Moderate Deviation and Central Limit Theorem for SDDEs with Polynomial Growth

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Abstract

In this paper, employing the weak convergence method, based on a variational representation for expected values of positive functionals of a Brownian motion, we investigate moderate deviation for a class of stochastic differential delay equations with small noises, where the coefficients are allowed to be highly nonlinear growth with respect to the variables. Moreover, we obtain the central limit theorem for stochastic differential delay equations which the coefficients are polynomial growth with respect to the delay variables.

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1 Introduction and Main Results

There has been extensive literature on the theory of large deviation principle (LDP) for stochastic differential equations (SDEs) with small noises since the pioneer work due to Freidlin-Wentzell [20]. As we know, the classical method to show LDP is based on an approximation argument and some exponential-type estimates; see, e.g., [12, 13, 19, 23, 35, 11]. As far as the classical method is concerned, the exponential-type estimate is a hard ingredient to deal with because different SDEs or stochastic partial differential equations (SPDEs) need different techniques. In recent years, the weak convergence method, (see, e.g., [5, 6, 8] and references therein) has been developed to study LDP problems for diverse setups, where the advantage of this method is that it avoids some exponential probability estimates; see, e.g., [9, 17, 26, 16, 31, 32] for SDEs/SPDEs driven by Brownian motion, and [2, 4, 11, 36] for SDEs/SPDEs driven by jump processes.

Recently, numerous mathematicians work on central limit theorem (CLT); see, e.g., [21, 18, 33]. Since moderate deviation principle (MDP) fills the gap between CLT scale and LDP scale, it has been gained much attention. With regard to MDP, we refer to, e.g., [7] for SDEs driven by a
Poisson random measure in finite and infinite dimensions, \cite{25} for stochastic heat equation driven by a Gaussian noise, \cite{33} for 2D stochastic Navier–Stokes equations, and \cite{34} for stochastic reaction-diffusion equations with multiplicative noise. Specially, \cite{27} is devoted to investigate moderate deviations for neutral stochastic differential delay equations with jump, the assumptions in it are those the coefficient is of quadratic growth with respect to the delay variables, inspired this, we try to construct weaker assumptions to investigate MDP.

It is worthy to point out that most of the literature focus on MDPs and CLTs for SDEs with linear growth; see, e.g., \cite{7, 33}. Whereas, in the present work, we are interested in MDPs for a wide range of SDEs with memory, which allow the coefficients are nonlinear growth with respect to the variables and CLTs which allow the coefficients to be of polynomial growth with respect to the delay variables. For more details on SDEs with memory, we refer to the monograph \cite{30}.

To begin, for any $\varepsilon \in (0, 1)$, consider the following stochastic differential delay equation (SDDE)

\begin{equation}
\begin{aligned}
\frac{dX^\varepsilon(t)}{dt} &= b(X^\varepsilon(t), X^\varepsilon(t-\tau))dt + \sqrt{\varepsilon}\sigma(X^\varepsilon(t), X^\varepsilon(t-\tau))dW(t), \\
\quad t > 0
\end{aligned}
\end{equation}

with the initial data $X^\varepsilon(\theta) = \xi(\theta), \theta \in [-\tau, 0]$, where $b : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$, and $\{W(t)\}_{t \geq 0}$ is an $m$-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Intuitively, as $\varepsilon \downarrow 0$, $\{X^\varepsilon(t)\}_{t \geq 0}$, the solution to (1.1), tends to $\{X^0(t)\}_{t \geq 0}$, which solves the following deterministic differential delay equation

\begin{equation}
\begin{aligned}
\frac{dX^0(t)}{dt} &= b(X^0(t), X^0(t-\tau))dt, \\
\quad t > 0, \\
\quad X^0(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0].
\end{aligned}
\end{equation}

In this paper, we shall investigate deviations of $X^\varepsilon$ from the deterministic solution $X^0$, as $\varepsilon \downarrow 0$. That is, we are interested in the asymptotic behavior of the trajectories:

\begin{equation}
\begin{aligned}
Z^\varepsilon(t) := \frac{1}{\sqrt{\varepsilon\lambda(\varepsilon)}}(X^\varepsilon(t) - X^0(t)), \\
\quad t \in [0, T],
\end{aligned}
\end{equation}

in which $\lambda(\varepsilon)$ is some deviation scale. In particular,

(1) For $\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$, it is corresponding to LDPs;

(2) For $\lambda(\varepsilon) \equiv 1$, it is associated with CLTs;

(3) For $\lambda(\varepsilon) \to \infty$ and $\sqrt{\varepsilon}\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$, it is concerned with MDPs.

Let $V : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}_+$ such that

\begin{equation}
\begin{aligned}
V(x, y) \leq K(1 + |x|^q + |y|^q), \\
\quad x, y \in \mathbb{R}^n
\end{aligned}
\end{equation}

holds for some constants $K, q \geq 1$.

For any $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, we assume that

(H1) There exists an $L > 0$ such that

$$
|b(x_1, y_1) - b(x_2, y_2)| + \|\sigma(x_1, y_1) - \sigma(x_2, y_2)\|_{HS} \leq L|x_1 - x_2| + V(y_1, y_2)|y_1 - y_2|,
$$

where $\| \cdot \|_{HS}$ stands for the Hilbert-Schmidt norm.
(H2) \( b(\cdot, \cdot) \) is Fréchet differentiable w.r.t. each component, and there exists an \( L_0 > 0 \) such that

\[
\| \nabla^{(1)} b(x_1, \cdot) - \nabla^{(1)} b(x_2, \cdot) \| \leq L_0 |x_1 - x_2|,
\]

and

\[
\| \nabla^{(2)} b(\cdot, y_1) - \nabla^{(2)} b(\cdot, y_2) \| \leq V(y_1, y_2) |y_1 - y_2|,
\]

in which \( \nabla^{(i)} b(\cdot, \cdot) \) denotes the gradient operator w.r.t. the \( i \)'th variable.

Under (H1), (1.1) admits a unique strong solution \( \{X(t)\}_{t \geq -\tau} \) (see, e.g., [1, Lemma 2.1]). For \( b(x, y) = 2x + 3y^2, \sigma(x, y) = 4y^2, x, y \in \mathbb{R} \), it is easy to see that (H1) and (H2) hold, respectively, with \( V(x, y) = 9(1 + x^2 + y^2) \) and \( L = L_0 = 2 \).

One of our main results in this paper is presented as below.

**Theorem 1.1.** Under (H1) and (H2),

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \frac{X^\epsilon(t) - X^0(t)}{\sqrt{\epsilon}} - Y(t) \right|^2 \right) \leq c \epsilon
\]

for some constant \( c > 0 \). Herein, \( Y(t) \) solves

\[
dY(t) = \{\nabla_Y^{(1)} b(X^0(t), X^0(t - \tau)) + \nabla_Y^{(2)} b(X^0(t), X^0(t - \tau))\}dt + \sigma(X^0(t), X^0(t - \tau))dW(t)
\]

where, \( \nabla_x^{(i)} \) is the gradient operator along the \( x \) direction, with the initial value \( Y(\theta) \equiv 0_n \), the zero vector in \( \mathbb{R}^n \), for any \( \theta \in [-\tau, 0] \), in which \( \{X^0(t)\}_{t \geq -\tau} \) is determined by (3.15).

