Borcherds lifts of harmonic Maass forms and modular integrals

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Abstract
We extend Borcherds’ singular theta lift in signature (1,2) to harmonic Maass forms of weight 1/2 whose non-holomorphic part is allowed to be of exponential growth at \(i\infty\). We determine the singularities of the lift and compute its Fourier expansion. It turns out that the lift is continuous but not differentiable along certain geodesics in the upper half-plane corresponding to the non-holomorphic principal part of the input. As an application, we obtain a generalization to higher level of the weight 2 modular integral of Duke, Imamoglu and Tóth. Further, we construct automorphic products associated to harmonic Maass forms.

1 Introduction

In [2], Borcherds constructed a regularized theta lift which maps weakly holomorphic modular forms of weight 1/2 to real analytic modular functions with logarithmic singularities at CM points. His results were generalized by Bruinier and Ono [8] to twisted lifts of harmonic Maass forms which map to cusp forms under the \(\xi\)-operator. In the present work, we extend the twisted Borcherds lift to general harmonic Maass forms (which may map to weakly holomorphic modular forms under the \(\xi\)-operator). By taking the derivative of the Borcherds lift of a suitable harmonic Maass form of
weight 1/2, we obtain modular integrals of weight 2 with rational period functions. Their Fourier coefficients are given by twisted traces of geodesic cycle integrals of harmonic Maass forms of weight 0. They generalize the modular integral of Duke, Imamoglu and Tóth [10] to higher level. In the introduction, we restrict to modular forms for the full modular group $\Gamma_1$ for simplicity, but in the body of the work we treat modular forms of arbitrary level $\Gamma_0(N)$ by using the language of vector valued modular forms for the Weil representation. Let us now describe our results in more detail.

1.1 The Borcherds lift of a harmonic Maass form

Recall from [5] that a harmonic Maass form of weight 1/2 for $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ which is annihilated by the invariant Laplace operator $\Delta_{1/2}$, transforms like a modular form of weight 1/2 for $\Gamma_0(4)$, and is at most of linear exponential growth at the cusps of $\Gamma_0(4)$. Such a form can be written as a sum $f = f^+ + f^-$ with a holomorphic part $f^+$ and a non-holomorphic part $f^-$ with Fourier expansions of the shape

$$f^+(\tau) = \sum_{D \in \mathbb{Z}} c_f^+(D)e(D\tau),$$

$$f^-(\tau) = c_f^-(0)\sqrt{v} + \sum_{D < 0} c_f^-(D)\sqrt{v}\beta_{1/2}(4\pi |D|v)e(D\tau) + \sum_{D > 0} c_f^-(D)\sqrt{v}\beta_{1/2}(-4\pi Dv)e(D\tau),$$

with $\tau = u + iv \in \mathbb{H}$, $e(x) = e^{2\pi ix}$ for $x \in \mathbb{C}$, coefficients $c_f^+(D) \in \mathbb{C}$, and

$$\beta_{1/2}(s) = \int_1^\infty e^{-st}t^{-1/2}dt, \quad \beta_{1/2}^c(s) = \int_0^1 e^{-s/t}t^{-1/2}dt.$$

We let $H_{1/2}$ denote the space of harmonic Maass forms of weight 1/2 which satisfy the Kohnen plus space condition, which means that the Fourier expansion is supported on indices $D \equiv 0, 1 \pmod{4}$. The antilinear differential operator

$$\xi_{1/2}f(\tau) = 2iv^{1/2}\frac{\partial}{\partial \bar{\tau}}f(\tau)$$

maps a harmonic Maass form $f \in H_{1/2}$ of weight 1/2 to a weakly holomorphic modular form of weight 3/2. We let $H_{1/2}^+$ be the subspace of $H_{1/2}$ consisting of forms which map to cusp forms under $\xi_{1/2}$, and we let $M_{1/2}^+$ be the subspace of weakly holomorphic modular forms.

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant. Following [2], we define the Borcherds lift $\Phi_\Delta(z, f)$ of a harmonic Maass form $f \in H_{1/2}$ by the regularized integral
\[ \Phi_\Delta(f, z) = \text{CT}_{s=0} \left[ \lim_{T \to \infty} \int_{\mathcal{F}_T(4)} f(\tau) \Theta_\Delta(\tau, z)v^{1/2-s} \frac{du \, dv}{v^2} \right], \]

where \( \Theta_\Delta(\tau, z) \) is a twisted Siegel theta function which transforms in \( \tau \) like a modular form of weight 1/2 for \( \Gamma_0(4) \) and is invariant in \( z \) under \( \Gamma \), \( \mathcal{F}_T(4) \) denotes a suitably truncated fundamental domain for \( \Gamma_0(4) \setminus \mathbb{H} \), and \( \text{CT}_{s=0} F(s) \) denotes the constant term in the Laurent expansion at \( s = 0 \) of a function \( F(s) \) which is meromorphic near \( s = 0 \). Borcherds [3] proved that for \( \Delta = 1 \) and a weakly holomorphic modular form \( f \in M^!_{1/2} \) the regularized theta lift \( \Phi_\Delta(f, z) \) defines a \( \Gamma \)-invariant real analytic function with logarithmic singularities at certain CM points in \( \mathbb{H} \), which are determined by the principal part of \( f \), i.e., by the coefficients \( c_f^+(D) \) with \( D < 0 \). Bruinier and Ono [8] showed that this result remains true for twisted Borcherds lifts of harmonic Maass forms \( f \in H_{1/2} \) which map to cusp forms under the \( \xi \)-operator, which means that \( c_f^+(0) = 0 \) and \( c_f^+(D) = 0 \) for \( D > 0 \). One of the main aims of the present work is to generalize the Borcherds lift \( \Phi_\Delta(z, f) \) to the full space \( H_{1/2} \).

Note that the twisted Siegel theta function \( \Theta_\Delta(\tau, z) \) considered in the introduction vanishes identically for \( \Delta < 0 \) for trivial reasons. Furthermore, to avoid some technical difficulties we omit the case \( \Delta = 1 \) in the introduction and assume \( \Delta > 1 \) for simplicity. However, in the body of the paper we allow arbitrary fundamental discriminants \( \Delta \), which also yields non-trivial results for \( \Delta < 0 \) in higher level cases.

For a discriminant \( D \) we let \( \mathcal{Q}_D \) be the set of integral binary quadratic forms \( \mathcal{Q} = [a, b, c] \) of discriminant \( D = b^2 - 4ac \). For \( D < 0 \) and \( \mathcal{Q} \in \mathcal{Q}_D \) there is an associated CM (or Heegner) point \( z_\mathcal{Q} \in \mathbb{H} \) which is characterized by \( \mathcal{Q}(z_\mathcal{Q}, 1) = 0 \). For \( D > 0 \) there is an associated geodesic in \( \mathbb{H} \) given by

\[ c_\mathcal{Q} = \{ z \in \mathbb{H} : a|z|^2 + bx + c = 0 \}, \]

with \( z = x + iy \in \mathbb{H} \). We let \( H^+_\Delta(f) \) be the set of all CM points \( z_\mathcal{Q} \) corresponding to quadratic forms \( \mathcal{Q} \in \mathcal{Q}_\Delta D \) with \( D < 0 \) such that \( c_f^+(D) \neq 0 \), and we let \( H^-_\Delta(f) \) be the union of all geodesics \( c_\mathcal{Q} \) corresponding to quadratic forms \( \mathcal{Q} \in \mathcal{Q}_\Delta D \) with \( D > 0 \) such that \( c_f^-(D) \neq 0 \). We obtain the following extension of the Borcherds lift on the full space \( H_{1/2} \).

**Theorem 1.1** Let \( \Delta > 1 \) be a fundamental discriminant. For \( f \in H_{1/2} \) the Borcherds lift \( \Phi_\Delta(f, z) \) defines a \( \Gamma \)-invariant harmonic function on \( \mathbb{H} \setminus (H^+_\Delta(f) \cup H^-_\Delta(f)) \). It has ‘logarithmic singularities’ at the CM points in \( H^+_\Delta(f) \) and ‘arcsin singularities’ along the geodesics in \( H^-_\Delta(f) \). More precisely, this means that for \( z_0 \in H^+_\Delta(f) \cup H^-_\Delta(f) \) the function

\[ \Phi_\Delta(f, z) - \sum_{D < 0} c_f^+(D) \sum_{\mathcal{Q} = [a, b, c] \in \mathcal{Q}_\Delta D} \chi_\Delta(\mathcal{Q}) \log |az^2 + bz + c| \]

\[ + \sum_{D > 0} \frac{c_f^-(D)}{\sqrt{D}} \sum_{\mathcal{Q} = [a, b, c] \in \mathcal{Q}_\Delta D} \chi_\Delta(\mathcal{Q}) \arcsin \left( \frac{1}{\sqrt{1 + \frac{1}{D^\tau} (|az|^2 + bx + c)^2}} \right). \]
can be continued to a real analytic function near \( z_0 \). Here \( \chi_\Delta \) is the usual genus character. Note that all the above sums are finite.

We refer the reader to Theorem 3.1 for the general result.

**Remark 1.2**  The logarithmic singularities imply that the Borcherds lift blows up at the Heegner points \( z_Q \in H_\Delta^+(f) \), and the arcsin singularities show that it is continuous but not differentiable at points on the geodesics \( c_Q \subset H_\Delta^-(f) \).

### 1.2 The Fourier expansion of the Borcherds lift

Using Maass–Poincaré series one can always write a harmonic Maass form \( f \in H_{1/2} \) as \( f = f_1 + f_2 \) where \( f_1, f_2 \in H_{1/2} \) satisfy \( c_f^+(D) = 0 \) for all \( D < 0 \) and \( c_f^-(D) = 0 \) for all \( D \geq 0 \). In particular, \( f_2 \) maps to a cusp form under the \( \xi \)-operator, and since the Borcherds lift of such harmonic Maass forms has already been investigated by Bruinier and Ono [8], we assume from now on that \( c_f^+(D) = 0 \) for all \( D < 0 \). In this case, the Borcherds lift \( \Phi_\Delta(f, z) \) only has singularities along the geodesics in \( H_\Delta^-(f) \).

For simplicity, we also assume that \( c_f^-(\Delta n^2) = 0 \) for all \( n > 0 \) in the introduction. This implies that \( \Phi_\Delta(f, z) \) has no singularities along vertical geodesics and simplifies the Fourier expansion of the Borcherds lift, which can be stated as follows.

**Proposition 1.3** Let \( \Delta > 1 \) be a fundamental discriminant and let \( f \in H_{1/2} \) such that \( c_f^+(D) = 0 \) for all \( D < 0 \) and \( c_f^-(\Delta n^2) = 0 \) for all \( n > 0 \). Then for \( z \in \mathbb{H} \setminus H_\Delta^-(f) \) the Borcherds lift of \( f \) has the Fourier expansion

\[
\Phi_\Delta(f, z) = -4 \sum_{m=1}^{\infty} c_f^+(\Delta m^2) \sum_{b(\Delta)} \left( \frac{\Delta}{b} \right) \log |1 - e(mz + b/\Delta)| \\
+ \sqrt{\Delta} L_\Delta(1) \left( 2c_f^+(0) + yc_f^-(0) \right) \\
- 4 \sum_{D > 0} \frac{c_f^-(D)}{\sqrt{D}} \sum_{Q \in \mathcal{Q}_{\Delta D}} \chi_\Delta(Q) \mathbf{1}_Q(z) \left( \arctan \left( \frac{y \sqrt{\Delta D}}{a|z|^2 + bx + c} \right) + \frac{\pi}{2} \right),
\]

where \( \mathbf{1}_Q(z) \) denotes the characteristic function of the bounded component of \( \mathbb{H} \setminus c_Q \), and \( L_\Delta(s) = \sum_{n \geq 1} \left( \frac{\Delta}{n} \right) n^{-s} \) for \( \text{Re}(s) > 1 \) is a Dirichlet \( L \)-function.

For the general result, see Proposition 4.2.

**Remark 1.4**  1. For \( Q \in \mathcal{Q}_{\Delta D} \) with \( a > 0 \) the corresponding geodesic \( c_Q \) is a semicircle centered at the real line which divides \( \mathbb{H} \) into a bounded and an unbounded connected component, so the characteristic function \( \mathbf{1}_Q \) makes sense.

2. The sum over \( D \) in the third line is finite since \( f \) has a finite principal part. The sum over \( Q \in \mathcal{Q}_{\Delta D} \) is locally finite since each point \( z \in \mathbb{H} \) lies in the bounded component of \( \mathbb{H} \setminus c_Q \) for finitely many geodesics \( c_Q \in \mathcal{Q}_{\Delta D} \), and it vanishes for \( y \gg 0 \) large enough since the imaginary parts of points lying on geodesics \( c_Q \) for \( Q \in \mathcal{Q}_{\Delta D} \) are bounded by \( \sqrt{\Delta D} \).

