Protection of quantum information from noise is a massive challenge. One avenue people have begun to explore is reducing the number of particles needing to be protected from noise and instead use systems with more states, so called qudit quantum computers. These systems will require codes which utilize the full computational space. In this paper we show that codes for these systems can be derived from already known codes, a result which could prove to be very useful.

1 Background

Having computational information conveyed without errors is an incredibly important problem. The ability to perform classical computation within an arbitrarily small error rate was shown by Shannon in the 40’s [13]. He provided a theoretical framework showing that modern classical computation would be possible. From that point, there arose a new challenge of finding actual codes that could implement Shannon’s result. This in turn pushed coding theory into a new realm, inspiring codes such as the Hamming code family and BCH codes [4], and later leading to incredible ideas such as Polar codes [2] and Turbo codes [3].

As computational power progressed, there began to be investigations into the potential power of using quantum phenomenon as a computational tool. This brought those same questions explored for classical computers back into question. These have been explored extensively, but still have plenty of directions to go. One of the major distinctions of quantum error-correction from classical error correction includes the issue of there being two error axes: bit-flip and phase-flip. No longer did it suffice to protect against bit-flips like classical codes. Although Hadamard gates could flip whether a particular error was a bit-flip or a phase-flip, still both axes need to be protected. This led to various ideas to try to bring over classical codes. Among some of the earlier ideas was the stabilizer formalism [7], CSS codes [5][16], and teleportation [9]. Many classical coding theory theorems have been generalized into this new quantum setting, such as MacWilliams’ Identity for dual codes [14], Polynomial codes (a generalization of BCH and cyclic codes) [1][6], Polar codes [18], Turbo codes [12][17]—including results such as a complete list of all perfect codes [11].

There is at least one family of codes that exist only for qudit systems, and so this provides hints as to there being other codes for qudits only. Maximally Distance Separated (MDS) codes are a family of codes generated with a tunable parameter trading off the number of qudits being transmitted for distance [10]. This is an example of a family that only exists for qudits with at least 3 levels. This brings up the question of when else qudits might outperform qubits or at least provide more freedom than qubits.

In this article we explore the ability to apply quantum error-correcting codes in smaller dimensional spaces onto systems with larger alphabets without having to discover codes for those systems through other methods, thus creating extensions of these already known codes into larger spaces.

Before we move on to discussing this problem, we must first define our mathematical language for working on these problems. Following that we introduce our results showing the ability to apply codes in larger spaces then show the condition required for preserving the distance of such codes. After this we move on to show how to generate the codewords for such codes, thus removing the challenge of finding these codewords through projections. Lastly, we propose some directions...
to carry this work.

2 Definitions

In this section we define the majority of the tools used in this paper. We recall common definitions and results for qudit operators.

A qubit is defined as a two level system with states $|0\rangle$ and $|1\rangle$. We define a qudit as being a quantum system over $q$ levels, where $q$ is prime.

Definition 1. Generalized Paulis for a space over $q$ orthogonal levels, where we assume $q$ is prime, is given by:

$$X_q|j\rangle = |(j + 1) \mod q\rangle, \quad Z_q|j\rangle = \omega^j|j\rangle$$

with $\omega = e^{2\pi i/q}$, where $j \in \mathbb{Z}_q$. These Paulis form a group, denoted $\mathbb{P}_q$.

When $q = 2$, these are the standard qubit operators. This group structure is preserved over tensor products since each of these Paulis has order $q$.

Definition 2. An $n$-qudit stabilizer $s$ is an $n$-fold tensor of generalized Pauli operators, such that there exists at least one state, $|\psi\rangle$ such that:

$$s|\psi\rangle = |\psi\rangle$$

where $|\psi\rangle \in \mathbb{C}^m$.

Definition 3. A stabilizer algebra $S$ with commuting generators $\{s_i\}$ is defined as the subgroup of all $n$-qudit generalized Paulis formed from all multiplicative compositions ($\circ$) of these generators. This subgroup must not contain $-I$.

Definition 4. We call the set of states $|\psi\rangle$ which satisfy this condition for the stabilizer algebra $S$ the codewords of the stabilizer.

Since each operator has order $k$, a collection of $k$ compositionally independent generators for this stabilizer algebra will have $q^k$ elements.

Measuring the eigenvalues of the members in our stabilizer algebra, called the syndrome, of our state gives us a way to determine what error might have occurred and then undo the determined error. This is just like the classical method of bit check syndromes.

We recall for the reader, the well-known result:

**Theorem 5.** For any stabilizer code with $k$ qudit stabilizers and $n$ physical qudits, there will be $q^{n-k}$ stabilizer states, or codewords.

For this work, we assume that all the states in our qudit system are degenerate in energy and so all power of Pauli operators are equally likely to occur as errors, a so-called egalitarian error model. With this, we have the following definition:

**Definition 6.** The weight of an $n$-qudit operator is given by the number of non-identity operators in it.

**Definition 7.** A stabilizer code, specified by its stabilizers and stabilizer states, is characterized by a set of values:

- $n$: the number of qudits that the states are over
- $n-k$: the number of encoded (logical) qudits, where $k$ is the number of stabilizers
- $d$ (for non-degenerate codes): the distance of the code, given by the lowest weight of an undetectable generalized Pauli error (commutes with all stabilizer generators)

These values are specified for a particular code as: $[[n, n-k, d]]_q$, where $q$ is the dimension of the qudit space.

