On the Dynamical Behaviour of the Generalized Ricci Flow

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Abstract
Motivated by Müller–Haslhofer results on the dynamical stability and instability of Ricci-flat metrics under the Ricci flow, we obtain dynamical stability and instability results for pairs of Ricci-flat metrics and vanishing 3-forms under the generalized Ricci flow.

Keywords Generalized Ricci flow · Dynamical stability

Mathematics Subject Classification 53E20 · 53C21 · 35K55

1 Introduction

The generalized Ricci flow, gRF for short, is a coupled geometric flow evolving a one-parameter family of Riemannian metrics \( g_t \) and closed 3-forms \( H_t \) on an oriented smooth manifold \( M \) as follows

\[
\begin{align*}
\frac{\partial}{\partial t} g_t &= -2 \operatorname{Rc}_{g_t} + \frac{1}{2} H_t^2, \\
\frac{\partial}{\partial t} H_t &= \Delta_{g_t} H_t.
\end{align*}
\]

In the system above, \( \Delta_g = -(dd^* + d^*d) \) is the negative of the Hodge Laplacian of the Riemannian metric \( g \), and the symmetric 2-tensor \( H^2 \) induced by the 3-form \( H \) is given by \( H^2(X, Y) = g(X \lrcorner H, Y \lrcorner H) \), for all \( X, Y \in C^\infty(M, TM) \).

The gRF was introduced in theoretical physics in the context of renormalization group flows of two-dimensional nonlinear sigma models [2,11]. It can also be interpreted as a generalization of the Ricci flow to connections with torsion [16], and as a
geometric flow of generalized metrics on exact Courant algebroids [17]. Recently, the gRF has been related to some geometric flows in Hermitian Geometry, like the pluriclosed flow and the generalized Kähler Ricci-flow [5,19,22], and it has been studied on nilpotent Lie groups [12]. We refer the reader to [5] for an extensive introduction to this topic.

In the compact case, the gRF is well posed and it enjoys various properties that are akin to well-known properties of the Ricci flow (see, e.g. [5,11] and compare with [7,9,13]). In particular, it is the gradient flow of the lowest eigenvalue \( \lambda(g,H) \) of the Schrödinger operator

\[
\Phi_{g,H} := -4 \Delta_g + R_g - \frac{1}{12} |H|^2_g,
\]

with respect to a suitable \( L^2 \) inner product on \( C^\infty(M, S^3_+) \times C^\infty(M, \Lambda^3) \) (cf. [11, Prop. 3.4]). This eigenvalue is characterized by the condition

\[
\lambda(g,H) = \inf_{\{ f \in C^\infty(M) \mid \int_M e^{-f} dV_g = 1 \}} \mathcal{F}(g,H,f),
\]

where the energy functional \( \mathcal{F} \) is defined as follows for any Riemannian metric \( g \), closed 3-form \( H \), and smooth function \( f \) on \( M \)

\[
\mathcal{F}(g,H,f) := \int_M \left( R_g - \frac{1}{12} |H|^2_g + |df|^2_g \right) e^{-f} dV_g.
\]

Moreover, the eigenfunction of \( \Phi_{g,H} \) corresponding to \( \lambda(g,H) \) is given by \( e^{-f_{g,H}/2} \), where \( f_{g,H} \) is the unique minimizer of \( \mathcal{F} \) under the constraint \( \int_M e^{-f} dV_g = 1 \). Equally, \( f_{g,H} \) solves the equation

\[
2 \Delta_g f_{g,H} - |df_{g,H}|^2_g + R_g - \frac{1}{12} |H|^2_g = \lambda(g,H).
\]

In [10], the authors obtained dynamical stability and instability results for Ricci-flat metrics under the Ricci flow, improving the results previously obtained in [8,15]. In detail, they showed that if a Ricci-flat metric \( \hat{g} \) is a local maximizer of Perelman’s lambda-functional, then the solution of the Ricci flow starting close to it exists for all times and converges to a nearby Ricci-flat metric modulo diffeomorphisms (dynamical stability). In the case when \( \hat{g} \) is not a local maximizer, they proved the existence of a non-trivial ancient solution \( g_t \) of the Ricci flow that converges to \( \hat{g} \) as \( t \to -\infty \) (dynamical instability).

Motivated by these results, in the present paper, we investigate the dynamical behaviour of the gRF on compact manifolds. Instead of focusing on the functional \( \lambda \) defined in (1.2), we fix a background closed 3-form \( \tilde{H} \) and we consider the functional

\[
\mu : C^\infty(M, S^3_+) \times C^\infty(M, \Lambda^3) \to \mathbb{R}, \quad \mu(g,b) := \lambda(g,\tilde{H} + db).
\]
The critical points of $\mu$ are solitons for the gRF, thus they are not fixed along the flow. However, if we choose $\tilde{H} = 0$, then the pairs of the form $(\hat{g}, 0)$, with $\hat{g}$ a Ricci-flat metric and $0 \in C^\infty(M, \Lambda^2)$, are critical points of $\mu$ and give rise to stationary points of the gRF. The linear stability of such pairs under the gRF was investigated in [11].

