Stability of McKean-Vlasov stochastic differential equations and applications

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Abstract

We consider McKean-Vlasov stochastic differential equations (MVSDEs), which are SDEs where the drift and diffusion coefficients depend not only on the state of the unknown process but also on its probability distribution. This type of SDEs was studied in statistical physics and represents the natural setting for stochastic mean-field games. We will first discuss questions of existence and uniqueness of solutions under an Osgood type condition improving the well known Lipschitz case. Then we derive various stability properties with respect to initial data, coefficients and driving processes, generalizing known results for classical SDEs. Finally, we establish a result on the approximation of the solution of a MVSDE associated to a relaxed control by the solutions of the same equation associated to strict controls. As a consequence, we show that the relaxed and strict control problems have the same value function. This last property improves known results proved for a special class of MVSDEs, where the dependence on the distribution was made via a linear functional.

Key words: McKean-Vlasov stochastic differential equation – Stability – Martingale measure - Wasserstein metric – Existence – Mean-field control – Relaxed control.

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1 Introduction

We will investigate some properties of a particular class of stochastic differential equations (SDE), called McKean-Vlasov stochastic differential equations (MVSDE) or mean-field stochastic differential equations. These are SDEs described by

\[
\begin{cases}
  dX_t = b(t, X_t, P_{X_t}) \, ds + \sigma(t, X_t, P_{X_t}) \, dB_s \\
  X_0 = x,
\end{cases}
\]

where \( b \) is the drift, \( \sigma \) is the diffusion coefficient and \( (B_t) \) is a Brownian motion. For this type of equations the drift and diffusion coefficient depend not only on the state variable \( X_t \), but also on its marginal distribution \( P_{X_t} \). This fact brings a non trivial additional difficulty compared to classical Itô SDEs. The solutions of such equation are known in the literature as non linear diffusions.

MVSDEs were first studied in statistical physics by M. Kac [19], as a stochastic counterpart for the Vlasov equation of plasma [28]. The probabilistic study of such equation has been performed by H.P. Mc Kean [22], see [27] for an introduction to this research field. These equations were obtained as limits of some weakly interacting particle systems as the number of particles tends to infinity.

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This convergence property is called in the literature as the propagation of chaos. The MVSDE, represents in some sense the average behavior of the infinite number of particles. One can refer to \([8, 11, 18]\) for details on the existence and uniqueness of solutions for such SDEs, see also \([6, 7]\) for the case of McKean-Vlasov backward stochastic differential equations (MVBSDE). Existence and uniqueness with less regularity on the coefficients have been established in \([9, 10, 11, 15, 25, 26]\). Recently there has been a renewed interest for MVSDEs, in the context of mean-field games (MFG) theory, introduced independently by P.L. Lions and J.M. Lasry \([20]\) and Huang, Malhamé, Caines \([16]\) in 2006. MFG theory has been introduced to solve the problem of existence of an approximate Nash equilibrium for differential games, with a large number of players (see \([5]\)). Since the earlier papers, MFG theory and mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks and management of oil resources. One can refer to the most recent and updated reference on the subject \([8]\) and the complete bibliographical list therein.

Our main objective in this paper is to study some properties of such equations such as existence, uniqueness, and stability properties. In particular, we prove an existence and uniqueness theorem for a class of MVSDEs under Osgood type condition on the coefficients, improving the well known globally Lipschitz case. It is well known that stability properties of deterministic or stochastic dynamical systems are crucial in the study of such systems. It means that the trajectories do not change too much under small perturbations. We study stability with respect to initial conditions, coefficients and driving processes, which are continuous martingales and bounded variation processes. These properties will be investigated under Lipschitz condition with respect to the state variable and the distribution and generalize known properties for classical Itô SDEs, see \([4, 17]\). Furthermore, we prove that in the context of stochastic control of systems driven by MVSDEs, the relaxed and strict control problems have the same value function. As it is well known when the Filipov type convexity condition is not fulfilled, there is no mean to prove the existence of a strict control. The idea is then to embedd the usual strict controls into the set of measure valued controls, called relaxed controls, which enjoys good compactness properties. So for the relaxed control to be a true extension of the initial problem, the value functions of both control problems must be the same. Under the Lipschitz condition we prove that the value functions are equal. Note that this result extends to general McKean-Vlasov equations known results \([2, 3]\) established for a special class of MVSDEs, where the dependence of the coefficient on the distribution variable is made via a linear form of the distribution.

2 Formulation of the problem and preliminary results

2.1 Assumptions

Let \((\Omega, \mathcal{F}, P)\) be a probability space, equipped with a filtration \((\mathcal{F}_t)\), satisfying the usual conditions and \((B_t)\) a \(d\)-dimensional \((\mathcal{F}_t, P)\)–Brownian motion. Let us consider the following McKean-Vlasov stochastic differential equation called also mean-field stochastic differential equation (MVSDE)

\[
\begin{aligned}
dX_t &= b(t, X_t, \mathbb{P}_{X_t})dt + \sigma(t, X_t, \mathbb{P}_{X_t})dB_t \\
X_0 &= x
\end{aligned}
\tag{2.1}
\]

Note that for this kind of SDEs, the drift \(b\) and diffusion coefficient \(\sigma\) depend not only on the position, but also on the marginal distribution of the solution.

The following assumption will be considered throughout this paper.

Let us denote \(\mathcal{P}_2(\mathbb{R}^d)\) the space of probability measures with finite second order moment. That is for each \(\mu \in \mathcal{P}_2(\mathbb{R}^d)\)

\[
\int |x|^2 \mu(dx) < +\infty.
\]
(H$_1$) Assume that

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$$

$$\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

are Borel measurable functions and there exist $C > 0$ such that for every $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:

$$|b(t, x, \mu)| + |\sigma(t, x, \mu)| \leq C (1 + |x|)$$

(H$_2$) There exist $L > 0$ such that for any $t \in [0, T], x, x' \in \mathbb{R}^d$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(t, x, \mu) - b(t, x', \mu')| \leq L [||x - x'|| + W_2(\mu, \mu')]$$

$$|\sigma(t, x, \mu) - \sigma(t, x', \mu')| \leq L [||x - x'|| + W_2(\mu, \mu')]$$

where $W_2$ denotes the 2-Wasserstein metric.

