HYPERBOLIC THREE-MANIFOLDS
WITH TRIVIAL FINITE TYPE INVARIANTS

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Abstract. We construct a hyperbolic three-manifold with trivial finite type invariants up to an arbitrarily given degree.

The concept of finite type invariants for integral homology three-spheres was introduced by T. Ohtsuki in [11] including the Casson invariant [1] for the first non-trivial example. It attracts not only mathematicians but also physicists since it is closely related to E. Witten’s quantum invariants for three-manifolds [15] in the following way.

It is proved by T.T.Q. Le [7] that the degree \( d \) term of the LMO invariant [8] is of finite type of degree \( d \) and conversely any finite type invariant of degree \( d \) comes from the degree \( d \) term of the LMO invariant. Since it is also known that the perturbative \( \text{PSU}(n) \) invariant can be obtained from the LMO invariant [8, 9], every coefficient of the perturbative \( \text{PSU}(n) \) invariant is of finite type of degree \( d \). (For \( n = 2 \) case was also proved in [9].) Therefore we may say that finite type invariants approximate quantum invariants from lower degree parts. We refer the reader to [10, 12] for more detail.

It is a natural question to ask how strong finite type invariants are. For any given integral homology three-sphere \( M \) and any positive integer \( k \), Le [7], and N. Habegger and K. Orr [2] gave infinitely many homology spheres which have the same finite type invariants as \( M \) up to degree \( k \) by using the LMO invariant. E. Kalfagianni [3] also obtained such manifolds using \( \pm 1 \)-surgery along knots.

In this paper we give at least one hyperbolic three-manifold with trivial finite type invariants up to an arbitrarily given degree by simple construction using a hyperbolic Brunnian link.

Acknowledgment. This work was done when the author was visiting the Mittag-Leffler Institute. He thanks the staffs for their hospitality: ‘Tack så mycket’. He also thanks Kazuo Habiro, Thang Le and Tomotada Ohtsuki for useful comments. He is grateful to Nathan Habegger, who sent him his preprint with K. Orr [3] where he found that the technique using a Brunnian link has been already used in the proof of Theorem 8.1, and Effie Kalfagianni, who informed him of her result [3].

Thanks are also due to the subscribers to the mailing list ‘knot’ (http://w3.to/oto/) run by Makoto Ozawa, where he could enjoy useful comments.

Date: March 30, 2022.

1991 Mathematics Subject Classification. 57N10, 57M50.

Key words and phrases. finite type invariant, hyperbolic three-manifold, Brunnian link.

Partially supported by Waseda University Grant for Special Research Projects (No. 98A-623) and Grant-in-Aid for Scientific Research (C) (No. 09640135), the Ministry of Education, Science, Sports and Culture.
1. Preliminaries

In this section we describe a filtration of integral homology three-spheres (closed, oriented three-manifolds with the same integral homology groups as the three-sphere) and define finite type invariants following T. Ohtsuki [11].

Let $\mathcal{M}$ be the $\mathbb{C}$-vector space spanned by all the integral homology three-spheres. A link $L$ in an integral homology three-sphere $M$ is called algebraically split if every linking number vanishes. A unit-framed link is a link with each framing 1 or $-1$. For an algebraically split, unit-framed link $L$ in an integral homology three-sphere $M$, we put

$$[M, L] = \sum_{\ell \in L} (-1)^{\mu(\ell)} \chi(M; \ell) \in \mathcal{M},$$

where the summation runs over all sublinks of $L$ including the empty link and $L$ itself, $\mu(\ell)$ is the number of components in $\ell$ and $\chi(M; \ell)$ is the manifold obtained from $M$ by surgery along $\ell$ respecting the framing. Note that since $L$ is algebraically split and unit-framed, $\chi(M; \ell)$ is again an integral homology three-sphere.

