Manifolds with the fixed point property and their squares

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Abstract

The Cartesian squares (powers) of manifolds with the fixed point property (f.p.p.) are considered. Examples of manifolds with the f.p.p. are constructed whose symmetric squares fail to have the f.p.p.

A topological space $X$ has the fixed point property (f.p.p.) if for every continuous map $f : X \to X$ there exists a fixed point, that is, a point $x \in X$ such that $f(x) = x$. There are plenty of examples of (nice) spaces which fail to have the f.p.p. and there are examples of spaces with the f.p.p.

The celebrated Theorem of Brouwer (cf. [7]) asserts that the $n$-dimensional cube $I^n$ has the f.p.p.. On the other hand the $n$-dimensional sphere $S^n$ fails to have the f.p.p.

The especially important role in the Fixed Point Theory is played by the Lefschetz Fixed Point Theorem (cf. [5]). To be more specific:

Let $X$ be a nice space, say a compact ANR (this includes finite CW-complexes and compact topological manifolds). Let $\Lambda$ be a field. A map $f : X \to X$ induces a homomorphism (linear transformation)

$$f_* : H_i(X, \Lambda) \to H_i(X, \Lambda), \quad i = 0, 1, 2, \cdots$$

The Lefschetz number $L(f, \Lambda)$ of a map $f : X \to X$ is defined as $L(f, \Lambda) = \sum (-1)^i \text{tr} f_* i$ where $\text{tr} f_* i$ is the trace of $f_* i$.

**Theorem 1 (Lefschetz Fixed Point Theorem).** Let $f : X \to X$ be a map. If $L(f, \Lambda) \neq 0$ then $f$ has a fixed point.

Now since $L(f, \mathbb{Q}) = 1$ for every continuous $f : I^n \to I^n$ and the field of rational numbers $\mathbb{Q}$, then the theorem of Brower is a very special case of the Lefschetz Fixed Point Theorem.

There are more direct proofs of the Brower Fixed Point Theorem, but all of them are surprisingly demanding given its elementary formulation.

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A basic calculus argument shows that the interval I has the f.p.p. and thus the most tempting attempt of proving the theorem of Brouwer would be mathematical induction.

This is in turn is directly related to the general question raised by Kuratowski in 1930 (cf. [12]).

**Question 1** Suppose \(X, Y\) are locally connected and compact spaces with the f.p.p., does the Cartesian product \(X \times Y\) have the f.p.p.? 

It turns out that the answer to the above question is NO. The case of polyhedra was treated by W. Lopez in [13] and the construction of corresponding example is far from simple.

More refined example of closed manifolds \(M, N\) with the f.p.p. such that \(M \times N\) admits a fixed point free map was provided by S. Husseini in [11]. The construction in [11] is quite involved and the technical difficulties are very substantial. In particular the crucial fact which makes the construction in [11] to work is that \(M \neq N\).

This led to the following question which is considered to be one of the most important open problems in the classical Fixed Point Theory (cf. [6]).

**Question 2** Does there exist a closed manifold \(M\) with the f.p.p. such that its Cartesian square \(M^2 = M \times M\) fails to have the f.p.p.?

The main purpose of this note is to rekindle the interest in the above question. Even though at present we are not able to answer this question we show that in the presence of an additional symmetry the answer is positive.

Namely, let \(X\) be a topological space. The quotient space \(X(n) = X^n/S_n\), where the symmetric group \(S_n\) acts on \(X^n = X \times \cdots \times X\) by coordinate permutation, is called the \(n\)-th symmetric product of \(X\). In particular the symmetric square \(X(2)\) is given by \(X(2) = (X \times X)/\mathbb{Z}_2\). The symmetric product plays an important role in the algebraic and geometric topology (cf. [1], [3], [7], [8]) as well as in algebraic geometry (cf. [1]).

If \(M\) is a \(k\)-dimensional closed smooth manifold, then for \(k \leq 2\), \(M(n)\) is a manifold (possibly with a boundary). For \(k > 2\), \(M(n)\) is not a manifold but it is a compact polyhedron.

Here are some examples (cf. [1], [14]):
- For \(M = \mathbb{R}P^2\), \(M(n) = \mathbb{R}P^{2n}\).
- For \(M = S^2\), \(M(n) = \mathbb{C}P^n\).
- For \(M = S^1\), \(M(n)\) is the total space of the non-orientable \(D^{n-1}\) disk bundle over \(S^1\).

