Characterization of equilibrium existence and purification in general Bayesian games

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This version: June 17, 2021

Abstract

This paper studies Bayesian games with general action spaces, correlated types and interdependent payoffs. We introduce the condition of “decomposable coarser payoff-relevant information”, and show that this condition is both sufficient and necessary for the existence of pure-strategy equilibria and purification from behavioral strategies. As a consequence of our purification method, a new existence result on pure-strategy equilibria is also obtained for discontinuous Bayesian games. Illustrative applications of our results to oligopolistic competitions and all-pay auctions are provided.

Keywords: General Bayesian game; Decomposable coarser payoff-relevant information; Pure-strategy equilibrium; Existence; Purification
1 Introduction

Bayesian games (à la Harsanyi (1967–1968)), where players have incomplete information about certain aspects of the environment, arise naturally in numerous real-life situations. Many economic applications are most conveniently formulated as Bayesian games with infinitely many choices. Consider, for instance, several bidders competing for a single object in an auction, where each bidder only knows her private information and chooses bids from some bid space. While it is true that the bid spaces in practical situations are in fact discrete, studying the setting with continuum bid spaces has appealing advantages. On the one hand, solving the case with finite bid spaces may involve complicated combinatorial arguments that need multiple steps of approximation. On the other hand, one often needs to use calculus to characterize equilibria and to conduct the comparative statics analysis in the setting of continuum bid spaces. The consideration of general action spaces allows one not only to simplify the analysis of Bayesian games, but also to uncover interesting results that cannot be found in the discrete framework.

The fundamental question of equilibrium existence for Bayesian games with general action spaces has been extensively studied in the literature. To ensure that players’ expected payoffs are continuous in strategy profiles, Milgrom and Weber (1985) worked with continuous payoffs and assumed the absolute continuity (AC) condition on the information structure. They provided an existence result for behavioral-strategy equilibria in general Bayesian games.²

Despite its wide use in the studies on Bayesian games, the notion of behavioral strategy has been criticized for various reasons. As noted in Radner and Rosenthal (1982) and Milgrom and Weber (1985), one rarely observes that individuals make decisions by using randomization devices in practical situations. In addition, economic applications of games with incomplete information often focus on pure-strategy equilibrium. If one adopts pure-strategy equilibrium as the solution concept, then several basic questions

¹Note that previous studies on auctions—one of the most successful applications of Bayesian games, generally allow bidders to choose from continuum bid spaces; see, for example, Krishna (2009).
²The AC condition is widely satisfied in economic applications, and plays an essential role in the proof of the equilibrium existence results in subsequent works. The recent studies on Bayesian games with discontinuous payoffs (e.g., Carbonell-Nicolau and McLean (2018) and Prokopovych and Yannelis (2019)) present various equilibrium existence results by proposing different payoff security-type conditions under the same AC condition on the information structure. Without the AC condition, a behavioral-strategy equilibrium may not exist even in the finite-action setting: Simon (2003) constructed a Bayesian game without any equilibrium; Hellman (2014) provided a simpler example without any approximate equilibrium; and Friedenberg and Meier (2017) presented an example associated with the universal type structure that does not have any equilibrium.
could be asked. Do pure-strategy equilibria generally exist in Bayesian games? If pure-strategy equilibria do exist, can players obtain the same equilibrium payoffs (and thus the same social welfare) as those based on behavioral-strategy equilibria; that is, will focusing on pure-strategy equilibria be without loss of generality? The answers to these questions are negative.\(^3\) Khan, Rath and Sun (1999) presented a simple two-player Bayesian game with private values and continuous payoffs,\(^4\) where both players’ action spaces are \([-1, 1]\), and the common prior is the uniform distribution on the square \([0, 1] \times [0, 1]\) \(i.e.,\) the two players have independent types with uniform distribution on their individual type space \([0, 1]\). They pointed out that this Bayesian game does not possess any pure-strategy equilibrium, while behavioral-strategy equilibria always exist.\(^5\) Then the question is, under what kind of suitable conditions, pure-strategy equilibria exist in Bayesian games with general action spaces.

We aim to address the above questions in this paper, and attempt to provide practitioners with a “toolkit” of relatively simple conditions that are useful in proving the existence of pure-strategy equilibria in applied works. The focus is on general Bayesian games, which allow players’ actions to be infinite, types to be correlated, and payoffs to be interdependent. Towards this end, we introduce the condition of “decomposable coarser payoff-relevant information” (DCPI in short) in the sense that the density-weighted payoff of each player is decomposed into a sum of finitely many components with each component being the product of a “coarser” action-relevant part and an action-irrelevant part.

Under DCPI, Theorem 1 establishes the existence of pure-strategy equilibria for Bayesian games with general action spaces. Fudenberg and Tirole (1991, p. 236)\(^3\) In Bayesian games with finitely many actions, the existence of pure-strategy equilibria has been well established. Based on Dvoretzky, Wald and Wolfowitz (1951)’s purification result, it is straightforward to obtain the existence of pure-strategy equilibria in the setting with private values and (conditionally) independent types; see Radner and Rosenthal (1982), Milgrom and Weber (1985), and Khan, Rath and Sun (2006). Khan and Zhang (2018) worked with finite-action Bayesian games with payoff-irrelevant correlated types. He and Sun (2019) characterized several important properties of pure-strategy equilibria via the “coarser inter-player information” condition in finite-action Bayesian games. Hellman and Levy (2017) proved the equilibrium existence result for Bayesian games with purely atomic types under a smoothness condition. Beißner and Khan (2019) studied games with incomplete information and non-expected utilities.

\(^3\)A player is said to have private values (resp. interdependent payoff) if her payoff function depends on her own type (resp. all players’ types) and the action profile.

\(^4\)The AC condition trivially holds in this example due to independent types. In addition, the condition of “coarser inter-player information” in He and Sun (2019) is also satisfied given the private values and independent types. The nonexistence of pure-strategy equilibria in this example indicates that the results in He and Sun (2019) cannot be applied to the infinite-action setting.
considered Bayesian games with continuous payoffs on compact action spaces, private values and conditionally independent types. If one imposes the condition that each player’s payoff-relevant private information is coarser than her full private information given any non-trivial event, then DCPI is satisfied. Thus, Theorem 1 implies the existence of pure-strategy equilibria in the particular setting; see Corollary 2 below.\footnote{The counterexample in Khan, Rath and Sun (1999) as mentioned above indicates that such an existence result may fail without the coarseness condition.} Remark 1 in Fudenberg and Tirole (1991, p. 236) also pointed out the need to find regularity conditions for working with pure-strategy equilibria in general Bayesian games without the restriction of conditionally independent types and private values. The DCPI condition serves this purpose by allowing us to consider Bayesian games with general action spaces, interdependent payoffs and correlated types as in Theorem 1. Furthermore, DCPI is shown to be a minimal regularity condition in the sense that it is satisfied if pure-strategy equilibria always exist in general Bayesian games; see Proposition 1. Example 1 provides an illustrative application of Theorem 1 to oligopolistic competitions.

In Theorem 2, we present a purification principle (called conditional purification) relating behavioral strategies to pure strategies. A conditional purification of a behavioral-strategy profile preserves the same expected payoffs and action distributions conditional on any non-trivial event in players’ types, and thus also preserves the equilibrium property. We show that the DCPI condition is both sufficient and necessary for the existence of conditional purification in general Bayesian games. In this sense, DCPI appears to be an appropriate condition showing that focusing on pure strategies is without loss of generality. To demonstrate the usefulness of the purification result, we apply it to discontinuous Bayesian games, which arise naturally in economic applications. For instance, auctions and price competitions are typical examples where players’ payoffs are not continuous when a tie occurs. Building on the result of Reny (1999) for normal form games, Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) established the equilibrium existence in behavioral strategies in the incomplete information setting. We present a new equilibrium existence result in pure strategies for discontinuous Bayesian games in Proposition 2, and demonstrate its applicability to all-pay auctions in Example 2.

There is a sizable literature studying the existence of pure-strategy equilibria and the purification method in Bayesian games. A major approach is to impose rich structures on the type spaces. Khan and Sun (1999) showed that a pure-strategy equilibrium exists
by modelling players’ type spaces as atomless Loeb spaces.\footnote{Loeb spaces were first introduced in Loeb (1975); see Loeb and Wolff (2015) for the construction.} The equilibrium existence result was further generalized to the setting with saturated probability spaces as the type spaces; see Wang and Zhang (2012) and Khan and Zhang (2014). Loeb and Sun (2006) obtained purification results with general action spaces by considering atomless Loeb spaces. The purification method was further studied in Podczeck (2009) based on saturated probability spaces. He and Sun (2014) relaxed the assumption of saturated probability spaces by the relative diffuseness condition for both the equilibrium existence and purification results. All these papers study Bayesian games with private values and (conditionally) independent types. In this paper, we allow for interdependent payoffs and correlated types, and dispense with the additional rich structures on the type spaces—Loeb spaces/saturated probability spaces/relative diffuseness.\footnote{Working with Loeb spaces/saturated probability spaces excludes Polish spaces as the type spaces, which are the widely used private type spaces in economic games. Our results do not have this restriction.} Therefore, our results cover the results in all those papers.

Another stream of literature studies the existence of pure-strategy equilibria in Bayesian games by assuming order structures on the payoffs. Vives (1990) presented an existence result of pure-strategy equilibria in supermodular games. By assuming the Spence-Mirrlees single crossing property, Athey (2001) proved that a monotone pure-strategy equilibrium exists, which was later extended to the setting with multidimensional and partially ordered type/action spaces in McAdams (2003). Reny (2011) further generalized the results in these two papers by allowing the action spaces to be compact locally complete metric semilattices and the type spaces to be partially ordered probability spaces.

The paper is organized as follows. Section 2 describes the formulation of general Bayesian games. Section 3 introduces the condition of DCPI. Section 4 shows that this condition is sufficient and necessary for the existence of pure-strategy equilibria. In Section 5, we characterize the existence of conditional purifications via DCPI, and present a new equilibrium existence result for discontinuous Bayesian games. The proofs are collected in Section 6.
2 General Bayesian games

We start by describing the model in terms of the players, types, actions, payoffs, strategies, and the equilibrium notion.

Players The set of players is $I = \{1, 2, \ldots, n\}$ with $n \geq 2$.

Types For each $i \in I$, player $i$ observes a private type $t_i$, whose value lies in some measurable space $(T_i, T_i)$. The set $T = \prod_{i \in I} T_i$ collects all the type profiles, which is endowed with $\mathcal{T} = \otimes_{i \in I} T_i$. Let $\lambda$ be the common prior in the game, which is a probability measure on the space of type profiles $(T, \mathcal{T})$.

For each $i \in I$, the marginal of the common prior $\lambda$ on player $i$’s private type space $(T_i, T_i)$ is denoted by $\lambda_i$. We shall work with the condition that $\lambda$ admits a density with respect to $\otimes_{i \in I} \lambda_i$; that is, types can be correlated as long as the common prior $\lambda$ is absolutely continuous with respect to $\otimes_{i \in I} \lambda_i$ with the corresponding Radon-Nikodym derivative $q$.

Let $(T, \mathcal{T}, \lambda)$ be a probability space. A finite measure $\nu$ is said to be absolutely continuous with respect to $\lambda$ if for any $D \in \mathcal{T}$, $\lambda(D) = 0$ implies $\nu(D) = 0$. In this case, there exists a ($\lambda$-almost) unique $\lambda$-integrable function $q$ such that $\nu(D) = \int_D q(t) \lambda(dt)$ for any $D \in \mathcal{T}$. Such a function $q$ is called the Radon-Nikodym derivative of $\nu$ with respect to $\lambda$; see Theorem 13.18 in Aliprantis and Border (2006).

That is, if $t_i = (t_{i1}, \ldots, t_{ik})$ is multi-dimensional, then different components $t_{i1}, \ldots, t_{ik}$ of $t_i$ can be arbitrarily correlated with each other.

Actions After observing the private type, player $i \in I$ chooses an action from the action space $A_i$. The set $A_i$ is a nonempty and compact metric space endowed with the Borel $\sigma$-algebra $\mathcal{B}(A_i)$. We denote the set of all action profiles by $A = \prod_{i \in I} A_i$.

Payoffs For each player $i \in I$, the payoff $u_i$ depends on the action profile $a \in A$ as well as the type profile $t \in T$. We shall assume that each $u_i$ is a well-behaved mapping. Specifically,

- $u_i$ is jointly measurable on $T \times A$;

9Let $(T, \mathcal{T}, \lambda)$ be a probability space. A finite measure $\nu$ is said to be absolutely continuous with respect to $\lambda$ if for any $D \in \mathcal{T}$, $\lambda(D) = 0$ implies $\nu(D) = 0$. In this case, there exists a ($\lambda$-almost) unique $\lambda$-integrable function $q$ such that $\nu(D) = \int_D q(t) \lambda(dt)$ for any $D \in \mathcal{T}$. Such a function $q$ is called the Radon-Nikodym derivative of $\nu$ with respect to $\lambda$; see Theorem 13.18 in Aliprantis and Border (2006).

10That is, if $t_i = (t_{i1}, \ldots, t_{ik})$ is multi-dimensional, then different components $t_{i1}, \ldots, t_{ik}$ of $t_i$ can be arbitrarily correlated with each other.
• $u_i$ is integrably bounded in the sense that there is a real-valued integrable mapping $h_i$ on $(T, \mathcal{T}, \lambda)$ with $|u_i(a, t)| \leq h_i(t)$ for all $(a, t) \in A \times T$.

Note that players’ payoffs are allowed to be interdependent.

Strategies  For player $i \in I$, let $\mathcal{M}(A_i)$ be the space of Borel probability measures on $A_i$ endowed with the topology of weak convergence. For each player $i \in I$, a behavioral strategy is a measurable mapping from the private type space $(T_i, \mathcal{T}_i)$ to $\mathcal{M}(A_i)$.\footnote{A distributional strategy of player $i$ is a probability measure on the product of her type and action spaces, with the marginal being $\lambda_i$ on $(T_i, \mathcal{T}_i)$. It is clear that every behavioral strategy corresponds to a natural distributional strategy, and every distributional strategy induces an equivalent class of behavioral strategies. We focus on behavioral strategies for simplicity, but all the results can be easily extended to distributional strategies.} Let $L_i^T$ be the set of all behavioral strategies for player $i$. Denote $L^T = \prod_{i \in I} L_i^T$. Similarly, we can define a pure strategy as a measurable mapping from $(T_i, \mathcal{T}_i)$ to $A_i$, which can be viewed as a behavioral strategy by taking it as a Dirac measure for all $t_i \in T_i$.

Equilibria  Given a strategy profile $h = (h_1, h_2, \ldots, h_n)$, player $i$’s expected payoff is

$$U_i(h) = \int_T \int_A u_i(a, t) \cdot \prod_{\ell \in I} h_\ell(t_\ell; da_\ell) \lambda(dt)$$
$$= \int_T \int_A u_i(a, t) \cdot q(t) \cdot \prod_{\ell \in I} h_\ell(t_\ell; da_\ell) \otimes \lambda_\ell(dt_\ell).$$

A behavioral-strategy equilibrium (resp. pure-strategy equilibrium) is a behavioral-strategy profile (resp. pure-strategy profile) $h^* = (h^*_1, h^*_2, \ldots, h^*_n)$ such that $h_i^*$ maximizes $U_i(h_i, h^*_{-i})$ for each player $i \in I$.

