Heisenberg model on a space with negative curvature: topological spin textures on the pseudosphere

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Abstract

Heisenberg-like spins lying on the pseudosphere (a 2-dimensional infinite space with constant negative curvature) cannot give rise to stable soliton solutions. Only fractional solutions can be stabilized on this surface provided that at least a hole is incorporated. We also address the issue of ‘in-plane’ vortices, in the XY regime. Interestingly, the energy of a single vortex no longer blows up as the excitation spreads to infinity. This yields a non-confining potential between a vortex and a antivortex at large distances so that the pair may dissociate at arbitrarily low temperature.
1 Introduction

As nanoscience and nanotechnology advance, materials with astonishing small sizes have appeared. Not only their sizes, but also their geometries have experienced a great challenge. Indeed, besides usual ones (cylinders, cones, spheres, and so forth), the manipulation of somewhat ‘exotic shapes’, like the Möbius stripe, have been also recently reported \[1\]. Actually, the relevance of geometric and/or topological features of the physical and/or the internal spaces has a long history in Physics. So, non-linear topological excitations (solitons, vortices etc) are important for understanding several properties of the system. For example, the depairing of vortices is intimately related to the so-called topological phase transition in a number of quasi-planar physical systems. On the other hand, the presence of solitons is traced back to a finite correlation length regime and the absence of any finite temperature phase transition. Therefore, we may ask ourselves how the underlying geometry affects the structure, stability and dynamical properties of topological objects and eventually some physical aspects of the system associated to them. Indeed, a number of works has addressed such an issue in the last years. For instance, in the context of magnetism, several aspects of solitonic solutions associated to the non-linear $\sigma$ model (NL$\sigma$M; which is the continuum limit of the classical isotropic Heisenberg model) have been studied in some geometries, like cylinders \[2, 3, 4\], cones\[5\], and so on, while a study of vortex-like excitations on a conical background has appeared in Ref.\[6\]. In all of these works, it became evident the influence of the underlying geometry on the features of such objects. In addition, solitons have also been investigated in non-simply connected surfaces, like the punctured plane and the truncated cone\[7, 8\]. There, it has been verified that a fractional ($\pi/2$ or half-) soliton emerges as the simplest non-trivial static solution of the sine-Gordon equation, whose energy is exactly one half of that associated with the usual $\pi$-soliton. It should be stressed that the study of such kind of excitations, in highly non-linear theories, is also important for biophysical and biological processes\[9\].

It is noteworthy that not only usual geometries have attracted the attention of Condensed Matter physicists. As we have already mentioned in the very beginning, samples of single crystals with Möbius stripes shape came to reality a few years ago. More recently, hyperbolic spaces (with negative Gaussian curvature) have also been considered in connection with Condensed Matter and Statistical Physics. For instance, in Ref. \[10\] the two-dimensional electron gas was studied on the pseudosphere (the simplest hyperbolic surface, whose curvature is constant), while in the works of Refs.\[11\] the authors studied the thermodynamics of the two-dimensional Ising model on this support, finding deviations in some of the critical exponents associated to the negative curvature. Here, we would like to consider a system of classical spins described by a continuum version of the Heisenberg model defined on the surface of a pseudosphere, in particular, how its geometrical features affect the structure of soliton and vortex-like excitations. Actually, we have realized that although the pseudosphere is infinite, the isotropic model does not support stable solitonic solutions: the negative curvature of the surface prevents the complete mapping of the spin sphere to the physical manifold. Therefore, stability of such excitations demands a non-trivial topological feature, like a hole on the surface, which avoids the collapse of the soliton. Furthermore, we also consider vortex-like solutions in the XY regime of the former model. Now, we have seen that the single-vortex energy is asymptotically ‘regularized’, i.e., its large distance term vanishes as long as the vortex spreads to infinite. It is also verified that a vortex-antivortex pair no longer experiences a confining potential asymptotically, it rather appears to be free at large distances so that entropy is expected to dominate at any arbitrary low temperature.
2 The model on the pseudosphere

Let us consider the Heisenberg model for nearest-neighbor interacting spins on a two-dimensional lattice, given by the Hamiltonian below:

\[ H_{\text{latt}} = -J' \sum_{<i,j>} \mathcal{H}_{i,j} = -J' \sum_{<i,j>} (S_i^x S_j^x + S_i^y S_j^y + (1 + \lambda) S_i^z S_j^z) , \]  

(1)

where \( J' \) is the exchange coupling between nearest-neighbor spins and \( \vec{S}_i = (S_i^x, S_i^y, S_i^z) \) is the spin operator at site \( i \). The parameter \( \lambda \) accounts for the anisotropy interaction amongst spins: for \( \lambda > 0 \) spins tend to align along the \( z \)-axis (easy-axis regime); for \( \lambda = 0 \) we have the isotropic case, while for \( -1 < \lambda < 0 \) one gets the easy-plane regime. Finally, \( \lambda = -1 \) yields the so-called \( XY \) model.