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3. We have $z \in c_Q$ for $Q = [a, b, c]$ if and only if $a|z|^2 + bx + c = 0$. Further, for $a > 0$ a point $z \in \mathbb{H}$ lies in the inside of the bounded component of $\mathbb{H} \setminus c_Q$ if and only if $a|z|^2 + bx + c < 0$. Since $\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$, we see from the Fourier expansion that $\Phi_\Delta(f, z)$ is continuous. However, computing the derivative of the above expansion for $z \in \mathbb{H} \setminus H^-_\Delta(f)$ shows that the third line is not differentiable at points $z \in H^-_\Delta(f)$. More precisely, the derivative of $\Phi_\Delta(f, z)$ has jumps along the geodesics in $H^-_\Delta(f)$.

1.3 The derivative of the Borcherds lift

We apply the (derivative of the) Borcherds lift to certain interesting harmonic Maass forms of weight 1/2 for $\Gamma_0(4)$, in order to construct modular integrals of weight 2 with rational period functions. In [10], Duke, Imamoglu and Tóth constructed a basis $\{h_d\}$ (indexed by discriminants $d > 0$) of $H_{1/2}$, which under $\xi_{1/2}$ maps to a basis $\{g_d\}$ of the space of weakly holomorphic modular forms of weight 3/2 for $\Gamma_0(4)$. More precisely, the $g_d$ are the generating series of traces of singular moduli, see [17]. The coefficients of the $h_d$ are given by traces of CM values and traces of (regularized) cycle integrals of weakly holomorphic modular functions for $\Gamma$. For example, the Fourier expansion of the function $h = h_1$ is given by

$$h(\tau) = \frac{1}{2\pi} \sum_{D > 0} \text{tr}_f(D)q^D + 2\sqrt{v}\beta_{1/2}(-4\pi v)q$$

$$-8\sqrt{v} + \sqrt{v} \sum_{D < 0} \text{tr}_f(D)\beta_{1/2}(4\pi|D|v)q^D,$$

where

$$\text{tr}_f(D) = \begin{cases} \sum_{Q \in \mathbb{Q}_D/\Gamma} \frac{J(\mathbb{Q})}{|\mathbb{Q}|}, & D < 0, \\ \sum_{Q \in \mathbb{Q}_D/\Gamma} \int_{\Gamma \mathbb{Q} \setminus \mathbb{Q}} J(z) \frac{dz}{Q(z, 1)}, & D > 0, \end{cases}$$

are traces of CM values and geodesic cycle integrals of $J = j - 744$, which need to be regularized as explained in [7] if $D > 0$ is a square. The harmonic Maass form $h$ does not map to a cusp form but to a weakly holomorphic modular form under $\xi_{1/2}$, so it is interesting to apply our extension of the Borcherds lift to it. The coefficients $c^h_\mathbb{Q}(D)$ for $D \leq 0$ vanish, so the Borcherds lift $\Phi_\Delta(h, z)$ is a harmonic $\Gamma$-invariant function on $\mathbb{H} \setminus H^-_\Delta(h)$ with arcnorm singularity along the geodesics in $H^-_\Delta(h)$. In this case, the latter set is just the union of all geodesics $c_Q$ for $Q \in \mathbb{Q}_\Delta$. Hence the derivative $\Phi'_\Delta(h, z) = \frac{\partial}{\partial z} \Phi_\Delta(h, z)$ is a holomorphic function on $\mathbb{H} \setminus H^-_\Delta(h)$ transforming like a modular form of weight 2 for $\Gamma$. Moreover, it turns out that $\Phi'_\Delta(h, z)$ has jump singularities along the geodesics in $H^-_\Delta(h)$, and admits a nice Fourier expansion.

**Proposition 1.5** Let $\Delta > 1$ be a fundamental discriminant. The derivative $\Phi'_\Delta(h, z)$ of the Borcherds lift of $h$ is a holomorphic function on $\mathbb{H} \setminus H^-_\Delta(h)$ which transforms
like a modular form of weight 2 for \( \Gamma \). For \( z \in \mathbb{H} \setminus H_{\Delta}(h) \) it has the expansion

\[
\frac{1}{4\pi i \sqrt{\Delta}} \Phi'_{\Delta}(h, z) = \frac{1}{2\pi} \text{tr}_1(\Delta) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left( \sum_{m|n} \left( \frac{\Delta}{n/m} \right) m \text{tr}_J(\Delta m^2) \right) e(nz) + \frac{1}{\pi} \sum_{Q \in \mathcal{Q}_\Delta} \mathbf{1}_Q(z) Q(z, 1),
\]

where \( \mathbf{1}_Q \) denotes the characteristic function of the bounded component of \( \mathbb{H} \setminus cQ \).

The result for general harmonic Maass forms \( f \in H_{1/2} \) of higher level is given in Proposition 5.3 and Corollary 5.4.

**Remark 1.6** The Fourier series over \( n \) is holomorphic on \( \mathbb{H} \), whereas the sum over \( Q \) has jump singularities along the geodesics \( cQ \) with \( Q \in \mathcal{Q}_\Delta \). Again, the sum over \( Q \) is locally finite and vanishes for \( y \gg 0 \) large enough.

### 1.4 Modular integrals

In [10, Theorem 5], the authors proved that the generating series

\[
F_{\Delta}(z) = \frac{1}{\pi} \sum_{m=0}^{\infty} \text{tr}_{J_m}(\Delta) e(mz),
\]

with \( J_m(z) = q^{-m} + O(q) \in M_0^! \), e.g. \( J_0 = 1 \) and \( J_1 = J \), defines a holomorphic function on \( \mathbb{H} \) which transforms as

\[
z^{-2}F_{\Delta} \left( -\frac{1}{z} \right) - F_{\Delta}(z) = \frac{2}{\pi} \sum_{Q \in \mathcal{Q}_\Delta \cap \mathcal{Q} = c<0< a} \frac{1}{Q(z, 1)},
\]

so \( F_{\Delta}(z) \) is a holomorphic modular integral of weight 2 with holomorphic rational period functions in the sense of [15].

Returning to the derivative of the Borcherds lift of \( h \), we note that

\[
\text{tr}_{J_m}(\Delta) = \sum_{d|m} \left( \frac{\Delta}{m/d} \right) d \text{tr}_J(\Delta d^2),
\]

compare [16, pp. 290–292], so \( F_{\Delta}(z) \) in fact agrees with \( \Phi'_{\Delta}(h, z) \) up to some constant factor if \( y \gg 0 \) is sufficiently large. The transformation behaviour of the singular part in the Fourier expansion of \( \Phi'_{\Delta}(h, z) \) can easily be determined, so we can recover (1) from Proposition 1.5. Further, using the Borcherds lift we generalize the construction of modular integrals of weight 2 with rational period functions from [10] to higher level, see Proposition 6.1. The coefficients of our modular integrals are linear combinations.
of Fourier coefficients of the holomorphic parts of harmonic Maass forms \( f \) of weight 1/2. Choosing \( f \) as the image of a theta lift of a harmonic Maass form \( F \) of weight 0 studied by Bruinier et al. [7], we obtain modular integrals whose coefficients are linear combinations of traces of cycle integrals of \( F \), see Example 6.3. In fact, the construction of \( F_\Delta \) as a theta lift and its generalizations to higher level were our main motivation to extend the Borcherds lift to the full space \( H_{1/2} \).

1.5 Borcherds products

Bruinier and Ono [8] defined a twisted Borcherds product associated to a harmonic Maass form \( f \in H_{1/2}^+ \) with real coefficients \( c^+_f(D) \) for all \( D \), and \( c^+_f(D) \in \mathbb{Z} \) for \( D \leq 0 \). For \( \Delta > 1 \) a fundamental discriminant and \( y \gg 0 \) sufficiently large the twisted Borcherds lift of \( f \) is given by

\[
\Psi_{\Delta}(f, z) = \prod_{m=1}^{\infty} \prod_{b(\Delta)} [1 - e(mz + b/\Delta)] \left( \frac{1}{\pi} \right) c^+_f(\Delta m^2).
\]

It has a meromorphic continuation to \( \mathbb{H} \) with roots and poles at CM points corresponding to the principal part of \( f \), and it transforms like a modular form of weight 0 with some unitary character for \( \Gamma \). We will define Borcherds products associated to general harmonic Maass forms \( f \in H_{1/2} \). For simplicity, in the introduction we only consider the harmonic Maass form \( \pi h \). The general result is given in Theorem 6.8.

**Theorem 1.7**  Let \( \Delta > 1 \) be a fundamental discriminant. Then the infinite product

\[
\Psi_{\Delta}(z) = e \left( -\sqrt{\Delta} \ tr_1(\Delta) z \right) \prod_{m=1}^{\infty} \prod_{b(\Delta)} [1 - e(mz + b/\Delta)] \left( \frac{1}{\pi} \right) w_f(\Delta m^2)
\]

converges to a holomorphic function on \( \mathbb{H} \). Its logarithmic derivative is given by

\[
\frac{d}{dz} \log(\Psi_{\Delta}(z)) = -2\pi^2 i \sqrt{\Delta} F_{\Delta}(z).
\]

Further, it transforms as

\[
\Psi_{\Delta}(z + 1) = e \left( -\sqrt{\Delta} \ tr_1(\Delta) \right) \Psi_{\Delta}(z),
\]

\[
\Psi_{\Delta} \left( -\frac{1}{z} \right) = e \left( -2 \sum_{Q \in Q_\Delta} \sum_{c < 0 < a} \left( \log \left( \frac{z - w_Q}{i - w_Q} \right) - \log \left( \frac{z - w'_Q}{i - w'_Q} \right) \right) \right) \Psi_{\Delta}(z),
\]

where \( w_Q > w'_Q \) denote the real endpoints of the geodesic \( c_Q \).

The work is organized as follows. We start with a section on the necessary preliminaries about the Grassmannian model of the upper half-plane, which is convenient for
the study of regularized theta lifts, and vector valued harmonic Maass forms for the Weil representation.

In Sect. 3, we define the Borcherds lift of a harmonic Maass form of weight 1/2 and prove its basic analytic properties.

In Sect. 4 we compute the Fourier expansion of the Borcherds lift. After that, in Sect. 5 we compute the Fourier expansion of the derivative of the Borcherds lift and show that it has jump singularities along geodesics.

In Sect. 6.1 we apply the derivative of the Borcherds lift to an interesting class of harmonic Maass forms which arise as images of regularized theta lifts of scalar valued weight 0 harmonic Maass forms \( F \). We show that the generating series of certain sums of traces of cycle integrals of \( F \) transform like modular forms of weight 2 for \( \Gamma_0(N) \) and are holomorphic up to jump singularities along geodesics. Their non-singular parts yield holomorphic modular integrals of weight 2 for \( \Gamma_0(N) \) with rational period functions.

Finally, in Sect. 6.2 we define the Borcherds product associated to a general harmonic Maass form of weight 1/2 and prove its modularity.

\section{Preliminaries}

\subsection{The Grassmannian model of the upper half-plane}

For a positive integer \( N \) we consider the quadratic space \( V \) of all rational traceless 2 by 2 matrices, equipped with the quadratic form \( Q(X) = N \det(X) \) and the associated bilinear form \( (X, Y) = -N \operatorname{tr}(XY) \). It has signature \((1, 2)\). We let \( D \) be the Grassmannian of positive definite lines in \( V(\mathbb{R}) = V \otimes \mathbb{R} \). We identify it with the complex upper half-plane \( \mathbb{H} \) by associating to \( z = x + iy \in \mathbb{H} \) the line spanned by

\[
X(z) = \frac{1}{\sqrt{2Ny}} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}.
\]

The group \( \text{SL}_2(\mathbb{R}) \) acts as isometries on \( V(\mathbb{R}) \) by conjugation and this action is compatible with the action by fractional linear transformations on \( \mathbb{H} \) under the above identification.

For \( X \in V \) and \( z \in D \) we let \( X_z \) and \( X_{z^\perp} \) denote the projection of \( X \) to \( z \) and its orthogonal complement \( z^\perp \), respectively.

\subsection{A lattice related to \( \Gamma_0(N) \)}

In \( V \) we consider the even lattice

\[
L = \left\{ \begin{pmatrix} -b & -c/N \\ a & b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.
\]
Its dual lattice is given by

\[ L' = \left\{ \left( \begin{array}{r} -b/2N \\ c/N \\ a \end{array} \right) : a, b, c \in \mathbb{Z} \right\} \].

We see that \( L'/L \cong \mathbb{Z}/2N \mathbb{Z} \), and we will use this identification without further notice in the following. For \( m \in \mathbb{Q} \) and \( h \in L'/L \) we let

\[ L_{m,h} = \{ X \in L + h : Q(X) = m \} \].

The group \( \Gamma_0(N) \) acts on \( L_{m,h} \), with finitely many orbits if \( m \neq 0 \).

