Explicit Numerical Iterative Methods Applied to the Three-Parameter Infiltration Equation

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Abstract. Finding explicit solutions to the partial differential equations governing the infiltration phenomenon tend to be a challenging issue. On account of such complexity, researchers have proposed algebraic equations to model this phenomenon. This paper deals with the solution of the general three-parameter infiltration equation, which is an implicit equation corresponding to the interpolation between the Green-Ampt and Talsma-Parlange infiltration models. By means of the Householders method, the cumulative infiltration is obtained by a rapidly converging iterative procedure. The results are further manipulated in order to obtain the infiltration rate explicitly. The effect of variables involved on the solution is also discussed. A case study is included to demonstrate the use of the proposed algorithm. The formulas proposed provide easy-to-use equations which enable calculations on the fly while considering practical applications.

Keywords: three-parameter infiltration formula, Green-Ampt, Householder’s Methods, Talsma-Parlange.

1. Introduction

Originally proposed by Parlange et al. (1982), the three parameter infiltration model unifies two other well-known models, introduced by Green-Ampt (1911) and Talsma & Parlange (1972). Green-Ampt’s proposition (Green & Ampt, 1911) figures as one of the most explored infiltration models. Green-Ampt’s model is based on Darcy’s equation and a few other assumptions, namely: the existence of a constant hydraulic pressure head at soil’s surface; the hydraulic conductivity in such zone $K_t$ is equivalent to the saturated one $K_o$; and the formation of a precisely defined wetting front leading to the so-called piston movement (Zonta et al. 2010).

The integrated form of Green-Ampt infiltration rate equation is given by Mein & Farrel (1974):

$$ t_s = I_s - \ln(1 + I_s) $$

where $t_s$ = nondimensional time; and $I_s$ = nondimensional cumulative infiltration, given by (Mein & Farrel, 1974):

$$ t_s = \frac{K_o(t - t_p + t_i)}{(\psi_s - p_s)(\theta_o - \theta_s)} $$

$$ I_s = \frac{I}{(\psi_s - p_s)(\theta_o - \theta_s)} $$

wherein $I$ is the cumulative infiltration; $t_s$ = ponding time; $t_i$ = theoretical time necessary to infiltrate the same volume infiltrated until the ponding time, under saturation of the superflcial layer of the soil; $\psi_s$ = average wetting front matric potential; $p_s$ = average surface pressure head and $\theta_s$ = saturated water content; $\theta_o$ = initial water content.

On the other hand Talsma & Parlange (1972) equation for the nondimensional cumulative infiltration and nondimensional time is:

$$ (I_s - t_s - 1) \exp(I_s - t_s - 1) + \exp(-I_s - 1) = 0 $$

Equations 1 and 4 were unified by Parlange et al. (1982) as

$$ t_s = I_s + (1 - \alpha)^{-1} \ln \left[ \frac{\alpha}{1 - (1 - \alpha) \exp(-\alpha I_s)} \right] $$

where $\alpha$ = a transition parameter lying in the range $0 \leq \alpha \leq 1$. When $\alpha = 0$, the Green-Ampt equation is recovered from Eq. 5. On the other hand, when $\alpha = 1$, Talsma-Parlange’s model follows from Eq. 5.

It has been shown by Parlange et al. (2002) that both the Green-Ampt and Talsma-Parlange equations can be explicitly solved in terms of Lambert W-function. On the other hand, Rathie et al. (2012) proposed the explicit solution of Eq. 5 in terms of the H-function. When implementations of such special functions are not available, standard trial and error procedures are used to solve Eq. 5.

Practical situations require neither complex nor precisely defined exact solutions. They do demand accurate solutions. This paper presents approximate solutions to the three-parameter infiltration equation, by means of an iterative algorithm solution whose single iteration order of convergence can be chosen as high as necessary.

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2. Solution of the Three-Parameter Infiltration Equation Using Householder’s Method

Consider a function \( f(x) \) whose root \( r \) needs to be determined. By choosing an initial point, \( x_0 \), Householder (1970) proposed the general iteration root-finding recurrence formula:

\[
x_{n+1} = x_n + (a + 1) \frac{[1/ f(x_n)]^{(a)}}{[1/ f(x_n)]^{(a+1)}}
\]

(6)

in which \( a \) is a real number which governs the rate at which the iterative scheme tends to the root of the nonlinear equation \( f(x) = 0 \) and \((1/f)^{(a)}\) represents the \( a \)-th derivative of \( 1/f(x) \).

In fact, it can be shown that \( |x_{n+1} - r| \leq |x_n - r|^{a+1} \) for some positive real number \( K \), thus if \( x_n \) is sufficiently close to \( r \), the iteration scheme converges to the value of \( r \) with rate \( a + 2 \). Note that this method is a generalization of the Newton-Raphson method, which can be easily obtained by taking \( a = 0 \) in Eq. 6.

