On the automorphism group of a smooth Schubert variety

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Abstract

Let \( G \) be a simple, simply connected algebraic group over the field \( \mathbb{C} \) of complex numbers. Let \( B \) be a Borel subgroup of \( G \) containing a maximal torus \( T \) of \( G \). Let \( w \) be an element of the Weyl group \( W \) and let \( X(w) \) be the Schubert variety in \( G/B \) corresponding to \( w \). Let \( \alpha_0 \) denote the highest long root of \( G \) with respect to \( T \) and \( B \) and let \( \mathcal{L}_{\alpha_0} \) denote the line bundle on \( X(w) \) associated to \( \alpha_0 \). Let \( G_{ad} \) denote the adjoint group corresponding to \( G \).

In this paper, we prove that if \( G \) is simply laced and \( X(w) \) is smooth, then the connected component of the automorphism group of \( X(w) \) is a parabolic subgroup of \( G_{ad} \) if and only if the set \( X(w^{-1})^{ss}_{v}(\mathcal{L}_{\alpha_0}) \) of semi-stable points is non-empty ( cf Corollary 3.8 ). We prove a partial result in the non simply laced case ( cf Theorem 4.7 ).

Keywords: Automorphism group, Schubert varieties, Tangent bundle.

1 Introduction

Let \( G \) be a simple, simply connected algebraic group over the field \( \mathbb{C} \) of complex numbers. Let \( B \) be a Borel subgroup of \( G \) containing a maximal torus \( T \) of \( G \). Let \( w \) be an element of the Weyl group \( W \). Let \( X(w) \) be the Schubert variety in \( G/B \) corresponding to \( w \).

The aim of this paper is to answer the following natural questions:

1. What is the automorphism group of a smooth Schubert variety \( X(w) \)?
2. For which Schubert variety \( X(w) \) in \( G/B \) all higher cohomologies \( H^i(X(w), \mathcal{T}_{G/B}) \) of the restriction of the tangent bundle \( \mathcal{T}_{G/B} \) to \( X(w) \) vanish?
More precisely, we prove the following Theorems.

**Theorem 1** Let $G$ be a simple, simply connected and simply laced algebraic group over $\mathbb{C}$. Let $w \in W$. Then, we have

1. $H^i(X(w), T_{G/B}) = (0)$ for every $i \geq 1$.
2. $H^0(X(w), T_{G/B})$ is the adjoint representation $\mathfrak{g}$ of $G$ if and only if the set of semi-stable points $X(w^{-1})^{ss}_{T}(\mathcal{L}_\alpha)$ is non-empty.

**Theorem 2**

Let $G$ be a simple, simply connected but not necessarily simply laced algebraic group over $\mathbb{C}$. Let $w \in W$. Then, we have

1. $H^i(X(w), T_{G/B}) = (0)$ for every $i \geq 1$.
2. The adjoint representation $\mathfrak{g}$ is a $B$-submodule of $H^0(X(w), T_{G/B})$ if and only if the set of semi-stable points $X(w^{-1})^{ss}_{T}(\mathcal{L}_\alpha)$ is non-empty.

For notation used in the above Theorems, see Theorem 3.7 and Theorem 4.7.

These questions arose in a natural way by the work of following Mathematicians:

Bott proved that all the higher cohomologies $H^i(G/B, T_{G/B})$ with respect to $T_{G/B}$ on the flag variety $G/B$ vanishes. He further showed that the $G$-module $H^0(G/B, T_{G/B})$ of global sections is the adjoint representation $\mathfrak{g}$ of $G$ (cf [3, Theorem VII]).

Also, there is a close connection of the smoothness of Schubert varieties with the representation theory of $B$-submodules of $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{b}$ is the Lie algebra of $B$. For instance, Carrell proved that if $G$ is not of type $G_2$, then $X(w)$ is smooth if and only if its Poincare polynomial is palindromic and the tangent space $TE(X(w))$ of the set $E(X(w))$ of $T$-curves to $X(w)$ is a $B$-submodule of $\mathfrak{g}/\mathfrak{b}$ (cf [5, Theorem 2]). For an exposition of results on smooth Schubert varieties, see the book of Billey-Lakshmibai [2].

Our vanishing results are similar in the spirit with the results relating the geometry of Schubert varieties and the representation theory of $B$-modules. For instance, the tangent space of $idB$ in $G/B$ is $\mathfrak{g}/\mathfrak{b}$. As a $T$-module, $\mathfrak{g}/\mathfrak{b}$ is a direct sum of weight spaces each of which is not dominant except the highest short root and highest long root.

Demazure character formula gives the Euler characteristic of the line bundle $\mathcal{L}_\lambda$ corresponding to a character $\lambda$ of $T$ (see [9, II, 14.18]) or (see [1, Theorem , page 617 ]). If $\lambda$ is dominant, using Frobenius split methods, Mehta-Ramanathan have proved that $H^i(X(w), \mathcal{L}_\lambda) = (0)$ for all $w \in W$ and for all $i \geq 1$ (cf [14] ). Hence, the character of the $T$-module $H^0(X(w), \mathcal{L}_\lambda)$ is given by the Demazure character formula.

Also, using Standard Monomial Theory methods, Lakshmibai-Musili-Seshadri proved cohomology vanishing results for Schubert varieties and they deduce geometric properties like Arithmetic Cohen Macaulay using the vanishing results (cf [16, Theorem 1.4.19]). Further, Littelmann constructed a basis of $H^0(X(w), \mathcal{L}_\lambda)$ corresponding to the set of all $LS$ paths of shape $\lambda$ which is compatible with the geometry of Schubert varieties. He used such a basis to deduce the projective normality property of Schubert varieties (cf [12]).

The organisation of the paper is as follows:
Section 2 consists of preliminaries from [4], [7], [8] and [9]. In section 3, we prove Theorem 1 and deduce Corollary 3.8. In section 4, we prove Theorem 2.

2 Notation and Preliminaries

The following notation will be maintained throughout this paper.

Let \( \mathbb{C} \) denote the field of complex numbers. Let \( G \) a simple, simply connected algebraic group over \( \mathbb{C} \). We fix a maximal torus \( T \) of \( G \) and let \( X(T) \) denote the group of all characters of \( T \). Let \( W = N(T)/T \) denote the Weyl group of \( G \) with respect to \( T \). Let \( R \) denote the set of roots of \( G \) with respect to \( T \).

Let \( R^+ \) denote the set of positive roots. Let \( B^+ \) be the Borel sub group of \( G \) containing \( T \) with respect to \( R^+ \). Let \( S = \{ \alpha_1, \ldots, \alpha_l \} \) denote the set of simple roots in \( R^+ \). Here \( l \) is the rank of \( G \). Let \( B \) be the Borel subgroup of \( G \) containing \( T \) with respect to the set of negative roots \( R^- = -R^+ \).

For \( \beta \in R^+ \) we also use the notation \( \beta > 0 \). The simple reflection in the Weyl group corresponding to \( \alpha_i \) is denoted by \( s_{\alpha_i} \).

Let \( \mathfrak{g} \) denote the Lie algebra of \( G \). Let \( \mathfrak{h} \) be the Lie algebra of \( T \). Let \( \mathfrak{b} \) be the Lie algebra of \( B \). We have \( X(T) \otimes \mathbb{R} = \text{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R}) \), the dual of the real form of \( \mathfrak{h} \). The positive definite \( W \)-invariant form on \( \text{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R}) \) induced by the Killing form of the Lie algebra \( \mathfrak{g} \) of \( G \) is denoted by \( (\ ,\ ) \). We use the notation \( \langle \ ,\ \rangle \) to denote \( \langle \nu, \alpha \rangle = \frac{2(\nu, \alpha)}{(\alpha, \alpha)} \).

Let \( x_\alpha, \alpha \in R, h_\alpha, \alpha \in S \), denote a Chevalley basis of the Lie algebra of \( G \). For a simple root \( \alpha \), we denote by \( \mathfrak{g}_\alpha \) (respectively \( \mathfrak{g}_{-\alpha} \)) the one dimensional root subspace of \( \mathfrak{g} \) spanned by \( x_\alpha \) (respectively \( x_{-\alpha} \)). Let \( sl_2, \alpha \) denote the 3 dimensional Lie subalgebra of \( \mathfrak{g} \) generated by \( x_\alpha \), and \( x_{-\alpha} \).