In [21], Streets and Tian introduced and studied a family of geometric flows for Hermitian metrics, obtaining in particular a dynamical stability result for Kähler Ricci-flat metrics under all these flows. This applies to the pluriclosed flow [20,22,23], which belongs to the family of Hermitian curvature flows and it is an instance of gRF for a suitable choice of the closed 3-form $H$ (cf. [23, Thm. 6.5]).

Adapting the proof of [10, Thm. 1] to this setting, we show the following dynamical stability result for the gRF.

**Theorem 1.1** Let $(M, \hat{g})$ be a compact Ricci-flat manifold. Assume that $(\hat{g}, 0) \in C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^2)$ is a local maximizer of $\mu(g, b) = \lambda(g, db)$. Then, there exists an open neighbourhood $U$ of $(\hat{g}, 0)$ in $C^\infty(M, S^2_+) \times dC^\infty(M, \Lambda^2)$ in the $C^\infty$-topology such that the generalized Ricci flow starting at any $(g_0, db_0) \in U$ has a long time solution which converges modulo diffeomorphisms to $(g_\infty, 0)$, with $g_\infty$ Ricci-flat.

The proof of Theorem 1.1 is discussed in Sect. 3, where we also show some preliminary results including a Lojasiewicz–Simon inequality for $\mu$ (Lemma 3.1).

As for the dynamical instability, we observe that if $(\hat{g}, 0)$ is not a local maximizer of $\mu$, then it is not dynamically stable under the gRF. This can be easily deduced from [10, Thm. 2]. Indeed, in such a case, there exists a sequence $\{(g_i, db_i)\}$ in $C^\infty(M, S^2_+) \times dC^\infty(M, \Lambda^2)$ converging to $(\hat{g}, 0)$ and such that $\lambda(g_i, db_i) > \lambda(\hat{g}, 0)$. Since $\lambda(g_i, 0) \geq \lambda(g_i, db_i)$, $(g_i)$ is a sequence in $C^\infty(M, S^2_+)$ converging to $\hat{g}$ and such that $\lambda(g_i, 0) > \lambda(\hat{g}, 0)$. Therefore, the metric $\hat{g}$ is not a local maximizer of Perelman’s lambda-functional and by [10, Thm. 2], there exists an ancient solution $g_t$ of the Ricci flow, with $t \in (-\infty, 0]$, that converges modulo diffeomorphisms to $\hat{g}$ as $t \to -\infty$. Consequently, the pair $(g_t, 0)$ is a an ancient solution of the gRF that converges modulo diffeomorphisms to $(\hat{g}, 0)$ as $t \to -\infty$.

**Notation** Given a vector bundle $\pi : E \to M$ over $M$, we denote by $C^\infty(M, E)$ the set of smooth sections of $E$ and by $C^\infty(M \times I, E)$ the set of smooth sections of $E$ depending on a real parameter $t \in I \subset \mathbb{R}$. We use similar notations for the spaces of $C^{k,\alpha}$ sections and $W^{q,p}$ sections. We use the shorthand $S^2_+$ to denote the bundle $S^2_+T^*M$ of positive definite symmetric 2-tensors on $M$, and $\Lambda^k$ to denote the bundle $\Lambda^kT^*M$ of exterior $k$-forms on $M$.

## 2 The Functional $\mu$

Let $M$ be a compact oriented smooth manifold and fix a background closed 3-form $\tilde{H} \in C^\infty(M, \Lambda^3)$. In this section, we review some useful properties of the functional

$$
\mu : C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^2) \to \mathbb{R}, \quad \mu(g, b) := \lambda(g, \tilde{H} + db).
$$

The reader may refer to [5,11] for further details. We begin computing the gradient of $\mu$. 

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Proposition 2.1 The gradient of $\mu$ at $(g, b) \in C^\infty(M, S^2) \times C^\infty(M, \Lambda^2)$ with respect to the $L^2(M, e^{-f_{g,b}} dV_g)$ inner product is given by

$$\nabla \mu(g, b) = \left( -Rc_g - \text{Hess}_g f_{g,H} + \frac{1}{4} H^2, -\frac{1}{2} d^a g H - \frac{1}{2} \nabla f_{g,H} \wedge H \right),$$

where $H = \widehat{H} + db$.

