Cube normalized symplectic capacities
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Abstract

We introduce a new normalization condition for symplectic capacities, which we call cube normalization. This condition is satisfied by the Lagrangian capacity and the cube capacity. Our main result is an analogue of the strong Viterbo conjecture for monotone toric domains in all dimensions. Moreover, we give a family of examples where standard normalized capacities coincide but not cube normalized ones. Along the way, we give an explicit formula for the Lagrangian capacity on a large class of toric domains.

1 Introduction

The study of symplectic embeddings is at the core of symplectic geometry. One of the most important tools in this study are symplectic capacities. A symplectic capacity is a function which assigns to each symplectic manifold \((X, \omega)\) of a fixed dimension \(2n\), possibly in some restricted class, a number \(c(X, \omega)\) satisfying the following conditions:

1. If there exists an embedding \(\varphi : X_1 \hookrightarrow X_2\) such that \(\varphi^* \omega_2 = \omega_1\), then 
   \[
   c(X_1, \omega_1) \leq c(X_2, \omega_2).
   \]

2. If \(r > 0\), then 
   \[
   c(X, r \cdot \omega) = r \cdot c(X, \omega).
   \]

After Gromov’s seminal work on symplectic embeddings [Gro85], many capacities were defined. The majority of these satisfy a normalization condition based on Gromov’s non-squeezing. More precisely, let \(B^{2n}(r) \subset \mathbb{C}^n\) denote the ball of radius \(r\) and let \(Z^{2n}(r) = B^2(r) \times \mathbb{C}^{n-1}\). As usual, the standard symplectic form on \(\mathbb{C}^n (= \mathbb{R}^{2n})\) is defined by

\[
\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.
\]

We say that a symplectic capacity is **ball normalized**\(^1\) if

\[
(3) \quad c(B^{2n}(r), \omega_0) = c(Z^{2n}(r), \omega_0) = \pi r^2.
\]

\(^1\)A capacity satisfying condition 3) is usually called **normalized** in the literature. We add the word “ball” in this paper because we will define another normalization condition below.
The central question about ball normalized capacities is the following conjecture, which apparently has been folklore since the 1990s.

**Conjecture 1** (strong Viterbo conjecture). *If $X$ is a convex domain in $\mathbb{R}^{2n}$, then all normalized symplectic capacities of $X$ are equal.*

We refer to [GHR20] for a presentation of known results around Conjecture 1. The strong Viterbo conjecture is, in particular, proven for all monotone toric domains in dimension 4.

Some of the main examples of symplectic capacities that do not satisfy this ball normalization 3) come in sequences, see [EH90, Hut11]. For all of these sequences, the first capacity is still ball normalized. Two other capacities stand alone not satisfying 3), namely the *Lagrangian capacity* and the *cube capacity*, defined by Cieliebak–Mohnke [CM18] and Gutt–Hutchings [GH18], respectively. In this paper we will introduce a new normalization condition (which we call cube normalization) which is satisfied by these latter capacities. Our main result is an equivalent of the strong Viterbo conjecture for cube normalized capacities.

**Theorem 2.** All cube normalized symplectic capacities coincide on all monotone toric domain in any dimension.

This paper is organized as follows. In Section 2, we define the cube normalization and prove Theorem 2. In Section 3, we provide an explicit formula for the Lagrangian capacity on a large class of toric domains encompassing monotone toric domains. In Section 4, we study cube normalized capacities of an interesting class of examples of non-monotone toric domains and we show that for some parameters, ball normalized capacities coincide while cube normalized do not. Finally, in Section 5, we find an upper bound for the cube capacity of a large class of weakly convex toric domains, which is used in Section 4.

## 2 A new normalization condition

Given a domain $\Omega \subset \mathbb{R}^n_{\geq 0}$, define the toric domain

$$X_{\Omega} = \mu^{-1}(\Omega) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid (\pi|z_1|^2, \ldots, \pi|z_n|^2) \in \Omega\}$$

where the map $\mu : \mathbb{C}^n \to [0, +\infty)^n : (z_1, \ldots, z_n) \mapsto (\pi|z_1|^2, \ldots, \pi|z_n|^2)$ is the periodic moment map. We let

$$\partial_+ \Omega = \{p = (p_1, \ldots, p_n) \in \partial \Omega \mid p_i > 0 \text{ for } i = 1, \ldots, n\}.$$ 

Recall from [GHR20] that a monotone toric domain is a compact toric domain with smooth boundary such that for every $p \in \partial_+ \Omega$, the outward pointing normal vector at $p$, $\nu = (\nu_1, \ldots, \nu_n)$ satifies $\nu_i \geq 0$ for $i = 1, \ldots, n$. Note that a monotone toric domain is the limit of toric domains $X_{\Omega'}$ where $\Omega'$ is bounded by the coordinate hyperplanes and the graph of a function whose partial derivatives are all negative, see the proof of [GHR20, Lemma 3.2].

