Adversarial Wiretap Channel with Public Discussion

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Abstract—We consider a model of adversarial wiretap channel where an adversary selects a fraction $\rho_r$ of a transmitted codeword to read, and selects a fraction $\rho_w$ of the codeword, to “add” adversarial error to. This model captures jamming adversaries with partial access to the wireless channel, as well as adversaries that partially control paths in networks. An $(\epsilon, \delta)$-AWTP protocol transmits messages from Alice to Bob, guaranteeing that the privacy loss of the transmitted message is bounded by $\epsilon$, and the success chance of the adversary in causing the decoder to output in error, is bounded by $\delta$. It was shown that secure and reliable communication (arbitrary small $\epsilon$ and $\delta$) over this channel is possible when $\rho_r + \rho_w < 1$.

In this paper we consider the case that $\rho_r + \rho_w > 1$, and show that secure communication is possible when the communicants have access to a public discussion channel, and not all codeword components are read and/or written to, by the adversary.

We, (i) formalize the model of AWTP$_{PD}$ protocol and define secrecy, reliability, and efficiency in terms of the rate of information transmission and round complexity, (ii) derive a tight upper bound on the rate, and a tight lower bound on the required number of rounds for an $(\epsilon, \delta)$-AWTP$_{PD}$, and (iii) give the construction of optimal protocols with minimum number of rounds, and with the rate approaching the upper bound. We show the implication of these results for Secure Message Transmission with Public Discussion, and discuss our results, and direction for future research.

I. INTRODUCTION

In Wyner’s [24] model of secure communication and its generalization [9], Alice is connected to Bob and Eve through two noisy channels, referred to as the main channel and the eavesdropper channel, respectively. The goal is to send a message from Alice to Bob with perfect secrecy and reliability. Wyner’s pioneering work showed that communication with (asymptotic) perfect secrecy and reliability is possible if the eavesdropper channel is noisier than the main channel. Here security is information theoretic and does not require a pre-shared secret key. In recent years wiretap model has attracted much attention [3], [5], [6], [8], [14], [19] because it provides a natural model for wireless communication when an adversary at the reception range of the transmitter is eavesdropping the communication. Extensions of wiretap channel have been studied by numerous authors [2], [8], [15], [16], [19]. In particular Maurer [16] considered a scenario where Alice and Bob are allowed to use a public discussion (PD) channel in addition to the wiretap channel. A public discussion channel is a reliable authenticated communication channel that can be used by the sender and the receiver, and is readable by everyone, including the adversary. Maurer proved that inclusion of the PD channel makes secure communication possible even when the noise in the eavesdropper channel is higher than the main channel, thus showing the importance of public discussion as a resource for secure communication.

Adversarial models of wiretap channel was first considered by Ozarow and Wyner [18]. In their model the main channel is noise free, and Eve can choose a constant fraction of the sent codeword to eavesdrop. More recently, wiretap channel with active adversaries has been considered by a number of authors [1], [7], [17], [23]. In these models the adversary not only eavesdrops the communication between the sender and the receiver, but also injects noise in the main channel. The model in [7] and [17] uses arbitrary varying channels to model adversarial wiretap. For adversarial wiretap model over arbitrary varying channel, the distributions of error on main channel and adversarial channel depend on the states of channel. The information that adversary observes, depends on the current state of eavesdropping channel, and the adversary can actively change the state of main channel to disturb the transmission between the sender and receiver.

We consider a model of adversarial wiretap channel (AWTP channel) proposed in [23], where the adversary chooses a subset of codeword coordinates, that is a fraction $\rho_r$ of the sent codeword, to read, and adds adversarial error to a fraction $\rho_w$ of the codeword. A binary version of model had been considered in [1]. This model can be seen as generalizing wiretap II to include active (jamming) adversarial noise. This model had been studied in [20], [21] with the goal of providing reliability for communication. In AWTP channel, the adversary’s observation and corruption are adaptive, and at each step uses all the observations and corruptions until that point. The goal of the adversary is to break the security and/or reliability of the communication.

A. Motivation

In [23], it was proved that the information rate of a AWTP code with perfect secrecy ($\epsilon = 0$) is positive, if and only if $\rho_r + \rho_w < 1$. The goal of this paper is to investigate if this restriction can be removed. Obviously, this needs extra resources for the communicants, or weakening the adversarial model. We will consider the same adversarial model, but assume a public discussion (PD) channel is available to the communicants, and study if it is possible to have secure communication over AWTP wiretap channel with public discussion (or AWTP$_{PD}$ for short), when $\rho_r + \rho_w > 1$. We assume
communication can be in multiple rounds, but communication over the AWTP channel is one-way, and always from Alice to Bob; that is, Bob cannot send messages to Alice over the AWTP channel. We however assume the communication over public discussion channel is two-way.

B. Our work

1) Model and Definitions: We define a multi-round message transmission protocol over AWTPPD and formalize its security and reliability of these protocols with the corresponding definitions for AWTP codes. We define secrecy in terms of the statistical distance between the adversary’s view of any two adversarially chosen messages, and reliability in terms of the probability (uniform distribution of messages) of the decoded message being different from the sent one. This definition of reliability is different from the one used for AWTP codes. Here the decoder is allowed to output incorrect message, while in [20], the decoder either outputs the correct message or fail symbol.

An AWTPPD protocol in general, has multiple rounds. In each round, either Alice transmits protocols messages to Bob over AWTP channel, or the PD channel, or Bob transmits protocols message to Alice over the PD channel. An \((\epsilon, \delta)\)-AWTPPD protocol guarantees that the information leaked about the message is bounded by \(\epsilon\) and probability of decoding wrong message is bounded by \(\delta\). The rate \(R\) of AWTPPD protocol is the number of information bits transmitted by the protocol, divided by the total number of transmitted bits over the AWTP channel. The capacity \(C^r\) \((C^0)\) of a AWTPPD channel is the maximum information rate that can be communicated over AWTP channel by a AWTPPD protocol family.

2) Bounds: Efficiency of AWTPPD protocols is in terms of the information rate \(R\), and the rounds complexity \(RC\). We derive a tight upper bound on \(R\) of AWTPPD protocols. We first give a bound on \(H(M)\), and use that to prove that the highest secrecy rate of an \((\epsilon, \delta)\)-AWTPPD protocol is bounded by, \(C^r \leq 1 - \rho + 2\epsilon \cdot (1 + \log_{|\Sigma|} \frac{1}{\delta}) + 2\epsilon n\), where \(n\) is the total length of transmission over PD channel, and \(\Sigma\) is the alphabet of AWTP channel. For perfect secrecy capacity we have, \(C^0 \leq 1 - \rho\), where \(\rho\) is the rate that adversary either read or write over AWTP channel. We note that, if \(S_r \cap S_w \neq \emptyset\), we have \(\rho < \rho_r + \rho_w\), and so the upper bound on the secrecy rate of AWTPPD protocols is higher than that of AWTP codes (Section 2V-A). This means secure communication may be possible in cases that \(\rho_r + \rho_w > 1\), but \(\rho < 1\).

A second efficiency measure for the code is the number of rounds. We derive a bound on the minimum number of rounds for a AWTPPD protocol that has nonzero rate when \(\rho_r + \rho_w > 1\). We show that it is impossible to have a AWTPPD protocol under above conditions, if public discussion is only used by Bob, or only by Alice. So the AWTPPD protocol must have at least three rounds: one round of transmission over AWTP channel, one round of transmission over PD channel used by Bob, and one round of PD channel used by Alice.

3) Construction of AWTPPD protocol: We construct a three round \((0, \delta)\)-AWTPPD protocol with perfect security, and probability of incorrect decoding bounded by \(\delta\). The rate of the AWTPPD protocol can be made arbitrarily close to the upper bound on the secrecy capacity. That is for any small \(\xi > 0\), there is \(N_0\), such that for all \(N > N_0\), the rate of the AWTPPD protocol satisfies, \(R \geq 1 - \rho - \xi\). The number of rounds meets the lower bound on the number of rounds. The construction consists of the following rounds: in the first round Alice sends to Bob over AWTP channel, a sequence of randomly selected elements from a set. In the second round, Bob uses an appropriately chosen universal hash family to calculate the hash values of the received elements, and sends them to Alice over the PD channel. In the third round, Alice, encrypts the message using a key that is extracted from the random values that are correctly received by Bob, and sends it over the PD channel to Bob, together with sufficient information that allows Bob to calculate the same key and recover the message.

