JOIN-SEMDISTRIBUTIVE LATTICES OF RELATIVELY
CONVEX SETS

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Abstract. We give two sufficient conditions for the lattice Co(ℝⁿ, X) of relatively convex sets of ℝⁿ to be join-semidistributive, where X is a finite union of segments. We also prove that every finite lower bounded lattice can be embedded into Co(ℝⁿ, X), for a suitable finite subset X of ℝⁿ.

1. Introduction

A lattice L is join-semidistributive, if

\[ x \lor y = x \lor z \text{ implies that } x \lor y = x \lor (y \land z), \]

for all \( x, y, z \in L \). Let \( X \subseteq \mathbb{R}^n \), and let Co(ℝⁿ, X) denote the lattice of convex subsets of ℝⁿ relative to X, that is,

\[ \text{Co}(\mathbb{R}^n, X) = \{ Y \subseteq \mathbb{R}^n \mid Y = \text{Co}(Y) \cap X \}, \]

where Co(Y) denotes the convex hull of Y, for any \( Y \subseteq \mathbb{R}^n \). For all \( X \subseteq \mathbb{R}^n \), the closure operator \( \phi: B_X \rightarrow B_X \), where \( \phi(Y) = \text{Co}(Y) \cap X \) for all \( Y \subseteq \mathbb{R}^n \), satisfies the so-called anti-exchange axiom that makes lattices of relatively convex sets just another example of a convex geometry (see the extensive monograph [7], also [2]). It is well known (cf. [2]) that a finite convex geometry is join-semidistributive, whence the lattice Co(ℝⁿ, X) is join-semidistributive, for any finite \( X \subseteq \mathbb{R}^n \).

Problem 3 in [2] asks about a description of lattices embeddable into lattices of the form Co(ℝⁿ, X) with finite X. Since any sublattice of a join-semidistributive lattice is join-semidistributive itself, all those lattices must also be join-semidistributive. Although the current paper does not provide a solution of the problem, it suggests some approaches to it. The main idea is to consider a more general setting for the problem dropping the requirement for X to be finite.

For a lattice L with the least element 0_L, let At(L) denote the set of atoms of L, that is, \( \text{At}(L) = \{ x \in L \mid 0_L \prec x \} \). While finite convex geometries are always join-semidistributive, a convex geometry L satisfies a weaker property:

\[ x \lor y = x \lor z \text{ implies that } x \lor y = x \lor (y \land z), \]

for all \( x \in L \) and all \( y, z \in \text{At}(L) \). In other words, if \( x \lor y = x \lor z \), for some \( x \in L \) and \( y, z \in \text{At}(L) \) the either \( y = z \) or \( y, z \leq x \). How weak this property is can be seen from the following result established in [3]: every finite lattice can be embedded into Co(ℝⁿ, X), for some \( n \in \omega \) and \( X \subseteq \mathbb{R}^n \). Thus we would like to generalize

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Problem 3 from [2], dropping the requirement for $X$ to be finite but still assuming $Co(\mathbb{R}^n, X)$ to be join-semidistributive:

**Problem 1.** Which finite lattices can be embedded into join-semidistributive lattices of the form $Co(\mathbb{R}^n, X)$?

It turns out that sets $X$ for which the corresponding lattice $Co(\mathbb{R}^n, X)$ is join-semidistributive are quite specific. The third section of the paper is mostly devoted to the case when $X$ is a finite union of segments, which seems to be a natural generalization of finiteness of $X$. We provide two sufficient conditions for $X$ to ensure $Co(\mathbb{R}^n, X)$ to be join-semidistributive.

The last section is devoted to an important proper subclass of the class of join-semidistributive lattices, the class of so-called *lower bounded lattices*. We prove that every finite lower bounded lattice embeds into a finite lower bounded lattice of the form $Co(\mathbb{R}^n, X)$. Another proof of this result can be found also in [10].

Here we use an essentially geometric idea, first constructing an embedding of the lattice $Sub_{\wedge}^{n+1}$ of meet-subsemilattices of the Boolean lattice $\mathcal{B}_{n+1}$ into the lattice of bounded convex subsets of $\mathbb{R}^n$, and then finding a finite set $X$ which provides an embedding into $Co(\mathbb{R}^n, X)$. We hope that this construction might give some additional insight into the question whether every finite join-semidistributive lattice embeds into a finite lattice $Co(\mathbb{R}^n, X)$.

2. Basic concepts

For any $a, b \in \mathbb{R}^n$, let $(a, b)$ denote the open segment and let $[a, b]$ denote the closed segment whose end points are $a$ and $b$, that is,

$$(a, b) = \{ x \in \mathbb{R}^n \mid x = \lambda a + (1 - \lambda)b \text{ for some } \lambda \in (0, 1) \},$$

$$[a, b] = \{ x \in \mathbb{R}^n \mid x = \lambda a + (1 - \lambda)b \text{ for some } \lambda \in [0, 1] \}.$$

It is straightforward to verify that for any $Y \subseteq \mathbb{R}^n$,

$$Co(Y) = \bigcup_{i \in \omega} Y^{(i)},$$

where $Y^{(0)} = Y$ and $Y^{(i+1)} = \{ [a, b] \mid a, b \in Y^{(i)} \}$, for all $i \in \omega$.

