On Fractional Langevin Equations with Stieltjes Integral Conditions

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Abstract: In this paper, we focus on the study of the implicit FDE involving Stieltjes integral boundary conditions. We first exploit some sufficient conditions to guarantee the existence and uniqueness of solutions for the above problems based on the Banach contraction principle and Schaefer’s fixed point theorem. Then, we present different kinds of stability such as UHS, GUHS, UHRS, and GUHRS by employing the classical techniques. In the end, the main results are demonstrated by two examples.

Keywords: Caputo fractional derivative; green function; multi-point integral boundary conditions; Ulam–Hyers stability

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1. Introduction

As is well known, Paul Langevin in 1908 introduced an equation of the form \( \eta \frac{du}{d\zeta} = \lambda \frac{du}{d\zeta} + \eta(\zeta) \), which is now called the Langevin equation. As far as we know, Langevin equations are widely applied to many fields, such as image processing, astronomy, physics, stochastic problems in chemistry, and mechanical engineering. It is worth pointing out that the Langevin equations describe Brownian motion in a reasonable way that the random oscillation force is assumed to be Gaussian noise. In order to remove noise and decrease the staircase effects, researchers employed fractional differential equations to replace ordinary differential equations. Therefore, it is significant to investigate the fractional Langevin equations. For additional information, we refer to earlier works [1–3].

In recent years, fractional differential equations have become increasingly significant in theory and applications. Here, we do not intend to list all of applicable areas involving the fractional operators, we just refer the interested reader to see the introduction in [2]. There are a lot of excellent works that offer the key theoretical instruments for the qualitative and quantitative characterization of fractional differential equations, see earlier works [4–12].

Since approximate solutions are frequently used in disciplines such as numerical analysis, optimization theory, and nonlinear analysis, it is important to determine how closely these solutions resemble the true solutions of the relevant system or systems. Other methods may be carried out for this; however, the UHS procedure may be easy to be understood. Ulam originally brought out the aforementioned stability in 1940 [13], and Hyers ingeniously responded in 1941 [14]. Rassias [15] established the mathematical UHS technique in 1978 by considering variables. After that, researchers expanded the ideas of functional differential and integral, and then FDEs were established by some authors, see for example [16–21].

The existence, uniqueness, and different types of UHS of the solutions of nonlinear implicit FDEs with Caputo fractional derivatives have recently attracted the interest of
many scholars; for more information, see [22–27]. In the following, we list some very related solutions:

- Muniyappan and Rajan [28] discussed \( UHS \) and \( UHRS \) for the following FDE:

\[
\begin{align*}
&\quad \mathcal{D}^\alpha u(\xi) = \Phi(\xi, u(\xi)), \quad 0 < \alpha \leq 1, \\
&\quad a u(0) + bu(T) = c,
\end{align*}
\]

where \( \mathcal{D}^\alpha \) is a Caputo fractional derivative of order \( \alpha \).

- Abbas [29] demonstrated the existence and uniqueness of solutions for the following FBVP:

\[
\begin{align*}
&\quad \mathcal{D}^\alpha u(\xi) = \Phi(\xi, u(\xi), \mathcal{D}^\beta u(\xi)), \quad \beta > 0, \\
&\quad u(0) = \lambda_1 u(\eta), \quad u'(0) = 0, \quad u''(0) = 0, \quad \cdots, \quad u''(0) = 0, \quad u(1) = \lambda_2 u(\eta),
\end{align*}
\]

where \( \alpha \in (m - 1, m], \beta \geq 2, \) and \( \mathcal{D}^\alpha, \mathcal{D}^\beta \) are the Caputo fractional derivatives.

- Ahmad and Nieto [30] investigate the existence and uniqueness of solutions for the following FBVP given by:

\[
\begin{align*}
&\quad \mathcal{D}^\alpha u(\xi) = \Phi(\xi, u(\xi)), \quad \xi \in [0, T], \quad T > 0, \quad \alpha \in (1, 2], \\
&\quad \mathcal{D}^{\alpha - 2} u(0^+) = \gamma \mathcal{D}^{\alpha - 2} u(T^-), \\
&\quad \mathcal{D}^{\alpha - 1} u(0^+) = \beta \mathcal{D}^{\alpha - 1} u(T^-),
\end{align*}
\]

where the function \( \Phi : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous and \( \beta, \gamma \neq 1 \).

- Ali et al. [31] investigated the existence and uniqueness of solutions, and four different types of Ulam stability for the implicit FBVP given by:

\[
\begin{align*}
&\quad \mathcal{D}^\alpha u(\xi) = \Phi(\xi, u(\xi), \mathcal{D}^\alpha u(\xi)), \quad \xi \in J = [0, T], \quad T > 0, \quad \alpha \in (1, 2], \\
&\quad \mathcal{D}^{\alpha - 2} u(0^+) = \gamma \mathcal{D}^{\alpha - 2} u(T^-), \\
&\quad \mathcal{D}^{\alpha - 1} u(0^+) = \beta \mathcal{D}^{\alpha - 1} u(T^-),
\end{align*}
\]

where \( \beta, \gamma \neq 1 \).

- Dai et al. [32] studied the \( UHS \) and \( UHRS \) of the following nonlinear FDE with integral boundary condition:

\[
\begin{align*}
&\quad u'(\xi) + \mathcal{D}^\alpha u(\xi) = \Phi(\xi, u(\xi)), \quad 0 < \alpha < 1, \quad \xi \in [0, 1], \\
&\quad u(1) = \mathcal{I}_0^\beta u(\eta) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta - 1} u(s) ds, \quad \beta > 0,
\end{align*}
\]

where \( \mathcal{D}^\alpha \) is Caputo derivative and \( \mathcal{I}_0^\beta \) is the Riemann–Liouville fractional integral.

Motivated by the above papers, we consider the following nonlinear implicit Langevin equations with mixed derivatives and Stieltjes integral conditions:

\[
\begin{align*}
&\quad \mathcal{D}^\alpha_0 (D + \lambda) u(\xi) = \Phi(\xi, u(\xi)), \quad \xi \in (0, 1], \\
&\quad u(0) = 0, \quad \mathcal{D}^\alpha_{0^+} u(1) = \sum_{i=1}^p \int_0^1 \mathcal{D}^\beta_{0^+} u(\xi) d\mu_i(\xi),
\end{align*}
\]

where \( \mathcal{D}^\alpha_0 \) represents a classical Caputo derivative, of order \( \alpha \), with the lower bound zero, \( 0 < \alpha < 1, \lambda \in \mathbb{R} \backslash \{0\}, \) \( p \in \mathbb{N}, \beta_i \in \mathbb{R} \) for all \( i = 0, \ldots, p, 0 \leq \beta_1 < \beta_2 < \cdots < \beta_p < \alpha, \beta_0 \in [0, \alpha], \Phi : J = [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous, and the integrals from the boundary condition are Riemann–Stieltjes integrals with \( \mu_i \) \( i = 1, \ldots, p \) function of bounded variation.