C. Relation with SMT-PD

In Secure message transmission with public discussion channel (SMT-PD) [12], Alice and Bob are connected by \(N\) node disjoint communication paths in a network, a subset of which is controlled by an adversary, and also an authenticated public discussion channel that can be read by everyone. The adversary chooses the subset of controlled wires and corrupt them arbitrarily. In Section VI we define \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD, a subset of SMT-PD protocols in which the wires are used by Alice only, and show our results for AWTPPD including bounds on rate and round complexity, and the construction of optimal AWTPPD protocol, implies similar results for \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD.

D. Related Work

Maurer’s introduction of PD channel was first in the context of key agreement protocols over wiretap channels; this was also independently considered in [2]. In a key agreement protocol, the goal of Alice and Bob is to generate a common random key. In wiretap with public discussion setting, once a shared key is established, since the PD channel usage is considered free and does not contribute to communication cost (similar to our model), secure message transmission can be obtained by encrypting the message and sending it over the public discussion channel. Our construction can be seen as a two steps construction: first a key agreement, followed by encrypting the message and sending it over the public discussion channel, as above.

Other models of adversarial wiretap, [1], [4], [7], [17], and their relationship to the model considered here, are discussed in [23].

One of the original motivations of AWTP channel was to establish the relation between adversaries in wiretap channels and Secure Message Transmission (SMT) [11] in networks. In SMT, Alice and Bob are connected by \(N\) node disjoint communication paths in a network, a subset of which
is corrupted by an adversary who can see what is sent on a corrupted path and can replace it with a value of their choosing. Security and reliability of SMT is defined similar to AWTP codes. It was shown that there is a one-to-one correspondence between one-round SMT and \((\epsilon, \delta)\)-AWTP codes.

E. Organization

In Section 2, we introduce the adversarial wiretap channel and public discussion channel. In Section 3, we introduce the AWTPPD protocol. In Section 4, we show the upper bound of rate and minimum requirement of round complexity. In Section 5, we give the construction of AWTPPD protocol. In Section 6, we discuss our results, open problems and future works.

II. PRELIMINARIES

We use, calligraphic symbols \(\mathcal{X}\) to denote sets, \(\Pr(X)\) to denote a probability distribution over \(\mathcal{X}\) with probability \(\Pr(X)\). The conditional probability of \(X\) given an event \(E\), is \(\Pr[X = x | E]\), \(\log()\) is logarithm in base two. Shannon entropy \(H(X) = \sum_{x} \Pr(x) \log \Pr(x)\), and conditional entropy of a distribution \(X\) given \(Y\) is defined by, \(H(X|Y) = \sum_{x,y} \Pr(x,y) \log \Pr(x|y)\). The mini-entropy of \(X\) over \(\mathcal{X}\) is defined as \(H_\infty(X) = \min_{x \in \mathcal{X}} - \log \Pr(X = x)\). Statistical distance between two random variables \(X_1, X_2\) defined over the same set is given by \(SD(X_1, X_2) = \frac{1}{2} \sum_{x} |\Pr(X_1 = x) - \Pr(X_2 = x)|\). Mutual information between random variables \(X\) and \(Y\) is given by, \(I(X, Y) = H(X) - H(X|Y)\). Hamming weight of a vector \(e\) is denoted by \(wt(e)\).

A. Channel Models

We consider two types of channels: AWTP channel and PD channel. A channel can be one-way or two-way.

Definition 1: The channel is called one-way channel from Alice to Bob (Bob to Alice), if the information can only be sent from Alice to Bob (Bob to Alice) through the channel. The channel is called two-way channel if the information can be sent in both directions, in each round either from Alice to Bob or from Bob to Alice.

Let \([N] = \{1, \ldots, N\}\), and \(S = \{i_1, \ldots, i_{\rho N}\} \subseteq [N]\) and \(S_w = \{j_1, \ldots, j_{\rho w}\} \subseteq [N]\) denote two subsets of the \(N\) coordinates, and \(\mathbb{SUPP}(x)\) of vector \(x \in \Sigma^N\) be the set of positions in which the component \(x_i\) is non-zero.

Definition 2: A \((\rho, \rho_w)\)-Adversarial Wiretap Channel \(\mathcal{A}(\rho, \rho_w)\) is an adversarial channel that is partially controlled by an adversary Eve, with two capabilities: Reading and Writing. For a codeword of length \(N\), Eve can select a subset \(S' \subseteq [N]\) of size \(|S'| = \rho N\) to read, and select a subset \(S'' \subseteq [N]\) of size \(|S''| = \rho w N\) for writing. The writing is by adding to an \(e\) error vector \(e\) with \(\mathbb{SUPP}(e) = S''\), resulting in \(e + e\) to be received. The adversary is adaptive and to select a component for reading and/or writing, it uses its knowledge of the codeword at the time. We denote the subset \(S = S' \cup S''\) with size \(|S| = \rho N\) be the set of components of codeword that adversary read or write.

We assume the adversarial wiretap channel is one-way. The AWTP channel is restricted-AWTP channel if \(S_e = S_w = S\).

Definition 3: (Public Discussion Channel (PD Channel)) is an error-free communication channel between Alice and Bob, that can be read by everyone including Eve.

We assume the PD channel is two-way and can be used in forward, or backward direction.

We consider AWTPPD protocols, each consisting of multiple rounds of communication. In each round, either Alice or Bob, use one the channels accessible to them: Alice can use the AWTP channel or the PD channel, and Bob can only use the PD channel.

Definition 4: The round complexity \(RC\) of a protocol is the total number of times that the AWTP channel and the PD channel, are invoked by Alice and Bob.

III. AWTPPD PROTOCOL

We consider secure and reliable message transmission over AWTP and PD channel (AWTPPD channel). Alice (sender) wants to send a message (also referred to as, information) \(m \in \mathcal{M}\) securely and reliably to Bob (receiver), using a multi-round protocol over a AWTPPD channel, referred to as AWTPPD protocols.

We give a general template for message flow in AWTPPD protocols. Note that each message can be “empty”, in which case the corresponding round can be ignored. The template consists of basic three round flows. For a protocol with \(\ell\) basic three round flow, there are \(2\ell\) invocations of the AWTP channel, and \(2\ell\) invocations of the PD channel. Let \(r_A\) and \(r_B\) denote the randomness of Alice and Bob. In the \(i\)th round, \(c_i(d_i)\) denotes the protocol message, also referred to as codeword, transmitted over the AWTP channel (PD channel).

We use \(e^i = \{e_1, \ldots, e_{\ell}\}\) \((d^i = \{d_1, \ldots, d_{\ell}\})\), to denote the concatenation of protocol messages, transmitted before the \((i + 1)\)th round of the AWTP channel (PD channel).

Let the protocol message alphabet for AWTP channel and PD channels, be \(\Sigma\) and \(\mathbb{F}_2\), respectively. In the round \(i\) invocation of the AWTP channel, Alice sends a codeword, of length \(N_i\). In the round \(i\) invocation of the PD channel, Alice or Bob, transmits a binary string of length \(n_i\). The total number of symbols sent over the AWTP channel is \(N = \sum_{i=1}^{\ell} N_i\), and the total number of bits transmitted over the PD is, \(n = \sum_{i=1}^{\ell} n_i\).

Let the view of Alice and Bob in a particular execution be, \(v_A\) and \(v_B\), respectively. We use the following notations to denote the protocol messages constructed by Alice and Bob in the \(i\)th round of the protocol:

\(c_i = \text{AWTPPD}(m, r_A, i, v_A, \text{AWTP})\);
\(d_i = \text{AWTPPD}(m, r_X, i, v_X, \text{PD})\), where \(X \in \{A, B\}\), is used to show \(d_i\) is the protocol message constructed by Alice or Bob.

