Studying New Universal Relations in the Black Holes Thermodynamics

J. Sadeghi\textsuperscript{a,c} ∗ B. Pourhassan\textsuperscript{b,c} ∗ S. Noori Gashti\textsuperscript{a} † S. Upadhyay\textsuperscript{d,e,f} § and E. Naghd Mezerji\textsuperscript{a} ¶

\textsuperscript{a} Department of Physics, Faculty of Basic Sciences, University of Mazandaran P. O. Box 47416-95447, Babolsar, Iran
\textsuperscript{b} School of Physics, Damghan University, Damghan, 3671641167, Iran
\textsuperscript{c} Canadian Quantum Research Center, 204-3002 32 Ave Vernon, BC V1T 2L7 Canada
\textsuperscript{d} Department of Physics, K.L.S. College, Nawada, Bihar 805110, India.
\textsuperscript{e} Department of Physics, Magadh University, Bodh Gaya, Bihar 824234, India and Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, Maharashtra 411007, India.

Our primary goal in this paper is to confirm new universal relations in black holes thermodynamics. We investigate two universal relations by selecting different black holes. First, we obtain the black holes thermodynamic relations assuming a small correction to the action. Then we confirm the universal relations by performing a series of direct calculations. It is noteworthy that according to each of the properties related to black holes, a new universal relation can be obtained according to this method. We confirm two different types of these universal relations for various black holes. Furthermore, we consider black holes in AdS space surrounded by perfect fluid also. We use the small correction to the action and obtain the modified thermodynamic quantities. We achieve two new universal relations which correspond to the parameters of perfect fluid and magnetic charge of the Bardeen AdS Black Hole. Finally, the new universal relation lead us to understand that the charge-to-mass ratio. We also find that weak gravity conjecture condition is satisfied for the black hole surrounded by perfect fluid.

Keywords: Universal relations; Black holes; Perfect fluid; Weak gravity conjecture.

\textsuperscript{*}Electronic address: pouriya@ipm.ir
\textsuperscript{†}Electronic address: b.pourhassan@du.ac.ir
\textsuperscript{‡}Electronic address: saeed.noorigashti@stu.umz.ac.ir
\textsuperscript{§}Electronic address: sudhakerupadhyay@gmail.com
\textsuperscript{¶}Electronic address: e.n.mezerji@stu.umz.ac.ir
One of the most important points that researchers have considered in various contexts for several years is the study of universal relations obtained from different methods. Physics is no exception. Recently, physicists have done many works in this context to be a beacon of hope for the unification of fundamental forces. One of the methods used in this theme, which is based on string theory, is the swampland program [1], which describes areas that are generally compatible with quantum gravity by stating some criteria. One of these criteria is the weak gravity conjecture WGC), which has been widely studied in the last few years [2]. This conjecture states that gravity is the weakest force, that is, there are states whose mass ($M$) is less than their charge ($Q$) which is $Q/M \leq 1$ [3–17]. For studies related to this issue, are used black holes with different mass, charge, etc. characteristics. The important point is that, this relation is not established for the black holes with naked singularity. The cosmic censorship is usually used to address this issue. One possible solution to this problem is to use correction to action that leads to modifying black hole solutions and in a way inverse the charge-to-mass ratio. For example, in Ref. [18] higher-derivative corrections are used for the mentioned issue. As mentioned, the result indicates that the charge-to-mass ratio is larger than 1 using these higher-derivative operators and satisfies the text’s relation.

The idea of correcting higher-derivatives has been widely used in the literature. On can find some works that has been done to limit effective theories using WGC in Refs. [19–31]. Concepts related to WGC and obtaining universal relations have been explored in many works in recent years. In recent investigation, a new example of WGC studies prove the concept of WGC in flat space concerning Wald entropy [22]. The concepts related to entropy changes due to the charge and mass of black holes and higher-derivative corrections have been studied to prove the notion of WGC about black holes. Goon and Penco[32] also proposed a universal relation. This universal relation has also been studied for charged AdS black holes in four-dimensional space, which exhibits black hole WGC-like behaviour [33]. Other work has also been done on the Goon-Penco universal relation, for example, in examining four-derivative corrections competencies to the geometry of charged AdS black holes. For further review of this relation, one can also look into Refs. [34, 35].
Furthermore, researchers have done a lot of works to achieve universal relations in recent years. For example, Davida and Nian [36] investigated the universal entropy and Hawking radiation of near-extremal AdS4 black holes. Chen, Hong and Tao [37] also studied universal thermodynamic extremality relations for charged AdS black hole surrounded by quintessence. In this work we study the new universal relations by investigating universal relation between corrections to entropy and extremality bounds with WGC verification as in previous work [38]. Given the universal relations implications, the Goon and Penco universal thermodynamic extremality relation have not been fully explored in studies related to AdS-rotating black holes. Therefore, in this work, we want to evaluate several black holes concerning this universal relation entirely. Thus, by selecting several black holes in the AdS space, using a small constant correction added to the action, we obtain the modified thermodynamic relations, and we examine the resulting in universal relations. A new type of universal relation will also be explored in rotating black holes.

In their previous work, the some authors of this article also examined black holes surrounded by quintessence dark energy and cloud of string which had very attractive results and led to the introduction of new universal relations [45]. Now, we want to examine another form of universal relation for three different black holes surrounded with perfect fluid that have not been studied in the literature. To obtain such universal relation, we add small constant correction to the corresponding action and obtain the modified thermodynamic quantities and relations. These modified quantities help us to perform a series of calculations and confirm the universal relations. We see also the effect of small correction to the thermodynamic relations. The corresponding correction an increase the charge-to-mass ratio, which indicates WGC behaviour. To study thermodynamic relationships as well as the WGC, we select Reissner-Nordstöm-AdS, rotating Bardeen and Kerr-Newman-AdS black holes surrounded by perfect fluid matter. As we know dark matter also includes cold dark matter (CDM), warm dark matter (WDM), scalar background dark matter (SFDM) and perfect fluid dark matter (PFDM). Here, we combine PFDM with black hole solution and investigate a particular type of universal relation as in Refs. [45–64]. Therefore, in addition to PFDM, we consider black holes with string fluid and obtain the new universal relation as in Refs. [65–73].

The paper is organized as following. In Sections II, III, IV, V, VI and VII, we discuss the universal
relations of AdS Schwarzschild black holes, charged BTZ black holes, charged rotating BTZ black holes, accelerating black holes, charged accelerating black holes and rotating accelerating black holes, respectively. In Section VIII, we confirm the universal thermodynamic relations for the Reissner-Nordström-AdS black hole with perfect fluid dark matter due to a small constant correction of the action. In Section IX, we consider Kerr-Newman-AdS black hole surrounded by perfect fluid matter. Here also we achieve the new universal relation and study the role of WGC on the corresponding system. In Section X, we consider rotating Bardeen black holes in AdS space surrounded by perfect fluid and we confirm a new universal relation. Finally, we draw conclusion in the last section.

II. ADS SCHWARZSCHILD BLACK HOLES

Concerning all the concepts discussed above, we examine the universal relations by considering different black holes. In this context, we first apply on the AdS Schwarzschild black hole [39]. The action of AdS Schwarzschild black hole is given by

$$S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} (R - 2\Lambda),$$  

(1)

where the gravitational constant $G = 1$ (for simplicity) and cosmological constant $\Lambda = -\frac{1}{l^2}$. The solution to the Einstein field equation for this black hole is written as follows

$$ds^2 = f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$  

(2)

where $f(r)$ has the following form:

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{2M_0}{r}.$$  

(3)

Here $M_0$ characterizes the mass of the black hole. The outer and inner horizons associated with a black hole are calculated from $f(r) = 0$, and thus the values of temperature, entropy, etc., can be easily calculated using the thermodynamic relations of the black hole.