Proof We follow the approach used in [8] to study the variational structure of Perelman’s lambda-functional. Let $\epsilon$ be a real parameter, choose $(h, \beta) \in C^\infty(M, S^2) \times C^\infty(M, \Lambda^2)$, and consider the variations

$$g_\epsilon = g + \epsilon h, \quad b_\epsilon = b + \epsilon \beta, \quad H_\epsilon = \widehat{H} + db_\epsilon, \quad \Phi_\epsilon = \Phi_{g_\epsilon, H_\epsilon}.$$  

Notice that $\mu(g_\epsilon, b_\epsilon)$ is the smallest eigenvalue of the Schrödinger operator $\Phi_0$ with corresponding normalized eigenvector $w_{g, b_\epsilon} := e^{-f_{\epsilon H}}$, where $H := H_0 = \widehat{H} + db$. Thus, $\mu(g_\epsilon, b_\epsilon)$ depends analytically on $\epsilon$ [14], and for $\epsilon$ small enough we can consider the $L^2(M, dV_g)$-orthogonal projection $P_\epsilon$ onto the one-dimensional $\mu(g_\epsilon, b_\epsilon)$-eigenspace of $\Phi_\epsilon$. In this way

$$\Phi_\epsilon P_\epsilon w_{g, b} = \mu(g_\epsilon, b_\epsilon) P_\epsilon w_{g, b},$$

and

$$\mu(g_\epsilon, b_\epsilon) = \mu(g) + \frac{\langle w_{g, b}, (\Phi_\epsilon - \Phi_0) P_\epsilon w_{g, b} \rangle_{L^2(M, dV_g)}}{\langle w_{g, b}, P_\epsilon w_{g, b} \rangle_{L^2(M, dV_g)}}.$$  

Consequently, we have

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mu(g_\epsilon, b_\epsilon) = \langle w_{g, b}, \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Phi_\epsilon w_{g, b} \rangle_{L^2(M, dV_g)}.$$  

Now, $\Phi_\epsilon = -4 \Delta_g + R_g - \frac{1}{12} |H_g|^2_{g_\epsilon}$. Using the variational formulae of the Laplacian operator and the scalar curvature, one has (cf. [8])

$$\langle w_{g, b}, \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (-4 \Delta_g + R_g) w_{g, b} \rangle_{L^2(M, dV_g)} = \int_M g(h, -Rc_g - \text{Hess}_g f_{g,H}) e^{-f_{g,H}} dV_g.$$  

Moreover, by [5, Lemma 5.3], the following identity holds

$$-\frac{1}{12} \langle w_{g, b}, \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} |H_\epsilon|^2_{g_\epsilon} w_{g, b} \rangle_{L^2(M, dV_g)} = \frac{1}{4} \langle w_{g, b}, g(h, H^2) w_{g, b} \rangle_{L^2(M, dV_g)} - \frac{1}{6} \langle w_{g, b}, g(d\beta, H) w_{g, b} \rangle_{L^2(M, dV_g)}.$$  

Finally, by [5, (6.9)], the second summand in the RHS of the previous identity can be rewritten as follows

$$-\frac{1}{6} \langle w_{g, b}, g(d\beta, H) w_{g, b} \rangle_{L^2(M, dV_g)} = \int_M g(\beta, -\frac{1}{2} (d^a g H + \nabla f_{g,H} \wedge H)) e^{-f_{g,H}} dV_g.$$  

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Hence
\[
\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mu(g_{\varepsilon}, b_{\varepsilon}) = \left( -\text{Rc}_g - \text{Hess}_g f_{g,H} + \frac{1}{4} H^2, h \right)_{L^2(M, e^{-f_{g,H}} dV_g)} + \left( -\frac{1}{2} (d^* g H + \nabla f, H \cdot H), \beta \right)_{L^2(M, e^{-f_{g,H}} dV_g)},
\]
and the statement follows.

Recall that Ricci-flat metrics are stationary points of the Ricci flow and they are the critical points of Perelman’s lambda-functional [13]. Moreover, this functional vanishes at any scalar-flat metric and thus at any Ricci-flat metric. The situation for the functional we are considering is slightly different. The stationary points of the gRF are given by pairs \((g, H) \in C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^3)\) satisfying the conditions
\[
\text{Rc}_g - \frac{1}{4} H^2 = 0, \quad \Delta_g H = 0, \quad (2.1)
\]
with \(dH = 0\). On the other hand, if we fix a background closed 3-form \(\hat{H}\) and we let \(H := \hat{H} + db\), then the critical points of the functional \(\mu(g, b) = \lambda(g, \hat{H} + db)\) satisfy
\[
\text{Rc}_g + \text{Hess}_g f_{g,H} - \frac{1}{4} H^2 = 0, \quad d^* g H + \nabla f, H \cdot H = 0, \quad (2.2)
\]
with \(f, H\) solving the equation (1.3). In particular, they are solitons for the gRF, and so they are not fixed along the flow (cf. [5, Prop. 4.28]).

Notice that \(\mu(g, b) = \lambda(g, \hat{H} + db)\) vanishes whenever
\[
R_g - \frac{1}{12} |H|_g^2 = 0. \quad (2.3)
\]
Indeed, in this case \(\Phi_{g,H} = -4\Delta_g\), whence it follows that \(\mu(g, b) = 0\) and \(f, H\) is constant. The condition (2.3) is not satisfied by the stationary points of the gRF, as one can see tracing the first equation in (2.1).

