A 'fat hyperplane section' weak Lefschetz (in arbitrary characteristic), and Barth-type theorems

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Abstract

The goal of this paper is to prove a certain 'algebraic fat hyperplane section' Weak Lefschetz-type result for étale cohomology (of a local complete intersection variety $X'$ that is not necessarily proper); somewhat similar results for $X' \to \mathbb{CP}^N$ were previously established by Goresky and MacPherson using stratified Morse theory. In contrast to their 'topological' statements (that relate $\pi_i(X')$ with $\pi_i(\mathbb{CP}^{N-b}_\varepsilon)$, where $\mathbb{CP}^{N-b}_\varepsilon$ is the $\varepsilon$-neighbourhood of $\mathbb{CP}^{N-b}$ in $\mathbb{CP}^N$ for a small $\varepsilon > 0$) we formulate and prove our (cohomological) result using purely sheaf-theoretic methods; this makes it independent from the base field characteristic. Our proof is quite short; we apply an argument similar to the one used by Beilinson in order to establish a Weak Lefschetz theorem for general hyperplane sections of a smooth $X' \subset \mathbb{P}^N$. Considering our main theorem as a substitute of the one of Goresky and MacPherson seems to be an important idea; it allows to combine the implications of the result mentioned (that were previously known for complex varieties only) with (very convenient) sheaf-theoretic methods (in particular, with proper and smooth base change). So we extend (generalizations of) the seminal results of Barth and others to base fields of arbitrary characteristic. We obtain certain statements on the lower cohomology for closed subvarieties of $\mathbb{P}^N$ of small codimensions and of their preimages with respect to proper morphisms (that are not necessarily finite); to this end we apply the

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methods developed by Deligne, Fulton, and Lazarsfeld. Note: whereas a certain Barth-type theorem for subvarieties of \( \mathbb{P}^N \) (over any field \( K \)) was proved by Lyubeznik, he proved nothing about their preimages.

Keywords: Weak Lefschetz, \( \acute{e} \)tale cohomology, hyperplane section, characteristic \( p \).

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Introduction

In contemporary algebraic geometry there are a lot of 'descendants' of the classical Weak Lefschetz theorem (a rich collection of those, that includes the theorem of Barth on the cohomology of closed subvarieties of \( \mathbb{P}^N \) of low codimensions, can be found in [Laz04]). Whereas for the 'ordinary' Weak Lefschetz theorem the Artin’s vanishing yields a 'purely algebraic' proof (that works over an arbitrary characteristic base field \( K \)), the proofs of several other results are often somewhat 'topological', or use Hodge theory, or De Rham cohomology. These methods of the proofs restrict the results obtained to the case \( \text{char } K = 0 \) (the literature for this case is so vast that the author will not attempt to mention all of these results).

In particular, one of powerful tools for studying these Weak Lefschetz-type questions for \( K = \mathbb{C} \) (as shown in particular in §9 of [FuL81]) is the 'fat (multiple) hyperplane section' weak Lefschetz theorem formulated in §III.1.2 of [GoM89]. For a quasi-finite morphism \( s : X' \to \mathbb{C} \mathbb{P}^N \) (\( N > 0 \), \( X' \) is a local complete intersection variety) and any small enough \( \varepsilon > 0 \) it states that the lower homotopy groups of \( X' \) are isomorphic to those of \( s^{-1}(\mathbb{C} \mathbb{P}^N \setminus \varepsilon^{-b}) \), where \( \mathbb{C} \mathbb{P}^N \setminus \varepsilon^{-b} \) is the \( \varepsilon \)-neighbourhood (in the sense of some Riemannian metric for \( \mathbb{C} \mathbb{P}^N \)) for the standard embedding \( i \) of \( \mathbb{C} \mathbb{P}^N \setminus \varepsilon^{-b} \) into \( \mathbb{C} \mathbb{P}^N \) (0 < \( b < N \)) (cf. the caution below). Note that even the formulation of this statement requires some 'topology', whereas the proof given in ibid. heavily relies on the Stratified Morse Theory. Thus, both the formulation and the proof of
loc. cit. are far from being 'algebraic' (and in the opinion of the author, the proof mentioned is very complicated, and is very long to write down in full detail).

The current paper grew our from the following easy observation: the direct limit of the $\mathbb{Z}/l^n\mathbb{Z}$-cohomology of $s^{-1}(\mathbb{C}P^N_{\varepsilon-b})$ when $\varepsilon \to 0$ is just the (hyper)cohomology of $\mathbb{C}P^N_{-b}$ with coefficients in $Ri^*R_s^*\mathbb{Z}/l^n\mathbb{Z}_{X'}$ (see Remark 2.3(3) below). Then it was an easy exercise to verify that the lower $\mathbb{Z}/l^n\mathbb{Z}$-cohomology of $X'$ is isomorphic to the one of $Ri^*R_s^*\mathbb{Z}/l^n\mathbb{Z}_{X'}$; this is true both for singular and for étale cohomology. Though our proof relies on certain results of [BBD82] and [KiW01] (since the perverse $t$-structure is a very convenient tool for our purposes), its base is just Artin’s vanishing. Certainly, the result obtained (in the setting of étale cohomology) does not depend on the choice of $K$ (and $p$); this is one of its major advantages over the results of [GoM89]. To the opinion of the author our proof below is much easier than the corresponding one of Goresky and MacPherson (though it is somewhat technical). Though our reasoning has nothing to do with the arguments of ibid., we should note that somewhat similar methods were previously used by Beilinson in order to establish a Weak Lefschetz-type result for general hyperplane sections of a smooth $X' \subset \mathbb{P}^N$ (see Lemma 3.3 of [Bei87] and Remark 4.2(3) below). Yet considering our main theorem as a substitute of the results of [GoM89] seems to be an important idea that has several nice consequences (some of them are really new, especially in the case $p > 0$). In particular, we immediately obtain a (generalized) Weak Lefschetz theorem for projective ('almost') locally set-theoretic complete intersections in arbitrary characteristic.

A disadvantage of our methods is that for $K = \mathbb{C}$ they yield no information on (the cohomology of) $s^{-1}(\mathbb{C}P^N_{\varepsilon-b})$ for any particular $\varepsilon > 0$. On the other hand, we do not demand $s$ to be quasi-finite (see also Remark 2.3 below for a further discussion on the comparison of our results with those of [GoM89], including Theorem II1.1 of ibid.). Besides, our 'sheaf-theoretic' result can be easily combined with proper and smooth base change; see (the proofs of) Proposition 2.4 and Corollary 2.6 below. This allows us to use it as a substitute of the theorem of [GoM89] in the argument of Deligne described in §9 of [FaL81]; as a consequence, we obtain certain cohomological analogues of the results of loc. cit. over arbitrary characteristic fields. In particular, we easily extend the theorems of Barth on the lower cohomology of closed subvarieties of $\mathbb{P}^N$ of small codimensions and of their preimages with respect to proper morphisms (we do not require those to be finite in contrast with Corollary 9.8 of [FaL81]) to the case of arbitrary characteristic. More generally, we compute the lower cohomology of the preimages of the diagonal in $(\mathbb{P}^N)^g$ with respect to proper morphisms.
Taking all of this into account, the author hopes that his methods will become a useful tool for studying (various) Weak Lefschetz-type questions (at least, 'higher degree' ones). We should certainly mention here that Lyubetskik has also applied certain 'sheaf-theoretic' methods to the study of the lower cohomology of subvarieties of $\mathbb{P}^N$ for an arbitrary $p$; in Theorem 10.5 of [Lyu93] he proved several results that are (somewhat) stronger than our Theorem 3.1(II1) (cf. also Remark 2.3(4) below). Yet it seems that the methods of ibid. (that are very interesting and quite distinct from our ones) cannot say much on the cohomology of the preimages of subvarieties in $\mathbb{P}^N$ (and in $(\mathbb{P}^N)^q$, with respect to proper morphisms; cf. Remark 3.2(5)).

A caution: for $K = \mathbb{C}$ the natural morphism $h^i : H^i(X',Z/l^nZ) \to H^i(s^{-1}(\mathbb{C}P^N-b),Z/l^nZ)$ certainly factorizes as the composition $H^i(X') \xrightarrow{f^i} \lim_{\epsilon \to 0} H^i(s^{-1}(\mathbb{C}P^N-b)) \xrightarrow{g^i} H^i(s^{-1}(\mathbb{C}P^N-b),Z/l^nZ)$ (for any $i \geq 0$). In the proof of Corollary 2.6(1) below we describe an algebraic analogue of this factorization (through $H^i(\mathbb{P}^{N-b},R^iR_*Z/l^nZ_{X'})$; unfortunately, the proof is somewhat formal). It turns out that this factorization simplifies the study of weak Lefschetz-type questions. In particular, the results of §II1.2 of [GoM89] (in the case $K = \mathbb{C}$) and our Theorem 2.2 (for the general case) yield that $f^i$ is bijective for 'small enough' $i$ (even if $s$ is not proper). Note that this result does not extend to $h^i$ (in the general case); in particular, $s^{-1}(\mathbb{C}P^N-b)$ could be empty. Yet if $s$ is proper over some open $U \subset X$, $Z \subset O$, then $g^i$ is necessarily an isomorphism for all $i$ (our proof of this result is a more or less easy combination of smooth and proper base change theorems). Thus one may think about $H^*(\mathbb{P}^{N-b},R^iR_*Z/l^nZ_{X'})$ as of an 'approximation' to the cohomology of $s^{-1}(\mathbb{C}P^N-b)$ that has several nice properties. Our Theorem 2.2 yields that $f^i$ is an isomorphism for lower $i$ (on may say that this is 'a Goresky-MacPherson-type' result); in Corollary 2.6(1) we use it in order to establish a ('true') Weak-Lefschetz-type result, which we actively apply in §III.

Lastly we note that our methods also yield (without any problem at all) a certain generalization of our basic result to the relative setting (for example, to schemes over $\text{Spec} \mathbb{Z}[[t]]$ instead of $K$-varieties). Yet this expansion would be stated in terms of perverse sheaves (over the base), and at the moment the author knows no 'visualizable' applications for it (still see Remark 2.3(7) below).

Now we list the contents of the paper. First we give a 'brief plan' of it. The main result of the paper is Theorem 2.2 (our 'Fat hyperplane section Weak Lefschetz'); we use it along with a simple (smooth and proper) base change argument in order to deduce Corollary 2.6 (that states that under certain restictions a 'True Weak Lefschetz-type result' holds). We apply
the latter result to certain $G_m$-bundles (constructed by Deligne) in Theorem 3.1 (I); then we prove a certain Barth-type theorem (in Theorem 3.1 (II)).

