Maclaurin Heat Coefficients and Associated Zeta Functions on Quaternionic Projective Spaces $\mathbb{P}^n(\mathbb{H})$ ($n \geq 1$)

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Abstract. The heat invariants or the Minakshisundaram-Pleijel heat coefficients $a_k^n = a_k^n (M)$ ($k \geq 0$) describe the asymptotic expansion of the heat kernel $H_M$ on any $N = 4n$-dimensional ($n \geq 1$) compact Riemannian manifold $M$; associated with the coefficients $a_k^n$ is the Minakshisundaram-Pleijel zeta function $\zeta_M = \zeta_M(s)$ ($s \in \mathbb{C}$). In this paper, we introduce and study a new class of heat coefficients, namely, the Maclaurin heat coefficients $b_{2m}^n = b_{2m}^n (t)$ ($t > 0, m \geq 0$) (i.e., the coefficients appearing in the Maclaurin expansion of the heat kernel $H_M(t, \theta)$ in terms of the classical and generalised Minakshisundaram-Pleijel coefficients $a_{k,j}^n$ and $a_{n,m}^n = a_{n,m}^n (M)$ ($0 \leq j \leq m$) respectively, when $M = \mathbb{P}^n(\mathbb{H})$ ($n \geq 1$), a quaternionic projective space. Remarkable asymptotic expansions for the Maclaurin spectral functions $b_{2m}^n (t)$ are established. We also introduce and construct new zeta functions $Z_{b_{2m}^n} (m \geq 0)$ associated with these Maclaurin heat coefficients (generalised Minakshisundaram-Pleijel zeta functions), and it is interesting to see that these generalised zeta functions can be explicitly understood in terms of the classical (Minakshisundaram-Pleijel) zeta functions.

1. Introduction

Let $\Delta_M$ denote the positive Laplace-Beltrami operator on a $N$-dimensional ($N \geq 1$) compact Riemannian manifold $M = (M, g)$ acting on smooth functions $f \in C^\infty (M)$; $g$ is the Riemannian metric associated with $M$. The heat equation, in particular, the heat kernel (fundamental solution of the heat equation) associated with $\Delta_M$ is a useful tool in spectral theory of Riemannian manifolds [19, 22].

By definition, the heat kernel $H_M = H_M(t, x, y)$ associated with $\Delta_M$ is given by the spectral sum

$$H_M(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k^N t} \varphi_k^N (x) \varphi_k^N (y),$$

(1.1)

where the sequence $\lambda_k^N = (\lambda_k^N : k \geq 0)$ represents the spectrum of $\Delta_M$ in $L^2 (M)$ which by basic spectral theory, is purely discrete, real, non-negative and satisfies the inequality $\lambda_0^N < \lambda_1^N \leq \lambda_2^N \leq \cdots$ ($\lambda_0^N = 0$) with $\lambda_j^N \rightarrow \infty$. Associated with the eigenvalue $\lambda_k^N$ are eigenfunctions which form a complete orthonormal basis ($\varphi_k^N : k \geq 0$) of $L^2 (M)$, i.e., $(\varphi_j^N, \varphi_k^N )_{L^2 (M)} = 0$ for $0 \leq j \neq k$ and $||\varphi_j^N ||_{L^2 (M)} = 1$ for all $j \geq 0$. For thorough discussions of heat kernels in Riemannian manifolds, see [10, 14, 15, 19, 22].
In their celebrated paper, Minakshisundaram & Pleijel ([24]) showed that the heat kernel $H_M$ satisfies the asymptotic expansion

$$H_M(t,x,x) \sim \frac{1}{(4\pi t)^{N/2}} \sum_{k=0}^{\infty} \tilde{u}_k^N(x,x)t^k, \quad t \searrow 0. \quad \text{(1.2)}$$

Here the functions $\tilde{u}_k^N$, the local heat invariants, can be explicitly expressed in terms of polynomials in the curvature tensor and its higher order derivatives (see, e.g., [11], [14], [15], [17] and [23] for further discussion in this regard).