Let \( \leq \) denote the partial order on \( X(T) \) given by \( \mu \leq \lambda \) if \( \lambda - \mu \) is a non negative integral linear combination of simple roots. We denote by \( X(T)^+ \) the set of dominant characters of \( T \) with respect to \( B^+ \). Let \( \rho \) denote the half sum of all positive roots of \( G \) with respect to \( T \) and \( B^+ \). For any simple root \( \alpha \), we denote the fundamental weight corresponding to \( \alpha \) by \( \omega_\alpha \).

For \( w \in W \), let \( l(w) \) denote the length of \( w \). We define the dot action by \( w \cdot \lambda = w(\lambda + \rho) - \rho \). Let \( \alpha_0 \) denote the highest long root. We set \( R^+(w) := \{ \beta \in R^+ : w(\beta) \in -R^+ \} \). Let \( w_0 \) denote the longest element of the Weyl group \( W \). For \( w \in W \), let \( X(w) := BwB/B \) denote the Schubert variety in \( G/B \) corresponding to \( w \).

Consider the left action of \( T \) on \( G/B \). Schubert varieties \( X(w) \) are stable under \( T \). Let \( \lambda \) be a dominant character of \( T \). We denote by \( L_\lambda \) denote the line bundle on \( G/B \) corresponding to the character \( \lambda \) of \( B \). We denote the restriction of the line bundle \( L_\lambda \) to \( X(w) \) as well by \( L_\lambda \).

We use the notion of semi-stable points introduced by Mumford [15]. We denote by \( X(w)_{ss}^T(L_\lambda) \) the set of all semi-stable points of \( X(w) \) with respect to the \( T \)-linearised line bundle \( L_\lambda \). So, in particular, we have the set \( X(w)_{ss}^T(L_{\alpha_0}) \) of semi-stable points with respect to the line bundle \( L_{\alpha_0} \) corresponding to \( \alpha_0 \).
2.1 Preliminaries on Bott-Samelson-Hansen-Demazure varieties

We recall some preliminaries on Bott-Samelson-Hansen-Demazure varieties and some application of Leray spectral sequences. Good references for this are [4] and [9].

We denote by $U$ the unipotent radical of $B$. We denote by $P_\alpha$ the minimal parabolic subgroup of $G$ containing $B$ and $s_\alpha$. Let $L_\alpha$ denote the Levi subgroup of $P_\alpha$ containing $T$. We denote by $B_\alpha$ the intersection of $L_\alpha$ and $B$. Then $L_\alpha$ is the product of $T$ and a homomorphic image $G_\alpha$ of $SL(2)$ via a homomorphism $\psi : SL(2) \to L_\alpha$. (cf. [9, II, 1.1.4]).

We make use of following points in computing cohomology groups.

Since $G$ is simply connected, the morphism $\psi : SL(2) \to G_\alpha$ is an isomorphism, and hence $\psi : SL(2) \to L_\alpha$ is injective. We denote this copy of $SL(2)$ in $L_\alpha$ by $SL(2,\alpha)$ We denote by $B'_\alpha$ the intersection of $B_\alpha$ and $SL(2,\alpha)$ in $L_\alpha$.

We also note that the morphism $SL(2,\alpha)/B'_\alpha \to L_\alpha/B_\alpha$ induced by $\psi$ is an isomorphism. Since $L_\alpha/B_\alpha \to P_\alpha/B$ is an isomorphism, to compute the cohomology $H^i(P_\alpha/B,V)$ for any $B$-module $V$, we treat $V$ as a $B_\alpha$-module and we compute $H^i(L_\alpha/B_\alpha,V)$.

Let $w \in W$. Let $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ be a reduced expression for $w$. Let $Z(w)$ be a Bott-Samelson-Demazure-Hansen variety corresponding to the reduced expression as above.

Let $V$ be a $B$-module. Let $\mathcal{L}_w(V)$ denote the pull back to $X(w)$ of the homogeneous vector bundle on $G/B$ associated to $V$. By abuse of notation we denote the pull back of $\mathcal{L}_w(V)$ to $Z(w)$ also by $\mathcal{L}_w(V)$, when there is no cause for confusion. We use the Bott-Samelson-Demazure-Hansen scheme $Z(w)$ for the computing and study of all the cohomology modules $H^i(X(w),\mathcal{L}_w(V))$.

We use the following ascending 1-step construction as a basic tool in computing cohomology modules.

Let $w$ in $W$. Let $\gamma$ be a simple root such that $l(w) = l(s_\gamma w) + 1$. Let $Z(w)$ be a Bott-Samelson-Demazure-Hansen variety corresponding to a reduced expression $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_r}}$ where $\alpha_{i_1} = \gamma$.

Then we have an induced morphism

$$g : Z(w) \to P_\gamma/B \simeq \mathbb{P}^1,$$

with fibres given by $Z(s_\gamma w)$.

By an application of the Leray spectral sequences together with the fact that the base is $\mathbb{P}^1$, we obtain for every $B$-module $V$, the following exact sequence of $P_\gamma$-modules:

$$(0) \to H^1(P_\gamma/B, R^{i-1}g_*\mathcal{L}_w(V)) \to H^i(Z(w),\mathcal{L}_w(V)) \to H^0(P_\gamma/B, R^ig_*\mathcal{L}_w(V)) \to (0).$$

This short exact sequence of $B$-modules will be used frequently in this paper. So, we denote this short exact sequence by $SES$ when ever this is being used.

Simplicity of Notation If $V$ is a $B$-module and $\mathcal{L}_w(V)$ is the induced vector bundle on $Z(w)$ we denote the cohomology modules $H^i(Z(w),\mathcal{L}_w(V))$ by $H^i(w,V)$. In particular, if $\lambda$ is a character of $B$ we denote the cohomology modules $H^i(Z(w),\mathcal{L}_\lambda)$ by $H^i(w,\lambda)$. 

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2.1.1 Some constructions from Demazure’s paper

We recall briefly two exact sequences from [6] that Demazure used in his short proof of the Borel-Weil-Bott Theorem (cf. [3]). We use the same notation as in [6].

Let \( \alpha \) be a simple root and let \( \lambda \in X(T) \) be a weight such that \( \langle \lambda, \alpha \rangle \geq 0 \). For such a \( \lambda \), we denote by \( V_{\lambda,\alpha} \) the module \( H^0(P_\alpha/B, L_{\lambda}) \). Let \( \mathbb{C}_\lambda \) denote the one dimensional \( B \)-module.

Here, we recall the following Lemma due to Demazure on a short exact sequence of \( B \)-modules:

1. If \( \langle \lambda, \alpha \rangle \geq 0 \), then \( H^i(w, \lambda) = H^i(\phi, V_{\lambda,\alpha}) \) for all \( j \geq 0 \).
2. If \( \langle \lambda, \alpha \rangle \geq 0 \), then \( H^i(w, \lambda) = H^{i+1}(w, s_\alpha \cdot \lambda) \). Further, if \( \langle \lambda, \alpha \rangle \leq -2 \), then \( H^i(w, \lambda) = H^{i-1}(w, s_\alpha \cdot \lambda) \).
3. If \( \langle \lambda, \alpha \rangle = -1 \), then \( H^i(w, \lambda) \) vanishes for every \( i \geq 0 \) (cf. [9, Prop 5.2]).

The following Lemma is an easy consequence of the Lemma 2.2 and [9, I, Proposition 4.8] which will be used to compute cohomology modules in this paper.

Let \( V \) be an irreducible \( L_\alpha \)-module. Let \( \lambda \) be a character of \( B_\alpha \).

