From pointwise convergence of evolutionary dynamics to average case analysis of decentralized algorithms

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Abstract

Nash’s proof of equilibrium existence has offered a tool of seminal importance to economic theory, however, a remaining, nagging hurdle is the possibility of multiple equilibria, perhaps with quite different properties. A wide range of approaches has been employed in an effort to reduce the study of games down to analyzing a single behavioral point (e.g., best or worst equilibria with respect to social welfare).

We move away from the search of a single, correct solution concept. We propose a quantitative framework where the predictions of different equilibria are weighted by their probability to arise under evolutionary dynamics given random initial conditions. This average case analysis is shown to offer the possibility of significantly tighter understanding of classic game theoretic settings, including quantifying the risk dominance of payoff dominated equilibria in stag-hunt games and reducing prediction uncertainty due to large gaps between price of stability and price of anarchy in coordination and congestion games.

This approach rests upon new deep topological results about evolutionary dynamics, including point-wise convergence results for replicator dynamics in linear congestion games and explicit computation of stable and unstable manifolds of equilibrium fixed points in asymmetric games. We conclude by introducing a concurrent convergent deterministic discrete-time dynamic for general linear congestion games and performing stability analysis on it.

1 Introduction

Nash’s theorem [24] on the existence of fixed points in game theoretic dynamics ushered in an exciting new era in the study of economics. At a high level, the inception of the (Nash) equilibrium concept allowed, to a large degree, the disentanglement between the study of complex behavioral dynamics and the study of games. Equilibria could be concisely described, independently from the dynamics that gave rise to them, as solutions of algebraic equations. Crucially, their definition was simple, intuitive, analytically tractable in many practical instances of small games, and arguably instructive about real life behavior. The notion of a solution to (general) games, which was introduced by the work of von Neumann in the special case of zero-sum games [34], would be solidified as a key landmark of economic thought. This mapping from games to their solutions, i.e., the set of equilibria, grounded economic theory in a solid foundation and allowed for a whole new class of questions in regards to numerous properties of these sets including their geometry, computability, and resulting agent utilities.
Unfortunately, unlike von Neumann’s essentially unique behavioral solution to zero-sum games, it became immediately clear that Nash equilibrium fell short from its role as a universal solution concept in a crucial way. It is non-unique. It is straightforward to find games\(^1\) with constant number of agents and strategies and uncountably many distinct equilibria with different properties in terms of support sizes, symmetries, efficiency, and practically any other conceivable attribute of interest. This raises a rather natural question. How should we analyze games with multiple Nash equilibria?

The centrality of the equilibrium selection problem can hardly be overestimated. Indeed, according to Ariel Rubinstein “No other task may be more significant within game theory. A successful theory of this type may change all of economic theory.” Accordingly, a wide range of radically different approaches has been explored by economists and computer scientists alike. Despite their differing points of view, they share a common high level goal. The goal is to reduce the number of admissible equilibria and, if possible, effectively pinpoint a single one as target for analytical inquiry. This way, the multi-valued equilibrium correspondence becomes a simple function and prediction uncertainty vanishes. Although no single approach stands out as providing the definitive answer, each has allowed for significant headways in specific classes of interesting games, and some have sprung forth standalone lines of inquiry. Next, we will focus on two approaches that have inspired our work: risk dominance and price of anarchy analysis.

Risk dominance is an equilibrium refinement process that centers around uncertainty about opponent behavior. A Nash equilibrium is considered risk dominant if it has the largest basin of attraction. Specifically, the more uncertainty agents have about the actions of others, the more likely it is that they will choose the corresponding strategy. The benchmark example is the Stag Hunt game, shown in figure 1(a). In such symmetric 2x2 coordination games a strategy is risk dominant if it is a best response to the uniformly random strategy of the opponent. Although risk dominance\(^2\) was originally introduced as a hypothetical model of the method by which perfectly rational players select their actions, it may also be interpreted\(^3\) as the result of evolutionary processes. Experimental evidence seems supportive about its practical relevance\(^4\). However, a critical shortcoming of this approach is that it is largely qualitative in nature. An equilibrium whose basin of attraction covers 99.9% of the state space has significantly more predictive power than one with 50.1%. More distressingly, as we move towards classes of games with exponentially or uncountably many equilibria such arguments become moot, since all equilibria may have attractors of vanishingly small size.

Price of anarchy\(^5\) follows a much more quantitative approach. The point of view here is that of optimization and the focus is on extremal equilibria. Price of anarchy, defined as the ratio between the social welfare of the worst equilibrium and that of the optimum tries to capture the loss in efficiency due to the lack of centralized authority. A plethora of similar concepts, based on normalized ratios, has been defined (e.g., price of stability\(^6\) focuses on best case equilibria). Tight bounds on these quantities have been established for large classes of games\(^7\). However, these bounds do not necessarily reflect the whole picture. They usually correspond to highly artificial instances. Even in these bad instances, typically there exist sizable gaps between their price of anarchy and price of stability, allowing for the possibility of significantly tighter analysis of system performance. More to the point, worst case equilibria maybe unlikely in themselves by having a negligible basin of attraction\(^8\).

\(^1\)An example of such a game can be found in section 6.1.
Our approach. We do not aim to solve the equilibrium selection problem, but to circumvent it. The high level intuition as follows: Each agent chooses a randomized strategy uniformly at random. We use this profile of randomized strategies as an initial condition for replicator, a deterministic evolutionary dynamic. As long as we can prove point-wise convergence, i.e., that given any initial condition the corresponding trajectory has a uniquely defined limit point, then the whole system can be viewed as a decentralized algorithm that maps initial conditions to their corresponding equilibrium. Each equilibrium has a corresponding basin of attraction, which is defined as the set of points that converge to it. We assign to each equilibrium a likelihood that is proportional to the volume of its respective basin of attraction.

Our results. In this paper we focus on potential games which are known to be isomorphic to congestion games [22]. We start by establishing the first to our knowledge point-wise convergence result for replicator dynamics in linear congestion games. The proof is based on local, information theoretic inspired Lyapunov functions around each equilibrium and not on the typical global potential functions used in establishing the standard set-wise convergence to equilibrium sets. This result, which is of independent interest, allows us to define properly the notion of average case system performance where the social welfare of each equilibrium is weighted by the size of its basin.

Next, we distinguish between equilibria whose region of attraction has positive/zero measure. All equilibria with no unstable eigenvectors must satisfy a necessary game theoretic condition, known as weak stability [15]. An equilibrium is weakly stable if given any two randomizing agents, fixing one of the agents to choosing one of his strategies with probability one, leaves the other agent indifferent between the strategies in his support. Given pointwise convergence of the replicator, all but a zero measurable subset of initial conditions converge to weakly stable equilibria. Two key technical hurdles in translating eigenvalue results regarding local stability/instability to global measure theoretic statements about basins of attraction is the possibility of uncountably many equilibria that exclude naive union bound treatments and the fact that the replicator in its usual form evolves over a simplex that defines a zero measure set in its native space. We circumvent the uncountably many equilibria case by producing countable open covers of the equilibrium set via Lindelöf’s lemma, producing zero measure statements over each open set in the cover and then applying union bound (proof can be found in the appendix).

Our approach can be thought of as a detailed, quantitative analysis of risk dominance. Specifically, instead of merely arguing that a specific equilibrium has size of basin of attraction at least 1/2, we aim to explicitly compute the sizes of the basin of attraction of each equilibrium. Thus, a natural starting point is the study of the Stag Hunt game itself. We show that the size of attraction of the risk dominant equilibrium is $\frac{1}{2\pi}(9 + 2\sqrt{3\pi}) \approx 0.7364$. Our analysis builds upon a combination of ideas. First, we apply weak stability to establish that the only equilibria with non-negligible region of attraction are the pure ones. Next, we construct an algebraic description of the stable and unstable manifolds of the mixed Nash equilibrium by building on information theoretic invariant properties of the replicator system. We solve these systems to provide an exact explicit description of these objects.

Next, we move to a class of two agent coordination games that generalizes Stag Hunt games/dynamics. Although we can still produce implicit parametric descriptions of the stable/unstable manifolds of the interior Nash equilibrium, the computation of explicit
parametric closed form solutions appears hopeless since the resulting systems contain parametric uncertainty in the exponent. Instead, we focus on computing approximately the volume of each basin. We exploit these parametric descriptions to extract geometric properties of these manifolds that remain valid for the whole parameter space. We use this understanding to produce tight coverings of the attractor basins via parametric families of polytopes. We compute both upper and lower bounds for the volume of these attractors. The average price of anarchy of this class of games, where the social welfare of each equilibrium is weighted by the size of its basin is shown to lie somewhere between 1.15 and 1.21. In contrast, the price of anarchy is unbounded.

Although distinctions between equilibria with zero and non-zero basin of attractions do not allow for as detailed performance analysis as the one presented in the case of two agent coordination games, one can still apply these results to get significant improvements over price of anarchy analysis. For example, we show that in the case of unweighted $n$-balls, $n$ bins games (singleton congestion games) with cost functions equal to the bins latency, the average price of anarchy both in terms of social cost as well as makespan is equal to one. In contrast the price of anarchy is $\Omega(\log(n)/\log \log(n))$. Finally, we show that the average price of anarchy of any linear congestion game is strictly less than $5/2$.

We conclude our analysis by introducing a replicator inspired discrete-time dynamic. This dynamic is both deterministic and concurrent and we establish set-wise convergence to equilibria for it in linear congestion games. The convergence result here is significantly more technical than that of the replicator, its continuous-time counterpart. To our knowledge, this is the first result of its kind since prior results on network congestion games are either based on agents moving one at a time or are based on probabilistic arguments [1]. We also perform local stability analysis, which similarly to the replicator, pinpoints weakly stable equilibria.

2 Related Work

Pointwise convergence in gradient/gradient-like systems: Pointwise convergence is much stronger than just convergence to a set. The gradient systems

$$\dot{x} = -\nabla V(x)$$

(or gradient-like systems like replicator dynamics) have the property of converging to equilibria sets because of the existence of the Lyapunov type function $V$. Lojasiewicz [18] proved an important inequality if $V$ is an analytic function $C^\infty$ a consequence of which is the pointwise convergence. Roughly, using this inequality one can show that the length of any trajectory is bounded. These results can be extended to systems called gradient-like $\dot{x} = x(t)$ [17] where there exists $V$ s.t $\nabla V \dot{x} \leq 0$ (with equality only at fixed points). It can be shown under the angle assumption ($\nabla V, \dot{x}$ have angle bounded away from 90 degrees) that we have pointwise convergence. However, in the replicator dynamics this is not the case. Even though someone can see that $V$ (potential function) is analytic, we cannot generally satisfy the angle assumption (an example is of 2 balls - 2 bins), i.e

$$\frac{\nabla V \cdot \dot{x}}{||x|| ||\nabla V||} \to 0$$

as you go to a fixed point. In any case, there are examples (mexican hat) where the limit points is a for example a circle.

