CONVERGENCE OF RANDOMIZED URN MODELS WITH IRREDUCIBLE AND REDUCIBLE REPLACEMENT POLICY

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Generalized Friedman urn is one of the simplest and most useful models considered in probability theory. Since Athreya and Ney (1972) showed the almost sure convergence of urn proportions in a randomized urn model with irreducible replacement matrix under the $L \log L$ moment assumption, this assumption has been regarded as the weakest moment assumption, but the necessary has never been shown. In this paper, we study the strong and weak convergence of generalized Friedman urns. It is proved that, when the random replacement matrix is irreducible in probability, the sufficient and necessary moment assumption for the almost sure convergence of the urn proportions is that the expectation of the replacement matrix is finite, which is less stringent than the $L \log L$ moment assumption, and when the replacement is reducible, the $L \log L$ moment assumption is the weakest sufficient condition. The rate of convergence and the strong and weak convergence of non-homogenous generalized Friedman urns are also derived.

1. Introduction. Urn models have long been considered powerful mathematical instruments in many areas, including the physical sciences, biological sciences, social sciences and engineering [Johnson and Kotz, 1977; Kotz and Balakrishnan, 1997]. The Pólya urn model was originally proposed to model the problem of contagious diseases [Eggenberger and Pólya, 1923]. Since then, there have been numerous generalizations and extensions. Among them, the generalized Friedman urn (also named as generalized Pólya urn in literature) is the most popular one in the literature [see Athreya and Karlin, 1968; Athreya and Ney, 1972; Higuerras et al, 2003, 2006; Janson, 2004; etc.]. The generalized Friedman urn is also a popular model of response-adaptive randomization in clinical trial studies [see Wei and Durham, 1978; Wei, 1979; Smythe, 1996; Bai and Hu, 1999, 2005; Hu and Zhang, 2004a; Hu and Rosenberger, 2006; Zhang, Hu and Cheung, 2006; Zhang et al, 2011; etc]. In the generalized Friedman urn, the urn starts with the urn composition $Y_0 = (Y_{0,1}, \ldots , Y_{0,d}) \in \mathbb{R}^d_+ \setminus \{0\}$. At the stage $m$ ($m = 1, 2, \ldots$), a ball is drawn from the urn with instant replacement. If the ball is of type $k$, then an additional random number $D_{k,q}(m)$ of balls of type $q$, $q = 1, \ldots , d$, are added to the urn. After $n$ draws and generations, the urn composition is denoted by the

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row vector $Y_n = (Y_{n,1}, \ldots, Y_{n,d})$, where $Y_{n,k}$ stands for the number of balls of type $k$ in the urn after the $n$th draw. This relation can be written as the following recursive formula:

(1.1) \quad Y_m = Y_{m-1} + X_mD_m,

where $D_m = (D_{k,q}(m))_{k,q=1}^d$, and $X_m$ is the result of the $m$th draw, distributed according to the urn composition at the previous stage, i.e., if the $m$th draw is a type $k$ ball, then the $k$th component of $X_m$ is 1 and other components are 0. The matrices $D_m$’s are named as the adding rules or replacement matrices. We assume that $D_m$, $m = 1, 2, \ldots$ are independent and identically distributed (i.i.d.), and let $H = E[D_m] = (E[D_{k,q}(m)])_{k,q=1}^d$, when the expectations in the brackets are finite. $H$ is said to be the generating matrix. In the original Pólya urn model and many of its generalizations, $D_n$ is a deterministic matrix. To distinguish whether $D_n$ is deterministic or not, the urn model is also called a randomized urn model when $D_n$ is random [see Bai and Hu, 2005].

When $D_{k,q}(m+1), Y_{m,k}, k, q = 1, \ldots, d, m = 0, 2, \ldots$, all take non-negative integer values, Athreya and Karlin (1968) and Athreya and Ney (1972) studied the convergence of $Y_n/n$ by embedding the urn process $Y_n$ into a multi-type branching process. Let $\{Z(t); t \geq 0\}$ be a $d$-type branching process for which (i) the life times of particles of all types are unit exponentials; and (ii) particles live and produce their offsprings independently of each other, and of the past; a $k$th type particle creates, on death, a random number of new particles of type $k$, and a random number of new particles of type $q$, where the random vector $(D_{k,1}, \ldots, D_{k,d})$ has the same distribution as $(D_{k,1}(m), \ldots, D_{k,d}(m))$ for each $k = 1, \ldots, d$. The matrix $D = (D_{k,l})_{k,l=1}^d$ represents the offspring producing rule. Let $\{\tau_n; n = 0, 1, 2, \ldots; \tau_0 = 0\}$ denote the split times for this process. Athreya and Ney (1972) presented the link between the urn model and the branching process by the following embedding theorem, see Theorem 9.2 of Athreya and Ney (1972).

**Theorem A** If $Z(0) = Y_0$, then the stochastic process $\{Y_n; n = 0, 1, 2, \ldots\}$ and $\{Z(\tau_n); n = 0, 1, 2, \ldots\}$ are equivalent.

By this equivalence and the limit theorems for branching processes, Athreya and Ney (1972) showed the almost sure convergence of $\frac{Y_n}{n}$ under the $L \log L$ moment assumption, namely,

(1.2) \quad E[D_{k,q} \log D_{k,q}] < \infty \quad \text{for all } k, q,

and an assumption on $H$ that $H^t$ is a matrix with positive entries for some integer $l \geq 1$, see also Athreya and Ney (2004). This assumption on $H$ implies that the nonnegative matrix $H$ is irreducible (for the definition see the next section) and so the largest real part $\lambda_H$ of all eigenvalues of $H$ is a simple and positive eigenvalue and, associated with $\lambda_H$ the nonnegative left eigenvector $v = (v_1, \ldots, v_d)$ with $v_1 + \cdots v_d = 1$ is unique and positive. The limit of $\frac{Y_n}{n}$ is just $\lambda_H v$. Because (1.2) is also a necessary condition for $e^{-\lambda_H t}Z(t)$ converging to a non-zero vector (c.f. Theorem 7.2 of Athreya and Ney, 1972), for a long history it has been expected that (1.2) is the almost sure moment condition for studying the almost sure convergence of the generalized Friedman urn models. By the supermartingale method, Zhang (2012) proved a similar almost sure convergence of $\frac{Y_n}{n}$ under (1.2) when $D_{k,q}(m+1), Y_{m,k}, k, q = 1, \ldots, d, m = 0, 2, \ldots$, take non-negative real, not necessary integer, values. Though the generalized Friedman urn model has been extended and studied in various ways [c.f. Bai and Hu, 1999, 2005; Benaim, Schreiber and Tarrès, 2004; Hugeras, et al, 2003, 2006; Janson, 2004; Laruelle and Pagès, 2013; Zhang, Hu and Cheung 2006; etc], the almost sure convergence is usually showed under moment conditions more stringent than (1.2). For examples, by applying the theory of branching process, Janson (2004) obtained the almost sure convergence and central limit theorems under the second moment finite, namely, $E[||D_n||^2] < \infty$. 
By applying the theory of matrices and the stochastic approximation (SA) algorithm, respectively, Bai and Hu (2005) and Laruelle and Pagés (2013) showed the almost sure convergence for the case of non-homogeneous replacement in which \( \{D_n\} \) may be not i.i.d., under the second moment finite, but the replacement is assumed to be balanced, namely, all the row sums of the (conditional) expectation of \( D_n \) are equal. For the general non-homogeneity case that the replacement may be unbalanced, Zhang (2016) proved the almost sure convergence by studying the stability of a stochastic approximation algorithm with a non-linear regression function, but the second moment finite is still assumed. The central limit theorems are also obtained in Bai and Hu (2005), Laruelle and Pagés (2013) and Zhang (2016) under the \((2 + \epsilon)\)-th moment finite. It seems that the almost sure convergence \( \frac{Y_n}{n} \) has not been proved under a moment condition less stringent than (1.2). Whether (1.2) is necessary has also never been shown in literature.

On the other hand, in the studies on the generalized Friedman urn models with random replacement matrices \( D_n \), the irreducibility of the mean matrix \( H \) is usually an essential condition. The randomized urn model with replacement reducible is seldom studied in literature. The purpose of this paper is to find the sufficient and necessary moment assumption for the almost sure convergence of \( \frac{Y_n}{n} \) in a generalized Friedman urn model with irreducible replacement or reducible replacement. We will find that, when the replacement is irreducible, the sufficient and necessary moment assumption is not (1.2), but that the expectations \( E[D_{k,q}(n)] \), \( k, q = 1, \ldots, d \), are finite; and when the replacement is reducible, the condition (1.2) is a sufficient moment condition that can not be weakened and the limit proportions are random. The results obtained in this paper give a full picture of the convergence for both the irreducible and reducible replacement cases. The rate of convergence, and weak and strong convergence of urn models with non-homogeneous replacement are also studied. The main results are given in the next section. The results for urns with non-homogenous replacement are given in Section 3. The proofs are stated in the last section. We will apply the method of the stochastic approximation algorithm to show the convergence as in Zhang (2016). For the stochastic approximation algorithm, the Kushner-Clark theorem [c.f. Kushner-Clark, 1978; Kushner and Yin, 2003; Duflo, 1997] is a usual tool to show the almost sure convergence. But now the equilibrium point of the regression of the stochastic approximation algorithm may be not unique and the related ordinary differential equation (ODE) may be not stable. We will use a direct way instead of the Kushner-Clark theorem to find the limit. For considering the properties of the limit proportions in the reducible replacement case, we will apply the supermartingale to show that when \( k \) is in an irreducible class corresponding the largest real eigenvalue of \( E[D_n] \), the limit of \( \frac{Y_{n,k}}{n} \) is a positive random variable and zero otherwise, and derive a conditional central limit theorem to show that the positive limit having no point probability mass.

2. Main results. Before we state the results. We first need some more notations and assumptions. As in Zhang (2016), to include various cases, we allow the numbers of balls to be non-integers and negative. For example, \( D_{k,l}(n) < 0 \) means that \( |D_{k,l}(n)| \) balls of type \( l \) is removed from the urn when a ball of type of \( k \) is drawn. We assume that a type of ball with a negative number will never be selected and so the selection probabilities are

\[
P(X_{n,k} = 1|\mathcal{F}_{n-1}) = \frac{Y_{n-1,k}^+}{\sum_{j=1}^d Y_{n-1,j}^+}, \quad k = 1, \ldots, d.
\]

Here \( Y_{n,k}^+ = \max\{Y_{n,k}, 0\} \) is the positive part of \( Y_{n,k} \), \( Y_n^+ = (Y_{n,1}^+, \ldots, Y_{n,d}^+) \), \( \mathcal{F}_n \) is the history sigma-field generated by \( X_1, \ldots, X_m, Y_1, \ldots, Y_m, D_1, \ldots, D_m \), and \( \mathcal{F}_0 \) is defined to be \((p_1, \ldots, p_d)\), which means that a \( k \)-type ball is selected with probability \( p_k \) when the urn has
no balls with a positive number, \( k = 1, \ldots, d \). Here \( p_1, \ldots, p_d \) are pre-specified probabilities with \( \sum_k p_k = 1 \). In this general framework, the urn allows the negative and/or non-integer number of balls, and the removal. Write \( N_n = (N_{n,1}, \ldots, N_{n,d}) \), where \( N_{n,k} \) is the number of times that a type \( k \) ball is drawn in the first \( n \) stages. Obviously, \( N_n = \sum_{k=1}^{n} X_n \).

In a general branching process \( Z(t) \), the life times of particles may not have the same distribution. If the life times of particles of type \( k \) are exponential with parameter \( \alpha_k > 0 \), \( k = 1, \ldots, d \), then the related urn processes is that with selection probabilities defined as

\[
P(X_{n,k} = 1 | \mathcal{F}_{n-1}) = \frac{\alpha_k Y_{n-1,k}}{\sum_{j=1}^{d} \alpha_j Y_{n-1,j}}, \quad k = 1, \ldots, d.
\]

Janson (2004) studied the properties of this kind of urn models. Under our framework, because the balls allow non-integer numbers, we can redefine the urn process as \((\alpha_1 Y_{n,1}, \ldots, \alpha_d Y_{n,k})\) with replacement matrices \( D_n \text{diag}(\alpha_1, \ldots, \alpha_d) \). The redefined urn process satisfies (1.1) with (2.1) and generating matrix \( H \text{diag}(\alpha_1, \ldots, \alpha_d) \).

For considering the asymptotic properties, we need assumptions on the replacement matrices.

**Assumption 2.1.** \( \{D_n\} \) is a sequence of independent and identically distributed random matrices.

**Assumption 2.2.** The expectations \( H_{k,q} = \mathbb{E}[D_{k,q}(n)] \), \( k, q = 1, \ldots, d \), are finite. Let \( H = (H_{q,k})_{k,q=1}^{d} \).

For a \( d \times d \)-matrix \( H \), we denote

\[
\lambda_H = \max \{ \Re(\lambda) : \lambda \text{ is an eigenvalue of } H \}
\]

to be the largest real part of its eigenvalues. If \( H_{k,q} \geq 0 \) for \( q \neq k \), then there is a \( \beta > 0 \) such that \((\beta I_d + \lambda H)\) is a nonnegative matrix, where \( I_d \) is the \( d \times d \) identity matrix, and so (i) \( \lambda_H \) is an eigenvalue of \( H \), (ii) if \( \lambda \neq \lambda_H \) is an eigenvalue of \( H \), then \( \Re(\lambda) < \lambda_H \), and (iii) \( H \) has nonnegative left eigenvectors and nonnegative right eigenvectors of \( H \) corresponding to \( \lambda_H \). We let \( S_H = \{v : vH = \lambda_H v, v \in \Delta^d\} \), where \( \Delta^d = \{v : \sum_{k=1}^{d} v_k = 1, v_k \geq 0, k = 1, \ldots, d\} \), to denote the space of scaled nonnegative left eigenvectors corresponding to \( \lambda_H \).