The previous observations lead us considering pairs \((\hat{g}, 0) \in C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^2)\), with \(\hat{g}\) a Ricci-flat metric. Indeed, if we choose \(\hat{H} = 0\), then \((\hat{g}, 0)\) is a stationary point of the gRF and a critical point of \(\mu, \mu(\hat{g}, 0)\) is zero and \(f, H\) is constant. On the other hand, we have the following.

**Lemma 2.2** Let \((g, b) \in C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^2)\) be a critical point of \(\mu\) and assume that \(\mu(g, b) = \lambda(g, \hat{H} + db) \leq 0\). Then, \(g\) is Ricci-flat and \(\hat{H} + db = 0\).

**Proof** Let \(H := \hat{H} + db\), \(f := f_{\hat{g}, H}\) and \(\mu := \mu(g, b)\). Tracing the first equation in (2.2), we obtain
\[
R_g + \Delta_g f - \frac{1}{4} |H|_g^2 = 0.
\]
Then, as \(f\) solves (1.3), we have
\[
\Delta_g f - |df|^2_g + \frac{1}{6} |H|^2_g = \mu.
\]
Multiplying both sides of the previous identity by $e^{-f}$ and integrating over $\mathcal{M}$ gives

$$
\int_{\mathcal{M}} \Delta g e^{-f} dV_g - \frac{1}{6} \int_{\mathcal{M}} |H|^2 g e^{-f} dV_g = -\mu \int_{\mathcal{M}} e^{-f} dV_g,
$$

whence

$$
\frac{1}{6} \int_{\mathcal{M}} |H|^2 g e^{-f} dV_g = \mu.
$$

Therefore, the assumption $\mu \leq 0$ implies that $\mu = 0$ and $\hat{H} + db = 0$. In particular, $f$ is constant and the first equation of (2.2) gives $Rc_g = 0$. $\square$

**Remark 2.3** Notice that the hypothesis $\mu(g, b) \leq 0$ in Lemma 2.2 is necessary, due to the existence of non-trivial (i.e., with non-constant $f_g, H$) solitons for the gRF [18,24].

We now determine the linearization of $\nabla \mu$ at $(\hat{g}, 0)$.

**Proposition 2.4** Let $\hat{g}$ be a Ricci-flat metric and $\hat{H} = 0$. Then, the linearization of $\nabla \mu$ at $(\hat{g}, 0)$ is given by

$$
L(h, \beta) = -\frac{1}{2} \left( \Delta^L g_h, d^\ast \hat{g} d\beta \right),
$$

for every $(h, \beta) \in C^\infty(\mathcal{M}, S^2) \times C^\infty(\mathcal{M}, \Lambda^2)$ such that $\text{div} \hat{g} h = 0$, where $\Delta^L g$ is the Lichnerowicz Laplacian of $\hat{g}$.

**Proof** Let $g_\epsilon := \hat{g} + \epsilon h$, with $h \in C^\infty(\mathcal{M}, S^2)$, and let $f_\epsilon := f_{g_\epsilon, \epsilon} d\beta$. From the proof of Proposition 2.1, we have

$$
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \nabla \mu(\hat{g} + \epsilon h, \epsilon \beta) = \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \left( -Rc_{g_\epsilon} - \text{Hess}_{g_\epsilon} f_\epsilon + \frac{1}{4} \epsilon^2 (d\beta)^2, -\frac{1}{2} \epsilon d^\ast g_\epsilon d\beta - \frac{1}{2} \epsilon \nabla f_\epsilon d\beta \right),
$$

Taking into account that $f_0 = f_{\hat{g}, 0}$ is constant, and using the variational formulae of the Ricci tensor and the Hessian operator, we obtain

$$
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \nabla \mu(\hat{g} + \epsilon h, \epsilon \beta) = \begin{cases} 
\left( -\frac{1}{2} \Delta^L g_h, -\frac{1}{2} d^\ast g d\beta \right), & \text{if } \text{div} \hat{g} h = 0, \\
\left( 0, -\frac{1}{2} d^\ast g d\beta \right), & \text{if } h \in (\ker \text{div} \hat{g})^\perp.
\end{cases}
$$

$\square$

### 3 Proof of Theorem 1.1

Before proving Theorem 1.1, we show two preliminary lemmas. The first one is a Lojasiewicz–Simon inequality for $\mu$, which is obtained applying [3, Thm. 6.3] in the same spirit of [10]. The second lemma involves a gauge fixing of the gRF and it is
obtained applying the Nash–Moser inverse function theorem [6] in the same fashion as in [1,7].

**Lemma 3.1** Let \((M, \hat{g})\) be a compact Ricci-flat manifold. Then, there exist \(\varepsilon > 0\) and \(\theta \in \left(0, \frac{1}{2}\right]\) such that

\[
\left\| \left( \text{Rc}_g + \text{Hess}_g f_g, H - \frac{1}{4}H^2, \frac{1}{2} \left( d^e_g H + \nabla f_g, \Lambda H \right) \right) \right\|_{L^2(M, e^{-f_g, \Lambda} dV_g)} \geq |\lambda(g, H)|^{1-\theta},
\]

for every \(g \in C^{2,\alpha}(M, S^2_+\Lambda)\) and \(H = db\), with \(b \in C^{2,\alpha}(M, \Lambda^2)\), such that \(\|(g - \hat{g}, b)\|_{C^{2,\alpha}} < \varepsilon\).

