GEOMETRIC EXPANSION, LYAPUNOV EXPONENTS AND
FOLIATIONS

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Abstract. We consider hyperbolic and partially hyperbolic diffeomorphisms
on compact manifolds. Associated with invariant foliation of these systems,
we define some topological invariants and show certain relationships between
these topological invariants and the geometric and Lyapunov growths of these
foliations. As an application, we show examples of systems with persistent non-
absolute continuous center and weak unstable foliations. This generalizes the
remarkable results of Shub and Wilkinson to cases where the center manifolds
are not compact.

1. Introduction

Let $M$ be an $n$-dimensional compact Riemannian manifold. Let $f \in \text{Diff}^r(M)$
be a $C^r$ diffeomorphism on $M$, $r \geq 1$. We will also consider volume-preserving
diffeomorphisms in our examples. Let $W$ be a $k$ dimensional foliation of $M$ with
$C^1$ leaves. We say that the foliation is invariant under $f$, if $f$ maps leaves of $W$
to leaves. We first define volume growth of $f$ on leaves. We will also assume that
the leaves are orientable. For any $x \in M$, let $W(x)$ be the leaf through $x$ and let
$W_r(x)$ be the $k$ dimensional disk on $W(x)$ centered at $x$, with radius $r$.

For most of the paper we will assume that the leaves of $W$ have uniform expo-
nential growth under the iterates of $f$, i.e., there are constants $\lambda > 1$ and $C > 0$
such that

$$\|df^n v\| \geq C\lambda^n \|v\|$$

for all $x \in M$, all $v \in T_x W(x)$ and all $n \in \mathbb{N}$, where $W(x)$ is the leaf of $W$
through the point $x$.

Examples of these expanding foliations can be found in hyperbolic and partially
hyperbolic diffeomorphisms. A map $f \in \text{Diff}^r(M)$ is said to be partially hyperbolic
if there is an invariant splitting of the tangent bundle of $M$, $TM = E^u \oplus E^c \oplus E^s$,
with at least two of them nontrivial, and there exist $\alpha > \alpha' > 1$, $\beta > \beta' > 1$ and
$C > 0$, $D > 0$, $C' > 0$, $D' > 0$ such that

1. $E^u$ is uniformly expanding:

$$\|Df^k(v_u)\| \geq C\alpha^k \|v_u\|, \forall v_u \in E^u, k \in \mathbb{N},$$

2. $E^s$ is uniformly contracting:

$$\|Df^k(v_s)\| \leq D\beta^{-k} \|v_s\|, \forall v_s \in E^s, k \in \mathbb{N},$$

3. $E^u$ dominates $E^c$, and $E^c$ dominates $E^s$:

$$D'(\beta')^{-k} \|v_c\| \leq \|Df^k(v_c)\| \leq C'(\alpha')^k \|v_c\|, \forall v_c \in E^c, k \in \mathbb{N}.$$
We remark that in some papers it is allowed that the bounds for the expansion rates depend on the points.

The unstable distribution $E^u$ is integrable and it integrates to the unstable foliation. The unstable foliation of a hyperbolic or partially hyperbolic diffeomorphism is certainly uniformly expanding. Likewise, the stable manifold of a hyperbolic and partially hyperbolic diffeomorphism is uniformly expanding for $f^{-1}$.

The growth rate of a uniformly expanding foliation can be measured in several different ways. We first define the geometric growth rate. This is related to the volumes growth used by Yomdin and Newhouse for the study of entropy of diffeomorphisms (see [Yo], [Ne]). The difference is that we consider only $k$-dimensional disks on the leaves of the foliation. Let

$$\chi(x, r) = \limsup_n \frac{1}{n} \ln \text{Vol}(f^n(W_r(x)))$$

$\chi(x, r)$ measures the volume growth of $W$ at $x$. Let

$$\chi = \chi(r) = \sup_{x \in M} \chi(x, r)$$

Then, $\chi$ is the maximum volume growth rate of the foliation $W$ under $f$. Obviously, the growth $\chi$ is independent of $r$. $\chi$ is also independent of the Riemannian metric on $M$.

The geometric growth is hard to compute and its dependence on the points and on the map itself is not very clear. We will define a topological growth rate for the foliation. This will depend on the homology that the invariant foliation carries and the action induced by $f$ on the homology. Typically this topological growth will be much easier to compute and it is a local constant for maps in $\text{Diff}^r(M)$. It turns out, as we will show, the geometric growth $\chi(x, r)$ and topological growth are the same for foliations carrying certain homological information. As a consequence, $\chi(x, r)$ is independent of $x$ and $r$ and remains the same under small perturbations.

We will define this homological invariant using De Rahm currents.

The third type of growth rate for an invariant foliation is measured by the Lyapunov exponents in the tangent spaces of the leaves of the foliations. The Lyapunov exponents are positive in the leaves of an expanding foliation. We can integrate, over the manifold, the sum of all Lyapunov exponents in the leaves and we call this integral the Lyapunov growth. We will show that, if the foliation is absolutely continuous, the Lyapunov growth is smaller than the geometric growth. As a consequence, if the Lyapunov growth is larger than the geometric growth, then the foliation must be singular.

Shub and Wilkinson showed some remarkable examples where the center foliations, whose leaves are circles, persistently fail to be absolutely continuous in some partially hyperbolic volume preserving diffeomorphisms (see [SW]). Moreover, every center leaf intersects a full measure set in a set of measure zero. They call these types of foliations pathological. Using our results, we will give examples of persistent pathological foliations with non-compact center leaves.

In section 2 we define the topological growth of an one dimensional foliation and we show how to relate it to the volume growth in some situations. In section 3 we generalize this results to higher dimensional foliations. In section 4 we talk about the Lyapunov growth and relate it to the volume growth in the case of absolutely continuous foliations. Section 5 contains the examples of non-absolutely continuous foliations.
As another application of the homological invariants we defined here, Hua, Saghin and Xia proved certain continuity properties of topological entropy for partially hyperbolic diffeomorphisms, see [HSX].

2. ONE DIMENSIONAL FOLIATIONS

We start with some simple cases where the leaves of the foliation are one dimensional. This case is geometrically more intuitive and it motivates the more general case treated in the next section. The basic idea goes back to Schwartzmann’s asymptotic cycles (see [Sc]).