Losert and Akin [19] proved pointwise convergence of replicator dynamics for a specific population model using relative entropy. Our proof for pointwise convergence in the

3 Concurrent means that all agents (or even an arbitrary number of agents) get to move simultaneously.
Learning as a refinement mechanism: A Nash equilibrium is evolutionarily stable: once it is fixed in a population, natural selection alone is sufficient to prevent alternative (mutant) strategies from invading successfully. Such fixed points are referred to as evolutionary stable equilibria or evolutionary stable strategies [20]. A related concept is that an evolutionary stable state [21]. This is a definition that arises in mathematical biology and explicitly in the study of replicator. A population is said to be in an evolutionarily stable state if its genetic composition is restored by selection after a disturbance, provided the disturbance is not too large. A stochastically stable equilibrium [7] is a further refinement of the evolutionarily stable states. An evolutionary stable state is also stochastically stable if under vanishing noise the probability that the population is in the vicinity of the state does not go to zero.

Replicator dynamics and system performance: This paper builds upon recent results that show favorable performance guarantees for the replicator dynamics (and discrete-time variants) in settings of interest. The key reference is [15], where many key ideas including the fact that replicator dynamics can significantly outperform worst case equilibria were introduced. In fact, in some games replicator dynamic can be shown to outperform even best case equilibria by converging to optimal cyclic attractors [13]. Finally, the use of information theoretic arguments as (weak) Lyapunov functions have been explored before (e.g., [13, 14]).

3 Preliminaries

3.1 Congestion Games

Congestion games [27] are non-cooperative games in which the utility of each agent depends only on the agent’s strategy and the number of other agents that either choose the same strategy, or some strategy that intersects/overlaps it. Formally, a congestion game is defined by the tuple \((N; E; (S_i)_{i \in N}; (c_e)_{e \in E})\) where \(N\) is the set of agents, \(E\) is a set of resources (also known as edges or bins or facilities), and each player \(i\) has a set \(S_i\) of subsets of \(E\) \((S_i \subseteq 2^E)\) and \(|S_i| \geq 2\). Each strategy \(s_i \in S_i\) is a set of edges (a path), and \(c_e\) is a cost (negative utility) function associated with facility \(e\). We will also use small greek characters like \(\gamma, \delta\) to denote different strategies/paths. For a strategy profile \(s = (s_1, s_2, \ldots, s_N)\), the cost of player \(i\) is given by \(c_i(s) = \sum_{e \in s_i} c_e(\ell_e(s))\), where \(\ell_e(s)\) is the number of players using \(e\) in \(s\) (the load of edge \(e\)). Congestion games admit a potential function \(\Phi(s) = \sum_{e \in E} \sum_{j=1}^{k_e(s)} c_e(j)\), which captures each player’s incentive to change his strategy [27]. Specifically, given a strategy profile \(s = (s_1, s_2, \ldots, s_N)\), and strategy \(s_i'\) of player \(i\), we have \(c_i(s_i', s_{-i}) - c_i(s) = \Phi(s_i', s_{-i}) - \Phi(s)\). As a result, starting from any strategy profile, any sequence of improving moves is bound to terminate. Such stable states \(s \in S\) where for each agent \(i\) and \(s_i' \in S_i\), \(c_i(s_i', s_{-i}) \geq c_i(s)\) are called Nash equilibria. The set of Nash equilibria correspond to local optima of \(\Phi(S)\). In linear congestion games where the latency functions are of the form \(c_e(x) = a_e x + b_e\) where \(a_e, b_e \geq 0\). Furthermore, the social cost will correspond to the sum of the costs of all the agents \(sc = \sum_i c_i(s)\). The price of anarchy is defined as: \(\text{PoA}(G) = \frac{\max_{\sigma \in \text{NE}} \text{Social Cost}(\sigma)}{\min_{\sigma^* \in \times_i S} \text{Social Cost}(\sigma^*)}\).
We will denote $\Delta(S_i) = \{p : \sum \gamma p_{i\gamma} = 1\}$, $\Delta = \times_i \Delta(S_i)$ and $M = \sum_i |S_i|$. Since $\Delta \subset \mathbb{R}^M$ sometimes we use the continuous function $g$ that is a natural projection of the points $p \in \mathbb{R}^M$ to $\mathbb{R}^{M-N}$ by excluding a specific (the "first") variable for each player (we know that the probabilities must sum up to one for each player). We denote the projection of $\Delta$ by $g(\Delta)$. Hence for a point $p \in \Delta$ we will denote $x = g(p) \in g(\Delta) \subset \mathbb{R}^{M-N}$ (for example $(p_{1,a},a,p_{1,b},p_{1,c},p_{2,a'},p_{2,b'}) \rightarrow g(p_{1,a},b,p_{1,c},p_{2,b'})$ where $p_{1,a} + p_{1,b} + p_{1,c} = 1$ and $p_{2,a'} + p_{2,b'} = 1$).

3.2 Average PoA in pointwise convergent systems

**Definition 1.** Given two manifolds $M, N$, a continuously differentiable map $f : M \rightarrow N$ is called $C^1$ - diffeomorphism if it is bijection and the inverse map $f^{-1}$ is continuously differentiable as well.

Let $\phi(t,p)$ be the flow of a differential equation $\dot{x} = f(x)$ where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Some properties of $\phi(t,p)$ are the following:

- $\phi(t,p)$ is a $C^1$ and local-diffeomorphism (around fixed points).
- $\phi(t,\phi(s,p)) = \phi(t+s,p)$

We have the assumption that the social cost is continuous and defined on $g(\Delta)$.

Assume that the dynamical system converges pointwise and let $\psi(x) = g(\lim_{t \rightarrow \infty} \phi_t(g^{-1}(x)))$, i.e., $\psi$ returns the limit point (always exists by assumption) of $p = g^{-1}(x)$ in $g(\Delta)$. We prove the following important lemma:

**Lemma 2.** $\psi(x)$ is Lebesgue measurable.

**Proof** See appendix.

We define a new performance measure that we call Average PoA. Essentially it compares the weighted average of all equilibrium points over the optimal solution. The respective weight of each equilibrium is just the measure of the points that converge to it. Formally:

**Definition 3.** (Average PoA) The Average PoA is the following Lebesgue integral ($\mu$ is the Lebesgue measure in $\mathbb{R}^{M-N}$, $sc$ is continuous)

$$\frac{1}{\mu(g(\Delta))} \int_{g(\Delta)} sc \circ \psi \, d\mu$$

The integral above is well-defined since $\psi(x)$ is a Lebesgue measurable function and $sc$ is a continuous function hence $sc \circ \psi$ is Lebesgue measurable.

\footnote{In the case of games that players want to maximize their utilities, we have the inverse fraction so that Average PoA $\geq 1$}
3.3 Replicator Dynamics

3.3.1 Continuous Replicator Dynamics

The replicator equation \([33, 30]\) is among the basic tools in mathematical ecology, genetics, and mathematical theory of selection and evolution. In its classic continuous form, it is described by the following differential equation:

\[
\frac{dp_i(t)}{dt} = \dot{p}_i = p_i[u_i(p) - \hat{u}(p)], \quad \hat{u}(p) = \sum_{i=1}^{n} p_i u_i(p)
\]

where \(p_i\) is the proportion of type \(i\) in the population, \(p = (p_1, \ldots, p_m)\) is the vector of the distribution of types in the population, \(u_i(p)\) is the fitness of type \(i\), and \(\hat{u}(p)\) is the average population fitness. The state vector \(p\) can also be interpreted as a randomized strategy of an adaptive agent that learns to optimize over its \(m\) possible actions, given an online stream of payoff vectors. As a result, it can be employed in any distributed optimization setting. An interior point of the state space is a fixed point for the replicator if and only if it is a fully mixed Nash equilibrium of the game. The interior (the boundary) of the state space \(\Delta = \times_i \Delta(S_i)\) are invariants for the replicator. We typically analyze the behavior of the replicator from a generic interior point, since points of the boundary can be captured as interior points of smaller dimensional systems. Summing all this up, our model is captured by the following system:

\[
\frac{dp_{i\gamma}(t)}{dt} = p_{i\gamma}(\hat{c}_i - c_{i\gamma})
\]

for each \(i \in N, \gamma \in S_i\), where \(c_{i\gamma} = E_{s_{-i} \sim p_{-i}} c_i(\gamma, s_{-i})\) and \(\hat{c}_i = \sum_{\delta} p_i \delta c_{i\delta}\).

The replicator dynamic enjoys numerous desirable properties such as universal consistency (no-regret) \([8, 11]\), connections to the principle of minimum discrimination information (Occam’s razor, Bayesian updating), disequilibrium thermodynamics \([12, 31]\), classic models of ecological growth (e.g. Lotka-Volterra equations\([10]\)), as well as several well studied discrete time learning algorithms (e.g. Multiplicative Weights algorithm\([15, 3]\)).

**Remark:** A fixed point of a flow is a point where the vector field is equal to zero. An interesting observation about the replicator is that its fixed points are exactly the set of randomized strategies such that each agent experiences equal costs across all strategies he employs with positive probability. This is a generalization of the notion of Nash equilibrium, since equilibria furthermore require that any strategy that is played with zero probability must have expected cost at least as high as those strategies which are played with positive probability.

3.3.2 Discrete Replicator Dynamics

The model of discrete replicator dynamics which is commonly used when we deal with utilities is the following:

\[
p_{i\gamma}(t + 1) = p_{i\gamma}(t) \frac{u_{i\gamma}(t)}{\hat{u}_i(t)}
\]

Observe that this dynamic has the same fixed points with the continuous replicator dynamic. We could not find in the literature a discrete replicator dynamics model for the case of congestion games, where we have cost functions. The model we consider is
slightly different.

\[ p_{i\gamma}(t + 1) = p_{i\gamma}(t) \frac{x - c_{i\gamma}(t)}{x - \hat{c}_i(t)} \]

where we consider \( x \) to be the following constant:

\[ x = \sum_e c_e(N) \]

\( x \) corresponds to the cost when all players use all edges with probability one, and we choose it in order to ensure that both numerator and denominator are positive. Observe that \( \Delta \) is invariant under the discrete dynamics defined above. If \( p_{i\gamma} = 0 \) it remains zero, if it is positive, it remains positive (both numerator and denominator are positive) and \( \sum_{\gamma} p_{i\gamma} = 1 \). The fixed points are the same with the continuous replicator setting, i.e if \( p_{i\gamma} > 0 \) then \( c_{i\gamma} = \hat{c}_i \).

