Coverings and crossed modules of topological groups with operations

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Abstract

It is a well known result in the covering groups that a subgroup $G$ of the fundamental group at the identity of a semi-locally simply connected topological group determines a covering morphism of topological groups with characteristic group $G$. In this paper we generalize this result to a large class of algebraic objects called topological groups with operations, including topological groups. We also give the cover of crossed modules within topological groups with operations.

Key Words: Covering group, universal cover, crossed module, group with operations, topological group with operations

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1 Introduction

The theory of covering spaces is one of the most interesting theories in algebraic topology. It is well known that if $X$ is a topological group, $p: \tilde{X} \to X$ is a simply connected covering map, $e \in X$ is the identity element and $\tilde{e} \in \tilde{X}$ is such that $p(\tilde{e}) = e$, then $\tilde{X}$ becomes a topological group with identity $\tilde{e}$ such that $p$ is a morphism of topological groups (see for example \cite{8}).

The problem of universal covers of non-connected topological groups was first studied in \cite{23}. He proved that a topological group $X$ determines an obstruction class $k_X$ in

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$H^3(\pi_0(X), \pi_1(X, e))$, and that the vanishing of $k_X$ is a necessary and sufficient condition for the lifting of the group structure to a universal cover. In [17] an analogous algebraic result was given in terms of crossed modules and group-groupoids, i.e., group objects in the category of groupoids (see also [6] for a revised version, which generalizes these results and shows the relation with the theory of obstructions to extension for groups and [16] for the recently developed notion of monodromy for topological group-groupoids).

In [7, Theorem 1] Brown and Spencer proved that the category of internal categories within the groups, i.e., group-groupoids, is equivalent to the category of crossed modules of groups. Then in [21, Section 3], Porter proved that a similar result holds for certain algebraic categories $C$, introduced by Orzech [19], which definition was adapted by him and called category of groups with operations. Applying Porter’s result, the study of internal category theory in $C$ was continued in the works of Datuashvili [10] and [11]. Moreover, she developed cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [2,12]. In a similar way, the results of [7] and [21] enabled us to prove that some properties of covering groups can be generalized to topological groups with operations.

If $X$ is a connected topological space which has a universal cover, $x_0 \in X$ and $G$ is a subgroup of the fundamental group $\pi_1(X, x_0)$ of $X$ at the point $x_0$, then by [22, Theorem 10.42] we know that there is a covering map $p: (\tilde{X}_G, \tilde{x}_0) \to (X, x_0)$ of pointed spaces, with characteristic group $G$. In particular if $G$ is singleton, then $p$ becomes the universal covering map. Further if $X$ is a topological group, then $\tilde{X}_G$ becomes a topological group such that $p$ is a morphism of topological groups. Recently in [2] this method has been applied to topological $R$-modules and obtained a more general result (see also [3] and [18] for groupoid setting).

The object of this paper is to prove that this result can be generalized to a wide class of algebraic categories, which include categories of topological groups, topological rings, topological $R$-modules and alternative topological algebras. This is conveniently handled by working in a category $TC$. The method we use is based on that used by Rothman in [22, Theorem 10.42]. Further we give the cover of crossed modules within topological groups with operations.

2 Preliminaries on groupoids and covering groups

As it is defined in [4,15] a groupoid $G$ has a set $G$ of morphisms, which we call just elements of $G$, a set $G_0$ of objects together with maps $d_0, d_1: G \to G_0$ and $\epsilon: G_0 \to G$ such that $d_0 \epsilon = d_1 \epsilon = 1_{G_0}$. The maps $d_0, d_1$ are called initial and final point maps respectively and the
map $e$ is called object inclusion. If $a, b \in G$ and $d_1(a) = d_0(b)$, then the composite $a \circ b$ exists such that $d_0(a \circ b) = d_0(a)$ and $d_1(a \circ b) = d_1(b)$. So there exists a partial composition defined by $G_{d_1} \times d_0 G \to G$, $(a, b) \mapsto a \circ b$, where $G_{d_1} \times d_0 G$ is the pullback of $d_1$ and $d_0$. Further, this partial composition is associative, for $x \in G_0$ the element $e(x)$ acts as the identity, and each element $a$ has an inverse $a^{-1}$ such that $d_0(a^{-1}) = d_1(a)$, $d_1(a^{-1}) = d_0(a)$, $a \circ a^{-1} = e d_0(a)$ and $a^{-1} \circ a = e d_1(a)$. The map $G \to G, a \mapsto a^{-1}$ is called the inversion.

In a groupoid $G$ for $x, y \in G_0$ we write $G(x, y)$ for the set of all morphisms with initial point $x$ and final point $y$. According to [4] for $x \in G_0$ the star of $x$ is defined as $\{a \in G \mid d_0(a) = x\}$ and denoted as $\text{St}_G x$.