The main result of this note is the following:

**Theorem 2** Let \(M = \mathbb{R}P^4 \# \mathbb{R}P^4 \# \mathbb{R}P^4\). Then \(M\) has the f.p.p. while \(M(2)\) admits a fixed point free map.
Here # stands for the connected sum operation.

**Proof** Our first observation is about the cohomology ring structure on $H^*(M; \mathbb{Z}_2)$. Namely, relatively simple but somewhat tedious considerations involving the Mayer-Vietoris exact sequence and the well known ring structure of $H^*(\mathbb{R}P^1; \mathbb{Z}_2)$ show that $H^*(M; \mathbb{Z}_2)$ is (ring) isomorphic with the ring

$$\mathbb{Z}_2[x_1, x_2, x_3]/\langle\{x_i^5|i = 1, 2, 3\}, \{x_i^4 + x_j^4|i \neq j\}, \{x_ix_j|i \neq j\}\rangle$$

with $|x_i| = 1$. Given the crucial role of the ring structure on $H^*(M; \mathbb{Z}_2)$ in our considerations, we include an appendix which contains the necessary computational details.

In particular the cohomology of $M$ has base $\{1, x_1^n, x_2^n, x_3^n(1 \leq n \leq 3), x_1^4\}$. Also this implies that $\chi(M) = -1$.

We show $L(f; \mathbb{Z}_2) = 1$ for each continuous map $f : M \to M$.

To see this let $f^*\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $A$ is a $3 \times 3$ matrix with entries $a_{ij}$, $i, j = 1, 2, 3$.

The trace of $f^*$ is given as follows

| Dimension | Trace |
|-----------|-------|
| 0         | 1     |
| 1         | $a_{11} + a_{22} + a_{33}$ |
| 2         | $a_{11}^2 + a_{22}^2 + a_{33}^2$ |
| 3         | $a_{11}^3 + a_{22}^3 + a_{33}^3$ |
| 4         | $a_{11}^4 = a_{22}^4 = a_{33}^4$ |

Table 1:

Now with the $\mathbb{Z}_2$-coefficients $a_{ij}^2 = a_{ij}$ and hence the equation $a_{11}^4 = a_{22}^4 = a_{33}^4$ implies $a_{11} = a_{22} = a_{33}$. This gives $L(f, \mathbb{Z}_2) = 1$.

Next we show that $M(2)$ admits a fixed point free map.

We start with the following observation:

**Lemma 1** *The Euler characteristic of $M(2)$ is trivial, i.e. $\chi(M(2)) = 0$.***

**Proof of Lemma 1:** The above claim follows from a very general formula cf. [4], Theorem 7.1 on p.145.

For the completeness of our paper we include a different, shorter and self-contained argument. Namely:

The $\mathbb{Z}_2$-action on $M \times M$ is obviously smooth and in particular (cf. [10]) simplicial, and hence cellular for some CW structure on $M \times M$.

Consider the equivariant cellular chain complex $C_*(M \times M)$. Let $\Delta \subset M \times M$ be the diagonal, then $\Delta = (M \times M)^{\mathbb{Z}_2}$ is the fixed point set of the $\mathbb{Z}_2$-action. Thus we have $C_*(M \times M) \cong C_*(\Delta) \oplus \widetilde{C}(M \times M)$ where $\widetilde{C}(M \times M)$ is an $\mathbb{Z}_2$-equivariant chain complex generated by cells in $M \times M$ which are not
\[ C_\ast(\mathcal{M} \times \mathcal{M}) \cong C_\ast(\Delta) \oplus \tilde{C}(\mathcal{M} \times \mathcal{M}) \]
\[ \mathcal{M}(2) \cong C_\ast(\Delta) \oplus \tilde{C}(\mathcal{M} \times \mathcal{M}) \]

in \( \Delta \). Let \( p : \mathcal{M} \times \mathcal{M} \to (\mathcal{M} \times \mathcal{M})/\mathbb{Z}_2 = \mathcal{M}(2) \) be the natural projection on the orbit space. Then we have a chain map \( p\# \) and the diagram

Here \( \tilde{C}(\mathcal{M} \times \mathcal{M}) \) is the quotient of \( \tilde{C}_\ast(\mathcal{M} \times \mathcal{M}) \) and \( (p_1)_\# \) are corresponding projections.

