Testing Factorization in $B \to D^{(*)} X$ Decays

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Abstract

In QCD the amplitude for $\bar{B}^0 \to D^{(*)+} \pi^-$ factorizes in the large $N_c$ limit or in the large energy limit $Q \gg \Lambda_{QCD}$ where $Q = \{m_b, m_c, m_b - m_c\}$. Data also suggests factorization in the processes $B \to D^{*+} \pi^- \pi^0$ and $B \to D^* \omega \pi^-$, however by themselves neither large $N_c$ nor large $Q$ can account for this. Noting that the condition for large energy release in $\bar{B}^0 \to D^+ \pi^-$ is enforced by the SV limit, $m_b \gg m_b - m_c \gg \Lambda$, we propose that the combined large $N_c$ and SV limits justify factorization in $B \to D^{(*)} X$. This combined limit is tested with the $B \to D^{(*)} X$ inclusive decay spectrum measured by CLEO. We also give exact large $N_c$ relations among isospin amplitudes for $\bar{B} \to D^{(*)} X$ and $\bar{B} \to D^{(*)} \bar{D}^{(*)} X$, which can be used to test factorization through exclusive or inclusive measurements. Predictions for the modes $B \to D^{(*)} \pi \pi$, $B \to D^{(*)} K \bar{K}$ and $B \to D^{(*)} \bar{D}^{(*)} K$ are discussed using available data.

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I. INTRODUCTION

There are two limits of QCD in which factorization in $B^0 \to D^+\pi^-$ (and related) decays can be proven rigorously. In the large $N_c$ limit, in which one takes the limit of infinite number of colors, one can show that all nonfactorizable contributions are suppressed by $1/N_c^2$. In the large energy limit, $Q \gg \Lambda_{QCD}$, where $Q$ denotes any one of the large scales $m_b, m_c$ or $E_{\pi}$, factorization has been proven rigorously \cite{1} using a soft-collinear effective theory \cite{2}. In this case the corrections to factorization are suppressed by powers of $1/Q$ \cite{3}.

The large $Q$ proof of factorization has its roots in a non-rigorous argument by Bjorken that justifies factorization using color transparency. Here the requirement is that the pion velocity in the $B$ rest frame be ultrarelativistic, so that the small color dipole of quarks that forms the $\pi^-$ grows to hadronic size $\Lambda$ only after a time delayed hadronization time $\gamma\Lambda$ much larger than the hadronic size of the $B$ meson. Although not systematic, color transparency therefore seems to suggest that violations to factorization are order $1/\gamma$. While both large $N_c$ and large $Q$ lead to factorization in a particular limit of QCD, it is not clear which is the most important for the phenomenologically observed factorization in real QCD, in which neither large $N_c$, large $Q$, nor large $\gamma$ are exactly realized. Each method gives similar predictions for the decay $B^0 \to D^+\pi^-$ and, while none predict the decay $B^0 \to D^{0}\pi^0$, they all render it suppressed. Hence these decays are not very useful for distinguishing among these arguments for factorization.

If large $N_c$ were the only requirement for factorization, one would expect factorization to hold equally well in $D$ decays and $K$ decays. Factorization does not work very well in $D$ decays \cite{4,5} and is even more strongly violated in $K$ decays. Clearly large $N_c$ by itself does not explain phenomenological factorization. A great deal of effort has been devoted to explaining this apparent puzzle \cite{6,7,8}.

On the other hand, if the large energy limit was solely responsible for explaining factorization in $B$ decays, one would expect \cite{3} corrections to grow as $m_X/Q$, where $m_X$ is the hadronic mass produced by the $(\bar{d}u)_{V-A}$ current. Recently, an analysis of the decays $B \to D^{*}\pi^+\pi^-\pi^-\pi^0$ and $B \to D^{*}\omega\pi^-$ was performed, investigating the applicability of factorization as a function of the invariant mass of the light hadrons \cite{9}. Using information from $\tau$ decays it was shown that up to $m_X \sim 1.7$ GeV there is no indication of violations to factorization, which indicates that the large $Q$ limit can not be solely responsible for explaining factorization. However, this data can not be explained solely from the large $N_c$ limit of QCD either. In order to calculate the predictions from factorization, Ref. \cite{9} used the factorized form

$$\langle XD^{(s)}|(\bar{c}b)_{V-A}(\bar{d}u)_{V-A}|B\rangle = \langle D^{(s)}|(\bar{c}b)_{V-A}|B\rangle\langle X|(\bar{d}u)_{V-A}|0\rangle.$$ \hspace{1cm} (1)

This neglects the possibility that part of the hadronic state containing the light particles can be created from the $b \to c$ current. In fact, large $N_c$ only gives

$$\langle XD^{(s)}|(\bar{c}b)_{V-A}(\bar{d}u)_{V-A}|\tilde{B}\rangle = \sum_{X',X''} \langle D^{(s)}|X'|(\bar{c}b)_{V-A}|B\rangle\langle X''|(\bar{d}u)_{V-A}|0\rangle,$$ \hspace{1cm} (2)

with $X'$ and $X''$ adding to give the final light hadron state $X$. The authors of Ref. \cite{9} addressed this issue by arguing that the contributions from non-zero $X'$ may be numerically small and that for $B \to D^{*}\omega\pi^-$ this can be tested using data from semileptonic $B \to D\omega\ell\nu$ decays. These effects can also be tested in the decay $\bar{B}^0 \to D^{*0}\pi^+\pi^+\pi^-\pi^-$ for which $X' = \{\pi^+,\pi^+\pi^+\pi^\pm\}$ and $X'' = \{\pi^+\pi^-\pi^-,\pi^-\}$. The CLEO measurement \cite{10} finds a small
but nonvanishing branching ratio for this mode \( Br(B^0 \to D^{*0}\pi^+\pi^+\pi^-) = (0.30 \pm 0.07 \pm 0.06)\% \).

