Critical Droplets and Phase Transitions in Two Dimensions

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In two space dimensions, the percolation point of the pure-site clusters of the Ising model coincides with the critical point $T_c$ of the thermal transition and the percolation exponents belong to a special universality class. By introducing a bond probability $p_B < 1$, the corresponding site-bond clusters keep on percolating at $T_c$ and the exponents do not change, until $p_B = p_{CK} = 1 - \exp(-2J/kT)$: for this special expression of the bond weight the critical percolation exponents switch to the 2D Ising universality class. We show here that the result is valid for a wide class of bidimensional models with a continuous magnetization transition: there is a critical bond probability $p_c$ such that, for any $p_B \geq p_c$, the onset of percolation of the site-bond clusters coincides with the critical point of the thermal transition. The percolation exponents are the same for $p_c < p_B \leq 1$ but, for $p_B > p_c$, they suddenly change to the thermal exponents, so that the corresponding clusters are critical droplets of the phase transition. Our result is based on Monte Carlo simulations of various systems near criticality.

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I. INTRODUCTION

The purely geometric percolation phenomenon \[1\] has astonishing analogies with ordinary second-order thermal phase transitions: the percolation variables show power-law behaviour near criticality, with relative exponents; the exponents are not all independent, but related to each other by scaling equalities which are formally identical to the corresponding thermal ones; the critical indices do not depend on the specific features of a system but they are grouped in universality classes, which in the percolation case are uniquely specified by the number of space dimensions of the system.

These analogies suggested many years ago that a continuous thermal transition might be nothing but a percolation transition \[2\], provided one defines suitable clusters, or droplets. The growth of the droplets describes naturally the propagation of correlations between different particles of the system and the formation of an infinite spanning structure represents the long-range order of the system in the new phase.

In this paper we focus on spin models with a continuous magnetization transition. In order to map the critical behaviour of a system onto a percolation picture, one must set a correspondence between thermal and geometric variables. The main percolation variables are:

- the percolation point must coincide with the thermal critical point;
- the connectedness length (average cluster radius) diverges as the thermal correlation length (same exponent);
- the percolation strength $P$ near the threshold varies like the order parameter $m$ of the model (same exponent);
- the average cluster size $S$ diverges as the physical susceptibility $\chi$ (same exponent).

It is not yet known whether, given a system, there are at all clusters satisfying such conditions, which are quite strict. The first studies concentrated on the simplest theory, the Ising model without field. A real breakthrough in these investigations was the discovery that the Ising model can be rewritten as a geometrical model \[3\], where the fundamental objects are special site-bond clusters, i.e. clusters built by joining nearest-neighbouring spins of the same sign with a temperature-dependent bond probability $p_{CK} = 1 - \exp(-2J/kT)$ ($J$ is the Ising spin-spin coupling). It is possible to prove that these site-bond clusters are just the critical droplets of the system. Starting from this result, which is valid in general for the $q$-state Potts model, it is possible to define the droplets for a wide variety of systems, like models with several spin-spin couplings, as long as they are all ferromagnetic \[4\], and $O(n)$ spin models \[5\]. A common feature of all these results is the fact that a geometrical bond (with a bond probability) is associated with each spin-spin coupling in the percolation picture. In this way, for models with several (ferromagnetic) interactions, the clusters are quite weird objects, consisting of geometrically disconnected parts which are joined to each other by virtue of invisible bridges due to long-ranged interactions. Besides, if not all couplings are ferromagnetic, it is not yet clear if and how one can define the droplets.
Very recent results show that the simple connections between nearest-neighbour spins have a close relationship with the phase transition of several bidimensional models: the percolation point of the pure-site clusters coincides with the critical point and the percolation exponents, which differ from the thermal exponents, are the same for models in the same universality class. In our opinion this property indicates that, in many cases, the simple geometrical connectivity between nearest-neighbours plays a crucial role in the mechanism of the phase transition.