In §1 we recall some basics on the derived categories of (constructible) $\mathbb{Z}/l^n\mathbb{Z}$-sheaves, on functors between them, and on the perverse $t$-structure.

In §2 we prove our ‘fat multiple hyperplane section’ weak Lefschetz-type Theorem 2.2. We also make several remarks (on possible modifications of loc. cit.; a reader that is only interested in the results of §3 may skip the rather long Remark 2.3), and show how one can use proper and smooth base change in order to calculate the (hyper)cohomology of $Ri^* Rs_* \mathbb{Z}/l^n\mathbb{Z}(r)_{X'}$.

In §3 we describe some applications of Theorem 2.2. Following §9 of [FuL81], we consider certain (locally trivial) $G_m$-bundles, and relate the cohomology of a $Y$ that is proper over $(\mathbb{P}^N)^g$ with the one of the corresponding $G_m$-bundle over the preimage of the diagonal. As a result, we extend the results of Barth to the case of an arbitrary $p$ (and to $\mathbb{Z}/l^n\mathbb{Z}$-coefficients).

In §4 we make several remarks on the calculating of the (hyper)cohomology of $Ri^*(-)$ via henselizations (that are related with author’s ideas on the proof of a certain Weak Lefschetz-type statement for torsion motivic cohomology).

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**Notation.** $K$ will be our base field of characteristic $p$ ($p$ could be zero). The reader may (always) assume that $K$ is algebraically closed; in particular, then one can ignore some of the statements from §1. We will denote by $X_{red}$ the reduced scheme associated to a scheme $X/K$.

$pt$ is a point, $\mathbb{A}^N$ is the $N$-dimensional affine space (over $K$), $\mathbb{P}^N$ is the projective space of dimension $N$, $G_m = \mathbb{A}^1 \setminus \{0\}$ is the multiplicative group scheme.

We will say that a variety has dimension $m$ only in the case when it is equidimensional of dimension $m$. We will say that a morphism of varieties has relative dimension $\leq e$ if the dimension of all of its fibres is $\leq e$.

We will call a equidimensional variety a local set-theoretic complete intersection (or just an LSTCI) if it possesses a Zariski cover each of whose components is isomorphic to an open subvariety of a set-theoretic complete intersection in $\mathbb{P}^N$ (for some $N \geq 0$).

Throughout the paper we will be interested in étale cohomology with $\mathbb{Z}/l^n\mathbb{Z}(r)$-coefficients, where $l$ is a prime $\neq p$, $n > 0$, and $r \in \mathbb{Z}$. One can assume $l, n$ to be fixed; alternatively, our results for the case of an arbitrary $n$ can be easily reduced to the case $n = 1$. Besides, the case $r = 0$ is already
interesting enough (and is equivalent to the general case if $K$ is an algebraically closed), though one may also consider cohomology with coefficients in arbitrary $l$-torsion locally constant étale sheaves (cf. Remark 2.3[11]).

Below we will consider a morphism $s_X : X' \to X$ and a closed embedding $i : Z \to X$; in the introduction and in the abstract we took $X = \mathbb{P}^N$, $Z = \mathbb{P}^{N-b}$, and wrote $s$ instead of $s_X$. Besides, following [BBD82] we will always omit $R$ in $Rf_*$, $Rf_!$, $Rf^*$ and $Rf^!$ (for $f$ being a finite type morphism of varieties); also, we will ignore the difference between cohomology (of sheaves) and hypercohomology (of complexes of sheaves). $t$ will denote the (self-dual) perverse $t$-structure for $D_c Sh^e_{\mathbb{Z}/l^nZ-mod}(-)$; see below.

$D^b Sh^e_{\mathbb{Z}/l^nZ-mod}(Y)$ will denote the derived category of complexes of étale $\mathbb{Z}/l^n\mathbb{Z}$-module sheaves over a variety $Y$ with constructible cohomology. We will introduce some (more) notation for these categories in Proposition 1.1.

For an (additive) category $C$ we will denote by $C(A, B)$ the group $\text{Mor}_C(A, B)$.

1 Some preliminaries on derived categories of sheaves and the perverse $t$-structure for them

In this section we introduce some notation and remind some of the properties of the categories $D_c Sh^e_{\mathbb{Z}/l^nZ-mod}(-)$ and functors between them; we relate those with the properties of the ‘usual’ étale (hyper)cohomology of complexes of sheaves. We also describe the properties of the perverse $t$-structure for $D_c Sh^e_{\mathbb{Z}/l^nZ-mod}(-)$. The statements of this section are well-known (possibly expect Proposition 1.3[11], which is quite easy). Most of them were (essentially) proved in SGA4, SGA41/2, or in [BBD82].

We do not mention the étale fundamental group here: though we consider it in Corollary 2.6(2,3) below, these results do not seem to be important to the author.

Proposition 1.1. Let $f : X \to Y$ be a morphism of $K$-varieties.

1. For $C \in \text{Obj} D^b(Sh^e(X, \mathbb{Z}/l^n\mathbb{Z}mod), i \in \mathbb{Z}$, we will denote $D^b(Sh^e(X, \mathbb{Z}/l^n\mathbb{Z}mod))(\mathbb{Z}/l^n\mathbb{Z}_X, C[i])$ by $H^i(X, C)$; this is the ‘usual’ $i$-th (hyper)cohomology of $X$ with coefficients in $C$.

2. For any variety $X/K$ there is a full tensor triangulated subcategory $D^b_c Sh^e_{\mathbb{Z}/l^n\mathbb{Z}mod}(X) \subset D^b(Sh^e(X, \mathbb{Z}/l^n\mathbb{Z}mod))$; the Tate twist $\mathbb{Z}/l^n\mathbb{Z}(s)$ of $\mathbb{Z}/l^n\mathbb{Z}$ belongs to $\text{Obj} D^b_c Sh^e_{\mathbb{Z}/l^n\mathbb{Z}mod}(X)$ for any $s \in \mathbb{Z}$.
For $C \in \text{Obj}D^b(\text{Sh}^\text{et}(X, \mathbb{Z}/l^n\mathbb{Z} - \text{mod}))$, $s \in \mathbb{Z}$, we will denote $C \otimes \mathbb{Z}/l^n\mathbb{Z}(s)$ by $C(s)$ (whereas $C[i]$ for $i \in \mathbb{Z}$ denotes the shift of $C$ by $i$ 'to the left').

3. The following pairs of adjoint functors are defined: $f^* : D(\text{Sh}^\text{et}(Y, \mathbb{Z}/l^n\mathbb{Z} - \text{mod})) \rightleftarrows D(\text{Sh}^\text{et}(X, \mathbb{Z}/l^n\mathbb{Z} - \text{mod})) : f_*$ and $f^! : D^b(\text{Sh}^\text{et}(Y, \mathbb{Z}/l^n\mathbb{Z} - \text{mod})) \rightleftarrows D^b(\text{Sh}^\text{et}(X, \mathbb{Z}/l^n\mathbb{Z} - \text{mod})) : f_!$; they restrict to functors $f^* : D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(Y)) \rightleftarrows D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(X)) : f_*$ and $f^! : D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(Y)) \rightleftarrows D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(X)) : f_!$. Any of these four types of functors (when $f$ varies) yields a 2-functor from the category of $K$-varieties to the 2-category of triangulated categories.

4. $D^b(\text{Sh}^\text{et}(\text{Spec} K, \mathbb{Z}/l^n\mathbb{Z} - \text{mod})) \cong D^b(\mathbb{Z}/l^n\mathbb{Z}[G] - \text{mod})$, where $G$ is the absolute Galois group of $K$; this is an isomorphism of tensor triangulated categories.

5. Consider the structure morphisms $x : X \to \text{Spec} K$ and $y : Y \to \text{Spec} K$. Then for any $C \in \text{Obj}D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(Y))$ we have: $H^*(Y, C) \cong H^*(\text{Spec} K, Y, C)$. Moreover, the 'usual' morphisms of cohomology $H^*(Y, C) \to H^*(x_\ast f^* C) \cong H^*(\text{Spec} K, x_\ast f^* C)$ come from the adjunction $f^* : D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(Y)) \rightleftarrows D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(X)) : f_*$.

6. $f^*$ is symmetric monoidal; $f^!(\mathbb{Z}/l^n\mathbb{Z}(s)) = \mathbb{Z}/l^n\mathbb{Z}(s)_X$ for any $s \in \mathbb{Z}$.

7. $f_\ast = f_!$, if $f$ is proper; if $f$ is an open immersion, then $f^! = f^*$. 

8. Let $K$ be algebraically closed. Then for any $s \in \mathbb{Z}$, $X/K$, one can choose an isomorphism $\mathbb{Z}/l^n\mathbb{Z}_X \cong \mathbb{Z}/l^n\mathbb{Z}(s)_X$ that is functorial in $X$. Besides, for $X, Y$ being varieties over $K$, $Z = X \times Y$, we have the Kunneth formula for the cohomology of $Z$, i.e., for the corresponding structure morphisms $x, y, z$ we have: $z_* \mathbb{Z}/l^n\mathbb{Z}_Z \cong x_* \mathbb{Z}/l^n\mathbb{Z}_X \otimes y_* \mathbb{Z}/l^n\mathbb{Z}_Y$.

9. If $i : Z \to X$ is a closed immersion, $U = X \setminus Z$, $j : U \to X$ is the complementary open immersion, then the functors given by assertion 3) yield gluing data for $D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(\cdot))$ (in the sense of §1.4.3 of [BBD83]; see also Definition 8.2.1 of [Bon10a]). That means that (in addition to the adjunctions given by assertion 3) the following statements are valid.

(i) $i_* \cong i^!$ is a full embeddings; $j^* = j^!$ is isomorphic to the localization (functor) of $D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(X))$ by $i_*(D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(Z)))$.

(ii) For any $M \in \text{Obj}D^b_c(\text{Sh}^\text{et}_{Z/l^n\mathbb{Z}-\text{mod}}(X)$ the pairs of morphisms

\[ jj^!(M) \to M \to ii^!(M) \] (1)
and
\[ i_*i^! M \to M \to j_*j^* M \] (2)
can be completed to distinguished triangles (here the connecting morphisms come from the adjunctions of assertion 3).

(iii) \( i^*j_* = 0; i^!j_* = 0 \).

(iv) All of the adjunction transformations \( i^*i_* \to 1_{D_c^b Sh_{Z/l^nZ-mod}(Z)} \) \( \to \) \( i^!i_* \) and \( j^*j_* \to 1_{D_c^b Sh_{Z/l^nZ-mod}(U)} \) \( \to \) \( j^*j_* \) are isomorphisms of functors.

