all, as it was already mentioned, we will be hunting only for the terms
containing \( \alpha_s \ln m_Q/\mu \) omitting all \( \alpha_s \) terms without logarithms. The logarithms
\( \ln m_Q/\mu \) have a dual nature – they appear from the loop integrations where the
integrand presents an infrared limit with respect to heavy quarks \( Q \) while presenting
simultaneously the ultraviolet limit with respect to gluons. That is why they were
called hybrid in Ref. [50], the paper where these logarithms were discovered. In the
language of HQET they are referred to as matching logarithms.

Secondly, in this perturbative calculation we will naturally discard all \( 1/m_Q \)
corrections.

The closed circle on the diagrams of Fig. 14 denotes the vertex \( (i/2)\sigma^{ij}G_{ij} =
-\bar{\sigma}B \). Let us consider for definiteness only one term with \( i, j = 1, 2 \), i.e. \(-\sigma_z B_z \)
keeping in mind that other terms will give the same.

It is absolutely obvious that the graph of Fig. 14e gives no contribution in our
approximation. Indeed, the very existence of this graph is due to the nonlinear term
\( [A_1A_2] \) in the definition of \( G_{12} \). However, neither \( A_1 \) nor \( A_2 \) interact with the heavy
quark in the leading in \( 1/m_Q \) approximation, as it is clear from Eq. (1.15), only \( A_0 \).
(We work in the rest frame of the heavy quark \( Q \).)
Next, let us analyze the diagrams $c$ and $d$. To this end it is convenient to write the gluon Green function in the background field. For a detailed exposition of the technique the reader is referred to the review paper [28]. For our purposes we need so little that it is quite in order to carry out all necessary derivations here. Let us split the four-potential $A_\mu$ in two parts – the external field $(A_\mu)^{\text{ext}}$ and the quantum part $a_\mu$ which will propagate in loops,

$$A_\mu = (A_\mu)^{\text{ext}} + a_\mu$$

(5.6)

As explained in Ref. [28] the gauge conditions on $(A_\mu)^{\text{ext}}$ and $a_\mu$ may be different, for instance, the Fock-Schwinger gauge with respect to the background field and the Feynman gauge with respect to the quantum field. Here we do not need to discuss the gauge condition on $(A_\mu)^{\text{ext}}$. The quantum field $a_\mu$ will be treated in the Feynman gauge. The definition of the gluon propagator in the background field is standard:

$$D_{\mu\nu}^{ab} = \langle T\{a_\mu^a(x), a_\nu^b(0)\} \rangle .$$

(5.7)

The Lagrangian of the quantum gluon field in the Feynman gauge has the form

$$\mathcal{L} = -\frac{1}{2}(D_\mu^{a\nu}a_\nu^a)^2 + ga_\mu^a(G_{\mu
u}^b)^{\text{ext}}a_\nu^c f^{abc} + \text{higher order terms},$$

(5.8)

plus cubic and higher order terms in $a_\mu$ plus the ghost terms – all irrelevant for the calculation at hand. Here

$$D_\mu^{a\nu}a_\nu^a = \partial_\mu a_\nu^a + gf^{abc}(A_\mu)^{\text{ext}}a_\nu^c .$$

The second term in the Lagrangian (5.8) describes the interaction of the magnetic moment of the gluon quantum with the background field. If we switch off this magnetic terms for a short while we immediately observe that both graphs, Fig. 14c and d, vanish. Indeed, the Lorentz structure of the first term in Eq. (5.8) is such that the Green function generated by it is obviously proportional to $g_{\mu\nu}$. Hence the loops displayed on Figs. $c$ and $d$ can not be formed. Say, the diagram $c$ requires converting the $a_i$ quantum leaving the vertex into the $a_0$ quantum coupled to the heavy quark. Let us now switch on the magnetic term and take into account the fact that the background field is chromomagnetic, not chromoelectric (I remind that we are interested in the vertex $-\bar{\sigma}\vec{B}$). This means that the graph $c$ still vanishes since the conversion of $a_i$ into $a_0$ can only take place in the chromoelectric background $(G_{\mu\nu}^{\text{ext}})$. The diagram $d$ is not vanishing, however, and is readily calculable. We start from the vertex