**Proof** By the Ebin slice theorem [4], there is an open neighbourhood \(\mathcal{V}\) of \(\hat{g}\) in \(C^{2,\alpha}(M, S^2_+)\) and a \(\sigma > 0\) such that every metric \(g \in \mathcal{V}\) can be written as \(g = \varphi^*(\hat{g} + h)\), for some \(\varphi \in \text{Diff}(M)\) and some \(h \in C^{2,\alpha}(M, S^2)\) satisfying \(\text{div}_g h = 0\) and \(\|h\|_{C^{2,\alpha}} < \sigma\).

Since (3.1) is \(\text{Diff}(M)\)-invariant and it only involves \(H = db\), we may choose some small \(\sigma' > 0\) and assume that \((g, b)\) belongs to the space

\[
\mathcal{S} := \left\{ \hat{g} + h \in C^{2,\alpha}(M, S^2_+) \mid \text{div}_g h = 0, \|h\|_{C^{2,\alpha}} < \sigma \right\}
\]

\[
\times \left\{ \beta \in C^{2,\alpha}(M, \Lambda^2) \mid d^e_g \beta = 0, \|\beta\|_{C^{2,\alpha}} < \sigma' \right\}.
\]

Let us consider the restriction of \(\mu\) to \(\mathcal{S}\)

\[
\mu_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}, \quad \mu_{\mathcal{S}}(g, b) = \lambda(g, db).
\]

We claim that \(\mu_{\mathcal{S}}\) satisfies the hypothesis of Colding–Minicozzi theorem [3, Thm. 6.3]. From standard perturbation theory [14], it follows that \(\mu_{\mathcal{S}}\) is analytic. By Proposition 2.1, the \(L^2(M, e^{-f_g, H} dV_g)\)-gradient of \(\mu_{\mathcal{S}}\) at \((g, b)\) is

\[
\nabla \mu_{\mathcal{S}}(g, b) = \pi \left( -\text{Rc}_g - \text{Hess}_g f_g, H + \frac{1}{4}H^2, -\frac{1}{2} \left( d^e_g H + \nabla f_g, \Lambda H \right) \right),
\]

where \(H = db\) and \(\pi\) is the \(L^2(M, e^{-f_g, H} dV_g)\)-orthogonal projection onto the tangent space

\[
T_{(\hat{g}, 0)\mathcal{S}} = \{ h \in C^{2,\alpha}(M, S^2) \mid \text{div}_g h = 0 \} \oplus \{ \beta \in C^{2,\alpha} \mid d^e_g \beta = 0 \}.
\]

Choosing \(\sigma\) and \(\sigma'\) small enough, standard elliptic theory implies that

\[
\| \nabla \mu_{\mathcal{S}}(g_1, b_1) - \nabla \mu_{\mathcal{S}}(g_2, b_2) \|_{C^{0,\alpha}} \leq C \| (g_1 - g_2, b_1 - b_2) \|_{C^{2,\alpha}},
\]

\[
\| \nabla \mu_{\mathcal{S}}(g_1, b_1) - \nabla \mu_{\mathcal{S}}(g_2, b_2) \|_{L^2} \leq C \| (g_1 - g_2, b_1 - b_2) \|_{W^{2,2}},
\]

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for every \((g_1, b_1)\) and \((g_2, b_2)\) in \(S\). Finally, from Proposition 2.4, the linearization of \(\nabla \mu_S\) at \((\hat{g}, 0)\) is

\[
L_S = -\frac{1}{2} \begin{pmatrix} \Delta_{\hat{g}} & 0 \\ 0 & d^*d \end{pmatrix}.
\]

In particular, \(L_S\) is symmetric (notice that this is not true for pairs \((\hat{g}, H)\) with \(H \neq 0\)).

Our claim then follows, and by [3, Thm. 6.3] there exists \(\theta \in (0, \frac{1}{2}]\) such that

\[
\|\nabla \mu_S(g, b)\|_{L^2(M, e^{-f_{g,b}} d\nu_g)} \geq |\mu_S(g, b)|^{1-\theta}.
\]

for every \((g, b) \in S\). This last formula implies (3.1). \(\square\)

**Lemma 3.2** Let \((\bar{g}, \bar{H}) \in C^\infty(M, S_2^2) \times C^\infty(M, \Lambda^3)\) be a stationary point of the gRF. For every \(T > 0, \varepsilon > 0\), there exists \(\delta > 0\) such that if \((g_0, H_0) \in C^\infty(M, S_2^2) \times C^\infty(M, \Lambda^3)\), with \(dH_0 = 0\), satisfies

\[
\| (g_0 - \bar{g}, H_0 - \bar{H}) \|_{C^\infty} < \delta,
\]

then the solution \((g_t, H_t)\) of the gRF starting at \((g_0, H_0)\) is defined for \(t \in [0, T')\), with \(T' > T\), and there exists a smooth family of diffeomorphisms \(\{\varphi_t\}_{t \in [0, T')}\) such that \(\varphi_0 = \text{Id}\) and

\[
\| (\varphi_t^*(g_t) - \bar{g}, \varphi_t^*(H_t) - \bar{H}) \|_{C^\infty} < \varepsilon.
\]

