On the multiplicity spaces for branching to a spherical subgroup of minimal rank

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Abstract

Let \( \hat{\mathfrak{g}} \) be a complex semi-simple Lie algebra and \( \mathfrak{g} \) be a semisimple subalgebra of \( \hat{\mathfrak{g}} \). Consider the branching problem of decomposing the simple \( \hat{\mathfrak{g}} \)-representations \( \hat{V} \) as a sum of simple \( \mathfrak{g} \)-representations \( V \). When \( \hat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{g} \), it is the tensor product decomposition. The multiplicity space \( \text{Mult}(V, \hat{V}) \) satisfies

\[
\hat{V} = \bigoplus \text{Mult}(V, \hat{V}) \otimes V,
\]

where the sum runs over the isomorphism classes of simple \( \mathfrak{g} \)-representations. In the case when \( \mathfrak{g} \) is spherical of minimal rank, we describe \( \text{Mult}(V, \hat{V}) \) as the intersection of kernels of powers of root operators in some weight space of the dual space \( V^* \) of \( V \). When \( \hat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{g} \), we recover by geometric methods a well known result.

1 Introduction

Let \( G \) be a connected reductive subgroup of a complex semisimple group \( \hat{G} \). The branching problem consists in decomposing irreducible representations of \( \hat{G} \) as sum of irreducible \( G \)-representations.

Fix maximal tori \( T \subset \hat{T} \) and Borel subgroups \( B \supset T \) and \( \hat{B} \supset \hat{T} \) of \( G \) and \( \hat{G} \) respectively. Let \( X(T) \) denote the group of characters of \( T \) and let \( X(T)^+ \) denote the set of dominant characters. For \( \nu \in X(T)^+ \), \( V_\nu \) denotes the irreducible representation of highest weight \( \nu \). Similarly, we use notation \( X(\hat{T}) \), \( X(\hat{T})^+ \), \( V_{\hat{\nu}} \) relatively to \( \hat{G} \). For any \( G \)-representation \( V \), the subspace of \( G \)-fixed vectors is denoted by \( V^G \). Given \( \nu \in X(T)^+ \) and \( \hat{\nu} \in X(\hat{T})^+ \), set

\[
\text{Mult}(\nu, \hat{\nu}) = \text{Hom}(V_\nu, V_{\hat{\nu}}^*)^G = (V_\nu^* \otimes V_{\hat{\nu}}^*)^G,
\]

where \( V_\nu^* \) and \( V_{\hat{\nu}}^* \) denote the dual representations of \( V_\nu \) and \( V_{\hat{\nu}} \) respectively. The branching problem is equivalent to the knowledge of these spaces. Indeed, there is a natural \( G \)-equivariant isomorphism:

\[
\bigoplus_{\nu \in X(T)^+} \text{Hom}(V_\nu, V_{\hat{\nu}}^*)^G \otimes V_\nu \rightarrow V_{\hat{\nu}}^* \quad \text{by} \quad f \otimes v \mapsto f(v).
\]

Let \( \hat{G}/\hat{B} \) denote the complete flag variety of \( \hat{G} \). In this article, we are interested in the case when the pair \( (\hat{G}, G) \) is spherical of minimal rank. In other words, we assume that there exists \( x \) in \( \hat{G}/\hat{B} \) such that the orbit \( G.x \) of \( x \) is open in \( \hat{G}/\hat{B} \) and the stabilizer \( G_x \) of \( x \) in \( G \) contains a maximal torus of \( G \). An important example is when \( \hat{G} = G \times G \) in which \( G \) is diagonally embedded. Then the point \( x = (B, B^-) \in \hat{G}/\hat{B} \) works, where \( B^- \) denotes the opposite Borel subgroup of \( B \)
containing $T$. More generally, the spherical pairs of minimal rank have been classified by the second author in [Res10]. The complete list, assuming in addition that $\hat{G}$ is semisimple simply connected, $G$ is simple and $G \neq \hat{G}$ is:

1. $G$ is simple, simply connected and diagonally embedded in $G = G \times G$;
2. $(\text{SL}_2n, \text{Sp}_{2n})$ with $n \geq 2$;
3. $(\text{Spin}_{2n}, \text{Spin}_{2n-1})$ with $n \geq 4$;
4. $(\text{Spin}_7, G_2)$;
5. $(E_6, F_4)$.

Our aim is to present a uniform description of the multiplicity spaces $\text{Mult}(\nu, \hat{\nu})$ for given $(\hat{G}, G)$ in this list. Let us first fix some notation. Recall that $\Delta$ (resp. $\hat{\Delta}$) denote the set of simple roots of $G$ (resp. $\hat{G}$). For each positive root $\alpha$ of $G$, we fix an $\mathfrak{sl}_2$-triple $(X_{\alpha}, H_{\alpha}, X_{-\alpha})$ such that $H_{\alpha}$ (resp. $X_{\alpha}$) belongs to the Lie algebra of $T$ (resp. $B$). Given $\mu \in X(T)$, we denote by $V_{\mu}(\mu) = \{v \in V_\nu : \forall t \in T \; t v = \mu(t)v\}$ the corresponding multiplicity space for the action of the maximal torus $T$.

Let $\hat{W}$ denote the Weyl group of $\hat{G}$. Fix also $\hat{\nu}_0 \in \hat{W}$ such that $G\hat{\nu}_0 \hat{B}/\hat{B}$ is dense in $\hat{G}/\hat{B}$ and such that $\hat{\nu}_0$ has minimal length with this property (see Section 2.3 for details).

For cases 2 to 4, we also denote by $\Phi^1$ the set of long roots of $G$. In case 1, we define $\Phi^1$ to be the empty set. Set

$$\mathcal{D} = \{\hat{\alpha} \in \hat{\Delta} : \rho(\hat{\nu}_0 \hat{\alpha}) \not\in \Phi^1\}.$$

**Theorem 1.** For $\nu \in X(T)^+$ and $\hat{\nu} \in X(\hat{T})^+$, there is a natural isomorphism from $\text{Mult}(\nu, \hat{\nu})$ onto the subspace of $V_{\nu}^*(\rho(\hat{\nu}_0 \hat{\nu}))$ consisting in the vectors $v$ such that

1. $X_{\alpha} \cdot v = 0$, for any $\alpha \in \Phi^+ \cap \Phi^1$;
2. $X_{-\rho(\hat{\nu}_0 \hat{\alpha})} \cdot v = 0 \quad \forall m > (\hat{\nu}, \hat{\alpha}^\vee), \text{ for any } \hat{\alpha} \in \mathcal{D}$.

Actually, the conditions 2 are not pairwise independent. For example, in the case of the tensor product, the conditions associated to $(\alpha, 0) \in \mathcal{D}$ and $(0, -w_0 \alpha) \in \mathcal{D}$ are equivalent for any simple root $\alpha$. Here $w_0$ denotes the longest element of the Weyl group. For each example, we describe in Section 4 an explicit subset of $\mathcal{D}$ giving an irredundant set of inequalities.

In the case of the tensor product, Theorem 1 is well known. See [PRR67, Theorem 2.1, p. 392] or [Z73, Theorem 5, p. 384]. The usual proofs are algebraic, based on properties of the enveloping algebra. Our geometric proof seems to be new even in this case.

The branching rule for $(\text{Spin}_{2n}, \text{Spin}_{2n-1})$ is multiplicity free and hence easy to determine (see e.g. [FH91]). That of $(\text{SL}_{2n}, \text{Sp}_{2n})$ has been the subject of much attention in the literature. The first positive rule in terms of dominos was obtained by Sundaram [Sun90]. Naito-Sagaki conjectured a rule in terms of Littelmann’s patches [NS03]. Later, B. Schumann and J. Torres proved this conjecture by obtaining a bijection with Sundaram’s model. A nonpositive rule for $(\text{Spin}_7, G_2)$ were obtained by McGovern in [McG90]. Hopefully, Theorem 1 could be the first step toward combinatorial rules for these branching problems.
2 Reminder and complements on spherical homogeneous spaces of minimal rank

2.1 Roots of \( G \) and \( \hat{G} \)

Fix a spherical pair of minimal rank \((G, \hat{G})\) with \( G \) and \( \hat{G} \) reductive. Choose a maximal torus \( T \) in \( G \) and a Borel subgroup \( T \subset B \subset G \). Denote by \( \Phi \) (resp. \( \Phi^+ \)) the set of roots (resp. positive roots). Recall that \( \Delta \) denotes the set of simple roots. Fix also a maximal torus \( \hat{T} \) of \( \hat{G} \) containing \( T \) and let \( \rho : X(\hat{T}) \to X(T) \) denote the restriction map. The set \( \hat{\Phi} \) of roots of \( \hat{G} \) maps onto \( \Phi \) (see \cite[Lemma 4.2]{Res10}). Let \( \hat{\rho} \) denote the restriction of \( \rho \) to \( \hat{\Phi} \). By setting \( \hat{\Phi}^+ = \hat{\rho}^{-1}(\Phi^+) \), one gets a choice of positive roots for \( \hat{G} \). Let \( \hat{B} \) denote the corresponding Borel subgroup. Then \( \hat{B} \) contains \( B \). By \cite[Lemma 4.6]{Res10}, \( \rho(\hat{\Delta}) = \Delta \), where \( \hat{\Delta} \) denote the set of simple roots of \( \hat{G} \).

On the following diagrams the restriction of \( \hat{\rho} \) to the set of simple roots is the vertical projection.

By \cite[Lemma 4.4]{Res10}, for any \( \alpha \in \Phi \), \( \hat{\rho}^{-1}(\alpha) \) has cardinality one or two. Hence, \( \Phi \) splits in the following two subsets

\[ \Phi^1 := \{ \alpha \in \phi_b : \sharp \hat{\rho}^{-1}(\alpha) = 1 \} \quad \text{and} \quad \Phi^2 := \{ \alpha \in \phi_b : \sharp \hat{\rho}^{-1}(\alpha) = 2 \}. \]

Set also \( \hat{\Phi}^1 = \hat{\rho}^{-1}(\Phi^1) \) and \( \hat{\Phi}^2 = \hat{\rho}^{-1}(\Phi^2) \). It is easy to check using the pictures, that this set \( \hat{\Phi}^1 \), coincides with that defined in the introduction.

2.2 Isotropies

Assume, in addition, that \( G \) is connected.

**Lemma 2.** Let \( x \in \hat{G}/\hat{B} \). Then the isotropy \( G_x \) is connected and contains a maximal torus of \( G \).

**Proof.** The fact that \( G_x \) contains a maximal torus \( T \) of \( G \) follows from the monotonicity properties of the rank of orbit closures of \( G \) in \( \hat{G}/\hat{B} \) (see \cite[Theorem 2.2]{Kno95}).

Let \( L \) be a Levi subgroup (maximal reductive subgroup) of \( G_x \). Then \( L \) is isomorphic to the quotient of \( G_x \) by its unipotent radical. A reference for the existence of \( L \) is \cite[Theorem 4. p286]{OV90}. Up to conjugacy, one may assume that \( T \subset L \).
But $L$ is a reductive subgroup of $\hat{G}_x = \hat{B}'$, that is a Borel subgroup of $\hat{G}$. Hence $L$ maps injectively into $\hat{B}'/\hat{U}' \simeq \hat{T}$ and $L$ is abelian. The torus $T$ being its own centralizer in $G$, we deduce that $L = T$.

Now $G_x$ is connected as the product of $T$ and its unipotent radical. 

