Solutions for a class of quasilinear Schrödinger equations with critical exponents term

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Abstract In this paper, we study a class of quasilinear Schrödinger equation of the form
\[-\varepsilon^2 \Delta u + V(x)u - \varepsilon^2 (\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u = \lambda|u|^{q-2}u + |u|^{2^*(2\alpha)-2}u, \quad \text{in} \mathbb{R}^N,\]
where \(\varepsilon > 0, \lambda > 0, q \geq 2, \alpha > 1/2\) are constants, \(N \geq 3\). By using change of variable and variational approach, the existence of positive solution which has a local maximum point and decays exponentially is obtained.

Keywords Critical exponent · Quasilinear Schrödinger equations · Concentration-compactness principle

Mathematics Subject Classification (2000) 35J10 35J20 35J25

1 Introduction

Let \(l\) and \(h\) be real functions of pure power forms, it is interesting to consider the existence of solutions to the following quasilinear Schrödinger equation
\[i\partial_t z = -\varepsilon^2 \Delta z + W(x)z - l(|z|^2)z - ke^2 \Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, N \geq 3, \quad (1.1)\]
where \(W(x)\) is a given potential, \(k\) is a real constant, \(\varepsilon > 0\) is a real parameter. It has many applications in physics according to different types of \(h\). For example, in [8], it was used in plasma physics with \(h(s) = s\), and was used in [19] to models the self-channeling of high-power ultrashort laser in matter with \(h(s) = (1+s)^{1/2}\). Readers can refer to [13,14] for references of more applications of it.

In this paper, we assume that \(h(s) = s^\alpha\), \(l(s) = \lambda s^{(q-2)/2} + s^{2\alpha/(\alpha-1)}\), where \(\lambda > 0, q \geq 2, \alpha > 1/2, 2^* = 2N/(N-2)\) are constants. If we consider standing waves solutions of

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the form \( z(x,t) = \exp(-iEt/\varepsilon)u(x) \), then \( z(x,t) \) satisfies equation (1.1) if and only if the function \( u(x) \) solves the equation

\[
-\varepsilon^2 \Delta u + V(x)u - k\alpha \varepsilon^2 (\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u
= \lambda |u|^{\alpha-2}u + |u|^{2^*(2\alpha)-2}u, \quad u > 0, x \in \mathbb{R}^N, \tag{1.2}
\]

where \( V(x) = W(x) - E \) is the new potential function.

In case \( k = 0 \), equation (1.2) is a semilinear elliptic equation which has been extensively studied in the past two decades. In recent years, the quasilinear case \( k \neq 0 \) arose a lot of interest to mathematical researchers. From [14], we know that the constant \( 2^*(2\alpha) > 2^* \) is thought to behave like the critical exponent to equation (1.2). When \( \alpha = 1 \) (i.e. \( l(s) = s \)) and \( \varepsilon = 1 \), this kind of problems with different types of nonlinearities \( l(s) \) at sub-"critical growth", i.e. at sub-\( 2^*(2\alpha) \) growth, have been widely studied, see [3,5,7,14] and so on.

In [13], by using a changing of variable, they transform the equation to a semilinear one, then the existence of solutions was obtained via variational methods under different types of potentials \( V(x) \). This method is significant and was widely used in the studies of this kind of problems. For general \( \alpha \), there are few results for this case, as far as we know, just [1,2,13,16]. In [13], for \( \alpha > \frac{1}{2} \), \( l(s) = \lambda s^{\frac{2^*}{2}} \) and \( 2 < p + 1 < 2^*(2\alpha) \), the existence of solution of equation (1.2) without critical term was obtained by using the method of Lagrange multiplier. In [16], the existence of at least one or sometimes two standing wave solutions for \( \alpha > \frac{1}{2} \) and \( l(s) = \mu f(x)s^{\frac{2^*}{2}} \) was obtained through fibreing method. Employing the change variable method just as [14], the authors of [1] obtained the existence of at least one positive solution for \( \alpha > \frac{1}{2} \) and general \( l(s) \) by using variational approach. Moreover, in [2], for \( V(x) = \lambda, \alpha > \frac{1}{2} \) and \( l(s) = s^{\frac{2^*}{2}} \), they obtained the unique existence of positive radial solution under some suitable conditions.

Problem at "critical growth", i.e. at \( 2^*(2\alpha) \) growth rate, was studied by Moameni [17] with a general \( \varepsilon > 0 \) and \( \alpha = 1 \). It was assumed in [17] that \( V(x) = 0 \) on an annule, which enable the avoidance of proving the compact embedding near the origin. In [15], the existence of radially symmetric solution was obtained for \( \varepsilon > 0 \) small enough. Other kinds of such problems at "critical growth" were studied in [6,15,21] and the references therein. But all these are studied for \( \alpha = 1 \), for general \( \alpha \), there is no results according to what we know.

In this paper, our aim is to study the existence of positive solutions of (1.2) with general \( \alpha > 1/2 \) and at \( 2^*(2\alpha) \) growth. Problem of (1.2) at \( 2^*(2\alpha) \) growth has two difficulties. Firstly, the embedding \( H^1(\mathbb{R}^N) \hookrightarrow L^{2^*(2\alpha)}(\mathbb{R}^N) \) is not compact, so it is hard to prove the Palais-Smale (PS) in short) condition. Secondly, even if we can obtain the compactness result of (PS) sequence, it is only holds at some level of positive upper bound, it is difficult for us to prove that the functional has such minimax level.

We assume that \( V(x) \) is locally Hölder continuous and

\[ (V) \quad \exists V_\alpha > 0 \quad \text{such that} \quad \min_{\alpha \in \mathbb{R}^N} V(x) = V_0 \quad \text{and} \quad \lim_{|x| \to \infty} V(x) = V_\alpha. \]

Note that assumption \( (V) \) allows zero be the minimum point of \( V(x) \). This is different to the assumptions on \( V(x) \) in [17].

Under assumption \( (V) \), we define a space

\[ X := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 < \infty \right\}, \]

with the norm \( ||u||_X^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(x) u^2 \).
For simplicity of notation, we let $m = 2\alpha$, $\bar{q} = q/m$ and $k\alpha = 1$ in this paper. Set

$$g(t) = \lambda |t|^{q-2}t + |t|^{2m-2}t \quad \text{and} \quad G(t) = \int_0^t g(s)ds.$$ 

We formulate problem (1.2) in variational structure in the space $X$ as follows:

$$I(u) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} (1 + m|u|^{2(m-1)})|\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} G(u).$$

Note that $I$ is lower semicontinuous on $X$, we define that $u \in X$ is a weak solution for (1.2) if $u \in X \cap L^\infty(\mathbb{R}^N)$ and it is a critical point of $I$.

Firstly, for an arbitrary $\varepsilon > 0$, we have:

**Theorem 1** Assume that $q \in (2m, 2^*m)$ and that condition (V) holds.

Case 1: $1 < m < 2$. Assume that one of the following conditions holds:

(i) $\bar{q} > 4 + \frac{2}{m}$ if $N = 3$;
(ii) $\bar{q} > 2 + \frac{2}{m}$ if $N = 4$;
(iii) $\bar{q} > \frac{1}{2} + \frac{2}{m}$ if $N = 5$;
(iv) $\bar{q} > 2$ if $N \geq 6$.

