Abstract

We study qualitative properties of two-dimensional freezing cellular automata with a binary state set initialized on a random configuration. If the automaton is also monotone, the setting is equivalent to bootstrap percolation. We explore the extent to which monotonicity constrains the possible asymptotic dynamics. We characterize the monotone automata that almost surely fill the space starting from any nontrivial Bernoulli measure. In contrast, we show the problem is undecidable if the monotonicity condition is dropped. We also construct examples where the space-filling property depends on the initial Bernoulli measure in a non-monotone way.

1 Introduction

Bootstrap percolation is a class of deterministic growth models in random environments. The basic premise is that we have a discrete universe of sites, typically arranged on a regular lattice such as $\mathbb{Z}^d$, a random subset of which are initially infected. A deterministic rule, typically uniform in space and time, allows the infection to spread into healthy sites that have enough infected neighbors. The main quantities of interest are then the probability of every site being eventually infected (called percolation), and the distribution of the time of infection, as a function of the initial distribution of infected sites. Bootstrap percolation was introduced by Chalupa, Leath and Reich in [4] as a model of impurities in magnetic materials. See [5] for an overview of subsequent literature.

Bootstrap percolation processes can be formalized as cellular automata (CA for short) on the binary state set $\{0, 1\}$ that are monotone ($x \leq y$ implies $f(x) \leq f(y)$) and freezing ($x \leq f(x)$ always holds) with respect to the cellwise partial order. The automaton is initialized on a random configuration $x \in \{0, 1\}^{\mathbb{Z}^d}$.
and the freezing property guarantees that the iterates $f^n(x)$ converge to a limit configuration. Percolation corresponds to this limit being the all-1 configuration. We say that $f$ trivializes the initial probability measure, if percolation happens almost surely.

Dropping the monotonicity requirement results in a richer set of possible asymptotic behaviors. Such automata may still be understood as models of physical or sociological phenomena. For example, if the cells of a graph represent agents with political leanings, then non-monotone rules can model individuals becoming suspicious of a sudden influx of opposing views among their peers. Examples of freezing non-monotone CA have been considered in the literature, like the “rule one” of S. Ulam [16] as an attempt to study models of crystal growth, or “life without death” [9] which is a freezing version of Conway’s Life. The dynamics of freezing cellular automata have been studied explicitly in e.g. [7, 6, 2]. We note that in the literature it is common to require freezing CA to be decreasing rather than increasing, but here we choose to follow the opposite convention of percolation theory.

In this article we study the variety of asymptotic behaviors exhibited by monotone and non-monotone freezing CA when initialized on Bernoulli random configurations. In the monotone case, we provide a characterization of those automata under which almost all initial configurations percolate with respect to at least one nontrivial Bernoulli measure. We state the characterization in terms of two criticality classifications of bootstrap percolation models. In [8] Gravner and Griffeath study *threshold growth dynamics*, which are a class of binary freezing monotone CA rules defined by a fixed neighborhood $N \subset \mathbb{Z}^2$ and a threshold $\theta \geq 0$, where the local rule turns a 0 into a 1 precisely when the number of 1-states in the neighborhood is at least $\theta$. They call such a CA *subcritical* if it has a fixed point with a nonzero but finite number of 0-states, *supercritical* if there is a configuration $x \in \{0,1\}^{\mathbb{Z}^2}$ with a finite number of 1-states and $\lim_{n} f^n(x)$ contains infinitely many 1-states, and *critical* if neither condition holds.

Bollobás, Smith and Uzzell provide in [3] an a priori different classification for the dynamics of arbitrary binary freezing monotone $\mathbb{Z}^2$-CA in terms of stable directions. Their definition of supercriticality agrees with that of Gravner and Griffeath, but their version of subcriticality is strictly weaker. By [3, 1], a binary freezing monotone $\mathbb{Z}^2$-CA trivializes every nontrivial Bernoulli measure if and only if it is critical or supercritical and the sense of [3], and the property is decidable. Our results in Section 4 concern the dual problem: given a binary freezing monotone CA, does there exist a nontrivial Bernoulli measure it trivializes? We show that this is equivalent to the stronger version of subcriticality defined by Gravner and Griffeath. In particular, the property is also decidable. As part of our proof, we give a characterization of subcriticality using stable directions, which was stated in [3] without proof.

In Section 5 we study the larger class of binary freezing CA that may not be monotone. Our results in this context have a different flavor, as they highlight the increase in complexity of the asymptotic dynamics that results from discarding the monotonicity constraint. First, we show that while the property of not trivializing any nontrivial Bernoulli measure is still equivalent to subcriticality, it is no longer decidable. Second, we show that the measure trivialization property may be non-monotone, in the sense that there exists a freezing CA that trivializes the Bernoulli measure of weight $p$ but not the one of weight $q$,
for some $0 < p < q < 1$. This can be interpreted as the system having at least two phase transitions.

Several open problems arise naturally from our investigation. First, in the context of cellular automata it is natural to ask whether the results extend to arbitrary finite state sets. We prove some of our auxiliary results in this context, but our main results concern the binary case. Do monotone freezing CA with three or more states have significantly more complex dynamics than binary CA? In particular, are the analogous trivialization properties decidable? Second, our example of a freezing CA with two phase transitions can likely be generalized to realize a wide range of exotic trivialization phenomena.

2 Definitions

For a finite alphabet $A$ and $d \geq 1$ (we will mostly be dealing with the case $d = 2$), the $d$-dimensional \textit{full shift} if the set $A^{\mathbb{Z}^d}$ equipped with the product discrete topology. Elements of $A^{\mathbb{Z}^d}$ are called \textit{configurations}. For $a \in A$, the \textit{$m$-uniform configuration} $x = a^{\mathbb{Z}^d} \in A^{\mathbb{Z}^d}$ is defined by $x_\vec{v} = a$ for all $\vec{v} \in \mathbb{Z}^d$. We have an action $\sigma : \mathbb{Z}^d \cap A^{\mathbb{Z}^d}$ of the additive group $\mathbb{Z}^d$ by homeomorphisms, called the \textit{shift action}, given by $\sigma_\vec{v}(x) = x_{\vec{v}+\vec{v}}$. If $A$ is a poset (partially ordered set), then we see $A^{\mathbb{Z}^d}$ as a poset with the cellwise order: $x \leq y$ means $x_\vec{v} \leq y_\vec{v}$ for all $\vec{v} \in \mathbb{Z}^d$.

A $d$-dimensional \textit{pattern} is a function $w \in A^D$ with $D \subset \mathbb{Z}^d$ finite. The topology of $A^{\mathbb{Z}^d}$ is generated by \textit{cylinder sets} of the form $[w]_\vec{v} = \{ x \in A^{\mathbb{Z}^d} | \sigma_\vec{v}(x)|_D = w \}$ for a pattern $w \in A^D$. If $\vec{v}$ is omitted, it is assumed to be $\vec{0}$. In a slight abuse of notation, each symbol $a \in A$ stands for the pattern $\vec{0} \mapsto a$ with domain $D = \{\vec{0}\}$, so that $[a]_\vec{0} = \{ x \in A^{\mathbb{Z}^d} | x_{\vec{0}} = a \}$.

A cellular automaton (CA for short) is a function $f : A^{\mathbb{Z}^d} \rightarrow A^{\mathbb{Z}^d}$ defined by a finite neighborhood $N \subset \mathbb{Z}^d$ and a local rule $F : A^N \rightarrow A$ with $f(x)_\vec{v} = F(\sigma_{-\vec{v}}(x)|_N)$. If $\|\vec{v}\|_{\infty} \leq r$ holds for all $\vec{v} \in N$, we say $r$ is a \textit{radius} for $f$. By the Curtis-Hedlund-Lyndon theorem, CA are exactly the continuous functions from $A^{\mathbb{Z}^d}$ to itself that commute with the shift action.