It is interesting to note that factorization in Eq. (1) is justified in the small velocity (SV) limit, \( m_b, m_c \gg m_b - m_c \gg \Lambda_{\text{QCD}} \) [1]. As shown in Ref. [2], in the SV limit the inclusive \( B \to X_c\ell\bar{\nu} \) branching ratio is saturated by \( D \) and \( D^* \) final states. Therefore, in the SV limit

\[
(X^\prime D^{(*)})|((\bar{c}b)_{V-A})|B) = \langle D^{(*)}|((\bar{c}b)_{V-A})|B)\delta_{X^\prime,0}. \tag{3}
\]

Thus, Eq. (2) reduces to Eq. (1) in the simultaneous limit of large \( N_c \) and SV and this combined limit is therefore capable of explaining the results of [4].

Although more stringent than the large \( Q \) limit, the SV limit implies large \( Q \) and is therefore fully consistent with it and also gives additional predictive power. For \( B \to D^{(*)} \) transitions the relevant kinematic limits can be summarized as

\[
\begin{align*}
\text{HQET:} & \quad m_b, m_c \gg \Lambda_{\text{QCD}}, \\
\text{Large } Q: & \quad m_b, m_c, m_b - m_c \gg \Lambda_{\text{QCD}}, \\
\text{SV limit:} & \quad m_b, m_c \gg m_b - m_c \gg \Lambda_{\text{QCD}},
\end{align*}
\]

(4)

each of which is a subset of the one above. The requirement that the light degrees of freedom in the \( D^{(*)} \) can be described by HQET requires that \( E_D \sim m_c \) or equivalently that the \( B \) and \( D \) velocities have \( v \cdot v' \) of order one. This implies that \( m_b - m_c \sim \sqrt{m_b m_c} \) or \( m_b - m_c \ll \sqrt{m_b m_c} \), which are allowed scalings in the large \( Q \) and SV limits respectively.

The purpose of this paper is twofold. In section [1] we show that even the more generally factorized form of the amplitude in Eq. (2) leads to experimentally testable predictions which are distinct from those following from large \( Q \) factorization. The \( B \to D^{(*)}X_u \) decays with charge eigenstates are parameterized by four independent isospin amplitudes. In the large \( N_c \) limit, they are given in terms of only two reduced matrix elements, compared with just one in large \( Q \) factorization. A similar result holds for \( B \to D\bar{D}X \) decays, for which we prove a similar reduction in isospin amplitudes from 7 to 5, in contrast with two in the combined large \( N_c \) and SV factorization. In section [1] we consider the simultaneous limit of large \( N_c \) and SV and calculate the inclusive differential decay rate \( B \to D^*X \) in this limit. The results obtained are compared with data available from the CLEO collaboration [3].

II. LARGE \( N_c \) RELATIONS FOR \( B \to DX \)

The final state \( X \) in the decay \( \bar{B} \to DX \) may have charm number \(-1\) or 0, depending on whether the underlying weak decay is \( b \to c\bar{c}s \) or \( b \to c\bar{c}d \). We analyze these separately, correspondingly labeling the final states \( X_c \) and \( X_u \). Taking into account the fact that the \( B \) and \( D \) states belong to isospin doublets and that the weak Hamiltonian

\[
\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left[ C_1(\mu) (\bar{c}u)_{V-A} (\bar{d}b)_{V-A} + C_2(\mu) (\bar{c}b)_{V-A} (\bar{d}u)_{V-A} \right]
\]

(5)

transforms as an isortriplet with \((I, I_3) = (1, +1)\), one obtains that the final light state \( X_u \) can have isospin \( I_{X_u} = 0, 1, 2 \). Specifying the isospin of the \( X_u \) system together with the total isospin of the final hadronic system gives rise to four isospin states: \([DX_0]_\frac{3}{2}, [DX_1]_\frac{3}{2}, [DX_1]_\frac{1}{2}\) and \([DX_2]_\frac{3}{2}\). The corresponding reduced isospin amplitudes are

\[
a_{\frac{3}{2}} = \langle [DX_0]_\frac{3}{2} | \mathcal{H}_W | \bar{B} \rangle, \quad b_{\frac{3}{2}} = \langle [DX_1]_\frac{3}{2} | \mathcal{H}_W | \bar{B} \rangle, \quad c_{\frac{3}{2}} = \langle [DX_2]_\frac{3}{2} | \mathcal{H}_W | \bar{B} \rangle.
\]

(6)
The isospin amplitudes depend on $X$ through its hadronic content and particle momenta. Squaring the amplitudes and summing over the isospin of the $X_u$ state one finds for the rates with $B$ and $D$ charge eigenstates

$$\Gamma(B_d \to D^+X_u^-) = \sum_X \left( \frac{2}{3} b_{1/2}(X) + \frac{1}{3} b_{3/2}(X) \right)^2 + \frac{1}{5} |c_{3/2}(X)|^2$$

$$\Gamma(B_d \to D^0X_u^0) = \sum_X \left( \frac{2}{3} a_{1/2}(X) + \frac{2}{9} b_{1/2}(X) - \frac{7}{9} b_{3/2}(X) \right)^2 + \frac{2}{15} |c_{3/2}(X)|^2$$

$$\Gamma(B^- \to D^0X_u^-) = \sum_X |b_{3/2}(X)|^2 + \frac{1}{5} |c_{3/2}(X)|^2$$

$$\Gamma(B^- \to D^+X_u^{--}) = \sum_X \frac{4}{5} |c_{3/2}(X)|^2.$$  \hspace{1cm}(7)

The phase space factors are implied. The corresponding sums over $X$ in the rate formulas include summation over the final states and integration over the phase space.

