Supersymmetric harmonic oscillator and nonlinear supercoherent states

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Abstract. We find and analyze the (nonlinear) supercoherent states associated with the generalized nonlinear supersymmetric annihilation operator (SAO) of the supersymmetric harmonic oscillator. We discuss as well the uncertainty relation for a special case in order to compare our results with those obtained for the linear supercoherent states.

1. Introduction

For the standard harmonic oscillator with angular frequency $\omega$, the commutation relations between the Hamiltonian and the annihilation and creation operators $\hat{H}, \hat{a}, \hat{a}^\dagger$ generate the well known Heisenberg-Weyl algebra:

$$[\hat{H}, \hat{a}] = -\hbar \omega \hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \hbar \omega \hat{a}^\dagger, \quad [\hat{a}, \hat{a}^\dagger] = 1.$$  \hspace{1cm} (1)

However, if a generalized form for the annihilation and creation operators is employed, which respects the commutation relations with the Hamiltonian, it is possible to establish a connection with the so-called polynomial deformations of the Heisenberg algebra.

Following the above ideas, the analysis of the algebraic commutation relations between the Hamiltonian $\hat{H}_s$ of the SUSY harmonic oscillator $[1, 2],$

$$\hat{H}_s = \omega \begin{pmatrix} \hat{a}^\dagger \hat{a} & 0 \\ 0 & \hat{a} \hat{a}^\dagger \end{pmatrix},$$  \hspace{1cm} (2)

and the corresponding creation $\hat{A}^\dagger$ and annihilation operators $\hat{A}$ of the system requires the knowledge of the explicit form of these operators. However, as it was shown in $[3, 4]$, this form is not unique.

Recently $[5]$, it was introduced a general expression for $\hat{A}$ and its eigenvectors $|Z\rangle$ with complex eigenvalues $z$ were found (called supercoherent states) which become expressed in terms of the standard harmonic oscillator coherent states $[6, 7]$.

Although the expression found in $[5]$ for $\hat{A}$ is very general, it is still not unique. In this work, we will consider some of its deformations $\hat{Q}$, which maintain however the structure given in $[5]$. We will analyze also a particular deformation of the SAO and we will find the explicit form of its eigenvectors $|X\rangle$ with complex eigenvalue $X$. Due to the deformation assumed, the eigenstates of $\hat{Q}$ turn out to be expressed in terms of nonlinear coherent states, associated to deformed Lie algebras (called nonlinear algebras) $[8, 9, 10, 11]$. Therefore, it is natural to call the states $|X\rangle$ nonlinear supercoherent states.
2. **Standard coherent states (CS)**

A standard coherent state $|\alpha\rangle$ can be defined as an eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue $\alpha$, i.e.,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}, \quad \langle \alpha|\alpha\rangle = 1. \tag{3}$$

In the Fock basis, the normalized coherent states read

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{4}$$

There are other definitions that can be used to build this type of quantum states, which are equivalent to each other for the harmonic oscillator; nevertheless, in this paper we will use the previous definition.

3. **Nonlinear coherent states (NLCS)**

The nonlinear coherent states $|z\rangle_f$ can be defined as eigenstates of the deformed annihilation operator $\tilde{a} = f(\hat{N})\hat{a}$ [12], i.e.,

$$\tilde{a}|z\rangle_f = f(\hat{N})\hat{a}|z\rangle_f = z|z\rangle_f, \quad z \in \mathbb{C}, \tag{5}$$

$f(\hat{N})$ being a well behaved real function of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. These states depend strongly on how many Fock states $|n\rangle$ are annihilated by $\tilde{a}$, which in turn depend on the explicit form of the function $f(\hat{N})$ since

$$\tilde{a}|n\rangle = \sqrt{n}f(n-1)|n-1\rangle, \quad n = 0, 1, 2, \ldots \tag{6}$$

For instance, for $f(\hat{N}) = \hat{N} + 1$ it turns out that $f(n-1) = n \neq 0, \forall n = 1, 2, \ldots$ which implies that $\tilde{a}|0\rangle = 0$. Thus:

$$|\alpha\rangle_{NL} = [_{0}F_{2}(1, 1; r^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(n+1)\sqrt{\Gamma(n+1)}} |n\rangle, \tag{7}$$

where $r = |\alpha|$ and $_pF_q$ is the generalized hypergeometric function

$$_{p}F_{q}(a_1, \ldots, a_p, b_1, \ldots, b_q; x) = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + n) \cdots \Gamma(a_p + n)}{\Gamma(b_1 + n) \cdots \Gamma(b_q + n)} \frac{x^n}{n!}. \tag{8}$$