**Proof** Let \(g\) be a Riemannian metric on \(M\). Following the proof of the short-time existence of the gRF [5, Sect. 5.2], we denote by \(X_g\) the vector field

\[
X_g = \text{tr}_g (D_g - D_{\bar{g}}),
\]

where \(D_g\) is the Levi–Civita connection of \(g\).

We let

\[
\mathcal{F} := C^\infty(M \times [0, T], S_2^2) \times C^\infty(M \times [0, T], \Lambda^3)
\]

and

\[
\mathcal{G} := \mathcal{F} \times C^\infty(M, S_2^2) \times C^\infty(M, \Lambda^3).
\]

Both \(\mathcal{F}\) and \(\mathcal{G}\) are tame Fréchet spaces with respect to the gradings

\[
\|(g, H)\|_n = \sum_{2j \leq n} \int_0^T \| (\partial_t^j g_t, \partial_t^j H_t) \|_{W^{n-2j, 2}} dt.
\]
and
\[
\| (g, H), (g', H') \|_n = \| (g, H) \|_n + \| (g', H') \|_{W^{n,2}},
\]
respectively. Consider the map \( F : \mathcal{F} \rightarrow \mathcal{G} \) defined by
\[
F(g, H) = \left( \partial_t g + 2 \text{Re}_g - \frac{1}{2} H^2 - \mathcal{L}_{X_g} g, \partial_t H + dd_g^* H - d(X_g \cdot H) \right), \quad (g|_{t=0}, H|_{t=0})
\]
where \( \mathcal{L} \) denotes the Lie derivative. Note that if \( H \) is closed, we have
\[
F(g, H) = \left( \partial_t g + 2 \text{Re}_g - \frac{1}{2} H^2 - \mathcal{L}_{X_g} g, \partial_t H - \Delta_g H - \mathcal{L}_{X_g} H \right), \quad (g|_{t=0}, H|_{t=0}).
\]
From the results of [5, Sect. 5.2], it follows that
\[
F|_{\{g, H\}}(h, K) = \left( \partial_t h - \Delta_g^2 h + \Phi(h, K), \partial_t K - \Delta_g K + d\Psi(h, K), (h|_{t=0}, K|_{t=0}) \right),
\]
for every closed 3-form \( K \), where \( \Phi \) and \( \Psi \) are first order in \( h \) and zeroth order in \( K \). Hence, the assumptions of the Hamilton–Nash–Moser Theorem [7, Thm. 5.1] with integrability condition \( dK = 0 \) are satisfied, and from the proof of [7, Thm. 5.1], it follows that \( F|_{\{g, H\}} \) is an isomorphism for every \( (g, H) \in C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^3) \).

Let \((\tilde{g}_t, \tilde{H}_t) \equiv (\tilde{g}, \tilde{H}) \in \mathcal{F}\). Since
\[
F(\tilde{g}, \tilde{H}) = ((0, 0), (\tilde{g}, \tilde{H})),
\]
the map \( F \) is invertible from an open neighbourhood of \((\tilde{g}, \tilde{H})\) in \( \mathcal{F} \) to an open neighbourhood of \(((0, 0), (\tilde{g}, \tilde{H}))\) in \( \mathcal{G} \). Thus, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \((\tilde{g}_0, \tilde{H}_0) \in C^\infty(M, S^2_+) \times C^\infty(M, \Lambda^3) \) satisfies \( \|(\tilde{g}_0 - \tilde{g}, \tilde{H}_0 - \tilde{H})\|_{C^\infty} < \delta \), then the initial value problem
\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{g}_t &= -2 \text{Re}_{\tilde{g}_t} + \frac{1}{2} \tilde{H}_t^2 + \mathcal{L}_{X_{\tilde{g}_t}} \tilde{g}_t, \\
\frac{\partial}{\partial t} \tilde{H}_t &= -dd_{\tilde{g}_t}^* \tilde{H}_t + d(X_{\tilde{g}_t} \cdot \tilde{H}_t), \\
\tilde{g}_t|_{t=0} &= \tilde{g}_0, \\
\tilde{H}_t|_{t=0} &= \tilde{H}_0,
\end{align*}
\]
has a unique solution \((\tilde{g}_t, \tilde{H}_t)\) defined for \( t \in [0, T') \), with \( T' > T \), and such that \( \|(\tilde{g}_t - \tilde{g}, \tilde{H}_t - \tilde{H})\|_{C^\infty} < \varepsilon \). If we further assume that \( d\tilde{H}_0 = 0 \), then \( \tilde{H}_t \) stays closed.
for every $t \in [0, T')$, and $(\tilde{g}_t, \tilde{H}_t)$ solves the generalized DeTurck–Ricci flow

