Correlations between large prime numbers

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It is shown that short-range correlations between large prime numbers (∼ 10^5 and larger) have a Poissonian nature. Correlation length \( \zeta \approx 4.5 \) for the primes ∼ 10^5 and it is increasing logarithmically according to the prime number theorem. For moderate prime numbers (∼ 10^4) the Poissonian distribution is not applicable (while the correlation length \( \zeta \) surprisingly continues to follow to the logarithmic law. A chaotic (deterministic) hypothesis has been suggested to explain the moderate prime numbers apparent randomness.

Key words: gaps between primes, short intervals, chaos

Introduction. The prime number distribution is apparently random. The apparent randomness, however, can be stochastic or chaotic (deterministic). The nature of the randomness is still unsolved problem. Using probabilistic methods we should take into account that there are different levels of averaging. One can consider the probabilities as representing a first level of averaging while the correlation functions represent a second one. Although the first level contains more details the second one can provide a more robust picture. In order to use the second level of averaging let us define a binary function \( v(n) \) of integers \( n = 3, 4, 5, \ldots \), which takes two values +1 or -1 and changes its sign passing any prime number. This function contains full information about the prime numbers distribution. Due to the prime number theorem, which states that the "average length" \( \tau_p \) of the gap between a prime \( p \) and the next prime number is proportional (asymptotically) to \( \ln p \), the \( v(n) \) is a statistically non-stationary function. However, for sufficiently large \( n \) this function can be considered (in a certain level of averaging) as a stationary function in the windows centered at \( n_0 \) with width \( \Delta/n_0 \ll 1 \). An heuristic estimate of a relation:

\[
\frac{\tau_{p+\Delta}}{\tau_p} \sim \frac{\ln(p+\Delta)}{\ln p} \approx 1 + \frac{\Delta}{p \ln p}
\]

for \( \Delta/p \ll 1 \) provides a basis for such consideration. We will show (numerically) that on the level of the correlation function this idea indeed seems to be rather well applicable for sufficiently large prime numbers. On the level of the probabilities, however, application of this idea demands certain additional average (after that the results obtained by the two different methods can be compared).

Large prime numbers. Before performing this analysis let us recall certain simple properties of the random telegraph signal. The random telegraph signal is a binary process \( x(n) \) which takes two values +1 or -1, and has Poissonian distribution of the values of \( n \) where \( x(n) \) changes its sign. For the stationary case the Poisson distribution results in the exponential distribution of gaps \( \tau \) between consecutive values of \( n \) where \( x(n) \) changes its sign

\[
P(\tau) = \frac{1}{\Theta} e^{-\tau/\Theta}
\]
with constant value of $\Theta \geq 0$. The autocorrelation function of the statistically stationary random telegraph signal $x(n)$

$$C(l) = \langle x(n)x(n+l) \rangle - \langle x(n) \rangle \langle x(n+l) \rangle = e^{-l/\zeta}$$

where the angular brackets denote averaging over realizations and

$$\zeta = \frac{\Theta}{2}$$

Figure 1 shows a correlation function computed for the $v(n)$ function in a 'window of stationarity' with the width $\Delta = 10^4$ centered at $n_0 \sim 10^5$. The semi-log axes has been used in this figure in order to indicate exponential decay, Eq. (3), of the autocorrelation function (the straight line). The correlation length $\zeta \simeq 4.5$. Figure 2 shows analogous autocorrelation function but in a window with the width $\Delta = 10^5$ centered at $n_0 \sim 10^7$. The correlation length $\zeta \simeq 6.3$ in this case. Figures 3 and 4 show the distribution of the gaps $\tau$ between consecutive primes: $P(\tau)$, for $v(n)$ in a 'window of stationarity' with the width $\Delta = 10^4$ centered at $n_0 \sim 10^5$ and in a window with the width $\Delta = 10^5$ centered at $n_0 \sim 10^7$ respectively. The straight lines in these figures represent the best fit to the data (also certain kind of averaging) and the slopes of these best fits provide us with the exponents of the Eq. (2): $\Theta \simeq 9$ and $\Theta \simeq 12.8$ respectively. Now one can compare the values of the correlation length $\zeta$ with values of the $\Theta$ and with the Poissonian relation Eq. (4). One can see that in the second level of averaging correspondence to the Poissonian distribution of the primes in the considered windows is rather good. Similarly to the "average gap" (see above) the second level's parameters $\zeta$ and $\Theta = 2\zeta$ are proportional to $\ln n_0$ due to the prime number theorem.