In the 2D Ising model, the fact that both the pure-site and the site-bond clusters with bond probability \( p_{CK} \) start to percolate at the critical temperature \( T_c \) necessarily implies that, by taking a bond probability \( p_B \) such that \( p_{CK} < p_B \leq 1 \), the percolation temperature of the site-bond clusters with bond weight \( p_B \) is always \( T_c \). A renormalization group analysis led to the conclusion that the percolation exponents are the same as for the pure-site clusters for any \( p_B > p_{CK} \), and that they suddenly switch to the thermal exponents for \( p_B = p_{CK} \). For \( p_B < p_{CK} \), the site-bond clusters percolate at some temperature \( T_p < T_c \) and the exponents switch to the 2D random percolation universality class. This analysis, though reliable, is however not rigorous. In this paper we will show that the result is true and valid for many bidimensional models. We will see that, given a model, there is a critical probability \( p_c \) such that, by taking a bond weight \( p_B > p_c \) (including the pure-site case \( p_B = 1 \)) the site-bond clusters keep percolating at \( T_c \) and the critical percolation exponents are the same: for \( p_B = p_c \) the exponents change abruptly to the thermal exponents. The probability \( p_c \), which is model-dependent, is the minimal probability which still allows the formation of a percolating cluster at \( T_c \), and this gives us a general prescription for the critical droplets.

II. RESULTS OF THE SIMULATIONS

Our investigations consisted of Monte Carlo simulations on square lattices of several models around the critical temperature. We took the same models that were studied in [2], i.e.

1. the Ising model, \( \mathcal{H} = -J \sum_{ij} s_i s_j \) \((J > 0, s_i = \pm 1)\);
2. a model with nearest-neighbour (NN) ferromagnetic coupling and a weaker next-to-nearest (NTN) antiferromagnetic coupling: \( \mathcal{H} = -J_1 \sum_{NN} s_i s_j - J_2 \sum_{NTN} s_i s_j \) \((J_1 > 0, J_2 < 0, |J_2/J_1| = 1/10, s_i = \pm 1)\);
3. the continuous Ising model, \( \mathcal{H} = -J \sum_{ij} S_i S_j \) \((J > 0, -1 \leq S_i \leq +1)\);
4. SU(2) pure gauge theory in 2+1 dimensions;
5. the 3-state Potts model, \( \mathcal{H} = -J \sum_{ij} \delta(s_i, s_j) \) \((J > 0, s_i = 1, 2, 3)\);
6. a model obtained by adding to 5) a weaker next-to-nearest (NTN) antiferromagnetic coupling: \( \mathcal{H} = -J_1 \sum_{NN} \delta(s_i, s_j) - J_2 \sum_{NTN} \delta(s_i, s_j) \) \((J_1 > 0, J_2 < 0, |J_2/J_1| = 1/10, s_i = 1, 2, 3)\);

The critical temperatures of all these systems are known with good precision (see [2]), for Models 1 and 5 the values are analytically known. We stress that the models belong to the 2D \( Z(2) \) (1-4) and \( Z(3) \) (5,6) universality classes. The estimates of [6] for the main critical indices of the percolation transition of the pure-site clusters for the two classes are in very good agreement with theoretical predictions [7, 10]; we list these analytical values in Table II

| Model | \( \gamma_p \) | \( \nu_p \) | Fractal Dim. | Cum. at \( T_p \) |
|-------|-------------|-------------|--------------|----------------|
| \( Z(2) \) | 5/96 | 91/48 | 1 | 187/96 | 0.9832(4) |
| \( Z(3) \) | 7/96 | 73/48 | 5/6 | 153/80 | 0.932(2) |

TABLE I: Critical percolation indices for pure-site clusters for the two universality classes we considered.