$$\begin{align*}
\frac{\partial}{\partial t} \tilde{g}_t &= -2Rc_{\tilde{g}_t} + \frac{1}{2} \tilde{H}_t^2 + \mathcal{L}_{\tilde{g}_t} \tilde{g}_t, \\
\frac{\partial}{\partial t} \tilde{H}_t &= \Delta_{\tilde{g}_t} \tilde{H}_t + \mathcal{L}_{\tilde{g}_t} \tilde{H}_t, \\
\tilde{g}_t|_{t=0} &= \tilde{g}_0, \\
\tilde{H}_t|_{t=0} &= \tilde{H}_0.
\end{align*}$$

The thesis follows by choosing $\{\psi_t\} \in \text{Diff}(M)$ solving

$$\partial_t \psi_t = -X_{g_t} \circ \psi_t, \quad \psi_0 = \text{Id},$$

so that $(g_t, H_t) = (\psi_t^*(\tilde{g}_t), \psi_t^*(\tilde{H}_t))$ is a solution of the gRF. \hfill \Box

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1** Let

$$B_r := \left\{ (h, db) \in C^\infty(M, S^2_+) \times dC^\infty(M, \Lambda^2) \mid \| (h - \hat{g}, db) \|_{C^k} < r, \forall k \geq 0 \right\},$$

where the $C^k$-norms are defined using $\hat{g}$. Furthermore, for every integer $k \geq 3$, let $B^k_\delta$ denote the ball of centre $(\hat{g}, 0)$ and radius $\varepsilon$ in $C^k(M, S^2_+) \times dC^{k+1}(M, \Lambda^2)$. Since $\mu(\hat{g}, 0) = 0$ and $(\hat{g}, 0)$ is a local maximizer of $\mu$, in view of Lemma 3.1 we choose $\varepsilon > 0$ so that for every $(g, db) \in B^3_\varepsilon$ we have $\mu(g, b) = \lambda(g, db) \leq 0$ and

$$\left\| \left( \text{Rc}_g + \text{Hess}_g f_{g,H} - \frac{1}{4} H^2, \frac{1}{2} d^*_g H + \frac{1}{2} \nabla H \right) \right\|_{L^2(M, e^{-f_{g,H}}dV_g)} \geq |\lambda(g, H)|^{1-\theta},$$

for some $\theta \in \left(0, \frac{1}{2}\right)$, with $H = db$.

By Lemma 3.2, there exists $0 < \delta < \varepsilon$ small enough so that for every $(g_0, H_0) \in B_\delta$ the gRF starting at $(g_0, H_0)$ has a solution $(g_t, H_t)$ defined for $t \in [0, T')$, with $T' > 1$, and there exists a smooth family of diffeomorphisms $\{\varphi_t\}$ such that $\varphi_0 = \text{Id}$ and $(\varphi^*_t g_t, \varphi^*_t H_t) \in B_{\varepsilon/4}$, for every $t \in [0, T')$.

Fix $k \geq 3$, $(g_0, H_0) \in B^3_\varepsilon$, and let $T > 0$ be the maximal time such that for every $t \in [0, T)$ there exists $\varphi_t \in \text{Diff}(M)$ such that $(\varphi^*_t g_t, \varphi^*_t H_t) \in B^k_\delta$. Our choice of $\delta$ implies that $T \geq 1$ and that there exists $\varphi_1 \in \text{Diff}(M)$ such that $\mu(\varphi^*_1 g_1, \varphi^*_1 H_1) \in B_{\varepsilon/4}$.

Let $X_t = -\nabla f_t$, where $f_t := f_{\varphi^*_t g_t, \varphi^*_t H_t}$ and the gradient is taken with respect to $\varphi^*_t g_t$, and let $\{\psi_t\}_{t \in [0, T)} \subseteq \text{Diff}(M)$ be the family of diffeomorphisms generated by $X_t$ and satisfying $\psi_1 = \text{Id}$. For $t \in [0, T)$, let $\tilde{\psi}_t = \varphi_1 \circ \psi_t$ and let $\tilde{g}_t = \tilde{\psi}_t^* g_t$, $\tilde{H}_t = \tilde{\psi}_t^* H_t$. Then

$$\partial_t \tilde{g}_t = -2(Rc_{\tilde{g}_t} + \text{Hess}_{\tilde{g}_t} \tilde{f}_t) + \frac{1}{2} \tilde{H}_t^2, \quad \partial_t \tilde{H}_t = \Delta_{\tilde{g}_t} \tilde{H}_t - d \left( \nabla \tilde{f}_t \downarrow \tilde{H}_t \right),$$