Case 2: $m \geq 2$. Assume that one of the following conditions holds:

(v) $\bar{q} > 5$ if $N = 3$;
(vi) $\bar{q} > 3$ if $N = 4$;
(vii) $\bar{q} > \frac{7}{2}$ if $N = 5$;
(viii) $\bar{q} > 2$ if $N \geq 6$.

Then for $\varepsilon > 0$ small enough, problem (1.2) has a positive weak solution $u_\varepsilon \in X \cap L^\infty(\mathbb{R}^N)$ with

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_X = 0, \quad \text{and} \quad u_\varepsilon(x) \leq C \exp\left(-\frac{\beta}{\varepsilon}|x - x_\varepsilon|\right).$$

where $C > 0$, $\beta > 0$ are constants, $x_\varepsilon \in \mathbb{R}^N$ is a local maximum point of $u_\varepsilon$.

Next, we consider the case $\varepsilon = 1$. We have the following result:

**Theorem 2** Assume that all conditions in Theorem 1 hold and that $\varepsilon = 1$, then problem (1.2) has a positive weak solution $u_1 \in X \cap L^\infty(\mathbb{R}^N)$.

This paper is organized as follows. In section 2, we first use a change of variable to reformulate the problem, then we modify the functional in order to regain the (PS) condition. In section 3, we prove that the functional satisfies the (PS) condition, this is a crucial job of this paper. Finally, in section 4, we prove the main theorems, which involves the construction of a mountain pass level at a certain high.
2 Preliminaries

Since $I$ is lower semicontinuous on $X$, we follow the idea in [5] and make the change of variables $v = f^{-1}(u)$, where $f$ is defined by

\[
\begin{align*}
  f(0) &= 0, \\
  f'(v) &= (1 + m|f(v)|^{2(m-1)})^{-1/2}, \text{ on } [0, +\infty); \\
  f(v) &= -f(-v), \quad \text{on } (-\infty, 0].
\end{align*}
\]

The above function $f(t)$ and its derivative satisfy the following properties (see [1,2,14]):

**Lemma 3** For $m > 1$, we have

1. $f$ is uniquely defined, $C^2$ and invertible;
2. $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
3. $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
4. $f(t)/t \to 1$ as $t \to 0$;
5. $|f(t)| \leq m^{1/2m}|t|^{1/m}$ for all $t \in \mathbb{R}$;
6. $\frac{1}{m} f(t) \leq f'(t) \leq f(t)$ for all $t > 0$;
7. $f(t)/\sqrt{t} \to m^{1/2m}$ as $t \to +\infty$;

According to [7] (see Corollary 2.1 and Proposition 2.2 in it), note that the embedding in Corollary 2.1 of [7] is also compact., we have:

**Lemma 4** The map: $v \mapsto f(v)$ from $X$ into $L^r(\mathbb{R}^N)$ is continuous for $1 \leq r \leq 2^* m$, and is locally compact for $1 \leq r < 2^* m$.

Using this change of variable, we rewrite the functional $I(u)$ to:

\[
J(v) = I(f(v)) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) - \int_{\mathbb{R}^N} G(f(v)).
\]

The critical point of $J$ is the weak solution of equation

\[
-\varepsilon^2 \Delta v + V(x) f(v) f'(v) = g(f(v)) f'(v), \quad x \in \mathbb{R}^N.
\]  

(2.1)

Now we define a suitable modification of the functional $J$ in order to regain the Palais-Smale condition. In this time, we make use of the method in [13].

Let $l$ be a positive constant such that

\[
l = \sup \{s > 0 : \frac{g(t)}{t} \leq \frac{V_0}{k} \text{ for every } 0 \leq t \leq s \}
\]

(2.2)

for some $k > \theta/(\theta - 2)$ with $\theta \in (2m,q)$. We define the functions:

\[
\begin{align*}
  \gamma(s) &= \begin{cases} g(s), & s > 0; \\
  0, & s \leq 0,
\end{cases} \\
  \varphi(s) &= \begin{cases} \varphi(s), & 0 \leq s \leq l; \\
  \frac{V_0}{k}, & s > l,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
p(x,s) &= \chi_R(x) \gamma(s) + (1 - \chi_R(x)) \varphi(s), \\
P(x,s) &= \int_{0}^{s} p(x,t)dt,
\end{align*}
\]

where $\chi_R$ denotes the characteristic function of the set $B_R$ (the ball centered at 0 and with radius $R$ in $\mathbb{R}^N$), $R > 0$ is sufficiently large and such that

\[
\min_{\partial B_R} V(x) < \min_{\partial B_R} V(x).
\]

By definition, the function $p(x,s)$ is measurable in $x$, of class $C$ in $s$ and satisfies:
Lemma 5

Assume that condition (V) holds and \( q \) in (2.3) is a solution of (1.2). In this section, we show that the functional
\[
J(v) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) - \int_{\mathbb{R}^N} P(x, f(v)).
\]

The corresponding functional of (2.3) is given by
\[
J(v) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) - \int_{\mathbb{R}^N} P(x, f(v)).
\]

For \( v \in X \), since
\[
|\nabla ((f(v))^m)|^2 = \frac{m^2 |f(v)|^{2(m-1)}}{1 + m |f(v)|^{2(m-1)}} |\nabla v|^2 \leq m |\nabla v|^2,
\]
we infer that \( |f(v)|^m \in X \). By Sobolev inequality, we have
\[
\|f(v)\|_{L^2} = \|f(v)|^m\|_{L^2^m}^{1/m} \leq C \|\nabla |f(v)|^{m}\|_{L^2}^{1/m} \leq C \|v\|^m_{L^2}.
\]

It results that \( f(v) \in L^{2^*(m)}(\mathbb{R}^N) \). Using interpolation inequality, we obtain that \( f(v) \in L^q(\mathbb{R}^N) \).

Thus \( J \) is well defined on \( X \). Let \((v_n) \subset X, v \in X \) with \( v_n \to v \) in \( X \). Then from Lemma 4, we infer that \( V(x) f^2(v_n) \to V(x) f^2(v) \) in \( L^1(\mathbb{R}^N) \) and that \( f(v_n) \to f(v) \) in \( L^q(\mathbb{R}^N) \). Thus \( J \) is continuous on \( X \). \( J \) is Gateaux-differentiable in \( X \) and the G-derivative is
\[
\langle J'(v), \phi \rangle = \varepsilon^2 \int_{\mathbb{R}^N} \nabla v \nabla \phi + \int_{\mathbb{R}^N} V(x) f(v) f'(v) \phi
\]
\[
- \int_{\mathbb{R}^N} P(x, f(v)) f'(v) \phi, \quad \forall \phi \in X.
\]

Then if \( v \in X \cap L^m(\mathbb{R}^N) \) is a critical point of \( \tilde{J} \), and \( v(x) \leq a := f^{-1}(l) \), \( \forall x \in B_R \), we have \( u = f(v) \in X \cap L^\infty(\mathbb{R}^N) \) (note that we have \( |u| \leq |v| \) and \( |\nabla u| \leq |\nabla v| \) by the properties of \( f \)) is a solution of (1.2).

3 Compactness of (PS) sequence

In this section, we show that the functional \( J \) satisfies (PS) condition, this is a crucial job, its proof is composed of four steps. Let \( S \) denotes the best Sobolev constant, we have

**Lemma 5** Assume that condition (V) holds and \( q \in (2m, 2^*m) \). Then \( J \) satisfies (PS) condition at level \( c_\varepsilon < \frac{\varepsilon^m}{m} e^{N/2} \).

