Models for Metal-Poor Stars with Different Initial Abundances of C, N, O, Mg, and Si. III. Grids of Isochrones for $-2.5 \leq [\text{Fe}/\text{H}] \leq -0.5$ and Helium Abundances $Y = 0.25$ and 0.29 at Each Metallicity

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ABSTRACT

Stellar evolutionary tracks for $0.12 \leq M/M_\odot \leq 1.0$ have been computed for each of several variations in the abundances of C, N, and O, assuming mass-fraction helium abundances $Y = 0.25$ and 0.29, and 11 metallicities in the range $-2.5 \leq [\text{Fe}/\text{H}] \leq -0.5$, in 0.2 dex increments. These computations are provided for mixtures with [O/Fe] between +0.4 and +0.8, for different C:N:O ratios at a fixed value of [CNO/Fe], and for enhanced C. Computer codes are provided to interpolate within these grids to produce isochrones for ages $\geq 7$ Gyr and to generate magnitudes and colours for many broad-band filters using bolometric corrections based on MARCS model atmospheres and synthetic spectra. The models are compared with (i) similar computations produced by other workers, (ii) observed UV, optical, and IR colour-magnitude diagrams (CMDs), (iii) the effective temperatures, $(V - I_{\text{C}})_0$, and $(V - K_{\text{S}})_0$ colours of Pop. II stars in the solar neighborhood, and (iv) empirical data for the absolute magnitude of the tip of the red-giant branch (TRGB). The isochrones are especially successful in reproducing the observed morphologies of optical CMDs and in satisfying the TRGB constraints. They also fare quite well in explaining the IR colours of low mass stars in globular clusters, indicating that they have [O/Fe] $\approx +0.6$, though some challenges remain.

Key words: globular clusters -- stars: abundances -- stars: binaries -- stars: evolution -- stars: Population II -- Hertzsprung-Russell and colour-magnitude diagrams

1 INTRODUCTION

The Hubble Space Telescope (HST) UV Legacy Survey (Piotto et al. 2015, Nardiello et al. 2018) obtained photometric data for 56 globular clusters (GCs) that make it possible to distinguish between the stellar populations within them that have different abundances of C, N, and O. This survey employed the F275W and F336W filters to measure the fluxes in passbands that include spectral features due to OH and NH, respectively, as well as the F438W filter, which samples a region of the electromagnetic spectrum with prominent CH and CN bands (see, e.g., Milone et al. 2012b, their Fig. 11). Indeed, the colour-magnitude diagrams (CMDs) presented by Piotto et al. provide a compelling demonstration of the power of UV photometry to separate GC stars into distinct sequences, due primarily to star-to-star variations in C:N:O at roughly constant C+N+O, that can often be traced from the lower main sequence (LMS) to the upper red-giant branch (RGB). Included in the Nardiello et al. catalogues are F606W and F814W magnitudes, as derived from a new reduction and calibration of the original HST Advanced Camera for Surveys (ACS) observations reported by Sarajedini et al. (2007). These data provide a valuable complement to the UV photometry insofar as the $(m_{\text{F606W}} - m_{\text{F814W}})_0$ colour index places strong constraints on the effective temperatures ($T_{\text{eff}}$) of stars and thereby on GC ages (e.g., Marin-Franch et al. 2009, VandenBerg et al. 2013, hereafter VBLC13) and star-to-star helium abundance variations within these systems (e.g., King et al. 2012, Milone et al. 2012).

At a given metallicity, variations in the total C+N+O abundance primarily affect the turnoff (TO) $T_{\text{eff}}$ and luminosity of an evolutionary track for a fixed mass on the H-R diagram (as well the isochrones derived from them at a fixed age), without altering the location of the RGB (see, e.g., Rood & Crocker 1985, Cassisi et al. 2008, VandenBerg et al. 2012). As a consequence, the difference in the colours of CN-weak and CN-strong giants on UV-optical CMDs is entirely a bolometric corrections (BCs) effect; i.e., it is the abundances of C, N, and O that were assumed in the synthetic spectra and computed BCs that matter rather than those adopted in the stellar models. On the other hand, TO luminosity versus age relations depend on the CNO abundances in the interiors of stars. As fully appreciated by, e.g., Pietrinferni et al. (2009), Sbordone et al. (2011), and Cassisi et al. (2013), it is therefore necessary to derive BCs from fully consistent atmospheres–interior models and syn-
thic spectra in order to make meaningful comparisons between theory and observations.

Because such self-consistent studies were previously quite limited in scope, VandenBerg et al. (2022a, hereafter Paper I) decided to generate relatively large grids of MARCS model atmospheres, synthetic spectra, and stellar evolutionary models for exactly the same chemical abundances, and to explore the implications of varying [CNO/Fe], as well as the ratio C:N:O at fixed values of [CNO/Fe]. This exploratory study, which also examined the effects of enhancing [Mg/Fe] and [Si/Fe] by 0.2 dex, considered three metallicities ([Fe/H] = −2.5, −1.5, −0.5) and two helium abundances (Y = 0.25, 0.29) at each metallicity. Moreover, BCs for most of the broad-band filters currently in use, including those employed in the HST UV Legacy Survey (except F275W) were calculated from the synthetic spectra. To make the best possible predictions of the fundamental properties of LMS stars, the outer boundary conditions of the stellar interior structures for masses in the range 0.45 ≤ M/⊙ ≤ 0.12 were derived from the MARCS model atmospheres. Importantly, Paper I showed that both the T_eff and IR colours of low-mass dwarf stars are quite dependent on the abundances of C, N, and O.

Encouragingly, isochrones that allow for CNO abundance variations are generally able to reproduce the morphologies of CMDs from the Nardiello et al. (2018) F336W, F438W, F606W, and F814W observations quite well; see VandenBerg et al. (2022a, hereafter Paper II), who considered 6 GCs with metallicities ranging from ≈ −2.3 (M92) to ≈ −0.7 (47 Tuc). The fits to the TO stars suggest that, with the exception of predicted F336W magnitudes, which appear to be too faint by ≈ 0.03–0.04 mag, the errors in the model fluxes at longer wavelengths are comparable with photometric zero-point uncertainties. Although the bluest stars along the lower RGB were universally found to have significantly bluer (M_F336W − M_F606W) colours than any of the model predictions, the colours spanned by most of redder giants on UV-optical CMDs could be explained by the expected abundance differences between CN-weak and CN-strong stars. It would clearly be worthwhile to consider more than a dozen different mixtures of the light elements that are allowed by the CNO cycle, but with improvements to the reduction and calibration. However, this error appears to be inconsequential, judging from the differences between the bolometric corrections given by Casagrande & VandenBerg (2014) for these two filters in the WFC3 and ACS photometric systems. For instance, if isochrones for [Fe/H] = −1.5, [α/Fe] = +0.4, and Y = 0.25 are transposed from the theoretical plane to the (M_F606W − M_F438W, M_F606W)-diagram, the differences between the ACS and WFC3 CMDs range from ≤ 0.001 mag along the main sequence to ≤ 0.005 mag along the giant branch at a fixed M_F606W magnitude. Clearly such small differences have no impact on the conclusions that were drawn from the studies of F606W and F814W observations that were presented in Papers I and II.

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2 Grids of evolutionary tracks for [Fe/H] = −2.5 to −0.5, in steps of 0.2 dex, were computed for the mixtures of C, N, and O that are listed in Table 1. In the case of the a4s21 mix, the log N_C abundances, on the scale log N_C = 12.0, correspond to the adopted Asplund et al. (2005) solar abundances with the addition of a 0.4 dex enhancement of O (and all of the other so-called “alpha elements”, though they are not tabulated explicitly) to be consistent with [α/Fe] = +0.4. The others assume the same abundances except for the adjustments, in dex, that are specified by upward or downward pointing arrows. For instance, the a4CNH mix has a lower abundance of carbon by 0.3 dex but a higher nitrogen abundance by 1.13 dex, resulting in log N_C = 8.13 and log N_N = 8.96, respectively. The tabulated numbers are equivalent to saying that the a4CNH mix assumes [C/Fe] = −0.3 and [N/Fe] = +1.13. Clearly, the CNO abundances at any metallicity can be obtained simply by adding the [Fe/H] value of interest to the adjusted log N_C values.

