\section{Introduction}

Ochiai proved an $\ell$-independence result for a variety over a local field. To be more precise, let $K$ be a henselian discrete valuation field, $\overline{K}$ a separable closure of $K$, and $X$ a variety over $K$. He proved that for an element $\sigma$ of the inertia subgroup $I_K$ of the absolute Galois group of $K$, the alternating sum

$$\sum_q (-1)^q \text{Tr} \left( \sigma, H^q_{\ell}(X_{\overline{K}}, \mathbb{Q}_\ell) \right)$$

is an integer independent of $\ell$ distinct from the residual characteristic [1, Theorem B].

By the same method, Vidal established an equivariant version of his result, that is, for a variety $X$ over $K$ with an action of a finite group $G$ and for an element $(g, \sigma) \in G \times I_K$, the alternating sum

$$\sum_q (-1)^q \text{Tr} \left( (g, \sigma), H^q_{\ell}(X_{\overline{K}}, \mathbb{Q}_\ell) \right)$$

is an integer independent of $\ell$ distinct from the residual characteristic [2, Proposition 4.2]. She further established the existence of a fixed geometric point when the alternating sum is nonzero [2, Proposition 5.1]; if the alternating sum is nonzero, then for every $G$-equivariant compactification $X$ of $X$ over the ring of integers $\mathcal{O}_K$, there exists a geometric point of $X$ fixed by $g$. She applied these results to comparing wild ramification of Galois representations; she deduced from them that, for constructible étale $\overline{\mathbb{F}}_{\ell}$-sheaves $\mathcal{F}_i$ ($i = 1, 2$) on a variety $Z$ over $K$, if they “have the same wild ramification”, then the elements $[R\Gamma_c(Z_{\overline{K}}, \mathcal{F}_i)]$ in the Grothendieck group of the category of Galois representations “have the same wild ramification” [2, Théorème 3.1]. See [2] for the definition of “having the same wild ramification”.

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She also worked in a relative situation. Let $f : Z \to Y$ be a morphism of varieties over a henselian discrete valuation field $K$ with excellent integer ring. In [3], she proved that if constructible étale $\mathbb{F}_\ell$-sheaves $\mathcal{F}_i$ ($i = 1, 2$) on $Z$ “have the same wild ramification”, then the elements $[Rf_*\mathcal{F}_i]$ in the Grothendieck group of the category of constructible étale $\mathbb{F}_\ell$-sheaves on $Y$ “have the same wild ramification” [3, Théorème 0.1]. The key ingredients to prove this result were an $\ell$-independence result and the existence of a fixed geometric point for relative curves [3, Proposition 2.2.1].

Although the case of relative curves is enough to deduce the result on wild ramification of étale sheaves, her results on $\ell$-independence and the existence of a fixed geometric point have independent interest themselves. So we generalize these results to the case of a general family. Vidal’s result [3, Proposition 2.2.1] mentioned above is the special case of the following theorem where $S = \text{Spec} \ O_K$ is a henselian trait, $\text{Spec} \ L \to S$ is the natural open immersion $\text{Spec} \ K \to \text{Spec} \ O_K$, and $\dim X \leq 1$.

**Theorem 1.1.** Let $S$ be an excellent noetherian scheme of dimension $\leq 2$. Let $L$ be a field and $\text{Spec} \ L \to S$ be a morphism of schemes such that $L$ is finitely generated over the residue field at the image. Let $X$ be a scheme separated and of finite type over $L$ on which a finite group $G$ acts admissibly. Let $L$ be a separable closure of $L$. Then, for every $(g, \sigma) \in G \times \text{Gal}(L/L)$ with $\sigma$ in Vidal’s ramified part, i.e., an element coming from the inertia group of a valuation ring over $S$,

1. The alternating sum

$$
\sum (-1)^q \text{Tr}((g, \sigma), H^q_c\left( X_L, \mathbb{Q}_\ell \right))
$$

is an integer independent of a prime number $\ell$ invertible on $S$,

2. If the alternating sum is nonzero, then for every $G$-equivariant “compactification” $\mathcal{X}$ of $X$ over $S$, there exists a geometric point of $\mathcal{X}$ fixed by $g$.

We make comments on Vidal’s ramified part. Vidal’s ramified part is a higher dimensional analogue of the inertia subgroup for a henselian discrete valuation field. It is defined for a morphism $\text{Spec} \ L \to S$ of schemes with $L$ a field and is a subset (not a subgroup in general) of $\text{Gal}(\overline{L}/L)$, where $\overline{L}$ is a separable closure of $L$ (Definition 2.1). For example, if $S$ is the spectrum $\text{Spec} \ O_K$ of a henselian discrete valuation ring $O_K$ and $\text{Spec} \ L \to S$ is the natural open immersion $\text{Spec} \ K \to \text{Spec} \ O_K$, then Vidal’s ramified part for $\text{Spec} \ L \to S$ coincides with the inertia subgroup of $K$. More generally, if $S$ is noetherian and if $L$ is the function field of a normal connected scheme $Y$ proper over $S$, then every element of (a dense subset of) Vidal’s ramified part belongs to the inertia subgroup at some geometric point $y$ of $Y$, i.e., the image of the morphism $\pi_1(\text{Spec} \ L \times Y, Y_{(y)}, \alpha) \to \text{Gal}(\overline{L}/L)$ for some geometric point $\alpha$ lying above $\text{Spec} \ L$ (Lemma 2.4). In other words, elements of Vidal’s ramified part are ramified along each “compactification” $\mathcal{Y}$ of $\text{Spec} L$ over $S$.

In equal characteristic case, there are some recent developments. In [4], Chiarellotto and Lazda discuss various forms of $\ell$-independence including $\ell = p$. Especially, in [4, Theorem 6.1], it is stated that, if $X$ is a smooth proper variety over a
local field of equal characteristic \( p > 0 \), then the trace of each element of the Weil group acting on each \( H^i \) is independent of \( \ell \), including \( \ell = p \). Though their proof included a mistake as pointed out in the introduction of [5], it has been corrected in [6].

Moreover, Q. Lu and W. Zheng obtained an \( \ell \)-independence result for an arbitrary equal characteristic base. They proved that, for arbitrary \( S \) of characteristic \( p > 0 \) and for smooth proper \( X/L \), the trace of each \( \sigma \in \text{Gal}(\overline{L}/L) \) in Vidal’s ramified part acting on the cohomology group of each degree is independent of \( \ell \neq p \) [5, Theorem 1.4 and Remark 2.18(3)]. Further, after the first version of this article was circulated, they informed us that Theorem 1.1.1 can be deduced from the results [5, Corollary 1.3] and [7, Proposition 4.15] on compatibility of \( \ell \)-adic sheaves and that the assumptions on \( S \) and \( \text{Spec} \, L \to S \) can be removed, i.e., Theorem 1.1.1 holds for an arbitrary scheme \( S \) and an arbitrary morphism \( \text{Spec} \, L \to S \) with \( L \) being a field. We include their comments as an “Appendix”.

Although the method of Q. Lu and W. Zheng is simpler and gives a more general result than ours, our method has an advantage that we can also get the existence of a fixed point (Theorem 1.1.2) at the same time.

Theorem 1.1 can be used to remove a reduction argument in the proof of the main result of [3] on comparison of wild ramification of étale sheaves. She decomposed a morphism into relative curves to reduce the problem to the case of relative curves. In other words, Theorem 1.1 gives a slightly simpler proof of the main result of [3] without decomposing a morphism into relative curves.

We explain the strategy of the proof of Theorem 1.1. First, we briefly recall the proof of Ochiai’s \( \ell \)-independence result for a variety over a henselian discrete valuation field. Ochiai reduced the proof to the semi-stable reduction case with a finite group action by taking an alteration using a result of de Jong. Then, he used the weight spectral sequence by Rapoport-Zink to describe the Galois action on the étale cohomology groups in terms of the geometry of the closed fiber and to deduce \( \ell \)-independence from the Lefschetz trace formula.

To work in a relative setting, Vidal modified Ochiai’s proof using log structures of Fontaine-Illusie instead of the weight spectral sequence. Our proof is based on her idea. We also take an alteration using a refinement of de Jong’s result due to Gabber (Lemma 3.6) to reduce the problem to the “semi-stable reduction” case with a finite group action (Proposition 6.10). Here, under the above notations, reduction is considered along a “compactification” of Spec \( L \) over \( S \) and “semi-stable” means that the variety \( X \) is the generic fiber of a “nodal fibration” (“pluri nodal fibration” in [8], see Sect. 3 for the precise definition) over a “compactification” of Spec \( L \) over \( S \). Then, we apply Nakayama’s log smooth and proper base change theorem to the “nodal fibration”, which we will denote by \( \mathcal{X} \to \mathcal{Y} \), with the natural log structures. It gives us a canonical isomorphism between the étale cohomology group of the generic fiber and the Kummer étale cohomology group of the log geometric fiber \( \mathcal{X}_y \) over a closed point \( y \) of \( \mathcal{Y} \). To describe the Kummer étale cohomology in terms of the usual étale cohomology, we compute the “log nearby cycle complex” \( R\varepsilon_*\mathbb{Q}_\ell \), where \( \varepsilon \) is the morphism forgetting the log structure of \( \mathcal{X}_y \). Then, we can deduce the \( \ell \)-independence and the existence of a fixed point from the Lefschetz trace formula.
This paper is organized as follows. In Sect. 2 we give a definition of Vidal’s ramified part in our setting and show some basic properties. We recall Gabber’s refinement (Lemma 3.6) of de Jong’s result in Sect. 3 and basic properties of log structures in Sect. 4. After formulating “monodromy action composed with a finite group action” in an equivariant setting in Sect. 5, we prove the main result in Sect. 6.

2. Vidal’s ramified part

**Definition 2.1.** Let $S$ be a scheme, $K$ be a field, and $\text{Spec } K \rightarrow S$ a morphism of schemes. We take a separable closure $\overline{K}$ of $K$. We define a subset $E_{K/S}$ of $\text{Gal} (\overline{K}/K)$, which we call Vidal’s ramified part, as follows: We consider a following commutative diagram:

\[
\begin{array}{cccc}
\text{Spec } F & \overset{i}{\longrightarrow} & \text{Spec } \overline{K} \\
\downarrow & & \downarrow \\
\text{Spec } F & \overset{i}{\longrightarrow} & \text{Spec } K \\
\downarrow & & \downarrow \\
\text{Spec } \mathcal{O}_F & \longrightarrow & S,
\end{array}
\]

where $\mathcal{O}_F$ is a strictly henselian valuation ring, $F$ its field of fraction, and $\overline{F}$ a separable closure of $F$. We define a subset $E_{(i,\overline{i})}$ of $\text{Gal} (\overline{K}/K)$ as the image of the natural map $\text{Gal} (\overline{F}/F) \rightarrow \text{Gal} (\overline{K}/K)$. We define $E_{K/S}$ to be the closure of the union of $E_{(i,\overline{i})}$ for all diagrams as above.

We study functorial properties of Vidal’s ramified part:

**Lemma 2.2.** Let $S$ be a scheme.

1. (c.f.[2, Proposition 2.1.1], [9, Lemma 2.4]) We consider a commutative diagram

\[
\begin{array}{cccc}
\text{Spec } \overline{K'} & \longrightarrow & \text{Spec } \overline{K} \\
\downarrow & & \downarrow \\
\text{Spec } K' & \longrightarrow & \text{Spec } K \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
\]

of schemes such that $K$ and $K'$ are fields and that $\overline{K}$ and $\overline{K'}$ are separable closures of $K$ and $K'$ respectively. Then, the natural morphism $\text{Gal} (\overline{K'}/K') \rightarrow \text{Gal} (\overline{K}/K)$ induces a map $E_{K'/S'} \rightarrow E_{K/S}$. 
2. Let $K$ be a field and $\text{Spec } K \to S$ be a morphism of schemes. Let $L$ be a finite purely inseparable extension of $K$. We take a separable closure $\overline{L}$ (resp. $\overline{K}$) of $L$ (resp. $K$) and fix an embedding $\overline{K} \to \overline{L}$ over $K$. Then, the map $E_{L/S} \to E_{K/S}$ defined in 1 is bijective.

