Odd pairs of cliques

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Abstract. A graph is Berge if it has no induced odd cycle on at least 5 vertices and no complement of induced odd cycle on at least 5 vertices. A graph is perfect if the chromatic number equals the maximum clique number for every induced subgraph. Chudnovsky, Robertson, Seymour and Thomas proved that every Berge graph either falls into some classical family of perfect graphs, or has a structural fault that cannot occur in a minimal imperfect graph. A corollary of this is the strong perfect graph theorem conjectured by Berge: every Berge graph is perfect. An even pair of vertices in a graph is a pair of vertices such that every induced path between them has even length. Meyniel proved that a minimal imperfect graph cannot contain an even pair. So even pairs may be considered as a structural fault. Chudnovsky et al. do not use them, and it is known that some classes of Berge graph have no even pairs.

The aim of this work is to investigate an “even-pair-like” notion that could be a structural fault present in every Berge graph. An odd pair of cliques is a pair of cliques \( \{K_1, K_2\} \) such that every induced path from \( K_1 \) to \( K_2 \) with no interior vertex in \( K_1 \cup K_2 \) has odd length. We conjecture that for every Berge graph \( G \) on at least two vertices, either one of \( G, \overline{G} \) has an even pair, or one of \( G, \overline{G} \) has an odd pair of cliques. We conjecture that a minimal imperfect graph has no odd pair of maximal cliques. We prove these conjectures in some special cases. We show that adding all edges between any 2 vertices of the cliques of an odd pair of cliques is an operation that preserves perfectness.

Keywords. Perfect graph, graph, even pair.

1. Introduction

In this paper graphs are simple, non-oriented, with no loop and finite. Several definitions that can be found in most handbooks (for instance [11]) will not be given. A graph \( G \) is perfect if every induced subgraph \( G' \) of \( G \) satisfies \( \chi(G') = \omega(G') \), where \( \chi(G') \) is the chromatic number of \( G' \) and \( \omega(G') \) is the maximum clique size in \( G' \). Berge [2, 3] introduced perfect graphs and conjectured that the
complement of a perfect graph is a perfect graph. This conjecture was proved by Lovász:

**Theorem 1.1 (Lovász, 19, 18).** The complement of every perfect graph is a perfect graph.

Berge also conjectured a stronger statement: a graph is perfect if and only if it does not contain as an induced subgraph an odd hole or an odd antihole (the Strong Perfect Graph Conjecture), where a hole is a chordless cycle with at least four vertices and an antihole is the complement of a hole. We follow the tradition of calling Berge graph any graph that contains no odd hole and no odd antihole. The Strong Perfect Graph Conjecture was the object of much research (see the book [24]), until it was finally proved by Chudnovsky, Robertson, Seymour and Thomas:

**Theorem 1.2 (Chudnovsky, Robertson, Seymour and Thomas 5).** Every Berge graph is perfect.

In fact Chudnovsky, Robertson, Seymour and Thomas [5] proved a stronger fact, conjectured by Conforti, Cornuèjols and Vušković [9]: every Berge graph either falls in a basic class or has a structural fault. Before stating this more precisely, let us say that a basic class of graphs is a class of graphs that are proved to be perfect by some classical coloring argument. A structural fault in a graph is something that cannot occur in a minimal counter-example to the perfect graph conjecture. The basic classes used by Chudnovsky et al. are the bipartite graphs, their complement, the line-graphs of bipartite graphs, their complement, and the double split-graphs. The structural faults used by Chudnovsky et al. are the 2-join (first defined by Cornuèjols and Cunningham [10]), the even skew partition (a refinement of Chvátal’s skew partition [7]) and the homogeneous pair (first defined by Chvátal and Sbihi [8]). We do not give here the precise definitions as far as we do not need them.