For any point \( x \in M \), let \( W_r(x) \) be a ball of radius \( r \) in \( W(x) \) centered at \( x \). \( W_r(x) \) inherits the orientation from \( W(x) \). For any positive integer \( i \), let \( l_i \) be the closed loop formed by adding to the line segment \( f^i(W_r(x)) \) an oriented curve \( b_i \) joining the two end points. The choice of curve, \( b_i \), joining the two end points is not unique, but its length can be uniformly bounded by the diameter of the manifold. Each \( l_i \) defines a first homology class in the manifold, \( [l_i] \in H_1(M, \mathbb{R}) \). For convenience, we also assume that \( H_1(M, \mathbb{R}) \) is non-trivial (which is the case for the interesting examples in our situation), and that, by properly choosing \( b_i \), \( [l_i] \neq 0 \). Let \( |l_i| \) be the length of the closed curve \( l_i \) with respect to the fixed Riemannian metric in \( M \). Now we consider the sequence of first homologies \( [l_i]/|l_i| \in H_1(M, \mathbb{R}) \), \( i = 1, 2, \ldots \). This sequence is uniformly bounded and hence has a convergent subsequences. We assume that the foliation is uniformly expanded, so \( \lim_{n \to \infty} |l_i| = \infty \), which means that the limits are independent on the choice of \( b_i \).

**Definition 2.1.** We say that the invariant foliation \( W \) carries a non-trivial homology \( h_W \in H_1(M, \mathbb{R}) \) if there are \( x \in M \), \( r > 0 \) and a subsequence \( n_i \to \infty \) such that

\[
\lim_{i \to \infty} \frac{[l_{n_i}]}{|l_{n_i}|} = h_W \in H_1(M, \mathbb{R}).
\]

We say that the invariant foliation \( W \) carries a unique non-trivial homology \( h_W \in H_1(M, \mathbb{R}) \) if \( h_W \) defined above is unique up to rescaling. Or more precisely, there are constants \( 0 < c_1 < c_2 \) and a unit vector \( h \in H_1(M, \mathbb{R}) \) such that the set of all homologies carried by the \( W \) is a subset of \( \{ch \in H_1(M, \mathbb{R}) \mid c_1 \leq c \leq c_2 \} \).

We remark that the homologies carried by an invariant foliation depend on the Riemannian metric we choose. The normalized homology vectors carried by the foliation are independent of the choice of the metric, but they depend of course on the choice of the norm on \( H_1(M, \mathbb{R}) \) (there is no natural way to choose the size of the homology of a foliation). By definition, if the foliation \( W \) carries a unique homology then the following limit exists

\[
\lim_{i \to \infty} \frac{[l_i]}{|[l_i]|} = h_W
\]

for some unit vector \( h_W \in H_1(M, \mathbb{R}) \).

We remark that if a foliation \( W \) carries no non-trivial homology, then for any \( x \in M \) and any \( r > 0 \), \( \lim_{i \to \infty} |l_i|/|l_i| = 0 \).

We have the following lemma.

**Lemma 2.2.** Let \( W \) be an one dimensional invariant foliation.
(1) Suppose $W$ carries some non-trivial homology and let $H \subset H_1(M, \mathbb{R})$ be the set of homologies carried by $W$. Then $H$ spans an invariant subset for $f_*: H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$.

(2) If $W$ carries a unique non-trivial homology $h_W \in H_1(M, \mathbb{R})$, then $h_W$ is an eigenvector of the induced map $f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$.

Proof. Let $W$ be an one dimensional invariant expanding foliation. Let $h \in H$ be a nontrivial homology carried by $W$, then there is a subsequence $n_i \to \infty$ such that

$$\lim_{i \to \infty} \frac{|l_{n_i}|}{|l_{n_i}|} = h$$

where we used the fact that each curve $b_i$, which joins the end points of $f^i(W_r(x))$ to form closed loops, is uniformly bounded and $|l_i| \to \infty$ as $i \to \infty$.

The sequence $\left\{ \frac{|l_{n_i}|}{|l_{n_i}|} \right\}_{i=1}^\infty$ is uniformly bounded both from above and away from zero, it has a convergent subsequence. Without loss of generality, we may assume that the sequence actually converges to a nonzero constant $\lambda^{-1} \neq 0$. i.e.,

$$\lim_{i \to \infty} \frac{|l_{n_i}|}{|l_{n_i}|} = \lambda^{-1}.$$  

Finally, we have

$$\lim_{i \to \infty} \frac{|l_{n_i}|}{|l_{n_i}|} = f_* h / \lambda$$

That is, if $h$ is carried by $W$, then so is $f_* h / \lambda$ for some $\lambda > 0$. Likewise, $\lambda f_*^{-1} h$ is also carried by $W$. This proves the first part of the lemma.

For the second part of the lemma, let $h_W$ be a homology carried by $W$, then from above, there exists $\lambda > 0$ such that $f_* h_W \lambda^{-1}$ is also carried the foliation. Since $W$ carries a unique homology up to rescale, there is a constant $c$, $c_1 \leq c \leq c_2$ such that

$$f_* h_W \lambda^{-1} = ch_W / ||h_W||,$$

or $h_W$ is an eigenvector for $f_*$ with corresponding eigenvalue $c \lambda / ||h_W||$, where $||h_W||$ is the norm of $h_W$ for any fixed norm defined on $H_1(M, \mathbb{R})$.

This completes the proof of the lemma. □
The key to proving that an invariant foliation carries a homology is to show that a sub-sequential limit of $[l_i]/|l_i|$ is nontrivial. One way to accomplish this is to find a closed one-form $\omega$ such that $\omega$ is non-degenerate on $T_x W(x)$ for any $x \in M$. This condition implies that the integral of $\omega$ over any oriented line segment of $W$ is nonzero. i.e.,

$$\int_l \omega \neq 0$$

for any line segment $l$ in a leaf of $W$. We may assume this integral is always positive by choosing $-\omega$ if necessary. To see this, observe that there is a constant $c > 1$ such that

$$c^{-1}|l| \leq \int_l \omega \leq c|l|$$

for any $l$ in leaves of $W$. This is due to the compactness of the manifold $M$.

Therefore,

$$c^{-1}|f^r W_r(x)| \leq \int_{f^r W_r(x)} \omega < c|f^r W_r(x)|.$$  

This implies that, if $h_W$ is any sub-sequential limit of $[l_i]/|l_i|$, then

$$c^{-1} (h_W, [\omega]) \leq c$$

where $(h_W, [\omega])$ is the canonical pairing. This implies that $h_W \neq 0$.

The following theorem shows the relationship between the geometric expansion and the topological expansion for foliations carrying a unique homology.