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2In this article, a domain is the closure of a non-empty open set.
Consider the following examples of toric domains:

- **The Ball** \( B_n(a) := \mu^{-1}(\Omega_{B_n(a)}) \), \( \Omega_{B_n(a)} := \{ x \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq a \} \),
- **The Cylinder** \( Z_n(a) := \mu^{-1}(\Omega_{Z_n(a)}) \), \( \Omega_{Z_n(a)} := \{ x \in \mathbb{R}^n_{\geq 0} \mid x_1 \leq a \} \),
- **The Cube** \( P_n(a) := \mu^{-1}(\Omega_{P_n(a)}) \), \( \Omega_{P_n(a)} := \{ x \in \mathbb{R}^n_{\geq 0} \mid \forall i = 1, \ldots, n: x_i \leq a \} \),
- **The NDUC** \( N_n(a) := \mu^{-1}(\Omega_{N_n(a)}) \), \( \Omega_{N_n(a)} := \{ x \in \mathbb{R}^n_{\geq 0} \mid \exists i = 1, \ldots, n: x_i \leq a \} \).

Here, NDUC stands for non-disjoint union of cylinders. Within those toric domains, the ball normalization condition reformulates as

\[ c\left( B_n(1) \right) = c\left( Z_n(1) \right) = 1. \]

This normalization stemmed out of Gromov’s non-squeezing theorem [Gro85] asserting that there exists a symplectic embedding \( B_n(a) \hookrightarrow Z_n(b) \) if and only if \( a \leq b \). The first examples of normalized capacities are the **Gromov width** \( c_B \) and the **cylindrical capacity** \( c^Z \) defined for any symplectic manifold \( (X, \omega) \).

\[
\begin{align*}
c_B(X, \omega) &:= \sup\{ a \mid \text{there exists a symplectic embedding } B_n(a) \hookrightarrow X \}, \\
c^Z(X, \omega) &:= \inf\{ a \mid \text{there exists a symplectic embedding } X \hookrightarrow Z_n(a) \},
\end{align*}
\]

Additional examples of normalized symplectic capacities are the Hofer-Zehnder capacity \( c_{HZ} \) defined in [HZ11] and the Viterbo capacity \( c_{SH} \) defined in [Vit99]. There are also useful families of symplectic capacities parametrized by a positive integer \( k \) including the Ekeland-Hofer capacities \( c^k_{EH} \) defined in [EH89, EH90] using calculus of variations; the “equivariant capacities” \( c^k_{CH} \) defined in [GH18] using positive equivariant symplectic homology; and in the four-dimensional case, the ECH capacities \( c^k_{ECH} \) defined in [Hut11] using embedded contact homology. For each of these families, the \( k = 1 \) capacities \( c^1_{EH} \), \( c^1_{CH} \), and \( c^1_{ECH} \) are normalized. For more about symplectic capacities in general we refer to [CHLS07, Sch18] and the references therein.

We now introduce a new normalization based on a “non-squeezing theorem” for the cube.

**Theorem 3** ([GH18, Proposition 1.20]). There exists a symplectic embedding \( P_n(a) \hookrightarrow N_n(b) \) if and only if \( a \leq b \).

This theorem, together with the previous discussion, motivates the following definition.

**Definition 4.** We say that a symplectic capacity \( c \) is **cube normalized** if

\[ c(P_n(1)) = c(N_n(1)) = 1. \]
We now wish to present examples of cube normalized symplectic capacities. The first example is the cube capacity \(c_P\) [GH18]
\[
c_P(X, \omega) := \sup \{ a \mid \text{there exists a symplectic embedding } P_n(a) \rightarrow X \}.
\]
A second example is the NDUC capacity \(c^N\)
\[
c^N(X, \omega) := \inf \{ a \mid \text{there exists a symplectic embedding } X \rightarrow N_n(a) \}.
\]
The first non immediate example of a cube normalized symplectic capacity was introduced by Cieliebak and Mohnke [CM18] and proved to be cube normalized by the second author in his PhD [Per22]. Let \((X, \omega)\) be a symplectic manifold and let \(L \subset X\) be a Lagrangian submanifold. The minimal area of \(L\) is given by
\[
A_{\min}(L) := \inf \left\{ \int_{\sigma} \omega \mid \sigma \in \pi_2(X, L), \int_{\sigma} \omega > 0 \right\}.
\]
The Lagrangian capacity of \((X, \omega)\) is defined as
\[
c_L(X, \omega) := \sup \{ A_{\min}(L) \mid L \text{ is an embedded Lagrangian torus} \}.
\]