We summarize the main highlights of this paper as follows:
1. It is the first time in literature that we work on this model (1) with pointwise Stieltjes integrals.

2. Different from [28–32], we obtain the solution of (1) we inquire about the existence and uniqueness of a solution for a class of FDE with Stieltjes integral conditions.

3. Different from the previous paper [32] which worked with Riemann integrals, we work with Stieltjes integral for better solutions.

The manuscript is organized as follows. In Section 2, we provide a uniform framework for the proposed model. Section 3 is devoted to employing different conditions and some well-known fixed point theorems to ensure the existence and uniqueness of solution for system (1). In Section 4, we present Ulam’s stabilities. In Section 5, we give two examples to demonstrate our main results.

2. Preliminary

Suppose $J = [0, 1]$ and $C(J, \mathbb{R})$ are the set of all continuous functions from $J$ to $\mathbb{R}$. Let $X = C(J, \mathbb{R})$ be a Banach space endowed with the following norm:

$$
\|u\|_X = \max_{\zeta \in J}\{|u(\zeta)| : \zeta \in J\}.
$$

Consider the linear form of (1) as follows:

$$
\begin{align*}
\left\{ \begin{array}{l}
cD_{0+}^{\alpha}(D + \lambda)u(\zeta) = \Phi(\zeta), \quad \zeta \in (0, 1], \\
u(0) = 0, \quad cD_{0+}^{\beta_0}u(1) = \sum_{i=1}^{n} \int_{0}^{1} cD_{0+}^{\beta_i}u(\xi)d\mu_i(\zeta).
\end{array} \right.
\end{align*}
$$

(2)

Next, we revisit some definitions of fractional calculus from [33] as follows:

**Definition 1.** The fractional integral of order $\alpha$ from 0 to $\zeta$ for the function $u$ is defined by:

$$
I_{0+}^{\alpha}u(\zeta) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\zeta} (\zeta - s)^{\alpha-1} u(s)ds, \quad \zeta > 0, \quad \alpha > 0,
$$

where $\Gamma(\cdot)$ is the Gamma function.

**Definition 2.** The Caputo derivative of fractional order $\alpha$ from 0 to $\zeta$ for a function $u$ can be defined as

$$
cD_{0+}^{\alpha}u(\zeta) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{\zeta} (\zeta - s)^{n-\alpha-1} u^{(n)}(s)ds, \quad \text{where} \quad n = [\alpha] + 1.
$$

**Lemma 1.** The FDE $cD_{0+}^{\alpha}u(\zeta) = 0$ with $\alpha > 0$, involving Caputo differential operator $cD_{0+}^{\alpha}$, has a solution in the following form:

$$
u(\zeta) = c_0 + c_1 \zeta + c_2 \zeta^2 + \cdots + c_m \zeta^{m-1},
$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots, m - 1$ and $m = [\alpha + 1]$.

**Lemma 2.** For each $\alpha > 0$, we have:

$$
I_{0+}^{\alpha}(cD_{0+}^{\alpha}u(\zeta)) = c_0 + c_1 \zeta + c_2 \zeta^2 + \cdots + c_m \zeta^{m-1},
$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots, m - 1$ and $m = [\alpha + 1]$.

**Lemma 3.** If $\Re(\alpha) > 0$ and $\lambda > 0$, then

$$
cD_{0+}^{\alpha}e^{\lambda \zeta} = \lambda^\alpha e^{\lambda \zeta} \quad \text{and} \quad cD_{0+}^{\alpha}e^{-\lambda \zeta} = \lambda^\alpha e^{-\lambda \zeta}.
$$
Theorem 1 (Arzela–Ascoli’s theorem). Let \( B \subset C(J,\mathbb{R}) \) be relatively compact, and if 
(i) \( B \) is uniformly bounded set such that there exists \( q > 0 \) with 
\[ \|u\| = \sup_{\xi \in J} |u(\xi)| < q \] for every \( u \in B \).

(ii) \( B \) is an equicontinuous set, i.e., for every \( \epsilon > 0 \), there exists \( \delta > 0 \), such that for any \( \zeta, \xi \in J, |\zeta - \xi| \leq \delta \Rightarrow |u(\zeta) - u(\xi)| \leq \epsilon \), for every \( u \in B \).

Theorem 2 (Banach’s fixed point theorem). Let \( B \) be a non-empty closed subset of \( X \), which is a Banach space. Any contraction that maps the \( \delta \) from \( B \) into itself has a unique fixed point.

Theorem 3 (Schaefer’s fixed point theorem). Suppose that \( X \) is a Banach space, and the operator \( \delta : X \rightarrow X \) is a continuous compact mapping (or completely continuous). Furthermore, assume:
\[ B = \{ u \in X | u = \eta \delta u, 0 < \eta < 1 \} \]
is a bounded set. Then, \( \delta \) has at least one fixed point in \( X \).

Ulam's Stabilities and Remark
The following definitions are taken from [34].

Definition 3. The \( FBVP \) (1) is named to be \( UH\mathcal{S} \) if there exists \( k_0 \in \mathbb{R}^+ \) such that for every \( \epsilon > 0 \) and for every solution \( u \in X \) of the inequality
\[ |^cD_{0^+} u(\xi) - \Phi(\xi, u(\xi))| \leq \epsilon, \quad \xi \in J, \] (3)
there exists a unique solution \( v \in X \) of the considered problem (1), such that
\[ |u(\xi) - v(\xi)| \leq k_0 \epsilon, \quad \xi \in J. \]

Definition 4. The \( FBVP \) (1) is called to be \( GU\mathcal{S} \) if there exists \( \varphi \in C(\mathbb{R}^+,\mathbb{R}^+), \varphi(0) = 0 \), such that for every solution \( u \in X \) of the inequality (3), there exists a unique solution \( v \in C(J,\mathbb{R}^+) \) of the considered problem (1), such that:
\[ |u(\xi) - v(\xi)| \leq \varphi(\epsilon), \quad \xi \in J. \]

Definition 5. The \( FBVP \) (1) is said to be \( U\mathcal{H}RS \) with respect to \( \varphi \in C(J,\mathbb{R}^+) \), if there exists a non-zero, positive, real number \( k_\varphi \), such that for every \( \epsilon > 0 \) and for every solution \( u \in X \) of the inequality
\[ |^cD_{0^+} u(\xi) - \Phi(\xi, u(\xi))| \leq \varphi(\xi) \epsilon, \quad \xi \in J, \] (4)
there exists a unique solution \( v \in C(J,\mathbb{R}^+) \) of the considered problem (1), such that
\[ |u(\xi) - v(\xi)| \leq k_\varphi \varphi(\xi) \epsilon, \quad \xi \in J. \]

