Some remarks on the homogenization of immiscible incompressible two-phase flow in double porosity media

B. Amaziane\textsuperscript{1}, M. Jurak\textsuperscript{2,*}, L. Pankratov\textsuperscript{1,3}, A. Vrbaški\textsuperscript{4}

May 5, 2018

\textsuperscript{1} Laboratoire de Mathématiques et de leurs Applications, CNRS-UMR 5142 Université de Pau, Av. de l’Université, 64000 Pau, France. E-mail: brahim.amaziane@univ-pau.fr

\textsuperscript{2} Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia. E-mail: jurak@math.hr

\textsuperscript{3} Laboratory of Fluid Dynamics and Seismic (RAEP.Stop100), Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russian Federation. E-mail: leonid.pankratov@univ-pau.fr

\textsuperscript{4} Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia. E-mail: anja.vrbski@rgn.hr

Abstract

This paper presents a study of immiscible incompressible two-phase flow through fractured porous media. The results obtained earlier in the pioneer work by A. Bourgeat, S. Luckhaus, A. Mikelić (1996) and L. M. Yeh (2006) are revisited. The main goal is to incorporate some of the most recent improvements in the convergence of the solutions in the homogenization of such models. The microscopic model consists of the usual equations derived from the mass conservation of both fluids along with the Darcy-Muskat law. The problem is written in terms of the phase formulation, i.e. the saturation of one phase and the pressure of the second phase are primary unknowns. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. The fractured medium consists of periodically repeating homogeneous blocks and fractures, the permeability being highly discontinuous. Over the matrix domain, the permeability is scaled by $\varepsilon^\theta$, where $\varepsilon$ is the size of a typical porous block and $\theta > 0$ is a parameter. The model involves highly oscillatory characteristics and internal nonlinear interface conditions. Under some realistic assumptions on the data, the convergence of the solutions, and the macroscopic models corresponding to various range of contrast are constructed using the two-scale convergence method combined with the dilation technique. The results improve upon previously derived effective models to highly heterogeneous porous media with discontinuous capillary pressures.

Keywords: Homogenization, double porosity media, two-scale convergence, dilation operator.

2010 Mathematics Subject Classification. Primary: 35B27, 35K55, 35K65, 74Q15; Secondary: 35Q35, 76S05.

1 Introduction

The modeling of displacement process involving two immiscible fluids in fractured porous media is important to many practical problems, including those in petroleum reservoir engineering, unsaturated zone hydrology,

*Corresponding author.
and soil science. More recently, modeling multiphase flow has received an increasing attention in connection with the disposal of radioactive waste and sequestration of \( CO_2 \). Furthermore, fractured rock domains corresponding to the so-called Excavation Damaged Zone (EDZ) receives increasing attention in connection with the behaviour of geological isolation of radioactive waste after the drilling of the wells or shafts, see, e.g., [36].

In this paper we use the homogenization theory to derive a double porosity model describing the flow of incompressible fluids in fractured reservoirs. The model corresponds physically to immiscible incompressible two-phase flow through fractured porous media. Naturally fractured reservoirs can be modeled by two superimposed continua, a connected fracture system and a system of topologically disconnected matrix blocks. The fracture system has low storage capacity but high conductivity, while the matrix block system has low conductivity and large storage capacity. The majority of fluid transport will occur along flow paths through the fissure system. When the system of fissures is so well developed that the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. For more details on the physical formulation of such problems see, e.g., [15, 33, 39].

This paper continues the research published in [16] and [40], and the goal is to reformulate in a more systematic manner and in somewhat more general context the homogenization problem for an immiscible incompressible two-phase flow in double porosity media by weakening the standing assumptions. Special attention is paid to developing a general approach to incorporating highly heterogeneous porous media with discontinuous capillary pressures.

During recent decades mathematical analysis and homogenization of multiphase flows in porous media have been the subject of investigation of many researchers owing to important applications in reservoir simulation. There is an extensive literature on this subject. We will not attempt a literature review here but will merely mention a few references. A recent review of the mathematical homogenization methods developed for incompressible immiscible two-phase flow in porous media and compressible miscible flow in porous media can be viewed in [5, 29, 30].

Let us now turn to a brief review of the homogenization in double porosity media. Here we restrict ourselves to the mathematical homogenization method as described in [30] for flow and transport in porous media. The interest for double porosity systems came at first from geophysics. The notion of double porosity, or double permeability is borne from studies carried out on naturally fractured porous rocks, such as oil fields. The double porosity model was first introduced in [14] and it is since used in a wide range of engineering specialities. The first rigorous mathematical result on the subject was obtained in [13], where a linear parabolic equation with asymptotically degenerating coefficients describing a single-phase flow in fractured media was considered. This result is then generalized in [17, 18, 32, 35] for non-periodic domains and various rates of contrast. Linear double porosity models with thin fissures were considered in [11, 34]. A singular double porosity model was considered in [19]. Notice that the works [11, 18, 32, 34] are done in the framework of Khruslov’s energy characteristic method which is close to the \( \Gamma \)-convergence method. Let us also notice that the double porosity model was obtained in [30] (see Chapter 3) using the two-scale convergence method. Non-linear double porosity models, elliptic and parabolic, including the homogenization in variable Sobolev spaces, were obtained in [7, 8, 23, 24, 26]. A study of discrete double-porosity models in the case of elastic energies has been recently done in [20]. Finally, in order to complete this brief review, we turn to the multiphase flow double porosity models. These models were obtained e.g., in [16, 23, 40] (see also [30] and the references therein) and recently in [2, 6] for immiscible compressible two-phase flows. A fully homogenized model for incompressible two-phase flow in double porosity media was obtained in [31].

This paper is concerned with a nonlinear degenerate system of diffusion-convection equations modeling the flow and transport of immiscible incompressible fluids through highly heterogeneous porous media, capillary and gravity effects being taken into account. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. The model to be presented herein is formulated in terms of the wetting phase saturation and the non-wetting phase
pressure, and the feature of the global pressure as introduced in [12, 21] for incompressible immiscible flows is used to establish a priori estimates. The governing equations are derived from the mass conservation laws of both fluids, along with constitutive relations relating the velocities to the pressures gradients and gravitational effects. Traditionally, the standard Muskat-Darcy law provides this relationship. Let us mention that the main difficulties related to the mathematical analysis of such equations are the coupling and the degeneracy of the diffusion term in the saturation equation. Moreover the transmission conditions are nonlinear and the saturation is discontinuous at the interface separating the two media.

We start with a microscopic model defined on a domain with periodic microstructure. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. The fractured medium consists of periodically repeating homogeneous blocks and fractures, the permeability being highly discontinuous. Over the matrix domain, the permeability is scaled by $\varepsilon^\theta$, where $\varepsilon$ is the size of a typical porous block and $\theta > 0$ is a parameter. Our aim is to study the macroscopic behavior of solutions of this system of equations as $\varepsilon$ tends to zero and give a rigorous mathematical derivation of upscaled models by means of the two-scale convergence method combined with the dilation technique. Thus, we extend the results of [16, 40] to the case of highly heterogeneous porous media with discontinuous capillary pressures.

The rest of the paper is organized as follows. In Section 2, we describe the physical model and formulate the corresponding mathematical problem. We also provide the assumptions on the data and a weak formulation of the problem firstly in terms of phase pressures and secondly in terms of the global pressure and the saturation. Section 3 is then devoted to the derivation of the basic a priori estimates of the problem under consideration. In Section 4, we formulate the two-scale convergence results which will be used in the derivation of the homogenized system. The key point here is the proof of the compactness result for the restriction-extension sequence of the wetting fluid saturation defined on the fracture set. It is done by using the ideas from [40]. Section 5 is devoted to the definition and the properties of the dilation operator and to the formulation of the convergence results for the diluted functions defined on the matrix part. The key point of the section is the proof of the compactness result for the diluted saturations which is done by using the compactness result from [5]. The formulation of the main results of the paper is given in Section 6. The resulting homogenized problem is a dual-porosity type model that contains a term representing memory effects which could be seen as source term or as a time delay for $\theta = 2$, and it is a single porosity model with effective coefficients for $0 < \theta < 2$ or $\theta > 2$. The proof of the convergence theorem in the critical case ($\theta = 2$) is done in subsection 6.1. The key point here is subsection 6.1.6 where we prove the uniqueness of the solution to the local problem. The proof is done by reducing the problem in the phase formulation to a boundary value problem for an imbibition equation and by using ideas from [38]. The proofs of the convergence theorems for non-critical cases ($\theta > 2$ or $0 < \theta < 2$) are given in subsections 6.2, 6.3. The effective model obtained in the case of moderate contrast ($0 < \theta < 2$, subsection 6.3), up to our knowledge, is for the first time proposed and rigorously justified here.

2 Formulation of the problem

The outline of this section is as follows. First, in subsection 2.1 we give a short description of the mathematical and physical model used in this study for immiscible incompressible two-phase flow in a periodic double porosity medium. The notion of the global pressure is briefly recalled in subsection 2.2. Finally, in subsection 2.3 we present the main assumptions on the data and we define the weak solution to our problem, first in terms of phase pressures and then an equivalent one in terms of the global pressure and saturation.
wetting and nonwetting fluids in of wetting and nonwetting fluids, respectively; \( \ell, n \) are the phase pressures of wetting and nonwetting fluids in \( \Omega_\ell \), respectively; \( p_{\ell,w}^\varepsilon(x,t), p_{\ell,n}^\varepsilon(x,t) \) are the phase pressures of wetting and nonwetting fluids in \( \Omega_\ell \), respectively. Here \( \ell = f, m \).

The conservation of mass in each phase can be written as (see, e.g., \([21, 22, 28]\)):

\[
\begin{align*}
\Phi^\varepsilon(x) & \frac{\partial}{\partial t} \left[ S_{\ell,w}^\varepsilon \varrho_w (p_{\ell,w}^\varepsilon) \right] + \text{div} \left[ \varrho_w (p_{\ell,w}^\varepsilon) \vec{q}_{\ell,w}^\varepsilon \right] = F_{\ell,w}^\varepsilon(x,t) \quad \text{in } \Omega_{\ell,T}^\varepsilon; \\
\Phi(x) & \frac{\partial}{\partial t} \left[ S_{\ell,n}^\varepsilon \varrho_n (p_{\ell,n}^\varepsilon) \right] + \text{div} \left[ \varrho_n (p_{\ell,n}^\varepsilon) \vec{q}_{\ell,n}^\varepsilon \right] = F_{\ell,n}^\varepsilon(x,t) \quad \text{in } \Omega_{\ell,T}^\varepsilon,
\end{align*}
\]

Figure 1: (a) The domain \( \Omega \). (b) The reference cell \( Y \).

### 2.1 Microscopic model

We consider a reservoir \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) which is assumed to be a bounded, connected Lipschitz domain with a periodic microstructure. More precisely, we will scale this structure by a parameter \( \varepsilon \) which represents the ratio of the cell size to the size of the whole region \( \Omega \) and we assume that \( 0 < \varepsilon \ll 1 \) is a small parameter tending to zero. Let \( Y = (0, 1)^d \) be a basic cell of a fractured porous medium. For the sake of simplicity and without loss of generality, we assume that \( Y \) is made up of two homogeneous porous media \( Y_m \) and \( Y_f \) corresponding to the parts of the domain occupied by the matrix block and the fracture, respectively (see Fig.1 (b)). Thus \( Y = Y_m \cup Y_f \cup \Gamma_{fm} \), where \( \Gamma_{fm} \) denotes the interface between the two media. Let \( \Omega^\varepsilon_\ell \) with \( \ell = "m" \) or "f" denote the open set corresponding to the porous medium with index \( \ell \). Then \( \Omega = \Omega^\varepsilon_m \cup \Omega^\varepsilon_f \cup \Omega^\varepsilon_{fm} \), where \( \Gamma^\varepsilon_{fm} \equiv \partial \Omega^\varepsilon_f \cap \partial \Omega^\varepsilon_m \cap \Omega \) and the subscripts "m", "f" refer to the matrix and fracture, respectively (see Fig.1(a)). For the sake of simplicity, we assume that \( \Omega^\varepsilon_m \cap \partial \Omega = \emptyset \). We also introduce the notation:

\[
\Omega_T^\varepsilon = \Omega \times (0,T), \quad \Omega_{\ell,T}^\varepsilon = \Omega_\ell^\varepsilon \times (0,T), \quad \Sigma_T^\varepsilon \equiv \Gamma_{fm}^\varepsilon \times (0,T), \quad \text{where } T > 0 \text{ is fixed.} \tag{2.1}
\]

Before describing the equations of the model, we give some notation: \( \Phi^\varepsilon(x) = \Phi(x, \bar{\varepsilon}) \) is the porosity of the reservoir \( \Omega \); \( K^\varepsilon(x) = K(x, \bar{\varepsilon}) \) is the absolute permeability tensor of \( \Omega \); \( \varrho_w, \varrho_n \) are the densities of wetting and nonwetting fluids, respectively; \( S_{\ell,w}^\varepsilon = S_{\ell,w}^\varepsilon(x,t), S_{\ell,n}^\varepsilon = S_{\ell,n}^\varepsilon(x,t) \) are the saturations of wetting and nonwetting fluids in \( \Omega_\ell^\varepsilon \), respectively; \( k_{\ell,w}^\varepsilon = k_{\ell,w}^\varepsilon(S_{\ell,w}^\varepsilon), k_{\ell,n}^\varepsilon = k_{\ell,n}^\varepsilon(S_{\ell,n}^\varepsilon) \) are the relative permeabilities of wetting and nonwetting fluids in \( \Omega_\ell^\varepsilon \), respectively; \( p_{\ell,w}^\varepsilon(x,t), p_{\ell,n}^\varepsilon(x,t) \) are the phase pressures of wetting and nonwetting fluids in \( \Omega_\ell^\varepsilon \), respectively. Here \( \ell = f, m \).
where the velocities of the wetting and nonwetting fluids $\bar{q}^e_{\ell,w}, \bar{q}^e_{\ell,n}$ are defined by Darcy-Muskat’s law:

$$\bar{q}^e_{\ell,w} \overset{\text{def}}{=} -K^e(\varepsilon)(x)\lambda_{\ell,w}(S^e_{\ell,w}) \left[ \nabla p^e_{\ell,w} - \varrho_w(p^e_{\ell,w}) \bar{g} \right], \quad \text{with} \quad \lambda_{\ell,w}(S^e_{\ell,w}) \overset{\text{def}}{=} \frac{k^e_{\ell,w}}{\mu_w}(S^e_{\ell,w}); \quad (2.3)$$

$$\bar{q}^e_{\ell,n} \overset{\text{def}}{=} -K^e(x)\lambda_{\ell,n}(S^e_{\ell,n}) \left[ \nabla p^e_{\ell,n} - \varrho_n(p^e_{\ell,n}) \bar{g} \right], \quad \text{with} \quad \lambda_{\ell,n}(S^e_{\ell,n}) \overset{\text{def}}{=} \frac{k^e_{\ell,n}}{\mu_n}(S^e_{\ell,n}). \quad (2.4)$$

Here $\bar{g}, \mu_w, \mu_n$ are the gravity vector and the viscosities of the wetting and nonwetting fluids, respectively. The source terms $F^e_{\ell,w}, F^e_{\ell,n}$ are given by:

$$F^e_{\ell,w} \overset{\text{def}}{=} q_w(p^e_{\ell,w})S^I_{\ell,w}f_I(x,t) - q_w(p^e_{\ell,w})S^e_{\ell,w}f_P(x,t); \quad (2.5)$$

$$F^e_{\ell,n} \overset{\text{def}}{=} q_n(p^e_{\ell,n})S^I_{\ell,n}f_I(x,t) - q_n(p^e_{\ell,n})S^e_{\ell,n}f_P(x,t), \quad (2.6)$$

where $f_I, f_P \geq 0$ are injection and productions terms and $S^I_{\ell,w}, S^I_{\ell,n}$ are known injection saturations.

From now on we deal with two incompressible fluids, that is the densities of the liquids, for the sake of simplicity and brevity, will be taken equal to one, i.e. $\varrho_w(p^e_{\ell,w}) = \varrho_n(p^e_{\ell,n}) = 1$. The model is completed as follows. By the definition of saturations, one has $S^e_{\ell,w} + S^e_{\ell,n} = 1$ with $S^e_{\ell,w}, S^e_{\ell,n} \geq 0$. We set $S^e_{\ell} = S^e_{\ell,w}$. Then the curvature of the contact surface between the two fluids links the difference in the pressures of the two phases to the saturation by the capillary pressure law:

$$P_{\ell,c}(S^e_{\ell}) \overset{\text{def}}{=} p^e_{\ell,w} - p^e_{\ell,n} \quad \text{with} \quad P'_{\ell,c}(s) < 0 \text{ for all } s \in (0, 1) \text{ and } P_{\ell,c}(1) = 0, \quad (2.7)$$

where $P'_{\ell,c}(s)$ denotes the derivative of the function $P_{\ell,c}(s)$.

Now due to the assumptions on the densities of the liquids, we rewrite the system (2.2) as follows:

$$\begin{cases}
0 \leq S^e \leq 1 \quad \text{in} \quad \Omega_T; \\
\Phi^e(x) \frac{\partial S^e}{\partial t} - \text{div} \left\{ K^e(x)\lambda_w(\frac{x}{\varepsilon}, S^e) \left( \nabla p^e_w - \bar{g} \right) \right\} = F^e_w \quad \text{in} \quad \Omega_T; \\
-\Phi^e(x) \frac{\partial S^e}{\partial t} - \text{div} \left\{ K^e(x)\lambda_n(\frac{x}{\varepsilon}, S^e) \left( \nabla p^e_n - \bar{g} \right) \right\} = F^e_n \quad \text{in} \quad \Omega_T; \\
P_{\ell,c}(\varepsilon, S^e) = p^e_w - p^e_n \quad \text{in} \quad \Omega_T,
\end{cases} \quad (2.8)$$

where $\lambda_{\ell,n}(S^e_{\ell}) := \lambda_{\ell,n}(1 - S^e_{\ell})$ and each function $u^e := S^e, p^e_w, p^e_n, F^e_w, F^e_n$ is defined as:

$$u^e = u^e_I(x,t) 1^e_I(x) + u^e_m(x,t) 1^e_m(x), \quad (2.9)$$

Here $1^e_\ell = 1_{I}(\varepsilon)$ is the characteristic function of the subdomain $\Omega^e_\ell$ for $\ell = f, m$. The exact form of the porosity function and the absolute permeability tensor corresponding to the double porosity model will be specified in conditions (A.1), (A.2) in subsection (2.3) below.