4 Continuous case

4.1 Pointwise convergence

**Theorem 4.** For any linear congestion game, continuous replicator dynamics converges to a fixed point (pointwise convergence).

**Proof** First of all, we observe that

\[ \Psi(p) = \sum_i \hat{c}_i + \sum_{i,\gamma} \left( \sum_{e \in \gamma} (b_e + a_e) p_{i\gamma} \right) \]

is a Lyapunov function for our game since

\[
\frac{\partial \Psi}{\partial p_{i\gamma}} = c_{i\gamma} + \sum_{j \neq i} p_{j\gamma'} \frac{\partial c_{j\gamma'}}{p_{i\gamma}} + \sum_{e \in \gamma} (b_e + a_e)
\]

\[ = c_{i\gamma} + \sum_{j \neq i} \sum_{\gamma'} \left( \sum_{e \in \gamma \cap \gamma'} a_e p_{j\gamma'} + \sum_{e \in \gamma} (b_e + a_e) \right) \]

\[ = 2c_{i\gamma} \]

and hence

\[ \frac{d\Psi}{dt} = \sum_{i,\gamma} \frac{\partial \Psi}{\partial p_{i\gamma}} \frac{dp_{i\gamma}}{dt} \]

\[ = -\sum_{i,\gamma,\gamma'} p_{i\gamma} p_{i\gamma'} (c_{i\gamma} - c_{i\gamma'})^2 \leq 0 \]

with equality at fixed points. Hence (as in [15]) we have convergence to equilibria sets (compact connected sets consisting of fixed points). We address the fact that this doesn’t suffice for pointwise convergence. To be exact it suffices only in the case the equilibria are discrete (which is not the case for linear congestion games - see [14]).
Let \( q \) be a limit point of the trajectory \( p(t) \) where \( p(t) \) is in the interior of \( \Delta \) for all \( t \in \mathbb{R} \) (since we started from an initial condition inside \( \Delta \)) then we have that \( \Psi(q) < \Psi(p(t)) \). We define the relative entropy

\[
I(p) = -\sum_i \sum_{\gamma: q_{i\gamma} > 0} q_{i\gamma} \ln(p_{i\gamma} / q_{i\gamma}) \geq 0 \quad \text{(Jensen's ineq.)}
\]

and \( I(p) = 0 \) iff \( p = q \). We get that

\[
\frac{dI}{dt} = -\sum_i \sum_{\gamma: q_{i\gamma} > 0} q_{i\gamma}(\hat{c}_i - c_i)
\]

\[
= -\sum_i \hat{c}_i + \sum_{i, \gamma} q_{i\gamma} c_{i\gamma}
\]

\[
= -\sum_i \hat{c}_i + \sum_{i, \gamma \in E} (b_e + a_e)q_{i\gamma} + \sum_{i, \gamma \in E} (b_e + a_e)q_{i\gamma} + \sum_{i, \gamma \in E} a_e q_{i\gamma} p_{j\gamma}
\]

\[
= -\sum_i \hat{c}_i + \sum_{i, \gamma \in E} (b_e + a_e)q_{i\gamma} - \sum_{i, \gamma \in E} (b_e + a_e)p_{i\gamma} + \sum_{i, \gamma \in E} p_{i\gamma}(d_i - d_{i\gamma})
\]

\[
= \sum_i \hat{d}_i - \sum_i \hat{c}_i + \sum_{i, \gamma \in E} (b_e + a_e)q_{i\gamma} - \sum_{i, \gamma \in E} (b_e + a_e)p_{i\gamma} - \sum_{i, \gamma \in E} p_{i\gamma}(d_i - d_{i\gamma})
\]

\[
= \Psi(q) - \Psi(p) - \sum_{i, \gamma} p_{i\gamma}(\hat{d}_i - d_{i\gamma})
\]

where \( d_{i\gamma}, \hat{d}_i \) correspond to the cost of player \( i \) if he chooses strategy \( \gamma \) and his expected cost respectively at point \( q \). The rest of the proof follows in a similar way to Losert and Akin.

We break the term \( \sum_{i, \gamma} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) \) to positive and negative terms (don’t care about zero terms), i.e., \( \sum_{i, \gamma} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) = \sum_{i, \gamma: d_i > d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) + \sum_{i, \gamma: d_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) \)

**Claim:** There exists an \( \epsilon > 0 \) so that the function

\[
Z(p) = I(p) + 2 \sum_{i, \gamma: d_i < d_{i\gamma}} p_{i\gamma}
\]

has \( \frac{dZ}{dt} < 0 \) for \( \|p - q\| < \epsilon \) and \( \Psi(q) < \Psi(p) \).

Assuming that \( p \to q \), we get \( \hat{c}_i - c_i \to \hat{d}_i - d_{i\gamma} \) for all \( i, \gamma \). Hence for small enough \( \epsilon > 0 \) with \( \|p - q\| < \epsilon \), we have that \( \hat{c}_i - c_i \leq 3/4 \hat{d}_i - d_{i\gamma} \) for the terms which \( \hat{d}_i - d_{i\gamma} < 0 \). Therefore

\[
\frac{dZ}{dt} = \Psi(q) - \Psi(p) - \sum_{i, \gamma: d_i > d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) - \sum_{i, \gamma: d_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) + 2 \sum_{i, \gamma: d_i > d_{i\gamma}} p_{i\gamma}(\hat{c}_i - c_i)
\]

\[
\leq \Psi(q) - \Psi(p) - \sum_{i, \gamma: d_i > d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) - \sum_{i, \gamma: d_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) + 3/2 \sum_{i, \gamma: d_i > d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma})
\]

\[
= \frac{\Psi(q) - \Psi(p)}{<0} + \sum_{i, \gamma: d_i > d_{i\gamma}} -p_{i\gamma}(\hat{d}_i - d_{i\gamma}) + 1/2 \sum_{i, \gamma: d_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) < 0
\]
where we substitute $\frac{p_i}{\partial t} = p_i (\hat{c}_i - c_{i\gamma})$ (replicator), and the claim is proved.

Notice that $Z(p) \geq 0$ (sum of positive terms and $I(p) \geq 0$) and is zero iff $p = q$.

(i) To finish the proof of the theorem, if $q$ is a limit point of $p(t)$, there exists an increasing sequence of times $t_n$ with $t_n \to \infty$ and $p(t_n) \to q$. We consider $\epsilon'$ such that the set $C = \{p : Z(p) < \epsilon'\}$ is inside $B = |p - q| < \epsilon$ where $\epsilon$ is from claim above. Since $p(t_n) \to q$, consider a time $t_N$ where $p(t_N)$ is inside $C$. From claim above we get that $Z(p)$ is decreasing inside $B$ (and hence inside $C$), thus $Z(p(t)) \leq Z(p(t_N)) < \epsilon'$ for all $t \geq t_N$, hence the orbit will remain in $C$. By the fact that $Z(p(t))$ is decreasing in $C$ (claim above) and also $Z(p(t_n)) \to Z(q) = 0$ it follows that $Z(p(t)) \to 0$ as $t \to \infty$. Hence $p(t) \to q$ as $t \to \infty$ using (i).

By showing pointwise convergence, we can perform quantitative average case analysis (average PoA) for linear congestion games. In [15] showed that in congestion games, every stable fixed point is a weakly stable Nash Equilibrium. The following theorem (that assumes pointwise convergence) has a corollary that for all but measure zero initial conditions, replicator dynamics converges to a weakly stable Nash Equilibrium.

**Theorem 5.** The set of initial conditions so that replicator dynamics converges to unstable fixed points has measure zero in $\Delta$.

**Proof** See appendix.

**Corollary 6.** For all but measure zero initial conditions, replicator dynamics converges to weakly stable Nash equilibria.

## 5 Discrete Case

### 5.1 Convergence and Stability

We prove that $\Psi$ consists a Lyapunov type also for the Discrete Replicator Dynamics. The proof uses Hölder’s inequality and is much more technical than the proof of the continuous case (which is just simple derivation)

**Theorem 7.** The potential function is decreasing, i.e. $\Psi(p(t+1)) \leq \Psi(p(t))$ and equality $\Psi(p(t+1)) = \Psi(p(t))$ holds only at fixed points.

**Proof** See appendix.

**Lemma 8.** The Jacobian is given by the equations below

\[
\begin{align*}
\frac{\partial \xi_{i\gamma}}{\partial p_{i\gamma}} &= \frac{x - c_{i\gamma}}{x - \hat{c}_i} + p_{i\gamma} \frac{(x - c_{i\gamma})c_{i\gamma}}{(x - \hat{c}_i)^2} \\
\frac{\partial \xi_{i\delta}}{\partial p_{i\delta}} &= p_{i\gamma} \frac{(x - c_{i\gamma})}{(x - \hat{c}_i)^2} c_{i\delta} \\
\frac{\partial \xi_{j\gamma}}{\partial p_{j\delta}} &= p_{i\gamma} \frac{(c_i - x) \sum_{e \in \gamma \cap \delta} a_e - (c_{i\gamma} - x) \sum_{e \in \gamma} p_{i\gamma} \sum_{e \in \gamma \cap \delta} a_e}{(x - \hat{c}_i)^2} 
\end{align*}
\]
In order to analyze the stability of the system we need to consider a projection to a lower dimensional space. One can observe that \( \max |J_{(i, \gamma)^T}(i, \gamma)| > 1 \) and hence there is always an eigenvalue with absolute value greater than 1. But in case our initial condition is in \( \Delta \) (i.e. we start from a probability distribution) this fact doesn’t give information about the stability in \( \Delta \). Let \( q \) be a fixed point of our state space \( \Delta^M \) and \( \Sigma = \bigcup_i S_i \). We define \( h_q : [N] \to [\Sigma] \) be a function such that \( h_q(i) = \gamma \) if \( q_{r_i} > 0 \) for some \( \gamma \in S_i \). We compute the Jacobian by substituting (variable) \( p_i, h_q(i) = 1 - \sum_{\gamma \neq h_q(i)} p_{i\gamma} \). Let \( J^q \) be ”projected” Jacobian at \( q \) (equations in appendix). The characteristic polynomial is

\[
\prod_{i, \gamma : q_{i\gamma} > 0} \left( \lambda - \frac{X - c_{i\gamma}}{x - c_i} \right) \times \text{det}(\lambda I - J^q)
\]

where \( J^q \) corresponds to \( J^q \) by deleting rows, columns \( i, \gamma \) such that \( q_{i\gamma} = 0 \).