For the simulations we made use of standard algorithms like Metropolis or heat bath; in some cases we could apply the Wolff cluster algorithm, which allowed us to reduce sensibly the correlation of the data. For each system we took four to six different lattice sizes. The clusters were identified by means of the algorithm devised by Hoshen and Kopelman [11]. We used everywhere free boundary conditions for the cluster labeling and say that a cluster percolates if it connects the top with the bottom side of the lattice. At each iteration we calculated the variables \( P \) and \( S \) and the size \( S_M \) of the largest cluster, from which one can determine the fractal dimension \( D \) of the percolating cluster at \( T_c \). Moreover, from the data sample of the percolation strength \( P \) one can extract a variable which turns out to be a formidable tool to investigate numerically the percolation transition. In fact, the number of configurations with (at least) a percolating cluster (i.e. for which \( P \neq 0 \)) for a given temperature and lattice size, divided by the total number of configurations, returns a scaling variable \( \Pi \) called percolation cumulant, whose properties are identical to those of the Binder cumulant in standard thermal transitions [12]. In particular, the value of \( \Pi \) at the critical threshold is a universal quantity, i.e. it is the same for models in the same universality class. We determined the percolation exponents through standard finite size scaling techniques at the critical temperature \( T_p \), considering simply the leading behaviour.
We began our studies starting from the Ising model. We tested three different expressions for the bond probability \( p_B \): \( CK_1 = 1 - \exp(-3J/kT) \), \( CK_2 = 1 - \exp(-2.5J/kT) \), \( CK_3 = 1 - \exp(-2.2J/kT) \). In all cases we found that, even for the smallest lattice size we examined (1000\(^2\)), the threshold value of the percolation cumulant II did not differ from the value relative to the pure-site clusters (0.9832(4)), which is a clear indication that the critical indices of the three percolation transitions remain in the universality class of the 2D \( Z(2) \) pure-site clusters. Figs. 1 and 2 show the situation for the probabilities \( CK_2 \) and \( CK_3 \), respectively.

\[
P(T_p) \propto L^{-\beta_p/\nu_p} \\
S(T_p) \propto L^{\gamma_p/\nu_p} \\
S_M(T_p) \propto L^D.
\]  

FIG. 1: 2D Ising model: value of the percolation cumulant II at the critical temperature \( T_c \) for various lattices. The bond weight is \( CK_2 = 1 - \exp(-2.5J/kT) \). All values agree with each other (so \( T_p = T_c \)) and with the (universal) pure-site value, represented with its error by the dashed lines in the plot.

Indeed, the values of the percolation exponents we extracted from finite size scaling fits of \( P \), \( S \) and \( S_M \) at the critical point are in good agreement with the \( Z(2) \) values reported in Table I. Next, we approached the critical value of \( p_{CK} \) at \( T_c (p_{CK}(T_c) = 0.58578... \) ) by successive numerical trials. We found that, up to a value greater than \( p_{CK}(T_c) \) by 4–5%, the threshold value of the cumulant did not change for any of the lattices we have taken, and the situation is the same as in Figs. 1 and 2. For still lower probabilities the cumulant decreases appreciably for the smaller lattices and the less the larger the size. That is the ”lattice reaction” to the fact that we are near an abrupt discontinuity. We know that the finite size of the lattice reduces divergences to finite peaks and discontinuities to smooth variations. Close to a sharp discontinuity, the real behaviour of the system is reproduced only on very large lattices; for small sizes, the system feels the proximity of the new state and that gives rise to intermediate configurations, which are unphysical. On the other hand, the threshold value of the percolation cumulant on our largest lattice (1000\(^2\)) remained fixed at the pure-site value until \( p_B/p_{CK}(T_c) \approx 1.02 \), and that clearly suggests that the transition from the pure-site behaviour to the droplet behaviour takes place just when \( p_B = p_{CK}(T_c) \).

By taking exactly \( p_B = p_{CK}(T_c) \), the threshold value of the percolation cumulant is again the same for all lattices, and equal to the value 0.585(1) that labels the universality class of the 2D \( Z(2) \) droplets. Next, we took some values of \( p_B < p_{CK}(T_c) \). Just below \( p_{CK}(T_c) \) we expected to see the same mixed behaviour observed when one approaches the critical probability from above, and that is indeed the case. However, already for \( p_{CK}(T_c) < p_B < 0.96 \) we could see again a ”clean” behaviour of the percolation variables on all lattices: the percolation cumulants no longer cross at \( T_c \) but at a somewhat lower temperature \( T_p \) which decreases by decreasing \( p_B \). The value of the cumulant at the crossing point is the same for any \( p_B \) and agrees with the value corresponding to the 2D random percolation universality class (0.450(2)), as shown in Fig. 3. Our findings confirm the scenario predicted in \( Z(2) \).