In the large $N_c$ limit these amplitudes simplify considerably. We assign the usual $N_c$ power counting to the Wilson coefficients $C_1(m_b) \sim 1/N_c$, $C_2(m_b) \sim 1$, in agreement with their perturbative expansion in $\alpha_s$ \[1\]. This gives that to leading order in $1/N_c$ the $B \to DX$ amplitude factors as

$$A_{N_c}^0(B \to DX_u) = \frac{G_F}{\sqrt{2}} V_{ub} V_{ud}^* C_2 \sum_{X',X''} \langle DX'||\bar{c}b|B\rangle \langle X''|\bar{u}u|0 \rangle + O(1/N_c)$$  \hspace{1cm}(8)

with $X_u = X' + X''$. Since the state $X''$ has to be in a state of isospin $|I, I_3\rangle = |1, -1\rangle$ and the state $X'$ can only have isospin $I = 0$ or 1, all possible decays $B \to DX_u$ are determined in the large $N_c$ limit by two amplitudes $F_I$ defined as

$$F_0(X) = \sum_{X',X''} \langle DX_0'||\bar{c}b|B\rangle \langle X''|\bar{u}u|0 \rangle,$$

$$F_1(X) = \sum_{X',X''} \langle DX_1'||\bar{c}b|B\rangle \langle X''|\bar{u}u|0 \rangle.$$

The four isospin amplitudes \[1\] can be written now in terms of $F_0$ and $F_1$ as\(^1\)

$$a_{1/2} = -F_1, \quad b_{1/2} = F_0 + \sqrt{2}F_1, \quad b_{3/2} = F_0 - \frac{1}{\sqrt{2}}F_1, \quad c_{3/2} = \sqrt{\frac{5}{2}}F_1.$$  \hspace{1cm}(10)

Using these expressions in the rate formulas, one finds the large $N_c$ predictions

$$\Gamma_{N_c}(B_d \to D^+X_u^-) = \sum_X |F_0(X)|^2 + \frac{1}{\sqrt{2}} |F_1(X)|^2 + \frac{1}{2} |F_1(X)|^2$$

$$\Gamma_{N_c}(B_d \to D^0X_u^0) = \sum_X 2|F_1(X)|^2$$

$$\Gamma_{N_c}(B^- \to D^0X_u^-) = \sum_X |F_0(X)|^2 - \frac{1}{\sqrt{2}} |F_1(X)|^2 + \frac{1}{2} |F_1(X)|^2$$

$$\Gamma_{N_c}(B^- \to D^+X_u^{--}) = \sum_X 2|F_1(X)|^2.$$  \hspace{1cm}(11)

\(^1\) Although in general $F_{0,1}(X) \sim O(N_c^0)$, for special states $X$ some of these amplitudes may vanish. For example, taking $X = \pi$ gives $F_1(\pi) = 0$. 

4
The two independent amplitudes can be extracted from the combinations of rates

\[ \Gamma(\bar{B}_d \to D^+X^-) + \Gamma(B^- \to D^0X^-) = \sum_X 2|F_1(X)|^2 + 2|F_0(X)|^2 \]  
(12)

\[ \Gamma(\bar{B}_d \to D^0X^0_u) = \Gamma(B^- \to D^+X^0_u^-) = \sum_X 2|F_1(X)|^2. \]  
(13)

The corrections to the predictions in Eq. (11) come at order $1/N_c$ and are parameterized by the subleading term in Eq. (8). This can be written again in a factorized form as

\[ A_{1/N_c}(B \to DX_u) = \frac{G_F}{\sqrt{2}}V_{cb}V_{ud}^* \left( C_1 + \frac{C_2}{N_c} \right) \sum_{X',X''} \langle DX'|\bar{c}u|0\rangle \langle X''|\bar{d}b|B \rangle + O(1/N_c^2). \]  
(14)

For this case both $X'$ and $X''$ can have isospin $I = 0, 1$. Thus four isospin amplitudes are required to describe all matrix elements (14), which shows that in general no reduction in the number of isospin amplitudes persists beyond leading order in $1/N_c$.

The large energy limit can be applied to these decays if the final state $X$ contains only particles which form a jet with energy much larger than its invariant mass (this includes the case of just a single light particle with large energy). In this case, the amplitude can be factored in a way similar to the large $N_c$ limit as

\[ A_E(\bar{B} \to DX_u) = \frac{G_F}{\sqrt{2}}V_{cb}V_{ud}^* \left( C_2 + \frac{C_1}{N_c} \right) \langle D|\bar{c}b|\bar{B} \rangle \langle X|\bar{d}u|0 \rangle + O \left( \alpha_s(Q), \left( \frac{m_X}{Q} \right)^n \right) \]  
(15)

where $m_X$ is the invariant mass of the state $X$ and $n > 0$. Since the isospin of $X$ is constrained to be $I = 1$, the only nonvanishing isospin amplitudes in this limit are $b_1/2, b_3/2$. This prediction can be tested for example by measuring the decay $B^- \to D^{\pi^-} \pi^-$ as a function of the angle between the two energetic pions, which should vanish as this angle decreases.