10. In the setting of the previous assertion, if \( i \) is a closed immersion of smooth varieties everywhere of some codimension \( s > 0 \), \( U = X \setminus Z \), \( j : U \to X \) is the corresponding embedding, then for any \( C \in \text{Obj}D_c^b(Sh^{et}(X, Z/l^nZ-mod)) \) there exists a distinguished triangle \( i_*i^*C(-s)[-2s] \to C \to j_*j^*C \).

Proof. The statements are well-known (see SGA4 and SGA41/2 for the proofs) and most of them were (actively) used in [BBD82] (see also Theorem 6.3 of [Eke90]). We will only give a few (more precise) references.

Assertion 2 is mostly given by Corollary 1.5 of SGA41/2.

Assertion 5 could be deduced from the ‘classical’ properties of cohomology by applying adjunctions.

Assertion 8 follows easily from Corollary 1.11 of loc. cit.

Assertion 10 can be obtained from the distinguished triangle (2) by applying purity; cf. §3.2 of SGA4.XVIII.

We will also need certain properties of base change transformations (closely following §4 of SGA4.XVII).

**Definition 1.2.** For a commutative square
\[ \begin{array}{ccc}
X' & \xleftarrow{i'} & Z' \\
\downarrow{s_X} & & \downarrow{s_Z} \\
X & \xleftarrow{i} & Z 
\end{array} \] (3)
of morphisms of varieties and \( C \in \text{Obj}D_c^b Sh_{Z/l^nZ-mod}(X') \) we call the composition of the morphisms \( i^*s_X_*C \to i^*s_X*i'^*C = i'^*s_Z*i^*C \to s_Z*i^*C \) coming from the corresponding adjunctions the base change morphism; we denote the corresponding natural transformation \( i^*s_X_* \implies s_Z*i^* \) by \( B(s_X, i) \).
Proposition 1.3. I In the setting of Definition 1.2 the following statements are fulfilled.

1. The morphism \( s_X \star C \to s_X \mathcal{i}^*C \) that comes from the adjunction \( \mathcal{i}^*: \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(X') \rightleftarrows \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(Z') : \mathcal{i}^* \) equals the composition of \( \mathcal{i}_*(B(s_X, \mathcal{i})(C)) \) with the morphism \( s_X \star C \to \mathcal{i}_! \mathcal{i}_*s_X \star C \) coming from the adjunction \( \mathcal{i}^*: \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(X) \rightleftarrows \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(Z) : \mathcal{i}_* \).

2. Let \( \square \) be a cartesian square. Then \( B(s_X, \mathcal{i}) \) is an isomorphism if \( \mathcal{i} \) is either smooth (this statement is called smooth base change), or if \( s_X \) is proper (this is proper base change).

II Base change transformations respect compositions, i.e., for a commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{i_1'} & O' & \xrightarrow{i_2'} & X' \\
\downarrow{s_Z} & & \downarrow{s_O} & & \downarrow{s_X} \\
Z & \xrightarrow{i_1} & O & \xrightarrow{i_2} & X
\end{array}
\]

we have: \( B(s_X, i_2 \circ i_1) = B(s_O, i_1)(i_2'(-)) \circ i_1'(B(s_X, i_2)) \).

Proof. II. So, we have the composition \( s_X \star C \to \mathcal{i}_! \mathcal{i}_*s_X \star C \to \mathcal{i}_! \mathcal{i}_*s_X \mathcal{i}^*C = \mathcal{i}_! \mathcal{i}_*s_X \mathcal{i}^*C \to \mathcal{i}_! s_Z \mathcal{i}^*C \) is the identity. The latter statement is immediate from the fact the the composition of transformations \( \mathcal{i}_* \implies \mathcal{i}_! \mathcal{i}^* \mathcal{i}_* \implies \mathcal{i}_* \) (comparing the adjunction \( \mathcal{i}^*: \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(X) \rightleftarrows \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(Z) : \mathcal{i}_* \)) is the identical transformation of \( \mathcal{i}_* \) (this is a basic general property of adjunctions).

II This is also a well-known statement (see §2 of [SGA4.XVI]). We should check that \( B(s_X, i_2 \circ i_1) \) equals the composition \( (i_2 \circ i_1)^*X \star s_X \implies (i_2 \circ i_1)^*s_X \mathcal{i}_! s_Z \mathcal{i}_! \mathcal{i}_! s_Z \mathcal{i}^*C = \mathcal{i}_! s_Z \mathcal{i}^*C \implies \mathcal{i}_! s_Z \mathcal{i}^*C \implies \mathcal{i}_! s_Z \mathcal{i}^*C \implies s_Z \mathcal{i}^*C \).

Now we recall some of the properties of the perverse t-structure \( t \) for \( \mathcal{D}^b_{\text{et}}Z_{/l^nZ-\text{mod}}(-) \) (that corresponds to the self-dual perversity denoted by \( p_{1/2} \) in [BBDS2]). We will not need much of them (we even will not need a definition for \( t \)); so we just recall some of the results of [BBDS2] and verify briefly that two of the results of [K-W01] can be carried over to our setting (of \( Z/l^nZ \)-module sheaves for an arbitrary \( K \)).
Proposition 1.4. For the perverse $t$-structure $t$ (given by the couple $(D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(-)^{p<0}, D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(-)^{p\geq 0})$; cf. ibid.) corresponding to the perversity $p_{1/2}$ the following statements are fulfilled.

1. If $X = \text{Spec} \, K$, $K$ is a field, then $t$ is just the canonical $t$-structure for $D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(\text{Spec} \, K)$ (that is compatible with the canonical $t$-structure for $D^b(\mathbb{Z}/l^n\mathbb{Z}[G] \text{- mod})$, where $G$ is the absolute Galois group of $K$). In particular, if $C \in \text{Obj} \, D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(\text{Spec} \, K)^{p \geq 0}$ then $H^i(\text{Spec} \, K, C) = 0$ for any $i < 0$.

2. Let $L$ be the algebraic closure of a field $K$, $f : X \to Y$ is a morphism of $K$-varieties; denote by $f_{L,\text{red}} : X_{L,\text{red}} \to Y_{L,\text{red}}$ the corresponding morphism of $L$-varieties.

   Suppose that for some $N, r \in \mathbb{Z}$ we have: $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f_{L,\text{red}})$ is bijective for all $i < N$, and is injective for $i = N$. Then $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is also bijective for all $i < N$, and is injective for $i = N$.

3. Suppose that a variety $U$ can be presented as the union of $b$ open affine subvarieties (for some $b$); $u : U \to \text{Spec} \, K$ is its structure morphism. Then $u_!(D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(U)^{p \geq 0}) \subset D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(K)^{p \geq 1-b}$.

4. Let $f : X \to Y$ be a morphism of relative dimension $\leq d$. Then $f_*(D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(X)^{p \geq 0}) \subset D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(Y)^{p \geq -d}$.

5. Let $X$ be a local set-theoretic complete intersection (in the sense of the Notation, i.e., suppose that it possesses a Zariski cover each of whose components is isomorphic to an open subvariety of a set-theoretic complete intersection in $\mathbb{P}^N$ for some $N \geq 0$) of dimension $a$. Then $\mathbb{Z}/l^n\mathbb{Z}(r)_X \in D_c^b \text{Sh}^\text{et}_{\mathbb{Z}/l^n\mathbb{Z}}(X)^{p \geq a}$ for any $r \in \mathbb{Z}$.

Proof. Most of these statements were proved in [BBD82] (note that the proofs work in the case of $\mathbb{Z}/l^n\mathbb{Z}$-coefficients; see §4.0 of [BBD82]).

1. Immediate from the definition of $t$ (see §2.2.12-13 of ibid.).

2. Applying Corollary of [SGA4.VIII], we can pass to the case of a perfect $K$ (so, $f_{L,\text{red}} : X_{L,\text{red}} \to Y_{L,\text{red}}$ equals $f_L : X_L \to Y_L$).

   We apply Proposition [1.1.5]; denote the corresponding structure morphisms $X, Y \to \text{Spec} \, K$ and $X_L, Y_L \to \text{Spec} \, L$ by $x, y, x_L, y_L$, respectively. We obtain: $H^i(\text{Spec} \, L, C_L) = 0$ for $i < N$, where $C_L = \text{Cone}(y_{L*}y_L^*\mathbb{Z}/l^n\mathbb{Z}(r)_L \to y_{L*}f_{L*}f_!y_L^*\mathbb{Z}/l^n\mathbb{Z}(r)_L)$. Obviously, this is...
equivalent to: $C_L \in D_c^b Sh^c_{Z/l^nZ \text{-mod}}(\text{Spec } L)^{p^a N}$. Consider $C_K = \text{Cone}(y_* y^* Z/l^nZ(r)_K \to y_* f_* f^* y^* Z/l^nZ(r)_K) \in D^b(Z/l^nZ[G] - \text{mod}).$

We note that $C_L$ can be obtained from $C_K$ via the forgetful functor $D^b(Z/l^nZ[G] - \text{mod}) \to D^b(Z/l^nZ - \text{mod})$ (this is a consequence of smooth base change; note that one can 'pass to the limit' in Proposition 1.3.1(ii); see Theorem 1.1 of [SGA4.XVI]). Hence $C_K \in D^b(Z/l^nZ[G] - \text{mod})^{p^a}$; so it belongs to $D_c^b Sh^c_{Z/l^nZ - \text{mod}}(\text{Spec } K)^{\leq N}$. It remains to apply the previous assertion.

3. Immediate from §4.2.3 of [BBD82].

4. The statement is contained in §4.2.4 of ibid.

5. This is 'the easier half of' the $Z/l^nZ$-coefficient analogue of Lemma III16.5 of [KiW01]; so we will only sketch the reduction of the statement to the results of [BBD82]. First we note that the property of belonging to $D_c^b Sh^c_{Z/l^nZ - \text{mod}}(-)^{p^a}$ is Zariski-local (it is even ´etale-local immediately from the definition of $t$); hence we can assume that $X$ is a set-theoretic complete intersection in $\mathbb{P}^N$ for some $N > 0$; we denote by $i$ the corresponding embedding.