In the continuum approach of spatial and spin variables, valid at sufficiently large wavelength and low temperature, the model above may be written as \((J \equiv J'/2)\):

\[ H_1 = J \int \int_\Omega \sum_{i,j=1}^2 \sum_{a,b=1}^3 g_{ij} h_{ab} (1 + \delta_{a3} \lambda) \left( \frac{\partial S^a}{\partial \eta_i} \right) \left( \frac{\partial S^b}{\partial \eta_j} \right) \sqrt{|g|} d\eta_1 d\eta_2 = J \int_\Omega (1 + \delta_{a3} \lambda)(\vec{D} S^a)^2 d\Omega , \]  

(2)

where \( \Omega \) is the surface with curvilinear coordinates \( \eta_1 \) and \( \eta_2 \) so that \( d\Omega = \sqrt{|g|} d\eta_1 d\eta_2 \), \( \vec{D} \) is the covariant derivative, \( \sqrt{|g|} = \sqrt{\text{det}[g]} \) and \( g_{ij} \) and \( h_{ab} = \delta_{ab} \) (\( \delta_{ab} \) is the usual Kronecker symbol) are the elements of the surface and the spin space metrics, respectively. In the static regime, expression (2) describes the classical properties of a number of ferro and antiferromagnets depending on whether the sign of \( J \) is positive or negative, respectively. The Hamiltonian above is an anisotropic non-linear \( \sigma \) model (NL\( \sigma \)M), lying on an arbitrary two-dimensional geometry, so that our considerations could have some relevance to other branches like Theoretical High Energy Physics and Cosmology.

Figure 1: The Poincaré disc method to obtain the pseudosphere (the upper hyperboloid sheet) from the projection point \( P = -1 \) on the \( z \) axis. Each point of the disc \((r, \varphi)\) is mapped to another on the sheet \((\tau, \varphi)\), so that the disc border is taken to infinite.
We shall deal with such a model on the pseudosphere, which is the simplest hyperbolic space, since it presents constant and negative (Gaussian) curvature. Let us briefly describe some of its geometrical features (further details may be found, for example, in Ref. [12]). First of all, let us recall that a sphere can always be embedded in a three-dimensional (3D) Euclidian space, such that \( x^2 + y^2 + z^2 = R^2 \) \((R^2 > 0\); in Cartesian rectangular coordinates). This global embedding cannot be generally performed for hyperbolic surfaces. Actually, in our case the manifold is defined by \( x^2 + y^2 - z^2 = -R^2 \) \((R^2 > 0)\), leading to two disjoint hyperboloids. For definiteness, we choose the pseudosphere to be the upper sheet limited by the upper “light-cone” \( x^2 + y^2 = z^2, z > 0 \) (Figure 1). There are several ways of “visualizing” the pseudosphere\(^{[12]}\). Here, we shall employ the Poincaré disc method: each point of this disc, whose radius is \( R \) (Figure 1), is mapped to an unique point on the pseudosphere (Figure 1). More explicitly, a point parametrized by \((r, \varphi)\) on the disc is mapped to \((x; y; z) = (R \sinh(\tau) \cos(\varphi); R \sinh(\tau) \sin(\varphi); R \cosh(\tau))\) on the pseudosphere, by means of:

\[
 r = R \frac{\sinh(\tau)}{1 + z/R} = R \tanh\left(\frac{\tau}{2}\right), \quad \varphi = \arctan\left(\frac{y}{x}\right).
\]

