Abstract. A pleasant family of graphs defined by Godsil and McKay is shown to have easily computed eigenvalues in many cases.

Let $G$ and $H$ be directed graphs on the respective vertices $U$ and $V$, and suppose that the vertex sets have each been partitioned into disjoint subsets $U = U_0 \cup U_1$ and $V = V_0 \cup V_1$. The partitioned tensor product $G \times H$ of $G$ and $H$ with respect to this partitioning is defined as follows:

a) Each vertex of $U_0$ is replaced by a copy of $H|V_0$, the subgraph of $H$ induced by $V_0$;
b) Each vertex of $U_1$ is replaced by a copy of $H|V_1$;
c) Each arc of $G$ that runs from $U_0$ to $U_1$ is replaced by a copy of the arcs of $H$ that run from $V_0$ to $V_1$;
d) Each arc of $G$ that runs from $U_1$ to $U_0$ is replaced by a copy of the arcs of $H$ that run from $V_1$ to $V_0$.

Figure 1. Partitioned tensor products, directed and undirected.
For example, Figure 1 shows two partitioned tensor products. The example in Figure 1b is undirected; this is the special case of a directed graph where each undirected edge corresponds to a pair of arcs in opposite directions. Arcs of \( G \) that stay within \( U_0 \) or \( U_1 \) do not contribute to \( G \times H \), so we may assume that no such arcs exist (i.e., that \( G \) is bipartite).

Figure 2 shows what happens if we interchange the roles of \( U_0 \) and \( U_1 \) in \( G \) but leave everything else intact. (Equivalently, we could interchange the roles of \( V_0 \) and \( V_1 \).) These graphs, which may be denoted \( G^R \times H \) to distinguish them from the graphs \( G \times H \) in Figure 1, might look quite different from their left-right duals, yet it turns out that the characteristic polynomials of \( G \times H \) and \( G^R \times H \) are strongly related.

\[ G^R \times H = \]

![Diagram](attachment:image.png)

Figure 2. Dual products after right-left reflection of \( G \).

Let \( E_{ij} \) be the arcs from \( U_i \) to \( U_j \) in \( G \), and \( F_{ij} \) the arcs from \( V_i \) to \( V_j \) in \( H \); multiple arcs are allowed, so \( E_{ij} \) and \( F_{ij} \) are multisets. It follows that \( G \times H \) has \( |U_0||V_0| + |U_1||V_1| \) vertices and \( |U_0||F_{00}| + |U_1||F_{11}| + |E_{01}| ||F_{01}| + |E_{10}| ||F_{10}| \) arcs. Similarly, \( G^R \times H \) has \( |U_1||V_0| + |U_0||V_1| \) vertices and \( |U_1||F_{00}| + |U_0||F_{11}| + |E_{10}| ||F_{01}| + |E_{01}| ||F_{10}| \) arcs.

The definition of partitioned tensor product is due to Godsil and McKay [3], who proved the remarkable fact that

\[
p(G \times H) p(H|V_0)^{|U_1| - |U_0|} = p(G^T \times H) p(H|V_1)^{|U_1| - |U_0|},
\]

where \( p \) denotes the characteristic polynomial of a graph. They also observed [4] that Figures 1b and 2b represent the smallest pair of connected undirected graphs having the same spectrum (the same \( p \)). The purpose of the present note is to refine their results by showing how to calculate \( p(G \times H) \) explicitly in terms of \( G \) and \( H \).

We can use the symbols \( G \) and \( H \) to stand for the adjacency matrices as well as for the graphs themselves. Thus we have

\[
G = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix}
\]
in partitioned form, where $G_{ij}$ and $H_{ij}$ denote the respective adjacency matrices corresponding to the arcs $E_{ij}$ and $F_{ij}$. (These submatrices are not necessarily square; $G_{ij}$ has size $|U_i| \times |U_j|$ and $H_{ij}$ has size $|V_i| \times |V_j|$.) It follows by definition that

$$G \times H = \begin{pmatrix} I_{|U_0|} \otimes H_{00} & G_{01} \otimes H_{01} \\ G_{10} \otimes H_{10} & I_{|U_1|} \otimes H_{11} \end{pmatrix}$$

where $\otimes$ denotes the Kronecker product or tensor product [7, page 8] and $I_k$ denotes an identity matrix of size $k \times k$.

