ON THE CAHN–HILLIARD EQUATION WITH MASS SOURCE FOR BIOLOGICAL APPLICATIONS

HUSSEIN FAKIH
Lebanese International University
School of Arts and Sciences, Department of Mathematics and Physics
Bekaa campus, Lebanon

Lebanese University, Faculty of Sciences, Department of Mathematics
Houch el Oumara, Zahle, Lebanon

RAGHEB MGHAMES*
Politehnica University of Bucharest
Splaiul Independentei 313, 060042, Bucharest, Romania

NOURA NASREDDINE
Lebanese International University
School of Arts and Sciences, Department of Mathematics and Physics
Bekaa campus, Lebanon

(Communicated by Alain Miranville)

Abstract. This article deals with some generalizations of the Cahn–Hilliard equation with mass source endowed with Neumann boundary conditions. This equation has many applications in real life e.g. in biology and image inpainting. The first part of this article, discusses the stationary problem of the Cahn–Hilliard equation with mass source. We prove the existence of a unique solution of the associated stationary problem. Then, in the latter part of this article, we consider the evolution problem of the Cahn–Hilliard equation with mass source. We construct a numerical scheme of the model based on a finite element discretization in space and backward Euler scheme in time. Furthermore, after obtaining some error estimates on the numerical solution, we prove that the semi discrete scheme converges to the continuous problem. In addition, we prove the stability of our scheme which allows us to obtain the convergence of the fully discrete problem to the semi discrete one. Finally, we perform the numerical simulations that confirm the theoretical results and demonstrate the performance of our scheme for cancerous tumor growth and image inpainting.

1. Introduction. We are interested in this article in the following initial and boundary value problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(x, u) &= 0, \quad \text{in } \Omega \times [0, T], \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \Gamma, \\
u|_{t=0} &= u_0, \quad \text{in } \Omega,
\end{align*}
\]

2020 Mathematics Subject Classification. Primary: 35Q92, 65M12, 65M60; Secondary: 35J60.

Key words and phrases. Cahn–Hilliard equation, image inpainting, cancerous tumor growth, well-posedness, stability, convergence, numerical simulations.

* Corresponding author.
in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, with boundary $\Gamma$ and $T > 0$. Here, the mass source term $g$ has the form $g(x, s) = h(x)L(s)$, where $h \in L^\infty(\Omega)$ and $L$ is a polynomial of odd degree. This equation (1.1) can be used as a model for the growth of cancerous tumors and other biological entities.

Typically, $g$ can be a linear function, $g(x, s) = \alpha s$, $\alpha > 0$, in which case (1.1) is known as the Cahn–Hilliard–Oono equation and accounts for long-ranged (nonlocal) interactions in the phase separation process (see [25, 27]; see also [20] for the study of the limit dynamics as $\alpha$ goes to zero). A second possibility is the quadratic function $g(x, s) = \alpha(s - 1)$, $\alpha > 0$; here, (1.1) has applications in biology [11, 17] and, more precisely, in the models of wound healing and tumor growth [19]. A third possibility, with applications in biology and more specifically in tumor growth [1], is the function $g(x, s) = \frac{1}{2}(s + 1) - \beta(1 - s)^2(1 + s)^2 + s(x, t)$, where $\alpha$ and $\beta$ are the death and growth coefficients, respectively. A fourth possibility is the function $g(x, u) = \chi_{\Omega\setminus D}(x)u$ (where $D \subset \subset \Omega$ and $\chi$ denotes the indicator function). In this case (1.1) has application in image inpainting (see [3, 4, 6, 8, 9, 13]; see also non-regular nonlinear terms in [7]).

Equation (1.1) is a generalization of the original Cahn–Hilliard equation which plays an essential role in material sciences as it describes the phase separation of binary systems in physics and chemistry. It was presented in the form of free energy in 1958 by J.W. Cahn and J.E. Hilliard [5]. Later, on the basis of the thermal dynamical principle, the Cahn–Hilliard equation was derived in the form of a partial differential equation (see, e.g., [24]). In fact, when a binary solution is cooled down sufficiently, phase separation may occur and then proceed in two ways: either by nucleation in which case nuclei of the second phase appear randomly and grow, or the whole solution appears to nucleate at once and then periodic or semiperiodic structures appear in the so-called spinodal decomposition. The pattern formation resulting from phase separation has been observed in alloys, glasses, and polymer solutions.

The function $f : \mathbb{R} \to \mathbb{R}$ belongs $C^2(\mathbb{R}, \mathbb{R})$ and it satisfies the following standard dissipativity assumption:

$$\liminf_{|s| \to \infty} f'(s) > 0.$$ 

Typical choice is

$$f(s) = s^3 - s.$$ 

The function $u(x, t)$ represents the concentration of one of the metallic components of the alloy.

From a mathematical point of view, problem (1.1)–(1.3) has been studied in [16]. The author proved the existence of a unique global solution, as well as the existence of a finite (fractal) dimension global attractor. Furthermore, the author in [21] and [22] considered (1.1) endowed with Dirichlet boundary conditions. In [21], he considered the problem with regular (polynomial) nonlinear term and he proved the existence of a long-time unique solution. Then, in [22], he considered the problem with singular (logarithmic) nonlinear term and he was also able to prove the existence of a long-time unique solution. Finally, we refer the readers to [23] for more details on the applications of the problem and the recent results.

In this article, we are interested to prove the existence of a unique weak solution of the steady state problem associated to (1.1)–(1.3), using alternative methods, namely by fixed point arguments. Then, a numerical scheme based on a finite element space discretization on space and Backward Euler discretization on time has been considered. After obtaining some error estimates on the semi-discrete solution, we obtain the convergence of the semi-discrete solution to the continuous one. In addition, we were able to prove the stability of the backward Euler scheme which allows us to obtain the convergence of fully discrete scheme to the continuous problem. Finally, we do some numerical simulations that confirm the theoretical results and show the efficiency of the scheme. The simulations were done using FreeFem++.

**Notations.** We now introduce the following spaces:

$$\overline{H}^{-1}(\Omega) = \{ \phi \in H^{-1}(\Omega), \langle \phi, 1 \rangle = 0 \},$$

$$\mathcal{P}(\Omega) = \{ \phi \in L^2(\Omega), \langle \phi \rangle = 0 \},$$

$$\mathcal{V}(\Omega) = \{ \phi \in H^1(\Omega), \langle \phi \rangle = 0 \},$$

which are the $H^{-1}$, $L^2$ and $H^1$ spaces with zero spatial average respectively.

Setting

$$\langle \phi \rangle = \frac{1}{\text{Vol}(\Omega)} \int \phi(x) \, dx.$$
2. Well-Posedness of the steady state problem. In this section, our aim is to prove the existence of a weak solution of the stationary problem. The stationary problem associated to (1.1)–(1.3) is given as follows:

\[ \Delta^2 u - \Delta f(u) + h(x)L(u) = 0, \quad \text{in } \Omega, \]
\[ \frac{\partial u}{\partial v} = 0, \quad \text{on } \Gamma. \]

Integrating (2.1) over \( \Omega \) and owing the boundary conditions, we have

\[ \langle h(x)L(u) \rangle = 0. \]

Refering to (1.4), we have

\[ f'(s) \geq -1. \]

In the proof of the existence of a solution to the variational problem of (2.1)-(2.2), we follow the subsequent strategy. We consider by the fixed point operator

\[ T : L^2(\Omega) \to L^2(\Omega), \quad \phi \to T(\phi) = u, \]

for a given \( \phi \) in \( L^2(\Omega) \), the equation

\[ \Delta^2 u - \Delta f(u) + h(x)L(u) + \left( \frac{1}{\alpha} + 1 \right) (u - \phi - \langle u - \phi \rangle) = 0, \quad \text{in } \Omega, \]
\[ \frac{\partial u}{\partial v} = 0, \quad \text{on } \Gamma, \]

where \( \alpha \) being a positive constant. Integrating (2.5) over \( \Omega \), we find (2.3).

