PENCILS OF SMALL DEGREE ON CURVES ON UNNODAL ENRIQUES SURFACES

NILS HENRY RASMUSSEN AND SHENGTIAN ZHOU

Abstract. We use vector-bundle techniques in order to compute \( \dim W^1_d(C) \) where \( C \) is general and smooth in a linear system on an unnodal Enriques surface. We furthermore find new examples of smooth curves on Enriques surfaces with an infinite number of \( g_{\text{gon}}(C) \)'s.

1. Introduction

Let \( S \) be a smooth surface over \( \mathbb{C} \), and \( L \) a line-bundle on \( S \). Let \( W^r_d(C) \) be the Brill–Noether variety, parametrising complete \( g^r_s \)'s on \( C \) for \( s \geq r \). We will be concerned with finding the dimension of \( W^1_d(C) \) for small \( d \) when \( S \) is an unnodal Enriques surface.

The theory on the dimension of \( W^r_d(C) \) dates back to 1874, when Alexander von Brill and Max Noether made an incomplete proof stating that \( \dim W^r_d(C) = \rho(g,r,d) := g - (r + 1)(g - d + r) \) provided \( C \) is general of genus \( g \). It was first much later that strict proofs for this were presented ([KL72], [Kem71], [GH80]). In 1987, a new proof was constructed by Lazarsfeld ([Laz86]) involving use of vector-bundle techniques for curves on K3 surfaces, exploiting the fact that for any \( g \geq 2 \), a K3 surface with Picard group \( \mathbb{Z} \) with \( C \) a smooth genus \( g \) curve can be constructed. These vector-bundle techniques, which were also developed by Tuyting ([Tyu87]), were later used to study the gonality and Clifford index of any smooth curve on an arbitrary K3 surface ([CP95], [Knu03], [Knu09b], [AFL1]). These methods have also lately been applied in the case of Enriques surfaces and rational surfaces with an anticanonical pencil ([Knu01], [KL09], [Knu09a], [LC12]).

The dimension of \( W^1_d(C) \) was studied in [AFL1] and [LC12] because of a result by Aprodu in 2005 ([Apr05]), stating that if \( \dim W^1_d(C) = d - \text{gon}(C) \) for \( d \leq g - \text{gon}(C) + 2 \), then the Green and Green–Lazarsfeld conjectures are satisfied. These conjectures state that the Clifford index and gonality can be read off minimal free resolutions of \( \bigoplus_n H^0(C, \mathcal{O}_C(nK_C)) \) and \( \bigoplus_n H^0(C, \mathcal{O}_C(nA)) \) for \( \text{deg}(A) \gg 0 \), respectively (see [Gre84] and [GL87]).

In this article, we make an attempt at finding the dimension of \( W^1_d(C) \) when \( C \) is a smooth curve on an unnodal Enriques surface \( S \). A smooth surface over \( \mathbb{C} \) is an Enriques surface if \( h^1(S, \mathcal{O}_S) = 0 \), \( 2K_S \sim 0 \) and \( K_S \sim 0 \). One defines

\[
\phi(L) := \min \{ L.E \mid E \in \text{Pic}(S), E^2 = 0, E \neq 0 \}
\]

and

\[
\mu(L) := \min \{ L.B - 2 \mid B \in \text{Pic}(S) \text{ with } B \text{ effective}, B^2 = 4, \phi(B) = 2, \text{ and } B \neq L \}.
\]

1991 Mathematics Subject Classification. 14H51, 14J28, 14J60.
By [KL09], the generic gonality for smooth curves in $|L|$, which we denote by $k$, is given by

$$k = \min \left\{ 2\phi(L), \mu(L), \left\lfloor \frac{L^2}{4} \right\rfloor + 2 \right\}.$$ 

Furthermore, $k = \mu(L) < 2\phi(L)$ precisely when:

- $L^2 = \phi(L)^2$ with $\phi(L) \geq 2$ and even, in which case $k = \mu(L) = 2\phi(L) - 2$; or
- $L^2 = \phi(L)^2 + \phi(L) - 2$ with $\phi(L) \geq 3$, $L \neq 2D$ for $D$ such that $D^2 = 10$, $\phi(D) = 3$, in which case $k = 2\phi(L) - 1$ for $\phi(L) \geq 5$ and $k = 2\phi(L) - 2$ for $\phi(L) = 3, 4$.

If $(L^2, \phi(L)) = (30, 5), (22, 4), (20, 4), (14, 3), (12, 3)$ or $(6, 2)$, then $k = \left\lfloor \frac{L^2}{4} \right\rfloor + 2 = \phi(L) - 1$.

In all other cases, $k = 2\phi(L)$.

Our main result is the following:

**Theorem 1.1.** Let $S$ be an unnodal Enriques surface, and let $|L|$ be an ample linear system with $L^2 \geq 2$ such that $k = 2\phi(L) < \mu(L)$. Then, for $k \leq d \leq g - k$ and $C$ general in $|L|$, 

$$\dim W^0_d(C) = d - k.$$ 

**Remark 1.2.** In the case where $L = n(E_1 + E_2)$ for $n \geq 3$ and $E_1.E_2 = 2$, we have $k = \mu(L) < 2\phi(L)$, by [KL09] Corollary 1.5 (a)]. In Example 5.1 we prove that there exists a sub-linear system $\mathfrak{d} \subseteq |L|$ of smooth curves such that for general $C \in \mathfrak{d}$, there exist infinitely many $g^1_{\text{gon}(C)}$'s. These curves are non-exceptional and are, as far as we know, new examples of curves with an infinite number of $g^1_{\text{gon}(C)}$’s.

**Remark 1.3.** A conjecture by Martens ([Mar84] Statement T, page 280) states that if $\dim W^1_{\text{gon}(C)}(C) = 0$, then $\dim W^1_d(C) = d - \text{gon}(C)$ for $d \leq g - \text{gon}(C) + 2$; and that if $\dim W^1_{\text{gon}(C)}(C) = 1$, then $\dim W^1_d(C) = d - \text{gon}(C) + 1$ for $d \leq g - \text{gon}(C) + 2$. We therefore expect that Theorem 1.1 is valid for $d \leq g - k + 2$, and hence that the Green and Green–Lazarsfeld conjectures are satisfied for the curves in question.

This paper is organised as follows: In Section 2 we introduce the basic results of Brill–Noether theory and the vector-bundles associated to the pairs $(C, A)$, where $|A|$ is a $g^1_d$ on $C$. In Section 3, we prove Theorem 1.1 in the case where the general vector-bundles are non-stable, while the stable case is covered in Section 4. We close with an example of a sub-linear system of curves with an infinite number of $g^1_{\text{gon}(C)}$’s in Section 5.

**Acknowledgments.** Thanks to Andreas Leopold Knutsen for introducing us to this subject, and for valuable comments and remarks.

## 2. Preliminaries

2.1. **Brill–Noether theory.** Let $C$ be a smooth curve over $\mathbb{C}$, and let $r$ and $d$ be non-negative integers. Then there is a variety $W^r_d(C)$ that parametrises all complete $g^r_d$’s on $C$, for all $s \geq r$.

Let $|A|$ be a complete $g^r_d$ on $C$, and let $\mu_{0,A} : H^0(C, \mathcal{O}_C(A)) \otimes H^0(C, \mathcal{O}_C(K_C - A)) \to H^0(C, \mathcal{O}_C(K_C))$ be the cup-product mapping. (This is known as the Petri map.) Then, from [ACGH85] IV, Proposition 4.2, we have

$$\dim T_{|A|} W^r_d(C) = \rho(g, r, d) + \dim \ker(\mu_{0,A}),$$

(1)
where $\rho(g, r, d) := g - (r + 1)(g - d + 1)$ is called the Brill–Noether Number, and also known as “the expected dimension of $W_d^r(C)$”.

Furthermore, if $|A|$ is base-point free and $h^0(C, \mathcal{O}_C(A)) = 2$, then the base-point free pencil trick (ACGH85, page 126) gives us

$$\ker \mu_{0,A} = H^0(C, \mathcal{O}_C(K_C - 2A)).$$

One defines the gonality of $C$ to be the smallest $d$ such that there exists a $g^1_d$ on $C$, and denotes it by $\text{gon}(C)$. It is known that for any smooth curve $C$ of genus $g$,

$$\text{gon}(C) \leq \left\lfloor \frac{g + 3}{2} \right\rfloor.$$  \hspace{1cm} (3)

For the general curve of genus $g$, we have equality in (3). Note that for curves on Enriques surfaces, since it is known that $\phi(C) \leq \sqrt{C^2} = \sqrt{2g - 2}$, the gonality is usually not maximal.

