Alleviating $H_0$ Tension with New Gravitational Scalar Tensor Theories

Shreya Banerjee,1,* Maria Petronikolou,2,† and Emmanuel N. Saridakis2,3,4,‡

1Institute for Quantum Gravity, FAU Erlangen-Nuremberg, Staudtstr. 7, 91058 Erlangen, Germany
2National Observatory of Athens, Lofos Nymfou, 11522 Athens, Greece
3CAS Key Laboratory for Researches in Galaxies and Cosmology, Department of Astronomy, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China
4Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla 1280 Antofagasta, Chile

We investigate the cosmological applications of new gravitational scalar-tensor theories and we analyze them in the light of $H_0$ tension. In these theories the Lagrangian contains the Ricci scalar and its first and second derivatives in a specific combination that makes them free of ghosts, thus corresponding to healthy bi-scalar extensions of general relativity. We examine two specific models, and for particular choices of the model parameters we find that the effect of the additional terms is negligible at high redshifts, obtaining a coincidence with ΛCDM cosmology, however as time passes the deviation increases and thus at low redshifts the Hubble parameter acquires increased values ($H_0 \approx 74 \mathrm{km/s/Mpc}$) in a controlled way. The mechanism behind this behavior is the fact that the effective dark-energy equation-of-state parameter exhibits phantom behavior, which implies faster expansion, which is one of the sufficient conditions that are capable of alleviating the $H_0$ tension. Lastly, we confront the models with Cosmic Chronometer (CC) data showing full agreement within 1σ confidence level.

PACS numbers:
I. INTRODUCTION

Although the concordance ΛCDM paradigm is very successful in describing early- and late-time cosmological evolution at both background and perturbation levels, nevertheless the last years there have appeared some potential tensions with specific datasets, such as the $H_0$ and $\sigma_8$ ones. In particular, the estimation for the present Hubble parameter $H_0$ according to the Planck collaboration and assuming ΛCDM scenario is $H_0 = (67.27 \pm 0.60) \mathrm{km/s/Mpc}$ [1], which is in tension at about 4.4σ with the direct measurement of the 2019 SH0ES collaboration (R19), namely $H_0 = (74.03 \pm 1.42) \mathrm{km/s/Mpc}$, obtained using long-period Cepheids [2]. On the other hand, the $\sigma_8$ tension arises from the fact that the parameter that quantifies the matter clustering within spheres of radius $8h^{-1}\mathrm{Mpc}$, is found to be different from Cosmic Microwave Background (CMB) estimation [1] and from SDSS/BOSS measurement [3–5]. These tensions, and especially the $H_0$ one, progressively seem not to be related to unknown systematics, opening the road to many modifications of the standard lore [6, 7] (for a review see [8]).

One may follow two main ways to alleviate the $H_0$ tension. The first is to modify the universe content and/or particle interactions while keeping general relativity as the underlying gravitational theory [9–41]. The second way is to construct gravitational modifications, which applied to cosmological framework would lead to altered expansion rate [42–56, 58–69]. We mention here that modified gravity has additional advantages too, such as the improvement of the renormalizability behavior of general relativity as well as the description of inflationary and dark-energy phases, and thus it might be more preferable. Finally, there is another way to alleviate $H_0$ tension, in the framework of the running vacuum models [70], based on quantum field theory in curved spacetime [71–73], without the need to acquire phantom behavior (for a review of both the theoretical and phenomenological situation see [74] and references therein).

In the present work we are interested in alleviating the $H_0$ tension in the framework of new gravitational scalar-tensor theories [75–77]. In such constructions one uses Lagrangians with the Ricci scalar as well as its first and second derivatives, nevertheless in combinations that result to ghost-free theories. These theories are found to have $2 + 2$ propagating degrees of freedom, and thus falling outside Horndeski/Galileon [78–80] and beyond-Horndeski theories [81]. However, although they are bi-scalar extensions of general relativity, they were named “new gravitational scalar-tensor theories” since they can still be expressed in pure geometrical terms [75].