3 Decomposable coarser payoff-relevant information

In this section, we propose a condition to describe the difference between the information conveyed via the types and the information conveyed via the payoff-relevant components of the types.\footnote{Players’ types may contain not only the payoff-relevant information, but also the payoff-irrelevant information. For example, the beliefs may differ given distinct types, and thus one could be able to design mechanisms eliciting players’ beliefs about others; see Cremer and McLean (1988), Heifetz and Neeman (2006), Chen and Xiong (2013), and Guo (2019) for more discussions.}

In the definition of the expected payoff $U_i$ for player $i$, it is strategically equivalent for each player $i$ to view $u_i(a, t) \cdot q(t)$ as the payoff and $\otimes_{\ell \in I} \lambda_\ell$ as the prior. Hereafter,
we shall work with the density-weighted payoff

\[ w_i(a, t) = u_i(a, t) \cdot q(t) \]

for each player \( i \in I \), action profile \( a \in A \), and type profile \( t \in T \). For a strategy profile \( f = (f_1, f_2, \ldots, f_n) \), player \( i \)'s expected payoff can be rewritten as

\[ U_i(f) = \int_T \int_A w_i(a, t) \cdot \prod_{\ell \in I} f_\ell(t_\ell; da_\ell) \otimes \lambda_\ell(dt_\ell). \]

Before stating the key condition of “decomposable coarser payoff-relevant information,” we first need to define the notion of “nowhere equivalence.” Let \((\Omega, \mathcal{F}, \mathbf{P})\) be an atomless finite positive measure space, and \( \mathcal{F} \) a sub-\( \sigma \)-algebra of \( \mathcal{T} \). For a set \( D \in \mathcal{T} \) with \( \mathbf{P}(D) > 0 \), let \( \mathcal{F}^D \) (resp. \( \mathcal{T}^D \)) be the restricted \( \sigma \)-algebra \( \{ D \cap D' \mid D' \in \mathcal{F} \} \) (resp. \( \{ D \cap D' \mid D' \in \mathcal{T} \} \)) on \( D \). The \( \sigma \)-algebra \( \mathcal{T} \) is said to be nowhere equivalent to \( \mathcal{F} \) under \( \mathbf{P} \) if the strong completion of \( \mathcal{F}^D \) in \( \mathcal{T}^D \) under \( \mathbf{P} \) is not equal to \( \mathcal{T}^D \) for any \( D \in \mathcal{T} \) of positive measure.\(^{13}\)

Let \( \mathcal{F}_i \) be a sub-\( \sigma \)-algebra of \( \mathcal{T}_i \). Throughout the paper, we assume that \((\mathcal{T}_i, \mathcal{F}_i, \lambda_i)\) is atomless for each \( i \in I \).\(^{14}\) We introduce the definition of decomposable coarser payoff-relevant information as follows.

**Definition 1.** A Bayesian game is said to have decomposable coarser payoff-relevant information (DCPI hereafter) if each \( \mathcal{T}_i \) is nowhere equivalent to \( \mathcal{F}_i \) under \( \lambda_i \), and there exists a positive integer \( J \) such that for player \( i \in I \),

\[ w_i(a, t) = \sum_{j=1}^J \left[ w_j^i(a, t) \cdot \prod_{\ell \in I} \rho_j^\ell(t_\ell) \right], \]

where for \( j = 1, 2, \ldots, J \), (1) \( w_j^i(a, \cdot) \) is \( \otimes_{\ell \in I} \mathcal{F}_\ell \)-measurable and \( \otimes_{\ell \in I} \lambda_\ell \)-integrable for each \( a \in A \), (2) \( \rho_j^\ell \) is nonnegative and integrable on \((T_\ell, \mathcal{T}_\ell, \lambda_\ell)\) for each \( \ell \in I \).

In a Bayesian game with DCPI, players have interdependent payoffs and correlated types, as players’ payoff functions can depend on the types of each other, and the density

\(^{13}\)The strong completion of \( \mathcal{F}^D \) in \( \mathcal{T}^D \) under \( \mathbf{P} \) is the collection of all sets in the form \( E \Delta E_0 \), where \( E \in \mathcal{F}^D \) and \( E_0 \) is a \( \mathbf{P} \)-null set in \( \mathcal{T}^D \), and \( E \Delta E_0 \) denotes the symmetric difference \( (E \backslash E_0) \cup (E_0 \setminus E) \).

\(^{14}\)In economic applications of Bayesian games, the type spaces are often modelled by intervals or rectangles with density functions, which are atomless probability spaces. Furthermore, it is natural to consider infinite types in Bayesian games, as the set of belief types may have the cardinality of the continuum; see, for example, the discussions in Brandenburger and Dekel (1993) and Hammond (2004).
function can be non-trivial. DCPI means that the density-weighted payoff of each player is decomposed into a sum of finitely many components with each component being the product of a “coarser” action-relevant part and an action-irrelevant part. When $J = 1$ and $\rho^1_i \equiv 1$, $w^J_i$ is simply the density-weighted payoff $w_i$. If the density-weighted payoffs are required to satisfy the DCPI condition for $J = 1$, then the functions $\{w_i\}_{i \in I}$ need to be measurable with respect to $\otimes_{\ell \in I} F_\ell$. In various economic applications of Bayesian games, players are often assumed to have conditionally independent types.\(^{15}\)

As demonstrated by Example 1 below, DCPI would fail for a simple two-player Bayesian game with conditionally independent payoff-irrelevant types if $J$ is required to be 1. To address this issue, we allow $J$ to be any positive integer. Indeed, Corollary 1 below presents a general model of Bayesian games with interdependent payoffs and conditionally independent types that satisfy the DCPI condition. In Sections 4.2 and 5.2 below, we also consider specific Bayesian games with conditionally independent types in the settings of oligopolistic competitions and all-pay auctions, and show that these games satisfy DCPI for a general $J \geq 1$.

Below, we present an example of a two-player Bayesian game to illustrate the DCPI condition.\(^{16}\)

**Example 1.** There are two players, $I = \{1, 2\}$. For $i \in I$, $T_i = [0, 1]$ endowed with the Borel $\sigma$-algebra $T_i = B([0, 1])$. The action space of each player is $[0, 1]$. Player $i$’s payoff function $u_i$ is bounded, continuous, and only depends on the action profile. The set of common states $T_0 = \{t_{01}, t_{02}\}$. The two common states are drawn with equal probability, which are unobservable to both players. At $t_{01}$, a pair $(t_1, t_2) \in T_1 \times T_2$ is drawn following the uniform distribution $\tilde{\lambda} = \eta \otimes \eta$. At $t_{02}$, $(t_1, t_2)$ is drawn under the distribution $\hat{\lambda}$, which has the density $6t_1t_2^2$ with respect to the uniform distribution. The space of type profiles is $T = T_1 \times T_2$, and the common prior is $\lambda = \frac{1}{2} \left( \tilde{\lambda} + \hat{\lambda} \right)$.

**Claim 1.** The Bayesian game in Example 1 satisfies DCPI only when $J \geq 2$.

\(^{15}\)For example, auctions with conditionally independent types have been studied in both the empirical and theoretical literatures; see Li, Perrigne and Vuong (2000) and Athey (2001).

\(^{16}\)This example is a variation of Example 1 in He and Sun (2019).
4 The existence result

4.1 Existence of pure-strategy equilibria

In this section, suppose that \( u_i(\cdot, t) \) is continuous in \( a \) for each \( t \in T \) and \( i \in I \). We shall prove that under DCPI, pure-strategy equilibria exist in general Bayesian games. Importantly, this condition is shown to be also necessary for the equilibrium existence result. The proofs are left in Appendix.

**Theorem 1.** Every Bayesian game with decomposable coarser payoff-relevant information has a pure-strategy equilibrium.

In the theorem above, we show that DCPI is sufficient for the equilibrium existence result. Below, we shall state the necessity part. Rather than proving that this condition must be satisfied if a pure-strategy equilibrium exists in every Bayesian game, we shall prove a stronger result: the necessity result holds even when we restrict our attention to a class of very simple games.

To state the necessity part, we repeat the description of the private type spaces for clarity.

1. The private type space of player \( i \in I \) is \((T_i, T_i/F_i, \lambda_i)\) with the payoff-relevant information \( F_i \).

   We focus on a special class of games with incomplete information as follows.

2. Players have type-irrelevant payoffs in the sense that the payoff function of each player does not depend on the type profile \( t \).

   Fix an arbitrary infinite compact set \( X \). Let \( \Gamma_n(X) \) be the collection of all Bayesian games with the player space \( I \), the common action set \( X \), the private type spaces \( \{(T_i, T_i/F_i, \lambda_i)\}_{i \in I} \), and type-irrelevant payoffs. A Bayesian game belonging to the class \( \Gamma_n(X) \) has a simple structure: players’ payoffs only depend on the action profile, but not on the types at all. One may wonder whether we need to construct some games with rather complicated information structures in order to prove the necessity part. Working with the class of simple games in \( \Gamma_n(X) \) addresses this concern.

**Proposition 1.** Fix any infinite compact set \( X \). If every Bayesian game in \( \Gamma_n(X) \) has a pure-strategy equilibrium, then the condition of decomposable coarser payoff-relevant information holds.
To illustrate the DCPI condition and Theorem 1, we consider Bayesian games with interdependent payoffs, where players’ private information are independent conditioned on finitely many states. This framework is commonly adopted in applications. We show that DCPI holds for this class of Bayesian games.

• For each \(i \in I\), player \(i\)'s private information space is \((T_i, T_i)\).

• Let \(T_0 = \{t_{01}, \ldots, t_{0J}\}\) be the space of unobservable common states that affect the payoffs of all players. The state \(t_{0j}\) happens with probability \(\tau_j > 0\) for \(1 \leq j \leq J\).

• Given each \(t_{0j} \in T_0\), let \(\lambda_j^i\) be the conditional prior on \((\prod_{i \in I} T_i, \otimes_{i \in I} T_i)\). The marginal \(\lambda_j^i\) of \(\lambda_j^i\) on \((T_i, T_i)\) is atomless and \(\lambda_j^i = \otimes_{i \in I} \lambda_j^i\).

• For each \(i\), player \(i\)'s payoff function \(u_i(a, t_0, t_1, \ldots, t_n)\) is \(\otimes_{\ell \in I} F_\ell\)-measurable for each \((a, t_0) \in A \times T_0\).

• The model is viewed as an \(n\)-player game, in which the space of type profiles is \(T = \prod_{1 \leq i \leq n} T_i\), the common prior is \(\hat{\lambda} = \sum_{1 \leq j \leq J} \tau_j \lambda_j^i\), and the marginal of \(\hat{\lambda}\) is \(\hat{\lambda}_\ell = \sum_{1 \leq j \leq J} \tau_j \lambda_j^\ell\) for each \(\ell \in I\).

Corollary 1. If \(T_\ell\) is nowhere equivalent to \(F_\ell\) under \(\hat{\lambda}_\ell\) for each \(\ell \in I\), then the above Bayesian game with interdependent payoff and conditionally independent types satisfies DCPI, and has pure-strategy equilibria.

When \(u_i(a, t_0, \cdot)\) does not depend on \(t_{-i}\) for each \(i \in I\), the game defined above is reduced to a Bayesian game with private values and conditionally independent types. If we impose the condition that \(T_\ell\) is nowhere equivalent to \(F_\ell\) for each \(\ell \in I\), then it follows directly from Corollary 1 that pure-strategy equilibria exist in Bayesian games with private values and conditionally independent types.

Corollary 2. Suppose that for each \(\ell \in I\), \(T_\ell\) is nowhere equivalent to \(F_\ell\) under \(\hat{\lambda}_\ell\) and \(u_\ell(a, t_0, \cdot)\) does not depend on \(t_{-\ell}\) for each \(a \in A, t_0 \in T_0\) and \(\ell \in I\). Then the Bayesian game with private values and conditionally independent types as defined above satisfies DCPI, and has pure-strategy equilibria.

Remark 1. A major idea to prove the existence of pure-strategy equilibria in Bayesian games with general action spaces is to assume rich structures on the type spaces; see Khan and Sun (1999) and Loeb and Sun (2006) for atomless Loeb spaces, Wang and Zhang (2012) and Khan and Zhang (2014) for saturated probability spaces, and He and Sun (2014) for the relative diffuseness condition. In this paper, we are able to relax these additional richness conditions. Furthermore, all those papers work with Bayesian games with private values and (conditionally) independent types. Our
Corollary 2 covers the corresponding results in those papers as special cases. Note that our Theorem 1 (and Corollary 1) allows for interdependent payoffs and correlated types.

4.2 A Cournot duopoly game

Below, we shall present a simple example of Cournot duopoly to illustrate the model and the condition of DCPI.

There are two firms, $I = \{1, 2\}$. The market could be in one of the two possible unknown status: $H$ or $L$, where $H$ represents boom and $L$ represents recession. That is, the set $T = \{H, L\}$ is the common state space, with $t_0 \in T$ being unobservable to both firms. Each firm $i$ will possess a bi-dimensional private type $t_i = (t_{i1}, t_{i2}) \in T_{i1} \times T_{i2}$, where $T_{i1} = T_{i2} = T_{21} = T_{22} = [0, 1]$ endowed with the Borel $\sigma$-algebra. The first component $t_{i1}$ summarizes firm $i$’s information about the basic structure of the market, including demand shock, regulation policy, production cost, etc. The second component $t_{i2}$ affects firm $i$’s belief about the market status $t_0$, though it does not directly enter the payoffs. Based on the private information $(t_{i1}, t_{i2})$, firm $i$ chooses a quantity $a_i \in [0, \bar{a}_i]$ to produce. For each firm $i$, the payoff function $u_i$ is bounded and continuous, and depends on $(a_1, a_2)$ and $(t_0, t_{i1}, t_{i2})$. The information structure is described as follows.

- The common states $H$ and $L$ are drawn with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively.
- The first components of firms’ types, $t_{i1}$ and $t_{21}$, are drawn according to the uniform distribution $\eta$ on $T_{i1} = T_{21} = [0, 1]$, and are independent of each other and also the other components $t_0, t_{i2}, t_{22}$.
- If $t_0 = H$, then a pair $(t_{i2}, t_{22}) \in T_{i2} \times T_{22} = [0, 1] \times [0, 1]$ will be drawn according to the distribution $\hat{\zeta}$, which has the density $(\frac{1}{2} + t_{i2})(\frac{1}{2} + t_{22})$ with respect to $\eta \otimes \eta$.
- If $t_0 = L$, then a pair $(t_{i2}, t_{22}) \in T_{i2} \times T_{22} = [0, 1] \times [0, 1]$ will be drawn according to the distribution $\hat{\zeta}$, which has the density $4t_{i2}t_{22}$ with respect to $\eta \otimes \eta$.

In the example above, the first component $t_{i1}$, together with the common state $t_0$, is payoff relevant to both firms. The second component $t_{i2}$ does not enter the payoffs directly. However, since $t_0$ is not observable, $t_{i2}$ is a signal that firm $i$ receives about the common state, which further affects firm $i$’s belief about the opponent’s type $t_{j2}$, $j \neq i$. Let $\hat{\lambda} = \eta \otimes \eta \otimes \hat{\zeta}$, and $\hat{\lambda} = \eta \otimes \eta \otimes \hat{\zeta}$. In this Cournot duopoly game, the space of type profiles is $T = T_1 \times T_2$ and the common prior is $\lambda = \frac{1}{3} \hat{\lambda} + \frac{2}{3} \hat{\lambda}$.

\footnote{Here we abuse the notion by using $\eta \otimes \eta \otimes \zeta$ to denote a probability measure on $T_{i1} \times T_{i2} \times T_{21} \times T_{22}$ such that $\eta \otimes \eta \otimes \zeta(B_1 \times B_2 \times D_1 \times D_2) = \eta(B_1) \cdot \eta(D_1) \cdot \zeta(B_2 \times D_2)$ for any Borel subsets $B_1$, $B_2$, $D_1$, and $D_2$ in $[0, 1]$. The notion $\eta \otimes \eta \otimes \zeta$ is similar.}
The following claim follows from Corollary 1.

**Claim 2.** The Cournot game above satisfies the condition of decomposable coarser payoff-relevant information, and possesses a pure-strategy equilibrium.

