Hirokazu NASU

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OBSTRUCTIONS TO DEFORMING CURVES ON A 3-FOLD, II: DEFORMATIONS OF DEGENERATE CURVES ON A DEL PEZZO 3-FOLD

by Hirokazu NASU (*)

Abstract. — We study the Hilbert scheme $\text{Hilb}^{sc}V$ of smooth connected curves on a smooth del Pezzo 3-fold $V$. We prove that any degenerate curve $C$, i.e. any curve $C$ contained in a smooth hyperplane section $S$ of $V$, does not deform to a non-degenerate curve if the following two conditions are satisfied: (i) $\chi(V, I_C(S)) \geq 1$ and (ii) for every line $\ell$ on $S$ such that $\ell \cap C = \emptyset$, the normal bundle $N_{\ell/V}$ is trivial (i.e. $N_{\ell/V} \cong \mathcal{O}_{\mathbb{P}^2}$). As a consequence, we prove an analogue (for $\text{Hilb}^{sc}V$) of a conjecture of J. O. Kleppe, which is concerned with non-reduced components of the Hilbert scheme $\text{Hilb}^{sc}\mathbb{P}^3$ of curves in the projective 3-space $\mathbb{P}^3$.

1. Introduction

This paper is a sequel to a joint work [13] with Shigeru Mukai. In [13] the embedded deformations of smooth curves $C$ on a smooth projective 3-fold $V$ have been studied under the presence of a smooth surface $S$ such that $C \subset S \subset V$, especially when $V$ is a uniruled 3-fold. In this paper, the same subject is studied in detail especially when $V$ is a del Pezzo 3-fold.

Keywords: Hilbert scheme, infinitesimal deformation, del Pezzo variety.
Math. classification: 14C05, 14H10, 14D15.

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It is known that even if the deformations of $C$ in $S$ and the deformations of $S$ in $V$ behave well, those of $C$ in $V$ behave badly in general. For example, even if $\text{Hilb} V$ and $\text{Hilb} S$ are nonsingular of expected dimension $\chi(N_{S/V})$ and $\chi(N_{C/S})$ at $[S]$ and $[C]$ respectively, $\text{Hilb} V$ can be generically non-reduced along some component passing through $[C]$ (cf. Mumford’s example in [14]). Such non-reduced components of the Hilbert scheme $\text{Hilb}^{sc} V$ of smooth connected curves on $V$ have been constructed for many uniruled 3-folds $V$ in [13]. The non-reducedness is originated from the non-surjectivity of the restriction map

$$H^0(S, N_{S/V}) \xrightarrow{\text{res}} H^0(C, N_{S/V}/C).$$

We say that $C$ is stably degenerate if every small global deformation of $C$ in $V$ is contained in a deformation $S'$ of $S$ in $V$ (cf. Definition 4.1). If (1.1) is surjective, then $C$ is stably degenerate (cf. Proposition 4.3). However if it is not surjective, then there exists a first order deformation $\tilde{C}$ of $C$ in $V$ which is not contained in any first order deformation $\tilde{S}$ of $S$. In this paper, we consider the following problem raised by Mukai:

**Problem 1.1.** — Suppose that (1.1) is not surjective and $\chi(V, I_C(S)) > 0$. Then (1) Is $C$ stably degenerate? (2) Is $\text{Hilb}^{sc} V$ singular at $[C]$?

Here $I_C$ denotes the ideal sheaf of $C$ in $V$ and $I_C(S) := I_C \otimes O_V(S)$.

J. O. Kleppe [8] and Ph. Ellia [2] considered Problem 1.1 for the case where $V$ is the projective 3-space $\mathbb{P}^3$, $S$ is a smooth cubic surface in $\mathbb{P}^3$ and $C$ is a smooth connected curve of degree $d$ lying on $S$. Kleppe gave a conjecture (cf. Conjectures 5.1), which can be reformulated as follows:

**Conjecture 1.2.** — Let $C \subset S \subset \mathbb{P}^3$ be as above and assume that $\chi(\mathbb{P}^3, I_C(3)) \geq 1$. Then:

1. If $C$ is linearly normal, then every small global deformation $C'$ of $C$ in $\mathbb{P}^3$ is contained in a cubic surface $S' \subset \mathbb{P}^3$, i.e., $C$ is stably degenerate, and

2. Suppose that $C$ is general and $d > 9$. Then $\text{Hilb}^{sc} \mathbb{P}^3$ is nonsingular at $[C]$ if and only if $H^1(\mathbb{P}^3, I_C(3)) = 0$.

As a testing ground of his conjecture, we consider Problem 1.1 for the case where $V$ is a smooth del Pezzo 3-fold (cf. §2.2), $S$ is a smooth member of the class $|H|$ of the polarization $H$ of $V$, i.e., a smooth del Pezzo surface in $V$, and $C$ is a smooth connected curve on $S$. The following theorem is an analogue of Kleppe’s conjecture.
Theorem 1.3.—Let $C \subset S \subset V$ be as above and assume that $\chi(V, I_C(S)) \geq 1$. If every line $\ell$ on $S$ such that $C \cap \ell = \emptyset$ is a good line on $V$ (i.e., the normal bundle $N_{\ell/V}$ of $\ell$ in $V$ is trivial), then:

1. $C$ is stably degenerate, and
2. $\text{Hilb}^{sc} V$ is nonsingular at $[C]$ if and only if $H^1(V, I_C(S)) = 0$.

If $\chi(V, I_C(S)) < 1$, then it follows from a dimension count that $C$ is not stably degenerate (Proposition 4.7). If some $\ell$ is a bad line on $V$ (i.e., $N_{\ell/V} \not\cong \mathcal{O}_{\mathbb{P}^1}(\mathbb{Z}^2)$) then $C$ is not necessarily stably degenerate (Proposition 5.4). As a corollary to Theorem 1.3, we give a sufficient condition for a maximal family $W$ of degenerate curves on $V$ to become an irreducible component of the Hilbert scheme $\text{Hilb}^{sc} V$ and determine whether $\text{Hilb}^{sc} V$ is generically non-reduced along $W$ or not (Theorem 4.14).

One of the main tools used in this paper is the infinitesimal analysis of the Hilbert scheme developed in [13]. As is well known, every infinitesimal deformation $\tilde{C}$ of $C$ in $V$ of the first order (i.e., over $\text{Spec } k[t]/(t^2)$) determines a global section $\alpha \in H^0(N_{C/V})$ and a cohomology class $\text{ob}(\alpha) \in H^1(N_{C/V})$ (called the obstruction) such that $\tilde{C}$ lifts to a deformation over $\text{Spec } k[t]/(t^3)$ if and only if $\text{ob}(\alpha) = 0$ (cf. §2.3). Let $\pi_{C/S} : N_{C/V} \to N_{S/V}|_C$ be the natural projection. In [13] Mukai and Nasu studied the exterior component of $\alpha$ and $\text{ob} (\alpha)$, i.e., the images of $\alpha$ and $\text{ob}(\alpha)$ by the induced maps $H^i(\pi_{C/S}) : H^i(N_{C/V}) \to H^i(N_{S/V}|_C)$ ($i = 0, 1$), respectively. They proved that if there exists a curve $E$ on $S$ such that $(E^2)_S < 0$ (e.g. $(-1)\mathbb{P}^1$ on $S$) and the exterior component of $\alpha$ lifts to a global section $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$, then the exterior component of $\text{ob}(\alpha)$ is nonzero provided that certain additional conditions on $E$, $C$ and $v$ hold (see [13, Theorem 2.2]). Such a rational section $v$ of $N_{S/V}$ admitting a pole along $E$ is called an infinitesimal deformation with a pole. In §3 we see that an infinitesimal deformation with a pole along $E$ induces an obstructed infinitesimal deformation of the open surface $S^o := S \setminus E$ in the open 3-fold $V^o := V \setminus E$ (Theorem 3.1). By using this fact, we prove Theorem 1.3 in §4. In §5 we give some examples of generically non-reduced components of the Hilbert scheme of curves on a del Pezzo 3-fold as an application.

Acknowledgements. I should like to express my sincere gratitude to Professor Shigeru Mukai. He showed me the example of non-reduced components of the Hilbert scheme of canonical curves in §5.2 as a simplification of Mumford’s example of a non-reduced component of $\text{Hilb}^{sc} \mathbb{P}^3$. This motivated me to research the topic of this paper. Throughout this research, he made many suggestions which are useful for obtaining and improving the
proofs. According to his suggestion, I studied the deformation theory of an open surface in an open 3-fold and organized §3. I am grateful to Professor Jan Oddvar Kleppe for giving me useful comments on Hilbert-flag schemes and for finding a gap in the proof of Lemma 4.8 in a earlier version. I should like to thank the referee, who showed me a straight proof of Proposition 4.3 and led me to a simplification of §4.1 and §4.2. According to his/her recommendation, I give the classes of non-reduced components of the Hilbert scheme explicitly in Proposition 5.5.

**Notation and Conventions.** We work over an algebraically closed field $k$ of characteristic 0. Let $V$ be a scheme over $k$ and let $X$ be a closed subscheme of $V$. Then $\mathcal{I}_X$ denotes the ideal sheaf of $X$ in $V$ and $N_{X/V}$ denotes the normal sheaf $(\mathcal{I}_X/\mathcal{I}_X^2)\vert_X$ of $X$ in $V$. For a sheaf $\mathcal{F}$ on $V$, we denote the restriction map $H^i(V, \mathcal{F}) \to H^i(X, \mathcal{F}\vert_X)$ by $\vert_X$. We denote the Euler-Poincaré characteristic of $\mathcal{F}$ by $\chi(V, \mathcal{F})$ or $\chi(\mathcal{F})$. $\text{Hilb}^{sc} V$ denotes the open subscheme of the Hilbert scheme $\text{Hilb} V$ whose point corresponds to a smooth connected curve on $V$.

2. Preliminaries

The results in this section will be used in § 4. Proposition 2.4 and Lemma 2.5 are important to our proof of Proposition 4.9 and 4.10, respectively.

2.1. Del Pezzo surfaces

A *del Pezzo surface* is a smooth surface $S$ whose anti-canonical divisor $-K_S$ is ample. Every del Pezzo surface is isomorphic to $\mathbb{P}^2$ blown up at fewer than 9 points or $\mathbb{P}^1 \times \mathbb{P}^1$. We denote the blow-up of $\mathbb{P}^2$ at $(9 - n)$-points by $S_n$. A curve $\ell \simeq \mathbb{P}^1$ on $S_n$ is called a *line* if $\ell \cdot (-K_S) = 1$. Every $(-1)\cdot \mathbb{P}^1$ on $S_n$ is a line and every line on $S_n$ is a $(-1)\cdot \mathbb{P}^1$. A curve $q$ on $S_n$ is called a *conic* if $q \cdot (-K_S) = 2$ and $q^2 = 0$.

**Lemma 2.1.** — Let $D$ be a divisor on a del Pezzo surface $S$. If $D$ is nef and $\chi(S, -D) \geq 0$, then $H^1(S, -D) = 0$.

\(\text{(1)}\) There exists no line on $\mathbb{P}^2$ and on $\mathbb{P}^1 \times \mathbb{P}^1$. 