2.3 Orbits of $G$ in $\hat{G}/\hat{B}$

We are now interested in the set $G\setminus \hat{G}/\hat{B}$ of $G$-orbits in $\hat{G}/\hat{B}$. Let $W$ and $\hat{W}$ be the Weyl groups of $G$ and $\hat{G}$ respectively. Since $T$ is a regular torus in $G$ (see [Bri99, Lemma 2.3]), $W$ naturally embeds in $\hat{W}$.

**Lemma 3.** The map

$$W\setminus \hat{W} \rightarrow G\setminus \hat{G}/\hat{B}$$

$$\hat{w} \mapsto G\hat{w}\hat{B}/\hat{B}$$

is a well-defined bijection.

**Proof.** It is clear that the $G$-orbit $G\hat{w}\hat{B}/\hat{B}$ does not depend on the representative $\hat{w}$ in its class $W\hat{w}$. Hence, the map of the lemma is well-defined.

Since $T$ is a regular torus in $\hat{G}$, its fixed points in $\hat{G}/\hat{B}$ are the points $\hat{w}\hat{B}/\hat{B}$ for $\hat{w} \in \hat{W}$. But Lemma 2 implies that each $G$-orbit in $\hat{G}/\hat{B}$ contains a $T$-fixed point. The surjectivity follows.

Let now $\hat{w}$ and $\hat{w}'$ in $\hat{W}$ such that $G\hat{w}\hat{B}/\hat{B} = G\hat{w}'\hat{B}/\hat{B}$. To get the injectivity and finish the proof, it remains to prove that $W\hat{w} = W\hat{w}'$. Choose $g \in G$ such that $g\hat{w}\hat{B}/\hat{B} = \hat{w}'\hat{B}/\hat{B}$. Let $H$ and $H'$ denote the isotropies in $G$ of the points $\hat{w}\hat{B}/\hat{B}$ and $\hat{w}'\hat{B}/\hat{B}$, respectively. Observe that $T$ is a maximal torus of both $H$ and $H'$. Then, $T$ and $gTg^{-1}$ are two maximal tori of $H'$, and there exists $h' \in H'$ such that $h'Th'^{-1} = gTg^{-1}$. Thus, $n := h'^{-1}g$ normalizes $T$ and satisfies $n\hat{w}\hat{B}/\hat{B} = h'^{-1}\hat{w}'\hat{B}/\hat{B}$. It follows that $\hat{w}' \in W\hat{w}$.

Let $\ell : \hat{W} \rightarrow \mathbb{N}$ denote the length function. It is well known that $\ell(\hat{w}) = \#(\hat{\Phi}^+ \cap \hat{w}^{-1}\hat{\Phi}^-)$, where $\hat{\Phi}^- = -\hat{\Phi}^+$. 

We fix, once for all, $\hat{y}_0 \in \hat{W}$ such that $G\hat{y}_0\hat{B}/\hat{B}$ is the open $G$-orbit in $\hat{G}/\hat{B}$ and such that $\hat{y}_0$ has minimal length with this property. Let $H_0$ denote the stabilizer of $\hat{y}_0\hat{B}/\hat{B}$ in $G$.

Given a root $\alpha$ (resp. $\hat{\alpha}$) of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$), denote by $\mathfrak{g}_\alpha$ (resp. $\hat{\mathfrak{g}}_{\hat{\alpha}}$) the corresponding root space.

**Lemma 4.** The Lie algebra of $H_0$ is

$$\text{Lie}(T) \oplus \bigoplus_{\alpha \in \hat{\Phi}^+ \cap \Phi^+} \mathfrak{g}_\alpha.$$ 

**Proof.** It is clear that $H_0$ contains $\hat{T} \cap G = T$. Then

$$\text{Lie}(H_0) = \text{Lie}(T) \oplus \bigoplus_{\alpha \in \mathcal{S}} \mathfrak{g}_\alpha,$$

for some subset $\mathcal{S}$ of $\Phi$.

Let now $\alpha \in \Phi^+ \cap \Phi^+$. We want to prove that $\alpha \in \mathcal{S}$. Let $\hat{\alpha} \in \hat{\Phi}$ such that $\rho(\hat{\alpha}) = \alpha$. We have $\mathfrak{g}_{\pm \hat{\alpha}} = \mathfrak{g}_{\pm \alpha}$. But, exactly one between $\mathfrak{g}_{\hat{\alpha}}$ and $\mathfrak{g}_{-\hat{\alpha}}$ is contained in the Borel Lie algebra $\text{Lie}(\hat{y}_0\hat{B}\hat{y}_0^{-1})$. Assume, for a contradiction that $\mathfrak{g}_{-\hat{\alpha}} \subset \text{Lie}(\hat{y}_0\hat{B}\hat{y}_0^{-1})$. 

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Set $\hat{\beta} = -\hat{y}_0^{-1}\hat{\alpha} \text{ and } \hat{y}_0' = s_{\hat{\alpha}}\hat{y}_0$. Since $\hat{\beta}$ is positive and $\hat{y}_0\hat{\beta}$ is negative, $\ell(\hat{y}_0') < \ell(\hat{y}_0)$. But, $g_{\pm\hat{\alpha}} = g_{\pm\alpha}$ implies that $s_{\alpha} = s_{\hat{\alpha}}$ and that $\hat{y}_0' \in W\hat{y}_0$. This contradicts the minimality assumption on the length of $\hat{y}_0$.

At this point, we proved that the Lie algebra of the lemma is contained in $\text{Lie}(H_0)$. But $\dim(G/H_0) = \dim(\hat{G}/\hat{B})$, hence

$$\dim(H_0) = \dim(G) - \dim(\hat{G}/\hat{B}) = \dim(T) + 2\#\Phi^+ - \#\Phi^+ = \dim(T) + \#(\Phi^+ \cap \Phi^1)$$

and we can conclude. \hfill $\Box$

**Lemma 5.** Fix $\beta \in \Phi^2$. Then there is exactly one root in $\rho^{-1}(\beta)$ which is sent in $\hat{\Phi}^+$ by the action of $\hat{y}_0^{-1}$.

**Proof.** Write $\rho^{-1}(\beta) = \{\hat{\beta}_1, \hat{\beta}_2\}$. Because of [Res10, Lemma 4.4], we have that $g_\beta \subseteq g_{\hat{\beta}_1} \oplus g_{\hat{\beta}_2}$ and $g_\beta \neq g_{\hat{\beta}_i}$ for $i = 1, 2$.

Lemma 3 implies that $g_\beta \not\subseteq \text{Lie}(H_0)$. But since $H_0 = H \cap \hat{y}_0\hat{B}\hat{y}_0^{-1}$, this means that $\hat{y}_0^{-1}\beta_1$ and $\hat{y}_0^{-1}\beta_2$ are not both positive roots, equivalently: $\{\hat{y}_0^{-1}\beta_1, \hat{y}_0^{-1}\beta_2\} \not\subseteq \hat{\Phi}^+$.

The same argument applied to $-\beta$ implies that $\{-\hat{y}_0^{-1}\beta_1, -\hat{y}_0^{-1}\beta_2\} \not\subseteq \hat{\Phi}^+$. The lemma follows. \hfill $\Box$

Lemma 5 allows us to distinguish between the roots in the fiber of $\rho$ by means of the action of $\hat{y}_0^{-1}$. In particular we introduce the following notation. If $\beta \in \Phi^2 \cap \Phi^+$, then $\hat{\beta}^+$ (resp. $\hat{\beta}^-$) is the unique element in $\rho^{-1}(\beta)$ that satisfies: $\hat{y}_0^{-1}\hat{\beta}^+ \in \hat{\Phi}^+$ (resp. $\hat{y}_0^{-1}\hat{\beta}^- \in \hat{\Phi}^-$).

### 2.4 The graph of $G$-orbits in $\hat{G}/\hat{B}$

We recall the definition given in [Res04] of a graph $\Gamma(\hat{G}/G)$ whose vertices are the elements of $G\backslash \hat{G}/\hat{B}$. The original construction of $\Gamma(\hat{G}/G)$ due to M. Brion is equivalent but slightly different (see [Br01]).

If $\hat{\alpha}$ belongs to $\hat{\Delta}$, we denote by $P_{\hat{\alpha}}$ the associated minimal standard parabolic subgroup of $\hat{G}$.

Consider the unique $\hat{G}$-equivariant map $\pi_{\hat{\alpha}} : \hat{G}/\hat{B} \rightarrow \hat{G}/P_{\hat{\alpha}}$ which is a $\mathbb{P}^1$-bundle. Let $O \in G\backslash \hat{G}/\hat{B}$ and $\hat{\alpha} \in \hat{\Delta}$. Consider $\pi_{\hat{\alpha}}^{-1}(\pi_{\hat{\alpha}}(O))$. Two cases occur.

- $G$ acts transitively on $\pi_{\hat{\alpha}}^{-1}(\pi_{\hat{\alpha}}(O))$.
- $\pi_{\hat{\alpha}}^{-1}(\pi_{\hat{\alpha}}(O))$ contains two $G$-orbits, one closed $V$ and one open $V'$. Then we says that $\hat{\alpha}$ raises $V$ to $V'$. In this case, $\dim(V') = \dim(V) + 1$, and for any $x \in V$, $\pi_{\hat{\alpha}}^{-1}(\pi_{\hat{\alpha}}(x)) \cap V = \{x\}$.

**Definition.** Let $\Gamma(\hat{G}/G)$ be the oriented graph with vertices the elements of $G\backslash \hat{G}/\hat{B}$ and edges labeled by $\hat{\Delta}$, where $V$ is joined to $V'$ by an edge labeled by $\hat{\alpha}$ if $\hat{\alpha}$ raises $V$ to $V'$.

**Lemma 6.** Let $\hat{w} \in \hat{W}$. We have

$$\dim G\hat{w}\hat{B}/\hat{B} - \dim G/B \leq \ell(\hat{w}).$$

Moreover, for any $G$-orbit $O$ in $\hat{G}/\hat{B}$ there exists $\hat{w} \in \hat{W}$ such that $O = G\hat{w}$ and

$$\dim G\hat{w}\hat{B}/\hat{B} - \dim G/B = \ell(\hat{w}).$$
Proof. We consider the $\hat{B}$-orbits in $\hat{G}/G$. The closed orbit is $\hat{B}/B$. Choose a reduced expression of $\hat{w} = s_{\hat{a}_1} \cdots s_{\hat{a}_1}$. Consider the quotient $P_{\hat{a}_1} \times \hat{B} \cdots \times \hat{B} P_{\hat{a}_s} \times \hat{B} \hat{B}/B$ of $P_{\hat{a}_1} \times \cdots \times P_{\hat{a}_s} \times \hat{B}/B$ by the action of $B^*$ given by $(b_1, \ldots, b_s)(p_1, \ldots, p_s, \hat{B}/B) = (p_1 b_1^{-1}, \ldots, b_{s-1} p_s b_s^{-1}, b_s \hat{B}/B)$ (with obvious notation). Consider the regular map

$$P_{\hat{a}_1} \times \hat{B} \cdots \times \hat{B} P_{\hat{a}_s} \times \hat{B} \hat{B}/B \longrightarrow \hat{G}/G$$

$$(p_1, \ldots, p_s, x) \longmapsto p_1 \cdots p_s x.$$

The dimension of the left hand side is $\ell(\hat{w}) + \dim(\hat{B}/B)$. The right hand side is a $\hat{B}$-orbit closure containing $\hat{B}\hat{w}^{-1}G/G$. The first inequality follows.

This equality is reached when the expression is obtained by reading the labels on some path from the closed orbit to $O$ in the graph $\Gamma(\hat{G}/G)$. Such a path exists by [Res10, Proposition 2.2].