**Theorem 5** ([Per22]).
\[
c_L(P_n(1)) = c_L(N_n(1)) = 1.
\]
The second author actually proved a stronger result. For any toric domain \(X_{\Omega} \subset \mathbb{C}^n\), define its diagonal to be
\[
\delta_{\Omega} := \sup \{ a \mid (a, \ldots, a) \in \Omega \}.
\]

**Theorem 6** ([Per22, Theorem 7.65]). If \(X_{\Omega}\) is a convex or concave toric domain then
\[
c_L(X_{\Omega}) = \delta_{\Omega}.
\]

**Remark 7.** The proof of Theorem 6 uses linearized contact homology, and this result is stated under some assumptions about this theory. For a more detailed discussion on these assumptions see [Sie20, Disclaimer 1.11] and [Per22, Section 7.1].

**Remark 8.** The proof of Theorem 6 uses other symplectic capacities, namely

1. the Gutt–Hutchings capacities from [GH18], denoted by \(c_{k}^{GH}\);
2. the higher symplectic capacities from [Sie20], denoted by \(c_{k}^{\leq 1}\);
3. the McDuff–Siegel capacities from [MS22], denoted by \(\tilde{c}_{k}^{\leq 1}\).

Inspecting the proof of this theorem, one sees that the proof extends word for word for any monotone toric domain, and that moreover
\[
c_L(X_{\Omega}) = \lim_{k \to +\infty} \frac{c_{k}^{\leq 1}(X_{\Omega})}{k} = \lim_{k \to +\infty} \frac{c_{k}^{\leq 1}(X_{\Omega})}{k} = \lim_{k \to +\infty} \frac{c_{k}^{GH}(X_{\Omega})}{k} = \delta_{\Omega}
\]
for any monotone toric domain \(X_{\Omega}\).
One can therefore define cube normalized symplectic capacities as follows.

**Definition 9.** For a nondegenerate Liouville domain \((X, \lambda)\), let

\[
\begin{align*}
    c_{\inf}^{\text{GH}}(X) &:= \liminf_k \frac{c_k^{\text{GH}}(X)}{k}, \\
    g_{\inf}^{\leq 1}(X) &:= \liminf_k \frac{g_k^{\leq 1}(X)}{k}, \\
    \tilde{g}_{\inf}^{\leq 1}(X) &:= \liminf_k \frac{\tilde{g}_k^{\leq 1}(X)}{k}.
\end{align*}
\]

By Remark 8 the symplectic capacities \(c_{\inf}^{\text{GH}}, g_{\inf}^{\leq 1}\) and \(\tilde{g}_{\inf}^{\leq 1}\) are cube normalized.

Using the main result of [GR] asserting that for all \(k \geq 1\) \(c_k^{\text{GH}} = c_k^{\text{EH}}\), we have another cube normalized symplectic capacity

\[
    c_{\inf}^{\text{EH}}(X) := \liminf_k \frac{c_k^{\text{EH}}(X)}{k}.
\]

Note that the main result of [GR] together with Remark 8 shows that for any monotone toric domain \(X_\Omega\)

\[
    c_L(X_\Omega) = \lim_{k \to +\infty} \frac{c_k^{\text{EH}}(X_\Omega)}{k}.
\]

This answers (for the monotone toric case) a Question by Cieliebak-Mohnke [CM18] who asks whether this equality holds for all convex domains in \(\mathbb{R}^{2n}\).

The following theorem, which is an analogue of Viterbo’s strong conjecture is our main result:

**Theorem 10.** All cube normalized capacities coincide on monotone toric domains in \(\mathbb{R}^{2n}\).

**Proof.** Let \(c\) be a cube normalized symplectic capacity and let \(X_\Omega\) be a monotone toric domain in \(\mathbb{R}^{2n}\). We are going to show that then the value of \(c(X_\Omega)\) is determined. The monotonicity of \(X_\Omega\) ensures that

\[
P_n(\delta_\Omega) \subset X_\Omega \subset N_n(\delta_\Omega).
\]

Then,

\[
\begin{align*}
    \delta_\Omega &= c(P_n(\delta_\Omega)) \quad \text{[since } c \text{ is cube normalized]} \\
    &\leq c(X_\Omega) \quad \text{[by monotonicity]} \\
    &\leq c(N_n(\delta_\Omega)) \quad \text{[by monotonicity]} \\
    &= \delta_\Omega \quad \text{[since } c \text{ is cube normalized].}
\end{align*}
\]

As a corollary of Theorem 10, we have the following formula for the value of cube normalized symplectic capacities on monotone toric domains.