We describe the general form of AWTPPD protocol in Figure III.

AWTPPD Protocol
1) A $\rightarrow$ B: $c_1$ over AWTP channel $c_1 = \text{AWTP}_D(m_S, r_A, 1, v_A, \text{AWTP})$; $v_A = \emptyset$.
   Eve reads a subset $S_e$ of $c_1$, and add error $e_1$ to $c_1$.
   At the end of the round, Bob receives $y_1 = c_1 + e_1$.
2) A $\rightarrow$ B: $d_1$ over PD channel; $v_A = \emptyset$.
3) B $\rightarrow$ A: $d_2$ over AWTP channel; $d_2 = \text{AWTP}_D(r_B, 2, v_B, PD)$, and $v_B = \{y_1, d_1\}$
4) Alice and Bob repeat the above three steps. In the $i^{th}$ three-round transmission:
   - $A$ $\rightarrow$ B: $c_i$; $v_A = \{c_{i-1}, d_{2i-2}\}$
   - $c_i = \text{AWTP}_D(m_S, r_A, i, v_A, \text{AWTP})$
   - Bob receives $y_i = c_i + e_i$.
   - $A$ $\rightarrow$ B: $d_{2i-1}$; $v_A = \{c_{i-1}, d_{2i-2}\}$
   - $d_{2i-1} = \text{AWTP}_D(m_S, r_A, 2i - 1, v_A, PD)$.
   - $B$ $\rightarrow$ A: $d_{2i}$; $v_B = \{y_i, d_{2i-1}\}$
   - $d_{2i} = \text{AWTP}_D(r_B, 2i, v_B, PD)$.
5) At the end of $\ell$ repetitions of basic three-round flow, Bob outputs a message, $m_R = \text{Dec}(y''^\ell, d''^\ell)$.

By allowing protocols messages to be empty, the above template covers all multi-round protocols.

Definition 5 ($\epsilon, \delta$-AWTP$_PD$ protocol): Let Alice and Bob be connected by a AWTP$_PD$ channel. An $\epsilon, \delta$-AWTP$_PD$ protocol for secure message transmission satisfies the following two properties:

1) Secrecy: For any two messages $m_1, m_2 \in M$, the statistical distance between Eve’s view, when the same random coins $r_E$ are used by Eve, is bounded by $\epsilon$.
   $$\max_{m_1, m_2} \text{SD} (\text{View}_E(\text{AWTP}_D(m_1), r_E), \text{View}_E(\text{AWTP}_D(m_2), r_E)) \leq \epsilon$$

2) Reliability: The probability that Bob outputs the message $m$ sent by Alice, is at least $1 - \delta$.
   $$\Pr(M_R \neq M_S) \leq \delta$$

Here probability is over all message (uniformly distributed) in message space, the randomness of Alice and Bob, and the adversary’s randomness in his/her corruption strategy.

The AWTP$_PD$ protocol provide perfect secrecy if $\epsilon = 0$. If adversary is passive, then Bob can always output correct message $m_S$ and $\Pr(M_R = M_S) = 1$. An AWTP$_PD$ protocol is restricted-$\epsilon, \delta$-AWTP$_PD$ protocol if the AWTP$_PD$ protocol is over restricted-AWTP channel and PD channel, and $N_i = N_j$ and $S_i = S_j$ for any $1 \leq i \leq j \leq \ell$.

We use the rate $R$ and round complexity $RC$ to measure the transmission efficiency of an $\epsilon, \delta$-AWTP$_PD$ protocol.

Definition 6: An $\epsilon$-secure AWTP$_PD$ protocol family for a $(\rho_r, \rho_w)$-AWTP channel, is a family $\Pi = \{\Pi^N\}_{N \in \mathbb{N}}$ of $\Pi^N = (\epsilon, \delta_N)$-AWTP$_PD$ protocols, indexed by $N \in \mathbb{N}$, for the $(\rho_r, \rho_w)$-AWTP channel.

A $\epsilon, \delta$-AWTP$_PD$ protocol family $\Pi$ achieves rate $R$, if for any $\xi > 0$, there exist $N_0$ such that for any $N \geq N_0$, there is $\delta < \xi$ and
   $$\frac{\log |\mathcal{M}|}{N \log |\mathcal{S}|} \geq R - \xi$$

The $\epsilon$-secrecy (perfect secrecy) capacity $C^\epsilon$ $(C^\delta)$ of a $(\rho_r, \rho_w)$-AWTP$_PD$ channel, is the largest achievable rate of all $\epsilon, \delta$-AWTP$_PD$ ($\epsilon, \delta$-AWTP$_PD$) protocol families, for the channel.

IV. BOUNDS ON ($\epsilon, \delta$)-AWTP$_PD$ PROTOCOLS

We give three bounds for $\epsilon, \delta$-AWTP$_PD$ protocols: an upper bound on the rate and a lower bound on the minimum number of rounds required for such protocols.

A. Upper Bound on Rate

Theorem 1: The rate of an $\epsilon, \delta$-AWTP$_PD$ protocol is bounded as,
   $$C^\epsilon \leq 1 - \rho + 2\epsilon \cdot \left(1 + \log |\mathcal{S}| \frac{1}{\epsilon} \right) + 2\epsilon n$$

We outline the proof of the upper bound on rate. First (Step1), we define a weaker adversary that, (i) chooses the reading and writing sets of all invocations of the AWTP channel before the start of the protocol, and (ii) uses random errors of appropriate weight for each invocation. For this adversary, we prove two lemmas (Lemmas 2 and 3, related to the entropy of the transmitted message. Second (Step 2), we use the lemmas to derive a bound on $\log |\mathcal{M}| / N \log |\mathcal{S}|$; Finally (Step 3) we prove the bound on the channel capacity.

Notations. We first give some notations that will be used to show the upper bound of rate over AWTP channel. Let $\{N_i\} = \{1, \ldots, N\}$ be the set of positions of codeword transmitted over $i^{th}$ round AWTP channel, and $\{N_i\} = \bigcup_{i=1}^{n} [N_i]$. Let $S_i^r$ and $S_i^w$ be the set of positions that adversary read/write over $i^{th}$ round of AWTP channel, with $S_i^r = \rho_r \cdot N_i$ and $S_i^w = \rho_w \cdot N_i$. Let $S_i^{r,w} = \{S_i^r, \ldots, S_i^w\}$ and $S_i^{*,w} = \{S_i^{*,r}, \ldots, S_i^{*,w}\}$ be the sets that adversary read and write after $i^{th}$ round of AWTP channel. Let $S_i^{*,r}$, $S_i^{*,w}$ be the set of only reading, $S_i^{*,r}$, $S_i^{*,w}$ be the set of reading and writing, $S_i^{*,r} = S_i^{*,w}$ be the set of only writing, and $S_i^{*,r} = [N_i] \setminus (S_i^{*,r} \cup S_i^{*,w})$ be the set of neither reading or writing on each $i^{th}$ round. Let the sets, $S_i^{r,a} = \bigcup_{j=1}^{\ell} S_i^{r,a}$, $S_i^{r,b} = \bigcup_{a=1}^{b} S_i^{r,a}$, $S_i^{w,a} = \bigcup_{j=1}^{\ell} S_i^{w,a}$, $S_i^{w,b} = \bigcup_{a=1}^{b} S_i^{w,a}$, and $S_i^{*,a} = \bigcup_{a=1}^{b} S_i^{*,a}$, be the sets of only read, read and write, only write, and unattended, respectively, for the AWTP$_PD$ protocol.

Let $C_i$, $D_i$ be the random variables of $c_i$, $d_i$ in round $i$; $C_{i,j}$ and $D_{i,j}$ be the random variables of $c_{i,j}$ and $d_{i,j}$; and $C^r$, $D^r$ be the random variables of $c^r$ and $d^r$. Let $C_i^{*,r}$ and $C_i^{*,w}$ be the random variables corresponding to protocol messages, on the sets $S_i^{*,r}$ and $S_i^{*,w}$. Let $C_i^{*,r}$, $C_i^{*,w}$ be the random variables on the set $S_i^{*,r}$, $S_i^{*,w}$, $S_i^{*,c}$, $S_i^{*,d}$.

