The $CP(N-1)$ model on a Disc and Decay of a Non-Abelian String

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Abstract

We consider the role of quantum effects in the non-perturbative decay of non-abelian string with orientational moduli in non-supersymmetric D=4 gauge theory. To this aim the effective action in the $CP(N-1)$ model on a disc at large N has been calculated. It exhibits phase transition at some radius, the "wrong sign" Luscher term and large boundary boojum-like negative contribution. The effect of $\theta$-term and the possibility of the spontaneous creation of the non-abelian string are briefly discussed.
1 Introduction

The non-abelian strings \([1, 2, 3]\) during the last decade become the prominent element in the landscape of the world of non-perturbative extended objects in supersymmetric and non-supersymmetric \([4]\) gauge theories with the different matter content. When tension of the nonabelian string tends to infinity it is known under the name of the surface operator providing the proper boundary conditions on some hyperplanes. Unlike the Abrikosov string (abelian string) which possesses only 2 translation moduli, non-abelian strings possess additional internal orientational degrees of freedom corresponding to the nontrivial embedding of the string in the peculiar global group. In the simplest non-supersymmetric model \([4]\) the low energy dynamics of orientational moduli is described by the \(CP(N - 1)\) sigma model (see \([5]\) for a review).

In this note we shall focus on the peculiar issue concerning the decay of non-abelian string with orientational moduli. The decay of the abelian strings has been considered in the cosmological context \([6, 7]\) and in the specific gauge theories \([8]\). The internal moduli makes the decay process more involved. This process has been briefly discussed in \([9]\) in the semiclassical setting. Here we shall take into account the quantum effects in the worldsheet theory modifying some aspects of the decay considerably. The worldsheet \(CP(N - 1)\) sigma model can be studied via \(1/N\) expansion and possesses, for example, dynamical mass generation, asymptotic freedom, confinement, non-trivial \(\theta\)-angle dependence \([13, 12]\).

Recently it was shown that the theory manifests a phase transition upon some deformation \([15]\) or at the interval \([10]\). Moreover the theory defined at the interval shows Casimir-type scaling in the vacuum energy despite of the mass gap \([10]\). The sign of the Luscher term is unusual but coincides with the one in some other models with the mass gap \([11]\). Another interesting point to be mentioned is that there are negative edge contributions to the energy of the finite non-abelian string \([10]\).

The decay process from the worldsheet viewpoint is similar to the false vacuum decay in two dimensions. However there are interesting features in this case. The Euclidean bounce in 1+1 corresponds to the \(CP(N - 1)\) sigma model on the disc and it turns out that the model on the disc undergoes the phase transition similar to one found in \([10]\) in the strip geometry. Therefore the decay rate is sensitive to the radius of the bounce which is fixed by the parameters of the problem. Moreover in addition to the false vacuum decay situation we shall find in a large \(N\) approximation the Luscher-type term affecting the probability rate.

In the \(CP(N - 1)\) sigma model the bulk \(\theta\) term generates the topological term in the worldsheet theory \([4]\). One could also consider axions both in the bulk and worldsheet theory. It was found in \([14]\) that the worldsheet axion amounts to the
deconfinement in the worldsheet theory. We shall discuss the effects of the $\theta$-term and axion on the string decay rate.

The paper is organized as follows. In Section 2 calculate the effective action in $CP(N-1)$ sigma model on the disc and show some unusual properties. In particular we demonstrate the phase transition in the $CP(N-1)$ sigma model at some disc radius. In Section 3 we recall the simplest field theory model supporting non-abelian string. In Section 4 we derive the rate of the non-abelian string decay. We will also briefly discuss the decay induced by the kink-antikink meson living on the string worldsheet. In Section 5 we discuss the impact of the $\theta$-term and the axion on the decay process. The final remarks can be found in the Conclusion.

2 CP(N-1) model on the disc

In this Section we shall perform some preliminary work for the decay probability rate calculation and derive the effective action for the $CP(N-1)$ model on the disc geometry with Dirichlet boundary conditions. We assume the sharp boundary of the disc which is equivalent to the thin-wall approximation in the false vacuum decay.

