Fidelity Susceptibility as Holographic PV-Criticality

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Abstract

It is well known that entropy can be used to holographically establish
a connection between geometry, thermodynamics and information theory. In this paper, we will use complexity to holographically establish
a connection between geometry, thermodynamics and information theory. Thus, we will analyse the relation between holographic complexity,
fidelity susceptibility, and thermodynamics in extended phase space. We will demonstrate that fidelity susceptibility (which is the informational
complexity dual to a maximum volume in AdS) can be related to the ther-
modynamical volume (which is conjugate to the cosmological constant in
the extended thermodynamic phase space). Thus, this letter establishes
a relation between geometry, thermodynamics, and information theory,
using complexity.

Studies done on various different branches of physics seem to indicate that
the physical laws are informational theoretical processes, as they can be rep-
resented by the ability of an observer to process relevant information \cite{1} \cite{2}. However, in such a process it is important to know the amount of information
that can be processes, and hence, the concept of loss of information in a information theoretical process become physically important. This loss of information
in a process is quantified using the concept of entropy. It may be noted that even
the structure of spacetime can be viewed as an emergent structure which occurs
due to a certain scaling behavior of entropy in the Jacobson formalism \cite{3} \cite{4}. In fact, in this formalism general relativity is obtained by using this scaling behav-
ior of maximum entropy i.e., the maximum entropy of a region of space scales
with its area. This scaling behavior of maximum entropy is motivated by the physics of black holes, and it in turn motivates the holographic principle [5, 6]. The holographic principle equates the number of degrees of freedom in a region of space to the number of degrees of freedom on the boundary surrounding that region of space. The AdS/CFT correspondence is one of the most important realizations of the holographic principle [7]. It relates the supergravity/string theory in AdS spacetime to the superconformal field theory on the boundary of that AdS spacetime. It may be noted that AdS/CFT correspondence has been used to obtain quantitively the entanglement by using the concept of quantum entanglement entropy, and this has in turn been used to address the black hole information paradox [8, 9]. Thus, for a subsystem $A$ (with its complement), it is possible to define $\gamma_A$ as the $(d-1)$-minimal surface extended into the AdS bulk with the boundary $\partial A$, and the holographic entanglement entropy for this system can be written as [10, 11]

$$\text{Entropy}_A = \frac{\mathcal{A}(\gamma_A)}{4G_{d+1}}$$

where $G$ is the gravitational constant in the AdS spacetime, and $\mathcal{A}(\gamma_A)$ is the area of the minimal surface. This relation can be viewed as a connection between a geometrical quantity (minimal volume), to a thermodynamical quantity (entropy), and which in turn related to information theory (loss of information).

In this paper, we will analyse this correspondence further, and establish such correspondence between volume in AdS, the difficulty to process information, with a thermodynamical quantity.

In an information theoretical process, it is not only important to know how much information is retained in a system, but also how easy is it to obtain that information. Just as entropy quantifies loss of information, a new quantity called the complexity quantifies the difficulty to obtain that information. Now as the physical laws are thought to be informational theoretical processes, complexity (just like entropy) is a fundamental quantity. In fact, complexity has been used to analyse condensed matter systems [12, 13], molecular physics [14], and even quantum computational systems [15]. Recently studies done on black hole physics seem to indicate that complexity might be very important in understanding the black hole information paradox, and this is because the information may not be ideally lost in a black hole, however, as it would be impossible to reconstruct it from the Hawking radiation, it would be effectively lost [16, 17, 18]. It has been suggested that the complexity would be dual to a volume in the bulk AdS spacetime, [19, 20, 21, 22],

$$\text{Complexity} = \frac{V}{8\pi RG_{d+1}}.$$  \hspace{1cm} (2)