Lemma 2.3. Then, we have

1. If \( \langle \lambda, \alpha \rangle \geq 0 \), then \( H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \) is isomorphic to the tensor product of \( V \) and \( H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda) \), and \( H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0 \) for every \( j \geq 1 \).
2. If \( \langle \lambda, \alpha \rangle \leq -2 \), then \( H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0 \), and \( H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \) is isomorphic to the tensor product of \( V \) and \( H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \lambda}) \).
3. If \( \langle \lambda, \alpha \rangle = -1 \), then \( H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0 \) for every \( j \geq 0 \).

We now state a combinatorial Lemma which is used in the computation of cohomology modules. We give a proof here for completeness.

Lemma 2.4. Let \( G \) be a simple simply laced algebraic group. Let \( \alpha \in S \), and \( \beta \) be a root different from both \( \alpha \) and \( -\alpha \). Then, \( \langle \beta, \alpha \rangle \in \{-1, 0, 1\} \).

Proof. Since \( \beta \) and \( \alpha \) are not proportional, we see that the product \( \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle \) is an integer lying in \( \{0, 1, 2, 3\} \).

Since \( G \) is simply laced, we have \( \langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle \). Since \( \langle \beta, \alpha \rangle \) is an integer, we have \( \langle \beta, \alpha \rangle \in \{-1, 0, 1\} \).
Lemma 3.1. Let $V$ given by

In this subsection, we prove that for any $B$-submodule $V$ of $g$, the evaluation map $ev : H^0(w, V) \to V$ given by $ev(s) = s(idB)$ is injective (cf Lemma 3.2).

We first prove the following basic Lemma.

**Lemma 2.5.** Every indecomposable $B_\gamma$-summand $V$ of $g$ is one of the following:

1. $V = \mathbb{C} \cdot h$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 0$.
2. $V = g_\beta \bigoplus g_{\beta - \gamma}$ for some root $\beta$ such that $\langle \beta, \gamma \rangle = 1$.
3. $V = sl_{2,\gamma}$, the three dimensional irreducible $L_\gamma$-module with highest weight $\gamma$.

**Proof.** Let $V$ be an indecomposable $B_\gamma$-summand of $g$. Let $\lambda$ be a maximal weight of $V$. Then, the direct sum $\bigoplus_{r \in \mathbb{Z}_{\geq 0}} V_{\lambda - r\gamma}$ is a $B_\gamma$-summand of $V$. Hence, we have $V = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} V_{\lambda - r\gamma}$. By Lemma 2.4, the dimension of $V$ must be at most two unless $V = sl_{2,\gamma}$.

Further, if the dimension of $V$ is one, $V = \mathbb{C} \cdot h$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 0$. Also, if the dimension of $V$ is two, then, we must have $V = g_\beta \bigoplus g_{\beta - \gamma}$ for some root $\beta$ such that $\langle \beta, \gamma \rangle = 1$. This completes the proof of the Lemma.

\[ \square \]

### 3 Proof of Theorem 1 - simply laced case

#### 3.1 Global sections $H^0(G/B, V)$ for the case when $V$ is a $G$-module

In this subsection, we prove that for any $B$-submodule $V$ of $g$, the evaluation map $ev : H^0(w, V) \to V$ given by $ev(s) = s(idB)$ is injective (cf Lemma 3.2).

We first prove the following basic Lemma.

**Lemma 3.1.** Let $V$ be a finite dimensional rational $G$-module.

1. Let $w \in W$. Then, the evaluation map $ev : H^0(w, V) \to V$ is an isomorphism of $B$-modules.
2. Let $w \in W$. Then, we have $H^i(w, V) = (0)$ for every $i \geq 1$.

**Proof.** Proof is by induction on $l(w)$.

If $l(w) = 0$, we are done. Otherwise, we choose a simple root $\gamma \in S$ be such that $l(s_\gamma w) = l(w) - 1$.

Since $V$ is a $P_\gamma$-module, by [9, I, Proposition 4.8], we have $H^i(P_\gamma/B, V) = V \otimes H^i(P_\gamma/B, \mathbb{C}_0)$. Since $H^i(P_\gamma/B, \mathbb{C}_0)$ is zero for every $i \geq 1$ and $H^0(P_\gamma/B, \mathbb{C}_0)$ is the trivial one dimensional $B$-module, we conclude that $H^i(P_\gamma/B, V) = 0$ for every $i \geq 1$ and the evaluation map $ev : H^0(P_\gamma/B, V) \to V$ is an isomorphism.

By induction, the evaluation map $ev : H^0(s_\gamma w, V) \to V$ is an isomorphism of $B$-modules and $H^i(s_\gamma w, V) = (0)$ for every $i \geq 1$. Now, using the short exact sequence $SES$ of $B$-modules, we see that for every $i \in \mathbb{Z}_{\geq 0}$,

$$H^i(w, V) = H^i(P_\gamma/B, H^0(s_\gamma w, V)) = H^i(P_\gamma/B, V).$$
Also, the evaluation map \( ev : H^0(w, V) \to V \) is the composition of \( ev : H^0(s, w, V) \to V \) and \( ev : H^0(P_s/B, H^0(s, w, V)) \to H^0(s, w, V) \). Since both the maps \( ev : H^0(s, w, V) \to V \) and \( ev : H^0(P_s/B, H^0(s, w, V)) \to H^0(s, w, V) \) are isomorphisms, \( ev : H^0(w, V) \to V \) is also an isomorphism. Proof of \( H^i(w, V) = 0 \) for every \( i \geq 1 \) is similar.

This completes the proof of the lemma.

Let \( w \in W \). Let \( \gamma \) be a simple root. Let \( V \) be a \( B \)-sub module of \( \mathfrak{g} \). Then, we have the following Lemma on the evaluation map.

**Lemma 3.2.** The evaluation map \( ev : H^0(w, V) \to V \) is injective.

**Proof.** Since \( V \) is a \( B \)-submodule of \( \mathfrak{g} \), \( H^0(w, V) \) is a \( B \)- submodule of \( H^0(w, \mathfrak{g}) \). Hence, we have the following commutative diagram of \( B \)- modules:

\[
\begin{array}{ccc}
H^0(w, V) & \longrightarrow & H^0(w, \mathfrak{g}) \\
\downarrow & & \downarrow \\
V & \longrightarrow & \mathfrak{g}
\end{array}
\]

Here, both the horizontal maps are the canonical inclusions and both the vertical maps are evaluation maps.

Since the \( B \)- module \( \mathfrak{g} \) is the restriction of a \( G \)- module, by using Lemma 3.1, we see that the evaluation map \( ev : H^0(w, \mathfrak{g}) \to \mathfrak{g} \) on the right hand side is an isomorphism of \( B \)- modules. Since the first horizontal map \( H^0(w, V) \to H^0(w, \mathfrak{g}) \) is injective, the composition \( H^0(w, V) \to \mathfrak{g} \) is also injective.

Thus, we conclude that vertical map \( ev : H^0(w, V) \to V \) on the left hand side is injective. This completes the proof of the Lemma.

\[\square\]

### 3.2 Proof of Theorem 1

In this subsection, we prove Theorem 1. The following notation will be maintained throughout this subsection.

Let \( G \) be a simple, simply connected and simply laced algebraic group over \( \mathbb{C} \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \). Let \( w \in W \). Let \( \gamma \) be a simple root. Let \( V \) be a \( B \)-submodule of \( \mathfrak{g} \) containing \( \mathfrak{b} \).

We now prove the following **Key Lemma** on the indecomposable \( B_\gamma \)-summand of \( H^0(w, V) \). In this Lemma, we prove that if \( V' \) is an indecomposable \( B_\gamma \)-summand of \( H^0(w, V) \), then, it must be one of the types described below.

**Lemma 3.3.** Every indecomposable \( B_\gamma \)- summand \( V' \) of \( H^0(w, V) \) is one of the following:

1. \( V' = \mathbb{C} \cdot h \) for some \( h \in \mathfrak{h} \) such that \( \gamma(h) = 0 \).
2. \( V' = \mathbb{C} \cdot h \bigoplus \mathfrak{g} - \gamma \) for some \( h \in \mathfrak{h} \) such that \( \gamma(h) = 1 \) and \( \nu(h) = 0 \) for every simple root \( \nu \) different from \( \gamma \).
3. \(V' = \mathfrak{g}_\beta\) for some root \(\beta\) such that \(\langle\beta, \gamma\rangle \in \{-1, 0\}\).