Recall that $CP(N-1)$ sigma model is described by Lagrangian

$$\mathcal{L} = \frac{N}{g^2} (\partial_{\mu} - i A_{\mu}) n_i (\partial^{\mu} + i A^{\mu}) n^{*i} - \lambda (n_{i}^{*} n^{i} - 1)$$

where $i = 1, ..., N$ and $\lambda$ and $A_{\mu}$ are Lagrange multipliers. $\lambda$ impose the constraint $n_{i}^{*} n^{i} = 1$, $A_{\mu}$ is just a dummy field that could be eliminated by equation of motion $A_{\mu} = i n_{i}^{*} \partial_{\mu} n^{i}$ but makes U(1) invariance obvious. In what follows the calculations are made in Euclidean space. As was mentioned above, we impose Dirichlet boundary conditions:

$$n_{i} = 0, i = 2, ..., N; n_{1} = 1, \sqrt{x_{0}^2 + x_{1}^2} = R$$

General strategy for the large $N$ expansion is to integrate over $n_{i}$ and then use the saddle-point approximation which is justified by the large $N$. In the leading order we could neglect the difference between $N$ and $N - 1$. Also we will assume that in the saddle-point $A_{\mu} = 0$ and $\lambda = const$. After the appropriate rescaling:

$$S_{eff} = N \log det (-\partial^2 + m^2) - \int d^2 x \lambda$$

where $m^2 = \frac{\lambda g^2}{N}$. In $\mathbb{R}^2$ the determinant can be calculated easily and the saddle-point
equation \( \frac{\partial S_{\text{eff}}}{\partial m} = 0 \) yields via the dimensional transmutation the IR scale

\[
m = \Lambda_{uv} \exp \frac{-2\pi}{g^2}
\]  

(2.4)

Calculating the determinant in the disc geometry is more complicated since explicit expression for eigenvalues involves roots of Bessel functions. Instead, we will use Gelfand-Yaglom method, which could be generalized for a multi-dimensional case. We can separate:

\[
\log \det(-\partial^2 + m^2) = \log \det(-\partial^2) - \log \frac{\det(-\partial^2)}{\det(-\partial^2 + m^2)}
\]

(2.5)

log \( \det(-\partial^2) \) - is irrelevant constant.

According to [23], in two dimensions with Dirichlet boundary conditions:

\[
\log \frac{\det(M + m^2)}{\det(M_{\text{free}} + m^2)} = \log \frac{\psi_0(R)}{\psi_{0, \text{free}}(R)} + 2 \sum_{l=1}^{\infty} \left( \log \frac{\psi_l(R)}{\psi_{0, \text{free}}(R)} - \frac{1}{2l} \int_0^R r V(r) \, dr \right) + \int_0^R r V(r) \left( \log \frac{\mu r}{2} + \gamma \right)
\]

(2.6)

where \( M \) - is Schrodinger-like operator

\[
M = -\partial^2 + V(r)
\]

(2.7)

\[
M_{\text{free}} = -\partial^2
\]

(2.8)

and

\[
(-\partial^2_r + \frac{l^2}{r^2} + V(r) + m^2)\psi_l(r) = 0, \quad \psi_l(r) \approx r^l, \text{ as } r \to 0.
\]

(2.9)

\( \psi_{l, \text{free}} \) is defined similarly with the \( V(r) \) omitted. \( \gamma \approx 0, 577... \) - Euler-Mascheroni constant and \( \mu \)-is renormalization scale - that is (2.6) is free of divergences. \( \gamma + \log 1/2 \) could be eliminated through the choice of \( \mu \). Actually, (2.6) is derived using zeta-function regularization for slightly modified determinant \( \det(\cdots) \) which is equivalent to renormalization at scale \( \mu \). That is we should write \( \frac{N\pi(mR)^2}{g_\mu^2} \) instead of \( \frac{N\pi(mR)^2}{g^2} \)
In our case \( V(r) = -m^2 \), hence

\[
\frac{\psi_l(r)}{\psi_{l}^{\text{free}}(r)} = \frac{(mr)^l}{I_l(mr)!!2^l}
\]  

(2.10)

where \( I_l(r) \) - is Infeld function (modified Bessel function of second kind).

