Semi-implicit energy-preserving numerical schemes for stochastic wave equation via SAV approach

Jianbo Cui‡, Jialin Hong†, and Liying Sun‡

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong.

†‡Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P.R.China

‡Institute of Applied Physics and Computational Mathematics, Beijing, 100094, China.

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Abstract

In this paper, we propose and analyze semi-implicit numerical schemes for the stochastic wave equation (SWE) with general nonlinearity and multiplicative noise. These numerical schemes, called stochastic scalar auxiliary variable (SAV) schemes, are constructed by transforming the considered SWE into a higher dimensional stochastic system with a stochastic SAV. We prove that they can be solved explicitly and preserve the modified energy evolution law and the regularity structure of the original system. These structure-preserving properties are the keys to overcoming the mutual effect of the noise and nonlinearity. By proving new regularity estimates of the introduced SAV, we establish the strong convergence rate of stochastic SAV schemes and the further fully-discrete schemes with the finite element method in spatial direction. To the best of our knowledge, this is the first result on the construction and strong convergence of semi-implicit energy-preserving schemes for nonlinear SWE.

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Key Words: stochastic wave equation, semi-implicit numerical scheme, energy evolution law, SAV approach, strong convergence

1 Introduction

In recent years, the stochastic wave equation has been widely exploited to characterize the sound propagation in the sea, the dynamics of the primary current density vector field within

Corresponding author: liyingsun@lsec.cc.ac.cn
the grey matter of the human brain, heat conduction around a ring, the dilatation of shock waves throughout the sun, the vibration of a string under the action of stochastic forces, etc., (see, e.g., [16–18,23,26]). As an intrinsic property of such stochastic partial differential equation (SPDE), the energy evolution law describes the longtime behaviour of the original system and is a keystone to prove the well-posedness of SWE with complex nonlinearities (see, e.g., [4,7,8]). Due to the loss of the explicit expression of the analytical solution, a lot of researchers have been concerned with numerical schemes to simulate SWE and the associated energy evolution law.

Up to now, various energy-preserving schemes have been developed for solving SWEs (see, e.g., [1,2,5,10,13,19]). For instance, the finite element method and the stochastic trigonometric scheme are shown in [10] to preserve the linear growth of the averaged energy for the linear SWE with additive noise. With respect to the SWE with globally Lipschitz continuous coefficient and additive noise, [2] presents that the discontinuous Galerkin finite element method satisfies the trace formula. By means of the linear finite element approximation and a stochastic trigonometric method, [1] designs a fully-discrete scheme for the nonlinear SWE driven by multiplicative noise, and proves that its numerical solution satisfies an almost trace formula in additive noise case. [19] constructs several fully-discrete schemes which preserve the averaged energy evolution law for nonlinear SWE driven by multiplicative noise based on the Padé approximation. For the SWE with superlinear coefficient and additive noise, [13] proposes an implicit fully-discrete scheme by adopting the spectral Galerkin method and the averaged vector field method, and establishes its strong convergence rate, where the exponential integrability and energy-preserving property of the numerical solution play key roles.

Despite fruitful results on energy-preserving numerical schemes for nonlinear SWEs, one has to solve the large stochastic algebraic system by iterative methods due to the implicitness of existing numerical schemes in general. In practice, it would be ideal to be able to treat the nonlinear term in SWE explicitly. However, it is known that the classical explicit and semi-implicit numerical schemes for stochastic differential equations with superlinear coefficients suffer from the strong or weak divergence phenomenon (see, e.g., [11,20]). To deal with this issue, the existing approach often exploits implicit schemes, or uses the tamed (or truncated) strategy to achieve explicit numerical schemes which could not preserve the energy evolution law. Up to now, there has not been any numerical scheme which is explicitly solvable and can preserve the averaged energy evolution law of nonlinear SWE. One main objective of the present work is to fill this gap for nonlinear SWE via a structure-preserving approach.

For deterministic conservative partial differential equations and gradient flow systems, the SAV approach has achieved a lot of successes in constructing semi-implicit numerical schemes which could preserve the energy decaying property or the modified energy conservation law (see, e.g., [24,25]). In contrast, less result on the SAV schemes is known for SPDEs. A natural question is how the noise influences the construction and analysis of numerical schemes based on the SAV approach. It turns out that several new challenges appear in this aspect for nonlinear SWE. First, the energy suffers from the random effect and satisfies the stochastic energy evolution law, rather than the energy conservation law. Second, the direct usage of SAV approach may not balance the diffusion coefficient and the new auxiliary variable,
which brings troubles in constructing numerical schemes preserving the modified energy evolution law. Last, due to the mutual effect of noise and nonlinearity on the stochastic auxiliary variable, the strong convergence analysis of stochastic SAV numerical schemes is more complicated than the standard error estimate.

In this paper, we first transform the considered SWE into a higher dimensional stochastic system (called the stochastic SAV reformulation) by introducing a stochastic SAV. By further exploiting the exponential Runge–Kutta methods and Runge–Kutta methods, as well as the spatial finite element method, we propose two novel kinds of semi-linear temporal semi-discrete stochastic SAV schemes and the further fully-discrete schemes. Then we prove that the proposed stochastic SAV schemes could preserve the averaged energy evolution law for the stochastic SAV reformulation, and overcome the divergence issue in the superlinear coefficient case. By establishing new regularity estimates of the stochastic auxiliary variable, we prove that the temporal strong convergence order of stochastic SAV numerical schemes is 1 under the globally Lipschitz continuous condition. In particular, this result is even new for the deterministic SWE. The convergence analysis of stochastic SAV numerical schemes in the non-globally Lipschitz case is beyond the scope of this paper, and will be studied in the future.

The rest of this paper is organized as follows. Section 2 presents an abstract formulation of SWE, and introduces its fundamental properties. In Section 3, the semi-implicit energy-preserving numerical schemes based on the SAV approach for SWEs are proposed. Section 4 is devoted to deducing the strong error estimate for the proposed numerical schemes. In Section 5, we present energy-preserving fully-discrete schemes based on the SAV approach and the finite element method. Numerical experiments are carried out in Section 6 to verify theoretical results.

2 Stochastic wave equation

In this section, we present the main assumptions and introduce the properties of the SWE with general nonlinearities and multiplicative noise, i.e.,

$$\begin{aligned}
\begin{cases}
  du(t) = v(t)dt, & \text{in } \mathcal{O} \times (0,T], \\
  dv(t) = \Lambda u(t)dt - f(u(t))dt + g(\Theta u(t))dW(t), & \text{in } \mathcal{O} \times (0,T], \\
  u(0) = u_0, & \text{in } \mathcal{O},
\end{cases}
\end{aligned}$$

where $\mathcal{O} \subset \mathbb{R}^d$, $d \leq 3$, is a bounded convex domain with polygonal boundary $\partial \mathcal{O}$, $T \in (0, \infty)$, $u_0, v_0 : \mathcal{O} \to \mathbb{R}$ are $\mathcal{F}_0$-measurable, and $\Theta u(t) := (u(t), \partial x_1 u(t), \cdots, \partial x_d u(t))$ with $\partial x_i = \frac{\partial}{\partial x_i}, i \leq d$. Here, $\Lambda = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator with the homogeneous Dirichlet boundary condition with eigenpairs $\{(\lambda_i, e_i)\}_{i=1}^{\infty}$, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and $\{e_i\}_{i=1}^{\infty}$ forms an orthonormal basis in $L^2 := L^2(\mathcal{O}; \mathbb{R})$. Moreover, $W(\cdot) = \sum_{k=1}^{\infty} Q^{\frac{d}{2}} e_k \beta_k(\cdot)$ is an $L^2$-valued $\mathbb{Q}$-Wiener process with respect to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\{\beta_k\}_{k \in \mathbb{N}+}$
is a sequence of i.i.d. real-valued Brownian motions and $Q$ is a symmetric, positive definite and finite trace operator. Denoting $L^p := L^p(\mathcal{O}, \mathbb{R})$, $p \geq 1$, and $\mathbb{H}^r := \mathcal{D}((\Lambda)^{\frac{r}{2}})$, $r \in \mathbb{R}$, equipped with the inner product $\langle x, y \rangle_{\mathbb{H}^r} = \langle (-\Lambda)^{\frac{r}{2}}x, (-\Lambda)^{\frac{r}{2}}y \rangle_{L^2}$, we introduce the mild assumptions on $f : \mathbb{H}^1 \to \mathbb{H}$ and $g : \mathbb{H}^1 \otimes \mathbb{H}^{\otimes d} \to L_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})$, where $\mathbb{H} := \mathbb{H}^0 = L^2$ and $L_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})$ is the space of Hilbert–Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}(\mathbb{H})$ into $\mathbb{H}$.

**Assumption 2.1.** Assume that $\|Q\|_{\mathcal{L}_2(\mathbb{H}, \mathbb{H})} < \infty$,

$$
\|f(u)\|_{\mathbb{H}} + \|g(\Theta u)\|_{\mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})} \leq c_0(\|u\|_{\mathbb{H}^1} + 1),
$$

$$
\|f(u) - f(\tilde{u})\|_{\mathbb{H}} + \|g(\Theta u) - g(\Theta \tilde{u})\|_{\mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})} \leq c_1\|u - \tilde{u}\|_{\mathbb{H}^1}
$$

for $u, \tilde{u} \in \mathbb{H}^1$, where $c_0, c_1 \geq 0$.

