An example of a simple double Lie algebra
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Abstract

We extend the correspondence between double Lie algebras and skew-symmetric
Rota—Baxter operators of weight 0 on the matrix algebra to the infinite-dimensional
case. We give the first example of a simple double Lie algebra.

Keywords: double Lie algebra, Rota—Baxter operator.

1 Introduction

In 2008 [23], M. Van den Bergh introduced the notion of a double Poisson algebra
developing noncommutative geometry. For this, he followed the Kontsevich—Rosenberg
principle saying that a structure on an associative algebra has geometric meaning if it
induces standard geometric structures on its representation spaces.

Given a finitely generated associative algebra $A$ and $n \in \mathbb{N}$, consider the representation
space $\text{Rep}_n(A) = \text{Hom}(A, M_n(F))$, where $F$ denotes the ground field. To equip $A$ with
a structure such that $\text{Rep}_n(A)$ is a Poisson variety for every $n$, M. Van den Bergh defined
a double bracket $\{\{,\}\} : A \otimes A \to A \otimes A$ satisfying the analogues of anti-commutativity,
Jacobi identity, and Leibniz rule. An associative algebra equipped with such a double
bracket is called a double Poisson algebra. One of the crucial examples of such structure
is a double Poisson algebra defined on a quiver algebra.

Double Poisson algebras are deeply connected with $H_0$-Poisson structures [7], pre-
Calabi—Yau algebras [12], vertex algebras [20].

The notion of a double Lie algebra naturally appeared from the very definition of dou-
ble Poisson algebra, it is a vector space endowed with a double bracket satisfying above
mentioned anti-commutativity and Jacobi identity. Every double Lie algebra structure
defined on a vector space $V$ can be uniquely extended to a double Poisson algebra struc-
ture on the free associative algebra $A_*(V)$. Thereby, A. Odesskii, V. Rubtsov, V. Sokolov
extended [15] linear and quadratic double Lie algebras defined on an $n$-dimensional vector
space to double Poisson algebras defined on the free $n$-generated associative algebra.

In [8], M. Goncharov and P. Kolesnikov proved that there are no simple finite-
dimensional double Lie algebras. This problem was stated by V. Kac during the confer-
ence “Lie and Jordan algebras, their representations and applications” dedicated to Efim
Zelmanov’s 60th birthday (Bento Gonçalves, Brasil, 2015). After this work the natural
question about constructing simple infinite-dimensional double Lie algebras has arisen.

It is known that the structure of a double Lie algebra on a finite-dimensional vector
space $V$ is equivalent to a skew-symmetric Rota—Baxter operator of weight 0 on the
matrix algebra $M_n(F)$, where $n = \dim(V)$ [8, 15, 18]. Recall that a linear operator $R$
deﬁned on an algebra $A$ is called a Rota—Baxter operator (RB-operator, for short) of
weight $\lambda$, if

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$
for all \( x, y \in A \). This notion for the first time appeared in the article [21] of F. Tricomi in 1951 and further was several times [5, 6] rediscovered, see the monograph [10]. Let us mention the bijection [11, 9, 18] between RB-operators of weight 0 on the matrix algebra \( M_n(F) \) and solutions of the associative Yang–Baxter equation (AYBE) on \( M_n(F) \) [3, 17, 24].

We generalize this correspondence between double Lie algebras and skew-symmetric Rota–Baxter operators for the infinite-dimensional case. We state such correspondence for a countable-dimensional double Lie algebra \( V \) and a Rota–Baxter operator acting from the space of matrices with finite numbers of nonzero elements to \( \text{End}(V) \) and satisfying some additional finiteness conditions. This correspondence allows us to construct new double Lie algebras. In particular, we show that the vector space \( F[t] \) with the double bracket

\[
\{\{t^n, t^m\}\} = -\frac{(t^n \otimes t^m - t^m \otimes t^n)}{t \otimes 1 - 1 \otimes t}
\]

is a simple double Lie algebra. As far as we know it is the first example of a simple double Lie algebra.

In terms of RB-operators we interpret the amazing double Lie algebra of V. Kac (see [8]) whose definition is very close to the definition of the Yangian \( Y(gl_N) \).

2 Preliminaries

2.1 Rota–Baxter operators

**Definition 1.** A linear operator \( R \) defined on a (not necessary associative) algebra \( A \) is called a Rota–Baxter operator (RB-operator, for short) of weight \( \lambda \in F \), if

\[
R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)
\]

holds for all \( x, y \in A \).

**Proposition 1 [9].** Let \( A \) be an algebra, let \( R \) be an RB-operator of weight \( \lambda \) on \( A \), and let \( \psi \) be either automorphism or antiautomorphism of \( A \). Then the operator \( R^{(\psi)} = \psi^{-1}R\psi \) is an RB-operator of weight \( \lambda \) on \( A \).