**Theorem 11.** Let \(c\) be a cube normalized symplectic capacity and let \(X_\Omega\) be a monotone toric domain in \(\mathbb{R}^{2n}\). Then

\[
c(X_\Omega) = \delta_\Omega.
\]
In view of Theorem 10, it is reasonable to conjecture the following:

**Conjecture 12.** All cube normalized capacities coincide on convex domains in \( \mathbb{C}^n \).

We wish now to make a few comments on what precedes:

**Remark 13.** The link between monotone toric and convex is studied intensively and is, at the moment, unclear. All monotone toric domains are dynamically convex\(^3\) toric domains; however, the converse is only true in \( \mathbb{R}^4 \). Examples of monotone toric domains not symplectomorphic to a convex domain were produced recently [DGZ, CE].

**Remark 14.** If \( c \) is a cube normalized symplectic capacity, then \( c \) is not normalized in the usual sense. Indeed, by Theorem 10, if \( c \) is cube normalized then \( c(B_n(1)) = 1/n \) and \( c(Z_n(1)) = 1 \). We have the following inequalities (for any 2n-dimensional symplectic manifold \((X, \omega)\)):

\[
c_P(X, \omega) \leq c_B(X, \omega) \leq n c_P(X, \omega).
\]

Those inequalities come from the optimal embeddings

\[
B_n(a) \subset P_n(a) \subset B_n(na)
\]

We also have

\[
c^n(X, \omega) \leq c^Z(X, \omega)
\]

coming from the inclusion \( Z_n(a) \subset N_n(a) \).

**Conjecture 15.**

\[
c^Z(X, \omega) \leq nc^N(X, \omega).
\]

The conjecture is true for \( n = 2 \). This is the main technical point of [GHR20]. This amounts to prove that there exists a symplectic embedding \( N_n(a) \hookrightarrow Z_n(na) \).

**Remark 16.** The minimal area of a Lagrangian torus, \( A_{\min}(L) \), is not continuous in \( L \). Indeed on a toric domain \( X_{\Omega}, \mu^{-1}(x) \) is a Lagrangian torus for \( x = (x_1, \ldots, x_n) \in (\text{int} \Omega \cup \partial_+ \Omega) \). By Lemma 17,

\[
A_{\min}(\mu^{-1}(x)) = \inf \{ k_1x_1 + \cdots + k_nx_n \mid k_1, \ldots, k_n \in \mathbb{Z} \}.
\]

### 3 Computing of the Lagrangian capacity for a more general family of toric domains

In this section, we will see how one can use Theorem 6 to compute the Lagrangian capacity for a larger class of toric domains which are not necessarily monotone (see Theorem 18 below). For a toric domain \( X_{\Omega} \), define

\[
\eta_{\Omega} := \inf \{ a \mid X_{\Omega} \subset N_n(a) \}.
\]

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\(^3\)Convexity is not a symplectically invariant property. This was already pointed out a long time ago but only a few symplectic substitutions have been suggested. The most prominent one is dynamical convexity, introduced in [HWZ98], where they show that strict convexity guarantees dynamical convexity.
Notice that if $X_\Omega$ is convex or concave, then $\delta_\Omega = \eta_\Omega$. To prove Theorem 18, we will make use of the following lemma:

**Lemma 17** ([Per22, Lemma 6.16]). Let $(X,\lambda)$ be an exact symplectic manifold and $L \subset X$ be a Lagrangian submanifold. If $\pi_1(X) = 0$, then

$$A_{\min}(L) = \inf \{\lambda(\rho) \mid \rho \in \pi_1(L), \lambda(\rho) > 0\}.$$

**Proof.** The diagram

$$\begin{array}{ccc}
\pi_2(X, L) & \xrightarrow{\partial} & \pi_1(L) \\
\omega & \downarrow & \pi_1(X) \\
 & \lambda & \\
\end{array}$$

commutes, where $\partial([\sigma]) = [\sigma|_{S^1}]$, and the top row is exact. ∎

