Nonlinear diffusion equations as asymptotic limits of Cahn-Hilliard systems on unbounded domains via Cauchy’s criterion

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Communicated by: P. Colli

MSC Classification: Primary: 35K59, 35K35; Secondary: 47H05

1 INTRODUCTION AND RESULTS

Nonlinear diffusion equations have been studied since a long time ago. In particular, the problems on bounded domains and \( \mathbb{R}^N \) have been often considered for the equations. This paper will focus on the case of unbounded domains.

In the case that \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), the nonlinear diffusion equation

\[
\frac{\partial u}{\partial t} - \Delta \beta(u) = g \quad \text{in} \quad \Omega \times (0, T),
\]  

\( (E) \)

is studied by many mathematicians, where \( \beta : \mathbb{R} \to \mathbb{R} \) is a maximal monotone function and \( T > 0 \). Recently, Colli and Fukao\(^1\)\(^2\) considered the Cahn-Hilliard type of approximate equation...
\[
\frac{\partial u_\epsilon}{\partial t} - \Delta (-\varepsilon \Delta u_\epsilon + \beta(u_\epsilon) + \pi_\epsilon(u_\epsilon)) = g \quad \text{in} \quad \Omega \times (0, T), \\
\]

where \( \pi_\epsilon \) is an antimonotone function, which goes to 0 in some sense as \( \varepsilon \searrow 0 \), and used one more approximation

\[
\frac{\partial u_{\epsilon, \lambda}}{\partial t} - \Delta \left( \lambda \frac{\partial u_{\epsilon, \lambda}}{\partial t} - \varepsilon \Delta u_{\epsilon, \lambda} + \beta_\lambda(u_{\epsilon, \lambda}) + \pi_\epsilon(u_{\epsilon, \lambda}) \right) = g \quad \text{in} \quad \Omega \times (0, T), \\
\]

where \( \beta_\lambda (\lambda > 0) \) is the Yosida approximation of \( \beta \). In Colli and Fukao,\(^1\) they proved existence of solutions to \((E)_{\epsilon, \lambda}\) by the compactness method for doubly nonlinear evolution inclusions (see, e.g., Colli and Visintin\(^2\)):

\[
Au'(t) + \partial \psi(u(t)) \ni k(t)
\]

with some bounded monotone operator \( A \) and subdifferential operator \( \partial \psi \) of a proper lower semicontinuous convex function \( \psi \). Next, in Colli and Fukao,\(^3\) they proved existence of solutions to \((E)\), and \((E)\) by passing to the limit in \((E)_{\epsilon, \lambda}\) as \( \lambda \searrow 0 \) and in \((E)\) as \( \epsilon \searrow 0 \) individually. It is known that existence of solutions to \((E)\) can be directly proved under a growth condition for \( \beta \) (see, e.g., Barbu\(^4\), p205). In Colli and Fukao\(^2\), section 6 they used the above approach that is based on the idea in Fukao\(^5\) to obtain existence and estimates of solutions for \((E)\) without the growth condition for \( \beta \). (The proof of existence of solutions to \((E)\) does not need this condition.) See also Fukao\(^6\) in the case of dynamic boundary conditions. A class of doubly nonlinear degenerate parabolic equations generalizing \((E)\) on bounded domains was studied by using maximal monotone operators in Damlaman\(^7\), Kenmochi\(^8\), Kubo and Lu\(^9\), and so on; see also Droniou, Eymard, and Talbot.\(^10\) Another approach to nonlinear diffusion equations via cross-diffusion systems was recently built by Murakawa,\(^11,12\) whose approach is versatile and easy to implement. In comparison with the Cahn-Hilliard approximation as in Colli and Fukao\(^2\) and Fukao,\(^6\) the methods by previous studies\(^7-12\) require the growth condition that \( \int_0^T \beta(s) \, ds \geq c |r|^2 - d \) for all \( r \in \mathbb{R} \) with some constants \( c, d > 0 \).

On the other hand, in the case that \( \Omega \) is an \textit{unbounded} domain in \( \mathbb{R}^N \), nonlinear diffusion equations are not so sufficiently studied from a viewpoint of the operator theory, whereas in the case that \( \Omega = \mathbb{R}^N \), the equations are studied by the method of real analysis (see, e.g., Daskalopoulos and Kenig\(^13\)). The case of unbounded domains would be important in both mathematics and physics. This paper is concerned the initial-boundary value problem for nonlinear diffusion equations

\[
\begin{cases}
\frac{\partial u}{\partial t} + (\Delta + 1) \beta(u) = g & \text{in} \quad \Omega \times (0, T), \\
\partial_t \beta(u) = 0 & \text{on} \quad \partial \Omega \times (0, T), \\
u(0) = u_0 & \text{in} \quad \Omega,
\end{cases}
\]

by passing to the limit in the following Cahn-Hilliard system as \( \epsilon \searrow 0 \):

\[
\begin{cases}
\frac{\partial u_\epsilon}{\partial t} + (\Delta + 1) \mu_\epsilon = 0 & \text{in} \quad \Omega \times (0, T), \\
\mu_\epsilon = \varepsilon (\Delta + 1) u_\epsilon + \beta(u_\epsilon) + \pi_\epsilon(u_\epsilon) - f & \text{in} \quad \Omega \times (0, T), \\
\partial_t \mu_\epsilon = \partial_t u_\epsilon = 0 & \text{on} \quad \partial \Omega \times (0, T), \\
u_\epsilon(0) = u_\epsilon & \text{in} \quad \Omega,
\end{cases}
\]