Therefore we can represent (2.5) in the following form:

\[
\log(I_0(mR)) + 2 \sum_{l=1}^{\infty} \left( \log \frac{I_l(mR)!!2^l}{(mR)^l} - \frac{(mR)^2}{4l} \right) + \frac{(mR)^2}{2} \log(\mu R) - \frac{(mR)^2}{4}
\]

(2.11)

and consider two limiting cases: \( mR >> 1 \) and \( mR << 1 \). The later is more simple, because Taylor expansion of \( I_l \) in the vicinity of 0 is valid for all \( l \) if \( mR << 1 \), while WKB-asymptotics \( I_l \approx e^x \) is valid only for \( mR >> l^2 \) which, of course, is not true for sufficiently large \( l \).

At small \( mR = x. \) the effective action reads as

\[
S_{\text{eff}} = N \left( -\frac{x^2}{2} - x^4 \left( \frac{\pi^2 \log \pi R + O(x^6)}{96} \right) + \frac{x^2 \mu R + O(x^6)}{2} - \frac{N \pi R^2}{g_\mu^2} \right)
\]

(2.12)

and

\[
\Lambda = \mu \exp \left( -\frac{2\pi}{g_\mu^2} \right)
\]

(2.13)

is dynamically generated scale. The saddle point equation with account of (2.13) yields

\[
2x^2 \left( \frac{\pi^2 - 9}{96} + \frac{1}{64} \right) = -\frac{1}{2} + \frac{1}{2} \log \Lambda R
\]

(2.14)

This equation does not have solutions for the sufficiently small \( R \) which indicates the phase transition at \( R \approx 1/\Lambda. \) Actually it means that \( n^1 \) receives non-zero VEV. The situation is very similar to one discussed in \([10]\). Dirichlet boundary conditions broke global \( SU(N) \) to \( SU(N - 1). \)

Now let us investigate the case \( x >> 1. \) The numerical calculations give

\[
S_{\text{eff}} = N \left( \frac{x^2}{2} \log x + Ax^2 - Bx + C \log x + \frac{x^2}{2} \mu R x^4 \right) - \frac{N \pi R^2}{g_\mu^2}
\]

(2.15)

with \( A \approx 1/2, B \approx 1.6, C \approx 1 \) Saddle point equation

\[
\frac{\partial S_{\text{eff}}}{\partial x^2} = 0 = -\frac{1}{2} \log x - \frac{1}{2} + A - \frac{B}{2x} + \frac{C}{2x^2} + \frac{1}{2} \log \Lambda R
\]

(2.16)
yields
\[ x = mR = \Lambda R - B + \frac{C - B^2/2}{\Lambda R} + O(1/R^2) \] (2.17)
and
\[ S_{\text{eff}} = N \left( \frac{\Lambda^2 R^2}{4} - B\Lambda R + C \log \Lambda R + O(1) \right) \] (2.18)

The area term \( \Lambda^2 R^2 / 4 \) reproduces the result for the flat \( \mathbb{R}^2 \) case (24).

The perimeter term reflects the contribution from the boundary excitation while the log term can be thought of as the contribution from the Luscher term. Let us emphasize that the signs of the different contributions are similar to the ones obtained in the model at the interval. The signs are unusual however and deserve for the comments. First, the Luscher logarithmic term has the sign which is opposite to the standard one but coincides with the results obtained in the other theories with the mass gap [11]. Let us emphasize that the previous discussion on this term in \( CP(N-1) \) model [17] seems to be incomplete. It was based on the argument that the mass gap results into the exponential suppression of the Luscher term resulted from the orientational moduli however it turns out that this argument does not work.

Another surprising point concerns the sign of the boundary contribution which corresponds to the localized negative energy of order \( O(N) \). At the first glance this looks extremely suspicious however it turns out that such boojums with the negative energy are known in some solid state examples and field theory models. In the most close situation the boojums occur at the intersection of the strings and domain walls [26, 27, 28] where the Dirichlet boundary conditions are imposed by domain walls. The detailed analysis in [26, 27, 28] demonstrated that the negative edge energy is consistent with the equation of motion and even with supersymmetry.