Set

$$\ell_m(\hat{w}) = \min \{ \ell(w\hat{w}) : w \in W \}.$$

By Lemma [Res10, Proposition 2.2], $\ell_m(\hat{w}) = \dim(G\hat{w}\hat{B}/B) - \dim(G/B)$. If an element in $\hat{w} \in \hat{W}$ satisfies $\ell(\hat{w}) = \ell_m(\hat{w})$, we say that $\hat{w}$ has minimal length.

Lemma 7.

1. There are $\#\Delta^2$ codimension one $G$-orbits in $\hat{G}/\hat{B}$.

2. Let $\hat{\alpha} \in \hat{\Delta}$. Then $G\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}$ has codimension one if and only if $\hat{y}_0 \hat{\alpha} \in \hat{\Phi}^2$.

3. Let $\hat{\alpha} \in \hat{\Delta}$ such that $\hat{y}_0 \hat{\alpha} \in \hat{\Phi}^2$. Then, the Lie algebra of the stabilizer of $\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}$ is

$$\text{Lie}(H_0) \oplus g_{-\rho(\hat{y}_0 \hat{\alpha})}.$$

Proof. By [Res10, Proposition 2.3], the number of codimension one $G$-orbits in $\hat{G}/\hat{B}$ is the number of $G$-orbits $O$ of dimension $\dim(G/B) + 1$. This is also

$$\#\{\{W s_{\hat{\alpha}} \in W \setminus \hat{W} : \hat{\alpha} \in \hat{\Delta}\} - \{W\}\}.$$

But $s_{\hat{\alpha}} \in W$ if and only if $\hat{\alpha} \in \hat{\Delta}^1$. Moreover, if $\hat{\alpha}_1 \neq \hat{\alpha}_2 \in \hat{\Delta}^2$ then $s_{\hat{\alpha}_1} s_{\hat{\alpha}_2} \in W$ if and only if $\rho(\hat{\alpha}_1) = \rho(\hat{\alpha}_2)$.

Since $G\hat{y}_0 \hat{B}/\hat{B}$ is dense, either $G\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B} = G\hat{y}_0 \hat{B}/\hat{B}$ or $G\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}$ has codimension one. But $\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}$ belongs to the open $G$-orbit if and only if $\hat{y}_0 s_{\hat{\alpha}} \in W \hat{y}_0$ if and only if $\hat{y}_0 \hat{\alpha} \in \hat{\Phi}^1$.

Let $\hat{\alpha}$ be like in the last assertion. Then

$$\hat{y}_0 s_{\hat{\alpha}} \Phi^+ = (\hat{y}_0 \Phi^+ \cup \{-\hat{y}_0 \hat{\alpha}\} - \{\hat{y}_0 \hat{\alpha}\}.$$

Now, the fact that $\text{Lie}(H_0)$ is contained in the Lie algebra of the stabilizer of $\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}$ follows from Lemma [Res10, Proposition 2.2]. The root $\hat{y}_0 \hat{\alpha}$ belonging to $\hat{\Phi}^2$, there exists a second root $\hat{\beta}$ in the fiber $\hat{\rho}^{-1}(\rho(-\hat{y}_0 \hat{\alpha}))$. But, Lemma [Res10, Proposition 2.2] implies that $\hat{\beta}$ belongs to $\hat{y}_0 \Phi^+$ and hence to $\hat{y}_0 s_{\hat{\alpha}} \Phi^+$. We deduce that

$$g_{\rho(\hat{y}_0 \hat{\alpha})} \subset g_{\hat{\beta}} \oplus g_{-\hat{y}_0 \hat{\alpha}} \subset \text{Lie}(\hat{y}_0 s_{\hat{\alpha}} \hat{B}(\hat{y}_0 \hat{\alpha})^{-1}).$$

In particular, $g_{\rho(\hat{y}_0 \hat{\alpha})}$ is contained in the Lie algebra of the stabilizer of $\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}$.

Now, the last assertion follows by equality of the dimensions. 

\[\square\]
2.5 Compatible $\mathfrak{sl}_2$-triples

Recall that we have fixed an $\mathfrak{sl}_2$-triple $(X_\beta, H_\beta, Y_\beta)$ associated to any positive root $\beta$ of $G$.

**Lemma 8.** If $\beta \in \Phi^2 \cap \Phi^+$ then there exist $\mathfrak{sl}_2$-triples $(X_{\hat{\beta}^+}, H_{\hat{\beta}^+}, X_{\hat{\beta}^-})$ and $(X_{\hat{\beta}^-}, H_{\hat{\beta}^-}, X_{\hat{\beta}^-})$ for $\hat{\beta}^+$ and $\hat{\beta}^-$ such that:

- $X_\beta = X_{\hat{\beta}^+} + X_{\hat{\beta}^-}; \quad H_\beta = H_{\hat{\beta}^+} + H_{\hat{\beta}^-}; \quad X_{-\beta} = X_{-\hat{\beta}^+} + X_{-\hat{\beta}^-}.$
- $\exp(tX_\beta) = \exp(tX_{\hat{\beta}^+}) \cdot \exp(tX_{\hat{\beta}^-})$ for any $t \in \mathbb{C}$

**Proof.** Since $\mathfrak{g}_\beta \subseteq \mathfrak{g}_{\hat{\beta}^+} \oplus \mathfrak{g}_{\hat{\beta}^-}$ and is not equal to any of the 2 root spaces on the right hand side, $X_\beta = X_{\hat{\beta}^+} + X_{\hat{\beta}^-}$ for nonzero $X_{\hat{\beta}^\pm} \in \mathfrak{g}_{\hat{\beta}^\pm}$. Then there exist $\mathfrak{sl}_2$-triples of the form $(X_{\hat{\beta}^+}, H_{\hat{\beta}^+}, X_{\hat{\beta}^-})$ and $(X_{\hat{\beta}^-}, H_{\hat{\beta}^-}, X_{\hat{\beta}^-})$ for $\hat{\beta}^+$ and $\hat{\beta}^-$. Moreover, $X_{-\beta} = aX_{-\hat{\beta}^+} + bX_{-\hat{\beta}^-}$ with nonzero constants $a$ and $b$, and

$$H_\beta = [X_\beta, X_{-\beta}] = [X_{\hat{\beta}^+} + X_{\hat{\beta}^-}, aX_{-\hat{\beta}^+} + bX_{-\hat{\beta}^-}] = aH_{\hat{\beta}^+} + bH_{\hat{\beta}^-}.$$ 

For the last equality we use that $[X_{\hat{\beta}^+}, X_{-\hat{\beta}^-}] = 0$ and $[X_{\hat{\beta}^-}, X_{-\hat{\beta}^+}] = 0$. Indeed, if $\hat{\beta}^+ - \hat{\beta}^-$ (resp. $\hat{\beta}^- - \hat{\beta}^+$) was a root in $\hat{\Phi}$, then $\rho(\hat{\beta}^+ - \hat{\beta}^-) = 0$ (resp. $\rho(\hat{\beta}^- - \hat{\beta}^+) = 0$). But this is absurd since $\rho$ sends $\hat{\Phi}$ to $\Phi$.

Using that $\rho(\hat{\beta}^+) = \rho(\hat{\beta}^-) = \beta$, we get that:

$$2 = \beta(H_\beta) = \hat{\beta}^+ (aH_{\hat{\beta}^+} + bH_{\hat{\beta}^-}) = a\hat{\beta}^+(aH_{\hat{\beta}^+}) = 2a.$$ 

Where for the second-last inequality we used that $\hat{\beta}^+$ and $\hat{\beta}^-$ are orthogonal, hence $\hat{\beta}^+(H_{\hat{\beta}^-}) = 0$. Then $a = 1$, and similarly we get $b = 1$. This proves the first point.

For the second one notice that $[X_{\hat{\beta}^+}, X_{\hat{\beta}^-}] = 0$. In fact if $\hat{\beta}^+ + \hat{\beta}^-$ was a root, then $\rho(\hat{\beta}^+ + \hat{\beta}^-) = 2\beta$. But this is absurd since $2\beta$ is not a root. Hence for any $t \in \mathbb{C}$, $\exp(tX_{\hat{\beta}^+} + tX_{\hat{\beta}^-}) = \exp(tX_{\hat{\beta}^+}) \cdot \exp(tX_{\hat{\beta}^-})$, and the second point follows immediately. \qed

**Remark.** For the proof we could also used [Res10] Lemma 4.1 and [Res10] Lemma4.3 to reduce the problem to the case of $\text{PSL}_2$ diagonally embedded in $\text{PSL}_2 \times \text{PSL}_2$.

3 Proof of Theorem [1]

3.1 An embedding of the multiplicity space

Fix dominant weights $\nu \in X(T)^+$ and $\tilde{\nu} \in X(\hat{T})^+$. Observe that $H_0$ is a subgroup of $\tilde{\gamma}_0 \tilde{B} \tilde{y}_0^{-1}$ and $\tilde{\gamma}_0 \tilde{\nu}$ is a character of this last group. In particular, $\tilde{\gamma}_0 \tilde{\nu}$ restricts as a character of $H_0$.

**Lemma 9.** In the $G$-representation $V_\nu^*$, the subspaces

$$(V_\nu^*)^{(H_0)_{\tilde{y}_0}} = \{v \in V_\nu^*: \forall h \in H_0 \quad h.v = (\tilde{y}_0 \tilde{\nu})(h)v\}$$

and

$$\{v \in V_\nu^*: \forall \alpha \in \Phi^+ \cap \Phi^1 \quad X_\alpha \cdot v = 0\}$$

coincide.
Now, Lemma 4 allows to conclude. □

Consider on $G/B$ the $G$-linearized line bundle $\mathcal{L}_\nu$ such that $B$ acts on the fiber over $B/B$ with weight $-\nu$. Similarly, define $\mathcal{L}_\tilde{\nu}$. By the Borel-Weyl theorem, the space

$$H^0(G/B \times \tilde{G}/\tilde{B}, \mathcal{L}_\nu \otimes \mathcal{L}_{\tilde{\nu}})^G \simeq (V_{\nu}^* \otimes V_{\tilde{\nu}}^*)^G$$

of $G$-invariant sections identifies with $\text{Mult}(\nu, \tilde{\nu})$. The orbit $G.\tilde{y}_0 \tilde{B}/\tilde{B}$ being open, the restriction map

$$H^0(G/B \times \tilde{G}/\tilde{B}, \mathcal{L}_\nu \otimes \mathcal{L}_{\tilde{\nu}})^G \to H^0(G/B \times G.\tilde{y}_0 \tilde{B}/\tilde{B}, \mathcal{L}_\nu \otimes \mathcal{L}_{\tilde{\nu}})^G$$

is injective. Moreover

$$H^0(G/B \times G.\tilde{y}_0 \tilde{B}/\tilde{B}, \mathcal{L}_\nu \otimes \mathcal{L}_{\tilde{\nu}})^G \simeq H^0(G/B, \mathcal{L}_\nu)^{(H_0)_{\tilde{y}_0}} \simeq (V_{\nu}^*)^{(H_0)_{\tilde{y}_0}}.$$

For $\varphi \in (V_{\nu}^*)^{(H_0)_{\tilde{y}_0}}$ and denote by $\tilde{\sigma} \in H^0(G/B \times G.\tilde{y}_0 \tilde{B}/\tilde{B}, \mathcal{L}_\nu \otimes \mathcal{L}_{\tilde{\nu}})^G$ the associated section. To describe the image of $H^0(G/B \times \tilde{G}/\tilde{B}, \mathcal{L}_\nu \otimes \mathcal{L}_{\tilde{\nu}})^G$ in $(V_{\nu}^*)^{(H_0)_{\tilde{y}_0}}$, we have to understand what sections $\tilde{\sigma}$ extend, and hence the order of the poles of $\tilde{\sigma}$ along the divisors of $(G/B \times \tilde{G}/\tilde{B}) - (G/B \times G.\tilde{y}_0 \tilde{B}/\tilde{B})$.

