On the geometry of the classical Rabi problem

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We investigate the motion of a classical spin precessing around a periodic magnetic field using Floquet theory as well as elementary differential geometry and considering a couple of examples. Under certain conditions the rôle of spin and magnetic field can be interchanged, leading to the notion of “duality of loops” on the Bloch sphere.

I. INTRODUCTION

The Rabi problem usually refers to the response of an atom to an applied harmonic electric field, with an applied frequency very close to the atom’s natural frequency [1], [2]. The corresponding classical model to describe this problem would be, for example, a driven, damped harmonic oscillator. In this paper, however, we will understand by frequency very close to the atom’s natural frequency [1], [2]. The corresponding classical model to describe this problem would be, for example, a driven, damped harmonic oscillator. In this paper, however, we will understand by a different approach: The atom can be approximated by a two-level system such that its “classical Rabi problem” a different approach: The atom can be approximated by a two-level system such that its semi-classical Hamiltonian assumes the form of Zeeman term in an $s = 1/2$ spin system:

$$H = \omega_0 S_z + h_1(t) S_x + h_2(t) S_y + h_3(t) S_z,$$

where the $S_x, S_y, S_z$ are the $s = 1/2$ spin operators. If $\psi(t)$ is a solution of the corresponding Schrödinger equation

$$i\hbar \frac{d}{dt} \psi(t) = H \psi(t),$$

then the projector $P(t) = |\psi(t)\rangle\langle \psi(t)|$ can be expanded as a linear combination of the spin operators:

$$P(t) = \frac{1}{2} \mathbb{I} + s_1(t) S_x + s_2(t) S_y + s_3(t) S_z.$$

It follows that $s(t) \equiv (s_1(t), s_2(t), s_3(t))^\top$ will be a unit vector that obeys the same equation of motion

$$\frac{d}{dt} s(t) = \mathbf{h}(t) \times s(t)$$

as a classical magnetic moment performing a Larmor precession around the time dependent periodic magnetic field $\mathbf{h}(t) \equiv (h_1(t), h_2(t), h_3(t) + \omega_0)^\top$. The study of this equation will be called the “classical Rabi problem” in what follows.

According to the preceding remarks it seems that the solutions of [1] provide the information about the corresponding solutions of [2] up to a (time-depending) phase factor. But it can be shown [3] that also this phase factor can be reconstructed from periodic solutions of [1] by means of certain integrals. Here we encounter the rare case where a quantum problem and the corresponding classical problem are essentially equivalent. This endows the classical Rabi problem with additional importance concerning quantum applications.

The differential equation [3] can be explicitly solved only in a few cases of physical interest, the most prominent one being a constant field superimposed by a monochromatic, circularly polarized field perpendicular to the constant one [1]. The analogous problem with a linearly polarized field component is solvable in terms of confluent Heun functions, see [4], [5] and [12] for the corresponding $s = 1/2$ Schrödinger equation. In this paper we will shift the problem of finding solutions of [1] to the study of geometric relations between such solutions and to the interplay between Floquet theory, differential geometry of the unit sphere and duality of loops. Not all results will be new, but we will provide new proofs that only use properties of solutions of the classical Rabi problem that are easier to visualize and do not resort to the mathematics of the underlying Schrödinger equation.

The structure of the paper will be as follows. After general remarks on the classical Rabi problem in section II we will in the main part (section III) specialize to periodic driving. The existence of periodic solutions of [1] is shown in subsection III A. This result can be re-phrased in term of Floquet theory, see subsection III B, where the central notion of the “quasienergy” is introduced. The “geometric part” of the quasienergy can be written as an integral involving the geodesic curvature of the periodic solution of [1] and, by virtue of the theorem of Gauss-Bonnet, will be related to the solid angle enclosed by this solution, see subsection III C. Alternatively, the geometric part of the quasienergy can be related to the phase shift of a second solution of [1] that is orthogonal to the periodic solution if the magnetic field is replaced by an equivalent “dual field”, see subsection III D. This confirms the well-known connection between the Rabi...
problem and the geometric phase introduced by M. V. Berry and others. The mentioned duality between spin loops and magnetic loops is further elaborated in subsection III E. It can be used to generate new classes of solutions, see subsection III F. Finally, we check the agreement between the integral representations for the quasienergy obtained in this paper and previous formulations, see subsection III G, and close with a summary.

II. GENERALITIES

We consider the equation of motion
\[ \frac{d}{d\tau} s(\tau) = h(\tau) \times s(\tau) \] (5)
describing the time evolution of a classical spin vector \( s(\tau) \) due to the precession around the time-dependent magnetic field
\[ h(\tau) = \begin{pmatrix} h_1(\tau) \\ h_2(\tau) \\ h_3(\tau) \end{pmatrix} , \] (6)
measured in units of (Larmor) frequency. In the following we will use \( \tau \) for the time variable and \( \sigma \) for another arbitrary parameter that is a function of \( \tau \). Differentiation w. r. t. \( \tau \) will be denoted by a dot. The arc length parameter of certain curves described by the spin vector or by the magnetic field vector will be denoted by \( s \) or \( r \), resp.

Since the scalar product \( s \cdot s = \|s\|^2 \) is conserved under (5) one usually assumes that
\[ \|s(\tau)\| = 1 , \] (7)
that is, \( s(\tau) \) is moving on the (unit) Bloch sphere.

Interestingly, the inverse problem of finding \( h(\tau) \) if the parametrized curve \( s(\tau) \) on the Bloch sphere is given, has the elementary solution
\[ h(\tau) = s(\tau) \times \dot{s}(\tau) + \lambda(\tau)s(\tau) , \] (8)
where \( \lambda(\tau) \) is an arbitrary smooth function. In order to prove (8) we consider (5) as an inhomogeneous, linear equation for the unknown \( h(\tau) \) if \( s(\tau) \) and \( \dot{s}(\tau) \) are given for any time \( \tau \) such that \( s(\tau) \cdot \dot{s}(\tau) = 0 \). A special solution of (5) is given by
\[ h^{(0)}(\tau) = s(\tau) \times \dot{s}(\tau) , \] (9)
since
\[ h^{(0)} \times s = (s \times \dot{s}) \times s = s \cdot \dot{s} - s \cdot \dot{s} = 0 \] (10)
\[ = \dot{s} . \] (11)

The homogenous equation corresponding to (5) reads
\[ 0 = h(\tau) \times s(\tau) \] (13)
and has the general solution
\[ h(\tau) = \lambda(\tau)s(\tau) . \] (14)

Adding \( h^{(0)}(\tau) \) and (14) yields the general solution (8) of the inverse problem (5).

Next we will consider rather arbitrary parametrizations of the curves described by \( s \) and \( h \) that are given by (locally) smooth \( 1 : 1 \) functions \( \tau \mapsto \sigma(\tau) \) the (local) inverse denoted in a somewhat sloppy but usual way by \( \sigma \mapsto \tau(\sigma) \). Conceptually, this means that we pass from the parametric curve given by \( \tau \mapsto s(\tau) \) to the curve \( S \) on the unit Bloch sphere without singling out a particular parametrization. \( S \) can be defined as the image of the map
FIG. 1: Illustration of the construction described in the text. The (blue) spin vector $s \in S$ and its tangent vector $s'$ define the plane $s'\perp S$ indicated by green lines. The (red) field vector $h$ is found as the intersection of $s'\perp S$ and $H$. It must be proportional to $\tilde{h} = s \times s' + \lambda s$ which yields the original $\tau$-parametrization by means of (22) and (23).

$s : \mathbb{R} \to S^2$, $\tau \mapsto s(\tau)$. Analogously, $H$ will be the curve swept by the magnetic field without assuming any special parametrization.

The equation of motion (5) is transformed under the local parameter change $\tau \mapsto \sigma(\tau)$ as follows:

$$\frac{d}{d\sigma} s(\tau(\sigma)) = \frac{d\tau}{d\sigma} \frac{d}{d\tau} s(\tau(\sigma)) = \frac{d\tau}{d\sigma} h(\tau(\sigma)) \times s(\tau(\sigma)) \quad (15)$$

This means that the transformed equation of motion has the form

$$\frac{d}{d\sigma} s(\sigma) = \tilde{h}(\sigma) \times s(\sigma) \quad (16)$$

with a modified magnetic field $\tilde{h}(\sigma) = \frac{d\tau}{d\sigma} h(\tau(\sigma))$ that has the same direction as the original field but possibly a different length. In case of $\frac{d\tau}{d\sigma} < 0$, $\tilde{h}(\sigma)$ may even point into the opposite direction of $h(\tau(\sigma))$ but it is still "projectively equivalent" to $h(\tau(\sigma))$.

As an application of the preceding equations we consider the constant magnetic field

$$h = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} \quad (17)$$
and the corresponding elementary solution of (5) describing a precessing spin with constant angular velocity $F$ and forming an angle $\theta = \arccos z$, $-1 < z < 1$ with the magnetic field:

$$s(\tau) = \begin{pmatrix} \sqrt{1 - z^2} \cos F\tau \\ \sqrt{1 - z^2} \sin F\tau \\ z \end{pmatrix}.$$  

(18)

The parameter change

$$\tau = \frac{1}{\omega} \sin \omega \sigma \Rightarrow \frac{d\tau}{d\sigma} = \cos \omega \sigma ,$$

(19)

gives rise to a modified field

$$\tilde{h}(\sigma) = \frac{d\tau}{d\sigma} h = \begin{pmatrix} 0 \\ 0 \\ F \cos \omega \sigma \end{pmatrix} ,$$

(20)

such that the spin vector $s$ can be written as a function of the new parameter $\sigma$, setting $f \equiv \frac{F}{\omega}$,

$$s(\sigma) = \begin{pmatrix} \sqrt{1 - z^2} \cos (f \sin \omega \sigma) \\ \sqrt{1 - z^2} \sin (f \sin \omega \sigma) \\ z \end{pmatrix} ,$$

(21)

and satisfies the transformed equation of motion (10), as can be easily confirmed by direct computation. This solution has been used to obtain results for the case of a monochromatic, linearly polarized magnetic field in the limit of a vanishing constant field component, see, e. g., [3].

