A SMALL GENERATING SET FOR THE BALANCED SUPERELLIPTIC HANDLEBODY GROUP

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Abstract. The balanced superelliptic handlebody group is the normalizer of the transformation group of the balanced superelliptic covering space in the handlebody group of the total space. We prove that the balanced superelliptic mapping class group is generated by four elements. To prove this, we also proved that the liftable Hilden group is generated by three elements. This generating set for the liftable Hilden group is minimal except for some hyperelliptic cases and the generating set for the balanced superelliptic mapping class group above is also minimal for several cases.

1. Introduction

Let \( H_g \) be an oriented 3-dimensional handlebody of genus \( g \geq 0 \) and \( B^3 = H_0 \) a 3-ball. We write \( \Sigma_g = \partial H_g \) and \( S^2 = \partial B^3 \), respectively. For a subset \( A \) of \( \Sigma_g \) \((g \geq 0)\), the mapping class group \( \text{Mod}(\Sigma_g, A) \) of the pair \((\Sigma_g, A)\) is the group of isotopy classes of orientation-preserving self-homeomorphisms on \( \Sigma_g \) which preserve \( A \) setwise. When \( A \) is a set of distinct \( n \) points, we denote the mapping class group by \( \text{Mod}_{g,n} \). We denote \( \text{Mod}_{g,0} \) simply by \( \text{Mod}_g \).

Humphries [10] proved that \( \text{Mod}_{g,n} \) is generated by \( 2g + 1 \) Dehn twists for \( g \geq 2 \) and \( n \in \{0, 1\} \) and the generating set is minimal in generating sets for \( \text{Mod}_{g,n} \) which consist of Dehn twists. We focus on the study of minimal generating sets for mapping class groups. Wajnryb [18] proved that \( \text{Mod}_{g,n} \) is generated by two elements for \( g \geq 1 \) and \( n \in \{0, 1\} \). Since \( \text{Mod}_{g,n} \) is not a cyclic group, this Wajnryb’s generating set is minimal. After that, Korkmaz [11] proved that \( \text{Mod}_{g,n} \) is generated by two elements whose one element is a Dehn twist for \( g \geq 1 \) and \( n \in \{0, 1\} \).

Monden [12] showed that \( \text{Mod}_{g,n} \) is generated by two elements for \( g \geq 3 \) and \( n \geq 0 \), and this generating set is also minimal.

The handlebody group \( H_g \) is the group of isotopy classes of orientation-preserving self-homeomorphisms on \( H_g \). Suzuki [16] gave a generating set for \( H_g \) which consists of six elements for \( g \geq 3 \) (actually, the number of the generators can be reduced to five). We have a well-defined injective homomorphism \( H_g \rightarrow \text{Mod}_g \) by restricting the actions of elements in \( H_g \) to \( \Sigma_g \) and using the irreducibility of \( H_g \). By this injective homomorphism, we regard \( H_g \) as the subgroup of \( \text{Mod}_g \) whose elements extend to \( H_g \).

For integers \( n \geq 1 \) and \( k \geq 2 \) with \( g = n(k-1) \), the balanced superelliptic covering map \( p = p_{g,k} : H_g \rightarrow B^3 \) is a \( k \)-fold branched covering map with the covering transformation group generated by the balanced superelliptic rotation \( \zeta = \zeta_{g,k} \) of order \( k \) (precisely defined in Section 2.1 and see Figure 1). The branch points set \( A \subset B^3 \) of \( p \) is the disjoint union of \( n + 1 \) proper arcs in \( B^3 \) and the restriction of \( p \) to the preimage \( \tilde{A} \) of \( A \) is injective (i.e. \( \tilde{A} \) is the fixed point set of \( \zeta \)). The restriction

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p|_{\Sigma_g}: \Sigma_g \to S^2 = \Sigma_0 is also a k-fold branched covering with the branch points set \( B = \partial A \) and we also call \( p|_{\Sigma_g} \) the balanced superelliptic covering map. When \( k = 2 \), \( \zeta|_{\Sigma_g} \) coincides with a hyperelliptic involution, and for \( k \geq 3 \), the balanced superelliptic covering space was introduced by Ghaswala and Winarski [6]. We often abuse notation and simply write \( p|_{\Sigma_g} = p \) and \( \zeta|_{\Sigma_g} = \zeta \), respectively.