2.2 Wasserstein metric

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ and for any $p > 1$, denote by $\mathcal{P}_p(\mathbb{R}^d)$ the subspace of $\mathcal{P}(\mathbb{R}^d)$ of the probability measures with finite moment of order $p$.

For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, define the $p$-Wasserstein distance $W_p(\mu, \nu)$ by:

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left[ \int_{E \times E} |x - y|^p d\pi(x, y) \right]^{1/p}$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginals are respectively $\mu$ and $\nu$.

In the case $\mu = \mathbb{P}_X$ and $\nu = \mathbb{P}_Y$ are the laws of $\mathbb{R}^d$-valued random variable $X$ and $Y$ of order $p$, then

$$W_p(\mu, \nu)^p \leq \mathbb{E}[|X - Y|^p].$$

Indeed

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left[ \int_{E \times E} |x - y|^p d\pi(x, y) \right]^{1/p}$$

$$W_p(\mu, \nu)^p = \inf_{\pi \in \Pi(\mu, \nu)} \left[ \int_{E \times E} |x - y|^p d\pi(x, y) \right]$$

$$\leq \int_{E \times E} |x - y|^p d(\mathbb{P}(X, Y))(x, y)$$

$$= \mathbb{E}[|X - Y|^p]$$

In the literature the Wasserstein metric is restricted to $W_2$ while $W_1$ is often called the Kantorovich-Rubinstein distance because of the role it plays in optimal transport.

3 Existence and uniqueness of solutions

3.1 The globally Lipschitz case

The following theorem states that under global Lipschitz condition, [27] admits a unique solution. Its complete proof is given in [27] for a drift depending linearly on the law of $X_t$ that is $b(t, x, \mu) =$
\[ \int b'(t, x, y) \mu(dy) \] and a constant diffusion. The general case as in (2.1) is treated in [8] Theorem 4.21 or [13] Proposition 1.2 and is based on a fixed point theorem on the space of continuous functions with values in \( \mathcal{P}_2(\mathbb{R}^d) \). Note that in [13] the authors consider MVSDEs driven by general Lévy process instead of a Brownian motion.

**Theorem 3.1.** Under assumptions \( (H_1), (H_2), (2.1) \) admits a unique solution such that \( E[\sup_{t \leq T} |X_t|^2] < +\infty \)

**Proof.**

Let us give the outline of the proof. Let \( \mu \in \mathcal{P}_p(\mathbb{R}^d) \) be fixed, the classical Itô’s theorem gives the existence and uniqueness of a solution denote by \( (X_t^\mu) \) satisfying \( E[\sup_{t \leq T} |X_t^\mu|^2] < +\infty \). Now let us consider the mapping

\[
\Psi : C([0, T], \mathcal{P}_2(\mathbb{R}^d)) \rightarrow C([0, T], \mathcal{P}_2(\mathbb{R}^d)) \\
\mu \rightarrow \Psi(\mu) = (\mathcal{L}(X_t^\mu))_{t \geq 0}, \text{ the distribution of } X_t^\mu.
\]

\( \Psi \) is well defined as \( X_t^\mu \) has continuous paths and \( E[\sup_{t \leq T} |X_t^\mu|^2] < +\infty \).

To prove the existence and uniqueness of (2.1), it is sufficient to check that \( \Psi \) has a unique fixed point. By using usual arguments from stochastic calculus and relation and the property of Wasserstein metric it is easy to show that:

\[
sup_{t \leq T} W_2((\Psi^k(\mu)), (\Psi^k(\nu)))^2 \leq C \frac{T^k}{k!} \sup_{t \leq T} W_2(\mu, \nu)^2
\]

For large \( k \), \( \Psi^k \) is a strict contraction which implies that \( \Psi \) admits a unique fixed point in the complete metric space \( C([0, T], \mathcal{P}_2(\mathbb{R}^d)) \).

The following version MVSDEs is also considered in the control literature

\[
\begin{aligned}
&dX_t = b(t, X_t, \int \varphi(y)\mathbb{P}_{X_t}(dy))dt + \sigma(t, X_t, \int \psi(y)\mathbb{P}_{X_t}(dy))dW_t \\
&X_0 = x
\end{aligned}
\]  

(3.1)

where

\((H_3)\) \( b, \sigma, \varphi \) and \( \psi \) are Borel measurable bounded functions such that \( b(t,..), \sigma(t,..), \varphi \) and \( \psi \) are globally lipshitz functions in \( \mathbb{R}^d \times \mathbb{R}^d \).

**Proposition 3.2.** Under assumptions \((H_1)\) and \((H_3)\) the MVSDE (3.1) has a unique strong solution. Moreover for each \( p > 0 \) we have \( E(|X_t|^p) < +\infty \).

**Proof.**

Let us define \( \overline{b}(t, x, \mu) \) and \( \overline{\sigma}(t, x, \mu) \) on \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\) by

\[
\overline{b}(t, x, \mu) = b(., \int \varphi(x)d\mu(x)), \overline{\sigma}(t, x, \mu) = \sigma(t, x, \int \psi(x)d\mu(x)).
\]

According to the last Theorem it is sufficient to check that \( \overline{b} \) and \( \overline{\sigma} \) are Lipschitz in \((x, \mu)\). Indeed since the coefficients \( b \) and \( \sigma \) are Lipschitz continuous in \( x \), then \( \overline{b} \) and \( \overline{\sigma} \) are also Lipschitz continuous in \( x \). Moreover one can verify easily that \( \overline{b} \) and \( \overline{\sigma} \) are also Lipschitz continuous in \( \mu \), with respect to the Wasserstein metric

\[
W_2(\mu, \nu) = \inf \left\{ (E^Q |X - Y|^2)^{1/2} ; Q \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d), \text{ with marginals } \mu, \nu \right\}
\]

\[
= \sup \left\{ \int hd(\mu - \nu) ; |h(x) - h(y)| \leq |x - y| \right\},
\]

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Note that the second equality is given by the Kantorovich-Rubinstein theorem [8]. Since the mappings $b$ and $\varphi$ in the the MFSDE are Lipschitz continuous in $y$ we have

$$\left| b(., \ldots, \int \varphi(y) d\mu(y), .) - b(., \ldots, \int \varphi(y) d\nu(y), .) \right| \leq K \left| \varphi(y) d(\mu(y) - \nu(y)) \right| \leq K' W_2(\mu, \nu).$$

Therefore $\mathbf{b}(t, ., .)$ is Lipschitz continuous in the variable $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ uniformly in $t \in [0, [T]]$.

