LARGE TIME BEHAVIOUR OF SOLUTIONS TO THE 3D-NSE IN $X^\sigma$ SPACES

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Abstract. In this paper we study the incompressible Navier-Stokes equations in $L^2(\mathbb{R}^3) \cap X^{-1}(\mathbb{R}^3)$. In the global existence case, we establish that if the solution $u$ is in the space $C([0,\infty),L^2(\mathbb{R}^3) \cap X^{-1}(\mathbb{R}^3))$, then for $\sigma > -3/2$ the decay of $\|u(t)\|_{X^\sigma}$ is at least of the order of $t^{-\sigma+3/2}$.

Fourier analysis and standard techniques are used.

Contents

1. Introduction 1
2. Notations and preliminary results 4
   2.1. Notations 4
   2.2. Preliminary results 4
3. Well posedness results in $L^2(\mathbb{R}^3) \cap X^{-1}(\mathbb{R}^3)$ 7
4. Proof of Theorem 1.6 11
5. Long time decay in $X^\sigma$ 14
References 15

1. Introduction

The 3D incompressible Navier-Stokes equations are given by:

$$
\begin{cases}
\partial_t u - \nu \Delta u + u \cdot \nabla u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) = u_0(x) & \text{in } \mathbb{R}^3,
\end{cases}
$$

where $\nu > 0$ is the viscosity of fluid, $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and $(u, \nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, while $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is an initial given velocity. If $u^0$ is quite regular, the divergence free condition determines the pressure $p$.

The Navier-Stokes system has the following scaling property: If $u = u(x, t)$ is a solution of (NS) with initial data $u^0 = u^0(x)$ on the interval $[0, T]$, then for all $\lambda > 0$, $u_\lambda = \lambda u(\lambda^2 t, \lambda x)$ is a solution of (NS) with initial data $u_\lambda(0, x) = \lambda u^0(\lambda x)$ on the interval $[0, T/\lambda^2]$. A functional space $(X, \| \cdot \|_X)$ is called critical space of (NS) system if

$$
\| f_\lambda \|_X = \| f \|_X; \quad \forall \lambda > 0, \quad \forall f \in X,
$$

where

$$
f_\lambda(x) = \lambda f(\lambda x).
$$

Particularly, $L^3(\mathbb{R}^3)$, $H^{1/2}(\mathbb{R}^3)$ and $X^{-1}(\mathbb{R}^3)$ are critical spaces for the system (NS). In order to explain the idea of studying the (NS) system in the space $L^2 \cap X^{-1}$, we introduce the following

2000 Mathematics Subject Classification. 35-xx, 35Bxx, 35Lxx.

Key words and phrases. Navier-Stokes Equations; Critical spaces; Long time decay.
notation: Two functional spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) are called "have the same scaling" if, there is a real number \(\alpha\) such that
\[
\|f_x\|_X = \lambda^\alpha \|f\|_X, \ \forall (\lambda, f) \in (0, \infty) \times X
\]
\[
\|g_x\|_Y = \lambda^\alpha \|g\|_Y, \ \forall (\lambda, g) \in (0, \infty) \times Y.
\]

In this case we note \(X \approx Y\). For example:
\[
\dot{H}^s(\mathbb{R}^3) \approx L^p(\mathbb{R}^3), \ \frac{1}{p} + \frac{s}{2} = \frac{1}{2}, \ 0 \leq s < 3/2
\]
\[
\mathcal{X}^s(\mathbb{R}^3) \approx \dot{H}^{s+\frac{2}{3}}(\mathbb{R}^3), \ \forall \sigma \in \mathbb{R}.
\]

The second is a counter-example of two functional spaces that have the same scaling and are not comparable (see [4] for \(\dot{H}^{1/2}(\mathbb{R}^3)\) and \(\mathcal{X}^{-1}(\mathbb{R}^3)\)). Now, We are ready to give the motivation for this work: Inspired by the works [9], [3] and [7] where they proved the decay results of a global solution of \((NS)\) in homogeneous Sobolev spaces by starting from the \(\dot{H}^s = L^2 \cap \dot{H}^s\) solutions. Here we study the Navier-Stokes system \((NS)\) starting from the \(L^2 \cap \mathcal{X}^{-1}\) solutions and proving some optimal decay results. Our first result is the following.

**Theorem 1.1.** Let \(u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)\) be a divergence free vector fields, then there is a time \(T > 0\) and unique solution \(u \in \mathcal{C}([0, T], \mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))\). Moreover \(u \in L^1([0, T], \mathcal{X}^1(\mathbb{R}^3))\). If \(\|u^0\|_{\mathcal{X}^{-1}} < \nu\), then \(u\) is global.

**Remark 1.2.** (i) If the maximal time \(T^*\) is finite then \(\int_{0}^{T^*} \|u(t)\|_{\mathcal{X}^1} = +\infty\). Indeed: The integral form of the system \((NS)\):
\[
u(t) = e^{t\Delta}u^0 - \int_{0}^{t} e^{(t-z)\Delta} P(u, \nabla u) dz
\]
implies
\[
\|u(t)\|_{L^2} \leq \|e^{t\Delta}u^0\|_{L^2} + \int_{0}^{t} \|e^{(t-z)\Delta} P(u, \nabla u)\|_{L^2} dz
\]
\[
\leq \|u^0\|_{L^2} + \int_{0}^{t} \|u\nabla u\|_{L^2} dz
\]
\[
\leq \|u^0\|_{L^2} + \int_{0}^{t} \|u\|_{L^2} \|\nabla u\|_{L^\infty} dz.
\]
Using the fact \(\|\nabla u\|_{L^\infty} \leq (2\pi)^{-3}\|u\|_{\mathcal{X}^1}\) and Gronwall lemma we get
\[
(1.1) \quad \|u(t)\|_{L^2} \leq \|u^0\|_{L^2} \exp\left((2\pi)^{-3} \int_{0}^{t} \|u\|_{\mathcal{X}^1}\right).
\]
Then, if \(\int_{0}^{T^*} \|u\|_{\mathcal{X}^1}\) is finite we get \(u \in \mathcal{C}([0, T^*], L^2 \cap \mathcal{X}^{-1}) \cap L^\infty([0, T^*], L^2 \cap \mathcal{X}^{-1})\). Then the solution lives beyond the time \(T^*\) which contradicts the fact that \(T^*\) is the maximum time of existence.

(ii) If \(\|u^0\|_{\mathcal{X}^{-1}} < \nu/2\), the above remark and [4] imply the global existence of solution \(u\) of \((NS)\) with \(u \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{-1}) \cap L^1(\mathbb{R}^+, \mathcal{X}^1) \cap \mathcal{C}(\mathbb{R}^+, L^2)\). Moreover,
\[
(1.2) \quad \|u(t)\|_{\mathcal{X}^{-1}} + \frac{\nu}{2} \int_{0}^{t} \|u\|_{\mathcal{X}^1} \leq \|u^0\|_{\mathcal{X}^{-1}}, \ \forall t \geq 0.
\]

(iii) Using (i)-(ii) and [4], we get if \(u \in \mathcal{C}(\mathbb{R}^+, L^2 \cap \mathcal{X}^{-1})\) is a global solution of \((NS)\), then \(u \in L^1(\mathbb{R}^+, \mathcal{X}^1(\mathbb{R}^3))\).