As a result, the heat operator $T_M(t) := \text{tr} e^{-t\Delta_M}$ (which is well-defined, finite and of trace class) given by

$$T_M(t) = \int_M H_M(t,x,x) \sqrt{\text{det} g} d^N x \quad \text{(1.3)}$$

satisfies the expansion

$$T_M(t) \sim \sum_{k=0}^{\infty} a_k^N t^k, \quad t \searrow 0. \quad \text{(1.4)}$$

The numbers $a_k^N$ (with $k \geq 0$) are called Minakshisundaram-Pleijel coefficients ([24]) with $a_0^N = \text{Vol}(M)$ and

$$a_k^N = \int_M \tilde{u}_k^N(x,x) \sqrt{\text{det} g} d^N x, \quad k \geq 1. \quad \text{(1.5)}$$

Another important spectral invariant associated with the heat kernel $H_M(t,x,y)$ on $M$ is the Minakshisundaram-Pleijel zeta function $\zeta_M(z)$ ($z \in \mathbb{C}, \Re z > N/2$). By definition, the Minakshisundaram-Pleijel zeta function $\zeta_M = \zeta_M(z)$ is given by the Dirichlet-type series

$$\zeta_M(z) = \sum_{k=1}^{\infty} \frac{1}{[\lambda_k^N]^z}, \quad \Re z > N/2. \quad \text{(1.6)}$$

Indeed, it is worth noting that the spectral zeta function $\zeta_M(z)$ is the Mellin transform of the heat trace $T_M(t)$, i.e.,

$$\zeta_M(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1}(T_M(t) - 1) dt. \quad \text{(1.7)}$$

The spectral zeta function $\zeta_M$ can be analytically continued to a meromorphic function on the complex plane with simple poles at $z_k = N/2 - k$ with $k = 0, 1, 2, \ldots$ (for $N$ odd) and $z_k = N/2 - k$ with $k = 0, 1, 2, \ldots, N/2 - 1$ (for $N$ even). The residues at these poles are given by ([24])

$$\text{Res} \zeta_M(z) \bigg|_{z = \frac{N}{2} - k} = \frac{a_k^N (4\pi)^{-N/2}}{\Gamma \left( \frac{N}{2} - k \right)}. \quad \text{(1.8)}$$

The afore-mentioned spectral invariants can be explicitly evaluated on rank one compact symmetric spaces, and it is the aim of this paper to investigate and examine such situations for a particular rank one compact symmetric space. In general, let $M$ be a $N$-dimensional ($N \geq 1$) rank one compact symmetric space, $H_M$ and $\Delta_M$ the associated heat kernel and Laplace-Beltrami operator (or simply Laplacian) respectively. Using the addition formula for the matrix
coefficients (see, e.g., [20 Ch. IV]) it is not difficult to see that the heat kernel $H_{M}(t, x, y)$ takes the form (see [5 Appendix A.1])

$$H_{M}(t, \vartheta) = \frac{1}{\omega_{N}} \sum_{k=0}^{\infty} \Lambda_{k}^{N} \Psi_{k}^{N}(\vartheta) e^{-\lambda_{k}^{N} t}. \quad (1.9)$$

The numbers $\lambda_{k}^{N}$ (with $k \geq 0$) are the numerically distinct eigenvalues of $\Delta_{M}$; $\Lambda_{k}^{N}$ is the dimension of the eigenspace associated with $\lambda_{k}^{N}$ (i.e., the multiplicity of the eigenvalue $\lambda_{k}^{N}$), $\Psi_{k}^{N}(\vartheta)$ is the spherical function on $M$ associated with the eigenvalue $\lambda_{k}^{N}$; $\vartheta$ is the geodesic distance between the points $x, y \in M$ and $\omega_{N} = \text{Vol}(M)$ is the volume of $M$. It is remarkable to know that the spherical functions $\Psi_{k}^{N}$ on $M$ can be explicitly given in terms of the normalised Jacobi polynomials $P_{k}^{(\alpha, \beta)}$ (with $k \geq 0; \alpha, \beta > -1$). (See [2], [18] and [29] for basic properties of Jacobi polynomials.) This paper specialises $M$ to the $n$-dimensional ($n \geq 1$) quaternionic projective space $M = \mathbb{P}^{n}(\mathbb{H})$ (with $N = 4n$). In Lie group theoretical notation, $\mathbb{P}^{n}(\mathbb{H}) = \text{Sp}(n+1)/\text{Sp}(n) \times \text{Sp}(1)$ (of real dimension $4n$). (See the monographs [11] and [30] for related material on Lie groups and symmetric spaces.)