**Assumption 2.2.** Assume that

$$
\|f(u)\|_{\mathbb{H}} \leq b_1(\|u\|_{\mathbb{H}^1}),
$$

$$
\|f(u) - f(u)\|_{\mathbb{H}} \leq b_2(\|u\|_{\mathbb{H}^1}, \|\tilde{u}\|_{\mathbb{H}^1})\|u - \tilde{u}\|_{\mathbb{H}^1},
$$

$$
\|g(\Theta u)\|_{\mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})} \leq c_2(1 + \|u\|_{\mathbb{H}^1}^{k+1} + \|u\|_{\mathbb{H}^1}^2),
$$

$$
\|g(\Theta u) - g(\tilde{u})\|_{\mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})} \leq b_3(\|u\|_{\mathbb{H}^1}, \|\tilde{u}\|_{\mathbb{H}^1})\|u - \tilde{u}\|_{\mathbb{H}^1},
$$

where $c_2 \geq 0$, and $b_1, b_2, b_3$ are polynomials with degree $2k + 1$, $2k$ and $k$, $k \in \mathbb{N}^+$, respectively. Furthermore, suppose that $\|Q\|_{\mathcal{L}_2(\mathbb{H}, \mathbb{H})} < \infty$, and that there exist positive constants $c_3, c_4, \delta_0 > 0$ such that the potential $F(u) = \int_{\mathbb{O}} \tilde{F}(u(x))dx$ satisfies

$$
f(u) = \tilde{F}'(u) \quad \text{and} \quad F(u) + \delta_0 \geq c_3\|u\|_{\mathbb{H}^1}^2 + c_4\|u\|_{\mathbb{H}^1}^{2k+2}.
$$

**Remark 2.1.** A typical example of $f$ satisfying Assumption 2.2 is the Nemitskii operator of $f(\xi) = \sum_{j=1}^{2k+1} a_j \xi^j$ with odd degree $2k + 1$ and $a_{2k+1} > 0$. More precisely, when $d \leq 2$, $k \in \mathbb{N}^+$ and when $d = 3, k = 1$ (see, e.g., [7]). It can be also found that Assumption 2.1 is a special case of Assumption 2.2.

### 2.1 Well-posedness and energy evolution law

To present spatial and temporal regularity estimates, as well as the averaged energy evolution law, of [1], we let $X = (u, v)^\top$. Then the compact form of [1] reads

$$
\frac{dX(t)}{dt} = AX(t)dt + \mathbb{F}(X(t))dt + \mathbb{G}(X(t))dW(t), \quad t \in (0, T],
$$

where

$$
X(0) = X_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \quad \mathbb{F}(X(t)) = \begin{bmatrix} 0 \\ -f(u(t)) \end{bmatrix}, \quad \mathbb{G} = \begin{bmatrix} 0 \\ g(\Theta u(t)) \end{bmatrix}
$$
with $I$ being the identity operator defined in $L^2$. Set the product space $\mathbb{H}^r := \dot{\mathbb{H}}^r \times \dot{\mathbb{H}}^{r-1}$, $r \in \mathbb{R}$, endowed with the inner product $\langle X_1, X_2 \rangle_{\mathbb{H}^r} = \langle x_1, x_2 \rangle_{\dot{\mathbb{H}}^r} + \langle y_1, y_2 \rangle_{\dot{\mathbb{H}}^{r-1}}$ for any $X_1 = (x_1, y_1)^T \in \mathbb{H}^r$ and $X_2 = (x_2, y_2)^T \in \mathbb{H}^r$. It can been shown that the domain of operator $A$ is given by

$$\mathcal{D}(A) = \left\{ X \in \mathbb{H} : AX = [v, \Lambda u]^T \in \mathbb{H} := L^2 \times \dot{\mathbb{H}}^{-1} \right\} = \dot{\mathbb{H}}^1 \times L^2,$$

and $A$ generates a unitary group $E(t) = \exp(tA) = \begin{bmatrix} C(t) & (-\Lambda)^{-\frac{1}{2}}S(t) \\ -(-\Lambda)^{\frac{1}{2}}S(t) & C(t) \end{bmatrix}$, $t \in \mathbb{R}$,

where $C(t) = \cos(t(-\Lambda)^{\frac{1}{2}})$ and $S(t) = \sin(t(-\Lambda)^{\frac{1}{2}})$ are cosine and sine operators, respectively.

Under Assumption 2.2, one could follow the proof of [7, Lemma 4.1 and Theorem 4.2] and get the global existence of the mild solution.

**Lemma 2.1** (Existence and uniqueness of mild solution). Let Assumption 2.2 hold and $X_0 \in \mathbb{H}^1$. Then there exists a unique mild solution of (2), i.e.,

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))ds + \int_0^t E(t-s)G(X(s))dW(s) \quad (3)$$

for $t \in [0, T]$. Moreover, for any $p \in [2, \infty)$, there exists $C(p, X_0, Q, T) > 0$ such that

$$\sup_{t \in [0,T]} \| X(t) \|_{L^p(\Omega; \mathbb{H}^1)} \leq C(p, X_0, Q, T).$$

The Lyapunov energy functional $V_1 : \mathbb{H}^1 \rightarrow \mathbb{R}$ of (2) is defined by

$$V_1(u, v) = \frac{1}{2} \| u \|_{\mathbb{H}^1}^2 + \frac{1}{2} \| v \|_{L^2}^2 + \int_O \tilde{F}(u)dx, \quad (4)$$

and plays a crucial role in the study of well-posedness and blow-up of (2). The following proposition introduces the averaged energy evolution law.

**Proposition 2.1.** Let Assumption 2.2 hold and $X_0 \in \mathbb{H}^1$. Then for $t \in [0, T]$, (2) admits

$$\mathbb{E}[V_1(u(t), v(t))] = \mathbb{E}[V_1(u_0, v_0)] + \frac{1}{2} \int_0^t \mathbb{E} \left[ \mathrm{Tr}(g(\Theta u(s))\mathbb{Q}^\frac{1}{2}(g(\Theta u(s))\mathbb{Q}^\frac{1}{2}))^* \right] ds, \quad (5)$$

where $\mathrm{Tr}(g(\Theta u)\mathbb{Q}^\frac{1}{2}(g(\Theta u)\mathbb{Q}^\frac{1}{2}))^* = \sum_{i=1}^{\infty} \langle g(\Theta u)\mathbb{Q}^\frac{1}{2}e_i, g(\Theta u)\mathbb{Q}^\frac{1}{2}e_i \rangle_{L^2}$.  

### 2.2 Regularity estimates

In this part, we provide the properties of $E(t), t \in \mathbb{R}$, and the regularity estimates of the mild solution of (2). The following lemma concerns with the temporal Hölder continuity of both sine and cosine operators, which have been discussed, for example, in [1].
Lemma 2.2. For $\gamma \in [0, 1]$, there exists a positive constant $C(\gamma)$ depending on $\gamma$, such that

\[
\|(S(t) - S(s))(-\Lambda)^{-\frac{3}{2}}\|_{L(H)} \leq C(\gamma)(t - s)^\gamma,
\]
\[
\|(C(t) - C(s))(-\Lambda)^{-\frac{3}{2}}\|_{L(H)} \leq C(\gamma)(t - s)^\gamma
\]

and $\|(E(t) - E(s))X\|_H \leq C(\gamma)(t - s)^\gamma\|X\|_H$, for all $t \geq s \geq 0$, where $L(H)$ denotes the space of all linear bounded operators from $H$ to $\overline{H}$.

Lemma 2.3. For any $t \geq 0$, $C(t)$ and $S(t)$ satisfy a trigonometric identity in the sense that $\|S(t)x\|_L^2 + \|C(t)x\|_L^2 = \|x\|_L^2$ for $x \in L^2$.

Based on the above trigonometric identity, it can be shown that $\|E(t)\|_{L(H)} = 1$, $t \in \mathbb{R}$. By means of Lemmas 2.2, 2.3, one could follow the similar arguments as in the proof of [13, Lemma 3.3] or [15, Theorem 7.4], and obtain the following regularity estimates of the mild solution.

Proposition 2.2. Let Assumption 2.2 hold and $T > 0$. Assume that $X_0 \in H^\beta$ for some $\beta \geq 1$ and that

\[
\|(-\Lambda)^{\frac{\beta-1}{2}}f(u)\|_{L^2} + \|(-\Lambda)^{\frac{\beta-1}{2}}g(\Theta u)\|_{L^2(Q^T_{H,\overline{H}})} \leq b_4(\|u\|_{H^1}, \|u\|_{H^{\beta-1}}), \ u \in H^\beta,
\]

where $b_4$ is a polynomial. Then it holds that for any $p \geq 2$,

\[
\sup_{t \in [0, T]} \mathbb{E}\left[\|X(t)\|_{H^\beta}^p\right] \leq C(p, X_0, Q, T),
\]

and for $0 \leq s \leq t \leq T$,

\[
\mathbb{E}\left[\|u(t) - u(s)\|_{L^2}^p\right] \leq C(p, X_0, Q, T)|t - s|^p,
\]
\[
\mathbb{E}\left[\|v(t) - v(s)\|_{H^{\beta-1}}^p\right] \leq C(p, X_0, Q, T)|t - s|^\beta.
\]

where $C(p, X_0, Q, T)$ is a positive constant depending on $p, X_0, Q$ and $T$.

Due to the loss of explicit expression of the exact solution, designing numerical schemes for nonlinear stochastic differential equations has become an active area of research (see, e.g., [6, 12, 14, 21]). Below we shall focus on the construction and analysis of the temporal and fully-discrete numerical schemes for the considered SWE.

3 SAV approach for stochastic wave equation

The existing structure-preserving numerical schemes for the nonlinear SWE are implicit in general. When implementing such numerical schemes, one has to solve pathwisely complex stochastic algebraic systems, which often require a lot of computational costs. To avoid such issue, we introduce the equivalent form of (1) with an auxiliary variable and construct efficient structure-preserving numerical schemes in this section.
3.1 Stochastic wave equation with auxiliary variable

In this part, we introduce a scalar auxiliary variable $q := \sqrt{F(u)} + \delta_0$. Here, $\delta_0$ is a constant to make $q$ well-posed, i.e., $F(u) + \delta_0 \geq c > 0$ for any $u \in \mathbb{H}^1$. Then (1) can be reformulated into the following equivalent form

$$
\begin{align*}
&du(t) = v(t)dt, \\
&dv(t) = \Lambda u(t)dt - \frac{f(u(t))}{\sqrt{F(u(t))} + \delta_0}q(t)dt + g(\Theta u(t))dW(t), \\
&dq(t) = \frac{\langle f(u(t)), \partial_t u(t) \rangle_{L^2}}{2 \sqrt{F(u(t))} + \delta_0} dt,
\end{align*}
$$

in $(0, T]$.