**Theorem 2.3.** If an invariant foliation $W$ carries a unique nontrivial homology $h_W$, then its geometric expansion $\chi(x, r)$ on $W$ is independent of the point $x \in M$ and independent of $r > 0$. Moreover, let $\lambda_W$ be the corresponding eigenvalue for the eigenvector $h_W$ for the linear map $f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$, then

$$\chi = \chi(x, r) = \limsup_{i \to \infty} \frac{1}{i} \ln |f^i(W_r(x))|$$

$$= \lim_{i \to \infty} \frac{1}{i} \ln |f^i(W_r(x))| = \ln \lambda_W$$

*Proof.* If $W$ carries a unique nontrivial homology $h_W$, then for any $x \in M$ and any $r > 0$,

$$\lim_{i \to \infty} \frac{|l_i|}{||l_i||} = h_W/||h_W||.$$  

Therefore,

$$\frac{h_W}{||h_W||} = \lim_{i \to \infty} \frac{|l_i+1|}{||l_i+1||}$$

$$= \lim_{i \to \infty} \left( \frac{|f(l_i)|}{||l_i+1||} \right) \left( \frac{|l_i|}{||l_i+1||} \right)$$

$$= \lim_{i \to \infty} \left( f_* \left( \frac{|l_i|}{||l_i||} \right) \right) \left( \frac{|l_i|}{||l_i+1||} \right)$$

$$= \lambda_W \frac{h_W}{||h_W||} \lim_{i \to \infty} \frac{||l_i||}{||l_i+1||}.$$  

This implies that

$$\lim_{i \to \infty} \frac{||l_i||}{||l_i+1||} = \lambda_W^{-1}.$$
On the other hand, since

\[ 0 < c_1 \leq \lim Inf_{i \to \infty} \frac{||l_i||}{|l_i|} \leq \lim Sup_{i \to \infty} \frac{||l_i||}{|l_i|} \leq c_2, \]

where \( c_1 \) and \( c_2 \) are from Definition 2.1, there is a constant \( c > 1 \) such that

\[ c^{-1}||l_i|| \leq |l_i| \leq c||l_i||, \]

for all \( i \in \mathbb{N} \).

\[
\chi = \chi(x, r) = \lim \sup_{i \to \infty} \frac{1}{i} \ln |f^i(W_r(x))|
\]
\[
= \lim_{i \to \infty} \frac{1}{i} \ln |f^i(W_r(x))|
\]
\[
= \lim_{i \to \infty} \frac{1}{i} \ln |l_i|
\]
\[
\leq \lim_{i \to \infty} \frac{1}{i} \ln(c||l_i||) = \lim_{i \to \infty} \frac{1}{i} \ln ||l_i||
\]
\[
= \lim_{i \to \infty} \frac{1}{i} \ln \left( ||l_0|| \prod_{j=1}^{i} \frac{||l_j||}{||l_{j-1}||} \right)
\]
\[
= \lim_{i \to \infty} \frac{1}{i} \ln ||l_0|| + \sum_{j=1}^{i} \frac{||l_j||}{||l_{j-1}||}
\]
\[
= \ln \lambda_W
\]

Here we used the elementary fact that if \( \lim_{i \to \infty} a_i = a \), then \( \lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i} a_i = a \).

Replacing \( c \) with \( c^{-1} \), we have \( \chi \geq \ln \lambda_W \).

This proves the theorem. \( \square \)

We now consider some specific examples. Let \( A \) be an integer \( 3 \times 3 \) matrix \( A \) with determinant one. We assume that \( A \) is hyperbolic and the three eigenvalues of \( A \) satisfy \( \lambda_1 > \lambda_2 > 1 > \lambda_3 \). Clearly \( \lambda_1 \lambda_2 \lambda_3 = 1 \). Let \( v_1, v_2 \) and \( v_3 \) be eigenvectors corresponding to \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) respectively.

Let \( T = T_A : \mathbb{T}^3 \to \mathbb{T}^3 \) be the hyperbolic toral automorphism defined by \( T_A x = Ax \mod \mathbb{Z}^3 \); for any \( x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3 \). \( T \) induces an isomorphism on the first homology of \( \mathbb{T}^3 \), \( T_* : H_1(\mathbb{T}^3, \mathbb{R}) \to H_1(\mathbb{T}^3, \mathbb{R}) \). With the right choice of basis, the map \( T_* \) is exactly \( A \) in its representation.

Let \( f : \mathbb{T}^3 \to \mathbb{T}^3 \) be a diffeomorphism on \( \mathbb{T}^3 \) that is close to \( T_A \). The map \( f \) is obviously homotopic to \( T_A \) and \( f \) induces the same map on the homology of \( \mathbb{T}^3 \). Moreover, \( f \) is Anosov and it is topologically conjugate to \( T_A \). For any such map \( f \), we let \( W^u, W^s \) be its stable foliation and its unstable foliation respectively. \( W^u \) is a two dimensional foliation. There are two additional invariant foliations: \( W^{uu} \), the strong unstable foliation and \( W^{wu} \) its weak unstable foliation. \( W^u \) is sub-foliated by \( W^{uu} \) and \( W^{wu} \). We may regard \( W^{uu} \) as the center foliation, if we regard \( f \) as a partially hyperbolic systems with stable, unstable and center distributions all one dimensional.

We claim that for such a map \( f \) and for its strong unstable and weak unstable manifolds \( W^{uu} \) and \( W^{wu} \), their geometric expansion \( \chi(x, r) \) is independent of \( x \) and \( r \). Moreover, \( \chi(x, r) \) is constant for all such maps close to \( T_A \) and is exactly \( \ln \lambda_1 \) and \( \ln \lambda_2 \) respectively, where \( \lambda_1 \) and \( \lambda_2 \) are unstable eigenvalues of \( A \).
To show this, it suffices to show that both $W^{uu}$ and $W^{wu}$ carry unique homologies, which are corresponding eigenvectors for $\lambda_1$ and $\lambda_2$ respectively. Let’s fix a foliation $W$ to be either the weak unstable foliation or the strong unstable foliation. $W$ is orientable and we fix an orientation on $W$. For the linear map $T_A$, there are many closed one forms $\omega$ on $T^3$ that are non-degenerate on all eigen directions of $A$. In fact, we may pick $\omega = \pm dx_1$, where $x_1$ is the first coordinate for $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Any such one form $\omega$ is also non-degenerate for invariant foliations for maps close to $T_A$, since the stable, center and unstable distributions are continuous with respect to the diffeomorphisms. This implies that $W$ carries a non-trivial homology. The fact that the homology it carries is unique is because that the expansion constant for maps close to $T_A$ is close to the expansion constant of $T_A$ and the eigenvector with eigenvalue close to these expansions is unique.