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Proof. — If $D^2 > 0$ then the assertion follows the Kawamata-Viehweg vanishing. Since $D$ is a nef divisor on a del Pezzo surface, we have $D^2 \geq 0$. Now we assume that $D^2 = 0$. If $S = S_n$, then $D$ is linearly equivalent to a multiple $mq$ ($m \geq 0$) of a conic $q$ on $S$. By the Riemann-Roch theorem, we have

$$
\chi(S, -D) = \frac{1}{2}(-mq) \cdot (-mq - KS) + \chi(O_S)
= -m + 1.
$$

Thus we have $m = 0$ or $1$ by assumption. This implies that $H^1(-mq) = 0$. If $S = \mathbb{P}^1 \times \mathbb{P}^1$, then $D$ is of bidegree $(m, 0)$ or $(0, m)$ with $m \geq 0$. Again by the Riemann-Roch theorem, we have $\chi(-D) = -m + 1 \geq 0$. Thus $H^1(O_{\mathbb{P}^1 \times \mathbb{P}^1}(-D)) = 0$. \hfill \Box

Lemma 2.2. — Let $D$ be an effective divisor on a del Pezzo surface $S$. Then the lines $\ell$ such that $D \cdot \ell < 0$ are mutually disjoint. The fixed part \textsuperscript{(2)} of the linear system $|D|$ on $S$ is equal to

$$
- \sum_{D \cdot \ell < 0} (D \cdot \ell)\ell.
$$

Proof. — We prove the two assertions at the same time. It is clear that any line $\ell$ satisfying $D \cdot \ell < 0$ is contained in $\text{Bs} |D|$. On the other hand, except for lines on $S$ every irreducible curve $C$ on $S$ can move on $S$ by the linearly equivalence since $\chi(C) \geq 2$ and $H^2(C) = 0$. Hence $|D|$ is decomposed into the sum

$$
|D| = |D'| + \sum_{i=1}^{k} m_i \ell_i,
$$

of a linear system $|D'|$ on $S$ such that $\text{Bs} |D'| = \emptyset$ and some lines $\ell_i$ on $S$ with coefficients $m_i \in \mathbb{Z}_{>0}$ ($1 \leq i \leq k$). If $\ell_i \cap \ell_j \neq \emptyset$ for some $i \neq j$, then $\ell_i + \ell_j$ is a (reducible) conic on $S$ and can move on $S$ by $\chi(\ell_i + \ell_j) = 2$. Thus $\ell_i$’s are mutually disjoint. Now we prove that $D \cdot \ell_i < 0$ for any $i$. Since $m_i = (D' - D) \cdot \ell_i > 0$, it suffices to show that $D' \cdot \ell_i = 0$. Since $D'$ is nef, we have $(D')^2 \geq 0$. Since $-K_S$ is ample, so is $D' - K_S$. Hence we have $H^1(D') = H^1((D' - K_S) + K_S) = 0$ by the Kodaira vanishing. If $D' \cdot \ell_i \geq 1$, then it follows from the exact sequence

$$
0 \longrightarrow O_S(D') \longrightarrow O_S(D' + \ell_i) \longrightarrow O_S(D' + \ell_i)|_{\ell_i} \longrightarrow 0
$$

that $h^0(D' + \ell_i) > h^0(D')$. Thus we have $D' \cdot \ell_i = 0$. \hfill \Box

\textsuperscript{(2)} the base locus of dimension one.
Lemma 2.3. — Let $E$ be a disjoint union of $m$ lines $(m \geq 0)$ on a del Pezzo surface $S$ and let $\varepsilon : S \to F$ be the blow-down of $E$ from $S$. If a divisor $D$ on $F$ satisfies $h^0(F, D) \geq m$, then we have the following:

1. $h^0(S, \varepsilon^*D - E) = h^0(F, D) - m$, and
2. If $H^1(S, \varepsilon^*D) = 0$, then $H^1(S, \varepsilon^*D - E) = 0$.

Proof. — (1) Let $\ell_i$ $(1 \leq i \leq m)$ be the disjoint lines on $S$ and let $E := \sum_{i=1}^m \ell_i$. We put $D_j := \varepsilon^*D - \sum_{1 \leq i \leq j} \ell_i$. Since the image of $\ell_i$ on $F$ is a point, we have $h^0(D_j) \geq h^0(D) - j$ for every $1 \leq j \leq m$. Moreover since $D_{j-1} \cdot \ell_j = 0$, Lemma 2.2 shows that $\ell_j$ is not contained in $Bs|D_{j-1}|$. Hence $\dim |D_j|$ decreases one by one as $j$ increases. Therefore we have $h^0(\varepsilon^*D - E) = h^0(D_m) = h^0(D) - m$.

(2) An exact sequence $0 \to \mathcal{O}_S(\varepsilon^*D - E) \to \mathcal{O}_S(\varepsilon^*D) \to \mathcal{O}_E \to 0$ on $S$ induces an exact sequence

$$H^0(S, \varepsilon^*D) \xrightarrow{\rho} H^0(E, \mathcal{O}_E) \to H^1(S, \varepsilon^*D - E) \to H^1(S, \varepsilon^*D)$$

of cohomology groups. Then $\rho$ is surjective by (1) and $H^1(S, \varepsilon^*D) = 0$ by assumption. Hence we have $H^1(S, \varepsilon^*D - E) = 0$. \hfill $\square$

Let $C$ be a smooth connected curve on a del Pezzo surface $S$. We consider the restriction to $C$ of the anti-canonical linear system $|-K_S|$ on $S$. The restriction map $H^0(-K_S) \to H^0(-K_S|_C)$ is not surjective in general. Let $\ell_i$ $(1 \leq i \leq m)$ be the lines on $S$ disjoint to $C$. Let us define an effective divisor $E$ on $S$ by the sum

$$E := \sum_{i=1}^m \ell_i$$

and we put $E := 0$ if there exists no such $\ell_i$. If $C$ is neither a line nor a conic, then the $\ell_i$‘s are mutually disjoint: indeed if $\ell_i \cap \ell_j \neq \emptyset$ for some $i \neq j$, then $q := \ell_i + \ell_j$ is a conic on $S$ and hence $C$ intersects $q$ by $C \cdot q > 0$.

Proposition 2.4. — Assume that $C$ is irrational and $\chi(S, -K_S - C) \geq 0$. Then we have $H^1(S, -K_S + E - C) = 0$ and the restriction map

$$(2.1) \quad H^0(S, -K_S + E) \xrightarrow{|C|} H^0(C, -K_S|_C)$$

is surjective. If $C$ is not elliptic either, then the map (2.1) is an isomorphism.

Proof. — It suffices to show that $H^1(-K_S + E - C) = 0$ by the exact sequence

$$\begin{align*}
0 & \to \mathcal{O}_S(-K_S + E - C) \to \mathcal{O}_S(-K_S + E) \to \mathcal{O}_S(-K_S)|_C \to 0.
\end{align*}$$

Claim. Put $D_1 := C + K_S - E$. Then $D_1$ is nef.
Since $S$ is regular (i.e., $H^1(K_S) = 0$), the restriction map $|C| : H^0(C + K_S) \to H^0(K_C)$ is surjective. Since $C \not\cong \mathbb{P}^1$, the linear system $|C + K_S|$ on $S$ is non-empty. Let $l$ be a line on $S$. Since $C$ is not a line, we have $C \cdot l \geq 0$ and hence $(C + K_S) \cdot l \geq -1$. By Lemma 2.2, $l$ is contained in $Bs |C + K_S|$ if and only if $C \cap l = \emptyset$. Thus we have $E = Bs |C + K_S|$ and $|D_1|$ does not have base components. In particular, $D_1$ is nef.

It follows from the exact sequence
\begin{align*}
0 \longrightarrow \mathcal{O}_S(-K_S - C) \longrightarrow \mathcal{O}_S(-K_S + E - C) \longrightarrow \mathcal{O}_S(-K_S + E)|_E \longrightarrow 0
\end{align*}
that $\chi(-D_1) = \chi(-K_S - C) + \chi(\mathcal{O}_E) \geq 0$. Hence we have $H^1(-D_1) = 0$ by Lemma 2.1.

Now we assume that $C$ is not elliptic. Then $K_C \not\cong 0$ and hence $C + K_S \not\cong E$ by adjunction. Thus $D_1 \not\cong 0$ and $H^0(-D_1) = 0$. Therefore (2.1) is injective.

**Lemma 2.5.** — If $C$ is not rational nor elliptic and $\chi(S, -K_S - C) \geq 0$, then the map
\[ H^1(S, -K_S + 3E) \xrightarrow{|c|} H^1(C, -K_S|_C) \]
induced by (2.2) $\otimes \mathcal{O}_S(2E)$ is injective.

**Proof.** — It suffices to show that $H^1(-K_S + 3E - C) = 0$. Let $\varepsilon : S \to F$ be the blow-down of $E$ from $S$. Then there exists a divisor $D_2$ on $F$ such that $\varepsilon^*D_2 \sim C + 2K_S - 2E$. By the Serre duality, it suffices to show that
\[ H^1(\varepsilon^*D_2 - E) = 0. \]

**Claim.** $H^i(S, \varepsilon^*D_2) = 0$ for $i = 1, 2$.

By (2.3)$\otimes \mathcal{O}_S(E)$, there exists an exact sequence
\[ H^1(S, -K_S + E - C) \longrightarrow H^1(S, -K_S + 2E - C) \longrightarrow H^1(E, (-K_S + 2E)|_E). \]
Since $H^1((-K_S + 2E)|_E) \simeq H^1(\mathcal{O}_E(E)) = 0$ and $H^1(-K_S + E - C) = 0$ by Proposition 2.4, we have $H^1(-K_S + 2E - C) = 0$. By the Serre duality, we have $H^1(\varepsilon^*D_2) = 0$. Similarly by the Serre duality, we have $H^2(\varepsilon^*D_2) \simeq H^0(K_S - \varepsilon^*D_2)^\vee$. Since $C$ is not rational nor elliptic, we have $(K_S - \varepsilon^*D_2) \cdot C = (-K_S - C) \cdot C = -\deg K_C < 0$. Hence we have $H^2(\varepsilon^*D_2) = 0$ because $C$ is nef. Thus the claim has been proved.

By this claim, we have $h^0(F, D_2) = h^0(S, \varepsilon^*D_2) = \chi(S, \varepsilon^*D_2)$. Then an easy calculation shows that $\chi(\varepsilon^*D_2) = \chi(-K_S - C) + \chi(\mathcal{O}_E)$. Since $\chi(-K_S - C) \geq 0$, we have $h^0(F, D_2) = \chi(S, \varepsilon^*D_2) \geq m$, where $m$ is the number of components of $E$. Since $H^1(\varepsilon^*D_2) = 0$, Lemma 2.3 (2) shows that $H^1(\varepsilon^*D_2 - E) = 0$. □
Let $S$ be a smooth projective surface and let $L$ be a line bundle on $S$.

**Lemma 2.6.** — Let $E$ be a disjoint union of irreducible curves $E_i$ ($i = 1, \ldots, m$) on $S$ such that $E_i^2 < 0$ and let $\iota : S^o := S \setminus E \hookrightarrow S$ be the open immersion. If $\deg(L|_{E_i}) \leq 0$ for every $i$, then the map

$$H^1(S, L) \to H^1(S^o, L|_{S^o})$$

induced by the sheaf inclusion $L \hookrightarrow L \otimes \iota^*\mathcal{O}_{S^o}$ is injective.

The proof is similar to that of [13, Lemma 2.5] and we omit it here. Lemma 2.6 allows us to identify $H^1(S, L(nE))$ ($n \geq 0$) with their images in $H^1(S^o, L|_{S^o})$. As a result, under the identification we obtain a natural filtration

$$H^1(S, L) \subset H^1(S, L(E)) \subset H^1(S, L(2E)) \subset \cdots \subset H^1(S^o, L|_{S^o})$$
on $H^1(S^o, L|_{S^o})$.

### 2.2. Del Pezzo threefolds

A del Pezzo threefold is a pair $(V, H)$ consisting of a (smooth) irreducible projective variety $V$ of dimension 3 and an ample Cartier divisor $H$ on $V$ such that $-K_V = 2H$. Here $H$ is called the polarization of $V$ and sometimes omitted. The self-intersection number $n := H^3$ is called the degree of $V$.