Let us emphasize that the boojum contribution in [26, 27, 28] was evaluated at weak coupling which was provided by the large twisted mass on the string worldsheet. The semiclassical consideration yields in \( SU(2) \) case the boundary energy equals to the half of the kink mass with the opposite sign. In our case we consider the pure quantum problem however for \( CP(1) \) case we reproduce at strong coupling the same value of the boundary boojum energy. At large \( N \) the edge contribution is proportional to \( N \). It would be very interesting to get the better physical explanation of the boojum negative energy.

In the Section 5 we shall use the effective action in the disc geometry to get the rate of the string decay.
3 Non-abelian strings

Here we review the simplest model with the gauge group $SU(N) \times U(1)$, which can be used to analyze non-abelian strings \cite{4}. The model contains $N$ scalar fields charged with respect to $U(1)$ which form $N$ fundamental representations of $SU(N)$. It is convenient to write these fields as $N \times N$ matrix $\Phi = \{ \varphi^{kA} \}$ where $k$ is the $SU(N)$ gauge index while $A$ is the flavor index,

$$\Phi = \begin{pmatrix} \varphi^{11} & \varphi^{12} & \ldots & \varphi^{1N} \\ \varphi^{21} & \varphi^{22} & \ldots & \varphi^{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi^{N1} & \varphi^{N2} & \ldots & \varphi^{NN} \end{pmatrix}. \quad (3.19)$$

The action of the model has the form

$$S = \int \! d^4x \left\{ \frac{1}{4g_1^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_2^2} (F_{\mu\nu})^2 \\
+ \text{Tr} (\nabla_\mu \Phi)^\dagger (\nabla^\mu \Phi) + \frac{g_2^2}{2} \left[ \text{Tr} (\Phi^\dagger T^a \Phi) \right]^2 + \frac{g_1^2}{8} \left[ \text{Tr} (\Phi^\dagger \Phi) - N\xi \right]^2 \\
+ \frac{i\theta}{32\pi^2} F_{\mu\nu}^a F^{a,\mu\nu} \right\}, \quad (3.20)$$

where $T^a$ stands for the generator of the gauge $SU(N)$,

$$\nabla_\mu \Phi \equiv \left( \partial_\mu - \frac{i}{\sqrt{2N}} A_\mu - iA_\mu^a T^a \right) \Phi, \quad (3.21)$$

and $\theta$ is the vacuum angle. The last term in the second line forces $\Phi$ to develop a vacuum expectation value (VEV) while the last but one term forces the VEV to be diagonal,

$$\Phi_{\text{vac}} = \sqrt{\xi} \text{diag} \{1, 1, \ldots, 1\}. \quad (3.22)$$

We assume that the parameter $\xi$ to be large,

$$\sqrt{\xi} \gg \Lambda_4, \quad (3.23)$$

where $\Lambda_4$ is the scale of the four-dimensional theory (3.20). That is we are in the weak coupling regime as both couplings $g_1^2$ and $g_2^2$ are frozen at a large scale.
The vacuum field (3.22) results in the spontaneous breaking of both gauge and flavor SU($N$)'s. A diagonal global SU($N$) survives

$$U(N)_{gauge} \times SU(N)_{flavor} \rightarrow SU(N)_{diag} \ .$$

yielding color-flavor locking in the vacuum.

The nontrivial topology providing the stability is based on

$$\pi_1 (SU(N) \times U(1)/\mathbb{Z}_N) \neq 0 \ .$$

and one can wind one element of $\Phi_{vac}$,

$$\Phi_{string} = \sqrt{\xi} \ \text{diag}(1, 1, ..., e^{i\alpha(x)}), \quad x \rightarrow \infty \ .$$