Model (2.3) have to be completed with appropriate interface, boundary and initial conditions. Interface conditions. The continuity at the interface $\Gamma^e_{fm}$ of the phase fluxes and the phase pressures, gives the following transmission conditions:

$$\begin{cases}
\bar{q}^e_{\ell,w} \cdot \bar{n} = \bar{q}^e_{m,w} \cdot \bar{n} \quad \text{and} \quad \bar{q}^e_{\ell,n} \cdot \bar{n} = \bar{q}^e_{m,n} \cdot \bar{n} \quad \text{on} \quad \Sigma^e_T; \\
p^e_{\ell,w} = p^e_{m,w} \quad \text{and} \quad p^e_{\ell,n} = p^e_{m,n} \quad \text{on} \quad \Sigma^e_T,
\end{cases} \quad (2.10)$$

where $\Sigma^e_T$ is defined in (2.1), $\bar{n}$ is the unit outer normal on $\Gamma^e_{fm}$, and the fluxes $\bar{q}^e_{\ell,w}, \bar{q}^e_{\ell,n}$, under the assumption on the densities of the liquids, are equal to the velocities (2.3), (2.4).
Remark 1 It is important to notice that in contrast to the functions $p_{n}^{ε}, p_{m}^{ε}$, the saturation $S^{ε}$ may have a jump at the interface $Γ_{fm}$. Namely, it is easy to see from the transmission conditions (2.10) for the phase pressures that $P_{t,c}(S^{ε}_{1}) = P_{m,c}(S^{ε}_{2})$ on $Σ_{T}^{ε}$ which gives a discontinuity of the saturation at the interface.

Now we specify the boundary and initial conditions. We suppose that the boundary $∂Ω$ consists of two parts $Γ_{1}$ and $Γ_{2}$ such that $Γ_{1} ∩ Γ_{2} = ∅$, $∂Ω = Γ_{1} ∪ Γ_{2}$.

Boundary conditions:
\[
\begin{align*}
 p_{w}^{ε}(x, t) &= p_{w}^{ε}(x, t) = 0 \quad \text{on } Γ_{1} \times (0, T); \\
 \tilde{q}_{t,w}^{ε} \cdot ν &= \tilde{q}_{t,n}^{ε} \cdot ν = 0 \quad \text{on } Γ_{2} \times (0, T).
\end{align*}
\]

(2.11)

Initial conditions:
\[
p_{w}^{ε}(x, 0) = p_{w}^{0}(x) \quad \text{and} \quad p_{n}^{ε}(x, 0) = p_{n}^{0}(x) \quad \text{in } Ω.
\]

(2.12)

2.2 A fractional flow formulation

In the sequel, we will use a formulation obtained after transformation using the concept of the global pressure introduced in [12][21]. For each subdomain $Ω^{ε}_{c}$, the global pressure, $P^{ε}_{c}$, is defined by:
\[
P^{ε}_{c} \overset{\text{def}}{=} P^{ε}_{c} + G^{ε}_{c}(S^{ε}_{c}) \quad \text{and} \quad p^{ε}_{c} \overset{\text{def}}{=} P^{ε}_{c} + G^{ε}_{n}(S^{ε}_{n}),
\]

(2.13)

where the functions $G^{ε}_{c}(s)$, $G^{ε}_{n}(s)$ are given by:
\[
G^{ε}_{c}(S^{ε}_{c}) \overset{\text{def}}{=} G^{ε}_{c}(0) + \int_{0}^{S^{ε}_{c}} \frac{λ^{ε}_{c}(s)}{λ^{ε}_{c}(s)} P^{c}_{c}(s) ds \quad \text{and} \quad G^{ε}_{n}(S^{ε}_{n}) \overset{\text{def}}{=} G^{ε}_{n}(S^{ε}_{n}) - P^{c}_{c}(S^{ε}_{c}),
\]

(2.14)

where $λ^{ε}_{c}(s) \overset{\text{def}}{=} λ^{ε}_{c}(s) + λ^{ε}_{n}(s)$ and $G^{ε}_{n}(0)$ is a constant chosen to ensure $p^{ε}_{c, w} \leq P^{ε}_{c} \leq p^{ε}_{c, n}$. Notice that from (2.14) we get:
\[
λ^{ε}_{c}(S^{ε}_{c}) \nabla G^{ε}_{c}(S^{ε}_{c}) = \nabla β^{ε}(S^{ε}_{c}) \quad \text{and} \quad λ^{ε}_{n}(S^{ε}_{n}) \nabla G^{ε}_{n}(S^{ε}_{n}) = -\nabla β^{ε}(S^{ε}_{c}),
\]

(2.15)

where
\[
β^{ε}(s) \overset{\text{def}}{=} \int_{0}^{s} α^{ε}(ξ) dξ \quad \text{with} \quad α^{ε}(s) \overset{\text{def}}{=} \frac{λ^{ε}_{n}(s)}{λ^{ε}_{c}(s)} |P^{c}_{c}(s)|.
\]

(2.16)

Furthermore, we have the following important relation:
\[
λ^{ε}_{n}(S^{ε}_{n})|∇p^{ε}_{c, n}|^{2} + λ^{ε}_{c}(S^{ε}_{c})|∇p^{ε}_{c, w}|^{2} = λ^{ε}(S^{ε}_{c})|∇P^{ε}_{c}|^{2} + |∇β^{ε}(S^{ε}_{c})|^{2},
\]

(2.17)

where
\[
b^{ε}(s) \overset{\text{def}}{=} \int_{0}^{s} a^{ε}(ξ) dξ \quad \text{with} \quad a^{ε}(s) \overset{\text{def}}{=} \sqrt{\frac{λ^{ε}_{n}(s)}{λ^{ε}_{c}(s)}} |P^{c}_{c}(s)|.
\]

(2.18)

Now if we use the global pressure and the saturation as new unknown functions then (2.8) reads:
\[
\begin{align*}
 0 \leq S^{ε}_{c} \leq 1 \quad \text{in } Ω^{ε}_{c,T}; \\
 \Phi^{ε}(x) \frac{∂S^{ε}_{c}}{∂t} - \text{div} \left\{ K^{ε}(x) \left[ λ^{ε}_{c}(S^{ε}_{c}) \nabla P^{ε}_{c} + \nabla β^{ε}(S^{ε}_{c}) - λ^{ε}_{c}(S^{ε}_{c}) \tilde{g} \right] \right\} = F^{ε}_{c, w} \quad \text{in } Ω^{ε}_{c,T}; \\
 -\Phi^{ε}(x) \frac{∂S^{ε}_{n}}{∂t} - \text{div} \left\{ K^{ε}(x) \left[ λ^{ε}_{n}(S^{ε}_{n}) \nabla P^{ε}_{c} - \nabla β^{ε}(S^{ε}_{c}) - λ^{ε}_{n}(S^{ε}_{n}) \tilde{g} \right] \right\} = F^{ε}_{c, n} \quad \text{in } Ω^{ε}_{c,T}.
\end{align*}
\]

(2.19)
The system (2.19) is completed by the following boundary, interface and initial conditions.

Boundary conditions:
\[
\begin{align*}
S^\varepsilon = 1 & \quad \text{and} \quad P^\varepsilon = P_{\Gamma_1} \quad \text{on} \quad \Gamma_1 \times (0, T); \\
\bar{q}_{\ell,w}^\varepsilon \cdot \bar{v} = \bar{q}_{\ell,n}^\varepsilon \cdot \bar{v} = 0 & \quad \text{on} \quad \Gamma_2 \times (0, T),
\end{align*}
\]
(2.20)

where \(P_{\Gamma_1}\) is a given constant and \(\bar{q}_{\ell,w}^\varepsilon, \bar{q}_{\ell,n}^\varepsilon\) are defined by
\[
\bar{q}_{\ell,w}^\varepsilon \overset{\text{def}}{=} -K^\varepsilon(x) [\lambda_{\ell,w}(S^\varepsilon)^{\varepsilon}] \nabla P^\varepsilon + \nabla \beta^\varepsilon(S^\varepsilon) - \lambda_{\ell,w}(S^\varepsilon)^{\alpha}];
\]
(2.21)
\[
\bar{q}_{\ell,n}^\varepsilon \overset{\text{def}}{=} -K^\varepsilon(x) [\lambda_{\ell,n}(S^\varepsilon)^{\varepsilon}] \nabla P^\varepsilon - \nabla \beta^\varepsilon(S^\varepsilon) - \lambda_{\ell,n}(S^\varepsilon)^{\alpha}].
\]
(2.22)

Interface conditions:
\[
\begin{align*}
\bar{q}_{\ell,w}^\varepsilon \cdot \bar{v} = \bar{q}_{\ell,n}^\varepsilon \cdot \bar{v} & \quad \text{and} \quad \bar{q}_{\ell,n}^\varepsilon \cdot \bar{v} = \bar{q}_{\ell,n}^\varepsilon \cdot \bar{v} \quad \text{on} \quad \Sigma_T; \\
P_f^\varepsilon + G_{f,j}(S_f^\varepsilon) & = P_m^\varepsilon + G_{m,j}(S_m^\varepsilon) \quad \text{on} \quad \Sigma_T^c \quad (j = w, n); \\
P_{f,c}(S_f^\varepsilon) & = P_{m,c}(S_m^\varepsilon) \quad \text{on} \quad \Sigma_T^c.
\end{align*}
\]
(2.23)

Note that the global pressure function might be discontinuous at the interface. This makes the compactness result in Section 4.2 non-trivial.

Initial conditions:
\[S^\varepsilon_\ell(x, 0) = S^\varepsilon_\ell(x) \quad \text{and} \quad P^\varepsilon_\ell(x, 0) = P^\varepsilon_\ell(0) \quad \text{in} \quad \Omega.\]
(2.24)

2.3 Weak formulations of the problem

Let us begin this subsection by stating the following assumptions.

(A.1) The porosity \(\Phi^\varepsilon\) is given by \(\Phi^\varepsilon(x) \overset{\text{def}}{=} \Phi^\varepsilon_\ell(x) 1^\varepsilon_\ell(x) + \Phi^\varepsilon_m(x) 1^\varepsilon_m(x) = \Phi^\varepsilon_\ell(x) 1^\varepsilon_\ell(x) + \Phi_m^\varepsilon 1^\varepsilon_m(x),\)
where \(\Phi^\varepsilon_\ell \in L^\infty(\Omega)\) and there are positive constants \(0 < \phi^-_\ell < \phi^+_\ell < 1, \ell = f, m\), that do not depend on \(\varepsilon\) and such that \(0 < \phi^-_m \leq \Phi^\varepsilon_m(x) \leq \phi^+_m < 1\) a.e. in \(\Omega\). Moreover, \(\Phi^\varepsilon_\ell \rightharpoonup \Phi^H \ell\) strongly in \(L^2(\Omega)\).

(A.2) The permeability \(K^\varepsilon(x) = K^\varepsilon(x, \frac{x}{\varepsilon})\) is defined as \(K^\varepsilon(x, y) \overset{\text{def}}{=} K(x, y) 1^\varepsilon_\ell(x) + \xi(x, y) 1^\varepsilon_m(x),\)
where \(\xi(x, y) \overset{\text{def}}{=} \varepsilon^\theta\) with \(\theta > 0\) and \(K \in (L^\infty(\Omega \times Y))^{d \times d}\). Moreover, there exist constants \(k_{\min}, k_{\max}\) such that \(0 < k_{\min} < k_{\max}\) and \(k_{\min}|\xi|^2 \leq (K(x, y) \xi, \xi) \leq k_{\max}|\xi|^2\) for all \(\xi \in \mathbb{R}^d\), a.e. in \(\Omega \times Y\).

(A.3) The capillary pressure function \(P_{f,c}(s) \in C^1([0, 1]; \mathbb{R}^+), \ell = f, m\). Moreover, \(P'_{f,c}(s) < 0\) in \([0, 1], P_{f,c}(1) = 0\) and \(P_{f,c}(0) = P_{m,c}(0)\).

(A.4) The functions \(\lambda_{\ell,w}, \lambda_{\ell,n}\) belong to the space \(C([0, 1]; \mathbb{R}^+)\) and satisfy the following properties:
(i) \(0 \leq \lambda_{\ell,w}, \lambda_{\ell,n} \leq 1\) in \([0, 1]\); (ii) \(\lambda_{\ell,w}(0) = 0\) and \(\lambda_{\ell,n}(1) = 0\); (iii) there is a positive constant \(L_0\) such that \(\lambda_{\ell}(s) = \lambda_{\ell,w}(s) + \lambda_{\ell,n}(s) \geq L_0 > 0\) in \([0, 1]\).

(A.5) The functions \(\alpha_{\ell} \in C([0, 1]; \mathbb{R}^+)\). Moreover, \(\alpha_{\ell}(0) = \alpha_{\ell}(1) = 0\) and \(\alpha_{\ell} > 0\) in \((0, 1)\).

(A.6) The functions \(\beta_{\ell}^{-1}\), inverse of \(\beta_{\ell}\) defined in (2.16), are H"older functions of order \(\gamma \in (0, 1)\) in \([0, \beta_{\ell}(1)]\). Namely, there exists a positive constant \(C_\beta\) such that for all \(s_1, s_2 \in [0, \beta_{\ell}(1)]\), we have:
\[|\beta_{\ell}^{-1}(s_1) - \beta_{\ell}^{-1}(s_2)| \leq C_\beta |s_1 - s_2|^{\gamma}.\]

(A.7) The initial data for the pressures are such that \(p^0_n, p^0_w \in L^2(\Omega)\).
The initial data for the saturation $S^0$ is given by $P_{\ell,c}(S^0) = p_{\ell,n}^0 - p_{\ell,w}^0$ and is such that $S^0 \in L^\infty(\Omega)$ and $0 \leq S^0 \leq 1$ a.e. in $\Omega$.

The source terms $F_{w}^\varepsilon$, $F_{n}^\varepsilon$ are equal to zero on the matrix part, i.e. $F_{w}^\varepsilon \equiv 1_{\varepsilon} f^\varepsilon \left[ S_{\varepsilon}^w f_I(x,t) - S_{\varepsilon}^w f_P(x,t) \right]$ and $F_{n}^\varepsilon \equiv 1_{\varepsilon} f^\varepsilon \left[ S_{\varepsilon}^n f_I(x,t) - (1 - S_{\varepsilon}^n) f_P(x,t) \right]$, where $f_I, f_P \in L^2(\Omega_T)$ and $0 \leq S_{\varepsilon}^w, S_{\varepsilon}^n \leq 1$.

The assumptions (A.1)-(A.9) are classical and physically meaningful for existence results and homogenization problems of two-phase flow in porous media. They are similar to the assumptions made in [12, 21] that dealt with the existence of a weak solution of the studied problem.

We next introduce the following Sobolev space: $H^1_\gamma(\Omega) \equiv \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1 \}$. The space $H^1_\gamma(\Omega)$ is a Hilbert space. The norm in this space is given by $\| u \|_{H^1_\gamma(\Omega)} = \| \nabla u \|_{(L^2(\Omega))^{d}}$.

**Definition 2.1** (Weak solution in terms of phase pressures) We say that the functions $(p^\varepsilon_w, p^\varepsilon_n, S^\varepsilon)$ is a weak solution of problem (2.8) if

(i) $0 \leq S^\varepsilon \leq 1$ a.e. in $\Omega_T$ and $P_{\ell,c}(S^\varepsilon) \equiv p_{\ell,n}^\varepsilon - p_{\ell,w}^\varepsilon$ for $\ell \in \{ t, m \}$.

(ii) The functions $p^\varepsilon_w, p^\varepsilon_n$ are such that

\[
\begin{align*}
  p^\varepsilon_w, p^\varepsilon_n, \lambda_w(x, S^\varepsilon) \nabla p^\varepsilon_w, \sqrt{\lambda_n(x, S^\varepsilon)} \nabla p^\varepsilon_n & \in L^2(\Omega_T).
\end{align*}
\]

(iii) The boundary conditions (2.11) and the initial conditions (2.12) are satisfied.

(iv) For any $\varphi_w, \varphi_n \in C^1([0, T]; H^1_\gamma(\Omega))$ satisfying $\varphi_w(T) = \varphi_n(T) = 0$, we have:

\[
\begin{align*}
- \int_{\Omega_T} \Phi^\varepsilon(x) S^0 \frac{\partial \varphi_w}{\partial t} \, dx \, dt - \int_{\Omega} \Phi^\varepsilon S^0 \varphi_w \, dx + \int_{\Omega_T} K^\varepsilon(x) \lambda_w \left( \frac{x}{\varepsilon}, S^\varepsilon \right) (\nabla p^\varepsilon_w - \bar{g}) \cdot \nabla \varphi_w \, dx \, dt = \int_{\Omega_T} F_{w}^\varepsilon \varphi_w \, dx \, dt \quad (2.25)
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega_T} \Phi^\varepsilon(x) S^0 \frac{\partial \varphi_n}{\partial t} \, dx \, dt + \int_{\Omega} \Phi^\varepsilon S^0 \varphi_n \, dx + \int_{\Omega_T} K^\varepsilon(x) \lambda_n \left( \frac{x}{\varepsilon}, S^\varepsilon \right) (\nabla p^\varepsilon_n - \bar{g}) \cdot \nabla \varphi_n \, dx \, dt = \int_{\Omega_T} F_{n}^\varepsilon \varphi_n \, dx \, dt \quad (2.26)
\end{align*}
\]

where $\varphi_w^0 \equiv \varphi_w(0, x)$, $\varphi_n^0 \equiv \varphi_n(0, x)$, and the function $S^0 = S^0(x)$ is defined by the initial condition (2.12) and the capillary pressure relation (2.7).

Let us also give an equivalent definition of a weak solution in terms of the global pressure and the saturation.

**Definition 2.2** (Weak solution in terms of global pressure and saturation) We say that the pair of functions $(S^\varepsilon, P^\varepsilon)$ is a weak solution of problem (2.19) if

(i) $0 \leq S^\varepsilon \leq 1$ a.e. in $\Omega_T$.

(ii) The global pressure function $P^\varepsilon \in L^2(0, T; H^1(\Omega^\varepsilon_T))$ and, for any $\varepsilon > 0$, the saturation function $S^\varepsilon_T$ is such that $\beta(\varepsilon_T) \in L^2(0, T; H^1(\Omega^\varepsilon_T))$.

(iii) The boundary conditions (2.20) and the initial conditions (2.24) are satisfied.

(iv) For any $\varphi_w, \varphi_n \in C^1([0, T]; H^1_\gamma(\Omega))$ satisfying $\varphi_w(T) = \varphi_n(T) = 0$, we have:
boundary conditions (2.11) after integration by parts, we get the following energy equality:

\[ - \int_{\Omega_T} \Phi^e(x) S^e \frac{\partial \phi^e}{\partial t} \, dx dt - \int_{\Omega} \Phi^e(x) S^o \phi^o \, dx + \int_{\Omega_{f,T}} K^e(x) \left\{ \lambda_{f,w}(S^e_f) (\nabla P^e_f - \bar{g}) + \nabla \beta_t(S^e_t) \right\} \cdot \nabla \phi^e \, dx dt + \]

\[ + \varepsilon(\varepsilon) \int_{\Omega_{m,T}} K^e(x) \left\{ \lambda_{m,w}(S^e_m) (\nabla P^e_m - \bar{g}) + \nabla \beta_m(S^e_m) \right\} \cdot \nabla \phi^e \, dx dt = \int_{\Omega_{f,T}} F^e_w \phi^e \, dx; \]

\[ \int_{\Omega_T} \Phi^e(x) S^e \frac{\partial \phi^e}{\partial t} \, dx dt + \int_{\Omega} \Phi^e(x) S^o \phi^o \, dx + \int_{\Omega_{f,T}} K^e(x) \left\{ \lambda_{f,n}(S^e_f) (\nabla P^e_f - \bar{g}) - \nabla \beta_t(S^e_t) \right\} \cdot \nabla \phi^e \, dx dt + \]

\[ + \varepsilon(\varepsilon) \int_{\Omega_{m,T}} K^e(x) \left\{ \lambda_{m,n}(S^e_m) (\nabla P^e_m - \bar{g}) - \nabla \beta_m(S^e_m) \right\} \cdot \nabla \phi^e \, dx dt = \int_{\Omega_{f,T}} F^e_n \phi^e \, dx. \]  

(2.27)

(2.28)

Existence theorem for the weak solutions defined in Definition 2.1 and Definition 2.2 is given in [10] in more general case of compressible fluids.