Now, we implement a minimal constant correction by introducing $\epsilon$ to the action as following:

$$S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} (R - \epsilon \times 2\Lambda).$$  

(4)
Due to the modification of the action, the black hole solution is also modified. Therefore, each of the thermodynamic values of the black holes will also be modified. Hence, the modified mass and temperature are obtained by considering a small constant correction as follows,

\[ M = \frac{S}{8\pi} + \frac{S^3(1 + \epsilon)}{128l^2\pi^3}, \]  
\[ T = \frac{1}{8\pi} + \frac{3S^2(1 + \epsilon)}{128l^2\pi^3}. \]  

For such black holes, the externality boundary is also modified. So with \( T = 0 \), the extremal entropy bounded by a small constant correction is given by,

\[ S = \frac{4\pi l}{\sqrt{-3(1 + \epsilon)}}. \]  

Here, upon solving the temperature equation, we chose the real positive entropy. Also, by solving the equation (5), the constant correction takes the following value

\[ \epsilon = \frac{128l^2M\pi^3 - 16l^2\pi^2S - S^3}{S^3}. \]  

The derivative of \( \epsilon \) with respect to \( S \) is given by

\[ \frac{\partial \epsilon}{\partial S} = -\frac{16l^2\pi^2 - 3S^2(1 + \epsilon)}{S^3}. \]  

By combining the equations (5), (7) and (9), we will have a relation at \( M \rightarrow M_{\text{ext}} \) as follows,

\[ -T \frac{\partial S}{\partial \epsilon} = \frac{1}{6\sqrt{3}l^2} \left( -\frac{l^2}{(1 + \epsilon)} \right)^{\frac{3}{2}}. \]  

After calculating the above relation, according to the equations (5) and (7), \( M_{\text{ext}} \) is given by

\[ M_{\text{ext}} = \frac{1}{3\sqrt{3}} \left( -\frac{l^2}{(1 + \epsilon)} \right)^{\frac{1}{2}}. \]  

The mass bound will increase with the constant correction parameter \( \epsilon \), and in a way, this added correction can satisfy the conditions related to weak gravity conjecture. According to the concepts, we take the derivative of \( M_{\text{ext}} \) with respect to this constant parameter, we have

\[ \frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{1}{6\sqrt{3}l^2} \left( -\frac{l^2}{(1 + \epsilon)} \right)^{\frac{3}{2}}. \]
Incidentally, the two equations (10) and (12) are precisely same. We first proved the Goon-Penco universal extremality relation for the AdS Schwarzschild black hole.

In the following, by selecting the other black holes and confirming the mentioned relation, we will examine the new universal relations that are somehow derived from the characteristics of black holes such as rotating.

III. CHARGED BTZ BLACK HOLES

In this section, we examine the universal relations for charged BTZ black holes. The corresponding action is given as [40].

\[ S = \frac{1}{16\pi} \int d^3x \sqrt{-g} (R - 2\Lambda) \]  

(13)

The singular solution to the Einstein field equation for this black hole is given by

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\phi^2, \]  

(14)

where \( f(r) \) has the following form:

\[ f(r) = -M_0 + \frac{r^2}{l^2} - \frac{Q^2}{2} \log(r). \]  

(15)

Here \( M_0 \) and \( Q \) show the mass and charge of the black hole. Following the above section, we introduce a minimal constant parameter \( \epsilon \) to the action (13) as follows,

\[ S = \frac{1}{16\pi} \int d^3x \sqrt{-g} (R - \epsilon \times 2\Lambda). \]  

(16)

This is obvious that the thermodynamics will also be modified due to modification in action. In this case, the modified mass and temperature are calculated concerning a small constant correction as follows,

\[ M = \frac{S^2(1 + \epsilon)}{16\pi^2 l^2} - \frac{Q^2}{2} \log \left[ \frac{S}{4\pi} \right], \]  

(17)

\[ T = \frac{Q^2}{2S} + \frac{S(1 + \epsilon)}{8\pi^2 l^2}. \]  

(18)
For charged BTZ black hole, the externality boundary is also modified. So with \( T = 0 \), the extremal entropy bound is obtained. After solving the temperature equation, we choose the real positive entropy, which is given by,

\[
S = \frac{2l\pi Q}{\sqrt{1 + \epsilon}}. \tag{19}
\]

Also, by simplifying the equation (17), the constant correction parameter takes following value in this case:

\[
\epsilon = \frac{16l^2M\pi^2 - S^2 + 8l^2\pi^2Q^2\log\left[\frac{S}{16}\right]}{S^2}. \tag{20}
\]

The derivative of \( \epsilon \) with respect to \( S \) yields

\[
\frac{\partial \epsilon}{\partial S} = \frac{8l^2\pi^2Q^2 - 2S^2(1 + \epsilon)}{S^3}. \tag{21}
\]

By combining the equations (18), (19) and (21), the mass takes \( M \to M_{\text{ext}} \) as follows,

\[
-T \frac{\partial S}{\partial \epsilon} = \frac{Q^2}{4(1 + \epsilon)}, \tag{22}
\]

After calculating the above relation, according to the equations (17) and (19), \( M_{\text{ext}} \) takes following value:

\[
M_{\text{ext}} = \frac{Q^2}{4} \left(1 - \log \left[\frac{l^2Q^2}{4(1 + \epsilon)}\right]\right). \tag{23}
\]

This added correction can satisfy the conditions related to weak gravity conjecture. Upon taking derivative with respect to the constant parameter, we will have

\[
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{Q^2}{4(1 + \epsilon)}. \tag{24}
\]

Here also, equations (22) and (24) are precisely the same. This justifies the Goon-Penco universal extremality relation for the charged BTZ black hole.

**IV. CHARGED ROTATING BTZ BLACK HOLES**

In this section, we chose the charged rotating BTZ black holes [43, 44] in order to obtain the modified thermodynamic relations. We also consider the universal thermodynamic relations. Therefore,
the modified thermodynamic parameters such as mass, temperature and angular velocity as follows

\[ M = \frac{4J^2\pi^2}{S^2} + \frac{S^2(\epsilon + 1)}{16l^2\pi^2} - \frac{1}{2}\pi Q^2 \log \left[ \frac{S}{4\pi} \right], \]  
\[ (25) \]

\[ T = -\frac{8J^2\pi^2}{S^3} - \frac{\pi Q^2}{2S} + \frac{S(1 + \epsilon)}{8l^2\pi^2}, \]  
\[ (26) \]

\[ \Omega = \frac{8J}{S^2}. \]  
\[ (27) \]

Now \( T = 0 \) from equation (26), the extremal entropy bound can be obtained. After solving this temperature equation, we choose the real positive entropy as

\[ S = \sqrt{\frac{2l^2\pi^3Q^2}{1 + \epsilon} + \frac{2\pi^2l(16J^2 + l^2\pi^2Q^4 + 16l^2\epsilon)}{1 + \epsilon}}. \]  
\[ (28) \]

From equation (25), we obtain

\[ \epsilon = -\frac{64S^2l^2\pi^4 + 16l^2M\pi^2S^2 - S^4 + 8l^2\pi^3Q^2S^2}{S^4}. \]  
\[ (29) \]

The derivative with respect to \( S \) leads to

\[ \frac{\partial \epsilon}{\partial S} = \frac{128J^2l^2\pi^4 + 8l^2\pi^3S^2Q^2 - 2S^4(1 + \epsilon)}{S^5}. \]  
\[ (30) \]

Exploiting relations (26), (28) and (30), we have a relation at \( M \rightarrow M_{ext} \) as follows

\[ -T \frac{\partial S}{\partial \epsilon} = \frac{l^2\pi Q^2 + \sqrt{l^4\pi^2Q^4 + 16J^2l^2(1 + \epsilon)}}{8l^2(1 + \epsilon)}. \]  
\[ (31) \]

The expression for \( M_{ext} \) in this case is given as

\[ M_{ext} = \frac{\sqrt{l^4\pi^2Q^4 + 16J^2l^2(1 + \epsilon)}}{4l^2} - \frac{\pi Q^2}{4} \log \left[ \frac{l^2\pi Q^2 + \sqrt{l^4\pi^2Q^4 + 16J^2l^2(1 + \epsilon)}}{8(1 + \epsilon)} \right]. \]  
\[ (32) \]

The derivative of \( M_{ext} \) with respect to \( \epsilon \) leads to

\[ \frac{\partial M_{ext}}{\partial \epsilon} = \frac{l^2\pi Q^2 + \sqrt{l^4\pi^2Q^4 + 16J^2l^2(1 + \epsilon)}}{8l^2(1 + \epsilon)}. \]  
\[ (33) \]

The equations (31) and (33) are the equal, so the universal relation for this black hole is also satisfied.

The concepts presented and the rotating nature of this black hole lead us to examine another universal
relation. We now study the new universal relation. Therefore, concerning equations (28) and (32), the shifting mass bound is given by

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{S^2}{16l^2 \pi^2}. \quad (34)$$

From equation (29), it is matter of calculation only to show

$$\frac{\partial J}{\partial \epsilon} = -\frac{S^4}{128l^2 \pi^4}. \quad (35)$$

This expression further simplifies to

$$-\Omega \frac{\partial J}{\partial \epsilon} = \frac{S^2}{16l^2 \pi^2}. \quad (36)$$

As one can see, the two equations (34) and (36) are the same. This relation is due to the rotating nature of black holes and, in this extremal limit, this equation is well proved. We observed that a new universal relation could be defined for a black hole based on its characteristics. In fact, black holes with other features can be considered and examined for the new universal relations. It is also possible to introduce new universal thermodynamic relation concerning the specific characteristics of black holes.