Therefore, (2.5) can be rewritten as

\[ -\Delta u + (f(u) - \langle f(u) \rangle) + (\Delta)^{-1}(h(x)L(u)) + \left( \frac{1}{\alpha} + 1 \right) (\Delta)^{-1}(u - \phi - \langle u - \phi \rangle) = 0. \]  

The variational formulation of (2.7) reads

\[ \langle (\nabla u, \nabla \psi) \rangle + \langle (f(u), \psi) \rangle + \left( \frac{1}{\alpha} + 1 \right) \langle (\Delta)^{-1/2}(u - \phi - \langle u - \phi \rangle), (\Delta)^{-1/2}\psi \rangle \]
\[ + \langle ((\Delta)^{-1/2}(h(x)L(u)), (\Delta)^{-1/2}\psi) \rangle = 0, \]

for \( \psi \in H^1(\Omega) \cap \mathcal{H}(\Omega) \). In addition, the functional of the variational formulation reads

\[ F(u, \phi) = E(u) + \left( \frac{1}{2\alpha} + \frac{1}{2} \right) \| u - \phi - \langle u - \phi \rangle \|^2_1 + \frac{1}{2} \| h(x)L(u) \|^2_1, \]

where \( E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_\Omega F(u) \, dx \) and \( \| \cdot \|_1 \) is the norm defined in \( \mathcal{H}^{-1} \).

**Lemma 2.1.** Assuming that \( F(s) = \frac{1}{4}(s^2 - 1)^2 \) (satisfies \( F'(s) = f(s) = s^3 - s \)), then

\[ F(u) + F(v) - 2F\left( \frac{u + v}{2} \right) > -\frac{1}{4} (u - v)^2, \]

for all \( u, v \in \mathbb{R} \) such that \( u \neq 0 \) (or \( v \neq 0 \)).

**Proof.** Refer to [26], page 9.

\[ \square \]

**Proposition 1.** Assuming that \( F(s) = \frac{1}{4}(s^2 - 1)^2 \) and

\[ h(x) \left( L(u) + L(v) - 2L\left( \frac{u + v}{2} \right) \right) > -c(u - v)^2, \]

such that \( 0 < c < \frac{1}{4\alpha} + \frac{1}{16} \). Then equation (2.7) has a unique weak solution in \( H^1(\Omega) \).

**Proof.**

\[ F(u, \phi) \geq \frac{1}{4} \| \nabla u \|^2_{L^2(\Omega)} + c_1 \| u \|^2_{L^2(\Omega)} - c_2 \text{Vol}(\Omega) + \left( \frac{1}{2\alpha} + \frac{1}{2} \right) \left( \frac{1}{2} \| \phi \|^2_1 - \| \phi - \langle \phi \rangle \|^2_1 \right), \]

\[ + \frac{1}{2} \| h(x)L(u) \|^2_1 \geq \frac{1}{4} \| \nabla u \|^2_{L^2(\Omega)} + c_1 \| u \|^2_{L^2(\Omega)} + \left( \frac{1}{2\alpha} + \frac{1}{4} \right) \| \phi \|^2_1 + c, \]

where we set \( v = u - \langle \phi \rangle \) and the constant \( c \) depends only on \( \phi, h, \) and \( \Omega \). We then deduce that the functional energy \( F \) is coercive in \( H^1(\Omega) \). Now, Considering a minimizing sequence \( u^n \in H^1(\Omega) \) of the functional \( F(u, \phi) \), and the fact that \( F \) is coercive in \( H^1(\Omega) \), implies that \( u^n \) is bounded in \( H^1(\Omega) \). Therefore, there exists \( u^* \in H^1(\Omega) \). Furthermore, \( u^n \to u^* \) weakly in \( H^1(\Omega) \) and \( u^n \to u^* \) strongly in \( L^2(\Omega) \), owing the compact embedding \( H^1(\Omega) \subset L^2(\Omega) \).
Recall that,
\[
\mathcal{F}(u^n, \phi) = \mathcal{E}(u^n) + \left(\frac{1}{2\alpha} + \frac{1}{2}\right)\|u^n - \phi - (u^n - \phi)\|^2_{L^2} + \frac{1}{2}\|h(x)L(u^n)\|^2_{L^2}.
\]  
(2.10)

Noting that
\[
\|u^n - \phi - (u^n - \phi)\|^2_{L^2} - \|u^n - \phi - (u^n - \phi)\|^2_{L^2} = (\mathcal{L}^{-1}(u^n - \phi - (u^n - \phi))) - (\mathcal{L}^{-1}(u^n - \phi - (u^n - \phi)))
\]
\[
= (\mathcal{L}^{-1}(u^n - \phi - (u^n - \phi))) - (\mathcal{L}^{-1}(u^n - \phi - (u^n - \phi)))
\]
\[
= (\mathcal{L}^{-1}(u^n - \phi - (u^n - \phi))) - (\mathcal{L}^{-1}(u^n - \phi - (u^n - \phi)))
\]
\[
\leq \|u^n - u^n - (u^n - u^n)\|^2_{L^2} - \|u^n - \phi - (u^n - \phi)\|^2_{L^2} + \|u^n - u^n - (u^n - u^n)\|^2_{L^2} - \|u^n - \phi - (u^n - \phi)\|^2_{L^2}.
\]

Since the function \(L\) is continuous and \(u^n\) strongly convergent to \(u\) in \(L^2(\Omega)\), we infer that
\[
\|h(x)L(u^n)\|^2_{L^2} \rightarrow \|h(x)L(u)\|^2_{L^2}
\]
\[
\|u^n - \phi - (u^n - \phi)\|^2_{L^2} \rightarrow \|u^n - \phi - (u^n - \phi)\|^2_{L^2}.
\]  
(2.11)

In addition, since the function \(\frac{1}{2}(u^2 - 1)^2\) is continuous, we can easily prove by Fatou’s lemma that \(\mathcal{E}(u)\) is weakly lower semi continuous. Thus, it gives that \(\mathcal{F}(u, \phi)\) is weakly lower continuous in \(H^1(\Omega)\),
\[
\mathcal{F}(u^n, \phi) \leq \lim \inf \mathcal{F}(u^n, \phi),
\]  
(2.13)

and \(\mathcal{F}\) has a minimizer in \(H^1(\Omega)\), i.e. there exists
\[
u^* \in H^1(\Omega) \text{ with } u^* = \arg\min_{u \in H^1(\Omega)} \mathcal{F}(u, \phi).
\]

Finally, by the standard results of the trace function with Neumann boundary condition, the equation to the minimization problem reads as
\[
\left(\frac{1}{\alpha} + 1\right)\mathcal{L}^{-1}(u^n) - (u^n - \phi) + (\mathcal{L}^{-1}(u^n) - \phi) + (\mathcal{L}^{-1}(u^n) - \phi) + (\mathcal{L}^{-1}(u^n) - \phi) = 0,
\]

which is equivalent to the variational formulation. Therefore, \(u^*\) is a weak solution of (2.1)-(2.2).