Similar arguments can be used for $\sigma$.

### 3.2 The uniqueness under Osgood type condition

In this section we relax the global Lipschitz condition in the state variable. We will prove the existence and uniqueness of a solution when the coefficients are globally Lipschitz in the distribution variable and satisfy an Osgood type condition in the state variable. To be more precise let us consider the following MVSDE

$$\begin{cases}
    dX_t = b(t, X_t, \mathbb{P}_{X_t}) ds + \sigma(t, X_t) dB_s \\
    X_0 = x
\end{cases} \tag{3.2}$$

Assume that $b$ and $\sigma$ are real valued bounded Borel measurable functions satisfying:

(H$_4$) There exist $C > 0$, such that for every $x \in \mathbb{R}$ and $(\mu, \nu) \in \mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R})$:

$$|b(t, x, \mu) - b(t, x, \nu)| \leq CW_1(\mu, \nu)$$

(H$_5$) There exists a strictly increasing function $\rho(u)$ on $[0, +\infty)$ such that $\rho(0) = 0$ and $\rho^2$ is convex satisfying $\int_0^{+\infty} \rho^{-2}(u) du = +\infty$, such that for every $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, $|\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|)$.

(H$_6$) There exists a strictly increasing function $\kappa(u)$ on $[0, +\infty)$ such that $\kappa(0) = 0$ and $\kappa$ is concave satisfying $\int_0^{+\infty} \kappa^{-1}(u) du = +\infty$, such that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R})$,

$$|b(t, x, \mu) - b(t, y, \mu)| \leq \kappa(|x - y|).$$

In the next Theorem we derive the pathwise uniqueness for (3.2) under an Osgood type condition in the state variable. This result improves [17], Theorem 3.2, established for classical Itô’s SDEs and [8] Theorem 4.21, at least for MVSDEs with a diffusion coefficient not depending on the distribution variable.

**Theorem 3.3.** Under assumptions (H$_4$)–(H$_6$), the MVSDE (3.2) enjoys the property of pathwise uniqueness.

**Proof.**

The following proof is inspired from [8] Theorem 4.21.

Since $\int_0^{+\infty} \rho^{-2}(u) du = +\infty$, there exist a decreasing sequence $(a_n)$ of positive real numbers such that $1 > a_1$ satisfying

$$\int_0^{a_n} \rho^{-2}(u) du = +\infty.$$
\[
\int_{a_1}^{1} \rho^{-2}(u)du = 1, \quad \int_{a_2}^{a_1} \rho^{-2}(u)du = 2, \ldots, \quad \int_{a_n}^{a_{n-1}} \rho^{-2}(u)du = n, \ldots
\]

Clearly \((a_n)\) converges to 0 as \(n\) tends to \(+\infty\).

The properties of \(\rho\) allow us to construct a sequence of functions \(\psi_n(u), n = 1, 2, \ldots\), such that

i) \(\psi_n(u)\) is a continuous function such that its support is contained in \((a_n, a_{n-1})\)

ii) \(0 \leq \psi_n(u) \leq \frac{2}{n} \rho^{-2}(u)\) and \(\int a_n \psi_n(u)du = 1\)

Let \(\varphi_n(x) = \int_{0}^{y} \psi_n(u)du, x \in \mathbb{R}\)

It is clear that \(\varphi_n \in C^2(\mathbb{R})\) such that \(|\varphi_n'| \leq 1\) and \((\varphi_n)\) is an increasing sequence converging to \(|x|\).

Let \(X_t^1\) and \(X_t^2\) two solutions of corresponding to the same Brownian motion and the same MVSDE

\[X_t^1 - X_t^2 = \int_{0}^{t} (\sigma(s, X_s^1) - \sigma(s, X_s^2))dW_s + \int_{0}^{t} (b(s, X_s^1, \mathbb{P}_{X_s^1}) - b(s, X_s^2, \mathbb{P}_{X_s^2}))dW_s\]

By using Itô’s formula we obtain

\[
\varphi_n(X_t^1 - X_t^2) = \int_{0}^{t} \varphi_n'(X_s^1 - X_s^2) (\sigma(s, X_s^1) - \sigma(s, X_s^2))dW_s
\]

\[+ \int_{0}^{t} \varphi_n'(X_s^1 - X_s^2) (b(s, X_s^1, \mathbb{P}_{X_s^1}) - b(s, X_s^2, \mathbb{P}_{X_s^2})) ds\]

\[+ \frac{1}{2} \int_{0}^{t} \varphi_n''(X_s^1 - X_s^2) (\sigma(s, X_s^1) - \sigma(s, X_s^2))^2 ds\]

\(\varphi_n'\) and \(\sigma\) being bounded, then the process under the sign integral is sufficiently integrable.