Lemma 3.1. Let Assumption 2.2 hold and $X_0 \in \mathbb{H}^1$. Then the averaged energy evolution law of $V(u(t), v(t), q(t)) := \frac{1}{2} \| u(t) \|^2_{\mathbb{H}^1} + \frac{1}{2} \| v(t) \|^2_{L^2} + q(t)$ has the following form

$$
E\left[V(u(t), v(t), q(t))\right] = E\left[V(u(0), v(0), q(0))\right] + \frac{1}{2} \int_0^t E\left[\text{Tr}(g(\Theta u(s))Q^{\frac{1}{2}}(g(\Theta u(s))Q^{\frac{1}{2}})^*\right] ds, 
$$

where $t \in [0, T]$. Furthermore, it holds that for $p \geq 2$ and $0 \leq s \leq t \leq T$,

$$
E\left[|q(t) - q(s)|^p\right] \leq C(p, X_0, Q, T)|t - s|^p,
$$

where $C(p, X_0, Q, T)$ is a positive constant depending on $p, X_0, Q,$ and $T$.

Proof. Proposition 2.1 implies the averaged evolution law (7). Notice that

$$
q(t) - q(s) = \int_s^t \frac{\langle f(u(r)), v(r) \rangle_{L^2}}{\sqrt{F(u(r))} + \delta_0} dr.
$$

By using Assumption 2.2 Hölder’s inequality and Lemma 2.1 we have

$$
E\left[|q(t) - q(s)|^p\right] \leq \frac{1}{c_p} \int_s^t E\left[b_1^p(\|u(r)\|_{\mathbb{H}^1})\|v(r)\|^p_{L^2}\right] dr \leq C(p, X_0, Q, T)|t - s|^p,
$$

where $b_1$ is a polynomial.

3.2 Semi-implicit stochastic SAV schemes

Let $\tau$ be the time step-size, and denote $t_n = n\tau$ for $n \in \{1, \ldots, N\}$ with $t_N = T$. Let $u_n$ be the numerical approximation of $u(t_n)$ for $n \in \{1, \ldots, N\}$. We propose two novel kinds of semi-implicit energy-preserving numerical schemes via the SAV approach. In the rest of this paper, the constant $C$ may be different from line to line but never depending on $N$ and $\tau$. 

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The first natural choice is based on the Runge–Kutta type discretization and the SAV approach. To make it clear, we employ the midpoint scheme to approximate the linear unbounded part of stochastic system (6) and obtain the midpoint SAV scheme, i.e.,

\[
\begin{align*}
    u_{n+1} &= u_n + \frac{\tau}{2} (v_n + v_{n+1}) + \frac{\tau}{2} g(\Theta u_n) \delta W_n, \\
    v_{n+1} &= v_n + \frac{\tau}{2} \Lambda (u_n + u_{n+1}) - \frac{f(\hat{u}_n)}{\sqrt{F(\hat{u}_n) + \delta_0}} \frac{q_n + q_{n+1}}{2} + g(\Theta u_n) \delta W_n, \\
    q_{n+1} &= q_n + \frac{\langle f(\hat{u}_n), u_{n+1} - u_n \rangle_{L^2}}{2 \sqrt{F(\hat{u}_n) + \delta_0}}.
\end{align*}
\]

Here, \( \hat{u}_n = u_n \) (or \( 3u_n - u_{n-1} \) with \( u_{-1} = u_0 \)), \( n \in \{0, 1, \ldots, N - 1\} \), and \( \delta W_n = W(t_{n+1}) - W(t_n) \). We would like to remark that the modified term \( \frac{\tau}{2} g(\Theta u_n) \delta W_n \) is used to balance the diffusion coefficient and the stochastic auxiliary variable. Without this modification, the direct discretization of (6) fails to preserve the modified energy evolution law.

An alternative way to construct semi-linear stochastic SAV schemes is inspired by the phenomenon that the exponential Runge–Kutta schemes often require less restriction for SPDEs (see, e.g., [13]) with regard to optimal strong convergence rate. By employing the exponential Euler scheme to linear unbounded part of stochastic system (6), we get

\[
\begin{align*}
    X_{n+1} &= E(\tau) X_n + A^{-1}(E(\tau) - I) \left( 0, -\frac{f(\hat{u}_n)}{\sqrt{F(\hat{u}_n) + \delta_0}} \frac{q_n + q_{n+1}}{2} \right)^\top \\
    q_{n+1} &= q_n + \frac{\langle f(\hat{u}_n), u_{n+1} - u_n \rangle_{L^2}}{2 \sqrt{F(\hat{u}_n) + \delta_0}},
\end{align*}
\]

where \( X_{n+1} = (u_{n+1}, v_{n+1})^\top, n \in \{0, 1, \ldots, N - 1\} \). We would like to mention that the proposed schemes [8] and [9] have a great potential for approximating other SPDEs, such as stochastic nonlinear Schrödinger equations and parabolic SPDEs.

One main advantage of the proposed schemes is that they can be solved efficiently. We take (9) as an example to illustrate this fact. Notice that (9) can be rewritten as

\[
\begin{align*}
    u_{n+1} &= e_{11} u_n + e_{12} v_n - a^1 \frac{f(\hat{u}_n)}{\sqrt{F(\hat{u}_n) + \delta_0}} \frac{q_n + q_{n+1}}{2} + e_{12} g(\Theta u_n) \delta W_n, \\
    v_{n+1} &= e_{21} u_n + e_{22} v_n - a^2 \frac{f(\hat{u}_n)}{\sqrt{F(\hat{u}_n) + \delta_0}} \frac{q_n + q_{n+1}}{2} + e_{22} g(\Theta u_n) \delta W_n, \\
    q_{n+1} &= q_n + \frac{\langle f(\hat{u}_n), u_{n+1} - u_n \rangle_{L^2}}{2 \sqrt{F(\hat{u}_n) + \delta_0}}
\end{align*}
\]

with \( n \in \{0, 1, \ldots, N - 1\} \), \( e_{11} = e_{22} = C(\tau), e_{12} = (-\Lambda)^{-\frac{1}{2}} S(\tau), e_{21} = -(-\Lambda)^{\frac{1}{2}} S(\tau), a^1 = -(-\Lambda)^{-1} (C(\tau) - I) \) and \( a^2 = (-\Lambda)^{-\frac{1}{2}} S(\tau) \). By eliminating \( \frac{q_n + q_{n+1}}{2} \), we deduce

\[
    u_{n+1} + \gamma_n \langle f(\hat{u}_n), u_{n+1} \rangle_{L^2} = \Gamma_n,
\]

where \( \gamma_n = e_{11} + e_{12} a^1 + e_{21} a^2 \).
where \( \gamma_n = \frac{a^1 f(\tilde{u}_n)}{4(F(\tilde{u}_n) + \delta_0)} \) and

\[
\Gamma_n = e_{11} u_n + e_{12} v_n - a^1 \frac{f(\tilde{u}_n)}{\sqrt{F(\tilde{u}_n) + \delta_0}} q_n + \gamma_n \langle f(\tilde{u}_n), u_n \rangle_{L^2} + e_{12} g(\Theta u_n) \delta W_n.
\]

Taking the inner product of (13) with \( f(\tilde{u}_n) \), we derive

\[
(1 + \langle f(\tilde{u}_n), \gamma_n \rangle_{L^2}) \langle f(\tilde{u}_n), u_{n+1} \rangle_{L^2} = \langle f(\tilde{u}_n), \Gamma_n \rangle_{L^2}.
\]

Here \( \langle f(\tilde{u}_n), \gamma_n \rangle_{L^2} \geq 0 \) due to \( a^1 = (-\Lambda)^{-1} (I - C(\tau)) \). Then it follows that

\[
\langle f(\tilde{u}_n), u_{n+1} \rangle_{L^2} = \frac{\langle f(\tilde{u}_n), \Gamma_n \rangle_{L^2}}{1 + \langle f(\tilde{u}_n), \gamma_n \rangle_{L^2}}.
\]

After solving \( \langle f(\tilde{u}_n), u_{n+1} \rangle_{L^2} \) from the linear system, \( u_{n+1} \) is then updated from (13). Subsequently, \( q_{n+1} \) is obtained from (12). Finally, we get \( v_{n+1} \) from (11). Utilizing the similar procedures, one can also verify that \( u_{n+1}, q_{n+1} \) and \( v_{n+1} \) in (8) can be solved explicitly.

Next we will show that the proposed schemes could overcome the divergent issue in the superlinear coefficient case and preserve the energy evolution law, which, together with the regularity estimates of the discrete auxiliary variable, plays the key role in the strong convergence analysis [4].

### 3.3 The preservation of energy evolution law

In this part, we address the important property of the proposed schemes, i.e., they could inherit the averaged energy evolution law of the SWE (1) (or (2)). The rigorous proof of these results requires some regularization techniques, like taking advantage of the spectral Galerkin approximation, such that the integration by parts formula makes sense, and then taking limit. For simplicity, we omit these tedious procedures.

**Proposition 3.1.** Let Assumption 2.2 hold and \( X_0 \in H^1 \). Numerical schemes (8) and (9) preserve the discrete averaged modified energy evolution law, for \( n \in \{0, 1, \ldots, N - 1\} \),

\[
\mathbb{E}[V(u_{n+1}, v_{n+1}, q_{n+1})] = \mathbb{E}[V(u_n, v_n, q_n)] + \tau \mathbb{E} \left[ \text{Tr} \left( g(\Theta u_n) Q^{\frac{1}{2}} (g(\Theta u_n) Q^{\frac{1}{2}})^* \right) \right],
\]

where \( V(u_n, v_n, q_n) = \frac{1}{2} \|u_n\|^2_{H^1} + \frac{1}{2} \|v_n\|^2_{L^2} + q_n^2 \).

