Modified Friedmann equation and survey of solutions in effective Bianchi-I loop quantum cosmology

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Abstract
In this article, we study the equations driving the dynamics of a Bianchi-I universe described by holonomy-corrected effective loop quantum cosmology (LQC). We derive the LQC-modified generalized Friedmann equation, which is used as a guide to find different types of solutions. It turns out that, in this framework, most solutions never reach the classical behavior.

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1. Introduction
Loop quantum gravity (LQG) is a tentative nonperturbative and background-independent quantization of general relativity. It uses Ashtekar variables, namely SU(2) valued connections and conjugate densitized triads. The quantization is obtained through holonomies of the connections and fluxes of the densitized triads (see, e.g., [1] for introductions). Basically, loop quantum cosmology (LQC) is the symmetry reduced version of LQG. In LQC, the big bang is generically replaced by a big bounce due to huge repulsive quantum geometrical effects (see, e.g., [2] for reviews).

In bouncing cosmologies, the issue of anisotropies is however crucial for a simple reason: the shear term basically scales as $1/a^6$ where $a$ is the scale factor of the universe. Therefore, when the universe is in its contraction phase, it is expected that the shear term eventually dominates and drives the dynamics. When spatial homogeneity is assumed, anisotropic hypersurfaces admit transitive groups of motion that must be three- or four-parameters isometry groups. The four-parameters groups admitting no simply transitive subgroups will not be considered here. There are nine algebraically inequivalent three-parameters simply transitive Lie groups, denoted Bianchi I through IX, with well known structure constants. The flat, closed and open generalizations of the FLRW model are respectively Bianchi-I, Bianchi-IX
and Bianchi-V. As the universe is nearly flat today and as the relative weight of the curvature term in the Friedmann equation is decreasing with decreasing values of the scale factor, it is reasonable to focus on the Bianchi-I model to study the dynamics around the bounce.

Many studies have already been devoted to Bianchi-I LQC [3–5]. In particular, it was shown that the bounce prediction is robust. As the main features of isotropic LQC are well captured by semi-classical effective equations, and it is a good guess that this remains true in the extended Bianchi-I case. The solutions of effective equations were studied in detail in [6]. In this work, we focus on slightly different aspects and derive the LQC-modified generalized Friedmann equation that was still missing. Thanks to this equation, we have systematically explored the full solution space in a way that has not been tired before.

2. Classical equations

The metric for a Bianchi-I spacetime reads as:
\[ ds := -N^2 dt^2 + a_1^2 dx^2 + a_2^2 dy^2 + a_3^2 dz^2, \] (1)
where \( a_i \) denote the directional scale factors. A dot means derivation with respect to the cosmic time \( t \), with \( dt = N d\tau \).

Classically, the evolution of this metric is described by the Hamiltonian
\[ \mathcal{H} = \mathcal{H}_G(c_i, p_i) + \mathcal{H}_M(p_i, \phi_n, \pi_n), \] (2)
where
\[ \mathcal{H}_G = \frac{N}{\kappa \gamma^2} \left( \sqrt{p_1^2 p_3^3 c_1 c_2} + \sqrt{p_2^2 p_3^3 c_2 c_2} + \sqrt{p_3^2 p_1^3 c_3 c_3} \right), \] (3)
and
\[ \mathcal{H}_M = N \sqrt{p_1 p_2 p_3} \rho, \] (4)
with the Poison brackets
\[ \{ c_i, p_j \} = \kappa \gamma \delta_{ij}, \quad \{ \phi_n, \pi_m \} = \delta_{nm}, \] (5)
where \( i, j \in \{ 1, 2, 3 \} \) and \( n, m \in \{ 1, 2, \ldots, M \} \) for \( M \) matter fields. In the following, we have chosen to consider a comoving volume of size \( 1 \times 1 \times 1 \). Since the universe is assumed to be homogenous, this will not affect the results. We denote by \( \phi_n \) the matter fields, \( \pi_n \) their conjugate momentum, and \( \rho \) the total matter density. The \( c_i \) and \( p_i \) entering equation (3) are the diagonal elements of the Ashtekar variables (\( p_i \) is assumed to always be positive).