We note that, so long as no ambiguity exists, we suppress $\otimes$. We only include $\otimes$ to make register changes explicit.

**Definition 8 (Egalitarian Error Model).** The errors occurring on a system is defined as egalitarian if all errors of a constant weight are equally likely to occur.

This means that $Z$ and $X^{-1}$ and $XZ^{-1}$ all occur with the same probability, as do all other single qudit errors. We assume that from this, all weight two errors occur with the same probability as each other. This is a reasonable error model since we could simply take the dominant error and overestimate the rate of the other types of errors using that.

Working with tensors of operators can be challenging, and so we make use of the following well-known mapping from these to vectors. By representing these operators as vectors at times the solution to a problem can become far more tractable.
Definition 9 (φ representation of a qudit operator). We define the surjective map:

\[ \phi_q : \mathbb{P}_q^n \rightarrow \mathbb{Z}_q^{2n} \]  

which carries an n-qudit Pauli in \( \mathbb{P}_q^n \) to a 2n vector mod \( q \), where we define this map as:

\[
\phi_q(\omega^a \otimes_{i-1} I \otimes X_q^a \otimes_{n-i} I) = (0^{\otimes(i-1)} a 0^{\otimes(n-i)} b 0^{\otimes(n-i)})
\]  

This mapping is also a homomorphism if we define: \( \phi_q(s_1 \circ s_2) = \phi_q(s_1) \oplus \phi_q(s_2) \), where \( \oplus \) is addition mod \( q \). We denote the first half of the vector as \( \phi_{q,x} \) and the second half as \( \phi_{q,z} \).

We may invert the map \( \phi_q \) to return to the original n-qudit Pauli operator with the leading coefficient being undetermined. We make note of a special case of the \( \phi \) representation:

Definition 10. Let \( q \) be the dimension of the initial system. Then we denote by \( \phi_\infty \) the mapping:

\[ \phi_\infty : \mathbb{P}_q^n \rightarrow \mathbb{Z}_q^{2n} \]  

where no longer are any operations taken mod \( \infty \), but instead carried over the integers.

This definition follows the previous one, since any number mod \( \infty \) is just the number itself. In general we will write a stabilizer as \( \phi_q \), perform some operations, then write it in \( \phi_\infty \). We shorten this to write it as \( \phi_\infty \), and can later select to write it as \( \phi_q \) for some prime \( q' \) by taking element-wise mod \( q' \). When we provide no subscript for the representation, that implies that the choice is irrelevant.

The commutator of two operators in this picture is given by:

Definition 11. Let \( s_i, s_j \) be two qudit Pauli operators over \( q \) bases, then these commute if and only if:

\[ \phi_q(s_i) \circ \phi_q(s_j) = 0 \mod q \]  

where \( \circ \) is the symplectic product, defined by:

\[
\phi_q(s_i) \circ \phi_q(s_j) = \oplus_b [\phi_{q,z}(s_j)_b \cdot \phi_{q,z}(s_i)_b - \phi_{q,x}(s_j)_b \cdot \phi_{q,y}(s_i)_b]
\]  

where \( \cdot \) is standard integer multiplication mod \( q \) and \( \oplus \) is addition mod \( q \).

Before finishing, we make a brief list of some possible operations we can perform on our \( \phi \) representation for a stabilizer algebra:

1. As remarked above, we may add rows together, which corresponds to composition of operators
2. We may swap rows, corresponding to permuting the stabilizers
3. We may multiply each row by any number in \( 1 \rightarrow q - 1 \), corresponding to composing a stabilizer with itself. Since all operations are done over a prime number of bases, each number has an inverse.
4. We may swap registers (qudits) in the following ways:

   (a) We may swap columns (Reg \( i \), Reg \( i + n \)) and (Reg \( j \), Reg \( j + n \)) for \( 0 < i, j \leq n \), corresponding to relabelling qudits.

   (b) We may swap columns Reg \( i \) and Reg \( i + n \), for \( 0 < i \leq n \), corresponding to conjugating by a Hadamard gate on register \( i \) (or Discrete Fourier Transforms in the qudit case [8]) thus swapping \( X \)'s and \( Z \)'s roles on that qudit.

All of these operations leave all properties of the code alone, but can be used in proofs.

At this point we have all the necessary definitions to prove our results and have a solid base in qudit operators.

3 Embedding Theorem

The promise of quantum computing seems great, but overcoming the challenges of errors on such systems has been an ongoing problem since its conception. Many stabilizer codes are defined for qubit systems, however, not nearly as many are defined for qudit systems. As qudit systems begin to be built, there will be a need for codes for these systems. We show that we may apply codes from one number of bases to a larger number of bases while preserving the code parameters, at least for sufficiently large number of bases. Prior to this only some select example were known, but this shows that this is generally possible and may even improve the quality of the original code. As qudit quantum computing devices begin to become
a reality, the ability to carry over codes directly may prove a valuable resource, and might aid in determining if a code is packing information better by utilizing the higher dimensionality.