A convex subset $F \subseteq P$ of a convex polytope $P$ is a face of $P$, if $(a, b) \cap F \neq \emptyset$ implies $[a, b] \subseteq F$, for all $a, b \in P$. An element $x$ of a convex set $X \subseteq \mathbb{R}^n$ is an *extreme point* of $X$ if $x \notin Co(X \setminus \{ x \})$. Let $Ex(X)$ denote the set of extreme points of $X$, for any $X \subseteq \mathbb{R}^n$.

For any $Y \subseteq \mathbb{R}^n$, we denote by $\overline{Y}$ the closure of $Y$ and by $int_n(Y)$ the interior of $Y$ in the Euclidean topology of $\mathbb{R}^n$.

**Lemma 2.1.** Let $X \subseteq \mathbb{R}^n$ be a finite union of segments. Then $Co(\overline{X}) = \overline{Co(X)}$. In particular, if $x \in Ex(\overline{Co(X)})$ then $x$ is an extreme point of a closure of a segment from $X$.

*Proof.* The proof is straightforward. \hfill \Box

**Lemma 2.2.** Let $P \subseteq \mathbb{R}^n$ be a convex polytope and let $F$ be a face of $P$. Then $Co(Y) \cap F = Co(Y \cap F)$, for any $Y \subseteq P$.

*Proof.* By induction on $k$, we prove that $Y^{(k)} \cap F \subseteq (Y \cap F)^{(k)}$, for all $k \in \omega$. For $k = 0$, the conclusion is obvious. Let $k > 0$ and let $x \in Y^{(k)} \cap F$. Then there
exist $a, b \in Y^{(k-1)}$ such that $x \in [a, b]$. If $x = a$ or $x = b$, then $x \in Y^{(k-1)} \cap F \subseteq (Y \cap F)^{(k-1)}$ by the induction hypothesis. Otherwise, $x \in (a, b) \cap F$, whence $a, b \in F$ since $F$ is a face of $P$. Therefore, $a, b \in Y^{(k-1)} \cap F \subseteq (Y \cap F)^{(k-1)}$ by the induction hypothesis, whence $x \in (Y \cap F)^{(k)}$. \hfill \Box

For any $Y \subseteq \mathbb{R}^n$, let $\psi_Y : \text{Co}(\mathbb{R}^n) \to \text{Co}(\mathbb{R}^n, Y)$ be the map defined by $\psi_Y(X) = X \cap Y$, for any $X \in \text{Co}(\mathbb{R}^n)$. Then $\psi_Y$ preserves meets, for any $Y \subseteq \mathbb{R}^n$.

**Lemma 2.3.** Let $P$ be a convex polytope and let $X \subseteq P$. Then the map $\psi_F : \text{Co}(\mathbb{R}^n, X) \to \text{Co}(\mathbb{R}^n, X \cap F)$ defined by $\psi_F(Y) = Y \cap F$ is a surjective lattice homomorphism, for any face $F$ of $P$.

**Proof.** The surjectivity of $\psi_F$ follows from the fact that if $A = \text{Co}(A) \cap X \cap F$ then $A = \psi_F(\text{Co}(A) \cap X)$. Let $A, B \in \text{Co}(\mathbb{R}^n, X)$. Evidently, $\psi_F$ preserves meets. Applying Lemma 2.2 we get

$$\psi_F(A \cup B) = \text{Co}(A \cup B) \cap X \cap F = \text{Co}((A \cap F) \cup (B \cap F)) \cap X = (\text{Co}(A \cap F) \cap X) \cup (\text{Co}(B \cap F) \cap X) = \psi_F(A) \cup \psi_F(B),$$

whence $\psi_F$ preserves joins. \hfill \Box

3. Join-semidistributivity of $\text{Co}(\mathbb{R}^n, X)$

If $X \subseteq \mathbb{R}^n$ is finite, then, as we mentioned above, the lattice $\text{Co}(\mathbb{R}^n, X)$ is a finite convex geometry; in particular, it is join-semidistributive. However, we do not know how far this fact can be extended.

**Problem 2.** Describe sets $X \subseteq \mathbb{R}^n$ such that the lattice $\text{Co}(\mathbb{R}^n, X)$ is join-semidistributive.

To remind that not every $X$ suits, we recall an example given in [4].

**Example 3.1.** Let $X$ contain the (2-dimensional) interior of some triangle $TML$. Pick any point $K$ inside that interior. Then the interior of each triangle $TMK$, $TLK$, and $MLK$ belongs to $\text{Co}(\mathbb{R}^n, X)$, and they form a modular sublattice isomorphic to $M_3$. In particular, $\text{Co}(\mathbb{R}^n, X)$ is not join-semidistributive.

A subset $X$ of $\mathbb{R}^n$ is sparse, if $\text{int}_2 \{ \text{int}(\text{H}) \} = \emptyset$, for any 2-dimensional affine subspace $H$ of $\mathbb{R}^n$. From Example 3.1 it follows that every set $X$ satisfying the requirement of Problem 2 has to be sparse.

Observe that if $X$ is a line in $\mathbb{R}^n$ then $\text{Co}(\mathbb{R}^n, X)$ is isomorphic to $\text{Co}(\mathbb{R})$, the lattice of order convex subsets of $\mathbb{R}$, and the latter is join-semidistributive (see Theorem 14 in [5]).