It is known that the linear system $|H|$ on $V$ determines a double cover $\varphi|_H : V \to \mathbb{P}^3$ if $n = 2$, and an embedding $\varphi|_H : V \hookrightarrow \mathbb{P}^{n+1}$ if $n \geq 3$. If $S$ is a smooth member of $|H|$, then the pair $(S, H|_S)$ is a del Pezzo surface of degree $n$. Every smooth del Pezzo 3-fold is one of $V_n$ ($1 \leq n \leq 8$) or $V'_6$ in Table 2.1, in which $\mathbb{L}^{(i)}$ denotes a linear subspace of dimension $i$, and $n$ and $\rho$ respectively denote the degree and the Picard number of $V_n$ (and of $V'_6$) (cf. [4],[5],[6]). It is known that a smooth 3-fold $V \subset \mathbb{P}^{n+1}$ ($n \geq 3$) is a del Pezzo 3-fold of degree $n$ if a linear section $[V \subset \mathbb{P}^{n+1}] \cap H_1 \cap H_2$ with two general hyperplanes $H_1, H_2 \subset \mathbb{P}^{n+1}$ is an elliptic normal curve in $\mathbb{P}^{n-1}$.

We briefly review the basics of the Hilbert scheme of lines on a del Pezzo 3-fold. We refer to Iskovskih ([6],[7]) for the details. Let $(V, H)$ be a smooth del Pezzo 3-fold of degree $n$. By a line on $(V, H)$, we mean a reduced irreducible curve $\ell$ on $V$ such that $(\ell \cdot H)_V = 1$ and $\ell \simeq \mathbb{P}^1$. If $n \leq 7$ then $V$ contains a line $\ell$. Then there are only the following possibilities for the
**Table 2.1. Del Pezzo 3-folds**

| Del Pezzo 3-folds                                      | n | ρ  | Remarks                                      |
|--------------------------------------------------------|---|-----|----------------------------------------------|
| $V_1 = (6) \subset \mathbb{P}(3, 2, 1, 1, 1)$          | 1 | 1  | a weighted hypersurface of degree 6          |
| $V_2 = (4) \subset \mathbb{P}(2, 1, 1, 1, 1)$          | 2 | 1  | a weighted hypersurface of degree 4 (a)      |
| $V_3 = (3) \subset \mathbb{P}^3$                      | 3 | 1  | a cubic hypersurface                         |
| $V_4 = (2) \cap (2) \subset \mathbb{P}^9$             | 4 | 1  | a complete intersection of two quadrics      |
| $V_5 = (\text{Gr}(2, 5) \hookrightarrow \mathbb{P}^9) \cap L(6)$ | 5 | 1  | a linear section of Grassmannian            |
| $V_6 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$ | 6 | 3  |                                            |
| $V_6' = [\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{\text{Segre}} \mathbb{P}^8] \cap L(7)$ | 6 | 2  |                                            |
| $V_7 = \text{Bl}_{\mathbb{P}^1} \mathbb{P}^3 \subset \mathbb{P}^7$ | 7 | 2  | the blow-up of $\mathbb{P}^3$ at a point (5) |
| $V_8 = \mathbb{P}^3 \hookrightarrow \mathbb{P}^9$     | 8 | 1  | the Veronese image of $\mathbb{P}^3$        |

(a) Another realization of $V_2$ is a double cover of $\mathbb{P}^3$ branched along a quartic surface.
(b) $V_7$ is realized as the projection of $V_8 \subset \mathbb{P}^9$ from one of its points.

In this paper, $\ell$ is called a **good line** if $N_{\ell/V}$ is trivial, and called a **bad line** otherwise. If $n \geq 3$, then every line on $V$ is of type (0, 0) or (1, −1). The Hilbert scheme $\Gamma$ of lines on $V$ is called the **Fano surface** of $V$, and in fact every irreducible (non-embedded) component of $\Gamma$ is of dimension two. Let $\Gamma_i \subset \Gamma$ be an irreducible component and let $S_i$ be the universal family of lines on $V$ over $\Gamma_i$. Then there exists a natural diagram

$$
\begin{array}{c}
S_i \\
\pi \downarrow \ \\
\Gamma_i,
\end{array}
\xrightarrow{p}
\begin{array}{c}
V
\end{array}
$$

By [7, Chap. III, Proposition 1.3 (iv)], if $n \geq 3$ then we have either

(a) $p$ is surjective; in this case a general line in $\Gamma_i$ is a good line; or
(b) $p(S_i) \simeq \mathbb{P}^2$ is a plane on $V \subset \mathbb{P}^{n+1}$; in this case every line in $\Gamma_i$ is a bad line.

We have either (a) or (b) also when $n \leq 2$. (See the proof (3) in [7], which works for $n \leq 2$.) If $n \neq 7$ then every irreducible component of $\Gamma$ is of type (a). If $n = 7$ then $\Gamma$ consists of two irreducible components $\Gamma_i \simeq \mathbb{P}^2(i = 0, 1)$, one of which is of type (a), while the other is of type (b). Consequently, we have

(3) In the proof, the assumption that $\text{char } k = 0$ is used.
**Lemma 2.7** (Iskovskih). — Every smooth del Pezzo 3-fold of degree \( n \neq 8 \) contains a good line.

**Lemma 2.8.** — Let \((V, H)\) be a smooth del Pezzo 3-fold of degree \( n \) and let \( S \) be a general member of \(|H|\). If \( n \neq 7 \) then every line on \( S \) is good. If \( n = 7 \) then there exist three lines \( \ell_0, \ell_1, \ell_2 \) on \( S \) forming the configuration in Figure 2.1. Then \( \ell_0 \) is bad, while \( \ell_1 \) and \( \ell_2 \) are good.

![Figure 2.1. \((-1)\mathbb{P}^1\)'s on \( S_7 \)](image)

**Proof.** — There exists no line on \( V_8 \). If \( n \neq 7 \), then the locus \( \mathcal{B} \) of bad lines in the Fano surface \( \Gamma \) is of dimension one. Let \( p_i \) denote the projection of 
\[
\{(\ell, S) \mid \ell \subset S\} \subset \Gamma \times |H|
\]
to the \( i \)-th factor. Since the fiber of \( p_1 \) is of dimension \( n - 1 \), \( p_2(p_1^{-1}(\mathcal{B})) \) is a proper closed subset of \(|H| \simeq \mathbb{P}^{n+1}\). Hence every line on a general member \( S \) of \(|H|\) is a good line.

Suppose that \( V = V_7 \), i.e., the blow-up of \( \mathbb{P}^3 \) at a point. Then \( S \) is a del Pezzo surface \( S_7 \), i.e., a blow-up of \( \mathbb{P}^2 \) at two distinct points, and hence there exist three lines \( \ell_0, \ell_1, \ell_2 \) on \( S \) as in Figure 2.1. Here \( \ell_0 \) is distinguished by the fact that it intersects both of the other lines. Let \( P \) be the exceptional divisor of the blow-up \( V_7 \to \mathbb{P}^3 \). Then \( P \simeq \mathbb{P}^2 \) is a unique plane on \( V_7 \) and \( \ell_0 \) is the intersection of \( S \) with \( P \) (cf. [7, Chap. II, §1.4]). Since \( N_{\ell_0/P} \simeq O_{\mathbb{P}^2}(1) \), \( \ell_0 \) is a bad line on \( V_7 \). On the other hand, \( \ell_1 \) and \( \ell_2 \) are good lines on \( V_7 \) since \( S \) is general. \( \square \)

### 2.3. Infinitesimal deformations and obstructions

Let \( V \) be a smooth variety and let \( X \) be a smooth closed subvariety of \( V \). An \textit{(embedded) first order infinitesimal deformation} of \( X \) in \( V \) is a closed subscheme \( \tilde{X} \subset V \times \text{Spec} \ k[t]/(t^2) \) which is flat over \( \text{Spec} \ k[t]/(t^2) \) and whose central fiber is \( X \). It is well known that there exists a one to one correspondence between the group of homomorphisms \( \alpha : \mathcal{I}_X \to \mathcal{O}_X \) and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure21.png}
\caption{\((-1)\mathbb{P}^1\)'s on \( S_7 \)}
\end{figure}
the first order infinitesimal deformations $\tilde{X}$ of $X$ in $V$. In what follows, we identify $\tilde{X}$ with $\alpha$ and abuse the notation. The standard exact sequence

\begin{equation}
0 \to \mathcal{I}_X \to \mathcal{O}_V \to \mathcal{O}_X \to 0
\end{equation}

induces $\delta: \text{Hom}(\mathcal{I}_X, \mathcal{O}_X) \to \text{Ext}^1(\mathcal{I}_X, \mathcal{I}_X)$ as a coboundary map. Then $\alpha \in \text{Hom}(\mathcal{I}_X, \mathcal{O}_X)$ (i.e., $\tilde{X}$) lifts to a deformation over Spec $k[t]/(t^3)$ if and only if

$$\text{ob}(\alpha) := \delta(\alpha) \cup \alpha \in \text{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$$

is zero, where $\cup$ is the cup product map

$$\text{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \times \text{Hom}(\mathcal{I}_X, \mathcal{O}_X) \xrightarrow{\cup} \text{Ext}^1(\mathcal{I}_X, \mathcal{O}_X).$$

(We refer to [11, Chap. I §2]. See also [15], [1], [3] and [10].) Then $\text{ob}(\alpha)$ is called the obstruction of $\alpha$ (i.e., $\tilde{X}$). Since both $X$ and $V$ are smooth, $\text{ob}(\alpha)$ is contained in $H^1(X, N_{X/V}) \subset \text{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$ (cf. [11, Chap. I, Prop. 2.14]). Since $\text{Hom}(\mathcal{I}_X, \mathcal{O}_X) \simeq H^0(N_{X/V})$, we regard $\alpha$ as a global section of $N_{X/V}$ from now on.

If $X$ is a hypersurface of $V$, i.e., of codimension one in $V$, then $\text{ob}(\alpha)$ becomes a simple cup product. Let $\delta_1: H^0(X, N_{X/V}) \to H^1(V, \mathcal{O}_V)$ be the coboundary map of the exact sequence $0 \to \mathcal{O}_V \to \mathcal{O}_V(X) \to N_{X/V} \to 0$. Let us define a map\(^{(4)}\)

\begin{equation}
d_X : H^0(X, N_{X/V}) \to H^1(X, \mathcal{O}_X)
\end{equation}

by the composition of $\delta_1$ and the restriction map $H^1(\mathcal{O}_V) \xrightarrow{\mid_X} H^1(\mathcal{O}_X)$. Then we have

**Lemma 2.9.** — *Let $X$ be a smooth hypersurface of $V$. Then $\text{ob}(\alpha)$ for $\alpha \in H^0(N_{X/V})$ is equal to the cup product $d_X(\alpha) \cup \alpha$, where $\cup$ is the cup product map

$$H^1(X, \mathcal{O}_X) \times H^0(X, N_{X/V}) \xrightarrow{\cup} H^1(X, N_{X/V}).$$

Proof. — Since $\mathcal{I}_X \simeq \mathcal{O}_V(-X)$ is a line bundle on $V$, we have $\text{Ext}^i(\mathcal{I}_X, \mathcal{O}_X) \simeq H^i(N_{X/V})$ ($i = 0, 1$) and $\text{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \simeq H^1(\mathcal{O}_V)$. Hence the coboundary map $\delta$ appearing in the definition of $\text{ob}(\alpha)$ is nothing but the coboundary map $\delta_1$ of (2.4)$\otimes \mathcal{O}_V(X)$. Since $\alpha$ is a cohomology class on $X$, the cup product map $H^1(\mathcal{O}_V) \to H^1(N_{X/V})$ with $\alpha$ factors through the restriction map $\mid_X$. □

\(^{(4)}\)The map $d_X$ is equal to the map $d_{X, \mathcal{O}_V(X)}$ defined in [13, §2.1 (2.3)].
We recall the definition of exterior component introduced in [13]. Let $X$ be a smooth closed subvariety of $V$ and let $Y$ be a smooth hypersurface of $V$ containing $X$. Then the natural projection $\pi_{X/Y} : N_{X/V} \to N_{Y/V} \big|_X \simeq O_X(Y)$ of normal bundles induces the maps $H^i(\pi_{X/Y}) : H^i(N_{X/V}) \to H^i(N_{Y/V} \big|_X)$, where $i = 0, 1$, of their cohomology groups. Let $\alpha$ be a global section of $N_{X/V}$.