$$(i/2)\bar{Q}\sigma^{12}G_{12}^{ci}t^c Q \rightarrow (i/2)\bar{Q}\sigma^{12}ga_1^a a_2^b f^{abc}t^c Q ,$$

make one insertion of the magnetic term in the Lagrangian (5.8),

$$iga_0^a(G_{\rho\nu}^{cb})^{\text{ext}}a_\nu^c f^{\hat{a}\hat{b}\hat{c}} ,$$

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and after that one can take the free gluon propagators, which yields

\[(i/2)2g^2\bar{Q}\sigma^{12}f^{abc}t^cQf^{abb}G_{12}^b(-i)^2i \int \frac{1}{k^4(2\pi)^4} \]  

(5.9)

where the factor 2 comes from two different ways of pairings. The integral over \(dk\) is evidently logarithmically divergent both at the upper and lower ends and should be cut off at \(m_Q\) from above and at \(\mu\) from below. This logarithmic divergence should be welcome since in this way we are going to get the desired hybrid logarithm. Equation (5.9) immediately leads to

\[-2\frac{N_cg^2}{16\pi^2} \frac{(i/2)\bar{Q}\sigma^{12}G_{12}^c t^cQ\ln \left(\frac{m_Q^2}{\mu^2}\right)}{\sqrt{m_Q^2 - \mu^2}}\]  

where \(N_c\) is the number of colors (\(N_c = 3\)). In other words the factor produced by the one-loop graph of Fig. 14d is

\[\gamma |_{\text{Fig. } 14d} = -2N_c\text{.} \quad (5.10)\]

The last step in our exercise is calculation of the diagrams of Fig. 14a and b. The Feynman integral for the diagram a is quite trivial,

\[g^2 \int \frac{d^4k}{k^4} \frac{1}{k_0} t^a (-\sigma_z B_z^b t^b) t^a \frac{1}{k_0} Q (-i)^{1 \over k^2} \]  

(5.11)

where \(k\) is the virtual gluon momentum. Now, a minute reflection shows that in the Abelian theory (i.e. if the gluons were photons and the diagram of Fig. 14a was considered in QED) this contribution must be exactly canceled by that coming from diagrams b. This assertion can be traced back to the nonrenormalization of the \(\bar{Q}\gamma\mu Q\) vertex in QED. (We should take into account the fact that the hybrid logarithms do not depend on the Lorentz structure of the vertex at all [50] and are the same for \(\gamma\mu\) and \(\sigma_{\mu\nu}\).) This observation implies that in QCD the net effect of the two diagrams 14b reduces to replacing Eq. (5.11) by

\[g^2 \int \frac{d^4k}{k^4} \frac{1}{2} \bar{Q} \left( t^{a[b} t^{a]} + [t^{a[b}] t^{a]} \right) (-\sigma_z B_z^b) Q (-i)^{1 \over k^2} \] =  

\[-\frac{N_cg^2}{2} \int \frac{d^4k}{k^4} Q(-\sigma_z B_z^b t^c) Q \frac{i}{(k_0 + i\epsilon)^2} \frac{1}{k^2 + i\epsilon} . \]  

(5.12)

The \(i\epsilon\) prescription indicated explicitly defines the integration contour (Fig. 15). We first do the \(k_0\) integration using the residue theorem, then the remaining \(d^3k\) integration and arrive at

\[-2\frac{N_cg^2}{16\pi^2} \frac{(i/2)\bar{Q}(-\sigma_z B_z^c t^c)Q\ln \left(\frac{m_Q^2}{\mu^2}\right)}{\sqrt{m_Q^2 - \mu^2}}\]  

(5.13)
leading to the following “dressing” factor due to diagrams 14b and c:

\[ \gamma |_{\text{Fig. } 14a+b} = N_c \ldots \] (5.14)

The diagram 14f must be discarded in the background field calculation – it merely renormalizes the gauge coupling constant included in the definition of \( O_G \).