A square matrix \( A \) is said to be reducible when there exists a permutation matrix \( P \) such that

\[
P^t A P = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are both square.}
\]

Otherwise, \( A \) is said to be an irreducible matrix. A \( d \times d \) matrix \( A \) with nonnegative off-diagonal entries is irreducible if and only if \( \beta I_d + A \geq 0 \) and \( (\beta I_d + H)^{d-1} > 0 \) for some \( \beta > 0 \). Here and in the sequel, \( A \geq 0 \) and \( A > 0 \) mean that all entries of \( A \) are nonnegative and positive, respectively. By the Perron-Frobenius theory, if \( H \) with \( H_{k,q} \geq 0 \) \((k \neq q)\) is irreducible, then (i) \( \lambda_H \) is a simple eigenvalue of \( H \); (ii) There exist an unique right eigenvector \( v = (v_1, \ldots, v_d) \) and left eigenvector \( u^t = (u_1, \ldots, u_d)^t \) corresponding to \( \lambda_H \) such that \( \sum_k v_k = \sum_k u_k k = 1 \) and \( v_k > 0, u_k > 0, k = 1, \ldots, K \), and so \( S_H = \{v\} \). In general, we assume the following assumption.

**Assumption 2.3.** (a) \( H_{q,k} \geq 0 \) for \( q \neq k \); (b) there exist a right eigenvector \( u^t = (u_1, \ldots, u_d)^t \) corresponding to \( \lambda_H \) such that \( u_k > 0 \), \( k = 1, \ldots, K \); and (c) \( \lambda_H > 0 \).
THEOREM 2.1. If Assumptions 2.1-2.3 are satisfied, then
\begin{equation}
\lim_{n \to \infty} \text{dist} \left( \frac{Y_n}{n}, \lambda_H S_H \right) = \lim_{n \to \infty} \text{dist} \left( \frac{Y_n^+}{n}, \lambda_H S_H \right) = 0 \ a.s.,
\end{equation}
(2.2)
\begin{equation}
\lim_{n \to \infty} \text{dist} \left( \frac{Y_n^+}{\sum_{j=1}^d Y_{n,j}^+}, S_H \right) = 0 \ a.s.
\end{equation}
(2.3)
\begin{equation}
\lim_{n \to \infty} \text{dist} \left( \frac{N_n}{n}, S_H \right) \ a.s.
\end{equation}
(2.4)
Here the distance \( \text{dist}(x, S) \) between a point \( x \) and a set \( S \) is defined by \( \text{dist}(x, S) = \inf \{ \| x - y \| : y \in S \} \).

The following theorem gives the necessity of the expectations \( \mathbb{E}[D_{k,q}(n)] \)'s being finite when the limit proportions are positive.

THEOREM 2.2. Suppose that Assumption 2.1 is satisfied and \( \mathbb{E}[D_{q,k}(n)] > -\infty \) for all \( q, k \).

(a) If there is a random vector \( V \) with \( \mathbb{P}(V > 0) > 0 \), such that
\begin{equation}
\frac{Y_n}{n} \to V \ \text{in probability},
\end{equation}
(2.5)
then Assumption 2.2 is satisfied.

(b) Suppose that there is a random vector \( V \) with \( \sum_{k=1}^d V_k = 1 \) and \( \mathbb{P}(V_k > 0) > 0 \), such that \( \frac{Y_{n,k}}{\sum_{j=1}^d Y_{n,j}} \to V_k \) or \( \frac{N_{n,k}}{n} \to V_k \) in probability for \( k = 1, \ldots, d \). If the entries in one column of \( H = \mathbb{E}[D_n] \) are finite, then all entries of \( H \) are finite.

Usually, as in Athreya and Karlin (1968) and Athreya and Ney (1972) etc., the values \( D_{k,q}(n) \)'s are assumed to be non-negative. Sometimes, as in Janson (2004) and Laruelle and Pagès (2013) etc., the drawn ball is allowed to be dropped, so the diagonal elements \( D_{k,k}(n) \)'s can take negative value \(-1\) or \(-c_k\) for some positive constant \( c_k \). Such replacement matrices satisfy the following assumption.

ASSUMPTION 2.4. \( D_{k,q}(n) \geq 0 \ a.s. \) for all \( k \neq q \), and \( D_{k,k}(n) \geq -c_0 \) for some \( c_0 \) and all \( k \).

A simple example of the replacement matrix satisfies Assumption 2.3 but not Assumption 2.4 is that \( \tilde{D}_{k,q}(n) = D_{k,q}(n) + \epsilon_{k,q}(n) \), where \( D_{k,q}(n) \)'s satisfy Assumption 2.4 and \( \epsilon_{k,q}(n) \) are replacement errors with mean zeros.

When Assumption 2.4 is satisfied, then for any \( C > 0 \) the matrix \( \mathbb{E}[D_n \wedge C] := (\mathbb{E}[C \wedge D_{q,k}(n)])_{q,k=1}^d \) satisfies Assumption 2.3(a). It is easily that, when \( \mathbb{E}[D_n \wedge C] \) is irreducible, then \( \mathbb{E}[D_n \wedge I] \) is irreducible for all \( l \geq C \), because \( \mathbb{E}[D_n \wedge l] \geq \mathbb{E}[D_n \wedge C] \) if \( l \geq C \). The irreducibility of the matrix \( \mathbb{E}[D_n] \) or \( \mathbb{E}[D_n \wedge C] \) only depends on the structure of what off-diagonal elements are nonzero. If define \( P_n = (p_{k,q}(n))_{k,q=1}^d \) by \( p_{k,q}(n) = \mathbb{P}(D_{k,q}(n) \neq 0) \), then \( P_n \) will have the same structure of \( \mathbb{E}[D_n \wedge C] \) when \( C \) is large. So, when \( H \) is not finite, we may define the irreducibility of the replacement by the irreducibility of \( \mathbb{E}[D_n \wedge C] \) or \( P_n \).
**Definition 2.1.** The replacement matrix $D_n$ is said to be irreducible in probability when $P_n$ is irreducible, where $P_n = (p_{k,q}(n))_{k,q=1}^d$ with $p_{k,q}(n) = P(D_{k,q}(n) \neq 0)$.

The replacement matrix $D_n$ satisfying Assumption 2.4 is said to be irreducible in mean when $E[D_n \wedge C]$ is irreducible for some $C > 0$.

It is obvious that, under Assumption 2.4, $D_n$ is irreducible in probability if and only if it is irreducible in mean, and when $E[D_n]$ is finite, the irreducibility in mean, irreducibility in probability and that $E[D_n]$ is irreducible are equivalent. So, in general, when $D_n$ is irreducible in probability, the replacement is said to be irreducible.

Now, we consider the case of irreducible replacement. The following corollary follows from Theorem 2.1 immediately.

**Corollary 2.1.** Suppose that Assumptions 2.1-2.3 are satisfied. Further, assume that $H$ is irreducible. Then

$$\lim_{n \to \infty} \frac{Y_n}{n} = \lim_{n \to \infty} \frac{Y_n^+}{n} = \lambda_H v \ a.s.,$$

$$\lim_{n \to \infty} \frac{Y_n^+}{\sum_{j=1}^d Y_n^+} = v \ a.s.$$

and

$$\lim_{n \to \infty} \frac{N_n}{n} = v \ a.s.,$$

where $v \in \Delta^d$ is the unique solution of the equation $vH = \lambda_H v$.

The following is the converse of Corollary 2.1.

**Corollary 2.2.** Suppose that Assumptions 2.1, 2.4 are satisfied, and $D_n$ is irreducible in probability. If there is a random vector $V$ with $P(V \geq 0, V \neq 0) > 0$ such that (2.5) holds, then Assumptions 2.2, 2.3 are satisfied, and $H$ is irreducible. Further, $V = \lambda_H v \ a.s.$, where $v \in \Delta^d$ is the unique solution of $vH = \lambda_H v$.

From Corollaries 2.1 and 2.2, we conclude that, in the case of irreducible replacement, the mean replacement matrix being finite is the sufficient and necessary condition for the proportions $\frac{Y_n}{n}$ to have a non-zero limit.

In literature, the studies on urn models with reducible replacements are very few. Gouet (1997) considered the case of fixed deterministic balanced, but not necessarily irreducible, replacement matrix. Abraham, Dhersin and Ycart (2007) considered a special urn scheme with reducible random replacements, in which, each time a ball is picked, another ball is added, and its type is chosen according to the transition probabilities of a reducible Markov chain. The vector of frequencies $Y_n/n$ is shown to converge almost surely to a random element of the set of stationary measures of the Markov chain.

Our next theorem shows that for the randomized urn model, when $H$ is reducible, $\frac{Y_n}{n}$ and $\frac{N_n}{n}$ will also converge almost surely under $L \log L$ moment assumption. To state the result, we need more notations for describing the structure of the matrix $H$. Let $\nu_1$ be the index of the eigenvalue $\lambda_H$. Under (a) and (b) in Assumption 2.3, $H$ has the following Jordan canonical form

$$H = T \text{diag}(\lambda_H I_{\nu_1}, J_2, \ldots, J_s)T^{-1},$$
where

\[ J_t = \begin{pmatrix}
\lambda_t & 1 & 0 & \ldots & 0 \\
0 & \lambda_t & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \lambda_t & 1 \\
0 & 0 & 0 & \ldots & \lambda_t
\end{pmatrix}_{\nu_x \times \nu_t}, \quad \text{Re}(\lambda_t) < \lambda_H.
\]

Further, in (2.9), \( T \) and \( T^{-1} \) can be chosen such that \( v_1, \ldots, v_{\nu_t} \in S_H \) be the first \( \nu_t \) rows of \( T^{-1} \), and \( u_j^t \geq 0, j = 1, \ldots, \nu_t \), be the first \( \nu_t \) columns of \( T \). Then \( H u_j^t = \lambda_H u_j^t \), \( v_j u_j^t = 1 \), \( v_j u_j^t = 0, i \neq j \). \( S_H \) has the form \( \{ v : v = \sum_{j=1}^{\nu_t} \beta_j v_j : \sum_{j=1}^{\nu_t} \beta_j = 1, \beta_j \geq 0, j = 1, \ldots, \nu_t \} \).

The set \( \{1, \ldots, d\} \) can be divided to several irreducible classes \( C_j, j = 1, \ldots, r \), such that (i) each principal submatrix \( H_{jj} = (H_{k,q}; k, q \in C_j) \) of \( H \) on the class \( C_j \) is irreducible or \((0)_{1 \times 1} \); (ii) \( \lambda_{H,ij} = \lambda_H, j = 1, \ldots, \nu_t \), and \( \lambda_{H,ij} < \lambda_H \) for other \( j \); (iii) the elements of \( v_j \) in the class \( C_j \) are positive and zeros otherwise, \( j = 1, \ldots, \nu_t \); (iv) the elements of \( u_j \) in the class \( C_j \) are positive and zeros in \( C_i \) for \( i \neq j, j = 1, \ldots, \nu_t \). Further, \( H \) has the following structure. If \( q \) is in one of the classes \( C_i, i \leq \nu_t \), and \( k \) is not in the same class, then \( H_{q,k} = 0 \). For any \( k \) in one of the classes of \( \nu_t \), \( C_j, j = 1, \ldots, \nu_t \), there is class \( C_j, j = 1, \ldots, \nu_t \), and a \( q \) in such that \( H_{k,q} \neq 0 \).

Let \( U = : \sum_{j=1}^{\nu_t} u_j^t v_j \). Then \( U \) is a project onto the space \( \{ v : v H = \lambda_H v \} = \{ v : v = \sum_{j=1}^{\nu_t} \beta_j v_j \} \). It is obvious that \( H \) is irreducible if and only if \( \nu_t = 1 \), and then \( S_H = \{ v \} \) has an unique point.

**Theorem 2.3.** Suppose that Assumptions 2.1-2.3 are satisfied. Further, assume \( E[||D_n|| \log(||D_n||)] < \infty \). Then there exists a random vector \( V \) which takes values in \( S_H \) such that

\[
Y_n \to \lambda_H V \quad \text{a.s.,}
\]

\[
\frac{Y_{n+}}{\sum_{j=1}^{d} Y_{n+}} \to V \quad \text{a.s.}
\]

and

\[
\frac{N_n}{n} \to V \quad \text{a.s.}
\]

If also \( D_n \) are non-negative matrices and \( Y_{0,k} > 0, k = 1, \ldots, d \), then the limit \( V \in S_H \) has the form \( V = \sum_{j=1}^{\nu_t} \alpha_j v_j \) with \( \sum_{j=1}^{\nu_t} \alpha_j = 1, 0 < \alpha_j \leq 1, j = 1, \ldots, \nu_t \), with probability one, and further, if \( E[||D_n||^2] < \infty \), then each \( \alpha_j \) has no point probability mass in \( (0,1) \), namely, \( P(\alpha_j = p) = 0 \) for any \( p \in (0,1) \).