Proof: The proof of upper bound of rate has three steps:

Step 1. We define an adversary $\text{Adv}_1$ and prove an upper bound on the rate of the protocols, assuming this adversary. The rate of the AWTP$_PD$ protocol against a general adversary cannot violate this bound, because $\text{Adv}_1$ is one of the possible adversarial strategies that can be used against the protocol.
Definition 7: $\text{Adv}_1$ works as follows:

1) Selects the reading and writing sets, $S^{\ell,r}$ and $S^{\ell,w}$, respectively, of the AWTP channel for all rounds before the start of the protocol.

2) For each round, chooses a random error vector $e_i$ with appropriate weight; that is, use chooses $e_i^T$, with uniform distribution from $\Sigma^{|M|}$; we have $Pr(e_i^T) = \frac{1}{|\Sigma|^{|M|}}$.

3) Read the transmission over PD channel.

$\text{Adv}_1$ can only read the communications on PD.

We give two lemmas that follow from $\epsilon$-secrecy and $\delta$-reliability of protocols with security against $\text{Adv}_1$. Let $V_E$ denote the random variable of adversary view.

Lemma 1: For an $(\epsilon, \delta)$-AWTP protocol, the following holds:

$$I(M; V_E) \leq 2\epsilon N \cdot \log \left( \frac{|\Sigma|}{\epsilon} \right) + 2\epsilon n$$

Proof is in Appendix [A].

Since $\text{Adv}_1$ selects the reading sets $S^{\ell,r}$ before the start of the protocol, we have, $V_E = \{ C^{\ell,r} D^{2\ell} \}$, and so,

Lemma 2: The view of $\text{Adv}_2$ in an $(\epsilon, \delta)$-AWTP protocol, satisfies:

$$I(M; C^{\ell,r} D^{2\ell}) \leq 2\epsilon N \cdot \log \left( \frac{|\Sigma|}{\epsilon} \right) + 2\epsilon n$$

Lemma 3: For an $(\epsilon, \delta)$-AWTP protocol, the following holds assuming $\text{Adv}_1$ adversary,

$$H(M|C^{\ell,a} D^{2\ell}) \leq H(\delta) + \delta(H(M) - 1)$$

Proof is in Appendix [B].

Lemma 2 and Lemma 3 are used to prove an upper bound on the rate of an $(\epsilon, \delta)$-AWTP protocol, assuming adversary $\text{Adv}_1$. This implies an upperbound for rate against a general adversary.

Step 2. We prove the upper bound,

$$\frac{\log |M|}{N \log |\Sigma|} \leq 1 - \rho + 2\epsilon \cdot (1 + \log_{|\Sigma|} \frac{1}{\epsilon}) + 2en + 2H(\delta) + \delta n$$

Here, $N$ is the total number of symbols sent over AWTP channel. Let $C^{\ell}$ and $D^{2\ell}$ denote the set of possible protocol messages that are sent over the AWTP channel and the PD channel, respectively. We have,

$$H(M) = I(M; C^{\ell,r} D^{2\ell}) + H(M|C^{\ell,r} D^{2\ell})$$

From Lemma 2 the first term can be upper bound as,

$$I(M; C^{\ell,r} D^{2\ell}) \leq 2\epsilon \cdot N \cdot \log \left( \frac{|\Sigma|}{\epsilon} \right) + 2\epsilon n$$

The upper bound on the second item $H(M|C^{\ell,r} D^{2\ell})$ is,

$$H(M|C^{\ell,r} D^{2\ell}) = H(M|C^{\ell,a} C^{\ell,b} D^{2\ell}) + H(C^{\ell,b} | M C^{\ell,a} D^{2\ell})$$

Using (1) and (2) we can bound $H(M|C^{\ell,r} D^{2\ell})$ as,

$$H(M|C^{\ell,r} D^{2\ell}) \leq (3) N(1 - \rho) \log |\Sigma| + \delta N \log |\Sigma| + n + H(\delta)$$

From (1), (2), and (3), the upper bound on $H(M)$ is,

$$H(M) \leq N(1 - \rho) \log |\Sigma| + 2\epsilon \cdot N \cdot \log \left( \frac{|\Sigma|}{\epsilon} \right) + 2en + 2H(\delta) + \delta n$$

The above inequality must hold for any distribution and in particular for a uniformly distributed message space, and so $H(M) = \log |M|$. Using $\delta \leq H(\delta)$ for $0 \leq \delta \leq 1/2$, we have,

$$\frac{\log |M|}{N \log |\Sigma|} \leq 1 - \rho + 2\epsilon \cdot (1 + \log_{|\Sigma|} \frac{1}{\epsilon}) + 2en + 2H(\delta) + \delta n$$
Step 3. We show that $\epsilon$-secrecy capacity of a $(\rho_r, \rho_w)$-AWTPPD is bounded as,

$$C' \leq 1 - \rho + 2\epsilon \cdot \left(1 + \log|\Sigma| \frac{1}{\epsilon}\right) + 2\epsilon n$$

Proof is by contradiction.

Let $C' = 1 - \rho + 2\epsilon \cdot \left(1 + \log|\Sigma| \frac{1}{\epsilon}\right) + 2\epsilon n + \hat{\xi}$, for some small constant $0 < \hat{\xi}$. From the Definition 8 for any $0 < \hat{\xi} < \min\left(\frac{1}{w}, H^{-1}(\frac{1}{\epsilon})\right)$, there is $N_0$, such that for any $N > N_0$, we have $\delta < \hat{\xi}$ and,

$$\frac{\log|\mathcal{M}|}{N \log|\Sigma|} \geq C' - \hat{\xi}$$

$$= 1 - \rho + 2\epsilon \cdot \left(1 + \log|\Sigma| \frac{1}{\epsilon}\right) + 2\epsilon n + 2H(\delta)$$

$$+ \delta n + \hat{\xi} - 2H(\delta) - \delta n$$

$$\geq 1 - \rho + 2\epsilon \cdot \left(1 + \log|\Sigma| \frac{1}{\epsilon}\right) + 2\epsilon n + 2H(\delta)$$

$$+ \delta n + \hat{\xi} > \frac{\log|\mathcal{M}|}{N \log|\Sigma|}$$

This contradicts the bound on $\frac{\log|\mathcal{M}|}{N \log|\Sigma|}$, and so,

$$C' \leq 1 - \rho + 2\epsilon \cdot \left(1 + \log|\Sigma| \frac{1}{\epsilon}\right) + 2\epsilon n$$

Corollary 1: The perfect secrecy capacity of a $(\rho_r, \rho_w)$-AWTPPD channel is bounded as,

$$C_0 \leq 1 - \rho$$

B. Lower Bound on Round Complexity

It was proved that $R \leq 1 - \rho_r - \rho_w$ and the construction of a $(0, \delta)$-AWTP code (1-round) with rate $R = 1 - \rho_r - \rho_w$, was given. This implies that secure communication over AWTP channel with 1-round protocols, is possible if $\rho_r + \rho_w < 1$. In Section IV-A we proved that for AWTPPD channels, $C_0 \leq 1 - \rho$ and so secure communication with $\rho_r + \rho_w > 1$ is possible, as long as $\rho < 1$. We prove that when $\rho_r + \rho_w \geq 1$ and $\rho < 1$, the $(\epsilon, \delta)$-AWTPPD protocols require at least three rounds.

Lemma 6 and Lemma 7 prove that $(\epsilon, \delta)$-AWTPPD protocols need invocation of the PD by both Alice and Bob. The bound is obtained by combing this, together with at least one invocation of AWTP channel.

Theorem 2: Perfect secure communication over AWTP channel requires,

(i) one round protocol, if $\rho_r + \rho_w < 1$.