As an application of Proposition 1, we will use the conjugation with transpose of an RB-operator defined on the matrix algebra.

The following definition has appeared by the name of relative Rota–Baxter operator or \( \mathcal{O} \)-operator [14] or as generalized RB-operator in the case of zero weight [22]. For simplicity, we also call it Rota–Baxter operator.

**Definition 2.** Let \( A \) be an algebra and \( I \) be an ideal of \( A \). A linear operator \( R: I \rightarrow A \) is called a Rota–Baxter operator of weight \( \lambda \), if

\[
R(i)R(j) = R(R(i)j + iR(j) + \lambda ij)
\]

holds for all \( i, j \in I \).

When \( I = A \), this definition coincides with Definition 1.

The next statement follows immediately.
Proposition 2. Let $A$ be an algebra and let $J$ be an ideal of $A$. Given an RB-operator $P: J \to A$ of weight $\lambda$ and an algebra $B$, the operator $Q = P \otimes \text{id}_B$ is again an RB-operator of the same weight $\lambda$ from $I \otimes B$ to $A \otimes B$.

Henceforth, we consider only Rota—Baxter operators of weight 0. It is well-known that, given an RB-operator $R$ of weight 0 and $\alpha \in F$, the operator $\alpha R$ is again an RB-operator $R$ of weight 0.

2.2 Double Lie algebras

Let $V$ be a linear space. Given $u \in V \otimes^n$ and $\sigma \in S_n$, $u^\sigma$ denotes the permutation of tensor factors. By a double bracket on $V$ we call a linear map from $V \otimes V$ to $V \otimes V$. Given an associative algebra $A$, we consider the outer bimodule action of $A$ on $A \otimes A$:

$$b(a \otimes a')c = (ba) \otimes (a'c).$$

Definition 3 [23]. A double Poisson algebra is an associative algebra $A$ equipped with a double bracket satisfying the following identities for all $a, b, c \in A$

$$\{\{a, b\}\} = -\{\{b, a\}\}^{(12)},$$
$$\{\{a, \{\{b, c\}\}\}\} - \{\{b, \{\{a, c\}\}\}\}^{(12)} = \{\{\{a, b\}, c\}\},$$
$$\{\{a, bc\}\} = \{\{a, b\}\}c + b\{\{a, c\}\},$$
where $\{\{a, b \otimes c\}\}_L = \{\{a, b\}\} \otimes c$, $\{\{a, b \otimes c\}\}_R = (b \otimes \{\{a, c\}\})^{(12)}$, and $\{\{a \otimes b, c\}\}_L = (\{\{a, c\}\} \otimes b)^{(23)}$.

Definition 4 [15, 18, 20]. A double Lie algebra is a linear space $V$ equipped with a double bracket satisfying the identities (2) and (3).

Due to [8], an ideal of a double Lie algebra $V$ is a subspace $I \subseteq V$ such that

$$\{\{V, I\}\} + \{\{I, V\}\} \subseteq I \otimes V + V \otimes I.$$ 

Given an ideal $I$ of a double Lie algebra $V$, we have a natural structure of a double Lie algebra on the quotient space $V/I$, i.e., $\{\{x + I, y + I\}\} = \{\{x, y\}\} + I \otimes V + V \otimes I$.

Let us define homomorphisms of double Lie algebras as follows. Let $L$ and $L'$ be double Lie algebras and let $\varphi: L \to L'$ be a linear map. Then $\varphi$ is called a homomorphism from $L$ to $L'$ if

$$(\varphi \otimes \varphi)(\{\{a, b\}\}) = \{\varphi(a), \varphi(b)\}$$
holds for all $a, b \in L$. Note that the kernel of any homomorphism from $L$ is an ideal of $L$.

Definition 5 [8]. A double Lie algebra $V$ is said to be simple if $\{\{V, V\}\} \neq (0)$ and there are no nonzero proper ideals in $V$.

3 Finite-dimensional double Lie algebras

Suppose that $V$ is a finite-dimensional space. In [8], it was shown that every double Lie algebra structure $\{\cdot, \cdot\}$ on $V$ is determined by a linear operator $R: \text{End}(V) \to \text{End}(V)$, precisely,

$$\{\{a, b\}\} = \sum_{i=1}^{N} c_i(a) \otimes R(e_i^*)(b), \quad a, b \in V,$$ (5)
where \(e_1, \ldots, e_N\) is a linear basis of \(\text{End}(V)\), \(e_1^*, \ldots, e_N^*\) is the corresponding dual basis relative to the trace form.

A linear operator \(P\) on \(\text{End}(V)\) is called skew-symmetric if \(P = -P^*\), where \(P^*\) is the conjugate operator on \(\text{End}(V)\) relative to the trace form.