Then \( \tau \in [0, +\infty) \) and \( \varphi \in [0, 2\pi] \), while the geodesic distance on the pseudosphere reads \( s = R \tau / 2 \). It should be noticed that both the pseudosphere and the Poincaré disc present the same constant negative curvature, \(-1/R^2\), as it is expected since they are topologically equivalent. The difference between them relies on the fact that the pseudosphere is an infinite surface while the disc is bounded at \( r = R \). In addition, the line element of the pseudosphere reads:

\[
ds^2 = R^2(d\tau^2 + \sinh^2(\tau)d\varphi^2),
\]

so that its metric tensor elements are \( g_{\tau\tau} = R^2 \), \( g_{\varphi\varphi} = R^2 \sinh^2(\tau) \) and \( g_{\tau\varphi} = 0 \). Alternatively, we have:

\[
ds^2 = 4 \left(1 - \frac{x^2}{R^2}\right)^{-2} (dr^2 + r^2d\varphi^2) = 4 \left(1 - \frac{(x^2+y^2)}{R^2}\right)^{-2} (dx^2 + dy^2),
\]

in usual polar and Cartesian coordinates, respectively. Now, the Hamiltonian \([2]\) on the pseudosphere can be written as follows:

\[
H_2 = J \int_0^{2\pi} d\varphi \int_0^\infty d\tau \left\{ \sinh(\tau) \left[ (1 + \lambda \sin^2(\theta))(\partial_r \theta)^2 + \sin^2(\theta)(\partial_r \Phi)^2 \right] + \frac{[\sinh(\tau)(\partial_r \theta)^2 + \sin^2(\theta)(\partial_r \Phi)^2]^2}{\sinh(\tau)} \right\},
\]

where \( \partial_i = \partial/\partial \eta_i \). The functions \( \theta = \theta(\tau, \varphi) \) and \( \Phi = \Phi(\tau, \varphi) \) are the spin angle variables, say, \( \vec{S} = (\sin \theta \cos \varphi; \sin \sin \sin \varphi; \cos \theta) \) so that \( |\vec{S}|^2 = S^2 \equiv 1 \). From Hamiltonian above we can derive the static Euler-Lagrange equations for \( \theta \) and \( \Phi \), that read, respectively:

\[
(1 + \lambda \sin^2(\theta)) \left[ \partial_r (\sinh \tau \partial_r \theta) + \frac{\partial^2 \theta}{\sinh \tau} \right] = \lambda \sin \theta \cos \theta \left[ \sinh \tau (\partial_r \theta)^2 + \frac{(\partial_r \theta)^2}{\sinh \tau} \right] + \
+ \sin \theta \cos \theta \left[ \sinh \tau (\partial_r \Phi)^2 + \frac{(\partial_r \Phi)^2}{\sinh \tau} \right],
\]

\[
\partial_r (\sinh \tau \sin^2(\theta) \partial_r \Phi) + \partial_\varphi \left( \frac{1}{\sinh \tau} \sin^2(\theta) \partial_\varphi \Phi \right) = 0.
\]

As expected, such static equations are highly non-linear and no general solutions for them are known up to the present. For proceeding further in our analysis we seek for special solutions.
3 Solitons on the pseudosphere

In order to obtain any suitable solution for Eqs. (6-7), we must impose some properties on them. First of all, we take the isotropic case, \( \lambda = 0 \). In addition, let us assume that \( \theta \) and \( \Phi \) describe a cylindrically symmetric solution, say, \( \partial_\theta \theta = 0 \) and \( \Phi = \Phi(\varphi) = \varphi \) (up to a constant). With such requirements, Eq. (7) identically vanishes while (6) reduces to:

\[
\partial_\tau (\sinh \tau \partial_\tau \theta) = \frac{\sin(2\theta)}{2 \sinh \tau}.
\] (8)

Defining \( u = \ln(\tanh(\tau/2)) \) so that \( u \in (-\infty, 0] \) (this maps the pseudosphere to a semi-infinite cylinder with axial coordinate \( u \)), we get a sine-Gordon equation:

\[
\partial^2_u \theta = \frac{\sin(2\theta)}{2},
\] (9)

whose simplest solution reads\(^{13}\):

\[
\theta(u) = 2 \arctan(e^{u-\overline{u}})
\] (10)

where \( \overline{u} \in (-\infty, \infty) \). Its energy is easily evaluated and gives:

\[
E_\theta = 2\pi J \int_{-\infty}^{0} \left[ (\partial_u \theta(u))^2 + \sin^2 \theta(u) \right] du = 8\pi J \frac{e^{2\overline{u}}}{1 + e^{2\overline{u}}} \in [0, 8\pi J].
\] (11)

Thus, the symmetric soliton solution lying on the pseudosphere is unstable and always decays to the ground-state, \( E = 0 \). Even though the pseudosphere is an infinite manifold, its negative curvature identifies it with the Poincaré disc, which like all finite discs cannot support Belavin-Polyakov-like excitations\(^4\) Note also that if the bottom ‘pseudosphere’ could be connected to the upper (our actual pseudosphere), then we would have a stable soliton. Thus, a ‘major’ negative curvature surface can, in principle, support this kind of excitation (the non-disjoint ‘space-like’ sheet defined by \( x^2 + y^2 - z^2 = R^2 > 0 \) is an example).