The above analysis can be carried through in an identical manner for $B \to D^*X$ decays. In the combined large $N_c$ and SV limit the amplitude $\langle D^{(*)}X|\bar{c}b|\bar{B} \rangle$ vanishes for $X \neq 0$. Thus, $F_1(X) = 0$ for all final states $X$. This prediction is similar to the one from large energy factorization, but without the kinematic requirement on the light hadronic system $X$. Separate measurements of the 4 rates with $B, D$ charge eigenstates would allow a test of these predictions, and distinguish between large $N_c$ factorization, large energy factorization or the combined limit of large $N_c$ and SV. In the next section we discuss a partial test along these lines, making use of the present limited experimental information on $B \to D^{(*)}X$ available from CLEO [13].

We stress that the large $N_c$ relations hold not only for the inclusive mode, but also for states $X$ with fixed hadronic content. For example, taking $X = \pi\pi$ gives that in the large $N_c$ limit, the amplitudes for the two pions in $B \to D^{(*)}(\pi\pi)_I$ to be emitted in states of isospin $I = 0$ and 2 are related. Eq. (13) gives a relation among rates to leading order in $1/N_c$ (note that $\Gamma(\bar{B}_d \to D^{(*)0}\pi^0\pi^0) \sim O(1/N_c^2)$)

\[ \Gamma(B_d \to D^{(*)0}\pi^+\pi^-) = \Gamma(B^- \to D^{(*)+}\pi^-\pi^-) + O(1/N_c). \]  
(16)

The branching ratios of these modes have been recently reported by the BELLE Collaboration [14]. For the $D$ modes they are $Br(\bar{B}_d \to D^0\pi^+\pi^-) = (7.5 \pm 0.7 \pm 1.5) \times 10^{-4}$ and $Br(B^- \to D^+\pi^-\pi^-) = (1.07 \pm 0.04 \pm 0.16) \times 10^{-3}$, and for the $D^*$ modes $Br(\bar{B}_d \to D^{*0}\pi^+\pi^-) = (6.2 \pm 1.2 \pm 1.7) \times 10^{-4}$, $Br(B^- \to D^{*+}\pi^-\pi^-) = (1.24 \pm 0.07 \pm 0.22) \times 10^{-3}$. 

5
The data agrees fairly well with the large $N_c$ prediction for the $D$ modes and suggests larger $1/N_c$ corrections for the $D^*$ modes. This agreement permits a determination of $|F_1(\pi\pi)|^2$ (integrated over phase space) using Eq. (12). A subtraction can then be performed to extract $|F_0(\pi\pi)|^2$ using Eq. (12). The widths on the left side of Eq. (12) are dominated by the two-body $B \to D\rho$ mode [17]. One finds (in units of $Br$)

$$
2|F_0(\pi\pi)|^2 = (20.4 \pm 2.3) \times 10^{-3}, \quad 2|F_1(\pi\pi)|^2 = (9.1 \pm 1.1) \times 10^{-4}.
$$

The large observed enhancement of $F_0$ over $F_1$ is a consequence of the inequality among two-body modes $\Gamma(D\rho) \gg \Gamma(D^{*\pi})$. Factorization for these two body modes can be explained either in large $N_c$ or in the large $Q$ limit. The suppression of $F_1$ over $F_0$ is in agreement with the SV limit.

Another test of the large $N_c$ relations is obtained by taking $X = K\bar{K}$, for which data is available from BELLE [16]. The isospin of the kaon pair can be only 0 and 1, which gives $c_{3/2}(K\bar{K}) = 0$. The large $N_c$ relations (10) imply $F_1(K\bar{K}) = 0$, which shows that in this limit the $B \to D(K\bar{K})_{I=0}$ amplitude is suppressed, and the decay rates satisfy

$$
\Gamma(\bar{B}_d \to D^+K^-\bar{K}^0) = \Gamma(B^- \to D^0K^-\bar{K}^0) + O(1/N_c)
$$
$$
\Gamma(\bar{B}_d \to D^0K^0\bar{K}^0) = \Gamma(\bar{B}_d \to D^0K^+K^-) = O(1/N_c^2).
$$

The first prediction agrees well with the BELLE results [16] $Br(\bar{B}_d \to D^+K^-\bar{K}^{*0}) = (8.8 \pm 1.1 \pm 1.5) \times 10^{-4}$, $Br(B^- \to D^0K^-\bar{K}^{*0}) = (7.5 \pm 1.3 \pm 1.1) \times 10^{-4}$ and $Br(\bar{B}_d \to D^{*+}K^-\bar{K}^{*0}) = (12.9 \pm 2.2 \pm 2.5) \times 10^{-4}$, $Br(B^- \to D^{*0}K^-\bar{K}^{*0}) = (15.3 \pm 3.1 \pm 2.9) \times 10^{-4}$.

