Evidence for radiative generation of lepton masses

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Abstract
We present a fit to the experimental charged lepton masses as coming from radiative corrections in QED.

1 Introduction
In November of the last year, one of us (HdV) noticed that the values of \(a_e \equiv \frac{(g_e - 2)}{2}\), the anomalous moment of electron, and of the difference \(a_\mu - a_e\) with the one of the muon, were amazingly close to the mass quotients \(m_\mu/m_Z\) and \(m_e/m_W\). This happened during an on-going internet quest for accurate empirical relationships between fundamental constants, but we felt that the accuracy of this particular case deserved further investigation:

| Value               | Description                        |
|---------------------|------------------------------------|
| 0.00115869          | muon / Z mass ratio                |
| 0.00115965          | electron magnetic anomaly          |
| 0.00000635          | electron / W mass ratio            |
| 0.00000626          | difference of muon and electron magnetic anomaly |

Of course the calculation of \((g - 2)/2\) involves the very well known series on the electromagnetic coupling \(\alpha\). A coincidence with simple combinations of lepton masses can be explained if such masses come themselves from expressions containing \(\alpha\). Then it strongly suggests that such masses are generated radiatively in such way that at low order both perturbative series can be related. It has been observed from time ago [1] that lepton masses have quotients of order \(\alpha\), and a whole industry of model-making starts from trying to fit it [1][3].

Getting the masses as radiative series on \(\alpha\). But until now, no new evidence had been observed for this kind of schemes

2 Self Energy and Vacuum Polarization

As we expect only a parallel between structures, we can do the ansatz of comparing the first quotient exclusively to self-energy graphs, and to ascribe all

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of the vacuum polarisation (v.p.) contribution to the second. This ansatz was
guided by Hans’ observation of the similarity between the quotient $m_W/m_Z$ and
the ratio of semiclassical velocities $\beta_1/\beta_s$. In any case, it amounts to excluding
the electron vacuum polarisation loop in the $\alpha^2$ order and, because precision
requires it, the v.p. and light by light diagrams in the third order. These are,
respectively

$$a_{e}^{vp} = \left(\frac{119}{36} - \frac{1}{3} \pi^2\right)(\frac{\alpha}{\pi})^2 - 0.099(\frac{\alpha}{\pi})^3 + 0.37(\frac{\alpha}{\pi})^3 = 88.0 \times 10^{-9} \quad (1)$$

Our table becomes

|                          |                                              |
|--------------------------|----------------------------------------------|
| 0.001 158 692 3          | $\mu$/Z mass ratio                          |
| 0.001 159 564 2          | exp. $a_e$ electron magnetic anomaly - $a_e^{vp}$ |
| 0.000 000 871 9          | difference                                   |
| 0.001 165 046 0          | $\mu$/Z mass ratio + electron / W mass ratio |
| 0.001 165 920 8          | exp muon magnetic anomaly $a_\mu$            |
| 0.000 000 874 8          | difference                                   |
| 0.000 006 353 7          | electron / W mass ratio                      |
| 0.000 006 356 7          | exp. $a_\mu$ - exp. $a_e + a_e^{vp}$         |
| 0.000 000 003 0          | difference                                   |

Table 2.

The uncertainty due to W mass is $2.99 \times 10^{-9}$. Actually, the third loop
correction, above incorporated, has a positive contribution $3.4 \times 10^{-9}$ against
a perfect match. In any case, it is empirically very satisfying to find oneself
inside the experimental error with only an ansatz on the diagrams. Still, the
$\mu$/Z ratio is accurate only up to $O(\frac{\alpha}{\pi})$, and it seems to ask for an additional
$O((\frac{\alpha}{\pi})^2)$ term.

3 First QED approximation

So, lets try this ansatz in a pure calculational setup, without recurring to the
experimental data, and lets see if -or how- the coincidence can be related to
a parallel of mathematical structures. The QED calculation of $a_e$ excluding
vacuum polarisation is [3],

$$a_e^{QED-v.p.} = \frac{1}{2} \alpha - 0.3441668(\frac{\alpha}{\pi})^2 + 0.943(\frac{\alpha}{\pi})^3 = 0.001 159 564 60 \quad (2)$$

while the whole QED result for $a_\mu$ is [3]

$$a_\mu^{QED} = 0.5\frac{\alpha}{\pi} + 0.765857388(\frac{\alpha}{\pi})^2 + 24.0505(\frac{\alpha}{\pi})^3 + 126.04(\frac{\alpha}{\pi})^4 = 0.001 165 847 00 \quad (3)$$

$^1 \beta_s$ is the velocity of a mass rotating on a orbit with angular momentum $\sqrt{s(s+1)}\hbar$ and
a frequency corresponding to its rest mass. The quotient $\beta_s/\beta_1$ is about 0.8814
The difference being \( a_\mu - a_e = 0.00000628240 \) The coincidences are thus initially of 99.92% and 98.88% and by themselves they should constitute at least collateral evidence of radiative terms for most part of the \( m_e, m_\mu \). Note that by betting for a mathematical structure with leptons only, we have lost the hadronic contribution, of order 67·10^{-9}, so now we are too in need of a corrective term for the missing 01.12% if we want to increase the order of accuracy.