The directional scale factors can be written as
\[ a_1 = \sqrt{\frac{p_2 p_3}{p_1}} \quad \text{and cyclic expressions.} \] (6)
The generalized Friedmann equation is
\[ H^2 = \sigma^2 + \frac{\kappa}{3} \rho, \] (7)
where
\[ H := \frac{\dot{a}}{a} = \frac{1}{3} (H_1 + H_2 + H_3), \] (8)
\[ a := (a_1 a_2 a_3)^{1/3}, \] (9)
\[ H_1 := \frac{\dot{a}_1}{a_1} = -\frac{\dot{p}_1}{2p_1} + \frac{\dot{p}_3}{2p_3} + \frac{\dot{p}_3}{2p_3} \quad \text{and cyclic expressions}, \] (10)
\[ \sigma^2 := \frac{1}{18} [(H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2]. \] (11)
It should be pointed out that the 1/18 factor is not used in similar studies.
If we assume isotropic matter, that is
\[ H_M(p_1, \phi, \pi) = H_M(\sqrt{p_1 p_2 p_3}, \phi, \pi), \]  
then the equations of motion for \( H_i \) become
\[ \dot{H}_i = -H_2 H_{i2} + H_3 H_{i3} - \frac{\kappa}{2} (\rho + P) \quad \text{and cyclic terms}, \]  
where \( P \) is defined to fulfill the equation \( \dot{\rho} = 3H(\rho + P) \), that is
\[ P := -\frac{\partial (H_m/N)}{\partial \sqrt{p_1 p_2 p_3}}. \]  
Several other relations will be useful:
\[ \dot{H}_i - \dot{H}_j = -3H(H_i - H_j) \Leftrightarrow H_i - H_j \propto a^{-3}, \]  
leading to
\[ \sigma^2 \propto a^{-6} \quad \text{and} \quad H_i - H_j = \text{constant}. \]  
Classically \( H_i \) can change sign, but \( H \) cannot. Many details about the classical behaviors of a Bianchi-I universe can be found, e.g., in [7].

3. Effective holonomy corrections

The holonomy correction in effective LQC is due to the fact that the Ashtekar connection cannot be promoted to be an operator but only its holonomy can. It is believed to capture most quantum effects at the semi-classical level. Following the usual prescription, we perform the substitution
\[ c_i \rightarrow \frac{\sin(\bar{\mu}_i c_i)}{\bar{\mu}_i} \]  
in the Hamiltonian given by equations (2) and (3). The \( \bar{\mu}_i \) are given by
\[ \bar{\mu}_1 = \lambda \sqrt{\frac{p_1}{p_2 p_3}} \quad \text{and cyclic expressions}, \]  
where \( \lambda \) is the square root of the minimum area eigenvalue of the LQG area operator (\( \lambda = \sqrt{\Delta} \)). This was first proposed in [3], and later derived in [4].

The effective holonomy-corrected gravitational Hamiltonian is
\[ \mathcal{H}_G = -\frac{N}{\kappa \gamma^2 \lambda^2} \left[ \sin(\bar{\mu}_1 c_1) \sin(\bar{\mu}_2 c_2) + \sin(\bar{\mu}_2 c_2) \sin(\bar{\mu}_3 c_3) + \sin(\bar{\mu}_3 c_3) \sin(\bar{\mu}_1 c_1) \right]. \]  
The matter Hamiltonian \( \mathcal{H}_M \) remains unchanged.

4. The LQC-modified generalized Friedmann equation

Various versions of the Friedmann equation—depending on the specific model considered—are used in cosmology. They allow to derive the key features of the dynamics in a simple way. The LQC-modified generalized Friedmann equation describing a holonomy-corrected Bianchi-I universe has so far been missing. It is derived in this section and, in more details, in the appendix.

The Friedmann equation is found by rewriting the constraint \( \mathcal{H} = 0 \) in terms of physical parameters. In our case, these parameters are: the total Hubble parameter, matter density and
Then, it is possible to derive the LQC-modified generalized Friedmann equation:

\[ \rho_1 = \frac{1}{N} \{ p_1, H \} = \frac{p_1}{\gamma^2} \cos(\mu_1 c_1) [\sin(\mu_2 c_2) + \sin(\mu_3 c_3)] \] and cyclic expressions. \hspace{1cm} (20)