where $R$ and $V$ are the radius of the curvature and the volume in the AdS bulk. As there are different ways to define a volume in AdS, different proposals for complexity have been proposed. For a subsystem $A$ (with its complement), it is possible to use the volume enclosed by the same minimal surface which was used to calculate the holographic entanglement entropy, $V = V(\gamma)$ [23]. This quantity is usually denoted by $C$. However, we can also define the complexity using the maximal volume in the AdS which ends on the time slice at the AdS
boundary, $V = V(\Sigma_{\text{max}})$ [24]. It has been demonstrated that the complexity calculated this way is actually the fidelity susceptibility of the boundary CFT. So, this quantity is called as the fidelity susceptibility even in the bulk, and it is denote by $\chi_F$ [24]. The fidelity susceptibility of the boundary theory can be used for analyzing the quantum phase transitions [25, 26, 27]. Thus, like the holographic entanglement entropy, this establishes a connection between geometry and information theory. As we want to distinguish between these two quantities, we shall call $C$ as holographic complexity [23], and $\chi_F$ as fidelity susceptibility [24].

In this paper, we would like to demonstrate that this connection can be extended even to thermodynamics. So, just like holographic entanglement entropy was used to establish a connection between information theory, geometry, and thermodynamics, we will demonstrate that fidelity susceptibility also establishes a connection between information theory, geometry and thermodynamics. The missing part of this connection is the connection between fidelity susceptibility and thermodynamics. To establish this connection for a concrete example, let us now consider the Schwarzschild black holes in AdS backgrounds (SAdS4).

The metric is given by

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where

$$f = 1 - \frac{2M}{r} + \frac{r^2}{l^2} = \frac{(r - r_+)(r_+^2 + r r_+ + l^2 + r^2)}{l^2},$$

$$= 1 + \frac{r^2}{l^2} \left(1 - \frac{2\epsilon r^3}{l^3}\right).$$

Following the standard procedure, the black hole mass $M$ is related to the temperature, through the Wick rotation $\tau = it$ (this requires the resulting Euclidean geometry to be free from a conical singularity). Denoting the position of the horizon by the largest root of $f(r_+) = 0$, the mass, temperature and the entropy of the black hole can be expressed as [28]

$$M_{\text{BH}} = \frac{r_+}{2} \left(1 + \frac{r^2}{l^2}\right),$$

$$T_{\text{BH}} = \frac{l^2 + 3r^2}{4\pi l^2 r_+},$$

$$S_{\text{BH}} = \frac{A}{4G} = \frac{\text{vol}(S^3)}{4G - r_+^2}.$$

Since we plan to use perturbative calculations, we assume $M$ is small and for the the horizon, we take $r_+ \propto M$. We can define the horizon size in terms of temperature $T$ (we set $l = 1$) as follows:

$$r_+ = l \left(\frac{2}{3} T \pi l - \frac{1}{3} \sqrt{4T^2\pi^2l^2 - 3}\right)$$

note that temperature has a minimum located at $T_{\text{min}} = \sqrt{3}/2\pi l$.

We define a perturbation parameter $\epsilon \equiv M/l$, and analyse all the expression up to first order in $\epsilon$, i.e., we neglect any $O(\epsilon)$ contribution. Now we can
obtain a thermodynamical quantity, which can be viewed as a volume in the bulk AdS spacetime. It has been observed that when a charge or rotation are added to a AdS black hole, their behavior qualitatively becomes analogous to a Van der Waals fluid [29, 30]. This analogy between AdS black holes and a Van der Waals fluid becomes more evident in extended phase space, where the cosmological constant is treated as the thermodynamics pressure [31, 28]. Thus, it is important to study the extended phase space for a system. In this paper, we will use the extended phase space, and relate it to the fidelity susceptibility of a system. Thus, in the extended phase space, the cosmological constant $\Lambda$ is treated as the thermodynamic pressure $P = -\Lambda/8\pi = 3/8\pi l^2$, and the first law of black hole thermodynamics is written as $\delta M = T\delta S + V\delta P$. The thermodynamic volume is defined as the quantity conjugate to $P$,