V. ACCELERATING BLACK HOLES

In previous two sections, we proved the universal relation for the Schwarzschild and charged BTZ black holes. In this section, we consider an accelerating black hole [41] to check the universal relation. The metric for the accelerating black hole is given by

$$ds^2 = \frac{1}{\omega^2} \left[ f(r)dt^2 - f^{-1}(r)dr^2 - r^2 \frac{d\theta^2}{g(\theta)} + r^2 g(\theta) \sin^2 \theta \frac{d\phi^2}{K^2} \right], \quad (37)$$

where $K$ is the conical deficit and

$$f(r) = (1 - A^2 r^2)(1 - \frac{2m}{r}) + \frac{r^2}{l^2}, \quad g(\theta) = 1 + 2mA \cos \theta, \quad \omega = 1 + Ar \cos \theta. \quad (38)$$

Here $A$ is the acceleration and $m$ is the mass scale of the black hole. When we modify the action with the small correction constant parameter, each of the thermodynamic values of the black holes also
gets modified. Hence, the modified mass and temperature are obtained as follows:

$$M = \frac{S^3 + l^2(16\pi^2 S(1 + \epsilon) - A^2 S^3(1 + \epsilon))}{8l^2(16\pi^2 - A^2 S^2)},$$  

(39)

$$T = \frac{48\pi^2 S^2 - A^2 S^4 + l^2(-16\pi^2 + A^2 S^2)^2(1 + \epsilon)}{8l^2\pi(-16\pi^2 + A^2 S^2)^2}. $$  

(40)

So with $T = 0$, the extremal entropy bounded by a small constant correction is given by

$$S = \sqrt{-\frac{-1 + A^2 l^2(1 + \epsilon)}{\sqrt{2} A^2 l(-1 + \sqrt{9 - 8 A^2 l^2(1 + \epsilon)})}} \cdot \frac{\sqrt{l^2(1 + \epsilon)}}{-6 + 4 A^2 l^2(1 + \epsilon) + 2\sqrt{9 - 8 A^2 l^2(1 + \epsilon)} - A^2 + A^4 l^2(1 + \epsilon)}.$$  

(41)

Here chose the real positive entropy. After simplifying equation (39), we get

$$\epsilon = \frac{S^3 - l^2(16\pi S - S)(16\pi - A^2 S^2)}{l^2(-16\pi^2 S + A^2 S^2)}. $$  

(42)

Now, the derivative of $\epsilon$ with respect to $S$ yields

$$\frac{\partial \epsilon}{\partial S} = \frac{-48\pi^2 S + A^2 S^3}{l^2(-16\pi^2 + A^2 S^2)^2} - \frac{(1 + \epsilon)}{S}.$$  

(43)

Exploiting equations (40), (41) and (43), we have a relation at $M \to M_{\text{ext}}$ as

$$-T \frac{\partial S}{\partial \epsilon} = \sqrt{\frac{l^2(1 + \epsilon)}{-6 + 4 A^2 l^2(1 + \epsilon) + 2\sqrt{9 - 8 A^2 l^2(1 + \epsilon)} - A^2 + A^4 l^2(1 + \epsilon)}}.$$  

(44)

According to the equations (39) and (41), $M_{\text{ext}}$ is given by

$$M_{\text{ext}} = \frac{(-1 + A^2 l^2(1 + \epsilon))(-3 + \sqrt{9 - 8 A^2 l^2(1 + \epsilon)})\sqrt{l^2(1 + \epsilon)}}{-6 + 4 A^2 l^2(1 + \epsilon) + 2\sqrt{9 - 8 A^2 l^2(1 + \epsilon)} - A^2 + A^4 l^2(1 + \epsilon)}.$$  

(45)

This added correction can satisfy the conditions related to weak gravity conjecture. We calculate the derivative of above relation according to this constant parameter $\epsilon$ and have

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \sqrt{\frac{l^2(1 + \epsilon)}{-6 + 4 A^2 l^2(1 + \epsilon) + 2\sqrt{9 - 8 A^2 l^2(1 + \epsilon)} - A^2 + A^4 l^2(1 + \epsilon)}}.$$  

(46)

The relations (44) and (46) coincide and the Goon-Penco universal extremality relation for this black hole is also justified.
VI. CHARGED ACCELERATING BLACK HOLES

In this section we generalize the result for the charged accelerating black hole [42]. According to the above concepts in the previous section for the accelerating black hole, we add the charge to this black hole. By introducing constant correction parameter $\epsilon$ to the action, the solution and thermodynamic parameters get modified. We obtain mass and temperature as follows

\[
M = \frac{2Q^2\pi}{S} + \frac{16l^2\pi^2 S(1 + \epsilon) + S^3 - A^2l^2S^3(1 + \epsilon)}{128l^2\pi^3 - 8A^2l^2\pi S^2},
\]

\[
T = \frac{1 + \epsilon}{8\pi} - \frac{2Q^2\pi}{S^2} + \frac{48\pi^2S^2 - A^2S^4}{8\pi l^2(-16\pi^2 + A^2S^2)^2}.
\]

For vanishing temperature, the extremal entropy bound is obtained. Four different values are obtained for entropy, two of which are imaginary, one is negative and the fourth is positive. After a very straightforward simplification and while maintaining the basic structure of entropy, it will be used in calculations related to proving the universal relation. First by solving the equation (47), we obtain

\[
\epsilon = \frac{16Q^2\pi^2(A^2 - \frac{16\pi^2}{S^2}) + M(128\pi^3 - 8A^2\pi S) - 16\pi^2 - \frac{S^2}{\pi^2} + A^2S^2}{16\pi^2 - A^2S^2}.
\]

By taking the derivative with respect to $S$ and concerning the positive value of entropy and implementing the equation (48) along with $M \to M_{\text{ext}}$, we have

\[
-T\frac{\partial S}{\partial \epsilon} = \frac{A^4l^2Q^2}{2\sqrt{2}(A^2l^2 - (1 + \epsilon))(A^4l^2Q^2 - (1 + \epsilon))} \sqrt{\frac{3}{A^2} + \frac{\rho(-1 + A^2Q^2)}{A^2l^2 - (1 + \epsilon)}}.
\]

According to the equation (47) one can obtain

\[
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{A^4l^2Q^2}{2\sqrt{2}(A^2l^2 - (1 + \epsilon))(A^4l^2Q^2 - (1 + \epsilon))} \sqrt{\frac{3}{A^2} + \frac{\rho(-1 + A^2Q^2)}{A^2l^2 - (1 + \epsilon)}}.
\]

In this case also, the universal relation is satisfied as equations (50) and (51) are the same.

VII. ROTATING ACCELERATING BLACK HOLES

In this section, we select a rotating accelerating black hole [42]. We follow the same procedure as in the previous sections. The point to be noted in this case is that the black hole under consideration is
rotating. We modify the action with a constant correction parameter and obtain each of the modified thermodynamic parameters such as mass, temperature, and angular velocity as follows,

\[
M = \frac{(16J^2\pi^2 + S^2)(S^2 + l^2(16\pi^2 - A^2S^2(1 + \epsilon)))}{8l^2(16\pi^3S - A^2\pi^3S^3)} \tag{52}
\]

\[
T = \frac{A \times B}{C}, \tag{53}
\]

\[
\Omega = \frac{4J\pi^2(S^2 + l^2(16\pi^2 - A^2S^2(1 + \epsilon)))}{l^2(16\pi^3S - A^2\pi^3S^3)}, \tag{54}
\]

where

\[
A = 48\pi^2S^4 - A^2S^6 + l^2S^2(256\pi^4 + A^4S^4(1 + \epsilon) - 16A^2\pi^2S^2(2 + 3\epsilon)), \tag{55}
\]

\[
B = -16J^2\pi^2(-S^2(16\pi^2 + A^2S^2) + l^2(256\pi^4 + 16A^2\pi^2S^2(-2 + \epsilon)) + A^4S^4(1 + \epsilon)), \tag{56}
\]

\[
C = 8l^2\pi S^2(-16\pi^2 + A^2S^2)^2. \tag{57}
\]

So concerning the equation (53) that means \(T = 0\), the extremal entropy bound is obtained. Still, according to the calculations, four different values are obtained for entropy, two of which are imaginary, one is negative, and the fourth is positive. After a very straightforward simplification and while maintaining the basic structure of entropy, it will be used in calculations related to proving the universal relation. Hence, first, by solving the equation (52), one can get

\[
\epsilon = -1 + \frac{1}{l^2A^2} + \frac{8\pi MS}{16J^2\pi^2 + S^2} + \frac{16\pi^2}{A^2} \left( \frac{16J^2\pi^2 - 8\pi MS + S^2}{16J^2\pi^2S^2 + S^4} \right). \tag{58}
\]

