Chordality, $d$-collapsibility, and componentwise linear ideals

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Abstract

Using the concept of $d$-collapsibility from combinatorial topology, we define chordal simplicial complexes and show that their Stanley-Reisner ideals are componentwise linear. Our construction is inspired by and an extension of “chordal clutters” which was defined by Bigdeli, Yazdan Pour and Zaare-Nahandi in 2017, and characterizes Betti tables of all ideals with linear resolution in a polynomial ring.

We show $d$-collapsible and $d$-representable complexes produce componentwise linear ideals for appropriate $d$. Along the way, we prove that there are generators that when added to the ideal, do not change Betti numbers in certain degrees.

We then show that large classes of componentwise linear ideals, such as Gotzmann ideals and square-free stable ideals have chordal Stanley-Reisner complexes, that Alexander duals of vertex decomposable complexes are chordal, and conclude that the Betti table of every componentwise linear ideal is identical to that of the Stanley-Reisner ideal of a chordal complex.

Introduction

Chordal simplicial complexes, as we call them here, arise from work of Bigdeli, Yazdanpour and Zaare-Nahandi [7] in 2017, where they defined chordal clutters in an attempt to give a combinatorial description of square-free monomial ideals that have linear resolution over all fields. The term “chordal” and the general approach stem from Fröberg’s 1990 paper [18] in which ideals generated by degree 2 monomials are characterized in terms of chordal graphs. Fröberg’s work initiated investigations by many authors find similar criteria for ideals with linear resolution generated by monomials of higher degree, which led to generalizations of chordality: the classes defined by Van Tuyl and Villarreal [36] in 2008, Emtander [16] in 2010, Woodroofe [38] in 2011, all produce ideals with linear resolution over all fields, and all these classes were shown to be contained in the class...
of chordal simplicial complexes in [7] (which we later found is equivalent to a class of simplicial complexes appearing in Cordovil, Lemos, and Sales [12] in 2009).

On the other hand Connon and Faridi [10] in 2013 gave a more general definition of chordality by focusing on necessary conditions for vanishing of simplicial homology, which forced a simplicial complex producing a linear resolution in any characteristic to belong to their class, and a more restrictive definition in [11] in 2015 characterized all simplicial complexes whose ideals have linear resolution over fields of characteristic 2. Adiprasito, Nevo, and Samper’s work [2] in 2016 characterized chordality by checking a smaller interval for the vanishing of simplicial homology, giving a homological characterization of chordality.

Since betti numbers depend on the characteristic of the ground field, for a combinatorial characterization of chordality, one should expect a definition that produces ideals that have linear resolution over all fields. So far neither of the above classes combinatorially characterizes monomial ideals with linear resolution, even when one considers ideals that have linear resolution over all fields.

However, it was shown by Bigdeli, Herzog, Yazdanpour and Zaare-Nahandi [5] that every Betti table of a graded ideal with linear resolution is the Betti table of an ideal coming from a chordal clutter, as defined in [7].

In this paper, we adapt the concept of chordal clutters from [7] and change the perspective from clutters to simplicial complexes. As a result, we show that chordality of the Stanley-Reisner complex of an ideal generated in degree $d + 1$ is equivalent to $d$-collapsibility, a notion well-known and well-used in algebraic topology and combinatorics which has specific homological consequences. Among other things, this perspective allows us to:

- show that $d$-chordal simplicial complexes (one of the largest known classes of complexes which produce ideals with linear resolution over all fields) are essentially, but not exactly, the same as $d$-collapsible ones (Theorem 3.4);
- introduce a large class of complexes, which we call chordal complexes, whose Stanley-Reisner ideals are componentwise linear (Theorem 4.6);
- show that, for a suitable $d$, $d$-collapsible and $d$-representable simplicial complexes are chordal and have componentwise linear Stanley-Reisner ideals (Theorem 4.6);
- show that square-free stable monomial ideals have chordal Stanley-Reisner complexes (Theorem 5.3);
- show that Alexander duals of vertex decomposable complexes are chordal (Theorem 5.2);
- show that Gotzmann square-free monomial ideals have chordal Stanley-Reisner complexes (Theorem 5.6);
- show that Betti tables of Stanley-Reisner ideals of chordal complexes encompass all Betti tables of componentwise linear ideals (Theorem 5.4);
- show that there are specific monomials we can add to the generators of a monomial ideal without affecting the Betti numbers in most degrees (Theorem 4.4).
• using induced subcomplexes, find useful inductive properties of componentwise linear ideals (Theorem 4.6).

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1 Basic definitions

A simplicial complex $\Gamma$ on the vertex set $[n] = \{1, \ldots, n\}$, is a set of subsets of $[n]$ such that if $F \in \Gamma$ and $F' \subseteq F$, then $F' \in \Gamma$. Each element of $\Gamma$ is called a face of $\Gamma$. A facet is a maximal face of $\Gamma$ (with respect to inclusion). The dimension of a face $F$ is $\dim F = |F| - 1$. We define $\dim \emptyset = -1$. A face $F$ of $\Gamma$ with $\dim F = t$ is called a $t$-face of $\Gamma$. Let $d = \max \{\dim F : F \in \Gamma\}$ and define the dimension of $\Gamma$ to be $\dim \Gamma = d$. We say that a simplicial complex is pure if all its facets have the same dimension.

A simplicial complex $\Gamma$ is uniquely determined by its facets. We denote the set of the facets of $\Gamma$ by $\text{Facets}(\Gamma)$ and when $\text{Facets}(\Gamma) = \{F_1, \ldots, F_m\}$, we write $\Gamma = \langle F_1, \ldots, F_m \rangle$. A simplicial complex with only one facet is called a simplex.

A subcomplex $\Sigma$ of $\Gamma$ is a simplicial complex with $\Sigma \subset \Gamma$. Let $E \subset [n]$. By $\Gamma \setminus E$ we mean $\Gamma \setminus E = \{F \in \Gamma : E \not\subseteq F\}$ which is a subcomplex of $\Gamma$. If $W \subset [n]$, we denote by $\Gamma_W$ the induced subcomplex of $\Gamma$ on the set $W$, in other words $\Gamma_W = \{F \in \Gamma : F \subset W\}$.

The Alexander dual $\Gamma^\vee$ of $\Gamma$ is the simplicial complex $\Gamma^\vee = \{F \subseteq [n] : [n] - F \not\in \Gamma\}$.

If $F$ is a face of $\Gamma$, then $\text{link}_\Gamma(F)$ is the simplicial complex on $[n] - F$ defined as $\text{link}_\Gamma(F) = \{G \in \Gamma : F \cap G = \emptyset \text{ and } G \cup F \in \Gamma\}$.

For a nonnegative integer $i \leq \dim \Gamma$, we define the pure $i$-skeleton $\Gamma^{[i]}$ of $\Gamma$ to be the simplicial complex $\Gamma^{[i]} = \{F \in \Gamma : \dim F = i\}$.

A nonface of $\Gamma$ is a subset $F$ of $[n]$ with $F \not\in \Gamma$.

Definition 1.1 (Stanley-Reisner ideal/complex). Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over the field $K$ with $n$ indeterminates.
• Let $\Gamma$ be a simplicial complex on $n$ vertices. The **Stanley-Reisner ideal** of $\Gamma$ is the monomial ideal $\mathcal{N}(\Gamma)$ of $S$ which is generated by the square-free monomials $x_F := \prod_{i \in F} x_i$ with $F \notin \Gamma$. In other words

$$\mathcal{N}(\Gamma) = (x_F : F \notin \Gamma).$$

The **Stanley-Reisner ring**, $K[\Gamma]$, is defined to be the quotient ring $S/\mathcal{N}(\Gamma)$.

• Let $I$ be a square-free monomial ideal in $S$. We define its **Stanley-Reisner complex** $\mathcal{N}(I)$ to be the simplicial complex

$$\mathcal{N}(I) = \{ F \subseteq [n] : x_F / \notin I \}.$$

It follows directly from the definitions that the Stanley-Reisner correspondence is a one-to-one correspondence between simplicial complexes on the vertex set $[n]$ and square-free monomial ideals in $S$.

Let $I \in S = K[x_1, \ldots, x_n]$ be a graded ideal and let

$$F : 0 \longrightarrow F_p \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

be its graded minimal free resolution with $F_i = \bigoplus_j S^{\beta_i^{S,j}(I)}(-j)$, for all $i$. For any pair of integers $(i, j)$, the **$(i, j)$-th graded Betti number** of $I$ in $S$ is defined to be

$$\beta_{i,j}^S(I) = \dim_K \text{Tor}_i^S(K, I)_j$$

for all $i$ and $j$. Throughout, we write $\beta_{i,j}(I)$ for $\beta_{i,j}^S(I)$. The ideal $I$ is called to have **$d$-linear resolution** if $\beta_{i,j}(I) = 0$ for all $i$ and all $j$ with $j \neq i + d$.

## 2 $d$-chordality

The definition below is a slight variation of that given in [10, Definition 5.4].

**Definition 2.1 ($d$-closure).** Let $\Gamma$ be a simplicial complex on the vertex set $[n]$ and $d$ a positive integer. The $d$-closure of $\Gamma$, denoted by $\Delta_d(\Gamma)$, is the simplicial complex on $[n]$ whose faces are given in the following way:

- the $d$-faces of $\Delta_d(\Gamma)$ are exactly the $d$-faces of $\Gamma$;
- all subsets of $[n]$ with at most $d$ elements are faces of $\Delta_d(\Gamma)$;
- a subset of $[n]$ with more than $d + 1$ elements is a face of $\Delta_d(\Gamma)$ if and only if all of its subsets of $d + 1$ elements are faces of $\Gamma$.

If $\Gamma$ is the $d$-closure of a simplicial complex, we simply say that $\Gamma$ is a **$d$-closure**.

To justify this terminology, note that all the simplicial complexes on $[n]$ which have the same pure $d$-skeleton, have the same $d$-closure. In particular, if $\Gamma = \Delta_d(\Sigma)$, by definition we have $\Gamma^{[d]} = \Sigma^{[d]}$ and it follows that
Figure 1: The simplicial complex $\Gamma$

$\Gamma = \Delta_d(\Sigma) \iff \Delta_d(\Sigma) = \Delta_d(\Gamma) \iff \Gamma = \Delta_d(\Gamma)$.

**Example 2.2.** Let $\Gamma = \langle \{2, 5\}, \{1, 4, 5\}, \{1, 2, 3, 4\} \rangle$ be a simplicial complex on $[5]$ in Figure 1.

Note that $\dim(\Gamma) = 3$. We have

$\Delta_1(\Gamma) = \langle \{1, 2, 4, 5\}, \{1, 2, 3, 4\} \rangle$,

$\Delta_2(\Gamma) = \langle \{2, 5\}, \{3, 5\}, \{1, 4, 5\}, \{1, 2, 3, 4\} \rangle$,

$\Delta_3(\Gamma) = \langle \{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\} \rangle$,

$\Delta_i(\Gamma) = \langle [5] \rangle^{[i-1]}$, for $i \geq 4$.

It is shown in [10, Proposition 5.6] that a square-free monomial ideal $I$ is equigenerated in degree $d + 1$ if and only if $\mathcal{N}(I)$ is a $d$-closure, i.e.

$\mathcal{N}(I) = \Delta_d(\mathcal{N}(I))$.

**Definition 2.3 (free face and simplicial face [30, Definition 2.13]).** A face $E$ of a simplicial complex $\Gamma$ is called a **free face** if it appears in a unique facet of $\Gamma$. Note that facets are automatically free faces.

If $\Gamma$ is a $d$-closure and $\dim E = d - 1$, then this free face is called **simplicial**. We denote the set of all simplicial faces of $\Gamma$ by $\text{Simp}(\Gamma)$.