Since $i_*$ is conservative and $t$-exact (see §4.2.4 of [BBD82]), it suffices to verify that $i_* Z/l^nZ(r)_X \cong i_* i^* Z/l^nZ(r)_{\mathbb{P}^N} \in D^b_c Sh^c_{Z/l^nZ - \text{mod}}((\mathbb{P}^N)^{p^a}).$

Now we note that $Z/l^nZ(r)_{\mathbb{P}^N} \in D^b_c Sh^c_{Z/l^nZ - \text{mod}}((\mathbb{P}^N)^{p^a N})$ (since $\mathbb{P}^N$ and $U$ are smooth, and so for $p: \mathbb{P}^N \to \text{Spec } K$ the functor $p^*[N]$ is $t$-exact; see Proposition 4.2.5 of ibid.); whereas for $j$ being the embedding of $U = \mathbb{P}^N \setminus X \to \mathbb{P}^N$ we have $j_! (D^b_c Sh^c_{Z/l^nZ - \text{mod}}(U)^{\geq 0}) \subset D^b_c Sh^c_{Z/l^nZ - \text{mod}}((\mathbb{P}^N)^{\geq 1 + N})$; see §4.2.3 of ibid. Next, consider the (rotation of the) distinguished triangle given by (I): $Z/l^nZ(r)_{\mathbb{P}^N} \to i_* i^* Z/l^nZ(r)_{\mathbb{P}^N} \to j_! j^* Z/l^nZ(r)_{\mathbb{P}^N}[1]$. We have $j_! j^* Z/l^nZ(r)_{\mathbb{P}^N}[1] \cong j_! Z/l^nZ(r)_{\mathbb{P}^N}[1] \in D^b_c Sh^c_{Z/l^nZ - \text{mod}}((\mathbb{P}^N)^{p^a})$; since $Z/l^nZ(r)_{\mathbb{P}^N} \in D^b_c Sh^c_{Z/l^nZ - \text{mod}}((\mathbb{P}^N)^{p^a})$, the same is true for $i_* Z/l^nZ(r)_X$ and we obtain the result in question.

Remark 1.5. Using devisage, we could have reduced some of the results of this section to the setting of sheaves of $Z/lZ$-vector spaces.
2 Our 'fat hyperplane section' weak Lefschetz-type theorem

Lemma 2.1. Let $s_X : X' \to X$ be a morphism of varieties; let $i : Z \to X$ be a closed immersion. Denote by $x : X \to \text{Spec} K$ the structure morphism of $X$.

Then we have: $H^*(X', \mathbb{Z}/l^n\mathbb{Z}(r)_X') \cong H^*(X, s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X)$, and $H^*(Z, i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X') \cong H^*(X, i_*i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X)$.

Proof. Denote by $x'$ and $z$ the structure morphisms of $X'$ and of $Z$, respectively.

We apply Proposition 1.1(1). We have $H^*(X', \mathbb{Z}/l^n\mathbb{Z}(r)_X') \cong H^*(\text{Spec} K, x'_*\mathbb{Z}/l^n\mathbb{Z}(r)_X') = H^*(\text{Spec} K, x_*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X) \cong H^*(X, s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X)$, and $H^*(Z, i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X') \cong H^*(\text{Spec} K, i_*i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X)$.

Consider the morphism $\mathcal{F}_r : s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X' \to i_*i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X'$ coming from the adjunction $i^* : D^b_{\text{Sh}}(\mathbb{Z}/l^n\mathbb{Z}(Z)_X) \cong D^b_{\text{Sh}}(\mathbb{Z}/l^n\mathbb{Z}(Z)_X) : i_*$. The base of the results of this paper is the following statement.

Theorem 2.2. In the setting of the previous lemma, suppose that $s_X$ is of relative dimension $\leq e$ (i.e., the dimension of all of its fibres is $\leq e$), $X$ is proper; assume that that $U' = s_X^{-1}(U)$ is locally a set-theoretic complete intersection (everywhere) of dimension $a$ (see the Notation section), and that $U$ can be presented as the union of $b$ open affine subvarieties (for some $a, b, e \geq 0$).

Then $H^i(X, -)(\mathcal{F}_r) : H^i(X', \mathbb{Z}/l^n\mathbb{Z}(r)_X') \to H^i(Z, i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X')$ is a bijection for $i < a - e - b$, and an injection for $i = a - e - b$.

Proof. We have the following commutative diagram

\[
\begin{array}{ccc}
U' & \xrightarrow{i'} & X' \\
\downarrow{s_U} & & \downarrow{s_X} \\
U & \xrightarrow{i} & X
\end{array}
\]

(both of the corner squares are cartesian); we denote the corresponding structure morphisms (whose target is $\text{Spec} K$) by $u, x, z, u', x'$, and $z'$, respectively.

Applying 1, we obtain that there exists a distinguished triangle $j^!s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X' \to s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X \to i_*i^*s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X'$.

By Proposition 1.1(1) it yields: we should prove that the cohomology of $\text{Spec} K$ with coefficients in $x, j^!s_{X*}\mathbb{Z}/l^n\mathbb{Z}(r)_X'$ vanishes in degrees $\leq$
a − e − b. To this end it suffices to verify that $x_3^*j^*s_XZ/l^nZ(r)_{X'} \in D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(\text{Spec } K)^{p \geq a−e−b+1}$ (see Proposition [1,4,1]).

By smooth base change (see Proposition [1,3,12]), the latter term is isomorphic to $x_3^*s_{U*}Z/l^nZ(r)_{X'}$. We transform this further into $x_3^*s_{U*}Z/l^nZ(r)_{U'} = u_3s_{U*}Z/l^nZ(r)_{U'}$ (we use the properness of $x$; see Proposition [1,1,7]).

We have $Z/l^nZ(r)_{U'} \in D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(U')^{p \geq a}$ (see Proposition [1,4,5]). Next, $s_{U*}$ sends $D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(U')^{p \geq a}$ into $D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(U)^{p \geq a−e}$ (since $s_U$ is of relative dimension $\leq e$; see part [1] of loc. cit.).

Now, $u_3$ sends $D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(U)^{p \geq a−e}$ into $D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(\text{Spec } K)^{p \geq a−e−b+1}$ by part [3] of loc. cit. Hence part [1] of loc. cit. yields the result.

$\square$

Remark 2.3. A reader that is only interested in the applications of our result described in §3 may skip this remark.

1. Instead of a general $i : Z \to X$ one can consider $i$ being the natural embedding of $\mathbb{P}^{N−b}$ into $\mathbb{P}^N$; this doesn’t decrease the generality of the result significantly.

Besides, instead of $Z/l^nZ(r)$ we could have considered any locally constant $l$-torsion sheaf on $X$ (cf. Theorem 10.5 of [Lyu93]). Indeed (as shown by our proof) it suffices to verify for any locally constant $Z/l^nZ$-sheaf $C/X'$ that $j^!C \in D^b_{\text{et}}Sh^c_{Z/l^nZ-\text{mod}}(U')^{p = a}$. Now, the latter statement can be verified étale-locally with respect to $U'$; so we can assume that $j^!C$ is constant and extend (the proof of) Proposition [1,4,5] to this setting.

2. The theorem was inspired by the ’fat multiple hyperplane section’ version of Weak Lefschetz proved by M. Goresky and R. MacPherson (for complex varieties; see the theorem in §II.1.2 of [GoM89]). For $i$ being the embedding of $Z = \mathbb{CP}^{N−b}$ into $X = \mathbb{CP}^N$, a quasi-finite $s_X$ (i.e., $e = 0$), and any small enough $\varepsilon > 0$ loc. cit. states: for the $\varepsilon$-neighbourhood (in the sense of some Riemannian metric for $X$) $Z_{\varepsilon}$ of $Z$ in $X$ the natural map $\pi_i(s_X^{-1}(Z_{\varepsilon})) \to \pi_i(X')$ is an isomorphism for $i < N−b$, and is an injection for $i = N−b$. Note that our methods can also be carried over to this ’topological’ context; they yield (cf. part [3] of this remark) a comparison of the cohomology of $X'$ with the limit for $\varepsilon \to 0$ of the cohomology of $s_X^{-1}(Z_{\varepsilon})$ (if one wants to consider cohomology with integral coefficients here, then one can apply our methods for $\mathbb{Q}$-coefficients and for $Z/lZ$-ones for all prime $l$ separately, and then combine the results obtained; see Theorem 5.2.16(ii) of [Dim04]). This statement is somewhat weaker than the result cited; on the other hand
the proof is much simpler than the one in ibid. (at least, it uses no stratified Morse theory).

We can also generalize our result in order to include the cases when \( U' \) is 'not quite a local complete intersection'. Similarly to [GoM89] (and to Theorem 10.5(iv) of [Lyu93]), this will decrease the highest degree in which we have a bijection (and an injection) for cohomology by a certain measure \( c \) of the failure for \( U' \) to be a LSTCI. In order to prove this statement only needs the corresponding generalization of Proposition 1.3(b) (cf. our argument above); the proof of loc. cit. can be easily extended to this setting.

3. We will discuss the difference between cohomological and homotopy formulations of Weak Lefschetz-type results in Remark 3.1 below.

We note here: if \( i : Z \rightarrow X \) is an embedding of paracompact topological spaces, \( F \) is a ‘topological’ sheaf (of sets or abelian groups) on \( X \), then
\[
(i^*F)(Z) = \lim_{Z_0 \supset Z, \text{\scriptsize Z_0 is open in X}} F(Z')
\]
(i.e., one does not need to sheafify here; see Corollary 1 in §3.3 of [God73]). Moreover, if \( Z \) is compact and \( X \) is a metric space then it suffices to take \( Z_\varepsilon \) (the \( \varepsilon \)-neighbourhood of \( Z \) in \( X \)) for all \( \varepsilon > 0 \) instead of all possible \( Z_0 \) in \( Z \). Hence we obtain (cf. Proposition 5.6.6 of [Kan88]): Theorem 2.2 yields indeed that a cohomological analogue of the result of [GoM89] (as discussed above) is fulfilled 'in the limit'. Besides, considering the preimages of 'fat multiple hyperplane sections' (i.e., \( s_Z^{-1}(Z_\varepsilon) \)) yields a method of computing \( H^*(Z, i^*s_X^*\mathbb{Z}/l^n\mathbb{Z}_X) \) when \( K = \mathbb{C} \). For a general \( K \) one can consider certain henselizations instead; see Remark 4.2 below.

4. Taking \( s_X = \text{id}_X \) we immediately obtain the ‘classical’ Weak Lefschetz (for a LSTCI; for an arbitrary \( X \) the corresponding bound on the degrees will decrease by the constant \( c \) mentioned in part 2 of this remark). The only other way to prove this fact (for \( p > 0 \)) that is known to the author is to deduce it from Theorem 9.3 of [Lyu93].

More generally, if \( s_X \) is proper then proper base change (see Proposition 1.3(b)) yields: \( i^*s_X^*\mathbb{Z}/l^n\mathbb{Z}(r)_X \cong s_Z^*i^*s_X^*\mathbb{Z}/l^n\mathbb{Z}(r)_X \cong s_Z^*\mathbb{Z}/l^n\mathbb{Z}(r)_Z \), i.e., we obtain an 'ordinary' Weak Lefschetz-type statement in this case also. The author was not able to find this result in literature.