**Definition 2.10.** — $\pi_{X/Y}(\alpha)$ and ob$_Y(\alpha)$ denote the images of $\alpha$ and ob$(\alpha)$ by the maps $H^0(\pi_{X/Y})$ and $H^1(\pi_{X/Y})$, respectively. They are called the exterior components of $\alpha$ and ob$(\alpha)$, respectively.

Roughly speaking, $\pi_{X/Y}(\alpha)$ is the projection of the normal vector $\alpha$ of $X$ in $V$ onto the normal directions to $Y$ in $V$. Then ob$_Y(\alpha)$ represents the obstruction to deforming $X$ into this direction. We recall a basic fact on exterior components.

**Lemma 2.11 ([13, Lemma 2.4]).** — Let $\pi_{X/Y}(\alpha)$ and ob$_Y(\alpha)$ be the exterior components of $\alpha$ and ob$(\alpha)$, respectively. If there exists a global section $v$ of $N_{Y/V}$ whose restriction $v \big|_X$ to $X$ coincides with $\pi_{X/Y}(\alpha)$, then we have

$$\text{ob}_Y(\alpha) = \text{ob}(v) \big|_X$$

where $\text{ob}(v) \big|_X \in H^1(X, N_{Y/V} \big|_X)$ is the restriction of $\text{ob}(v) \in H^1(Y, N_{Y/V})$ to $X$.

Lemma 2.11 together with Lemma 2.9 shows that ob$_Y(\alpha) = d_Y(v) \big|_X \cup \pi_{X/Y}(\alpha)$, where $d_Y$ is the map (2.5) for $Y$ and $\cup$ is the cup product map

$$H^1(X, O_X) \times H^0(X, N_{Y/V} \big|_X) \xrightarrow{\cup} H^1(X, N_{Y/V} \big|_X).$$

Let $E$ be an effective divisor of $Y$ disjoint to $X$ (i.e., $X \cap E = \emptyset$). Let $Y^\circ$ and $V^\circ$ denote the two complements of $E$ in $Y$ and $V$, respectively. Every rational section $v$ of $N_{Y/V} \simeq O_Y(Y)$ having poles only along $E$ determines a global section $v^\circ$ of the normal bundle $N_{Y^\circ/V^\circ}$ of $Y^\circ$ in $V^\circ$ and hence the obstruction ob$(v^\circ) \in H^1(N_{Y^\circ/V^\circ})$ to deforming $Y^\circ$ in $V^\circ$. Let $i$ denote the open immersion of $Y^\circ \hookrightarrow Y$. Then a natural homomorphism $i* N_{Y^\circ/V^\circ} \to N_{Y/V} \big|_X (= [i* O_{Y^\circ} \to O_X] \otimes N_{Y/V})$ of sheaves on $Y$ induces a map $H^1(N_{Y^\circ/V^\circ}) \big|_X \to H^1(N_{Y/V} \big|_X)$. Since ob$(\alpha)$ is (and hence ob$_Y(\alpha)$ is) determined by a neighborhood of $X$, we have the following variant of Lemma 2.11.

**Lemma 2.12.** — Let $\alpha$ be a global section of $N_{X/V}$. If there exists a rational section $v$ of $N_{Y/V}$ whose only poles are along $E$ and whose restriction
to $X$ coincides with $\pi_{X/Y}(\alpha)$, then we have

$$\text{ob}_Y(\alpha) = \text{ob}(v^\circ)|_X,$$

where $\text{ob}(v^\circ)|_X$ is the image of $\text{ob}(v^\circ)$ by the map $H^1(Y^\circ, N_{Y/V^\circ}) \xrightarrow{|_X} H^1(X, N_{Y/V}|_X)$.

### 3. Infinitesimal deformations with a pole

Let $V$ be a smooth projective 3-fold, $S$ a smooth connected curve on $S$. We put $V^\circ := V \setminus E$ and $S^\circ := S \setminus E$, the complementary open subvarieties. In this section, we study the first order infinitesimal deformations of $S^\circ$ in $V^\circ$, when the self-intersection number of $E$ on $S$ is negative. We are interested in a rational section $v$ of $N_{S/V}$ having a pole only along $E$ and of order one, that is, $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$. Let $\iota : S^\circ \hookrightarrow S$ be the open immersion. Then $\iota_*\mathcal{O}_{S^\circ}$ contains $\mathcal{O}_S(nE)$ as a subsheaf for any $n \geq 0$. Hence the natural sheaf injection $N_{S/V}(nE) \hookrightarrow \iota_*N_{S^\circ/V^\circ}$ induces $H^0(S, N_{S/V}(nE)) \hookrightarrow H^0(S^\circ, N_{S^\circ/V^\circ})$ for each $n$. Therefore $v$ determines a first order infinitesimal deformation of $S^\circ$ in $V^\circ$. The main theorem of this section is the following.

**Theorem 3.1.** — Let $v$ be as above and assume that $E^2 < 0$ and $\det N_{E/V} := \Lambda^2 N_{E/V}$ is trivial. If the exact sequence

$$0 \to N_{E/S} \to N_{E/V} \to N_{S/V}|_E \to 0 \tag{3.1}$$

does not split, then the first order infinitesimal deformation of $S^\circ \subset V^\circ$ determined by $v$ does not lift to a deformation over $\text{Spec } k[t]/(t^3)$.

Let $n$ be a non-negative integer. In what follows, we identify $H^0(N_{S/V}(nE))$ with its image in $H^0(N_{S^\circ/V^\circ})$. We shall prove that the obstruction $\text{ob}(v)$ is nonzero in $H^1(N_{S^\circ/V^\circ})$. Let $d_{S^\circ}$ denote the map (2.5) for $X = S^\circ$. Then by Lemma 2.9, $\text{ob}(v)$ is equal to the cup product $d_{S^\circ}(v) \cup v$, where $\cup$ is the cup product map

$$H^1(S^\circ, \mathcal{O}_{S^\circ}) \times H^0(S^\circ, N_{S^\circ/V^\circ}) \xrightarrow{\cup} H^1(S^\circ, N_{S^\circ/V^\circ}).$$

The inclusion $\mathcal{O}_S(nE) \hookrightarrow \iota_*\mathcal{O}_{S^\circ}$ of sheaves induces a map $H^1(S, \mathcal{O}_S(nE)) \to H^1(S^\circ, \mathcal{O}_{S^\circ})$ of cohomology groups. Suppose that $E^2 < 0$. Then this map is injective by Lemma 2.6. Hence we identify $H^1(\mathcal{O}_S(nE))$ with its image in $H^1(S^\circ, \mathcal{O}_{S^\circ})$. Under this identification, there exists a natural filtration

$$H^1(S, \mathcal{O}_S) \subset H^1(S, \mathcal{O}_S(E)) \subset H^1(S, \mathcal{O}_S(2E)) \subset \cdots \subset H^1(S^\circ, \mathcal{O}_{S^\circ})$$
on $H^1(S^\circ, \mathcal{O}_{S^\circ})$. Suppose now that $\det N_{E/V}$ is trivial. Then under similar identifications, there exists a natural filtration

$$H^1(S, N_{S/V}(E)) \subset H^1(S, N_{S/V}(2E)) \subset \cdots \subset H^1(S^\circ, N_{S^\circ/V^\circ})$$
on $H^1(S^\circ, N_{S^\circ/V^\circ})$, because we have $\deg N_{S/V}(nE)\big|_E = \deg(\det N_{E/V}) + (n-1)E^2 = (n-1)E^2 \leq 0$ for $n \geq 1$. Then it follows from [13, Proposition 2.6 (1)] that the image of $d_{S^\circ}$ over $H^0(N_{S/V}(E))$ is contained in $H^1(\mathcal{O}_S(2E))$. By the commutative diagram

$$H^1(\mathcal{O}_{S^\circ}) \times H^0(N_{S^\circ/V^\circ}) \cup \cdots \cup H^1(N_{S^\circ/V^\circ})$$

the image of the obstruction map $ob$ over $H^0(N_{S/V}(E))$ is contained in $H^1(N_{S/V}(3E))$. The following lemma is essential to the proof of Theorem 3.1. Let $d_S$ denote the restriction of the map $d_{S^\circ}$ to $H^0(S, N_{S/V}(E))$.

**Lemma 3.2** ([13, Proposition 2.6 (2)]). — Let $\partial : H^0(N_{S/V}(E))\big|_E \to H^1(\mathcal{O}_E(2E)) \simeq H^1(N_{E/S}(E))$ be the coboundary map of the exact sequence $(3.1) \otimes \mathcal{O}_S(E)$. Then the diagram

$$\begin{array}{ccc}
H^0(S, N_{S/V}(E)) & \xrightarrow{d_S} & H^1(S, \mathcal{O}_S(2E)) \\
\big|_E & \downarrow & \big|_E \\
H^0(E, N_{S/V}(E))\big|_E & \xrightarrow{\partial} & H^1(E, \mathcal{O}_E(2E))
\end{array}$$

is commutative.

**Proof of Theorem 3.1.** It suffices to show that the image $\operatorname{ob}(v)\big|_E \in H^1(N_{S/V}(3E))\big|_E$ of $\operatorname{ob}(v) \in H^1(N_{S/V}(3E))$ is nonzero. By the definition of $v$, we have $v_{\big|_E} \neq 0$ in $H^0(N_{S/V}(E))\big|_E$. Then the line bundle $N_{S/V}(E)\big|_E \simeq \det N_{E/V}$ on $E$ is trivial. Since $(3.1)$ does not split by assumption, we have $\partial(v_{\big|_E}) \neq 0$. Hence by Lemma 3.2, we conclude that $\operatorname{ob}(v)\big|_E = d_S^\circ (v)\big|_E \cup v_{\big|_E} = \partial(v_{\big|_E}) \cup v_{\big|_E} \neq 0$. \hfill \Box

If $E$ is a $(-1)^{-\mathbb{P}^1}$ on $S$ with $\det N_{E/V} \simeq \mathcal{O}_{\mathbb{P}^1}$, then the exact sequence $(3.1)$ does not split if and only if $N_{E/V}$ is trivial.

**Example 3.3.** — Let $V_n$ be a smooth del Pezzo 3-fold of degree $n \neq 8$ and let $E$ be a good line on $V_n$, i.e., $N_{E/V_n}$ is trivial (cf. §2.2). If $S_n$ is a smooth hyperplane section of $V_n$ containing $E$, then there exists an obstructed infinitesimal deformation of $S_n^\circ := S_n \setminus E$ in $V_n^\circ := V_n \setminus E$. Indeed, let $\varepsilon : S_n \to S_{n+1}$ be the blow-down of $E$ from $S_n$. Since $N_{S_n/V_n} \simeq -K_{S_n}$, $N_{S_n/V_n}(E) \simeq \varepsilon^*(-K_{S_{n+1}})$, and $h^0(-K_{S_{n+1}}) > h^0(-K_{S_n})$, there exists a global section $v \in H^0(N_{S_n/V_n}(E)) \setminus H^0(N_{S_n/V_n})$. Then by Theorem 3.1,
the first order infinitesimal deformation of \( S_n^o \) in \( V_n^o \) determined by \( v \) is obstructed.

