Markov process representation of semigroups whose generators include negative rates

F. Völlering*

February 6, 2020

Generators of Markov processes on a countable state space can be represented as finite or infinite matrices. One key property is that the off-diagonal entries corresponding to jump rates of the Markov process are non-negative. Here we present stochastic characterizations of the semigroup generated by a generator with negative rates. This is done by considering a larger state space with one or more particles and antiparticles, with antiparticles being particles carrying a negative sign.

1 Introduction

Consider the generator $L$ of a Markov jump process $(X_t)_{t \geq 0}$ on a countable state space $E$. It is characterized by jump rates $r(x, y)$ for jumps from $x$ to $y$, and for $f : E \to \mathbb{R}$

$$Lf(x) = \sum_{y \in E} r(x, y) [f(y) - f(x)].$$

(1)

The relationship between the probabilistic process $(X_t)_{t \geq 0}$, its semi-group $(P_t)_{t \geq 0}$ with $P_t f(x) = \mathbb{E}_x f(X_t)$ and generator describing the rules for jumps is very fruitful. One essential restriction is that the jump rates are non-negative. If $r(x, y) < 0$ is allowed, then (1) is still a perfectly valid operator which under reasonable conditions will be the generator of a semi-group $S_t = e^{tL}$, but the probabilistic interpretation is lost. The aim of this note is to recover some probabilistic meaning.

*florian.voellering@math.uni-leipzig.de, University of Leipzig
2 Switching between particles and antiparticles

Let us write $r^+(x,y) = \max(r(x,y),0)$ and $r^-(x,y) = \max(-r(x,y),0)$, and consider the Markov process $(\tilde{X}_t, Z_t)_{t \geq 0}$ on $E \times \{-1, +1\}$ with generator

$$\hat{L} f(x,s) = \sum_{y \in E} r^+(x,y) [f(y,s) - f(x,s)] + \sum_{y \in E} r^-(x,y) [f(y,-s) - f(x,s)].$$

We interpret $\tilde{X}_t$ as the position of the Markov process, and $Z_t$ indicates whether it is a particle ($Z_t = +1$) or an anti-particle ($Z_t = -1$). Then the interpretation of negative rates is that these jumps involve the flip from particle to anti-particle or vice versa.

Theorem 2.1. Assume $\sup_{x \in E} \sum_{y \in E} |r(x,y)| < \infty$. Then $L$ is a bounded operator w.r.t. the supremum-norm, $S_t = e^{tL}$ is well-defined and for any $f : E \to \mathbb{R}$ bounded, we have

$$S_t f(x) = \mathbb{E}_{x,+1} \left[ Z_t f(\tilde{X}_t) e^{2 \int_0^t \sum_{y \in E} r^-(\tilde{X}_u,y) du} \right].$$

Proof. Write $\hat{f}(x,s) = sf(x)$ and $V(x,s) = 2\sum_{y \in E} r^-(x,y)$. Then the right hand side of (2) is the Feynman-Kac formulation of the solution of

$$\frac{\partial \phi_t}{\partial t}(x,s) = \hat{L} \phi_t(x,s) + V(x,s) \phi_t(x,s),
\phi_0 = \hat{f}.$$

On the other hand, $\tilde{\phi}_t(x,s) := sS_t f(x)$ also satisfies

$$\frac{\partial \tilde{\phi}_t}{\partial t}(x,s) = sLS_t f(x) = \hat{L} \tilde{\phi}_t(x,s) + V(x,s) \tilde{\phi}_t(x,s),$$

and since $\tilde{\phi}_0 = \phi_0$ the claim (2) follows.

3 Branching particles and antiparticles

Consider a system of particles $\eta_+^t$ and antiparticles $\eta_-^t$, where $\eta_+^t(x)$ is the number of particles/anti-particles at site $x$ and time $t$. These particles move independently with jump rates $r^+(x,y)$. Additionally there is the following branching mechanism: a particle at site $x$ branches into two particles at $x$ and an anti-particle at site $y$ at rate $r^-(x,y)$. The same is true for antiparticles at $x$, which branch into two at $x$ plus a particle at $y$. The generator describing the movement and branching of particles is

$$L^+ \eta^+, \eta^- = \sum_{x,y} r^+(x,y) [f(\eta^+, \delta_y - \delta_x, \eta^-) - f(\eta^+, \eta^-)]$$

$$+ \sum_{x,y} r^-(x,y) [f(\eta^+, \delta_x, \eta^- + \delta_y) - f(\eta^+, \eta^-)].$$
and the generator describing the movement and branching of antiparticles is
\[
L_{\uparrow} f(\eta^+, \eta^-) = \sum_{x,y} r^+(x,y)\eta^-(x)[f(\eta^+, \eta^- + \delta_y - \delta_x) - f(\eta^+, \eta^-)] \\
+ \sum_{x,y} r^-(x,y)\eta^-(x)[f(\eta^+ + \delta_y, \eta^- + \delta_x) - f(\eta^+, \eta^-)].
\]