Now, let \(u_1, u_2 \in H^1(\Omega)\) and setting \(u = u_1 - u_2\), we have owing to Lemma 2.1 and (2.9),
\[
\mathcal{F}(u_1, \phi) + \mathcal{F}(u_2, \phi) - 2\mathcal{F}\left(\frac{u_1 + u_2}{2}\right) > 0.
\]

On account to the interpolation inequality
\[
\|u\|^2_{L^2(\Omega)} \leq \|v\|^2_{-1}\|\nabla u\|^2_{L^2(\Omega)},
\]

we find,
\[
\mathcal{F}(u_1, \phi) + \mathcal{F}(u_2, \phi) - 2\mathcal{F}\left(\frac{u_1 + u_2}{2}\right) > 0.
\]

So that \(\mathcal{F}(u_1, \phi) + \mathcal{F}(u_2, \phi) - 2\mathcal{F}\left(\frac{u_1 + u_2}{2}\right) > 0\). Thus \(\mathcal{F}\) is strictly convex and a unique minimizer has been obtained for the minimization problem.

\textbf{Proposition 2.} We assume that \(\Omega \subset \mathbb{R}^2\) and there exists a constant \(\beta > 1\) such that
\[
(h(x)L(u, u)) \geq \beta\|u\|^2_{L^2(\Omega)}.
\]  
(2.14)

Set \(T : L^2(\Omega) \rightarrow L^2(\Omega)\), \(T(\phi) = u^*\), where \(u^* \in H^1(\Omega)\) is the unique weak solution. Then \(T\) has a fixed point \(\tilde{u} \in H^1(\Omega)\) if \(\alpha > 0\).
Remark 1. Note that $L$ is a polynomial of odd degree, then $L(u)u$ is a polynomial of even degree greater or equal 2. Then, if the function $h$ has a positive lower bound over $\Omega$, assumption (2.14) holds.

Proof. In this proof, for simplicity and without loss of generality of the proof, we will set $u^* = u$. Then we can write the problem as
\[
\left( \frac{1}{\alpha} + 1 \right)(u - \phi, u) + \|\Delta u\|^2_{L^2(\Omega)} + (\nabla f(u), \nabla u) + ((h(x)L(u), u)) = 0. \tag{2.15}
\]
Thanks to (2.4) and (2.14), we find
\[
\left( \frac{1}{\alpha} + \beta + 1 \right)\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2} \left( \frac{1}{\alpha} + 1 \right)\|\Delta u\|^2_{L^2(\Omega)}. \tag{2.16}
\]
Therefore,
\[
\left( \frac{1}{2\alpha} + \beta + 1 \right)\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2} \left( \frac{1}{\alpha} + 1 \right)\|\Delta u\|^2_{L^2(\Omega)}. \tag{2.17}
\]
Besides, by applying the interpolation inequality and then using Young’s inequality, we have that
\[
\|\nabla u\|^2_{L^2(\Omega)} \leq \|\Delta u\|^2_{L^2(\Omega)} \leq \|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\phi\|^2_{L^2(\Omega)} + c,
\]
it then follows that
\[
\left( \frac{1}{2\alpha} + \beta + 1 \right)\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\phi\|^2_{L^2(\Omega)} + c + \left( \frac{1}{\alpha} + 1 \right)\|\phi\|^2_{L^2(\Omega)}, \tag{2.18}
\]
which yields
\[
\left( \frac{1}{2\alpha} + \beta \right)\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} \leq \left( \frac{1}{2\alpha} + 1 \right)\|\phi\|^2_{L^2(\Omega)} + c. \tag{2.19}
\]
Thus,
\[
\|\nabla u\|^2_{L^2(\Omega)} \leq \|\Delta u\|^2_{L^2(\Omega)} + c. \tag{2.20}
\]
where $\xi < 1$ and $\zeta$ is an arbitrary constant. Hence,
\[
\|T(\phi)\|^2_{L^2(\Omega)} \leq \|\phi\|^2_{L^2(\Omega)} + c. \tag{2.21}
\]
Therefore, $u$ is bounded in $L^2(\Omega)$ and $T$ is a map from the closed ball
\[ K = B[0, M] = \{ u \in L^2(\Omega); \|u\|^2_{L^2(\Omega)} \leq M \} \]
into itself for an appropriate constant $M > 0$. Moreover, owing to the stationary problem, we have
\[
\|\Delta u\|^2_{L^2(\Omega)} \leq c\|\phi\|^2_{L^2(\Omega)} + c’. \tag{2.22}
\]
We then conclude that $u$ is also uniformly bounded in $H^2(\Omega)$. It thus follows that $K$ is a closed bounded and convex subset of $L^2(\Omega)$. Now, considering a sequence $\phi_n$ such that,
\[
\phi_n \to \phi \text{ in } L^2(\Omega),
\]
$T(\phi_n) = u_n$ bounded in $H^1(\Omega)$ for all $n$. Then, by taking a subsequence (that is not relabeled),
\[
u_n \to u \text{ weakly in } H^1(\Omega).
\]
Since $H^1(\Omega) \subset L^p(\Omega), 1 \leq p < \infty$, we obtain
\[
u_n \to u \text{ strongly in } L^p(\Omega)
\]
and
\[
u_n \to u \text{ a.e. in } \Omega.
\]
Moreover, since $f$ is continuous,
\[
f(u_n) \to f(u) \text{ weakly in } L^2(\Omega),
\]
by obeying the weak dominated convergence theorem. We conclude that $u$ is a weak solution of the problem. Finally, the solution is unique provided $\alpha > 0$, we have $T(\phi) = u$ and the operator $T$.
is continuous. We finally deduce from Schauder’s Theorem that the operator $T$ has a fixed point $	ilde{u} \in L^2(\Omega)$ and $\tilde{u}$ is the unique solution.

As a consequence from Propositions 1 and 2, we have the following theorem.

**Theorem 2.2.** We assume that $\Omega \subset \mathbb{R}^2$ and there exists a constant $\beta > 1$ such that

$$((h(x)L(u), u))_{L^2(\Omega)}^2 > \beta \|u\|_{L^2(\Omega)}^2.$$  

Then the problem (2.1)-(2.2) has a unique weak solution in $H^1(\Omega)$.

3. **Numerical analysis of the evolution problem.** We recall that the problem is given by

$$u_t = \Delta u + g(x, u), \quad \text{in } \Omega,$$  

$$w = f(u) - \Delta u, \quad \text{in } \Omega,$$  

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial \nu} = 0, \quad \text{on } \partial \Omega.$$  

The variational formulation of (3.1)-(3.3) reads

$$((u_t, \phi)) = -((\nabla w, \nabla \phi)) + ((g(x, u), \phi)), (3.4)$$

$$((w, \chi)) = ((f(u), \chi)) + ((\nabla u, \nabla \chi)), (3.5)$$

for all $\phi, \chi \in H^1(\Omega)$. We use a quasi-uniform family of decomposition $\Omega^h$ of $\Omega$ into k-simplices. For a given triangulation $\Omega^h = \bigcup_{T \in \Omega^h} T$. We define the bilinear form

$$a(\phi, \chi) = ((\nabla \phi, \nabla \chi)). (3.16)$$

We define the bilinear form
which is coercive on $\overline{V}(\Omega)$ i.e there exists $c_0 > 0$, such that

$$a(\phi, \phi) \geq c_0 \|\phi\|^2_{\overline{V}},$$

for all $\phi \in \overline{V}(\Omega).$ \hfill (3.17)

We will start by estimating $\rho^v$ and $\rho^w$.