Some more interesting cases when Theorem 2.2 yields that \( H^*(X') \cong H^*(Z') \) are given by Corollary 2.6 (where we consider \( s_X \) that is proper only over an open neighbourhood of \( Z \) in \( X \)) and by Theorem 3.1 below.
5. In contrast to §II.1.2 of [GoM89], we do not demand \( s_X \) to be quasi-finite. Actually, in §II.1.1 of ibid. there is also a version of a 'fat multiple hyperplane section weak Lefschetz' for a not necessarily quasi-finite \( s_X \). Yet loc. cit. requires \( X' \) to be smooth. This setting has certain advantages: instead of subtracting \( e \) (i.e., the maximum of the dimensions of fibres of \( s_X \)) from \( a - b \) to get the bound on the homotopy degrees, in loc. cit. one only subtracts a certain measure \( e' \) of the failure for \( s_X \) to be semi-small. Being more precise, if \( \phi(k) \) for \( k \geq 0 \) denotes \( \dim(\{ x \in X : \dim s_X^{-1}(x) = k \}) \) (here we set \( \dim(\emptyset) = -\infty \)), then one takes

\[
e' = \max_{k \geq 0}(2k + \phi(k) - a + \min(\phi(k), b - 1)) + 1 - b \quad (5)
\]

(this gives a better bound when \( s_X \) is not equidimensional). Yet one has to pay a price for this refinement of the result: as can be seen from the example in §II.8.4 of ibid., (both the cohomological and the homotopy version of) the statement with the improved bound do not extend to the case of an arbitrary local complete intersection \( X' \).

We note that Theorem 2.2 implies the cohomological version of the statement mentioned (we have written it down in Remark 4.1(2) below). Indeed, for a smooth closed \( Y \subset X' \) of (constant) codimension \( c \) one can apply the Gysin distinguished triangle

\[
v_*\mathbb{Z}/l^n\mathbb{Z}(r)_{Y}(-c)[-2c] \to \mathbb{Z}/l^n\mathbb{Z}(r)(X) \to u_*\mathbb{Z}/l^n\mathbb{Z}(r)_{X\setminus Y} \quad (6)
\]

(see (2)); here \( v : Y \to X' \) and \( u : X' \setminus Y \to X' \) are the corresponding embeddings). Hence instead of verifying the statement for \( X' \) it suffices to verify it for \( X' \setminus Y \) 'with the same \( e' \)' (i.e., up to the cohomological degree \( a - b - e' \)) and to verify it for \( Y \) up to the cohomological degree \( a - b - e' - 2c \) (certainly, there is nothing to check if \( a - b - e' - 2c < 0 \)). Considering a smooth stratification of \( X' \) such that the restriction of \( s_X \) to each stratum is equidimensional, one easily deduces from Theorem 2.2 the result in question (by induction on the number of strata).

Also note that the Verdier dual to (the \( \mathbb{Z}/l^n\mathbb{Z} \)-module version of) Lemma III.7.4(1) of [KiW01] yields: \( s_{U*}\mathbb{Z}/l^n\mathbb{Z}(r)_U \in D^{b}_{cSh}_{\mathbb{Z}/l^n\mathbb{Z}-\text{mod}}(U)^{p \geq a - e'} \) in the case when the maximum in (5) is attained for some value of \( k \) such that \( \phi(k) \geq b - 1 \). Hence one can prove the 'regular' version of Theorem 2.2 in the case when this assumption on the maximum is fulfilled by slightly modifying the proof of loc. cit.

Certainly, this 'regular' version of our main result immediately implies the corresponding versions of Corollary 2.6 and of Theorem 3.1 below.
6. The only restriction on \( U \) that we actually applied in the proof above is the restriction on its ‘perverse’ cohomological amplitude (note: the bound on the cohomological amplitude of \( u_t \) from below is equivalent to the bound on the cohomological amplitude of \( u_s \) from above by Verdier duality). In particular, one can consider a \( U \) such that there exists an affine bundle \( p : U' \to U \) such that \( U' \) is also affine (indeed, then we have \( p_*p^* \cong 1_{D^b \text{Sh}_{Z/l^n Z}-mod(U)} \)). Therefore the proof of Theorem 7.1.1 in [Laz04] yields: one can take for \( U \) the complement of the zero locus of a section of an ample vector bundle (or rank \( b \)) over a projective \( P \) (since we do not have to demand the transition maps of the corresponding bundle to be affine).

Thus one can prove a certain extension of the Sommese’s theorem; this result also implies the corresponding analogue of Corollary 2.6 below. The latter statement seems to be quite new even in the case \( K = \mathbb{C} \).

7. The argument used in the proof of the Theorem can also be carried over to the relative context, i.e., for \( X \) and everything else being schemes over a (more or less ‘reasonable’) base scheme \( S/\text{Spec} \mathbb{Z} \). Certainly, the conclusion will be formulated in terms of the \( t(S) \)-cohomology of the corresponding total derived direct images. Then one can proceed to prove the corresponding (relative) analogues of Corollary 2.6 and of Theorem 3.1(I) (below).

8. The author has also benefited from the proof of Theorem 6.1.1 of [ArS00] (note that the result itself is not quite correct; cf. [ArS07]). Following loc. cit., we could have (directly) applied the Verdier duality instead of the perverse \( t \)-structure in the proof of Theorem 2.2.

Yet then we would (probably) need the smoothness of \( U' \). In fact, the perverse \( t \)-structure seems to be a very convenient tool for our purposes.

Now we prove two easy statements, that often simplify the calculation of \( H^*(Z, i^*s_X, Z/\mathbb{Z}) \).

**Proposition 2.4.** Adopt the setting of Lemma 2.1.

1. Suppose that there exists an open \( O \subset X \) such that \( Z \subset O \) and the restriction of \( s_X \) to the preimage \( O' \) of \( O \) is proper. Then the base change transformation \( B(s_X, i) : i^*s_X \to s_Z, i^* \) is an isomorphism.

2. Let \( i = t \circ v \), where \( t : T \to X \) is a smooth morphism. Then \( i^*s_X, Z/\mathbb{Z} \cong v^*s_T, Z/\mathbb{Z} \), here \( s_T : T' \to T \) is the base change for \( s_X \).
Proof. 1. Consider the commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'_1} & O' \xrightarrow{i'_2} & X' \\
\downarrow{s_Z} & & \downarrow{s_O} & \xrightarrow{s_X} \\
Z & \xrightarrow{i_1} & O \xrightarrow{i_2} & X
\end{array}
\]

(7)

We have \(i^* = i^*_1i^*_2\). Proposition 1.3(II) yields that \(B(s_X, i)\) is the composition of transformations \(i^*s_{X,*} \xrightarrow{B_1} i^*_1s_{O,*}i^*_2 \xrightarrow{B_2} s_{Z,*}i^*_1i^*_2 = s_{Z,*}i'^*\). It remains to note that \(B_1\) is an isomorphism by smooth base change (for the right hand side square of (7)), whereas \(B_2\) is an isomorphism by proper base change (for the left hand side square of (7)); see Proposition 1.3(12).

2. Denote the base change of \(t\) via \(s_X\) by \(t'\), and note that \(i^* = v^*t^*\). Then smooth base change yields: \(i^*s_{X,*}Z/l^nZ(r)_{X'} = v^*s_{T,*}v^*Z/l^nZ(r)_{X'} = v^*s_{T,*}Z/l^nZ(r)_{T'}\).

\[\square\]

Remark 2.5. Part 1 of the Proposition yield our (very simple) substitute of §II5A of [GoM89] (that is sufficient for our applications below, and makes sense for any \(p = \text{char} K\)); note that loc. cit. relies on the Thom’s first isotopy lemma whose proof is really long (see §II.5 of ibid.).

Now we combine the previous result with Theorem 2.2. \(\pi_1(-)\) will denote the étale fundamental group functor. For simplicity in parts 2 and 3 of the Corollary below we will assume that \(X'\) is connected. It follows (by part 1 of the Corollary) that \(H^i(Z', Z/l^nZ) \cong Z/l^nZ\); hence \(Z'\) is connected also, and we have no need to keep track of the corresponding base points.

**Corollary 2.6.** Adopt the setting of Theorem 2.2 and Proposition 2.4(1). Then the following statements are fulfilled.

1. \(H^i(-, Z/l^nZ(r))((i'))\) is bijective for all \(i < a - e - b\), and is injective for \(i = a - e - b\).

2. Let \(a - e - b \geq 1\). Then \(\pi_1(i')\) is surjective.

3. Let \(a - e - b \geq 2\). Then the kernel of \(\pi_1(i')\) has no factors isomorphic to \(Z/l\mathbb{Z}\) (for all prime \(l \neq p\)).

**Proof.** 1. First note that \(H^i(X', Z/l^nZ(r)_{X'}) \cong H^i(\text{Spec } K, x'_*Z/l^nZ(r)_{X'}) = H^i(\text{Spec } K, x_*s_{X,*}Z/l^nZ(r)_{X'})\) and \(H^i(Z', Z/l^nZ(r)_{Z'}) \cong H^i(\text{Spec } K, z'_*Z/l^nZ(r)_{Z'}) = H^i(\text{Spec } K, x_*s_{X,*}v^*Z/l^nZ(r)_{X'})\) (by Proposition 1.4(5)). By Proposition 1.3(11) we obtain that \(H^i(-, Z/l^nZ(r))((i'))\) is given by applying \(H^i(\text{Spec } K, x_*(-))\) to the composition of \(i_*B(s_{X,*})(Z/l^nZ(r)_{X'})\) with \(\mathcal{F}_r\).

Hence the statement can be applied immediately by combining Theorem 2.2 with the first part of Proposition 2.4.
2. Consider a finite connected etale cover $X'' \to X'$, and the corresponding cartesian square

\[\begin{array}{ccc}
Z'' & \xrightarrow{i''} & X'' \\
\downarrow & & \downarrow \\
Z' & \xrightarrow{i'} & X'
\end{array}\] (8)

Now, we can apply assertion 1 for $i''$ instead of $i'$; hence $H^0(-, \mathbb{Z}/l^n\mathbb{Z})(i'')$ is bijective, i.e., $Z''$ is connected also.

Next we recall that finite connected etale covers of any variety $Y$ are given by open subgroups of $\pi_1(Y)$, and that the degree of such a cover equals the index of the corresponding subgroup (in $\pi_1(Y)$). Hence for the subgroup $H$ of $\pi_1(X')$ corresponding to $X''$, we obtain a bijection of cosets $\pi_1(Z')/(\pi_1(i')^{-1}(H)) \to \pi_1(X')/H$.

Considering $H$ running through a projective system of open subgroups of $\pi_1(X')$ such that their limit is $\{0\}$, we obtain the result.