Notational convention. In what follows \( C, C_1, \ldots \) denote generic constants that do not depend on \( \varepsilon \).

3 A priori uniform estimates

The uniform estimates for the initial system (2.8) or the equivalent one (2.19) are given by the following lemma:

Lemma 3.1 Let \( (p^e_w, p^e_n, S^e) \) be a solution to problem (2.8). Then under assumptions (A.1)-(A.9) the following uniform in \( \varepsilon \) estimates hold true:

\[ \| \sqrt{\lambda_{f,w}} (S^e_f) \nabla p^e_{f,w} \|_{L^2(\Omega_{f,T})} + \| \sqrt{\lambda_{f,n}} (S^e_f) \nabla p^e_{f,n} \|_{L^2(\Omega_{f,T})} + \]

\[ + \varepsilon^2(\varepsilon) \| \sqrt{\lambda_{m,w}} (S^e_m) \nabla P^e_{m,w} \|_{L^2(\Omega_{m,T})} + \] \[ + \varepsilon^2(\varepsilon) \| \nabla \beta_t(S^e_t) \|_{L^2(\Omega_{f,T})} \leq C; \]  

(3.1)

\[ \| \nabla \beta_t(S^e_t) \|_{L^2(\Omega_{f,T})} \leq C, \]  

(3.2)

where \( \varepsilon(\varepsilon) \equiv \varepsilon^\theta \) with \( \theta > 0 \).

Proof of Lemma 3.1 Notice that the uniform boundedness results (3.1), (3.2) were already proved by many authors (see, e.g., [40] and the references therein) in the case when the source terms in (2.8) were assumed to be zero. We also refer here to [5] and the references therein, where the uniform boundedness results were obtained in the case of compressible two-phase flows in porous media. Here, for reader’s convenience, we recall the proof of the bounds (3.1), (3.2) focusing on the terms involving the source functions \( F^e_w, F^e_n \).

We start our analysis by obtaining the uniform bound (3.1). To this end we multiply the first equation in (2.8) by \( p^e_{w} \), the second equation in (2.8) by \( p^e_{n} \) and then integrate over the domain \( \Omega \). Taking into account the boundary conditions (2.11) after integration by parts, we get the following energy equality:

\[ - \frac{d}{dt} \int_{\Omega} \Phi^e(x) F(S^e) \, dx + \int_{\Omega} \left\{ K^e(x) \lambda_w \left( \frac{x}{\varepsilon}, S^e \right) (\nabla p^e_w - \bar{g}) \right\} \cdot \nabla p^e_w \, dx + \]

\[ + \int_{\Omega} \left\{ K^e(x) \lambda_n \left( \frac{x}{\varepsilon}, S^e \right) (\nabla p^e_n - \bar{g}) \right\} \cdot \nabla p^e_n \, dx = \int_{\Omega} \left[ F^e_w(x,t) p^e_w + F^e_n(x,t) p^e_n \right], \]

(3.3)
where

\[
F(S^\varepsilon) \overset{\text{def}}{=} F_f(S^\varepsilon) 1_f^\varepsilon(x) + F_m(S^\varepsilon_m) 1_m^\varepsilon(x) \overset{\text{def}}{=} \int F_{1,c}(u) \, du + \int F_{m,c}(u) \, du.
\]

The equality (3.3) is the desired energy equality which will be used below to obtain the necessary bounds that are uniform in \(\varepsilon\). To this end we integrate (3.3) over the interval \((0, T)\) to get:

\[
- \int_\Omega \Phi^\varepsilon(x) F(S^\varepsilon) \, dx + \int_{\Omega_T} \left\{ K^\varepsilon(x) \lambda_w \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \left( \nabla p_w^\varepsilon - \bar{g} \right) \right\} \cdot \nabla p_w^\varepsilon \, dxdt + \int_{\Omega_T} \left\{ K^\varepsilon(x) \lambda_n \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \left( \nabla p_n^\varepsilon - \bar{g} \right) \right\} \cdot \nabla p_n^\varepsilon \, dxdt = \mathcal{J}_{w,n}^\varepsilon - \int_\Omega \Phi^\varepsilon(x) F(S^\varepsilon(x, 0)) \, dx,
\]

where

\[
\mathcal{J}_{w,n}^\varepsilon \overset{\text{def}}{=} \int_{\Omega_T} \left[ F_w^\varepsilon(x, t) p_w^\varepsilon + F_n^\varepsilon(x, t) p_n^\varepsilon \right] \, dxdt.
\]

First, we notice that due to the positiveness of the porosity function \(\Phi^\varepsilon\) and the definition of the function \(F(S^\varepsilon)\) we have that the first term on the left-hand side of (3.5) is bounded from below by a constant which does not depend on \(\varepsilon\). It is also easy to see from conditions (A.1), (A.3) that the second term on the right-hand side of (3.5) is uniformly bounded in \(\varepsilon\). Then from (3.5) we get the following inequality:

\[
\int_{\Omega_T} K^\varepsilon(x) \lambda_w \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \nabla p_w^\varepsilon \cdot \nabla p_w^\varepsilon \, dxdt + \int_{\Omega_T} K^\varepsilon(x) \lambda_n \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \nabla p_n^\varepsilon \cdot \nabla p_n^\varepsilon \, dxdt \leq C + \int_{\Omega_T} K^\varepsilon(x) \lambda_w \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \bar{g} \cdot \nabla p_w^\varepsilon \, dxdt + \int_{\Omega_T} K^\varepsilon(x) \lambda_n \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \bar{g} \cdot \nabla p_n^\varepsilon \, dxdt + \mathcal{J}_{w,n}^\varepsilon.
\]

With the help of Young’s inequality the second and the third terms in the right-hand side of (3.7) can be absorbed by the first and second term in the left-hand side of (3.7). Namely, we get:

\[
\int_{\Omega_T} K^\varepsilon(x) \lambda_w \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \nabla p_w^\varepsilon \cdot \nabla p_w^\varepsilon \, dxdt + \int_{\Omega_T} K^\varepsilon(x) \lambda_n \left( \frac{x}{\varepsilon}, S^\varepsilon \right) \nabla p_n^\varepsilon \cdot \nabla p_n^\varepsilon \, dxdt \leq C [1 + \mathcal{J}_{w,n}^\varepsilon].
\]

Now it remains to estimate \(\mathcal{J}_{w,n}^\varepsilon\). Due to condition (A.9), it can be written as:

\[
\mathcal{J}_{w,n}^\varepsilon = \int_{\Omega_{I,T}} \left[ S_{f}^{I,T} f_I(x, t) - S_{f}^{I} f_P(x, t) \right] p_{f,w}^\varepsilon \, dxdt + \int_{\Omega_{I,T}} \left[ S_{f}^{I,n} f_I(x, t) - (1 - S_{f}^{I}) f_P(x, t) \right] p_{f,n}^\varepsilon \, dxdt \overset{\text{def}}{=} \mathcal{J}_{w}^\varepsilon + \mathcal{J}_{n}^\varepsilon.
\]

Consider, first, the term \(\mathcal{J}_{w}^\varepsilon\). From the boundedness of the saturation functions, Cauchy’s inequality and condition (A.9), we get:

\[
\left| \mathcal{J}_{w}^\varepsilon \right| \leq \left[ \| f_I \|_{L^2(\Omega_T)} + \| f_P \|_{L^2(\Omega_T)} \right] \| p_{f,w}^\varepsilon \|_{L^2(\Omega_{I,T})} \leq C_1 \| p_{f,w}^\varepsilon \|_{L^2(\Omega_{I,T})}.
\]

In a similar way,

\[
\left| \mathcal{J}_{n}^\varepsilon \right| \leq C_2 \| p_{f,n}^\varepsilon \|_{L^2(\Omega_{I,T})}.
\]
Now using condition (A.2), (3.9), (3.10), and (3.11), from the inequality (3.8), we get:

\[
\begin{align*}
\mathbb{L}^\varepsilon & \equiv k_{\min} \int_{\Omega_{i,T}^\varepsilon} \lambda_{\varepsilon,w}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,w}^f \right|^2 \, dx dt + k_{\min} \int_{\Omega_{i,T}^\varepsilon} \lambda_{\varepsilon,n}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,n}^f \right|^2 \, dx dt + \\
&+ \varepsilon(\varepsilon) k_{\min} \int_{\Omega_{m,T}^\varepsilon} \lambda_{m,w}(S_m^\varepsilon) \left| \nabla p_{m,w}^\varepsilon \right|^2 \, dx dt + \varepsilon(\varepsilon) k_{\min} \int_{\Omega_{m,T}^\varepsilon} \lambda_{m,n}(S_m^\varepsilon) \left| \nabla p_{m,n}^\varepsilon \right|^2 \, dx dt \\
&\leq C_3 \left[ 1 + \| p_{\varepsilon,w}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} + \| p_{\varepsilon,n}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} \right]. \\
\end{align*}
\]

(3.12)

Consider the right-hand side of (3.12). From (2.13) we have:

\[
\begin{align*}
\| p_{\varepsilon,w}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} + \| p_{\varepsilon,n}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} & \leq \\
&\leq \left[ \| P_{\varepsilon,w}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} + \| G_{\varepsilon,w}(S_f^\varepsilon) \|_{L^2(\Omega_{i,T}^\varepsilon)} + \| P_{\varepsilon,n}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} + \| G_{\varepsilon,n}(S_f^\varepsilon) \|_{L^2(\Omega_{i,T}^\varepsilon)} \right]. \\
\end{align*}
\]

(3.13)

Then, taking into account that the functions $G_{\varepsilon,w}(S_f^\varepsilon)$, $G_{\varepsilon,n}(S_f^\varepsilon)$ are uniformly bounded in $\varepsilon$, the inequality (3.12) takes the form:

\[
\mathbb{L}^\varepsilon \leq C_4 \left[ 1 + \| P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} \right]. \\
\]

(3.14)

Taking into account the boundary condition $P_{\varepsilon}^f = P_{\Gamma_1} = \text{Const}$ on $\Gamma_1 \times (0,T)$ and applying Friedrich’s inequality we obtain that

\[
\| P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} \leq C_5 \left[ 1 + \| \nabla P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} \right]. \\
\]

(3.15)

Finally, in view of (3.15), the inequality (3.14) takes the form:

\[
\begin{align*}
&\int_{\Omega_{i,T}^\varepsilon} \lambda_{\varepsilon,w}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,w}^f \right|^2 \, dx dt + \int_{\Omega_{i,T}^\varepsilon} \lambda_{\varepsilon,n}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,n}^f \right|^2 \, dx dt + \varepsilon(\varepsilon) \int_{\Omega_{m,T}^\varepsilon} \lambda_{m,w}(S_m^\varepsilon) \left| \nabla p_{m,w}^\varepsilon \right|^2 \, dx dt + \\
&+ \varepsilon(\varepsilon) \int_{\Omega_{m,T}^\varepsilon} \lambda_{m,n}(S_m^\varepsilon) \left| \nabla p_{m,n}^\varepsilon \right|^2 \, dx dt \leq C_6 \left[ 1 + \| \nabla P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} \right]. \\
\end{align*}
\]

(3.16)

In order to complete the derivation of the uniform estimate, we make use of the Cauchy inequality as follows:

\[
C_6 \| \nabla P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)} \leq C_6 \frac{\eta}{2} \| \nabla P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)}^2 + C_6 \frac{1}{2\eta}, \\
\]

where $\eta > 0$ is an arbitrary number. Moreover, it follows from (2.17) that

\[
\lambda_f(S_f^\varepsilon) \left| \nabla P_{\varepsilon}^f \right|^2 \leq \lambda_{f,n}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,n}^f \right|^2 + \lambda_{f,w}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,w}^f \right|^2. \\
\]

(3.18)

Now (3.17) allows us to rewrite (3.16) in the form:

\[
\begin{align*}
&\int_{\Omega_{i,T}^\varepsilon} \lambda_{\varepsilon,w}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,w}^f \right|^2 \, dx dt + \int_{\Omega_{i,T}^\varepsilon} \lambda_{\varepsilon,n}(S_f^\varepsilon) \left| \nabla p_{\varepsilon,n}^f \right|^2 \, dx dt + \varepsilon(\varepsilon) \int_{\Omega_{m,T}^\varepsilon} \lambda_{m,w}(S_m^\varepsilon) \left| \nabla p_{m,w}^\varepsilon \right|^2 \, dx dt + \\
&+ \varepsilon(\varepsilon) \int_{\Omega_{m,T}^\varepsilon} \lambda_{m,n}(S_m^\varepsilon) \left| \nabla p_{m,n}^\varepsilon \right|^2 \, dx dt \leq C_6 + C_6 \frac{\eta}{2} \| \nabla P_{\varepsilon}^f \|_{L^2(\Omega_{i,T}^\varepsilon)}^2 + C_6 \frac{1}{2\eta}. \\
\end{align*}
\]

(3.19)
Let us estimate the second term on the right-hand side of (3.19). From condition (A.4) and (3.18), we have:

\[
C_0 \frac{\eta}{2} \| \nabla P_0^e \|_{L^2(\Omega^e_{m,T})}^2 \leq C_0 \frac{\eta}{2L_0} \int_{\Omega^e_{m,T}} \lambda_S(S^e) \| \nabla P_0^e \|^2 \, dx dt
\]

\[
\leq C_0 \frac{\eta}{2L_0} \int_{\Omega^e_{m,T}} \left[ \lambda_{t,w}(S^e) \| \nabla p_{0,w}^e \|^2 + \lambda_{t,n}(S^e) \| \nabla p_{0,n}^e \|^2 \right] \, dx dt.
\]

(3.20)

We set \( \eta = \frac{L_0}{C_0} \) and, finally, obtain from (3.19) the desired inequality (3.1).

Now we turn to the uniform bound (3.2). It immediately follows from (3.1) equality (2.17) and the following inequality: \( \| \nabla \beta_{\ell}(S^e) \| \leq C \| \nabla b_{\ell}(S^e) \| \). This completes the proof of Lemma 3.1.

\( \square \)

**Lemma 3.2** Let \((p_0^e, p_{1,n}^e, S^e)\) be a solution to problem (2.8) and \(\varepsilon = \varepsilon^0\) with \(\theta \leq 2\). Then under assumptions (A.1)-(A.9) the following uniform in \(\varepsilon\) estimate holds true:

\[
\| P_{m}^e \|_{L^2(\Omega^e_{m,T})} \leq C.
\]

(3.21)

**Proof of Lemma 3.2** In contrast to the papers [16, 40], where the standing assumptions allow to prove the continuity of the global pressure on the interface \(\Sigma^e_{m,T}\), in our case the global pressure is discontinuous on \(\Sigma^e_{m,T}\). So the method which allowed to prove (3.21) by use of the extension operator from the subdomain \(\Omega^e_{m}\) to the whole \(\Omega^e\) cannot be applied here. To avoid this difficulty we make use of the ideas from [27] (see also [9]). Since \(P_{m}^e \in L^2(0, T; H^1(\Omega^e_{m}))\) and \(P_{0}^e - P_{1,n} \in L^2(0, T; H^1(\Omega^e_{m,T}))\), then we have:

\[
\| P_{m}^e \|_{L^2(\Omega^e_{m,T})} \leq C \left[ \varepsilon \| \nabla P_{m}^e \|_{L^2(\Omega^e_{m,T})} + \sqrt{\varepsilon} \| P_{m}^e \|_{L^2(\Sigma^e_{m,T})} \right].
\]

(3.22)

Then due to the definition of the global pressure \(P_{m}^e\), (2.13), and the interface condition (2.10) written in terms of the global pressure, one obtains the following estimate:

\[
\| P_{m}^e \|_{L^2(\Sigma^e_{m,T})} \leq \| P_{m}^e + G_{m,w}(S^e_m) \|_{L^2(\Sigma^e_{m,T})} + \| G_{m,w}(S^e_m) \|_{L^2(\Sigma^e_{m,T})} = \| P_{1}^e + G_{t,w}(S^e_0) \|_{L^2(\Sigma^e_{m,T})} + \| G_{t,w}(S^e_0) \|_{L^2(\Sigma^e_{m,T})}.
\]

(3.23)

Now, taking into account the boundedness of \(G_{t,w}(S^e_0)\), the geometry of \(\Omega^e_{m,T}\), (3.23), and the estimate:

\[
\sqrt{\varepsilon} \| P_{1}^e \|_{L^2(\Sigma^e_{m,T})} \leq C \left[ \varepsilon \| \nabla P_{1}^e \|_{L^2(\Omega^e_{m,T})} + \| P_{1}^e \|_{L^2(\Omega^e_{m,T})} \right].
\]

(3.24)

we obtain

\[
\| P_{m}^e \|_{L^2(\Omega^e_{m,T})} \leq C \left( \varepsilon \| \nabla P_{m}^e \|_{L^2(\Omega^e_{m,T})} + 1 \right) = C \left( \frac{\varepsilon}{\varepsilon^2} \right) \| \nabla P_{m}^e \|_{L^2(\Omega^e_{m,T})} + 1.
\]

(3.25)

By using (3.2), from (3.25) we get

\[
\| P_{m}^e \|_{L^2(\Omega^e_{m,T})} \leq C \left( \varepsilon \varepsilon^2 + 1 \right),
\]

(3.26)

which means that for \(\varepsilon = \varepsilon^0\) with \(\theta \leq 2\) the desired inequality (3.21) is obtained. Lemma 3.2 is proved.

\( \square \)

**Lemma 3.3** Let \((p_0^e, p_{1,n}^e, S^e)\) be a solution to problem (2.8). Then under assumptions (A.1)-(A.9) the following uniform in \(\varepsilon\) estimate holds true:

\[
\{ \partial_\ell \Phi^{e}_{0}(S^e) \}_{\varepsilon > 0} \text{ uniformly bounded in } L^2(0, T; H^{-1}(\Omega^e_{\ell})),
\]

(3.27)

where the functions \(\Phi^{e}_{0}, \Phi^{e}_{m}\) are defined in condition (A.1).
4 Compactness and convergence results

The outline of this section is as follows. First, in subsection 4.1 we extend the function \( S^\varepsilon_f \) from the subdomain \( \Omega^\varepsilon_f \) to the whole \( \Omega \) and obtain uniform estimates for the extended function \( \tilde{S}^\varepsilon_f \). Then in subsection 4.2, using the uniform estimates for the function \( \tilde{P}^\varepsilon_f \) and the corresponding bounds for \( S^\varepsilon_f \), we prove the compactness result for the family \( \{S^\varepsilon_f\}_{\varepsilon>0} \). Finally, in subsection 4.3 we formulate the two-scale convergence which will be used in the derivation of the homogenized system.