From this, we get the directional Hubble parameters \( H_1 \) and total Hubble parameter \( H \):

\[ H_1 = -\frac{\dot{p}_1}{2p_1} + \frac{\dot{p}_2}{2p_2} + \frac{\dot{p}_3}{2p_3} = \frac{1}{2\gamma^2} [\sin(\mu_2 c_2 + \mu_3 c_3) + \sin(\mu_1 c_1 - \mu_2 c_2) + \sin(\mu_3 c_3 - \mu_1 c_1)] \] and cyclic, \hspace{1cm} (21)

\[ H = \frac{1}{3} (H_1 + H_2 + H_3) \]

\[ = \frac{1}{6\gamma^2} [\sin(\mu_1 c_1 + \mu_2 c_2) + \sin(\mu_2 c_2 + \mu_3 c_3) + \sin(\mu_3 c_3 + \mu_1 c_1)]. \] \hspace{1cm} (22)

We also define the 'quantum shear' as:

\[ \sigma_0^2 := \frac{1}{3\lambda^2\gamma^2} \left( 1 - \frac{1}{3} [\cos(\mu_1 c_1 - \mu_2 c_2) + \cos(\mu_2 c_2 - \mu_3 c_3) + \cos(\mu_3 c_3 - \mu_1 c_1)] \right). \] \hspace{1cm} (23)

Then, it is possible to derive the LQC-modified generalized Friedmann equation:

\[ H^2 = \sigma_0^2 + \frac{\kappa}{3\rho} - \lambda^2\gamma^2 \left( \frac{3}{2} \sigma_0^2 + \frac{\kappa}{3\rho} \right)^2. \] \hspace{1cm} (24)

The details of how to obtain this non-trivial equation are given in the appendix. It should be pointed out that

\[ \lim_{\lambda \to 0} \sigma_0^2 = \lim_{\lambda \to 0} \sigma^2, \] \hspace{1cm} (25)

so that in the limit \( \lambda \to 0 \) the classical Friedmann equation is recovered. On the other hand, in the limit \( \sigma_0^2 \to 0 \), the isotropic holonomy-corrected Friedmann equation is recovered.

From equation (24), we can easily find the upper bounds for \( \rho \) and \( \sigma_0^2 \):

\[ \rho \leq \rho_c := \frac{3}{\kappa \lambda^2\gamma^2}, \] \hspace{1cm} (26)

\[ \sigma_0^2 \leq \sigma_{0c}^2 := \frac{4}{9} \lambda^2\gamma^2. \] \hspace{1cm} (27)

5. Equations of motion

In the gravitational sector, the information is contained in the combined objects \( h_i \):

\[ h_1 := \mu_1 c_1 = \lambda \sqrt{\frac{p_1}{p_2 p_3}} c_1 \] and cyclic expressions. \hspace{1cm} (28)

It is expected that the six gravitational degrees of freedom \( (c_i, p_i) \) account for only three physical degrees of freedom \( h_i \). This is because three degrees of freedom are just rescaling of the scale factors which have no physical meaning.

Just as in the classical calculations, we assume isotropic matter. Then we can derive:

\[ \dot{h}_1 = \frac{1}{N} [h_1, \mathcal{H}] = \frac{1}{2\gamma^2} [(h_2 - h_1)(\sin h_1 + \sin h_3) \cos h_2 + (h_3 - h_1)(\sin h_1 + \sin h_2) \cos h_3] \]

\[ -\frac{\kappa \gamma^2}{2} (\rho + P) \] and cyclic expressions, \hspace{1cm} (29)
where we have used the constraint $H_G + H_M = 0$. Thus we have

$$ (h_i - h_j) = -3H(h_i - h_j), $$

which means that

$$ (h_i - h_j) \propto a^{-3} \quad \text{and} \quad \frac{h_i - h_j}{h_i - h_k} = \text{constant}. $$

This should be compared with the classical results given by equations (15) and (16).

6. Symmetries of the effective quantum equations

Equations (21)–(24) are invariant under the discrete symmetry

$$
\begin{cases}
    h_1 \to h_1 + (2\tilde{n}_1 + \tilde{m})\pi, \\
    h_2 \to h_2 + (2\tilde{n}_2 + \tilde{m})\pi, \\
    h_3 \to h_3 + (2\tilde{n}_3 + \tilde{m})\pi,
\end{cases}
\forall \tilde{n}_1, \tilde{n}_2, \tilde{n}_3 \in \mathbb{Z} \quad \forall \tilde{m} \in \{0, 1\}.

(32)

However, equation (29) is only invariant under the smaller symmetry

$$
\begin{cases}
    h_1 \to h_1 + \tilde{n}\pi, \\
    h_2 \to h_2 + \tilde{n}\pi, \\
    h_3 \to h_3 + \tilde{n}\pi
\end{cases}
\forall \tilde{n} \in \mathbb{Z}.