Proof. 1. Clear from the definition.
2. Since the injectivity of $E_{L/S} \to E_{K/S}$ follows from the bijectivity of the map between the absolute Galois groups, it suffices to prove that the map $E_{L/S} \to E_{K/S}$ is surjective. Let

\[
\begin{array}{ccc}
\text{Spec } \overline{F} & \longrightarrow & \text{Spec } \overline{L} \\
\downarrow & & \downarrow \\
\text{Spec } F & \longrightarrow & \text{Spec } K \\
\downarrow & & \downarrow \\
\text{Spec } \mathcal{O}_F & \longrightarrow & S
\end{array}
\]

be a diagram such that $\mathcal{O}_F$ is a strictly henselian valuation ring, $F$ its field of fractions, and $\overline{F}$ an algebraic closure of $F$. Let $E$ be the minimum subfield of $\overline{F}$ containing $F$ and $L$. Then, $E$ is finite and purely inseparable over $F$. Further, the normalization of $\mathcal{O}_F$ in $E$ is a strictly henselian valuation ring. Thus, the map in the assertion is surjective.

\[\square\]

Let $\text{Spec } K \to S$ be as above and assume it is “essentially of finite type” (see Definition 2.3 below). In Lemma 2.4 below, we characterize Vidal’s ramified part $E_{K/S}$ as the subset of $\text{Gal}(\overline{K}/K)$ consisting of elements which have a fixed geometric point for every compactification over $S$ and every finite Galois extension of $K$. To be more precise, we make the following definition:

**Definition 2.3.** Let $S$ be a scheme, $K$ a field, and $\text{Spec } K \to S$ a morphism of schemes.

1. We say that $\text{Spec } K \to S$ is **essentially of finite type** if the field $K$ is a finitely generated extension of the residue field at the image.
2. Assume that $\text{Spec } K \to S$ is essentially of finite type. A **compactification of $K$ over $S$** is an integral scheme $Y$ proper over $S$ with an $S$-morphism $\text{Spec } K \to Y$ which induces an isomorphism between $\text{Spec } K$ and the generic point of $Y$.

Logically, in the proof of the main theorem, we use only the implication $1 \Rightarrow 2$ in Lemma 2.4 below and do not use the other implication $2 \Rightarrow 1$. But, this characterization shows that Vidal’s ramified part $E_{K/S}$ is the largest subset to which we can apply the argument in Sect. 6.

**Lemma 2.4.** (cf. [3, 6.1, 6.3]) Let $S$ be a noetherian scheme and $K$ be a field with a morphism $\text{Spec } K \to S$ which is essentially of finite type. We take a separable closure $\overline{K}$ of $K$. For $\sigma \in \text{Gal}(\overline{K}/K)$, the following are equivalent.
1. \( \sigma \in E_{K/S} \).
2. For every normal compactification \( \mathcal{Y} \) of \( K \) over \( S \) and every finite Galois extension \( L \) of \( K \) contained in \( \overline{K} \), there exists a geometric point \( \bar{v} \) of the normalization \( \overline{\mathcal{V}} \) of \( \mathcal{Y} \) in \( L \) such that \( \sigma \bar{v} = \bar{v} \) under the natural action of \( \text{Gal}(\overline{K}/K) \) on \( \mathcal{V} \).

The proof is essentially the same as that in [3, 6.1]. But we include the proof for the completeness.

**Proof.** 1 \( \Rightarrow \) 2. Let \( \sigma \in E_{K/S} \) and let \( \bar{\sigma} \in \text{Gal}(L/K) \) be the image of \( \sigma \) by \( \text{Gal}(\overline{K}/K) \to \text{Gal}(L/K) \). Then, \( \bar{\sigma} \) is in the image of \( \text{Gal}(\overline{F}/F) \to \text{Gal}(\overline{K}/K) \to \text{Gal}(L/K) \) for some diagram (2.1). Let \( \overline{E} \) be the minimum subfield of \( \overline{F} \) containing \( L \) and \( F \) and let \( \overline{O}_E \) be the normalization of \( \overline{O}_F \) in \( E \). Then, since \( \overline{O}_F \) is a henselian valuation ring, \( \overline{O}_E \) is also a henselian valuation ring by [10, Chapitre VI 8.6 Proposition 6]. Then, by the valuative criterion of proper morphisms [11, Théorème 7.3.8], there exists a unique \( S \)-morphism \( \text{Spec} \overline{O}_E \to \mathcal{V} \) extending the composite \( \text{Spec} E \to \text{Spec} L \to \mathcal{V} \). Then, since \( \overline{O}_F \) is strictly henselian, the geometric point \( \bar{v} \) of \( \mathcal{V} \) defined by the closed point of \( \text{Spec} \overline{O}_E \) is fixed by \( \sigma \).

2 \( \Rightarrow \) 1. Let \( \sigma \in \text{Gal}(\overline{K}/K) \) be an element satisfying the condition 2. To show \( \sigma \in E_{K/S} \), it suffices to show that for every finite Galois extension \( L \) of \( K \) contained in \( \overline{K} \), the image of \( \sigma \) in \( \text{Gal}(L/K) \) is in the image of \( \text{Gal}(\overline{F}/F) \to \text{Gal}(\overline{K}/K) \to \text{Gal}(L/K) \) for some diagram (2.1).

We take an inverse system \( (\mathcal{V}_i)_{i \in I} \), indexed by a directed set \( I \), of normal compactifications of \( K \) over \( S \) which is cofinal in the category of normal compactifications of \( K \) over \( S \). We denote the normalization of \( \mathcal{V}_i \) in \( L \) by \( \mathcal{V}_i \).

**Claim 2.5.** There exists a compatible system \( (\bar{v}_i \to \mathcal{V}_i)_{i \in I} \) of geometric points fixed by \( \sigma \), that is, each \( \bar{v}_i \) is a geometric point of \( \mathcal{V}_i \) fixed by \( \sigma \) and if there is a morphism \( \mathcal{V}_i \to \mathcal{V}_j \) of compactifications of \( K \) over \( S \), then \( \bar{v}_j \) is the image of \( \bar{v}_i \) by the induced morphism \( \mathcal{V}_i \to \mathcal{V}_j \).

**Proof.** Let \( \gamma_{\sigma} : \mathcal{V}_i \to \mathcal{V}_i \times_{\mathcal{V}_j} \mathcal{V}_j \) be the graph of \( \sigma : \mathcal{V}_i \to \mathcal{V}_j \). We define, for each \( i \in I \), a closed subscheme \( \mathcal{V}_i^{\sigma} \) of \( \mathcal{V}_i \) to be the pullback \( \gamma_{\sigma}^*(\Delta_{\mathcal{V}_j}) \), where \( \Delta_{\mathcal{V}_j} \) is the diagonal in \( \mathcal{V}_i \times_{\mathcal{V}_j} \mathcal{V}_j \). Then, a point \( v \in \mathcal{V}_i \) is in \( \mathcal{V}_i^{\sigma} \) if and only if a geometric point over \( v \) is fixed by \( \sigma \). Here, \( \mathcal{V}_i^{\sigma} \) is nonempty by the condition 2. Note that \( \mathcal{V}_i^{\sigma} \) is compact and Hausdorff with respect to the constructible topology \( 1 \)[12, Exercise 27–28 of Chap. 3]. Thus, the inverse limit \( \lim_{i \in I} \mathcal{V}_i^{\sigma} \) in the category of sets is nonempty. We take an element \( (v_i)_{i \in I} \) of the inverse limit \( \lim_{i \in I} \mathcal{V}_i^{\sigma} \) and a separable closure \( \Omega \) of the inductive limit \( \lim_{i \in I} \kappa(v_i) \) of the residue fields \( \kappa(v_i) \). Then \( \text{Spec} \Omega \to (\mathcal{V}_i)_{i \in I} \) is a compatible system of fixed geometric points. \( \square \)

We denote the image of \( \bar{v}_i \) in \( \mathcal{V}_i \) by \( \bar{y}_i \). Let \( \bar{\mathcal{Y}} \) (resp. \( \hat{\mathcal{Y}} \)) be the inverse limit \( \lim_{i \in I} \mathcal{Y}_i(\bar{y}_i) \) (resp. \( \lim_{i \in I} \mathcal{Y}_i(\bar{y}_i) \)) and let \( \hat{K} \) (resp. \( \hat{L} \)) be the fraction field of \( \hat{\mathcal{Y}} \) (resp. \( \hat{\mathcal{Y}} \)). Since \( \sigma \) fixes every \( \bar{v}_i \), its image is in the image of the map \( \text{Gal}(\hat{L}/\hat{K}) \to \text{Gal}(L/K) \). Here we note that \( \mathcal{Y} \) is the spectrum of a strictly henselian valuation ring. In fact, if we write \( \mathcal{Y} = \text{Spec} \hat{A} \), then the natural morphism \( \hat{A} \to (\lim_{i \in I} \overline{O}_\mathcal{Y}(\bar{y}_i))^{sh} \)

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1 The constructible topology on a noetherian scheme is the topology generated by Zariski open subsets and Zariski closed subsets.
is an isomorphism, where $y_i$ is the point of $\mathcal{Y}_i$ lying under $\tilde{y}_i$. Since $\lim_{i \in I} \mathcal{O}_{\mathcal{Y}_i, y_i}$ is a valuation ring by Lemma 2.6 below, $\tilde{A}$ is also a valuation ring ([13, Tag 0ASK, Lemma 15.112.5]). Thus, the assertion follows.

Lemma 2.6. Let $S$ be a noetherian scheme and $\text{Spec } K \to S$ be a morphism which is essentially of finite type. We take an inverse system $(\mathcal{Y}_i)_{i \in I}$, indexed by a directed set $I$, of normal compactifications of $K$ over $S$ which is cofinal in the category of normal compactifications of $K$ over $S$. Then, for every element $(y_i)_{i \in I}$ of the set $\lim_{i \in I} \mathcal{Y}_i$, the ring $\lim_{i \in I} \mathcal{O}_{\mathcal{Y}_i, y_i}$ is a valuation ring.

This seems to be well known, but the author could not find a suitable reference. So we include a proof.

Note that the assumption that $S$ is noetherian assures the existence of a compactification, by Nagata’s compactification theorem.

Proof. We put $O = \lim_{i \in I} \mathcal{O}_{\mathcal{Y}_i, y_i}$. We check that for any $x \in K$, we have either $x \in O$ or $x^{-1} \in O$. We fix an index $i_0 \in I$. We may assume that $\mathcal{Y}_{i_0}$ is affine; $\mathcal{Y}_{i_0} = \text{Spec } A$. Then, any $x \in K$ can be written as $x = f/g$ for some $f, g \neq 0 \in A$. We take an index $i \in I$ such that $\mathcal{Y}_i$ dominates the blow up of $\mathcal{Y}_{i_0}$ along the ideal $I$ of $A$ generated by $f$ and $g$. Then, either $f/g$ or $g/f$ belongs to $\mathcal{O}_{\mathcal{Y}_i, y_i}$, and hence, the assertion follows.

Finally, we mention that Vidal’s ramified part defined here recovers Vidal’s original one. We assume that $S$ is an excellent trait. Let $Y$ be an $S$-scheme separated of finite type which is normal and connected. Let $K$ be the function field of $Y$. We take a separable closure $K$ of $K$. Let $E_{K/S}(Y)$ be the subset of $\pi_1(Y, \text{Spec } K)$ which Vidal calls “la partie génériquement ramifiée” in [3, 1.2]. By Gabber’s valuative criterion [3, 6.3], we see the following.

Lemma 2.7. The natural surjection $\text{Gal}(K/K) \to \pi_1(Y, \text{Spec } K)$ induces a surjection $E_{K/S}(Y)$.

Remark 2.8. In [3, 1.2], a subset $E_{Y/S}$ of $\pi_1(Y, \text{Spec } K)$, which she calls “la partie ramifiée”, is defined. This subset contains the subset $E_{K/S}(Y)$. By Gabber’s valuative criterion [3, 6.1], it coincides with the subset defined by replacing Spec $K$ by $Y$ in Definition 2.1. In the case $Y$ is regular, we have $E_{Y/S} = E_{K/S}(Y)$ [3, 6.4], but we do not know whether this equality holds or not in the general case, i.e., for a normal $Y$.

3. Alterations and nodal fibrations

We introduce some terminologies used in the proof of the main theorem.

Definition 3.1. ([14, 7.1]) Let $X$ be a regular scheme with an action of a finite group $G$ and $D$ a divisor of $X$ with simple normal crossings which is $G$-stable. We say that $D$ is a divisor with $G$-strict normal crossings if the $G$-orbit of every irreducible component of $D$ is a disjoint union of irreducible components of $D$. 
Definition 3.2. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism of schemes which is flat and of finite presentation.

1. [14, 2.21] We say \( f \) is a nodal curve if every geometric fiber of \( f \) is a connected curve whose singularities are at most ordinary double points.

2. [2, 4.4.1] (cf. [14, 2.22]) Let \( G \) be a finite group acting on \( \mathcal{X} \) and \( \mathcal{Y} \) so that \( f : \mathcal{X} \to \mathcal{Y} \) is \( G \)-equivariant. We say \( f \) is a \( G \)-split nodal curve if \( f \) is a nodal curve and if for every point \( y = \text{Spec} \kappa(y) \) of \( \mathcal{Y} \),
   
   (a) every singular point of \( \mathcal{X} \times_{\mathcal{Y}} y \) is rational over \( \kappa(y) \),
   
   (b) every irreducible component of \( \mathcal{X} \times_{\mathcal{Y}} y \) is smooth over \( \kappa(y) \),
   
   (c) the \( G \)-orbit of any irreducible component of \( \mathcal{X} \times_{\mathcal{Y}} y \) is a disjoint union of irreducible components of fibers of \( f : \mathcal{X} \to \mathcal{Y} \) (not necessarily the fiber \( \mathcal{X} \times_{\mathcal{Y}} y \)).