### 3. Higher dimensional foliations

In this section, we generalize our results on one dimensional foliations to high dimensions. The natural objects used to define homologies of foliations are the closed currents supported on the foliation (see [Pl], [Su]). This approach was used for example in the study of entropy of axiom A diffeomorphisms (see [SW], [RS]). Here we will restrict our attention to some specific currents supported on the foliation, which are related to the dynamics of $f$.

Let $W$ be a $k$ dimensional foliation of $M$, invariant under $f$. For any positive integer, we define the currents:

$$C_n(\omega) = \frac{1}{\text{Vol}(f^n(W_r(x)))} \int_{f^n(W_r(x))} \omega,$$

for any $k$-form $\omega$ on $M$. These currents depend on $x$ and $r$. The currents are uniformly bounded so there must be subsequences with weak limits. Let $C$ be such a limit, i.e., we have a sequence $n_i \to \infty$ such that for any $k$-form $\omega$ we have $\lim_{i \to \infty} C_{n_i}(\omega) = C(\omega)$.

A current $C$ is said to be closed if for any exact $k$-form $\omega = d\alpha$, we have $C(\omega) = C(d\alpha) = 0$. If $C$ is closed, it has a homology class $[C] = h_C \in H_k(M, \mathbb{R})$. This homology class is nontrivial if there exist a closed $k$-form $\omega$ such that $C(\omega) \neq 0$.

We would like to investigate the conditions under which the sub-sequential limits of the currents $C_n$ is closed. In general, $C_n$ itself is not closed. From Stokes’ Theorem, we have:

$$C_n(\omega) = \frac{1}{\text{Vol}(f^n(W_r(x)))} \int_{f^n(W_r(x))} d\alpha$$

$$= \frac{1}{\text{Vol}(f^n(W_r(x)))} \int_{W_r(x)} (f^n)^* \alpha$$

$$= \frac{1}{\text{Vol}(f^n(W_r(x)))} \int_{\partial W_r(x)} (f^n)^* \alpha$$

If the above sequence approaches zero as $n \to \infty$, then every sub-sequential limit of the currents $C_n$ is closed. In many situations, the volume growth of $f^n(W_r(x))$ is larger than the lower dimensional volume growth of its boundary.

The first case is that when the dimension of the foliation is one. In this case, $\alpha$ is a real valued function and hence $\int_{\partial W_r(x)} (f^n)^* \alpha$ is the difference of that function
evaluated at the two end points of \( f^n(W_r(x)) \) and therefore it is uniformly bounded. Thus \( C_n(\omega) \to 0 \) as \( n \to \infty \).

Another case is that when \( f \) is close to a linear map on the torus \( \mathbb{T}^n \) and \( W \) is any of the expanding foliations close to the linear one. We will consider this case in more details later.

A more general condition is just to have some convenient uniform bounds for the expansion rates:

\[
\sup_{v^{k-1} \in \Lambda^{k-1}} \frac{f_{rn}v^{k-1}}{\|v^{k-1}\|} \leq \inf_{v^{k} \in \Lambda^{k}} \frac{f_{rn}v^{k}}{\|v^{k}\|}
\]

for some number \( n > 0 \). This is an open condition, it is verified also for small perturbations of \( f \) and \( W \).

**Definition 3.1.** We say that a \( k \)-dimensional invariant foliation \( W \) carries a non-trivial homology \( h \in C_k(M, \mathbb{R}) \) if for some \( x \in M \), \( r > 0 \) the currents \( C_n \) defined above have a closed sub-sequential limit \( C \) and \( h_C = [C] \neq 0 \).

We say that a \( k \)-dimensional invariant foliation \( W \) carries a unique non-trivial homology (up to rescale) if all sub-sequential limits of the currents \( C_n \) are closed and the homologies it carries are unique up to scalar multiplication and are uniformly bounded away from zero, for all \( x \in M \) and all \( r > 0 \).

It is easy to see that the above definition is consistent with what we defined for the one-dimensional case.

A closed current is non-trivial if there is a closed \( k \)-form \( \omega \) such that \( C(\omega) \neq 0 \). The homology class of a non-trivial closed current is non-trivial. Again, one way to show that the closed current \( C \) is non-trivial is to show that there is a closed \( k \)-form \( \omega \) such that \( \omega \) is non-degenerate on \( T_xW(x) \) for any \( x \in M \). This condition implies that the integral of \( \omega \) over any oriented segment of \( W \) is nonzero. i.e.,

\[
\int_D \omega \neq 0
\]

for any piece \( D \) on a leaf of the foliation \( W \), with its orientation inherited from the leaf. We may assume that the integral is positive by choosing \( -\omega \) if necessary. When we have a non-degenerate \( k \)-form on the leaves of \( W \), by compactness of the manifold, there exists a constant \( c > 1 \) such that

\[
c^{-1}\text{Vol}(D) \leq \int_D \omega \leq c\text{Vol}(D)
\]

for any segment \( D \) on the leaves of \( W \) and therefore

\[
c^{-1}\text{Vol}(f^n(W_r(x))) \leq \int_{f^n(W_r(x))} \omega \leq c\text{Vol}(f^n(W_r(x)))
\]

This implies that \( C(\omega) > 0 \).

Assume that an invariant foliation \( W \) carries a unique non-trivial homology and let \( h_C = [C] \in C_k(M, \mathbb{R}) \), where \( C \) is the current as defined above. The next proposition shows that \( h_C \) is actually an eigenvector of the induced linear map by \( f \) on the homology of \( M \).

**Proposition 3.2.** Let \( W \) be a \( k \)-dimensional expanding invariant foliation that carries a unique non-trivial homology \( h_C \). Then \( h_C \) is an eigenvector of the induced linear map:

\[
f_* : C_k(M, \mathbb{R}) \to C_k(M, \mathbb{R}).
\]
Proof. First we observe that the map $f$ naturally induces an action on the currents, defined by:

$$f_\ast C(\omega) = C(f^\ast \omega)$$

for any $k$ current $C$ and $k$ form $\omega$. Obviously, if $C$ is closed, then $f_\ast C$ is closed too and

$$[f_\ast C] = f_\ast [C] \in H_k(M, \mathbb{R}).$$

Let a current $C$ be a sub-sequential limit of $C_n(x, r)$, then

$$C(\omega) = \lim_{i \to \infty} \frac{1}{\text{Vol}(f^{n_i}(W_{r_i}(x)))} \int_{f^{n_i}(W_{r_i}(x))} \omega,$$

for any $k$-form on $M$. Therefore

$$(f_\ast C)(\omega) = \lim_{i \to \infty} \frac{1}{\text{Vol}(f^{n_i}(W_{r_i}(x)))} \int_{f^{n_i}(W_{r_i}(x))} f^\ast \omega$$

$$= \lim_{i \to \infty} \frac{1}{\text{Vol}(f^{n_i}(W_{r_i}(x)))} \int_{f^{n_i+1}(W_{r_i}(x))} \omega$$

$$= \lim_{i \to \infty} \frac{\text{Vol}(f^{n_i+1}(W_{r_i}(x)))}{\text{Vol}(f^{n_i}(W_{r_i}(x)))} \cdot \frac{1}{\text{Vol}(f^{n_i+1}(W_{r_i}(x)))} \int_{f^{n_i+1}(W_{r_i}(x))} \omega$$