Let $W$ be a component of $W_d^2(C)$ containing $A$. Then,

$$\text{dim ker } \mu_{0,A} = 0 \text{ and } d \leq g - \text{gon}(C) + 2, \text{ then dim } W \leq d - \text{gon}(C).$$  \hspace{1cm} (4)

Also, note that if the general $g^1_d$ in $W$ has base-points, then we can obtain these $g^1_d$’s by considering $g^1_{d-1}$’s and add base-points. It follows that

$$\text{if the general } g^1_d \text{ in } W \text{ have base-points, then dim } W \leq \text{dim } W^1_d(C) + 1.$$  \hspace{1cm} (5)

The following definition, which was introduced in Mar68, generalises the notion of gonality for a curve $C$:

**Definition 2.1.** Let $C$ be a smooth curve of genus $g \geq 4$. The Clifford index of $C$ is defined to be

$$\text{Cliff}(C) := \min\{\deg(A) - 2(h^0(C, \mathcal{O}_C(A)) - 1) \mid h^0(C, \mathcal{O}_C(A)) \geq 2 \text{ and } h^1(C, \mathcal{O}_C(A)) \geq 2\}.$$  \hspace{1cm} (6)

If $A$ is a divisor on $C$ satisfying $h^0(C, \mathcal{O}_C(A)) \geq 2$ and $h^1(C, \mathcal{O}_C(A)) \geq 2$, then one says that $A$ contributes to the Clifford index of $C$, and $A$ is then defined to have Clifford index $\text{Cliff}(A) := \deg(A) - 2(h^0(C, \mathcal{O}_C(A)) - 1)$.

If $C$ is hyperelliptic of genus 2 or 3, one defines $\text{Cliff}(C) = 0$; and if $C$ is non-hyperelliptic of genus 3, one defines $\text{Cliff}(C) = 1$.

It was proved in CM91 Theorem 2.3 that $\text{Cliff}(C) \in \{k - 2, k - 3\}$, where $k = \text{gon}(C)$. We have $\text{Cliff}(C) = k - 2 = \left\lfloor \frac{g - 1}{2} \right\rfloor$ if $C$ is general in $\mathcal{M}_g$ for $g \geq 2$. If $\text{Cliff}(C) = k - 3$, then $C$ is said to be exceptional.

**2.2. Vector-bundle techniques.** Let $S$ be an Enriques surface, and let $L$ be a line-bundle on $S$. One defines $W_d^1|L| := \{(C, A) \mid C \in |L|, A \in W_d^1(C)\}$, and $\pi : W_d^1|L| \rightarrow |L|$, the natural projection map, where $|L|$ denotes the smooth curves of $|L|$. Each fibre of $\pi$ is isomorphic to $W_d^1(C)$.

Let $W$ be an irreducible component of $W_d^1|L|$ such that $\pi$ restricted to $W$ dominates. By (3), we can assume that for general $(C, A)$ in $W$, $|A|$ is base-point free. It thus makes sense to study the associated Lazarsfeld–Mukai vector bundles, $\mathcal{F}_{C,A}$ and $\mathcal{E}_{C,A}$ (see Laz86).

Let $A \in W_d^1(C) \setminus W_d^2(C)$ be base-point free. The vector-bundle $\mathcal{F}_{C,A}$ is defined by

$$\begin{array}{c}
0 \longrightarrow \mathcal{F}_{C,A} \longrightarrow H^0(S, \mathcal{O}_S(A)) \otimes \mathcal{O}_S \longrightarrow \mathcal{O}_S(A) \longrightarrow 0.
\end{array}$$  \hspace{1cm} (6)
One denotes the dual of \( \mathcal{F} \) by \( \mathcal{F}^\vee = \mathcal{E}_{C,A} \). Dualising (1), one gets
\[
0 \to H^0(S, \mathcal{O}_S(A))^\vee \otimes \mathcal{O}_S \to \mathcal{E}_{C,A} \to \mathcal{O}_C(K_C - A + K_S|C) \to 0.
\]

Note that because we are assuming \( d \leq g - \gon(C) \), then \( h^0(C, \mathcal{O}_C(K_C - A + K_S|C)) > 0 \), by Riemann–Roch. Hence, the vector-bundles \( \mathcal{E}_{C,A} \) are globally generated away from a finite set of points, those points being the possible base-points of \( \mathcal{O}_C(K_C - A + K_S|C) \). One has the following properties of \( \mathcal{E}_{C,A} \):
\[
\begin{align*}
&\cdot \ c_1(\mathcal{E}_{C,A}) = L \\
&\cdot \ c_2(\mathcal{E}_{C,A}) = d \\
&\cdot \ h^0(S, \mathcal{E}_{C,A}^\vee) = h^1(S, \mathcal{E}_{C,A}) = 0, h^2(S, \mathcal{E}_{C,A}) = 0 \\
&\cdot \ h^1(S, \mathcal{E}_{C,A}) = h^0(C, \mathcal{O}_C(A + K_S|C))
\end{align*}
\]

Given a vector-bundle \( \mathcal{E} \) of rank 2, with \( c_1(\mathcal{E}) = L, c_2(\mathcal{E}) = d \), and \( h^2(S, \mathcal{E}) = 0 \), and which is finitely generated away from a finite set of points, then given a two-dimensional subspace \( \Lambda \) in \( H^0(S, \mathcal{E}) \), the cokernel of \( \Lambda \otimes \mathcal{O}_S \to \mathcal{E} \) is isomorphic to \( \mathcal{O}_{C\Lambda}(B) \) for some \( C\Lambda \in |L| \), and where \( B \) is a torsion-free sheaf of rank 1 on \( C\Lambda \). If \( C\Lambda \) is smooth, then \( B \cong \mathcal{O}_{C\Lambda}(K_{C\Lambda} - A\Lambda + K_S|C\Lambda) \) for some \( |A\| \in W_d^1(C\Lambda) \), giving us an exact sequence
\[
0 \to \Lambda \otimes \mathcal{O}_S \to \mathcal{E} \to \mathcal{O}_{C\Lambda}(K_{C\Lambda} - A\Lambda + K_S|C\Lambda) \to 0.
\]

An important tool for us will be the following:

**Proposition 2.2.** Suppose that \( \mathcal{W} \) is a component of \( W_d^2|L| \) such that \( \pi : \mathcal{W} \to |L| \) dominates. Let \( (C, A) \) be sufficiently general in \( \mathcal{W} \), and suppose that \( |A| \) is base-point free for these \( A \). Then there exists an exact sequence
\[
0 \to H^0(C, K_S|C) \to H^0(C, \mathcal{E}_{C,A}^\vee \otimes \mathcal{O}_C(K_C - A)) \to H^0(C, \mathcal{O}_C(K_C - 2A)) \to 0.
\]

In particular, \( h^0(C, \mathcal{E}_{C,A}^\vee \otimes \mathcal{O}_C(K_C - A)) = \dim \ker \mu_{0,A} \).

**Proof.** We follow the proof of [Par95, Theorem 2]. (See also [LC12, Proposition 3.2].)

Since \( |A| \) is base-point free and \( h^0(C, \mathcal{O}_C(A)) = 2 \), we have an exact sequence
\[
0 \to \mathcal{O}_C(2A) \to H^0(C, \mathcal{O}_C(A)) \otimes \mathcal{O}_C \xrightarrow{ev} \mathcal{O}_C(A) \to 0,
\]
where \( ev \) is the evaluation morphism.

The diagram
\[
\begin{array}{cccccc}
0 & \to & \mathcal{E}_{C,A}^\vee & \to & H^0(C, \mathcal{O}_C(A)) \otimes \mathcal{O}_S & \to & \mathcal{O}_C(A) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{O}_C(2A) & \to & H^0(C, \mathcal{O}_C(A)) \otimes \mathcal{O}_C & \to & \mathcal{O}_C(A) & \to & 0 \\
\end{array}
\]

yields a surjection \( \mathcal{E}_{C,A}^\vee|C \to \mathcal{O}_C(-A) \to 0 \), and since \( \bigwedge^2 \mathcal{E}_{C,A}^\vee|C = \mathcal{O}_C(-K_C + K_S|C) \), the kernel must be \( \mathcal{O}_C(A - K_C + K_S|C) \), and we get the sequence
\[
0 \to \mathcal{O}_C(A - K_C + K_S|C) \to \mathcal{E}_{C,A}^\vee|C \to \mathcal{O}_C(-A) \to 0.
\]
We tensor with \( \mathcal{O}_C(K_C - A) \) and get
\[
0 \to \mathcal{O}_C(K_S|C) \to \mathcal{E}_{C,A}|_C \otimes \mathcal{O}_C(K_C - A) \to \mathcal{O}_C(K_C - 2A) \to 0.
\]

Taking global sections gives us
\[
0 \to H^0(C, \mathcal{O}_C(K_S|C)) \to H^0(C, \mathcal{E}_{C,A}|_C \otimes \mathcal{O}_C(K_C - A)) \to H^0(C, \mathcal{O}_C(K_C - 2A))
\]
\[
\to H^1(C, \mathcal{O}_C(K_S|C)).
\]

Note that from \( \text{(2)} \) we have \( H^0(C, \mathcal{O}_C(K_C - 2A)) = \ker \mu_{0,A} \). Following an argument identical to \( \text{[Par95] Lemma 1} \), we have that the coboundary-map \( H^0(C, \mathcal{O}_C(K_C - 2A)) \to H^1(C, \mathcal{O}_C(K_S|C)) \) up to constant factors is equal to the map \( \mu_{1,A,S} : \ker_{0,A} \to H^1(C, \mathcal{O}_C(K_S|C)) \) which is given as follows:

The map \( \mu_{1,A,S} \) is the composition of the Gaussian map \( \mu_{1,A} : H^0(C, \mathcal{O}_C(K_C - 2A)) \to H^0(C, \mathcal{O}_C(2K_C)) \) with the transpose of the Kodaira–Spencer map \( \delta_{C,S}^\vee : H^0(C, \mathcal{O}_C(2K_C)) \to (T_C|L)^\vee = H^1(C, N_{C,S}^\vee \otimes \mathcal{O}_C(K_C)) = H^1(C, \mathcal{O}_C(K_S|C)) \).

