ENERGY OF A ROTATING BOSE-EINSTEIN CONDENSATE IN A HARMONIC TRAP

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Abstract. The state of a rotating Bose-Einstein condensate in a harmonic trap is modeled by a wave function that minimizes the Gross-Pitaevskii functional. The resulting minimization problem has two new features compared to other similar functionals arising in condensed matter physics, such as the Ginzburg-Landau functional. Namely, the wave function is defined in all the plane and is normalized relative to the $L^2$-norm. This paper deals with the situation when the coupling constant tends to 0 (Thomas-Fermi regime) and the rotation speed is large compared with the first critical speed. It is given the leading order estimate of the ground state energy together with the location of the vortices of the minimizing wave function in the bulk of the condensate.

1. Introduction

The analysis of energy functionals modeling rotating Bose-Einstein condensation is currently an important field of mathematical physics. A lot of mathematical papers addressed several questions related to this physical phenomenon. In [13], it is proved that the Gross-Pitaevskii framework is a valid approximation of the $N$-body model of rotating Bose-Einstein condensation. The monograph [1] contains original results as well as many open questions regarding various models in the subject (see also the papers [2, 3, 4] and the references therein). A series of important contributions ([7, 14] and references therein) contain a deep analysis that describes the various critical speeds of rotating Bose-Einstein condensates in a flat trap.

In the recent papers [8, 9, 11], it is developed methods to analyze the three dimensional Ginzburg-Landau functional, and it is pointed that the same methods should prove useful when applied to similar models. That is confirmed here as the energy of a rotating Bose-Einstein condensate will be studied along an approach similar to [11]. When the atoms of the condensate are confined in a harmonic trap, the Gross-Pitaevskii functional to study is:

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left( |(\nabla - i\Omega A_0)u|^2 + \frac{1}{2\varepsilon^2} \left( |a(x) - |u|^2|^2 - |a_-(x)|^2 \right) - \frac{\Omega^2}{4} |x|^2 |u|^2 \right) dx. \quad (1.1)$$

The functional in (1.1) is defined for functions satisfying the mass constraint,

$$\int_{\mathbb{R}^2} |u|^2 dx = 1. \quad (1.2)$$

The constant $\varepsilon > 0$ is the coupling constant, $\Omega$ is the rotational speed, $A_0(x) = (-x_2/2, x_1/2)$, $a(x) = a_0 - |x|^2$, $a_0 = \sqrt{2\Lambda/\pi}$ and $|x|_\Lambda = \sqrt{x_1^2 + \Lambda x_2^2}$. The condensate is confined in the region

$$\mathcal{D} = \{ x \in \mathbb{R}^2 : a(x) > 0 \} . \quad (1.3)$$

The ground state energy is:

$$E_{gs}(\varepsilon, \Omega) = \inf \{ F_\varepsilon(u) : u \in H^1(\mathbb{R}^2), |x|^2 u \in L^2(\mathbb{R}^2) \& \int_{\mathbb{R}^2} |u|^2 dx = 1 \} . \quad (1.4)$$

The minimization problem in (1.4) is studied in [10] when $\varepsilon \to 0_+$ and $\Omega \approx |\ln \varepsilon|$. Among other things, it is found a critical speed $\Omega_c = \omega_c |\ln \varepsilon|$ such that minimizers start to have zeros when
In [10, Thm. 2.2], it is established that (1.6) has a positive minimizer u. Theorem 1.1. Let u be a unique up to a multiplicative complex constant of unit modulus. Following an idea of [12] and Remark 1.2, it is obtained the reduced functional:

\[ E_\varepsilon(u) = \int_{\mathbb{R}^2} \left( |\nabla u|^2 + \frac{1}{2\varepsilon^2} \left( |a(x) - |u|^2|^2 - |a_-(x)| \right) \right) \, dx. \]  

The ground state energy of this functional is:

\[ e_\varepsilon = \inf \{ E_\varepsilon(u) : u \in H^1(\Omega), |x|^2 u \in L^2(\Omega) \& \int_{\mathbb{R}^2} |u|^2 \, dx = 1 \}. \]  

In [10, Thm. 2.2], it is established that (1.6) has a positive minimizer \( \tilde{\eta}_\varepsilon \). This minimizer is unique up to a multiplicative complex constant of unit modulus. Following an idea of [12] and writing \( u = \tilde{\eta}_\varepsilon v \), there holds the following decomposition:

\[ F_\varepsilon(u) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \mathcal{G}_\varepsilon(v), \]  

with

\[ \mathcal{G}_\varepsilon(v) = \int_{\mathbb{R}^2} \left( \tilde{\eta}_\varepsilon^2 |(\nabla - i\Omega A_0)v|^2 + \frac{\tilde{\eta}_\varepsilon^4}{2\varepsilon^2} (1 - |v|^2)^2 - \frac{\Omega^2}{4} \tilde{\eta}_\varepsilon^2 |x|^2 |v|^2 \right) \, dx. \]  

Also, if \( u \) is selected as a minimizer of (1.4), then \( v \) will be a minimizer of \( \mathcal{G}_\varepsilon \) under the weighted mass constraint,

\[ \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |v|^2 \, dx = 1. \]  

More precisely, the minimization problem (1.4) is equivalent to

\[ C_0(\varepsilon, \Omega) = \inf \{ \mathcal{G}_\varepsilon(v) : v \in H^1(\Omega), \, \tilde{\eta}_\varepsilon |x|^2 v \in L^2(\Omega) \& \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |v|^2 \, dx = 1 \}. \]  

The main theorem of this paper is:

**Theorem 1.1.** Let \( M > 0 \) and \( b : (0, 1) \to (0, \infty) \) satisfies \( \lim_{\varepsilon \to 0^+} b(\varepsilon) = \infty \). Suppose that the rotational speed satisfies:

\[ b(\varepsilon) |\ln \varepsilon| \leq \Omega \leq \frac{M}{\varepsilon^{2/3}}, \quad (\varepsilon \in (0, 1)). \]

There exist a constant \( \varepsilon_0 > 0 \) and a function \( \text{err} : (0, \varepsilon_0] \to \mathbb{R} \) such that,

\[ \lim_{\varepsilon \to 0^+} \text{err}(\varepsilon) = 0, \]

and

\[ E_{gs} = e_\varepsilon - \Omega^2 c_\varepsilon + \Omega \left[ \ln \frac{1}{\varepsilon \sqrt{\Omega}} \right] \left( 1 + \text{err}(\varepsilon) \right), \quad (\varepsilon \in (0, \varepsilon_0)). \]  

Here \( E_{gs} \) is introduced in (1.4), \( e_\varepsilon \) in (1.6) and

\[ c_\varepsilon = \frac{1}{4} \int_{\mathbb{R}^2} |x|^2 \tilde{\eta}_\varepsilon(x) \, dx. \]

**Remark 1.2.** In light of the decomposition in (1.7), the proof of Theorem 1.1 is done by establishing that:

\[ C_0(\varepsilon, \Omega) = -\Omega^2 c_\varepsilon + \Omega \left[ \ln \frac{1}{\varepsilon \sqrt{\Omega}} \right] \left( 1 + \text{err}(\varepsilon) \right). \]
Remark 1.3. Along the proof of Theorem 1.1, one gets information about the qualitative behavior of the minimizers. More precisely, it is possible to get information about the arrangement of vortices. This is discussed in Section 6.

Remark 1.4. The assumption on $\Omega$ is technical. To derive a lower bound, the condition $\Omega \leq \frac{M}{\varepsilon^{2/3}}$ is needed. Comparing with [7], it is expected that the theorem remains true under the relaxed assumption that $\Omega \leq \frac{M}{\varepsilon}$.

Remark 1.5. The letter $C$ denotes a positive constant independent of $\varepsilon$ and $\Omega$, and whose value is not the same when seen in different formulas. The quantity $O(B)$ is any expression that remains in the interval $(-C|B|, C|B|)$. Writing $A \ll B$ means that $A = \delta B$ and $\delta \to 0$.