3. We use the notation and the statements mentioned in the proof of the previous assertion. Moreover, applying assertion 1 to $X''$ instead of $X'$ we obtain that $H^1(-, \mathbb{Z}/l^n\mathbb{Z})(i'')$ is bijective also.

Now recall that $H^1(Y, \mathbb{Z}/l^n\mathbb{Z}) \cong \text{Hom}(\pi_1(Y), \mathbb{Z}/l^n\mathbb{Z})$. Hence taking a 'small enough' $H$ (i.e., letting it to run through a system of open subgroups of $\pi_1(X')$ such that their limit is $\{0\}$) again we obtain the result.

\[\square\]

Remark 2.7. We conjecture that in the setting of Corollary 2.6(3) the homomorphism $\pi_1(i')$ is actually bijective. Possibly, one can prove this conjecture in general by combining our methods with those of [Cut97] (somehow).

3 Applications: certain Barth-type theorems

It turns out that Corollary 2.6(1) allows to study the cohomology of the intersection of a (closed LSTCI) subvariety of $(\mathbb{P}^N)^q$ with the diagonal (as well as the cohomology of the preimages of the diagonal with respect to proper morphisms whose target is $(\mathbb{P}^N)^q$). As a consequence, we can calculate the lower cohomology of (the preimages of) subvarieties of $\mathbb{P}^N$ (of small codimension); so we extend the seminal results of Barth and others to the case of arbitrary characteristic.

Our exposition closely follows the one of §9 of [FuL81]. The main distinctions are due to the fact that we consider the cohomology groups of varieties instead of the homotopy ones. In particular, this makes the proof of our Theorem 3.1(I1,2) quite different and somewhat more complicated than the
proofs of Corollaries 9.7 and 9.8 of \[FuL81\]. On the other hand, in our Theorem 3.1(II2) we do not require \(v\) to be finite (in contrast with Corollary 9.8 of \[FuL81\]).

We will need some notation. We fix some \(l, n, r, \) and also some \(N > 0\).

We recall a construction of certain locally trivial \(G^s\)-bundles (for some \(s \geq 0\)); it is described in more detail in \(\S 3\) of ibid (for the case \(q = 2\)).

For any \(q > 0\) we note that the natural projection \(a_q : (\mathbb{A}^{N+1} \setminus \{0\})^q \to (\mathbb{P}^N)^q\) factorizes through the quotient \(V_q\) of \((\mathbb{A}^{N+1} \setminus \{0\})^q\) by the diagonal action of \(G_m\) on \(\mathbb{A}^{Nq+q+1}\). So, we obtain a (locally trivial) \(G_{m}^{-1}\)-bundle \(p_q : V_q \to (\mathbb{P}^N)^q\), and a \(G_m\)-bundle \(b_q : (\mathbb{A}^{N+1} \setminus \{0\})^q \to V_q\), whereas \(V_q\) is an open subvariety of \(\mathbb{P}^{Nq+q-1}\).

For a morphism of varieties \(g : Y \to (\mathbb{P}^N)^q\) we will denote by \(g_q : Y_q \to V_q\) (resp. \(g'_q : Y'_q \to (\mathbb{A}^{N+1} \setminus \{0\})^q\)) the base change of \(g\) with respect to \(p_q\) (resp. with respect to \(a_q\)).

**Theorem 3.1.** Let \(q > 1\).

1. Let \(g : Y \to (\mathbb{P}^N)^q\) be a proper morphism of varieties of relative dimension \(\leq e\), where \(Y\) is a LSTCI (see the Notation section) of dimension \(a\). Denote the diagonal of \((\mathbb{P}^N)^q\) by \(\Delta(\cong \mathbb{P}^N)\). Then there is a natural morphism \(c : g^{-1}(\Delta) \to Y_q\) such that \(H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(c)\) is bijective for \(i < a - e - qN + N\) and is injective for \(i = a - e - qN + N\).

2. In the setting of the previous assertion, assume that \(Y = (\prod Y_j)_{\text{red}}\) (see the Notation section) for some proper morphisms \(g_j : Y_j \to \mathbb{P}^N\) (1 \(\leq j \leq q\), and \(g\) is the restriction of \(\prod g_j\) to \(Y\) (if \(K\) is perfect, then we always have \(g = \prod g_j\) and \(Y = \prod Y_j\). Then \(Y_q' \cong \prod Y_{j, \text{red}}\) and for the base change \(c' : (g^{-1}(\Delta))' \to \prod Y_{j, \text{red}}\) of the morphism \(c\) (that is given by the previous assertion) with respect to \(b_q\) we have: \(H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(c')\) is bijective for \(i < a - e - qN + N\) and is injective for \(i = a - e - qN + N\).

If \(t : T \to \mathbb{P}^N\) be a closed embedding, where \(T\) is a LSTCI of dimension \(d\). Then the following statements are valid.

1. \(H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(t)\) is bijective for \(i \leq 2d - N\) and is injective for \(i = 2d - N + 1\); the same is true for the corresponding \(G_m\)-bundle morphism \(t'_1 : T'_1 \to \mathbb{A}^{N+1} \setminus \{0\}\).

2. Let \(v : V \to \mathbb{P}^N\) be a proper morphism of relative dimension \(\leq b\), where \(V\) is a LSTCI of dimension \(u\). Then for the morphism \(w : v^{-1}(T) \to V\) we have: \(H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(w)\) is bijective for \(i \leq \min(d + u - b - N - 1, 2d - N)\) and is injective for \(i = \min(d + u - b - N, 2d - N + 1)\). Besides, the same is true for \(w'_1\) being the base change of \(w\) with respect to \(v'_1\).

**Proof.** 1. The diagonal embedding \(\mathbb{A}^{N+1} \setminus \{0\} \to (\mathbb{A}^{N+1} \setminus \{0\})^q\) yields (after the factorization by the diagonal action of \(G_m\)) a subvariety \(L_q \subset V_q\) such
that the restriction $p_L$ of $p_q$ onto $L_q$ gives an isomorphism $L_q \to \Delta$. Note also that $L_q$ is a closed subvariety in $\mathbb{P}^{N_q+q-1} \supset V_q$.

We denote the embedding $L_q \to \mathbb{P}^{N_q+q-1}$ (resp. $L_q \to V_q$) by $i$ (resp. by $i_1$). Take for $c$ the base change of $i_1$ with respect to $g$, and denote the composite morphism $Y_q \to V_q \to \mathbb{P}^{N_q+q-1}$ by $s$.

Now we apply Corollary 2.6(1): we take $s_X = s$, $Z = L_q$, $O = V_q$. Since $g_q^{-1}(L_q) \cong g^{-1}(\Delta)$, we obtain the result.

2. The first part of the assertion is obvious. Next we note that $\Delta \cong \mathbb{P}^N$, and the base change of $c$ with respect to $b_q$ does yield $c' : (\prod_{\mathbb{Z}} Y_j)_{t, \text{red}} \to (\prod Y'_j)_{\text{red}}$.

The cohomological part of our statement follows immediately from the previous assertion (applied for all $r \in \mathbb{Z}$) together with Lemma 3.3 (below).

If By Proposition 1.1(1), we can assume that $K$ is algebraically closed. In this case we can (and will) set $r = 0$ (see Proposition 1.1(8)). We denote the functor $H^*(-, \mathbb{Z}/l^n\mathbb{Z})$ by $H^*$.

1. Lemma 3.3 yields that it suffices to prove the second part of the assertion. Since the étale cohomology of $\mathbb{A}^{N+1} \setminus \{0\} (K)$ vanishes in all degrees between 1 and $2N$, to this end it suffices to verify that $H^i(t'_1)$ is an isomorphism for all $i \leq 2d - N$.

Now in the notation of assertion I2 we take $q = 2$ and $g_1 = g_2 = t$. For the corresponding $Y = T \times T$ (that also equals $(T \times T)_{\text{red}}$) we have $a = 2d$, $e = 0$.

Hence assertion I2 yields that for the corresponding $c' : T'_1 \to T'_1 \times T'_1$ we have: $H^i(c')$ is bijective for $i < 2d - N$, and is injective for $i = 2d - N - 1$ and a surjection in degree $2d - N$.

Now we note that the projection $pr : T'_1 \times T'_1 \to T'_1$ (via the first factor) splits $c'$; hence $H^*(pr)$ yields an isomorphism up to degree $2d - N - 1$ and a surjection in degree $2d - N$.

Now we rewrite the result obtained in terms of $D^b(Sh^{et}(\text{Spec } K, \mathbb{Z}/l^n\mathbb{Z} - \text{mod})) \cong D^b(\mathbb{Z}/l^n\mathbb{Z} - \text{mod})$ and of the total derived functors $RH(-) \cong p_*$ (where $p : T'_1 \to pt$ is the structure morphism; see Proposition 1.1(4)). Choose some section $s : pt \to T'_1$. Since $p \circ s = id_{pt}$, we obtain a splitting $RH(T'_1) \cong \mathbb{Z}/l^n\mathbb{Z} (= RH(pt)) \bigoplus D$, where $D$ is the 'image' of the idempotent morphism $(id_{RH(T'_1)} - RH(s \circ p)) \in D(\mathbb{Z}/l^n\mathbb{Z} - \text{mod})(RH(T'_1), RH(T'_1))$. Now, the Kunneth formula (see Proposition 1.1(8)) yields that $RH(pr)$ can be described as the tensor product of the split morphism $\mathbb{Z}/l^n\mathbb{Z} \to \mathbb{Z}/l^n\mathbb{Z} \bigoplus D$ by $RH(T'_1) \cong \mathbb{Z}/l^n\mathbb{Z} \bigoplus D$. Hence $H^i(pr)$ is injective for any $i \geq 0$ (including $i = 2d - N$). Moreover, we obtain that the cohomology of $D$ is concentrated in degrees $> 2d - N$, since $D$ is a direct summand of $\text{Cone}(RH(pr)) \cong D \bigoplus D \otimes D$; this concludes the proof.

2. Again, it suffices to prove the second part of the assertion. We set $q = 2$ (again) and take $g_1 = t$, $g_2 = v$. For the corresponding $Y = T \times V$ we
have \( a = d + u, \ e = b. \)

Hence assertion I2 yields that for the corresponding \( c' : (v^{-1}(T))'_1 \rightarrow V'_1 \times T'_1 \) we have: \( H^i(c') \) is bijective for \( i < d + u - b - N \) and is injective for \( i = d + u - b - N \).

Now, assertion II1 (combined with Proposition 1.1(8)) yields that for the projection \( pr : V'_1 \times T'_1 \rightarrow V'_1 \) we have: \( H^i(pr) \) is a bijection for \( i \leq 2d - N \) and is an injection for \( i = 2d - N + 1. \) Since \( H^i(w'_1) = H^i(c') \circ H^i(pr), \) we obtain the result.