Since the ratio $\text{Vol}(f^{n_i+1}(W_{r_i}(x)))/\text{Vol}(f^{n_i}(W_{r_i}(x)))$ is uniformly bounded, both from above and away from zero, there is a convergent subsequence. Without loss of generality, we may assume that the sequence actually converges and there is a constant $\lambda > 0$ such that

$$\lim_{i \to \infty} \frac{\text{Vol}(f^{n_i+1}(W_{r_i}(x)))}{\text{Vol}(f^{n_i}(W_{r_i}(x)))} = \lambda$$

This implies that $f_\ast C/\lambda$ is also a sub-sequential limit of the current $C_n(x, r)$. Since $W$ carries a unique non-trivial homology, the homology of this limit must be a scalar multiple of $[C]$. Therefore, there is a constant $c$ such that we have $[f_\ast C \lambda^{-1}] = [cC]$. This implies that

$$f_\ast h_C = c\lambda h_C$$

i.e., $h_C$ is an eigenvector of

$$f_\ast : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R})$$

with corresponding eigenvalue $c\lambda$.

This proves the proposition. $\square$

Let $\lambda_W$ be the eigenvalue of $f_\ast$ corresponding to the eigenvector $h_C$, as in the above proposition. We call $\lambda_W$ the topological growth of the foliation $W$. We will see below that the topological growth and the volume growth are the same for a foliation that carries a unique non-trivial homology, except that the volume growth we defined here is an exponent, while the topological growth is a multiplier.

**Theorem 3.3.** Let $W$ be an expanding invariant foliation that carries a unique non-trivial homology $h_W$. Let $\lambda_W$ be the topological growth of the foliation. Then the volume growth defined before,

$$\chi(x, r) = \lim_{i \to \infty} \frac{1}{i} \ln(\text{Vol}(f^i(W_{r_i}(x))))$$

$$= \lim_{i \to \infty} \frac{1}{i} \ln(\text{Vol}(f^i(W_{r_i}(x)))) = \ln \lambda_W,$$
for any $x \in M$ and any $r > 0$.

Proof. The volume of a piece of leaf in a foliation depends on the Riemannian metric defined on $M$. So in general, the volume does not grow uniformly with each iteration. We will rescale the volume at each step so that there will be uniform growth. Let $h_W \in H_k(M, \mathbb{R})$ be a homology carried by $W$. Let $\omega_W$ be a closed $k$-form such that the pairing between $h_W$ and $[\omega_W]$ is nonzero. For any $x \in M$ and $r > 0$, we choose a sequence of numbers $d_i$, $i \in \mathbb{N}$ such that

$$
\lim_{i \to \infty} d_i C_i(\omega_W) = (h_W, [\omega_W]).
$$

Moreover, there are numbers $0 < c_1 \leq c_2$ such that $d_i$ can be chosen with $c_2^{-1} \leq d_i \leq c_1$. Then, because of the uniqueness of homologies carried by the foliation, every limit current of $\{d_i C_i\}_{i \in \mathbb{N}}$ must have the homology $h_W$. This implies that the relation

$$
\lim_{i \to \infty} d_i C_i(\omega) = (h_W, [\omega])
$$

Holds for every closed form $\omega$, so also for $f^* \omega$. Therefore,

$$(h_W, [f^* \omega]) = \lim_{i \to \infty} \frac{d_i}{\Vol(f^i(W_r(x)))} \int_{f^i(W_r(x))} f^* \omega
= \lim_{i \to \infty} \frac{d_i}{\Vol(f^i(W_r(x)))} \int_{f^{i+1}(W_r(x))} \omega
= \lim_{i \to \infty} \frac{\Vol(f^{i+1}(W_r(x))/d_{i+1})}{\Vol(f^i(W_r(x))/d_i)} \cdot \frac{d_{i+1}}{\Vol(f^{i+1}(W_r(x)))} \int_{f^{i+1}(W_r(x))} \omega
= \lim_{i \to \infty} \frac{\Vol(f^{i+1}(W_r(x))/d_{i+1})}{\Vol(f^i(W_r(x))/d_i)} \cdot (h_W, [\omega])
$$

Therefore

$$
\lim_{i \to \infty} \frac{\Vol(f^{i+1}(W_r(x))/d_{i+1})}{\Vol(f^i(W_r(x))/d_i)} = (h_W, [f^* \omega])/(h_W, [\omega])
= (f_* h_W, [\omega])/(h_W, [\omega]) = \lambda_W
$$

This implies that

$$
\chi(x, r) = \lim \sup_{i \to \infty} \frac{1}{i} \ln(\Vol(f^i(W_r(x))))
= \lim \frac{1}{i} \ln \Vol(f^i(W_r(x)))
= \lim \frac{1}{i} \ln(d_i^{-1} \Vol(f^i(W_r(x))))
= \lim \frac{1}{i} \ln \left( d_0^{-1} \Vol(W_r(x)) \cdot \prod_{j=1}^{i} d_j^{-1} \Vol(f^j(W_r(x))) \right)
= \lim \frac{1}{i} \sum_{j=1}^{i} \ln \frac{d_j^{-1} \Vol(f^j(W_r(x)))}{d_{j-1}^{-1} \Vol(f^{j-1}(W_r(x)))}
= \ln \lambda_W
$$
Here again we used the elementary fact that if \( \lim_{i \to \infty} a_i = a \), then
\[
\lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i} a_j = a.
\]

This proves the theorem. \( \square \)

The next proposition discusses the situation where a foliation carries more than one non-trivial homologies.