We note the curiosity that the system $(S, H)$ of two curves that is derived from a solution of (5) in the way described above can be considered as a kind of “clock” in the sense that it allows, at least locally, the reconstruction of the original parametrization $\tau \mapsto s(\tau)$ and $\tau \mapsto h(\tau)$. To show this we start with an arbitrary local parametrization $s(\sigma) \in S$, denoting the derivative $\frac{ds}{d\sigma}$ by a prime $'$, and define $h(\sigma)$ as the intersection of the two-dimensional subspace $s'^\perp$ of all vectors orthogonal to the vector $s' = \frac{ds}{d\sigma}(\sigma)$, and $H$. This is sensible since, according to (5), $h(\tau)$ must be orthogonal to $s(\tau)$ and hence to $s'$, the tangent to the curve $S$ being independent of the parametrization. We assume for simplicity that the intersection of $s'^\perp$ and $H$ is unique and leave it to the reader to consider the more general case of multiple intersections. Hence we have obtained a kind of (local) synchronization between the two curves $S$ and $H$, see Figure 1. Then we consider the equation of motion (10) that necessarily holds and employ the solution (8) of the inverse problem. This entails

$$\tilde{h} = s \times s' + \lambda(s)s \frac{d\tau}{d\sigma} h ,$$

(22)

Taking scalar products of (22) with $s(\sigma)$ and $h(\sigma)$ yields $\frac{ds}{d\sigma}$ uniquely as

$$\frac{ds}{d\sigma} = \frac{(s \times s') \cdot h}{h \cdot h - (s \cdot h)^2} ,$$

(23)

neglecting special cases. Finally, $\tau(\sigma)$ is obtained as the integral $\tau(\sigma) = \int \frac{d\sigma}{d\tau} d\sigma$.

The general solution of (5) can be obtained as a linear superposition of three “fundamental solutions” $s^{(i)}(\tau)$, $i = 1, 2, 3$. We will slightly generalize the initial point of time to $\tau = \tau_0$. Since scalar products are invariant under time evolution according to (5) it follows that the fundamental solutions will form a right-handed orthonormal frame for all $\tau \in \mathbb{R}$, i. e.,

$$R(\tau, \tau_0) = \begin{pmatrix} s^{(1)}(\tau) \\ s^{(2)}(\tau) \\ s^{(3)}(\tau) \end{pmatrix}$$

(24)

will be a rotational matrix with unit determinant, in symbols, $R(\tau, \tau_0) \in SO(3)$. The three equations of motion (5) for the $s^{(i)}(\tau)$, $i = 1, 2, 3$ can be compactly written in matrix form as

$$\frac{d}{d\tau} R(\tau, \tau_0) = H(\tau) R(\tau, \tau_0) ,$$

(25)
where $H(\tau)$ is the real, antisymmetric $3 \times 3$-matrix

$$H(\tau) = \begin{pmatrix} 0 & -h_3(\tau) & h_2(\tau) \\ h_3(\tau) & 0 & -h_1(\tau) \\ -h_2(\tau) & h_1(\tau) & 0 \end{pmatrix},$$

(26)

and the $h_i(\tau)$ are the components of the magnetic field according to (6). Moreover, we choose the initial values $s^{(i)}(\tau_0)$ as the corresponding unit standard vectors such that

$$R(\tau_0, \tau_0) \equiv \begin{pmatrix} s^{(1)}(\tau_0), s^{(2)}(\tau_0), s^{(3)}(\tau_0) \end{pmatrix} = \mathbb{I}.$$  

(27)

The general solution $s(\tau)$ of (5) with initial value $s(\tau_0)$ can be obtained from the three fundamental solutions by means of $s(\tau) = R(\tau, \tau_0) s(\tau_0)$. More generally, let $t \mapsto \tilde{R}(\tau, \tau_0) \in SO(3)$ be a one-parameter family of rotational matrices satisfying the differential equation

$$\frac{d}{d\tau} \tilde{R}(\tau, \tau_0) = H(\tau) \tilde{R}(\tau, \tau_0)$$

(28)

with initial condition

$$\tilde{R}(\tau_0, \tau_0) = S \in SO(3),$$

(29)

then it follows that

$$\tilde{R}(\tau, \tau_0) = R(\tau, \tau_0) S$$

(30)

for all $\tau \in \mathbb{R}$ since both sides of (30) satisfy the same differential equation with the same initial value.

For the sake of reference we recapitulate the following well-known method of transforming (28) into a rotating frame. Set $\tau_0 = 0$ and let $U(\tau) \in SO(3)$ be such that

$$\dot{U} = KU,$$

(31)

where $K(\tau)$ is an anti-symmetric $3 \times 3$-matrix, moreover

$$\tilde{R} \equiv U R.$$  

(32)

$\tilde{R}$ can be thought to consist of the columns of $R$ transformed to a frame rotating with $U^\top$. We obtain for the transformed equation of motion

$$\frac{d}{d\tau} \tilde{R} = \dot{U} R + U \dot{R} \equiv K U R + U H R = (K + U H U^\top) \tilde{R} \equiv \tilde{H} \tilde{R},$$

(33)

where $\tilde{H}$ will be again an anti-symmetric $3 \times 3$-matrix representing the transformed magnetic field.

III. MAIN PART

For the remainder of the paper we consider the case where the three components $h_i(\tau)$, $i = 1, 2, 3$ of the magnetic field $h(\tau)$ are $T \equiv \frac{2\pi}{\omega}$-periodic functions of $\tau$. For example, periodicity holds if the magnetic field is generated by a plane electro-magnetic wave. It does not follow that all solutions $s(\tau)$ of the equation of motion (5) will also be $T$-periodic. However, for suitable initial conditions, at least one periodic solution $s(\tau)$ always exists. Note that, in general, this does not hold for the underlying $s = 1/2$ Schrödinger equation.

A. Periodic solutions

1. Proof and further remarks

In order to prove the existence of a periodic solution, we consider the fundamental matrix solution $R(\tau)$ according to (24) with initial value $R(0) = \mathbb{I}$. Its value after one period will be the rotational matrix $R(T) \in SO(3)$. As any
rotational matrix with unit determinant it will be a rotation about an axis with an angle $\delta$. Accordingly, $R(T)$ will have the eigenvalues $1, e^{\pm i \delta}$, corresponding to a real eigenvector $a$ and two generally complex eigenvectors, resp. The eigenvector $a$ satisfying

$$R(T) a = a$$

(34)

represents the axis of rotation and, after normalization $\|a\| = 1$, will be unique up to a sign. The angle of rotation is also only defined up to a sign: If $\delta$ is the angle of rotation about $a$ then $-\delta$ will be the angle of rotation about $-a$. Since the trace of $R(T)$ is the sum of its eigenvalues it follows that

$$\text{Tr} \ R(T) = 1 + e^{i \delta} + e^{-i \delta} = 1 + 2 \cos \delta ,$$

(35)

and hence $\pm \delta$ can immediately be obtained from $R(T)$. In the special case where $R(T) = I$ it follows that $\pm \delta = 0$ and $R(T)$ has a three-dimensional degenerate eigenspace.

For later reference we note that $R(T)$ can be written as the matrix exponential

$$R(T) = e^{\delta A} ,$$

(36)

where, analogously to (26),

$$A \equiv \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} ,$$

(37)

and the $a_i$ are the components of the normalized eigenvector $a$ according to (34). Note that the eigenvalues of $A$ are of the form $0, \pm i$.

Now we consider the special solution $s(\tau)$ of (31) with the initial value given by the real eigenvector of $R(T)$, namely $s(0) = a$. According to the remark after (26) in Section II this solution can be written as $s(\tau) = R(\tau) a$ for all $\tau \in \mathbb{R}$. It follows that

$$s(T) = R(T) a \overset{(34)}{=} a = s(0) ,$$

(38)

and, more generally,

$$s(\tau + T) = s(\tau) ,$$

(39)

for all $\tau \in \mathbb{R}$. Hence $s(\tau)$ is the $T$-periodic solution we are looking for.

In order to prove (38) we restore the notation of $\tau = \tau_0$ for the initial point of time and write, for example, $R(T + \tau_0, \tau_0)$ instead of $R(T)$. We consider the one-parameter family $\tau \mapsto R(T + \tau, \tau_0)$ of rotational matrices with initial value $R(\tau_0 + T, \tau_0)$. It satisfies the differential equation

$$\frac{d}{d\tau} R(T + \tau, \tau_0) \overset{(26)}{=} H(T + \tau) R(T + \tau, \tau_0) = H(\tau) R(T + \tau_0, \tau_0) ,$$

(40)

where we have used that $H(T)$ is $T$-periodic. Applying (39) and the preceding remarks we conclude

$$R(T + \tau, \tau_0) = R(\tau, \tau_0) R(\tau_0 + T, \tau_0) .$$

(41)

Then it follows that

$$s(\tau + T) = R(T + \tau, \tau_0) s(\tau_0) \overset{(41)}{=} R(\tau, \tau_0) R(\tau_0 + T, \tau_0) a \overset{(34)}{=} R(\tau, \tau_0) a = R(\tau, \tau_0) s(0) = s(\tau) ,$$

(42)

thereby completing the proof of (39).

In general, there will only be two $T$-periodic solutions of unit length, namely $s(\tau)$ and $-s(\tau)$. Only in the special case of $R(T + \tau_0, \tau_0) = I$ every solution $\tilde{s}(\tau)$ will be $T$-periodic since every unit vector $a = \tilde{s}(0)$ will satisfy (34) and the preceding arguments may be repeated for $\tilde{s}(\tau)$.