**Proof** Let \((v_n) \in E \) be a (PS) sequence of \( \tilde{J} \) at level \( c_\varepsilon \), that is, \((v_n) \) satisfies:

\[
\tilde{J}(v_n) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_n)
\]
\[
- \int_{\mathbb{R}^N} P(x, f(v_n)) = c_\varepsilon + o(1),
\]
and

\[
\langle \tilde{J}'(v_n), \phi \rangle = \varepsilon^2 \int_{\mathbb{R}^N} \nabla v_n \nabla \phi + \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) \phi
\]
\[
- \int_{\mathbb{R}^N} P(x, f(v_n)) f'(v_n) \phi = o(1) \|\phi\|_X, \quad \forall \phi \in X.
\]
We divide the proof into four steps.

Step 1: the sequence \( \int_{B_R} (|\nabla v_n|^2 + V(x)f^2(v_n)) \) is bounded. Multiplying (3.1) by \( \theta \) (\( \theta \) is given in section 2) and using (p1)-(p2), we get

\[
\frac{\theta}{2} \epsilon^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{\theta}{2} \int_{\mathbb{R}^N} V(x)f^2(v_n)
\]

\[
\leq \int_{B_R} p(x,f(v_n))f(v_n) + \frac{\theta}{2k} \int_{B_R} V(x)f^2(v_n) + \theta c_\epsilon + o(1).
\]

On the other hand, by Hölder inequality, we consider a cut-off function \( \psi \) satisfying (3.2), we get

\[
\int_{\mathbb{R}^N} \epsilon^2 \left( 1 + \frac{m(m-1)f(v_n)}{1+m|f(v_n)|^{2(m-1)}} \right) |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x)f^2(v_n)
\]

\[
= \int_{\mathbb{R}^N} p(x,f(v_n))f(v_n) + \epsilon \|v_n\|_X \geq \int_{B_R} p(x,f(v_n))f(v_n) + o(1)\|v_n\|_X.
\]

Combining the above two inequalities, we get

\[
\left( \frac{\theta}{2} - m \right) \epsilon^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 + \left( \frac{\theta}{2k} - 1 \right) \int_{\mathbb{R}^N} V(x)f^2(v_n)
\]

\[
\leq \frac{\theta}{2} \epsilon^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 - \int_{\mathbb{R}^N} \epsilon^2 \left( 1 + \frac{m(m-1)f(v_n)}{1+m|f(v_n)|^{2(m-1)}} \right) |\nabla v_n|^2
\]

\[
+ \frac{\theta}{2k} \int_{\mathbb{R}^N} V(x)f^2(v_n) - \int_{\mathbb{R}^N} V(x)f^2(v_n)
\]

\[
\leq \theta c_\epsilon + o(1) + o(1)\|v_n\|_X. \tag{3.3}
\]

Since \( \theta > 2m \) and \( k > \frac{\theta}{2m} \), we get the conclusion from (3.3).

Step 2: for every \( \delta > 0 \), there exists \( R_1 \geq R > 0 \) such that

\[
\limsup_{n \to \infty} \int_{B_{R_1}} (|\nabla v_n|^2 + V(x)f^2(v_n)) < \delta. \tag{3.4}
\]

We consider a cut-off function \( \psi_{R_1} = 0 \) on \( B_{R_1} \), \( \psi_{R_1} = 1 \) on \( B_{2R_1} \), \( |\nabla \psi_{R_1}| \leq C/R_1 \) on \( \mathbb{R}^N \) for some constant \( C > 0 \). On one hand, taking \( \varphi = f(v_n)/f'(v_n) \), we compute \( \langle f'(v_n), \varphi \psi_{R_1} \rangle \) and get

\[
o(1)\|v_n\|_X = \int_{\mathbb{R}^N} \epsilon^2 \left( 1 + \frac{m(m-1)f(v_n)}{1+m|f(v_n)|^{2(m-1)}} \right) |\nabla v_n|^2 \psi_{R_1}
\]

\[
+ \int_{\mathbb{R}^N} \epsilon^2 \varphi \nabla v_n \nabla \psi_{R_1} + \int_{\mathbb{R}^N} V(x)f^2(v_n)\psi_{R_1}
\]

\[
- \int_{\mathbb{R}^N} p(x,f(v_n))f(v_n)\psi_{R_1}
\]

\[
\geq \int_{\mathbb{R}^N} \epsilon^2 |\nabla v_n|^2 \psi_{R_1} + \int_{\mathbb{R}^N} \epsilon^2 \varphi \nabla v_n \nabla \psi_{R_1}
\]

\[
+(1 - \frac{1}{R}) \int_{\mathbb{R}^N} V(x)f^2(v_n)\psi_{R_1}. \tag{3.5}
\]

On the other hand, by Hölder inequality,

\[
\left| \int_{\mathbb{R}^N} \varphi \nabla v_n \nabla \psi_{R_1} \right| \leq \frac{C}{R_1} \|\nabla v_n\|_{L^2(B_{R_1})} \|\varphi\|_{L^2(B_{R_1})}. \tag{3.6}
\]
Note that $\| \nabla v_n \|_{L^2(\mathbb{R}^N)}$ is bounded, and

$$
\| \phi \|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} f^2(v_n)(1 + m|f(v_n)|^{2(m-1)})
= \int_{\mathbb{R}^N} f^2(v_n) + m \int_{\mathbb{R}^N} |f(v_n)|^{2m},
$$

by (3.5), $\| \phi \|_{L^2(\mathbb{R}^N)}$ is bounded also. Therefore, it follows from (3.5)-(3.7) that

$$
\limsup_{n \to \infty} \int_{B_{2R_1}^c} (|\nabla v_n|^2 + V(x)f^2(v_n)) \leq \frac{C}{R_1}
$$

for $R_1$ sufficiently large, this yields (3.3).

Step 3, there exists $v \in X$ such that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} p(x, f(v_n)) f(v_n) = \int_{\mathbb{R}^N} p(x, f(v)) f(v).
$$

Firstly, by step 1, there exists $v \in X$ such that up to a subsequence, $v_n \to v$ weakly in $X$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. Since we may replace $v_n$ by $|v_n|$, we assume $v_n \geq 0$ and $v \geq 0$. By (3.4), for any $\delta > 0$, there exists $R_1 > 0$ sufficiently large such that

$$
\limsup_{n \to \infty} \int_{B_{2R_1}^c} (|\nabla v_n|^2 + V(x)f^2(v_n)) \leq k\delta.
$$

Therefore, by (p_2) we have

$$
\limsup_{n \to \infty} \int_{B_{R_1}^c} p(x, f(v_n)) f(v_n) \leq \limsup_{n \to \infty} \int_{B_{R_1}^c} \frac{V(x)}{k} f^2(v_n) \leq \delta,
$$

and by Fatou Lemma,

$$
\int_{B_{R_1}^c} p(x, f(v)) f(v) \leq \delta.
$$

Secondly, we prove that

$$
\int_{B_{R_1}^c} p(x, f(v_n)) f(v_n) \to \int_{B_{R_1}^c} p(x, f(v)) f(v).
$$

Then from this, (3.9)-(3.10), and the arbitrariness of $\delta$, we get (3.8). In fact, since $(v_n)$ is bounded in $X$, we have $(f(v_n))$ is bounded also. Thus there exists a $w \in X$ such that $f(v_n) \to w$ in $X$, $f(v_n) \to w$ in $L^r(B_{R_1})$ for $1 \leq r < 2^*$ and $f(v_n) \to w$ a.e. in $B_{R_1}$. According to (2.4), $(|f(v_n)|^m)$ is also bounded in $X$. By a normal argument, we have $|f(v_n)|^m \to |w|^m$ in $X$, $|f(v_n)|^m \to |w|^m$ in $L^r(B_{R_1})$ for $1 \leq r < 2^*$ and $|f(v_n)|^m \to |w|^m$ a.e. in $B_{R_1}$. Applying Lions’ concentration compactness principle [12] to $(|f(v_n)|^m)$ on $B_{R_1}$, we obtain that there exist two nonnegative measures $\mu, \nu$, a countable index set $K$, positive constants $\{|\nu_k|\}$, $\{|\nu_k|\}$, $k \in K$ and a collection of points $\{x_k\}$, $k \in K$ in $B_{R_1}$ such that for all $k \in K$,