(33)

Remember that $h_i = \tilde{\mu} c_i$.

All observable quantities, and their evolution, are invariant under equation (33). This suggests that equation (33) is a gauge symmetry. However, this might not be the case, if more degrees of freedom are taken into account.

More consequences of these symmetries will be discussed later.

7. Classical limit

As one would expect, the classical equations are recovered in the limit $\lambda \to 0$. But one also expects to find a classical limit in the far future and in the remote past, far away from the bounce. We will therefore investigate for what values of $h_i$ and $\rho$ classical equations are recovered.

For $\sigma_Q^2 \ll \sigma_Q^2$ and $\rho \ll \rho_c$, equation (24) becomes

$$ H^2 = \sigma_Q^2 + \frac{\kappa}{3}\rho, $$

(34)

to first order in $\sigma_Q^2$ and $\rho$. The above equation is equivalent to equation (7) if and only if $\sigma_Q^2 = \sigma^2$. It is trivial to check that this is the case, to lowest order in $h_i$ if $h_i \ll 1$. But since $\sigma_Q^2$ and $\sigma^2$ are cyclic expressions of $h_i$, this is not the only region where equation (7) is recovered from (24).

Equation (7) is not enough to completely describe the classical system. To say that we have a classical limit, we also need to recover equation (13). The matter equations are assumed to be unaffected by the holonomy corrections.

The symmetries, equations (32) and (33), suggest the existence of more than one classical limit. And the knowledge of these symmetries could of course be used in the search for such limits. However, to be absolutely certain that we find all regions of classical behavior, we will search in the full parameter space.

We will try to recover the classical equations, equations (7) and (13), from the quantum modified equations (24), in the perturbative regime. But, instead of assuming, for example,
that \( h_i \) and \( \rho \) are small, an thus make an expansion around \((h_i, \rho) = (0, 0, 0, 0)\), we will expand around the more general point \((h_i, \rho) = (h_i^{(0)}, \rho^{(0)})\). We define
\[
\delta h_i := h_i - h_i^{(0)},
\delta \rho := \rho - \rho^{(0)}.
\] (35)

In this section, we will find all points \((h_i^{(0)}, \rho^{(0)})\), such that, for \(\delta h_i \ll 1\) and \(\delta \rho \ll \rho_c\), equations (7) and (13) are recovered from the expressions given in sections 4 and 5.

All the following calculations in this section will be carried out to lowest order in \(\delta h_i\) and \(\delta \rho\). We will also use the notations
\[
\delta \sigma^2 := \sigma^2 - (\sigma^2)^{(0)} := \sigma^2(h_i) - \sigma^2(h_i^{(0)}),
\delta \sigma_Q^2 := \sigma_Q^2 - (\sigma_Q^2)^{(0)} := \sigma_Q^2(h_i) - \sigma_Q^2(h_i^{(0)}).
\] (36)

It should be pointed out at this stage that, even though we assume \(\delta \rho \ll \rho_c\), and indirectly \(\delta \sigma^2, \delta \sigma_Q^2 \ll \sigma_Q^2\), this does not mean that the energy density and shear have to be small in a classical sense. This is because \(\rho_c\) and \(\delta \sigma_Q^2\) have very large values.

Combining equations (7) and (24) we find that in the classical limit
\[
\sigma^2 = \sigma_Q^2 - \lambda^2 \gamma^2 \left( \frac{3}{2} \sigma_Q^2 + \frac{\kappa}{3} \rho \right)^2.
\] (37)

Expanded, this becomes
\[
(\sigma^2)^{(0)} + \delta \sigma^2 = (\sigma_Q^2)^{(0)} + \delta \sigma_Q^2 - \lambda^2 \gamma^2 \left( \frac{3}{2} (\sigma_Q^2)^{(0)} + \frac{\kappa}{3} \rho^{(0)} \right)^2 - 2 \lambda^2 \gamma^2 \left( \frac{3}{2} (\sigma_Q^2)^{(0)} + \frac{\kappa}{3} \rho^{(0)} \right) \left( \frac{3}{2} \delta \sigma_Q^2 + \frac{\kappa}{3} \delta \rho \right).
\] (38)