In this special case of $M = \mathbb{P}^{n}(\mathbb{H})$, the Laplacian $\Delta_{\mathbb{P}^{n}(\mathbb{H})}$ has eigenvalues given explicitly by $\lambda_{k}^{n} = (k+2n+1) : k \geq 0$) each of which with multiplicity

$$\Lambda_{k}^{n} = \frac{(2k+2n+1)\Gamma(k+2n)\Gamma(k+2n+1)}{\Gamma(2n+2)\Gamma(2n+1+1)\Gamma(k+2)} \quad k \geq 0. \quad (1.10)$$

Corresponding to the eigenvalue $\lambda_{k}^{n}$ are the (normalised) eigenfunctions (called spherical functions) $\Psi_{k}^{n}$ given explicitly by the normalised Jacobi polynomials (29)

$$\Psi_{k}^{n}(\vartheta) := P_{k}^{(2n-1,1)}(\cos \vartheta) = \frac{P_{k}^{(2n-1,1)}(1)}{P_{k}^{(2n-1,1)}(1)} P_{k}^{(\alpha, \beta)}(1) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(k+1)}, \quad (1.11)$$

where $P_{k}^{(\alpha, \beta)}(t)$ ($k \geq 0; \alpha, \beta > -1$) is the Jacobi polynomial which is a solution to the Jacobi differential equation

$$(1-t^{2}) \frac{d^{2}y}{dt^{2}} - (\alpha - \beta + (\alpha + \beta + 2)t) \frac{dy}{dt} + k(k+\alpha + \beta + 1)y = 0. \quad (1.12)$$

Clearly, as a result of (1.11), $\Psi_{0}^{n}(0) = 1$.

It is now straightforward to see that the heat kernel on $\mathbb{P}^{n}(\mathbb{H})$ admits the spectral series representation

$$H_{\mathbb{P}^{n}(\mathbb{H})}(t, \vartheta) = \frac{1}{\omega_{n}} \sum_{k=0}^{\infty} \Lambda_{k}^{n} \Psi_{k}^{n} e^{-\lambda_{k}^{n} t}$$

$$= \sum_{k=0}^{\infty} \omega_{n} \Gamma(2n+1+1)\Gamma(k+2n+1)\Gamma(k+2n+1+1)\Gamma(k+2) P_{k}^{(2n-1,1)}(\cos \vartheta) e^{-\lambda_{k}(2n+1) t}, \quad (1.13)$$

where $\omega_{n} = \text{Vol}(\mathbb{P}^{n}(\mathbb{H})) = (4\pi)^{2n}/\Gamma(2n+2)$.

1 Note that $\mathbb{P}^{n}(\mathbb{H})$ as a real manifold has dimension $4n$. For the sake of convenience of notation here and in future we slightly abuse notation and denote its associated spectral and geometric data by $\lambda_{k}^{n}, \Lambda_{k}^{n}, \omega_{n}, a_{k}^{n}, \Psi_{k}^{n}$, instead of $\lambda_{k}^{4n}, \Lambda_{k}^{4n}, \omega_{4n}, a_{k}^{4n}, \Psi_{k}^{4n}$ respectively.
As a result the Minakshisundaram-Pleijel zeta function $\zeta_{P^0(\mathbb{H})}(s)$ has the formulation

$$\zeta_{P^0(\mathbb{H})}(s) = \sum_{k=1}^{\infty} \frac{A_k^0}{|A_k^0|^s} = \sum_{k=1}^{\infty} \frac{(2k + 2n + 1)\Gamma(k + 2n)\Gamma(k + 2n + 1)}{\Gamma(2n + 2)\Gamma(2n)\Gamma(k + 1)\Gamma(k + 2)(k + 2n + 1)} t^s. \quad (1.14)$$

It follows immediately that $\zeta_{P^0(\mathbb{H})}$ can be analytically continued to a meromorphic function on $\mathbb{C}$ with simple poles located at the points $s = 2n, 2n - 1, \ldots, 2$ (see [13]). It is one of the aims of this paper to prove that the heat trace $T_{P^0(\mathbb{H})}(t)$ given by

$$T_{P^0(\mathbb{H})}(t) = \sum_{k=0}^{\infty} \frac{(2k + 2n + 1)\Gamma(k + 2n)\Gamma(k + 2n + 1)}{\Gamma(2n + 2)\Gamma(2n)\Gamma(k + 1)\Gamma(k + 2)} e^{-k(k + 2n + 1)t} \quad (1.15)$$

satisfies the asymptotic expansion ($t \searrow 0$)

$$T_{P^0(\mathbb{H})}(t) \sim \frac{1}{(4\pi t)^{2n}} \sum_{k=0}^{\infty} a_k^n t^k, \quad a_k^n = \omega_n \sum_{j=0}^{k} \left[ \frac{(2n + 1)^{2j}}{4^{k-j}} \right] (k-j)! u_j^n, \quad (1.16)$$

for some coefficients $u_j^n$ ($k \geq 0$). (The cases of the complex projective space and the Cayley projective plane are respectively discussed in [3] and [6], see also [5].)