On the other hand, for $f(\hat{N}) = \hat{N}$ we have that $f(0) = 0$. This assumption implies that $\tilde{a}|0\rangle = \tilde{a}|1\rangle = 0$. Hence, the corresponding NLCS become now

$$|\alpha\rangle_{NL} = [_{0}F_{2}(1, 2; r^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(n+1)\sqrt{\Gamma(n+2)}} |n+1\rangle. \tag{9}$$

This result indicates that the contribution of the ground state is removed from $|\alpha\rangle_{NL}$ since this eigenstate $|0\rangle$ is isolated from the remaining ones due to $\tilde{a}|0\rangle = \tilde{a}^\dagger|0\rangle = 0$.

Note that, if the form of the function $f(\hat{N})$ is chosen appropriately, it is possible to isolate more energy eigenstates.
3.1. Heisenberg uncertainty relation

Knowing the explicit form of the standard coherent states $|\alpha\rangle$ and the nonlinear ones $|\alpha\rangle_{NL}$, the corresponding Heisenberg uncertainty relations can be straightforwardly calculated.

For the standard coherent states, the mean values of the operators $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$ become (with $\hbar = m = \omega = 1$):

$$
\langle \hat{x} \rangle = \sqrt{2} \text{Re}(\alpha), \quad \langle \hat{x}^2 \rangle = \frac{1}{2} + 2|\text{Re}(\alpha)|^2,
$$

$$
\langle \hat{p} \rangle = \sqrt{2} \text{Im}(\alpha), \quad \langle \hat{p}^2 \rangle = \frac{1}{2} + 2|\text{Im}(\alpha)|^2.
$$

Hence, the Heisenberg uncertainty relation turns out to be given by $(\sigma_x)^2_\alpha (\sigma_p)^2_\alpha = 1/4$.

For the nonlinear coherent states $|\alpha\rangle_{NL}$, the mean values of the operators $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$ are:

$$
\langle \hat{x} \rangle_{NL} = \sqrt{2} \text{Re}(\alpha) \left[ \frac{0 F_2(1, 2; r^2)}{0 F_2(1, 1; r^2)} \right], \quad \langle \hat{x}^2 \rangle_{NL} = \frac{1}{2} + |\text{Re}(\alpha)|^2 \beta(r) + |\text{Im}(\alpha)|^2 \sigma(r),
$$

$$
\langle \hat{p} \rangle_{NL} = \sqrt{2} \text{Im}(\alpha) \left[ \frac{0 F_2(1, 2; r^2)}{0 F_2(1, 1; r^2)} \right], \quad \langle \hat{p}^2 \rangle_{NL} = \frac{1}{2} + |\text{Re}(\alpha)|^2 \sigma(r) + |\text{Im}(\alpha)|^2 \beta(r),
$$

where

$$
\beta(r) = \left[ \frac{0 F_2(2, 2; r^2)}{0 F_2(1, 1; r^2)} \right] + \frac{1}{2} \left[ \frac{0 F_2(1, 3; r^2)}{0 F_2(1, 1; r^2)} \right], \quad \sigma(r) = \left[ \frac{0 F_2(2, 2; r^2)}{0 F_2(1, 1; r^2)} \right] - \frac{1}{2} \left[ \frac{0 F_2(1, 3; r^2)}{0 F_2(1, 1; r^2)} \right].
$$

The explicit expression for the Heisenberg uncertainty relation is (see Figure 1):

$$
(\sigma_x)^2_\alpha (\sigma_p)^2_\alpha = \left[ \frac{1}{2} - |\text{Re}(\alpha)|^2 \tau(r) + |\text{Im}(\alpha)|^2 \sigma(r) \right] \left[ \frac{1}{2} - |\text{Im}(\alpha)|^2 \tau(r) + |\text{Re}(\alpha)|^2 \sigma(r) \right],
$$

where

$$
\tau(r) = 2 \left[ \frac{0 F_2(1, 2; r^2)}{0 F_2(1, 1; r^2)} \right]^2 - \beta(r).
$$