It should be noticed that \(\delta \sigma\) and \(\delta \sigma_Q^2\) are not independent variables since they both depend on \(\delta h_i\). However, \(\delta \rho\) is independent of \(\delta \sigma\) and \(\delta \sigma_Q^2\). The left-hand side of the above equation does not depend on \(\delta \rho\), and since this equation has to be identically fulfilled in the classical limit, the pre-factor in front of \(\delta \rho\) on the right-hand side must vanish,
\[
\frac{3}{2} (\sigma_Q^2)^{(0)} + \frac{\kappa}{3} \rho^{(0)} = 0.
\] (39)

As \(\sigma_Q^2 \geq 0\) and \(\rho \geq 0\) at any time, the only solution is
\[
(\sigma_Q^2)^{(0)} = \rho^{(0)} = 0.
\] (40)

Combing the above equation with the definition of \(\sigma_Q^2\) in equation (23), we find:
\[
\cos \left( h_i^{(0)} - h_j^{(0)} \right) = 1,
\] (41)

which can be translated into
\[
h_2^{(0)} = h_1^{(0)} + n_2 2\pi, \quad n_2 \in \mathbb{Z},
\]
\[
h_3^{(0)} = h_1^{(0)} + n_3 2\pi, \quad n_3 \in \mathbb{Z}.
\] (42)

The other equation that has to be satisfied in the classical limit is equation (13). The left-hand side of equation (13) is calculated from \(H_i = \sum_j \frac{\partial}{\partial h_j} h_j\), where \(h_j\) is given by equation (29). The right-hand side is calculated by inserting expressions for \(H_j\) given by equation (21).

Equation (13) should be fulfilled for all \(\delta h_i \ll 1\), and therefore also for \(\delta h_i = 0\). Applying equation (42) and \(\delta h_i = 0\) to equation (13), we get
\[
\frac{\pi (n_2 + n_3)}{2\gamma^2 \lambda^2} \left[ 3 - \cos (2h_1^{(0)}) \right] \sin (2h_1^{(0)}) - \cos (2h_1^{(0)}) \frac{\kappa}{2} (\rho + P) = -\frac{\kappa}{2} (\rho + P).
\] (43)
Since this equation has to be identically fulfilled for all matter states, the pre-factor in front of \((\rho + P)\) has to be the same on both sides. Therefore \(\cos(2h_1^{(0)}) = 1\), which is equivalent to
\[
 h_1^{(0)} = n_1\pi, \quad n_1 \in \mathbb{Z}.
\] (44)
This also solves the rest of equation (43).

We also need to recover equation (13) for all \(\delta h_i \ll 1\), not equal to zero. Expanding equation (13) to first order in \(\delta h_i\) and using equations (42) and (44) we get
\[
 \frac{\pi}{\gamma^2 \lambda^2} [(n_2 + n_3)\delta h_1 + n_3 \delta h_2 + n_2 \delta h_3] = 0.
\] (45)
For this to be identically fulfilled for all \(\delta h_i \ll 1\), we must have \(n_2 = n_3 = 0\). Finally, we find that
\[
 h_1^{(0)} = h_2^{(0)} = h_3^{(0)} = n\pi, \quad n \in \mathbb{Z}.
\] (46)
In the classical limit, equation (21) becomes
\[
 H_i = \frac{\delta h_i}{\gamma \lambda} \ll \frac{1}{\gamma \lambda}
\] (47)
and
\[
 \sigma^2_Q = \sigma^2, \\
= \frac{1}{18\gamma^2 \lambda^2} [(h_1 - h_2)^2 + (h_2 - h_3)^2 + (h_3 - h_1)^2], \\
\ll \sigma^2_{Qc}.
\] (48)
We also have
\[
 \rho = \rho^{(0)} + \delta \rho = \delta \rho \ll \rho_c.
\] (49)
This means that if we are in the classical limit, the Hubble parameters, the shear and the energy density, are small compared to the scale of quantum effects. We want to remind the reader that \(\sigma^2_Q\) and \(\rho_c\) are of the order of Plank values, which are very large compared to anything expected during most of the evolution of the universe.

However, \(\sigma^2_Q \ll \sigma^2_{Qc}\) and \(\rho \ll \rho_c\) do not guarantee the classical behavior. This can be seen from the symmetries presented in section 6. A change of \(h_i\) belonging to the symmetry group equation (32) but not to equation (33) will give unchanged values of \(\sigma^2_Q\) and \(\rho\) but will change the dynamics away from the classical one. For example, the evolutions in figures 7 and 8 have low energy density through the whole simulation, and passes trough regions of low share and Hubble rates, but never behaves classically.