$$P = \left(\frac{\partial M}{\partial V}\right)_{S,\ldots},$$  

(9)

where all other quantities like $S,\ldots$ are fixed. Thus, it is also possible to write the black hole equation of state using, $P = P(V,T)$, and compare it to the corresponding fluid mechanical equation of state. It may be noted that it is also possible to construct a quantity thermodynamically conjugate to pressure, and this quantity represents the thermodynamical volume. In this paper, we will demonstrate that this thermodynamical volume corresponds to the fidelity susceptibility, thus establishing the connection between thermodynamics, information theory, and geometry. So, now using metric (3), we observe that the thermodynamic volume can be written as $V = V(T,P)$, and this equation of state is given by the following expression,

$$V = \frac{1}{48} \left( T\pi - \sqrt{T^2\pi - 2P}\sqrt{\pi} \right)^3$$  

(10)

We plot (10) in Fig. 11. We plot $P$ as thermodynamic pressure, defined in Eq. (9) versus thermodynamic volume given by Eq. (10). This graph shows EoS of black hole for different isothermal lines when $T = \text{constant}$. Note that due to the EoS, temperature is always bounded to $T \geq \frac{1}{2\sqrt{3}}$. This graph demonstrates that the pressure initially increases with volume, and then after reaching a maximum volume it slowly decreases with the further increase in volume. We will now compare this behavior of thermodynamic pressure and its conjugate volume to the pressure and volume defined for different information theoretical complexity, and observe that this behaviour of thermodynamic pressure and volume matches the behavior of the pressure and volume defined from the fidelity susceptibility of the system.

In condensed matter physics, the fidelity susceptibility has been calculated for a many-body quantum Hamiltonian $H(\lambda) = H_0 + \lambda H_I$, where $\lambda$ is an external excitation parameter [23, 26, 27]. This Hamiltonian can be diagonalized by an appropriate set of eigenstates $|n(\lambda)\rangle$ and eigenvalues $E_n(\lambda)$, $H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle$. These eigenstates are usually taken as orthonormal basis for the Hilbert space for CFT system. Now if two states $\lambda, \lambda' = \lambda + \delta\lambda$ are close to each other, then it is possible to define the distance between two states as $F(\lambda, \lambda + \delta\lambda) = 1 - \delta\lambda^2 \chi_F(\lambda)/2 + O(\delta\lambda^3)$, where the fidelity susceptibility of the system is $\chi_F(\lambda)$ [23, 26, 27]. This quantity $\chi_F(\lambda)$ can be holographically
Figure 1: Graph of thermodynamical $P V$, given by Eqs. (9, 10).

Figure 2: Graph of $p_{fid} v_{fid}$, given by Eqs. (13, 12).
Figure 3: Graph of $p_{ent} v_{ent}$, given by Eqs. (18, 17).
calculated, and it is the equal to the informational complexity, when the volume is taken as the maximum volume in AdS, i.e., \( V = V(\Sigma_{\text{max}}) \) \[24\]. Now we can use this expression for any deformation of AdS, and here we will use it for SAdS\(_4\). So, we can write the volume term in this expression as

\[ V(\Sigma_{\text{max}}) = 4\pi \int_{r_+}^{r_\infty} \frac{r^2 dr}{\sqrt{f}} \] \(11\)

It may be noted that if we expand this expression in series, the zeroth order term is divergent even for pure AdS. So, we can define a fidelity volume \( v_{\text{fid}} \), by subtracting a volume term for pure AdS, \( V(\Sigma_{\text{max}})_{\text{AdS}} \), from a volume term for AdS black holes, \( V(\Sigma_{\text{max}})_{\text{SAdS}} \).

\[ v_{\text{fid}} = V(\Sigma_{\text{max}})_{\text{SAdS}} - V(\Sigma_{\text{max}})_{\text{AdS}}. \] \(12\)

