Ergodic-localized junctions in periodically-driven spin chains

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Quantum phases of matter have many relevant applications in quantum computation and quantum information processing. Current experimental feasibilities in diverse platforms allow us to couple two or more subsystems in different phases. In this letter, we investigate the situation where one couples two domains of a periodically-driven spin chain where one of them is ergodic while the other is fully localized. By combining tools of both graph and Floquet theory, we show that the localized domain remains stable for strong disorder, but as this disorder decreases the localized domain becomes ergodic.

One of the most intriguing aspects of physics is the nontrivial collective behavior of matter at low temperatures [1]. In contrast to classical phase transitions, quantum fluctuations can induce changes between different quantum phases of matter at temperatures close to the absolute zero [1]. For instance, quantum interference is responsible for the metal-to-insulator transition in the Anderson model of electrons moving in disordered potential [2]. Since its discovery, Anderson localization has been a paradigmatic phenomenon in condensed matter physics [3, 4] and it has had dramatic consequences. In one and two dimensions, uncorrelated disorder leads to a massive localization of all the energy states and the system behaves always as an insulator in the thermodynamic limit [2–4]. However, there are well-known exceptions to this behavior, including when the disorder has certain correlations [5–9], or if there is long-range hopping along the lattice [2, 10, 11]. Recently there has been an enormous amount of interest on the effects of interactions on localization properties of the states. The interplay between disorder and interactions can be exploited to avoid thermalization in manybody systems, which is referred to as manybody localization [12–15]. This phenomenon is closely related to the Anderson model in random graphs [16–21] and has been observed experimentally in trap ions [22], cold atoms [23, 24], and superconducting qubits [25, 26].

In recent years, there is an increasing interest in the exploration of localization properties of manybody systems under the effect of an external driving [27–32]. The drive can produce unexpected effects [33, 34] and states of matter that are absent in undriven systems such as discrete time crystals become available [35–37]. In addition, an external drive can suppress tunneling in a coherent way, which is referred to as coherent destruction of tunneling [38, 39]. This can be used to generate a nonequilibrium version of the Mott-insulator transition [40], which has observed in driven optical lattices [41]. Moreover, in contrast to undriven models, manybody systems can absorb energy from the external drive and so heat up to infinite temperature [42, 43]. This phenomenon is accompanied by the divergence of celebrated high-frequency expansions such as the Floquet Magnus expansion [44, 45]. Hence the stroboscopic dynamics cannot be described using a local Hamiltonian [42, 43].

In this letter, we investigate the interplay between driving and disorder in a spin system that can be realized currently in arrays of superconducting qubits [25, 46]. We consider a one-dimensional spin chain that is divided into two domains: half of the chain is disordered while the second half is driven. The total system can be thought of as an ergodic-localized (EL) junction where the drive leads to ergodicity in one domain, while disorder induces localization in the other one. In the context of undriven systems, this situation resembles the so-called manybody localization proximity effect, where a manybody-localized subsystem is coupled to a thermalized one [47, 48]. While in some cases the thermalized system becomes localized [47], there is numerical evidence of thermalization of the whole system [49]. Recently, the stability of the localized phase has been the focus of active theoretical [50–53] and experimental [54–57] research. The theoretical description of localization properties of a EL junction changes dramatically in the context of periodically-driven systems and there are nontrivial effects that do not appear in the undriven case [31, 33, 34]. In this work, we show that if one looks at the total system stroboscopically, the dynamics is generated by a highly non-local Hamiltonian. We provide a geometrical interpretation of the total system by using graph theory tools. When the disorder is weak, there is a proximity effect where the localized domain becomes unstable. The stability of the latter increases by increasing the disorder. The EL junction can be visualized as a graph with two clusters: the localized domain is a cluster with low connectivity sites, whereas the ergodic domain is highly connected. The proximity effect is related to an increasing connectivity between the two clusters. In terms of spins, this has nontrivial consequences, because the effective Hamiltonian contains long-range strings of Pauli matrices. We show that the connectivity of the graph is related to the participation ratio, that is, sites with high connectivity in the graph have also a high participation ratio.