Definition 6. The \( FBVP \) (1) is named to be \( GU\mathcal{H}RS \) with respect to \( \varphi \in C(J,\mathbb{R}) \), if there exists \( k_\varphi \in \mathbb{R}^+ \), such that for every solution \( u \in X \) of the inequality (4), there exists a unique solution \( v \in C(J,\mathbb{R}) \) of the considered problem (1), such that
\[ |u(\xi) - v(\xi)| \leq k_\varphi \varphi(\xi), \quad \xi \in J. \]

Remark 1. A function \( u \in X \) is a solution for the inequality (3) if there exists a function \( \psi \in C(J,\mathbb{R}) \) depending on \( u \), such that:
(ii) \( cD_{0+}^\alpha (D + \lambda)u(\zeta) = \Phi(\zeta, u(\zeta)) + \psi(\zeta), \quad \zeta \in J. \)

**Lemma 4.** The function \( u \in C(J, \mathbb{R}) \) is a solution of (2) if and only if

\[
u(\zeta) = \int_0^\zeta G(\zeta, s)\Phi(s)ds,
\]

where

\[
G(\zeta, s) = G_1(\zeta, s) + G_2(\zeta, s),
\]

\[
G_1(\zeta, s) = \begin{cases} e^{-\lambda(\zeta-s)}I_{0+}^\alpha(1) + \frac{(1 - e^{-\lambda s})e^{-\lambda(1-s)}I_{0+}^\alpha(1)}{\Delta}, & 0 \leq s \leq \zeta \leq 1, \\ \frac{(1 - e^{-\lambda s})e^{-\lambda(1-s)}I_{0+}^\alpha(1)}{\Delta}, & 0 \leq \zeta \leq s \leq 1, \end{cases}
\]

\[
G_2(\zeta, s) = e^{-\lambda s} - 1 \sum_{i=1}^p \int_0^\zeta e^{-\lambda(s-i)}I_{0+}^\alpha(1)d\mu_i(\zeta)
\]

and

\[
\Delta = \sum_{i=1}^p \int_0^1 (-\lambda)^{\beta_i}e^{-\lambda s}d\mu_i(\zeta) - (-\lambda)^{\beta_0}e^{-\lambda \zeta} \neq 0.
\]

**Proof.** Consider

\[
cD_{0+}^\alpha (D + \lambda)u(\zeta) = \Phi(\zeta), \quad \alpha \in (0, 1), \quad \zeta \in J,
\]

where \( D \) denotes an ordinary differential operator. In light of Lemma 2 and an ordinary integration, we obtain

\[
u(\zeta) = \int_0^\zeta e^{-\lambda(\zeta-s)}I_{0+}^\alpha \Phi(s)ds + c_0 \left( \frac{1 - e^{-\lambda \zeta}}{\lambda} \right) + c_1 e^{-\lambda \zeta}.
\]

By the condition \( u(0) = 0 \), we obtain \( c_1 = 0 \). Then, we have

\[
u(\zeta) = \int_0^\zeta e^{-\lambda(\zeta-s)}I_{0+}^\alpha \Phi(s)ds + c_0 \left( \frac{1 - e^{-\lambda \zeta}}{\lambda} \right).
\]

(5)

Furthermore, we obtain:

\[
cD_{0+}^{\beta_0} u(1) = \int_0^1 e^{-\lambda(1-s)}I_{0+}^{\beta_0} \Phi(s)ds + c_0 \left( \frac{(-\lambda)^{\beta_0}e^{-\lambda}}{\lambda} \right)
\]

(6)

and

\[
cD_{0+}^{\beta_i} u(\zeta) = \int_0^\zeta e^{-\lambda(\zeta-s)}I_{0+}^{\beta_i} \Phi(s)ds + c_0 \left( \frac{(-\lambda)^{\beta_i}e^{-\lambda \zeta}}{\lambda} \right).
\]

Then, we obtain:

\[
\sum_{i=1}^p \int_0^1 cD_{0+}^{\beta_i} u(\zeta)d\mu_i(\zeta)
\]

\[
= \sum_{i=1}^p \int_0^\zeta e^{-\lambda(\zeta-s)}I_{0+}^{\beta_i} \Phi(s)dsd\mu_i(\zeta) + c_0 \sum_{i=1}^p \int_0^1 \left( \frac{(-\lambda)^{\beta_i}e^{-\lambda \zeta}}{\lambda} \right)d\mu_i(\zeta).
\]

(7)
From (6) and (7), we have
\[
c_0 = \frac{\lambda}{\Delta} \left[ \int_0^1 e^{-\lambda(1-s)} \int_0^{\xi} \Phi(s) ds - \sum_{i=1}^{p} \int_0^1 \int_0^\xi e^{-\lambda(\xi-s)} \int_0^{\xi} \Phi(s) ds d\mu_i(\xi) \right].
\] (8)

Using (8) in (5), we deduce that
\[
u(\xi) = \int_0^\xi e^{-\lambda(\xi-s)} \int_0^{\xi} \Phi(s) ds + \frac{1-e^{-\lambda\xi}}{\Delta} \int_0^1 e^{-\lambda(1-s)} \int_0^{\xi} \Phi(s) ds
- \frac{1-e^{-\lambda\xi}}{\Delta} \sum_{i=1}^{p} \int_0^1 \int_0^\xi e^{-\lambda(\xi-s)} \int_0^{\xi} \Phi(s) ds d\mu_i(\xi)
\]
\[
u(\xi) = \int_0^\xi \left( e^{-\lambda(\xi-s)} \int_0^{\xi} \Phi(s) ds + \frac{1-e^{-\lambda\xi}}{\Delta} \int_0^1 e^{-\lambda(1-s)} \int_0^{\xi} \Phi(s) ds - \frac{1-e^{-\lambda\xi}}{\Delta} \sum_{i=1}^{p} \int_0^1 \int_0^\xi e^{-\lambda(\xi-s)} \int_0^{\xi} \Phi(s) ds d\mu_i(\xi) \right)
\]
\[
u(\xi) = \int_0^1 \sum_{i=1}^{p} \int_0^\xi \Phi(s) ds - \frac{1-e^{-\lambda\xi}}{\Delta} \sum_{i=1}^{p} \int_0^1 \int_0^\xi e^{-\lambda(\xi-s)} \int_0^{\xi} \Phi(s) ds d\mu_i(\xi)
\]
\[
u(\xi) = \int_0^1 (G_1(\xi,s) + G_2(\xi,s)) \Phi(s) ds - \int_0^1 G(\xi,s) \Phi(s) ds.
\]

Hence, the proof of this lemma is finished.  

3. Existence and Uniqueness

In this section, we will build up adequate conditions to ensure the existence and uniqueness of the solution to the FBVP (1).