To underline the latter statement let $S(\tau) \equiv \{ e^{(1)}(\tau), e^{(2)}(\tau), e^{(3)}(\tau) \}$ be a solution of $\dot{S}(\tau) = H(\tau) S(\tau)$ such that $S(0)$ is a right-handed orthonormal frame and $e^{(1)}(\tau) = s(\tau)$ for all $\tau \in \mathbb{R}$, i. e., $e^{(1)}(\tau)$ is the periodic solution considered above. It follows that $S(T) = (a, e^{(2)}(T), e^{(3)}(T)) = R(T) S(0)$ will also be a right-handed orthonormal frame and hence $e^{(2)}(T)$ and $e^{(3)}(T)$ will lie in the plane $a^\perp$ orthogonal to $a$. But since $R(T)$ is a rotation about the
axis through \( \mathbf{a} \) with an angle \( \delta \), the vectors \( \mathbf{e}^{(2)}(T) \) and \( \mathbf{e}^{(3)}(T) \) will, in general, be different from \( \mathbf{e}^{(2)}(0) \) and \( \mathbf{e}^{(3)}(0) \) resp. More precisely,

\[
\mathbf{e}^{(2)}(T) = \cos(\delta) \mathbf{e}^{(2)}(0) + \sin(\delta) \mathbf{e}^{(3)}(0),
\]

\[
\mathbf{e}^{(3)}(T) = -\sin(\delta) \mathbf{e}^{(2)}(0) + \cos(\delta) \mathbf{e}^{(3)}(0),
\]

and hence \( \mathbf{e}^{(1)}(\tau) \) and \( \mathbf{e}^{(2)}(\tau) \) will not be \( T \)-periodic except for \( \delta = 0 \).

As a preparation for the next subsection we again consider the fundamental solution \( R(\tau, \tau_0) \) satisfying \( R(T + \tau_0, \tau_0) = e^{\delta A} \) and define

\[
\epsilon \equiv \frac{\delta}{T}.
\]

Then we proceed by defining \( P(\tau, \tau_0) \in SO(3) \) as

\[
P(\tau, \tau_0) \equiv R(\tau, \tau_0) e^{-\epsilon (\tau - \tau_0) A},
\]

and show that it will be \( T \)-periodic:

\[
P(\tau + T, \tau_0) = R(\tau + T, \tau_0) e^{-\epsilon (\tau - \tau_0) A} = R(\tau, \tau_0) e^{-\epsilon (\tau - \tau_0) A} P(\tau, \tau_0),
\]

2. Example 1

As an example we consider the circularly polarized Rabi problem where

\[
\mathbf{h}(\tau) = \begin{pmatrix} F \cos(\omega \tau) \\ F \sin(\omega \tau) \\ \omega_0 \end{pmatrix}.
\]

Let \( \mathbf{H}(\tau) \) denote the corresponding anti-symmetric matrix according to (26). We will transform the equation of motion into a rotating frame as described in Section II and set

\[
\mathbf{U}(\tau) = \begin{pmatrix} \cos(\tau \omega) & \sin(\tau \omega) & 0 \\ -\sin(\tau \omega) & \cos(\tau \omega) & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

such that

\[
\dot{\mathbf{U}} = K \mathbf{U},
\]

where

\[
K = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The transformed equation of motion corresponding to (23) will be written as

\[
\frac{d}{d\tau} \mathbf{\tilde{S}} = \mathbf{\tilde{H}} \mathbf{\tilde{S}},
\]

and \( \mathbf{\tilde{H}} \) is obtained after a short calculation as

\[
\mathbf{\tilde{H}} = \begin{pmatrix} 0 & \omega - \omega_0 & 0 \\ \omega_0 - \omega & 0 & -F \\ 0 & F & 0 \end{pmatrix},
\]
no longer depending on $\tau$. Hence (53) can be immediately solved with the result
\[
\tilde{S} = \begin{pmatrix}
\frac{F}{\Omega} & \left(1 - \frac{\omega}{\Omega}\right) \sin(\tau \Omega) & \left(1 - \frac{\omega}{\Omega}\right) \cos(\tau \Omega) \\
0 & \frac{\Omega}{\Omega} & -\sin(\tau \Omega) \\
\omega_0 - \omega & \frac{\Omega}{\Omega} & \frac{\Omega}{\Omega}
\end{pmatrix},
\]
where
\[
\Omega = \Omega_{\text{Rabi}} \equiv \sqrt{F^2 + (\omega_0 - \omega)^2},
\]
denotes the so-called Rabi frequency. The remaining reverse transformations yield
\[
S = U^T \tilde{S},
\]
and, in order to obtain a fundamental solution,
\[
R = SS(0)^{-1}.
\]
The result is of the somewhat complicated form
\[
R(\tau) = \begin{pmatrix}
\cos(\tau \Omega)(F^2 + (\omega - \omega_0)^2 \cos(\tau \Omega)) + (\omega_0 - \omega) \Omega \sin(\tau \Omega) \sin(\tau \Omega) \\
(\omega_0 - \omega) \Omega \sin(\tau \Omega) \cos(\tau \Omega) - \sin(\tau \Omega) \cos(\tau \Omega) \\
F\Omega \sin(\tau \Omega) \sin(\tau \Omega) + 2(\omega_0 - \omega) \cos(\tau \Omega) \sin^2(\tau \Omega)
\end{pmatrix},
\]
After one period the fundamental solution reads
\[
R(T) = \begin{pmatrix}
F^2 + (\omega - \omega_0)^2 \cos(2\tau \Omega) & (\omega_0 - \omega) \sin(2\tau \Omega) & 2F(\omega_0 - \omega) \sin(2\tau \Omega) \\
(\omega_0 - \omega) \sin(\tau \Omega) \cos(2\tau \Omega) & \cos(2\tau \Omega) & \frac{F\sin(2\tau \Omega)}{\Omega} \\
2F(\omega_0 - \omega) \sin^2(\tau \Omega) & -\frac{F\sin(2\tau \Omega)}{\Omega} & \frac{F^2 \cos(2\tau \Omega) + (\omega - \omega_0)^2}{\Omega}
\end{pmatrix}.
\]
This matrix has the eigenvalues $1, e^{\pm i\delta}$ with
\[
\delta = \frac{2\pi \Omega}{\omega},
\]
and the normalized eigenvector
\[
a = \begin{pmatrix}
\frac{\omega}{\Omega} \\
0 \\
\omega_0 - \omega
\end{pmatrix},
\]
corresponding to the eigenvalue 1. This implies the well-known result for the quasienergy
\[
\epsilon = \frac{\delta}{T} = \Omega \mod \omega.
\]
We consider the right-handed orthonormal system with $a$ as its first vector and write this system as the columns of a rotational matrix
\[
A = \frac{1}{\Omega} \begin{pmatrix}
F & 0 & \omega - \omega_0 \\
0 & \Omega & 0 \\
\omega_0 - \omega & 0 & F
\end{pmatrix},
\]
For $A$ as the initial value of the differential equation (25) the solution reads, according to the argument in connection with (23), (29), (30):
\[
S(\tau) = R(\tau) A,
\]
\[
= \frac{1}{\Omega} \begin{pmatrix}
F \cos(\tau \Omega) & (\omega - \omega_0) \cos(\tau \Omega) + \Omega \sin(\tau \Omega) \cos(\tau \Omega) & \Omega \sin(\tau \Omega) \sin(\tau \Omega) + (\omega - \omega_0) \cos(\tau \Omega) \cos(\tau \Omega) \\
F \sin(\tau \Omega) & (\omega - \omega_0) \sin(\tau \Omega) \sin(\tau \Omega) + \Omega \cos(\tau \Omega) \cos(\tau \Omega) & \Omega \sin(\tau \Omega) \sin(\tau \Omega) + (\omega - \omega_0) \cos(\tau \Omega) \sin(\tau \Omega) \\
\omega_0 - \omega & \omega_0 - \omega & F \cos(\tau \Omega)
\end{pmatrix}.
\]
Its first column represent the $T$-periodic solution of (25), unique up to a sign, whereas the second and third columns are not periodic but after one period will advance by the angle $\delta = T \Omega$. 


B. Floquet theory

It will be instructive to rephrase the results of the last subsection in terms of Floquet theory, see [4] or [5] for a more recent reference. Floquet theory deals with linear differential equations of the form (25) with a $T$-periodic matrix function $H(\tau)$ but for arbitrary finite dimensions and real or complex matrix functions $R(\tau, \tau_0)$. Its main result is that the fundamental solution of the differential equation can be written as the product of a $T$-periodic matrix function and a special exponential matrix function of $t$. In our case of the differential equation (25) this means that the fundamental solution $R(\tau, \tau_0)$ with initial condition $R(\tau_0, \tau_0) = I$ can be cast in the “Floquet normal form”

$$R(\tau, \tau_0) = P(\tau, \tau_0) e^{\epsilon (\tau - \tau_0) A} ,$$

(67)

such that $\tau \mapsto P(\tau, \tau_0) \in SO(3)$ is $T$-periodic, $\epsilon$ is a real number and $A$ a real anti-symmetric $3 \times 3$-matrix.

$\epsilon$ is called a “Floquet exponent” or, in physical applications, also a “quasienergy”. For $\tau = \tau_0$ we have $I = R(\tau_0, \tau_0) = P(\tau_0, \tau_0)$. For $\tau - \tau_0 = T$ it follows that

$$R(\tau_0 + T, \tau_0) = P(\tau_0 + T, \tau_0) e^{\epsilon T A} = P(\tau_0, \tau_0) e^{\epsilon T A} \equiv e^{\delta A} = e^{\delta A} .$$

(68)

Recall that we have defined $A, \delta$ and $P(\tau, \tau_0)$ in accordance with these requirements in subsection III A, and thus we have proven the main result of Floquet theory for the special case of the classical Rabi problem.

The quasienergy $\epsilon$ is only defined up to integer multiples of $\omega$ because, for $n \in \mathbb{Z}$,

$$R(\tau, \tau_0) = \left( P(\tau, \tau_0) e^{-n \omega (\tau - \tau_0) A} \right) \left( e^{(\epsilon + n \omega) (\tau - \tau_0) A} \right)$$

(69)

would also be a product of a periodic and an exponential matrix function and hence also of the Floquet normal form. Here we have used the fact that the eigenvalues of $A$ are $0, \pm i$. Similarly, the replacement $a \mapsto -a$ and hence $A \mapsto -A$ shows that together with $\epsilon$, $-\epsilon$ is also a possible quasienergy. Taking into account the non-uniqueness of the quasienergy we also consider the equivalence class

$$[\epsilon] \equiv \{ \epsilon + n \omega | n \in \mathbb{Z} \} .$$

(70)

In applications to the Schrödinger equation the operator $e^{\epsilon \tau (\tau - \tau_0) A}$ is often decomposed in terms of its eigenvalues and eigenvectors. In the context of the classical Rabi problem we will only consider its real eigenvector $a$ corresponding to the eigenvalue $e^{\epsilon (\tau - \tau_0)}$ and not the other two complex eigenvectors and eigenvalues since they have no direct geometric interpretation.