The plan of the work is the following: In Section II we briefly review the new gravitational scalar-tensor theories, and in Section III we apply them to a cosmological framework, extracting the modified Friedmann equations. Then, in Section IV we construct specific models that can alleviate the $H_0$ tension, and we compare the induced behavior to that of ΛCDM scenario as well as to Cosmic Chronometers (CC) data. Finally, in Section V we provide the conclusions.

---

*Electronic address: shreya.banerjee@fau.de
†Electronic address: petronikoloumaria@mail.ntua.gr
‡Electronic address: msaridak@noa.gr
II. OVERVIEW

In this section we give a brief overview of the gravitational scalar-tensor theories. The action of such constructions is given as \([75, 76]\)

\[
S = \int d^4\sqrt{-g} f \left( R, (\nabla R)^2, \Box R \right),
\]

with \((\nabla R)^2 = g^{\mu\nu}\nabla_\mu \nabla_\nu R\). In the following we set the Planck mass \(M_P = 1/\kappa = 1\), where \(\kappa\) is the gravitational constant, for simplicity. One can rewrite the above action by converting the Lagrangian using double Lagrange multipliers, resulting to actions of multi-scalar fields coupled minimally to gravity. In order to achieve it, one fixes the dependence of \(f\) on \(\Box R = \beta\).

In the present work, we consider theories with the following \(f\) form:

\[
f(R, (\nabla R)^2, \Box R) = K((R, (\nabla R)^2) + G(R, (\nabla R)^2)\Box R,
\]

thus maintaining a linear form in \(\Box R = \beta\). Generalizations to non-linear forms are straightforward, although more complicated. In this case, (1) transforms to

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{6} e^{-\sqrt{3}x} \hat{g}^{\mu\nu} \nabla_\mu \chi \nabla_\nu \phi + \frac{1}{e^{-\sqrt{3}x} K} \nabla_\mu \chi \nabla_\nu \phi \right],
\]

where \(K = K(\phi, B)\) and \(G = G(\phi, B)\), with \(B = 2e\sqrt{2}x g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi\). The \(\chi\) and \(\phi\) fields are introduced through the conformal transformations \(g_{\mu\nu} = \frac{1}{2} e^{-\sqrt{2}x} \hat{g}_{\mu\nu}\), \(\varphi \equiv f_{\beta}\), and they enter in a specific combination in a way that the final form of the action is equivalent to the original higher-derivative gravitational action.

Varying the action (3) with respect to the metric leads to the following field equations in Einstein frame \([75, 76]\):

\[
\mathcal{E}_{\mu\nu} = \frac{1}{2} G_{\mu\nu} + \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \chi \nabla_\beta \chi - \frac{1}{2} \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{6} \nabla_\mu \chi \nabla_\nu \phi
\]

\[
+ \frac{1}{2} g_{\mu\nu} \nabla_\alpha \chi \nabla_\beta \phi \nabla_\alpha \chi \nabla_\beta \phi
\]

\[
- \frac{1}{2} \sqrt{\frac{2}{3}} e^{-\sqrt{2}x} \hat{g}^{\mu\nu} \nabla_\mu \chi \nabla_\nu \phi
\]

\[
- \sqrt{\frac{2}{3}} \hat{g}^{\mu\alpha} \nabla_\alpha \chi \nabla_\beta \phi \nabla_\beta \phi
\]

\[
- \frac{1}{4} g_{\mu\nu} e^{-\sqrt{2}x} \hat{g} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \nabla_\mu \left( e^{-\sqrt{2}x} \hat{g} \nabla_\mu \phi \nabla_\nu \phi \right)
\]

\[
+ \frac{1}{2} \sqrt{\frac{2}{3}} \hat{g} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4} \nabla_\mu \left( e^{-\sqrt{2}x} \hat{g} \nabla_\mu \phi \nabla_\nu \phi \right)
\]

\[
+ \frac{1}{4} \nabla_\mu \left( e^{-\sqrt{2}x} \hat{g} \nabla_\mu \phi \nabla_\nu \phi \right)
\]

\[
+ \frac{1}{8} g_{\mu\nu} e^{-2\sqrt{2}x} \hat{g} K
\]

\[
+ \frac{1}{2} e^{-\sqrt{2}x} K B \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{8} g_{\mu\nu} e^{-\sqrt{2}x} \phi = 0.
\]