Despite those breakthroughs, some conjectures about Berge graphs remain open. An even pair in a graph \(G\) is a pair of non-adjacent vertices such that every chordless path between them has even length (number of edges). Given two vertices \(x, y\) in a graph \(G\), the operation of contracting them means removing \(x\) and \(y\) and adding one vertex with edges to every vertex of \(G\) \(\setminus\{x, y\}\) that is adjacent in \(G\) to at least one of \(x, y\); we denote by \(G/xy\) the graph that results from this operation. Fonlupt and Uhry proved the following:

**Theorem 1.3 (Fonlupt and Uhry 15).** If \(G\) is a perfect graph and \(\{x, y\}\) is an even pair in \(G\), then the graph \(G/xy\) is perfect and has the same chromatic number as \(G\).

Meyniel also proved the following:

**Theorem 1.4 (Meyniel, 21).** Let \(G\) be a minimal imperfect graph. Then \(G\) has no even pair.
Odd pairs of cliques

So even pairs can be considered as a “structural fault”, with respect to a proof of perfectness for some classes of graphs. This approach for proving perfectness has been formalised by Meyniel [21]: a strict quasi-parity graph is a graph such that every induced subgraph either is a clique or has an even pair. By Theorem 1.4, every strict quasi-parity graph is perfect. Many classical families of perfect graphs, such as Meyniel graphs, weakly chordal graphs, perfectly orderable graphs, Artemis graphs, are strict quasi-parity, see [12, 20]. A quasi-parity graph is a graph $G$ such that for every induced subgraph $G'$ on at least two vertices, either $G'$ has an even pair, or $\overline{G'}$ has an even pair. By Theorems 1.4 and 1.1, we know that quasi-parity graphs are perfect. Quasi-parity graphs graphs include every strict quasi-parity graphs, and also other classes of graphs: bull-free Berge graphs [13], bull-reducible Berge graphs [13].

There are interesting open problems about quasi-parity graphs. Say that a graph is a prism if it consists of two vertex-disjoint triangles (cliques of size 3) with three vertex-disjoint paths between them, and with no other edges than those in the two triangles and in the three paths. (Prisms were called stretchers in [12] and 3PC($\Delta$, $\Delta$)’s in [9]). A prism is said to be long if it has at least 7 vertices. The double-diamond and $L(K_{3,3} \setminus e)$ are the graphs depicted in Figure 1. Let us now recall a definition: a graph is bipartisan [6] if in $G$ and $\overline{G}$ there is no odd hole, no long prism, no double-diamond and no $L(K_{3,3} \setminus e)$. The last 50 pages of the strong perfect theorem paper [5] are devoted to a proof of perfectness for bipartisan graphs. This part could be replaced by a proof of the following conjecture:

**Conjecture 1.5 (Maffray, Thomas).** Every bipartisan graph is a quasi-parity graph.

Why not conjecture that every Berge graph is a quasi-parity graph? Simply because this is false. Some counter-examples (like the smallest one: $L(K_{3,3} \setminus e)$), were known since the very beginning of the study of even-pairs. Hougardy found an infinite class of counter-examples:

**Theorem 1.6 (Hougardy, [17]).** Let $G$ be the line-graph of a 3-connected graph. Then $G$ and $\overline{G}$ have no even pair.

The aim of this paper is to investigate the following question: is there an “even-pair-like” notion that could be a structural fault present in every Berge
Conjecture 1.7. Let $G$ be a Berge graph on at least two vertices. Then either:

- $G$ or $\overline{G}$ has an even pair.
- $G$ or $\overline{G}$ has an odd pair of cliques $\{K_1, K_2\}$ such that $K_1, K_2$ are maximal cliques of $G$.

Conjecture 1.8. Let $G$ be a mimimal imperfect graph. Then $G$ has no odd pair of cliques $\{K_1, K_2\}$, such that $K_1, K_2$ are maximal cliques of $G$.

Clearly, between two maximal cliques of an odd hole, there exists an external induced path of even length. Between two maximal cliques of an odd antihole, there exists an external induced path of length 2. But by the strong perfect graph theorem, the only minimal imperfect are the odd holes and the odd antiholes. Thus the conjecture above is true. But we would like a proof that does not use the strong perfect graph theorem.