We take the derivative of \(S\) with respect to \(\epsilon\) in (58). By combining the equation (53) and the positive entropy value as well as according to the condition \(M \to M_{\text{ext}}\), we will have,

\[
-T\frac{\partial S}{\partial \epsilon} = \frac{2J^3A^2(25 + 9Al(-2 + Al(1 + \sqrt{\epsilon}))(1 + \sqrt{\epsilon}))}{3(-16J^2A^2 + 9(-1 + Al(1 + \sqrt{\epsilon}))^2)(-1 + Al(1 + \sqrt{\epsilon}))}. \tag{59}
\]

We consider the equation (52) and the positive value of entropy one can write

\[
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{2J^3A^2(25 + 9Al(-2 + Al(1 + \sqrt{\epsilon}))(1 + \sqrt{\epsilon}))}{3(-16J^2A^2 + 9(-1 + Al(1 + \sqrt{\epsilon}))^2)(-1 + Al(1 + \sqrt{\epsilon}))}. \tag{60}
\]
Here, we observe that equations (59) and (60) are exactly same and the universal relation for this black hole are also satisfied. As mentioned above, the rotating nature of this black hole leads us to examine another universal relation. According to the concepts presented, we now study the new universal relation. Therefore, with respect to equation (52) and mass bound with respect to the condition, $M \to M_{\text{ext}}$ and solve the entropy in terms of $J$, we can obtain the new relation with regard to $\epsilon$ derivative as follows,

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{A^2(16J^2 \pi^2 + S^2)^2(-S^2 + l^2(-16\pi^2 + A^2 S^2(1 + \epsilon)))}{64l^2 M \pi^2(-16\pi^2 + A^2 S^2)^2}.$$  
(61)

Now considering the equations (61) and (58) we have at $M \to M_{\text{ext}},$

$$-\Omega \left( \frac{\partial J}{\partial \epsilon} \right) = \frac{A^2(16J^2 \pi^2 + S^2)^2(-S^2 + l^2(-16\pi^2 + A^2 S^2(1 + \epsilon)))}{64l^2 M \pi^2(-16\pi^2 + A^2 S^2)^2}.$$  
(62)

Comparing the equations (61) and (62), one can see that these two equations are the same. This relation is due to the rotating nature of black holes, and in this extremal limit, these equations are well confirmed.

In the next section, we will study both of these universal relations examined in this section for a charged rotating BTZ black hole in AdS space-time. Of course, the interesting point is that such a new universal relation can be confirmed by considering each black holes specific feature.

### VIII. REISSNER-NORDSTÅRM ADS BLACK HOLE WITH PFDM

In this section, we wish to study the Goon-Penco universal extremality relation for Reissner-Nordström AdS black hole with PFDM and obtain the other new universal relation. So, we consider Reissner-Nordström AdS black hole. We use a small correction $\epsilon$ to the action, and obtain the modified thermodynamic quantities and relations. Here we first write the action which is given by [73],

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{\Lambda}{8\pi G} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{DM} \right],$$  
(63)

where $G$, $\Lambda$, $F_{\mu\nu}$ and $L_{DM}$ are the Newton gravity constant, cosmological constant, tensor of electromagnetic field and dark matter Lagrangian density, respectively. So, the solution of action with
PFDM is as following \[74\],

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \] (64)

where \( f(r) \) is

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{1}{3} \Lambda r^2 + \frac{\alpha}{r} \ln \left( \frac{r}{|\alpha|} \right). \] (65)

Here, the \( M, Q \) and \( \alpha \) are the mass, charge and the intensity of the PFDM, respectively. \( f(r) = 0 \) determine the outer and inner horizons. The thermodynamic relation such as temperature, entropy, etc can be easily calculated. The mass and temperature of Reissner-Nordström AdS black hole with PFDM are given by,

\[ M = \frac{\sqrt{\frac{2}{3}} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{2\sqrt{2\pi}} - \frac{S^\frac{3}{2}}{4\sqrt{2}l^2\pi^2} + \frac{1}{2} \alpha \log \left( \frac{\sqrt{S}}{\sqrt{2\pi\alpha}} \right), \] (66)

and

\[ T = -\frac{\sqrt{\frac{2}{3}} Q^2}{\sqrt{S}} + \frac{1}{4\sqrt{2\pi}S} - \frac{3\sqrt{S}}{8\sqrt{2}l^2\pi^2} + \frac{\alpha}{4S}. \] (67)

Here, we consider a small constant correction as \( \epsilon \) to the action. So, we have following modified action:

\[ S = \int d^4x\sqrt{-g} \left[ \frac{R}{16\pi G} + (1 + \epsilon)(-\frac{\Lambda}{8\pi G}) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + L_{DM} \right]. \] (68)

The modified mass and temperature are given, respectively, by

\[ M = \frac{\sqrt{\frac{2}{3}} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{2\sqrt{2\pi}} - \frac{S^\frac{3}{2}(1 + \epsilon)}{4\sqrt{2}l^2\pi^2} + \frac{1}{2} \alpha \log \left( \frac{\sqrt{S}}{\sqrt{2\pi\alpha}} \right), \] (69)

and

\[ T = -\frac{\sqrt{\frac{2}{3}} Q^2}{\sqrt{S}} + \frac{1}{4\sqrt{2\pi}S} - \frac{3\sqrt{S}(1 + \epsilon)}{8\sqrt{2}l^2\pi^2} + \frac{\alpha}{4S}. \] (70)

We see here that the modified mass and temperature of the black hole are written in terms of entropy \( S \), charge \( Q \), \( \alpha \) and correction parameter \( \epsilon \). By solving equation (69), we obtain constant correction parameter \( \epsilon \) as

\[ \epsilon = \frac{4\sqrt{2}l^2\pi^2 Q^2 - 8l^2 M \pi^{\frac{3}{2}} \sqrt{S} + 2\sqrt{2}l^2 S \pi - \sqrt{2} S^2 + 4l^2 \sqrt{2} S \alpha \pi^{\frac{3}{2}} \log \left( \frac{\sqrt{S}}{\sqrt{2\pi\alpha}} \right)}{\sqrt{2} S^2}. \] (71)
Now, we take derivative with respect to $S$, which yields

$$\frac{\partial \epsilon}{\partial S} = l^2 \pi (-8\pi Q^2 - 2S + \sqrt{2\pi}S(6M + \alpha - 3\alpha \log(\sqrt{2\pi\alpha})))/S^3. \quad (72)$$

We use the equations (70) and (72) and then the corresponding limit is taken, which simplifies equation as follows

$$- T \frac{\partial S}{\partial \epsilon} = \frac{S^{\frac{3}{2}}}{4\sqrt{2} l^2 \pi \frac{3}{2}}. \quad (73)$$

To obtain the second part of universal relation, we solve the temperature equation and obtain the corresponding entropy. We use equations (69), (70) and corresponding entropy, to obtain

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = - \frac{S^{\frac{3}{2}}}{4\sqrt{2} l^2 \pi \frac{3}{2}}. \quad (74)$$

We see that the equations (73) and (74) are exactly same. So, we first confirm the Goon-Penco universal extremality relation for Reissner-Nordström AdS black hole with PFDM.

In the second step, we investigate another universal relation. So, we use the relation (71) and calculate

$$\frac{\partial \epsilon}{\partial Q} = \frac{8l^2 \pi^2 Q}{S^2}. \quad (75)$$

By considering equation (75), electric potential $\Phi = \frac{\sqrt{2\pi}Q}{\sqrt{S}}$ and extremal bound, we get

$$- \Phi \frac{\partial Q}{\partial \epsilon} = - \frac{S^{\frac{3}{2}}}{4\sqrt{2} l^2 \pi \frac{3}{2}}. \quad (76)$$

Here, also two equations (76) and (74) are same.