Paper I considered more than a dozen different mixtures of the metals, but most of the mixes for [CNO/Fe] = +0.28, with various ratios of C:N:O, were dropped from consideration because the eclipsing binaries in GCs favour stellar models that assume [O/Fe] ≥ +0.6 (see Paper II), in good agreement with spectroscopic determinations of the O abundances in solar neighbourhood Pop. II stars (e.g., Fabian et al. 2009, Nissen et al. 2014, Amarsi et al. 2019). In fact, if [CNO/Fe] were as low as +0.28, assuming the Asplund et al. (2005) scale of solar abundances, the maximum possible value of [N/Fe] would be +1.30 if all of the carbon and oxygen implied by an initial mixture with [C/Fe] = 0.0, [N/Fe] = 0.0 and [O/Fe] = [α/Fe] = +0.4 were converted to nitrogen. Since GCs are typically observed to have C+N+O = constant (e.g., Smith et al. 1999, Cohen & Meléndez 2005) and for some fraction of the member stars to have [N/Fe] ≥ +1.5 (e.g., Briley et al. 2022).
at very low metallicities (Richard et al. 2002). A significant role in our understanding of the evolution of GC stars (Spite et al. 1984, Ryan et al. 1999) and the measured abundance constraints as the observed Li abundances in field halo dwarfs (e.g., Richard et al. 2002) have shown that additional mixing at the inner boundaries of surface convection zones, possibly due to turbulence, appears to be necessary to satisfy such empirical relations prior to the turnoff and for predicted TO luminosity versus age relations (also see VandenBerg et al. 2002) have been treated, along with extra mixing below envelope convection zones when they occur.\(^a\) Fully diffusive computations by e.g., Richard et al. (2002) have shown that additional mixing at the inner boundaries of surface convection zones, possibly due to turbulence, appears to be necessary to satisfy such empirical constraints as the observed Li abundances in field halo dwarfs (Spite et al. 1984, Ryan et al. 1999) and the measured variations between TO and lower RGB stars (see, e.g., Gratton et al. 2001, Ramirez & Cohen 2002). Indeed, more recent studies by Nordlander et al. (2012), and Gruyters et al. (2014), among others, have found that TO stars in GCs, which should show the strongest signature of diffusion, have surface iron abundances that are only ~ 0.1 dex lower than those derived for stars just beginning their ascent of the RGB. This is far smaller than the 0.5–0.7 dex variation that is predicted by fully diffusive models that neglect extra mixing at very low metallicities (Richard et al. 2002).

Although radiative accelerations probably do not play a very significant role in our understanding of the evolution of GC stars (see the discussion of this point by Pietrinferni et al. 2021), the gravitational settling of the metals should, in principle, be taken into account. However, in practice, the neglect of this process will have no more than minor consequences, at least for stellar models that are applied to GC CMDs.\(^a\) As a general rule of thumb, a 0.3 dex (i.e., a factor of two, or a 100%) increase of the C+N+O abundance in the nuclear burning regions of low mass, metal-poor stars results in a reduction of the predicted age at a given TO luminosity by ~ 1 Gyr. Since the settling of the metals will increase the central abundances of CNO and the other metals by a few to several percent over MS lifetimes of ~10 Gyr, it can be expected that this process alone will reduce TO ages by ~ 0.1 Gyr (at most). This is insignificant compared the uncertainties associated with cluster distances and chemical abundances. The settling of helium is much more important for GC ages because He is so abundant compared with any of the metals.

The veracity of these remarks is borne out by the comparison given below of V-R isochrones with those from the latest BaSTI grids (Pietrinferni et al. 2021). Even though the latter, but not the former, follow the settling of the metals, both sets of isochrones predict virtually identical TO luminosities when the same [Fe/H], [CNO/Fe], and age are assumed. It should be appreciated, however, that there are differences in the respective treatments of diffusive processes that may have compensated for the expected small offset of these computations. Whereas a code very similar to the one

\(^a\) The diffusion of the metals should not be ignored in more exacting investigations, such as comparisons of predicted sound speed profiles for the solar interior with that inferred from helioseismic observations (e.g., Babcock et al. 2005), studies of metal abundance variations between MS and lower RGB stars in GCs (e.g., Korn et al. 2007), or the determination of the minimum mass, at a fixed metallicity, that is able to retain a convective core throughout its core H-burning phase (e.g., Michaud et al. 2004).
Figure 2. Differences of the temperatures, pressures, and radii at $\tau = 100$ (top, middle, and bottom rows of plots, respectively) between the predictions from MARCS model atmospheres for the a4s21 mix and, in turn, the a4x0_p2, a4CNN, and a4xC0 mixtures (left-hand, middle, and right-hand columns of plots) as a function of $T_{\text{eff}}$ and the indicated [Fe/H] values. All model atmospheres were computed for $\log g = 5.0$. Crosses indicate those atmospheres that failed to converge; these points were obtained by interpolating in, or extrapolating, the results from the converged models. Note that the ordinate values along the right-hand edge apply only to the right-hand column of plots.

written by Thoul et al. (1994) is used in the BaSTI stellar evolution program to solve the applicable transport equations (Burgers 1969), somewhat improved versions of the subroutines developed by Proffitt & Michaud (1991), see the Appendix of their paper) are employed in the Victoria code.

There are some differences in the predicted $T_{\text{eff}}$s of TO stars, but they are relatively minor ($\sim 60$ K) and they may be due in part to other differences between the Victoria and BaSTI evolutionary programs besides the treatments of diffusion. In fact, because the predicted $T_{\text{eff}}$ scale is subject to so many uncertainties that are hard to evaluate, such as known deficiencies in the treatments of convection and the atmospheric boundary condition, one must rely on comparisons with empirical constraints to evaluate the reliability of the temperatures predicted by any stellar models. As also shown below, V-R isochrones provide rather good fits, not only to the temperatures of field Pop. II stars as derived from the Infrared Flux Method (Casagrande et al. 2011), but also to the TO colours and morphologies of observed CMDS, especially when derived from the measured fluxes in optical passbands.

As in all previous presentations of V-R isochrones since the study by VandenBerg et al. (2014), MARCS model atmospheres at an optical depth $\tau = 100$ have been attached to the interior structures of stellar models for masses $\lesssim 0.45 M_\odot$ in order to make the best possible predictions of the $T_{\text{eff}}$s of LMS stars. In higher mass models, the photosphere was taken to be the outer boundary and the pressure at $T = T_{\text{eff}}$ was determined by integrating the hydrostatic equation in conjunction with an assumed $T$-$\tau$ relation — specifically, the fit given by VandenBerg & Poll (1989) to the semi-empirical solar atmosphere derived by Holweger & Müller (1974) — from very small optical depths to the photospheric value of $\tau$. The justification for this procedure, the transition between the two mass regimes, and the whole issue of the atmospheric boundary condition has already been discussed quite extensively by VandenBerg et al. (2014); consequently, little else needs to be mentioned here concerning this aspect of the models. As illustrated in Figure 2, isochrones generated from the evolutionary tracks that have been computed for this project (for the a4s21 mix, in this particular case) are very smooth over the full range in luminosity between the LMS and the RGB tip. Those for the other metal abundance mixtures listed in Table 1 look qualitatively very similar, though there are small systematic differences between them as a function of $T_{\text{eff}}$ and luminosity.

The only modeling results that have not been discussed previously involve the dependence of the atmospheric properties at $\tau = 100$ on the assumed mix of the metals and their consequences for predicted $T_{\text{eff}}$s. Figure 2 plots the differences in $\log T$, $\log P$, and $\log R$ at $\tau = 100$ between the a4s21 mix and the a4x0_p2, a4CNN, and a4xC0 mixtures in the left-hand, middle, and right-hand panels, respectively, as a function of $T_{\text{eff}}$ and [Fe/H]. (B. Edvardsson kindly computed additional sets of model atmospheres for [Fe/H] = −2.0 and −1.0 and $\log g > 4.7$ for each of the metal abundance mixtures in Table 1 to complement those for [Fe/H] = −2.5, −1.5, and −0.5, which were produced for Paper I.) Not surprisingly, the differences increase with increasing [Fe/H] and usually with decreases.
enhancements result in higher values of $T_P$ in CN-strong stars (the mixture has higher N and lower C similar to the abundances found as in the $a4xCO$ these two cases, the structural properties change only slightly if a occurs if the abundances of C and O are increased by large amounts, for $[\text{Fe}/\text{H}]$ .

The top panel of Figure 4 compares V-R and BaSTI isochrones for $Y = 0.249, [\text{Fe}/\text{H}] = -1.30, \log(N_\text{e} + N_\text{N} + N_\text{O}) = 7.96$, and ages of 10.0, 11.5, and 13.0 Gyr. Although they overlay one another along the MS (at $M_\text{bol} \lesssim 5.8$), the BaSTI isochrones have somewhat hotter TOs and RGBs than the V-R isochrones. However, the more interesting plot is the one in the bottom panel. If the BaSTI isochrones are adjusted to cooler temperatures by only $\delta \log T_\text{eff} = 0.0045$, their TOs superimpose those of the V-R counterparts almost exactly, and there are only slight differences between the two along the subgiant branch (SGB) and lower RGB. Of course, the MS portions of the two sets of isochrones no longer coincide if such a horizontal adjustment is applied to the BaSTI loci. However, the main point of this plot is simply to show that the V-R and BaSTI isochrones predict exactly the same turnoff luminosities at the same ages when both sets assume almost the same abundances. There are, after all, small differences in the abundances of most of the metals, including, e.g., Ne, Mg, and Si, in the solar distributions of the elements tabulated by Asplund et al. (2009) and Caffau et al. (2011).