**Theorem 18.** Let $X_\Omega$ be a toric domain. If $(\eta_\Omega, \ldots, \eta_\Omega) \in \partial \Omega$ then

$$c_L(X_\Omega) = \eta_\Omega.$$ 

**Proof.** By definition of $\eta_\Omega$, we have $X_\Omega \subset N_n(\eta_\Omega)$. Define $T := \mu^{-1}(\eta_\Omega, \ldots, \eta_\Omega)$. Then $T$ is an embedded Lagrangian torus in $X_\Omega$ (see Fig. 2 for an illustration of $\eta_\Omega$, $T$, $\Omega$ and $\Omega_{N_n(\eta_\Omega)}$). Therefore,

$$\eta_\Omega = A_{\min}(T) \quad \text{[by Lemma 17]}
\leq c_L(X_\Omega) \quad \text{[by definition of $c_L$]}
\leq c_L(N_n(\eta_\Omega)) \quad \text{[by monotonicity]}
\leq \eta_\Omega \quad \text{[by Theorem 6].}$$

Note that Theorem 18 extends mutatis mutandis, using Eq. (1), to the following

**Theorem 19.** Let $X_\Omega \subset N_n(\eta_\Omega)$ be a toric domain in $\mathbb{R}^{2n}$ such that there exist a point $x \in \overline{\partial \Omega} \cap \partial N_n(\eta_\Omega)$ of the form $x = (k_1 \eta_\Omega, \ldots, k_n \eta_\Omega)$ where the $k_i \in \mathbb{N}$ (see Fig. 2). Then,

$$c_L(X_\Omega) = \eta_\Omega.$$ 

### 4 An interesting nonexample

We now study a family of examples coming from [GHR20] of non-monotone toric domains, and we determine when they satisfy the conclusion of Theorem 10.

For $0 < a < 1/2$, let $\Omega_a$ be the convex polygon with corners $(0,0), (1-2a,0), (1-a,a), (a,1-a)$ and $(0,1-2a)$, and write $X_a = X_{\Omega_a}$; see Fig. 3. Then $X_a$ is a weakly convex (but not monotone) toric domain.
Proposition 20. The cubic, Lagrangian and NDUC capacities of \( X_a \) are given as follows.

\[
\begin{align*}
    c_P(X_a) &= \min \left( 1 - 2a, \frac{1}{2} \right), \\
    c_L(X_a) &= c_N(X_a) = \frac{1}{2}.
\end{align*}
\]

Remark 21. It follows from Proposition 20 that \( c_P(X_a) \neq c_N(X_a) \) for \( a > 1/4 \). But in [GHR20] it was shown that \( c_B(X_a) = c^Z(X_a) \) for all \( a \leq 1/3 \). So for \( 1/4 < a \leq 1/3 \), the Gromov and cylindrical capacities of \( X_a \) coincide, but not the cubic and NDUC capacities.

Proof. We note that \( \eta_{\Omega_a} = 1/2 \) for all \( a \leq 1/2 \) and that \( (1/2, 1/2) \in \Omega_a \). So it follows from Theorem 18 that \( c_L(X_a) = 1/2 \). Since \( X_a \subset N_2(a) \), it follows that

\[
\frac{1}{2} = c_L(X_a) \leq c_N(X_a) \leq \frac{1}{2}.
\]

So \( c_N(X_a) = 1/2 \).
To compute the cubic capacities, we first observe that
\[ P_n \left( \frac{1}{2} \right) \subset X_a, \text{ for } 0 < a \leq 1/4, \]
\[ P_n(1 - 2a) \subset X_a, \text{ for } 1/4 < a < 1/2. \]
So \( c_p(X_a) \geq \min(1 - 2a, 1/2). \) Since \( c_p(X_a) \leq c^N(X_a) = 1/2, \) it follows that \( c_p(X_a) = 1/2 = \min(1 - 2a, 1/2) \) for \( 0 < a \leq 1/4. \)

The fact that \( c_p(X_a) \leq 1 - 2a \) for \( 1/4 < a < 1/2 \) follows from Theorem 22 below.

5 The cubic capacity of some weakly convex toric domains

In this section we obtain an upper bound for the cubic capacity of some non-monotone toric domains, which will not in general coincide with their NDUC capacity.

A four-dimensional toric domain \( X_\Omega \) is said to be weakly convex if \( \Omega \subset \mathbb{R}^2_{\geq 0} \) is convex and \( \partial_+ \Omega \) is a piecewise smooth curve connecting the two coordinate axes, see Figure 4.