We establish same results as in continuous case\(^{15}\) for stability of discrete replicator dynamics.

**Definition 9.** We call a fixed point \( q \) stable if \( J^q \) has eigenvalues with absolute value at most 1.

**Lemma 10.** Every stable fixed point is a Nash Equilibrium

**Proof** Assume that a fixed point \( p \) is not a Nash equilibrium. Then there exists a player \( i \) and a strategy \( \gamma \in S_i \) such that \( p_{i\gamma} = 0 \) and \( c_{i\gamma} < c_i \). Then the characteristic polynomial has \( \frac{X - c_{i\gamma}}{x - c_i} > 1 \) as a root.

Let \( t \times t \) be the size of \( J^q \) and \( M^\gamma,\gamma',\delta,\delta' \)\(^{2}\) be \( \sum_{p \in \Delta} a_p - \sum_{p \in \Delta} a_p - \sum_{p \in \Delta} a_p + \sum_{p \in \Delta} a_p \). We show the following lemma:

**Lemma 11.**

\[
\text{tr}((J^q)^2) = t + \sum_{i,j} \frac{1}{(x - c_i)(x - c_j)} \sum_{\gamma \neq \gamma'} \sum_{s, s' \neq h_q(i), h_q(j)} p_{i\gamma} p_{i\gamma'} p_{j\delta} p_{j\delta'} (M^\gamma,\gamma',\delta,\delta')^2
\]

\[
+ \sum_{i,j} \frac{1}{(x - c_i)(x - c_j)} \sum_{\gamma \neq \gamma'} \sum_{s, s' \neq h_q(i), h_q(j)} p_{i\gamma} p_{i\gamma'} p_{j\delta} p_{j\delta'} (M^\gamma,\gamma',\delta,\delta')^2
\]

\[
+ 2 \sum_{i,j} \frac{1}{(x - c_i)(x - c_j)} \sum_{\gamma \neq \gamma', \gamma \neq h_q(i)} p_{i\gamma} p_{i\gamma'} p_{j\delta} p_{j\delta'} (M^\gamma,\gamma',\delta,\delta')^2
\]

**Proof** See appendix.

**Corollary 12.** Every stable fixed point is a weakly stable Nash.

**Proof** See appendix.
6 Applications

6.1 Balls and Bins

In this section we consider the classic game of $n$ identical balls, with $n$ identical bins. Each ball chooses a distribution over the bins selfishly and we assume that the ”delay” of bin $\gamma$ is equal to $\gamma$’s load. We know for this game that the PoA is $\Omega(\frac{\log n}{\log \log n})$ \[6\]. We will prove that the Average PoA is actually 1 (in the replicator setting).

Lemma 13. In the problem of $n$ identical balls and $n$ identical bins every weakly stable Nash Equilibrium is pure.

From theorem lemma \[13\] we get that for all but measure zero starting points of $g(\Delta)$, the replicator converges to pure Nash Equilibria. Every pure Nash Equilibrium (each ball chooses a distinct bin) has social cost (makespan) 1 which is also the optimal. Hence the Average PoA is trivially 1.

Remark: The lemma below shows how crucial is Lindelöf lemma (essentially separability of $R^m$ for all $m$) in the proof of \[5\] that allow us to ”group” the equilibrium in countably many open balls because the set of Nash equilibria might be uncountable.

Lemma 14. Let $n \geq 4$ then the set of Nash equilibria of the $n$ balls $n$ bins game is uncountable.

6.2 Exact Quantitative Analysis of Risk Dominance in the Stag Hunt Game

The Stag Hunt game has two pure Nash equilibria, $(\text{Stag, Stag})$ and $(\text{Hare, Hare})$ and a symmetric mixed Nash equilibrium with each agent choosing strategy Hare with probability 2/3. Stag Hunt is payoff equivalent to a coordination game\[6\]. Coordination games are potential games where the potential function in each state is equal to the utility of each agent. Replicator dynamic converges to pure Nash equilibria in this setting with probability 1. When we study the replicator dynamic here, it suffices to examine its projection in the subspace $p_{1s} \times p_{2s} \subset (0,1)^2$ which captures the evolution of the probability that each agent assigns to strategy Stag. We compute using an invariant property of relative entropy the size of each region of attraction in this space and thus

\[5\] Union bound doesn’t hold to prove that a set has measure zero if you have uncountable union of measure zero sets

\[6\] If each agent reduces their payoff in their first column by 4, these results to the depicted coordination game.
provide a quantitative analysis of risk dominance in the classic Stag Hunt game (details can be found in appendix).

**Theorem 15.** The region of attraction of \((Hare, Hare)\) is the subset of \((0,1)^2\) that satisfies \(p_{2s} < \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})\) and has Lebesgue measure \(\frac{1}{27}(9 + 2\sqrt{3}) \approx 0.7364\). The region of attraction of \((Stag, Stag)\) is the subset of \((0,1)^2\) that satisfies \(p_{2s} > \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})\) and has Lebesgue measure \(\frac{1}{27}(18 - 2\sqrt{3}) \approx 0.2636\). The stable manifold of the mixed Nash equilibrium satisfies the equation \(p_{2s} = \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})\) and has 0 Lebesgue measure.

**Proof** See appendix.

### 6.3 Average Price of Anarchy Analysis in Coordination Games via Polytope Approximations of Regions of Attraction

In the previous section, we used the algebraic equations implied from lemma 26 to find an explicit description of the stable/unstable manifold of the mixed Nash equilibrium. These in turn were used to compute exactly the size of the regions of attraction for each of the two pure Nash equilibria. As we move away from single instance games towards classes of games, the task of finding exact explicit descriptions of the topology of the attractor landscape becomes infeasible quickly.

We focus on the following generalized class of Stag Hunt game, as described in figure 1(b). The \(w\) parameter is greater or equal to 1. We denote an instance of such a game as \(G(w)\). It is straightforward to check that for \(G(2)\) is equivalent to the standard stag hunt game (modulo payoff shifts to the agents). The invariant property of the replicator dynamics for \(G(w)\) translates to \(p_{1s}^w(1 - p_{1s}) = p_{2s}^w(1 - p_{2s})\). The presence of the parameter \(w\) on the exponent precludes the existence of a simple, explicit, parametric description of all the solutions. We analyze the topology of the basins of attractions and produce simple subsets/supersets of them. The volume of these polytope approximations can be computed explicitly and these measures can be used to provide upper and lower bounds on the average system performance and average price of anarchy.

For any \(w\), \(G(w)\) is a coordination/potential game and therefore it is payoff equivalent to a congestion game. The only two weakly stable equilibria are the pure ones, hence in order to understand the average case system performance it suffices to understand the size of regions of attraction for each of them. As in the case of Stag Hunt game, we focus on the projection of the system to the subspace \((p_{1s}, p_{2s}) \subset [0,1]^2\). We show the following:

**Theorem 16.** The average price of anarchy of \(G(w)\) with \(w \geq 1\) is at most \(\frac{w^2+w}{w(w+1)^2-2w+2}\) and at least \(\frac{w}{w(w+1)^2} \leq 1\).

**Proof** See appendix for details and figure.

By combining the exact analysis of the standard Stag Hunt game (theorem 15) and theorem 16 we derive that:

**Corollary 17.** The average price of anarchy of the class of \(G(w)\) games with \(w > 0\) is at least \(\frac{2}{1 + \frac{3\sqrt{3}}{4}} \approx 1.15\) and at most \(\frac{4+3\sqrt{3}}{4+2\sqrt{2}} \approx 1.21\). In comparison, the price of anarchy for this class of games is unbounded.

---

\(^7\)It is easy to see that for any \(0 < w < 1\), \(G(w)\) is isomorphic to \(G(1/w)\) after relabeling of strategies.
Our analysis essentially shows that as \( w \) grows, the size of the attraction basin of the optimal equilibrium grows at a sufficiently fast pace to counteract the presence of any increasingly suboptimal equilibrium.

### 6.4 Average PoA \( 5/2 - \epsilon \)

We prove the following lemma which raises an interesting question for future work discussed in the conclusion.

**Lemma 18.** For any congestion game \( G \) with linear costs there exists a constant \( \epsilon(G) \) so that the Average PoA is at most \( 5/2 - \epsilon \) for replicator dynamic.

**Proof** We consider among all the pure Nash Equilibrium of the game \( G \), one \( q \) with minimum \( \Phi \). Since for all pure strategies \( s \) we have that \( \Phi(s) \leq sc(s) \leq 2\Phi(s) \), the following inequalities holds for all mixed strategies \( p \)

\[
\Psi(p) \leq sc(p) \leq 2\Psi(p) \tag{3}
\]

\[
sc(q) \leq 2\Phi(q) = 2\Psi(q) \leq 2\Psi(p) \leq 2sc(p) \tag{4}
\]

We consider the open interval \( U = (0, 1.24 \cdot \Psi(q)) \). Hence the set \( V = g(\Psi^{-1}(U)) \) must be open since \( \Psi \circ g^{-1} \) is continuous. Finally \( \text{int}(g(\Delta)) \cap V \) is open (intersection of open sets) and nonempty, hence it has positive measure \( \epsilon' > 0 \) in \( \mathbb{R}^{M-N} \). Assume that the minimum social cost has value \( \text{opt} \) and also by (3) it follows that \( \Psi(q) \leq \text{opt} \) (since \( \Psi(q) \leq sc(p) \) for all mixed strategies \( p \)).

For all \( x \in V \) we have that \( \Psi(g^{-1}(x)) \leq 1.24\Psi(q) \) and the fact that \( \Psi \) is decreasing (w.r.t time) we get that \( \omega \)-limit point of \( x \) will have potential at most \( 1.24\Psi(q) \), therefore social cost at most \( 2.48\Psi(q) \) (using (3), thus at most \( 2.48\text{opt} \). For all \( x \in \text{int}(g(\Delta)), x \notin V \), the limit point of \( x \) will be a Nash Equilibrium with cost at most \( \frac{5}{2}\text{opt} \) (PoA \( \leq 5/2 \) in [4, 28]). Therefore the Average PoA is at most

\[
\frac{2.48\text{opt} \cdot \epsilon' + 2.5\text{opt} \cdot (\mu(g(\Delta)) - \epsilon')}{\text{opt} \cdot \mu(g(\Delta))} = 2.5 - \frac{0.02\epsilon'}{\mu(g(\Delta))}\]

\[\square\]

### 7 Conclusion and Future Work

#### 7.1 Summary

In this paper, we attempt a novel approach to the problem of equilibrium selection. Our approach does not exclude apriori any equilibrium points, but instead weighs all equilibria according to their likelihood, as it is captured by the size of their attractors. This approach raises exciting new questions and shows the promise of introducing powerful new tools and techniques in algorithmic game theory.