To compare fidelity susceptibility with thermodynamics, we can use this fidelity volume \( v_{\text{fid}} \). It may be noted that in the extended phase space \[31, 28\], a thermodynamic volume was defined as the quantity thermodynamically conjugate to the thermodynamic pressure (which was obtained using the cosmological constant). Here we will use the same argument to obtain the pressure conjugate to the fidelity volume. So, we can define a new quantity, which we call as the fidelity pressure. This quantity will be defined to be thermodynamically conjugate to the fidelity volume,

\[ p_{\text{fid}} = -\frac{\partial M}{\partial M} - \frac{\partial v_{\text{fid}}}{\partial M}. \] \(13\)

We numerically plotted it in Fig. (2). This graph shows that there is a criticality in the \( p_{\text{fid}} v_{\text{fid}} \). This has the same form as the graph for the thermodynamic pressure and volume. This graph has been plotted using numerical integration to obtain \( v_{\text{fid}}, p_{\text{fid}} \). To obtain numerical solutions the cutoff parameter \( r_\infty \) we will set it equal to \( 1/\delta, \delta \ll 0.05 \) in the numerical computations. We numerically constructed an EoS for fidelity concept. To compare the results with thermodynamic description given in Fig. (1), we plotted the fidelity pressure and fidelity volume based on this EoS in the same isothermal regimes. It is observed that the fidelity pressure again increases with the fidelity volume till it reaching a maximum value. After reaching this maximum value, it reduces with further increase in the volume. Thus, the thermodynamics of black holes and the fidelity susceptibility seem to represent the same physical process. However, the fidelity susceptibility is well defined in terms of a boundary conformal field theory \[25, 26, 27, 24\], and this would in principle imply that the thermodynamics of black hole would be well defined in terms of boundary field theory. In fact, the fidelity susceptibility represents the difficulty to extract information from a process, so it is more important to understand the difficulty to extract information during the evaporation of a black hole, rather than the loss of information during the evaporation of a black hole. The fidelity volume measures this difficulty to extract information during the evaporation of a black hole. The fidelity pressure would be the quantity conjugate to this quantity, and would measure the flow of this quantity with the change in the mass of the black hole, during its evaporation. Thus, the fidelity volume and fidelity pressure can be important quantity which could be used to analyze such a process. It may be noted that
recent studied on black hole information have suggested that even though the information may not be actually lost in a black hole, it would be effectively lost, as it would be impossible to obtain it back from Hawking radiation [16, 17, 18]. This again seems to indicate the information paradox in a black hole should be represented by fidelity volume and fidelity pressure, and it is more important to understand the difficulty to recover the information which could be expressed in terms of these quantities.

As alternative definition for the informational complexity of the boundary theory have been made using a different definitions for the volume in the bulk AdS, we will also use this definition to compare it to thermodynamical volume. Thus, we will also use the volume enclosed by the same minimal surface which was used to calculate the holographic entanglement entropy, \( V = V(\gamma) \), and compare this to the thermodynamic volume [23]. Now for \( M \neq 0 \), the area integral for metric (3) is defined as

\[
\mathcal{A}(\gamma_A) = 2\pi \int_0^\theta_0 r \sin \theta \sqrt{r^2 + \frac{r'^2}{f}} d\theta
\]  

(14)

here \( \cos \theta_0 = \frac{\rho_0}{\sqrt{1 + \rho_0^2}}, \rho_0 \sim l \) The Euler-Lagrange equation for \( r = r(\theta) \) is given by,

\[
r'' = r'^2 \left( \frac{f'}{2f} - \frac{r' \cot \theta}{f r^2} \right) - r' \cot \theta + 2rf
\]

(15)

here prime denotes derivative with respect to the \( \theta \). So, the volume integral can now be written as

\[
V(\gamma) = 2\pi \int_0^\theta_0 d\theta \sin \theta \left( \int_{r^+}^{r(\theta)} \frac{r^2 dr}{\sqrt{f}} \right)
\]