Denote by $\mathcal{M}(A^{\mathbb{Z}^d})$ the set of Borel probability measures on $A^{\mathbb{Z}^d}$, and by $\mathcal{M}_d(A^{\mathbb{Z}^d})$ the \textit{$\sigma$-invariant ones} (which satisfy $\mu(\sigma_\vec{v}(X)) = \mu(X)$ for all Borel sets $X$). We equip $\mathcal{M}(A^{\mathbb{Z}^d})$ with the weak-$*$ topology, or convergence on cylinder sets. The support $\text{Supp}(\mu)$ of $\mu \in \mathcal{M}(A^{\mathbb{Z}^d})$ is the unique smallest closed set $K \subset A^{\mathbb{Z}^d}$ with $\mu(K) = 1$. We can apply a CA $f$ to a measure $\mu \in \mathcal{M}(A^{\mathbb{Z}^d})$ by $f(\mu)(X) = \mu(f^{-1}X)$.

For a probability vector $\pi : A \rightarrow [0,1]$ (that satisfies $\sum_{a \in A} \pi(a) = 1$), the Bernoulli or product measure $\mu_\pi \in \mathcal{M}_d(A^{\mathbb{Z}^d})$ is the unique Borel measure with $\mu([P]) = \prod_{\vec{v} \in D} \pi(P_\vec{v})$ for all patterns $P \in A^D$. If $A = \{0,1\}$ and $0 \leq p \leq 1$, then $\mu_p = \mu_\pi$ is the Bernoulli measure with $\pi(1) = p$. For $x \in A^{\mathbb{Z}^d}$, denote by $\delta_x$ the unique measure with $\delta_x(\{x\}) = 1$.

The convex hull of a set $K \subset \mathbb{R}^d$ is denoted $\text{CHull}(K)$. The notation $\forall x \in X$ means “for all but finitely many $x \in X$”, and $\exists x \in X$ means “exists infinitely many $x \in X$.”
3 Freezing, monotonicity and measures

Definition 3.1. Let $P$ be a finite poset. A cellular automaton $f$ on $P^{Z^d}$ is freezing, if $x \leq f(x)$ for all $x \in P^{Z^d}$. It is monotone, if $f(x) \leq f(y)$ for all $x \leq y \in P^{Z^d}$.

In this paper, when considering freezing cellular automata on $\{0,1\}^{Z^d}$, we always implicitly refer to the poset with elements 0 and 1 such that $\{0 < 1\}$.

Lemma 3.2. A cellular automaton $f$ on $\{0,1\}^{Z^d}$ is freezing and monotone if and only if there exists a finite family $E$ of finite subsets of $Z^2 \setminus \{\emptyset\}$ with the following property. For all $x \in \{0,1\}^{Z^d}$, we have $f(x)_g = 1$ if and only if $x_0 = 1$ or there exists $N \in E$ with $x|_N = 1^N$.

Proof. Given a freezing and monotone $f$, choose $E$ as the family of minimal subsets of $Z^2 \setminus \{\emptyset\}$ such that $x_N = 1^N$ implies $f(x)_0 = 1$. The other direction is clear.

Definition 3.3. For a freezing monotone cellular automaton $f$ on $\{0,1\}^{Z^2}$, we write $E(f)$ for the set $E$ given by Lemma 3.2. We also denote

$$F(f) = \{N \in E(f) \mid \emptyset \notin \text{CHull}(N)\},$$

and $G(f) = E(f) \setminus F(f)$. For a finite family $E$ of incomparable subsets of $Z^2 \setminus \{\emptyset\}$, we denote by $f_E$ the cellular automaton defined by $E(f_E) = E$. If $E = \{N\}$ is a singleton, we may also abuse notation and write $f_N$ for $f_E$.

Definition 3.4. Let $\mu \in \mathcal{M}(S^{Z^2})$ be a measure. The $\mu$-limit set of a cellular automaton $f$ on $S^{Z^2}$ is

$$\Omega^\mu_f = \bigcup_{\nu \in \mathcal{F}} \text{Supp}(\nu),$$

where $\mathcal{F}$ is the set of limit points of the sequence $(f^n(\mu))_{n \in \mathbb{N}}$.

If $\mu \in \mathcal{M}_s(S^{Z^2})$ is shift-invariant, then $\Omega^\mu_f$ is the set of configurations $x$ such that no pattern $w$ occurring in $x$ satisfies $\lim_n \mu(f^{-n}([w]_G)) = 0$. $\mu$-limit sets were first defined in [13] in the shift-invariant case using this characterization. The measures that occur in our results are shift-invariant, but in some proofs we work with intermediate measures that are not.

Definition 3.5. Let $\mu \in \mathcal{M}(S^{Z^2})$ and let $f$ be a cellular automaton on $S^{Z^2}$. We say $f$ trivializes $\mu$, if $|\Omega^\mu_f| = 1$.

In cellular automata literature, a CA $f$ is called $\mu$-nilpotent if $\Omega^\mu_f = \{x\}$ for some unary configuration $x \in S^{Z^2}$.

If $P$ is a poset with a maximal element $m$, and $f$ is a freezing CA on $P^{Z^2}$ that trivializes a full-support Bernoulli measure, then the limit measure must of course be concentrated on the $m$-uniform point.

Lemma 3.6. Let $P$ be a finite poset, $\mu \in \mathcal{M}_s(P^{Z^2})$ of full support and $f$ a freezing cellular automaton on $P^{Z^2}$. The following conditions are equivalent:

- $f$ trivializes $\mu$
\[ \lim_{n \to \infty} f^n(\mu) = \delta_z \text{ for some (unary) } x \in P^{\mathbb{Z}^2} \]

- for some \( p \in P \) and \( \mu \)-almost every \( x \), we have \( f^n(x)_z = p \) for all \( z \in \mathbb{Z}^2 \) and all large enough \( n \in \mathbb{N} \) (depending on \( z \)).

**Proof.** Suppose that \( f \) trivializes \( \mu \), so that \( \Omega^\mu_f = \{ x \} \) for some \( x \in P^{\mathbb{Z}^2} \). Since \( \mu \) is shift-invariant, for each \( \vec{v} \in \mathbb{Z}^2 \) we have

\[
\{ \sigma_{\vec{v}}(x) \} = \Omega^{\vec{v}}_f = \Omega^\mu_f = \{ x \}
\]

and hence there exists \( p \in P \) such that \( x = p^{\mathbb{Z}^2} \) is unary. Thus, for each finite pattern \( w \) containing an occurrence of some \( q \in P \setminus \{ p \} \) we have \( \lim_{n \to \infty} f^n\mu([w]) = 0 \), and for each all-\( p \) pattern \( w' \) we have \( \lim_{n \to \infty} f^n\mu([w']) = 1 \). This implies \( \lim_{n \to \infty} f^n\mu = \delta_z \), the second item. The converse is clear, so the first two items are equivalent.