2. Preliminaries

Some basic properties of the positive minimizer $\tilde{v}_e$ of (1.6) as well as of minimizers of the modified problem (1.10) will be used along the proof of Theorem 1.1. These properties are recalled here.

The estimates in Theorem 2.1 are obtained in [10]. Notice however that the boundedness of $\tilde{v}_e$ (as stated in (4)-(5) below) is obtained after collecting formulas (2.14), (2.15) & (2.27) of [10]. Recall that $\mathcal{D}$ is the region introduced in (1.3).

Theorem 2.1. There exist positive constants $\varepsilon_0 > 0$, $C > 0$ and $\delta_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then the real-valued minimizer $\tilde{v}_e$ of (1.6) satisfies:

1. $E_e(\tilde{v}_e) \leq C|\ln \varepsilon|$;
2. $0 < \tilde{v}_e(x) \leq C \varepsilon^{1/3} |\ln \varepsilon|$ if $|x|_A \geq \sqrt{a_0} + \varepsilon$;
3. $\tilde{v}_e(x) \leq C \varepsilon^{2/3} |\ln \varepsilon|$ if $|x|_A \leq \sqrt{a_0} - \delta_0 \varepsilon^{2/3}$.

Furthermore, to every compact set $K \subset \mathcal{D}$, there exist two positive constants $C_K$ and $\varepsilon_K$ such that, if $\varepsilon \in (0, \varepsilon_K)$, then,

$$||\tilde{v}_e - \sqrt{\varepsilon}||_{C^1(K)} \leq C_K \varepsilon^2 |\ln \varepsilon|.$$

The analysis in [10] and [2] can be generalized to obtain three useful estimates that are collected in Theorem 2.2 below. The proofs will be reproduced for the convenience of the reader.

Theorem 2.2. Let $M > 0$ and $\alpha \in (0, 1)$. There exist positive constants $C$, $\delta$, $\lambda$ and $\varepsilon_0$ such that, if $\varepsilon \in (0, \varepsilon_0)$ and $0 < \Omega \leq \frac{M}{\varepsilon^\alpha}$, then every minimizer $v_e$ of (1.10) satisfies:

1. $\int_{\mathbb{R}^2} \left( \frac{\tilde{v}_e^2}{2}\varepsilon_2 |\nabla v|^2 + \frac{\tilde{v}_e^4}{2}\varepsilon^2 (|v|^2 - 1)^2 \right) dx \leq C \Omega^2$;
2. $|\tilde{v}_e v_e(x)| \leq C$ if $x \in \mathbb{R}^2$; and
3. $|\tilde{v}_e v_e(x)| \leq C \exp(-\delta/\varepsilon^{\alpha/2})$ if $|x|_A \geq \sqrt{a_0} + \lambda \varepsilon^{1-\alpha} + \varepsilon^{\alpha/2}$.

Proof. Under the assumption on the rotational speed, Proposition 3.2 in [10] implies that the problem (1.4) has a minimizer $u_e$. In light of the decomposition in (1.7), it follows that $v_e = u_e/\tilde{v}_e$ is a minimizer of the problem (1.10). Theorem 2.2 will be proved by establishing properties of $u_e$. The function $u_e$ satisfies

$$-\Delta u_e + 4i\Omega A_0 \cdot \nabla u_e = \frac{1}{\varepsilon^2} (a(x) - |u_e|^2) + \ell_e u_e \quad \text{in} \; \mathbb{R}^2,$$

where $\ell_e$ is the lagrange multiplier. Furthermore, it holds (see the derivation of [10, (3.7)&(3.11)]):

$$F_e(u_e) \leq C|\ln \varepsilon|, \quad |\ell_e| \leq C \varepsilon^{-1}\Omega, \quad \int_{\mathbb{R}^2 \setminus \mathcal{D}} |u_e|^4 dx \leq C \varepsilon^2 \Omega^2.$$

Let $U_e = |u_e|^2$ and $\Theta_e = \{x \in \mathbb{R}^2 \setminus \mathcal{D} : a_-(x) \geq 2(\varepsilon^2 |\ell_e| + \varepsilon^2 \Omega^2 |x|^2)\}$. There exists $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$ and $\varepsilon$ is sufficiently small then $\partial \Theta_e \subset B_{\lambda, r_e}$ and $\mathcal{E}_e \subset \Theta_e$. Here $\mathcal{E}_e = \mathbb{R}^2 \setminus B_{\lambda, r_e}$.

$B_{\lambda, r_e} = \{x \in \mathbb{R}^2 \setminus \mathcal{D} : |x|_\lambda \leq r_e\}$ and $r_e = \sqrt{a_0 + \lambda \varepsilon^{1-\alpha}}.$
Let $x_0 \in \mathbb{E}_\varepsilon$ be an arbitrary point. Clearly, $B(x_0, \lambda \varepsilon^{1-\alpha}/2) \subset \mathbb{E}_\varepsilon$. The function $U_\varepsilon$ is subharmonic in $\Theta_\epsilon$, and its $L^2$-norm is estimated in (2.2). As a consequence, there is a constant $C_* > 0$ such that,

$$0 \leq U_\varepsilon(x) \leq \frac{1}{|B(x_0, \lambda \varepsilon^{1-\alpha}/2)|} \int_{B(x_0, \lambda \varepsilon^{1-\alpha}/2)} U_\varepsilon^2(x) \, dx \leq C_* \varepsilon^{1-\alpha}.$$

The function $U_\varepsilon$ becomes a subsolution of the problem:

$$-\varepsilon^2 \Delta w + a_-(x) w = 0 \text{ in } \mathbb{E}_\varepsilon, \quad \text{and, } w = C_* \varepsilon^{1-\alpha} \text{ on } \partial \mathbb{E}_\varepsilon. \tag{2.3}$$

A supersolution of (2.3) is $v_{out}(x) = C_* \varepsilon^{1-\alpha} \Omega \exp[(a_0 + \lambda \varepsilon^{1-\alpha} - |x|^2)/\varepsilon^\alpha]$. By the comparison principle, one obtains that $U_\varepsilon(x) \leq v_{out}(x)$ in $\mathbb{E}_\varepsilon$. Given the definition of $\mathbb{E}_\varepsilon$, the assumption on $\Omega$ and $u_\varepsilon = \tilde{\eta}_\varepsilon v_\varepsilon$, this establishes (3) in Theorem 2.2.

The estimate in (2) is obtained in a similar manner. Define,

$$v_{in}(x) = \begin{cases} a(x) + |\ell_\varepsilon|\varepsilon^2 + \frac{\varepsilon^2 \Omega^2}{\Lambda^2} |x|_\Lambda^2 \\ a_0 - \tilde{\rho}_\varepsilon(1 - \varepsilon^2 \Omega^2/\Lambda^2)(2|x|_\Lambda - \tilde{\rho}_\varepsilon) + |\ell_\varepsilon|\varepsilon^2 & \text{if } |x|_\Lambda \leq \tilde{\rho}_\varepsilon := \sqrt{a_0 - \lambda \varepsilon^{1-\alpha}}, \\ \tilde{\rho}_\varepsilon & \text{if } \tilde{\rho}_\varepsilon \leq |x|_\Lambda \leq \sqrt{a_0 + \lambda \varepsilon^{1-\alpha}}. \end{cases}$$

It is possible to select $\lambda$ and a positive constant $C_\Lambda > 0$ so that one has the additional properties:

$$v_{in} \geq C_* \varepsilon^{1-\alpha} \text{ in } B_{\Lambda, \tilde{\rho}_\varepsilon}, \quad m(x) = a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2 \Omega^2 |x|^2 > 0 \text{ in } B_{\Lambda, \tilde{\rho}_\varepsilon},$$

and $b(x) = U_\varepsilon(x) + v_{in}(x) - m(x) \geq 0$ in $B_{\Lambda, \tilde{\rho}_\varepsilon}$. As a consequence, $V_\varepsilon = U_\varepsilon - v_{in}$ satisfies,

$$-\varepsilon^2 V_\varepsilon + b(x)V_\varepsilon \leq 0 \text{ in } B_{\Lambda, \varepsilon}, \quad \text{and } V_\varepsilon \leq 0 \text{ on } \partial B_{\Lambda, \varepsilon}.$$

By the comparison principle, it holds that $V_\varepsilon \leq 0$ in $B_{\Lambda, \varepsilon}$. This proves (2) in Theorem 2.2.