C. Quasienergy I

It has been shown [3] that the quasienergy of the $s = 1/2$ Schrödinger equation with a periodic magnetic field can be expressed in terms of integrals using the periodic solution of the analogous classical Rabi problem. Here we will re-derive the analogous result for the classical Rabi problem without employing the reference to the Schrödinger equation, solely by using the periodic solution $s(\tau)$ considered in subsection III A.

To this end we consider the time-dependent right-handed orthonormal frame, shortly called “e-frame”, defined by

$$e^{(1)}(\tau) = s(\tau),$$

(71)

$$e^{(2)}(\tau) = \frac{s(\tau)}{\|s(\tau)\|},$$

(72)

$$e^{(3)}(\tau) = e^{(1)}(\tau) \times e^{(2)}(\tau).$$

(73)

Further, let

$$S(\tau) = \left( s^{(1)}(\tau), s^{(2)}(\tau), s^{(3)}(\tau) \right) \in SO(3)$$

(74)

be a solution of (25) with initial conditions

$$s^{(i)}(0) = e^{(i)}(0), \quad i = 1, 2, 3 ,$$

(75)
and hence
\[ s^{(1)}(\tau) = e^{(1)}(\tau) = s(\tau) \]  
for all \( \tau \in \mathbb{R} \). It follows that the other two components of \( S(\tau) \) can be expanded w. r. t. the e-frame in the form
\[ s^{(2)}(\tau) = \cos(\alpha(\tau)) e^{(2)}(\tau) + \sin(\alpha(\tau)) e^{(3)}(\tau), \]  
(77)
\[ s^{(3)}(\tau) = -\sin(\alpha(\tau)) e^{(2)}(\tau) + \cos(\alpha(\tau)) e^{(3)}(\tau), \]  
(78)
where \( \alpha(\tau) \) is a smooth function given by an integral that we will derive below. We use the abbreviation
\[ v(\tau)^2 \equiv \dot{s}(\tau) \cdot \dot{s}(\tau) \Rightarrow v \dot{v} = \ddot{s} \cdot \ddot{s}, \]  
(79)
and expand the magnetic field w. r. t. the e-frame:
\[ h(\tau) = \sum_{i=1}^{3} k_i(\tau) e^{(i)}(\tau). \]  
(80)
The equation
\[ \dot{s} = v e^{(2)} = h \times s = \left( k_1 e^{(1)} + k_2 e^{(2)} + k_3 e^{(3)} \right) \times e^{(1)} = -k_2 e^{(3)} + k_3 e^{(2)} \]  
(81)
immediately implies
\[ k_1 = h \cdot s, \quad k_2 = 0, \quad k_3 = v. \]  
(82)
We will also expand \( \ddot{s} \) w. r. t. the e-frame. First, we obtain
\[ \dot{s} \cdot s = 0 \Rightarrow \dot{s} \cdot \ddot{s} + \ddot{s} \cdot s = 0 \Rightarrow \ddot{s} \cdot s = -v^2. \]  
(83)
Second,
\[ \ddot{s} \cdot e^{(3)} = \ddot{s} \cdot \left( e^{(1)} \times e^{(2)} \right) = \ddot{s} \cdot \left( s \times \frac{1}{v} \ddot{s} \right) = \frac{1}{v} s \cdot (\ddot{s} \times \ddot{s}) \equiv g \frac{\ddot{v}}{v}, \]  
(84)
where we have abbreviated the triple product \( s \cdot (\ddot{s} \times \ddot{s}) \) by \( g \). Together with (79) the last two equations yield
\[ \ddot{s} = -v^2 e^{(1)} + \dot{v} e^{(2)} + g \frac{\ddot{v}}{v} e^{(3)}. \]  
(85)
With this we can evaluate the \( \tau \)-derivatives of the frame vectors in the following way:
\[ \dot{e}^{(2)} = \frac{d}{d\tau} \dot{s} = -\frac{\dot{v}}{v^2} \dot{s} + \frac{1}{v} \ddot{s} \]  
(86)
\[ \dot{e}^{(3)} = \dot{e}^{(1)} \times e^{(2)} + e^{(1)} \times \dot{e}^{(2)} = \frac{1}{v} \dot{s} \times \ddot{s} + g \frac{\ddot{v}}{v^2} e^{(1)} \times e^{(3)} = -g \frac{\ddot{v}}{v^2} e^{(2)}. \]  
(87)
The \( \tau \)-derivative of \( s^{(2)}(\tau) \) can now be calculated in two different ways:
\[ \dot{s}^{(2)} = h \times s^{(2)} \]  
(88)
\[ \dot{s}^{(2)} = \left( h \cdot s \right) e^{(1)} + v e^{(3)} \times \left( \cos \alpha e^{(2)} + \sin \alpha e^{(3)} \right) \]  
(89)
\[ = -v \cos \alpha e^{(1)} - \left( h \cdot s \right) \sin \alpha e^{(2)} + \left( h \cdot s \right) \sin \alpha e^{(3)}, \]  
(90)
and
\[ \dot{s}^{(2)} = -\dot{\alpha} \sin \alpha e^{(2)} + \cos \alpha \dot{\alpha} e^{(3)} + \dot{\alpha} \cos \alpha e^{(3)} + \sin \alpha \dot{e}^{(3)} \]  
(91)
\[ \dot{s}^{(2)} = \dot{\alpha} \left( -\sin \alpha e^{(2)} + \cos \alpha e^{(3)} \right) + \cos \alpha \left( -v e^{(1)} + g \frac{\ddot{v}}{v^2} e^{(3)} \right) - \sin \alpha g \frac{\ddot{v}}{v^2} e^{(2)}. \]  
(92)
Comparing the $e^{(2)}$-components of (90) and (92) yields the expression for $\dot{\alpha}$ we are looking for:

$$\dot{\alpha} = (\mathbf{h} \cdot \mathbf{s}) - \frac{g}{\hbar^2} = (\mathbf{h} \cdot \mathbf{s}) - \frac{\mathbf{s} \cdot (\dot{\mathbf{s}} \times \mathbf{s})}{\mathbf{s} \cdot \mathbf{s}}. \tag{93}$$

This shows that $\alpha(\tau)$ can be obtained as an integral over the r. h. s. of (93) that is a function of $\mathbf{s}(\tau)$ and its first two derivatives. Note that we have not used the fact that $\mathbf{s}(\tau)$ would be $T$-periodic. These calculations show that if one solution $\mathbf{s}(\tau)$ of (5) is given, then the other two solutions with orthogonal initial conditions can be obtained by means of certain integrals.

In particular we obtain the quasienergy as

$$\epsilon = \frac{\delta}{T} = \frac{1}{T} \int_{0}^{T} \left( (\mathbf{h} \cdot \mathbf{s}) - \frac{\mathbf{s} \cdot (\dot{\mathbf{s}} \times \mathbf{s})}{\mathbf{s} \cdot \mathbf{s}} \right) \, d\tau, \tag{94}$$

now assuming that $\mathbf{s}(\tau)$ will be $T$-periodic. This result slightly improves the corresponding equation (62) in [3] in so far as it is manifestly independent of a coordinate system. The explicit accordance with [3] will be shown later in subsection III G. We note that the form of (94) suggest the following splitting of the quasienergy

$$\epsilon = \epsilon_d + \epsilon_g = \frac{1}{T} \int_{0}^{T} \mathbf{h} \cdot \mathbf{s} \, d\tau + \frac{1}{T} \int_{0}^{T} \left( -\frac{\mathbf{s} \cdot (\dot{\mathbf{s}} \times \mathbf{s})}{\mathbf{s} \cdot \mathbf{s}} \right) \, d\tau \tag{95}$$

into a “dynamical part” $\epsilon_d$ and a “geometrical part” $\epsilon_g$. The dynamical part $\epsilon_d$ is obviously the time average of the energy $\mathbf{h} \cdot \mathbf{s}$, and depends on the dynamics, i. e., how fast the angle between $\mathbf{s}$ and $\mathbf{h}$ changes over one period.

For the geometrical part $\epsilon_g$ we note that the corresponding integral $\int_{0}^{T} \ldots \, d\tau$ is invariant under an arbitrary parameter transformation $\tau \mapsto \sigma(\tau)$ that leads to a new period $\tilde{T}$:

$$\int_{0}^{T} \left( -\frac{\mathbf{s} \cdot (\dot{\mathbf{s}} \times \mathbf{s})}{\mathbf{s} \cdot \mathbf{s}} \right) \, d\tau = \int_{0}^{\tilde{T}} \left( -\frac{\mathbf{s}' \cdot (\dot{\mathbf{s}}' \times \mathbf{s}'')}{\mathbf{s}' \cdot \mathbf{s}''} \right) \, d\sigma, \tag{96}$$

where we have denoted the $\sigma$-derivative by a prime $'$. This transformation produces a factor $\left( \frac{d\sigma}{d\tau} \right)^3$ in the numerator of the integrand of the l. h. s. of (96) and a factor $\left( \frac{d\sigma}{d\tau} \right)^2$ in the denominator; after cancelling the remaining factor $\frac{d\sigma}{d\tau}$ is used to transform the $d\tau$-integration into a $d\sigma$-integration. This means that this integral is independent of the dynamics of the spin precession and depends only on the geometry of the curve $\mathcal{S}$, thereby justifying the notation as “geometrical part of the quasienergy”. Note, however, that $\epsilon_g$ still depends on the period $T$ according to the pre-factor $\frac{1}{T}$ in (95). The geometrical meaning of $\epsilon_g$ will be considered below.

Next we remark that the splitting $\epsilon = \epsilon_d + \epsilon_g$ can be connected to the form of the magnetic field according to the solution of the inverse problem [3],

$$\mathbf{h}(\tau) = \mathbf{s} \times \dot{\mathbf{s}} + \lambda(\tau) \mathbf{s}, \tag{97}$$

where now the function $\lambda(\tau)$ must be $T$-periodic. First we note that due to $\mathbf{s} \cdot (\mathbf{s} \times \dot{\mathbf{s}}) = 0$

$$\mathbf{h} \cdot \mathbf{s} = \lambda(\tau) \mathbf{s} \cdot \mathbf{s} = \lambda(\tau), \tag{98}$$

and hence the dynamical part $\epsilon_d$ can be obtained by an integration that involves only the second part of $\mathbf{h}(\tau)$ according to (77). Recall that for a given spin function $\mathbf{s}(\tau)$ there are different magnetic fields $\mathbf{h}(\tau)$ satisfying the equation of motion [3], corresponding to different functions $\lambda(\tau)$ in (77). Upon the choice of these magnetic fields one can realize any value of $\epsilon_d \in \mathbb{R}$.