where \( \Omega \) is an \textit{unbounded} domain in \( \mathbb{R}^N \) \((N \in \mathbb{N})\) with smooth bounded boundary \( \partial \Omega \) (e.g., \( \Omega = \mathbb{R}^N \setminus B(0, R) \)), where \( B(0, R) \) is the open ball with center 0 and radius \( R > 0 \) or \( \Omega = \mathbb{R}^N \) or \( \Omega = \mathbb{R}^*_+, \quad T > 0, \) and \( \partial_t \) denotes differentiation with respect to the outward normal of \( \partial \Omega \), under the conditions \((C1)-(C4)\) given later. Note that the additional term \( \beta(u) \) on the left-hand side of the equation in \((P)\) is technically caused by unbounded settings (see Remark 1.2 below). Taking account of unbounded settings, we can deal with diffusion phenomena in wider settings, e.g., on exterior domains. In this context, there are 2 recent works\(^14,15\) that dealt with \((P)\) and \((P)\) on unbounded domains. In Kurima and Yokota,\(^14,15\) existence and estimates of solutions for \((P)\) could be directly proved by regarding \((P)\) as nonlinear evolution equations of the form

\[
u'(t) + \partial \phi(u(t)) = \ell(t) \quad \text{in} \quad (H^1(\Omega))^* \]

with a proper lower semicontinuous convex function \( \phi \) defined well and by applying monotonicity methods (Brézis\(^16\)), which are useful methods for unbounded domains. In Kurima and Yokota,\(^15\) the growth condition for \( \beta \) was imposed as
\[ \int_0^r \beta(s) ds \geq c|r|^2 \quad \text{for all } r \in \mathbb{R} \]  

(1)

with some constant \( c > 0 \), and \( \beta \) admits the example \( \beta(r) = |r|^{q-1}r + r \), where \( q > 0, q \neq 1 \). In Kurima and Yokota,\textsuperscript{14} the growth condition for \( \beta \) was assumed as follows:

\[ \int_0^r \beta(s) ds \geq c|r|^m \quad \text{for all } r \in \mathbb{R}, \]  

(2)

with some constant \( c > 0 \) and \( m > 1 \), and \( \beta \) includes the typical example

\[ \beta(r) = |r|^{q-1}r, \]

where \( q > 0 \) (\( q > 1 \): the porous media equation [see, eg, previous studies\textsuperscript{17-20}], \( 0 < q < 1 \): the fast diffusion equation [see, eg, other studies\textsuperscript{19,21,22}]). However, the examples in Kurima and Yokota\textsuperscript{14,15} exclude the Stefan problem (see, eg, literature\textsuperscript{5,7,23-26}):

\[ \beta(r) = \begin{cases} 
    k_r r & \text{if } r < 0, \\
    0 & \text{if } 0 \leq r \leq L, \\
    k_r (r - L) & \text{if } r > L,
\end{cases} \]

since this \( \beta \) does not satisfy (1) and (2). This is due to a direct approach to (P) in Kurima and Yokota.\textsuperscript{14,15}

The purpose of this paper is to remove the growth condition for \( \beta \) such as (1) and (2) completely and provide a new existence result for (P). To this end, we turn our eyes to the fact that (P) is solvable without such growth condition for \( \beta \) by the help of the approximation term \( \epsilon (\Delta + 1) u_\epsilon + \pi_\epsilon(u_\epsilon) \) and regard (P) as an asymptotic limit of (P)\( \epsilon \), as \( \epsilon \searrow 0 \). As a consequence, the Stefan problem can be included in examples of (P) even if \( \Omega \) is \textit{unbounded}. To describe the result, we introduce conditions, notations, and definitions. We will assume the following 4 conditions:

(C1) The following conditions (C1a) and (C1b) hold:

(C1a) \( \beta : \mathbb{R} \to \mathbb{R} \) is a single-valued maximal monotone function and

\[ \beta(r) = \hat{\beta}'(r) = \partial \hat{\beta}(r), \]

where \( \hat{\beta}' \) and \( \partial \hat{\beta} \), respectively, denote the differential and subdifferential of a differentiable convex function \( \hat{\beta} : \mathbb{R} \to [0, +\infty) \) satisfying \( \hat{\beta}(0) = 0 \). This entails \( \beta(0) = 0 \).

(C1b) For all \( z \in H^1(\Omega) \), if \( \hat{\beta}(z) \in L^1(\Omega) \), then \( \hat{\beta}(z) \in L^1_{\text{loc}}(\Omega) \). For all \( z \in H^1(\Omega) \) and all \( \psi \in C^\infty_0(\Omega) \), if \( \hat{\beta}(z) \in L^1(\Omega) \), then \( \hat{\beta}(z + \psi) \in L^1(\Omega) \).

(C2) \( g \in L^2(0, T; L^2(\Omega)) \). Then we fix a solution \( f \in L^2(0, T; H^2(\Omega)) \) of

\( \left\{ \begin{array}{l}
\nabla f(t) \cdot \nabla z dx + \int_\Omega f(t) z dx = \int_\Omega g(t) z dx \\
\partial_\tau f(t) = 0
\end{array} \right. \)

for a.a. \( t \in (0, T) \), that is,

\[ \int_\Omega \nabla f(t) \cdot \nabla z dx + \int_\Omega f(t) z dx = \int_\Omega g(t) z dx \quad \text{for all } z \in H^1(\Omega). \]

(C3) \( \pi_\epsilon : \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous function and \( \pi_\epsilon(0) = 0 \) for all \( \epsilon \in (0, 1] \). Moreover, there exist a constant \( c_1 > 0 \) and a strictly increasing continuous function \( \sigma : [0, 1] \to [0, 1] \) such that \( \sigma(0) = 0, \sigma(1) = 1, c_1 \sigma(\epsilon) < \epsilon \) and

\[ |\pi_\epsilon'|_{L^\infty(\mathbb{R})} \leq c_1 \sigma(\epsilon) \quad \text{for all } \epsilon \in (0, 1]. \]

Moreover, \( r \to \frac{r^2}{2} + \hat{\pi}_\epsilon(r) \) is convex for all \( \epsilon \in (0, 1] \), where \( \hat{\pi}_\epsilon(r) := \int_0^r \pi_\epsilon(s) ds \).