The strings with this boundary condition can be called elementary $Z_N$ strings; their tension is $1/N$-th of that of the ANO string. The ANO string can be viewed as a bound state of $N$ $Z_N$ strings. The $Z_N$ string solution can be written as follows:

$$\Phi = \begin{pmatrix} \phi(r) & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \phi(r) & 0 \\ 0 & 0 & \ldots & e^{i\alpha}\phi_N(r) \end{pmatrix} ,$$

$$A^{SU(N)}_i \ = \ \frac{1}{N} \left( \begin{array}{cccc} 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -(N-1) \end{array} \right) (\partial_i \alpha) \left[ -1 + f_{NA}(r) \right] ,$$

$$A^{U(1)}_i \ = \ \frac{1}{N} (\partial_i \alpha) \left[ 1 - f(r) \right] , \quad A^{SU(N)}_0 = A^{SU(N)}_0 = 0 \ ,$$

where $i = 1, 2$ labels coordinates in the plane orthogonal to the string axis and $r$ and $\alpha$ are the polar coordinates in this plane. The profile functions satisfy the following boundary conditions:

$$\phi_N(0) = 0 ,$$

$$f_{NA}(0) = 1 , \quad f(0) = 1 \ ,$$

$$7$$
at \( r = 0 \), and
\[
\phi_N(\infty) = \sqrt{\xi}, \quad \phi(\infty) = \sqrt{\xi}, \\
f_{NA}(\infty) = 0, \quad f(\infty) = 0 \tag{3.29}
\]
at \( r = \infty \).

The tension of this elementary string is
\[
T_1 = 2\pi \xi. \tag{3.30}
\]
while the tension of the ANO string is
\[
T_{ANO} = 2\pi N \xi \tag{3.31}
\]
which confirms its composite nature.

To obtain the non-Abelian string solution from the \( Z_N \) string (3.27) we apply the diagonal color-flavor rotation preserving the vacuum (3.22). In singular gauge we have
\[
\Phi = U \begin{pmatrix}
\phi(r) & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \phi(r) & 0 \\
0 & 0 & \ldots & \phi_N(r)
\end{pmatrix} U^{-1},
\]
\[
A_{i}^{SU(N)} = \frac{1}{N} U \begin{pmatrix}
1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & -(N-1)
\end{pmatrix} U^{-1} (\partial_i \alpha) f_{NA}(r),
\]
\[
A_{i}^{U(1)} = -\frac{1}{N} (\partial_i \alpha) f(r), \quad A_{0}^{U(1)} = A_{0}^{SU(N)} = 0, \tag{3.32}
\]
where \( U \) is a matrix \( \in \text{SU}(N) \). This matrix parameterizes orientational zero modes of the string associated with flux embedding into \( \text{SU}(N) \).

Let us discuss the worldsheet description of the nonabelian string. It is important that there are two independent contribution from "space" and "internal" terms. To obtain the kinetic term in the "internal" action we follow the standard logic
in the derivation of the low-energy action in the moduli approximation. That is we substitute our solution, which depends on the moduli $n^l$, in the action, assuming that the fields acquire a dependence on the coordinates $x_k$ via $n^l(x_k)$. Then we arrive at the $CP(N-1)$ sigma model,

$$S^{(1+1)}_{CP(N-1)} = 2f \int dt dz \left\{ (\partial_k n^* \partial_k n) + (n^* \partial_k n)^2 \right\},$$

(3.33)

where the coupling constant $f$ is given by a normalizing integral defined in terms of the string profile functions which yields

$$f = \frac{2\pi}{g_2^2}.$$  

(3.34)

that is two-dimensional coupling constant is determined by the four-dimensional non-Abelian coupling.

The sigma model (3.33) is asymptotically free hence at large distances it gets into the strong coupling regime. The running coupling constant as a function of the energy scale $E$ at one loop is given by

$$4\pi f = N \ln \left( \frac{E}{\Lambda_{CP(N-1)}} \right) + \cdots,$$

(3.35)

where $\Lambda_{CP(N-1)}$ is the dynamical scale of the $CP(N-1)$ model. The UV cut-off of the sigma model at hand is determined by $g_2\sqrt{\xi}$, hence,

$$\Lambda_{CP(N-1)}^N = g_2^N \xi^{N/2} e^{-\frac{8\pi^2}{g_2^2}}.$$  

(3.36)

In the bulk theory, due to the VEV’s of the scalar fields, the coupling constant is frozen at $g_2\sqrt{\xi}$. There are no logarithms in the bulk theory below this scale and the logarithms of the world-sheet theory take over.