Denote by \( m \) a maximal element of \( P \). Since \( f \) is freezing,

\[
[m]_{\vec{z}} \subseteq f^{-n}(m)_{\vec{z}}
\]

for all \( n \in \mathbb{N} \) and \( \vec{z} \in \mathbb{Z}^2 \). From (1) and the full support of \( \mu \) it follows that no other state than \( m \) can be chosen as \( p \) in the third item. Let

\[
B = \{ x \in P^{\mathbb{Z}^2} \mid \forall \vec{v} \in \mathbb{Z}^2 \forall n \in \mathbb{N} : f^n(x)_\vec{v} = m \}
\]

be the set of configurations that satisfy the condition of the third item. Consider \( E_{\vec{z},n} = f^{-n}(P^{\mathbb{Z}^2} \setminus [m]_{\vec{z}}) \). From (1) we have \( E_{\vec{z},n+1} \subseteq E_{\vec{z},n} \) for all \( \vec{z} \) and \( n \), and

\[
B = \bigcup_{\vec{z} \in \mathbb{Z}^2} \bigcap_{n \in \mathbb{N}} E_{\vec{z},n}.
\]

Suppose that \( f \) does not trivialize \( \mu \). Then there must be \( \epsilon > 0 \) such that \( \mu(E_{\vec{z},n}) \geq \epsilon \) for all \( n \geq 0 \) (otherwise we would have \( \lim_{n \to \infty} \mu(f^{-n}([m])) = 1 \) since sets \( E_{\vec{z},n} \) are decreasing, a contradiction). By continuity of \( \mu \) from above, we deduce \( \mu(\bigcap_{n \in \mathbb{N}} E_{\vec{z},n}) \geq \epsilon \), so \( \mu(B) \leq 1 - \epsilon < 1 \). Therefore the third item does not hold.

Suppose then that the third item does not hold, so that \( \mu(B) < 1 \). Since \( P^{\mathbb{Z}^2} \setminus B = \bigcup_{\vec{z} \in \mathbb{Z}^2} \bigcap_{n \in \mathbb{N}} E_{\vec{z},n} \) has positive measure, \( \epsilon := \mu(\bigcap_{n \in \mathbb{N}} E_{\vec{z},n}) > 0 \) for some \( \vec{z} \). For each \( n \), we then have \( \sum_{p \neq m} \mu(f^{-n}([p]_{\vec{z}})) = \mu(E_{\vec{z},n}) \geq \epsilon \), and in particular \( \mu(f^{-n}([p]_{\vec{z}})) \geq \epsilon/|P| \) for some \( p \in P \setminus \{ m \} \). For some \( p \) this holds for infinitely many \( n \), so some limit point \( \nu \) of \( (f^n\mu)_{n \in \mathbb{N}} \) satisfies \( \nu(f^{-n}([p]_{\vec{z}}) \geq \epsilon/|P| \). Thus \( f \) does not trivialize \( \mu \). We have shown that the third item is equivalent to the first.

We note that the first two items of Lemma 3.6 are equivalent even without the freezing hypothesis, and for the third item we only need the condition that some state \( m \in P \) is persistent, that is, \( x_\vec{v} = m \) implies \( f(x)_\vec{v} = m \).

**Example 3.7.** Let \( h = f_{((0,1),(1,1))} \), and let \( x \in \{0,1\}^{\mathbb{Z}^2} \). Then \( h^n(x)_{\vec{z}} = 0 \) for all \( n \in \mathbb{N} \) if and only if there exists a path \( (\vec{z}_i)_{i \in \mathbb{N}} \) in \( \mathbb{Z}^2 \) such that \( \vec{z}_0 = \vec{0}, \vec{z}_{i+1} \in \{ \vec{z}_i + (0,1), \vec{z}_i + (1,1) \} \) and \( x_{\vec{z}_i} = 0 \) for all \( i \geq 0 \). Indeed, if such a path exists, then every cell in it will always have the state 0, including the origin. On the other hand, if an infinite path does not exist, by König’s lemma there is a bound for the length of the paths. If the maximal length of a path starting from the origin in \( x \) is \( k \), then that in \( h(x) \) is \( k - 1 \). Inductively, we see that \( h^k(x)_{\vec{z}} = 1 \).
Take the product measure \( \mu_p \in \mathcal{M}(\{0,1\}^\mathbb{Z}^2) \) for \( 0 < p < 1 \), and consider the probability \( \theta(p) \) that in a \( \mu_p \)-random configuration \( x \in \{0,1\}^{\mathbb{Z}^2} \) there exists an infinite path \((\vec{z}_i)_{i \in \mathbb{N}}\) of 0-states as above. This probability is clearly nonincreasing with respect to \( p \), and the infimum of those \( p \) for which it equals 0 is \( 1 - p_c \), where \( p_c \) is the critical probability of nearest-neighbor oriented site percolation on \( \mathbb{N}^2 \); see [10, Section 12.8] for discussion on the analogous bond percolation model. If \( p > 1 - p_c \), then \( h \) trivializes \( \mu_p \), and if \( p < 1 - p_c \), then it does not.

4 Freezing monotone CA

We begin our study of freezing monotone cellular automata on the binary alphabet by recalling the definitions of sub- and supercriticality from [5, 3]. Since the definitions are not equivalent, we rename them for consistency’s sake.

**Definition 4.1.** Let \( f \) be a freezing and monotone CA on \( \{0,1\}^{\mathbb{Z}^2} \). For a unit vector \( \vec{v} \in S^1 \), write \( x_{\vec{v}} \in \{0,1\}^{\mathbb{Z}^2} \) for the configuration defined by

\[
x_{\vec{v}} = \begin{cases} 1, & \text{if } \vec{v} \cdot \vec{z} < 0, \\ 0, & \text{otherwise.} \end{cases}
\]

A direction \( \vec{v} \in S^1 \) is **stable** for \( f \), if \( f(x_{\vec{v}}) = x_{\vec{v}} \), and **strongly stable** if it is contained in an open interval of stable directions. We say \( f \) is

- **strongly subcritical**, if every direction is stable,
- **subcritical**, if every open semicircle contains a strongly stable direction,
- **weakly subcritical**, if it subcritical but not strongly subcritical,
- **supercritical**, if some open semicircle contains no stable directions, and
- **critical**, if none of the above hold.

**Lemma 4.2.** Let \( f \) be a freezing and monotone CA on \( \{0,1\}^{\mathbb{Z}^2} \). Then the following are equivalent:

- \( f \) is strongly subcritical.
- \( F(f) = \emptyset \).
- There exists \( x \in \{0,1\}^{\mathbb{Z}^2} \) such that \( f(x) = x \) and \( C = \{ \vec{z} \in \mathbb{Z}^2 \mid x_{\vec{z}} = 0 \} \) is nonempty and finite.

The equivalence of the first two items was claimed in [3]. We include a proof for completeness, but postpone it until after a few auxiliary results. The first one allows us to separate convex compact sets with a rational line.

**Lemma 4.3.** For any disjoint convex compact sets \( A, B \subset \mathbb{R}^2 \), there exists a closed rational half-plane \( H \) such that \( A \subset H \) and \( B \cap H = \emptyset \).

**Proof.** By a standard hyperplane separation theorem, such as [11] Theorem 4.4(i)], there exist \( \vec{v} \in \mathbb{R}^2 \) and \( a < b \) such that the two half-planes \( G = \{ \vec{z} \in \mathbb{R}^2 \mid \vec{z} \cdot \vec{v} \leq a \} \) and \( G' = \{ \vec{z} \in \mathbb{R}^2 \mid \vec{z} \cdot \vec{v} \leq b \} \) satisfy \( A \subset G \) and \( B \cap G' = \emptyset \). We approximate \( \vec{v} \) by a rational vector \( \vec{q} \in \mathbb{Q}^2 \). By continuity of the dot product, if the approximation is accurate enough, the half-plane \( H = \{ \vec{z} \in \mathbb{R}^2 \mid \vec{z} \cdot \vec{q} \leq (a + b)/2 \} \) satisfies the claim. □
Lemma 4.4. Let $A, B \subset \mathbb{R}^2$ be intersecting convex polygons such that for every face of $A$, there exist strictly longer faces of $B$ parallel and antiparallel to it. Then the intersection $A \cap B$ contains a vertex of $A$.