where $\tilde{f}_t = f_{\tilde{\psi}_t^* g_t, \tilde{\psi}_t^* H_t}$.\hfill \(\Box\) Springer
Now, we can write \( \tilde{H}_t = d \tilde{b}_t \), with \( \tilde{b}_t \) solving
\[
\partial_t \tilde{b}_t = -d_{\tilde{g}_t}^* d \tilde{b}_t - \nabla \tilde{f}_t \omega d \tilde{b}_t.
\]
By interpolation, we then have
\[
\|(\partial_t \tilde{g}_t, \partial_t \tilde{H}_t)\|_{C^k} = \|(\partial_t \tilde{g}_t, d \partial_t \tilde{b}_t)\|_{C^k} \leq \|(\partial_t \tilde{g}_t, \partial_t \tilde{b}_t)\|_{C^{k+1}} \leq C \|(\partial_t \tilde{g}_t, \partial_t \tilde{b}_t)\|_{L^2}^{1-\eta},
\]
for some \( \eta \in (0, 1) \). We let
\[
T'' := \sup \left\{ t \in [1, T] \mid (\tilde{g}_t, \tilde{H}_t) \in B^k_t \right\},
\]
and \( \sigma := \theta - \eta + \theta \eta > 0 \). Then, by Lemma 3.1 and the inequality (3.2), we obtain
\[
-\frac{d}{dt} |\mu(\tilde{g}_t, \tilde{b}_t)|^\sigma = \sigma |\mu(\tilde{g}_t, \tilde{b}_t)|^{\sigma-1} \frac{d}{dt} |\mu(\tilde{g}_t, \tilde{b}_t)|
= 2\sigma |\mu(\tilde{g}_t, \tilde{b}_t)|^{(\sigma-1)(1+\eta)}
\left\| \left( \text{Re}_{\tilde{g}_t} + \text{Hess}_{\tilde{g}_t}, \tilde{f}_t - \frac{1}{4} \tilde{H}_t^2, \frac{1}{2} d_{\tilde{g}_t}^* \tilde{H}_t + \frac{1}{2} \nabla \tilde{f}_t \omega \tilde{H}_t \right) \right\|_{L^2}^{1+\eta} \|(\partial_t \tilde{g}_t, \partial_t \tilde{b}_t)\|_{L^2}^{1-\eta}
\geq \frac{\sigma}{C} \|(\partial_t \tilde{g}_t, \partial_t \tilde{b}_t)\|_{C^{k+1}}.
\]
Therefore, for \( k \) sufficiently large, we have
\[
\int_1^{T''} \|(\partial_t \tilde{g}_t, \partial_t \tilde{H}_t)\|_{C^k} dt \leq \int_1^{T''} \|(\partial_t \tilde{g}_t, \partial_t \tilde{b}_t)\|_{C^{k+1}} dt \leq \frac{C}{\sigma} |\mu(\tilde{g}_1, \tilde{b}_1)|^\sigma
\leq \frac{C}{\sigma} |\mu(\tilde{g}_0, \tilde{b}_0)|^\sigma = \frac{C}{\sigma} |\lambda(0, db)|^\sigma,
\]
where the last inequality follows since
\[
\partial_t \tilde{g}_t = -\text{Re}_{\tilde{g}_t}, - \text{Hess}_{\tilde{g}_t}, \tilde{f}_t + \frac{1}{4} \tilde{H}_t^2, \quad \partial_t \tilde{b}_t = -\frac{1}{2} d_{\tilde{g}_t}^* d \tilde{b}_t - \frac{1}{2} \nabla \tilde{f}_t \omega d \tilde{b}_t,
\]
is the gradient flow of \( \mu \) (cf. Proposition 2.1), while the last equality follows from the diffeomorphism invariance of \( \mu \).

Now, up to shrinking \( \delta \), we may assume that \( \frac{C}{\sigma} |\lambda(0, H_0)|^\sigma \leq \frac{\epsilon}{4} \). This implies that \( T'' = \infty \), as otherwise we would have \( (\tilde{g}_{T''}, \tilde{H}_{T''}) \in B^k_{\epsilon/2} \), which is a contradiction. Hence, \( (\tilde{g}_t, db_t) \) is defined for every positive \( t \), and it converges in the \( C^k \)-norm to a pair \( (\tilde{g}_\infty, db_\infty) \in B^k_{\epsilon} \). In particular, \( \mu(\tilde{g}_\infty, db_\infty) = \lambda(\tilde{g}_\infty, db_\infty) \leq 0 \). In view of the smoothing estimates for the gRF [5, Sect. 5.4], \( (\tilde{g}_t, db_t) \) converges in the \( C^\infty \)-topology to \( (\tilde{g}_\infty, db_\infty) \). Since \( (\partial_t \tilde{g}_t, \partial_t \tilde{b}_t) \to 0 \) in the \( C^\infty \)-topology, we also have \( \nabla \mu(\tilde{g}_\infty, db_\infty) = 0 \). We can then apply Lemma 2.2 to conclude that \( db_\infty = 0 \) and \( \text{Re}_{\tilde{g}_\infty} = 0 \). \( \square \)
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