(C4) \( u_0 \in L^2(\Omega) \) and \( \hat{\beta}(u_0) \in L^1(\Omega) \). Also, \( u_\epsilon \in H^1(\Omega) \) fulfills \( \hat{\beta}(u_\epsilon) \in L^1(\Omega) \), \( |u_\epsilon|_{L^2(\Omega)}^2 \leq c_2 \), \( \int_\Omega \hat{\beta}(u_\epsilon) dx \leq c_2 \), \( \epsilon |\nabla u_\epsilon|_{L^2(\Omega)}^2 \leq c_2 \) for all \( \epsilon \in (0, 1] \), where \( c_2 > 0 \) is a constant independent of \( \epsilon \); in addition, \( u_\epsilon \rightharpoonup u_0 \) in \( L^2(\Omega) \) as \( \epsilon \searrow 0 \).
Remark 1.1. The condition (C1b) and the convexity of $r \mapsto \frac{\xi}{2} r^2 + \tilde{r}_\varepsilon(r)$ in the condition (C3) are useful in proving that $(\Delta + 1)\mu_\varepsilon$ in (P) can be represented by a subdifferential of some convex function when $\Omega$ is unbounded (see Kurima and Yokota, lemma 4.2). Also, in this paper, it is an essential assumption that $\beta$ is single valued. The multivalued case will be discussed in our future work. Moreover, the condition that $\partial \Omega$ is smooth and bounded or $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$ is assumed to use the elliptic regularity.

We put the spaces $H$, $V$, and $W$ as follows:

$$ H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \left\{ z \in H^2(\Omega) \mid \partial_z z = 0 \; \text{a.e. on} \; \partial \Omega \right\}. $$

Then $H$ and $V$ are Hilbert spaces with inner products $(u, v)_H := \int_\Omega u \overline{v} \, dx$ $(u, v) \in H \times \overline{H}$ and $(v_1, v_2)_V := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx$ $(v_1, v_2) \in V \times \overline{V}$, respectively, and with norms $|v|_H := (v, v)^{1/2}_H$ $(v) \in H$ and $|w|_V := (w, w)^{1/2}_V$ $(w) \in V$, respectively. The notation $V^*$ denotes the dual space of $V$ with duality pairing $(\cdot, \cdot)_{V^*, V}$. Moreover, we define a bijective mapping $F : V \to V^*$ and an inner product in $V^*$ as

$$ (Fv_1, v_2)_{V^*, V} := (v_1, v_2)_V \quad \text{for all} \quad v_1, v_2 \in V, \quad (3) $$

$$ (v^1_1, v^1_2)_{V^*} := (v^1_1, F^{-1}v_2^1)_{V^*, V} \quad \text{for all} \quad v_1^1, v_2^1 \in V^*, \quad (4) $$

note that $F : V \to V^*$ is well-defined by the Riesz representation theorem. Also, $V^*$ has the norm $|v^*|_{V^*} := (v^*, v^*)^{1/2}_{V^*}$ $(v^*) \in V^*)$. We remark that (C2) implies $F\mu(t) = g(t)$ for a.a. $t \in (0, T)$.

We define weak solutions of (P) as follows.

**Definition 1.1.** A pair $(u, \mu)$ with

$$ u \in H^1(0, T; V^*) \cap L^\infty(0, T; H), \quad \mu \in L^2(0, T; V) $$

is called a **weak solution** of (P) if $(u, \mu)$ satisfies

$$ \langle u'(t), z \rangle_{V^*, V} + (\mu(t), z)_V = 0 \quad \text{for all} \quad z \in V \quad \text{and a.a.} \quad t \in (0, T), \quad (5) $$

$$ \mu(t) = \beta(u(t)) - f(t) \quad \text{in} \quad V \quad \text{for a.a.} \quad t \in (0, T), \quad (6) $$

$$ u(0) = u_0 \quad \text{a.e. on} \quad \Omega. \quad (7) $$

Now, the main result reads as follows:

**Theorem 1.** Let $T > 0$. Assume (C1) to (C4). Then there exists a unique weak solution $(u, \mu)$ of (P), satisfying

$$ u \in H^1(0, T; V^*) \cap L^\infty(0, T; H), \quad \mu \in L^2(0, T; V) $$

and there exists a constant $M > 0$ such that

$$ |u(t)|_H^2 \leq M, \quad (8) $$

$$ \int_0^t |u'(s)|^2_{V^*} \, ds \leq M, \quad (9) $$

$$ \int_0^t |\mu(s)|^2_V \, ds \leq M, \quad (10) $$

$$ \int_0^t |\beta(u(s))|^2_V \, ds \leq M \quad (11) $$

for all $t \in [0, T]$. In (C4), assume further that

$$ |u_0 - u_1|^2_{V^*} \leq c_3 \varepsilon^{3/2} $$

for some constant $c_3 > 0$ and let $(u_\varepsilon, \mu_\varepsilon)$ be a weak solution of (P) for $\varepsilon \in (0, \varepsilon]$. Then there exists a constant $C^* > 0$ such that for all $\varepsilon \in (0, \varepsilon]$. 

\[ |u_\varepsilon - u|_{C([0,T];V^*)}^2 + 2 \int_0^T (\beta(u_\varepsilon(s)) - \beta(u(s)), u_\varepsilon(s) - u(s))_H ds \leq C^* \varepsilon^{1/2}. \quad (12) \]

**Remark 1.2.** The operator \(-\Delta + 1\) in (P) and (P)_\varepsilon corresponds to the Riesz isomorphism from V onto V^*. In the case of bounded domains, “+1” of the operator \(-\Delta + 1\) can be removed by virtue of the Poincaré-Wirtinger inequality (see, eg, previous works\(^{1,8,9}\)). However, since the domain \(\Omega\) is unbounded and the function \(\beta\) is nonlinear in this paper, then it would be difficult to remove “+1” of the operator \(-\Delta + 1\) in (P) and (P)_\varepsilon, which is an open question; note that the methods of previous works\(^{1,8,9}\) cannot be applied in this paper because \(|\Omega|\) appears in these methods, for example, the projection

\[ Pz = z - \frac{1}{|\Omega|} \int_\Omega z(x) dx, \quad z \in H \]

was effectively used. Thus, “+1” plays a major role mathematically and to remove “+1” is an open problem.