4. \(V' = \mathfrak{g}_\beta \bigoplus \mathfrak{g}_{\beta-\gamma}\) for some root \(\beta\) such that \(\langle\beta, \gamma\rangle = 1\).

5. \(V' = sl_{2,\gamma}\), the restriction of the three dimensional irreducible \(L_\gamma\)-module with highest weight \(\gamma\).

**Proof.** Let \(V'\) be an indecomposable \(B_\gamma\)-summand of \(H^0(w, V)\). If the weight of the \(B_\gamma\)-stable line in \(V'\) is different from \(-\gamma\), then, using Lemma 2.5 and Lemma 3.2, we see that \(V'\) must be one of the types (1), (3) or (4).

Otherwise, \(\mathfrak{g}_{-\gamma}\) is a \(B_\gamma\)-submodule of \(V'\). In this case, we need to show that \(\mathfrak{g}_{-\gamma}\) is a proper subspace of \(V'\). That is, either \(V' = \mathbb{C} \cdot h \bigoplus \mathfrak{g}_{-\gamma}\) for some \(h \in \mathfrak{h}\) such that \(\gamma(h) = 1\) and \(\nu(h) = 0\) for every simple root \(\nu\) different from \(\gamma\) or \(V' = sl_{2,\gamma}\).

We prove this by induction on \(l(w)\).

If \(l(w) = 0\), then, \(w = id\) and so we are done. Otherwise, choose a simple root \(\alpha\) such that \(l(w) = 1 + l(s_\alpha w)\).

By induction on \(l(w)\), we assume that for any simple root \(\nu\), if \(V'\) is an indecomposable \(B_\nu\) summand of \(H^0(s_\alpha w, V)\) containing \(\mathfrak{g}_{-\nu}\), then, either \(V' = \mathbb{C} \cdot h \bigoplus \mathfrak{g}_{-\nu}\) for some \(h \in \mathfrak{h}\) such that \(\nu(h) = 1\) and \(\mu(h) = 0\) for every simple root \(\mu\) different from \(\nu\) or \(V' = sl_{2,\nu}\).

We now fix a simple root \(\gamma\). We give the details of proof in 3 different cases as follows.

**Case 1:** We first assume that \(\gamma = \alpha\).

Now, let \(V_1'\) be an indecomposable \(B_\gamma\)-summand of \(H^0(w, V)\) containing \(\mathfrak{g}_{-\gamma}\). Then, using Lemma 3.2, we see that there is an indecomposable \(B_\gamma\)-summand \(V'\) of \(H^0(s_\gamma w, V)\) containing \(\mathfrak{g}_{-\gamma}\). Since \(l(s_\gamma w) = l(w) - 1\), by induction on length of \(w\), we see that \(V'\) must be of type (2) or of type (5).

If \(V'\) is of type (2), then, \(H^0(s_\gamma, V') = 0\). Hence, \(\mathfrak{g}_{-\gamma}\) can not be a subspace of \(H^0(s_\gamma, V')\).

On the other hand, using SES, we have \(H^0(s_\gamma, H^0(s_\gamma w, V)) = H^0(w, V)\). Thus, \(\mathfrak{g}_{-\gamma}\) can not be a subspace of \(H^0(w, V)\). This is a contradiction.

If \(V'\) is of type (5), then, we have \(H^0(L_\gamma/B_\gamma, V') = V'\). Hence, \(H^0(L_\gamma/B_\gamma, V')\) is also of type (5). This completes the proof for the case when \(\alpha = \gamma\).

**Case 2:** We assume that \(\alpha\) is different from \(\gamma\) and \(\langle\gamma, \alpha\rangle \neq 0\). By using Lemma 2.4, we have \(\langle\gamma, \alpha\rangle = -1\).

By a similar argument as in **Case 1**, we may assume that there is a \(B_\gamma\)-summand \(V'\) of \(H^0(s_\gamma w, V)\) containing \(\mathfrak{g}_{-\gamma}\). By induction, \(V'\) must be of type (2) or of type (5).

**Sub case 1:** If \(V'\) is of type (2), then, \(V' = \mathbb{C} \cdot h \bigoplus \mathfrak{g}_{-\gamma}\) for some \(h \in \mathfrak{h}\) such that \(\gamma(h) = 1\) and \(\nu(h) = 0\) for every simple root \(\nu\) different from \(\gamma\).

Since \(\langle\gamma, \alpha\rangle = -1\), an indecomposable \(B_\alpha\)-summand \(V_1\) of \(H^0(s_\alpha w, V)\) containing \(\mathfrak{g}_{-\gamma}\) must be of the form \(\mathfrak{g}_{-\gamma} \bigoplus \mathfrak{g}_{-\gamma-\alpha}\). So, by Lemma 3.1, we have \(H^0(s_\alpha, V_1) = V_1\). Hence, \(\mathfrak{g}_{-\gamma}\) must be a subspace of \(H^0(s_\alpha, H^0(s_\alpha w, V))\).

Since \(\alpha \neq \gamma\), we have \(\alpha(h) = 0\). Hence, \(\mathbb{C} \cdot h\) is a \(B_\alpha\)-direct summand of \(H^0(s_\alpha w, V)\). Hence, \(\mathbb{C} \cdot h\) must be a \(B_\alpha\)-submodule of \(H^0(s_\alpha, H^0(s_\alpha w, V))\). Hence, \(V' = \mathbb{C} \cdot h \bigoplus \mathfrak{g}_{-\gamma}\) must be a subspace of \(H^0(s_\alpha, H^0(s_\alpha w, V))\).
Thus, by using $\text{SES}$, we conclude that $V' = \mathbb{C} \cdot h \bigoplus \mathfrak{g}_{-\gamma}$ is a subspace of $H^0(w, V)$.

Sub case 2: Let $V'$ be of type (5). Then, we have $V' = sl_{2,\gamma}$.

Now, if $\mathfrak{g}_{\alpha+\gamma}$ is a subspace of $H^0(s_\alpha w, V)$, then, $\mathfrak{g}_{\alpha+\gamma} \bigoplus \mathfrak{g}_\gamma$ is an indecomposable $B_\alpha$-summand of $H^0(s_\alpha w, V)$. Since $\mathfrak{g}_{\alpha+\gamma} + \mathfrak{g}_\gamma$ is a $L_\gamma$-module, using Lemma 3.1, we see that $H^0(s_\alpha, \mathfrak{g}_{\alpha+\gamma} \bigoplus \mathfrak{g}_\gamma) = \mathfrak{g}_{\alpha+\gamma} \bigoplus \mathfrak{g}_\gamma$. Hence, we have

Observation:

$\mathfrak{g}_{\alpha+\gamma} \bigoplus \mathfrak{g}_\gamma$ is a subspace of $H^0(s_\alpha w, V)$. On the other hand, since $H^0(s_\alpha, H^0(s_\alpha w, V))$ is a $B_\alpha$-module, using Observation, we see that the $B_\gamma$-span $sl_{2,\gamma}$ of $\mathfrak{g}_\gamma$ must be a $B_\gamma$-submodule of $H^0(s_\alpha, H^0(s_\alpha w, V))$.

Using $\text{SES}$, we conclude that $H^0(w, V)$ contains $sl_{2,\gamma}$. This proves that $H^0(w, V)$ contains an indecomposable $B_\gamma$-summand of type (5).

Now, if $\mathfrak{g}_{\alpha+\gamma}$ is not a subspace of $H^0(s_\alpha w, V)$, then, $\mathfrak{g}_\gamma$ is an indecomposable $B_\alpha$-direct summand of $H^0(s_\alpha w, V)$. Since $\langle \gamma, \alpha \rangle = -1$, by Lemma 2.2, we have $H^i(s_\alpha, \gamma) = 0$ for every $i \in \mathbb{Z}_{>0}$.