Then the first term is a true martingale, so that its expectation is 0. Therefore

\[E (\varphi_n(X_t^1 - X_t^2)) = E \left[ \int_{0}^{t} \varphi_n'(X_s^1 - X_s^2) (b(s, X_s^1, \mathbb{P}_{X_s^1}) - b(s, X_s^2, \mathbb{P}_{X_s^2})) ds \right]\]

\[+ \frac{1}{2} E \left[ \int_{0}^{t} \varphi_n''(X_s^1 - X_s^2) (\sigma(s, X_s^1) - \sigma(s, X_s^2))^2 ds \right]\]

\[= I_1 + I_2\]

But we know that \(W_t(\mathbb{P}_{X_s^1}, \mathbb{P}_{X_s^2}) = E (|X_t^1 - X_t^2|)\)

Then

\[|I_1| \leq E \int_{0}^{t} \kappa(|X_s^1 - X_s^2|)ds + \int_{0}^{t} CE (|X_s^1 - X_s^2|) ds\]

Then by Growall lemma, there exist a constant \(M\) such that \(|I_1| \leq M E \int_{0}^{t} \kappa(|X_s^1 - X_s^2|)ds\)

On the other hand
Assume that the convergence of the Picard successive approximation implies the existence and uniqueness of a strong solution of equation (3.2) (see [21] example 2.14, page 10), the pathwise uniqueness proved in the last theorem (see [18] Proposition 1.10). Then by the well-known Yamada-Watanabe theorem applied to the solution of (2.1). Let \( \phi_1 \) of the Picard iteration scheme. This scheme is useful for numerical computations of the unique solution of (2.1). Letting \( \phi_1 \) tending to \(+\infty\) as \( n \rightarrow \infty \) it holds that: \(|X^n_t - X^n_s|^2 \leq \int_0^t \varphi_n'(X^n_s - X^n_t) (\sigma(s, X^n_s) - \sigma(s, X^n_t))^2 ds\), \( \varphi_n = \frac{1}{n} \int_0^t \varphi_n'(X^n_s - X^n_t) \rho_n(X^n_s - X^n_t) ds \). Then \(|I_2|\) tends to 0 as \( n \) tends to \(+\infty\).

Then \(|I_2|\) tends to 0 as \( n \) tends to \(+\infty\).

Letting \( n \) tending to \(+\infty\) it holds that: \( E(|X^n_t - X^n_s|^2) \leq M.E \int_0^t \kappa(|X^n_s - X^n_t|) ds\). Since \( \int_0^{+\infty} \kappa^{-1}(u) du = +\infty \) we conclude that \( E(|X^n_t - X^n_s|^2) = 0 \).

**Remark.** The continuity and boundedness of the coefficients imply the existence of a weak solution (see [18] Proposition 1.10). Then by the well-known Yamada-Watanabe theorem applied to equation (3.2) (see [21] example 2.14, page 10), the pathwise uniqueness proved in the last theorem implies the existence and uniqueness of a strong solution.

### 4 Convergence of the Picard successive approximation

Assume that \( b(t, x, \mu) \) and \( \sigma(t, x, \mu) \) satisfy assumptions (H\(_1\)), (H\(_2\)). We will prove the convergence of the Picard iteration scheme. This scheme is useful for numerical computations of the unique solution of (2.1). Let \( (X^n_t) = x \) for all \( t \in [0, T] \) and define \( (X^{n+1}_t) \) as the solution of the following SDE

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\dot{X}^{n+1}_t &= b(t, X^n_t, \mathbb{P}_{X^n_t}) dt + \sigma(t, X^n_t, \mathbb{P}_{X^n_t}) dB_t \\
X^{n+1}_0 &= x
\end{array}
\right.
\end{align*}
\]

**Theorem 4.1.** Under assumptions (H\(_1\)), (H\(_2\)), the sequence \((X^n)\) converges to the unique solution of (2.1)

\[ E[\sup_{t \leq T} |X^n_t - X_t|^2] \rightarrow 0 \]

**Proof.** Let \( n \geq 0 \), by applying usual arguments such as Schwartz inequality and Burkholder-Davis Gundy inequality for the martingale part, we get

\[
|X^{n+1}_t - X^n_t|^2 \leq 2 \int_0^t [b(s, X^n_s, \mathbb{P}_{X^n_s}) - b(s, X^{n-1}_s, \mathbb{P}_{X^{n-1}_s})] ds^2 \\
+ 2 \int_0^t [\sigma(s, X^n_s, \mathbb{P}_{X^n_s}) - \sigma(s, X^{n-1}_s, \mathbb{P}_{X^{n-1}_s})] dB_s^2 \\
E[\sup_{t \leq T} |X^{n+1}_t - X^n_t|^2] \leq 2T E\left[ \int_0^T [b(s, X^n_s, \mathbb{P}_{X^n_s}) - b(s, X^{n-1}_s, \mathbb{P}_{X^{n-1}_s})]^2 ds \right] \\
+ 2C_2 E\left[ \int_0^T [\sigma(s, X^n_s, \mathbb{P}_{X^n_s}) - \sigma(s, X^{n-1}_s, \mathbb{P}_{X^{n-1}_s})]^2 ds \right]
\]

the coefficients \( b \) and \( \sigma \) being Lipschitz continuous in \((x, \mu)\) we get
\[ E[\sup_{t \leq T} |X_{t+1}^n - X_t^n|^2] \leq 2(T + C_2)L^2 \int_0^T E[|X_s^n - X_s^{n-1}|^2] + W_2(\mathbb{P}_{X_t^n}, \mathbb{P}_{X_{t-1}^n}) ds \]

\[ \leq 4(T + C_2)L^2 \int_0^T E[|X_s^n - X_s^{n-1}|^2] ds \]

\[ \leq 4(T + C_2)L^2 \int_0^T E[\sup_{t \leq T} |X_s^n - X_s^{n-1}|^2] ds \]

Then for all \( n \geq 1, \) and \( t \leq T \)

\[ E[\sup_{t \leq T} |X_t^n - X_0^n|^2] \leq 2(T + C_2)L^2 \int_0^T b(s, x, \mu)^2 ds + C_2 \int_0^T \sigma(s, x, \mu)^2 ds \]

\[ \leq 2(C_2 + T)M(1 + E[|x|^2])T \]

\[ \leq A_1 T \]

where the constant \( A_1 \) only depends on \( C_2, M, T \) and \( E[|x|^2] \). So by induction on \( n \) we obtain

\[ E[\sup_{t \leq T} |X_{t+1}^n - X_t^n|^2] \leq \frac{A_2^{n+1}T^{n+1}}{(n+1)!} \]

This implies in particular that \((X_t^n)\) is a Cauchy sequence in \( L^2(\Omega, C([0, T], \mathbb{R}^d)) \) which is complete. Therefore \((X_t^n)\) converges to a limit \((X_t)\) which is the unique solution of (2.1) \( \square \)

5 Stability with respect to initial condition

In this section, we will study the stability of MFSDEs with respect to small perturbations of the initial condition.