From the symmetry, equation (33), we can also conclude that all classical limits are equivalent within this framework.

It should be noticed that we have not assumed anything about the pressure. That means that any pressure is allowed in the classical limit.

Finally, it should be stressed that this analysis does not claim that the shear cannot be large when compared to the other terms in the classical limit of the Friedmann equations. Usual Bianchi-I can appear as the classical limit of quantum Bianchi-I. Rather, the shear and density have to be small when compared to their maximum allowed values.

8. Allowed regions in parameter space

Figure 1 displays the parameter space projected down on to \((h_2 - h_1, h_3 - h_1)\). In this projection, the space is devised into allowed and forbidden regions by the requirement \(\sigma^2_Q \ll \sigma^2_{Qc}\). The boundaries of those regions, e.g. when \(\sigma^2_Q = \sigma^2_{Qc}\), correspond to
Figure 1. $\sigma_\alpha^2$ as a function of $h_2 - h_1$ (x-axis) and $h_3 - h_1$ (y-axis). The white areas correspond to $\sigma_\alpha^2 > \sigma_\alpha^2$, which is forbidden by the modified Friedman equation (24). The black lines are $\sigma_\alpha^2 = \frac{1}{4}\sigma_\alpha^2, \frac{1}{2}\sigma_\alpha^2, \frac{3}{4}\sigma_\alpha^2, \sigma_\alpha^2$.

\[ h_i - h_j = (2m + 1)\pi, \quad i \neq j, \quad m \in \mathbb{Z}. \quad (50) \]

The pattern showed in figure 1 goes on infinitely in all directions, which means that there is an infinite number of allowed regions. But, from equation (46), one can see that there is only one point in this projection near which it is possible to recover the classical limit, and that is $(h_2 - h_1, h_3 - h_1) = (0, 0)$.

An interesting question one can ask is: is it possible, within this framework, to dynamically pass between allowed regions? The answer is no, as we shall show in this section.

The allowed regions are only connected by points, therefore any evolution between regions has to pass through these points, defined by:

\[
\begin{align*}
h_i - h_j &= (2m_1 + 1)\pi, & i \neq j, \quad m_1 \in \mathbb{Z} \quad (51) \\
h_i - h_j &= (2m_2 + 1)\pi', & i \neq j, \quad m_2 \in \mathbb{Z} \\
h_i - h_j &= (2m_3 + 1)\pi'', & i \neq j, \quad m_3 \in \mathbb{Z}.
\end{align*}
\]

Any point on the boundary of the allowed regions, including the points connecting regions can only be reached when $\rho = 0$. But even without matter dynamical transitions between regions are impossible. The argument is as follows,

\[
\rho = 0 \iff H_M = 0 \iff H_G = 0 \quad (52)
\]

which is equivalent to

\[
\sin h_1 \sin h_2 + \sin h_2 \sin h_3 + \sin h_3 \sin h_1 = 0. \quad (53)
\]

Combining the above expression with equations (51) gives

\[
0 = \sin h_1(- \sin h_1) + (\sin h_1) \sin h_1 + (- \sin h_1)(- \sin h_1) = - \sin^2 h_1. \quad (54)
\]

By once again using equations (51) with the above relation, one gets:

\[
\begin{align*}
h_i &= (m_3 - 1)\pi, \\
h_i &= (2m_1 + m_3 + 1)\pi, & i \neq j, \quad m_1, m_2, m_3 \in \mathbb{Z}. \\
h_i &= (2m_2 + m_3 + 1)\pi.
\end{align*}
\]
Inserting this into equation (29), we obtain $\dot{h} = 0$ in all the connection points. Therefore those points can never be dynamically reached. Transitions between the allowed regions displayed in figure 1 are not possible, even without matter.

Whatever the region chosen by initial conditions, the solution will stay in that region. In other words, there are infinitely many solutions that never reach a classical limit. However if we assume that the universe starts out in the classical limit of a contracting universe, then the correct region is picked up from the beginning and the evolution will end up in the classical limit of an expanding universe.

It is however meaningful to wonder what happened to all the solutions that live in regions without classical limits. We find a clue in equation (31). Since in all the non-classical regions there is a lower bound for at least two of the differences $h_i - h_j$, there must also be an upper bound on $a$. This leaves two possibilities, either the solution approaches a constant $a$ or the solution oscillates forever, leading to multiple bounces. Simulations favor the second hypothesis.

