Scale Invariance, New Inflation and Decaying Λ-Terms

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Abstract

Realizations of scale invariance are studied in the context of a gravitational theory where the action (in the first order formalism) is of the form

\[ S = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x \]

where \( \Phi \) is a density built out of degrees of freedom, the "measure fields" independent of \( g_{\mu\nu} \) and matter fields appearing in \( L_1, L_2 \). If \( L_1 \) contains the curvature, scalar potential \( V(\phi) \) and kinetic term for \( \phi \), \( L_2 \) another potential for \( \phi \), \( U(\phi) \), then the true vacuum state has zero energy density, when theory is analyzed in the conformal Einstein frame (CEF), where the equations assume the Einstein form. Global Scale invariance is realized when \( V(\phi) = f_1 e^{\alpha \phi} \) and \( U(\phi) = f_2 e^{2\alpha \phi} \). In the CEF the scalar field potential energy \( V_{\text{eff}}(\phi) \) has in, addition to a minimum at zero, a flat region for \( \alpha \phi \to \infty \), with non zero vacuum energy, which is suitable for either a New Inflationary scenario for the Early Universe or for a slowly rolling decaying Λ-scenario for the late universe, where the smallness of the vacuum energy can be understood as a kind of see-saw mechanism.

1 Introduction

Recent developments in cosmology have been influenced to a great extent by the idea of inflation\(^1\), which provides an attractive scenario for solving some
of the fundamental puzzles of the standard Big Bang model, like the horizon and the flatness problems as well as providing a framework for sensible calculations of primordial density perturbations.

However, although the inflationary scenario is very attractive, it has been recognized that a successful implementation requires some very special restrictions on the dynamics that drive inflation. In particular, in New Inflation\(^2\), a potential with a large flat region, which then drops to zero (or almost zero) in order to reproduce the vacuum with almost zero (in Planck units) cosmological constant of the present universe, is required. It is hard to find a theory that gives a potential of this type naturally.

In addition to this, it is worthwhile pointing out that a potential with a very flat region, slowly approaching zero could be of use as a model for a decaying cosmological constant being considered as a model for the "accelerating universe", now preferred by observations\(^3\). This is of course, at a totally different scale to that of Inflation.

Here we want to see whether such shapes of potentials can be obtained from first principles, i.e. whether there is some fundamental principle that produces this type of behavior for a scalar field.

We find indeed that this is possible and the fundamental principle in question is none other than scale invariance. However, scale invariance has to be discussed in a more general framework than that of the standard Lagrangian formulation of generally relativistic theories. Before going into the question of scale invariance it is necessary therefore to discuss first the general framework where this discussion will be set.

### 2 The Non Gravitating Vacuum Energy (NGVE) Theory. Strong and Weak Formulations.

When formulating generally covariant Lagrangian formulations of gravitational theories, we usually consider the form

\[ S_1 = \int L \sqrt{-g} d^4x, \quad g = \text{det}g_{\mu\nu} \]

As it is well known, \(d^4x\) is not a scalar but the combination \(\sqrt{-g} d^4x\) is a scalar. Inserting \(\sqrt{-g}\), which has the transformation properties of a density, produces a scalar action (1), provided \(L\) is a scalar.
One could use nevertheless other objects instead of $\sqrt{-g}$, provided they have the same transformation properties and achieve in this way a different generally covariant formulation.

For example, given 4-scalars $\varphi_a$ ($a = 1,2,3,4$), one can construct the density

$$\Phi = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\mu \varphi_a \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d$$

and consider instead of $(1)$ the action

$$S_2 = \int L \Phi d^4x.$$  \hspace{1cm} (3)

$L$ is again some scalar, which contains the curvature (i.e. the gravitational contribution) and a matter contribution, as is standard also in $(1)$.

In the action $(3)$ the measure carries degrees of freedom independent of that of the metric and that of the matter fields. The most natural and successful formulation of the theory is achieved when the connection is also treated as an independent degree of freedom. This is what is usually referred to as the first order formalism.