**Lemma 3.1.** for all $u \in H^2(\Omega)$, the function $w^h \in V^h$ defined by (3.14) satisfies

$$\|u^h - u\|_{L^2(\Omega)} + h\|u^h - u\|_{H^1(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)}.$$ \hfill (3.18)

**Proof.** First, we have that

$$a(w^h, \chi) = a(u, \chi),$$

where $a(., .)$ as given by (3.16). Since $u^h - I^h \in V^h$, we get

$$a(u^h - u, w^h - u) = a(u^h - u, u^h - I^h) + a(u^h - u, I^h - u),$$

hence

$$a(u^h - u, w^h - u) \geq c_0\|u^h - u\|_{H^1(\Omega)}^2,$$

which yields

$$c_0\|u^h - u\|_{H^1(\Omega)}^2 \leq a(u^h - u, u^h - u) \leq \|u^h - u\|_{\overline{V}} I^h - u\|_{\overline{V}}.$$

Therefore,

$$\|u^h - u\|_{\overline{V}} \leq c_0 \|I^h - u\|_{\overline{V}},$$

it then follows from (3.6) that

$$\|u^h - u\|_{\overline{V}} \leq Ch\|u\|_{H^2(\Omega)}.$$ \hfill (3.20)

Besides, for $z \in L^2(\Omega)$, let $\phi$ be the unique solution of

$$a(\phi, \chi) = (z, \chi),$$

for all $\chi \in \overline{V}$. \hfill (3.21)

Thus, we obtain

$$\|\phi\|_{H^2(\Omega)} \leq C\|z\|_{L^2(\Omega)},$$ \hfill (3.22)

such that $c$ does not depend on $z$.

Take $\chi = u^h - u$ in (3.21), we find

$$(z, u^h - u) = a(\phi, u^h - u) = a(\phi - I^h, u^h - u) \leq \|\phi - I^h\|_{\overline{V}} \|u^h - u\|_{\overline{V}}.$$

Furthermore, choosing $z = u^h - u$ and owing to (3.6) and (3.20), we have

$$\|u^h - u\|_{L^2(\Omega)} \leq Ch\|\phi\|_{H^2(\Omega)}Ch\|u\|_{H^2(\Omega)} \leq Ch^2\|u^h - u\|_{L^2(\Omega)}\|u\|_{H^2(\Omega)}.$$

Thus,

$$\|u^h - u\|_{L^2(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)}. \hfill \Box$$

In a similar way to Lemma 3.1, there exists $c$ that relies only on $\Omega^h$, in such a way as for all $w \in H^2(\Omega)$, the function $w^h \in V^h(\Omega)$ defined by (3.12)-(3.13) satisfies

$$\|w^h - w\|_{L^2(\Omega)} + h\|w^h - w\|_{H^1(\Omega)} \leq Ch^2\|w\|_{H^2(\Omega)}.$$ \hfill (3.23)

Next, we define the discrete inverse Laplacian $T^h : \overline{V} \rightarrow \nabla$ by $T^h f = v^h$, where $f \in \overline{H}(\Omega)$ and $v^h \in \nabla$ solves,

$$((\nabla v^h, \nabla \chi^h)) = ((f, \chi^h)), \hfill (3.24)$$

for all $\chi^h \in V^h(\Omega)$. Note that $T^h$ is self adjoint and positive semi-definite on $\nabla$ since

$$(g, T^h f) = ((\nabla T^h g, \nabla T^h f)) = ((f, T^h g)), \hfill (3.25)$$

for all $f, g \in \overline{H}(\Omega)$.

By stating the discrete negative semi norm as follows,

$$\|v\|_{1,1,h} = \|(T^h v, v)\|^{1/2} = \|\nabla T^h v\|_{L^2(\Omega)},$$

for all $v \in \overline{H}(\Omega)$.

Using an orthonormal basis of $\nabla$ for the $L^2(\Omega)$-scalar product, we can clearly see that the following interpolation inequality holds:

$$\|v^h\|_{L^2(\Omega)} \leq \|v^h\|_{1,1,h} \|\nabla^h\|_{H^1(\Omega)},$$

for all $v^h \in \nabla$.

It is also seen that,

$$\|f\|_{1,1,h} \leq c_0 \|f\|_{L^2(\Omega)},$$

for all $f \in \overline{H}(\Omega)$. \hfill (3.26)
where \( c_p \) is the Poincaré constant. Moreover, we have
\[
\delta(t) = \frac{1}{\text{Vol}(\Omega)} (\langle \theta^w(0), 1 \rangle), \quad \text{for all } t \geq 0. \tag{3.27}
\]
So that \( (\theta^w - \delta(t), 1) = 0 \). In the remaining part of this section, the final time \( T \in (0, \infty) \). We have the following Lemma.

**Lemma 3.2.** Let \((u, w)\) be a solution of (3.4)-(3.5) with sufficient regularity and let \((u^h, w^h)\) be a solution of (3.8)-(3.9). Supposing that
\[
\sup_{t \in [0, T]} \|u(t)\|_{C^0(\Omega)} < R,
\]
\[
\sup_{t \in [0, T]} \|u(t)\|_{C^0(\Omega)} \leq R,
\]
\[
\sup_{t \in [0, T]} \|u^h(0)\|_{C^0(\Omega)} < R, \text{ such that } R < \infty.
\]
The maximal time is \( T^h \in (0, T], \) such that \( \|u^h(t)\|_{L^\infty(\Omega)} \leq R, \) for every \( t \in [0, T^h] \) where \( \mathcal{N}(t) = \|\theta^u\|_{H^1(\Omega)}^2 + \|\theta^w - \delta(t)\|_{L^2,1,h}^2. \)

Then
\[
\mathcal{N}(t) + \int_0^t \|\theta^u\|_{H^1(\Omega)}^2 + \frac{1}{2} \|\theta^w\|_{L^2(\Omega)}^2 ds \\
\leq C \mathcal{N}(0) + C' \int_0^t \|\rho^w\|_{L^2(\Omega)}^2 + \|\rho^w\|_{L^2(\Omega)}^2 + \|\rho^w\|_{L^2(\Omega)}^2 ds \\
+ C' \int_0^t \|\rho^w\|_{L^2(\Omega)}^2 + \|\rho^w\|_{L^2(\Omega)}^2 ds, \quad \text{for all } t \in [0, T^h]. \tag{3.28}
\]
Moreover,
\[
|\delta(t| \leq \text{Vol}(\Omega)^{-\frac{1}{2}} (\|\rho^w\|_{L^2(\Omega)} + \|\rho^w\|_{L^2(\Omega)} + \|\rho^w\|_{L^2(\Omega)}), \tag{3.29}
\]
and
\[
\|\theta^w, 1\| \leq C \mathcal{N}(t)^{\frac{1}{2}} + \|\theta^w\|_{L^2(\Omega)}, \quad \text{for all } t \in [0, T]. \tag{3.30}
\]