For the nonlinear coherent states $|\alpha\rangle_{NL}$ the mean values of the operators $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$ are:
\[\langle \hat{x} \rangle_{NL} = \sqrt{2} \text{Re}(\alpha) \begin{pmatrix} 0 F_2(2, 2; r^2) \\ 0 F_2(1, 2; r^2) \end{pmatrix}, \quad \langle \hat{x}^2 \rangle_{NL} = \frac{3}{2} + [\text{Re}(\alpha)]^2 \begin{pmatrix} 0 F_2(2, 3; r^2) \\ 0 F_2(1, 2; r^2) \end{pmatrix}, \]

\[\langle \hat{p} \rangle_{NL} = \sqrt{2} \text{Im}(\alpha) \begin{pmatrix} 0 F_2(2, 2; r^2) \\ 0 F_2(1, 2; r^2) \end{pmatrix}, \quad \langle \hat{p}^2 \rangle_{NL} = \frac{3}{2} + [\text{Im}(\alpha)]^2 \begin{pmatrix} 0 F_2(2, 3; r^2) \\ 0 F_2(1, 2; r^2) \end{pmatrix}. \]  

The expression for the Heisenberg uncertainty relation is now (see Figure 2) [13]:

\[(\sigma_x)^2(\sigma_p)^2 = \left[ \frac{3}{2} - [\text{Re}(\alpha)]^2 \rho(r) \right] \left[ \frac{3}{2} - [\text{Im}(\alpha)]^2 \rho(r) \right],\]

where

\[
\rho(r) = 2 \left[ \frac{0 F_2(2, 2; r^2)}{0 F_2(1, 2; r^2)} \right]^2 - \left[ \frac{0 F_2(2, 3; r^2)}{0 F_2(1, 2; r^2)} \right].
\]

4. Nonlinear supercoherent states

The generalized form for the SAO \( \hat{A} \) [5] and its deformation are respectively:

\[
\hat{A} = \begin{pmatrix} k_1 \hat{a} & k_2 \\ k_3 \hat{a}^2 & k_4 \hat{a} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} k_1 f(\tilde{N}) \hat{a} & k_2 \\ k_3 [f(\tilde{N}) \hat{a}]^2 & k_4 f(\tilde{N}) \hat{a} \end{pmatrix},
\]

where \( k_i \in \mathbb{C} \) and \( f(\tilde{N}) = \tilde{N} + \hat{1} \) or \( f(\tilde{N}) = \tilde{N} \). We will label \( \hat{Q} = \hat{A}' \) when \( f(\tilde{N}) = \tilde{N} + \hat{1} \), and \( \hat{Q} = \hat{A}'' \) when \( f(\tilde{N}) = \tilde{N} \). Note even that \( \hat{Q} = \hat{A} \) when \( f(\tilde{N}) = \tilde{1} \).

Both linear and nonlinear supercoherent states are defined in the generic way as

\[
\hat{Q}|\chi \rangle = X|\chi \rangle, \quad |\chi \rangle = \left( \sum_{n=0}^{\infty} a_n |n \rangle \right) \left( \sum_{n=1}^{\infty} c_n |n-1 \rangle \right), \quad X \in \mathbb{C}.
\]

From this equation, after several calculations and simplifications, an overall matrix relationship is found as

\[
K \begin{pmatrix} \tilde{a}_{n+1} \\ \tilde{c}_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ c_n \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},
\]

where the quantities \( \tilde{a}_n \) and \( \tilde{c}_n \) are defined as

\[
\tilde{a}_n = \beta_n \sqrt{n!} X^{1-n} a_n, \quad \tilde{c}_n = \beta_{n-1} \sqrt{(n-1)!} X^{1-n} c_n,
\]

with

\[
X = \begin{cases} z, & \text{for } \hat{Q} = \hat{A} \\ Y, & \text{for } \hat{Q} = \hat{A}' \end{cases}, \quad \beta_n = \begin{cases} 1, & \text{for } \hat{Q} = \hat{A} \text{ and } n \geq 1 \\ n!, & \text{for } \hat{Q} = \hat{A}' \text{ and } n \geq 1 \end{cases}.
\]

As can be seen, both the linear and nonlinear supercoherent states depend on the eigenvalues \( \psi_{\pm} \) of the matrix \( K \). This induces a classification of the supercoherent states \( |\chi \rangle \) into three different families as follows: degenerate \( (\psi_+ = \psi_- \equiv \psi \neq 0) \); singular \( (\psi_+ \psi_- = 0) \); generic \( (\text{everything else}) \).