**Proposition 3.4.** Let \( W \) be an expanding invariant foliation and let \( H \subset H_k(M, \mathbb{R}) \) be the set of non-trivial homologies carried by \( W \). Then \( H \) spans a linear space, invariant under
\[
f_* : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R}).
\]

**Proof.** We first observe that \( H \subset H_k(M, \mathbb{R}) \) is a bounded set. Let \( h \in H \) be a homology carried by the foliation \( W \). It follows from the proof of Proposition 3.2 that there exists a constant \( c > 0 \) such that \( f_* h / c \) is also carried by \( W \). The proposition follows. \( \square \)

As an example, we consider maps on \( n \)-torus \( \mathbb{T}^n \) close to a linear map. Consider an \( n \times n \) matrix \( A \) with determinant one and with integer entries. The matrix \( A \) induces a toral automorphism: \( T_A : \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \to \mathbb{T}^n \) defined by \( T_A x = Ax \mod \mathbb{Z}^n \). If all eigenvalues are away from the unit circle, then \( T_A \) is a hyperbolic toral automorphism. If the eigenvalues of \( A \) are mixed, with some on the unit circle and some away from unit circle, then \( T_A \) is partially hyperbolic.

In both hyperbolic or partially hyperbolic cases, let \( E^u \) be the unstable distribution of \( T_A \). At each point \( x \in \mathbb{T}^n \), \( E^u(x) \subset T_x \mathbb{T}^n \) is the unstable subspace for \( dT_A : T_x \mathbb{T}^n \to T_x \mathbb{T}^n \). Let \( W^u \) be the unstable foliation generated by \( E^u \). Then it is easy to see that the currents \( C_n \) converge to a unique closed current \( C \) and \( C \) is non-trivial. Moreover, the eigenvalue corresponding to \( h_C \) is the product of all eigenvalues outside of the unit circle. i.e., \( \lambda_W = \prod_{|\lambda_i| > 1} \lambda_i \).

For maps close to \( T_A \), all the sub-sequential limits of the currents \( C_n \) are closed. This is because that the \( k \) dimensional volume \( \text{Vol}(f^n(W_x(x))) \) grows like the product of \( k \) eigenvalues \( \left( \prod_{|\lambda_i| > 1} \lambda_i \right)^n \), while the \( k-1 \)-form \( (f^*)^n \alpha \) grows approximately at the rate of the product of \( k-1 \) eigenvalues. In this case, it is also easy to see that every sub-sequential limit of the currents is non-trivial. Having a non-degenerate form on leaves of \( W \) is an open condition.

To show that \( C_n \) actually converges to a unique current \( C \), we first observe that the map is homotopic to the linear map and hence the induced map on the homology is exactly the same as that of the linear map. By Proposition 3.3, the set of all homologies carried by \( W^u \) span an invariant subspace in \( H_k(T^n, \mathbb{R}) \). Every eigenvector of \( f_* \) in this subspace has an eigenvalue close to \( \prod_{|\lambda_i| > 1} \lambda_i \). However, and there is only one (up to a constant multiple) eigenvector with the eigenvalue \( \prod_{|\lambda_i| > 1} \lambda_i \). This implies that the limit is unique and the eigenvalue is exactly \( \prod_{|\lambda_i| > 1} \lambda_i \).

4. Lyapunov Exponents

The expansion of an invariant foliation \( W \) can also be described by the Lyapunov exponents. In this section, we will consider this analytical description and show its relations with the geometric expansion we described in the first section.
Let $f$ be a diffeomorphism of $M$ with an invariant probability measure $\mu$, then for $\mu$-a.e. $x \in M$, there exist real numbers $\lambda_1(x) > \ldots > \lambda_l(x)$ ($l \leq n$): positive integers $n_1, \ldots, n_l$ such that $n_1 + \ldots + n_l = n$; and a measurable invariant splitting $T_x M = E_x^1 \oplus \cdots \oplus E_x^l$, with dimension $\dim(E_x^i) = n_i$ such that
\[
\lim_{j \to \infty} \frac{1}{j} \log \|D_x f^j(v_i)\| = \lambda_i(x),
\]
whenever $v_i \in E_x^i, v \neq 0$.

These numbers $\lambda_1(x), \ldots, \lambda_l(x)$ are called the Lyapunov exponents for $x \in M$. If the probability measure $\mu$ is ergodic, then these exponents are constants for a.e. $(\mu) x \in M$. The existence of these Lyapunov exponents are the result of Oseledec’s Multiplicative Ergodic Theorem.

Let $E$ be an invariant sub-bundle of $TM$. For example it can be $TW$, the tangent spaces of leaves are preserved under the map. i.e., for any $x \in M, D_x f(T_x W(x)) = T_{f(x)} W(f(x))$. For any invariant probability measure $\mu$ and for a.e. $(\mu) x \in M$, a subset of the Lyapunov splitting $E^i_x, i = 1, \ldots, l$ spans $E_x$. Let $\Lambda_E(x)$ be the sum (counting multiplicity $n_i$) of the Lyapunov exponents corresponding to the splittings in $E_x$. $\Lambda_E(x)$ is defined a.e. $(\mu)$ and it is also given by the formula:
\[
\Lambda_E(x) = \lim_{j \to \infty} \frac{1}{j} \log \|\Lambda^k D_x f^j|_{\Lambda^k E_x}\|.
\]

We also define the integrated Lyapunov exponent of $E$ to be
\[
\Lambda_E = \int_M \Lambda_E(x) d\mu.
\]

This is also equal to the integral over the manifold $M$ of the logarithm of the Jacobian of $f$ restricted to the sub-bundle $E$:
\[
\Lambda_E(x) = \int_M \log(\|\Lambda^k D_x f|_{\Lambda^k E_x}\|) d\mu.
\]

When $\mu$ is ergodic, $\Lambda_E(x) = \Lambda_E$ a.e. $(\mu)$. If $E = TW$ we will denote $\Lambda_W(x) = \Lambda_{TW}(x)$ and $\Lambda_W = \Lambda_{TW}$.

For the following result, we need to define the concept of absolute continuity. For simplicity, we use a stronger version of absolute continuity. For any $x \in M$, let $D_1$ and $D_2$ be sufficiently small $(n-k)$ dimensional smooth disks transverse to $W(x)$. One can locally define a map, called holonomy map for the foliation, from $D_1$ to $D_2$, $y_1 \mapsto y_2$ with $y_1 \in D_1$ and $y_2 = D_2 \cap W(y_1)$. The holonomy map is said to be absolutely continuous if it maps sets of measure zero in $D_1$ to sets of measure zero in $D_2$. The foliation is said to be absolutely continuous if the holonomy maps are absolutely continuous. If a foliation is absolutely continuous, a full measure set for a smooth measure intersects almost all leaves in full measure. Here the measure on the leaves is the Riemannian volume restricted to $W$, and almost all leaves is with respect to Riemannian volume on transversals.