The overall one-loop dressing factor is obtained by adding up Eqs. (5.10) and (5.14),

\[ \gamma = -N_c = -3. \] (5.15)

Now the renormalization group allows us to sum up all leading log terms, in a standard manner; the summation leads to Eq. (1.4). The same result can be rephrased as follows: in the 1/m_Q expanded effective Lagrangian \( L_{\text{heavy}}(\mu) \) the overall coefficient in front of \( O_G \) is

\[ C(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(m_Q)} \right)^{-\frac{3}{8}}. \] (5.16)

This is nothing else than the reflection of the hybrid anomalous dimension of the operator \( O_G \) found in Ref. [8].

It is curious to note that \( O_\pi \), the second operator of dimension 5 (see Eq. (3.6)), has vanishing anomalous dimension which can be proven with no calculations in no time.

To see that this is indeed the case we merely repeat the argument preceding and following Eq. (5.5). Let us assume for a short while that the hybrid anomalous dimension of the operator \( O_\pi \) is non-vanishing. Then after evolving to a low normalization point its coefficient gets renormalized, and there is no way one could absorb \( O_\pi \) back into a Lorentz invariant expression \( \bar{Q}(\not{P} - m_Q) \) in the effective Lagrangian. Needless to say that \( L_{\text{heavy}}(\mu) \) (before the 1/m_Q expansion) must be expressible in terms of the Lorentz invariant structures.

A close line of reasoning leading to the same conclusion takes advantage of the expansion (3.3),

\[ \bar{Q}Q - \frac{1}{2m_Q^2} \bar{Q}i\gamma_5 GQ = \bar{Q}\gamma_0 Q - \frac{1}{2m_Q^2} \bar{Q}\not{\pi}^2 Q + \ldots \] (5.17)

where the dots denote terms of the higher order in 1/m_Q. The left-hand side is Lorentz scalar while the right-hand side is written as a sum of terms that are not Lorentz scalars individually. The matrix element of \( \bar{Q}\gamma_0 Q \) has the meaning of energy \( E \) (which at small velocities reduces to \( m + \not{p}^2/2m \)) while that of the second term on the right-hand side has the meaning of \( -\not{p}^2/2m \). The first term is not renormalized by the gluon dressings, of course. If the coefficient of the second term was distorted by the anomalous dimension, the cancellation of the Lorentz noninvariant part would be ruined, and the right-hand side could not be equal to the left-hand side.

Concluding this section let us discuss the impact of the hard gluons on the scaling law of, say, pseudoscalar coupling \( f_P \) defined in Sect. 3.4. In this section it was
shown that \( f_P \sim m_Q^{-1/2} \) modulo logarithmic corrections. Now we address the issue of the logarithmic corrections due to the hybrid anomalous dimension of the current \( \bar{Q}\gamma_\mu\gamma_5q \). Let us add to the original QCD Lagrangian the term \( \Delta L = A_\mu \bar{Q}\gamma_\mu\gamma_5q \) where \( A_\mu \) is an auxiliary \( c \)-number field and evolve \( \Delta L \) down to \( \mu \). The result of this evolution is the anomalous dimension

\[
(\bar{Q}\gamma_\mu\gamma_5q)_{m_Q} = \left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{2/3} (\bar{Q}\gamma_\mu\gamma_5q)_{\mu};
\]

the subscript here indicates the normalization point. The corresponding calculation is even simpler than that of the anomalous dimension of \( \mathcal{O}_G \) and will not be discussed here. The interested reader is referred to Ref. [50] or to review papers [4] [12]. Correspondingly the complete asymptotic scaling law of \( f_P \) is \( f_P \sim m_Q^{-1/2}(\alpha_s(m_Q))^{-2/b} \).