**Proof.** To simplify the presentation, we only present the proof of (9) since that of (8) is similar. Fix \( n \in \{0, 1, \ldots, N - 1\} \), and denote \( M = \begin{bmatrix} -\Lambda & 0 \\ 0 & I \end{bmatrix} \), \( Id = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \), \( J = \begin{bmatrix} 0 & J \\ -I & 0 \end{bmatrix} \),

\[
\tilde{g}_n = \begin{bmatrix} 0 \\ g(\Theta u_n) \delta W_n \end{bmatrix} \quad \text{and} \quad \tilde{f}_n = \begin{bmatrix} 0 \\ -\frac{f(\tilde{u}_n)}{\sqrt{F(\tilde{u}_n) + \delta_0}} q_n + q_{n+1} \end{bmatrix}.
\]

Then using the integration by parts formula, it follows that

\[
\frac{1}{2} \|u_n\|^2_{H^1} + \frac{1}{2} \|v_n\|^2_{L^2} = \frac{1}{2} \int \mathcal{O} X_n^T M X_n dx.
\] (14)
Notice that

\[
X_{n+1}^\top MX_{n+1} - X_n^\top MX_n
= X_n^\top (E(\tau)^\top ME(\tau) - M)X_n + 2X_n^\top E(\tau)^\top M(E(\tau) - Id)A^{-1}\tilde{f}_n
+ 2X_n^\top E(\tau)^\top ME(\tau)\tilde{g}_n + (\tilde{f}_n)^\top (A^{-1})^\top (E(\tau) - Id)^\top M(E(\tau) - Id)A^{-1}\tilde{f}_n
+ 2(\tilde{f}_n)^\top (A^{-1})^\top (E(\tau) - Id)^\top ME(\tau)\tilde{g}_n + (\tilde{g}_n)^\top E(\tau)^\top ME(\tau)\tilde{g}_n,
\]

and

\[
q_{n+1}^2 - q_n^2 = \frac{\langle f(\hat{u}_n), u_{n+1} \rangle_{L^2}}{\sqrt{F(\hat{u}_n) + \delta_0}^2} - \frac{\langle f(\hat{u}_n), u_n \rangle_{L^2} + q_n + q_{n+1}}{\sqrt{F(\hat{u}_n) + \delta_0}^2}
= \int_{\partial} (X_n)^\top (E(\tau) - Id)^\top J^{-1}\tilde{f}_n dx + \int_{\partial} (\tilde{f}_n)^\top (A^{-1})^\top (E(\tau) - Id)^\top J^{-1}\tilde{f}_n dx
+ \int_{\partial} (\tilde{g}_n)^\top E(\tau)^\top J^{-1}\tilde{f}_n dx.
\]

As a consequence, due to $MA^{-1} = J^{-1}$ and the fact that $\int_{\partial} \tilde{f}_n^\top (A^{-1})^\top ME(\tau)\tilde{g}_n dx = \int_{\partial} \tilde{g}_n^\top E(\tau)^\top MA^{-1}\tilde{f}_n dx$, we obtain

\[
\frac{1}{2} \int_{\partial} (X_{n+1}^\top MX_{n+1} - X_n^\top MX_n) dx + q_{n+1}^2 - q_n^2
= \frac{1}{2} \int_{\partial} X_n^\top (E(\tau)^\top ME(\tau) - M)X_n dx + \frac{1}{2} \int_{\partial} (\tilde{g}_n)^\top E(\tau)^\top ME(\tau)\tilde{g}_n dx
+ \int_{\partial} X_n^\top (E(\tau)^\top M(E(\tau) - Id)A^{-1} + (E(\tau) - Id)^\top J^{-1}) \tilde{f}_n dx
+ \int_{\partial} X_n^\top E(\tau)^\top ME(\tau)\tilde{g}_n dx
+ \frac{1}{2} \int_{\partial} (\tilde{f}_n)^\top (A^{-1})^\top (E(\tau) - Id)^\top M(E(\tau) + Id)A^{-1}\tilde{f}_n dx
+ \int_{\partial} (\tilde{f}_n)^\top (A^{-1})^\top ((E(\tau) - Id)^\top ME(\tau) + ME(\tau)) \tilde{g}_n dx.
\]

From $(\tilde{f}_n)^\top (A^{-1})^\top M\tilde{g}_n = 0$, it follows that

\[
\frac{1}{2} \int_{\partial} (X_{n+1}^\top MX_{n+1} - X_n^\top MX_n) dx + q_{n+1}^2 - q_n^2
= \frac{1}{2} \int_{\partial} (X_n + A^{-1}\tilde{f} + \tilde{g}_n)^\top (E(\tau)^\top ME(\tau) - M)(X_n + A^{-1}\tilde{f} + \tilde{g}_n) dx
+ \frac{1}{2} \int_{\partial} (\tilde{f}_n)^\top (A^{-1})^\top (E(\tau)^\top M - ME(\tau))A^{-1}\tilde{f}_n dx
+ \int_{\partial} X_n^\top M\tilde{g}_n dx + \frac{1}{2} \int_{\partial} (\tilde{g}_n)^\top M\tilde{g}_n dx.
\]
According to the unitary property of $E(\tau)$ and the skew symmetry of the matrix $E(\tau)^T M - ME(\tau)$, taking expectation leads to

$$
\mathbb{E} \left( \frac{1}{2} \left\| u_{n+1} \right\|^2_{H^1} + \frac{1}{2} \left\| v_{n+1} \right\|^2_{L^2} + q_{n+1}^2 \right)
= \mathbb{E} \left( \frac{1}{2} \left\| u_n \right\|^2_{H^1} + \frac{1}{2} \left\| v_n \right\|^2_{L^2} + q_n^2 \right) + \frac{\tau}{2} \text{Tr} \left( g(\Theta u_n) Q^2 (g(\Theta u_n) Q^2)^* \right),
$$

which completes the proof.

As a result of Proposition 3.1, we are in a position to present a priori estimates of numerical solutions.

**Corollary 3.1.** Let Assumption 2.2 hold and $X_0 \in H^1$. In addition suppose that $\|g(\Theta u)\|_{L_2(Q^{1/2}(\tilde{H},\tilde{H}))} \leq c_2(1 + \|u\|_{H^1})$ with $c_2 > 0$. Then it holds that for any $p \geq 1$,

$$
\sup_{n \leq N} \mathbb{E} [V^p(u_n, v_n, q_n)] \leq C(p, X_0, Q, T),
$$

where $(u_n, v_n, q_n)$ is the numerical solution of (8) or (9), and $C(p, X_0, Q, T) > 0$.

**Proof.** By (15), $\|g(\Theta u)\|_{L_2(Q^{1/2}(\tilde{H},\tilde{H}))} \leq c_2(1 + \|u\|_{H^1})$, the Burkholder inequality, as well as Hölder’s and Young’s inequalities, we get that for $p \geq 1$,

$$
\begin{align*}
\mathbb{E} [V^p(u_{n+1}, v_{n+1}, q_{n+1})] &\leq \mathbb{E} [V^p(u_n, v_n, q_n)](1 + C(p)\tau) + p\mathbb{E} [V^{p-1}(u_n, v_n, q_n)\langle v_n, g(\Theta u_n)\delta W_n \rangle_{L^2}] \\
&+ C(p)\mathbb{E} \left[ \langle g(\Theta u_n)\delta W_n, g(\Theta u_n)\delta W_n \rangle_{L^2} \right] + C(p)\tau\mathbb{E} \left[ 1 + \|u_n\|_{H^1}^{2p} \right] \\
&\leq \mathbb{E} [V^p(u_n, v_n, q_n)](1 + C(p)\tau) + C(p)\tau\mathbb{E} \left[ \|v_n\|_{L^2}^{2p} \right] + C(p)\tau\mathbb{E} \left[ 1 + \|u_n\|_{H^1}^{2p} \right],
\end{align*}
$$

where we have used the independent increment property of $W(\cdot)$ and Proposition 3.1 in the last inequality. The discrete Grönwall’s inequality yields the desired result. \hfill \Box

## 4 Strong convergence rate of stochastic SAV schemes

In this section, we provide a generic approach to study the strong convergence rates of the proposed stochastic SAV schemes (8)-(9).

### 4.1 Properties of the discrete auxiliary variable

In the following, we first show a priori estimates of both $q_n$ and $u_n$ for $n \in \{1, \ldots, N\}$, which are of vital importance in the study of strong error estimates of the proposed stochastic energy-preserving schemes via the SAV approach.
Lemma 4.1. Under the condition of Corollary 3.1 it holds that
\[ \sup_{j \in \{0, 1, \ldots, N - 1\}} \mathbb{E}[\|u_{j+1} - u_j\|_{L^2}^p] \leq C \tau^p, \]
where \( u_j \) is the numerical solution of (8) or (9), \( N \in \mathbb{N}^+, N \tau = T, \) and \( C := C(p, X_0, Q, T) > 0. \)

**Proof.** Fix \( j \in \{0, 1, \ldots, N - 1\}. \) Based on the properties that \( |\cos(x) - 1| \leq C|x|^2 \) and \( \left| \frac{\sin(x)}{x} \right| \leq C \) for some \( C > 0, \) we have
\[
\begin{align*}
\|u_{j+1} - u_j\|_{L^2} & \leq \|\cos(\tau(-\Lambda)^{1/2} - I)u_j\|_{L^2} + \|(-\Lambda)^{-1/2} \sin(\tau(-\Lambda)^{1/2})v_j\|_{L^2} \\
& \quad + \|(-\Lambda)^{-1}(\cos(\tau(-\Lambda)^{1/2} - I) - 1)\|_{L^2} \frac{f(\hat{u}_j)}{F(\hat{u}_j)} q_j + q_{j+1} \|_{L^2} \\
& \quad + \|(-\Lambda)^{-1/2} \sin(\tau(-\Lambda)^{1/2})g(\Theta u_j)\delta W_j\|_{L^2} \\
& \leq C\tau(\|u_j\|_{L^2} + \|v_j\|_{L^2}) + C\tau^2 \frac{\|f(\hat{u}_j)\|_{L^2}}{\sqrt{F(\hat{u}_j)} + \delta_0} (|q_j| + |q_{j+1}|) + \tau g(\Theta u_j)\delta W_j\|_{L^2}.
\end{align*}
\]

Thanks to \( \hat{u}_j = u_j \) or \( \frac{3u_j - u_{j-1}}{2}, \) taking the \( p \)th moment, and using Corollary 3.1 Young’s and Hölder’s inequalities, we obtain
\[
\begin{align*}
\mathbb{E}\left[\|u_{j+1} - u_j\|_{L^2}^p\right] & \leq C(p)\tau^p \left( \mathbb{E}[\|u_j\|_{L^2}^p] + \mathbb{E}[\|v_j\|_{L^2}^p] + \mathbb{E}[b(\|u_j\|_{L^2}, \|\hat{u}_j\|_{L^2})] + \mathbb{E}[q_j^p + q_{j+1}^p] \right) \\
& \leq C(p, X_0, Q, T)\tau^p,
\end{align*}
\]
where \( b \) is a polynomial. This completes the proof. \( \square \)