To prove (1), notice that (2) and (3) together give that:

$$\int_{\mathbb{R}^2} |x|^2 \tilde{\eta}_\varepsilon^2 v_\varepsilon^2 \, dx \leq C.$$

The estimate in (1) now follows in light of (2.2) and the decomposition in (1.7). \hfill \Box

**Remark 2.3.** As a consequence of (3) in Theorem 2.1 and Theorem 2.2, it follows that if $K \subset D$ is a compact subset of $D$ and $\varepsilon$ is sufficiently small, then $|v_\varepsilon(x)| \leq C$ in $K$.

### 3. Reduced Ginzburg-Landau Energy

Let $K = (-1/2, 1/2) \times (-1/2, 1/2)$ be a square of unit side length, $\lambda$, $h_{ex}$ and $\varepsilon$ be positive parameters. Consider the functional defined for all $u \in H^1(K; \mathbb{C})$,

$$E^{2D}_\lambda(u) = \int_K \left( |(\nabla - i h_{ex} \mathbf{A}_0) u|^2 + \frac{\lambda}{2\varepsilon^2} (1 - |u|^2)^2 \right) \, dx. \tag{3.1}$$

Here $\mathbf{A}_0$ is the vector potential whose curl is equal to $1$,

$$\mathbf{A}_0(x_1, x_2) = \frac{1}{2}(-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{3.2}$$

Notice that the functional $E^{2D}_\lambda$ is a simplified version of the full Ginzburg-Landau functional considered in [16], as the magnetic potential in (3.1) is given and not an unknown of the problem.

Minimization of the functional $E^{2D}_\lambda$ arises naturally over ‘magnetic periodic' functions. Let us introduce the following space,

$$E_{h_{ex}} = \{ u \in H^1_{loc}(\mathbb{R}^2; \mathbb{C}) : u(x_1 + 1, x_2) = e^{ih_{ex} x_2/2} u(x_1, x_2), \quad u(x_1, x_2 + 1) = e^{-ih_{ex} x_1/2} u(x_1, x_2) \}, \tag{3.3}$$

together with the ground state energy,

$$m_{p}(h_{ex}, \varepsilon) = \inf\{ E^{2D}_\lambda(u) : u \in E_{h_{ex}} \}. \tag{3.4}$$
Minimization of $E_{3D}^2$ over configurations without prescribed boundary conditions will be needed as well. The ground state energy of this problem is,

$$m_0(h_{\text{ex}}, \varepsilon) = \inf \{ E_{3D}^2(u) : u \in H^1(K) \}.$$  

The ground state energies $m_0(h_{\text{ex}}, \varepsilon)$ and $m_p(h_{\text{ex}}, \varepsilon)$ are estimated in [11] by borrowing tools from [15] and [16]. This is recalled in the next theorem.

**Theorem 3.1.** Assume that $\lambda_2 > \lambda_1 > 0$ are given constants, $\lambda \in (\lambda_1, \lambda_2)$ and $h_{\text{ex}}$ is a function of $\varepsilon$ such that

$$|\ln \varepsilon| \ll h_{\text{ex}} \ll \frac{1}{\varepsilon^2}, \quad \text{as } \varepsilon \to 0.$$  

As $\varepsilon \to 0$, the ground state energies $m_0(h_{\text{ex}}, \varepsilon)$ and $m_p(h_{\text{ex}}, \varepsilon)$ satisfy,

$$m_0(h_{\text{ex}}, \varepsilon) = h_{\text{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}} (1 + o(1)) \quad \text{and} \quad m_p(h_{\text{ex}}, \varepsilon) = h_{\text{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}} (1 + o(1)).$$

In the forthcoming section, it will be needed a trial state satisfying the mass constraint ($L^2$-norm equal to 1) and having an energy close to $m_p(h_{\text{ex}}, \varepsilon)$. The next Lemma provides one with a useful trial state whose $L^2$-norm is close to 1.

**Lemma 3.2.** Suppose that $\lambda > 0$, $h_{\text{ex}}$ and $\varepsilon$ are as in Theorem 3.1. There exists a function $f_\varepsilon$ in $H^1(K)$ such that

$$|f_\varepsilon| \leq 1 \quad \text{in } K,$$

$$\{x \in K : |f_\varepsilon(x)| < 1\} \subset \bigcup_{i=1}^n B(a_i, \varepsilon) \quad \text{and} \quad n = O(h_{\text{ex}}),$$

$$1 - O(\varepsilon^2 h_{\text{ex}}) \leq \int_K |f_\varepsilon(x)|^2 \, dx \leq 1,$$

and

$$E_{3D}^2(f_\varepsilon) \leq h_{\text{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}} (1 + o(1)).$$

as $\varepsilon \to 0_+$. Furthermore, $f_\varepsilon$ is independent of $\lambda$, and $O$ is uniform with respect to $\lambda$.

**Proof.** For the convenience of the reader, the construction of $f_\varepsilon$ is outlined. Details can be found in [6]. Let $N$ be the largest positive integer satisfying $N \leq \sqrt{h_{\text{ex}}}/2\pi < N + 1$. Divide the square $K$ into $N^2$ disjoint squares $(K_j)_{0 \leq j < N^2-1}$ each of side length equal to $1/N$ and center $a_j$. Let $h$ be the unique solution of the problem,

$$-\Delta h + h_{\text{ex}} = 2\pi \delta_{a_j} \quad \text{in } K_0, \quad \frac{\partial h}{\partial \nu} = 0 \quad \text{on } \partial K_0,$$

$$\int_{K_0} h \, dx = 0.$$

Here $\nu$ is the unit outward normal vector of $K_0$. The function $h$ satisfies periodic conditions on the boundary of $K_0$, and

$$\int_{K_0 \setminus B(a_0, \varepsilon)} |\nabla h|^2 \, dx \leq 2\pi \ln \frac{1}{\varepsilon N} + O(1) = 2\pi \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}} + O(1), \quad \text{as } \varepsilon \to 0_+.$$

The function $h$ is extended by periodicity in the square $K$. Let $\phi$ be a function (defined modulo $2\pi$) satisfying in $K \setminus \{a_j : 0 \leq j \leq N^2 - 1\},$

$$\nabla \phi = -\nabla \cdot h + h_{\text{ex}} A_0, \quad (\nabla \cdot = (-\partial_{x^2}, \partial_{x^1})).$$

Here $A_0$ is the magnetic potential in (3.2). If $x \in K_0$, let $\rho(x) = \min(1, |x - a_0|/\varepsilon)$. The function $\rho$ is extended by periodicity in the square $K$. Put $f_\varepsilon(x) = \rho(x)e^{i\phi(x)}$ for all $x \in K$. The function $f_\varepsilon$ can be extended as a function in the space $E_{h_{\text{ex}}}$ in (3.3), see [5, Lemma 5.11] for details.
The energy of \( f_\varepsilon \) is easily computed, since \( f_\varepsilon \) is 'magnetic periodic' and \( N = \sqrt{h_{ex}/2\pi (1 + o(1))} \). Clearly, in the square \( K_0, |f_\varepsilon(x)| < 1 \) if and only if \( |x - a_0| < \varepsilon \). Thus, it is easy to check that \( f_\varepsilon \) satisfies the requirements in Lemma 3.2.