The strategy of the proof of Theorem 1 is as follows. The advantage of our approach from the Cahn-Hilliard system as in Colli and Fukao\(^7\) and Fukao\(^6\) is to obtain estimates, independent of \(\varepsilon > 0\), for solutions to (P)_\varepsilon without any growth condition for \(\beta\) (Lemma 3.1). The main part of this paper is to confirm Cauchy’s criterion for solutions of (P)_\varepsilon (Lemma 4.1) and to obtain existence and estimates of solutions for (P) without the growth condition for \(\beta\) by passing to the limit in the approximate problem (P)_\varepsilon as \(\varepsilon \searrow 0\).

The plan of this paper is as follows. Section 2 presents the porous media equation, the fast diffusion equation, and the Stefan problem as examples. Section 3 provides the result for (P)_\varepsilon. In Section 4, we verify Cauchy’s criterion of solutions to (P)_\varepsilon and prove Theorem 1.

### 2 | EXAMPLES

**Example 2.1.** (The porous media equation, the fast diffusion equation)

In (P) and (P)_\varepsilon, we consider

\[ \beta(r) = |r|^{q-1}r \quad (q > 0), \quad \pi_\varepsilon(r) = -\frac{\varepsilon}{2}r. \]

In the case that \(q > 1\), the above function \(\beta\) appears in the porous media equation (see, eg, previous studies\(^{17-20}\)). In the case that \(0 < q < 1\), \(\beta\) is the function in the fast diffusion equation (see, eg, other studies\(^{19,21,22}\)). Also, \(\pi_\varepsilon\) is the function appearing in the Cahn-Hilliard equations. In both examples, \(\beta\) and \(\pi_\varepsilon\) satisfy (C1) and (C3) and for \(\varepsilon > 0\), there exists \(u_{0\varepsilon}\) satisfying (C4) and the assumption of Theorem 1 (see Kurima and Yokota\(^{14}\), section 6).

**Example 2.2.** (The Stefan problem)

The Stefan problem mathematically describes the solid-liquid phase transition. The problem is described by (P) with

\[ \beta(r) = \begin{cases} k_s r & \text{if } r < 0, \\ 0 & \text{if } 0 \leq r \leq L, \\ k_l (r - L)^2 & \text{if } r > L, \end{cases} \]

\[ \pi_\varepsilon(r) = -\frac{\varepsilon}{2} r \]

for all \(r \in \mathbb{R}\), where \(k_s, k_l > 0\) stand for the heat conductivities on the solid and liquid regions, respectively; \(L > 0\) is the latent heat coefficient. In this model, \(u\) and \(\beta(u)\) represent the enthalpy and the temperature, respectively, (see, eg, literature\(^{5,7,23-26}\)). In this case, we can confirm that \(\beta\) and \(\pi_\varepsilon\) satisfy (C1) and (C4) as follows.

It follows from a direct computation that

\[ \beta(r) = \beta'(r) = \partial_\varepsilon \beta(r), \]

where

\[ \beta(r) := \begin{cases} \frac{k_s}{2} r^2 & \text{if } r < 0, \\ 0 & \text{if } 0 \leq r \leq L, \\ \frac{k_l}{2} (r - L)^2 & \text{if } r > L. \end{cases} \]
Let $z \in V = H^1(\Omega)$ and let $K \subset \Omega$ be compact. Then we have
\[
\int_K \hat{\beta}(z) \, dx = k_s \int_{|z| < \varepsilon} z \, dx + k_\varepsilon \int_{|z| > \varepsilon} (z - L) \, dx \leq (k_s + k_\varepsilon)|K|^{1/2} |z|_{L^2(\Omega)} < \infty,
\]
where $[z < 0] := \{ x \in \Omega | z(x) < 0 \}$ and $[z > L]$ etc. are defined in the same manner. Thus, $\beta(z) \in L^1_{0\varepsilon}(\Omega)$. Also, letting $z \in V$ and $\psi \in C^\infty_c(\Omega)$, we derive that
\[
\int_V \hat{\beta}(z + \psi) \, dx = \frac{k_s}{2} \int_{[z+\psi<0]} (z + \psi)^2 \, dx + \frac{k_\varepsilon}{2} \int_{[z+\psi>L]} (z + \psi - L)^2 \, dx
\leq \frac{k_s}{2} \int_{\Omega} (z + \psi)^2 \, dx + \frac{k_\varepsilon}{2} \int_{[z+\psi>L]} (z + \psi)^2 \, dx
\leq (k_s + k_\varepsilon)(|z|_{L^2(\Omega)}^2 + |\psi|_{L^2(\Omega)}^2)
\leq \infty,
\]
which implies $\hat{\beta}(z + \psi) \in L^1(\Omega)$. Hence, (C1) holds.