One can notice that $\Phi$ is the total derivative of something, for example, one can write

$$\Phi = \partial_\mu (\varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d).$$

This means that a shift of the form

$$L \rightarrow L + \text{constant}$$

just adds the integral of a total divergence to the action $(3)$ and it does not affect therefore the equations of motion of the theory. The same shift, acting on $(1)$ produces an additional term which gives rise to a cosmological constant. Since the constant part of $L$ does not affect the equations of motion, this theory is called the Non Gravitating Vacuum Energy (NGVE) Theory\(^4\).

One can generalize this structure and allow both geometrical objects to enter the theory and consider

$$S_3 = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x.$$  \hspace{1cm} (6)

Now instead of $(5)$, the shift symmetry can be applied only on $L_1$ ($L_1 \rightarrow L_1 + \text{constant}$). Since the structure has been generalized, we call this formulation the weak version of the NGVE - theory. Here $L_1$ and $L_2$ are $\varphi_a$ independent.
There is a good reason not to consider mixing of $\Phi$ and $\sqrt{-g}$, like for example using

$$\frac{\Phi^2}{\sqrt{-g}}$$

(7)

This is because (6) is invariant (up to the integral of a total divergence) under the infinite dimensional symmetry

$$\varphi_a \to \varphi_a + f_a(L_1)$$

(8)

where $f_a(L_1)$ is an arbitrary function of $L_1$ if $L_1$ and $L_2$ are $\varphi_a$ independent. Such symmetry (up to the integral of a total divergence) is absent if mixed terms (like (7)) are present. Therefore (6) is considered for the case when no dependence on the measure fields (MF) appears in $L_1$ or $L_2$.

In this paper we will see that the existence of two independent measures of integrations as in (6) allows new realizations of global scale invariance with most interesting consequences when the results are viewed from the point of view of cosmology.

3 Dynamics of a Scalar Field and the Requirement of Scale Invariance in the Weak NGVE - Theory

4 The Action Principle

We will study now the dynamics of a scalar field $\phi$ interacting with gravity as given by the following action

$$S_\phi = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x$$

(9)

$$L_1 = -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

(10)

$$L_2 = U(\phi)$$

(11)

$$R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma), R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda}$$

(12)

$$R^\lambda_{\mu\nu\sigma}(\Gamma) = \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu,\sigma,\nu} + \Gamma^\lambda_{\alpha\sigma,\mu} \Gamma^\alpha_{\mu\nu} - \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\sigma}.$$
In the variational principle $\Gamma_{\mu\nu}^\lambda, g_{\mu\nu}$, the measure fields scalars $\varphi_a$ and the "matter" - scalar field $\phi$ are all to be treated as independent variables although the variational principle may result in equations that allow us to solve some of these variables in terms of others.

5 Global Scale Invariance

If we perform the global scale transformation ($\theta = \text{constant}$)

$$g_{\mu\nu} \to e^\theta g_{\mu\nu} \quad (14)$$

then (9) is invariant provided $V(\phi)$ and $U(\phi)$ are of the form

$$V(\phi) = f_1 e^{\alpha \phi}, U(\phi) = f_2 e^{2\alpha \phi} \quad (15)$$

and $\varphi_a$ is transformed according to

$$\varphi_a \to \lambda_a \varphi_a \quad (16)$$

(no sum on a) which means

$$\Phi \to \left(\prod_a \lambda_a\right) \Phi \equiv \lambda \Phi \quad (17)$$

such that

$$\lambda = e^\theta \quad (18)$$

and

$$\phi \to \phi - \frac{\theta}{\alpha}. \quad (19)$$

6 The Equations of Motion

We will now work out the equations of motion for arbitrary choice of $V(\phi)$ and $U(\phi)$. We study afterwards the choice (15) which allows us to obtain the results for the scale invariant case and also to see what differentiates this from the choice of arbitrary $U\phi$ and $V\phi$ in a very special way.