The format of this section is as follows. We begin by defining invariant codes, which are the codes which can be embedded, then proceed to show that all qudit codes are invariant codes over larger spaces. This only shows that codes are valid over higher spaces, we then show that at least for sufficiently sized spaces all parameters of the code—particularly the distance—is at least preserved, if not even improved. We provide an argument about when the distance of the code will be improved upon embedding. Along with these we provide a bound on the number of bases considered sufficient for this problem—although propose as an important continuation showing that the distance is at least preserved for all larger prime dimension spaces.

We begin by defining what property all embedded codes need to satisfy:

**Definition 12 (Invariant codes).** A stabilizer code is invariant iff:

\[ \phi(s_i)_q \odot \phi(s_j)_q = 0, \quad \forall i, j \]

holds for all primes \( q \).

This can equivalently be stated as \( \phi_{\infty}(s_i) \odot \phi_{\infty}(s_j) = 0 \), for all stabilizers \( s_i \) and \( s_j \) in the stabilizer algebra \( S \).

### 3.1 Motivating Examples

Consider the following example of generators for a stabilizer algebra: \( \langle XX, ZZ \rangle \). As a qubit code this forms a valid stabilizer code with codeword:

\[ |00\rangle + |11\rangle \]

and the commutator of these generators can be seen to be: \( (1) + (1) = 2 \equiv 0 \mod 2 \). Now suppose we wish to use this code for a qutrit system. In order to do that we must transform these generators into ones which have commutator 0, this can be achieved with \( \langle XX^{-1}, ZZ \rangle \). In this case \( \phi_{\infty}(X \otimes X^{-1}) \odot \phi_{\infty}(Z \otimes Z) = 0 \). This means that not only can this be used for qutrits, but for all prime number of bases. The codeword in the qutrit case is:

\[ |00\rangle + |1|2\rangle + |21\rangle \]

and the generalization of this for the codewords is a simple extension. We would simply make each code letter in the codeword have the entries sum to a multiple of the qudit dimension so that the \( ZZ \) operator has a +1 eigenvalue:

\[ \frac{1}{\sqrt{q}} \left( \sum_{j=1}^{q} |j \mod q, q - j \mod q\rangle \right) \]

If we look at the generators of this code, there is no single qudit operator that commutes with the generators, thus the distance of this invariant form of the code is still \( d = 2 \).

This is not the only example of a code that can be turned into invariant form. Another great example is the 5-qubit code. In fact, we don’t even need to make any changes:

\[ \langle XZZXI, IXZZX, XIXZZ, ZXIXZ \rangle \]

From inspection this can be seen to have commutators 0, and so this is a valid stabilizer code for qudits, and it can also be checked that this code will always have distance 3.

It is helpful to have a couple of examples, however, it has been unknown whether it is always possible to put stabilizer codes into invariant form. We move forward from here to show that this can be done, and how to do this. We also show that for infinitely many primes the distance of the code is at least preserved when applied to larger spaces.

### 3.2 Embedding Theorem Statement and Proof

We begin by defining and proving our embedding theorem in two steps: first showing that all codes can be embedded into larger spaces, and second by showing that all parameters of the code are at least preserved in sufficiently high dimensions. The embedding theorem then allows us to transform qudit codes into forms that can be used as qudit codes in larger spaces while still keeps all the parameters of the original code.

We now show that all qudit stabilizer codes can be written in an invariant form\(^1\). This shows that we can apply these codes directly over any number of bases, but says nothing about the distance of these codes. This aspect is treated in the section immediately following.

\(^1\)We acknowledge Andrew Jena for his contributions in the form of the below theorem and corollary.
Theorem 13. All qudit stabilizer codes can be transformed into invariant codes.

Proof. Let \( \{s_1, \ldots, s_k\} \) be a basis of qudit stabilizers for a code, \( S \), with \( k \leq n \) and a prime, \( q \). We must construct a set of stabilizers, \( \{s'_1, \ldots, s'_k\} \), such that:

1. \( \phi_\infty(s'_i) \equiv \phi_q(s_i) \) mod \( q \), for all \( i \)
2. \( \phi_\infty(s'_i) \circ \phi_\infty(s'_j) = 0 \), for all \( i \neq j \).

Without loss of generality, we assume that our stabilizers are given in canonical form:

\[
\begin{pmatrix}
\phi(s_1) \\
\vdots \\
\phi(s_k)
\end{pmatrix} = \begin{pmatrix} I_k & X_2 & Z_1 & Z_2 \end{pmatrix}.
\]

We define the strictly lower diagonal matrix, \( L \), with entries:

\[
L_{ij} = \begin{cases} 0 & i \leq j \\ \phi(s_i) \circ \phi(s_j) & i > j \end{cases}
\]

and define \( s'_1, \ldots, s'_k \) such that:

\[
\begin{pmatrix} \phi(s'_1) \\
\vdots \\
\phi(s'_k) \end{pmatrix} = \begin{pmatrix} I_k & X_2 & Z_1 & L \end{pmatrix}.
\]

We show that \( s'_1, \ldots, s'_k \) satisfy the conditions.