## 5 Purification

In this section, we consider the purification method for general Bayesian games. In Section 5.1, we establish a general purification principle relating behavioral strategies to pure strategies, which preserves the same expected payoffs and action distributions, and thus the equilibrium property. It is shown that DCPI is both necessary and sufficient for the purification in Bayesian games. Importantly, the purification result does not impose the continuity condition on payoffs. As a result, we are able to apply it to discontinuous Bayesian games and obtain a new equilibrium existence result in pure strategies.

### 5.1 A purification result

In this section, we shall define the notion of conditional purification, and prove that DCPI is sufficient and necessary for the existence of conditional purifications.

**Definition 2.** Let \( f = (f_1, f_2, \ldots, f_n) \) and \( g = (g_1, g_2, \ldots, g_n) \) be two behavioral-strategy profiles.

1. The strategy profiles \( f \) and \( g \) are said to be payoff equivalent for player \( i \) if \( U_i(f) = U_i(g) \), and \( U_i(h_i, f_{-i}) = U_i(h_i, g_{-i}) \) for any given behavioral strategy \( h_i \) of player \( i \).
2. The strategy profiles \( f \) and \( g \) are said to be conditional distribution equivalent for player \( i \) if for any \( D \in \mathcal{F}_i \) and Borel subset \( B \subseteq X \), \( \int_D f_i(t_i; B) \lambda_i(dt_i) = \int_D g_i(t_i; B) \lambda_i(dt_i) \).
3. When \( f_i \) is a pure-strategy for player \( i \), \( f \) and \( g \) are said to be belief consistent for player \( i \) if \( f_i(t_i) \in \text{supp} g_i(t_i) \) for \( \lambda_i \)-almost all \( t_i \in T_i \), where \( \text{supp} g_i(t_i) \) is the support of the probability measure \( g_i(t_i) \).

A pure-strategy profile \( f \) is said to be a *conditional purification* of a behavioral-strategy profile \( g \) if \( f \) and \( g \) are payoff equivalent, conditional distribution equivalent, and belief consistent for every player.

The notion of conditional purification requires that from each player’s perspective, a behavioral-strategy profile and its purification are the same when the player evaluates
the payoff and action distribution conditioned on any non-trivial payoff-relevant event. It is clear that if a behavioral-strategy profile \( g \) is an equilibrium, then its conditional purification \( f \) is a pure-strategy equilibrium.

We focus on Bayesian games with private type spaces \( \{(T_i, \mathcal{F}_i, \lambda_i)\}_{i \in I} \). The following result characterizes the existence of conditional purifications for these games via DCPI.

**Theorem 2.** In Bayesian games with private type spaces \( \{(T_i, \mathcal{F}_i, \lambda_i)\}_{i \in I} \), every behavioral-strategy profile \( g \) possesses a conditional purification \( f \) if and only if the condition of decomposable coarser payoff-relevant information holds.

### 5.2 Discontinuous Bayesian games

Payoff discontinuity in actions is a natural feature in various economic applications of Bayesian games; e.g., auctions, contests, price competitions, etc.\(^{18}\) Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) proved the existence of equilibria in behavioral strategies for Bayesian games with discontinuous payoffs. In this section, we shall demonstrate the usefulness of the purification result via discontinuous Bayesian games. In particular, by verifying DCPI in these games, we obtain a new existence result on pure-strategy equilibria through the purification principle from Theorem 2. As an illustrative example, we present a model of common-value all-pay auctions with general value functions and tie-breaking rules, and obtain a new existence result for pure-strategy equilibria.

To guarantee the existence of behavioral-strategy equilibria, Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) proposed the conditions of “uniform payoff security” and “random disjoint payoff matching,” respectively.\(^{19}\) Both conditions are to ensure that the ex ante payoff \( U_i \) is payoff secure for each player \( i \in I \), which is the key condition in Reny (1999). Thus, one can readily apply Reny (1999)’s result to conclude the equilibrium existence. To be concrete, the conditions are stated in the following.

**Definition 3** (Uniform payoff security). The Bayesian game is said to be **uniformly payoff secure** if for any \( \epsilon > 0 \), each \( i \in I \), and each pure strategy \( f_i \), there exists another

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\(^{18}\)There is a large literature studying discontinuous games in the last two decades. For some recent developments, see, for example, Prokopovych (2011), Carbonell-Nicolau and McLean (2019) and Prokopovych and Yannelis (2021). The literature is too vast to be discussed in the context of this paper, we refer the interested readers to the recent survey by Reny (2020).

\(^{19}\)Carbonell-Nicolau and McLean (2018) also considered the equilibrium existence in distributional strategies under the same condition.
pure strategy $f'_i$ such that for all $(t, a_{-i})$, there exists a neighborhood $O_{a_{-i}}$ of $a_{-i}$ such that for all $y_{-i} \in O_{a_{-i}},$

$$u_i(t, f'_i(t_i), y_{-i}) - u_i(t, f_i(t_i), a_{-i}) > -\epsilon.$$ 

**Definition 4** (Random disjoint payoff matching). Consider the points at which a player’s payoff function is discontinuous in other players’ strategies. Let $D_i : T_i \times A_i \rightarrow T_{-i} \times A_{-i}$ be defined by

$$D_i(t_i, a_i) = \{(t_{-i}, a_{-i}) \in T_{-i} \times A_{-i} \mid u_i(a_i, \cdot, t_i, t_{-i}) \text{ is discontinuous in } a_{-i}\}.$$ 

Suppose that $D_i$ has a $B(A_i) \otimes T_i$-measurable graph for each $i \in I$. Given a pure strategy $f_i$ of player $i$, denote $D_{f_i}(t_i) = D_i(t_i, f_i(t_i)).$

A Bayesian game $G$ is said to satisfy the condition of **random disjoint payoff matching** if for each player $i \in I$ and for any pure strategy $f_i$, there exists a sequence of pure-strategy deviations $\{h^k_i\}_{k=1}^\infty$ such that

1. for $\lambda$-almost all $t = (t_i, t_{-i}) \in T$ and for all $a_{-i} \in A_{-i},$

$$\liminf_{k \to \infty} u_i(f^k_i(t_i), a_{-i}, t_i, t_{-i}) \geq u_i(f_i(t_i), a_{-i}, t_i, t_{-i});$$

2. $\limsup_{k \to \infty} D_i(t_i, f^k_i(t_i)) = \emptyset$ for each $i \in I$ and for $\lambda_i$-almost all $t_i \in T_i.$

The following lemma summarizes Theorem 1 in Carbonell-Nicolau and McLean (2018) and Theorem 2 in He and Yannelis (2016).

**Lemma 1.** Suppose that the aggregate payoff $\sum_{i \in I} u_i(\cdot, t) : A \rightarrow \mathbb{R}$ is upper semicontinuous for each $t \in T$. If any of the following conditions holds,

1. the uniform payoff security,

2. the random disjoint payoff matching,

then the game has a behavioral-strategy equilibrium.

It is clear that based on Theorem 2 and Lemma 1 above, one can obtain the existence of pure-strategy equilibria by assuming the DCPI condition in discontinuous Bayesian games.

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20For a sequence of sets $\{X_k\}$, $\limsup_{k \to \infty} X_k = \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty X_j$ and $\liminf_{k \to \infty} X_k = \bigcup_{k=1}^\infty \bigcap_{j=k}^\infty X_j$. 

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Proposition 2. Suppose that the aggregate payoff \( \sum_{i \in I} u_i(\cdot, t) : A \to \mathbb{R} \) is upper semicontinuous for each \( t \in T \), and one of the following conditions holds,

1. the uniform payoff security,
2. the random disjoint payoff matching.

If the discontinuous game has decomposable coarser payoff-relevant information, then it possesses a pure-strategy equilibrium.

Below, we provide an example of a common-value all-pay auction with general value functions and tie-breaking rules. Carbonell-Nicolau and McLean (2018, Section 6.1) studied an all-pay auction with general value functions and standard tie-breaking rules. He and Yannelis (2016, Section 4) presented an all-pay auction with quasi-linear payoffs and general tie-breaking rules. The following example cannot be covered by the all-pay auctions as considered in those two papers.

Example 2. Suppose that \( n \geq 2 \) bidders compete for an object. Let \( I = \{1, 2, \ldots, n\} \).

For each \( i \in I \), bidder \( i \) observes a private type \( t_i = (t_{i1}, t_{i2}) \in T_i \) and submits a bid \( a_i \) from the bid space \( A_i \), where \( T_i = T_{i1} \times T_{i2} \) is a compact rectangle in \( \mathbb{R}^2 \) endowed with the Borel \( \sigma \)-algebra \( T_i = \mathcal{B}(T_i) \), and \( A_i = [0, \bar{a}] \subseteq \mathbb{R}_+ \) with \( \bar{a} > 0 \). An unobservable common type \( t_0 \) is drawn from a finite type space \( T_0 = \{t_0^1, t_0^2, \ldots, t_0^J\} \) with probability \( \tau^j > 0 \), and \( \sum_{j=1}^J \tau^j = 1 \). Given \( t_0 \), the types \( (t_1, t_2, \ldots, t_n) \) is drawn from \( \prod_{i \in I} T_i \) independently according to a Borel probability measure \( \otimes_{i \in I} \lambda_i^t \). For each \( i \in I \) and for each \( j = 1, 2, \ldots, J \), the density of \( \lambda_i^j \) is \( q_i^j > 0 \) (with respect to the Lebesgue measure on \( T_i \)).

The common prior is \( \lambda = \sum_{j=1}^J \tau^j \otimes_{i \in I} \lambda_i^j \). Let \( T = \prod_{i \in I} T_i \) and \( A = \prod_{i \in I} A_i \).

For convenience, denote \( \tilde{T} = T_0 \times T \).

Given \( \tilde{t} = (t_0, t_1, \ldots, t_n) \) and \( a = (a_1, a_2, \ldots, a_n) \), let

\[
v_i(\tilde{t}, a) = \begin{cases} \psi_1(\tilde{t}, a) + \varphi_i(\tilde{t}, a), & \text{if bidder } i \text{ is the unique winning bidder,} \\ \psi_2(\tilde{t}, a) + \varphi_i(\tilde{t}, a), & \text{if bidder } i \text{ is a losing bidder.} \end{cases}
\]

Intuitively, one can interpret \( \psi_1(\tilde{t}, a) \) as the common value for winning the object, \( \psi_2(\tilde{t}, a) \) as the common outside option when losing it, and \( \varphi_i(\tilde{t}, a) \) as the cost of bidding. The bidder with the highest bid wins the object. Ties are broken as follows: if \( a_i = \max_{k \in I} a_k \),

\[
v_i(\tilde{t}, a) = \sigma_i(a)\psi_1(\tilde{t}, a) + (1 - \sigma_i(a))\psi_2(\tilde{t}, a) + \varphi_i(\tilde{t}, a),
\]

where \( \sigma_i(a_1, a_2, \ldots, a_n) = \frac{\xi_i(a_1, a_2, \ldots, a_n)}{\xi_i(a_1, a_2, \ldots, a_n)} \) and \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) : [0, \bar{a}]^n \to \)
is a continuous function measuring the relative importance of each bidder’s position when breaking a tie. We assume that \( \xi_i \) is increasing in \( a_i \) and decreasing in \( a_{-i} \); that is, when bidder \( i \) increases his bid, he is more likely to win, while the others are more likely to lose.

For any \( \tilde{t} = (t_0, t_1, \ldots, t_n) \), we further assume that \( \psi_1(\tilde{t}, a) = \psi_1(\tilde{t}^1, a) \), \( \psi_2(\tilde{t}, a) = \psi_2(\tilde{t}^1, a) \), and \( \varphi_i(\tilde{t}, a) = \varphi_i(\tilde{t}^1, a) \) for each \( i \in I \), where \( \tilde{t}^1 = (t_0, t_{11}, t_{21}, \ldots, t_{n1}) \). Thus, the first coordinate \( t_{i1} \) of \( t_i \) directly affects the payoffs of all the players. Although the second coordinate \( t_{i2} \) of \( t_i \) is payoff irrelevant, it influences player \( i \)’s belief about other players’ types.

We assume that

1. \( \psi_1, \psi_2, \) and \( \{ \varphi_i \}_{i \in I} \) are continuous on \( \tilde{T} \times A \);
2. winning the object is always better than losing it, that is, for any \( (t_0, t_{11}, \ldots, t_{n1}, a) \),

\[
\psi_1(t_0, t_{11}, \ldots, t_{n1}, a) \geq \psi_2(t_0, t_{11}, \ldots, t_{n1}, a).
\]

Define a transition probability \( \kappa \) from \( T = \prod_{i \in I} T_i \) to \( \mathcal{M}(T_0) \) such that for \( j = 1, 2, \ldots, J \),

\[
\kappa(\{ t_{j0} \} \mid t) = \frac{\tau^j \prod_{i \in I} q_i^j(t_i)}{\sum_{r=1}^J \tau^r \prod_{i \in I} q_i^r(t_i)},
\]

which is the conditional probability of the common type \( t_{j0} \) given the realized type profile \( t \in T \). When the type profile is \( t = (t_1, t_2, \ldots, t_n) \) and the bidding profile is \( a = (a_1, a_2, \ldots, a_n) \), bidder \( i \)’s payoff is

\[
u_i(t_1, t_2, \ldots, t_n, a_1, a_2, \ldots, a_n) = \sum_{j=1}^J v_i(t_{j0}, t_1, t_2, \ldots, t_n, a_1, a_2, \ldots, a_n) \kappa(\{ t_{j0} \} \mid t).
\]

In the all-pay auction above, we consider the environment in which bidders may have interdependent value functions, outside options, and general tie-breaking rules. We show that the condition of uniform payoff security is satisfied and the aggregate payoff is upper semicontinuous. Then there exists a behavioral-strategy equilibrium. We conclude the existence of a pure-strategy equilibrium by applying the purification result.

**Claim 3.** A pure-strategy equilibrium exists in the above all-pay auction with general value functions and tie-breaking rules.

\(^{21}\text{If } \xi_i \equiv 1 \text{ for each } i, \text{ then a tie is broken via the standard equal proportion rule.}\)
6 Appendix

6.1 Technical preparations

In this section, we present several results as the mathematical preparations for the proofs of Theorems 1 and 2, and Proposition 1.

Let \((T, \hat{T})\) be a measurable space, \(\hat{\mathcal{F}}\) a countably-generated sub-\(\sigma\)-algebra of \(\hat{T}\), and \(\mu_j\) an atomless finite positive measure on \((T, \hat{T})\) for \(j = 1, 2, \ldots, J\). Denote \(\mu = (\mu_1, \mu_2, \ldots, \mu_J)\). Suppose that each \(\mu_j\) is absolutely continuous with respect to some atomless probability measure \(\mu_0\).

Let \(X\) be a Polish space (complete metrizable topological space), \(\mathcal{B}(X)\) the Borel \(\sigma\)-algebra of \(X\), and \(\mathcal{M}(X)\) the space of all Borel probability measures on \(X\) endowed with the topology of weak convergence.

An \(\hat{\mathcal{F}}\)-measurable transition probability from \(T\) to \(X\) is a mapping \(\phi: T \to \mathcal{M}(X)\) such that for every \(B \in \mathcal{B}(X)\), the mapping \(\phi(\cdot; B): t \mapsto \phi(t; B)\) is \(\mathcal{F}\)-measurable, where \(\phi(t; B)\) is the value of the probability measure \(\phi(t)\) on the Borel set \(B \subseteq A\). For each \(j = 1, 2, \ldots, J\), we use \(\mathcal{R}(\hat{\mathcal{F}}, \mu_j)(X)\), or \(\mathcal{R}(\hat{\mathcal{F}}, \mu_j)\) when it is clear, to denote the set of all \(\hat{\mathcal{F}}\)-measurable transition probabilities from \(T\) to \(X\) under \(\mu_j\). The set \(\mathcal{R}(\hat{\mathcal{F}}, \mu_j)\) is endowed with the following weak topology.