Let us begin by considering the equations which are obtained from the variation of the fields that appear in the measure, i.e. the $\varphi_a$ fields. We obtain then

$$A_\alpha^a \partial_\mu L_1 = O \quad (20)$$
where \( A^a_\mu = \varepsilon^{\mu
u\alpha\beta} \varepsilon_{abcd} \partial_\nu \phi \partial_\alpha \phi \partial_\beta \phi \partial_\delta \phi \) (21). Since it is easy to check that 
\[ A^\mu_\mu \partial_\mu \phi \partial_\nu \phi = \frac{\delta \Phi}{4!}, \]
it follows that 
\[ \text{det}(A^\mu_\mu) = 4! \Phi, \]
if \( \Phi \neq O \). Therefore if \( \Phi \neq O \) we obtain that 
\[ L_1 = -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu
u} \partial_\mu \phi \partial_\nu \phi - V = M \] (21)
where \( M \) is constant.

Let us study now the equations obtained from the variation of the connections \( \Gamma^\lambda_\mu \). We obtain then
\[ -\Gamma^\lambda_\mu - \Gamma^\alpha_\beta \mu g^{\beta \lambda} g_{\alpha \nu} + \delta^\lambda_\nu \Gamma^\alpha_\mu + \delta^\mu_\alpha \Gamma^\gamma_\alpha \beta g_{\gamma \nu} - g_{\alpha \nu} \partial_\mu g^{\alpha \lambda} + \delta^\lambda_\nu \Phi - \Phi \] (22)
If we define \( \Sigma^\lambda_\mu \) as \( \Sigma^\lambda_\mu = \Gamma^\lambda_\mu - \{^\lambda_\mu \} \) where \( \{^\lambda_\mu \} \) is the Christoffel symbol, we obtain for \( \Sigma^\lambda_\mu \) the equation
\[ -\sigma_\lambda g_{\mu \nu} + \sigma_\mu g_{\nu \lambda} - g_{\nu \alpha} \Sigma^\alpha_\lambda + g_{\mu \nu} \Sigma^\alpha_\lambda + g_{\nu \lambda} g_{\alpha \mu} g^{\beta \gamma} \Sigma^\alpha_\beta \gamma = O \] (23)
where \( \sigma = 1/n, \chi = \sqrt{-g} \).

The general solution of (23) is
\[ \Sigma^\alpha_\mu = \delta^\alpha_\mu \lambda_\nu + \frac{1}{2} (\sigma_\mu \delta_\nu \sigma_\mu - \sigma_\nu \sigma_\mu g^{\alpha \beta}) \] (24)
where \( \lambda \) is an arbitrary function due to the \( \lambda \)-symmetry of the curvature (5)
\[ R^\lambda_\mu \nu \alpha (\Gamma), \]
\[ \Gamma^\alpha_\mu \nu \rightarrow \Gamma^\alpha_\mu \nu = \Gamma^\alpha_\mu \nu + \delta^\alpha_\mu Z_\nu \] (25)
\( Z \) being any scalar (which means \( \lambda \rightarrow \lambda + Z \)).

If we choose the gauge \( \lambda = \frac{2}{\sqrt{-g}} \), we obtain
\[ \Sigma^\alpha_\mu (\sigma) = \frac{1}{2} (\delta^\alpha_\mu \sigma_\nu + \delta^\alpha_\nu \sigma_\mu - \sigma_\beta g_{\mu \nu} g^{\alpha \beta}). \] (26)

Considering now the variation with respect to \( g^{\mu \nu} \), we obtain
\[ \Phi = (\frac{-1}{\kappa} R_{\mu \nu} (\Gamma) + \frac{1}{2} \phi_{\mu} \phi_{\nu}) - \frac{1}{2} \sqrt{-g} U(\phi) g_{\mu \nu} = O \] (27)
solving for \( R = g^{\mu \nu} R_{\mu \nu} (\Gamma) \) and introducing in (22), we obtain
\[ M + V(\phi) - \frac{2U(\phi)}{\chi} = O \] (28)
a constraint that allows us to solve for $\chi$,

$$
\chi = \frac{2U(\phi)}{M + V(\phi)}.
$$

(29)