Furthermore, by rewriting \( a_n \) and \( c_n \) in terms of \( \psi_{\pm} \), the eigenstates of \( \hat{A} \) turn out to be expressed in terms of the CS in (4) while those of \( \hat{A}' \) and \( \hat{A}'' \) are determined by the ones of

\[
\begin{align*}
&X = \begin{cases} z, & \text{for } \hat{Q} = \hat{A} \\ Y, & \text{for } \hat{Q} = \hat{A}' \end{cases}, \quad \beta_n = \begin{cases} 1, & \text{for } \hat{Q} = \hat{A} \text{ and } n \geq 1 \\ n!, & \text{for } \hat{Q} = \hat{A}' \text{ and } n \geq 1 \end{cases}, \\
&\quad \text{for } \hat{Q} = \hat{A}'' \text{ and } n \geq 2.}
\end{align*}
\]
(7) or (9), when choosing \( f(\hat{N}) = \hat{N} + \hat{1} \) or \( f(\hat{N}) = \hat{N} \) respectively. Then we will identify the respective supercoherent states as follows:

\[
|\mathcal{X}\rangle = \begin{cases} 
|Z\rangle & \text{for } \hat{Q} = \hat{A} \\
|Y\rangle & \text{for } \hat{Q} = \hat{A'} \\
|Z\rangle & \text{for } \hat{Q} = \hat{A''}
\end{cases}
\]  

\[ (23) \]

4.1. Supercoherent states classification

4.1.1. Generic. The general solution for the matrix Eq. (20) is

\[
|\mathcal{X}\rangle = B_1|\mathcal{X}_A\rangle + B_2|\mathcal{X}_C\rangle,
\]

where

\[
B_1 = \begin{cases} 
a_0/k_1 & \text{for } \hat{Q} = \hat{A}, \hat{A'} \\
b_1/k_1 & \text{for } \hat{Q} = \hat{A''}
\end{cases}, \quad B_2 = \begin{cases} 
c_1/(Xk_1) & \text{for } \hat{Q} = \hat{A}, \hat{A'} \\
d_2/(Xk_1) & \text{for } \hat{Q} = \hat{A''}
\end{cases},
\]

\[ (25a) \]

\[
|\mathcal{X}_A\rangle = \frac{1}{\psi_+ - \psi_-} G_A \left( \begin{array}{c} \chi_+ \\ \chi_- \end{array} \right), \quad |\mathcal{X}_C\rangle = \frac{1}{\psi_+ - \psi_-} G_C \left( \begin{array}{c} \chi_+ \\ \chi_- \end{array} \right),
\]

\[ (25b) \]

with

\[
\chi_{\pm} = X_{\psi_{\pm}^{-1}}, \quad |\chi_{\pm}\rangle = \begin{cases} 
|\beta_{\pm}\rangle & \text{for } \hat{Q} = \hat{A} \\
|\phi_{\pm}\rangle_{nl} & \text{for } \hat{Q} = \hat{A'} \\
|\phi_{\pm}\rangle_{NL} & \text{for } \hat{Q} = \hat{A''}
\end{cases}
\]

\[ (26a) \]

and

\[
G_A = \begin{pmatrix} 
\psi_+(\psi_+ - k_4) & -\psi_-(\psi_- - k_4) \\
k_3X & -k_3X
\end{pmatrix}, \quad G_C = \begin{pmatrix} 
k_2\psi_+\psi_- & -k_2\psi_+\psi_- \\
X[k_1\psi_+ - (k_1^2 + k_2k_3)] & -X[k_1\psi_- - (k_1^2 + k_2k_3)]
\end{pmatrix}.
\]

\[ (26b, 26c) \]

Besides the set \( \{ |\mathcal{X}_A\rangle, |\mathcal{X}_C\rangle \} \), we can choose a new basis for the state space formed by the elements

\[
|\mathcal{X}_{\pm}\rangle = \begin{pmatrix} 
k_2\psi_+|\chi_{\pm}\rangle \\
(\psi_+ - k_1)X|\chi_{\pm}\rangle
\end{pmatrix},
\]

\[ (27) \]

whereby it is possible to pass from a parameter space \( \{ k_1, k_2, k_3, k_4 \} \) to a new one, formed by \( \{ \psi_+ , \psi_- , k_1, k_2 \} \).