Let $G$ and $H$ be groupoids. A morphism from $H$ to $G$ is a pair of maps $f : H \to G$ and $f_0 : H_0 \to G_0$ such that $d_0 f = f_0 d_0$, $d_1 f = f_0 d_1$, $f e = e f_0$ and $f(a \circ b) = f(a) \circ f(b)$ for all $(a, b) \in H_{d_1} \times d_0 H$. For such a morphism we simply write $f : H \to G$.

Let $p : \tilde{G} \to G$ be a morphism of groupoids. Then $p$ is called a covering morphism and $\tilde{G}$ a covering groupoid of $G$ if for each $\tilde{x} \in \tilde{G}_0$ the restriction $\text{St}_{\tilde{G}} \tilde{x} \to \text{St}_G p(\tilde{x})$ is bijective.

We assume the usual theory of covering maps. All spaces $X$ are assumed to be locally path connected and semi-locally 1-connected, so that each path component of $X$ admits a simply connected cover. Recall that a covering map $p : \tilde{X} \to X$ of connected spaces is called universal if it covers every covering of $X$ in the sense that if $q : \tilde{Y} \to X$ is another covering of $X$ then there exists a map $r : \tilde{X} \to \tilde{Y}$ such that $p = q r$ (hence $r$ becomes a covering). A covering map $p : \tilde{X} \to X$ is called simply connected if $\tilde{X}$ is simply connected. Note that a simply connected covering is a universal covering.

A subset $U$ of a space $X$, which has a universal cover, is called liftable if it is open, path connected and lifts to each covering of $X$, that is, if $p : \tilde{X} \to X$ is a covering map, $i : U \to X$ is the inclusion map and $\tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = x \in U$, then there exists a map (necessarily unique) $\tilde{i} : U \to \tilde{X}$ such that $\tilde{p} \tilde{i} = i$ and $\tilde{i}(x) = \tilde{x}$. It is an easy application that $U$ is liftable if and only if it is open, path connected and for all $x \in U$, the fundamental group $\pi_1(U, x)$ is mapped to the singleton by the morphism $i_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $i : (U, x) \to (X, x)$.

A space $X$ is called semi-locally simply connected if each point has a liftable neighborhood and locally simply connected if it has a base of simply connected sets. So a locally simply connected space is also semi-locally simply connected.

For a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of pointed topological spaces, the subgroup $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ of $\pi_1(X, x_0)$ is called characteristic group of $p$, where $p_*$ is the morphism induced by $p$ (see [4], p.379) for the characteristic group of a covering map in terms of covering morphism of groupoids). Two covering maps $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and $q : (\tilde{Y}, \tilde{y}_0) \to (X, x_0)$ are called equivalent if their characteristic groups are isomorphic, equivalently there is a homeo-
morphism \( f : (\tilde{X}, \tilde{x}_0) \to (\tilde{Y}, \tilde{y}_0) \) such that \( qf = p \).

We recall a construction from [22, p.295] as follows: Let \( X \) be a topological space with a base point \( x_0 \) and \( G \) a subgroup of \( \pi_1(X, x_0) \). Let \( P(X, x_0) \) be the set of all paths of \( \alpha \) in \( X \) with initial point \( x_0 \). Then the relation defined on \( P(X, x_0) \) by \( \alpha \simeq \beta \) if and only if \( \alpha(1) = \beta(1) \) and \([\alpha \circ \beta^{-1}] \in G\), is an equivalence relation. Denote the equivalence relation of \( \alpha \) by \( \langle \alpha \rangle_G \) and define \( \tilde{X}_G \) as the set of all such equivalence classes of the paths in \( X \) with initial point \( x_0 \). Define a function \( p : \tilde{X}_G \to X \) by \( p(\langle \alpha \rangle_G) = \alpha(1) \).

Let \( \alpha_0 \) be the constant path at \( x_0 \) and \( \tilde{x}_0 = \langle \alpha_0 \rangle_G \in \tilde{X}_G \). If \( \alpha \in P(X, x_0) \) and \( U \) is an open neighbourhood of \( \alpha(1) \), then a path of the form \( \alpha \circ \lambda \), where \( \lambda \) is a path in \( U \) with \( \lambda(0) = \alpha(1) \), is called a continuation of \( \alpha \). For an \( \langle \alpha \rangle_G \in \tilde{X}_G \) and an open neighbourhood \( U \) of \( \alpha(1) \), let \( (\langle \alpha \rangle_G, U) = \{ \langle \alpha \circ \lambda \rangle_G : \lambda(I) \subseteq U \} \). Then the subsets \( (\langle \alpha \rangle_G, U) \) form a basis for a topology on \( \tilde{X}_G \) such that the map \( p : (\tilde{X}_G, \tilde{x}_0) \to (X, x_0) \) is continuous.