(ii) at least three round protocol, if $\rho_r + \rho_w \geq 1$. That is,

$$RC \begin{cases} \geq 1 & \text{if } \rho_r + \rho_w < 1; \\ \geq 3 & \text{if } \rho_r + \rho_w \geq 1. \end{cases}$$

We use the same notations as in Section IV-A. Let $S^{e,a}$, $S^{e,b}$, $S^{e,c}$, $S^{e,d}$ denote the sets that the adversary, read only, read and write, write only, and neither read nor write, on the AWTP channel. We denote by $c^{e,a}, c^{e,b}, c^{e,c}, c^{e,d}$, the component of the protocol message on the sets, $S^{e,a}, S^{e,b}, S^{e,c}, S^{e,d}$, respectively. Let $\epsilon' = 2\epsilon N \cdot \log(\frac{1}{\epsilon}) + 2\epsilon n$.

1) PD Channel invoked by Bob: We show that it is impossible to have $(\epsilon, \delta)$-AWTPPD protocol with public discussion channel invoked by Bob, only. The proof is by constructing an adversary $A_{PD}$, and showing that for this adversary, one can pair executions of the protocol on all messages in the message space, such that for a paired execution, Bob will have the same view (receive the same vector over the AWTPPD), and so will output the same value. By repeating the argument for all basic round pairs (one invocation of AWTP by Alice and one invocation of PD by Bob), one can conclude that Bob will output the same at the end of the protocol and so, one incorrect output. This will be used to show that the probability of incorrect output is high.

Let the set of messages with the same adversarial view on $e^{e,a}$ be $\mathcal{M}(e^{e,a})$; that is, $\mathcal{M}(e^{e,a}) = \{m : \Pr(m|e^{e,a}) > 0\}$. Denote by $\mathcal{V}_1$, the set of adversarial views on $e^{e,a}$ that does not leak too much information about the message. That is,

$$\mathcal{V}_1 = \{e^{e,a} : \mathcal{H}(M|e^{e,a}) \leq \mathcal{H}(M) - 2\epsilon'\}$$

We define a group of adversary $A_2$ of AWTPPD protocol, and show that if PD channel is invoked by Bob only, it is impossible to have a $(\epsilon, \delta)$-AWTPPD protocol with security against adversary in $A_2$.

Definition 8: Adversary in $A_2$ works as follows:

1) Adversary $Adv_2 \in A_2$ selects reading set $S^{e,r}$ before AWTPPD protocol.

2) Adversary $Adv_2 \in A_2$ selects a special form of error $\epsilon_r$ on each round of AWTP channel, and adds error to transmission.

3) Adversary $Adv_2 \in A_2$ read the transmission over PD channel.

Lemma 4: An $(\epsilon, \delta)$-AWTPPD protocol against the adversary $Adv_2 \in A_2$, will have, $\Pr(C^{e,a} \in \mathcal{V}_1) \geq \frac{1}{2}$.

Proof is in Appendix D

Lemma 5: If $\mathcal{H}(M|e^{e,a}) \geq \mathcal{H}(M) - 2\epsilon'$, then $\Pr(M \in \mathcal{M}(e^{e,a})) \geq 1 - 2\epsilon'$.

Proof is in Appendix D

Lemma 4 and Lemma 5 are used to prove the following lemma.

Lemma 6: In an $(\epsilon, \delta)$-AWTPPD protocol with $\rho_r + \rho_w \geq 1$, if PD is only invoked by Bob, will have,

$$2\epsilon' + 4\delta \geq 1 - \frac{1}{|\mathcal{M}|}$$

Proof is in Appendix D

2) PD Channel Invoked by Alice: We show that it is impossible to have $(\epsilon, \delta)$-AWTPPD protocol over AWTP channel if public discussion channel is invoked by Alice, only. The proof is similar to the proof in Section IV-B1 and by constructing a group of adversary $A_2$ that results in pairing of the executions, and resulting in th relation between $\epsilon$ and $\delta$ as follows. The proof is omitted because of space.

Lemma 7: In a $(\epsilon, \delta)$-AWTPPD protocol with $\rho_r + \rho_w \geq 1$, if PD is used by Alice only, we have,

$$4\epsilon + 2\epsilon' \geq 1 - \frac{1}{|\mathcal{M}|}$$
Proof is in Appendix F.

Theorem 2 follows, by combining Lemmas 6 and 7 and noting that the protocol needs at least one round of AWTPPD.

V. \((0, \delta)\)-AWTPPD Protocol

We first introduce the building blocks of the AWTPPD protocol, and then describe the construction. The rate of the protocol meets the upper bound. The protocol has three rounds and so meets the minimum round complexity. The construction is inspired to Shi et al. 13.

A. Universal Hash Family

Definition 9: \([22]\) An \((N, n, m)\)-hash family is a set of \(N\) functions \(\mathcal{F}\) such that \(f : \mathcal{X} \rightarrow \mathcal{T}\) for each \(f \in \mathcal{F}\), where \(|\mathcal{X}| = n\) and \(|\mathcal{T}| = m\).

Without loss of generality, we assume \(n \geq m\).

Definition 10: \([22]\) Suppose that the \((N, n, m)\)-hash family \(\mathcal{F}\) has range \(\mathcal{T}\) which is an additive Abelian group. \(\mathcal{F}\) is called \(\epsilon\)-universal provided that for any two distinct elements \(x_1, x_2 \in \mathcal{X}\), and for any element \(t \in \mathcal{T}\), there exist at most \(\epsilon n\) functions \(f \in \mathcal{F}\) such that \(f(x_1) - f(x_2) = t\), were the operation is over the Abelian group.

We will use the classic construction of \(\frac{n}{q}\)-universal hash family \([22]\). Let \(q\) be a prime and \(n \leq q - 1\). Let the message be \(x = \{x_1, \cdots, x_n\}\). For \(\alpha \in \mathbb{F}_q\), define the universal hash function \(h(x, \cdot)\), by the rule,

\[
t = h(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + \cdots + x_n \alpha^n \mod q \tag{10}
\]

Then \((h(x, \cdot) : \alpha \in \mathbb{F}_q)\) is a \(\frac{n}{q}\)-universal \((q, q^n, q)\)-hash family.

B. Randomness Extractor

A randomness extractor is a function, which is applied to a weakly random entropy source (i.e., a non-uniform random variable), and results in a uniformly distributed source.

Definition 11: \([10]\) A (seeded) \((n, m, r, \delta)\)-strong extractor is a function \(\text{Ext} : q^m \times q^r \rightarrow q^m\) such that for any source \(X\) with \(H_\infty(X) \geq r\), there is

\[
\text{SD}((\text{Ext}(X, \text{Seed}), \text{Seed}), (U, \text{Seed})) \leq \delta
\]

with the seed uniformly distributed over \(\mathbb{F}_q^m\).

A function \(\text{Ext} : q^m \rightarrow q^m\) is a (seedless) \((n, m, r, \delta)\)-extractor if for any source \(X\) with \(H_\infty(X) \geq r\), the distribution \(\text{Ext}(X)\) satisfies \(\text{SD}(\text{Ext}(X), U) \leq \delta\).

There is a construction of seedless extractor from Reed-Solomon code \([8]\), which is easy to construct and extracts uniform randomness. The construction works only for a restricted class of sources known as symbol-fixing sources.

Definition 12: An \((n, m)\) symbol-fixing source is a tuple of independent random variables \(X = (X_1, \cdots, X_n)\), defined over a set \(\Omega\), such that \(m\) of the variables take values uniformly and independently from \(\Omega\), and the rest have fixed values.

We show the construction of seedless \((n, m, m \log q, 0)\)-extractor. Let \(q \geq n + m\). Let an \((n, m)\) symbol-fixing source \(X = (X_1, \cdots, X_n)\) \(\in \mathbb{F}_q^n\) with \(H(X) \geq m \log q\). The extractor has two steps:

1) The extractor first constructs a polynomial \(f(x) \in \mathbb{F}_q[X]\) of degree \(\leq n - 1\), such that \(f(i) = x_i\) for \(i = 0, \cdots, n - 1\).

