Generic twisted $T$-adic exponential sums of binomials

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Received January 11, 2009; accepted August 19, 2010; published online March 23, 2011

Abstract The twisted $T$-adic exponential sum associated with $x^d + \lambda x$ is studied. If $\lambda \neq 0$, then an explicit arithmetic polygon is proved to be the Newton polygon of the $C$-function of the twisted $T$-adic exponential sum. It gives the Newton polygons of the $L$-functions of twisted $p$-power order exponential sums.

Keywords twisted, $T$-adic exponential sums, binomials

MSC(2000): 11L07, 14F30

Citation: Liu C L, Niu C Z. Generic twisted $T$-adic exponential sums of binomials. Sci China Math, 2011, 54(5): 865–875, doi: 10.1007/s11425-011-4203-z

1 Introduction

1.1 Classical twisted exponential sums

Let $p$ be a fixed prime number, $\mathbb{Z}_p$ the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let $q = p^a$ be a power of $p$, $\mathbb{F}_q$ the finite field of $q$ elements, $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and $\mathbb{Z}_q$ the integer ring of $\mathbb{Q}_q$.

Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial of degree $d$. Let $\mu_{q-1}$ be the group of $(q-1)$-th roots of unity in $\mathbb{Z}_q$ and $\chi = \omega^{-u}$ with $u \in \mathbb{Z}/(q-1)$ a fixed multiplicative character of $\mathbb{F}_q^\times$ into $\mu_{q-1}$, where $\omega : x \to \hat{x}$ is the Teichmüller character. Let $\psi$ be a character of $\mathbb{Z}_p$ of order $p^m$ and $\pi_\psi = \psi(1) - 1$.

Definition 1.1. The sum

$$S_{f,u}(k, \psi) = \sum_{x \in \mathbb{F}_{q^k}^\times} \chi(\text{Norm}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x))\psi(\text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(\hat{f}(\hat{x}))) \in \mathbb{Z}_q[\pi_\psi]$$

is called a twisted $p^m$-order exponential sum. And the function

$$L_{f,u}(s, \psi) = \exp\left(\sum_{k=1}^{\infty} S_{f,u}(k, \psi) \frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_q[\pi_\psi][[s]]$$

is called an $L$-function of a twisted $p^m$-order exponential sum.

The $L$-function $L_{f,u}(s, \psi)$ is well known to be rational in $s$. However, if $f$ is non-degenerate, then it is a polynomial of degree $p^{m-1}d$, as was shown by Adolphson-Sperber [3,4] for $m = 1$ and Liu [6] for all $m$.

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1.2 Twisted T-adic exponential

Let $T$ be a variable. We now define the twisted $T$-adic exponential sum and then state our main results.

**Definition 1.2.** The sum

$$S_{f,u}(k,T) = \sum_{x \in \mathbb{Z}_{q^k}} \chi(\text{Norm}_{F_{q^k}/F_q}(x))(1 + T)^{Tr\gamma_{q^k}/q^m}(f(x)) \in \mathbb{Z}_q[[T]]$$

is called a twisted $T$-adic exponential sum. And the function

$$L_{f,u}(s,T) = \exp\left(\sum_{k=1}^{\infty} S_{f,u}(k,T) \frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_q[[T]][[s]]$$

is called an $L$-function of a twisted $T$-adic exponential sum.

**Definition 1.3.** The function

$$C_{f,u}(s,T) = \exp\left(\sum_{k=1}^{\infty} -(q^k - 1)^{-1} S_{f,u}(k,T) \frac{s^k}{k}\right),$$

is called a C-function of a twisted $T$-adic exponential sum.

The $L$-function and the $C$-function determine each other:

$$L_{f,u}(s,T) = C_{f,u}(s,T)^{-1}$$

and

$$C_{f,u}(s,T) = \prod_{j=0}^{\infty} L_{f,u}(q^j s,T).$$

By the last identity, one sees that

$$C_{f,u}(s,T) \in 1 + s\mathbb{Z}_q[[T]][[s]].$$

The $T$-adic exponential sums were first introduced by Liu-Wan [10] and the theory of twisted $T$-adic exponential sums was developed by Liu [7]. We view $L_{f,u}(s,T)$ and $C_{f,u}(s,T)$ as power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_q(T)$. The C-function $C_{f,u}(s,T)$ was shown to be $T$-adic entire in $s$ by Liu-Wan [10] for $u = 0$ and Liu [6] for all $u$.

Let $\zeta_p$ be a primitive $p^n$-th root of unity, and $\pi_m = \zeta_p^m - 1$. Then $L_{f,u}(s,\pi_m) = L_{f,u}(s,\psi)$ is the classical $L$-function of the $p$-power order exponential sums $S_{f,u}(k,\pi_m)$ studied by Adolphson-Sperber [1–4] for $m = 1$, by Liu-Wei [9] and Liu [6] for $m \geq 1$. By a result of Li [8], we see that if $p \nmid d$, then $L_{f,u}(s,\pi_m)$ is a polynomial of degree $p^{m-1}d$.

Let $b$ be the least positive integer such that $p^b u \equiv u \pmod{q - 1}$. Let $a = \log_p q$. Write $u = u_0 + u_1 p + \cdots + u_{b-1} p^{b-1}$ with $0 \leq u_i \leq p - 1$. Then we have

$$\frac{u}{q - 1} \equiv -(u_0 + u_1 p + \cdots), \quad u_i = u_{b+i}.$$

**Definition 1.4.** The infinite $u$-twisted Hodge polygon $H_{\infty,0,d,u}$ is the convex function on $[0, +\infty)$ with initial point 0 which is linear between consecutive integers and whose slopes are

$$\frac{u_0 + u_1 + \cdots + u_{b-1}}{b(d(p - 1))} + \frac{l}{d}, \quad l = 0, 1, \ldots$$

Write NP for the short of Newton polygon. As was shown by Liu [6], we have

$$T\text{-adic NP of } C_{f,u}(s,T) \geq \text{ord}_p(q)(p - 1)H_{\infty,0,d,u},$$

i.e., the infinite $u$-twisted Hodge polygon is a lower bound of the $T$-adic Newton polygon of $C_{f,u}(s,T)$.

In the rest of this paper, $f(x) = x^d + \lambda x \in \mathbb{F}_q[x]$, where $\lambda \in \mathbb{F}_q^*$. We fix $0 \leq u \leq q - 1$. We shall study the twisted exponential sum of $f(x)$. 