4.1.2. Degenerate. Explicitly the supercoherent states are

\[
|\mathcal{X}\rangle = B_1|\mathcal{X}_A^d\rangle + B_2|\mathcal{X}_C^d\rangle,
\]

where

\[
|\mathcal{X}_A^d\rangle = G_A^{(d)} \left( \begin{array}{c} |\chi\rangle \\ |\chi'\rangle \end{array} \right), \quad |\mathcal{X}_C^d\rangle = G_C^{(d)} \left( \begin{array}{c} |\chi\rangle \\ |\chi'\rangle \end{array} \right),
\]

\[ (29) \]

with
4.2.1. Uncertainties for squares.

\[ G_A^{(d)} = \left( \begin{array}{cc} k_1 & -\psi - k_4 \\ 0 & -k_3 \chi^2 \end{array} \right), \quad G_C^{(d)} = \psi \left( \begin{array}{c} 0 \\ k_1 \chi - \frac{-k_2 \chi}{k_1 - k_4} \chi^2 \end{array} \right). \] (30a)

4.1.3. Singular. The corresponding supercoherent states are now

\[ |\chi'_s\rangle = \left( \begin{array}{c} k_1 |\chi\rangle \\ k_3 \chi |\chi\rangle \end{array} \right), \quad \chi' = \frac{X}{k_1 + k_4}. \] (31)

4.2. Superposition and uncertainties

Consider a superposition of states \(|\chi_\pm\rangle\) with parameters \(\eta\) and \(\lambda\) as follows:

\[ |\chi_m\rangle = \cos \eta |\chi_+\rangle + e^{i\lambda} \sin \eta |\chi_-\rangle = \left( \begin{array}{c} \gamma_1+ |\chi_+\rangle + \gamma_1- |\chi_-\rangle \\ \gamma_2+ |\chi_+\rangle + \gamma_2- |\chi_-\rangle \end{array} \right), \] (32)

where \(\gamma_{1\pm}\) and \(\gamma_{2\pm}\) are given by

\[ \begin{align*}
\gamma_{1+} &= k_2 \psi_+ \cos \eta, \\
\gamma_{1-} &= k_2 \psi_- e^{i\lambda} \sin \eta, \\
\gamma_{2+} &= (\psi_+ - k_1) \cos \eta, \\
\gamma_{2-} &= (\psi_- - k_1) e^{i\lambda} \sin \eta.
\end{align*} \] (33)

The mean value of an arbitrary observable \(\hat{c}\) is then

\[ \langle \hat{c} \rangle = \frac{\langle \chi_m | \hat{c} | \chi_m \rangle}{\langle \chi_m | \chi_m \rangle}, \] (34a)

\[ \langle \chi_m | \hat{c} | \chi_m \rangle = \Gamma^+ \langle \chi_+ | \hat{c} | \chi_+ \rangle + \Gamma^- \langle \chi_- | \hat{c} | \chi_- \rangle + 2 \text{Re} (\Gamma^{+\pm} \langle \chi_+ | \hat{c} | \chi_- \rangle), \] (34b)

with

\[ \Gamma^+ = |\gamma_{1+}|^2 + |\gamma_{2+}|^2, \quad \Gamma^- = |\gamma_{1-}|^2 + |\gamma_{2-}|^2, \quad \Gamma^{+\pm} = \gamma_{1+}^* \gamma_{1-} + \gamma_{2+}^* \gamma_{2-} |X|^2. \] (35)

This allows us to find the expressions for the uncertainties of the operators \(\hat{x}\) and \(\hat{p}\) and their squares.

4.2.1. Uncertainties for \(\hat{Q} = \hat{A}\). The expressions for the uncertainties are [5]:

\[ \sigma_x^2 = \frac{\Gamma^+ \frac{1}{2}[(\beta_+ + \beta_+^*)^2 + 1] e^{\beta_+^+ |\beta_+|^2} + \Gamma^- \frac{1}{2}[(\beta_- + \beta_-^*)^2 + 1] e^{\beta_-^- |\beta_-|^2} + 2 \text{Re} \Gamma^{\pm -\pm} \frac{1}{2}[(\beta_- + \beta_-^*)^2 + 1] e^{\beta_-^- |\beta_-|^2}}{\Gamma^+ e^{\beta_+^+ |\beta_+|^2} + \Gamma^- e^{\beta_-^- |\beta_-|^2} + 2 \text{Re} \Gamma^{\pm -\pm} e^{\beta_-^- |\beta_-|^2}} \] (36)