2) Then, the extractor evaluate polynomial on \(i = \{n, \cdots, n + m - 1\}\). That is,

\[
\text{Ext}(x) = (f(n), f(n + 1), \cdots, f(n + m - 1))
\]

C. AWTPPD Protocol

Let AWTP channel have alphabet \(\Sigma = \mathbb{F}_q^m\) with \(q > 2^u N^2\), and the message be \(m = (m_1, \cdots, m_t) \in \mathcal{M}\), where \(m_i \in \mathbb{F}_q\). Let \(N\) denote the transmission length over the AWTP channel. We use the \(\frac{n}{q}\)-universal \((q, q^n-1, q)\)-hash family and the seedless \((uN, \ell, \ell \log q, 0)\)-extractor.

| AWTPPD Protocol |
|------------------|
| **Round 1:** Alice \(\xrightarrow{\text{AWTP}}\) Bob. For \(i \in N\):
| Alice randomly chooses a vector \(r_i = (r_{i1}, \cdots, r_{iu-1}) \in \mathbb{F}_q^{u-1}\), and \(\beta_i \in \mathbb{F}_q\). Alice sends \(c = (c_1, \cdots, c_N) \in \mathbb{F}_q^N\) with \(c_i = (r_{i1}, \beta_i)\) to Bob, over the AWTP channel.
| Bob receives \(y = (y_1, \cdots, y_N)\), where \(y_i = (r_i', \beta_i')\).
| **Round 2:** Bob \(\xrightarrow{\text{PD}}\) Alice.
| Bob generates random keys, \((\alpha_1, \cdots, \alpha_N), \alpha_i \in \mathbb{F}_q\), for the hash family, and generates \(t = (t_1, \cdots, t_N)\) where, \(t_i = h_{\alpha_i}(r_i') + \beta_i' \mod q\). Bob maps \(d_1 = \{\alpha_1, \cdots, \alpha_N, t_1, \cdots, t_N\}\) to a binary vector over \(\mathbb{F}_2\), and sends \(d_1\) to Alice, over the PD channel. Alice receives \(d_1\).
| **Round 3:** Alice \(\xrightarrow{\text{PD}}\) Bob.
| - Alice checks,
| \[h_{\alpha_i}(r_i') + \beta_i' = t_i \mod q, \ i = 1 \cdots N\]
| and constructs a binary vector \(v = (v_1, \cdots, v_N)\), where with \(v_i = 1\) if \(h_{\alpha_i}(r_i') + \beta_i' = t_i \mod q\), and \(v_i = 0\), otherwise.
| - Let, \(v_{i1} = \cdots = v_{i\ell} = 1\). Alice concatenates all \(r_i\) for which \(v_{ij} = 1\), and obtains \((r_{i1} || \cdots || r_{i\ell})\) over \(\mathbb{F}_q\). Alice uses the extractor on this string, and obtains a uniformly random string, \(k = \text{Ext}(r_{i1} || \cdots || r_{i\ell})\).
| - Alice encrypts the message \(m\) and obtains, \(c = (c_1, \cdots, c_{\ell})\) where, \(c_i = k_i + m_i \mod q\) for \(i = 1, \cdots, \ell\). Alice maps \(d_2 = (c, v)\) (over \(\mathbb{F}_q\)) into a binary vector and sends it to Bob over the PD channel. Bob receives \(d_2\).
| - Bob decodes \(\text{Dec}(y_1, d_1, d_2)\) as follows:
| - Constructs the vector \((r_i' || \cdots || r_i')\) with \(r_i' \in \mathbb{F}_q, \ \text{for all } v_{ij} = 1 \in v\). He uses the extractor to obtain, \(k_i' = \text{Ext}(r_i' || \cdots || r_i')\).
| - Recovers the message \(m'\) with \(m' = c_i - k_i' \mod q\) for \(i = 1, \cdots, \ell\).
Lemma 8: The AWTPPD protocol above, provides perfect secrecy if \( \ell \leq (u - 1)(1 - \rho)N \).

Proof is in Appendix [3].

Lemma 9: The probability of decoding error in AWTPPD protocol is \( \delta \leq uN \).

Proof is in Appendix [4].

Lemma 10: The rate of the AWTPPD protocol family is \( R = 1 - \rho \).

Proof: For a small \( \xi > 0 \), let the parameters of AWTPPD protocol be chosen \( u = \frac{1}{2}, q > 2uN^2, \ell = (u - 1)(1 - \rho)N, N_0 \geq \frac{1}{\xi} \) and \( \Sigma = \mathbb{F}_q^u \). For uniform message distribution, we have \( \log |M| = \ell \log q \), and so for any \( N > N_0 \), the rate of AWTPPD protocol family is given by,

\[
\frac{\log |M|}{N \log |E|} = \frac{(u - 1)(1 - \rho)N \log q}{uN \log q} = \frac{1 - \xi}{\ell - \xi} \geq 1 - \rho - \xi
\]

The probability of decoding error is bounded by,

\[
\delta \leq \frac{uN}{q} \leq \frac{1}{2N} \leq \frac{\xi}{2} \leq \xi
\]

Theorem 3: The AWTPPD protocol family provides perfect security. The rate of the protocol approaches \( 1 - \rho \) as, \( N \to \infty \). The round complexity is three, and the required computation is \( O((N \log q)^2) \).

VI. AWTPPD PROTOCOL AND SMT-PD

In [23] the relationship between AWTP and SMT was outlined. In this Section we develop the relationship between AWTPPD and SMT-PD models.

In an SMT-PD there is a sender \( S \) (Alice) and receiver \( R \) (Bob), that can interact over \( N \) node disjoint paths in a synchronous network, referred to as wires, and a public discussion channel. Wires and the PD provide two-way communication. An SMT-PD protocol proceeds in rounds. In each round, Alice (Bob) sends protocol messages over wires and/or the PD channel, which will be received by Bob (Alice) before the end of the round. A computationally unbounded adversary (Eve) can corrupt up to \( t \) wires. Eve can eavesdrop, modify or block messages sent over a corrupted wire. Adversary is adaptive and can corrupt wires any time during the protocol execution and after observing communications over the wires that she has corrupted so far.

Definition 13: A protocol between \( S \) and \( R \) is an \( (\epsilon, \delta) \)-secure message transmission with public discussion ((\( \epsilon, \delta) \)-SMTPD) protocol if the following two conditions are simultaneously satisfied.

- Privacy: For every two messages \( m_1, m_2 \in M \) and random coin \( r_E \) used by Eve, it has

\[
\max_{m_1, m_2} \text{SD}(\text{View}_E(\text{SMTPD}(m_1, r_E)), \text{View}_E(\text{SMTPD}(m_2, r_E))) \leq \epsilon
\]

where the probability is over the randomness of \( S, R \).
- Reliability: \( R \) recovers the message \( M_S \) with probability larger than \( 1 - \delta \), or formally

\[
\Pr(M_R \neq M_S) \leq \delta
\]

where the probability is over the choice of \( M_S \), the randomness of players \( S, R \) and Eve.

Remark 1: In the above definition of SMT-PD, (i) wires provide two-way communication, and (ii) in each round of the protocol, Alice (Bob) can invoke both types of channels simultaneously (wires and the PD). This is different to the model we considered in Section [11] where, (i) AWTP is from Alice to Bob only, and (ii) in each round one type of channel (AWTP , or PD) can be invoked.

Efficiency parameters of an SMT-PD protocol are, Round Complexity \( RC \), Transmission Rate \( TR \), and computational complexity.

- \( RC \) is the total number of rounds of a protocol.
- \( TR \) is the number of communicated bits for transmitting a single message bit. Let \( W_i \) denote the set of possible transmissions on wire \( i \). The transmission rate of an SMT-PD protocol is given by,

\[
TR = \frac{\log |M|}{\sum_{i=1}^{N} \log |W_i|}
\]

An SMT-PD protocol is optimal if the transmission rate is of the order \((\log N)^2\) notation) of the lower bound.
- An SMT-PD protocol is computationally efficient if the computational complexity of the sender and the receiver algorithms, is polynomial in \( N \).

A. Relation Between AWTPPD and SMT-PD

In a general SMT-PD protocol, the transmission over wires is two-way. We consider protocols where Alice wants to send a message to Bob.

Definition 14: A one-way secure message transmission with public discussion ((\( \epsilon, \delta) \)-SMT[1]PD) protocol is an SMT-PD protocol where transmission over wires is one-way (from Alice to Bob, or Bob to Alice).