(iv) Using (i)-(ii)-(iii) and [4], we get if \(u \in \mathcal{C}(\mathbb{R}^+, L^2 \cap \mathcal{X}^{-1})\) is a global solution of \((NS)\), then \(u \in \mathcal{C}_b(\mathbb{R}^+, L^2(\mathbb{R}^3))\). Indeed: By [4] there is a time \(t_0 \geq 0\) such that \(\|u(t_0)\|_{\mathcal{X}^{-1}} < \nu/2\). Then
Theorem 1.4. (see [4]) Let \( u \in C(\mathbb{R}^+, X^{-1}(\mathbb{R}^3)) \) be a global solution of Navier-Stokes system. Then
\[
\lim_{t \to \infty} \|u(t)\|_{X^{-1}} = 0.
\]

Theorem 1.5. (see [5]) For any initial data \( u^0 \in H^s(\mathbb{R}^3) \) with \( \text{div} u^0 = 0 \), there exists a unique solution \( u \in C([0, T_0], H^s(\mathbb{R}^3)) \) such that \( T_0 = T_0(s, \|u^0\|_{H^s}) \).

Our second result is the following.

Theorem 1.6. Let \( u \in C(\mathbb{R}^+, X^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \) be a global solution of Navier-Stokes system. Then
\[
\lim_{t \to +\infty} \|u(t)\|_{X^{-1} \cap L^2} = 0,
\]

Precisely,
\[
\|u(t)\|_{X^{-1}} = o(t^{-\frac{3}{2}}); \quad t \to +\infty.
\]

Using theorem 1.6 and theorem 1.4 which characterizes the regularizing effect of the Navier-Stokes equations, we get the following decay result of \( \|u(t)\|_{X^s} \).

Corollary 1.7. Let \( u \in C(\mathbb{R}^+, X^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \) be a global solution of Navier-Stokes system.
Then, for all \( \sigma > -3/2 \), we have \( u \in C((0, +\infty), X^\sigma) \) and
\[
\|u(t)\|_{X^\sigma} = o(t^{-\frac{\sigma+3}{2}}); \quad t \to +\infty.
\]
2. Notations and preliminary results

2.1. Notations. In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as
  \[ \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix.\xi} f(x) dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \]
- The inverse Fourier formula is
  \[ \mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix.\xi} g(\xi) d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3. \]
- The convolution product of a suitable pair of function \( f \) and \( g \) on \( \mathbb{R}^3 \) is given by
  \[ (f * g)(x) := \int_{\mathbb{R}^3} f(y) g(x-y) dy. \]
- If \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \) are two vector fields, we set
  \[ f \otimes g := (g_1 f_1, g_2 f_2, g_3 f_3), \]
  and
  \[ \text{div} (f \otimes g) := (\text{div} (g_1 f_1), \text{div} (g_2 f_2), \text{div} (g_3 f_3)). \]

Moreover, if \( \text{div} g = 0 \) we obtain
\[ \text{div} (f \otimes g) := g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f := g \nabla f. \]
- Let \((B, \|.,\|_B)\), be a Banach space, \(1 \leq p \leq \infty\) and \(T > 0\). We define \( L^p_T(B) \) the space of all measurable functions \([0, t] \ni t \mapsto f(t) \in B \) such that \( t \mapsto \| f(t) \|_B \in L^p([0, T]) \).
- The Sobolev space \( H^s(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^3) \} \).
- The homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); \hat{f} \in L^1_{\text{loc}} \text{ and } |\xi|^s \hat{f} \in L^2(\mathbb{R}^3) \} \).
- The Lei-Lin space \( X^\sigma(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); \hat{f} \in L^1_{\text{loc}} \text{ and } |\xi|^\sigma \hat{f} \in L^1(\mathbb{R}^3) \} \).

2.2. Preliminary results. In this section, we recall some classical results and we give new technical lemmas.

Lemma 2.1. We have \( X^{-1}(\mathbb{R}^3) \cap X^1(\mathbb{R}^3) \hookrightarrow X^0(\mathbb{R}^3) \). Precisely, we have
\[ \| f \|_{X^0(\mathbb{R}^3)} \leq \| f \|_{X^0(\mathbb{R}^3)}^{1/2} \| f \|_{X^1(\mathbb{R}^3)}^{1/2}, \quad \forall f \in X^{-1}(\mathbb{R}^3) \cap X^1(\mathbb{R}^3). \]

Proof. We can write
\[ \| f \|_{X^0} = \int_{\mathbb{R}^3} |\hat{f}(\xi)| d\xi \leq \int_{\mathbb{R}^3} |\xi|^{-1/2} |\hat{f}(\xi)|^{1/2} \| \hat{f}(\xi) \|^{1/2} |\xi|^{1/2} d\xi. \]

Cauchy-Schwarz inequality gives the result.

Lemma 2.2. Let \( \sigma, s \in \mathbb{R} \) such that \( 0 < \sigma + \frac{3}{2} < s \). Then \( H^s(\mathbb{R}^3) \hookrightarrow X^\sigma(\mathbb{R}^3) \). Precisely, there is a constant \( C = C(s, \sigma) \) such that
\[ \| f \|_{X^\sigma(\mathbb{R}^3)} \leq C \| f \|_{L^2(\mathbb{R}^3)}^{\sigma + \frac{3}{2}} \| f \|_{H^s(\mathbb{R}^3)}^{\frac{3}{2} - \sigma}, \quad \forall f \in H^s(\mathbb{R}^3). \]

Proof. For \( \lambda > 0 \), we have
\[ \| f \|_{X^\sigma} = I_\lambda + J_\lambda, \]
with
\[ I_\lambda = \int_{|\xi| < \lambda} |\xi|^\sigma |\hat{f}(\xi)| d\xi, \quad J_\lambda = \int_{|\xi| > \lambda} |\xi|^\sigma |\hat{f}(\xi)| d\xi. \]
We have
\begin{align*}
I_\lambda & \leq \left( \int_{|\xi|<\lambda} |\xi|^{2\gamma} d\xi \right)^{1/2} \|f\|_{L^2} \\
& \leq \frac{1}{\sqrt{2\sigma + 3}} \lambda^{\sigma + \frac{3}{2}} \|f\|_{L^2}
\end{align*}
\begin{align*}
J_\lambda & \leq \left( \int_{|\xi|>\lambda} |\xi|^{2(\sigma-s)} d\xi \right)^{1/2} \|f\|_{H^s} \\
& \leq \frac{C}{\sqrt{s-\sigma}} \lambda^{(\sigma + \frac{3}{2})-s} \|f\|_{H^s}.
\end{align*}

For \( \lambda = (\|f\|_{H^s}/\|f\|_{L^2})^{1/s} \), we obtain the desired result.