Let $\Gamma$ be a simplicial complex on $[n]$ and $E \subset [n]$. The **deletion** of $E$ from $\Gamma$, is the simplicial complex

$\Gamma \setminus_E = \{ F \in \Gamma : E \not\subset F \} = \begin{cases} \Gamma & \text{if } E \notin \Gamma \\ (\Gamma \setminus E) \cup \{E\} & \text{if } E \in \Gamma. \end{cases}$

Note that if $E \in \Gamma$, the face $E$ is not deleted in this operation. In case $E$ is a simplicial face of a $d$-closure $\Gamma$, this operation is called **simplicial deletion** of $E$ from $\Gamma$. The simplicial complex obtained from a simplicial deletion is again a $d$-closure. Note also that all $(d - 1)$-faces of a $d$-closure $\Gamma$ which are its facets are simplicial. Indeed, for a $d$-closure $\Gamma$, $\Gamma \setminus_E = \Gamma$ if and only if $E$ is a facet of $\Gamma$. 

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Let $E = E_1, \ldots, E_t$ be a sequence of $(d - 1)$-faces of a $d$-closure $\Gamma$. The sequence $E$ is called a **simplicial sequence** of $\Gamma$ if $E_1 \in \text{Simp}(\Gamma)$, and $E_i \in \text{Simp}(\Gamma \l_{E_1} \cdots \l_{E_{i-1}})$ for all $i \geq 2$. The sequence $E$ is called a **simplicial order** of $\Gamma$ if $E_1$ is not a facet of $\Gamma$, $E_i$ is not a facet in $\Gamma \l_{E_1} \cdots \l_{E_{i-1}}$, and

$$
\Gamma \l_{E_1} \cdots \l_{E_t} = ([n])^{[d-1]}.
$$

In order to shorten the notation, we often use $\Gamma \l_{E_1, \ldots, E_t}$ instead of $\Gamma \l_{E_1} \cdots \l_{E_t}$.

**Example 2.4.** Consider $\Delta_2(\Gamma)$ in Example 2.2 and let $E_1 = \{1, 5\}$. Since $E_1$ is uniquely contained in the facet $\{1, 4, 5\}$, it is a simplicial face of $\Delta_2(\Gamma)$. We have

$$
\Sigma_1 := \Delta_2(\Gamma) \l_{E_1} = \langle \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3, 4\} \rangle.
$$

Now let $E_2 = \{1, 2\}$. Since the only facet in $\Sigma_1$ containing $E_2$ is $\{1, 2, 3, 4\}$, $E_2$ is simplicial in $\Sigma_1$. Then

$$
\Sigma_2 := \Sigma_1 \l_{E_2} = \langle \{1, 2\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 3, 4\}, \{2, 3, 4\} \rangle.
$$

Now $E_3 = \{1, 3\}$ is simplicial in $\Sigma_2$ and

$$
\Sigma_3 := \Sigma_2 \l_{E_3} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{2, 3, 4\} \rangle.
$$

Finally $E_4 = \{2, 3\}$ is simplicial in $\Sigma_3$ and

$$
\Sigma_3 \l_{E_4} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \rangle
$$

$$
= \langle [5] \rangle^{[1]}.
$$

Therefore $E_1, \ldots, E_4$ is a simplicial order of $\Delta_2(\Gamma)$.

**Lemma 2.5.** Let $d$ be a positive integer, $\Gamma$ a simplicial complex and $E$ a $(d - 1)$-dimensional face of $\Delta_d(\Gamma)$. Then $\Delta_d(\Gamma) \l_E = \Delta_d(\Gamma \l_E)$.

**Proof.** It is clear that the two complexes have the same faces of dimension $\leq d - 1$. By Definition 2.1 if $\text{dim}(F) = d$

$$
F \in \Delta_d(\Gamma) \l_E \iff F \in \Gamma \text{ and } F \not\supset E \iff F \in \Gamma \l_E \iff F \in \Delta_d(\Gamma \l_E)
$$

and if $\text{dim}(F) > d$

$$
F \in \Delta_d(\Gamma) \l_E \iff \forall G \subset F \text{ if } \text{dim}(G) = d \text{ then } G \in \Gamma \text{ and } F \not\supset E
$$

$$
\iff \forall G \subset F \text{ if } \text{dim}(G) = d \text{ then } G \in \Gamma \text{ and } G \not\supset \not\supset E
$$

$$
\iff \forall G \subset F \text{ if } \text{dim}(G) = d \text{ then } G \in \Gamma \l_E
$$

$$
\iff F \in \Delta_d(\Gamma \l_E).
$$
Lemma 2.5 allows us to define a chordal simplicial complex with two equivalent conditions. Below we define chordal simplicial complexes using the concept of chordal clutters as defined by the first author and the coauthors in [7].

**Definition 2.6 (d-chordal and chordal simplicial complex, see [7]).** Let \( \Gamma \) be a simplicial complex on the vertex set \([n]\) and \( d \) a positive integer. We say that \( \Gamma \) is **d-chordal** if it satisfies one of the following equivalent conditions:

1. either \( \Delta_d(\Gamma) = \langle [n] \rangle^{d-1} \), or else \( \Delta_d(\Gamma) \) admits a simplicial order.
2. either \( \Delta_d(\Gamma) = \langle [n] \rangle^{d-1} \), or else there is \( E \in \text{Simp}(\Delta_d(\Gamma)) \) such that \( E \) is not a facet of \( \Delta_d(\Gamma) \) and \( \Gamma \wr E \) satisfies condition (\( \ast' \)).

We say that \( \Gamma \) is **chordal** if it is \( d \)-chordal for every \( d \geq 1 \).

Later in Proposition 3.12 we will show that to prove \( \Gamma \) is chordal, it is sufficient to check it is \( d \)-chordal for a finite number of values of \( d \).

Definition 2.6 of a "\( d \)-chordal simplicial complex" is a Stanley-Reisner equivalent of "chordal \((d + 1)\)-uniform clutters" in [7]. The following statement follows directly from the definitions, we include it for the sake of comparison.

**Proposition 2.7.** Let \( \Gamma \) be a simplicial complex on \([n]\), and let \( C \) be the \((d + 1)\)-uniform clutter
\[
C = \{ A \subseteq [n] : |A| = d + 1, A \in \Gamma \} = \text{Facets}(\Gamma^{[d]}).
\]
Then \( C \) is chordal in the sense of [7] if and only if \( \Gamma \) is \( d \)-chordal.

In particular, as in the case of [7], our definition of chordality for simplicial complexes extends that of graphs. Given a simplicial complex \( \Gamma \), its 1-closure \( \Delta_1(\Gamma) \) is the clique complex of a graph \( G = \Gamma^{[1]} \). It is clear that \( G \) is chordal (i.e. has no minimal cycles of length greater than 3) if and only if its clique complex \( \Delta_1(\Gamma) = \Delta_1(G) \) is 1-chordal.

### 3 \( d \)-collapsing

In this section we show how the concept of **elementary \( d \)-collapsing** introduced by Wegner [57] relates directly to simplicial deletion. Elementary \( d \)-collapsing is a special case of the better known operation of **simplicial collapsing** (see for example [28 Definition 6.13]), which when applied to a simplicial complex produces a new simplicial complex which is homotopy equivalent to the original one. The main difference between the two operations is that in the case of \( d \)-collapsing a free face is allowed to be facet.

Recall that \( \Gamma \setminus E \) refers to the operation of deleting all faces of the simplicial complex \( \Gamma \) containing the face \( E \) (including \( E \) itself). In the case where \( E \) is a free face we denote this complex by \( \Gamma \setminus \rhd E \), that is
\[
\Gamma \setminus \rhd E = \Gamma \setminus E.
\]
A sequence of faces $E = E_1, \ldots, E_t$ is called a **free sequence** of $\Gamma$ if $E_1$ is a free face in $\Gamma$, and $E_i$ is a free face in $\Gamma \setminus E_1 \ldots \setminus E_{i-1}$ for all $i > 1$. We shorten the notation for the series of deletions, by using

$$\Gamma \setminus E_1, \ldots, E_t = \Gamma \setminus E_1 \ldots \setminus E_t.$$  

**Definition 3.1** ($d$-collapsing). If $\Gamma$ is a simplicial complex with a free face $E$, and $d$ is a positive integer with $\dim E < d$, then the operation $\Gamma \setminus E$ is called an **elementary $d$-collapsing**. The simplicial complex $\Gamma$ is called **$d$-collapsible** if it can be reduced to the void complex $\emptyset$ after a finite number of elementary $d$-collapsings.

Suppose now $\Gamma$ is a $d$-closure and $E$ is a simplicial face of $\Gamma$. Then, by definition, $E$ is a free face with $\dim E = d - 1$ and

$$\Gamma \setminus E = \Gamma \setminus E \cup \{E\}.$$ 

Suppose $E = E_1, \ldots, E_t$ is a simplicial order of $\Gamma$. Then it is a free sequence of $\Gamma$ and

$$\Gamma \setminus E_1, \ldots, E_t = \{(n)\}^{d-1} - \{E_1, \ldots, E_t\}. \tag{1}$$

We now start working our way towards Theorem 3.4 where we show that there is a direct relation between the $d$-chordal simplicial complexes and $d$-collapsible ones.

A very useful tool when considering $d$-collapsings is Lemma 3.2 below, which we proved independently and then found later in Tancer’s work 35. We refer the reader there for a full proof.

**Lemma 3.2** (Tancer 35, Lemma 5.1). Let $\Sigma$ be a simplicial complex, $d$ a positive integer, $E \subseteq E'$ free faces of $\Sigma$ of dimension $< d$. Then $\Sigma \setminus E'$ $d$-collapses to $\Sigma \setminus E$. In particular, if $\Sigma \setminus E$ is $d$-collapsible, then so is $\Sigma \setminus E'$.

**Lemma 3.3.** Let $\Sigma$ be a simplicial complex on the vertex set $[n]$ and let $E = E_1, \ldots, E_r$ be a free sequence of $\Sigma$ with the property that $\dim E_r = d - 1$ and $E_r$ is the only element in this sequence such that the unique facet containing it has dimension $\geq d$. Then $E_r$ is a free face of $\Sigma$ and $E_1, \ldots, E_{r-1}$ is a free sequence of $\Sigma \setminus E_r$. Moreover,

$$\Sigma \setminus E_r, E_1, \ldots, E_{r-1} = \Sigma \setminus E_1, \ldots, E_r.$$  

**Proof.** Suppose $F$ is the unique facet in $\Sigma \setminus E_1, \ldots, E_{r-1}$ which contains $E_r$. If $T \in \Sigma$ has dimension $\geq d$, then $E_i \not\subset T$ for $i < r$. Thus $T \in \Sigma \setminus E_1, \ldots, E_{r-1}$. It follows that $F$ is a facet in $\Sigma$. Suppose $G$ is another facet in $\Sigma$ containing $E_r$. Since $F$ is the unique facet in $\Sigma \setminus E_1, \ldots, E_{r-1}$ containing $E_r$, we conclude that $G$ contains some $E_i$ with $i < r$. Hence $\dim G < d$ and so $G = E_r \setminus F$, a contradiction. Thus $E_r$ is a free face of $\Sigma$.