### 5.4 \( \mu \) dependence of the basic parameters of the heavy quark theory. Measuring \( \Lambda(\mu) \)

The Lagrangian (1.3) summarizes the evolution from a high normalization point down to \( \mu \). Since all operators in this Lagrangian are normalized at \( \mu \) it is perfectly natural that their matrix elements are also \( \mu \) dependent. In particular, the matrix elements of \( \mathcal{O}_G \) and \( \mathcal{O}_\pi \) denoted by \( \mu^2_G \) and \( \mu^2_\pi \) in Lecture 3 depend on \( \mu \). Actually, \( \mu^2_G \) depends on \( \mu \) rather strongly, through logarithms of \( \mu \) – this is explicitly demonstrated by the fact that \( (\mathcal{O}_G)_{m_Q} = (\alpha_s(\mu)/\alpha_s(m_Q))^{-3/2}(\mathcal{O}_G)_{\mu} \). In this case hardly anybody would even think about tending \( \mu \to 0 \). As for the operator \( \mathcal{O}_\pi \), the situation here is trickier. As we saw, it has no diagonal anomalous dimension, still some \( \mu \) dependence appears through mixing with \( \bar{Q}Q \), see [25] for details.

Let us discuss now \( \bar{\Lambda} \), another basic parameter of the heavy quark theory. The issue of its \( \mu \) dependence was at the epicenter of a heated debate recently. By itself \( \bar{\Lambda} \) never appears in \( \mathcal{L}_{\text{heavy}} \); moreover the quark mass \( m_Q \) appears in the \( 1/m_Q \) expanded effective Lagrangian (1.11) only through \( 1/m_Q \) corrections. Therefore, in the limit \( m_Q \to \infty \) (which is often identified with HQET) it is quite tempting to say that \( \bar{\Lambda} = M_{H_Q} - m_Q \) is a universal constant. For a few years it was taken for granted that such a constant exists. Within the framework of our approach based on the Wilso-nean treatment of full QCD it is perfectly clear that this is not the case. The quark mass in Eq. (1.3) explicitly depends on \( \mu \) resulting in a \( \mu \) dependence of \( \bar{\Lambda} \).

Since the issue is of importance let us rephrase this statement as follows. Since quarks are permanently confined the notion of the heavy quark mass becomes ambiguous. To eliminate this ambiguity one must explicitly specify the procedure of measuring “the heavy quark mass”. The definition through the effective Lagrangian

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12The wording in these reviews is somewhat different. You will read about the matching logarithms of HQET for the axial current. Technically this is perfectly the same as the anomalous dimension within our approach.
(1.3) is consistent. Other definitions are certainly conceivable; any consistent procedure will necessarily involve a cut-off parameter \( \mu \), and then \( \tilde{\Lambda}(\mu) = M_Q - m_Q(\mu) \).

The effective Lagrangian (1.3) is not something you directly measure, neither is \( m_Q \). Defining \( \tilde{\Lambda} \) or \( m_Q \) is equivalent to saying how they are measured, how the parameters in the effective Lagrangian are related to measurable quantities. To this end one can use any suitable prediction of the heavy quark theory, in particular, Voloshin’s sum rule (2.35). To avoid inessential technicalities I will discuss the issue in the framework of the toy model of Sect. 2.1. All results can be immediately extended to the real QCD. Equation (2.35) gives a nice definition of \( \tilde{\Lambda} \) in terms of a measurable quantity, the average value of \( E^\text{phys}_0 - E \) where \( E \) is the energy of the \( \phi \) quantum. The problem is that in Sect. 2.1 we discussed the question switching off all hard gluons, so that the above average value looked like a \( \mu \) independent number. To see where the \( \mu \) dependence comes from we must include hard gluon corrections.

If the gluon field is treated only as a soft medium the spectrum of the decay \( H_Q \to X_q + \phi \) looks roughly as on Fig. 9. The shoulder to the left of the elastic peak arises due to production of the excited states. It is important that in this approximation the spectrum rapidly (exponentially) decreases outside the end point domain, so that the entire region of \( E \) from zero up to \( E^\text{phys}_0 - \) several units \( \times \Lambda_{QCD} \) remains unpopulated. The average

\[
\int_0^{E^\text{phys}_0} dE (E^\text{phys}_0 - E) \frac{1}{\Gamma_0} \frac{d\Gamma}{dE}
\]

is then a well-defined number independent of \( m_Q \) or any cut-offs.