**Theorem 1** [8]. Let \(V\) be a finite-dimensional vector space with a double bracket \(\{\cdot, \cdot\}\) determined by an operator \(R: \text{End}(V) \rightarrow \text{End}(V)\) by (5). Then \(V\) is a double Lie algebra if and only if \(R\) is a skew-symmetric RB-operator of weight 0 on \(\text{End}(V)\).

**Remark 1.** Theorem 1 was stated in [18] in terms of skew-symmetric solutions of the associative Yang–Baxter equation (AYBE). Since there is a one-to-one correspondence between solutions of AYBE and Rota–Baxter operators of weight 0 on the matrix algebra [9], Theorem 1 follows from [18]. Actually, Theorem 1 was mentioned also in [15].

Let us consider several examples of double Lie algebras and corresponding RB-operators. We will use the linear basis \(e_{ij}, 1 \leq i, j \leq \dim(V), \text{of} \ \text{End}(V)\). So, we have \(e_{ij}^* = e_{ji}\) relative to the trace form.

In the case of a one-dimensional double Lie algebra \(L\), we have by (2) only zero double bracket.

**Example 1** [8, 15, 23]. The space \(F^2 = Fe_1 \oplus Fe_2\) equipped with a double product \(\{e_1, e_1\} = e_1 \otimes e_2 - e_2 \otimes e_1\) (others are zero) is a double Lie algebra. The corresponding RB-operator on \(M_2(F)\) is \(R_1(e_{11}) = e_{21}\) and \(R_1(e_{12}) = -e_{11}\) (others are zero).

**Example 2** [8, 15, 16]. The space \(F^2\) with a double product \(\{e_1, e_2\} = e_1 \otimes e_1 = -\{e_2, e_1\}\) is again a double Lie algebra. The corresponding RB-operator on \(M_2(F)\) is \(R_2(e_{11}) = e_{12}\) and \(R_2(e_{21}) = -e_{11}\).

The RB-operators \(R_1\) and \(R_2\) are conjugate with the transpose of matrices, i.e., \(R_2 = R_1^T\), where \(T\) denotes the transpose. However, the algebraic properties of the double Lie algebras from Examples 1 and 2 are quite different, see [8].

Note that all RB-operators (including skew-symmetric) of weight 0 on \(M_2(F)\) were classified by M. Aguiar [2] in 2000 and all skew-symmetric RB-operators of weight 0 on \(M_3(\mathbb{C})\) were described by V. V. Sokolov [19] in 2013.

**Example 3.** Consider the restriction of the double bracket defined in [23] §6.5] on the infinite-dimensional path algebra over a field \(F\) arisen from the quiver \(Q\) with the vertex set \(\{1, 2\}\) and the edge set \(\{e_1, e_2, a, a^*\}\), where \(a = (1, 2)\) and \(a^* = (2, 1)\). We put \(L = \text{Span}\{e_1, e_2, a, a^*\}\), and the double bracket on \(L\) equals
\[
\{a, a^*\} = e_2 \otimes e_1, \quad \{a^*, a\} = -e_1 \otimes e_2,
\]
all other double brackets are zero. Let us identify \(e_3 = a\) and \(e_4 = a^*\). By [23] we get the RB-operator \(R\) on \(M_4(F)\) defined as follows, \(R(e_{32}) = e_{14}\), \(R(e_{41}) = -e_{23}\).

### 4 Infinite-dimensional double Lie algebras

Consider a countable-dimensional double Lie algebra \(\langle V, \{\cdot, \cdot\}\rangle\). We fix a linear basis \(u_i, i \in \mathbb{N}, \text{of } V\). Define \(e_{ij} \in \text{End}(V)\) by the formula \(e_{ij}u_k = \delta_{jk}u_i\). Let \(\varphi \in \text{End}(V)\), then we may write \(\varphi = \sum_{ij} a_{ij}e_{ij}\). We identify \(\varphi\) with an infinite matrix \([\varphi] = (a_{ij})_{i,j \geq 0}\). Since
\( \varphi \in \operatorname{End}(V) \) is well-defined, there is only a finite number of nonzero elements in every column of the matrix \([\varphi]\), i.e., \(a_{ik} = 0\) for almost all \(i\) when \(k\) is fixed.

Let us define the subalgebra \(\operatorname{End}_f(V)\) of \(\operatorname{End}(V)\) as follows,

\[
\operatorname{End}_f(V) = \{ \varphi \in \operatorname{End}(V) \mid \text{for every } i, \ [\varphi]_{ij} = 0 \text{ for almost all } j \}.
\]

Introduce \(I\) as an ideal in \(\operatorname{End}_f(V)\) linearly spanned by matrix unities \(e_{ij}\).