A sequence \(\{\phi_n\}_{n=1}^{\infty}\) in \(\mathcal{R}(\hat{\mathcal{F}}, \mu_j)\) is said to weakly converge to \(\phi\) in \(\mathcal{R}(\hat{\mathcal{F}}, \mu_j)\), if for every bounded Carathéodory function \(c: T \times X \to \mathbb{R}\),

\[
\lim_{n \to \infty} \int_T \left[ \int_X c(t, x) \phi_n(t; dx) \right] \mu_j(dt) = \int_T \left[ \int_X c(t, x) \phi(t; dx) \right] \mu_j(dt).
\]

The weak topology on \(\mathcal{R}(\hat{\mathcal{F}}, \mu_j)\) is defined as the weakest topology for which the functional

\[
\phi \mapsto \int_T \left[ \int_X c(t, x) \phi(t; dx) \right] \mu_j(dt)
\]

is continuous for every bounded Carathéodory function \(c: T \times X \to \mathbb{R}\).

Denote \(\mathcal{R}(\hat{\mathcal{F}}, \mu) = \prod_{j=1}^{J} \mathcal{R}(\hat{\mathcal{F}}, \mu_j)\). Similarly, one can define \(\mathcal{R}(\hat{T}, \mu_k)\) for each \(k = 1, 2, \ldots, J\) and \(\mathcal{R}(\hat{T}, \mu) = \prod_{j=1}^{J} \mathcal{R}(\hat{T}, \mu_j)\).

Next, we review the notion of regular conditional distribution. Let \(f\) be a \(\hat{T}\)-measurable mapping from \(T\) to \(X\). A mapping \(\mu_j^f: T \times \mathcal{B}(X) \to [0, 1]\) is said to...
be a regular conditional distribution of \( f \) given \( \hat{F} \) under \( \mu_j \), if

1. for \( \mu_j \)-almost all \( t \in T \), \( \mu_j^{f|\hat{F}}(t, \cdot) \) is a probability measure on \( X \);
2. for each Borel subset \( B \subseteq X \), \( \mu_j^{f|\hat{F}}(\cdot, B) \) is a version of \( \mathbb{E}[1_B(f) \mid \hat{F}] \), where \( \mathbb{E}[1_B(f) \mid \hat{F}] \) is the conditional expectation of the indicator function \( 1_B(f) \) given \( \hat{F} \) under \( \mu_j \).

Let \( F \) be a correspondence from \( T \) to \( X \). We use

\[
\mathcal{R}_F^{(\hat{T},\hat{F},\mu)} = \left\{ \mu_j^{f|\hat{F}} = (\mu_j^{f|\hat{F}_1}, \mu_j^{f|\hat{F}_2}, \ldots, \mu_j^{f|\hat{F}_J}) \mid f \text{ is a } \hat{T}\text{-measurable selection of } F \right\}
\]

to denote the set of all regular conditional distributions induced by \( \hat{T}\)-measurable selections of \( F \) conditional on \( \hat{F} \) under the vector measure \( \mu = (\mu_1, \mu_2, \ldots, \mu_J) \).

The following result is a direct corollary of He and Sun (2021, Theorem 2), which presents several desirable properties for the conditional expectation of Banach-valued correspondences.\(^{23}\)

**Lemma 2.** If \( \hat{T} \) is nowhere equivalent to \( \hat{F} \) under \( \mu_0 \), then we have the following results.

1. For any compact-valued \( \hat{T}\)-measurable correspondence \( F \) from \( T \) to \( X \), \( \mathcal{R}_F^{(\hat{T},\hat{F},\mu)} \) is convex and weakly compact.
2. Let \( F \) be a compact-valued \( \hat{T}\)-measurable correspondence from \( T \) to \( X \), \( Z \) a metric space, and \( G \) a closed-valued correspondence from \( T \times Z \) to \( X \) such that
   - for each \( (t, z) \in T \times Z \), \( G(t, z) \subseteq F(t) \);
   - for each \( z \in Z \), \( G(\cdot, z) \) is \( F \)-measurable from \( T \) to \( X \);
   - for each \( t \in T \), \( G(t, \cdot) \) is upper-hemicontinuous from \( Z \) to \( X \).

Then \( H(z) = \mathcal{R}_G^{(\hat{T},\hat{F},\mu)} \) is upper-hemicontinuous in \( z \) from \( Z \) to \( \mathcal{R}^{(\hat{F},\mu)} = \prod_{j=1}^J \mathcal{R}^{(\hat{F},\mu_j)} \).
3. For any \( g = (g_1, g_2, \ldots, g_J) \in \mathcal{R}^{(\hat{F},\mu)} = \prod_{j=1}^J \mathcal{R}^{(\hat{F},\mu_j)} \), there exists a \( \hat{T}\)-measurable mapping \( f : T \to X \) such that \( g = \mu_f^{f|\hat{F}} \), that is, for each \( j = 1, 2, \ldots, J \), \( g_j = \mu_j^{f|\hat{F}} \).

### 6.2 Proofs of Theorem 1 and Corollary 1

The following lemma will be useful for proving Theorem 1.\(^{23}\)

\(^{23}\)The theory of correspondences has been well developed based on Loeb spaces and saturated probability spaces; see, for example, Sun (1996), Podczeck (2008) and Sun and Yannelis (2008). These results exclude many widely-adopted probability spaces, including the Euclidean spaces. Lemma 2 does not have this restriction.
Lemma 3. Let \((T, \mathcal{T}, \mu)\) be a probability space, \(\mathcal{F}\) a sub-\(\sigma\)-algebra of \(\mathcal{T}\), \(h\) a \(T\)-measurable mapping from \(T\) to a Polish space \(X\) with the Borel \(\sigma\)-algebra \(\mathcal{B}\), and \(\mu^{h|\mathcal{F}}\) a regular conditional distribution of \(h\) given \(\mathcal{F}\) under \(\mu\). Let \(\psi\) be a \(\mathcal{B} \otimes \mathcal{F}\)-measurable function from \(X \times T\) to \(\mathbb{R}\) which is integrably bounded; that is, \(|\psi(x, t)| \leq \varphi(t)\) for all \((x, t) \in X \times T\), where \(\varphi\) is integrable on \((T, \mathcal{T}, \mu)\). Then, for \(\mu\)-almost all \(t \in T\),

\[
E[\psi(h(t), t) \mid \mathcal{F}] = \int_X \psi(x, t) \mu^{h|\mathcal{F}}(t; dx).
\]

(1)

Proof. Let \(\Psi\) be the class of all nonnegative \(\mathcal{B} \otimes \mathcal{F}\)-measurable functions from \(X \times T\) to \(\mathbb{R}\) such that Equation (1) holds for \(\mu\)-almost all \(t \in T\).

For any sets \(B \in \mathcal{B}\) and \(D \in \mathcal{F}\), let \(\psi(x, t) = 1_{B \times D}(x, t) = 1_B(x) \cdot 1_D(t)\). Since \(\mu^{h|\mathcal{F}}\) is a regular conditional distribution of \(h\) given \(\mathcal{F}\) under \(\mu\), we have that for \(\mu\)-almost all \(t \in T\),

\[
E[1_B(h(t)) \mid \mathcal{F}] = \mu^{h|\mathcal{F}}(t; B) = \int_X 1_B(x) \mu^{h|\mathcal{F}}(t; dx).
\]

Since \(D\) is \(\mathcal{F}\)-measurable, for \(\mu\)-almost all \(t \in T\),

\[
E[1_B(h(t))1_D(t) \mid \mathcal{F}] = 1_D(t) \cdot E[1_B(h(t)) \mid \mathcal{F}] = \int_X 1_B(x)1_D(t) \mu^{h|\mathcal{F}}(t; dx),
\]

which implies that Equation (1) holds for \(\psi = 1_{B \times D}\). Hence, \(1_{B \times D} \in \Psi\).

By the properties of conditional expectation, it is obvious that \(\Psi\) is a \(\lambda\)-system in the sense that (1) the constant function \(1\) is in \(\Psi\); (2) for any nonnegative real numbers \(\alpha_1, \alpha_2\), any \(\psi_1, \psi_2 \in \Psi, \alpha_1 \psi_1 + \alpha_2 \psi_2 \in \Psi\); (3) for any increasing sequence of functions \(\{\psi_k\}_{k=1}^{\infty} \in \Psi\) with a limit function \(\psi\), one has \(\psi \in \Psi\). Since the class of measurable rectangles \(B \times D\) with \(B \in \mathcal{B}\) and \(D \in \mathcal{F}\) is a \(\pi\)-class (i.e., closed under the operation of finite intersections), the usual \(\pi\)-\(\lambda\) Theorem implies that Equation (1) holds for every nonnegative \(\mathcal{B} \otimes \mathcal{F}\)-integrable function \(\psi\) from \(X \times T\) to \(\mathbb{R}\); see, for example, Theorem 1.4.3 in Chow and Teicher (1997, p.16).

For a \(\mathcal{B} \otimes \mathcal{F}\)-measurable function from \(X \times T\) to \(\mathbb{R}\) satisfying the conditions of the lemma, one can consider the positive and negative parts of \(\psi\) separately. The rest is clear.

\[\square\]

Proof of Theorem 1. For each \(j = 1, 2, \ldots, J\), let \(\mu_{ij}\) be a probability measure on \((T_i, \mathcal{T}_i)\) so that it is absolutely continuous with respect to \(\lambda_i\) with the density \(\rho_i^j\). Denote \(\mu_{i0} = \lambda_i\)
and $\mu_i = (\mu_{i0}, \mu_{i1}, \ldots, \mu_{iJ})$.

For each $i \in I$, recall that $\mathcal{R}^{(F_i, \mu_{ij})}$ is the set of $\mathcal{F}_i$-measurable transition probabilities from $T_i$ to $A_i$ under the probability measure $\mu_{ij}$ for each $j = 0, 1, \ldots, J$, and $\mathcal{R}^{(F_i, \mu_i)} = \prod_{j=0}^{J} \mathcal{R}^{(F_i, \mu_{ij})}$. Clearly, each $\mathcal{R}^{(F_i, \mu_{ij})}$ is nonempty, convex and weakly compact (under the topology of weak convergence), so is $\mathcal{R}^{(F_i, \mu_i)}$. Let $\mathcal{R}^{\mathcal{F}} = \prod_{i \in I} \mathcal{R}^{(F_i, \mu_i)} = \prod_{i \in I} \prod_{j=0}^{J} \mathcal{R}^{(F_i, \mu_{ij})}$, which is endowed with the product topology.

Fix a pure-strategy profile $f = (f_1, f_2, \ldots, f_n)$. For any two distinct players $i$ and $\ell$, $j = 1, 2, \ldots, J$, types $t_{-\ell} \in T_{-\ell}$, and actions $a_{-\ell} \in A_{-\ell}$, the Fubini property implies that the function $w_i^j(a_{-\ell}, \cdot, t_{-\ell}, \cdot)$ is $\mathcal{B}(A_i) \otimes \mathcal{F}_\ell$-measurable in $(a_{\ell}, t_{\ell})$. Lemma 3 above implies that for $\mu_{ij}$-almost all $t_{\ell} \in T_{\ell}$,

$$E^{\mu_{ij}}[w_i^j(a_{-\ell}, f_\ell(t_{\ell}), t_{-\ell}, t_{\ell}) \mid \mathcal{F}_\ell] = \int_{A_\ell} w_i^j(a_{-\ell}, a_{\ell}, t_{-\ell}, t_{\ell}) \mu_{ij}^{f_\ell | \mathcal{F}_\ell}(t_{\ell}; da_{\ell}), \quad (2)$$

where the left hand side is the conditional expectation given $\mathcal{F}_\ell$ under $\mu_{ij}$.

Fix player 1. For $j = 1, 2, \ldots, J$, $t_1 \in T_1$, and $a_1 \in A_1$, we have that

$$\int_{T_{-1}} w_1^j(a_1, f_{-1}(t_{-1}), t_1, t_{-1}) \prod_{\ell \neq 1} \rho_\ell^j(t_{\ell}) \lambda_{-1}(dt_{-1})$$

$$= \int_{T_{-1,2}} \int_{T_2} w_1^j(a_1, f_2(t_2), f_{-1,2}(t_{-1,2}), t_2, t_{-2}) \mu_{2j}(dt_2) \mu_{-1,2j}(dt_{-1,2})$$

$$= \int_{T_{-1,2}} \int_{T_2} E^{\mu_{2j}}[w_1^j(a_1, f_2(t_2), f_{-1,2}(t_{-1,2}), t_2, t_{-2}) \mid \mathcal{F}_2] \mu_{2j}(dt_2) \mu_{-1,2j}(dt_{-1,2})$$

$$= \int_{T_{-1,2}} \int_{A_2} w_1^j(a_1, a_2, f_{-1,2}(t_{-1,2}), t_2, t_{-2}) \mu_{2j}^{f_2 \mid \mathcal{F}_2}(t_2; da_2) \mu_{2j}(dt_2) \mu_{-1,2j}(dt_{-1,2})$$

$$= \ldots$$

$$= \int_{T_{-1}} w_1^j(a_1, a_{-1}, t_1, t_{-1}) \prod_{\ell \neq 1} \mu_{ij}^{f_\ell | \mathcal{F}_\ell}(t_{\ell}; da_{\ell}) \mu_{-1j}(dt_{-1}),$$

where the subscript $-(1,2)$ denotes all the players except players 1 and 2, and $\mu_{-1,2j} = \otimes_{i \neq 1,2} \mu_{ij}$. The first equality is due to the definition of $\mu_{ij}$, which has density $\rho_\ell^j$ with respect to $\lambda_i$. The second equality holds by taking the conditional expectation. The third equality follows from Equation (2). The last equality follows by repeating these three steps from $T_1$ to $T_n$, which is omitted in the fourth equality. By the definition of
the density-weighted payoff, we have that

$$
\int_{T_{i-1}} u_1(a_1, f_{i-1}(t_{i-1}), t_{i-1}, t_{i-1}) q(t_{i-1}, t_{i-1}) \lambda_{-1}(dt_{i-1})
$$

$$=
\int_{T_{i-1}} w_1(a_1, f_{i-1}(t_{i-1}), t_{i-1}, t_{i-1}) \lambda_{-1}(dt_{i-1})
$$

$$=
\int_{T_{i-1}} \sum_{j=1}^{J} \left[ w^j_1(a_1, f_{i-1}(t_{i-1}), t_{i-1}, t_{i-1}) \prod_{t \neq t_j} \rho^j_t(t_t) \right] \lambda_{-1}(dt_{i-1})
$$

$$=
\sum_{j=1}^{J} \rho^j_t(t_t) \int_{T_{i-1}} w^j_1(a_1, f_{i-1}(t_{i-1}), t_{i-1}, t_{i-1}) \prod_{t \neq t_j} \rho^j_t(t_t) \lambda_{-1}(dt_{i-1})
$$

$$=
\sum_{j=1}^{J} \rho^j_t(t_t) \int_{T_{i-1}} \int_{A_{i-j}} w^j_1(a_1, a_{i-j}, t_{i-j}, t_{i-j}) \prod_{t \neq t_j} \mu^j_{t_j}(t_i; da_t) \mu_{-i-j}(dt_{i-j}).
$$

One can repeat the argument for each player \(i \in I\) such that for each \(t_i \in T_i\) and \(a_i \in A_i\),

$$
\int_{T_{i-1}} u_i(a_i, f_{-i}(t_{i-1}), t_{i-1}, t_{i-1}) q(t_{i-1}, t_{i-1}) \lambda_{-1}(dt_{i-1})
$$

$$=
\sum_{j=1}^{J} \rho^j_t(t_t) \int_{T_{i-1}} \int_{A_{i-j}} w^j_i(a_i, a_{i-j}, t_{i-j}, t_{i-j}) \prod_{t \neq t_j} \mu^j_{t_j}(t_i; da_t) \mu_{-i-j}(dt_{i-j}). \quad (3)
$$

For each \(i \in I\), let \(F_i\) be a mapping from \(T_i \times A_i \times \mathcal{R}^F\) to \(\mathbb{R}\), which is defined as follows:

$$
F_i(t_i, a_i, g_1, g_2, \ldots, g_n) = \sum_{j=1}^{J} \rho^j_t(t_t) \int_{T_{i-j}} \int_{A_{i-j}} w^j_i(a_i, a_{i-j}, t_{i-j}, t_{i-j}) \prod_{t \neq t_j} \mu^j_{t_j}(t_i; da_t) \mu_{-i-j}(dt_{i-j}),
$$

where each \(g_t = (g_0, g_{1t}, \ldots, g_{lt}) \in \mathcal{R}(F_{t-}, \mu_t) = \prod_{j=0}^{J} \mathcal{R}(F_{t-j}, \mu_{t-j})\). It is clear that \(F_i\) is \(\mathcal{F}_{t}\)-measurable on \(T_i\) and continuous on \(A_i \times \mathcal{R}^F\).