In this paper we shall examine the role played by these heat coefficients in describing a new class of heat coefficients – the Maclaurin heat coefficients, i.e., the coefficients associated with the Maclaurin expansion of the heat kernel, and then introduce the associated zeta functions. Apart from giving a precise and relatively simple description of the heat coefficients $a_k^n = a_k^n (P^n(\mathbb{H}))$, we study further the heat coefficient $a_k^n$ and describe its role in the series expansion of the heat kernel, thereby revealing that the Minakshisundaram-Pleijel coefficients can be used to describe the Maclaurin heat coefficients. Finally we use these Maclaurin coefficients to set up a new construction for the associated zeta functions. The results presented in this paper are an extension and improvement over the works of [12], [13], [16], [21], [24], [26, 27] and thus provide a meaningful and deep insight into Minakshisundaram-Pleijel coefficients and create a new path in research concerning the heat kernel coefficients associated with symmetric spaces.

2. The Minakshisundaram-Pleijel Coefficients $a_k^n = a_k^n (P^n(\mathbb{H}))$

In this section we give a more explicit and clearer calculations of the heat coefficients $a_k^n$ associated with $P^n(\mathbb{H})$ in a way slightly different from that of Cahn and Wolf ([12]). The idea is to first express the heat trace $T_{P^0(\mathbb{H})}(t)$ associated with $P^n(\mathbb{H})$ purely in terms of a Jacobi theta function and its higher order derivatives, and then employ the asymptotics of the Jacobi theta function (as $t \searrow 0$) in the spirit of [25].

**Theorem 2.1.** The heat trace $T_{P^0(\mathbb{H})}(t)$ associated with $P^n(\mathbb{H})$ (given by (1.15)) admits the Minakshisundaram-Pleijel asymptotic expansion

$$T_{P^0(\mathbb{H})}(t) \sim \frac{1}{(4\pi t)^{2n}} \sum_{k=0}^{\infty} a_k^n t^k \quad \text{as} \ t \searrow 0, \quad (2.1)$$

where the heat coefficients $a_k^n$ are given by

$$a_k^n = \omega_n \sum_{j=0}^{k} \frac{(2n + 1)^{2k-2j}}{4^{k-j}(k-j)!} u_j^n, \quad a_k^n = \omega_n \sum_{j=0}^{k} \frac{(2n + 1)^{2k-2j}}{4^{k-j}(k-j)!} u_j^n, \quad 0 \leq k \leq 2n - 1;$$

$$u_j^n = \frac{(2n - j - 1)!}{(2n - 1)!} B_{2n-1-j}, \quad 0 \leq k \leq 2n - 1; \quad (2.2)$$

$$a_k^n = \omega_n \sum_{j=0}^{k} \frac{(2n + 1)^{2k-2j}}{4^{k-j}(k-j)!} u_j^n, \quad u_j^n = \sum_{\ell=0}^{2n-1} \frac{(-1)^{\ell}}{(j-2n)! (2n-1)!} B_{j+\ell-2n}, \quad k \geq 2n;$$

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with \( u_0^0 = 1 \) and \( a_0^0 = \omega_n \). Here the coefficient \( \mathcal{A}_m^n \) and \( B_m \) are respectively defined by

\[
\left[ \eta^2 - \left( \frac{2n - 1}{2} \right)^2 \right] \prod_{j=1/2}^{2n-3} \left[ \eta^2 - j^2 \right]^2 = \sum_{m=0}^{2n-1} \mathcal{A}_m^n \eta^{2m}, \quad \text{and} \quad B_m = \frac{(-1)^m (1 - 2^{-2n-1})}{(m + 1)} B_{2m+2},
\]

where \( B_m \) is the well-known \( m \)th Bernoulli number.

**Proof.** Writing the multiplicity \( \Lambda_k^n \) in a polynomial form we have

\[
\Lambda_k^n = \frac{(2k + 2n + 1) \Gamma(k + 2n) \Gamma(k + 2n + 1)}{\Gamma(2n + 2) \Gamma(2n + 1) \Gamma(k + 1) \Gamma(k + 2)} = \frac{(k + 2n)(2k + 2n + 1)}{2n(2n + 1)(k + 2)} \prod_{j=1}^{2n-3} \left( \frac{k + j}{j} \right)^2
\]

\[
= \frac{2}{(2n + 1)! (2n - 1)!} \sum_{m=0}^{2n-1} \mathcal{A}_m^n \left( \frac{k + 2n + 1}{2} \right)^{2m+1}.
\]

