Monodromy Map and Classical r-matrices

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Abstract

We compute the Poisson bracket relations for the monodromy matrix of the auxiliary linear problem. If the basic Poisson brackets of the model contain derivatives, this computation leads to a peculiar kind of symmetry breaking which accounts for a ‘spontaneous quantization’ of the underlying global gauge group. A classification of possible patterns of symmetry breaking is outlined.

1 Introduction

Computation of the Poisson bracket relations for the monodromy matrix of the auxiliary linear problem is one of the keypoints in the Classical Inverse Scattering Method [1]. Its quantum counterpart, the computation of the commutation relations for the quantum monodromy matrix, plays an equally important role in Quantum Inverse Scattering Method. Usually, this computation is carried out under the technical assumption of ‘ultralocality’ of

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the basic Poisson brackets (a precise definition is given below in Section 1, 
Lemma 2.)

It is the aim of the present note to discuss the general case. As we shall 
see, dropping out of the ultralocality condition leads to new physical effects 
(some kind of spontaneous symmetry breaking). In quantum setting these 
effects manifest themselves in spontaneous quantization (i.e. deformation of 
special kind) of certain symmetry groups. Similar effects were discovered 
recently in Conformal Field Theory where they have drawn much attention 
[6], [7], [8]. We shall see that possible patterns of symmetry breaking may 
be fully classified: an easy reformulation reduces the problem to the classi-
fication of solutions of the modified classical Yang–Baxter equation for the 
square of the original Lie algebra and hence to the classification theorem of 
Belavin and Drinfeld [2].

2 Monodromy Matrices and Zero Curvature 
Equations

Zero curvature equations are compatibility conditions for the linear system 
of differential equations

\[
\begin{align*}
\partial_x \psi &= L \psi, \\
\partial_t \psi &= A \psi.
\end{align*}
\]

(1)

In applications to integrable systems the coefficients \((L, A)\) of the flat 
connection usually take values in the loop algebra \(g = \mathcal{L}(a)\) of an auxiliary 
finite-dimensional Lie semi-simple algebra \(a\). The main content of the Inverse 
Scattering Method as applied to zero curvature equations is given by the 
following two assertions:

1. **Integrals of motion for the zero curvature equations are spectral invar-
aiants of the ordinary differential operator \(\partial_x - L\).**

2. **These integrals are in involution with respect to some natural Poisson 
bracket on the phase space.** (The notion of ‘natural’ Poisson Brackets will be 
precised below.)

Let us suppose for concreteness that the coefficients of \(L\) are periodic 
functions of \(x\) with period \(2\pi\). In that case, according to the classical Floquet 
Theorem, spectral invariants of \(L\) depend only on the eigenvalues of the 
monodromy matrix. By definition, the monodromy matrix of the differential
equation
\[ \partial_x \psi = L\psi \]  \hspace{1cm} (2)

is given by

\[ M = \psi(2\pi)\psi(0)^{-1}, \]  \hspace{1cm} (3)

where \( \psi \) is a fundamental solution of (2). It is natural to regard \( M \) as an element of the Lie group \( G \) which corresponds to \( \mathfrak{g} \). Let us consider the monodromy map \( M : \mathcal{M} \longrightarrow G \) which assigns to a point \( L \in \mathcal{M} \) of the phase space the corresponding monodromy matrix. Since the eigenvalues of \( M \) are integrals of motion, the monodromy itself satisfies an evolution equation of the form

\[ \partial_t M = [M, N] \]

which is usually called the Novikov equation.

Let now \( \mathcal{F}_t : \mathcal{M} \longrightarrow \mathcal{M} \) be the dynamical flow associated with the zero curvature equation, and \( F_t : G \longrightarrow G \) the flow on \( G \) determined by the Novikov equation. The relation between the two flows is given by the commutative diagram

\[ \begin{array}{ccc}
\mathcal{F}_t & : & \mathcal{M} \longrightarrow \mathcal{M} \\
M & \downarrow & F_t \\
\mathcal{G} & \longrightarrow & G
\end{array} \]  \hspace{1cm} (4)

In other words, the flow \( \mathcal{F}_t \) factorizes over \( G \).

Let us recall now that the phase space \( \mathcal{M} \) is equipped with a Poisson bracket. It is natural to expect that the group \( G \) may also be equipped with a Poisson structure in such a way that the monodromy map preserves the Poisson brackets and the diagram (4) consists of Poisson mappings. In order to make this picture more precise let us recall first the hamiltonian interpretation of the spectral invariants of the monodromy [3], [4].