#### 7.2 Questions

As far as continuous replicator is concerned, it would be interesting to generalize the pointwise convergence result to a larger class of congestion games (generally we know...
that continuous replicator dynamics don’t converge), maybe for congestion games with polynomial cost functions. Another interesting question is to prove something stronger than \[ \epsilon \] i.e if there is a global constant \( \epsilon \) (not depending on the game) so that every linear congestion game has average price of anarchy less than \( \frac{1}{2} - \epsilon \). Finally (probably most difficult one) would be to find an algorithm that approximates the average price of anarchy for other classes of linear congestion/potential games (essentially approximates the regions of attraction).

Another interesting question is whether pointwise convergence to equilibria also holds for this discrete time variant of the replicator using similar or different techniques. We conjecture pointwise convergence to be true. This will allow us to perform average case analysis as well (with the current machinery).

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A Appendix

A.1 Missing proofs

A.1.1 Proof of lemma 2

Proof Clearly \( \psi(x) = g(\lim_{n \to \infty} \phi_n(g^{-1}(x))) = \lim_{n \to \infty} g \circ \phi_n \circ g^{-1}(x) \) where \( n \) is over positive integers (since we have always convergence). For an arbitrary \( c \in \mathbb{R} \) we have that

\[
\{ x : \psi(x) < c \} = \bigcup_{k=1}^{\infty} \cup_{m=1}^{\infty} \cap_{n>m} \{ x : g \circ \phi_n \circ g^{-1}(x) < c - \frac{1}{k} \}
\]

The set \( \{ x : g \circ \phi_n \circ g^{-1}(x) < c - \frac{1}{k} \} \) is measurable since \( g \circ \phi_n \circ g^{-1}(x) \) is measurable, by continuity of \( g, g^{-1}, \phi \) (so continuous is the composition). Therefore \( \psi(x) \) is a measurable function and so it is \( \psi(x) \).

A.1.2 Proof of theorem 5

Let \( q \) be a point of our state space \( \Delta^M \) and \( \Sigma = \bigcup S_i \). Let \( h_q : [N] \to [\Sigma] \) be a function such that \( h_q(i) = \gamma \) if \( q_\gamma > 0 \) for some \( \gamma \in S_i \) (same definition with discrete case). We consider the mapping \( z_q : R^M \to R^{M-N} \) so that we exclude from each player \( i \) the variable \( p_{i,h_q(i)} \) (\( z_q \) plays the same role as \( g \) but we drop variables with specific property this time). We substitute the variables \( p_{i,h_q(i)} \) with \( 1 - \sum_{\gamma \notin S_i} p_{i,\gamma} \). Let \( J_q \) be the reduced Jacobian at a fixed point \( z_q(q) \).

Definition 19. A fixed point \( q \in \Delta \subset \mathbb{R}^M \) is called unstable if \( J_q \) has an eigenvalue with positive real part. Otherwise is called stable.

To prove theorem 5 we will make use of the following important theorem in dynamical systems.

Theorem 20. (Center and Stable Manifolds, p. 65 of [32]) Let 0 be a fixed point for the \( C^r \) local differomorphism \( f : U \to \mathbb{R}^n \) where \( U \subset \mathbb{R}^n \) is a neighborhood of zero in \( \mathbb{R}^n \) and \( r \geq 1 \). Let \( E^s \oplus E^c \oplus E^u \) be the invariant splitting of \( \mathbb{R}^n \) into generalized eigenspaces of \( Df(0) \) corresponding to eigenvalues of absolute value less than one, equal to one, and greater than one. To the \( Df(0) \) invariant subspace \( E^s \oplus E^c \) there is associated a local \( f \) invariant \( C^r \) embedded disc \( W^{sc}_{loc} \) tangent to the linear subspace at 0 and a ball \( B \) around zero such that:

\[
f(W^{sc}_{loc}) \cap B \subset W^{sc}_{loc}. \text{ If } f^n(x) \in B \text{ for all } n \geq 0, \text{ then } x \in W^{sc}_{loc}
\]
For \( t = 1 \) and an unstable fixed point \( p \) we consider the function \( \psi_{1,p}(x) = z_p \circ \phi_t \circ z_p^{-1}(x) \) which is \( C^1 \) local diffeomorphism. Let \( B_{z_p(p)} \) be the ball that is derived from \( 20 \) and we consider the union of these balls (transformed in \( \mathbb{R}^M \))

\[
A = \bigcup_p A_{z_p(p)}
\]

where \( A_{z_p(p)} = z_p(B_{z_p(p)}) \) (\( z_p^{-1} \) “returns” the set \( B_{z_p(p)} \) back to \( \mathbb{R}^M \)). Taking advantage of separability of \( \mathbb{R}^M \) we have the following theorem.

**Theorem 21.** (Lindelöf’s lemma) For every open cover there is a countable subcover.

Therefore we can find a countable subcover for \( A = \bigcup_p A_{z_p(p)} \), i.e., \( A = \bigcup_{m=1}^{\infty} A_{z_m(p_m)} \).

Let \( \psi_{n,p}(x) = z_p \circ \phi_n \circ z_p^{-1}(x) \). If a point \( x \in \text{int} g(\Delta) \) (which corresponds to \( g^{-1}(x) \) in our original \( \Delta \)) has as unstable fixed point as a limit, there must exist a \( n_0 \) and \( m \) so that \( \psi_{m,p_m} \circ z_{m_0} \circ g^{-1}(x) \in B_{z_{m_0}(p_m)} \) for all \( n \geq n_0 \) and therefore again from \( 20 \) we get that \( \psi_{m_0,p_m} \circ z_{m_0} \circ g^{-1}(x) \in W_{\text{loc}}^{s,c} z_{m_0}(p_m) \), hence \( x \in g \circ z_{m_0} \circ \psi_{m_0,p_m}^{-1}(W_{\text{loc}}^{s,c} z_{m_0}(p_m)) \).

Hence the set of points in \( \text{int} g(\Delta) \) whose \( \omega \)-limit has an unstable equilibrium is a subset of

\[
C = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} g \circ z_{m_0} \circ \psi_{m_0,p_m}^{-1}(W_{\text{loc}}^{s,c} z_{m_0}(p_m))
\]

(6)

Observe that the dimension of \( W_{\text{loc}}^{s,c} z_{m_0}(p_m) \) is at most \( M - N - 1 \) since we assume that \( p_m \) is unstable (\( J_{p_m} \) has an eigenvalue with positive real part) \(^8\) and thus \( \dim E^u \geq 1 \), hence the manifold has Lebesgue measure zero in \( \mathbb{R}^{M-N} \). Finally since \( g \circ z_{m_0} \circ \psi_{m_0,p_m} : \mathbb{R}^{M-N} \to \mathbb{R}^{M-N} \) is continuously differentiable (\( \psi_{m_0,p_m} \) is a \( C^1 \) it is locally Lipschitz (see [25] p.71) and it preserves the null-sets (see lemma 22)\(^9\)). Namely, \( C \) is a countable union of measure zero sets, i.e., is measure zero as well and the theorem 5 follows \(^9\).

**Lemma 22.** Let \( g : \mathbb{R}^n \to \mathbb{R}^n \) be a locally Lipschitz function, then \( g \) is null-set preserving, i.e., for \( E \subset \mathbb{R}^n \) if \( E \) has measure zero then \( g(E) \) has also measure zero.

**Proof** Let \( B_\gamma \) be an open ball such that \( ||g(y) - g(x)|| \leq K_\gamma ||y - x|| \) for all \( x, y \in B_\gamma \). We consider the union \( \bigcup \gamma B_\gamma \) which cover \( \mathbb{R}^n \) by the assumption that \( g \) is locally lipschitz. By Lindelöf’s lemma we have a countable subcover, i.e., \( \bigcup_{i=1}^{\infty} B_{\gamma_i} \). Let \( E_i = E \cap B_{\gamma_i} \). We will prove that \( g(E_i) \) has measure zero. Fix an \( \epsilon > 0 \). Since \( E_i \subset E \), we have that \( E_i \) has measure zero, hence we can find a countable cover of open balls \( C_1, C_2, \ldots \) for \( E_i \), namely \( E_i \subset \bigcup_{j=1}^{\infty} C_j \) so that \( C_j \subset B_i \) for all \( j \) and also \( \sum_{j=1}^{\infty} \mu(C_j) < \frac{\epsilon}{K_\gamma^n} \). Since \( E_i \subset \bigcup_{j=1}^{\infty} C_j \) we get that \( g(E_i) \subset \bigcup_{j=1}^{\infty} g(C_j) \), namely \( g(C_j) \) \( \cap \) \( g(C_j) \) ... cover \( g(E_i) \) and also \( g(C_j) \subset B_i \) for all \( j \). Assuming that ball \( C_j \equiv B(x, r) \) (center \( x \) and radius \( r \) then it is clear that \( g(C_j) \subset B(g(x), K_\gamma r) \) (\( g \) maps the center \( x \) to \( g(x) \) and the radius \( r \) to \( K_\gamma r \) because of lipschitz assumption). But \( \mu(B(x, K_\gamma r)) = K_\gamma^n \mu(B(x, r)) = K_\gamma^n \mu(C_j) \), therefore \( \mu(g(C_j)) \leq K_\gamma^n \mu(C_j) \) and so we conclude that

\[
\mu(g(E_i)) \leq \sum_{j=1}^{\infty} \mu(g(C_j)) \leq K_\gamma^n \sum_{j=1}^{\infty} \mu(C_j) < \epsilon
\]

Since \( \epsilon \) was arbitrary, it follows that \( \mu(g(E_i)) = 0 \).

To finish the proof, observe that \( g(E) = \bigcup_{i=1}^{\infty} g(E_i) \) therefore \( \mu(g(E)) \leq \sum_{i=1}^{\infty} \mu(g(E_i)) = 0 \).