In Theorem 3.7 we generalize the following result to topological groups with operations.

**Theorem 2.1.** [22, Theorem 10.34] Let \( (X, x_0) \) be a pointed topological space and \( G \) a subgroup of \( \pi_1(X, x_0) \). If \( X \) is connected, locally path connected and semi-locally simply connected, then \( p : (\tilde{X}_G, \tilde{x}_0) \to (X, x_0) \) is a covering map with characteristic group \( G \).

**Remark 2.2.** Let \( X \) be a connected, locally path connected and semi locally simply connected topological space and \( q : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) a covering map. Let \( G \) be the characteristic group of \( q \). Then the covering map \( q \) is equivalent to the covering map \( p : (\tilde{X}_G, \tilde{x}_0) \to (X, x_0) \) corresponding to \( G \).

So from Theorem 2.1 the following result is obtained.

**Theorem 2.3.** [22, Theorem 10.42] Suppose that \( X \) is a connected, locally path connected and semi-locally simply connected topological group. Let \( e \in X \) be the identity element and \( p : (\tilde{X}, \tilde{e}) \to (X, e) \) a covering map. Then the group structure of \( X \) lifts to \( \tilde{X} \), i.e., \( \tilde{X} \) becomes a topological group such that \( \tilde{e} \) is identity and \( p : (\tilde{X}, \tilde{e}) \to (X, e) \) is a morphism of topological groups.

### 3 Universal covers of topological groups with operations

In this section we apply the methods of Section 2 to the topological groups with operations and obtain parallel results.

The idea of the definition of categories of groups with operations comes from [14] and [19] (see also [20]) and the definition below is from [21] and [13, p.21], which is adapted from [19].
Definition 3.1. Let $C$ be a category of groups with a set of operations $\Omega$ and with a set $E$ of identities such that $E$ includes the group laws, and the following conditions hold for the set $\Omega_i$ of $i$-ary operations in $\Omega$:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively $0$, $-$ and $+$ are the elements of $\Omega_0$, $\Omega_1$ and $\Omega_2$ respectively. Let $\Omega'_2 = \Omega_2 \setminus \{+\}$, $\Omega'_1 = \Omega_1 \setminus \{-\}$ and assume that if $\star \in \Omega'_2$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in $\Omega'_2$. Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega'_2$, $E$ includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega'_1$ and $\star \in \Omega'_2$, $E$ includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

The category $C$ satisfying the conditions (a)-(d) is called a category of groups with operations.

In the paper from now on $C$ will denote the category of groups with operations.

A morphism between any two objects of $C$ is a group homomorphism, which preserves the operations of $\Omega'_1$ and $\Omega'_2$.

Remark 3.2. The set $\Omega_0$ contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3. The categories of groups, rings generally without identity, $R$-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

 Remark 3.4. The set $\Omega_0$ contains exactly one element, the group identity; hence for instance associative rings with unit are not groups with operations.

The category of topological groups with operations is defined in [1] as follows:

Definition 3.5. A category $TC$ of topological groups with a set $\Omega$ of continuous operations and with a set $E$ of identities such that $E$ includes the group laws such that the conditions (a)-(d) in Definition 3.1 are satisfied, is called a category of topological groups with operations and the object of $TC$ are called topological groups with operations.

In the rest of the paper $TC$ will denote a category of topological groups with operations.

A morphism between any two objects of $TC$ is a continuous group homomorphism, which preserves the operations in $\Omega'_1$ and $\Omega'_2$.

The categories of topological groups, topological rings, topological $R$-modules and alternative topological algebras are examples of categories of topological groups with operations.

Proposition 3.6. If $X$ is a topological group with operations, then the fundamental group $\pi_1(X, 0)$ becomes a group with operations.
Proof: Let $X$ be an object of $TC$ and $P(X, 0)$ the set of all paths in $X$ with initial point $0$ as described in Section 1. There are binary operations on $P(X, 0)$ defined by

$$(\alpha \star \beta)(t) = \alpha(t) \star \beta(t)$$

for $\star \in \Omega_2$ and $t \in I$, unit interval, and unary operations defined by

$$(\omega \alpha)(t) = \omega(\alpha(t))$$

for $\omega \in \Omega_1$. Hence the operations (1) induce binary operations on $\pi_1(X, 0)$ defined by

$$[\alpha] \star [\beta] = [\alpha \star \beta]$$

for $[\alpha], [\beta] \in \pi_1(X, 0)$. Since the binary operations $\star$ in $\Omega_2$ are continuous it follows that the binary operations (3) are well defined. Similarly the operations (2) reduce the unary operations defined by

$$\omega[\alpha] = [\omega \alpha].$$

By the continuity of the unary operations $\omega \in \Omega_1$, the operations (4) are also well defined. The other details can be checked and so $\pi_1(X, x_0)$ becomes a group with operations, i.e., an object of $C$. \hfill \Box