With an extra assumption, we can compute an upper bound for the cubic capacity of \( X_\Omega \).

**Theorem 22.** Let \( X_\Omega \) be a weakly convex toric domain, where \( \partial_+ \Omega \) is parametrized by the curve \( (x, y) : [0, 1] \rightarrow \mathbb{R}^2_{\geq 0} \) such that \( y(0) = 0 \) and \( x(1) = 0. \) Suppose that

\[ \max \left( \frac{x'(0)}{y'(0)}, \frac{y'(1)}{x'(1)} \right) \leq 1. \]

Then

\[ c_p(X_\Omega) \leq \frac{x(0) + y(1)}{2}. \]

The proof of Theorem 22 uses embedded contact homology. Namely, we need a version of [Hut16, Theorem 1.20] for weakly convex toric domains. We now explain the context.

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4 Cristofaro-Gardiner defined this to be a convex toric domain in [CG19], but usually a convex toric domain is defined to be a particular case of this, see [GHR20], for example.
and the modifications that need to be made in the proof of [Hut16, Theorem 1.20] for our purposes here.

We need some definitions to state a more general version of [Hut16, Theorem 1.20]. Let $X_\Omega$ be a weakly convex toric domain. We define a combinatorial Reeb orbit to be a pair $(v, s)$, where $v = (x_v, y_v)$ is a primitive vector in $\mathbb{Z}^2$ and $s = \{0, 1\}$ such that $x_v \geq 0$ or $y_v \geq 0$. A combinatorial orbit set is a finite formal product

$$\alpha = \prod_{i=1}^k (v_i, s_i)^{m_i},$$

where $(v_i, s_i)$ are distinct combinatorial Reeb orbits and $m_i \in \mathbb{Z}_{\geq 1}$ such that $m_i = 1$ whenever $s_i = 0$. We define the following numbers.

$$x(\alpha) = \sum_{i=1}^k m_i x_{v_i},$$  \hspace{1cm} (2)

$$y(\alpha) = \sum_{i=1}^k m_i y_{v_i},$$  \hspace{1cm} (3)

$$I(\alpha) = x(\alpha) + y(\alpha) + \sum_{i,j=1}^k m_i m_j \max(x_{v_i} y_{v_j}, x_{v_j} y_{v_i}) + \sum_{i=1}^k s_i m_i,$$  \hspace{1cm} (4)

$$m(\alpha) = \sum_{i=1}^k m_i,$$  \hspace{1cm} (5)

$$h(\alpha) = \sum_{i=1}^k (1 - s_i).$$  \hspace{1cm} (6)

We note that none of those numbers depend on $\Omega$. The number $I(\alpha)$ is called the combinatorial ECH index of $\alpha$. We define the combinatorial action of $\alpha$ to be

$$A_\Omega(\alpha) = \sum_{i=1}^k m_i \max\{v_i \cdot p \mid p \in \partial_+ \Omega\}.$$

We now state a version of [Hut16, Definition 1.18] for weakly convex toric domains.

**Definition 23.** Let $X_\Omega$ and $X_{\Omega'}$ be weakly convex toric domains and let $\alpha$ and $\alpha'$ be combinatorial orbit sets. We write $\alpha \leq_{\Omega, \Omega'} \alpha'$ if the following conditions hold:

(i) $I(\alpha) = I(\alpha')$,  

(ii) $A_\Omega(\alpha) \leq A_{\Omega'}(\alpha')$,  

(iii) $x(\alpha) + y(\alpha) - h(\alpha)/2 \geq x(\alpha') + y(\alpha') + m(\alpha') - 1$.

The version of [Hut16, Theorem 1.20] that we need is the following result.

**Theorem 24.** Let $X_\Omega$ and $X_{\Omega'}$ be weakly convex toric domains such that $X_\Omega \hookrightarrow X_{\Omega'}$. Let $\alpha'$ be an orbit set such that $I(\alpha') > 0$ and $h(\alpha') = 0$. Then there is an orbit set $\alpha$ with $I(\alpha) = I(\alpha')$ and product decompositions

$$\alpha = \prod_{j=1}^l \alpha_j, \quad \alpha' = \prod_{j=1}^l \alpha'_j,$$

such that:
(a) $\alpha_j \leq_{\Omega, \Omega'} \alpha'_j$.

(b) Given $i, j$, if $\alpha_i = \alpha_j$ or $\alpha'_i = \alpha'_j$, then $\alpha_i$ and $\alpha_j$ have no combinatorial Reeb orbits in common with $s = 1$.

