Formation of singularities in solutions to ideal hydrodynamics of freely cooling inelastic gases

Olga Rozanova

Department of Differential Equations & Mechanics and Mathematics Faculty, Moscow State University, Moscow 119992, Russia
E-mail: rozanova@mech.math.msu.su

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Abstract

We consider solutions to the hyperbolic system of equations of ideal granular hydrodynamics with conserved mass, total energy and finite momentum of inertia and prove that these solutions generically lose the initial smoothness within a finite time in any space dimension $n$. Furthermore, in the one-dimensional case we introduce a solution depending only on the spatial coordinate outside of a ball containing the origin and prove that this solution under rather general assumptions on initial data cannot be global in time too. Then we construct an exact axially symmetric solution with separable time and space variables having a strong singularity in the density component beginning from the initial moment of time, whereas other components of solution are initially continuous.

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1. Introduction

Granular gases are now a popular subject of experimental, numerical and theoretical investigation (e.g. [3, 8, 13] and references therein). In contrast to ordinary molecular gases, granular gases cool spontaneously because of the inelastic collisions between the particles. The inelasticity of the collisions generally causes the granular gas to form dense clusters. The formation of complex structure of clusters has been investigated by means of molecular dynamics simulations and hydrodynamic simulations.

A granular gas can be modelled with a dilute assembly of identical hard spheres, which lose energy at instantaneous binary inelastic collisions. It is natural to use the Boltzmann kinetic equation to describe the complete dynamics. In recent years, an accurate derivation of Navier–Stokes order hydrodynamics has been given from the Boltzmann equation for granular gases
using an adaptation of the Chapman–Enskog method for normal gases (see, e.g., [2, 3]). The Navier–Stokes granular hydrodynamics is the natural language for a theoretical description of granular macroscopic flows. A characteristic feature of time-dependent solutions of the continuum equations is the formation of finite-time singularities: the density blowup signals the formation of close-packed clusters.

The Navier–Stokes system contains the so-called cooling rate term. It consists of the viscous and heat conduction terms together with a term describing energy loss due to inelastic collisions. The viscous and heat conduction terms depend on spatial gradients of the velocity and temperature, whereas the energy loss term is independent of the gradients. When the characteristic hydrodynamic length scale of the flow is sufficiently large, the viscous and heat conduction terms can be neglected, while the energy loss term should be kept. Thus, we obtain the system of equations of ideal granular hydrodynamics (see, e.g., [7, 8] and references therein for details).

This system has the following form:

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho u) = 0, \]
\[ \frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u \otimes u) = -\nabla p, \]
\[ \frac{\partial}{\partial t} T + (u, \nabla T) + (\gamma - 1) \nabla \cdot u = -\Lambda \rho T^{3/2}, \]

where \( \rho(t, x) \) is the gas density, \( u(t, x) = (u_1, ..., u_n) \) is the velocity, \( T(t, x) \) is the temperature, \( p(t, x) = \rho T \) is the pressure, \( t \in \mathbb{R}_+, x \in \mathbb{R}^n \). The positive constants \( \gamma \) and \( \Lambda \) are chosen for physical reasons, \( \gamma = 1 + \frac{2}{n} \). We denote by \( \text{Div} \) and \( \text{div} \) the divergence of tensor and vector with respect to the space variables, and \((\cdot, \cdot)\) is the inner product. The only difference between equations (1.1)–(1.3) and the standard ideal gas dynamic equations (where elastic collision of particles is supposed) is the presence of the inelastic energy loss term \(-\Lambda \rho T^{3/2}\) in (1.3).

System (1.1)–(1.3) can be written in a hyperbolic symmetric form in variables \( \rho, u, K = \rho p^{\gamma-1} \) and therefore the Cauchy problem

\[ \rho, u, p^{\gamma-1} |_{t=0} \in H^m(\mathbb{R}^n), \quad m \geq 1 + \left\lceil \frac{n}{2} \right\rceil \]

has a solution as smooth as the initial data at least for small \( t > 0 \) [11].

We will call the solution to (1.1)–(1.3) classical if \( \rho > 0, p > 0 \) and the components of solution belong to \( C^1([0, T), H^m(\mathbb{R}^n)) \), \( T \leq \infty \).