In particular, $\mathfrak{g}_\gamma$ can not be a subspace of $H^0(s_\alpha, H^0(s_\alpha w, V))$. Hence, $sl_{2,\gamma}$ is not a $B_\gamma$-submodule of $H^0(s_\alpha, H^0(s_\alpha w, V))$.

We now show that $H^0(s_\alpha, H^0(s_\alpha w, V))$ contains a $B_\gamma$-summand of type 2.

Let $S_1$ be the set of all simple roots $\beta$ such that $sl_{2,\beta}$ is a $B_\beta$-summand of $H^0(s_\alpha w, V)$. Then, clearly, $\gamma \in S_1$. Since $S_1$ is a linearly independent subset of $\text{Hom}(\mathfrak{h}, \mathbb{C})$, there is a $h_1$ in $\bigoplus_{\beta \in S_1}(\mathbb{C} \cdot h_\beta \cap H^0(s_\alpha w, V))$ such that $\gamma(h_1) = 1$ and $\beta(h_1) = 0$ for every simple root $\beta$ in $S_1$ different from $\gamma$.

If $\nu(h_1) = 0$ for every $\nu \in S$, we are done. Otherwise, let $S_2$ be the set of all simple roots $\nu \notin S_1$ such that $\nu(h_1) \neq 0$. Let $\nu \in S_2$. Then, there is a $\beta \in S_1$ such that $\nu(h_\beta) = -1$. Since $\beta \in S_1$, we have $h_\beta \in H^0(s_\alpha w, V)$. Hence, the Lie bracket $x_{-\nu} = [x_{-\nu}, h_\beta]$ is in $H^0(s_\alpha w, V)$. Hence, $\mathfrak{g}_{-\nu}$ must be a subspace of $H^0(s_\alpha w, V)$.

Therefore, by induction applying to the simple root $\nu$, either there is an indecomposable $B_{\nu}$-summand $H^0(s_\alpha w, V)$ of type (2) or of type (5) containing $\mathfrak{g}_{-\nu}$. Since $\nu \notin S_1$, we conclude that there is an indecomposable $B_{\nu}$-module $\mathbb{C} \cdot h(\nu) \bigoplus \mathfrak{g}_{-\nu}$ for some $h(\nu) \in \mathfrak{h}$ such that $\nu(h(\nu)) = 1$ and $\mu(h(\nu)) = 0$ for every simple root $\mu$ different from $\nu$.

Let $h_2 = \sum_{\nu \in S_2} \nu(h_1) h(\nu)$. Take $h = h_1 - h_2$. Then, we have $\gamma(h) = 1$ and $\mu(h) = 0$ for every simple root $\mu$ different from $\gamma$. Therefore, $\mathbb{C} \cdot h$ is a $B_\alpha$-summand of $H^0(s_\alpha w, V)$. Hence, we see that $H^0(s_\alpha, \mathbb{C} \cdot h) = \mathbb{C} \cdot h$ is a subspace of $H^0(s_\alpha, H^0(s_\alpha w, V))$.

Using $\text{SES}$, we conclude that $\mathbb{C} \cdot h \bigoplus \mathfrak{g}_{-\gamma}$ is a $B_\gamma$-direct summand of $H^0(w, V))$.

Case 3: We assume that $\langle \gamma, \alpha \rangle = 0$. Proof in this case is similar and actually simpler than that of Case 2.

\[\square\]

Let $G$ be simply laced. Let $V$ be a $B$-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$. Then, we have

**Lemma 3.4.** Let $w \in W$. Then, we have $H^i(w, V) = (0)$ for every $i \geq 1$.

**Proof.** Proof is by induction on $l(w)$.
If \( l(w) = 0 \), we are done. Otherwise, we choose a simple root \( \gamma \in S \) be such that \( l(s, w) = l(w) - 1 \). By Lemma 3.3, every indecomposable \( B_\gamma \)-summand \( V' \) of \( H^0(s, w, V) \) must be one of the 5 types given in Lemma 3.3.

Hence, using Lemma 2.3, we conclude that \( H^i(L_\gamma/B_\gamma, V') \) is zero for every indecomposable \( B_\gamma \)-summand \( V' \) of \( H^0(s, w, V) \) and for every \( i \geq 1 \).

Thus, we see that

\[
\text{Observation :} \quad H^i(P_\gamma/B, H^0(s, w, V)) = (0) \quad \text{for all} \quad i \geq 1.
\]

By induction on \( l(w) \), we have \( H^i(s, w, V) \) is zero for all \( i \geq 1 \). Now, using Observation and using the short exact sequence SES of \( B \) modules, we conclude that \( H^i(w, V) \) is zero for all \( i \geq 1 \). This completes the proof of Lemma.

\[
\square
\]

Let \( V_1 \) be a \( B \)-submodule of \( g \) containing \( \mathfrak{b} \). Let \( V_2 \) be a \( B \)-submodule of \( V_1 \) containing \( \mathfrak{b} \). Let \( w \in W \). The natural projection \( \Pi : V_1 \rightarrow V_1/V_2 \) of \( B \)-modules induces a homomorphism of \( B \)-modules \( \Pi_w : H^0(w, V_1) \rightarrow H^0(w, V_1/V_2) \) of \( B \)-modules.

We now deduce the following Lemma as a consequence of the Lemma 3.4.

**Lemma 3.5.**

1. \( H^i(w, V_1/V_2) \) is zero for all \( i \geq 1 \).
2. \( \Pi_w : H^0(w, V_1) \rightarrow H^0(w, V_1/V_2) \) is a surjective homomorphism of \( B \)-modules whose kernel is \( H^0(w, V_2) \).

**Proof.** Proof of (1):

We have the short exact sequence \((0) \rightarrow V_2 \rightarrow V_1 \rightarrow V_1/V_2 \rightarrow (0)\) of \( B \)-modules.

Applying \( H^i(w, -) \) to this short exact sequence of \( B \)-modules, we obtain the following long exact sequence of \( B \)-modules:

\[
\cdots \rightarrow H^i(w, V_2) \rightarrow H^i(w, V_1) \rightarrow H^i(w, V_1/V_2) \rightarrow H^{i+1}(w, V_2) \rightarrow \cdots
\]

By Lemma 3.4, \( H^i(w, V_2) \), \( H^i(w, V_1) \) and \( H^{i+1}(w, V_2) \) are all zero for every \( i \geq 1 \). Thus, we conclude that \( H^i(w, V_1/V_2) = (0) \) for every \( i \geq 1 \). This proves (1).

Proof of (2):

Taking \( i = 0 \) in Observation and using \( H^1(w, V_2) = (0) \), we obtain the following short exact sequence

\[
(0) \rightarrow H^0(w, V_2) \rightarrow H^0(w, V_1) \rightarrow H^0(w, V_1/V_2) \rightarrow (0).
\]

This proves (2).

\[
\square
\]

We have
Corollary 3.6. Let \( w \in W \). Let \( \alpha \) be a positive root. Then, \( H^i(w, \alpha) = (0) \) for every \( i \geq 1 \).

Proof. Let \( V_1 := \bigoplus_{\mu \leq \alpha} \mathfrak{g}_\mu \) denote the direct sum of the weight spaces of \( \mathfrak{g} \) of weights \( \mu \) satisfying \( \mu \leq \alpha \).

Let \( V_2 := \bigoplus_{\mu < \alpha} \mathfrak{g}_\mu \) denote the direct sum of the weight spaces of \( \mathfrak{g} \) of weights \( \mu \) satisfying \( \mu < \alpha \).

It is clear that \( V_2 \) is a \( B \)-submodule of \( \mathfrak{g} \) containing \( \mathfrak{b} \) and \( V_1 \) is a \( B \)-submodule of \( \mathfrak{g} \) containing \( V_2 \). Since \( \mathfrak{g}_\alpha \) is one dimensional and is isomorphic to the quotient \( V_1 / V_2 \), we have \( H^i(w, \alpha) = H^i(w, \mathfrak{g}_\alpha) = H^i(w, V_1 / V_2) \) for every \( i \geq 1 \). Hence, by Lemma 3.5, \( H^i(w, \alpha) = (0) \) for every \( i \geq 1 \).