From Lemma 4, we can determine the integral equation of problem (1) as follows:
\[
u(\xi) = \int_0^1 G(\xi,s) \Phi(s, \nu(s)) ds.
\] (9)

Throughout the paper, we assume that
\[
(H_1) \Delta = \sum_{i=1}^{p} \int_0^1 (-\lambda)^{\beta_i} e^{-\lambda s} d\mu_i(\xi) - (-\lambda)^{\beta_0} e^{-\lambda \xi} \neq 0.
\]
\[
(H_2) \Phi : J \times \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous.}
\]
\[
(H_3) \text{For } \xi \in J \text{ and } u \in X, \text{ there are } \pi_1, \pi_2 \in C(J, \mathbb{R}^+), \text{ such that}
\]
\[
|\Phi(\xi, u(\xi))| \leq \pi_1(\xi) + \pi_2(\xi)|u(\xi)|
\]
with \( \pi_1^* = \sup_{\xi \in J} \pi_1(\xi) \) and \( \pi_2^* = \sup_{\xi \in J} \pi_2(\xi) < 1. \)
\[
(H_4) \text{For any } u, \bar{u} \in \mathbb{R} \text{ and for all } \xi \in J, \text{ there exists a constant } K > 0, \text{ such that}
\]
\[
|\Phi(\xi, u) - \Phi(\xi, \bar{u})| \leq K|u - \bar{u}|
\]
\[
(H_5) \text{If } \varphi \in C(J, \mathbb{R}) \text{ is increasing, then there exists } \Omega_\varphi > 0 \text{ such that for all } \xi \in J, \text{ the following integral inequality}
\]
\[
\int_0^{\xi} \varphi(\xi) \leq \Omega_\varphi \varphi(\xi)
\]
Lemma 5. The Green’s function \( \mathbb{G}(\zeta, s) \), which is found in Lemma 4, has the following properties:

1. \( \mathbb{G}(\zeta, s) \) is a continuous function over \( J \):
2. \( \max_{\zeta \in J} \int_{0}^{1} |\mathbb{G}(\zeta, s)| ds \leq \mathcal{Y} \),

where

\[
\mathcal{Y} = \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(\alpha + 1)} + \frac{|(1 - e^{-\lambda})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} + \sum_{i=1}^{p} \frac{1 - e^{-\lambda}}{\lambda \Gamma(\alpha - \beta_i + 1)} |1 - e^{-\lambda |\rho^\alpha - \beta_i q_i|}|
\]

and

\[
\nabla = \sum_{i=1}^{p} (-\lambda) \beta_i e^{-\lambda \rho q_i} - (-\lambda) \beta_0 e^{-\lambda}.
\]

Proof. It is easy to see that (1) holds true, so we omit the proof.

(2): Because the Green’s function has the following expression:

\[
\int_{0}^{1} |\mathbb{G}(\zeta, s)| ds = \left[ \int_{0}^{\zeta} e^{-\lambda(s-\zeta)} \left( \frac{1}{\Gamma(\alpha)} \int_{0}^{s} (s-t)^{\alpha-1} dt \right) ds \
+ \frac{1 - e^{-\lambda \zeta}}{\Delta} \int_{0}^{1} e^{-\lambda(1-s)} \left( \frac{1}{\Gamma(\alpha - \beta_0)} \int_{0}^{s} (s-t)^{\alpha-\beta_0-1} dt \right) ds \
- \frac{1 - e^{-\lambda \zeta}}{\Delta} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(1-s)} \left( \frac{1}{\Gamma(\alpha - \beta_i)} \int_{0}^{s} (s-t)^{\alpha-\beta_i-1} dt \right) ds d\mu_i(s) \right].
\]

Using Mean Value Theorem [35] with \( \rho \in [0, 1] \) and \( \mu_i(1) = \Phi_i > 0 \), we obtain:

\[
\max_{\zeta \in J} \int_{0}^{1} |\mathbb{G}(\zeta, s)| ds \leq \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(\alpha + 1)} + \frac{|(1 - e^{-\lambda})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} + \sum_{i=1}^{p} \frac{1 - e^{-\lambda}}{\lambda \Gamma(\alpha - \beta_i + 1)} = \mathcal{Y}.
\]

Hence, the proof of (2) is complete. \( \square \)

If \( u \) is the solution of the FBPVP (1), then

\[
u(\zeta) = \int_{0}^{\zeta} e^{-\lambda(\zeta-s)} l_{0,\zeta}^s \Phi(s, u(s))ds + \frac{1 - e^{-\lambda \zeta}}{\Delta} \int_{0}^{1} e^{-\lambda(1-s)} l_{0,\zeta}^s \Phi(s, u(s))ds \
- \frac{1 - e^{-\lambda \zeta}}{\Delta} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(1-s)} l_{0,\zeta}^s \Phi(s, u(s))d\mu_i(s).
\]

Define the operator \( \delta : X \to X \) as

\[
\delta u(\zeta) = \int_{0}^{\zeta} e^{-\lambda(\zeta-s)} l_{0,\zeta}^s \Phi(s, u(s))ds + \frac{1 - e^{-\lambda \zeta}}{\Delta} \int_{0}^{1} e^{-\lambda(1-s)} l_{0,\zeta}^s \Phi(s, u(s))ds \
- \frac{1 - e^{-\lambda \zeta}}{\Delta} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(1-s)} l_{0,\zeta}^s \Phi(s, u(s))d\mu_i(s).
\]

Theorem 4. Under the hypotheses (H1)–(H4), the operator \( \delta \) is compact.

Proof. Consider the operator \( \delta \) defined in (10). We have to show that the operator \( \delta \) is compact. For this, the proof will be given in several steps.

Step 1: We assert that the operator \( \delta \) is continuous. Let \( \{u_n\} \subset X \) such that \( u_n \to u \) in \( X \); then, for each \( \zeta \in J \), we have:
As a result, we have:

\[
\left| \delta(u_n)(\xi) - \delta(u)(\xi) \right| \leq K|u_n(\xi) - u(\xi)|.
\]

Since we supposed that \(u_n \to u\), then \(\Phi(\xi, u_n(\xi)) \to \Phi(\xi, u(\xi))\) as \(n \to \infty\) for each \(\xi \in J\). So, by Lebesgue Dominated Convergence Theorem [36], (11) implies that

\[
|\delta(u_n)(\xi) - \delta(u)(\xi)| \to 0 \quad \text{as} \quad n \to \infty;
\]

hence,

\[
\|\delta(u_n) - \delta(u)\|_X \to 0 \quad \text{as} \quad n \to \infty.
\]

As a result, \(\delta\) is continuous.