In the sequel, we shall extend Theorems 1.1 to SDDEs of neutral type:

\[
d\{X^\epsilon(t) - G(X^\epsilon(t - \tau))\} = b(X^\epsilon(t), X^\epsilon(t - \tau))dt + \sqrt{\epsilon}\sigma(X^\epsilon(t), X^\epsilon(t - \tau))dW(t), \quad t > 0,
\]

with the initial data \( X^\epsilon(\theta) = \xi(\theta), \theta \in [0, 0] \), where \( G : \mathbb{R}^n \mapsto \mathbb{R}^n \) and the other parameters are defined exactly as in (1.1).

As \( \epsilon \downarrow 0 \), \( X^\epsilon(t) \), the solution to (1.7), tends to \( X^0(t) \), which solves the deterministic differential delay equation of neutral type

\[
d\{X^0(t) - G(X^0(t - \tau))\} = b(X^0(t), X^0(t - \tau))dt, \quad t > 0, \quad X^0(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0].
\]

Besides (H1) and (H2), we further suppose that

(H3) \( G(\cdot) \) is Fréchet differentiable, and for any \( x, y \in \mathbb{R}^n \),

\[
|G(x) - G(y)| \leq V(x, y)|x - y|,
\]

and

\[
|\nabla G(x) - \nabla G(y)| \leq V(x, y)|x - y|,
\]

where \( V(\cdot, \cdot) \) such that (1.4) holds.

Concerning (1.7), Theorem 1.1 can be generalized as below.
Theorem 1.2. Under (H1)-(H3),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \left| \frac{X^\varepsilon(t) - X^0(t)}{\sqrt{\varepsilon}} - Y(t) \right|^2 \right) \leq c\varepsilon,
\]

for some constant \( c > 0 \). Herein, \( Y(t) \) solves

\[
d\{Y(t) - \nabla_{Y(t-\tau)}G(X^0(t-\tau))\} = \{\nabla^{(1)}_{Y(t)} b(X^0(t), X^0(t-\tau)) \}
\]

\[
+ \nabla^{(2)}_{Y(t-\tau)} b(X^0(t), X^0(t-\tau)) dt
\]

\[
+ \sigma(X^0(t), X^0(t-\tau)) dW(t), \quad t > 0
\]

with the initial value \( Y(\theta) \equiv 0_n \), for any \( \theta \in [-\tau, 0] \), in which \( \{X^0(t)\}_{t \geq -\tau} \) is determined by (1.8).

The outline of this work is organized as follows: In section 2, we give the proofs of the Theorems 1.1 and 1.2; Section 3 is devoted to the moderate deviation principle for SDDEs, which allow the coefficients are highly nonlinear growth with respect to the variables; In section 4, we give two examples, which the coefficients are polynomial growth with respect to the variables. Throughout the paper, \( C \) is a generic constant, whose value may be different from line to line by convention, and we use the shorthand notation \( a \lesssim b \) to mean \( a \leq cb \).

2 Proofs of Theorems 1.1 and 1.2

Before we complete proofs of our main results, we prepare several auxiliary lemmas. Throughout this section, we point out that \( \{X^\varepsilon(t)\}, \{X^0(t)\} \) and \( \{Y(t)\} \) below solve (1.7), (1.8), and (1.12), respectively.

The lemma below show that \( \{X^\varepsilon(t)\}, \{X^0(t)\} \), the solutions to (1.7), (1.8), respectively, are uniformly bounded in \( p \)-th moment sense in a finite horizon.

Lemma 2.1. Under (H1) and (1.9),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X^\varepsilon(t)|^p \right) \vee \left( \sup_{0 \leq t \leq T} |X^0(t)|^p \right) \leq C, \quad p \geq 2, \ \varepsilon \in (0, 1),
\]

where \( C \) is a constant, which depends on \( \|\xi\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\xi(\theta)| \).

Proof. From (H1), a straightforward calculation gives that

\[
|b(x, y)| + \|\sigma(x, y)\|_{HS} \lesssim 1 + |x| + |y|^{q+1}, \quad x, y \in \mathbb{R}^n.
\]

Hereinafter, \( q \geq 1 \) is given in (1.4). Set \( r := 1 + q \) for notational simplicity and let \( t \in [0, T] \) be
arbitrary. From (2.2) and (1.9), we obtain that for any \( p \geq 2 \) and \( \epsilon \in (0, 1) \),
\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |X^\epsilon(s)|^p \right) \leq \mathbb{E}|\xi(0) - G(\xi (-\tau))|^p + \mathbb{E}\left( \sup_{0 \leq s \leq t} |G(X^\epsilon(s - \tau)|^p) \right.
\]
\[+ \mathbb{E}\left( \sup_{0 \leq s \leq t} \left| \int_0^s b(X^\epsilon(u), X^\epsilon(u - \tau))du \right|^p \right)
\]
\[+ \epsilon^{p/2}\mathbb{E}\left( \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(X^\epsilon(u), X^\epsilon(u - \tau))dW(u) \right|^p \right) \]
\[\lesssim 1 + \|\xi\|_{\|p\|\infty}^p + \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau)\vee 0} |X^\epsilon(s)|^{pr} \right)
\]
\[+ \int_0^t \mathbb{E}|b(X^\epsilon(s), X^\epsilon(s - \tau))|^p ds + \epsilon^{p/2} \int_0^t \mathbb{E}\|\sigma(X^\epsilon(s), X^\epsilon(s - \tau))\|_{HS}^p ds
\]
\[\lesssim 1 + \|\xi\|_{\|p\|\infty}^p + \int_0^t \mathbb{E}|X^\epsilon(s)|^p ds + \int_0^{(t-\tau)\vee 0} \mathbb{E}|X^\epsilon(s)|^{pr} ds + \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau)\vee 0} |X^\epsilon(s)|^{pr} \right),
\]
where \( a \vee b := \max\{a, b\} \) for \( a, b \in \mathbb{R} \). By the Gronwall inequality, one has
\[
(2.3) \quad \mathbb{E}\left( \sup_{0 \leq s \leq t} |X^\epsilon(s)|^p \right) \lesssim 1 + \|\xi\|_{\|p\|\infty}^p + \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau)\vee 0} |X^\epsilon(s)|^{pr} \right) + \int_0^{(t-\tau)\vee 0} \mathbb{E}|X^\epsilon(s)|^{pr} ds.
\]
Let
\[ p_i = ([T/\tau] + 2 - i)pr^{[T/\tau] + 1 - i}, \quad i = 1, 2, \cdots, [T/\tau] + 1. \]
It is easy to see that \( p_i \geq 2 \) such that
\[ rp_{i+1} < p_i \quad \text{and} \quad p_{[T/\tau]+1} = p, \quad i = 1, 2, \cdots, [T/\tau]. \]
We obtain from (2.3) that
\[ \mathbb{E}\left( \sup_{0 \leq s \leq \tau} |X(s)|^{p_1} \right) \lesssim 1 + \|\xi\|_{\|p\|\infty}^{p_1}. \]
This further yields by Hölder’s inequality that
\[ \mathbb{E}\left( \sup_{0 \leq s \leq 2\tau} |X^\epsilon(s)|^{p_2} \right) \lesssim 1 + \mathbb{E}\left( \sup_{0 \leq s \leq \tau} |X^\epsilon(s)|^{p_{2r}} \right) + \int_0^\tau \mathbb{E}|X^\epsilon(s)|^{p_{2r}} ds
\]
\[\lesssim 1 + \mathbb{E}\left( \sup_{0 \leq s \leq \tau} |X^\epsilon(s)|^{p_1} \right)^{\frac{p_{2r}}{p_1}} + \int_0^\tau \left( \mathbb{E}|X^\epsilon(s)|^{p_1} \right)^{\frac{p_{2r}}{p_1}} ds
\]
\[\lesssim 1 + \|\xi\|_{\|p\|\infty}^{p_2}. \]
Repeating the previous procedures yields that
\[ (2.4) \quad \mathbb{E}\left( \sup_{0 \leq t \leq T} |X^\epsilon(t)|^p \right) \lesssim 1 + \|\xi\|_{\|p\|\infty}^{p(1+q)[T/\tau] + 1}, \quad p \geq 2, \quad \epsilon \in (0, 1), \]
which further leads to
\[ \left( \sup_{0 \leq t \leq T} |X^0(t)|^p \right) \lesssim 1 + \|\xi\|_{\|p\|\infty}^{p(1+q)[T/\tau] + 1}, \quad p \geq 2 \]
by letting \( \epsilon \) go to zero. The proof is therefore complete.
The following lemma provides the order of deviation between \( X^\epsilon \) and \( X^0 \).