In the following, we try to find another universal relationship between mass and pressure ($P = \frac{3}{8\pi l^2} = -\frac{\Lambda}{8\pi}$). So with respect to equation (69) we will have

$$\frac{\partial P}{\partial \epsilon} = \frac{4\sqrt{2} P^2 S^2}{3S^2(1+\epsilon) \sqrt{2} l^2 \pi}. \quad (77)$$

Therefore, according to $V = -\frac{1}{3} \sqrt{\frac{2}{\pi}} S^{\frac{3}{2}}(1+\epsilon)$ and extremal bound, we have following expression,

$$-V \frac{\partial P}{\partial \epsilon} = - \frac{S^{\frac{3}{2}}}{4\sqrt{2} l^2 \pi \frac{3}{2}}. \quad (78)$$
Two equations (76) and (78) are exactly same and universal relation is proved. Now our goal is to get a new universal relation, the meaning of being new is to be related to a newly introduced parameter such as perfect fluid ($\alpha$). To achieve such new universal relation, we use (71) and obtain the following equation:

$$\frac{\partial \epsilon}{\partial \alpha} = -\frac{4l^2 \sqrt{S \pi^2} + 4l^2 \sqrt{S \pi^2} \log(\frac{\sqrt{S}}{\alpha \sqrt{2\pi}})}{\sqrt{2S^2}}.$$  \hspace{1cm} (79)

Hence, by using equation (79) and expression $\xi = -\frac{1}{2} + \frac{1}{2} \log(\frac{\sqrt{S}}{\alpha \sqrt{2\pi}})$ which is the conjugate the parameter $\alpha$, one can obtain

$$-\xi \frac{\partial \alpha}{\partial \epsilon} = -\frac{S^2}{4\sqrt{2l^2 \pi^2}}.$$  \hspace{1cm} (80)

We see here two equations (80) and (78) are exactly same. In fact, we obtained a new universal relation for the Reissner-Nordström AdS black hole with PFDM, so this relationship is confirmed correctly. Now we draw some figures and compare the mass charge ratio and also thermodynamic quantities in the modified and unmodified modes. We fix some parameters and draw figure of the mass-to-charge, we have three cases as charge-mass ratio be bigger, equal and smaller than one. So, when we have extremality the charge-mass ratio is one which is shown in the figure 1 by dashed line. For different

![FIG. 1: The plot of unmodified $M$ in term of $Q$ with respect to $l = 0.1, 0.2, 0.3$ in fig. (a) and the plot of modified $M$ in term of $Q$ with respect to $l = 0.1$ and $\epsilon = -0.1, 0, 0.1$ in fig. (b)](image)
radius $l$ of AdS in the uncorrected mass mode, the mass-to-charge ratio is greater than one as shown in the figure [I(a)]. For the modified case we have figure [I(b)], when the constant correction is always a negative value, the mass of the black hole decreases, and when the small correction is positive, the mass also increases. Therefore, when we consider small negative correction, the charge-to-mass ratio always increases. As we mentioned in the text, it can behave like WGC. So, we see that the small correction and its parameter in the metric background play an important role for the existence of WGC.

In the next section, we calculate same universal relation for Kerr-Newman AdS black hole surrounded by perfect fluid matter and also obtain a new universal relation. Then we compare the corresponding results of two section with respect to each other.

**IX. KERR-NEWMAN ADS BLACK HOLE SURROUNDED BY PERFECT FLUID MATTER**

In this section, we want to study the universal relations for Kerr-Newman AdS black hole surrounded by perfect fluid matter in Rastall gravity [75]. The concepts of general relativity and its modification have always been discussed in various branch of physics. Some researchers use it’s modified form in concepts of conservation condition of the energy-momentum tensor. One of these modified theories was also introduced by Rastall. Rastall gravity is based on the hypothesis introduced by $T_{\mu \nu} = \lambda R_{\nu}^{\mu}$, where $T_{\mu \nu}$ and $\lambda$ are energy-momentum tensor and the Rastall parameter, respectively [76, 77]. The Kerr-Newman AdS black hole solution is given by [76]

$$ds^2 = \frac{\Sigma^2}{f(r)} dr^2 + \frac{\Sigma^2}{f(\theta)} d\theta^2 + \frac{f(\theta) \sin^2 \theta}{\Sigma^2} \left( a \frac{dt}{\Xi} - (r^2 + a^2) \frac{d\phi}{\Xi} \right)^2 - \frac{f(r)}{\Sigma^2} \left( \frac{dt}{\Xi} - a \sin^2 \frac{d\phi}{\Xi} \right)^2,$$

$$f(r) = r^2 - 2Mr + a^2 + Q^2 - \frac{\Lambda}{3} r^2 (r^2 + a^2) - \alpha r^{1 - \frac{3\omega}{1 - 3\alpha(1 + \omega)}}$$

$$f(\theta) = 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta, \quad \Xi = 1 + \frac{\Lambda}{3} a^2,$$

where $M$, $Q$, $a$, $\alpha$ and $\lambda$ are the mass, charge, rotational parameter, perfect fluid parameter and Rastall parameter, respectively. Likewise the previous sections the outer and inner horizons are calculated by
\( f(r) = 0 \). With respect to entropy of black hole, the thermodynamic quantities of Kerr-Newman AdS black hole surrounded by perfect fluid such as mass, temperature and angular velocity are given by

\[
M = \frac{a^2 \sqrt{\pi} \sqrt{\pi} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{\sqrt{\pi}} + \frac{a^2 \sqrt{S}}{4\sqrt{\pi}} + \frac{S^\frac{3}{4}}{16l^2 \pi^\frac{7}{2}} - 2\pi - 2\frac{3(\kappa \lambda + (-1 + \kappa \lambda) \omega)}{\pi - 2 + 6 \kappa \lambda (1 + \omega)} \alpha, \tag{83}
\]

\[
T = -\frac{a^2 \sqrt{\pi}}{\sqrt{S}} - \frac{\sqrt{\pi} Q^2}{2S^\frac{3}{4}} + \frac{1}{8\sqrt{\pi} S} + \frac{a^2}{8l^2 \pi S} + \frac{3\sqrt{S}}{32l^2 \pi^\frac{7}{2}} + 3 \times 2^{\frac{1 - 3 \omega}{1 + 3 \kappa \lambda (1 + \omega)}} \times \pi - 2 + 6 \kappa \lambda (1 + \omega), \tag{84}
\]

\[
\Omega = \frac{2a \sqrt{\pi}}{\sqrt{S}} + \frac{a \sqrt{S}}{2l^2 \sqrt{\pi}}. \tag{86}
\]

Now, we consider a small constant correction \( \epsilon \), to Kerr-Newman AdS black hole surrounded perfect fluid action. So, the modified thermodynamic quantities as mass, temperature and angular velocity are given by

\[
M = \frac{a^2 \sqrt{\pi} \sqrt{\pi} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{\sqrt{\pi}} + \frac{a^2 \sqrt{S}(1 + \epsilon)}{4\sqrt{\pi}} + \frac{(1 + \epsilon) S^\frac{3}{4}}{16l^2 \pi^\frac{7}{2}} - 2\pi - 2\frac{3(\kappa \lambda + (-1 + \kappa \lambda) \omega)}{\pi - 2 + 6 \kappa \lambda (1 + \omega)} \alpha, \tag{87}
\]

\[
T = -\frac{a^2 \sqrt{\pi}}{\sqrt{S}} - \frac{\sqrt{\pi} Q^2}{2S^\frac{3}{4}} + \frac{1}{8\sqrt{\pi} S} + \frac{a^2(1 + \epsilon)}{8l^2 \pi S} + \frac{3\sqrt{S}(1 + \epsilon)}{32l^2 \pi^\frac{7}{2}} + 3 \times 2^{\frac{1 - 3 \omega}{1 + 3 \kappa \lambda (1 + \omega)}} \times \pi - 2 + 6 \kappa \lambda (1 + \omega), \tag{88}
\]

and

\[
\Omega = \frac{2a \sqrt{\pi}}{\sqrt{S}} + \frac{a \sqrt{S}(1 + \epsilon)}{2l^2 \sqrt{\pi}}. \tag{89}
\]

Here we note that thermodynamic relation lead us to consider \( \alpha \) as a conjugate to \( \eta = -2\pi - 2\frac{3(\kappa \lambda + (-1 + \kappa \lambda) \omega)}{\pi - 2 + 6 \kappa \lambda (1 + \omega)} \alpha \), charge conjugate to electric potential as \( \Phi = \frac{2a \sqrt{\pi} Q}{\sqrt{S}} \) and volume

\[
V = \frac{2}{3} a^2 \sqrt{\pi} + \frac{S^\frac{3}{4}}{6 \sqrt{\pi}}(1 + \epsilon) \text{ conjugate to } P = -\frac{\lambda}{8 \pi} \text{. We use such conjugate quantities and prove universal relations. On the other hand, the changes of mass and entropy in terms of small constant}.
\]
corrections $\epsilon$ can actually be a clue to the WGC. Therefore, by solving equation (87), the constant correction parameter $\epsilon$ is calculated by

$$\epsilon = -1 + \frac{1}{\frac{a^2 \sqrt{\pi}}{4l^2 \sqrt{\pi}} + \frac{s^2}{16l^2 \pi^2}} \left( \frac{M - a^2 \sqrt{\pi}}{\sqrt{S}} - \frac{\sqrt{\pi} Q^2}{\sqrt{S}} - \frac{\sqrt{S}}{4 \sqrt{\pi}} \right) + 2 \frac{1}{2 + 3 \kappa \lambda (1 + \omega)} \times \pi \frac{\kappa \lambda + (-1 + \kappa \lambda) \omega}{S} \frac{3(\kappa \lambda + (-1 + \kappa \lambda) \omega)}{2 + 3(\kappa \lambda + (-1 + \kappa \lambda) \omega)} \alpha \right).$$