Let \( X \in L_{m,h} \). If \( m > 0 \) we let \( z_X = \text{span}(X) \in D \) be the associated CM (or Heegner) point. If \( m < 0 \) we let \( c_X = \{ z \in D : z \perp X \} \) be the associated geodesic in \( D \). We use the same symbols for the corresponding points and geodesics in \( \mathbb{H} \). We can identify \( X = [a, b, c] \in L' \) with the binary quadratic form

\[ Q_X = [aN, b, c] \].

Under this identification, the set \( L_{m,h} \) corresponds to the set of all binary quadratic forms \( [aN, b, c] \) with \( a, b, c \in \mathbb{Z} \) of discriminant \( -4Nm \) and \( b \equiv h \) (mod \( 2N \)). Furthermore, CM points and geodesics associated to vectors \( X \in L_{m,h} \) correspond to the usual CM points and geodesics associated to \( Q_X \). We define the quantity

\[ p_X(z) = -\frac{aN|z|^2 + bx + c}{\gamma \sqrt{N}}, \]

which vanishes exactly at the geodesic \( c_X \).

### 2.3 Vector valued harmonic Maass forms for the Weil representation

Let \( \tilde{\Gamma} = \text{Mp}_2(\mathbb{Z}) \) be the integral metaplectic group, realized as the set of pairs \((M, \phi)\) with \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) and \( \phi : \mathbb{H} \to \mathbb{C} \) holomorphic with \( \phi^2(\tau) = c\tau + d \). We let \( \tilde{\Gamma}_\infty \) be the subgroup of \( \tilde{\Gamma} \) generated by \( \tilde{T} = \left( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), 1 \right) \). Let \( \mathbb{C}[L'/L] \) be the group ring of \( L'/L \), generated by the formal basis vector \( e_\gamma \) for \( \gamma \in L'/L \). We let \( \langle \cdot, \cdot \rangle \) be the inner product on \( \mathbb{C}[L'/L] \) which is antilinear in the second variable and satisfies \( \langle e_\gamma, e_\beta \rangle = \delta_{\gamma,\beta} \). The Weil representation \( \rho_L \) is a unitary representation of \( \tilde{\Gamma} \) on \( \mathbb{C}[L'/L] \), see [3, Section 4]. We let \( \rho_L^* \) be the dual Weil representation.

A harmonic Maass form of weight 1/2 for \( \rho_L \) is a harmonic function \( f : \mathbb{H} \to \mathbb{C}[L'/L] \) which transforms like a modular form of weight 1/2 for \( \rho_L \) and is at most of linear exponential growth at \( i\infty \). We let \( H_{1/2,\rho_L} \) be the space of all harmonic Maass forms.
forms of weight 1/2 for $\rho_L$. Every $f \in H_{1/2,\rho_L}$ can be written as a sum $f = f^+ + f^-$ of a holomorphic and a non-holomorphic part having Fourier expansions of the form

$$
f^+ = \sum_{h \in L'/L} \sum_{n \gg -\infty} c_f^+(n, h) e(n \tau) \varepsilon_h,
$$

$$
f^- = \sum_{h \in L'/L} \left( c_f^-(0, h) \sqrt{v} + \sum_{n < 0} c_f^-(n, h) \sqrt{v} \beta_{1/2}(-4\pi n v) e(n \tau) \right) \varepsilon_h,
$$

with coefficients $c_f^\pm(D) \in \mathbb{C}$, $e(x) = e^{2\pi ix}$ for $x \in \mathbb{C}$, and

$$
\beta_{1/2}(s) = \int_1^\infty e^{-st} t^{-1/2} dt, \quad \beta_{1/2}^c(s) = \int_0^1 e^{-st} t^{-1/2} dt.
$$

Remark 2.1 For $N = 1$ the space $H_{1/2,\rho_L}$ of vector valued harmonic Maass forms is isomorphic to the space $H_{1/2}$ of scalar valued harmonic Maass forms from the introduction. The isomorphism is given by the map $f_0(\tau) e_0 + f_1(\tau) e_1 \mapsto f_0(4\tau) + f_1(4\tau)$, compare [12, Theorems 5.1 and 5.4]. This identification can be used to translate the results from the body of the paper to the scalar valued setup in the introduction. More generally, by the results of [12, Section 5], vector valued functions which transform like modular forms of weight 1/2 for $\rho_L$ or $\rho_L^*$ (for arbitrary $N$) can be identified with functions on $\mathbb{H} \times \mathbb{C}$ which transform like (skew-holomorphic) Jacobi forms of weight 1 and index $N$.

3 Analytic properties of the Borcherds lift

In this section we extend Borcherds’ regularized theta lift to general harmonic Maass forms of weight 1/2.

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant (possibly 1) and let $r \in \mathbb{Z}/2N\mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4} N$. We set

$$
\tilde{\rho}_L = \begin{cases} 
\rho_L, & \text{if } \Delta > 0, \\
\rho_L^*, & \text{if } \Delta < 0.
\end{cases}
$$

$$
Q_\Delta(X) = \frac{1}{|\Delta|} Q(X), \quad (X, Y)_\Delta = \frac{1}{|\Delta|} (X, Y).
$$

We consider the twisted Siegel theta function

$$
\Theta_{\Delta, r}(\tau, z) = v \sum_{h \in L'/L} \sum_{X \in L' + rh \mathbb{C}} \chi_{\Delta}(X) e \left( \tau Q_\Delta(Xz) + \bar{\tau} Q_\Delta(Xz^\perp) \right) \varepsilon_h, \quad (3)
$$
which is $\Gamma_0(N)$-invariant in $z$ and transforms like a modular form of weight $-1/2$ for $\tilde{\rho}_L$, compare [8, Theorem 4.1]. Here $\chi_\Delta$ is the genus character defined as in [8, Section 4].

For $f \in H_{1/2, \tilde{\rho}_L^*}$, we let

$$H_{\Delta, r}^+(f) = \bigcup_{h \in L'/L, n < 0} \{ z_X : X \in L_{-|\Delta| n, rh} \}$$

and

$$H_{\Delta, r}^-(f) = \bigcup_{h \in L'/L, n > 0} \bigcup_{X \in L_{-|\Delta| n, rh}} c_X,$$

be the sets of Heegner points and geodesics associated to $f$.

Following Borcherds [3], we define the regularized theta lift of $f \in H_{1/2, \tilde{\rho}_L^*}$ by

$$\Phi_{\Delta, r}(f, z) = \text{CT}_{s=0} \left( \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), \Theta_{\Delta, r}(\tau, z) \rangle v^{-s} \frac{du \: dv}{v^2} \right),$$

where

$$\mathcal{F}_T = \{ \tau = u + iv \in \mathbb{H} : |\tau| \geq 1, |u| \leq 1/2, v \leq T \}$$

is a truncated fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and $\text{CT}_{s=0} F(s)$ denotes the constant term in the Laurent expansion of the analytic continuation of $F(s)$ at $s = 0$.

We say that a complex-valued function $f$ defined on some subset of $\mathbb{R}^n$ has a singularity of type $g$ (written $f \approx g$) at a point $z_0$ if there is an open neighbourhood $U$ of $z_0$ such that $f$ and $g$ are defined on a dense subset of $U$ and $f - g$ can be continued to a real analytic function on $U$.

Theorem 3.1 For $f \in H_{1/2, \tilde{\rho}_L^*}$ the Borcherds lift $\Phi_{\Delta, r}(f, z)$ defines a $\Gamma_0(N)$-invariant real analytic function on $\mathbb{H} \setminus (H_{\Delta, r}^+(f) \cup H_{\Delta, r}^-(f))$ with

$$\Delta_0 \Phi_{\Delta, r}(f, z) = \begin{cases} -2c_f^+(0, 0), & \text{if } \Delta = 1, \\ 0, & \text{if } \Delta \neq 1. \end{cases}$$

At a point $z_0 \in H_{\Delta, r}^+(f) \cup H_{\Delta, r}^-(f)$ it has a singularity of type

$$- \sum_{h \in L'/L} \sum_{n < 0} c_f^+(n, h) \sum_{X \in L_{-|\Delta| n, rh}} \chi_\Delta(X) \log(-Q_\Delta(X_{\perp} z_0))$$

$$+ \sum_{h \in L'/L} \sum_{n > 0} c_f^-(n, h) n^{-1/2} \sum_{X \in L_{-|\Delta| n, rh}} \chi_\Delta(X) \arcsin \left( \frac{Q_\Delta(X)}{Q_\Delta(X_{\perp} z_0)} \right).$$
Remark 3.2 1. For $X = \left( -\frac{b}{2N} -\frac{c}{N} \right) \in L'$ we have

$$-Q_\Delta(X_{z\perp}) = \frac{1}{4N|\Delta|y^2} |Q_X(z)|^2 = \frac{1}{4N|\Delta|y^2} |aNz^2 + bz + c|^2,$$

which yields a more explicit formula for the singularities. Since $0 < \frac{Q_\Delta(X)}{Q_\Delta(X_{z\perp})} \leq 1$ for all $X$ with $Q_\Delta(X) < 0$, and $Q_\Delta(X)/Q_\Delta(X_{z\perp}) = 1$ exactly for $z \in c_X$, we see that the Borcherds lift extends to a continuous function on $H_\Delta \setminus H^+_\Delta, r(f)$, which is not differentiable along the geodesics in $H^-_\Delta, r(f)$. Note that we can also write the singularities in the form

$$\arcsin \left( \sqrt{\frac{Q_\Delta(X)}{Q_\Delta(X_{z\perp})}} \right) = \arctan \left( \sqrt{\frac{Q_\Delta(X)}{-Q_\Delta(X_{z\perp})}} \right).$$

2. Recall from Sect. 2.2 that we may identify $L'/L$ with $\mathbb{Z}/2N \mathbb{Z}$. For every exact divisor $d \mid |N$ (i.e., $d \mid N$ and $(d, N/d) = 1$) there is an involution $w_d$ on $\mathbb{Z}/2N \mathbb{Z}$ defined by the equations

$$w_d(h) \equiv -h \pmod{2d} \quad \text{and} \quad w_d(h) \equiv h \pmod{2N/d}.$$

Moreover, $w_d$ respects the finite quadratic form $x \mapsto -x^2/4N$ on $\mathbb{Z}/2N \mathbb{Z}$, and hence defines an orthogonal map. It acts on vector valued modular forms $f = \sum_h f_h e_h$ for $\rho_L$ or $\rho_L^*$ by $f^{w_d} = \sum_h f_h e_{w_d(h)}$, i.e., by permuting the components. Under the natural identification between vector valued modular forms and Jacobi forms alluded to in Remark 2.1, the map $w_d$ corresponds to the Atkin–Lehner operator on Jacobi forms defined in [12, Theorem 5.2].

It is not hard to show that the usual Atkin–Lehner involution $W_d$ acts on the Siegel theta function by

$$\Theta_{\Delta, r}(\tau, W_d z) = \Theta_{\Delta, r}(\tau, z)^{w_d}.$$

This implies that the Borcherds lift satisfies

$$\Phi_{\Delta, r}(f, W_d z) = \Phi_{\Delta, r}(f^{w_d}, z).$$

Proof of Theorem 3.1 We first show that for $z \in H_\Delta \setminus (H^+_\Delta, r(f) \cup H^-_\Delta, r(f))$ the integral in (4) converges absolutely and locally uniformly for $\Re(s) > 1/2$ and has a meromorphic continuation to $s = 0$. The proof follows the arguments of [4, Proposition 2.8].

The integral over the compact set $F_1 = \{ \tau \in H : |\tau| \geq 1, |u| \leq 1/2, v \leq 1 \}$ converges absolutely and locally uniformly for all $s \in \mathbb{C}$ and $z \in H$. We consider the remaining integral

$$\varphi(z, s) = \int_{v=1}^{\infty} \int_{u=0}^{1} \left( f(\tau), \Theta_{\Delta, r}(\tau, z) \right) v^{-s} \frac{du dv}{v^2}.$$
Inserting the Fourier expansions of \( f(\tau) \) and \( \Theta_{\Delta,r}(\tau, z) \) and carrying out the integral over \( u \), we obtain

\[
\varphi(z, s) = \chi_\Delta(0) \left( c_f^+(0, 0) \int_{v=1}^\infty v^{-1-s} dv + c_f^-(0, 0) \int_{v=1}^\infty v^{-1/2-s} dv \right) \\
+ \int_{v=1}^\infty \sum_{h, X} \chi_\Delta(X) c_f^+ (-Q_\Delta(X), h) \exp \left( 4\pi Q_\Delta(X_z) v \right) v^{-s-1} dv \\
+ \int_{v=1}^\infty \sum_{Q(X)=0} \chi_\Delta(X) c_f^- (-Q_\Delta(X), h) \exp \left( 4\pi Q_\Delta(X_z) v \right) v^{-s-1/2} dv \\
+ \int_{v=1}^\infty \sum_{Q(X)>0} \chi_\Delta(X) c_f^- (-Q_\Delta(X), h) \beta_{1/2} (4\pi Q_\Delta(X) v) \\
\times \exp \left( 4\pi Q_\Delta(X_z) v \right) v^{-s-1/2} dv \\
+ \int_{v=1}^\infty \sum_{Q(X)<0} \chi_\Delta(X) c_f^- (-Q_\Delta(X), h) \beta^\prime_{1/2} (4\pi Q_\Delta(X) v) \\
\times \exp \left( 4\pi Q_\Delta(X_z) v \right) v^{-s-1/2} dv
\]

where the sums run over \( h \in L'/L \) and \( X \in (L + rh) \setminus \{0\} \) with \( Q(X) \equiv \Delta Q(h) \) (mod \( \Delta \)).