**Lemma 2.3.** Let \( \sigma_0 > -3/2 \). If we have
\[ X^{\sigma_0}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \hookrightarrow X^{\sigma}(\mathbb{R}^3); \quad \forall -3/2 < \sigma \leq \sigma_0. \]

Precisely
\[ \|f\|_{X^\sigma} \leq c_0 \|f\|_{L^2}^{1-\theta} \|f\|_{X^{\sigma_0}}, \quad \forall c_0 = c(\sigma_0, \sigma), \quad \theta = \frac{\sigma + \frac{3}{2}}{\frac{3}{2} + \sigma_0}. \]

**Proof.** For \( \lambda > 0 \), we have
\[ \|f\|_{X^\sigma} = A(\lambda) + B(\lambda), \]
with
\begin{align*}
A(\lambda) & = \int_{\{\xi \in \mathbb{R}^3/|\xi|<\lambda\}} |\xi|^\sigma \tilde{f}(\xi) d\xi \\
B(\lambda) & = \int_{\{\xi \in \mathbb{R}^3/|\xi|>\lambda\}} |\xi|^\sigma |\tilde{f}(\xi)| d\xi.
\end{align*}
We have
\begin{align*}
A(\lambda) & \leq \int_{\{\xi \in \mathbb{R}^3/|\xi|<\lambda\}} |\xi|^\sigma d\xi \|f\|_{L^2} \\
& \leq \frac{1}{\sqrt{2\sigma + 3}} \lambda^{\sigma + \frac{3}{2}} \|f\|_{L^2}
\end{align*}
\begin{align*}
B(\lambda) & \leq \int_{\{\xi \in \mathbb{R}^3/|\xi|>\lambda\}} |\xi|^{(\sigma-\sigma_0)} |\xi|^\sigma |\tilde{f}(\xi)| d\xi \\
& \leq \lambda^{\sigma-\sigma_0} \|f\|_{X^{\sigma_0}},
\end{align*}
which imply
\[ \|f\|_{X^\sigma} \leq \frac{e}{\sqrt{2\sigma + 3}} \lambda^{\sigma + \frac{3}{2}} \|f\|_{L^2} + \lambda^{\sigma-\sigma_0} \|f\|_{X^{\sigma_0}}. \]
For \( \lambda = (\|f\|_{X^{\sigma_0}}/\|f\|_{L^2})^{1/(3/2+\sigma_0)} \), we obtain
\[ \|f\|_{X^\sigma} \leq c_{\sigma, \sigma_0} \|f\|_{L^2}^{\sigma_0-\sigma} \|f\|_{X^{\sigma_0}}^{\frac{\sigma_0-\sigma}{\frac{3}{2}+\sigma_0}}. \]

**Lemma 2.4.** Let \( f, g \in L^\infty_T(\mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{1}(\mathbb{R}^3)) \) such that \( \text{div} \ f = 0 \) almost everywhere. Then
\begin{align*}
(2.4) & \sup_{0 \leq t \leq T} \|\int_0^t e^{(t-s)\Delta} \mathbb{P}(f,\nabla g) dz\|_{X^{-1}} \leq \|f\|_{L^2}^{1/2} \|f\|_{L^1_T(\mathcal{X}^{1})} \|g\|_{L^2_T(\mathcal{X}^{1})}^{1/2}, \\
(2.5) & \sup_{0 \leq t \leq T} \|\int_0^t e^{(t-s)\Delta} \mathbb{P}(f,\nabla g) dz\|_{L^2} \leq (2\pi)^{-3/2} \|f\|_{L^2_T(\mathcal{X}^{-1})} \|g\|_{L^2_T(\mathcal{X}^{1})}, \\
(2.6) & \int_0^T \|\int_0^t e^{(t-s)\Delta} \mathbb{P}(f,\nabla g) dz\|_{X^{1}} dt \leq \nu^{-1} \|f\|_{L^2}^{1/2} \|f\|_{L^1_T(\mathcal{X}^{1})} \|g\|_{L^2_T(\mathcal{X}^{-1})}^{1/2} \|g\|_{L^2_T(\mathcal{X}^{1})}^{1/2}.
\end{align*}
Proof.

- Proof of (2.4): We can write

\[
\| \int_0^t e^{\nu(t-z)} \Delta P(f, \nabla g)dz \|_{X^{-1}} \leq \int_0^T \| e^{\nu(t-z)} \Delta P(f, \nabla g) \|_{X^{-1}} \frac{dz}{dt} \\
\leq \int_0^T \| f, \nabla g \|_{X^{-1}} \frac{dz}{dt} \\
\leq \int_0^T \| \text{div} (f \otimes g) \|_{X^{-1}} \frac{dz}{dt} \\
\leq \int_0^T \| f \otimes g \|_{X^0} \frac{dz}{dt} \\
\leq \int_0^T \| f \|_{X^1} \frac{dz}{dt} \\
\leq \int_0^T \| f \|_{X^2} \frac{dz}{dt} \\
\leq \int_0^T \| f \|_{L^2_T(X^1)} \frac{dz}{dt} \\
\leq \left( \int_0^T \frac{1}{\nu} \| f \|_{L^2_T(X^1)} \right) dt \\
\leq \left( \int_0^T \frac{1}{\nu} \| f \|_{L^2_T(X^1)} \right) dt.
\]

- Proof of (2.5): We can write

\[
\| \int_0^t e^{\nu(t-z)} \Delta P(f, \nabla g)dz \|_{L^1} \leq \int_0^T \| e^{\nu(t-z)} \Delta P(f, \nabla g) \|_{L^1} \frac{dz}{dt} \\
\leq \int_0^T \| f, \nabla g \|_{L^1} \frac{dz}{dt} \\
\leq \int_0^T \| f \otimes g \|_{L^1} \frac{dz}{dt} \\
\leq \int_0^T \| f \|_{L^2_T(X^1)} \| g \|_{L^1_T(X^2)} \| f \|_{L^2_T(X^2)} \| g \|_{L^1_T(X^2)} \| f \|_{L^2_T(X^2)} \| g \|_{L^1_T(X^2)} \| f \|_{L^2_T(X^2)} \| g \|_{L^1_T(X^2)}.
\]

- Proof of (2.6): We can write

\[
\int_0^T \| \int_0^t e^{\nu(t-z)} \Delta P(f, \nabla g)dz \|_{X^1} dt \\
\leq \int_0^T \int_0^T \| e^{\nu(t-z)} \Delta P(f, \nabla g) \|_{X^1} dzdt \\
\leq \int_0^T \int_0^T \| e^{\nu(t-z)} \Delta P(f, \nabla g) \|_{X^1} dzdt \\
\leq \left( \int_0^T \frac{1}{\nu} \| f \|_{L^2_T(X^1)} \right) dt \\
\leq \left( \int_0^T \frac{1}{\nu} \| f \|_{L^2_T(X^1)} \right) dt.
\]

Lemma 2.5. Let \( T > 0 \) and \( f : [0, T] \to \mathbb{R}_+ \) be continuous function such that

\[
(2.7) \quad f(t) \leq M_0 + \theta_1 f(\theta_2 t); \quad \forall 0 \leq t \leq T.
\]

with \( M_0 \geq 0 \) and \( \theta_1, \theta_2 \in (0, 1) \). Then

\[
f(t) \leq \frac{M_0}{1 - \theta_1}; \quad \forall 0 \leq t \leq T.
\]

Proof. As \( f \) is a positive and continuous function, then there is a time \( t_0 \in [0, T] \) such that

\[
0 \leq f(t_0) = \max_{0 \leq t \leq T} f(t).
\]
Applying (3.5) at \( t = t_0 \) we get
\[
f(t_0) \leq M_0 + \theta_1 f(\theta_2 t_0) \leq M_0 + \theta_1 f(t_0)
\]
which implies \( f(t_0) \leq \frac{M_0}{1 - \theta_1} \). As \( f(t_0) = \max_{0 \leq t \leq T} f(t) \), we get the desired result.