4.1 Extensions of the functions \( P^\varepsilon_f, S^\varepsilon_f \)

The goal of this subsection is to extend the functions \( P^\varepsilon_f, S^\varepsilon_f \) defined in the subdomain \( \Omega^\varepsilon_f \) to the whole \( \Omega \) and derive the uniform in \( \varepsilon \) estimates for the extended functions.

Extension of the function \( P^\varepsilon_f \). First, we introduce the extension operator from the subdomain \( \Omega^\varepsilon_f \) to the whole \( \Omega \). Taking into account the results of [11], we conclude that there exists a linear continuous extension operator \( \Pi^\varepsilon : H^1(\Omega^\varepsilon_f) \rightarrow H^1(\Omega) \) such that: (i) \( \Pi^\varepsilon u = u \) in \( \Omega^\varepsilon_f \) and (ii) for any \( u \in H^1(\Omega^\varepsilon_f) \),

\[
\|\Pi^\varepsilon u\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega^\varepsilon_f)} \quad \text{and} \quad \|\nabla (\Pi^\varepsilon u)\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega^\varepsilon_f)},
\]

where \( C \) is a constant that does not depend on \( u \) and \( \varepsilon \). Now it follows from (3.2) and the Dirichlet boundary condition on \( \Gamma_1 \), that

\[
\|\nabla (\Pi^\varepsilon P^\varepsilon_f)\|_{L^2(\Omega_T)} + \|\Pi^\varepsilon P^\varepsilon_f\|_{L^2(\Omega_T)} \leq C.
\]

Notational convention. In what follows the extension of any function \( f \) will be denoted by \( \tilde{f} \) instead of \( \Pi^\varepsilon f \).

Extension of the function \( S^\varepsilon_f \). In order to extend \( S^\varepsilon_f \), following the ideas of [16], we make use of the function \( \beta_f \) defined in (2.16). It is evident that \( \beta_f \) is a monotone function of \( s \). Let us introduce the function:

\[
\beta^\varepsilon_f(x,t) := \beta_f(\tilde{S}^\varepsilon_f) = \int_0^{\tilde{S}^\varepsilon_f} \alpha_f(u) \, du.
\]

Then it follows from condition (A.5) that

\[
0 \leq \beta^\varepsilon_f \leq \max_{s \in [0,1]} \alpha_f(s) \quad \text{a.e. in } \Omega^\varepsilon_f, \tag{4.4}
\]

It is also clear from (3.2) that

\[
\|\nabla \beta^\varepsilon_f\|_{L^2(\Omega^\varepsilon_f,T)} \leq C. \tag{4.5}
\]

Hence,

\[
0 \leq \beta^\varepsilon_f \leq \|\Pi^\varepsilon \beta^\varepsilon_f\|_{L^2(\Omega^\varepsilon_f,T)} \leq \max_{s \in [0,1]} \alpha_f(s) \quad \text{a.e. in } \Omega_T \quad \text{and} \quad \|\nabla \beta^\varepsilon_f\|_{L^2(\Omega_T)} \leq C. \tag{4.6}
\]

Now we can extend \( S^\varepsilon_f \) from \( \Omega^\varepsilon_f \) to the whole \( \Omega \). We denote this extension by \( \tilde{S}^\varepsilon_f \) and define it as follows:

\[
\tilde{S}^\varepsilon_f \defeq (\beta_f)^{-1}(\beta^\varepsilon_f). \tag{4.7}
\]

This implies that

\[
\int_{\Omega_T} |\nabla \beta_f(\tilde{S}^\varepsilon_f)|^2 \, dx \, dt = \int_{\Omega_T} |\nabla \beta^\varepsilon_f|^2 \, dx \, dt \leq C \quad \text{and} \quad 0 \leq \tilde{S}^\varepsilon_f \leq 1 \quad \text{a.e. in } \Omega_T. \tag{4.8}
\]
4.2 Compactness results for the sequence \( \{ \tilde{S}^\varepsilon_f \}_{\varepsilon > 0} \)

In this subsection we establish the compactness and corresponding convergence results for the sequence \( \{ \tilde{S}^\varepsilon_f \}_{\varepsilon > 0} \) constructed in the previous section.

**Proposition 4.1** Under our standing assumptions there is a function \( S \) such that \( 0 \leq S \leq 1 \) in \( \Omega_T \) and (up to a subsequence)

\[
\tilde{S}^\varepsilon_f \rightarrow S \quad \text{strongly in } L^q(\Omega_T) \quad \text{for all } 1 \leq q < +\infty.
\] (4.9)

**Proof of Proposition 4.1** In the proof of Proposition 4.1 we follow the lines of [16, 40]. Namely, first, we establish the modulus of continuity in time for \( \tilde{\beta}^\varepsilon_f \) and then apply the compactness result from [37]. The derivation of the modulus of continuity in time is based on the lemma obtained earlier in [40], (see also [6]).

**Lemma 4.1** For \( h \) sufficiently small, we have:

\[
\int \int \frac{T}{h} [S^\varepsilon_f(t) - S^\varepsilon_f(t - h)] [\beta^\varepsilon_f(t) - \beta^\varepsilon_f(t - h)] dx \, dt \leq C h \quad \text{with } \beta^\varepsilon_f \overset{\text{def}}{=} \beta_f(S^\varepsilon_f),
\] (4.10)

where \( C \) is a constant that does not depend on \( \varepsilon, h \).

**Corollary 4.2** For \( h \) sufficiently small, we have:

\[
\int \int \frac{T}{h} \left| \tilde{\beta}^\varepsilon_f(t) - \tilde{\beta}^\varepsilon_f(t - h) \right|^2 dx \, dt \leq C h \quad \text{with } \Omega^h_T \overset{\text{def}}{=} \Omega \times (h, T).
\] (4.11)

**Proof of Corollary 4.2** First, let us show that the bound (4.10) implies:

\[
\int \int \frac{T}{h} \left| \tilde{\beta}^\varepsilon_f(t) - \tilde{\beta}^\varepsilon_f(t - h) \right|^2 dx \, dt \leq C h.
\] (4.12)

In fact, it is clear that due to the definition of the function \( \beta_f \) and condition (A.6) we have:

\[
|\beta_f(S^\varepsilon_f(t)) - \beta_f(S^\varepsilon_f(t - h))| = \left| \int_{S^\varepsilon_f(t)(t - h)} \alpha_f(\xi) d\xi \right| \leq \max_{s \in [0,1]} \alpha_f(s) |S^\varepsilon_f(t) - S^\varepsilon_f(t - h)|.
\]

Then from (4.10) we get:

\[
\int \int \frac{T}{h} \left| \tilde{\beta}^\varepsilon_f(t) - \tilde{\beta}^\varepsilon_f(t - h) \right|^2 dx \, dt \leq C \int \int \frac{T}{h} \left[ S^\varepsilon_f(t) - S^\varepsilon_f(t - h) \right] \left[ \beta^\varepsilon_f(t) - \beta^\varepsilon_f(t - h) \right] dx \, dt \leq C h
\]

and the desired bound (4.12) is obtained.

Now using the property (4.1) of the extension operator, from (4.12) we get (4.11). This completes the proof of Corollary 4.2.

Now we are in position to complete the proof of Proposition 4.1. First, we observe that the sequence \( \{ \tilde{\beta}^\varepsilon_f \}_{\varepsilon > 0} \) is uniformly bounded in the space \( L^2(0, T; H^1_{\Gamma, 1}(\Omega)) \) and this sequence satisfies (4.11). Then it follows
from \[37\] that $\left\{ \tilde{\beta}_T^\varepsilon \right\}_{\varepsilon > 0}$ is a compact set in the space $L^2(\Omega_T)$ and we have that $\tilde{\beta}_T^\varepsilon \rightarrow \beta^*$ strongly in $L^2(\Omega_T)$ and due to the uniform boundedness of the function $\tilde{\beta}_T^\varepsilon$ in the space $L^\infty(\Omega_T)$,

$$\tilde{\beta}_T^\varepsilon \rightarrow \beta^* \text{ strongly in } L^q(\Omega_T) \text{ for all } 1 \leq q < +\infty. \quad (4.13)$$

Now we recall that the extended saturation function $S_T^\varepsilon$ is defined by $S_T^\varepsilon \overset{\text{def}}{=} (\beta_T^\varepsilon)^{-1}(\tilde{\beta}_T^\varepsilon)$. We set

$$S \overset{\text{def}}{=} (\beta_T)^{-1}(\beta^*). \quad (4.14)$$

Then from condition (A.6) we have:

$$\|S_T^\varepsilon - S\|_{L^q(\Omega_T)} = \|(\beta_T)^{-1}(\tilde{\beta}_T^\varepsilon) - (\beta_T)^{-1}(\beta^*)\|_{L^q(\Omega_T)} \leq C \|\tilde{\beta}_T^\varepsilon - \beta^*\|_{L^r(\Omega_T)}.$$ 

This inequality along with (4.13) implies (4.9) and Proposition 4.1 is proved. \(\square\)

### 4.3 Two-scale convergence results

In this subsection, taking into account the compactness results from the previous section, we formulate the convergence results for the sequences $\left\{ P_T^\varepsilon \right\}_{\varepsilon > 0}$, $\left\{ S_T^\varepsilon \right\}_{\varepsilon > 0}$. In this paper the homogenization process for the problem is rigorously obtained by using the two-scale approach, see, e.g., [3]. For the reader's convenience, let us recall the definition of the two-scale convergence.

**Definition 4.3** A sequence of functions $\left\{ v^\varepsilon \right\}_{\varepsilon > 0} \subset L^2(\Omega_T)$ two-scale converges to $v \in L^2(\Omega_T \times Y)$ if $\|v^\varepsilon\|_{L^2(\Omega_T)} \leq C$, and for any test function $\varphi \in C^\infty(\overline{\Omega_T}; C_\#(Y))$ the following relation holds:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} v^\varepsilon(x, t) \varphi \left( x, \frac{x}{\varepsilon}, t \right) \, dx \, dt = \int_{\Omega_T \times Y} v(x, y, t) \varphi(x, y, t) \, dy \, dx \, dt.$$ 

This convergence is denoted by $v^\varepsilon(x, t) \overset{2s}{\rightharpoonup} v(x, y, t)$.

Following [4] we also introduce the two-scale convergence on periodic surfaces:

**Definition 4.4** A sequence of functions $\left\{ v^\varepsilon \right\}_{\varepsilon > 0} \subset L^2(\Sigma_T^\varepsilon)$ two-scale converges to $v \in L^2(\Omega_T; L^2(\Gamma fm))$ on $\Gamma fm$ if for any test function $\varphi \in C^\infty(\overline{\Omega_T}; C_\#(Y))$ the following relation holds:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_T^\varepsilon} v^\varepsilon(x, t) \varphi \left( x, \frac{x}{\varepsilon}, t \right) \, dH^{d-1}(x) \, dt = \int_{\Omega_T \times \Gamma fm} v(x, y, t) \varphi(x, y, t) \, dH^{d-1}(y) \, dx \, dt,$$

where, as before $\Sigma_T^\varepsilon \overset{\text{def}}{=} \Gamma fm \times (0, T)$, and $dH^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

This convergence is denoted by $v^\varepsilon(x, t) \overset{2s-\Gamma fm}{\rightharpoonup} v(x, y, t)$.

Now we summarize the convergence results for the sequences $\left\{ P_T^\varepsilon \right\}_{\varepsilon > 0}$ and $\left\{ S_T^\varepsilon \right\}_{\varepsilon > 0}$. We have:

**Lemma 4.2** For any rate of contrast there exist a function $S$ such that $0 \leq S \leq 1$ a.e. in $\Omega_T$, $\beta_T(S) - \beta_T(1) \in L^2(0, T; H_1^1(\Omega))$, and functions $P - P_{\Gamma 1} \in L^2(0, T; H_1^1(\Omega))$, $w_p, w_s \in L^2(\Omega_T; H_{per}^1(Y))$ such that up to a subsequence:

$$\tilde{S}_T^\varepsilon(x, t) \rightarrow S(x, t) \text{ strongly in } L^q(\Omega_T) \text{ for all } 1 \leq q < +\infty; \quad (4.15)$$
\[ \tilde{P}_f(x,t) \rightarrow P(x,t) \text{ weakly in } L^2(0,T;H^1(\Omega)); \]  
(4.16)  
\[ \nabla \tilde{P}_f(x,t) \xrightarrow{2 \times} \nabla P(x,t) + \nabla_y w_p(x,t,y); \]  
(4.17)  
\[ \beta_t(S_f^\varepsilon) \rightarrow \beta_t(S) \text{ strongly in } L^q(\Omega_T) \forall 1 \leq q < +\infty; \]  
(4.18)  
\[ \nabla \beta_t(S_f^\varepsilon)(x,t) \xrightarrow{2 \times} \nabla \beta_t(S)(x,t) + \nabla_y w_q(x,t,y); \]  
(4.19)  
\[ \tilde{P}_f(x,t) \xrightarrow{2^{n-1}\Gamma_{mf}} P(x,t); \]  
(4.20)  
\[ \beta_t(S_f^\varepsilon(x,t)) \xrightarrow{2^{n-1}\Gamma_{mf}} \beta_t(S(x,t)). \]  
(4.21)

The **Proof of Lemma 4.2** is based on the *a priori estimates* for the functions \( \beta_t(S_f^\varepsilon) \) and \( P_f \) obtained in Section 3, the extension results from Subsection 4.1, and Proposition 4.1. The two-scale convergence results (4.17) and (4.19) are obtained by arguments similar to those in [3]. The two-scale convergence (4.20) and (4.21) can be proved by applying Proposition 2.6 in [4]. Lemma 4.2 is proved.

Note also that the notion of strong two-scale convergence on periodic surfaces can be introduced in analogy with the ordinary strong two-scale convergence.

**Definition 4.5** A sequence \( \{v^\varepsilon\}_{\varepsilon>0} \subset L^2(\Sigma_f^\varepsilon) \) converges the two-scale strongly to \( v \in L^2(\Omega_T;L^2(\Gamma_{mf})) \) on \( \Gamma_{mf} \) if
\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma_f^\varepsilon} |v^\varepsilon(x,t) - v\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)|^2 dH^{d-1}(x) \, dt = 0.
\]

It is easy to verify that the strong two-scale convergence on periodic surfaces implies the two-scale convergence on periodic surfaces with the same limit.

Using the strong convergence (4.18) and the boundedness of \( \nabla \beta_t(S_f^\varepsilon) \) given in Lemma 3.1, we get:
\[
\varepsilon \|\beta_t(S_f^\varepsilon) - \beta_t(S)\|^2_{L^2(\Sigma_f^\varepsilon)} \leq C \left[ \varepsilon^2 \|\nabla \beta_t(S_f^\varepsilon) - \nabla \beta_t(S)\|^2_{L^2(\Omega_{mf,T})} + \|\beta_t(S_f^\varepsilon) - \beta_t(S)\|^2_{L^2(\Omega_{mf,T})} \right],
\]
which tends to zero on a given subsequence as \( \varepsilon \to 0 \). Therefore, we conclude that the sequence \( \{\beta_t(S_f^\varepsilon)\}_{\varepsilon>0} \) converges strongly two-scale on the surface \( \Gamma_{mf} \) to \( \beta_t(S) \). Furthermore, we have:

**Lemma 4.3** Let \( \{\beta_t(S_f^\varepsilon)\} \) be a subsequence from Lemma 4.2 Then for any Lipschitz function \( M : [0,\beta_t(1)] \to \mathbb{R} \) the sequence \( \{M(\beta_t(S_f^\varepsilon))\}_{\varepsilon>0} \) converges strongly two-scale on the surface \( \Gamma_{mf} \) to \( M(\beta_t(S)) \).

Lemma 4.3 follows immediately from the estimate
\[
\|M(\beta_t(S_f^\varepsilon)) - M(\beta_t(S))\|^2_{L^2(\Sigma_f^\varepsilon)} \leq L_M^2 \|\beta_t(S_f^\varepsilon) - \beta_t(S)\|^2_{L^2(\Sigma_f^\varepsilon)},
\]
where \( L_M \) is the Lipschitz constant which does not depend on \( \varepsilon \).

## 5 Dilation operator and convergence results

It is known that due to the nonlinearities and the strong coupling of the problem, the two-scale convergence does not provide an explicit form for the source terms appearing in the homogenized model, see for instance [16, 23, 40]. To overcome this difficulty the authors make use of the dilation operator. Here we refer to [13, 16, 23, 40] for the definition and main properties of the dilation operator. Let us also notice that the notion
of the dilation operator is closely related to the notion of the unfolding operator. We refer here, e.g., to [25] for the definition and the properties of this operator.

The outline of this section is as follows. First, in subsection 5.1, we introduce the definition of the dilation operator and describe its main properties. Then in subsection 5.2, we obtain the equations for the dilated saturation and the global pressure functions and the corresponding uniform estimates. Finally, in subsection 5.3, we consider the convergence results for the dilated functions.

5.1 Definition and preliminary results

**Definition 5.1** For a given \( \varepsilon > 0 \), we define a dilation operator \( \mathcal{D}^\varepsilon \) mapping measurable functions defined in \( \Omega_{m,T}^\varepsilon \) to measurable functions defined in \( \Omega_T \times Y_m \) by

\[
(\mathcal{D}^\varepsilon \varphi)(x,y,t) \overset{\text{def}}{=} \begin{cases} 
\varphi(x+y,t), & \text{if } x+y \in \Omega_{m}^\varepsilon; \\
0, & \text{elsewhere,}
\end{cases}
\]

where \( c^\varepsilon(x) \overset{\text{def}}{=} k \) if \( x \in \varepsilon(Y + k) \) with \( k \in \mathbb{Z}^d \) denotes the lattice translation point of the \( \varepsilon \)-cell domain containing \( x \).

The basic properties of the dilation operator are given by the following lemma (see [13, 40]).

**Lemma 5.1** Let \( \varphi, \psi \in L^2(0, T; H^1(\Omega^\varepsilon_m)) \). Then we have:

\[
\nabla_y \mathcal{D}^\varepsilon \varphi = \varepsilon \mathcal{D}^\varepsilon (\nabla_x \varphi) \quad \text{a.e. in } \Omega_T \times Y_m;
\]

\[
|| \mathcal{D}^\varepsilon \varphi ||_{L^2(\Omega_T \times Y_m)} = || \varphi ||_{L^2(\Omega_{m,T}^\varepsilon)}; \quad || \nabla_y \mathcal{D}^\varepsilon \varphi ||_{L^2(\Omega_T \times Y_m)} = \varepsilon || \mathcal{D}^\varepsilon \nabla_x \varphi ||_{L^2(\Omega_{m,T}^\varepsilon)} = \varepsilon || \nabla_x \varphi ||_{L^2(\Omega_{m,T}^\varepsilon)};
\]

\[
(\mathcal{D}^\varepsilon \varphi, \mathcal{D}^\varepsilon \psi)_{L^2(\Omega_T \times Y_m)} = (\varphi, \psi)_{L^2(\Omega_{m,T}^\varepsilon)}.
\]

The following lemma gives the link between the two-scale and the weak convergence (see, e.g., [16]).