For \( g = n(k-1) \geq 1 \), an orientation-preserving self-homeomorphism \( \varphi \) on \( \Sigma_g \) or \( H_g \) is symmetric for \( \zeta = \zeta_{g,k} \) if \( \varphi(\zeta)\varphi^{-1} = \langle \zeta \rangle \). The balanced superelliptic mapping class group (or the symmetric mapping class group) \( S\text{Mod}_{g,k} \) is the subgroup of \( \text{Mod}_g \) which consists of elements represented by symmetric homeomorphisms. In particular, \( S\text{Mod}_{g,2} \) is called the hyperelliptic mapping class group.

Birman and Hilden [2] showed that \( S\text{Mod}_{g,k} \) coincides with the group of symmetric isotopy classes of symmetric homeomorphisms on \( \Sigma_g \). We call the intersection \( S\mathcal{H}_{g,k} = S\text{Mod}_{g,k} \cap H_g \) the balanced superelliptic handlebody group (or the symmetric handlebody group). By Lemma 1.21 in [8] and Lemma 2.1 in [14], \( S\mathcal{H}_{g,k} \) is also isomorphic to the group of symmetric isotopy classes of symmetric homeomorphisms on \( H_g \). When \( k = 2 \), Stukow [15] gave a minimal generating set for \( S\text{Mod}_{g,2} \) which consists of two torsion elements. For the case \( k \geq 3 \), the author [13] proved that \( S\text{Mod}_{g,k} \) is generated by three elements and this generating set is minimal except for the case of even \( n \).

We regard \( \text{Mod}(S^2; \mathcal{B}) \) as \( \text{Mod}_{0,2n+2} \) and let \( H_{2n+2} \) be the group of isotopy classes of orientation-preserving self-homeomorphisms on \( B^3 \) fixing \( \mathcal{A} \) setwise. It is a well-known result that \( \text{Mod}_{0,2n+2} \) is generated by two elements and this generating set is minimal. The group \( H_{2n+2} \) is introduced by Hilden [7] and is called the Hilden group. He gave a finite generating set for \( H_{2n+2} \) whose generating set is smaller than Hilden’s generating set. By restricting the actions of elements in \( H_{2n+2} \) to \( S^2 \), we have an injective homomorphism \( H_{2n+2} \hookrightarrow \text{Mod}_{0,2n+2} \) (see [3] p. 157) and regard \( H_{2n+2} \) as the subgroup of \( \text{Mod}_{0,2n+2} \) whose elements extend to homeomorphisms on \( B^3 \) which preserve \( \mathcal{A} \) by this injective homomorphism. Since elements in \( S\text{Mod}_{g,k} \) (resp. in \( S\mathcal{H}_{g,k} \)) are represented by elements which preserve \( \mathcal{B} \) (resp. \( \mathcal{A} \)) by the definitions, we have homomorphisms \( \theta: S\text{Mod}_{g,k} \to \text{Mod}_{0,2n+2} \) and \( \theta: S\mathcal{H}_{g,k} \to H_{2n+2} \) that are introduced by Birman and Hilden [3]. They also proved that \( \theta(\text{Mod}_{g,2}) = \text{Mod}_{0,2n+2}, \) and Hirose and Kin [9] showed that \( \theta(S\mathcal{H}_{g,2}) = H_{2n+2} \).

A self-homeomorphism \( \varphi \) on \( \Sigma_0 \) (resp. on \( B^3 \)) is \textit{liftable} with respect to \( p = p_{g,k} \) if there exists a self-homeomorphism \( \tilde{\varphi} \) on \( \Sigma_g \) (resp. on \( H_g \)) such that \( p \circ \tilde{\varphi} = \varphi \circ p \), namely, the following diagrams commute:

\[
\begin{array}{ccc}
\Sigma_g & \xrightarrow{\varphi} & \Sigma_g \\
p \downarrow & \circ & \downarrow p \\
\Sigma_0 & \xrightarrow{\varphi} & \Sigma_0,
\end{array}
\quad
\begin{array}{ccc}
H_g & \xrightarrow{\varphi} & H_g \\
p \downarrow & \circ & \downarrow p \\
B^3 & \xrightarrow{\varphi} & B^3.
\end{array}
\]