Now on the chain complex level

\[
\chi(C_\ast(\mathcal{M} \times \mathcal{M})) = \chi(C_\ast(\Delta)) + \chi(\tilde{C}_\ast(\mathcal{M} \times \mathcal{M}))
\]

and analogously

\[
\chi(C_\ast(\mathcal{M}(2))) = \chi(C_\ast(\Delta)) + \chi(\tilde{C}_\ast(\mathcal{M} \times \mathcal{M}))
\]

Note that \( \chi(\tilde{C}_\ast(\mathcal{M} \times \mathcal{M})) = 2\chi(\tilde{C}_\ast(\mathcal{M} \times \mathcal{M})) \), and hence on the level of topological spaces one obtains

\[
2\chi(M(2)) = \chi(M) + \chi(M \times M) = \chi(M)(1 + \chi(M)) = 0
\]

and hence \( \chi(M(2)) = 0 \) as claimed.

Finally the symmetric square \( M(2) \) is obviously a simplicial complex of dimension 8.

It is a rational homology manifold (cf. [2]). In particular it means that for each vertex \( v \in M(2) \) the link \( \text{Ln}(v) = \partial \text{St}(v) \) has the rational homology of \( S^7 \), where \( \text{St}(v) \) is the star of \( v \). This implies that \( M(2) \) is a polyhedron of type \( W \) in the sense of [5] p.143, with \( \chi(M(2)) = 0 \).

Consequently by the Theorem 1 (the converse of the Lefschetz Deformation Theorem) in [5] p.143, \( M(2) \) admits a fixed point free deformation. \( \square \)

**Remarks and comments**

The example of closed manifolds \( \mathcal{M}, \mathcal{N} \) with the f.p.p. for which \( \mathcal{M} \times \mathcal{N} \) fails to have the f.p.p. presented in [11] is surprisingly complicated. One attempt to construct "simple" examples of this sort could be to consider products of basic manifolds with the f.p.p.

These basic examples are: \( \mathbb{R}P^{2n}, \mathbb{C}P^{2n}, n = 1, 2, \ldots \) and \( \mathbb{H}P^n, n = 2, 3, 4, \ldots \) i.e., the corresponding real, complex and quaternionic projective spaces.

It turns out that mixing different projective spaces, i.e., forming

(a) \( \mathbb{R}P^{2m} \times \mathbb{C}P^{2n} \)
(b) \( \mathbb{R}P^{2m} \times \mathbb{H}P^n \)
(c) \( \mathbb{C}P^{2m} \times \mathbb{H}P^n \)
one ends up with manifold with the f.p.p. cf. [9], Theorem 4.7.
It appears that a more involved argument would show that the Cartesian
powers of these manifolds have the f.p.p.
The case of $\mathbb{R}P^{2n}$ is simple (use the Lefschetz Fixed Point Theorem with therational coefficients). The considerations involving $\mathbb{C}P^{2n}$ and $\mathbb{H}P^{n}$ are more
involved. As an example we check the following crucial case.

**Theorem 3** The Cartesian power $(\mathbb{C}P^{2})^{n} = \mathbb{C}P^{2} \times \mathbb{C}P^{2} \times \cdots \times \mathbb{C}P^{2}$ has the
f.p.p..

**Proof** Let $f : (\mathbb{C}P^{2})^{n} \rightarrow (\mathbb{C}P^{2})^{n}$ be a map. We show that the Lefschetz
number computed with the $\mathbb{Z}_{2}$-coefficient is given by $L(f; \mathbb{Z}_{2}) = 1$.

Consider the induced homomorphism

$$f^{*} : H^{*}((\mathbb{C}P^{2})^{n}) \rightarrow H^{*}((\mathbb{C}P^{2})^{n})$$

By the Kunneth Formula, $H^{*}((\mathbb{C}P^{2})^{n})$ can be identified with the $n$-fold tensor
product $H^{*}((\mathbb{C}P^{2}) \otimes \cdots \otimes H^{*}((\mathbb{C}P^{2})$. Let $X_{i}, 1 \leq i \leq n$ be the generator of
$H^{2}((\mathbb{C}P^{2})^{n})$ corresponding to $1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots 1$, where $x$ in the $i$th
place is a fixed generator of $H^{2}(\mathbb{C}P^{2})$.