### 3.2 Local description along the divisors

Fix $\tilde{\alpha} \in \tilde{\Delta}$ that is a label of some edge descending from the open orbit in $\Gamma(\tilde{G}/G)$. Then $D_{\tilde{\alpha}} := G.\tilde{y}_0 s_{\tilde{\alpha}} \tilde{B}/\tilde{B}$ is a divisor of $\tilde{G}/\tilde{B}$. By Lemma 7 this happens if and only if $\tilde{y}_0 s_{\tilde{\alpha}} \in \tilde{\Phi}^2$. Set $\beta = \pm \rho(\tilde{y}_0 \tilde{\alpha})$ the sign being chosen to get $\beta \in \Phi^+$. Lemma 8 allows to distinguish two situations that lead to two slightly different local descriptions of the corresponding divisor.

1) $\tilde{y}_0 \tilde{\alpha}$ is negative. Then $\beta = -\rho(\tilde{y}_0 \tilde{\alpha})$ and $\tilde{y}_0 \tilde{\alpha} = -\beta^-$.

2) $\tilde{y}_0 \tilde{\alpha}$ is positive. Then $\beta = \rho(\tilde{y}_0 \tilde{\alpha})$ and $\tilde{y}_0 \tilde{\alpha} = \beta^+$. Given a root $\alpha$ (resp. $\tilde{\alpha}$) of $G$ (resp. $\tilde{G}$), we denote by $U_{\alpha}$ (resp. $U_{\tilde{\alpha}}$) the corresponding subgroup isomorphic to $(\mathbb{C}, +)$. The aim of this section is to describe an open subset of $\tilde{G}/\tilde{B}$ intersecting the divisor $D_{\tilde{\alpha}}$. In particular we will prove that $U_{\tilde{\alpha}}$ is a common transverse slice to $D_{\tilde{\alpha}}$ at any point of this subset. Before doing so we need some preparatory work.

**Lemma 10.** Let $\tilde{\alpha}$, $\beta$ and $D_{\tilde{\alpha}}$ be as above. Set $S(\tilde{\alpha}) := (\Phi^+ \cap \Phi^2) \setminus \{\beta\}$. Index the elements of $S(\tilde{\alpha}) = \{\gamma_1, \ldots, \gamma_s\}$. Then:
1) If \( \hat{y}_0 \hat{\alpha} \) is negative then the map
\[
i_{\hat{\alpha}} : U^- \times \prod_{\gamma \in S(\hat{\alpha})} U_\gamma \longrightarrow G\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}
\]
\[
(u^-, (u_\gamma)_\gamma) \longmapsto (u^- \prod_{i=1}^s u_{\gamma_i}) \hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}
\]
is an open immersion.

2) If \( \hat{y}_0 \hat{\alpha} \) is positive then the map
\[
i_{\hat{\alpha}} : U^- \times \prod_{\gamma \in S(\hat{\alpha})} U_\gamma \longrightarrow G\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}
\]
\[
(u^-, (u_\gamma)_\gamma) \longmapsto s_\beta(u^- \prod_{i=1}^s u_{\gamma_i}) s_\beta \hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}
\]
is an open immersion.

Our proof of Lemma 10 depends on the following well-known lemma.

**Lemma 11.**

1. The product induces an open immersion
\[
U^- \times U \times T \longrightarrow G
\]
\[
(u^-, u, t) \longmapsto u^- u t.
\]

2. Let \( H \) be a \( T \)-stable closed subgroup of \( U \). Number \( \gamma_1, \ldots, \gamma_s \) the positive roots of \( G \) that are not roots of \( H \). Then the map
\[
U_{\gamma_1} \times \cdots \times U_{\gamma_s} \times H \longrightarrow U
\]
\[
((u_\gamma), u) \longmapsto u_1 \cdots u_s u
\]
is an isomorphism.

**Proof.** The first assertion is a part of the Bruhat decomposition (see e.g. [Hum75, Section 28.5]). The second assertion is the content of [Hum75, Section 28.1]. \( \square \)

**Proof.** We begin by the negative case. Notice that Lemma 7 implies that \( \gamma \in S(\hat{\alpha}) \) if and only if \( U_\gamma \not\subseteq U_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} \). Applying the second assertion of Lemma 11 with \( H = U_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} \), one gets an isomorphism
\[
\prod_{\gamma \in S(\hat{\alpha})} U_\gamma \times U_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} \longrightarrow U.
\]

Now, again by Lemma 7, \( G_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} = TU_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} \). Then the first assertion of Lemma 11 implies that
\[
U^- \times \prod_{\gamma \in S(\hat{\alpha})} U_\gamma \times G_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} \longrightarrow G
\]
is an open embedding. Taking the quotient by \( G_{\hat{y}_0 s_{\hat{\alpha}} \hat{B}/\hat{B}} \) we conclude.

The second case is obtained similarly by applying Lemma 11 to the maximal unipotent subgroups \( s_\beta U^\pm s_\beta \) in place of \( U^\pm \). \( \square \)
Proposition 12. Keep the setting as in Lemma \[10\]

1) If \(\hat{y}_{0}\tilde{\alpha}\) is negative then the map:

\[
f_{\tilde{\alpha}} : U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times U_{-\tilde{\alpha}} \longrightarrow \hat{G}/\hat{B}
\]

\[
(u^{-}, (u_{\gamma}, u_{-\tilde{\alpha}}) \longmapsto (u^{-} \prod_{i=1}^{s} u_{\gamma_{i}} \hat{y}_{0}s_{\tilde{\alpha}}u_{-\tilde{\alpha}} \hat{B})
\]

is an open immersion. Furthermore the image of \(f_{\tilde{\alpha}}\) is contained in \(G\hat{y}_{0}\hat{B}/\hat{B} \cup D_{\tilde{\alpha}}\), and \(f_{\tilde{\alpha}}^{-1}(D_{\tilde{\alpha}}) = U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times \{1\}\).

2) If \(\hat{y}_{0}\tilde{\alpha}\) is positive then the map:

\[
f_{\tilde{\alpha}} : U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times U_{-\tilde{\alpha}} \longrightarrow \hat{G}/\hat{B}
\]

\[
(u^{-}, (u_{\gamma}, u_{-\tilde{\alpha}}) \longmapsto s_{\beta}(u^{-} \prod_{i=1}^{s} u_{\gamma_{i}} s_{\beta}\hat{y}_{0}s_{\tilde{\alpha}}u_{-\tilde{\alpha}} \hat{B})
\]

is an open immersion. Furthermore the image of \(f_{\tilde{\alpha}}\) is contained in \(G\hat{y}_{0}\hat{B}/\hat{B} \cup D_{\tilde{\alpha}}\), and \(f_{\tilde{\alpha}}^{-1}(D_{\tilde{\alpha}}) = U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times \{1\}\).

Proof. We prove part 1). The proof of the positive case is obtained from the following one by replacing every appearance of \(\hat{y}_{0}\) with \(s_{\beta}\hat{y}_{0}\). Identify \(\hat{G}/\hat{B}\) with the fibered product \(\hat{G} \times_{P_{\tilde{\alpha}}} P_{\tilde{\alpha}}/\hat{B}\). Via this identification \(f_{\tilde{\alpha}}\) is the map:

\[
f_{\tilde{\alpha}} : U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times U_{-\tilde{\alpha}} \longrightarrow \hat{G} \times_{P_{\tilde{\alpha}}} P_{\tilde{\alpha}}/\hat{B}
\]

\[
(u^{-}, (u_{\gamma}, u_{-\tilde{\alpha}}) \longmapsto [(u^{-} \prod_{i=1}^{s} u_{\gamma_{i}} \hat{y}_{0}s_{\tilde{\alpha}}) : u_{-\tilde{\alpha}} \hat{B}/\hat{B})].
\]

Since \(\hat{y}_{0}s_{\alpha}\hat{B}/\hat{B}\) and \(\pi_{\alpha}(\hat{y}_{0}s_{\alpha} \hat{B}/\hat{B})\) have the same isotropy in \(G\), we can identify their \(G\)-orbits. In particular we will think about \(i_{\tilde{\alpha}}\), the open immersion of Lemma \[10\] as a map to \(G\hat{y}_{0}s_{\alpha}P_{\tilde{\alpha}}/P_{\tilde{\alpha}} \subseteq \hat{G}/P_{\tilde{\alpha}}\). Call \(V_{\tilde{\alpha}} \subseteq \hat{G}/P_{\tilde{\alpha}}\) the image of \(i_{\tilde{\alpha}}\). By Lemma \[10\] \(V_{\tilde{\alpha}}\) is open in \(\hat{G}/P_{\tilde{\alpha}}\). Denote the two components of

\[
i_{\tilde{\alpha}}^{-1} : V_{\tilde{\alpha}} \longrightarrow U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma},
\]

by \(j_{1} : V_{\tilde{\alpha}} \longrightarrow U^{-}\) and \(j_{2} : V_{\tilde{\alpha}} \longrightarrow \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma}\), respectively.

Observe that \(U_{-\tilde{\alpha}} \simeq U_{-\tilde{\alpha}} \hat{B}/\hat{B} = P_{\tilde{\alpha}}/\hat{B} \setminus (\hat{B}/\hat{B})\) is open in \(P_{\tilde{\alpha}}/\hat{B}\). The image of \(f_{\tilde{\alpha}}\) is

\[
\Omega_{\tilde{\alpha}} := \{ [[\hat{g} : x]] \in \hat{G} \times_{P_{\tilde{\alpha}}} P_{\tilde{\alpha}}/\hat{B} : [\hat{g}] \in V_{\tilde{\alpha}} \text{ and } s_{\alpha}^{-1}\hat{y}_{0}^{-1}j_{2}([\hat{g}])^{-1}j_{1}([\hat{g}])^{-1}g \in U_{-\tilde{\alpha}} \hat{B}/\hat{B})\},
\]

where \([\hat{g}] = \hat{g}P_{\alpha}/\hat{P}_{\tilde{\alpha}} \in \hat{G}/P_{\tilde{\alpha}}\). We deduce that \(\Omega_{\tilde{\alpha}}\) is open in \(\hat{G} \times_{P_{\tilde{\alpha}}} P_{\tilde{\alpha}}/\hat{B}\).

Finally we prove that \(f_{\tilde{\alpha}}\) is an isomorphism. If we call \(\phi_{\tilde{\alpha}} : U_{-\tilde{\alpha}} \hat{B}/\hat{B} \longrightarrow U_{-\tilde{\alpha}}\) the inverse of the natural projection, then it’s easy to see that the map

\[
\Omega_{\tilde{\alpha}} \longrightarrow U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times U_{-\tilde{\alpha}}
\]
that sends \([\hat{g}, x]\) to:

\[
(i_{\tilde{\alpha}}^{-1})_1([\hat{g}]) \mathbin{,} (i_{\tilde{\alpha}}^{-1})_2([\hat{g}]) \mathbin{,} \phi_{\tilde{\alpha}}(s_{\tilde{\alpha}}^{-1}y_0^{-1}f_2([\hat{g}])^{-1}j_1([\hat{g}])^{-1}gx)
\]

is the inverse of \(f_{\tilde{\alpha}}\).