As far as the other models with an Ising-like transition are concerned we remark that, except for the continuous Ising model \( \xi \), the problem of the existence of eventual critical droplets is open. Since in each case the pure-site clusters begin to percolate at \( T_c \), it was natural for us to check whether also here one has the same situation like in the Ising model. We then sat at \( T_c \) in every case, and introduced progressively decreasing bond weights. We remark that for the continuous spin models 3 and 4 we bound nearest-neighbour spins of the same sign, independently of their absolute value.

FIG. 2: 2D Ising model: value of the percolation cumulant II at the critical temperature \( T_c \) for various lattices. The bond weight is \( CK_3 = 1 - \exp(-2.2J/kT) \). All values agree with each other (so \( T_p = T_c \)) and with the (universal) pure-site value, represented with its error by the dashed lines in the plot.
We found that the scenario is indeed the same. In every case, we found a critical probability $p_c$, which equals 0.583(1), 0.6115(9) and 0.6275(7) for Model 2, 3 and 4, respectively. We adopted several values for $p_B$, both above and below $p_c$. For $p_B > p_c$ we have always found that the percolation point coincides with the critical temperature $T_c$ of the model, and that the critical percolation indices coincide (within errors) with the $Z(2)$ percolation indices of Table II. In all cases we could show analogous pictures as Fig. 1 or 2. If $p_B < p_c$, the percolation temperature $T_p < T_c$ and the exponents belong to the 2D random percolation universality class (as in Fig. 3). Like in the Ising model $p_c$ is then the minimal probability for which the site-bond clusters still percolate at $T_c$. Fig. 4 shows the threshold value of the percolation cumulant for all $Z(2)$ models when $p_B = p_c$: all values agree within errors. In this case, the critical indices are in accord with the ones of the 2D Ising droplets, except that the bond weight $p_{FK} = 1 - \exp(-J/kT)$, but the situation is identical. It is then not surprising that for the 2D $q$-state Potts model [12] the same scenario as for Ising was predicted [10]. The results of our simulations confirm such prediction. The situation does not change for Model 6: we found a critical bond probability $p_c = 0.61(1)$, at which the site-bond clusters become critical droplets for the model, as one can see in Table III. In Fig. 5 we compare the threshold values of $\Pi$ for the two $Z(3)$ models when $p_B = p_c$.

Finally we investigated Models 5 and 6. We know that the $q$-state Potts model can be transformed into a cluster model by means of the Fortuin-Kasteleyn transformation [3]. The clusters are defined in the same way as for Ising, except that the bond weight $p_{FK} = 1 - \exp(-J/kT)$, but the situation is identical. It is then not surprising that for the 2D $q$-state Potts model [12] the same scenario as for Ising was predicted [10]. The results of our simulations confirm such prediction. The situation does not change for Model 6: we found a critical bond probability $p_c = 0.61(1)$, at which the site-bond clusters become critical droplets for the model, as one can see in Table III. In Fig. 5 we compare the threshold values of $\Pi$ for the two $Z(3)$ models when $p_B = p_c$.

![Image 1](https://via.placeholder.com/150)

**FIG. 3:** 2D Ising model: value of $\Pi$ at the percolation temperature $T_p$ for various lattices. The bond weight is $1 - \exp(-J/kT) < p_c = 1 - \exp(-2J/kT)$. In this case is $T_p < T_c$. The data points agree with the (universal) value for 2D random percolation, represented with its error by the dashed lines in the plot.

![Image 2](https://via.placeholder.com/150)

**FIG. 4:** Threshold value of the percolation cumulant $\Pi$ at the critical temperature $T_c$ by using the “minimal” probability $p_c$ for all $Z(2)$ models. The X-coordinates 1, 2, 3 and 4 label the four models we investigated. There is a clear accord of the four data points with each other. The dashed lines indicate the bounds of the more precisely measured Ising value (first point to the left), which labels the universality class of the 2D Ising droplets.