---

\(^8\)Here we used the fact that the eigenvalues with absolute value less than one, one and greater than one of \( e^{\lambda t} \) correspond to eigenvalues with negative real part, zero real part and positive real part respectively of \( A \)

\(^9\)we used “silently” that \( \mu(g(\Delta)) = \mu(\text{int} g(\Delta)) \)
A.1.3 Proof of theorem 7

Proof Observe that $x = N\sum_{e} a_e + \sum_{e} b_e = \sum_{j \neq i} \sum_{\delta, e} a_{\delta, e} p_{j\delta} + \sum_{e} (a_e + b_e)$ and hence

$$x - c_{i\gamma} = \sum_{j \neq i} \sum_{\delta} \sum_{e \notin \gamma \cap \delta} a_{\delta, e} p_{j\delta} + \sum_{e \notin \gamma} (a_e + b_e)$$

Therefore we get

$$\sum_{i} (x - \hat{c}_i) = \sum_{i, \gamma} p_{i\gamma} (x - c_{i\gamma}) = 2Q - \sum_{i, \gamma} \sum_{e \notin \gamma} (a_e + b_e)$$

Thus $2Q + \Psi = Nx + N\sum_{e} (a_e + b_e)$ which is constant. It suffices to prove that $Q(p(t+1)) \ge Q(p(t))$ and $Q(p(t+1)) = Q(p(t))$ only at fixed points, i.e $Q$ is increasing with equality at fixed points.

$$Q(p(t)) = \frac{1}{2} \sum_{i, \gamma} \sum_{j \neq i} \sum_{\delta} \sum_{e \notin \gamma \cap \delta} a_{\delta, e} p_{i\gamma} p_{j\delta} + \sum_{i, \gamma} (\sum_{e \notin \gamma} (a_e + b_e)) p_{i\gamma}$$

$$= \sum_{i, \gamma} \sum_{j \neq i} \sum_{\delta} \sum_{e \notin \gamma \cap \delta} \left( \frac{1}{2} a_{\delta, e} \frac{x - c_{i\gamma}}{x - \hat{c}_i} \frac{x - c_{j\delta}}{x - \hat{c}_j} p_{i\gamma} p_{j\delta} \right)^{\frac{1}{3}} \left( \frac{1}{2} a_{\delta, e} \frac{x - \hat{c}_i}{x - c_{i\gamma}} \frac{x - \hat{c}_j}{x - c_{j\delta}} p_{i\gamma} p_{j\delta} \right)^{\frac{2}{3}}$$

$$+ \sum_{i, \gamma} \left( \sum_{e \notin \gamma} (a_e + b_e) p_{i\gamma} \frac{x - c_{i\gamma}}{x - \hat{c}_i} \right) \left( \sum_{e \notin \gamma} (a_e + b_e) p_{i\gamma} \frac{x - \hat{c}_i}{x - c_{i\gamma}} \right)^{\frac{1}{3}}$$

$$\le \left( \sum_{i, \gamma} \sum_{j \neq i} \sum_{\delta} \sum_{e \notin \gamma \cap \delta} \frac{1}{2} a_{\delta, e} \frac{x - c_{i\gamma}}{x - \hat{c}_i} \frac{x - c_{j\delta}}{x - \hat{c}_j} p_{i\gamma} p_{j\delta} + \sum_{i, \gamma} (\sum_{e \notin \gamma} (a_e + b_e)) \frac{x - c_{i\gamma}}{x - \hat{c}_i} p_{i\gamma} \right)^{\frac{1}{3}}$$

$$\times \left( \sum_{i, \gamma} \sum_{j \neq i} \sum_{\delta} \sum_{e \notin \gamma \cap \delta} a_{\delta, e} \frac{1}{2} p_{i\gamma} p_{j\delta} \sqrt{\frac{x - \hat{c}_i}{x - c_{i\gamma}} \frac{x - \hat{c}_j}{x - c_{j\delta}}} + \sum_{i, \gamma} (\sum_{e \notin \gamma} (a_e + b_e)) p_{i\gamma} \sqrt{\frac{x - \hat{c}_i}{x - c_{i\gamma}}} \right)^{\frac{2}{3}}$$

where we used Hölder’s inequality in the following form

$$\sum_{i} x_i^{1/3} y_i^{2/3} \le \left( \sum_{i} x_i \right)^{1/3} \left( \sum_{i} y_i \right)^{2/3}$$

We will show that the second term of the last product is at most $Q(p(t))$. Using the fact that $\sqrt{ab} \le \frac{a+b}{2}$ (with equality iff $a = b$) we get that
Equality holds iff $p = 0$ or $\hat{c}_i = c_i$ for all $i, \gamma$ (namely at fixed points)

**A.1.4** Equations of the $J^q$ at section 5.1

\[
J^q_{(i, \gamma), (i, \gamma)} = \frac{x - c_i}{x - \hat{c}_i} + \frac{q_{i\gamma}(x - c_i) c_{i\gamma}}{(x - \hat{c}_i)^2} - \frac{q_{i\gamma}(x - c_i) c_i h_q(i)}{x - \hat{c}_i} = \frac{x - c_i}{x - \hat{c}_i}
\]

\[
J^q_{(i, \gamma), (i, \delta)} = q_{i\gamma} \frac{(x - c_i) c_{i\delta}}{(x - \hat{c}_i)^2} - q_{i\gamma} \frac{(x - c_i) c_i h_q(i)}{x - \hat{c}_i} = \frac{(x - c_i) c_{i\delta}}{(x - \hat{c}_i)^2} - \frac{(x - c_i) c_i h_q(i)}{x - \hat{c}_i}
\]

\[
J^q_{(i, \gamma), (j, \delta)} = q_{i\gamma} \frac{(c_j - x) \sum_{e\in\Gamma\cap\delta} a_e - (c_i - x) \sum_{e\in\Gamma\cap\delta} \sum_{e\in\gamma\cap\delta} a_e}{(x - \hat{c}_i)^2} = \frac{(c_j - x) \sum_{e\in\Gamma\cap\delta} a_e - (c_i - x) \sum_{e\in\Gamma\cap\delta} \sum_{e\in\gamma\cap\delta} a_e}{(x - \hat{c}_i)^2}
\]

**A.1.5** Proof of theorem 11

**Proof** Since $\mathbb{J}^{q}_{i\gamma, i\delta} = 0$ for $\gamma \neq \delta$ and $\mathbb{J}^{q}_{i\gamma, i\delta} = 1$ we get that

\[
tr((\mathbb{J}^q)^2) = t + \sum_{\gamma \in S_i} \sum_{\delta \in S_j} \mathbb{J}^{q}_{i\gamma, j\delta} \mathbb{J}^{q}_{j\delta, i\gamma}
\]

We consider the following cases:

- Let $i < j$, $\gamma < \gamma'$ with $\gamma, \gamma' \neq h_q(i)$ and $\delta < \delta'$ with $\delta, \delta' \neq h_q(j)$ and we examine the term $\frac{1}{(x - c_i)(x - c_j)} p_{i\gamma} p_{j\delta} p_{j\delta'}$ in the sum and we get that it appears

\[
\left[[M^\gamma, \gamma', \delta, h_q(j)] \times [M^\gamma, h_q(i), \delta', \delta]ight] + \left[[M^\gamma, \gamma', \delta, h_q(j)] \times [M^\gamma, h_q(i), \delta', \delta]ight]
\]

\[
+ \left[[M^\gamma, \gamma', \delta, h_q(j)] \times [M^\gamma, h_q(i), \delta', \delta]ight] + \left[[M^\gamma, \gamma', \delta, h_q(j)] \times [M^\gamma, h_q(i), \delta', \delta]ight] = (M^\gamma, \gamma', \delta', \delta')^2
\]
• Let \( i < j \), \( \gamma \neq h_q(i) \) and \( \delta \neq h_q(j) \). The term \( \frac{1}{(x-c_i)(x-c_j)}P_{\gamma,P_i,h_q(i)}P_{\gamma,P_j,h_q(j)} \) in the sum appears

\[
[M^{h_q(i),\gamma,h_q(j)}] \times [M^{\gamma,h_q(i),h_q(j),\delta}] = (M^{\gamma,h_q(i),\delta,h_q(j)})^2
\]

• Let \( \gamma < \gamma' \) with \( \gamma, \gamma' \neq h_q(i) \) and \( \delta \neq h_q(j) \). The term \( \frac{1}{(x-c_i)(x-c_j)}P_{\gamma,P_i,h_q(i)}P_{\gamma,P_j,h_q(j)} \) in the sum appears

\[
2[M^{\gamma',\gamma,h_q(j)}] \times [M^{\gamma,h_q(i),h_q(j),\delta}] + 2[M^{\gamma',\delta,h_q(j)}] \times [M^{\gamma',h_q(i),h_q(j),\delta}]
= 2(M^{\gamma',\delta,h_q(j)})^2
\]

\[\blacksquare\]

**Corollary 23.** Every stable fixed point is a weakly stable Nash equilibrium (discrete case).

**Proof** If the trace of \((J^q)^2\) (matrix has size \(t \times t\)) is larger than \(t\) then exists an eigenvalue of absolute value greater than one. Hence for a stable fixed point we must have \(M^{\gamma,\gamma',\delta,\delta'} = 0\). The rest follows from same argument as \[15\], theorem 3.8.


\[\blacksquare\]

**A.1.6 Proof of theorem \[15\]**

**Proof** Since Stag Hunt is payoff equivalent to a coordination game and has a fully mixed Nash equilibrium, 
\[
d\left(\frac{2}{3} \ln(\phi_{1s}(p,t)) + \frac{1}{3} \ln(1 - p_{1s}) - \frac{2}{3} \ln(p_{2s}) - \frac{1}{3} \ln(1 - p_{2s}) = \frac{2}{3} \ln(q_{1h}) + \frac{1}{3} \ln(1 - q_{1h}) - \frac{2}{3} \ln(q_{2h}) - \frac{1}{3} \ln(1 - q_{2h}) \right) = 0,
\]

then the fully mixed Nash equilibrium is symmetric. This condition is equivalent to \(p_{1s}^2(1 - p_{1s}) = p_{2s}^2(1 - p_{2s})\), where \(0 < p_{1s}, p_{2s} < 1\). It is straightforward to verify that this algebraic equation is satisfied by the following two distinct solutions, the diagonal line \((p_{2s} = p_{1s})\) and \(p_{2s} = \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})\).

Below, we show that these manifolds correspond indeed to the state and unstable manifold of the mixed Nash equilibrium, by showing that the Nash equilibrium satisfies those equations and by establishing that the vector field is tangent everywhere along them.