**Step 2:** We claim that the operator \(\delta\) is bounded in \(X\). For this, we have to demonstrate that for any \(\xi^* > 0\), there is \(\rho > 0\) such that for all \(u \in E^* = \{u \in X : \|u\|_X \leq \xi^*\}\), we have:

\[
\|\delta(u)\|_X \leq \rho.
\]

From (10), for each \(\xi \in J\), we have:

\[
\delta(u)(\xi) = \int_0^\xi e^{-\lambda(\xi-s)} I_{0,\xi}^a \Phi(s, u(s))ds + \frac{1 - e^{-\lambda\xi}}{\Delta} \int_0^1 e^{-\lambda(1-s)} I_{0,\xi}^a \Phi(s, u(s))ds
\]

\[
- \frac{1 - e^{-\lambda\xi}}{\Delta} \sum_{i=1}^p \int_0^\xi e^{-\lambda(\xi-s)} I_{0,\xi}^a \Phi(s, u(s))ds d\mu_i(\xi).
\]

(12)

Now, by (H3), we have:

\[
\|\Phi(\xi, u)\|_X \leq \pi_1^* + \pi_2^* \xi^* = M_0.
\]

(13)

In this way, (12) becomes

\[
\|\delta(u)\|_X \leq M_0 \left( \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(a + 1)} + \frac{|(1 - e^{-\lambda})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} + \sum_{i=1}^p \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(a - \beta_i + 1)} \right)
\]

\[
= \rho.
\]

Hence \(\delta(E^*)\) is uniformly bounded.
Step 3: Now, we claim that the operator \( \delta \) is equicontinuous in \( X \). Let \( \zeta_1, \zeta_2 \in J \) with \( \zeta_1 > \eta_2 \), since \( E^* \) is a bounded set in \( X \), and let \( u \in E^* \). Then we have:

\[
\|\delta(u)(\zeta_1) - \delta(u)(\zeta_2)\| \\
= \left\| \int_0^{\zeta_1} e^{-\lambda(s-s)} I_{\zeta_1}^{\zeta_2} \Phi(s, u(s)) ds + \frac{1 - e^{-\lambda s}}{\Delta} \int_0^1 e^{-\lambda(1-s)} I_{\zeta_1}^{\zeta_2} \Phi(s, u(s)) ds \\
- \frac{1 - e^{-\lambda s}}{\Delta} \sum_{i=1}^p \int_0^{\zeta_1} e^{-\lambda(s-s)} I_{\zeta_1}^{\zeta_2} \Phi(s, u(s)) ds d\mu_i(\zeta_1) \\
- \int_0^{\zeta_2} e^{-\lambda(s-s)} I_{\zeta_1}^{\zeta_2} \Phi(s, u(s)) ds + \frac{1 - e^{-\lambda s}}{\Delta} \int_0^1 e^{-\lambda(1-s)} I_{\zeta_1}^{\zeta_2} \Phi(s, u(s)) ds \\
+ \frac{1 - e^{-\lambda s}}{\Delta} \sum_{i=1}^p \int_0^{\zeta_2} e^{-\lambda(s-s)} I_{\zeta_1}^{\zeta_2} \Phi(s, u(s)) ds d\mu_i(\zeta_2) \right\| \\
\leq M_0 \left( \left\| \frac{|1 - e^{-\lambda s}|}{\lambda \Gamma(\alpha + 1)} \pi_1 + \frac{|(1 - e^{-\lambda s})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} \pi_1 \right\| + \sum_{i=1}^p \left\| \frac{|1 - e^{-\lambda s}|}{\lambda \Gamma(\alpha - \beta_0 + 1)} |(1 - e^{-\lambda s})| \rho^{\alpha - \beta_0} \pi_1 \right\| \right).
\]

The right-hand side of the aforementioned inequality has tended to zero since \( \zeta_1 \to \zeta_2 \). Because of this, \( \delta(E^*) \) is equicontinuity. As a solution of Step 1 to 3, the operator \( \delta \) is completely continuous. Therefore, the operator \( \delta \) is compact in light of the Arzela–Ascoli theorem.

In the next theorem, the following notations are used:

\[
Q = \frac{|1 - e^{-\lambda s}|}{\lambda \Gamma(\alpha + 1)} \pi_1 + \frac{|(1 - e^{-\lambda s})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} \pi_1 + \sum_{i=1}^p \frac{|(1 - e^{-\lambda s})|}{\lambda \Gamma(\alpha - \beta_0 + 1)} |(1 - e^{-\lambda s})| \rho^{\alpha - \beta_0} \pi_1
\]

and

\[
S = \frac{|1 - e^{-\lambda s}|}{\lambda \Gamma(\alpha + 1)} \pi_2 + \frac{|(1 - e^{-\lambda s})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} \pi_2 + \sum_{i=1}^p \frac{|(1 - e^{-\lambda s})|}{\lambda \Gamma(\alpha - \beta_0 + 1)} |(1 - e^{-\lambda s})| \rho^{\alpha - \beta_0} \pi_2.
\]

Theorem 5. Let the hypothesis \((H_3)\) hold if \( S < 1 \). Then, problem (1) has at least one solution in \( X \).

Proof. We first consider a set \( B \subset X \) which is defined as follows:

\[
B = \{ u \in X : u = \eta \delta u, 0 < \eta < 1 \}.
\]

We prove that \( B \) is bounded. Let \( u \in B \) such that

\[
u(\xi) = \eta \delta u(\xi), \text{ where } \eta \in (0, 1).
\]

Then, for each \( \zeta \in J \), we have:

\[
|u(\zeta)| = |\eta \left( \int_0^\zeta e^{-\lambda(s-s)} I_{\zeta}^{\zeta} \Phi(s, u(s)) ds + \frac{1 - e^{-\lambda s}}{\Delta} \int_0^1 e^{-\lambda(1-s)} I_{\zeta}^{\zeta} \Phi(s, u(s)) ds \\
- \frac{1 - e^{-\lambda s}}{\Delta} \sum_{i=1}^p \int_0^\zeta e^{-\lambda(s-s)} I_{\zeta}^{\zeta} \Phi(s, u(s)) ds d\mu_i(\zeta) \right) | \\
\leq L(\zeta) \Phi(s, u(s)),
\]

(14)
where

\[
L(\zeta) = \frac{|1 - e^{-\lambda \zeta}|^z_{\alpha}}{\lambda \Gamma(\alpha + 1)} + \frac{|(1 - e^{-\lambda \zeta})(1 - e^{-\lambda})|^z_{\alpha - \beta_0}}{\lambda \Gamma(\alpha - \beta_0 + 1)} + \sum_{i=1}^{p} \frac{|(1 - e^{-\lambda \zeta})(1 - e^{-\lambda \rho})|^z_{\alpha - \beta_i} \Phi_{\beta_i}}{\lambda \Gamma(\alpha - \beta_i + 1)}.
\]

Now, by (H_3), we have:

\[
|\Phi(\xi, u(\xi))| \leq a(\xi) + b(\xi)|u(\xi)| \leq \pi_1^* + \pi_2^*|u(\xi)|.
\]

(15)

Plugging (15) in (14) and taking sup_{\xi \in J}, we obtain:

\[
\|u\|_X \leq \left[ \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(\alpha + 1)} + \frac{|(1 - e^{-\lambda})|^2}{\lambda \Gamma(\alpha - \beta_0 + 1)} + \sum_{i=1}^{p} \frac{|(1 - e^{-\lambda})(1 - e^{-\lambda \rho})|^z_{\alpha - \beta_i} \Phi_{\beta_i}}{\lambda \Gamma(\alpha - \beta_i + 1)} \right] [\pi_1^* + \pi_2^*]\|u\|_X.
\]

(16)

So, (16) becomes:

\[
\|u\|_X \leq Q + S\|u\|_X,
\]

from which we achieve:

\[
\|u\|_X \leq \frac{Q}{1 - S},
\]

which means that B is bounded. By Theorems 4 and 3, we know that the operator \(\delta\) has one fixed point in \(X\), as desired. \(\square\)

**Theorem 6.** Let (H_1), (H_2) and (H_4) hold with \(\mathbb{K}Y < 1\). Then, problem (1) has a unique solution in \(X\).