The lemma follows from considering a commutative diagram
\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C(K_S|C) \\
| & & | \\
\| & \overline{=} & \| \\
0 & \to & \mathcal{O}_C(K_S|C) \\
| & & | \\
0 & \to & \mathcal{E}_{C,A}|_C \otimes \mathcal{O}_C(K_C - A) \\
| & & | \\
\| & \overline{=} & \| \\
0 & \to & \mathcal{O}_C(K_C - 2A) \\
| & & | \\
\| & \overline{=} & \| \\
0 & \to & \Omega_S^1 \otimes \mathcal{O}_C(K_C) \\
| & & | \\
\| & \overline{=} & \| \\
0 & \to & \mathcal{O}_C(2K_C) \\
\end{array}
\]

where \( \mu_{1,A} \) is found by considering \( s \) on the global sections level, and \( \delta_{C,S}^\vee \) is the coboundary map \( H^0(C, \mathcal{O}_C(2K_C)) \to H^1(C, \mathcal{O}_C(K_S|C)) \).

In \( \text{[Par95] page 197} \), it is argued that
\[
\text{Im}(d\pi_{C,A}) \subset \text{Ann}(\text{Im}(\mu_{1,A,S})).
\]

We also have a natural inclusion
\[
\text{Ann}(\text{Im}(\mu_{1,A,S})) \subset H^1(C, \mathcal{O}_C(K_S|C))^\vee,
\]
and the latter has dimension \( g - 1 \).

Since by assumption \( \pi \) dominates \( |L| \), then by Sard’s lemma, \( d\pi_{C,A} \) is surjective for general \( \langle C, A \rangle \), and so \( \text{Im}(d\pi_{C,A}) \) also has dimension \( g - 1 \).

It follows that \( \text{Ann}(\text{Im}(\mu_{1,A,S})) = H^1(C, \mathcal{O}_C(K_S|C))^\vee \), and so \( \text{Im}(\mu_{1,A,S}) = 0 \). Hence, the sequence
\[
0 \to H^0(C, \mathcal{O}_C(K_S|C)) \to H^0(C, \mathcal{E}_{C,A}|_C \otimes \mathcal{O}_C(K_C - A)) \to H^0(C, \mathcal{O}_C(K_C - 2A)) \to 0
\]
is exact. \( \square \)

We will prove the main theorem by considering the case where the general \( \mathcal{E}_{C,A} \)'s are \( \mu_L \)-stable and non-\( \mu_L \)-stable.

**Definition 2.3.** Given a line-bundle \( L \) on a surface \( S \), a vector-bundle \( \mathcal{E} \) is said to be \( \mu_L \)-stable if for any sub-vector bundle \( \mathcal{E}' \) of rank \( 0 < \text{rk}(\mathcal{E}') < \text{rk}(\mathcal{E}) \), we have
\[
\frac{c_1(\mathcal{E}')L}{\text{rk}(\mathcal{E}')} < \frac{c_1(\mathcal{E})L}{\text{rk}(\mathcal{E})}.
\]
A vector-bundle $E$ is said to be non-$\mu_L$-stable if there exists a sub-vector bundle $E'$ of rank $0 < \text{rk}(E') < \text{rk}(E)$ satisfying

$$\frac{c_1(E').L}{\text{rk}(E')} \geq \frac{c_1(E).L}{\text{rk}(E)}.$$  

2.3. Assumptions. Throughout the article, we will be using the following assumptions:

(13) \hspace{0.5cm} \bullet \hspace{0.5cm} \dim W^1_{d-1}(C) = d - 1 - k \text{ for } C \text{ general in } |L| \text{ (by induction).}
(14) \hspace{0.5cm} \bullet \hspace{0.5cm} \text{The general } g^1_a \text{'s are base-point free for general } (C, A) \text{ in a component}
(15) \hspace{0.5cm} \bullet \hspace{0.5cm} k \geq 3 \text{ (since linear growth is always satisfied for hyperelliptic curves. This implies that } L^2 \geq 4)
(16) \hspace{0.5cm} \bullet \hspace{0.5cm} \text{and } \mathcal{W} \text{ is a component of } W^1_d[L] \text{ such that } \pi : \mathcal{W} \to |L| \text{ dominates}
(17) \hspace{0.5cm} \bullet \hspace{0.5cm} \text{and for general } C \in |L| \text{ the fibre over } C \text{ has dimension } \dim W^1_d(C)

3. The case where the $E_{C,A}$'s are non-$\mu_L$-stable

In this section, we will assume that for general $(C, A) \in \mathcal{W}$, the vector-bundles $E_{C,A}$ are non-$\mu_L$-stable. The main result of this section is Proposition 3.9, where we do a parameter count of all possible non-$\mu_L$-stable vector-bundles that satisfy the properties of $E_{C,A}$.

We start by recalling two results, one from $[KL07]$ and one from $[KL09]$, which we will be using several times throughout this section:

**Theorem 3.1 ($[KL07]$ Theorem, case of Enriques surfaces).** Let $S$ be an Enriques surface, and $O_S(D)$ a line-bundle on $S$ such that $D > 0$ and $D^2 \geq 0$. Then $H^1(S, O_S(D)) \neq 0$ if and only if one of the following occurs:

(i) $D \sim nE$ for $E > 0$ nef and primitive with $E^2 = 0$, $n \geq 2$ and $h^1(S, O_S(D)) = \lfloor \frac{n}{2} \rfloor$;
(ii) $D \sim nE + KS$ for $E > 0$ nef and primitive with $E^2 = 0$, $n \geq 3$ and $h^1(S, O_S(D)) = \lfloor \frac{n-1}{2} \rfloor$;
(iii) there is a divisor $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.D \leq -2$.

Note that since the Enriques surfaces in question in our article are assumed to be unmodal, then part (iii) of Theorem 3.1 cannot occur.

**Lemma 3.2 ($[KL09]$ Lemma 2.12).** Let $L > 0$ be a line bundle on an Enriques surface $S$ with $L^2 \geq 0$. Then there is an integer $n$ such that $1 \leq n \leq 10$ and, for any $i = 1, \ldots, n$, there are primitive divisors $E_i > 0$ with $E_i^2 = 0$ and integers $a_i > 0$ such that

$$L \equiv a_1E_1 + \cdots + a_nE_n$$

and one of the following intersection sets occurs:

(i) $E_i.E_j = 1$ for $1 \leq i < j \leq n$.
(ii) $n \geq 2$, $E_1.E_2 = 2$ and $E_i.E_j = 1$ for $2 \leq i < j \leq n$ and for $i = 1$, $3 \leq j \leq n$.
(iii) $n \geq 3$, $E_1.E_2 = E_1.E_3 = 2$ and $E_i.E_j = 1$ for $3 \leq i < j \leq n$, for $i = 1$, $4 \leq j \leq n$ and for $i = 2$, $3 \leq j \leq n$. 
The following proposition is crucial to our result. The fact that we can assume that the vector-bundles are contained in a short-exact sequence as in (15), where \( M.L \geq N.L \), will eventually ensure that the dimensions of extensions of various \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \otimes I_\xi \) is small enough to give us the desired result (see Lemma 3.7).

**Proposition 3.3.** Suppose \( \mathcal{E}_{C,A} \) is non-\( \mu_L \)-stable. Then there exist line-bundles \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \), and a 0-dimensional subscheme \( \xi \), such that \( \mathcal{E}_{C,A} \) sits inside an exact sequence (18)

\[
0 \to \mathcal{O}_S(M) \to \mathcal{E}_{C,A} \to \mathcal{O}_S(N) \otimes I_\xi \to 0,
\]

satisfying the following conditions:

(a) We have \( M + N \sim C \), \( \text{length}(\xi) = d - M.N \), and \( |N| \) is non-trivial and base-component free (implying that \( h^0(S, \mathcal{O}_S(N)) \geq 2 \)). Furthermore, \( h^2(S, \mathcal{O}_S(M - N)) = 0 \) and \( h^0(S, \mathcal{O}_S(N - M)) = 0 \), unless \( M \sim N + K_S \) or \( M \sim N \), respectively.

(b) We have \( h^2(S, \mathcal{O}_S(M)) = 0 \) and \( h^0(S, \mathcal{O}_S(M)) \geq 2 \).

(c) We have \( h^1(S, \mathcal{O}_S(M)) = 0 \).

(d) We have \( N|_C \geq A \).

(e) If \( \xi \neq \emptyset \), then \( h^1(S, \mathcal{O}_S(N)) = 0 \) and \( N^2 > 0 \).

Note that the points where \( \mathcal{E}_{C,A} \) is not globally generated lie along the curve \( C \).