(c) For any $\emptyset \neq S \subset \{1, \ldots, l\}$,

$$I \left( \prod_{j \in S} \alpha_j \right) = I \left( \prod_{j \in S} \alpha'_j \right) > 0.$$ 

Proof. The proof is essentially the same as the one of [Hut16, Theorem 1.20]. As in the proof of [GHR20, Theorem 5.6], we first approximate $\Omega$ by a domain $\tilde{\Omega} \subset \Omega$ such that $\partial_x \tilde{\Omega}$ is a smooth curve and the slopes of the tangent lines at the intersections with the x-axis and y-axis are $\varepsilon$ and $\varepsilon^{-1}$. We observe that for a given orbit set $\alpha$ and $\delta > 0$, we can define $\tilde{\Omega}$ so that $|A_{\tilde{\Omega}}(\alpha) - A_{\tilde{\Omega}}(\alpha)| < \delta$. We define $\Omega' \supset \Omega'$ satisfying the same properties as above, c.f. [Hut16, Lemma 5.4]. In particular $X_{\tilde{\Omega}} \to X_{\tilde{\Omega}'}$.

We now briefly recall the embedded contact homology (ECH) chain complex. Let $(x, y) : [0, 1] \to \mathbb{R}^2$ be a parametrization of $\partial_x \tilde{\Omega}$ such that $y(0) = x(1) = 0$. So $y'(0)/x'(0) = x'(1)/y'(1) = \varepsilon$. We assume that $\varepsilon$ is a small irrational number and that $(x''(t), y''(t)) \neq 0$ for $t \in [0, 1]$. Then the standard Liouville form on $\mathbb{R}^4$ restricts to a contact form $\lambda_0$ on $\partial X_{\tilde{\Omega}}$ whose Reeb flow foliates $\mu^{-1}((x(t), y(t)))$ for each $t \in [0, 1]$. Then for each $t \in [0, 1]$ such that $x'(t)/y'(t) \in \mathbb{Q} \cup \{\infty\}$, there is a unique $(p, q) \in \mathbb{Z}^2$ such that $p$ and $q$ are relatively prime and

$$(x'(t), y'(t)) = c \cdot (p, q), \quad \text{for } c > 0.$$ 

So the torus $T_{p, q} := \mu^{-1}((x(t), y(t)))$ is foliated by closed Reeb orbits. Note that $T_{(p, q)}$ is uniquely determined by $(p, q)$ since $X_{\tilde{\Omega}}$ is weakly convex. For a Reeb orbit $\gamma \in T_{p, q}$, its symplectic action is defined by

$$A_{\tilde{\Omega}}(\gamma) := \int_{\gamma} \lambda_0.$$ 

It is straight-forward to check that this action doesn’t depend on $\gamma$. Indeed for every $\gamma \in T_{p, q}$, it follows from a simple calculation that

$$A_{\tilde{\Omega}}(\gamma) = \max \{(p, q) \cdot x \mid x \in \partial_x \tilde{\Omega}\} = A_{\tilde{\Omega}}((p, q), 1).$$ 

The only other Reeb orbits of $\lambda_0$ are the two circles $\mu^{-1}((x(0), y(0)))$ and $\mu^{-1}((x(1), y(1)))$. One can check that $\lambda_0$ is Morse–Bott. Given $L > 0$, we can perturb the contact form in neighborhoods of the tori $T_{p, q}$ for which $A_{\tilde{\Omega}}(\gamma) < L$ for $\gamma \in T_{p, q}$, thus obtaining an elliptic and a hyperbolic Reeb orbit, denoted by $e_{(p, q)}$ and $h_{(p, q)}$, respectively. This is explained in more detail in [Hut11] and [CCGF+14], for example. Let $\tilde{\lambda}$ denote the perturbed contact form. The only other closed Reeb orbits of $\lambda$ with action less than $L$ are the two circles fibering above $(x(0), 0)$ and $(0, y(1))$, which are elliptic. We denote them by $e_0$ and $e_1$.