For the reducible case \( (\nu_t > 1) \), from Theorem 2.3, we find that if \( D_n \) is nonnegative and has second moment finite, then \( \frac{Y_{n,k}}{\lambda_{n,k}} \) and \( \frac{N_n}{\lambda_{n,k}} \) converge to a random variable \( V_k \) in \( (0,1) \) when \( k \) belongs to one of the irreducible classes \( C_j, j = 1, \ldots, \nu_t \), associated with the largest eigenvalue \( \lambda_{H_j} \), and this random variable \( V_k \) has no point mass. When \( k \) belongs to other classes, \( \frac{Y_{n,k}}{\lambda_{n,k}} \) and \( \frac{N_n}{\lambda_{n,k}} \) converge to zero. We conjecture that the condition \( E[||D||^2] < \infty \) can be weakened to \( E[||D_n||^{\log(||D_n||)}] < \infty \), and \( \alpha_j, j = 1, \ldots, \nu_t \) are continuous random variables having densities with support \( (0,1) \).

For the urn considered in Abraham, Dbersin and Ycart (2007), the replacement matrices \( D_n \) take zero or one entries with \( P(D_{k,q}(n) = 1) = p_{k,q}; \sum_{q=1}^{d} p_{k,q} = 1 \), and \( H = E[D_n] = (p_{k,q})_{k,q=1}^{d} \) possibly reducible. This model satisfies Assumptions 2.1-2.4 with \( H = P \),
$\lambda_H = 1$ and $u = (1, \cdots, 1)$. Abraham, Dhersin and Ycart (2007) characterized the limit probability distribution as the solution to a fixed point problem. Examples showed that the limit probability distribution has density.

The next example shows that, the condition $\mathbb{E}[\|D_n\| \log(\|D_n\|)] < \infty$ can not be weakened for the reducible case.

**Example 1.** Suppose that $\{D_n = \text{diag}(D_{1,1}(n), \cdots, D_{d,d}(n)); n \geq 1\}$ is a sequence of i.i.d. diagonal and non-negative matrices. This is a reinforced urn studied by Zhang et al (2014). Suppose that there is a random vector $V$ with $P(V > 0) > 0$ such that (2.5) holds. Then by Theorem 2.2 (a), $m_k = \mathbb{E}[D_{k,k}(1)] < \infty$ for all $k$.

It is obvious that $m_k \neq 0$, for otherwise we will have $D_{k,k}(n) = 0$ a.s., and then $Y_{n,k} = Y_{0,k}$ a.s. for all $n$. By Theorem 2.3 of Zhang et al (2014), $m_1 = \cdots = m_d$. So, Assumptions 2.1-2.3 are satisfied with $\lambda_H = m_1$ and $u = (1, \cdots, 1)$.

Further, by Theorem 2.3 of Zhang et al (2014), if one of $\mathbb{E}[|D_{k,k}(1)| \log|D_{k,k}(1)|]$ is finite then all of them are finite.

From Theorems 2.1 and 2.3, we get the following corollary on the branching process.

**Corollary 2.3.** Suppose $Z(t)$ is a branching process with nonnegative offspring producing rule $D$ and life time parameters $\alpha_1, \cdots, \alpha_d$. Let $M = (\alpha_i \mathbb{E}[D_{i,j}])_{i,j=1}^d$ and suppose the largest real part $\lambda_M$ of the eigenvalues of $M$ is positive. If $M$ is irreducible, then as $t \to \infty$,

$$\frac{Z(t)}{\sum_{j=1}^d Z_j(t)} \to \nu \text{ a.s.,}$$

(2.13)

$$\frac{N_k(t)}{\sum_{j=1}^d N_j(t)} \to \frac{\alpha_k v_k}{\sum_{j=1}^d \alpha_j v_j} \text{ a.s. } k = 1, \cdots, d,$$

(2.14)

where $N_j(t)$ is number of $k$th type particles died up to time $t$, and $\nu \in \Delta^d$ is the unique solution of the equation $\nu M = \lambda_M \nu$ and positive. If $M$ is reducible but has a positive right eigenvector corresponding to $\lambda_H$, and (1.2) is satisfied, then there exists an random vector $\nu$ taking values in the space $S_M$ such that (2.13) and (2.14) holds.

The last theorem in this section gives the rate of the convergence.

**Theorem 2.4.** Suppose that Assumptions 2.1-2.3 are satisfied. Further, assume $\mathbb{E}[\|D_n\|] < \infty$. Then there exists a random vector $V$ which takes values in $S_H$ such that

$$\left\| \frac{Y_n}{n} - \lambda_H V \right\| = O(b_n) \text{ a.s.,}$$

(2.15)

$$\left\| \frac{Y^+}{\sum_{j=1}^d Y^+_{n,j}} - V \right\| = O(b_n) \text{ a.s.}$$

(2.16)

and

$$\left\| \frac{N_n}{n} \to V \right\| = O(b_n) \text{ a.s.}$$

(2.17)

where $b_n$ is defined as follows. Let $\rho = \max\{\Re(\lambda_j)/\lambda_H : j = 2, \cdots, s\}$ be the ratio of the second largest real part of the eigenvalues of $H$ to the largest one, and $\nu_{\text{sec}} = \max\{\nu_j : \nu_j > 0\}$.
$Re(\lambda_j) = \rho \lambda_H$ be the largest algebraic multiplicity of the eigenvalues with the second largest real part. Define $b_n$ by

$$b_n = \begin{cases} n^{\rho-1}(\log n)^{\nu_{sec}-1}, & \text{if } \rho > 1/2, \\ n^{-1/2}(\log n)^{\nu_{sec}-1}(\log \log n)^{1/2}, & \text{if } \rho = 1/2, \\ n^{-1/2}(\log n)^{1/2}, & \text{if } \rho < 1/2. \end{cases} \quad (2.18)$$

We will show Theorems 2.1, 2.2, 2.3 and 2.4 in the last section after establishing the results for the general models with non-homogenous replacements. Here we give the proofs of Theorem 2.2 and Corollary 2.2. To prove the results, we need a lemma at first.

**Lemma 2.1.** (a) Let $\xi_{n,k} = f(D_{k,1}(n), \ldots, D_{k,d}(n))$ be a Borel function of $(D_{k,1}(n), \ldots, D_{k,d}(n))$. Then on the event $\{N_{n,k} \to \infty\}$,

$$\lim_{n \to \infty} \frac{\sum_{m=1}^{n} X_{m,k} \xi_{m,k}}{N_{n,k}} = E[\xi_{1,k}] \text{ a.s. if } E[|\xi_{1,k}|] < \infty,$$

and

$$\frac{|\sum_{m=1}^{n} X_{m,k}(\xi_{m,k} - E[\xi_{1,k}])|}{\sqrt{N_{n,k} \log \log N_{n,k}}} = O(1) \text{ a.s. if } E[|\xi_{1,k}|^2] < \infty.$$

(b) Let $\Delta M_{n,1} = X_n - E[X_n|\mathcal{F}_{n-1}]$. Then

$$||M_{n,1}|| = O\left(\sqrt{n \log \log n}\right) \text{ a.s., } \max_{m \leq n} ||M_{n,1}|| = O(\sqrt{n}) \text{ in probability;}$$

(c) Let $\Delta M_{n,2} = X_n(D_n - E[D_n])$. Under Assumptions 2.1 and 2.2,

$$\frac{M_{n,2}}{n} \to 0 \text{ a.s.}$$

Further, if $E[\|D_m\|^2] < \infty$, then

$$||M_{n,2}|| = O\left(\sqrt{n \log \log n}\right) \text{ a.s.}$$

**Proof.** The proof of (a) can be found in Hu and Zhang (2004b) (c.f. their Lemma A.4). (b) is obvious because $\Delta M_{n,1}$ is a sequence of bounded martingale differences. (c) is a direct conclusion of (a). □.

**Proof of Theorem 2.2.** For a vector $\theta = (\theta_1, \cdots, \theta_d)$, we write $\alpha(\theta) = \sum_{j=1}^{d} \theta_j$.

(a) Suppose (2.5) holds. Then on the event $\{V > 0\}$, $\frac{Y_{\alpha(V)}^{+}}{\alpha(V)} \to \frac{V}{\alpha(V)}$ in probability, and so

$$\frac{N_{n,k}}{n} = \frac{1}{n} \sum_{m=1}^{n} \frac{Y_{\alpha(Y_{m-1})}^{+}}{\alpha(Y_{m-1})} + \frac{1}{n} \sum_{m=1}^{n} (X_{m,k} - E[X_{m,k}|\mathcal{F}_{m-1}])$$

$$= \frac{1}{n} \sum_{m=1}^{n} \frac{Y_{\alpha(Y_{m-1})}^{+}}{\alpha(Y_{m-1})} + o(1) \text{ a.s.} \quad (2.19)$$

$$\to \frac{V_k}{\alpha(V)} \text{ in probability,}$$
\( k = 1, \ldots, d, \) by Lemma 2.1(b). Note for each constant \( M > 0, \)
\[
\frac{Y_{n,k}}{n} = \frac{\sum_{m=1}^{n} \sum_{q=1}^{d} X_{m,q} D_{q,k}(m)}{n} \\
\geq \frac{\sum_{m=1}^{n} X_{m,q} D_{q,k}^+(m)}{n} - \frac{\sum_{m=1}^{n} X_{m,k} D_{q,k}^-(m)}{n} \\
\geq \frac{\sum_{m=1}^{n} X_{m,q} (M \wedge D_{q,k}^+(m))}{n} - \frac{\sum_{m=1}^{n} X_{m,k} E[D_{q,k}^-]}{n} + o(1) \ a.s.
\]
\[
\geq \frac{N_{n,q} E[M \wedge D_{q,k}^+] - \min_{q,k} E[D_{q,k}^-]}{n} + o(1) \ a.s.
\]
\[
\rightarrow \frac{V_{q} \alpha(V)}{\alpha(V)} E[M \wedge D_{q,k}^+] - \min_{q,k} E[D_{q,k}^-] \text{ in probability}
\]
on the event \( \{ V > 0 \} \), by Lemma 2.1 (a). If \( E[D_{q,k}] = \infty \), then by letting \( M \rightarrow \infty \) we conclude that \( \frac{Y_{n,k}}{n} \rightarrow +\infty \) in probability on the event \( \{ V > 0 \} \) which contradicts the assumption (2.5). So, Assumption 2.2 is satisfied.

(b) Note that, if \( \frac{Y_{n,k}}{\alpha(Y_{n,q})} \rightarrow V_k \) in probability, then on the event \( \{ V > 0 \} \), \( \frac{Y_{n,k}^+}{\alpha(Y_{n,q}^+)} \rightarrow V_k \) in probability and \( \frac{N_{n,k}}{n} \rightarrow V_k \) in probability by (2.19). So, it is sufficient to consider the case that \( \frac{N_{n,k}}{n} \rightarrow V_j \) in probability for all \( j = 1, \ldots, d. \)

Suppose that the entries in \( k \)th column of \( H \) are finite, and there is an element of \( H, \) say \( H_{l,q}, \) such that \( H_{l,q} = E[D_{l,q}(m)] = \infty. \) Then,
\[
\frac{|Y_{n,k}|}{n} \leq \frac{\sum_{m=1}^{n} \sum_{j=1}^{d} X_{m,j} |D_{j,k}(m)|}{n} = \frac{\sum_{j=1}^{d} N_{n,j} E[|D_{j,k}|]}{n} + o(1) \ a.s.
\]
\[
\rightarrow \frac{\sum_{j=1}^{d} V_j E[|D_{j,k}|]}{n} \text{ in probability},
\]
by Lemma 2.1 (a) and the fact that \( \frac{N_{n,j}}{n} \rightarrow V_j \) in probability, and
\[
\frac{Y_{n,q}}{n} \rightarrow \infty \text{ in probability on the event } \{ V > 0 \}.
\]
It follows that
\[
\frac{Y_{n,k}^+}{\alpha(Y_{n,q}^+)} \leq \frac{|Y_{n,k}|}{Y_{n,q}} \rightarrow 0 \text{ in probability on the event } \{ V > 0 \}.
\]
Hence, by (2.19) \( \frac{N_{n,k}}{n} \rightarrow 0 \) in probability on the event \( \{ V > 0 \}, \) which contradicts the assumption that \( \frac{N_{n,k}}{n} \rightarrow V_k \) in probability. So, we conclude that all entries of \( H \) are finite. \( \Box \)