**Lemma 2.2.** Under (H1) and (1.9), for any \( p \geq 2 \) there is a constant \( C_{p,T} > 0 \), such that

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \left| \frac{X^\epsilon(t) - X^0(t)}{\sqrt{\epsilon}} \right|^p \right) \leq C_{p,T},
\]

(2.5)

**Proof.** For notational simplicity, set

\[
Y^\epsilon := \frac{X^\epsilon - X^0}{\sqrt{\epsilon}}.
\]

(2.6)

Since (1.7) and (1.8) share the same initial value, one has \( Y^\epsilon(\theta) \equiv 0 \) for any \( \theta \in [-\tau, 0] \). In terms of (2.1), one gets from (1.4) that, for each \( l \geq 1 \), there exists a constant \( C_{l,T} > 0 \) such that

\[
\mathbb{E}\left( \sup_{-\tau \leq t \leq T} |V^l(X^\epsilon(t), X^0(t))| \right) \leq C_{l,T}.
\]

(2.7)

By the elementary inequality:

\[
(a_1 + a_2 + \cdots + a_m)^l \leq m^{l-1}(a_1^l + a_2^l + \cdots + a_m^l), \quad l \geq 1, \quad a_i \geq 0,
\]

the B-D-G inequality as well as the Hölder inequality, we obtain (H1) and (1.9) that for \( p \geq 2 \) and \( t \in [0, T] \)

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |Y^\epsilon(s)|^p \right) \leq \mathbb{E}\left( \sup_{0 \leq s \leq t} \left| \frac{G(X^\epsilon(s) - \tau) - G(X^0(s) - \tau)}{\sqrt{\epsilon}} \right|^p \right)
\]

\[
+ \mathbb{E}\left( \sup_{0 \leq s \leq t} \left| \int_0^s \frac{b(X^\epsilon(r), X^\epsilon(r - \tau)) - b(X^0(r), X^0(r - \tau))}{\sqrt{\epsilon}} \, dr \right|^p \right)
\]

\[
+ \mathbb{E}\left( \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(X^\epsilon(r), X^\epsilon(r - \tau)) \, dW(r) \right|^p \right)
\]

\[
\lesssim \left( \mathbb{E}\left( \sup_{-\tau \leq s \leq t} V^{2p}(X^\epsilon(s), X^0(s)) \right) \right)^{\frac{1}{2}} \left( \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau)\vee 0} |Y^\epsilon(s)|^{2p} \right) \right)^{\frac{1}{2}}
\]

\[
+ \int_0^t \mathbb{E}|Y^\epsilon(s)|^p \, ds + \int_0^{(t-\tau)\vee 0} (\mathbb{E}V^{2p}(X^\epsilon(s), X^0(s)))^{\frac{1}{2}} (\mathbb{E}|Y^\epsilon(s)|^{2p})^{\frac{1}{2}} \, ds
\]

\[
+ \int_0^t \left\{ 1 + \mathbb{E}|X^\epsilon(s)|^p + \mathbb{E}|X^\epsilon(s - \tau)|^{p(q+1)} \right\} \, ds
\]

\[
\lesssim 1 + \left( \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau)\vee 0} |Y^\epsilon(s)|^{2p} \right) \right)^{\frac{1}{2}} + \int_0^t \mathbb{E}|Y^\epsilon(s)|^p \, ds + \int_0^{(t-\tau)\vee 0} (\mathbb{E}|Y^\epsilon(s)|^{2p})^{\frac{1}{2}} \, ds,
\]

where in the last display we have used (2.1) and (2.7). So, Gronwall’s inequality gives that

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |Y^\epsilon(s)|^p \right) \lesssim 1 + \left( \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau)\vee 0} |Y^\epsilon(s)|^{2p} \right) \right)^{\frac{1}{2}} + \int_0^{(t-\tau)\vee 0} (\mathbb{E}|Y^\epsilon(s)|^{2p})^{\frac{1}{2}} \, ds.
\]

(2.9)

In what follows, by mimicking the argument of (2.4), the proof of Lemma 2.2 can be done. \( \square \)
Lemma 2.3. Under (H1) and (H2), for any $p \geq 2$ there exists a constant $\tilde{C}_{p,T} > 0$ such that

\begin{equation}
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y(t)|^p \right) \leq \tilde{C}_{p,T}.
\end{equation}

Proof. In view of (H1), we find that for any $x, y, z \in \mathbb{R}^n$,

\begin{equation}
|\nabla_z b(x, y)| \leq L|x| \quad \text{and} \quad |\nabla_z^2 b(x, y)| \leq V(y, y)|z|.
\end{equation}

On the other hand, from (1.9) we have

\begin{equation}
|\nabla_z G(y)| \leq V(y, y)|z|, \quad y, z \in \mathbb{R}^n.
\end{equation}

Recall that $Y(\theta) = 0_n$ for any $\theta \in [-\tau, 0]$. Then, by B-D-G’s inequality and Hölder’s inequality, in addition to (2.2), (2.11) as well as (2.12), we deduce that for any $p \geq 2$

\begin{align*}
\mathbb{E} \left( \sup_{0 \leq s \leq t} |Y(s)|^p \right) &\leq \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)\vee 0} V^p(X^0(s), X^0(s))|Y(s)|^p \right) \\
&\quad + \int_0^t \mathbb{E} \{1 + |X^0(s)| + |X^0(s-\tau)|^{q+1}\}^p ds \\
&\quad + \int_0^t \mathbb{E}|Y(s)|^p ds + \int_0^{(t-\tau)\vee 0} \mathbb{E}V^p(X^0(s), X^0(s))|Y(s)|^p ds \\
&\quad \leq 1 + \mathbb{E} \left( \sup_{0 \leq s \leq t-\tau} |Y(s)|^p \right) + \int_0^t \mathbb{E}|Y(s)|^p ds + \int_0^{(t-\tau)\vee 0} \mathbb{E}|Y(s)|^p ds \\
&\quad \leq 1 + \mathbb{E} \left( \sup_{0 \leq s \leq t-\tau} |Y(s)|^p \right) + \int_0^t \mathbb{E}|Y(s)|^p ds,
\end{align*}

where in the last second inequality we have utilized (2.1). Then, the desired assertion is available by the Gronwall’s inequality and an induction argument. \hfill \square

