The Maximum Number of Triangles in a Graph of Given Maximum Degree

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Received XX Month 20XX; Revised XX Month 20XX; Published XX Month 20XX

Abstract: We prove that any graph on $n$ vertices with max degree $d$ has at most $q\left(\frac{d+1}{3}\right) + \binom{r}{3}$ triangles, where $n = q(d+1) + r$, $0 \leq r \leq d$. This resolves a conjecture of Gan-Loh-Sudakov.

1 Introduction

Fix positive integers $d$ and $n$ with $d + 1 \leq n \leq 2d + 1$. Galvin [7] conjectured that the maximum number of cliques in an $n$-vertex graph with maximum degree $d$ comes from a disjoint union $K_{d+1} \cup K_r$ of a clique on $d+1$ vertices and a clique on $r := n-d-1$ vertices. Cutler and Radcliffe [4] proved this conjecture. Engbers and Galvin [6] then conjectured that, for any fixed $t \geq 3$, the same graph $K_{d+1} \cup K_r$ maximizes the number of cliques of size $t$, over all $(d+1+r)$-vertex graphs with maximum degree $d$. Engbers and Galvin [6]; Alexander, Cutler, and Mink [1]; Law and McDiarmid [11]; and Alexander and Mink [2] all made progress on this conjecture before Gan, Loh, and Sudakov [9] resolved it in the affirmative. Gan, Loh, and Sudakov then extended the conjecture to arbitrary $n \geq 1$ (for any $d$).

Conjecture (Gan-Loh-Sudakov Conjecture). Any graph on $n$ vertices with maximum degree $d$ has at most $q\left(\frac{d+1}{3}\right) + \binom{r}{3}$ triangles, where $n = q(d+1) + r$, $0 \leq r \leq d$.

They showed their conjecture implies that, for any fixed $t \geq 4$, any max-degree $d$ graph on $n = q(d+1) + r$ vertices has at most $q\left(\frac{d+1}{t}\right) + \binom{r}{t}$ cliques of size $t$. In other words, considering triangles is enough to resolve the general problem of cliques of any fixed size.

*The author is partially supported by Ben Green’s Simons Investigator Grant 376201 and gratefully acknowledges the support of the Simons Foundation.
The Gan-Loh-Sudakov conjecture (GLS conjecture) has attracted substantial attention. Cutler and Radcliffe [5] proved the conjecture for $d \leq 6$ and showed that a minimal counterexample, in terms of number of vertices, must have $q = O(d)$. Gan [8] proved the conjecture if $d + 1 - \frac{9}{4096} d \leq r \leq d$ (there are some errors in his proof, but they can be mended). Using Fourier analysis, the author [3] proved the conjecture for Cayley graphs with $q \geq 7$. Kirsch and Radcliffe [10] investigated a variant of the GLS conjecture in which the number of edges is fixed instead of the number of vertices (with still a maximum degree condition).

In this paper, we fully resolve the Gan-Loh-Sudakov conjecture.

**Theorem 1.** For any positive integers $n,d \geq 1$, any graph on $n$ vertices with maximum degree $d$ has at most $q\left(\binom{d+1}{3} + \binom{d}{3}\right)$ triangles, where $n = q(d+1) + r$, $0 \leq r \leq d$.

Analyzing the proof shows that $qK_{d+1} \sqcup K_r$ is the unique extremal graph if $r \geq 3$, and that $qK_{d+1} \sqcup H$, for any $H$ on $r$ vertices, are the extremal graphs if $0 \leq r \leq 2$.

The heart of the proof is the following Lemma, of independent interest, which says that, in any graph, we can find a closed neighborhood whose removal from the graph removes few triangles. Theorem 1 will follow from its repeated application.

**Lemma 1.** In any graph $G$, there is a vertex $v$ whose closed neighborhood meets at most $\binom{d(v)+1}{3}$ triangles.