**Remark 2.6.** Applying Lemma 2.5 to a positive continuous function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfying
\[
f(t) \leq M_0 + \theta_1 f(\theta_2 t) ; \quad \forall t \geq 0
\]
with \( M_0 \geq 0 \) and \( \theta_1, \theta_2 \in (0, 1) \), we obtain
\[
\limsup_{t \rightarrow +\infty} f(t) \leq \frac{M_0}{1 - \theta_1}.
\]

3. **Well posedness results in \( L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3) \)**

In this section we prove Theorem 1.1. To prove the existence result we need the following remark:
For \( f \in L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3) \) and \( \varepsilon_0 > 0 \) there is \( \lambda > 0 \) such that
\[
\| \lambda f(\lambda) \|_{\mathcal{X}^{-1}} = \| f \|_{\mathcal{X}^{-1}} \quad \text{and} \quad \| \lambda f(\lambda) \|_{L^2} < \varepsilon_0.
\]

Precisely, just take \( \lambda = \frac{\varepsilon_0^2}{4 \| f \|_{L^2}^2 + 1} \). Then we can choose \( \lambda_0 > 0 \) such that
\[
\| \lambda_0 u^0(\lambda_0) \|_{\mathcal{X}^{-1}} = \| u^0 \|_{\mathcal{X}^{-1}} \quad \text{and} \quad \| \lambda_0 u^0(\lambda_0) \|_{L^2} < \frac{1}{48}.
\]

Consider then the Navier-Stokes system
\[
(S_{\lambda_0}) \begin{cases}
\partial_t v - \nu \Delta v + v \cdot \nabla v = -\nabla q & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } v = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
v(0, x) = \lambda_0 u^0(x) & \text{in } \mathbb{R}^3.
\end{cases}
\]

If the system \( (NS_{\lambda_0}) \) has a unique solution \( v \) in \( C([0, T], L^2 \cap \mathcal{X}^{-1}) \), then \( u = \lambda_0^{-1} v(\lambda_0^{-2} t, \lambda_0^{-1} x) \) is a solution of Navier-Stokes system starting by \( u^0 \). Therefore, we can assume in the following that
\[
\| u^0 \|_{L^2} < \frac{1}{48}
\]

Let’s go back to the proof of Theorem 1.1. A uniqueness in \( L^2 \cap \mathcal{X}^{-1} \) is given by the uniqueness in \( \mathcal{X}^{-1} \). (see [8]). It remains a proven existence, for this let \( k_0 \in \mathbb{N}^+ \) such that
\[
\int_{\{ \xi \in \mathbb{R}^3 / |\xi| > k_0 \}} \frac{|\hat{u}^0(\xi)|}{|\xi|} d\xi < \min\left(\frac{\nu}{16}, \frac{1}{16}\right).
\]

Put
\[
a^0 = F^{-1}(1_{|\xi| < k} \hat{u}^0(\xi))
\]
\[
b^0 = F^{-1}(1_{|\xi| \geq k} \hat{u}^0(\xi)).
\]

We have \( a^0 \in H^s(\mathbb{R}^3) \), (for all \( s \geq 0 \))
\[
a^0 \in H^s(\mathbb{R}^3), \quad \forall s \geq 0,
\]
\[
\| b^0 \|_{\mathcal{X}^{-1}} < \min\left(\frac{\nu}{16}, \frac{1}{16}\right).
\]

Moreover
\[
\| a^0 \|_{L^2} \leq \| u^0 \|_{L^2} \quad \text{and} \quad \| b^0 \|_{L^2} \leq \| u^0 \|_{L^2}.
\]

There is a time \( T_0 > 0 \) such that the system \( (NS) \) has a unique solution \( a \) in \( C([0, T_0], H^4(\mathbb{R}^3)) \) with initial condition \( a^0 \) (see [8]). Using the fact (see Lemma 2.2)
\[
H^4(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3),
\]

we get
\[
a \in C([0, T_0], L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3)).
\]
Using the regularity of the function $a$ and inequality (3.4), we obtain

$$
(3.6) \quad \|a(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla a(z)\|_{L^2}^2 = \|a(0)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2, \forall t \in [0, T_0].
$$

Put $b = a - a$, $b$ satisfies the following system

$$
(RNS) \quad \begin{cases}
\partial b - \nu \Delta b + b \nabla b + b \nabla a + a \nabla b = -\nabla q \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
b(0, x) = b^0(x) \text{ in } \mathbb{R}^3.
\end{cases}
$$

The integral form of $(RNS)$ is

$$
b = \psi(b) = e^{\nu t} \Delta b^0 - \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(a, \nabla b) - \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(b, \nabla a) - \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(b, \nabla b).
$$

Put

$$
f_0 = e^{\nu t} \Delta b^0
$$

$$
L(b) = -\int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(a, \nabla b) - \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(b, \nabla a)
$$

$$
Q(b) = -\int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(b, \nabla b).
$$

For $T > 0$ put the space

$$
X_T = C([0, T], L^2(\mathbb{R}^3) \cap X^{-1}(\mathbb{R}^3)) \cap L^1([0, T], X^1(\mathbb{R}^3)).
$$

This vector space is equipped with the norm

$$
\|f\|_{X, T} = \|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R}^3)}.
$$

For $\epsilon, T > 0$ (fixed later), such that $T \leq T_0$, put the closed subset of $X_T$ defined by

$$
B(\epsilon, T) = \left\{ f \in X_T : \begin{array}{l}
\|f\|_{L^2(\mathbb{R}^3)} \leq 2\|b^0\|_{L^2} \\
\|f\|_{L^2(\mathbb{R}^3)} \leq 2\|b^0\|_{X^{-1}} \\
\|f\|_{L^1(\mathbb{R}^3)} \leq \epsilon
\end{array} \right\}
$$

Explanation of the choice of $\epsilon$ and $T$ : We have

$$
\|f_0\|_{L^\infty(\mathbb{R}^3)} \leq \|b^0\|_{X^{-1}}
$$

$$
\|f_0\|_{L^2(\mathbb{R}^3)} \leq \|b^0\|_{L^2}
$$

$$
\|f_0\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} e^{-\nu t}\|\xi\|^2|\hat{\psi}(\xi)|d\xi = \int_{\mathbb{R}^3} \left(\int_0^T e^{-\nu t}\|\xi\|^2 dt\right)|\xi||\hat{\psi}(\xi)|d\xi
$$

$$
= \int_{\mathbb{R}^3} \left(1 - e^{\nu T}\|\xi\|^2\right)|\xi||\hat{\psi}(\xi)|d\xi
$$

$$
= \int_{\mathbb{R}^3} (1 - e^{\nu T}\|\xi\|^2)|\hat{\psi}(\xi)|d\xi.
$$

Dominates Convergence Theorem implies

$$
(3.7) \quad \lim_{t \to T_0^+} \|f_0\|_{L^1(\mathbb{R}^3)} = 0.
$$

Let $0 < \epsilon < 1/24$ and $0 < T \leq T_0$ such that

(C1) \[\|a\|_{L^2(\mathbb{R}^3)} \leq \nu^{-1/2} \|a\|_{L^1(\mathbb{R}^3)} \leq \frac{\|b^0\|_{X^{-1}}}{4}\]

(C2) \[\|a^0\|_{L^2(\mathbb{R}^3)} + 2\|a\|_{L^2(\mathbb{R}^3)} \leq \frac{\|b^0\|_{L^2}}{4}\]

(C3) \[\nu^{-1/2} \|a\|_{L^2(\mathbb{R}^3)} \leq \frac{\|b^0\|_{X^{-1}}}{2}\]

(C4) \[\epsilon + 2\|b^0\|_{L^2} \leq 1/12\]