As already mentioned, it is easy to see that Conjecture 1.7 holds for bipartite graphs, line-graphs of bipartite graphs, and their complement. Let us prove that it holds also for the last basic class: double split graphs. A double split graph (defined in [5]) is any graph $G$ that can be constructed as follows. Let $m, n \geq 2$ be integers. Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$, $D = \{d_1, \ldots, d_n\}$ be four disjoint sets. Let $G$ have vertex set $A \cup B \cup C \cup D$ and edges in such a way that:

- $a_i$ is adjacent to $b_i$ for $1 \leq i \leq m$. There are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$.
- $c_j$ is non-adjacent to $d_j$ for $1 \leq i \leq m$. There are all four edges between $\{c_j, d_{j'}\}$ and $\{c_{j'}, d_j\}$ for $1 \leq j < j' \leq n$.
- There are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ and these two edges are disjoint.
Odd pairs of cliques

If $G$ is a double split graph with the notation of the definition, we may assume up to a relabeling of the $c_j, d_j$'s that $a_1$ sees every $c_j$ and that $b_1$ sees every $d_j$ (if this fails for some $j$, just swap $c_j, d_j$). Now it is easy to see that $K_a = \{a_1, c_1, \ldots, c_n\}$ and $K_b = \{b_1, d_1, \ldots, d_n\}$ are both maximal cliques of $G$. The only possible external induced paths of length greater than 1 from $K_a$ to $K_b$, are paths from $c_j$ to $d_j$ for some $j$. But such a path must start in $c_j$, and then go to some $a_i$, and then the only option is to go to $b_i$, and then back to $d_j$. So, every external path from $K_a$ to $K_b$ has length 1 or 3. So, $\{K_a, K_b\}$ is an odd pair of maximal cliques. Note that there exist double split graphs that have no even pair: $L(K_{3,3} \setminus e)$ is an example and arbitrarily large examples exist. However, Conjecture 1.7 holds for every basic graph.

2. Odd pairs of cliques in line-graphs of bipartite graphs

We observed in the introduction that every line-graph of bipartite graph has an odd pair of cliques. In this section, we will see that we can say something much stronger. But we first need some information on the structure of line-graphs of bipartite graphs.

The facts stated in this paragraph need careful checking, but we do not prove them since they are well known (see [11] and [16]). Let us consider a graph $G$ that contains no claw and no diamond. Let $v$ be a vertex of $G$. Either $v$ belongs to exactly one maximal clique of $G$, or $v$ belongs to exactly two maximal cliques of $G$. In the second case, the intersection of the two cliques is exactly $\{v\}$. Let us build a new graph $R$. Every maximal clique of $G$ is a vertex of $R$. Such vertices of $R$ are called the clique vertices of $R$. Every vertex of $G$ that belongs to a single clique of $G$ is also of vertex of $R$. Such vertices of $R$ are called pendent vertices of $R$. We add an edge between two clique vertices of $R$ whenever the two corresponding cliques in $G$ do intersect. We add an edge between a clique vertex $u$ of $R$ and a pendent vertex $v$ of $R$ whenever the pendent vertex $v$ is a vertex of $G$ that belongs to $u$ seen as a clique of $G$. Note that a pendent vertex of $R$ has always degree 1 (the converse is not true when $G$ has a connected component that consists in a single vertex). One can check that $R$ has no triangle and that $G$ is isomorphic to $L(R)$. This leads us to the following well known theorem:

**Theorem 2.1.** Let $G$ be a graph. There exists a triangle-free graph $R$ such that $G = L(R)$ if and only if $G$ contains no claw and no diamond.
The following theorem shows that the maximal cliques of the line-graph of a bipartite graph behave like the vertices of a bipartite graph in quite a strong sense.

**Theorem 2.2.** Let $G$ be a graph with no claw and no diamond. Then $G$ is the line-graph of a bipartite graph if and only if the maximal cliques of $G$ may be partitioned into two sets $A$ and $B$ such that for every distinct maximal cliques $K_1, K_2$ of $G$ we have:

- If $K_1 \in A$ and $K_2 \in A$, then $\{K_1, K_2\}$ is an odd pair of cliques.
- If $K_1 \in B$ and $K_2 \in B$, then $\{K_1, K_2\}$ is an odd pair of cliques.
- If $K_1 \in A$ and $K_2 \in B$, then $\{K_1, K_2\}$ is an even pair of cliques.