In contrast, the integral defining the geometrical part $\epsilon_g$ depends only on the curve $\mathcal{S}$, hence for its calculation we may use any parametrization of $\mathcal{S}$ and any magnetic field that satisfies the corresponding equation of motion.

The following choice considerably simplifies the geometry: As a parameter of $\mathcal{S}$ we will use the arc length that will always be denoted by $s$ in what follows. Differentiation w. r. t. $s$ will again be denoted by a prime $'$ without danger of confusion. The length of the curve $\mathcal{S}$ will be denoted by $|\mathcal{S}|$. This has the consequence that

$$\mathbf{s}'(s) = v = 1 \quad \text{for all } s \in [0, |\mathcal{S}|]. \tag{99}$$

Further we will choose as a corresponding magnetic field the first part of (77), namely

$$\mathbf{h}(s) = \mathbf{s}(s) \times \mathbf{s}'(s), \tag{100}$$
which will always be a unit vector,

\[ \| \mathbf{h}(s) \| = 1 \]  

(101)
as the vector product (100) of two orthogonal unit vectors. Due to \( \mathbf{h} \cdot \mathbf{s} = 0 \) the dynamical part \( \epsilon_d \) of the quasienergy always vanishes. For reasons that will become clear later we call (100) the “dual magnetic field” of the spin vector function \( \mathbf{s}(s) \) and \( \mathcal{H} \) the “dual loop” of the loop \( \mathcal{S} \). With the above choice the geometric part of the quasienergy can be written as

\[ \epsilon_g = \frac{1}{T} \int_0^{||S||} \left( -\mathbf{s} \cdot (\mathbf{s}' \times \mathbf{s}'') \right) ds . \]  

(102)

It is known from elementary differential geometry that the “geodesic curvature” \( k_g \) of a curve \( \mathcal{S} \) on a surface parametrized by its arc length is defined as the triple product

\[ k_g \equiv \mathbf{s} \cdot (\mathbf{s}' \times \mathbf{s}'') \]  

(103)

measuring the component of the acceleration \( s'' \) in the tangent plane of the curve, see, e. g., [14]. It can have positive or negative values and vanishes at the inflection points of the curve. It follows that the integrand in (102) can be, up to a sign, interpreted as the geodesic curvature \( k_g \) of \( \mathcal{S} \). Then we will apply the theorem of Gauss-Bonnet [14] for the unit sphere that may be written as

\[ \int_M K dA + \int_{\partial M} k_g ds = 2\pi . \]  

(104)

Here \( M \) denotes a two-dimensional submanifold of \( S^2 \) with boundary \( \partial M \) and (constant) Gaussian curvature \( K \). In our case we set \( \partial M = \mathcal{S} \) and can identify the surface integral \( \int_M K dA \) with the (signed) solid angle \( \mathcal{A}(\mathcal{S}) \) encircled by the loop \( \mathcal{S} \) and thus re-write (104) in the form

\[ \int_0^{||S||} \left( -\mathbf{s} \cdot (\mathbf{s}' \times \mathbf{s}'') \right) = - \int_{\mathcal{S}} k_g ds = \mathcal{A}(\mathcal{S}) - 2\pi . \]  

(105)
The last term \(-2\pi\) is irrelevant since the quasienergy is only defined modulo \( \omega = \frac{2\pi}{T} \). Thus we thus have re-established the result

\[ \epsilon_g = \frac{1}{T} \mathcal{A}(\mathcal{S}) \text{ modulo } \omega , \]  

(106)

that endows \( \epsilon_g \) with a geometric meaning.

One may ask whether the full strength of the theorem of Gauss-Bonnet is really necessary to prove (106). For its essential content is the validity for general surfaces whereas we need only its application to the sphere \( S^2 \). We are not aware of a direct precursor of the Gauss-Bonnet theorem restricted to \( S^2 \). But, as has been pointed out in [13], analogous results for geodesic polygons in spherical geometry, especially for triangles, have been known long before: The area \( \mathcal{A}(\Delta) \) of a spherical triangle \( \Delta \) equals its spherical excess,

\[ \mathcal{A}(\Delta) = \alpha_1 + \alpha_2 + \alpha_3 - \pi , \]  

(107)

where the \( \alpha_i \) are the interior angles of \( \Delta \). This theorem is usually ascribed to the Flemish mathematician Albert Girard (1595 – 1632). It can be immediately generalized to spherical \( N \)-polygons \( \Pi \): Since \( \Pi \) can be decomposed into \( N-2 \) triangles and the area as well as the sum over the interior angles are additive, we obtain

\[ \mathcal{A}(\Pi) = \sum_{i=1}^N \alpha_i - (N-2)\pi = \sum_{i=1}^N (\alpha_i - \pi) + 2\pi . \]  

(108)

From this (104) follows if the curve \( \partial M \) is arbitrarily close approximated by \( N \)-polygons and if

\[ \sum_{i=1}^N (\alpha_i - \pi) \to - \int_{\partial M} k_g ds . \]  

(109)

We will use this line of reasoning, following [13], to give a simplified account of the geometric phase associated to the classical Rabi problem in the next section.
D. Geometric phases

It is well-known \[11\], \[3\] that, in the case of the underlying \( s = 1/2 \) Schrödinger equation, the geometrical part of the quasienergy is closely related to the geometric phase \[9\], \[10\] arising from closed curves on the unit sphere. Hence it is very plausible that the analogous relation also holds in the case of the classical Rabi problem thus extending the geometric interpretation given in the last subsection. It will be instructive to recapitulate this connection without resorting to the Schrödinger equation. Although the pertaining differential-geometrical tools have become somewhat familiar to theoretical physicists due to their applications in general relativity and gauge theories it is a challenge to give a more elementary account going back to older results of spherical geometry, partially following \[13\]. Our treatment will not be completely rigorous w. r. t. the standards of pure mathematics, but this seems to be adequate in view of the purpose.

For two points \( r \) and \( s \) on the unit sphere \( S^2 \) let \( T_r S^2 \) and \( T_s S^2 \) denote the respective tangent spaces. They are isomorphic as two-dimensional, oriented Euclidean vector spaces but there is no natural isomorphism \( \rho : T_r S^2 \to T_s S^2 \). Now consider a smooth curve \( \mathcal{P} = \mathcal{P}(r, s) \subset S^2 \) connecting \( r \) and \( s \). In contrast to the general situation, the
The parallel transport along $\mathcal{P}$ as an isomorphism

$$\rho_{\mathcal{P}} : T_r S^2 \rightarrow T_{s} S^2 .$$

Upon introducing arbitrary right-handed, orthonormal basis in $T_r S^2$ and $T_s S^2$, $\rho_{\mathcal{P}}$ can be described by a rotation with an angle $\varphi(\mathcal{P})$ that will be called a “phase shift”. The extension of parallel transport to the case where $\mathcal{P}$ is only piecewise smooth is in a straightforward way defined by composition of the corresponding transport isomorphisms for smooth curves.

$\rho_{\mathcal{P}}$ is uniquely determined by the requirement, additionally to being an isomorphism of two-dimensional, oriented Euclidean vector spaces, that it maps unit tangent vectors onto unit tangent vectors if $\mathcal{P}$ is a (segment of a) great circle of $S^2$. More precisely, let $p(s)$, $s \in [s_1, s_2]$ be the parametrization of a segment of a great circle by the arc length $s$ such that

$$p'(s_1) \in T_{p(s_1)} S^2, \quad \text{and} \quad p'(s_2) \in T_{p(s_2)} S^2 ,$$

are the corresponding tangent vectors, necessarily satisfying

$$\|p'(s_1)\| = \|p'(s_2)\| = 1 ,$$

then we require

$$\rho_{\mathcal{P}} (p'(s_1)) = p'(s_2) .$$

It is clear that (113) uniquely determines the parallel transport along great circles. In fact, if $v \in T_{p(s_1)} S^2$ is a vector that is obtained from $p'(s_1)$ by a clock-wise rotation with an angle $\gamma$ then $\rho_{\mathcal{P}}(v)$ must be obtained from $p'(s_2)$ in the same way.

Next consider a general smooth curve $\mathcal{P}$. It can be approximated arbitrarily close by a polygon $\Pi \in S^2$ consisting piecewise of segments of great circles $\Pi_{\nu}$, $\nu = 1, \ldots, N$. Recall that great circles are exactly the geodesics of $S^2$ equipped with the natural Riemannian metric. As mentioned above, $\rho_{\Pi}$ will be uniquely defined as the composition

$$\rho_{\Pi} = \rho_{\Pi_{N}} \circ \ldots \circ \rho_{\Pi_{2}} \circ \rho_{\Pi_{1}} .$$

The parallel transport $\rho_{\mathcal{P}} : T_r S^2 \rightarrow T_{s} S^2$ will then be defined as the limit of the $\rho_{\Pi}$ for $N \rightarrow \infty$ and is such uniquely determined.

We will have a closer look at the geometry of the parallel transport along polygons. Let $u_{\nu}$ be the point of intersection between $\Pi_{\nu}$ and $\Pi_{\nu+1}$ and $d_{\nu} \equiv d(u_{\nu}, u_{\nu+1})$ the spherical distance of $u_{\nu}$ and $u_{\nu+1}$, i.e., the arc length of $\Pi_{\nu+1}$ between $u_{\nu}$ and $u_{\nu+1}$. Further let $\varphi_{\nu}$ be the angle between the respective tangent vectors at $u_{\nu}$, see Figure 2. Then the tangent vector $a \in T_{u_{\nu-1}} S^2$ will be parallel transported into $b \in T_{u_{\nu}} S^2$ and further into $c \in T_{u_{\nu+1}} S^2$, see Figure 2. We see that the angles between the transported vector and the respective tangent vectors add up as partial sums of the $\varphi_{\nu}$.