\( \square \)

**Remark 3.2.** 1. The corresponding Leray spectral sequence yields a relation of the cohomology of \( Y_q \) with the one of \( Y \); for \( q = 2 \) one obtains a a Gysin long exact sequence (see (9)).

2. Applying our cohomological results to \( H^*(-, \mathbb{Z}/l^n\mathbb{Z}(1)) \), one immediately obtains certain statements on the Picard and Brauer groups of varieties mentioned (if the corresponding bound on the cohomology degrees where we have an isomorphism is not too small). One can also apply our results to get some information on the number of points of varieties over finite fields.

The author does not know (at the moment) whether the corresponding results are interesting (and whether these results for Brauer and Picard groups are new; cf. §11 of [Lyu93] for some results on the Picard groups).

3. Note that in contrast to §9 of [FuL81] we do not demand \( f \) to be finite; cf. Remark 2.3(2) above.

4. Certainly, our results can be extended to the case when the variety \( Y \) considered above (this includes \( Y = T \times T \) and \( Y = T \times V \) in the proofs of assertions III and II2 of the theorem) is not (‘quite’) a LSTCI; the corresponding degree bounds should be decreased by a certain measure \( c \) of their failure to be so (cf. Remark 2.3(2)). Also, it is not really necessary to assume that \( Y \) is equi-dimensional. A certain extension of our Theorem 3.1(II1) of this sort along with several other results (on the cohomology of subvarieties of \( \mathbb{P}^N \) of low codimension) can be found in Theorem 10.5 of [Lyu93].

5. In particular, one can easily prove the following statement: if \( q > 1 \) and \( T_j \subset \mathbb{P}^N \) are closed LSTCI of dimensions \( d_j \) for \( 1 \leq j \leq q, \) then \( H^i(\cap_{1 \leq j \leq q} T_j, \mathbb{Z}/l^n\mathbb{Z}(r)) \cong H^i(\mathbb{P}^N, \mathbb{Z}/l^n\mathbb{Z}(r)) \) for \( i \leq D = \sum_{j=1}^q d_j - (q-1)N. \) Indeed, we can assume that \( d_1 \) is the largest of \( d_j; \) take \( T = T_1 \) and \( V = \cap_{2 \leq j \leq q} T_j. \) Then \( H^i(T, \mathbb{Z}/l^n\mathbb{Z}(r)) \cong H^i(\mathbb{P}^N, \mathbb{Z}/l^n\mathbb{Z}(r)) \) for \( j \leq 2d_1 - N \) by part III1 of our theorem. In the case when \( T_j, \ 2 \leq j \leq q, \) intersect properly (i.e. the dimension of their intersection is \( \sum_{j=2}^q d_j - (q-2)N = D + N - d_1 \)) it remains to apply part II2 of our theorem; otherwise one can apply the ’defect’ version of this result mentioned above.

Thus we are able to extend the statement mentioned after Remark 9.9 of
to the case of an arbitrary $K$; it seems that this assertion does not follow from the results of [Lyu93].

6. Possibly there exist some other (locally trivial) $G$-bundle constructions (where $G$ is some algebraic group) similar to the $G_m$-bundles considered above, such that our methods can be used for them in order to obtain certain Barth-type results.

In order to conclude the proof of Theorem 3.1, it remains to prove the following statement.

**Lemma 3.3.** Let $f': A' \to B'$ be a morphism of locally trivial $G_m$-bundles over the base $f : A \to B$; fix a $j \in \mathbb{Z}$. Then we have: $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is bijective for all $i < j$ and is injective for $i = j$ and all $r \in \mathbb{Z}$ whenever the same is true for $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f')$.

**Proof.** We prove the statement by induction in $j$.

If $j < 0$, then the statement is vacuous. Now assume that it is fulfilled for $j = s - 1$, and that $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r'))(f)$ and $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(r'))(f')$ are bijective for all $i < s - 1$ and are injective for $i = s - 1$ (for some $s \geq 0$, and for all $r' \in \mathbb{Z}$). We should verify (for some fixed $r \in \mathbb{Z}$) that $H^{s-1}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is surjective and $H^{s}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is injective whenever the same is true for $f'$.

Now assume that for any $G_m$-bundle $b : X' \to X$ there exists a certain (Gysin) long exact sequence

$$H^{s-1}(X', \mathbb{Z}/l^n\mathbb{Z}(r)) \to H^{s-2}(X, \mathbb{Z}/l^n\mathbb{Z}(r - 1)) \to H^{s}(X, \mathbb{Z}/l^n\mathbb{Z}(r)) \to \cdots$$

and that this sequence is functorial with respect to $b$.

Then the five lemma yields: if $H^{s-1}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is bijective and $H^{s}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is injective, then $H^{s-1}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f')$ is bijective; if $H^{s-1}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f')$ is bijective, then $H^{s-1}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is bijective also. Next, the four lemma on monomorphisms yields: if $H^{s}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is injective and $H^{s}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f')$ is injective also; if $H^{s-1}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f')$ is injective, then $H^{s}(-, \mathbb{Z}/l^n\mathbb{Z}(r))(f)$ is injective.

Hence we obtain the inductive assumption for $j = s$.

Now it remains to verify the existence and the functoriality of (9). We note that $X'$ can be presented as the complement to $X$ for the line bundle $X'' = X \times \mathbb{A}^1/G_m$ over $X$ (we consider the diagonal action of $G_m$ on the product and the zero section embedding $X \to X''$). Hence it remains to apply Corollary 1.5 of [SGA5.VII].
Remark 3.4. Now we say more on the relation of our results with those of [FuL81] (where for $K = \mathbb{C}$ the homotopy analogues of all of the results of this section were proved), and also explain why our results are formulated in the way they are, and how could they be modified.

1. §9 of [FuL81] relies on the results of [GoM89]. So loc. cit. has three features that distinguish it from our Theorem 3.1:
   (i) it treats only complex varieties;
   (ii) all the corresponding morphisms to $\mathbb{P}^N$ and $(\mathbb{P}^N)^q$ are required to be quasi-finite;
   (iii) the results are formulated in terms of homotopy instead of cohomology.

   Certainly, (i) is a serious disadvantage of [FuL81]; the restriction (ii) probably could be removed via generalizing Theorem II1.2 of [GoM89].

2. Now we discuss the difference between treating cohomology and homotopy.

   Certainly, cohomology is much easier to compute. On the other hand, homotopy groups have certain theoretical advantages. One of them is that the relation between the (higher) homotopy groups of a (locally trivial) $G_m(\mathbb{C})$-bundle $X' \to X$ with those of $X$ are much easier to use than (9). Using this, it was proved (in Theorem 9.2 of loc. cit., whose setting corresponds to that of our Theorem 3.1(II) for $q = 2$) that the homotopy groups of $Y$ are isomorphic to those of $g^{-1}(\Delta)$ in degrees between 2 and $a - N$ (note that $e = 0$ in loc. cit.). Moreover, in the setting of our assertion I2 (with $q = 2$) this result reformulates as an isomorphism (in the corresponding degrees) of relative homotopy groups for the pairs $(Y_1, Y_1 \times_{\mathbb{P}^N} Y_2)$ and $(\mathbb{P}^N, Y_2)$. The homotopical analogues of our assertions II(2,1) follow easily.

   So, it would be interesting to obtain certain homotopy analogues of the results above. Now we discuss possible ways to do this. First we note that even for complex varieties it is impossible to recover the ‘topological’ homotopy groups staying inside the category of algebraic varieties. Therefore, one has to consider the homotopy groups of the corresponding étale homotopy types.

   To the knowledge of the author, the existing homotopy-theoretic technique does not allow to modify the proof of Theorem 2.2 so that it would yield information on (étale) homotopy groups directly. For this reason, in order to obtain the homotopy analogues desired one would probably have to apply the results above, and then extract the information on homotopy from the one on cohomology using étale analogues of the Hurewicz theorem.

   An obvious obstruction to do so is the (possible) non-triviality of $\pi_1$ of the varieties in question. The author hopes that Corollary 2.6(2,3) or the methods of its proof can help here (cf. also Remark 2.7).
3. Now we describe a possible ‘cohomological’ way to get rid from the $G_m$-bundles in (the formulation of) Theorem 3.1(I). As can be shown by simple examples, there is no ‘easy’ way to do this. Yet the author suspects that (similarly to the case of a pullback square of topological spaces, as studied by Eilenberg and Moore) in the setting of Theorem 3.1(I2) the differential graded algebra that computes the cohomology of $\prod_{P,N} Y_i$ is quasi-isomorphic up to degree $a - e - qN + N$ to the tensor product of the corresponding algebras for $Y_i$ over the one for $\mathbb{P}^N$ (maybe, only when $K$ is separably closed). Note that for the corresponding $G_m$-bundles (i.e., for $(\prod_{P,N} Y_i)'_1$ and $\prod Y_i''_1$) this fact is given by loc. cit.; then one probably has to ’descend’ using an (induction) argument similar to the one in the proof of Lemma 3.3.

4 Some other remarks: henselizations as ’small neighbourhoods’

Remark 4.1. 1. As we have already noted, Theorem 2.2 generalizes the ’classical’ Weak Lefschetz. Moreover, the author knows no way to deduce loc. cit. from the ’ordinary’ Weak Lefschetz (or even to reduce it to the case when $s_X$ is proper).

Yet we note that the ’regular version’ of loc. cit. mentioned in Remark 2.3(5) could be reduced to its very partial case.

2. First we formulate the ’regular version’ here.

In the setting of Theorem 2.2 we assume that $X'$ is regular. For any $k \geq 0$ denote by $\phi(k)$ the dimension of $\{x \in X : \dim s_X^{-1}(x) = k\}$ (here we set $\dim(\emptyset) = -\infty$). Then for $\beta = 2a - 1 - \max_{k \geq 0} (2k + \phi(k) - a + \min(\phi(k), b - 1))$ the homomorphism $H^i(X, -)(\mathcal{F}_r)$ is a bijection if $i < \beta$, and is an injection if $i = \beta$.

3. Now we verify briefly that the general case of the statement above follows from its ’étale-local’ case when we take $X'$ being a geometric point of $X$ (considered as an ind-étale scheme over the corresponding closed subvariety of $X$).

Indeed, using the Gysin distinguished triangle (6) and passing to the limit we obtain: the general case of the statement follows from the case when $X'$ is a (regular) variety over a Zariski point $P \in X$. Moreover, applying étale descent one obtains: it suffices to prove the statement for $X'$ being a variety over a geometric point $P_{geom}/P$.