Let \( \mathfrak{g} \) be a Lie algebra equipped with a nondegenerate invariant inner product. Let \( \mathcal{G} = C^\infty(S^1; \mathfrak{g}) \) be the corresponding current algebra (the algebra of smooth periodic functions with values in \( \mathfrak{g} \) and with pointwise commutator). The bilinear form

\[ \omega(X,Y) = \int \langle X, \partial_x Y \rangle \, dx \]  \hspace{1cm} (5)
is a 2-cocycle on $G$ and defines a central extension $G^\wedge = G \oplus \mathbb{R}$ of $G$. Using the inner product on $G$ we may identify the dual of $G^\wedge$ with $G \oplus \mathbb{R}$; the coadjoint representation of $G$ in $G^{\wedge^*}$ is then given by

$$ad^*X(L,e) = ([X,L] - e\partial_x X, 0).$$  \hspace{1cm} (6)

The number $e \in \mathbb{R}$ (the central charge) is a coadjoint invariant; without loss of generality we may assume that $e = 1$ (i.e. fix an invariant hyperplane $G^{\wedge^*}_1 = (G, 1) \subset G^{\wedge^*}$). As usual, the dual of $G^\wedge$ is equipped with a natural Poisson bracket, the Lie–Poisson bracket of $G^\wedge$. Recall that a function is called a Casimir function of a given Poisson structure if its Poisson brackets with any other function are identically zero (i.e. if it lies in the center of the Lie algebra of functions with respect to the Poisson bracket). The Casimir functions form a ring with respect to pointwise multiplication.

**Theorem 1** The ring of Casimir functions on $G^{\wedge^*}_1$ is generated by the spectral invariants of the monodromy matrix of equation (2).

More precisely, let $\varphi \in C^\infty(G)$ be a central function on $G$ (i.e. $\varphi(xy) = \varphi(yx)$ for any $x, y \in G$). Then $L \mapsto \varphi(M[L])$ is Casimir function on $G^{\wedge^*}_1$, and the ring of Casimir functions is generated by functions of this form.

In order to get nontrivial equations of motion from the spectral invariants of monodromy we need a different Poisson bracket. The corresponding construction is basic in the theory of classical r-matrices. Recall that classical r-matrix on a Lie algebra $g$ is a linear operator $R \in \text{End}(g)$ such that the bracket on $g$ given by

$$[X,Y]_R = \frac{1}{2} [RX,Y] + \frac{1}{2} [X,RY]$$  \hspace{1cm} (7)

satisfies the Jacobi identity. In this case there are two structures of a Lie algebra on the linear space $g$ given by the original Lie bracket and by the $R-$bracket, respectively. Let $G_R = C^\infty(S^1; g)$ be the corresponding current algebra.

**Proposition 1** [4] The bilinear form on $G_R$ given by

$$\omega_R(X,Y) = \frac{1}{2} (\omega(RX,Y) + \omega(X,RY))$$  \hspace{1cm} (8)

is a 2-cocycle on $G_R$.  \hspace{1cm} (8)
Let $G_R$ be the corresponding central extension. Clearly, the dual spaces of $G^\wedge$ and $G_R^\wedge$ coincide. Hence the space $G^{\wedge^*}$ (and even the hyperplane $G_i^{\wedge^*}$) is equipped with two different Lie–Poisson brackets which correspond to the original Lie bracket in $G^\wedge$ and to the $R$–bracket, respectively.

**Theorem 2** The Casimir functions of $G^\wedge$ are in involution with respect to the $R$–bracket and give rise to zero curvature equations on $G_i^{\wedge^*}$.

**Sketch of a proof.** We shall start with the following Lemma which will be useful in the sequel. Let $\varphi \in C^\infty (G)$. Consider the functional $\varphi^M$ on $G_i^{\wedge^*}$ given by $\varphi^M : L \mapsto \varphi (M [L] )$. The Frechet derivative of $\varphi^M$ is a function $\text{grad} \varphi^M = X_\varphi$ on $[0, 2\pi]$ with values in $g$ defined by the relation

$$\int \langle X_\varphi(x), \xi(x) \rangle \, dx = \left( \frac{d}{dt} \right)_{t=0} \varphi (M + t\xi)$$

for any $\xi \in G$.