\[ \square \]

4. Upper Bound

4.1. The test configuration. Recall the definition of the ground state energy \( C_0(\varepsilon, \Omega) \) in (1.10). The assumption on the rotational speed \( \Omega \) is \( |\ln \varepsilon| \ll \Omega \leq M/\varepsilon^\alpha \) with \( M > 0 \) and \( \alpha \in (0, 1) \). Let \( L \in (0, a_0) \) and define,

\[ \mathcal{D}_L = \{ x \in \mathcal{D} : |x|_\Lambda < L \}. \]

Define

\[ \ell = \left( \frac{\Omega}{|\ln \varepsilon|} \right)^{1/4} \frac{1}{\sqrt{\Omega}} \]

\[ h_{ex} = \frac{1}{\ell^2}. \]

Recall the ground state energy \( m_{\text{xt}}(h_{ex}, \varepsilon) \) and the space \( E_{h_{ex}} \) introduced in (3.4) and (3.3) respectively. Let \( f_\varepsilon \in E_{h_{ex}} \) be the test function defined in Lemma 3.2. In particular, \( f_\varepsilon \) satisfies \( E_\lambda^D(f_\varepsilon) \leq h_{ex} \ln \frac{1}{\varepsilon \sqrt{h_{ex}}} (1 + o(1)) \) for any \( \lambda \) varying between two positive constants \( \lambda_1 \) and \( \lambda_2 \).

Define,

\[ v(x) = \chi(x) f_\varepsilon(\ell \sqrt{\Omega} x) \quad (x \in \mathbb{R}^2), \]

where \( \chi \) is a cut-off function satisfying,

\[ 0 \leq \chi \leq 1 \text{ in } \mathbb{R}^2, \quad \chi = 0 \text{ when } |x|_\Lambda \geq \sqrt{a_0} + 1, \quad \chi = 1 \text{ when } |x|_\Lambda \leq \sqrt{a_0} + \frac{1}{2}, \]

and

\[ |\nabla \chi| \leq C \quad \text{in } \mathbb{R}^2. \]

Let \( (K_j) \) be the lattice of \( \mathbb{R}^2 \) generated by the cube,

\[ K = \left( -\frac{1}{2\ell \sqrt{\Omega}}, \frac{1}{2\ell \sqrt{\Omega}} \right) \times \left( -\frac{1}{2\ell \sqrt{\Omega}}, \frac{1}{2\ell \sqrt{\Omega}} \right). \]

Let \( \mathcal{J} = \{ K_j : K_j \cap \mathcal{D}_L \neq \emptyset \} \) and \( N = \text{Card} \mathcal{J} \). As \( \varepsilon \to 0_+ \), the number \( N \) satisfies

\[ N = |\mathcal{D}_L| \times (\ell \sqrt{\Omega})^2 \left( 1 + o(1) \right). \]

In light of Lemma 3.2 and the exponential decay of \( \tilde{\eta}_\varepsilon \) in Lemma 2.1, the function \( v \) satisfies

\[ 1 - \mathcal{O}(\varepsilon^2 \Omega) \leq \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |v|^2 \, dx \leq 1. \]

(4.1)

Define the test function,

\[ \tilde{v}(x) = \frac{v(x)}{\sqrt{\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |v|^2 \, dx}}. \]

(4.2)

Clearly, the function \( \tilde{v} \) satisfies the weighted mass constraint,

\[ \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\tilde{v}|^2 \, dx = 1, \]

(4.3)

and consequently, there holds the upper bound \( C_0(\varepsilon, \Omega) \leq \mathcal{G}_\varepsilon(\tilde{v}) \). The rest of the section will be devoted to estimating the energy \( \mathcal{G}_\varepsilon(\tilde{v}) \). It will be established that:

\[ \lim_{L \to (a_0)_-} \left[ \lim_{\varepsilon \to 0_+} \sup \left( \frac{\mathcal{G}_\varepsilon(\tilde{v}) - \left( -\Omega^2 c_\varepsilon \right)}{2\Omega} \right) \right] \leq 0. \]

(4.4)

The next estimate (4.5) is a consequence of (4.4),

\[ C_0(\varepsilon, \Omega) \leq -\Omega^2 c_\varepsilon + \Omega \left[ \ln \frac{1}{\varepsilon \sqrt{\Omega}} \right] (1 + \text{err}(\varepsilon)). \]

(4.5)
4.2. Energy of the test configuration: Proof of (4.4). According to the definition of $G_\varepsilon(\tilde{v})$ in (1.10), this term will be decomposed as the sum of two terms $B_\varepsilon$ and $C_\varepsilon$, see (4.7) and (4.8).

**The principal term:** Recall the definition of $c_\varepsilon$ in Theorem 1.1. It will be shown that the term

$$B_\varepsilon = -\frac{1}{4} \Omega^2 \int_{\mathbb{R}^2} \tilde{\eta}_e^2 |x|^2 |\tilde{v}|^2 \, dx ,$$

(4.6)

satisfies

$$B_\varepsilon = -c_\varepsilon \Omega^2 + o(\Omega) \quad \text{as } \varepsilon \to 0_+ .$$

(4.7)

In light of (4.1) and the definition of $\tilde{v}$ in (4.2), there holds the pointwise bound $|\tilde{v}| \geq |v|$ and consequently,

$$4B_\varepsilon \leq -\Omega^2 \int_{\mathbb{R}^2} \tilde{\eta}_e^2 |x|^2 |v|^2 \, dx$$

$$= -\Omega^2 \int_{\mathbb{R}^2} \tilde{\eta}_e^2 |x|^2 \, dx + \Omega^2 \int_{\mathbb{R}^2} \tilde{\eta}_e^2 |x|^2 (1 - |v|^2) \, dx$$

$$= -4\Omega^2 c_\varepsilon + \Omega^2 \int_{\mathbb{R}^2} \tilde{\eta}_e^2 |x|^2 (1 - |v|^2) \, dx .$$

Recall that $|v| \leq 1$ everywhere and $\tilde{\eta}_e$ decays exponentially fast as described in Theorem 2.1. As a consequence of this, we have

$$\Omega^2 \int_{\mathbb{R}^2} \tilde{\eta}_e^2 |x|^2 (1 - |v|^2) \, dx = \Omega^2 \int_{\{ |x| < \sqrt{\eta_0} + 1 \}} \tilde{\eta}_e^2 |x|^2 (1 - |v|^2) \, dx + o(1) \quad \text{as } \varepsilon \to 0_+ .$$

The function $\tilde{\eta}_e$ satisfies the mass constraint $\int_{\mathbb{R}^2} \tilde{\eta}_e^2 \, dx = 1$, the function $v$ satisfies (4.1) and the term $1 - |v|^2$ is always non-negative. In light of this, it is possible to write,

$$\int_{\{ |x| < \sqrt{\eta_0} + 1 \}} \tilde{\eta}_e^2 |x|^2 (1 - |v|^2) \, dx \leq (\sqrt{\eta_0} + 1)^2 \int_{\{ |x| < \sqrt{\eta_0} + 1 \}} \tilde{\eta}_e^2 (1 - |v|^2) \, dx = O(\varepsilon^2 \Omega) .$$

As a consequence, there holds $B_\varepsilon \leq -\Omega^2 c_\varepsilon + O(\varepsilon^2 \Omega^3)$ and the term $O(\varepsilon^2 \Omega^3)$ is of order $o(\Omega)$, thanks to the assumption on $\Omega$.