**Proof.** It follows from (3.4) and (3.8) that
\[
((u^h_1, \phi)) - ((u_1, \phi)) = -((\nabla u^h, \nabla \phi)) + ((\nabla u, \nabla \phi)) + ((g(x, u^h), \phi)) - ((g(x, u), \phi)).
\]
Therefore,
\[
((\theta^u, \phi)) + ((\nabla \theta^w, \nabla \phi)) = -((\rho^w, \phi)) + ((g(x, u^h) - g(x, u), \phi)), \tag{3.31}
\]
In particular \( \phi \equiv 1 \), we obtain
\[
((\theta^w, 1)) = \delta(t) = -((\rho^w, 1)) + \left( h(x)(L(u^h) - L(u)), 1 \right). \tag{3.32}
\]
Owing to (3.15), we have
\[
((\rho^w, 1)) = 0,
\]
hence,
\[
\delta(t) = \left( h(x)(L(u^h) - L(u)), 1 \right). \tag{3.33}
\]
Differentiating (3.32) with respect to time, we find
\[
((\theta^w, 1)) = \delta(t) = -((\rho^w, 1)) + \left( h(x)(L'(u^h)u^h_t - L'(u))u_t, 1 \right), \tag{3.34}
\]
which yields from (3.15) again that
\[
\delta(t) = \left( h(x)(L'(u^h)u^h_t - L'(u))u_t, 1 \right). \tag{3.35}
\]
Subtracting (3.5) from (3.9). Similarly, using the definition (3.14) of \( w^h \),
\[
((w^h, \chi)) - ((w, \chi)) = (f(u^h), \chi)) - (f(u), \chi)) + (\nabla u^h, \nabla \chi)) - ((\nabla u, \nabla \chi))
\]
and
\[
-((\theta^w, \chi)) + ((\nabla \theta^w, \nabla \chi)) = ((\rho^w, \chi)) - (f(u^h) - f(u), \chi)), \tag{3.36}
\]
on \([0, T]\) for all \( \chi \in V^h \). Taking \( \phi = \theta^w \) in (3.31) and \( \chi = \theta^w \) in (3.36), we get
\[
\|\nabla \theta^w\|_{L^2(\Omega)}^2 + \frac{d}{dt} \|\nabla \theta^w\|_{L^2(\Omega)}^2 \\
= -((\rho^w, \theta^w)) + ((\theta^w, \theta^w)) + (g(x, u^h) - g(x, u), \theta^w) - ((f(u^h) - f(u), \theta^w)).
\]
Because there is an $L^\infty$ bound on $u$ and $u^h$, we have
\begin{equation}
\|f(u^h) - f(u)\|_{L^2(\Omega)} \leq L_f \|u^h - u\|_{L^2(\Omega)}
\end{equation}
and
\begin{equation}
\|g(x, u^h) - g(x, u)\|_{L^2(\Omega)} \leq \|h\|_{L^\infty(\Omega)} (L_f(u^h) - L(u)) \|u^h - u\|_{L^2(\Omega)},
\end{equation}
where $L_f$ and $L_L$ are the Lipschitz constants of $f$ and $L$ respectively on $[-R, R]$. Thus,
\begin{equation}
\| \theta^w \|^2_{H^1(\Omega)} + \frac{1}{2} \frac{d}{dt} \| \theta^w \|^2_{H^1(\Omega)} \\
\leq \rho^\theta \| \theta^w \|^2_{L^2(\Omega)} (|\Omega|^{-\frac{1}{2}} \tau^w + \frac{1}{2} |\theta^w, \theta^w|) + c_p \| \theta^w \|^2_{H^1(\Omega)} + \| \rho^\omega \|_{L^2(\Omega)} \| \theta^w \|_{L^2(\Omega)} + L_f \| \theta^w \|_{L^2(\Omega)} \\
+ \| \rho^\omega \|_{L^2(\Omega)} \| \theta^w \|_{L^2(\Omega)} + |h|_{L^\infty(\Omega)} L_L \| \theta^w \|_{L^2(\Omega)} \| \theta^w \|_{L^2(\Omega)} \| \theta^w \|_{L^2(\Omega)} \| \theta^w \|_{L^2(\Omega)},
\end{equation}
on $[0, T^h]$.
Now, we will estimate $(\langle \theta^w, 1 \rangle)$. We will choose $\chi \equiv 1$ in (3.36) and use $(\langle \theta^w, 1 \rangle) = 0$, and the estimates (3.37)-(3.38) yields,
\begin{equation}
\| (\langle \theta^w, 1 \rangle) \| \leq L_f \| \theta^w \|_{L^2(\Omega)} + \| \rho^\omega \|_{L^2(\Omega)} |\Omega|^{-\frac{1}{2}} \tau^w, \text{ on } [0, T^h].
\end{equation}
By using (3.40), the triangular inequality, (3.33) and the modified Poincaré inequality,
\begin{equation}
\| \theta^w \|^2_{L^2(\Omega)} \leq C'(\| \theta^w \|^2_{H^1(\Omega)}), \text{ for all } v \in H^1(\Omega).
\end{equation}
We deduce (3.39). In the right side of (3.39), by using the inequality
\begin{equation}
abla \leq \alpha a^2 + (4\epsilon)^{-1} b^2, \text{ for all } a, b \geq 0, \forall \epsilon > 0.
\end{equation}
Obviously, with (3.40) to find,
\begin{equation}
\| \theta^w \|^2_{H^1(\Omega)} + \frac{d}{dt} \| \theta^w \|^2_{H^1(\Omega)} \\
\leq C(\| \rho^\omega \|^2_{L^2(\Omega)} + \| \rho^\omega \|^2_{L^2(\Omega)} + \| \rho^\omega \|^2_{L^2(\Omega)} + \| \rho^\omega \|^2_{L^2(\Omega)} + \| \rho^\omega \|^2_{L^2(\Omega)}), \text{ on } [0, T^h],
\end{equation}
for constants $C_1$ and $C_2$ that depend only on $|\Omega|, c_p, L_f, L_L, |h|_{L^\infty(\Omega)}$.
Now, we need to estimate $\theta^w_t$, we first differentiate (3.31) with respect to time,
\begin{equation}
\langle (\theta^w_t, \chi) \rangle + \langle (\nabla \theta^w, \nabla \chi) \rangle = \langle (\rho^w \chi) \rangle + \langle (g(x, u^h) - g(x, u)) r, \chi \rangle,
\end{equation}
and we differentiate (3.36) with respect to time,
\begin{equation}
\langle (\theta^w_t, \chi) \rangle + \langle (\nabla \theta^w, \nabla \chi) \rangle = \langle (\rho^w \chi) \rangle - \langle (f(u^h) - f(u)) r, \chi \rangle.
\end{equation}
We choose $\chi = T^h(\theta^w - \delta_t)$ in (3.44) where $\delta$ (as defined on (3.27)) and $\chi = \theta^w_t - \delta_t$ in (3.43), and by adding the resulting equations, and using (3.23)), we get
\begin{equation}
\langle (\theta^w_t, T^h(\theta^w - \delta_t)) \rangle + \langle (\theta^w_t)^\rho_{\delta_t} \|_{H^1(\Omega)} = \langle (\rho^w \chi) \rangle - \langle (f(u^h) - f(u)) r, \delta_t \rangle - \langle (f(u^h) - f(u)) r, \chi \rangle.
\end{equation}
In the first term of the left hand side, we write,
\begin{equation}
\theta^w_t = (\theta^w_t - \delta_t) + \delta_t
\end{equation}
and note that $\langle (\delta_t + \rho^w_t \chi, 1) \rangle = \langle (g(x, u^h) - g(x, u)) r, 1 \rangle$ by (3.32). For the non linear terms we have
\begin{equation}
[f(u^h) - f(u)] r = f'(u^h)[u^h - u] + [f'(u^h) - f(u)] u t
\end{equation}
and
\begin{equation}
\langle [g(x, u^h) - g(x, u)] r, \theta^w_t \rangle = \langle h(x) \theta^w([u^h - u]) + h(x) \theta^w([u^h - u]) \rangle [u^h - u].
\end{equation}
Thus, (3.46) implies,
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \theta^w_t - \delta_t \|^2_{L^2(\Omega)} + \| \theta^w_t \|^2_{H^1(\Omega)} \\
\leq \| \delta_t + \rho^w \|_{L^2(\Omega)} \| \theta^w_t - \delta_t \|_{L^2(\Omega)} + \| \rho^w \|_{L^2(\Omega)} \| \theta^w_t - \delta_t \|_{L^2(\Omega)} + \| \delta_t \|_{L^2(\Omega)} \| \theta^w_t - \delta_t \|_{L^2(\Omega)} + \| \delta_t \|_{L^2(\Omega)} \| \theta^w_t - \delta_t \|_{L^2(\Omega)}
\end{equation}
where \( L_{f'} \) and \( L_{L'} \) are the Lipschitz constants of \( f' \), \( L' \) respectively on \([-R, R]\). With the help of the interpolation inequality (3.25), applied to \( \theta^h = \theta^h - \delta t \) i.e.
\[
\|\theta^h_t - \delta t\|_{L^2(\Omega)} \leq \|\theta^h_t - \delta t\|_{L^2(\Omega)} \leq 1, h \|\theta^h_t\|_{H^1(\Omega)},
\]
and inequality (3.26), Poincaré Inequality and (3.40) in an obvious way, we find,
\[
\frac{d}{dt}\|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{H^1(\Omega)}^2 \leq C_3(\|\Delta t + \rho^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_{tt}\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2)
\]
\[
+ C_4(\|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2), \text{ on } [0, T^h],
\]
for some constants \( C_3 \) and \( C_4 \) which depend only on \( R, c_p, L_{f'}, L_{L'}, \sup |L'| \) and \( \sup |f'| \).