System (1.1)–(1.3) has no constant solution except for the trivial one \( (p \equiv 0) \), therefore the solution with components substantially decreasing as \( |x| \to \infty \) can be considered as a natural perturbation if this is steady state in the case of mass conservation.

Let us note that there exists a solution (the homogeneous cooling state) with constant \( \rho, u \) and \( p \neq 0 \). In this case the temperature \( T = T(t) = (\frac{\Lambda \rho_0 t + T(0)^{-1}}{\Lambda})^{-2} \), where \( T(0) \) is the initial value of temperature (Haff’s law). Another trivial solution is \( u = p = T \equiv 0, \rho(t, x) = \rho_0(x) \).

We introduce the following integrals: the total mass

\[ M(t) = \int_{\mathbb{R}^n} \rho \, dx, \]

the momentum

\[ P(t) = \int_{\mathbb{R}^n} \rho u \, dx, \]

and the total energy

\[ \mathcal{E}(t) = \int_{\mathbb{R}^n} \left( \frac{1}{2} \rho |u|^2 + \rho T \right) \, dx = E_k(t) + E_i(t). \]
Here $E_k(t)$ and $E_i(t)$ are the kinetic and internal components of energy, respectively. Let us also introduce the functionals

$$G(t) = \frac{1}{2} \int_{\mathbb{R}^n} \rho(t, x)|x|^2 \, dx,$$
$$F(t) = \int_{\mathbb{R}^n} \rho(t, x) \langle u(t, x), x \rangle \, dx,$$

where $\langle u, x \rangle$ in the latter integrand denotes the inner product of the vector of velocity $u$ and the radius-vector $x$ of a point $x \in \mathbb{R}^n$.

The integral $G(t)$ is the momentum of inertia of a continuous dense medium. In classical mechanics, moment of inertia (also called mass moment of inertia) is a measure of an object’s resistance to changes in its rotation [12]. Mathematically, the reason for considering this quantity is its relatively simple dependence on time (see lemma 2 below). The analogous integral is useful in other fields, in the framework of the formation of singularities (see, e.g., [10] for the nonlinear Schrödinger equation and [4] for chemotaxis models). As we will show below, $G(t)$ tends to rise with time, which corresponds to an expansion of continuum mostly concentrated at the origin (recall that the convergence of the integral requires a decay of density as $|x| \to \infty$). On the other hand, we are going to show that for the system of ideal granular hydrodynamics one can find a restriction in the growth of $G(t)$ from above, which signifies that there exists an obstacle for the expansion of continuum that causes the formation of singularities.

We consider below the solutions to (1.1)–(1.3) such that the integrals $M$, $P$, $E$ and $G$ converge and call them solutions with finite momentum of inertia (FMI). It is easy to verify that for this class of solutions $M(t) = M = \text{const}$, $P(t) = P = \text{const}$,

$$E'(t) = -\frac{\Lambda}{\gamma - 1} \int_{\mathbb{R}^n} \rho^{1/2} p^{3/2} \, dx. \quad (1.4)$$

The latter equation expresses the inelastic energy loss per collision.

Further we consider a function $K = p \rho^{-\gamma}$. System (1.1)–(1.3) results in

$$\frac{dK}{dt} = -\Lambda K^3 \rho^{\gamma - 1} \leq 0, \quad (1.5)$$

therefore

$$K(t, x) \leq K_\ast = \sup_{x \in \mathbb{R}^n} K(0, x). \quad (1.6)$$

2. Main theorem: nonexistence of global smooth solutions

**Theorem 1.** Let $P \neq 0$ and $M$ be sufficiently small. Then there exists no global in time classical FMI solution to the Cauchy problem for (1.1)–(1.3).

To prove the theorem we need to first obtain certain estimates of energy.

**Lemma 1.** For the classical FMI solutions to (1.1)–(1.3) the following estimates hold:

$$E_k(t) \geq \frac{|P|^2}{2M} = \text{const}, \quad (2.1)$$

$$E'(t) \leq -\Lambda C_1 E^{\frac{2\gamma - 1}{\gamma - 1}}(t), \quad (2.2)$$

where $C_1 = K_\ast^{-\frac{1}{\gamma - 1}} (\gamma - 1)^{\frac{2\gamma - 1}{\gamma - 1}} M^{-\frac{\gamma - 1}{\gamma - 1}}$. 