With Lemmas 2.1-2.3 in hand, we are now in position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $Y^\epsilon$ be defined as in (2.6). Thanks to $Y^\epsilon(\theta) = Y(\theta) \equiv 0_n$ for any
\( \theta \in [-\tau, 0] \), it follows that

\[
Y^\epsilon(t) - Y(t) = \frac{G(X^\epsilon(t-\tau)) - G(X^0(t-\tau))}{\sqrt{\epsilon}} - \nabla_{Y^\epsilon(t-\tau)}G(X^0(t-\tau))
\]

\[+ \int_0^t \left\{ \frac{b(X^\epsilon(s), X^0(s-\tau)) - b(X^0(s), X^0(s-\tau))}{\sqrt{\epsilon}} - \nabla_{Y^\epsilon(s)}b(X^0(s), X^0(s-\tau)) \right\} ds \]

\[+ \int_0^t \left\{ \frac{b(X^\epsilon(s), X^\epsilon(s-\tau)) - b(X^\epsilon(s), X^0(s-\tau))}{\sqrt{\epsilon}} - \nabla_{Y^\epsilon(s-\tau)}b(X^\epsilon(s), X^0(s-\tau)) \right\} ds \]

\[+ \int_0^t \{ \sigma(X^\epsilon(s), X^\epsilon(s-\tau)) - \sigma(X^0(s), X^0(s-\tau)) \} dW(s) \]

\[+ \nabla_{Y^\epsilon(t-\tau)-Y(t-\tau)}G(X^0(t-\tau)) \]

\[+ \int_0^t \nabla_{Y^\epsilon(s)-Y(s-\tau)}b(X^\epsilon(s), X^0(s-\tau)) ds \]

\[+ \int_0^t \nabla_{Y^\epsilon(s-\tau)-Y(s-\tau)}b(X^\epsilon(s), X^0(s-\tau)) ds \]

\[+ \int_0^t \left\{ \nabla_{Y^\epsilon(s-\tau)}b(X^\epsilon(s), X^0(s-\tau)) - \nabla_{Y^\epsilon(s-\tau)}b(X^0(s), X^0(s-\tau)) \right\} ds \]

\[=: \sum_{i=1}^8 I_i(t). \]

Observe from (H2) that for any \( x, y, z \in \mathbb{R}^n \),

\[
|\nabla_z^{(1)} \nabla_z^{(i)} b(x, y)| \leq L_0|z|^2, \quad i = 1, 2, \quad |\nabla_z^{(2)} \nabla_z^{(2)} b(x, y)| \leq V(y, y)|z|^2,
\]

and from (H3) that

\[
|\nabla_y \nabla_y G(x)| \leq V(x, x)|y|^2.
\]

For notational simplicity, set \( J_i(t) := \mathbb{E} \left( \sup_{0 \leq s \leq t} |I_i(s)|^2 \right) \). To achieve the desired assertion, in what follows we intend to estimate \( J_i(t) \) one-by-one. By Taylor’s expansion and Hölder’s inequality, together with (2.12) and (2.14), we infer from (2.11) and (2.10) that

\[
J_1(t) + J_5(t) \leq \frac{\epsilon}{2} \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} \left| \nabla_{Y^\epsilon(s)} \nabla_{Y^\epsilon(s)} G(X^0(s) + u^\epsilon(s)) \right|^2 \right)
\]

\[+ \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} \left| \nabla_{Y^\epsilon(s)-Y(s)} G(X^0(s)) \right|^2 \right) \]

\[\leq \frac{\epsilon}{2} \left( \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} V^2(X^0(s) + u^\epsilon(s), X^0(s) + u^\epsilon(s)|Y^\epsilon(s)|^4) \right) \right)
\]

\[+ \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} \left| Y^\epsilon(s) - Y(s) \right|^2 V^2(X^0(s), X^0(s)) \right) \]

\[\leq \frac{\epsilon}{2} \left( \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} V^4(X^0(s) + u^\epsilon(s), X^0(s) + u^\epsilon(s)) \right) \right)^{1/2} \left( \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} |Y^\epsilon(s)|^8 \right) \right)^{1/4}
\]

\[+ \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} \left| Y^\epsilon(s) - Y(s) \right|^2 \right) \]

\[\leq \epsilon + \mathbb{E} \left( \sup_{0 \leq s \leq (t-\tau)v_0} \left| Y^\epsilon(s) - Y(s) \right|^2 \right), \]
where
\begin{equation}
(2.15) \quad u^\epsilon(s) := \theta^\epsilon(s)(X^\epsilon(s) - X^0(s)), \quad s \in [0, t]
\end{equation}
with $\theta^\epsilon \in (0, 1)$ being a random variable. Also, with the aid of the Taylor expansion and the Hölder inequality, along with (2.5), (2.7), (2.10), (2.13), and (2.15), we derive that

\begin{align*}
J_2(t) + J_3(t) + J_6(t) & \lesssim \epsilon \int_0^t \mathbb{E}|\nabla^{(1)}_{Y^\epsilon(s)} \nabla^{(1)}_{Y^\epsilon(s)} b(X^0(s) + u^\epsilon(s), X^0(s - \tau))|^2 ds \\
& \quad + \epsilon \int_0^t \mathbb{E}|\nabla^{(2)}_{Y^\epsilon(s)} \nabla^{(2)}_{Y^\epsilon(s)} b(X^\epsilon(s), X^0(s - \tau) + u^\epsilon(s - \tau))|^2 ds \\
& \quad + \epsilon \mathbb{E} \int_0^t |\nabla^{(1)}_{Y^\epsilon(s)} \nabla^{(2)}_{Y^\epsilon(s)} b(X^0(s) + u^\epsilon(s), X^0(s - \tau))|^2 ds \\
& \lesssim \epsilon \int_0^t \{\mathbb{E}|Y^\epsilon(s)|^4 + \mathbb{E}|Y^\epsilon(s) \cdot Y(s - \tau)|^2\} ds \\
& \quad + \epsilon \int_0^{(t-\tau)\vee 0} (\mathbb{E}(V^4(X^0(s) + u^\epsilon(s), X^0(s) + u^\epsilon(s)))^{1/2}(\mathbb{E}|Y^\epsilon(s)|^8)^{1/2} ds \\
& \lesssim \epsilon.
\end{align*}

Using B-D-G’s inequality and Hölder’s inequality and taking (2.5) and (2.7) into consideration yields that

\begin{align*}
J_4(t) & \lesssim \int_0^t \mathbb{E}||\sigma(X^\epsilon(s), X^\epsilon(s - \tau)) - \sigma(X^0(s), X^0(s - \tau))||^2_{H^2} ds \\
& \lesssim \int_0^t \mathbb{E}|X^\epsilon(s) - X^0(s)|^2 ds + \int_0^{(t-\tau)\vee 0} \mathbb{E}V^2(X^\epsilon(s), X^0(s))|X^\epsilon(s) - X^0(s)|^2 ds \\
& \lesssim \epsilon \int_0^t \mathbb{E}|Y^\epsilon(s)|^2 ds + \epsilon \int_0^{(t-\tau)\vee 0} (\mathbb{E}V^4(X^\epsilon(s), X^0(s)))^{1/2}(\mathbb{E}|Y^\epsilon(s)|^4)^{1/2} ds \\
& \lesssim \epsilon.
\end{align*}

With the help of (2.1) and (2.11),

\begin{align*}
\sum_{i=6}^7 J_i(t) & \lesssim \int_0^t \mathbb{E}|Y^\epsilon(s) - Y(s)|^2 ds + \int_0^{(t-\tau)\vee 0} \mathbb{E}(V^2(X^0(s), X^0(s))|Y^\epsilon(s) - Y(s)|^2) ds \\
& \lesssim \int_0^t \mathbb{E}|Y^\epsilon(s) - Y(s)|^2 ds.
\end{align*}

Hence, we arrive at
\begin{equation}
(2.16) \quad \mathbb{E}\left(\sup_{0 \leq s \leq t} |Y^\epsilon(s) - Y(s)|^2\right) \lesssim \epsilon + \mathbb{E}\left(\sup_{0 \leq s \leq (t-\tau)\vee 0} |Y^\epsilon(s) - Y(s)|^2\right).
\end{equation}

Then, with the induction argument, we get the desired assertion. \hfill \Box

**Proof of Theorem 1.1.** The proof of Theorem 1.1 can be done by taking $G \equiv 0_n$ in the argument of Theorem 1.2. \hfill \Box
3 Moderate deviation principle

In what follows, we recall some basic notions concerned with LDPs (see, e.g., [15, Chapter 1]). Let $S$ be a Polish space (i.e., a complete separable metrizable topological space) and $\mathcal{B}(S)$ the Borel $\sigma$-algebra generated by all open sets in $S$.