Next we turn our attention to the $b \to c\bar{c}s$ process. For this case, large $Q$ arguments can not be used to justify factorization, which leaves large $N_c$ as the sole possible explanation. The Hamiltonian $H^c_W$ responsible for these decays is identical to (10) with the substitution $u \to c$. We neglect the penguin operators because of their small CKM factors and Wilson coefficients. We first consider the case in which experiments may tag the $B$ meson and distinguish the charm from anti-charm. There are two isospin amplitudes $h_0, h_1$ corresponding to the two possible values of the isospin $I$ of the state $X_c$

$$
h_0(X) = \langle [DX_{c0}]_+ | H^c_W | B \rangle, \quad h_1(X) = \langle [DX_{c1}]_\pm | H^c_W | B \rangle.
$$

The corresponding rates are given by

$$
\Gamma(\bar{B}_d \to D^+X^-_c) = \sum_X |h_0(X)|^2 + \frac{1}{3} |h_1(X)|^2
$$
$$
\Gamma(\bar{B}_d \to D^0X^0_c) = \sum_X \frac{2}{3} |h_1(X)|^2
$$
$$
\Gamma(B^- \to D^+X^-_c) = \sum_X \frac{2}{3} |h_1(X)|^2
$$
$$
\Gamma(B^- \to D^0X^0_c) = \sum_X |h_0(X)|^2 + \frac{1}{3} |h_1(X)|^2.
$$

No simplifications are expected for these modes in the large $N_c$ limit. Due to the identical isospin structure of their Hamiltonian, similar relations can be written down for the semi-inclusive semileptonic decays $\bar{B} \to DXe\bar{\nu}$, in terms of another two amplitudes $g_{0,1}(X)$.
The definition of the SV limit for decays containing two charm quarks in the final state is somewhat different from the one introduced in Eq. (4). Requiring that both charmed hadrons move slowly gives (8)

$$m_b, m_c \gg m_b - 2m_c \gg \Lambda_{QCD}.$$ (21)

In this combined large $N_c$ and SV limit, the state $X_c$ is produced by the $(\bar{s}c)$ current and must have isospin 0, which requires the amplitude $h_1(X)$ to vanish $h_1(X) \to 0$.

Making explicit the charm and anticharm in the final states gives many more modes. This type of analysis is necessary if the experimental inclusive measurement relies on the presence of a charmed meson and separates them only according to whether they are charged or neutral. Taking into account the fact that the Hamiltonian responsible for these decays is an isospin singlet $I = 0$, one finds seven independent isospin amplitudes describing these decays. Three reduced amplitudes describe decays into nonstrange $D$ mesons

$$a_1 = \langle D(\bar{D}X_{1/2})_0 \| H_W^0 \| \bar{B} \rangle \quad a_2 = \langle D(\bar{D}X_{1/2})_1 \| H_W^0 \| \bar{B} \rangle \quad a_3 = \langle D(\bar{D}X_{3/2})_1 \| H_W^0 \| \bar{B} \rangle$$ (22)

and another four reduced amplitudes parameterize decays into $D_s$

$$b_1 = \langle D(D_s^- X_0)_0 \| H_W^0 \| \bar{B} \rangle \quad b_2 = \langle D(D_s^- X_1)_1 \| H_W^0 \| \bar{B} \rangle \quad c_1 = \langle \bar{D}(D_s^+ X_0)_0 \| H_W^0 \| \bar{B} \rangle \quad c_2 = \langle \bar{D}(D_s^+ X_1)_1 \| H_W^0 \| \bar{B} \rangle.$$ (23)

Finally, another amplitude $d$ describes $\bar{B} \to D_s^+ D_s^- X_{1/2}$ decays. The isospin decomposition of the most general $B \to D\bar{D}X$ decay amplitude is shown in Table I.

In the large $N_c$ limit, the amplitudes for $\bar{B} \to D\bar{D}X$ factor in a similar way as for $\bar{B} \to DX_u$ (8), (14). Keeping terms up to $O(1/N_c)$, the factorizable terms read

$$A(\bar{B} \to D\bar{D}X) = \frac{G_F}{\sqrt{2}} V_{cb}V_{cs}^* C_2 \sum_{X', X''} \langle DX'|\bar{c}\bar{b}|\bar{B}\rangle \langle \bar{D}X''|\bar{s}\bar{c}|0 \rangle$$ (24)
$\Gamma(\bar{B}_d \to D^0 \bar{D}^0 K^-) = \Gamma(\bar{B}_d \to D^0 D^- X) = \Gamma(B^- \to D^+ \bar{D}^0 X) = \Gamma(B^- \to D^+ D^- X) = \Gamma(B_d \to D^+ \bar{D}^0 X) = \Gamma(B_d \to D^+ D^- X) = \Gamma(B^- \to D_s^+ \bar{D}^0 X) = \Gamma(B^- \to D_s^+ D^- X)$. 

(27)
the SV limit predicts a vanishing $B_0$. However, since this amplitude is subleading in $1/N_c$, these modes can not be used to test the SV limit. Therefore separate measurements of other modes in Table I are needed to distinguish which of these limits (if any) are actually realized in nature.

III. FACTORIZATION FOR $\bar{B} \to D^* X$

As a further test of factorization we propose studying the inclusive hadronic decay $B \to D^* X$ as a function of the invariant mass of the state $X$, $q^2 = m_X^2$. CLEO has measured this decay spectrum from $m_X^2 \simeq 0$ out to the maximum hadronic mass $m_X^2 = (m_B - m_{D^*})^2 \simeq 10.7 \text{ GeV}^2$ [13]. Thus, the inclusive spectrum allows for a test of factorization over a much larger range than the $m_X^2 \simeq 0$ to $3.2 \text{ GeV}^2$ region considered in Ref. [9].

As explained in the introduction, the inclusive $B \to D^* X$ amplitude in the large $N_c$ limit is given by

$$A(\bar{B} \to D^* X) = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{cb} \sum_{X',X''} C_2(\mu)(D^* X'|(\bar{c}b)_{V-A}|B)(X''|(\bar{d}u)_{V-A}|0), \quad (28)$$

with a sum over $X = X' + X''$ in the rate, plus additional analogous contributions involving the $(s\bar{c})_{V-A}$ current and Cabibbo suppressed terms. Imposing the SV limit as well gives rise to two important simplifications to Eq. (28). First, as in Eq. (3), the contributions from hadrons $X'$ produced together with the $D^*$ from the $(\bar{c}b)_{V-A}$ current are suppressed. Second, the contributions from non-prompt $D^*$, that is, $D^*$ that arise from production of higher resonances which decay into $D^*$, are suppressed.