The situation drastically changes once we include hard gluon emission. In calculating radiative gluon correction we can disregard, in the leading approximation, nonperturbative effects, like the difference between \( m_Q \) and \( M_{H_Q} \) or the motion of the initial quark inside \( H_Q \). Thus we deal with the decay of the free quark \( Q \) at rest into \( q + \phi + \) gluon. The virtual gluon contribution merely renormalizes the constant \( h \) in the analysis presented above and is irrelevant.

The effects from real gluon emission are most simply calculated in the Coulomb gauge, where only the graph shown in Fig. 13 contributes. A straightforward computation yields \([25, 31]\) to leading order in \( \vec{v}^2 \)

\[
\frac{d\Gamma^{(1)}}{dE} = \Gamma_0 \frac{8\alpha_s}{9\pi} \frac{E^3}{E_0 m_Q^2} \frac{1}{E_0 - E}.
\]

\(^{13}\)In the literature you can find assertions that an “absolute” heavy quark mass, or the so-called pole mass, can be defined and can be shown to be a universal number independent of any cut-offs. These assertions are false. The notion of the pole mass exists only to a given finite order of perturbation theory. No consistent definition of the pole mass can be given already at the level of the leading nonperturbative corrections \( O(1/m_Q) \). The notion of the pole mass is absolutely foreign to the approach I present here, therefore, I do not want to go into details, see \([73]\). I will only say that it assumes it is possible to separate perturbative contributions from nonperturbative (?!?) in contradiction with our approach which separates soft contributions from hard.
Hard gluon emission obviously contributes to the spectrum in the entire interval $0 < E < E_0^{\text{phys}}$ creating a long “radiative” tail to the left of the end point domain. (note that in this calculation one can put $E_0^{\text{phys}}$ to $E_0$). In the first order calculation $\alpha_s$ does not run, of course. Its scale dependence shows up only in the two-loop calculation; it is quite evident, however, that it is $\alpha_s(E - E_0)$ that enters. Therefore, strictly speaking, one cannot apply Eq. (5.18) too close to $E_0$. Even leaving aside the blowing up of $\alpha_s(E - E_0)$, there exists another reason not to use Eq. (5.18) in the vicinity of $E_0$: if $E$ is close to $E_0$, the emitted gluon is soft; such gluons are to be treated as belonging to the soft gluon medium in order to avoid double counting. The separation between soft and hard gluons is achieved by explicitly introducing a normalization point $\mu$. The value of $\mu$ should be large enough to justify a small value for $\alpha_s(\mu)$. On the other hand we would like to choose $\mu$ as small as possible. We then draw a line: to the left of $E_0 - \mu$ the gluon is considered to be hard, to the right soft. At $E < E_0 - \mu$ the experimentally measured spectrum must follow the one-loop formula (5.18), see Fig. 16.

Let us return now to Voloshin’s sum rule, i.e. the first moment of $E_0^{\text{phys}} - E$, with radiative corrections included. A qualitative sketch of how $d\Gamma / dE$ looks now is presented in Fig. 16. Because of the tail to the left of the end point domain we can not define $\bar{\Lambda}$ as the value of $E_0^{\text{phys}} - E$ averaged over the entire range of the $\phi$ energy, $0 < E < E_0^{\text{phys}}$. The integral would be proportional to $\alpha_s m_Q$ because of the domain of small $E$. Besides, this would contradict the physical meaning of what we want to define. By evolving the effective Lagrangian down to $\mu$ we include all gluons harder than $\mu$ in $m_Q$, thus excluding them from $\bar{\Lambda}$. Thus, we must accept that

$$\bar{\Lambda}(\mu) = \int_{E_0^{\text{phys}} - \mu}^{E_0^{\text{phys}}} \frac{2}{\nu_0^2} \frac{1}{\Gamma_0} \frac{d\Gamma}{dE} (E_0^{\text{phys}} - E) dE.$$  (5.19)