Additionally, varying (3) with respect to \(\chi\) and \(\phi\) gives rise to field equations as

\[
\mathcal{E}_\chi = \Box \chi + \frac{1}{3} e^{-\sqrt{2}x} \hat{g}^{\mu\nu} \nabla_\mu \chi \nabla_\nu \phi
\]

\[
- \frac{2}{3} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \phi \nabla_\phi g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi
\]

\[
+ \frac{1}{2} \sqrt{\frac{2}{3}} \nabla_\mu \left( e^{-\sqrt{2}x} \hat{g}^{\mu\nu} \nabla_\nu \phi \right)
\]

\[
- \frac{1}{2} \sqrt{\frac{2}{3}} e^{-\sqrt{2}x} \hat{g} \Box \phi + \sqrt{\frac{2}{3}} \hat{g} \nabla_\mu \phi \nabla_\nu \phi g^{\mu\nu} \Box \phi
\]

\[
- \frac{1}{2} \sqrt{\frac{2}{3}} e^{-2\sqrt{2}x} K + \frac{1}{2} e^{-\sqrt{2}x} K B \sqrt{\frac{2}{3}} \hat{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi
\]

\[
+ \frac{1}{4} \sqrt{\frac{2}{3}} e^{-\sqrt{2}x} \phi = 0,
\]

where for simplicity we have neglected the hats. Here, the subscripts in \(G\) and \(K\) denote the partial derivatives and the symmetrization is indicated by the parentheses in spacetime indices. The above equations reduce to GR for \(K = \phi/2\) and \(G = 0\), with the conformal transformation in this case being \(\chi = -\sqrt{\frac{2}{3}} \ln 2\). As we can see, the above equations do not contain any higher derivative terms, and therefore the present theory is well-behaved. Lastly, note that since we have set the Planck mass to one, the field \(\chi\) is dimensionless while \(\phi\) has dimensions of \(\sqrt{M}\).

III. COSMOLOGICAL BEHAVIOUR

We can now proceed to the study of the cosmological behaviour of the present model. For this we consider a
flat Friedmann-Robertson-Walker (FRW) metric
\[ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \] (7)

with \( a(t) \) the scale factor. We further assume that the two scalars are time-dependent only.

Including the matter sector, considered to correspond to a perfect fluid, the metric field equations (4) become
\[ \mathcal{E}_{\mu\nu} = \frac{1}{2} T_{\mu\nu}, \] (8)
with \( T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \) representing the matter energy-momentum tensor.

With the above substitutions into equations (4), we obtain the following Friedmann equations:
\[ 3H^2 - \rho_m - \frac{1}{2} \chi^2 + \frac{1}{4} e^{-2\sqrt{\chi} K} \]
\[ + \frac{2}{3} \phi^2 \left[ \phi \left( \sqrt{6} \chi - 9H \right) - 3 \dot{\phi} \right] G_B \]
\[ - \frac{1}{2} e^{-\sqrt{\chi} K} \left[ B \phi G_B + \frac{\phi}{2} + \dot{\phi}^2 (G_{\phi} - 2 K_B) \right] = 0, \] (9)

\[ 3H^2 + 2 \dot{H} + \rho_m + \frac{1}{2} \chi^2 + \frac{1}{4} e^{-2\sqrt{\chi} K} \]
\[ + \frac{1}{2} e^{-\sqrt{\chi} K} \left( - \frac{\phi}{2} + B \phi G_B + \frac{\phi^2}{2} G_{\phi} \right) = 0, \] (10)

with \( B(t) = 2e^{\sqrt{\chi} \phi\mu^\nu \nabla_\nu \phi \nabla_\mu \phi} = -2e^{\sqrt{\chi} \phi^2} \), and \( H = \dot{a}/a \) the Hubble parameter, where dots denoting differentiation with respect to \( t \). Similarly, the two scalar field equations (5) and (6) lead to:
\[ \mathcal{E}_\chi = \ddot{\chi} + 3H \dot{\chi} - \frac{1}{3} \phi^2 \left[ \phi \left( 3\sqrt{6} H - 2 \dot{\chi} \right) + \sqrt{6} \dot{\phi} \right] G_B \]
\[ + \frac{1}{2\sqrt{6}} e^{-\sqrt{\chi} K} \left[ B \phi G_B - \phi + 2 \dot{\phi}^2 (K_B + G_{\phi}) \right] \]
\[ + \frac{1}{\sqrt{6}} e^{-\sqrt{\chi} K} K = 0, \] (11)