To conclude, notice that for \(\alpha - \tilde{\alpha} = 1\), \(f_{\tilde{\alpha}}\) restricts to \(i_{\tilde{\alpha}}\), hence \(\Omega_{\tilde{\alpha}} \cap D_{\tilde{\alpha}}\) is an open dense subset of the divisor.

For \(\alpha - \tilde{\alpha} \neq 1\), \(y_0s_{\tilde{\alpha}}u_{\alpha - \tilde{\alpha}}\hat{B}/\hat{B} \neq y_0s_{\tilde{\alpha}}B/\hat{B}\). Hence \(y_0s_{\tilde{\alpha}}u_{\alpha - \tilde{\alpha}}\hat{B}/\hat{B}\) has to be a point of the open \(G\)-orbit, that is the orbit of \(y_0\hat{B}/\hat{B}\).

We conclude that \(f_{\tilde{\alpha}}\) maps \(U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times \{1\}\) in the open \(G\)-orbit and that \(f_{\tilde{\alpha}}^{-1}(D_{\tilde{\alpha}}) = U^{-} \times \prod_{\gamma \in S(\tilde{\alpha})} U_{\gamma} \times \{1\}\).

\[\square\]

### 3.3 Conclusion

In the setting of the end of Section 3.1, we are now in position to characterize the image of the embedding:

\[H^0(G/B \times \hat{G}/\hat{B}, \mathcal{L}_\nu \otimes \mathcal{L}_\varphi)^G \longrightarrow H^0(G/B, \mathcal{L}_\nu)^{(H_0)\hat{\varphi}}.\]  \hspace{1cm} (2)

Fix \(\varphi \in (V_\nu^*)^{(H_0)\hat{\varphi}}\) and follow it through the following isomorphisms:

\[
(V_\nu^*)^{(H_0)\hat{\varphi}} \mathbin{\overset{\varphi}{\longrightarrow}} H^0(G/B, \mathcal{L}_\nu)^{(H_0)\hat{\varphi}} \mathbin{\overset{\sigma}{\longrightarrow}} H^0(G/B \times G, \hat{\varphi}_{\gamma_0}\hat{B}/\hat{B}, \mathcal{L}_\nu \otimes \mathcal{L}_\varphi)^G \mathbin{\overset{\tilde{\sigma}}{\longrightarrow}} \tilde{\sigma}.
\]

Fix \(v_0 \in V_\nu(B)\) and \(\tilde{\gamma}_0 \in (\mathcal{L}_\varphi)_{\tilde{\gamma}_0} - \{0\}\). Explicitly, \(\sigma\) and \(\tilde{\sigma}\) are given by the formulas

\[\forall g \in G \quad \sigma(gB/B) = [g : \varphi(gv_0)]\]

\[\forall g_1, g_2 \in G \quad \tilde{\sigma}(g_1B/B, g_2\hat{\gamma}_0\hat{B}/\hat{B}) = [g_1 : \varphi(g_2^{-1}g_1v_0)] \otimes g_2\hat{\gamma}_0.
\]

We want to determine when \(\tilde{\sigma}\) extends to a global section. Take \(\tilde{\alpha} \in \hat{\Delta}\) that is a label of some edge descending from the open orbit. Then \(D_{\tilde{\alpha}} := \overline{G\hat{\gamma}_0s_{\tilde{\alpha}}\hat{B}/\hat{B}}\) is a divisor of \(\hat{G}/\hat{B}\) along which, we want to determine the vanishing order of \(\tilde{\sigma}\). Consider the image \(V_{\tilde{\alpha}}\) of the map \(i_{\tilde{\alpha}}\) defined by Proposition [12]

**Proposition 13.** 1. Assume that \(\tilde{\gamma}_0\tilde{\alpha}\) is negative. Then, the section \(\tilde{\sigma}\) extends to a regular section on \(G/B \times V_{\tilde{\alpha}}\) if and only if

\[X_{\beta}^m \varphi = 0 \text{ for } m > (\tilde{\gamma}_0 \cdot \hat{\nu}, (\hat{\beta}^-)^\vee).
\]

2. Assume that \(\tilde{\gamma}_0\tilde{\alpha}\) is positive. Then, the section \(\tilde{\sigma}\) extends to a regular section on \(G/B \times V_{\tilde{\alpha}}\) if and only if

\[X_{\beta}^m \varphi = 0 \text{ for } m > (\tilde{\gamma}_0 \cdot \hat{\nu}, (\hat{\beta}^+)^\vee).
\]

**Proof.** Denote by \(\epsilon_{\pm \beta} : \mathbb{C} \longrightarrow \mathfrak{g}_{\pm \beta} \xrightarrow{\exp} U_{\pm \beta}\) the additive one-parameter subgroups, and think about \(\beta^\vee : \mathbb{C}^* \longrightarrow T \subset G\) as a one-parameter subgroup. These three morphisms glue to give
a group homomorphism \( \phi_\beta : \text{SL}_2(\mathbb{C}) \rightarrow G \). The same notation is used, in the obvious way, also for \( \hat{G} \). Now fix an \( \mathfrak{sl}_2 \)-triple \((X_\delta, H_\delta, X_{-\delta})\) for \( \delta \), with \( X_\delta \in \mathfrak{g}_\delta \).

Suppose first that we are in the negative case, that is \( \hat{\gamma}_0 \delta = -\hat{\beta}^- \), where \( \beta = -\rho(\hat{\gamma}_0 \delta) \). Because of Proposition \([12]\) \( \delta \) extends to a section on \( G/B \times V_\delta \) if and only if the map:

\[
G/B \times U^- \times \prod_{\gamma \in S(\delta)} U_\gamma \times \mathbb{C}^* \longrightarrow \mathcal{L}_\nu \otimes \mathcal{L}_\varphi \quad (gB/B, u^-, (u_\gamma)_\gamma, t) \mapsto \hat{\sigma}((gB/B, u^- \prod_{\gamma=1}^s \hat{\gamma}_0 \delta \epsilon_{-\delta}(t) \hat{B}/\hat{B}))
\]

extends at \( t = 0 \). Notice also that \( U^- \) and \( U_\gamma \) are subgroups of \( G \) and \( \hat{\sigma} \) is \( G \)-invariant, hence the function above extend at \( t = 0 \) if and only if the following map, that with a little abuse will still be called \( \hat{\sigma} \), extends at \( t = 0 \).

\[
G/B \times \mathbb{C}^* \longrightarrow \mathcal{L}_\nu \otimes \mathcal{L}_\varphi \quad (gB/B, t) \mapsto \hat{\sigma}((gB/B, \hat{\gamma}_0 \delta \epsilon_{-\delta}(t) \hat{B}/\hat{B})).
\]

Now fix \( \mathfrak{sl}_2 \)-triples for \( \hat{\beta}^\pm \) as in Lemma \([8]\) so that for any \( t \in \mathbb{C}^* \), \( \epsilon_{\pm \beta}(t) = \epsilon_{\pm \hat{\beta}^\pm}(t) \epsilon_{\pm \hat{\beta}^-}(t) \).

In \( \text{SL}_2 \), we have

\[
\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t^{-1} & 1 \end{bmatrix} \begin{bmatrix} -t & 0 \\ 0 & -t^{-1} \end{bmatrix} \begin{bmatrix} 1 & -t^{-1} \\ 0 & 1 \end{bmatrix},
\]

for any \( t \in \mathbb{C}^* \). Hence

\[
\epsilon_{\delta}(t)s_\delta = \epsilon_{-\delta}(t^{-1})\delta^\lor(-t)\epsilon_{\delta}(-t^{-1}). \tag{5}
\]

Where \( \delta \) can be replaced by any positive root of \( G \) or \( \hat{G} \) for which a corresponding \( \mathfrak{sl}_2 \)-triple has been fixed. Now take \((gB/B, t) \in G/B \times \mathbb{C}^* \),

\[
(gB/B, \hat{\gamma}_0 \delta \epsilon_{-\delta}(t) \hat{B}/\hat{B}) = (gB/B, \hat{\gamma}_0 \epsilon_{-\delta}(t) s_\delta \hat{B}/\hat{B}) = (gB/B, \hat{\gamma}_0 \epsilon_{-\delta}(t^{-1}) \hat{B}/\hat{B}) = (gB/B, \epsilon_{\delta^-}(c^{-1}t^{-1}) \hat{\gamma}_0 \hat{B}/\hat{B})
\]

where \( c \) is the nonzero constant satisfying \( \hat{\gamma}_0 X_\delta = cX_{-\hat{\beta}^-} \). But since \( U_{\beta^+} \subset \hat{\gamma}_0 \hat{B}\hat{\gamma}_0^{-1} \):

\[
\epsilon_{\delta^-}(c^{-1}t^{-1}) \hat{\gamma}_0 \hat{B}/\hat{B} = \epsilon_{\hat{\beta}^-}(c^{-1}t^{-1}) \epsilon_{\hat{\beta}^+}(c^{-1}t^{-1}) \hat{\gamma}_0 \hat{B}/\hat{B} = \epsilon_{\beta}(c^{-1}t^{-1}) \hat{\gamma}_0 \hat{B}/\hat{B}.
\]

Now, Formula \([6]\) gives

\[
\hat{\sigma}((gB/B, \hat{\gamma}_0 \delta \epsilon_{-\delta}(t) \hat{B}/\hat{B})) = [g : \varphi(\epsilon_{\beta}(-c^{-1}t^{-1})gv_0)] \otimes \epsilon_{\beta}(c^{-1}t^{-1}) \hat{\gamma}_0.
\]

Moreover,

\[
\epsilon_{\beta}(c^{-1}t^{-1}) \hat{\gamma}_0 = \epsilon_{\hat{\beta}^-}(c^{-1}t^{-1}) \epsilon_{\hat{\beta}^+}(c^{-1}t^{-1}) \hat{\gamma}_0 = \epsilon_{\hat{\beta}^-}(ct)(s_{\hat{\beta}^-})^{-1}(\hat{\beta}^-)^\lor(-ct)\epsilon_{\hat{\beta}^-}(c^{-1}t^{-1}) \hat{\gamma}_0 \quad \text{by formula } [5]
\]

since \( U_{\hat{\beta}^-} \subset \hat{\gamma}_0 \hat{U} \hat{\gamma}_0^{-1} \) and \( T \) acts with weight \( -\hat{\gamma}_0 \hat{\nu} \).
Rewrite the term $\varphi(\epsilon_\beta(-c^{-1}t^{-1})gv_0)$ as
\[
\varphi(\epsilon_\beta(-c^{-1}t^{-1})gv_0) = \langle \epsilon_\beta(c^{-1}t^{-1})\varphi, gv_0 \rangle = \sum_{n \geq 0} \frac{(c^{-1}t^{-1})^n}{n!} (X^n_\beta \varphi, gv_0).
\]
Finally, we get
\[
\dot{\sigma}((gB/B, \tilde{y}_0 s_\alpha \tilde{\epsilon}_-\tilde{\alpha}(t)\tilde{B}/\tilde{B})) = [g : \sum_{n \geq 0} \frac{(c^{-1}t^{-1})^n(\tilde{y}_0, (\tilde{\beta}^-)^{\vee})}{n!} (X^n_\beta \varphi, gv_0) \otimes (\epsilon_\beta-\tilde{(ct)}(s_\beta^-)^{-1}\tilde{y}_0).
\]
The term $\epsilon_\beta-\tilde{(ct)}(s_\beta^-)^{-1}\tilde{y}_0$ is regular on $C$. Hence $\dot{\sigma}$ has no pole along $t = 0$ if and only if
\[
\forall n > \langle -\tilde{y}_0, (\tilde{\beta}^-)^{\vee} \rangle \implies \langle X^n_\beta \varphi, gv_0 \rangle = 0 \in C[G].
\]
But $Gv_0$ spans $V_\alpha$ that is irreducible. As a consequence, $\langle X^n_\beta \varphi, gv_0 \rangle = 0$ if and only if $X^n_\beta \varphi = 0$.