**Proof.** By the discussion above, we know that $G$ is isomorphic to the line-graph of a triangle-free graph $R$. So, we may assume that $R$ is built from $G$ like in the construction described in the discussion above.

If $R$ is bipartite, then the clique vertices of $R$ are partitioned into two stable sets $A$ and $B$. This partition is also a partition of the maximal cliques of $G$. So, let $K_1 \in A$ and $K_2 \in A$ be two maximal cliques of $G$. Note that $K_1$ and $K_2$ are also non-adjacent vertices of $R$. So, we know that $K_1$ and $K_2$ are disjoints cliques of $G$. If there exists an induced path of $G$ of even length, external from $K_1$ to $K_2$, then the interior vertices of this path (which has length at least 2) are the edges of the interior of a path of $R$ of odd length, linking the vertex $K_1$ to the vertex $K_2$. This contradicts the bipartition of $R$. So, every external induced path in $G$ between $K_1$ and $K_2$ is of odd length, in other words, $K_1$ and $K_2$ form an odd pair of cliques.

By the same way, we prove that if $K_1 \in B$ and $K_2 \in B$, then $K_1$ and $K_2$ form an odd pair of cliques. Similarly, if $K_1 \in A$ and $K_2 \in B$, then $K_1$ and $K_2$ form an even pair of cliques.

If $R$ is not bipartite, then $R$ has an odd hole $H$ of length at least 5 (because $R$ is triangle-free). Let $v_1, v_2, \ldots v_{2k+1}$ be the vertices of $H$ in their natural order. Every vertex of $H$ has degree at least 2, and therefore is a clique vertex of $R$. So, every vertex $v_i$ is in fact a maximal clique of $G$. If one manages to partition the maximal cliques of $G$ into two sets $A$ and $B$ as indicated in the lemma, two consecutive cliques $v_i$ and $v_{i+1}$ are not disjoint. So, they cannot be both in $A$ or both in $B$. So in the sequence $(v_1, \ldots, v_{2k+1}, v_1, \ldots)$ every second clique is in $A$ and the other ones are in $B$. But this is impossible because there is an odd number of $v_i$’s. □

3. An operation that preserves perfectness

We know that the contraction of an even pair $\{x, y\}$ in a perfect graph $G$ yields another perfect graph. What would be the corresponding operation in $L(G)$ for an odd pair of cliques? The edges incident to $x$ form a clique $K_x$ of $L(G)$, and those incident to $y$ form a clique $K_y$. The contracted vertex $xy$ in $G/xy$ is incident to the edges that were incident to $x$ or $y$ in $G$, and so becomes in $L(G)$ a clique obtained
by adding an edge between every vertex of \( K_1 \) and every vertex of \( K_2 \). So let us define the following operation for any graph \( G \) and any pair \( \{K_1, K_2\} \) of disjoint cliques of \( G \): just add an edge between every vertex in \( K_1 \) and every vertex in \( K_2 \) (if they are not adjacent). The graph obtained is denoted by \( G_{K_1 \equiv K_2} \). We will see that this operation preserves perfectness when applied to an odd pair of cliques. Before this, we need a technical lemma, roughly saying that in \( G_{K_1 \equiv K_2} \), there is no other big clique than the clique induced by \( V(K_1) \cup V(K_2) \):

**Lemma 3.1.** Let \( \{K_1, K_2\} \) be an odd pair of cliques in a graph \( G \). Let \( K \) be a clique of \( G_{K_1 \equiv K_2} \). There are then only two possibilities:

- \( K \) is a clique of \( G \).
- \( V(K) \subseteq V(K_1) \cup V(K_2) \).