Let $H \uparrow \sigma$ denote the graph obtained from $H$ by $\sigma$-fold repetition of each arc that joins $V_0$ to $V_1$. In matrix form

$$H \uparrow \sigma = \begin{pmatrix} H_{00} & \sigma H_{01} \\ \sigma H_{10} & H_{11} \end{pmatrix}.$$  

This definition applies to the adjacency matrix when $\sigma$ is any complex number, but of course $H \uparrow \sigma$ is difficult to “draw” unless $\sigma$ is a nonnegative integer. We will show that the characteristic polynomial of $G \times H$ factors into characteristic polynomials of graphs $H \uparrow \sigma$, times a power of the characteristic polynomials of $H_{00}$ or $H_{11}$. The proof is simplest when $G$ is undirected.

**Theorem 1.** Let $G$ be an undirected graph, and let $(\sigma_1, \ldots, \sigma_l)$ be the singular values of $G_{01} = G_{10}^T$, where $l = \min(|U_0|, |U_1|)$. Then

$$p(G \times H) = \begin{cases} \prod_{j=1}^l p(H \uparrow \sigma_j) p(H_{00})^{(|U_0| - |U_1|)} & \text{if } |U_0| \geq |U_1|; \\
\prod_{j=1}^l p(H \uparrow \sigma_j) p(H_{11})^{(|U_1| - |U_0|)} & \text{if } |U_1| \geq |U_0|. \end{cases}$$

**Proof.** Any real $m \times n$ matrix $A$ has a singular value decomposition

$$A = QSR^T$$

where $Q$ is an $m \times m$ orthogonal matrix, $R$ is an $n \times n$ orthogonal matrix, and $S$ is an $m \times n$ matrix with $S_{ij} = \sigma_j \geq 0$ for $1 \leq j \leq \min(m, n)$ and $S_{ij} = 0$ for $i \neq j$ [6, page 16]. The numbers $\sigma_1, \ldots, \sigma_{\min(m,n)}$ are called the singular values of $A$.

Let $m = |U_0|$ and $n = |U_1|$, and suppose that $QSR^T$ is the singular value decomposition of $G_{01}$. Then $(\sigma_1, \ldots, \sigma_l)$ are the nonnegative eigenvalues of the bipartite graph $G$, and we have

$$\begin{pmatrix} Q^T \otimes I_{|V_0|} & O \\ O & R^T \otimes I_{|V_1|} \end{pmatrix} G \times H \begin{pmatrix} Q \otimes I_{|V_0|} & O \\ O & R \otimes I_{|V_1|} \end{pmatrix} = \begin{pmatrix} I_{|U_0|} \otimes H_{00} & S \otimes H_{01} \\ S^T \otimes H_{10} & I_{|U_1|} \otimes H_{11} \end{pmatrix}$$

because $G_{10} = RS^TQ^T$. Row and column permutations of this matrix transform it into the block diagonal form

$$\begin{pmatrix} H \uparrow \sigma_1 \\ \vdots \\ H \uparrow \sigma_l \\ \begin{pmatrix} \ldots & \ldots \\ & D \end{pmatrix} \end{pmatrix},$$

where $D$ consists of $m - n$ copies of $H_{00}$ if $m \geq n$, or $n - m$ copies of $H_{11}$ if $n \geq m$. □