In the rest of this section, we discuss a generalization of Theorem 3.1, which will be used in the proof of Theorem 1.3. Let \( E \) be a disjoint union of smooth connected curves \( E_i \) \( (i = 1, \ldots, m) \) on \( S \) such that \( E_i^2 < 0 \) and \( \det N_{E_i/V} \) is trivial. By the same symbol \( E \) we also denote the divisor \( \sum_{i=1}^m E_i \) on \( S \). Let us define \( V^o \) and \( S^o \) as above and identify \( H^0(N_{S^o/V}(E)) \) with its image in \( H^0(N_{S^o/V^o}(E)) \). We compute the restriction to \( H^0(N_{S/V}(E)) \) of the obstruction map \( \text{ob} : H^0(N_{S^o/V}(E)) \to H^1(N_{S^o/V^o}(E)) \). Lemma 2.6 allows us to regard \( H^1(\mathcal{O}_{S}(2E)) \) and \( H^1(\mathcal{O}_{S}(3E)) \) as subgroups of \( H^1(\mathcal{O}_{S^o}) \) and \( H^1(N_{S^o/V^o}(E)) \), respectively. Then an argument similar to [13, Proposition 2.6 (1)] shows that the image of \( H^0(N_{S^o/V}(E)) \) by \( d_{S^o} \) is contained in \( H^1(\mathcal{O}_{S}(2E)) \). Therefore we conclude that

**Lemma 3.4.** — The image of \( H^0(N_{S/V}(E)) \) by \( \text{ob} \) is contained in \( H^1(N_{S/V}(3E)) \subset H^1(N_{S^o/V^o}(E)) \).

Let \( v \) and \( v' \) be any global sections of \( N_{S/V}(E) \) and \( N_{S/V} \), respectively. Then we have \( \text{ob}(v + v')|_E = \text{ob}(v)|_E \) in \( H^1(N_{S/V}(3E)|_E) \). Indeed it follows from the definition of \( d_{S^o} \) (cf. (2.5)) that \( d_{S^o}(v') \) is contained in \( H^1(\mathcal{O}_{S}) \) and hence

\[
\text{ob}(v + v') = (d_{S^o}(v) + d_{S^o}(v')) \cup (v + v') \\
= \text{ob}(v) + d_{S^o}(v) \cup v' + d_{S^o}(v') \cup v + d_{S^o}(v') \cup v'.
\]

Therefore the obstruction map \( \text{ob} \) induces a map

\[
(3.2) \quad \tilde{\text{ob}} : H^0(N_{S/V}(E))/H^0(N_{S/V}) \to H^1(N_{S/V}(3E)|_E).
\]

**Proposition 3.5.** — If \( H^1(N_{S/V}) = 0 \) and the exact sequence

\[
(3.3) \quad 0 \to N_{E_i/S} \to N_{E_i/V} \to N_{S/V}|_{E_i} \to 0
\]

does not split for every \( i \), then \( \tilde{\text{ob}} \) is injective.

This is an immediate consequence of the next lemma.

**Lemma 3.6.** — Under the assumption of Proposition 3.5, \( \tilde{\text{ob}} \) is equivalent to the quadratic map

\[
k^m \to k^n, \quad (a_1, \ldots, a_m) \mapsto (a_1^2, \ldots, a_m^2, 0, \ldots, 0)
\]

of diagonal type, where \( n = \dim H^1(N_{S/V}(3E)|_E) \).
Proof. — Since $H^1(N_{S/V}) = 0$, the source of the map $\overline{\alpha}$ is isomorphic to $H^0(N_{S/V}(E))$. Moreover there exist global sections $v_i$ of $N_{S/V}(E_i)$ such that $v_i|_E \neq 0$ in $H^0(N_{S/V}(E_i)|_{E_i})$ for all $i$. Since $E_i$’s are mutually disjoint, we have $N_{S/V}(E)|_E \cong \bigoplus_{i=1}^{m} N_{S/V}(E_i)|_{E_i} \cong \bigoplus_{i=1}^{m} O_{E_i}$. Then there exists a commutative diagram

$$0 \to H^0(N_{S/V}) \to H^0(N_{S/V}(E)) \to H^0(N_{S/V}(E)|_E) \to 0$$

$$0 \to \bigoplus_i H^0(N_{S/V}) \to \bigoplus_i H^0(N_{S/V}(E_i)) \to \bigoplus_i H^0(N_{S/V}(E_i)|_{E_i}) \to 0,$$

where the two horizontal sequences are exact and $a_i$ for $1 \leq i \leq 3$ are defined by addition. Since $a_1$ and $a_3$ are surjective, so is $a_2$. Hence every element $v \in H^0(N_{S/V}(E))$ is written as a $k$-linear combination $\sum_{i=1}^{m} c_i v_i$ of $v_i \in H^0(N_{S/V}(E_i))$ and the expression is unique modulo $H^0(N_{S/V})$. By the commutative diagram

$$H^1(O_S(2E)) \times H^0(N_{S/V}(E)) \xrightarrow{\cup} H^1(N_{S/V}(3E))$$

we have

$$\text{ob}(v)|_E = (d_{S^o}(v) \cup v)|_E = d_{S^o}(v)|_E \cup v|_E = \sum_i c_i^2 d_{S^o}(v_i)|_{E_i} \cup v_i|_{E_i}.$$ 

By Lemma 3.2, $d_{S^o}(v_i)|_{E_i}$ is equal to $\partial_i(v|_{E_i})$ in $H^1(O_{E_i}(2E_i))$, where $\partial_i$ is the coboundary map of (3.3). Since (3.3) does not split by assumption, we have $\partial_i(v|_{E_i}) \neq 0$ and hence $d_{S^o}(v_i)|_{E_i} \neq 0$ for any $i$. As a result, $d_{S^o}(v_i)|_{E_i} \cup v_i|_{E_i}$ ($1 \leq i \leq m$) form a sub-basis of $H^1(N_{S/V}(3E)|_E)$. \hfill \square

Suppose now that $E_i$ is a $(-1)$-$\mathbb{P}^1$ on $S$ and $N_{E_i/V}$ is trivial for every $1 \leq i \leq m$. Then Proposition 3.5 shows that

**Corollary 3.7.** — Let $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$ be a global section. If $H^1(N_{S/V}) = 0$, then we have $\text{ob}(v) \neq 0$ in $H^1(N_{S/V}(3E)) \subset H^1(N_{S^o/V^o})$.

### 4. Obstructions to deforming curves

Let $V$ be a smooth projective 3-fold. In this section we study the deformation of smooth curves $C$ on $V$ under the presence of smooth surface $S$ such that $C \subset S \subset V$. 

**ANNALES DE L’INSTITUT FOURIER**
4.1. \( S \)-normal curves and \( S \)-maximal families

In what follows, we assume that \( \text{Hilb} V \) is nonsingular at \([S]\). Then there exists a unique irreducible component \( U_S \) of \( \text{Hilb} V \) passing through \([S]\). We use the following convention.

**Definition 4.1.**

1. \( C \) is said to be stably degenerate if there exists an (Zariski) open neighborhood \( U \subset \text{Hilb} V \) of \([C]\) such that for any member \([C']\) \( \in U \), there exists a deformation \( S' \) of \( S \) in \( V \) such that \( C' \subset S' \) and \([S']\) \( \in U_S \).

2. \( C \) is said to be \( S \)-normal if the restriction map (1.1) is surjective.

Let

\[
V \times U_S \supset S \xrightarrow{p_2} U_S
\]

be the universal family of \( U_S \). Let us denote the Hilbert scheme of smooth connected curves in \( S \) by \( \text{Hilb}^{sc} S \), which is the relative Hilbert scheme of \( S/U_S \). \( \text{Hilb}^{sc} S \) is regarded as an open subscheme of the Hilbert-flag scheme of \( V \) (see [8] for the definition), which parametrizes all flat families of pairs \((C, S)\) of a curve \( C \) and a surface \( S \) in \( V \) such that \( C \subset S \). The projection \( p_1 : S \rightarrow V \) induces a natural morphism

\[
(4.1) \quad p_{1} : \text{Hilb}^{sc} S \longrightarrow \text{Hilb}^{sc} V,
\]

which is the forgetful morphism \((C, S) \rightarrow C\). Then by definition \( C \) is stably degenerate if and only if \( p_{1} \) is surjective in a neighborhood of \([C] \in \text{Hilb}^{sc} V\).

The next lemma plays an important role in our proof of Theorem 1.3 later (cf. § 4.3).

**Lemma 4.2.** — Assume that \( H^1(C, N_{C/S}) = 0 \). Then:

1. The kernel and the cokernel of the tangential map

\[
(4.2) \quad \kappa_{C,S} : H^0(C, N_{C/S}) \longrightarrow H^0(C, N_{C/V}).
\]

of \( p_{1} \) at \((C, S)\) are isomorphic to those of the restriction map (1.1), respectively.

2. \( \text{Hilb}^{sc} S \) is nonsingular at \((C, S)\).

For the proof we refer to [13, Lemma 3.1] for (1) and [9, Lemma 1.10] for (2). We can also prove (1) by using the “fundamental exact sequence relating \( A^i(C \subset S) \) and \( H^{i-1}(N_{C/V}) \)” in [9].

In what follows, we assume that \( H^1(C, N_{C/S}) = 0 \). If \( C \) is \( S \)-normal, then \( \kappa_{C,S} \) is surjective by Lemma 4.2 (1). Then by (2) of the same lemma,
Hilb^{ac} V is nonsingular at [C] and furthermore pr_1 is surjective in a neighborhood of [C]. In fact, if C is S-normal, then the morphism pr_1 is smooth at (C, S) (cf. [8, Lemma A10]). Thus we conclude that

**Proposition 4.3** (cf. [8],[9]). — If C is S-normal, then C is stably degenerate and Hilb^{ac} V is nonsingular at [C].

We recall the S-maximal family introduced in [13, §3.2]. By the smoothness of Hilb^{ac} S, there exists a unique irreducible component W_{S,C} of Hilb^{ac} S containing (C, S).

**Definition 4.4.** — We define the S-maximal family of curves containing C to be the image of W_{S,C} in Hilb^{ac} V and denote it by W_{S,C}.

By Proposition 4.3, if C is S-normal then W_{S,C} is an irreducible component of Hilb^{ac} V and Hilb^{ac} V is generically smooth along W_{S,C}.

### 4.2. Deformation of curves on a del Pezzo 3-fold

Let V be a smooth del Pezzo 3-fold with the polarization H, S a smooth member of |H|, and C a smooth connected curve on S. Let n denote the degree of V and let d and g denote the degree (:= (C · H)_V) and the genus of C, respectively.