**Proof.** If \( K \) is not a clique of \( G \), then \( K \) contains at least a vertex \( v_1 \) of \( K_1 \) and a vertex \( v_2 \) of \( K_2 \) that are not adjacent in \( G \). Moreover, if \( V(K) \) if not included in \( V(K_1) \cup V(K_2) \), then \( K \) contains a vertex \( v \) that is neither in \( K_1 \), nor in \( K_2 \), and that sees \( v_1 \) and \( v_2 \). But then, \( v_1 - v - v_2 \) is an external induced path of \( G \), of even length from \( K_1 \) to \( K_2 \), a contradiction. \( \Box \)

The proof of the next theorem looks like the proof of Fonlupt and Uhry for Theorem 1.3. For Theorem 1.3 it is needed to prove by a bichromatic exchange that some vertices may have the same color in some optimal coloring of a graph. The only possible obstruction to this exchange is a path of odd length between them, contradicting the definition of an even pair of vertices. In our theorem, at a certain step we will need to prove that there is an optimal coloring that gives different colors to some vertices. The only obstruction to this will be an induced path of even length, contradicting the definition of odd pairs of cliques.

**Theorem 3.2.** Let \( G \) be a perfect graph and let \( \{K_1, K_2\} \) be an odd pair of cliques of \( G \). Then \( G_{K_1 \equiv K_2} \) is a perfect graph.

**Proof.** Let \( H' \) be an induced subgraph of \( G_{K_1 \equiv K_2} \). Let \( H \) be the induced subgraph of \( G \) that has the same vertex-set than \( H' \). Clearly, \( V(K_1) \cap V(H) \) and \( V(K_2) \cap V(H) \) form an odd pair of cliques in \( H \) and \( H' = H_{(K_1 \cap H) \equiv (K_2 \cap H)} \). So, to prove the theorem, it suffices to check \( \chi(G_{K_1 \equiv K_2}) = \omega(G_{K_1 \equiv K_2}) \). Let us suppose that \( G \) is colored with \( \omega(G) \) colors. We look for a coloring of \( G_{K_1 \equiv K_2} \) with \( \omega(G_{K_1 \equiv K_2}) \) colors.

Let us first color the vertices that are neither in \( K_1 \) nor in \( K_2 \): we give them their color in \( G \). If \( \omega(G_{K_1 \equiv K_2}) > \omega(G) \), then by Lemma 3.1 we know that \( V(K_1) \cup V(K_2) \) induces the only maximum clique of \( G_{K_1 \equiv K_2} \). So, whatever the sizes of \( K_1, K_2 \), we take \( \gamma = \max(\{|V(K_1) \cup V(K_2)| - \omega(G)\}) \) new colors. We use them to color \( \gamma \) vertices in \( V(K_1) \cup V(K_2) \). So, we are left with \( |V(K_1) \cup V(K_2)| - \gamma \) vertices in \( V(K_1) \cup V(K_2) \): let us give them their color in \( G \). We may assume that there is a vertex \( v_1 \) in \( K_1 \) and a vertex \( v_2 \) in \( K_2 \) with the same color (say red) for otherwise we have an \( \omega(G_{K_1 \equiv K_2}) \)-coloring of \( G_{K_1 \equiv K_2} \) and the conclusion of the lemma holds.
So there is a color used in $G$ (say blue) that is used neither in $K_1$ nor in $K_2$. Let $C$ be the set of vertices of $G$ that are red or blue. The set $C$ induces a bipartite subgraph of $G$ and we call $C_1$ the connected component of $v_1$ in this subgraph. If $v_2 \in C_1$, then a shortest path in $C_1$ from $v_1$ to $v_2$ is an induced path of $G$, of even length from $K_1$ to $K_2$. This path is external because there is no blue vertex in $V(K_1) \cup V(K_2)$. This contradicts the definition of an odd pair of cliques, so $v_1$ and $v_2$ are not in the same connected component $C_1$. So, we can exchange the colors red and blue in $C_1$, and give the color blue to $v_1$, without changing the color of $v_2$. We can do this again as long as there are vertices of the same color in $K_1 \cup K_2$. Finally, we obtain an $\omega(G_{K_1 \equiv K_2})$-coloring of $G_{K_1 \equiv K_2}$. □

4. Odd pairs of cliques in minimal imperfect graphs

In this section, we will see that in a minimal imperfect graph $G$, there is no pair of odd cliques $(K_1, K_2)$ with $|K_1| + |K_2| = \omega(G)$. This will be proven without using the strong perfect graph theorem. We first need some results on minimal imperfect graphs.