Also plotted in Fig. 4 is a DSEP isochrone (Dotter et al. 2008) for $[\text{Fe}/\text{H}] = -1.30, Y = 0.248$, and an age of 11.5 Gyr. (As Dotter et al. adopt a slightly different enrichment law, $\Delta Y/\Delta Z$, for the variation of $Y$ with the mass-fraction abundance of the metals, $Z$, the helium abundance of the DSEP isochrone at the value of $Z$ corresponding to $[\text{Fe}/\text{H}] = -1.30$ is smaller by $\delta Y = 0.001$ that of the BaSTI isochrone.) Once again, it is the bottom panel that is the most instructive one. It reveals that the morphology of the DSEP isochrone in the vicinity of the turnoff is quite different from those of the others that have been plotted; in particular, the isochrone has an odd shape in the luminosity range $4.6 \lesssim M_\text{bol} \lesssim 4.0$. Furthermore, because the DSEP models adopted the reference solar abundances given by Grevesse & Sauval (1998), which has higher C, N, and O than the Caffau et al. (2011) scale, one would have expected that at the same TO luminosity, the DSEP isochrone would predict a younger age, by $\sim 0.2$ Gyr, than a BaSTI isochrone for

\[^{5}\] http://basti-iac.oa-teramo.inaf.it
\[^{6}\] http://stellar.dartmouth.edu/models
isochrones (Pietrinferni et al. 2021) also do a better job of reproducing the TO, but the top panel shows that this is not the case. The apparent inconsistencies cannot be attributed to differing treatments of diffusion because the methods described by Thoul et al. (1994) have been employed in both the BaSTI and DSEP codes to follow the settling of helium and the metals. However, there must be some differences in the physics incorporated in these evolutionary codes in order to explain the offsets in both the predicted luminosities and temperatures of the respective isochrones.

There is ample evidence that the V-R isochrones provide good fits to the CMDs of GCs (e.g., VandenBerg et al. 2014, VandenBerg & Demissenko 2018, Paper II), while DSEP isochrones have some difficulties in this regard (see, e.g., Dotter et al. 2014, their Figs. 4 and 5). Indeed, the latest BaSTI isochrones (Pietrinferni et al. 2021) also do a better job of reproducing cluster CMDs than DSEP models, though they appear to be somewhat less successful than V-R computations in explaining BV1C observations, or HST photometry of GCs at optical wavelengths. This is shown in Figure 4 where the same 11.5 Gyr isochrones that were intercompared in Fig. 4 are fitted to various CMDs for NGC 5904 (M5), which has a metallicity close to [Fe/H] = −1.30 (e.g., Carretta et al. 2009). If $E(B - V) = 0.035$ is adopted for M5, which corresponds to the average of the studies by Schlegel et al. (1998) and Schlafly & Finkbeiner (2011), the cluster TO stars can be matched quite well by 11.5 Gyr isochrones if $(m - M)_V = 14.40$, which is close to the apparent distance modulus implied by ZAHB models (see, e.g., VBLC13). The sources of the F336W, F438W, F606W, and F814W observations, on the one hand, and the BVIc data, on the other, are Nardiello et al. (2018) and Stetson et al. (2019).

Note that, in this investigation, the $E(B - V)$ values that are mentioned in the text or in figure legends are so-called “nominal” reddening indices; i.e., they are applicable to early-type stars, which is the usual convention for many reddening determinations in the literature, including those derived from the Schlegel et al. (1998) dust maps. The actual colour excess, $E(\zeta - \eta)$ for filters $\zeta$ and $\eta$, that applies to the stars in a given GC can be calculated to sufficient accuracy (especially when the reddening is low) using the values of $R_\zeta$ and $R_\eta$ given by Casagrande & VandenBerg (2014, their Table A1), where $R_\zeta = A_\zeta / E(B - V)$, $A_\zeta$ is the extinction and $E(B - V)$ is the nominal reddening (similarly for $R_\eta$). Since the reddening produced by a given amount of dust is less for stars of later spectral types, it follows that $R_\zeta - R_\eta < 1$; to be specific, Casagrande & VandenBerg’s Table A1 gives $R_\zeta - R_\eta = 0.922$ for F-type TO stars. One could equivalently specify the actual $E(B - V)$, but then the colour excess $E(\zeta - \eta)$, in general, would have to be calculated by multiplying this value of $E(B - V)$ by $(R_\zeta - R_\eta)/(R_\zeta - R_\eta)$. It is simpler just to give the nominal reddening and to adopt the $R_\zeta$ values, as tabulated, to derive the colour excesses of interest.

Furthermore, apparent distance moduli as measured in the $V$ magnitude are generally provided instead of, say, $(m - M)_V$ when fitting isochrones to the $(M_{F606W} - M_{F814W}) - (M_{F606W})$ CMD, because they are commonly used to specify GC distances. However, they can be easily converted to $(m - M)_V$, where $\zeta$ is the filter of interest, using $(m - M)_V = (m - M)_\zeta + (R_\zeta - R_\eta)E(B - V)$, where $E(B - V)$ is the nominal reddening and $R_\zeta$ has the value provided by Casagrande & VandenBerg (2014). Note that the standard $R_\eta = 3.1$ $(E(B - V)$ reddening law was assumed in these computations. To match the cluster TO stars, the various isochrones had to be shifted in colour by the amounts indicated by the $\delta X$ values. The loci in black, purple, and orange represent, in turn, the V-R, BaSTI and DSEP models to be consistent with the colours adopted in Fig. 4. (It should be kept in mind that comparisons with observed photometry involve both the predicted $T_{\text{eff}}$ and the adopted bolometric corrections.) In the case of the V-R isochrones, the BCs were derived from the the transformations provided by

![Figure 4](https://example.com/figure4.png)

**Figure 4.** Top panel: Comparison of V-R and BaSTI isochrones for the indicated metallicity, helium abundance, and ages. Also shown is an 11.5 Gyr isochrone for the same [Fe/H] value, but for $Y = 0.248$. Bottom panel: As in the top panel, except that the isochrones have been adjusted horizontally by the indicated $\delta X$ values, where $X = \log T_{\text{eff}}$.
Figure 5. Comparisons of the same isochrones that appear in the previous figure with the CMDs of M5 that have been derived from HST UV Legacy Survey observations (the top four panels) and from ground-based BVIC photometry (the bottom two panels); these data were released for general use by Nardiello et al. (2018) and Stetson et al. (2019), respectively. The adopted cluster reddening and the V-band apparent distance modulus are specified in the top left-hand panel. The δX values in each panel indicate the colour offsets that were applied to the isochrones in order to obtain best fits to the turnoff stars. The filled circles in black represent median fiducial points for the different CMDs, while the solid curves in black, purple, and orange represent, in turn, the V-R, BaSTI, and DSEP isochrones.
as discussed in Paper II, such colours appear to be insensitive to variations in the abundances of C, N, and O, which would seem to leave high $Y$ as the most likely explanation for bluer colours. However, the horizontal branch (HB) of M5 does not have an extended blue tail; consequently, star-to-star variations in $Y$ are probably not very large. This is suggested by recent studies of the M3 HB (Denissenkov et al. 2017, Tailo et al. 2019), which is morphologically quite similar to the HB of M5. Although so-called ‘chromosome maps’ appear to favor larger He abundance variations than those inferred from HB simulations, especially in the case of M3 (Milone et al. 2018), Tailo et al. present a number of quite compelling arguments against this possibility. Errors in the model $T_\text{eff}$s may be partly responsible for the discrepancies between the predicted and observed ($V-I_\text{C}$) colours (and similar HST colours), but the indications for such errors are less apparent when other colours are considered. It may be worthwhile to explore the implications of even more extreme chemical abundance variations (in particular, [N/Fe] > +1.5) than those examined in Paper II.

### 3 EMPIRICAL CONSTRAINTS

CMD studies, such as those just discussed, provide useful tests of stellar models, especially if the basic cluster parameters (distance, reddening, metal abundances) are derived by independent methods; i.e., if such fundamental properties are not based on the fits themselves of the isochrones to the photometric data (as in the work by, e.g., Wagner-Kaiser et al. 2013, Goncharov et al. 2021). For instance, recent papers that have used V-R isochrones to derive cluster ages (e.g., VandenBerg et al. 2016, Denissenkov et al. 2017), generally adopted distance moduli that were obtained by fitting zero-age horizontal branch (ZAHB) loci to their HB populations on the assumption of reddenings close to those given by dust maps (Schlegel et al. 1998, Schlafly & Finkbeiner 2011) and spectroscopically derived metallicities (Carretta et al. 2009). These particular investigations also used fully consistent evolutionary tracks for the core He-burning phase to generate synthetic HBs for direct comparisons with observed HBs and/or to predict the periods of the $ab$- and $c$-type RR Lyrae variables in the target GCs using the latest theoretical calibrations for log $P_{ab}$ and log $P_c$ as a function of luminosity, mass, $T_\text{eff}$, and metallicity (Marconi et al. 2015). Encouragingly, the models usually reproduced the observed periods to within the uncertainties of the factors upon which periods depend, though not in the case of M13 (see Denissenkov et al. 2017). However, it would be possible to explain the periods of its $c$-type variables if a reduced distance and/or an increased reddening were adopted. MS-fits to local subdwarfs were undertaken in some studies as well (e.g., VandenBerg & Denissenkov 2018); they also supported the ZAHB-based distance scale.