Consider that Eq. 3 can be rewritten as:

\[
f(I_p) = t_p - I_p + (1 - \alpha) \ln \left[ \frac{1 - (1 - \alpha) \exp(-\alpha I_p)}{\alpha} \right]
\]

(7)

The first and second derivatives of \( f(I_p) \) with respect to \( I_p \) are:

\[
\frac{\partial}{\partial I_p} f(I_p) = -1 + \frac{\alpha}{\exp(\alpha I_p) - (1 - \alpha)}
\]

(8)

\[
I_p = t_p - \frac{2[\exp(\alpha I_p)](\exp(\alpha I_p) - (1 - \alpha))}{\alpha \exp(\alpha I_p) + 2[1 - \exp(\alpha I_p)](1 - \alpha)^{-1} \ln \left[ \frac{1 - (1 - \alpha) \exp(-\alpha I_p)}{\alpha} \right]}
\]

(11)

It can be noticed that the solution is an iterative algorithm.

In order to compare Eq. 11 to the exact solution of Eq. 5, consider the absolute difference between both solutions. Such departure is shown in Fig. 1. The exact solution has been obtained by trial and error procedures.

Figure 1 reveals that Eq. 11 behaves similarly for different values of the interpolation value \( \alpha \). The latter figure also shows that the maximum departure happens at low values of the nondimensional time, which is fully justified by the “long term” initial guess for nondimensional cumulative infiltration.

It is worth noticing that one may find an optimal initial guess in order to diminish the absolute departure over the whole domain. Also, the results in Fig. 1 correspond to a single iteration of the algorithm. Thus, if more iterations where taken into account, the error would be mitigated. The methodology is, this way, verified to produce good estimations of the solutions.

3. Determination of Some of the Equation Parameters

In order to obtain the values of \( t_p \) and \( t_r \), one shall consider the following equation, obtained by means of implicit differentiation of Eq. 5 with respect to \( t_r \):

\[
I = \left( \psi_w - p_0 \right) \frac{\theta_s - \theta_w}{\alpha} \ln \left( 1 + \frac{\alpha K_s}{r - K_s} \right)
\]

(12)

in which \( r \) is the infiltration rate. If one wants to find the total infiltration until ponding time, \( I_p \), one shall make the substitution \( r = i_r \) in Eq. 12, thus

\[
I_p = \left( \psi_w - p_0 \right) \frac{\theta_s - \theta_w}{\alpha} \ln \left( 1 + \frac{\alpha K_s}{i_r - K_s} \right)
\]

(13)

where \( i_r \) is the rainfall intensity. It’s simple to get \( t_r \) from Eq. 13 since \( t_r = I/i_r \), which provides:
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\[ t_p = \frac{(\psi_u - p_f)(\theta_s - \theta_r)}{\alpha i_p} \ln \left(1 + \frac{\alpha K_s}{i_p - K_s}\right) \]  

(14)

One shall note that when \( \alpha \) tends to 0, the same relation obtained by Mein & Farrel (1974) is retrieved for a Green-Ampt case, which is:

\[ t_s = \frac{I_p}{K_s} + \frac{(1-\alpha)^{-1}(\psi_u - p_f)(\theta_s - \theta_r)}{K_s} \ln \left[\frac{\alpha}{1-(1-\alpha) \exp(\alpha I_p)} \right] \]  

(15)

On the other hand, in order to obtain \( \psi_u \), the following equation suggested in Cecílio et al. (2007) shall be used:

\[ \psi_u = \frac{2 + 3\lambda}{[K_s(\theta_s) - K_s(\theta_r)][3\lambda + 1]} \]  

(16)

where \( \psi_u \) is the air-entry value (mm); \( \lambda \) is the pore size distribution index (nondimensional) and \( K_s \) is the relative hydraulic conductivity (nondimensional), expressed as \( K(\theta)/K_s \). Based on Brooks & Corey (1964) model, the relative hydraulic conductivity is:

\[ K_s(\theta_i) = \left(\frac{\theta_i - \theta_r}{\theta_s - \theta_r}\right)^{1+\lambda} \]  

(17)

where, \( \theta_i \) is the water content whose relative hydraulic conductivity is sought at; \( \theta_s \) is the residual water content.

Some propositions related to choosing appropriate parameters which better represent the properties of the soil must be taken into account. Transition zone’s water content \( \theta_s \) can be used instead of the saturation water content \( \theta_s \), since \( \theta_s \) is the water content that most of the soil mass reaches while the infiltration process occurs. Also, the experiments available to obtain the value of \( K_s \) often give a considerable dispersion of the results, what can be minimized by choosing the steady infiltration rate, \( \dot{T}_{ie} \), as representative of the hydraulic conductivity of the soil profile (Cecílio et al., 2007).

4. Examples of the Application of the Proposed Method

The experimental data used in this study are given by Cecílio (2005). Three types of soil will be analyzed, namely: Red Clay Soil (PV), Red Laterite Soil (LV) and Red-Yellow Laterite Soil (LVA). The characteristics of these soils and experimental data obtained (Cecílio, 2005) are presented in Tables 1 and 2 and Figs. 2 and 3.

In order to provide a better characterization of the soils studied, each one is graphically analyzed. It can be verified by experimental data (Cecílio, 2005) that the relative hydraulic conductivity can be graphically related to the volumetric water content as seen on Fig. 2.