Now consider a closed polygon $\Pi$, set $u_0 = u_N \equiv u$ and consider the parallel transport isomorphism $\rho_{\Pi} : T_{u} S^2 \rightarrow T_{u} S^2$. The total phase shift $\varphi(\Pi)$ will be independent of any local frames and is obtained as

$$\varphi(\Pi) = \sum_{\nu=1}^{N} \varphi_{\nu} .$$

This phase shift can be identified with the “total geodesic curvature” of the closed polygon $\Pi$. It vanishes for a full great circle.

A mentioned above, it is at least plausible that for $N \rightarrow \infty$ the quotient $\frac{\varphi_{\nu}}{d_{\nu}}$ approaches a limit which can be identified with the negative geodesic curvature $k_g$ introduced above

$$\frac{\varphi_{\nu}}{d_{\nu}} \rightarrow -k_g(r) .$$

The negative sign in (116) is due to the that a positively oriented polygon has negative phase shifts $\varphi_{\nu}$, see Figure 2. Then the r. h. s. of

$$\varphi(\Pi) = \sum_{\nu=1}^{N} \varphi_{\nu} \rightarrow - \int_{0}^{||\mathcal{P}||} k_g(s) \, ds = \delta(\mathcal{P})$$

(117)
can be considered as the total geodesic curvature of $\mathcal{P}$ that also appears in (104).

Next we will translate and extend these considerations into the realm of the classical Rabi problem. Let, as in subsection III C, $\mathcal{S} \subset S^2$ be a loop of spin vectors parametrized as $s(s)$ by the arc length $s$ for $s \in [0, |\mathcal{S}|)$ and $h(s) = s(s) \times s'(s)$ its dual magnetic field such that

$$s'(s) = h(s) \times s(s).$$

The loop of magnetic field vectors $h(s)$ will again be denoted by $\mathcal{H}$. We will make it plausible that the parallel transport along the curve $\mathcal{S}$ can be realized by a solution of the classical Rabi problem using the dual magnetic field. As above, this will be achieved by approximating $\mathcal{S}$ by a geodesic polygon.

To this end we will divide the parameter range $[0, |\mathcal{S}|)$ into $N$ equal parts of length $\delta s = \frac{|\mathcal{S}|}{N}$ and define

$$s_\nu \equiv s(s_\nu) \equiv s(\nu \delta s), \quad s'_\nu \equiv s'(s_\nu),$$

for $\nu = 1, \ldots, N$. We will approximate the loop $\mathcal{S}$ by the closed polygon $\Pi$ consisting of segments of great circles $\Pi_\nu$ that are tangent to $\mathcal{S}$ at $s_\nu$ and oriented into the direction $s'_\nu$. As usual, the normal $n_\nu$ of $\Pi_\nu$ is defined as

$$n_\nu \equiv s_\nu \times s'_\nu \quad (118)$$

It follows that the normal can be identified with the dual magnetic field vector,

$$n_\nu = h_\nu = h(r_\nu) \quad (121)$$

for all $\nu = 1, \ldots, N$. Let, as above, $u_\nu$ denote the point of intersection between $\Pi_\nu$ and $\Pi_{\nu+1}$ and $\varphi_\nu$ the angle between $\Pi_\nu$ and $\Pi_{\nu+1}$ at $u_\nu$. It follows that, up to a sign, $\varphi_\nu$ is identical with the angle between the normals $h_\nu$ and $h_{\nu+1}$,

$$|\varphi_\nu| = d(h_\nu, h_{\nu+1}).$$

Hence the total geodesic curvature of $\Pi$ will be

$$\varphi(\Pi) = \sum_{\nu=1}^{N} \varphi_\nu = \pm \sum_{\nu=1}^{N} d(h_\nu, h_{\nu+1}),$$

the $\pm$ sign depending on the orientation of $\mathcal{S}$. In the limit $N \to \infty$ we will, similarly as above, obtain

$$\varphi(\Pi) = \sum_{\nu=1}^{N} \varphi_\nu \to \delta(\mathcal{S}) = -\int_{0}^{|\mathcal{S}|} k_g \, ds. \quad (124)$$

The curve $\mathcal{S}$ will be called simple iff its geodesic curvature does not change its sign. In this case we may rewrite the r. h. s. of (124) in the form

$$-\int_{0}^{|\mathcal{S}|} k_g \, ds = \pm |\mathcal{H}|,$$

where we have used that $\sum_{\nu=1}^{N} d(h_\nu, h_{\nu+1})$ approaches the length of $\mathcal{H}$ for $N \to \infty$.

The additional results thus obtained are, first, that the parallel transport along $\mathcal{S}$ can be obtained as a solution of the classical Rabi problem with time $\tau$ replaced by arc length $s$ using the dual magnetic field. Second, we can geometrically interpret the phase shift $\delta = \delta(\mathcal{S})$, not only as the solid angle $A(\mathcal{S})$ (modulo $2\pi$), but also, for simple spin curves, as the length of the dual loop $\mathcal{H}$, up to a sign, and hence

$$\epsilon_g = \frac{\delta}{T} = \pm \frac{|\mathcal{H}|}{|\mathcal{S}|} \mod \omega. \quad (126)$$
E. Duality of loops

We start with a loop $S$ and its dual loop $H$ on the unit Bloch sphere parametrized by the arc length $s$ of $S$ via $s(s)$ and $h(s)$ such that

$$\frac{ds}{ds} = h \times s \quad (127)$$

and

$$h = s \times \frac{ds}{ds} \quad (128)$$

hold. (We will avoid the use of primes for derivatives in this subsection in order to avoid misunderstandings.) Hence the triple $(h, s, \frac{ds}{ds})$ will be a right-handed orthonormal frame for all values of the parameter $s$. It follows that

$$\frac{d}{ds} h = s \times \frac{d^2 s}{ds^2} \quad (129)$$

is orthogonal to $h$ and $s$ and hence

$$\frac{d}{ds} h = a(s) \frac{ds}{ds} \quad (130)$$

Moreover,

$$\frac{d^2}{ds^2} s = \frac{d}{ds} (h \times s) = \left( \frac{d}{ds} h \right) \times s + h \times \left( \frac{d}{ds} s \right) = a \left( \frac{d}{ds} s \right) \times s - s = -a h - s, \quad (131)$$

and hence

$$g = \left( s \times \frac{ds}{ds} \right) \cdot \frac{d^2 s}{ds^2} s = -a h \cdot h = -a. \quad (132)$$

Now (127), (130) and (132) imply

$$\frac{d}{ds} h = -g \frac{d}{ds} s = (gs) \times h \equiv \hat{s} \times h. \quad (133)$$

The latter equation has the form of (5) and hence can be interpreted in such a way that the “spin vector” $h$ moves according to (5) under the influence of the “magnetic field” $\hat{s}$. In this sense the rôle of classical spin and magnetic field is interchanged. However, in general $\hat{s}$ will not be a unit vector and $s$ will not be the arc length of the loop $H$.

The situation will be more symmetric if we additionally pass from $s$ to the arc length parameter of $H$, denoted by $r$. Then the equation of motion for $h(r)$ assumes the form

$$\frac{d}{dr} h = \frac{ds}{dr} \frac{d}{ds} h = \left( \frac{ds}{dr} g s \right) \times h \equiv \bar{s} \times h. \quad (134)$$

Now the new “magnetic field” $\bar{s}$ has unit length since it is the vector product of two orthogonal unit vectors:

$$h \times \frac{d}{dr} h = h \times (\bar{s} \times h) = \bar{s} h \cdot h - h \bar{s} \cdot h = \bar{s}. \quad (135)$$

This means

$$||\bar{s}|| = \left| \frac{ds}{dr} g s \right| = ||s|| = 1, \quad (136)$$

and hence

$$\frac{ds}{dr} = \frac{1}{g}. \quad (137)$$
Together with (135) this implies
\[ \mathfrak{S}(r) = \pm s(r). \] (138)

If the rôle of \( S \) and \( H \) is interchanged, we obtain
\[ \frac{d r}{d s} = |g| = \left| \frac{1}{G} \right|, \] (139)
where \( G \) is the geodesic curvature of \( H \):
\[ G \equiv h \cdot \left( \frac{d h}{d r} \times \frac{d^2 h}{d r^2} \right). \] (140)

Summarizing, we have two loops \( S \) and \( H \) on the unit Bloch sphere that give rise to two different solutions of (5): Either \( S \) consists of spin vectors and \( H \) of magnetic field vectors and the time parameter \( \tau \) in (5) is chosen as the arc length \( s \) of \( S \). Or, \( H \) consists of spin vectors and \( \pm S \) of magnetic field vectors and the time parameter \( \tau \) in (5) is chosen as the arc length \( r \) of \( H \). This symmetry between \( S \) and \( H \) justifies the denotation as “dual loops”. The possible sign change indicated by \( \pm S \) signifies that the proper mathematical framework would be to consider loops, not on the Bloch sphere \( S^2 \), but in the projective plane \( \mathbb{P}_2(\mathbb{R}) \) obtained by identifying anti-podal points of \( S^2 \).

For both realizations of solutions of (5) we can calculate the quasienergy denoted by \( \epsilon_S \) and \( \epsilon_H \), resp. It consists only of its geometric part since spin vector and magnetic field will be orthogonal in both realizations. For the case of simple spin curves, (126) immediately implies that there are representatives of the classes \([\epsilon_S]\) and \([\epsilon_H]\), resp., such that
\[ \epsilon_S = \pm \frac{1}{\epsilon_H}, \] (141)

further illustrating the duality between \( S \) and \( H \).