**The second correction term:** It will be shown that the term

$$C_\varepsilon = \int_{\mathbb{R}^2} \left( \tilde{\eta}_e^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\tilde{\eta}_e^4}{2\varepsilon^2} (1 - |\tilde{v}|^2)^2 \right) \, dx ,$$

is of leading order equal to $L_\varepsilon = \Omega \left[ \ln \frac{1}{\varepsilon \sqrt{\varepsilon \Omega}} \right]$. It is useful to write $C_\varepsilon$ as the sum of five terms,

$$C_\varepsilon = C_{\varepsilon,1} + C_{\varepsilon,2} + C_{\varepsilon,3} + C_{\varepsilon,4} + C_{\varepsilon,5} ,$$

(4.8)

where

$$C_{\varepsilon,1} = \int_{D_\varepsilon} \left( \tilde{\eta}_e^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\tilde{\eta}_e^4}{2\varepsilon^2} (1 - |\tilde{v}|^2)^2 \right) \, dx ,$$

(4.9)

$$C_{\varepsilon,2} = \int_{L \leq |x| \leq \sqrt{\eta_0}} \left( \tilde{\eta}_e^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\tilde{\eta}_e^4}{2\varepsilon^2} (1 - |\tilde{v}|^2)^2 \right) \, dx ,$$

(4.10)

$$C_{\varepsilon,3} = \int_{\sqrt{\eta_0} \leq |x| \leq \sqrt{\eta_0} + 1/2} \left( \tilde{\eta}_e^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\tilde{\eta}_e^4}{2\varepsilon^2} (1 - |\tilde{v}|^2)^2 \right) \, dx ,$$

(4.11)

$$C_{\varepsilon,4} = \int_{\sqrt{\eta_0} + 1/2 \leq |x| \leq \sqrt{\eta_0} + 1} \left( \tilde{\eta}_e^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\tilde{\eta}_e^4}{2\varepsilon^2} (1 - |\tilde{v}|^2)^2 \right) \, dx ,$$

(4.12)

$$C_{\varepsilon,5} = \int_{|x| \geq \sqrt{\eta_0} + 1} \left( \tilde{\eta}_e^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\tilde{\eta}_e^4}{2\varepsilon^2} (1 - |\tilde{v}|^2)^2 \right) \, dx .$$

(4.13)
The term $C_{\varepsilon,1}$: Recall that $L$ is selected independently of $\varepsilon$. In light of Theorem 2.1 and the Mean Value Theorem applied to the function $a$, there holds in every square $K_j \subset D_L$,

$$\tilde{h}_\varepsilon(x) \leq \sqrt{a(x)} + C\varepsilon^2|\ln \varepsilon| \quad \text{and} \quad \sqrt{a(x)} \leq \sqrt{a(x_j) + \frac{C}{\varepsilon \sqrt{\Omega}}},$$

where $x_j$ is an arbitrary point in $K_j$. The above two estimates applied successively yield an upper bound of the term $C_{\varepsilon,1}$ as follows:

$$C_{\varepsilon,1} \leq \sum_j \left[ a(x_j) + C\varepsilon^2|\ln \varepsilon| + \frac{C}{\varepsilon \sqrt{\Omega}} \right] \int_{K_j} \left( |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\lambda_e}{2\varepsilon^2}(1 - |\tilde{v}|^2)^2 \right) dx,$$

where

$$\lambda_e = \max_j \left( \frac{a(x_j)}{\sqrt{a(x_j)} + C\varepsilon^2|\ln \varepsilon| + \frac{C}{\varepsilon \sqrt{\Omega}}}. \right)$$

In the domain $D_L$, the function $\chi$ is equal to 1 and $v(x) = f_\varepsilon(\ell \sqrt{\Omega} x)$. By using successively the estimate in (4.1), the ‘magnetic’ periodicity of $v$ over the lattice $(K_j)_j$ and the bound $|v| \leq 1$, one gets the following upper bound,

$$\int_{K_j} \left( |(\nabla - i\Omega A_0)\tilde{v}|^2 + \frac{\lambda_e}{2\varepsilon^2}(1 - |\tilde{v}|^2)^2 \right) dx$$

$$\leq (1 + C\varepsilon^2\Omega) \int_{K_j} \left( |(\nabla - i\Omega A_0)v|^2 + \frac{\lambda_e}{2\varepsilon^2}(1 - |v|^2)^2 \right) dx + C\Omega \int_{K_j} |v|^2 dx$$

$$\leq (1 + C\varepsilon^2\Omega) \int_{K} \left( |(\nabla - i\Omega A_0)v|^2 + \frac{\lambda_e}{2\varepsilon^2}(1 - |v|^2)^2 \right) dx + C\Omega|K_j|. \quad (4.14)$$

The integral term in (4.14) is computed by the change of variable $y = \ell \sqrt{\Omega} x$ that transforms it to

$$\int_{\widetilde{K}} \left( |(\nabla - ih_{\text{ex}} A_0)f_{\varepsilon x}|^2 + \frac{\lambda_e}{2\varepsilon^2}(1 - |f_{\varepsilon x}|^2)^2 \right) dx$$

where $\varepsilon = \varepsilon \ell \sqrt{\Omega}$ and $h_{\text{ex}} = \frac{1}{{\varepsilon}^2}$. As $\varepsilon \to 0_+$, $\varepsilon \gg \varepsilon$ and $h_{\text{ex}}$ satisfies $|\ln \varepsilon| \ll h_{\text{ex}} \ll \varepsilon^{-2}$. Also, $\lambda_e$ remains inside a fixed interval $[\lambda_1, \lambda_2]$. Consequently, it is possible to use Lemma 3.2 and get that $(1 + o(1))h_{\text{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}}$ is an upper bound of the term in (4.15). As a consequence, it is obtained the following upper bound of $C_{\varepsilon,1}$,

$$C_{\varepsilon,1} \leq (1 + C\varepsilon^2\Omega) \sum_j \left[ a(x_j) + C\varepsilon^2|\ln \varepsilon| + \frac{C}{\varepsilon \sqrt{\Omega}} \right] \left( (1 + o(1))h_{\text{ex}} \ln \frac{1}{{\varepsilon}^2h_{\text{ex}}} + C\Omega|K_j| \right). \quad (4.16)$$

Recall that, as $\varepsilon \to 0_+$, the number of squares $K_j$ satisfies $N = |D_L| \times \ell^2\Omega(1 + o(1))$. Since $|K_j| = \frac{1}{\ell^2\Omega}$ for every $j$, then $\sum_j |K_j| = |D_L|(1 + o(1))$. Also, all the extra terms appearing in (4.16) are $o(1)$ as $\varepsilon \to 0_+$, and this leads one to,

$$C_{\varepsilon,1} \leq (1 + o(1)) \sum_j \frac{1}{|K_j|}a(x_j)\ell^2\Omega h_{\text{ex}} \ln \frac{1}{{\varepsilon}^2h_{\text{ex}}}$$

$$= (1 + o(1))\Omega \ln \frac{1}{{\varepsilon}^2\Omega} \sum_j \frac{1}{|K_j|}a(x_j).$$

Since each point $x_j$ is arbitrarily selected in the square $K_j$, then the sum $\sum_j \frac{1}{|K_j|}a(x_j)$ becomes a Riemann sum and, as $|K_j| \to 0$, there holds, $\sum_j \frac{1}{|K_j|}a(x_j) = \int_{D_L} a(x) dx + o(1)$. As a
consequence, the term $C_{ε,1}$ satisfies,

$$C_{ε,1} \leq (1 + o(1))Ω \ln \frac{1}{ε \sqrt{Ω}} \int_{D_L} a(x) \, dx \quad \text{as } ε \to 0_+ . \quad (4.17)$$

The term $C_{ε,2}$: To estimate the term $C_{ε,2}$, it is used that the function $\tilde{η}_ε$ is bounded independently of $ε$ to get that,

$$C_{ε,2} \leq C \int_{L \leq |x|_A \leq \sqrt{ω_0}} \left( |(∇ - iΩA_0)\tilde{v}|^2 + \frac{1}{2ε^2} (1 - |\tilde{v}|^2)^2 \right) \, dx .$$