(90)

Then, we take derivative with respect to $S$ and get

$$\frac{\partial \epsilon}{\partial S} = \frac{A + B}{C},$$

$$A = - \left( \frac{a^2}{8l^2 \sqrt{\pi} S} + \frac{3 \sqrt{S}}{32l^2 \pi^2} \right) (1 + \epsilon) + \frac{a^2 \sqrt{\pi}}{2 S} + \frac{\sqrt{\pi} Q^2}{2 S} - \frac{1}{8 \sqrt{\pi} S},$$

$$B = - 3 \times 2 \frac{1}{1 + 3 \kappa \lambda (1 + \omega)} \times \pi \frac{\kappa \lambda + (-1 + \kappa \lambda) \omega}{S} \frac{3(\kappa \lambda + (-1 + \kappa \lambda) \omega)}{2 + 3(\kappa \lambda + (-1 + \kappa \lambda) \omega)} \alpha(\kappa \lambda + (-1 + \kappa \lambda) \omega),$$

$$C = \frac{a^2 \sqrt{\pi}}{4l^2 \sqrt{\pi}} + \frac{s^2}{16l^2 \pi^2}. $$

(91)

Now by combining two equations (88) and (91), we have

$$-T \frac{\partial S}{\partial \epsilon} = \frac{\sqrt{S}(4a^2 \pi + S)}{16l^2 \pi^2}. $$

(92)

To get the second part of universal relation, we set $T = 0$. By solving the temperature, obtaining the entropy and using equations (87) and (88), we obtain following equation:

$$\frac{\partial M_{ext}}{\partial \epsilon} = \frac{\sqrt{S}(4a^2 \pi + S)}{16l^2 \pi^2}. $$

(93)

Here we see that two equations (92) and (93) are exactly same. So, we confirmed the Goon-Penco universal extremality relation.

To investigate another universal relation, we use the equation (90) to obtain following relation:

$$\frac{\partial \epsilon}{\partial Q} = \frac{-2 \sqrt{\pi} Q}{\sqrt{S}(\frac{a^2 \sqrt{\pi}}{4l^2 \sqrt{\pi}} + \frac{s^2}{16l^2 \pi^2})}. $$

(94)

From (94) and electric potential $\Phi$ as well as extremality bound, we have,

$$-\Phi \frac{\partial Q}{\partial \epsilon} = \frac{\sqrt{S}(4a^2 \pi + S)}{16l^2 \pi^2}. $$

(95)
Also here, we see that the equation (95) and (98) are same, so the universal relation is also proved. Therefore, with respect to pressure $P = \frac{3}{8\pi l^2} = -\frac{\Lambda}{8\pi}$ and equation (90), we have

$$\frac{\partial P}{\partial \epsilon} = -\frac{P^2(\frac{3}{2}a^2\sqrt{\pi S} + \frac{S^3}{6\sqrt{\pi}})}{16l^2\pi^{\frac{7}{2}}}.$$ (96)

The above equation and extremal bound lead us to obtain following equation:

$$-\sqrt{\epsilon} \frac{\partial P}{\partial \epsilon} = \frac{\sqrt{S}(4a^2\pi + S)}{16l^2\pi^{\frac{7}{2}}}.$$ (97)

Here also we see that two equations (97) and (93) are exactly same.

We also study the other universal relation, hence according to relation (90) and using the equation (89) one can obtain

$$-\Omega \frac{\partial a}{\partial \epsilon} = \frac{\sqrt{S}(4a^2\pi + S)}{16l^2\pi^{\frac{7}{2}}}.$$ (98)

We see, two equations (98) and (93) are exactly same. Now we will study the new universal relation with respect to $\eta$ which is conjugate to the perfect fluid parameter $\alpha$. So, according to equation (90), we have

$$\frac{\partial \epsilon}{\partial \alpha} = \frac{2}{1 + \frac{1}{m} \left(1 + \frac{1}{m}\right)} = \frac{\pi}{2 + \frac{1}{m} \left(1 + \frac{1}{m}\right)} = \frac{\pi}{2 + \frac{1}{m} \left(1 + \frac{1}{m}\right)}.$$(99)

Now by using the $\eta$ and the equation (99), we confirmed another universal relation which is calculated by,

$$-\eta \frac{\partial \alpha}{\partial \epsilon} = \frac{\sqrt{S}(4a^2\pi + S)}{16l^2\pi^{\frac{7}{2}}} = \frac{\partial M_{ext}}{\partial \epsilon}.$$ (100)

According to the mass charge plot of the modified thermodynamic relations for the Kerr Newman AdS black hole as shown in Fig. 2, we want to describe the concepts of WGC. We compare the mass charge ratio and thermodynamic quantities of modified and unmodified cases to each other. The charge and mass are equal in the extremal state or the mass charge ratio is unit as shown in the figure by dashed line. For the uncorrected mass mode, the mass-to-charge ratio is greater than one, we have plotted for different radius $l$ of AdS, as shown in the figure 2(a). Now we look at the modified state and use the
FIG. 2: The plot of unmodified $M$ in term of $Q$ with respect to $l = 0.1, 0.2, 0.3$ in (a) and the plot of modified $M$ in term of $Q$ with respect to $l = 0.1, \omega = -\frac{2}{3}$ and $\epsilon = -0.1, 0, 0.1$ in (b).

corrected thermodynamic parameters. We evaluate the amount of change in this correction as shown in figure 2(b). When the constant correction is negative, the mass of the black hole decreases, and when the small correction is positive, the mass increases. Therefore, when we consider small negative correction, the charge-to-mass ratio always increases. So, the negative $\epsilon$ correction to the action show us this black hole can behave like WGC. Thus, we confirm that the corrections play a very important role in the concept of WGC.

X. ROTATING BARDEEN BLACK HOLES IN ADS SPACE SURROUNDED BY PERFECT FLUID

In this section, we check universal relations for the rotating Bardeen black holes in AdS space surrounded by perfect fluid [77]. We note here that black holes have singularity [78] and in such case space-time, densities and curvatures tend to infinity [79, 80]. The physical predictions in these points face serious problems and these singularities always cause serious problems for general relativity. But Bardeen black hole solution includes a new description of black holes without singularity [81, 82]. Now, in order to investigate new universal relation here we introduce the magnetic charge $\vartheta$ to the
theory and see the effect of such parameter to universal relation. The rotating Bardeen black holes in AdS space surrounded by perfect fluid is described by

\[ ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2d\Omega^2, \] (101)

with

\[ f(r) = r^2 + a^2 - \frac{2Mr^4}{(r^2 + \varrho^2)^{\frac{1}{2}}} + \frac{r^2}{l^2} + \alpha r \ln \frac{r}{|\alpha|}, \] (102)

where \( M, a, \alpha \) and \( \varrho \) denote mass, rotational parameter, perfect fluid parameter and magnetic charge parameter, respectively. With respect to entropy of black hole, we investigate the thermodynamic relation of rotating Bardeen black holes in AdS space surrounded by perfect fluid. Here, we consider a small constant correction to the action and calculate the modified thermodynamic quantities as given by

\[ M = \frac{(S + \pi \varrho^2)^{\frac{3}{2}}}{2S^2 \sqrt{\pi}} ((a^2 \pi + S) + \frac{S(1 + \epsilon)}{l^2}) + \alpha \sqrt{S\pi} \log \left( \frac{\sqrt{S}}{\alpha \sqrt{\pi}} \right), \] (103)

\[ T = \frac{(S + \pi \varrho^2)^{\frac{3}{2}}}{2S^2 \sqrt{\pi}} \left( -1 - \frac{2a^2 \pi}{S} + \frac{\sqrt{\pi} \alpha}{2S} + \frac{3(a^2 \pi + S)}{2(S + \pi \varrho^2)} + \frac{(S - 2\pi \varrho^2)(1 + \epsilon)}{2l^2(S + \pi \varrho^2)} - \frac{3\varrho^2 \alpha \pi^{\frac{3}{2}} \log \left( \frac{\sqrt{S}}{\alpha \sqrt{\pi}} \right)}{2\sqrt{S}(S + \pi \varrho^2)} \right), \] (104)

and

\[ \Omega = \frac{a\sqrt{\pi}(S + \pi \varrho^2)^{\frac{3}{2}}}{S^2}. \] (105)