Assume that $f^{*} \left( \begin{array}{c} x_{1} \\ \vdots \\ x_{n} \end{array} \right) = A \cdot \left( \begin{array}{c} x_{1} \\ \vdots \\ x_{n} \end{array} \right)$ for a matrix $A$ given by $A = \{a_{ij}\}$

Then $X = \bigcup_{k+l \leq n} X_{k,l}$ is a basis for $H^{*}((\mathbb{C}P^{2})^{n})$, where $X_{0,0}$ is the basis for

$H^{0}((\mathbb{C}P^{2})^{n}) = \mathbb{Z}_{2}$. Now $L(f)$ is the trace of $f^{*}$ with respect to $X$.

Let $T_{k,l}$ be the trace of $f^{*}$ generated by $X_{k,l}$. Then we claim the following:

(1) $T_{0,0} = 1$

(2) $T_{k,0} = T_{l,k}$

(3) $T_{k,k} = 0$ for $k \geq 1$

Note that these claims imply $L(f; \mathbb{Z}_{2}) = 1$ completing the proof of Theorem
3. With respect to the proof of (1), (2) and (3):

The claim (1) is obvious.

**Proof of the claim (2):**

Let $t x_{i_{1}}^{j_{1}} \cdots x_{i_{k}}^{j_{k}}$ be the trace generated by the element $x_{i_{1}}^{j_{1}} \cdots x_{i_{k}}^{j_{k}} x_{j_{1}} \cdots x_{j_{l}}$.
It suffices to show that $t x_{i_{1}}^{j_{1}} \cdots x_{i_{k}}^{j_{k}} x_{j_{1}} \cdots x_{j_{l}} = t x_{j_{1}}^{j_{1}} \cdots x_{j_{k}}^{j_{k}} x_{i_{1}} \cdots x_{i_{l}}$, for any distinct $i_{1}, \cdots , i_{k}, j_{1}, \cdots , j_{l}$.

We have

$$f^{*}(x_{i_{s}}) = \sum_{r=1}^{n} a_{s,r} x_{r}, 1 \leq s \leq k,$$
\[ f^*(x_{i_t}) = \sum_{r=1}^{n} a_{j_r} x_r, 1 \leq t \leq l. \]

Thus \[ f^*(x^2_{i_t}) = f^*(x^2_{i_t}) = \sum_{r=1}^{n} a_{i_r}^2 x_r^2 = \sum_{r=1}^{n} a_{i_r} x_r^2. \] Similarly \[ f^*(x^2_{j_t}) = \sum_{r=1}^{n} a_{j_r} x_r^2. \]

So \[ f^*(x^2_{i_1} \cdots x^2_{i_k} x_{j_1} \cdots x_{j_l}) = \left( \prod_{s=1}^{k} \sum_{r=1}^{n} a_{i_r} x_r^2 \right) \cdot \left( \prod_{t=1}^{l} \sum_{r=1}^{n} a_{j_r} x_r \right) \]

and analogously \[ f^*(x^2_{j_1} \cdots x^2_{j_k} x_{i_1} \cdots x_{i_l}) = \left( \prod_{t=1}^{l} \sum_{r=1}^{n} a_{i_r} x_r^2 \right) \cdot \left( \prod_{s=1}^{k} \sum_{r=1}^{n} a_{j_r} x_r \right) \]

From this it is not difficult to see that
\[ t_{x^2_{i_1} \cdots x^2_{i_k} x_{j_1} \cdots x_{j_l}} = t_{x^2_{j_1} \cdots x^2_{j_k} x_{i_1} \cdots x_{i_l}} \]

Namely, both of them are given by
\[
\begin{array}{ccc}
  & a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i_k i_1} & \cdots & \cdots & a_{i_k i_k} \\
\end{array}
\quad \quad
\begin{array}{ccc}
  & a_{j_1 j_1} & \cdots & a_{j_1 j_l} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{j_l j_1} & \cdots & \cdots & a_{j_l j_l} \\
\end{array}
\]

**Proof of the claim (3):**

We have
\[ T_{k,k} = \sum_{i_1, \ldots, i_k} t_{x^2_{i_1} \cdots x^2_{i_k} x_{j_1} \cdots x_{j_k}} \]

But \[ t_{x^2_{i_1} \cdots x^2_{i_k} x_{j_1} \cdots x_{j_k}} + t_{x^2_{j_1} \cdots x^2_{j_k} x_{i_1} \cdots x_{i_k}} = 2t_{x^2_{i_1} \cdots x^2_{i_k} x_{j_1} \cdots x_{j_k}} = 0. \]