and
\[ \mathcal{E}_{\phi} = \frac{1}{3} e^{-\sqrt{\chi} K} \left[ \phi \left( -9H + \sqrt{6} \dot{\chi} \right) - 3 \dot{\phi} \right] K_B \]
\[ + \frac{1}{6} B \left\{ 3e^{-\sqrt{\chi} K} B + 4 \dot{\phi} \left[ \phi \left( 9H - \sqrt{6} \dot{\chi} \right) + 3 \dot{\phi} \right] \right\} GB \]
\[ + \frac{1}{3} e^{-\sqrt{\chi} K} \left[ \phi \left( 9H - \sqrt{6} \dot{\chi} \right) + 3 \dot{\phi} \right] G_{\phi} \]
\[ + \left\{ e^{-\sqrt{\chi} K} B \phi + \frac{2}{3} \phi^2 \left[ \phi \left( 9H - \sqrt{6} \dot{\chi} \right) + 3 \dot{\phi} \right] \right\} G_{B\phi} \]
\[ - e^{-\sqrt{\chi} K} \phi^2 K_{B\phi} + \frac{4}{3} e^{-\sqrt{\chi} K} \phi^2 d \phi \phi - e^{-\sqrt{\chi} K} B \phi K_{BB} \]
\[ + \left[ \frac{4}{3} \phi B \right] \left( 9H - 2 \sqrt{6} \dot{\chi} \right) \phi - \frac{1}{\sqrt{6}} e^{-\sqrt{\chi} K} B \phi \]
\[ + \frac{\phi^2}{2} \left( 18H^2 + 6 \dot{H} - 3 \sqrt{6} \dot{H} + \frac{2}{3} H^2 + 3 \sqrt{6} \dot{\chi} \right) G_B \]
\[ - \frac{1}{4} e^{-2\sqrt{\chi} K} K + \frac{1}{4} e^{-\sqrt{\chi} K} = 0, \] (12)

with \( G_{B\phi} = G_{\phi B} = \frac{\rho_{DE}}{\rho_{DE} + \rho_m} \), etc.

The above Friedmann equations (9),(10) can be rewritten as
\[ H^2 = \frac{1}{3} (\rho_{DE} + \rho_m) \]
\[ 2\dot{H} + 3H^2 = -(\rho_{DE} + \rho_m), \]
with the effective dark energy and pressure defined as
\[ \rho_{DE} = \frac{1}{2} \chi^2 - \frac{1}{4} e^{-2\sqrt{\chi} K} \]
\[ - \frac{1}{2} \phi^2 \left[ \phi \left( \sqrt{6} \chi - 9H \right) - 3 \dot{\phi} \right] G_B \]
\[ + \frac{1}{2} e^{-\sqrt{\chi} K} \left[ B \phi G_B + \phi + \dot{\phi}^2 (G_{\phi} - 2 K_B) \right], \] (15)

\[ p_{DE} = \frac{1}{2} \chi^2 + \frac{1}{4} e^{-2\sqrt{\chi} K} \]
\[ + \frac{1}{2} e^{-\sqrt{\chi} K} \left( - \frac{\phi}{2} + B \phi G_B + \frac{\phi^2}{2} (G_{\phi} - \frac{\phi}{2}) \right). \] (16)

Hence, one can show that in the new gravitational scalar-tensor theories the effective dark-energy density satisfies
\[ \dot{\rho}_{DE} + 3H (\rho_{DE} + p_{DE}) = 0, \] (17)
while one can define the corresponding dark-energy equation-of-state parameter as
\[ w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}. \] (18)

## IV. HUBBLE TENSION

In this section we construct specific models of the theory in order to be able to alleviate the \( H_0 \) tension. We mention here that in modified gravity theories one typically has arbitrary functions, and thus she has a huge freedom to determine both their forms as well as their parameters. This freedom is similar to the freedom of choosing the arbitrary potentials in scalar-field cosmology. Hence, in the end of the day the obtained models are phenomenological, aiming to be in agreement with observations. In the theories examined in the present manuscript, we consider specific ansätze for the functions \( K(\phi, B) \) and \( G(\phi, B) \) and we select models that lead to higher Hubble function at low redshifts, while introducing negligible deviations in the Hubble parameter at high redshifts as compared to ΛCDM. The two phenomenological models with the best behavior related to the \( H_0 \) tension are presented in the following.