**Proof.** Since \( \mathcal{E}_{C,A} \) by assumption is non-\( \mu_L \)-stable, there exists a line bundle \( \mathcal{O}_S(M) \) of slope \( \geq g - 1 \) on \( C \) that injects into \( \mathcal{E}_{C,A} \). We can assume that the injection is saturated, and so we obtain the sequence (15). Note that since \( M.C \geq g - 1 \), then \( N.C \leq g - 1 \).

We have \( M + N \sim C \), \( \text{length}(\xi) = d - M.N \) and \( |N| \) base-point free because of (8) and (9), and the fact that \( \mathcal{E}_{C,A} \) is globally generated away from a finite set of points. We have \( N \) non-trivial because otherwise, \( M.N = 0 \), implying that \( \xi \neq \emptyset \), and this would imply that \( h^0(S, \mathcal{O}_S(N) \otimes I_\xi) = 0 \), which contradicts \( \mathcal{E}_{C,A} \) being globally generated away from a finite set.

We have \( h^2(S, \mathcal{O}_S(M - N)) = 0 \) and \( h^0(S, \mathcal{O}_S(N - M)) = 0 \) by the Nakai–Moishezon criterion, using that \( L \) by assumption is ample, and that \( M.L \geq N.L \).

We now show part (b). Since \( M.C \geq g - 1 \), then since \( L \) is ample, \( -M + K_S \) cannot be effective. It follows that \( h^0(S, \mathcal{O}_S(-M + K_S)) = 0 \), and this equals \( h^2(S, \mathcal{O}_S(M)) \) by Serre duality.

To prove that \( h^0(S, \mathcal{O}_S(M)) \geq 2 \), by part (a), we have \( M.N \leq d \leq g - k < g - 1 \), and this gives us \( g - 1 \leq M.C = M^2 + M.N < M^2 + g - 1 \), yielding \( M^2 > 0 \). The result now follows from Riemann–Roch.

Part (c) follows from the fact that \( M^2 > 0 \) (proven in part (b)) together with Theorem 3.1.

To prove (d), note that by tensoring (15) with \( \mathcal{O}_S(-M) \) and taking global sections, we get \( h^0(S, \mathcal{E}_{C,A} \otimes \mathcal{O}_S(-M)) \geq 1 \). Rewrite (1) as

\[
0 \to \mathcal{O}_S^{\otimes 2} \to \mathcal{E}_{C,A} \to \mathcal{O}_C(C|_C - A) \to 0,
\]
tensor with \( \mathcal{O}_S(-M) \) and take global sections. This gives us an injection \( H^0(S, \mathcal{E}_{C,A} \otimes \mathcal{O}_S(-M)) \hookrightarrow H^0(C, \mathcal{O}_C(N|_C - A)) \), proving that \( N|_C - A \geq 0 \).

As for (e), suppose that \( h^1(S, \mathcal{O}_S(N)) > 0 \). By Theorem 3.1 it follows that \( N^2 = 0 \). From (d), we then have \( d \leq N.C = N(M + N) = M.N \), contradicting part (a), which states that \( d = M.N + \text{length}(\xi) \).

\( \square \)

The following lemma gives us an upper bound on \( h^0(S, \mathcal{E}) \), because of (15).
Lemma 3.4. Suppose that for general \((C, A) \in W\), the associated vector-bundle \(E_{C,A}\) is non-\(\mu_L\)-stable, so that we have a short-exact sequence as in Proposition 3.3 where \(M\) and \(N\) are fixed. Then, for general \((C, A)\), we have \(h^1(S, E_{C,A}) \leq 2\) and \(h^1(S, O_S(N) \otimes I_{C,A}) \leq 2\).

Proof. Note that from (17), we are assuming that \(\pi : W \to |L|\) dominates, and that for general \(C \in |L|\), the fibre over \(C\) has dimension \(W^3_d(C)\).

Suppose \(h^1(S, O_S(N) \otimes I_C) \geq 3\). Then taking cohomology of (18), we get a surjection \(H^1(S, E_{C,A}) = H^1(S, O_S(N) \otimes I_C) \to 0\), implying that \(h^1(S, E_{C,A}) \geq 3\).

By (11), \(h^1(S, E_{C,A}) = h^0(C, O_C(A + KS|_C))\), giving us \(W^3_d(C)\) dimensions of \(g^2_s\)’s, which is impossible.

The following lemma is necessary for the proof of Proposition 3.8 where we prove that \(M.N \geq k - 1\). This lemma is (in the Enriques surface case) an improvement of a similar result in [AF11], where it is shown that \(M|_C\) contributes to the Clifford index. By using \(M|_C\) instead of \((M + E)|_C\) in Proposition 3.8, we would only obtain \(M.N \geq k - 2\).

Lemma 3.5. Suppose we have a sequence as in Proposition 3.3 with \((M - N)^2 \geq 0\). If there exists a primitive elliptic curve \(E\) such that \((M - N).E > 0\) and \(h^0(S, O_S(N - E + KS)) \geq 2\), then \((M + E)|_C\) contributes to \(\text{Cliff}(C)\).

Proof. By (15), we have \(k \geq 3\), and so there exist line-bundles on \(C\) that contribute to \(\text{Cliff}(C)\).

We show that \(h^i(C, O_S(M + E)|_C) \geq 2\) for \(i = 0, 1\).

Consider the exact sequence
\[
(19) \quad 0 \to O_S(-C) \to O_S \to O_C \to 0
\]
tensored with \(O_S(M + E)\), giving us
\[
0 \to O_S(-N + E) \to O_S(M + E) \to O_S(M + E)|_C \to 0.
\]
Because \(h^0(S, O_S(N)) \geq 2\) by Proposition 3.3, we must have \(h^0(S, O_S(-N + E)) = 0\). By the same proposition, it follows that \(h^0(S, O_S(M + E)) \geq 2\), and so also \(h^0(C, O_S(M + E)|_C) \geq 2\), as desired.

We have \(h^1(C, O_S(M + E)|_C) = h^0(C, O_C(K_C - M|_C - E|_C)) = h^0(C, O_S(C + KS - M - E)|_C) = h^0(C, O_S(N - E + KS)|_C)\). By considering the sequence (19) tensored with \(O_S(N - E + KS)\), we get
\[
0 \to O_S(-M - E + KS) \to O_S(N - E + KS) \to O_S(N - E + KS)|_C \to 0.
\]
Since \(h^0(S, O_S(M)) \geq 2\) by Proposition 3.3, \(h^0(S, O_S(-M + KS)) = 0\), and so \(H^0(C, O_S(N - E + KS)) \to H^0(C, O_S(N - E + KS)|_C)\). We have \(h^0(S, O_S(N - E + KS)) \geq 2\) by assumption, and so \(h^0(C, O_S(N - E + KS)|_C) \geq 2\) as well. \(\square\)

In the following proposition, we obtain a connection between \(M.N\) and the generic gonality in \(|L|\). This is used when we make the parameter count of extensions of \(O_S(M)\) and \(O_S(N) \otimes I_{C,A}\) in the proof of Proposition 3.9.

Here we use that the general curves in \(|L|\) are non-exceptional. This is a consequence of [KL13] Corollary 1.2, where we find that the Clifford index for the general curve in \(|L|\) is \(k - 2\).

Note that Proposition 3.6 in the case of \(N^2 = 4\), is the only place where we use the assumption that \(\mu(L) > 2\phi(L)\).

Proposition 3.6. Suppose that for general \((C, A)\) in \(W\), the vector-bundle \(E_{C,A}\) is non-\(\mu_L\)-stable. Suppose furthermore that \(\mu(L) > 2\phi(L) = k\). Then \(M.N \geq k - 1\).
Proof. Suppose first that \((M - N)^2 \leq 0\). Then \(2M.N \geq M^2 + N^2\), and so \(2g - 2 = (M + N)^2 = M^2 + 2M.N + N^2 \leq 4M.N\), yielding \(M.N \geq \frac{g - 1}{2}\). Since \(k \leq \frac{g}{2}\) by assumption (10), the result follows.

Now suppose \((M - N)^2 > 0\).

We start by considering three special cases, namely \(N^2 = 0\), \(N^2 = 2\), and \(N^2 = 4\) with \(\phi(N) = 2\).

Special case 1. If \(N^2 = 0\), then we know from Proposition 3.3 that \(d \leq N.C = N.(M + N) = M.N\), and so it follows in particular that \(k - 1 \leq M.N\).

Special case 2. If \(N^2 = 2\), note that by Theorem 3.1 \(h^1(S, \mathcal{O}_S(N)) = 0\), so that \(h^0(S, \mathcal{O}_S(N)) = \frac{1}{2}N^2 + 1 = 2\). Since \(E_S\) is globally generated away from a finite set of points, then so must \(|\mathcal{O}_S(N) \otimes \mathcal{I}_\xi|\), and so all points of \(\xi\) must be along base-points of \(|N|\) (or else \(\dim |\mathcal{O}_S(N) \otimes \mathcal{I}_\xi| \leq 0\)). Since \(E_S\) is globally generated outside of \(C\), this implies that the base-points of \(|N|\), and hence also the points of \(\xi\), must lie along \(C\). However, \(h^1(S, \mathcal{O}_S(N) \otimes \mathcal{I}_\xi)\) indicates (in this particular case) how many points of \(\xi\) that lie along base-points of \(|N|\), and by Lemma 3.4 it follows that \(\text{length}(\xi) \leq 2\).