\[ \sigma_p^2 = \frac{\Gamma^+ \frac{1}{2}[-(\beta_+ - \beta_+^*)^2 + 1] e^{\beta_+^+ |\beta_+|^2} + \Gamma^- \frac{1}{2}[-(\beta_- - \beta_-^*)^2 + 1] e^{\beta_-^- |\beta_-|^2} + 2 \text{Re} \Gamma^{\pm -\pm} \frac{1}{2}[-(\beta_- - \beta_-^*)^2 + 1] e^{\beta_-^- |\beta_-|^2}}{\Gamma^+ e^{\beta_+^+ |\beta_+|^2} + \Gamma^- e^{\beta_-^- |\beta_-|^2} + 2 \text{Re} \Gamma^{\pm -\pm} e^{\beta_-^- |\beta_-|^2}} \] (37)

where the quantities in Eq. (35) are evaluated taking \(X = z\).
\[ \sigma_2^{\text{err}} = \Delta_1^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+)^2 \frac{a_{F_2(2, 2); |\phi_+|^2}}{a_{F_2(1, 1); |\phi_-|^2}} + \frac{1}{2} a_{F_2(1, 1); |\phi_-|^2} \right] \right. + \Gamma^- \left[ \text{Re}(\phi_-)^2 \frac{a_{F_2(2, 2); |\phi_-|^2}}{a_{F_2(1, 1); |\phi_-|^2}} + \frac{1}{2} a_{F_2(1, 1); |\phi_-|^2} \right] + \text{im}(\phi_+)^2 \frac{a_{F_2(2, 2); |\phi_+|^2}}{a_{F_2(1, 1); |\phi_-|^2}} + \frac{1}{2} a_{F_2(1, 1); |\phi_-|^2} \right\} + 2 \text{Re} \left( \Gamma^+ \frac{1}{2\sqrt{a_{F_2(2, 1); |\phi_+|^2}}} \right) \left[ \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right] \left( \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \right) \]

\[ \sigma_p^{\text{err}} = \Delta_1^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+)^2 \frac{a_{F_2(2, 2); |\phi_+|^2}}{a_{F_2(1, 1); |\phi_-|^2}} + \frac{1}{2} a_{F_2(1, 1); |\phi_-|^2} \right] \right. + \Gamma^- \left[ \text{Re}(\phi_-)^2 \frac{a_{F_2(2, 2); |\phi_-|^2}}{a_{F_2(1, 1); |\phi_-|^2}} + \frac{1}{2} a_{F_2(1, 1); |\phi_-|^2} \right] + \text{im}(\phi_+)^2 \frac{a_{F_2(2, 2); |\phi_+|^2}}{a_{F_2(1, 1); |\phi_-|^2}} + \frac{1}{2} a_{F_2(1, 1); |\phi_-|^2} \right\} + 2 \text{Re} \left( \Gamma^+ \frac{1}{2\sqrt{a_{F_2(2, 1); |\phi_+|^2}}} \right) \left[ \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right] \left( \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \right) \]

where \( X = Y \) in Eq. (35) and

\[ \Delta_1 = \Gamma^+ + \Gamma^- + 2 \text{Re} \left( \Gamma^+ \frac{a_{F_2(1, 1); |\phi_+|^2}}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \left( \frac{a_{F_2(2, 1); |\phi_-|^2}}{\sqrt{a_{F_2(1, 1); |\phi_-|^2}}} \right) \right) \]

\[ \sigma_x^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+)^2 \frac{a_{F_2(2, 3); |\phi_+|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] \right. + \Gamma^- \left[ \text{Re}(\phi_-)^2 \frac{a_{F_2(2, 3); |\phi_-|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] + 2 \text{Re} \left( \Gamma^+ \frac{1}{2\sqrt{a_{F_2(2, 1); |\phi_+|^2}}} \right) \left[ \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right] \left( \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \right) \]

\[ \sigma_x^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+)^2 \frac{a_{F_2(2, 3); |\phi_+|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] \right. + \Gamma^- \left[ \text{Re}(\phi_-)^2 \frac{a_{F_2(2, 3); |\phi_-|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] + 2 \text{Re} \left( \Gamma^+ \frac{1}{2\sqrt{a_{F_2(2, 1); |\phi_+|^2}}} \right) \left[ \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right] \left( \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \right) \]