As mentioned above, Theorem 1, together with the work of Gan, Loh, and Sudakov [9], yields the general result, for cliques of any fixed size.

**Theorem 2.** Fix $t \geq 3$. For any positive integers $n,d \geq 1$, any graph on $n$ vertices with maximum degree $d$ has at most $q\left(\binom{d+1}{3} + \binom{d}{t}\right)$ cliques of size $t$, where $n = q(d+1) + r$, $0 \leq r \leq d$.

Theorem 2 gives another proof of (the generalization of) Galvin’s conjecture (to $n \geq 2d + 2$) that a disjoint union of cliques maximizes the total number of cliques in a graph with prescribed number of vertices and maximum degree.

Finally, the author would like to point out a connection to a related problem, that of determining the minimum number of triangles that a graph of fixed number of vertices $n$ and prescribed minimum degree $\delta$ can have. The connection stems from a relation, observed in [2] and [9], between the number of triangles in a graph and the number of triangles in its complement:

$$|T(G)| + |T(G^c)| = \binom{n}{3} - \frac{1}{2} \sum_v d(v)[n-1-d(v)].$$

Lo [12] resolved this “dual” problem when $\delta \leq \frac{4n}{5}$. His results resolve the GLS conjecture for regular graphs for $q = 2, 3$, and the GLS conjecture implies his results, up to an additive factor of $O(\delta^2)$, for $q = 2, 3$, and yields an extension of his results for $q \geq 4$ — these are the optimal results asymptotically, in the natural regime of $\frac{\delta}{n}$ fixed, and $n \to \infty$.
2 Notation

Denote by \( E \) the edge set of \( G \); for two vertices \( u, v \), we write “\( uv \in E \)” if there is an edge between \( u \) and \( v \) and “\( uv \notin E \)” otherwise — in particular, for any \( u, uv \notin E \). For a vertex \( v \), let \( |T_{N[v]}| \) denote the number of triangles with at least one vertex in the closed neighborhood \( N[v] := \{ u : uv \in E \} \cup \{ v \} \), and let \( |T(G - N[v])| \) denote the number of triangles with all vertices in the graph \( G - N[v] \) (the subgraph induced by the vertices not in \( N[v] \)). Finally, \( d(v) \) denotes the degree of \( v \).

3 Proof of Theorem 1

For a graph \( G \), let \( W(G) = \{ (x, u, v, w) : ux, vx, wx \in E, uv, uw, vw \notin E \} \).

Lemma 2. For any graph \( G \), \( 6 \sum_v |T_{N[v]}| + |W(G)| = \sum_v d(v)^3 \).

Proof. Let \( \Omega = \{ (z, u, v, w) : uv, uw, vw \in E \text{ and } [zu \in E \text{ or } zv \in E \text{ or } zw \in E] \} \), \( \Sigma = \{ (x, u, v, w) : ux, vx, wx \in E \} \), and \( W = W(G) \). Note that repeated vertices in the 4-tuples are allowed. First observe that, since there are 6 ways to order the vertices of a triangle, \( \sum_v 6|T_{N[v]}| = |\Omega| \). Any 4-tuple in \( \Sigma, W \), or \( \Omega \) gives rise to one of the induced subgraphs shown below, since one vertex must be adjacent to all the others.

![Graph subgraphs](image)

Since \( |\Sigma| = \sum_v d(v)^3 \), it thus suffices to show that for each of the induced subgraphs above, the number of times it comes from a 4-tuple in \( \Sigma \) is the sum of the number of times it comes from 4-tuples in \( \Omega \) and \( W \). Any fixed copy of \( A \), say on vertices \( u \) and \( v \), comes 0 times from a 4-tuple in \( \Omega \) (since it has no triangles), and 2 times from each of \( W \) and \( \Sigma \); \( (u, v, v, v), (v, u, u, u) \). Any fixed copy of \( B \), say on vertices \( u, v, w \) with \( vu, vw \in E \), comes 0 times from \( \Omega \), and 6 times from each of \( W \) and \( \Sigma \); \( (v, u, u, w), (v, u, w, u), (v, u, u, w), (v, w, u, u), (v, w, w, u) \). Any fixed copy of \( C \) comes 18 times from each of \( \Omega \) and \( \Sigma \) (3 choices for the first vertex and then 6 for the ordered triangle), and 0 times from \( W \). Similarly, any fixed copy of \( D \) comes 6 times from each of \( W \) and \( \Sigma \), and 0 times from \( \Omega \); finally, \( F, H, I \) come 6, 12, 24 times, respectively, from each of \( \Omega \) and \( \Sigma \), and 0 times from \( W \).