**Definition 3.1.** A function $I: S \to [0, \infty]$ is called a rate function, if for each $M < \infty$, the level set $\{x \in S : I(x) \leq M\}$ is a compact subset of $S$.

**Definition 3.2.** A family $\{X^\epsilon\}$ of $S$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy the large deviation principle on $S$ with the rate function $I$ and the speed function $\{\lambda(\epsilon)\}_{\epsilon > 0}$, if the following conditions hold:

(i) (Upper bound) For each closed subset $F$ of $S$,
\[
\limsup_{\epsilon \to 0} \frac{1}{\epsilon} \log \mathbb{P}(X^\epsilon \in F) \leq - \inf_{x \in F} I(x);
\]

(ii) (Lower bound) For each open subset $G$ of $S$,
\[
\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \log \mathbb{P}(X^\epsilon \in G) \geq - \inf_{x \in G} I(x).
\]

we need to introduce some notation.

Define the Cameron-Martin space $\mathcal{F}$ by
\[
\mathcal{F} = \{h : [0, T] \to \mathbb{R}^m | h \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}^m) - \text{measurable},
\]
\[
h(t) = \int_0^t h(s)ds, t \in [0, T], \text{ and } \int_0^T |h(s)|^2ds < \infty, \text{ } \mathbb{P} - \text{a.s.}\},
\]

where the dot denotes the generalized derivative, and define
\[
L_T(h) := \frac{1}{2} \int_0^T |\dot{h}(s)|^2ds.
\]

For each $N > 0$, let
\[
S_N = \{h : [0, T] \to \mathbb{R}^m : L_T(h) \leq N\}.
\]

Let $S = \cup_{N \geq 1} S_N$ and $\mathcal{F}_N = \{h \in \mathcal{F} : h(\omega) \in S_N, \mathbb{P} - \text{a.s.}\}$.

Recall the SDDEs of neutral type
\[
d\{X^\epsilon(t) - G(X^\epsilon(t - \tau))\} = b(X^\epsilon(t), X^\epsilon(t - \tau))dt + \sqrt{\epsilon} \sigma(X^\epsilon(t), X^\epsilon(t - \tau))dW(t), \quad t > 0,
\]

In this section, we consider the Moderate deviation principles for this kind of SDDEs, which allow the coefficients are nonlinear growth with all the variables, specifically, we assume the following assumptions on the coefficients of (3.3) hold. For more details, please refer to [3, Theorem 1.1] and [28, Corollary 3.5].

Recall the polynomial function $V(x, y) \leq K(1 + |x|^{q_1} + |y|^{q_1})$. 

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(A1) Assume there are constants \( q > p \geq 2, a_1 \geq 0, a_2 > a_3 \geq 0 \) and \( a_4 > a_5 > 0 \) such that

\[
p| x - G(y)|^{p-2} \left( (x - G(y), b(x, y)) + \frac{(p-1)}{2} \| \sigma(x, y) \|_{HS}^2 \right) \\
\leq a_1 - a_2 |x|^p + a_3 |y|^p - a_4 |x|^q + a_5 |y|^q
\]

for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\).

(A2) \(b\) and \(\sigma\) are continuous and bounded on bounded subsets of \(\mathbb{R}^n \times \mathbb{R}^n\),

\[
|G(x) - G(y)| \leq V(x, y)|x - y|.
\]

We also assume the assumptions for the gradients of the coefficients.

(A3) \(b(\cdot, \cdot)\) is Fréchet differentiable w.r.t. each component, and the gradient satisfy follows,

\[
\|\nabla^{(1)}b(x_1, \cdot) - \nabla^{(1)}b(x_2, \cdot)\| \leq V(x_1, x_2)|x_1 - x_2|, \quad x_1, x_2 \in \mathbb{R}^n,
\]

and

\[
\|\nabla^{(2)}b(\cdot, y_1) - \nabla^{(2)}b(\cdot, y_2)\| \leq V(y_1, y_2)|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}^n.
\]

and

\[
\|\nabla G(x) - \nabla G(y)\| \leq V(x, y)|x - y|, \quad x, y \in \mathbb{R}^n.
\]

Remark 3.1. Reference [27] investigated the moderate deviation principle for this kind of neutral stochastic differential delay equations with jumps. Inspired by the assumptions therein, we give more weakly assumptions (A1)-(A2), specifically, the drift and diffusion coefficients in this paper are allowed to be highly nonlinear growth with respect to the variables. We remark also that under (A3), the gradients are also polynomial growth with respect to variables. The assumptions in [27] are the special case of this work.

The main result of this section is stated as follows.

Theorem 3.1. Under (A1)-(A3), \(\{Z^t\}\), defined by (1.3), satisfies an LDP in \(C([0, T]; \mathbb{R}^n)\) with speed \(\lambda^2(\epsilon)\) such that \(\lambda(\epsilon) \to \infty\) and \(\sqrt{\epsilon} \lambda(\epsilon) \to 0\) as \(\epsilon \to 0\) and the rate function given by

\[
I(f) = \inf_{h \in S_f} L_T(h),
\]

Herein, \(S_f := \left\{ h \in S : f = G^0 \left( \int_0^t \hat{h}(s) ds \right) \right\} \), and \(Z^h(t)\) solves the deterministic differential delay equation of neutral type

\[
\frac{d}{dt} \{Z^h(t) - \nabla Z^h(t-\tau) G(X^0(t - \tau))\} = \{\sigma(X^0(t), X^0(t - \tau)) \hat{h}(t) + \nabla^{(1)} Z^h(t) b(X^0(t), X^0(t - \tau)) \\
+ \nabla^{(2)} Z^h(t - \tau) b(X^0(t), X^0(t - \tau))\} dt
\]

with the initial value \(Z^h(0) \equiv 0_n\), for any \(\theta \in [-\tau, 0]\).
For the solution $X^v(t)$ to \([3.3]\), by the Yamada-Watanabe theorem \([24]\), there exists a measurable map $\mathcal{G}$ such that

$$X^v(\cdot) = \mathcal{G}(\sqrt{\varepsilon}W).$$

Then, by the Girsanov theorem, for any $v \in \mathcal{T}$, $X^{\varepsilon,v} := \mathcal{G}\left(\sqrt{\varepsilon}W + \sqrt{\varepsilon}\lambda(\varepsilon) \int_0^t \dot{\varepsilon}(s)\,ds\right)$ solves

$$
d(X^{\varepsilon,v}(t) - G(X^{\varepsilon,v}(t - \tau))) = b(X^{\varepsilon,v}(t), X^{\varepsilon,v}(t - \tau))dt + \sqrt{\varepsilon}\sigma(X^{\varepsilon,v}(t), X^{\varepsilon,v}(t - \tau))dW(t)
+ \sqrt{\varepsilon}\lambda(\varepsilon)\sigma(X^{\varepsilon,v}(t), X^{\varepsilon,v}(t - \tau))\dot{\varepsilon}(t)dt, \quad t > 0.
$$