**Proposition 4.1.** Let the condition of Corollary 3.1 hold and \( |q_0 - \sqrt{F(\hat{u}_0) + \delta_0}| \leq C\tau. \)

Suppose that
\[ \langle f'(u)v, w \rangle \leq b_5(\|u\|_{H^1}, |v|_{H^1}, \|w\|_{H^1})\|v\|_{L^2} \|w\|_{L^2} \]
for some \( \gamma_1 > \frac{1}{2} \) and some polynomial \( b_5. \) Then for (8) and (9), it holds that
\[ \sup_{j \in \{1, \ldots, N\}} \mathbb{E}\left[\left|\sqrt{F(\hat{u}_j) + \delta_0 - q_j}\right|^p\right] \leq C\tau^{\min(1, 2\gamma_1 - 1)p}, \]
where \( C := C(p, X_0, Q, T) > 0, N \in \mathbb{N}^+, N \tau = T. \)
Proof. Fix $j \in \{1, \ldots, N-1\}$. The definitions of $q_j$ and $F(\hat{u}_j)$, and the Taylor expansion yield that

$$
\sqrt{F(\hat{u}_{j+1}) + \delta_0 - q_{j+1}}
=\sqrt{F(\hat{u}_j) + \delta_0 - q_j - \tau \frac{\langle f(\hat{u}_j), \frac{u_{j+1}-u_j}{\tau} \rangle_{L^2}}{2\sqrt{F(\hat{u}_j) + \delta_0}} + \frac{F(\hat{u}_{j+1}) - F(\hat{u}_j)}{\sqrt{F(\hat{u}_j) + \delta_0 + \sqrt{F(\hat{u}_{j+1}) + \delta_0}}}}
$$

By applying Hölder’s inequality, we obtain

$$
|\sqrt{F(\hat{u}_{j+1}) + \delta_0 - q_{j+1}}|
\leq |\sqrt{F(\hat{u}_j) + \delta_0 - q_j}|
+ \tau \left| \langle f(\hat{u}_j), \frac{\hat{u}_{j+1} - \hat{u}_j}{\tau} \rangle_{L^2} \right| \frac{\sqrt{F(\hat{u}_{j+1}) + \delta_0} - \sqrt{F(\hat{u}_j) + \delta_0}}{2(F(\hat{u}_j) + \delta_0) + 2\sqrt{F(\hat{u}_j) + \delta_0} \sqrt{F(\hat{u}_{j+1}) + \delta_0}}
+ \tau \left| \int_0^1 f'(\hat{u}_j + \theta(\hat{u}_{j+1} - \hat{u}_j)) d\theta(\hat{u}_{j+1} - \hat{u}_j), \frac{\hat{u}_{j+1} - \hat{u}_j}{\tau} \right|_{L^2}
\sqrt{F(\hat{u}_j) + \delta_0 + \sqrt{F(\hat{u}_{j+1}) + \delta_0}}
=: |\sqrt{F(\hat{u}_j) + \delta_0 - q_j}| + A_1 + A_2.
$$

Due to the assumption of $f$, there exists a polynomial $b$ such that

$$
A_1 \leq \tau \left| \langle f(\hat{u}_j), \frac{\hat{u}_{j+1} - \hat{u}_j}{\tau} \rangle_{L^2} \right| \frac{F(\hat{u}_{j+1}) - F(\hat{u}_j)}{2\sqrt{F(\hat{u}_j) + \delta_0} \sqrt{F(\hat{u}_{j+1}) + \delta_0}}
\leq Cb(\|\hat{u}_{j+1}\|_{H^1}, \|\hat{u}_{j+1}\|_{H^1}) \|\hat{u}_{j+1} - \hat{u}_j\|_{L^2}^2.
$$

For the term $A_2$, using $|\langle f'(u)v, w \rangle| \leq b_5(\|u\|_{H^1}, \|v\|_{H^1}, \|w\|_{H^1}) \|v\|_{L^2} \|w\|_{L^2}$, it holds that

$$
A_2 \leq Cb_5(\|\hat{u}_{j+1}\|_{H^1}, \|\hat{u}_{j+1}\|_{H^1}) \|\hat{u}_{j+1} - \hat{u}_j\|_{L^2}^{2\gamma_1}.
$$

In sum,

$$
|\sqrt{F(\hat{u}_{j+1}) + \delta_0 - q_{j+1}}|
\leq |\sqrt{F(\hat{u}_j) + \delta_0 - q_j}| + C \left( b(\|\hat{u}_{j+1}\|_{H^1}, \|\hat{u}_{j+1}\|_{H^1}) + b_5(\|\hat{u}_{j+1}\|_{H^1}, \|\hat{u}_{j+1}\|_{H^1}) \right) \|\hat{u}_{j+1} - \hat{u}_j\|_{L^2}^{2\gamma_1}
\times (1 + \|\hat{u}_{j+1} - \hat{u}_j\|_{L^2}^{2-2\gamma_1}).
$$
Taking the $p$th moment for $p \geq 1$, using Lemma 4.1 Corollary 3.1 and the definition of $\hat{u}_j$, we conclude that
\[
\|\sqrt{F(\hat{u}_{j+1}) + \delta_0 - \hat{u}_{j+1}}\|_{L^p(\Omega)} \leq \|q_0 - \sqrt{F(\hat{u}_0) + \delta_0}\|_{L^p(\Omega)} + C \sum_{k=0}^j \|\hat{u}_{j+1} - \hat{u}_j\|_{L^{4\gamma_1}(\Omega)}^{2\gamma_1} \\
\leq C \tau + C \tau^{2\gamma_1-1} \leq C \tau^{2\gamma_1-1}.
\]

It can be seen that if Assumption 2.1 holds, one could take $\gamma_1 = 1$ in Proposition 4.1.

4.2 Strong convergence analysis of stochastic SAV schemes

Now, we are in a position to prove the strong convergence rate of the proposed stochastic SAV schemes under Assumption 2.1.

**Theorem 4.1.** Let $X_0 \in \mathbb{H}^1$ and Assumption 2.1 hold. Suppose that
\[
\|(-\Lambda)^{-\frac{1}{2}} (f(u) - f(\bar{u}))\|_{L^2} + \|(-\Lambda)^{-\frac{1}{2}} (g(\Theta u) - g(\Theta \bar{u}))\|_{\mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}(\mathbb{H},J))} \leq C\|u - \bar{u}\|_{L^2},
\]
where $u, \bar{u} \in \mathbb{H}^1$. Then for $p \geq 1$, the numerical scheme (9) satisfies
\[
\sup_{n \in \{1, \ldots, N\}} \mathbb{E} \left[\|X(t_n) - X_n\|_{\mathbb{H}}^{2p}\right] \leq C \tau^{2p},
\]
where $C := C(p, X_0, Q, T) > 0$, $N \in \mathbb{N}^+$, $N \tau = T$.

**Proof.** We only present the proof for the case $p = 1$ since the proof for other cases is similar. Fix $n \in \{1, \ldots, N\}$. Notice that the solution of (9) can be rewritten as
\[
X_n = E(t_n)X_0 + \sum_{j=0}^{n-1} E(t_{n-j}) \begin{bmatrix} 0 \\ g(\Theta u_j) \delta W_j \end{bmatrix} \\
+ \sum_{j=0}^{n-1} E(t_{n-1-j})A^{-1}(E(\tau) - I) \begin{bmatrix} 0 \\ -\frac{f(\hat{u}_j)}{\sqrt{F(\hat{u}_j) + \delta_0}} - q_j + q_{j+1} \end{bmatrix}.
\]

Recall that the mild solution of (1) satisfies
\[
X(t_n) = E(t_n)X_0 + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} E(t_n - s)[F(X(s))ds + G(X(s))dW(s)].
\]
Let $\varepsilon = X(t_i) - X_i$ for $i \in \{0, 1, \ldots, N\}$. Then

$$
\|\varepsilon_n\|_H^2 \leq C \left( \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (E(t_n - s) - E(t_n - t_j)) \left[ g(\Theta u(s)) \right] dW(s) \right)^2 + C \left( \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} E(t_n - t_j) \left[ g(\Theta u(s)) - g(\Theta u_j) \right] dW(s) \right)^2 + C \left( \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left( E(t_n - s) \left[ -\frac{f(u(s)q(s))}{\sqrt{F(u(s)) + \delta_0}} + \frac{f(\hat{u}_jq_j)}{\sqrt{\hat{F}(\hat{u}_j) + \delta_0}} \right] \right) ds \right)^2
$$

\[ =: Err_{n,1} + Err_{n,2} + Err_{n,3}. \]

Then by Assumption 2.1, Lemmas 2.1 and 2.2, we obtain

$$
\mathbb{E}[Err_{n,1}] \leq C \int_0^{t_n} \mathbb{E} \left[ \left( E(s - \left[ \frac{s}{T} \right] \tau) - I \right) \left[ g(\Theta u(s)) \right] \right] L_2(Q^{\frac{1}{2}}(H, H)) \ ds
$$

\[ \leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} |s - \left[ \frac{s}{T} \right] \tau|^2 \mathbb{E} \left[ \left\| g(\Theta u(s)) \right\|^2 \right] L_2(Q^{\frac{1}{2}}(H, H)) \ ds \leq C \tau^2. \]