**Theorem 4.1 (Lovász, [18]).** A graph $G$ is perfect if and only if for every induced subgraph $G'$ we have $\alpha(G') \omega(G') \geq |V(G')|$.

Lovász also introduced an important notion. Let $p, q \geq 1$ be two integers. A graph $G$ is $(p, q)$-partitionable if and only if for every vertex $v$ of $G$, the graph $G \setminus v$ can be partitioned into $p$ cliques of size $q$ and also into $q$ stable sets of size $p$. The theorem to come follows from Theorem 4.1.

**Theorem 4.2 (Lovász, [18]).** Let $G$ be a minimal imperfect graph. Then $G$ is partitionable.

Partitionable graphs have several interesting properties (see [23] for a survey). Padberg [22] proved the following in the particular case of minimal imperfect graphs:

**Theorem 4.3 (Bland, Huang, Trotter [4]).** Let $G$ be a graph $(p, q)$-partitionable with $n = pq + 1$ vertices. Then:

1. $\alpha(G) = p$ and $\omega(G) = q$.
2. $G$ has exactly $n$ cliques of size $\omega$.
3. $G$ has exactly $n$ stable sets of size $\alpha$.
4. Every vertex of $G$ belongs to exactly $\omega$ cliques of size $\omega$.
5. Every vertex of $G$ belongs to exactly $\alpha$ stable sets of size $\alpha$.
6. Every clique of $G$ of size $\omega$ is disjoint from exactly one stable set of $G$ of size $\alpha$.
7. Every stable set of $G$ of size $\alpha$ is disjoint from exactly one clique of $G$ of size $\omega$.
8. For every vertex $v$ of $G$, there is a unique coloring of $G \setminus v$ with $\omega$ colors.
If $K_1$ and $K_2$ are two disjoint subcliques of a clique $K$, then they form an odd pair of cliques. In this case, we say that $K_1$ and $K_2$ form a trivial odd pair of cliques. The following theorem is a particular case of Conjecture 1.8.

**Theorem 4.4.** Let $G$ be a minimal imperfect graph. Let $\{K_1, K_2\}$ be a non trivial odd pair of cliques of $G$. Then $|K_1| + |K_2| \neq \omega(G)$.

**Proof.** Suppose $|K_1| + |K_2| = \omega(G)$. By Lemma 3.1 $\omega(G_{K_1 \equiv K_2}) = \omega(G)$. Moreover, $\alpha(G_{K_1 \equiv K_2}) \leq \alpha(G)$. And by Theorem 3.1 we have $\alpha(G) \omega(G) < |V(G)|$.

By the definition, every induced subgraph of $G$ is perfect. So, by Theorem 4.2, every induced subgraph of $G_{K_1 \equiv K_2}$ is perfect. Note that the $\omega$-clique $K_1 \cup K_2$ of $G_{K_1 \equiv K_2}$ is not a clique of $G$ since $\{K_1, K_2\}$ is not a trivial odd pair of cliques. So, by counting the cliques and by the fact that $G$ is partitionable, we know that $G_{K_1 \equiv K_2}$ is not partitionable (because of Property 2 of Theorem 4.3). All its subgraphs are perfect, so by Theorem 1.2 we know it is perfect. But we have:

$$\alpha(G_{K_1 \equiv K_2}) \omega(G_{K_1 \equiv K_2}) \leq \alpha(G) \omega(G) < |V(G)| = |V(G_{K_1 \equiv K_2})|$$

This contradicts Theorem 4.1. □

By the preceding theorem, if $\{K_1, K_2\}$ is an odd pair of cliques in a minimal imperfect graph $G$, there are two cases:

- $|K_1| + |K_2| < \omega(G)$
  
  In this case, interestingly, the edges that we add when constructing $G_{K_1 \equiv K_2}$ do not create any $\omega$-clique by Lemma 3.1. Moreover, these edges do not destroy any $\alpha$-stable. Let us prove this:

  **Proof.** Suppose that an $\alpha$-stable set of $G$ is destroyed. This means that there exists two vertices $v_1 \in K_1$ and $v_2 \in K_2$ that are in some $\alpha$-stable set $S$ of $G$. By Property 2 of Theorem 4.3 there exists one $\omega$-clique $K$ disjoint from $S$. Let $v \in V(K)$. By the definition of partitionable graphs, $G \setminus v$ can be partitioned into $\omega$ stable sets of size $\alpha$. At least one of these stable sets (say $S'$) is disjoint from $K$, since $K \setminus v$ contains $\omega - 1$ vertices. By Property 6 of Theorem 4.3 we know that $S' = S$. So we have found in $G$ a vertex $v$ such that $G \setminus v$ can be optimally colored giving to $v_1$ and $v_2$ the same color, say red. But since $|K_1| + |K_2| < \omega(G)$, there exists a color (say blue) that is not used in $K_1 \cup K_2$. By a bichromatic exchange (like in the proof of Theorem 3.2), we can find a coloring of $G \setminus v$ that gives the same red color to $v_1$ and color blue to $v_2$ (if such an exchange fails, there is an external induced path of even length between $K_1$ and $K_2$, a contradiction). Finally we found two different colorings of $G \setminus v$. This contradicts Property 8 of Theorem 4.3. □

So $G_{K_1 \equiv K_2}$ is a partitionable graph. Seemingly, this does not lead to a contradiction.

- $|K_1| + |K_2| > \omega(G)$
  
  In this case, by Lemma 3.1 $G_{K_1 \equiv K_2}$ has a unique maximum clique: $K_1 \cup K_2$. 

Odd pairs of cliques
This graph is not partitionable, all its induced subgraphs are perfect, so it is perfect. One more time, this does not seem to lead to contradiction.

5. Odd pairs of cliques in Berge graphs

To prove Conjecture 1.7, one could try to use the approach that worked for the decomposition of Berge graphs [5]: first, consider the case when \( G \) has a “substantial” line-graph \( H \) as an induced subgraph. We know that \( H \) has an odd pair of cliques (by Theorem 2.2). Then, one could hope that this pair of cliques is likely to somehow “grow” to an odd pair of cliques of the whole graph. A star-cutset in a graph \( G \) is a set \( C \) of vertices such that \( G \setminus C \) is disconnected and such that there exists a vertex in \( C \) that sees all the other vertices of \( C \). Star cutsets have been introduced by Chvátal [7], who proved that they are a “structural fault” that cannot occur in minimal imperfect graph. It is known however that some non-basic Berge graphs have no star-cutset. The following lemma shows that there is something wrong in the idea of making the odd pair cliques “grow”: it can work only in graphs that have a star-cutset.

Lemma 5.1. Let \( \{K_1, K_2\} \) be an odd pair of cliques of a graph \( G \). Suppose that \( K_2 \) is a maximal clique of \( G \). Let \( K_1' \neq K_1 \) be a sub-clique of \( K_1 \). If \( \{K_1', K_2\} \) is an odd pair of cliques, then \( G \) has a star cutset.

Proof. Let \( a \in K_1' \) and \( b \in K_2 \) be non adjacent vertices (they exist because \( K_2 \) is maximal). Let \( c \) be any vertex of \( V(K_1) \setminus V(K_1') \). We are going to show that \( \{a \} \cup N(a) \setminus \{c\} \) is a cutset of \( G \) separating \( c \) from \( b \). To prove this, we check that every induced path \( P \) from \( c \) to \( b \) that has no interior vertex in \( K_1 \) contains a neighbour of \( a \) different of \( c \). Indeed:

If the interior of \( P \) contains no vertex of \( K_2 \), then \( P \) has odd length because \( \{K_1, K_2\} \) is an odd pair of cliques. Since \( \{K_1', K_2\} \) is an odd pair of cliques, there is a chord in the even-length path \( (a,c,\ldots,b) \), and this chord is between \( a \) and a vertex of the interior of \( P \).