As in the proof of Theorem 2.2 we assume that $n = 1$. Denoting the corresponding morphism $X' \to P_{geom}$ by $s_{geom}$, and denoting the morphism $P_{geom} \to X$ by $j_{geom}$, we obtain: we should compare $A = H^r(X, j_{geom}^*s_{geom}^*\mathbb{Z} / l^n\mathbb{Z}(r))_{X'}$. 

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with $B = H^*(Z, i^* s_{geom} \mathbb{Z}/l^n \mathbb{Z}(r)_X^r)$ (cf. the proof of Theorem 2.2).

Now, the (small) étale site of $P_{geom}$ is trivial, and $s_{geom} \mathbb{Z}/l^n \mathbb{Z}(r)_X^r \in D^b \text{Sh}^\text{et}_{Z/l^n \mathbb{Z} - \text{mod}}(P_{geom})^{p \geq 0}$; hence $s_{geom} \mathbb{Z}/l^n \mathbb{Z}(r)_X^r$ splits as a direct sum of $\mathbb{Z}/l^n \mathbb{Z}_{P_{geom}}[r_i]$ for some $r_i \leq 0$. Hence instead of $s_{geom} \mathbb{Z}/l^n \mathbb{Z}(r)_X^r$, both in $A$ and $B$ one can put $\mathbb{Z}/l^n \mathbb{Z}_{P_{geom}}$ (that is isomorphic to $\mathbb{Z}/l^n \mathbb{Z}(r)_{P_{geom}}$ for any $r \in \mathbb{Z}$); this finishes the reduction in question.

This observation fits nicely with the ’henselization’ methods for calculating $i^*$, that we will consider in the next remark.

**Remark 4.2.** Now we describe some methods for studying the cohomology of $i^! s_X \mathbb{Z}/l^n \mathbb{Z}(r)_X^r$ using certain henselizations. They hardly have any computational value; their advantage is that they are somewhat ’motivic’ and can be applied when no properness assumptions on $s_X$ (cf. Corollary 2.6) are fulfilled.

By the main result of [Gab94] and [Hub91], if $h : A \to B'$ is a henselian pair (of affine schemes) and $C'$ is a complex of étale sheaves over $B'$, then $H^*(B', C') \cong H^*(A, h^*(C'))$. Therefore if we decompose a closed embedding $i : A \to B$ of affine schemes as $A \to B' \to B$, where $B$ is the henselization of $A$ in $B$ and $e$ is the corresponding pro-étale morphism, then for a complex $C$ of étale sheaves over $B$ we have $H^*(A, i^* C) \cong H^*(B', e^* C)$. Now, since $e$ is a pro-étale morphism, $e^*$ is just the corresponding ’restriction’ functor, and it ’commutes with base change’.

Thus, henselizations in $X$ could be thought about as of algebraic analogues of ’very small’ neighbourhoods of (closed) submanifolds $Z$ of a manifold $X$ (the latter can be used in order to compute the ’topological’ functor $i^*$; see Remark 2.3). Unfortunately, ’nice’ henselizations exist only for affine schemes, whereas $X$ and $Z$ are proper. We describe some possible methods for overcoming this difficulty.

1. The first way is to choose an étale affine hypercovering $X. \to X$ (certainly, $X$ could also be a Zariski or a Nisnevich hypercovering; it could be a Čech hypercovering). Then for the corresponding morphisms $c_j : X_j \to X$ and their restrictions $c'_j : c_j^{-1}(Z) \to Z$ (for $j \geq 0$) we would have a spectral sequence $H^i(c_j^{-1}(Z), c'_j i^! s_{X_j} \mathbb{Z}/l^n \mathbb{Z}(r)_{X_j}^r) \implies H^{i+j}(Z, i^* s_X \mathbb{Z}/l^n \mathbb{Z}(r)_X^r)$, whereas smooth base change yields that the $E_1$-terms of this spectral sequence are the corresponding cohomology of the henselizations of $c_j^{-1}(Z)$ in $X_j$ (for non-connected schemes the henselizations are defined component-wise).

2. Another way to reduce the computation in question to a one for affine schemes is to apply Jouanolou’s device.

In the setting of Theorem 2.2 let $b_X : \hat{X} \to X$ be an affine vector bundle torsor such that $\hat{X}$ is affine; we denote the pullback of $b_X$ to $X'$ and $Z$ by
\( b_{X'} : \hat{X}' \to X' \) and \( b_Z : \hat{Z} \to Z \), respectively; we will also use similar notation for morphisms. Since the cohomology of the fibres of \( b_Z \) over geometric points is trivial, we have \( b_Z^* b_Z^* \cong 1_{D^b \text{Sh}_{Z/l^nZ\text{-mod}}(Z)} \). For \( C \in \text{Obj} D^b \text{Sh}_{Z/l^nZ\text{-mod}}(X) \) we obtain: \( H^*(Z, i^* C) \cong H^*(\hat{Z}, b_Z^* i^* C) \cong H^*(\hat{Z}, \hat{i}^* b_X^* C) \) (we used smooth base change in the last isomorphism). Now, \( \hat{i} \) is a closed embedding of affine varieties, and one obtains that the cohomology groups in question are isomorphic to \( H^*(\hat{Z}_h, \hat{z}_h^* b_X^* C) \), where \( \hat{Z}_h \) is the henselization of \( \hat{Z} \) in \( \hat{X} \), \( \hat{z}_h : \hat{Z}_h \to \hat{X} \) (the corresponding pro-étale morphisms).

Then we obtain:

\[
H^*(Z, i^* s_X^* \mathbb{Z}/l^n \mathbb{Z}(r)_{X'}) \cong H^*(\hat{Z}_h, \hat{z}_h^* b_X^* s_X^* \mathbb{Z}/l^n \mathbb{Z}(r)_{X'}) \\
\cong H^*(\hat{Z}_h, s_{\hat{Z}_h} \hat{z}_h^* b_X^* \mathbb{Z}/l^n \mathbb{Z}(r)_{X'}) \cong H^*(\hat{Z}_h, \hat{T}_h^* \mathbb{Z}/l^n \mathbb{Z}(r))
\]

where \( \hat{Z}_h \xrightarrow{\hat{z}_h} \hat{X} \xrightarrow{b_h} X \) is obtained by base change via \( s_X \) from \( \hat{Z}_h \xrightarrow{\hat{z}_h} \hat{X} \xrightarrow{b_h} X \) (we apply smooth base change again).

3. One more application of the smooth base change (in the setting of Theorem \([2,2]\)) can be obtained by applying Proposition \([2,3](2)\). In particular, if \( Z \) is a (multiple) hyperplane section of \( X \) (corresponding to some embedding of \( X \) into \( \mathbb{P}^N \) for some \( N > 0 \)) then it can be presented as a member of a 'smooth family' of (multiple) hyperplane sections of \( X \). This means: for some variety \( B \) and some closed \( T \subset B \times X \) the projection \( t : T \to X \) is a smooth morphism and it restricts to an isomorphism \( b^{-1}(p) \to Z \), where \( p \) is some closed point \( B \), \( b \) is the projection \( T \to B \) (whereas all closed fibres of \( b \) yield multiple hyperplane sections of \( X \) via \( t \)). Now, one can assume that \( B \) is affine; denote by \( p_h \) the henselization of \( p \) in \( B \), and denote by \( Z_h \) the base change of \( p_h \) with respect to \( b \). Then (a very simple case of) Corollary 1 of [Gab94] yields (in the notation of Proposition \([2,4](2)\)): \( H^*(Z, v^* s_X^* \mathbb{Z}/l^n \mathbb{Z}(r)_{X'}) \cong H^*(Z, v^* s_T^* \mathbb{Z}/l^n \mathbb{Z}(r)_{X'}) \cong H^*(Z_h, v_h^* s_T^* \mathbb{Z}/l^n \mathbb{Z}(r)) \), where \( v^* \) is the isomorphism \( Z \to b^{-1}(p) \), \( v_h \) is the corresponding pro-étale morphism \( Z_h \to T \), and \( T_h \) is the base change of \( Z_h/T \) with respect to \( s_T \).

For \( K = \mathbb{C} \) it can be easily verified that one can take an 'infinitely small ball' around \( p \) instead of \( p_h \) (cf. Remark \([2,3]\)). So, our observation above could be thought of as being one more version of an (algebraic) 'fat multiple hyperplane section' approach to Weak Lefschetz-type questions. It is also closely related with the classical yoga of general hyperplane sections (of not necessarily projective varieties; cf. Lemma 3.3 of [Bei87]), especially in the case when \( X \) is smooth and \( s_X \) is an open embedding.

For all of the methods listed in this remark it seems interesting to consider \( X' \) that runs through geometric points of \( X \) (cf. Remark \([4,1]\)).
4. The current paper grew out from an attempt to prove a certain weak Lefschetz-type result for \( \mathbb{Z}/l^n\mathbb{Z} \)-motivic cohomology (at least, of complex varieties). The idea was to join Theorem II1.2 of [GoM89] together with a formula that relates the motivic cohomology with the étale one (see Corollary 7.5.2(2) of [Bon10a]; this is a more or less simple consequence of the Beilinson-Lichtenbaum conjecture) in order to study torsion motivic cohomology. Unfortunately, it turned out that in order to realize this program (which certainly becomes more realistic if one replaces Theorem II1.2 of [GoM89] with our Theorem 2.2) one needs a certain ’henselian model’ for \( \mathbb{Z} \) in \( X \) (a sort of ’motivic tubular neighbourhood’; cf. [Lev07]) that is ind-étale over \( X \). Parts 2 and 3 of this remark do not yield such a model, whereas in part 1 we obtain a ’complex’ of ind-étale \( X \)-schemes. One could also try to consider some \( \mathbb{G}_m \)-bundles over the varieties in question (similarly to §3; cf. also §10 of [Lyu93]).

Yet the author has some ideas for overcoming this difficulty (and to prove at least that the lower motivic cohomology of \( X \) is isomorphic to the one of the scheme \( \tilde{Z}_h \) that was considered in part 2 of this remark). It also seems to make sense to pass to the limit with respect to Nisnevich hypercoverings \( X. \rightarrow X \) (in part 1 of this remark), and use the fact that \( i_{\text{Nis}}^*(\mathbb{Z}/l^n\mathbb{Z}(r)_{\text{Nis},X}) \cong \mathbb{Z}/l^n\mathbb{Z}(r)_{\text{Nis},Z} \) (here \( \mathbb{Z}/l^n\mathbb{Z}(r)_{\text{Nis},-} \) is a complex of Nisnevich sheaves that computes the corresponding motivic cohomology; this isomorphism is given by a certain ’rigidity’ argument).

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