By adjunction we have \( N_{S/V} \simeq -K_V|_S + K_S \) and \( N_{C/S} \simeq -K_S|_C + K_C \). Since \(-K_V\) and \(-K_S\) are ample, we have \( H^1(N_{S/V}) = H^1(N_{C/S}) = 0 \). Hence Hilb V and Hilb S are nonsingular at [S] and [C], respectively. Thus if C is S-normal, then by Proposition 4.3, C is stably degenerate and Hilb^{ac} V is nonsingular at [C]. Because \( H^1(N_{S/V}) = 0 \), it follows from the exact sequence

\[
0 \rightarrow N_{S/V}(-C) \rightarrow N_{S/V} \rightarrow N_{S/V}|_C \rightarrow 0
\]

that C is S-normal if and only if \( H^1(N_{S/V}(-C)) = 0 \). There exists a natural exact sequence

\[
0 \rightarrow N_{C/S} \rightarrow N_{C/V} \xrightarrow{\pi_{C/S}} N_{S/V}|_C \rightarrow 0.
\]

Since \( H^1(N_{C/S}) = 0 \), we have \( H^1(N_{C/V}) \simeq H^1(N_{S/V}|_C) \). Thus every obstruction to deforming C is contained in the cohomology group \( H^1(N_{S/V}|_C) \).

Since \( \chi(N_{C/V}) = (-K_V \cdot C)_V = 2d \), we also have

**Lemma 4.5.** — If \( H^1(N_{S/V}|_C) = 0 \), then Hilb^{ac} V is nonsingular of expected dimension 2d at [C].
In particular, if $C$ is rational ($g = 0$) or elliptic ($g = 1$), then the $\text{Hilb}^c V$ is nonsingular at $[C]$ because $H^1(N_{S/V}|_C) \simeq H^1(-K_S|_C) = 0$.

Let $W_{S,C}$ be the $S$-maximal family $W_{S,C}$ of curves containing $C$. We compute the dimension of $W_{S,C}$. Let $\text{pr}_1 : \text{Hilb}^c S \to \text{Hilb}^c V$ be the morphism (4.1).

**Lemma 4.6.**

1. $\text{Hilb}^c S$ is nonsingular of dimension $d + g + n$ at $(C, S)$.
2. If $g \geq 2$ or $d \geq n + 1$, then $\text{pr}_1$ is a closed embedding in a neighborhood of $(C, S)$ and $\dim W_{S,C} = d + g + n$.

**Proof.** — (1) Let $W_{S,C}$ be the irreducible component of $\text{Hilb}^c S$ containing $(C, S)$. By the Riemann-Roch theorem on $S$, we have $\dim |O_S(C)| = d + g - 1$. Then $W_{S,C}$ is birationally equivalent to $\mathbb{P}^{d+g-1}$ over an open subset of $|H| \simeq \mathbb{P}^{n+1}$. Hence we have $\dim W_{S,C} = d + g + n$.

(2) By assumption, we have $(-K_S - C) \cdot C = 2 - 2g < 0$ or $(-K_S - C) \cdot (-K_S) = n - d < 0$. Since both $C$ and $-K_S$ are nef, we have $H^0(N_{S/V}(-C)) \simeq H^0(-K_S - C) = 0$. By Lemma 4.2 (1), $\text{pr}_1$ is a closed embedding near $(C, S)$. Hence we have $\dim W_{S,C} = \dim W_{S,C}$. □

We denote by $\text{Hilb}_{d,g}^c V$ the open and closed subscheme of $\text{Hilb}^c V$ of curves of degree $d$ and genus $g$. It is known that the dimension of every irreducible component of $\text{Hilb}_{d,g}^c V$ is greater than or equal to the expected dimension $\chi(N_{C/V}) = 2d$ (cf. [11, Chap. I, Theorem 2.8]).

**Proposition 4.7.** — If $\chi(V, \mathcal{I}_C(S)) < 1$, then $C$ is not stably degenerate, i.e., there exists a deformation $C'$ of $C$ in $V$ which is not contained in any deformation $S'$ of $S$ in $V$.

**Proof.** — There exists an exact sequence $[0 \to \mathcal{I}_C \to O_V \to O_C \to 0] \otimes O_V(S)$ on $V$. We see that $\chi(O_C(S)) = d + 1 - g$ and $\chi(O_V(S)) = n + 2$. Hence $\chi(\mathcal{I}_C(S)) < 1$ is equivalent to $g < d - n$. Then we have $\dim W_{S,C} \leq \dim W_{S,C} = d + g + n < 2d$. Hence there exists an irreducible component $W' \supset W_{S,C}$ of $\text{Hilb}^c V$ such that $\dim W' > \dim W_{S,C}$. A general member $C'$ of $W' \setminus W_{S,C}$ is such a deformation of $C$ in $V$. □

### 4.3. Stably degenerate curves

We devote this subsection to the proof of Theorem 1.3. Notation is the same as in the previous subsection. The following are equivalent: (i) $\chi(V, \mathcal{I}_C(S)) \geq 1$, (ii) $\chi(S, N_{S/V}(-C)) \geq 0$ and (iii) $g \geq d - n$. Indeed
we have already seen in the proof of Proposition 4.7 that (i) and (iii) are equivalent. Also (i) and (ii) are equivalent because we have \( \chi(N_{S/V}(-C)) = \chi(T_C(S)) - 1 \) by the exact sequence

\[
(4.5) \quad 0 \to T_S \to T_C \to O_S(-C) \to 0 \otimes O_V(S).
\]

Throughout this subsection, we assume one of (i), (ii) and (iii) (and hence all).

**Lemma 4.8.** — If \( H^1(C, N_{S/V}|_C) = 0 \) then \( C \) is \( S \)-normal.

**Proof.** — It suffices to show that \( H^1(N_{S/V}(-C)) = 0 \). Since \( H^2(N_{S/V}) \simeq \chi(N_{S/V}(-C)) - 1 \) by (4.3), we obtain \( H^1(N_{S/V}(-C)) = 0 \) by (4.3). Then by assumption, we have \( \chi(N_{S/V}(-C)) = h^0(N_{S/V}(-C)) - h^1(N_{S/V}(-C)) \). Therefore if \( H^0(N_{S/V}(-C)) = 0 \), then we have \( H^1(N_{S/V}(-C)) = 0 \). Suppose that \( H^0(N_{S/V}(-C)) \neq 0 \). There exists an effective divisor \( D \) on \( S \) such that \( N_{S/V}(-C) \simeq O_S(D) \). If \( D = 0 \), then \( H^1(N_{S/V}(-C)) = 0 \). Suppose that \( D \neq 0 \). Let \( h \) be a general member of \( |-K_S| \). Then \( h \) is a smooth elliptic curve on \( S \). Since \( -K_S \) is ample, we have \( \deg O_S(D)|_h = D \cdot (-K_S) > 0 \) and hence \( H^1(O_S(D)|_h) = 0 \). Since \( C \) is connected, we obtain \( H^1(D - h) \simeq H^1(-C) = 0 \) from the exact sequence \( 0 \to O_S(-C) \to O_S \to O_C \to 0 \). Therefore it follows from the exact sequence

\[
0 \to O_S(D - h) \to O_S(D) \to O_S(D)|_h \to 0
\]

that \( H^1(N_{S/V}(-C)) \simeq H^1(D) = 0 \). \(\square\)

Let \( E_1, \ldots, E_m \) be lines on \( S \) disjoint to \( C \). We define an effective divisor \( E \) on \( S \) by \( E := \sum_{i=1}^m E_i \). If \( C \) is not \( S \)-normal, then \( E \) is responsible for the abnormality.

**Proposition 4.9.** — Suppose that \( C \) is not rational nor elliptic.

1. The restriction map \( H^0(S, N_{S/V}(E)) \xrightarrow{|_C} H^0(C, N_{S/V}|_C) \) is an isomorphism.
2. \( C \) is \( S \)-normal if and only if there exists no line \( \ell \) such that \( C \cap \ell = \emptyset \) (i.e., \( E = 0 \)).

**Proof.** — (1) Since \( N_{S/V} \simeq -K_S \), we have the assertion by Proposition 2.4.

(2) The “if” part follows from (1). We prove the “only if” part. Suppose that there exist such lines on \( S \). Let \( \varepsilon : S \to F \) be the blow-down of \( E \) from \( S \). Then \( F \) is also a del Pezzo surface and \( \varepsilon^*(-K_F) = -K_S + E \). Since \( \deg F > \deg S \), we have \( h^0(-K_F) > h^0(-K_S) \). Hence it follows from \( N_{S/V} \simeq -K_S \) that \( N_{S/V}(E) \) has more global sections than \( N_{S/V} \). Hence
we have $h^0(N_{S/V}|_C) = h^0(N_{S/V}(E)) > h^0(N_{S/V})$ by (1). Therefore $C$ is not $S$-normal. 

Let $\kappa_{C,S} : H^0(N_{C/S}) \rightarrow H^0(N_{C/V})$ denote the tangential map (4.2).

**Proposition 4.10.** — Suppose that $C$ is not $S$-normal. If every $E_i$ is a good line on $V$, then the obstruction $\text{ob}(\alpha)$ is nonzero for any $\alpha \in H^0(C, N_{C/V}) \setminus \text{im} \kappa_{C,S}$.

**Proof.** — Let $\pi_{C/S}(\alpha) \in H^0(N_{S/V}|_C)$ and $\text{obs}(\alpha) \in H^1(N_{S/V}|_C)$ be the exterior component of $\alpha$ and $\text{ob}(\alpha)$, respectively (cf Definition 2.10). We compute $\text{obs}(\alpha)$ instead of $\text{ob}(\alpha)$ itself. Since $C$ is not $S$-normal, by Lemma 4.8, we have $H^1(N_{S/V}|_C) \neq 0$. In particular, $C$ is not rational nor elliptic. By Proposition 4.9 (1), there exists a global section $v$ of $N_{S/V}(E)$ whose restriction $v\big|_C \in H^0(N_{S/V}|_C)$ to $C$ coincides with $\pi_{C/S}(\alpha)$. Since $\alpha$ is not contained in the image of $\kappa_{C,S}$, $\pi_{C/S}(\alpha)$ is not contained in the image of (1.1) by Lemma 4.2 (1). Hence $v$ is not a global section of $N_{S/V}$, in other words, an infinitesimal deformation with a pole (cf. §3).

Let $S^o$ and $V^o$ respectively denote the two complements $S\setminus E$ and $V\setminus E$ of $E$. There exists a natural injection $H^0(S, N_{S/V}(E)) \hookrightarrow H^0(S^o, N_{S^o/V^o})$ of cohomology groups. In what follows, we identify $v \in H^0(S, N_{S/V}(E))$ with its image $v^o \in H^0(S^o, N_{S^o/V^o})$, i.e., a first order infinitesimal deformation of $S^o$ in $V^o$. Now we prove that the obstruction $\text{ob}(v) \in H^1(S^o, N_{S^o/V^o})$ is nonzero. By Lemma 3.4, $\text{ob}(v)$ is contained in the subgroup $H^1(S, N_{S/V}(3E))$ of $H^1(S^o, N_{S^o/V^o})$. Every component $E_i$ of $E$ is a $(-1)$-$\mathbb{P}^1$ on $S$ and its normal bundle $N_{E_i/V}$ in $V$ is trivial by assumption. Therefore by virtue of Corollary 3.7, we have $\text{ob}(v) \neq 0$ in $H^1(S, N_{S/V}(3E))$.

Finally we show that $\text{obs}(\alpha) \neq 0$ in $H^1(C, N_{S/V}|_C)$. There exists an exact sequence

$$0 \rightarrow N_{S/V}(3E - C) \rightarrow N_{S/V}(3E) \xrightarrow{|C} N_{S/V}|_C \rightarrow 0.$$ 

Since $N_{S/V} \simeq -K_S$, the restriction map $H^1(S, N_{S/V}(3E)) \rightarrow H^1(C, N_{S/V}|_C)$ is injective by Lemma 2.5. Therefore we have $\text{obs}(\alpha) = \text{ob}(v)|_C \neq 0$ by Lemma 2.12.

Now we prove Theorem 1.3. Let $C$ be as in the theorem. Then we have

**Lemma 4.11.** — Every small global deformation of $C$ in $V$ is contained in the $S$-maximal family $W_{S,C}$ of curves containing $C$.