Finally, we add (3.49) and (3.43), using the modified Poincaré inequality (3.41), the triangular inequality,
\[
\|\theta^h_t\|_{L^2(\Omega)}^2 \leq \|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + |\delta t|^2,
\]
the interpolation inequality (3.48), and inequality (3.42) we find,
\[
\frac{1}{2} \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_{tt}\|_{H^1(\Omega)}^2 + \frac{d}{dt} \mathcal{N}(t)
\]
\[
\leq C_5(\|\rho^h_{tt}\|_{L^2(\Omega)} + \|\rho^h_{tt}\|_{L^2(\Omega)} + |\delta t|^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2)
\]
\[
+ \|\theta^h_{tt}\|_{L^2(\Omega)}^2 + C_6(\|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2)
\]
\[
+ \|\theta^h_{tt}\|_{L^2(\Omega)}^2 + C_6(\|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2),
\]
\[
Owing to (3.33), we have
\[
|\delta t| \leq \int_\Omega \left| h(x) \right| \left| L(u^h) - L(u) \right| dx
\]
\[
\leq \|h\|_{L^\infty(\Omega)} L_L \int_\Omega |u^h - u| \leq \|h\|_{L^\infty(\Omega)} L_L \int_\Omega \left( \text{Vol}(\Omega) \right)^\delta \|u^h - u\|_{L^2(\Omega)}
\]
\[
\leq \|h\|_{L^\infty(\Omega)} L_L \left( \text{Vol}(\Omega) \right)^\delta \left( \|\theta^h_t\|_{L^2(\Omega)} + \|\rho^h_t\|_{L^2(\Omega)} \right) \leq C \left( \|\theta^h_t\|_{L^2(\Omega)} + \|\rho^h_t\|_{L^2(\Omega)} \right),
\]
where \( C \) here and below is a general constant that may change from line to another one, hence
\[
|\delta t|^2 \leq C \left( \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 \right).
\]

Besides, take into account (3.44), equation (3.38) yields to
\[
|\delta t| \leq \|h\|_{L^\infty(\Omega)} \sup L' \|u^h - u\|_{L^2(\Omega)} \left( \text{Vol}(\Omega) \right)^\delta
\]
\[
+ \|h\|_{L^\infty(\Omega)} L_L \sup \|u_t\|_{L^2(\Omega)} \|u^h - u\|_{L^2(\Omega)}
\]
\[
\leq C \left( \|\theta^h_t\|_{L^2(\Omega)} + \|\rho^h_t\|_{L^2(\Omega)} + \|\theta^h_t\|_{L^2(\Omega)} + \|\rho^h_t\|_{L^2(\Omega)} \right),
\]
which yields to
\[
|\delta t|^2 \leq C \left( \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 \right)
\]
\[
\leq C \left( \|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 \right)
\]
\[
\leq C \left( \|\theta^h_t - \delta t\|_{H^1(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 \right)
\]
\[
\leq \frac{1}{4} \|\theta^h_t\|_{H^1(\Omega)}^2 + C \left( \|\theta^h_t - \delta t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\theta^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 \right).
\]

It then follows that
\[
\frac{1}{4} \|\theta^h_t\|_{H^1(\Omega)}^2 + \|\theta^h_t\|_{H^1(\Omega)}^2 + \frac{d}{dt} \mathcal{N}(t)
\]
\[
\leq C \left( \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 + \|\rho^h_t\|_{L^2(\Omega)}^2 \right) + C' \mathcal{N}(t).
\]

We then conclude (3.28) by applying Gronwall's lemma.

\[\square\]
Theorem 3.3. Let \((u, w)\) be a solution of (3.4)-(3.5) such that \(u, \ u_t, \ u_{tt}, \ w, \ w_t \in L^2(0, T, H^2(\Omega))\) and let \((u^h, w^h)\) be the solution of (3.8)-(3.9). If
\[
\theta^a(0) = 0, \ \theta^b(0) = 0, \ \text{and} \ \rho^s(0) = 0,
\]
then
\[
\sup_{[0, T]} \left( \left\| u^h \right\|_{L^2(\Omega)} + \left\| u^h_t \right\|_{-1, H} \right) \leq C h^2,
\]
\[
\left( \int_0^T \left\| u^h - w^h \right\|_{L^2(\Omega)}^2 \, ds \right)^{\frac{1}{2}} \leq C h^2,
\]
\[
\sup_{[0, T]} \left\| u^h - u^h \right\|_{H^1(\Omega)} \leq C h,
\]
\[
\left( \int_0^T \left\| u^h - w^h \right\|_{H^2(\Omega)}^2 + \left\| u^h_t - w^h_t \right\|_{H^2(\Omega)}^2 \, ds \right)^{\frac{1}{2}} \leq C h.
\]