Equation (31) can be seen as an independent proof of the fact that there is no classical limit in regions not containing $h_i - h_j$ for all $i, j = 1, 2, 3$. Classically, $a$ is unbounded, and this is only possible if $h_i - h_j$ is allowed to be arbitrarily close to zero.

9. Numerical solutions

In this section we present some typical examples of numerically generated solutions, both with and without classical limit. In all simulations the matter is taken to be a single massive scalar field, $V(\phi) = m^2\phi^2/2$, $m = 10^{-3}$. The equations used in the simulations are equations (29) and the matter equation

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0,$$

with $H$ expressed as a function of $h_i$.

Figures 2–5 are all plots from the same numerical simulations with parameters in the region containing the classical limit. In figure 2, we see that, initially, all the directional scale factors are negative but, still in the classical region, one of them changes sign. After the bounce $H_1 \approx H_2 \approx H_3 \approx H$. This is because the matter caused a short inflation—as
can be seen more clearly in figure 3 which is a zoom around the bounce. Figure 4 is an even closer zoom. Here the quantum effects can be seen. The classical equations are a good approximation until $h_i/(\gamma \lambda)$ deviates from $H_i$. The classical equations become a good approximation again when $H_i \approx (h_i - \pi)/(\gamma \lambda)$. During the bounce all the $h_i$ are shifted by $\pi$ compared to $\gamma \lambda H_i$. Simulations suggest that this shift always occurs. This specific solution exhibits a shear-dominated bounce. This can be seen in figure 5 since $\sigma^2_Q \gg \frac{1}{2} \rho$ at the bounce.

Simulations show that $\sigma^2$ is typically not symmetric around the bounce. For solutions with a classical limit, it appears to be the case that $\sigma^2$ is symmetric around the bounce if and only
Figure 5. The full line is $\sigma_Q^2$, the dashed line is $\sigma^2$ and the dotted line is $\rho$, as a function of time.

Figure 6. Upper: a solution with maximally symmetric anisotropy. Lower: a solution with maximally asymmetric anisotropy. Left: the full lines are $\gamma H_i$, the dashed lines are $h_i$ and the dotted lines are $h_i + \pi$, as a function of time. Right: the full line is $\sigma_Q^2$, the dashed line is $\sigma^2$ and the dotted line is $\rho$, as a function of time.

if $h_i - h_j = h_j - h_k$ for some value of $i \neq j \neq k \neq i$. We therefore call this case maximally symmetric anisotropy. The opposite case is when $h_i = h_j \neq h_k$ for some $i \neq j \neq k \neq i$, and we call this maximally asymmetric anisotropy. Plots similar to figures 4 and 5, for the maximally symmetric and asymmetric cases are showed in figure 6.

Figures 7 and 8 are both plots from the same numerical simulations but with parameters in a region with no classical limit. One can see that the behavior is oscillatory and does not resemble anything classically expected.
Figure 7. The full line is the total Hubble factor $H$ and the tree dashed lines are the directional Hubble factors $H_i$, as a function of time.

Figure 8. $h_i$ as a function of time in a region without classical limit.

The simulation shown in figures 9–11 is generated by taking, as initial conditions, the values for $h_i$, $\phi$ and $\dot{\phi}$, as given by the solution shown in figure 2–5 at the bounce, with the only difference that $h_i \rightarrow h_i + 2\pi$ for $i = 1, 2, 3$ respectively.

Figures 9–11 all show oscillatory solutions with periods in the range 1–2 Plank times. These simulations clearly illustrate that equation (32) is not a symmetry of the full system. There are some similarities between figures 4 and 9, just around the bounce but the time scale is different by about a factor 4. The solutions in figures 10 and 11 are very different from anything classical.

10. Discussion

The results presented in this paper raise an important question for loop quantum cosmology. If the initial conditions are to be put at the bounce, as advocated e.g. in [8], we face a delicate problem: there are infinitely many more cases leading to universes that do not resemble ours than cases leading to a classically expanding universe. On the other hand, if we set the initial
Figure 9. This solution is generated by taking the initial conditions at the bounce with the same values as the solution in figure 2–5 but adding $2\pi$ to $h_1$. The full line are $\gamma \lambda H_i$, the dashed lines are $h_i$ and the dotted line is $h_1 - 2\pi$.