Now we show that $E_1, \ldots, E_{r-1}$ is a free sequence of $\Sigma \setminus E_r$. Suppose $F_1$ is the unique facet in $\Sigma$ containing $E_1$ and for $1 < i < r$, $F_i$ is the unique facet in $\Sigma \setminus E_1, \ldots, E_{i-1}$ containing $E_i$. Then since $\dim F_i < d$, we have $E_r \subseteq F_i$ if and only if $E_r = F_i$. If $E_r \subseteq F_i$, then $F_i \subseteq F$ and since $F \in \Sigma \setminus E_1, \ldots, E_{i-1}$, it contradicts the fact that $F_i$ is a facet. Therefore $E_r \not\subset F_i$ for $1 \leq i \leq r - 1$. Hence $F_1$ is a facet in $\Sigma \setminus E_r$ and $F_i$ is a facet in $\Sigma \setminus E_r, \setminus E_1, \ldots, E_{i-1}$ for $1 < i < r$. Moreover, since $\Sigma \setminus E_r, \setminus E_1, \ldots, E_{i-1} \subseteq \Sigma \setminus E_1, \ldots, E_{i-1}$ it follows that $F_i$ is the only facet in $\Sigma \setminus E_r, E_1, \ldots, E_{i-1}$ containing $E_i$. Thus $E_1$ is a free face in $\Sigma \setminus E_r$ and $E_i$ is a free face in $\Sigma \setminus E_r, E_1, \ldots, E_{i-1}$.
An immediate consequence of Lemma 3.2 and Lemma 3.3 is that \( d \)-collapsibility and \( d \)-chordality are intimately connected.

**Theorem 3.4 (\( d \)-collapsible is equivalent to \( d \)-chordal for \( d \)-closures).** Let \( \Gamma \) be a \( d \)-closure on the vertex set \([n]\) for a positive integer \( d \). Then \( \Gamma \) is \( d \)-chordal if and only if \( \Gamma \) is \( d \)-collapsible. In particular, a simplicial complex \( \Sigma \) is chordal if and only if for all \( d \geq 1 \) the simplicial complex \( \Delta_d(\Sigma) \) is \( d \)-collapsible.

**Proof.** It is enough to prove the first statement. Suppose \( \Gamma \) is \( d \)-collapsible. We prove by induction on the number of faces of \( \Gamma \) that \( \Gamma \) is \( d \)-chordal. The base case of the induction is the smallest \( d \)-closure \( \Gamma = \langle [n] \rangle^{[d-1]} \) which is \( d \)-chordal by definition.

Suppose \( \Gamma \supseteq \langle [n] \rangle^{[d-1]} \) is \( d \)-collapsible. Hence there is a free sequence \( E_1, \ldots, E_t \) such that \( \dim E_i < d \) for all \( i \) and \( \Gamma \backslash E_1, \ldots, E_t = \emptyset \). Suppose \( r \) is the smallest integer in \( 1, \ldots, t \) such that the facet \( F \) of \( \Gamma \backslash E_1, \ldots, E_r \) uniquely containing \( E_r \) has dimension \( \geq d \). Note that since \( \Gamma \neq \langle [n] \rangle^{[d-1]} \), such \( r \) exists. We may assume by Lemma 3.2 that \( \dim E_r = d - 1 \). By Lemma 3.3 we know that \( E_r \in \text{Simp}(\Gamma) \) and \( E_1, \ldots, E_{r-1} \) is a free sequence of \( \Gamma \backslash E_r \).

Since \( \Gamma \backslash E_r \backslash E_1, \ldots, E_{r-1} = \Gamma \backslash E_1, \ldots, E_r \) is \( d \)-collapsible, so is \( \Gamma \backslash E_r \). On the other hand \( \Gamma \backslash E_r = (\Gamma \backslash E_r) \backslash E_r \). So \( \Gamma \backslash E_r \) is \( d \)-collapsible, and hence \( d \)-chordal by the induction hypothesis, which implies that \( \Gamma \) is \( d \)-chordal.

Suppose now that \( \Gamma \) is \( d \)-chordal and admits a simplicial order \( E_1, \ldots, E_t \). By Equation (1) the simplicial complex \( \Gamma \), \( d \)-collapses along this sequence to \( \langle [n] \rangle^{[d-1]} - \{ E_1, \ldots, E_t \} \), and since all faces of this simplicial complex have dimension \( < d \), it collapses into \( \langle [n] \rangle^{[d-2]} \) by elementary \( d \)-collapsings along its facets. Continuing this process one sees that \( \langle [n] \rangle^{[d-1]} - \{ E_1, \ldots, E_t \} \) collapses into \( \emptyset \) by a sequence of elementary \( d \)-collapsings. Hence \( \Gamma \) is \( d \)-collapsible.

Given a simplicial complex \( \Gamma \) on \([n]\), the set of all \( d \)-faces of \( \Gamma \) forms a “\((d+1)\)-uniform clutter” which we call \( C \). In [7] the authors defined a chordal \((d+1)\)-uniform clutter. It is straightforward to check that \( \Gamma \) is \( d \)-chordal if and only if \( C \) is a chordal \((d+1)\)-uniform clutter. It is also proved in [7] that if \( \Gamma = \langle [n] \rangle \), then \( C \) is chordal. It follows that \( \Gamma = \langle [n] \rangle \) is \( d \)-chordal for any \( d \). In the following lemma we give a direct short proof for this fact using \( d \)-collapsibility.

**Lemma 3.5.** [See also [7] Corollary 3.11] The simplicial complex \( \langle [n] \rangle \) is chordal.

**Proof.** Let \( \Gamma = \langle [n] \rangle \) and observe that \( \Gamma \) is a \( d \)-closure. By Theorem 3.4 it is enough to show that \( \Gamma \) is \( d \)-collapsible for all \( d \geq 1 \). The face \( E = \{ n \} \) appears in the unique facet \([n]\), and is therefore a free face of \( \Gamma \) of dimension \( 0 < d \) for any \( d \geq 1 \). We now use induction on \( n \). If \( n = 1 \), then \( \Gamma \backslash E = (\emptyset) \) and we are done. If \( n > 1 \), then \( \Gamma \backslash E = \langle [n-1] \rangle \) which is \( d \)-collapsible by induction hypothesis, settling our claim.

The condition of being a \( d \)-closure is necessary in the statement of Theorem 3.4, as can be seen in the example below.

**Example 3.6.** If \( \Gamma = \{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \} \) is the hollow tetrahedron and \( d = 1 \), then \( \Delta_1(\Gamma) = \{ \{1, 2, 3, 4\} \} \) is the full tetrahedron, and so \( \Gamma \) is 1-chordal by Lemma 3.5. But \( \Gamma \) has no free face of dimension \( < 1 \), and therefore \( \Gamma \) is not 1-collapsible.
This example shows that $d$-collapsibility of $\Delta_d(\Gamma)$ is not a sufficient condition for $\Gamma$ to be $d$-collapsible. It is, however, a necessary condition, as we show in Theorem [3.9] which implies, in particular, that every $d$-collapsible complex is $d$-chordal, though the converse is not true in general (Proposition 3.10(a)). To show Theorem 3.9 we need the following two lemmas.

**Lemma 3.7.** Let $\Sigma$ be a $d$-collapsible simplicial complex on $[n]$ and let $E$ be a subset of $[n]$ with the property that all facets of $\Sigma$ containing $E$ have dimension $\leq d - 1$. Then $\Sigma \setminus E$ is $d$-collapsible.

**Proof.** If $E = \emptyset$, then $\Sigma \setminus E = \emptyset$ which is $d$-collapsible by definition. Suppose $E \neq \emptyset$. If $E \notin \Sigma$, then $\Sigma \setminus E = \Sigma$ is $d$-collapsible. Suppose $E \in \Sigma$ and $E = E_1, \ldots, E_t$ is a free sequence of $\Sigma$ with $\dim E_i \leq d - 1$ and $\Sigma \setminus E = \emptyset$.

Suppose $E = \{F \in \Sigma : E \subseteq F\}$. Then $\Sigma \setminus E = \Sigma - E$ and all maximal elements of $E$ are facets of $\Sigma$ with dimension $\leq d - 1$.

Suppose $r$ is the smallest element in $1, \ldots, t$ with $E_r \subseteq G$ for some $G \in E$. Then $E_1, \ldots, E_{r-1}$ is a free sequence in $\Sigma \setminus E$, and we have

$$(\Sigma \setminus E) \setminus_{E_1,\ldots,E_{r-1}} = (\Sigma \setminus_{E_1,\ldots,E_{r-1}}) \setminus E.$$  

So without loss of generality we may assume that $r = 1$.

We now proceed with induction on the number of faces of $\Sigma$. If $\Sigma = \emptyset$, then there is nothing to prove. Consider $\Sigma = \langle \{1\} \rangle$ as the base case of induction. Then $E = \{1\}$ and $\Sigma \setminus E = \emptyset$ which is $d$-collapsible. Let $G$ be a facet of $\Sigma$ in $E$ containing $E_1$. Then $G$ is a free face of $\Sigma$ of dimension $\leq d - 1$, and by Lemma [3,2] since $\Sigma \setminus_{E_1}$ is $d$-collapsible, so is $\Sigma \setminus_{E_1}$. By induction hypothesis

$$\Sigma \setminus_{E_1} \setminus E = \Sigma \setminus E$$

is $d$-collapsible, and we are done. \qed

**Lemma 3.8.** Let $E$ be a free face of a simplicial complex $\Gamma$ with $\dim E \leq d - 1$. Then there is a free sequence $E = E_1, \ldots, E_r$ of dimension $\leq d - 1$ for $\Sigma := \Delta_d(\Gamma)$ such that

$$\Sigma \setminus_{E} = \Sigma \setminus E.$$  

**Proof.** If $E$ is a free face of $\Sigma$ we set $E = E$ and we are done.

Suppose that $E$ is not a free face for $\Sigma$ and $E$ is contained in a unique facet $F$ of $\Gamma$. Then $\dim E < d - 1$, because if $\dim E = d - 1$, then either $E$ is a facet of $\Gamma$ in which case it will be a facet of $\Sigma$, or all $d$-faces of $\Gamma$ containing $E$ are contained in $F$, which makes $F$ also the unique facet of $\Sigma$ containing $E$.

Let $E^1 = \{E^1_1, \ldots, E^1_{m_1}\}$ be the set of $(d - 1)$-faces of $\Sigma$ which contain $E$ but are not subsets of $F$. Since all $d$-faces of $\Gamma$ containing $E$ are in $F$, there is no face of dimension $\geq d$ in $\Sigma$ which contains $E^1_i$ for $1 \leq i \leq m_1$. This implies that $E^1_i$ is a facet in $\Sigma$ for any $i$. So $E^1$ gives a free sequence of $\Sigma$.

If $F$ is the only facet in $\Sigma \setminus_{E^1}$ which contains $E$, then $E$ is a free face and we set

$$E = E^1_1, \ldots, E^1_{m_1}, E$$

so that $\Sigma \setminus_{E} = \Sigma \setminus E$.  

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Otherwise, let $E_2$ be the set of all $(d-2)$-faces of $\Sigma \setminus E_1$ which contain $E$ but are not subsets of $F$. Once again, $E_2$ is a free sequence in $\Sigma \setminus E_1$, and continuing in this way after a finite number of steps, we get the free sequence

$$E = E^1, \ldots, E^s, E$$

of $\Sigma$ of dimension $\leq d - 1$ for which $\Sigma \setminus E = \Sigma \setminus E$.

\textbf{Theorem 3.9 (The $d$-closure of a $d$-collapsible complex is $d$-collapsible).} Let $\Gamma$ be a $d$-collapsible simplicial complex for some $d \geq 1$. Then $\Delta_d(\Gamma)$ is $d$-collapsible.

\textbf{Proof.} Let $\Sigma := \Delta_d(\Gamma)$ and $r$ be the length of the shortest free sequence of $\Gamma$ which $d$-collapses it into $\langle \emptyset \rangle$. We proceed by induction on $r$. If $r = 0$ then $\Gamma = \langle \emptyset \rangle$ and $\Delta_d(\Gamma) = ([n])^{[d-1]}$, which is $d$-collapsible, because the facets are free faces and all of them have dimension $< d$.