Proof. Inequality (2.1) follows immediately from the Hölder inequality [6]. Indeed,
\[
P^2(t) = \left( \int_{\mathbb{R}^n} \rho u^2 \, dx \right)^2 = \int_{\mathbb{R}^n} \rho u^2 \, dx \int_{\mathbb{R}^n} \rho \, dx = 2E_k(t)M(t).
\]
To prove (2.2) we first use the Jensen inequality [6] as follows:
\[
\left( \int_{\mathbb{R}^n} p \, dx \right)^{\frac{\gamma-1}{\gamma}} \leq \int_{\mathbb{R}^n} K\rho^{\gamma-1} \, dx \leq \int_{\mathbb{R}^n} \rho^{1/2} p^{3/2} \, dx.
\]
Together with (1.4) inequality (2.3) gives (2.2).

The following lemmas establish the properties of the momentum of inertia. Acting as in [5] we get

Lemma 2. For classical FMI solutions to (1.1)–(1.3) the equalities
\[
G'(t) = F(t), \quad F'(t) = 2E_k(t) + n(\gamma - 1)E_i(t).
\]
take place.

Proof. The lemma can be proved by direct calculation using the general Stokes formula. □

Then we get two-sided estimates of \(G(t)\).

Lemma 3. If \(\gamma \leq 1 + \frac{2}{n}\), then for the classical FMI solutions to (1.1)–(1.3) the estimates
\[
\frac{|P|^2}{2M} t^2 + F(0)t + G(0) \leq G(t) \leq \mathcal{E}(0)t^2 + F(0)t + G(0)
\]
hold.

Proof. First of all (2.6) and (2.7) result in
\[
G''(t) = 2E_k(t) + n(\gamma - 1)E_i(t) = 2\mathcal{E}(t) - (2 - n(\gamma - 1))E_i(t).
\]
Therefore together with (2.1) we have
\[
\frac{|P|^2}{M} \leq G''(t) \leq 2\mathcal{E}(0),
\]
after integration this gives (2.6). □

Now we get an upper estimate of \(E_i(t)\).

Lemma 4. If \(\gamma \leq 1 + \frac{2}{n}\) and \(P \neq 0\), then for the classical FMI solutions the following estimate is true:
\[
E_i(t) \leq \frac{C_2}{G^4(\gamma-1)^2},
\]
where \(C_2 = \left(4G\mathcal{E}(0) - F(0)G^{(\gamma-1)/2}(0)\right)\).
Proof. The method of obtaining the upper estimate of \( E_i(t) \) is similar to [5]. Namely, let us consider the function \( Q(t) = 4G(t)E(t) - F^2(t) \). The Hölder inequality gives \( F^2 \leq 4G(t)E_k(t) \), therefore \( E(t) = E_k(t) + E_i(t) \geq E_i(t) + \frac{E_k(t)}{4G(t)} \) and

\[
E_i(t) \leq \frac{Q(t)}{4G(t)}. \tag{2.10}
\]

We also note that \( Q(t) > 0 \) provided the pressure is not equal to zero identically. Then taking into account (2.4), (2.5) and (2.7) we have

\[
Q'(t) = 4G(t)E(t) - 2G'(t)G''(t) + 4G(t)E'(t) - 2G(t)E_k(t) + 4G(t)E'(t). \tag{2.11}
\]

Further, one can see from (2.8) that \( G'(t) > 0 \) beginning from a positive \( t_0 \) for all initial data. Thus, for \( \gamma \leq 1 + \frac{2}{n} \) we have from (2.10) and (2.11)

\[
Q'(t) \leq \frac{2 - n(\gamma - 1)}{2} \frac{G'(t)}{G(t)}. \tag{2.12}
\]

Then (2.10) and (2.12) give

\[
E_i(t) \leq \frac{C_2}{G^{(\gamma - 1)n/2}(t)}, \quad C_2 = \frac{Q(0)G^{(\gamma - 1)n/2}(0)}{4}.
\]

The proof is complete. \( \square \)

Remark 1. Inequalities (2.1) and (2.2) result in

\[
E'(t) \leq -\Lambda C_1 \left( E(t) - \frac{p^2}{2M} \right)^{\frac{\gamma - 1}{\gamma n}}.
\]

Integrating this inequality we get the following upper estimate of \( E_i(t) \):

\[
E_i(t) \leq (c_1 t + c_2)^{\frac{2 - n(\gamma - 1)}{1 + n}}.
\]

where \( c_1 = \frac{\Delta C_1}{\gamma} \) and \( c_2 = (E(0) - |P|)^{\frac{2 - n(\gamma - 1)}{1 + n}} \) are positive constants. However, for \( \gamma \leq 1 + \frac{2}{n} \) this estimate is less exact than (2.9) and is not enough for our proof.