If the interior of \( P \) contains a vertex of \( K_2 \), then this vertex is the neighbour of \( b \) in \( P \): we denote it by \( d \). We see that \( c-P-d \) has odd length because \( \{K_1, K_2\} \) is an odd pair of cliques. So the path \( (a,c,\ldots,d) \) has even length, and there is a chord between \( a \) and a vertex of the interior of \( P \) (this chord can be \( ad \)). \( \square \)

References

[1] L. W. Beineke, Characterisation of derived graphs, Journal of Combinatorial Theory 9 (1970), 129–135.
[2] C. Berge, Les problèmes de coloration en théorie des graphes, Publ. Inst. Stat. Univ. Paris, 1960.
[3] C Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung), Wiss. Z. Martin Luther Univ. Math.-Natur. Reihe (Halle-Wittenberg), 1961.
[4] R. G. Bland, H. C. Huang, and L. E. Trotter, Jr., Graphical properties related to minimal imperfection, Discrete Math. 27 (1979), 11–22.

[5] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, Manuscript, 2002.

[6] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, Progress on perfect graphs, Manuscript, 2002.

[7] V. Chvátal, Star-cutsets and perfect graphs, J. Combin. Ser. B 39 (1985), 189–199.

[8] V. Chvátal and N. Sbihi, Bull-free Berge graphs are perfect, Graphs and Combinatorics 3 (1987), 127–139.

[9] M. Conforti, G. Cornuéjols, and K. Vušković, Square-free perfect graphs, Jour. Comb. Th. Ser. B 90 (2004), 257–307.

[10] G. Cornuéjols and W. H. Cunningham, Composition for perfect graphs, Disc. Math. 55 (1985), 245–254.

[11] R. Diestel, Graph theory, second ed., Springer, New York, 2000.

[12] H. Everett, C.M.H. de Figueiredo, C. Linhares Sales, F. Maffray, O. Porto, and B.A. Reed, Even pairs, in Ramírez Alfonsín and Reed [24], pp. 67–92.

[13] C. M. H. de Figueiredo, F. Maffray, and O. Porto, On the structure of bull-free perfect graphs, Graphs Combin. 13 (1997), 31–55.

[14] C. M. H. de Figueiredo, F. Maffray, and C. R. Vilela Maciel, Even pairs in bull-reducible graphs, Manuscript, 2004. Res. Report 117, Laboratoire Leibniz.

[15] J. Fonlupt and J.P. Uhry, Transformations which preserve perfectness and h-perfectness of graphs, Ann. Disc. Math. 16 (1982), 83–85.

[16] F. Harary and C. Holzmann, Line graphs of bipartite graphs, Rev. Soc. Mat. Chile 1 (1974), 19–22.

[17] S. Hougardy, Even and odd pairs in line-graphs of bipartite graphs, European J. Combin. 16 (1995), 17–21.

[18] L. Lovász, A characterization of perfect graphs, J. Combin. Theory Ser. B 13 (1972), 95–98.

[19] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (1972), 253–267.

[20] F. Maffray and N. Trotignon, A class of perfectly contractile graphs, Submitted to Comb. Th. Ser. B (2003).

[21] H. Meyniel, A new property of critical imperfect graphs and some consequences, European J. Comb. 8 (1987), 313–316.

[22] M. W. Padberg, Almost integral polyhedra related to certain combinatorial optimization problems, Math. Programming 6 (1974), 180–196.

[23] M. Preissmann and A. Sebő, Some aspects of minimal imperfect graphs, in Ramírez Alfonsín and Reed [24], pp. 185–214.

[24] J. L. Ramírez Alfonsín and B. A. Reed (eds.), Perfect graphs, Series in Discrete Mathematics and Optimization, Wiley-Interscience, 2001.
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