(16)

To analyse the relation between the holographic complexity and thermodynamics, we define a volume as the entanglement volume \( v_{ent} \) and relate it to \( V(\gamma) \). In fact, just as the fidelity volume, we define this entanglement volume \( v_{ent} \) by subtracting this volume for pure AdS, \( V(\gamma)_{AdS} \), from this volume for AdS black hole, \( V(\gamma)_{SAdS} \),

\[
v_{ent} = V(\gamma)_{SAdS} - V(\gamma)_{AdS}.
\]

(17)

It may be noted that from the argument used in defining extended phase space [31, 28], fidelity pressure was defined to be thermodynamically conjugate to fidelity volume. So, using the same argument, we can define the entangled pressure \( p_{ent} \) as a new quantity thermodynamically conjugate to the entanglement volume,

\[
p_{ent} = -\frac{\partial M}{\partial v_{ent} \partial r^+}
\]

(18)

Now we numerically plot \( v_{ent} - p_{ent} \) for different values of temperature in Fig. 3. It may be noted that the entanglement pressure can get negative, and so we use the absolute value of the pressure, \( |p_{ent}| \) in such a plot. To obtain the numerical solution for holographic complexity, we use the the initial conditions
\( r(0) = \rho_0 \), and \( r'(0) = 0 \). We solve the Euler-Lagrange equation to find \( r(\theta) \), and obtain the holographic complexity.

Thus, we plot the \( v_{\text{ent}} - p_{\text{ent}} \), using numerical solutions for complexity pressure given by Eq. (18) and its conjugate volume. It may be noted that at peaks, we observed that \( \frac{d p_{\text{ent}}}{d v_{\text{ent}}} \to \infty \). It may be noted that this graph diverges at points, and this behavior is expected, as we are using the same minimal surface which was used to calculate the holographic entanglement entropy, and such divergences have been observed to occur in holographic entanglement entropy \([32, 33, 34, 35, 36]\). As we are using the same minimal surface, we would expect similar behavior for holographic complexity. It would be interesting to find an explicit relation between the holographic complexity and entanglement entropy, as both these quantities are defined using the same minimal surface. Such a relation could be used to define the holographic complexity of a boundary theory. It is expected that it would measure that the entanglement volume could be used to analyze the difficulty to extract information during a phase transition, and the entanglement pressure would indicate a holographic flow in such a quantity, when the geometry describing such a quantities changes holographically. It may be noted that this quantity does not resemble the behaviour of the thermodynamic volume and pressure. Thus, it would be more interesting to analyze the phase transition of a boundary theory using this quantity, after defining its boundary dual rather than analyzing the black hole information paradox. However, as fidelity susceptibility does resemble the behavior of thermodynamic volume and pressure, the fidelity susceptibility would be the quantity to use for studying the black hole information paradox.

Thus, we have plotted various quantities which are represented by different definitions of the volume in AdS, and the conjugate to these definitions of volume in AdS. For each of these cases, we plotted the \( PV \) graph for the same deformed AdS solution. Now we can compare the behavior of these different quantities using the graphs Figs. (2,3) and (1). It was observed that behavior of the \( PV \) graph for holographic complexity was totally different from the \( PV \) graphs for the thermodynamic volume and fidelity susceptibility. However, it was also observed that the \( PV \) graph obtained from fidelity susceptibility and thermodynamic volume had almost the same behavior. So, it was conclude that to the thermodynamical volume in extended phase and fidelity susceptibility represent the same physical quantity. The fidelity susceptibility can identified with informational complexity of the boundary theory, which is obtained geometrically using maximum volume in AdS. So, in this paper, informational complexity was related to the thermodynamic volume of a theory using the the maximum volume in AdS spacetime. Thus, the results of this paper established a connection between geometry, thermodynamics, and information theory. It would be interesting to investigate this relation further, and analyse it for other deformations of the AdS spacetime.

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