Proof of Corollary 2.2. Under Assumption 2.4, \( Y_{n,k} \) will not decrease when a type \( q \) ball is drawn \( q \neq k \) and it will not decrease when it becomes negative. So, \( Y_{n,k} \geq -c_0. \) It follows
that $Y_{n,k} \to Y$ in probability, $S_{n,k} \to D$ in probability, and so
\[
\frac{Y_n}{\alpha(Y_n)} \to \frac{Y}{\alpha(Y)} \quad \text{in probability,}
\]
by (2.19). It follows that, one the event $\{V \geq 0, V \neq 0\}$,
\[
\frac{Y_n}{n} = \frac{1}{n} \sum_{m=1}^{n} X_mD_m \geq \frac{1}{n} \sum_{m=1}^{n} X_m(D_m + M)
\]
\[
= \frac{N_n}{n} E[D_1 + M] + \alpha_p(1) \to \frac{V}{\alpha(V)} E[D_1 + M] \quad \text{in probability,}
\]
by Lemma 2.1 (a). It follows that $\alpha(V)V \geq V[D_1 + M]$ on the event $\{V \geq 0, V \neq 0\}$. By the assumption that $D_1$ is irreducible in probability which is equivalent to that it is irreducible in mean, there exist $M > 0$ and $\beta > 0$ such that $\beta I_d + E[D_1 + M] \geq 0$ and $(\beta I_d + E[D_1 + M])^d \geq 0$. Hence $(\beta + \alpha(V))^d V \geq V(\beta I_d + E[D_1 + M])^d \geq 0$ on the event $\{V \geq 0, V \neq 0\}$. So, $P(V > 0) = P(V \geq 0, V \neq 0) > 0$. Therefore, by Theorem 2.2 (a) which has been proved, $H = E[D_1] \leq \mu$ is finite. Then $H$ is also irreducible and so, its unique left eigenvector $\omega \in \Delta^d$ corresponding to $\lambda_H$ is positive. By Lemma 2.1 (c) again,
\[
\frac{Y_n}{n} = \frac{N_n}{n} H + \frac{M_{n,2}}{n} = \frac{N_n}{n} H + o(1) \quad \text{in probability,}
\]
which implies that $V = \frac{V}{\alpha(V)} H$ on the event $\{V \geq 0, V \neq 0\}$. So, $\frac{V}{\alpha(V)} = \omega$ and $\lambda_H = \alpha(V)$ on the event $\{V \geq 0, V \neq 0\}$. Hence, Assumptions 2.1-2.3 are satisfied. Finally, combing (2.6) and (2.5) yields $V = \lambda_H \omega$ a.s. □

3. Non-homogenous replacement. In this section, we consider the case that $\{D_n\}$ are not i.i.d. matrices. Let
\[
H = E[D_n|F_{n-1}] = \left( H_{q,k}(n) \right)_{q,k=1}^d.
\]

THEOREM 3.1. Suppose that there exists a random matrix $H$ such that
\[
\sum_{m=1}^{n} \|H_n - H\| = o(n) \quad \text{a.s.,}
\]
and with probability one, $H(\omega)$ satisfies Assumption 2.3 in which $\lambda_H, S_H, \omega$ may depend on $\omega$. Assume
\[
\|M_{n,2}\| = o(n) \quad \text{a.s.}
\]
Then (2.2)-(2.4) hold. Further, if $H$ is irreducible almost surely, then $S_H$ has an unique point $\omega$ and so (2.6)-(2.8) hold.

The condition (3.2) is satisfied if, for all $q, k$, there is a sequence $c_m = c_{m, q, k} > 0$ such that
\[
\sum_{m=1}^{\infty} P(D_{q,k}(m) \geq c_m) < \infty \quad \text{a.s.,}
\]
and
\[
\sum_{m=1}^{n} E[D_{q,k}(m) I(D_{q,k}(m) \geq c_m) | F_{m-1}] = o(n) \quad \text{a.s.}
\]
and

\[
\sum_{m=1}^{\infty} \frac{\mathbb{E} [\tilde{D}_{q,k}(m)^2 | \mathcal{F}_{m-1}]}{m^2} < \infty \ a.s.,
\]

where \(\tilde{D}_{q,k}(m) = D_{q,k}(m)I\{|D_{q,k}(m)| \leq c_m\} - \mathbb{E} [D_{q,k}(m)I\{|D_{q,k}(m)| \leq c_m\} | \mathcal{F}_{m-1}]\).

**Remark 3.1.** It is easily seen that (3.3)-(3.5) with \((c_m = m)\) are implied by

\[
\begin{align*}
\text{either } & \sup_m \mathbb{E} [\|D_m\| \log^{1+\epsilon} (\|D_m\|) | \mathcal{F}_{m-1}] < \infty \ a.s. \\
\text{or } & \sup_m \mathbb{E} [\|D_m\| \log^{1+\epsilon} (\|D_m\|) < \infty \\
\text{for some } & \epsilon > 0. \text{ When } \{D_m\} \text{ is a sequence of i.i.d. matrices, then } \mathbb{E} [|D_{k,q}(m)|] < \infty \text{ implies (3.1) and (3.3)-(3.5) with } c_m = m.
\end{align*}
\]

For the non-homogeneity case with \(H\) irreducible, Bai and Hu (2005) proved (2.6)-(2.8) under the conditions that

\[
H_m \geq 0 \text{ and } H_m 1' = c 1' \ a.s.,
\]

\[
\sum_{m=1}^{\infty} \frac{\|H_m - H\|}{m} < \infty \ a.s.,
\]

\[
\sup_m \mathbb{E} [\|D_m\|^{2+\epsilon} | \mathcal{F}_{m-1}] < \infty \ a.s. \text{ for some } \epsilon > 0.
\]

By applying a stochastic approximation algorithm, Larielle and Pagés (2013) relaxed the conditions (3.8) and (3.9) to \(H_n \rightarrow H\) a.s. and the second moment finite that \(\sup_m \mathbb{E} [||D_m||^2 | \mathcal{F}_{m-1}]) < \infty\) which is still more stringent than (3.6). The condition (3.7) means that the updating of the urn at each stage is balanced. Zhang (2016) removed this condition and proved (2.6)-(2.8) under (3.1) and a moment condition that \(\sum_{m=1}^{n} \mathbb{E} [D_{q,k}^2(m)] = O(n)\). This moment condition implies (3.3)-(3.5) with \(c_m = \infty\). Zhang (2012) proved (2.6)-(2.8) under (3.1) and (3.6), but assumed that the entries of the replacement matrices \(D_n\) are non-negative.

It is easily shown that the condition (3.8) implies (3.1). Under this condition and moment assumptions a little stricter than those in Theorem 3.1, the urn proportions also converge almost surely to a random point in \(\lambda H S_H\) even \(H\) is reducible.

**Theorem 3.2.** Suppose that there exists a random matrix \(H\) satisfying (3.8), and with probability one, \(H(\omega)\) satisfies Assumption 2.3 in which \(\lambda_H, S_H, \mathbf{u}\) may depend on \(\omega\). Assume

\[
\sum_{m=1}^{\infty} \frac{\max_{1 \leq i \leq m} \|M_{i,2}\|}{m^2} < \infty \ a.s.
\]

Then there exists a random vector \(V\) taking values in \(S_H\) such that (2.10)-(2.12) hold.

The condition (3.10) is satisfied if, for all \(q, k\), there is a sequence \(c_m = c_{m,q,k} > 0\) such that

\[
\sum_{m=1}^{\infty} \mathbb{E} [D_{q,k}(m)I\{|D_{q,k}(m)| \geq c_{m}\} | \mathcal{F}_{m-1}] < \infty \ a.s.
\]
and

$$\sum_{m=1}^{\infty} \frac{(\log m)^{1+\epsilon} \mathbb{E}[\tilde{D}_{q,k}(m) | \mathcal{F}_{m-1}]}{m^2} < \infty \text{ a.s. for some } \epsilon > 0,$$

where \( \tilde{D}_{q,k}(m) = D_{q,k}(m) I\{|D_{q,k}(m)| \leq c_m\} - E[D_{q,k}(m) I\{|D_{q,k}(m)| \leq c_m\} | \mathcal{F}_{m-1}]\).

**Remark 3.2.** It is easily seen that (3.6) implies (3.11) and (3.12) with \( c_m = m/(\log m)^{1+\epsilon} \). When \( \{D_n; n \geq 1\} \) are i.i.d., the condition that \( \mathbb{E}[\|D_n\| \log(\|D_n\|)] < \infty \) implies (3.11) and (3.12) with \( c_m = m/(\log m)^{1+\epsilon} \).

In fact, write \( c_m = m/(\log m)^{1+\epsilon} \) and \( \sigma_i^2 = \mathbb{E}[\tilde{D}_{q,k}(i)^2 | \mathcal{F}_{i-1}] \). Then

the left hand of (3.11)

$$\leq \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{(\log c_m)^{1+\epsilon}} \mathbb{E}[|D_{q,k}(m)| \log^{1+\epsilon}(|D_{q,k}(m)|) | \mathcal{F}_{m-1}]$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{(\log m)^{1+\epsilon}} \mathbb{E}[|D_{q,k}(m)| \log^{1+\epsilon}(|D_{q,k}(m)|) | \mathcal{F}_{m-1}]$$

$$\leq c \sup_m \mathbb{E}[|D_{q,k}(m)| \log^{1+\epsilon}(|D_{q,k}(m)|) | \mathcal{F}_{m-1}],$$

and

the left hand of (3.12) = \( \sum_{m=1}^{\infty} \frac{\sigma_i^2 (\log m)^{1+\epsilon/2}}{m^2} \)

$$\leq \sum_{m=1}^{\infty} \frac{(\log m)^{1+\epsilon/2}}{m^2} \mathbb{E}[|D_{q,k}(m)|^2 I\{|D_{q,k}(m)| \leq m/(\log m)^{1+\epsilon}\} | \mathcal{F}_{m-1}]$$

$$\leq c \sum_{m=1}^{\infty} \frac{m/(\log m)^{1+\epsilon/2}}{m^2} \mathbb{E}[|D_{q,k}(m)| \log(|D_{q,k}(m)|) | \mathcal{F}_{m-1}]$$

$$\leq c \sup_m \mathbb{E}[|D_{q,k}(m)| \log(|D_{q,k}(m)|) | \mathcal{F}_{m-1}],$$

which both are finite when the first condition in (3.6) is satisfied. When the second condition in (3.6) is satisfied, taking the expectation instead of the conditional expectation get the same conclusion. If \( \{D_m\} \) are i.i.d., then

left hand of (3.11) = \( \sum_{m=1}^{\infty} \frac{1}{m} \mathbb{E}[D_{q,k}(1) I\{|D_{q,k}(1)| \geq c_m\}] \)

$$= \mathbb{E}[D_{q,k}(1) \sum_{m=1}^{\infty} \frac{1}{m} I\{|D_{q,k}(1)| \geq c_m\}] \leq c \mathbb{E}[D_{q,k}(1) \log(|D_{q,k}(1)|)].$$

The next theorem gives the rate of convergence.

**Theorem 3.3.** Suppose that there exists a random matrix \( H \) satisfying

$$\sum_{m=1}^{n} \|H_m - H\| = O(\sqrt{n \log \log n}) \text{ a.s.,}$$

(3.13)
and with probability one, $H(\omega)$ satisfies Assumption 2.3 in which $\lambda_H$, $S_H$, $u$ may depend on $\omega$. Assume

\begin{equation}
\|M_{n,2}\| = O(\sqrt{n \log \log n}) \text{ a.s.}
\end{equation}

Then there exists a random vector $V$ taking values in $S_H$ such that (2.15)-(2.17) hold, where $b_n$ by defined as in (2.18), but now $\rho$ and $\nu_{\sec}$ are random variables.

The condition (3.14) is satisfied if (3.9) is satisfied.

The condition (3.1) means that the conditional expectation $H_n$ of the random replacement matrix $D_n$ will converge almost surely to $H$ in Cesaro sense. If it does not holds, then we may have no result on the convergence even high moments are assumed finite. Recently, Gangopadhyay and Maulik (2017) showed that when the almost sure convergence in (3.1) is replaced by the convergence in probability, then $Y_n$ converges in probability under the conditions that $H$ is irreducible, $\{D_n; n \geq 1\}$ is uniformly integrable, and $Y_0$ has first moment finite. The last theorem of this paper gives a general result on the weak convergence of $Y_n$.

**Theorem 3.4.** Suppose that there exists a random matrix $H$ such that

\begin{equation}
\sum_{m=1}^{n} \|H_n - H\| = o(n) \text{ in probability},
\end{equation}

and with probability one, $H(\omega)$ satisfies Assumption 2.3 in which $\lambda_H$, $S_H$, $u$ may depend on $\omega$. Assume

\begin{equation}
\max_{m \leq n} \|M_{m,2}\| = o(n) \text{ in probability}.
\end{equation}

Then, for any $T > 0$,

\begin{equation}
\lim_{n \to \infty} \max_{n \leq m \leq nT} \text{dist}(\frac{Y_m}{m}, \lambda_H S_H) = 0,
\end{equation}

\begin{equation}
\lim_{n \to \infty} \max_{n \leq m \leq nT} \text{dist}(\frac{Y^+_m}{m}, \lambda_H S_H) = 0,
\end{equation}

\begin{equation}
\lim_{n \to \infty} \max_{n \leq m \leq nT} \text{dist}(\frac{Y^+_m}{\alpha(Y^+_m)}, S_H) = 0
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \max_{n \leq m \leq nT} \text{dist}(\frac{N_m}{m}, S_H) = 0
\end{equation}

in probability. In particular, if $H$ is irreducible almost surely, then $S_H$ has an unique random point $v$ and so

\begin{equation}
\lim_{n \to \infty} \frac{Y_n}{n} = \lim_{n \to \infty} \frac{Y^+_n}{n} = \lambda_H v \text{ in probability},
\end{equation}

\begin{equation}
\lim_{n \to \infty} \frac{Y^+_n}{\sum_{j=1}^{d} Y^+_{n,j}} = v \text{ in probability}
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \frac{N_n}{n} = v \text{ in probability}.
\end{equation}
The condition (3.16) is satisfied, if

\[(3.23) \quad \sum_{m=1}^{n} E\left[\|D_m\| |\mathcal{F}_{m-1}\right] = O(n) \text{ in probability}
\]

and for any \( \epsilon > 0 \)

\[(3.24) \quad \sum_{m=1}^{n} E\left[\|D_m\| I\left\{\|D_m\| \geq \epsilon n\right\} |\mathcal{F}_{m-1}\right] = o(n) \text{ in probability.}
\]

**Remark 3.3.** If \( D_{q,k}(n) \geq -c_0 \) for some \( c_0 \), then (3.23) is implied by (3.15) and can be removed. The uniform integrability of \( \{D_n\} \) implies (3.24).

