Corrections to the book “Vertex algebras for beginners”, second edition, by Victor Kac.

p. 39, ℓ. 3↑; p. 49, ℓ. 11↑; p. 56, ℓ. 10↑: should be $n \gg 0$ instead of $n g 0$

p. 50, ℓ. 5: should be $N \gg 0$ instead of $N g 0$

p. 56, ℓ. 3↑ reads: Now, choose a system of generators $\{ a^\alpha \}_{\alpha \in I}$ of $R$ viewed as a $\mathbb{C}[\partial]$

p. 57, ℓ. 1: should be $\{ a_j^\alpha | \alpha \in I, j \geq n \}$

p. 67, ℓ. 8↑ reads: $= \sum_{i=1}^{n+1} (-1)^{i+1} a_i \gamma_{\lambda_1, \ldots, \lambda_i, \ldots, \lambda_{n+1}}(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1})$

p. 81: before formula (3.1.1) a line is missing: Note that in the expansion (cf. (2.2.5))

p. 100, ℓ. 13: should be $(\varphi | \psi)$ instead of $(a | b)$

pp. 130–131: Theorem 4.11 and Proposition 4.11 are false. A corrected version of Section 4.11 is given below.

### 4.11. Field algebras

Field algebras generalize vertex algebras in the same way as unital associative algebras generalize unital commutative associative algebras.

A field algebra $V$ is defined by the same data as a vertex algebra, but weaker axioms (cf. Proposition 4.8(b)): 
(partial vacuum): $Y([0], z) = I_V$, $a_{(-1)}[0] = a$,

(n-th product): $Y(a_{(n)} b, z) = Y(a, z)_{(n)} Y(b, z)$, $n \in \mathbb{Z}$.

Note that the n-th product axiom is nothing else but Borcherds identity in the form (4.8.1) for $F = (z - w)^n$. As in the proof of Theorem 4.8, it follows that (4.8.1) holds for $F = z^m$ with $m \in \mathbb{Z}_+$. Hence the n-th product axiom implies (4.6.4) for $m \in \mathbb{Z}_+$, and in particular, the axiom (C3) of conformal algebra.

As in the case of vertex algebra, the axioms of a field algebra imply:

(4.11.1) $Y(a, z)[0]_{z=0} = a$, $Y([0], z) = I_V$

(4.11.2) $Y(T a, z) = \partial Y(a, z) = [T, Y(a, z)]$

where $T \in \text{End } V$ is defined by $T a = a_{(-1)}[0]$. The n-th product axiom for $n \gg 0$ implies weak locality:

\begin{equation}
\text{Res } z (z - w)^N [Y(a, z), Y(b, w)] = 0 \text{ for } N \gg 0.
\end{equation}

Note that weak locality of fields $a(z)$ and $b(z)$ means that $a(z)_{(n)} b(z) = 0$ for $n \geq N$, some $N$. (Unlike the usual locality, this is not a symmetric property.) Then, clearly, $(za(z))_{(n)} b(z) = 0$ for $n \geq N$. Using this remark, one can extend the proof of Dong’s lemma to the weakly local case (assuming that all ordered pairs are weakly local).

Example 4.11. Recall that any two local fields satisfy the skewsymmetry relation (3.3.6). This, however, fails for weakly local fields. In order to construct a counterexample, consider the free bosonic field $\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}$ (cf. Example 3.5), and let $\beta(z) = \sum_{n>0} n^{-1} \alpha_n z^{-n}$. Then we have:

$[\alpha(z), \beta(w)] = i_{w, z} (z - w)^{-1}$.

Hence for $j \in \mathbb{Z}_+$ we have:

$\alpha(z)_{(j)} \beta(z) = 0$, $\beta(z)_{(j)} \alpha(z) = \delta_{j0}$. 
Therefore both pairs \((\alpha, \beta)\) and \((\beta, \alpha)\) are weakly local, but (3.3.6) fails for \(a = \alpha, b = \beta, n = 0\).

Recall that the \(-1\)st product axiom means:

\[(4.11.4) \quad Y(a_{(-1)} b, z) =: Y(a, z)Y(b, z) : .\]

Replacing \(a\) by \(T^n a\) and using (4.11.2), we see that (4.11.4) implies the \(n\)-th product axiom for \(n < 0\).