A similar result holds when $G$ is directed, but we cannot use the singular value decomposition because the eigenvalues of $G$ might not be real and the elementary divisors of $\lambda I - G$ might not be linear. The following lemma can be used in place of the singular value decomposition in such cases.
Lemma. Let $A$ and $B$ be arbitrary matrices of complex numbers, where $A$ is $m \times n$ and $B$ is $n \times m$. Then we can write

$$A = QSR^{-1}, \quad B = RTQ^{-1},$$

where $Q$ is a nonsingular $m \times m$ matrix, $R$ is a nonsingular $n \times n$ matrix, $S$ is an $m \times n$ matrix, $T$ is an $n \times m$ matrix, and the matrices $(S,T)$ are triangular with consistent diagonals:

$$S_{ij} = T_{ij} = 0 \quad \text{for} \ i > j;$$
$$S_{jj} = T_{jj} \quad \text{or} \quad S_{jj}T_{jj} = 0, \quad \text{for} \ 1 \leq j \leq \min(m,n).$$

Proof. We may assume that $m \leq n$. If $AB$ has a nonzero eigenvalue $\lambda$, let $\sigma$ be any square root of $\lambda$ and let $x$ be a nonzero $m$-vector such that $ABx = \sigma^2 x$. Then the $n$-vector $y = Bx/\sigma$ is nonzero, and we have

$$Ay = \sigma x, \quad Bx = \sigma y.$$ 

On the other hand, if all eigenvalues of $AB$ are zero, let $x$ be a nonzero vector such that $ABx = 0$. Then if $Bx \neq 0$, let $y = Bx$. If $Bx = 0$, let $y$ be any nonzero vector such that $Ay = 0$; this is possible unless all $n$ columns of $A$ are linearly independent, in which case we must have $m = n$ and we can find $y$ such that $Ay = x$. In all cases we have therefore demonstrated the existence of nonzero vectors $x$ and $y$ such that

$$Ay = \sigma x, \quad Bx = \tau y, \quad \sigma = \tau \quad \text{or} \quad \sigma \tau = 0.$$ 

Let $X$ be a nonsingular $m \times m$ matrix whose first column is $x$, and let $Y$ be a nonsingular $n \times n$ matrix whose first column is $y$. Then

$$X^{-1}AY = \begin{pmatrix} \sigma & a \\ 0 & A_1 \end{pmatrix}, \quad Y^{-1}BX = \begin{pmatrix} \tau & b \\ 0 & B_1 \end{pmatrix}$$

where $A_1$ is $(m-1) \times (n-1)$ and $B_1$ is $(n-1) \times (m-1)$. If $m = 1$, let $Q = X$, $R = Y$, $S = (\sigma a)$, and $T = (\sigma b)$. Otherwise we have $A_1 = Q_1 S_1 R_1^{-1}$ and $B_1 = R_1 T_1 Q_1^{-1}$ by induction, and we can let

$$Q = X \begin{pmatrix} 1 & 0 \\ 0 & Q_1 \end{pmatrix}, \quad R = Y \begin{pmatrix} 1 & 0 \\ 0 & R_1 \end{pmatrix}, \quad S = \begin{pmatrix} \sigma & aR_1 \\ 0 & S_1 \end{pmatrix}, \quad T = \begin{pmatrix} \tau & BQ_1 \\ 0 & T_1 \end{pmatrix}.$$ 

All conditions are now fulfilled. \qed

Theorem 2. Let $G$ be an arbitrary graph, and let $(\sigma_1, \ldots, \sigma_l)$ be such that $\sigma_j = S_{jj} = T_{jj}$ or $\sigma_j = 0 = S_{jj}T_{jj}$ when $G_{01} = QSR^{-1}$ and $G_{10} = RTQ^{-1}$ as in the lemma, where $l = \min(|V_0|, |V_1|)$. Then $p(G H)$ satisfies the identities of Theorem 1.