Let \(\varphi \in \operatorname{End}_f(V) = \sum_{ij} a_{ij} e_{ij}\). We define the symmetric non-degenerate bilinear trace form \(\langle \cdot, \cdot \rangle\) on \(I \times \operatorname{End}_f(V) \cup \operatorname{End}_f(V) \times I\) as follows,

\[
\langle e_{kl}, \varphi \rangle = \langle \varphi, e_{kl} \rangle = \operatorname{tr}(e_{kl} \varphi) = a_{lk}.
\]

Moreover, the form is associative, i.e., \(\langle a, bc \rangle = \langle ab, c \rangle\), where at least one of \(a, b, c\) lies in \(I\) and others are from \(\operatorname{End}_f(V)\).

Given a double bracket algebra \(\{\{\cdot, \cdot\}\}\) on a space \(V\), we may define a linear operator \(R: I \to \operatorname{End}(V)\) by the formula

\[
\{\{a, b\}\} = \sum_{i,j \geq 0} e_{ij}(a) \otimes R(e_{ji})(b), \quad a, b \in V.
\]  \hspace{1cm} (6)

Conversely, given an operator \(R: I \to \operatorname{End}(V)\), one can define a double bracket on \(V\) by the formula (6). Note that the correspondence does not work if \(R: \operatorname{End}(V) \to \operatorname{End}(V)\).

Moreover, we define a conjugate operator \(R^*: I \to \operatorname{End}(V)\) as follows,

\[
\{\{b, a\}\}^{(12)} = \sum_{i,j \geq 0} e_{ij}(a) \otimes R^*(e_{ji})(b), \quad a, b \in V.
\]  \hspace{1cm} (7)

Denote \(R(e_{st}) = \sum_{k,l} a_{kl}^{st} e_{kl}\). By (6), \(a_{kl}^{st}\) equals the coefficient by \(u_t \otimes u_k\) of the double product \(\{\{u_s, u_t\}\}\). Analogously, put \(R^*(e_{st}) = \sum_{k,l} b_{kl}^{st} e_{kl}\). Then \(b_{kl}^{st}\) equals the coefficient by \(u_k \otimes u_t\) of the double product \(\{\{u_t, u_s\}\}\). Hence,

\[
\langle R(e_{st}), e_{kl} \rangle = \{\{u_s, u_k\}\}|_{u_t \otimes u_l} = \langle R^*(e_{kl}), e_{st} \rangle.
\]

Generally we have

\[
\langle R(x), y \rangle = \langle x, R^*(y) \rangle, \quad x, y \in I.
\]  \hspace{1cm} (8)

**Remark 2.** It is not clear how to introduce objects defined above in invariant manner. For example, consider a linear basis \(u_s, s \in \mathbb{N}\), of \(V\) and a linear map \(\psi \in \operatorname{End}_f(V)\) defined as follows, \(\psi(u_0) = u_0\) and \(\psi(u_s) = 0\), \(s > 0\). Let us consider the basis \(w_s, s \in \mathbb{N}\), of \(V\), where \(w_0 = u_0\) and \(w_s = u_0 + u_s\), \(s > 0\). Then \(\psi(w_s) = w_0\) for all \(s\). Thus, \(\psi \notin \operatorname{End}_f(V)\). Hence, the change of the basis does not preserve the condition \(R: I \to \operatorname{End}_f(V)\).

**Theorem 2.** Let \(V\) be a countable-dimensional vector space with a fixed linear basis \(u_i\) and with a double bracket \(\{\{\cdot, \cdot\}\}\) determined by a linear map \(R \in \operatorname{End}_f(V)\) by \(6\). Then \(V\) is a double Lie algebra if and only if \(R\) is a skew-symmetric RB-operator of weight 0 from \(I\) to \(\operatorname{End}_f(V)\).

**Proof.** By (6) and (7), the identity (2) holds if and only if \(R = -R^*\).
Define $F_{12} \in \text{End}(V^{\otimes 3})$ by

$$F_{12}(a \otimes b \otimes c) = \{\{a, \{b, c\}\}\}_L = \sum_{i,j} e_j(a) \otimes R(e_j^*) R(e_i)(b) \otimes R(e_j^*)(c), \quad a, b, c \in V.$$  

For $x, y \in I$, we compute applying associativity of the form $\langle \cdot, \cdot \rangle$

$$(\langle x, \cdot \rangle \otimes \langle y, \cdot \rangle \otimes \text{id})F_{12} = \sum_{i,j} \langle x, e_j \rangle \langle y, R(e_j^*)e_i \rangle R(e_i^*)$$

$$= \sum_i \left( \sum_j \langle x, e_j \rangle R(e_j^*)e_i \right) R(e_i^*) = \sum_i \langle y, R(x)e_i \rangle R(e_i^*)$$

$$= \sum_i \langle yR(x), e_i \rangle R(e_i^*) = R(yR(x)). \quad (9)$$

Analogously, put

$$F_{23}(a \otimes b \otimes c) = \{\{b, \{a, c\}\}\}^{\otimes 12}_R = \sum_{i,j} e_j(a) \otimes e_i(b) \otimes R(e_j^*)(R(e_j^*)(c)),$$

$$G_{12}(a \otimes b \otimes c) = \{\{a, b\}, c\}_L = \sum_{i,j} e_i(e_j(a)) \otimes R(e_j^*)(b) \otimes R(e_i^*)(c).$$