This completes the proof of corollary.

We now prove Theorem 1.

Let \( G \) be simple, simply connected and simply laced algebraic group over \( \mathbb{C} \). Let \( T_{G/B} \) denote the tangent bundle of the flag variety \( G/B \). Let \( \alpha_0 \) denote the highest root of \( G \) with respect to \( T \) and \( B \). Let \( \mathcal{L}_{\alpha_0} \) denote the line bundle on \( X(w) \) associated to \( \alpha_0 \).

Then, we have

**Theorem 3.7.** Let \( w \in W \).

1. \( H^i(X(w), T_{G/B}) = (0) \) for every \( i \geq 1 \).
2. \( H^0(X(w), T_{G/B}) \) is the adjoint representation \( \mathfrak{g} \) of \( G \) if and only if the set \( X(w^{-1})^{ss}(\mathcal{L}_{\alpha_0}) \) of semi-stable points is non-empty.

Proof. Since the tangent space of \( G/B \) at the point \( idB \) is \( \mathfrak{g}/\mathfrak{b} \), the tangent bundle \( T_{G/B} \) is the homogeneous vector bundle \( \mathcal{L}(\mathfrak{g}/\mathfrak{b}) \) on \( G/B \) associated to the \( B \)-module \( \mathfrak{g}/\mathfrak{b} \).

Hence, it is sufficient to prove the following:

1. \( H^i(w, \mathfrak{g}/\mathfrak{b}) = (0) \) for every \( i \geq 1 \).
2. \( H^0(w, \mathfrak{g}/\mathfrak{b}) \) is the adjoint representation \( \mathfrak{g} \) of \( G \) if and only if the set \( X(w^{-1})^{ss}(\mathcal{L}_{\alpha_0}) \) of semi-stable points is non-empty.

We prove this now.

Let \( V_1 := \mathfrak{g} \) and let \( V_2 := \mathfrak{b} \). The natural projection \( \Pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{b} \) of \( B \)-modules induces a homomorphism \( \Pi_w : H^0(w, \mathfrak{g}) \rightarrow H^0(w, \mathfrak{g}/\mathfrak{b}) \) of \( B \)-modules.

Proof of (1) follows from Lemma 3.5.

Since the evaluation map \( ev : H^0(w, \mathfrak{g}) \rightarrow \mathfrak{g} \) is an isomorphism, using Lemma 3.5, we have the following short exact sequence of \( B \)-modules:

\[
(0) \rightarrow H^0(w, \mathfrak{b}) \rightarrow \mathfrak{g} \rightarrow H^0(w, \mathfrak{g}/\mathfrak{b}) \rightarrow (0).
\]

Taking \( -\alpha_0 \)-weight spaces, we obtain the following short exact sequence of vector spaces:

\[
(0) \rightarrow H^0(w, \mathfrak{b})_{-\alpha_0} \rightarrow \mathfrak{g}_{-\alpha_0} \rightarrow H^0(w, \mathfrak{g}/\mathfrak{b})_{-\alpha_0} \rightarrow (0).
\]

Since \( \mathfrak{g}_{-\alpha_0} \) is one dimensional, \( H^0(w, \mathfrak{b})_{-\alpha_0} \) is zero if and only if \( H^0(w, \mathfrak{g}/\mathfrak{b})_{-\alpha_0} \) is non-zero. Also, since there is a unique \( B \)-stable line in \( \mathfrak{g} \) and that is of weight \( -\alpha_0 \), we see that \( H^0(w, \mathfrak{b}) \) is zero if and only if \( H^0(w, \mathfrak{b})_{-\alpha_0} \) is zero.
On the other hand, $H^0(w, g/b)_{-\alpha_0}$ is non-zero if and only if $w^{-1}(-\alpha_0) \in R^+$. Also, by [11, Lemma 2.1], $w^{-1}(-\alpha_0) \in R^+$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non empty.

Summarising the above arguments, we conclude that $H^0(w, b)$ is zero if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non empty.

Since $\text{Ker}(\Pi_w) = H^0(w, b)$, we see that $\text{Ker}(\Pi_w)$ is zero if and only if the set $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ of semi-stable points is non-empty. This completes the proof of (2).

Let $w$ in $W$ be such that $X(w)$ is smooth. Let $H$ denote connected component of identity of the group of all automorphisms of $X(w)$. Let $G_{ad}$ denote the adjoint group corresponding to $G$.

Then, we have

**Corollary 3.8.** The set $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ of semi-stable points is non-empty if and only if $H$ is isomorphic to a parabolic subgroup of $G_{ad}$.

**Proof.** Let $T_w$ denote the tangent bundle of $X(w)$. By [13, Theorem 3.7], we see that $H$ is an algebraic group. It is well known that the Lie algebra of $H$ is isomorphic to the global sections $H^0(X(w), T_w)$.

Since $T_w$ is a vector subbundle of $T_{G/B}$, $H^0(X(w), T_w)$ is a subspace of $H^0(X(w), T_{G/B})$. Since $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty, we have $H^0(X(w), T_{G/B}) = g$.

Taking $V_1 = g$ and $V_2 = b$ in Lemma 3.5, we see that $\Pi_w : g \rightarrow H^0(X(w), T_{G/B})$ is surjective. Since $H^0(G/B, T_{G/B}) = g$, the restriction map $r : H^0(G/B, T_{G/B}) \rightarrow H^0(X(w), T_{G/B})$ is also surjective.

Since $H^0(X(w), T_w)$ is a subspace of $H^0(X(w), T_{G/B})$, $r^{-1}(H^0(X(w), T_w))$ is a subspace of $g$. Hence, $r^{-1}(H^0(X(w), T_w))$ is a Lie subalgebra of $g$ and the map $r : r^{-1}(H^0(X(w), T_w)) \rightarrow H^0(X(w), T_w)$ is a homomorphism of Lie algebras.

Thus, Lie($H$) is a subquotient of $g$. Further, Lie($H$) is a subalgebra of $g$ if and only if $\Pi_w : g \rightarrow H^0(w, T_{G/B})$ is an isomorphism.

By Theorem 3.7, $\Pi_w : g \rightarrow H^0(w, T_{G/B})$ is an isomorphism if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty.

The following Corollary connects the set $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ of semi-stable points with the vanishing of all cohomologies of the vector bundle $\mathcal{L}_0$ on $X(w)$.

**Corollary 3.9.** The set $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ of semi-stable points is non-empty if and only if $H^i(w, b) = 0$ for every $i \in \mathbb{Z}_{\geq 0}$.

**Proof.** We take $V_1 = g$ and $V_2 = b$ in Lemma 3.5. We then have the following long exact sequence of $B$-modules:

$$\cdots \rightarrow H^i(w, g) \rightarrow H^i(w, g/b) \rightarrow H^{i+1}(w, b) \rightarrow H^{i+1}(w, g) \cdots$$

By Theorem 3.7, we have $H^{i+1}(w, g) = 0$ and $H^i(w, g/b) = 0$ for every $i \geq 1$. Hence, we have $H^j(w, b) = 0$ for every $j \geq 2$. 

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The rest of the proof follows from part 2 of Theorem 3.7.

4 Proof of Theorem 2

Throughout this section, we assume that $G$ is simple, simply connected algebraic group over $\mathbb{C}$ which is not simply laced.

We first prove that for such $G$, there is a positive root $\beta$ and a simple root $\alpha$ such that $s_\alpha \cdot \beta$ is the highest short root.

Lemma 4.1. Let $G$ be a simple algebraic group which is not simply laced. Then, there is a positive root $\beta$ and a simple root $\alpha$ such that $s_\alpha \cdot \beta$ is the highest short root.

Proof. If $G$ is of type $G_2$, then, the simple roots $\alpha_1$ and $\alpha_2$ satisfy the following:

$$\langle \alpha_1, \alpha_2 \rangle = -1 \text{ and } \langle \alpha_2, \alpha_1 \rangle = -3.$$ Here, we follow the convention in [7].