For each \(i \in I\), we consider the best response correspondence \(G_i\) from \(T_i \times \mathcal{R}^F\) to \(A_i\), which is given by

$$
G_i(t_i, g_1, g_2, \ldots, g_n) = \arg \max_{a \in A_i} F_i(t_i, a_i, g_1, g_2, \ldots, g_n).
$$

For each \(t_i \in T_i\), since \(A_i\) is compact and \(F_i\) is continuous on \(A_i \times \mathcal{R}^F\), Berge’s maximal theorem (see Theorem 17.31 in Aliprantis and Border (2006) for example) implies that \(G_i(t_i, \cdot)\) is nonempty, compact-valued, and upper-hemicontinuous on
$R^F$. We have already known that for any $a_i \in A_i$ and $(g_1, g_2, \ldots, g_n) \in R^F$, $F_i(\cdot, a_i, g_1, g_2, \ldots, g_n)$ is $F_i$-measurable. Then measurable maximal theorem (see Theorem 18.19 in Aliprantis and Border (2006) for example) implies that the correspondence $G_i(\cdot, g_1, g_2, \ldots, g_n)$ admits an $F_i$-measurable selection. Thus, $R^{(T_i, F_i; \mu_i)}_{G_i(\cdot, g_1, g_2, \ldots, g_n)}$, the set of regular conditional distributions induced by $T_i$-measurable selections of correspondence $G_i(\cdot, g_1, g_2, \ldots, g_n)$ on $F_i$ under the vector measure $\mu_i = (\mu_{i0}, \mu_{i1}, \ldots, \mu_{ij})$, is nonempty. Since $T_i$ is nowhere equivalent to $F_i$ under $\lambda_i = \mu_{i0}$ and $\mu_{ij}$ is absolutely continuous with respect to $\lambda_i$ for each $j = 1, 2, \ldots, J$, we have that $T_i$ is nowhere equivalent to $F_i$ under $\mu_{ij}$ for each $j = 0, 1, \ldots, J$; see Lemma 3 in He and Sun (2019). Lemma 2 then implies that $R^{(T_i, F_i; \mu_i)}_{G_i(\cdot, g_1, g_2, \ldots, g_n)}$ is convex, weakly compact-valued, and weakly upper-hemicontinuous on $R^F = \prod_{i \in I} R^F_i$.

Consider the correspondence $\Phi$ from $R^F$ to itself as follows:

$$\Phi(g_1, g_2, \ldots, g_n) = \prod_{i \in I} R^{(T_i, F_i; \mu_i)}_{G_i(\cdot, g_1, g_2, \ldots, g_n)},$$

It is clear that $\Phi$ is nonempty, convex, weakly compact-valued, and upper-hemicontinuous on $R^F$. By Fan-Glicksberg’s fixed-point theorem, there exists a fixed point $(g_1^*, g_2^*, \ldots, g_n^*)$ of $\Phi$. That is, for each $i \in I$, $g_i^* = (g_{i0}, g_{i1}, \ldots, g_{ij}) \in R^{(T_i, F_i; \mu_i)}_{G_i(\cdot, g_1, g_2, \ldots, g_n)}$. Thus, for each $i \in I$, there exists a $T_i$-measurable selection $f_i^*$ of $G_i(\cdot, g_1^*, g_2^*, \ldots, g_n^*)$ such that $g_{ij}^* = \mu_{ij}^{f_i^*F_i}$ for each $j = 0, 1, \ldots, J$.

Under the pure-strategy profile $(f_1^*, f_2^*, \ldots, f_n^*)$, the payoff of player $i$ is

$$U_i(f^*) = \int_T u_i(f_i^*(t), f_{-i}^*(t-i), t_i, t_{-i}) \lambda(dt)$$

$$= \int_{T_i} \int_{T_{-i}} u_i(f_i^*(t_i), f_{-i}^*(t-i), t_i, t_{-i}) g(t_i, t_{-i}) \lambda_{-i}(dt_{-i}) \lambda_i(dt_i)$$

$$= \int_{T_i} \sum_{j=1}^J \rho_i^j(t_i) \int_{T_{-i}} \int_{A_{-i}} w_i^j(f_i^*(t_i), a_{-i}, t_i, t_{-i}) \prod_{ \ell \neq i} g_{ij}^\ell(t_\ell; da_\ell) \mu_{-ij}(dt_{-i}) \lambda_i(dt_i)$$

$$= \int_{T_i} F_i(t_i, f_i^*(t_i), g_1^*, \ldots, g_n^*) \lambda_i(dt_i).$$

The first equality is true due to the definition of $U_i$, and the second equality holds based on the Fubini property. The third equality relies on Equation (3), and the fourth equality holds due to the definition of $F_i$. By the choices of $(f_1^*, f_2^*, \ldots, f_n^*)$, we have that for each $i \in I$, $f_i^*$ maximizes $U_i(f_i, f_{-i})$, and hence $(f_1^*, f_2^*, \ldots, f_n^*)$ is a pure-strategy equilibrium. \qed
Below, we prove Corollary 1.

**Proof of Corollary 1.** We only need to verify that DCPI holds. The equilibrium existence result follows from Theorem 1.

Recall that the marginal \( \hat{\lambda}_i \) of \( \hat{\lambda} \) on \( (T_i, \mathcal{T}_i) \) is \( \sum_{1 \leq j \leq J} \tau^j \lambda_i^j \). Since \( \tau^j > 0 \), \( \lambda_i^j \) is absolutely continuous with respect to \( \hat{\lambda}_i \) for each \( i \in I \) and \( 1 \leq j \leq J \). We assume that the Radon-Nikodym derivative is \( q_i^j \). It is clear that \( \hat{\lambda} \) is absolutely continuous with respect to \( \otimes_{i \in I} \hat{\lambda}_i \) with the Radon-Nikodym derivative \( q = \sum_{1 \leq j \leq J} \tau^j \prod_{i \in I} q_i^j \). For any \( t_{0j} \in T_0 \), and \( D_l \in \mathcal{T}_l \) for \( 1 \leq l \leq n \),

\[
\hat{\lambda}(\{t_{0j}\} \times D_1 \times \cdots \times D_n) = \tau^j \int_{T_1} \cdots \int_{T_n} \prod_{1 \leq l \leq n} 1_{D_l}(t_l) \lambda_i^l (dt_n) \cdots \lambda_1^l (dt_1) \\
= \int_{T_1} \cdots \int_{T_n} \tau^j \prod_{1 \leq l \leq n} [1_{D_l}(t_l) q_i^l (t_l)] \hat{\lambda}_n (dt_n) \cdots \hat{\lambda}_1 (dt_1) \\
= \int_{T_1} \cdots \int_{T_n} \prod_{1 \leq l \leq n} 1_{D_l}(t_l) \cdot \frac{\tau^j \prod_{1 \leq l \leq n} q_i^l (t_l)}{\sum_{1 \leq j \leq J} \tau^j \prod_{i \in I} q_i^j (t_j)} \hat{\lambda}(\{t_{0j}, \ldots, t_n\}).
\]

Define a transition probability \( \nu \) from \( \prod_{1 \leq l \leq n} T_l \) to \( \mathcal{M}(T_0) \) such that for \( 1 \leq j \leq J \),

\[
\nu(\{t_{0j}\}|t) = \frac{\tau^j \prod_{1 \leq l \leq n} q_i^l (t_l)}{\sum_{1 \leq j \leq J} \tau^j \prod_{i \in I} q_i^j (t_j)},
\]

Define a new payoff function \( v_i \) on \( A \times \prod_{1 \leq l \leq n} T_l \) as

\[
v_i(a, t) = \sum_{1 \leq j \leq J} u_i(a, t_{0j}, t) \nu(\{t_{0j}\}|t).
\]

Consider the \( n \)-player game in which player \( i \) has private information space \( (T_i, \mathcal{T}_i) \), action space \( A_i \), and payoff function \( v_i \). The common prior is \( \hat{\lambda} \). Then the density weighted payoff in this game is

\[
v_i(a, t) q(t) = \left( \sum_{1 \leq j \leq J} u_i(a, t_{0j}, t) \nu(\{t_{0j}\}|t) \right) \cdot \left( \sum_{1 \leq j \leq J} \tau^j \prod_{i \in I} q_i^j (t_i) \right) \\
= \left( \sum_{1 \leq j \leq J} u_i(a, t_{0j}, t) \cdot \frac{\tau^j \prod_{1 \leq l \leq n} q_i^l (t_l)}{\sum_{1 \leq j \leq J} \tau^j \prod_{i \in I} q_i^j (t_j)} \right) \cdot \left( \sum_{1 \leq j \leq J} \tau^j \prod_{i \in I} q_i^j (t_i) \right)
\]
Let \( w_j^i(a,t) = \tau^j \cdot u_i(a,t_0j,t), \) and \( p_l^i = q_l^i \) for \( 1 \leq j \leq J \) and \( 1 \leq l \leq n. \) It is clear that the DCPI condition is satisfied.

6.3 Proof of Proposition 1

To prove Proposition 1, we consider the following auxiliary games.

Let \( d \) be a metric on the compact metric space \( X. \) For a fixed integer \( m \geq 2, \) pick \( m \) distinct elements in \( X, \) which are denoted as \( a_1, a_2, \ldots, a_m. \) Choose a positive real number \( r < 1 \) such that the closed balls \( \overline{B}(a_k,r) = \{ a \in X \mid d(a_k,a) \leq r \} \) are disjoint \( (k = 1, 2, \ldots, m). \) By Urysohn’s Lemma (see Lemma 2.46 in Aliprantis and Border (2006)) and the property that every closed set is a \( G_\delta \) set, there exist continuous functions \( \{\beta_1, \beta_2, \ldots, \beta_m, \gamma \} \) from \( X \) to \([0,1]\) satisfying the following properties:

- for each \( k = 1, 2, \ldots, m, \) \( \beta_k(a) = 1 \) for \( a \in \overline{B}(a_k, \frac{r}{2}), \) \( \beta_k(a) = 0 \) for \( a \in B(a_k, r)^c, \) and \( \beta_k(a) \in (0,1) \) for \( a \in B(a_k, r) \cap \overline{B}(a_k, \frac{r}{2})^c; \)
- \( \gamma(a) = 0 \) for \( a \in \bigcup_{k=1}^m \overline{B}(a_k, \frac{r}{2}) \) and \( \gamma(a) = -5 \) for \( (\bigcup_{k=1}^m B(a_k, r))^c. \)

Step 1 We construct a 2-player Bayesian games \( G_2 \) as follows. The common action space for the two players is \( X. \) For each \( i = 1, 2, \) player \( i \) has a private type space \( L_i = [0,1] \) with Borel \( \sigma \)-algebra. The information structure \( \tau \) on \( L_1 \times L_2 \) is the uniform distribution on the triangle \( \{(l_1, l_2) \mid 0 \leq l_1 \leq l_2 \leq 1\}. \) Given the type profile \( (l_1, l_2) \in L_1 \times L_2, \) when player 1 chooses action \( s_1 \) and player 2 chooses action \( s_2, \) their type-irrelevant payoffs are given by

\[
\begin{align*}
u_1(s_1, s_2, l_1, l_2) &= \sum_{k=1}^m \beta_k(s_1) \cdot \beta_k(s_2) \cdot (3 - d(s_1, a_k)) + \sum_{k=1}^m \beta_k(s_1) \cdot \beta_{k+1}(s_2) \cdot (1 - d(s_1, a_k)) \\
&\quad + \sum_{k=1}^m \sum_{n \neq k, k+1} \beta_k(s_1) \cdot \beta_n(s_2) \cdot (2 - d(s_1, a_k)) + \gamma(s_1) - 2, \\
u_2(s_1, s_2, l_1, l_2) &= \sum_{k=1}^m \beta_k(s_1) \cdot \beta_{k+1}(s_2) \cdot (3 - d(s_2, a_{k+1})) + \sum_{k=1}^m \beta_k(s_1) \cdot \beta_k(s_2) \cdot (1 - d(s_2, a_k))
\end{align*}
\]

\( ^{24} \)We use \( B(a, r) \) to denote the open ball with the center \( a \) and the radius \( r, \) \( \overline{B}(a, r) \) to denote the closed ball with the center \( a \) and the radius \( r, \) and \( B^c \) to denote the complement of a set \( B. \)
\[ + \sum_{k=1}^{m} \sum_{\ell \neq k, k+1} \beta_k(s_1) \cdot \beta_\ell(s_2) \cdot \left( 2 - d(s_2, a_k) \right) + \gamma(s_2) - 2, \]

where we adopt the convention \( \beta_{m+1} = \beta_1 \) and \( a_{m+1} = a_1 \).

In the following, we will show that only the actions in \( \{a_1, a_2, \ldots, a_m\} \) can be chosen with positive probabilities in an equilibrium for both players.

For any \((l_1, l_2) \in L_1 \times L_2\) and any \(s_2 \in X\), we consider the player 1’ payoff when she chooses various actions.

When player 1 chooses an action \( x \in \left( \cup_{k=1}^{m} B(a_k, r) \right)^c \), then each \( \beta_k(x) = 0 \) and \( \gamma(x) = -5 \), and hence her payoff is \( u_1(x, s_2, l_1, l_2) = \gamma(x) - 2 = -7 \).

When player 1 chooses the action \( a_1 \), her payoff is

\[
\begin{cases}
-2, & \text{if } s_2 \notin \left( \cup_{k=1}^{m} B(a_k, r) \right)^c; \\
3 \cdot \beta_1(s_2) - 2, & \text{if } s_2 \in B(a_1, r); \\
\beta_2(s_2) - 2, & \text{if } s_2 \in B(a_2, r); \\
2 \cdot \beta_\ell(s_2) - 2, & \text{if } s_2 \in B(a_\ell, r) \text{ for some } \ell \neq 1, 2. \\
\end{cases}
\]

Since for each \( k = 1, 2, \ldots, m \), \( 0 \leq \beta_k(a) \leq 1 \) for any \( a \in X\), the action \( x \in \left( \cup_{k=1}^{m} B(a_k, r) \right)^c \) is strictly dominated by \( a_1 \) for player 1. Therefore, every action in \( \left( \cup_{k=1}^{m} B(a_k, r) \right)^c \) will not be chosen with positive probability in an equilibrium for player 1. That is, only actions in \( X' = \cup_{k=1}^{m} B(a_k, r) \) can be chosen with positive probabilities in an equilibrium for player 1.

Similarly, only the actions in \( X' = \cup_{k=1}^{m} B(a_k, r) \) can be chosen with positive probabilities in an equilibrium for player 2.

For any \((l_1, l_2) \in L_1 \times L_2\) and any \(s_2 \in X'\), we also consider the player 1’ payoff when she chooses various actions in \( X'\).