Proof. Proceeding as in the proof of Theorem 1, we have

$$\left( Q^{-1} \otimes I_{|V_0|} \quad O \right) \left( O \quad R^{-1} \otimes I_{|V_1|} \right) G \times H \left( O \quad R \otimes I_{|V_1|} \right) = \left( I_{|V_0|} \otimes H_{00} \quad O \right) \left( T \otimes H_{10} \quad S \otimes H_{01} \right) \left( O \quad I_{|V_1|} \otimes H_{11} \right).$$
This time a row and column permutation converts the right-hand matrix to a block triangular form, with zeroes below the diagonal blocks. Each block on the diagonal is either \( H \uparrow \sigma_j \) or \( H_{00} \) or \( H_{11} \), or of the form

\[
\begin{pmatrix}
H_{00} & \sigma H_{01} \\
\tau H_{10} & H_{11}
\end{pmatrix}, \quad \sigma \tau = 0.
\]

In the latter case the characteristic polynomial is clearly \( p(H_{00})p(H_{11}) = p(H \uparrow 0) \), so the remainder of the proof of Theorem 1 carries over in general. \( \square \)

The proof of the lemma shows that the numbers \( \sigma_1^2, \ldots, \sigma_p^2 \) are the characteristic roots of \( G_{01}G_{10} \), when \( |U_0| \leq |U_1| \), otherwise they are the characteristic roots of \( G_{10}G_{01} \). Either square root of \( \sigma_j^2 \) can be chosen, since the matrix \( H \uparrow \sigma \) is similar to \( H \uparrow (-\sigma) \).

We have now reduced the problem of computing \( p(G \times H) \) to the problem of computing the characteristic polynomial of the graphs \( H \uparrow \sigma \). The latter is easy when \( \sigma = 0 \), and some graphs \( G \) have only a few nonzero singular values. For example, if \( G \) is the complete bipartite graph having parts \( U_0 \) and \( U_1 \) of sizes \( m \) and \( n \), all singular values vanish except for \( \sigma = \sqrt{mn} \).

If \( H \) is small, and if only a few nonzero \( \sigma \) need to be considered, the computation of \( p(H \uparrow \sigma) \) can be carried out directly. For example, it turns out that

\[
\begin{pmatrix}
\lambda & -1 & -\sigma & 0 & 0 \\
-1 & \lambda & 0 & 0 & -\sigma \\
-\sigma & 0 & \lambda & -1 & 0 \\
0 & 0 & -1 & \lambda & -1 \\
0 & -\sigma & 0 & -1 & \lambda
\end{pmatrix} = (\lambda^2 + \lambda - \sigma^2)(\lambda^3 - \lambda^2 - (2 + \sigma^2)\lambda + 2);
\]

so we can compute the spectrum of \( G \times H \) by solving a few quadratic and cubic equations, when \( H \) is this particular 5-vertex graph (a partitioned 5-cycle). But it is interesting to look for large families of graphs for which simple formulas yield \( p(H \uparrow \sigma) \) as a function of \( \sigma \).

One such family consists of graphs that have only one edge crossing the partition. Let \( H_{00} \) and \( H_{11} \) be graphs on \( V_0 \) and \( V_1 \), and form the graph \( H = H_{00} \uparrow H_{11} \) by adding a single edge between designated vertices \( x_0 \in V_0 \) and \( x_1 \in V_1 \). Then a glance at the adjacency matrix of \( H \) shows that

\[
p(H \uparrow \sigma) = p(H_{00})p(H_{11}) - \sigma^2 p(H_{00}|V_0 \setminus x_0)p(H_{11}|V_1 \setminus x_1).
\]

(The special case \( \sigma = 1 \) of this formula is Theorem 4.2(ii) of [5].)