If \(d > k\), then this yields \(2 \geq \text{length}(\xi) = d - M.N > k - M.N\), which leads to \(M.N \geq k - 1\).

If \(d = k\), then note that since \(h^0(C, \mathcal{O}_S(N)) \geq 2\), then \(N.C \geq k + \text{length}(\xi)\), or else we get a contradiction on the gonality of \(C\). But this gives us \(N.C = N^2 + M.N = 2 + M.N \geq k + k - M.N\), yielding \(M.N \geq k - 1\).

General case. Now suppose \(M \geq N\), \(N^2 \geq 4\) and \((M - N)^2 > 0\). If \(N^2 = 4\), we suppose that \(\phi(N) \neq 2\). We first show that there exists an elliptic curve \(E\) such that the conditions of Lemma 3.3 are satisfied.

By Lemma 3.2 \(M - N = a_1 E_1 + \cdots + a_m E_m\) for some elliptic curves \(E_i\) satisfying \(E_i, E_j > 0\) for \(i \neq j\), and integers \(a_i > 0\). Since \((M - N)^2 > 0\), we must have \(m \geq 2\), and so \((M - N).E > 0\) for any elliptic curve \(E\).

Applying Lemma 3.2 again, we see that there exist positive integers \(b_i\) and elliptic curves \(E'_i\) satisfying \(1 \leq E'_i, E'_j \leq 2\) for \(i \neq j\), such that \(M \equiv b_1 E'_1 + \cdots + b_m E'_m\). This implies that \(N - (b_1 E'_1 + \cdots + b_m E'_m)\) is linearly equivalent to either 0 or \(K_S\). If \(h^0(S, \mathcal{O}_S(N - E)) = 1\), where \(E < N\), the only way this can happen is that \(N - E\) is linearly equivalent to a stationary elliptic curve or a sum of two elliptic curves \(E''_1, E''_2\) satisfying \(2E''_1 \sim 2E''_2\). But the first case implies \(N^2 \leq 4\) with \(\phi(N) = 2\), which is a contradiction; and in the second case, \(N \sim E + E''_1 + E''_2\) such that \(h^0(S, \mathcal{O}_S(N - E''_i)) \geq 2\) for \(i = 1, 2\), so that Lemma 3.3 can still be applied.

It follows that there exists an elliptic curve \(E\) such that \((M + E)|_C\) contributes to \(\text{Cliff}(C)\). By [KL13, Corollary 1.2], the general curve \(C\) in \(|L|\) has Clifford index \(k - 2\). Recalling from
the proof of Lemma 3.3 that \( H^0(S, \mathcal{O}_S(M + E)) \hookrightarrow H^0(C, \mathcal{O}_C(M + E)) \), we get
\[
k - 2 = \text{Cliff}(C)
\leq \text{Cliff}((M + E)|_C) = (M + E).C - 2(h^0(C, \mathcal{O}_C(M + E)) - 1)
\leq M.(M + N) + E.C - (M + E)^2
= M.N + E.C - 2M.E
= M.N + E.M + E.N - 2M.E
= M.N - E.(M - N)
\leq M.N - 1,
\]
as desired. \(\Box\)

The two following lemmas are used in the proof of Proposition 3.9. The first one gives a parameter space for the extensions of \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \otimes \mathcal{I}_\xi \), and is the second place in this article where we use the assumption that \( L \) is ample. The second lemma is important when we count the dimensions of possible pairs \((C, A)\) that can arise from the same vector-bundle \( \mathcal{E} \).

**Lemma 3.7.** Suppose \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \) are two line-bundles on \( S \) satisfying \( M.L \geq N.L \), and let \( \xi \) be a non-empty zero-dimensional subscheme on \( S \) of length \( \ell > 0 \). Then all isomorphism-classes of extensions of \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \otimes \mathcal{I}_\xi \) are parametrised by
\[
\mathbb{P}Ext^1(\mathcal{O}_S(N) \otimes \mathcal{I}_\xi, \mathcal{O}_S(M)) \cong \mathbb{P}H^1(S, \mathcal{O}_S(N - M + K_S) \otimes \mathcal{I}_\xi)^\vee,
\]
which has dimension
\[
\ell + h^1(S, \mathcal{O}_S(M - N)) - h^2(S, \mathcal{O}_S(M - N)) - 1.
\]

**Proof.** The isomorphism classes of extensions of \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \otimes \mathcal{I}_\xi \) are parametrised by \( \mathbb{P}Ext^1(\mathcal{O}_S(N) \otimes \mathcal{I}_\xi, \mathcal{O}_S(M)) \cong \mathbb{P}H^1(S, \mathcal{O}_S(N - M + K_S) \otimes \mathcal{I}_\xi)^\vee \), by [Fru98] pages 36 and 39.

To find an expression for \( h^1(S, \mathcal{O}_S(N - M + K_S) \otimes \mathcal{I}_\xi) \), we tensor the exact sequence
\[
0 \to \mathcal{I}_\xi \to \mathcal{O}_S \to \mathcal{O}_S(\mathcal{I}_\xi) \to 0
\]
with \( \mathcal{O}_S(N - M + K_S) \) and take global sections, yielding
\[
0 \to H^0(S, \mathcal{O}_S(N - M + K_S) \otimes \mathcal{I}_\xi) \to H^0(S, \mathcal{O}_S(N - M + K_S)) \to \mathbb{C}^\ell
\to H^1(S, \mathcal{O}_S(N - M + K_S) \otimes \mathcal{I}_\xi) \to H^1(S, \mathcal{O}_S(N - M + K_S)) \to 0.
\]
We have \( h^0(S, \mathcal{O}_S(N - M + K_S) \otimes \mathcal{I}_\xi) = 0 \) because \( (N - M + K_S).L \leq 0 \) by assumption, and using that \( L \) is ample together with the Nakai–Moishezon criterion.

The result now follows by Serre duality. \(\Box\)

**Lemma 3.8.** Suppose \( \mathcal{E} \) is an extension of \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \otimes \mathcal{I}_\xi \) so that we have a sequence as in Proposition 3.3 Then \( h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) \geq h^0(S, \mathcal{O}_S(M - N)) \).

**Proof.** If \( M \neq N \) or \( M \sim N + K_S \), we have \( h^0(S, \mathcal{O}_S(M - N)) \leq 1 \), so there is nothing to prove. So assume that \( M \geq N \).

Tensor the sequence
\[
0 \to \mathcal{O}_S(M) \to \mathcal{E} \to \mathcal{O}_S(N) \otimes \mathcal{I}_\xi \to 0
\]
with \( \mathcal{O}_S(M - N) \).

We have
\[
h^0(S, \mathcal{O}_S(M - N) \otimes \mathcal{I}_\xi) = 0
\]
by Serre duality. Then
\[
h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) \geq h^0(S, \mathcal{O}_S(M - N)) \]
by $\mathcal{E}^\vee$. Taking global sections, we see that $h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) \geq h^0(S, \mathcal{E}^\vee \otimes \mathcal{O}_S(M))$. By Serre duality, we have $h^0(S, \mathcal{E}^\vee \otimes \mathcal{O}_S(M)) = h^2(S, \mathcal{E} \otimes \mathcal{O}_S(-M + K_S))$. It thus suffices to prove that $h^2(S, \mathcal{E} \otimes \mathcal{O}_S(-M + K_S)) \geq h^0(S, \mathcal{O}_S(M - N))$.

Tensoring $\mathcal{O}_S(-M + K_S)$. Taking cohomology, we get

$$H^2(S, \mathcal{E} \otimes \mathcal{O}_S(-M + K_S)) \to H^2(S, \mathcal{O}_S(N - M + K_S) \otimes I_\xi) \to 0.$$ 

So we have $h^2(S, \mathcal{E} \otimes \mathcal{O}_S(-M + K_S)) \geq h^2(S, \mathcal{O}_S(N - M + K_S) \otimes I_\xi)$.

But if we consider

$$0 \to I_\xi \to \mathcal{O}_S \to O_\xi \to 0$$

tensored with $\mathcal{O}_S(N - M + K_S)$ and take cohomology, we see that $h^2(S, \mathcal{O}_S(N - M + K_S) \otimes I_\xi) = h^2(S, \mathcal{O}_S(N - M + K_S))$, which by Serre duality equals $h^0(S, \mathcal{O}_S(M - N))$. The result follows.

We are now ready to state and prove the main result of this section.

**Proposition 3.9.** Suppose that for general $(C, A)$ in $W$, the vector-bundles $\mathcal{E}_{C,A}$ are non-$\mu_L$-stable, and suppose that $\mu(L) > 2\phi(L) = k$. Then $\dim W \leq g - 1 + d - k$.

**Proof.** By assumption, for general $(C, A)$ in $W$, $\mathcal{E}_{C,A}$ sits inside an exact sequence as in Proposition 6.3. We prove the proposition by making a parameter-count of all pairs $(C, A)$ such that $\mathcal{E}_{C,A}$ is non-$\mu_L$-stable, making a similar construction as the one done in [AFT11, Section 3] in the case of non-simple vector-bundles on K3-surfaces.