We consider a one-dimensional spin chain with time-dependent coupling between nearest-neighbors given by the
Hamiltonian
\[ \hat{H}(t) = \frac{\hbar}{2} \sum_{l=1}^{L} h_l(t) \hat{Z}_l + \frac{\hbar}{2} \sum_{l=1}^{L-1} g_l(t)(\hat{X}_l \hat{X}_{l+1} + \hat{Y}_l \hat{Y}_{l+1}), \]
where \( \hat{X}_l, \hat{Y}_l, \hat{Z}_l \) are the usual Pauli matrices with \( h_l(t) \) and \( g_l(t) \) being the strengths of the transverse field and the spin-spin interaction, respectively. Current experimental feasibilities in arrays of superconducting qubits allow for a high degree of control of these [25, 46]. By using the GMON architecture [25, 46], one has control of the on-site energies \( h_l(t) \), that can be tuned to define different spatial profiles. Further, one can consider the case where the coupling strengths \( g_l(t) \) are modulated in time in certain domains of the array. Motivated by the current experimental possibilities, in this work we investigate what happens when a localized system couples to a driven system that is ergodic, to form an ergodic-localized (EL) junction.

Intuition tells us that if the drive is strong enough, it should overcome the effect of disorder and create delocalized states along the whole spin chain. In this case, there is a proximity effect: the ergodic system influences the localized one. Further, the disorder within one domain, influences the ergodic behavior of the other domain. To investigate this, we consider a partition of the lattice Eq. (1) into two domains with \( M \) sites each, such that \( L = 2M \). By using this representation, the total system (1) can be decomposed in terms of a localized system \( \hat{H}_{\text{Loc}} \) coupled to an ergodic “bath” \( \hat{H}_{\text{Erg}}(t) \), as follows
\[ \hat{H}(t) = \hat{H}_{\text{Loc}} + \hat{H}_{\text{Erg}}(t) + \hat{H}_{\text{int}}(t). \]
Here the interaction between the localized and ergodic domains is given by \( \hat{H}_{\text{int}}(t) = h_g(t)(\hat{X}_M \hat{X}_{M+1} + \hat{Y}_M \hat{Y}_{M+1}) \). In our case, however, the “bath” is a spin chain that is externally driven. We do not assume a priori, that the spin chain is in a thermal state, but consider a domain that becomes ergodic due to the drive [27, 42, 43].

Before we investigate the dynamics of the EL junction, let us discuss in detail the main features of the individual domains. We begin by defining the Hamiltonian of the localized domain as
\[ \hat{H}_{\text{Loc}} = \frac{\hbar}{2} \sum_{l=1}^{M} h_l \hat{Z}_l + \frac{\hbar g_0}{2} \sum_{l=1}^{M-1} (\hat{X}_l \hat{X}_{l+1} + \hat{Y}_l \hat{Y}_{l+1}), \]
which is a sublattice with sites \( l = 1, 2, \ldots, M \). For the purposes of this work, we are interested in the spatial dependence \( h_l = h \cos(2\pi l/M) + \Delta_l \). Here, \( \Delta_l \in [-W, W] \) denotes the disorder drawn from a uniform distribution with strength \( W \). We also consider a time-independent coupling \( g_0 \) between neighboring sites. Let us discuss the physics of the clean system, i.e., in the absence of disorder \( W = 0 \). As a consequence of the spatial profile of the on-site energies, the quantum states are localized between the energy branches \( E_{\pm}(l)/\hbar = \frac{1}{\sqrt{2}} \cos(\frac{\pi l}{2M}) \mp 2g_0 \), a phenomenon known as Bragg localization [58, 59]. \( E_{-}(l) \) and \( E_{+}(l) \) give us information about the classical (for long wavelength modes) and Bragg (short wavelength modes) turning points [58], respectively. However, when the disorder is stronger than the coupling \( g_0 \), the states are localized due to Anderson localization [2–4].