**Lemma 5.2** Let \( \{ \varphi^\varepsilon \}_{\varepsilon>0} \) be a uniformly bounded sequence in \( L^2(\Omega_{m,T}^\varepsilon) \) satisfying: (i) \( \mathcal{D}^\varepsilon \varphi^\varepsilon \rightharpoonup \varphi^0 \) weakly in \( L^2(\Omega_T; L^2_{\text{per}}(Y_m)) \); (ii) \( 1_m(x)\varphi^\varepsilon \overset{2s}{\rightharpoonup} \varphi^* \in L^2(\Omega_T; L^2_{\text{per}}(Y_m)) \). Then \( \varphi^0 = \varphi^* \) a.e. in \( \Omega_T \times Y_m \).

Finally, we also have the following result (see, e.g., [23, 40]).

**Lemma 5.3** If \( \varphi^\varepsilon \in L^2(\Omega_{m,T}^\varepsilon) \) and \( 1_m(x)\varphi^\varepsilon \overset{2s}{\rightharpoonup} \varphi \in L^2(\Omega_T; L^2_{\text{per}}(Y_m)) \) then \( \mathcal{D}^\varepsilon \varphi^\varepsilon \) converges to \( \varphi \) strongly in \( L^2(\Omega_T \times Y_m) \). Here \( \overset{2s}{\rightharpoonup} \) denotes the strong two-scale convergence. If \( \varphi \in L^2(\Omega_T) \) is considered as an element of \( L^2(\Omega_T \times Y_m) \) constant in \( y \), then \( \mathcal{D}^\varepsilon \varphi \) converges strongly to \( \varphi \) in \( L^2(\Omega_T \times Y_m) \).

The dilation operator shows the same properties with respect to the two-scale convergence on periodic surfaces. For a given function \( v \in L^2(\Sigma_T^\varepsilon) \) and from definition of the dilation operator we have \( \mathcal{D}^\varepsilon(v) \in L^2(\Omega_T; L^2(\Gamma_m)) \) and

\[
\sqrt{\varepsilon} ||v||_{L^2(\Sigma_T^\varepsilon)} = ||\mathcal{D}^\varepsilon(v)||_{L^2(\Omega_T; L^2(\Gamma_m))}.
\]

We have also the following lemma:
Lemma 5.4  If \( \{ v^\varepsilon \}_{\varepsilon > 0} \subset L^2(\Sigma_T^\varepsilon) \) is a sequence that converges to \( v \in L^2(\Omega_T; L^2(\Gamma_{fm})) \) in the two-scale sense on \( \Gamma_{fm} \), then the sequence \( \{ \mathcal{D}^\varepsilon(v^\varepsilon) \}_{\varepsilon > 0} \) converges weakly to the same limit, that is \( \mathcal{D}^\varepsilon(v^\varepsilon) \rightharpoonup v \) in \( L^2(\Omega_T; L^2(\Gamma_{fm})) \). If \( \{ v^\varepsilon \}_{\varepsilon > 0} \subset L^2(\Sigma_T^\varepsilon) \) converges strongly to \( v \in L^2(\Omega_T; L^2(\Gamma_{fm})) \) in the two-scale sense on \( \Gamma_{fm} \), then the sequence \( \{ \mathcal{D}^\varepsilon(v^\varepsilon) \}_{\varepsilon > 0} \) converges strongly to the same limit in \( L^2(\Omega_T; L^2(\Gamma_{fm})) \).

Due to Lemma 4.3 one can apply Lemma 5.4 to the sequence \( \{ \mathcal{M}(\beta_\varepsilon(\tilde{S}_t^\varepsilon)) \}_{\varepsilon > 0} \) and find a subsequence, such that
\[
\int_{\Omega_T} \int_{\Gamma_{fm}} \left| \mathcal{M}(\beta_\varepsilon(\mathcal{D}^\varepsilon(\tilde{S}_t^\varepsilon))) - \mathcal{M}(\beta_\varepsilon(S)) \right|^2 dH^{d-1}(y) \, dx \, dt \to 0
\]
when \( \varepsilon \to 0 \), for any Lipschitz function \( \mathcal{M} \). As a consequence we have.

Corollary 5.2 Let \( \mathcal{M} : [0, \beta_\varepsilon(1)] \to \mathbb{R} \) be a Lipschitz function. Then there is a subsequence \( \varepsilon = \varepsilon_k \) of the sequence \( \{ \mathcal{M}(\beta_\varepsilon(\tilde{S}_t^\varepsilon)) \}_{\varepsilon > 0} \), still denoted by \( \varepsilon \), such that for a.e. \( x \in \Omega \)
\[
\int_0^T \int_{\Gamma_{fm}} \left| \mathcal{M}(\beta_\varepsilon(\mathcal{D}^\varepsilon(\tilde{S}_t^\varepsilon)(x, y, t))) - \mathcal{M}(\beta_\varepsilon(S(x, y, t))) \right|^2 dH^{d-1}(y) dt \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

5.2 The dilated functions \( \mathcal{D}^\varepsilon S^\varepsilon_m, \mathcal{D}^\varepsilon P^\varepsilon_m \) and their properties

In this section we derive the equations for the dilated functions \( \mathcal{D}^\varepsilon S^\varepsilon_m, \mathcal{D}^\varepsilon P^\varepsilon_m \) and obtain the corresponding uniform estimates. In what follows we also make use of the notation:
\[
\mathcal{D}^\varepsilon S^\varepsilon_m \overset{\text{def}}{=} s^\varepsilon_m \quad \text{and} \quad \mathcal{D}^\varepsilon P^\varepsilon_m \overset{\text{def}}{=} p^\varepsilon_m.
\]
The equations for the dilated functions \( s^\varepsilon_m, p^\varepsilon_m \) are given by the following lemma.

Lemma 5.5 For \( x \in \Omega \), the functions \( s^\varepsilon_m, p^\varepsilon_m \) satisfy the following system of equations:
\[
\begin{align*}
\Phi_m(y) \frac{\partial s^\varepsilon_m}{\partial t} - \frac{\kappa(\varepsilon)}{\varepsilon^2} \text{div}_y \left\{ K(x, y) \left[ \lambda_{m,w}(s^\varepsilon_m) \nabla_y p^\varepsilon_m + \nabla_y \beta_m(s^\varepsilon_m) - \varepsilon \lambda_{m,w}(s^\varepsilon_m) \tilde{g} \right] \right\} &= 0; \\
\Phi_m(y) \frac{\partial \tilde{s}^\varepsilon_m}{\partial t} - \frac{\kappa(\varepsilon)}{\varepsilon^2} \text{div}_y \left\{ K(x, y) \left[ \lambda_{m,n}(s^\varepsilon_m) \nabla_y \tilde{p}^\varepsilon_m - \nabla_y \beta_m(s^\varepsilon_m) - \varepsilon \lambda_{m,n}(s^\varepsilon_m) \tilde{g} \right] \right\} &= 0,
\end{align*}
\]
in the space \( L^2(0, T; H^{-1}(Y_m)) \).

The Proof of Lemma 5.5 is given in [16, 40].

The system of equations (5.3)-(5.4) is provided with the following boundary conditions:
\[
\beta_m(s^\varepsilon_m) = \mathcal{M}(\beta_\varepsilon(\mathcal{D}^\varepsilon \tilde{S}_t^\varepsilon)) \quad \text{on} \quad \Gamma_{fm}
\]
for \( (x, t) \in \Omega_m^\varepsilon \times (0, T) \), where
\[
\mathcal{M} \overset{\text{def}}{=} \beta_\varepsilon \circ (P_{m,c})^{-1} \circ P_{\varepsilon,c} \circ (\beta_\varepsilon)^{-1}.
\]
Note that under our hypothesis function \( \mathcal{M} \) is Lipschitz continuous. We also have
\[
p^\varepsilon_m + G_{m,w}(s^\varepsilon_m) = \mathcal{D}^\varepsilon P^\varepsilon_m + G_{\varepsilon}(\mathcal{D}^\varepsilon \tilde{S}_t^\varepsilon) \quad \text{and} \quad p^\varepsilon_m + G_{m,n}(s^\varepsilon_m) = \mathcal{D}^\varepsilon P^\varepsilon_m + G_{\varepsilon,n}(\mathcal{D}^\varepsilon \tilde{S}_t^\varepsilon)
\]
(5.7)
on $\Gamma_{\text{m}}$ for $(x,t) \in \Omega_{m}^{\epsilon} \times (0,T)$.

The initial conditions are

$$s_{m}^{\epsilon}(x,y,0) = (\mathcal{D}^{\epsilon}S_{m}^{0})(x,y) \quad \text{and} \quad p_{m}^{\epsilon}(x,y,0) = (\mathcal{D}^{\epsilon}P_{m}^{0})(x,y) \quad \text{in} \quad \Omega_{m}^{\epsilon} \times Y_{m},$$

(5.8)

where $S_{m}^{0}, P_{m}^{0}$ are the restrictions to the domain $\Omega_{m}^{\epsilon}$ of the functions $S^{0}, P^{0}$ defined in (2.24) and the dilations of the functions defined on the fracture system can be defined in a way similar to one already used for the functions defined on the matrix part.

Now we establish a priori estimates for the functions $s_{m}^{\epsilon}, p_{m}^{\epsilon}$. They are given by the following lemma.

**Lemma 5.6** Let $(s_{m}^{\epsilon}, p_{m}^{\epsilon})$ be a solution to problem (5.3)-(5.4). Then:

(i) For any rate of contrast $(\theta > 0)$,

$$0 \leq s_{m}^{\epsilon} \leq 1 \quad \text{a.e. in} \quad \Omega_{T} \times Y_{m};$$

(5.9)

$$\|\partial_{t}(\Phi_{m} s_{m}^{\epsilon})\|_{L^{2}(\Omega_{T}; H_{\text{per}}^{1}(Y_{m}))} \leq C.$$  \hspace{1cm} (5.10)

(ii) For the high contrast in the critical case $(\theta = 2)$,

$$\|\nabla_{y} \beta_{m}(s_{m}^{\epsilon})\|_{L^{2}(\Omega_{T}; L_{\text{per}}^{2}(Y_{m}))} \leq C;$$

(5.11)

$$\|p_{m}^{\epsilon}\|_{L^{2}(\Omega_{T}; L_{\text{per}}^{2}(Y_{m}))} \leq C.$$  \hspace{1cm} (5.12)

(iii) For the moderate contrast $(0 < \theta < 2)$,

$$\varepsilon^{\frac{\theta-1}{2}} \|\nabla_{y} \beta_{m}(s_{m}^{\epsilon})\|_{L^{2}(\Omega_{T}; L_{\text{per}}^{2}(Y_{m}))} + \|\beta_{m}(s_{m}^{\epsilon})\|_{L^{2}(\Omega_{T}; L_{\text{per}}^{2}(Y_{m}))} \leq C;$$

(5.13)

$$\varepsilon^{\frac{\theta-1}{2}} \|\nabla_{y} p_{m}^{\epsilon}\|_{L^{2}(\Omega_{T}; L_{\text{per}}^{2}(Y_{m}))} + \|p_{m}^{\epsilon}\|_{L^{2}(\Omega_{T}; L_{\text{per}}^{2}(Y_{m}))} \leq C.$$  \hspace{1cm} (5.14)

**Proof of Lemma 5.6** Statement (5.9) is evident. The bound (5.10) with $\Phi_{m} = \Phi_{m}(y)$ follow from Lemma 3.3 and Lemma 5.1. The uniform estimates for $p_{m}^{\epsilon}$ in (5.12) and (5.14) follow from the uniform bound (3.21) and Lemma 5.1. The uniform estimates for the gradients of the functions $\beta_{m}(s_{m}^{\epsilon})$ and $p_{m}^{\epsilon}$ easy follow from the uniform bounds (3.2) and Lemma 5.1. Lemma 5.6 is proved. \hfill $\Box$

**Remark 2** Notice that in what follows we do not need the uniform estimates for the dilated functions in the case of the very high contrast.

### 5.3 Convergence results for the dilated functions

In this subsection we establish convergence results which will be used below to obtain the homogenized system. From Lemmas 5.2, 5.6 we get the following convergence results.

**Lemma 5.7** Let $(s_{m}^{\epsilon}, p_{m}^{\epsilon})$ be a solution to problem (5.3)-(5.4), (5.5)-(5.8). Then (up to a subsequence),
(i) For the high contrast in the critical case \((\theta = 2)\),
\[
1^\varepsilon_m(x)S^\varepsilon_m \xrightarrow{2} s \in L^2(\Omega_T; L^2_{\text{per}}(Y_m)) \quad \text{and} \quad s^\varepsilon_m \rightharpoonup s \text{ weakly in } L^2(\Omega_T \times Y_m);
\]
\[
1^\varepsilon_m(x)P^\varepsilon_m \xrightarrow{2} p \in L^2(\Omega_T; L^2_{\text{per}}(Y_m)) \quad \text{and} \quad p^\varepsilon_m \rightharpoonup p \text{ weakly in } L^2(\Omega_T; H^1(Y_m));
\]
\[
1^\varepsilon_m(x)\nabla_x P^\varepsilon_m \xrightarrow{2} \nabla_y p \in L^2(\Omega_T; L^2_{\text{per}}(Y_m));
\]
\[
1^\varepsilon_m(x)\beta_m(S^\varepsilon_m) \xrightarrow{2} \beta^* \quad \text{and} \quad \beta_m(s^\varepsilon_m) \rightarrow \beta^* \text{ weakly in } L^2(\Omega_T; H^1(Y_m));
\]
\[
1^\varepsilon_m(x)\nabla_x \beta_m(S^\varepsilon_m) \xrightarrow{2} \nabla_y \beta^*.
\]

(ii) For the very high contrast \((\theta > 2)\),
\[
1^\varepsilon_m(x)S^\varepsilon_m \xrightarrow{2} s \in L^2(\Omega_T; L^2_{\text{per}}(Y_m)).
\]

(iii) For the moderate contrast \((0 < \theta < 2)\),
\[
1^\varepsilon_m(x)S^\varepsilon_m \xrightarrow{2} s \in L^2(\Omega_T; L^2_{\text{per}}(Y_m)) \quad \text{and} \quad s^\varepsilon_m \rightharpoonup s \text{ weakly in } L^2(\Omega_T \times Y_m);
\]
\[
1^\varepsilon_m(x)\beta_m(S^\varepsilon_m) \xrightarrow{2} \beta^*_1 \quad \text{and} \quad 1^\varepsilon_m(x)\varepsilon^\theta \nabla \beta_m(S^\varepsilon_m) \xrightarrow{2} \beta_1;
\]
\[
\beta_m(s^\varepsilon_m) \rightarrow \beta^*_1 \text{ weakly in } L^2(\Omega_T; H^1(Y_m)).
\]

It is important to notice that the convergence results of Lemma \[5.7\] are not sufficient for derivation of the equations for the limit functions \((s, p)\) which involve only these functions and not the undefined limits appearing in \((5.18)\), \((5.19)\), \((5.22)\) and \((5.23)\). In order to overcome this difficulty, we introduce the restrictions of the functions \(s^\varepsilon_m, p^\varepsilon_m\) which are defined below. For these functions we obtain more estimates which allow us to obtain the desired equations. For this, we make use of the density arguments. Namely, following \[23\] (see also \[4\]), we fix \(x \in \Omega\) and define the restrictions of \(s^\varepsilon_m, p^\varepsilon_m\) to the \(\varepsilon\)-cell containing the point \(x\). These functions are defined in the domain \(Y_m \times (0, T)\) and are constants in the slow variable \(x\). In order to obtain the uniform estimates for the restricted functions (they are similar to the corresponding estimates for \(P^\varepsilon_m, S^\varepsilon_m\) from Section \[3\]) we make use of the estimates \((5.9)-(5.14)\).

The scheme is as follows. First, for any natural \(n\), we introduce the set of points \(x \in \Omega\) such that the corresponding norms for the restricted functions are not uniformly bounded in \(\varepsilon\). It turns out that the measure of this set is asymptotically small as \(n \rightarrow +\infty\) (see Propositions \[5.3, 5.4\] below). Then taking into account this fact and using the estimates \((5.9)-(5.14)\), we, finally, obtain the desired uniform estimates for the restricted functions (see Lemma \[5.8\] below).

Let us first denote a periodicity cell \(\varepsilon(Y + k)\) which contains point \(x_0\) by \(K^\varepsilon_{x_0}\). For given \(x_0\) and \(\varepsilon\) the index \(k \in \mathbb{Z}^d\) which defines the cell \(K^\varepsilon_{x_0}\) can be uniquely defined and therefore we have a well defined function \(k(x_0, \varepsilon) \in \mathbb{Z}^d\) such that \(K^\varepsilon_{x_0} \overset{\text{def}}{=} \varepsilon(Y + k(x_0, \varepsilon))\). Due to the definition of the dilation operator dilated functions are constant in \(x\) on \(K^\varepsilon_{x_0}\). The restricted functions are given by:

\[
s^\varepsilon_{m,x_0}(y, t) \overset{\text{def}}{=} \begin{cases} 
 s^\varepsilon_m, & \text{for } x \in K^\varepsilon_{x_0}; \\
 0, & \text{if not};
\end{cases}
\]
\[
p^\varepsilon_{m,x_0}(y, t) \overset{\text{def}}{=} \begin{cases} 
 p^\varepsilon_m, & \text{for } x \in K^\varepsilon_{x_0}; \\
 0, & \text{if not}.
\end{cases}
\]

For any \(\varepsilon > 0\), the pair \((s^\varepsilon_{m,x_0}, p^\varepsilon_{m,x_0})\) is a solution to problem \((5.3)-(5.4), (5.5)-(5.8)\) in \(Y_m \times (0, T)\).

Now we estimate the measure of the set of points \(x \in \Omega\) such that the corresponding norms for the restricted functions are not uniformly bounded in \(\varepsilon\). The following result holds true.
Proposition 5.3 Let \( f_m^\varepsilon = f_m^\varepsilon(x, y, t) \) be a dilated function such that
\[
\| f_m^\varepsilon \|_{L^2(O_T; L^2_{\text{per}}(Y_m))} \leq C \tag{5.25}
\]
and let \( A_n \) be a set of points defined by
\[
A_n \overset{\text{def}}{=} \left\{ x \in \Omega : \lim_{\varepsilon \to 0} \| \hat{f}_{m,k}(x, \varepsilon) \|_{L^2(0, T; L^2_{\text{per}}(Y_m))} \geq n \right\}, \tag{5.26}
\]
where for fixed \( k \in \mathbb{Z}^d \)
\[
\hat{f}_{m,k}^\varepsilon(y, t) \overset{\text{def}}{=} \begin{cases} f_m^\varepsilon(\varepsilon k, y, t), & \text{if } k \text{ is such that } \varepsilon(Y_m + k) \cap \Omega \neq \emptyset; \\ 0, & \text{if not}. \end{cases} \tag{5.27}
\]
Then \( \sqrt{|A_n|} \leq C/n \).