To verify (C4), we let $u_0 \in H = L^2(\Omega)$ with $\hat{\beta}(u_0) \in L^1(\Omega)$ and put
\[
A := -\Delta + 1 : D(A) := W \subset H \rightarrow H,
J_\varepsilon := (I + \varepsilon A)^{-1}, \quad \varepsilon > 0.
\]
Then there exists $u_{0\varepsilon} \in H^2(\Omega)$ such that
\[
\begin{cases}
    u_{0\varepsilon} + \varepsilon(-\Delta + 1)u_{0\varepsilon} = u_0 & \text{in } \Omega, \\
    \partial_\nu u_{0\varepsilon} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
that is,
\[
u_{0\varepsilon} = J_\varepsilon u_0.
\]
The properties of $J_\varepsilon$ yield that
\[
|u_{0\varepsilon}|_H = |J_\varepsilon u_0|_H \leq |u_0|_H,
\]
and hence,
\[
\int_{\Omega} \hat{\beta}(u_{0\varepsilon}) \, dx \leq \frac{1}{2} \max\{k_s, k_\varepsilon\} |u_{0\varepsilon}|_H^2 \leq \frac{1}{2} \max\{k_s, k_\varepsilon\} |u_0|_H^2,
\]
\[
\varepsilon|u_{0\varepsilon}|_{V^\varepsilon}^2 = (\varepsilon(-\Delta + 1)u_{0\varepsilon}, u_{0\varepsilon})_H = (u_0 - u_{0\varepsilon}, u_{0\varepsilon})_H \leq |u_0|_H^2.
\]
Thus, there exists $u_{0\varepsilon}$ satisfying (C4). Moreover, we observe that
\[
|u_{0\varepsilon} - u_0|_{V^\varepsilon} \leq \varepsilon^{1/2} |u_0|_H.
\]
Indeed, it follows from (13) that
\[
|u_{0\varepsilon} - u_0|_{V^\varepsilon}^2 = |\varepsilon(-\Delta + 1)u_{0\varepsilon}|_{V^\varepsilon}^2 = \varepsilon^2 |Fu_{0\varepsilon}|_{V^\varepsilon}^2 = \varepsilon^2 |u_{0\varepsilon}|_{V^\varepsilon}^2 \leq \varepsilon |u_0|^2.
\]
Finally, letting $g \in L^2(0, T; L^2(\Omega))$, we can see that (C2) is satisfied. Also, we can confirm (C3) in view of the definition of $\pi_\varepsilon$.
Therefore, (C1) to (C4) hold, and we can apply Theorem 1 for the above $\beta$ and $\pi_\varepsilon$.

3 | PRELIMINARIES

In this section, we introduce the definition of weak solutions to (P), and show the result for existence of weak solutions to (P), with uniform estimates in $\varepsilon$. 
**Definition 3.1.** Let $T > 0$. A pair $(u_\varepsilon, \mu_\varepsilon)$ with

$$u_\varepsilon \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W),$$

$$\mu_\varepsilon \in L^2(0, T; V)$$

is called a weak solution of $(P)$, if $(u_\varepsilon, \mu_\varepsilon)$ satisfies

$$\langle u'_\varepsilon(t), z \rangle_{V^*, V} + (\mu_\varepsilon(t), z)_V = 0 \quad \text{for all} \quad z \in V \quad \text{and a.a.} \quad t \in (0, T),$$

(14)

$$\mu_\varepsilon(t) = \varepsilon (-\Delta + 1)u_\varepsilon(t) + \beta(u_\varepsilon(t)) + \pi_\varepsilon(u_\varepsilon(t)) - f(t) \quad \text{in} \quad V \quad \text{for a.a.} \quad t \in (0, T),$$

(15)

$$u_\varepsilon(0) = u_{0\varepsilon} \quad \text{a.e. on} \quad \Omega.$$  

(16)

**Lemma 3.1.** Let $T > 0$. Assume (C1) to (C4). Then for every $\varepsilon \in (0, 1]$, there exists a unique weak solution $(u_\varepsilon, \mu_\varepsilon)$ of $(P)$, satisfying

$$u_\varepsilon \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad \mu_\varepsilon \in L^2(0, T; V)$$

and there exist constants $M > 0$ and $\bar{\varepsilon} \in (0, 1]$ such that

$$|u_\varepsilon(t)|_H^2 + \varepsilon \int_0^T |(-\Delta + 1)u_\varepsilon(s)|_H^2 ds \leq M,$$

(17)

$$\int_0^T |u'_\varepsilon(s)|_{V^*}^2 ds + \varepsilon |u_\varepsilon(t)|_V^2 \leq M,$$

(18)

$$\int_0^T |\mu_\varepsilon(s)|_V^2 ds \leq M,$$

(19)

$$\int_0^T |\beta(u_\varepsilon(s))|_H^2 ds \leq M$$

(20)

for all $t \in [0, T]$ and all $\varepsilon \in (0, \bar{\varepsilon}]$.

**Proof.** We can prove existence and estimates for $(P)$, by setting the proper lower semicontinuous convex function $\phi_\varepsilon : V^* \to \mathbb{R}$ as

$$\phi_\varepsilon(z) := \begin{cases} 
\frac{\varepsilon}{2} \int_\Omega \left( |z(x)|^2 + |\nabla z(x)|^2 \right) dx + \int_\Omega \beta(z(x)) dx + \int_\Omega \pi_\varepsilon(z(x)) dx \\
+\infty 
\end{cases}$$

if $z \in D(\phi_\varepsilon) := \{z \in V^* \mid \hat{\beta}(z) \in L^1(\Omega)\},$

otherwise

and by applying the monotonicity method for

$$\begin{align*}
&u'_\varepsilon(t) + \partial \phi_\varepsilon(u_\varepsilon(t)) = g(t) \quad \text{in} \quad V^* \quad \text{for a.a.} \quad t \in [0, T], \\
u_\varepsilon(0) = u_{0\varepsilon} \quad \text{in} \quad V^* 
\end{align*}$$