**Appendix**

**Theorem 4** The cohomology ring \( H^* \left( \frac{n \# P^{2k}; \mathbb{Z}_2}{}; \right), k \geq 2 \) is isomorphic to \( \mathbb{Z}_2[x_1, \cdots, x_n]/(x_1^{2k+1}, \{x_i^{2k} + x_j^{2k} | i \neq j\}, \{x_i x_j | i \neq j\}), |x_i| = 1 \)

**Proof** We shall omit the coefficients since it will always be \( \mathbb{Z}_2 \).

The additive structure comes easily from the integral homology and Universal Coefficient Theorem. We only need to determine the multiplicative structure.

To do this we will proceed by induction.

For inductive purpose we shall prove a stronger version of the above theorem.

Denote \( \# P^{2k} \) by \( P_n \), treated as \( S^{2k} \# P^{2k} \) where all disks cut from \( S^{2k} \) have positive distance between each other. For \( n = 1 \), write \( P_1 = P^{2k} \) as \( P \).
Define a map \( p_i^0 : P_n \to P \) by fixing the \( i \)th copy of \( P \) (with the open disk removed) in \( P_n \), mapping an “annulus” in \( S^{2k} \) near the boundary of this disk via radial projection onto the open disk in \( P \) and sending the remainder onto the center of that disk. Let \( x \) be the generator of \( H^1(P; \mathbb{Z}_2) \). We claim in addition that in the Theorem 7, \( x_i \) can be chosen as \( p_i^n(x) \).

The case \( n = 1 \) is well-known.

For any \( n \), let \( \overline{P}_n \) be \( P_n \) with yet another open disk (disjoint with the existing ones) removed from \( S^{2k} \). Denote \( \overline{P}_1 \) as \( P \). Note that \( \overline{P} \) is \( P^{2k} \) with an open disk removed.

Now assume that the stronger version of the above theorem holds for \( n \) copies of \( P^{2k} \), i.e., for \( P_n \). We shall prove it for \( P_{n+1} \).

By definition, \( P_{n+1} = \overline{P}_n \cup \overline{P}_n \cap P = S^{2k-1} \), where \( \overline{P}_n \) corresponds to the first \( n \) copies of \( P \) in \( P_{n+1} \).

From the Mayor-Vietoris Sequence of \( P_n = \overline{P}_n \cup D^{2k} \) and using the fact that \( H^{2k}(\overline{P}_n) = 0 \) (this is because \( \overline{P}_n \) is homotopy equivalent to a non-compact 2k-manifold), one can see that the inclusion \( \overline{P}_n \to P_n \) induces isomorphisms on \( H^m \) for \( 0 \leq m \leq 2k-1 \) and for any \( n \).

An argument by M-V sequence with respect to \( P_{n+1} = \overline{P}_n \cup \overline{P} \) similar to the one above shows that

\[ H^m(P_{n+1}) \xrightarrow{i_n^* \oplus i_1^*} H^m(\overline{P}_n) \oplus H^m(\overline{P}) \]

is an isomorphism for \( 0 \leq m \leq 2k-1 \), where \( i_1, i_n \) are canonical inclusions.

There is a projection \( q_n : P_{n+1} \to P_n \) (defined similarly as \( p_i^n \) above) that is identity on \( \overline{P}_n \) and maps \( P \) onto the disk \( D^{2k} \). It is not hard to show \( p_i^{n+1} = p_i^n \circ q_n, 1 \leq i \leq n \).

Now consider the composition:

\[ H^m(P_n) \oplus H^m(P) \xrightarrow{q_n^* \oplus p_i^{n+1}*} H^m(P_{n+1}) \xrightarrow{i_n^* \oplus i_1^*} H^m(\overline{P}_n) \oplus H^m(\overline{P}) \]

for \( 1 \leq m \leq 2k-1 \).

We have proven that the inclusions \( i_n \circ q_n \) and \( i_1 \circ p_i^{n+1} \) induce isomorphism on \( H^m \). On the other hand, \( i_n \circ p_i^{n+1} \) and \( i_1 \circ q_n \) are null-homotopic, whence \( (q_n^* \oplus p_i^{n+1}*) \circ (i_n^* \oplus i_1^*) = (q_n^* \circ i_n^*) \oplus (p_i^{n+1} \circ i_1^*) \) is an isomorphism. We have seen that \( i_n^* \oplus i_1^* \) is an isomorphism, thus the same is true for \( q_n^* \oplus p_i^{n+1} \).