The generator \( L^+ = L_{\uparrow}^+ + L_{\downarrow}^+ \) then describes the total system. This system is well-defined under the assumption that \( \sup_{x \in \mathcal{E}} |r(x,y)| = M < \infty \), which guarantees that there is no explosion: if \( N_t = \sum_x \eta^+_t(x) + \sum_x \eta^-_t(x) \) is the total number of particles and anti-particles in the system, then \( N_t \) is dominated by a jump process with jumps from \( n \) to \( n+2 \) at rate \( nM \), which leads to exponential growth but no explosion. Also note that under the dynamics the number \( \sum_x \eta^+_t(x) - \sum_x \eta^-_t(x) \) is preserved in time. In particular, for the system starting with a single particle at \( x \), i.e., \( \eta^+_0 = \delta_x \) and \( \eta^-_0 = 0 \), the sum is always 1.

**Theorem 3.1.** Assume \( \sup_{x \in \mathcal{E}} |r(x,y)| < \infty \). Given \( f : \mathcal{E} \to \mathbb{R} \) bounded, define
\[
f^\uparrow(\eta^+, \eta^-) = \sum_{x \in \mathcal{E}} (\eta^+(x) - \eta^-(x))f(x).
\]

Then the semigroup \( S_t \) generated by \( \square \) has the stochastic description
\[
S_tf(x) = \mathbb{E}[(\delta_x,0)f^\uparrow(\eta^+_t, \eta^-_t)].
\]

**Proof.** Let \( (\eta^+_i, \eta^-_i)_{i \geq 0}, i = 1, \ldots, n \) be independent realizations of the particle system started at \( (\eta^+_0, \eta^-_0) \). Then, by the independence of the branching and movement of particles, \( (\sum_{i=1}^n \eta^+_i, \sum_{i=1}^n \eta^-_i)_{t \geq 0} \) has the same law as a system started in \( (\sum_{i=1}^n \eta^+_0, \sum_{i=1}^n \eta^-_0) \). As a consequence, since \( f^\uparrow \) is linear in \( \eta^+, \eta^- \), and anti-symmetric under exchange of \( \eta^+ \) and \( \eta^- \),
\[
\mathbb{E}_{\eta^+_0, \eta^-_0} f^\uparrow(\eta^+_t, \eta^-_t) = \sum_x \eta^+_0(x)\mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) + \sum_x \eta^-_0(x)\mathbb{E}_{0,\delta_x} f^\uparrow(\eta^+_t, \eta^-_t) \\
= \sum_x \eta^+_0(x)\mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) - \sum_x \eta^-_0(x)\mathbb{E}_{0,\delta_x} f^\uparrow(\eta^+_t, \eta^-_t). \quad (3)
\]
and in particular
\[
\mathbb{E}_{2\delta_x, \delta_y} f^\uparrow(\eta^+_t, \eta^-_t) - \mathbb{E}_{\delta_y,0} f^\uparrow(\eta^+_t, \eta^-_t) = \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) - \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t).
\]
If we write \( u_t(x) = \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) \), then
\[
\frac{d}{dt} u_t(x) = \left[ L^\uparrow \mathbb{E} f^\uparrow(\eta^+_t, \eta^-_t) \right] (\delta_x, 0)
\]
\[
= \sum_y r^+(x,y) \left[ \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) - \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) \right]
\]
\[
+ \sum_y r^-(x,y) \left[ \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) - \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) \right]
\]
\[
= \sum_y r^+(x,y) \left[ \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) - \mathbb{E}_{\delta_x,0} f^\uparrow(\eta^+_t, \eta^-_t) \right]
\]
\[
= L u_t(x).
\]
Hence \( u_t(x) \) is the unique solution of
\[
\begin{align*}
\frac{\partial u_t}{\partial t}(x) &= Lu_t(x), \\
u_0 &= f(x).
\end{align*}
\]