Note that any base change of a nodal curve is also a nodal curve and that any base change of a \( G \)-split nodal curve by any \( G \)-equivariant morphism is also a \( G \)-split nodal curve.

Definition 3.3. Let \( G \) be a finite group.

1. A \( G \)-split nodal fibration datum is a following datum:
   
   (a) a sequence of \( G \)-split nodal curves \( f_i : \mathcal{X}_i \to \mathcal{X}_{i-1} \) \( (i = 1, \ldots, d) \);
   
   \[ \mathcal{X}_d \to \mathcal{X}_{d-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0, \]
   
   (b) a \( G \)-stable proper closed subset \( Z_0 \) of \( \mathcal{X}_0 \),
   
   (c) finitely many disjoint sections \( \sigma_{ij} : \mathcal{X}_{i-1} \to \mathcal{X}_i \) \( (i = 1, \ldots, d) \) into the smooth locus of \( f_i : \mathcal{X}_i \to \mathcal{X}_{i-1} \) which are permuted by \( G \),
   
   satisfying the following condition; if we define for \( i = 1, \ldots, d \) a closed subset \( Z_i \) of \( \mathcal{X}_i \) inductively by
   
   \[ Z_i = f_i^{-1}(Z_{i-1}) \cup \bigcup_j \sigma_{ij}(Z_{i-1}), \]
   
   the nodal curve \( f_i : \mathcal{X}_i \to \mathcal{X}_{i-1} \) is smooth over \( \mathcal{X}_{i-1} \setminus Z_{i-1} \) for every \( i \).

2. We say a \( G \)-split nodal fibration datum \( (\mathcal{X}_d \to \mathcal{X}_{d-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0, Z_0, \{\sigma_{ij}\}) \) is strictly \( G \)-split if \( \mathcal{X}_i \) is regular and the closed subscheme \( Z_i \) defined as above is a divisor with \( G \)-strict normal crossings for every \( i = 0, \ldots, d \).

Definition 3.4. A \( G \)-split nodal fibration (resp. strictly \( G \)-split nodal fibration) is a \( G \)-equivariant morphism \( f : \mathcal{X} \to \mathcal{Y} \) of excellent schemes such that there exists a \( G \)-split nodal fibration datum (resp. strictly \( G \)-split nodal fibration datum)

\[ (\mathcal{X} = \mathcal{X}_d \to \mathcal{X}_{d-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{Y}, Z_0, \{\sigma_{ij}\}) \]

such that the composite \( \mathcal{X} = \mathcal{X}_d \to \mathcal{X}_{d-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{Y} \) coincides with \( f \).
We note that any base change of a $G$-split nodal fibration by any $G$-equivariant morphism is also a $G$-split nodal fibration.

When the group $G$ is trivial, we omit $G$ from these terminologies; for example, a $G$-split nodal fibration for trivial $G$ will be simply called a split nodal fibration.

Recall that an alteration is a proper generically finite surjection of integral noetherian schemes \([14, 2.20]\). We use a following refinement Lemma 3.6 of de Jong’s theorem \([8, \text{Theorem 5.9}]\) due to Gabber, to reduce the proof of our main theorem to the case where we can take a nodal fibration as an “integral model”.

**Definition 3.5.** (\([8, 5.3]\)) Let $X$ (resp. $X'$) be an integral noetherian scheme with an action of a finite group $G$ (resp. $G'$). A Galois alteration $(\phi, \alpha) : (X', G') \to (X, G)$ is a pair of a surjective homomorphism $\alpha : G' \to G$ of finite groups and an alteration $\phi : X' \to X$ which is $G'$-equivariant via $\alpha$ such that, writing $\Gamma$ for the kernel of the natural map $G' \to \text{Aut}(X)$, the fixed part $K(X')^{\Gamma}$ of the function field $K(X')$ of $X'$ is purely inseparable over the function field $K(X)$ of $X$.

**Lemma 3.6.** (\([2, \text{Lemme 4.4.3}]\)) Let $X$ and $Y$ be integral excellent noetherian schemes with an action of a finite group $G$ and $f : X \to Y$ a $G$-equivariant morphism separated and of finite type. Let $Z$ be a proper closed subset of $X$. Assume that the geometric generic fiber of $f$ is irreducible of dimension $d$. Then, there exist

- A surjective homomorphism $\alpha : G' \to G$ of finite groups
- Projective Galois alterations $(\psi, \alpha) : (Y', G') \to (Y, G)$ and $(\phi, \alpha) : (X', G') \to (X, G)$,
- A $G'$-split nodal fibration datum (Definition 3.3)

$$(X_d \to X_{d-1} \to \cdots \to X_1 \to X_0, Z_0, \{\sigma_{ij}\})$$

with $X_d = X'$ and $X_0 = Y'$, such that $Z_d$ defined as in Definition 3.3 contains the pullback $f^{-1}(Z)$ and that the following diagram commutes:

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}$$

**Lemma 3.7.** Let

$$(X_d \to X_{d-1} \to \cdots \to X_1 \to X_0, Z_0, \{\sigma_{ij}\})$$

be a $G$-split nodal fibration datum such that $X_0$ is regular and $Z_0$ is a divisor with $G$-strict normal crossings. Then, there exist a sequence of $G$-equivariant blowups

$$X'_1 \to X_1,$$

$$X'_2 \to X_2 \times_{X_1} X_1'^{\prime} ,$$

$$\cdots$$

$$X'_d \to X_d \times_{X_{d-1}} X_{d-1}'$$
such that the center of \(X'_i \to \mathcal{X}'_i \times \mathcal{X}'_{i-1}\) is outside the smooth locus of \(\mathcal{X}'_i \times \mathcal{X}'_{i-1}\) and that

\[
(\mathcal{X}'_d \to \mathcal{X}'_{d-1} \to \cdots \to \mathcal{X}'_1 \to \mathcal{X}_0, Z_0, \{\sigma_{ij}\})
\]

is a strictly \(G\)-split nodal fibration datum.

**Proof.** The \(d = 1\) case is proved in [2, Lemme 4.4.4]. We argue by induction on the relative dimension \(d\). If \(d = 0\), we have nothing to do. We assume that the assertion holds for \(G\)-split nodal fibration data of relative dimension \(d - 1\). Then, we may assume that

\[
(\mathcal{X}_{d-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0, Z_0, \{\sigma_{ij}\})
\]

is a strictly \(G\)-split nodal fibration datum. Then, since \(\mathcal{X}_{d-1}\) is regular and \(Z_{d-1}\) is a divisor with \(G\)-strict normal crossings, we can apply the \(d = 1\) case to the \(G\)-split nodal curve \(\mathcal{X}_d \to \mathcal{X}_{d-1}\).

\[\square\]

**Corollary 3.8.** Let \(S\) be an excellent noetherian scheme which is irreducible and of dimension \(\leq 2\). Let \(X\) be an \(S\)-scheme of finite type with an action of a finite group \(G\) and \(Z\) a proper closed subset of \(X\) which is \(G\)-stable. We assume that the generic geometric fiber of \(X \to S\) is irreducible. Then, there exists a Galois alteration \((\phi, \alpha) : (X', G') \to (X, G)\) with \(X'\) regular and a divisor with \(G'\)-strict normal crossings containing the pullback \(\phi^{-1}(Z)\).

**Proof.** We consider the trivial action of \(G\) on \(S\). By Lemma 3.6, we may assume that \(X \to S\) is given by a \(G\)-split nodal fibration datum \((X = X_d \to \cdots \to X_0 = S, Z_0, \{\sigma_{ij}\})\) with \(Z = Z_d\) in the notation in Definition 3.3. Here, by [15], we may assume that \(S\) is regular. Further, by [16, Corollary 0.4], we may assume that \(Z_0\) is a divisor of \(S_0\) with simple normal crossings. Then, the assertion follows from Lemma 3.7.

\[\square\]

4. Log structures and nodal curves

We refer to [17] for log structures. In particular, log structures are always considered on the étale sites of schemes. For a log structure \(\mathcal{M}_X \to \mathcal{O}_X\) on a scheme \(X\), we denote the quotient \(\mathcal{M}_X / \mathcal{O}_X^*\) by \(\overline{\mathcal{M}}_X\).

Let \(X\) be a regular noetherian scheme and \(D\) a divisor with simple normal crossings. We consider the log structure \(\mathcal{M}_X \to \mathcal{O}_X\) defined by \(D\), that is, \(\mathcal{M}_X = j_* \mathcal{O}_U^* \cap \mathcal{O}_X\) where \(U = X \setminus D\) and \(j\) is the open immersion \(U \to X\). We write \(D\) as the sum of the irreducible components: \(D = \bigcup_{i=1}^n D_i\). For a subset \(I \subset \{1, \ldots, n\}\), we put \(D_I = \bigcap_{i \in I} D_i\) and \(\overline{D}_I = D_I \setminus \bigcup_{i \notin I} D_{I \cup \{i\}}\).

**Lemma 4.1.** In the above settings, we have a canonical isomorphism \(\overline{\mathcal{M}}_X|_{\overline{D}_I} \cong \mathbb{N}^I\).
Proof. In the case where \( X = \text{Spec} \, A \) is affine and \( D \) is defined by \( h_1 \cdots h_n \) for some prime elements \( h_i \in A \) \((i = 1, \ldots, n)\), the composite of the map \( \mathbb{N}^I \rightarrow \mathcal{M}_X|_{\hat{D}_j} : e_i \mapsto h_i \) and the natural map \( \mathcal{M}_X|_{\hat{D}_j} \rightarrow \mathcal{M}_X|_{\hat{D}_j} \) is an isomorphism. Note that it does not depend on the choice of the prime elements \( h_i \). Thus, in general case, by gluing we get the desired isomorphism. \( \square \)

Further, the isomorphism in Lemma 4.1 is functorial in the following sense (Lemma 4.2): We consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow j & & \downarrow j' \\
U & \xrightarrow{j} & V
\end{array}
\]

such that \( X \) and \( Y \) are regular, \( f \) is flat, \( j \) and \( j' \) are open immersions, and \( D = X \setminus U \) and \( E = Y \setminus V \) are divisors with simple normal crossings. We consider the log structure \( \mathcal{M}_X \) (resp. \( \mathcal{M}_Y \)) defined by \( D \) (resp. \( E \)). Then, we have a natural morphism \( (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y) \) of log schemes. We write \( D \) and \( E \) as the sums of their irreducible components: \( D = \sum_{i \in I} D_i \) and \( E = \sum_{j \in J} E_j \). Let \( I_f \) be the subset of \( I \) consisting of elements \( i \in I \) such that \( f(D_i) \subset E \). Here, since \( f \) is flat, we can consider the pullback \( f^* E \) of the divisor \( E \) and a map \( \varphi : I_f \rightarrow J \) sending \( i \) to \( j \) such that \( f(D_i) \subset E_j \). We write the divisor \( f^* E \) as \( \sum_{i \in I} m_i D_i \) for some \( m_i \in \mathbb{Z}_{\geq 0} \).

**Lemma 4.2.** In the above notations, for every subset \( I' \subset I \), we have a commutative diagram

\[
\begin{array}{ccc}
(f^{-1}\mathcal{M}_Y)|_{\hat{D}_{I'}} & \xrightarrow{\cong} & \mathcal{M}_X|_{\hat{D}_{I'}} \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{N}^{\varphi(I')} & \xrightarrow{=} & \mathbb{N}^{I'},
\end{array}
\]

where the lower horizontal arrow is the morphism sending \( e_j \) to \( \sum_{i \in I' \cap \varphi^{-1}(j)} m_i e_i \).

**Proof.** This follows from the fact that \( f^* E_j \) is locally defined by the ideal generated by \( \prod_{i \in \varphi^{-1}(j)} h_i^{m_i} \), where \( h_i \) is a defining equation of \( D_i \). \( \square \)

**Lemma 4.3.** ([14, 2.23]) Let \( A \) be a strictly henselian noetherian local ring. Let \( f : \mathcal{X} \rightarrow \text{Spec} \, A \) be a split nodal curve. Then, étale locally on \( \mathcal{X} \), there exists an étale \( A \)-morphism \( \mathcal{X}' \rightarrow \text{Spec} \, A[x, y]/(xy - t) \) for some \( t \in A \).

We give a preliminary on log structure on a nodal curve. Let \( A \) be a \( d \)-dimensional regular local ring with a system of regular parameters \( t_1, \ldots, t_d \). We put \( t = t_1 \cdots t_r \) for some \( r \) with \( 1 \leq r \leq d \). We consider the split nodal curve \( f : \mathcal{X} = \text{Spec} \, A[x, y]/(xy - t) \rightarrow \mathcal{Y} = \text{Spec} \, A \), which is smooth outside the divisor \( Z \) of \( \mathcal{Y} \) defined by \( t = 0 \).