Since the explicit form of the tail to the left of the end point domain is known (for small $\alpha_s(\mu)/\pi$ the physical spectrum is supposed to tend to the perturbative result) the $\mu$ dependence of $\bar{\Lambda}$ becomes obvious,

$$\delta \bar{\Lambda} = \delta \mu \frac{16}{9} \frac{\alpha_s(\mu)}{\pi}. \quad (5.20)$$

Equation (5.19) provides us with one possible physical definition of $\bar{\Lambda}(\mu)$ (among others) relating this quantity to an integral over a physically measurable spectral density. The pole-mass based definition, being applied to our example, would involve three steps: (i) Take the radiative perturbative tail to the left of the shoulder and extrapolate it all the way to the point $E = E_0$; (ii) subtract the result from the measured spectrum; (iii) integrate the difference over $dE$ with the weight function $(E - E_0^{\text{phys}})$. The elastic peak drops out and the remaining integral is equal to $\Gamma_0 (\nu^2/2) \bar{\Lambda}$. It is quite clear that this procedure cannot be carried out consistently – there exists no unambiguous way to extrapolate the perturbative tail too close to $E_0^{\text{phys}}$, the end point of the spectrum. Our procedure, with the normalization point $\mu$ introduced explicitly, is free from this ambiguity.
In practice, the $\mu$ dependence of $\bar{\Lambda}(\mu)$ may turn out to be rather weak. This is the case if the spectral density is such as shown in Fig. 16, where the contribution of the first excitations (lying within $\sim \Lambda_{\text{QCD}}$ from $E_0^{\text{phys}}$) is numerically much larger than the radiative tail representing high excitations. It is quite clear that if the physical spectral density resembles that of Fig. 16 and $\mu = \text{several units} \times \Lambda_{\text{QCD}}$, the running $\bar{\Lambda}(\mu)$ is rather insensitive to the particular choice of $\mu$.

It remains to be added that a similar definition of $\bar{\Lambda}(\mu)$ works in real QCD. Here it may be defined through an integral over the spectrum in the decay $B \to X_c l\nu$ measured in the domain where the recoil of the hadronic system is small, $|\vec{q}| \ll m_c$, i.e. in the SV limit.

5.5 Hard gluons and the line shape

Actually we have already started considering this question in the previous section where the radiative correction to the spectrum in the decay $H_Q \to X_q + \phi$ was found in the SV limit. In this case the impact of the hard gluons is mild – they provide a long but squeezed tail outside the end point domain which could be evaluated in the leading (one-loop) approximation. The reason why there are no violent distortions of the spectrum is simple: the $q$ quark produced is slow, and slow quarks do not like to emit hard gluons. If the final quark was fast it would produce gluons like crazy through bremsstrahlung, and the impact of such bremsstrahlung on the line shape would be much more drastic. As a matter of fact, if $m_q \to 0$ one can not limit oneself to any finite number of gluons – an infinite sequence of the so called Sudakov (or double-log) corrections must be summed over.

By definition the Sudakov corrections are those in which each power of $\alpha_s$ is accompanied by two powers of logarithm $\ln m_Q/(m_Q - 2E)$. When one approaches the end point domain the logarithm inevitably becomes large, and overcompensates the smallness of the gauge coupling constant $\alpha_s$. So, the more gluons emitted the higher the probability. The phenomenon is classical in nature and has a transparent physical interpretation. Indeed, in the initial state $H_Q$ the color field in the light cloud corresponds to a static source. The final quark produced is very fast. The stationary state of the color field corresponding to a fast-moving color charge is strongly different from that of the stationary charge. Therefore, the excess of the color filed is just shaken off in the form of the multiple emission of gluons. If you forbid to emit a large number of gluons and insist that the final state is just “one quark” (this would correspond to the two-body decay kinematics and the delta-function-like narrow spectrum) then the probability of such an improbable event is terribly suppressed. This explains why for the massless final quarks the narrow peak in the end point domain obtained in Sect. 4.2 will be drastically distorted, and a well-developed tail to the left of the end point domain will appear.