4 Decay of the string

4.1 Decay rate

Consider now the decay rate. In the simple ANO-like abelian string the probability can be estimated using the total effective action for the classical Euclidean bounce(hole)

$$S_{eff}^{U(1)} = T_0 \pi R^2 - M_0 2\pi R$$  

(4.37)
where the origin of two terms is evident. The logarithmic Luscher term in the abelian case can be neglected with. The extremization with respect to the radius of the hole yields the value of the effective action evaluated at the critical bounce. Taking into account the exact answer for the preexponential factor in 1+1 dimensional false vacuum decay calculated in \[20, 18\] the probability rate reads as

\[ w = \frac{T_0}{2\pi} \exp\left(-\frac{M_0^2}{T_0}\right) \]  

(4.38)

There are a few new issues for the non-abelian string. The total effective action involves now two terms

\[ S_{\text{eff}} = S_{\text{eff}}^{U(1)} + S_{\text{eff}}^{CP(N-1)} \]  

(4.39)

As we have shown in Section 2 there are two phases in the \( CP(N-1) \) model and the \( S_{\text{eff}}^{CP(N-1)} \) depends on the solution \( R_{\text{crit}} \) to the extremization equation. If the \( R_{\text{crit}} \) is large one could use the asymptotic behavior \( (2.18) \) and the total action reads as

\[ S_{\text{eff}} = T_{\text{eff}}\pi R^2 - M_{\text{eff}} 2\pi R + NC\log \Lambda R \]  

(4.40)

where we neglect small abelian contribution to the Luscher term and

\[ T_{\text{eff}} = T_0 + N\Lambda^2/4, \quad M_{\text{eff}} = M_0 - NBA. \]  

(4.41)

The validity of this approximation can be justified with the proper choice of the parameters from the abelian sector. In this regime the abelian component of the string dominates which however needs for the very large values of the abelian tension and mass \( T_0, M_0 \). The decay rate has the form \( (4.38) \) corrected by the additional preexponential factor due to the Luscher term.

It is interesting to discuss the opposite limit when the orientational moduli dominates that is question if there are stable Euclidean configurations with the radius near the phase transition. The saddle point solution for the non-abelian part of the action with respect to the radius can be analyzed exactly because the mass gap is also obtained as the extremum of the action with respect to the mass. Indeed the action has the form \( (C_1, C_2 \) are some constants)

\[ S = \log \det(-\partial^2 + m^2) - C_1 mL/g^2 = \log \det(-\partial^2 + m^2)_{\text{reg}} + C_2 mR \log \Lambda R \]  

(4.42)

where \( \log \det(-\partial^2 + m^2)_{\text{reg}} \) - is well-defined function of \( mR \) only. \( m(R) \) is the solution to the equation

\[ \frac{\partial S(m, R)}{\partial m} = 0 \]  

(4.43)
Critical radius is the solution to

\[ \frac{dS(m(R), R)}{dR} = 0 \]  \hspace{1cm} (4.44)

Now it is not difficult to see that this equation reduces to \( m = 0 \), that is, the critical radius coincides with the phase transition radius. When the abelian part of the action is added only the numerical analysis is available. It demonstrates that by the choice of the abelian parameters the critical radius could move in both directions from the transition point. However if it is less then \( R_{\text{tran}} \) the analysis becomes inconsistent.

One could also imagine more involved situation when only the non-abelian flux terminates at the boundary of the disc on the string worldsheet in the Euclidean space while the abelian flux remains intact. To some extend such process would mean that string non-perturbatively drops off its nonabelian hair and becomes the conventional abelian string. This means that the boundary excitations with the negative energy are created and propagate along the string in the opposite directions separating two regions.

### 4.2 Spontaneous creation of the string

One could also comment on the opposite problem. Namely let us assume that we are in the vacuum state in the 4D theory which admits the non-abelian string and question if the non-abelian string can be created spontaneously. The question is reasonable since there are competing area and perimeter terms in the \( CP(N-1) \) model in the disc geometry. It looks a bit unusual since contrary to the standard case the boundary term in the energy of the long string is negative. In the standard case both area and perimeter terms are positive and the saddle point is absent.