1. Since \( \phi(s_i) \circ \phi(s_j) \equiv 0 \) for all \( i \neq j \), we observe that \( L \equiv 0_k \) mod \( q \). By adding rows of \( L \) to our stabilizers, we have not changed the code modulo \( q \).
2. For \( i > j \), we observe that:

\[
\phi(s'_i) \circ \phi(s'_j) = (\phi(s_i) + (0 \mid L_i \ 0)) \circ (\phi(s_j) + (0 \mid L_j \ 0))
\]

\[
= \phi(s_i) \circ \phi(s_j) + \phi(s_i) \circ (0 \mid L_j \ 0)
\]

\[
+ (0 \mid L_i \ 0) \circ \phi(s_j) + (0 \mid L_i \ 0) \circ (0 \mid L_j \ 0)
\]

\[
= \phi(s_i) \circ \phi(s_j) + 0 - \phi(s_i) \circ \phi(s_j) + 0
\]

\[
= 0.
\]

Example 14. Consider the 7-qubit Steane code with parameters \([7,1,3]\), denote it by \( \Xi \). The \( \phi \) representation is given by:

\[
\phi_2(\Xi) = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}
\]

where \( H \) is the parity-check matrix for the classical Hamming code given by:

\[
H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}
\]

We begin by putting this in standard form. First we apply Hadamards on registers 4, 5, and 7, then diagonalizing those rows, this is:

\[
\phi_2(\Xi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\]

Next, performing addition mod 2, we perform the following row operations: \( r_4 \leftarrow r_4 + r_5 \), \( r_6 \leftarrow r_6 + r_4 \), \( r_5 \leftarrow r_5 + r_4 + r_6 \). This then provides our matrix as:

\[
\phi_2(\Xi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}
\]
Now we swap registers: (reg5, reg4), (reg6, reg7). Now our matrix is:

\[
\phi_2(\Xi) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

We remind the reader that none of these operations need to be actually applied, they are simply mathematically applied—indeed these applications aren’t even needed. Our code \(\Xi\) is now in standard form. For the following operations, we no longer take our operations over \(\mod 2\).

The following is the anti-symmetric matrix \([\circ]\) representing the commutators between the stabilizers and the \(L_{ij}\) matrix for this code:

\[
[\circ] = \begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \Rightarrow L_{ij} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Adding this to our standard form, we have an invariant form for the Steane code given by:

\[
\phi(\Xi) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

We will want to know the size of the maximal entry in this invariant form for our bound on ensuring the distance of the code is at least preserved. The bound on the maximal entry is provided from the above proof:

**Corollary 15.** The maximal element in \(\phi_\infty(S)\), \(B\), is upper bounded by:

\[(2 + (n - k)(q - 1))(q - 1)\]  \hspace{1cm} (13)

**Proof.** For any \(i \neq j\), there are at most \(n - k\) entries in which both \(\phi_q(s_i)\) and \(\phi_q(s_j)\) are non-zero and bounded above by \(q - 1\), and a single entry in which one is 1 whereas the other is bounded above by \(q - 1\). This gives us a bound on the inner product of: \((n - k)(q - 1)^2 + (q - 1)\). This is a bound on the size of an entry in our invariant stabilizer of \(q - 1 + (n - k)(q - 1)^2 + (q - 1) = (2 + (n - k)(q - 1))(q - 1)\).

**Example 16.** In this example we show that CSS codes remain CSS codes under this transformation. Consider a general CSS code given by:

\[
\phi(\Xi) = \begin{bmatrix}
I_{k_1} & X_{k_2} & X_{n-(k_1+k_2)} & 0 & 0 & 0 \\
0 & 0 & 0 & Z_{k_1} & I_{k_2} & Z_{n-(k_1+k_2)}
\end{bmatrix}
\]

where we have put the two block matrices into approximately standard form. Now, we perform Hadamards (or discrete Fourier transforms) on the \(k_2\) sized middle blocks. We then have:

\[
\phi(\Xi) = \begin{bmatrix}
I_{k_1} & 0 & X_{n-(k_1+k_2)} & 0 & 0 & X_{k_2} & 0 \\
0 & I_{k_2} & 0 & Z_{k_1} & 0 & Z_{n-(k_1+k_2)} & 0
\end{bmatrix}
\]
Now, we note that the first $k_1$ stabilizers commute with each other and likewise for the $k_2$ other stabilizer generators. Now we simply need to consider the case where we pick generators from each of the halves. We consider the matrix $[\circ]$, as above. The has nonzero entries for rows in $k_2$ when the columns are in $k_1$. Likewise for when the rows are in $k_1$, the entries are nonzero for columns in $k_2$, thus we only add entries to $\mathbb{Z}_{k_1}$ and $X_{k_2}$ with $[\circ]$ and so certainly do for our $L$ matrix. We may now invert our initialization and we will still have a CSS code.

### 3.3 Distance Preserving Condition

Now that we know that all qudit codes can be put into an invariant form, we now prove that at least for most sizes of the space we can ensure that the distance of the code is at least preserved. We can find the cutoff on the number of bases in the underlying space needed to at least preserve the distance.

**Theorem 17.** There exist infinitely many primes $p > p^*$, with $p^*$ a cutoff value greater than $q$, such that the distance of an embedding of a non-degenerate stabilizer code $[[n, n-k, d]]_p$ into $p$ bases, $[[n, n-k, d]]_p$, still has at least the same distance.

Before proving this theorem we make a couple of definitions:

**Definition 18.** An unavoidable error is an error that commutes with all stabilizers and produces the $\overline{0}$ syndrome over the integers.

**Remark 19.** The distance of a code over the integers is given by the minimal weight member in the set of unavoidable errors. The distance over the integers is represented by $d^*$, and so $d^* \geq d$. This value is also the number of columns that are linearly independent over the integers (or equivalently over the rationals).