If we relax the cylindrically symmetric requirement on the solution, say, \( \partial_\varphi \theta \neq 0 \), then we now have the following differential equation (with \( \lambda = 0 \)): \( \partial_\varphi^2 \theta + \partial_\theta^2 \theta = \sin(2\theta)/2 \), with \( u \) defined like above (this takes the time-dependent sine-Gordon equation if we replace \( \varphi = it \)). Its simplest solution reads:

\[
\theta_{u\varphi} = 2 \arctan \left[ \frac{\sin(a\varphi/\sqrt{1-a^2})}{a \cosh((u-\overline{u})/\sqrt{1-a^2})} \right],
\] (12)

which is well-defined provided that \( a/\sqrt{1-a^2} \equiv m \) is an integer. The parameter \( a \) is (for \( \varphi = it \)) the “speed of the excitation” divided by the “speed of light”, so that \( a \in [0, 1] \), while \( \overline{u} \) is related to its radius, like in the former case. In addition, note that:

\[
\theta_{u\varphi} \to 0 \quad \text{as} \quad u \to -\infty,
\] (13)

\[
\theta_{u\varphi} \to 2 \arctan \left[ \frac{\sin(m\varphi)}{a \cosh(-ma\overline{u})} \right] \quad \text{as} \quad u \to 0^-,\] (14)

\(^1\)Note, however, that the sphere does support these solitons\(^4\): the spin sphere covers the physical surface exactly \( N \) times, which is bound and compact but (stereographically) equivalent to the compactified infinite flat plane.
and, therefore, no complete mapping of the spin sphere to the physical support is possible. Indeed, since \( \cosh(x) \geq 1 \) then \( \theta_{ue} \) never equals \( \pi \). Hence, like their cylindrically symmetric counterparts the solution above is unstable and decays.

Actually, stability for these excitations may be provided by means of topological obstructions. For example, if we consider a circular-type hole with radius \( s_0 = R\tau_0 \) centered at the origin (equivalently, we remove a disc of radius \( r_0 = R\tanh(\tau_0/2) \in (0, R) \) from the Poincaré disc), then Eq. (8) or (9) is now solved, with the requirement that \( \theta(u = r_0) = \theta(\tau = \tau_0) = constant \), by (see, for example, Ref. [7]):

\[
\theta_{\tau_0} = 2 \arctan \left( \frac{\tanh(\tau/2)}{\tanh(\tau_0/2)} \right),
\]

whose associated energy reads:

\[
E|_{\tau_0} = 2\pi J \int_{\tau_0}^{\infty} d\tau \left[ \sinh(\tau) \left( \partial_\tau \theta_{\tau_0} \right)^2 + \frac{\sin^2(\theta_{\tau_0})}{\sinh(\tau)} \right] = 4\pi J \left( 1 - \frac{2}{1 + \cotanh(\tau_0/2)} \right) \in (0, 4\pi J). \tag{16}
\]

In this case, we have a fractional solution with charge interpolating between a null \( (\tau_0 \to \infty) \) and a \( \pi/2 \)-soliton \( (\tau_0 \to 0) \) whose energy runs from 0 to \( 4\pi J \), depending on the excitation (or on the hole) size, since scale invariance no longer holds. Note that, the hole on the support now prevents the collapse of the soliton, similarly to what happens in the annulus and finite truncated cone[7].