We now generalize Theorem 2.1 to topological groups with operations. We first make the following preparation:

Let $X$ be a topological group with operations. By the evaluation of the compositions and operations of the paths in $X$ such that $\alpha_1(1) = \beta_1(0)$ and $\alpha_2(1) = \beta_2(0)$ at $t \in I$, we have the following interchange law

$$(\alpha_1 \circ \beta_1) \star (\alpha_2 \circ \beta_2) = (\alpha_1 \star \alpha_2) \circ (\beta_1 \star \beta_2)$$

for $\star \in \Omega_2$, where $\circ$ denotes the composition of paths, and

$$(\alpha \star \beta)^{-1} = \alpha^{-1} \star \beta^{-1}$$

for $\alpha, \beta \in P(X, 0)$ where, say $\alpha^{-1}$ is the inverse path defined by $\alpha^{-1}(t) = \alpha(1 - t)$ for $t \in I$. Further we have that

$$(\omega \alpha)^{-1} = \omega \alpha^{-1}.$$
When \( \alpha(1) = \beta(0) \).

Parallel to Theorem 2.1 in the following theorem we prove a general result for topological groups with operations.

**Theorem 3.7.** Let \( X \) be a topological group with operations, i.e., an object of \( \mathcal{TC} \) and let \( G \) be the subobject of \( \pi_1(X, 0) \). Suppose that the underlying space of \( X \) is connected, locally path connected and semi-locally simply connected. Let \( p: (\tilde{X}_G, 0) \rightarrow (X, 0) \) be the covering map corresponding to \( G \) as a subgroup of the additive group of \( \pi_1(X, 0) \) by Theorem 2.1. Then the group operations of \( X \) lift to \( \tilde{X}_G \), i.e., \( \tilde{X}_G \) is a topological group with operations and \( p: \tilde{X}_G \rightarrow X \) is a morphism of \( \mathcal{TC} \).

**Proof:** By the construction of \( \tilde{X}_G \) in Section 1, \( \tilde{X}_G \) is the set of equivalence classes defined via \( G \). The binary operations on \( P(X, 0) \) defined by (1) induce binary operations

\[
\langle \alpha \rangle_G \star \langle \beta \rangle_G = \langle \alpha \star \beta \rangle_G
\]  

(9)

and the unary operations on \( P(X, x_0) \) defined by (2) induce unary operations

\[
\omega \langle \alpha \rangle_G = \langle \omega \alpha \rangle_G
\]  

(10)

on \( \tilde{X}_G \).

We now prove that these operations (9) and (10) are well defined: For \( \star \in \Omega_2 \) and the paths \( \alpha, \beta, \alpha_1, \beta_1 \in P(X, 0) \) with \( \alpha(1) = \alpha_1(1) \) and \( \beta(1) = \beta_1(1) \), we have that

\[
[\langle \alpha \star \beta \rangle \circ (\alpha_1 \star \beta_1)^{-1}] = [\langle \alpha \star \beta \rangle \circ (\alpha_1^{-1} \star \beta_1^{-1})]
\]  

(by 5)

\[
= [\langle \alpha \circ \alpha_1^{-1} \rangle \star (\beta \circ \beta_1^{-1})]
\]  

(by 5)

\[
= [\alpha \circ \alpha_1^{-1}] \star [\beta \circ \beta_1^{-1}]
\]  

(by 3)

So if \( \alpha_1 \in \langle \alpha \rangle_G \) and \( \beta_1 \in \langle \beta \rangle_G \), then \( [\alpha \circ \alpha_1^{-1}] \in G \) and \( [\beta \circ \beta_1^{-1}] \in G \). Since \( G \) is a subobject of \( \pi_1(X, 0) \), we have that \( [\alpha \circ \alpha_1^{-1}] \star [\beta \circ \beta_1^{-1}] \in G \). Therefore the binary operations (9) are well defined.