**Definition 20.** An artifact error is an error that commutes with all stabilizers but produces at least one syndrome that is only zero modulo the base.

**Proof.** The ordering of the stabilizers and the ordering of the registers does not alter the distance of the code. With this, $\phi_\infty$ for the stabilizer generators over the integers can have the rows and columns arbitrarily swapped.

Let us begin with a code over $q$ bases and extend it to $p$ bases. The errors for the original code are the vectors in the nullspace of $\phi_q$. These errors are either unavoidable errors or are artifact errors. We may rearrange the rows and columns so that the stabilizers and registers that generate these nonzero entries are the upper left $2d \times 2d$ minor, padding with identities if needed. The factor of 2 occurs due to weights in $\phi_\infty$ are up to double those of the Pauli. The stabilizer(s) that generate these multiples of $q$ entries in the syndrome are members of the null space of the minor formed using the corresponding stabilizer(s).

Now, consider the extension of the code to $p$ bases. Building up the qudit Pauli operators by weight $j$, we consider the minors of the matrix composed through all row and column swaps. These minors of size $2j \times 2j$ can have a nontrivial null space in two possible ways:

- If the determinant is 0 over the integers then this is either an unavoidable error or an error whose existence did not occur due to the choice of the number of bases.
- If the determinant is not 0 over the integers, but takes the value of some multiple of $p$, then it’s 0 mod $p$ and so a nullspace exists.

Thus we can only introduce artifact errors to decrease the distance. By bounding the determinant by $p^*$, any choice of $p > p^*$ will ensure that the determinant is a unit in $\mathbb{Z}_p$, and hence have a trivial nullspace since the matrix is invertible.

Now, in order to guarantee that the value of $p$ is at least as large as the determinant, we can use Hadamard’s inequality to obtain:

$$p > p^* = B^{2(d-1)}(2(d-1))^{(d-1)}$$

where $B$ is the maximal entry in $\phi_\infty$. Since we only need to ensure that the artifact induced nullspace is trivial for Paulis with weight less than $d$, we used this identity with $2(d-1) \times 2(d-1)$ matrices.

When $j = d$, we can either encounter an unavoidable error, in which case the distance of the code is $d$ or we could obtain an artifact error, also causing the distance to be $d$. It is possible that neither of these occur at $j = d$, in which case the distance becomes some $d'$ with $d < d' \leq d^*$. □
Example 21. In our example of the Steane code, we then have $B = 1$ and $d = 3$, so for all primes larger than $1^{22}(22)^2 = 16$ we are guaranteed that the distance is preserved. For primes below that value, we can manually check and apply alternate manipulations if needed. Given the structure of the matrix, we know that the determinant of all the minors is bounded by 4, all primes at least as large as 5 preserve the distance and through manual checking 3 also works, so all primes preserve the distance for our invariant form of the Steane code.

We alluded prior to this proof that the code over the integers has distance at least as large. This begs the question of at what point will this distance $d^*$ be ensured? To this, we simply extend our above result slightly to obtain the cutoff expression, whereby no further distance improvements can be obtained from embedding the code—suggesting that another code ought to be used.

Corollary 22. We obtain the integer distance $d^*$ when:

$$ p > B^2(d^*-1)(2(d^*-1))d^*-1 $$

after this value, the distance cannot be improved through embedding. If $d^*$ is unknown, this can be upper bounded by using $k$ in place of $d^*$.

Proof. This follows from the above proof. The looser bound comes from $d^* \leq k$, so we can evaluate this at $d^* = k$ to obtain the loosest condition.

3.4 The Hierarchy of Stabilizer Codes

Definition 23. Denote the set of all stabilizers codes over $n$ qudits over $q$ levels as $S_n[\mathbb{C}^q]$ and the set of all qubit stabilizer codes as: $S[\mathbb{C}^q] = \bigcup_{n=1}^{\infty} S_n[\mathbb{C}^q]$.

The previous proofs tell us that all qudit codes can be embedded into qubit codes over larger spaces, however, it says nothing about inscribing them into smaller spaces. We address this now, showing that it cannot always be done while preserving the distance of the code.

Theorem 24 (Inscribing Theorem). Let all stabilizers be written in invariant form, and let $p < q$ both be primes, then:

$$ S_n[S_p] \prec S_n[S_q], \quad \forall n \geq 2 $$

and moreover:

$$ S[S_p] \prec S[S_q] $$

where $\prec$ indicates that there are some codes for which $[[n,n-k,d]]_q$ become $[[n,n-k,d']]_p$ with $d' < d$.

This shows that the quality of a code won’t go down upon being embedded (at least for sufficiently large spaces), but might if inscribed. Of particular note is the case where $p = 2$ where this implies that we can apply qubit codes in larger spaces and that there are some qudit codes which will decrease in distance upon being inscribed.