Proof. Differentiating equations (3.12)-(3.14) with respect to time, we reach that the elliptic projections of \(u_t\) and \(w_t\) are respectively \((u_e_t), (w_e_t)\). Similarly, this holds for \(u_{tt}\) and \(w_{tt}\). Since \(u \in C^1([0, T], H^2(\Omega))\) and by Sobolev continuous injection \(H^2(\Omega) \subset C^0(\Omega)\), we have
\[
u, \ u_t \in C^0([0, T], C^0(\Omega)).
\]
Thus,
\[
\sup_{t \in [0, T]} \left\| u(t) \right\|_{C^0(\Omega)} < R
\]
and
\[
\sup_{t \in [0, T]} \left\| u_t(t) \right\|_{C^0(\Omega)} \leq R, \ \text{for some} \ R > 0.
\]
Using the inverse estimate (3.7), we have
\[
\left\| u^h(0) - u(0) \right\|_{C^0(\Omega)} \leq C_0 h^{\frac{\gamma}{2}} \left( \left\| u^h(0) - u(0) \right\|_{L^2(\Omega)} + \left\| u(0) - I^h u(0) \right\|_{L^2(\Omega)} \right)
\]
\[
\quad + C h^\gamma \left\| u(0) \right\|_{H^2(\Omega)},
\]
where \(\gamma \in (0, 1)\) is such that \(H^2(\Omega) \subset C^{0, \gamma}(\Omega)\). Thanks to Lemma 3.1, (3.6), and (3.51), we get
\[
\left\| u^h(0) - u(0) \right\|_{C^0(\Omega)} < (C_0 C h^{\frac{\gamma}{2}} + C_0^2 h^\gamma) \left\| u(0) \right\|_{H^2(\Omega)}.
\]
Taking \(h\) small enough,
\[
\left\| u^h(0) \right\|_{C^0(\Omega)} < R.
\]
By claiming that \(N(t) \leq Ch^4\), where \(N\) is given by the lemma 3.2. Noting that
\[
N(t) = \left\| \theta_t^a(0) - \delta_t(0) \right\|_{-1, H}^2,
\]
We will argue in a similar way to the proof of 3.2. Having the equation (3.33) correct at \(t=0\). Taking \(\chi = T_h \theta_t^a(0) - \delta_t(0)\), in (3.31), and we used (3.51) to obtain,
\[
\left( (\theta_t^a(0), T_h \theta_t^a(0) - \delta_t(0)) + \left( \nabla \theta_t^a(0), \nabla T_h \theta_t^a(0) - \delta_t(0) \right) \right) = \quad
\]
\[
- (\rho_t^h(0), T_h \theta_t^a(0) - \delta_t(0)) + ((g(x, u_t^h(0)) - g(x, u(0)), T_h \theta_t^a(0) - \delta_t(0)) = \quad
\]
\[
- (\rho_t^h(0), T_h \theta_t^a(0) - \delta_t(0)) = - (\rho_t^h(0), T_h \theta_t^a(0) - \delta_t(0)),
\]
where we have used the fact \((\rho_t^h + \delta_t, 1) = (g(x, u_t^h) - g(x, u), 1))\). Therefore,
\[
\theta_t^a(0) + \delta_t(0) = (g(x, u_t^h(0)) - g(x, u(0)), 1) = 0.
\]
and thanks to (3.51), we obtain,
\[
\left\| \theta_t^a(0) - \delta_t(0) \right\|_{-1, H} \leq \left\| \theta_t^a(0) + \delta_t(0) \right\|_{H^2(\Omega)} \leq C \left\| u_t(0) \right\|_{H^2(\Omega)} \leq C h^2 \left\| u_t(0) \right\|_{H^2(\Omega)}.
\]
where in the last inequality, we used Lemma 3.1, (3.26) and (3.29). Thus, \(N(t) \leq Ch^4\) that proves our claim. Lemma 3.2, Lemma 3.1, estimate (3.23) and the regular assumption on \(u\) and \(w\) implicate that,
\[
N(t) \leq C h^4, \quad \text{for all} \ t \in [0, T^h].
\]
Particularly, this gives that,
\[
\left\| \theta^a(t) \right\|_{L^2(\Omega)} \leq C h^2, \quad \text{for all} \ t \in [0, T^h].
\]
We will argue as in (3.52), and we conclude from this that,
\[
\sup |u^h(t) - u(t)||_{C^0(\Omega)} \to 0 \text{ as } h \to 0.
\]
Therefore, taking \( h \) small enough \( T^h = T \). Lemma 3.1, Lemma 3.2 and (3.23) conclude the result.

3.2. Stability of the Backward Euler scheme. By considering the backward Euler that we apply to the space semi-discrete scheme. The time step \( \delta t > 0 \) is fixed. The scheme can be written as follows:

Let \( u^n_h \in V^h \) and for \( n = 1, 2, \ldots \). Find \( (u^n_h, w^n_h) \in V^h \times V^h \) verifying
\[
((u^n_h - u^{n-1}_h, \phi)) = -((\nabla u^n_h, \nabla \phi)) + ((h(x)L(u^n_h), \phi)), \tag{3.53}
\]
\[
((u^n_h, \chi)) = ((f(u^n_h), \chi)) + ((\nabla u^n_h, \nabla \chi)), \tag{3.54}
\]
for all \( \phi, \chi \in V^h \).

In the results that follows, we show the existence, stability and uniqueness of sequences \((u^n_h, w^n_h)\).

**Theorem 3.4.** Knowing that \( G^*(u) = L(u) \). Suppose that \( h \) and \( G \) are nonnegative functions. Then, for every \( u^n_h \in V^h \) there exists a sequence \((u^n_h, w^n_h)\) generated by (3.53)-(3.54) and satisfies
\[
\mathcal{E}(u^n_h) + \frac{1}{2\delta t} ||u^n_h - u^{n-1}_h||_{L^1,h}^2 + \frac{1}{2} \int h(x)G(u^n_h)dx
\]
\[
\leq \mathcal{E}(u^{n-1}_h) + \frac{1}{2} \int h(x)G(u^{n-1}_h)dx, \text{ for all } n \geq 1. \tag{3.55}
\]

In addition, if \( \delta_t < \delta_t^* \) where \( \delta_t^* = \frac{2}{C_z + 2C_zL_L} \) and \( C_z = \frac{\delta_t ||h||^2 L^2}{2} + L_f + \delta_t L_f^2 + ||h||_{L^\infty(\Omega)} L_L^2 \delta_t^2 \), then this sequence is unique.

**Proof.** Consider the minimization problem:
\[
J^u = \inf_{v \in K^h} J^h(v), \tag{3.56}
\]
where
\[
K^h = \{v \in V^h; (v - u^{n-1}_h, 1) = 0\}, \tag{3.57}
\]
and
\[
J^h(v) = \mathcal{E}(v) + \frac{1}{2\delta_t} ||v - u^{n-1}_h||_{L^1,h}^2 + \frac{1}{2} \int h(x)G(v)dx. \tag{3.58}
\]
\[
J^h(v) \geq \frac{1}{2} ||\nabla v||_{L^2(\Omega)}^2 + C||v||_{L^2(\Omega)}. \tag{3.59}
\]
Since \( J^h(\cdot) \) is continuous, it follows that there exists a solution to the variational problem (3.56). This solution satisfies the equation
\[
0 = ((\nabla u, \nabla \chi)) + ((h(x)L(u), \chi)) + ((f(u), \chi)) + \frac{1}{\delta_t} ((h(x)(u - u^{n-1}_h), \chi)) - \lambda((1, \chi)), \tag{3.60}
\]
for all \( \chi \in V^h \).

We set \( w^n_h = u \) and \( u^n_h = \lambda - T^h(\frac{1}{\delta_t}(u - u^{n-1}_h) - h(x)L(u)) \), and we see that \((u^n_h, w^n_h)\) satisfies (3.53)-(3.54). By construction, we have
\[
J^h(u^n_h) \leq J^h(u^{n-1}_h),
\]
we then obtain (3.55).

To prove the uniqueness, let \( \theta^u = (u^n_h)^1 - (u^n_h)^2 \) and \( \theta^w = (w^n_h)^1 - (w^n_h)^2 \) signifies the difference of two solutions \((u^n_h)^i, (w^n_h)^i\) \(i = 1, 2\) of (3.53)-(3.54) for a given \( u^{n-1}_h \). Then \( \theta^u, \theta^w \), satisfies
\[
((\theta^u, \phi)) = -\delta_t ((\nabla \theta^u, \nabla \phi)) + \delta_t ((h(x)[L((u^n_h)^1) - L((u^n_h)^2)], \phi)), \tag{3.61}
\]
\[
((\theta^w, \chi)) = ((f((u^n_h)^1) - f((u^n_h)^2), \chi)) + ((\nabla \theta^w, \nabla \chi)), \tag{3.62}
\]
for all \( \phi, \chi \in V^h \).