(3.11)

So there exists a measurable map $\mathcal{G} : \mathcal{T} \mapsto \mathbb{R}^n$ such that

$$Z^{\varepsilon,v} := \mathcal{G}\left(\sqrt{\varepsilon}W + \sqrt{\varepsilon}\lambda(\varepsilon) \int_0^t \dot{\varepsilon}(s)\,ds\right) = \frac{X^{\varepsilon,v} - X^0}{\sqrt{\varepsilon}\lambda(\varepsilon)}.
$$

As a result,

$$
d\left(Z^{\varepsilon,v}(t) - G(X^{\varepsilon,v}(t - \tau)) - G(X^0(t - \tau))\right)
\begin{align*}
= & \frac{1}{\lambda(\varepsilon)}\sigma(X^{\varepsilon,v}(t), X^{\varepsilon,v}(t - \tau))dW(t) + \sigma(X^{\varepsilon,v}(t), X^{\varepsilon,v}(t - \tau))\dot{\varepsilon}(t)dt \\
+ & \frac{b(X^{\varepsilon,v}(t), X^{\varepsilon,v}(t - \tau)) - b(X^0(t), X^{\varepsilon,v}(t - \tau))}{\sqrt{\varepsilon}\lambda(\varepsilon)}dt \\
+ & \frac{b(X^0(t), X^{\varepsilon,v}(t - \tau)) - b(X^0(t), X^0(t - \tau))}{\sqrt{\varepsilon}\lambda(\varepsilon)}dt.
\end{align*}
$$

(3.13)

with the initial condition $Z^{\varepsilon,v}(\theta) = 0_n$, for any $\theta \in [-\tau, 0]$.

**Lemma 3.2.** Assume (A2) holds, there is a constant $C > 0$ and $\epsilon_0 \in (0, 1)$ such that

$$
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X^{\varepsilon,v}(t)|^p \vee \sup_{0 \leq t \leq T} |X^0(t)|^p\right) \leq C, \quad p \geq 2, \varepsilon \in (0, \epsilon_0),
$$

(3.14)

where $C$ depends on $\|\xi\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\xi(\theta)|$.

**Proof.** Let $q' = q_1 + 1$ and Set $M_\varepsilon(t) := X^{\varepsilon,v}(t) - G(X^{\varepsilon,v}(t - \tau))$. For any $p \geq 2$,

$$
\sup_{0 \leq s \leq t} |X^{\varepsilon,v}(s)|^p \leq 1 + \sup_{-\tau \leq s \leq t-\tau} |X^{\varepsilon,v}|^{pq'} + M_\varepsilon(t)
+ p \int_0^t |M_\varepsilon(s)|^{p-2} \langle M_\varepsilon(s), b(X^{\varepsilon,v}(s), X^{\varepsilon,v}(s - \tau))\rangle ds
+ \frac{p(p-1)}{2} \int_0^t |M_\varepsilon(s)|^{p-2}\|\sigma(X^{\varepsilon,v}(s), X^{\varepsilon,v}(s - \tau))\|_{HS}^2 ds
+ p\sqrt{\varepsilon}\lambda(\varepsilon) \int_0^t |M_\varepsilon(s)|^{p-2} \langle M_\varepsilon(s), \sigma(X^{\varepsilon,v}(s), X^{\varepsilon,v}(s - \tau)) \cdot \dot{\varepsilon}(s)\rangle ds,
$$

(3.16)
where
\[ M_{1,\epsilon}(t) := p\epsilon \sup_{0 \leq s \leq t} \left| \int_0^s |M_\epsilon(z)|^{p-2} \langle M_\epsilon(z), \sigma(X^{\epsilon,v}(z), X^{\epsilon,v}(z - \tau)) \rangle dW(z) \right|. \]

Firstly, we consider the \( M_{1,\epsilon}(t) \), utilizing the B-D-G’s inequality, it follows that
\[
\mathbb{E} M_{1,\epsilon}(t) \leq 4p\sqrt{2}\epsilon \mathbb{E} \left( \int_0^t |M_\epsilon(s)|^{p-2} \| \sigma(X^{\epsilon,v}(s), X^{\epsilon,v}(s - \tau)) \|_{H^S}^2 ds \right)^{\frac{1}{2}}.
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} |M_\epsilon(s)|^p + 16p^2 \epsilon \mathbb{E} \int_0^t |M_\epsilon(s)|^{p-2} \| \sigma(X^{\epsilon,v}(s), X^{\epsilon,v}(s - \tau)) \|_{H^S}^2 ds.
\]

\[
p \sqrt{\epsilon} \lambda(\epsilon) \int_0^t |M_\epsilon(s)|^{p-2} \langle M_\epsilon(s), \sigma(X^{\epsilon,v}(s), X^{\epsilon,v}(s - \tau)) \rangle \cdot \dot{v}(s) ds
\]

\[
\leq p \sqrt{\epsilon} \lambda(\epsilon) \left( \int_0^t |M_\epsilon(s)|^{p-2} \| \sigma(X^{\epsilon,v}(s), X^{\epsilon,v}(s - \tau)) \|_{H^S}^2 ds + \int_0^t \sup_{0 \leq r \leq s} |M_\epsilon(r)|^p \| \dot{v}(s) \|_{H^S}^2 ds \right).
\]

Combining the above estimates and (3.13), \( 16p^2 \epsilon_0 \lor p \epsilon_0 \lambda(\epsilon) \leq \frac{p(p-1)}{2} \), \( \epsilon \in (0, \epsilon_0) \), one gets that
\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X^{\epsilon,v}(s)|^p \right) \lesssim 1 + \mathbb{E} \left( \sup_{-\tau \leq s \leq t-\tau} |X^{\epsilon,v}(s)|^{pq} \right)
\]

\[
+ \mathbb{E} \int_0^t \left( -a_2 |X^{\epsilon,v}(s)|^p + a_3 |X^{\epsilon,v}(s - \tau)|^p - a_4 |X^{\epsilon,v}(s)|^q + a_5 |X^{\epsilon,v}(s - \tau)|^q \right) ds
\]

Then, for \( \epsilon \in (0, \epsilon_0) \), we drive that
\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X^{\epsilon,v}(s)|^p \right) \lesssim 1 + \mathbb{E} \left( \sup_{-\tau \leq s \leq t-\tau} |X^{\epsilon,v}(s)|^{pq} \right).
\]

Thus, the assertion (3.14) is established by induction argument. \( \square \)

For any \( h \in S \), define \( \mathcal{G}^0(h) = Z^h \), where \( Z^h \) solves (3.10).