**Lemma 1** (i) The Frechet derivative of $\varphi^M$ is equal to

$$X_\varphi = \psi \nabla_\varphi \psi^{-1},$$

where $\psi$ is the fundamental solution of (1) normalized so that $\psi(0) = 1$, and $\nabla_\varphi$ is the left gradient of $\varphi$ on $G$ defined by

$$\langle \nabla_\varphi(x), \eta \rangle = \left( \frac{d}{dt} \right)_{t=0} \varphi(e^{t\eta}x).$$

(ii) It satisfies the differential equation

$$\partial_x X_\varphi = [L, X_\varphi]$$

and the boundary condition

$$X_\varphi(2\pi) = \text{Ad} L \cdot X_\varphi(0).$$

(iii) If $\varphi$ is central on $G$, then $X_\varphi$ is a smooth function on the circle.
The proof of the lemma consists in the standard use of variation of coefficients in equation (1). Assertions (ii, iii) immediately follow from the explicit formula (9).

The Poisson bracket of two functionals is a bilinear form of their Frechet derivatives. The linear operator associated with this bilinear form is called the Poisson operator. It is easy to write down its explicit expression for the $R-$bracket in question.

**Lemma 2** (i) The Poisson operator associated with the $R-$bracket is given by

$$\mathcal{H}_{R} = (R + R^{*})\partial_{x} + (R^{*} \circ adL + adL \circ R).$$  \hspace{1cm} (12)

(ii) This operator is bounded if and only if $R = -R^{*}$.

The latter case plays an important role; we say then that the Poisson bracket is ultralocal and the classical $R-$matrix satisfies the unitarity condition.

In the general case in order to get a bona fide linear operator we must add to the differential expression (12) some boundary conditions.

**Lemma 3** Operator $\mathcal{H}_{R}$ is essentially skew-selfadjoint on the space of smooth periodic functions on $[0,2\pi]$.

**Definition 1** A functional $\Phi$ on $\mathcal{G}^{\star}_{1}$ is called smooth if its Frechet derivative lies in the domain of $\mathcal{H}_{R}$.

Lemma 1 (ii) shows that if a function $\varphi \in C^{\infty}(G)$ is central, the corresponding functional $\varphi^{M}$ is smooth; hence the Poisson bracket of such functionals is well defined. Put $X_{i} = grad\varphi_{i}^{M}$, $i = 1,2$. To prove that the functionals $\varphi^{M}$ are in involution with each other we have to compute the bilinear form $\left(\mathcal{H}_{R}X_{1}, X_{2}\right)$. By Lemma 1 (ii, iii) the integrand is a total derivative of a periodic function.

Assume that $R = -R^{*}$. In this case the Poisson bracket $\left\{\varphi_{1}^{M}, \varphi_{2}^{M}\right\}$ is well defined for any two functions $\varphi_{1}, \varphi_{2} \in C^{\infty}(G)$. Using Lemma 1 it is easy to check that in this case $\left(\mathcal{H}_{R}X_{1}, X_{2}\right)$ is still a total derivative; however, in this case the boundary terms in general do not vanish. We get

$$\left\{\varphi_{1}^{M}, \varphi_{2}^{M}\right\}_{R} = \left\langle R \nabla \varphi_{1}, \nabla \varphi_{2}\right\rangle - \left\langle R(AdM \cdot \nabla \varphi_{1}), AdM \cdot \nabla \varphi_{2}\right\rangle.$$  \hspace{1cm} (13)
The right hand side in (13) is the *Sklyanin bracket* on $G$ determined by $R$. Thus we get the following well known result.

**Theorem 3** Assume that $R = -R^*$. Equip $G$ with the Sklyanin bracket determined by $R$. Then the monodromy map preserves Poisson brackets.

### 3 Nonultralocal Case

Let us now pass to the study of the general case when $R \neq -R^*$. Lemma 1 shows that in general the Frechet derivative of a functional $\varphi^M$ is a discontinuous functions. Thus the problem is to extend the bilinear form associated with the unbounded operator $\mathcal{H}_R$ to functions with a jump. The problem of this kind is well known in elementary quantum mechanics (zero range potentials). The difference is that in quantum mechanics the free parameters of the model are the boundary conditions imposed on the wave function. In our case, by contrast, the boundary conditions are fixed in advance (as in Lemma 1 (iii)). The 'physical' freedom consists in adding to the bilinear form of $\mathcal{H}_R$ an interaction term which is sensitive to the jump of the Frechet derivative at $0 \equiv 2\pi$. The resulting bilinear form usually has a lower symmetry than the formal differential expression (12); hence one can speak of the 'spontaneous symmetry breaking'.