**Proof.** Let \(u, \bar{u}\) be the solution of (1) and for \(\xi \in J\). Then, we have:

\[
|\delta(u)(\xi) - \delta(\bar{u})(\xi)| \leq \int_{0}^{\xi} e^{-\lambda(\xi - s)} \int_{0}^{\xi} \Phi(s, u(s)) - \Phi(s, \bar{u}(s))ds
\]

\[
+ \frac{|1 - e^{-\lambda \xi}|}{|\Delta|} \int_{0}^{1} e^{-\lambda(1-s)} \int_{0}^{\xi} \Phi(s, u(s)) - \Phi(s, \bar{u}(s))ds\|ds
\]

\[
+ \frac{|1 - e^{-\lambda \xi}|}{|\Delta|} \sum_{i=1}^{p} \int_{0}^{1} \int_{0}^{\xi} e^{-\lambda(\xi - s)} \int_{0}^{\xi} \Phi(s, u(s)) - \Phi(s, \bar{u}(s))ds d\mu_i(\xi).
\]

(17)

Now, by (H_4), we have:

\[
|\Phi(\xi, u(\xi)) - \Phi(\xi, \bar{u}(\xi))| \leq \mathbb{K}|u(\xi) - \bar{u}(\xi)|.
\]

So, (17) becomes:

\[
|\delta(u)(\xi) - \delta(\bar{u})(\xi)| \leq \mathbb{K} \left[ \int_{0}^{\xi} e^{-\lambda(\xi - s)} \int_{0}^{\xi} |u(\xi) - \bar{u}(\xi)|ds
\]

\[
+ \frac{|1 - e^{-\lambda \xi}|}{|\Delta|} \int_{0}^{1} e^{-\lambda(1-s)} \int_{0}^{\xi} |u(\xi) - \bar{u}(\xi)|ds\|ds
\]

\[
+ \frac{|1 - e^{-\lambda \xi}|}{|\Delta|} \sum_{i=1}^{p} \int_{0}^{1} \int_{0}^{\xi} e^{-\lambda(\xi - s)} \int_{0}^{\xi} |u(\xi) - \bar{u}(\xi)|ds d\mu_i(\xi) \right].
\]

Now, taking sup_{\xi \in J} on both sides, we obtain:
This implies that
\[ \|\delta(u) - \delta(\bar{u})\| \leq K \left( \frac{1 - e^{-\lambda}}{\lambda (\alpha + 1)} + \frac{(1 - e^{-\lambda})^2}{\lambda \sqrt{\Gamma(\alpha - \beta_0) + 1}} + \sum_{i=1}^{p} \frac{1 - e^{-\lambda |1 - e^{-\lambda p}| p^{a-\beta_i} q_i}}{\lambda \sqrt{\Gamma(\alpha - \beta_i + 1)}} \right) \|u - \bar{u}\| \]

Hence, the operator \( \delta \) is a contraction mapping. Therefore, according to Theorem 2, the operator \( \delta \) possesses a unique fixed point, which is a unique solution to (1). \( \square \)

4. Ulam Stability Analysis

In this section, we investigate stability results in the sense of Ulam for the proposed problem (1).

Lemma 6. Let \( \alpha \in (0, 1) \). If \( u \in C(\mathcal{J}, \mathbb{R}) \) is a solution of (3), then \( u \) is also a solution of the following inequality:
\[ |u(\xi) - w(\xi)| \leq \mathcal{U} \epsilon. \]

Proof. Let \( u \) be the solution of inequality (3). So in view of (i) of Remark (1), we have:
\[ \begin{cases} \mathcal{C}D_{0+}^\alpha (D + \lambda) u(\xi) = \Phi(\xi, u(\xi)) + \psi(\xi), & \xi \in (0, 1), \\ u(0) = 0, & \mathcal{C}D_{0+}^\beta u(1) = \sum_{i=1}^{p} \int_{0}^{1} \mathcal{C}D_{0+}^\beta_i u(\xi) d\mu_i(\xi). \end{cases} \]

Hence, the solution of (18) will be in the following form:
\[ u(\xi) = \int_{0}^{1} \mathcal{G}(\xi, s) \Phi(s, u(s)) ds + \int_{0}^{1} \mathcal{G}(\xi, s) \psi(s) ds. \]

From (19), we have:
\[ u(\xi) = \int_{0}^{\xi} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^\alpha \Phi(s, u(s)) ds + \frac{1 - e^{-\lambda \xi}}{\lambda} \int_{0}^{1} e^{-\lambda(1-s)} \mathcal{I}_{0+}^{\alpha-\beta_0} \Phi(s, u(s)) ds \\
- \frac{1 - e^{-\lambda \xi}}{\lambda} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^{\alpha-\beta_i} \Phi(s, u(s)) ds d\mu_i(\xi) \\
+ \int_{0}^{\xi} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^{\alpha-\beta_0} \psi(s) ds + \frac{1 - e^{-\lambda \xi}}{\lambda} \int_{0}^{1} e^{-\lambda(1-s)} \mathcal{I}_{0+}^{\alpha-\beta_0} \psi(s) ds \\
- \frac{1 - e^{-\lambda \xi}}{\lambda} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^{\alpha-\beta_i} \psi(s) ds d\mu_i(\xi). \]

For simplicity, let us denote the sum of terms free of \( \psi \) by \( w(\xi) \), we have:
\[ w(\xi) = \int_{0}^{\xi} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^\alpha \Phi(s, u(s)) ds + \frac{1 - e^{-\lambda \xi}}{\lambda} \int_{0}^{1} e^{-\lambda(1-s)} \mathcal{I}_{0+}^{\alpha-\beta_0} \Phi(s, u(s)) ds \\
- \frac{1 - e^{-\lambda \xi}}{\lambda} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^{\alpha-\beta_i} \Phi(s, u(s)) ds d\mu_i(\xi). \]