(i) $\nu = |w|^{2m} + \sum_{k \in K} \nu_k \delta_{x_k}$;

(ii) $\mu = |\nabla (|w|^m)|^2 + \sum_{k \in K} \mu_k \delta_{x_k}$;

(iii) $\mu_k \geq S \nu_k^{2/2}$,
where $\delta_{x_k}$ is the Dirac measure at $x_k$, $S$ is the best Sobolev constant. We claim that $v_k = 0$ for all $k \in K$. In fact, let $x_k$ be a singular point of measures $\mu$ and $\nu$, as in [10], we define a function $\phi \in C^\infty_0(\mathbb{R}^N)$ by

$$
\phi(x) = \begin{cases} 1, & B_0(x_k); \\ 0, & \mathbb{R}^N \setminus B_{2\rho}(x_k); \\ \phi \geq 0, |\nabla \phi| \leq \frac{1}{\rho}, B_{2\rho}(x_k) \setminus B_\rho(x_k). \end{cases}
$$

where $B_\rho(x_k)$ is a ball centered at $x_k$ and with radius $\rho > 0$. We take $\phi = \phi f(v_n)/f'(v_n)$ as test functions in $\mathcal{D}(v_n), \phi)$ and get

$$
\int_{\mathbb{R}^N} e^2 \left( 1 + \frac{m(m-1)|f(v_n)|^{2(m-1)}}{1 + m|f(v_n)|^{2(m-1)}} \right) |\nabla v_n|^2 \cdot \phi \\
+ \int_{\mathbb{R}^N} e^2 \nabla v_n \cdot f(v_n)/f'(v_n) + \int_{\mathbb{R}^N} V(x) f^2(v_n) \phi \\
- \int_{\mathbb{R}^N} p(x, f(v_n)) f(v_n) \phi = o(1) \|v_n \phi\|_X. \quad (3.12)
$$

Then Lions’ concentration compactness principle implies that

$$
\int_{B_{R_1}} |\nabla f(v_n)|^m \phi \to \int_{B_{R_1}} \phi d\mu, \quad \int_{B_{R_1}} |f(v_n)|^{2m} \phi \to \int_{B_{R_1}} \phi d\nu. \quad (3.13)
$$

Since $x_k$ is singular point of $\nu$, by the continuity of $f$, we have

$$
f(v_n(x))|_{(B_{2\rho} \setminus \{x_k\})} \to \infty \quad \text{as } \rho \to 0.
$$

Thus

$$
1 + \frac{m(m-1)|f(v_n)|^{2(m-1)}}{1 + m|f(v_n)|^{2(m-1)}} = m - o(\rho).
$$

on $B_{2\rho}$ for $\rho$ sufficiently small. Then by (2.4), we get from (3.12) that

$$
\int_{B_{R_1}} e^2 \phi d\mu - \int_{B_{R_1}} \phi d\nu \\
= \lim_{n \to \infty} \int_{B_{R_1}} e^2 |\nabla f(v_n)|^m \phi - \int_{B_{R_1}} |f(v_n)|^{2m} \phi \\
\leq \lim_{n \to \infty} \int_{B_{R_1}} m e^2 |\nabla v_n|^2 \phi - \int_{B_{R_1}} |f(v_n)|^{2m} \phi \\
\leq \lim_{n \to \infty} \int_{B_{R_1}} e^2 \left( 1 + \frac{m(m-1)|f(v_n)|^{2(m-1)}}{1 + m|f(v_n)|^{2(m-1)}} \right) |\nabla v_n|^2 \phi \\
+ o(\rho) \int_{B_{R_1}} e^2 |\nabla v_n|^2 \phi - \int_{B_{R_1}} |f(v_n)|^{2m} \phi \\
\leq \lim_{n \to \infty} \int_{B_{R_1}} e^2 \nabla v_n \cdot f(v_n)/f'(v_n) + \int_{B_{R_1}} |f(v_n)|^q \phi \\
+ o(\rho) \int_{B_{R_1}} e^2 |\nabla v_n|^2 \phi + o(1) \|v_n \phi\|_X, \quad (3.14)
$$
We prove that the last inequality in (3.14) tends to zero as \( \rho \to 0 \). By Hölder inequality, we have

\[
\lim_{n \to \infty} \left| \int_{B_R} \nabla v_n \nabla \phi \cdot f(v_n) / f'(v_n) \right| \\
\leq \lim_{n \to \infty} \sup \left( \int_{B_R} |\nabla v_n|^2 \right)^{1/2} \cdot \left( \int_{B_R} |f(v_n) / f'(v_n)| \cdot |\nabla \phi|^2 \right)^{1/2}.
\]

(3.15)

Since \( |f(v_n) / f'(v_n)|^2 = f^2(v_n) + m|f(v_n)|^{2m} \), using Hölder inequality we have

\[
\lim_{n \to \infty} \int_{B_R} |f(v_n) / f'(v_n)| \cdot |\nabla \phi|^2 \\
\leq C \rho \left( \|w\|^2_{L^2(B_R)} + \|w\|^2_{L^2(B_R)} \right) \to 0
\]
as \( \rho \to 0 \). Thus we obtain that the right hand side of (3.15) tends to 0. On the other hand, since \( q \in (2m, 2^* m) \), by Lemma 3, we can prove that \( g(x, h(v_n))h(v_n) \phi \to g(x, w)w \phi \) in \( L^1(B_R) \) and \( f_{B_1} g(x, w)w \phi \to 0 \) as \( \rho \to 0 \). All these facts imply that the last inequality in (3.14) tends to zero as \( \rho \to 0 \). Thus \( \nu_k \geq \varepsilon^2 \mu_k \). This means that either \( \nu_k = 0 \) or \( \nu_k \geq \varepsilon^2 \mu_k \) by virtue of Lions’ concentration compactness principle. We claim that the latter is impossible. Indeed, if \( \nu_k \geq \varepsilon^2 \mu_k \) holds for some \( k \in K \), then

\[
c_k = \lim_{n \to \infty} \left\{ J(v_n) - \frac{1}{2m} \int_{\mathbb{R}^N} |f(v_n)|^{2m} \right\} \\
\geq \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^N} \left[ \frac{1}{2m} - \frac{1}{2} \right] |w|^m \right\} \\
\geq \left( \frac{1}{2} - \frac{1}{2^* m} \right) \int_{\mathbb{R}^N} |w|^m + \frac{1}{2^* m} S^{N/2} e^N \geq 1 \frac{N}{N} e^N S^{N/2},
\]

which is a contradiction. Thus \( \nu_k = 0 \) for all \( k \in K \), and it implies that \( \|f(v_n)\|_{L^{2m}(B_R)} \to \|w\|_{L^{2m}(B_R)} \). By the uniform convexity of \( L^{2m}(B_R) \), we have \( f(v_n) \to w \) strongly in \( L^{2m}(B_R) \). Finally, since \( \rho(x, f(v_n)) / f(v_n) \) is sub-\( (2^* m) \) growth on \( B_{2R} \setminus B_R \), we conclude that (3.11) holds. This proves (3.3).