When player 1 chooses an action \( x' \in B(a_1, r) \setminus \{a_1\} \), her payoff is

\[
\begin{cases}
\beta_1(x') \cdot \beta_1(s_2) \cdot (3 - d(x', a_1)) + \gamma(x') - 2, & \text{if } s_2 \in B(a_1, r); \\
\beta_1(x') \cdot \beta_2(s_2) \cdot (1 - d(x', a_1)) + \gamma(x') - 2, & \text{if } s_2 \in B(a_2, r); \\
\beta_1(x') \cdot \beta_\ell(s_2) \cdot (2 - d(x', a_1)) + \gamma(x') - 2, & \text{if } s_2 \in B(a_\ell, r) \text{ for some } \ell \neq 1, 2.
\end{cases}
\]
On the other hand, when player 1 chooses the action $a_1$, her payoff is

$$
\begin{align*}
&\begin{cases}
3 \cdot \beta_1(s_2) - 2, & \text{if } s_2 \in B(a_1, r); \\
\beta_2(s_2) - 2, & \text{if } s_2 \in B(a_2, r); \\
2 \cdot \beta_\ell(s_2) - 2, & \text{if } s_2 \in B(a_\ell, r) \text{ for some } \ell \neq 1, 2.
\end{cases}
\end{align*}
$$

Since for each $k = 1, 2, \ldots, m$, $0 \leq \beta_k(a) \leq 1$ for any $a \in X$, the action $x' \in B(a_1, r) \setminus \{a_1\}$ is strictly dominated by $a_1$ for player 1. For $k = 2, 3, \ldots, m$, similar arguments show that every action in $B(a_k, r) \setminus \{a_k\}$ is strictly dominated by $a_k$ for player 1. Therefore, every action in $X' \setminus \{a_1, a_2, \ldots, a_m\}$ will not be chosen with positive probability in an equilibrium for player 1. As a result, only the actions in $X'' = \{a_1, a_2, \ldots, a_m\}$ can be chosen with positive probabilities in an equilibrium for player 1.

Similarly, only the actions in $X'' = \{a_1, a_2, \ldots, a_m\}$ can be chosen with positive probabilities in an equilibrium for player 2.

We can focus on the case that the two players only choose actions from the common action space $X'' = \{a_1, a_2, \ldots, a_m\}$. The payoff matrix restricted on the action set $X''$ is as follows.

| Player 1 | $a_1$ | $a_2$ | $a_3$ | $\cdots$ | $a_m$ |
|----------|-------|-------|-------|-----------|-------|
| $a_1$    | 1, -1 | -1, 1 | 0, 0  | $\cdots$  | 0, 0  |
| $a_2$    | 0, 0  | 1, -1 | -1, 1 | $\cdots$  | 0, 0  |
| $a_3$    | 0, 0  | 0, 0  | 1, -1 | $\cdots$  | 0, 0  |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $a_m$    | -1, 1 | 0, 0  | $\cdots$ | 0, 0       | 1, -1 |

In each cell, the first number is the payoff for player 1 and the second number is the payoff for player 2.

**Step 2** Next, we construct a new auxiliary 2-player game $\Gamma_2$. Recall that $\tau$ is the uniform distribution on the triangle $\{(l_1, l_2) \mid 0 \leq l_1 \leq l_2 \leq 1\}$. We use $\tau_i$ to denote the marginal of $\tau$ on $L_i$. Let

$$q(t_1, t_2) = \begin{cases}
\frac{1}{2(1 - \phi_1(t_1))\phi_2(t_2)}, & \text{if } 0 < \phi_1(t_1) \leq \phi_2(t_2) < 1, \\
0, & \text{otherwise},
\end{cases}$$

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where $\phi_i$ is a measure-preserving mapping from $(T_i, \mathcal{F}_i, \lambda_i)$ to $([0, 1], \mathcal{B}, \tau_i)$ such that for any $E \in \mathcal{F}_i$ there exists a set $E' \in \mathcal{B}$ with $\lambda_i(E \Delta \phi_i^{-1}(E')) = 0$.

The components of game $\Gamma_2$ is as follows: (1) Players 1 and 2’s action spaces and payoffs are the same as in the game $G_2$; (2) The private type space for each player $i$ is $(T_i, \mathcal{T}_i, \lambda_i)$; (3) The common prior $\lambda$ has the Radon-Nikodym derivative $q$ with respect to $\lambda_1 \otimes \lambda_2$.

It can be easily checked that each $\lambda_i$ is the marginal of $\lambda$ on $T_i$. Based on the analysis in Step 1, both players only choose actions from the set $X''$ in an equilibrium. Then $\Gamma_2$ is reduced to the game constructed in He and Sun (2019, p.31). Following the arguments therein, we have that $\mathcal{T}_i$ is nowhere equivalent to $\mathcal{F}_i$ under $\lambda_i$ for each $i = 1, 2$. Thus, players 1 and 2 have decomposable coarser payoff-relevant information.

The proof is then completed by adding dummy players. In particular, one can consider an $n$-player game in which only players 1 and $i$ are active for some $2 \leq i \leq n$, while all other players are inactive. The payoffs, action sets and private type spaces of players 1 and $i$ are the same as those of players 1 and 2 in $\Gamma_2$. Then the above argument shows that players 1 and $i$ have decomposable coarser payoff-relevant information. This further implies that all the players have decomposable coarser payoff-relevant information.

### 6.4 Proof of Theorem 2

In the following two subsections, we provide the proofs for the sufficiency part and necessity part of Theorem 2, respectively.

#### 6.4.1 Proof for the sufficiency part of Theorem 2

To prove the sufficiency part of Theorem 2, we first present a new purification result under a vector measure.

**Lemma 4.** Let $\mu_0, \mu_1, \ldots, \mu_J$ be probability measures on some measurable space $(T, \mathcal{T})$ such that $\mu_j$ is absolutely continuous with respect to $\mu_0$ with bounded density for $j = 1, 2, \ldots, J$. Let $A$ be a compact metric space, and $\{v_k\}_{k=1}^K$ be $\mathcal{B}(A) \otimes \mathcal{F}$-measurable mappings from $A \times T$ to $\mathbb{R}$ such that $v_k$ is integrably bounded under $\mu_0$ for each $a \in A$ and $k = 1, 2, \ldots, K$. If $\mathcal{T}$ is nowhere equivalent to $\mathcal{F}$ under $\mu_0$, then for any $\mathcal{T}$-measurable transition probability $g$ from $T$ to $\mathcal{M}(A)$, there exists a $\mathcal{T}$-measurable mapping $f$ from
T to A such that for \( j = 0, 1, \ldots, J \),

1. for \( k = 1, 2, \ldots, K \),

\[
\int_T \int_A v_k(a, t) g(t; da) \mu_j(dt) = \int_T v_k(f(t), t) \mu_j(dt);
\]

2. for any \( D \in \mathcal{F} \) and any \( B \in \mathcal{B}(A) \),

\[
\int_D g(t; B) \mu_j(dt) = \int_D 1_B(f(t)) \mu_j(dt);
\]

3. for any \( \mathcal{F} \)-measurable mapping \( g_1 \) from \( T \) to some Polish space \( Y \) and \( k = 1, 2, \ldots, K \),

\[
\mu_j \circ (g, g_1)^{-1} = \mu_j \circ (f, g_1)^{-1},
\]

where \( \mu_j \circ (g, g_1)^{-1} \) and \( \mu_j \circ (f, g_1)^{-1} \) denote the joint distributions on \( A \times Y \):

\[
\mu_j \circ (g, g_1)^{-1}(B \times B_1) = \int_{\{t \in T|g_1(t) \in B_1\}} g(t; B) \mu_j(dt)
\]

and

\[
\mu_j \circ (f, g_1)^{-1}(B \times B_1) = \int_{\{t \in T|g_1(t) \in B_1\}} 1_B(f(t)) \mu_j(dt)
\]

for any \( B \in \mathcal{B}(A) \) and \( B_1 \in \mathcal{B}(Y) \).

4. \( f(t) \in \text{supp } g(t) \) for \( \mu_0 \)-almost all \( t \in T \).

Proof. By Lemma 2, there exists a mapping \( f \) from \( T \) to \( A \) such that

\[
\int_D g(t; B) \mu_j(dt) = \int_D 1_B(f(t)) \mu_j(dt)
\]

for any \( D \in \mathcal{F}, B \in \mathcal{B}(A) \), and \( j = 0, 1, \ldots, J \). This proves part 2.

This equality further implies that

\[
\int_T \int_A 1_D(t) 1_B(a) g(t; da) \mu_j(dt) = \int_T 1_D(t) 1_B(f(t)) \mu_j(dt).
\]

Fix a \( \mathcal{B}(A) \otimes \mathcal{F} \)-measurable mapping \( v_k \). Without loss of generality, we assume that it is nonnegative. Then \( v_k \) is an increasing limit of a sequence of simple functions. By the
monotone convergence theorem, we have that
\[
\int_T \int_A v_k(a, t) g(t; da) \mu_j(dt) = \int_T v_k(f(t), t) \mu_j(dt).
\]
This proves part 1.

In addition, part 4 is shown via replacing \( v_k \) by the mapping \( c(a, t) = \mathbf{1}_{\text{supp} g(t)}(a) \).

Part 3 follows by noting that the set \( \{ t \in T \mid g_1(t) \in B_1 \} \in \mathcal{F} \) for any \( \mathcal{F} \)-measurable mapping \( g_1 \) from \( T \) to some Polish space \( Y \) and \( B_1 \in \mathcal{B}(Y) \).

Remark 2. The above lemma covers several purification results in the literature. Sun (1996) and Loeb and Sun (2006)\textsuperscript{25} worked with atomless Loeb spaces as the probability spaces, which was extended to saturated probability spaces in Podczeck (2009) and Wang and Zhang (2012). He and Sun (2014) presented a purification result based on a probability measure, which generalizes these earlier results. Lemma 4 above obtains a further generalization in the setting with vector measures.

Proof for the sufficiency part of Theorem 2. Recall that the measure \( \mu_{ij} \) on \((T_i, \mathcal{T}_i)\) is absolutely continuous with respect to \( \lambda_i \) with the density \( \rho_{ij}^k \) for \( j = 1, 2, \ldots, J, \mu_{i0} = \lambda_i \), and \( \mu_i = (\mu_{i0}, \mu_{i1}, \ldots, \mu_{iJ}) \). Fix a behavioral-strategy profile \( g \). For each \( i \in I, k = 1, 2, \ldots, J, a_i \in A_i \), and \( t_i \in T_i \), let
\[
V_{ik}^g(a_i, t_i) = \int_{T_{-i}} \int_{A_{-i}} w_{ik}^k(a_i, a_{-i}, t_i, t_{-i}) \prod_{\ell \neq i} g_{\ell}(t_{\ell}; da_{\ell}) \mu_{-ik}(dt_{-i}).
\]

By Lemma 4, for each \( i \in I \), there exists a \( \mathcal{T}_i \)-measurable mapping \( f_i: T_i \to A_i \) such that for each \( j = 0, 1, \ldots, J \),
\begin{enumerate}
\item for each \( k = 1, 2, \ldots, J \),
\[
\int_{T_i} \int_{A_i} V_{ik}^g(a_i, t_i) g_i(t_i; da_i) \mu_{ij}(dt_i) = \int_{T_i} V_{ik}^g(f_i(t_i), t_i) \mu_{ij}(dt_i); \]
\end{enumerate}

\textsuperscript{25}The proof of Loeb and Sun (2006, Theorem 2.2) needs to be slightly modified as follows. Let \( \mathcal{D} = \{ \phi_n \}_{n \geq 1} \). In the third line of p. 751, let \( \nu \) be the vector measure on \((T, \mathcal{T})\) with values in \( \mathbb{R}^{mk_m} \) for which the \((i-1)k_m + k\)-th component is \( \nu_{k}^{\phi_{M_{\phi}}} \). Note that for each \( \phi \in \mathcal{D} \), there exists a positive integer \( M_{\phi} \) such that \( \phi = \phi_{M_{\phi}} \). The rest of the argument would go through by working with \( m \geq M_{\phi} \) on p. 751.
(2) for each \( D \in \mathcal{F}_i \) and \( B \in \mathcal{B}(A_i) \),

\[
\int_D g_i(t_i; B)\mu_{ij}(dt_i) = \int_D 1_B(f_i(t_i))\mu_{ij}(dt_i);
\]

(3) \( f_i(t_i) \in \text{supp} g_i(t_i) \) for \( \lambda_i \)-almost all \( t_i \in T_i \).

Then (2) implies that \( f \) and \( g \) are conditional distribution equivalent, and (3) means that \( f \) and \( g \) are belief consistent.

Given \( i \in I, j = 1, 2, \ldots, J, t_i \in T_i \) and \( a_i \in A_i \), we have that

\[
V^g_{ij}(a_i, t_i) = \int_{T_{-i}} \int_{A_{-i}} w_i^j(a_i, a_{-i}, t_i, t_{-i}) \prod_{\ell \neq i} g_{\ell}(t_\ell; da_\ell)\mu_{-ij}(dt_{-i})
\]

\[
= \int_{T_{-i}} \int_{A_{-i}} w_i^j(a_i, a_{-i}, t_i, t_{-i}) \prod_{\ell \neq i} \mu_{\ell ij}^{f_{\ell}X_i}(t_\ell; da_\ell)\mu_{-ij}(dt_{-i})
\]

\[
= \int_{T_{-i}} w_i^j(a_i, f_{-i}(t_{-i}); t_i, t_{-i})\mu_{-ij}(dt_{-i})
\]

\[
= V^f_{ij}(a_i, t_i).
\]

(4)

The second equality holds because of Property (2) above, and the third equality is due to Equation (3). Thus, for any behavioral strategy \( h_i \) and \( t_i \in T_i \),

\[
\int_{A_i} V^f_{ij}(a_i, t_i)h_i(t_i; da_i) = \int_{A_i} V^g_{ij}(a_i, t_i)h_i(t_i; da_i).
\]

(5)

We have that

\[
U_i(g) = \sum_{j=1}^J \int_{T_i} \int_{A_i} \rho_i^j(t_i)V^g_{ij}(a_i, t_i)g_i(t_i; da_i)\lambda_i(dt_i)
\]

\[
= \sum_{j=1}^J \int_{T_i} \int_{A_i} V^g_{ij}(a_i, t_i)g_i(t_i; da_i)\mu_{ij}(dt_i)
\]

\[
= \sum_{j=1}^J \int_{T_i} V^g_{ij}(f_i(t_i), t_i)\mu_{ij}(dt_i) \quad \text{due to Property (1)}
\]

\[
= \sum_{j=1}^J \int_{T_i} V^f_{ij}(f_i(t_i), t_i)\mu_{ij}(dt_i) = U_i(f), \quad \text{due to Equation (4)}
\]

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and

\[ U_i(h_i, g_{-i}) = \sum_{j=1}^{J} \int_{T_i} \int_{A_i} \mu^j_i(t_i) V_{ij}^g(a_i, t_i) h_i(t_i; da_i) \lambda_i(dt_i) \]

\[ = \sum_{j=1}^{J} \int_{T_i} \int_{A_i} V_{ij}^g(a_i, t_i) h_i(t_i; da_i) \mu_{ij}(dt_i) \]

\[ = \sum_{j=1}^{J} \int_{T_i} \int_{A_i} V_{ij}^f(a_i, t_i) h_i(t_i; da_i) \mu_{ij}(dt_i) = U_i(h_i, f_{-i}). \text{ due to Equation (5)} \]

Thus, \( f \) and \( g \) are payoff equivalent. Therefore, \( f \) is a conditional purification of \( g \) \( \square \)

### 6.4.2 Proof for the necessity part of Theorem 2

Below, we present an equivalence result for the notion of nowhere equivalence (see Lemma 2 in He and Sun (2021)), which is useful for deriving the necessity part of Theorem 2.