Before proceeding to a more detailed test, we note that some evidence for the validity of the SV limit can be obtained by comparing the inclusive branching ratios measured by CLEO

$$Br(B \to D^{*0} X) = (0.247 \pm 0.028), \quad Br(B \to D^{*\pm} X) = (0.239 \pm 0.019). \quad (29)$$

These numbers are summed over both charged and neutral $B$ decays, and include semileptonic and $cd \bar{u}$, $c \bar{c}s$ final states. Both semileptonic and $b \to c \bar{c}s$ decay mechanisms are isospin symmetric, so they can not produce an asymmetry between the rates (29). (Although this is manifest from (20), it holds also for the charge-averaged rates, with multiplicity factors added to account for identical particles in the final state.) The only source for such an asymmetry is the $b \to cd \bar{u}$ decay mechanism. The large $N_c$ predictions (11) together with (20) imply

$$\Gamma(\bar{B}_d \to D^{*+} X^-) + \Gamma(B^- \to D^{*+} X^-) = \sum_X |F_0(X)|^2 + 3|F_1(X)|^2 + \sqrt{2} \text{Re}(F_0(X)F_1^*(X)) + \sum_{i=0,1} (|h_i|^2 + |g_i|^2) + O(1/N_c) \quad (30)$$

$$\Gamma(\bar{B}_d \to D^{*0} X^0) = \Gamma(B^- \to D^{*0} X^-) = \sum_X |F_0(X)|^2 + 3|F_1(X)|^2 - \sqrt{2} \text{Re}(F_0(X)F_1^*(X)) + \sum_{i=0,1} (|h_i|^2 + |g_i|^2) + O(1/N_c). \quad (31)$$

Neglecting the $1/N_c$ corrections, the approximate equality of the measured rates (29) can be interpreted as evidence for a small ratio $F_1/F_0$, which coincides with the expectation from
It is well known\[18\] that in the large $N_c$ minimum possible number of final state mesons. Each additional meson comes at the price of a $1/N_c$ suppression factor. Thus, naively one should include in Eq. (28) only states in which $X$ is a single meson. However, in the SV limit the energy available for the decay becomes arbitrarily large and phase space effects invalidate the naive conclusion. To see this, consider large but fixed $b$ and $c$ masses, and let $n$ be the number of single resonances $X$ kinematically allowed in the decay, that is, lighter than about $m_b - m_c$. The number of combinations of $m$ mesons in the final state scales as $n^{am}$, where $a$ is a fixed constant$^3$. So we see that, in relation to the width into one light meson, the width into $m$ light mesons roughly scales as $(n/N_c)^m$. Thus, there is no suppression of multimeson states for $m_b - m_c$ large enough so that, roughly, $n > N_c$.

Hence in the combined large $N_c$ and SV limits the ratio of the inclusive and semileptonic $B \to D^*\ell\nu$ rates can be predicted in terms of the spectral functions $v_1$ and $a_1$ for the vector and axial currents. The spectral functions are defined through

\[
(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_1^I(q^2) + q_\mu q_\nu \Pi_0^I = i \int d^4x \, e^{iqx} \langle 0 | T J_\mu^I(x) J_\nu(0) | 0 \rangle ,
\]

\[
v_1(q^2) = 2\pi \text{Im} \, \Pi_1^{J=V}(q^2) , \quad a_1(q^2) = 2\pi \text{Im} \, \Pi_1^{J=A}(q^2) ,
\]

and have been measured$^4$ by the ALEPH collaboration in $\tau$-decays up to $q^2 = 3.0 \text{ GeV}^2$. Above the resonant region the $v_1 + a_1$ data displays a plateau, in excellent agreement with the operator product expansion prediction. Therefore one can safely extrapolate the data for $q^2 > 3.0 \text{ GeV}^2$ using perturbative QCD, $(v_1 + a_1) = 1.1$. This extrapolation is independent of the mechanism responsible for factorization and therefore does not bias our analysis in any way.

For hadronic states $X_u$ not including charmed hadrons the prediction from factorization is

\[
\frac{d}{dq^2} \frac{\Gamma(B \to D^*X_u)}{\Gamma(B \to D^*\ell\nu)} = 3G_2^2 |V_{ud}|^2 [v_1(q^2) + a_1(q^2)] + \ldots ,
\]

where the ellipses denote terms suppressed by $1/N_c$ or two powers of the SV expansion parameter. Note that although the numerical results for $B \to D^*\ell\nu$ in the SV limit are distorted by the modification in the kinematical factors, such modifications occur in a similar way for the inclusive and exclusive decays and therefore cancel out in the ratio. We emphasize that Eq. (33) can be used to give a very clean factorization prediction for the $d\Gamma(B \to D^*X_u)/dq^2$ decay rate with input from the $\tau$-spectral functions$^4$ and from the measured $B \to D^*\ell\nu$ form factor.

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2 We neglected for this argument the non-prompt $D^*$ production, which is itself justified in the SV limit.