Then for $x, y \in I$ we have

$$(\langle x, \cdot \rangle \otimes \langle y, \cdot \rangle \otimes \text{id})F_{23} = R(y)R(x), \quad (\langle x, \cdot \rangle \otimes \langle y, \cdot \rangle \otimes \text{id})G_{12} = R(R^*(y)x).$$

Thus, the identities (2), (3) hold if and only if $R$ is a skew-symmetric RB-operator of weight 0 from $I$ to $\text{End}_f(V)$. \hfill \Box

**Remark 3.** We restrict $R$ in Theorem 2 as an operator from $I$ to $\text{End}_f(V)$ instead of $\text{End}(V)$, since otherwise the term $R(yR(x))$ in (1) is not well-defined.

**Example 4.** The space $V = F[t]$ equipped with

$$\{\{t^n, t^m\}\} = \frac{(t^n \otimes 1 - 1 \otimes t^n)(t^m \otimes 1 - 1 \otimes t^m)}{t \otimes 1 - 1 \otimes t}$$

is a double Lie algebra $L_1$.

Compute the operator $R_1 : I \to \text{End}(V)$ corresponding to the double Lie algebra $L_1$,

$$R_1(e_{ij}) = \begin{cases} -(e_{i,j+1} + e_{i+1,j+2} + \ldots), & i > j \\ e_{0,j-i+1} + e_{1,j-i+2} + \ldots + e_{i-1,j}, & i \leq j, \end{cases} \quad (10)$$

where the sum is infinite when $i > j$. By the formula, $R_1 \in \text{End}'(V)$ and by Theorem 2, $R_1$ is a skew-symmetric RB-operator from $I$ to $\text{End}_f(V)$.

Let us identify the matrix algebra $M_n(F)$ of order $n$ with $e_{ij} \in I$, $0 \leq i, j \leq n - 1$. Given an operator $P$ from $I$ to $\text{End}_f(V)$, by the projection $P_n$ we mean a linear operator of the space $\text{Span}\{e_{ij} \mid 0 \leq i, j \leq n - 1\}$ acting as follows: $P(e_{ij}) - P_n(e_{ij}) \in \text{Span}\{e_{kl} \mid n \leq k \text{ or } n \leq l\}$.
One can check that the linear operator \((R_1)_n\) of \(M_n(F)\) is an RB-operator of weight 0 on \(M_n(F)\) for each \(n\). Moreover, \(((R_1)_n(\psi_n))^{(T)}\) coincides with the RB-operator from [9, Example 5.15] and it appears in [3, Example 2.3.3] in terms of the solution of associative Yang–Baxter equation. Here \(\psi_n\) is the automorphism of \(M_n(F)\) defined as follows,

\[
\psi(e_{ij}) = e_{n-1-i,n-1-j}.
\]

**Example 5.** Consider \(R_2 \in \text{End}'(V)\) such that

\[
R_2(e_{ij}) = \begin{cases} 
-(e_{i-1,j} + e_{i-2,j-1} + \ldots + e_{i-1-j,0}) & i > j, \\
 e_{i,j+1} + e_{i+1,j+2} + \ldots, & i \leq j. 
\end{cases}
\]  

(11)

We have defined \(R_2\) in such a way that \((R_2)_n = (((R_1)_n(\psi_n))^{(T)}\). Such definition does not guarantee that we necessarily obtain a skew-symmetric RB-operator of weight 0 from \(I\) to \(\text{End}_f(V)\). Thus, we have to state this property of \(R_2\).

**Proposition 3.** The operator \(R_2\) is a skew-symmetric RB-operator of weight 0 from \(I\) to \(\text{End}_f(V)\).

**Proof.** Firstly, we check the identity \(R_2(e_{ij})R_2(e_{kl}) = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl}))\) considering different cases of the values of indices.

**Case 1:** \(i > j, k > l\). Then

\[
\alpha = R_2(e_{ij})R_2(e_{kl}) = (e_{i-1,j} + \ldots + e_{i-1-j,0})(e_{k-1,l} + \ldots + e_{k-1-l,0}),
\]

\[
\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl}))
\]

\[
= R_2((e_{i-1,j} + \ldots + e_{i-1-j,0})e_{kl} + e_{ij}(e_{k-1,l} + \ldots + e_{k-1-l,0})).
\]