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

$$
\bar{g}_{\mu\nu} = \chi g_{\mu\nu}
$$

(30)

and $\chi$ given by (29b). In terms of $\bar{g}_{\mu\nu}$ the non Riemannian contribution $\Sigma^\alpha_{\mu\nu}$ dissappears from the equations, which can be written then in the Einstein form ($R_{\mu\nu}(\bar{g}_{\alpha\beta}) = $ usual Ricci tensor)

$$
R_{\mu\nu}(\bar{g}_{\alpha\beta}) - \frac{1}{2} \bar{g}_{\mu\nu} R(\bar{g}_{\alpha\beta}) = \frac{\kappa}{2} T^{\text{eff}}_{\mu\nu}(\phi)
$$

(31)

where

$$
T^{\text{eff}}_{\mu\nu}(\phi) = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \phi_{,\alpha} \phi_{,\beta} \bar{g}^{\alpha\beta} + \bar{g}_{\mu\nu} V^{\text{eff}}(\phi)
$$

(32)

and

$$
V^{\text{eff}}(\phi) = \frac{1}{4U(\phi)}(V + M)^2.
$$

(33)

In terms of the metric $\bar{g}^{\alpha\beta}$, the equation of motion of the Scalar field $\phi$ takes the standard General - Relativity form

$$
\frac{1}{\sqrt{-\bar{g}}} \partial_\mu (\bar{g}^{\mu\nu} \sqrt{-\bar{g}} \partial_\nu \phi) + V^{\prime}_{\text{eff}}(\phi) = 0.
$$

(34)

Notice that if $V + M = O$, $V^{\text{eff}} = O$ and $V^{\prime}_{\text{eff}} = O$ also, provided $V^{\prime}$ is finite and $U \neq O$ there. This means the zero cosmological constant state is achieved without any sort of fine tuning. This is the basic feature that characterizes the NGVE - theory and allows it to solve the cosmological constant problem.

In what follows we will study (33) for the special case of global scale invariance, which as we will see displays additional very special features which makes it attractive in the context of cosmology.

Notice that in terms of the variables $\phi, \bar{g}_{\mu\nu}$, the ”scale” transformation becomes only a shift in the scalar field $\phi$, since $\bar{g}_{\mu\nu}$ is invariant (since $\chi \rightarrow \lambda^{-1}\chi$ and $g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$)

$$
\bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}, \phi \rightarrow \phi - \frac{\theta}{\alpha}.
$$

(35)
7 Analysis of the Scale - Invariant Dynamics

If $V(\phi) = f_1 e^{\alpha \phi}$ and $U(\phi) = f_2 e^{2\alpha \phi}$ as required by scale invariance (14), (16), (17), (18), (19), we obtain from (33)

$$V_{\text{eff}} = \frac{1}{4f_2} (f_1 + Me^{-\alpha \phi})^2$$

(36)

Since we can always perform the transformation $\phi \rightarrow -\phi$ we can choose by convention $\alpha > 0$. We then see that as $\phi \rightarrow \infty$, $V_{\text{eff}} \rightarrow \frac{f_1^2}{4f_2} = \text{const.}$ providing an infinite flat region. Also a minimum is achieved at zero cosmological constant for the case $\frac{f_1}{M} < 0$ at the point

$$\phi_{\text{min}} = \frac{-1}{\alpha} \ln \left| \frac{f_1}{M} \right| .$$

(37)

Finally, the second derivative of the potential $V_{\text{eff}}$ at the minimum is

$$V''_{\text{eff}} = \frac{\alpha^2}{2f_2} | f_1 |^2 > 0$$

(38)

if $f_2 > 0$, there are many interesting issues that one can raise here. The first one is of course the fact that a realistic scalar field potential, with massive exitations when considering the true vacuum state, is achieved in a way consistent with the idea (although somewhat generalized) of scale invariance.

The second point to be raised is that there is an infinite region of flat potential for $\phi \rightarrow \infty$, which makes this theory an attractive realization of the improved inflationary model\(^2\).

A peculiar feature of the potential (36), is that the constant $M$, provided it has the correct sign, i.e. that $f_1/M < 0$, does not affect the physics of the problem. This is because if we perform a shift

$$\phi \rightarrow \phi + \Delta$$

(39)

in the potential (36), this is equivalent to the change in the integration constant $M$

$$M \rightarrow Me^{-\alpha \Delta}.$$  

(40)

We see therefore that if we change $M$ in any way, without changing the sign of $M$, the only effect this has is to shift the whole potential. The physics
of the potential remains unchanged, however. This is reminiscent of the
dilatation invariance of the theory, which involves only a shift in \( \phi \) if \( \mathcal{T}_{\mu \nu} \) is
used (see eq. (35)).