Let \( \mathcal{M}_Y \) be the log structure defined by the divisor \( Z \), or equivalently, the log structure associated to the pre-log structure \( \mathbb{N}^r \rightarrow \mathcal{O}_Y; e_i \mapsto t_i \), where \( e_i \) denotes
the standard basis of \( \mathbb{N}^r \). To give \( \mathcal{X} \) a log structure, we consider the (commutative) monoid \( M \) defined by the pushout square

\[
\begin{array}{c}
\mathbb{N}^2 \\
\downarrow \quad \Delta \\
\mathbb{N}^r \quad \downarrow \quad \Delta \\
\end{array}
\]

(4.1)

where \( \Delta : \mathbb{N} \to \mathbb{N}^2 \) and \( \Delta : \mathbb{N} \to \mathbb{N}^r \) are given by \( 1 \mapsto (1, 1) \) and by \( 1 \mapsto (1, \ldots, 1) \) respectively. Then we have the natural morphism \( \alpha : \mathcal{O}_X \to \mathcal{O}_X \) of sheaves of monoids on \( X \) defined by the morphisms \( \mathbb{N}^2 \to \mathcal{O}_X; e_1 \mapsto x, e_2 \mapsto y \) and \( \mathbb{N}^r \to \mathcal{O}_X; e_i \mapsto t_i \). We consider the log structure \( \mathcal{M}_X \) on \( X \) associated to the pre-log structure \( \alpha \). Let \( U \) denote the maximum open subscheme of \( X \) on which the log structure \( \mathcal{M}_X \) is trivial, or equivalently, \( U = X \setminus f^{-1}(Z) \). Let \( j : U \to X \) denote the canonical open immersion.

**Lemma 4.4.** 1. The morphism \( f : (\mathcal{X}, \mathcal{M}_X) \to (\mathcal{Y}, \mathcal{M}_Y) \) of log schemes is log smooth and saturated.

2. The morphism \( \mathcal{M}_X \to \mathcal{O}_X \) is injective and its image is identified with \( j_*\mathcal{O}_U \cap \mathcal{O}_X \).

For the definition of saturated morphisms of log schemes, see [18, I.3.5, I.3.7, I.3.12, II.2.10].

**Proof.** 1. We consider the Cartesian diagram of log schemes

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \text{Spec} \mathbb{Z}^2 \\
\downarrow f & & \downarrow h \\
\mathcal{Y} & \xrightarrow{\psi} & \text{Spec} \mathbb{Z}^r,
\end{array}
\]

induced from the diagram (4.1). Since \( Z \to \mathbb{Z}^2 : 1 \mapsto (1, 1) \) is an injective map with torsion-free cokernel, \( f \) is log smooth. Further \( f \) is integral (see [17, Proposition 4.1] for the definition), since \( h \) is flat. Since every fiber of \( f \) is reduced, we can use [18, Theorem 4.2] to see that \( f \) is saturated.

2. By [19, Proposition 2.6], it suffices to check that the log scheme \( (\mathcal{X}, \mathcal{M}_X) \) is log regular in the sense of [19, Definition 2.2] (c.f. [18, Definition II.4.5], [20, Definition 2.1]).

The log regularity of \( (\mathcal{X}, \mathcal{M}_X) \) follows from [18, Proposition II.4.8] (c.f. [20, Theorem 8.2]) since the log scheme \( (\mathcal{Y}, \mathcal{M}_Y) \) is log regular and \( f \) is log smooth.

**Lemma 4.5.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be regular noetherian schemes. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a split nodal curve, \( Z \) a divisor of \( \mathcal{Y} \) with simple normal crossings such that \( f \) is smooth over \( \mathcal{Y} \setminus Z \), and \( \sigma : \mathcal{Y} \to \mathcal{X} \) a section of \( f \) into the smooth locus. We consider the divisor \( D = f^{-1}(Z) \cup \sigma(\mathcal{Y}) \). We consider the log structure \( \mathcal{M}_\mathcal{X} \) (resp. \( \mathcal{M}_\mathcal{Y} \)) on \( \mathcal{X} \) (resp. \( \mathcal{Y} \)) defined by \( D \) (resp. \( Z \)). Then, the morphism \( (\mathcal{X}, \mathcal{M}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{M}_\mathcal{Y}) \) of log schemes is log smooth and saturated.
Proof. First we note that the question is étale local. We prove the assertion on an étale neighborhood of each point $p$ of $\mathcal{X}$.

If $p \notin \sigma(\mathcal{Y})$, we may assume, by Lemma 4.3, that $\mathcal{Y}$ is of the form $\text{Spec } A$ for a regular local ring $A$ and there exists an $A$-isomorphism $\mathcal{X} \cong \text{Spec } A[x, y]/(xy - t)$ for some $t = t_1 \cdots t_r \in A$ with $t_1, \ldots, t_r$ a part of a system of regular parameters. Then the assertion follows from Lemma 4.4.

If $p \in \sigma(\mathcal{Y})$, we may assume that $f$ is smooth. Further, by replacing $\mathcal{X}$ and $\mathcal{Y}$ by open neighborhoods of $p$ and $f(p)$ respectively, we may assume that we have a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \text{Spec } \mathbb{Z}[\mathbb{N}^2] \\
\downarrow f & & \downarrow \text{pr}_1 \\
\mathcal{Y} & \longrightarrow & \text{Spec } \mathbb{Z}[\mathbb{N}],
\end{array}
$$

of log schemes, where $\text{pr}_1$ is the morphism induced by $\mathbb{N} \rightarrow \mathbb{N}^2; 1 \mapsto (1, 0)$. Then we obtain the assertion in the same way as in the proof of Lemma 4.4.1. More precisely, $f$ is log smooth since $\mathbb{Z} \rightarrow \mathbb{Z}^2; 1 \mapsto (1, 0)$ is an injective map with torsion-free cokernel. Then, note that $\text{pr}_1$ is a flat morphism with reduced fibers and use [18, Theorem 4.2] to see that $f$ is saturated. \hfill \Box

5. Monodromy action composed with a finite group action

Let $Y$ be a noetherian scheme, $\ell$ be a prime number invertible on $Y$, and $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/\ell^n$-modules on $Y$. We consider an admissible action of a finite group $G$ on the scheme $Y$ and a $G$-sheaf structure on $\mathcal{F}$, that is a family of morphisms $\{\varphi_g : g^*\mathcal{F} \rightarrow \mathcal{F}\}_{g \in G}$ satisfying the cocycle condition; for $g, h \in G$, the composite $(gh)^*\mathcal{F} \cong h^*g^*\mathcal{F} \xrightarrow{h^*\varphi_g} h^*\mathcal{F} \xrightarrow{\varphi_h} \mathcal{F}$ coincides with $\varphi_{gh}$. We assume that $Y$ is connected and that $Y \rightarrow Y_0 = Y/G$ is étale, and hence a Galois finite étale cover. We denote the Galois group $\text{Aut}(Y/Y_0)$ by $H$ and take a geometric point $y$ of $Y$. Then, we have a natural action of $G \times_H \pi_1(Y_0, y)$ on the stalk $\mathcal{F}_y$ defined as follows: For a pointed connected finite étale cover $(Y', y')$ of $(Y, y)$ which is Galois over $(Y_0, y)$, an element $(g, \sigma) \in G \times_H \pi_1(Y_0, y)$ induces a commutative diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y' \\
\downarrow p & & \downarrow p \\
Y & \longrightarrow & Y.
\end{array}
$$

We define an action of $(g, \sigma)$ on $\Gamma(Y', \mathcal{F})$ by the composite $\Gamma(Y', \mathcal{F}) \cong \Gamma(Y', p^*\mathcal{F}) \xrightarrow{\sigma^*} \Gamma(Y', \sigma^*p^*\mathcal{F}) \cong \Gamma(Y', g^*\mathcal{F}) \xrightarrow{\varphi_g} \Gamma(Y', \mathcal{F})$. It defines an action of $G \times_H \pi_1(Y_0, y)$ on $\Gamma(Y', \mathcal{F})$ which is functorial with respect to $(Y', y')$, and hence we obtain an action $G \times_H \pi_1(Y_0, y)$ on $\mathcal{F}_y \cong \varprojlim_{(Y', y')} \Gamma(Y', \mathcal{F})$, where $(Y', y')$ runs over all pointed connected finite étale covers which are Galois over $(Y_0, y)$. 

\$\ell$-independence of the trace of local...
This action is defined also in \( \ell \)-adic settings: Let \( \mathcal{F} \) be a lisse \( \mathbb{Z}_\ell \)-sheaf on \( Y \) with a \( G \)-sheaf structure. We write \( \mathcal{F} \) as an inverse system \( (\mathcal{F}_n)_{n \geq 1} \), where \( \mathcal{F}_n \) is a locally constant constructible sheaf of \( \mathbb{Z}/\ell^n \)-modules such that \( \mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \cong \mathcal{F}_n \). Then, we have a natural action of \( G \times H \pi_1(Y_0, y) \) on the stalk \( \mathcal{F}_y \cong \lim_n (\mathcal{F}_{n,y}) \). We also have, for a lisse \( \mathbb{Q}_\ell \)-sheaf \( \mathcal{F} \) on \( Y \) with a \( G \)-sheaf structure, a natural action of \( G \times H \pi_1(Y_0, y) \) on the stalk \( \mathcal{F}_y \).

Let \( \mathcal{Y} \) be the spectrum of a strictly henselian regular local ring with an action of a finite group \( G \). Let \( Z \) be a divisor of \( \mathcal{Y} \) with simple normal crossings which is \( G \)-stable. We denote by \( \mathcal{M}_Y \) the log structure on \( \mathcal{Y} \) defined by \( Z \). Let \( \ell \) be a prime number invertible on \( \mathcal{Y} \). Let \( \mathcal{F} \) be a constructible sheaf of \( \mathbb{Z}/\ell^n \)-modules on the Kummer étale toposes. See [21, Sect. 2] for the details of Kummer étale toposes. We denote the fraction field of \( \mathcal{Y} \) by \( K \). We write \( K_0 \), for the fixed subfield \( K^G \) of Kummer étale toposes. We denote the fraction field of \( Y \) by \( k \). We fix an embedding \( \mathcal{Y} \rightarrow \mathcal{Y} \) over the log point \( \mathcal{Y} \), where \( y \) is the closed point of \( \mathcal{Y} \) and the log structure \( \mathcal{M}_Y \) is the pullback of the log structure \( \mathcal{M}_Y \) by the closed immersion \( y \rightarrow \mathcal{Y} \). Let \( (\mathcal{Y}, \mathcal{M}_Y) \) be the log strict localization [21, 4.5] of \( (\mathcal{Y}, \mathcal{M}) \) at \( y \). We write \( \bar{K} \) for the fraction field of \( \bar{Y} \).

**Lemma 5.1.** \( \bar{K} \) is Galois over \( K_0 \).

**Proof.** Let \( \bar{K} \) be a separable closure of \( K \). We fix an embedding \( \bar{K} \rightarrow \bar{K} \). Then \( \bar{K} \) is the union of all finite sub-extensions \( K' \) of \( \bar{K}/K \) such that \( K' \) is isomorphic over \( \mathcal{Y} \) to the fraction field of some integral scheme with a log structure which is finite and Kummer étale over \( \mathcal{Y} \). Thus, for every \( \sigma \in \text{Gal}(\bar{K}/K_0) \), we have \( \sigma(\bar{K}) \subset \bar{K} \).

We define a natural action \( G \times H \text{Gal}(\bar{K}/K_0) \) on the stalk \( \mathcal{F}_{\bar{Y}} \). Let \( (\mathcal{Y}', \bar{Y}') \) be a pointed connected finite Kummer étale cover of \( (\mathcal{Y}, \bar{Y}) \) such that the fraction field \( K' \) of \( \mathcal{Y}' \) is Galois over \( K_0 \). Note that \( K' \) can be viewed as a subfield of \( \bar{K} \) by the natural morphism \( \mathcal{Y}' \rightarrow \bar{Y} \). Then, an element \( (g, \sigma) \in G \times H \text{Gal}(\bar{K}/K_0) \) induces a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{\sigma} & \mathcal{Y}' \\
\downarrow p & & \downarrow p \\
\mathcal{Y} & \xrightarrow{g} & \mathcal{Y}.
\end{array}
\]

We define an action of \( (g, \sigma) \) on \( \Gamma(\mathcal{Y}', \mathcal{F}) \) by the composite \( \Gamma(\mathcal{Y}', \mathcal{F}) \cong \Gamma(\mathcal{Y}', p^*\mathcal{F}) \xrightarrow{\sigma^*} \Gamma(\mathcal{Y}', \sigma^*p^*\mathcal{F}) \cong \Gamma(\mathcal{Y}', g^*\mathcal{F}) \xrightarrow{g_{\mathcal{F}}} \Gamma(\mathcal{Y}', \mathcal{F}) \). It defines an action of \( G \times H \text{Gal}(\bar{K}/K_0) \) on \( \Gamma(\mathcal{Y}', \mathcal{F}) \) which is functorial with respect to \( (\mathcal{Y}', \bar{Y}') \). Now we have \( \mathcal{F}_{\bar{Y}} \cong \lim_{\rightarrow} (\mathcal{Y}', \bar{Y}') \Gamma(\mathcal{Y}', \mathcal{F}) \), where \( (\mathcal{Y}', \bar{Y}') \) runs over all pointed finite Kummer étale covers of \( (\mathcal{Y}, \bar{Y}) \). Further, by Lemma 5.1, in the filtered inverse system of pointed finite Kummer étale covers of \( (\mathcal{Y}, \bar{Y}) \), pointed finite Kummer étale covers \( (\mathcal{Y}', \bar{Y}') \) such that \( \text{Spec } K \times \mathcal{Y}' \mathcal{Y}' \) is Galois over \( \text{Spec } K_0 \) form a cofinal system. Thus, we obtain an action \( G \times H \text{Gal}(\bar{K}/K_0) \) on \( \mathcal{F}_{\bar{Y}} \). As before, this action is defined also in \( \ell \)-adic settings.
Lemma 5.2. Let \( \overline{K} \) be a separable closure of \( K \). We fix an embedding \( \iota : \tilde{K} \rightarrow \overline{K} \). Then the cospecialization map \( F_{\tilde{Y}} \rightarrow F_{\overline{K}} \) induced by \( \iota \) is \( G \times_H \text{Gal}(\overline{K}/K_0) \)-equivariant, where on \( F_{\tilde{Y}} \) we consider the action defined through the surjection \( G \times_H \text{Gal}(\overline{K}/K_0) \rightarrow G \times_H \text{Gal}(\overline{K}/K_0) \) induced by \( \iota \).