Suppose now that we are in the positive case, hence $\tilde{y}_0 \cdot \tilde{\alpha} = \tilde{\beta}^+$ with $\beta = \rho(\tilde{y}_0, \tilde{\alpha})$. The outline of the proof doesn’t change. By the same argument of the previous case, $\dot{\sigma}$ extends to a section on $G/B \times V_\alpha$ if and only if the following map, that will still be called $\dot{\sigma}$, extends at $t = 0$.

\[
\begin{align*}
G/B \times C^* & \longrightarrow L_\alpha \otimes L_\beta \\
(gB/B, t) & \mapsto \dot{\sigma}((gB/B, s_\beta\tilde{y}_0 s_\alpha \epsilon_\alpha^{-1}(t)\tilde{B}/\tilde{B}))
\end{align*}
\]
Again fix $\mathfrak{sl}_2$-triples for $\tilde{\beta}^\pm$ as in Lemma \(\blacksquare\). Take $(gB/B, t) \in G/B \times C^*$, then
\[
\begin{align*}
(gB/B, s_\beta\tilde{y}_0 s_\alpha \epsilon_\alpha^{-1}(t)\tilde{B}/\tilde{B}) &= (gB/B, s_\beta\tilde{y}_0 s_\alpha \epsilon_\alpha^{-1}(t)\tilde{B}/\tilde{B}) \\
&= (gB/B, s_\beta \epsilon_\beta^{-1}(c^{-1}t^{-1})\tilde{y}_0) \tilde{B}/\tilde{B}
\end{align*}
\]
where now $c$ is the nonzero constant satisfying $\tilde{y}_0X_\alpha = cX_{\beta^+}$. Hence:
\[
\dot{\sigma}((gB/B, s_\beta\tilde{y}_0 s_\alpha \epsilon_\alpha^{-1}(t)\tilde{B}/\tilde{B})) = [g : \varphi(\epsilon_\beta(-c^{-1}t^{-1})(s_\beta^-)^{-1}gv_0) \otimes s_\beta \epsilon_\beta^{-1}(c^{-1}t^{-1})\tilde{y}_0].
\]
Similarly to the previous computations:
\[
\epsilon_\beta(-c^{-1}t^{-1})\tilde{y}_0 = \epsilon_\beta^+(c^{-1}t^{-1})\epsilon_\beta^-(-c^{-1}t^{-1})\tilde{y}_0 = \epsilon_\beta^+(ct)s_\beta^+ \epsilon_\beta^+(c^{-1}t^{-1})(\tilde{\beta}^\vee)(-c^{-1}t^{-1})\tilde{y}_0 = \epsilon_\beta^+(ct)s_\beta^+(-c^{-1}t^{-1})(-\tilde{y}_0, (\tilde{\beta}^\vee)^\vee).
\]
While for the other term
\[
\varphi(\epsilon_\beta(-c^{-1}t^{-1})s_\beta^{-1}gv_0) = \langle \epsilon_\beta(c^{-1}t^{-1})\varphi, s_\beta^{-1}gv_0 \rangle = \sum_{n \geq 0} \frac{(c^{-1}t^{-1})^n}{n!} (X^n_\beta \varphi, s_\beta^{-1}gv_0).
\]
Finally we get
\[
\dot{\sigma}((gB/B, s_\beta\tilde{y}_0 s_\alpha \epsilon_\alpha^{-1}(t)\tilde{B}/\tilde{B})) = [g : \sum_{n \geq 0} \frac{(c^{-1}t^{-1})^n(-\tilde{y}_0, (\tilde{\beta}^\vee)^\vee)}{n!} (X^n_\beta \varphi, s_\beta^{-1}gv_0) \otimes (s_\beta \epsilon_\beta^+(ct)s_\beta^+)\tilde{y}_0].
\]
By the same argument of the previous case we conclude that \( \tilde{\sigma} \) has no pole along \( t = 0 \) if and only if

\[
\forall n > \langle \tilde{y}_0 \hat{\nu}, (\tilde{\beta}^+)\rangle \implies X^*_\beta \varphi = 0.
\]

\[\square\]

**Remark.** If \( \tilde{y}_0 \hat{\alpha} = -\beta^- \), then \( \tilde{y}_0 \cdot \hat{\alpha}^\vee = -(\tilde{\beta}^-)^\vee \), hence:

\[
\langle -\tilde{y}_0 \cdot \hat{\nu}, (\tilde{\beta}^-)^\vee \rangle = \langle \tilde{y}_0 \cdot \hat{\nu}, \tilde{y}_0 \cdot \hat{\alpha}^\vee \rangle = \langle \hat{\nu}, \hat{\alpha}^\vee \rangle.
\]

Similarly, if \( \tilde{y}_0 \hat{\alpha} = \tilde{\beta}^+ \), then \( \tilde{y}_0 \cdot \hat{\alpha}^\vee = (\tilde{\beta}^+)^\vee \), which implies:

\[
\langle \tilde{y}_0 \cdot \hat{\nu}, (\tilde{\beta}^+)\rangle = \langle \tilde{y}_0 \cdot \hat{\nu}, \tilde{y}_0 \cdot \hat{\alpha}^\vee \rangle = \langle \hat{\nu}, \hat{\alpha}^\vee \rangle.
\]

Recall that in the introduction we set \( D = \{ \hat{\alpha} \in \hat{\Phi} : \rho(\tilde{y}_0 \hat{\alpha}) \notin \Phi^1 \} \), for \( \hat{\alpha} \in D \), let \( V_{\hat{\alpha}} \) be as in Proposition \[\underline{13} \] We prove the theorem of the introduction.

**Proof of Theorem 1.** The first condition of the theorem is implied by Lemma \[\underline{9} \] Then set \( V := \bigcup_{\hat{\alpha} \in D} G/B \times V_{\hat{\alpha}} \), which is open in \( G/B \times \widehat{G/B} \). The complement of \( V \) is of codimension strictly larger than 1 and \( G/B \times \widehat{G/B} \) is normal, hence the restriction map

\[
H^0(G/B \times \widehat{G/B}, L_\nu \otimes L_{\hat{\nu}}) \rightarrow H^0(V, L_\nu \otimes L_{\hat{\nu}})
\]

is an isomorphism. Then, from Proposition \[\underline{13} \] and from the previous remark we deduce that \( \varphi \in (V^*_\nu)^{(H^0)_{\tilde{y}_0 \nu}} \) is in the image of

\[
H^0(G/B \times \widehat{G/B}, L_\nu \otimes L_{\hat{\nu}}) \rightarrow (V^*_\nu)^{(H^0)_{\tilde{y}_0 \nu}}
\]

if and only if the second condition of the theorem holds.

\[\square\]

As remarked in the introduction, the conditions of Theorem \[\underline{1} \] are in general redundant. This is linked to the fact in \( \Gamma(\widehat{G}/G) \) we may have different edges between the same vertices. It seems not easy to determine a minimal set of conditions in a uniform way. The following lemma checks that, for the tensor product case, a minimal set of conditions is described by the set:

\[
\{ (\alpha, 0) \in \Phi \times \Phi : \alpha \in \Delta \}.
\]

**Lemma 14.** For \( \beta \in \Phi^+ \), and \( \sigma \in V^*_\nu(\tilde{y}_0 \cdot \hat{\nu}) \), the following are equivalent:

1) \( X^m_\beta \cdot \sigma = 0 \) for \( m > \langle -\tilde{y}_0 \cdot \hat{\nu}, (\tilde{\beta}^-)^\vee \rangle \).

2) \( X^m_{-\beta} \cdot \sigma = 0 \) for \( m > \langle \tilde{y}_0 \cdot \hat{\nu}, (\tilde{\beta}^+)\rangle \).

**Remark.** If \( \beta \in \Phi^+ \) and \( \hat{\alpha}_1, \hat{\alpha}_2 \in \hat{\Delta} \) satisfy:

\[
\tilde{y}_0 \cdot \hat{\alpha}_1 = -\tilde{\beta}^- \text{ and } \tilde{y}_0 \cdot \hat{\alpha}_1 = \tilde{\beta}^+
\]

then \( \tilde{y}_0 s_{\hat{\alpha}_1} = s_{\beta} \tilde{y}_0 s_{\hat{\alpha}_2} \), hence their \( G \)-orbit is the same, and by the proof we have done we expect that \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) give the same condition in Theorem \[\underline{1} \] This is directly checked in previous lemma.

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Proof. Call \(\mathfrak{s}\mathfrak{l}_2\) the subalgebra of \(\mathfrak{g}\) spanned by \(X_\beta, H_\beta, X_{-\beta}\). Decompose \(V_\nu^*\) into a direct sum of \(\mathfrak{s}\mathfrak{l}_2\) irreducible representations: \(V_\nu^* = \bigoplus V_\delta\). Write \(\sigma = \sum_\delta \sigma_\delta\) accordingly to this decomposition. Observe that \(\sigma_\delta \in V_\delta(\hat{y}_0 \cdot \tilde{\nu} |_{\hat{H}})\). Since \(V_\delta\) is an \(\mathfrak{s}\mathfrak{l}_2\) irreducible representation, if \(\sigma_\delta \neq 0\), then:

\[
X^m_\beta \cdot \sigma_\delta = 0 \iff \langle \hat{y}_0 \cdot \tilde{\nu}, \beta^\vee \rangle + 2m > \langle \delta, \beta^\vee \rangle, \quad \text{and} \quad
X^m_{-\beta} \cdot \sigma_\delta = 0 \iff \langle \hat{y}_0 \cdot \tilde{\nu}, \beta^\vee \rangle - 2m < -\langle \delta, \beta^\vee \rangle.
\]

Hence condition 1) is equivalent to:

\[
\forall \delta : \sigma_\delta \neq 0, \iff \langle \hat{y}_0 \cdot \tilde{\nu}, \beta^\vee \rangle - 2\langle \hat{y}_0 \cdot \tilde{\nu}, (\hat{\beta}^-)^\vee \rangle + 2 > \langle \delta, \beta^\vee \rangle.
\]

Similarly condition 2) is equivalent to:

\[
\forall \delta : \sigma_\delta \neq 0, \iff \langle \hat{y}_0 \cdot \tilde{\nu}, \beta^\vee \rangle - 2\langle \hat{y}_0 \cdot \tilde{\nu}, (\hat{\beta}^+)^\vee \rangle - 2 < -\langle \delta, \beta^\vee \rangle.
\]

But Lemma \(\S\) implies that \(\langle \hat{y}_0 \cdot \tilde{\nu}, \beta^\vee \rangle = \langle \hat{y}_0 \cdot \tilde{\nu}, (\hat{\beta}^-)^\vee \rangle + \langle \hat{y}_0 \cdot \tilde{\nu}, (\hat{\beta}^+)^\vee \rangle\), then we easily conclude that 1) and 2) are equivalent. \(\square\)

## 4 Explicit description on the examples

For each example, we determine a working \(\hat{y}_0\) and a set of simple roots parametrizing the \(G\)-stable divisors of \(\tilde{G}/\tilde{B}\). With the notation of the introduction, we give a subset \(D_0\) of \(\mathcal{D}\) such that the map \(\hat{\alpha} \mapsto \tilde{G}\hat{y}_0\hat{\alpha}\tilde{B}/\tilde{B}\) is a bijection from \(D_0\) onto the set of \(G\)-stable divisors.