The \textit{liftable mapping class group} \( \text{LMod}_{2n+2,k} \) is the subgroup of \( \text{Mod}_{0,2n+2} \) which consists of elements represented by liftable homeomorphisms on \( S^2 \) for \( p|_{S^2} \), and the \textit{liftable Hilden group} \( \text{LH}_{2n+2,k} \) is the subgroup of \( H_{2n+2} \) which consists of elements represented by liftable homeomorphisms on \( B^3 \) for \( p \). As a homomorphic image in \( \text{Mod}_{0,2n+2} \), we have \( \text{LH}_{2n+2,k} = \text{LMod}_{2n+2,k} \cap H_{2n+2} \) by Lemma 2.2 in [14]. By the definitions, we have homomorphisms \( \theta: S\text{Mod}_{g,k} \to \text{LMod}_{2n+2,k} \) and \( \theta: S\mathcal{H}_{g,k} \to \text{LH}_{2n+2,k} \). Birman and Hilden [1] proved that \( \theta \) induces an isomorphism \( \text{LMod}_{2n+2,k} \cong S\text{Mod}_{g,k} / \langle \zeta \rangle \), and Hirose and Kin [9] showed that \( \theta|_{S\mathcal{H}_{g,2}} \) induces an isomorphism \( \text{LH}_{2n+2,2} \cong H_{2n+2} \cong S\mathcal{H}_{g,2} / \langle \zeta_{g,2} \rangle \) (remark that...
LMod_{2n+2,2} = Mod_{0,2n+2}). For the case of the handlebody subgroup and \( k \geq 3 \), the author and Yoshida [14, Lemma 2.3] showed that \( \theta|_{\text{SH}_{g,k}} \) induces an isomorphism \( \text{LH}_{2n+2,k} \cong \text{SH}_{g,k}/\langle \zeta \rangle \). For \( k \geq 3 \), the author [13] proved that \( \text{LMod}_{2n+2,k} \) is generated by three elements and this generating set is minimal except for the case of even \( n \).

The main theorem in this paper is as follows.

**Theorem 1.1.** For \( k \geq 2 \), \( \text{LH}_{2n+2,k} \) is generated by three elements.

Theorem 1.1 is proved in Sections 3. We remark that \( \text{LH}_{2n+2,2} = \text{H}_{2n+2} \) and \( \text{LH}_{2n+2,k} = \text{H}_{2n+2,l} \) for \( k, l \geq 3 \) by Lemma 2.3 (that is Lemma 3.6 in [3]). Hence we omit “\( k \)” in the notation of the liftable Hilden group for \( k \geq 3 \) (i.e., we express \( \text{LH}_{2n+2,k} \) as \( \text{LH}_{2n+2} \) for \( k \geq 3 \)). The next corollary follows immediately from Theorem 1.1 and the following exact sequence which is obtained from Theorem 2.11 in [9] and Lemma 2.3 in [14]:

\[
1 \to \langle \zeta \rangle \to \text{SH}_{g,k} \xrightarrow{\theta} \text{LH}_{2n+2,k} \to 1.
\]

**Corollary 1.2.** Assume that \( g = n(k-1) \) for \( n \geq 1 \) and \( k \geq 2 \). Then, \( \text{SH}_{g,k} \) is generated by four elements.

For explicit generators for \( \text{LH}_{2n+2,k} \) in Theorem 1.1 and their lifts to \( H_g \) in Corollary 1.2, see Proposition 3.1, Appendix in [9], and Section 5.1 in [14].

The integral first homology group \( H_1(G) \) of a group \( G \) is isomorphic to the abelianization of \( G \). By Corollary A.9 in [9], Theorems 1.1 and 1.2 in [14], and a computation of aberianization of \( \text{H}_{2n+2} \) from the presentation in [17], the integral first homology groups of \( \text{LH}_{2n+2,k} \) and \( \text{SH}_{g,k} \) for \( k \geq 2 \) are as follows (remark that Tawn gave a presentation for the Hilden group of one marked disk case in [17]. The group \( \text{H}_{2n+2} \) is a quotient of its Hilden group):

**Theorem 1.3** [17] for the case of \( k = 2 \) and [14] for the case of \( k \geq 3 \). For \( n \geq 1 \) and \( k \geq 2 \),

\[
H_1(\text{LH}_{2n+2,k}) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } k = 2 \text{ and } n \text{ is odd}, \\
\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{otherwise}.
\end{cases}
\]