We denote by \((X_t^x)\) the unique solution of (2.1) such that \( X_0^x = x\)

\[
\begin{align*}
    dX_t^x &= b(t, X_t^x, \mathbb{P}_{X_t^x}) dt + \sigma(t, X_t^x, \mathbb{P}_{X_t^x}) dB_t \\
    X_0^x &= x
\end{align*}
\]

**Theorem 5.1.** Assume that \( b(t, x, \mu) \) and \( \sigma(t, x, \mu) \) satisfy \((H_1), (H_2)\), then the mapping

\[
\Phi : \mathbb{R}^d \rightarrow L^2(\Omega, C([0, T], \mathbb{R}^d))
\]

defined by \( \Phi(x)_t = (X_t^x) \) is continuous.

**Proof.**

Let \((x_n)\) be a sequence in \( \mathbb{R}^d \) converging to \( x \). Let us prove that \( \lim_{n \to +\infty} E \left[ \sup_{t \leq T} |X_t^n - X_t|^2 \right] = 0 \),

where \( X_t^n = X_t^{x_n} \). We have

\[
|X_t^n - X_t|^2 = |x_n - x| + \int_0^t (b(s, X_s^n, \mathbb{P}_{X_s^n}) - b(s, X_s, \mathbb{P}_{X_s})) ds \\
+ \int_0^t (\sigma(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma(s, X_s, \mathbb{P}_{X_s})) dB_s
\]
In this section, we will establish the stability of the MVSDE with respect to small perturbation of the coefficients $b$ and $\sigma$. Let us consider sequences of functions $(b_n)$ and $(\sigma_n)$ and consider the corresponding MFSDE:

$$dX^n_t = b_n(t, X^n_t, \mathbb{P}_{X^T}) dt + \sigma_n(t, X^n_t, \mathbb{P}_{X^T}) dB_t$$

(6.1)

$$X^n_0 = x$$

The following theorem gives us the continuous dependence of the solution with respect to the coefficients.

$$E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] \leq 3|x_n - x|^2 + 3 \left( \int_0^t |b(s, X^n_s, \mathbb{P}_{X^T}) - b(s, X^n_s, \mathbb{P}_{X^T})| ds \right)^2 + 3 \left( \int_0^t |\sigma(s, X^n_s, \mathbb{P}_{X^T}) - \sigma(s, X^n_s, \mathbb{P}_{X^T})| dB_s \right)^2$$

Therefore

$$\lim_{n \to \infty} E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] = 0.$$

By the Lipschitz condition, we have

$$E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] \leq 3|x_n - x|^2 + 3T \left( C_2 \int_0^T E|X^n_s - X_s|^2 ds \right)$$

and

$$E \left[ \sup_{s \leq t} |b(s, X^n_s, \mathbb{P}_{X^T}) - b(s, X^n_s, \mathbb{P}_{X^T})|^2 ds \right] \leq 6(T + c_2)L^2 \int_0^T E|X^n_s - X_s|^2 ds.$$

Finally, applying the Gronwall lemma, we conclude that

$$E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] \leq 3|x_n - x|^2 \exp\{6(T + c_2)L^2t\}$$

Therefore $\lim_{n \to \infty} x_n = x$ implies that $\lim_{n \to \infty} E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] = 0.$

6 Stability with respect to the coefficients

The previous theorem gives us the continuous dependence of the solution with respect to the coefficients.
Theorem 6.1. Assume that the functions $b(t, x, \mu)$, $b_n(t, x, \mu)$, $\sigma(t, x, \mu)$ and $\sigma_n(t, x, \mu)$ satisfy $(H_1)$, $(H_2)$. Further suppose that for each $T > 0$, and each compact set $K$ there exists $C > 0$ such that

\begin{enumerate} 
  \item $\sup_{t \leq T} (|b_n(t, x, \mu)| + |\sigma_n(t, x, \mu)|) \leq C(1 + |x|)$,
  \item $\lim_{n \to \infty} \sup_{t \leq T} \sup_{x \in K} \sup_{\mu \in \mathbb{P}_2(\mathbb{R}^d)} (|b_n(t, x, \mu) - b(t, x, \mu)| + |\sigma_n(t, x, \mu) - \sigma(t, x, \mu)|) = 0$
\end{enumerate}

then

\[ \lim_{n \to \infty} E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] = 0 \]

where $(X^n_t)$ and $(X_t)$ are respectively solutions of $(6.1)$ and $(2.1)$.

**Proof.**

For each $n \in \mathbb{N}$, let $(X^n_t)$ be a solution of $(6.1)$, then by using

\[
|X^n_t - X_t|^2 \leq 3 \left( \int_0^t |b_n(s, X^n_s, \mathbb{P}X^n_s) - b_n(s, X_s, \mathbb{P}X_s)| ds \right)^2 \\
+ 3 \left( \int_0^t |b_n(s, X_s, \mathbb{P}X_s) - b(s, X_s, \mathbb{P}X_s)| ds \right)^2 \\
+ 3 \left( \int_0^t (\sigma_n(s, X^n_s, \mathbb{P}X^n_s) - \sigma_n(s, X_s, \mathbb{P}X_s)) dB_s \right)^2 \\
+ 3 \left( \int_0^t (\sigma_n(s, X_s, \mathbb{P}X_s) + \sigma(s, X_s, \mathbb{P}X_s)) dB_s \right)^2
\]