Proof of Proposition 5.3 Let \( f_m^\varepsilon = f_m^\varepsilon(x, y, t) \) be a function that satisfies (5.25). Then we can write
\[
\| f_m^\varepsilon \|_{L^2(O_T; L^2_{\text{per}}(Y_m))}^2 = \sum_{k=1}^{N_k} |eY_m| \| \hat{f}_{m,k}^\varepsilon \|_{L^2(0, T; L^2_{\text{per}}(Y_m))}^2, \tag{5.28}
\]
where, due to (5.25), we have that
\[
\sum_{k=1}^{N_k} |eY_m| \| \hat{f}_{m,k}^\varepsilon \|_{L^2(0, T; L^2_{\text{per}}(Y_m))}^2 \leq C^2. \tag{5.29}
\]
Now, for any \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), let us introduce the set of “bad points” \( A_n^\varepsilon \) defined by:
\[
A_n^\varepsilon \overset{\text{def}}{=} \left\{ x \in \Omega : \| \hat{f}_{m,k}(x, \varepsilon) \|_{L^2(0, T; L^2_{\text{per}}(Y_m))} > n \right\}. \tag{5.30}
\]
Let us estimate the measure of the set \( A_n^\varepsilon \). It follows from (5.29) and (5.30) that
\[
C^2 \geq \sum_{k=1}^{N_k} |eY_m| \| \hat{f}_{m,k}(x, \varepsilon) \|_{L^2(0, T; L^2_{\text{per}}(Y_m))}^2 \geq \sum_{x \in A_n^\varepsilon} |eY_m| n^2 = |A_n^\varepsilon| n^2. \tag{5.31}
\]
Therefore, \( |A_n^\varepsilon| \leq C^2/n^2 \). By definition of limit inferior, for any \( \eta > 0 \) we have \( A_n \subseteq \lim_{\varepsilon \to 0} A_n^\varepsilon - \eta \) (where \( \varepsilon \) denotes a sequence of real numbers). Due to the continuity of the measure we get
\[
|A_n| \leq \liminf_{\varepsilon \to 0} |A_n^\varepsilon| - \eta \leq C^2/(n - \eta)^2. \tag{5.32}
\]
Proposition 5.3 is proved.

We note that previously defined restricted functions are linked to ones appearing in Proposition 5.3 by the following relation:
\[
f_{m, \varepsilon}(x_0, t) = \hat{f}_{m,k}(x_0, \varepsilon)(y, t).
\]
In a similar way, taking into account the uniform estimate (5.10), we prove the following proposition.

Proposition 5.4 Let \( f_m^\varepsilon = f_m^\varepsilon(x, y, t) \) be a dilated function such that \( \| f_m^\varepsilon \|_{L^2(O_T; H^1_{\text{per}}(Y_m))} \leq C \) and let \( B_n \) be a set of points defined by
\[
B_n \overset{\text{def}}{=} \left\{ x \in \Omega : \lim_{\varepsilon \to 0} \| \hat{f}_{m,k}(x, \varepsilon) \|_{L^2(0, T; H^1_{\text{per}}(Y_m))} \geq n \right\},
\]
where the function \( \hat{f}_{m,k}^\varepsilon \) is defined in (5.27). Then \( \sqrt{|B_n|} \leq C/n \).
Now let us introduce $A_n$, the set of "bad points" for the functions appearing in (5.9)-(5.14). We set:

$$A_{1,n} = \left\{ x \in \Omega : \lim_{\varepsilon \to 0} \varepsilon^{\theta/2 - 1} \| \nabla y / \beta_m(s_{m,x}) \|_{L^2(0,T;L^2_{per}(Y_m))} \geq n \right\};$$

$$A_{2,n} = \left\{ x \in \Omega : \lim_{\varepsilon \to 0} \| \rho_{m,x}^{\varepsilon} \|_{L^2(0,T;L^2_{per}(Y_m))} \geq n \right\};$$

$$A_{3,n} = \left\{ x \in \Omega : \lim_{\varepsilon \to 0} \varepsilon^{\theta/2 - 1} \| \nabla y \rho_{m,x}^{\varepsilon} \|_{L^2(0,T;L^2_{per}(Y_m))} \geq n \right\};$$

$$A_{4,n} = \left\{ x \in \Omega : \lim_{\varepsilon \to 0} \| \partial_t (\Phi_m s_{m,x}^{\varepsilon}) \|_{L^2(0,T;H^{-1}_{per}(Y_m))} \geq n \right\}.$$

Here $s_{m,x}, \rho_{m,x}$ are defined in (5.24). Then

$$A_n = \bigcup_{\ell=1}^{4} A_{\ell,n}$$

(5.31)

and, due to Propositions 5.3, 5.4 the measure of this set satisfies the estimate $\sqrt{|A_n|} \leq C/n$.

The following result holds.

**Lemma 5.8** Let $s_{m,x_0}, \rho_{m,x_0}$ be the functions defined in (5.24) and $0 < \theta \leq 2$. Then for any $x_0 \in \Omega \setminus A_n$, there is a subsequence $\varepsilon = \varepsilon_k$ still denoted by $\varepsilon$ such that:

$$0 \leq s_{m,x_0} \leq 1 \quad \text{a.e. in } Y_m \times (0,T);$$

(5.32)

$$\| \nabla y / \beta_m(s_{m,x_0}) \|_{L^2(0,T;L^2_{per}(Y_m))} \leq C \varepsilon^{1-\theta/2};$$

(5.33)

$$\| \rho_{m,x_0} \|_{L^2(0,T;L^2_{per}(Y_m))} \leq C; \quad \| \nabla y \rho_{m,x_0} \|_{L^2(0,T;L^2_{per}(Y_m))} \leq C \varepsilon^{1-\theta/2};$$

(5.34)

$$\| \partial_t (\Phi_m s_{m,x_0}^{\varepsilon}) \|_{L^2(0,T;H^{-1}_{per}(Y_m))} \leq C;$$

(5.35)

where $C = C(n)$ is constant that does not depend on $x_0$ and $\varepsilon$, and $n$ is an arbitrary natural number.

**Proof of Lemma 5.8** First, we notice that the estimate (5.32) follows immediately from (5.9). Let us prove, for example, (5.33). Taking into account that $x_0 \in \Omega \setminus A_n$, from the definition of the set $A_{1,n}$, we obtain immediately the existence of a subsequence on which (5.33) holds with constant $C$ depending only on $n$. The estimates (5.34), (5.35) are obtained in a similar way. Lemma 5.8 is proved.

Using these estimates and applying Lemma 4.2 from [5], we obtain the following compactness result.

**Proposition 5.5** Assume $0 < \theta \leq 2$. For any $x_0 \in \Omega \setminus A_n$, on a subsequence extracted in Lemma 5.8 the family $\{ s_{m,x_0}^{\varepsilon} \}_{\varepsilon > 0}$ is a compact set in the space $L^q(Y_m \times (0,T))$ for all $q \in [1, \infty)$. In the case $\theta < 2$ every limit point of the sequence $\{ s_{m,x_0}^{\varepsilon} \}_{\varepsilon > 0}$ is independent of the fast variable $y$.  

---

22
6 Homogenization results

In this section we formulate and prove the main results of the paper corresponding to the homogenized models for various rates of contrast. First, we introduce the notation.

- $S$, $P_w$, $P_n$ denote the homogenized wetting liquid saturation, wetting liquid pressure, and nonwetting liquid pressure, respectively.
- $\Phi^* = \Phi^*(x)$ denotes the effective porosity and is given by:
  \[ \Phi^*(x) \equiv \Phi^H(x) \frac{|Y_f|}{|Y_m|}, \]
  where $\Phi^H$ is defined in condition (A.1) and $|Y_f|$ is the measure of the set $Y_f (\ell = f, m)$.
- $F_w^*$, $F_n^*$ denote the effective source terms and are given by:
  \[ F_w^*(x, t) \equiv F_w^H(x, t) \frac{|Y_f|}{|Y_m|} \quad \text{and} \quad F_n^*(x, t) \equiv F_n^H(x, t) \frac{|Y_f|}{|Y_m|}, \]
  where
  \[ F_w^H(x, t) \equiv S_{t,w}^f f_I(x, t) - S f_P(x, t) \quad \text{and} \quad F_n^H(x, t) \equiv S_{t,n}^f f_I(x, t) - (1 - S) f_P(x, t) \]
  and where the functions $S_{t,w}^f, S_{t,n}^f, f_I, f_P$ are defined in (2.5), (2.6), respectively (see also (A.9)).
- $\mathbb{K}^* = \mathbb{K}^*(x)$ is the homogenized tensor with the entries $\mathbb{K}_{ij}^*$ defined by:
  \[ \mathbb{K}_{ij}^*(x) \equiv \frac{1}{|Y_m|} \int_{Y_f} K(x, y) \left[ \nabla_y \xi_j + \bar{\epsilon}_j \right] \cdot \left[ \nabla_y \xi_i + \bar{\epsilon}_i \right] \, dy, \]
  where $\xi_j = \xi_j(x, y)$ ($j = 1, \ldots, d$) is a $Y$-periodic solution to the auxiliary cell problem:
  \[ \begin{cases} 
  -\text{div}_y \left\{ K(x, y) \nabla_y \xi_j \right\} = 0 & \text{in } Y_f; \\
  \nabla_y \xi_j \cdot \bar{\nu}_y = -\bar{\epsilon}_j \cdot \bar{\nu}_y & \text{on } \Gamma_{fm}; \\
  y \mapsto \xi_j(y) & Y - \text{periodic.}
  \end{cases} \]

6.1 High contrast media: critical case, $\theta = 2$

We study the asymptotic behavior of the solution to problem (2.8), (2.11)-(2.12) in the case $\theta(\varepsilon) = \varepsilon^2$ as $\varepsilon \to 0$. In particular, we are going to show that the effective model, expressed in terms of the homogenized phase pressures, reads:

\[ \begin{cases} 
  0 \leq S \leq 1 & \text{in } \Omega_T; \\
  \Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ \mathbb{K}^*(x) \lambda_{t,w}(S)(\nabla P_w - \bar{g}) \right\} = \Omega_w + F_w^* & \text{in } \Omega_T; \\
  -\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ \mathbb{K}^*(x) \lambda_{t,n}(S)(\nabla P_n - \bar{g}) \right\} = \Omega_n + F_n^* & \text{in } \Omega_T; \\
  P_{t,c}(S) = P_n - P_w & \text{in } \Omega_T.
\]
For almost every point \( x \in \Omega \) a matrix block \( Y_m \subset \mathbb{R}^d \) is suspended topologically. The system for flow in a matrix block is given by the so-called imbibition equation:

\[
\begin{align*}
\Phi_m(y) \frac{\partial s}{\partial t} - \text{div}_y \{ K(x,y) \nabla_y \beta_m(s) \} &= 0 \quad \text{in} \ Y_m \times \Omega_T; \\
\varphi(x,y,t) &= \mathcal{P}(S(x,t)) \quad \text{on} \ \Gamma_{fm} \times \Omega_T; \\
\varphi(x,y,0) &= S_0^m(x) \quad \text{in} \ Y_m \times \Omega.
\end{align*}
\]

(6.7)

Here \( s \) denotes the wetting liquid saturation in the block \( Y_m \) and the function \( \mathcal{P}(S) \) is defined by

\[
\mathcal{P}(S) \overset{\text{def}}{=} (P^{-1}_{c,m} \circ P_{c,f})(S).
\]

(6.8)

For any \( x \in \Omega \) and \( t > 0 \), the matrix-fracture sources are given by:

\[
\mathcal{Q}_w \overset{\text{def}}{=} -\frac{1}{|Y_m|} \int_{Y_m} \Phi_m(y) \frac{\partial s}{\partial t}(x,y,t) \, dy = -\mathcal{Q}_n.
\]

(6.9)

The boundary conditions for the effective system (6.6) are given by:

\[
\begin{align*}
P_w &= P_n = 0 \quad \text{on} \ \Gamma_1 \times (0,T); \\
\mathbb{K}^* \lambda_n(S) (\nabla P_w - \vec{g}) \cdot \vec{v} = \mathbb{K}^* \lambda_w(S)(\nabla P_n - \vec{g}) \cdot \vec{v} &= 0 \quad \text{on} \ \Gamma_2 \times (0,T).
\end{align*}
\]

(6.10)

Finally, the initial conditions read:

\[
P_w(x,0) = p^0_w(x) \quad \text{and} \quad P_n(x,0) = p^0_n(x) \quad \text{in} \ \Omega.
\]

(6.11)

The first main result of the paper is given by the following theorem.

**Theorem 6.1** Let \( \varepsilon(x) = \varepsilon^2 \) and let assumptions (A.1)-(A.9) be fulfilled. Then the solution of the initial problem (2.8), (2.10)-(2.12) converges (up to a subsequence) in the two-scale sense to a weak solution of the homogenized problem (6.6), (6.7), (6.9)-(6.11).

**Proof of Theorem 6.1** It is done in several steps. We start our analysis by considering the system (2.8). The main difficulty with the initial unknown functions \( p^\varepsilon, p^\varepsilon_n \) in this system is that they do not possess the uniform \( H^1 \)-estimates (see Lemma 3.1). It is important to notice that in the case of two-phase incompressible flow it is possible to find appropriate but rather strong conditions which allow us to deal directly with the phase pressures in a space wider than \( H^1 \) (see [40]). To overcome the difficulties appearing due to the absence of the uniform \( H^1 \)-estimates, the authors usually pass to the equivalent formulation of the problem in terms of the global pressure and saturation. In our case it is done in subsection 2.2 and the corresponding weak formulation of the problem is then given in subsection 2.3. Then using the convergence and compactness results from subsection 4 we pass to the limit in equations (2.27), (2.28). This is done in subsections 6.1.1 and 6.1.2. In order, to pass to the homogenized phase pressures we make use of the change of the unknown functions. Namely, we set, by the definition of the global pressure: \( P_w \overset{\text{def}}{=} P + G_{f,w}(S) \) and \( P_n \overset{\text{def}}{=} P + G_{f,n}(S) \). Then we rewrite the limit system obtained in terms of the global pressure and saturation in terms of the homogenized phase pressures. The passage to the limit in the matrix blocks makes use of the dilation operator (see Section 5 above). Then we pass to the equivalent problem for the imbibition equation and, finally, obtain system (6.7).
6.1.1 Passage to the limit in equation (2.27)

We set:
\[
\varphi_w(x, \frac{x}{\varepsilon}, t) \overset{\text{def}}{=} \varphi(x, t) + \varepsilon \zeta(x, \frac{x}{\varepsilon}, t) = \varphi(x, t) + \varepsilon \zeta_1(x, t) \zeta_2 \left( \frac{x}{\varepsilon} \right) \overset{\text{def}}{=} \varphi(x, t) + \varepsilon \zeta^\varepsilon(x, t),
\]
(6.12)
where \( \varphi \in \mathcal{D}(\Omega_T), \zeta_1 \in \mathcal{D}(\Omega_T), \zeta_2 \in C^\infty_{\text{per}}(Y) \), and plug the function \( \varphi_w \) in (2.27). This yields:
\[
- \int_{\Omega_T} 1_T^w(x) \Phi^e_f(x) S^e_f \left[ \frac{\partial \varphi}{\partial t} + \varepsilon \frac{\partial \zeta^\varepsilon}{\partial t} \right] \, dx \, dt +
+ \int_{\Omega_T} 1_T^w(x) K \left( x, \frac{x}{\varepsilon} \right) \left\{ \lambda_{f,w}(\overline{S}^e_f) \left( \nabla \overline{P}^e_f - \overline{g} \right) + \nabla \beta_t(S^e_f) \right\} \cdot \left[ \nabla \varphi + \varepsilon \nabla_x \zeta^\varepsilon + \nabla_y \zeta^\varepsilon \right] \, dx \, dt =
- \int_{\Omega_{\text{m,T}}} \Phi^e_m \left( \frac{x}{\varepsilon} \right) S^e_m \left[ \frac{\partial \varphi}{\partial t} + \varepsilon \frac{\partial \zeta^\varepsilon}{\partial t} \right] \, dx \, dt +
+ \varepsilon^2 \int_{\Omega_{\text{m,T}}} K \left( x, \frac{x}{\varepsilon} \right) \left\{ \lambda_{m,w}(S^e_m) \left( \nabla P^e_{m} - \overline{g} \right) + \nabla \beta_m(S^e_m) \right\} \cdot \left[ \nabla \varphi + \varepsilon \nabla_x \zeta^\varepsilon + \nabla_y \zeta^\varepsilon \right] \, dx \, dt =
= \int_{\Omega_{\text{T}}} \left( S^e_{f,w} f_1(x, t) - S^e_f f_p(x, t) \right) [\varphi + \varepsilon \zeta^\varepsilon] \, dx \, dt.
\]
(6.13)

Taking into account Lemma 3.1 and the convergence results of Lemma 4.2 and Lemma 5.7, we pass to the limit in (6.13) as \( \varepsilon \to 0 \) and obtain the following homogenized equation:
\[
- |Y_t| \int_{\Omega_T} \Phi^H_f(x) S(x, t) \frac{\partial \varphi}{\partial t} \, dx \, dt +
+ \int_{\Omega_T \times Y_t} K(x, y) \left\{ \lambda_{f,w}(S) \left[ \nabla P + \nabla_y w_p - \overline{g} \right] + \nabla \beta_f(S) + \nabla_y w_s \right\} \cdot \left[ \nabla \varphi + \zeta_1 \nabla \zeta_2 \right] \, dy \, dx \, dt =
= \int_{\Omega_T \times Y_m} \Phi^e_m(y) s(x, y, t) \frac{\partial \varphi}{\partial t} \, dy \, dx \, dt + |Y_t| \int_{\Omega_T} F^H_w \varphi \, dx \, dt,
\]
(6.14)
where \( F^H_w \) is given by (6.3).

6.1.2 Passage to the limit in equation (2.28)

Equation (2.28) is treated in the same way as equation (2.27). Taking the test function of the form (6.12) and using the same arguments we can pass to a limit \( \varepsilon \to 0 \) and obtain the following homogenized equation:
\[
|Y_t| \int_{\Omega_T} \Phi^H_f(x) S(x, t) \frac{\partial \varphi}{\partial t} \, dx \, dt +
+ \int_{\Omega_T \times Y_t} K(x, y) \left\{ \lambda_{f,w}(S) \left[ \nabla P + \nabla_y w_p - \overline{g} \right] - \nabla \beta_t(S) - \nabla_y w_s \right\} \cdot \left[ \nabla \varphi + \zeta_1 \nabla \zeta_2 \right] \, dy \, dx \, dt =
= - \int_{\Omega_T \times Y_m} \Phi^e_m(y) s(x, y, t) \frac{\partial \varphi}{\partial t} \, dy \, dx \, dt + |Y_t| \int_{\Omega_T} F^H_n \varphi \, dx \, dt.
\]
(6.15)
6.1.3 Identification of the corrector functions \( w_p, w_s \) and homogenized equations

In this section we identify the corrector functions \( w_p, w_s \) appearing in the equations (6.14), (6.15) and obtain the desired homogenized system (6.6).