Proof. It suffices to show that, in the case of torsion coefficients, for every finite Kummer étale cover \( Y' \) of \( Y \) such that the fraction field \( K' \) is Galois over \( K_0 \), the morphism \( \Gamma(Y', \mathcal{F}) \rightarrow \Gamma(\text{Spec } K', \mathcal{F}) \) is \( G \times_H \text{Gal}(K'/K_0) \)-equivariant. This follows from the following commutative diagram (we write \( Y' \) for \( \text{Spec } K' \) to simplify the notations):

\[
\begin{array}{c}
\Gamma(Y', \mathcal{F}) \xrightarrow{=} \Gamma(Y', p^*\mathcal{F}) \xrightarrow{\sigma^*} \Gamma(Y', \sigma^* p^* \mathcal{F}) \xrightarrow{\varphi_g} \Gamma(Y', \mathcal{F}) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\Gamma(Y', \mathcal{F}) \xrightarrow{=} \Gamma(Y', p^*\mathcal{F}) \xrightarrow{\sigma^*} \Gamma(Y', \sigma^* p^* \mathcal{F}) \xrightarrow{\varphi_g} \Gamma(Y', \mathcal{F}).
\end{array}
\]

\[\square\]

6. \( \ell \)-independence

We consider a monodromy action composed with a finite group action defined as follows. Let \( K \) be a field with an action of a finite group \( G \). Let \( X \) be a scheme separated of finite type over \( K \) with an action of \( G \) which makes the structural morphism \( X \rightarrow \text{Spec } K \) equivariant. Let \( \ell \) be a prime number different from the characteristic of \( K \). We write \( H \) for the Galois group \( \text{Gal}(K/K_0) \), where \( K_0 \) is the fixed subfield \( K^G \) of \( K \). We take a separable closure \( \overline{K} \) of \( K \). Then, as in Sect. 5, we have a natural continuous action of the fiber product \( G \times_H \text{Gal}(\overline{K}/K_0) \) on \( H^q_{K}(X \times_K \overline{K}, \mathbb{Q}_\ell) \) for every \( q \).

Theorem 6.1. Let \( S \) be an excellent noetherian scheme of dimension \( \leq 2 \). Let \( K \) be a field with an action of a finite group \( G \) and \( \text{Spec } K \rightarrow S \) a morphism which is essentially of finite type (in the sense of Definition 2.3.1). Let \( X \) be a scheme separated of finite type over \( K \) with an admissible action of \( G \) which makes the structural morphism \( X \rightarrow \text{Spec } K \) equivariant. We take a separable closure \( \overline{K} \) of \( K \) and a prime number \( \ell \) invertible on \( S \). We write \( H \) for the Galois group \( \text{Gal}(K/K_0) \), where \( K_0 \) is the fixed subfield \( K^G \) of \( K \). Then, for every element \( (g, \sigma) \in G \times_H E_{K_0/S} \),

1. The eigenvalues of the action of \( (g, \sigma) \) on \( H^q_{K}(X \times_K \overline{K}, \mathbb{Q}_\ell) \), for each \( q \), are roots of unity,
2. The alternating sum of the traces

\[
\sum_q (-1)^q \text{Tr}((g, \sigma), H^q_{K}(X \times_K \overline{K}, \mathbb{Q}_\ell)) \tag{6.1}
\]

is an integer independent of a prime number \( \ell \) invertible on \( S \).
We first note that the alternating sum (6.1) does not change even if we replace $H^i_c$ by $H^i$:

**Lemma 6.2.** In the setting of Theorem 6.1, for every $(g, \sigma) \in G \times H E_{K_0/S}$, we have an equality

$$\sum_q (-1)^q \text{Tr}((g, \sigma), H^q(X \times_K K, \mathbb{Q}_\ell)) = \sum_q (-1)^q \text{Tr}((g, \sigma), H^q_c(X \times_K K, \mathbb{Q}_\ell)).$$

**Proof.** In fact, [22, Theorem 2.2] implies that the two objects $R\Gamma(X \times_K K, \mathbb{Q}_\ell)$ and $R\Gamma_c(X \times_K K, \mathbb{Q}_\ell)$ in the Grothendieck ring of continuous $\mathbb{Q}_\ell$-representations of $G \times H \text{Gal}(\overline{K}/K_0)$ coincide up to Tate twists, i.e., the difference is contained in the ideal generated by $[\mathbb{Q}_\ell(1)] - [\mathbb{Q}_\ell]$. Thus, it suffices to remark that $\sigma \in E_{K_0/S}$ acts trivially on $\mathbb{Q}_\ell(1)$. For this, it suffices to show that, for an integer $n$ invertible on $S$, every $\sigma \in E_{K_0/S}$ acts trivially on the $n$-th power roots of unity in $K$, i.e., the image of $E_{K_0/S} \subset \text{Gal}(\overline{K}/K_0) \rightarrow \text{Gal}(K_0(\mu_n)/K_0)$ is trivial. By the definition of Vidal’s ramified part $E_{K_0/S}$, this is equivalent to the following: for any strictly henselian valuation ring $O_F$ over $S$ with an $S$-morphism $K_0 \rightarrow F$ (and an embedding of separable closures), the image of $\text{Gal}(\overline{F}/F) \rightarrow \text{Gal}(\overline{K}/K_0) \rightarrow \text{Gal}(K_0(\mu_n)/K_0)$ is trivial. But, this is true because $F$ contains $n$-th power roots of unity. $\square$

In the proof of the theorem, we use the following terminology:

**Definition 6.3.** Let $S$ be a noetherian scheme. Let $K$ be a field with an action of a finite group $G$ and $\text{Spec } K \rightarrow S$ a morphism which is essentially of finite type. Let $X$ be a scheme separated of finite type over $K$ with an admissible action of $G$ which makes the structural morphism $X \rightarrow \text{Spec } K$ equivariant. A $G$-integral model over $S$ of $X \rightarrow \text{Spec } K$ is a $G$-equivariant morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of proper $S$-schemes together with a $G$-equivariant commutative diagram

$$\begin{array}{ccc}
X & \rightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec } K & \rightarrow & \mathcal{Y}
\end{array}$$

such that the lower horizontal arrow is a compactification of $K$ over $S$ in the sense of Definition 2.3 and that the natural morphism $X \rightarrow \mathcal{X} \times \mathcal{Y}$ Spec $K$ is a dense open immersion.

**Lemma 6.4.** Let $S$ and $X \rightarrow \text{Spec } K$ be as above. Then a $G$-integral model of $X \rightarrow \text{Spec } K$ over $S$ exists.

**Proof.** By Nagata’s compactification theorem, we can find an $e$-integral model $\mathcal{X}_1 \rightarrow \mathcal{Y}_1$ of $X \rightarrow \text{Spec } K$, where $e$ is the trivial group. We write $\iota_X$ (resp. $\iota_Y$) for the morphism $X \rightarrow \mathcal{X}_1$ (resp. $\text{Spec } K \rightarrow \mathcal{Y}_1$). Take the closure of the image of the $G$-equivariant morphism $X \rightarrow \prod_{g \in G} \mathcal{X}_1 : x \mapsto (\iota_X(gx))_{g \in G}$ (resp. $\text{Spec } K \rightarrow \prod_{g \in G} \mathcal{Y}_1 : y \mapsto (\iota_Y(gy))_{g \in G}$), where the products are taken over $S$. $\square$

At the same time that we prove Theorem 6.1, we also prove the existence of a fixed geometric point when the alternating sum is nonzero:
Theorem 6.5. Let notations be as in Theorem 6.1 and let \( \mathcal{X} \to \mathcal{Y} \) be a G-integral model of \( X \to \text{Spec } K \) over \( S \). We assume that the alternating sum (6.1) is nonzero. Then there exists a geometric point of \( \mathcal{X} \) which is fixed by \( g \).

Proof of Theorems 6.1 and 6.5. We prove the assertions by induction on \( d = \dim X \). By the arguments as in [3, 2.2.3], the proof is reduced to the case where \( X \) is normal connected and geometrically irreducible over \( K \). For the completeness, we include the reduction arguments. In the reduction arguments, we use the following two claims frequently.

Claim 6.6. Let \( K' \) be a finite quasi-Galois extension over \( K \) which is also quasi-Galois over \( K_0 \). Write \( H' \) for \( \text{Aut}(K'/K_0) \). Then, the assertions of Theorem 6.1 for \( X' = (X \times_K K')_{\text{red}} \to \text{Spec } K' \) with the \( G' = G \times_H H' \)-action are equivalent to those for \( X \to \text{Spec } K \) with the \( G \)-action. Further, the assertion of Theorem 6.5 for \( X' = (X \times_K K')_{\text{red}} \to \text{Spec } K' \) with the \( G' = G \times_H H' \)-action implies the assertion for \( X \to \text{Spec } K \) with the \( G \)-action.

Proof of Claim 6.6. Let \( \overline{K'} \) be a separable closure of \( K' \) containing \( \overline{K} \) and let \( K'_0 \) be the fixed subfield \( (K')^{H'} \). Then, we have a natural isomorphism \( H^d_c(X' \times_K \overline{K'}, \mathbb{Q}_\ell) \cong H^d_c(X \times_K \overline{K}, \mathbb{Q}_\ell) \), which is equivariant with respect to the map \( \alpha : G' \times_H E_{K'_0/S} \to G \times_H E_{K_0/S} \). Since \( K'_0 \) is purely inseparable over \( K_0 \), the map \( \alpha \) is bijective by Lemma 2.2.2. Thus, the assertions of Theorem 6.1 for \( X' = (X \times_K K')_{\text{red}} \to \text{Spec } K' \) with the \( G' = G \times_H H' \)-action are equivalent to those for \( X \to \text{Spec } K \) with the \( G \)-action.

Let \( \mathcal{X} \to \mathcal{Y} \) be a G-integral model of \( X \to \text{Spec } K \) over \( S \) and \( \mathcal{Y}' \) the normalization of \( \mathcal{Y} \) in \( K' \). Then \( \mathcal{X}' = (\mathcal{X} \times_\mathcal{Y} \mathcal{Y}')_{\text{red}} \) is a G'-integral model of \( X' \to \text{Spec } K' \) over \( S \). Let \( (g, \sigma) \) be an element of \( G \times H E_{K'_0/S} \) and \( (g', \sigma) \) be the corresponding element of \( G' \times H E_{K'_0/S} \). We assume that Theorem 6.5 holds for \( X' \to \text{Spec } K' \). Then we find a geometric point of \( \mathcal{X}' \) fixed by \( g' \), whose image in \( \mathcal{X} \) is also fixed by \( g \).

Claim 6.7. Let \( U \) be a G-stable dense open subscheme of \( X \). Then, under the induction hypothesis, the assertions for \( U \) and those for \( X \) are equivalent.

We can take a finite Galois extension \( K' \) over \( K \) which is also Galois over \( K_0 \) such that every irreducible component of \( X \times_K K' \) is geometrically irreducible over \( K' \). Then, by Claim 6.6, we may assume that \( X \) is reduced and every irreducible component of \( X \) is geometrically irreducible over \( K \). Further by Claim 6.7, we may assume that \( X \) is normal.