The next step is a lower estimate of \( E_i(t) \).

Lemma 5. Let \( P \neq 0 \). Then for the classical FMI solutions the estimate

\[
E_i(t) \geq \frac{C_3}{G^{(\gamma - 1)n/2}(t)} \tag{2.13}
\]

holds with a positive constant

\[
C_3 = \frac{1}{2^{(\gamma - 1)n/2} K_\gamma^{(\gamma - 1)(1+n/2)} (\gamma - 1)} C_\gamma^{(\gamma - 1)n/2} (C_6(\gamma) - \frac{1}{2^{(\gamma - 1)n/2}}),
\]

the value of \( C_6 \) is written in (2.19).

Proof. The proof is based on the inequality

\[
\| f \|_{L^1(\mathbb{R}^n; \, dx)} \leq C_{\gamma,n} \| f \|_{L^{2\gamma/(n-1)}(\mathbb{R}^n; \, dx)} \| f \|_{L^{2\gamma/(n-1)}(\mathbb{R}^n; \, |x|^2 \, dx)},
\]

\[
C_{\gamma,n} = \left( \frac{2\gamma}{n(n-1)} \right)^{\frac{n-1}{2\gamma(n-1)}} + \left( \frac{2\gamma}{n(n-1)} \right)^{\frac{n-1}{2\gamma(n-1)}},
\]

established in [5].
Namely, we have for \( f = K\rho \)
\[
E_i(t) = \frac{1}{\gamma - 1} \int_{\mathbb{R}^n} \frac{p}{\rho^\gamma} \frac{(K\rho)^\gamma}{K^\gamma} \, dx \geq \frac{1}{K_4^{\gamma - 1}} \left( \int_{\mathbb{R}^n} (K\rho)^\gamma \, dx \right)^{\frac{\gamma - 1}{\gamma}} \frac{1}{K_4^{\gamma - 1}} \left( \int_{\mathbb{R}^n} (K\rho)^\gamma \, dx \right)^{\frac{\gamma - 1}{\gamma}} \geq \frac{1}{2^{\gamma - 1} K_4^{\gamma - 1}} \left( \int_{\mathbb{R}^n} (K\rho)^\gamma \, dx \right)^{\frac{\gamma - 1}{\gamma}}.
\]
(2.14)
where we denoted \( S(t) = \int_{\mathbb{R}^n} K\rho \, dx \).

Further, from (1.1) and (1.5) we have
\[
\partial_t (K\rho) + \text{div}_x (K\rho u) = -\Lambda K^2 \rho^{\frac{\gamma - 1}{\gamma}}.
\]

From the Jensen inequality we get
\[
\int_{\mathbb{R}^n} K^2 \rho^{\frac{\gamma - 1}{\gamma}} \, dx \leq K^2 M \left( \frac{\int_{\mathbb{R}^n} K\rho \, dx}{S(t)} \right)^{\frac{1}{\gamma - 1}}.
\]
(2.15)
where \( C_4 = K^2 M (\gamma - 1)^{\frac{1}{\gamma - 1}}. \)

Further, from (2.9), (2.15) and (2.16) we obtain
\[
K^2 \rho^{\frac{\gamma - 1}{\gamma}} \leq K^2 M \left( \frac{\int_{\mathbb{R}^n} K\rho \, dx}{S(t)} \right)^{\frac{1}{\gamma - 1}}.
\]
(2.17)
Now we take into account the lower estimate in (2.6) together with the fact that beginning from a certain \( t_0 \) the value of \( F(t) \) becomes positive if \( P \neq 0 \) (see (2.1) and (2.5)) and integrate (2.17). Thus, for \( t \geq t_0 \) we get in the case \( \gamma < 3 \)
\[
S(t) \geq \left( S(t_0) + \Lambda \frac{4(\gamma - 1)}{(3 - \gamma)(\gamma - 1) - 2} C_5^{\frac{1}{3 - 2\gamma + 1}} \right)^{\frac{2 - \gamma}{\gamma - 1}},
\]
(2.18)
where \( C_5 = M (F(t_0))^{-\frac{1}{3 - 2\gamma + 1}} |P|^{-2} \), and for \( \gamma = 3 \) (\( n = 1 \))
\[
S(t) \geq S(t_0) \exp \left( \frac{2\Lambda}{C_5} \right).
\]
(2.19)
We denote the constant in the right-hand side of (2.18) and (2.19) by \( C_6(\gamma) \).