Consider the equations (6.14), (6.15). Setting \( \varphi \equiv 0 \), we get:

\[
\int_{Y} f K(x,y) \left\{ \lambda f \nabla P + \nabla y w_p - \vec{g} \right\} \cdot \nabla y \zeta_2(y) \, dy = 0 \quad (6.16)
\]

and

\[
\int_{Y} f K(x,y) \left\{ \lambda f \nabla P + \nabla y w_s - \vec{g} \right\} \cdot \nabla y \zeta_2(y) \, dy = 0. \quad (6.17)
\]

Now adding (6.16) and (6.17) and taking into account condition (A.4) and the fact that the saturation \( S \) does not depend on the fast variable \( y \), we obtain:

\[
\int_{Y} f K(x,y) \left\{ \nabla P + \nabla y w_p - \vec{g} \right\} \cdot \nabla y \zeta_2(y) \, dy = 0. \quad (6.18)
\]

Then we proceed in a standard way (see, e.g., [30]). Let \( \xi_j = \xi_j(x,y) \) \((j = 1, \ldots, d)\) be the \( Y \)-periodic solution of the auxiliary cell problem (6.5). Then the function \( w_p \) can be represented as:

\[
w_p(x,y,t) = d \sum_{j=1}^{d} \xi_j(x,y) \left( \frac{\partial P}{\partial x_j}(x,t) - g_j \right). \quad (6.19)
\]

Now we turn to the identification of the function \( w_s \). From (6.16) and (6.18), we get:

\[
\int_{Y} f K(x,y) \left\{ \nabla \beta f + \nabla y w_s \right\} \cdot \nabla y \zeta_2(y) \, dy = 0. \quad (6.20)
\]

Then as in the previous case, we obtain that

\[
w_s(x,y,t) = \sum_{j=1}^{d} \xi_j(x,y) \frac{\partial \beta f(S)}{\partial x_j}(x,t). \quad (6.21)
\]

6.1.4 Effective equations in terms of the global pressure and saturation

We start by obtaining the corresponding homogenized equation for the wetting phase. Choosing \( \zeta_2 = 0 \) in (6.14), we get:

\[
\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ K^*(x) \left[ \lambda f,w(S) \nabla P + \nabla \beta f(S) - \lambda f,w(S) \vec{g} \right] \right\} = - \frac{1}{|Y_m|} \int_{Y_m} \Phi_m(y) \frac{\partial S}{\partial t}(x,y,t) \, dy + F_w^*(x,t), \quad (6.22)
\]

where the effective porosity \( \Phi^* \), the effective source term \( F_w^* \), and the homogenized permeability tensor \( K^* \) are defined in (6.1), (6.2) and (6.4), respectively.

In a similar way, choosing \( \zeta_2 = 0 \) in equation (6.15), we derive the second homogenized equation:

\[
-\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ K^*(x) \left[ \lambda f,n(S) \nabla P + \nabla \beta f(S) - \lambda f,n(S) \vec{g} \right] \right\} = \]

26
\begin{equation}
\frac{1}{|Y_m|} \int_{Y_m} \Phi_m(y) \frac{\partial s}{\partial t}(x, y, t) \, dy + F_n^*(x, t), \tag{6.23}
\end{equation}

where $F_n^*$ denotes the effective source term defined in (6.2).

### 6.1.5 Effective equations in terms of the phase pressures

Let us introduce now the functions that is naturally to call the homogenized phase pressures. Namely, we set,

\begin{equation}
P_w \overset{\text{def}}{=} P + G_{f,w}(S) \quad \text{and} \quad P_n \overset{\text{def}}{=} P + G_{f,n}(S),
\end{equation}

where the functions $G_{f,w}, G_{f,n}$ are defined in Section 2.2. Then it easy to see that the homogenized equations can be rewritten as follows:

\begin{align*}
\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ K^*(x) \lambda_{f,w}(S) (\nabla P_w - \bar{g}) \right\} &= \Omega_w + F_w^* \quad \text{in } \Omega_T; \\
-\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ K^*(x) \lambda_{f,n}(S) (\nabla P_n - \bar{g}) \right\} &= \Omega_n + F_n^* \quad \text{in } \Omega_T; \tag{6.25}
\end{align*}

\begin{equation}
P_c(S) = P_n - P_w \quad \text{in } \Omega_T.
\end{equation}

### 6.1.6 Flow equations in the matrix block

In this section, following the ideas of the papers [6,16,23], we obtain the system (6.7) describing the behavior of the function $s$ which is involved in the definition of the matrix-fracture source term. Briefly, we pass to the limit in the equations for the dilated functions for fixed $k$ and then by density arguments the limit equations will be obtained. We recall that the equations for the dilated functions are already obtained in Lemma 5.5 from subsection 5.2. Namely, for almost all $x \in \Omega$, the functions $s_m^\varepsilon, p_m^\varepsilon$ satisfy the following variational problem: for all $\phi_n, \phi_w \in L^2(0, T; H^1_0(Y_m)) \cap H^1(0, T; L^2(Y_m))$, $\phi_n(T) = \phi_w(T) = 0$,

\begin{align*}
-\int_0^T \int_{Y_m} \Phi_m(y) s_m^\varepsilon \frac{\partial \phi_w}{\partial t} \, dy - \int_0^T \int_{Y_m} \Phi_m(y) (\mathcal{D}^\varepsilon S_m^0) \frac{\partial \phi_w}{\partial t} \, dy \\
+ \int_0^T \int_{Y_m} \left\{ K(x, y) \left[ \lambda_{m,w}(s_m^\varepsilon) \nabla_y p_m^\varepsilon + \nabla_y \beta_m(s_m^\varepsilon) - \varepsilon \lambda_{m,w}(s_m^\varepsilon) \bar{g} \right] \right\} \cdot \nabla_y \phi_w \, dy &= 0; \tag{6.26}
\end{align*}

\begin{align*}
\int_0^T \int_{Y_m} \Phi_m(y) s_m^\varepsilon \frac{\partial \phi_n}{\partial t} \, dy + \int_0^T \int_{Y_m} \Phi_m(y) (\mathcal{D}^\varepsilon S_m^0) \frac{\partial \phi_n}{\partial t} \, dy \\
+ \int_0^T \int_{Y_m} \left\{ K(x, y) \left[ \lambda_{m,n}(s_m^\varepsilon) \nabla_y p_m^\varepsilon - \nabla_y \beta_m(s_m^\varepsilon) - \varepsilon \lambda_{m,n}(s_m^\varepsilon) \bar{g} \right] \right\} \cdot \nabla_y \phi_n \, dy &= 0 \tag{6.27}
\end{align*}

with the boundary conditions 5.5.

The uniform estimates for the functions $s_m^\varepsilon, p_m^\varepsilon$ imply the convergence results of $\langle s_m^\varepsilon, p_m^\varepsilon \rangle$ to $\langle s, p \rangle$ in a weak sense (see Lemma 5.7). Thus, the limit behavior of the dilated functions $s_m^\varepsilon, p_m^\varepsilon$ is determined. However, the convergence results of Lemma 5.7 are not sufficient for derivation of the equations for the limit functions $\langle s, p \rangle$. To overcome this difficulty, in Section 5.3 we pass to the restrictions of the functions $s_m^\varepsilon, p_m^\varepsilon$ to $K_{\varepsilon 0}^\varepsilon$. 

27
Lemma 6.1 Let \( x_0 \in \Omega \setminus A_n \). There exist functions \( s_{x_0}, p_{x_0} \), and \( \beta_m(s_{x_0}) \) such that up to a subsequence:

\[
s^{\varepsilon}_{m,x_0} \to s_{x_0} \text{ strongly in } L^q(Y_m \times (0,T)) \quad \forall \ 1 \leq q < +\infty; \\
p^{\varepsilon}_{m,x_0} \to p_{x_0} \text{ weakly in } L^2(0,T; H^1_{\text{per}}(Y_m)); \\
\beta_m(s^{\varepsilon}_{m,x_0}) \to \beta_m(s_{x_0}) \text{ weakly in } L^2(0,T; H^1_{\text{per}}(Y_m)); \\
\beta_m(s^{\varepsilon}_{m,x_0}) \to \beta_m(s_{x_0}) \text{ strongly in } L^q(Y_m \times (0,T)) \quad \forall \ 1 \leq q < +\infty; \\
\beta_m(s^{\varepsilon}_{m,x_0}) \big|_{\Gamma_{mf}} \to \beta_m(s_{x_0}) \big|_{\Gamma_{mf}} \text{ weakly in } L^2(0,T; L^2(\Gamma_{mf})); \\
p^{\varepsilon}_{m,x_0} \big|_{\Gamma_{mf}} \to p_{x_0} \big|_{\Gamma_{mf}} \text{ weakly in } L^2(0,T; L^2(\Gamma_{mf})).
\]

As in subsections 6.1.1, 6.1.2 we can easily pass to the limit in (6.26) and (6.27). We get the following system of equations:

\[
- \int_0^T \int_{Y_m} \Phi_m(y) s_{x_0} \frac{\partial \phi_w}{\partial t} \, dy - \int_0^T \int_{Y_m} \Phi_m(y) S^{\emptyset}_{x_0} \frac{\partial \phi_w}{\partial t} (0) \, dy \\
+ \int_0^T \int_{Y_m} \left\{ K(x,y) \left[ \lambda_{m,w}(s_{x_0}) \nabla_y p_{x_0} + \nabla_y \beta_m(s_{x_0}) \right] \right\} \cdot \nabla_y \phi_w \, dy = 0;
\]

\[
- \int_0^T \int_{Y_m} \Phi_m(y) s_{x_0} \frac{\partial \phi_n}{\partial t} \, dy - \int_0^T \int_{Y_m} \Phi_m(y) S^{\emptyset}_{x_0} \frac{\partial \phi_n}{\partial t} \, dy \\
+ \int_0^T \int_{Y_m} \left\{ K(x,y) \left[ \lambda_{m,n}(s_{x_0}) \nabla_y p_{x_0} - \nabla_y \beta_m(s_{x_0}) \right] \right\} \cdot \nabla_y \phi_n \, dy = 0,
\]

where we have used the fact that \( \mathbb{D}^{\varepsilon} S^{\emptyset}_{x_0} \to S^{\emptyset}_{x_0} \) strongly in \( L^2(Y_m) \) for almost all \( x_0 \in \Omega \).

Now we turn to the boundary condition for \( s_{x_0} \) on \( \Gamma_{mf} \). From Corollary 5.2 we know that for a.e. \( x_0 \),

\[
\mathcal{M}(\beta_{\mathbb{D}}(\mathbb{D}^{\varepsilon}(S^{\emptyset}_{x_0}(x_0, \cdot, \cdot)))) \to \mathcal{M}(\beta_{\mathbb{D}}(S(x_0, \cdot, \cdot))) \quad \text{strongly in } L^2(0,T; L^2(\Gamma_{mf})),
\]

where \( \mathcal{M} \) is the function given in (5.6). Therefore, for a.e. \( x_0 \in \Omega \setminus A_n \), from (5.5) and (6.32) we have:

\[
\beta_m(s_{x_0}) \big|_{\Gamma_{mf}} = \mathcal{M}(\beta_{\mathbb{D}}(S(x_0, \cdot, \cdot))) \big|_{\Gamma_{mf}},
\]

or, equivalently

\[
s_{x_0} = \mathcal{P}(S(x_0, \cdot)) \quad \text{on } \Gamma_{mf} \times (0,T).
\]
Note also that it follows from (5.5) that the convergence in (6.32) is strong in $L^2(0, T; L^2(\Gamma_{mf}))$. This, together with convergence (6.33) and Lipschitz continuity of the functions $G_{\ell,q}$, $G_{\ell,w}$, enables us to pass to the limit in the boundary condition for dilated global pressure (5.7) using the two-scale convergence on $\Gamma_{mf}$, and get

$$p_{x_0} + G_{m,w}(s_{x_0}) = \mathcal{P}(x_0, \cdot) + G_{f,w}(S(x_0, \cdot)) \quad \text{on} \quad \Gamma_{mf} \times (0, T).$$  \hspace{1cm} (6.37)

In the same way we also get

$$p_{x_0} + G_{m,g}(s_{x_0}) = \mathcal{P}(x_0, \cdot) + G_{f,g}(S(x_0, \cdot)) \quad \text{on} \quad \Gamma_{mf} \times (0, T).$$  \hspace{1cm} (6.38)

Thus the system which is satisfied by the limit $\langle s_{x_0}, p_{x_0} \rangle$ is obtained for any $x_0 \in \Omega \setminus A_n$. Now it remains to make the link between the functions $s_{x_0}, p_{x_0}$ and the limits $s, p$ of the sequences $\{s^{\varepsilon}_m\}_{\varepsilon > 0}, \{p^{\varepsilon}_m\}_{\varepsilon > 0}$. First, we observe that the convergent subsequence in Lemma 6.1 depends on $x_0 \not\in A_n$. To avoid this difficulty we will prove (see subsection 6.1.6 below) that the problem (6.34), (6.35) with the corresponding boundary conditions (6.36), (6.37), and (6.38) has a unique weak solution. Then the convergence results from Lemma 6.1 hold for the whole sequences, as $\varepsilon \to 0$. Since the functions $s_{x_0} = s(x_{x_0}, y, t), p_{x_0} = p(x_{x_0}, y, t)$ satisfy (6.34)-(6.38) for almost all $x_0 \in \Omega \setminus A_n$, we conclude that $s$ and $p$ are weak solution of the following system of equations:

$$\begin{cases}
0 \leq s \leq 1 & \text{in} \ Y_m \times \Omega_T; \\
\Phi_m(y) \frac{\partial s}{\partial t} - \text{div}_y \left( K(x, y) \left[ \lambda_{m,w}(s) \nabla_y p + \nabla_y \beta_m(s) \right] \right) = 0 & \text{in} \ Y_m \times \Omega_T; \\
-\Phi_m(y) \frac{\partial s}{\partial t} - \text{div}_y \left( K(x, y) \left[ \lambda_{m,n}(s) \nabla_y p - \nabla_y \beta_m(s) \right] \right) = 0 & \text{in} \ Y_m \times \Omega_T.
\end{cases}$$  \hspace{1cm} (6.39)

The system is completed by the corresponding boundary and initial conditions:

$$\begin{cases}
\mathcal{P} + G_{f,w}(S) = p + G_{m,w}(s) & \text{on} \ \Gamma_{mf} \times \Omega_T; \\
\mathcal{P} + G_{f,n}(S) = p + G_{m,n}(s) & \text{on} \ \Gamma_{mf} \times \Omega_T; \\
s(x, y, t) = \mathcal{P}(S(x, t)) & \text{on} \ \Gamma_{mf} \times \Omega_T, \\
s(x, y, 0) = S^0(x) & \text{in} \ Y_m \times \Omega.
\end{cases}$$  \hspace{1cm} (6.40)

Thus, we have identified $s$ and $p$ for $x \in \Omega \setminus A_n$. Since by Propositions 5.3, 5.4 the measure of the set $A_n$ goes to zero as $n \to \infty$ we conclude that our conclusion holds a.e. in $\Omega$.

The proof of the uniqueness of the solution to problem (6.39) will be done as follows. First, we reduce the system (6.39) to a boundary value problem for the so-called imbibition equation and then make use of the uniqueness result from [38]. Equation (6.7) is the well known generalized porous medium equation (see, e.g., [38]).

**Lemma 6.2** Let $s = s(x, y, t)$ be the solution of the cell problem (6.39)-(6.40). Then $s$ satisfies the boundary value problem (6.7).

**Proof of Lemma 6.2** First we observe that it follows from the boundary conditions (6.40) that the function $s$ does not depend on $y$ on $\Gamma_{mf} \times \Omega_T$. Then the global pressure $p$ does not depend on $y$ on $\Gamma_{mf} \times \Omega_T$. Namely, we can write that

$$p(x, y, t) = p_T(x, t) \quad \text{on} \ \Gamma_{mf} \times \Omega_T.$$  \hspace{1cm} (6.41)
By summing the two equations in (6.39) we get:

$$\text{div} \left\{ K(x, y) \lambda_m(s) \nabla_y p \right\} = 0 \quad \text{in} \quad Y_m \times \Omega. \quad (6.42)$$

Then multiplying the equation (6.42) by \((p - p_T)\) and integrating over \(Y_m \times \Omega_T\), using (6.41) and conditions (A.2), (A.4) we obtain:

$$0 = \int_{Y_m \times \Omega_T} K(x, y) \lambda_m(s) \nabla_y p \cdot \nabla_y p \, dx \, dy \, dt \geq k_{\min} L_0 \int_{Y_m \times \Omega_T} |\nabla_y p|^2 \, dx \, dy \, dt,$$

which gives \(\nabla_y p = 0\) a.e. in \(Y_m \times \Omega_T\). This result allows us to reduce the two equations in the problem (6.39) to only one, as announced in (6.7). This completes the proof of Lemma 6.2.

Now we turn to the proof of the uniqueness of the solution to (6.7). This proof is given in Theorem 5.3 from [38]. For reader’s convenience we discuss it briefly in the following lemma.

**Lemma 6.3** Under our standing assumptions, there is a unique weak solution to problem (6.7).

**Proof of Lemma 6.3** First, we introduce the weak formulation of problem (6.7). Omitting, for the sake of simplicity, the dependence on the slow variable \(y\), we have:

*for any function \(\eta \in C^1(\bar{Y}_{m,T})\), where \(Y_{m,T} \equiv Y_m \times (0, T)\), vanishing on \(\Gamma_m\) and such that \(\eta(x, T) = 0\), we have:

$$\int_{Y_{m,T}} \left\{ K(y) \nabla_y \beta_m(s) \cdot \nabla_y \eta - s \frac{\partial \eta}{\partial t} \right\} \, dy \, dt = \int_{Y_m} S_m^0(x) \eta(y, 0) \, dy, \quad (6.43)$$

Suppose now that we have two solutions \(s_1\) and \(s_2\) satisfying (6.43). Then denoting \(W_i \equiv \beta_m(s_i)\), from (6.43), we have:

$$\int_{Y_{m,T}} \left\{ K(y) \nabla_y (W_1 - W_2) \cdot \nabla_y \eta - (s_1 - s_2) \frac{\partial \eta}{\partial t} \right\} \, dy \, dt = 0 \quad (6.44)$$

for all \(\eta\). Then we use as a special test function \(\eta = \widehat{\eta}\), see e.g. [38]:

$$\widehat{\eta} \equiv \begin{cases} \int_t^T [W_1(x, \varsigma) - W_2(x, \varsigma)] \, d\varsigma & \text{if } 0 < t < T; \\ 0, & \text{if } t \geq T. \end{cases} \quad (6.45)$$

Then, plugging (6.45) in (6.44), we get:

$$\int_{Y_{m,T}} (s_1 - s_2) (W_1 - W_2) \, dy \, dt + \int_{Y_{m,T}} K(y) \nabla_y (W_1 - W_2) \cdot \left\{ \int_t^T \nabla_y (W_1 - W_2) \, d\varsigma \right\} \, dy \, dt = 0. \quad (6.46)$$

Integration of the last term leads to the following relation:

$$\int_{Y_{m,T}} (s_1 - s_2) (\beta_m(s_1) - \beta_m(s_2)) \, dy \, dt + \frac{1}{2} \int_{Y_m} K(y) \left[ \int_0^T \nabla_y (\beta_m(s_1) - \beta_m(s_2)) \, d\varsigma \right]^2 \, dy \, dt = 0. \quad (6.47)$$

Due to the monotonicity of the function \(\beta_m\), the first term in (6.44) is non-negative. Therefore we can conclude that \(s_1 = s_2\) a.e. in \(Y_{m,T}\). Lemma 6.3 is proved.