Proof. Consider the following fragments of stabilizer generators over $n$ registers for any integer choice of $p$:

$$ \alpha = \langle X^p X, ZZ^p \rangle, \quad \beta = \langle Z^p X^p Z, XZ^2 X^p \rangle $$

these fragments $\alpha$ and $\beta$ are both invariant and so concatenation (tensor product of registers) of these will still form an invariant code. Thus strings such as:

$$ \alpha\beta\alpha\alpha\alpha, \quad \alpha\beta\alpha\beta\beta $$

are valid pairs of stabilizer generators over $n = 13$ registers. Since any integer $n$ is either even or odd, we can write a valid code of length $n$ as either: 1. $\beta$ and $\frac{n-3}{2}$ $\alpha$ or as $\frac{n}{2}$ iterations of $\alpha$. This is not all codes of length $n$, but is a valid code and in invariant form. In addition, $\alpha$ has distance $d = 2$ so long as we don’t have $p$ bases, otherwise it has distance 1 only. Therefore, the distance of this code decreases whenever we have $p$ bases. Now, we may select all primes below $q$ as stabilizers of this form and so we know that there will always exist stabilizers whose distance can decrease upon being inscribed into a smaller space.

This then suggests a hierarchy for the stabilizers of qudit operators: all qubit codes are able to be transformed into invariant forms for dimension two or greater, and all qutrit codes can be used for qubits with dimension three or greater, and all qudit codes with five levels can be used for qubits with dimension five or greater, and so forth. In order to show this, we must ensure that embedding can never decrease the distance of a code, an extension of the work here. If such a
hierarchy is shown it might suggest that as the underlying space increases in size it opens more codes up as options, perhaps allowing for additional nice properties and rates not priorly possible for lower dimensional codes.

**Remark 25.** Unfortunately, we have only been able to show a weaker version of this where $q > p^s$, but believe this to always be true, although might require care in the choice of the invariant form.

Now that we know that we may embed codes, we now show that we can immediately and explicitly construct the codewords for these embeddings.

### 4 Explicit Construction of the Embedded Codewords

In this section we describe how to generate the codewords for a code that has been embedded using the stabilizer generators and a single element within each codeword–this latter restriction can be removed if all encoded $X$ operators are known. This can be viewed as a dual interpretation of the stabilizer generators and a single element.

The traditional way to determine the codewords for $q$ bases is to project onto the stabilizer algebra using:

$$
\Pi = \prod_i \left( \sum_{j=0}^{q-1} s_i^j \right)
$$

(20)

This method: 1) requires multiplication and 2) needs to know $q$ to determine this projector–and is not valid when $q = \infty$. It could be powerful to have an additive method that doesn’t require knowledge of $q$. We now show this alternative way of looking at the codewords.

**Definition 26.** The simultaneous weight of an $n$-qudit operator over $q$ levels, $s$, is given by:

$$
\xi := \xi(s) = \phi_{q,x}(s) \cdot \phi_{q,z}(s)
$$

(21)

**Definition 27.** For some stabilizer code $S$, let $C$ be an orthonormal +1 eigenstate of this stabilizer algebra. Each $C$ is a codeword. Each $C$ is a superposition of states in the computational basis, termed codeletters, $c$:

$$
C = \sum a_c|c\rangle
$$

with $a_c \in \mathbb{C}$ and $\|a\|^2 = 1$, where $a_c = 0$ is allowed.

The codeletters $c$ may be represented by an $n$-digit $q$-ary number, or equivalently as a vector in $\mathbb{Z}_q^n$.

A Pauli operator only has $X, Z$ operators, so the $\phi$ representation of these operators will be a vector in $\mathbb{Z}_2^{2n}$, where each entry in the vector corresponds to the power of the operator in the Pauli as given by $\phi_q$. Likewise, for an $n$-qudit operator, the $\phi_q$ representation of an operator will be a vector in $\mathbb{Z}_q^{2n}$.

The action of a stabilizer on each codeletter $c$ is to either map it to itself again or to map it to another codeletter in the same codeword with perhaps a different coefficient.

We begin by proving a lemma about the structure of the codewords in terms of their codeletters.

**Lemma 28.** Let $S$ be a stabilizer algebra and $\{C_m\}$ be the codewords for this code. Let $k$ be the number of generators for $S$ over $q$ levels and denote each generator by $s_i$. If the stabilizer algebra $S$ has logical $Z$ operators that have disjoint supports and are solely composed of $Z$ operators, then:

- Each codeletter has length $n$, and these can be written so that there are $q^k$ codeletters in each codeword–allowing for coefficients of 0 where needed
- Each codeword is formed by $q^{k-1}$ disjoint cycles of length $q$, or by $q^k$ disjoint cycles of length 1 under the action of $s_i$.

**Proof.** From theorem 5, there are $q^{n-k}$ code words and the entire space has size $q^n$, so each codeword has $q^k$ codeletters. Each codeletter has length $n$ since the code is over $n$ qudits.

Next, any generator for $S$ has order $q$. Let $s$ be some generator for $S$. If $s_{q,x}(s) \neq 0$, then $sc \propto c'$ and $s^q c = c$, for a pair of codeletters $c, c'$. This implies that $C$ can be broken into cycles of length $q$. Next, we note that in this case, the cycles must be formed from unique codeletters,
since we have logical \( Z \) operators that are products of \( Z \) operators only, and thus are disjoint. This implies that there are \( q^{k-1} \) disjoint cycles of length \( q \).

Next, we consider the case that \( \phi_{r_{c_k}}(s) = 0 \). These are the generators that are composed solely of \( Z \) operators. These cause the codeletters to form cycles of length 1, \( sc = c \) or \( ac = 0 \), and thus all the codeletters are disjoint.