The theory of the Sudakov corrections constitutes a noticeable part of the perturbative QCD, and here, of course, I have no possibility even to scratch the surface. I will give just a few hints referring the interested reader to the original papers and
textbooks [74].

The first order probability of emission of the massless gluon in the $b \rightarrow s\gamma$ decay is

$$\frac{1}{\Gamma} \frac{d^2\Gamma_g}{d\omega d\vartheta} = \frac{2\alpha_s}{3\pi\omega(1 - \cos \vartheta)}$$

(5.21)

where $\omega$ is the gluon momentum and $\vartheta$ is its angle relative to the momentum of the $q$ quark. In this expression it is assumed that $\omega \ll m_Q$. As a matter of fact it is perfectly legitimate to make this assumption since the double logarithm comes only from this domain of integration. The $\phi$ energy in the presence of a gluon in the final state is given by

$$E = \frac{m_b^2 - 2\omega(m_b - \omega)(1 - \cos \theta)}{2m_Q - 2\omega(1 - \cos \theta)} \approx \frac{m_Q}{2} - \frac{k_\perp^2}{4\omega},$$

(5.22)

$$k_\perp \approx \omega \vartheta.$$

One starts from computing the (first order) probability $w(E)$ for the gluon to be emitted with such momentum that the $\phi$ quantum gets energy below given $E$. This probability is obtained by integrating the distribution (5.21) with the constraint that $(m_Q/2) - (k_\perp^2/4\omega)$ is less than the given $E$,

$$w(E) = \int d\omega d\vartheta^2 \frac{1}{\Gamma} \frac{d^2\Gamma_g}{d\omega d\vartheta^2} \theta \left( \frac{k_\perp^2}{4\omega} - \left( \frac{m_Q}{2} - E \right) \right) = \frac{2\alpha_s}{3\pi} \ln^2 \frac{m_Q}{m_Q - 2E}.$$

(5.23)

I integrated over $\vartheta^2$ first; the upper limit of integration is of order one, the lower limit is determined from the $\theta$ function in Eq. (5.23). Then we can carry out the $\omega$ integration. The upper limit is $m_Q/2$ while the lower limit is seen from the same $\theta$ function,

$$\omega \lesssim \frac{m_Q}{2} - E.$$

The function $w(E)$ has the meaning of the probability of emission of a sufficiently hard gluon lowering the $\phi$ energy below $E$. The all-order summation of double logs amounts then to merely exponentiating this probability [74],

$$S(E) = e^{-w(E)}.$$  

(5.24)

The spectrum then takes the form

$$\frac{1}{\Gamma} \frac{d\Gamma}{dE} = -\frac{dS}{dE}.$$  

(5.25)

We see that as $E$ approaches the end point, $E$ close to $m_Q/2$, the spectrum gets suppressed, in full accord with our expectations, since the presence of the very hard $\phi$ does not allow, purely kinematically, the gluon shower to develop and the color field to restructure itself. Notice that the Sudakov corrections merely redistribute
the probability, since the full integral over the spectrum remains unchanged. They pump events out from the end point domain to lower values of $E$.

The double log approximation *per se* does not allow us to determine the scale of $\alpha_s$ in the Sudakov exponent $S(E)$. For practical purposes the scale setting is of course very important since $\exp(-w(E))$ is a steep function. The question goes far beyond the scope of this lecture. Some partial answers can be found in Refs. [75, 76], see also [63]; suffice it to mention here that $\alpha_s$ in Eq. (5.23) turns out to be $\alpha_s(\sqrt{(m_Q - 2E)m_Q})$. Another point deserving stressing is that with the classical Sudakov formula one cannot travel over the energy axis too close to the end point $E = m_Q/2$ (even after the scale setting). Indeed, if $E > (m_Q/2) - \mu$ the gluons emitted become too soft; such gluons constitute the soft gluon medium and have nothing to do with the perturbative calculation; they have to be referred to the primordial distribution function. Equation (5.23) is applicable provided that

$$\bar{\Lambda} \ll m_Q - 2E \ll m_Q.$$ 

If we come closer to the end point domain the classical Sudakov factor must be modified by cutting off and discarding the contribution of the soft gluons. This idea gained recognition only recently; it is obviously premature to further immerse into this topic for the time being.