Proof. First, $B$ cannot be contained in $A$: to every face of $A$ with a positive $x$-component corresponds a face of $B$ with a larger positive $x$-component, and hence the sum of these $x$-components is larger than the width of $A$.

If $A \subset B$, then we are done. Otherwise some faces $F$ of $A$ and $G$ of $B$ intersect. If they are parallel, then one endpoint of $F$ lies in $G$. Otherwise, let $F_p$ and $F_a$ be the faces of $B$ parallel and antiparallel to $F$. Since $G$ is a face of $B$, the intersection point of $F$ and $G$ is not in the interior of the trapezoid $T = \text{CHull}(F_p \cup F_a) \subset B$. Both $F_p$ and $F_a$ are strictly longer than $F$, so one endpoint of $F$ lies in the interior of $\text{CHull}(T \cup G) \subset B$. □

Lemma 4.5. Let $V$ be a finite set of directions, and let $r > 0$. Then there exists a polygon $B \subset \mathbb{R}^2$ such that

- every face of $B$ is strictly longer than $r$, and
- for each direction in $V$, there exists faces of $B$ parallel and antiparallel to it.

Proof. Let $W = V \cup (-V)$. For each $w \in W$, let $p_w = e^{2\pi i \alpha_w} \in S^1 \subset \mathbb{C}$, where $\alpha_w \in [0, 1)$, be a point of the complex unit circle such that the tangent of $S^1$ at $p_w$, taken in the clockwise direction, is parallel to $w$. Let

$$\epsilon = \min_{w \neq w \in W : b \in \{0, 1\}} |\alpha_v - \alpha_w - b| > 0.$$ 

Order the $\alpha_w$ increasingly as $\alpha_0, \ldots, \alpha_{k-1}$, and define $q_{2i} = e^{2\pi i (\alpha - \epsilon) / 3}$ and $q_{2i+1} = e^{2\pi i (\alpha + \epsilon / 3)}$ for all $i \in [0, k - 1]$. Now $\text{CHull}(\{ s \cdot q_j \mid j \in [0, 2k - 1]\})$ satisfies the required conditions for large enough $s > 0$.

Proof of Lemma 4.2. Suppose first that $f$ is strongly subcritical. If there exists $N \in F(f)$, then by Lemma 1.3 there is a closed rational half-plane $H$ with $\tilde{0} \in H$ and $N \cap H = \emptyset$. Translating $H$ so that $\tilde{0}$ lies on its border, we have $N \subset \{ \tilde{z} \in \mathbb{Z}^2 \mid \tilde{z} \cdot \tilde{v} < 0 \}$ for some $\tilde{v} \in S^1$. This implies $f(\tilde{x}^3)_{\tilde{q}} = 1$, contradicting the stability of $\tilde{v}$. Hence $F(f) = \emptyset$.

Suppose then that $F(f) = \emptyset$. For all $A \subset \mathbb{Z}^2$, define $K(A) \in [0, 1] \mathbb{Z}^2$ by $K(A)_{\tilde{q}} = 0 \iff \tilde{q} \in A$. For all $N \in G = G(f)$, define $A_N = \text{CHull}(N) \subset \mathbb{R}^2$. Let $V_N$ be the set of directions of the faces of the polygon $A_N$, and $r_N = \text{diam}(A_N)$. Let $B \subset \mathbb{R}^2$ be given by Lemma 1.3 for the parameters $V = \bigcup_{N \in G} V_N$ and $r = \max_{N \in G} r_N$. Let $C = B \cap \mathbb{Z}^2$. By Lemma 1.3 we have $f(N)(K(C)) = K(C)$ for all $N \in G$, and it follows that $f(K(C)) = K(C)$. Thus we can choose $x = K(C)$.

Suppose that the third condition holds, and let $\tilde{v} \in S^1$. Let $\tilde{w} \in C$ be such that $\tilde{w} \cdot \tilde{v}$ is minimal. For each $\tilde{z} \in \mathbb{Z}^2$ with $x^2_{\tilde{z}} = 0$ we then have $x^2_{\tilde{x}} \leq \sigma_{\tilde{w} \cdot \tilde{x}}(x)$ (because $\tilde{z} \cdot \tilde{v} \geq 0$ and by choice of $w$). By monotonicity $f(x^2) \leq f(\sigma_{\tilde{w} \cdot \tilde{x}}(x)) = \sigma_{\tilde{w} \cdot \tilde{x}}(x)$, and in particular $f(x^2)_{\tilde{z}} \leq \sigma_{\tilde{w} \cdot \tilde{x}}(x)_{\tilde{z}} = x_{\tilde{w}} = 0$. Since $\tilde{z}$ was arbitrary under condition $x^2_{\tilde{z}} = 0$ and since $f$ is freezing, $f(x^2) = x^2$ and $\tilde{v}$ is stable for $f$. □

Next, we study infinite paths in $\phi$-images of Bernoulli random configurations, where $\phi$ is a local transformation of $\{0, 1\} \mathbb{Z}^2$. 


Definition 4.6. For a map \( \phi : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2} \), the dependence neighborhood \( \mathcal{N}_\phi(z) \) at position \( z \in \mathbb{Z}^2 \) is the minimal set \( N \subseteq \mathbb{Z}^2 \) such that \( \phi(x) \) is determined by \( x_N \), i.e. \( x_N = y_N \) implies \( \phi(x) = \phi(y) \). We say \( \phi \) is \( k \)-dependent \( (k \in \mathbb{N}) \) if for any \( z, z' \) with \( \|z - z'\|_\infty > k \) it holds \( \mathcal{N}_\phi(z) \cap \mathcal{N}_\phi(z') = \emptyset \).

Definition 4.7. For \( b \in \{0,1\} \) and \( z \in \mathbb{Z}^2 \), denote by \( P_b^+(z) \) the set of configurations \( x \in \{0,1\}^{\mathbb{Z}^2} \) such that there is an infinite path \( (a_i)_{i \in \mathbb{N}} \) in \( \mathbb{Z}^2 \) such that \( a_0 = z \), \( a_{i+1} - a_i \in \{(0,1), (1,1)\} \) and \( x_{a_i} = b \) for all \( i \geq 0 \). Denote its complement by \( P_b^-(z) \). We may abbreviate \( P_b^+(z) = P_b^+ \) and \( P_b^-(z) = P_b^- \) when \( z \) is clear from the context.

Lemma 4.8. For any \( k \in \mathbb{N} \), there is \( 0 < p_k < 1 \) such that for any \( 0 < p \leq 1 \) and for any \( k \)-dependent map \( \phi : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2} \) verifying \( \mu_p(\phi^{-1}([1|z])) > p_k \) for all \( z \in \mathbb{Z}^2 \) it holds:

- \( \mu_p(\phi^{-1}(P_0^-(z))) = 1 \)
- \( \mu_p(\phi^{-1}(P_1^+(z))) > 0 \)

Proof. Let us first consider the case where \( \phi \) is the identity map. Let \( p_\ast = \sup\{p : \mu_p(P_1^+) = 0\} \), which is the critical probability of directed percolation on the square lattice. It is a classical result from percolation theory that \( 0 < p_\ast < 1 \). By symmetry between states 0 and 1, for \( p > \max(p_\ast, 1 - p_\ast) \) both \( \mu_p(P_0^-) = 1 \) and \( \mu_p(P_1^+) > 0 \) hold.