It is easy to see that for sufficiently large \( t \) the value of \( S(t) \) is separated from zero.

Thus, from (2.14), (2.18) and (2.19) we obtain (2.13).

\textbf{Proof of theorem 1.} Taking into account (2.2) and (2.13), and the right-hand side of (2.6) we get
\[
\mathcal{E}'(t) \leq -\Lambda C_1 \frac{2\gamma - 1}{\gamma - 1} (G(t))^{-\frac{\gamma - 1}{\gamma - 1}}
\]
(2.20)
\[
\leq -\Lambda C_1 \frac{2\gamma - 1}{\gamma - 1} (\mathcal{E}(0)t^2 + F(0)t + G(0))^{-\frac{\gamma - 1}{\gamma - 1}}.
\]
As follows from (2.1) and (2.4) beginning from \( t_0 > 0 \) the value of \( F(t) \) becomes positive. Integrating (2.20) we obtain for \( t > t_0 \)
\[
\mathcal{E}(t) \leq \mathcal{E}(t_0) - \Lambda C_1 C_3 \frac{2^n(3y-1)/2(F(t_0)) - n(3y-1)/2+1(\mathcal{E}(0))^{\rho(y-1)/4-1}}{n(3y-1) - 2} + \lambda(t),
\]
where \( \lambda(t) = O(t^{-n(3y-1)/2+1}), t \to \infty \). Since \( C_1 \) tends to a positive constant as \( M \to 0 \) and \( C_1 \) contains \( M \) in a negative degree (see the statements of lemmas 1 and 5, then choosing \( M \) sufficiently small, we can always get a contradiction to inequality (2.1). This proves the theorem. \( \square \)

**Remark 2.** The main idea of this paper can be found in [14], where the nonexistence of global smooth solutions to the compressible Navier–Stokes equations was proved. For the freely cooling gas, it would be possible to add viscous terms (with constant viscosity coefficients) to apply the same technique and obtain the analogous nonexistence result.

### 3. One-dimensional case

As was noted, system (1.1)–(1.3) has no constant solution except for the trivial one \( p \equiv 0 \). However, it is possible to construct the nontrivial steady-state solution \( \bar{\rho}(0,x), \bar{v}(0,x), \bar{p}(0,x) \) for regions \( |x| > R(t) > 0 \). For \( |x| \leq R(0) \) we chose the functions \( \rho(0,x), v(0,x), p(0,x) \) arbitrarily to get the initial data, smooth on the whole real axis. We are going to show that such a solution necessarily loses its initial smoothness.

#### 3.1. Self-similar solution

Let us find a solution that depends on the variable \( \xi = x - at \), \( a = \text{const} \). The continuity equation (1.1) gives the connection between velocity and density as follows:
\[
u(\xi) = \frac{c_1}{\rho(\xi)} + a, \quad c_1 = \text{const} \neq 0. \tag{3.1}
\]
Equations (1.2) and (3.1) result in
\[
(c_1 u(\xi) + p(\xi)) = c_2, \quad c_2 = \text{const}. \tag{3.2}
\]
From (3.1) and (3.2) we have \( c_1^2 + \rho p = \rho(c_2 - ac_1) \), therefore \( c_2 - ac_1 > 0 \).