By using the Lipschitz continuity and Burkholder Davis Gundy inequality, it holds that

\[
E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] \leq 3(T + C_2)L^2 \int_0^T E[|X^n_s - X_s|^2] + W_2(\mathbb{P}X^n_s, \mathbb{P}X_s)^2] ds \\
+ 3(T + C_2)E[\int_0^t |b_n(s, X_s, \mathbb{P}X_s) - b(s, X_s, \mathbb{P}X_s)|^2 ds] \\
+ 3(T + C_2)E[\int_0^t |\sigma_n(s, X_s, \mathbb{P}X_s) - \sigma(s, X_s, \mathbb{P}X_s)|^2 ds] \\
\leq 6(T + C_2)L^2 \int_0^T E[|X^n_s - X_s|^2] ds + K_n \\
\leq 6(T + C_2)L^2 \int_0^T E \left[ \sup_{s \leq t} |X^n_s - X_s|^2 \right] dt + K_n
\]

such that

\[
K_n = 3(T + C_2)E[\int_0^T (|b_n(s, X_s, \mathbb{P}X_s) - b(s, X_s, \mathbb{P}X_s)|^2 + |\sigma_n(s, X_s, \mathbb{P}X_s) + \sigma(s, X_s, \mathbb{P}X_s)|^2) ds]
\]

An application of Gronwall lemma allows us to get

\[
E \left[ \sup_{t \leq T} |X^n_t - X_t|^2 \right] \leq K_n \exp(6(T + C_2)L^2 \cdot T)
\]

By using assumptions i) and ii) it is easy to see that $K_n \to 0$ as $n \to +\infty$, which achieves the proof.
7 Stability with respect to the driving processes

In this section, we consider McKean-Vlasov SDE driven by continuous semi-martingales.

Let \( b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \) and \( \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d} \) be bounded continuous functions.

We consider MVSDEs driven by continuous semi-martingales of the following form

\[
\begin{cases}
    dX_t = b(t, X_t, \mathbb{P}_{X_t})dA_t + \sigma(t, X_t, \mathbb{P}_{X_t})dM_t \\
    X_0 = x
\end{cases}
\]  

(7.1)

where \( A_t \) is an adapted continuous process of bounded variation and \( M_t \) is a continuous local martingale.

Let us consider the following sequence of MVSDEs

\[
\begin{cases}
    dX^n_t = b(t, X^n_t, \mathbb{P}_{X^n_t})dA^n_t + \sigma(t, X^n_t, \mathbb{P}_{X^n_t})dM^n_t \\
    X^n_0 = x
\end{cases}
\]

(7.2)

where \((A^n)\) is a sequence of \(\mathcal{F}_t\)-adapted continuous process of bounded variation and \(M^n\) is continuous \((\mathcal{F}_t, \mathbb{P})\)-local martingales.

Let us assume that \((A, A^n, M, M^n)\) satisfy:

\((\mathcal{H}_7)\)

1) The family \((A, A^n, M, M^n)\) is bounded in \(C([0, 1])^4\).
2) \((M^n - M)\) converges to 0 in probability in \(C([0, 1])\) as \(n\) tends to +\(\infty\).
3) The total variation \((A^n - A)\) converges to 0 in probability as \(n\) tends to +\(\infty\).

**Theorem 7.1.** Let \(b(t, x, \mu)\) and \(\sigma(t, x, \mu)\) satisfy \((\mathcal{H}_1), (\mathcal{H}_2)\). Further assume that \((A, A^n, M, M^n)\) satisfy \((\mathcal{H}_7)\). Then for each \(\varepsilon > 0\)

\[
\lim_{n \to \infty} E[\sup_{t \leq T} |X^n_t - X_t|^2] = 0
\]

where \((X^n_t)\) and \((X_t)\) are respectively solutions of (7.2) and (7.1).

**Proof.**

Let \(n \in \mathbb{N}\), then by using similar arguments as in the preceding theorems, we have

\[
E[\sup_{t \leq T} |X^n_t - X_t|^2] \leq 3E[\sup_{t \leq T} \int_0^t |b(s, X^n_s, \mathbb{P}_{X^n_s}) - b(s, X^n_s, \mathbb{P}_{X^n_s})|dA^n_s|^2]
\]

\[+ 3E \left[ \sup_{t \leq T} \left| \int_0^t (\sigma(s, X^n_s, \mathbb{P}_{X^n_s}) - \sigma(s, X^n_s, \mathbb{P}_{X^n_s}))dM^n_s \right|^2 \right]
\]

\[+ 3E[\sup_{t \leq T} \int_0^t |b(t, X_t, \mathbb{P}_{X_t})|d|A^n_s - A_s|^2] + \sup_{t \leq T} \left| \int_0^t \sigma(t, X_t, \mathbb{P}_{X_t})d(M^n_s - M_s) \right|^2 \]

Let

\[
K_n = 3E[\sup_{t \leq T} \int_0^t |b(t, X_t, \mathbb{P}_{X_t})|d|A^n_s - A_s|^2] + \sup_{t \leq T} \left| \int_0^t \sigma(t, X_t, \mathbb{P}_{X_t})d(M^n_s - M_s) \right|^2
\]

By using Schwartz and Burkholder Davis Gundy inequalities along with the Lipschitz condition, we obtain
\[
\mathbb{E}[\sup_{t \leq T} |X^n_t - X_t|^2] \leq C(T) \left[ \int_0^T (E \left( \sup_{s \leq t} |X^n_s - X_s|^2 \right) + W_2(\mathbb{P}_{X^n}, \mathbb{P}_{X_s})^2)(dA^n_s + d < M^n, M^n >_s) \right] + K_n
\]

\[
\leq 2C(T) \int_0^T E[\sup_{s \leq t} |X^n_s - X_s|^2](dA^n_s + d < M^n, M^n >_s) + K_n
\]

where \( C(T) \) is a positive constant which may change from line to line.

Since \((A^n_s + d < M^n, M^n >_s) \) is an increasing process, then according to the Stochastic Gronwall lemma [24] Lemma 29.1, page 202, we have

\[
\mathbb{E}[\sup_{t \leq T} |X^n_t - X_t|^2] \leq 2K_n C(E(A^n_T + < M^n, M^n >_T)) < +\infty,
\]

where \( C \) is a constant.