If the effect of the hard (perturbative) gluon emission is known the full physical spectrum is obtained by convoluting the perturbative one with the primordial distribution function,

$$\frac{d\Gamma(E)}{dE} = \theta(E) \int dy F(y) \frac{d\Gamma^{\text{pert}}_Q(E - (\bar{\Lambda}/2)y)}{dE}.$$ (5.26)

Integration over $y$ runs from $-\infty$ to 1 (more exactly, the lower limit of integration is $y_0 = -m_Q/\bar{\Lambda}$ but this difference can be ignored). One should keep in mind that $d\Gamma^{\text{pert}}_Q/dE$ is nonvanishing only in the interval $(0, m_Q/2)$. The convolution formula above is legitimate only as long as one does not apply it to the very low energy part, $E \sim \bar{\Lambda}$; further details are presented in Ref. [63].

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6 Figure Captions

Fig. 1. The forward scattering amplitude \( Q \to Q \). The dashed line denotes the \( \phi \) quantum. The solid line connecting two vertices is the \( q \) quark Green function in the background gluon field. The thick solid lines describe the \( Q \) quarks in the background field.

Fig. 2. The transition operator relevant to the total semileptonic width of the heavy mesons. The dashed lines denote the leptons, \( l \) and \( \nu \). Other notations are the same as on Fig. 1.

Fig. 3. The two-point function (2.63). The wavy line is the external scalar or pseudoscalar current; the quark propagator is in the background field.

Fig. 4. The two-point function of two scalar currents in the scalar QCD.

Fig. 5. The four-point function appearing in Eq. (2.64). The dashed line denotes the current \( Q^\dagger G_{\alpha\beta} Q \).

Fig. 6. The three-point function relevant to the proof of the Isgur-Wise formula.

Fig. 7. The “photon” spectrum in the decay \( H_Q \to X_q \phi \). The final quark is assumed to be massless. The thick line represents the delta-function spectrum of the free quark approximation. The solid line is a sketch of the actual hadronic spectrum in the end point domain (possible radiation of hard gluons is neglected).

Fig. 8. Evolution of the spectrum of Fig. 7 as the mass of the final quark increases (the schematic plot refers to \( m_q \sim m_Q/2 \). The effects of the hard gluon bremsstrahlung are not included.

Fig. 9. The photon spectrum in the SV limit, \( \vec{v}^2 \ll 1 \). The dashed line shows the would be spectrum of the free quark decay.

Fig. 10. Kinematically allowed domain in the transition \( B \to X_u l \nu \). The thick line indicates the populated phase space in the free quark decay. The shaded area of width \( \sim \bar{\Lambda} \) is the end point domain populated due to the primordial motion of the \( b \) quark inside \( B \). The shaded square in the lower right corner is the exclusive resonance domain where the inclusive approach developed here is inapplicable.

Fig. 11. More or less realistic light-cone primordial distribution function versus \( x \) (borrowed from Ref. [53]).

Fig. 12. Kinematically allowed domain in the decay \( B \to X_c l \nu \). Two large
circles show the domains where description should be based on the light cone and temporal distribution functions, respectively.

Fig. 13. Correction of the first order in $\alpha_s$ to the transition operator of Fig. 1. Shown is the only graph contributing to the imaginary part in the Coulomb gauge. The gluon is denoted by the curly line.

Fig. 14. One-loop diagrams determining the coefficient $c_G$. The wavy line denotes the gluon quanta, dashed line background gluon field.

Fig. 15. Integration contour in the $k_0$ plane.

Fig. 16. A sketch of the photon spectrum in the SV limit with the hard gluon radiation included.
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