Since \( \chi_\Delta(0) = 0 \) for \( \Delta \neq 1 \) the integrals in the first line only appear if \( \Delta = 1 \). They can be evaluated for \( \Re(s) > 1/2 \) by

\[
\int_{v=1}^\infty v^{-1-s} dv = \frac{1}{s}, \quad \int_{v=1}^\infty v^{-1/2-s} dv = \frac{1}{s - 1/2},
\]

giving their meromorphic continuations to \( s = 0 \). Note that this shows that for \( \Delta = 1 \) the regularization involving the extra parameter \( s \) is really necessary.

The integral in the second line involving the coefficients \( c_f^+(n, h) \) converges locally uniformly and absolutely for \( s \in \mathbb{C} \) and \( z \in \mathbb{H} \setminus \mathbb{H}_\Delta, (f) \) by the same arguments as in the proof of [4, Proposition 2.8]. The integrals over the sums corresponding to \( Q(X) = 0 \) and \( Q(X) > 0 \) in the third and fourth line can be treated in the same way, and they converge locally uniformly and absolutely for \( s \in \mathbb{C} \) and \( z \in \mathbb{H} \).

The remaining integral in the fifth line can be written as

\[
\sum_{h \in L'/L, n>0} c_f^-(n, h) \int_{v=1}^\infty \sum_{X \in L' \setminus \mathbb{H} \setminus \mathbb{H}_\Delta, (f)} \chi_\Delta(X) \beta^\prime_{1/2} (4\pi Q_\Delta(X) v) \\
\times \exp \left( 4\pi Q_\Delta(X_z) v \right) v^{-s-1/2} dv,
\]

where the first two sums are finite. Hence, estimating

\[
\beta^\prime_{1/2} (4\pi Q_\Delta(X) v) \leq 2 \exp(-4\pi Q_\Delta(X) v)
\]
and using $Q_\Delta(X) = Q_\Delta(X_z) + Q_\Delta(X_{z\perp})$, it suffices to consider the integral
\begin{equation}
\int_{v=1}^{\infty} \sum_{X \in L_{-|\Delta|n, rh}} \exp\left(-4\pi Q_\Delta(X_z)v\right)v^{-\Re(s)-1/2}dv.
\end{equation}

For any $C \geq 0$ and any compact subset $K \subset \mathbb{H}$ the set
\[ \{ X \in L_{-|\Delta|n, rh} : \exists z \in K \text{ with } |Q_\Delta(X_z)| \leq C \} \]
is finite, so if $z \in K \subset \mathbb{H}\setminus H^{-}_{\Delta,r}(f)$ then there is some $\varepsilon > 0$ such that $Q_\Delta(X_z) > \varepsilon$ for all $X \in L_{-|\Delta|n, rh}$. We can now estimate
\begin{align*}
\sum_{X \in L_{-|\Delta|n, rh}} \exp\left(-4\pi Q_\Delta(X_z)v\right) &\leq e^{-2\pi \varepsilon v}e^{\pi n} \\
&\times \sum_{X \in L_{-|\Delta|n, rh}} \exp\left(-\pi\left|Q_\Delta(X_z) - Q_\Delta(X_{z\perp})\right|\right)
\end{align*}
for $v \geq 1$. The series on the right-hand side converges since $X \mapsto Q_\Delta(X_z) - Q_\Delta(X_{z\perp})$ is a positive definite quadratic form. In particular, the integral in (5) converges absolutely and locally uniformly for $s \in C$ and $z \in \mathbb{H}\setminus H^{-}_{\Delta,r}(f)$. This shows that the regularized theta integral exists.

By similar arguments as above we see that all iterated partial derivatives of $\Phi_{\Delta,r}(f, z)$ converge absolutely and locally uniformly on $\mathbb{H}\setminus (H^{+}_{\Delta,r}(f) \cup H^{-}_{\Delta,r}(f))$, so the Borcherds lift is a smooth function. The statement concerning the Laplacian can now be proven by interchanging $\Delta_0 = \Delta_{0,z}$ with the integral, using the differential equation
\[ \Delta_{0,z} \Theta_{\Delta,r}(\tau, z) = 4v^{1/2}\Delta_{1/2,r}v^{-1/2}\Theta_{\Delta,r}(\tau, z), \]
(which can be checked by a direct calculation) and then applying Stokes’ theorem to move $\Delta_{1/2,r}$ from the theta function to $f(\tau)v^{-s}$ in the integral (compare [4, Lemma 4.3]). It is easy to verify that the appearing boundary integrals vanish. By computing $\Delta_{1/2}(f(\tau)v^{-s})$ explicitly and using that $f$ is harmonic, we obtain
\[ \Delta_0 \Phi_{\Delta,r}(f, z) = -2\text{Res}_{s=0} \lim_{T \to \infty} \int_{F_T} f(\tau) \Theta_{\Delta,r}(\tau, z) v^{-s} \frac{du \, dv}{v^2}. \]

We have seen above that the integral on the right-hand side is holomorphic at $s = 0$ if $\Delta \neq 1$, and has a simple pole with residue $a_f^+(0, 0)$ if $\Delta = 1$, coming from the first integral in the first line of $\varphi(z, s)$. This shows the Laplace equation for $\Phi_{\Delta,r}(f, z)$, which also implies that the Borcherds lift is real analytic by a standard regularity result for elliptic differential equations.

The singularities of $\Phi_{\Delta,r}(f, z)$ can be determined using the following lemma with $n = -Q_\Delta(X)$ and $r = -Q_\Delta(X_{z\perp})$. \[ \square \]
Lemma 3.3 1. The function

\[ I^+(t) = \int_{v=1}^{\infty} e^{-4\pi tv} \frac{dv}{v} \]

is real analytic for \( t > 0 \) and has a singularity of type \(-\log(t)\) at \( t = 0 \).

2. For \( n > 0 \) the function

\[ I_n^-(t) = \int_{v=1}^{\infty} \sqrt{v} \beta_{1/2}(-4\pi nv)e^{-4\pi tv} \frac{dv}{v} \]

is real analytic for \( t > n \) and has a singularity of type \( n^{-1/2} \arcsin\left(\frac{n}{t}\right)\) at \( t = n \).

Proof We follow the proof of [3, Lemma 6.1]. Using partial integration and the fact that \( \log(v) \) is integrable near \( v = 0 \), we see that

\[ I^+(t) \approx 4\pi t \int_{v=0}^{\infty} e^{-4\pi tv} \log(v)dv = \int_{v=0}^{\infty} e^{-v} \log\left(\frac{v}{4\pi t}\right)dv \approx -\log(t). \]

For \( n > 0 \), we use that \( \sqrt{v} \beta_{1/2}(-4\pi nv) = O(\sqrt{v}) \) as \( v \to 0 \) and compute

\[
I_n^-(t) \approx \int_{v=0}^{\infty} \left(2\sqrt{v} \int_{w=0}^{1} e^{4\pi n vw^2} dw\right) e^{-4\pi tv} \frac{dv}{v} \\
= \int_{w=0}^{1} \frac{1}{\sqrt{t - nw^2}} dw \\
= n^{-1/2} \arcsin\left(\sqrt{\frac{n}{t}}\right).
\]

This finishes the proof of the lemma and of Theorem 3.1.

4 The Fourier expansion of the Borcherds lift

Next, we compute the Fourier expansion of the Borcherds lift. To this end, we first need to introduce a special function which captures the arcsin singularities of \( \Phi_{\Delta,r}(f, z) \) along vertical geodesics.

For \( a \geq 1 \) and \( \Re(s) > -1 \) we define

\[
\arcsin_s\left(\frac{1}{\sqrt{a}}\right) = \int_{0}^{1} \frac{1}{\sqrt{a - t^2}} \left(\frac{1 - t^2}{a - t^2}\right)^s dt. \tag{6}
\]

The function \( \arcsin_s \) is holomorphic in \( s \) and satisfies

\[ \arcsin_0(1/\sqrt{a}) = \arcsin(1/\sqrt{a}). \]
The factor $(1 - t^2)^s$ ensures that the integral converges at $a = 1$ if $\text{Re}(s) \geq 1/2$, and the factor $(a - t^2)^s$ in the denominator was added to make the estimate

$$| \arcsin_s(1/\sqrt{a}) | \leq (a - 1)^{-\text{Re}(s) - 1/2}$$

(7)

for $a > 1$ and $\text{Re}(s) > 0$ hold. Note that for $\text{Re}(s) > -1$ we can write

$$\arcsin_s \left( \frac{1}{\sqrt{a}} \right) = \frac{\sqrt{\pi}}{2\Gamma(s + 1/2)} B(1/a; s + 1/2, 1/2),$$

(8)

where

$$B(z; \alpha, \beta) = \int_0^z u^{\alpha - 1} (1 - u)^{\beta - 1} du$$

is the incomplete beta function. The following lemma will be needed for the computation of the Fourier expansion of the Borcherds lift.

**Lemma 4.1** For $z = x + iy \in \mathbb{H}$ with $x \notin \mathbb{Z}$ and $\text{Re}(s) > 0$ we have the Fourier expansion

$$\sum_{\ell \in \mathbb{Z}} \arcsin_s \left( \frac{y}{\sqrt{(x + \ell)^2 + y^2}} \right) = y \frac{\sqrt{\pi}}{\Gamma(s + 1/2)} + 2y \frac{\sqrt{\pi}}{\Gamma(s + 1/2)} \sum_{n \neq 0} (\pi |n| y)^s \left( \int_0^1 (1 - t^2)^{s/2} K_s \left( 2\pi |n| y \sqrt{1 - t^2} \right) dt \right) \cos(2\pi nx),$$

(9)

where $K_s$ denotes the $K$-Bessel function of order $s$. In particular, we have

$$\lim_{s \to 0} \left( \sum_{\ell \in \mathbb{Z}} \arcsin_s \left( \frac{y}{\sqrt{(x + \ell)^2 + y^2}} \right) - y \frac{\sqrt{\pi}}{\Gamma(s + 1/2)} \right) = 2y \sum_{n \neq 0} \left( \int_0^1 K_0 \left( 2\pi |n| y \sqrt{1 - t^2} \right) dt \right) \cos(2\pi nx).$$

(10)

The sum on the right-hand side converges absolutely for all $z \in \mathbb{H}$.

**Proof** The estimate (7) shows that the series over $\ell$ in (9) converges absolutely and locally uniformly for all $z \in \mathbb{H}$ for $\text{Re}(s) > 0$. A short calculation using the representation (8) shows that for $x \notin \mathbb{Z}$ we have

$$\frac{\partial}{\partial x} \arcsin_s \left( \frac{y}{\sqrt{(x + \ell)^2 + y^2}} \right) = -y^{2s+1} \frac{\sqrt{\pi} \Gamma(s + 1)}{\Gamma(s + 1/2)} \frac{\text{sgn}(x + \ell)}{((x + \ell)^2 + y^2)^{s+1}}.$$

In particular, the sum over $\ell$ in (9) is differentiable for $x \in \mathbb{R} \setminus \mathbb{Z}$ since the series itself and the series of its derivatives converge absolutely and locally uniformly. The series
over \( \ell \) in (9) is also 1-periodic and even in \( x \) and hence has a Fourier expansion of the form \( \sum_{n \in \mathbb{Z}} a(n, y) \cos(2\pi nx) \) with coefficients

\[
a(n, y) = \int_{-\infty}^{\infty} \arcsin_s \left( \frac{1}{\sqrt{(u/y)^2 + 1}} \right) \cos(2\pi nu) du.
\]

We plug in the definition of \( \arcsin_s \) and interchange the order of integration to find

\[
a(n, y) = \int_{0}^{1} \left( \int_{-\infty}^{\infty} \frac{\cos(2\pi nu)}{(u/y)^2 + 1 - t^2} du \right) (1 - t^2)^s dt
\]

\[
= y \int_{0}^{1} \left( \int_{-\infty}^{\infty} \frac{\cos(2\pi nu\sqrt{1 - t^2})}{(u^2 + 1)^{s+1/2}} du \right) dt.
\]

By estimating \(| \cos(x) | \leq 1\) for all \( x \in \mathbb{R} \) we see that the last double integral converges, hence interchanging the order of integration is justified. For \( n = 0 \) the inner integral can be evaluated as

\[
\int_{-\infty}^{\infty} \frac{1}{(u^2 + 1)^{s+1/2}} du = \frac{\sqrt{\pi} \Gamma(s)}{\Gamma(s + 1/2)},
\]

by a direct calculation using the definition of the Gamma function. For \( n \neq 0 \) we can replace \( n \) by \(|n|\), and then the inner integral can be computed using the representation

\[
K_s(x) = \frac{2^{s-1} \Gamma(s + 1/2)}{\sqrt{\pi} x^s} \int_{-\infty}^{\infty} \frac{\cos(xu)}{(u^2 + 1)^{s+1/2}} du,
\]

which is valid for \( \text{Re}(s) > -1/2 \) and \( x > 0 \) (see [1, 9.6.25]). This gives the stated Fourier expansion for \( \text{Re}(s) > 0 \).