4 Vortex-like excitations on the pseudosphere

A magnetic vortex is commonly thought to be a spin profile with non-vanishing vorticity. Indeed, its core may develop spin component out of the surface whenever this is demanded for regulating its energy. However, the XY anisotropy is frequently invoked for ensuring that spins will lie on the surface. This holds for the planar case, once that the \( Z \)-axis of the (internal space) spin sphere is everywhere aligned perpendicular to the surface. Therefore, spins lying on the internal equator (XY plane) is equivalent to lie on the physical surface. Nevertheless, this is not valid for an arbitrary surface curvature, torsion, and other geometrical features. In these cases, a new type of anisotropy may be demanded. For example, a term like \( b(\hat{n} \cdot \vec{S})^2 \), with \( b > 0 \) and \( \hat{n} \) being a unity vector directed normally to the surface everywhere, can guarantee that the spins will not develop component out of the surface, provided that \( b \) is large enough (for further details, see, for example, Ref. [6]).

A question naturally arises: how is the scenario in the present surface? Although the pseudosphere is curved, it shares a special property with the plane: their metric are conformal each other[12] (the sphere, \( S^2 \), also belongs to this group). This is to say that the difference between them relies in measuring distance along geodesics in both supports, so that while on the plane we have straight lines as the least way between two distinct points, on the pseudosphere they are joined by a hyperbole. In order to simplify our analysis, we shall employ in this section the equivalence between the pseudosphere and the Poincaré disc, which is a disc of radius \( R \) endowed with a metric that yields a constant and negative curvature. Once our calculations are performed on this disc, where the centers of the vortices are better identified by using Cartesian coordinates, we move to pseudospherical coordinates for finding further properties of the solutions on the actual support.

Taking \( \lambda = -1 \) in the Hamiltonian (5) and its associated differential equations, the spins will tend to align only on the (internal) XY plane and equivalently on the surface of the Poincaré disc.
Now, in order to look for the simplest in-plane solution, we take \( \theta = \pi/2 \) and \( \partial_r \Phi = 0 \). For such a case, the differential equations (6)-(7) are highly simplified, leaving us with

\[
\partial^2_r \Phi = 0 \quad \Rightarrow \quad \Phi(\varphi) = Q \varphi + \varphi_0 ,
\]

where \( Q \) is the charge of the vortex (its vorticity) centered at the origin while \( \varphi_0 \) is a constant related to its global profile. The energy is easily calculated, and reads:

\[
E_v = E_c + J \int_0^{2\pi} d\varphi \int_\tau_a^{\tau_L} d\tau \frac{1}{\sinh(\tau)} (\partial_\varphi \Phi)^2 = E_c + 2\pi J Q^2 \ln \left( \frac{\tanh(\tau_L/2)}{\tanh(\tau_a/2)} \right) .
\]

Here, \( E_c \) is the vortex core energy which diverges in the continuum limit for in-plane solutions (for out-of-plane vortices, it can be estimated analytically likewise usual cases); \( \tau_a \) and \( \tau_L \) are the small and large scale cutoff parameters, related to the sizes of the inner core and outer region of the excitation, respectively, by \( a = R \tau_a/2 \) and \( L = R \tau_L/2 \). As it is well-known, at least in planar and conical geometries, the vortex energy logarithmically blows up as the vortex spreads without limit to infinite (an infrared-like divergence). What should be stressed is that in the present case such a divergence is naturally ruled out by means of the negative curvature of the geometrical support. Actually, as long as \( \tau_L \) is raised the vortex energy increases but goes asymptotically to:

\[
E_v|_{\infty} = +2\pi J Q^2 \ln (\cotanh(\tau_a/2)) ,
\]

which is finite (see Figure 2). To our knowledge this is the first time an infrared-like divergence associated with topological excitations is ruled out by geometrical properties. This fact will be related to another interesting result on the pseudosphere whenever two-vortex solutions are considered. For that, it is more convenient to work in Cartesian \( xy \) coordinates. Since the linear combination of solutions of \( \nabla^2 \Phi = 0 \) is another one, we take:

\[
\Phi_{2v} = Q_1 \arctan \left( \frac{y - y_1}{x - x_1} \right) + Q_2 \arctan \left( \frac{y - y_2}{x - x_2} \right) ,
\]

where \( Q_1 \) and \( Q_2 \) are the charges of the vortices centered at \((x_1, y_1)\) and \((x_2, y_2)\) on the Poincaré disc (or at the corresponding points on the pseudosphere). The energy of this configuration may be analytically evaluated, giving:

\[
3
\]