### A. Model I

As a first example we consider the following forms for \( K(\phi, B) \) and \( G(\phi, B) \):
\[ K(\phi, B) = \frac{1}{2} \phi - \frac{\phi}{2} B \quad \text{and} \quad G(\phi, B) = 0, \] (19)
with $\zeta$ a coupling constant with dimensions $[M]^{-4}$. The corresponding Friedmann equations (9),(10) read as

$$3H^2 - \rho_m - \frac{1}{2} \dot{\chi}^2 + \frac{1}{8} e^{-2\sqrt{2}\chi} \phi - \frac{1}{4} e^{-\sqrt{2}\chi} \left( \phi - \zeta \dot{\phi}^2 \right) = 0,$$

(20)

$$3H^2 + 2\dot{H} + p_m + \frac{1}{2} \chi^2 + \frac{1}{8} e^{-2\sqrt{2}\chi} \phi - \frac{1}{4} e^{-\sqrt{2}\chi} \left( \phi - \zeta \dot{\phi}^2 \right) = 0,$$

(21)

while the two scalar field equations (11) and (12) become

$$\dot{\chi} + 3H\chi + \frac{1}{2\sqrt{6}} e^{-2\sqrt{2}\chi} \phi - \frac{1}{2\sqrt{6}} e^{-\sqrt{2}\chi} \left( \phi - \zeta \dot{\phi}^2 \right) = 0,$$

(22)

$$\dot{\phi} + \frac{1}{3} \zeta \dot{\phi} \left( 9H - \sqrt{6} \chi \right) - \frac{1}{4} e^{-\sqrt{2}\chi} \frac{3}{H} + \frac{1}{2} = 0.$$  

(23)

The corresponding effective dark-energy energy density and pressure (15),(16) become

$$\rho_{DE} = \frac{1}{2} \dot{\chi}^2 - \frac{1}{8} e^{-2\sqrt{2}\chi} \phi + \frac{1}{4} e^{-\sqrt{2}\chi} \left( \phi - \zeta \dot{\phi}^2 \right),$$

(24)

$$p_{DE} = \frac{1}{2} \dot{\chi}^2 + \frac{1}{8} e^{-2\sqrt{2}\chi} \phi - \frac{1}{4} e^{-\sqrt{2}\chi} \left( \phi - \zeta \dot{\phi}^2 \right).$$

(25)

In order to obtain the behaviour of the Hubble parameter, we first set $z = -1 + a_0/a$, with the current value of the scale factor being set to $a_0 = 1$. It is well know that the behaviour of the Hubble parameter in $\Lambda$CDM cosmology is given by

$$H_{\Lambda\text{CDM}}(z) = H_0 \sqrt{\Omega_{m_0}(1+z)^3 + 1 - \Omega_{m_0}},$$

(26)

where $H_0$ is the present value of the Hubble parameter and $\Omega_{m_0}$ is the present value of matter density parameter defined as $\Omega_{m_0} = \frac{\rho_m}{3H^2}$ in Planck units. We set $\Omega_{m_0} = 0.31$ and $H_0 = 67.3 \text{km/s/Mpc}$. We then solve Eq. (20)-(23) numerically to obtain the solutions for the scale factor and hence for the Hubble parameter. In order to achieve this we set the initial conditions such that the evolution of $H(z)$ that we obtain for $z = z_{CMB} \approx 1100$ coincides with $H_{\Lambda\text{CDM}}$, namely $H(z \rightarrow z_{CMB}) \approx H_{\Lambda\text{CDM}}$ while $H(z \rightarrow 0) > H_{\Lambda\text{CDM}}(z \rightarrow 0)$. For our present analysis we have one model parameter, i.e. $\zeta$, which determines the late-time deviation of the model from $\Lambda$CDM scenario.