**Theorem 1.4** [9] for the case of \( k = 2 \) and [14] for the case of \( k \geq 3 \). For \( n \geq 1 \) and \( k \geq 2 \) with \( g = n(k-1) \),

\[
H_1(\text{SH}_{g,k}) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k \geq 4 \text{ is even and } n \text{ is odd}, \\
\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{otherwise}.
\end{cases}
\]

For a group \( G \), the minimal number of generators for \( H_1(G) \) gives a lower bound of the minimal number of generators for \( G \). By Theorem 1.3 we see that the generating set for \( \text{LH}_{2n+2,k} \) in Theorem 1.1 is minimal except for \( k = 2 \) and odd \( n \). Similarly, the generating set for \( \text{SH}_{g,k} \) in Corollary 1.2 is minimal for even \( k \geq 4 \) and odd \( n \) by Theorem 1.4.

2. Preliminaries

2.1. The balanced superelliptic covering space. In this section, we review the definition of the balanced superelliptic covering space from Section 2.1 in [14]. For integers \( n \geq 1 \), \( k \geq 2 \), and \( g = n(k-1) \), we describe the handlebody \( H_g \) as follows. We take the unit 3-ball \( B(1) \) in \( \mathbb{R}^3 \) and \( n \) mutually disjoint parallel copies \( B(2), B(3), \ldots, B(n+1) \) of \( B(1) \) by translations along the x-axis such that

\[
\max(\text{min}(B(i) \cap (\mathbb{R} \times \{0\} \times \{0\}))) < \min(\text{max}(B(i+1) \cap (\mathbb{R} \times \{0\} \times \{0\})))
\]

in \( \mathbb{R} = \mathbb{R} \times \{0\} \times \{0\} \) for \( 1 \leq i \leq n \) (see Figure 1). Let \( \zeta \) be the rotation of \( \mathbb{R}^3 \) by \(-\frac{2\pi}{k}\) about the x-axis. Then for each \( 1 \leq i \leq n \), we connect \( B(i) \) and
B(i + 1) by k 3-dimensional 1-handles such that the union of the k 3-dimensional 1-handles are preserved by the action of ζ as in Figure 1. Since the union of B(1) ∪ B(2) ∪ ⋯ ∪ B(n + 1) and the attached n × k 3-dimensional 1-handles is homeomorphic to Hg(= Hn(k-1)), we regard this union as Hg.

Figure 1. The balanced superelliptic covering map \( p = p_{g,k} : H_g \to B^3 \).

By the construction above, the action of ζ on \( \mathbb{R}^3 \) induces the action on Hg and the fixed points set of ζ = \( \zeta|_{H_g} \) is \( \tilde{A} = H_g \cap (\mathbb{R} \times \{0\} \times \{0\}) \). We call ζ the balanced superelliptic rotation on Hg. We can see that the intersection \( \tilde{a}_i = B(i) \cap \tilde{A} \) for \( 1 \leq i \leq n + 1 \) is a proper simple arc in Hg and \( \tilde{A} = \tilde{a}_1 \cup \tilde{a}_2 \cup ⋯ \tilde{a}_{n+1} \) (see Figure 1). The quotient space \( H_g/\langle \zeta \rangle \) is homeomorphic to \( B^3 \) and the induced quotient map \( p = p_{g,k} : H_g \to B^3 \) is a branched covering map with the branch points set \( A = p(\tilde{A}) = p(\tilde{a}_1) \cup p(\tilde{a}_2) \cup ⋯ \cup p(\tilde{a}_{n+1}) \subset B^3 \). We call p the balanced superelliptic covering map. Put

- \( a_i = p(\tilde{a}_i) \) for \( 1 \leq i \leq n + 1 \),
- \( \tilde{p}_{2i-1} = \min \tilde{a}_i \) and \( \tilde{p}_{2i} = \max \tilde{a}_i \) in \( \mathbb{R} = \mathbb{R} \times \{0\} \times \{0\} \) for \( 1 \leq i \leq n + 1 \),
- \( p_i = p(\tilde{p}_i) \) for \( 1 \leq i \leq 2n + 2 \),
- \( B = \partial A = \{p_1, p_2, ⋯, p_{2n+2}\} \),
- \( \Sigma_g = \partial H_g \), and \( S^2 = \partial B^3 \) (see Figure 1).