\[ \sigma_x^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+)^2 \frac{a_{F_2(2, 3); |\phi_+|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] \right. + \Gamma^- \left[ \text{Re}(\phi_-)^2 \frac{a_{F_2(2, 3); |\phi_-|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] + 2 \text{Re} \left( \Gamma^+ \frac{1}{2\sqrt{a_{F_2(2, 1); |\phi_+|^2}}} \right) \left[ \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right] \left( \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \right) \]

\[ \sigma_x^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+)^2 \frac{a_{F_2(2, 3); |\phi_+|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] \right. + \Gamma^- \left[ \text{Re}(\phi_-)^2 \frac{a_{F_2(2, 3); |\phi_-|^2}}{a_{F_2(1, 2); |\phi_-|^2}} + \frac{3}{2} \right] + 2 \text{Re} \left( \Gamma^+ \frac{1}{2\sqrt{a_{F_2(2, 1); |\phi_+|^2}}} \right) \left[ \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right] \left( \frac{\phi_+^* \phi_-}{\sqrt{a_{F_2(2, 1); |\phi_-|^2}}} \right) \right) \]
\[ \sigma_p^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Im}(\phi_+)^2 \frac{0F_2(2, 3; |\phi_+|^2)}{0F_2(1, 2; |\phi_+|^2)} + \frac{3}{2} \right] + \Gamma^- \left[ \text{Im}(\phi_-)^2 \frac{0F_2(2, 3; |\phi_-|^2)}{0F_2(1, 2; |\phi_-|^2)} + \frac{3}{2} \right] \right\} \\
+ 2\Re \left[ \Gamma^{+-} \frac{1}{2\sqrt{0F_2(1, 2; |\phi_+|^2)0F_2(1, 2; |\phi_-|^2)}} \right. \\
\left. \left\{ -\frac{(\phi_- - \phi_+^*)^2}{2} \frac{0F_2(2, 3; \phi_+^* \phi_-)}{0F_2(1, 2; |\phi_+|^2)} + 3[0F_2(1, 2; \phi_+^* \phi_-)] \right\} \right] \right\} \\
- \Delta_2^{-2} \left\{ \sqrt{2}\Gamma^+ \text{Im}(\phi_+) \frac{0F_2(2, 2; |\phi_+|^2)}{0F_2(1, 2; |\phi_+|^2)} + \sqrt{2}\Gamma^- \text{Im}(\phi_-) \frac{0F_2(2, 2; |\phi_-|^2)}{0F_2(1, 2; |\phi_-|^2)} \right\} \\
+ \sqrt{2}\Re \left[ -i\Gamma^{+-}(\phi_- - \phi_+^*) \frac{[0F_2(2, 2; \phi_+^* \phi_-)]}{\sqrt{0F_2(1, 2; |\phi_+|^2)0F_2(1, 2; |\phi_-|^2)}} \right] \right\}^2 \right) (42) \\
\text{with } X = Z \text{ in Eq. (35) and} \\
\Delta_2 = \Gamma^+ + \Gamma^- + 2\Re \left( \Gamma^{+-} \frac{0F_2(1, 2; \phi_+^* \phi_-)}{\sqrt{0F_2(1, 2; |\phi_+|^2)0F_2(1, 2; |\phi_-|^2)}} \right) \right) \right). (43) \\

5. A particular case 
To analyze the behavior of the supercoherent states (linear and nonlinear) the following particular values for the parameters \( k_i \) are taken: \( k_1 = k_4 = 1 \), \( k_2 = \cos \theta \) and \( k_3 = \sin \theta \). Thus, the deformed SAO in (18) takes the generic form: 
\[ \hat{Q} = \begin{pmatrix} f(\hat{N})\hat{a} & \cos \theta \\ \sin \theta[f(\hat{N})\hat{a}] & f(\hat{N})\hat{a} \end{pmatrix}, \] (44) 
meanwhile the superposition (32) (with \( \eta = \lambda = \pi/4 \)) becomes 
\[ |X_{\hat{a}}\rangle = \frac{1}{\sqrt{2}} (|X_{+\theta}\rangle + |X_{-\theta}\rangle e^{i\pi/4}). \] (45) 

On the other hand, the eigenvalues \( \psi_{\pm} \) of the matrix \( K \) of Eq. (20) become now: 
\[ \psi_{\pm, \theta} = 1 \pm \sqrt{\frac{1}{2} \sin(2\theta)}, \] (46) 
so that for \( 0 < \theta < \pi/2 \) both \( \psi_{\pm} \) are real while for \( \pi/2 < \theta < \pi \) they turn out to be complex. By substituting equations (45) and (46) in the expressions (36)-(42), the Heisenberg uncertainty relation for each form of the operator \( \hat{Q} \) is found.