The general case follows from the results of [14]. Simplicifying the setting a bit to match our needs, we say a measure \( \mu \in \mathcal{M}(\{0,1\}^{\mathbb{Z}^2}) \) is \( k \)-dependent if for any \( A, B \subseteq \mathbb{Z}^2 \) with \( \min(\|z_A - z_B\|_\infty \mid z_A \in A, z_B \in B) > k \), the random variables \( x_A \) and \( x_B \) are independent when \( x \in \{0,1\}^{\mathbb{Z}^2} \) is drawn from \( \mu \). On the other hand, we say \( \mu \) dominates another measure \( \mu' \in \mathcal{M}(\{0,1\}^{\mathbb{Z}^2}) \) if for any upper-closed measurable set \( E \) (i.e. \( x \in E \) and \( x_Z \leq y_Z \) for all \( Z \in \mathbb{Z}^2 \)) implies \( y \in E \) we have \( \mu(E) \geq \mu'(E) \). [14] Theorem 0.0] implies that for any \( k \geq 0 \) and \( 0 < p < 1 \), if \( \mu \in \mathcal{M}(\{0,1\}^{\mathbb{Z}^2}) \) is a \( k \)-dependent measure and \( \min(\mu([1|z]) \mid z \in \mathbb{Z}^2) < 1 \) is large enough, then \( \mu \) dominates the product measure \( \mu_p \). Note that \( \mu \) need not be shift-invariant.

For any \( k \)-dependent map \( \phi \) and any product measure \( \mu_p \) it is the case that \( \mu_p \circ \phi^{-1} \) is \( k \)-dependent. The domination result above allows to conclude since both \( P_0^+ \) and \( P_1^+ \) are upper-closed sets.

Lemma 4.9. Let \( B \subset \mathbb{R}^2 \setminus \{0\} \) be a convex compact set. Then there exists \( \vec{v} = (a, b) \in \mathbb{Z}^2 \) such that \( \gcd(a, b) = 1 \) and for any \( \vec{w} \in \mathbb{R}^2 \) with \( \vec{v} \parallel \vec{w} \), we have either \( B \subset \mathbb{R}\vec{v} + \mathbb{R}_{>0}\vec{w} \) or \( B \subset \mathbb{R}\vec{v} + \mathbb{R}_{<0}\vec{w} \).

Proof. Let \( A = \{0\} \) and \( B \). By rescaling, we may assume \( \vec{q} = (i, j) \in \mathbb{Z}^2 \) with \( \gcd(i, j) = 1 \). Choose \( \vec{v} = (-j, i) \), so that \( \vec{v} \cdot \vec{q} = 0 \). Then any \( \vec{w} \parallel \vec{v} \) forms a basis of \( \mathbb{R}^2 \) with \( \vec{v} \) and satisfies \( \vec{v} \cdot \vec{q} \neq 0 \), and we have either \( H = \mathbb{R}\vec{v} + \mathbb{R}_{\geq 0}\vec{w} \) or \( H = \mathbb{R}\vec{v} + \mathbb{R}_{<0}\vec{w} \).

Theorem 4.10. Let \( f \) be a freezing monotone cellular automaton on \( \{0,1\}^{\mathbb{Z}^2} \). Then \( f \) trivializes some nontrivial Bernoulli measure if and only if it is not strongly subcritical.
Suppose first that $f$ is subcritical. By Lemma 4.12 and monotonicity, there exists a nonempty finite set $C \subseteq Z^2$ such that $x|_C = 0^C$ implies $f^n(x)|_C = 0^C$ for all $x \in \{0, 1\}^{Z^2}$ and $n \geq 0$. Under any nontrivial Bernoulli measure $\mu$, the event $x|_C = 0^C$ has a nonzero probability, so $f$ does not trivialize $\mu$.

Suppose then that $f$ is not strongly subcritical. Let $N \in P(f)$, and define $g = f_N$. We will show that $g$ trivializes a nontrivial Bernoulli measure $\mu \in M((\{0, 1\})^{Z^2})$, and then so does $f$. First, let $\vec{v} = (x_1, y_1) \in Z^2$ be the vector given by Lemma 4.13 for the set CHull$(N) \subseteq R^2$. Since gcd$(x_1, y_1) = 1$, Bezout’s identity gives another vector $\vec{w} = (x_2, y_2) \in Z^2$ such that $x_1y_2 - x_2y_1 = \pm 1$. Let $T = (x_2, x_2, y_2, y_2)$, which now has determinant $\pm 1$. By the assumption on $\vec{v}$, we have either $T^{-1}(N) \subseteq Z \times Z_{\geq 0}$ or $T^{-1}(N) \subseteq Z \times Z_{< 0}$, and we may choose the former case by negating $\vec{w}$ if necessary. We replace $g$ by the automaton $T^{-1} \circ g \circ T$, which clearly trivializes a given Bernoulli measure if and only if $g$ does.

We apply another automorphism of $Z^2$, a shear transformation $((t, 0) \mapsto (t, f(t)))$ for some $t \in N$, to guarantee $N \subseteq Z_{\geq 0} \times Z_{> 0}$. Now there exist $a, b \in N$, computable from $N$, such that if $x_2a = 1$ for all $\vec{n} \in [0, 2b - 1] \times [a, 2a - 1]$, then $g^a(x)_{\vec{n}} = 1$ for all $\vec{m} \in [0, b - 1] \times [0, a - 1]$. Define the map $\phi : \{0, 1\}^{Z^2} \to \{0, 1\}^{Z^2}$ by

$$\phi(x)(i,j) = \begin{cases} 1 & \text{if } x(b_i, a_j) + \vec{m} = 1 \text{ for all } \vec{m} \in [0, b - 1] \times [0, a - 1], \\ 0 & \text{otherwise.} \end{cases}$$

As in Example 3.7 it follows from König’s Lemma that if $x \in P_0^-(\vec{0})$ then there is $t$ such that $g^t(x)_{\vec{0}} = 1$. Note that $\phi$ is 0-dependent. Note also that $\mu_p(\phi^{-1}([1]_{\vec{z}}))$ can be arbitrarily close to 1 for all $\vec{z}$ when $p$ is chosen large enough. We deduce from the first item of Lemma 4.8 that $g$ trivializes some Bernoulli measure $\mu_p$ with $0 < p < 1$.

**Corollary 4.11.** Given a freezing and monotone cellular automaton $f$ on $\{0, 1\}^{Z^2}$, it is decidable whether $f$ trivializes a nontrivial Bernoulli measure.

**Theorem 4.12.** Let $f$ be a freezing monotone cellular automaton on $\{0, 1\}^{Z^2}$. If $G(f)$ is empty and there exist two linearly independent coordinates $\vec{n}, \vec{m} \in Z^2$ such that every $N \in P(f)$ contains either $\vec{n}$ and $\vec{m}$, or $-\vec{n}$ and $-\vec{m}$, then $f$ does not trivialize every nontrivial Bernoulli measure.

**Proof.** The proof is the same as in the second direction of Theorem 4.10 but switching the roles of 1 and 0 in the reduction.

## 5 Freezing non-monotone CA

The goal of this section is to study the same classical question of bootstrap percolation (equivalently $\mu$-nilpotency by Lemma 5.1) but dropping the monotone assumption in the CA considered. We show that it gives rise to much more complexity through two results: an undecidability result and a construction of an example with two phase transitions.

**Theorem 5.1.** The problem of whether a given binary freezing CA trivializes some nontrivial Bernoulli measure is undecidable.
Proof. We first give a reduction from the halting problem of Turing machines to the above problem for freezing CA with arbitrary number of states. Then we show how to recode the construction with only two states.