Similar arguments, together with Burkholder–Davis–Gundy’s inequality, yield that

\[ \mathbb{E}[Err_{n,2}] \leq C \mathbb{E} \left[ \left\| \int_0^{t_n} E(t_n - \left[ \frac{s}{T} \right] \tau) \left[ g(\Theta u(s)) - g(\Theta u(\left[ \frac{s}{T} \right] \tau)) \right] dW(s) \right\|_H^2 \right]
\]

\[ \leq C \int_0^{t_n} \mathbb{E} \left[ \left\| E(t_n - \left[ \frac{s}{T} \right] \tau) \left[ g(\Theta u(s)) - g(\Theta u(\left[ \frac{s}{T} \right] \tau)) \right] \right\|^2 \right] L_2(Q^{\frac{1}{2}}(H, H)) \ ds
\]

\[ \leq C \int_0^{t_n} \mathbb{E} \left[ \left\| E(t_n - \left[ \frac{s}{T} \right] \tau) \right\|^2 \right] L_2(Q^{\frac{1}{2}}(H, H)) \ ds
\]

\[ + C \int_0^{t_n} \mathbb{E} \left[ \left\| \left[ -\Lambda \right]^{\frac{1}{2}} \left( g(u(s)) - g(u(\left[ \frac{s}{T} \right] \tau)) \right) \right\|^2 \right] L_2(Q^{\frac{1}{2}}(H, \hat{H})) \ ds
\]

\[ \leq C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E} \left[ \left\| u(s) - u(t_j) \right\|^2 \right] L_2 ds + C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E} \left[ \left\| u(t_j) - u_j \right\|^2 \right] L_2 ds
\]

\[ \leq C \tau^2 + C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u(t_j) - u_j \right\|^2 \right] L_2. \]
Now we turn to consider \( Err_{n,3} \). Based on Lemma \ref{lem:corollary3.1}, we obtain
\[
E[Err_{n,3}] \leq C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left( \left\| f(u(s)) - u(t) \right\|_{L^2}^2 - \left( \frac{f(\hat{u}_j)q_j}{\sqrt{F(\hat{u}_j) + \delta_0}} \right) q_j \right)^2 ds \right] + C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left( \frac{f(\hat{u}_j)q_j}{\sqrt{F(\hat{u}_j) + \delta_0}} \right) q_j \right. \left. \left\| f(u(s)) - f(u(t_j)) \right\|_{\mathbb{H}^{-1}}^2 \right] ds \right]
\]
\[
= : \Pi^1_n + \Pi^2_n + \Pi^3_n + \Pi^4_n.
\]
According to \[16\] and Proposition \ref{prop:prop2.2}, we derive
\[
\Pi^1_n \leq C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\| u(s) - u(t_j) \right\|_{L^2}^2 ds \right] \leq C \tau^2,
\]
\[
\Pi^2_n \leq C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u(t_j) - u_j \right\|_{L^2}^2 \right] + C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u_j - \hat{u}_j \right\|_{L^2}^2 \right].
\]
If \( \hat{u}_j = u_j \), then \( \Pi^2_n \leq C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u(t_j) - u_j \right\|_{L^2}^2 \right] \). When \( \hat{u}_j = \frac{3u_j - u_{j-1}}{2} \), thanks to \( u_0 = u_{-1} \), we have
\[
\Pi^2_n \leq C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u(t_j) - u_j \right\|_{L^2}^2 \right] + C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u_j - u_{j-1} \right\|_{L^2}^2 \right]
\]
\[
\leq C \tau \sum_{j=0}^{n-1} \mathbb{E} \left[ \left\| u(t_j) - u_j \right\|_{L^2}^2 \right] + C \tau \sum_{j=0}^{n-2} \mathbb{E} \left[ \left\| u_{j+1} - u_j \right\|_{L^2}^2 \right].
\]
For the term \( \Pi^4_n \), we obtain
\[
\Pi^4_n \leq C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left( \frac{f(\hat{u}_j)}{\sqrt{F(\hat{u}_j) + \delta_0}} \right) q_j + \delta_0 \right] q_j^2 ds \right]
\]
\[
\leq C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left( \frac{f(\hat{u}_j)}{\sqrt{F(\hat{u}_j) + \delta_0}} \right) \left( \frac{f(\hat{u}_j)}{\sqrt{F(\hat{u}_j) + \delta_0}} \right)^2 \left\| f(\hat{u}_j) \right\|_{L^2}^2 \left\| u_{j+1} - u_j \right\|_{L^2}^2 ds \right].
\]
Since \( \left\| f(\hat{u}_j) \right\| \leq b_1(\left\| \hat{u}_j \right\|_{\mathbb{H}^1}) \) and \( F(\hat{u}_j) + \delta_0 \geq c > 0 \), the Hölder’s inequality, Lemma \ref{lem:lem4.1} and Corollary \ref{cor:cor3.1} yield that
\[
\Pi^4_n \leq C \tau \sum_{j=0}^{n-1} \sqrt{\mathbb{E} \left[ \left\| u_{j+1} - u_j \right\|_{L^2}^4 \right]} \leq C \tau^2.
\]

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By applying Proposition 4.1, Corollary 3.1 and Hölder’s inequality, we get

\[ \Pi_n^3 \leq C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\| f(\tilde{u}_j) - \frac{f(\tilde{u}_j)q_j}{\sqrt{F(\tilde{u}_j) + \delta_0}} \right\|^2 ds \right] \]

\[ \leq C \mathbb{E} \left[ \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\| f(\tilde{u}_j) \right\|_{\mathbb{H}^{-1}}^2 \left( \frac{1}{F(\tilde{u}_j) + \delta_0} \right) ds \right] \]

\[ \leq C \tau^2. \]

Combining the above estimates together, we arrive at

\[ \| \varepsilon_n \|_{\mathbb{H}}^2 \leq C \tau^2 + C \tau \sum_{j=0}^{n-1} \mathbb{E} \| u(t_j) - \tilde{u}_j \|_{L^2}^2. \]

By discrete Grönwall’s inequality and Lemma 4.1, we complete the proof.

Next, we present the strong convergence rate of (8), which can be shown by using the similar steps as in Theorem 4.1 [13, Lemmas 4.1 and 4.2] and Proposition 4.1. We omit its proof for convenience.

**Theorem 4.2.** Let \( X_0 \in \mathbb{H}^\gamma, \gamma \in [1, 2], \) and Assumption 2.1 hold. Suppose that

\[ \| (-\Lambda)^{-\frac{1}{2}} (f(u) - f(\tilde{u})) \|_{L^2} + \left\| (-\Lambda)^{-\frac{1}{2}} (g(\Theta u) - g(\Theta \tilde{u})) \right\|_{L^2(\mathbb{Q}^{1}\mathbb{H}, \mathbb{H})} \leq C \| u - \tilde{u} \|_{L^2}, \]

\[ \| (-\Lambda)^{-\frac{1}{2}} f(u) \|_{L^2} + \left\| (-\Lambda)^{-\frac{1}{2}} g(u) \right\|_{L^2(\mathbb{Q}^{1}\mathbb{H}, \mathbb{H})} \leq b_6(\| u \|_{\mathbb{H}^1}), \]

where \( b_6 \) is a polynomial, \( u, \tilde{u} \in \mathbb{H}^1 \). Then for \( p \geq 1 \), the numerical scheme (8) satisfies

\[ \sup_{n \in \{1, \ldots, N\}} \mathbb{E} \left[ \| X(t_n) - X_n \|_{\mathbb{H}}^{2p} \right] \leq C \tau^{p\gamma}, \]

where \( C := C(p, X_0, Q, T) > 0, N \in \mathbb{N}^+, N\tau = T. \)

**Remark 4.1.** Schemes (8) and (9) can also be used to numerically solve the deterministic wave equation, which corresponds to the case of \( g = 0 \). In this case, the above convergence result of SAV schemes is also new.

## 5 Semi-implicit energy-preserving fully-discrete schemes

In this section, we combine the stochastic SAV schemes with a spatial finite element method (see, e.g., [1, 22]) to propose implementary and energy-preserving fully-discrete schemes for (2). We first briefly recall the definition of the linear finite element method. Let \( \{T_h\} \) be a quasi-uniform family of triangulations of the convex polygonal domain \( \mathcal{O} \) with \( h_K = \text{diam}(K) \).
and \( h = \max_{k \in \mathcal{K}_h} h_K \). Let \( V_h \subset \mathbb{H}^1 \) be the space of piecewise linear continuous functions with respect to \( \mathcal{T}_h \) which are zero on the boundary of \( \mathcal{D} \), and let \( \mathcal{P}_h : \mathbb{H} \to V_h \) denote the \( \mathbb{H} \)-orthogonal projector and \( \mathcal{R}_h : \mathbb{H}^1 \to V_h \) denote the \( \mathbb{H}^1 \)-orthogonal projector (Ritz projector). Thus,

\[
(\mathcal{P}_h v, w_h) = (v, w_h), \quad (\nabla \mathcal{R}_h u, \nabla w_h) = (\nabla u, \nabla w_h) \quad \forall \; v \in \mathbb{H}, \; u \in \mathbb{H}^1, \; w_h \in V_h.
\]

The discrete Laplace operator \( \Lambda_h : V_h \to V_h \) is then defined by

\[
(\Lambda_h v, w_h) = - (\nabla v, \nabla w_h) \quad \forall \; w_h \in V_h.
\]

Notice that \( \mathcal{R}_h = (-\Lambda_h)^{-1}\mathcal{P}_h(-\Lambda) \) (see, e.g., [22]). We define discrete norms and interpolation spaces by

\[
\|v_h\|_{h,\alpha} = \|(-\Lambda_h)^{\frac{\alpha}{2}}v_h\| \quad \forall \; v_h \in V_h
\]

and \( \mathbb{H}^0_h = V_h \) equipped with the norm \( \| \cdot \|_{h,\alpha} \), respectively. Then the finite element method of \([2]\) becomes

\[
\begin{aligned}
\left\{
\begin{array}{l}
dX^h(t) = A_h X^h(t) dt + \mathcal{P}_h \mathbb{P}(X^h(t)) dt + \mathcal{P}_h \mathbb{G}(X^h(t)) dW(t), \quad t > 0, \\
X^h(0) = X^{h,0}.
\end{array}
\right.
\end{aligned}
\]

(23)

where \( X^{h,0} = (u^{h,0}, v^{h,0})^\top \), \( u^{h,0} = \mathcal{R}_h u_0, v^{h,0} = \mathcal{P}_h v_0 \),

\[
A_h := \begin{bmatrix} 0 & I \\ \Lambda_h & 0 \end{bmatrix} \quad \text{and} \quad X^h := \begin{bmatrix} u_h \\ v_h \end{bmatrix}.
\]