Let \( \{X_i\}_{i=1}^r \) be the set of connected components of \( X \). We may assume that \( g \) transitively permutes the irreducible components. To show the assertion 1 of Theorem 6.1, it suffices to show that the eigenvalues of the \( r \)-th iteration of \( (g, \sigma) \) acting on \( H^d_c(X_i \overline{K}, \mathbb{Q}_\ell) \) are roots of unity. For the assertion 2 of Theorem 6.1 and the assertion of Theorem 6.5, we have

\[
\text{Tr} \left((g, \sigma), H^d_c(X_i \overline{K}, \mathbb{Q}_\ell)\right) = \sum_{i \in I} \text{Tr} \left((g, \sigma), H^d_c(X_i \overline{K}, \mathbb{Q}_\ell)\right).
\]
Thus, we may further assume that $X$ is connected.

As in [3, 2.2.3], we will reduce the proof to the case where $X \to \text{Spec} \ K$ admits a $G$-integral model $\mathcal{X} \to \mathcal{Y}$ over $S$ which is a nodal fibration.

We take any $G$-integral model $\mathcal{X} \to \mathcal{Y}$ of $X \to \text{Spec} \ K$ over $S$, which exists by Lemma 6.4. By Lemma 3.6, we can find projective Galois alterations $(\mathcal{X}', G') \to (\mathcal{X}, G)$ and $(\mathcal{Y}', G') \to (\mathcal{Y}, G)$ with a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
\downarrow f' & & \downarrow f \\
\mathcal{Y}' & \longrightarrow & \mathcal{Y}
\end{array}
$$

such that $f'$ is a $G'$-split nodal fibration. Let $K'$ be the function field of $\mathcal{Y}'$. Applying Claim 6.8 below to the diagram

$$
\begin{array}{ccc}
X \times_X \mathcal{X}' & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec} \ K' & \longrightarrow & \text{Spec} \ K,
\end{array}
$$

we may assume that there exists a $G$-integral model of $X \to \text{Spec} \ K$ over $S$ which is a $G$-split nodal fibration.

**Claim 6.8.** Let $(X', G') \to (X, G)$ be a Galois alteration with a $G'$-equivariant commutative diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec} \ K' & \longrightarrow & \text{Spec} \ K
\end{array}
$$

such that $K'$ is a quasi-Galois extension of $K$ which is also quasi-Galois over $K_0$. Then, under the induction hypothesis, the assertions for $X' \to \text{Spec} \ K'$ with the $G'$-action imply those for $X \to \text{Spec} \ K$ with the $G$-action.

**Proof of Claim 6.8.** By Claim 6.6, we may assume that $K' = K$. By Claim 6.7, we may assume that $X' \to X$ is finite. We write $\Gamma$ for the kernel of the map $G' \to \text{Aut}(X)$. Then, by the definition of Galois alterations, $X'/\Gamma \to X$ is a finite radicial surjection. Then, by Lemma 6.9 below, we have $H_c^q(X_K, \mathbb{Q}_\ell) \cong H^q_c(X'_K, \mathbb{Q}_\ell)^\Gamma$. Thus, the assertion 1 of Theorem 6.1 for $X' \to \text{Spec} \ K'$ with the $G'$-action implies the assertion 1 for $X \to \text{Spec} \ K$ with the $G$-action. Further, we have

$$
\text{Tr} \left( (g, \sigma), H^q_c(X \times_K \overline{K}, \mathbb{Q}_\ell) \right) = \frac{1}{|\Gamma|} \sum_{g'} \text{Tr} \left( (g', \sigma), H^q_c(X' \times_K \overline{K}, \mathbb{Q}_\ell) \right).
$$

Here, $g'$ runs elements of $G'$ which define the same element in $\text{Aut}(X)$ as $g$. Thus, the assertion 2 of Theorems 6.1 and 6.5 for $X' \to \text{Spec} \ K'$ with the $G'$-action also implies those for $X \to \text{Spec} \ K$ with the $G$-action. □
**Lemma 6.9.** Let $K$ be a field and $X$ a scheme separated of finite type over $K$. Let $G$ be a finite group acting admissibly on $X$ over $K$. Then, the natural morphism

$$H^q_c((X/G)_{\overline{K}}, \mathbb{Q}_{\ell}) \rightarrow H^q_{c}(X_{\overline{K}}, \mathbb{Q}_{\ell})^G$$

is an isomorphism, where $\overline{K}$ is a separable closure of $K$ and $\ell$ is a prime number distinct from the characteristic of $K$.

**Proof.** See the proof of [1, Lemma 2.3].

Further, we reduce the proof to the case where there exists a $G$-integral model over $S$ which is a strictly $G$-split nodal fibration (Definition 3.4).

We take a $G$-integral model $\mathcal{X} \rightarrow \mathcal{Y}$ of $X \rightarrow \text{Spec} \ K$ over $S$ which is a $G$-split nodal fibration and a $G$-split nodal fibration datum $(\mathcal{X} = \mathcal{X}_d \rightarrow \cdots \rightarrow \mathcal{X}_0, Z_0, \{\sigma_{ij}\})$ realizing $\mathcal{X} \rightarrow \mathcal{Y}$. By Corollary 3.8, we can find a Galois alteration $(\psi, \alpha) : (\mathcal{Y}', G') \rightarrow (\mathcal{Y}, G)$ with $\mathcal{Y}'$ regular and a divisor $Z'_0$ with $G'$-strict normal crossings containing $\psi^{-1}(Z_0)$. Then, by Lemma 3.7, we can find a $G'$-equivariant commutative diagram

$$\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{Y}' & \longrightarrow & \mathcal{Y}
\end{array}$$

such that $\mathcal{X}' \rightarrow \mathcal{Y}'$ is a strictly $G'$-split nodal fibration. Thus, by Claim 6.8, we may assume that $X \rightarrow \text{Spec} \ K$ admits a $G$-integral model $\mathcal{X} \rightarrow \mathcal{Y}$ over $S$ which is a strictly $G$-split nodal fibration.

Let $(g, \sigma) \in G \times H \ E(K_0/S)$. We may assume that $G$ is a cyclic group and is generated by $g$. Then, by Lemma 2.4, there exists a geometric point $y$ of $\mathcal{Y}$ fixed by $g$, and hence, $G$ naturally acts on the strict localization $\mathcal{Y}(y)$. Further, by Claim 6.7, we may assume that $X = (\mathcal{X} \setminus Z_d) \times_{\mathcal{Y}} \text{Spec} \ K$. Thus, by Lemma 6.2, the proof of the main results follows from Proposition 6.10 below.

**Proposition 6.10.** Let $\mathcal{Y}$ be the spectrum of an excellent regular strictly local ring with an action of a finite group $G$ which is trivial on the residue field of $\mathcal{Y}$. We consider a strictly G-split proper nodal fibration datum $(\mathcal{X} = \mathcal{X}_d \rightarrow \cdots \rightarrow \mathcal{X}_0, Z = Z_0, \{\sigma_{ij}\})$ with $\mathcal{X}_0 = \mathcal{Y}$. Let $K$ be the function field of $\mathcal{Y}$ with a separable closure $\overline{K}$. Let $\ell$ be a prime number invertible on $\mathcal{Y}$. We write $H$ for the Galois group of $K$ over the fixed subfield $K_0 = K^G$. Let $Z_d$ be the closed subset of $\mathcal{X} = \mathcal{X}_d$ as in Definition 3.3. Then, for every $(g, \sigma) \in G \times H \text{ Gal}(\overline{K} / K_0)$,

1. The eigenvalues of the action of $(g, \sigma)$ on $H^q((\mathcal{X} \setminus Z_d) \times_{\mathcal{Y}} \text{Spec} \overline{K}, \mathbb{Q}_{\ell})$, for each $q$, are roots of unity,
2. The alternating sum

$$\sum_q (-1)^q \text{Tr}((g, \sigma), H^q((\mathcal{X} \setminus Z_d) \times_{\mathcal{Y}} \text{Spec} \overline{K}, \mathbb{Q}_{\ell}))$$

is an integer independent of $\ell$, (6.2)
3. If the alternating sum (6.2) is nonzero, then there exists a geometric point of $X$ fixed by $g$.

**Proof.** Let $\mathcal{M}_{\chi}$ be the log structure on $\mathcal{X}$ defined by $Z_i$, where $Z_i$ is the closed subscheme defined as in Definition 3.3. By purity for log regular log schemes [21, Theorem 7.4], we have a canonical isomorphism

$$H^q((\mathcal{X} \setminus Z_d) \times \mathcal{Y} \text{ Spec } \overline{K}, \mathbb{Q}_\ell)) \cong H^q(\mathcal{X}_{\mathcal{K}, \text{két}}, \mathbb{Q}_\ell)$$

for each $q$. Here, $\mathcal{X}_{\mathcal{K}, \text{két}}$ denotes the Kummer étale topos of the log scheme $(\mathcal{X}, \mathcal{M}_{\chi}) \times (\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$ Spec $\overline{K}$, where we regard Spec $\overline{K}$ as a log scheme with the trivial log structure.

By Lemma 4.5, $f^{\log} : (\mathcal{X}, \mathcal{M}_{\chi}) \to (\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$ induced by the composite $\chi \to \mathcal{X}$ is a log smooth and saturated morphism of fs log schemes. By a proper log smooth base change theorem [23, Proposition 4.3], $Rf_*^{\log} \mathbb{Q}_\ell$ is lisse with respect to the Kummer étale topology. Here, for each $q$ the lisse sheaf $R^q f_*^{\log} \mathbb{Q}_\ell$ has a canonical $G$-sheaf structure. Thus, using the proper exact base change theorem [24, Theorem 5.1] and the construction in Sect. 5, we obtain a $G \times_H \text{Gal}(\overline{K}/K_0)$-equivariant isomorphism

$$H^q(\mathcal{X}_{\mathcal{Y}, \text{két}}, \mathbb{Q}_\ell) \cong H^q(\mathcal{X}_{\mathcal{Y}_{\text{két}}, \mathbb{Q}_\ell}).$$

Here, $\mathcal{X}_{\mathcal{Y}, \text{két}}$ denotes the Kummer étale topos of the log scheme $(\mathcal{X}, \mathcal{M}_{\chi}) \times (\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$ $\mathcal{Y}$, where $\mathcal{Y}$ is a log geometric point over $(\mathcal{Y}, \mathcal{M}_{\mathcal{Y}}|_y)$ and where the fiber product is taken in the category of saturated log schemes.

We denote the underlying scheme of the log scheme $\mathcal{X}_{\mathcal{Y}}$ by $(\mathcal{X}_{\mathcal{Y}})^o$. Since $f^{\log}$ is saturated, the natural morphism $(\mathcal{X}_{\mathcal{Y}})^o \to \mathcal{X}_{\mathcal{Y}}$ of usual schemes is an isomorphism by [18, Proposition II.2.13]. We consider the natural morphism $\varepsilon : \mathcal{X}_{\mathcal{Y}} \to (\mathcal{X}_{\mathcal{Y}})^o$ of log schemes. We identify it with the morphism $\mathcal{X}_{\mathcal{Y}} \to \mathcal{X}_{\mathcal{Y}}$, where we regard $\mathcal{X}_{\mathcal{Y}}$ as a log scheme with the trivial log structure. Since $G \times_H \text{Gal}(\overline{K}/K_0)$ acts naturally on $\mathcal{X}_{\mathcal{Y}}$, the sheaf $R^q \varepsilon_* \mathbb{Q}_\ell$ for each $q$ has a canonical $G \times_H \text{Gal}(\overline{K}/K_0)$-sheaf structure and we have a $G \times_H \text{Gal}(\overline{K}/K_0)$-equivariant spectral sequence

$$E_2^{p,q} = H^p(\mathcal{X}_{\mathcal{Y}}, R^q \varepsilon_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\mathcal{X}_{\mathcal{Y}, \text{két}}, \mathbb{Q}_\ell). \quad (6.3)$$

Thus, we have an equality

$$\sum_q (-1)^q \text{Tr}((g, \sigma), H^q(\mathcal{X}_{\mathcal{Y}, \text{két}}, \mathbb{Q}_\ell))$$

$$= \sum_q (-1)^q \sum_p (-1)^p \text{Tr}((g, \sigma), H^p(\mathcal{X}_{\mathcal{Y}}, R^q \varepsilon_* \mathbb{Q}_\ell)).$$