The case of the diagonal is trivial and follows from the symmetric nature of the game. We verify the claims about \(p_{2s} = \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})\). Indeed, the mixed equilibrium point in which \(p_{1s} = p_{2s} = 2/3\) satisfies the above equation. We establish that the vector filed is tangent to this manifold by showing in lemma \[24\] that

\[
\frac{\partial p_{2s}}{\partial p_{1s}} = \frac{c_{2s}}{c_{1s}} = \frac{p_{2s}(u_2(s)(1 - p_{2s}^2) + (1 - p_{2s})(u_2(h)))}{p_{1s}(u_1(s)(1 - p_{1s}^2) + (1 - p_{1s})(u_1(h)))},
\]

Finally, this manifold is indeed attracting to the equilibrium. Since the function \(p_{2s} = y(p_{1s}) = \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})\) is a strictly decreasing function of \(p_{1s}\) in \([0,1]\) and satisfies \(y(2/3) = 2/3\), this implies that its graph is contained in the subspace \((0 < p_{1s} < 2/3 \cap 2/3 < p_{2s} < 1) \cup (2/3 < p_{1s} < 1 \cap 0 < p_{2s} < 2/3)\). In each of these subsets \((0 < p_{1s} < 2/3 \cap 2/3 < p_{2s} < 1), (2/3 < p_{1s} < 1 \cap 0 < p_{2s} < 2/3)\) the replicator vector field coordinates have fixed signs that "push" \(p_{1s}, p_{2s}\) towards their respective equilibrium values.
The unstable manifold partitions the set $0 < p_{1s}, p_{2s} < 1$ into two subsets, each of which is flow invariant since the unstable manifold itself is flow invariant. Our convergence analysis for the generalized replicator flow implies that in each subset all but a measure zero of initial conditions must converge to its respective pure equilibrium. The size of the lower region of attraction\textsuperscript{10} is equal to the following definite integral $\int_0^1 \frac{1}{2} (1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{2s}^2}) dx = \left[ \frac{1}{2} \left( p_{1s} - \frac{p_{1s}^2}{2} + \left( -\frac{1}{2} + \frac{p_{1s}}{2} \right) \sqrt{1 + 2p_{1s} - 3p_{1s}^2} - \frac{2 \arcsin \left( \frac{1}{3} (1 - 3p_{1s}) \right)}{3\sqrt{3}} \right) \right]_0^1 = \frac{1}{2\pi} (9 + 2\sqrt{3\pi}) = 0.7364$ and the theorem follows.

We conclude by providing the proof of the following technical lemma:

**Lemma 24.** For any $0 < p_{1s}, p_{2s} < 1$, with $p_{2s} = \frac{1}{2} (1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})$ we have that:

$$
\frac{\partial p_{2s}}{\partial p_{1s}} = \frac{\zeta_{2s}}{\zeta_{1s}} = \frac{p_{2s}(u_2(s) - (p_{2s}u_2(s) + (1 - p_{2s})u_2(h)))}{p_{1s}(u_1(s) - (p_{1s}u_1(s) + (1 - p_{1s})u_1(h)))}
$$

**Proof** By substitution of the stag hunt game utilities, we have that:

$$
\frac{\zeta_{2s}}{\zeta_{1s}} = \frac{p_{2s}(u_2(s) - (p_{2s}u_2(s) + (1 - p_{2s})u_2(h)))}{p_{1s}(u_1(s) - (p_{1s}u_1(s) + (1 - p_{1s})u_1(h)))} = \frac{p_{2s}(1 - p_{2s})(3p_{1s} - 2)}{p_{1s}(1 - p_{1s})(3p_{2s} - 2)}
$$

However, $p_{2s}(1 - p_{2s}) = \frac{1}{2} p_{1s}(p_{1s} - 1 + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})$. Combining this with (7),

$$
\frac{\zeta_{2s}}{\zeta_{1s}} = \frac{1}{2} \left( p_{1s} - 1 + \sqrt{1 + 2p_{1s} - 3p_{1s}^2} \right)(3p_{1s} - 2) = \frac{1}{2} \left( \sqrt{1 + 3p_{1s} - \sqrt{1 - p_{1s}}} (3p_{1s} - 2) \right)
$$

Similarly, we have that $3p_{2s} - 2 = \frac{1}{2} \sqrt{1 + 3p_{1s} - \sqrt{1 - p_{1s}} (3p_{1s} - 2)}$. By multiplying and dividing equation (8) with $\sqrt{1 + 3p_{1s} + 3\sqrt{1 - p_{1s}}}$ we get:

$$
\frac{\zeta_{2s}}{\zeta_{1s}} = \frac{1}{2} \left( \sqrt{1 + 3p_{1s} + 3\sqrt{1 - p_{1s}}} (\sqrt{1 + 3p_{1s} - \sqrt{1 - p_{1s}}} (3p_{1s} - 2)) \right) = \frac{1}{2} \left( \frac{\sqrt{1 + 3p_{1s} + 3\sqrt{1 - p_{1s}}} (\sqrt{1 + 3p_{1s} - \sqrt{1 - p_{1s}}} (3p_{1s} - 2))}{\sqrt{1 + 3p_{1s} + 3\sqrt{1 - p_{1s}}}} \right)
$$

$$
= \frac{1}{2} \left( 1 - 3p_{1s} \right) \left( \frac{\partial}{\partial p_{1s}} \left( \frac{1}{2} (1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2}) \right) \right) = \frac{\partial p_{2s}}{\partial p_{1s}}.
$$

\[ A.1.7 \] **Proof of theorem 16**

We denote by $\zeta, \psi$, the projected flow and vector field respectively.

**Lemma 25.** All but a zero measure of initial conditions in the polytope $(P_{Hare})$:

$$
p_{2s} \leq -wp_{1s} + w
$$

$$
p_{2s} \leq \frac{1}{w}p_{1s} + 1
$$

$$
0 \leq p_{1s}, p_{2s} \leq 1
$$

\textsuperscript{10}This corresponds to the risk dominant equilibrium $(Hare, Hare)$. 

\[ 22 \]
converges to the \((Hare, Hare)\) equilibrium. All but a zero measure of initial conditions in the polytope \((P_{Stag})\):

\[
p_{2s} \geq -p_{1s} + \frac{2w}{w + 1}
\]

\[
0 \leq p_{1s}, p_{2s} \leq 1
\]

converges to the \((Stag, Stag)\) equilibrium.

**Proof** First, we will prove the claimed property for polytope \((P_{Stag})\). Since the game is symmetric, the replicator dynamics are similarly symmetric with \(p_{2s} = p_{1s}\) axis of symmetry. Therefore it suffices to prove the property for the polytope \(P_{Hare}' = P_{Hare} \cap \{p_{2s} \leq p_{1s}\} \cap \{p_{2s} \leq -wp_{1s} + w\} \cap \{0 \leq p_{1s} \leq 1\} \cap \{0 \leq p_{2s} \leq 1\}\) We will argue that this polytope is forward flow invariant, i.e., if we start from an initial condition \(x \in P_{Hare}'\) \(\psi(t, x) \in P_{Hare}'\) for all \(t > 0\). On the \(p_{1s}, p_{2s}\) subspace \(P_{Hare}'\) defines a triangle with vertices \(A = (0, 0), B = (1, 0)\) and \(C = \left(\frac{w}{w+1}, \frac{w}{w+1}\right)\) (see figure 2). The line segments \(AB, AC\) are trivially flow invariant. Hence, in order to argue that the \(ABC\) triangle is forward flow invariant, it suffices to show that everywhere along the line segment \(BC\) the vector field does not point “outwards” of the \(ABC\) triangle. Specifically, we need to show that for every point \(p\) on the line segment \(BC\) (except the Nash equilibrium \(C\)),

\[
\frac{\lvert \zeta_{1s}(p) \rvert}{\lvert \zeta_{2s}(p) \rvert} \geq \frac{1}{w}.
\]

\[
\frac{\lvert \zeta_{1s}(p) \rvert}{\lvert \zeta_{2s}(p) \rvert} = \frac{p_{1s}p_{2s} - (p_{1s}p_{2s} + w(1-p_{1s})(1-p_{2s}))}{p_{2s}p_{1s} - (p_{1s}p_{2s} + w(1-p_{1s})(1-p_{2s}))} = \frac{p_{1s}(1-p_{1s})(w - (w+1)p_{2s})}{p_{2s}(1-p_{2s})(-w + (w+1)p_{1s})}
\]

However, the points of the line passing through \(B, C\) satisfy \(p_{2s} = w(1-p_{1s})\).

\[
\frac{\lvert \zeta_{1s}(p) \rvert}{\lvert \zeta_{2s}(p) \rvert} = \frac{wp_{1s}(1-p_{1s})(1 - (w + 1)(1-p_{1s}))}{w(1-p_{1s})(1-w(1-p_{1s}))(w - (w+1)p_{1s})}
\]

\[
= \frac{wp_{1s}}{1-w+wp_{1s}} \geq \frac{p_{1s}}{wp_{1s}} = \frac{1}{w}
\]

Figure 2: Vector field of replicator dynamics in Stag Hunt
We have established that the \(A\) is a potential game, all but a zero measurable set of initial conditions converge to one of the two pure equilibria. Since \(A\) is forward invariant, all but a zero measure of initial conditions converge to \((Hare, Hare)\). A symmetric argument holds for the triangle \(A'B'C\) with \(B' = (0, 1)\). The union of \(A\) and \(A'B'C\) is equal to the polygon \(P_{Hare}\), which implies the first part of the lemma.

Next, we will prove the claimed property for polytope \((P_{Stag})\). Again, due to symmetry, it suffices to prove the property for the polytope \(P_{Stag}' = P_{Stag} \cap \{p_2s \leq p_1s\} \subseteq \{p_2s \leq p_1s\}\) where \(p_2s \geq -p_1s + \frac{2w}{w+1}\). All but a zero measurable set of initial conditions converge to one of the two pure equilibria. Since \(A\) is forward invariant, it suffices to show that everywhere along the line segment \(CD\) is forward invariant. On the \(p_1s, p_2s\) subspace \(P_{Stag}'\) defines a triangle with vertices \(D = (1, \frac{w-1}{w+1}), E = (1, 1)\) and \(C = (\frac{w}{w+1}, \frac{w}{w+1})\). The line segments \(CD, DE\) are trivially forward flow invariant. Hence, in order to argue that the \(CDE\) triangle is forward flow invariant, it suffices to show that everywhere along the line segment \(CD\) the vector field does not point “outwards” of the \(CDE\) triangle (see figure 2). Specifically, we need to show that for every point \(p\) on the line segment \(CD\) (except the Nash equilibrium \(C\)),

\[
\frac{|g_{1s}(p)|}{|g_{2s}(p)|} \leq 1.
\]

However, the points of the line passing through \(C, D\) satisfy \(p_2s = -p_1s + \frac{2w}{w+1}\).

\[
\frac{|g_{1s}(p)|}{|g_{2s}(p)|} = \frac{p_1s(1-p_1s)(-w + (w+1)p_1s)}{(-p_1s + \frac{2w}{w+1})(-\frac{w}{w+1} + p_1s)(-w + (w+1)p_1s)} = \frac{p_1s(1-p_1s)}{(-p_1s + \frac{2w}{w+1})(-\frac{w}{w+1} + p_1s)} \leq 1
\]

We have established that the \(CDE\) triangle is forward flow invariant. Since \(A\) is a potential game, all but a zero measurable set of initial conditions converge to one of the two pure equilibria. Since \(CDE\) is forward invariant, all but a zero measure of initial conditions converge to \((Stag, Stag)\). A symmetric argument holds for the triangle \(CD'E\) with \(D' = (\frac{w-1}{w+1}, 1)\). The union of \(CDE\) and \(CD'E\) is equal to the polygon \(P_{Stag}\), which implies the second part of the lemma.