An orbit set is a finite formal product $\alpha = \prod_i \alpha_i^{m_i}$, where $\alpha_i$ is a simple Reeb orbit and $m_i$ is positive integer. We always assume that $\alpha_i \neq \alpha_j$ if $i \neq j$ and $m_i = 1$ if $\alpha_i$ is hyperbolic. The action of an orbit set is defined by

$$A_{\tilde{\Omega}}(\alpha) = \sum_i m_i A_{\tilde{\Omega}}(\alpha_i).$$
The filtered ECH chain complex $ECC^L(\partial X_{\tilde{\Omega}}, \tilde{\lambda})$ is the $\mathbb{Z}/2$ vector space generated by all orbit sets $\alpha$ such that
\[ A_{\tilde{\Omega}}(\alpha) < L. \]
Under the identification $e_{p,q} = ((p,q),1)$ and $h_{p,q} = ((p,q),1)$, we can see orbit sets as combinatorial orbit sets and their symplectic actions coincide\(^5\). The differential of $ECC^L(\partial X_{\tilde{\Omega}}, \tilde{\lambda})$ is obtained by counting pseudo-holomorphic curves in $\mathbb{R} \times \partial X_{\tilde{\Omega}}$ whose ECH index is 1. We will not define the ECH index here. Instead it suffices to recall that in this setting the ECH index gives rise to an absolute index such that for each orbit set $\alpha$, $I(\alpha)$ is simply the combinatorial ECH index defined in (4). The fact that the original definition and the combinatorial definition coincide follows from very similar calculations to the one in the proof of [Hut16, Lemma 5.4], which uses previous calculations from the proof of [CCGF+14, Lemma 3.3]. Here we have a max instead of a min, because of the opposite concavity, as in [Hut16, Lemma 5.4]. It is worth noting that the calculation of the first Chern class $\tilde{CCGF}$ to the one in the proof of [Hut16, Lemma 5.4], which uses previous calculations from the definition and the combinatorial definition coincide follows from very similar calculations as defined in (2) and (3).

The rest of the argument uses the cobordism map in ECH and the $J_0$-invariant. It is identical to the proof of [Hut16, Theorem 1.20] using (7), where we note that the original and the combinatorial definitions of $h$ and $m$ coincide.

We can now prove Theorem 22.

**Proof of Theorem 22.** Suppose that $P_2(a) \hookrightarrow X_{\Omega}$. We can find a weakly convex toric domain $X_{\Omega'} \supset X_{\Omega}$ such that the tangent lines to the curve $\partial_+ \Omega'$ at the $x$ and $y$ axes have slopes $1 - \delta$ and $1 + \delta$ for some small $\delta > 0$, respectively. For each $L > 0$ sufficiently large and $\varepsilon > 0$, we can choose $X_{\Omega'}$ so that
\[ |A_{\Omega'}(e_{1,-1}) - x(0)| < \varepsilon \quad \text{and} \quad |A_{\Omega'}(e_{-1,1}) - y(1)| < \varepsilon, \]
and that
\[ |A_{\Omega}(e_{p,q}) - A_{\Omega'}(e_{p,q})| < \varepsilon, \]
for all $(p,q)$ such that $A_{\Omega}(e_{p,q}) < L$.

Now let $\alpha' = e_{1,-1}^d e_{-1,1}^d e_{1,1}^2$. It follows from Theorem 24 that there exists an orbit set $\alpha$ and factorizations
\[ \alpha = \prod_{j=1}^l \alpha_j, \quad \alpha' = \prod_{j=1}^l \alpha'_j, \]
satisfying (a), (b) and (c). Using (b) and (c), we conclude that $l \leq 3$ and that $\alpha_i = e_{1,-1}^{d_i} e_{-1,1}^{d_i} e_{1,1}^k$ for some $k \in \{0,1,2\}$ such that $d_i \geq d/3$. Using (a), it follows from properties (ii) and (iii) from Definition 23 that
\[ 3k + 2d_i - 1 = x(\alpha'_i) + y(\alpha'_i) + m(\alpha'_i) - 1 \leq x(\alpha_i) + y(\alpha_i) \]
\[ \leq \frac{A_{P_2(a)}(\alpha_i)}{a} \leq A_{\Omega}(\alpha_i) \leq \frac{(d_i(x(0) + y(1)) + k)(1 + \varepsilon)}{a}. \]

\(^{5}\)To be precise, the symplectic actions with respect to the perturbed contact form is bounded from the combinatorial action by a small constant which can be as small as desired for a given $L$. 

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Hence
\[
a < \frac{(d_i(x(0) + y(1)) + k)(1 + \varepsilon)}{2d_i + 3k - 1}.
\]
Taking the limit as \(d \to \infty\) and then as \(\varepsilon \to 0\), it follows that
\[
a \leq \frac{x(0) + y(1)}{2}.
\]
Therefore
\[
c_P(X_\Omega) \leq \frac{x(0) + y(1)}{2}.
\]

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