Let us now explore in detail the most relevant aspects of the ergodic domain. In particular, we will explain the subtle relation between parametric resonance [60, 61] and the emergence of ergodicity [27]. The Hamiltonian describing the ergodic domain of the spin chain (1) can be written as
\[ \hat{H}_{\text{Erg}}(t) = \frac{\hbar}{2} \sum_{l=M}^{L} h \cos\left(\frac{2\pi l}{M}\right) \hat{Z}_l + \frac{\hbar g(t)}{2} \sum_{l=M+1}^{L-1} (\hat{X}_l \hat{X}_{l+1} + \hat{Y}_l \hat{Y}_{l+1}), \]
with sites \( l = M, M + 1, \ldots, L \). Within this domain, we ignore the effects of disorder (\( W = 0 \)), and consider a time-dependent coupling \( g(t) = g_0 + g_1 \cos(\omega t) \) between the qubits. An external drive can lead to parametric resonance [60, 61]. In the continuum limit of the spin chain Eq. (4), by following the same procedure as in Ref. [59], one can obtain the classical Hamiltonian
\[ \mathcal{H}(x, k) = h \cos\left(\frac{2\pi x}{M}\right) - 2g(t) \cos k, \]
where the kinetic energy is nonlinear, and the particle moves in a cosine potential. From this we obtain the frequency of small oscillation \( \Omega^2_0 = 8\hbar^2g_0/M^2 \). This is the frequency of the periodic orbits surrounding the classical equilibrium position \( x = 3M/2 \) that is relevant to describe long-wavelength spin excitations with momentum \( k \approx 0 \). The phenomenon of parametric resonance appears when the driving frequency is
twice the frequency of small oscillations [60, 61]. One can immediately observe that the oscillation frequency \( \Omega_0 \) is inversely proportional to the system size \( M \). This means that for a large spin chain, one must drive with a very low frequency to achieve this resonance condition. In this regime, if the amplitude of the drive is strong, the classical Hamiltonian (5) becomes chaotic. The onset of chaos in this system can be understood in terms of conservation laws. In the absence of drive \( (g_1 = 0) \), the energy is conserved and all the orbits are periodic in phase space. The system has two degrees of freedom, position \( x \) and momentum \( k \). Therefore, due to conservation of energy, the system is effectively one-dimensional. This is not the case when the external drive is acting on the system, because the energy is not conserved. Due to the drive, the long-wavelength oscillations around the equilibrium position become unstable for \( g_1 > 0 \). When the drive is strong enough \( g_1 \approx g_0 \), the regular structures in phase space disappear and the system becomes fully chaotic [27].

Now we explore what happens when the localized and ergodic domains are coupled via the interaction Hamiltonian \( \hat{H}_{\text{int}}(t) \). One of the most natural questions to ask is to which extent the localized domain is stable when it is coupled to the ergodic one. This resembles a common situation in the theory of open quantum systems: a system is coupled to a thermal bath at a given temperature. In that context, one would expect that if the system-bath coupling is weak, and if the correlation time of the bath is very short, the system thermalizes [62]. Of course, there are some caveats in this argument arising from symmetries preventing thermalization [62]. Symmetries are responsible for level crossings in the spectrum, and the system fails to reach a diagonal ensemble in the long-time limit [27–30].