**Lemma 3.3.** Assume (A1)-(A2) hold, and suppose that \( h_n \to h \) as \( n \to \infty \) for any \( h_n, h \in S_N \). Then, as \( n \to \infty \),
\[ \mathcal{G}^0(h_n) \to \mathcal{G}^0(h). \]

**Proof.** From the notion of \( \mathcal{G}^0 \), it suffices to show that
\[
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} |Z^{h_n}(t) - Z^h(t)| = 0
\]
for any \( h_n, h \in S_N \). Whereas, to derive (3.17), according to the Arzelà-Ascoli theorem (see, e.g., [22] Theorem 4.9)), we need only show that

(i) \( \{Z^{h_n}(\cdot)\}_{\epsilon \in (0,1)} \) is uniformly bounded, i.e., \( \sup_{\epsilon \in (0,1)} \sup_{t \in [0,T]} |Z^{h_n}(t)| < \infty \);

(ii) \( \{Z^{h_n}(\cdot)\}_{\epsilon \in (0,1)} \) is equicontinuous, i.e., \( \lim_{\delta \to 0} \sup_{\epsilon \in (0,1)} |Z^{h_n}(t + \delta) - Z^{h_n}(t)| = 0 \).
In what follows, we verify that (i) and (ii) hold one-by-one. With the aid of (3.4), (3.14), and $X'(\theta) = X^0(\theta)$ for any $\theta \in [-\tau, 0]$, we derive from Hölder’s inequality that for any $h_n \in S_N$,

$$|Z^{h_n}(t)| \lesssim V(X^0(t - \tau), X^0(t - \tau))|Z^{h_n}(t - \tau)|$$

$$+ \left( \int_0^t \|\sigma(X^0(s), X^0(s - \tau))\|_{H^0}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t |\dot{h}_n(s)|^2 ds \right)^{\frac{1}{2}}$$

$$+ \int_0^t V(X^0(s), X^0(s))|Z^{h_n}(s)|ds + \int_{(t-\tau)^\circ}^t V(X^0(s), X^0(s))|Z^{h_n}(s)|ds$$

$$\lesssim 1 + \sup_{0 \leq s \leq (t-\tau)^\circ} |Z^{h_n}(s)| + \int_0^t |Z^{h_n}(s)|ds.$$

By the Gronwall inequality, one has

$$|Z^{h_n}(t)| \lesssim 1 + \sup_{0 \leq s \leq (t-\tau)^\circ} |Z^{h_n}(s)|.$$

Then (i) follows from an induction argument.

In the sequel, without loss of generality, we assume $\delta \in (0, 1)$. From (A1), (3.4) and (3.14), one has that

$$|X^0(t + \delta) - X^0(t)| \lesssim |X^0(t + \delta - \tau) - X^0(t - \tau)| + \int_t^{t+\delta} |b(X^0(s), X^0(s - \tau))|ds$$

$$\lesssim V(X^0(t + \delta - \tau), X^0(t - \tau))|X^0(t + \delta - \tau) - X^0(t - \tau)|$$

$$+ \int_t^{t+\delta} \{1 + |X^0(s - \tau)|^2 + |X^0(s - \tau)|^9\}^{\frac{1}{2}} ds$$

$$\lesssim |X^0(t + \delta - \tau) - X^0(t - \tau)| + \delta.$$
where in the first inequality, we have used (A)3, the second inequality, we have used (3.8), (3.18) and \( h_n, h \in S_N \). As a result, (ii) is established by an induction argument and the continuity of the initial data.

Since \((Z^h_n(\cdot))_{n \in \mathbb{N}}\) is pre-compact in \(C([0, T]; \mathbb{R}^n)\), every sequence, which is still denoted by \((Z^h_n(\cdot))_{n \in \mathbb{N}}\), has a convergent subsequence. So we conclude that \(Z^h(\cdot)\) be the limit point of \((Z^h_n(\cdot))_{n \in \mathbb{N}}\) from the uniqueness.

**Lemma 3.4.** Let \(v, v_\epsilon \in \mathcal{A}_N\) such that \(v_\epsilon\) converges weakly to \(v\), as \(\epsilon \to 0\). Then

\[
\mathcal{G}^\epsilon\left(\sqrt{\epsilon}W + \sqrt{\epsilon} \lambda(\epsilon) \int_0^\cdot \dot{v}_\epsilon(s) ds\right) \Rightarrow \mathcal{G}^0\left(\int_0^\cdot \dot{v}(s) ds\right),
\]

where \(\Rightarrow\) stands for convergence in distribution of random variables.

**Proof.** To begin, we show that \(\{Z^{\epsilon,v_\epsilon}\}_{\epsilon \in (0,1)}\) is tight in \(C([0, T]; \mathbb{R}^n)\). By virtue of the Arzel`a-Ascoli theorem (see, e.g., [22, Theorem 4.11]), it is sufficient to verify that

(i) \(\sup_{\epsilon \in (0,1)} \mathbb{E}|Z^{\epsilon,v_\epsilon}(t)|^\gamma < \infty\);

(ii) \(\sup_{\epsilon \in (0,1)} \mathbb{E}|Z^{\epsilon,v_\epsilon}(t) - Z^{\epsilon,v_\epsilon}(s)|^\alpha \leq C_T|t-s|^{1+\beta},\quad 0 \leq s, t \leq T, \quad |t-s| < 1\)

for some positive constants \(\alpha, \beta, \gamma\) and \(C_T\).

By the chain rule and the Taylor expansion, in addition to (3.4), (3.6), (3.7), and (3.8) for \(\alpha, \beta, \gamma\) and \(C_T\), we derive that

\[
\mathbb{E}|Z^{\epsilon,v_\epsilon}(t)|^p \lesssim \mathbb{E}\left|\nabla Z^{\epsilon,v_\epsilon}(t-\tau) G(X^0(t-\tau) + u^\epsilon(t-\tau))\right|^p
+ \lambda^{-p}(\epsilon)\mathbb{E}\left(\int_0^t \|\sigma(X^{\epsilon,v_\epsilon}(s), X^{\epsilon,v_\epsilon}(s-\tau))\|^p_{H_S} ds\right)
+ \mathbb{E}\left(\int_0^t |\dot{v}_\epsilon(s)|^2 ds\right)^\frac{p}{2} \left(\int_0^t \|\sigma(X^{\epsilon,v_\epsilon}(s), X^{\epsilon,v_\epsilon}(s-\tau))\|^2_{H_S} ds\right)^\frac{p}{2}
+ \int_0^t \mathbb{E}\left|\nabla Z^{\epsilon,v_\epsilon}_1(s) b(X^0(s) + u^\epsilon(s), X^{\epsilon,v_\epsilon}(s-\tau))\right|^p ds
+ \int_0^t \mathbb{E}\left|\nabla Z^{\epsilon,v_\epsilon}_2(s-\tau) b(X^0(s), X^0(s-\tau) + u^\epsilon(s-\tau))\right|^p ds
\lesssim 1 + \lambda^{-p}(\epsilon) + \sup_{0 \leq s \leq (t-\tau) \wedge 0} \left(\mathbb{E}|Z^{\epsilon,v_\epsilon}(s)|^{2p}\right)^\frac{1}{2p} + \int_0^t \mathbb{E}|Z^{\epsilon,v_\epsilon}(s)|^p ds
+ \int_0^t (\mathbb{E}|Z^{\epsilon,v_\epsilon}(s-\tau)|^{2p})^{\frac{1}{2p}} ds, \quad t \in [0, T], \quad p \geq 2.
\]

In the last step, we utilize the (3.11) and the H"older inequality, Then taking Gronwall’s inequality into consideration, one has that

\[
(3.19)\quad \mathbb{E}|Z^{\epsilon,v_\epsilon}(t)|^p \lesssim 1 + \lambda^{-p}(\epsilon) + \sup_{0 \leq s \leq (t-\tau) \wedge 0} \left(\mathbb{E}|Z^{\epsilon,v_\epsilon}(s)|^{2p}\right)^\frac{1}{2p} + \int_0^t (\mathbb{E}|Z^{\epsilon,v_\epsilon}(s-\tau)|^{2p})^{\frac{1}{2p}} ds.
\]

Hereinafter, by mimicking the argument of Theorem 1.2, we get that
\[
E|Z^{\varepsilon,\nu}(t)|^p \lesssim 1 + \lambda^{-p}(\varepsilon), \quad t \in [0, T].
\]
Hence (i) holds with \( \gamma = p \).