Let $T$ be any Turing machine and let $Q_T$ be a set of Wang tiles that simulates its valid space-time diagrams in the classical sense (see for example [15 §4]) with $q_0 \in Q_T$ and $q_h \in Q_T$ representing the initial and halting states respectively. Without loss of generality concerning undecidability of the halting problem, we consider Turing machines that only use a right semi-infinite tape and that always go to the rightmost non-blank position before entering halting state. We construct a freezing CA $F_T$ with a single maximum state $M$ satisfying the following implications:

1. if $T$ halts starting from the empty tape, then there exists a finite “obstacle” pattern of states of $F_T$ which is not $M$-uniform and that stays unchanged under the dynamics of $F_T$ whatever the context;

2. if $T$ does not halt, then, for any configuration $x$ such that the maximal rectangle containing the origin and without occurrence of state $M$ inside is finite, $F^n_T(x)_{q_0} = M$ for some finite $n$.

In the first case, $F_T$ does not trivialize any Bernoulli measure, because a cell has a positive probability to be in state $M$ in the initial configuration (and then will never change) and also has a positive probability to be in a state different from $M$ inside the finite “obstacle” pattern (and will never change either). In the second case, $\mu_{\nu_T}$-almost all configurations $x$ have the described property for any nontrivial Bernoulli measure $\mu_{\nu_T}$, because there are almost surely occurrences of $M$ on each axis, both for positive and negative coordinates. Hence $F_T$ trivializes all nontrivial Bernoulli measures in this case.

We now describe the CA $F_T$. Its dynamics is basically to check that the configuration belongs to some SFT of “valid” configurations, to turn any cell into $M$ when a local error is detected and to propagate the state $M$ to neighboring cells according to certain conditions.

Denote $D = \{N, NE, E, SE, S, SW, W, NW\}$. We interpret is as a set of cardinal and diagonal directions (north, north-east, east, etc). The state set of $F_T$ is $Q_F = Q_T \cup \{M\} \cup D$. We call a frame any rectangular pattern whose perimeter is made with states from $D$ in the following way: the north, east, south, west sides are respectively in state $N$, $E$, $S$, $W$, and the north-east, south-east, south-west, north-west corners are respectively in states $NE$, $SE$, $SW$, $NW$. We say that a frame contains a valid halting computation if (see Figure 1):

- the bottom-left cell of the interior is $q_0$ and the bottom line of the interior represents an empty tape;
- the upper-right position of the interior is $q_h$;
- any other position of the interior is in $Q_T$ and they respect the tiling conditions of $Q_T$, and the constraint that the head must not escape the frame.

Note in particular that our assumption on Turing machines ensures that a halting computation always fits inside a rectangular space-time diagram with initial state at the lower-left corner and halting state at the upper-right one.
Then, valid configurations are those made of frames containing a valid halting computation, and with only state $M$ outside frames. All these conditions can be defined by a set a valid $2 \times 2$ patterns (intuitively, the $2 \times 2$ patterns appearing in Figure 1).

![Diagram of a frame encoding a valid halting computation](image)

Figure 1: A frame encoding a valid halting computation

Depending of its state, we define the active neighbors of a cell:

- a cell in a state from $Q_F \setminus D$ has four active neighbors, one in each cardinal direction;
- a cell in a state $d \in D$ has as active neighbors in the cardinal directions which do not appear in $d$: e.g. a cell in state $NE$ has active neighbours to the south and west.

The dynamics of $F_T$ is precisely the following:

1. if a cell belongs to some invalid $2 \times 2$ pattern, then it becomes $M$;
2. if a cell has an occurrence of $M$ among its active neighbors, then it becomes $M$;
3. otherwise the state doesn’t change.

The CA $F_T$ has neighborhood $\{-1,0,1\}^2$ and its local rule can be algorithmically determined from $T$.

From this definition, it is clear that when $T$ halts starting from the empty tape, then $F_T$ admits a finite obstacle pattern: a frame encoding a valid computation and surrounded by state $M$ like in Figure 1. Now suppose that $T$ does
not halt and consider any configuration \(x\) with an occurrence of \(M\) on each semi-axis, both for positive and negative coordinates.

We want to show that \(F^n_T(x)_{\vec{0}} = M\) for some \(n \geq 0\). Suppose that \(x_{\vec{0}} = M\) and consider the (finite) maximal rectangle \(R \subseteq \mathbb{Z}^2\) whose interior contains the origin but no occurrence of state \(M\) in \(x\). We proceed by induction on the size of \(R\). If \(R\) is \(1 \times 1\), then the origin is surrounded by \(M\)-cells and becomes \(M\) in one step.

Suppose then that the claim holds for all strictly smaller rectangles. Each side of \(R\) has a neighbor outside of \(R\) in state \(M\) (if not \(R\) would not be maximal). Hence, if \(R\) is not a valid finite frame encoding a halting configuration then in one step at least one cell in the interior of \(R\) becomes \(M\) (either because the SFT condition is locally violated somewhere or because some cell has an active neighbor in state \(M\)). Since \(T\) does not halt, this means that some \(M\) appears in the interior \(R\), and in \(F_T(x)\) we either have the cell at the origin in state \(M\), or a smaller maximal rectangle of non-\(M\) states so we can conclude by induction.

So far we have proved the undecidability result for freezing CA with an arbitrary number of states. We now adapt the construction to binary freezing CA. Precisely, for any \(T\) we recursively construct a freezing CA \(G_T\) with only two states satisfying the following:

1. if \(T\) halts starting from the empty tape, then there exists a finite “obstacle” pattern of states of \(F_T\) which is not 1-uniform and that stays unchanged under the dynamics of \(G_T\) whatever the context;

2. if \(T\) does not halt, then there exist a constant \(N\) such that for any configuration \(x\) such that the maximal rectangle containing the origin and without occurrence of an \(N \times N\) block of 1s inside is bounded, we have \(G^n_T(x)_{\vec{0}} = 1\) for some \(n\).

For the same reasons as before, a construction with the above properties proves the undecidability result claimed in the statement of the theorem.

The CA \(G_T\) is constructed from \(F_T\) by a standard block encoding. Let \(N\) be large enough to recode any state of \(F_T\) as a \(N \times N\) block of 0s and 1s in the following way:

- \(M\) is coded by a \(N \times N\) block of 1s;
- any state in \(Q_{F_T} \setminus \{M\}\) is coded by an \(N \times N\) block made of an outer \(N \times N\) annulus of 1s, an inner \((N - 2) \times (N - 2)\) annulus of 0s, and inside them a uniquely defined \((N - 4) \times (N - 4)\) pattern of 0s and 1s.

A given \(N \times N\) block over alphabet \(\{0, 1\}\) is called valid if it is one of the coding blocks above, and invalid otherwise. The dynamics of \(G_T\) is the following:

- if a cell is not inside the central block of some \(3N \times 3N\) pattern of nine valid blocks, then it turns into 1;
- otherwise, if the local rule of \(F_T\) applied to the pattern \(w \in Q_T^{3 \times 3}\) encoded by the blocks yields \(M\), then it turns into 1;
- otherwise the cell retains its state.
Note that by choice of the coding (frame of 0s inside a frame of 1s) there is always at most one way to find a valid block around a cell which is in state 0, hence the second case of the dynamics above is well-defined. Moreover, if a cell in state 0 inside a valid block turns into 1, then the entire block turns into $1^{N \times N}$. Finally, by construction, on properly encoded configurations, $G_T$ exactly simulates $F_T$.

Therefore, if $T$ halts, then the block encoding of a valid frame of $F_T$ (containing a halting computation) surrounded by $N \times N$ blocks of 1s clearly forms a finite obstacle pattern under the dynamics of $G_T$. Now suppose that $T$ does not halt and consider a configuration $x$ with a finite maximal rectangle $R$ around the origin not containing any $N \times N$ block of 1s. Like for $F_T$ we proceed by induction on the size of $R$. If $R$ doesn’t contain any valid block then every cell it contains turns into 1 in one step and we are done. If $R$ contains a valid block with an invalid neighborhood, it turns into a $N \times N$ block of 1s in one step and we can apply the induction hypothesis. If $R$ is entirely made of valid blocks with valid neighborhoods of blocks, we can apply the analysis of $F_T$ and show that $G_T(x)$ has a smaller maximal rectangle. In any case, we show that after some time the central cell turns into 1.