We shall describe $R^q \varepsilon_* \mathbb{Q}_\ell$ in terms of log structures. Let $\overline{\mathcal{M}}_{\mathcal{X}_{\mathcal{Y}}}$ be the cokernel of the morphism $f^{-1} \overline{\mathcal{M}}_{\mathcal{Y}} \to \overline{\mathcal{M}}_{\mathcal{X}_{\mathcal{Y}}}$. It has a natural $G$-sheaf structure coming from the action of $G$ on the log schemes $\mathcal{X}_{\mathcal{Y}}$ and $y$. By [3, Corollaire 5.4], we have a canonical $G \times_H \text{Gal}(\overline{K}/K_0)$-equivariant isomorphism

$$R^q \varepsilon_* \mathbb{Q}_\ell \cong \bigwedge^q (\overline{\mathcal{M}}_{\mathcal{X}_{\mathcal{Y}}}) \otimes \mathbb{Q}_\ell(-1)).$$
Here, we consider the $G \times H \text{Gal}(\overline{K}/K_0)$-sheaf structure on $\mathcal{M}_{X/y}^{\text{gp}}$ defined by the surjection $G \times H \text{Gal}(\overline{K}/K_0) \to G$ and the $G$-sheaf structure on it. Thus, the action of $G \times H \text{Gal}(\overline{K}/K_0)$ on $E_2^{p,q} \cong H^p(X_y, \bigwedge^q (\mathcal{M}_{X/y}^{\text{gp}} \otimes \mathbb{Q}_\ell(-1)))$ factors through the surjection $G \times H \text{Gal}(\overline{K}/K_0) \to G$, and hence, the eigenvalues of $(g, \sigma)$ acting on $E_2^{p,q}$ are roots of unity. Therefore, the assertion 1 of Theorem 6.1 follows. Further, we have an equality

$$\sum_p (-1)^p \text{Tr}((g, \sigma), H^p(X_y, R^q j_* \mathbb{Q}_\ell))$$

$$= \sum_p (-1)^p \text{Tr}(g, H^p(X'_y, \bigwedge^q (\mathcal{M}_{X'_y}^{\text{gp}} \otimes \mathbb{Q}_\ell)))$$

for each $q$. Then, the rest of the assertions follows from Proposition 6.11 below. \hfill \Box

**Proposition 6.11.** Let $\mathcal{Y}$ be the spectrum of an excellent regular strictly local ring with an action of a finite group $G$ which is trivial on the residue field of $\mathcal{Y}$. We consider a $G$-equivariant commutative diagram

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
j \downarrow & & \downarrow j' \\
U & \longrightarrow & V
\end{array}$$

such that $\mathcal{X}$ is regular, $f$ is flat and proper, $j$ and $j'$ are open immersions, and $D = \mathcal{X}\setminus U$ and $Z = \mathcal{Y}\setminus V$ are divisors with simple normal crossings. We endow $\mathcal{X}$ (resp. $\mathcal{Y}$) with the log structure $\mathcal{M}_\mathcal{X}$ (resp. $\mathcal{M}_\mathcal{Y}$) defined by $D$ (resp. $Z$). Let $y$ be the closed point of $\mathcal{Y}$ and let $M_{\mathcal{X}y}$ be the cokernel of the morphism $f^{-1}\mathcal{M}_y^{\text{gp}} \to \mathcal{M}_X^{\text{gp}}$ with a natural action of $G$, where $\mathcal{M}_{X'_y}$ (resp. $\mathcal{M}_{Y_y}$) is the log structure on $X'_y$ (resp. $y$) obtained from $\mathcal{M}_X$ (resp. $\mathcal{M}_Y$) via pullback. Then, for every $g \in G$ and $q \geq 0$,

1. The alternating sum

$$\sum_p (-1)^p \text{Tr}(g, H^p(X'_y, \bigwedge^q (\mathcal{M}_{X'_y}^{\text{gp}} \otimes \mathbb{Q}_\ell))) (6.4)$$

is an integer independent of a prime number $\ell$ distinct from the residual characteristic of $\mathcal{Y}$,

2. If the alternating sum (6.4) is nonzero, then there exists a geometric point of $\mathcal{X}$ fixed by $g$.

**Proof.** We write $D$ and $Z$ as the sums of their irreducible components: $D = \sum_{i \in I} D_i$ and $Z = \sum_{j \in J} Z_j$. For a subset $I' \subset I$, we put $D_{I'} = \bigcap_{i \in I'} D_i$ and $D_{I'}' = D_{I'} \setminus \bigcup_{i \in I \setminus I'} D_i$. Let $I_f$ be the subset of $I$ consisting of elements $i \in I$ such that $f(D_i) \subset Z$. Here, since $f$ is assumed to be flat, we can consider the
pullback $f^*Z$ of the divisor $Z$ and a map $\varphi : I_f \to J$ sending $i$ to $j$ such that $f(D_i) \subset Z_j$. We write the divisor $f^*Z$ as $\sum_{i \in I} m_i D_i$ for some $m_i \in \mathbb{Z}_{\geq 0}$.

Let $I'$ be a $G$-stable subset of $I$. Then, by Lemma 4.1, we have a $G$-equivariant commutative diagram:

$$
\begin{array}{ccc}
(f^{-1}M_{y/I_f})_{\varphi_I'} & \xymatrix{\cong \ar[r] & } & M_{\varphi_I'} \\
\mathbb{Z}^J & \xymatrix{\varphi_I' \ar[r] & } & \mathbb{Z}^{I'}
\end{array}
$$

where $\varphi_I'$ is the homomorphism sending $e_j$ to $\sum_{i \in I' \cap \varphi^{-1}(j)} m_i e_i$. Thus, we get an isomorphism of $G$-sheaves

$$
(M_{X/\mathcal{Y}_y})_{\varphi_I'} \cong M_{I'}, \quad (6.5)
$$

where $M_{I'}$ is the constant sheaf given by the cokernel of the map $\varphi_I'$ with the $G$-sheaf structure defined by the action of $G$ on $I'$.

We may and do assume that $G$ is a cyclic group generated by $g$. We consider the power set $\mathcal{P}(I)$ of $I$, that is, the set of all subsets of $I$, and the natural action of $G$ on $\mathcal{P}(I)$. For $A \in G \setminus \mathcal{P}(I)$, i.e., for a $G$-orbit $A \subset \mathcal{P}(I)$, we set $\check{D}_A = \bigcup_{I' \in A} \check{D}_{I'}$.

Then $(\check{D}_A)_y$ give a stratification $X_y = \bigsqcup_{A \in G \setminus \mathcal{P}(I)} (\check{D}_A)_y$ by $G$-stable locally closed subsets, and the alternating sum (6.4) is equal to

$$
\sum_{A \in G \setminus \mathcal{P}(I)} \sum_p (-1)^p \text{Tr} \left( g, H^p_c \left( (\check{D}_A)_y, \bigwedge^q \left( M_{X_y/\mathcal{Y}} \otimes \mathbb{Q}_\ell \right) \right) \right).
$$

Here, $\text{Tr}(g, H^p_c ((\check{D}_A)_y, -))$ is nonzero only if $A$ is the orbit of a $G$-stable subset of $I$. Thus, it suffices to show that for a $G$-stable subset $I'$ of $I$, the alternating sum

$$
\sum_p (-1)^p \text{Tr} \left( g, H^p_c ((\check{D}_{I'})_y, \bigwedge^q \left( M_{X_y/\mathcal{Y}} \otimes \mathbb{Q}_\ell \right) \right). \quad (6.6)
$$

is an integer independent of $\ell$ distinct from the residual characteristic of $\mathcal{Y}$ and that, if the alternating sum (6.6) is nonzero, there exists a geometric point of $X$ fixed by $g$. By the above description (6.5) of $M_{X_y/\mathcal{Y}}$, we obtain a $G$-equivariant isomorphism

$$
H^p_c ((\check{D}_{I'})_y, \bigwedge^q \left( M_{X_y/\mathcal{Y}} \otimes \mathbb{Q}_\ell \right)) \cong H^p_c ((\check{D}_{I'})_y, \mathbb{Q}_\ell) \otimes \bigwedge^q M_{I'}.
$$

Hence, the alternating sum (6.6) is equal to

$$
\text{Tr} \left( g, \bigwedge^q M_{I'} \otimes \mathbb{Q} \right) \cdot \sum_p (-1)^p \text{Tr} \left( g, H^p_c ((\check{D}_{I'})_y, \mathbb{Q}_\ell) \right).
$$
Thus, it suffices to show the following: Let $X$ be a scheme separated of finite type over a separably closed field $K$ with an action of a finite group $G$. Then for every $g \in G$, the alternating sum $\sum_q (-1)^q \Tr(g, H^q_c(X, \mathbb{Q}_\ell))$ is an integer independent of $\ell$ different from the characteristic of $K$, and if the alternating sum is nonzero, then for every $G$-equivariant compactification $\mathcal{X}$ of $X$, there exists a geometric point of $\mathcal{X}$ fixed by $g$. Note that these are nothing but Theorems 6.1 and 6.5 in the case where $K$ is separably closed, $\text{Spec} K = S$, and the $G$-action on $K$ is trivial. As we did in the beginning of the proof, we can reduce, using alterations, the proof to the case where $X$ is smooth and projective over $K$. But, in this case, the assertion follows from the Lefschetz trace formula. The following is an argument for reduction to the smooth projective case.

First, we may assume that $K$ is algebraically closed and $X$ is reduced. Then we may assume that $X$ is normal (by Claim 6.7) and connected. Take a $G$-equivariant normal compactification $\mathcal{X}$ of $X$ (Lemma 6.3). By [14, Theorem 7.3], we can find a surjection $G' \to G$ of finite groups and a Galois alteration $(\mathcal{X}', G') \to (\mathcal{X}, G)$ over $K$ with $\mathcal{X}'$ smooth and projective. Thus, by Claim 6.8, we may assume that $X$ has a smooth projective compactification. Using Claim 6.7 again, we reduce the proof to the case where $X$ is smooth and projective. $\square$

Remark 6.12. We make comments on $G$-strictness of divisors with simple normal crossings.

1. We do not use $G$-strictness in the proof of Theorem 6.1: In the proof of Theorem 6.1, we reduce the problem to the case where $X \to \text{Spec} K$ admits a $G$-integral model which is a strictly $G$-split nodal fibration, and in the definition (Definition 3.3, 3.4) of a strictly $G$-split nodal fibration, we impose the condition that $Z_j$ is a divisor with $G$-strict normal crossings. But, for the argument after the reduction to work, it suffices to reduce to the case where $X \to \text{Spec} K$ admits a $G$-integral model which is a strictly split nodal fibration.

2. The $G$-strictness assumption makes the argument simpler in the sense that we can avoid to see combinatorial action of $G$ on the set $I$ of irreducible components of the divisor $D$, which appears in the proof of Proposition 6.11. In fact, the $G$-strictness assures that the intersection $D_{I'}$ is nonempty only if the action of $G$ on $I'$ is trivial. Further, since we have $\sum_q (-1)^q \Tr(g, \bigwedge^q M_{I'}) = 0$ for such $I'$ if $I'$ is nonempty, we get a simpler description of the alternating sum (6.2): it is equal to $\sum_p (-1)^p \Tr(g, H^p((\hat{D}_{\emptyset})_{\bar{y}}, \mathbb{Q}_\ell))$.

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8. Appendix: \(\ell\)-independence over Henselian valuation fields (by Qing Lu\(^2\) and Weizhe Zheng \(^3\))

In this “Appendix” we prove results on \(\ell\)-independence and integrality of \(\ell\)-adic cohomology over Henselian valuation fields with not necessarily discrete valuations, for the action of inertia and Weil subgroups of the Galois group. We deduce these results from relative results on compatible systems and integral sheaves over discrete valuation fields (\([7,25]\)) using the valuative criteria of \([5]\). Our result for inertia action (Theorem A.1) slightly generalizes Hiroki Kato’s Theorem 6.1 and our method is different from his. We thank him for allowing us to present our results in this “Appendix”.

Let \(K\) be a field equipped with an action of a finite group \(G\). Let \(X\) be an algebraic space of dimension \(d\) of finite presentation over \(K\), equipped with an equivariant action of \(G\). Let \(K_0 = K^G\), \(H = \text{Gal}(K/K_0)\) and let \(\bar{K}\) denote a separable closure of \(K\). For every prime number \(\ell\) invertible in \(K\) and every integer \(i\), \(G \times_H \text{Gal}(\bar{K}/K_0)\) acts continuously on \(H^i_{\text{c}}(X_{\bar{K}}, \mathbb{Q}_\ell)\) and \(H^i(X_{\bar{K}}, \mathbb{Q}_\ell)\).

**Theorem A.1.** Let \(\mathbb{L}\) be a set of prime numbers not containing the characteristic of \(K\). Then for every \((g, \sigma) \in G \times_H E_{K_0/\text{Spec}(\mathbb{Z}[\mathbb{L}^{-1}])}\) and every \(i\), the eigenvalues of \((g, \sigma)\) acting on \(H^i_{\text{c}}(X_{\bar{K}}, \mathbb{Q}_\ell)\) and \(H^i(X_{\bar{K}}, \mathbb{Q}_\ell)\) are roots of unity for \(\ell \in \mathbb{L}\), and

\[
\sum_i (-1)^i \text{tr}((g, \sigma), H^i_{\text{c}}(X_{\bar{K}}, \mathbb{Q}_\ell)) = \sum_i (-1)^i \text{tr}((g, \sigma), H^i(X_{\bar{K}}, \mathbb{Q}_\ell))
\]

is a rational integer independent of \(\ell \in \mathbb{L}\).