To show that the Fourier series on the right-hand side of (10) converges absolutely, we use the estimate \( K_0(x) \leq \frac{\sqrt{\pi}}{\sqrt{2}} x^{-1/2} e^{-x} \) for \( x > 0 \), compare [14, Corollary 3.4]. A change of variables yields

\[
\int_{0}^{1} K_0 \left( 2\pi |n| y \sqrt{1 - t^2} \right) dt \leq \frac{\sqrt{\pi}}{\sqrt{2} \sqrt{2\pi |n| y}} \int_{0}^{1} \frac{\sqrt{t}}{\sqrt{1-t^2}} e^{-2\pi |n| y t} dt.
\]

We split the integral at \( t = 1/2 \). Then

\[
\int_{1/2}^{1} \frac{\sqrt{t}}{\sqrt{1-t^2}} e^{-2\pi |n| y t} dt \leq e^{-\pi |n| y} \int_{1/2}^{1} \frac{\sqrt{t}}{\sqrt{1-t^2}} dt = O \left( e^{-\pi |n| y} \right)
\]
and
\[
\int_0^{1/2} \frac{\sqrt{t}}{\sqrt{1-t^2}} e^{-2\pi |n|yt} dt \leq 2 \int_0^{1/2} \sqrt{t} e^{-2\pi |n|yt} dt
= 2|n|^{-3/2} \int_0^{|n|/2} \sqrt{t} e^{-2\pi |n|yt} dt \leq 2|n|^{-3/2} \int_0^{\infty} \sqrt{t} e^{-2\pi y^2 t} dt = O \left( |n|^{-3/2} \right)
\]
as \( |n| \to \infty \). In total, we obtain
\[
\int_0^1 K_0 \left( 2\pi |n|y\sqrt{1-t^2} \right) dt = O \left( |n|^{-2} \right)
\]
as \( |n| \to \infty \), which shows that the Fourier series in (10) converges absolutely. \(\square\)

**Proposition 4.2** Let \( f \in H_{1/2, \mathbb{R}_+} \). For \( z \in \mathbb{H} \setminus H_{\Delta, r}(f) \) with \( y \gg 0 \) sufficiently large, the Borcherds lift of \( f \) has the Fourier expansion
\[
\Phi_{\Delta, r}(f, z) = -4 \sum_{m=1}^{\infty} c_f^+ (|\Delta|m^2/4N, rm) \sum_{b(\Delta)} \left( \frac{\Delta}{b} \right) \log |1 - e(mz + b/\Delta)|
+ 2 \sum_{m=1}^{\infty} c_f^- (|\Delta|m^2/4N, rm) \left( \frac{|\Delta|m^2}{4N} \right)^{-1/2} \sum_{b(\Delta)} \left( \frac{\Delta}{b} \right) \mathcal{F}(mz + b/\Delta)
\]
\[
+ \begin{cases} 
\sqrt{N} y \phi_{1/2}(0, 0) & \text{if } \Delta = 1, \\
- \sqrt{N} y c_f^- (0, 0) & \text{if } \Delta > 1, \\
2\sqrt{\Delta} L_{\Delta}(1) (c_f^+ (0, 0) + \sqrt{N} y c_f^- (0, 0)) & \text{if } \Delta < 0,
\end{cases}
\]
where \( \phi_{1/2}(\tau) = \sum_{h \in \mathcal{L}/L} \sum_{n \in \mathbb{Z}} e(n^2 \tau/4N) \psi_n \) is the Jacobi theta function and \( L_{\Delta}(s) = \sum_{n \geq 1} \left( \frac{\Delta}{n} \right) n^{-s} \) for \( \text{Re}(s) > 1 \) is a Dirichlet \( L \)-function. Here the function \( \mathcal{F}(z) : \mathbb{H} \to \mathbb{R} \) is defined by
\[
\mathcal{F}(z) = \lim_{s \to 0} \left( \sum_{\ell \in \mathbb{Z}} \arcsin \left( \frac{y}{\sqrt{(x + \ell)^2 + y^2}} \right) - y \frac{\sqrt{\pi}}{\Gamma(s + 1/2)} \right),
\]
compare Lemma 4.1.

**Remark 4.3** 1. The singularities of \( \Phi_{\Delta, r}(f, z) \) at Heegner points and geodesics given by semi-circles centered at the real line are not reproduced in the Fourier expansion above, but the part involving the function \( \mathcal{F} \) captures the singularities along vertical geodesics.
2. By Dirichlet’s class number formula we have

\[ L_\Delta(1) = \frac{1}{\sqrt{\Delta}} h(\Delta) \log(\epsilon_\Delta) = \frac{1}{2} \text{tr}_1(\Delta) \]

for \( \Delta > 1 \), where \( h(\Delta) \) is the narrow class number of \( \mathbb{Q}(\sqrt{\Delta}) \), \( \epsilon_\Delta \) is the smallest unit \( > 1 \) of norm 1, and \( \text{tr}_1(\Delta) \) is the \( \Delta \)th trace of the constant 1 function as defined in the introduction.

**Proof of Proposition 4.2** The proof follows the arguments of [8, Theorem 5.3]. First, by [8, Theorem 4.8], we can write

\[ v^{-1/2} \Theta_{\Delta, r}(\tau, z) = \delta_{\Delta=1} \frac{\sqrt{N_y}}{\sqrt{|\Delta|}} \theta_{1/2}(\tau) + \frac{\sqrt{N_y}}{\sqrt{|\Delta|}} \sum_{n \geq 1} \sum_{M \in \tilde{\Gamma}/\Gamma} \left[ \exp\left( -\frac{\pi n^2 N_y^2}{|\Delta|v} \right) \Xi(\tau, \mu, n, 0) \right] |1/2, \rho^*_L(\mu)| \]

where \( \mu = \left( \begin{array}{c} x \\ -x^2 \end{array} \right) \) and

\[ \Xi(\tau, \mu, n, 0) = \left( \begin{array}{c} \Delta \\ n \end{array} \right) e^{\sqrt{\Delta} \sum_{h \in K'/K} \sum_{X \in K + rh} e (-Q_\Delta(X)\tau + n(X, \mu)\Delta) \epsilon_h}, \]

with \( \epsilon = 1 \) if \( \Delta > 0 \) and \( \epsilon = i \) if \( \Delta < 0 \). Further, \( K \) denotes the one-dimensional negative definite sublattice

\[ K = \left\{ \left( \begin{array}{cc} b & 0 \\ 0 & -b \end{array} \right) : b \in \mathbb{Z} \right\} \]

of \( L \). Its dual lattice is given by

\[ K' = \left\{ \left( \begin{array}{cc} b/2N & 0 \\ 0 & -b/2N \end{array} \right) : b \in \mathbb{Z} \right\}. \]

Inserting this into the definition of the theta lift, the unfolding argument yields

\[ \Phi_{\Delta, r}(f, z) = \delta_{\Delta=1} \frac{\sqrt{N_y}}{\sqrt{|\Delta|}} (f, \theta_{1/2})_{\text{reg}} + \text{CT}_{s=0} \Phi^0_{\Delta, r}(f, z, s), \]

where

\[ \Phi^0_{\Delta, r}(f, z, s) = 2\sqrt{N_y} \sum_{n \geq 1} \int_{v=0}^\infty \int_{u=0}^1 \exp\left( -\frac{\pi n^2 N_y^2}{|\Delta|v} \right) \langle f, \Xi(\tau, \mu, n, 0) \rangle \frac{dv}{v^{s+3/2}}. \]
The unfolding is justified for $y \gg 0$ by the same arguments as in [3, Theorem 7.1]. Let us write

$$f(\tau) = \sum_{h \in L'/L} \sum_{n \in \mathbb{Z}} c_f(n, h, v) e(n \tau) e_h$$

for the Fourier expansion of $f$ for the moment. Since $\Delta$ is fundamental, the conditions $X \in K + rh$ and $Q(X) \equiv \Delta Q(h) \pmod{\Delta}$ are equivalent to $X = \Delta X' + rh$ for some $X' \in K'$. Plugging in the definition of $\Xi(\tau, n, \mu, 0)$, and evaluating the integral over $u$, we obtain

$$\Phi_{0, r}^0(f, z, s) = 2\sqrt{Ny} e \sum_{X \in K' \setminus \{0\}} \sum_{n \geq 1} \left( \frac{\Delta}{n} \right) e(-\operatorname{sgn}(\Delta)n(X, \mu))$$

$$\times \int_{v=0}^{\infty} c_f(-|\Delta|Q(X), rX, v)$$

$$\times \exp \left(-\frac{\pi n^2 N y^2}{|\Delta| v} + 4\pi |\Delta|Q(X)v \right) \frac{dv}{v^{s+3/2}}.$$

Now we use the explicit form of the Fourier coefficients of $f$. The summand for $X = 0$ in $\Phi_{0, r}^0(f, z, s)$ is given by

$$2\sqrt{Ny} e \sum_{n \geq 1} \left( \frac{\Delta}{n} \right) \int_{v=0}^{\infty} \left( c_f^+(0, 0) + c_f^-(0, 0)v^{1/2} \right) \exp \left(-\frac{\pi n^2 N y^2}{|\Delta| v} \right) \frac{dv}{v^{s+3/2}}$$

$$= 2e \left( Ny^2 \right)^{-s} \left( c_f^+(0, 0) \left( \frac{\pi}{|\Delta|} \right)^{-s-1/2} \Gamma(s + 1/2)L_\Delta(2s + 1) \right.$$

$$+ \sqrt{Ny} c_f^-(0, 0) \left( \frac{\pi}{|\Delta|} \right)^{-s} \Gamma(s)L_\Delta(2s) \right).$$

For $\Delta < 0$ the harmonic Maass form $f$ transforms with $\rho_L$, which implies that its zero component vanishes, so $c_f^-(0, 0) = 0$. For $\Delta > 0$ the completed Dirichlet $L$-function

$$\Lambda_\Delta(s) = (\pi/\Delta)^{-s/2} \Gamma(s/2)L_\Delta(s)$$

satisfies the functional equation $\Lambda_\Delta(1 - s) = \Lambda_\Delta(s)$. It is holomorphic at $s = 1$ if $\Delta > 1$. Taking the constant term at $s = 0$, we get the contribution in the large bracket in the proposition.

For $X \in K'$ with $X \neq 0$ we have $-|\Delta|Q(X) > 0$. We can write

$$c_f(n, h, v) = c_f^+(n, h) + c_f^-(n, h)\sqrt{v}B_1^{1/2}(-4\pi nv)$$

for $n > 0$. The contribution coming from the coefficients $c_f^+(n, h)$ can be computed as in [8, Theorem 5.2], and yields the first line of the Fourier expansion. Plugging in
the definition of $\beta^{c}_{1/2}(s)$, it remains to compute

$$\begin{align*}
4\sqrt{N}ye & \sum_{\substack{X \in K' \setminus \{0\} \\text{ s.t. } X \neq 0}} \sum_{n \geq 1} \left( \frac{\Delta}{n} \right) e\left(-\text{sgn}(\Delta)n(X, \mu)\right) \\
\times & \int_{v=0}^{\infty} \left( \int_{w=0}^{1} \exp(-4\pi |\Delta|Q(X)w^2v)dw \right) \\
\times & \exp\left(-\frac{\pi n^2Nv^2}{|\Delta|v} + 4\pi |\Delta|Q(X)v\right) \frac{dv}{v}.
\end{align*}$$

If we change the order of integration, the inner integral can be computed in terms of the $K$-Bessel function by [13, (3.471.9)], giving

$$\int_{v=0}^{\infty} \exp\left(4\pi |\Delta|Q(X)(1 - w^2)v - \frac{\pi n^2Nv^2}{|\Delta|v}\right) \frac{dv}{v} = 2K_0\left(2\pi y|n|\sqrt{-4NQ(X)(1 - w^2)}\right).$$

Write $X = \left(\frac{m/2N}{0}, -\frac{m/2N}{0}\right) \in K'\setminus\{0\}$ with $m \in \mathbb{Z}$, $m \neq 0$. Then $-Q(X) = m^2/4N$ and $-(X, \mu) = mx$. We use the evaluation of the Gauss sum

$$\sum_{b(\Delta)} \left( \frac{\Delta}{b} \right) e(bn/|\Delta|) = \varepsilon\left(\frac{\Delta}{n}\right) \sqrt{|\Delta|}.$$  