We now prove our key lemma, previously mentioned in the introduction.

Lemma 1. In any graph \( G \), there is a vertex \( v \) whose closed neighborhood meets at most \( \binom{d(v)+1}{3} \) triangles, i.e. \( |T_{N[v]}| \leq \binom{d(v)+1}{3} \).

Proof. By Lemma 2, since \( |W(G)| \geq |\{ (x, u, u, u) : ux \in E \}| = \sum_v d(x) \), we have \( \sum_v |T_{N[v]}| \leq \sum_v \frac{1}{6}[d(v)^3 - d(v)] \). By the pigeonhole principle, there is some \( v \) with

\[
|T_{N[v]}| \leq \frac{1}{6}[d(v)^3 - d(v)] = \left( \frac{d(v)+1}{3} \right).
\]

□

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Lemma 3. For any positive integers $a \geq b \geq 1$, it holds that $\binom{a}{3} + \binom{b}{3} \leq \binom{a + 1}{3} + \binom{b - 1}{3}$. Consequently, for any positive integers $a, b$ and any positive integer $c$ with $\max(a, b) \leq c \leq a + b$, it holds that $\binom{a}{3} + \binom{b}{3} \leq \binom{\frac{a+b}{3}}{3} + \binom{\frac{a+b-c}{3}}{3}$.

Proof. $(\frac{a+1}{3}) - \binom{a}{3} = \binom{a}{2}$, and $(\frac{b-1}{3}) - \binom{b}{3} = \binom{b-1}{2}$. Iterate to get the consequence.

We now finish the proof of Theorem 1.

Proof of Theorem 1. With a fixed $d$, we induct on $n$. For $n = 1$, the result is obvious. Take some $n \geq 2$, and suppose the theorem holds for all smaller values of $n$. Let $G$ be a max-degree $d$ graph on $n$ vertices. By Lemma 1, we may take $v$ with $|T(N[v])| \leq d(v) + 1$. Write $n = q(d + 1) + r$ for $0 \leq r \leq d$. Note $|T(G)| = |T(G - N[v])| + |T_N[v]|$. Since $G - N[v]$ has maximum degree (at most) $d$, if $d(v) + 1 \leq r$, then induction and Lemma 3 give

$$|T(G)| \leq q\left(\binom{d+1}{3} + \binom{r - (d(v) + 1)}{3}\right) + \left(\binom{d(v) + 1}{3}\right) \leq q\left(\binom{d+1}{3} + \binom{r}{3}\right),$$

and if $d(v) + 1 > r$, then induction and Lemma 3 give

$$|T(G)| \leq (q-1)\left(\binom{d+1}{3} + \binom{d+1+r - (d(v) + 1)}{3}\right) + \left(\binom{d(v) + 1}{3}\right) \leq q\left(\binom{d+1}{3} + \binom{r}{3}\right).$$

The maximum degree condition ensured $d + 1 + r - (d(v) + 1) \geq 0$ and $d(v) + 1 \leq d + 1$.

Acknowledgments

I would like to thank Po-Shen Loh for telling me the Gan-Loh-Sudakov conjecture and my advisor Ben Green for encouragement. I also thank Daniel Korandi for a cleaner proof of Lemma 2 and for helpful suggestions on the paper’s presentation.

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