(C5) \[(1 + \nu^{-1/2}) \|b^0\|_{X^{-1}} \leq 1/12\]

(C6) \[\|f_0\|_{L^1(\mathbb{R}^3)} \leq \epsilon/3\]

(C7) \[2\sqrt{\|b^0\|_{X^{-1}}} \leq 1/12\]

(C8) \[\|a\|_{L^2(\mathbb{R}^3)} \leq 1/12\]

(C9) \[\|a\|_{L^1(\mathbb{R}^3)} \leq 1/24.\]
These choices are possible just use the equations \[3.3, 3.5, 3.7\]. Now we want to prepare to apply the Fixed Point Theorem, for this we prove the following

\begin{equation}
\psi(B(\varepsilon, T)) \subset B(\varepsilon, T).
\end{equation}

\begin{equation}
\|\psi(\alpha_1) - \psi(\alpha_2)\|_{\varepsilon, T} \leq \frac{1}{2} \|\alpha_1 - \alpha_2\|_{\varepsilon, T}, \forall \alpha_1, \alpha_2 \in B(\varepsilon, T).
\end{equation}

**Proof of (3.8):** Using inequality (2.4), we obtain

\begin{align*}
\|L(b)\|_{L^\infty X^{-1}} & \leq \|a\|_{L^\infty X^{-1}}^{1/2} \|a\|_{L^1 X^1}^{1/2} \|b\|_{L^1 X^{-1}}^{1/2} \\
& \leq \|a\|_{L^\infty X^{-1}}^{1/2} \|a\|_{L^1 X^1} \sqrt{2\varepsilon \|b\|_{X^{-1}}^{1/2}} \\
& \leq \frac{\|b\|_{X^{-1}}}{4}, \text{ (by (C1))}
\end{align*}

\begin{align*}
\|Q(b)\|_{L^\infty X^{-1}} & \leq \|b\|_{L^\infty X^{-1}} \|b\|_{L^1 X^1} \\
& \leq 2\varepsilon \|b\|_{X^{-1}} \\
& \leq \frac{\|b\|_{X^{-1}}}{4}.
\end{align*}

Then

\begin{equation}
\|\psi(b)\|_{L^\infty X^{-1}} \leq 2\|b\|_{X^{-1}}, \forall b \in B(\varepsilon, T).
\end{equation}

Similarly, inequality (2.5) gives

\begin{align*}
\|L(b)\|_{L^\infty L^2} & \leq \|a\|_{L^\infty L^2} \|b\|_{L^1 L^1} + \|b\|_{L^\infty L^2} \|a\|_{L^1 L^1} \\
& \leq \|a\|_{L^\infty L^2} + 2\|a\|_{L^1 L^1} \|b\|_{L^2} \\
& \leq \frac{\|b\|_{L^2}}{4}, \text{ (by (C2))}
\end{align*}

\begin{align*}
\|Q(b)\|_{L^\infty L^2} & \leq \|b\|_{L^\infty L^2} \|b\|_{L^1 L^1} \\
& \leq 2\varepsilon \|b\|_{L^2} \\
& \leq \frac{\|b\|_{L^2}}{4}.
\end{align*}

Then

\begin{equation}
\|\psi(b)\|_{L^\infty L^2} \leq 2\|b\|_{L^2}, \forall b \in B(\varepsilon, T).
\end{equation}

Finally, inequality (2.6) gives

\begin{align*}
\|L(b)\|_{L^1 X^1} & \leq \nu^{-1} \|a\|_{L^\infty X^{-1}}^{1/2} \|a\|_{L^1 X^1}^{1/2} \|b\|_{L^1 X^{-1}}^{1/2} \|b\|_{L^1 X^{-1}}^{1/2} \\
& \leq \nu^{-1} \|a\|_{L^\infty X^{-1}}^{1/2} \|a\|_{L^1 X^1} \sqrt{2\varepsilon \|b\|_{X^{-1}}^{1/2}} \\
& \leq \frac{\varepsilon}{3}, \text{ (by (C3))}
\end{align*}

\begin{align*}
\|Q(b)\|_{L^1 X^1} & \leq \nu^{-1} \|b\|_{L^\infty X^{-1}} \|b\|_{L^1 X^1} \\
& \leq 2\nu^{-1} \|b\|_{X^{-1}} \\
& \leq \frac{\varepsilon}{3}, \text{ (by (3.3)).}
\end{align*}

Then,

\begin{equation}
\|\psi(b)\|_{L^1 X^1} \leq \varepsilon, \forall b \in B(\varepsilon, T).
\end{equation}
Therefore inequalities (3.13) - (3.14) imply (3.12).

**Proof of (3.9):** Using inequality (2.4), we obtain

\[
\|L(\alpha_1) - L(\alpha_2)\|_{L^p_t(X)} \leq \|L(\alpha_1 - \alpha_2)\|_{L^p_t(X)} \\
\leq \|a\|_{L^p_t(X)} \|a\|_{L^q_t(X)}\|\alpha_1 - \alpha_2\|_{L^1_t(L^q(X))} \\
\leq \|a\|_{L^p_t(X)} \|a\|_{L^q_t(X)}\|\alpha_1 - \alpha_2\|_{\varepsilon,T} \\
\leq \frac{1}{\varepsilon}\|\alpha_1 - \alpha_2\|_{\varepsilon,T}, \quad \text{(by (C9))}
\]

Then

\[
(3.13) \quad \|\psi(\alpha_1) - \psi(\alpha_2)\|_{L^p_t(X)} \leq \frac{1}{\varepsilon}\|\alpha_1 - \alpha_2\|_{\varepsilon,T}, \forall \alpha_1, \alpha_2 \in B(\varepsilon, T).
\]

Similarly, inequality (2.5) gives

\[
\|L(\alpha_1) - L(\alpha_2)\|_{L^2_t(L^2)} \leq \|L(\alpha_1 - \alpha_2)\|_{L^2_t(L^2)} \\
\leq \|a\|_{L^2_t(L^2)} \|a\|_{L^q_t(X)}\|\alpha_1 - \alpha_2\|_{L^1_t(L^q(X))} \\
\leq \left(\|a\|_{L^2_t(L^2)} + \|a\|_{L^1_t(X)}\right)\|\alpha_1 - \alpha_2\|_{\varepsilon,T} \\
\leq \left(\|a\|_{L^2_t(L^2)} + \|a\|_{L^1_t(X)}\right)\|\alpha_1 - \alpha_2\|_{\varepsilon,T} \\
\leq \frac{1}{\varepsilon}\|\alpha_1 - \alpha_2\|_{\varepsilon,T}, \quad \text{(by (C9))}
\]