**Moderate prime numbers.** Figure 5 shows an exponentially decaying part of the autocorrelation computed for the $v(n)$ function in a 'window of stationarity' with the width $\Delta = 4000$ centered at $n_0 \sim 18000$ (moderate values of the primes). Naturally, the exponential range here is rather short. However, the computed value of the correlation length $\zeta \simeq 3.83$ follows quite precisely to the logarithmic dependence on $n_0$. It is surprising if we take into account that the distribution of the gaps between consecutive primes (the parameter $\Theta$) does not, as one can see from figure 6 (the $\Theta \simeq 9$ here). This means, in particular, that the $v(n)$ function in this window is not a random telegraph signal. Something wrong is going with the Poissonian hypothesis for these values of the primes. But if previously we related the logarithmical dependence of the correlation length $\zeta$ on $n_0$ just with the logarithmical dependence of $\Theta$, then how can we explain it now?

The problem with the Poissonian distribution for the moderate primes may be related to the strong and persis-
tent periodic oscillations in the distribution of the gaps. Figure 7 shows spectrum of these oscillations (seen in Fig. 6) after a detrending, and the strong peak in Fig. 7 corresponds to the period equal to 6 (as well as for the larger $n_0$). In the Refs. [1]-[3] this period has been studied both numerically and theoretically. We are more interested, however, in nature of the randomness of the moderate primes. The spectrum of the function $v(n)$ in the moderate window of stationarity, shown in figure 8 in the semi-log scales, may help in solving this problem. One can see a pronounced range of an exponential decay of the spectrum. Many of the well known chaotic attractors (‘Lorenz’, ‘Rossler’, etc.) exhibit the exponentially decaying spectra and this spectrum is considered as a strong indication of a chaotic behavior [4]-[7]. Here, for comparison with the Fig. 8, we will consider a chaotic spectrum generated by the Kaplan-Yorke map [8]. In the Langevin approach to Brownian motion the equation of motion is

$$\dot{y} = -\gamma y + F(t)$$

(5)

where the fluctuating kick force on the particle is a Gaussian white noise: $F(t) = \sum_n \eta_n \delta(t - n\tau)$ and $g(t)$ to take values in $R^m$. One can assume [8] that the evolution of the kick strengths is determined by a discrete time dynamical system $T$ on the phase space and projected onto $R^m$ by a function $g$:

$$\eta_n = g(x_{n-1}), \quad x_{n+1} = Tx_n$$

(6)

Then the solution of Eq. (5) is

$$y(t) = e^{-\gamma(t-n\tau)}y_n$$

(7)

where $n$ equals the integer part of the relation $t/\tau$ and the recurrence

$$x_{n+1} = Tx_n, \quad y_{n+1} = \alpha y_n + g(x_n)$$

(8)

provides value of $y_n$ (with $\alpha = e^{-\gamma\tau}$). In certain sense the dynamical system (8) is equivalent to the stochastic differential equation (5). In the generalization related to the Eq. (8) the force $F(t)$ can be considered as a non-Gaussian process which is determined by $g$ and $T$.

The Kaplan-Yorke map [8],[10],[11] is a particular simple case for this generalization:

$$Tx = 2x \mod 1, \quad g(x) = \cos 4\pi x$$

(9)

An optimal computational algorithm for the Kaplan-Yorke map is

$$a_{n+1} = 2a_n \mod (p), \quad x_{n+1} = a_n/p, \quad y_{n+1} = \alpha y_n + \cos 4\pi x$$

(10)

where $p$ is a large prime number.