Now, we show that $\phi_\varepsilon$ is proper lower semicontinuous convex on $V^*$ in a similar way to Kurima and Yokota,\textsuperscript{14}, lemma 4.1. The above function $\phi_\varepsilon$ is convex since $\hat{\beta}$ is convex and $r \mapsto \frac{\varepsilon}{2} r^2 + \hat{\pi}_\varepsilon(r)$ is convex. Thus, it suffices to confirm that $[\phi_\varepsilon \leq \lambda] := \{z \in V^* \mid \phi_\varepsilon(z) \leq \lambda\}$ is closed in $V^*$ for each $\lambda \in \mathbb{R}$. Let $\{z_n\}$ be a sequence in $[\phi_\varepsilon \leq \lambda]$ such that $z_n \to z$ in $V^*$ as $n \to +\infty$. Then from the nonnegativity of $\beta$, we have

$$\lambda \geq \phi_\varepsilon(z_n) = \frac{\varepsilon}{2} |z_n|_V^2 + \int_\Omega \hat{\beta}(z_n) dx + \int_\Omega \hat{\pi}_\varepsilon(z_n) dx \geq \frac{\varepsilon}{2} |z_n|_V^2 + \int_\Omega \hat{\pi}_\varepsilon(z_n) dx.$$
Here, (C3) implies
\[ |\mathbf{\kappa}_e(r)| = \left| \int_0^r (\pi_x(s) - \pi_x(0)) \, ds \right| \leq \left| \pi_x' \right|_{L^\infty([0, \infty])} \int_0^r |s| \, ds \leq \frac{1}{2} c_1 \sigma(\varepsilon) |r|^2, \]
and hence,
\[ \left| \int_\Omega \mathbf{\kappa}_e(z_n) \, dx \right| \leq \frac{1}{2} c_1 \sigma(\varepsilon) |z_n|_V^2 \leq \frac{1}{2} c_1 \sigma(\varepsilon) |z|_V^2. \]
Thus, we obtain \( \lambda \geq \frac{1}{2}(\varepsilon - c_1 \sigma(\varepsilon)) |z|_V^2 \), and so there exists a subsequence \( \{ z_{n_k} \} \) of \( \{ z_n \} \) such that \( z_{n_k} \to z \) weakly in \( V \) as \( k \to +\infty \). Now, we set
\[
\Phi_e : V \ni w \mapsto \frac{\varepsilon}{2} \int_\Omega \left( |w|^2 + |\nabla w|^2 \right) \, dx + \int_\Omega \beta(w) \, dx + \int_\Omega \mathbf{\kappa}_e(w) \, dx \in \mathbb{R}.
\]
Then this function \( \Phi_e \) is proper l.s.c. convex on \( V \) because of the convexity and lower semicontinuity of \( r \mapsto \beta(r) \) and \( r \mapsto \frac{\varepsilon}{2} r^2 + \mathbf{\kappa}_e(r) \), and so \( \Phi_e \) is weakly l.s.c. on \( V \). Therefore, we derive
\[
\Phi_e(z) \leq \liminf_{k \to +\infty} \Phi_e(z_{n_k}) = \liminf_{k \to +\infty} \phi_e(z_{n_k}) \leq \lambda < +\infty.
\]
Hence, we see that \( z \in D(\phi_e) \) and \( \phi_e(z) = \Phi_e(z) \leq \lambda \). Thus, we can prove this lemma in the same way as in Kurima and Yokota\textsuperscript{14} section 4 (without growth conditions for \( \beta \)).

\[ \square \]

4 PROOF OF THEOREM 1

This section gives the proof of Theorem 1 by confirming that the solution of \((P)_e\) converges to a function as \( \varepsilon \searrow 0 \), which constructs the solution of \((P)\). The key is to show the following lemma that asserts Cauchy’s criterion for solutions of \((P)_e\).

**Lemma 4.1.** Let \( \bar{\varepsilon}, (u_\varepsilon, \mu_\varepsilon) \), and \( M \) be as in Lemma 3.1. Then we have
\[
|u_\varepsilon - u_\bar{\varepsilon}|^2_{L^2([0, T])} + 2 \int_0^T \left( \beta(u_\varepsilon(s)) - \beta(u_\bar{\varepsilon}(s)), u_\varepsilon(s) - u_\bar{\varepsilon}(s) \right)_H \, ds \leq |u_{0_e} - u_{0_{\bar{\varepsilon}}}|^2_V + 2M(\varepsilon^{1/2} + \gamma^{1/2}) + 2MT(\varepsilon^{1/2} + \gamma^{1/2} + 2c_1(\sigma(\varepsilon) + \sigma(\gamma)))
\]
for all \( \varepsilon, \gamma \in (0, \bar{\varepsilon}] \).