**Remark 3.4.** In Theorem 3.4, (3.17)-(3.19) give the results for the case of the reducible replacement, but we don’t know whether there is a \( v \) such that (3.20)-(3.22) hold in probability.

**4. Proofs.** We apply the stochastic approximation algorithm method as we did in Zhang (2016). Note that \( Y_{n+1} = Y_n + X_{n+1}D_{n+1}, \ E[X_{n+1}|\mathcal{F}_n] = \frac{Y_n^+}{\alpha(Y_n^+)} \) and \( E[D_{n+1}|\mathcal{F}_n] = H_{n+1} \). Recall \( \Delta M_{n,1} = X_n - E[X_n|\mathcal{F}_{n-1}] \) and \( \Delta M_{n,2} = X_n(D_n - E[D_n|\mathcal{F}_{n-1}]) \). We have

\[
Y_{n+1} = Y_n + \left(\frac{Y_n^+}{\alpha(Y_n^+)}\right)H + \Delta M_{n+1,1}H + \Delta M_{n+1,2} + X_{n+1} (H_{n+1} - H).
\]

Write \( \theta_n = \frac{Y_n^+}{n} \) and

\[
r_n = (\Delta M_{n,1}H + \Delta M_{n,2}) + X_{n+1} (H_n - H) + (\Delta Y_n^+ - \Delta Y_n),
\]

\[
s_n = \sum_{m=1}^{n} r_m = M_{n,1}H + M_{n,2} + \sum_{m=1}^{n} X_m (H_{m-1} - H) + (Y_n^+ - Y_n)
\]

and

\[(4.1) \quad h(\theta) = \theta \left( I_d - \frac{H}{\alpha(\theta)} \right).
\]

where \( \alpha(\theta) = \sum_{k=1}^{d} \theta_k \). Then \( \theta_n \) satisfies the stochastic approximation algorithm:

\[(4.2) \quad \theta_{n+1} = \theta_n - \frac{h(\theta_n)}{n+1} + \frac{r_{n+1}}{n+1}.
\]

Its related ordinary differential equation (ODE) is

\[(4.3) \quad \frac{d}{dt}\theta(t) = -h(\theta(t)).
\]

It is obvious that every point \( \lambda_H v \) in \( \lambda_H S_H \) is an equilibrium point of \( h(\theta) \), namely, \( h(\lambda_H v) = 0 \). When \( H \) is irreducible, Zhang (2016) show that \( \theta_n \) a.s. converges to the unique equilibrium point by applying the Kushner-Clark theorem [c.f. Kushner-Clark, 1978; Kushner and Yin, 2003; Duflo, 1997]. But now, it fails to use the Kushner-Clark theorem because, when \( H \) is reducible, for an equilibrium point \( \theta^* \), its any neighborhood is not a region of attraction for \( \theta^* \). However, the following lemma shows that if a solution of the ODE (4.3) has path bounded and bounded from zero, then it must be in \( \lambda_H S_H \).
LEMMA 4.1. Suppose Assumption 2.3 is satisfied. Let \( \Theta_0 = \{ \theta \geq 0 : \alpha(\theta) \leq C_0 < 0, \theta(t)u^t \geq c_0 > 0 \} \supset \lambda_HS_H. \) If a solution of the ODE (4.3) satisfies that \( \theta(t) \in \Theta_0 \) for all \( t \in (-\infty, \infty) \), then \( \theta(0) \in \lambda_HS_H \) and \( \theta(t) \equiv \theta(0) \).

Proof. Suppose \( \theta(t) \) is a solution of (4.3) with the whole path in \( \Theta_0 \). Let \( f(t) = \int_0^t \frac{1}{\alpha(\theta(s))} ds \). Then \( f(t) \to \infty \) and \( f(-t) \to -\infty \) as \( t \to \infty \), since \( \alpha(\theta(s)) \) is positive and bounded. From (4.3), it follows that

\[
\theta(t) = \theta(0) \exp \{-t + f(t)H\},
\]

and

\[
\frac{d}{dt} \theta(t)u^t = -\theta(t)u^t \left( 1 - \frac{\lambda_H}{\alpha(\theta(t))} \right),
\]

(4.5)

(4.6)

\[
\theta(t)u^t = \theta(0)u^t \exp \{-t + \lambda_H f(t)\}.
\]

Let \( H \) have the Jordan canonical form (2.9), and

\[
U = T \text{diag}(I_{\nu_1}, 0, \cdots, 0)T^{-1} = \sum_{j=1}^{\nu_1} u_j^tv_j; \quad \tilde{H} = H - \lambda_HU.
\]

Then by (4.4) and (4.6),

\[
\theta(t)U \frac{\theta(t)U}{\theta(t)u^t} = \frac{\theta(0)U}{\theta(0)u^t},
\]

(4.8)

\[
\frac{\theta(t)(I_d - U)}{\theta(t)u^t} = \frac{\theta(0)(I_d - U)}{\theta(0)u^t} \exp \{-f(t)(\lambda_H I_d - \tilde{H})\}.
\]

(4.9)

Note \( \theta(t) \in \Theta \) which implies the right hand of (4.9) is bounded, \( f(t) \to -\infty \) as \( t \to -\infty \), and that all eigenvalues of \( \lambda_H I_d - \tilde{H} \) have positive real parts. Letting \( t \to -\infty \) in (4.9) yields \( \theta(0)(I_d - U) = 0 \), and then \( \theta(t)(I_d - U) \equiv 0 \). Hence

\[
\frac{\theta(t)}{\theta(t)u^t} = \frac{\theta(t)U}{\theta(t)u^t} = \frac{\theta(0)U}{\theta(0)u^t} = \frac{\theta(0)}{\theta(0)u^t} \in (0, \infty).
\]

(4.10)

\[
\frac{\alpha(\theta(t))}{\theta(t)u^t} = \frac{\alpha(\theta(t)U)}{\theta(t)u^t} = \frac{\alpha(\theta(0)U)}{\theta(0)u^t} = \frac{\alpha(\theta(0))}{\theta(0)u^t} \in (0, \infty).
\]

(4.11)

Combing the above equalities with (4.5) yields

\[
\frac{d}{dt} \alpha(\theta(t)) = -\alpha(\theta(t)) \left( 1 - \frac{\lambda_H}{\alpha(\theta(t))} \right) = -\alpha(\theta(t)) + \lambda_H.
\]

So, \( e^t \alpha(\theta(t)) - \alpha(\theta(0)) = (e^t - 1)\lambda_H \). Since \( \alpha(\theta(t)) \) is bounded, letting \( t \to -\infty \) yields \( \alpha(\theta(0)) = \lambda_H \), and then \( \alpha(\theta(t)) \equiv \lambda_H \). Now, from (4.10) and (4.11) it follows that

\[
\theta(t) = \lambda_H \frac{\theta(t)U}{\alpha(\theta(t))} = \lambda_H \frac{\theta(0)U}{\alpha(\theta(0))} \in \lambda_HS_H. \quad \square
\]

Next, we show that the remainder term \( r_{n+1} \) in the stochastic approximation algorithm (4.2) can be neglected. To do so we need a lemma first.

LEMMA 4.2. (a) Under Conditions (3.23) and (3.24) in Theorem 3.4, (3.16) is satisfied.
(b) Under Conditions (3.15) and (3.16), for any $T > 0$,

\[
\max_{n \leq m \leq nT} \frac{\alpha(Y_m)}{m} \leq d \max_{k,q} |H_{k,q}| + o(1),
\]

\[
\min_{n \leq m \leq nT} \frac{Y_m u^t}{m} \geq \lambda_H \min_k u_k + o(1)
\]

and

\[
\max_{n \leq m \leq nT} \frac{\alpha(Y_m^-)}{m} \to 0
\]

in probability.

(c) Under Conditions (3.3)-(3.5) in Theorem 3.1, (3.2) is satisfied. Under (3.1) and (3.2), (4.12)-(4.14) holds almost surely.

(d) Under Conditions (3.11) and (3.12) in Theorem 3.2, (3.10) is satisfied.

**Proof.** (a) For (3.16), it is sufficient to show that

\[
\max_{m \leq n} \left\| \sum_{i=1}^{m} X_{i,q} (D_{q,k}(i) - H_{q,k}(i)) \right\| = o(n) \quad \text{in probability.}
\]

By (3.24), there is a sequence $0 < \epsilon_n \searrow 0$ such that

\[
\frac{1}{\epsilon_n n} \sum_{m=1}^{n} \text{E} \left[ \left\| D_m \right\| I\{ \left\| D_m \right\| \geq \epsilon_n n \} \right| \mathcal{F}_{m-1} \right] \to 0 \quad \text{in probability.}
\]

Denote $\xi_i^{(n)} = D_{q,k}(i) I\{ \left\| D_i \right\| < \epsilon_n n \}$. Then

\[
\sum_{i=1}^{n} P(\xi_i^{(n)} \neq D_{q,k}(i) | \mathcal{F}_{i-1}) \leq \sum_{i=1}^{n} P(\left\| D_i \right\| \geq \epsilon_n n | \mathcal{F}_{i-1})
\]

\[
\leq \frac{1}{\epsilon_n n} \sum_{i=1}^{n} \text{E} \left[ \left\| D_i \right\| I\{ \left\| D_i \right\| \geq \epsilon_n n \} \right| \mathcal{F}_{i-1} \right] \to 0 \quad \text{in probability,}
\]

by (4.16). So, by Lemma 2.5 of Hall and Heyde (1980),

\[
P \left( \xi_i^{(n)} \neq D_{q,k}(i) \text{ for some } i = 1, \ldots, n \right) \to 0.
\]

Also,

\[
\sum_{i=1}^{n} |H_{q,k}(n) - \text{E}[\xi_i^{(n)}] | \mathcal{F}_{i-1} | \]

\[
\leq \sum_{i=1}^{n} \text{E} \left[ \left\| D_i \right\| I\{ \left\| D_i \right\| \geq \epsilon_n n \} \right| \mathcal{F}_{i-1} \right] = o(n) \quad \text{in probability,}
\]

by (4.16). Hence, for (4.15) it is sufficient to show that

\[
\max_{m \leq n} \left| \sum_{i=1}^{m} X_{i,q} (\xi_i^{(n)} - \text{E}[\xi_i^{(n)}] | \mathcal{F}_{i-1} ) \right| = o(n) \quad \text{in probability.}
\]
Note that \( \sum_{i=1}^{m} X_{i,q}(\xi^{(n)}_i - E[\xi^{(n)}_i|\mathcal{F}_{i-1}]) \), \( m = 1, \ldots, n \), are martingales with
\[
\sum_{i=1}^{n} E \left[ X_{i,q}^2(\xi^{(n)}_i - E[\xi^{(n)}_i|\mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1} \right] \leq \sum_{i=1}^{n} \epsilon_n E \left[ |D_{q,k}(i)| | \mathcal{F}_{i-1} \right] = \epsilon_n O_P(n) = o(n) \quad \text{in probability},
\]
which implies (4.17). (3.16) is proved.

(b) By Lemma 2.1 (b), and the conditions (3.15) and (3.16),
\[
(4.18) \quad \max_{n \leq m \leq nT} \left\| \frac{Y_m}{m} - \frac{1}{m} \sum_{i=1}^{m} \frac{Y_{i-1}^+}{\alpha(Y_{i-1}^+)} \right\| \rightarrow 0 \quad \text{in probability}.
\]
Note that
\[
\alpha \left( \left| \frac{Y_{i-1}^+}{\alpha(Y_{i-1}^+)} H \right| \right) \leq d \max_{q,k} |H_{q,k}|
\]
and
\[
\frac{Y_{i-1}^+}{\alpha(Y_{i-1}^+)} H u^t = \frac{Y_{i-1}^+}{\alpha(Y_{i-1}^+)} \lambda_H u^t \geq \lambda_H \min_k u_k.
\]
So, (4.12) and (4.13) are proved.