**Proof.** — Let $C_T \subset V \times T$ be a small global deformation of $C$, i.e., a flat family $C_T$ over a small open variety $T$, having a point $0 \in T$ with $C_0 = C$. Given an element of the Zariski tangent space of $T$ at $0$, we obtain
a morphism \( \text{Spec} \, k[t]/(t^2) \to T \) and a first order infinitesimal deformation \( \tilde{C} \to \text{Spec} \, k[t]/(t^2) \) of \( C \) by base extension. Then by Proposition 4.10 \( \tilde{C} \) is contained in the image of the map \( \kappa_{(C, S)} \). Hence there exists a first order infinitesimal deformation \( \tilde{S} \) of \( S \) such that \( \tilde{S} \supset \tilde{C} \). Since \( \text{Hilb}^{ac} S \) is nonsingular at \( (C, S) \), the first order infinitesimal deformation \( (\tilde{C}, \tilde{S}) \) of \( (C, S) \) lifts to a global deformation \( (C_T, S_T) \) over \( T \).

Therefore \( C \) is stably degenerate. The rest of the proof is as follows. If \( C \) is \( S \)-normal, then \( \text{Hilb}^{ac} V \) is nonsingular at \( [C] \) by Proposition 4.3. Otherwise, there exists a first order infinitesimal deformation \( \tilde{C} \) of \( C \) not contained in the image of \( \kappa_{(C, S)} \). Then \( \text{Hilb}^{ac} V \) is singular at \( [C] \) by Proposition 4.10. We have an isomorphism \( H^1(S, N_{S/V}(C)) \cong H^1(V, \mathcal{I}_C(S)) \) by the exact sequence (4.5) together with that \( H^i(V, \mathcal{I}_S(S)) = H^i(V, \mathcal{O}_V) = 0 \) for \( i = 1, 2 \). Hence \( C \) is \( S \)-normal if and only if \( H^1(V, \mathcal{I}_C(S)) = 0 \). The proof of Theorem 1.3 has been completed.

**Remark 4.12.** — We give two remarks on Theorem 1.3.

1. Suppose that \( V \) is not isomorphic to a blow-up \( V_7 \) of \( \mathbb{P}^3 \) at a point.
   If \( S \in |H| \) is general, then by Lemma 2.8, every line on \( S \) is a good line on \( V \). Hence every curve \( C \) on \( S \) is stably degenerate by the theorem. Meanwhile there exists a non-stably degenerate curve \( C \) on \( V_7 \) which is contained in a general member \( S \) of \( |H| \) (cf. Proposition 5.4).

2. There exists no line on a del Pezzo 3-fold \( V_8 \simeq \mathbb{P}^3 \). Hence if \( V = V_8 \), then the assumption of the theorem concerning lines \( \ell \) on \( S \) such that \( C \cap \ell = \emptyset \) is empty. In fact, if \( g \geq d - 8 \) then every curve \( C \) on \( V_8 \) is \( S \)-normal and hence stably degenerate. This coincides with the previous result [15, Appendix, Proposition 4.11], which proved that every curve of degree \( e \) and genus \( p \geq 2e - 8 \) in \( \mathbb{P}^3 \) lying on a smooth quadric surface \( Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) is stably degenerate.

The following proposition is more practical than Proposition 4.10 in showing that \( \text{Hilb}^{ac} V \) is singular at \( [C] \).

**Proposition 4.13.** — Suppose that \( C \) is not rational nor elliptic. If there exists a good line \( \ell \) on \( V \) such that \( \ell \subset S \) and \( C \cap \ell = \emptyset \), then \( \text{Hilb}^{ac} V \) is singular at \( [C] \).

The proofs of Proposition 4.10 and Proposition 4.13 are very similar. Take a global section \( v \in H^0(N_{S/V}(\ell)) \setminus H^0(N_{S/V}) \) and put \( \alpha \in H^0(N_{C/V}) \) as a lift of \( v \big|_C \in H^0(N_{S/V}) \) by the surjective map \( \pi_{C/S} : H^0(N_{C/V}) \to H^0(N_{S/V} \big|_C) \). Then it is enough to show that \( \text{ob}_S(\alpha) \neq 0 \) in \( H^1(N_{S/V} \big|_C) \).
by reducing it to \( \text{ob}(v) |_\ell \neq 0 \) as in the proof of Proposition 4.10. We omit the details.

The following is an analogue of Conjecture 5.1 due to Kleppe and Ellia.

**Theorem 4.14.** — Let \( C \) be the curve in Theorem 1.3. Then:

1. The \( S \)-maximal family \( W_{S,C} \subset \text{Hilb}^{sc} V \) of curves containing \( C \) is an irreducible component of \( (\text{Hilb}^{sc} V)_{\text{red}} \).
2. \( \text{Hilb}^{sc} V \) is generically smooth along \( W_{S,C} \) if \( H^1(V, I_C(S)) = 0 \), and generically non-reduced along \( W_{S,C} \) otherwise.

**Proof.** — (1) By definition \( W_{S,C} \) is an irreducible closed subset of \( \text{Hilb}^{sc} V \). By Lemma 4.11, \( W_{S,C} \) is maximal among all such subsets.

(2) Let \( C' \) be a general member of \( W_{S,C} \). Then \( C' \) is contained in a smooth surface \( S' \sim S \) in \( V \). Since \( C' \) is general, so is \( S' \) in \( |S| \). Suppose that \( H^1(I_C(S)) = 0 \). Then since \((C', S') \) is a generalization of \((C, S) \), we have \( H^1(I_C(S')) = H^1(I_C(S)) = 0 \) by the upper semicontinuity. Hence \( \text{Hilb}^{sc} V \) is nonsingular at \([C']\) and hence generically smooth along \( W_{S,C} \).

Suppose that \( H^1(I_C(S)) \neq 0 \), i.e., \( C \) is not \( S \)-normal. Then Lemma 4.8 shows that \( H^1(N_{S/V} |_{C'}) \neq 0 \) and hence \( g \geq 2 \). By Proposition 4.9 (2), there exists a line \( \ell \) on \( S \) such that \( C \cap \ell = \emptyset \). Since \( H^1(\mathcal{O}_S) = 0 \), the Picard group of \( S \) does not change under the smooth deformation of \( S \) and hence \( \text{Pic} S \simeq \text{Pic} S' \). Since \( H^1(\mathcal{O}_S(\ell)) = 0 \), the line \( \ell \) is deformed to a line \( \ell' \) on \( S' \). Then we have \( C' \cap \ell' = \emptyset \). Moreover since \( \ell \) is a good line, so is \( \ell' \).

Hence \( \text{Hilb}^{sc} V \) is singular at \([C']\) by Proposition 4.13. Since \( C' \) is a general member of \( W_{S,C} \), \( \text{Hilb}^{sc} V \) is everywhere singular along \( W_{S,C} \) and hence generically non-reduced along \( W_{S,C} \). \( \square \)

**5. Original motivation and examples**

**5.1. Kleppe’s conjecture**

The original motivation of the present work was to show the following conjecture due to Kleppe. We denote by \( \text{Hilb}^{sc}_{d,g} \mathbb{P}^3 \) the open and closed subscheme of \( \text{Hilb}^{sc} \mathbb{P}^3 \) consisting of curves of degree \( d \) and genus \( g \).

**Conjecture 5.1 (Kleppe, Ellia).** — Let \( W \) be a maximal irreducible closed subset of \( \text{Hilb}^{sc}_{d,g} \mathbb{P}^3 \) whose general member \( C \) is contained in a smooth cubic surface. If

\[
d \geq 14, \quad g \geq 3d - 18, \quad H^1(\mathbb{P}^3, I_C(3)) \neq 0 \quad \text{and} \quad H^1(\mathbb{P}^3, I_C(1)) = 0,
\]

then \( W \) is a component of \( (\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}} \) and \( \text{Hilb}^{sc} \mathbb{P}^3 \) is generically non-reduced along \( W \).
In the original conjecture [8, Conjecture 4] of Kleppe, the assumption of the linearly normality of $C$ (i.e., $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$) was missing. However Ellia [2] pointed out that the conjecture does not hold for linearly non-normal curves $C$ by a counterexample, and suggested restricting the conjecture to linearly normal ones. The most crucial part to prove this conjecture is the proof of the maximality of $W$ in $(\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}}$. Once we prove that $W$ is a component of $(\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}}$, then the non-reducedness of $\text{Hilb}^{sc} \mathbb{P}^3$ along $W$ naturally follows. Therefore Conjecture 5.1 follows from Conjecture 1.2 (1), where the condition $\chi(\mathbb{P}^3, \mathcal{I}_C(3)) \geq 1$ is equivalent to $g \geq 3d - 18$. Recently it has been proved in [15] that Conjecture 5.1 is true if $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$. Kleppe and Ellia gave a proof for the conjecture under some other conditions, however the whole conjecture is still open.

5.2. Hilbert scheme of canonical curves

Let $(V, H)$ be a polarized variety. We say that a curve $C \subset V$ is canonical if $f^*H = K_C$, where $f : C \hookrightarrow V$ is the embedding, or equivalently $C$ is embedded into $V$ by a linear subsystem of $|K_C|$. We apply Theorem 4.14 to prove the following:

**Proposition 5.2** (cf. [13]). — Let $V$ be a smooth del Pezzo 3-fold of degree $n$. If $n \leq 7$ then the Hilbert scheme $\text{Hilb}^{sc} V$ of smooth connected curves on $V$ has a generically non-reduced component $W$, whose general member is a canonical curve on $V$.

**Proof.** — Since $n \leq 7$, there exists a good line $\ell$ on $V$ by Lemma 2.7. Let $S_n$ be a smooth member of $|H|$ containing $\ell$. We consider the complete linear system $\Lambda := |-2K_{S_n} + 2\ell|$ on $S_n$. Let $S_{n+1}$ be the the blow-down of $\ell$ from $S_n$, which is a del Pezzo surface of degree $n + 1$. Then $\Lambda$ is the pull-back of $|−2K_{S_{n+1}}| \simeq \mathbb{P}^{3n+3}$ on $S_{n+1}$. Since $\Lambda$ is base point free, every general member $C$ of $\Lambda$ is a smooth connected curve of degree $d = 2n + 2$ and genus $g = n + 2$. Therefore we have $g = d - n$ and hence $\chi(V, \mathcal{I}_C(S)) = 1$. Then $\ell$ does not intersect $C$ by $\langle -2K_{S_n} + 2\ell, \ell \rangle = 2-2 = 0$. Moreover $\ell$ is the only such line on $S_n$. By Theorem 4.14 (1), $W_{S_n, C}$ is an irreducible component of $(\text{Hilb}^{sc} V)_{\text{red}}$. Since $C \cap \ell = \emptyset$, $C$ is not $S_n$-normal by Proposition 4.9 (2). Therefore $\text{Hilb}^{sc} V$ is generically non-reduced along $W_{S_n, C}$ by Theorem 4.14 (2). By construction, $C$ is the image of a canonical curve $C' \sim -2K_{S_{n+1}}$ on $S_{n+1}$ by the projection $S_{n+1} \cdots \rightarrow S_n$ from a point $p \in S_{n+1} \setminus C'$. □
Remark 5.3. — The dimension of the irreducible component \( W_{S_n,C} \) is equal to \( d + g + n = 4n + 4 \) by Lemma 4.6 (2). The tangential dimension of \( \text{Hilb}^n V \) at a general point \([C]\) of \( W_{S_n,C} \) is equal to \( h^0(N_{C/V}) = 4n + 5 \). Indeed the exact sequence (4.4) is

\[
0 \to \mathcal{O}_C(2K_C) \to N_{C/V} \to \mathcal{O}_C(K_C) \to 0,
\]

since \( N_{S/V}|_C \simeq -K_S|_C \simeq K_C \). Hence we have

\[
h^0(N_{C/V}) = h^0(2K_C) + h^0(K_C) = (3n + 3) + (n + 2) = 4n + 5.
\]

The next example shows that the curve \( C \) in Theorem 1.3 is not necessarily stably degenerate if there exists a bad line \( \ell \) on \( S \) such that \( C \cap \ell = \emptyset \).