**Lemma 5.** The following two conditions are equivalent.

- The \( \sigma \)-algebra \( \mathcal{T} \) is nowhere equivalent to \( \mathcal{F} \) under a probability measure \( \mu \).
- The sub-\( \sigma \)-algebra \( \mathcal{F} \) admits an asymptotic independent supplement in \( \mathcal{T} \) under \( \mu \); that is, for some strictly increasing sequence \( \{k_m\}_{m=1}^{\infty} \) and each \( m \geq 1 \), there exists a \( \mathcal{T} \)-measurable partition \( \{E_1, E_2, \ldots, E_{k_m}\} \) of \( T \) such that for \( j = 1, 2, \ldots, k_m \), (a) \( \mu(E_j) = \frac{1}{k_m} \), and (b) \( E_j \) is independent of \( \mathcal{F} \) under \( \mu \).

**Proof for the necessity part of Theorem 2.** Fix a positive integer \( m \geq 2 \). We consider the following game. The set of players is \( I = \{1, 2, \ldots, n\} \). Each player \( i \) has the private type space \( (T_i, \mathcal{F}_i, \lambda_i) \). The common prior is \( \lambda = \otimes_{i \in I} \lambda_i \). The common action space is \( A = [0, m] \).

The payoffs are defined as follows. Let \( ([0, 1], \mathcal{B}, \eta) \) be the Lebesgue unit interval, where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \([0, 1]\) and \( \eta \) is the Lebesgue measure. As is well known, for each \( i \in I \), there is a measure-preserving mapping \( \phi_i \) from \((T_i, \mathcal{F}_i, \lambda_i)\) to \(([0, 1], \mathcal{B}, \eta)\) such that for any \( E \in \mathcal{F}_i \), there exists a set \( E' \in \mathcal{B} \) with \( \lambda_i(E \Delta \phi_i^{-1}(E')) = 0 \). For the action profile \((a_1, a_2, \ldots, a_n) \in \prod_{i \in I} A \) and type profile \((t_1, t_2, \ldots, t_n) \in \prod_{i \in I} T_i \), the
payoff of player $i$ is given by

$$u_i(a_1, a_2, \ldots, a_n, t_1, t_2, \ldots, t_n) = -m - \prod_{j=0}^{m-1} (a_i - \phi_i(t_i) - j)^2.$$ 

Notice that the payoff of player $i$ does not depend on the actions and the types of her opponents.

Define a behavioral strategy $g_1: T_1 \rightarrow \mathcal{M}(A)$ for player 1 as

$$g_1(t_1) = \frac{1}{m} \sum_{j=0}^{m-1} \delta_{\phi_1(t_1)+j},$$

where $\delta_{\phi_1(t_1)+j}$ is the Dirac measure on $A$ at the points $\phi_1(t_1) + j$. Let $\hat{g}_1: [0, 1] \rightarrow \mathcal{M}(A)$ be a function on the unit interval as

$$\hat{g}_1(l_1) = \frac{1}{m} \sum_{j=0}^{m-1} \delta_{l_1+j}.$$ 

Then $g_1 = \hat{g}_1 \circ \phi_1$. Let $\tau_1$ be a probability measure on $A$ such that for any Borel subset $B$ in $A$,

$$\tau_1(B) = \int_{T_1} g_1(t_1; B) \lambda_1(dt_1),$$

which implies that

$$\tau_1(B) = \int_{T_1} \hat{g}_1(\phi_1(t_1); B) \lambda_1(dt_1) = \int_{[0,1]} \hat{g}_1(l_1; B) \eta(dl_1).$$

Given the definition of $\hat{g}_1$, it is clear that $\tau$ is the uniform distribution on $A$.

Due to the assumption of necessity part of Theorem 2, the behavioral strategy $g_1$ has a conditional purification $f_1$. By the definition, $f_1$ and $g_1$ are distribution equivalent. As a result, $\lambda_1 f_1^{-1}(B) = \int_{T_1} g_1(t_1; B) \lambda_1(dt_1) = \tau(B)$; that is, $\lambda_1 f_1^{-1} = \tau$. Based on the belief consistency condition, $f_1(t_1) \in \{\phi_1(t_1), \phi_1(t_1) + 1, \ldots, \phi_1(t_1) + m - 1\}$ for $\lambda_1$-almost all $t_1 \in T_1$. Thus, $f_1(t_1) = \phi_1(t_1) + j$ on a $T_1$-measurable set $C_j \subseteq T_1$ for $j = 0, 1, \ldots, m - 1$.

Recall that for any $E \in \mathcal{F}_1$, there exists a set $E' \in \mathcal{B}$ with $\lambda_1(\phi_1^{-1}(E')) = 0$. Then for each $j = 0, 1, \ldots, m - 1$, we have that

$$\lambda_1(C_j \cap E) = \lambda_1(C_j \cap \phi_1^{-1}(E')) = \lambda_1(f_1 \in E' + j)$$

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\[
\tau_1(E' + j) = \frac{1}{m} \eta(E') = \frac{1}{m} \lambda_1(\phi_1^{-1}(E')) = \frac{1}{m} \lambda_1(E),
\]

where \(E' + j\) denotes the set \(\{l + j \mid l \in E'\}\). Thus, we have that \(\lambda_1(C_j) = \frac{1}{m}\). Therefore, \(\{C_0, C_1, \ldots, C_{m-1}\}\) is a \(T_i\)-measurable partition of \(T_1\) such that \(C_j\) is independent of \(\mathcal{F}_i\) for \(j = 0, 1, \ldots, m - 1\). Since \(m\) is arbitrary, \(\mathcal{F}_1\) admits an asymptotic independent supplement in \(\mathcal{T}_1\) under \(\lambda_1\). By Lemma 5, \(\mathcal{T}_1\) is nowhere equivalent to \(\mathcal{F}_1\) under \(\lambda_1\). Based on an analogous argument, \(\mathcal{T}_i\) is nowhere equivalent to \(\mathcal{F}_i\) under \(\lambda_i\) for each \(i \in I\). \(\square\)

### 6.5 Proofs of Claims 1 and 3

**Proof of Claim 1.** We first explicitly give the marginal \(\lambda_i\) of \(\lambda\), \(i = 1, 2\). For any \(D \in \mathcal{B}([0, 1])\), we have

\[
\lambda_1(D) = \frac{1}{2} \int_D \int_{T_2} 1 \eta(dt_2)\eta(dt_1) + \frac{1}{2} \int_D \int_{T_2} 6t_1t_2^2\eta(dt_2)\eta(dt_1)
\]

\[
= \int_D (\frac{1}{2} + t_1)\eta(dt_1).
\]

The density of \(\lambda_1\) with respect to \(\eta\) is \(\frac{1}{2} + t_1\). Similarly, the density of \(\lambda_2\) on \(\eta\) is \(\frac{1}{2} + \frac{3}{2}t_2^2\).

We need to verify that \(\lambda\) is absolutely continuous with respect to \(\lambda_1 \otimes \lambda_2\). For any \(D_1, D_2 \in \mathcal{B}([0, 1])\),

\[
\lambda(D_1 \times D_2) = \frac{1}{2} \int_{D_1} \int_{D_2} 1 \eta(dt_2)\eta(dt_1) + \frac{1}{2} \int_{D_1} \int_{D_2} 6t_1t_2^2\eta(dt_2)\eta(dt_1)
\]

\[
= \int_{D_1} \int_{D_2} \left(\frac{1}{2} + t_1\right)\left(\frac{1}{2} + \frac{3}{2}t_2^2\right)\eta(dt_2)\eta(dt_1)
\]

\[
= \int_{D_1} \int_{D_2} \frac{1}{2} + 3t_1t_2^2\lambda_2(dt_2)\lambda_1(dt_1).
\]

That is, \(\lambda\) has the density \(q(t_1, t_2) = \frac{1}{2} + 3t_1t_2^2\) on \(\lambda_1 \otimes \lambda_2\). The mapping \(q(t_1, \cdot)\) induces the full \(\sigma\)-algebra \(\mathcal{B}([0, 1])\) on \(T_2\) when \(t_1 > \frac{1}{2}\), as \(q(t_1, \cdot)\) is continuous and strictly increasing in \(t_2\). Recall that \(u_1\) only depends on the action profile \(a\). Thus, \(w_1(a, t_1, \cdot)\) also induces \(\mathcal{B}([0, 1])\) on \(T_2\) when \(t_1 > \frac{1}{2}\), implying that DCPI is not satisfied for \(J = 1\).
One can write
\[ w_1(a, t_1, t_2) = \frac{1}{2}u_1(a) - \frac{1}{2}t_1 - \frac{1}{2}t_2 + 3u_1(a) - \frac{t_1}{t_2} + \frac{t_2}{t_2}. \]

Let \( w_1^1(a, t) = \frac{1}{2}u_1(a), \) \( \rho_1^1(t_1) = \frac{1}{1+t_1} \), \( \rho_2^1(t_2) = \frac{1}{t_1+\frac{t_2}{2}}, \) \( w_1^2(a, t) = 3u_1(a), \) \( \rho_1^2(t_1) = \frac{1}{3+t_1}, \) \( \rho_2^2(t_2) = \frac{t_2}{3+t_2} \). Note that \( w_1^1 \) and \( w_1^2 \) do not depend on the type profile \( t \). By Lemma 3 in He and Sun (2019), there exists a sub-\( \sigma \)-algebra \( G_i \subseteq T_i \) such that \((T_i, G_i, \lambda_i)\) is atomless and \( T_i \) is nowhere equivalent to \( G_i \) for \( i = 1, 2 \). Thus, DCPI is satisfied for \( J = 2 \). For the case \( J > 2 \), one simply lets \( \rho_i^j \equiv 0 \) for \( j > 2 \) and \( i = 1, 2 \). This completes the proof. \( \square \)

**Proof of Claim 3.** We first check that the condition of uniform payoff security is satisfied.

Recall that \( \psi_1, \psi_2 \), and \( \{\phi_i\}_{i \in I} \) are all continuous on the compact set \( \hat{T} \times A \). For any \( \epsilon > 0 \), there exists some \( \delta > 0 \) such that for any \((\tilde{t}, a), (\tilde{t}', a') \in \hat{T} \times A \) with \( \| (\tilde{t}, a) - (\tilde{t}', a') \|_2 < \delta \), and for each \( i \in I \),

\[ |\psi_1(\tilde{t}, a) - \psi_1(\tilde{t}', a')| < \frac{\epsilon}{3}, \quad |\psi_2(\tilde{t}, a) - \psi_2(\tilde{t}', a')| < \frac{\epsilon}{3}, \quad |\phi_i(\tilde{t}, a) - \phi_i(\tilde{t}', a')| < \frac{\epsilon}{3}, \]

where \( \| \cdot \|_2 \) denotes the Euclidean norm in \( \mathbb{R}^3 \).

Fix a bidding strategy profile \((b_1, b_2, \ldots, b_n)\). For each bidder \( i \), we construct a new bidding strategy as follows: \( b_i^*(t_i) = \min\{b_i(t_i) + \frac{\epsilon}{2}, a\} \). To verify the condition of uniform payoff security, we shall show that for each \( i \in I \) and for all \((t, a_{-i})\), there exists a neighborhood \( O_{a_{-i}} \) of \( a_{-i} \) such that for all \( y_{-i} \in O_{a_{-i}} \),

\[ u_i(t, b_i^*(t_i), y_{-i}) - u_i(t, b_i(t_i), a_{-i}) > -\epsilon. \]

Suppose that bidder \( i \) follows the bidding strategy \( b_i \). Fix the type profile \( t \) and other bidders' bids \( a_{-i} \). There are three cases to consider.

**Case 1** Bidder \( i \) is a losing bidder. In this case, \( b_i(t_i) < \max_{k \neq i} a_k \). Bidder \( i \)'s payoff is

\[ u_i(t, b_i(t_i), a_{-i}) = \sum_{j=1}^{n} \left( \psi_2(t'_0, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) + \phi_i(t'_0, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) + 3u_1(a) - \frac{t_1}{t_2} + \frac{t_2}{t_2} \right) \kappa(t'_0 | t). \]

Let \( O_{a_{-i}}(a_{-i}) \) be the ball in \( A_{-i} \) with center \( a_{-i} \) and radius \( \frac{\epsilon}{2} \). For any \( y_{-i} \in O_{a_{-i}}(a_{-i}) \),
\[ \| (b_i(t_i), a_{-i}) - (b_i^*(t_i), y_{-i}) \|_2 < \delta. \]

Thus, for each \( j = 1, 2, \ldots, J, \)
\[ \psi_1(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \geq \psi_2(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) > \psi_2(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) - \frac{\varepsilon}{3}, \]
and
\[ \varphi_i(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \geq \varphi_i(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) - \frac{\varepsilon}{3}. \]

For any \( y_{-i} \in O_{\delta'}(a_{-i}), \) the two inequalities above imply that
\[
\begin{align*}
&u_i(t, b_i^*(t_i), y_{-i}) \\
&\geq \sum_{j=1}^{J} \left( \psi_2(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) + \varphi_i(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \right) \kappa(\{ t_0^j \} | t) \\
&> \sum_{j=1}^{J} \left( \psi_2(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), a_{-i}) + \varphi_i(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), a_{-i}) \right) \kappa(\{ t_0^j \} | t) - \frac{2\varepsilon}{3} \\
&> u_i(t, b_i(t_i), a_{-i}) - \varepsilon.
\end{align*}
\]

**Case 2** Bidder \( i \) is the unique winning bidder. In this case, \( b_i(t_i) > \max_{k \neq i} a_k. \)

Bidder \( i' \)’ payoff is
\[
u_i(t, b_i(t_i), a_{-i}) = \sum_{j=1}^{J} \left( \psi_1(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) + \varphi_i(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) \right) \kappa(\{ t_0^j \} | t).
\]

Let \( O_{\delta'}(a_{-i}) \) be the \( \delta' \)-ball of \( a_{-i} \) such that \( \delta' < \frac{\delta}{2} \) and
\[ b_i(t_i) > \max_{y_{-i} \in O_{\delta'}(a_{-i}), k \neq i} y_k. \]

Since \( b_i^*(t_i) \geq b_i(t_i) \), bidder \( i \) is still the unique winner by adopting the bidding strategy \( b_i^* \) if others bid \( y_{-i} \in O_{\delta'}(a_{-i}) \). For any \( y_{-i} \in O_{\delta'}(a_{-i}) \), since \( \| (b_i(t_i), a_{-i}) - (b_i^*(t_i), y_{-i}) \|_2 < \delta, \)
\[ \psi_1(t_0^j, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) > \psi_1(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) - \frac{\varepsilon}{3} \]
for each \( j = 1, 2, \ldots, J. \) It implies that
\[ u_i(t, b_i^*(t_i), y_{-i}) - u_i(t, b_i(t_i), a_{-i}) > -\varepsilon. \]
Case 3  Bidder $i$ is one of the winning bidders. In this case, $b_i(t_i) = \max_{\ell \neq i} a_\ell$. Bidder $i$’s payoff is

$$u_i(t, b_i(t_i), a_{-i})$$

$$= \sum_{j=1}^J \left[ \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_j(t_i), a_{-i}) + \sum_{\ell \neq i: a_\ell = b_j(t_i)} \xi_\ell(b_i(t_i), a_{-i})} \psi_1(t_j^0, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) \right. \right.$$  

$$+ \left. \left( 1 - \frac{\xi_i(b_j(t_i), a_{-i})}{\xi_i(b_j(t_i), a_{-i}) + \sum_{\ell \neq i: a_\ell = b_j(t_i)} \xi_\ell(b_j(t_i), a_{-i})} \right) \psi_2(t_j^0, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) \right.$$  

$$+ \varphi_i(t_j^0, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) \left] \kappa(t_j^0 | t). \right.$$