**Proof** The measure/size of \(\mu(P_{Hare}) = 2|ABC| = \frac{w}{w+1}\), and similarly the measure of \(\mu(P_{Stag}) = 2|CDE| = \frac{2w}{(w+1)^2}\). In terms of the average limit performance of the replicator dynamics \(\int g(\omega)sw(\psi(x))d\mu \geq 2w \cdot \mu(P_{Hare}) + 2(1 - \mu(P_{Hare})) = \frac{2w^2+1}{(w+1)^2}\). Furthermore, \(\int g(\omega)sw(\psi(x))d\mu \leq 2w(1 - \mu(P_{Stag})) + 2 \cdot \mu(P_{Stag}) = 2w(1 - \frac{2}{(w+1)^2}) + 2 \cdot \frac{2}{(w+1)^2} = 2w - 4\frac{w-1}{(w+1)^2}\). This implies that \(\frac{w(w+1)^2}{w(w+1)^2} \geq APoA \leq \frac{w^2+w}{w^2+1}\).

**A.1.8 Proof of lemma 13**

**Proof** Assume we have a weakly Nash equilibrium \(p\). We have the following facts:

- Fact 1: For every bin \(\gamma\), if a player \(i\) chooses \(\gamma\) with probability \(1 > p_{\gamma i} > 0\), he must be the only player that chooses that bin with nonzero probability. Let \(i, j\) two
players that choose bin $\gamma$ with nonzero probabilities and also $p_i\gamma, p_j\gamma < 1$. Clearly if player $i$ changes his strategy and chooses bin $\gamma$ with probability one, then player $j$ doesn’t stay indifferent (his cost $c_i\gamma$ increases).

- Fact 2: If player $i$ chooses bin $\gamma$ with probability one, then he is the only player that chooses bin $\gamma$ with nonzero probability. This is true because every player $j \neq i$ can find a bin with load less than 1 to choose.

From Facts 1.2 and since the number of balls is equal to the number of bins we get that $p$ must be pure.

A.1.9 Proof of lemma 14

Proof We will prove it for $n = 4$ and the generalization is then easy, i.e., if $n > 4$ then the first 4 players will play as shown below in the first 4 bins and each of the remaining $n - 4$ players will choose a distinct remaining bin. Below we give matrix $A$ where $A_{i\gamma} = p_{i\gamma}$.

Observe that for any $x \in [\frac{1}{4}, \frac{3}{4}]$ we have a Nash equilibrium.

$A = \begin{pmatrix} x & 1 - x & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & x & 1 - x \end{pmatrix}$

A.2 Information Theory

Entropy is a measure of the uncertainty of a random variable and captures the expected information value from a measurement of the random variable. The entropy $H$ of a discrete random variable $X$ with possible values $\{1, \ldots, n\}$ and probability mass function $p(X)$ is defined as $H(X) = -\sum_{i=1}^{n} p(i) \ln p(i)$.

Given two probability distributions $p$ and $q$ of a discrete random variable their K-L divergence (relative entropy) is defined as $D_{KL}(p\|q) = \sum_{i} \ln \left( \frac{p(i)}{q(i)} \right) p(i)$. It is the average of the logarithmic difference between the probabilities $p$ and $q$, where the average is taken using the probabilities $p$. The K-L divergence is only defined if $q(i) = 0$ implies $p(i) = 0$ for all $i$. K-L divergence is a "pseudo-metric" in the sense that for it is always non-negative and is equal to zero if and only if the two distributions are equal (almost everywhere). Other useful properties of the K-L divergence is that it is additive for independent distributions and that it is jointly convex in both of its arguments; that is, if $(p_1, q_1)$ and $(p_2, q_2)$ are two pairs of distributions then for any $0 \leq \lambda \leq 1$: $D_{KL}(\lambda p_1 + (1 - \lambda) p_2\|\lambda q_1 + (1 - \lambda) q_2) \leq \lambda D_{KL}(p_1\|q_1) + (1 - \lambda) D_{KL}(p_2\|q_2)$.

A closely related concept is that of the cross entropy between two probability distributions, which measures the average number of bits needed to identify an event from a set of possibilities, if a coding scheme is used based on a given probability distribution $q$, rather than the “true” distribution $p$. Formally, the cross entropy for two distributions $p$ and $q$ over the same probability space is defined as follows: $H(p, q) = -\sum_{i=1}^{n} p(i) \ln q(i) = H(p) + D_{KL}(p\|q)$. For more details and proofs of these basic facts the reader should refer to the classic text by Cover and Thomas [5].

11The quantity $0 \ln 0$ is interpreted as zero because $\lim_{x \to 0} x \ln(x) = 0$. 

25
We will start by arguing a simple technical lemma about an information theoretic invariant property of (bipartite networks) of coordination games. A coordination game is a two agent game that in each outcome both agents receive the same utility. A graphical polymatrix game is defined via an undirected graph $G = (V, E)$, where $V$ corresponds to the set of agents of the game and where every edge corresponds to a bimatrix game between its two endpoints/agents. We denote by $S_i$ the set of strategies of agent $i$. We denote the bimatrix game on edge $(i, k) \in E$ via a pair of payoff matrices: $A^{i,k}$ of dimension $|S_i| \times |S_k|$ and $A^{k,i}$ of dimension $|S_k| \times |S_i|$. Let $s \in \times_i S_i$ be a strategy profile of the game, then we denote by $s_i \in S_i$ the respective strategy of agent $i$. Similarly, we denote by $s_{-i} \in \times_{j \in V \setminus i} S_j$ the strategies of the other agents. The payoff of agent $i \in V$ in strategy profile $s$ is equal to the sum of the payoffs that agent $i$ receives from all the bimatrix games she participates in. Specifically, $u_i(s) = \sum_{(i,k) \in E} A^{i,k}_{s_i,s_k}$. If the case of a bipartite network we denote by $V_{\text{left}}$, $V_{\text{right}}$ the respective vertices of each size of the bipartite graph.

We will show that the cross entropy between a fully mixed Nash $q$ and an evolving interior state $\sum_{i \in V_{\text{left}}} \sum_{\gamma \in S_i} q_{i\gamma} \ln(p_{i\gamma}) - \sum_{i \in V_{\text{right}}} \sum_{\gamma \in S_i} q_{i\gamma} \ln(p_{i\gamma})$ is an invariant of the dynamics. This results follows in the line of similar invariants \cite{23,24}. When $x, y \in \times_i \Delta(S_i)$ we will use $H(x, y), D_{\text{KL}}(x \mid \mid y)$ to denote respectively the $\sum_i H(x_i, y_i), \sum_i D_{\text{KL}}(x_i \mid \mid y_i)$.

**Lemma 26.** Let $\phi$ denote the flow of the replicator dynamic when applied to a bipartite network of coordination games that has a fully mixed Nash equilibrium then given any starting point $p \in \times_i \text{int} (\Delta(S_i))$, $\sum_{i \in V_{\text{left}}} \frac{dH(\phi(p), (p_t))}{dt} = \sum_{i \in V_{\text{right}}} \frac{dH(\phi(p), (p_t))}{dt}$.

**Proof.** The derivative of $\sum_{i \in V_{\text{left}}} \sum_{\gamma \in S_i} q_{i\gamma} \cdot \ln(p_{i\gamma}) - \sum_{i \in V_{\text{right}}} \sum_{\gamma \in S_i} q_{i\gamma} \cdot \ln(p_{i\gamma})$ has as follows:

$$
\sum_{i \in V_{\text{left}}} \sum_{\gamma \in S_i} \frac{d}{dt} \left( q_{i\gamma} \ln(p_{i\gamma}) \right) - \sum_{i \in V_{\text{right}}} \sum_{\gamma \in S_i} \frac{d}{dt} \left( q_{i\gamma} \ln(p_{i\gamma}) \right) = \sum_{i \in V_{\text{left}}} \sum_{\gamma \in S_i} \frac{d}{dt} \left( q_{i\gamma} \ln(p_{i\gamma}) \right) - \sum_{i \in V_{\text{right}}} \sum_{\gamma \in S_i} \frac{d}{dt} \left( q_{i\gamma} \ln(p_{i\gamma}) \right) =
$$

$$
= \sum_{i \in V_{\text{left}}} \sum_{j \in E} \left( q_i^T A^{i,j} p_j - q_i^T A^{i,j} p_j \right) - \sum_{i \in V_{\text{right}}} \sum_{j \in E} \left( q_i^T A^{i,j} p_j - q_i^T A^{i,j} p_j \right) =
$$

$$
= \sum_{i \in V_{\text{left}}} \sum_{j \in E} \left( q_i^T A^{i,j} p_j - q_i^T A^{i,j} p_j \right) =
$$

$$
= \sum_{(i,j) \in E_{\text{left}}} \sum_{j \in E} \left( q_i^T - p_i^T A^{i,j} (p_j - q_j) \right) - \sum_{(i,j) \in E_{\text{right}}} \sum_{j \in E} \left( q_i^T - p_i^T A^{i,j} (p_j - q_j) \right) =
$$

$$
= \sum_{(i,j) \in E_{\text{left}}} \sum_{j \in E} \left( q_i^T - p_i^T A^{i,j} (p_j - q_j) \right) - \sum_{(i,j) \in E_{\text{right}}} \sum_{j \in E} \left( q_i^T - p_i^T A^{i,j} (p_j - q_j) \right) = 0
$$

The cross entropy between the Nash $q$ and the state of the system, however is equal to the summation of the K-L divergence between these two distributions and the entropy of $q$. Since the entropy of $q$ is constant, we derive the following corollary:

**Corollary 27.** Let $\phi$ denote the flow of the replicator dynamic when applied to a bipartite network of coordination games that has a fully mixed Nash equilibrium then given any (interior) starting point $p \in \times_i \Delta(S_i)$, the K-L divergence between $\phi(q, t)$ and the $p$ is constant, i.e., does not depend on $t$. 

26