5.1. Heisenberg uncertainty relation for \( \hat{Q} = \hat{A} \).
The Heisenberg uncertainty relation associated with the linear supercoherentes states of [5] is shown in Figures 3 and 4. For \( z \) real (left plot) the uncertainty reaches a maximum value equal to 0.83 at \( |z| \sim 0.5 \) for real eigenvalues \( \psi_{\pm} \) \( (0 < \theta < \pi/2) \), while it shows a growing behavior as \( |z| \) increases for complex eigenvalues \( \psi_{\pm} \) \( (\pi/2 < \theta < \pi) \). For complex \( z \) (right plot), the uncertainty relation behaves similarly as for the real case.
5.2. Heisenberg uncertainty relation for $\hat{Q} = \hat{A}'$.

Figures 5 and 6 show the Heisenberg uncertainty relation $\sigma_x^2(\text{nl}) \sigma_p^2(\text{nl})$ for the nonlinear supercoherent states $|\gamma\rangle$ of the operator $\hat{A}'$. For $Y$ real (left plot), for both real ($0 < \theta < \pi/2$) as well as complex ($\pi/2 < \theta < \pi$) eigenvalues $\psi_{\pm}$, the uncertainty starts from a minimum value and grows slowly. In the region of degenerate eigenvalues, for $\theta = \pi/2$, the uncertainty reaches a maximum value. For complex $Y$ (right plot), the uncertainty relation behaves similarly as for the real case.

5.3. Heisenberg uncertainty relation for $\hat{Q} = \hat{A}''$.

Figures 7 and 8 show the uncertainty relations $\sigma_x^2(\text{NL}) \sigma_p^2(\text{NL})$ for the nonlinear supercoherent states $|\zeta\rangle$ of the operator $\hat{A}''$. For $Z$ real (left plot), for both real ($0 < \theta < \pi/2$) as well as complex ($\pi/2 < \theta < \pi$) eigenvalues $\psi_{\pm}$, the uncertainty starts from a value equal to 2.25, then it decreases smoothly and grows again. In the region of degenerate eigenvalues, for $\theta = \pi/2$ eigenvalues $\psi_{\pm}$, the uncertainty reaches a minimum value close to 0.25. For complex $Z$ (right plot)
As shown in the Figures above, the behavior of the Heisenberg uncertainty relation for the supercoherent states $|Z\rangle$ becomes more involved than for the nonlinear ones, $|Y\rangle$ and $|Z\rangle$. Moreover, due to the nonlinear supercoherent states $|Z\rangle$ are expressed in terms of the nonlinear coherent states $|\alpha\rangle_{NL}$, which isolate the eigenstate $|0\rangle$, the corresponding uncertainty relation seems to start from a different value than its counterparts associated to $A$ and $A'$.

6. Conclusion
The introduction of a deformation in the form of the supersymmetric annihilation operator $\hat{A}$ given in [5] allowed us to find interesting results related with the nonlinear supercoherent states. As it was noted, the form of the operator $\hat{a} = f(\hat{N})\hat{a}$ enabled us to isolate the ground state contribution, which has interesting implications for the properties of the nonlinear supercoherent states and their associated Heisenberg uncertainty relation.

Let us note that the results shown in this paper obey the Heisenberg uncertainty principle ($\sigma_x^2 \sigma_p^2 \geq 1/4$), which ensures that the states constructed here are consistent with quantum theory.

Finally, the freedom we have to choose the form of the function $f(\hat{N})$ will allow to consider a more general algebraic study, focused on the commutation relations of the operators $\{\hat{H}_s, \hat{A}, \hat{A}^\dagger\}$ of the supersymmetric harmonic oscillator and the possible relation with the polynomial deformations of the Heisenberg algebra, as it happens with the operators $\{\hat{H}, \hat{a}, \hat{a}^\dagger\}$ for the standard harmonic oscillator.

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