Similarly for the paths \( \alpha, \alpha_1 \in P(X, 0) \) with \( \alpha(1) = \alpha_1(1) \) and \( \omega \in \Omega_1 \) we have that

\[
[(\omega \alpha) \circ (\omega \alpha_1^{-1})] = [(\omega \alpha) \circ (\omega \alpha_1^{-1})]
\]  

(by 7)

\[
= [(\omega \alpha \circ \alpha_1^{-1})]
\]  

(by 8)

\[
= \omega [\alpha \circ \alpha_1^{-1}]
\]  

(by 4)
Since $G$ is a subobject of $\pi_1(X,0)$, if $[\alpha \circ \alpha_1^{-1}] \in G$ and $\omega \in \Omega_1$ then $\omega[\alpha \circ \alpha_1^{-1}] \in G$. Hence the unary operations $\blacksquare$ are also well defined.

The axioms (a)-(d) of Definition 3.1 for $\tilde{X}_G$ are satisfied and therefore $\tilde{X}_G$ becomes a group with operations. Further by Theorem 2.1 $p: (\tilde{X}_G, \tilde{0}) \to (X,0)$ is a covering map, $\tilde{X}_G$ is a topological group and $p$ is a morphism of topological groups. In addition to this we need to prove that $\tilde{X}_G$ is an object of $TC$ and $p$ is a morphism of $TC$. To prove that the operations $\blacksquare$ for $* \in \Omega_2$ are continuous let $\langle \alpha \rangle_G, \langle \beta \rangle_G \in \tilde{X}_G$ and $(W, \langle \alpha * \beta \rangle_G)$ be a basic open neighbourhood of $\langle \alpha * \beta \rangle_G$. Here $W$ is an open neighbourhood of $(\alpha * \beta)(1) = \alpha(1) * \beta(1)$.

Since the operations $*: X \times X \to X$ are continuous there are open neighbourhoods $U$ and $V$ of $\alpha(1)$ and $\beta(1)$ respectively in $X$ such that $U * V \subseteq W$. Therefore $(U, \langle \alpha \rangle_G)$ and $(V, \langle \beta \rangle_G)$ are respectively base open neighbourhoods of $\langle \alpha \rangle_G$ and $\langle \beta \rangle_G$ and

$$(U, \langle \alpha \rangle_G) * (V, \langle \beta \rangle_G) \subseteq (W, \langle \alpha * \beta \rangle_G).$$

Therefore the binary operations $\blacksquare$ are continuous.

We now prove that the unary operations $\blacksquare$ for $\omega \in \Omega_1$ are continuous. For if $(V, \langle \omega \alpha \rangle)$ is a base open neighbourhood of $\langle \omega \alpha \rangle$, then $V$ is an open neighbourhood of $\omega \alpha(1)$ and since the unary operations $\omega: X \to X$ are continuous there is an open neighbourhood $U$ of $\alpha(1)$ such that $\omega(U) \subseteq V$. Therefore $(U, \langle \alpha \rangle)$ is an open neighbourhood of $\langle \alpha \rangle$ and $\omega(U, \langle \alpha \rangle) \subseteq (V, \langle \omega \alpha \rangle)$.

Moreover the map $p: \tilde{X}_G \to X$ defined by $p(\langle \alpha \rangle_G) = \alpha(1)$ preserves the operations of $\Omega_2$ and $\Omega_1$. \hfill $\square$

From Theorem 3.7 the following result can be restated.

**Theorem 3.8.** Suppose that $X$ is a topological group with operations whose underlying space is connected, locally path connected and semi-locally simply connected. Let $p: (\tilde{X}, \tilde{0}) \to (X,0)$ be a covering map such that $\tilde{X}$ is path connected and the characteristic group $G$ of $p$ is a subobject of $\pi_1(X,0)$. Then the group operations of $X$ lifts to $\tilde{X}$.

**Proof:** By assumption the characteristic group $G$ of the covering map $p: (\tilde{X}, \tilde{0}) \to (X,0)$ is a subobject of $\pi_1(X,0)$. So by Remark 2.2 we can assume that $\tilde{X} = \tilde{X}_G$ and hence by Theorem 3.7 the group operations of $X$ lift to $\tilde{X}$ as required. \hfill $\square$

In particular, in Theorem 3.7 if the subobject $G$ of $\pi_1(X,0)$ is chosen to be the singleton, then the following corollary is obtained.

**Corollary 3.9.** Let $X$ be a topological group with operations such that the underlying space of $X$ is connected, locally path connected and semi-locally simply connected. Let $p: (\tilde{X}, \tilde{0}) \to (X,0)$ be a universal covering map. Then the group structures of $X$ lifts to $\tilde{X}$.

The following proposition is useful for Theorem 3.12.
Proposition 3.10. Let $X$ be a topological group with operations and $V$ a liftable neighbourhood of 0 in $X$. Then there is a liftable neighbourhood $U$ of 0 in $X$ such that $U \ast U \subseteq V$ for $\ast \in \Omega_2$.