To unveil the dynamics of the EL junction, we invoke Floquet theory for time-periodic Hamiltonians [39]. This is a natural choice because the Hamiltonian (1) of the total system is periodic, that is, \( \hat{H}(t + T) = \hat{H}(t) \), where \( T = 2\pi/\omega \) is the period of the drive. We will now use the Floquet operator \( \hat{F} = \hat{U}(T) \), which is the evolution operator \( \hat{U}(t) \) in one period of the drive [33, 39, 44]. The most relevant information can be obtained by solving the eigenvalue problem \( \hat{F}|\Phi_\mu\rangle = e^{-i\epsilon_\mu T/\hbar}|\Phi_\mu\rangle \). The eigenvectors \( |\Phi_\mu\rangle \) are known as the Floquet states and \(-\hbar\omega/2 \leq \epsilon_\mu \leq \hbar\omega/2 \) are the quasienergies. At discrete times \( t_n = nT \), an initial state \( |\Psi(0)\rangle \) evolves stroboscopically as \( |\Psi(nT)\rangle = \hat{F}^n|\Psi(0)\rangle \). This motivates the use of the effective Hamiltonian \( \hat{H}_{\text{eff}} \), a generator of the stroboscopic dynamics \( \hat{F} = e^{-i\hat{H}_{\text{eff}}T/\hbar} \). It gives us important information of the effective interactions that appear due to the drive and it can be interpreted in terms of quantum simulation [31, 33, 34]. Nonetheless, it is a difficult task to obtain \( \hat{H}_{\text{eff}} \) analytically. As a matter of fact, there are high-frequency expansions [33, 44] that allow one to obtain analytical expressions up to a finite order in \( \omega^{-1} \). In this work we are interested in the low-frequency regime, where all the high frequency expansions are known to diverge [45]. Our approach to solve this problem is to numerically calculate the effective Hamiltonian by taking the logarithm of the Floquet operator \( -i\hat{H}_{\text{eff}}T/\hbar = \log_2 \hat{F} \). To do so, we map the Hamiltonian (1) to a quadratic fermionic Hamiltonian by using the Jordan-Wigner transformation [63]. We calculate the effective Hamiltonian in the fermionic representation and then apply the inverse Jordan-Wigner transformation to be able to write it in terms of spins. After this procedure, we obtain the effective Hamiltonian

\[
\hat{H}_{\text{eff}} = \hbar \sum_{l=1}^L M_{l,j}\hat{Z}_l + \hbar \sum_{l=1}^L M_{l,j} \left( \hat{X}_l \hat{O}_{l,j} \hat{X}_l + \hat{Y}_l \hat{O}_{l,j} \hat{Y}_l \right),
\]

where \( \hat{O}_{l,j} = \hat{Z}_{l+1} \cdots \hat{Z}_{l+j-1} \) are highly non-local terms weighted by the matrix elements \( M_{l,j} \) that appear due to the Jordan-Wigner strings [63]. All the information we are interested in is contained in the matrix \( \mathcal{M} \), which can be used to visualize the total system as a graph using TULIP5 [64], and further allows us to determine localization properties of the individual domains. In addition, one can represent the matrix as a graph, which allows us to unveil the formation of clusters and communities [65]. In so doing, we construct a adjacency matrix \( \mathcal{A} \) by following the rules \( \mathcal{A}_{l,j} = 0 \) and \( \mathcal{A}_{l,j} = 1 \) if \( |M_{l,j}| < C \), where \( C = 10^{-4}g_0 \) is a cutoff that we introduce to have a better visualization. We consider 1000 realizations of disorder and for each realization, we obtain a different adjacency matrix. Figure 1 shows the results for the ensemble-averaged adjacency matrix. There, one can recognize two domains with different behavior. When the disorder is moderate, \( W = g_0 \), there is a proximity effect and a region close to the interface becomes ergodic. When the disorder is strong, \( W = 5g_0 \), the
localized domain is stable and fails to become ergodic. Note that in general, as depicted in Figs. 1 (a) and (b), the connectivity of the graph is high in the ergodic domain and low in the localized one. In figure (c) there is a formation of two clusters in the graph. Due to the proximity effect, there is certain connectivity between these two clusters. In contrast, although figure (d) shows two clusters, the connectivity is low between them. One way to quantify the connectivity is to calculate the degree $K$ of a node, which is the number of links to other nodes in the graph [66]. As we consider 1000 realizations of disorder, we can calculate the probability distribution $P(K)$, which gives us the probability of a node to have a degree $K$ [66]. In Fig. 1 (c) one can see that this probability is unimodal in the case of moderate disorder and Fig. 1 (d) shows that it is bimodal in the case of strong disorder. The connectivity of the ergodic domain resembles the emergence of a giant component and percolation in the theory of random networks [66].