In this case, we take $\beta = \alpha_2$ and $\alpha = \alpha_1$. Hence, $s_\alpha \cdot \beta = \alpha_2 + 2\alpha_1$ is the highest short root. Hence, we may assume that $G$ is a simple algebraic group of type $B_n$, $C_n$ or $F_4$.

Let $\nu$ be the highest short root. We now show that there is a simple root $\alpha$ such that $\nu + \alpha$ is a root and $\langle \nu, \alpha \rangle = 0$. To show that $\nu + \alpha$ is a root, it is sufficient to show that the weight space $g_{\nu+\alpha}$ is non-zero.

Since $G$ is simple and the base field is $\mathbb{C}$, $g$ is an irreducible $G$-module. Hence, $g_{\alpha_0}$ is the unique $B^+$-stable line in $g$. Hence, there is a simple root $\alpha$ such that the Lie bracket $[g_\alpha, g_\nu]$ is non-zero. Thus, $\nu + \alpha$ is a root.

Since $\nu$ is dominant, we have

Observation 1 $\langle \nu, \alpha \rangle \geq 0$.

On the other hand, since $G$ is not of type $G_2$, $\langle \nu + \alpha, \alpha \rangle \leq 2$. Hence, we have $\langle \nu, \alpha \rangle \leq 0$ By Observation , we have $\langle \nu, \alpha \rangle = 0$.

Proof of the Lemma follows by taking $\beta = s_\alpha \cdot \nu$.

Let $\alpha$ and $\beta$ be as in Lemma 4.1. Let $w \in W$ be such that $s_\alpha \leq w$. Let $V_1 := \bigoplus_{\mu \leq \beta} g_\mu$ be the direct sum of all weight spaces of weights $\mu$ satisfying $\mu \leq \beta$. Also, let $V_2 := \bigoplus_{\mu < m\beta} g_\mu$ be the direct sum of all weight spaces of weights $\mu$ satisfying $\mu < m\beta$. We note that $V_2$ is a $B$-submodule of $g$ containing $b$ and $V_1$ is a $B$-submodule of $g$ containing $V_2$.

We then have:

Lemma 4.2. $H^1(w, V_1/V_2)$ is a non-zero.

Proof. Since $s_\alpha \cdot \beta$ is the highest short root, $s_\alpha \cdot \beta$ is a dominant character of $T$. Hence, by the Borel-Weil-Bott’s theorem, $H^1(w_0, \beta)$ is an irreducible representation of $G$ with highest weight $s_\alpha \cdot \beta$. 

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On the other hand, by [10, Proposition 4.2], $H^1(s_\alpha, \beta)$ is non-zero. Also, by [10, corollary 4.3], the restriction map $H^1(w_0, \beta) \rightarrow H^1(s_\alpha, \beta)$ is surjective.

Thus, the restriction map $H^1(w, \beta) \rightarrow H^1(s_\alpha, \beta)$ is also surjective. Hence, $H^1(w, \beta)$ is non-zero. Since, we have $V_1/V_2 = \{ \beta \}$, we have $H^1(w, V_1/V_2) = H^1(w, \beta)$. Hence $H^1(w, V_1/V_2)$ is non-zero. This completes the proof.

Let $G$ be a simple, simply connected algebraic group over $\mathbb{C}$ which is not simply laced.

Let $V_1$ be a $B$-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$. Let $V_2$ be a $B$-submodule of $V_1$ containing $\mathfrak{b}$. Let $w \in W$.

The natural projection $\Pi : V_1 \rightarrow V_1/V_2$ of $B$-modules induces a homomorphism of $B$-modules $\Pi_w : H^0(w, V_1) \rightarrow H^0(w, V_1/V_2)$ of $B$-modules.

We now deduce the following Lemma as a consequence of the above Lemma.

Let $w \in W$. Let $\gamma$ be a simple root. Let $V$ be a $B$-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$. Then, we have the following Lemma on indecomposable submodules of $H^1(w, V)$ similar to Lemma (3.4) except that (2) and (5) are possible only if $\gamma$ is a short root

**Lemma 4.3.** Every indecomposable $B_{\gamma}$-summand $V'$ of $H^1(w, V)$ is one of the following:

1. $V' = \mathbb{C} \cdot h$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 0$.
2. $V' = \mathbb{C} \cdot h \oplus \mathfrak{g}_{-\gamma}$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 1$ and $\nu(h) = 0$ for every simple root $\nu$ different from $\gamma$.
3. $V' = \mathfrak{g}_{\beta}$ for some short root $\beta$ different from $\gamma$.
4. $V' = \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{\beta-\gamma}$ for some short root $\beta$.
5. $V' = \mathfrak{sl}_{2,\gamma}$, the three dimensional irreducible $L_{\gamma}$-module with highest weight $\gamma$.

**Proof.** Proof is by induction on $l(w)$. If $l(w) = 0$, then, $w = id$ and so we are done. Otherwise, choose a simple root $\alpha$ such that $l(w) = 1 + l(s_\alpha w)$.

By induction, every indecomposable $B_{\gamma}$-summand $V'$ of $H^1(s_\alpha w, V)$ must be one of the above mentioned 5 types.

We first assume that $\gamma = \alpha$. Now, using Lemma 2.3, we see that if $V'$ is one of the types (1), (4), and (5), then, $H^0(s_\gamma, V')$ must be of the same type in $H^0(s_\gamma, H^1(s_\alpha w, V))$. In $V'$ is of type (2), using Lemma 2.3, we see that $H^0(s_\gamma, V') = (0)$. In type (3), using Lemma 2.3, we see that $H^0(s_\gamma, V')$ is either zero or is one of the types (3) or (4). This completes the proof for the case when $\alpha = \gamma$.

We may therefore assume that $\alpha \neq \gamma$. We further assume that $\langle \gamma, \alpha \rangle \neq 0$ since the subcase $\langle \gamma, \alpha \rangle = 0$ is easy to handle. In this case, we use Lemma 2.3 to see that if $V'$ is of type different from type (5), then $H^0(s_\alpha, V')$ must be of the same type in $H^0(s_\alpha, H^1(s_\alpha w, V))$.

If $V'$ is of type (5), we again use Lemma 2.3 to conclude that $H^0(s_\alpha, V')$ contains an indecomposable $B_{\gamma}$-summand of type (2) in $H^0(s_\alpha, H^1(s_\alpha w, V))$. (Here, we use the induction hypothesis that $V'$ can be of type(5) only if $\gamma$ is a short root. Hence, we have $\langle \gamma, \alpha \rangle = -1$)

We now recall $\text{SES}$:
The description of the indecomposable $B_\gamma$-summands of $H^1(w,V)$ follows from above discussion and the above short exact sequence.

We have the following Lemma on the evaluation map.

**Lemma 4.4.** Let $V$ be a $B$-submodule of $\mathfrak{g}$ containing $b$. Then, we have

1. The evaluation map $ev : H^0(w,V) \to V$ is injective.
2. $H^i(w,V)$ is zero for all $i \geq 2$.

**Proof.** Proof of (1) is similar to that of Lemma 3.3.

Proof of (2): Proof is by induction on $l(w)$.

If $l(w) = 0$, we are done. Otherwise, choose a simple root $\gamma \in S$ be such that $l(s_\gamma w) = l(w) - 1$.

By Lemma 4.3, every indecomposable $B_\gamma$-summand $V'$ of $H^1(s_\gamma w,V)$ must be one of the 5 types given in Lemma 4.3. Hence, using Lemma 2.3, we conclude that $H^i(P_\gamma/B,V')$ is zero for every indecomposable $B_\gamma$-summand $V'$ of $H^1(s_\gamma w,V))$ and for every $i \geq 1$.

Hence, we have $H^i(P_\gamma/B,H^1(s_\gamma w,V)) = (0)$ and for every $i \geq 1$. Since $\dim(P_\gamma/B) = 1$, $H^i(P_\gamma/B,H^0(s_\gamma w,V)) = (0)$ for every $i \geq 2$.

Thus, we see that

**Observation 1**

1. $H^i(P_\gamma/B,H^1(s_\gamma w,V)) = (0)$ for all $i \geq 1$.
2. $H^i(P_\gamma/B,H^0(s_\gamma w,V)) = (0)$ for every $i \geq 2$.