For the general case, suppose $E$ is the first element in the shortest free sequence of length $r$ for $\Gamma$. Then $\Gamma \setminus \setminus E$ is $d$-collapsible using a sequence of length $r - 1$, so by induction hypothesis $\Delta_d(\Gamma \setminus \setminus E)$ is $d$-collapsible. Now

$$\Sigma \setminus E = \Delta_d(\Gamma \setminus \setminus E) \setminus E = \Delta_d(\Gamma \setminus \setminus E) \setminus \{ G \subset [n] : E \subseteq G, \dim G \leq d - 1 \}.$$

Since the maximal elements of $\{ G \subset [n] : E \subseteq G, \dim G \leq d - 1 \}$ are facets of $\Delta_d(\Gamma \setminus \setminus E)$ and no other facet of $\Delta_d(\Gamma \setminus \setminus E)$ contains $E$, Lemma 3.7 implies that $\Sigma \setminus E$ is $d$-collapsible. By Lemma 3.8 there is a free sequence $E = E_1, \ldots, E_t$ of $\Sigma$ of dimension $< d$ such that

$$\Sigma \setminus \setminus E = \Sigma \setminus E$$

which implies that $\Sigma$ is $d$-collapsible.

\textbf{Proposition 3.10 ($d$-collapsible implies $t$-chordal for $t \geq d$).} Let $\Gamma$ be a simplicial complex on $[n]$, and let $d \geq 1$.

(a) If $\Gamma$ is $d$-collapsible then $\Gamma$ is $t$-chordal for all $t \geq d$.

(b) If $\Gamma$ is $d$-chordal then the simplicial complex $\Delta_d(\Gamma)$ is $t$-chordal for all $t \geq d$.

(c) If $\Gamma$ is a $d$-closure then $\Gamma$ is chordal if and only if $\Gamma$ is $d$-chordal.

\textbf{Proof.} (a) By the definition of $d$-collapsibility, we know that $\Gamma$ is $t$-collapsible for all $t \geq d$. Using Theorem 3.9 for all $t \geq d$, $\Delta_t(\Gamma)$ is $t$-collapsible. Hence by Theorem 3.4 for all $t \geq d$, $\Delta_t(\Gamma)$ is $t$-chordal. Therefore $\Gamma$ is $t$-chordal for $t \geq d$.
(b) Since $\Gamma$ is $d$-chordal for some $d \geq 1$, by Theorem 3.4 the simplicial complex $\Sigma = \Delta_d(\Gamma)$ is $d$-collapsible. Hence $\Sigma$ is t-collapsible for all $t \geq d$. Theorem 3.9 implies that $\Delta(t)(\Sigma)$ is $t$-collapsible, and by Theorem 3.4 $\Sigma$ is $t$-chordal for all $t \geq d$.

(c) Suppose $\Gamma$ is $d$-chordal, and let $t < d$. Since $\Gamma$ contains all $(t + 1)$-subsets of $[n]$ we have $\Delta(t)(\Gamma) = \langle [n] \rangle$ which by Lemma 3.5 is $t$-chordal. For $t \geq d$, since $\Gamma = \Delta_d(\Gamma)$ Part (b) implies the assertion. □

Note that in the assumption of Proposition 3.10(a), we cannot replace $d$-collapsibility of $\Gamma$ with $d$-collapsibility of $\Delta_d(\Gamma)$ (or, equivalently, by $d$-chordality of $\Gamma$). Let $\Gamma$ be a simplicial complex which is not chordal and let $d \geq 1$ be an integer with $d < r$, where $r$ is the smallest dimension of the nonfaces of $\Gamma$. Then $\Delta_d(\Gamma) = \langle [n] \rangle$, which is $d$-chordal and hence by Theorem 3.4 it is $d$-collapsible. But since $\Gamma$ is not chordal, there exists $t \geq r > d$ such that $\Gamma$ is not $t$-chordal.

We now examine the relation between free faces of dimension $d - 1$ in a simplicial complex $\Gamma$ and its $d$-closure. We saw in Example 3.6 that it is possible for $\Gamma$ to be $d$-chordal without having a free face of dimension $d - 1$. In other words $\Delta_d(\Gamma)$ having a free face of dimension $d - 1$ does not imply that $\Gamma$ has a free face of dimension $d - 1$. But the converse is true.

**Proposition 3.11.** Let $\Gamma$ be a simplicial complex on the vertex set $[n]$, and let $E = E_1, \ldots, E_t$ be a sequence of $(d - 1)$-faces of $\Gamma$ with the property that $E_1$ is a free face in $\Gamma$ and $E_i$ is a free face in $\Gamma\{E_1, \ldots, E_{i-1}\}$ for $i > 1$. Then

(a) $E$ is a simplicial sequence for $\Delta_d(\Gamma)$;

(b) If $\Gamma\{E\} \subseteq \langle [n] \rangle^{d-1}$, then $E$ contains a simplicial order for $\Delta_d(\Gamma)$.

**Proof.** (a) Let $F$ be the unique facet in $\Gamma$ containing $E_1$. Since $\Gamma \subseteq \Delta_d(\Gamma)$ we have $F \in \Delta_d(\Gamma)$. Suppose $E_1 \cup \{v\} \in \Delta_d(\Gamma)$ for some $v \in [n] - E$. Then $E_1 \cup \{v\} \in \Gamma$ because any $d$-face of $\Delta_d(\Gamma)$ belongs to $\Gamma$. It follows that $v \in F$. Hence $F$ is the only facet containing $E_1$ in $\Delta_d(\Gamma)$. Thus $E_1$ is simplicial in $\Delta_d(\Gamma)$, and by Lemma 2.5 $E$ is a simplicial sequence for $\Delta_d(\Gamma)$.

(b) Let $E'$ be the subsequence of $E$ which contains all elements in $E$ which are not facets in $\Delta_d(\Gamma)$. Note that if $E_i$ in $E$ is a facet of $\Delta_d(\Gamma)$, then it is a facet in $\Gamma$ too. Hence $\Gamma\{E_1, \ldots, E_i\} = \Gamma\{E_1, \ldots, E_{i-1}\}$. Now, Part (a) implies that $E'$ is a simplicial sequence for $\Delta_d(\Gamma)$ and the assertion follows from the fact that $\Delta_d(\Gamma) = \langle [n] \rangle^{d-1}$, whenever $\Gamma \subseteq \langle [n] \rangle^{d-1}$. □

As promised earlier, in Proposition 3.12 we show that to check chordality, it is enough to check $d$-chordality for a finite number of positive integers $d$. Note that Proposition 3.10(c) can be also deduced from Proposition 3.12

**Proposition 3.12.** Let $\Gamma$ be a simplicial complex with vertex set $[n]$ and dimension $r$, let

$$t = \min \{ \dim F : F \subseteq [n], \text{a minimal nonface of } \Gamma \}$$

and

$$s = \max \{ \dim F : F \subseteq [n], \text{a minimal nonface of } \Gamma \}.$$ 

The following conditions are equivalent.
(i) \( \Gamma \) is chordal;

(ii) \( \Gamma \) is \( d \)-chordal for \( t \leq d \leq \min\{r, s\} \).

**Proof.** The implication \((i) \implies (ii)\) follows from the definition of chordality.

For \((ii) \implies (i)\), note that if \( d < t \), then since all \( F \subseteq [n] \) with \( \dim F = d \), are in \( \Gamma \), we have \( \Delta_d(\Gamma) = \langle [n] \rangle \), which is \( d \)-chordal by Lemma 3.5. So \( \Gamma \) is \( d \)-chordal for \( d < t \).

If \( d > r \), then there is no \( d \)-face in \( \Gamma \) and so we will automatically have \( \Delta_d(\Gamma) = \langle [n] \rangle^{[d−1]} \), which satisfies condition \((*)\) and is therefore \( d \)-chordal.

Now let \( \min\{r, s\} = s \) and \( d > s \). We claim that

\[
\Delta_d(\Gamma) = \Delta_d(\Delta_s(\Gamma)).
\]

Then since by assumption \( \Delta_s(\Gamma) \) is \( s \)-chordal, it follows from Proposition 3.10(b) that \( \Delta_d(\Gamma) \) is \( d \)-chordal for \( d \geq s \). This implies that \( \Gamma \) is chordal, as desired.

Next we prove the claim. To do this we show that for \( d \geq s \)

\[
\Delta_d(\Gamma) = \langle [n] \rangle^{[d−1]} \cup \Delta_s(\Gamma), \quad \text{and} \quad \Delta_d(\Delta_s(\Gamma)) = \langle [n] \rangle^{[d−1]} \cup \Delta_s(\Gamma).
\]

To prove Equation (3), we first observe that by definition \( \langle [n] \rangle^{[d−1]} \subseteq \Delta_d(\Gamma) \), and so we need to only worry about faces of dimension \( \geq d \). Let \( F \subseteq [n] \) with \( \dim F \geq d \).

If \( F \in \Delta_d(\Gamma) \) then any \( d \)-face of \( F \) belongs to \( \Gamma \), and since \( d \geq s \), it follows that any \( s \)-face of \( F \) is in \( \Gamma \). Hence \( F \in \Delta_s(\Gamma) \).

If \( F \in \Delta_s(\Gamma) \) and \( F \notin \Delta_d(\Gamma) \), then there is \( G \subseteq F \) with \( \dim G = d \) and \( G \notin \Gamma \). Since the minimal nonfaces of \( \Gamma \) have dimension \( \leq s \), there exists \( H \subseteq G \), where \( H \notin \Gamma \) and \( \dim H = s \). But \( F \in \Delta_s(\Gamma) \), and any \( s \)-face of \( F \) belongs to \( \Gamma \), hence \( H \in \Gamma \), a contradiction. So \( F \in \Delta_d(\Gamma) \). This settles Equation (3).

Now we prove Equation (4). The containment \( \supseteq \) holds by definition. Suppose \( F \in \Delta_d(\Delta_s(\Gamma)) \). If \( \dim F < d \), then \( F \in \langle [n] \rangle^{[d−1]} \). If \( \dim F \geq d \), then any \( d \)-face \( G \) of \( F \) belongs to \( \Delta_s(\Gamma) \). Hence any \( s \)-face \( H \) of \( G \) (and hence \( F \)) is in \( \Gamma \). It follows that \( F \in \Delta_s(\Gamma) \), as desired. This settles Equation (4).

Equation (2) now follows, and the proof is complete. \( \square \)

The following theorem shows that to check the chordality of a simplicial complex, it is enough to check its \( d \)-collapsibility for one appropriate \( d \).

**Theorem 3.13.** If \( \Gamma \) is a \( d \)-collapsible simplicial complex and \( \dim F \geq d \geq 1 \) for all nonfaces \( F \) of \( \Gamma \), then \( \Gamma \) is chordal.

**Proof.** By assumption \( d \leq r = \min\{\dim F : F \text{ a nonface of } \Gamma\} \). By Proposition 3.10(a), \( \Gamma \) is \( t \)-chordal for all \( t \geq d \), and in particular for all \( t \geq r \), and so from Proposition 3.12 \( \Gamma \) is chordal. \( \square \)
**Example 3.14.** We continue with $\Gamma$ as in Example 2.2. Consider 

$$\Delta_1(\Gamma) = \langle \{1, 2, 4, 5\}, \{1, 2, 3, 4\}\rangle$$

calculated in Example 2.2. Then $E_1 = \{5\}$ is contained in only one facet and hence is simplicial. So 

$$\Delta_1(\Gamma)\cap E_1 = \langle \{1, 2, 3, 4\}\rangle \cup \langle \{5\}\rangle.$$ 

In order to see $\Gamma$ is 1-chordal, now it is enough to find a simplicial order for $\langle \{1, 2, 3, 4\}\rangle$. But it follows from Lemma 3.5 that $\langle \{1, 2, 3, 4\}\rangle$ admits a simplicial order.

The work done in Example 2.4 shows that $\Gamma$ is 2-chordal. Since $\max\{\dim F : F \subseteq [n]\}$, a minimal nonface of $\Gamma$ = 2, it follows from Proposition 3.12 that $\Gamma$ is chordal.

Note that $\Gamma$ is not 1-collapsible and hence we cannot make use of Theorem 3.13 to prove that $\Gamma$ is chordal.