Not only does such work give one added confidence in the ages that are derived, but the resultant overlays of the isochrones onto the observed CMDs also provide meaningful comparisons of the predicted morphologies of the isochrones with those observed. While small zero-point shifts to the synthetic colours (typically $\lesssim$0.01–0.02 mag) have to be applied to the models in order to match the observed TOs, such offsets are not necessarily an indication of deficiencies in the isochrones as they could easily be due to errors in the photometric zero-points or to the assumption of a reddening that is somewhat too large or too small or metal abundances that are too high or too low by $\sim 0.1$ dex. The ACS observations originally obtained by Sarajedini et al. (2007) provide a good example in support of this assertion. When VBLC13 compared their isochrones to these observations, they generally had to shift the models by $\delta(M_{F606W} - M_{F814W})_0 \sim -0.015$ mag in order to match the TO colours. No such offsets, or at least ones that amount to no more than a a few thousandths of a magnitude, are needed when the
same isochrones are applied to the new reduction and calibration of the same data by Nardiello et al. (2018). In fact, a favourable outcome of the 2018 calibration is that the fits of V-R isochrones to F606W, F814W observations are now much more consistent with similar fits to V′I′C′ observations (see Fig. 5).

Regardless, it should be appreciated that small colour offsets have no impact on the inferred ages. Rather, such offsets must be applied to match the TO colours in order to derive the best estimate of the cluster age — because it is the superposition of the isochrones onto the arc of stars in the vicinity of the TO that identifies which one has the same TO as the cluster, for a discussion of this point, see VBLC13. Isochrones from different sources must be similarly registered to each other, as demonstrated in Fig. 4 to ascertain whether the models for the same age and chemical abundances predict the same TO luminosity.

In what follows, V-R isochrones are compared with recent IR observations of GCs to assess their reliability in the regime of very low masses. The next subsection examines how well V-R isochrones match the temperatures and selected colours of nearby subdwarfs with accurate distances that are located along the upper MS and near the TO. Finally, the constraints provided by a recent empirical calibration of the absolute IC magnitude at the tip of the RGB (TRGB) as a function of (V − Ic)0 are studied.

3.1 IR Photometry

The LMS portions of IR CMDs have a distinctive appearance insofar as such colours as J − Ks or m′F110W − m′F160W remain nearly constant or become bluer as the MS stars fall below ∼ 0.45 M⊙, thereby producing a knee-like morphology. Because the absolute magnitude of this feature is age-independent, it can be used to constrain GC ages, as first pointed out by Calamida et al. (2009) and subsequently investigated by many researchers (e.g., Bono et al. 2010; Monelli et al. 2015; Correnti et al. 2014; Saracino et al. 2018). However, aside from the concern raised by Saracino et al. that BaSTI, DSEP, and V-R isochrones predict absolute magnitudes of the MS-knee that differ by a few tenths of a mag, which calls into question the reliability of this reference point, more recent work has shown that the MS-knee is complicated by the presence of multiple stellar populations. This was revealed in the truly spectacular IR CMD of NGC6752 that was obtained by Milone et al. (2019), see their Fig. 1). Whereas the upper MS in this system was found to be very narrow and well defined, the width of the LMS increased dramatically with increasing magnitude beginning at the location of the MS knee. In fact, such CMDs appear to be quite typical of GCs; see the survey results by Dondoglio et al. (2022).

Milone et al. (2019) attributed the observed spread in the colours of the LMS stars of NGC6752 below the MS knee to star-to-star variations in the abundance of oxygen. Since isochrones have been generated for several O-enhanced mixtures (e.g., a4×O−p2, a4×O−p4) as well as one mix with a very low O abundance (a400R), it is of considerable interest to find out how well those models are able to explain the IR CMDs of GCs. The primary goal here is to overlay isochrones for the different mixtures of the metals onto the CMDs of a few clusters in order to test the most basic predictions of the models. More detailed studies, especially of the superb photometry obtained by Dondoglio et al. (2022), is well outside the scope of this particular paper. To have a first look at isochrones that span a wide range in metallicity, the decision was made to examine F110W, F160W observations of NGC6397 (Correnti et al. 2018), NGC6752 (Milone et al. 2019), 47 Tuc (Correnti et al. 2016), which have [Fe/H] values ranging from −2.0 to −0.7.

To constrain the distances of NGC6397 and NGC6752, their HB populations have been fitted to ZAHBs for Y = 0.25 and [Fe/H] = −2.0 and −1.5, respectively; see the left-hand panels of Figure 6, which also illustrate how well the lower boundaries of the distributions of HB stars in M92 and M3 can be matched by the appropriate ZAHBs for these two GCs. Because VBLC13 was able to obtain rather good fits to the Sarajedini et al. (2007) observations of the cluster HB stars — without any apparent difficulties whatsoever — using ZAHBs that differ only slightly from those computed for this project, the F606W, F814W data that appear in Fig. 6 have been taken from the same source. Indeed, the VBLC13 study left the impression that the Sarajedini et al. observations are quite homogeneous (as expected). For instance, even though the isochrones required small zero-point offsets to match the observed TO colours, these offsets were usually in the range from −0.01 to −0.025 mag, independently of the cluster metallicity. Cluster-to-cluster differences in applied colour shifts could easily be due in part to small errors in the adopted reddenings or metal abundances.

As noted in the introductory remarks of this section, there is considerable evidence in support of the ZAHB-based distances that are obtained when metallicities given by Carretta et al. (2009) and reddenings close to those derived from the Schlegel et al. (1998) dust maps are adopted (also see VBLC13). To be sure, without additional constraints, the uncertainties are very large in the case of clusters that only have very blue HBs, such as NGC6397 and NGC6752, since small differences in the adopted reddenings can have a huge impact on the distance moduli that are derived from such fits (see Fig. 6). However, the assumed values of E(B − V) and (m − M)0 must be such that the main sequences of the four clusters that are considered in Fig. 6 are located relative to one another approximately as shown in the right-hand panel in order to be consistent with the differences in their metallicities.

At low metal abundances, neither the model Teff nor the predicted (M_F606W − M_F814W)0 colours along the MS are very dependent on [Fe/H]. In fact, the horizontal separation between the loci for M92 and M3 at at M_F606W ≥ 5.5 in the right-hand panel is consistent with the predictions of the V-R isochrones in a differential sense. This was accomplished by adopting reddenings from the Schlegel et al. (1998) dust maps for M3. NGC6397, and NGC6752 (the values of E(B − V) that are specified in the left-hand panels of Fig. 6), along with E(B − V) = 0.028 for M92, which is higher than the dust map value by only 0.006 mag. These reddenings, coupled with the apparent distance moduli from the fits of the cluster HB populations to the relevant ZAHBs resulted in the very agreeable comparison of the cluster fiducials in the right-hand panel.

One might question whether ZAHBs for Y = 0.25 are relevant to GCs with HBs that are entirely to the blue of the instability strip, but the spectroscopic study by Villanova et al. (2009) found that the HB stars in NGC6752 with 8500 < Teff < 11,500 K (near the top of the blue tail, where gravitational settling is not effective) have close to the primordial He abundance. Thus, it seems to be a safe assumption that at least some of reddest HB stars in globular clusters have Y = 0.25 (also see the study of M4 by Villanova et al. 2012). This is, anyway, of little concern as modest variations in Y (or [Fe/H]; see Fig. 5) do not affect the location of the blue HB tail in optical CMDs very much.

In the case of 47 Tuc, (m − M)0 = 13.30 has been adopted so to be within 0.03 mag of the values found from fits of a suitable ZAHB and synthetic HB to the observed HB stars (Denissenkov et al. 2017), from the eclipsing binary V69 by Brogaard et al. (2017).
Figure 6. Left-hand panels: Fits of the HB populations of 4 GCs, as identified in each panel, to ZAHB loci for \([\text{Fe/H}] = -2.35, -2.0,\) and \(-1.5\) (in the order of increasing \(M_{F606W}\) at a fixed colour) on the assumption of the indicated values of \(E(B-V)\) and \((m-M)_V\). The ZAHBs assume \(Y = 0.25, [\text{O/Fe}] = +0.6,\) and \([\text{m/Fe}] = +0.4\) for the other \(\alpha\) elements. Right-hand panel Comparison of the locations of the median fiducial sequences for the upper MS, TO, and SGB stars in the same 4 clusters. As in the left-hand panels, the black, purple, orange, and blue points represent M92, NGC 6397, M3, and NGC 6752, respectively.

Figure 7. Fits of isochrones for the indicated ages and chemical abundances to the MS and TO portions of the NGC6397, NGC6752, and 47 Tuc, on the assumption of the same values of \(E(B-V)\) and \((m-M)_V\) that were adopted in the previous figure. The photometry is from Sarajedini et al. (2007). The small filled circles in black represent the median fiducial sequences.

and from Gaia DR2 parallaxes (Chen et al. 2018). Thompson et al. (2020) also analyzed V69, as well as a second eclipsing binary, E32, from which they obtained \((m-M)_0 = 13.29\), which is 0.08 mag larger than the determination by Brogaard et al. The difference in these results seems to be mostly due to the adoption of different \(T_{eff}\)s for the binary components, according to K. Brogaard (private communication); consequently, it would be reasonable to adopt the average distance modulus from these two investigations as the best estimate from the 47 Tuc binaries. Nevertheless, consistency with the ZAHB-based distances that have been adopted for the more...
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metal-deficient GCs would favour an apparent modulus near 13.30. As for the other clusters, the reddening from the Schlegel et al. (1998) dust maps, $E(B - V) = 0.032$, has been adopted for 47 Tuc.

