Scattering by a Nihility Cylinder
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Abstract The total scattering and the extinction efficiencies of a nihility cylinder of infinite length and circular cross-section are identical and independent of the polarization state of a normally incident plane wave.

Keywords Extinction efficiency, Negative refraction, Nihility, Scattering efficiency

1. Introduction

The emergence of nihility as an electromagnetic medium can be attributed to the rather extraordinary developments on negatively refracting materials during this decade. Much of the impetus for this development has been provided by the prospect of the so-called perfect lens. Any perfect lens in the present context is required to simulate nihility.

The relative permittivity and the relative permeability of nihility are null-valued. Clearly, nihility is unachievable, but it may be approximately simulated in some narrow frequency range — hence, its attraction. Reflection and refraction of plane waves due to nihility half-spaces has been studied in some detail, as well as the scattering of plane waves by nihility spheres. Along the same lines, this communication focuses on the canonical problem of the scattering response of a nihility cylinder of circular cross-section and infinite length to a normally incident plane wave. An \( \exp(-i\omega t) \) time-dependence is implicit in the following sections.

2. Boundary Value Problem

The geometry of the canonical problem is best stated using the cylindrical coordinate system \((\rho, \phi, z)\). The cylinder \( \rho \leq a \) is oriented parallel to the \( z \) axis; and, with its wave vector parallel to the \(-x\) axis, a plane wave is normally incident on this cylinder. Two different cases must be considered: (i) The incident magnetic field phasor is parallel to the \( z \) axis, (ii) the incident electric field phasor is parallel to the \( z \) axis.

Direct derivation for a nihility cylinder being evidently intractable, a limiting procedure has to be resorted to. Therefore, let the relative permittivity of the cylinder be denoted by \( \epsilon_r \), and its relative permeability by \( \mu_r \). The standard results for this cylinder can be manipulated for nihility cylinders.

2.1 Case (i)

The incident field exists everywhere in the region of interest when the scatterer is absent. Therefore, the incident electric and magnetic field phasors may be stated as follows:

\[
\begin{align*}
E_{\text{inc}}(\rho, \phi) &= \sum_{n=-\infty}^{\infty} \alpha_n M_n^{(1)}(\rho; k_0) \\
H_{\text{inc}}(\rho, \phi) &= (i\eta_0)^{-1} \times \sum_{n=-\infty}^{\infty} \alpha_n N_n^{(1)}(\rho; \phi; k_0) \\
\rho &\geq 0,
\end{align*}
\]

where \( k_0 \) is the wavenumber in and \( \eta_0 \) is the intrinsic impedance of free space (i.e., vacuum), while the coefficients

\[
\alpha_n = (-i)^{n+1}/k_0.
\]

The wavefunctions used in the foregoing equations and hereafter are defined as follows:

\[
\begin{align*}
M_n^{(1)}(\rho, \phi; k) &= k \left( in \frac{J_n(k\rho)}{k\rho} \hat{\rho} \\
&- \frac{dJ_n(k\rho)}{dk\rho} \hat{\phi} \right) e^{in\phi}, \\
N_n^{(1)}(\rho, \phi; k) &= k J_n(k\rho) e^{in\phi} \hat{z}, \\
M_n^{(3)}(\rho, \phi; k) &= k \left( in \frac{H_n^{(1)}(k\rho)}{k\rho} \hat{\rho} \\
&- \frac{dH_n^{(1)}(k\rho)}{dk\rho} \hat{\phi} \right) e^{in\phi}, \\
N_n^{(3)}(\rho, \phi; k) &= k H_n^{(1)}(k\rho) e^{in\phi} \hat{z}.
\end{align*}
\]

Whereas \( J_n(\xi) \) are Bessel functions, \( H_n^{(1)}(\xi) \) are Hankel functions of the first kind, with \( \xi \) denoting the argument. The scattered field phasors are given by

\[
\begin{align*}
E_{\text{scat}}(\rho, \phi) &= \sum_{n=-\infty}^{\infty} \alpha_n a_n M_n^{(3)}(\rho; \phi; k_0) \\
H_{\text{scat}}(\rho, \phi) &= (i\eta_0)^{-1} \times \sum_{n=-\infty}^{\infty} \alpha_n a_n N_n^{(3)}(\rho; \phi; k_0) \\
\rho &\geq a,
\end{align*}
\]

where

\[
a_n = - \left[ \eta_r J_n(k_0a) - J_n'(k_0a) L_n(k_0a r) \right]
\left[ \eta_r H_n^{(1)}(k_0a) - H_n^{(1)'}(k_0a) L_n(k_0a r) \right]^{-1},
\]
the prime denotes differentiation with respect to the argument, the functions
\[ L_n(\xi) = \frac{J_n(\xi)}{J_n'(\xi)}, \quad (9) \]
the relative impedance \( \eta_r = \sqrt{\mu_r/\epsilon_r} \), and the refractive index \( n_r = \sqrt{\epsilon_r/\mu_r} \).

The total scattering efficiency is the sum
\[ Q_{sca}^{(i)} = \frac{2}{k_0a} \sum_{n=-\infty}^{\infty} |a_n|^2, \quad (10) \]
and the extinction efficiency may be derived from the optical theorem [11] as
\[ Q_{ext}^{(i)} = -\frac{2}{k_0a} \Re \left( \sum_{n=-\infty}^{\infty} a_n \right), \quad (11) \]
where \( \Re \) stands for ‘the real part of’.