Using the multiplicity (2.4) in the trace formula (1.15) we have

\[
T_{\mathcal{P}_n(H)}(t) = \frac{2e^{(2n+1)^2 t / 4}}{(2n + 1)! (2n - 1)!} \sum_{k=0}^{2n-1} \sum_{m=0}^{2n-1} \mathcal{A}_m^n \left( \frac{k + 2n + 1}{2} \right)^{2m+1} e^{-(k + 2n + 1)^2 t / 4}
\]

\[
= \frac{e^{(2n+1)^2 t / 4}}{(2n + 1)! (2n - 1)!} \sum_{m=0}^{2n-1} \mathcal{A}_m^n \sum_{s=(2n+1)/2}^{\infty} s^{2m+1} e^{-s^2 t / 4}
\]

\[
= \frac{e^{(2n+1)^2 t / 4}}{(2n + 1)! (2n - 1)!} \sum_{m=0}^{2n-1} (-1)^m \mathcal{A}_m^n \theta^{(m)}(t),
\]

where the Jacobi theta function \( \theta \) is given by

\[
\theta(t) = \sum_{j=0}^{\infty} (2j + 1) e^{-(j+1/2)^2 t} \lesssim \frac{1}{t} + \sum_{j=0}^{\infty} B_j t^j \quad \text{(see, e.g., [25])},
\]

with \( B_j = B_j / j! \). In general, for \( p \geq 1 \), we have

\[
\theta^{(p)}(t) \lesssim \frac{(-1)^p p!}{t^{p+1}} + \sum_{j=p}^{\infty} B_j^p t^{j-p},
\]

where \( B_j^p = B_j / (j - p)! \). Substituting the generalised Jacobi theta function (2.7) in the trace formula (2.5), we have

\[
T_{\mathcal{P}_n(H)}(t) \lesssim \frac{e^{(2n+1)^2 t / 4}}{(2n + 1)! (2n - 1)!} \left[ \frac{(2n - 1)!}{t^{2n}} + \sum_{\ell=0}^{2n-2} \frac{\mathcal{A}_\ell^n \ell!}{t^{\ell+1}} + \sum_{\ell=0}^{2n-1} \frac{\mathcal{A}_\ell^n \ell!}{j=\ell} B_j^p t^{j-\ell} \right].
\]
The aim of this section is to develop the Maclaurin expansion of the heat kernel of the Hurwitz zeta function as

\[ A_n = \frac{\omega_n}{(4\pi t)^{2n}} \sum_{\ell=0}^{2n-2} \frac{2n-2-\ell}{\ell!} \frac{A_{\ell}^{(2n+\ell-1)} + \sum_{j=\ell}^{2n-1} B_j^{(2n+j-\ell)} }{\ell!} = \frac{1}{(4\pi t)^{2n}} \sum_{n=0}^{\infty} a_k^n t^k, \]

where \( A_n = \frac{\omega_n n!}{(2n-1)!} \), \( \tilde{A}_\ell = \frac{\omega_{\ell} (2\ell)!}{(2\ell-1)!} \). Consequently, we obtain the heat coefficients \( a_k^n \) given in (2.2) as required.

\[ \square \]

3. Spectral Relations Involving Heat Coefficients \( a_k^n \) and Zeta Functions \( \zeta_{\mathbb{P}^n(\mathbb{H})} \)

In this section we relate the heat coefficients \( a_k^n \) to the residues of the zeta functions \( \zeta_{\mathbb{P}^n(\mathbb{H})} \). That is, we show that the Minakshisundaram-Pleijel formula (1.8) also holds for the special case of the quaternionic projective space \( \mathbb{P}^n(\mathbb{H}) \). This is done by first expressing the Minakshisundaram-Pleijel zeta function \( \zeta_{\mathbb{P}^n(\mathbb{H})} \) in terms of the Hurwitz zeta function \( \zeta(z, \cdot) \) (\( z \in \mathbb{C} \)) of number theory.

By definition, the Hurwitz zeta function \( \zeta(z, a) \) is defined, for \( a > 0 \), by (see [1], [28, Sec. 2.2])

\[ \zeta(z, a) = \sum_{j=0}^{\infty} \frac{1}{(j+a)^z}, \quad \text{Re} \, z > 1. \]  

(3.1)

The Hurwitz zeta function \( \zeta(z, \cdot) \) can be analytically continued to a meromorphic function on all of \( \mathbb{C} \) with its only (simple) pole located at \( z = 1 \) and the residue at this pole is one.