Let \(k > j\), i.e., \(j = k - 1 - p\) for some \(p \geq 0\). Then

\[
\beta = -\chi_{j+l+1-k\geq 0}R_2(e_{ij}e_{k-1-p,l-p}) = -\chi_{j+l+1-k\geq 0}R_2(e_{ij+l+1-k})
\]

\[
= \chi_{j+l+1-k\geq 0}(e_{i-1,j+l+1-k} + \ldots + e_{i-j-l+k-2,0}),
\]

since \(i > j + l + 1 - k\). Here \(\chi_P = 1\), if \(P\) is true, and \(\chi_P = 0\), else. Moreover,

\[
\alpha = \chi_{j+l+1-k\geq 0}(e_{i-1,j+l+1-k} + \ldots + e_{i-j-l+k-2,0}) = \beta.
\]

Let \(k \leq j\), then applying the inequality \(i > j + l + 1 - k\), we compute

\[
\beta = -R_2(e_{i-1-j+k,k}e_{kl}) = -R_2(e_{i-1-j+k,l}) = e_{i-2-j+k,l} + \ldots + e_{i-2-j+k-l,0}.
\]

On the other hand,

\[
\alpha = e_{i-2-j+k,l} + e_{i-2-j+k-1,l-1} + \ldots + e_{i-2-j+k-l,0} = \beta.
\]

**Case 2:** \(i \leq j, k \leq l\). Then

\[
\alpha = R_2(e_{ij})R_2(e_{kl}) = (e_{i,j+1} + \ldots)(e_{k,l+1} + \ldots),
\]

\[
\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) = R_2((e_{i,j+1} + \ldots)e_{kl} + e_{ij}(e_{k,l+1} + \ldots)).
\]
Let $k > j$, i.e., $j = k - p$ for some $p > 0$. Then $\alpha = e_{i+p-1,l+1} + \ldots = e_{i+k-j-1,l+1} + \ldots$. Also, 
\[
\beta = R_2(e_{i+p-1,k}e_{kl}) = R_2(e_{i+k-j-1,l}) = e_{i+k-j-1,l+1} + \ldots = \alpha,
\]
since $i + k \leq j + l + 1$.

Let $j \geq k$, i.e., $j = p + k$ for some $p \geq 0$. Then $\alpha = e_{i,l+p+2} + \ldots = e_{i,j+l+2-k} + \ldots$.

Further, 
\[
\beta = R_2(e_{ij}e_{k+p,l+p+1}) = R_2(e_{i,l+p+1}) = R_2(e_{i,l+1}) = e_{i,l-j+2} + \ldots = \alpha,
\]
since $i + k < l + j + 2$.

**Case 3:** $i > j$, $k \leq l$. Then 
\[
\alpha = R_2(e_{ij})R_2(e_{kl}) = -(e_{i-1,j} + \ldots + e_{i-1,0})(e_{k,l+1} + \ldots),
\]
\[
\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) = R_2(-(e_{i-1,j} + \ldots + e_{i-1,0})e_{kl} + e_{ij}(e_{k,l+1} + \ldots)).
\]

When $k > j$, we get $\alpha = \beta = 0$. Let $j = k + p$ for some $p \geq 0$. We compute $\alpha = -(e_{i-1,l+j-k+1} + \ldots + e_{i-1,j+k,l+1})$. On the other hand, $\beta = R_2(-e_{i-1-j+k,l} + e_{i,l+j-k+1})$.

If $i + k \leq j + l + 1$, then 
\[
\beta = -(e_{i-1-j+k,l+1} + \ldots) + (e_{i,l+j-k+2} + \ldots) = -(e_{i-1,l+j-k+1} + \ldots + e_{i-1-j+k,l+1}) = \alpha.
\]

Else, 
\[
\beta = (e_{i-2-j+k,l} + \ldots + e_{i-2-j+k-l,0}) - (e_{i-1,l+j-k+1} + \ldots + e_{i-2-j+k-l,0}) = \alpha.
\]

**Case 4:** $i \leq j$, $k > l$. Then 
\[
\alpha = R_2(e_{ij})R_2(e_{kl}) = -(e_{i,j+1} + \ldots)(e_{k-1,l} + \ldots + e_{k-1,0}),
\]
\[
\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) = R_2((e_{i+1,j} + \ldots)e_{kl} - e_{ij}(e_{k-1,l} + \ldots + e_{k-1,0})).
\]

When $j + 1 \geq k$, we have $\alpha = 0 = \beta$.