The definition of $\tilde{v}$ and the estimate in (4.1) together yield,

$$C_{ε,2} \leq C(1 + Cε^2Ω) \int_{L \leq |x|_A \leq \sqrt{ω_0}} \left( |(∇ - iΩA_0)v|^2 + \frac{1}{2ε^2} (1 - |v|^2)^2 \right) \, dx$$

$$+ CΩ \int_{L \leq |x|_A \leq \sqrt{ω_0}} |v|^2 \, dx .$$

The function $χ$ is equal to 1 when $L \leq |x|_A \leq \sqrt{ω_0} + 1/2$. As a consequence $v(x) = f_ε(ℓ\sqrt{Ω}x)$. As is done for the term $C_{ε,1}$, one gets that,

$$C_{ε,2} \leq C(1 + o(1)) \left( \int_{L \leq |x|_A \leq \sqrt{ω_0}} dx \right) \Omega \ln \frac{1}{ε \sqrt{Ω}} . \quad (4.18)$$

The term $C_{ε,3}$: When $\sqrt{ω_0} \leq |x|_A \leq \sqrt{ω_0} + 1/2$, the function $\tilde{η}_ε$ is bounded by $Cε^{1/3}$ (Theorem 2.1) and the function $χ$ is constant and equal to 1. As is done for the terms $C_{ε,1}$ and $C_{ε,2}$, this allows one to write,

$$C_{ε,3} \leq Cε^{1/3} \int_{\sqrt{ω_0} \leq |x|_A \leq \sqrt{ω_0} + 1/2} \left( |(∇ - iΩA_0)\tilde{v}|^2 + \frac{1}{2ε^2} (1 - |\tilde{v}|^2)^2 \right) \, dx$$

$$\leq Cε^{1/3}(1 + Cε^2Ω) \int_{\sqrt{ω_0} \leq |x|_A \leq \sqrt{ω_0} + 1/2} \left( |(∇ - iΩA_0)v|^2 + \frac{1}{2ε^2} (1 - |v|^2)^2 \right) \, dx$$

$$+ Cε^{1/2}Ω \int_{L \leq |x|_A \leq \sqrt{ω_0}} |v|^2 \, dx$$

$$\leq Cε^{1/3}Ω \ln \frac{1}{ε \sqrt{Ω}} . \quad (4.19)$$

The term $C_{ε,4}$: Recall the definition of this term in (4.13). This term is going to be an error term of negligible order compared to the principal terms. Notice here that the function $χ$ is no more constant and the function $v$ is small. As a consequence, it is not useful to estimate the ‘Ginzburg-Landau’ energy of $v$ along the same procedure as done before. However, as Theorem 2.1 states, the function $\tilde{η}_ε$ decays exponentially, and this will be the key to estimate the term $C_{ε,4}$. Thanks to (4.1), the function $\tilde{v}$ satisfies the uniform inequality $|1 - |\tilde{v}|^2| \leq 1 + O(ε^2Ω)$. This and the exponential decay of $\tilde{η}_ε$ in Theorem 2.1 together yield when $ε \to 0_+$,

$$\frac{1}{2ε^2} \int_{\sqrt{ω_0} + 1/2 \leq |x|_A \leq \sqrt{ω_0} + 1} \tilde{η}_ε^4 |1 - |\tilde{v}|^2|^2 dx \leq C \frac{1}{ε^2} \exp \left(-\frac{δ}{ε^{α/2}}\right) \int_{\sqrt{ω_0} + 1/2 \leq |x|_A \leq \sqrt{ω_0} + 1} dx = o(1) .$$
Using a similar reasoning, the kinetic energy term is estimated as follows,
\[
\int_{\sqrt{\alpha_0} + 1/2 \leq |x|, \Lambda \leq \sqrt{\alpha_0} + 1} \tilde{n}_\varepsilon^2 |(\nabla - i\Omega A_0)\tilde{v}|^2 \, dx
\leq C \exp \left( -\frac{\delta}{\varepsilon^{n/2}} \right) \int_{\sqrt{\alpha_0} + 1/2 \leq |x|, \Lambda \leq \sqrt{\alpha_0} + 1} \left( |(\nabla - i\Omega A_0)v|^2 + |\nabla \chi|^2 |v|^2 \right) \, dx
\leq C \exp \left( -\frac{\delta}{\varepsilon^{n/2}} \right) \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} = o(1),
\]
thereby obtaining that \(C_{\varepsilon,4} = o(1)\) as \(\varepsilon \to 0_+\).

The term \(C_{\varepsilon,5}\): Recall the definition of this term in (4.13) and that the function \(\tilde{v} = 0\) here. As a consequence,
\[
C_{\varepsilon,5} = \int_{|x|, \Lambda \geq \sqrt{\alpha_0} + 1} \frac{\tilde{n}_\varepsilon^2}{2\varepsilon^2} \, dx
\]
and this is equal to \(o(1)\) as \(\varepsilon \to 0_+\) after using the exponential decay of \(\tilde{n}_\varepsilon\).

Conclusion: Collecting the estimates \(C_{\varepsilon,4} = o(1)\), \(C_{\varepsilon,5} = o(1), (4.17), (4.18)\) and \((4.19)\) and inserting them into (4.8) yields an upper bound of \(C_{\varepsilon}\). Inserting this bound and (4.7) into the expression of \(G_{\varepsilon}(\tilde{v})\) yields the upper bound
\[
C_0(\varepsilon, \Omega) \leq -c_\varepsilon \Omega^2 + \left( \int_{D_L} a(x) \, dx + \int_{L \leq |x|, \Lambda \leq \sqrt{\alpha_0}} \, dx + o(1) \right) \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} + o(1),
\]
as \(\varepsilon \to 0_+\). This yields (4.5) by taking the successive limits as \(\varepsilon \to 0_+\) and then as \(L \to \sqrt{\alpha_0}\).

5. LOWER BOUND

Suppose that \(v\) is a minimizer of the functional \(G_{\varepsilon}\) introduced in (1.8), and that the rotational speed \(\Omega\) satisfies the assumption of Theorem 1.1. The aim of this section is to write a lower bound of \(G_{\varepsilon}(v)\).

Approximation of the term \(\Omega^2 \int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 |v|^2 \, dx\): The starting point is the simple decomposition,
\[
\int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 |v|^2 \, dx = \int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 \, dx - \int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 (1 - |v|^2) \, dx
= 4c_\varepsilon - \int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 (1 - |v|^2) \, dx.
\]
The second term on the right side of (5.1) is of order \(\varepsilon \Omega\). This is observed as follows. Write
\[
\int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 (1 - |v|^2) \, dx = \int_{|x| \leq 2\alpha_0} |x|^2 \tilde{n}_\varepsilon^2 (1 - |v|^2) \, dx + \int_{|x| \geq 2\alpha_0} |x|^2 \tilde{n}_\varepsilon^2 (1 - |v|^2) \, dx.
\]
The first term is estimated by using Cauchy-Schwartz inequality and (1) in Theorem 2.2. It is of order \(\varepsilon \Omega\). The second term is of order \(o(1)\). This follows from the boundedness of \(|v|\) ((2)-Theorem 2.2) and the exponential decay of \(\tilde{n}_\varepsilon\) described in Theorem 2.1.