Lastly, these cycles mix upon taking elements from \( S \) that are compositions of generators, however, the resulting behavior still follows.

The prior lemma has given us insight into how the codewords for a stabilizer code can be broken into cycles. We now put our stabilizers into a standard form so that we can illuminate the periodic behavior of the codewords, allowing for this alternative method for producing the codewords.

**Definition 29 (XZ Form).** All \( n \)-qudit stabilizers, \( s \), over \( q \) levels can be written in the form:

\[
s = \omega^a_q \otimes \omega^b_q X_q^{a_i} \omega^{c_h}_q
\]

for some \( a, b, r \in \mathbb{Z}_q \). This is the XZ form for the stabilizer \( s \).

**Theorem 30.** Let \( S \) be a stabilizer algebra with generators \( \{s_i\}_1 \) for a stabilizer code, where it has been written over the integers, and let \( \{X_i\}_l \) be the set of logical \( X \) operators for this code. Then we can write the codeletters in each codeword as vector additions to the all 0 codeletter and applications of \( X_i \).

This says that we do not need to preemptively know the dimension of the underlying space but can still determine the codewords, and in an efficient manner.

**Proof.** Assume the stabilizer \( s \) is in XZ form, then the action of this stabilizer on a codeletter cycle \( c_t \) is given by:

\[
s \sum_l \omega^a_l |c_l \rangle = s \sum_l \omega^a_l |c_0 + l \phi_x(s) \rangle
\]

Applying only the \( Z \) portion of the operator, we collect a phase from those registers that are not altered by the \( X \) portion, denoted \( \phi_0(s) \), as well as \( \xi(s) \) phase from the simultaneous operators:

\[
\sum_l \omega^{a_l + \phi_0(s) + \xi} |c_0 + (l + 1) \phi_x(s) \rangle
\]

which means that in order for this cycle to be stabilized, we must have:

\[
a_1 = a_0 + \phi_0(s),
\]

\[
a_2 = a_1 + \xi
\]

\[
= a_0 + \phi_0(s) + (2 - 1)\xi,
\]

\[
a_1 = a_0 + \phi_0(s) + (l - 1)\xi
\]

This gives us an equation set that each cycle must satisfy for this stabilizer.

We wish for there to be coefficients to satisfy our condition above for each of the \( k \) stabilizers. Before proceeding, we note a reduction:

\[
a_q = a_0 = a_0 + \phi_0(s) + (q - 1)\xi
\]

\[
\Rightarrow \phi_0(s) + (q - 1)\xi = 0
\]

\[
\Rightarrow \phi_0(s) = \xi
\]

This is equivalent to \( s \) being a stabilizer. Using this, we have:

\[
a_l^l = a_0^l + b\xi
\]

meaning that we have 2 parameters, or in total \( 2k \) variables which need to satisfy \( k \) equations and each cycle may overlap with another code only once. Since \( \xi \) is constant for each stabilizer across all cycles, we will always have a solution so long as the original \( s \) had one.

The condition of the logical \( Z \) operators being composed solely of \( Z \) operators is not nearly as restrictive as first appearance. We can ensure this condition for all codes with disjoint supports for the logical \( Z \) operators.

**Lemma 31.** Let \( g_i \) be the logical \( Z \) operators for a stabilizer code \( S \), with each \( g_i \) having disjoint support. Let \( g = g_i \), for some \( g_i \), then we can write \( g \) as a product of all \( Z \) operators through Clifford conjugations, for each \( i \).

The Clifford group for qudits contains qudit Paulis and the following operators (these are from [8], with specification of the power of the S gate):

**Definition 32.** The Hadamard is replaced by the discrete Fourier transform \( \mathcal{F} \):

\[
\mathcal{F} X \mathcal{F}^\dagger = Z, \quad \mathcal{F} Z \mathcal{F}^\dagger = X^{-1}
\]
Definition 33. The phase gate $P$ performs:
\[
PZP^\dagger = Z, \quad PXP^\dagger = XZ
\] (33)

Definition 34. The power gate $S$ performs:
\[
SX^2S^\dagger = X^2, \quad SZ^2S^\dagger = Z^2 - 1
\] (34)

Proof. We work component-wise. WLOG we write the term as: $X^\alpha Z^\beta$. Then conjugating by $P^{-\beta}$ then $F$ turns this term to $Z^\alpha$. Lastly, conjugate by $S^{2\alpha-1}$. Applying this form each term, we obtain a logical $Z$ operator that only contains $Z$ terms.

We can ensure full support for the logical operator since non-degenerate codes have full support over the qudits. All of these Clifford operations preserve the distance of the code.

Remark 35. Combining the results in this paper implies the ability to define stabilizer codes for systems defined over the integers (with countably-infinite number of bases). This could be useful for future systems such as bosonic systems.