Then we also call the restriction \( p|_{\Sigma_g} : \Sigma_g \to S^2 \) the balanced superelliptic covering map and we often simply write \( p|_{\Sigma_g} = p \). We note that the branch points set of \( p : \Sigma_g \to S^2 \) coincides with B.

2.2. Generators for the Hilden group and the spherical wicket group. In this section, we review generators for \( H_{2n+2} = LH_{2n+2,2} \) via homomorphic images from the spherical wicket group.

Let \( SB_{2n+2} \) be the spherical braid group of \( 2n + 2 \) strands. We regard an element in \( SB_{2n+2} \) as a \( (2n + 2) \)-tangle in \( S^2 \times [0,1] \) which consists of \( 2n + 2 \) simple proper arcs whose one of the endpoints lies in \( B \times \{0\} \) and the other one lies in \( B \times \{1\} \), and we also regard A as a \( (n + 1) \)-tangle in \( B^3 \). Such A is called a wicket. For
b ∈ SB_{2n+2}, denote by bA the (n + 1)-tangle in B^3 which is obtained from b by attaching a copy of B^3 to S^2 × {0} such that p_i ∈ ∂B^3 is attached to the end of i-th strand in b (see Figure 2), where we regard the manifold obtained by attaching the copy of B^3 to S^2 × [0, 1] along S^2 × {0} as B^3. The spherical wicket group SW_{2n+2} is the subgroup of SB_{2n+2} whose element b satisfies the condition that bA is isotopic to A relative to ∂B^3 = S^2 × {1}. Brendle and Hatcher [3] introduced the group SW_{2n+2} and gave its finite presentation.

Figure 2. A spherical braid b and the tangles A and bA in B^3.

The spherical wicket group SW_{2n+2} is generated by σ_1, σ_2, ..., σ_{2n+1}. We can check that σ_{2i−1} ∈ SW_{2n+2} for 1 ≤ i ≤ n + 1. For b_1, b_2 ∈ SB_{2n+2}, the product b_1b_2 is a braid as on the right-hand side in Figure 3.

Figure 3. The half-twist σ_i ∈ SB_{2n+2} (1 ≤ i ≤ 2n + 1) and the product b_1b_2 in SB_{2n+2} for b_i (i = 1, 2).

Let s_i and r_i for 1 ≤ i ≤ n be the 2n + 2 strands braids as in Figure 4. We see that s_i and r_i lie in SW_{2n+2}. Remark that the relations

s_i = σ_2σ_{2i+1}σ_{2i−1}σ_{2i} and r_i = σ_{2i}^{-1}σ_{2i+1}σ_{2i−1}σ_{2i}
for \(1 \leq i \leq n\) hold in \(SB_{2n+2}\). Brendle and Hatcher \[3\] gave the following proposition.

**Proposition 2.1.** For \(n \geq 1\), \(SW_{2n+2}\) is generated by \(s_i, r_i\) for \(1 \leq i \leq n\), and \(\sigma_{2j-1}\) for \(1 \leq j \leq n+1\).

Let \(l_i\) (\(1 \leq i \leq 2n+1\)) be an oriented simple arc on \(S^2\) whose endpoints are \(p_i\) and \(p_{i+1}\) as in Figure 5. Put \(L = l_1 \cup l_2 \cup \cdots \cup l_{2n+1}\). The isotopy class of a homeomorphism \(\varphi\) on \(\Sigma_0\) (resp. \(B^3\)) relative to \(B\) (resp. \(A\)) is determined by the isotopy class of the image of \(L\) by \(\varphi\) relative to \(B\). We identify \(B^3\) with the 3-manifold with a sphere boundary on the lower side in Figure 5 by some homeomorphism.

Let \(\sigma[l_i]\) for \(1 \leq i \leq 2n+1\) be a self-homeomorphism on \(S^2\) which is described as the result of anticlockwise rotation of \(l_i\) by \(\pi\) in the regular neighborhood of \(l_i\) in \(S^2\) as in Figure 6. The self-homeomorphism \(\sigma[l_i]\) is called the half-twist along \(l_i\). It is well-known that \(\text{Mod}_{0,2n+2}\) is generated by \(\sigma[l_1], \sigma[l_2], \ldots, \sigma[l_{2n+1}]\) (see for
instance Section 9.1.4 in [4]. For maps or mapping classes $f$ and $g$, the product $gf$ means that $f$ is applied first. Then we have the surjective homomorphism