One can even see that the induced subcomplexes of a simplicial complex inherit chordality.

**Proposition 3.15 (Chordality of induced subcomplexes).** Let $\Gamma$ be a simplicial complex on the vertex set $[n]$, $d$ be a positive integer and let $W \subset [n]$.

(a) For a face $E$ of $\Gamma$ one has $(\Gamma \setminus E)_W = (\Gamma_W \setminus E)$.

(b) As simplicial complexes on the vertex set $W$ we have $\Delta_d(\Gamma)_W = \Delta_d(\Gamma_W)$, and in particular if $\Gamma$ is a $d$-closure on $[n]$, then so is $\Gamma_W$ on $W$.

(c) If $E \subseteq W$ is a free face of $\Gamma$ then $E$ is a free face of $\Gamma_W$.

(d) If $E \subseteq W$ with $E \in \text{Simp}(\Delta_d(\Gamma))$, then $E \in \text{Simp}(\Delta_d(\Gamma_W))$.

(e) If $\Gamma$ is $d$-chordal then $\Gamma_W$ is $d$-chordal.

(f) If $\Gamma$ is chordal then $\Gamma_W$ is chordal.

**Proof.** (a) We have 

$$F \in (\Gamma \setminus E)_W \iff F \subseteq W \text{ and } F \in \Gamma \setminus E \iff F \subseteq W, F \in \Gamma \text{ and } E \not\subseteq F \iff F \in \Gamma_W \text{ and } E \not\subseteq F \iff F \in (\Gamma_W \setminus E).$$

(b) By definition of $d$-closure both simplicial complexes $\Delta_d(\Gamma)_W$ and $\Delta_d(\Gamma_W)$ contain all subsets of $W$ of cardinality $\leq d$. Let $F \subseteq W$ with $|F| > d$. Then by definition of $d$-closures 

$$F \in \Delta_d(\Gamma)_W \iff \text{all } d\text{-faces of } F \text{ are in } \Gamma \iff \text{all } d\text{-faces of } F \text{ are in } \Gamma_W \iff F \in \Delta_d(\Gamma_W).$$
which settles our claim.

(c) Suppose $E$ is contained in the unique facet $F$ of $\Gamma$. Since the facets of $\Gamma_W$ are the maximal elements of $\{ G \cap W : G \in \text{Facets}(\Sigma) \}$, we see that $E$ is contained in the unique facet $F \cap W$ of $\Gamma_W$. Hence $E$ is a free face of $\Gamma_W$.

(d) Follows from Part (b) and Part (c).

(e) Since $\Gamma$ is $d$-chordal, Theorem 3.4 implies that $\Delta_d(\Gamma)$ is $d$-collapsible. It follows from [37, Lemma 2] that $\Delta_d(\Gamma_W) = \Delta_d(\Gamma)_W$ is $d$-collapsible, and hence by Theorem 3.4 $\Gamma_W$ is $d$-chordal.

(f) Since $\Gamma$ is chordal, it is $d$-chordal for all $d \geq 1$. Part (e) implies the assertion.

\[ \square \]

**d-representable complexes**

Let $A = \{ A_1, A_2, \ldots, A_n \}$ be a family of sets. Consider the following family of subsets of $A$

\[ N(A) := \{ F \subset [n] : \cap_{i \in F} A_i \neq \emptyset \}. \]

This finite family is a simplicial complex which is called the **Nerve Complex** of $A$. A simplicial complex which is the nerve complex of some finite family of convex sets in $\mathbb{R}^d$ is called **$d$-representable**. One of the main problems regarding nerve complexes is to characterize $d$-representable complexes. This problem is solved in case $d = 1$, see [29]. For $d > 1$ the problem is still open. The reader may consult with [37] for more information about $d$-representable complexes.

**Theorem 3.16** (d-representable complexes are $d$-chordal). Let $\Gamma$ be a $d$-representable simplicial complex on the vertex set $[n]$. Then $\Delta_d(\Gamma) = \Gamma \cup ([n])^{[d-1]}$. Moreover, $\Gamma$ is $d$-chordal.

**Proof.** The inclusion $\Gamma \cup ([n])^{[d-1]} \subseteq \Delta_d(\Gamma)$ always holds. For the converse, we use a celebrated theorem of Helly, [19], which states that if each $d + 1$ members of a finite family of at least $d + 1$ convex sets in $\mathbb{R}^d$ have nonempty intersection, then the whole family intersects. This implies that if $F \subset [n]$, $|F| \geq d + 1$ and each $(d + 1)$-subset of $F$ belongs to $\Gamma$, then $F \in \Gamma$. Hence any $t$-face of $\Delta_d(\Gamma)$ with $t \geq d$ is a face in $\Gamma$. It follows that $\Delta_d(\Gamma) \subseteq \Gamma \cup ([n])^{[d-1]}$. This proves the equality.

Wegner [37] proved that $d$-representable complexes are $d$-collapsible, so $\Gamma$ is $d$-collapsible, and by Theorem 3.9 $\Delta_d(\Gamma)$ is $d$-collapsible as well. Now Theorem 3.4 yields the result. \[ \square \]

**Remark 3.17.** The converse of Theorem 3.16 does not hold in general. Let $\Gamma$ be a simplicial complex of dimension $< d$ which is not $d$-representable, for example the complex $C_2$ in [37, Figure 2]. Then $\Delta_d(\Gamma) = ([n])^{[d-1]}$ and hence $\Gamma$ is $d$-chordal by definition.

The converse of Theorem 3.16 is not true even for $d$-closures: there are simple examples of $d$-closures which are $d$-chordal but not $d$-representable: Figure 2 illustrates a chordal graph which can be viewed as a 1-closure simplicial complex (and hence 1-chordal). But since it is not an interval graph it is not 1-representable.

Also, it is possible for a simplicial complex to not be $d$-representable, while its $d$-closure is $d$-representable. For example, $([n])^{[d-1]}$ which is $d$-closure of all simplicial complexes of dimension
<d is d-representable: Let \( A = \{A_1, \ldots, A_n\} \), where \( A_i \) are affine hyperplanes in \( \mathbb{R}^d \). Then any \( d \) of them intersect in a point, and no \( d + 1 \) of them intersect, as the dimension of the intersection reduces by one each time we intersect with a new affine hyperplane. Hence \( \langle [n] \rangle^{d-1} = N(A) \).

4 Applications to monomial ideals

We now apply the combinatorial results in the previous sections to minimal free resolutions of monomial ideals. Let \( I \) be a square-free monomial ideal in the polynomial ring \( K[x_1, \ldots, x_n] \) over a field \( K \), with Stanley-Reisner complex \( \Gamma \). We write \( I_{(j)} \) for the ideal generated by all homogeneous polynomials of degree \( j \) belonging to \( I \). We say that \( I \) is \textbf{componentwise linear} [20] if \( I_{(j)} \) has a linear resolution for all \( j \). Componentwise linear ideals generalize ideals with linear resolution, in the sense that an ideal with linear resolution is componentwise linear: if \( I \) is generated in a fixed degree \( d \) and has linear resolution, then all \( I_{(k)} \) have linear resolutions. This is the perspective we take when chordality is being considered; see Proposition 3.12.

If \( I \) is a square-free monomial ideal, then by \( I_{[j]} \) we mean the square-free monomial ideal generated by all the square-free monomials of degree \( j \) belonging to \( I \). The ideal \( I \) is called \textbf{square-free componentwise linear} if \( I_{[j]} \) has a linear resolution for all \( j \). Herzog and Hibi [20] proved that a square-free monomial ideal is componentwise linear if and only if it is square-free componentwise linear.

For \( E \subseteq [n] \), we set

\[ x_E = \prod_{i \in E} x_i. \]

The main tool used in this section is examining, for a free face \( E \) of \( \Gamma \), how adding \( x_E \) to the generating set of \( I \) affects the Betti numbers of \( I \). As a consequence, among other things, we are able to produce large classes of componentwise linear ideals.

We begin with some basic observations.

**Lemma 4.1.** Let \( I \) be a square-free monomial ideal in \( K[x_1, \ldots, x_n] \), \( K \) a field, and let \( \Gamma = N(I) \).

(a) \( N(I_{[d+1]}) = \Delta_d(\Gamma) \) for all \( d \).
(b) If $E \subseteq [n]$, then $\mathcal{N}(I + (x_E)) = \Gamma \setminus E$.

(c) If $E$ is a free face of $\Gamma$, then $\mathcal{N}(I + (x_E)) = \Gamma \setminus E$.

**Proof.** (a) First note that both $\mathcal{N}(I_{[d+1]})$ and $\Delta_d(\Gamma)$ contain all possible faces of dimension $< d$. Suppose $F \subseteq [n]$ and $|F| \geq d + 1$. Then

$$F \in \mathcal{N}(I_{[d+1]}) \iff x_F \notin I_{[d+1]}$$

$$\iff \forall G \subseteq F \text{ with } |G| = d + 1, x_G \notin I_{[d+1]}$$

$$\iff \forall G \subseteq F \text{ with } |G| = d + 1, G \in \Gamma$$

$$\iff F \in \Delta_d(\Gamma).$$

(b) If $\sigma \subseteq [n]$, then

$$\sigma \in \mathcal{N}(I + x_E) \iff x_\sigma \notin (I + x_E) \iff x_\sigma \notin I \text{ and } x_E \nmid x_\sigma \iff \sigma \in \Gamma \setminus E.$$

(c) This statement follows directly from Part (b). \qed

We now turn to the effect of the operation of $d$-collapsing on the reduced homology modules of a simplicial complex. It is well known that simplicial collapsing preserves reduced homology modules (see for example [28, Theorem 6.6, Definition 6.13 and Proposition 6.14]). In the special case of $d$-collapsing this is true only for higher reduced homology modules, since we allow facets as free faces.

We write a proof for this fact, since we could not find one in the literature, but it is folklore (see also [6, Proposition 2.3]).

**Proposition 4.2.** If $\Gamma$ is a simplicial complex with a free face $E$, then

$$\tilde{H}_i(\Gamma; K) \cong \tilde{H}_i(\Gamma \setminus E; K) \quad \text{for} \quad i > \begin{cases} \dim E & \text{if } E \in \text{Facets}(\Gamma) \\ 0 & \text{if } E \notin \text{Facets}(\Gamma). \end{cases}$$

**Proof.** This follows from a simple application of the Mayer-Vietoris sequence: if $F$ is the unique facet in $\Gamma$ containing $E$, then $\langle F \rangle = \langle E \rangle \ast \langle G \rangle$, where the operation $\ast$ denotes simplicial join and $G = F - E$. Then, setting $\Gamma' = \Gamma \setminus E$, we have:

$$\Gamma' \cup \langle F \rangle = \Gamma \text{ and } \Gamma' \cap \langle F \rangle = \partial(E) \ast \langle G \rangle,$$

where $\partial(E)$ is the boundary complex of $E$. The Mayer-Vietoris sequence (e.g. [28, Theorem 5.17]) gives

$$\cdots \to \tilde{H}_i(\partial(E) \ast \langle G \rangle; K) \to \tilde{H}_i(\Gamma'; K) \oplus \tilde{H}_i(\langle F \rangle; K) \to \tilde{H}_i(\Gamma; K) \to \tilde{H}_{i-1}(\partial(E) \ast \langle G \rangle; K) \to \cdots.$$

(5)

Note that $\tilde{H}_i(\langle F \rangle; K) = 0$ for all $i$. If $G \neq \emptyset$, then $\partial(E) \ast \langle G \rangle$ is a cone and hence acyclic, and (5) gives the isomorphism of the homology modules for all $i > 0$ (this is the better-known case of an elementary collapse). If $G = \emptyset$ (this is the case when $E$ is a facet of $\Gamma$), then the same argument gives us the isomorphism of the homology modules for $i > \dim E$. \qed
Our statement about Betti numbers in Theorem 4.4 is a generalization of [7, Theorem 2.1]. For the proof we will need the following statement form [20].