In order to describe the boundary bilinear form let us introduce the following definition.

Let $g \oplus g$ be the direct sum of two copies of $g$. We shall equip $g \oplus g$ with the inner product

$$\langle\langle (X_1, Y_1), (X_2, Y_2) \rangle\rangle = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle.$$  

(14)

Let us define a mapping

$$\partial : C^\infty([0, 2\pi] ; g) \rightarrow g \oplus g : X \mapsto (X(0), X(2\pi)).$$  

(15)

Observe that for $X = \text{grad} \varphi^M$ we have

$$\partial X = (X(0), AdL \cdot X(0)),$$

and hence $\partial X$ is isotropic with respect to the inner product (14).
Let us define the bracket of two (in general, non smooth) functionals \( \varphi_1, \varphi_2 \) by the formula
\[
\{ \varphi_1, \varphi_2 \} = \frac{1}{2} \int \left( \langle H X_1, X_2 \rangle - \langle H X_2, X_1 \rangle \right) dx + \langle \langle B \partial X_1, X_2 \rangle \rangle,
\]
where \( B \in \text{End}(g \oplus g) \). The bracket (16) is skew if \( B \) is skew with respect to the inner product (14) on \( g \oplus g \). Let us discuss the conditions to be imposed on \( B \) so as to make (16) a \textit{bona fide} Poisson bracket.

The necessary conditions on \( B \) are as follows:

(i) The boundary form should vanish on the diagonal subalgebra \( g^\delta \subset g \oplus g \). (Indeed, if \( \partial \varphi \in g^\delta \), the functional \( \varphi \) is smooth.)

(ii) ('Weak field approximation') The Poisson bracket vanishes identically for \( M = 1 \); its linearization at the unit element \( 1 \in G \) should coincide with the Lie–Poisson bracket of the Lie algebra \( g_R \).

The second condition is less obvious and deserves some comment. Observe first of all that if the potential \( L = 0 \), the monodromy is equal to identity. In this case the gradient of any functional \( \varphi^M \) is smooth (cf. the boundary condition (11)), and the Poisson bracket is identically zero (as before, the integrand in the formula for the Poisson bracket is a total derivative of a periodic function). Assume that there exists a Poisson bracket on \( G \) which is compatible with the monodromy map; in that case this bracket should, for consistency, also vanish at the unit element of \( G \). Moreover, if the potential \( L \) in the auxiliary linear equation (2) is close to zero, we may find the monodromy perturbatively, and this allows to compute the linearization of the bracket at \( M = 1 \). The result is quite obvious: the linearized bracket coincides with the Lie–Poisson bracket of the Lie algebra \( g_R \). If the \( r \)-matrix \( R \) is skew, the Poisson bracket on \( G \) satisfying this condition is obviously the Sklyanin bracket. It is natural to impose this condition in general case as well; the existence of such a Poisson bracket is of course nontrivial.

**Proposition 2** An operator \( B \) satisfying the above condition has the following form in block notation
\[
B = \begin{pmatrix} \alpha & \alpha + s \\ -\alpha + s & -\alpha \end{pmatrix},
\]
where \( s = \frac{1}{2}(r + r^*) \), \( \alpha \in \text{End}(g) \) is a skew symmetric operator.
We see in particular that the boundary form is needed in order to correctly reproduce the linearized bracket. Operator $\alpha$ is a free parameter which characterizes the interaction term in our bilinear form; further restrictions on $\alpha$ are imposed by the Jacobi identity.

**Proposition 3** Let the bracket $\{,\}$ be defined by formula (16) with $H$ given by (12) and the boundary form $B$ chosen as above. Then the bracket of two functionals $\varphi_1^M, \varphi_2^M$ (where as usual $\varphi_1^M, \varphi_2^M$ are smooth functions of the monodromy) is given by

$$\{\varphi_1^M, \varphi_2^M\} = \langle \langle R \partial X_1, \partial X_2 \rangle \rangle,$$

where

$$R = \begin{vmatrix} a + \alpha & \alpha + s \\ -\alpha + s & a - \alpha \end{vmatrix}, \quad a = \frac{1}{2} (R - R^*), \quad s = \frac{1}{2} (R + R^*),$$

and $X_i = \text{grad} \varphi_i^M$.