Finally, we show (4.14). For \( m \) and \( k \), let \( l_m = \max\{l \leq m : Y_{l,k} \geq 0 \} \) be the largest integer for which \( Y_{l,k} \geq 0 \). Then for \( n \leq m \leq nT \),
\[
Y_{m,k} = Y_{l_m,k} + \sum_{i=l_m+1}^{m} \sum_{q=1}^{d} X_{i,q}D_{q,k}(i)
\]
\[
= Y_{l_m,k} + \sum_{i=l_m+1}^{n} \sum_{q=1}^{d} X_{i,q}[D_{q,k}(i) - H_{q,k}(i)]
\]
\[
+ \sum_{i=l_m+1}^{m} \sum_{q=1}^{d} X_{i,q}[H_{q,k}(i) - H_{q,k}] + \sum_{i=l_m+1}^{m} \sum_{q=1}^{d} X_{i,q}H_{q,k}.
\]
Note \( H_{q,k} \geq 0 \) for \( q \neq k \). It follows that
\[
(4.19) \quad Y_{m,k}^- \leq d \max_{i \leq m} \|M_{i,2}\| + d \sum_{i=l_m+1}^{m} \|H_i - H\| + \sum_{i=l_m+1}^{m} X_{i,k}|H_{k,k}|.
\]
For any given \( 0 < \delta < 1 \), on the event \{ \min_{\delta n \leq m \leq Tn} \frac{Y_m u^t}{m} > 0 \}, \( \min_{\delta n \leq m \leq Tn} \alpha(Y_m^+) > 0 \). So, if \( l_m + 1 \geq \delta n \), then \( X_{i,k} = 0 \) for \( i = l_m + 2, \ldots, m \), and if \( l_m + 1 < \delta n \), then \( X_{i,k} = 0 \) for \( \delta n + 1 \leq i \leq m \), because \( \frac{Y_{l_m+1,k}^+}{\alpha(Y_{l_m+1}^+)} = 0 \) for such \( i \). It follows that, on the event \{ \min_{\delta n \leq m \leq Tn} \frac{Y_m u^t}{m} > 0 \},
\[
\sum_{i=l_m+1}^{m} X_{i,k}|H_{k,k}| \leq \delta n|H_{k,k}|.
\]
By (4.13), \( P\left( \min \frac{Y^{-\infty}_n}{m} > \lambda_H \min_k u_k / 2 \right) \to 1. \) From (3.16) and (4.12) it follows that

\[
\max_{n \leq m \leq T_n} \frac{Y^{-\infty}_{m,k}}{m} \leq \delta + o(1) \quad \text{in probability.}
\]

The proof of (4.14) is completed.

(c) By the strong law of large numbers of martingales, the condition (3.5) implies

\[
\sum_{m=1}^{n} X_{m,k} \bar{D}_{q,k}(m) = o(n) \quad \text{a.s.}
\]

The conditions (3.3) and (3.4) imply

\[
\sum_{m=1}^{n} X_{m,k} |D_{q,k}(m) - \bar{D}_{q,k}(m)| = o(n) \quad \text{a.s.}
\]

(3.2) is proved. The reminder of the proof is similar to that of (b).

(d) It is sufficient to consider each \( X_{m,k}(D_{q,k}(m) - H_{q,k}(m)) \). Denote \( \bar{D}_{q,k}(m) = D_{q,k}(m) - \bar{D}_{q,k}(m) \). Then

\[
\sum_{m=1}^{\infty} \mathbb{E}\left[ \frac{|\bar{D}_{q,k}(m)|^2}{m} \right] \leq 2 \mathbb{E}\left[ \frac{|D_{q,k}(m)| |I\{|D_{q,k}| \geq c_m\}|}{m} \right] < \infty,
\]

by the condition (3.11), which implies that \( \sum_{m=1}^{\infty} \frac{|D_{q,k}(m)|}{m} < \infty \quad \text{a.s.} \), and then \( \sum_{m=1}^{\infty} \frac{|\bar{D}_{q,k}(m)|}{m} < \infty \quad \text{a.s.} \). For the martingale differences \( \xi_m =: X_{m,k} \bar{D}_{q,k}(m) \), by the condition (3.12) and the stop-time method, without loss of generality, we can assume that

\[
\sum_{m=1}^{\infty} \frac{(\log m)^{1+\epsilon} \mathbb{E}\left[ \frac{|\bar{D}_{q,k}(m)|^2}{m} \right]}{m^2} < M < \infty.
\]

Let \( S_m = \sum_{i=1}^{m} \xi_i \). Then \( \mathbb{E}[\max_{i \leq m} |S_i|^2] \leq 2 \sum_{i=1}^{m} \mathbb{E}[\bar{D}_{q,k}(m)] \). Hence

\[
\mathbb{E}\left[ \sum_{m=1}^{\infty} \frac{\max_{i \leq m} |S_i|^2}{m} \right]^2 \leq \mathbb{E}\left[ \sum_{m=1}^{\infty} \frac{1}{m(\log m)^{1+\epsilon}} \sum_{m=1}^{\infty} \frac{(\log m)^{1+\epsilon} \max_{i \leq m} |S_i|^2}{m^3} \right] \leq C \sum_{m=1}^{\infty} \frac{(\log m)^{1+\epsilon} \mathbb{E}[\max_{i \leq m} |S_i|^2]}{m^3} \leq C M.
\]

Hence

\[
\sum_{m=1}^{\infty} \frac{\max_{i \leq m} |S_i|}{m^2} < \infty \quad \text{a.s.}
\]

The proof is completed. \( \square \)
Now, recall that \( \theta_n = \frac{Y_n^+}{n} \) satisfies the stochastic approximation algorithm (4.2) with regression function defined as in (4.1). For the remainder \( r_{m+1} \), by Lemma 4.2, we have for any \( T \),

\[
\max_{n \leq m \leq nT} \frac{|s_m|}{m} \to 0,
\]

in probability under the assumptions in Theorem 3.4, and almost surely under the assumptions in Theorem 3.1 or 3.2, where \( s_n = \sum_{m=1}^n r_m \). We begin to prove the theorems. We first prove the weak convergence and then show the almost sure convergence for the non-homogeneity case. At last, we prove the theorems for the i.i.d. case.

**Proof of Theorem 3.4.** Note (4.20), and for any \( T \),

\[
\max_{n \leq m \leq nT} \alpha(\theta_n) \leq d \max_{k,q} |H_{k,q}| + o(1),
\]

(4.22)

\[
\min_{n \leq m \leq nT} \theta_m u^t \geq \lambda_H \min_k u_k + o(1)
\]

in probability by (4.12) and (4.13).

Setting \( t_0 = 0, t_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \), the ’interpolation function’ for a sequence \( \{u_n; n \geq 1\} \) in \( \mathbb{R}^d \) is the function from \( \mathbb{R}^+ \) to \( \mathbb{R}^d \) defined, by setting

\[
u_0 = 0, \quad \bar{u}(t) = \frac{u_n(t_n + 1 - t) + u_{n+1}(t - t_n)}{t_{n+1} - t_n} \quad \text{for} \quad t_n \leq t \leq t_{n+1},
\]

\[
u(t) = u_0 \quad \text{for} \quad t \leq t_0.
\]

Denote the function which interpolates \( \{\theta_n\} \) by \( \bar{\theta} \) and that which interpolates \( \{\sigma_n\} \) where \( \sigma_n = \sum_{k=1}^n \frac{r_k}{k} \) by \( \bar{\sigma} \). We also denote \( \bar{\theta}(t) = \theta_n \) if \( t_n \leq t < t_{n+1} \) and \( = u_0 \) if \( t < t_0, h(u_0) = 0 \). Then

\[
\theta(t) = \theta(0) + \int_0^t h(\bar{\theta}(s)) \, ds + \bar{\sigma}(t),
\]

(4.23)

\[
\bar{\theta}(t_n + t) = \bar{\theta}(t_n) + \int_0^t h(\bar{\theta}(t_n + s)) \, ds + \bar{\sigma}(t_n + t) - \bar{\sigma}(t_n).
\]

We will show that for any \( T > 0 \),

\[
\max_{t \in [-T,T]} \text{dist} \left( \bar{\theta}(t_n + t), \lambda_H S_H \right) \to \text{in probability}
\]

Let \( m(n, T) = \inf \{k : k \geq n, t_{k+1} - t_n \geq T\} \). Then \( n \leq m(n, T) \leq 2neT \). Note

\[
\sum_{k=n}^j \frac{r_{k+1}}{k+1} = \sum_{k=n+1}^j \frac{s_k}{k+1} \frac{1}{k} + \frac{s_{j+1}}{j+1} - \frac{s_n}{n+1}.
\]

It follows that, in probability,

\[
\max_{0 \leq t \leq T} \|\bar{\sigma}(t_n + t) - \bar{\sigma}(t_n)\| \leq \max_{n \leq j \leq m(n, T)} \left\| \sum_{k=n}^j \frac{r_{k+1}}{k+1} \right\| \\
\leq \max_{n \leq j \leq m(n, T)} \frac{\|s_j\|}{j} \sum_{k=n+1}^{m(n, T)} \frac{1}{k} + 2 \max_{n \leq j \leq m(n, T)+1} \frac{\|s_j\|}{j} \\
\leq (T + 3) \max_{n \leq j \leq 3ne^T} \frac{\|s_j\|}{j} \to 0,
\]

where
\[
\max_{0 \leq t \leq T} \alpha \left( \tilde{\theta}(t_n + t) \right) = \max_{n \leq j \leq m(n,T) + 1} \alpha (\theta_j) \\
\leq \max_{n \leq j \leq 3ne^T} \alpha (\theta_j) \leq d \max_{q,k} |H_{q,k}| + o(1),
\]

where \(\max_{n \leq j \leq m(n,T) + 1} \alpha (\theta_j)\) is the maximum of the sequence of functions \(\alpha (\theta_j)\). By (4.20)-(4.22), on the other hand, for any given \(T\), when \(t < T\), there is an \(n'\) and \(s_{n'}\) with \(t_n - T = t_{n'} + s_{n'}\), \(0 \leq s_{n'} \to 0\) and \(n' \to \infty\). Therefore, for \(t \geq -T\), we have \(\tilde{\theta}(t_n + t) = \tilde{\theta}(t_{n'} + s_{n'} + T + t) = \tilde{\theta}(t_{n'} + s_{n'} + T)\) and

\[
\tilde{\sigma}(t_n + t) - \tilde{\sigma}(t_n) = \tilde{\sigma}(t_{n'} + s_{n'} + T + t) - \tilde{\sigma}(t_{n'}) - [\tilde{\sigma}(t_{n'} + s_{n'} + T) - \tilde{\sigma}(t_{n'})].
\]

It follows that

\[
\max_{t \in [-T,T]} \| \tilde{\sigma}(t_n + t) - \tilde{\sigma}(t_n) \| \to 0,
\]

\[
\max_{t \in [-T,T]} \| \tilde{\theta}(t_n + t) - \tilde{\theta}(t_n) \| \to 0,
\]

\[
\max_{t \in [-T,T]} \alpha \left( \tilde{\theta}(t_n + t) \right) \leq d \max_{q,k} |H_{q,k}| + o(1),
\]

\[
\max_{t \in [-T,T]} \tilde{\theta}(t_n + t) \| u_t \| \geq \lambda_H \min_k u_k + o(1)
\]

in probability for any \(T > 0\). Write

\[
\tilde{\theta}(t_n + t) = \tilde{\theta}(t_n) + \int_0^t h(\tilde{\theta}(t_n + s)) ds + \sigma^{(n)}(t).
\]

Then by (4.26) and the continuity of \(h(\cdot)\),

\[
\max_{t \in [-T,T]} \| \sigma^{(n)}(t) \| \to 0
\]

in probability for any \(T > 0\).

Now, we are ready to prove (4.24). It is sufficient to show for any subsequence \(\{n'\}\) there is a further subsequence \(\{n_j\} \subset \{n'\}\) such that (4.24) holds for this subsequence \(\{n_j\}\) almost surely. We can choose the subsequence \(\{n_j\}\) such that (4.25)-(4.28) and (4.30) holds for this subsequence almost surely for any \(T > 0\). Except on a null event \(\Omega_0\), for each fixed \(\omega\), the real functions \(\tilde{\theta}(t_{n_j} + \cdot, \omega)\) and \(\tilde{\sigma}^{(n_j)}(\cdot, \omega)\) satisfy (4.27)-(4.30) for any \(T > 0\). Hence the sequence of functions \(\{\tilde{\theta}(t_{n_j} + \cdot, \omega)\}\) is bounded and equicontinuous on any bounded interval \([-T,T]\); therefore it relatively compact for the topology of uniform convergence on all compact subsets, and every limit point \(\tilde{\theta}(\cdot, \omega) = \tilde{\theta}(\cdot, \omega)\) in sense that

\[
\max_{t \in [-T,T]} \| \tilde{\theta}(t_{n_j} + t, \omega) - \tilde{\theta}(t, \omega) \| \to 0 \quad \text{for any} \ T > 0,
\]
will satisfy the ODE (4.3) with the path \( \theta(t) \) \(( -\infty < t < \infty )\) in \( \Theta = \{ \theta \geq 0 : \alpha(\theta) \leq 2d \max_{q,k} |H_{q,k}|, \theta u^t \geq \lambda_H \min_k u_k / 2 \} \). By Lemma 4.1, \( \theta(t) \equiv \theta(0) \in \lambda_H S_H \). Therefore
\[
\max_{t \in [-T,T]} \text{dist}\left( \tilde{\theta}(t), \lambda_H S_H \right) \to 0 \quad \text{for any } T > 0 \text{ and } \omega \notin \Omega_0.
\]
That is
\[
\max_{t \in [-T,T]} \text{dist}\left( \tilde{\theta}(t), \lambda_H S_H \right) \to 0 \quad \text{a.s. for any } T > 0,
\]
and (4.24) is proved.