Due to thermodynamic relation, we consider \( \eta = \frac{(S + \pi \varrho^2)^{\frac{3}{2}}(-1 + \log \left( \frac{\sqrt{S}}{\alpha \sqrt{\pi}} \right))}{2S^2} \) conjugate to perfect fluid \( \alpha \), volume \( V = \frac{4\sqrt{\pi}(S + \pi \varrho^2)^{\frac{3}{2}}(-1 + \log \left( \frac{\sqrt{S}}{\alpha \sqrt{\pi}} \right))}{3S} \) conjugate to \( P = -\frac{\Lambda}{8\pi} \) and \( \zeta = \frac{3M\pi \varrho}{S + \pi \varrho^2} \) conjugate to \( \varrho \). To study the universal relation we solve the equation (103). So the constant correction parameter \( \epsilon \) is given by

\[ \epsilon = -1 + \frac{2l^2 MS \sqrt{\pi}}{(S + \pi \varrho^2)^{\frac{3}{2}}} - \frac{l^2 (a^2 \pi + S + \sqrt{S} \alpha \log \left( \frac{\sqrt{S}}{\alpha \sqrt{\pi}} \right))}{S}. \] (106)

The derivative of \( \epsilon \) with respect to \( S \) gives

\[ \frac{\partial \epsilon}{\partial S} = -\frac{l^2 M \sqrt{\pi}(S - 2\pi \varrho^2)}{(S + \pi \varrho^2)^{\frac{3}{2}}} + \frac{l^2 \sqrt{\pi}(2a^2 \sqrt{\pi} - \alpha \sqrt{S} + \alpha \sqrt{S} \log \left( \frac{\sqrt{S}}{\alpha \sqrt{\pi}} \right))}{2S^2}. \] (107)
We combine two equations (104) and (107), one can obtain

$$- T \frac{\partial S}{\partial \epsilon} = \frac{(S + \pi \partial^2)^{\frac{3}{2}}}{2l^2 S \sqrt{\pi}}. \quad (108)$$

By setting $T = 0$ and using equations (103) and (104), we get

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{(S + \pi \partial^2)^{\frac{3}{2}}}{2l^2 S \sqrt{\pi}}. \quad (109)$$

We see here two equations (108) and (109) are exactly same.

Now, we use the pressure $P = \frac{3}{8\pi l^2} = -\frac{\Lambda}{8\pi}$ and equation (106) to obtain

$$\frac{\partial \epsilon}{\partial P} = \frac{3 \left( \frac{1}{\pi} + \frac{a^2}{S} \right) - \frac{2MS}{\sqrt{\pi} \sqrt{(S + \pi \partial^2)}} + \frac{\alpha \log(\sqrt{S} / \sqrt{\pi})}{\sqrt{\pi}S}}{8P^2}. \quad (110)$$

According to volume and extremal bound, we have

$$- V \frac{\partial P}{\partial \epsilon} = \frac{(S + \pi \partial^2)^{\frac{3}{2}}}{2l^2 S \sqrt{\pi}}. \quad (111)$$

Here two equations (111) and (109) are exactly same.

Also with respect to equation (106), we get

$$\frac{\partial \epsilon}{\partial a} = -\frac{2al^2\pi}{S}. \quad (112)$$

So from (105) and (112), we obtain following relation:

$$- \Omega \frac{\partial a}{\partial \epsilon} = \frac{(S + \pi \partial^2)^{\frac{3}{2}}}{2l^2 S \sqrt{\pi}}. \quad (113)$$

We see here, two equations (111) and (113) are exactly same.

We use equation (106) for obtaining the another universal repletion. Here, we have

$$\frac{\partial \epsilon}{\partial \alpha} = \frac{l^2 \sqrt{\pi}}{\sqrt{S}} - \frac{l^2 \sqrt{\pi} \log(\sqrt{S} / \sqrt{\pi})}{\sqrt{S}}. \quad (114)$$

So by using the $\eta$ and (114), we obtain the other universal relation as given by

$$- \eta \frac{\partial \alpha}{\partial \epsilon} = \frac{(S + \pi \partial^2)^{\frac{3}{2}}}{2l^2 S \sqrt{\pi}}. \quad (115)$$
Also here, two equation (115) and (113) are extremely same.

Again we use relation (106) and obtain

\[
\frac{\partial \epsilon}{\partial \vartheta} = -\frac{6l^2 M \pi \frac{2}{S} \vartheta}{(S + \pi \vartheta^2)^{\frac{3}{2}}}. \tag{116}
\]

So, with respect the above equation and \( \zeta \), we obtain the new universal relation of rotating Bardeen black hole as given by

\[
-\zeta \frac{\partial \vartheta}{\partial \epsilon} = \frac{(S + \pi \vartheta^2)^{\frac{3}{2}}}{2l^2 S \sqrt{\pi}} = \frac{\partial M_{ext}}{\partial \epsilon}. \tag{117}
\]

The most important thing here is that, the universal relation related to magnetic charge is also confirmed.

We thoroughly investigated the universal relations for the three different black holes in the AdS space surrounded by perfect fluid, such as Kerr-Newman, rotating Bardeen and Reissner-Nordström. We calculated the universal relationships of each black hole separately. We also introduced new universal relations with respect to the concepts of perfect fluid and string fluid. We also observed that when a small correction constant is added to the action, the modified thermodynamic quantities and relations can be calculated. This constant correction can lead to a decrease of mass and increase the charge-to-mass ratio, which is a clue of WGC behaviour. The concepts in this paper can be evaluated for other black holes with different properties, as well as considering the higher dimensions and black holes in different structures.

**XI. CONCLUSIONS**

In the last few years, black holes have acquired new universal relations from multiple methods. These universal relations can be an excellent impetus for integrating different sciences and possibly a great solution to the path of quantum gravity for physicists. In this paper, we confirmed new universal relations for black holes thermodynamics. We investigated each of these universal relations by selecting different black holes such as AdS Schwarzschild, charged BTZ, charged rotating BTZ, accelerating and charged accelerating black holes together with AdS black hole surrounded by perfect fluid. First, we
obtained the modified thermodynamic relations of the black holes assuming a small correction to the action. We confirmed the universal relations by performing a series of direct calculations. It is noteworthy that according to each of the properties related to black holes, such as rotating, charged, accelerating, etc., a new universal relation can be obtained according to this method. So, we have confirmed two different types of these universal relations for various block holes. One of the most valuable results is that using the unique feature of black holes, a new universal relation between different black holes thermodynamics can be investigated.

For Kerr-Newman, rotating Bardeen and Reissner-Nordström black holes in the AdS space surrounded by perfect fluid we also introduced small constant correction to the action and computed modified thermodynamic quantities and relations. Then, by using a series of calculations, we obtained the universal relations of these black holes. We have investigated two new universal relation related to the parameter of perfect fluid and magnetic charge. Also, according to the constant correction and universal relations we have studied the effect of prefect fluid and magnetic parameters to the charge-to-mass ratio. Here, we have seen that the WGC condition is satisfied by the black hole system. And this also help us to obtain a simple way to obtain new universal relation. This work will be interesting to investigate the universal relation of black hole in higher dimension as well. Also it may interesting to do correction the Einstein-Gauss-Bonnet action and obtain the new modified thermodynamic relations. Given the relationship between the universal relation and the WGC, it may be interesting to obtain the relation between the correction parameter to the action and Gauss-Bonnet parameter.