Another extreme case is when $X$ is the boundary of a ball; in this case, the lattice $\text{Co}(\mathbb{R}^n, X)$ is Boolean (cf. an example of section 9 in [4]); in particular, it is distributive. This gives two natural examples of sparse sets which qualify for Problem 2. Unfortunately, being a sparse set is a necessary condition but not sufficient.

**Example 3.2.** Let $X$ be the union of three lines $A$, $B$, and $C$ which are on the same plane and have a common intersection. Then $A \cup B = A \cup C = X$ but $A \cup (B \cap C) = A$ in $\text{Co}(\mathbb{R}^n, X)$. 


On the other hand, if we take *segments* instead of lines, then the corresponding lattice turns out to be join-semidistributive. Thus the following question is rather natural: *if $X$ is a finite union of segments, is the lattice $\text{Co}(\mathbb{R}^n, X)$ join-semidistributive?* Unfortunately, even this simplest generalization of finiteness of $X$ does not ensure that $\text{Co}(\mathbb{R}^n, X)$ is join-semidistributive, as the example below demonstrates.

**Example 3.3.** Let $T$ be a triangle in $\mathbb{R}^2$ with the set of extreme points $\{a, b, c\}$ and let $p, m \in \text{int}_2 T$, $p \neq m$. Without loss of generality, we may assume that $p, m$, and $a$ are not collinear. We put $X = [b, c] \cup [p, a] \cup [m, a]$ and $A = [b, c]$, $B = (p, a)$, $C = (m, a)$. Then $A \lor B = A \lor C = X \setminus \{a\} \neq A \lor (B \land C) = A$ in $\text{Co}(\mathbb{R}^2, X)$. Thus this lattice is not join-semidistributive.

We note that the failure of join-semidistributivity in the example above is due to the fact that closed segments $[p, a]$ and $[m, a]$ have a common point. Also, it is essential that $(p, a)$ and $(m, a)$ are subsets of $\text{int}_2 T$. Were points $p$ and $m$ chosen, say, on faces $[a, b]$ and $[a, c]$ of the triangle $T$, respectively, the lattice $\text{Co}(\mathbb{R}^n, X)$ would be join-semidistributive.

For the rest of this section, we assume $X$ to be a finite union of segments. The following theorem provides two sufficient conditions for $\text{Co}(\mathbb{R}^n, X)$ to be join-semidistributive. Each of them eliminates at least one condition that plays role in Example 3.3.

**Theorem 3.4.** Let $n, k \in \omega$ and let $X = \bigcup \{I_j \mid j < k\}$, where $I_j \subseteq \mathbb{R}^n$ is a segment, for all $j < k$. Consider the following two conditions:

(i) $\overline{I_s} \cap \overline{I_t} = \emptyset$, for all $s, t < k$, $s \neq t$;

(ii) there exists a convex polytope $P \subseteq \mathbb{R}^n$ such that for any $j < k$, $I_j$ is a subset of a face of $P$.

If $X$ satisfies either (i) or (ii) then the lattice $\text{Co}(\mathbb{R}^n, X)$ is join-semidistributive.

**Proof.** We argue by induction on $n$. Let $n = 1$. For any $X \subseteq \mathbb{R}$, the lattice $\text{Co}(\mathbb{R}, X)$ is the lattice of order-convex subsets of $X$ endowed with the standard (linear) order, thus it is join-semidistributive (see [5, Theorem 14]).

Let $n > 1$. Suppose that $X$ satisfies either (i) or (ii) and $A \lor B = A \lor C > A \lor (B \land C)$, for some $A, B, C \in \text{Co}(\mathbb{R}^n, X)$. Let $Y = \text{Co}(A \lor (B \land C))$. Then $B, C \not\subseteq Y$. We prove that there are a convex polytope $Q$ and a face $F$ of $Q$ such that $B \cap F \not\subseteq Y$ and $Y \subseteq Q$.

Suppose first that $X$ satisfies (i). By Lemma 2.4, we get

$$K = \text{Co}(A \cup B) = \text{Co}(A \lor B) = \text{Co}(A \lor C) = \text{Co}(A \lor C).$$

If $K \not\subseteq Y$, then there exists an extreme point $a \in \text{Ex}(K)$ such that $a \not\in Y$. Since $\overline{A} \subseteq Y$, by Lemma 2.1, $a \in \overline{B} \cap \overline{C}$ contradicting (i). Thus, $B \subseteq K \subseteq Y$ but $B \not\subseteq Y$. Therefore, there exists a face $F$ of $Y$ such that $B \cap F \not\subseteq Y$. We take $Q = Y$ in this case.

Suppose that $X$ satisfies (ii). Since $B \not\subseteq Y$, there is a face $F$ of $P$ such that $B \cap F \not\subseteq Y$. We take $Q = P$ in this case.