Now, note for \( T > 1 \),
\[
\max_{n \leq m \leq nT} \text{dist}(\theta_m, \lambda_H S_H) \leq \max_{t \in [0,T]} \text{dist}(\theta(t), \lambda_H S_H).
\]
(3.17) is proved. (3.18) follows from (3.17) immediately. Finally, note
\[
\frac{N_n}{n} = \frac{1}{n} \sum_{m=1}^{n} \frac{Y_{m-1}^+}{\alpha(Y_{m-1}^+)} + \frac{M_{m,1}}{n}.
\]
So, for \( n \leq m \leq nT \),
\[
dist\left( \frac{N_m}{m}, S_H \right) \leq \frac{1}{m} \sum_{i=1}^{n} \text{dist}\left( \frac{Y_{i-1}^+}{\alpha(Y_{i-1}^+)} , S_H \right) + \frac{\|M_{m,1}\|}{m} \\
\leq \frac{1}{m} \sum_{i=1}^{\delta_n} 2 + \frac{1}{m} \sum_{i=\delta_n+1}^{n} \max_{\delta_n \leq i \leq nT} \text{dist}\left( \frac{Y_{i}^+}{\alpha(Y_{i}^+)} , S_H \right) + \frac{\|M_{m,1}\|}{m} \\
\leq 2\delta + \max_{\delta_n \leq i \leq nT} \text{dist}\left( \frac{Y_{i}^+}{\alpha(Y_{i}^+)} , S_H \right) + \max_{n \leq m \leq nT} \frac{\|M_{m,1}\|}{m}.
\]
(3.19) follows from (3.18) immediately. \( \square \)

**Proof of Theorem 3.1.** Now, by Lemma 4.2 (c) (4.20)-(4.30) holds almost surely for any \( T > 0 \). So, almost surely, every limit point \( \theta(\cdot) \) of the sequence \( \{ \tilde{\theta}(t, \cdot) \} \) will satisfy \( \theta(t) \in \lambda_H S_H \), which implies
\[
\lim_{n \to \infty} \text{dist}\left( \frac{Y_{n}^+}{n} , \lambda_H S_H \right) = \lim_{n \to \infty} \text{dist}\left( \tilde{\theta}(t_n) , \lambda_H S_H \right) = 0 \quad \text{a.s.}
\]
The proof is completed by noting \( \frac{Y_{n}^+}{n} \to 0 \) a.s. \( \square \)

**Proof of Theorem 3.2.** (3.2) remains true. (3.8) implies (3.1) by (3.10). By Theorem 3.1, (2.2) holds. Let \( U = \sum_j \mathbf{w}_j^t v_j \) be defined as in (4.7), then \( \frac{\theta U}{\alpha(\theta U)} \) is a project from \( \mathbb{R}^d \) to \( S_H \). So, by (2.2),
\[
\theta_n - \lambda_H \frac{\theta_n U}{\alpha(\theta_n U)} \to 0 \quad \text{a.s.}
\]
It is sufficient to show that
\[
\frac{\theta_n U}{\alpha(\theta_n U)} \text{ converges a.s.}
\]
By the stochastic approximation algorithm (4.2),
\begin{align}
(4.31) \quad \theta_{n+1} U - \theta_n U &= -\frac{(1 - \lambda_H / \alpha(\theta_n)) \theta_n U}{n+1} + \frac{r_{n+1} U}{n+1}.
\end{align}

Write \( x_n = \theta_n U \) and \( \delta_n = s_n U / n \). We can rewrite the above equality as
\begin{align*}
x_{n+1} - \delta_{n+1} - (x_n - \delta_n) &= -\frac{(1 - \lambda_H / \alpha(\theta_n)) x_n}{n+1} + \frac{\delta_n}{n+1}.
\end{align*}

It is obvious that \( \delta_n \to 0 \) a.s. Let \( f(x) = \frac{x}{\alpha(x)} \). Then
\[
\frac{\partial f(x)}{\partial x} = \frac{f(x)}{\alpha(x)} - \frac{1}{\alpha^2(x)} - \frac{x \partial f(x)}{\alpha^2(x)} \equiv 0.
\]

Note that \( f(x), \frac{\partial f(x)}{\partial x} \) and the derivations of \( \frac{\partial f(x)}{\partial x} \) are all bounded on a neighborhood of \( S_H \). Let \( x' = x_n - \delta_n \). It follows that
\begin{align}
f(x_{n+1} - \delta_{n+1}) - f(x_n - \delta_n) &= \left( -\frac{(1 - \lambda_H / \alpha(\theta_n)) x_n}{n+1} + \frac{\delta_n}{n+1} \right) \frac{\partial f(x)}{\partial x} \bigg|_{x=x_n-\delta_n} \\
&\quad + \frac{O(1)}{(n+1)^2} + \frac{O(\|\delta_n\|)^2}{(n+1)^2} \\
&= -\frac{(1 - \lambda_H / \alpha(\theta_n)) x_n}{n+1} \frac{\partial f(x)}{\partial x} \bigg|_{x=x_n} + \frac{O(1)}{(n+1)^2} + \frac{O(\|\delta_n\|)}{n+1} \\
&\quad -\frac{(1 - \lambda_H / \alpha(\theta_n)) x_n}{n+1} \left( \frac{\partial f(x)}{\partial x} \bigg|_{x=x_n-\delta_n} - \frac{\partial f(x)}{\partial x} \bigg|_{x=x_n} \right) \\
&= \frac{O(1)}{(n+1)^2} + \frac{O(\|\delta_n\|)}{n+1}.
\end{align}

(4.32)

We will prove that
\begin{align}
(4.33) \quad \sum_{m=1}^{\infty} \frac{\|\delta_m\|}{m+1} < \infty \text{ a.s.}
\end{align}

Then \( f(x_n - \delta_n) \) is convergent a.s., which implies that \( f(x_n) \) is convergent a.s. because \( \delta_n \to 0 \) a.s.

Note \( \liminf_{n \to \infty} Y_i^+ t_i / n > 0 \) a.s. Without loss of generality, we can assume that \( \alpha(Y_i^+) \geq c Y_i^+ t_i \neq 0 \) a.s. for all \( i \). By (4.19),
\[
Y^{-}_{m,k} \leq \max_{i \leq m} \|M_{i,2}\| + \sum_{i=1}^{m} \|H_i - H\| + X_{t_{m+1,i}} |H_{k,i}|.
\]

So,
\[
\|s_m\| \leq \max_{i \leq m} \|M_{i,2}\| + C \max_{i \leq m} \|M_{i,1}\| + C \sum_{i=1}^{m} \|H_i - H\| + C \text{ a.s.}
\]

Note
\[
\sum_{m=1}^{\infty} \sum_{i=1}^{m} \frac{\|H_i - H\|}{m(m+1)} = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \frac{\|H_i - H\|}{m(m+1)} \leq \sum_{i=1}^{\infty} \|H_i - H\| < \infty \text{ a.s.}
\]

by the assumption (3.8),
\[
\sum_{m=1}^{\infty} \max_{i \leq m} \|M_{i,1}\| \leq C \sum_{m=1}^{\infty} \sqrt{m \log \log m} \frac{m}{m(m+1)} < \infty \text{ a.s.}
\]
by Lemma 2.1 (b), and the fact (3.10). So, (4.33) is satisfied. □

**Proof of Theorem 3.3.** Under the assumptions in the theorem, \( \theta_n = \frac{Y_n}{n} \) satisfies the stochastic approximation algorithm (4.2) with

\[
\|s_n\| \leq C \max_{m \leq n} \|M_{n,1}\| + C \max_{m \leq n} \|M_{n,2}\| + C \sum_{m=1}^{n} \|H_m - H\| + C
\]

\[
= O\left( \sqrt{n \log \log n} \right) \text{ a.s.}
\]

By (4.32),

\[
\|f(x_n - \delta_n) - V\| \leq \sum_{m=n-1}^{\infty} \left( \frac{O(1)}{(m+1)^2} + \frac{O(\|\delta_m\|)}{m(m+1)} \right)
\]

\[
\leq \sum_{m=n-1}^{\infty} \frac{O(\sqrt{m \log m})}{m(m+1)} = O\left( n^{-1/2} \sqrt{\log \log n} \right) \text{ a.s.,}
\]

where \( x_n = \theta_n U \), \( \delta_n = s_n U / n \) and \( f(x) = \frac{x}{\alpha(x)} \). It follows that

\[
\|f(x_n) - V\| \leq \|f(x_n - \delta_n) - V\| + O(\|\delta_n\|)
\]

\[
= O\left( n^{-1/2} \sqrt{\log \log n} \right) \text{ a.s.}
\]

By the stochastic approximation algorithm (4.2),

\[
\theta_{n+1}(I_d - U) = \theta_n(I_d - U) \left( I_d - \frac{I_d - \tilde{H}/\alpha(\theta_n)}{n+1} \right) + \frac{r_{n+1}(I - U)}{n+1}.
\]

Let \( H_{j+1} = I_d - \tilde{H}/\alpha(\theta_j) \) and

\[
\Pi_m^n = \left( I_d - \frac{H_{m+1}}{m+1} \right) \cdots \left( I_d - \frac{H_n}{n} \right), \quad \tilde{\Pi}_m^n = \frac{n}{j=m+1} \left( I_d - \frac{I_d - \tilde{H}}{\lambda_H j} \right).
\]

Then \( H_j \to I_d - \tilde{H}/\lambda_H \) a.s. as \( j \to \infty \) because we have shown that \( \alpha(\theta_j) \to \lambda_H \alpha(V) = \lambda_H \) a.s. in Theorem 3.2. Note that the smallest real part of the eigenvalues of \( I_d - \tilde{H}/\lambda_H \) is \( 1 - \rho > 0 \). It follows that \( \|\Pi_m^n\| \leq C_\delta (n/m)^{\rho-1+\delta} \) for any \( \delta > 0 \) and \( \|\tilde{\Pi}_m^n\| \leq C_\delta (n/m)^{\rho-1}(\log n/m)^{\nu_{\rho,\delta}^{-1}} \) by Lemma B.1 of Zhang (2016). Similar to (2.17) and (2.22) of Zhang (2016), we have for \( \delta > 0 \) small enough,

\[
\theta_n - x_n = \theta_n(I_d - U) = \theta_0(I_d - U) \Pi_0^n + \sum_{m=1}^{n} \frac{r_m(I_d - U)}{n} \Pi_m^n
\]

\[
= \theta_0(I_d - U) \Pi_0^n + \frac{s_n(I_d - U)}{n} \Pi_n^n + \sum_{m=1}^{n-1} s_m(I_d - U) \frac{I_d - H_{m+1}}{m(m+1)} \Pi_{m+1}^n
\]

\[
= O\left( n^{\rho-1+\delta} \right) + O\left( \sqrt{n \log \log n} \right) + \sum_{m=1}^{n-1} O\left( \sqrt{m \log m (m+1)} \right) \left( \frac{n}{m} \right)^{\rho-1+\delta}
\]

\[
= \begin{cases} O\left( n^{\rho-1+\delta} \right) \text{ a.s.} & \text{if } \rho \geq 1/2, \\ O\left( n^{-1/2} \sqrt{\log \log n} \right) \text{ a.s.} & \text{if } \rho < 1/2 \end{cases} =: O\left( \tilde{\Pi}_n \right) \text{ a.s.}
\]

(4.36)
So,

\[(4.37) \quad f(\theta_n) - f(x_n) = O(||\theta_n - x_n||) = O(\bar{b}_n) \ a.s.\]

Combining (4.34) and (4.37) yields

\[
\frac{\theta_n}{\alpha(\theta_n)} - V = f(\theta_n) - f(x_n) + (f(x_n) - V) = O(\bar{b}_n) \ a.s.
\]

By the stochastic approximation algorithm (4.2) again,

\[
\theta_n - \lambda_H V = \frac{1}{n} \sum_{m=1}^{n} \left( \frac{\theta_{m-1}}{\alpha(\theta_{m-1})} - V \right) H + \frac{s_n}{n} \sum_{m=1}^{n} \bar{b}_m + O(n^{-1/2} \sqrt{\log \log n}) = O(\bar{b}_n) \ a.s.
\]

Hence

\[
\alpha(\theta_n) - \lambda_H = O(\bar{b}_n) \ a.s.
\]

Then \(H_{j+1} = I_d - \frac{\bar{H}}{\lambda_H} + O(\bar{b}_j) \ a.s.,\) and we can rewrite the stochastic approximation algorithm for \(\theta_n(I - U)\) as

\[
\theta_{n+1}(I - U) = \theta_n(I - U) \left( I_d - \frac{I_d - \frac{\bar{H}}{\lambda_H}}{n+1} \right) + \frac{r_{n+1}}{n+1},
\]

with \(r_{n+1} = r_{n+1}(I_d - U) + O(\bar{b}_n) \ a.s.\) Repeating the above arguments yields

\[
\theta_n - x_n = \theta_0(I_d - U) \bar{\Pi}_0^n + \frac{s_n}{n} \bar{\Pi}_n^n + \sum_{m=1}^{n-1} s_m^* \frac{\bar{H}}{\lambda_H} \bar{\Pi}_{m+1}^n + O(\bar{b}_n^2 + \sqrt{n \log \log n})
\]

\[
= O(n^{\rho - 1}(\log n)^{\nu_{sec} - 1}) + \frac{O(n^{\rho - 1}(\log n)^{\nu_{sec} - 1})}{n}
\]

\[
+ \sum_{m=1}^{n-1} O(m \bar{b}_m + \sqrt{m \log \log m}) \left( \frac{n}{m} \right)^{\rho - 1}(\log n/m)^{\nu_{sec} - 1}
\]

\[
= \begin{cases} 
O(n^{\rho - 1}(\log n)^{\nu_{sec} - 1}) & \text{a.s.} \quad \text{if } \rho > 1/2, \\
O(n^{-1/2}(\log n)^{\nu_{sec} - 1}(\log \log n)^{1/2}) & \text{a.s.} \quad \text{if } \rho = 1/2, \\
O(n^{-1/2}\sqrt{\log \log n}) & \text{a.s.} \quad \text{if } \rho < 1/2
\end{cases}
\]

\[= O(b_n) \ a.s.,\]

\[
\frac{\theta_n}{\alpha(\theta_n)} - V = f(\theta_n) - f(x_n) + (f(x_n) - V) = O(b_n) \ a.s.
\]

and

\[
\theta_n - \lambda_H V = O(b_n) \ a.s.
\]

Note \(||Y_n^-|| = O(\sqrt{n \log \log n}) \ a.s.\) and

\[
\frac{N_n}{n} - V = \frac{1}{n} \sum_{m=1}^{n} \left( \frac{\theta_{m-1}}{\alpha(\theta_{m-1})} - V \right) + M_{n,1}.
\]
The proof is completed. □

The theorems for non-homogeneity case have been proved. Finally, we show the main results in Section 2.