Here the notation \( \mathcal{P}_h \mathbb{P}(X^h) = (0, \mathcal{P}_h f(u^h))^\top \) and similarly for \( \mathcal{P}_h \mathbb{G}(X^h) \). Like \([3]\), the mild form of \( X^h \) reads

\[
X^h(t) = E_h(t) X^{h,0} + \int_0^t E_h(t-s) \mathcal{P}_h \mathbb{P}(X^h(s))ds + \int_0^t E_h(t-s) \mathcal{P}_h \mathbb{G}(X^h(s))dW(s),
\]

where \( E_h = \begin{bmatrix} C_h(t) & (-\Lambda_h)^{-\frac{1}{2}} S_h(t) \\ -(\Lambda_h)^{\frac{1}{2}} S_h(t) & C_h(t) \end{bmatrix} \) is a \( \mathbb{C}_0 \)-semigroup generated by \( A_h \) on \( \mathbb{H}_h := \mathbb{H}^0_h \times \mathbb{H}^{-1}_h \) with \( C_h(t) = \cos(t(-\Lambda_h)^{\frac{1}{2}}) \) and \( S_h(t) = \sin(t(-\Lambda_h)^{\frac{1}{2}}) \). For \([23]\), the corresponding energy is defined by

\[
H(X^h) = \frac{1}{2} \left\| (-\Lambda_h)^{\frac{1}{2}} u^h \right\|^2 + \frac{1}{2} \left\| v^h \right\|^2 + F(u^h),
\]

since \( \| \nabla u^h \|^2 = \left\| (-\Lambda_h)^{\frac{1}{2}} u^h \right\|^2 \). By using the Itô formula, one can obtain the following averaged energy evolution of the finite element solution \( X^h \). Similar to the continuous case, the regularity estimate of the mild solution follows.
Proposition 5.1. Let Assumption 2.2 hold and $X_0 \in \mathbb{H}^1$. The solution $X^h$ of the finite element approximation (23) satisfies the averaged energy evolution law,

$$
\begin{align*}
\mathbb{E} \left[ H(X^h(t)) \right] &= \mathbb{E} \left[ H(X^h(0)) \right] \\
&+ \frac{1}{2} \int_0^t \mathbb{E} \left[ \text{Tr} \left( \mathcal{P}_h g(\Theta u^h(s)\mathbb{Q}^{1/2})(\mathcal{P}_h g(\Theta u^h(s))\mathbb{Q}^{-1/2})^* \right) \right] ds, \quad t \geq 0.
\end{align*}
$$

Furthermore, suppose that $X_0 \in \mathbb{H}^\beta$ for some $\beta \in [1, 2]$ and that

$$
\|(-\Lambda)^{\frac{\beta-1}{2}} f(u)\|_{L^2} + \|(-\Lambda)^{\frac{\beta-1}{2}} g(\Theta u)\|_{L^2(\mathbb{Q}^{1/2}(\mathbb{H}), \mathbb{H})} \leq b_4(\|u\|_{\mathbb{H}^{1/2}}), \ u \in \mathbb{H}^\beta,
$$

where $b_4$ is a polynomial. Then it holds that for any $p \geq 2$,

$$
\sup_{t \in [0, T]} \mathbb{E} \left[ \|X^h(t)\|_{\mathbb{H}^{1/2}}^p \right] \leq C(p, X_0, Q, T),
$$

and for $0 \leq s \leq t \leq T$,

$$
\begin{align*}
\mathbb{E} \left[ \|u^h(t) - u^h(s)\|_{\mathbb{H}^{p/2}}^p \right] &\leq C(p, X_0, Q, T)|t - s|^p, \\
\mathbb{E} \left[ \|v^h(t) - v^h(s)\|_{\mathbb{H}^{1/2}}^p \right] &\leq C(p, X_0, Q, T)|t - s|^\frac{p}{2},
\end{align*}
$$

where $C(p, X_0, Q, T)$ is a positive constant depending on $p, X_0, Q, \text{ and } T$.

Next, we apply the SAV scheme (9) to (23) and obtain the following fully-discrete scheme

$$
\begin{align*}
X_{n+1}^h &= E_h(\tau)X_n^h + A_h^{-1}(E_h(\tau) - I)\mathcal{P}_h \left( 0, -\frac{f(\hat{u}_n^h)}{F(\hat{u}_n^h) + \delta_0} \frac{q_n + q_{n+1}}{2} \right)^T \\
&+ E_h(\tau)\mathcal{P}_h(0, g(\Theta u_n^h)\delta W_n)^T, \\
q_{n+1} &= q_n + \frac{(f(\hat{u}_n^h), u_{n+1}^h - u_n^h)}{2\sqrt{F(\hat{u}_n^h) + \delta_0}}.
\end{align*}
$$

Here $\hat{u}_n^h = u_n^h$ (or $\frac{3u_n^h - u_{n-1}^h}{2}$, with $u_0^h = u_0^h$), $T > 0$, $N \in \mathbb{N}^+$, $N\tau = T$, $n \leq N$. Following the procedures in the proof of Theorem 4.1 we could get the following strong convergence rate result of (25). Below we only present a sketch proof due to the limitation of pages.

Theorem 5.1. Under Assumption 2.1, let $T > 0$, $N \in \mathbb{N}^+$, $N\tau = T$, $n \leq N$. Assume that $f, g$ satisfy (24) and (16) with some $\beta \geq 1$, and that $X_0 \in \mathbb{H}^\gamma$, $\gamma \geq 1 + \frac{2}{3}\beta$. Then the numerical scheme (25) satisfies that for $p \geq 2$,

$$
\mathbb{E} \left[ \|X_n^h - X(t_n)\|_{\mathbb{H}_h}^p \right] \leq C(p, X_0, Q, T)(h^{\frac{2}{3}\beta p} + \tau^p).
$$
Proof. We decompose $X_n^h - X(t_n)$ by

$$X_n^h - X(t_n) = X_n^h - X^h(t_n) + X^h(t_n) - X(t_n).$$

The estimate of $\mathbb{E}\left[\|X_n^h(t_n) - X(t_n)\|_{\mathbb{H}_h}^p\right] \leq C(p, X_0, Q, T)h^{\frac{3}{2}}\tau^p$ could be established by using similar arguments as in the proof of \([1, \text{Theorem 3.1}]\). For the term $X_n^h - X^h(t_n)$, one may follow same steps as in proving Theorem 4.1, together with the properties of the discrete cosine and sine operators (see, e.g., \([1]\)), and obtain

$$\mathbb{E}\left[\|X_n^h - X^h(t_n)\|_{\mathbb{H}_h}^p\right] \leq C(p, X_0, Q, T)\tau^p.$$ 

Combining the above estimates, we complete the proof. 

Following the above approach, one could obtain the analogous result for the fully-discrete scheme based on \([8]\) and linear finite element method, i.e.,

$$u_{n+1}^h = u_n^h + \frac{h}{2}(v_n^h + v_{n+1}^h) + \frac{h}{2}P_h g(\Theta u_n^h)\delta W_n,$$

$$v_{n+1}^h = v_n^h + \frac{h}{2}A_h(u_n^h + u_{n+1}^h) - \frac{h}{2}P_h f(u_n^h) - \frac{h}{2}P_h g(\Theta u_n^h)\delta W_n,$$

$$q_{n+1} = q_n + \frac{(f(up_n^h) - v_{n+1}^h - f_u^h)}{2\sqrt{F(u_n^h) + \delta_0}},$$

where $\hat{u}_n^h = u_n^h$ (or $\frac{3u_n^h - u_{n-1}^h}{2}$ with $u_{n-1}^h = u_0^h$), $T > 0, N \in \mathbb{N}^+, N\tau = T, n \leq N$. In the end of this section, we present the discrete energy evolution of \([25]\) and its weak error estimate.

**Proposition 5.2.** Let the condition of Theorem 5.1 hold and $X_0 \in \mathbb{H}, T > 0, N \in \mathbb{N}^+, N\tau = T, n \leq N$. The proposed scheme \([25]\) preserves the discrete averaged modified energy evolution law, i.e., for $n \in \{0, 1, \ldots, N - 1\}$,

$$\mathbb{E}[\tilde{H}(u_{n+1}^h, v_{n+1}^h, q_{n+1})] = \mathbb{E}[\tilde{H}(u_n^h, v_n^h, q_n)] + \frac{T}{2} \mathbb{E}\left[\text{Tr}\left(P_h g(\Theta u_n^h)\mathbf{Q}^{\frac{1}{2}}(P_h g(\Theta u_n^h)\mathbf{Q}^{\frac{1}{2}})^*)\right)\right].$$

Here $\tilde{H}(u_n^h, v_n^h, q_n) = \frac{1}{2}\|(-A_h)\frac{1}{2}u_n^h\|^2 + \frac{1}{2}\|v_n^h\|^2 + q_n^2$. In addition, assume that

$$\|g(\Theta u) - g(\Theta \tilde{u})\|_{\mathcal{L}_2(\mathbf{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})} \leq C\|u - \tilde{u}\|,$$

$$\|(-A)\frac{\beta_1}{2}g(\Theta u)\|_{\mathcal{L}_2(\mathbf{Q}^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})} \leq C\|u\|_{\mathbb{H}^{\beta_1}}, \quad \beta_1 \in [1, 2].$$

Then \([25]\) satisfies

$$\mathbb{E}\left[\tilde{H}(u_n^h, v_n^h, q_n) - V_1(u(t_n), v(t_n))\right] \leq C(p, T, X_0)(|err_0| + h^{\beta_1} + h^{\frac{3}{2}\beta} + \tau),$$

where $\beta \in [1, 3], err_0 = |\tilde{H}(u_0^h, v_0^h, q_0) - V_1(u(0), v(0))|$.
Proof. Following similar arguments as in the proof of Proposition 3.1, one can get the discrete energy evolution of \( (25) \),

\[
\mathbb{E}[\bar{H}(v_{n+1}^h, v_{n+1}^h, q_{n+1})] = \mathbb{E}[\bar{H}(v_n^h, v_n^h, q_n)] + \frac{\tau}{2} \mathbb{E} \left[ \text{Tr} \left( \mathcal{P}_h g(\Theta u_n^h) Q^{\frac{1}{2}} (\mathcal{P}_h g(\Theta u_n^h) Q^{\frac{1}{2}})^* \right) \right],
\]

which implies the first assertion. According to \( (5) \), we have

\[
\mathbb{E} \left[ \bar{H}(u_n^h, v_n^h, q_n) - V_1(u(t_n), v(t_n)) \right] = err_0 + \frac{1}{2} \int_0^{t_n} \mathbb{E} \left[ \text{Tr} \left( \mathcal{P}_h g(\Theta u_{n+1}^h) Q^{\frac{1}{2}} (\mathcal{P}_h g(\Theta u_{n+1}^h) Q^{\frac{1}{2}})^* \right) \right] ds.
\]