![Image 3](https://via.placeholder.com/150)

**FIG. 5:** Threshold value of the percolation cumulant $\Pi$ at the critical temperature $T_c$ by using the “minimal” probability $p_c$ for the $Z(3)$ models. The X-coordinates 5 and 6 label the two models we investigated. The values agree well within errors. The dashed lines indicate the bounds of the more precisely measured 3-state Potts value (point to the left), which labels the universality class of the 2D $Z(3)$ droplets.

| Model Label | $\beta_p/\nu_p$ | $\gamma_p/\nu_p$ | Fractal Dim. D | Cum. at $T_p$ |
|-------------|-----------------|------------------|----------------|--------------|
| 2D Ising    | 1/8=0.125       | 7/4=1.75         | 15/8=1.875     | 0.585(1)     |
| Model 2     | 0.131(10)       | 1.742(12)        | 1.862(20)      | 0.583(4)     |
| Model 3     | 0.121(9)        | 1.764(14)        | 1.870(11)      | 0.587(3)     |
| Model 4     | 0.140(19)       | 1.761(17)        | 1.882(18)      | 0.586(5)     |

**TABLE II:** Critical percolation indices for the site-bond clusters of the models with an Ising-like transition when $p_B = p_c$, compared with the values of the 2D Ising droplets.

We found that the scenario is indeed the same. In every case, we found a critical probability $p_c$, which equals 0.583(1), 0.6115(9) and 0.6275(7) for Model 2, 3 and 4, respectively. We adopted several values for $p_B$, both above and below $p_c$. For $p_B > p_c$ we have always found that the percolation point coincides with the critical temperature $T_c$ of the model, and that the critical percolation indices coincide (within errors) with the $Z(2)$ percolation indices of Table II. In all cases we could show analogous pictures as Fig. 1 or 2. If $p_B < p_c$, the percolation temperature $T_p < T_c$ and the exponents belong to the 2D random percolation universality class (as in Fig. 3). Like in the Ising model $p_c$ is then the minimal probability for which the site-bond clusters still percolate at $T_c$. Fig. 4 shows the threshold value of the percolation cumulant for all $Z(2)$ models when $p_B = p_c$: all values agree within errors. In this case, the critical indices are in accord with the ones of the 2D Ising droplets, as shown in Table II.
### III. CONCLUSIONS

We found a general criterion to identify the critical droplets of various bidimensional systems. The droplets are the clusters obtained by joining nearest-neighbour spins of the same sign (-state for Potts-like systems) with the minimal bond probability $p_c$ that still allows the clusters to percolate at the critical point of the thermal transition. For any $p_B > p_c$ the onset of percolation for the relative site-bond clusters keeps coinciding with the thermal threshold and the critical indices are the same up to the pure-site case $p_B = 1$. We stress that this prescription seems to work also for the cases in which one can rigorously define the droplets. For the continuous Ising model, for instance, the droplets are defined by introducing local bond weights which also depend on the length of the spins and not only on their sign \[14\]. These clusters certainly differ in detail from the droplets defined in this paper; nevertheless their behaviour at criticality is identical. The matter would be even more involved for models with several ferromagnetic interactions, for which a rigorous definition of droplets requires the presence of longer-range connections than just between nearest-neighbour spins \[4\]. The fact that the critical behaviour of the system is reproduced by simple site-bond clusters, no matter how complicated the model is, suggests that the long-range fluctuations of the system, which are responsible for the phase transition, are embodied in the simple magnetic domains of the system; the bond probability $p_c$ is necessary in order to destroy additional spin correlations due to purely geometrical effects, as already remarked in \[7\]. Besides, we have seen that the result is also valid in cases where a rigorous definition of droplets is, at present, missing (Models 2, 4 and 6). In particular, it remains true for $SU(2)$ pure gauge theory, which is a very complicated model involving many different interactions, like multispin couplings, long-ranged couplings (ferromagnetic and antiferromagnetic) and self-interactions. This shows that the result has some generality, and it would be interesting to check to which extent it is true for bidimensional models with a continuous magnetization transition. It would be also interesting to check whether simple site-bond clusters play a role in the description of critical behaviour at higher dimensions as well.

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\[\text{TABLE III: Critical percolation indices for the site-bond clusters of Model 6 when } p_B = p_c, \text{ compared with the corresponding values of the 2D 3-state Potts droplets.}\]

| 2D 3S Potts | 2D 3S Potts | D | Cum. at $T_c$ |
|-------------|-------------|---|--------------|
| $\beta_{\nu}/\nu_\rho$ | $\gamma_{\nu}/\nu_\rho$ | $D$ | Cum. at $T_c$ |
| $2/15$ | $26/15$ | $28/15$ | $0.649(9)$ |