Figure 9 shows spectrum of a chaotic solution of the Kaplan-Yorke map ($\alpha = 0.2$). We used the semi-log axes in this figure in order to indicate exponential decay of the spectrum (the straight line).
FIG. 9: Spectrum of a chaotic solution of the Kaplan-Yorke map.

One can compare Fig. 9 and Fig. 8 in order to conclude on the chaotic nature of the "window defined" distribution of the moderate primes.

A rigorously minded reader can find a lot of good mathematics related the above discussed approach: such as Gallagher theorem for short intervals [12] (about Poisson distribution in the limit case of logarithmically short windows), Cramer’s probabilistic model [13] and the difficulties of this model in the short intervals (Maier’s theorem [14],[15],[16]). Since present consideration was devoted to the correlations between large primes and the correlation length $\zeta$ is rather small (see Figs. 1,2 and 5) the consideration was restricted by the windows of statistic stationarity where the Poisson distribution seems to be reigning on the second level of statistical averaging at sufficiently large primes. Above these windows distributions of the primes can be different (see, for instance, Ref. [17] and references therein). For moderate values of primes the Poisson distribution seems to be not applicable and a chaotic hypothesis has been suggested instead.

[1] M. Wolf, Applications of statistical mechanics in number theory, Physica A, 274, 149-157 (1999)
[2] P. Kumar, P.Ch. Ivanov, and H.E. Stanley, Information Entropy and Correlations in Prime Numbers, arXiv:cond-mat/0303110v4 [cond-mat.stat-mech], (2003).
[3] S. Aresa, and M. Castro, Hidden structure in the randomness of the prime number sequence?, Physica A, 360, 285-296 (2006).
[4] D.E. Sigeti, Survival of deterministic dynamics in the presence of noise and the exponential decay of power spectra at high frequency, Phys. Rev. E, 52, 2443; Physica D, 82, 136 (1995).
[5] N. Ohtomo, K. Tokiwano, Y. Tanaka et. al., Exponential Characteristics of Power Spectral Densities Caused by Chaotic Phenomena, J. Phys. Soc. Jpn., 64 1104 (1995).
[6] J. D. Farmer, Chaotic Attractors of an Infinite-Dimensional Dynamical System, Physica D, 4, 366 (1982).
[7] U. Frisch and R. Morf, Intermittency in nonlinear dynamics and singularities at complex times, Phys. Rev., 23, 2673 (1981).
[8] J.L. Kaplan and J.A. Yorke, in Functional Differential Equations and Approximations of Fixed Points, Lecture Notes in Mathematics, 730, p. 204. (New York: Springer 1979).
[9] C. Beck, Ergodic Properties of a Kicked Damped Particle, Commun. Math. Phys., 130, 51 (1990).
[10] R.V. Jensen and C.R. Oberman, Calculation of the Statistical Properties of Strange Attractors, Phys. Rev. Lett., 46, 1547 (1981).
[11] J. C. Sprott, Chaos and Time-Series Analysis (Oxford Univ. Press, 2003)
[12] P.X. Gallagher, On the distribution of primes in short intervals, Mathematika, 23, 4-9, (1976).
[13] H.Cramer, On the order of magnitude of the difference between consecutive prime numbers. Acta Arith., 2, 23-46 (1936).
[14] H. Maier, Primes in short intervals, The Michigan Math. Journal, 32 221 (1985).
[15] A. Granville, Proc. Intern. Congress of Math., 1, 388 (1995) (Zurich, Switzerland, 1994).
[16] H.L. Montgomery, and K. Soundararajan, Primes in short intervals, Commun. Math. Phys., 252, 589-617 (2004).
[17] K. Soundararajan, The distribution of the prime numbers, NATO Science Series II: Mathematics, Physics and Chemistry, , 237, 59-83 (2007).