Then

\[
(3.14) \quad \|\psi(\alpha_1) - \psi(\alpha_2)\|_{L^p_t(L^2)} \leq \frac{1}{\varepsilon}\|\alpha_1 - \alpha_2\|_{\varepsilon,T}, \forall \alpha_1, \alpha_2 \in B(\varepsilon, T).
\]
Finally, inequality (2.10) gives
\[
\|L(\alpha_1) - L(\alpha_2)\|_{L^1_t(X^1)} = \|L(\alpha_1) - L(\alpha_2)\|_{L^1_t(X^1)} \\
\leq \nu^{-1}\|a\|_{L^2_t(\chi^{-1})}\|a\|_{L^1_t(X^1)}^{1/2}\|\alpha_1 - \alpha_2\|_{L^2_t(\chi^{-1})}^{1/2}\|\alpha_1 - \alpha_2\|_{L^1_t(X^1)}^{1/2} \\
\leq \nu^{-1}\|a\|_{L^2_t(\chi^{-1})}\|a\|_{L^1_t(X^1)}^{1/2}\|\alpha_1 - \alpha_2\|_{\varepsilon, T}^{1/2} \\
\leq \frac{1}{12}\|\alpha_1 - \alpha_2\|_{\varepsilon, T}.
\]

\[
\|Q(\alpha_1) - Q(\alpha_2)\|_{L^1_t(X^1)} = \|\int_0^t e^{-\nu(t-s)}\Delta\mathbb{P}((\alpha_1 - \alpha_2)\nabla\alpha_1 + \alpha_2\nabla(\alpha_1 - \alpha_2))\|_{L^1_t(X^1)} \\
\leq \nu^{-1}\left(\sum_{i=1}^2\|\alpha_i\|_{L^2_t(\chi^{-1})}\|\alpha_i\|_{L^1_t(X^1)}^{1/2}\right)\|\alpha_1 - \alpha_2\|_{L^2_t(\chi^{-1})}\|\alpha_1 - \alpha_2\|_{L^1_t(X^1)}^{1/2} \\
\leq \nu^{-1}\left(\sum_{i=1}^2\|\alpha_i\|_{L^2_t(\chi^{-1})}\|\alpha_i\|_{L^1_t(X^1)}^{1/2}\right)\|\alpha_1 - \alpha_2\|_{\varepsilon, T} \\
\leq \nu^{-2}\|\mathbb{P}\|_{\chi^{-1}}\|\alpha_1 - \alpha_2\|_{\varepsilon, T} \\
\leq \frac{1}{12}\|\alpha_1 - \alpha_2\|_{\varepsilon, T}.
\]

Then,
\[
(3.15) \quad \|\psi(\alpha_1) - \psi(\alpha_2)\|_{L^1_t(X^1)} \leq \frac{1}{2}\|\alpha_1 - \alpha_2\|_{\varepsilon, T}, \forall \alpha_1, \alpha_2 \in B(\varepsilon, T).
\]

Therefore inequalities (3.13) - (3.11) - (3.12) gives
\[
(3.16) \quad \|\psi(\alpha_1) - \psi(\alpha_2)\|_{B(\varepsilon, T)} \leq \frac{1}{2}\|\alpha_1 - \alpha_2\|_{B(\varepsilon, T)}, \forall \alpha_1, \alpha_2 \in B(\varepsilon, T).
\]

Fixed Point Theorem gives the existence and uniqueness of solution of (RNS) in $C_T(L^2 \cap \chi^{-1}) \cap L^1_t(X^1)$. Therefore, we can deduce the existence and uniqueness of a local solution for Navier-Stokes system.

4. Proof of Theorem 1.6

**Proof of (1.3) :** In this subsection we want to prove the long time decay in $L^2 \cap \chi^{-1}$. Let $u \in C(\mathbb{R}^+; L^2 \cap \chi^{-1})$ be global solution of (NS). By [3] we have
\[
\lim_{t \to \infty} \|u(t)\|_{\chi^{-1}} = 0.
\]

Now, prove that $\lim_{t \to \infty} \|u(t)\|_{L^2} = 0$. For a strictly positive real number $\delta$ and a given distribution $f$, we define the operators $A_\delta(D)$ and $B_\delta(D)$, respectively, by the following:
\[
A_\delta(D)f = \mathcal{F}^{-1}(1_{\{|\xi|<\delta\}}\hat{f}), \\
B_\delta(D)f = \mathcal{F}^{-1}(1_{\{|\xi|>\delta\}}\hat{f}).
\]

Let $u$ be a solution of (NS). Denote by $w_3 = A_\delta(D)u$ and $v_3 = B_\delta(D)u$, respectively, the low-frequency part and the high-frequency part of $u$ and so on $\xi_3$ and $\xi_3^1$ for the initial data $u^0$. Applying the pseudo-differential operator $A_\delta(D)$ to the (NS), we get
\[
(4.1) \quad \partial_t w_3 - \nu \Delta w_3 + A_\delta(D)\mathbb{P}(u, \nabla u) = 0
\]

Taking the $L^2(\mathbb{R}^3)$-inner product and using the fact $A_\delta(D)^2 = A_\delta(D)$, we obtain
\[
\frac{1}{2} \frac{d}{dt}\|w_3(t)\|_{L^2}^2 + \nu\|\nabla w_3(t)\|_{L^2}^2 \\
= \langle A_\delta(D)\mathbb{P}(u, \nabla u)(t)/w_3(t)\rangle_{L^2} \\
= \langle \mathbb{P}(u, \nabla u)(t)/A_\delta(D)w_3(t)\rangle_{L^2} \\
= \langle \mathbb{P}(u, \nabla u)(t)/w_3(t)\rangle_{L^2} \\
= \langle u, \nabla u(t)/w_3(t)\rangle_{L^2} \\
= \langle (\text{div}(u \otimes u))(t)/w_3(t)\rangle_{L^2} \\
= \langle u \otimes u(t)/\nabla w_3(t)\rangle_{L^2} \\
= \|u \otimes u(t)\|_{L^1}\|\nabla w_3(t)\|_{L^\infty} \\
= (2\pi)^{-3}\|u(t)\|_{L^2}^2\|w_3(t)\|_{\chi^1}.
\]
Integrating with respect to time and using Remark 1.2-(iv), we obtain
\[ \|w_\delta(t)\|_{L^2}^2 \leq \|w_\delta^0\|_{L^2}^2 + m_0 \int_0^t \|w_\delta(s)\|_{X^1} ds, \]
where \( m_0 = (2\pi)^{-3} \|u\|_{L^\infty(\mathbb{R}^+, L^2)} \). Also using Remark 1.2-(iii) we get \( \|w_\delta(t)\|_{L^2}^2 \leq M_\delta \), where
\[ M_\delta = \|w_\delta^0\|_{L^2}^2 + m_0 \int_0^\infty \|w_\delta(s)\|_{X^1} ds. \]
On the one hand, it is clear that \( \lim_{\delta \to 0} \|w_\delta(t)\|_{L^2}^2 = 0 \). On the other, we have \( \lim_{\delta \to 0} \|w_\delta(t)\|_{X^1} = 0 \) and \( \|w_\delta(t)\|_{X^1} \leq \|u(t)\|_{X^1} \in L^1([0, \infty)) \). Then Dominating Convergence Theorem implies that
\[ \lim_{\delta \to 0} \int_0^\infty \|w_\delta(s)\|_{X^1} ds = 0. \]
Hence, \( \lim_{\delta \to 0} M_\delta = 0 \) and thus
\[ \lim_{\delta \to 0} \sup_{t \geq 0} \|w_\delta(t)\|_{L^2}^2 = 0. \]