This is very different from the situation for two generic functions \( U(\phi) \)
and \( V(\phi) \) in (34). There, \( M \) appears in \( V_{\text{eff}} \) as a true new parameter that
generically changes the shape of the potential \( V_{\text{eff}} \), i.e. it is impossible then
to compensate the effect of \( M \) with just a shift. For example \( M \) will appear
in the value of the second derivative of the potential at the minimum, unlike
what we see in eq. (38), where we see that \( V''_{\text{eff}}(\text{min}) \) is \( M \) independent.

In conclusion, the scale invariance of the original theory is responsible
for the non appearance (in the physics) of a certain scale, that associated to
\( M \). However, masses do appear, since the coupling to two different measures
of \( L_1 \) and \( L_2 \) allow us to introduce two independent couplings \( f_1 \) and \( f_2 \), a
situation which is unlike the standard formulation of globally scale invariant
theories, where usually no stable vacuum state exists.

Notice that we have not considered all possible terms consistent with
global scale invariance. Additional terms in \( L_2 \) of the form \( e^{\alpha \phi} R \)
and \( e^{\alpha \phi} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \) are indeed consistent with the global scale invariance (14), (16), (17), (18),
(19) but they give rise to a much more complicated theory, which will be studied
in a separate publication. There it will be shown that for slow rolling
and for \( \phi \to \infty \) the basic features of the theory are the same as what has
been studied here. Let us finish this section by comparing the appearance of the
potential \( V_{\text{eff}}(\phi) \), which has privileged some point depending on \( M \) (for
example the minimum of the potential will have to be at some specific point),
although the theory has the “translation invariance” (35), to the physics of
solitons.

In fact, this very much resembles the appearance of solitons in a space-
translation invariant theory: The soliton solution has to be centered at some
point, which of course is not determined by the theory. The soliton of course
breaks the space translation invariance spontaneously, just as the existence of
the non trivial potential \( V_{\text{eff}}(\phi) \) breaks here spontaneously the translations
in \( \phi \) space, since \( V_{\text{eff}}(\phi) \) is not a constant.

Notice however, that the existence for \( \phi \to \infty \), of a flat region for \( V_{\text{eff}}(\phi) \)
can be nicely described as a region where the symmetry under translations
(35) is restored.
8 Cosmological Applications of the Model

Since we have an infinite region in which $V_{eff}$ as given by (36) is flat ($\phi \to \infty$), we expect a slow rolling (new inflationary) scenario to be viable, provided the universe is started at a sufficiently large value of the scalar field $\phi$.

One should point out that the model discussed here gives a potential with two physically relevant parameters $\frac{f_2^2}{4f_2}$, which represents the value of $V_{eff}$ as $\phi \to \infty$, i.e. the strength of the false vacuum at the flat region and $\frac{\alpha^2 f_1^4}{2f_2}$, representing the mass of the excitations around the true vacuum with zero cosmological constant (achieved here without fine tuning).

When a realistic model of reheating is considered, one has to give the strength of the coupling of the $\phi$ field to other fields. It remains to be seen what region of parameter space provides us with a realistic cosmological model.

Furthermore, one can consider this model as suitable for the very late universe rather than for the early universe, after we suitably reinterpret the meaning of the scalar field $\phi$.

This can provide a long lived almost constant vacuum energy for a long period of time, which can be small if $f_1^2/4f_2$ is small. Such small energy density will eventually disappear when the universe achieves its true vacuum state.

Notice that a small value of $\frac{f_2^4}{f_2}$ can be achieved if we let $f_2 >> f_1$. In this case $\frac{f_2^4}{f_2} << f_1$, i.e. a very small scale for the energy density of the universe is obtained by the existence of a very high scale (that of $f_2$) the same way as a small fermion mass is obtained in the see-saw mechanism from the existence also of a large mass scale.

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