So from above, we have:
\[ |u(\xi) - w(\xi)| \leq \int_{0}^{\xi} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^\alpha |\psi(s)| ds + \frac{1 - e^{-\lambda \xi}}{|\lambda|} \int_{0}^{1} e^{-\lambda(1-s)} \mathcal{I}_{0+}^{\alpha-\beta_0} |\psi(s)| ds \\
+ \frac{1 - e^{-\lambda \xi}}{|\lambda|} \sum_{i=1}^{p} \int_{0}^{1} e^{-\lambda(\xi-s)} \mathcal{I}_{0+}^{\alpha-\beta_i} |\psi(s)| ds d\mu_i(\xi). \]
In terms of (2) of Lemma 5 and (ii) of Remark 1, we have:

\[
|u(\xi) - w(\xi)| \leq \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(\alpha + 1)} \frac{|(1 - e^{-\lambda})^2|}{\lambda \Gamma(\alpha - \beta_0 + 1)} + \sum_{i=1}^p \frac{|1 - e^{-\lambda}| |1 - e^{-\lambda \eta}| |\rho^a - \beta_i|}{\lambda \Gamma(\alpha - \beta_i + 1)} \epsilon \\
\leq \mathcal{Y} \epsilon,
\]

as desired. \( \square \)

**Theorem 7.** Under the hypotheses \((H_1), (H_2), \) and \((H_4)\) with \( \mathcal{Y} \mathcal{K} < 1, \) system (1) is \( \mathcal{UHS} \) and, consequently, \( \mathcal{GUHS}. \)

**Proof.** Suppose \( u \in C(J, \mathbb{R}) \) is a solution of the inequality (3) and \( v \) is the unique solution of the given system:

\[
\begin{cases}
\frac{d}{d\xi} \Omega(D + \lambda) \nu(\xi) = \Phi(\xi, \nu(\xi)), & \xi \in (0, 1], \\
v(0) = 0, & \frac{d}{d\xi} \Omega(D + \lambda) \nu(1) = \sum_{i=1}^p \int_0^1 \frac{d}{d\xi} \Omega(D + \lambda) \nu(\xi) d\mu_i(\xi).
\end{cases}
\]

Then, for \( \xi \in J, \) the solution of (20) is:

\[
v(\xi) = \int_0^1 G(\xi, s) \Phi(s, v(s)) ds.
\]

Consider:

\[
|u(\xi) - v(\xi)| \leq |u(\xi) - w(\xi)| + |w(\xi) - v(\xi)|.
\]

By using Lemma 6 in (21), we have:

\[
|u(\xi) - v(\xi)| \leq \mathcal{Y} \epsilon + \int_0^\xi e^{-\lambda(\xi - s)} \frac{d}{d\xi} \Omega(D + \lambda) \Phi(s, u(s)) - \Phi(s, v(s)) ds
\]

\[
+ \frac{|1 - e^{-\lambda \xi}|}{\Delta} \int_0^1 e^{-\lambda(1-s)} \frac{d}{d\xi} \Omega(D + \lambda) \Phi(s, u(s)) - \Phi(s, v(s)) ds
\]

\[
+ \frac{|1 - e^{-\lambda \xi}|}{\Delta} \sum_{i=1}^p \int_0^1 e^{-\lambda(1-s)} \frac{d}{d\xi} \Omega(D + \lambda) \Phi(s, u(s)) - \Phi(s, v(s)) ds d\mu_i(\xi).
\]

Now, by \((H_4),\) we have:

\[
|\Phi(\xi, u(\xi)) - \Phi(\xi, v(\xi))| \leq \mathcal{K} |u(\xi) - v(\xi)|.
\]

Using (2) of Lemma 5 and (23) in (22), we have:

\[
\|u - v\|_X \leq \frac{\mathcal{Y} \epsilon}{1 - \mathcal{Y} \mathcal{K}} = K_0 \epsilon
\]

where

\[
K_0 = \frac{\mathcal{Y}}{1 - \mathcal{Y} \mathcal{K}}
\]

such that

\[
\mathcal{Y} \mathcal{K} < 1.
\]

Thus, problem (1) is \( \mathcal{UHS}. \)

Now, by setting \( \varphi(\epsilon) = K_0 \epsilon, \varphi(0) = 0 \) in (26) yields that the problem (1) is \( \mathcal{GUHS}. \) \( \square \)

**Lemma 7.** Let \((H_3)\) hold. If \( u \in C(J, \mathbb{R})\) is a solution of (4), then \( u \) is also a solution of the following inequality:

\[
|u(\xi) - w(\xi)| \leq \mathcal{Y} \Omega \varphi(\xi) \epsilon.
\]
Proof. From Lemma 6, we have:
\[
|u(\zeta) - w(\zeta)| \leq \int_0^\zeta e^{-\lambda(\zeta-s)} f_0^s |\psi(s)| ds + \frac{|1 - e^{-\lambda \zeta}|}{|\Delta|} \int_0^1 e^{-\lambda(1-s)} f_0^s |\psi(s)| ds
\]
\[
+ \frac{|1 - e^{-\lambda \zeta}|}{|\Delta|} \sum_{i=1}^p \int_0^\zeta e^{-\lambda(\zeta-s)} f_0^s |\psi(s)| ds d\mu_i(\zeta).
\]
Using (2) of Lemma 5, (ii) of Remark 1, and \((\text{H}_5)\), we obtain:
\[
|u(\zeta) - w(\zeta)| \leq \left( \frac{|1 - e^{-\lambda}|}{\lambda \Gamma(a + 1)} + \frac{|(1 - e^{-\lambda})^2|}{\lambda \Gamma'(a - \beta_0 + 1)} + \sum_{i=1}^p \frac{|1 - e^{-\lambda}|}{\lambda \Gamma'(a - \beta_i + 1)} \right) \Omega_\phi \phi(\zeta) e
\]
\[
\leq \mathcal{K}_\mathcal{Y} \phi(\zeta) e.
\]
This completes the proof. \(\square\)

**Theorem 8.** Let the hypotheses \((\text{H}_1)\), \((\text{H}_2)\), \((\text{H}_4)\), and \((\text{H}_5)\) hold alongside with the condition \(\mathcal{K}_\mathcal{Y} < 1\). Then, the FBVP (1) will be \(UHR\) and \(GHR\).