Let $\mathcal{M}_d$ be the vector subspace of $\mathcal{M}$ spanned by all $[M, L]$ with $\mu(L) = d$. Now we have the following filtration of $\mathcal{M}$:

$$\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots \supset \mathcal{M}_d \supset \mathcal{M}_{d+1} \supset \cdots.$$ 

An invariant $T$ of integral homology three-spheres is said to be of finite type of degree $d$ if $T|_{\mathcal{M}_{d+1}} = 0$. Here we extend $T$ linearly over $\mathcal{M}$.

2. Main result

In this section we state our main result and give the proof. Main tools are hyperbolic Brunnian links due to T. Kanenobu and hyperbolic Dehn surgery due to W. Thurston.

**Theorem 1.** For any positive integer $k$, there exists (at least) one hyperbolic three-manifold $M$ such that for any finite type invariant $T_d$ of degree $d < k$, $T_d(M) = T_d(S^3)$. Note that $M$ is not homeomorphic to $S^3$ since it is hyperbolic.

**Proof.** From the definition of finite type invariants it is sufficient to construct a hyperbolic three-manifold $M$ satisfying $M - S^3 \in \mathcal{M}_k$ for a given $k$.

By [3], there exists a hyperbolic Brunnian link $L$ with $k$ components, that is, the exterior of $L$ is hyperbolic and each proper sublink of $L$ is trivial. Let $M_1/n_1, 1/n_2, \ldots, 1/n_k$ be the closed three-manifold obtained from $S^3$ by Dehn surgery along $L$ with surgery coefficient $(1/n_1, 1/n_2, \ldots, 1/n_k)$ for positive integers $n_i$. Here we use the notation described in [3]. By W. Thurston’s hyperbolic Dehn surgery theorem [3], $M_1/n_1, 1/n_2, \ldots, 1/n_k$ is hyperbolic if $n_i$ are sufficiently large for all $i$. Let $M = M_1/n_1, 1/n_2, \ldots, 1/n_k$ be a hyperbolic three-manifold obtained in this way.

We show that $M - S^3 \in \mathcal{M}_k$. Since each component of $L$ is trivial, $M$ is obtained by Dehn surgery along the link $L^{n_1, n_2, \ldots, n_k}$ with every coefficient one, where $L^{n_1, n_2, \ldots, n_k}$ is the $(n_1 + n_2 + \cdots + n_k)$-component link obtained by replacing $i$th component of $L$ with $n_i$ parallels.

Now one can easily see that for $d < k$ each $d$-component sublink of $L^{n_1, n_2, \ldots, n_k}$ is trivial. Let $L_k$ be a $k$-component sublink of $L^{n_1, n_2, \ldots, n_k}$. Then since any proper sublink in $L_k$ is trivial, we have

$$\mathcal{M}_k \ni [S^3, L_k] = (-1)^k \left( \chi(L_k) - S^3 \right),$$

where $\chi(L_k)$ is the Euler characteristic of $L_k$.
where $\chi(L_k)$ is the three-manifold obtained by Dehn surgery along $L_k$ with every coefficient one. Next let $L_{k+1}$ be a $(k+1)$-component sublink of $L^{n_1,n_2,\ldots,n_k}$. Then $\mathcal{M}_{k+1} \ni [S^3, L_{k+1}] = (-1)^{k+1} \chi(L_{k+1}) + (-1)^k \sum_{\ell_k \subset L_{k+1}, \sharp(\ell_k) = k} \chi(\ell_k) + (-1)^{k+1} k S^3$

But from (1) we see that for any $k$-component sublink $\ell_k$ of $L_{k+1}$, $\chi(\ell_k) - S^3 \in \mathcal{M}_k$. Noting that $\mathcal{M}_k \supset \mathcal{M}_{k+1}$, we see that $\chi(L_{k+1}) - S^3 \in \mathcal{M}_k$. Continuing this argument we can show that $M (= \chi(L^{n_1,n_2,\ldots,n_k}) - S^3 \in \mathcal{M}_k$, completing the proof.

Remark 1. During the preparation of this paper, the author was informed that K. Habiro obtained a stronger result by using his clasper theory.

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