Then the expression in (11) becomes

$$\begin{align*}
2\sum_{m=1}^{\infty} & a_f(|\Delta|m^2/4N, rm) \left(\frac{|\Delta|m^2}{4N}\right)^{-1/2} \sum_{b(\Delta)} \left( \frac{\Delta}{b} \right) \\
\times & 2my \sum_{n \neq 0} e(n(mx + b/\Delta)) \int_{0}^{1} K_0\left(2\pi my|n|\sqrt{1 - w^2}\right) dw.
\end{align*}$$

By Lemma 4.1 the second line agrees with $\mathcal{F}(mz + b/\Delta)$, which finishes the proof.

\[\square\]

If the coefficients $c^+_f(n, h)$ vanish for $n < 0$, then $\Phi_{\Delta,r}(f, z)$ does not have singularities at Heegner points, and extends to a continuous function on $\mathbb{H}$ which is not differentiable along the geodesics in $H_{\Delta,r}^{-}(f)$. In this case, we can derive the Fourier expansion of $\Phi_{\Delta,r}(f, z)$ on $\mathbb{H}$, without assuming $y \gg 0$ to be large enough.
Corollary 4.4 Let $f \in H_{1/2, \tilde{\rho}_L^r}$, and suppose that $c_f^+(n, h) = 0$ for all $n < 0$ and $h \in L'/L$. Then the Fourier expansion of the Borcherds lift $\Phi_{\Delta, r}(f, z)$ on $\mathbb{H} \setminus H_{\Delta, r}(f)$ is given by the formula from Proposition 4.2 plus the expression

$$-2 \sum_{h \in L'/L} \sum_{n > 0} c_f^-(n, h)n^{-1/2} \sum_{X \in L-|\Delta|n, rh \atop a \neq 0} \chi\Delta(X)1_X(z) \left( \arctan \left( \frac{\sqrt{4|\Delta|n}}{-\text{sgn}(a)p_X(z)} \right) + \frac{\pi}{2} \right),$$

where $1_X(z)$ denotes the characteristic function of the bounded component of $\mathbb{H}\setminus c_X$.

Remark 4.5 1. Recall that for $X = \left(\begin{array}{cc} -b/2N & -c/N \\ a & b/2N \end{array}\right) \in L'$ we defined

$$p_X(z) = -\frac{aN|z|^2 + bx + c}{y\sqrt{N}},$$

which vanishes exactly along the geodesic $c_X$. Further, if $a \neq 0$ then a point $z$ lies inside the bounded component of $\mathbb{H}\setminus c_X$ if and only if $\text{sgn}(a)p_X(z) > 0$. In particular, we see that if $z \in \mathbb{H}\setminus c_X$ approaches $c_X$, then the expression

$$1_X(z) \left( \arctan \left( \frac{\sqrt{4|\Delta|n}}{-\text{sgn}(a)p_X(z)} \right) + \frac{\pi}{2} \right)$$

goes to 0. In this sense, the above Fourier expansion can be extended on all of $\mathbb{H}$.

2. The sum in (13) is locally finite since for fixed $n$ each point $z$ lies in the bounded component of $\mathbb{H}\setminus c_X$ for only finitely many $X \in L-|\Delta|n, rh$ with $a \neq 0$.

Proof Let $\tilde{\Phi}_{\Delta, r}(f, z)$ denote $\Phi_{\Delta, r}(f, z)$ minus the expression in (13). Then we have $\tilde{\Phi}_{\Delta, r}(f, z) = \Phi_{\Delta, r}(f, z)$ for $y \gg 0$ large enough since the imaginary parts of points lying on geodesics $c_X$ for $X \in L-|\Delta|n, rh$ with $a \neq 0$ are bounded by a constant depending on $n$, and the sum over $n$ is finite.

Further, for $a \neq 0$ and $z \notin c_X$ we can write

$$-2 \cdot 1_X(z) \left( \arctan \left( \frac{\sqrt{4|\Delta|n}}{-\text{sgn}(a)p_X(z)} \right) + \frac{\pi}{2} \right)$$

$$= \arctan \left( \frac{\sqrt{4|\Delta|n}}{p_X(z)} \right) - \arctan \left( \frac{\sqrt{4|\Delta|n}}{-\text{sgn}(a)p_X(z)} \right) + 1_X(z)\pi$$

$$= \arctan \left( \sqrt{\frac{Q\Delta(X)}{-Q\Delta(X)}} \right) - \arccot \left( \frac{-\text{sgn}(a)p_X(z)}{\sqrt{4|\Delta|n}} \right).$$

Using that the function $\arccot$ is real analytic at the origin, and the shape of the singularities of $\Phi_{\Delta, r}(f, z)$ determined in Theorem 3.1, we see that $\Phi_{\Delta, r}(f, z)$ extends to a real analytic function on all of $\mathbb{H}$. In particular, the Fourier expansion of $\Phi_{\Delta, r}(f, z)$ given in Proposition 4.2, which a priori only converges for $y \gg 0$ sufficiently large,
3.1 and the formula 4.2 and Corollary 4.4 above. Let $f$ be the Borcherds lift.

We consider the derivative

$$f' = f_0(\tau) \varepsilon_0 + f_1(\tau) \varepsilon_1$$

of the Borcherds lift. The derivative of the Borcherds lift is also the Fourier expansion of the real analytic function $\Phi_{\Delta, r}(f, z)$ on all of $\mathbb{H}$, and hence converges on all of $\mathbb{H}$. We obtain the stated Fourier expansion.

We briefly explain how Proposition 1.3 in the introduction follows from Proposition 4.2 and Corollary 4.4 above. Let $N = 1$ and $\Delta > 0$. As mentioned in Remark 2.1, we can identify a vector valued harmonic Maass form $f(\tau) = f_0(\tau) \varepsilon_0 + f_1(\tau) \varepsilon_1$ of weight $1/2$ for $\rho_L$ with the scalar valued harmonic Maass form $\tilde{f}(\tau) = f_0(4\tau) + f_1(4\tau)$ of weight $1/2$ for $\Gamma_0(4)$. Their Fourier coefficients are related by

$$c^+_f(n, h) = c^+_f(4n) \quad \text{and} \quad c^-_f(n, h) = \frac{1}{2} c^-_f(4n)$$

for all $n \in \mathbb{Z} + h^2/4$. Furthermore, as explained in Sect. 2.2, we can identify $X \in L_{-\Delta n, rh}$ with a binary quadratic form $Q = [a, b, c] \in Q_{4\Delta n}$ of discriminant $4\Delta n$. Then $\chi_\Delta(X) = \chi_\Delta(Q)$ and $1_X(z) = 1_Q(x)$. If we write $D = 4n$, then the sums over $n$ become sums over integers $D$ with $D \equiv 0, 1 \pmod{4}$. The second line in the Fourier expansion in Proposition 4.2 vanishes due to our assumption $c^-_f(\Delta n^2) = 0$ for all $n > 0$. Finally, we plug in $p_X(z) = -(a|z|^2 + bx + c)/y$ in (13) and replace all $X$ with $a < 0$ by $-X$, which gives a factor 2. We obtain the Fourier expansion in Proposition 1.3.

5 The derivative of the Borcherds lift

We consider the derivative

$$\Phi'_{\Delta, r}(f, z) = \frac{\partial}{\partial z} \Phi_{\Delta, r}(f, z)$$

of the Borcherds lift.

**Theorem 5.1** Let $f \in H_{1/2, \rho_L^*}$. The derivative $\Phi'_{\Delta, r}(f, z)$ of the Borcherds lift is harmonic on $\mathbb{H}\backslash (H^+_{\Delta, r}(f) \cup H^-_{\Delta, r}(f))$ and transforms like a modular form of weight 2 under $\Gamma_0(N)$. If $\Delta \neq 1$ or if $c^+_f(0, 0) = 0$, then $\Phi'_{\Delta, r}(f, z)$ is holomorphic on its domain.

At a point $z_0 \in H^+_{\Delta, r}(f) \cup H^-_{\Delta, r}(f)$ it has a singularity of type

$$i\sqrt{N} \sum_{h \in \mathcal{L}} \sum_{L \cap h < 0} c^+_f(n, h) \sum_{X \in L_{-\Delta n, rh}} \chi_\Delta(X) \frac{p_X(z)}{Q_X(z)}$$

$$+ i\sqrt{N|\Delta|} \sum_{h \in \mathcal{L}} \sum_{L \cap h > 0} c^-_f(n, h) \sum_{X \in L_{-\Delta n, rh}} \chi_\Delta(X) \frac{\text{sgn}(p_X(z))}{Q_X(z)}.$$

**Proof** The analytic properties of $\Phi'_{\Delta, r}(f, z)$ follow from the Laplace equation in Theorem 3.1 and the formula $\Delta_0 = -4y^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}}$. The types of singularities of $\Phi'_{\Delta, r}(f, z)$
are obtained as the derivatives of the types of singularities of $\Phi_{\Delta,r}(f,z)$ given in Theorem 3.1.

\begin{remark}
Let $X = \left(\frac{-b/2N}{a}, \frac{-c/N}{b/2N}\right) \in L_{-|\Delta|n,rh}$. For $n < 0$ we have

$$Q_X(z) = aNz^2 + bz + c = 0$$

exactly for the Heegner point $z = z_X$. Hence $\Phi'_{\Delta,r}(f,z)$ has simple poles at the Heegner points in $H_{\Delta,r}^+(f)$. For $n > 0$ the sign of

$$p_X(z) = -\frac{aN|z|^2 + bx + c}{y\sqrt{N}}$$

changes if $z$ crosses the geodesic $c_X$. This means that $\Phi'_{\Delta,r}(f,z)$ has jump singularities along the geodesics in $H_{\Delta,r}^-(f)$.

\begin{proposition}
Let $f \in H_{1/2,\tilde{\rho}_L}$. For $z \in \mathbb{H}\setminus H_{\Delta,r}^-(f)$ with $y \gg 0$ sufficiently large we have the Fourier expansion

$$\Phi'_{\Delta,r}(f,z) = 4\pi i \sqrt{|\Delta|\varepsilon} \sum_{n=1}^{\infty} \left( \sum_{d|n} \left( \frac{\Delta}{n/d} \right) d \ c_f^+(|\Delta|d^2/4N, rd) \right) e(nz)$$

$$+ 2 \sum_{m=1}^{\infty} c_f^-(|\Delta|m^2/4N, rm) \left( \frac{|\Delta|}{4N} \right)^{-1/2} \sum_{b(\Delta)} \left( \frac{\Delta}{b} \right) \mathcal{F}'(mz + b/\Delta)$$

\begin{align*}
&\begin{cases}
- \frac{i\sqrt{N}}{2}(f, \theta_{1/2})^{\text{reg}} + \frac{i}{y} c_f^+(0, 0) & \text{if } \Delta = 1, \\
+ \frac{i}{2\sqrt{N}} c_f^-(0, 0) \left( \log(4\pi) - \log(Ny^2) - 2 + \Gamma'(1) \right) & \text{if } \Delta > 1, \\
- i\sqrt{N} \Delta L_\Delta(1)c_f^-(0, 0) & \text{if } \Delta < 0.
\end{cases}
\end{align*}

where $\varepsilon = 1$ if $\Delta > 0$ or $\varepsilon = i$ for $\Delta < 0$, and

$$\mathcal{F}'(z) = -\frac{i}{2} \lim_{s \to 0} \left( y^{2s} \Gamma(s + 1) \sum_{\ell \in \mathbb{Z}} \frac{\text{sgn}(x + \ell)(\bar{z} + \ell)}{|z + \ell|^{2s+2}} - \Gamma(s) \right).$$

\begin{proof}
This follows by computing the derivative of the Fourier expansion of $\Phi'_{\Delta,r}(f,z)$ given in Proposition 4.2. The derivative of $\mathcal{F}(z)$ can be computed most easily using the representation (8) of arcsin$_x$ as an incomplete beta function. Using the formula (12) for the Gauss sum, the calculation of the remaining derivatives is straightforward.

Again, we consider the special case that $c_f^+(n, h) = 0$ for all $n < 0$. 

\end{proof}

\end{proposition}
Corollary 5.4 Let $f \in H_{1/2, \rho^+}$, and suppose that $c_f^+(n, h) = 0$ for all $n < 0$ and $h \in L'/L$. Then the Fourier expansion of the derivative $\Phi'_{\Delta, r}(f, z)$ of the Borcherds lift on $\mathbb{H} \setminus \mathbb{H}_{\Delta, r}$ is given by the formula from Proposition 5.3 plus the expression

$$-2i \sqrt{|\Delta|} \sum_{h \in L'/L, n > 0} c_f^-(n, h) \sum_{a \neq 0, X \in \mathbb{L} - |\Delta|n, rh} \chi_{\Delta}(X) \frac{1_X(z) \text{sgn}(a)}{Q_X(z)},$$

where $1_X(z)$ denotes the characteristic function of the bounded component of $\mathbb{H} \setminus cX$.