**Proof.** Suppose that \(u \in C(J, \mathbb{R})\) is any solution of the inequality (4), and let \(v\) be the unique solution of the system:
\[
\begin{cases}
\mathcal{C}D^a_{0^+} (D + \lambda) v(\zeta) = \Phi(\zeta, v(\zeta)), & \zeta \in (0, 1), \\
v(0) = 0, & \mathcal{C}D^a_{0^+} v(1) = \sum_{i=1}^p \mathcal{C}D^a_{0^+} v(\zeta) d\mu_i(\zeta).
\end{cases}
\]
(25)
Then, for \(\zeta \in J\), the solution of (25) is:
\[
v(\zeta) = \int_0^1 G(\zeta, s) \Phi(s, v(s)) ds.
\]
Consider:
\[
|u(\zeta) - v(\zeta)| \leq |u(\zeta) - w(\zeta)| + |w(\zeta) - v(\zeta)|.
\]
(26)
Using \((\text{H}_4)\), in a same way as in Theorem 7, we obtain:
\[
|w(\zeta) - v(\zeta)| \leq \mathcal{K}_\mathcal{Y} |u(\zeta) - v(\zeta)|.
\]
(27)
Now, by Lemma 7 and (27), (26) becomes:
\[
\|u - v\| \leq \mathcal{K}_\mathcal{Y} \phi(\zeta) e + \mathcal{K}_\mathcal{Y} \|u - v\|,
\]
which implies
\[
\|u - v\| \leq \frac{\mathcal{K}_\mathcal{Y} \phi(\zeta) e}{1 - \mathcal{K}_\mathcal{Y}} = K_\phi \phi(\zeta) e
\]
(28)
where
\[
K_\phi = \frac{\mathcal{K}_\mathcal{Y} \phi(\zeta)}{1 - \mathcal{K}_\mathcal{Y}} \Omega_\phi
\]
such that
\[
\mathcal{K}_\mathcal{Y} < 1.
\]
Thus, the $FBVP$ (1) is $UHRS$.

Now, if we plug $e = 1$ in (28), then, by Definition 6, the problem is $GUHRS$. This finishes the proof. □

5. Examples

In this section, we give two examples to support our main results.

Example 1. Suppose that $FBVP$:

\[
\begin{aligned}
\begin{cases}
\frac{c^2D}{0,\xi}^2 (D - 1) u(\xi) = \frac{2 + |u(\xi)|}{12\xi^{p+1} (1 + |u(\xi)|)}, & \xi \in (0, 1], \\
u(0) = 0, & \frac{c^pD}{0,\xi}^{\beta_0} u(1) = \sum_{i=1}^{2} \int_{0}^{1} \frac{c^pD}{0,\xi}^{\beta_i} u(\xi) d\mu_i(\xi),
\end{cases}
\end{aligned}
\]

where $\alpha = \frac{3}{4}$, $\lambda = -1$, $p = 2$, $\Psi_1 = 10$, $\Psi_2 = 20$, $\beta_0 = \frac{1}{4}$, $\beta_1 = \frac{1}{2}$, and $\beta_2 = \frac{7}{10}$.

Set

\[\Phi(\xi, u) = \frac{2 + |u|}{12\xi^{p+1} (1 + |u|)}, \quad \xi \in [0, 1], \quad u \in \mathbb{R}.\]

The above function $\Phi$ is jointly continuous. Now, for every $u, \bar{u} \in \mathbb{R}$ and $\xi \in [0, 1]$, we have:

\[|\Phi(\xi, u) - \Phi(\xi, \bar{u})| = \frac{1}{12\xi} (|u - \bar{u}|).\]

Thus, ($H_4$) is satisfied with $K = \frac{1}{12\xi}$.

In addition, we have:

\[\Phi(\xi, u) = \frac{1}{12\xi^{p+1}} (2 + |u|).\]

Thus, ($H_3$) is satisfied with $\pi_1(\xi) = \frac{1}{6\xi^{p+1}}$, $\pi_2(\xi) = \frac{1}{12\xi^{p+1}}$, where $\pi_1^* = \frac{1}{6\xi}$ and $\pi_2^* = \frac{1}{12\xi}$.

From Theorem 6, we use the inequality which is found:

\[\mathbb{K} \mathbb{Y} \approx -0.0948682 < 1.\]

Hence, the solution of the problem (1) is unique.

Furthermore, from Theorem 7 with $k_0 > 0$, we use the inequality found:

\[\mathbb{K} \mathbb{Y} \approx -0.0948682 < 1.\]

Then, Theorem 7 gives that system (1) is $UHRS$ and hence $GUHRS$. Moreover, by Theorem 8, we know that problem (1) is $UHRS$ and $GUHRS$.

Example 2. Suppose that $FBVP$:

\[
\begin{aligned}
\begin{cases}
\frac{c^3D}{0,\xi}^{\frac{1}{2}} (D - \frac{3}{2}) u(\xi) = \frac{1}{90} (\xi \cos(u(\xi)) - u(\xi) \sin(\xi)), & \xi \in (0, 1], \\
u(0) = 0, & \frac{c^pD}{0,\xi}^{\beta_0} u(1) = \sum_{i=1}^{3} \int_{0}^{1} \frac{c^pD}{0,\xi}^{\beta_i} u(\xi) d\mu_i(\xi),
\end{cases}
\end{aligned}
\]

where $\alpha = \frac{1}{4}$, $\lambda = -\frac{3}{2}$, $p = 3$, $\Psi_1 = 10$, $\Psi_2 = 20$, $\Psi_3 = 30$, $\beta_0 = \frac{1}{16}$, $\beta_1 = \frac{1}{8}$, $\beta_2 = \frac{3}{16}$, and $\beta_3 = \frac{5}{16}$.

Set:

\[\Phi(\xi, u) = \frac{1}{90} (\xi \cos u - u \sin(\xi)), \quad \xi \in [0, 1], \quad u \in \mathbb{R}.\]
The above function $\Phi$ is jointly continuous. Now, for every $u, \bar{u} \in \mathbb{R}$ and $\zeta \in [0, 1]$, we have:

$$|\Phi(\zeta, u) - \Phi(\zeta, \bar{u})| \leq \frac{1}{90}|\zeta||\cos u - \cos \bar{u}| + \frac{1}{90}|\sin(\zeta)||u - \bar{u}|,$$

$$\leq \frac{1}{90}|u - \bar{u}| + \frac{1}{90}|u - \bar{u}|,$$

$$\leq \frac{1}{45}|u - \bar{u}|.$$

Thus, $(H_4)$ is satisfied with $K = \frac{1}{45}$.

From Theorem 6, we use the inequality which is found:

$$K\mathcal{V} \approx -0.0945716 < 1.$$

This implied that the solution of problem (2) is unique.

Furthermore, from Theorem 7 with $k_0 > 0$, we reach the conclusion:

$$K\mathcal{V} \approx -0.0945716 < 1.$$

Then, Theorem 7 implies that system (1) is UHS and, hence, GUHS. Moreover, from Theorem 8, we can verify that problem (2) is UHRS and GUHRS.

6. Conclusions

In this paper, we present some sufficient conditions to obtain the existence, uniqueness, and Ulam’s stabilities of solutions to system (1). The desired results are investigated by employing the Banach contraction mapping principle, Schaefer’s fixed point theorem, and Arzela–Ascoli theorem. Meanwhile, we derive various types of Ulam’s stabilities for solutions to system (1). Finally, two examples are provided to support the main results.

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