We divide this proof into three cases. We first consider the case where the vector-bundles $\mathcal{E}_{C,A}$ are indecomposable with $\ell > 0$, followed by the indecomposable case when $\ell = 0$. Finally, we consider the case where the $\mathcal{E}_{C,A}$’s are decomposable.

**The case where the general $\mathcal{E}_{C,A}$’s are indecomposable with $\ell > 0$.** Fix a line-bundle $\mathcal{O}_S(N)$ such that $|N|$ is base-component free, and which satisfies the following conditions: $(L - N).L \geq N.L$, $d \geq (L - N).N$, and $d - (L - N).N \leq h^0(S, \mathcal{O}_S(N))$. Set $M := L - N$ and $\ell := d - M.N$. Note that these conditions imply that $h^1(S, \mathcal{O}_S(M)) = 0$, $h^0(S, \mathcal{O}_S(M)) \geq 2$ and $h^2(S, \mathcal{O}_S(M)) = 0$.

Let $\mathcal{P}_{N,e}$ be the family of vector-bundles that are extensions of $\mathcal{O}_S(M)$ and $\mathcal{O}_S(N) \otimes I_\xi$ where $\xi$ is a zero-dimensional subscheme of length $\ell$. For $0 \leq i \leq 2$ (see Lemma 3.4), define

$$\mathcal{P}_{N,e,i} := \{[\mathcal{E}] \in \mathcal{P}_{N,e} | h^2(S, \mathcal{E}) = 0, h^1(S, \mathcal{E}) = i\},$$

and $\mathcal{E}$ is globally generated away from a finite set of points.$\}$

We can think of $\mathcal{P}_{N,e,i}$ as extensions of $\mathcal{O}_S(M)$ and $\mathcal{O}_S(N) \otimes I_\xi$ where $\xi$ imposes $\ell - i$ conditions on $|N|$. Note that this puts restrictions on the dimensions of possible $\xi$’s that can be considered. Whereas the Hilbert scheme $S^{[\ell]}$ parametrises all possible $\xi$’s of length $\ell$, the $\xi$’s that impose $\ell - i$ conditions on $|N|$ can be found by considering elements $\eta$ of $S^{[\ell-i]}$ and add base-points of $|\mathcal{O}_S(N) \otimes I_\eta|$. Since $\mathcal{E}$ is globally generated away from a finite set of points, then $|\mathcal{O}_S(N) \otimes I_\xi|$ is base-component free, and so there are only a finite set of base-points in $|\mathcal{O}_S(N) \otimes I_\xi|$.

(* It follows that there are at most $2\ell - 2i$ dimensions of $\xi$’s in $S^{[\ell]}$ that impose $\ell - i$ conditions on $|N|$.

Still following the construction of [AFT11, Section 3], we let $\mathcal{G}_{N,e,i}$ be the Grassmann bundle over $\mathcal{P}_{N,e,i}$ classifying pairs $([\mathcal{E}], \Lambda)$ with $[\mathcal{E}] \in \mathcal{P}_{N,e,i}$ and $\Lambda \in \text{Gr}(2, h^0(S, \mathcal{E}))$. (Note that $h^0(S, \mathcal{E}) = h^0(S, \mathcal{O}_S(M)) + h^0(S, \mathcal{O}_S(N)) - \ell + 1$, and is thus constant.)
By assumption, we have a rational map

\[ h_{N,\ell,i} : \mathcal{G}_{N,\ell,i} \to \mathcal{W}_d^1(\{L\}) \]

given by \( h_{N,\ell,i}(\mathcal{E}, \Lambda) := (C_A, A_\Lambda) \) (see sequence (12)). The dimension of each fibre of \( h_{N,\ell,i} \) is found by finding the dimension of all surjections \( \mathcal{E} \to \mathcal{O}_C(K_{DA} - A_\Lambda + K_S|C_A) \) and subtract the dimension of all morphisms from \( \mathcal{O}_C(K_{DA} - A_\Lambda + K_S|C_A) \) to itself (which is 1).

By Proposition 3.6, all extensions of single \( P \) and \( g \) is construct the same family \( P \) there can only be finitely many different \( 2 \) dim bounded by \( \geq 1 + 2 \).

By Lemma 3.8, this is \( \geq h^0(S, \mathcal{O}_S(M - N)) - 1 \).

Letting \( e \) be \( h^0(S, \mathcal{E}) \) for any vector-bundle \( \mathcal{E} \) in \( \mathcal{P}_{N,\ell,i} \), we conclude that \( \dim \mathcal{W}_d^1|L| \) is bounded by \( \dim \mathcal{P}_{N,\ell,i} + \dim \text{Gr}(2, e) - h^0(S, \mathcal{O}_S(M - N)) + 1 \).

By (a) combined with Lemma 3.7 and using that \( \ell = d - M.N \), we have \( \dim \mathcal{P}_{N,\ell,i} \leq 2\ell - 2i + \ell + h^1(S, \mathcal{O}_S(M - N)) - h^2(S, \mathcal{O}_S(M - N)) = 1 = 3d - 3M.N - 2i + h^1(S, \mathcal{O}_S(M - N)) - h^2(S, \mathcal{O}_S(M - N)) - 1 \). We furthermore have \( \dim \text{Gr}(2, e)^2 = 2\ell - 2 = 2e - 4 = 2(\chi(S, \mathcal{E}) + 2i - 4 = 2(g + 1 - d) + 2i - 4 = 2g - 2d + 2i - 2 \).

This gives us in total

\[
\dim \mathcal{W} \leq 3d - 3M.N - 2i - \chi(S, \mathcal{O}_S(M - N)) - 1 + 2g - 2d + 2i - 1
= 2g - 3M.N + d - 2 - \chi(S, \mathcal{O}_S(M + N)) + 2M.N
= 2g - M.N + d - 2 - g = g - 2d + d - M.N.
\]

By Proposition 3.6, \( M.N \geq k - 1 \), and it follows that

\[
\dim \mathcal{W} \leq g - 1 + d - k.
\]

**The case where the general \( \mathcal{E}_{C,A} \)'s are indecomposable with \( \ell = 0 \).**

In this case, we also construct the same family \( \mathcal{P}_{N,\ell,i} = \mathcal{P}_{N,0,i} \) of vector-bundles as in the previous case. By [129], all extensions of \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \) are parametrised by \( \mathbb{P}H^1(S, \mathcal{O}_S(N - M + K_S)|\mathbb{P}^N = \mathbb{P}H^1(S, \mathcal{O}_S(M - N)) \). As in the previous case, we consider the same family of vector-bundles \( \mathcal{P}_{N,0,i} \), where \( i \leq 2 \), together with the grassmannian bundle \( \mathcal{G}_{N,0,i} \). This gives us the bound

\[
\dim \mathcal{W}_d^1|L| \leq h^1(S, \mathcal{O}_S(M - N)) - 1 + \dim \text{Gr}(2, e) - h^0(S, \mathcal{O}_S(M - N)) + 1,
\]

where \( e = h^0(S, \mathcal{E}) \) for the extensions \( \mathcal{E} \) with \( h^1(S, \mathcal{E}) = i \).

Since there are no indecomposable extensions of \( \mathcal{O}_S(M) \) and \( \mathcal{O}_S(N) \) when \( M \sim N + K_S \) (since then \( h^1(S, \mathcal{O}_S(M - N)) = 0 \), we can by Proposition 3.3 (a) assume that \( h^2(S, \mathcal{O}_S(M - N)) = 0 \). We have, as before, \( \dim \text{Gr}(2, e) = 2g - 2d + 2i - 2 \), and \( \chi(\mathcal{O}_S(M - N)) = g - 2M.N \).

Note that \( \ell = d - M.N \), we have \( d = M.N \) in this case. It follows that

\[
\dim \mathcal{W}_d^1|L| \leq -\chi(\mathcal{O}_S(M - N)) + 2g - 2d + 2i - 2 + 1 = -g + 2M.N + 2g - 2d + 2i - 1 = g - 1 + 2i.
\]

Now, if \( i = 0 \), we are done. So suppose \( i > 0 \). Since \( h^1(S, \mathcal{O}_S(M)) = 0 \) (by Proposition 3.3), it follows that \( h^1(S, \mathcal{O}_S(N)) = i \), and so by Theorem 3.1 \( N^2 = 0 \). But then, \( N.C = N.M = d \), and since \( N|C \geq A \) (by Proposition 3.3), it follows that these vector-bundles only yield one single \( g_d^1 \) for each curve \( C \).

**The case where the general \( \mathcal{E}_{C,A} \)'s are decomposable.**

Now suppose \( \mathcal{E}_{C,A} \) is decomposable for general \( (A, C) \). In that case, we must have \( \ell = 0 \), and so \( M.N = d \). Note also that there can only be finitely many different \( \mathcal{E}_{C,A} \) in this case, and so we will here show that the
image of the map $f_{E} : \text{Gr}(h^{0}(S, \mathcal{E}), 2) \rightarrow \mathcal{W}_{d}^{1}|L|$ is of dimension at most $g - 1$, thus implying that $\dim W_{d}^{1}(C) = 0$ (given the assumptions in the proposition).