3 For example, $a = 1/2$ in the ’t Hooft model in the case of decays of a heavy meson into light mesons$^{19, 20}$.

4 The $\tau$ decay data in this context was also used in$^{22}$ for a test of duality in nonleptonic $B$ decays.
FIG. 1: Inclusive differential decay rate \( d\Gamma(B \to D^{*0}X_u)/dx \). The dashed line shows the result in the \( B \) rest frame. Here \( x \) is the rescaled \( D^* \) momentum, \( x = |\vec{p}_D|/(4.95 \text{ GeV}) \) and we normalize the rate as in [13]. To compare to the CLEO [13] data we boost to the \( \Upsilon \) rest frame which gives the solid line. Only three data points (shown) are available that are not contaminated by charm contributions above the \( D_s \) threshold.

FIG. 2: a) Breakdown of contributions to the \( B \to D^*X \) spectrum in the \( B \) rest frame from the \( \tau \) (dotted line), \( s\bar{c} \) (dot-dashed line), \( \{e, \mu\} \) (thin solid line), and \( d\bar{u} \) (dashed line) final states. The thick solid line shows the total result. b) Inclusive differential decay rate \( d\Gamma/dp \) for the process \( B \to D^{0*}X \) using factorization from the large \( N_c \) and SV limits. The two curves are as in Fig. 1. Charmed states in \( X \) are included using perturbative QCD to “model” the contributions of the \((s\bar{c})_{V-A}\) spectral density. For large \( q^2 \) (small \( x \)) this approximation should be reasonable.

The CLEO data is presented as a function of the rescaled lab momentum of the \( D^* \) meson \( x = |\vec{p}|/(4.95 \text{ GeV}) \). In Fig. 1 we show the boosted\(^5\) spectrum assuming that the \( B \) mesons

\(^5\) In Ref. [25] an analysis of the CLEO data was performed using factorization as in Eq. (8) plus a model
are produced monochromatically and isotropically, with velocity $\beta = 0.06$ in the lab frame, and using
\[
\frac{d\Gamma_{\text{lab}}}{d|\vec{p}|} = \frac{1}{2\beta\gamma^2} \frac{1}{E} \int_{s_{\text{min}}}^{s_{\text{max}}} ds \frac{1}{P(s)} \frac{d\Gamma_{\text{CM}}}{ds}.
\]
Here $\gamma = 1/\sqrt{1-\beta^2}$ and $E = \sqrt{\vec{p}^2 + m_{D^*}^2}$, and $P(s)$ and $d\Gamma_{\text{CM}}/ds$ are the momentum of the $D^*$ and the decay rate in the rest frame of the $B$ meson, respectively. The limits of integration are $s_{\text{max}} = m_B^2 + m_{D^*}^2 + 2m_B\gamma(E - \beta|\vec{p}|)$ and $s_{\text{min}} = max(0, m_B^2 + m_{D^*}^2 - 2m_B\gamma(E + \beta|\vec{p}|))$.

Fig. 1 shows our result for the $x$ spectrum as a continuous curve. The CLEO data is shown as data points. The semileptonic rate was taken from the BELLE fit to a form factor using a unitarity constrained parametrization. The available data is in good agreement with the factorization prediction.

Unfortunately, for large $q^2$ it is necessary to include charmed hadrons in $X$ to enable a comparison with the experimental data that is currently available from CLEO. If we include charmed hadrons in the $X$ produced through a $(\bar{s}c)_{V-A}$ current then it becomes harder to test factorization in a clean way. One problem is that no data for the $(\bar{s}c)_{V-A}$ spectral functions is available. To make predictions for the $(\bar{s}c)_{V-A}$ current we model the spectral functions using perturbative QCD. This neglects the resonant structures in the region above $q^2 \sim m_{D^*}^2$, but should be fine for testing factorization at large invariant masses.

A second potential problem is that the $(\bar{s}\bar{c})$ and $(\tau\bar{\nu}_e)$ contributions require $b \to c$ form factors which are not accessible in $B \to D^{(*)}\ell\nu$ decay. However, since the SV limit implies heavy quark symmetry, this can be used to predict these form factors. Finally, an additional source of uncertainty is introduced by the value of the charm quark mass. We choose $m_c = 1.5$ GeV in our calculations. Note that since the $(\bar{s}c)_{V-A}$ current is phase space suppressed it only contributes roughly half as much to the decay rate relative to $(\bar{d}u)_{V-A}$ current. This helps to reduce the model dependence of our $B \to D^*X$ predictions.

In Fig. 2 we show a breakdown of the lepton and quark contributions to the total $B \to D^*X$ rate in the $B$ rest frame. The prediction from factorization is in moderate agreement with the data. The disagreement does not seem to scale with the invariant mass of the system, however due to the large theoretical uncertainties in our calculation, the results in the region above the $c\bar{c}s$ threshold are inconclusive. Improved measurements of the energy spectrum in Fig. 2 which disentangle the states with different charm quark numbers would help to clarify this issue.

\section{IV. DISCUSSION AND CONCLUSIONS}

Data suggests that the amplitude for $\bar{B}^0 \to D^+\pi^-$ factorizes. This can be understood via large $N_c$ counting or via large $Q$ as an expansion in powers of $1/Q$, the inverse of the energy released. Thus, deviations from factorization in this process are doubly suppressed. However, data also suggests factorization in the exclusive processes $B \to D^{(*)}\pi^-\pi^-\pi^0$ and $B \to D^*\omega\pi^-$ as well as in the inclusive $B \to D^*X$. Neither large $N_c$ nor large $Q$ explain by themselves these results. Indeed, large $N_c$ predicts instead a sum of factorizable terms. On the other hand the corrections to factorization in large $Q$ are order $m_X/Q$, and in these decays, particularly in $B \to D^*X$, this ratio can be comparable with unity.

\[\text{for the } X' \text{ contributions. Our approach differs from this by the use of the SV limit, } \tau\text{-decay data, and the inclusion of the boost to the lab frame.}\]
We have pointed out that the condition for large energy release in $B^0 \to D^+\pi^-$ can be understood as a consequence of the SV limit, $m_b, m_c \gg m_b - m_c \gg \Lambda$. In the combined large $N_c$ and SV limits we can justify factorization in $B \to D^*X$ out to arbitrary invariant hadronic mass.