**Lemma 4.3** ([20, Lemma 1.2]). Let \( I \subseteq S \) be a graded ideal, and for a nonnegative integer \( k \) let \( I_{\leq k} \) denote the ideal generated by all homogeneous polynomials of \( I \) whose degree is less than or equal to \( k \). Then for all \( k \) and all \( j \leq k \) we have

\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(I_{\leq k}).
\]

**Theorem 4.4** (Betti numbers from free faces). Let \( I \) be a square-free monomial ideal of \( S = K[x_1, \ldots, x_n] \) where \( K \) is a field, \( \Gamma = \mathcal{N}(I) \), and \( E \subseteq [n] \) with \( |E| = d \).

(a) If \( E \) is a free face of \( \Gamma \), then for every \( i \)

\[
\beta_{i,i+j}(I + (x_E)) = \beta_{i,i+j}(I) \text{ for all } j \neq d, d + 1.
\]

Moreover, if \( E \notin \text{Facets}(\Gamma_W) \) for every \( W \subseteq [n] \) with \( |W| = a > 2 \), then

\[
\beta_{i,i+j}(I + (x_E)) = \beta_{i,i+j}(I)
\]

for all \( i \) and all \( j > 2 \) such that \( i + j = a \).

(b) If \( E \) is a free face of \( \Gamma \) or of \( \Delta_d(\Gamma) \) and

\[
d + 1 \geq \max \{ \deg u : u \text{ a minimal generator of } I \}
\]

and \( A \subseteq [n] - E \) such that \( E \cup \{m\} \in \Gamma \) for all \( m \in A \), then for every \( i \) we have

\[
\beta_{i,i+j}(I + (x_m x_E : m \in A)) = \beta_{i,i+j}(I) \text{ for all } j \neq d + 1.
\]

Moreover, if \( I \) is minimally generated by monomials of degree \( d + 1 \), and

\[
E \notin \text{Facets}((\mathcal{N}(I + (x_m x_E : m \in A)))_W)
\]

for every \( W \subseteq [n] \) with \( |W| = a > 2 \), then

\[
\beta_{i,i+j}(I + (x_E)) = \beta_{i,i+j}(I + (x_m x_E : m \in A)) = \beta_{i,i+j}(I)
\]

for all \( i \) and all \( j > 2 \) such that \( i + j = a \).

**Proof.** (a) By Hochster’s formula [25] (See also [22, Theorem 8.1.1]) and Lemma 4.1(c), for all \( i \) and \( j \)

\[
\beta_{i,i+j}(I + (x_E)) = \sum_{W \subseteq [n]} \dim_K \widetilde{H}_{j-2}(\mathcal{N}(I + (x_E))_W; K)
\]

\[
= \sum_{W \subseteq [n]} \dim_K \widetilde{H}_{j-2}(\mathcal{N}(I \setminus (x_E))_W; K).
\]
If \( E \not\subset W \), then by abuse of notation \( \Gamma_W \setminus _E = \Gamma_W \). If \( E \subset W \), then by Proposition 3.15 parts (c) and (a), \( E \) is a free face of \( \Gamma_W \), and \( (\Gamma \setminus _E)W = (\Gamma_W) \setminus _E \). By Proposition 4.2

\[
\beta_{i,i+j}(I + (x_E)) = \sum_{W \subset [n] \mid |W| = i+j} \dim_K \tilde{H}_{j-2}(\Gamma_W; K) = \beta_{i,i+j}(I)
\]

for \( j > d + 1 \).

Since \( \deg x_E = d \) we have \( I_{\leq d-1} = (I + (x_E))_{\leq d-1} \). Hence by Lemma 4.3

\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(I_{\leq d-1}) = \beta_{i,i+j}((I + (x_E))_{\leq d-1}) = \beta_{i,i+j}(I + (x_E))
\]

for all \( i \) and all \( j \leq d - 1 \).

Moreover, if for every \( W \) with \( |W| = a > 2 \) we have \( E \notin \text{Facets}(\Gamma_W) \), then using Equation (8) and by Proposition 4.2

\[
\beta_{i,i+j}(I + (x_E)) = \beta_{i,i+j}(I)
\]

for all \( i \) and all \( j > 2 \) such that \( i + j = a \).

(b) We first deal with the case where all generators of \( I \) have degree \( d + 1 \). Let

\[
\Gamma = \Delta_d(\Gamma) = \langle G_1, \ldots, G_t \rangle
\]

where \( G_t \) is the unique facet of \( \Gamma \) containing \( E \), and let

\[
\Sigma = \Gamma \setminus F_1 \setminus \cdots \setminus F_r,
\]

where \( F_k = E \cup \{m_k\} \) for each \( m_k \in A \). Now

\[
\Sigma = \langle G_1, \ldots, G_{t-1} \rangle \cup \langle G_t - \{i\} : i \in E \rangle \cup \langle G_t - A \rangle.
\]

It follows that \( E \) is uniquely contained in \( G_t - A \), and hence it is a free face of \( \Sigma \). Applying Part (a) to \( \Sigma \), we see that

\[
\beta_{i,i+j}(\mathcal{N}(\Sigma)) = \beta_{i,i+j}(\mathcal{N}(\Sigma) + (x_E)) \quad \text{for all } i \text{ and all } j > d + 1.
\]

On the other hand, \( \Sigma \setminus _E = \Gamma \setminus _E \). Therefore,

\[
\mathcal{N}(\Sigma) + (x_E) = \mathcal{N}(\Sigma \setminus E) = \mathcal{N}(\Gamma \setminus _E) = I + (x_E).
\]

This implies that for all \( i \) and all \( j > d + 1 \),

\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(I + (x_E)) \quad \text{(using Part (a))}
\]

\[
= \beta_{i,i+j}(\mathcal{N}(\Sigma) + (x_E)) \quad \text{(using Equation (10))}
\]

\[
= \beta_{i,i+j}(\mathcal{N}(\Sigma)) \quad \text{(using Equation (9))}
\]

\[
= \beta_{i,i+j}(I + (x_m x_E : m \in A)) \quad \text{(using Lemma 4.1(b)).}
\]
Since the ideals \( I \) and \( I + (x_m x_E : m \in A) \) are minimally generated in degree \( d + 1 \) they both have Betti numbers equal to 0 when \( j \leq d \).

For \( W \subseteq [n] \) if we have \( E \notin \text{Facets}(\Sigma_W) \), then \( E \notin \text{Facets}(\Gamma_W) \). Hence if for every \( W \) with \( |W| = a > 2 \) we have \( E \notin \text{Facets}(\Gamma_W) \), then
\[
\beta_{i,i+j}(I + (x_E)) = \beta_{i,i+j}(I + (x_m x_E : m \in A)) = \beta_{i,i+j}(I)
\]
for all \( i \) and all \( j > 2 \) such that \( i + j = a \).

This settles the equigenerated case. Now suppose all generators of \( I \) have degree \( \leq d + 1 \). By Lemma 4.1(a), \( \Delta_d(\Gamma) = \mathcal{N}(I_{[d+1]}) \). Observe that if \( E \) is free in \( \Gamma \), then it is also a free face of \( \Delta_d(\Gamma) \), otherwise it would be contained in at least two facets of \( \Gamma \) which would contradict it being free. By our discussions above
\[
\beta_{i,i+j}(I_{[d+1]}) = \beta_{i,i+j}(I_{[d+1]} + (x_E)) = \beta_{i,i+j}(I_{[d+1]} + (x_m x_E : m \in A)), \tag{11}
\]
for all \( i \) and all \( j \neq d, d + 1 \).

Set \( t = \max\{\deg u : u \text{ minimal generator of } I\} \). It is proved in [6, Lemma 4.2] that if \( d + 1 \geq t \), then for all \( i \) and all \( j > d + 1 \)
\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(I_{[d+1]}). \tag{12}
\]
It follows from Equation (11) and Equation (12) that for all \( i \) and all \( j > d + 1 \)
\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(I_{[d+1]} + (x_m x_E : m \in [n] - E)). \tag{13}
\]
Let \( J := I + (x_E) \). Then
\[
I_{[d+1]} = I_{[d+1]} + (x_m x_E : m \in [n] - E).
\]
So by Equation (13) for all \( i \) and all \( j > d + 1 \)
\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(J_{[d+1]}). \tag{14}
\]
Now set \( s = \max\{\deg u : u \text{ minimal generator of } J\} \). Since \( \deg x_E = d \) we have \( t \geq s \) and hence \( d + 1 \geq s \). Again [6, Lemma 4.2] implies that for all \( i \) and all \( j > d + 1 \)
\[
\beta_{i,i+j}(J) = \beta_{i,i+j}(J_{[d+1]}). \tag{15}
\]
Equation (14) and Equation (15) yield the following result
\[
\beta_{i,i+j}(I) = \beta_{i,i+j}(I + (x_E)) \quad \text{for all } i \text{ and all } j > d + 1. \tag{16}
\]
By Lemma 4.3 Equation (16) also holds for \( j \leq d - 1 \).

Now let \( L = I + (x_m x_E : m \in A) \) for \( A \subseteq [n] - E \). Then
\[
L_{[d+1]} = I_{[d+1]} + (x_m x_E : m \in A).
\]
Using Equation (11) and Equation (12) one has
\[ \beta_{i,i+j}(L_{[d+1]}) = \beta_{i,i+j}(I) \quad \text{for all } i \text{ and all } j > d + 1. \tag{17} \]

Since \( d + 1 \geq \max \{ \deg u : u \text{ minimal generator of } L \} \), by [6, Lemma 4.2] we have
\[ \beta_{i,i+j}(L) = \beta_{i,i+j}(L_{[d+1]}) \quad \text{for all } i \text{ and all } j > d + 1. \tag{18} \]

Consequently, using Equation (17) and Equation (18)
\[ \beta_{i,i+j}(I + (x_m x_E : m \in A)) = \beta_{i,i+j}(I), \quad \text{for all } i \text{ and all } j > d + 1. \tag{19} \]

Now
\[ I_{\leq d} = (I + (x_m x_E : m \in A))_{\leq d} \quad \text{and} \quad I_{\leq d-1} = (I + (x_E))_{\leq d-1} \]
so our assertions follows from Lemma 4.3.

Recall that for a graded ideal \( I \) of the polynomial ring \( S \) the regularity of \( I \) the maximum of all \( j \) such that \( \beta_{i,i+j}(I) \neq 0 \).

**Corollary 4.5 (Adding generators to componentwise linear ideals).** Let \( I \) be a square-free monomial ideal in \( K[x_1, \ldots, x_n] \), and suppose the degree of each minimal monomial generator of \( I \) is \( \leq d + 1 \). Let \( E \) be a \((d-1)\)-dimensional free face of \( \Gamma := N(I) \) or \( \Delta_d(\Gamma) \), and \( A \subseteq [n] - E \), \( A \neq \emptyset \), with \( E \cup \{m\} \in \Gamma \) for each \( m \in A \). If \( I \) is componentwise linear, then
\[ I + (x_m x_E : m \in A) \]
is componentwise linear of regularity \( d + 1 \).

**Proof.** We show that \( (I + (x_m x_E : m \in A))_{[k]} \) has \( k \)-linear resolution for all \( k \).

If \( k < d + 1 \) we have
\[ (I + (x_m x_E : m \in A))_{[k]} = I_{[k]}, \]
and since the latter has linear resolution, we are done. Suppose \( k \geq d + 1 \). Then
\[ (I + (x_m x_E : m \in A))_{[k]} = \left( m^{k-d-1}(I_{[d+1]} + (x_m x_E : m \in A)) \right)^{s_q}, \tag{20} \]
where \( m^{k-d-1} \) denotes \((k - d - 1)\)-st power of the graded maximal ideal \( m \) of \( S \) and by \( J^{s_q} \) we mean the ideal generated by square-free generators of \( J \).