**Proof.** We have from (3), (4), and (14) that for all \( s \in (0, T) \),
\[
\frac{1}{2} \frac{d}{ds} |u_\varepsilon(s) - u_\bar{\varepsilon}(s)|^2_{V'} = \langle u'_\varepsilon(s) - u'_\bar{\varepsilon}(s), F^{-1}(u_\varepsilon(s) - u_\bar{\varepsilon}(s)) \rangle_{V', V'}
\]
\[
= -(F^{-1}(u_\varepsilon(s) - u_\bar{\varepsilon}(s)), \mu_\varepsilon(s) - \mu_\bar{\varepsilon}(s))_V
\]
\[
= -(u_\varepsilon(s) - u_\bar{\varepsilon}(s), \mu_\varepsilon(s) - \mu_\bar{\varepsilon}(s))_{V, V}
\]
\[
= -(u_\varepsilon(s) - u_\bar{\varepsilon}(s), \mu_\varepsilon(s) - \mu_\bar{\varepsilon}(s))_H.
\]
Here, (15) yields that for all \( s \in (0, T) \),
\[
-(u_\varepsilon(s) - u_\bar{\varepsilon}(s), \mu_\varepsilon(s) - \mu_\bar{\varepsilon}(s))_H
\]
\[
= (u_\varepsilon(s) - u_\bar{\varepsilon}(s), -\varepsilon(-\Delta + 1)u_\varepsilon(s) + \gamma(-\Delta + 1)u_\bar{\varepsilon}(s))_H
\]
\[
- (\beta(u_\varepsilon(s)) - \beta(u_\bar{\varepsilon}(s)), u_\varepsilon(s) - u_\bar{\varepsilon}(s))_H
\]
\[
+ (u_\varepsilon(s) - u_\bar{\varepsilon}(s), -\pi_\varepsilon(u_\varepsilon(s)) + \pi_\bar{\varepsilon}(u_\bar{\varepsilon}(s)))_H.
\]
Combination of (22) and (23) together with the Schwarz inequality gives that
for all \(s \in (0, T)\). Moreover, it follows from (17) and (C3) that
\[
|u_{\epsilon}(s)|_H \leq \sqrt{M},
\]
\[
|\pi_\epsilon(u_{\epsilon}(s))|_H \leq c_1\sigma(\epsilon)|u_{\epsilon}(s)|_H \leq c_1\sigma(\epsilon)\sqrt{M}
\]
for all \(s \in [0, T]\) and all \(\epsilon \in (0, \bar{\epsilon})\). Thus, we have from (24), (25), and (26) that
\[
\frac{1}{2} \frac{d}{ds} |u_{\epsilon}(s) - u_{\epsilon}(s)|^2_{V'} + (\beta(u_{\epsilon}(s)) - \beta(u_{\epsilon}(s)), u_{\epsilon}(s) - u_{\epsilon}(s))_H
\]
\[
\leq 2\sqrt{M(\epsilon|(-\Delta + 1)u_{\epsilon}(s)|_H + \gamma|(-\Delta + 1)u_{\epsilon}(s)|_H + c_1\sqrt{M}\sigma(\epsilon) + c_1\sqrt{M}\sigma(\gamma))}
\]
\[
\leq M(\epsilon^{1/2} + \epsilon^{1/2}) + \epsilon^{3/2}|(-\Delta + 1)u_{\epsilon}(s)|^2_H + \gamma^{3/2}|(-\Delta + 1)u_{\epsilon}(s)|^2_H
\]
\[
+ 2c_1M(\sigma(\epsilon) + \sigma(\gamma)),
\]
for all \(s \in (0, T)\). Hence, integrating this inequality, we conclude from (17) that (21) holds.

We are now in a position to complete the proof of Theorem 1.

**Proof of Theorem 1.** (Existence and uniqueness)
Lemma 4.1, the monotonicity of \(\beta\), (C3), (C4) imply that \(\{u_{\epsilon}\}_{\epsilon \in (0, \bar{\epsilon})}\) satisfies Cauchy’s criterion in \(C([0, T]; V^*)\), and hence, there exists a function \(u \in C([0, T]; V^*)\) such that
\[
u_{\epsilon} \to u \quad \text{in} \quad C([0, T]; V^*)
\]
as \(\epsilon \searrow 0\). We have from (27) and (C4) that
\[
\begin{align*}
\epsilon \to u_0 & \quad \text{in} \quad V^* \\
u(0) &= u_0 \quad \text{and} \quad \Omega \\
\end{align*}
\]
and, since \(u_0 \in H\), it holds that
\[
\begin{align*}
\epsilon(0) &= u_0 \quad \text{a.e. on} \quad \Omega. \\
\end{align*}
\]
The estimates (17) to (20) yield that there exist a subsequence \(\{\epsilon_k\}_{k \in \mathbb{N}}\), with \(\epsilon_k \searrow 0\) as \(k \to \infty\), and some functions \(v \in H^1(0, T; V^*) \cap L^\infty(0, T; H), \mu \in L^2(0, T; V)\) and \(\xi \in L^2(0, T; H)\) satisfying
\[
u_{\epsilon_k} \to v \quad \text{weakly* in} \quad H^1(0, T; V^*) \cap L^\infty(0, T; H),
\]
\[
\epsilon_k(-\Delta + 1)u_{\epsilon_k} \to 0 \quad \text{in} \quad L^2(0, T; H),
\]
\[
\mu_{\epsilon_k} \to \mu \quad \text{weakly in} \quad L^2(0, T; V),
\]
\[
\beta(u_{\epsilon_k}) \to \xi \quad \text{weakly in} \quad L^2(0, T; H)
\]
as \(k \to \infty\). Now, we will confirm that
\[
u = v \quad \text{a.e. on} \quad \Omega \times (0, T).
\]
Let \(\psi \in C^\infty_c(\Omega \times [0, T])\). Then we see from (27) that
\[
\int_0^T \int_\Omega (u_{\epsilon_k} - v)\psi dxdt \to \int_0^T \int_\Omega (u - v)\psi dxdt
\]
as \(k \to \infty\). On the other hand, from (29), we have
\[
\int_{0}^{T} \int_{\Omega} (u_{\varepsilon_{k}} - v) \psi \, dx \, dt = \int_{0}^{T} \langle u_{\varepsilon_{k}}(t) - v(t), \psi(t) \rangle_{V', V} \, dt \to 0
\]  
(35)

as \( k \to \infty \). Thus, it follows from (34) and (35) that

\[
\int_{0}^{T} \int_{\Omega} (u - v) \psi \, dx \, dt = 0
\]

for all \( \psi \in C^{\infty}_{c}(\Omega \times [0, T]) \). This implies that (33) holds. Consequently, we derive that \( u \in H^{1}(0, T; V') \cap L^{\infty}(0, T; H) \) and

\[
u_{\varepsilon_{k}} \to u \quad \text{weakly* in } H^{1}(0, T; V') \cap L^{\infty}(0, T; H)
\]

(36)
as \( k \to \infty \).