It is worth pointing out that the derived cluster distances agree rather well with those reported by Baumgardt & Vasiliev (2021), who did an exhaustive literature review of distance modulus determinations for most of the Galactic GCs, as well as employing a number of different methods to obtain additional estimates of $(m - M)_0$. If the adopted apparent moduli are converted to $(m - M)_0$, on the assumption of the aforementioned reddenings, one obtains 14.65 (14.65) for M92, where the number enclosed by parentheses is from the Baumgardt & Vasiliev study, 11.96 (11.97) for NGC 6397, 13.12 (13.08) for NGC 6752, 14.98 (15.04) for M3, and 13.21 (13.28) for 47 Tuc. Recall that the measurement of the direct trigonometric parallax of NGC 6397 by Brown et al. (2018) yielded $(m - M)_0 = 11.89 \pm 0.07$ for this system, which is consistent with the other determinations to within $\pm 1 \sigma$. Clearly, the use of ZAHBs to derive cluster distances is very competitive with other methods, even when dealing with very blue HB populations.

Before turning to the IR CMDs, it is worthwhile to show that the V-R isochrones are able to provide exceedingly good fits to the MS populations of NGC 6397, NGC 6752, and 47 Tuc on CMDs constructed from the F606W, F814 filters. This is illustrated in Figure 7 which shows that the stellar models accurately reproduce the observed morphologies of the cluster CMDs from the SGB down to $M_{F606W} < 8.0$. Although small colour offsets had to be applied to the models to achieve the results that are shown, they are of no consequence. As already mentioned, significantly smaller adjustments are required to obtain comparable fits of the same isochrones to the updated versions of the same CMDs by Nardiello et al. (2018). Regardless, there are bound to be small differences in the zero points of the observed and synthetic photometry, small errors in the assumed cluster properties, and errors that are difficult to evaluate in the stellar models due to inadequacies in the treatment of convection, the atmospheric boundary condition, etc. In fact, one would not have expected the agreement between theory and observations to be anywhere near as good in the examples shown in Fig. 7.

It should be appreciated that the isochrones have effectively been overlaid onto the observed CMDs (aside from the application of small colour shifts) on the assumption of well supported estimates of the cluster properties that are independent of the isochrone fits to the MS and TO stars. The basis for the adopted values of $E(B - V)$ and $(m - M)_V$ has already been explained. The metallicities of NGC 6397 and NGC 6752 are within 0.01 dex of the spectroscopic values given by Carretta et al. (2009), while $[\text{Fe}/H] = -0.70$ has been assumed for 47 Tuc because it seems to be difficult to satisfy the constraints from its eclipsing binaries if it has a lower metallicity (see Brogaard et al. 2017, Paper II). The evidence in support of Pop. II stars having $[\text{O}/\text{Fe}] = +0.6$ and $[\text{m}/\text{Fe}] = +0.4$ for the other α elements being provided in §2 Insofar as the mean He abundances are concerned, variations amounting to $\Delta Y \sim 0.01$ appear to be present in NGC 6397 (Milone et al. 2012a), while they are apparently closer to $-0.03$ in both NGC 6752 (Milone et al. 2012b, Dotter et al. 2015) and 47 Tuc (Salaris et al. 2016, Demisеников et al. 2017). Accordingly, $(Y_0) = 0.255$ should be reasonably close to the mean value in NGC 6397, while $(Y_0) = 0.27$ should be a good estimate for NGC 6752 and 47 Tuc. Given that there is little ambiguity in the fits of the selected isochrones to the TO stars, the inferred cluster ages should be quite accurate, subject to the reliability of the adopted cluster properties.

The main results of this section are shown in Figure 8 which superimposes isochrones for the mixtures of the metals that are identified in the left-hand panel onto IR observations of NGC 6397, NGC 6752, and 47 Tuc. Since the star-to-star variations of $Y$ are small in NGC 6397, all of the isochrones that have been compared with its CMD assume the same He abundance ($Y_0 = 0.255$). To illustrate the effects of $Y = 0.03$–0.04 in the other two GCs, isochrones for just the a40ON mix have been plotted for $Y = 0.29$ as well as for $Y = 0.25$ in the middle and right-hand panels. (The minimum value of $Y_0$ is expected to be somewhat larger in 47 Tuc than in NGC 6752 due to Galactic chemical evolution.) Had isochrones for $Y = 0.29$ been plotted for other mixtures (i.e., a4x0_p8, a4x0_p2, and a4x0_p4), they would have coincided with the long-dashed curve in purple above the knee, and been nearly coincident with the loci for the same mixtures and the low value of $Y_0$ below the knee. The location of an isochrone for the a4x0C mix, assuming $Y_0 = 0.255$, is represented by small filled circles in orange in only the right-hand panel. Note that the a4x0_p2 and a40ON mixtures have the same value of [CNO/Fe]; hence, the spread in LMS colours between the respective isochrones (the solid and long-dashed loci) is expected for stars of the same C+Ne/O abundance but with variations in the ratio C:N:O that result from CN- and ON-cycling. A few isochrones for very low C and O and high N with the same [CNO/Fe] as the a4x0_p9 mixture were presented in Paper I. As illustrated in Fig. 12 of that investigation, they superimpose the isochrones for the a40ON mixture at the same $[\text{Fe}/H]$ almost exactly.

These few remarks and the information provided in the figure caption should provide a sufficiently detailed description of what has been plotted; but before assessing the performance of the stellar models, some of the features of the isochrones should be highlighted. First, at a fixed value of $Y$, the isochrones for all of the metal abundance mixtures that have been considered overlay one another above the MS knee, but they are widely separated below the knee. Varying $Y$ has the opposite effect; predicted $(M_{F110W} - M_{F160W})$ colours are dependent on the He abundance only above the knee. Second, the morphologies of the isochrones are fairly sensitive functions of both $[\text{Fe}/H]$ and the mixture of the metals. Below the knee, isochrones are nearly vertical at low metallicities and at low oxygen abundances, but they bend strongly back to the blue at high $[\text{Fe}/H]$ values and/or high O abundances. Third, the $M_{F150W}$ magnitude of the knee becomes progressively brighter with increasing $[\text{Fe}/H]$ (compare the location of the knee in the three panels of Fig. 8) and with increasing [O/Fe] at a fixed metallicity, which is most evident in the third panel, but also apparent in the other panels if magnified versions of the figure are viewed.

As regards the capabilities of the stellar models in reproducing IR CMDs, their most obvious deficiency is that they are unable to explain the reddest stars below the knee. The bluest stars do not present a problem and, indeed, it is encouraging that the models indicate a preference for [O/Fe] = +0.6 at low metallicities and a lower value by 0.1 dex or so at $[\text{Fe}/H] = -0.70$, which is very similar to the observed trend in nearby Pop. II stars (see, e.g., Nissen et al. 2014). The IR observations thus provide a further justification for such O abundances in addition to those given in §2 Higher N might help to reduce the discrepancies, though the a40ON mix assumes $[\text{N}/\text{Fe}] = +1.48$ (along with very low C and O, see Table 1), which is close to the maximum value found in spectroscopic studies of GCs (e.g., Briley et al. 2004, Cohen et al. 2005). It seems more likely that either the reddest LMS stars have chemical abundances that are quite different from those assumed in any of the mixtures considered in this study or that the MARCS models do not treat the molecular sources of opacity well enough in cool, LMS stars. Milone et al. (2019) had better success reproducing the
spread in the observed colours below the knee in NGC6752 (see their Fig. 3), but they did not allow for the effects of different O abundances on the $T_{\text{eff}}$ of low-mass stellar models (because of the lack of suitable isochrones at the time). It would be interesting to find out how much their results would change if they allowed for the effects of O abundance variations on $T_{\text{eff}}$ of stars.

Other than this, the fits of the stellar models to the NGC6752 CMD look quite good; even the width of the MS above the knee is well matched by isochrones for $Y_0 = 0.25$ and 0.29. In a systematic sense, the CMD of 47 Tuc is also in satisfactory agreement with the model predictions, though there seems to be a zero point offset between them. There is, in fact, some evidence in support of this possibility. As part of the same survey that resulted in the 47 Tuc observations, Correnti et al. (2016) obtained an IR CMD for NGC6752. An examination of those data revealed that the median fiducial points, which are tabulated in their paper, are in superb agreement with those of low-mass stellar models (because of the lack of suitable isochrones at the time). It would be interesting to find out how much their results would change if they allowed for the effects of O abundance variations on $T_{\text{eff}}$ of stars.

3.2 Field Halo Stars

Nearby subdwarfs with accurate distances provide one of the primary constraints on the temperatures and colours of stellar models, and they have frequently been used to check the reliability of Victoria computations and the isochrones derived from them. Such studies have shown, for instance, that V-R isochrones reproduce the locations of solar neighbourhood Pop. II stars on the $(V-I)$. The same models are able to reproduce the observed $(V-I)$ colours (usually to within 0.0-0.02 mag) if the colour transformations based on MARCS model atmospheres and synthetic spectra are adopted.