**Proof:** Since $X$ is a topological group with operations and hence the binary operations $\ast \in \Omega_2$ are continuous, there is an open neighbourhood $U$ of 0 in $X$ such that $U \ast U \subseteq V$. Further if $V$ is liftable, then $U$ can be chosen as liftable. For if $V$ is liftable, then for each $x \in U$, the fundamental group $\pi_1(U, x)$ is mapped to the singleton by the morphism induced by the inclusion map $i: U \rightarrow X$. Here $U$ is not necessarily path connected and hence not necessarily liftable. But since the path component $C_0(U)$ of 0 in $U$ is liftable and satisfies these conditions, $U$ can be replaced by the path component $C_0(U)$ of 0 in $U$ and assumed that $U$ is liftable.

Definition 3.11. Let $X$ and $Y$ be topological groups with operations and $U$ an open neighbourhood of 0 in $X$. A continuous map $\phi: U \rightarrow S$ is called a local morphism in TC if $\phi(a \ast b) = \phi(a) \ast \phi(b)$ when $a, b \in U$ such that $a \ast b \in U$ for $\ast \in \Omega_2$.

Theorem 3.12. Let $X$ and $\tilde{X}$ be topological groups with operations and $q: \tilde{X} \rightarrow X$ a morphism of TC, which is a covering map. Let $U$ be an open, path connected neighbourhood of 0 in $X$ such that for each $\ast \in \Omega_2$, the set $U \ast U$ is contained in a liftable neighbourhood $V$ of 0 in $X$. Then the inclusion map $i: U \rightarrow X$ lifts to a local morphism $\hat{i}: U \rightarrow \tilde{X}$ in TC.

**Proof:** Since $V$ lifts to $\tilde{X}$, then $U$ lifts to $\tilde{X}$ by $\hat{i}: U \rightarrow \tilde{X}$. We now prove that $\hat{i}$ is a local morphism of topological groups with operations. We know by the lifting theorem that $\hat{i}: U \rightarrow \tilde{X}$ is continuous. Let $a, b \in U$ be such that for each $\ast \in \Omega_2$, $a \ast b \in U$. Let $\alpha$ and $\beta$ be the paths from 0 to $a$ and $b$ respectively in $U$. Let $\gamma = \alpha \ast \beta$. So $\gamma$ is a path from 0 to $a \ast b$. Since $U \ast U \subseteq V$, the paths $\gamma$ is in $V$. So the paths $\alpha, \beta$ and $\gamma$ lift to $\tilde{X}$. Suppose that $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ are the liftings of $\alpha, \beta$ and $\gamma$ in $\tilde{X}$ respectively. Then we have

$$q(\tilde{\gamma}) = \gamma = \alpha \ast \beta = q(\tilde{\alpha}) \ast q(\tilde{\beta}).$$

But $q$ is a morphism of topological group with operations and so we have,

$$q(\tilde{\alpha} \ast \tilde{\beta}) = q(\tilde{\alpha}) \ast q(\tilde{\beta})$$

for $\ast \in \Omega_2$. Since the paths $\tilde{\gamma}$ and $\tilde{\alpha} \ast \tilde{\beta}$ have the initial point $\tilde{0} \in \tilde{X}$, by the unique path lifting

$$\tilde{\gamma} = \tilde{\alpha} \ast \tilde{\beta}$$

On evaluating these paths at $1 \in I$ we have

$$\hat{i}(a \ast b) = \hat{i}(a) \ast \hat{i}(b).$$
4 Covers of crossed modules within topological groups with operations

If $A$ and $B$ are objects of $C$, an extension of $A$ by $B$ is an exact sequence

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0 \quad (11)$$

in which $p$ is surjective and $i$ is the kernel of $p$. It is split if there is a morphism $s : B \to E$ such that $ps = id_B$. A split extension of $B$ by $A$ is called a $B$-structure on $A$. Given such a $B$-structure on $A$ we get actions of $B$ on $A$ corresponding to the operations in $C$. For any $b \in B$, $a \in A$ and $\star \in \Omega'_2$ we have the actions called derived actions by Orzech [19, p.293]

$$b \cdot a = s(b) + a - s(b)$$
$$b \star a = s(b) \star a. \quad (12)$$

In addition to this we note that topologically if an exact sequence (11) in $TC$ is a split extension, then the derived actions (12) are continuous. So we can state Theorem [19, Theorem 2.4] in topological case, which is useful for the proof of Theorem 4.6, as follows.

**Theorem 4.1.** A set of actions (one for each operation in $\Omega_2$) is a set of continuous derived actions if and only if the semidirect product $B \rtimes A$ with underlying set $B \times A$ and operations

$$(b, a) + (b', a') = (b + b', a + (b \cdot a'))$$
$$(b, a) \star (b', a') = (b \star b', a \star a' + b \star a' + a \star b')$$

is an object in $TC$.