Proof This can either be proved by similar arguments as in the proof of Corollary 4.4, or by computing the derivative of the expression (13). □

6 Applications: modular integrals with rational period functions and Borcherds products of harmonic Maass forms

For simplicity, we assume in this section that $N$ is square free. Then the cusps of $\Gamma_0(N)$ can be represented by the fractions $1/c$ with $c | N$. Note that $\infty$ corresponds to $1/N$. The width of $1/c$ is given by $\alpha_{1/c} = N/c$. We choose the matrix $\sigma_{1/c} \in SL_2(\mathbb{Z})$ sending $\infty$ to $1/c$ in the form

$$\sigma_{1/c} = \begin{pmatrix} 1 & \beta \\ c & N \gamma/c \end{pmatrix}$$

where $\beta, \gamma \in \mathbb{Z}$ are such that $N\gamma/c - c\beta = 1$. Then we can take the Atkin–Lehner involution corresponding to $N/c$ as

$$W_{N/c} = \sigma_{1/c} \begin{pmatrix} N/c & 0 \\ 0 & 1 \end{pmatrix}.$$ 

We see that $W_{N/c}\infty = 1/c$, so the Atkin–Lehner involutions act transitively on the cusps. Further, the expansion at the cusp $1/c$ of a function $F$, which is modular of weight $k \in \mathbb{Z}$, is given by

$$(F|k\sigma_{1/c}')(z) = (c/N)^k : (F|k W_{N/c})(cz/N).$$

Since

$$\Phi_{\Delta, r}(f, z)|_{1} W_{N/c} = \Phi_{\Delta, r}(f^{w_{N/c}}, z)$$

and consequently

$$\Phi'_{\Delta, r}(f, z)|_{2} W_{N/c} = \Phi'_{\Delta, r}(f^{w_{N/c}}, z),$$

the expansion of $\Phi'_{\Delta, r}(f, z)$ at the cusp $1/c$ is essentially given by $\Phi'_{\Delta, r}(f^{w_{N/c}}, z)$. 
6.1 Modular integrals with rational period functions

As an application of our extension of the Borcherds lift, we construct modular integrals of weight 2 for \( \Gamma_0(N) \) with rational period functions from harmonic Maass forms of weight 1/2. Following Knopp [15], we call a holomorphic function \( F : \mathbb{H} \rightarrow \mathbb{C} \) a modular integral of weight \( k \in \mathbb{Z} \) for \( \Gamma_0(N) \) with rational period functions if

\[
q_M(z) = F(z) - (F|_k M)(z)
\]

is a rational function of \( z \) for each \( M \in \Gamma_0(N) \), and if \( F \) is holomorphic at the cusps of \( \Gamma_0(N) \), in the sense that \( \lim_{y \to \infty} (F|_k M)(z) \) exists for every \( M \in SL_2(\mathbb{Z}) \). Then the map \( M \mapsto q_M \) defines a weight \( k \) cocycle for \( \Gamma_0(N) \) with values in the rational functions which are holomorphic on \( \mathbb{H} \), i.e., it satisfies

\[
q_{MM'} = q_M q_{M'} + q_{M'}
\]

for all \( M, M' \in \Gamma_0(N) \). Conversely, it follows from a more general result of Knopp [15] that every such cocycle admits a holomorphic modular integral. Knopp’s modular integrals are Poincaré series built from the cocycles. It was shown in [9,10] that certain generating series of (traces of) cycle integrals of weakly holomorphic modular functions for \( SL_2(\mathbb{Z}) \) are modular integrals of weight 2 with rational period functions.

Using the Borcherds lift we generalize their construction to higher level.

**Proposition 6.1** Let \( \Delta \neq 1 \) be a fundamental discriminant. Let \( f \in H_{1/2, \rho^*} \) with \( c_f(n, h) = 0 \) for all \( n < 0 \) and \( h \in L'/L \). Further, assume that \( c_f(|\Delta| m^2/4N, rm) = 0 \) for all \( m \in \mathbb{Z}, m > 0 \). Then the function

\[
F_{\Delta,r}(f, z) = -\frac{1}{4\pi} L_{\Delta}(1) c_f^-(0, 0) \sum_{d|n} \left( \sum_{n/d} \left( \frac{\Delta}{n/d} \right) d c_f^+(|\Delta| d^2/4N, rd) \right) e(nz)
\]

is holomorphic on \( \mathbb{H} \) and at the cusps of \( \Gamma_0(N) \), and satisfies the transformation rule

\[
F_{\Delta,r}(f, z)|_2 M - F_{\Delta,r}(f, z) = -\frac{1}{\pi} \sum_{h \in L'/L} \sum_{n > 0} c_f^-(n, h) \sum_{X \in L-|\Delta|n, rh} \chi_{\Delta}(X) \frac{\chi_X(z)}{Q_X(z)}
\]

for all \( M \in \Gamma_0(N) \), where \( a_X \) denotes the a entry of \( X \). In particular, \( F_{\Delta,r}(f, z) \) is a modular integral of weight 2 for \( \Gamma_0(N) \).

**Remark 6.2** 1. The requirement \( c_f^-(|\Delta| m^2/4N, rm) = 0 \) for all \( m \in \mathbb{Z}, m > 0 \), ensures that \( \Phi_{\Delta,r}(f, z) \) does not have singularities along vertical geodesics, and implies that the second line of the Fourier expansion in Proposition 5.3 vanishes.
2. The proof of the transformation behaviour works for arbitrary positive integers \( N \), but the assumption that \( N \) is square free is used to obtain the Fourier expansions of \( \Phi'_{\Delta,r}(f,z) \) at different cusps via Atkin–Lehner operators. One could compute the expansion at a cusp \( \ell \) by choosing an appropriate sublattice \( K_\ell \) instead of \( K \) in Proposition 4.2 and modify the computation of the expansion at \( \infty \) correspondingly. However, the above result is certainly true without the assumption that \( N \) is square free, but the computations become much more technical.

**Proof of Proposition 6.1** Let \( z \in \mathbb{H} \setminus \mathbb{H}_{\Delta,r}(h) \), and let

\[
F_{\Delta,r}^+(f,z) = -\frac{1}{2\pi} \sum_{h \in L'/L} \sum_{n > 0} c_f^-(n,h) \sum_{X \in L'-\Delta_{jn,r}h} \chi_\Delta(X) \frac{1_X(z) \operatorname{sgn}(a)}{Q_X(z)}.
\]

By Corollary 5.4 we have

\[
\Phi'_{\Delta,r}(f,z) = 4\pi i \sqrt{N|\Delta|} \left( F_{\Delta,r}(f,z) + F_{\Delta,r}^+(f,z) \right).
\]

Since \( \Phi'_{\Delta,r}(f,z) \) transforms like a modular form of weight 2 for \( \Gamma_0(N) \), we obtain

\[
F_{\Delta,r}(f,z)|_{2M} - F_{\Delta,r}(f,z) = -F_{\Delta,r}^+(f,z)|_{2M} + F_{\Delta,r}^+(f,z).
\]

Using \( Q_X(z)|_{-2M} = Q_{M-1}X(z) \), we obtain that the right-hand side of the last formula equals

\[
-\frac{1}{2\pi} \sum_{h \in L'/L} \sum_{n > 0} c_f^-(n,h) \sum_{X \in L'-\Delta_{jn,r}h} \chi_\Delta(X) \frac{1_X(z) \operatorname{sgn}(aX) - 1_{MX}(Mz) \operatorname{sgn}(a_{MX})}{Q_X(z)}.
\]

The characteristic functions \( 1_X \) and \( 1_{MX} \) are related by

\[
1_{MX}(Mz) = \begin{cases} 1_X(z), & \text{if } a_X \cdot a_{MX} > 0, \\ 1 - 1_X(z), & \text{if } a_X \cdot a_{MX} < 0. \end{cases}
\]

In particular, all summands with \( a_X \cdot a_{MX} > 0 \) cancel out. In the remaining sum over \( X \) with \( a_X \cdot a_{MX} < 0 \), we replace \( X \) with \( -X \) if \( a_X < 0 \), giving a factor 2. This proves the transformation behaviour of \( F_{\Delta,r}(f,z) \) for \( z \in \mathbb{H} \setminus \mathbb{H}_{\Delta,r}(f) \). Since all the functions appearing in the transformation formula are holomorphic on \( \mathbb{H} \), we obtain the transformation law by analytic continuation.

Using \( \Phi'_{\Delta,r}(f,z)|_{2W_d} = \Phi'_{\Delta,r}(f^{\ell_0d},z) \) we obtain

\[
F_{\Delta,r}(f,z)|_{2W_d} = F_{\Delta,r}(f^{\ell_0d},z) + F_{\Delta,r}^+(f^{\ell_0d},z) + F_{\Delta,r}^+(f,z)|_{2W_d}.
\]

Since \( F_{\Delta,r}(f^{\ell_0d},z) \) is holomorphic at \( \infty \), and \( F_{\Delta,r}^+(f^{\ell_0d},z) \) and \( F_{\Delta,r}^+(f,z)|_{2W_d} \) vanish as \( y \to \infty \), we see that \( F_{\Delta,r}(f,z) \) is holomorphic at the cusps. \( \square \)
**Example 6.3** Let $\Delta > 1$. We apply Proposition 6.1 to a harmonic Maass form $f \in H_{1/2, \rho_L^*}$ arising as the image of the regularized theta lift studied by Bruinier et al. [7] of a harmonic Maass form $F \in H_0^+ (\Gamma_0(N))$. We assume that the constant coefficients $a_\ell^+ (0)$ of $F$ vanish at all cusps. By Theorem 4.1 in [7] the Fourier expansion of the $h$-th component of $f$ is given by

$$f_h (\tau) = -2 \text{tr}_F (0, h) \sqrt{v} + \sum_{n < 0} \text{tr}_F (-n, h) \sqrt{v} \beta_{1/2} (4\pi |n| v) e(n\tau) + \sum_{n > 0} \frac{\sqrt{N}}{\pi} \text{tr}_F (-n, h) e(n\tau) + \sum_{n > 0} \text{tr}_F^c (-n^2 / 4N, h) \sqrt{v} \beta_{1/2}^c (-4\pi n^2 v / 4N) e(n^2 \tau / 4N),$$

with the traces

$$\text{tr}_F (-n, h) = \begin{cases} \sum_{X \in \Gamma_0(N) \setminus \Gamma_{-n,h}} \frac{1}{|\Gamma X|} F(zX), & \text{if } n < 0, \\ -\delta_{0,h} \frac{1}{2\pi} \int_{\Gamma_0(N) \setminus \mathbb{H}} \text{reg} F(z) \frac{dx \, dy}{y^2}, & \text{if } n = 0, \\ \sum_{X \in \Gamma_0(N) \setminus \Gamma_{-n,h}} \int_{\Gamma X \setminus cX} \text{reg} F(z) \frac{dz}{Q(z, 1)}, & \text{if } n > 0, \end{cases}$$

and the so-called complementary trace $\text{tr}_F^c (-n^2 / 4N, h)$, which is defined in [7, Section 3]. Our definition of the traces of cycle integrals equals $\pi / \sqrt{N}$ times the traces of cycle integrals defined in [7], and the traces for $|n| / N$ being a square need to be regularized as explained in [7, Section 3]. Note that the trace of index 0 and the complementary trace can be evaluated explicitly in terms of the principal parts of $F$ at the cusps of $\Gamma_0(N)$, see [6, Remark 4.9], and that the complementary trace is nonzero only for finitely many $n$, see [6, Proposition 4.7]. Observe that $c_f^+ (n, h) = 0$ for $n < 0$ and $c_f^- (\Delta m^2 / 4N, rm) = 0$ for $m \in \mathbb{Z}, m > 0$, if $\Delta > 1$. The Duke–Imamoglu–Tóth harmonic Maass form $h(\tau) = h_1 (\tau)$ from the introduction can be constructed as the Bruinier–Funke–Imamoglu lift of $\frac{1}{2} J$.

By Proposition 6.1, for $\Delta > 1$ a fundamental discriminant the function

$$F_{\Delta, r} (f, z) = \frac{1}{2\pi} L_\Delta (1) \text{tr}_F (0, 0) + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \sum_{d|n} \left( \frac{\Delta}{n/d} \right) d \text{ tr}_F \left( -\Delta d^2 / 4N, rd \right) \right) e(nz)$$

is a holomorphic function on $\mathbb{H}$, which transforms under the weight 2 slash operation of $M \in \Gamma_0(N)$ by

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\[
F_{\Delta,r}(f, z)|_{2M} - F_{\Delta,r}(f, z) = -\frac{1}{\pi} \sum_{h \in L' / L} \sum_{n > 0} \text{tr}^r((-n^2/4N, h)) \sum_{X \in L \Delta n^2/4N, rh} \frac{1}{QX(z)}. 
\]

Since \( \chi_\Delta(X) = 1 \) for \( X \in L \Delta n^2/4N, rh \) we dropped it from the notation.