Another case where \( p(H \uparrow \sigma) \) has a simple form arises when the matrices

\[
H_0 = \begin{pmatrix} H_{00} & 0 \\ 0 & H_{11} \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 0 & H_{01} \\ H_{10} & 0 \end{pmatrix}
\]

commute with each other. Then it is well known [2] that the eigenvalues of \( H_0 + \sigma H_1 \) are \( \lambda_j + \sigma \mu_j \), for some ordering of the eigenvalues \( \lambda_j \) of \( H_0 \) and \( \mu_j \) of \( H_1 \). Let us say that \( (V_0, V_1) \) is a compatible partition of \( H \) if \( H_0H_1 = H_1H_0 \), i.e., if

\[
H_{00}H_{01} = H_{01}H_{11} \quad \text{and} \quad H_{11}H_{10} = H_{10}H_{00}.
\]

When \( H \) is undirected, so that \( H_{00} = H_{00}^T \) and \( H_{11} = H_{11}^T \) and \( H_{10} = H_{01}^T \), the compatibility condition boils down to the single relation

\[
H_{00}H_{01} = H_{01}H_{11}.
\]
Let \( m = |V_0| \) and \( n = |V_1| \), so that \( H_{00} \) is \( m \times m \), \( H_{01} \) is \( m \times n \), and \( H_{11} \) is \( n \times n \). One obvious way to satisfy (\( \ast \)) is to let \( H_{00} \) and \( H_{11} \) both be zero, so that \( H \) is bipartite as well as \( G \). Then \( H \uparrow \sigma \) is simply \( \sigma H \), the \( \sigma \)-fold repetition of the arcs of \( H \), and its eigenvalues are just those of \( H \) multiplied by \( \sigma \). For example, if \( G \) is the \( M \)-cube \( P_M^2 \) and \( H \) is a path \( P_N \) on \( N \) points, and if \( U_0 \) consists of the vertices of even parity in \( G \) while \( V_0 \) is one of \( H \)’s bipartite parts, the characteristic polynomial of \( G \times H \) is

\[
\prod_{1 \leq j \leq M} \prod_{1 \leq k \leq N} \left( \lambda - (2N - 4j) \cos \frac{k\pi}{N+1} \right)^{(M/2)},
\]

because of the well-known eigenvalues of \( G \) and \( H \) [1]. Figure 3 illustrates this construction in the special case \( M = N = 3 \). The smallest pair of cospectral graphs, \( \bigcirc \) and \( \square \), is obtained in a similar way by considering the eigenvalues of \( P_3 \times P_3 \) and \( P_T^3 \times P_3 \) [4].

Another simple way to satisfy the compatibility condition (\( \ast \)) with symmetric matrices \( H_{00} \) and \( H_{11} \) is to let \( H_{01} \) consist entirely of 1s, and to let \( H_{00} \) and \( H_{11} \) both be regular graphs of the same degree \( d \). Then the eigenvalues of \( H_0 \) are \( (\lambda_1, \ldots, \lambda_m, \lambda'_1, \ldots, \lambda'_n) \), where \( (\lambda_1, \ldots, \lambda_m) \) belong to \( H_{00} \) and \( (\lambda'_2, \ldots, \lambda'_n) \) belong to \( H_{11} \) and \( \lambda_1 = \lambda'_1 = d \). The eigenvalues of \( H_1 \) are \((\sqrt{mn}, -\sqrt{mn}, 0, \ldots, 0)\). We can match the eigenvalues of \( H_0 \) properly with those of \( H_1 \) by looking at the common eigenvectors \((1, \ldots, 1)^T\) and \((1, \ldots, 1, -1, \ldots, -1)^T\) that correspond to \( d \) in \( H_0 \) and \( \pm \sqrt{mn} \) in \( H_1 \); the eigenvalues of \( H \uparrow \sigma \) are therefore

\[
(d + \sigma \sqrt{mn}, \lambda_2, \ldots, \lambda_m, d - \sigma \sqrt{mn}, \lambda'_2, \ldots, \lambda'_n).
\]