**Theorem 3.1** ([3]). The Minakshisundaram-Pleijel zeta function \( \zeta_{\mathbb{P}^n(\mathbb{H})} \) can be written in terms of the Hurwitz zeta function as

\[ \zeta_{\mathbb{P}^n(\mathbb{H})}(s) = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m! \Gamma(s)} \sum_{\ell=0}^{2n-1} \frac{2\omega_{\ell} \left( \frac{2n+\ell}{2} \right)^{2m}}{(2n+1)! (2n-1)!} \zeta \left( 2(s-\ell + m) - 1, \frac{2n+3}{2} \right). \]

(3.2)

By noting that the residue of the Hurwitz zeta function in (3.2) is \( 1/2 \), and that \( \zeta_{\mathbb{P}^n(\mathbb{H})}(s) \) has simple poles at \( s = 2n - k \), we obtain the following result:

**Corollary 3.2.** The following relation holds:

\[ a_k^n = \frac{\omega_n}{\Gamma(2n)} \sum_{\ell=0}^{k} \frac{\omega_{\ell} 2n-1-k}{\ell!} \Gamma(2n-k+\ell) \left( \frac{2n+1}{2} \right)^{2\ell}. \]

(3.3)

4. The Maclaurin Spectral Functions \( b_{2m}^n(t) \) and the Heat Coefficients \( a_{k,j}^{n,m} \)

The aim of this section is to develop the Maclaurin expansion of the heat kernel \( H_{\mathbb{P}^n(\mathbb{H})}(t, \theta) \) \( (t > 0) \) and introduce the associated heat coefficients \( b_{2m}^n = b_{2m}^n(t) \) \( (m \geq 0) \) (now referred to as the Maclaurin heat coefficients) in terms of the heat trace \( \text{tr} e^{-t \Delta_{\mathbb{P}^n(\mathbb{H})}} \) and its higher order derivatives; the latter are then described by the Minakshisundaram-Pleijel heat coefficients \( (a_k^n : k \geq 0) \).

By definition, the Maclaurin expansion of the heat kernel \( H_{\mathbb{P}^n(\mathbb{H})}(t, \theta) \) \( (t > 0) \) refers to the formal expansion

\[ H_{\mathbb{P}^n(\mathbb{H})}(t, \theta) = \sum_{m=0}^{\infty} \frac{\theta^{2m}}{(2m)!} \frac{\partial^{2m}}{\partial \theta^{2m}} H_{\mathbb{P}^n(\mathbb{H})}(t, \theta) \bigg|_{\theta=0}, \]

(4.1)
where we have taken note of the fact that $H_{P^n(\mathbb{H})}(t, \theta)$ is an even function in the $\theta$-variable [cf. (1.13)]. Clearly the first term in the expansion (4.1) is given by the classical heat trace $\text{tr} e^{-tP^n(\mathbb{H})}/\omega_n$. For the higher order terms and their description in terms of the heat trace and its higher derivatives we need the following proposition:

**Proposition 4.1 (Jacobi coefficients [1, 7, 8, 9]).** Let $P_k^{(\alpha,\beta)}(t) = P_k^{(\alpha,\beta)}(t)/P_k^{(\alpha,\beta)}(1)$ denote the normalised Jacobi polynomial with $k \geq 0$ and $\alpha, \beta > -1$. Then we have the differential identity

$$\frac{d^{2m}}{d\theta^{2m}} P_k^{(\alpha,\beta)}(\cos \theta) \Bigg|_{\theta=0} = \sum_{j=1}^{m} c_j^m [k(k+\alpha+\beta+1)]^j, \quad m \geq 1. \quad (4.2)$$

The Jacobi coefficients $(c_j^m : 1 \leq j \leq m)$ are scalars depending on the parameters $\alpha$ and $\beta$, while $(k(k+\alpha+\beta+1) : k \geq 0)$ are the eigenvalues of the Jacobi operator.

Now, as a consequence of Proposition (4.1), we obtain the following nice description for the derivatives of the heat kernel $H_{P^n(\mathbb{H})}(t, \theta)$ at the diagonal.

**Theorem 4.2 (Spectral expansion ).** The Maclaurin expansion of the heat kernel associated with the quaternionic projective space $P^n(\mathbb{H})$ $(n \geq 1)$ given by the series (4.1) can be written as

$$H_{P^n(\mathbb{H})}(t, \theta) = \frac{1}{\omega_n} \sum_{m=0}^{\infty} b^n_{2m}(t) \theta^{2m}, \quad (4.3)$$

where the Maclaurin heat coefficients $b^n_{2m} = b^n_{2m}(t)$ are given in terms of the heat trace by

$$b^n_0(t) = \text{tr} e^{-t\Delta P^n(\mathbb{H})}, \quad b^n_{2m}(t) = \sum_{j=1}^{m} \frac{r_j^m}{(2m)!} \left( -\frac{d}{dt} \right)^j b^n_0(t) \quad m \geq 1, \quad (4.4)$$

and the constants $(c_j^m ; 1 \leq j \leq m)$ are the Jacobi coefficients with $\alpha = 2n - 1$ and $\beta = 1$.