If we consider the effective action of the \( CP(N-1) \) model on the disc discussed above which involves area, perimeter and Luscher terms the saddle point equation for the radius of the disc has to be found. In the absence of the abelian component the critical radius coincides with the phase transition point and the claim on the existence of the consistent saddle point can not be made. Hence the abelian component has to be taken into account. It is necessary to take \( M_0 \) small enough to keep the effective perimeter term negative. The numerical analysis demonstrates that the saddle point does exist in some region of the parameter space.

Let us question on the related problem. Is there the stable finite length non-abelian string? If it exists it could be considered as the part of the spectrum. To answer this question consider the extremization equation for the energy which in-
volves the abelian terms and the non-abelian contribution found in [10]:

\[ E = T_0 L + 2M_0 + E_{\text{eff}}^{CP(N-1)}(L) \]  

(4.45)

It turns out that the solution to the extremization equation does not exist hence there are no short magnetic strings in the spectrum.

### 4.3 Induced decay

Another issue concerns the induced decay of the non-abelian string. This process is analogous to the induced false vacuum decay in two dimensions [19, 16]. The decay can be induced by the particle excitation above the vacuum state. The same process can be interpreted as the non-perturbative decay of the particle in the false vacuum. The physical reason for the enhanced decay is as follows. The external particle usually has zero mode on the kink therefore the initial particle gets localized as zero mode at the boundary of the bounce and the tunneling occurs not at zero but at energy equals the particle mass. The bounce itself gets deformed into a fish-like configuration with two cusps.

In our case the decay can be induced by the kink-antikink meson. The classical action for the deformed bounce solution reads as [19]

\[ S_{\text{ind}} = \frac{2M^2}{T} \arcsin \frac{m}{2M} - \frac{mM}{T} \sqrt{1 - \frac{m^2}{4M^2}} \]

(4.46)

where \( m \) is of order of the meson mass and \( M \) is the mass of the boundary state. The preexponential factor has been calculated as well [16]. In our case the induced process could be interpreted as the decay of the meson into constituents. The kink and antikink get localized at the hole boundary. The kink masses are of order \( \Lambda \) hence we can certainly claim that the tunneling happens at the energy of order \( \Lambda \) which has to be substituted as the mass of the external particle in (4.46). However more detailed analysis is required to get the induced decay rate.

### 5 Axion and \( \theta \)-term

In this Section we shall discuss how the \( \theta \) term and axion affect the decay rate. First, let us recall the results of [14]. Two key properties were found; the 2d axion results in the deconfinement phase transition on the string worldsheet while the 4d axion does not. On the other hand the four-dimensional \( \theta \)-term penetrates into the worldsheet theory

\[ \theta_{D=4} = \theta_{D=2} \]

(5.47)
The mechanism of deconfinement is quite transparent. The worldsheet Lagrangian involves the photon-axion mixing. Upon taking into account this mixing the propagating particle becomes massive and there is no linear confinement between kink and antikink. Therefore if the worldsheet axion is present the kink is liberated and becomes the particle in the physical spectrum of the worldsheet theory. The deconfinement phase transition has happened at $\theta = \pi$ as well due to the degeneration of the ground state of the worldsheet theory. Let us also mention that kinks became the dyonic particles if the $\theta$ term is included.

Turn now to the decay of the non-abelian string. The $\theta$-term in Minkowski space provides the constant electric field along the string in the gauged formulation of the $CP(N-1)$ model similar to [21] hence one could ask how it is screened at the ends of the string at the hole boundary. The naive answer would be that dyonic kinks are created at the ends of the string and screen the effective electric field. However as we have seen above the boundary state has negative energy and does not have naive kink interpretation. Nevertheless this boundary energy is proportional to the non-perturbative scale therefore it becomes $\theta$-dependent when the $\theta$-term is switched on. This $\theta$-dependent boundary state should screen the electric field along the string. We plan to discuss this subtle issue elsewhere.