1. Example 2

In order to illustrate the notion of duality considered in this subsection we consider two examples. The first one is a special case of the Rabi problem with circular polarization. Let
\[ s(s) = \begin{pmatrix} \sqrt{1 - z^2} \cos \left( \frac{s}{\sqrt{1 - z^2}} \right) \\ \sqrt{1 - z^2} \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \\ z \end{pmatrix}, \] (142)
be the arc length parametrization of a circle \( S \) on \( S^2 \) lying in the plane \( z = \text{const.} \) with \(-1 < z < 1 \) and \( z \neq 0 \). This leads to
\[ |S| = 2\pi \sqrt{1 - z^2} \equiv \frac{2\pi}{\omega}. \] (143)

Then (128) yields the parametrization of the dual loop \( H \):
\[ h(s) = \begin{pmatrix} -z \cos \left( \frac{s}{\sqrt{1 - z^2}} \right) \\ -z \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \\ \sqrt{1 - z^2} \end{pmatrix}, \] (144)
satisfying
\[ |H| = 2\pi |z| \equiv \frac{2\pi}{\Omega}. \] (145)

After some elementary calculations we obtain
\[ g = \left( s \times \frac{d}{d s} s \right) \cdot \frac{d^2 s}{d s^2} s = \frac{z}{\sqrt{1 - z^2}}, \] (146)
and hence the first expression for the quasienergy $\epsilon_S$ reads

$$\epsilon_S^{(1)} \equiv \frac{1}{|S|} \int_0^{|S|} (-g) \, ds = -g = -\frac{z}{\sqrt{1 - z^2}}. \tag{147}$$

The solid angle enclosed by $S$ will be

$$\mathcal{A}(S) = 2\pi \,(1 - z) \equiv 2\pi \, |\mathcal{H}|, \tag{148}$$

and hence a second expression for $\epsilon_S$ will be

$$\epsilon_S^{(2)} \equiv \frac{\mathcal{A}(S)}{|S|} = \frac{\sqrt{1 - z}}{\sqrt{1 + z}}. \tag{149}$$

It satisfies

$$\epsilon_S^{(2)} = \epsilon_S^{(1)} + \omega, \tag{150}$$
FIG. 4: Plot of the geodesic curvature $g$ of $S$ (blue curve), $G$ of $H$ (orange curve), and the product $gG$ (green curve) as a function of $\tau$. The data are the same as for Figure 3. According to (139) the product $gG$ must have the absolute value 1.

and hence both expressions (147) and (149) agree modulo $\omega$.

The arc length $r$ corresponding to $H$ is obtained as

$$r = g \, s \left( \frac{z}{\sqrt{1 - z^2}} s \right). \quad (151)$$

After some elementary calculations we obtain

$$G \equiv \left( h \times \frac{d}{dr} h \right) \cdot \left( \frac{d^2}{dr^2} h \right) = \frac{\sqrt{1 - z^2}}{z} = \frac{1}{g}, \quad (152)$$

and hence the first expression for the quasienergy $\epsilon_H$ reads

$$\epsilon_H^{(1)} = \frac{1}{|H|} \int_{|H|}^{\tau} (-G) \, d\tau = -G = -\frac{\sqrt{1 - z^2}}{z}. \quad (153)$$

The solid angle enclosed by $H$ will be

$$A(H) = 2\pi \left( 1 - \sqrt{1 - z^2} \right) \equiv 2\pi - |S|, \quad (154)$$

and hence a second expression for $\epsilon_S$ will be

$$\epsilon_S^{(2)} = \frac{A(H)}{|H|} = \frac{1 - \sqrt{1 - z^2}}{|z|}. \quad (155)$$

It satisfies

$$\epsilon_H^{(2)} = \pm \epsilon_H^{(1)} + \Omega, \quad (156)$$

the $\pm$ sign depending on the sign of $z$. Hence both expressions (147) and (149) agree up to a sign and modulo $\Omega$. (141) holds for the present example since the triple products $g$ and $G$ are constant and, due to (152), inverses of each other.
In the case of \(0 < z < 1\) the curve \(S\) of our example generates a closed, convex cone \(C(S) \subset \mathbb{R}^3\), analogously for the dual curve \(H\). Then it follows that \(C(H)\) is the dual cone of \(C(S)\) and vice versa. Here the dual cone \(C'\) of a cone \(C\) is defined by

\[
C' \equiv \{ x \in \mathbb{R}^3 \mid x \cdot y \geq 0 \text{ for all } y \in C \},
\]

see, e. g., \([13]\). In this sense our definition of dual curves is compatible with the established notion of dual cones in \(\mathbb{R}^3\).

Finally, we note that the magnetic field \((144)\) can be understood as a special case of the Rabi problem with circularly polarized driving \((49)\) if we set \(F = -z, \omega = \frac{1}{\sqrt{1 - z^2}}\) and \(\omega_0 = \frac{1}{\sqrt{1 - z^2}}\). It is well-known \([1], [3]\) that for this problem the quasienergy \(\epsilon_c\) assumes the form

\[
\epsilon_c = \Omega_{\text{Rabi}} \equiv \sqrt{F^2 + (\omega_0 - \omega)^2},
\]

(158)

taking into account that the quasienergy of the classical Rabi problem is twice the quasienergy of the \(s = 1/2\) quantum Rabi problem modulo \(\omega\), see also \([63]\). In our case it follows that

\[
\Omega_{\text{Rabi}} = \sqrt{z^2 + \left(\frac{1}{\sqrt{1 - z^2}} - \sqrt{1 - z^2}\right)^2} = \frac{|z|}{\sqrt{1 - z^2}},
\]

(159)

which agrees with \((147)\) up to a possible sign.

2. Example 3

For the second example we take a case where \(S\) is not simple, but of the form of the figure “8” with a double point. This example also illustrates that we need not explicitly calculate the arc length parameters \(s\) of \(S\) or \(r\) of \(H\) but may work within the initial parametrization. Let

\[
s(\tau) = \frac{1}{\sqrt{\sin^2(\tau) + \sin^2(2\tau) + 1}} \begin{pmatrix} \sin \tau \\ \sin 2\tau \\ 1 \end{pmatrix},
\]

(160)

We calculate the dual loop \(H \in S^2\) by

\[
h(\tau) = \frac{1}{\|ds\|} s \times \frac{ds}{d\tau},
\]

(161)

analogously to \((128)\) but without directly using the arc length parameter \(s\). This and the following expressions can be easily obtained by a computer algebra software but are too involved to be displayed here. The loop \(H\) is displayed in Figure 3. It shows two cusps corresponding to the double point of \(S\). These cusps necessarily occur according to the following reasoning: At the double point corresponding to the values \(\tau = 0, \pi\) of the parameter, the geodesic curvature \(g\) of \(S\) changes its sign, see Figure 4. According to \((128)\) the geodesic curvature \(G\) of \(H\) must diverge at \(\tau = 0, \pi\) which explains the two cusps.

The curve \(S\) can be divided into the parts \(S_1\) and \(S_2\) that have in common only the double point. The corresponding parts of \(H\) are denoted by \(H_i, i = 1, 2\). Both parts \(S_i\) encircle solid angles \(A(S_i) \approx \pm 4.64172\) that correspond to the length of the \(H_i\). But due to the different signs the total solid angle and the corresponding quasienergy vanishes.

Interestingly, if we calculate the “bi-dual” loop \(R\) according to

\[
r(\tau) = \frac{1}{\|dh\|} h \times \frac{dh}{d\tau},
\]

(162)

then we obtain two disjoint curves that locally coincide with \(-S_1\) and \(S_2\), see Figure 3.

F. New solutions

In this subsection we will show how the results of the preceding subsection III E can be used to generate new solutions of \((5)\). We start with two dual loops \(S\) and \(H\) on the unit Bloch sphere parametrized by the arc length \(s\) of \(S\) such that the equations \((127) - (137)\) of subsection III E are satisfied.
Then we consider a second curve $\tilde{S} \subset S^2$ parametrized by

$$\tilde{s}(s) = \cos \alpha(s) s(s) + \sin \alpha(s) h(s), \quad (163)$$

where $s \mapsto \alpha(s)$ is an arbitrary smooth function. Solving (163) for $s$ gives

$$s = (\sec \alpha) \tilde{s} - (\tan \alpha) h. \quad (164)$$

We want to derive an equation of motion for $\tilde{s}$ and consider the following equations:

$$\frac{d}{ds} \tilde{s} = (\cos \alpha) s' + (\sin \alpha) h' + \alpha' (-\sin \alpha) s + (\cos \alpha) h \quad (165)$$

$$= (\cos \alpha) h \times s - g (\sin \alpha) h \times s + \alpha' (-\sin \alpha) s + (\cos \alpha) h \quad (166)$$

$$= (1 - g \tan \alpha) h \times s + \alpha' (-\sin \alpha) s + (\cos \alpha) h \equiv (1 - g \tan \alpha) h \times \tilde{s} + v. \quad (167)$$

The vector $v$ will be orthogonal to $\tilde{s}$ by

$$v \cdot \tilde{s} = (\alpha' (-\sin \alpha) s + (\cos \alpha) h) \cdot ((\cos \alpha) s + (\sin \alpha) h) = (\cos \alpha)(\sin \alpha)(-\alpha' + \alpha') = 0. \quad (168)$$

Using $\tilde{s} \cdot \tilde{s} = 1$ we obtain

$$(\tilde{s} \times v) \times \tilde{s} \equiv (\tilde{s} \cdot v) \tilde{s} = v, \quad (169)$$

and

$$\tilde{s} \times v = ((\cos \alpha) s + (\sin \alpha) h) \times (\alpha' (-\sin \alpha) s + (\cos \alpha) h) = -\alpha' ((\cos \alpha)^2 + (\sin \alpha)^2) h \times s = -\alpha' h \times s. \quad (170)$$

Finally,

$$\frac{d}{ds} \tilde{s} = ((1 - g \tan \alpha) h - \alpha' h \times s) \times \tilde{s} \equiv \tilde{h} \times \tilde{s}, \quad (171)$$

and hence $\tilde{s}$ satisfies a Rabi type equation of motion with a modified magnetic field $\tilde{h}$. Upon a suitable choice of $\alpha(s)$ the new solution $\tilde{s}$ will also be periodic in the original arc length parameter $s$ of $S$.