The internal category in $C$ is defined in [21] as follows. We follow the notations of Section 1 for groupoids.

**Definition 4.2.** An internal category $C$ in $C$ is a category in which the initial and final point maps $d_0, d_1 : C \to C_0$, the object inclusion map $\epsilon : C_0 \to C$ and the partial composition $\circ : C_{d_1} \times_{d_0} C \to C$, $(a, b) \mapsto a \circ b$ are the morphisms in the category $C$. □

Note that since $\epsilon$ is a morphism in $C$, $\epsilon(0) = 0$ and that the operation $\circ$ being a morphism
implies that for all $a, b, c, d \in C$ and $\star \in \Omega_2$,

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d) \quad (13)$$

whenever one side makes sense. This is called the interchange law [21].

We also note from [21] that any internal category in $C$ is an internal groupoid since given $a \in C$, $a^{-1} = ed_1(a) - a + ed_0(a)$ satisfies $a^{-1} \circ a = ed_1(a)$ and $a \circ a^{-1} = ed_0(a)$. So we use the term internal groupoid rather than internal category and write $G$ for an internal groupoid. For the category of internal groupoids in $C$ we use the same notation $\text{Cat}(C)$ as in [21]. Here a morphism $f : H \to G$ in $\text{Cat}(C)$ is morphism of underlying groupoids and a morphism in $C$.

In particular if $C$ is the category of groups, then an internal groupoid $G$ in $C$ becomes a group-groupoid and in the case where $C$ is the category of rings, an internal groupoid in $C$ is a ring object in the category of groupoids [18].

**Definition 4.3.** An internal groupoid in the category $T(C)$ of topological groups with operations is called a topological internal groupoid.

So a topological internal groupoid is a topological groupoid $G$ in which the set of morphisms and the set $G_0$ of objects are objects of $T(C)$ and all structural maps of $G$, i.e, the source and target maps $s, t : G \to G_0$, the object inclusion map $\epsilon : G_0 \to G$ and the composition map $\circ : G_t \times_s G \to G$, are morphisms of $T(C)$.

If $T(C)$ is the category of topological groups, then an internal topological groupoid becomes a topological group-groupoid.

For the category of internal topological groupoids in $T(C)$ we use the notation $\text{Cat}(T(C))$. Here a morphism $f : H \to G$ in $\text{Cat}(T(C))$ is morphism of underlying groupoids and a morphism in $T(C)$.

**Theorem 4.4.** Let $X$ be an object of $T(C)$ such that the underlying space is locally path connected and semi-locally simply connected. Then the fundamental groupoid $\pi X$ is a topological internal groupoid.

**Proof:** Let $X$ be a topological group with operations as assumed. By [5] Theorem 1, $\pi X$ has a topology such that it is a topological groupoid. We know by [5] Proposition 3 that when $X$ and $Y$ are endowed with such topologies, for a continuous map $f : X \to Y$, the induced morphism $\pi(f) : \pi X \to \pi Y$ is also continuous. Hence the continuous binary operations $\star : X \times X \to X$ for $\star \in \Omega_2$ and the unary operations $\omega : X \to X$ for $\omega \in \Omega_1$ respectively induce continuous binary operations $\tilde{\star} : \pi X \times \pi X \to \pi X$ and unary operations $\tilde{\omega} : \pi X \to \pi X$. So the set of morphisms becomes a topological group with operations. The groupoid structural maps are morphisms of groups with operations, i.e., preserve the operations. Therefore $\pi X$ becomes a topological internal groupoid. \[\square\]
As similar to the crossed module in $C$ formulated in [21, Proposition 2], we define a crossed module in $TC$ as follows:

**Definition 4.5.** A crossed module in $TC$ is a morphism $\alpha: A \to B$ in $TC$, where $B$ acts topologically on $A$ (i.e. we have a continuous derived action in $TC$) with the conditions for any $b \in B, a, a' \in A$, and $* \in \Omega_2'$:

1. $CM_1\ \alpha(b \cdot a) = b + \alpha(a) - b$;
2. $CM_2\ \alpha(a) \cdot a' = a + a' - a$;
3. $CM_3\ \alpha(a) \star a' = a \star a'$;
4. $CM_4\ \alpha(b \star a) = b \star \alpha(a) \text{ and } \alpha(a \star b) = \alpha(a) \star b$.

\[\square\]

A morphism from $\alpha: A \to B$ to $\alpha': A' \to B'$ is a pair $f_1: A \to A'$ and $f_2: B \to B'$ of the morphisms in $TC$ such that

1. $f_2\alpha(a) = \alpha'f_1(a)$,
2. $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$,
3. $f_1(b \star a) = f_2(b) \star f_1(a)$

for any $x \in B, a \in A$ and $* \in \Omega_2'$. So we have a category $\text{XMod}(TC)$ of crossed modules in $TC$.