Let $j + 2 \leq k$, i.e., $k = j + 2 + p$ for some $p \geq 0$. Thus, $\alpha = -(e_{i,l-k-j+2} + \ldots + e_{i+k-j-2,l})$. Also, $\beta = R_2(e_{i+k-j-1,l}) - R_2(e_{i,l-k+j+1})$. If $i + k > j + l + 1$, we have 
\[
\beta = -(e_{i+k-j-2,l} + \ldots + e_{i+k-j-l-2,0}) + (e_{i-1,l-k+j+1} + \ldots + e_{i+k-j-l-2,0})
\]
\[
= -(e_{i+k-j-2,l} + \ldots + e_{i,l-k+j+2}) = \alpha.
\]

If $i + k \leq j + l + 1$, we have 
\[
\beta = (e_{i+k-j-1,l+1} + \ldots) - (e_{i,l-k+j+2} + \ldots) = -(e_{i+k-j-2,l} + \ldots + e_{i,l-k+j+2}) = \alpha.
\]

Now, we check that $R_2$ is also skew-symmetric operator from $I$ to $\text{End}_f(V)$. Thus, we have to show that $R_2(e_{ij}) + R_2^*(e_{ij}) = 0$ for all $i, j \geq 0$. By the definition, 
\[
R_2^*(e_{ij}) = \sum_{k,l \geq 0} R(e_{ik})|e_{ij}e_{kl},
\]

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where $R(e_{lk})|_{e_{ji}} = (R(e_{lk}), e_{ij})$ denotes the $e_{ji}$-coordinate of $R(e_{lk})$.

**Case 1:** $i \leq j$. Then $R(e_{lk})|_{e_{ji}}$ is nonzero only when $(l, k) \in \{(j+1+p, i+p) \mid p \geq 0\}$. Thus, $R_2^s(e_{ij}) = -(e_{i,j+1} + \ldots) = -R_2(e_{ij})$, as required.

**Case 2:** $i > j$. Then $R(e_{lk})|_{e_{ji}}$ is nonzero only for $(l, k) \in \{(j-p, i-1-p) \mid p = 0, \ldots, j\}$. Therefore, $R_2^s(e_{ij}) = e_{i-1,j} + \ldots + e_{i-1-j,0} = -R_2(e_{ij})$.

**Corollary.** We have a double Lie algebra structure $L_2$ on $V$ defined due to (5) by $R_2$,

$$\llbracket t^n, t^m \rrbracket = -\frac{(t^n \otimes t^m - t^m \otimes t^n)}{t \otimes 1 - 1 \otimes t}.$$ 

Now, we prove that the obtained double Lie algebra $L_2$ is simple. It is the first example of a simple double Lie algebra.

**Theorem 3.** The double Lie algebra $L_2$ is simple.

**Proof.** Suppose that $J$ is a nonzero proper ideal in $L_2$. Define $n$ as the minimal degree in $t$ of elements from $J$. Let us show that $n = 0$. If $n > 0$, then consider $f = t^n + \sum_{j=0}^{n-1} \alpha_j t^j \in J$. We have that the product

$$\llbracket 1, f \rrbracket = t^{n-1} \otimes 1 + t^{n-2} \otimes t + \ldots + 1 \otimes t^{n-1} + \sum_{j=1}^{n-1} \alpha_j (t^{j-1} \otimes 1 + \ldots + 1 \otimes t^{j-1})$$

lies in $V \otimes J + J \otimes V$.

Consider the map $\psi : V \otimes V \to V/J \otimes V/J$ acting as follows: $\psi(v \otimes w) = (v+J) \otimes (w+J)$. On the one hand, $1 + J, t + J, \ldots, t^{n-1} + J$ are linearly independent elements of $V/J$. On the other hand, $J \otimes V + V \otimes J = \ker(\psi)$. Thus, $\llbracket 1, f \rrbracket$ is at the same time zero and nonzero element of $V/J \otimes V/J$. We obtain a contradiction. So, $n = 0$ and $1 \in J$.

Let us prove by induction on $s \geq 0$ that $t^s \in J$. For $s = 0$, it is true. Suppose that $s > 0$ and we have proved that $t^q \in J$ for all $q < s$. Since $1 \in J$, we have

$$\llbracket 1, t^{2s+1} \rrbracket = t^s \otimes 1 + t^{2s-1} \otimes t + \ldots + t^{s+1} \otimes t^{s-1} + t^s \otimes t^s + \ldots + 1 \otimes t^{2s} \in V \otimes J + J \otimes V.$$ 

So, $t^s \otimes t^s \in V \otimes J + J \otimes V$. Hence, $\psi(t^s \otimes t^s) = 0$, it means that $t^s \in J$.

In the next two examples we consider conjugation of $R_1$ and $R_2$ with transpose and corresponding double Lie algebras.