Recall from Definition 2.1 that \(E_{K_0/\text{Spec}(\mathbb{Z}[\mathbb{L}^{-1}])} \subseteq \text{Gal}(\bar{K}/K_0)\) denotes the ramified part, namely the closure of the union of the images of \(\text{Gal}(\bar{L}/L) \to \text{Gal}(\bar{K}/K_0)\) for all commutative diagrams

\[
\begin{array}{cccc}
\bar{L} & \to & \bar{K} \\
\uparrow & & \uparrow \\
L & \to & K_0 \\
\uparrow & & \uparrow \\
\mathcal{O}_L & \to & \mathbb{Z}[\mathbb{L}^{-1}]
\end{array}
\]

with \(\mathcal{O}_L\) a strictly Henselian valuation ring of fraction field \(L\), and \(\bar{L}\) a separable closure of \(L\). In particular, if \(K_0\) is the fraction field of a Henselian valuation

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ring of residue characteristic \( \not\in \mathbb{L} \), then \( E_{K_0/\text{Spec}(\mathbb{L}[L^{-1}])} \) is the inertia subgroup of \( \text{Gal}(\overline{K}/K_0) \).

**Theorem A.2.** Assume that \( K_0 \) is the fraction field of a Henselian valuation ring \( \mathcal{O} \) of finite residue field \( \mathbb{F}_q \). Then for all \((g, \sigma) \in G \times_H W(\overline{K}/K_0)\) with \( \sigma \) of degree \( v \geq 0 \), \( i \in \mathbb{Z}, \ell \not\mid q \), and all eigenvalues \( \alpha \) of \((g, \sigma)\) acting on \( H^i_c(X_{\overline{K}}, \mathbb{Q}_\ell) \) and \( H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \), the numbers \( q^{v(d-i)}\alpha, q^{vd}\alpha^{-1}, q^v\alpha^{-1} \) are algebraic integers, and

\[
\sum_i (-1)^i \text{tr}((g, \sigma), H^i_c(X_{\overline{K}}, \mathbb{Q}_\ell)), \quad \sum_i (-1)^i \text{tr}((g, \sigma), H^i(X_{\overline{K}}, \mathbb{Q}_\ell))
\]

are rational integers independent of \( \ell \neq q \). Here \( W(\overline{K}/K_0) \) denotes the inverse image of the Weil group \( W(\overline{\mathbb{F}_q}/\mathbb{F}_q) \) under the reduction map \( r: \text{Gal}(\overline{K}/K_0) \to \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \), and that \( \sigma \) has degree \( v \) means \( r(\sigma) = \text{Fr}_q^v \), where \( \text{Fr}_q \) denotes the geometric Frobenius \( a \mapsto a^{1/q} \).

Theorem A.2 was previously known under the additional assumption that \( \mathcal{O} \) is a finite field or a discrete valuation ring ([1,26], [7,25,27, Appendix]).

**Proof of the theorems.** In Theorem A.1, we may assume by continuity (see Lemma A.3 below) that \( K_0 = L \) is the fraction field of a strictly Henselian valuation ring \( \mathcal{O} \) over \( \mathbb{Z}[L^{-1}] \). In this case, the equality follows from the equivariant form of a theorem of Laumon [22, Theorem 2.2]. We will prove the rest of Theorem A.1 and Theorem A.2 in parallel. In Theorem A.2 we take \( \mathbb{L} \) to be the set of primes \( \ell \not\mid q \).

Let us prove that the alternating sums in the theorems are in \( \mathbb{Q} \) and independent of \( \ell \in \mathbb{L} \). The first step is to take care of the finite group action. Up to replacing \( G \) by \( \langle g \rangle \), we may assume that \( G \) is abelian. We fix \( \sigma, \) of image \( \overline{\sigma} \in H \). Let \( t_{g, \ell}^c \) and \( t_{g, \ell} \) denote the alternating sums in the theorems. Let \( \overline{\mathbb{Q}} \) be an algebraic closure of \( \mathbb{Q} \) and let \( I \) be the set of pairs \((\ell, i)\), where \( \ell \in \mathbb{L} \) and \( \iota: \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_{\ell} \) is an embedding. It suffices to show the existence of \( t_{g, \ell}^c \in \overline{\mathbb{Q}} \) such that \( t_{g, \ell}(\iota) = t_{g, \ell}^c \) for all \((\ell, i) \in I \) (cf. [7, Remarque 1.18 (iv)]). For any character \( \chi: G \to (\overline{\mathbb{Q}})\times \), let \( V_{\chi_{\iota}} \) denote the one-dimensional \( \overline{\mathbb{Q}}_{\ell}\)-representation of character \( \iota \chi \) and let \( L_{\chi_{\iota}} \) denote the corresponding lisse \( \overline{\mathbb{Q}}_{\ell}\)-sheaf on the quotient stack \([X/G] \). Let \([X/G]_{\overline{K}} := [X/G] \otimes_{K_0} \overline{K} \) and let \( \pi: G \to H \). Then \( H^*(c)_c([X/G]_{\overline{K}}, L_{\chi_{\iota}}) \simeq (H^*(c)_c(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_{\ell}} V_{\chi_{\iota}})^{\text{Ker}(\pi)} \), so that

\[
\text{tr}(\sigma, H^*(c)_c([X/G]_{\overline{K}}, L_{\chi_{\iota}})) = \frac{1}{\#\text{Ker}(\pi)} \sum_{g \in \pi^{-1}(\overline{\sigma})} t_{g, \ell}^c(\chi(g)).
\]

Since every function \( \pi^{-1}(\overline{\sigma}) \to \overline{\mathbb{Q}} \), in particular every indicator function, is a \( \overline{\mathbb{Q}} \)-linear combination of the \( \chi |_{\pi^{-1}(\overline{\sigma})} \)'s, it suffices to show that \( (\text{tr}(\sigma, H^*(c)_c([X/G]_{\overline{K}}, L_{\chi_{\iota}}))) \) is \( I \)-compatible.

The rest of the proof is similar to the proof of [5, Theorem 1.4], except that the base here may have mixed characteristics. By standard limit arguments, there exists a finitely generated sub-algebra \( R \subseteq K_0 \) over \( \mathbb{Z} \) such that \( X \) and the action of \( G \) are defined over \( B = \text{Spec}(R) \); there exists an algebraic space \( \mathcal{X} \) of finite presentation
over \( B \), equipped with an action of \( G \) by \( B \)-automorphisms, such that we have a \( G \)-equivariant isomorphism \( X \simeq \mathcal{X} \times_B \text{Spec}(K) \) over \( K \). Let \( f : [\mathcal{X}/G] \to B \). If the residue field of \( \mathcal{O} \) has characteristic \( > 0 \), let \( p \) be its characteristic. Otherwise (this case may only occur in Theorem 1), noting that it suffices to show \( J \)-compatibility for every finite subset \( J \subseteq I \), we may assume that \( \mathbb{L} \) is not the set of all primes and we choose \( p \not\in \mathbb{L} \). In both cases, we obtain a commutative square

\[
\begin{array}{ccc}
\text{Spec}(K_0) & \longrightarrow & B(p) \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{O}) & \longrightarrow & \text{Spec}(\mathbb{Z}(p)),
\end{array}
\]

where \( \mathbb{Z}(p) \) denotes the Henselization of \( \mathbb{Z} \) at the ideal \( (p) \), which is an excellent Henselian discrete valuation ring, and \( B(p) = \text{Spec}(R \otimes_{\mathbb{Z}} \mathbb{Z}(p)) \). Let \( h : [\mathcal{X}(p)/G] \to B(p) \) denote the base change of \( f \) to \( B(p) \). We still denote by \( \mathcal{L}_{\mathcal{X}} \) the lisse \( \mathcal{O}_\ell \)-sheaf on \([\mathcal{X}(p)/G]\) given by \( V_{\mathcal{X}} \). By [7] (as summarized in [5, Theorem 2.3]), \( (Rh_{i!}\mathcal{L}_{\mathcal{X}}) \) and \( (Rh_{*}\mathcal{L}_{\mathcal{X}}) \) are \( J \)-compatible systems on \( B(p) \). Moreover, since \( \text{Spec}(K_0) \) maps to the generic point of \( B(p) \), \( H^1((X/G)_{\kappa}, \mathcal{L}_{\mathcal{X}}) \simeq (R^1h_{*}\mathcal{L}_{\mathcal{X}})_{\kappa} \) and similarly for \( H^1 \). Thus by the valuative criterion for compatible systems [5, Corollary 1.3], \( (\text{tr}(\sigma, H^1_{\mathcal{O}}((X/G)_{\kappa}, \mathcal{L}_{\mathcal{X}}))) \) is \( J \)-compatible. This finishes the proof of rationality and \( \ell \)-independence.

It remains to show the assertions on eigenvalues. For these assertions we may replace \( (g, \sigma) \) by \( (g, \sigma)^n \) for any \( n \geq 1 \). Thus we may assume \( G = \{1\} \). We conclude the proof of Theorem A.1 by the last assertion of [5, Corollary 3.10] applied to \( R^i h_{i!}\overline{\mathcal{O}}_\ell \) and \( R^i h_{*}\overline{\mathcal{O}}_\ell \).

Finally, we prove the assertion on eigenvalues in Theorem A.2. If \( K_0 \) has characteristic \( 0 \), we may remove the fiber at \( p \) from \( B \). Thus we may assume that \( B(p) \) is above either the closed point or the generic point of \( \text{Spec}(\mathbb{Z}(p)) \). Further shrinking \( B \) if necessary, we may assume that \( B(p) \) is regular of pure dimension \( d' \), \( \mathcal{X}(p) \) has dimension \( d'' := d + d' \), and \( R^i h_{i!}\overline{\mathcal{O}}_\ell \) and \( R^i h_{*}\overline{\mathcal{O}}_\ell \) are lisse. By [25, Proposition 6.4], \( R^i h_{i!}\overline{\mathcal{O}}_\ell \), \( R^i h_{i!}\overline{\mathcal{O}}_\ell (i - d) \), \( (R^i h_{i!}\overline{\mathcal{O}}_\ell (i))^\vee \), \( (R^i h_{i!}\overline{\mathcal{O}}_\ell (d))^\vee \), \( R^i h_{*}\overline{\mathcal{O}}_\ell \), \( (R^i h_{*}\overline{\mathcal{O}}_\ell (i))^\vee \) are integral sheaves on \( B(p) \). Here \( \mathcal{F}^\vee := \mathcal{H}\text{om}(\mathcal{F}, \overline{\mathcal{O}}_\ell) \) for \( \mathcal{F} \) lisse. Moreover, \( Rh_{*}\overline{\mathcal{O}}_\ell \simeq R\mathcal{H}\text{om}(Rh_{!}\mathcal{X}(p)_{\overline{\mathcal{O}}_\ell}, \overline{\mathcal{O}}_\ell(d)[2d'']) \), where we adopted the usual normalization of dualizing functor \( Dy := R\mathcal{H}\text{om}(\mathcal{O} \to \mathcal{O}_\ell) \) for \( a : Y \to \text{Spec}(k) \). By [25, Propositions 6.2, 6.4], \( (R^i h_{i!}\overline{\mathcal{O}}_\ell (i - d))^\vee \) and \( (R^i h_{i!}\overline{\mathcal{O}}_\ell (i + d''))^\vee \) are integral. It follows that \( R^i h_{*}\overline{\mathcal{O}}_\ell (i - d) \) and \( (R^i h_{*}\overline{\mathcal{O}}_\ell (d))^\vee \) are integral. We conclude by the valuative criterion for integrality [5, Corollary 3.10].

The following continuity lemma is extracted from the proof of [5, Lemma 4.5]. Let \( P^n_\ell \simeq \overline{\mathbb{Q}}_\ell^n \) denote the space of monic polynomials in \( \overline{\mathbb{Q}}_\ell[T] \) of degree \( n \).

**Lemma A.3.** Let \( I \) be a finite set. For each \( i \), let \( \ell_i \) be a prime and \( n_i \geq 0 \) an integer. Let \( E \) be a topological space equipped with continuous maps \( \rho_i : E \to \text{GL}_{n_i}(\overline{\mathbb{Q}}_{\ell_i}) \) and let \( U \subseteq E \) be a dense subspace. Let \( S \subseteq \prod_i P^n_{\ell_i} \) be a subset. Assume that \( \rho_i(\sigma) \) is quasi-unipotent and \( (\det(T \cdot 1 - \rho_i(\sigma)))_i \in S \) for all \( \sigma \in U \). Then the same holds for all \( \sigma \in E \).
Proof. Consider the continuous map \( \lambda: E \to \prod_i P_{\ell_i}^{n_i} \) carrying \( \sigma \) to \( (\det(\rho_i(\sigma) - T \cdot 1))_i \). As in [5, Remark 2.11], let \( P_{\ell_i}^{n_i,\text{qu}} \subseteq P_{\ell_i}^{n_i} \) denote the subset of polynomials whose roots are all roots of unity, which is discrete and closed. By assumption, \( \lambda(U) \subseteq S \cap \prod_i P_{\ell_i}^{n_i,\text{qu}} \). Thus \( \lambda(U) = \lambda(E) \). \( \Box \)

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