Next, we show that

\[
\pi_{\varepsilon_{k}}(u_{\varepsilon_{k}}) \to 0 \quad \text{in } L^{\infty}(0, T; H)
\]

(37)
as \( k \to \infty \). It follows from (C3) that

\[
|\pi_{\varepsilon_{k}}(u_{\varepsilon_{k}})| \leq c_{1} \sigma(\varepsilon_{k}) |u_{\varepsilon_{k}}| \quad \text{a.e. on } \Omega \times (0, T),
\]

and (17) enables us to see that (37) holds.

Moreover, we prove that

\[
\xi = \beta(u) \quad \text{a.e. on } \Omega \times (0, T).
\]

(38)

To this end, it suffices to confirm that

\[
limsup_{k \to +\infty} \int_{0}^{T} \langle \beta(u_{\varepsilon_{k}}(t)), u_{\varepsilon_{k}}(t) \rangle_{H} \, dt \leq \int_{0}^{T} \langle \xi(t), u(t) \rangle_{H} \, dt
\]

(39)

(see Barbu\textsuperscript{27}, lemma 2.3, p.38). We infer from (15), (27), (31), (36), and (37) that

\[
\int_{0}^{T} \langle \beta(u_{\varepsilon_{k}}(t)), u_{\varepsilon_{k}}(t) \rangle_{H} \, dt
\]

\[
= \int_{0}^{T} \langle \mu_{\varepsilon_{k}}(t) + f(t), u_{\varepsilon_{k}}(t) \rangle_{H} \, dt - \varepsilon_{k} \int_{0}^{T} |u_{\varepsilon_{k}}(t)|_{V'}^{2} \, dt
\]

\[
- \int_{0}^{T} \langle \pi_{\varepsilon_{k}}(u_{\varepsilon_{k}}(t)), u_{\varepsilon_{k}}(t) \rangle_{H} \, dt
\]

\[
\leq \int_{0}^{T} \langle u_{\varepsilon_{k}}(t), \mu_{\varepsilon_{k}}(t) + f(t) \rangle_{V', V} \, dt - \int_{0}^{T} \langle \pi_{\varepsilon_{k}}(u_{\varepsilon_{k}}(t)), u_{\varepsilon_{k}}(t) \rangle_{H} \, dt
\]

\[
- \int_{0}^{T} \langle u(t), \mu(t) + f(t) \rangle_{V', V} \, dt = \int_{0}^{T} \langle u(t), \mu(t) + f(t) \rangle_{H} \, dt
\]

(40)
as \( k \to \infty \). Here, from (15), (30)-(32), and (37), we have

\[
\mu = \xi - f \quad \text{a.e. on } \Omega \times (0, T).
\]

(41)

Thus, combination of (40) and (41) leads to (39), i.e., (38).

Next, we confirm that there exists a constant \( C_{1} > 0 \) such that

\[
\int_{0}^{t} |\beta(u(s))|_{V'}^{2} \, ds \leq C_{1}
\]

(42)

for all \( t \in [0, T] \). We derive from (19) and (31) that there exists a constant \( C_{2} > 0 \) such that

\[
\int_{0}^{t} |\mu(s)|_{V'}^{2} \, ds \leq C_{2}
\]

(43)
for all $t \in [0, T]$. Hence, noting that $\mu \in L^2(0, T; V)$ and (C2) implies $f \in L^2(0, T; V)$, we see from (38), (41), and (43) that $\beta(u) \in L^2(0, T; V)$ and

$$
\int_0^t |\beta(u(s))|^2_V \, ds = \int_0^t |\mu(s) + f(s)|^2_V \, ds \\
\leq 2 \int_0^t |\mu(s)|^2_V \, ds + 2 \int_0^t |f(s)|^2_V \, ds \\
\leq 2C_2 + 2\|f\|^2_{L^2(0, T; V)},
$$

which implies (42). Thus, from (14), (28), (31), (36), (38), and (41), we have shown that $(u, \mu)$ is a solution of (P). Moreover, by (17), (18), (36), (42), and (43), we obtain (8) to (11).

Finally, we check that the solution $(u, \mu)$ of the problem (P) is unique. Assume that $(u_1, \mu_1)$ and $(u_2, \mu_2)$ are the solutions of (P) with the same initial data. Then it follows from (3) to (6) that

$$(u'_1(t) - u'_2(t), F_z)_V + \langle F_z, \beta(u_1(t)) - \beta(u_2(t)) \rangle_V = 0$$

for all $z \in V$. Choosing $z = F^{-1}(u_1(t) - u_2(t)) \in V$, we derive from (7) that

$$\frac{1}{2} |u_1(t) - u_2(t)|^2_V + \int_0^t (\beta(u_1(s)) - \beta(u_2(s)), u_1(s) - u_2(s))_H \, ds = 0$$

for all $t \in [0, T]$. Then the second term on the left-hand side is nonnegative by virtue of the monotonicity of $\beta$, so that

$$|u_1(t) - u_2(t)|^2_V \leq 0$$

for all $t \in [0, T]$. Hence, it holds that $u_1 = u_2$. Furthermore, we infer from (6) that

$$\mu_1(t) = \beta(u_1(t)) - f(t) = \beta(u_2(t)) - f(t) = \mu_2(t)$$

for all $t \in [0, T]$. $\Box$

**Proof of Theorem 1.** (Error estimate)
The estimate (12) can be proved by the same argument as in the proof of Kurima and Yokota\textsuperscript{14}, theorem 1.3. $\Box$

**ACKNOWLEDGEMENT**
The authors would like to thank the anonymous referees for helpful comments and suggestions.

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**How to cite this article:** Fukao T, Kurima S, Yokota T. Nonlinear diffusion equations as asymptotic limits of Cahn-Hilliard systems on unbounded domains via Cauchy's criterion. *Math Meth Appl Sci*. 2018;41:2590–2601. https://doi.org/10.1002/mma.4760