Step 4: \( \nu_k \) is compact in \( X \). Since we have (3.3), the proof of the compactness is trivial. This completes the proof of the lemma. \( \square \)

4 Proof of main results

Before we prove Theorem 1, we will show firstly some properties about the change variable \( f \).

Lemma 6 Let \( f_1(v) = |f(v)|^m / v, v \neq 0 \) and \( f_1(0) = 0 \), then \( f_1 \) is continuous, odd, nondecreasing and

\[
\lim_{v \to 0^+} f_1(v) = 0, \quad \text{and} \quad \lim_{|v| \to +\infty} |f_1(v)| = \sqrt{m}.
\]

(4.1)
Proof. In fact, by (6) of Lemma 3, 
\[ f_1'(v) = v^{-2}(m|f(v)|^{m-2}f(v)f'(v)v - |f(v)|^{m}) \geq 0, \]
so \( f_1 \) is nondecreasing. By (4) of Lemma 3, \( f_1(v) \to 0 \) as \( v \to 0 \). Finally, according to Hospital Principle,
\[ \lim_{v \to +\infty} f_1(v) = \lim_{v \to +\infty} \frac{|f(v)|^m}{v} = \lim_{v \to +\infty} m|f(v)|^{m-2}f(v)f'(v) = \sqrt{m}. \]
This shows that (4.1) holds. \( \square \)

Lemma 7. There exists \( d_0 > 0 \) such that
\[ \lim_{v \to +\infty} (\sqrt{mv} - f^m(v)) \geq d_0. \]

Proof. Assume that \( v > 0 \). Since by (6) of Lemma 3, \( f(v) \leq mf'(v)v \), we have
\[
\sqrt{mv} - f^m(v) \geq \sqrt{mv} - mf^{m-1}(v)f'(v)v = \frac{\sqrt{1 + mf^{2(m-1)}(v)} - \sqrt{mv}f^{m-1}(v)}{\sqrt{1 + mf^{2(m-1)}(v)}} \sqrt{mv} \\
\geq \frac{\sqrt{mv}}{2(1 + mf^{2(m-1)}(v))} \geq \frac{f^m(v)}{4mf^{2(m-1)}(v)} = \frac{1}{4mf^{2(m-2)}(v)} := d(m,v). \tag{4.2}
\]
In the last inequality, we have used the fact that \( \sqrt{mv} \geq f^m(v) \) and that \( mf^{2(m-1)}(v) > 1 \) for \( v > 0 \) sufficiently large.

If \( 1 < m < 2 \), then \( d(m,v) \to +\infty \) as \( v \to +\infty \). If \( m = 2 \), then \( d(m,v) = 1/8 \). If \( m > 2 \), we claim that \( \sqrt{mv} - f^m(v) \to 0 \) as \( v \to +\infty \) is impossible. In fact, assume on the contrary, then using Hospital Principle, we get
\[
0 \leq \lim_{v \to +\infty} \frac{\sqrt{mv} - f^m(v)}{f^{2-m}(v)} = \lim_{v \to +\infty} \frac{\sqrt{m} - mf^{m-1}(v)f'(v)}{(2-m)f^{1-m}(v)f'(v)} \\
= \lim_{v \to +\infty} \frac{m}{(2-m)f^{1-m}(v)} \left( \frac{1}{\sqrt{m} \sqrt{1 + mf^{2(m-1)}(v)} + mf^{m-1}(v)} \right) \\
= \frac{1}{2(2-m)} < 0.
\]
This is a contradiction. Thus for all \( m > 1 \), there exists \( d_0 > 0 \) such that there holds
\[ \lim_{v \to +\infty} (\sqrt{mv} - f^m(v)) \geq d_0. \]
This completes the proof. \( \square \)
Lemma 8 We have 
(i) If $1 < m < 2$, then 
\[
\lim_{v \to +\infty} \frac{\sqrt{mv} - f^m(v)}{f^{2-m}(v)} = \frac{1}{2(2-m)}.
\]

(ii) If $m \geq 2$, then 
\[
\lim_{v \to +\infty} \frac{\sqrt{mv} - f^m(v)}{\log f(v)} \leq \left\{ \begin{array}{ll}
\frac{1}{m}, & m = 2; \\
0, & m > 2.
\end{array} \right.
\]

Proof Firstly, we prove part (i). According to (4.2) in Lemma 7, we have $\sqrt{mv} - f^m(v) \to +\infty$ as $v \to +\infty$. Thus by Hospital Principle, we get 
\[
\lim_{v \to +\infty} \frac{\sqrt{mv} - f^m(v)}{f^{2-m}(v)} = \lim_{v \to +\infty} \frac{\sqrt{m} - mf^{m-1}(v)f'(v)}{(2-m)f^{1-m}(v)f'(v)} = \frac{1}{2(2-m)}.
\]

Next, we prove part (ii). If there exists a constant $C > 0$ such that $\sqrt{mv} - f^m(v) \leq C$, then the conclusion holds. Otherwise, assume that $\sqrt{mv} - f^m(v) \to +\infty$ as $v \to +\infty$. Then again by Hospital Principle, we have 
\[
\lim_{v \to +\infty} \frac{\sqrt{mv} - f^m(v)}{\log f(v)} = \lim_{v \to +\infty} \frac{\sqrt{m} - mf^{m-1}(v)f'(v)}{f'(v)/f(v)} = \left\{ \begin{array}{ll}
\frac{1}{m}, & m = 2; \\
0, & m > 2.
\end{array} \right.
\]

This completes the proof. \[\square\]

To prove Theorem 1, it is crucial to prove that $J$ has the mountain pass level $c_\varepsilon < \frac{1}{\lambda_m}e^{\alpha N^{N/2}}$. Let us consider the following family of functions in $[3]$ 
\[
v_\varepsilon(x) = \frac{n(n-2)\alpha^2(n-2)/4}{\omega^2 + |x|^2(n-2)/2},
\]
which solves the equation $-\Delta u = u^{2^* - 1}$ in $\mathbb{R}^N$ and satisfies $\|\nabla v_\varepsilon\|_{L^2}^2 = \|v_\varepsilon\|_{L^2}^{2^*} = S^{N/2}$. Let $\omega$ be such that $2\omega < R$ and let $\eta_\varepsilon(x) \in [0, 1]$ be a positive smooth cut-off function with $\eta_\varepsilon(x) = 1$ in $B_{\omega}$, $\eta_\varepsilon(x) = 0$ in $B_R \setminus B_{2\omega}$. Let $v_\varepsilon = \eta_\varepsilon v_\varepsilon$. For all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $J(t_\varepsilon v_\varepsilon) < 0$ for all $t > t_\varepsilon$. Define the class of paths 
\[
\Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = t_\varepsilon v_\varepsilon \},
\]
and the minimax level 
\[
c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(t\gamma).
\]