Further, we substitute the functions \( u \) and \( p \), found from (3.1) and (3.2), and expressed through \( \rho \), in the equation
\[
\partial_t \rho + u \partial_x \rho + \rho \gamma \partial_x u = -\Lambda \rho^{1/2} \rho^{3/2},
\]
which is a corollary of (1.1), (1.3) and the state equation \( p = \rho T \) (recall that we consider a rarified gas). Thus, we get an ordinary differential equation
\[
\rho'(\xi) = \frac{\Lambda}{c_1} \frac{\rho^2(\xi)((c_2 - ac_1)\rho(\xi) - c_1^2)^{3/2}}{c_1^2(\gamma + 1) - \gamma(c_2 - ac_1)\rho(\xi)},
\]
or
\[
z'(\xi) = -\frac{\Lambda}{c_1(c_2 - ac_1)^2} \frac{z^{3/2}(\xi)(z(\xi) + c_1^2)^2}{c_1^2 - \gamma z(\xi)}, \tag{3.3}
\]
where \( z = (c_2 - ac_1)\rho - c_1^2 \). The case \( c_2 - ac_1 = 0 \), which seems simpler, corresponds to a negative pressure and we do not consider it. Equation (3.3) has a solution
\[
\xi = f(z) := \frac{(c_2 - ac_1)^2}{c_1 \Lambda} \left[ \frac{(\gamma + 3) \arctan \frac{\sqrt{z}}{c_1}}{c_1} + \frac{(\gamma + 1) \sqrt{z}}{z + c_1^2} + \frac{2}{\sqrt{z}} \right] + c_3, \tag{3.4}
\]
Let us consider a stationary solution for $\alpha = 0$, $\xi = x$, choose a point $x_0 \geq 0$, a constant $c_1 = u(x_\ast)\rho(x_\ast) > 0$, and construct on the semi-axis $x > x_\ast$ a solution $z_\ast(x)$. Analogously for the semi-axis $x \leq x_\ast$ we choose $c_1 = u(x_\ast)\rho(x_\ast) < 0$ and construct a solution $z_\ast(x)$.

Thus, outside of the segment $[x_-, x_\ast]$ we define a solution $\bar{z}(x) = \begin{cases} z_\ast(x), & x \in (-\infty, x_-), \\ z_\ast(x), & x \in (x_\ast, +\infty). \end{cases}$

For the sake of simplicity we set $x_- = -x_\ast$, $c_1 = -k$ for $z_\ast(x)$ and $c_1 = k$ for $z_\ast(x)$, where $k = \text{const} > 0$.

In their turn, the density, pressure and velocity can be found as

$$\bar{\rho} = \frac{\bar{z} + k^2}{c_2}, \quad \bar{p} = c_2 - \frac{k^2}{\bar{\rho}}, \quad \bar{u} = \frac{k \text{sign } x}{\bar{\rho}}. \quad (3.5)$$

It is very attractive to choose $x_\ast = x_- = 0$ and to construct a piecewise continuous solution like a solution of a ‘Riemann problem’ with nonconstant left and right states. Nevertheless, it can be readily shown that the Rankine–Hugoniot jump conditions (see, e.g., [17]) do not hold on the jump. Indeed, the components $\bar{\rho}$ and $\bar{p}$ are continuous at the point $x = 0$, having a jump in the derivative; however, the velocity itself has a jump $[\bar{u}] = \frac{2k}{\bar{\rho}} \geq \frac{2c_2}{\gamma + 1}$ (see figures 1–3). The value of $[\bar{u}]$ tends to zero as $k \to \infty$; however, the Hugoniot condition $[\bar{\rho} \bar{u}] = 0$ does not implement in the origin $x = 0$. Of course, we can choose $c_1$ such that the density and pressure have jumps in the origin and consider $\alpha \neq 0$, nevertheless a careful analysis shows that the Hugoniot conditions do not hold anyway.

Therefore we choose $R(0) > x_\ast$, and define smooth initial data such that for $|x| > R(0)$ they coincide with $\bar{\rho}(x)$, $\bar{u}(x)$, $\bar{p}(x)$ and on the segment $|x| \leq R(0)$ they are arbitrary smooth functions $\rho(0, x) > 0$, $p(0, x) > 0$, $u(0, x)$. We will call this type of initial data the compact perturbation of nontrivial final steady state. Let us note that for $k = 0$ we get the trivial zero-state solution.
3.3. Breakdown of the compact perturbation of the nontrivial ‘final steady state’