We suggest the following physical picture. The corrections to factorization are parametrically small, of order $1/N_c$, but depend on kinematic variables in a way that can amplify the magnitude of the corrections if the recoiling particle is not moving fast. The SV limit ensures that this kinematic enhancement is absent: for cases for which we have control like $\bar{B}^0 \to D^+\pi^-$, the correction term is $\sim 1/Q$ so it depends on the kinematics but in such a way as to further suppress the correction to factorization.

This picture is additionally supported by $D$ and $K$ decay data. When two body $D$ decay amplitudes are written in terms of weak transitions into definite isospin states, $A_I$, and final state interaction phases, $\exp(i\delta_I)$, it is found that the amplitudes $A_I$ do factorize, provided one uses a modified $N_c$ counting $C_1(m_c) \sim C_2(m_c) \sim O(1)$ \footnote{We take this opportunity to comment on the recent measurement by BELLE \footnote{We take this opportunity to comment on the recent measurement by BELLE \cite{14} of the ratio $Br(B^+ \to D_s^{(*)}\pi^+)/Br(B^+ \to D_s^{(*)}\pi^-) = 0.89 \pm 0.14$, which has been interpreted as a test of factorization. However, as pointed out in Ref. \cite{27} this ratio depends sensitively on unknown subleading Isgur-Wise functions, which can accommodate values in the range $0 - 1.5$ within factorization.}}. The strong phases, taken from experiment, are not small but should vanish in the large $N_c$ limit. We interpret this as the expected kinematic enhancement of the correction to factorization, and it suggests that factorization fails precisely because the kinematic enhancement shows up mostly in $\delta_I \sim 1/N_c$ rather than in $A_I \sim (1/N_c)^0$.

Is this kinematic suppression due to large velocity of the products or large energy of the products? In the first case, which corresponds to the color transparency argument, the suppression is expected to behave as $1/\gamma$. In the second the suppression should be given by the larger of $\Lambda/Q$ or $m_X/Q$, where $\Lambda$ is a typical hadronic scale, $m_X$ is the invariant mass of state produced from the current by the factorizing current, and $Q$ is its energy. We see that in $K \to \pi\pi$ decays one is bound to have $\Lambda/Q \sim 1$, while $1/\gamma = 2m_\pi/m_K \approx 1/2$ and is parametrically suppressed in the chiral limit. Unfortunately, the actual pion mass is too far from the chiral limit to distinguish between the two alternatives.

We studied in this paper the factorization predictions for $B \to D^{(*)}X$, focusing on methods which can distinguish among the various possible explanations for factorization. In Sec. II we derived large $N_c$ relations among isospin amplitudes, which lead to observable predictions among decay rates. Imposing additionally the SV limit gives even more relations, since certain amplitudes allowed in the pure $N_c$ limit are suppressed. The large $Q$ limit by itself gives predictions similar to the combined limit of large $N_c$ and $SV$, however these predictions should fail outside of a limited kinematical range. We presented several such predictions that can be tested experimentally. Using available data from BELLE and Babar we discussed such predictions for the modes $B \to D^{(*)}\pi\pi$, $B \to D^{(*)}KK$ and $B \to D^{(*)}D^{(*)}K$.

In Sec. III we calculated the inclusive decay rate $B \to D^*X$ in the combined limit of large $N_c$ and $SV$. In this limit, factorization is expected to work out to arbitrary invariant hadronic mass, in contrast to the predictions from large $Q$ factorization in which factorization breaking corrections should scale as $m_X/Q$. The cleanest theoretical prediction is for the $B \to D^*X_{\tau}$ decay, for which we can use $\tau$-decay data to extract the spectral function in the resonance region\footnote{We take this opportunity to comment on the recent measurement by BELLE \cite{14} of the ratio $Br(B^- \to D_s^{(*)}\pi^-)/Br(B^- \to D_s^{(*)}\pi^-) = 0.89 \pm 0.14$, which has been interpreted as a test of factorization. However, as pointed out in Ref. \cite{27} this ratio depends sensitively on unknown subleading Isgur-Wise functions, which can accommodate values in the range $0 - 1.5$ within factorization.}°. The comparison of the prediction with the CLEO data is shown in Fig. I and shows good agreement. Inclusive data for this process is available over a much
larger kinematic range, however the data then includes contributions from the \((sc)_{V-A}\) part of the current. No data is available for the spectral function of this current, and we therefore have to rely on perturbative calculations. The result is shown in Fig. 4, however the large theoretical uncertainties preclude us from drawing definite conclusions.

We emphasize that it would be desirable to separate the \(b \rightarrow cd\bar{u}\) and \(b \rightarrow c\bar{c}s\) final states in the \(B \rightarrow D^{(*)}X\) data, including separate measurements for \(B\) and \(D\) charge eigenstates. The methods described in this paper would then allow a more detailed study of factorization in this decay, and should help shed light on the mechanism underlying the observed factorization in \(B\) decays.

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