Let $t_\varepsilon$ be such that 
\[
J(t_\varepsilon v_\varepsilon) = \max_{t \geq 0} J(tv_\varepsilon).
\]

Note that the sequence $(v_\varepsilon)$ is uniformly bounded in $X$, then if $J(t_\varepsilon v_\varepsilon) \to 0$ as $t_\varepsilon \to 0$, we are done; on the other hand, if $t_\varepsilon \to +\infty$, then $J(t_\varepsilon v_\varepsilon) \to -\infty$, which is impossible, so it remains to consider the case where the sequence $(t_\varepsilon)$ is upper and lower bounded by two positive constants. According to [3], we have, as $\varepsilon \to 0$, 
\[
\|\nabla v_\varepsilon\|_{L^2}^2 = S^{N/2} + O(\omega^{N-2}), \quad \|v_\varepsilon\|_{L^{2^*}}^{2^*} = S^{N/2} + O(\omega^N).
\]
Let \( a \in (0, \frac{c^{N-2}/2}{2m}) \), \( b \in (\frac{c^{N-2}/2}{2m}, +\infty) \) be such that \( t_\omega \in [a, b] \), \( \forall \omega \in (0, \omega_0) \), where \( \omega_0 > 0 \) small enough. By computing \( \frac{d}{dt} f(t_\omega v) = 0 \), we obtain \( t_\omega = \frac{c^{N-2}/2}{2m} + o(1) \). Let
\[
H(v) = -\frac{1}{2} V(x) f^2(v) + \frac{c}{q} |f(v)|^q - \frac{1}{2m} |\nabla q(v)|^2 + \frac{1}{2m} |f(v)|^2m,
\]
then by (4.4) and (4) of Lemma 3 for \( m > 1 \), we have
\[
\lim_{|v| \to +\infty} \frac{H(v)}{|v|^2} = 0, \quad \text{and} \quad \lim_{v \to 0} \frac{H(v)}{|v|^2} = -\frac{1}{2} V(x).
\]
Thus \( H(v) \) is sub-\((2^*)\) growth.

The following proposition is important to the computation of a mountain pass level \( \epsilon_c < \frac{1}{Nm} e^{\omega \sqrt{N}/2} \).

**Proposition 9** Under the assumptions of Theorem 7 there exists a function \( \tau = \tau(\omega) \) such that \( \lim_{\omega \to 0} \tau(\omega) = +\infty \) and for \( \omega \) small enough,
\[
\int_{\mathbb{R}^N} H(t_\omega v_\omega) \geq \tau(\omega) \cdot \omega^{-N-2}.
\]

**Proof** We divide the proof into three steps.

Step 1: we prove that
\[
\frac{1}{\omega^{N-2}} \int_{B_\omega} H(t_\omega v_\omega) \geq \tau_1(\omega)
\]
with \( \lim_{\omega \to 0} \tau_1(\omega) = +\infty \).

By the definition of \( v_\omega \), for \( x \in B_\omega \), there exist constants \( c_2 \geq c_1 > 0 \) such that for \( \omega \) small enough, we have
\[
c_1 \omega^{-(N-2)/2} \leq v_\omega(x) \leq c_2 \omega^{-(N-2)/2}.
\]
and
\[
c_1 \omega^{-(N-2)/2} \leq f^m(v_\omega(x)) \leq c_2 \omega^{-(N-2)/2}.
\]
On one hand, by (7) of Lemma 3 (4.3) and the continuity of \( V(x) \) in \( B_\omega \), there exists \( C_1 > 0 \) such that
\[
\int_{B_\omega} V(x) f^2(t_\omega v_\omega) \leq C_1 \omega^{N-2 \cdot \frac{N-2}{m}} = C_1 \omega^{\frac{2m-N}{m} \cdot (N-2)}.
\]
Similarly, there exists \( C_2 > 0 \) such that
\[
\int_{B_\omega} f^q(t_\omega v_\omega) \geq C_2 \omega^{\frac{2q-N}{m} \cdot (N-2)} = C_2 \omega^{\frac{2q-N}{m} \cdot (N-2)}.
\]
where \( q = q/m \). On the other hand, using Hölder inequality, we have
\[
\frac{1}{2m} \int_{B_\omega} \left[ (\sqrt{m} t_\omega v_\omega)^2 - (f^m(t_\omega v_\omega))^2 \right]
\leq \frac{1}{m} \int_{B_\omega} (\sqrt{m} t_\omega v_\omega)^2 - \sqrt{m} t_\omega v_\omega - f^m(t_\omega v_\omega)
\leq \frac{1}{m} \left( \int_{B_\omega} (\sqrt{m} t_\omega v_\omega)^{2/(2-1)} \right)^{2/(2-1)} \left( \int_{B_\omega} [\sqrt{m} t_\omega v_\omega - f^m(t_\omega v_\omega)]^{2} \right)^{1/2}.
\]
that in Theorem 1 holds, there exists a such that
\[
\frac{1}{2^m} \int_{B_\omega} \left[ (\sqrt{m} \omega v_\omega)^2 - (f^m(t_\omega v_\omega))^2 \right] \leq C_3 \omega^{N - (\frac{2}{N} - \frac{1}{2})N - 2} = C_3 \omega^{\frac{1}{3} \frac{2}{N} - \frac{1}{2}} (N - 2).
\]
(4.8)

Combining (4.5), (4.6) and (4.8), we have
\[
\frac{1}{\omega^{N - 2}} \int_{B_\omega} H(t_\omega v_\omega)
\geq -C_1 \omega^{\frac{2}{N} - \frac{1}{2} + 1(N - 2) + C_2 \omega^{\frac{2}{N} - \frac{1}{2} - 1(N - 2) - C_3 \omega^{-\frac{1}{2}}}} (N - 2) := \tau_1(\omega).
\]
It is obvious that \(\frac{2}{N} - \frac{1}{2} - 1 > -\frac{1}{m}\). No matter which one of conditions (ii)-(iv) in Theorem 1 holds, we all have \(\frac{2}{N} - \frac{1}{2} - 1 < -\frac{1}{m}\). It results that \(\tau_1(\omega) \to +\infty\) as \(\omega \to 0\).

Case 2: \(m \geq 2\). Note that for any \(\delta \in (0, m)\), \(\lim_{\omega \to +\omega} \log f^\delta(\omega) f^\delta(\omega) = 0\), we have \(\log f^\delta(\omega) \leq f^\delta(\omega)\) for \(\nu > 0\) large enough. Thus for \(\omega > 0\) small enough, from (4.7) and (ii) of Lemma 8 we get
\[
\frac{1}{\omega^{N - 2}} \int_{B_\omega} \left[ (\sqrt{m} \omega v_\omega)^2 - (f^m(t_\omega v_\omega))^2 \right] \leq C_3 \omega^{\frac{1}{3} \frac{2}{N} - \frac{1}{2} \nu^2} = C_3 \omega^{\frac{1}{3} \frac{2}{N} - \frac{1}{2} \nu^2}.
\]
(4.9)

Combining (4.5), (4.6) and (4.9), we have
\[
\frac{1}{\omega^{N - 2}} \int_{B_\omega} H(t_\omega v_\omega)
\geq -C_1 \omega^{\frac{2}{N} - \frac{1}{2} + 1(N - 2) + C_2 \omega^{\frac{2}{N} - \frac{1}{2} - 1(N - 2) - C_3 \omega^{-\frac{1}{2}}}} (N - 2) := \tau_1(\omega).
\]
Since \(m \geq 2\), we have \(\frac{2}{N} - \frac{1}{2} - 1 > -\frac{1}{2}(1 + \frac{2}{N})\). No matter which one of conditions (v)-(viii) in Theorem 1 holds, there exists a \(\delta = \delta(N, \bar{q}) > 0\) (depends on \(N\) and \(\bar{q}\) small enough such that \(\frac{2}{N} - \frac{1}{2} - 1 < -\frac{1}{2}(1 + \frac{2}{N})\). It results that \(\tau_1(\omega) \to +\infty\) as \(\omega \to 0\).