In the $\theta$ enriched theory there is the non-vanishing vacuum density of the topological charge

$$<F> = \frac{d\log Z}{d\theta} \propto \Lambda^2 \sin\left(\frac{\theta}{N}\right) \quad (5.48)$$

Therefore one could ask what happens with the topological charge stored in the disc which is removed in the Euclidean bounce solution $Q_{\text{top}} = <F> \pi R^2$. Upon the rotation to the Euclidean space the electric field gets transformed into the effective magnetic field transverse to the string worldsheet. There are two $\theta$-dependent boundary terms: the screening charge which provides the Wilson loop along the boundary and the $\theta$-dependent contribution to the boundary energy. We expect that these terms are collected together to satisfy the Stokes theorem

$$ P \exp \oint_{\text{bound}} A = \exp(<F> R^2) \quad (5.49) $$

Note that at large $N$ we can expand $\sin\left(\frac{\theta}{N}\right)$ dependence and keep the linear term. This fits with the linear $\theta$-dependence of the effective electric field. However to make these arguments precise it is necessary to get the exact $\theta$-dependence of the effective action.

Hence the total effect of the $\theta$ term is twofold. First, the vacuum energy density is $\theta$-dependent therefore the decay rate is modified. Secondly, there are dyonic-like
boundary states and there is nontrivial Wilson loop of the auxiliary gauge field along the hole boundary. If the axion field localized at the string worldsheet is included from the very beginning then one has to take into account the axion-photon mixing when calculating the vacuum energy. The decay is also accomplished by the strong axion emission discussed in [14].

Note in this context that the $CP(N - 1)$ model in Euclidean space was used for description of the QHE droplets at strong magnetic field [29]. In that description the $\theta$ term provides the filling fraction in QHE $\frac{\theta}{2\pi} = \nu$ or equivalently the Hall conductivity. The phase transition at $\theta = \pi$ is assumed to be related with the transition between the platoes. The most interesting issue concerns the possible relation between the boojum-like energy with the edge state in QHE and the possible phase transition at some radius of the droplet we have found. We hope to discuss these issues elsewhere.

6 Conclusion

In this note we have considered the quantum features of non-abelian string decay in non-supersymmetric gauge theory. To this aim the effective action of the $CP(N - 1)$ model on Euclidean disc has been calculated and the phase transition similar to one discovered in [10] has been found. The boundary boojum energy has been identified and the decay rate has been evaluated. We also comment on the opposite process of spontaneous creation of a non-abelian string.

The calculation of the action in the disc geometry shows a few general phenomena. First, the Luscher logarithmic term has the "wrong" sign - the phenomena seems to be common for the gapped theories [11]. Secondly there is boojum negative energy localized at the boundary which seems to be common phenomena as well. Moreover it can be argued that two "wrong sign" phenomena are closely related. This relation inspires the possible speculations on the relevance of the picture similar to the topological insulators. In that case one has the gapped bulk theory and the massless modes localized at the boundary. It is not clear if there are massless boundary modes in our case as well and this point certainly deserves the special attention.

The third general feature concerns the phase transition as the function of the length of the interval or the disc radius. This phase transition is the non-supersymmetric counterpart of the marginal stability walls in the twisted mass space in the supersymmetric theories. It that case there is no phase transition but the spectrum of the stable particles get changed at the wall. In the brane framework the particles are represented by the strings or branes stretched between branes and the position of the marginal stability wall roughly corresponds to the particular distance between
the "background" branes in the internal space. It would be interesting to relate the phase transition at some scale in the physical space found in our case and the wall crossing phenomena for the stretched strings at some scale in the internal space.

It would be interesting to discuss more general boundary conditions along the lines developed in [25]. We used the Dirichlet boundary condition but the nontrivial set of open Wilson lines could be included into consideration both for the strip and disc geometries. These additional parameters could be very useful in investigation of the role of non-commutativity of large volume and large N limits discussed in [30]. Another possibility to generalize the boundary condition is to include into the game the boundary S-matrix. Finally, it would be also interesting to investigate the effects discussed in this paper in the framework of cosmic strings reconnection [31].

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