1. **Example 4**

As an example we choose $s(s)$ according to (142) and $h(s)$ according to (144), further

$$\alpha(s) = \sin \left( \frac{s}{\sqrt{1 - z^2}} \right). \quad (172)$$

This entails the new solution of (171) of the form

$$\tilde{s}(s) = \left( \begin{array}{c}
\cos \left( \frac{s}{\sqrt{1 - z^2}} \right) \\
\sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \\
\sqrt{1 - z^2} \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) + z \cos \left( \frac{s}{\sqrt{1 - z^2}} \right)
\end{array} \right)
\left( \begin{array}{c}
\sqrt{1 - z^2} \cos \left( \frac{s}{\sqrt{1 - z^2}} \right) - z \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \\
\sqrt{1 - z^2} \cos \left( \frac{s}{\sqrt{1 - z^2}} \right) - z \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \\
\sqrt{1 - z^2} \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) + z \cos \left( \frac{s}{\sqrt{1 - z^2}} \right)
\end{array} \right), \quad (173)$$

with

$$\tilde{h}(s) = \left( \begin{array}{c}
\cos \left( \frac{s}{\sqrt{1 - z^2}} \right) \left( \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \right) + z \left( \tan \left( \frac{s}{\sqrt{1 - z^2}} \right) \right) \\
z \sin \left( \frac{s}{\sqrt{1 - z^2}} \right) \left( \tan \left( \frac{s}{\sqrt{1 - z^2}} \right) \right) - \sqrt{1 - z^2} \\
\sqrt{1 - z^2} - z \tan \left( \frac{s}{\sqrt{1 - z^2}} \right)
\end{array} \right), \quad (174)$$

see Figures 9 and 10 where $z$ has been chosen as $z = \frac{1}{\sqrt{2}}$. 
G. Quasienergy II

In this subsection we are going to show that our result (94) for the quasienergy of a periodic solution of (5) is equivalent to the integral obtained in [3]. The corresponding calculations are elementary but somewhat lengthy. Let \( s(\tau) \) and \( r(\tau) \) be two solutions of (5) such that \( h(\tau) \) and \( s(\tau) \) are \( T \)-periodic and \( r(0) \cdot s(0) = 0 \) and hence

\[
    r(\tau) \cdot s(\tau) = 0 \quad \text{for all } \tau \in \mathbb{R} . 
\]

Further, let \( f \) be a constant unit vector that is not contained in the curve \( S \). We consider the right-handed orthogonal, not necessarily normalized, “f-frame”

\[
    (s(\tau), f \times s(\tau), s(\tau) \times (f \times s(\tau))) ,
\]

and expand \( r(\tau) \) w. r. t. this frame:

\[
    r(\tau) = a(\tau) f \times s(\tau) + b(\tau) s(\tau) \times (f \times s(\tau)) ,
\]
taking into account (175). Differentiating (177) w. r. t. time, using the equation of motion (5) and expanding the multiple vector products gives

\[
\dot{\mathbf{r}} = \dot{a} \mathbf{f} \times \mathbf{s} + \dot{b} \mathbf{s} \times (\mathbf{f} \times \mathbf{s}) + a \mathbf{f} \times \mathbf{s} + b (\mathbf{s} \times (\mathbf{f} \times \mathbf{s}) + \mathbf{s} \times (\mathbf{f} \times \mathbf{s}))
\]

(178)

Another expression for \(\dot{\mathbf{r}}\) is obtained by inserting (177) into the equation of motion:

\[
\dot{\mathbf{r}} = \mathbf{h} \times \mathbf{r} = \mathbf{h} \times (a \mathbf{f} \times \mathbf{s} + b \mathbf{s} \times (\mathbf{f} \times \mathbf{s}))
\]

(181)

\[
= a (\mathbf{h} \cdot \mathbf{s} \mathbf{f} - \mathbf{h} \cdot \mathbf{f} \mathbf{s}) + b (\mathbf{h} \times \mathbf{f} - \mathbf{f} \times \mathbf{h} \mathbf{s} \times \mathbf{h}) .
\]

(182)

We will expand (180) and (182) w. r. t. the \(\mathbf{f}\)-frame. The \(\mathbf{s}\)-components give no new results. For the \(\mathbf{f} \times \mathbf{s}\)-components we obtain

\[
\dot{\mathbf{r}} \cdot (\mathbf{f} \times \mathbf{s}) \overset{180}{=} \dot{a} (\mathbf{f} \times \mathbf{s}) \cdot (\mathbf{f} \times \mathbf{s}) + a \mathbf{f} \cdot \mathbf{s} \mathbf{h} \cdot (\mathbf{f} \times \mathbf{s}) + b \mathbf{f} \cdot \mathbf{s} (\mathbf{s} \times \mathbf{h}) \times (\mathbf{f} \times \mathbf{s})
\]

(183)

\[
= \dot{a} (1 - (\mathbf{f} \cdot \mathbf{s})^2) + a \mathbf{f} \cdot \mathbf{s} \mathbf{h} \cdot (\mathbf{f} \times \mathbf{s}) + b \mathbf{f} \cdot \mathbf{s} (\mathbf{s} \times \mathbf{h}) \times (\mathbf{f} \times \mathbf{s})
\]

(184)

\[
\overset{182}{=} b ((\mathbf{h} \times \mathbf{f}) \cdot (\mathbf{f} \times \mathbf{s}) - \mathbf{f} \cdot \mathbf{s} (\mathbf{h} \times \mathbf{s}) \cdot (\mathbf{f} \times \mathbf{s}))
\]

(185)

\[
= b (\mathbf{f} \cdot \mathbf{h} \mathbf{f} \cdot \mathbf{s} - \mathbf{s} \cdot \mathbf{h} - \mathbf{f} \cdot \mathbf{h} \mathbf{s} \times \mathbf{h}) .
\]

(186)

Subtracting (180) from (182) yields

\[
\dot{a} (1 - (\mathbf{f} \cdot \mathbf{s})^2) = -a \mathbf{f} \cdot \mathbf{s} \mathbf{h} \cdot (\mathbf{f} \times \mathbf{s}) - b (\mathbf{s} \cdot \mathbf{h} - \mathbf{f} \cdot \mathbf{h} \mathbf{f} \cdot \mathbf{s}) .
\]

(187)

The analogous calculation of the \(\mathbf{s} \times (\mathbf{f} \times \mathbf{s})\)-components that will not be given in detail yields

\[
\dot{b} (1 - (\mathbf{f} \cdot \mathbf{s})^2) = -b \mathbf{f} \cdot \mathbf{s} \mathbf{h} \cdot (\mathbf{f} \times \mathbf{s}) + a (\mathbf{s} \cdot \mathbf{h} - \mathbf{f} \cdot \mathbf{h} \mathbf{f} \cdot \mathbf{s}) .
\]

(188)
The two equations (187) and (188) together imply
\[
\frac{\dot{b}a - \dot{a}b}{a^2 + b^2} = \frac{s \cdot h - f \cdot h f \cdot s}{1 - (f \cdot s)^2}.
\] (189)

Recall that \( f \times s \) and \( s \times (f \times s) \) have the same length hence the angle \( \varphi(\tau) \) between \( r(\tau) \) and \( f \times s(\tau) \) satisfies
\[
\tan \varphi = \frac{b}{a} \equiv u,
\] (190)
which entails
\[
\frac{\dot{b}a - \dot{a}b}{a^2 + b^2} = \frac{\dot{u}}{1 + u^2} = \frac{d}{d\tau} \arctan u = \dot{\varphi}.
\] (191)

The total phase shift \( \delta \) between \( r(\tau) \) and \( f \times s(\tau) \) over one period can thus be written as
\[
\delta = \int_0^T \dot{\varphi} d\tau \]
\[
= \int_0^T \frac{s \cdot h - f \cdot h f \cdot s}{1 - (f \cdot s)^2} d\tau.
\] (192)

Consequently, the quasienergy can be written as
\[
\epsilon = \frac{\delta}{T} = \frac{1}{T} \int_0^T \frac{s \cdot h - f \cdot h f \cdot s}{1 - (f \cdot s)^2} d\tau.
\] (193)

For the comparison with the analogous expression in [3] we choose
\[
f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
and write \( s = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \),

which yields
\[
\delta = \int_0^T \frac{h_1 X + h_2 Y}{1 - Z^2} d\tau.
\] (195)

Let \( \phi(\tau) \) be the azimuthal angle of \( s(\tau) \) w. r. t. the constant standard frame such that
\[
\phi = \frac{X \dot{Y} - Y \dot{X}}{X^2 + Y^2}.
\] (196)

The integral \( \int_0^T \phi d\tau \) will assume the value \( 2\pi n \) where \( n \in \mathbb{Z} \) is the winding number of the periodic solution \( s \) around the 3-axis. Since \( \delta \) is only relevant up to integer multiples of \( 2\pi \) we may freely add \( \phi \) to the integrand of (194) and obtain:
\[
\frac{h_1 X + h_2 Y}{1 - Z^2} + \phi = \frac{1}{1 - Z^2} \left( h_1 X + h_2 Y + X \dot{Y} - Y \dot{X} \right)
\] (197)
\[
= \frac{1}{1 - Z^2} \left( h_1 X + h_2 Y + X (h_3 X - h_1 Z) - Y (h_2 Z - h_3 Y) \right)
\] (198)
\[
= \frac{1}{1 - Z^2} \left( h_1 X + h_2 Y + h_3 (X^2 + Y^2) - (h_1 X + h_2 Y) Z \right)
\] (199)
\[
= h_3 + \frac{1}{1 - Z^2} (h_1 X + h_2 Y)(1 - Z)
\] (200)
\[
= h_3 + \frac{h_1 X + h_2 Y}{1 + Z},
\] (201)
which agrees with Eq. (62) of [3] up to a factor of \( \frac{1}{2} \) as discussed above.
IV. SUMMARY

The Rabi oscillations of a periodically driven two-level system can be translated into certain orbits on the Bloch sphere that are solutions of the classical Rabi problem $\dot{s} = \mathbf{h} \times s$. The latter differential equation has various physical and geometric ramifications. It can be analyzed in terms of Floquet theory leading to the notion of a “quasienergy” $\epsilon$. The geometric part $\epsilon_g$ of the quasienergy in turn can be related to the solid angle encircled by the closed curve $S$ given by the solution $s(\tau)$ via the theorem of Gauss-Bonnet, or, alternatively, understood in terms of the geometric phase similarly as for the Foucault pendulum. These well-known connections are recapitulated in the present paper and illustrated by a couple of examples. A probably new result is that $\epsilon_g$ can also be expressed through the length of the “dual loop” $H$, and that the rôle of $S$ and $H$ can, in a certain sense, be interchanged. For the special case of simple curves the quasienergies of a dual pair of curves are reciprocal. The closer study of this “duality of loops” is devoted to future papers.

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