By choosing \( \phi = \theta^u, \chi = \theta^w \), and subtracting the resulting equations. This yields:
\[
\delta_t ||\nabla \theta^u||_{L^2(\Omega)}^2 + ||\nabla \theta^w||_{L^2(\Omega)}^2 - \delta_t ((h(x)[L((u^n_h)^1) - L((u^n_h)^2)], \theta^w))
\]
\begin{equation}
\phi \text{ which yields } \quad \text{Cancerous tumor growth applications.}
\end{equation}

\section*{4. Numerical simulations.} We consider the proposed scheme \((3.8)-(3.9).\) The numerical simulations are performed with the software Freefem++ (see [18]).

\subsection*{4.1. Cancerous tumor growth applications.} In figure 1, the initial datum

\[ u_0 = -\tanh \left( \frac{1}{\sqrt{2\pi}} \sqrt{(x-0.5)^2 + \frac{1}{8} (y+0.5)^2 - 0.05} \right) \in [-1,1], \]

leading to \( \langle u_0 \rangle = -0.225. \) Besides,

\[ g(s) = 30(s+1) - 250(s-1)^2(s+1)^2 \]

and in Figure 1 the variation of the solution \( u. \) The time step is taken as \( \delta t = 10^{-5}. \) We further take \( f(s) = s^3 - s. \)
Figure 1. (a) Initial datum at $t = 0$. (b) Solution after 2000 iterations. (c) Solution after 3000 iterations. (d) Solution after 4000 iterations. (e) Solution after 5000 iterations. (f) Solution after 6000 iterations. (g) Solution after 8000 iterations. (h) Solution after 9000 iterations.

Figure 2. (a) initial image. (b) mask. (c) Inpainting result.

Figure 3. (a) original image. (b) mask. (c) Inpainting result.
4.2. Binary inpainting image applications. We consider
\[ h(x)L(u) = \lambda_0 \chi_{\Omega \setminus D}(u - h), \]
where \( h \) is the damaged image.

Figure 4.1 and Figure 4.1 are for vandalized image and masked image, respectively. By running the modified Cahn–Hilliard system with \( \varepsilon = 0.03 \). Next, when we get near to a steady state at \( t = 0.15 \), we substitute the dominant color by 1 and the other colors by 0 to get the final inpainting in Figure 4.1. In this case, \( \lambda_0 = 10^5 \) and \( \Delta t = 0.05 \).

Moreover, in Figure 3, we give an example of removing a text from image. Figure 4.1 and Figure 4.1 are for vandalized image and masked image, respectively. By running the modified Cahn–Hilliard system with \( \varepsilon = 0.03 \). Then, when we are close to a steady state at \( t = 0.15 \), we replace the dominant color by 1 and the other colors by 0 to obtain the final inpainting in Figure 4.1. In this case, \( \lambda_0 = 10^5 \) and \( \Delta t = 0.05 \). In these examples, \( f(s) = 4s^3 - 6s^2 + 2s \).

Acknowledgments. The authors wish to thank the referees for their careful reading of the article and useful comments.

REFERENCES

[1] A. C. Aristotelous, O. A. Karakashian and S. M. Wise, Adaptive, second order in time, primitive-variable discontinuous Galerkin schemes for a Cahn–Hilliard equation with a mass source, IMA J. Numer. Anal., 35 (2015), 1167–1198.
[2] A. Bertozzi, S. Esedoglu and A. Gillette, Inpainting of binary images using the Cahn–Hilliard equation, IEEE Trans. Imaging Proc., 16 (2007), 285–291.
[3] A. Bertozzi, S. Esedoglu and A. Gillette, Analysis of a two-scale Cahn–Hilliard model for binary image inpainting, Multiscale Model. Simul., 6 (2007), 913–936.
[4] M. Burger, L. He and C. Schönnlieb, Cahn–Hilliard inpainting and a generalization for gray-value images, SIAM J. Imaging Sci., 3 (2009), 1129–1167.
[5] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, J. Chem. Phys., 28 (1958), 258–267.
[6] L. Cherfils, H. Fakih and A. Miranville, Finite-dimensional attractors for the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation in image inpainting, Inv. Prob. Imaging, 9 (2015), 105–125.
[7] L. Cherfils, H. Fakih and A. Miranville, On the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation with logarithmic nonlinear terms, SIAM J. Imag. Sci., 8 (2015), 1123–1140.
[8] L. Cherfils, H. Fakih and A. Miranville, A Cahn–Hilliard system with a fidelity term for color image inpainting, J. Math. Imaging Vis., 54 (2016), 117–131.
[9] L. Cherfils, H. Fakih and A. Miranville, A complex version of the Cahn-Hilliard equation for grayscale image inpainting, J. Multiscale Model. Simul., 15 (2017), 575–605.
[10] L. Cherfils, A. Miranville and S. Zelik, The Cahn–Hilliard equation with logarithmic potentials, Milan J. Math., 79 (2011), 561–596.
[11] L. Cherfils, A. Miranville and S. Zelik, On a generalized Cahn–Hilliard equation with biological applications, Discrete Cont. Dyn.-B, 19 (2014), 2013–2026.
[12] L. Cherfils, M. Petcu and M. Pierre, A numerical analysis of the Cahn–Hilliard equation with dynamic boundary conditions, Discrete Cont. Dyn. S., 27 (2010), 1511–1533.
[13] I. C. Dolcetta, S. F. Vita and R. March, Area-preserving curve-shortening flows: from phase separation to image processing, Interface. Free Bound., 4 (2002), 325–343.
[14] C. M. Elliott, D. A. French and F. A. Milner, A second order splitting method for the Cahn–Hilliard equation, Numer. Math., 54 (1989), 575–590.
[15] A. Ern and J. L. Guermond, Elements finis: theorie, applications, mise en oeuvre, Springer-Verlag, Berlin, 2002.
[16] H. Fakih, Asymptotic behavior of a generalized Cahn–Hilliard, equation with a mass source, Appl. Anal., 96 (2016), 324–348.
[17] H. Fakih, A Cahn–Hilliard equation with a proliferation term for biological and chemical applications, Asympt. Anal., 94 (2015), 71–104.
[18] F. Hecht, New development in FreeFem++, J. Numer. Math., 20 (2012), 251–265.
[19] E. Khain and L. M. Sander, A generalized Cahn–Hilliard equation for biological applications, Phys. Rev. E, 77 (2008), 51–129.
[20] A. Miranville, Asymptotic behavior of the Cahn–Hilliard–Oono equation, J. Appl. Anal. Comp., 1 (2011), 523–536.
[21] A. Miranville, Asymptotic behavior of a generalized Cahn–Hilliard equation with a proliferation, *Appl. Anal.*, 92 (2013), 1308–1321.

[22] A. Miranville, Existence of solutions to a Cahn–Hilliard type equation with a logarithmic nonlinear term, *Mediterr. J. Math.*, 16 (2019), 1–18.

[23] A. Miranville, The Cahn–Hilliard equation and some of its variants, *AIMS Math.*, 2 (2017), 479–544.

[24] A. Novick-Cohen and L. A. Segal, Nonlinear Cahn–Hilliard equation *Proc. Roy. Soc. London Ser. A.*, 422 (1989), 261–278.

[25] Y. Oono and S. Puri, Computationally efficient modeling of ordering of quenched phases, *Phys. Rev. Lett.*, 58 (1987), 836–839.

[26] C. B. Schönlieb and A. Bertozzi, Unconditionally stable schemes for higher order inpainting, *Commun. Math. Sci.*, 9 (2011), 413–457.

[27] S. Villain-Guillot, *Phases modulées et dynamique de Cahn–Hilliard*, Habilitation thesis, Université Bordeaux 1, 2010.

Received December 2019; revised September 2020.

E-mail address: hussein.fakih@liu.edu.lb
E-mail address: ragheb.mghames@mathem.pub.ro
E-mail address: nourarn@gmail.com