This completes the proof of Theorem 6.1.
6.2 Very high contrast media: \( \theta > 2 \)

We study the asymptotic behavior of the solution to problem (2.8), (2.11)-(2.12) as \( \varepsilon \to 0 \) in the case \( \varepsilon(\varepsilon) = \varepsilon^\theta \) with \( \theta > 2 \). In particular, we are going to show that the effective model reads:

\[
\begin{aligned}
0 \leq S \leq 1 & \quad \text{in } \Omega_T; \\
\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ K^*(x) \lambda_{f,w}(S)(\nabla P_w - \vec{g}) \right\} = F^*_w & \quad \text{in } \Omega_T; \\
-\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ K^*(x) \lambda_{f,n}(S)(\nabla P_n - \vec{g}) \right\} = F^*_n & \quad \text{in } \Omega_T; \\

P_{f,c}(S) = P_n - P_w & \quad \text{in } \Omega_T,
\end{aligned}
\]

(6.48)

where the effective porosity \( \Phi^* \), the effective source terms \( F^*_w, F^*_n \), and the homogenized permeability tensor \( K^* \) in (6.48) are defined in (6.1), (6.2) and (6.4), respectively.

The boundary and the initial conditions for the system (6.48) are given by (6.10), (6.11).

We see that in this case the matrix blocks have a vanishing, as \( \varepsilon \to 0 \), influence on the effective flow. This means that in the case of very high contrast, the medium behaves as a perforated one.

The second main result of the paper is as follows.

**Theorem 6.2** Let \( \varepsilon(\varepsilon) = \varepsilon^\theta \) with \( \theta > 2 \) and let assumptions (A.1)-(A.9) be fulfilled. Then the solution of the initial problem (2.8), (2.10)-(2.12) converges (up to a subsequence) in the two-scale sense to a weak solution of the homogenized problem (6.48), (6.10), (6.11).

**Proof of Theorem 6.2** Let \( \theta > 2 \). In the proof of Theorem 6.2 we follow the lines of the proof of Theorem 6.1. Namely, arguing as in Sections 6.1.1, 6.1.2, 6.1.3, we obtain the homogenized equations (6.22), (6.23).

Now we want to show that in the case of the very high contrast, the model behaves as in the perforated media, i.e., the matrix blocks are totally impermeable and the additional matrix-source term equals zero. As in the paper [40], we prove the following result.

**Lemma 6.4** The following equation holds true:

\[
\Phi_m(y) \frac{\partial S}{\partial t}(x,y,t) = 0 \quad \text{in } Y_m \times \Omega_T.
\]

**Proof of Lemma 6.4** Let us define the function:

\[
\mathcal{F}^\varepsilon(x,t) \equiv \varepsilon^{\frac{\theta}{2}} K^\varepsilon(x) \left\{ \lambda_{m,w}(S^\varepsilon_m) (\nabla P^\varepsilon_m - \vec{g}) + \nabla \beta_m(S^\varepsilon_m) \right\}.
\]

By using the estimate (3.2) and the assumptions (A.2), (A.4), we get the uniform bound:

\[
\|\mathcal{F}^\varepsilon\|_{L^2(\Omega_m,T)} \leq C. \tag{6.49}
\]

Let define a function:

\[
\varphi(w) \left( x, \frac{x}{\varepsilon}, t \right) \in \mathcal{D}(\Omega_T; C^\infty_{\text{per}}(Y)) \quad \text{such that } \varphi = 0 \quad \text{for } y \in Y_t.
\]

Plugging \( \varphi_w \) in (2.27), and taking into account condition (A.9), we get:

\[
- \int_{\Omega_T} \mathcal{F}^\varepsilon \left( x, \frac{x}{\varepsilon}, t \right) \Phi_m^\varepsilon(x) S^\varepsilon_m \frac{\partial \varphi_w}{\partial t} \, dx \, dt + \varepsilon^{\frac{\theta}{2}} \int_{\Omega_m,T} \mathcal{F}^\varepsilon \nabla_x \varphi \, dx \, dt + \varepsilon^{\frac{\theta}{2}-1} \int_{\Omega_m,T} \mathcal{F}^\varepsilon \nabla_y \varphi \, dx \, dt = 0, \tag{6.50}
\]

31
We pass to the two-scale limit in (6.50) using (6.49). We obtain:

\[ \int_{\Omega_T \times Y_m} \Phi_m(y) s(x, y, t) \frac{\partial \varphi_w}{\partial t} \, dx \, dt \, dy = 0. \]  \hspace{1cm} (6.51)

This completes the proof of Lemma 6.4. \hfill \Box

Finally, from the equations (6.22), (6.23) in view of Lemma 6.4, arguing as in subsection 6.1.5 we arrive to the desired system (6.48). This completes the proof of Theorem 6.2. \hfill \Box

6.3 Moderate contrast media: \( \theta < 2 \)

We study the asymptotic behavior of the solution to problem (2.8) as \( \varepsilon \to 0 \) in the case \( \varkappa(\varepsilon) = \varepsilon^\theta \) with \( 0 < \theta < 2 \). In particular, we are going to show that the effective model reads:

\[
\begin{align*}
0 \leq S &\leq 1 \quad \text{in } \Omega_T; \\
\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ \kappa^*(x) \lambda_{f,w}(S) (\nabla P_w - \vec{g}) \right\} &\equiv F^*_w \quad \text{in } \Omega_T; \\
-\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ \kappa^*(x) \lambda_{f,n}(S) (\nabla P_n - \vec{g}) \right\} &\equiv F^*_n \quad \text{in } \Omega_T; \\
P_{f,c}(S) &= P_n - P_w \quad \text{in } \Omega_T,
\end{align*}
\]  \hspace{1cm} (6.52)

where the effective porosity \( \Phi^* \), the effective source terms \( F^*_w, F^*_n \), and the homogenized permeability tensor \( \kappa^* \) in (6.52) are defined in (6.1), (6.2) and (6.4), respectively.

The boundary conditions and the initial conditions for the system (6.52) are given by (6.10), (6.11).

In this case we observe a complete decoupling between microscale and macroscale, which is not the case for the critical scaling \( \theta = 2 \).

The third main result of the paper is as follows.

**Theorem 6.3** Let \( \varkappa(\varepsilon) = \varepsilon^\theta \) with \( 0 < \theta < 2 \) and let assumptions (A.1)-(A.9) be fulfilled. Then the solution of the initial problem \( (2.8), (2.10)-(2.12) \) converges (up to a subsequence) in the two-scale sense to a weak solution of the homogenized problem (6.52), (6.10), (6.11).

Let \( 0 < \theta < 2 \). In the proof of Theorem 6.3 we follow the lines of the proof of Theorem 6.1. Namely, arguing as in Sections 6.1.1, 6.1.2, 6.1.3 we obtain the homogenized equations (6.22), (6.23). Namely, in the case of the moderate contrast we have:

\[
\begin{align*}
0 \leq S &\leq 1 \quad \text{in } \Omega_T; \\
\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ \kappa^*(x) \lambda_{f,w}(S) (\nabla P_w - \vec{g}) \right\} &\equiv \tilde{Q}_w + F^*_w \quad \text{in } \Omega_T; \\
-\Phi^*(x) \frac{\partial S}{\partial t} - \text{div}_x \left\{ \kappa^*(x) \lambda_{f,n}(S) (\nabla P_n - \vec{g}) \right\} &\equiv \tilde{Q}_n + F^*_n \quad \text{in } \Omega_T; \\
P_{f,c}(S) &= P_n - P_w \quad \text{in } \Omega_T,
\end{align*}
\]  \hspace{1cm} (6.53)

where the effective porosity \( \Phi^* \), the effective source terms \( F^*_w, F^*_n \), and the homogenized permeability tensor \( \kappa^* \) in (6.53) are defined in (6.1), (6.2) and (6.4), respectively. For any \( x \in \Omega \) and \( t > 0 \), the matrix-fracture
source terms $\hat{\mathcal{Q}}_w, \hat{\mathcal{Q}}_n$ in (6.52) have the form:

$$\hat{\mathcal{Q}}_w \equiv -\Phi_m \frac{\partial \hat{s}}{\partial t}(x, t) = -\hat{\mathcal{Q}}_n \quad \text{with} \quad \Phi_m \overset{\text{def}}{=} \frac{1}{|Y_m|} \int_{Y_m} \Phi_m(y) \, dy.$$ 

In order to complete the proof of Theorem 6.3 we have to identify the saturation function $s$ appearing on the right-hand side of equations in (6.53). The following result holds true:

**Lemma 6.5** Let $s$ be the weak limit of $\{\mathcal{D}^\varepsilon(S^n_m)\}_{\varepsilon>0}$ and $S$ is the saturation function defined in (4.15). Then

$$s = \mathcal{P}(S) \quad \text{a.e. in } \Omega_T \quad \text{with} \quad \mathcal{P}(S) = (P_{c,m}^{-1} \circ P_{c,f})(S). \quad (6.54)$$

**Proof of Lemma 6.5** Applying Lemma 5.8 and Proposition 5.5 we conclude that, for any $x_0 \in \Omega \setminus A_n$, $\beta_m(s^\varepsilon_{m,x_0}) \rightarrow \beta_m(s_{x_0})$ weakly in $L^2(0,T;H^1(Y_m))$, $s^\varepsilon_{m,x_0} \rightarrow s_{x_0}$ a.e. in $Y_m \times (0,T)$, and the limit $s_{x_0}$ does not depend of the fast variable $y$. Due to continuity of the trace operator we also have:

$$\beta_m(s^\varepsilon_{m,x_0})|_{\Gamma_{mf}} \rightarrow \beta_m(s_{x_0})|_{\Gamma_{mf}} \quad \text{weakly in } L^2(0,T;L^2(\Gamma_{mf})).$$

On the other hand we know that, for a.e. $x_0 \in \Omega$,

$${\mathcal{M}(\beta_f(\mathcal{D}^\varepsilon(S^n_m(x_0,\cdot))))}|_{\Gamma_{mf}} = \beta_m(s^\varepsilon_{m,x_0})|_{\Gamma_{mf}} \quad \text{with} \quad \mathcal{M} \overset{\text{def}}{=} \beta_m \circ (P_{m,c})^{-1} \circ P_{c,f} \circ (\beta_f)^{-1}$$

a.e. on $\Gamma_{mf} \times (0,T)$. For a.e. $x_0 \in \Omega$, from Corollary 5.2 we have that

$${\mathcal{M}(\beta_f(\mathcal{D}^\varepsilon(S^n_m(x_0,\cdot))))} \rightarrow {\mathcal{M}(\beta_f(S(x_0,\cdot)))} \quad \text{strongly in } L^2(0,T;L^2(\Gamma_{mf}))$$

and therefore, for a.e. $x_0 \in \Omega \setminus A_n$,

$$\beta_m(s_{x_0})|_{\Gamma_{mf}} = {\mathcal{M}(\beta_f(S(x_0,\cdot)))}|_{\Gamma_{mf}}.$$ 

Since these functions are independent of $y$ we have that $\beta_m(s_{x_0}) = {\mathcal{M}(\beta_f(S(x_0,\cdot)))}$ in $L^2(0,T)$, or, equivalently, $s_{x_0} = \mathcal{P}(S(x_0,\cdot))$. Now, for a chosen $x_0 \in \Omega \setminus A_n$, we can find a subsequence such that

$$s^\varepsilon_{m,x_0} \rightarrow \mathcal{P}(S(x_0,\cdot)) \quad \text{a.e. in } Y_m \times (0,T).$$

Since the limit is uniquely defined by the limit $S$ of the sequence $\mathcal{D}^\varepsilon(S^n_f)$ we conclude that the whole sequence converge to the same limit (that is the whole subsequence for which $\mathcal{D}^\varepsilon(S^n_f)$ converges). Now we can repeat our procedure for almost any $x_0 \in \Omega \setminus A_n$ and conclude that $s = \mathcal{P}(S)$ a.e. in $(\Omega \setminus A_n) \times (0,T)$. Thanks to Propositions 5.3, 5.4 the measure of the set $A_n$ goes to zero as $n \rightarrow \infty$ and the desired equality (6.54) is proved.

Now we complete easily the proof of Theorem 6.3. Taking into account (6.54) we can rewrite (6.53) and thereby obtain (6.52). Theorem 6.3 is proved.

**Acknowledgments**

The work of M. Jurak and A. Vrbaški was funded by Croatian science foundation project no 3955. The work of L. Pankratov was funded by the RScF research project N 15-11-00015. This work was partially supported by ISIFoR (http://www.carnot-isifor.eu) France. The supports are gratefully acknowledged. Most of the work on this paper was done when M. Jurak and L. Pankratov were visiting the Applied Mathematics Laboratory of the University of Pau & CNRS.
References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso, D. Percival, An extension theorem from connected sets, and homogenization in general periodic domains, *J. Nonlinear Analysis*, 18, (1992), 481–496.

[2] L. Ait Mahiout, B. Amaziane, A. Mokrane, L. Pankratov, Homogenization of immiscible compressible two-phase flow in double porosity media, *Electronic Journal of Differential Equations*, 2016:52 (2016), 1–28.

[3] G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.*, 28, (1992), 1482–1518.

[4] G. Allaire, A. Damlamian, U. Hornung, Two-scale convergence on periodic surfaces and applications, In Proceedings of the International Conference on Mathematical Modelling of Flow through Porous Media, A. Bourgeat et al. eds., pp. 15–25, World Scientific Pub., Singapore (1996).

[5] B. Amaziane, S. Antontsev, L. Pankratov, A. Piatnitski, Homogenization of immiscible compressible two-phase flow in porous media: application to gas migration in a nuclear waste repository, *SIAM MMS*, 8, (2010), 2023–2047.

[6] B. Amaziane, L. Pankratov, Homogenization of a model for water-gas flow through double-porosity media, *Math. Meth. Appl. Sci.*, 39, (2016), 425–451.

[7] B. Amaziane, L. Pankratov, A. Piatnitski, Homogenization of a class of quasilinear elliptic equations in high-contrast fissured media, *Proc. of the Royal Society of Edinburgh*, 136A (2006), 1131–1155.

[8] B. Amaziane, L. Pankratov, A. Piatnitski, Nonlinear flow through double porosity media in variable exponent Sobolev spaces, *Nonlinear Analysis: Real World Applications*, 10 (2009), 2521–2530.

[9] B. Amaziane, L. Pankratov, A. Piatnitski, Homogenization of immiscible compressible two-phase flow in highly heterogeneous porous media with discontinuous capillary pressures, *Math. Models and Methods in Applied Sci.*, 24, (2014), 1421–1451.

[10] B. Amaziane, L. Pankratov, A. Piatnitski, The existence of weak solutions to immiscible compressible two-phase flow in porous media: the case of fields with different rock-types, *DCDS B* 18, 5 (2013), 1217–1251.

[11] B. Amaziane, L. Pankratov, V. Rybalko, On the homogenization of some double porosity models with periodic thin structures, *Appl. Anal.*, 88 (2009), 1469–1492.

[12] S. N. Antontsev, A. V. Kazhikhov, V. N. Monakov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, North-Holland, Amsterdam, 1990.

[13] T. Arbogast, J. Douglas, U. Hornung, Derivation of the double porosity model of immiscible two-phase flow via homogenization theory, *SIAM J. Math. Anal.*, 21, (1990), 823–826.

[14] G. I. Barenblatt, Yu. P. Zheltov, I. N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. Mech.*, 24 (1960), 1286–1303.

[15] J. Bear, C.F. Tsang, G. de Marsily, *Flow and Contaminant Transport in Fractured Rock*, Academic Press Inc, London, 1993.

[16] A. Bourgeat, S. Luckhaus, A. Mikelić, Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow, *SIAM J. Math. Anal.*, 27, (1996), 1520–1543.

[17] A. Bourgeat, A. Mikelić, A. Piatnitski, Modèle de double porosité aléatoire, *C. R. Acad. Sci. Paris, Sér. I*, 327 (1998), 99–104.

[18] A. Bourgeat, M. Goncharenko, M. Panfilov, L. Pankratov, A general double porosity model, *C. R. Acad. Sci. Paris, Série IIb*, 327 (1999), 1245–1250.

[19] A. Bourgeat, G. Chechkin, A. Piatnitski, Singular double porosity model, *Appl. Anal.*, 82 (2003), 103–116.

[20] A. Braides, V. Chiadò Piat, A. Piatnitski, Homogenization of discrete high-contrast energies, *SIAM J. Math. Anal.*, 47 (2015), 3064–3091.

[21] G. Chavent, J. Jaffré, *Mathematical Models and Finite Elements for Reservoir Simulation*, North-Holland, Amsterdam, 1986.

[22] Z. Chen, G. Huan, Y. Ma, *Computational Methods for Multiphase Flows in Porous Media*, SIAM, Philadelphia, 2006.

[23] C. Choquet, Derivation of the double porosity model of a compressible miscible displacement in naturally fractured reservoirs, *Appl. Anal.*, 83, (2004), 477–500.

[24] C. Choquet, L. Pankratov, Homogenization of a class of quasilinear elliptic equations with non-standard growth in high-contrast media, *Proc. of the Royal Society of Edinburgh*, 140 (2010), 495–539.

[25] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, *C. R. Acad. Sci. Paris, Ser. I*, 335, (2002), 99–104.
[26] G. W. Clark, R.E. Showalter, Two-scale convergence of a model for flow in a partially fissured medium, *Electronic Journal of Differential Equations*, **1999** (1999), 1–20.

[27] H. I. Ene, D. Polisevski, Model of diffusion in partially fissured media, *Z. angew. Math. Phys.*, **53**, (2002), 1052–1059.

[28] R. Helmig, *Multiphase Flow and Transport Processes in the Subsurface*, Springer, Berlin, 1997.

[29] P. Henning, M. Ohlberger, B. Schweizer, Homogenization of the degenerate two-phase flow equations, *Math. Models Methods Appl. Sci.*, **23**, (2013), 2323–2352.

[30] U. Hornung, *Homogenization and Porous Media*, Springer-Verlag, New York, 1997.

[31] M. Jurak, L. Pankratov, A. Vrbaški, A fully homogenized model for incompressible two-phase flow in double porosity media, *Applicable Analysis* (2015), DOI: 10.1080/00036811.2015.1031221.

[32] V. A. Marchenko, E. Ya. Khruslov, *Homogenization of Partial Differential Equations*, Boston, Birkhäuser, 2006.

[33] M. Panfilov, *Macroscale Models of Flow Through Highly Heterogeneous Porous Media*, Dordrecht-Boston-London, Kluwer Academic Publishers, 2000.

[34] L. Pankratov, V. Rybalko, Asymptotic analysis of a double porosity model with thin fissures, *Mat. Sbornik*, **194**, (2003), 121–146.

[35] G. Sandrakov, Homogenization of parabolic equations with contrast coefficients, *Izvestiya: Mathematics*, **63**, (1999), 1015–1061.

[36] R. P. Shaw, Gas Generation and Migration in Deep Geological Radioactive Waste Repositories. Geological Society, 2015.

[37] J. Simon, Compact sets in the space $L^p(0, t; B)$, *Ann. Mat. Pura Appl. IV. Ser.*, **146**, (1987), 65–96.

[38] J. L. Vázquez, *The Porous Medium Equation*, Oxford University Press Inc., New York, 2007.

[39] T. D. Van Golf-Racht, *Fundamentals of Fractured Reservoir Engineering*, Elsevier Scientific Publishing Company, Amsterdam, 1982.

[40] L. M. Yeh, Homogenization of two-phase flow in fractured media, *Math. Models and Methods in Applied Sci.*, **16**, (2006), 1627–1651.