By Lemma 2.3, the map $\psi_P: \text{Co}(\mathbb{R}^n, X \cap Q) \to \text{Co}(\mathbb{R}^n, X \cap Q \cap F)$ is a lattice homomorphism. Thus, $\psi_P(A) \lor \psi_P(B) = \psi_P(A) \lor \psi_P(C)$. Also, the lattice $\text{Co}(\mathbb{R}^n, X \cap F)$ is isomorphic to the lattice $\text{Co}(\mathbb{R}^m, X \cap F)$, where $m \in \omega$ is the dimension of an affine subspace of $\mathbb{R}^n$ containing $F$. Moreover, $X \cap F$ is a finite union of segments. By the induction hypothesis, the lattice $\text{Co}(\mathbb{R}^m, X \cap F)$ is...
join-semidistributive, whence
\[ B \cap F = \psi_F(B) \subseteq \psi_F(A \lor B) = \]
\[ \psi_F(A) \lor (\psi_F(B) \cap \psi_F(C)) = \]
\[ \psi_F(A \lor (B \cap C)) = \psi_F(Y) \subseteq Y, \]
a contradiction. \qed

4. Lower bounded lattices as sublattices of finite \( \text{Co}(\mathbb{R}^n, X) \)

In this section, we consider sublattices of lattices of the form \( \text{Co}(\mathbb{R}^n, X) \), where \( X \subseteq \mathbb{R}^n \) is finite. As was observed in [2], we do not know yet any special type of finite convex geometries which admit any finite join-semidistributive lattice as a sublattice. We have a partial confirmation that lattices of the form \( \text{Co}(\mathbb{R}^n, X) \) could be such a "universal" class of convex geometries for the class of finite join-semidistributive lattices.

The main result of this section shows that, at least, this class is universal for the class of finite lower bounded lattices which is a proper subclass in the class of finite join-semidistributive lattices. We recall that a (finite) lattice is lower bounded, if it is an image of a finitely generated free lattice under a lower bounded homomorphism, that is, the preimage of every element under this homomorphism has a least element. We refer the reader to the comprehensive monograph on the topic [6]. There exist at least two other particular classes of finite convex geometries which admit every finite lower bounded lattice as a sublattice: suborder lattices of finite partial orders [9] and subsemilattice lattices of finite semilattices [1, 8].

Unlike these known examples, lattices of relatively convex subsets are not necessarily lower bounded. The simplest example is \( \text{Co}(\mathbb{R}, X) \), where \( X \) consists of four different points on the same line. The other common feature of many types of convex geometries is that they are biatomic. Due to [5], a lattice \( L \) with the least element \( 0_L \) is biatomic if for any \( x \in \text{At}(L) \) and any \( y, z \in \text{At}(L) \), the inequality \( x \leq y \lor z \) implies that there are \( y', z' \in \text{At}(L) \) such that \( y' \leq y \), \( z' \leq z \), and \( x \leq y' \lor z' \).

A result from [3] shows that not every finite join-semidistributive lattice embeds into a finite biatomic join-semidistributive lattice. The counter-example from [3] is the lattice \( \text{Co}(\mathbb{R}^2, X) \), where \( X \) is a 5-element set of points on a plane. In particular, this emphasizes that lattices of relatively convex subsets are essentially non-biatomic, thus might serve as a "universal" class of convex geometries for the class of finite join-semidistributive lattices.

Observe that an alternate approach which leads to the result that every finite lower bounded lattice is a sublattice of \( \text{Co}(\mathbb{R}^n, X) \) with finite \( X \) is presented in [10]. The authors of [10] find an embedding of every finite lower bounded lattice into the lattice of convex polytopes of a finite-dimensional vector space, from where the result easily follows.

**Proposition 4.1.** For every \( n < \omega \), the lattice \( \text{Sub} \cap \mathcal{B}_{n+1} \) embeds into the lattice of bounded convex sets of \( \mathbb{R}^n \).

**Proof.** Let \( S_{n+1} \) denote a regular polytope in \( \mathbb{R}^n \) with \( n + 1 \) vertices. It is not that important to have a regular polytope, but it is easier to deal with because of the total symmetry of the argument. Thus, in \( \mathbb{R}^2 \) it is an equilateral triangle, in \( \mathbb{R}^3 \) it is a regular tetrahedron, etc.
Let $\text{Ex}(S_{n+1}) = \{ p_i \mid i \leq n + 1 \}$. We define the map $\psi: \mathcal{B}_{n+1} \rightarrow \text{Co}(\mathbb{R}^n)$ by the rule

$$
\psi(t) = \begin{cases} 
\emptyset, & \text{if } t = n + 1, \\
\{ p_i \}, & \text{if } n + 1 \setminus t = \{ i \}, \\
\text{int}_A \text{Co} \{ \{ p_i \mid i \in A = n + 1 \setminus t \} \}, & \text{if } |t| < n.
\end{cases}
$$

(1)

Claim 1. For any $a, b \in \mathcal{B}_{n+1}$, $\text{Co}(\psi(a) \cup \psi(b)) = \psi(a) \cup \psi(b) \cup \psi(a \cap b)$.

Proof of Claim. Without loss of generality, we may assume that $a$ and $b$ are non-comparable. By induction on $i$, we prove that $(\psi(a) \cup \psi(b))^{(i)} \subseteq \psi(a) \cup \psi(b) \cup \psi(a \cap b)$, for all $i \in \omega$. For $i = 0$, the conclusion is obvious. Suppose that $i < \omega$ and that $z \in (\psi(a) \cup \psi(b))^{(i+1)} \setminus (\psi(a) \cup \psi(b))^{(i)}$. Then there are $\lambda \in (0, 1)$, $x, y \in (\psi(a) \cup \psi(b))^{(i)}$ such that $z = \lambda x + (1-\lambda)y$. By the induction hypothesis, $x, y \in \psi(a) \cup \psi(b) \cup \psi(a \cap b)$.