It’s clear from Fig. 2 that LV soil has a different behavior from LVA and PV soils, which is mainly related to their composition (LV has more sand in its composition while both LVA and PV have more clay), as seen on Table 1.

On the other hand, Fig. 3 shows graphically the relation between the relative hydraulic conductivity and the average wetting front matric potential based on Brooks and Corey model.

Figure 3 also shows that each soil has a unique behavior which mainly depends on the pore size distribution index. This dependence is obtained due to the adoption of Brooks and Corey model for the prediction of the wetting front matric potential.

Based on the data presented by Cecílio (2005) and considering Eq. 11 for predicting the temporal evolution of the cumulative infiltration, the results obtained for the modeling of each soil are presented in Fig. 4.

It can be seen that Eq. 11 provides good predictions of the infiltration behavior. For all the soils analyzed, crescent, however discontinuous, functions have been adjusted.

Table 1 - Grain-size distribution of the soils considered.

| Soil | Coarse sand (%) | Fine sand (%) | Silt (%) | Clay (%) |
|------|----------------|---------------|---------|---------|
| LVA  | 13             | 10            | 7       | 70      |
| LV   | 26             | 52            | 2       | 20      |
| PV   | 7              | 9             | 25      | 59      |

Table 2 - Physical attributes of the soils considered.

| Soil | \( \theta_s \) | \( \theta_d \) | \( \theta_r \) | \( \rho \) (10^3 kg/m³) | \( K_n \) (10³ m/s) | \( \dot{T}_{ie} \) (10³ m/s) | \( i_s \) (10³ m/s) | \( \psi_u \) (10³ m/s) | \( \lambda \) |
|------|----------------|---------------|-------------|-------------------------|-------------------|--------------------------|----------------|----------------|-------|
| LVA  | 0.327          | 0.541         | 0.521       | 1.05                    | 31.14             | 36.19                    | 486.0         | 0.087         | 0.4032|
| LV   | 0.113          | 0.479         | 0.398       | 0.085                   | 28.50             | 70.31                    | 530.0         | 0.132         | 0.7470|
| PV   | 0.300          | 0.543         | 0.523       | 1.19                    | 9.86              | 21.92                    | 211.0         | 0.104         | 0.3572|
for each situation. By choosing $\alpha$ close to 0, the graph tends to overestimate the values of the cumulative infiltration. On the other hand, if such parameter is closer to 1, the values of the cumulative infiltration decrease. It can be seen that the function which predicts the infiltration has a discontinuity when $t$ is equal to $t_p$. This comes from the assumption that $t_p$ and $t_f$ are obtained by means of the exact three-parameter equation. In order to work out this issue, one shall obtain the explicit equations for each parameter based on the approximated solution proposed. Since such discontinuities do not affect the values of $\alpha$, this procedure is out of the scope of the present effort. The results are summarized in Table 3.

5. Conclusions

The three-parameter equation for infiltration prediction has been solved by means of an iterative algorithm whose single-iteration convergence ratio can be chosen as high as desired.

The methodology has been validated by comparing the approximate solutions to the exact ones. The maximum absolute departure of the approximated nondimensional cumulative infiltration is of about 0.23 and happens at low nondimensional times. The results presented considered a single iteration of the algorithm, thus the error could be mitigated by taking more iterations into account.

Based on the three-parameter infiltration equations and the solutions developed, three real cases were studied and the value of the interpolation parameter $\alpha$ has been obtained for each one. For a red clay soil $\alpha = 0.55$; for a red laterite $\alpha = 0.99$ and finally, for a red-yellow laterite soil $\alpha = 0.50$. These values may vary mainly due to the type of soil analyzed.

It's known that exact solutions are not necessary for practical purposes. The tree-parameter infiltration equation can only be exactly solved in terms of special functions,
which ultimately reduces its applicability to practical situations. In the present paper, simple yet powerful approximated solutions with good accuracy are presented. This way, it becomes easier to apply the three-parameter infiltration equation to field studies.

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Table of Symbols

- $a$: convergence parameter
- $I$: cumulative infiltration
- $I_n$: nondimensional cumulative infiltration
- $i_p$: cumulative infiltration until ponding time
- $i_r$: rainfall intensity
- $K_t$: hydraulic conductivity in transmission zone
- $K_r$: relative hydraulic conductivity
- $K_s$: saturated hydraulic conductivity
- $p_s$: average surface pressure head
- $r$: infiltration rate
- $t$: time
- $t_n$: nondimensional time
- $t_p$: ponding time
- $t_i$: time necessary to infiltrate the same amount infiltrated until the ponding time, under saturation of the superficial layer of the soil
- $Tie$: steady infiltration rate
- $\alpha$: interpolation parameter;
- $\theta_s$: saturated water content
- $\theta_i$: initial water content
- $\theta_r$: residual water content
- $\lambda$: pore size distribution index.
- $\psi_w$: average wetting front matric potential
- $\psi_a$: absolute value of the matric potential of air entrance.

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