More surprisingly, the huge improvement to the parallaxes of nearby stars resulting from Gaia observations, and the consequent large increase in the number of stars with very accurate distances, has only served to confirm such findings. To demonstrate this, Gaia EDR3 parallaxes have been used in the modelling of such data, it is clear that much remains to be done.
Figure 9. Overlays of nearby subdwarfs with \([\text{Fe/H}]\) values within the indicated ranges by 12.5 Gyr V-R isochrones for \(Y = 0.25\), \([\text{O/Fe}] = +0.6\), \([\text{m/Fe}] = +0.4\) for the other \(\alpha\) elements, and metallicities given by the upper and lower limits of those ranges. The temperatures and their uncertainties are from the study by Casagrande et al. (2010). The absolute visual magnitudes and their uncertainties, which are smaller than the sizes of the filled circles that are used to represent the stars, were derived from EDR3 parallaxes (see the text for details).

Plots of \(M_V\) as a function of \(\log T_{\text{eff}}\) for the binned subdwarfs are shown in Figure 9. For instance, panel (a) in the top left-hand corner contains the stars with \(-2.3 \leq [\text{Fe/H}] < -2.1\) along with 12 Gyr V-R isochrones for \([\text{Fe/H}] = -2.3\) and \(-2.1\). All of the isochrones in this figure assume \(Y = 0.25\), \([\text{O/Fe}] = +0.6\), and \([\text{m/Fe}] = +0.4\) for all of the other \(\alpha\) elements. For the most part, the isochrones provide good fits to the subdwarfs, though there are a few stars in this bin that are redder than the stellar models at \(M_V \sim 4.5\). In the next metallicity bin (see panel b), the stars define a much tighter sequence and their locations on the \((\log T_{\text{eff}}, M_V)\)-diagram are reproduced particularly well by the relevant isochrones. Although there are a few outliers in most of the panels, the MS and TO portions of the isochrones that are plotted in each bin are approximately where they should be based on the subdwarfs that have been plotted. The discrepant stars could well
be unrecognized binary components or there may be some issues with their metallicities and/or temperatures. Most of the TO stars and subgiants are reasonably well fitted, suggesting that they have ages near 12.5 Gyr, though it would be worthwhile to fit isochrones directly to the subgiants in order to obtain best estimates of their ages for comparisons with the ages of the Galactic GCs. The most metal-rich stars (see panel i) do appear to be younger than 12.5 Gyr as the TO stars are somewhat brighter and bluer than the isochrones that have been plotted.

Figure [10] is similar to the previous figure, except that the comparisons of the subdwarfs with the isochrones involve \( V - K_s \) colours (the top row of panels) or \( V - I_C \) colours (the bottom row of panels). As Casagrande et al. [2011] tabulate JHK\(_S\) magnitudes for all of the stars, but BVRI\(_C\) magnitudes for only a relatively small subset of them, a few additional stars with \( V - I_C \) colours were drawn from the investigations by O’Malley et al. [2017] and Chaboyer et al. [2017]. In order to reduce the space taken by these plots and still sample the full range of metallicity from \(-2.30\) to \(-0.5\), only the metallicity bins along the main diagonal in Fig. [9] that runs from the upper left-hand corner to the lower right-hand corner are shown. Indeed, they are sufficient to serve the purpose of demonstrating that V-R isochrones reproduce the CMDs of local halo stars rather well. Note that the reddenings of the subdwarfs were derived from the 3D maps given by Capitanio et al. [2017].

For the few stars that are subject to small amounts of reddening (most are unreddened), the \( E(B-V) \) values tend to be quite close to those determined from analyses of the interstellar Na I D-lines by, e.g., Meléndez et al. [2010, see their Table 1].

A forthcoming paper will examine the implications of local subdwarfs for GC distances, as derived from the MS-fitting method, and determine the ages of the few subdwarfs with accurate distances that are located between the MSTO and the RGB.

### 3.3 TRGB Luminosities

Jang & Lee [2017] have obtained some of the most accurate calibrations to date of the absolute magnitude of the tip of the RGB (TRGB) as a function of colour, based on observations of several nearby galaxies with the zero-point determined from photometry of NGC 4258 and the LMC, which have precise geometric distances. Their relation for \( M_I \) as a function of \( (V - I)_0 \), in the

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**Figure 10.** As in the previous figure, except that the comparisons of the isochrones involve \( V - K_s \) colours (the top row of panels) or \( V - I_C \) colours (the bottom row of panels). Some of the stars that appear in panels (d)-(f) were taken from the investigations by O’Malley et al. [2017] and Chaboyer et al. [2017] given that \( V - I_C \) colours are provided for only a small number of subdwarfs in the Casagrande et al. [2010] data set. The comparisons have been limited to the metallicity bins along the main diagonal in Fig. 9 that runs from the upper left-hand corner to the lower right-hand corner, in order to reduce the space taken by the plots and still sample the full range of metallicity from \(-2.30\) to \(-0.5\). The reddenings of the subdwarfs were derived from the 3D maps given by Capitanio et al. [2017].
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Johnson-Cousins system, is represented by the solid curve in orange in the left-hand panel of Figure 1 along with the associated 1σ uncertainties (the orange dashed loci that are offset from the solid curve by ±0.058 mag at a fixed colour). Also shown are the estimated locations of TRGB stars in ω Cen, M5, and 47 Tuc, though the $M_I$ values are not the same as those reported by Bellazzini et al. (2004, 2001) for ω Cen and 47 Tuc or by Vaux et al. (2013) for M5. Only the apparent magnitudes of the TRGB, which includes corrections for under-sampling due to the limited number of very bright RGB stars in GCs, have been taken from these sources. According to Bellazzini et al., the TRGB occurs in ω Cen and 47 Tuc at $I = 9.84$ and $9.40$, respectively, whereas Vaux et al. give $I_{TRGB} = 10.27$ for M5.

Because fits of Victoria ZAHB models to cluster HB populations yield distance moduli that are comparable with current best estimates (e.g., Baumgardt & Vasiliev 2021), the aforementioned apparent magnitudes have been converted to best estimates (e.g., Baumgardt & Vasiliev 2021), the aforementioned apparent magnitudes have been converted to $M_I$ values assuming ZAHB-based determinations of $(m-M)_V$ and nominal $E(B-V)$ values given by the Schlegel et al. (1998) dust maps. Although Bellazzini et al. (2004) argued that the reddening of ω Cen is $E(B-V) = 0.11 ± 0.01$, it is simply not possible to obtain a consistent superposition of the blue HB populations of M3, which is nearly unreddened, and ω Cen if such a low reddening is adopted for the latter. This is illustrated in the right-hand panel of Figure 12: If $ω$ Cen has $E(B-V) = 0.11$ and $(m-M)_V = 14.06$, as derived by Thompson et al. (2003) from the application of an empirical IR versus surface brightness relation by Di Benedetto (1998) to observations of the eclipsing binary OGLE17, its blue HB tail lies below both the M3 HB and a ZAHB for $[Fe/H] = −1.50$. The extended blue tail of a ZAHB on optical CMDs is predicted to be nearly independent of metallicity and, because it is almost vertical, to be a very good constraint on the cluster reddening. Since $ω$ Cen has a mean metallicity of $[Fe/H] = −1.64$ (Carretta et al. 2009), it is a further difficulty with the adopted properties in the right-hand panel that the RGB of the more metal rich cluster, M3, lies on the blue side of the ω Cen giant branch.

All of these problems are resolved if ω Cen has $E(B-V) = 0.14$ from the Schlegel et al. (1998) dust maps, which corresponds to an actual $E(B-V) = 0.13$ for the stellar populations that are portrayed in Figure 12 and the distance modulus is obtained by matching a ZAHB to the bluest HB stars. Encouragingly, Bono et al. (2019) derived almost exactly the same reddening and distance modulus from their analysis of optical/near-IR observations of RR Lyrae stars in ω Cen. Insofar as the other two GCs are concerned: the HB of M5, which extends well to the red, is well reproduced by a ZAHB for $[Fe/H] = −1.33$ (Carretta et al. 2009) if $E(B-V) = 0.038$ (Schlegel et al. 1998) and $(m-M)_V = 14.38$ (see VBLC13). As already discussed, the preferred parameters for 47 Tuc are $E(B-V) = 0.032$, from the same dust maps, and $(m-M)_V = 13.30$. If these properties are adopted along with $R_I = A_I/E(B-V) = 1.885$ from Casagrande & Vandenberg (2014), it is a straightforward exercise to convert the apparent $I_{TRGB}$ magnitudes given above to the values of $M_{I_{TRGB}}$ that have been plotted in Figure 11. The uncertainties of $M_I$ are the same as those given by Bellazzini et al. (2004, 2001) for ω Cen and 47 Tuc, and by Vaux et al. (2013) for M5. The TRGB magnitudes for the three GCs are clearly in good agreement with the JL17 relation to well within their respective 1σ uncertainties. This consistency can be viewed as further evidence in support of Victoria ZAHB models. For instance, the significantly larger true distance modulus that was adopted by Vaux et al. (2013), $(m-M)_0 = 14.43$, from their review of published determinations, results in the considerably less satisfactory value of $M_{I_{TRGB}} = −4.17 ± 0.13$ from the perspective of the JL17 results.