We consider several cases:

Case 1. $x, y \in \psi(a)$ or $x, y \in \psi(b)$. In this case, $z \in \psi(a) \cup \psi(b)$ since both $\psi(a)$ and $\psi(b)$ are convex.

Case 2. $x \in \psi(a)$ and $y \in \psi(b)$. In this case, there are $\lambda_k \in (0, 1)$, $k \in n + 1 \setminus a$, and $\mu_l \in (0, 1)$, $l \in n + 1 \setminus b$, such that

$$
\sum_{k} \{ \lambda_k \mid k \in n + 1 \setminus a \} = \sum_{l} \{ \mu_l \mid l \in n + 1 \setminus b \} = 1
$$

$$
x = \sum_{k} \{ \lambda_k p_k \mid k \in n + 1 \setminus a \}, \quad y = \sum_{l} \{ \mu_l p_l \mid l \in n + 1 \setminus b \}.
$$

Then

$$
z = \sum \{ \lambda_k \lambda_k p_k \mid k \in n + 1 \setminus a \} + \sum \{ (1-\lambda) \mu_l p_l \mid l \in n + 1 \setminus b \}.
$$

Moreover, $\lambda_k \lambda_k, (1-\lambda) \mu_l \in (0, 1)$, for all $k \in n + 1 \setminus a$ and all $l \in n + 1 \setminus b$, and

$$
\sum \{ \lambda_k \mid k \in n + 1 \setminus a \} + \sum \{ (1-\lambda) \mu_l \mid l \in n + 1 \setminus b \} = \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.
$$

Thus, $z \in \psi(a \cap b)$.

Case 3. $x \in \psi(a)$, $y \in \psi(a \cap b)$. In this case, there are $\lambda_k \in (0, 1)$, $k \in n + 1 \setminus a$, and $\mu_l \in (0, 1)$, $l \in n + 1 \setminus (a \cap b)$, such that

$$
\sum_{k} \{ \lambda_k \mid k \in n + 1 \setminus a \} = \sum_{l} \{ \mu_l \mid l \in n + 1 \setminus (a \cap b) \} = 1
$$

$$
x = \sum_{k} \{ \lambda_k p_k \mid k \in n + 1 \setminus a \}, \quad y = \sum_{l} \{ \mu_l p_l \mid l \in n + 1 \setminus (a \cap b) \}.
$$

Then

$$
z = \sum \{ \lambda_k \lambda_k + (1-\lambda) \mu_l \mid k \in n + 1 \setminus a \} + \sum \{ (1-\lambda) \mu_l p_l \mid l \in a \setminus b \}.
$$

Again, all the coefficients are from $(0, 1)$, and

$$
\sum \{ \lambda_k \lambda_k + (1-\lambda) \mu_k \mid k \in n + 1 \setminus a \} + \sum \{ (1-\lambda) \mu_l \mid l \in a \setminus b \} = \lambda \sum \{ \lambda_k \mid k \in n + 1 \setminus a \} + \sum \{ (1-\lambda) \mu_l \mid l \in a \setminus b \}
$$

$$
= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.
$$

Thus, $z \in \psi(a \cap b)$. Therefore, we have proved that $\text{Co}(\psi(a) \cup \psi(b)) \subseteq \psi(a) \cup \psi(b) \cup \psi(a \cap b)$.
We prove the inverse inclusion. It suffices to show that $\psi(a \cap b) \subseteq \text{Co}(\psi(a) \cup \psi(b))$.

Let $z \in \psi(a \cap b)$. There are $\lambda_k \in (0, 1)$, $k \in n + 1 \setminus (a \cap b)$ such that \( \sum \{ \lambda_k \mid k \in n + 1 \setminus (a \cap b) \} = 1 \) and
\[
z = \sum \{ \lambda_k p_k \mid k \in n + 1 \setminus (a \cap b) \}.
\]

We put\[
\lambda = \left( \sum \{ \lambda_k \mid k \in b \setminus a \} + \frac{1}{2} \sum \{ \lambda_k \mid k \in n + 1 \setminus (a \cup b) \} \right)^{-1};
\]
\[
x = \sum \{ \frac{\lambda_k}{\lambda} p_k \mid k \in b \setminus a \} + \sum \{ \frac{\lambda_k}{2\lambda} p_k \mid k \in n + 1 \setminus (a \cup b) \};
\]
\[
y = \sum \{ \frac{\lambda_k}{1-\lambda} p_k \mid k \in a \setminus b \} + \sum \{ \frac{\lambda_k}{2(1-\lambda)} p_k \mid k \in n + 1 \setminus (a \cup b) \}.
\]

We get\[
\sum \{ \frac{\lambda_k}{\lambda} \mid k \in b \setminus a \} + \sum \{ \frac{\lambda_k}{2\lambda} \mid k \in n + 1 \setminus (a \cup b) \} = \frac{1}{\lambda} \left( \sum \{ \lambda_k \mid k \in b \setminus a \} + \frac{1}{2} \sum \{ \lambda_k \mid k \in n + 1 \setminus (a \cup b) \} \right) = \frac{1}{\lambda} \cdot \lambda = 1;
\]
\[
\sum \{ \frac{\lambda_k}{1-\lambda} \mid k \in a \setminus b \} + \sum \{ \frac{\lambda_k}{2(1-\lambda)} \mid k \in n + 1 \setminus (a \cup b) \} = \frac{1}{1-\lambda} \left( \sum \{ \lambda_k \mid k \in a \setminus b \} + \frac{1}{2} \sum \{ \lambda_k \mid k \in n + 1 \setminus (a \cup b) \} \right) = \frac{1}{1-\lambda} \cdot (1-\lambda) = 1.
\]

Thus, $x \in \psi(a)$ and $y \in \psi(b)$. Moreover, $z = \lambda x + (1-\lambda)y$, whence $z \in \text{Co}(\psi(a) \cup \psi(b))$. \(\square\) Claim 1.