Let \( V_7 \subset \mathbb{P}^8 \) be a smooth del Pezzo 3-fold of degree 7, \( S_7 \) a smooth hyperplane section of \( V_7 \). Let \( \ell_0, \ell_1, \ell_2 \) be the three lines on \( S_7 \) explained in Lemma 2.8, i.e., \( \ell_0 \) is bad and \( \ell_1 \) and \( \ell_2 \) are good. Consider a general member \( C \) of \( \Lambda := | -2K_{S_7} + 2\ell_0 | \). Then \( C \) is a smooth connected curve of degree 16 and genus 9 = 16 - 7 and not \( S_7 \)-normal by \( C \cap \ell_0 = \emptyset \).

**Proposition 5.4.** — Let \( C \) be as above. Then there exists a smooth deformation \( C' \subset V_7 \) of \( C \) not contained in any hyperplane section. In other words, \( C \) is not stably degenerate.

**Proof.** — Recall that \( V_7 \) is isomorphic to the blow-up of \( \mathbb{P}^3 \) at a point \( p \). It is realized as the projection of the Veronese image \( V_8 \subset \mathbb{P}^9 \) of \( \mathbb{P}^3 \) from \( p \in V_8 \) (cf. §2.2). Then \( S_7 \) is the image by the projection of a hyperplane section \( Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) of \( V_8 \) containing \( p \). Hence we have a diagram

\[
\begin{array}{ccc}
S_7 \simeq \text{Bl}_{2\text{pts}} \mathbb{P}^2 & \subset & V_7 \simeq \text{Bl}_p \mathbb{P}^3 \subset \mathbb{P}^8 \\
Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 & \subset & V_8 \simeq \mathbb{P}^3 \subset \mathbb{P}^9,
\end{array}
\]

where the down arrows (resp. the up arrows) are the blow-up morphisms at (resp. the projections from) \( p \in Q_2 \subset V_8 \subset \mathbb{P}^9 \). Let \( P \simeq \mathbb{P}^2 \) denote the exceptional divisor of \( \pi_p \). Then its intersection with \( S_7 \) is equal to the bad line \( \ell_0 \).

Since \( C \cap \ell_0 = \emptyset \) and \( C \cdot \ell_i = 4 \) for each \( i = 1, 2 \), \( \pi_p \) maps \( C \) isomorphically onto a curve of bidegree \((4, 4)\) on \( Q_2 \). Let \( Q_2' \) be a general hyperplane section of \( V_8 \). Then \( Q_2' \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) is mapped isomorphically onto a surface \( Q_2'' \) on \( V_7 \) by \( \Pi_p \). Here \( Q_2'' \) is linearly equivalent to \( S_7 + P \) as a divisor of \( V_7 \) and contains a smooth deformation \( C' \) of \( C \). Then there exists no hyperplane section of \( V_7 \) containing \( C' \). Suppose that there exists such a hyperplane section \( S_7' \). Then the image \( \pi_p(S_7') \) is contained in the intersection of two hyperplane sections \( \pi_p(S_7') \) and \( Q_2' \) of \( V_8 \). Hence the pull-back of \( \pi_p(C') \)
in \( \mathbb{P}^3 \) by the Veronese embedding is contained in a complete intersection of two quadrics. This is impossible since the degree of the inverse image is equal to \( 8 > 4 \).

\[ \square \]

### 5.3. Non-reduced components of the Hilbert scheme

In this subsection, we give the classes of irreducible components of the Hilbert scheme of del Pezzo 3-folds which are non-reduced by Theorem 4.14 more precisely (cf. Proposition 5.5).

Let \( V_n \) be a smooth del Pezzo 3-fold of degree \( n \leq 7 \) and let \( S \subset V_n \) be a smooth member of the class \([H] \) of the polarization of \( V_n \), i.e., a del Pezzo surface. Put \( r := 9 - n \) (\( \geq 2 \)). Then \( S \) is isomorphic to a \( \mathbb{P}^2 \) blown up at \( r \) points in general position, i.e., no three are on a line, no six are on a conic and any cubic containing eight points is smooth at each of them. The class of the pullback \( l \) of a line in \( \mathbb{P}^2 \) and the \( r \) exceptional curves \( e_i \) \((1 \leq i \leq r)\) form a free \( \mathbb{Z} \)-basis of the Picard group \( \text{Pic} S \simeq \mathbb{Z}^{r+1} \) of \( S \). Thus given a divisor \( D \) on \( S \), we obtain a \((r + 1)\)-tuple \((a; b_1, \ldots, b_r)\) of integers as the coefficients of the divisor class \( D = al - \sum_{i=1}^r b_ie_i \). On the other hand, for each \( r \geq 2 \) there exists a Weyl group \( W_r \subset \text{Aut}(\text{Pic} S) \). Here \( W_r \) is the subgroup generated by the permutations of \( e_i \) \((1 \leq i \leq r)\) and \((r \geq 3)\) by the additional Cremona element \( \sigma \) given by \( \sigma(l) = 2l - e_1 - e_2 - e_3 \), \( \sigma(e_1) = l - e_2 - e_3 \), \( \sigma(e_2) = l - e_1 - e_3 \), \( \sigma(e_3) = l - e_1 - e_2 \) and \( \sigma(e_i) = e_i \) for \( i \notin \{1, 2, 3\} \). The root systems corresponding to the Weyl group \( W_r \) \((r = 2, 3, 4, 5, 6, 7, 8)\) are \( A_1, A_1 \times A_2, A_4, D_5, E_6, E_7, E_8 \), respectively (See [12] for the details). Every element of \( W_r \) induces a base change of \( \text{Pic} S \). By virtue of the Weyl groups and this base change, given a divisor \( D \) on \( S \) there exists a suitable blow-up \( S \to \mathbb{P}^2 \) (in other words, a suitable choice of \( r \) exceptional curves on \( S \)) such that we have

\[ b_1 \geq \cdots \geq b_r \quad \text{and} \quad a \geq b_1 + b_2 + b_3 \quad \text{(only for} \ r \geq 3) \]  

(5.2)

When (5.2) holds, we say the basis \( \{l, e_1, \ldots, e_r\} \) of \( \text{Pic} S \) is standard for \( D \). For the standard basis of \( \text{Pic} S \) for \( D \), the linear system \(|D| \) on \( S \) contains a smooth connected curve \( C \) of degree \( \geq 2 \) if and only if \( a > b_1 \) and \( b_r \geq 0 \) (and \( a \geq b_1 + b_2 \) for \( r = 2 \)). The degree \( d \) and genus \( g \) of \( C \) is computed as

\[ d = 3a - \sum_{i=1}^r b_i \quad \text{and} \quad g = \binom{a - 1}{2} - \sum_{i=1}^r \binom{b_i}{2}. \]  

(5.3)

Let \((d, g)\) be a pair of integers with \( d \geq 2 \) and let \((a; b_1, \ldots, b_r)\) be a \((r + 1)\)-tuple of integers satisfying (5.2), (5.3), \( a > b_r \) and \( b_r \geq 0 \) (and
Suppose that \( a \geq b_1 + b_2 \) as well for \( r = 2 \)). Then the linear system \(|a l - \sum_{i=1}^{r} b_i e_i| \) on \( S \) contains a smooth connected member \( C \) of degree \( d \) and genus \( g \). Then we denote by \( W_{(a; b_1, \ldots, b_r)} \) the \( S \)-maximal family \( W_{S,C} \subset \text{Hilb}_{d,g}^{\text{sc}} V_n \) of curves containing \( C \) (cf. Definition 4.4). By definition, \( W_{(a; b_1, \ldots, b_r)} \) contains every smooth connected curve \( C' \) on \( V_n \) such that \( C' \) is contained in a smooth member \( S' \in |H| \) and such that \( C' \sim a l' - \sum_{i=1}^{r} b_i e'_i \) on \( S' \) for a standard basis \( \{l', e'_1, \ldots, e'_r\} \) of \( \text{Pic } S' \) for \( C' \).

**Proposition 5.5.** — Suppose that \( g \geq 2 \) and \( g \geq d - n \). If \( b_r = 0 \), then \( W_{(a; b_1, \ldots, b_r)} \) is an irreducible component of \( (\text{Hilb}_{d,g}^{\text{sc}} V_n)_{\text{red}} \) of dimension \( d + g + n \) and \( \text{Hilb}_{d,g}^{\text{sc}} V_n \) is generically non-reduced along \( W_{(a; b_1, \ldots, b_r)} \).

**Proof.** — Let \( C \) denote a general member of \( W_{(a; b_1, \ldots, b_r)} \). Then \( C \) is contained in a smooth member \( S \in |H| \). Since \( C \) is general, so is \( S \) in \( |H| \). By Lemma 2.8 every line on \( S \) is good except for the bad line \( \ell_0 \) on \( V_7 \). If \( n = 7 \) then \( \ell_0 \) is linearly equivalent to \( l - e_1 - e_2 \). Since \( b_2 = 0 \), \( C \) intersects \( \ell_0 \) by \( C \cdot \ell_0 = (a l - b_1 e_1) \cdot \ell_0 = a - b_1 > 0 \). We recall that \( g \geq d - n \) is equivalent to \( \chi(V, \mathcal{I}_C(S)) \geq 1 \). Therefore \( W_{(a; b_1, \ldots, b_r)} \) is an irreducible component of \( (\text{Hilb}_{d,g}^{\text{sc}} V_n)_{\text{red}} \) by Theorem 4.14 (1), and of dimension \( d + g + n \) by Lemma 4.6.

Since \( b_r = 0 \), the line \( e_r \) on \( S \) does not intersect \( C \). Since \( g \geq 2 \), \( C \) is not \( S \)-normal by Proposition 4.9 (2), and hence we have \( H^1(V, \mathcal{I}_C(S)) \neq 0 \). Thus \( \text{Hilb}_{d,g}^{\text{sc}} V_n \) is generically non-reduced along \( W_{(a; b_1, \ldots, b_r)} \) by Theorem 4.14 (2).

The next example shows that for every integer \( d \geq 12 \) the Hilbert scheme of smooth connected curves of degree \( d \) on a smooth cubic 3-fold \( V_3 \) has a generically non-reduced component.

**Example 5.6.** — Let \( \lambda \in \mathbb{Z}_{\geq 0} \) and let \( W \) be one of the \( S \)-maximal families

\[
W_{(\lambda+6; \lambda+1,1,1,1,0)} \subset \text{Hilb}_{d,2d-16}^{\text{sc}} V_3 \quad (d = 2\lambda + 13) \quad \text{and}
\]

\[
W_{(\lambda+6; \lambda+2,1,1,1,0)} \subset \text{Hilb}_{d,\frac{5}{2}d-9}^{\text{sc}} V_3 \quad (d = 2\lambda + 12).
\]

Then \( W \) is an irreducible component of \( (\text{Hilb}_{3}^{\text{sc}} V_3)_{\text{red}} \) and \( \text{Hilb}_{d}^{\text{sc}} V_3 \) is generically non-reduced along \( W \).

It was shown in [13, Theorem 1.4] that for many uniruled 3-folds \( V \) the Hilbert scheme \( \text{Hilb}_{d}^{\text{sc}} V \) has infinitely many generically non-reduced components.
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