The next result is a standard way to prove non-absolute continuity of foliations:

**Lemma 4.1.** Let $f \in \text{Diff}^r_\mu(M)$ be a diffeomorphism on $M$, preserving a smooth volume $\mu$. Let $W$ be a $k$ dimensional foliation of $M$, invariant under $f$ and
\[
\chi(x, r) = \lim_{i \to \infty} \frac{1}{i} \ln \text{Vol}(f^i(W_r(x))),
\]
and let
\[ \chi = \chi(r) = \sup_{x \in M} \chi(x, r). \]

Finally, let \( \Lambda_W \) be the integrated Lyapunov exponent of the foliation \( W \) for the invariant measure \( \mu \). If the foliation \( W \) is absolutely continuous, then
\[ \Lambda_W \leq \chi. \]

\textbf{Proof.} Let \( A \subset M \) be the set of Lyapunov generic points. i.e., for any \( x \in A \), there exist the sum of the Lyapunov exponents for \( x \) on \( T_x W \) and is equal to \( \Lambda_W(x) \). This is a full measure set with respect to \( \mu \). We have that
\[ \Lambda_W = \int_M \Lambda_W(x) d\mu, \]
so there exists a positive measure set \( B \subset M \) such that for any \( x \in B \) we have \( \Lambda_W(x) \geq \Lambda_W \). The absolute continuity of \( W \) implies that there exists at least a leaf \( W(y) \) for some \( x \in M \) such that \( W(y) \) intersects \( B \) in a set of positive measure (actually there is a positive set of such leaves). Denote by \( m_W \) the Riemannian volume on \( W(x) \) and fix a disk \( W_r(x) \) such that \( m_W(W_r(x) \cap B) > 0 \).

For any small \( \epsilon > 0 \), for any \( y \in W_r(x) \cap B \) there exist \( N_y \in \mathbb{N} \) such that for all \( i \geq N_y \) we have
\[ \text{Jac}_y(f^i) \geq (\Lambda_W - \epsilon)^i, \]
where \( \text{Jac}_y(f^i) = \| \Lambda^k D_y f^i \| \Lambda^k T_y W \| \) is the Jacobian of the function at \( x \) restricted to \( W \). Let \( B_N \subset W_r(x) \cap B \) be the set of points \( y \) such that \( N_y \leq N \). Then \( B_N \) is an increasing sequence of sets and the union is \( W_r(x) \cap B \) which has positive measure, so there is an \( N \in \mathbb{N} \) such that \( m_W(B_N) > 0 \). It follows that for any \( y \in B_N \) and any \( i > N \) we have
\[ \text{Jac}_y(f^i) \geq (\Lambda_W - \epsilon)^i \geq (\Lambda_W - \epsilon)^i. \]

We will use this to estimate the volume of \( f^i(W_r(x)) \) if \( i > N \):
\[
\begin{align*}
\text{Vol}(f^i(W_r(x))) & = \int_{f^i(W_r(x))} dm_W \\
& = \int_{W_r(x)} \text{Jac}_y(f^i) dm_W \\
& \geq \int_{B_N} \text{Jac}_y(f^i) dm_W \\
& \geq (\Lambda_W - \epsilon)^i m_W(B_N)
\end{align*}
\]
Therefore,
\[ \chi \geq \chi(x, r) = \limsup_{i \to \infty} \frac{1}{i} \ln \text{Vol}(f^i(W_r(x))) \geq \Lambda_W - \epsilon \]
Since \( \epsilon > 0 \) is arbitrary, we have \( \chi \geq \Lambda_W \).

\[ \square \]

\textbf{Corollary 4.2.} Let \( f \in \text{Diff}_\mu^r(M) \) be diffeomorphism on \( M \), preserving a smooth volume \( \mu \). Let \( W \) be a \( k \) dimensional foliation of \( M \), invariant under \( f \) and let \( \Lambda_W \) be the integrated Lyapunov exponent of the foliation \( W \) for the measure \( \mu \).
If $\chi < \Lambda_W$, then the foliation $W$ is not absolutely continuous. Moreover, if $\mu$ is ergodic, then there is a full measure set $A \in M$ such that every leaf $W(x)$ of the foliation $W$ intersect $A$ in a zero measure set,

$$\mu_W(W(x) \cap A) = 0,$$

for all $x \in M$, where $\mu_W$ is the conditional measure of $\mu$ on the leaves of $W$.

In the last statement of the corollary $A$ is the set of Lyapunov regular points.

If a leaf of $W$ intersects $A$ in a positive measure set then the same argument from the proof of the lemma gives a contradiction.

5. Perturbations and examples

In this section we show how to perturb a linear map of the torus in order to make an intermediate foliation non-absolute continuous in a persistent way. The main tool used here is a result of A. Baraviera and C. Bonatti (see [BB]). Before we state it, we have to define dominated splittings.

We say that $T_M = E \oplus F$ is a dominated splitting for the diffeomorphism $f$ if the sub-bundles $E$ and $F$ are invariant under $Df$ and there is an $l \in \mathbb{N}$ such that for each $x \in M$ and each nonzero vectors $u \in E_x, v \in F_x$ we have

$$\frac{\|D_x f^l(u)\|}{\|u\|} < \frac{1}{2} \frac{\|D_x f^l(v)\|}{\|v\|}.$$

An invariant splitting is continuous and it persists after perturbations, meaning that any $g$ which is $C^1$ close to $f$ will also have a dominated splitting $TM = E' \oplus F'$ close to the dominated splitting $TM = E \oplus F$ for $f$. The definition can be of course extended for splittings with more than two sub-bundles.