For any $S \in \text{Sub}_\lambda B_{n+1}$, we put\[
\varphi(S) = \bigcup \{ \psi(t) \mid t \in S \}. \quad (2)
\]

According to Claim 11, $\varphi(S) \in \text{Co}(\mathbb{R}^n)$, for any $S \in \text{Sub}_\lambda B_{n+1}$. We verify that $\varphi$ is a lattice homomorphism from $\text{Sub}_\lambda B_{n+1}$ to $\text{Co}(\mathbb{R}^n)$. It is straightforward that $\varphi$ is one-to-one. Moreover, $\varphi$ preserves meets.

Let $S_0, S_1 \in \text{Sub}_\lambda B_{n+1}$ and let $S = S_1 \vee S_2$. If $t \in S \setminus (S_0 \cup S_1)$, then $t = t_0 \cap t_1$, for some $t_i \in S_i$, $i < 2$. Hence, by Claim 11, $\psi(t) \subseteq \text{Co}(\psi(t_0) \cup \psi(t_1)) \subseteq \varphi(S_0) \cup \varphi(S_1)$. Thus $\varphi(S_0 \vee S_1) \subseteq \varphi(S_0) \cup \varphi(S_1)$, whence $\varphi$ preserves joins. \(\square\)

For any $k < \omega$, for any $\lambda \geq 0$ small enough, and for any convex polytope $P \subseteq \mathbb{R}^k$, let $P^\lambda$ denote the (nonempty) convex polytope which is a subset of $P$, whose faces are parallel to the corresponding faces of $P$, and $\rho(P^\lambda, P) = \lambda$, where $\rho(A, B)$ denotes the distance between $A$ and $B$ defined by the standard Euclidean metric $\rho$. For any $x \in \text{Ex} P$, let $x^\lambda$ denote the corresponding extreme point of $P^\lambda$.

We fix $\omega \in \omega P$, and consider the polytope $S_{n+1}$ defined in the proof of Proposition 11. Let $\lambda > 0$ be small enough.
If \( A \subseteq \mathbb{n} + 1 \) and \(|A| = k + 1\), for some \( k < \omega\), then \( S_A\) denotes the regular polytope in \( \mathbb{R}^k \) with the set of extreme points \( \text{Ex}S_A = \{ p_i \mid i \in A \} \). For any \( B \subseteq A \), we put
\[
H_B = \{ \sum_{i \in B} \lambda_i p_i \mid \lambda_i \in \mathbb{R} \text{ for all } i \in B \}.
\]

For any different \( i, j \in A \), let \( p(i, A, j) \) be a unique point from the intersection \([p_i, p_j] \cap H_{A \setminus \{j\}}\). We put
\[
T(A, \lambda, j) = \text{Co}\{ p_i, p(i, A, j) \mid i \in A, i \neq j \}.
\]

For any \( j \in A \), the convex polytope \( T(A, \lambda, j) \) has two parallel faces: one is the face \( S_{A \setminus \{j\}} \) of the polytope \( S_A \), the other is the face \( S'_{A \setminus \{j\}} = \text{Co}\{ p(i, A, j) \mid i \in A, i \neq j \} \).

**Lemma 4.2.** For any \( j \in A \), \( T(A, \lambda, j) \cap S^\lambda_A \subseteq S'_{A \setminus \{j\}} \).

**Proof.** The proof is straightforward. \( \square \)

We also put \( U(A, \lambda, i) = \text{Co}\{ p_i \cup \{ p(i, A, j) \mid j \in A, j \neq i \} \} \).

**Lemma 4.3.** For any \( i \in A \), \( U(A, \lambda, i) \subseteq \bigcap\{ T(A, \lambda, j) \mid j \in A, j \neq i \} \).

**Proof.** For any \( j \in A \), \( j \neq i \), the convex polytope \( T(A, \lambda, j) \) contains the point \( p_i \) and the point \( p(i, A, j) \). Moreover, it contains the whole face \( S_{A \setminus \{j\}} \) whence all the points \( p(i, A, k), k \neq i, j \). Therefore, \( U(A, \lambda, i) \subseteq T(A, \lambda, j) \), for all \( j \in A, j \neq i \). \( \square \)

**Lemma 4.4.** For any \( i, j \in A \) such that \( i \neq j \), \( U(A, \lambda, i) \cap S'_{A \setminus \{j\}} = \{ p(i, A, j) \} \).