The second result of this section is the following.

**Theorem 5.2.** There exists a freezing CA $f$ and two Bernoulli measures $\mu_{\epsilon_1}$ and $\mu_{\epsilon_2}$ on $\{0,1\}^2$ such that $0 < \epsilon_1 < \epsilon_2 < 1$, and $f$ trivializes $\mu_{\epsilon_1}$ but not $\mu_{\epsilon_2}$.

The CA of the above theorem can only turn 0s into 1s, and starting from a $\mu_{\epsilon_1}$-random configuration it will converge towards the all-1 configuration, but starting from a $\mu_{\epsilon_2}$-random configuration, which has a strictly higher density of 1s, it will leave some cells in state 0 forever. To understand the fundamental use of non-monotony in the construction and resolve this apparent paradox, the basic idea is the following: If a freezing CA has a large enough neighborhood, it can locally “see” a good enough approximation of the density of 1s in the initial configuration. Then, if this local estimate of density is close to $\epsilon_1$ it can produce a lot of 1s locally, while if it is close to $\epsilon_2$ it doesn’t.

This density jump when starting from $\epsilon_1$ is chosen so that it passes the critical probability of some percolation process (a modification of Example 3.7), while the system stays below this critical probability when starting close enough to $\epsilon_2$. There are thus two processes going on: a density modification and a percolation process. Our main trick is to use a block encoding to avoid interactions between the two. A block of 1s encodes a single cell in state 1 of the percolation process of Example 3.7 and, when such a block appears in the neighborhood, the density modification is inhibited. Thanks to this trick, the CA behaves as if the density modification was only applied once at the initial step, and then successive steps just reproduce the percolation process on a modified initial configuration.

Recall Hoeffding’s inequality [12], which will be used to quantify the local estimated density of 1s: if $P(n, p, \epsilon)$ is the probability that the average of $n$ binary Bernoulli trials, where the probability of 1 is $p$, does not lie in $[p - \epsilon, p + \epsilon]$, then

$$P(n, p, \epsilon) \leq 2 \exp \left(-2\epsilon^2 n\right)$$

for all $\epsilon > 0$. 

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Proof of Theorem \[\text{5.3}\] We first define the automaton \(f\) using some undefined parameters, and then show that it is correct for some choice of said parameters. The parameters are \(N \in \mathbb{N}\) and \(\epsilon_1, \epsilon_2, \delta \in \mathbb{R}\) subject to \(0 < \delta < \epsilon_1 < \epsilon_2 - \delta\) and \(\epsilon_2 < 1\), so that \(f = f_{N, \epsilon_1, \epsilon_2, \delta}\). We will define it in two phases, such that \(f = h \circ g\) for some other CA \(g\) and \(h\).

In the remainder of this proof, \(x_D \equiv 1\) for a configuration \(x\) and finite \(D \subset \mathbb{Z}^2\) means that \(x_Z = 1\) for all \(Z \in D\). We first define an auxiliary CA \(g'\) by

\[
g'_1(x) = \begin{cases} 
1, & \text{if } x_{\vec{z}+[0,N-1]^2} \neq 1 \text{ for all } \vec{z} \in [-4N,3N]^2 \\
& \text{and } |x_{-N,N}^2|/2N + 1) \in (\epsilon_1 - \delta, \epsilon_1 + \delta), \\
0, & \text{otherwise.}
\end{cases}
\]

The role of \(g'\) is to indicate regions in which the density modification process should take place. The CA \(g\), which implements said process, is then defined by

\[
g(x) = \begin{cases} 
1, & \text{if } x_{\vec{z}} = 1 \text{ or } g'_1(x)_{\vec{z}+\{0,N-1\}^2} \equiv 1 \\
& \text{for some } \vec{z} \in [-N+1,0]^2, \\
0, & \text{otherwise.}
\end{cases}
\]

The CA \(h\), which implements the percolation process, is defined by

\[
h(x) = \begin{cases} 
1, & \text{if } x_{\vec{z}} = 1 \text{ or } x_{\vec{z}+\{0,2N-1\} \times [N,2N-1]} \equiv 1 \\
& \text{for some } \vec{z} \in [-N+1,0]^2, \\
0, & \text{otherwise.}
\end{cases}
\]

Claim 5.3. For all \(n \geq 0\) we have \(g \circ h^n \circ g = h^n \circ g\).

In particular, \(f_n = h^n \circ g\) for any integer \(n \geq 1\).

**Proof.** Say that a configuration \(x \in \{0,1\}^{\mathbb{Z}^2}\) is nice if for all \(\vec{z} \in \mathbb{Z}^2\) such that \(x_Z = 0\), we have \(g'_1(x)_{\vec{z}+\{0,N-1\}^2} \neq 1\) for all \(\vec{v} \in \vec{z} - [0,N-1]^2\). It is now enough to prove for all \(x \in \{0,1\}^{\mathbb{Z}^2}\) that

1. \(g(x)\) is nice,
2. if \(x\) is nice, then \(x = g(x)\), and
3. if \(x\) is nice, then \(h(x)\) is nice.

The second item is clear by the definition of \(g\). We prove the first and third items using the fact that \(g\) and \(h\) only change cells by creating large batches of 1s, which suppress the density modification process.

For the first item, suppose for a contradiction that \(g(x)\) is not nice: for some \(\vec{z} \in \mathbb{Z}^2\) and \(\vec{z}_1 \in \vec{z} - [0,N-1]^2\) we have \(g(x)_{\vec{z}} = 0\) and \(g'_1(x)_{\vec{z}+\{0,N-1\}^2} \equiv 1\). By the definition of \(g\), we in particular have \(g'_1(x)_{\vec{z}+\{0,N-1\}^2} \neq 1\), so there is some \(\vec{z}_2 \in \vec{z}_1 + [0,N-1]^2\) with \(g'(x)_{\vec{z}_2} = 0\) and \(g'(g(x))_{\vec{z}_2} = 1\). The latter equation implies

\[
g(x)_{\vec{v}+\{0,N-1\}^2} \neq 1
\]

for all \(\vec{v} \in \vec{z}_2 + [-4N,3N]^2\), and since \(g\) is freezing, the same holds for \(x\) in place of \(g(x)\). Thus the density of 1s in \(g(x)_{\vec{z}_2+[-N,N]}\) lies in \((\epsilon_1 - \delta, \epsilon_1 + \delta)\), but that of \(x_{\vec{z}_2+[-N,N]}\) does not. Hence \(g(x)_{\vec{z}_2} \neq x_{\vec{z}_2}\) for some \(\vec{z}_2 \in \vec{z}_2 + [-N,N]\).
By hypothesis and by definition of $\mu$,

**Proof.**

If the parameters of our construction so that the marginals of both $\mu$ and $\beta$ are nice, then it is true for all $\mu(x) = 0$, and by niceeness $g'(x)\zeta_x \equiv 0$. As in the previous paragraph, we have

$$h(x)_{\mathbb{Z}} \equiv 1$$

for all $\vec{v} \in \mathbb{Z} + [-4N, 3N]^2$ and similarly for $x$, and $h(x)_{\mathbb{Z}} \equiv 0$ for some $\zeta_x \in \mathbb{Z} + [-N, N]$. By the definition of $h$, there now exists $\zeta_x \in \mathbb{Z} + [-N, N]^2$ with $x \zeta_x \equiv 0$. Thus, by the definition of $A$, we have

$$A(x)_{(a, b)} = \begin{cases} 1, & \text{if } g(x)_{(aN, bN)} = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$B(x)_{(a, b)} = \begin{cases} 1, & \text{if } g(x)_{(aN, bN)} \equiv 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since the radius of $g$ is at most $5N$, the two functions $A$ and $B$, when seen as random variables over a Bernoulli measure on $\{0, 1\}^{\mathbb{Z}^2}$, are $5$-dependent as defined in Lemma 4.8. Following the notations of Definition 4.7, we make the following claim.