Let us denote the perturbed region by $B(t) := \{ x \mid |x| \leq R(t) \}$, and consider the analogues of functionals used in the previous section:

\[
\tilde{G}(t) = \frac{1}{2} \int_{B(t)} \rho(t, x)|x|^2 \, dx, \quad \tilde{F}(t) = \int_{B(t)} \rho(t, x)(u(t, x), x) \, dx,
\]
\[
\tilde{E}_k(t) = \frac{1}{2} \int_{B(t)} \rho(t, x)|u(t, x)|^2 \, dx, \quad \tilde{M}(t) = \int_{B(t)} \rho(t, x) \, dx.
\]
\[
\tilde{P}(t) = \int_{B(t)} \rho(t, x)u(t, x) \, dx, \quad \tilde{S}(t) = \int_{B(t)} K(t, x)\rho(t, x) \, dx.
\]

The following theorem holds:

**Theorem 2.** Let $\tilde{P}^2(0) > \frac{8M_c}{k}$, $n = 1$. Then there exists no globally in $t$ smooth perturbation of the nontrivial steady state for system (1.1)–(1.3).

**Proof.** First of all we note that system (1.1)–(1.3) is hyperbolic and therefore the speed of the boundary of the perturbations is equal to $|\vec{u}| + V_t$, where $V_t = \sqrt{\tilde{P}/\tilde{\rho}}$, the sound speed. As follows from (3.5),

\[
V_t = \frac{k}{\tilde{\rho}} \leq \frac{c_2}{k} := \sigma,
\]

where we use the estimate $\tilde{\rho} \geq \frac{\tilde{P}}{c_2^2}$.
Then we have
\[ \dot{F}(t) = 2\dot{E}_k + \int_{B(t)} (p(t, x) - \bar{p}(t, R(t))) \, dx, \]
\[ \dot{M}(t) = 0, \quad \dot{P}(t) = 0. \]

Further, the Hölder inequality implies
\[ \dot{E}_k(t) \geq \frac{\dot{F}^2(t)}{4\dot{G}(t)}, \]
and one can estimate
\[ \dot{G}(t) \leq \frac{1}{2} R^2(t) \dot{M}(t) \leq \frac{1}{2} R^2(t) \dot{M}(0). \]

Further, the Jensen inequality yields
\[ \int_{B(t)} p(t, x) \, dx \geq (2R(t))^{1-\gamma} K_+^{\gamma-1} (\dot{S}(t))^\gamma. \]

As in the proof of lemma 5 we can show that for sufficiently large \( t \) the function \( S(t) \geq S_0 = \text{const} > 0 \). Then we note that (3.3) implies
\[ \bar{z}(x) \sim \frac{4c_l^2}{\Lambda^2} x^{-2}, \quad x \to \infty. \]

Thus, from (3.6)–(3.12) and the estimate
\[ \dot{F}(t) \geq \frac{\dot{P}(t)}{2\dot{M}(t)} = \frac{\dot{P}(0)}{2\dot{M}(0)} \]
we have
\[ \dot{F}(t) \geq \dot{E}_k(t) + \frac{\dot{F}^2(t)}{4\dot{M}(0)(R(0) + \sigma t)^2} + \frac{K_+^{\gamma-1} S_0^\gamma}{(R(0) + \sigma t)^{\gamma-1}} - \frac{4c_l^2}{\Lambda^2 (R(0) + \sigma t)} \]
\[ \geq \frac{\dot{P}(0)}{2\dot{M}(0)} + \frac{\dot{F}^2(t)}{4\dot{M}(0)(R(0) + \sigma t)^2} + \lambda(t), \]
where \( \lambda(t) \to 0, t \to \infty. \)

As we can see from (3.13), beginning from a certain \( t_0 > 0 \) the function \( \dot{F}(t) > 4 \dot{M}(0) R(t_0) \sigma \), moreover,
\[ \dot{F}(t) \geq \frac{\dot{F}^2(t)}{4\dot{M}(0)(R(0) + \sigma t)^2}, \quad t > t_0. \]

Integrating (3.14) from \( t = t_0 \) we get
\[ \dot{F}(t) \geq \frac{4\dot{M}(0) \dot{F}(t_0) R(t_0)(R(t_0) + \sigma t)}{4\dot{M}(0) R^2(t_0) + (4\dot{M}(0) R(t_0) \sigma - \dot{F}(t_0)) t}. \]

Thus, \( \dot{F}(t) \) blows up at a finite time. This contradicts the inequality \( \dot{F}^2(t) \leq 4\dot{G}(t)\dot{E}(0) \). The theorem is proved.