By induction on $l(w)$, we have

**Observation 2** $H^i(s_\gamma w,V)$ is zero for all $i \geq 2$.

Now, using **Observation 1**, we conclude that $H^i(w,V)$ is zero for all $i \geq 2$. We now recall SES: For every $i \geq 1$, we have:

$$
(0) \to H^1(s_\gamma,H^{i-1}(s_\gamma w,V)) \to H^i(w,V) \to H^0(s_\gamma,H^i(s_\gamma w,V)) \to (0).
$$

Using **Observation 1** and **Observation 2** in the above short exact sequence, we conclude that $H^i(w,V) = (0)$ for every $i \geq 2$. This completes the proof of (2).

Let $V_1$ be a $B$-submodule of $\mathfrak{g}$ containing $b$. Let $V_2$ be a $B$-submodule of $V_1$ containing $b$. Let $w \in W$. The natural projection $\Pi : V_1 \to V_1/V_2$ of $B$-modules induces a homomorphism of $B$-modules $\Pi_w : H^0(w,V_1) \to H^0(w,V_1/V_2)$ of $B$-modules.

We now deduce the following Lemma as a consequence of the Lemma 4.4.
Lemma 4.5. $H^i(w, V_1/V_2)$ is zero for all $i \geq 2$.

Proof. Proof is similar to that of Lemma 3.5.

We have the following corollary as an application of Lemma 4.5.

Corollary 4.6. Let $w \in W$. Let $\alpha$ be a positive root. Then, $H^i(w, \alpha) = (0)$ for every $i \geq 2$.

Proof. Let $V_1 := \bigoplus_{\mu \leq \alpha} g_\mu$ denote the direct sum of the weight spaces of $g$ of weights $\mu$ satisfying $\mu \leq \alpha$.

Let $V_2 := \bigoplus_{\mu < \alpha} g_\mu$ denote the direct sum of the weight spaces of $g$ of weights $\mu$ satisfying $\mu < \alpha$. It is clear that $V_2$ is a $B$-submodule of $g$ containing $b$ and $V_1$ is a $B$-submodule of $g$ containing $V_2$.

Since $g_\alpha$ is one dimensional and is isomorphic to the quotient $V_1/V_2$, we have

Observation $H^i(w, \alpha) = H^i(w, g_\alpha) = H^i(w, V_1/V_2)$ for every $i \geq 2$.

Proof of corollary follows from Lemma 4.5 and the above Observation.

We now prove Theorem 2. Let $G$ be simple, simply connected and not simply laced algebraic group over $\mathbb{C}$.

Let $T_G/B$ denote the tangent bundle of the flag variety $G/B$. Let $\alpha_0$ denote the highest long root of $G$ with respect to $T$ and $B$. Let $L_{\alpha_0}$ denote the line bundle on $X(w)$ associated to $\alpha_0$.

Theorem 4.7. Let $w \in W$. Then, we have

1. $H^i(X(w), T_G/B) = (0)$ for every $i \geq 1$.
2. The adjoint representation $g$ is a $B$-submodule of $H^0(X(w), T_G/B)$ if and only if the set of semi-stable points $X(w^{-1})^{ss}_T(L_{\alpha_0})$ is non-empty.

Proof. Proof is similar to that of Theorem 3.7. We provide a proof here for completeness.

As in proof of Theorem 3.7, it is sufficient to prove the following:

1. $H^i(w, g/b) = (0)$ for every $i \geq 1$.
2. The adjoint representation $g$ of $G$ is a $B$-submodule of $H^0(w, g/b)$ if and only if the set of semi-stable points $X(w^{-1})^{ss}_T(L_{\alpha_0})$ is non-empty.

We prove this now.

Let $V_1 := g$ and let $V_2 := b$. The natural projection $\Pi : g \rightarrow g/b$ of $B$-modules induces a homomorphism $\Pi_w : H^0(w, g) \rightarrow H^0(w, g/b)$ of $B$-modules.

Proof of (1): As in the proof of Theorem 3.7, we take $V_1 := g$ and let $V_2 := b$. The natural projection $\Pi : g \rightarrow g/b$ of $B$-modules induces a homomorphism $\Pi_w : H^0(w, g) \rightarrow H^0(w, g/b)$ of $B$-modules.
We have the short exact sequence \( 0 \to b \to g \to g/b \to 0 \) of \( B \)-modules.

Applying \( H^i(w, -) \) to this short exact sequence of \( B \)-modules, we obtain the following long exact sequence of \( B \)-modules:

\[
\cdots H^i(w, b) \to H^i(w, g) \to H^i(w, g/b) \to H^{i+1}(w, b) \to \cdots
\]

On the other hand, by Lemma 3.1, we have \( H^i(w, g) = 0 \) for every \( i \geq 1 \). Further, by Lemma 4.4, we have \( H^{i+1}(w, b) = 0 \) for every \( i \geq 1 \). Applying this in the above long exact sequence of \( B \)-modules, we conclude that \( H^i(w, g/b) = 0 \) for every \( i \geq 1 \).

This proves (1).

Proof of 2 is similar to that of Theorem 3.7.

\[ \blacksquare \]

Remark 1: The part 2 of Theorem 1 does not hold for an arbitrary \( w \) if \( G \) is not simply laced. However, part 2 of Theorem 1 is still true when \( G \) is of type \( B_2 \).

Now, let \( G \) be of type \( B_2 \). Let \( \alpha_1 \) and \( \alpha_2 \) be two simple roots such that \( \langle \alpha_1, \alpha_2 \rangle = -2 \) and \( \langle \alpha_2, \alpha_1 \rangle = -1 \). Using the result of Carrell [5, Theorem 2], we see that \( X(w) \) is smooth if and only if \( w \neq s_{\alpha_2}s_{\alpha_1}s_{\alpha_2} \) (cf [2] also).

We now show that \( H^1(s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}, b) \) is the one dimensional representation \( C_{-\langle \alpha_1+\alpha_2 \rangle} \) of \( B \) and \( H^1(w, b) = 0 \) for every \( w \neq s_{\alpha_2}s_{\alpha_1}s_{\alpha_2} \).

Proof. Basically, the proof depends on studying the indecomposable \( B_{\alpha_i} \)-modules.

We have \( H^0(s_{\alpha_1}, b) = g_{-\alpha_2} \bigoplus \mathbb{C} \cdot h \bigoplus g_{-(\alpha_1+\alpha_2)} \bigoplus g_{-(\alpha_1+2\alpha_2)} \) for some \( h \in \mathfrak{h} \) such that \( \alpha_2(h) = 1 \) and \( \alpha_1(h) = 0 \) and \( H^1(s_{\alpha_1}, b) = 0 \). Hence, by using Lemma 2.3, we have \( H^i(s_{\alpha_2}, H^0(s_{\alpha_1}, b)) = 0 \) for every \( i \geq 0 \).

Similarly, we also see that \( H^0(s_{\alpha_2}, b) = g_{-\alpha_1} \bigoplus \mathbb{C} \cdot h \bigoplus g_{-(\alpha_1+\alpha_2)} \bigoplus g_{-(\alpha_1+2\alpha_2)} \) for some \( h \in \mathfrak{h} \) such that \( \alpha_1(h) = 1 \) and \( \alpha_2(h) = 0 \) and \( H^1(s_{\alpha_1}, b) = 0 \). Hence, by using Lemma 2.3, we see that \( H^0(s_{\alpha_1}, H^0(s_{\alpha_2}, b)) = g_{-(\alpha_1+2\alpha_2)} \) and \( H^i(s_{\alpha_1}, H^0(s_{\alpha_2}, b)) = 0 \) for every \( i \geq 1 \).

Since \( -(\alpha_1+2\alpha_2), \alpha_2 \) = -2, we have \( H^1(s_{\alpha_2}, H^0(s_{\alpha_1}s_{\alpha_2}, b)) = g_{-(\alpha_1+\alpha_2)} \). The rest of the proof follows using the short exact sequence \( SES \).

\[ \blacksquare \]

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