By using assumption \((H_7)\) it is easy to that

\[
\lim_{n \to \infty} K_n = 0
\]

Therefore

\[
\lim_{n \to \infty} \mathbb{E}[\sup_{t \leq T} |X^n_t - X_t|^2] = 0
\]

\section{Approximation of relaxed control problems}

It is well known that in the deterministic as well as in stochastic control problems, an optimal control does not necessarily exist in the space of strict controls, in the absence of convexity conditions. The classical method is then to introduce measure valued controls which describe the introduction of a stochastic parameter see [13] and the references therein. These measure valued controls called relaxed controls generalize the strict controls in the sense that the set of strict controls may be identified as a dense subset of the set of the relaxed controls. The relaxed control problem is a true extension of the strict control problem if they have the same value function. That is the infimum among strict controls is equal to the infimum among relaxed controls. This last property is based on the continuity of the dynamics and the cost functional with respect to the control variable.

We show that under Lipschitz condition and continuity with respect to the control variable of the coefficients that the strict and relaxed control problems have the same value function. Our result extends those in [2, 3], to general MFSDEs of the type 8.1.

Let \( \mathfrak{A} \) be some compact metric space called the action space. A strict control \((u_t)\) is a measurable, \( \mathcal{F}_t \)-adapted process with values in the action space \( \mathfrak{A} \). We denote \( \mathcal{U}_{ad} \) the space of strict controls.

The state process corresponding to a strict control is the unique solution, of the following MFSDE

\[
\left\{ \begin{array}{l}
    dX_t = b(t, X_t, \mathbb{P}_{X_t}, u_t)dt + \sigma(t, X_t, \mathbb{P}_{X_t}, u_t)dB_t \\
    X_0 = x
\end{array} \right.
\]

and the corresponding cost functional is given by

\[
J(u) = E \left[ \int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t)dt + g(X_T, \mathbb{P}_{X_T}) \right].
\]
The problem is to minimize $J(u)$ over the space $U_{ad}$ of strict controls and to find $u^* \in U_{ad}$ such that $J(u^*) = \inf \{ J(u), u \in U_{ad} \}$.

Let us consider the following assumptions in this section.

$$(H_4) \ b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \rightarrow \mathbb{R}^d, \ \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d,$$
are continuous bounded functions.

$$(H_5) \ b(t, \ldots, a) \ \text{and} \ \sigma(t, \ldots, a) \ \text{are Lipschitz continuous uniformly in} \ (t, a) \in [0, T] \times \mathbb{A}.$$

$$(H_6) \ h : [0, T] \times \mathbb{R} \times \mathbb{A} \rightarrow \mathbb{R} \ \text{and} \ g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \ \text{are bounded continuous functions}, \ \text{such that} \ (h(t, \ldots, a) \ \text{is Lipschitz in} \ (x, \mu)).$$

It is clear that under assumptions $(H_4)$ and $(H_6)$ and according to Theorem 3.1, for each $u \in U_{ad}$, the MFSDE (8.1) has a unique strong solution, such that for every $p > 0$, $E(\|X_t\|^p) < +\infty$. Moreover for each $u \in U_{ad}$, $|J(u)| < +\infty$.

Let $\mathbb{V}$ be the set of product measures $\mu$ on $[0, T] \times \mathbb{A}$ whose projection on $[0, T]$ coincides with the Lebesgue measure $dt$. $\mathbb{V}$ as a closed subspace of the space of positive Radon measures $\mathbb{M}_+(\mathbb{R} \times \mathbb{A})$ is compact for the topology of weak convergence.

**Definition 8.1.** A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each $t, 1_{(0, t]} \mu$ is $\mathcal{F}_t$-measurable.

**Remark 8.2.** The set $U_{ad}$ of strict controls is embedded into the set of relaxed controls by identifying $u_t$ with $dt\delta_{u_t}(da)$.

It was proved in [12] for classical control problems and in [3] that the relaxed state process corresponding to a relaxed control must satisfy a MFSDE driven by a martingale measure instead of a Brownian motion. That is the relaxed state process satisfies

$$dX_t = \int_{\mathbb{A}} b(t, X_t, \mathbb{P}_{X_t}, a) \mu_t(da) dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t}, a) M(da, dt)$$

$$X_0 = x,$$

where $M$ is an orthogonal continuous martingale measure, with intensity $dt\mu_t(da)$. Using the same tools as in Theorem 3.1, it is not difficult to prove that (8.2) admits a unique strong solution.

The following Lemma, known in the control literature as Chattering Lemma states that the set of strict controls is a dense subset in the set of relaxed controls.

**Lemma 8.3.** i) Let $(\mu_t)$ be a relaxed control. Then there exists a sequence of adapted processes $(u^n_t)$ with values in $\mathbb{A}$, such that the sequence of random measures $(\delta_{u^n_t}(da) \, dt)$ converges in $\mathbb{V}$ to $\mu_t(da) \, dt$, $P$-a.s.

ii) For any $g$ continuous in $[0, T] \times \mathbb{M}_1(\mathbb{A})$ such that $g(t, \cdot)$ is linear, we have

$$\lim_{n \to +\infty} \int_0^t g(s, \delta_{u^n_t}(da) \, ds = \int_0^t g(s, \mu_t) \, ds$$

uniformly in $t \in [0, T], P$-a.s.

**Proof.** See [13]

Let $X^n_t$ be the solution of the state equation (8.1) corresponding to $u^n$, where $u^n$ is a strict control defined as in the last Lemma. If we denote $M^n(t, F) = \int_0^t \int_{\mathbb{A}} \delta_{u^n_t}(da) dW_s$, then $M^n(t, F)$ is an orthogonal martingale measure and $X^n_t$ may be written in a relaxed form as follows

$$dX^n_t = \int_{\mathbb{A}} b(t, X^n_t, \mathbb{P}_{X^n_t}, a) \delta_{u^n_t}(da) \, dt + \int_{\mathbb{A}} \sigma(t, X^n_t, \mathbb{P}_{X^n_t}, a) M^n(dt, da)$$

$$X_0 = x$$

Therefore $X^n_t$ may be viewed as the solution of (8.2) corresponding to the relaxed control $\mu^n = dt\delta_{u^n_t}(da)$. 