In Fig. 1 we plot the evolution of the dark-energy equation-of-state parameter $w_{DE}$ as a function of the redshift. As we can see from the figure, $w_{DE} < -1$ most of the time, thereby depicting phantom evolution which implies faster expansion. The phantom behavior is one of the mechanisms that can lead to the Hubble tension alleviation [82, 83] (see also the discussion in [8]), and as we will see in the following, this is exactly what happens.

In Fig. 2, we present the normalised combination $H(z)/(1+z)^{3/2}$ as a function of the redshift for $\Lambda$CDM cosmology (blue dotted line), and for Model I for $\zeta = -12$ (solid blue line), for $\zeta = -10$ (solid black line), and for $\zeta = -8$ (solid red line), in Planck units.

In Fig. 2, we present the normalised combination $H(z)/(1+z)^{3/2}$ as a function of the redshift for $\Lambda$CDM cosmology, and for Model I for different values of $\zeta$. Here we used $\zeta = -8, -10, -12$ in Planck units. We find that the present value of $H_0$ depends on the model parameter $\zeta$ as expected. For $\zeta = -10$, the present value of the
Hubble parameter is around \( H_0 \approx 74\text{km/s/Mpc} \), which is consistent with the direct measurement of the present Hubble parameter. Values of \( \zeta \) higher or lower than this lead to higher or lower values for \( H_0 \) respectively, and positive \( \zeta \) corresponds to \( H_0 \) values lower than the value of \( H_0 \) in \( \Lambda \)CDM scenario, thus they are not relevant for our present analysis. Note that in natural units \( \zeta \approx 10^{-10} \) corresponds to a typical value \( \zeta^{1/4} \approx 10^{-19} \text{GeV}^{-1} \). Hence, such values are the ones that needed in order to bring \( H_0 \) from its \( \Lambda \)CDM value to the local-measurement value, in other words the magnitude and the sign of the modified gravity modification is phenomenologically determined by the distance of \( H_0 = 67.3\text{km/s/Mpc} \) and \( H_0 \approx 74\text{km/s/Mpc} \).

For completeness, in Fig. 3 we depict the evolution of the deceleration parameter \( q \equiv -1 - H/H^2 \) as a function of the redshift. As we see, the redshift at which the transition from deceleration to acceleration occurs is around \( z_{tr} = 0.68 \), in agreement with current observations.

![Graph of deceleration parameter q as a function of redshift z for Model I with \( \zeta = -10 \) in Planck units](image)

In summary, as we observe, there exist a range of the free model parameter \( \zeta \) that is able to reproduce a Hubble function evolution that coincides with \( \Lambda \)CDM cosmology at high redshifts, but at late times it alleviates the \( H_0 \) tension. The reason that this happens is the fact that the effective dark-energy equation-of-state parameter exhibits a phantom behavior (following the general requirements of \([8, 83]\)).

\[ \begin{align*}
3H^2 - \rho_m &= \frac{1}{2} \dot{\chi}^2 + \frac{1}{8} e^{-2\sqrt{\frac{\phi}{\chi}}} \left( 1 - 2e^{\sqrt{\frac{\phi}{\chi}}} \right) \phi \\
&\quad + \xi \dot{\phi}^3 \left( \sqrt{6\dot{\chi}} - 6H \right) = 0, \\
3H^2 + 2\dot{H} + p_m &= \frac{1}{2} \dot{\chi}^2 + \frac{1}{8} e^{-2\sqrt{\frac{\phi}{\chi}}} \left( 1 - 2e^{\sqrt{\frac{\phi}{\chi}}} \right) \phi \\
&\quad - \frac{1}{3} \xi \dot{\phi}^2 \left( \sqrt{6\dot{\phi}} \dot{\chi} + 6\dot{\phi} \right) = 0,
\end{align*} \]