Let $R \in \text{End} (g \oplus g)$, define the bilinear map

$$[[R, R]] : \wedge^2 (g \oplus g) \rightarrow g \oplus g$$

by

$$[[R, R]] (X, Y) = [RX, RY] - R([RX, Y] + [X, RY]).$$

The inner product on $g$ allows to identify $[[R, R]]$ with an element of $\otimes^3 (g \oplus g)$; it is easy to see that if $R$ is skew, then actually $[[R, R]] \in \wedge^3 (g \oplus g)$.

**Proposition 4** The bracket (18) satisfies the Jacobi identity if and only if the element $[[R, R]] \in \wedge^3 (g \oplus g)$ is $\text{ad} (g \oplus g)$-invariant.

As usual (cf. [4]), it is convenient to replace this necessary and sufficient condition with the following sufficient one.

**Proposition 5** Assume that $R$ satisfies the modified classical Yang–Baxter equation

$$[RX, RY] - R([RX, Y] + [X, RY]) + [X, Y] = 0$$

for any $X, Y \in g \oplus g$. Then the bracket (18) satisfies the Jacobi identity.
It is useful to write down equation (20) in terms of the matrix coefficients of
\[ R = \begin{vmatrix} A & B \\ B^* & D \end{vmatrix}. \]

**Proposition 6**  
(i) Equation (20) is equivalent to the following relations
\[
\begin{align*}
[Au, Av] &= A([Au, v] + [u, Av]) - [u, v], \\
[Du, Dv] &= D([Du, v] + [u, Dv]) - [u, v], \\
[Bu, Bv] &= B([Du, v] + [u, Dv]), \\
[B^* u, B^* v] &= B^* ([Au, v] + [u, Av])
\end{align*}
\]
for any \( u, v \in g \).

(ii) If relations (21) are satisfied and moreover \( A + B = B^* + D \), then \( r = A + B \) satisfies the modified classical Yang–Baxter equation.

Relations (21) on the matrix coefficients of \( R \) were considered in [11], [12]; however, it passed unnoticed that they are equivalent to the modified CYBE for the square of \( g \) which reduces the classification of solutions to a standard problem.

Let us recall the fundamental classification theorem of Belavin and Drinfeld [2].

**Theorem 4**  
Let \( g \) be an affine Lie algebra, \( h \subset g \) its Cartan subalgebra, \( P \subset h^* \) the set of its simple roots. (i) To each solution of equation (20) on \( g \) one can assign a triple \( (\Gamma_1, \Gamma_2, \tau) \), where \( \Gamma_1, \Gamma_2 \subset P \) and \( \tau \) is an isometry \( \Gamma_1 \rightarrow \Gamma_2 \) such that \( \tau^k \alpha \notin \Gamma_1 \) for any \( \alpha \in \Gamma_1 \) and for sufficiently large \( k \) (expression \( \tau^k \alpha \) makes sense if \( \tau \alpha, \tau^2 \alpha, ..., \tau^{k-1} \alpha \in \Gamma_1 \cap \Gamma_2 \)). (ii) For each triple \( (\Gamma_1, \Gamma_2, \tau) \) the solutions are parametrized by tensors \( r \in h \otimes h \) such that
\[ r_{12} + r_{21} = t_0 \]
is a Casimir element in \( h \otimes h \) and for each \( \alpha \in \Gamma_1 \)
\[ (\tau \alpha \otimes id + id \otimes \alpha)r = 0. \]

The system of simple roots of \( g \oplus g \) is the union of two copies of \( P \) (we shall denote the second copy by \( P \)). Let us give three important examples of r-matrices on \( g \oplus g \) satisfying the additional condition \( A + B = B^* + D \).
(1) $\Gamma_1 = \Gamma_2 = \emptyset$; in this case $R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, where $r$ is the standard 'trigonometrical' $r$-matrix associated with $g$.

(2) Let $\Gamma_1 = P, \Gamma_2 = \tilde{P}$ and let $\tau$ be a natural isometry $P \rightarrow \tilde{P}$. The corresponding $r$-matrix has the form

$$R = \begin{pmatrix} r & r_+ \\ r_- & -r \end{pmatrix}$$

where $r$ is the same as above and $r_\pm = (r \pm id)$. The $r$-matrix (22) is the canonical $r$-matrix of the double of the Lie bialgebra $(g, g_r)$.