Case 1 and case 2 show that (4.3) holds.

Step 2: we prove that there exists \(C_4 > 0\) such that
\[
\frac{1}{\omega^{N - 2}} \int_{B_\omega} H(t_\omega v_\omega) \geq -C_4 \omega^{\frac{2}{N} - \frac{1}{2} + 1(N - 2)} := \tau_2(\omega).
\]
(4.10)

Note that for \(x \in B_{2\omega},\), we have
\[
v_\omega(x) \leq v^*_\omega(x) \leq C_2 \omega^{-\frac{N - 2}{2}}.
\]
(4.11)

Since \(\eta_{\omega}\) is a positive smooth cut-off function, without lost of generality, we may assume that \(\eta_{\omega}\) is such that
\[
\frac{1}{\omega^{N - 2}} \int_{B_\omega} |v_\omega|^2 \leq \int_{B_{2\omega}} V(x) f^2(v_\omega).
\]
Thus by (4.1) and (4.11), we have

\[
\begin{align*}
\frac{1}{2\omega^{N-2}} \int_{B_{2\omega}} H(t_0 v_{\omega}) 
\geq & -\frac{1}{2\omega^{N-2}} \int_{B_{2\omega}} V(x) f^2(t_0 v_{\omega}) - \frac{1}{2m\omega^{N-2}} \int_{B_{2\omega}} |\sqrt{mt} v_{\omega}|^{2^*} \\
\geq & -\frac{C_5}{\omega^{N-2}} \int_{B_{2\omega}} V(x) f^2(t_0 v_{\omega}) \\
\geq & -C_4 \omega^{N-2} \frac{2-N}{N-2} (N-2) = -C_4 \omega^{\frac{2-N}{2} - 1} (N-2),
\end{align*}
\]

where \( C_4 > 0, C_5 > 0 \) are constants. This shows that (4.10) holds.

Step 3: to conclude, let \( \tau(\omega) = \tau_1(\omega) + \tau_2(\omega) \), we have \( \tau(\omega) \to +\infty \) as \( \omega \to 0 \). This implies the conclusion of the proposition. \( \Box \)

Proof of Theorem 1

By (4) and (7) of Lemma 3, it is easy to verify that \( \bar{J} \) has the Mountain Pass Geometry. Lemma 5 shows that \( \bar{J} \) satisfies (PS) condition. We prove that \( \bar{J} \) has the mountain pass level \( c_{\epsilon} < \frac{1}{Nm} \epsilon^N S^N/2 \). Let

\[
F(t) = \frac{\epsilon^2}{2} \|\nabla (tv_{\omega})\|^2_2 - \frac{1}{2^*m} \|\sqrt{mt} v_{\omega}\|^{2^*}_{L^{2^*}}.
\]

Then we have

\[
F(t) \leq F(t_0) = \frac{1}{Nm} \epsilon^N S^N/2 + O(\omega^{N-2}), \quad \forall t \geq 0,
\]

where \( t_0 = \frac{\epsilon^{(N-2)/2}}{\sqrt{m}} \). By (4.12) and (9), we have

\[
\bar{J}(t_0 v_{\omega}) = F(t_0 v_{\omega}) - \int_{\mathbb{R}^N} H(t_0 v_{\omega}) 
\leq & \frac{1}{Nm} \epsilon^N S^N/2 + O(\omega^{N-2}) - \tau(\omega) \omega^{N-2} \\
< & \frac{1}{Nm} \epsilon^N S^N/2.
\]

This shows that \( \bar{J}(v) \) has a nontrivial critical point \( v_{\epsilon} \in X \), which is a weak solution of (2.3).

We prove that \( v_{\epsilon} \) is also a weak solution of (2.1). Firstly, we can argue as the proof of Proposition 2.1 in [18] to obtain that

\[
\lim_{\epsilon \to 0} \max_{x \in \partial B_R} v_{\epsilon}(x) = 0.
\]

Thus there exists \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \), we have \( v_{\epsilon}(x) \leq a := f^{-1}(l), \quad \forall |x| = R \), where \( l \) is given in (4.2). Secondly, we prove that

\[
v_{\epsilon}(x) \leq a, \quad \forall \epsilon \in (0, \epsilon_0) \text{ and } \forall x \in \mathbb{R}^N \setminus B_R.
\]

Taking

\[
\varphi = \begin{cases} 
(v_{\epsilon} - a)^+, & x \in \mathbb{R}^N \setminus B_R; \\
0, & x \in B_R.
\end{cases}
\]
as a test function in \( J(v_\varepsilon) \), \( \varphi = 0 \), we get

\[
e^2 \int_{\mathbb{R}^N \setminus B_R} |\nabla (v_\varepsilon - a)^+|^2 \\
+ e^2 \int_{\mathbb{R}^N \setminus B_R} \left( V(x) - \frac{p(x, f(v_\varepsilon))}{f(v_\varepsilon)} \right) f(v_\varepsilon) f'(v_\varepsilon) (v_\varepsilon - a)^+ = 0.
\]  (4.14)

By (p2), we have

\[
V(x) - \frac{p(x, f(v_\varepsilon))}{f(v_\varepsilon)} > 0, \quad \forall x \in \mathbb{R}^N \setminus B_R.
\]

Therefore, all terms in (4.14) must be equal to zero. This implies \( v_\varepsilon \leq a \) in \( \mathbb{R}^N \setminus B_R \). This proves (4.13). Thus \( v_\varepsilon \) is a solution of Problem (2.1).

To complete the proof, we deduce as the proof for Theorem 4.1 in [11] to obtain that \( v_\varepsilon \mid_{\partial B_R} \in L^\infty(B_R) \). Thus \( u_\varepsilon = f(v_\varepsilon) \in X \cap L^\infty(\mathbb{R}^N) \) is a nontrivial weak solution of (1.2). Finally, by Proposition 10 in the following, we have \( \lim_{\varepsilon \to 0} ||u_\varepsilon||_X = 0 \) and \( u_\varepsilon(x) \leq Ce^{-\frac{C}{\varepsilon} |x-x_0|^\beta} \).

This completes the proof. □

We prove the norm estimate and the exponential decay.

**Proposition 10** Let \( v_\varepsilon \in X \cap L^\infty(\mathbb{R}^N) \) be a solution of (2.7) and let \( u_\varepsilon = f(v_\varepsilon) \), then we have

\[
\lim_{\varepsilon \to 0} ||u_\varepsilon||_X = 0, \quad \text{and} \quad u_\varepsilon(x) \leq Ce^{-\frac{C}{\varepsilon} |x-x_0|^\beta},
\]

where \( C > 0, \beta > 0 \) are constants.