**Example 6.** For $R_3 = R_1^{(T)}$, we get a double Lie algebra $L_3$ with the double bracket

$$\llbracket t^n, t^m \rrbracket = \frac{(t^{n+1} \otimes t^{m+1} - t^{m+1} \otimes t^{n+1})}{t \otimes 1 - 1 \otimes t}.$$ 

**Example 7.** For $R_4 = R_2^{(T)}$, we get a double Lie algebra $L_4$ with the double bracket

$$\llbracket t^n, t^m \rrbracket = -\frac{(t^{n+1} \otimes 1 - 1 \otimes t^{n+1})(t^{m+1} \otimes 1 - 1 \otimes t^{m+1})}{t \otimes 1 - 1 \otimes t}.$$ 

In [23], it was stated that each homogeneous double Poisson algebra on $F[t]$ up to an equivalence is either $L_1$ or $L_4$. It is easy to show that the double Lie algebras $L_2$ and $L_3$ do not satisfy (4), for example, since $\llbracket t, 1 \rrbracket \neq 0$, and therefore do not define the
structure of a double Poisson algebra on $F[t]$. Note the following connections between double brackets in $L_1, L_2, L_3, L_4$:

$$\{\{t^n, t^m\}\}_{L_3} = -(t \otimes t) \{\{t^n, t^m\}\}_{L_2}, \quad \{\{t^n, t^m\}\}_{L_4} = -\{\{t^{n+1}, t^{m+1}\}\}_{L_1}.$$ 

**Example 8** (V. Kac, see [3]). Consider the double Poisson algebra $dY(N) = F[t] \otimes M_N(F)$. Its double bracket relative to the basis $T_{ij}^n = t^n \otimes e_{ij}$, $n \geq 0$, $i, j = 1, \ldots, N$, has the following form:

$$\{\{T_{ij}^m, T_{kl}^n\}\} = \sum_{r=0}^{\min\{m,n\}-1} (T_{r}^{kj} \otimes T_{m+n-r-1}^{il} - T_{m+n-r-1}^{kj} \otimes T_{r}^{il}),$$

the inner bimodule $dY(N)$-action is the associative product.

It is worth mentioning that these relations are similar to the defining relations of the Yangian $Y(gl_N)$:

$$[T_{ij}^m, T_{kl}^n] = \sum_{r=0}^{\min\{m,n\}-1} (T_{r}^{kj} T_{m+n-r-1}^{il} - T_{m+n-r-1}^{kj} T_{r}^{il}).$$

We get an RB-operator $R$: $I \otimes M_N(F) \rightarrow \text{End}_f(V) \otimes M_N(F)$ such that the double bracket on $dY(N)$ is defined by (3) with the help of $R$. We have

$$R(e_{ij} \otimes e_{st}) = \begin{cases} (e_{i,j+1} + e_{i+1,j+2} + \ldots) \otimes e_{st}, & i > j \\ -(e_{0,j-i+1} + e_{1,j-i} + \ldots + e_{i-1,j}) \otimes e_{st}, & i \leq j, \end{cases}$$

where $e_{ij} \in I$, $e_{st} \in M_N(F)$. Actually, $R = (-R_1) \otimes \text{id}$.

**Remark 4.** We may extend the double Lie algebra structures $L_1, L_2, L_3, L_4$ on $F[t, t^{-1}]$ and $dY(N)$ on $F[t, t^{-1}] \otimes M_N(F)$ respectively. It is enough to let both sums in the definition of the corresponding RB-operator $R_1, R_2, R_3, R_4$, and $R$ be infinite. Introduce $e_{ij} \in \text{End}(V)$, where $V = F[t, t^{-1}]$, $i, j \in \mathbb{Z}$, in such a way that $e_{ij}t^k = \delta_{jk}t^i$. For example, let us extend $R_2$. We define

$$\tilde{R}_2(e_{ij}) = \begin{cases} -\sum_{p=0}^{\infty} e_{i-p,j-p}, & i > j, \\ \sum_{p=0}^{\infty} e_{i+p,j+1+p}, & i \leq j. \end{cases}$$

Analogously to the proof of Proposition 3, one can check that $\tilde{R}_2$ is a skew-symmetric RB-operator from $\text{Span}\{e_{ij} \mid i, j \in \mathbb{Z}\}$ to $\text{End}_f(V)$. By Theorem 2, we get a double Lie algebra structure $\tilde{L}_2$ on $F[t, t^{-1}]$. Analogously we get double Lie algebras $\tilde{L}_1, \tilde{L}_3, \tilde{L}_4,$ and $d\tilde{Y}(N)$.

**Remark 5.** The operator $R_2$ is injective. Moreover, $I \subset \text{Im}(R_2)$. So, we may define the inverse map $d = R_2^{-1}$ from $I$ to $I$. Then $d(e_{ij}) = e_{i,j-1} - e_{i+1,j}$ is a derivation of $I$. By [13], every such derivation is an inner derivation, i.e., of the form $x \rightarrow x a - a x$, where $a \in \text{End}(V)$ such that $a^{-1}[U]$ is finite-dimensional for each finite-dimensional subspace $U \subset V$. 

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It is easy to show that $d(x) = Ax - xA$ for $A = e_{10} + e_{21} + \ldots$. Given a positive integer $k$, we may define $d_k : I \to I$ such that $d_k(x) = A^k x - x A^k$. By $d_k$, we may define $P_k \in \text{End}'(V)$ as an analogue of the operator $R_2$. And all of $P_k$ provide by $d_k$ some double Lie algebra structures on $F[t]$.

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