The algebraic case of the following theorem was proved in $C$ in [21, Theorem 1]. We can state the topological version as follows.

**Theorem 4.6.** The category $\text{XMod}(TC)$ of crossed modules in $TC$ and the category $\text{Cat}(TC)$ of internal groupoids in $TC$ are equivalent.

**Proof:** We give a sketch proof based on that of algebraic case. A functor $\delta: \text{Cat}(TC) \to \text{XMod}(TC)$ is defined as follows: For a topological internal groupoid $G$, let $\delta(G)$ be the topological crossed module $(A, B, d_1)$ in $TC$, where $A = \text{Ker}d_0, B = G_0$ and $d_1: A \to B$ is the restriction of the target point map. Here $A$ and $B$ inherit the structures of topological group with operations from that of $G$, and the target point map $d_1: A \to B$ is a morphism in $TC$.

Further the actions $B \times A \to A$ on the topological group with operations $A$ given by

\[
\begin{align*}
b \cdot a & = \epsilon(b) + a - \epsilon(b) \\
b \star a & = \epsilon(b) \star a
\end{align*}
\]
for $a \in A$, $b \in B$ are continuous by the continuities of $\epsilon$ and the operations in $\Omega_2$; and the axioms of Definition 4.5 are satisfied. Thus $(A, B, d_1)$ becomes a crossed module in $\text{TC}$.

Conversely define a functor $\eta: \text{XMod}(\text{TC}) \to \text{Cat}(\text{TC})$ in the following way. For a crossed module $(A, B, \alpha)$ in $\text{TC}$, define a topological internal groupoid $\eta(A, B, \alpha)$ whose set of objects is the topological group with operations $B$ and set of morphisms is the semi-direct product $B \ltimes A$ which is a topological group with operations by Theorem 4.1. The source and target point maps are defined to be $d_0(b, a) = b$ and $d_1(b, a) = \alpha(a) + b$ while the object inclusion map and groupoid composition is given by $\epsilon(b) = (b, 0)$ and

$$(b, a) \circ (b_1, a_1) = (b, a_1 + a)$$

whenever $b_1 = \alpha(a)b$. These structural maps are all continuous and therefore $\eta(A, B, \alpha)$ is a topological internal groupoids.

The other details of the proof is obtained from that of [21, Theorem 1].

By Theorem 4.6 we obtain the cover of a crossed module in $\text{TC}$. If $f: H \to G$ is a covering morphism in $\text{Cat}(\text{TC})$ and $(f_1, f_2)$ is the morphism of crossed modules corresponding to $f$, then $f_1: A \to A'$ is an isomorphism in $\text{TC}$, where $A = \text{St}_H0$, $A' = \text{St}_G0$ and $f_1$ is the restriction of $f$. Therefore we call a morphism $(f_1, f_2)$ of crossed modules form $\alpha: A \to B$ to $\alpha': A' \to B'$ in $\text{TC}$ as cover if $f_1: A \to A'$ is an isomorphism in $\text{TC}$.

Let $G$ be a topological internal groupoid, i.e., an object of $\text{Cat}(\text{TC})$. Let $\text{Cov}_{\text{Cat}(\text{TC})}/G$ be the category of covers of $G$ in the category $\text{Cat}(\text{TC})$. So the objects of $\text{Cov}_{\text{Cat}(\text{TC})}/G$ are the covering morphisms $p: \tilde{G} \to G$ over $G$ in $\text{Cat}(\text{TC})$ and a morphism from $p: \tilde{G} \to G$ to $q: \bar{G} \to G$ is a morphism $f: \tilde{G} \to \bar{G}$ in $\text{Cat}(\text{TC})$ such that $qf = p$.

The algebraic case of the following theorem was proved in [1, Theorem 5.3]. We give the topological case of this theorem in $\text{TC}$ as follows. The proof is obtained by Theorem 4.6 and Theorem 4.7.

**Theorem 4.7.** Let $G$ be an object of $\text{Cat}(\text{TC})$ and $\alpha: A \to B$ the crossed module in $\text{TC}$ corresponding to $G$ by Theorem 4.6. Let $\text{Cov}_{\text{XMod}(\text{TC})}/(\alpha: A \to B)$ be the category of covers of $A \to B$ in $\text{TC}$. Then the categories $\text{Cov}_{\text{Cat}(\text{TC})}/G$ and $\text{Cov}_{\text{XMod}(\text{TC})}/(\alpha: A \to B)$ are equivalent.

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