while the two scalar field equations (11) and (12) read as

\[ \begin{align*}
\ddot{\chi} + 3H \dot{\chi} + \frac{1}{2\sqrt{6}} e^{-2\sqrt{\frac{\phi}{\chi}}} \left( 1 - e^{\sqrt{\frac{\phi}{\chi}}} \right) \phi \\
&\quad - \sqrt{6} \xi \dot{\phi}^2 \left( H \dot{\phi} + \dot{\phi} \right) = 0, \\
\xi \dot{\phi} \left\{ 2 \left( -6H + \sqrt{6\dot{\chi}} \right) \dot{\phi} \\
&\quad + \dot{\phi} \left[ -6\dot{H} + 3H \left( -6H + \sqrt{6\dot{\chi}} \right) + \sqrt{6\dot{\chi}} \right] \right\} \\
&\quad + \frac{1}{8} e^{-2\sqrt{\frac{\phi}{\chi}}} \left( 1 - 2e^{\sqrt{\frac{\phi}{\chi}}} \right) \phi = 0.
\end{align*} \]

Therefore, in this case the effective dark-energy energy density and pressure (15),(16) write as

\[ \begin{align*}
\rho_{DE} &= \frac{1}{2} \dot{\chi}^2 - \frac{1}{8} e^{-2\sqrt{\frac{\phi}{\chi}}} \left( 1 - 2e^{\sqrt{\frac{\phi}{\chi}}} \right) \phi \\
&\quad - \xi \dot{\phi}^3 \left( \sqrt{6\dot{\chi}} - 6H \right), \\
p_{DE} &= \frac{1}{2} \dot{\chi}^2 + \frac{1}{8} e^{-2\sqrt{\frac{\phi}{\chi}}} \left( 1 - 2e^{\sqrt{\frac{\phi}{\chi}}} \right) \phi \\
&\quad - \frac{1}{3} \xi \dot{\phi}^2 \left( \sqrt{6\dot{\phi}} \dot{\chi} + 6\dot{\phi} \right).
\end{align*} \]

Let us now proceed to the numerical investigation of the above equations. Similarly to the previous Model I, we choose the initial conditions such that our scenario matches \( \Lambda \)CDM cosmology for \( z \approx 1100 \). In Fig. 4, we depict the evolution of the dark-energy equation-of-state parameter with the redshift. As in the case of the previous subsection, here we also see that \( w_{DE} < -1 \) for most redshifts, thereby depicting phantom evolution, thus serving as a mechanism for Hubble tension alleviation.
In Fig. 5, we present the normalised $H(z)/(1 + z)^{3/2}$ as a function of the redshift for $\Lambda$CDM cosmology, and for model II for different values of $\xi$, namely $\xi = -8, -10, -12$. As expected, we find that the present Hubble value $H_0$ depends on the model parameter $\xi$. Specifically, for $\xi = -10$ it is around $H_0 \approx 74$ km/s/Mpc, which is consistent with the directly measured value of the Hubble parameter. Values of $\xi$ higher or lower than this give higher or lower values for $H_0$ respectively. Note that in natural units $\xi \sim -10$ corresponds to a typical value $\xi^{1/8} \sim 10^{-19}$ GeV$^{-1}$. Hence, similarly to Model I above, such values are the ones that are phenomenologically needed in order to bring $H_0$ from $H_0 = 67.3$ km/s/Mpc to $H_0 \approx 74$ km/s/Mpc.

In Fig. 6, we depict the evolution of the deceleration parameter $q$ in terms of $z$. The transition redshift between deceleration and acceleration for this case is around $z_{tr} = 0.65$, in agreement with current observations, too.

We close our analysis by confronting the two examined models with Cosmic Chronometer (CC) cosmological data. This datasets is based on the $H(z)$ measurements through the relative ages of passively evolving galaxies and the corresponding estimation of $dz/dt$ [84]. In Fig. 7 we confront the predicted $H(z)$ evolution of our models, alongside the one of $\Lambda$CDM scenario, with the $H(z)$ Cosmic Chronometer Data [85] at 1$\sigma$ confidence level. As we deduce, the agreement is very good, and the theoretical $H(z)$ evolution lies within the direct measurements of the $H(z)$ from the CC data.

Figure 4: The effective dark-energy equation-of-state parameter $w_{DE}$ as a function of the redshift, for Model II for $\xi = -10$ in Planck units.