More precisely, we have that \( \nabla^2 \Phi = 0 \). In pseudospherical coordinates \((\tau, \varphi)\) the Laplacian reads:

\[
\nabla^2_{\tau \varphi} \Phi(\tau, \varphi) = \frac{4}{R^2} \left( \frac{1}{\sinh \tau} \partial_{\tau}(\sinh \tau \partial_{\tau}) + \frac{1}{\sinh^2 \tau} \partial^2_{\varphi} \right) \Phi(\tau, \varphi) .
\]

This reduces to Eq. (17) if \( \partial_{\tau} \Phi = 0 \). Equivalently:

\[
\nabla^2_{xy} \Phi = \frac{1}{4} \left( 1 - \frac{x^2 + y^2}{R^2} \right)^2 \left( \partial^2_x + \partial^2_y \right) , \quad \nabla^2_{r\varphi} = \frac{1}{4} \left( 1 - \frac{r^2}{R^2} \right)^2 \left( \frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial^2_{\varphi} \right) ,
\]

in Cartesian and polar coordinates, respectively.

Instead of evaluating an integral similar to that of Eq. (18) for two vortices explicitly, we may use the fact that \( \nabla \Phi_{2v} \) is analytic everywhere, except at the vortices centers, around which \( \Phi_{2v} \) is a multivalued function. By this method, the result of Eq. (21) is much easier and elegantly obtained. For details, see Ref. [15].
Figure 2: The energy of a single vortex (in units of $JQ^2$) as a function of its outer size $L$. Upper (dashed) curve illustrates the usual planar-like case, while the another concerns the present geometry. Note that, in our case, the energy goes asymptotically to $E_v|_{\infty} = 2\pi JQ^2 \ln 2 \approx 4.85 JQ^2$ (for $\tau_a = 2a/R = 1$). Therefore, its infrared-like divergence is cured in this framework.

$$E_{2v} = E_{v1} + E_{v2} - 2\pi JQ_1 Q_2 \ln \left( \frac{\tanh(\tau_d/2)}{\tanh(\tau_L/2)} \right),$$

where $E_{v1}$ and $E_{v2}$ are the energies of the isolated vortices (see Eq. (18)). The last term represents the effective potential $V_{\text{eff}}$ between the two vortices, separated by $d = R\tau_d/2$ (measured along the hyperbolic geodesic joining them), and $\tau_d \geq 2\tau_a$. This potential presents a remarkable property: as $\tau_d \to \infty$, then $V_{\text{eff}} \to 0$, which is constant. Therefore, vortices move without appreciable interaction whenever they are sufficiently apart one from another. [This should be contrasted to the strong logarithmic potential of usual planar-type cases]. For the case of a vortex-antivortex pair (for definiteness with $Q_1 = -Q_2 = +1$), the potential is attractive but it plays an effective role only for enough small distances. Therefore, even though a pair keeps tied at zero temperature, an arbitrary weak thermal excitation may be sufficient to dissociate it. This leads us to claim that a topological phase transition \cite{16} will take place at any temperature close to zero.

5 Conclusions and Prospects

Static and isotropic classical Heisenberg spins lying on the geometry of a pseudosphere does not support a stable soliton. Indeed, stability is verified if, for instance, we consider a punctured support (a pseudosphere with a hole, rendering it a non-simply connected feature) in which a fractional excitation interpolating between a null and $\pi/2$-soliton is obtained.

Taking into account the XY regime, we have seen that vortices also present an interesting property whenever compared to their usual counterparts: their energy no longer blows up as they spread to infinite. We have also pointed out some possible consequences of this fact for the two-vortex solution and related issues, like topological phase transition. [We expect that similar scenario would take place]
to the so-called ‘out-of-plane’ vortices, with some suitable modifications for taking into account their internal core energy]. Actually, we have seen that the cure of the infrared divergence associated to the vortex energy on the pseudosphere implies, for instance, that a pair of vortices no longer interacts through a confining potential at large distances. This fact indicates that the depairing of vortices may occur at any arbitrary low temperature. This is another example of how the geometry of the underlying support may affect this transition (a previous one was provided by the conical surface[6]). Besides of verifying this conjecture by means of numeric/simulation techniques, other prospects for future investigation include how solitons and vortices structure and dynamics are affected by defects, like holes and/or impurities, on this surface. In a more general framework it remains the issue of how topological phase transitions are sensitive to geometrical parameters, like curvature, torsion etc. Some additional light to this problem could be shed by studying the dissociation of magnetic vortices pairs on the sphere (this study has been carried out and the results will be communicated elsewhere[17]).

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