2.2 Case (ii)

The incident electric and magnetic field phasors may be stated as follows [11]:
\[ \begin{align*}
\mathbf{E}_{inc}(\rho, \phi) &= \sum_{n=-\infty}^{\infty} \beta_n \mathbf{N}_n^{(1)}(\rho, \phi; k_0) \\
\mathbf{H}_{inc}(\rho, \phi) &= (i\eta_n)^{-1} \times \sum_{n=-\infty}^{\infty} \beta_n \mathbf{M}_n^{(1)}(\rho, \phi; k_0) \\
& \quad \rho \geq 0,
\end{align*} \quad (12) \]
where
\[ \beta_n = i\alpha_n. \quad (13) \]

The scattered field phasors are given by
\[ \begin{align*}
\mathbf{E}_{sca}(\rho, \phi) &= \sum_{n=-\infty}^{\infty} \beta_n b_n \mathbf{N}_n^{(3)}(\rho, \phi; k_0) \\
\mathbf{H}_{sca}(\rho, \phi) &= (i\eta_n)^{-1} \times \sum_{n=-\infty}^{\infty} \beta_n b_n \mathbf{M}_n^{(3)}(\rho, \phi; k_0) \\
& \quad \rho \geq a,
\end{align*} \quad (14) \]
where
\[ \begin{align*}
b_n &= -[J_n(k_0a) - \eta_r J_n'(k_0a)L_n(k_0n_r a)] \\
& \quad \left[ H_n^{(1)}(k_0a) - \eta_r H_n^{(1)}(k_0a)L_n(k_0n_r a) \right]^{-1}
\end{align*} \quad (15) \]

The total scattering efficiency and the extinction efficiency, respectively, are as follows:
\[ \begin{align*}
Q_{sca}^{(ii)} &= \frac{2}{k_0a} \sum_{n=-\infty}^{\infty} |b_n|^2, \quad (16) \\
Q_{ext}^{(ii)} &= -\frac{2}{k_0a} \Re \left( \sum_{n=-\infty}^{\infty} b_n \right). \quad (17)
\end{align*} \]

2.3 Limiting Procedure for Nihility Cylinder

Now, the refractive index of nihility must be null–valued because \( \epsilon_r = \mu_r = 0 \). For the functions \( L_n(\xi) \), we have
\[ \begin{align*}
\lim_{\xi \to 0} \xi L_0(\xi) &= -2, \\
\lim_{\xi \to 0} \xi^{-1} L_n(\xi) &= n^{-1}, \quad n \neq 0.
\end{align*} \quad (18) \quad (19) \]
Therefore, after taking the limit \( n_r \to 0 \), (8) and (15) for a nihility cylinder simplify to
\[ \begin{align*}
a_0 &= b_0 = -\frac{J_1(k_0a)}{H_1^{(1)}(k_0a)}, \\
a_n &= b_n = -\frac{J_n(k_0a)}{H_n^{(1)}(k_0a)}, \quad n \neq 0.
\end{align*} \quad (20) \quad (21) \]

3. Discussion

From [11], [11], [15], [17], [20], and [21], it follows that
\[ Q_{sca}^{(i)} = Q_{sca}^{(ii)} = Q_{ext}^{(i)} = Q_{ext}^{(ii)}, \quad (22) \]
because
\[ \left| \frac{J_n(\xi)}{H_n^{(1)}(\xi)} \right|^2 = \Re \left( \frac{J_n(\xi)}{H_n^{(1)}(\xi)} \right), \quad (23) \]
The equality of extinction and total scattering efficiencies for either case is an affirmation of the nondissipative nature of nihility. The equality of efficiencies for both cases (i) and (ii) emerges from the identity \( a_n = b_n \forall n \in (-\infty, \infty) \) for nihility cylinders.

An remarkable consequence of (22) is that the extinction and the total scattering efficiencies of a nihility cylinder do not change with the polarization state of the incident plane wave. In other words, if the incident plane wave is arbitrarily polarized such that
\[ \mathbf{E}_{inc} = (A_x \hat{z} + A_y \hat{y}) \exp(-ik_0x), \quad (24) \]
the total scattering and the extinction efficiencies are independent of the ratio \( A_x/A_y \). The extinction efficiency is shown in Figure [11] as a function of the normalized size parameter \( k_0a \).

Furthermore, on examining the scattering function
\[ \mathbf{F}_{sca}(\phi) = \lim_{k_0a \to \infty} (k_0\rho)^{1/2} \exp(-ik_0\rho) \mathbf{E}_{sca}(\rho, \phi), \quad (25) \]
it can be deduced that the scattering pattern \( |\mathbf{F}_{sca}(\phi)| \) is independent of the polarization state of the incident plane wave. Again, this is because \( a_n = b_n \forall n \).

The equality of scattering coefficients for cases (i) and (ii) is a curious result, at first glance. From (8) and (15), it can be shown that \( a_n = b_n \) for all \( n \) if and only if \( \eta_r^2 = 1 \). On writing \( \mu_r = \eta_r^2 \), it becomes clear that nihility is impedance–matched to free space (i.e., \( \eta_r = 1 \)). Indeed,
nihility is impedance–matched to any isotropic, homogeneous, dielectric–magnetic medium (i.e., with both $\varepsilon_r \neq 1$ and $\mu_r \neq 1$), so that the results derived in Sec. 2.3 are very general. Parenthetically, we note that on repeating the exercise in Section 2 for obliquely incident plane waves led to expressions that could not be unambiguously interpreted after the limiting procedure was implemented.

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