**Proof.** It is seen that

$$H_{P^n(\mathbb{H})}(t, \theta) = \sum_{m=0}^{\infty} \frac{\theta^{2m}}{(2m)!} \frac{\partial^{2m}}{\partial \theta^{2m}} H_{P^n(\mathbb{H})}(t, \theta) \Bigg|_{\theta=0}, \quad (4.5)$$

where

$$\frac{\partial^{2m}}{\partial \theta^{2m}} H_{P^n(\mathbb{H})}(t, \theta) \Bigg|_{\theta=0} = \sum_{k=0}^{\infty} \frac{(2k+2n+1)!\Gamma(k+2n)\Gamma(k+2n+1)}{\omega_n \Gamma(2n)\Gamma(2n+2)\Gamma(k+1)\Gamma(k+2)} \times$$

$$\times e^{-k(k+2n+1)t} \frac{\partial^{2m}}{\partial \theta^{2m}} P_k^{(2n-1,1)}(\cos \theta) \Bigg|_{\theta=0}. \quad (4.6)$$

Proposition 4.1 now gives

$$\frac{\partial^{2m}}{\partial \theta^{2m}} H_{P^n(\mathbb{H})}(t, \theta) \Bigg|_{\theta=0} = \sum_{j=1}^{m} \frac{c_j^m}{\omega_n} \left( -\frac{d}{dt} \right)^j \text{tr} e^{-t\Delta P^n(\mathbb{H})}. \quad (4.7)$$

□
Theorem 4.3 (Maclaurin heat expansion). The Maclaurin heat coefficient $b_{2m}^n(t)$ admits the asymptotic expansion (as $t \to 0$)

$$b_{2m}^n(t) \sim \sum_{j=1}^{m} \frac{1}{(4\pi t)^{2n+j}} \sum_{k=0}^{\infty} a_{k,j}^{n,m} t^k, \quad a_{k,j}^{n,m} = (-1)^j c_j^m (4\pi)^j (k-2n)!/(2m)! (k-2n-j)! a_k^n. \quad (4.8)$$

Here $a_{k,j}^{n,m}$ are called the generalised (Minakshisundaram-Pleijel) heat coefficients with $a_{k,0}^n = a_k^n$.

Proof. For any integer $j \geq 1$, the $j$th derivative of the heat trace as expressed above is given by

$$\left( \frac{d}{dt} \right)^j \sum_{k=0}^{\infty} a_k^n \frac{1}{(4\pi t)^{2n}} = \sum_{k=0}^{\infty} a_k^n \frac{1}{(4\pi t)^{2n}} \left( \frac{d}{dt} \right)^j t^{k-2n} = \sum_{k=0}^{\infty} \frac{(k-2n)! a_k^n k^{2n-j}}{(4\pi)^{2n}(k-2n-j)!}. \quad (4.9)$$

This therefore results in

$$b_{2m}^n(t) = \sum_{j=1}^{m} c_j^m \left( \frac{1}{2m!} \right)^j \left( -\frac{d}{dt} \right)^j b_0^n(t) \sim \sum_{j=1}^{m} \frac{(-1)^j c_j^m}{(2m)!} \frac{(k-2n)! a_k^n k^{2n-j}}{(4\pi)^{2n}(k-2n-j)!}. \quad (4.10)$$

5. Spectral Zeta Functions Associated with the Maclaurin Heat Coefficients $b_{2m}^n(t)$

In this section, we introduce and construct a new zeta function associated with the Maclaurin coefficient $b_{2m}^n = b_{2m}^n(t)$, and establish interesting spectral formulae for this zeta function.

It is remarkable to see that this zeta function can be studied in terms of the classical (Minakshisundaram-Pleijel) zeta function.

Theorem 5.1. The zeta function $Z_{P^m(H)}^n = Z_{P^m(H)}(s)$ associated with the Maclaurin heat coefficients $b_{2m}^n = b_{2m}^n(t)$ is given by the spectral sum

$$Z_{P^m(H)}(s) = \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \zeta_{P^m(H)}(s-j). \quad (5.1)$$

In particular, $Z_{P^m(H)}(s) = \zeta_{P^m(H)}(s)$.