From \( (27) \), Proposition 2.2, Corollary 3.1, H"older’s inequality and \( \|(I - P_h)w\| \leq Ch^\beta \|w\|_{\tilde{H}^s} \), \( \beta \in [1, 2] \), it follows that

\[
\left| \mathbb{E} \left[ \bar{H}(u_n^h, v_n^h, q_n) - V_1(u(t_n), v(t_n)) \right] \right| \\
\leq \left| err_0 \right| + C \int_0^{t_n} \mathbb{E} \left[ \text{Tr} \left( (I - \mathcal{P}_h) g(\Theta u(s)) Q^{\frac{1}{2}} ((I + \mathcal{P}_h) g(\Theta u(s)) Q^{\frac{1}{2}})^* \right) \right] ds \\
+ C \int_0^{t_n} \mathbb{E} \left[ \left| \left| \text{Tr} \left( \mathcal{P}_h (g(\Theta u(s)) - g(\Theta u_{n+1}^h)) Q^{\frac{1}{2}} (\mathcal{P}_h (g(\Theta u(s)) + g(\Theta u_{n+1}^h)) Q^{\frac{1}{2}})^* \right) \right| \right| ds \\
\leq \left| err_0 \right| + C \int_0^{t_n} \mathbb{E} \left[ \left| \left| g(\Theta u(s)) \right| \right|_{\mathcal{L}_2(Q^{\frac{1}{2}}(\tilde{H}, H))} \left| \left| g(\Theta u(s)) \right| \right|_{\mathcal{L}_2(Q^{\frac{1}{2}}(\tilde{H}, H))} ds \\
+ C \int_0^{t_n} \mathbb{E} \left[ \left| \left| g(\Theta u(s)) - g(\Theta u_{n+1}^h) \right| \right|_{\mathcal{L}_2(Q^{\frac{1}{2}}(\tilde{H}, H))} \left| \left| g(\Theta u(s)) + g(\Theta u_{n+1}^h) \right| \right|_{\mathcal{L}_2(Q^{\frac{1}{2}}(\tilde{H}, H))} ds \\
\leq \left| err_0 \right| + Ch^{\beta_1} + \int_0^{t_n} \sqrt{\mathbb{E} \left[ \left| u_{n+1}^h - u(s) \right|^2 \right]} ds.
\]

Applying the temporal regularity estimate in Proposition 2.2 and Theorem 5.1, we conclude that

\[
\left| \mathbb{E} \left[ \bar{H}(u_n^h, v_n^h, q_n) - V_1(u(t_n), v(t_n)) \right] \right| \leq \left| err_0 \right| + Ch^{\beta_1} + C(\tau + h^{\frac{1}{2}\beta}).
\]

It can be seen that the weak error of the energy for \( (25) \) is only determined by the spatial discretization in the additive noise case since there is no weak error in the temporal direction in this case (see Proposition 3.1).

6 Numerical experiments

This section presents numerical experiments to illustrate the strong convergence order and the preservation of energy evolution law of the proposed numerical schemes \( (25) \) and \( (26) \).
with \( \hat{u}_n = u_n, n \in \{0, 1, \ldots, N\} \), for 1-dimensional nonlinear SWE under the homogeneous Dirichlet boundary condition,

\[
\begin{cases}
    du = v dt, & (x, t) \in (0, 1) \times (0, T], \\
v = u_{xx} dt - f(u) dt + g(u) dW(t), & (x, t) \in (0, 1) \times (0, T], \\
u(x, 0) = \sin(\pi x), & v(x, 0) = 0.
\end{cases}
\]

(28)

Figure 1: The strong convergence order in temporal direction (left) \( f(u) = u, g(u) = \sin(u) \), (right) \( f(u) = \sin(u), g(u) = \sin(u) \)

Fig. 1 displays the temporal approximation errors \( ||u(T) - U^h_N||_{L^2} \) against \( \tau \) on log-log scale with \( \tau = 2^{-5}, 8, 9, 10, 11, 12 \) at time \( T = 1 \) for multiplicative noise, respectively. We fix the spatial step size \( h = 2^{-6} \) and simulate the exact solution with the numerical one by using a small step size \( \tau = 2^{-14} \). It can be observed that the slopes of numerical schemes are close to 1, which implies that the temporal convergence order of the proposed numerical schemes (25) and (26) is 1. Here the expectation is approximated by taking average over 1000 realizations.

Figure 2: Averaged energy evolution relationship (left) \( f(u) = u, \) (right) \( f(u) = \sin(u) \)

Fig. 2 presents the evolution of discrete averaged energies for the proposed numerical schemes (25) and (26). Here, the expectation is approximated by taking average over 5000 realizations. From Fig. 2, it can be seen that averaged energies associated with numerical solutions grow linearly with the time raising when the SWE is driven by additive noise. The numerical results are consistent with the theoretical results.
References

[1] R. Anton, D. Cohen, S. Larsson, and X. Wang. Full discretization of semilinear stochastic wave equations driven by multiplicative noise. *SIAM J. Numer. Anal.*, 54(2):1093–1119, 2016.

[2] L. Banjai, G. Lord, and J. Molla. Strong convergence of a Verlet integrator for the semilinear stochastic wave equation. *SIAM J. Numer. Anal.*, 59(4):1976–2003, 2021.

[3] C. E. Bréhier, J. Cui, and J. Hong. Strong convergence rates of semidiscrete splitting approximations for the stochastic Allen-Cahn equation. *IMA J. Numer. Anal.*, 39(4):2096–2134, 2019.

[4] Z. Brzeźniak, M. Ondreját, and Seidler J. Invariant measures for stochastic nonlinear beam and wave equations. *J. Differential Equations*, 260:4157–4179, 2016.

[5] Y. Cao and L. Yin. Spectral Galerkin method for stochastic wave equations driven by space-time white noise. *Commun. Pure Appl. Anal.*, 6(3):607–617, 2007.

[6] C. Chen, D. Cohen, R. D’Ambrosio, and A. Lang. Drift-preserving numerical integrators for stochastic hamiltonian systems. *Adv. Comput. Math.*, 46(2):1019–7168, 2020.

[7] P. Chow. Stochastic wave equations with polynomial nonlinearity. *Ann. Appl. Probab.*, 12(1):361–381, 2002.

[8] P. Chow. Asymptotics of solutions to semilinear stochastic wave equations. *Ann. Appl. Probab.*, 16(2):757–789, 2006.

[9] D. Cohen, J. Cui, J. Hong, and L. Sun. Exponential integrators for stochastic Maxwell’s equations driven by Itô noise. *J. Comput. Phys.*, 410:109382, 21, 2020.

[10] D. Cohen, S. Larsson, and M. Sigg. A trigonometric method for the linear stochastic wave equation. *SIAM J. Numer. Anal.*, 51(1):204–222, 2013.

[11] J. Cui. Explicit numerical methods for high dimensional stochastic nonlinear schroedinger equation: Divergence, regularity and convergence. *arXiv:2112.10177*, 2021.

[12] J. Cui and J. Hong. Analysis of a splitting scheme for damped stochastic nonlinear Schrödinger equation with multiplicative noise. *SIAM J. Numer. Anal.*, 56(4):2045–2069, 2018.

[13] J. Cui, J. Hong, L. Ji, and L. Sun. Energy-preserving exponential integrable numerical method for stochastic cubic wave equation with additive noise. *arXiv:1909.00575*, 2019.

[14] J. Cui, J. Hong, Z. Liu, and W. Zhou. Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations. *J. Differential Equations*, 266(9):5625–5663, 2019.
[15] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[16] R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao. *A minicourse on stochastic partial differential equations*. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2009.

[17] P. Etter. Advanced applications for underwater acoustic modeling. *Adv. Acoust. Vib.*, 2012:1–28, 2012.

[18] A. Galka, T. Ozaki, H. Muhle, U. Stephani, and M. Siniatchkin. A data-driven model of the generation of human eeg based on a spatially distributed stochastic wave equation. *Cogn. Neurodyn.*, 2(2):101–113, 2008.

[19] J. Hong, B. Hou, and L. Sun. Energy-preserving fully-discrete schemes for nonlinear stochastic wave equations with multiplicative noise. *J. Comput. Phys.*, 451:Paper No. 110829, 20, 2022.

[20] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong and weak divergence in finite time of euler’s method for stochastic differential equations with non-globally lipschitz continuous coefficients. *Proc. R. Soc. A.*, 467:1563–1576, 2011.

[21] C. Kelly and G. Lord. Adaptive time-stepping strategies for nonlinear stochastic systems. *IMA J. Numer. Anal.*, 38(3):1523–1549, 2018.

[22] M. Kovács, S. Larsson, and F. Saedpanah. Finite element approximation of the linear stochastic wave equation with additive noise. *SIAM J. Numer. Anal.*, 48(2):408–427, 2010.

[23] E. Orsingher. Randomly forced vibrations of a string. *Ann. Inst. H. Poincaré Sect. B (N.S.*), 18(4):367–394, 1982.

[24] J. Shen and J. Xu. Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows. *SIAM J. Numer. Anal.*, 56(5):2895–2912, 2018.

[25] J. Shen, J. Xu, and J. Yang. A new class of efficient and robust energy stable schemes for gradient flows. *SIAM Rev.*, 61(3):474–506, 2019.

[26] L. E. Thomas. Persistent energy flow for a stochastic wave equation model in nonequilibrium statistical mechanics. *J. Math. Phys.*, 53(9):095208, 10, 2012.