Figure 5: The normalized combination $H(z)/(1 + z)^{3/2}$ as a function of the redshift, for $\Lambda$CDM cosmology (blue dotted line), and for Model II for $\xi = -12$ (solid blue line), for $\xi = -10$ (solid black line), and for $\xi = -8$ (solid red line), in Planck units.

Figure 6: The deceleration parameter $q$ as a function of redshift $z$, for Model II with $\xi = -10$ in Planck units.
V. CONCLUSIONS

New gravitational scalar-tensor theories are novel modifications of gravity, consisting of a Lagrangian with the Ricci scalar and its first and second derivatives in a specific combination that makes the theory free of ghosts. Such constructions propagate 2+2 degrees of freedom, thus forming a subclass of bi-scalar extensions of general relativity.

In the present work we investigated the possibility of resolving the Hubble tension using these new gravitational scalar tensor theories. Considering a homogeneous and isotropic background, we extracted the Friedmann equations, as well as the evolution equations of the new extra scalar degrees of freedom. We obtained an effective dark energy sector that consists of both extra scalar degrees of freedom.

We then studied the cosmological behaviour of two specific models, imposing as initial conditions at high redshifts the coincidence of the behaviour of the Hubble function with that predicted by ΛCDM cosmology. However, we showed that as time passes, the effect of bi-scalar modifications become important and thus at low redshifts the Hubble function acquires increased values in a controlled way. In particular, the present value of the Hubble parameter is sensitive to the choice of the model parameters.

In both models we showed that at high and intermediate redshifts the Hubble function behaves identically to that of ΛCDM scenario, however at low redshifts it acquires increased values, resulting to \( H_0 \approx 74 \text{km/s/Mpc} \) for particular parameter choices. Hence, these new gravitational scalar tensor theories can alleviate the Hubble tension. The mechanism behind this behavior is the fact that the effective dark-energy equation-of-state parameter exhibits phantom behavior, which implies faster expansion, and it is one of the sufficient theoretical requirements that are capable of alleviating the Hubble tension [8, 83] (although it is not a necessary requirement as we mention in the Introduction). Finally, we further confronted our models with Cosmic Chromometer data and we found they are viable and in agreement with observations.

It would be interesting to investigate what is the situation of the other famous tension, namely the \( \sigma_8 \) one (there seems to be a disagreement between the amount of matter clustering, quantified by \( \sigma_8 \), predicted by ΛCDM cosmology and the local measurements of the matter distribution [8]) in the scenario at hand. In particular, a suggested solution for the Hubble tension does not guarantee an alleviation for the \( \sigma_8 \) one. There are models in which \( H_0 \) alleviation does impinge positively on the \( \sigma_8 \) tension, such as the running vacuum ones [71, 86, 87] or \( f(T) \) gravity ones [82, 88], however there are others in which it leads to a worsening of the latter. That is why it is necessary to perform a \( \sigma_8 \) analysis, too. Since such an analysis requires the investigation of perturbations and the evolution of matter overdensity \( \delta \), it is left for a separate project, however the obtained phantom behavior is expected to lead to an increase in the friction term in the Jeans equation for \( \delta \), which is qualitatively expected to lead to a smaller \( \sigma_8 \).

In conclusion, in this first work on the subject we deduced that the \( H_0 \) tension can be alleviated in the framework of new geometric gravitational theories. Definitely, the full verification of the above result requires a complete observational analysis, using data from Supernovae type Ia (SNIa), Baryonic Acoustic Oscillations (BAO), Redshift Space Distortion (RSD), and Cosmic Microwave Background (CMB) observations. Such a full and detailed observational confrontation, is left for a future project.

Acknowledgments

M. P. is supported by the Basic Research program of the National Technical University of Athens (NTUA, PEVE) 65232600-ACT-MTG: Alleviating Cosmological Tensions Through Modified Theories of Gravity. The authors would like to acknowledge the contribution of the COST Action CA18108 “Quantum Gravity Phenomenology in the multi-messenger approach”, as well as the contribution of the COST Action CA21136 “Addressing observational tensions in cosmology with systematics and fundamental physics (CosmoVerse)”. 