Perhaps somewhat fortuitously given the uncertainties, the TRGB predictions from the V-R isochrones for the same O abundance that is favoured from other considerations (i.e., $[O/Fe] = +0.6$) provides the closest match to the JL17 relation between $M_{I_{TRGB}}$ and $(V-I)_0$: see the left-hand panel of Figure 11. Table 2 lists the properties of the stellar models along this particular sequence, including predicted TRGB absolute magnitudes for the Johnson-Cousins VI filters, the HST ACS F606W, F814W passbands, and the 2MASS $K_s$ filter. Data files containing the information in this table and for the other theoretical results that have been plotted, which illustrate the effects of varying Y and $[O/Fe]$, may be downloaded from the web site that is specified in the Data Availability section of this paper.

The left-hand panel of Figure 11 also shows the TRGB $M_I$ versus $(V-I)_0$ relations from BaSTI and DSEP computations. The TRGB models were taken to be the most luminous points along isochrones that were generated via their respective web sites (see footnotes 5 and 6) for an age of 12.5 Gyr, $[α/Fe] = +0.4$, $−2.5 < [Fe/H] < −0.5$, and initial He abundances that vary with the mass-fraction abundance of the metals, $Z$, according to an adopted enrichment ratio $ΔY/ΔZ = 1.31$ (BaSTI) or 1.54 (DSEP) with, in turn, $Y_p = 0.247$ and 0.245 for the primordial abundances. (In the right-hand panel, the various theoretical and GC results have been plotted as a function of $[Fe/H]$ instead of $(V-I)_0$ to illustrate the near constancy of $M_{I_{TRGB}}$ over quite a wide range in metallicity, $−2.5 < [Fe/H] < −0.9$.)

Because of differences in the reference solar abundances, DSEP isochrones for $[α/Fe] = +0.4$, where $α$ includes all of the $α$ elements (including oxygen), have almost the same absolute C+N+O abundance at a given $[Fe/H]$ value as V-R models for $[O/Fe] = +0.6$ and $[m/Fe] = +0.4$ for all other $α$ elements. Accordingly, one would expect that they should predict nearly the same stellar properties at the TRGB, and indeed they do, though the dependence of $M_{I_{TRGB}}$ on $(V-I)_0$ given by these models does not follow the empirical relation (the solid orange curve in the left-hand panel) as well as the predictions from V-R isochrones. On the other hand, BaSTI isochrones for $[α/Fe] = +0.4$ assume a lower C+N+O abundance at a fixed $[Fe/H]$ than either the V-R or DSEP models; consequently, they would be expected to lie somewhat below the black solid curve instead of $≈ 0.06–0.07$ mag above it in order to be consistent with the other results.

Serrenelli et al. (2017) recently carried a study of the brightness of the TRGB, using both the BaSTI and GARSTEC (Weiss & Schlattl 2008) stellar evolution codes, with a focus on the input physics and the transformations of the models from the H-R diagram to observational CMDs. When updated nuclear reaction rates, radiative and conductive opacities, neutrino cooling processes, etc. are adopted, their computations predict $M_{I_{TRGB}}$ magnitudes that are just outside of the error bars of the JL17 calibration, on the high side. V-R isochrones do not have this difficulty even though the same or very similar basic physics has been incorporated into the Victoria code (this was checked). The largest difference is probably associated with the equation of state (EOS), but if the

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10 As reported in their paper, Thompson et al. (2001) found a much smaller distance modulus, $(m-M)_V = 13.78$, from the measured bolometric luminosities of the binary components.
Figure 11. Left-hand panel: Plot of $M_I$ as a function of $(V-I)_0$ at the TRGB from JL17 (the solid curve in orange). The parallel dashed loci indicate the 1σ uncertainties associated with the solid curve. The absolute $I$ magnitudes of RGB tip stars in ω Cen, M5, and 47 Tuc are represented by filled circles and the attached error bars (see the text). The black curves, as identified in the lower left-hand corner of the plot, were derived from V-R isochrones for the indicated initial He abundances ($Y_0$) and values of $[O/Fe]$ for each of the metallicities from $-2.5$ to $-0.5$ (the small open circles in the direction from left to right). Predictions from the latest BaSTI and DSEP isochrones have been plotted by filled circles in purple and in orange, respectively. Right-hand panel: Similar to the left-hand panel, except that the results from the various isochrones and for the three GCs are plotted as a function of $[Fe/H]$.

Figure 12. Overlay of the CMDs for the HB and adjacent RGB populations in M3 and ω Cen on the assumption of two different estimates of $E(B-V)$ and $(m-M)_V$ for the latter. The photometry is from Sarajedini et al. (2007). The large filled circles in blue define the median points along the lower RGB of M3. As indicated, the same properties of M3 are assumed in both panels. The solid curve in black represents the same ZAHB for $[Fe/H]=-1.50$ that was fitted to the M3 HB in Fig. 6.

highly regarded FreeEOS is adopted instead of the default EOS, the TRGB luminosity is increased by only $\Delta \log L/L_\odot = 0.002$ (or 0.005 mag). To further investigate this issue, a prediction of the TRGB luminosity from the MESA code (Paxton et al. 2011) was obtained for a 0.8$M_\odot$ model with $[Fe/H] = -1.5$, $[\alpha/Fe] = +0.4$, and $Y = 0.25$ from P. Denisenkov (private communication). This differed from the value of $L_{\text{TRGB}}$ that is obtained for the same case using the Victoria code by only $\Delta \log L/L_\odot = 0.001$, despite small differences in the adopted nuclear reactions and the treatment of diffusion. MESA employs JINA nuclear reaction rates (Cyburt et al. 2010) and treats diffusion using a code similar to that developed by Thoul et al. (1994).

In their presentation of a new method to measure TRGB magnitudes, Durbin et al. (2020) also examined some recent theoretical predictions of the TRGB luminosity; specifically, those derived from PARSEC (Bressan et al. 2012, Marigo et al. 2017) and MIST

11 http://freeeos.sourceforge.net
Masses are specified in solar units (\(M_\odot\)). The mass interior to the center of the H-burning shell, where the H abundance is one-half of the surface abundance, is predicted to be brighter absolute TRGB magnitudes than the MIST isochrones by solar abundances of the metals instead of a mixture with enhanced chemical abundances are taken into account. In fact, it has already been mentioned in the preceding paragraph that the TRGB luminosities which are obtained from MESA and Victoria stellar models are in very good agreement when both assume [\(\alpha/Fe\)] = +0.4.

Small differences in the adopted stellar physics may not be primarily responsible for the larger variations in the predicted values of \(L_{\text{TRGB}}\) mentioned above. Serenelli et al. (2017) have already shown the importance of employing very small timesteps in following the evolution along the RGB when the mass is used as the independent variable. It is also possible that the way in which the stellar structure and the nucleosynthesis equations are solved could have an impact on the predicted TRGB luminosities. For instance, in the Victoria and MESA codes, these equations are solved implicitly; i.e., the chemical abundances and the structure variables (normally consisting of the radius, luminosity, pressure, and temperature) are converged together so that the chemical profiles and the stellar structure are always fully consistent with each other.

A different approach is used by the GARSTEC code (Weiss & Schlatt 2008), and possibly some others, insofar as the abundance changes that occur in a given timestep are calculated on the assumption of nuclear reaction rates based on the abundances and thermodynamic properties of the gas given by the last converged model, and then the stellar structure equations are solved without updating the solution of the nucleosynthesis equations. To avoid significant errors, an explicit treatment of the chemical abundances is required. Whether or not this suggestion is part of the explanation for some of the discordant results that are obtained from different codes is not known, but it would be worthwhile to investigate this possibility.

In any event, it has been shown that the relation between \(M_T^{\text{TRGB}}\) and \((V-I)_0\) colour given by V-R isochrones is in excellent agreement when both assume [\(\alpha/Fe\)] = +0.4.

This and similar tables for the other He and metal abundance mixtures may be downloaded from the web site given in the Data Availability Section.