**Proof** Firstly, let \( x_0 \in B_R \) be such that \( V(x_0) = V_0 \). Define \( J_0 : X \to \mathbb{R} \) by

\[
J_0(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_0 f^2(v) - \int_{\mathbb{R}^N} G(f(v)).
\]

Let

\[
c_0 = \inf_{\gamma \in I_0} \sup_{r \in [0,1]} J_0(\gamma(r)),
\]

\[
I_0 = \{ v \in C([0,1], X) : \gamma(0) = 0, J_0(\gamma(1)) < 0 \}.
\]

Similar to the proof for estimate (2.4) in [13] (or Lemma 3.1 in [20]), we can show that \( c_\varepsilon \leq e^{Nc_0} + o(\varepsilon^N) \) by using the change of coordinates \( y = (x - x_0)/\varepsilon \). Arguing as for (3.3), and by virtue of this energy estimate, we obtain

\[
||v_\varepsilon||_X \leq \frac{\theta c_\varepsilon}{\min\left( \frac{(N-m)\varepsilon^2}{2}, \frac{(N-k)^2}{2\varepsilon^2} - 1 \right)} \leq \frac{2\theta c_0}{\theta - 2m} e^{\frac{N-k}{2\varepsilon^2}} + o(\varepsilon^{N-2})
\]

for \( \varepsilon > 0 \) sufficient small. Let \( u_\varepsilon = f(v_\varepsilon) \), then \( u_\varepsilon \neq 0 \). Note that \( |\nabla u_\varepsilon| \leq |v_\varepsilon| \) and \( |u_\varepsilon| \leq |v_\varepsilon| \), we get \( \lim_{\varepsilon \to 0} ||u_\varepsilon||_X = 0 \).

Secondly, similar to the proof for Theorem 4.1 in [11], we conclude that \( v_\varepsilon \in L^\infty(\mathbb{R}^N) \) and by [2], we have \( v_\varepsilon \in C^{1,\alpha}(B_R) \). Now let \( x_\varepsilon \) denote the maximum point of \( v_\varepsilon \) in \( B_R \) and let

\[
\sigma := \sup\{ s > 0 : g(t) < V_0 \text{ for every } t \in [0,s] \}.
\]

Then \( v_\varepsilon(x_\varepsilon) \geq f^{-1}(\sigma) \) for \( \varepsilon > 0 \) small. In fact, assume that \( v_\varepsilon(x_\varepsilon) < f^{-1}(\sigma) \) for some \( \varepsilon > 0 \) sufficiently small. According to the definition of \( l \) (see (2.2) and \( \sigma \), we have \( v_\varepsilon(x) \leq f^{-1}(l) < f^{-1}(\sigma) \) (note that \( k > 1 \) in (2.2)), \( \forall x \in \mathbb{R}^N \setminus B_R \). Thus

\[
V(x) - \frac{g(f(v_\varepsilon))}{f(v_\varepsilon)} > 0, \quad \forall x \in \mathbb{R}^N.
\]
Since $v_e = f^{-1}(u_e)$ is a critical point of $J_e$, we choose $\phi = f(v_e)/f'(v_e)$ as a test function in $(J'_e(v_e), \phi) = 0$ and get
\[
0 = e^2 \int_{\mathbb{R}^N} \left( 1 + \frac{m(m-1)|f(v_e)|^{2(m-1)}}{1 + m|f(v_e)|^{2(m-1)}} \right) |\nabla v_e|^2 \\
+ \int_{\mathbb{R}^N} V(x) f^2(v_e) - \int_{\mathbb{R}^N} g(f(v_e)) f(v_e)
\]
\[
= e^2 \int_{\mathbb{R}^N} \left( 1 + \frac{m(m-1)|f(v_e)|^{2(m-1)}}{1 + m|f(v_e)|^{2(m-1)}} \right) |\nabla v_e|^2 \\
+ \int_{\mathbb{R}^N} (V(x) - \frac{g(f(v_e))}{f(v_e)}) f^2(v_e).
\]

It turns out that all terms in the above equality must be equal to zero, which means that $v_e \equiv 0$, a contradiction.

Now let $w_e(x) = v_e(x_e + e^2 x)$, then $w_e$ solves the equation
\[-\Delta w_e + V(x_e + e^2 x)f(w_e) f'(w_e) = g(f(w_e)) f'(w_e), \quad x \in \mathbb{R}^N.
\]
Note that $\lim_{t \to 0} \frac{\int (f(t)/t)}{t} = 1$ by the properties of $f$ and that $w_e(x) \to 0$ as $|x| \to +\infty$, we have, there exists $R_0 > 0$ such that for all $|x| \geq R_0$,
\[
f(w_e(x)) f'(w_e(x)) \geq \frac{3}{4} w_e(x)
\] (4.15)
and
\[
g(f(w_e(x))) f'(w_e(x)) \leq \frac{V_0}{2} w_e(x).
\] (4.16)

Let $\phi(x) = M e^{-\beta |x|}$ with $\beta^2 < \frac{V_0}{4}$ and $M e^{-\beta R_0} \geq w_e(x)$ for all $|x| = R_0$. It is easy to verify that for $x \neq 0$,
\[
\Delta \phi \leq \beta^2 \phi.
\] (4.17)

Now define $\psi_e = \phi - w_e$. Using (4.15)-(4.17), we have
\[
\left\{\begin{array}{ll}
-\Delta \psi_e + \frac{V_0}{2} \psi_e \geq 0, & \text{in } |x| \geq R_0; \\
\psi_e \geq 0, & \text{in } |x| = R_0; \\
\lim_{|x| \to \infty} \psi_e = 0.
\end{array}\right.
\]

By the maximum principle, we have $\psi_e \geq 0$ for all $|x| \geq R_0$. Thus, we obtain that for all $|x| \geq R_0$,
\[
w_e(x) \leq \phi(x) \leq M e^{-\beta |x|}.
\]

Using the change of variable, we have that for all $|x| \geq R_0$,
\[
v_e(x) = w_e(e^{-1}(x - x_e)) \leq M e^{-\frac{\beta}{e} |x - x_e|}.
\]

Then by the regularity of $v_e$ on $B_R$ and note that $f(t) \leq t$ for all $t \geq 0$, we have
\[
u_e(x) \leq C e^{-\frac{\beta}{e} |x - x_e|}
\]
for some $C > 0$. This completes the proof. \(\square\)

**Proof of Theorem** We consider the following equation


\[
-\Delta u + V(x)u - k\alpha(x)|u|^{2\alpha - 2}u = \lambda|u|^{q - 2}u + |u|^{2(\alpha - 1)}u, \quad u > 0, x \in \mathbb{R}^N. \tag{4.18}
\]

Let \( y = \varepsilon x \) with \( \varepsilon \in (0, \varepsilon_0) \), \( \varepsilon_0 \) is given by Theorem [1], then we can transform (4.18) into

\[
-\varepsilon^2\Delta u + \bar{V}(y)u - k\alpha \varepsilon^2(\Delta(|u|^{2\alpha}))|u|^{2\alpha - 2}u = \lambda|u|^{q - 2}u + |u|^{2(\alpha - 1)}u, \quad u > 0, y \in \mathbb{R}^N. \tag{4.19}
\]

Here \( \bar{V}(y) = V(\frac{y}{\varepsilon}) \) still has the properties given in assumption (V). Thus according to Theorem [1] (4.19) has a positive weak solution \( u_\varepsilon(y) \) in \( X \cap L^\infty(\mathbb{R}^N) \), this implies that (4.18) has a positive weak solution \( u_1(x) = u_\varepsilon(\varepsilon x) \). \( \square \)

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