We consider \((\epsilon, \delta)\)-SMT[1]PD protocols with transmission over wires from Alice to Bob. This is not without loss of generality- since the goal is to send a message from Alice to Bob, we are assuming one-way is from Alice to Bob also.

Theorem 4: A restricted \((\epsilon, \delta)\)-AWTPPD protocol is equivalent to \((\epsilon, \delta)\)-SMT[1]PD protocol. The following results on the latter protocols, follow from the former ones.

- The lower bound on the transmission rate of a 1-SMT-PD protocol is,

\[
TR \geq \frac{1}{1 - \rho + 2N(1 + \log |W|^{1/2}) + 2N}
\]

For protocols with perfect secrecy \((\epsilon = 0)\) we have,

\[
TR \geq \frac{1}{1 - \rho}
\]
The lower bound on the round complexity of \((\epsilon, \delta)\)-SMT\^[1]-PD protocol is three.

There is an efficient \((\epsilon, \delta)\)-SMT\^[1]-PD protocol with minimum number of rounds.

Proof of the theorem is given in Lemmas [11] and [14].

**Lemma 11:** A restricted \((\epsilon, \delta)\)-AWTP\^[PD] protocol is equivalent to \((\epsilon, \delta)\)-SMT\^[1]-PD protocol.

Proof of Lemma [II]

**Lemma 12:** The rate of a AWTP\^[PD] protocol is the inverse of the transmission rate of the corresponding \((\epsilon, \delta)\)-SMT\^[1]-PD protocol. The upper bound on the rate of AWTP\^[PD] protocols implies a lower bound on the transmission rate of \((\epsilon, \delta)\)-SMT\^[1]-PD protocol. That is if an adversary corrupts \(\rho = \frac{1}{N}\) fraction of \(N\) wires, the lower bound on the transmission rate is

\[
TR \geq \frac{1}{1 - \rho + 2N\epsilon(1 + \log_2|W|) + 2\rho N}
\]

Proof of Lemma [II]

**Lemma 13:** If \(r \geq \frac{N}{2}\), then an \((\epsilon, \delta)\)-SMT\^[1]-PD protocol requires at least three rounds.

Proof is in Appendix [K].

1) Construction: A \((\epsilon, \delta)\)-AWTP\^[PD] protocol can be used directly to obtain a restricted-(\(\epsilon, \delta)\)-AWTP\^[PD] protocol, where \(\rho = \rho_r + \rho_w\). This latter, using the protocol conversion in Lemma [I] can be used to construct \((\epsilon, \delta)\)-SMT\^[PD] protocol. In Section [V], we gave the construction of a \((0, \delta)\)-AWTP\^[PD] protocol with minimum number of round and rate approaching the capacity of the \((\rho_r, \rho_w)\)-AWTP channel. This implies the following.

**Lemma 14:** There exists a three-round There is a three round \((\epsilon, \delta)\)-SMT\^[1]-PD protocol, with transmission rate, \(O(\frac{N}{\epsilon})\), and computational complexity of decoding equal to, \(O((N \log q)^2)\).

**VII. CONCLUSION**

We motivated and introduced AWTP\^[PD], where Alice and Bob, in addition to the AWTP channel, have access to a public discussion channel, and showed that with this new resource, secure communication is possible even when \(\rho_r + \rho_w \geq 1\) as long as \(\rho < 1\). We derived bound on the information rate and the number of rounds of the protocols that provide \(\epsilon\)-secrecy and \(\delta\)-reliability, and constructed an optimal protocol family that achieve both lower bounds. We also showed the relationship between AWTP\^[PD] and a subclass of SMT-PD in which wires are used by Alice only, and gave the construction of an optimal \((\epsilon, \delta)\)-SMT\^[1]-PD protocol with minimum number of rounds, with transmission rate approaching the lower bound as \(N\) approaches infinity. A three-round protocol SMT-PD (two-way wires) with the same rate had been constructed in [13]. This suggests that the performance of our protocol is as good as this latter protocol in the SMT-PD setting, suggesting that the same efficiency can be obtained if wires are one-way.

In our model, the AWTP channel is one-way. Deriving similar results for two-way AWTP channel, bound on the communication rate and the number of rounds, and construction of optimal protocols, are interesting open questions.

**REFERENCES**

[1] V. Aggarwal, L. Lai, A. Calderbank, and H. Poor, “Wiretap Channel Type II with an Active Eavesdropper”, *ISIT*, pp. 1944–1948, 2009.

[2] R. Ahlswede, I. Csiszar, “Common randomness in information theory and cryptography - I: Secret sharing”, IEEE Transactions on Information Theory, vol. 39, no. 4: 1121–1132, 1993.

[3] J. Bellare, S. Tessaro, and A. Vardy, “Semantic Security for the Wiretap Channel”, *CRYPTO*, pp. 294–311, 2012.

[4] I. Bjelakovi, H. Boche, and J. Sommerfeld, “Capacity Results for Arbitrarily Varying Wiretap Channels”, *Information Theory, Combinatorics, and Search Theory*, pp. 123–144, 2013.

[5] M. Bloch, J. Barros, M. Rodrigues, and S. McLaughlin, “Wireless Information Theoretic Security”, *IEEE Transactions on Information Theory*, vol. 54(6), pp. 2515-2534, 2008.

[6] H. Boche, and R. Scharf, “Capacity Results and Super-Activation for Wiretap Channels With Active Wiretappers”, *IEEE Transactions on Information Forensic and Security*, vol. 8(9) pp. 1482–1496, 2013.

[7] M. Cheraghchi, F. Didier, and A. Shokrollahi, “Invertible Extractors and Wiretap Protocols”, *IEEE Transactions on Information Theory*, vol. 58, pp. 1254–1274, 2012.

[8] I. Csiszar and J. Körner, “Broadcast Channels with Confidential Messages”, *IEEE Transaction on Information Theory*, vol. 24(3), pp. 339–348, 1978.

[9] Y. Dodis, R. Ostrovsky, L. Reyzin, A. Smith, “Puzzy Extractors: How to Generate Strong Keys from Biometrics and Other Noisy Data,” *SIAM Journal on Computing*, vol. 38(1), pp. 97–139, 2008.

[10] L. Ozarow, and A. Wyner, “Wire-Tap Channel II”, *Bell System Technical Journal*, vol. 54, pp. 1751–1761, 1975.

[11] I. Csiszar and J. Körner, “Broadcast Channels with Confidential Messages”, *IEEE Transaction on Information Theory*, vol. 24(3), pp. 339–348, 1978.

[12] E. MolavianJazi, M. Bloch, and J. N. Laneman, “Arbitrary Jamming Can Adversarially Vary the Wiretap Channel’s Security”, *IEEE Transactions on Information Theory*, 2013.

[13] A. Pye, and T. Yoshida, “Optimal Codes for Limited View Adversarial Channels”, *IEEE Transactions on Information Theory*, vol. 54(11), pp. 5059-5067, 2008.

[14] Leung-Yan-Cheong, and E. Hellman, “The Gaussian wire-tap channel”, *IEEE Transactions on Information Theory*, 24(4), pp. 451 – 456, 1978.

[15] U. Maurer, “protocols for Secret Key Agreement by Public Discussion Based on Common Information”, *Crypto*, pp. 461–470, 1992.

[16] E. MolavianJazi, M. Bloch, and J. N. Laneman, “Arbitrary Jamming Can Adversarially Vary the Wiretap Channel’s Security”, *IEEE Transactions on Information Theory*, 2013.

[17] L. Ozarow, and A. Wyner, “Wire-Tap Channel II”, *EUROCRYPT*, pp. 33–50, 1984.

[18] C. Popper, N. Tippenhauer, B. Danev, and S. Capkun, “Investigation of Signal and Message Manipulations on the Wireless Channel”, *ESORICS*, pp. 40–59, 2011.

[19] R. Safavi-Naini, and P. Wang, “Codes for Limited View Adversarial Channels”, *ISIT*, pp. 266-270, 2013.

[20] R. Safavi-Naini, and P. Wang, “Efficient Codes for Limited View Adversarial Channels”, *IEEE Conference on Communications and Network Security*, 2013.