Table 2. Properties of 12.5 Gyr Stellar Models at the RGB Tip

| [Fe/H]  | \(N^e\) | \(\log L/L_\odot\) | \(\log T_{\text{eff}}\) | \(\log g\) | \(\log M_{\text{tr}}\) | \(Y_{\text{surf}}\) | \(M_V\) | \(M_V^d\) | \(M_{F606W}\) | \(M_{F814W}\) | \(M_{Ks}\) |
|---------|--------|------------------|----------------|--------|----------------|-------------|------|------------|--------------|-------------|-----------|
| -2.50   | 0.7897 | 3.2236           | 3.6414         | 0.6304 | 0.4998        | 0.2442      | -2.6140 | -3.9611    | -2.9728      | -3.9829     | -5.6252   |
| -2.30   | 0.7910 | 3.2402           | 3.6362         | 0.5932 | 0.4974        | 0.2448      | -2.6169 | -3.9895    | -2.9841      | -4.0118     | -5.7053   |
| -2.10   | 0.7930 | 3.2566           | 3.6294         | 0.5510 | 0.4948        | 0.2461      | -2.6048 | -4.0103    | -2.9827      | -4.0329     | -5.7956   |
| -1.90   | 0.7963 | 3.2732           | 3.6207         | 0.5018 | 0.4923        | 0.2475      | -2.5734 | -4.0229    | -2.9658      | -4.0460     | -5.8996   |
| -1.70   | 0.8019 | 3.2896           | 3.6104         | 0.4467 | 0.4900        | 0.2489      | -2.5195 | -4.0267    | -2.9313      | -4.0503     | -6.0126   |
| -1.50   | 0.8094 | 3.3061           | 3.5989         | 0.3884 | 0.4877        | 0.2508      | -2.4409 | -4.0239    | -2.8783      | -4.0478     | -6.1324   |
| -1.30   | 0.8220 | 3.3224           | 3.5861         | 0.3276 | 0.4856        | 0.2523      | -2.3267 | -4.0141    | -2.7963      | -4.0374     | -6.2576   |
| -1.10   | 0.8394 | 3.3391           | 3.5719         | 0.2645 | 0.4835        | 0.2545      | -2.1504 | -4.0004    | -2.6523      | -4.0201     | -6.3904   |
| -0.90   | 0.8673 | 3.3552           | 3.5566         | 0.2000 | 0.4815        | 0.2568      | -1.8250 | -3.9755    | -2.3505      | -3.9871     | -6.5272   |
| -0.70   | 0.9042 | 3.3702           | 3.5408         | 0.1405 | 0.4795        | 0.2592      | -1.1765 | -3.8934    | -1.7286      | -3.9008     | -6.6653   |
| -0.50   | 0.9508 | 3.3847           | 3.5246         | 0.0823 | 0.4773        | 0.2615      | -0.0209 | -3.6169    | -0.6656      | -3.6473     | -6.8054   |

**Footnotes:**
a This and similar tables for the other He and metal abundance mixtures may be downloaded from the web site given in the Data Availability Section.
b Masses are specified in solar units (\(M_\odot\)). Mass loss during RGB evolution has not been treated.
c The mass interior to the center of the H-burning shell, where the H abundance is one-half of the surface abundance.
d Predicted I magnitudes are in the Johnson-Cousins photometric system.
e Predicted F606W and F814W magnitudes are in the HST Advanced Camera for Surveys (ACS) photometric system.

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**Figure 13.** As in the left-hand panel of Fig. 11, except that the theoretical relations are compared with the J17 empirical relation between \(M_{F814W} - M_{F606W}\) and \((V-I)_0\) colour.

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2 A similar procedure was used in the computation of BaSTI stellar models until recently. According to S. Cassisi (private communication), the current version of the BaSTI code, as used by Hidalgo et al. (2018) and others since 2017, does iterate between the solution of the stellar structure and nucleosynthesis equations during each timestep.
lent agreement with empirical determinations from galactic and GC studies, and that similar consistency is found using MESA and DSEP models. This success is not limited to $V/I$ data. As shown in Figure 13, which considers ACS F606W, F814 photometry, V-R isochrones also reproduce the observed relation between $M_{F814W}$ and the $M_{F606W} - M_{F814W}$ colour index to within its 1 or uncertainty over most of the colour range that has been plotted. The observed colour dependence is not matched by the models quite as well as in Fig. [11] which is presumably a consequence of systematic differences in the respective bolometric corrections as a function of $T_{\text{eff}}$. A final point: given the considerable evidence in support of Victoria ZAHB models, the He core mass and the envelope helium abundance at the TRGB must be quite close to the values given by V-R isochrones since it is these properties that largely determine where ZAHB models are located on the H-R diagram.

4 SUMMARY

Grids of stellar evolutionary tracks have been computed for different abundances of C, N, and O (the mixtures listed in Table 11) primarily to provide a set of models that are relevant to the chemically distinct stellar populations that reside in GCs. Whereas Papers I and II considered stellar models for only a few metallicities, the present grids were generated for sufficiently fine spacings in [Fe/H] and $Y$ to permit the calculation of isochrones for any metallicity and He abundance within the ranges $-2.5 \leq [\text{Fe/H}] \leq -0.5$ and $0.25 \leq Y \leq 0.29$, respectively. These isochrones are limited to ages $\gtrsim 7$ Gyr because the evolutionary sequences were computed only for masses from 0.12 to 1.0 $M_{\odot}$. The same bolometric corrections (BCs) that were presented in Paper I, which are based on fully consistent MARCS model atmospheres and synthetic colours, are used to transform the isochrones to CMDs that involve the magnitudes and colours which characterize many of the broad-band photometric systems currently in use — including, in particular, Johnson-Cousins $UBVRI$ and most of the HST ACS and WFC3 filters.

Paper II has already demonstrated that V-R isochrones are reasonably successful in fitting the UV/optical CMDs that are derived from the HST UV Legacy Survey (Piotto et al. 2013, Nardiello et al. 2018). There are certainly some unexplained features, such as the populations of lower RGB stars with very blue ($M_{F336W} - M_{F438W} \sim 2$) colours that seem to be common to all GCs, but the observed colour spreads encompassed by most of the cluster stars are similar to those predicted for CN-weak and CN-strong stars. Moreover, the morphologies of the observed CMDs are reproduced quite well by the models, including the development of the hook feature in the vicinity of the TO in ($M_{F336W} - M_{F438W}$), $M_{F606W}$ diagrams as the metallicity decreases below [Fe/H] $\sim 2$. While predicted $M_{F336W}$ magnitudes appear to be too bright by about 0.03 mag, there are no obvious zero-point offsets between the observed and synthetic magnitudes that are derived from redder passbands.

In this study, the comparisons with observations have been extended to the IR. In general, V-R isochrones appear to provide good fits to the bluest stars below the knee in e.g., ($M_{F110W} - M_{F160W}$), $M_{F160W}$ diagrams, indicating that they have [O/Fe] $\approx +0.6$, but not the reddest populations. Whether this is due to deficiencies in the BCs for mixtures with very low O abundances or the stars have chemical properties that differ in some significant way from those assumed in the present models is not known. Errors in the model $T_{\text{eff}}$ scale seem unlikely given that the isochrones provide excellent fits to observed ($M_{F606W} - M_{F814W}$) colours down to $M_{F606W} \gtrsim 8$ ($\sim$ 4 magnitudes below the TO). The IRFM temperatures that have been derived by Casagrande et al. (2010) for TO and upper MS field halo stars with accurate Gaia parallaxes, as well as their $(V - I)_{0}$ and $(V - K_S)_{0}$ colours, are also well matched by the V-R isochrones. The same computations predict TRGB absolute magnitudes and their colour dependencies that agree very well with the empirical results by Jane & Lee (2017). Indeed, Victoria stellar models and the isochrones derived from them appear to be particularly successful in satisfying the available observational constraints.

ZAHB models have played an important role in this investigation. Grids of such models for $-2.5 \leq [\text{Fe/H}] \leq -0.5$ and at least two values of $Y$ and [O/Fe] at each metallicity will be provided in a forthcoming paper.

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DATA AVAILABILITY

The evolutionary tracks for all of the metal abundance mixtures that have been considered in this project, and the means to interpolate in these tracks to produce isochrones for ages $\gtrsim 7$ Gyr and for any metallicity and He abundance within the respective ranges $-2.5 \leq [\text{Fe/H}] \leq -0.5$ and $0.25 \leq Y \leq 0.29$, may be obtained from https://www.canfar.net/storage/list/VRmodels. The files relevant to this investigation are: (i) vriso.zip, which contains not only the stellar models computed for this project, but also those presented by VandenBerg et al. (2014) that allow for variations in [$\alpha$/Fe] at a given metallicity, (ii) README_vriso, which provides detailed instructions on how to generate isochrones and to transform them to colour-magnitude diagrams using user-friendly computer programs that are provided, and (iii) TRGB_vriso.data, which lists the properties of stellar models at the tip of the RGB (in small tables similar to that shown in Table 2), allowing for the effects of varying $Y$ and [O/Fe] at metallicities ranging from $-2.5 \leq [\text{Fe/H}] \leq -0.5$ in steps of 0.2 dex.

REFERENCES

Amarsi, A. M., Nissen, P. E., & Skuladóttir, Á. 2019, A&A, 630, A104
Asplund, M., Grevesse, N., Sauval, A. J., & Scott, P. 2009, ARA&A, 47, 481
Bahcall, J. N., Basu, S., Pinsonneault, M., & Serenelli, A. M. 2005, ApJ, 618, 1049
Baumgardt, H., & Vasiliev, E. 2021, MNRAS, 505, 5957
Bellazzini, M., Ferraro, F. R., & Pancino, E. 2001, ApJ, 556, 635
Bellazzini, M., Ferraro, F. R., Sollima, A., Pancino, E., & Origlia, L. 2004, A&A, 424, 199
Bono, G., Iannicola, G., Braga, V. F., et al. 2019, ApJ, 870, 115

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