As argued in the indecomposable case, we have

$$\dim \im f_{E} = 2(h^{0}(S, \mathcal{E}) - 2) - h^{0}(S, \mathcal{E} \otimes \mathcal{E}^{\vee}) + 1.$$  

Since $\mathcal{E}$ is decomposable, we have $\mathcal{E} \otimes \mathcal{E}^{\vee} \cong \mathcal{O}_{S}^{\mathbb{Z}} \oplus \mathcal{O}_{S}(M - N) \oplus \mathcal{O}_{S}(N - M)$.

By Proposition 3.3, we either $M \sim N + K_{S}$ or $h^{2}(S, \mathcal{O}_{S}(M - N)) = 0$.

If $M \sim N + K_{S}$, then both $M^{2} > 0$ and $N^{2} > 0$, and so $h^{1}(S, \mathcal{E}) = h^{1}(S, \mathcal{O}_{S}(M)) + h^{1}(S, \mathcal{O}_{S}(N)) = 0$, by Theorem 3.1. In this case, we have $h^{0}(S, \mathcal{E}) = g - d + 1$ and $d = M.N = \frac{g - d}{2}$, and so

$$\dim \im f_{E} = 2g - 2 \cdot \frac{g - 1}{2} - 3 - h^{0}(M - N) - h^{0}(N - M) \leq g - 2.$$  

If $h^{2}(S, \mathcal{O}_{S}(M - N)) = 0$, then let $i = h^{1}(S, \mathcal{E})$. As in the case where the $\mathcal{E}$‘s are indecomposable with $\ell = 0$, we also here get $N^{2} = 0$ if $i > 0$, and hence that $N|_{C} = A$. We thus get one single $g_{d}^{1}$ for each curve $C$.

Now suppose $i = 0$. This implies that $h^{0}(S, \mathcal{E}) = g - d + 1$, and so

$$\dim \im f_{E} \leq 2(g - d - 1) - h^{0}(S, \mathcal{O}_{S}(M - N)).$$  

Using Riemann–Roch together with the assumption that $h^{2}(S, \mathcal{O}_{S}(M - N)) = 0$, we get $h^{0}(S, \mathcal{O}_{S}(M - N)) \geq \frac{1}{2}C^{2} - 2M.N + 1 = g - 2M.N = g - 2d$, which gives us

$$\dim \im f_{E} \leq g - 3.$$  

$\square$

4. THE CASE WHERE THE $\mathcal{E}_{C, A}$‘S ARE $\mu_{L}$-STABLE

In this section, we cover the cases where $\mathcal{E}_{C, A}$ is $\mu_{L}$-stable for general $(C, A)$ in $\mathcal{W}$. It is here not possible to do a parameter count in order to obtain a suitable bound, but we prove here instead that $\dim \ker(\mu_{0, A}) \leq 2$, yielding that $\dim W_{d}^{1}(C) = d - k$ for $d \leq g - k$ for the curves in question.

Note that, by Assumption (17), we cannot have $h^{0}(C, \mathcal{O}_{C}(A + K_{S}|_{C})) \geq 3$ for general $(C, A)$ in $\mathcal{W}$, since otherwise, by subtracting points, we would have more than $\dim W_{d}^{1}(C)$ dimensions of $g_{d}^{1}$‘s, which is impossible. In the following propositions, we therefore only need to consider the cases where $h^{0}(C, \mathcal{O}_{C}(A + K_{S}|_{C})) = 2$ or $\leq 1$, respectively.

**Proposition 4.1.** Suppose that $\mathcal{E}_{C, A}$ is $\mu_{L}$-stable, and that $h^{0}(C, \mathcal{O}_{C}(A + K_{S}|_{C})) = 2$, for general $(C, A)$ in $\mathcal{W}$. Then $\dim \mathcal{W} \leq g - 1 + d - k$.

**Proof.** If $h^{0}(C, \mathcal{O}_{C}(A + K_{S}|_{C})) = 2$ for the general pairs $(C, A)$ in $\mathcal{W}$, then by (5), we can assume that $\mathcal{O}_{C}(A + K_{S}|_{C})$ is also base-point free for general $A$, and so these pairs $(C, A + K_{S}|_{C})$ define vector-bundles $\mathcal{E}_{C, A + K_{S}|_{C}}$. If these vector-bundles are non-$\mu_{L}$-stable for general $(C, A)$ in $\mathcal{W}$, then by Proposition 3.3 we get at most $g - 1 + d - k$ dimensions of pairs $(C, A + K_{S}|_{C})$, and so there must also be at most that many dimensions of pairs $(C, A)$. So suppose the vector-bundles are $\mu_{L}$-stable.

The vector-bundles $\mathcal{E}_{C, A + K_{S}|_{C}}$ lie inside a sequence

$$0 \rightarrow H^{0}(S, \mathcal{O}_{S}(A + K_{S}|_{C}))^{\vee} \otimes \mathcal{O}_{S} \rightarrow \mathcal{E}_{C, A + K_{S}|_{C}} \rightarrow \mathcal{O}_{C}(K_{C} - A) \rightarrow 0.$$
Now, tensoring this sequence with $\mathcal{E}^\vee_{C,A}$ and taking global sections, we get

$$0 \rightarrow H^0(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \rightarrow H^0(S,\mathcal{E}^\vee_{C,A+K_S} \otimes \mathcal{E}^\vee_{C,A}) \rightarrow H^0(C,\mathcal{O}_C(K_C - A) \otimes \mathcal{E}^\vee_{C,A}) \rightarrow H^1(S,\mathcal{E}^\vee_{C,A}) \rightarrow H^1(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \rightarrow \cdots$$

Since $h^0(S,\mathcal{E}^\vee_{C,A}) = h^1(S,\mathcal{E}^\vee_{C,A}) = 0$, then by Proposition 2.2, $h^0(S,\mathcal{E}^\vee_{C,A+K_S} \otimes \mathcal{E}^\vee_{C,A}) = \dim \ker \mu_{0,A}$.

Now, suppose first that $\mathcal{E}_{C,A} \cong \mathcal{E}_{C,A+K_S}$. Since we are assuming stability, then it follows that the vector-bundles are simple, and so $h^0(S,\mathcal{E}_{C,A} \otimes \mathcal{E}^\vee_{C,A}) = 1$, and it follows that $\dim \ker \mu_{0,A} = 1$. By (1), $\dim W^1(C) = -g + 2d - 1$, and by putting $d \leq g - k$, we have $\dim W^1(C) = d - k - 1$.

Now assume that $\mathcal{E}_{C,A} \not\cong \mathcal{E}_{C,A+K_S}$. Since both $\mathcal{E}_{C,A}$ and $\mathcal{E}_{C,A+K_S}$ are $\mu_L$-stable, then (noting that $\mu_L(\mathcal{E}_{C,A}) = \mu_L(\mathcal{E}_{C,A+K_S})$), we have $h^0(S,\mathcal{E}_{C,A+K_S} \otimes \mathcal{E}^\vee_{C,A}) = 0$, and so $\dim \ker \mu_{0,A} = 0$, and $\dim W^1(C) = d - k$ by (1) and (1).

**Proposition 4.2.** Suppose that $\mathcal{E}_{C,A}$ is $\mu_L$-stable, and that $h^0(C,\mathcal{O}_S(A + K_S|C)) \leq 1$, for general $(C,A)$ in $W$. Then $\dim \ker \mu_{0,A} \leq 2$. It follows that if $d \leq g - k$, then $\dim W \leq g - 1 + d - k$.

**Proof.** Tensoring the sequence (7) by $\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)$ and taking cohomology, one gets

$$0 \rightarrow H^0(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \rightarrow H^0(S,\mathcal{E}^\vee_{C,A+K_S} \otimes \mathcal{O}_S(K_S)) \rightarrow H^1(C,\mathcal{E}^\vee_{C,A+K_S} \otimes \mathcal{O}_C(K_C - A)) \rightarrow H^1(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \rightarrow \cdots$$

We have $H^0(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) = 0$ by the stableness assumption, and so it follows that $H^0(C,\mathcal{E}^\vee_{C,A+K_S} \otimes \mathcal{O}_C(K_C - A))$ injects into $H^1(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \otimes \mathcal{O}_S(K_S)) \rightarrow H^1(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \otimes \mathcal{O}_S(K_S)) \rightarrow H^1(S,\mathcal{E}^\vee_{C,A} \otimes \mathcal{O}_S(K_S)) \otimes \mathcal{O}_S(K_S)) \rightarrow \cdots$.

By (11), it follows that if $d \leq g - k$, then $\dim W = -g + 2d$. By putting $d \leq g - k$, the result follows.

**Remark 4.3.** It is interesting to note that the same result can be obtained by considering the moduli-space $M$ of $\mu_L$-stable vector-bundles of rank 2 with $c_1 = L$ and $c_2 = d$ on $S$. It is known (see e.g. [Kim06, Remark, page 768]) that the dimension of the tangent space at $E$ is given by

$$\dim T_E M = 4c_2 - c_1^2 - 3 + h^2(S,\mathcal{E} \otimes \mathcal{E}^\vee).$$

Since $h^2(S,\mathcal{E} \otimes \mathcal{E}^\vee) = 0$ in our case, it follows that the dimension is given by $4d - L^2 - 3 = 4d - 2g - 1$.