4 Additional Terms

Our first research must be how the fit to \( m_\mu/m_Z \) can be improved by using additional terms. There is no very much playroom using only electroweak mass data, but a bit surprisingly, there are possibilities of improvement. Keeping with simple quotients of \( Z \), it is possible to enter into the one-sigma experimental precision of \( Z \), 26.68 · 10^{-9}, by using \((1/2\pi)m_e/m_Z\). If we are willing to admit more higher powers of mass quotients, a term \( m_\mu^2/2m_W^2 \) drives the estimate up to almost full coincidence with the central values. And if we do not like extra coefficients, we can instead use \( m_\mu^2/m_X^2 \) for an undiscovered mass \( X \) of 114.5 GeV. Let us compare these possibilities:

\[
\begin{align*}
  a_\mu^{QED-v.p} & = 0.00115956460 \\
  \frac{m_\mu}{m_Z} + \frac{1}{2\pi} \frac{m_e}{m_Z} & = 0.00115958417 : -0.000 000 019 57 \\
  \frac{m_\mu}{m_Z} + \frac{m_e^2}{2m_W^2} & = 0.00115955526 : 0.000 000 009 34 \\
  \frac{m_\mu}{m_Z} + \frac{m_\mu^2}{m_X^2} & = .00115954381 : 0.000 000 020 79 \\
  Z \text{ error} & : 0.000 000 026 68
\end{align*}
\]

The last column shows differences, to be compared with the uncertainty induced from the experimental measurement of \( Z_0 \).

The fit at \( X \) is appealing because \( Z \) is a neutral particle, and the experimental hint of CERN at this value was for the neutral scalar. While waiting for news in the experimental front, we can happily admit the correction of (6).

Another motivation to prefer quadratic correction terms is that we can use also a term in \( m_e m_\tau \) to recover almost completely the precision we lost for the second quotient when we decided to do not include the hadronic (quark loop) contributions. We have

\[
\begin{align*}
  a_\mu^{QED} - a_e^{QED-v.p} & = 0.00000628240 \\
  \frac{m_e}{m_W} - \frac{m_e m_\tau}{2m_W^2} & = 0.00000628354 : 0.000 000 001 14 \\
  \frac{m_e}{m_W} - \frac{m_e m_\tau}{m_X^2} & = 0.00000628447 : 0.000 000 002 07 \\
  W \text{ error} & : 0.000 000 002 99
\end{align*}
\]

Again, the last column shows differences, to be compared with the uncertainty induced from the experimental measurement of \( W^+ \). And besides the already
mentioned hadronic contribution, $6.7 \cdot 10^{-9}$, we could consider also the pure electroweak contribution, $1.51 \cdot 10^{-9}$, to be added to $a_\mu$. We mention it separately to show that we can not decide if we are comparing against the structure of a pure QED kind of series or against an electroweak series.

5 Remarks

Remark 1. It can be asked if there is a role for the tau anomalous moment in this scheme. It is a touchy issue, because while the tau lives at order $\alpha$ of the electroweak vacuum\(^2\), it is also at the mass scale typical of SU(3) colour, while the next lepton, the muon, lives at the mass scale of the chiral breaking (whose goldstone boson is in some sense the pion). We can suspect things are not very clear cut in its radiative process, and in fact one could prefer to admit quarks in the calculation instead of using the correction of formula (10) above, and then to adjust the $a_e$ term with formula (4).

As for the $a_\tau$ correction, it refers, it is tempting to try to guess if a simple expression does it exist. This value is not known experimentally, but from Samuel et al \cite{8}, we know its calculated QED value, 0.0011732. If we ask for a simple quotient, we would again to use the electron mass over some particle $X^+$, which we could expect (but not necessarily) to be a charged one, to imitate the use of W. The total expression

$$\frac{m_\mu}{m_Z} + \frac{m_e}{m_W} + \frac{m_\tau^2 - m_\tau m_e}{m_X^2}$$

actually matches $a_\tau$ for a mass of $X^+$ about 68 GeV \(^3\).

Remark 2. In principle, if all the three formulae above are taken seriously, a matching order-by-order in $\alpha$ could be done to estimate the corresponding coefficients of the radiative series for each lepton mass. But without further understanding of the role of the electroweak bosons, or of the full electroweak scale and the role of $\tau$, such matching becomes merely a mathematical exercise.

Remark 2.5 Another consequence of taking seriously the quadratic formulae is that their simultaneous use gives an hyperbolic relationship between electroweak masses. It should be interesting if some family of GUT models were able to generate this kind of relationship:

$$\frac{m_\tau}{m_Z} + \frac{m_\mu}{m_W} = \frac{m_\tau}{m_\mu} a_{\tau,e} + \frac{m_\mu}{m_e} a_{\mu,p}$$

Note that $a_{\mu,p}$, containing the vacuum polarisation (and light by light) terms, has also an internal dependence on the quotients $m_e/m_\mu$, $m_\tau/m_\mu$.

Remark 3 As the pure self-energy contributions do not depend (in QED) of lepton mass, it is indifferent if we extract them from $a_e$ or $a_\mu$. Along this note we have kept with $a_e$ due to historical reasons, but it results more symmetric to refer to $a_{\mu,e}^{QED,s.e.}$ and $a_{\mu,p}^{QED,v.p.}$, as we have done in formula (14) above.

\(^2\)As Jay R Yablon reminded us recently

\(^3\)It is perhaps worth to note here that the existence of a charged scalar at this value was pursued \cite{7} in the LEP, while the final evaluation reduced the value of the events down to a two sigma deviation. So in some sense this scalar has presently the same experimental status that the events at 115 GeV assigned to a neutral scalar.
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