Example 36. We again work with the Steane code. In total there are $q^7$ possible codeletters, most of which will have coefficient 0. We recall our invariant form for the code $\Xi$:
\[
\phi(\Xi) =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

We perform discrete Fourier transforms, $F$, on registers 1 and 7 to obtain:
\[
\phi(\Xi) =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

We now have a stabilizer that is composed solely of $Z$ operators which makes finding $|0\rangle_l$ far more simple. Beginning with $|0\rangle^{\otimes 7}$ we obtain:
\[
|0\rangle_l \propto \omega^Y |f - d, b, c, d, e, f, d + e\rangle
\] (35)

\[
Y = a \cdot (f - d) + (d + e + f) \cdot b + (f - e) \cdot c + (b + c) \cdot d + (a + b) \cdot e + (a + b + c) \cdot f + (a + c) \cdot (d + e)
\] (36)

where $a, b, c, d, e, f \in \mathbb{Z}$. Having the logical $X$ operator for this matrix would allow us to have the logical $|1\rangle$ state. Restricting all variables to $\mathbb{Z}_2$, we obtain the standard $|0\rangle_l$ codeword for the Steane code. In this way, we now have a version of the Steane code with parameters $[[7,1,3]]_{\infty}$.

We note that this technique may prove in its own right a useful tool for future analysis of sta-
bilinear codes.

5 Conclusion and Discussion

Although in this work we find some critical value above which all primes preserve the distance of the code, we believe that this result carries to all primes at least as large as the initial dimension. Proving this, or at least tightening the bound on the critical value, seems like an important extension of this result, since the current bound can be quite large. In addition, it would make the hierarchy for stabilizer codes more explicit. In addition, there is the question of whether these results also hold for degenerate codes. Beyond this, there is the important question of whether there exists some property of qudit systems that could allow for some aspect of error correction not accountable for by qubits, beyond those of MDS codes as mentioned in the introduction. Although at first glance, the answer would appear to be no, it’s is worth considering that truncating a Hilbert space by neglecting certain values might result in some defects (such as not truly having a prime number of bases) preventing the qudit advantage from appearing.

In this paper we have achieved the following. We have shown that qudit codes can be embedded into larger spaces, and at least for sufficiently large number of bases, all properties are preserved or improved, thus hinting at a hierarchy of stabilizer codes whereby code from each space are a strict subset of codes possible for the larger sized space. This result may aid in error correction schemes for qudit based quantum computing by firstly providing immediate codes for these devices by applying already known codes, and secondly perhaps aiding in the search for codes for these qudit devices that better utilize the underlying higher dimensionality of these systems. Here we have provided a method for producing the codewords of these embedded codes without having to compute them from scratch or using another method, but simply adding to the registers and applying an associated phase.

Some important directions to carry these results include the following. Firstly, some additional applications of this result showing additional utility beyond those discussed already. Secondly, upon testing many codes, we found that distance was at least preserved for embedded codes over all primes greater than or equal to the original number of bases, so long as care was taken in performing the embedding. We were not able to show this result here, however, this would be an important result, thus showing that embedding can never harm the distance of the code. Lastly, as alluded to earlier, this allows for immediate creation of embedded codes, however, it is not clear whether there is, or how greatly there is, an advantage of using stabilizer codes within the corresponding level of our proposed hierarchy. Perhaps embedded codes are approximately equivalent to codes optimized for the number of bases in the space, however, we suspect that it’s unlikely, but leave that as another open problem.

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References

[1] Dorit Aharonov and Michael Ben-Or. Fault-tolerant quantum computation with constant error rate. arXiv preprint quant-ph/9906129, 1999.

[2] Erdal Arikan. Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels. IEEE Transactions on information Theory, 55(7):3051–3073, 2009.

[3] Claude Berrou, Alain Glavieux, and Punya Thitimajshima. Near shannon limit error-correcting coding and decoding: Turbo-codes. 1. In Proceedings of ICC’93-IEEE In-
ternational Conference on Communications, volume 2, pages 1064–1070. IEEE, 1993.

[4] Raj Chandra Bose and Dwijendra K Ray-Chaudhuri. On a class of error correcting binary group codes. Information and control, 3(1):68–79, 1960.

[5] A Robert Calderbank and Peter W Shor. Good quantum error-correcting codes exist. Physical Review A, 54(2):1098, 1996.

[6] Richard Cleve, Daniel Gottesman, and Hoi-Kwong Lo. How to share a quantum secret. Physical Review Letters, 83(3):648, 1999.

[7] Daniel Gottesman. Stabilizer codes and quantum error correction. arXiv preprint quant-ph/9705052, 1997.

[8] Daniel Gottesman. Fault-tolerant quantum computation with higher-dimensional systems. In NASA International Conference on Quantum Computing and Quantum Communications, pages 302–313. Springer, 1998.

[9] Daniel Gottesman and Isaac L Chuang. Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations. Nature, 402(6760):390, 1999.

[10] Markus Grassl, Thomas Beth, and Martin Roetteler. On optimal quantum codes. arXiv e-prints, pages quant–ph/0312164, Dec 2003.

[11] Zhuo Li and Lijuan Xing. No more perfect codes: Classification of perfect quantum codes. arXiv preprint arXiv:0907.0049, 2009.

[12] David Poulin, Jean-Pierre Tillich, and Harold Ollivier. Quantum serial turbo codes. IEEE Transactions on Information Theory, 55(6):2776–2798, 2009.

[13] Claude Elwood Shannon. A mathematical theory of communication. Bell system technical journal, 27(3):379–423, 1948.

[14] Peter Shor and Raymond Laflamme. Quantum analog of the macwilliams identities for classical coding theory. Physical review letters, 78(8):1600, 1997.

[15] Andrew Steane. Multiple-particle interference and quantum error correction. Proceedings of the Royal Society of London. Series