By considering all possible injections $\Lambda \hookrightarrow H^0(S,\mathcal{E})$, as done in the proof of Proposition 3.9, we obtain $\dim W \leq 2d - 1$ using this approach, or equivalently, $\dim W^1(C) \leq 2d - g$. We have $2d - g \leq d - k$ precisely when $d \leq g - k$.

**Proof of Theorem 1.1.** Suppose that $\pi : W \rightarrow |L|$ dominates. From (5), we can assume that for general $(C,A) \in W$, we have that $|A|$ is base-point free. We can therefore for these $(C,A)$ consider vector-bundles $\mathcal{E}_{C,A}$. 
If for general \((C, A)\) in \(W\) we have \(\mathcal{E}_{C,A}\) non-\(\mu_L\)-stable, then by Proposition \([3.9]\) we have \(\dim W \leq g - 1 + d - k\). If for general \((C, A)\) in \(W\) we have \(\mathcal{E}_{C,A}\) \(\mu_L\)-stable, then we have the same bound by Propositions \([4.1]\) and \([4.2]\).

Since \(\dim |L| = g - 1\), the result follows. \(\square\)

5. Example of curves on Enriques surfaces with an infinite number of \(g^1_{\text{gon}(C)}\)’s

We here present an example of curves with an infinite number of \(g^1_{\text{gon}(C)}\)’s.

**Example 5.1.** Let \(S\) be any Enriques surface (which is possibly nodal). Let \(L = n(E_1 + E_2)\) for \(n \geq 3\), where \(E_1, E_2 = 2\), in which case, \(k = \mu(L)\) by \([\text{KL09}, \text{Corollary 1.5 (a)}]\). Then there exists a sub-linear system \(\mathfrak{d} \subseteq |L|\) of smooth curves such that for general \(C \in \mathfrak{d}\), there exist infinitely many \(g^1_{\text{gon}(C)}\)’s.

Indeed, let \(B = \mathcal{O}_S(E_1 + E_2)\), consider the map \(f_B : S \to \mathbb{P}^2\), and let \(\mathfrak{d} = f^*|\mathcal{O}_{\mathbb{P}^2}(n)|\). This is then a sub-linear system of \(|L|\), consisting of all curves that map 4–1 onto curves of \(|\mathcal{O}_{\mathbb{P}^2}(n)|\). By Bertini’s theorem, since this linear system is base-point free, the generic elements are smooth.

One constructs infinitely many \(g^1_{B,L-4}\)’s on a generic smooth curve \(C \in \mathfrak{d}\) in the following way: Let \(C = f^{-1}(C')\), where \(C'\) is smooth in \(\mathcal{O}_{\mathbb{P}^2}(n)\). Then \(C\) is also smooth. We let the \(g^1_{B,L-4}\)’s be \(f_B^*(\mathcal{O}_{C'}(1) \otimes \mathcal{O}_{C'}(-P))\), where \(P\) is any point on \(C'\). (On \(C\), this is the same as subtracting one point \(Q\) on \(B\) and noting that \(|B - Q|\) has three base-points \(f^{-1}(f(Q)) - Q\) that can also be subtracted.)

By \([\text{KL09}, \text{Corollary 1.6}]\), the minimal gonality is always at most 2 less than the generic gonality, and the generic gonality is given by \(B.L - 2\) by \([\text{KL09}, \text{Corollary 1.5}]\), so in our case, it follows that \(\text{gon}(C) = B.L - 4\). Since \(n \geq 3\), we are ensured that the \(g^1_{B,L-4}\)’s are distinct.

These \(g^1_{B,L-4}\)’s are, as far as we know, new examples of curves \(C\) with infinitely many \(g^1_{\text{gon}(C)}\)’s. The curves are furthermore non-exceptional.

These curves \(C\) are 4–1 coverings of plane curves, and the \(g^1_{\text{gon}(C')}\)’s are induced from the \(g^1_{\text{gon}(C')}\)’s. According to the Castelnuovo–Severi inequality (see e.g. \([\text{Kan84}]\)), whenever we have an \(m\)–1 covering from a curve \(C\) to a curve \(C'\), if \(g(C) > mg(C') + (m - 1)(d - 1)\), then any base-point free \(g^1_{\mathcal{E}}\) on \(C\) is induced by a base-point free linear system on \(C'\). In particular, if \(d = \text{gon}(C)\) and \(C'\) has infinitely many \(g^1_{\text{gon}(C')}\)’s, then \(C\) also has infinitely many \(g^1_{\text{gon}(C)}\)’s. However, in this example, \(g(C) \leq mg(C') + (m - 1)(d - 1)\).

Furthermore, by \([\text{CM91}, \text{Corollary 2.3.1}]\), any exceptional curve \(C\) has infinitely many \(g^1_{\text{gon}(C)}\)’s. However, by \([\text{KL13}, \text{Theorem 1.1}]\), the only exceptional curves \(C\) on Enriques surfaces are isomorphic to smooth plane quintics and satisfy \(C^2 = 10\). It follows that the curves in our example are non-exceptional.

**References**

[A11] A. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, **Geometry of algebraic curves**, vol. 1, Springer Verlag, 1985.

[AF11] M. Aprodu and G. Farkas, **Green’s conjecture for curves on arbitrary K3 surfaces**, Compositio Math 147 (2011), 839–851.

[Apr05] M. Aprodu, **Remarks on syzygies of d-gonal curves**, Math. Res. Lett. 12 (2005), 387–400.

[CM91] M. Coppens and G. Martens, **Secant spaces and Clifford’s theorem**, Compositio Mathematica 78 (1991), 193–212.
[CP95] C. Ciliberto and G. Pareschi, Pencils of minimal degree on curves on a K3 surface, Journal für die reine und angewandte Mathematik 460 (1995), 14–36.

[Fri98] R. Friedman, Algebraic surfaces and holomorphic vector bundles, Springer Verlag, 1998.

[GH80] P. Griffiths and J. Harris, The dimension of the variety of special linear systems on a general curve, Duke Math. J 47 (1980), 233–272.

[GL87] M. Green and R. Lazarsfeld, Special divisors on curves on a K3 surface, Inventiones mathematicae 89 (1987), no. 2, 357–370.

[Gre84] M. Green, Koszul cohomology and the geometry of projective varieties (appendix by M. Green and R. Lazarsfeld), Journal of differential geometry 6 (1984), 125–171.

[Kan84] E. Kani, On Castelnuovo’s equivalence defect, Journal für die reine und angewandte Mathematik 352 (1984), 24–70.

[Kem71] G. Kempf, Schubert methods with an application to algebraic curves, Stichting Mathematisch Centrum, 1971.

[Kim06] H. Kim, Stable vector bundles of rank two on Enriques surfaces, J. Korean Math. Soc 43 (2006), no. 4, 765–782.

[KL72] S.L. Kleiman and D. Laksov, On the existence of special divisors, American Journal of Mathematics 94 (1972), no. 2, 431–436.

[KL07] A.L. Knutsen and A.F. Lopez, A sharp vanishing theorem for line bundles on K3 or Enriques surfaces, Proceedings of the American Mathematical Society 135 (2007), no. 11, 3495–3498.

[KL09] ———, Brill-Noether theory for curves on Enriques surfaces, I: the positive cone and gonality, Mathematische Zeitschrift 261 (2009), 659–690.

[KL13] ———, Brill-Noether theory of curves on Enriques surfaces, II. The Clifford index, arXiv preprint arXiv:1308.1074 (2013).

[Knu01] A.L. Knutsen, On kth-order embeddings of K3 surfaces and Enriques surfaces, manuscripta mathematica 104 (2001), no. 2, 211–237.

[Knu03] ———, Gonality and Clifford index of curves on K3 surfaces, Archiv der Mathematik 80 (2003), no. 3, 235–238.

[Knu09a] ———, On secant spaces to Enriques surfaces, Bulletin of the Belgian Mathematical Society-Simon Stevin 16 (2009), no. 5, 907–931.

[Knu99b] ———, On two conjectures for curves on K3 surfaces, International Journal of Mathematics 20 (2009), 1547–1560.

[Laz86] R. Lazarsfeld, Brill-Noether-Petri without degenerations, J. Diff. Geom. 23 (1986), 299–307.

[LC12] M. Lelli-Chiesa, Green’s conjecture for curves on rational surfaces with an anticanonical pencil, Mathematische Zeitschrift (2012), 1–12.

[Mar68] H.H. Martens, On the varieties of special divisors on a curve. II, Journal für die reine und angewandte Mathematik (Crelles Journal) 1968 (1968), no. 233, 89–100.

[Mar84] G. Martens, On dimension theorems of the varieties of special divisors on a curve, Mathematische Annalen 267 (1984), no. 2, 279–288.

[Par95] G. Pareschi, A proof of Lazarsfeld’s theorem on curves on K3 surfaces, J. Alg. Geom. 4 (1995), 195–200.

[Tyu87] A.N. Tyurin, Cycles, curves and vector bundles on an algebraic surface, Duke Math. J 54 (1987), no. 1, 1–26.

Nils Henry Rasmussen and Shengtian Zhou
nils.h.rasmussen@hit.no, shengtian.zhou@hit.no
Telemark University College, Dept. of Teacher Education
Lærerskolevegen 40, 3679 NOTODDEN, Norway