### 4.1 Tensor product case

Here \(\hat{G} = G \times G\), \(\hat{T} = T \times T\) and \(\hat{B} = B \times B\). Moreover \(\Phi^1\) is empty, \(X(\hat{T}) = X(T) \times X(T)\) and \(\rho(\lambda, \mu) = \lambda + \mu\). Set \(\hat{y}_0 = (e, w_0) \in \hat{W} = W \times W\). It is clear that \(G\hat{y}_0\) is open in \(\tilde{G}/\tilde{B}\) and that \(\ell(\hat{y}_0) = \dim(\tilde{G}/\tilde{B}) - \dim(G/B)\). By the Bruhat decomposition \(\{G(B/B, w_0\hat{\alpha}B/B) : \alpha \in \Delta\}\) is the collection of codimension one \(G\)-orbits in \(\tilde{G}/\tilde{B}\). In particular the set

\[
\mathcal{D}_0^- = \{(0, \alpha) : \alpha \in \Delta\}
\]

works. Another possible choice is

\[
\mathcal{D}_0^+ = \{(\alpha, 0) : \alpha \in \Delta\}.
\]

Note that \(\hat{y}_0 = \hat{y}_0^{-1}\) and that, for \(\alpha \in \Delta\), \(\hat{y}_0(0, \alpha) = (0, w_0\alpha) \in \tilde{\Phi}^-,\) while \(\hat{y}_0(\alpha, 0) = (\alpha, 0) \in \tilde{\Phi}^+\). So, according to our notations, \((0, \alpha) = \hat{\alpha}^-\) and \((\alpha, 0) = \hat{\alpha}^+\). If \(\nu \in \tilde{X}(T)^+\) and \(\lambda, \mu \in \tilde{X}(T)\) we define two sets:

\[
V^+(\nu, \lambda, \mu) := \{v \in V_\nu(\lambda) : \forall \alpha \in \Delta, X^m_\alpha v = 0 \text{ for } m > \langle \mu, \alpha^\vee \rangle\}
\]

\[
V^-(\nu, \lambda, \mu) := \{v \in V_\nu(\lambda) : \forall \alpha \in \Delta, X^m_{-\alpha} v = 0 \text{ for } m > \langle \mu, \alpha^\vee \rangle\}
\]

From now on, fix \(\nu, \nu_1, \nu_2 \in \tilde{X}(T)^+\) and set \(\tilde{\nu} = (\nu_1, \nu_2) \in \tilde{X}(T \times T)^+\). We denote by \(\nu^* := -w_0\nu\), so that \(V_{\nu^*} \simeq V_{\nu}^*\). In Theorem 2.1 of \[\text{PRRV}67\] the authors realized, by algebraic methods isomorphisms:

\[
\text{Mult}(\nu, \tilde{\nu}^*) \simeq V^-((\nu, \nu_1 - \nu_2, \nu_1) \quad (7)
\]
\[ \text{Mult}(\nu, \nu^*) \simeq V^+(\nu, \nu_2 - \nu_1^*, \nu_1^*) \]  

We explain how this can be recovered from Theorem 11 using \( D_0^+ \) and \( D_0^- \) as parametrizations of the \( G \)-stable divisors. Notice that, in this case, the conditions 1 of the theorem are empty. Then \( \rho(\gamma_0 \nu) = (\nu_1 - \nu_2^*) \) and for \( \alpha^+ = (\alpha, 0) \in D_0^+ \), \( \rho(\gamma_0 \alpha^+) = \alpha \) and \( \langle \nu, (\alpha)^\vee \rangle = \langle \nu_1, \alpha^\vee \rangle \). Hence, from Theorem 11, we deduce that

\[ \text{Mult}(\nu^*, \nu) \simeq V^-(\nu, \nu_1 - \nu_2^*, \nu_1). \]

Since \( \text{Mult}(\nu, \nu^*) \) is isomorphic to \( \text{Mult}(\nu^*, \nu) \), we recover (7).

Now, \( \rho(\gamma_0(0, -w_0 \alpha)) = -\alpha \), and \( \langle \nu, (0, -w_0 \alpha)^\vee \rangle = \langle \nu_2, (-w_0 \alpha)^\vee \rangle = \langle \nu_2^*, \alpha^\vee \rangle \). Then, if we use \( D_0^- \) to get a minimal number of conditions in the second point of the theorem, we deduce that:

\[ \text{Mult}(\nu^*, \nu) \simeq V^+(\nu, \nu_1 - \nu_2^*, \nu_2^*). \]

And since \( \text{Mult}(\nu, \nu^*) \simeq \text{Mult}(\nu^*, \nu) \simeq \text{Mult}(\nu^*, (\nu_2, \nu_1)) \), we recover (8).

4.2 \( \text{Sp}_{2n} \) in \( \text{SL}_{2n} \)

Fix \( n \geq 2 \). Let \( V \) be a \( 2n \)-dimensional vector space with fixed basis \( B = (e_1, \ldots, e_{2n}) \). Consider the following matrices

\[ J_n = \begin{pmatrix} \ldots & 1 \\ 1 & \ldots \end{pmatrix}; \quad \text{and} \quad \omega = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}. \]  

of size \( n \times n \) and \( 2n \times 2n \) respectively. View \( \omega \) as a symplectic bilinear form of \( V \).

Let \( G \) be the associated symplectic group. Set \( T = \{ \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{C}^* \} \). Let \( B \) be the Borel subgroup of \( G \) consisting of upper triangular matrices of \( G \). For \( i \in [1, n] \), let \( \varepsilon_i \) denote the character of \( T \) that maps \( \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \) to \( t_i \); then \( X(T)^+ = \oplus_{i=1}^n \mathbb{Z} \varepsilon_i \). Here

\[ \Phi^+ = \{ \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_i : 1 \leq i \leq n \}, \]
\[ \Delta = \{ \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = 2\varepsilon_n \}, \]
\[ X(T)^+ = \{ \sum_{i=1}^n \lambda_i \varepsilon_i : \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \}. \]

For \( i \in [1; 2n] \), set \( \tau = 2n + 1 - i \). The Weyl group \( W \) of \( G \) is a subgroup of the Weyl group \( S_{2n} \) of \( \hat{G} = \text{SL}(V) \). More precisely

\[ W = \{ w \in S_{2n} : w(\tau) = \overline{w(\tau)} \forall i \in [1; 2n] \}. \]

Note that \( \dim G = \dim \text{GL}_{2n}(\mathbb{C}) - \dim \wedge^2 V^* = 2n^2 + n \) and \( \dim \hat{G} / \hat{B} = n(n-1) \). Hence we are looking for \( \gamma_0 \in \hat{W} \) such that

\[ \dim G_{\gamma_0 \hat{B} / \hat{B}} = 2n. \]

This is consistent with the fact that \( \sharp \Phi = n \) (\( \Phi = \{ \varepsilon_i : 1 \leq i \leq n \} \)).

An element \( \hat{w} \in \hat{W} \) is written as a word \( [\hat{w}(1) \hat{w}(2) \ldots \hat{w}(2n)] \). Set

\[ \gamma_0 = [1122 \ldots]. \]

Since the \( \omega \)-orthogonal of \( \langle e_1, e_1, \ldots, e_k \rangle \) is \( \langle e_{k+1}, \ldots, e_n \rangle \) the stabilizer of \( \gamma_0 \hat{B} / \hat{B} \) is diagonal by blocks with 2 blocks of size \( 2k \) and \( 2(n-k) \) (in the basis ordered according to \( \gamma_0 \)). Since this is
true for any \( k \), this stabilizer consists in block diagonal matrices with blocks of size 2. Moreover, in each block the matrix has to be triangular because of its action on \( e_1, \ldots, e_n \). Moreover, these blocks belong to \( \text{Sp}(2) = \text{SL}(2) \). We just proved that \( G_{\hat{y}_0B/B} \) is contained in a group of dimension \( 2n \). For dimension reasons, we deduce that \( G_{\hat{y}_0B/B} \) is equal to this subgroup and that \( G_{\hat{y}_0B/B} \) in open in \( \hat{G}/\hat{B} \).

The length of \( \hat{y}_0 \) is the number of inversions, that is the set of pairs \( (i < j) \) such that \( j \) occurs before \( i \) in the word. Fix \( i \in \{1, \ldots, n\} \). The number of \( j > i \) such that \( j \) occurs before \( i \) in the word \( \hat{y}_0 \) is \( i - 1 \). The contribution to these pairs to \( \ell(\hat{y}_0) \) is \( \frac{n(n-1)}{2} \). Similarly, the contribution of the pairs \( j > \hat{i} \) with \( \hat{i} \in \{n+1, \ldots, 2n\} \) is \( \frac{n(n-1)}{2} \). Hence \( \ell(\hat{y}_0) = n(n-1) = \dim \hat{G}/\hat{B} - \dim(G/B) \) and \( y_0 \) has minimal length.

Let \( \hat{T} \) be the maximal torus of \( \hat{G} \) consisting in diagonal matrices. Write \( X(\hat{T}) = \bigoplus_{i=1}^{2n} \mathbb{Z} \hat{e}_i / (\hat{e}_1 + \cdots + \hat{e}_2n) \) with the usual notation. Let \( \hat{\alpha}_i = \hat{e}_i - \hat{e}_{i+1} \) be a simple root of \( \hat{G} \). If \( i = 2k \) is even then \( \hat{y}_0 \hat{\alpha}_i = \hat{e}_k - \hat{e}_{k+1} \in \hat{\Phi}^1 \). If \( i = 2k+1 \) is odd then \( \hat{y}_0 \hat{\alpha}_i = \hat{e}_{k+1} - \hat{e}_{k+1} \in \hat{\Phi}^2 \). By Lemma 7

\[
\mathcal{D}_0 = \{ \hat{\alpha}_{2k} : k = 1, \ldots, n-1 \}
\]

works. Moreover

\[-\rho(\hat{\alpha}_{2k}) = \epsilon_k + \epsilon_{k+1} \quad \forall k = 1, \ldots, n-1.\]

### 4.3 Spin\(_{2n-1}\) in Spin\(_{2n}\)

Let \( V \) be a \( 2n \)-dimensional vector space endowed with a basis \( B = (e_1, \ldots, e_{2n}) \). Denote by \( (x_1, \ldots, x_{2n}) \) the dual basis. For \( i \in [1; 2n] \), set \( \hat{i} = 2n + 1 - i \). Let \( \hat{G}_0 \) be the orthogonal group associated to the quadratic form

\[ Q = \sum_{i=1}^{n} x_i x_i^\ast. \]

Set \( \hat{T}_0 = \{ \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) : t_i \in \mathbb{C}^* \} \) in \( \hat{G}_0 \). Let \( \hat{B}_0 \) be the Borel subgroup of \( \hat{G}_0 \) consisting in upper triangular matrices of \( \hat{G}_0 \). Let \( \hat{e}_i \) denote the character of \( \hat{T}_0 \) that maps \( \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \) to \( t_i \); then \( X(\hat{T}_0) = \sum_{i=1}^{n} \mathbb{Z} \hat{e}_i \). Here

\[
\hat{\Phi}^+ = \{ \hat{e}_i \pm \hat{e}_j : 1 \leq i < j \leq n \}, \quad \text{and} \quad \hat{\Delta} = \{ \hat{\alpha}_1 = \hat{e}_1 - \hat{e}_2, \hat{\alpha}_2 = \hat{e}_2 - \hat{e}_3, \ldots, \hat{\alpha}_{n-1} = \hat{e}_{n-1} - \hat{e}_n, \hat{\alpha}_n = \hat{e}_{n-1} + \hat{e}_n \}.
\]

The Weyl group \( \hat{W} \) of \( \hat{G}_0 \) is a subgroup of the Weyl group \( S_{2n} \) of \( \text{SL}(V) \). More precisely

\[
\hat{W} = \{ w \in S_{2n} : \left\{ \begin{array}{l} w(\hat{i}) = \overline{w(\hat{i})} \quad \forall i \in [1; 2n] \\
\exists w([1; n]) \cap [n+1; 2n] \text{ is even} \end{array} \right. \}
\]

For \( i = 1, \ldots, n-1 \), \( s_{\hat{\alpha}_i} = (i, i+1)(\hat{i} + 1 \; \hat{i}) \). Moreover \( s_{\hat{\alpha}_n} = (n-1 \; \bar{n})(n \; n - 1) \).