**Proof of Theorem 2.1.** Note that \( \{D_n\} \) are i.i.d. and (3.1) is obvious since \( H_n = H \). Also, \( \mathbb{E}[|D_{k,q}(m)|] < \infty \) implies (3.3)-(3.5) with \( c_m = m \). Theorem 2.1 follows from Theorem 3.1 immediately.

**Proof of Theorem 2.3.** In Remark 3.2, we have shown that \( \mathbb{E}[|D_n| \log(|D_n|)] < \infty \) implies (3.11) and (3.12) in Theorem 3.2 with \( c_m = m/(\log m)^{1+\epsilon} \). So, (2.10)-(2.12) hold.

Now, assume \( D_{k,q}(m) \geq 0 \) for all \( q, k \) and \( n \). The goal is to show that \( \varpi_j > 0 \). Note that \( u_j^t \) is a right eigenvector of \( H \) corresponding to \( \lambda_H \). Then

\[
\mathbb{E}[|Y_{n+1}u_j^t|] = Y_n u_j^t \left(1 + \frac{\lambda_H}{\alpha(Y_n)}\right) \leq Y_n u_j^t \exp \left\{ \frac{\lambda_H}{\alpha(Y_n)} \right\}.
\]

It follows that \( Y_n u_j^t \exp \{-\lambda_H q_{n-1}\} \) is a nonnegative supermartingale, where \( q_n = \sum_{i=0}^{n-1} \frac{1}{\alpha(Y_j)} \), and so it converges to a nonnegative random variable, say \( \varphi_j \), almost surely. On the other hand, note

\[
\frac{Y_n u_j^t}{Y_{n+1}u_j^t} = 1 - \frac{X_{n+1}D_{n+1}u_j^t}{Y_n u_j^t} + \frac{(X_{n+1}D_{n+1}u_j^t)^2}{Y_n u_j^t + X_{n+1}D_{n+1}u_j^t}.
\]

It follows that

\[
\mathbb{E} \left[ \frac{Y_n u_j^t}{Y_{n+1}u_j^t} \mid \mathcal{F}_n \right] = 1 - \frac{\lambda_H}{\alpha(Y_n)} + \frac{1}{Y_n u_j^t} \sum_{k=1}^{d} \frac{Y_n u_j^t}{\alpha(Y_n)} \mathbb{E} \left[ \frac{(d_k^{(k)} u_j^t)^2}{Y_n u_j^t + d_k^{(k)} u_j^t} \mid \mathcal{F}_n \right] \]

where \( d_k^{(k)} \) is the \( k \)-th row of \( D_n \). Since \( \mathbb{E}[D_n u_j^t] = \lambda_H u_j^t \), if the \( k \)-th element \( u_j^{(k)} \) of \( u_j \) is zero, then \( \mathbb{E}[d_k^{(k)} u_j^t] = u_j^{(k)} = 0 \), and then \( d_k^{(k)} u_j^t \) must be zero because it is a nonnegative random variable. So, the summation over \( k \) is taken over those \( k \) for which the \( k \)-th element of \( u_j \) is not zero. Then \( \sum_k Y_{n,k} \leq c Y_n u_j^t \) where \( c = \max \{1/\mu_j^{(k)} : \mu_j^{(k)} \neq 0, k = 1, \cdots, d\} \). It follows that

\[
\mathbb{E} \left[ \frac{Y_n u_j^t}{Y_{n+1}u_j^t} \mid \mathcal{F}_n \right] \leq 1 - \frac{\lambda_H}{\alpha(Y_n)} + c \frac{1}{\alpha(Y_n)} \max_k \mathbb{E} \left[ \frac{(d_k^{(k)} u_j^t)^2}{Y_n u_j^t + d_k^{(k)} u_j^t} \mid \mathcal{F}_n \right] \]

\[
\leq \exp \left\{ -\frac{\lambda_H}{\alpha(Y_n)} + c \frac{1}{\alpha(Y_n)} \max_k f_k(Y_n u_j^t, u_j) \right\},
\]

where \( f_k(l, y) = \mathbb{E} \left[ \frac{(d_k^{(k)} y_j)^2}{l + d_k^{(k)} y_j} \right] \). It follows that

\[
\frac{1}{Y_n} \exp \left\{ \lambda_H q_{n-1} - \sum_{m=0}^{n-1} \frac{c \max_k f_k(Y_m u_j^t, u_j)}{\alpha(Y_m)} \right\}
\]

is a nonnegative supermartingale, and so it also converges to a nonnegative random variable almost surely. Next, we will show that \( \sum_{m=0}^{\infty} \frac{f_k(Y_m u_j^t, u_j)}{\alpha(Y_m)} < \infty \) a.s. Then both \( \frac{1}{Y_n} \exp \{\lambda_H q_{n-1}\} \) and \( Y_n u_j^t \exp \{-\lambda_H q_{n-1}\} \) converge to nonnegative random variables. Hence \( P(\varphi_j > 0) = 1 \).
First, note
\[
\sum_{m=2}^{\infty} P(X_{m+1,k} = 1|\mathcal{F}_m) \geq \sum_{m=2}^{\infty} \frac{Y_{0,k}}{\alpha(Y_{m-1})} \geq c \sum_{m=1}^{\infty} \frac{1}{m} = +\infty.
\]
So, \( P(X_{m,k} = 1 \ i.o.) = 1 \), which implies \( N_{n,k} = \sum_{m=1}^{n} X_{m,k} \to \infty \) a.s. If the \( k \)-th element \( u_j^{(k)} \) of \( u_j \) is not zero, then \( E[d_1^k u_j^k] = \lambda_H u_j^{(k)} > 0 \). Hence, \( Y_n u_j^k \geq \sum_{m=1}^{n} X_{m,k} d_m^{(k)} u_j^k \sim N_{n,k} E[d_1^k u_j^k] \to \infty \) a.s.

Next, from \( Y_n u_j^k \to \infty \) a.s., we conclude that \( f_k(Y_n u_j^k, u_j) \to 0 \) a.s. and then
\[
c \exp\{-\lambda_H (1-\delta) q_{n-1}\} \leq Y_n u_j^k \leq C \exp\{-\lambda_H q_{n-1}\}.
\]
Taking the summation over \( j \) and noting that \( \left( \sum_{j=1}^{n} Y_n u_j^k \right)/n \to \lambda_H \) a.s., yields that
\[
c \exp\{-\lambda_H (1-\delta) q_{n-1}\} \leq n \leq C \exp\{-\lambda_H q_{n-1}\} \text{ a.s.}
\]
Finally, by noting \( f_k(Y_n u_j^k, u_j) \leq \frac{n^{1/2}}{Y_n u_j^k} + \max_k E[d_1^k u_j^k I\{d_1^k u_j^k \geq n^{1/2}\}] \) and \( \frac{\alpha(Y_n)}{n} \to \lambda_H \) a.s., we have
\[
\sum_{m} \frac{1}{\alpha(Y_m)} f(Y_n u_j^k, u_j) \leq C \sum_{m} \frac{m^{1/2}}{m^{1-\delta}} + C \sum_{m} \max_k E[d_1^k u_j^k I\{d_1^k u_j^k \geq m^{1/2}\}] \leq C + C \max_k E[(d_1^k u_j) \log(d_1^k u_j)] < \infty.
\]
So, \( Y_n u_j^k \exp\{-\lambda_H q_{n-1}\} \to \varphi_j > 0 \) a.s., which implies that
\[
\frac{Y_n u_j^k}{n} = \frac{\sum_{i=1}^{\nu_i} Y_n u_i^k}{\sum_{i=1}^{\nu_i} u_i^k} \to \frac{\varphi_j}{\sum_{i=1}^{\nu_i} \varphi_i} \quad \text{a.s.}
\]
We conclude that
\[
\varpi_j = V u_j^k = \frac{\varphi_j}{\sum_{i=1}^{\nu_i} \varphi_i} > 0 \quad \text{a.s.,} \quad j = 1, \ldots, \nu_1.
\]
For showing that \( \varphi_j \) has no point probability mass in \([0, 1]\) we apply the conditional central limit theorem. Now we have a more condition that \( E[\|D_n\|^2] < \infty \). Consider (4.31) again, where \( r_{n+1} = \varpi \Delta M_{n+1,1} H + \varpi \Delta M_{n+1,2} \) since \( H_n = H \) and \( Y_n \geq Y_0 > 0 \). Similar to (4.32) for \( x_n = \varpi_n U \) and \( f(x) = x/\alpha(x) \), we have
\[
f(x_{n+1}) - f(x_n) = \left(-\frac{1-\lambda_H/\alpha(\theta_n)}{n+1} x_n + \frac{r_{n+1} U}{n+1}\right) \frac{\partial f(x)}{\partial x} \bigg|_{x=x_n}
\]
\[
+ \frac{O(1)}{(n+1)^2} + \frac{O(\|r_{n+1}\|^2)}{(n+1)^2}
\]
\[
- r_{n+1} U \frac{\partial f(x)}{\partial x} \bigg|_{x=x_n} + \frac{O(1)}{(n+1)^2} + \frac{O(\|D_{n+1}\|^2)}{(n+1)^2}.
\]
It is easily shown that \( \sqrt{n} \sum_{m=n}^{\infty} \frac{\|D_m\|^2}{m^2} = o(1) \) a.s. Write \( \eta_n = r_n U \frac{\partial f(x)}{\partial x} \bigg|_{x=x_{n-1}} \). Then \( \{\eta_n\} \) is a sequence of martingale differences with
\[
E[\eta_{n+1}^k | \mathcal{F}_n]
\]
Further, the conditional Lindeberge condition is satisfied, i.e., for any $\epsilon > 0$,

$$\sum_{m=n}^{\infty} E \left[ n \| \eta_m \|^2 \right] \leq e \sum_{m=n}^{\infty} \frac{n \| \eta_m \|^2}{m^2} E \left[ \| D_1 \|^2 I \{ \| D_1 \| \geq \sqrt{\epsilon} \} \right] \to 0 \text{ a.s.}$$

With the above results, we can show that given $\mathcal{F}_n$, the conditional distribution of $\sqrt{n} \sum_{m=1}^{\infty} \frac{u_m}{m}$ will almost surely converge to a multi-normal distribution $N(0, \Sigma(\omega))$. It follows that

$$E \left[ e^{it\sqrt{n}(V-f(x_n))u_j} \right] \to e^{-\frac{t^2}{2} \sigma_j^2} \text{ a.s.,}$$

where $\sigma_j^2 = U_j \Sigma U_j$. For any $0 < p < 1$, let $E = \{ \varphi_j = p \} = \{ V u_j = p \}$, $I_n = E[I_E|\mathcal{F}_n]$. Then $I_n \to I_E$ a.s., $E[I_n - I_E|\mathcal{F}_n] \to 0$ in $L_1$, and so

$$\lim_n E \left[ e^{it\sqrt{n}(V-f(x_n))u_j} I_n \right] = \lim_n E \left[ e^{it\sqrt{n}(V-f(x_n))u_j} I_n |\mathcal{F}_n \right] = \lim_n E \left[ e^{it\sqrt{n}(V-f(x_n))u_j} I_n |\mathcal{F}_n \right] = e^{-\frac{t^2}{2} \sigma_j^2} I_n \text{ in } L_1.$$

Note on the event $E$, $(V-f(x_n))u_j = p - f(x_n)u_j$ is $\mathcal{F}_n$ measurable. We conclude

$$I_E = \lim_n E[I_E|\mathcal{F}_n] = \lim_n E \left[ e^{it\sqrt{n}(V-f(x_n))u_j} I_E |\mathcal{F}_n \right] = e^{-\frac{t^2}{2} \sigma_j^2} I_E \text{ in } L_1.$$ 

So, $I_E = e^{-\frac{t^2}{2} \sigma_j^2} I_E$ a.s. Next, it is sufficient to prove that on the event $E$, $\sigma_j^2 > 0$, which implies $I_E = 0$, and so $P(\varphi_j = p) = P(V u_j = p) = 0$. We denote $\alpha_i = u_i \text{diag}(\omega_i)u_i$. Then $\alpha_i > 0$, $U^t \text{diag}(V) U = \sum_{i=1}^{n} \alpha_i \omega_i \varphi_i v_i$. Here, we use the fact that $\text{diag}(\omega_i)u_j = 0$ for $i \neq j$ due to $\omega_i \varphi_j = 0$. So,

$$\sigma_j^2 \geq U_j \Sigma U_j = 2 \omega_j (1 - \omega_j)^2 \alpha_j + 2 \omega_j \sum_{i \neq j} \omega_i \alpha_i > 0 \text{ when } 0 < \omega_j < 1.$$ 

Because we have shown that $\omega_j > 0$ a.s., the proof is now completed. $\square$
Proof of Theorem 2.4. Note that (3.13) and (3.14) are satisfied. Theorem 2.4 follows from Theorem 3.3 immediately. □

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