Figure 10. This solution is generated by taking the initial conditions at the bounce with the same values as the solution in figure 2–5 but adding $2\pi$ to $h_2$. The full line are $\gamma \lambda H_i$, the dashed lines are $h_i$ and the dotted line is $h_2 - 2\pi$.

conditions in the classically contracting phase, as advocated in [9], we escape this problem. But we face another one: what is the ‘natural’ initial shear? Or, according to which measure—and at which time—should we assume a flat probability distribution function for variables quantifying the shear? In any case, this requires a deep rethinking of the initial conditions problem.
However, we do not yet know what is the physical meaning of the solutions without classical limit. To understand this better the results presented here, should be compared with, e.g., results found when quantizing this system.

It may also be the case that transitions between different regions in figure 1 are possible when including a non-zero curvature.

This work should also be extended so as to generalize the results presented in [9]: how will the prediction of the duration of inflation be modified by including anisotropies? This question has been partly addressed already in [6], however, only for a very narrow range of initial conditions.

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Appendix. Derivation of the modified generalized Friedmann equation

We define:

\[ s_+ = \frac{1}{3} \left[ \sin(h_1 + h_2) + \sin(h_2 + h_3) + \sin(h_3 + h_1) \right], \]
\[ c_\pm = \frac{1}{3} \left[ \cos(h_1 \pm h_2) + \cos(h_2 \pm h_3) + \cos(h_3 \pm h_1) \right]. \]

(A.1) \hspace{1cm} \text{(A.2)}

The average Hubble parameter can now be written as:

\[ H = \frac{s_+}{2 \gamma \lambda}. \]

(A.3)

By using elementary trigonometric relations \( \sin(a) \sin(b) = (\cos(a - b) - \cos(a + b))/2 \) and \( \cos(a) \cos(b) = (\cos(a - b) + \cos(a + b))/2 \), we find:
\[ s_+^2 + c_+^2 = \frac{1 + 2c_-}{3}. \]  

(A.4)

and

\[ \mathcal{H}_G = \frac{3\sqrt{p_1p_2p_3}}{2\kappa\gamma^2\lambda^2}(c_+ - c_-). \]  

(A.5)

The constraint \( \mathcal{H}_G + \mathcal{H}_M = 0 \) then becomes

\[ c_+ - c_- = -\frac{2\gamma^2\lambda^2\kappa\rho}{3}. \]  

(A.6)

One can now use equations (A.4) and (A.6) to rewrite \( H^2 \) as a function of \( c_- \). It will turn out to be useful to expand this expression in terms of \( (1 - c_-) \). We re-express equations (A.4) and (A.6) as

\[ s_+^2 = 1 - \frac{2}{3}(1 - c_-) - c_+^2, \]  

(A.7)

\[ c_+ = 1 - \left[(1 - c_-) + 2\gamma^2\lambda^2\frac{\kappa\rho}{3}\right]. \]  

(A.8)

This allows us to write:

\[ H^2 = \frac{s_+^2}{4\gamma^2\lambda^2} = \frac{1}{4\gamma^2\lambda^2}\left(1 - \frac{2}{3}(1 - c_-) - c_+^2\right) \]  

\[ = \frac{1}{4\gamma^2\lambda^2}\left(-\frac{2}{3}(1 - c_-) + 2\left((1 - c_-) + 2\gamma^2\lambda^2\frac{\kappa\rho}{3}\right)\right) \]  

\[ - \left[(1 - c_-) + 2\gamma^2\lambda^2\frac{\kappa\rho}{3}\right]^2 \]  

\[ = \frac{1 - c_-}{3\gamma^2\lambda^2} + \frac{\kappa\rho}{3} - \gamma^2\lambda^2\left(\frac{1 - c_-}{2\gamma^2\lambda^2} + \frac{\kappa\rho}{3}\right)^2, \]  

(A.9)

where we have used equation (A.7) for the second equality, and equation (A.8) for the third equality.

It can now be seen that, to first order, \( (1 - c_-)/(3\gamma^2\lambda^2) \) appears just like the shear in the classical equation (7). It can therefore be labeled the quantum shear

\[ \sigma^2_Q := \frac{1 - c_-}{3\gamma^2\lambda^2}, \]  

(A.10)

which is exactly equation (23). Re-inserting this definition into equation (A.9), we find exactly equation (24).

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