$$
\Gamma: SB_{2n+2} \rightarrow \text{Mod}_{0,2n+2}
$$

which is defined by $\Gamma(\sigma_i) = \sigma[i]$ for $1 \leq i \leq 2n + 1$. The homomorphism $\Gamma$ has a kernel with order 2 which is generated by the full twist braid (see for instance Section 9.1.4 in [4]). We abuse notation and simply denote $\Gamma(b)$ for $b \in SB_{2n+2}$ by $b$ (i.e. we write $\sigma[i] = \sigma_i$, $\Gamma(s_i) = s_i$, and $\Gamma(r_i) = r_i$ in Mod$_{0,2n+2}$). Since $\Gamma(SW_{2n+2}) = H_{2n+2}$ by Theorem 2.6 in [9], we have the following proposition.

**Proposition 2.2.** For $n \geq 1$, $H_{2n+2}$ is generated by $s_i$, $r_i$ for $1 \leq i \leq n$, and $\sigma_{2j-1}$ for $1 \leq j \leq n+1$.

![Figure 6. The half-twist $\sigma[i] = \sigma_i$ for $1 \leq i \leq 2n + 1$.](image)

2.3. Generators for the liftable Hilden group. Assume that $k \geq 3$ in Section 2.2. In this section, we review the generating set for the liftable Hilden group $LH_{2n+2} = LH_{2n+2}^k$ for $k \geq 3$ of the presentation in Theorem 4.1 of [14] as homomorphic images from $SW_{2n+2}$. First, we review Ghaswala-Winarski’s necessary and sufficient condition for lifting a homeomorphism on $S^2$ with respect to $p = p_{g,k}$ for $k \geq 3$. Since Mod$_{0,2n+2}$ is naturally acts on $B = \{p_1, p_2, \ldots, p_{2n+2}\}$, we have a surjective homomorphism

$$
\Psi: \text{Mod}_{0,2n+2} \rightarrow S_{2n+2}
$$

given by $\Psi(\sigma_i) = (i \ i + 1)$, where $S_{2n+2}$ is the symmetric group of degree $2n + 2$.

Put $B_o = \{p_1, p_3, \ldots, p_{2n+1}\}$ and $B_e = \{p_2, p_4, \ldots, p_{2n+2}\}$. An element $\sigma$ in $S_{2n+2}$ is parity-preserving if $\sigma(B_o) = B_o$, and is parity-reversing if $\sigma(B_o) = B_e$. An element $f$ in Mod$_{0,2n+2}$ is parity-preserving (resp. parity-reversing) if $\Psi(f)$ is parity-preserving (resp. parity-reversing). Let $W_{2n+2}$ be the subgroup of $S_{2n+2}$ which consists of parity-preserving or parity-reversing elements. Ghaswala and Winarski [5] proved the following lemma.

**Lemma 2.3** (Lemma 3.6 in [5]). Let LMod$_{2n+2}$ be the liftable mapping class group for the balanced superelliptic covering map $p_{g,k}$ for $n \geq 1$ and $k \geq 3$ with $g = n(k-1)$. Then we have

$$
\text{LMod}_{2n+2} = \Psi^{-1}(W_{2n+2}).
$$

Lemma 2.3 implies that a mapping class $f \in \text{Mod}_{0,2n+2}$ lifts with respect to $p = p_{g,k}$ if and only if $f$ is parity-preserving or parity-reversing. By Lemma 2.2 in [14] we have $\text{LH}_{2n+2} = H_{2n+2} \cap \text{LMod}_{2n+2}$, namely the liftability for $p$ is equivalent to one for $p_{g,k}$. Moreover, by Lemma 2.3 when $k \geq 3$, the liftability of a homeomorphism on $S^2$ does not depend on $k$. Hence we omit “$k$” in the notation of the liftable mapping class group and the liftable Hilden group for $k \geq 3$ (i.e. we express LMod$_{2n+2,k} = \text{LMod}_{2n+2}$ and LH$_{2n+2,k} = \text{LH}_{2n+2}$ for $k \geq 3$).
Recall that $s_i$, $r_j$ for $1 \leq i \leq n$, and $\sigma_{2j-1}$ for $1 \leq j \leq n+1$ generates $H_{2n+2}$ by Proposition 2.2. We denote $t_j = \sigma_{2j-1}$ for $1 \leq j \leq n+1$ and $r = \sigma_1 \sigma_3 \cdots \sigma_{2n+1}$. Then we can see that $\Psi(s_i) = \Psi(r_j) = (2i - 1 2i + 1)(2i 2i + 2)$ for $1 \leq i \leq n$, $\Psi(t_j) = 1$ for $1 \leq j \leq n+1$, and $\Psi(r) = (1 2)(3 4) \cdots (2n+1 2n+2)$. Hence $s_i$, $r_j$, and $t_j$ are parity-preserving and $r$ is parity-reversing, and these elements lie in $LH_{2n+2}$ by Lemma 2.3. By Theorem 4.1 in [14], we have the following proposition:

**Proposition 2.4.** For $n \geq 1$, $LH_{2n+2}$ is generated by $s_i$, $r_j$ for $1 \leq i \leq n$, $t_j$ for $1 \leq j \leq n+1$, and $r$.

### 3. Proof of Main Theorem

In this section, we prove Theorem 1.1. Assume that $n \geq 1$ and $k \geq 2$. Denote $s = s_n \cdots s_2 s_1 \in SW_{2n+2}$ and $s = \Gamma(s) \in LH_{2n+2,k}$ (see Figure 7). Theorem 1.1 follows from the next proposition.

**Proposition 3.1.**

1. For $n \geq 1$, $LH_{2n+2;2} = H_{2n+2}$ is generated by $s \sigma_1$, $s_1$, and $r_1$.
2. For $n \geq 1$ and $k \geq 3$, $LH_{2n+2;k} = LH_{2n+2}$ is generated by $s r$, $s_1$, and $r_1$.

![Figure 7. The braid $s = s_n \cdots s_2 s_1$ and $s^{-1}$.](image)

**Proof of Proposition 3.1 for $k = 2$.** Let $G$ be a subgroup of $LH_{2n+2;2} = H_{2n+2}$ which is generated by $s \sigma_1$, $s_1$, and $r_1$. By Proposition 2.2, $H_{2n+2}$ is generated by $s_i$, $r_j$ for $1 \leq i \leq n$, and $\sigma_{2j-1}$ for $1 \leq j \leq n+1$. Since $(s \sigma_1)^{-1} s_1 (s \sigma_1)^{-1} = s_i$ for $2 \leq i \leq n$, we have $s_i \in G$ for $2 \leq i \leq n$. Hence $s = s_n \cdots s_2 s_1 \in G$ and $\sigma_1 = s^{-1} (s \sigma_1) \in G$. We can check that $r_1 = s^{-1} r_1 s^{-1} \in G$ for $2 \leq i \leq n$ and $\sigma_{2j-1} = s^{-1} (s \sigma_1) s^{-1} \in G$ for $2 \leq j \leq n+1$. Therefore $H_{2n+2} = G$ and we have completed the proof of Proposition 3.1 for $k = 2$.

**Proof of Proposition 3.1 for $k \geq 3$.** Let $G$ be a subgroup of $LH_{2n+2;k} = LH_{2n+2}$ which is generated by $s r$, $s_1$, and $r_1$. By Proposition 2.4, $LH_{2n+2}$ is generated by $s_i$, $r_j$ for $1 \leq i \leq n$, $t_j$ for $1 \leq j \leq n+1$, and $r$. Since $(s r)^{-1} s_1 (s r)^{-1} = s_i$ for $2 \leq i \leq n$, we have $s_i \in G$ for $2 \leq i \leq n$. Hence $s = s_n \cdots s_2 s_1 \in G$ and $r = s^{-1} (s r) \in G$. We can check that $r_1 = s^{-1} r_1 s^{-1} \in G$ for $2 \leq i \leq n$. Finally, by the relation (4) of Theorem 4.1 in [14], we have the relation $r_1 t_2 \cdots r_n s_n \cdots s_2 s_1 t_1 = 1$. Thus, $t_1 = s_1^{-1} s_2^{-1} \cdots s_{n-1}^{-1} \cdots s_1^{-1} \cdots s_2^{-1} r_1^{-1} \in G$ and $t_2 = s^{-1} (s r) s^{-1} \in G$ for $2 \leq j \leq n+1$. Therefore $H_{2n+2} = G$ and we have completed the proof of Proposition 3.1 for $k \geq 3$.

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