Let us investigate the high-frequency part. To do so, one applies the pseudo-differential operator \( B_\delta(D) \) to the \((NS)\) to get
\[ \partial_t v_\delta - \nu \Delta v_\delta + B_\delta(D)P(u, \nabla u) = 0. \]
The integral form of \( v_\delta \) is
\[ v_\delta(t) = e^{\nu t \Delta} v_\delta^0 - \int_0^t e^{\nu(t-\tau) \Delta} B_\delta(D)P(u, \nabla u) d\tau. \]
Taking the \( L^2(\mathbb{R}^3) \) norm, we obtain
\[ \|v_\delta(t)\|_{L^2} \leq \|e^{\nu t \Delta} v_\delta^0\|_{L^2} + \int_0^t \|e^{\nu(t-\tau) \Delta} B_\delta(D)P(u, \nabla u)\|_{L^2} d\tau \]
\[ \leq e^{-\nu t \Delta^2} \|v_\delta^0\|_{L^2} + \int_0^t e^{-\nu(t-\tau) \Delta^2} \|u\|_{L^2} d\tau \]
\[ \leq e^{-\nu t \Delta^2} \|v_\delta^0\|_{L^2} + \int_0^t e^{-\nu(t-\tau) \Delta^2} \|u\|_{L^2} d\tau. \]

Then
\[ \|v_\delta(t)\|_{L^2} \leq e^{-\nu t \Delta^2} \|v_\delta^0\|_{L^2} + m_0 \int_0^t e^{-\nu(t-\tau) \Delta^2} \|u(\tau)\|_{X^1} d\tau := G_\delta(t). \]
We have
\[ \int_0^\infty G_\delta(t) dt \leq \|v_\delta^0\|_{L^2} + \frac{m_0}{\nu |\Delta|^2} \|u\|_{L^1(\mathbb{R}^+, L^{1+})} < \infty. \]
This leads to the fact that the function \( (t \to \|v_\delta(t)\|_{L^2}) \) is both continuous and Lebesgue integrable over \( \mathbb{R}^+ \). Let \( \varepsilon > 0 \) be positive real number small enough. Firstly, equation (4.2) implies that some \( \delta_\varepsilon > 0 \) exists such that
\[ \|w_\delta(t)\|_{L^2} \leq \varepsilon/2, \quad \forall t \geq 0. \]
Secondly, consider the set \( R_\delta \) defined by
\[ R_\delta := \{ t > 0, \|v_\delta(t)\|_{L^2} > \varepsilon/2 \}. \]
If we denote by \( \lambda_1(R_\delta) \) the Lebesgue measure of \( R_\delta \), we have
\[ \int_0^\infty \|v_\delta(t)\|_{L^2} dt \geq \int_{R_\delta} \|v_\delta(t)\|_{L^2(\mathbb{R}^3)} dt \geq \frac{\varepsilon}{2} \lambda_1(R_\delta). \]
By this, we can deduce that \( \lambda_1(R_\delta) \leq T_\varepsilon \), where \( T_\varepsilon = (2/\varepsilon) \int_0^\infty \|v_\delta(t)\|_{L^2(\mathbb{R}^3)} dt \). Then, there is \( t_\varepsilon \in [0, T_\varepsilon + 1] \) such that \( t_\varepsilon \) does not belong to \( R_\delta \). This implies that
\[ \|v_\delta(t_\varepsilon)\|_{L^2(\mathbb{R}^3)} \leq \varepsilon/2. \]
Equations (4.4) and (4.6) together with triangular inequality imply that \( \|u(t)\|_{L^2} < \varepsilon \). For \( t \geq t_\varepsilon \), we have
\[
\|u(t)\|_{L^2} \leq \|u(t_\varepsilon)\|_{L^2} (2\pi)^{-3} \int_{t_\varepsilon}^\infty \|u(z)\|_{\dot{X}^1} dz \\
\leq \varepsilon (2\pi)^{-3} \int_0^\infty \|u\|_{\dot{X}^1} dz.
\]
It suffices to replace \( \varepsilon \) by \( \varepsilon \exp(-(2\pi)^{-3} \int_0^\infty \|u\|_{\dot{X}^1}) \) in (4.4)–(4.5)–(4.6) we get the desired result.

**Proof of (4.4):** In this subsection we want to give a precision for the decay of \( \|u(t)\|_{\dot{X}^1} \) at \( \infty \). Let \( \varepsilon > 0 \) such that \( \varepsilon < \varepsilon_0(\varepsilon) \) is given by Theorem (4.4), by (1.3) we can suppose that,
\[
\|u_0\|_{\dot{X}^1} < \min(\varepsilon, \frac{\nu}{2}) \text{ and } \|u_0\|_{L^2} < \varepsilon/2.
\]
Then, by Remark (1.2) (ii)-(iv) we get \( \|u(t)\|_{L^2} \leq 2\|u_0\|_{L^2} < \varepsilon \) for all \( t \geq 0 \) and
\[
\|u(t)\|_{\dot{X}^1} + \frac{\nu}{2} \int_0^t \|u(z)\|_{\dot{X}^1} dz \leq \|u_0\|_{\dot{X}^1} < \frac{\nu}{2}, \forall t \geq 0.
\]
For \( \lambda > 0 \) and \( t > 0 \), we have
\[
\|u(t)\|_{\dot{X}^1} = I_\lambda(t) + J_\lambda(t),
\]
with
\[
I_\lambda(t) = \int_{\xi \in \mathbb{R}^3/|\xi| < \lambda} \frac{|\hat{u}(t, \xi)|}{|\xi|} d\xi,
\]
\[
J_\lambda(t) = \int_{|\xi| \geq \lambda} \frac{|\hat{u}(t, \xi)|}{|\xi|} d\xi.
\]
We have
\[
I_\lambda(t) \leq \left( \int_{\xi \in \mathbb{R}^3/|\xi| < \lambda} \frac{1}{|\xi|} d\xi \right)^{1/2} \|\hat{u}\|_{L^2} \leq c_0 \left( \int_0^\lambda d\xi \right)^{1/2} \|\hat{u}\|_{L^2} \leq c_0 \sqrt{\lambda} \|\hat{u}\|_{L^2} \\
\leq c_0 \sqrt{\lambda} \|u_0\|_{L^2} \leq c_1 \sqrt{\lambda} \|u_0\|_{L^2}
\]
and
\[
J_\lambda(t) \leq \int_{|\xi| \geq \lambda} e^{-\sqrt{\nu t/2}|\xi|} |\hat{u}(t, \xi)| |\xi| d\xi \leq e^{-\sqrt{\nu t/2}|\xi|} \int_{|\xi| \geq \lambda} |\hat{u}(t, \xi)| |\xi| d\xi.
\]
For a fixed time \( t > 0 \) the \( v : (z, x) \rightarrow u(\frac{x}{2} + z, x) \) satisfies \( \|v(0)\|_{\dot{X}^1} \leq \varepsilon_0 \) and it is the unique global solution of the following system,
\[
\begin{align*}
\partial_t v - \nu \Delta v + v \cdot \nabla v &= -\nabla q \\
v(0, x) &= u(\frac{x}{2}, x).
\end{align*}
\]
By Theorem (1.4) we get
\[
\int_{\mathbb{R}^3} e^{\sqrt{\nu t/2}|\xi|} |\hat{u}(z, \xi)| \frac{|\xi|}{|\xi|} d\xi + \frac{\nu}{2} \int_0^z \int_{\mathbb{R}^3} e^{\sqrt{\nu t/2}|\xi|} |\hat{u}(s, \xi)| \frac{|\xi|}{|\xi|} d\xi ds \leq 2\|v(0)\|_{\dot{X}^1}
\]
or
\[
\int_{\mathbb{R}^3} e^{\sqrt{\nu t/2}|\xi|} |\hat{u}(z + \xi, \xi)| \frac{|\xi|}{|\xi|} d\xi + \frac{\nu}{2} \int_0^z \int_{\mathbb{R}^3} e^{\sqrt{\nu t/2}|\xi|} |\hat{u}(s + \xi, \xi)| \frac{|\xi|}{|\xi|} d\xi ds \leq 2 \int_{\mathbb{R}^3} |\hat{u}(z, \xi)| \frac{|\xi|}{|\xi|} d\xi.
\]
For \( z = \frac{x}{2} \), we get \( \int_{\xi} e^{\sqrt{\nu t/2}|\xi|} \frac{|\hat{u}(t, \xi)|}{|\xi|} d\xi \leq \|u(t/2)\|_{\dot{X}^1} \), which implies
\[
J_\lambda(t) \leq e^{-\sqrt{\nu t/2}\lambda} \|u(t/2)\|_{\dot{X}^1}.
\]
Then
\[
\|u(t)\|_{\dot{X}^1} \leq c_1 \sqrt{\lambda} \|u_0\|_{L^2} + e^{-\sqrt{\nu t/2}\lambda} \|u(t/2)\|_{\dot{X}^1}.
\]
Multiplying this inequality by \( t^{1/4} \)
\[
t^{1/4} \|u(t)\|_{\dot{X}^1} \leq t^{1/4} c_1 \sqrt{\lambda} \|u_0\|_{L^2} + 2^{1/4} e^{-\sqrt{\nu t/2}\lambda} (\frac{1}{2})^{1/4} \|u(t/2)\|_{\dot{X}^1}.
\]
and choose \( \lambda > 0 \) such that
\[
2^{1/4}e^{-\sqrt{\nu t}/2\lambda} = 1/2 \Rightarrow \sqrt{\nu t}/2\lambda = 5/4 \ln 2 \Rightarrow \lambda = \frac{5\sqrt{2}\ln 2}{4\sqrt{\nu}}
\]
we obtain
\[
t^{1/4}\|u(t)\|_{X^{-1}} \leq M_0 + \frac{1}{2} \left( \frac{t}{2} \right)^{1/4}\|u(t/2)\|_{X^{-1}}
\]
with
\[
M_0 = c_0 \left( \frac{5\sqrt{2}\ln 2}{4\sqrt{\nu}} \right)^{1/2}\|u_0\|_{L^2}.
\]
Applying Lemma 2.5 and Remark 2.6 with
\[
f(t) = t^{1/4}\|u(t)\|_{X^{-1}}, \quad \theta_1 = \theta_2 = 1/2,
\]
we get
\[
\limsup_{t \to +\infty} t^{1/4}\|u(t)\|_{X^{-1}} \leq 2M_0.
\]
Applying this result to the solution of the following system, for \( a \geq 0 \)
\[
\begin{align*}
\partial_t w - \nu \Delta w + w.\nabla w &= -\nabla h \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div} v &= 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
w(0, x) &= u(a, x) \quad \text{in} \quad \mathbb{R}^3,
\end{align*}
\]
we obtain
\[
\limsup_{t \to +\infty} t^{1/4}\|u(t)\|_{X^{-1}} \leq c_0 \left( \frac{5\sqrt{2}\ln 2}{4\sqrt{\nu}} \right)^{1/2}\|u(a)\|_{L^2}.
\]
Then the fact \( \lim_{a \to +\infty} \|u(a)\|_{L^2} = 0 \) implies the desired result.