Inserting these estimates into (5.1) yields,
\[
\Omega^2 \int_{\mathbb{R}^2} |x|^2 \tilde{n}_\varepsilon^2 |v|^2 \, dx \leq 4c_\varepsilon \Omega^2 + O(\varepsilon \Omega^3).
\]
Approximation of the term $\Omega^2 \int_{R^d} |x|^2 |\tilde{\eta}_j^2| |v|^2 \, dx$ with a refined error: The error in (5.3) can be improved once the upper bound of $G_\varepsilon(v)$ obtained in Section 4 is used. Actually, by writing,

$$\int_{R^d} \left( \tilde{\eta}_j^2 |(\nabla - i\varepsilon \mathbf{A}_0)v|^2 + \frac{\tilde{\eta}_j^4}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx \leq \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} (1 + o(1)) + O(\varepsilon^{3}) ,$$

(5.4)

it is obtained the improved upper bound $\int_{R^d} \tilde{\eta}_j^2 (1 - |v|^2)^2 \leq O(\varepsilon^2 \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}}) + O(\varepsilon^3)$. By Cauchy-Schwartz inequality, the first term on the right side of (5.2) becomes of the order of $\varepsilon \Omega^{1/2} \sqrt{\ln \frac{1}{\varepsilon \sqrt{\Omega}}} + (\varepsilon \Omega)^{3/2}$. As pointed earlier, the second term on the right side of (5.2) is of order $o(1)$. Consequently, the error in (5.3) is improved as follows,

$$\Omega^2 \int_{R^d} |x|^2 \tilde{\eta}_j^2 |v|^2 \, dx \leq 4c_\varepsilon \Omega^2 + O \left( \varepsilon \Omega^{3/2} \sqrt{\ln \frac{1}{\varepsilon \sqrt{\Omega}}} + (\varepsilon \Omega)^{3/2} \Omega^2 \right) + o(1) .$$

(5.5)

Bootstrapping this argument leads to, if $p \in \mathbb{N}$, then there exists a constant $C_p > 0$ such that,

$$\Omega^2 \int_{R^d} |x|^2 \tilde{\eta}_j^2 |v|^2 \, dx \leq 4c_\varepsilon \Omega^2 + C_p \left( \varepsilon \Omega^{3/2} \sqrt{\ln \frac{1}{\varepsilon \sqrt{\Omega}}} + (\varepsilon \Omega)^{(2p+1)/2} \right) + o(1).$$

(5.6)

Recall that the assumption on $\Omega$ implies that $\varepsilon \Omega^{3/2} = O(1)$ and $\sqrt{\ln \frac{1}{\varepsilon \sqrt{\Omega}}} \gg 1$. Also, selecting $p$ sufficiently large gives that $(\varepsilon \Omega)^{(2p+1)/2} \ll 1$. Therefore, it is inferred from (5.6),

$$\Omega^2 \int_{R^d} |x|^2 \tilde{\eta}_j^2 |v|^2 \, dx \leq 4c_\varepsilon \Omega^2 + \left( \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \right) o(1) \quad \text{as} \quad \varepsilon \to 0_+ .$$

(5.7)

Lower bound of the ‘Ginzburg-Landau’ energy: Consider an arbitrary constant $L \in (0, a_0)$. Recall the lattice of squares $K_j$ introduced in Section 4. Put

$$\mathcal{J} = \{ j : K_j \subset D_L \} .$$

(5.8)

There holds the obvious lower bound,

$$\int_{R^d} \left( \tilde{\eta}_j^2 |(\nabla - i\varepsilon \mathbf{A}_0)v|^2 + \frac{\tilde{\eta}_j^4}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx \geq \int_{K_j} \left( \tilde{\eta}_j^2 |(\nabla - i\varepsilon \mathbf{A}_0)v|^2 + \frac{\tilde{\eta}_j^4}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx \geq \sum_{j \in \mathcal{J}} \int_{K_j} \left( \tilde{\eta}_j^2 |(\nabla - i\varepsilon \mathbf{A}_0)v|^2 + \frac{\tilde{\eta}_j^4}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx .$$

(5.9)

For each $j \in \mathcal{J}$, it will be obtained a lower bound of the term,

$$G_\varepsilon(v, K_j) = \int_{K_j} \left( \tilde{\eta}_j^2 |(\nabla - i\varepsilon \mathbf{A}_0)v|^2 + \frac{\tilde{\eta}_j^4}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx .$$

(5.10)

By Theorem 2.1, one can write for an arbitrary point $x_j$ in $K_j$,

$$G_\varepsilon(v, K_j) \geq (1 - C\varepsilon^2 |\ln \varepsilon|) \int_{K_j} \left( a(x_j) |(\nabla - i\varepsilon \mathbf{A}_0)v|^2 + \frac{a(x_j)^2}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx .$$

(5.11)

Let $y_j$ be the center of the square $K_j$, $K = (-1/2, 1/2)^2$, $\varepsilon = \ell \sqrt{\Omega} \varepsilon$ and $h_{ex} = 1/\ell^2$. Using the re-scaled function $f(x) = v(y_j + \ell \sqrt{\Omega} x)$, $(x \in K)$, it is possible to express (5.11) in the following form,

$$G_\varepsilon(v, K_j) \geq (1 - C\varepsilon^2 |\ln \varepsilon|) a(x_j) \int_{K} \left( |(\nabla - ih_{ex} \mathbf{A}_0)f|^2 + \frac{a(x_j)^2}{2\varepsilon^2} (1 - |f|^2)^2 \right) \, dx .$$

(5.12)
Notice that the term $a(x_j)$ remains in a constant interval $[\lambda_1, \lambda_2]$ as $j$ and $\varepsilon$ vary. Also, as $\varepsilon \to 0$, $\breve{\varepsilon}$ and $h_{\text{ex}}$ satisfy $|\ln \breve{\varepsilon}| \ll h_{\text{ex}} \ll \breve{\varepsilon}^{-2}$. Thus, it is possible to bound the integral on the right side of (5.12) by the ground state energy $m_0(h_{\text{ex}}, \breve{\varepsilon})$ in (3.5), which is estimated from below in Theorem 3.1. Therefore, it is inferred from (5.12),

$$
G_{\varepsilon}(v, K_j) \geq (1 + o(1)) a(x_j) h_{\text{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}} = (1 + o(1)) a(x_j) \frac{1}{L^2} \ln \frac{1}{\varepsilon \sqrt{\Omega}}. \tag{5.13}
$$

Inserting this into (5.10) and then into (5.9) yields,

$$
\int_{\mathbb{R}^2} \left( \frac{\eta_{\varepsilon}^2}{2} |(\nabla - i\Omega A_0)v|^2 + \frac{\eta_{\varepsilon}^2}{2\varepsilon^2} (1 - |v|^2) \right) dx \geq (1 + o(1)) \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \sum_{j \in J'} \frac{1}{L^2 \Omega} a(x_j). \tag{5.14}
$$

The sum on the right side of (5.14) is estimated as follows. As $\varepsilon \to 0_+$, the term $\sum_{j \in J'} \frac{1}{L^2 \Omega} a(x_j)$ is a Riemann sum converging to $\int_{D_L} a(x) dx$. As a consequence, there holds,

$$
\sum_{j \in J'} a(x_j) h_{\text{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\text{ex}}}} = \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \sum_{j \in J'} \frac{1}{L^2 \Omega} a(x_j)
= \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \left( \int_{D_L} a(x) dx + o(1) \right) \quad \text{as } \varepsilon \to 0_+.
$$

Therefore, it results from (5.14),

$$
\int_{\mathbb{R}^2} \left( \frac{\eta_{\varepsilon}^2}{2} |(\nabla - i\Omega A_0)v|^2 + \frac{\eta_{\varepsilon}^2}{2\varepsilon^2} (1 - |v|^2) \right) dx \geq \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \left( \int_{D_L} a(x) dx + o(1) \right). \tag{5.15}
$$

**Conclusion:** It is obtained by collecting the estimates in (5.3) and (5.15),

$$
C_0(\varepsilon, \Omega) \geq -c_\varepsilon \Omega^2 + \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \left( \int_{D_L} a(x) dx + o(1) \right).
$$

As a consequence, it is obtained by taking the limit as $\varepsilon \to 0_+$,

$$
\liminf_{\varepsilon \to 0_+} \frac{C_0(\varepsilon, \Omega) - c_\varepsilon \Omega^2}{\Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}}} \geq \int_{D_L} a(x) dx.
$$

By Taking $L \to (a_0)_-$, it results the lower bound:

$$
\liminf_{\varepsilon \to 0_+} \frac{C_0(\varepsilon, \Omega) - c_\varepsilon \Omega^2}{\Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}}} \geq \int_D a(x) dx = 1.
$$

The conclusion of this section and Section 4 finishes the proof of Theorem 1.1.