**Proof.** \( p(i, A, j) \in U(A, \lambda, i) \cap S'_{A \setminus \{j\}} \) by the definition of \( U(A, \lambda, i) \) and \( S'_{A \setminus \{j\}} \). To prove the reverse inclusion, we suppose that \( z \in U(A, \lambda, i) \cap S'_{A \setminus \{j\}} \). Then there are \( \mu_j \in [0, 1], j \in A \), such that \( \sum \{ \mu_j \mid j \in A \} = 1 \) and \( z = \mu_i p_i + \sum \{ \mu_j p(i, A, j) \mid j \in A, j \neq i \} \). Since \( S'_{A \setminus \{j\}} \) is a face and \( p_i \notin S'_{A \setminus \{j\}} \), we have \( \mu_i = 0 \) and
\[
\{ p(i, A, j) \mid j \in A, j \neq i, \mu_j \neq 0 \} \subseteq S'_{A \setminus \{j\}}.
\]

Obviously, \( p(i, A, k) \notin S'_{A \setminus \{j\}} \), for all \( k \neq i, j \). Thus, \( \mu_k = 0 \), for all \( k \neq i, j \), whence \( \mu_j = 1 \) and \( z = p(i, A, j) \). \( \square \)

**Lemma 4.5.** If \( q_i \in U(A, \lambda, i) \setminus \{ p(i, A, j) \mid j \in A, j \neq i \} \), for all \( i \in A \), then \( S^\lambda_A \subseteq \text{int}_{|A|} \text{Co}\{ q_i \mid i \in A \} \).

**Proof.** For any \( i \in A \), we put \( B_i = \text{Co}\{ q_j \mid j \in A, j \neq i \} \). Then \( B_i \subseteq T(A, \lambda, i) \), for all \( i \in A \), by Lemma 1.4. \( \square \) Moreover, if \( B_i \cap S'_{A \setminus \{i\}} \neq \emptyset \), then there exists \( j \in A \setminus \{i\} \) such that \( q_j \in S'_{A \setminus \{i\}} \cap U(A, \lambda, j) \) since \( S'_{A \setminus \{i\}} \) is a face of \( T(A, \lambda, i) \). By Lemma 1.4, this implies that \( q_j = p(j, A, i) \), a contradiction with the choice of \( q_j \). Therefore, \( B_i \subseteq T(A, \lambda, i) \setminus S'_{A \setminus \{i\}} \).

By Lemma 1.2, we get \( S^\lambda_A \cap B_i = \emptyset \), for all \( i \in A \). Thus, for any \( i \in A \), \( S^\lambda_A \) is a subset of the open half-space \( X_i \) defined by the hyperplane which contains \( B_i \). Hence, \( S^\lambda_A \subseteq \bigcap\{ X_i \mid i \in A \} = \text{int}_{|A|} \text{Co}\{ q_i \mid i \in A \} \). \( \square \)

**Lemma 4.6.** There is \( \varepsilon(\lambda) > 0 \) such that \( S^\lambda_A \subseteq \text{int}_{|A|} \text{Co}\{ S'_{A \setminus \{i\}} \cup S'_{A \setminus \{j\}} \} \), for any \( \varepsilon \in (0, \varepsilon(\lambda)] \) and any \( i, j \in A, i \neq j \).
Proof. We pick $\varepsilon(\lambda) > 0$ with respect to the property that the extreme point $p_k^i(\lambda)$ of the polytope $S_{i\{ i \}}^{<}(\lambda)$ (of the polytope $S_{i\{ j \}}^{<}(\lambda)$, respectively) belongs to $U(A, \lambda, k)$, for all $k \in A\{ i \}$ (for all $k \in A\{ j \}$, respectively). The desired conclusion follows then from Lemma 4.5. \hfill $\Box$  

We construct the finite set $X$ which provides an embedding of the lattice $\text{Sub}_{X} \mathcal{P}_{n+1}$ into the lattice $\text{Co}(\mathbb{R}^n, X)$. Let $v$ be the center of $S_{n+1}$. Let $\lambda_0 > 0$ be small enough. Suppose that $k < n - 1$ and we have already found $\lambda_0, \ldots, \lambda_k > 0$ such that $\lambda_j \in (0, \varepsilon(\lambda_{j-1}))$, for all $0 < j \leq k$. By Lemma 4.4 there exists $\lambda_{k+1} \in (0, \varepsilon(\lambda_k))$ such that, for any $A \subseteq n + 1$ with $|A| = n + 1 - k > 2$ and any $i, j \in A$, $i \neq j$, we have $S_A^{\lambda_k} \subseteq \text{int}_{|A|} \text{Co}(S_{A\{ i \}}^{<} \cup S_{A\{ j \}}^{<})$. We put $\lambda_n = 0$. For any nonempty $A \subseteq n + 1$ and any $i \in A$, we also put

$$P_A = S_A^{\lambda_k}, \quad U(A, i) = U(A, \lambda_k, i), \quad p(i, A) = p_i^{\lambda_k}$$

where $k < n + 1$ is such that $|A| + k = n + 1$.

**Lemma 4.7.** For any $A \subseteq B \subseteq n + 1$ and any $i \in A$, we have $U(A, i) \subseteq U(B, i)$.

Proof. We argue by induction on $|B\setminus A|$. If $|B\setminus A| = 0$ then $U(B, i) = U(A, i)$, and we are done. Let $j \in B\setminus A$. By the induction hypothesis, $U(A, i) \subseteq U(B\setminus \{ j \}, i)$. All the extreme points of the polytope $U(B\setminus \{ j \}, i)$ are in the interior of the face of $U(B, i)$ which is the convex hull of the set $\{ p_i \} \cup \{ p(i, B, k) \mid k \in B, k \neq i, j \}$. Therefore, $U(B\setminus \{ j \}, i) \subseteq U(B, i)$.