**Claim 5.4.** Let $x \in \{0, 1\}^{\mathbb{Z}^2}$.

- If $A(x) \in P_0^+$ then $f^t(x)_0 = 0$ for all $t$.
- If $B(x) \in P_0^-$ then $f^t(x)_0 = 1$ for some $t$.

**Proof.** If $A(x) \in P_0^+$ then there is an infinite path $(\vec{z}_i)_{i \in \mathbb{N}}$ as in Definition 4.7 with $A(x)_{\vec{z}_i} = 1$ for all $i \in \mathbb{N}$ and $\vec{z}_0 = \vec{0}$. Let us show by induction on $t$ that $h^t(g(x))_{\mathbb{Z}} = 0$ for all $i \in \mathbb{N}$.

By hypothesis and by definition of $A$ it is true for $t = 0$. Suppose now that it holds for some $t \geq 0$. For any $i \in \mathbb{N}$, we have $N\vec{z}_{i+1} = N\vec{z}_i + \{(0, N), (N, N)\}$. By hypothesis $h^t(g(x))_{N\vec{z}_i} = h^t(g(x))_{N\vec{z}_{i+1}} = 0$, so we deduce $h^{t+1}(g(x))_{\mathbb{Z}} = 0$ by the definition of $h$.

Suppose that $B(x) \in P_0^-$ and consider finite paths $(\vec{z}_i)_{0 \leq i \leq m}$ with $\vec{z}_0 = \vec{0}$ and $\vec{z}_{i+1} - \vec{z}_i \in \{(0, 1), (1, 1)\}$ for each $0 \leq i < m$. By König’s lemma there is a bound $\beta(x)$ on the length of those paths with $B(x)_{\vec{z}} = 0$ for all $0 \leq i \leq m$. By definition of $B$ and $\beta$, we have $h(g(x)) \in P_0^+$ and $\beta(h(g(x))) \leq \beta(x) - 1$ (recall that $g(h(g(x))) = h(g(x))$ by Claim 5.3). By immediate induction we have $f^\beta(x)_0 = h^\beta(x)(g(x))_0 = 1$ and the second item of the claim follows.

In order to apply Lemma 4.8 simultaneously to $A$ and $B$, we need to choose the parameters of our construction so that the marginals of both $\mu_{\epsilon_1}A^{-1}([1])$ and $\mu_{\epsilon_1}B^{-1}([1])$ are close enough to $1$. We claim that it is possible.
Claim 5.5. For any $0 < \epsilon < 1$ there are parameters $N$, $\epsilon_1$, $\epsilon_2$ and $\delta$ such that we have simultaneously $\mu_{z_{2}}A^{-1}([1]_{g}) > 1 - \epsilon$ and $\mu_{z_{2}}B^{-1}([1]_{g}) > 1 - \epsilon$ for all $z \in \mathbb{Z}^2$.

Proof. It is sufficient to prove the inequalities for $z = 0$ since Bernoulli measures are translation invariant and translations are turned into translations through $A^{-1}$ and $B^{-1}$. We first show that $\epsilon_1$, $\epsilon_2$ and $\delta$ can be chosen so that for all $N$ large enough we have $\mu_{z_{2}}A^{-1}([1]_{g}) > 1 - \epsilon$. By definition $A(x)_{q} = 1$ iff $g(x)_{q} = 0$. So $\mu_{z_{2}}A^{-1}([1]_{g}) \geq 1 - \mu_{z_{2}}g^{-1}([1]_{g})$. Moreover, by the definition of $g$ and $g'$ we have $g^{-1}([1]_{g}) \subseteq [1]_{g} \cup D_{g}$, where $D_{g}$ is the set of configurations $x \in \{0, 1\}^{\mathbb{Z}^2}$ such that the finite pattern $x_{\mathbb{Z}^2\times[-N,N]^2}$ has a density of $1$-symbols strictly between $\epsilon_1 - \delta$ and $\epsilon_1 + \delta$. We deduce that

$$\mu_{z_{2}}A^{-1}([1]_{g}) \geq 1 - \epsilon_2 - \mu_{z_{2}}(D_{g}).$$

Let us fix $\epsilon_2 = \epsilon/2$ and any values of $\epsilon_1$ and $\delta$ such that $0 < \delta < \epsilon_1 < \epsilon_2 - \delta$. From (2) we know that $\mu_{z_{2}}(D_{g}) \rightarrow 0$ as $N$ grows. We deduce that for large enough $N$, we have $\mu_{z_{2}}A^{-1}([1]_{g}) > 1 - \epsilon$.

Let us now prove that for the parameters $\epsilon_1$, $\epsilon_2$ and $\delta$ fixed above and for large enough $N$, we also have $\mu_{p_{1}}([1]_{0}) > 1 - \epsilon$. We compute

$$\mu_{z_{1}}B^{-1}([1]_{g}) \geq \mu_{z_{2}}(\{x : g'((x)_{0,N-1}^2) \equiv 1\})$$

$$\geq 1 - \sum_{z \in \{z \in [0,N-1]^2 \}} \mu_{z_{2}}(\{x : x_{\mathbb{Z}^2\times[0,N-1]^2} \equiv 1\})$$

$$- \sum_{z \in [0,N-1]^2} \mu_{z_{2}}(\{0,1\}^{\mathbb{Z}^2} \setminus D_{g})$$

$$\geq 1 - 81 \cdot N^2 \cdot \epsilon_1^N - N^2(1 - \mu_{z_{2}}(D_{g})).$$

The last expression goes to $1$ as $N$ goes to infinity because by (2), $\mu_{z_{2}}(D_{g}) \rightarrow 1$ exponentially fast as $N$ grows.

From Claim 5.4, Claim 5.5 and Lemma 4.8 we deduce that $f$ trivializes $\mu_{z_{1}}$, but not $\mu_{z_{2}}$.

6 Future directions

In our study of trivialization properties of freezing monotone CA, we have concentrated on the binary alphabet. It remains open whether the characterization of Lemma 4.2 can be generalized to arbitrary poset alphabets, and whether the trivialization properties are decidable.

Question 6.1. Given a finite poset $P$ and a freezing monotone CA $f$ on $P^{\mathbb{Z}^2}$, is it decidable whether

1. $f$ trivializes all full-support Bernoulli measures?
2. $f$ trivializes some full-support Bernoulli measure?

Theorem 5.2 shows that the property of trivializing a Bernoulli measure need not be monotone with respect to the measure for a fixed binary freezing CA. We believe our construction only scratches the surface of the measure trivialization
property, and that much more intricate constructions are possible. More explicitly, for a CA $f$ on $\{0,1\}^{\mathbb{Z}^2}$, let $T(f) = \{0 \leq p \leq 1 \mid f$ trivializes $\mu_p\}$. If $f$ is freezing, then $1 \in T(f)$, and Theorem\eqref{thm:freezing} shows that in this case $T(f) \setminus \{0\}$ need not be an interval. Apart from these facts, it is not clear to us how intricate the structure of $T(f)$ can be.

**Question 6.2.** What is the class of sets $T(f)$ for freezing CA $f$ on $\{0,1\}^{\mathbb{Z}^2}$?

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