In the sequel, note that (3.6), (3.7), (3.8) and (3.14). Then utilizing the Taylor expansion fields that
\[
E|Z^{\varepsilon,\nu}(t) - Z^{\varepsilon,\nu}(s)|^\alpha
\lesssim E|\nabla Z^{\varepsilon,\nu}(t)G(X^0(t - \tau) + u^{\varepsilon}(t - \tau)) - \nabla Z^{\varepsilon,\nu}(s)G(X^0(s - \tau) + u^{\varepsilon}(s - \tau))|^\alpha
+ E|\nabla Z^{\varepsilon,\nu}(t) - Z^{\varepsilon,\nu}(s)|G(X^0(s - \tau) + u^{\varepsilon}(s - \tau))|^\alpha
+ \frac{|t - s|^{\alpha - 2}}{\lambda^\alpha(\varepsilon)} \int_s^t E|\sigma(X^{\varepsilon,\nu}(r), X^{\varepsilon,\nu}(r - \tau))|_{HS}^\alpha dr
+ |t - s|^{\alpha - 2} \int_s^t E|\nabla^{(1)} Z^{\varepsilon,\nu}(r) b(X^0(r) + u^{\varepsilon}(r), X^{\varepsilon,\nu}(r - \tau))|^\alpha dr
+ |t - s|^{\alpha - 2} \int_s^t E|\nabla^{(2)} Z^{\varepsilon,\nu}(r) b(X^0(r), X^0(r - \tau) + u^{\varepsilon}(r - \tau))|^\alpha dr
\lesssim |t - s|^{\frac{2\alpha - 1}{\alpha - 1}} + \frac{|t - s|^{\frac{2}{2\alpha}}}{{\lambda^\alpha(\varepsilon)}} + (E|Z^{\varepsilon,\nu}(t) - Z^{\varepsilon,\nu}(s)|^2)^{\frac{1}{2}}, \quad 0 \leq s, t \leq T.
\]

Next, taking the induction argument has that
\[
E|Z^{\varepsilon,\nu}(t) - Z^{\varepsilon,\nu}(s)|^\alpha \lesssim |t - s|^{\frac{2\alpha - 2}{\alpha - 1}} + \frac{|t - s|^{\frac{2}{2\alpha}}}{{\lambda^\alpha(\varepsilon)}}.
\]
Therefore, (ii) holds with \( \alpha = 2(1 + \beta) \). Thus \( \{Z^{\varepsilon,\nu}\}_{\varepsilon \in (0, 1)} \) is tight in \( C([0, T]; \mathbb{R}^n) \).

In the sequel, it suffices to show that \( Z^c \) is the unique limit point of \( \{Z^{\varepsilon,\nu}\}_{\varepsilon \in (0, 1)} \). Let
\[
M^c(t) = \frac{1}{\lambda(\varepsilon)} \int_0^t \sigma(X^{\varepsilon,\nu}(s), X^{\varepsilon,\nu}(s - \tau))dW(s).
\]
Since \( \{Z^{\varepsilon,\nu}\}_{\varepsilon \in (0, 1)} \) is tight in \( C([0, T]; \mathbb{R}^n) \), we can choose a subsequence of \( \{Z^{\varepsilon,\nu}, v, M^c\} \) convergent weakly to \( (Y, v, 0) \) as \( \varepsilon \to 0 \). Without loss of generality, by the Skorokhod representation theorem [14], there exists a probability space \( (\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{F}) \), on this basis, an Brownian motion \( \mathbb{W} \) and a family of \( \mathbb{F} \)-predictable process \( \{\mathbb{v}_\varepsilon; \varepsilon > 0\}, \mathbb{v} \in \mathcal{A} \) taking values on \( \mathcal{A}_N, \mathbb{F} \)-a.s., such that the joint law of \( (v, W) \) under \( \mathbb{P} \) coincides with that of \( (\mathbb{v}, \mathbb{W}) \) under \( \mathbb{P} \) and
\[
\lim_{\varepsilon \to 0} \int_0^T < \mathbb{v}_\varepsilon - \mathbb{v}, g > ds = 0, \quad \forall g \in \mathcal{A}, \mathbb{F} \text{-a.s.}
\]
Without confusion, we drop the bars in the notation. Thus, we may assume
\[
(Z^{\varepsilon,\nu}, v, M^c) \to (Y, v, 0), \quad \mathbb{P} \text{-a.s.}
\]
Taking \( \varepsilon \to 0 \) on both sides of (3.13), we infer that \( Y \) also satisfies (3.10). Thus the desired assertion follows from the uniqueness.

**The proof of Theorem 3.1**

**Proof.** With Lemmas 3.3 and 3.4 in hand and by taking [29, Theorem 4.4] into account, the proof of Theorem 3.1 can be completed.
4 Examples

Theorem 3.1 covers many highly nonlinear SDDEs. Let us discuss two examples at the end of this section.

Example 4.1. Consider a one-dimensional SDDE

\begin{equation}
 dx(t) = [x^2(t-\tau) - 2x(t) - x^3(t)]dt + \sqrt{\epsilon}x^2(t-\tau)dB(t),
\end{equation}

where \( B(t) \) is a one dimensional Brownian motion.

For simplicity, let \( y = x(t - \tau), x = x(t) \),

\[
 2x(-x^3 - 2x + y^2) + y^4 \leq -2x^4 - 3x^2 + y^2 + y^4.
\]

where \( p = 2, q = 4 \) and \( a_1 = 0, a_2 = 2, a_3 = 1, a_4 = 3, a_5 = 1 \),

\[
\|\nabla^{(1)}b(x_1, \cdot) - \nabla^{(1)}b(x_2, \cdot)\| \leq 3(x_1 + x_2)|x_1 - x_2|,
\]

\[
\|\nabla^{(2)}b(\cdot, y_1) - \nabla^{(2)}b(\cdot, y_2)\| \leq 2|y_1 - y_2|.
\]

Assumptions (A1)-(A3) are therefore satisfied, then the conclusion of theorem 3.1 is established.

Example 4.2. We consider another equation

\begin{equation}
 dx(t) = -x^3(t) - 2x(t) + x(t - \tau)dt + \sqrt{\epsilon}x^2(t)dB(t)
\end{equation}

For simplicity, we set \( y = x(t - \tau), x = x(t) \),

\[
 2x(-x^3 - 2x + y) + x^4 \leq -2x^4 - 3x^2 + y^2,
\]

where \( p = 2, q = 4 \) and \( a_1 = 0, a_2 = 3, a_3 = 1, a_4 = 2, a_5 = 0 \),

\[
\|\nabla^{(1)}b(x_1, \cdot) - \nabla^{(1)}b(x_2, \cdot)\| \leq 3(x_1 + x_2)|x_1 - x_2|,
\]

\[
\|\nabla^{(2)}b(\cdot, y_1) - \nabla^{(2)}b(\cdot, y_2)\| = 0
\]

Assumptions (A1)-(A3) are therefore satisfied, then the conclusion of theorem 3.1 is established.

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