Proof. The Maclaurin spectral function $b_{2m}^n(t)$ can be written in the form

$$b_{2m}^n(t) = \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{k=1}^{\infty} \Lambda_k^n [k(k+2n+1)]^j e^{-k(k+2n+1)t}. \quad (5.2)$$

Thus we have

$$Z_{P^m(H)}(s) := \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{k=1}^{\infty} \Lambda_k^n [k(k+2n+1)]^j \int_0^{\infty} e^{-k(k+2n+1)t} s^{-1} dt$$

$$= \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{k=1}^{\infty} \Lambda_k^n [k(k+2n+1)]^j s^{-s} \Gamma(s). \quad (5.3)$$

Now let $Z_{P^m(H)}^n = Z_{P^m(H)}^n(s)$ denote the spectral zeta function associated with the Maclaurin heat coefficients $b_{2m}^n = b_{2m}^n(t)$ (see (1.7)). Then

$$Z_{P^m(H)}^n(s) := \frac{Z_{P^m(H)}^n(s)}{\Gamma(s)} = \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{k=1}^{\infty} \frac{\Lambda_k^n}{[k(k+2n+1)]^{s-j}}, \quad (5.4)$$

and this completes the proof of the theorem.
**Theorem 5.2.** The zeta function $Z_{\mathbb{P}^n(\mathbb{H})}^m(s)$ admits the series representation

\[
Z_{\mathbb{P}^n(\mathbb{H})}^m(s) = \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{\ell=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{\ell+l} \binom{j}{\ell} \binom{2n+1}{\ell} \Gamma\left(\frac{2n+1}{2}\right)^{2(\ell+l)} \times \sum_{m=0}^{2n-1} \omega_m^n \zeta \left(2(s-j-m+\ell+l) - 1, \frac{2n+3}{2}\right). \tag{5.5}
\]

*Proof.* By Mellin transform we have

\[
Z_{\mathbb{P}^n(\mathbb{H})}^m(s) := \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{k=1}^{\infty} \Lambda_k^0 [k(k+2n+1)]^j \int_0^{\infty} e^{-(k+2n+1)^2} \frac{t^{2n+1}}{4} t^{s-1} dt = \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{\ell=0}^{\infty} \frac{\Gamma(s+\ell)}{\ell!} \left(\frac{2n+1}{2}\right)^{2\ell} \times \sum_{k=1}^{\infty} \Lambda_k^0 \left[\left(k + \frac{2n+1}{2}\right)^2 - \left(\frac{2n+1}{2}\right)^2\right]^j \left(k + \frac{2n+1}{2}\right)^{-2(\ell+s)}. \tag{5.6}
\]

Using the explicit binomial expansion

\[
\left[\left(k + \frac{2n+1}{2}\right)^2 - \left(\frac{2n+1}{2}\right)^2\right]^j = \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} \left(k + \frac{2n+1}{2}\right)^{-2l+2j} \left(\frac{2n+1}{2}\right)^{2l}, \tag{5.7}
\]

we see that

\[
\sum_{\ell=0}^{\infty} \frac{\Gamma(s+\ell)}{\ell!} \left(\frac{2n+1}{2}\right)^{2\ell} \sum_{k=1}^{\infty} \Lambda_k^0 \left[\left(k + \frac{2n+1}{2}\right)^2 - \left(\frac{2n+1}{2}\right)^2\right]^j \left(k + \frac{2n+1}{2}\right)^{-2(\ell+s)} = \sum_{\ell=0}^{\infty} \frac{\Gamma(s+\ell)}{\ell!} \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} \left(\frac{2n+1}{2}\right)^{2l+2l-2n-1} \sum_{m=0}^{2n-1} \omega_m^n \zeta \left(2(s-j-m+\ell+l) - 1, \frac{2n+3}{2}\right), \tag{5.8}
\]

where $\omega_m^n = 2\omega_m^n/(2n+1)!(2n-1)!$. By taking note of the fact that $(-1)^\ell \binom{-s}{\ell} = \Gamma(s+\ell)/\ell! \Gamma(s)$ we obtain the required result. \hfill \Box

**Theorem 5.3.** The zeta function $Z_{\mathbb{P}^n(\mathbb{H})}^m = Z_{\mathbb{P}^n(\mathbb{H})}^m(s)$ associated with the Maclaurin heat coefficients $b_{2m}^n = b_{2m}^n(t)$ admits the spectral formula

\[
Z_{\mathbb{P}^n(\mathbb{H})}^m(s) = \sum_{j=1}^{m} \frac{c_j^m}{(2m)!} \sum_{m=0}^{\infty} \frac{\Gamma(s-j+m)}{m! \Gamma(s-j)} \left(\frac{2n+1}{2}\right)^{2m-2n} \sum_{l=0}^{2n-1} \omega_l^n \zeta \left(2(s-j-\ell+m) - 1, \frac{2n+3}{2}\right). \tag{5.9}
\]

*Proof.* We substitute the zeta function $\zeta_{\mathbb{P}^n(\mathbb{H})}^m(s)$ given in (3.2) into the equation (5.1) to obtain the required formula. \hfill \Box
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