If bidder $i$ becomes the unique winner by bidding $b_i^*(t_i)$ when others bid $a_{-i}$, then $b_i^*(t_i) > b_i(t_i)$. One can identify a neighbourhood $O_{\delta'}(a_{-i})$ of $a_{-i}$ such that $\delta' < \frac{\delta}{2}$ and

$$b_i^*(t_i) > \max_{y_{-i} \in O_{\delta'}(a_{-i}), \ell \neq i} y_\ell.$$

For any $y_{-i} \in O_{\delta'}(a_{-i})$, since $\| (b_i(t_i), a_{-i}) - (b_i^*(t_i), y_{-i}) \|_2 < \delta$,

$$u_i(t, b_i^*(t_i), y_{-i})$$

$$= \sum_{j=1}^J \left[ \psi_1(t_j^0, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) + \varphi_i(t_j^0, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \right] \kappa(t_j^0 | t)$$

$$= \sum_{j=1}^J \left[ \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_j(t_i), a_{-i}) + \sum_{\ell \neq i: a_\ell = b_j(t_i)} \xi_\ell(b_i(t_i), a_{-i})} \psi_1(t_j^0, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \right.$$  

$$+ \left. \left( 1 - \frac{\xi_i(b_j(t_i), a_{-i})}{\xi_i(b_j(t_i), a_{-i}) + \sum_{\ell \neq i: a_\ell = b_j(t_i)} \xi_\ell(b_j(t_i), a_{-i})} \right) \psi_2(t_j^0, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \right.$$  

$$+ \varphi_i(t_j^0, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}) \left] \kappa(t_j^0 | t) \right. \left. - \frac{\delta}{2} \right.$$  

$$> u_i(t, b_i(t_i), a_{-i}) - \epsilon.$$  

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If bidder $i$ is still one of winning bidders by bidding $b_i^*(t_i)$ when others bid $a_{-i}$, then $b_i^*(t_i) = b_i(t_i) = \bar{a}$. That is, no matter what the other bidders bid, bidder $i$ must be a winning bidder by bidding $b_i^*(t_i)$. Pick $\delta'' < \delta'$ such that for any $y_{-i} \in O_{\delta'}(a_{-i})$, (1) $y_{\ell} < \bar{a}$ if $a_{\ell} < \bar{a}$ for $\ell \neq i$, and (2)

$$
\left| \frac{\xi_i(b_i^*(t_i), y_{-i})}{\xi_i(b_i^*(t_i), y_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i^*(t_i), y_{-i}) \right| < \epsilon_1,
$$

where $0 < \epsilon_1 < \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_i(t_i), a_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}}$ and

$$\epsilon_1 \sum_{j=1}^{J} \left( \psi_1(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i(t_i), a_{-i}) - \psi_2(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i(t_i), a_{-i}) \right) \kappa(\{t_0^j\} | t) < \frac{\epsilon}{3}.
$$

For any $y_{-i} \in O_{\delta'}(a_{-i})$,

$$u_i(t, b_i^*(t_i), y_{-i})$$

$$= \sum_{j=1}^{J} \left[ \frac{\xi_i(b_i^*(t_i), y_{-i})}{\xi_i(b_i^*(t_i), y_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i^*(t_i), y_{-i}) \psi_1(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i^*(t_i), y_{-i}) 
$$

$$+ \left(1 - \frac{\xi_i(b_i^*(t_i), y_{-i})}{\xi_i(b_i^*(t_i), y_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i^*(t_i), y_{-i}) \right) \psi_2(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i^*(t_i), y_{-i}) 
$$

$$+ \varphi_i(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i^*(t_i), y_{-i}) \right] \kappa(\{t_0^j\} | t)$$

$$> \sum_{j=1}^{J} \left[ \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_i(t_i), a_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i(t_i), a_{-i}) \psi_1(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i(t_i), a_{-i}) - \frac{\epsilon_1}{3} \right) 
$$

$$+ \left(1 - \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_i(t_i), a_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i(t_i), a_{-i}) \right) \psi_2(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i(t_i), a_{-i}) - \frac{\epsilon_1}{3} \right)$$

$$+ \varphi_i(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i^*(t_i), y_{-i}) \right] \kappa(\{t_0^j\} | t)$$

$$\geq \sum_{j=1}^{J} \left[ \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_i(t_i), a_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i(t_i), a_{-i}) \psi_1(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i(t_i), a_{-i}) - \frac{\epsilon_1}{3} \right) 
$$

$$+ \left(1 - \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_i(t_i), a_{-i}) + \sum_{\ell \neq i: \ell \neq a}=\bar{a}} \xi_i(b_i(t_i), a_{-i}) \right) \psi_2(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i(t_i), a_{-i}) - \frac{\epsilon_1}{3} \right)$$

$$+ \varphi_i(t_0^j, t_{11}, \ldots, t_{n_{1}}, b_i^*(t_i), y_{-i}) \right] \kappa(\{t_0^j\} | t)$$
\[
+ \varphi_i(t_0^i, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) - \frac{e}{3} \kappa(\{t_0^i\} \mid t)
\]

\[= u_i(t, b_i(t_i), a_{-i}) - \frac{2e}{3}
- \epsilon_1 \sum_{j=1}^{J} \left[ \psi_1(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) - \psi_2(t_0^j, t_{11}, \ldots, t_{n1}, b_i(t_i), a_{-i}) \right] \kappa(\{t_0^i\} \mid t)
\]

\[> u_i(t, b_i(t_i), a_{-i}) - \epsilon.
\]

The first inequality holds since

\[
\frac{\xi_i(b_i^*(t_i), y_{-i})}{\xi_i(b_i^*(t_i), y_{-i}) + \sum_{\ell \neq i: y_{t_\ell} = \bar{a}} \xi_\ell(b_i^*(t_i), y_{-i})} \geq \frac{\xi_i(b_i^*(t_i), y_{-i})}{\xi_i(b_i^*(t_i), y_{-i}) + \sum_{\ell \neq i: a_{t_\ell} = \bar{a}} \xi_\ell(b_i^*(t_i), y_{-i})}
\]

\[> \frac{\xi_i(b_i(t_i), a_{-i})}{\xi_i(b_i(t_i), a_{-i}) + \sum_{\ell \neq i: a_{t_\ell} = \bar{a}} \xi_\ell(b_i(t_i), a_{-i})} - \epsilon_1,
\]

and

\[
\psi_1(t_0^1, t_{11}, \ldots, t_{n1}, b_i(t_i), y_{-i}) \geq \psi_2(t_0^1, t_{11}, \ldots, t_{n1}, b_i^*(t_i), y_{-i}).
\]

The other equalities and inequalities are due to simple algebras.

To summarize, in all the cases above, for any bidder \(i\) and all \((t, a_{-i})\), there exists a neighborhood \(O_{a_{-i}}\) of \(a_{-i}\) such that for all \(y_{-i} \in O_{a_{-i}}\),

\[u_i(t, b_i^*(t_i), y_{-i}) - u_i(t, b_i(t_i), a_{-i}) > -\epsilon.
\]

That is, the all-pay auction game satisfies the condition of uniform payoff security.

Note that

\[
\sum_{i \in I} u_i(t, a) = \sum_{i \in I} \sum_{j=1}^{J} v_2(t_0^i, t_1, \ldots, t_n, a_1, \ldots, a_n) \kappa(\{t_0^i\} \mid t)
\]

\[= \sum_{j=1}^{J} \left[ \psi_1(t_0^j, t_{11}, \ldots, t_{n1}, a) + (n - 1) \psi_2(t_0^j, t_{11}, \ldots, t_{n1}, a) + \sum_{i \in I} \varphi_i(t_0^j, t_{11}, \ldots, t_{n1}, a) \right] \kappa(\{t_0^j\} \mid t),
\]

which is continuous in \(a\) for every \(t \in T\). By Lemma 1, the game possesses a behavioral-strategy equilibrium.
Recall that $\mathcal{T}_i = \mathcal{B}(T_i)$, the Borel $\sigma$-algebra on $T_i$. Let $\mathcal{F}_i = T_i^{\mathcal{B}(\mathbb{R}) \otimes \{\emptyset, \mathbb{R}\}}$; that is, $\mathcal{F}_i$ is the restriction of the $\sigma$-algebra $\mathcal{B}(\mathbb{R}) \otimes \{\emptyset, \mathbb{R}\}$ to $T_i$, where the trivial $\sigma$-algebra $\{\emptyset, \mathbb{R}\}$ is imposed on the dimension $t_{i2}$ for each $i \in I$. Then $\mathcal{T}_i$ is nowhere equivalent to $\mathcal{F}_i$. Due to Corollary 1, DCPI is satisfied. By Proposition 2, a pure-strategy equilibrium exists.

References

Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker’s Guide*, 3rd edition, Berlin, Springer, 2006.

Susan Athey, Single crossing properties and the existence of pure strategy equilibria in games of incomplete information, *Econometrica* 69 (2001), 861–889.

Patrick Beißner and M. Ali Khan, On Hurwicz–Nash equilibria of non-Bayesian games under incomplete information, *Games and Economic Behavior* 115 (2019), 470–490.

Adam Brandenburger and Eddie Dekel, Hierarchies of beliefs and common knowledge, *Journal of Economic Theory* 59 (1993), 189–198.

Oriol Carbonell-Nicolau and Richard P. McLean, On the existence of Nash equilibrium in Bayesian games, *Mathematics of Operations Research* 43 (2018), 100–129.

Oriol Carbonell-Nicolau and Richard P. McLean, Nash and Bayes–Nash equilibria in strategic-form games with intransitivities, *Economic Theory* 68 (2019), 935–965.

Yi-Chun Chen and Siyang Xiong, Genericity and robustness of full surplus extraction, *Econometrica* 81 (2013), 825–847.

Yuan Shih Chow and Henry Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd edition, New York, Springer, 1997.

Jacques Cremer and Richard P. McLean, Full extraction of the surplus in Bayesian and dominant strategy auctions, *Econometrica* 56 (1988), 1247–1257.

Aryeh Dvoretzky, Abraham Wald and Jacob Wolfowitz, Elimination of randomization in certain statistical decision procedures and zero-sum two-person games, *Annals of Mathematical Statistics* 22 (1951), 1–21.

Amanda Friedenberg and Martin Meier, The context of the game, *Economic Theory* 63 (2017), 347–386.

Drew Fudenberg and Jean Tirole, *Game Theory*, MIT Press, Cambridge, 1991.

Huiyi Guo, Mechanism design with ambiguous transfers: An analysis in finite dimensional naive type spaces, *Journal of Economic Theory* 183 (2019), 76–105.

Peter J. Hammond, Expected utility in non-cooperative game theory, in *Handbook of Utility Theory* (Salvador Barbera, Peter J. Hammond, and Christian Seidl, eds.), vol. 2, ch. 18, 982–1063, Boston, Kluwer Academic Publishers, 2004.

John C. Harsanyi, Games with incomplete information played by ‘Bayesian’ players, Parts I–III, *Management Science* 14 (1967–1968), 159–182, 320–334, and 486–502.
Wei He and Xiang Sun, On the diffuseness of incomplete information game, *Journal of Mathematical Economics* **54** (2014), 131–137.

Wei He and Yeneng Sun, Pure-strategy equilibria in Bayesian games, *Journal of Economic Theory* **180** (2019), 11–49.

Wei He and Yeneng Sun, Conditional expectations of Banach valued correspondences, working paper, 2021. Available at [https://arxiv.org/abs/2105.08552](https://arxiv.org/abs/2105.08552).

Wei He and Nicholas C. Yannelis, Existence of equilibria in discontinuous Bayesian games, *Journal of Economic Theory* **162** (2016), 181–194.

Aviad Heifetz and Zvika Neeman, On the generic (im)possibility of full surplus extraction in mechanism design, *Econometrica* **74** (2006), 213–233.

Ziv Hellman, A game with no Bayesian approximate equilibria, *Journal of Economic Theory* **153** (2014), 138–151.

Ziv Hellman and Yehuda John Levy, Bayesian games with a continuum of states, *Theoretical Economics* **12** (2017), 1089–1120.

Tong Li, Isabelle Perrigne and Quang Vuong, Conditionally independent private information in OCS wildcat auctions, *Journal of Econometrics* **98** (2000), 129–161.

Peter A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, *Transactions of the American Mathematical Society* **211** (1975), 113–122.

Peter A. Loeb and Yeneng Sun, Purification of measure-valued maps, *Illinois Journal of Mathematics* **50** (2006), 747–762.

Peter A. Loeb and Manfred P. H. Wolff, eds., *Nonstandard Analysis for the Working Mathematician*, 2nd edition, Springer, Berlin, 2015.

M. Ali Khan, Kali P. Rath and Yeneng Sun, On a private information game without pure strategy equilibria, *Journal of Mathematical Economics* **31** (1999), 341–359.

M. Ali Khan, Kali P. Rath and Yeneng Sun, The Dvoretzky-Wald-Wolfowitz theorem and purification in atomless finite-action games, *International Journal of Game Theory* **34** (2006), 91–104.

M. Ali Khan and Yeneng Sun, Non-cooperative games on hyperfinite Loeb spaces, *Journal of Mathematical Economics* **31** (1999), 455–492.

M. Ali Khan and Yongchao Zhang, On the existence of pure-strategy equilibria in games with private information: A complete characterization, *Journal of Mathematical Economics* **50** (2014), 197–202.

M. Ali Khan and Yongchao Zhang, On pure-strategy equilibria in games with correlated information, *Games and Economic Behavior* **111** (2018), 289–304.

Vijay Krishna, *Auction Theory*, Academic press, 2009.

David McAdams, Isotone equilibrium in games of incomplete information, *Econometrica* **71** (2003), 1191–1214.

Paul R. Milgrom and Robert J. Weber, Distributional strategies for games with incomplete information, *Mathematics of Operations Research* **10** (1985), 619–632.

Konrad Podczeck, On the convexity and compactness of the integral of a Banach space valued correspondence, *Journal of Mathematical Economics* **44** (2008), 836–852.
Konrad Podczeck, On purification of measure-valued maps, *Economic Theory* **38** (2009), 399–418.

Pavlo Prokopovych, On equilibrium existence in payoff secure games, *Economic Theory* **48** (2011), 5–16.

Pavlo Prokopovych and Nicholas C. Yannelis, On monotone approximate and exact equilibria of an asymmetric first-price auction with affiliated private information, *Journal of Economic Theory* **184** (2019), 1–29.

Pavlo Prokopovych and Nicholas C. Yannelis, On nondegenerate equilibria of double auctions with several buyers and a price floor, *Economic Theory*, forthcoming, 2021.

Roy Radner and Robert W. Rosenthal, Private information and pure-strategy equilibria, *Mathematics of Operations Research* **7** (1982), 401–409.

Philip J. Reny, On the existence of pure and mixed strategy Nash equilibria in discontinuous games, *Econometrica* **67** (1999), 1029–1056.

Philip J. Reny, On the existence of monotone pure strategy equilibria in Bayesian games, *Econometrica* **79** (2011), 499–553.

Philip J. Reny, Nash equilibrium in discontinuous games, *Annual Review of Economics*, forthcoming, 2020.

Robert Samuel Simon, Games of incomplete information, ergodic theory, and the measurability of equilibria, *Israel Journal of Mathematics* **138** (2003), 73–92.

Yeneng Sun, Distributional properties of correspondences on Loeb spaces, *Journal of Functional Analysis* **139** (1996), 68–93.

Yeneng Sun and Nicholas C. Yannelis, Saturation and the integration of Banach valued correspondences, *Journal of Mathematical Economics* **44** (2008), 861–865.

Xavier Vives, Nash equilibrium with strategic complementarities, *Journal of Mathematical Economics* **19** (1990), 305–321.

Jianwei Wang and Yongchao Zhang, Purification, saturation and the exact law of large numbers, *Economic Theory* **50** (2012), 527–545.

Yishu Zeng, Essays on Game Theory, Ph.D Dissertation, National University of Singapore, 2016.