Let \( \mathcal{H} = \langle e_1, \ldots, e_{n-1}, e_n + e_{n+1} / \sqrt{2}, e_{n+1}, \ldots, e_{2n} \rangle \) with coordinates \( (x_i)_{1 \leq i \leq n-1} \cup (y) \cup (x_i)_{1 \leq i \leq n-1} \). The restriction of \( Q \) on \( \mathcal{H} \) is

\[ Q|_{\mathcal{H}} = y^2 + \sum_{i=1}^{n-1} x_i x_i^\ast. \]
The stabilizer of $\mathcal{H}$ in $\hat{G}_0$ is $G_0 = SO_{2n-1}(\mathbb{C})$. Its maximal torus $T_0$ is
\[ \{ \text{diag}(t_1, \ldots, t_{n-1}, 1, t_{n-1}^{-1}, \ldots, t_1^{-1}) : t_i \in \mathbb{C}^* \}. \]

The groups $G$ and $\hat{G}$ are the universal covers of $G_0$ and $\hat{G}_0$ respectively. Here
\[ \Phi^+ = \{ \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n-1 \} \cup \{ \varepsilon_i : 1 \leq i \leq n-1 \}, \quad \text{and} \]
\[ \Delta = \{ \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-2} = \varepsilon_{n-2} - \varepsilon_{n-1}, \alpha_{n-1} = \varepsilon_{n-1} \}. \]

Note that $\Delta^2 = \{ \alpha_{n-1} \}$ and there is only one $G_0$-stable divisor in $\hat{G}/\hat{B}$. We have
\[ \dim \hat{G}/\hat{B} = n(n-1) \quad \text{dim}(G/B) = (n-1)^2. \]

Set
\[ y_0 = s_{\hat{\alpha}_{n-1}} \cdots s_{\hat{\alpha}_2}s_{\hat{\alpha}_1} = (1 \ n \ n-1 \ldots 2)(\bar{1} \bar{n} \ldots \bar{2}). \]

We have $\hat{y}_0 \hat{e}_i = \hat{e}_{i-1}$ for $i = 2, \ldots, n$ and $\hat{y}_0 \hat{e}_n = \hat{e}_n$. In particular
\[ \hat{y}_0 \Phi^+ = \{ \hat{\varepsilon}_n \pm \hat{\varepsilon}_j : j = 1, \ldots, n-1 \} \cup \{ \hat{\varepsilon}_i \pm \hat{\varepsilon}_j : 1 \leq i < j \leq n-1 \}. \]

One easily deduce that the stabilizer of $\hat{y}_0$ in $G_0$ is $H_0$.

The only descent is $\hat{\alpha}_1$. Then $D_0 = D = \{ \hat{\alpha}_1 \}$ and $-\rho(\hat{y}_0 \hat{\alpha}_1) = -\rho(\hat{\varepsilon}_n - \hat{\varepsilon}_1) = \varepsilon_1$.

### 4.4 $G_2$ in $\text{Spin}_7$

Here $G$ is the group of type $G_2$ embedded in $\text{Spin}_7(\mathbb{C}) = \hat{G}$ using the first fundamental representation which is 7-dimensional. We label the simple roots as follows:

\[
\begin{array}{ccc}
B_3 & \overset{1}{\longrightarrow} & \overset{2}{\longrightarrow} \overset{3}{\longrightarrow} \\
1 & 2 & 3 \\
G_2 & \overset{1}{\longrightarrow} \overset{2}{\longrightarrow} \\
1 & 2
\end{array}
\]

The map $\rho$ is characterized by
\[ \rho(\hat{\alpha}_2) = \alpha_2 \quad \rho(\hat{\alpha}_1) = \rho(\hat{\alpha}_3) = \alpha_1, \]
and satisfies
\[ \rho(\hat{\alpha}_1 + \hat{\alpha}_2) = \rho(\hat{\alpha}_2 + \hat{\alpha}_3) = \alpha_1 + \alpha_2 \]
\[ \rho(\hat{\alpha}_2 + 2\hat{\alpha}_3) = \rho(\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3) = 2\alpha_1 + \alpha_2 \]

In particular
\[ \Phi^+ \cap \Phi^1 = \{ \hat{\alpha}_2, \hat{\alpha}_1 + \hat{\alpha}_2 + 2\hat{\alpha}_3, \hat{\alpha}_1 + 2\hat{\alpha}_2 + 2\hat{\alpha}_3 \}. \]

The working $\hat{y}_0$ are
\[ \hat{y}_0 = s_{\hat{\alpha}_3} s_{\hat{\alpha}_2} s_{\hat{\alpha}_1} \quad \text{and} \quad \hat{y}_0 = s_{\hat{\alpha}_3} s_{\hat{\alpha}_2} s_{\hat{\alpha}_1} \]

Indeed, one can check that $\ell(\hat{y}_0) = 3$ and $\dim(G \cap \hat{y}_0 \hat{B} \hat{y}_0^{-1}) = 3$.

Moreover, the only simple roots $\hat{\alpha}$ such that $\dim(G \cap \tilde{w} \hat{B} \tilde{w}^{-1}) = 4$ where $\tilde{w} = \hat{y}_0 \hat{B}$ is $\hat{\alpha}_3$. The so obtained divisor is the only $G$-stable divisor accordingly to $2\Delta^2 = 1$.

There are $\frac{\dim G}{\dim \hat{G}} = 4$ $G$-orbits in $\hat{G}/\hat{B}$. The closed orbit has dimension 6 and the open one 9: hence there is one orbit in each dimension from 6 to 9. Computing the dimensions of the stabilizers we get the graph $\Gamma(\hat{G}/G)$.
Here $\mathcal{D}_0 = \mathcal{D} = \{\hat{\alpha}_3\}$ and $-\rho(\hat{y}_0\hat{\alpha}_3) = \alpha_1 + \alpha_2$.

4.5 $F_4$ in $E_6$

Here $G$ is the group of type $F_4$ embedded in $\hat{G}$ of type $E_6$. We label the simple roots as follows:

The root system $E_6$ lies in $\mathbb{R}^8$ with basis $(\hat{\epsilon}_i)_{1 \leq i \leq 8}$. More precisely it spans the space $x_6 = x_7 = -x_8$. See [Bou02]. The roots are

$$\pm \hat{\epsilon}_i \pm \hat{\epsilon}_j \quad 1 \leq i < j \leq 5 \quad \text{and} \quad \pm \frac{1}{2}(\hat{\epsilon}_8 - \hat{\epsilon}_7 - \hat{\epsilon}_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} \hat{\epsilon}_i) \sum \nu(i) \text{ even.}$$

Set $\hat{\epsilon}_6 = \hat{\epsilon}_8 - \hat{\epsilon}_6 - \hat{\epsilon}_7$. Then, the simple roots are

$$\hat{\alpha}_1 = \frac{1}{2}\left(\hat{\epsilon}_8 - \hat{\epsilon}_7 - \hat{\epsilon}_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} \hat{\epsilon}_i\right) \sum \nu(i) \text{ even.}$$

and

$$\hat{\alpha}_2 = \hat{\epsilon}_1 + \hat{\epsilon}_2 \quad \hat{\alpha}_3 = \hat{\epsilon}_2 - \hat{\epsilon}_1 \quad \hat{\alpha}_4 = \hat{\epsilon}_3 - \hat{\epsilon}_2 \quad \hat{\alpha}_5 = \hat{\epsilon}_4 - \hat{\epsilon}_3 \quad \hat{\alpha}_6 = \hat{\epsilon}_5 - \hat{\epsilon}_4.$$ 

**Root system $F_4$.** The roots are

$$\frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \quad \text{and} \quad \pm \epsilon_i \pm \epsilon_j \quad 1 \leq i < j \leq 4.$$ 

The simple roots are

$$\alpha_1 = \epsilon_2 - \epsilon_3 \quad \alpha_2 = \epsilon_3 - \epsilon_4 \quad \alpha_3 = \epsilon_4 \quad \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

The map $\rho$ is characterized by

$$\rho(\hat{\alpha}_2) = \alpha_1 \quad \rho(\hat{\alpha}_4) = \alpha_2 \quad \rho(\hat{\alpha}_3) = \rho(\hat{\alpha}_5) = \alpha_3 \quad \rho(\hat{\alpha}_1) = \rho(\hat{\alpha}_6) = \alpha_4.$$ 

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It is the quotient by

\[ \epsilon_2 + \epsilon_3 - \epsilon_1 - \epsilon_4 = \frac{1}{2} \left( \epsilon_1 + \epsilon_6 + \epsilon_4 - (\epsilon_2 + \epsilon_3 + 3\epsilon_5) \right) \]

Moreover

\[ \#\Phi^+ = 36 \quad \#\hat{W} = 51,840 \quad \#\Delta^2 = 2 \quad \#W = 1152 \quad \#\Phi^+ = 24 \quad \#\hat{W}/W = 45. \]

We have 12 short positive roots in \( F_4 \): \( \epsilon_i \quad \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \)

A working \( \hat{y}_0 \) is

\[ \hat{y}_0 = \hat{s}_1 \hat{s}_5 \hat{s}_3 \hat{s}_4 \hat{s}_2 \hat{s}_5 \hat{s}_4 \hat{s}_3 \hat{s}_1 \hat{s}_6. \]

Indeed, one can check that \( \ell(\hat{y}_0) = 12 \) and \( \dim(G \cap \hat{y}_0 \hat{B}\hat{y}_0^{-1}) = 16. \)

Moreover, the only simple roots \( \hat{\alpha} \) such that \( \dim(G \cap \hat{w}\hat{B}\hat{w}^{-1}) = 17 \) where \( \hat{w} = \hat{y}_0s_6 \) are \( \hat{\alpha}_1 \) and \( \hat{\alpha}_6 \). The so obtained divisors are distinct since \( \#\Delta^2 = 2. \) Set

\[ D_0 = \{ \hat{\alpha}_1, \hat{\alpha}_6 \}. \]

One checks that

\[ -\hat{y}_0\hat{\alpha}_1 = \hat{\alpha}_1 + \hat{\alpha}_2 + 2\hat{\alpha}_3 + 2\hat{\alpha}_4 + \hat{\alpha}_5 \quad \rho \rightarrow \alpha_1 + \alpha_2 + 3\alpha_3 + \alpha_4; \]

\[ -\hat{y}_0\hat{\alpha}_6 = \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 + 2\hat{\alpha}_4 + 2\hat{\alpha}_5 + \hat{\alpha}_6 \quad \rho \rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4. \]

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