By Theorem 4.4(b) for all \( i \) and all \( j \neq d + 1 \)
\[ \beta_{i,i+j}(I_{[d+1]} + (x_m x_E : m \in A)) = \beta_{i,i+j}(I_{[d+1]}) = 0. \]

Therefore \( I_{[d+1]} + (x_m x_E : m \in A) \) has a \((d + 1)\)-linear resolution. It follows from [22, Lemma 8.2.10] that
\[ J = m^{k-d-1} \left( I_{[d+1]} + (x_m x_E : m \in A) \right) \]
has a $k$-linear resolution. Therefore, the square-free component $J^{sq}$ of $J$ has $k$-linear resolution [22, Proposition 8.2.17], and so by Equation (20) $(I + (x_m x_E : m \in A))_{[k]}$ has a $k$-linear resolution as desired.

By [22, Corollary 8.2.14] the regularity of the componentwise linear ideal $I + (x_m x_E : m \in A)$ is the highest degree of its minimal generators, which in this case is equal to $d + 1$.

\[\square\]

**Theorem 4.6 (Chordal complexes produce componentwise linear ideals).** Let $I$ be a nonzero square-free monomial ideal, $d$ a positive integer and let $\Gamma = \mathcal{N}(I)$. Then, over all fields, we have

(a) If $\Gamma$ is $d$-chordal, then $I_{[d+1]} = \mathcal{N}((\Delta_d(\Gamma)))$ has a $(d+1)$-linear resolution ([7, Theorem 3.3]).

(b) If $\Gamma$ is $d$-chordal and $W \subseteq \{n\}$, then $\mathcal{N}(\Gamma_W)_{[d+1]}$ has a $(d+1)$-linear resolution.

(c) If $\Gamma$ is $d$-collapsible, then $I_{[d+1]} = \mathcal{N}((\Delta_d(\Gamma)))$ has a $(d+1)$-linear resolution.

(d) If $\Gamma$ is $d$-representable, then $I_{[d+1]} = \mathcal{N}((\Delta_d(\Gamma)))$ has a $(d+1)$-linear resolution.

(e) If $\Gamma$ is chordal, then $I$ is componentwise linear.

(f) If $\Gamma$ is $d$-chordal for all $t - 1 \leq d \leq s - 1$ where $t$ and $s$ are, respectively, the smallest and the largest degrees of the minimal monomial generators of $I$, then $I$ is componentwise linear.

(g) If $\Gamma$ is $d$-collapsible and $\deg u > d$ for all $u \in I$, then $I$ is componentwise linear.

(h) If $\Gamma$ is $d$-representable and $\deg u > d$ for all $u \in I$, then $I$ is componentwise linear.

(i) If $\Gamma$ is chordal and $W \subseteq \{n\}$, then $\mathcal{N}(\Gamma_W)$ is componentwise linear.

**Proof.**

(a) By Lemma 4.1(a) $\mathcal{N}(I_{[d+1]}) = \Delta_d(\Gamma)$. Since $\Gamma$ is $d$-chordal, $\Delta_d(\Gamma)$ admits a simplicial order $E = E_1, \ldots, E_t$. It follows from Theorem 4.4(a) that for all $i$ and all $j > d + 1$

$$\beta_{i,i+j}(I_{[d+1]}) = \beta_{i,i+j}(I + (x_{E_1}, \ldots, x_{E_t})) = \beta_{i,i+j}(\mathcal{N}((\{n\})^{[d-1]}))$$

The ideal $\mathcal{N}((\{n\})^{[d-1]})$, generated by all square-free monomials of degree $d + 1$ in $S$, has $(d + 1)$-linear resolution over all fields (Herzog and Hibi [21]). Hence $\beta_{i,i+j}(I_{[d+1]}) = 0$ for all $i$ and all $j > d + 1$. Since $I_{[d+1]}$ is generated by monomials of degree $d + 1$, for each $i$, the $i$th syzygies are of degree $\geq i + d + 1$, and so $\beta_{i,i+j}(I_{[d+1]}) = 0$ for all $i$ and all $j < d + 1$. Therefore $I_{[d+1]}$ has $(d + 1)$-linear resolution over all fields.

(b) Follows from Part (a) and Proposition 3.15(e).

(c) Follows from Part (a), Theorem 3.4 and Theorem 3.9.

(d) Follows from Part (a) and Theorem 3.16.

(e) By assumption $\Gamma$ is $d$-chordal for all $d \geq 1$. Hence $I_{[d+1]}$ has $(d+1)$-linear resolution over all fields using Part (a). Since by [22, Proposition 8.2.17] a square-free monomial ideal $I$ is componentwise linear if and only if $I$ is square-free componentwise linear, our assertion follows.
(f) Follows from Part (e) and Proposition 3.12.

(g) Follows from Part (e) and Theorem 3.13.

(h) Follows from Part (g), and the fact that $d$-representable complexes are $d$-collapsible [37].

(i) Follows from Part (e) and Proposition 3.15(f).

\[ \square \]

Note that one can prove Theorem 4.6(a) independently: Since $\Gamma$ is $d$-chordal, $\Delta_d(\Gamma)$ is $d$-collapsible using Theorem 3.4. It is shown in [37, Lemma 3] that any $d$-collapsible complex is $d$-Leray. Hence $\tilde{H}_j(\Delta_d(\Gamma)_W; K) = 0$ for all $j \geq d$. This yields the desired conclusion.

The following example, which was suggested by Eric Babson in a communication with Ali Akbar Yazdan Pour [4], shows that the converses of none of the parts of Theorem 4.6 holds.

**Example 4.7.** Let $\Gamma$ be a triangulation of a Dunce hat, see Figure 3, and let $\Sigma := \Delta_2(\Gamma)$ be its 2-closure. Then it is seen that $\Sigma$ is not 2-collapsible, and hence it is not 2-chordal or 2-representable, while $\mathcal{N}(\Sigma)$ has 3-linear resolution over all fields.

![Figure 3: A triangulation of the Dunce hat](image)

In the next section we show that the Betti numbers of all componentwise linear ideals appear as Betti numbers of Stanley-Reisner ideals of chordal complexes.

**Cohen-Macaulay properties**

Let $\Gamma$ be a simplicial complex on the vertex set $[n]$, $I = \mathcal{N}(\Gamma)$ be an ideal of $S = K[x_1, \ldots, x_n]$ where $K$ is a field, and let $K[\Gamma] = S/I$ be the Stanley-Reisner ring of $\Gamma$. 

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A pure complex $\Gamma$ is called **Cohen-Macaulay** over $K$ if $K[\Gamma]$ is a Cohen-Macaulay ring, or, equivalently, by Eagon and Reiner [15, Theorem 3], if $I^\vee = N(\Gamma^\vee)$ has a linear resolution.

Stanley [34] generalized the Cohen-Macaulay property to all simplicial complexes, calling this new class of complexes **sequentially Cohen-Macaulay**. In Duval’s [14] characterization, the complex $\Gamma$ is sequentially Cohen-Macaulay over $K$ if and only if $\Gamma^{[d]}$ is Cohen-Macaulay (over $K$) for all $d \leq \dim \Gamma$. Herzog and Hibi [20, Theorem 2.1(a)] then extended the criterion of Eagon and Reiner showing that a square-free monomial ideal $I$ is componentwise linear if and only if $N(I)^\vee$ is sequentially Cohen-Macaulay.

Combining these facts with Corollary 4.5 and Theorem 4.6, we make the following observation.

**Corollary 4.8 (Chordal complexes have sequentially Cohen-Macaulay duals).** Let $\Gamma$ be a simplicial complex on $[n]$. If $\Gamma$ is either $d$-chordal or $d$-collapsible or $d$-representable, then $(\Gamma^\vee)^{[n-d-2]}$ is Cohen-Macaulay. In particular, if $\Gamma$ is chordal, then $\Gamma^\vee$ is sequentially Cohen-Macaulay.

**Proof.** Setting $I = N(\Gamma)$, it follows from [17, page 131] that

$$I_{[d+1]} = N(((\Gamma^\vee)^{[n-d-2]})^\vee).$$

Hence $I_{[d+1]}$ has linear resolution if and only if $(\Gamma^\vee)^{[n-d-2]}$ is Cohen-Macaulay. Our statements now follow from Theorem 4.6.

---

**5 More chordal complexes and Betti tables of componentwise linear ideals**

In this section we focus on well-known classes of componentwise linear ideals with, and of simplicial complexes which arise from them. It is still not known whether Alexander duals of shellable complexes (Björner and Wachs [8]), which provide a large class of componentwise linear ideals containing most other such ideals, are chordal (see Herzog and Hibi [20], and also Eagon and Reiner [15]).

We also show in this section that the Betti table of every componentwise linear ideal is equal to that of the Stanley-Reisner ideal of a chordal complex.

**5.1 Alexander duals of vertex decomposable complexes**

One large class of ideals with linear resolution is the class of Stanley-Reisner ideals of the Alexander duals of vertex decomposable complexes (Björner and Wachs [9], Provan and Billera [33]). Nikseresht [31] showed that if a pure $d$-dimensional simplicial complex on $n$ vertices is vertex decomposable, then its Alexander dual is $(n - d - 2)$-chordal. Here we use this result to show that the Alexander dual of any vertex decomposable simplicial complex is chordal.

The main idea is that, similar to the property of sequential Cohen-Macaulayness, vertex decomposability of a simplicial complex reduces to that of its pure skeletons, [38, Lemma 3.10].
Definition 5.1 (vertex decomposable simplicial complex). A simplicial complex $\Gamma$ on the vertex set $[n]$ is called vertex decomposable if it is a simplex, including $\emptyset$ and $\{\emptyset\}$, or it contains a vertex $v$ such that

(i) $v$ is a shedding vertex of $\Gamma$, i.e. no face of $\text{link}_\Gamma(v)$ is a facet of $\Gamma \setminus \{v\}$, and

(ii) both $\Gamma \setminus \{v\}$ and $\text{link}_\Gamma(v)$ are vertex decomposable.

Nikseresht [31, Lemma 3.1] shows that for a pure $d$-dimensional complex $\Delta$, a vertex $v$ is a shedding vertex if and only if $\Delta \setminus \{v\}$ is also pure of dimension $d$. This fact will be used in the arguments below.

Theorem 5.2 (Alexander duals of vertex decomposable complexes are chordal). Let $\Gamma$ be a vertex decomposable complex on $[n]$. Then its Alexander dual $\Gamma^\vee$ is chordal.

Proof. Let $d \geq 1$. We need to show that $\Delta_d(\Gamma^\vee)$ is $d$-chordal. From [17, page 131], we have that

$$\mathcal{N}((\Gamma^{[n-d-2]})^\vee) = \mathcal{N}(\Gamma^\vee)[d+1]$$

which by Lemma 4.1 implies that

$$(\Gamma^{[n-d-2]})^\vee = \Delta_d(\Gamma^\vee).$$

Woodroofe proves in [38, Lemma 3.10] that all the skeletons of a vertex decomposable simplicial complex are vertex decomposable. Since $\Gamma$ is vertex decomposable, it follows that $\Gamma^{[n-d-2]}$ is vertex decomposable too. On the other hand Nikseresht [31, Theorem 3.10] proved that the dual of any pure $t$-dimensional vertex decomposable complex is $(n-t-2)$-chordal. Therefore $(\Gamma^{[n-d-2]})^\vee = \Delta_d(\Gamma^\vee)$ is $d$-chordal, as desired.

5.2 Square-free (strongly) stable ideals

Square-free stable ideals, defined by Aramova, Herzog and Hibi [3] form a large class of component-wise linear ideals. This class contains the class of square-free strongly stable ideals and lexsegment ideals.