4 Branching and annihilating particles and antiparticles

The process in Section 3 tends to have an exponentially growing number of particles. It turns out that we can introduce annihilation of particles and antiparticles to reduce this number. We do so by letting any pair of particle and antiparticle which are at the same site annihilate at rate \( \lambda \in [0, \infty] \), where infinite rate corresponds to instant annihilation. Let
\[
L^\uparrow,\lambda f^\uparrow(\eta^+_t, \eta^-_t) = L^\uparrow f^\uparrow(\eta^+_t, \eta^-_t) + \lambda \sum_x \eta^+(x)\eta^-(x) \left[ f(\eta^+_t - \delta_x, \eta^-_t - \delta_x) - f(\eta^+_t, \eta^-_t) \right]
\]
be the generator of the particle system which includes annihilation.

Theorem 4.1. Theorem 3.1 is also valid when there is annihilation for any \( \lambda \in [0, \infty] \).

Proof. Write \( P_t^{\uparrow,\lambda} f^\uparrow(\eta^+_t, \eta^-_t) = \mathbb{E}_{\eta^+_t, \eta^-_t} f(\eta^+_t, \eta^-_t) \) for the semigroup generated by \( L^\uparrow,\lambda \), with \( \lambda = 0 \) being the system without annihilation. By (3), if \( \eta^+(x) > 0 \) and \( \eta^-(x) > 0 \),
\[
P_t^{\uparrow,0} f^\uparrow(\eta^+, \eta^-) = P_t^{\uparrow,0} f^\uparrow(\eta^+_t - \delta_x, \eta^-_t - \delta_x).
\]
Hence
\[
(L^\uparrow,\lambda - L^\uparrow,0) P_t^{\uparrow,0} f^\uparrow(\eta^+, \eta^-) = 0
\]
and it follows that \( P_t^{\uparrow,\lambda} f^\uparrow = P_t^{\uparrow,0} f^\uparrow \). \( \qed \)
5 Applications to duality of Markov processes

A very brief introduction to duality of Markov processes is as follows. Two Markov processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ on state spaces $E$ and $F$ are said to be dual with duality function $H : E \times F \to \mathbb{R}$, if for all $x \in E$ and $y \in F$,

$$E_x H(X_t; y) = E_y H(x; Y_t).$$  \hfill (4)

A sufficient condition is that the generators $L_X$ and $L_Y$ satisfy

$$[L_X H(\cdot; y)](x) = [L_Y H(x; \cdot)](y), \quad \forall \ x \in E, \ y \in F.$$  \hfill (5)

Duality has proven fruitful in many applications. For a survey on duality, see [2]. The challenge with duality is that given a Markov process $X_t$ of interest, how to find a Markov process $Y_t$ and duality function $H$ so that (4) holds. One can make an educated guess on $H$, and then find a generator $L_Y$ which satisfies (5). Or one can use symmetries of $L_X$ to identify a suitable Lie algebra representation whose building blocks can build $L_X$, and then find a dual representation, which then allows to build $L_Y$, see [1] and [3] for an introduction to this method. However, neither method guarantees that the dual generator $L_Y$ is actually a Markov generator. If $F$ is countable, as is the case in many applications of duality, then $L_Y$ can be represented as a finite or infinite matrix. A stochastic representation of the semigroup generated by such an $L_Y$ is desirable, and with Theorem 2.1, Theorem 3.1 or Theorem 4.1 this is possible.

**Theorem 5.1.** Assume that there is a duality function $H$ and generator $L_Y$ satisfying (5), with $F$ countable. Further assume that the matrix representation of $L_Y$ has row sums 0, so that it can be written in the form of (1), and sup$_{y \in F} \sum_{z \in F} |r(y, z)| < \infty$. Then the Markov process $(X_t)_{t \geq 0}$ is dual to the process $(\eta^+, \eta^-)_{t \geq 0}$ with duality function

$$H^\uparrow(x; \eta^+, \eta^-) = \sum_{y \in F} (\eta^+(y) - \eta^-(y)) H(x; y).$$