We define the desired set $X$ by

$$X = \{ v \} \cup \bigcup \{ \text{Ex} P_A \mid A \subseteq n + 1 \}.$$  

First, we notice the important property of the lattice $\text{Co}(\mathbb{R}^n, X)$.

We remark that the join dependency relation $D$ is defined for join irreducible elements $a, b$ of a lattice $L$, $a D b$, if $a \neq b$, and there is a $p \in L$ with $a \leq b \vee p$ and $a \nleq c \vee p$ for $c < p$. A $D$-sequence is a finite sequence $a_0, \ldots, a_{n-1}$ ($n \geq 2$) of join irreducible elements of $L$ such that $a_i D a_{i+1}$ for all $i < n$, where the subscripts are computed modulo $n$. It is well-known that a finite lattice $L$ is lower bounded if it contains no $D$-cycles (see, for example, Corollary 2.39 in [4]).

**Lemma 4.8.** The finite lattice $\text{Co}(\mathbb{R}^n, X)$ is lower bounded.

Proof. Suppose $a, b \in X\setminus \{ v \}$, then there are $A, B \subseteq n + 1$ such that $a \in \text{Ex} P_A$ and $b \in \text{Ex} P_B$. In this case, $\{ a \} D \{ b \}$ implies that $|B| < |A|$. Moreover, $\{ v \} D \{ a \}$, for any $a \in X\setminus \{ v \}$, and $\{ a \} D \{ v \}$ holds for no $a \in X$. Thus, the lattice $\text{Co}(\mathbb{R}^n, X)$ does not contain a $D$-cycle whatsoever it is lower bounded. \hfill $\Box$  

Secondly, we observe that the composition of $\psi_X$ defined in section 2, and $\varphi$ given by (2) is a a desired mapping of lattices.

**Proposition 4.9.** The map $\psi_X \varphi: \text{Sub}_{X} \mathcal{P}_{n+1} \rightarrow \text{Co}(\mathbb{R}^n, X)$ is a lattice embedding.

Proof. Since both $\psi_X$ and $\varphi$ preserve meets, the composition $\psi_X \varphi$ also does.

If $A \in B_0 \setminus B_1$, for some $B_0, B_1 \in \text{Sub}_{X} \mathcal{P}_{n+1}$, then $x \in \psi_X \varphi(B_0) \setminus \psi_X \varphi(B_1)$, where $x \in \text{Ex} P_{n+1\setminus A}$ in the case $A \subseteq n + 1$ and $x = v$ in the case $A = n + 1$. Therefore, the map $\psi_X \varphi$ is one-to-one.
To prove that $\psi_X \varphi$ preserves joins, it suffices to show that, for any noncomparable sets $A_0, A_1 \subseteq n + 1$,

$$\psi(A_0 \cap A_1) \cap X \subseteq \operatorname{Co}(\psi(A_0) \cup \psi(A_1)) \cap X,$$

where $\psi$ is the map defined by (1). By the definition, we have

$$\psi(A_0 \cap A_1) \cap X = \operatorname{Ex} P_{A_0 \cup A_1} = \{ p(i, A_0 \cup A_1) \mid i \in A_0 \cup A_1 \},$$

when $A_0 \cup A_1 \subseteq n + 1$, and

$$\psi(A_0 \cap A_1) \cap X = \{ v \},$$

when $A_0 \cup A_1 = n + 1$. By Lemma 4.7, for any $j_i \in A_i, i < 2$, we have $p(j_i, A_i) \in U(A_i \cup \{ j_1-i \}, j_i) \subseteq U(A_0 \cup A_1, j_i)$. Thus, by Lemma 4.8, we get

$$\psi(A_0 \cap A_1) \cap X \subseteq \operatorname{Co}\{ \{ p(i, A_0) \mid i \in A_0 \} \cup \{ p(i, A_1) \mid i \in A_1 \} \} \cap X$$

$$= \operatorname{Co}(\psi(A_0) \cup \psi(A_1)) \cap X.$$  

Moreover, for any $A_0, A_1 \subseteq n + 1$ such that $A_0 \cup A_1 = n + 1$, we have that $v \in \operatorname{Co}(\psi(A_0) \cup \psi(A_1))$. The proof of the lemma is complete.  

Now we state the main result of this section.

**Theorem 4.10.** For any finite lower bounded lattice $L$, there is $n \in \omega$ and a finite set $X \subseteq \mathbb{R}^n$ such that the lattice $\operatorname{Co}(\mathbb{R}^n, X)$ is lower bounded and $L$ embeds into both $\operatorname{Co}(\mathbb{R}^n)$ and $\operatorname{Co}(\mathbb{R}^n, X)$.

**Proof.** According to [13], for any finite lower bounded lattice $L$, there is $n \in \omega$ such that $L$ is isomorphic to a sublattice of $\operatorname{Sub}(\mathbb{B}_{n+1})$. The desired conclusion follows from Propositions 4.1 and 4.2.  

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