Now let us consider the localization properties of the Floquet states using the participation ratio [3, 25]. This quantity measures how localized is a state in a given basis. From now on, we work in the single-excitation subspace of the spin chain and choose a basis corresponding to spin states of the form $|l⟩ = |↓_1, ↓_2, \ldots, ↑_l, \ldots, ↓_L⟩$ for the total system. Any Floquet state can be decomposed as a quantum superposition of position states, as follows $|Φ_p⟩ = \sum_{l=1}^{L} c_{p,l}|l⟩$. From this decomposition one can see that the number of coefficients $c_{p,l}$ determines how extended is $|Φ_p⟩$ in the basis $|l⟩$. Conversely, one can also quantify how extended is a position state in the energy eigenbasis

$$PR(l) = 1/\sum_{l=1}^{L} |c_{p,l}|^4 . \quad (7)$$

The participation ratio measures how many states “participate” in a quantum superposition. One might be tempted to interpret the participation ratio as a localization length, but this is not correct. In fact, the participation ratio is commonly interpreted as the “Radius” of the wave function [3]. In contrast, the localization length has to do with the rate of spatial exponential decay of the state along the lattice and plays an important role in the theory of Anderson and manybody localization [3]. For single realization of disorder, Figs. 2(a) and (b) show the probability amplitudes $|c_{p,l}|^2$ as a function of the position $l$ and the quasienergies $e_p$ for moderate ($W = g_0$) and strong disorder ($W = 5g_0$), respectively. This figure nicely reflects the geometrical representation shown in figure 1. Figures 2(c) and (d) show the ensemble average [67] over 1000 realizations of disorder for $W = g_0$ and $W = 5g_0$, respectively. In addition to this, Figs. 3(a) and (b) show the participation ratio for a single realization of disorder and Figs. 3(c) and (d) the corresponding ensemble averages. The high connectivity between nodes in the ergodic domain is related to localization properties of the Floquet states. In the case of strong disorder, the localized domain remains stable in despite of being coupled to the ergodic one. For weaker disorder, the proximity effect is stronger and the states becomes delocalized. When the disorder of order $W < g_0$, most of the states become delocalized. Furthermore, the effective Hamiltonian is highly non-local and the clusters of the associated graph disappear, i.e., the graph becomes almost fully connected.

To summarize, in this letter we explored the situation where one couples two domains of a spin chain: one of them is ergodic due to the external drive and the other one is fully localized. The total system constitutes an ergodic-localized (EL) junction. By using Floquet theory, we show that the localized domain remains stable for strong disorder. When the disorder is decreased, there is a proximity effect and the localized domain becomes ergodic. We provide a geometrical interpretation of this phenomenon by representing the effective Hamiltonian as a graph. The connectivity of the associated graph reveals if certain domains of the system are localized or ergodic. This behavior can be quantified by using the degree distribution $P(K)$ that becomes unimodal when the localized domain is unstable. We anticipate that our work will open a new avenue of research and inspire the use of graph theory to unveil the dynamics of periodically-driven quantum systems. A possible application of our approach is to perform stroboscopic quantum simulation [31, 33, 34] of complex network topologies[68] by driving a system of qubits with a simple topology. Furthermore, our methodology should be generalizable to investigate periodic random circuits [69], ergodic systems [42, 43] and manybody localized states such as time crystals [35–37].

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