Multiplying both sides of the \(n\)-th product axiom by \((-w)^{-n-1}\) and taking summation over \(n \in \mathbb{Z}\), we obtain its equivalent form in the domain \(|z| > |w|:\)

\[(4.11.5) \quad Y(Y(a, z)b, -w)c = Y(a, z-w)Y(b, -w)c - p(a, b)Y(b, -w)\sum_{j \geq 0} \partial_{a}^{j} \delta(z - w) \text{Res}_x x^{(j)} Y(a, x)c .\]

This is immediate by the following special case of Taylor’s formula in the domain \(|z| > |w|:\)

\[i_w, x\delta((w + x) - z) = \sum_{j \geq 0} x^{(j)} \partial_{a}^{j} \delta(z - w) .\]

Formula (4.11.5) implies the associativity property in the domain \(|z| > |w|:\)

\[(4.11.6)(z - w)^N Y(Y(a, z)b, -w)c = (z - w)^N Y(a, z-w)Y(b, -w)c \quad \text{for} \quad N \gg 0 .\]

As in Section 1.4, it is easy to show that all holomorphic field algebras are obtained by taking a unital associative algebra \(V\) and its derivation \(T\), and letting

\[Y(a, z)b = e^{zT}(a)b, \quad a, b \in V .\]

The general linear field algebra \(g\ell f(U)\) defined in Section 3.2 is not a field algebra since the field property

\[(4.11.7) \quad a_{(n)}b = 0 \quad \text{for} \quad n \gg 0 \]

fails in general. However, if we take a collection of mutually weakly local fields \(\{a^\alpha(z)\} \subset g\ell f(U)\), they generate a linear field algebra which is a field algebra.
The $n$-th product axiom for $n \geq 0$ is implied by (3.3.7). Next, it is immediate to check (4.11.1) and (4.11.2). Weak locality is proved in the same way as Proposition 3.2. The $n$-th product axiom for $n < 0$ follows from (4.11.4) as explained above. Finally, the $-1$st product axiom is checked by a direct calculation.

We have the following field algebra analogs of the uniqueness and existence theorems (obtained jointly with Bojko Bakalov).

**Theorem 4.11.** (a) Let $V$ be a field algebra. For each field $Y(a, z)$ define the “opposite” field $X(a, z)$ by the formula (cf. (4.2.1)):

\begin{equation}
X(a, z)b = p(a, b)e^{zT}Y(b, -z)a.
\end{equation}

Let $B(z)$ be a field which is mutually local with all fields $X(a, z), a \in V$, on any $v \in V$, i.e.,

\[(z - w)^N[B(z), X(a, z)]v = 0 \text{ for } N \gg 0.\]

Suppose that (4.4.1) holds for some $b \in V$. Then $B(z) = Y(b, z)$.

(b) Let $V$ be a vector superspace, let $|0\rangle$ be an even vector and $T$ an even endomorphism of $V$. Let $\{a^{\alpha}(z)\}_{\alpha \in A}$ and $\{b^{\beta}(z)\}_{\beta \in B}$ ($A,B$ index sets) be two collections of fields such that each of them satisfies conditions (i)–(v) of Theorem 4.5 except that in (iii) “local” is replaced by “weakly local”. Suppose, in addition, that all pairs $(a^{\alpha}(z),b^{\beta}(z))$ are local on any $v \in V$. Then formula (4.5.1) defines a unique structure of a field algebra on $V$ such that $|0\rangle$ is the vacuum vector, $T$ is the infinitesimal translation operator and (4.5.2) holds. The same conclusion holds if the family $\{a^{\alpha}(z)\}$ is replaced by the family $\{b^{\beta}(z)\}$ and the fields $Y$ are replaced by the fields $X$. The two field algebra structures on $V$ are related by (4.11.8).

**Proof:** It is similar to that of Theorems 4.4 and 4.5 using the observation that the associativity property (4.11.6) is equivalent to the locality of the pair $(Y(a, z), X(c, z))$ on $b \in V$. □

Taking in this theorem all fields of a field algebra, we obtain the following corollary.
Corollary 4.11. (a) Vacuum and translation covariance axioms along with weak locality (4.11.3) and associativity (4.11.6) form an equivalent system of axioms of a field algebra.

(b) If \((V,|0\rangle, T, Y(a, z))\) is a field algebra, then \((V,|0\rangle, T, X(a, z))\), where \(X(a, z)\) are defined by (4.11.8), is a field algebra as well.

Remark 4.11a. It follows from the above discussion that a field algebra with \(n\)-th products for \(n \in \mathbb{Z}_+\) and \(\partial = T\) satisfies all axioms of a conformal algebra, except the skewsymmetry axiom (C2), which may fail in view of Example 4.11.

Remark 4.11b. It follows from the proof of Proposition 3.3(b) that two weakly local fields \(a(z)\) and \(b(z)\) for which the skewsymmetry property (3.3.6) holds, are local. Hence a field algebra satisfying the skewsymmetry property (4.2.2) is a vertex algebra. This follows also from Corollary 4.11(a).