**Remark 5.1.** If $U \subset D$ and $u \in H^1(U)$, define the local energy:

$$
\mathcal{E}_\varepsilon(u; U) = \int_U \left( \frac{\eta_{\varepsilon}^2}{2} |(\nabla - i\Omega A_0)v|^2 + \frac{\eta_{\varepsilon}^4}{2\varepsilon^2} (1 - |v|^2)^2 \right) dx.
$$

The analysis of this section allows one to prove the following. If $v$ is a minimizer of (1.10), $U \subset D$ is open and $|\partial U| = 0$ then,

$$
\mathcal{E}_\varepsilon(v; U) \geq \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \left( \int_U a(x) dx + o(1) \right) \quad \text{as } \varepsilon \to 0_+.
$$

Combine this lower bound with (5.7) and the upper bound (4.5) to obtain the ‘local’ energy asymptotics:

$$
\mathcal{E}_\varepsilon(v; U) = \Omega \ln \frac{1}{\varepsilon \sqrt{\Omega}} \left( \int_U a(x) dx + o(1) \right) \quad \text{as } \varepsilon \to 0_+.
$$
6. Vortices and their density

The assumption on the rotational speed is as in Theorem 1.1. Recall the definition of the domain \( D \) in (1.3). Suppose that \( U \) is an open set in \( \mathbb{R}^2 \) and \( U \subset D \). According to Theorem 2.1, the function \( \tilde{\eta}_\varepsilon \) satisfies the pointwise bound \( \tilde{\eta}_\varepsilon \geq \eta_0(U) > 0 \) in \( U \). The constant \( \eta_0(U) \) depends only on \( U \). As a consequence, if \( v \) is a minimizer of (1.10), then

\[
\int_U \left( (\nabla - i\Omega A_0) v \right)^2 + \frac{1}{\varepsilon^2} (1 - |v|^2)^2 \, dx \leq C \varepsilon^\alpha \tag{6.1}
\]

with \( \alpha \in (2/3, 1) \). This bound follows from the indications in Remark 5.1 and the aforementioned lower bound of \( \tilde{\eta}_\varepsilon \) in \( U \). By borrowing the results of \([15, 16]\), it will be given some details regarding the location and ‘density’ of the zeros of the minimizer \( v \) inside \( U \).

The first step is to notice that (6.1) together with the fact that \( |v| \) is bounded in \( U \) (see Remark 2.3) allows one to apply Theorem 4.1 in \([15]\). That way, there exists a family of disjoint closed balls \( (B(a_i, r_i)) \) satisfying:

1. \( \sum_i r_i = \varepsilon^\alpha/4 \);
2. \( \{ x \in U \mid |v(x)| < 1 - \varepsilon^\alpha/2 \} \subset \bigcup_i B(a_i, r_i) \);
3. If \( d_i \) is the winding number of \( v/|v| \) when \( B(a_i, r_i) \subset U \) and 0 otherwise, then,

\[
\int_{U \setminus \bigcup_i B(a_i, r_i)} \left( |(\nabla - i\Omega A_0) v|^2 + \frac{1}{\varepsilon^2} (1 - |v|^2)^2 \right) \, dx \geq 2\pi D \left( \frac{\ln \varepsilon^\alpha/4}{D \varepsilon} - C \right),
\]

where \( D = \sum_i |d_i| \) and \( C \) a universal constant.

This shows that if \( v \) has zeros (vortices), then these should live inside the balls \( B(a_i, r_i) \). Since \( v = u/\tilde{\eta}_\varepsilon \) and \( u \) is a minimizer of (1.4), then this gives immediately the location of the zeros of \( u \).

Define the measure

\[
\mu_\varepsilon = \sum_i d_i \delta_{a_i}, \tag{6.2}
\]

where \( \delta_{a_i} \) is the dirac measure supported at \( a_i \). The measure \( \mu_\varepsilon \) is called the vorticity measure in \( U \): It indicates the existence of vortices (when \( \mu_\varepsilon \neq 0 \)), its support indicates the location of vortices, and its norm indicates their density.

It is possible to prove that:

**Theorem 6.1.** Under the assumption of Theorem 1.1, the vorticity measure in \( U \) fulfills the weak convergence:

\[
\frac{1}{|\Omega|} \mu_\varepsilon \rightharpoonup 1_U \, dx \quad \text{as } \varepsilon \to 0_+,
\]

where \( dx \) is the Lebesgue measure in \( U \) and \( 1_U \) the characteristic function of \( U \).

**Proof.** Notice that the lower bound in (3) and the asymptotics in Remark 5.1 together yield that \( \Omega^{-1} \sum_i |d_i| \) is bounded independently of \( \varepsilon \) and \( \Omega \). Consequently, by passing to a subsequence, one can suppose that \( \Omega^{-1} \mu_\varepsilon \) converges weakly to a measure \( \mu \). It suffices to prove that \( \mu = 1_U \, dx \).

Consider the lattice of squares \( (K_j) \) generated by the square \( \delta = (-\delta, \delta) \times (-\delta, \delta) \), where \( \delta = \frac{1}{2} \left( \ln |\varepsilon|/|\Omega| \right)^{-1/4} \). Suppose that \( x_j \) is the center of the square \( K_j \). One distinguishes between good squares and bad squares in \( U \): good squares are those satisfying that

\[
\text{GL}_\varepsilon(v; K_j) := \int_{K_j} \left( |(\nabla - i\Omega A_0) v|^2 + \frac{\eta_0^2(x_j)}{2\varepsilon^2} (1 - |v|^2)^2 \right) \, dx \leq (1 + o(1)) \Omega \delta^2 \ln \frac{1}{\varepsilon \sqrt{|\Omega|}},
\]

while bad squares satisfy the reverse condition that \( \text{GL}_\varepsilon(v; K_j) - \Omega \delta^2 \ln \frac{1}{\varepsilon \sqrt{|\Omega|}} \gg h_{ex} \ln \frac{1}{\varepsilon \sqrt{|\Omega|}} \). The number of bad squares \( N_b \) is small compared to the number of good squares \( N_g \), namely \( N_b \ll N_g \).
as \( \varepsilon \to 0_+ \). An easy adjustment of Proposition 5.1 in [16] allows one to prove the following. If \( K_j \) is a good square and \( J_{j,g} = \{ i : B(a_i, r_i) \subset K_j \} \), then

\[
\text{GL}_\varepsilon(v; K_j) \geq (1 + o(1)) \Omega \delta^2 \ln \frac{1}{\varepsilon \sqrt{\Omega}}, \quad \sum_{i \in J_{j,g}} d_i \geq \Omega \delta^2 (1 + o(1)), \quad \text{and}
\]

\[
\sum_{i \in J_{j,g}} |d_i| \leq \Omega \delta^2 (1 - o(1)) \quad \text{as} \quad \varepsilon \to 0_+.
\]

Since the number of good squares satisfies \( N_g \times \delta^2 = |U| + o(1) \) as \( \varepsilon \to 0_+ \), then the above two-sided estimate of \( \sum d_i \) leads to the following. If \( S \) is an open set in \( U \) and \( |\partial S| = 0 \), then

\[
\Omega |S|(1 + o(1)) \leq \sum_{i \in \bigcup J_{j,g}} d_i \leq (1 + o(1)) \mu_c(S)
\]

\[
\leq (1 + o(1)) \sum_{j \in \bigcup J_{j,g}} |d_j| \leq \Omega |S|(1 + o(1)), \quad \text{as} \quad \varepsilon \to 0_+.
\]

This proves that \( \Omega^{-1} \mu_c \) converges weakly to the Lebesgue measure restricted to \( U \). \( \square \)

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