**Remark 3.** The idea of the method is due to [16], where it was proved that the compact smooth perturbation of a constant state of a gas dynamic equation cannot be globally smooth in time.
4. Exact solution with singularity

Naturally, there arises a question on the type of predicted singularity. In particular, in the remarkable papers [7, 8] for the one-dimensional case the authors employ Lagrangian coordinates and derive a broad family of exact nonstationary nonself-similar solutions. These solutions exhibit a singularity, where the density blows up in a finite time when starting from smooth initial conditions. Moreover, the velocity gradient also blows up while the velocity itself develops a cusp discontinuity (rather than a shock) at the point of singularity. This approach is partially extended to the 2D case in [9].

Here for any spatial dimension we construct a simple family of solutions to system (1.1)–(1.3) having a singularity in density whereas other components are continuous. Indeed, if we substitute in (1.1)–(1.3)

\[ u(t, x) = \alpha(t)x, \quad \rho(t, x) = \beta(t)|x|^q, \quad p(t, x) = s(t)|x|^l, \]  (4.1)

where \( x \) is a radius-vector of point, we obtain

\[ q = -1, \quad l = 1, \quad \beta(t) = \beta_0 = \text{const} \geq 0, \]  (4.2)

and \( (\alpha(t), s(t)) \geq 0 \) satisfy the following system of nonlinear ODE:

\[ \alpha'(t) + \alpha^2(t) + \frac{s(t)}{\beta_0} = 0, \]  (4.3)

\[ s'(t) + (\gamma + 1)ns(t)\alpha(t) = -\Lambda\beta^{1/2}(t)s^{3/2}(t). \]  (4.4)

This system has a unique equilibrium \((\alpha(t) = 0, s(t) = 0)\), which is unstable. One of its solutions is very simple: \( s(t) = 0 (\rho \equiv 0), \alpha(t) = (t + \alpha^{-1}(0))^{-1} \). An analysis of the phase portrait shows that if \( s(0) > 0 \), then \( \alpha(t) \to -\infty, s(t) \to +\infty \) for all \( \alpha(0) \).

We prove this fact in a different way. We consider a symmetric material volume \( B(t) \) containing the origin \( x = 0 \) and use the denotation of section 3.3. We can see that in spite of the singularity in the component of density all integrals below exist. Thus, due to the structure of solution (4.1) and (4.2) we have

\[ \tilde{G}'(t) = \tilde{F}(t) = 2\alpha(t)\tilde{G}(t), \]  (4.5)

\[ \tilde{F}'(t) = 2\tilde{E}_k(t) + \int_{B(t)} (p(t, x) - \bar{p}(t, R(t))) \, dx \leq 2\tilde{E}_k(t) = 2\alpha^2(t)\tilde{G}(t). \]  (4.6)

As follows from (4.5) and (4.6), the velocity gradient obeys the inequality

\[ \alpha'(t) \leq -\alpha^2(t), \]

therefore \( \alpha(t) \leq (t + \alpha^{-1}(t_0))^{-1} \), and in the case \( \alpha(t_0) < 0 \) we can see that \( \alpha(t) \to -\infty \) as \( t \to -\alpha^{-1}(t_0) \). Since (4.3) and (4.4) result in \( \alpha'(t) < -s(t)/\beta_0 \leq -\epsilon < 0 \) (the latter inequality follows from the uniqueness theorem), \( \alpha(t) \) becomes negative in a finite time. The proof is complete.

We see that in the presence of a stationary singularity in the component of density, a balance between velocity and pressure arises. Generically the velocity and pressure blow up in a finite time. Thus, the components of this ‘black hole’ solution can collapse at different moments of time. The singularity of density in the origin is integrable for the dimension \( n \geq 2 \).

Let us remark that for usual gas dynamics the solutions of such kind with a ‘linear profile’ of velocity are well investigated (e.g. [15], chapter IV, section 15).

It is worth mentioning that there are papers where steady-state solutions to equations related to granular hydrodynamics are obtained. For example, in [1] it is shown that the Boltzmann equation for smooth inelastic hard spheres admits a steady state characterized by
uniform pressure and linear temperature profile. It seems to contradict our result concerning
the nonexistence of a nontrivial steady state. Nevertheless, we are dealing with a different
object. The system under consideration was derived from the Boltzmann equation under a
number of assumptions. In particular, we neglect the viscous and heat conduction terms. If
those terms were included, a variety of time independent solutions would be obtained.

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