Now define \( x_i = p_i^{n+1+1*}(x) \in H^1(P_{n+1}), 1 \leq i \leq n+1 \), then \( x_i = q_n^* \circ p_i^{n+1*}(x) \) for \( 1 \leq i \leq n \).

The inductive assumption implies that for \( 1 \leq m \leq 2k-1 \), \( \{p_i^n(x)^m \} \) is a basis for \( H^m(P_n) \).

Since \( q_n^* \oplus p_i^{n+1+1} \) is an isomorphism, \( H^m(P_{n+1}) \) has basis \( \{x_1^m, \ldots, x_n^m, x_{n+1}^m \} \), \( 1 \leq m \leq 2k-1 \).

Next we turn to dimension \( 2k \).

**Claim:** Both \( q_n \) and \( p_i^{n+1} \) induce isomorphism on \( H^{2k} \).

**Proof of the claim:** Consider the commutative diagram:
where $\tilde{p}$ is induced by $p_{n+1}^{n+1}$ and the vertical maps are canonical inclusions or projections. $\tilde{p}$ is a homeomorphism and the two lower vertical maps induce isomorphism on cohomology. Consider the long exact sequence

$$\cdots \rightarrow H^{2k+1}(P_{n+1}) \rightarrow H^{2k}(P_{n+1}, P_n) \rightarrow H^{2k}(P_{n+1}) \rightarrow H^{2k}(P_n) \rightarrow \cdots$$

Since $H^{2k+1}(P_n) = 0 = H^{2k}(P_n)$, the upper left map in the above diagram induces isomorphism on cohomology. Trivially, $P \rightarrow (P, D^{2k})$ induces isomorphism on $H^{2k}$.

Combining the above arguments and using commutativity, we have shown that $p_{n+1}^{n+1} : P_{n+1} \rightarrow P$ induces an isomorphism on $H^{2k}$.

In much the same way one can show that $q_n$ induces isomorphism on $H^{2k}$. This finishes the proof of the claim.

The claim together with the inductive assumption implies that $H^{2k}(P_{n+1})$ is generated by $x_1^{2k} = x_2^{2k} = \cdots = x_n^{2k} = x_{n+1}^{2k}$.

It remains to show that $x_i x_j = 0$, $i \neq j$, $1 \leq i, j \leq n + 1$.

For the case $n = 1$, let $x_1 x_2 = ax_1^2 + bx_2^2$ (this is because $\{x_1^2, x_2^2\}$ are basis) for some $a, b$. Since one can exchange the role of $x_1$ and $x_2$ (by exchanging the two copies of $P$ in $P_2$), we must have $a = b$.

Suppose $a = b = 1$, then $x_1^2 x_2 = x_1(x_1 x_2) = x_1^3 + x_1 x_2^2$, whence $x_1^3 = x_1^2 x_2 + x_1 x_2^2$. Similarly $x_2^3 = x_1^2 x_2 + x_1 x_2^2$. This contradicts to $\{x_1^3, x_2^3\}$ being basis.

Consequently $a = b = 0$ and the claim is proven for $n = 1$.

For the case $n > 1$, we decompose $p_{i+1}^{n+1}, p_{j+1}^{n+1}$ into commutative diagrams:

$$\begin{array}{ccc}
P_{n+1} & \xrightarrow{p_{i+1}^{n+1}} & P \\
\downarrow & & \downarrow \\
(P_{n+1}, P_n) & \xrightarrow{p_{i+1}^{n+1}} & (P, D^{2k}) \\
\downarrow & & \downarrow \\
(P_{n+1}/P_n, *) & \xrightarrow{\tilde{p}} & (P/D^{2k}, *)
\end{array}$$

Table 3:

Table 4:

Here $p_{ij}$ preserves the $i$th and $j$th copy of $P$ in $P_n$ while project other copies of $P$ onto disks, and $p_{ij}'$ (resp. $p_{ij}''$) preserves the $i$th (resp. $j$th) copy of $P$ while projects the other onto respective disks.
Then $x_i x_j = p_i^{n+1}(x)p_j^{n+1}(x) = p_{ij}^*[p_i^*(x) \cdot p_j^*(x)] = p_{ij}^*(0) = 0$ by the inductive assumption.

This finishes both the inductive step and the proof of Theorem 4. □

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