In the special case \( N = 1 \) and \( F = \frac{1}{2} J \) (with \( \text{tr}_F(0, 0) = 4 \) and \( \text{tr}_F(-1/4, 1) = 2 \)) we recover the transformation behaviour of the modular integral \( F_\Delta(z) \) of Duke et al. [10] stated in the introduction.

### 6.2 Borcherds products

In this section we construct twisted Borcherds products of harmonic Maass forms \( f \in H_{1/2, \tilde{\rho}}^* \). For simplicity we assume \( \Delta \neq 1 \).

In order to generalize the Borcherds product to the full space \( H_{1/2, \tilde{\rho}}^* \) we first recall the construction of certain weight 0 and weight 2 cocycles from [11], which will appear in the transformation rule of the Borcherds product.

**Lemma 6.4** Let \( n > 0 \) such that \( N | \Delta | n \) is not a square, and let \( A \in \Gamma_0(N) \setminus L \Delta |n, rh \). Then the function

\[
q^A_M(z) = \sum_{X \in A \atop aMX < 0 < aX} \frac{1}{QX(z)}
\]

defines a weight 2 cocycle with values in the rational functions which are holomorphic on \( \mathbb{H} \).

**Proof** As in the proof of Proposition 6.1 we compute

\[
\sum_{X \in A \atop aMX < 0 < aX} \frac{1}{QX(z)} = \sum_{X \in A \atop a > 0} \frac{1}{QX(z)} \text{sgn}(a) - \sum_{X \in A \atop a > 0} \frac{1}{QX(z)} \text{sgn}(a) \Bigg|_2 M
\]

for \( z \) not lying on any geodesic \( c_X \) with \( X \in A \). This easily implies that the map \( M \mapsto q^A_M \) is a weight 2 cocycle. \( \square \)

Next, we would like to construct a weight 0 cocycle \( R^A_M(z) \) with values in the holomorphic functions on \( \mathbb{H} \) such that \( \frac{\partial}{\partial z} R^A_M(z) = q^A_M(z) \). The following proposition gives such a construction for general cocycles with values in rational functions which are holomorphic on \( \mathbb{H} \).

**Proposition 6.5** ([11], Theorem 2.1) Let \( F(z) = \sum_{n \geq 0} a(n)x(nz) \) be a holomorphic modular integral of weight 2 for \( \Gamma_0(N) \) with rational period functions \( q_M = F|_2M - F \). Assume that \( a(n) \ll n^\alpha \) for some \( \alpha > 0 \). For \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \) with \( c \neq 0 \) we let

\[ \text{sgn}(a) = \text{sgn}(d) \neq 1 \text{ for } (a, d) = 1 \]

\[ \text{sgn}(a) = \text{sgn}(d) = 1 \text{ for } (a, d) = 1 \]

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\[ \text{sgn}(a) = \text{sgn}(d) = 1 \text{ for } (a, d) = 1 \]
\[ \Lambda \left( s, \frac{a}{c} \right) = \left( \frac{2\pi}{c} \right)^{-s} \Gamma(s) \sum_{n \geq 1} a(n) e \left( \frac{an}{c} \right) n^{-s} \]

and

\[ H \left( s, \frac{a}{c} \right) = \Lambda \left( s, \frac{a}{c} \right) + \int_{1}^{\infty} q_{M}(-d/c + it/c)t^{1-s}dt + \frac{a(0)}{s} - \frac{a(0)}{2-s}. \]

Then \( H \left( s, \frac{a}{c} \right) \) is entire and satisfies the functional equation \( H \left( s, \frac{a}{c} \right) = -H \left( 2 - s, -\frac{d}{c} \right) \).

Further, for \( c \neq 0 \) we set

\[ R_{M}(z) = -\frac{i}{c} H \left( 1, \frac{a}{c} \right) + \int_{-\frac{d}{c} + \frac{i}{c}}^{z} q_{M}(w)dw + a(0)\frac{a + d}{c}, \]

and for \( M = \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \) we let \( R_{M}(z) = na(0) \). Then \( R_{M}(z) \) defines a weight 0 cocycle for \( \Gamma_{0}(N) \) with values in the holomorphic functions on \( \mathbb{H} \), and which satisfies \( \frac{\partial}{\partial z} R_{M}(z) = q_{M}(z) \) for every \( M \in \Gamma_{0}(N) \).

**Proof** The proof is exactly the same as that of [11, Theorem 2.1], so we only give a sketch. By a standard computation we obtain for \( c \neq 0 \) the integral representation

\[ H \left( s, \frac{a}{c} \right) = -\int_{1}^{\infty} (F(z_{1/t}) - a(0))t^{1-s}dt + \int_{1}^{\infty} (F(Mz_{t}) - a(0))t^{s-1}dt, \]

where \( z_{t} = -\frac{d}{c} + \frac{i}{c}t \). Since \( z_{1/t} = -\frac{d}{c} + \frac{it}{c} \) and \( Mz_{t} = \frac{a}{c} + \frac{it}{c} \), we see that \( H \left( s, \frac{a}{c} \right) \) is entire and satisfies the claimed functional equation. Further, we let

\[ G(z) = a(0)z + \sum_{n \geq 1} \frac{a(n)}{2\pi in}e(nz) \]

be a primitive of \( F(z) \). By taking the limit \( s \to 1 \) in \( H \left( s, \frac{a}{c} \right) \) we obtain after a short calculation

\[ R_{M}(z) = G(Mz) - G(z), \]

which is valid for all \( M \in \Gamma_{0}(N) \) and defines a weight 0 cocycle with values in the holomorphic functions on \( \mathbb{H} \), and \( \frac{\partial}{\partial z} R_{M}(z) = q_{M}(z) \).

**Lemma 6.6** Let \( q_{M}^{A} \) be the weight 2 cocycle associated to \( A \in \Gamma_{0}(N) \setminus L_{-1|\Delta_{|n, rh}} \) as above. For \( X \in \mathcal{A} \) let \( w_{X} > w_{X}' \) denote the two real endpoints of the geodesic \( c_{X} \). Let \( F(z) = \sum_{n \geq 0} a(n)q^{n} \) be a modular integral for \( q_{M}^{A} \) with \( a(n) \ll n^{\alpha} \) for some \( \alpha > 0 \) and let \( M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_{0}(N) \). Further, for \( c \neq 0 \) let

\[ L_{F} \left( s, \frac{a}{c} \right) = \sum_{n \geq 1} a(n)e \left( \frac{an}{c} \right) n^{-s}. \]
and

\[ R^A_M(z) = \frac{1}{\sqrt{4N|\Delta|n}} \sum_{X \in A \atop a_{MX} < 0 < a_X} \left( \log(z - w_X) - \log(z - w'_X) \right) \]

\[ + \frac{1}{2\pi i} L_F \left( 1, \frac{a}{c} \right) + a(0) \frac{a + d}{c}, \]

and for \( M = \pm \left( \begin{smallmatrix} 1 & n \\ 0 & \Delta \end{smallmatrix} \right) \) we let \( R^A_M(z) = n a(0) \). Then \( R^A_M(z) \) is a weight 0 cocycle with values in the holomorphic functions on \( \mathbb{H} \) which satisfies \( \frac{\partial}{\partial z} R^A_M(z) = q^A_M(z) \).

**Proof** Note that

\[ q^A_M(z) = \frac{1}{\sqrt{4N|\Delta|n}} \sum_{X \in A \atop a_{MX} < 0 < a_X} \left( \log(z - w_X) - \log(z - w'_X) \right). \]

Thus if we choose

\[ \frac{1}{\sqrt{4N|\Delta|n}} \sum_{X \in A \atop a_{MX} < 0 < a_X} \left( \log(z - w_X) - \log(z - w'_X) \right) \]

as a primitive for \( q^A_M(z) \), the formula for \( R^A_M(z) \) follows from Proposition 6.5. \( \square \)

**Example 6.7** Let \( N = 1, \Delta > 1, \) and \( M = S = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \). We have

\[ q^A_S(z) = \sum_{X \in A \atop c < 0 < a} \frac{1}{Q_X(z)}. \]

It easily follows from the definition and the functional equation of \( H(s, 0) \) given in Proposition 6.5 that

\[ L_F(1, 0) = -\frac{2\pi i}{\sqrt{4\Delta n}} \sum_{X \in A \atop c < 0 < a} \left( \log(i - w_X) - \log(i - w'_X) \right) \]

independently of the modular integral \( F \) for \( q^A \). In particular, we obtain

\[ R^A_S(z) = \frac{1}{\sqrt{4\Delta n}} \sum_{X \in A \atop c < 0 < a} \left( \log \left( \frac{z - w_X}{i - w_X} \right) - \log \left( \frac{z - w'_X}{i - w'_X} \right) \right). \]

We can now state the transformation behaviour of the Borcherds product associated to \( f \in H_{1/2, \rho^*_L} \).
Theorem 6.8 Let \( \Delta \neq 1 \) be a fundamental discriminant. Let \( f \in H_{1/2, r_L^*} \) and suppose that \( c_f^+(|\Delta|m^2/4N, rm) \in \mathbb{R} \) for all \( m \in \mathbb{Z}, m > 0 \). Further, assume that \( c_f^+(n, h) = 0 \) for all \( n < 0, h \in L'/L \), and that \( c_f^-(|\Delta|m^2/4N, rm) = 0 \) for all \( m \in \mathbb{Z}, m > 0 \). Then the infinite product

\[
\Psi_{\Delta, r}(f, z) = e\left(\frac{\sqrt{|\Delta|N}}{4\pi} \Delta(1)c_f^-(0, 0)z\right) \times \prod_{m=1}^{\infty} \prod_{b(\Delta)} \prod_{n} [1 - e(mz + b/\Delta)] \left(\frac{\pi}{\sqrt{|\Delta|}}\right) c_f^+(|\Delta|m^2/4N, rm)
\]

converges to a holomorphic function on \( \mathbb{H} \) transforming as

\[
\Psi_{\Delta, r}(f, Mz) = \chi(M) \mu_{\Delta, r}(f, M, z) \Psi_{\Delta, r}(f, z)
\]

for all \( M \in \Gamma_0(N) \), where \( \chi \) is a character of \( \Gamma_0(N) \) and

\[
\mu_{\Delta, r}(f, M, z) = \prod \prod \prod_{\tau \in L'/L} e\left(-\frac{\sqrt{|\Delta|N}}{2\pi} c_f^-(n, h) \chi_A(A) R_M^A(z)\right),
\]

where \( R_M^A(z) \) is the weight 0 cocycle with \( \frac{\partial}{\partial z} R_M^A(z) = q_M^A(z) \). Further, its logarithmic derivative is given by

\[
\frac{\partial}{\partial z} \log \left(\Psi_{\Delta, r}(f, z)\right) = -2\pi i \sqrt{|\Delta|N} F_{\Delta, r}(f, z),
\]

where \( F_{\Delta, r}(f, z) \) is the modular integral defined in Proposition 6.1.

**Proof** Using Proposition 6.1 we see after a short calculation that the logarithmic derivatives of \( \Psi_{\Delta, r}(f, Mz) \) and \( \mu_{\Delta, r}(f, M, z) \Psi_{\Delta, r}(f, z) \) agree. Further, both functions are holomorphic and non-vanishing on \( \mathbb{H} \). Hence they are constant multiples of each other. This proves the transformation behaviour.

The fact that \( R_M^A(z) \) is a weight 0 cocycle together with the transformation formula of the Borcherds product implies that \( \chi \) is a character of \( \Gamma_0(N) \).

**Example 6.9** Let \( \Delta > 1 \), and let \( f \in H_{1/2, r_L^*} \) be the Bruinier–Funke–Imamoglu lift of a harmonic Maass form \( F \in H_0^+(\Gamma_0(N)) \) with vanishing constant coefficients \( a_{l}^+(0) \) at all cusps as in Example 6.3. Its Borcherds lift is given by

\[
\Psi_{\Delta, r}\left(\frac{\pi}{\sqrt{|N|}}, f, z\right) = \prod_{m=1}^{\infty} \prod_{b(\Delta)} \prod_{n} [1 - e(mz + b/\Delta)] \left(\frac{\pi}{\sqrt{|\Delta|}}\right) c_f^+(|\Delta|m^2/4N, rm)
\]

\[
\times e\left(-\frac{\sqrt{|\Delta|}}{2} \Delta(1) \text{tr}_F(0, 0)z\right).
\]
For $N = 1$ and $F = J = j - 744$ (with $\text{tr}_J(0, 0) = 4$ and $\text{tr}_J(-1/4, 1) = 2$) we obtain the theorem in the introduction. Note that the relations $S^4 = 1$, $(ST)^6 = 1$ and $\chi(T) = 1$ imply that $\chi = 1$ for $N = 1$.

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