Yet another easy way to satisfy (\( \ast \)) is to assume that \( m = n \) and to let \( H_{00} = H_{11} \) commute with \( H_{01} \). One general construction of this kind arises when the vertices of \( V_0 \) and \( V_1 \) are the elements of a group, and when \( H_{00} = H_{11} \) is a Cayley graph on that group. In other words, two elements \( \alpha \) and \( \beta \) are adjacent in \( H_{00} \) iff \( \alpha \beta^{-1} \in X \), where \( X \) is an arbitrary set of group elements closed under inverses. And we can let \( \alpha \in V_0 \) be adjacent to \( \beta \in V_1 \) iff \( \alpha \beta^{-1} \in Y \), where \( Y \) is any normal subgroup. Then \( H_{00} \) commutes with \( H_{01} \). The effect is to make the cosets of \( Y \) fully interconnected between \( V_0 \) and \( V_1 \), while retaining a more interesting Cayley graph structure inside \( V_0 \) and \( V_1 \). If \( Y \) is the trivial subgroup, so that \( H_{01} \) is simply the identity
matrix, our partitioned tensor product $G \times H$ becomes simply the ordinary Cartesian product $G \oplus H = I_{|U|} \otimes H + G \otimes I_{|V|}$. But in many other cases this construction gives something more general.

A fourth family of compatible partitions is illustrated by the following graph $H$ in which $m = 6$ and $n = 12$:

$$
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}
$$

In general, let $C_{2k}$ be the matrix of a cyclic permutation on $2k$ elements, and let $m = 2k$, $n = 4k$. Then we obtain a compatible partition if

$$
H_{00} = (C_{2k}^j + C_{2k}^{-j} + C_{2k}^{-j}), \quad H_{01} = (I_{2k} \ C_{2k}), \quad H_{11} = \begin{pmatrix}
C_{2k}^j + C_{2k}^{-j} & C_{2k}^{k+1} \\
C_{2k}^{k-1} & C_{2k}^j + C_{2k}^{-j}
\end{pmatrix}.
$$

The $18 \times 18$ example matrix is the special case $j = 2$, $k = 3$. The eigenvalues of $H \uparrow \sigma$ in general are

$$
\omega^{jl} + \omega^{-jl} + 1, \quad \omega^{jl} + \omega^{-jl} - 1 + \sqrt{2} \sigma, \quad \omega^{jl} + \omega^{-jl} - 1 - \sqrt{2} \sigma
$$

for $0 \leq l < 2k$, where $\omega = e^{\pi i/k}$.

Compatible partitionings of digraphs are not difficult to construct. But it would be interesting to find further examples of undirected graphs, without multiple edges, that have a compatible partition.

References

[1] Dragoš M. Cvetković, Michael Doob, and Horst Sachs, *Spectra of Graphs* (New York: Academic Press, 1980).
[2] G. Frobenius, “Über vertauschbare Matrizen,” *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* (1896), 601–614. Reprinted in his *Gesammelte Abhandlungen 2* (Berlin: Springer, 1968), 705–718.

[3] C. Godsil and B. McKay, “Products of graphs and their spectra,” in *Combinatorial Mathematics IV*, edited by A. Dold and B. Eckmann, *Lecture Notes in Mathematics* 560 (1975), 61–72.

[4] C. Godsil and B. McKay, “Some computational results on the spectra of graphs,” in *Combinatorial Mathematics IV*, edited by A. Dold and B. Eckmann, *Lecture Notes in Mathematics* 560 (1975), 73–82.

[5] C. D. Godsil and B. D. McKay, “Constructing cospectral graphs,” *Æquationes Mathematicae* 25 (1982), 257–268.

[6] Gene H. Golub and Charles F. Van Loan, *Matrix Computations* (Baltimore: Johns Hopkins University Press, 1983).

[7] Marvin Marcus and Henrik Minc, *A Survey of Matrix Theory and Matrix Inequalities* (Boston: Allyn and Bacon, 1964).