Here $(\eta^+, \eta^-)_{t \geq 0}$ is the branching (and annihilating) particle system introduced in sections 3 and 4 with arbitrary annihilation rate $\lambda \in [0, \infty]$. In other words

$$E_x H^\uparrow(X_t; (\eta^+, \eta^-)) = E_{\eta^+, \eta^-} H^\uparrow(x; (\eta^+, \eta^-)).$$  \hfill (6)

**Proof.** By the proof of Theorem 4.1 the right hand side of (6) does not depend on the annihilation rate, so we can restrict ourself to the case of no annihilation. By (5) we have $E_x H(X_t; y) = [S_t H(x; \cdot)](y)$, where $S_t$ is the semigroup generated by $L_Y$. Then, by Theorem 3.1 we have

$$E_x H^\uparrow(X_t; (\delta_y, 0)) = E_x H(X_t; y) = [S_t H(x; \cdot)](y) = E_{\delta_y, 0} H^\uparrow(x; (\eta^+, \eta^-)).$$

Finally, with (5) we can extend the above from $(\delta_y, 0)$ to arbitrary starting configurations. \qed
6 Example: Double Laplacian on the integers

Let \( \Delta f(x) = \frac{1}{2}f(x+1) - f(x) + \frac{1}{2}f(x-1) \) be the discrete Laplacian on \( \mathbb{Z} \). Then the double Laplacian is given by

\[
\Delta \Delta f(x) = \frac{1}{4}(f(x+2) - f(x)) + \frac{1}{4}(f(x-2) - f(x)) - (f(x+1) - f(x)) - (f(x-1) - f(x)) ,
\]

which is of the form \( \mathbb{1} \) with negative rates. Let \( S_t \) be the semigroup generated by the double Laplacian \( \Delta \Delta \). We will apply Theorem 2.1. So let \( \hat{X}_t \) be the random walk on \( \mathbb{Z} \) which performs the jumps \( \pm 1 \) at rate 1 and \( \pm 2 \) at rate \( \frac{1}{4} \). Since jumps using the rates \( r^- \) involve flipping the sign of \( Z_t \), we have that \( Z_t = (-1)^{N_t} \), where \( N_t \) is the number of nearest neighbour jumps performed by \( \hat{X}_t \). Note that \( N_t \) is even iff \( \hat{X}_t - \hat{X}_0 \) is even. Hence

\[
Z_t = 2 \mathbb{1}_{N_t \text{ is even}} - 1 = 2 \mathbb{1}_{\hat{X}_t - \hat{X}_0 \text{ is even}} - 1. \tag{7}
\]

Finally we observe that by spacial homogeneity \( \sum_y r^-(x,y) = 2 \). By Theorem 2.1

\[
S_t f(x) = e^{4t}E_x \left( Z_t f(\hat{X}_t) \right). \tag{8}
\]

Note that \( N_t \) is Poisson(2t)-distributed, and therefore \( \mathbb{P}(N_t \text{ is even}) = \frac{1}{2}(1 + e^{-4t}) \) and \( E Z_t = e^{-4t} \). Alternatively, \( E Z_t = e^{-4t} \) follows from (8) applied to the constant function \( 1 \), since \( S_t 1 = 1 \). For a more complex example consider \( f \) of the form \( f(x) = g(x) \mathbb{1}_x \) is even. Then, by (7) and (8),

\[
S_t f(x) = \begin{cases} 
\frac{1}{2}(e^{4t} + 1)E_x [g(\hat{X}_t) | \hat{X}_t \text{ even}], & \text{ } x \text{ even}; \\
-\frac{1}{2}(e^{4t} - 1)E_x [g(\hat{X}_t) | \hat{X}_t \text{ odd}], & \text{ } x \text{ odd.}
\end{cases}
\]

The conditional expectations are reasonably well approximated by integrating \( g \) against a normal distribution with variance \( \text{Var}(\hat{X}_t) = 4t \) assuming \( g \) is smooth enough and \( t \) not too small.

References

[1] Cristian Giardinà, Jorge Kurchan, Frank Redig, and Kiamars Vafayi. Duality and hidden symmetries in interacting particle systems. *Journal of Statistical Physics*, 135(1):25–55, 2009.

[2] Sabine Jansen and Noemi Kurt. On the notion(s) of duality for markov processes. *Probab. Surveys*, 11:59–120, 2014.

[3] Anja Sturm, Jan M. Swart, and Florian Villerling. The algebraic approach to duality: An introduction. *arXiv preprint*, 2018.