[1] C. Vafa, arXiv: [hep-th/0509212]
[2] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, JHEP. 06, 060 (2007).
[3] M. Orellana, F. Garcia, F. T. Pannia and G. Romero, Gen. Rel. Grav 45, 771 (2013).
[4] S. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011).
[5] S. Capozziello, M. De Laurentis, S. D. Odintsov and A. Stabile, Phys. Rev. D 83, 064004 (2011).
[6] S. Capozziello, M. Faizal, M. Hameeda, B. Pourhassan, V. Salzano and S. Upadhyay, Mon. Not. Roy. Astron. Soc. 474, 2430 (2018).
[7] A. Arapoglu, C. Deliduman and K. Y. Eksi, JCAP 1107, 020 (2011).
[8] S. Capozziello, R. D’Agostino, O. Luongo, Int. J. Mod. Phys. D, 28, 1930016 (2019).
[9] S. Capozziello, R. D’Agostino, O. Luongo, JCAP 1805, 008 (2018).
[10] M. Khurshudyan, A. Pasqua, and B. Pourhassan, Can. J. Phys. 1107, 449 (2015).
[11] P. Channuie, Eur. Phys. J. C, 79, 508 (2019).
[12] S. Capozziello, R. D’Agostino, O. Luongo, Gen. Rel. Grav. 51, 2 (2019).
[13] J. Sadeghi, E. Naghd Mezerji and S. Noori Gashti, arxiv: 1910.11676.
[14] R. Myrzakulov, L. Sebastian and S. Vagnozzi, Eur. Phys. J.C 75, 444 (2015).
[15] J. Sadeghi, B. Pourhassan, A. S. Kubeka and M. Rostami, Int. J. Mod. Phys. D, 25, 1650077 (2015).
[16] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. 36, 1765 (2004).
[17] S. D. Odintsov, V. K. Oikonomou and L. Sebastiani, Nucl. Phys. B, 923, 608 (2017).
[18] Y. Kats and P. Petrov, JHEP 01, 044 (2009).
[19] C. Cheung and G. N. Remmen, Phys. Rev. Lett. 113, 051601 (2014).
[20] G. Shiu, P. Soler, and W. Cottrell, Sci. China Phys. Mech. Astron. 62, 110412 (2019).
[21] C. Cheung, J. Liu, and G. N. Remmen, JHEP 10, 004 (2018).
[22] S. Andriolo, D. Junghans, T. Noumi, and G. Shiu, Fortsch. Phys. 66, 1800020 (2018).
[23] Y. Hamada, T. Noumi, and G. Shiu, Phys. Rev. Lett. 123, 051601 (2019).
[24] H. S. Reall and J. E. Santos, JHEP 1904, 021 (2019).
[25] B. Bellazzini, M. Lewandowski, and J. Serra, Phys. Rev. Lett. 123, 251103 (2019).
[26] C. Cheung, J. Liu, and G. N. Remmen, Phys. Rev. D 100, 046003 (2019).
[27] L. Aalsma, A. Cole, and G. Shiu, JHEP 08, 022 (2019).
[28] A. M. Charles, Phys. Rev. Lett. 123, 051601 (2019).
[29] C. R. T. Jones and B. McPeak, JHEP 06, 140(2020).
[30] G. J. Loges, T. Noumi, and G. Shiu, Phys. Rev. D 102 4, 046010 (2020).
[31] G. J. Loges, T. Noumi, and G. Shiu, JHEP. 2001, 003 (2020).
[32] G. Goon and R. Penco, Phys. Rev. Lett. 124, 101103 (2020).
[33] S. Cremonini, C. R.T. Jones, J. T. Liu, and B. McPeak, JHEP 09, 003 (2020).
[34] P. A. Cano, T. Ortin, and P. F. Ramirez, JHEP 02, 175 (2020).
[35] P. A. Cano, S. Chimento, R. Linares, T. Ortin, and P. F. Ramirez, JHEP 2002, 031 (2020).
[36] M. David and J. Nian, arXiv:2009.12370.
[37] Q. Chen, W. Hong and J. Tao, arXiv:2005.00747.
[38] J. Sadeghi, S. Noori Gashti, and E. Naghd Mezerji. Phys. Dark Univ 30, 100626 (2020).
[39] Z.-Ming Xu, B. Wu and W.-Li Yang, Phys. Rev. D. 101, 024018 (2020).
[40] S. Soroushfar, R. Saffari, and A. Jafari, Phys. Rev. D 93, 104037 (2016).
[41] A. Ball and N. Miller, arXiv: 2008.03682.
[42] K. Jafarzade and B. Eslam Panah, arXiv: 1906.09478.
[43] M. Akbar and A. A. Siddiqui, Phys. Lett. B 656, 217 (2007).
[44] M. Akbar, H. Quevedo, K. Saifullah, A. Sanchez and S. Taj, Phys. Rev. D 83, 084031 (2011).
[45] J. Sadeghi, S. Noori Gashti, E. Naghd Mezerji and B. Pourhassan, arXiv: 2011.05109.
[46] J. Dubinski and R.G. Carlberg, ApJ 378, 496 (1991).
[47] J. F. Navarro, C. S. Frenk and S. D. M. White, ApJ 490, 493 (1997).
[48] J. F. Navarro, C. S. Frenk and S. D. M. White, ApJ 462, 563 (1996).
[49] M. Viel, J. Lesgourgues, M.G. Haehnelt, S. Matarrese and A. Riotto, Phys. Rev. D 71, 063534 (2005).
[50] L. Amendola and R. Barbieri, Phys. Lett. B 642, 192 (2006).
[51] J. Goodman, New Astronomy Reviews 5, 103 (2000).
[52] W. Hu, R. Barkana and A. Gruzinov, Phys. Rev. Lett. 85, 1158 (2000).
[53] L. Hui, J. P. Ostriker, S. Tremaine and E. Witten, Phys. Rev. D 95, 043541 (2017).
[54] D. J. E. Marsh, Phys. Rep. 643, 1 (2016).
[55] P. J. E. Peebles, ApJL 534, L127 (2000).
[56] W. H. Press, B. S. Ryden and D. N. Spergel, Phys. Rev. Lett. 64, 1084 (1990).
[57] H.-Y. Schive, T. Chiueh and T. Broadhurst, Nat. Phys. 10, 496 (2014).
[58] S.-J. Sin, Phys. Rev. D 50, 3650 (1994).
[59] M. S. Turner, Phys. Rev. D 28, 1243 (1983).
[60] F.S. Guzman, T. Matos, D. Nunez and E. Ramirez, Rev. Mex. Fis. 49, 203 (2003).
[61] V. V. Kiselev, arXiv: gr-qc/0303031
[62] F. Rahaman, P. K. F. Kuhfittig, K. Chakraborty, M. Kalam and D. Hossain, Int. J. Theor. Phys. 50, 2655 (2011).
[63] F. Rahaman, K. K. Nandi, A. Bhadra, M. Kalam and K. Chakraborty, Phys. Lett. B 694, 10 (2010).
[64] Z. Xu and J. Wang, Phys. Rev. D 95, 064015 (2017).
[65] A. Strominger and C. Vafa, Phys. Lett. B 379, 99 (1996).
[66] P. S. Letelier, Phys. Rev. D 20, 1294 (1979).
[67] M. G. Richarte and C. Simeone, Int. J. Mod. Phys. D 17, 1179 (2008).
[68] A. K. Yadav, V. K. Yadav and L. Yadav, Int. J. Theor. Phys. 48, 568 (2009).
[69] A. Ganguly, S. G. Ghosh and S. D. Maharaj, Phys. Rev. D 90, 064037 (2014).
28

[70] K. A. Bronnikov, S. W. Kim and M. V. Skvortsova, Class. Quant. Grav. 33, 195006 (2016).
[71] S. G. Ghosh and S. D. Maharaj, Phys. Rev. D 89, 084027 (2014).
[72] S. G. Ghosh, U. Papnoi and S. D. Maharaj, Phys. Rev. D 90, 044068 (2014).
[73] T. H. Lee, D. Baboolal and S. G. Ghosh, Eur. Phys. J. C 75, 297 (2015); M. Chabab and S Iraoui, Gen. Rel. Grav. 52, 875 (2020).
[74] X. Zhaoyi, H. Xian, W. Jiancheng and L. Yi, Adv. High Energy Phys. 2019, 2434390 (2019).
[75] Z. Xu, X. Hou, X. Gong and J. Wang, Eur. Phys. J. C78 513 (2018); M. Chabab and S. Iraoui, Gen. Rel. Grav. 52 (2020) 75.
[76] P. Rastall, Phys. Rev. D 6, 3357 (1972).
[77] P. Rastall, Can. J. Phys. 54, 66 (1976).
[78] Z. He-Xu, C. Yuan, H. Peng-Zhang, F. Qi-Qi and D. Jian-Bo, arXiv: 2007.09408.
[79] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time, volume 1. Cambridge University Press (1973).
[80] S. Hawking and R. Penrose, The Nature of Space and Time, Princeton University Press (2010).
[81] J. M. Bardeen, Int. Conf. GR5, Tbilisi 174, 22 (1968).
[82] E. A.-Beato and A. Garcia, Phys. Lett. B 493, 149 (2000).
[83] S. A. Hayward, Phys. Rev. Lett. 96, 031103 (2006).
[84] E. A.-Beato and A. Garcia, Phys. Rev. Lett. 80, 5056 (1998).
[85] W. Berej, J. Matyjasek, D. Tryniecki and M. Woronowicz, Gen. Rel. Grav. 38, 885 (2006).