5. Long time decay in \( X^\sigma \)

In this section we want to prove Corollary 1.7

First case: \(-3/2 < \sigma < -1\). For \( \lambda > 0 \) and \( t > 0 \), we have
\[
\|u(t)\|_{X^\sigma} = I_1(t, \lambda) + I_2(t, \lambda)
\]
\[
I_1(t, \lambda) = \int_{\{\xi \in \mathbb{R}^3, |\xi| < \lambda\}} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi
\]
\[
I_2(t, \lambda) = \int_{\{\xi \in \mathbb{R}^3, |\xi| > \lambda\}} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi.
\]
We have
\[
I_1(t, \lambda) \leq \left( \int_{\{\xi \in \mathbb{R}^3, |\xi| < \lambda\}} |\xi|^{2\sigma} d\xi \right)^{1/2} \|\hat{u}(t)\|_{L^2}
\leq c_1 \lambda^{\sigma+3/2} \|u(t)\|_{L^2}
\]
and
\[
I_2(t, \lambda) \leq \int_{\{\xi \in \mathbb{R}^3, |\xi| > \lambda\}} |\xi|^{\sigma+1} \left| \frac{\hat{u}(t, \xi)}{|\xi|} \right| d\xi
\leq \lambda^{\sigma+1} \int_{\mathbb{R}^3} \left| \frac{\hat{u}(t, \xi)}{|\xi|} \right| d\xi
\leq \lambda^{\sigma+1} \|u(t)\|_{X^{-1}}.
\]
We get \( \|u(t)\|_{X^\sigma} = A\lambda^{\sigma+3/2} + B\lambda^{\sigma+1} := \varphi(\lambda) \), with
\[
A = c_0 \|u(t)\|_{L^2} \quad \text{and} \quad B = \|u(t)\|_{X^{-1}}.
\]
The study of the function \( \varphi \) gives
\[
\varphi'(\lambda) = (\sigma + 3/2)A\lambda^{\sigma+1/2} + (\sigma + 1)B\lambda^{\sigma},
\]
then
\[
\varphi'(\lambda) = 0 \Rightarrow \lambda = \lambda_0 = \frac{-(1 + \sigma)B}{(\sigma + 3/2)A^2}.
\]
For $\lambda = \lambda_0$, we get
\[
\|u(t)\|_{X^s} \leq A\left(\frac{-\lambda(1+\sigma)B}{(\sigma+j/2)A}\right)^{2\sigma+3} + B\left(\frac{-\lambda(1+\sigma)B}{(\sigma+j/2)A}\right)^{2\sigma+2} \\
\leq \epsilon_\sigma A^{2\sigma-2}B^{1+2\sigma}.
\]
Then
\[
\|u(t)\|_{X^s} \leq C'_\sigma\left(\|u(t)\|_{L^2}\right)^{-2\sigma-2}\left(\|u(t)\|_{X^{-1}}\right)^{3+2\sigma}.
\]
Theorem 1.5 implies
\[
\|u(t)\|_{X^{-1}} = o(t^{-1/4}) \quad \text{and} \quad \|u(t)\|_{L^2} \to 0,
\]
which gives the desired result.

Second case: $-1 < \sigma$. By Theorem 1.5 we can assume that $\|u^0\|_{X^{-1}} < \epsilon_0$ and Theorem 1.4 gives,
\[
\|u(t)\|_{X^s} = \int_{\mathbb{R}^3} e^{-\sqrt{\nu t}/|\xi|} |\xi|^\sigma e^{\sqrt{\nu t}/|\xi|} \frac{|\hat{u}(t,\xi)|}{|\xi|^s} d\xi \\
\leq C_\nu \nu^{-\frac{s+1}{2}} \int e^{\nu t/2} \frac{|\hat{u}(t,\xi)|}{|\xi|^s} d\xi \\
\leq 2C_\nu \nu^{-\frac{s+1}{2}} \|u(t/2)\|_{X^{-1}},
\]
with $C_\nu = \nu^{-\frac{s+1}{2}} \sup_{z \geq 0} z^{\sigma+1} e^{-rz}$. Combining this result with the fact $\|u(t/2)\|_{X^{-1}} = o(t^{-1/4})$ we get the desired result.

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