**Theorem 5.1.** Let $M$ be a compact Riemannian manifold and $\mu$ a smooth volume form on $M$. Let $f$ be a $C^1$ diffeomorphism of $M$ preserving $\mu$ and admitting a dominated splitting $TM = E^1 \oplus E^2 \oplus E^3$. Then there are arbitrarily small volume preserving $C^1$ perturbations $g$ of $f$ such that, if $TM = \tilde{E}^1 \oplus \tilde{E}^2 \oplus \tilde{E}^3$ is the new dominated splitting for $g$, then the integrated Lyapunov exponent of $\tilde{E}^2$ with respect to $g$ is strictly larger than the integrated Lyapunov exponent of $E^2$ with respect to $f$:

$$\Lambda_{\tilde{E}^2}(g) > \Lambda_{E^2}(f).$$

The idea of the proof is the following. One has to make a small perturbation to 'mix' the direction of $E^2$ with the direction of $E^3$, while keeping the coordinates corresponding to $E^1$ almost unchanged. This mixing almost doesn’t change the direction of $E^2 \oplus E^3$ and the Jacobian restricted to it, so the integrated Lyapunov exponent of $E^2 \oplus E^3$ is very close to the one of $E^2 \oplus E^3$. The perturbation will be local, supported on a small ball with very large returning time. This perturbation will change the direction of $E^3$ towards $E^2$ at the image of the ball, but then the dynamics of the map will tend to correct this perturbation, and if the return time is large enough then this perturbation becomes negligible for estimating the Jacobian along $E^3$ for the further iterates. Then, analyzing the change on the small ball where the perturbation is supported, one can prove that the integrated exponent corresponding to the new $\tilde{E}^3$ is ‘significantly’ smaller than the one of $E^3$. As a consequence, the integrated exponent corresponding to the new $\tilde{E}^2$ becomes larger than the one of $E^2$. For the details of the proof we send the reader to [BB].
A. Baraviera and C. Bonatti show that a consequence of this result is the fact that a $C^1$ generic small perturbation of the time one map of an volume preserving Anosov flow has a non-absolutely continuous central foliation. Previously M. Shub and A. Wilkinson gave some examples of perturbations of skew products where the central foliation is again non-absolutely continuous in a persistent way. In their situation the central foliation consists of circles (see [SWk]). Recently M. Hirayama and Y. Pesin proved that $C^1$ generically a partially hyperbolic map with compact center leaves has the central foliation non-absolutely continuous (see [HP]). All this results lead to the following conjecture:

**Conjecture 1.** Generically the central foliation (if it exists) of a volume preserving partially hyperbolic diffeomorphism is non-absolutely continuous.

We want to give another example of persistent non-absolutely continuous central and intermediate foliations of volume preserving partially hyperbolic diffeomorphisms that supports this conjecture.

Consider a linear automorphism $A$ of the torus $T^n$ such that the tangent bundle has a dominated invariant splitting $TT^n = E^1 \oplus E^2 \oplus E^3$ with the corresponding integrated Lyapunov exponents $\Lambda_1, \Lambda_2, \Lambda_3$, which are the logarithm of the absolute value of the product of the eigenvalues of $Df$ (with their multiplicity) corresponding to the eigenvectors in each sub-bundle ($A$ preserves the Lebesgue measure on $T^n$).

We can also denote by $J_1, J_2, J_3$ the Jacobians of $A$ on $E^1, E^2, E^3$ and we have

$$\Lambda_i = \log J_i, i \in \{1, 2, 3\}.$$

We will also have the invariant foliations by planes $W^1, W^2, W^3$. We assume that $W^2$ and $W^3$ are uniformly expanding, or $W^3$ is a strong unstable foliation, $W^2$ is a weak unstable foliation and $W^1$ is a stable or a center stable foliation. We also have $W^{23}$, which is an unstable foliation, and $W^{12}$, which can be seen as a center foliation. All these foliations have unique non-trivial homology which is an eigenvector of the map induced by $A$ in the corresponding homology group. We denote this eigenvectors $h_1, h_2, h_3$. The topological growth of $W^1, W^2, W^3$ will be exactly the corresponding eigenvalues $J_1, J_2, J_3$. Because the map is linear, for all these foliations the volume growth, the Lyapunov growth and the logarithm of the topological growth coincide.

For any $f$ a $C^1$ small perturbation of $A$ we will have an $f$-invariant dominated splitting $TT^n = E^1 \oplus E^2 \oplus E^3$ an corresponding $f$-invariant foliations $W^1, W^2, W^3$ which are close to the dominated splitting and the foliations for $A$. This foliations persist because $W^{23}$ and $W^3$ are (strong) unstable foliations, $W^1$ and $W^{12}$ are normally hyperbolic foliations.

For simplicity we also assume that $J_2$ is a simple eigenvalue of

$$A_\ast : H_k(T^n, \mathbb{R}) \to H_k(T^n, \mathbb{R}).$$

**Theorem 5.2.** For any such linear automorphism $A$ of the torus $T^n$ there exist an open set of volume preserving diffeomorphisms $U$, $C^1$ arbitrarily close to $A$, such that, for any $f \in U$, the foliation $W^2$ is non-absolutely continuous.

**Proof.** By the previous theorem we can make an arbitrarily $C^1$ small perturbation of $A$ to obtain a volume preserving diffeomorphism $f$ such that

$$\Lambda E^2(f) < \Lambda_2.$$ 

We remark that this property is true also for small $C^1$ perturbations of $f$. 

Now we want to conclude the non-absolute continuity of $\hat{W}^2$. For this we need to show that the volume growth of $\hat{W}^2$ is $\Lambda_2$, which is strictly greater than $\Lambda_{\hat{F}^2}$, and the conclusion will follow from the results in the previous section.

Suppose that $\hat{W}^2$ is $k$-dimensional. We can choose $f$ sufficiently close to $A$ so that the Jacobian of $f$ on $W^2$ is inside a small neighborhood of $J_2$ which doesn’t contain any other eigenvalue of the map $A_\ast : H_k(\mathbb{T}^n, \mathbb{R}) \to H_k(\mathbb{T}^n, \mathbb{R})$.

Because $f$ is close to a linear map on the torus, every limit current on $\hat{W}^2$ is closed and nontrivial. Suppose that $\hat{W}^2$ doesn’t have the unique unique nontrivial homology $h_2$. Then there exist a disk $\hat{W}_2^2(x)$ in $W^2$ and a subsequence of corresponding currents $C_n$ such that $\lim_{n \to \infty} C_n = C$ and $h_C \neq h_2$. Because the homologies of the limit currents of $\{C_n\}_{n \in \mathbb{N}}$ form a closed invariant set, we may assume that $h_C$ is an eigenvector of $A_\ast = f_\ast : H_k(\mathbb{T}^n, \mathbb{R}) \to H_k(\mathbb{T}^n, \mathbb{R})$ corresponding to an eigenvalue different that $J_2$. Then the condition on the Jacobian of $f$ on $W^2$ will give a contradiction.

So we know that $\hat{W}^2$ has unique nontrivial homology which is $h_2$. Then the volume growth of $\hat{W}^2$ will have to be

$$\chi(f, \hat{W}^2) = \log J_2 = \Lambda_2 < \Lambda_{\hat{F}^2}(f) = \Lambda_{\hat{W}^2}(f)$$

so the foliation is non-absolutely continuous. The same is true for all sufficiently $C^1$ close maps to $f$.

\[\square\]

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