(3) Let $g = L(a)$ be the loop algebra of a semisimple Lie algebra $a$; let $\alpha$ be the root of $g$ which corresponds to the additional vertex of the extended Dynkin diagram of $a$. Put $\Gamma_1 = P \setminus \{\alpha\}, \Gamma_2 = \tilde{P} \setminus \{\tilde{\alpha}\}$ and let $\tau$ be a natural isometry $\Gamma_1 \rightarrow \Gamma_2$. Let us denote by $r^0$ the standard $r$-matrix on $a \subset L(a)$. Then

$$R = \begin{pmatrix} r + r^0 & r^0 \\ r^0 & r - r^0 \end{pmatrix}$$

Observe that in case (1) the bracket (18) is the standard Sklyanin bracket; in case (2) it is the dual bracket on $G$ (cf. [5] and the discussion in the next Section below); case (3) is an interpolation between the first two. The symmetric part of the linearized bracket is zero in case (1), in case (2) we have $s = id$, and in case (3) we have $s = P^0$, where $P^0$ is the projection operator onto the subalgebra of constant loops $a \subset L(a)$.

### 4 Conclusion. A few Words on Symmetry Breaking

It is probably worth saying a few words on the resulting breakdown of global gauge symmetry and 'spontaneous quantization' of the global gauge group. Let $G = C^\infty(S^1, G)$ be the loop group of $G$. We may identify $G$ with the subgroup of constant loops. Let $G_0$ be the subgroup of $G$ consisting of loops satisfying $g(0) = e$. Clearly, $G_0$ is normal in $G$ and $G_0 / G = G$. The group $G$ acts on the space of the first order differential operators by conjugations, and this induces the gauge action of $G$ on the phase space.
\( \mathcal{M} \). Let \( G \times G \rightarrow G \) be the action of \( G \) on itself by conjugations. Clearly, we get a commutative diagram

\[
\begin{array}{ccc}
G \times \mathcal{M} & \rightarrow & \mathcal{M} \\
\pi \times M & \downarrow & \downarrow M \\
G \times G & \rightarrow & G
\end{array}
\]

Suppose now that \( G \) is a finite-dimensional simple Lie group and the \( r \)-matrix is chosen as in Example 2 of the previous section, i.e., it is given by (22). The Poisson bracket for the monodromy matrices we get in this way is essentially that of the dual group \( G^* \) which is identified with \( G \) via the canonical factorization map. Let us equip the group \( G \) of global gauge transformations with the standard Sklyanin bracket. According to the well-known results of the Poisson Lie groups theory [5], there is a canonical Poisson action \( G \times G^* \rightarrow G^* \) called dressing transformations. As explained in [5], if we identify \( G^* \) with \( G \), dressing transformations correspond to conjugations in \( G \); thus in order to maintain the gauge covariance of our monodromy map (which should, we recall, be a morphism in the category of Poisson manifolds) we must equip the global gauge group with a nontrivial Poisson bracket. By contrast, it is consistent to assume that the subgroup \( G_0 \) remains classical, i.e. carries a trivial Poisson bracket. The implications for quantization are obvious: to preserve the gauge covariance on the quantum level we have to assume that the global gauge group becomes quantum (while the subgroup \( G_0 \) remains classical. If we replace functions on the circle with the more physical case of functions on the line, the same discussion will apply to the subgroup of rapidly decreasing gauge transformations.

In applications to completely integrable systems we usually need a Lax operator with a spectral parameter, i.e. our Lie algebra \( \mathfrak{g} \) is a loop algebra in auxiliary parameter \( \lambda \). The model situation here is illustrated by Example 3 of the previous section. It is natural to demand in this case the Poisson covariance of the monodromy map with respect to the global gauge group consisting of functions which do not depend neither on \( x \) nor on \( \lambda \). This is again made consistent with the regularized Poisson bracket for the monodromy provided that we equip the global gauge group with the Sklyanin bracket which corresponds to the constant \( r \)-matrix \( r^0 \).

Let us outline our conclusions. If the input \( r \)-matrix of the model is not skew, the Poisson bracket for the monodromy requires regularization; the regularized Poisson bracket is determined by a solution of the modified
classical Yang–Baxter equation on the square of the Lie algebra $g$. All such solutions may be completely classified. Applications to concrete examples (in particular, to lattice systems and difference Lax equations) will be considered in a separate paper. (A special case corresponding to example (2) above was studied in [9],[10].)

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