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Proposition 8.4. Proof. \( \varphi : \Omega \times [0,T] \times \mathbb{A} \to \mathbb{R} \), such that \( \varphi(\omega,t,.) \) is continuous, we have

\[
E \left[ \left( \int_0^T \int_{\mathbb{A}} \varphi(\omega,t,a)M^n(dt,da) - \int_0^T \int_{\mathbb{A}} \varphi(\omega,t,a)M(dt,da) \right)^2 \right] \to 0 \quad \text{as} \quad n \to +\infty. \tag{8.3}
\]

see \([1, 23]\).

The following proposition gives the continuity of the dynamics \( X_t \) with respect to the control variable.

Proposition 8.4. i) If \( X_t, X^n_t \) denote the solutions of state equation \( (8.2) \) corresponding to \( \mu \) and \( \mu^n \), then \( \lim_{n \to +\infty} \|X^n_t - X_t\| = 0 \).

ii) Let \( J(u^n) \) and \( J(\mu) \) be the expected costs corresponding respectively to \( u^n \) and \( \mu \), then \( J(u^n) \to J(\mu) \).

Proof. 1) Let \( X_t, X^n_t \) the solutions of the MVSDE \( (8.2) \) corresponding to \( \mu \) and \( u^n \). We have

\[
|X_t - X^n_t| \leq \left| \int_0^t \int_{\mathbb{A}} b(s,X_s,\mathbb{P}_{X_t},u) \mu_s(da)ds - \int_0^t \int_{\mathbb{A}} b(s,X^n_s,\mathbb{P}_{X^n_t},u) \delta_{u^n}(da)ds \right| \\
+ \left| \int_0^t \int_{\mathbb{A}} \sigma(s,X_s,\mathbb{P}_{X_t},a) M(ds,da) - \int_0^t \int_{\mathbb{A}} \sigma(s,X^n_s,\mathbb{P}_{X^n_t},a) M^n(ds,da) \right| \\
\leq \left| \int_0^t \int_{\mathbb{A}} b(s,X_s,\mathbb{P}_{X_t},u) \mu_s(da)ds - \int_0^t \int_{\mathbb{A}} b(s,X^n_s,\mathbb{P}_{X^n_t},a) \delta_{u^n}(da)ds \right| \\
+ \left| \int_0^t \int_{\mathbb{A}} \sigma(s,X_s,\mathbb{P}_{X_t},a) \mu_s(da)ds - \int_0^t \int_{\mathbb{A}} \sigma(s,X^n_s,\mathbb{P}_{X^n_t},a) \delta_{u^n}(da)ds \right| \\
+ \left| \int_0^t \int_{\mathbb{A}} \sigma(v,X_v,\mathbb{P}_{X^n_t},a) M^n(ds,da) - \int_0^t \int_{\mathbb{A}} \sigma(v,X^n_v,\mathbb{P}_{X^n_t},a) M^n(ds,da) \right| \\
+ \left| \int_0^t \int_{\mathbb{A}} \sigma(v,X_v,\mathbb{P}_{X_t},a) M^n(ds,da) - \int_0^t \int_{\mathbb{A}} \sigma(v,X^n_v,\mathbb{P}_{X^n_t},a) M^n(ds,da) \right|
\]

Then by using Burkholder-Davis-Gundy inequality for the martingale part and the fact that all the functions in equation \( (8.2) \) are Lipschitz continuous, it holds that

\[
E \left( |X_t - X^n_t|^2 \right) \leq C \int_0^T E \left( |X_s - X^n_s|^2 + W_2(\mathbb{P}_{X^n_s},\mathbb{P}_{X_s})^2 \right) dt + K_n,
\]

where \( C \) is a nonnegative constant and

\[
K_n = E \left( \left| \int_0^t \int_{\mathbb{A}} b(s,X_s,\mathbb{P}_{X_t},u) \mu_s(da)ds - \int_0^t \int_{\mathbb{A}} b(s,X^n_s,\mathbb{P}_{X^n_t},a) \delta_{u^n}(da)ds \right|^2 \right) \\
+ E \left( \left| \int_0^t \int_{\mathbb{A}} \sigma(s,X_s,\mathbb{P}_{X_t},a) M(ds,da) - \int_0^t \int_{\mathbb{A}} \sigma(s,X^n_s,\mathbb{P}_{X^n_t},a) M^n(ds,da) \right|^2 \right) \\
= I_n + J_n
\]

Using the fact that \( W_2(\mathbb{P}_{X^n_t},\mathbb{P}_{X_t})^2 \leq E \left( |X_s - X^n_s|^2 \right) \), we get

\[
E \left( |X_t - X^n_t|^2 \right) \leq 2C \int_0^T E \left( |X_s - X^n_s|^2 \right) dt + K_n. \tag{8.4}
\]
Since the sequence \((\delta_{u^n}(da)\,dt)\) converges weakly to \(\mu_t(da)\,dt\), \(P - a.s.\) and \(b\) is bounded and continuous in the control variable, then by applying the Lebesgue dominated convergence theorem we get \(\lim_{n \to +\infty} I_n = 0\). On the other hand since \(\sigma\) is bounded and continuous in \(a\), applying (8.3) we get \(\lim_{n \to +\infty} J_n = 0\). We conclude by using Gronwall’s Lemma.

ii) Let \(u^n\) and \(\mu\) as in i) then

\[
|J(u^n) - J(\mu)| \leq E \left[ \int_0^T h(t, X^n_t, \mathbb{P}X^n_t, a) - h(t, X_t, \mathbb{P}X_t, a) \delta_{u^n_t}(da) \, dt \right] \\
+ E \left[ \int_0^T h(t, X_t, \mathbb{P}X_t, a) \delta_{u^n_t}(da) \, dt - \int_0^T h(t, X_t, \mathbb{P}X_t, a) \mu_t(da) \, dt \right] \\
+ E \left[ |g(X^n_T, \mathbb{P}X^n_T) - g(X_T, \mathbb{P}X_T)| \right]
\]

The first assertion implies that the sequence \((X^n_t)\) converges to \(X_t\) in probability, then by using the assumptions on the coefficients \(h\) and \(g\) and the dominated convergence theorem it is easy to conclude.

Remark 8.5. According to the last Proposition, it is clear that the infimum among relaxed controls is equal to the infimum among strict controls, which implies the value functions for the relaxed and strict models are the same.

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