For a monomial $u \in S = K[x_1, \ldots, x_n]$ we define $m(u) = \max\{i : x_i \mid u\}$. A square-free monomial ideal $I$ is called square-free stable if for all square-free monomials $u \in I$

$$x_i \left(\frac{u}{x_{m(u)}}\right) \in I \text{ for all } i < m(u) \text{ such that } x_i \nmid u,$$

and $I$ is called square-free strongly stable if for all square-free monomials $u \in I$ and $x_j \mid u$

$$x_i \left(\frac{u}{x_j}\right) \in I \text{ for all } i < j \text{ such that } x_i \nmid u.$$

It turns out that the defining property for square-free (strongly) stable ideals $I$ needs only be checked for the monomials in the minimal monomial generating set $G(I)$ [22, Problem 6.9].
Theorem 5.3 (Stanley-Reisner complexes of square-free stable ideals are chordal). Let $I$ be a square-free stable ideal in $S = K[x_1, \ldots, x_n]$, $K$ a field. Then $\mathcal{N}(I)$ is chordal.

Proof. First note that for each $d \geq 1$ the ideal $I_{d+1}$ is square-free stable, for if $u \in G(I_{d+1}) \subseteq I$ and $i < m(u)$ with $x_i | u$, the monomial $x_i(u/x_{m(u)}) \in I$ and $\deg(x_i(u/x_{m(u)})) = d + 1$ which implies that $x_i(u/x_{m(u)}) \in I_{d+1}$. By Nikseresht and Zaare-Nahandi’s work \cite[Theorem 2.5]{22} the complex $\Delta_d(\mathcal{N}(I)) = \mathcal{N}(I_{d+1})$ is $d$-chordal. Therefore $\mathcal{N}(I)$ is chordal. \hfill $\square$

Recall that a simplicial complex $\Gamma$ is called shifted if for any face $F \in \Gamma$, any $i \in F$ and $j \in [n]$ with $j > i$ one has $(F - \{i\}) \cup \{j\} \in \Gamma$.

Theorem \ref{thm:Stanley-Reisner complexes of square-free stable ideals are chordal} in particular implies that square-free strongly stable ideals have chordal Stanley-Reisner complexes. This statement can also be deduced from the fact that and ideal $I$ is square-free strongly stable if and only if $(\mathcal{N}(I))'$ is shifted, and therefore vertex decomposable by \cite[Theorem 11.3]{9}. Hence $\mathcal{N}(I)$ is chordal by Theorem \ref{thm:Chordal complexes give Betti tables of all componentwise linear ideals}.

We now show that the study of the Betti tables of componentwise linear ideals reduces to the study of the Betti tables of Stanley-Reisner ideals of chordal complexes, generalizing a similar result of Bigdeli and coauthors in the case of equigenerated ideals \cite[Theorem 3.3]{5}.

For the proof we use the square-free operator \cite{22} which takes a monomial $u = x_{i_1}x_{i_2}\cdots x_{i_t} \in S$ with $i_1 \leq \cdots \leq i_t$, to the square-free monomial $u' = x_{i_1}x_{i_2+1}\cdots x_{i_t+(t-1)}$. If $I$ is a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$, then $I'$ is the square-free monomial ideal

$$I' = (u'_1, \ldots, u'_m).$$

Theorem 5.4 (Chordal complexes give Betti tables of all componentwise linear ideals). Let $K$ be a field and $I \subset S = K[x_1, \ldots, x_n]$ be a graded ideal which is componentwise linear. Then there exists a chordal complex $\Gamma$ such that the Betti table of $I$ coincides with that of $\mathcal{N}(\Gamma)$.

Proof. It follows from Herzog and coauthors \cite[Proposition 2.1]{23} that the Betti table of a componentwise linear ideal $I$ coincides with the Betti table of a strongly stable ideal $J$ (not necessarily square-free). By \cite[Lemma 11.2.5]{22} $J'$ is square-free strongly stable. Moreover, \cite[Lemma 11.2.6]{22} implies that $J'$ has the same Betti table as of $J$. Hence the Betti tables of $I$ and $J'$ coincide. Since square-free strongly stable ideals are square-free stable, Theorem \ref{thm:Stanley-Reisner complexes of square-free stable ideals are chordal} implies that $\mathcal{N}(J')$ is chordal, as desired. \hfill $\square$

5.3 Square-free Gotzmann ideals

A homogeneous ideal $I$ in a polynomial ring $S = K[x_1, \ldots, x_n]$ over a field $K$ is a Gotzmann ideal if its “growth” in degrees is similar to a lex ideal. More precisely, let $S_1$ be the first graded piece of $S$ (generated by $x_1, \ldots, x_n$ as a $K$-vector space), and similarly, let $I_u$ be the $u$-th graded piece of $I$ (generated by all degree $u$ monomials in $I$), and $L$ be a lex ideal with the same Hilbert function as $I$. Then $I$ is Gotzmann if and only if $\dim_K(S_1I_u) = \dim_K(S_1L_u)$ for all $u \geq 0$. 

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Herzog and Hibi [20] proved that Gotzmann monomial ideals are componentwise linear. Below we use a characterization of Gotzmann square-free monomial ideals due to Hoefel and Mermin [27] to show that the Stanley-Reisner complex of these ideals is chordal.

**Theorem 5.5** (Hoefel [26], Theorem 5.9; Hoefel-Mermin [27], Theorem 3.9). Let $K$ be a field, $S = K[x_1, \ldots, x_n]$ and ideal $I$ be a square-free monomial ideal in $S$. Then $I$ is a Gotzmann ideal if and only if $I$ is generated by one variable or

$$I = m_1(z_{1,1}, \ldots, z_{1,r_1}) + m_1m_2(z_{2,1}, \ldots, z_{2,r_2}) + \cdots + m_1m_2 \cdots m_s(z_{s,1}, \ldots, z_{s,r_s})$$

for some square-free monomials $m_1, \ldots, m_s$ and variables $z_{i,j}$ all having pairwise disjoint support and satisfying

- $\deg(m_i) \geq 1$ for $1 \leq i \leq s$,
- $r_i \geq 1$ for $1 \leq i < s$,
- $r_s \neq 1$ and
- $\deg(m_s) \geq 2$ when $r_s = 0$.

**Theorem 5.6 (Gotzmann ideals are chordal).** Let $I$ be a Gotzmann square-free monomial ideal in $S = K[x_1, \ldots, x_n]$, $K$ a field. Then $\mathcal{N}(I)$ is chordal.

**Proof.** With notation as in Theorem 5.5 let

$$I = m_1(z_{1,1}, \ldots, z_{1,r_1}) + m_1m_2(z_{2,1}, \ldots, z_{2,r_2}) + \cdots + m_1m_2 \cdots m_s(z_{s,1}, \ldots, z_{s,r_s}),$$

where $m_i = y_{i,1} \cdots y_{i,t_i}$ for $1 \leq i \leq s$, and all the $z_{i,j}$ and $y_{i,j}$ are distinct variables in $\{x_1, \ldots, x_n\}$.

Now we re-order the variables, so that setting $\alpha_i = \sum_{j=1}^{i-1} t_j + r_j$ for $i > 1$ and $\alpha_1 = 0$, for $1 \leq i \leq s$ we have

$$x_{\alpha_i+1} = y_{i,1}, \ldots, x_{\alpha_i+t_i} = y_{i,t_i}, \quad x_{\alpha_i+t_i+1} = z_{i,1}, \ldots, x_{\alpha_i+t_i+r_i} = z_{1,r_i}.$$

So the relabeled form of $I$ is

$$m'_1(x_{t_1+1}, \ldots, x_{t_1+r_1}) + \cdots + m'_1m'_2 \cdots m'_s(x_{\alpha_s+t_s+1}, \ldots, x_{\alpha_s+t_s+r_s}),$$

where $m'_i = x_{\alpha_i+1} \cdots x_{\alpha_i+t_i}$ for $1 \leq i \leq s$.

This latter ideal is clearly square-free strongly stable. To see this, take any monomial generator of the form $M = m'_1m'_2 \cdots m'_w x_{\alpha_w+t_w+u}$. Suppose $x_i|M$, $j < i$ and $x_j \nmid M$. Then $j = \alpha_w + t_w + l$, where $w \leq v$ and $\begin{cases} 1 \leq l < u & \text{if } w = v, \\ 1 \leq l \leq r_w & \text{if } w < v. \end{cases}$

Since $m'_1m'_2 \cdots m'_w x_{\alpha_w+t_w+l}$ is a generator, the monomial $x_j(M/x_i) = m'_1m'_2 \cdots m'_w x_{\alpha_w+t_w+l}$ belongs to $I$, and we are done.

Now $\mathcal{N}(I)$ is isomorphic to the Stanley-Reisner complex of a square-free strongly stable ideal, and is therefore chordal by Theorem 5.3. \qed
Theorem 5.6 can also be proved directly, because of the nice inductive structure that square-free Gotzmann ideals have.

6 Further questions and remarks

Remark 6.1. It is well-known [13] that any chordal graph has at least two simplicial vertices. Equivalently, the flag complex of a chordal graph (which is a 1-closure) has at least two simplicial faces which are not facets. One may ask if the same holds for the d-closure of an arbitrary d-chordal simplicial complex. Theorem 2.3 of [1] implies that for any $d > 1$ there is a $d$-dimensional simplicial complex $\Gamma$ which is $d$-collapsible and has only one free face $E$ of dimension $d - 1$ which is not a facet. It turns out that $E$, being contained in a single $d$-dimensional facet of $\Gamma$, is a simplicial face of $\Delta_d(\Gamma)$ which is not a facet. By Theorem 3.9 we know that $\Delta_d(\Gamma)$ is $d$-collapsible. Now Theorem 3.4 implies that $\Delta_d(\Gamma)$ is $d$-chordal with $E$ as its non-facet simplicial face.

Figure 4 is an example of the complexes constructed in Theorem 2.3 of [1]. It is a 2-dimensional 2-collapsible complex $\Gamma$ with $\{1, 2\}$ as its unique free face. Then $\Delta_2(\Gamma) = \Gamma \cup \{\{3, 5\}, \{5, 7\}\}$ is 2-chordal with $\{1, 2\}$ as a simplicial face, by above argument. It is easy to check that indeed, $\{1, 2\}$ is the unique non-facet simplicial face of the complex $\Delta_2(\Gamma)$. So the answer to the above question is negative in general.

![Figure 4: A 2-chordal complex with \{1, 2\} as the unique simplicial (non-facet) face of its 2-closure](image)

Remark 6.2. The class of $d$-chordal complexes includes nonshellable ones. Setting $\Gamma$ to be the triangulation of the dunce hat in Figure 3, it is well known that $\Gamma$ is a Cohen-Macaulay non-shellable complex while [7, Example 3.14] implies that $\Gamma^\lor$, which is a 4-closure, is chordal.

The following question is then a natural one.

Question 6.3. A large combinatorial class of simplicial complexes whose Stanley-Reisner ideals are componentwise linear are Alexander duals of shellable complexes. Are duals of shellable complexes chordal? Since shellability reduces to the pure skeletons [8, Theorem 2.9], it is enough to ask the question in pure case. Equivalently one can ask: is the Stanley-Reisner complex of an
ideal equigenerated in degree $d + 1$ with linear quotients ([24]) $d$-chordal? (See also [22] Proposition 8.2.5].)

**Question 6.4.** Not all free faces of a $d$-collapsible complex $\Gamma$ can be the starting face of a free sequence which reduces $\Gamma$ to $\emptyset$. Tancer [35] constructs $d$-collapsible complexes $\Gamma$ with a free face $E$ (which he calls a “bad” face) such that $\Gamma \setminus E$ is not $d$-collapsible. What about the case of $d$-chordal complexes: given a $d$-chordal $d$-closure $\Gamma$ and a simplicial face $E$, is $\Gamma \setminus E$ always $d$-chordal?

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