[21] D. Stinson, “Universal hash families and the leftover hash lemma, and applications to cryptography and computing”, *J. Combin. Math. Combin. Comput.*, vol. 42, pp. 3–31, 2002.

[22] P. Wang, R. Safavi-Naini, “A Model for Adversarial Wiretap Channel”, *CoRR abs/1312.6452*, 2013.

[23] A. Wyner, “The wire-tap channel”, *Bell System Technical Journal*, vol. 54, pp. 1355-1387, 1975.

**APPENDIX**

**A. Proof of Lemma [II]**

**Proof:** The proof is similar to Theorem 4.9 [3]. The proof uses Pinsker’s Lemma:
Lemma 15: Let $P$, $Q$ be probability distributions. Let $SD(P, Q) \leq \epsilon$. Then

$$H(P) - H(Q) \leq 2\epsilon \log\left(\frac{|P \cup Q|}{\epsilon}\right)$$

Denote the distribution of adversarial view be $V_E$, and the space of adversarial view be $V$. According to the definition of $\epsilon$-secrecy (Definition 5), for any pair of message $m_1, m_2 \in M$, the statistical distance between the distribution of $V_E$ when Alice sends $m_1$, and the distribution of $V_E$ when Alice sends $m_2$, is no more than $\epsilon$. That is

$$\epsilon \geq \max_{m_1, m_2} SD(V_E|M = m_1, V_E|M = m_2)$$

It implies,

$$SD(V_E, V_E|M = m) = \frac{1}{2} \sum_{v \in V} |Pr(v|m) - Pr(v)|$$

$$= \frac{1}{2} \sum_{v \in V} |Pr(v|m) - \sum_{m'} Pr(v|m')Pr(m')|$$

$$= \frac{1}{2} \sum_{v \in V} \sum_{m'} |Pr(m')|Pr(v|m) - Pr(v|m')|$$

$$\leq \frac{1}{2} \sum_{v \in V} \sum_{m'} Pr(m')|Pr(v|m) - Pr(v|m')|$$

$$= \sum_{m'} Pr(m') \max_{m_1, m_2} SD(V_E|M = m_1, V_E|M = m_2)$$

$$\leq \epsilon$$

From Pinsker Lemma and (11), it implies,

$$H(V_E) - H(V_E|M = m) \leq 2\epsilon \cdot \log\left(\frac{|V_E|}{\epsilon}\right)$$

From $|V_E| \leq 2^n \times |\Sigma|^N$, it implies,

$$H(V_E) - H(V_E|M = m) \leq 2\epsilon \cdot \log\left(\frac{|\Sigma|^N}{\epsilon}\right) + 2\epsilon n$$

So the difference between $H(M)$ and $H(M|V_E)$ is

$$H(M) - H(M|V_E) = H(V_E) - H(V_E|M)$$

$$= \sum_{m \in M} Pr(m)(H(V_E) - H(V_E|m))$$

$$\leq 2\epsilon N \cdot \log\left(\frac{|\Sigma|}{\epsilon}\right) + 2\epsilon n$$

B. Proof of Lemma 3

Proof: Let $\delta' = H(\delta) + \delta(H(M) - 1)$. We give the upper bound of $H(M|C^{t,a}C^{d,t}D^{2t}) \leq \delta'$. The proof is consisted by three steps.

1) In the first step, we show that $H(M|C^{t,a}Y^{t,w}C^{d,t}D^{2t}) \leq \delta'$. From $Pr(M_S \neq M_R) \leq \delta$, we apply Fano’s inequality and get

$$H(M_S|M_R) \leq H(\delta) + \delta(H(M) - 1)$$

Since $m_R = Dec(y^t, d^{2t})$ and $y^t = \{c^{t,a}, y^{t,w}, y^{t,d}\}$, it implies,

$$H(M|C^{t,a}Y^{t,w}C^{d,t}D^{2t}) \leq H(M_S|Dec(y^t, d^{2t}))$$

$$\leq H(M_S|Dec(y^t, d^{2t})) \leq \delta'$$

2) In second step, we show that $H(M|C^{t,a}C^{d,t}D^{2t}) \leq \delta' + I(Y^{t,w}C^{t,a}C^{d,t}D^{2t})$. From

$$H(M|C^{t,a}Y^{t,w}C^{d,t}D^{2t}) = H(M|C^{t,a}Y^{t,w}C^{d,t}D^{2t}) + H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

It implies,

$$H(M|C^{t,a}C^{d,t}D^{2t}) = H(M|C^{t,a}Y^{t,w}C^{d,t}D^{2t}) + H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

If there is no error, Bob will always output the correct message, that is $m_R = m_S$. It implies $m_S = m_R = Dec(c^{t,d^{2t}})$. So there is

$$H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t}) = H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t}Dec(C^{t,a}C^{d,t}D^{2t}))$$

$$\geq H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

From (13), (14) and (15), it implies

$$H(M|C^{t,a}C^{d,t}D^{2t})$$

$$= H(M|C^{t,a}Y^{t,w}C^{d,t}D^{2t} + H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

$$\leq \delta' + H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

If there is no error, Bob will always output the correct message, that is $m_R = m_S$. It implies $m_S = m_R = Dec(c^{t,d^{2t}})$. So there is

$$H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

$$= H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t}Dec(C^{t,a}C^{d,t}D^{2t}))$$

$$\geq H(Y^{t,w}|C^{t,a}C^{d,t}D^{2t})$$

$$\leq \delta' + I(Y^{t,w}C^{t,a}C^{d,t}D^{2t})$$

3) In third step, we show that

$$I(Y^{t,w}C^{t,a}C^{d,t}D^{2t}) = 0$$

So from (16) and (17), it implies the upper bound of $H(M|C^{t,a}C^{d,t}D^{2t}) \leq \delta'$. 
Denote the space of components of received word $y^{ℓ, w}$ on set $S^{ℓ, w}$ as $Y^{ℓ, w}$. From $H(y^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w} y^{ℓ, w}) = 0$ and $H(E^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w} y^{ℓ, w}) = 0$, and

$$H(Y^{ℓ, w} E^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w} y^{ℓ, w}) = H(Y^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w} y^{ℓ, w})$$

It implies,

$$H(E^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w}) = H(Y^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w})$$

So there is

$$H(Y^{ℓ, w} | C^{ℓ, a} D^{2ℓ}) = H(Y^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w})$$

(1) is from the error $e^{ℓ, w}$ on set $S^{ℓ, w}$ is randomly chosen from $\Sigma^{S^{ℓ, w}}$. (2) is from that the error $e^{ℓ, w}$ is uniformly distributed, so $H(E^{ℓ, w}) = n \log |\Sigma|$.

It implies

$$H(Y^{ℓ, w} | C^{ℓ, a} D^{2ℓ}) = H(Y^{ℓ, w} | C^{ℓ, a} D^{2ℓ} C^{ℓ, w})$$

and

$$1 \{Y^{ℓ, w} C^{ℓ, a} D^{2ℓ} C^{ℓ, w} \} = 0$$

C. Proof of Lemma 2

Proof: The Adv$_1$ and Adv$_2$ has the same reading capability. From Lemma 2 it implies for $(ε, δ)$-AWTP$_{PD}$ protocols with adversary Adv$_2$,

$$H(M | C^{ℓ, a}) ≥ H(M | C^{ℓ, a} D^{2ℓ}) ≥ H(M) - ε'$$

(19)

Assume $Pr(C^{ℓ, a} ∈ V_1) = p < \frac{1}{2}$. Then we have,

$$H(M | C^{ℓ, a}) = \sum_{c^{ℓ, a} ∈ V_1} Pr(c^{ℓ, a}) H(M | C^{ℓ, a}) = \sum_{c^{ℓ, a} ∈ V_1} Pr(c^{ℓ, a}) H(M) + \sum_{c^{ℓ, a} ∈ V_1} Pr(c^{ℓ, a}) H(M | c^{ℓ, a})$$

$$≤ pH(M) + (1 - p)(H(M) - 2ε')$$

This contradicts (19) and so we have, $Pr(c^{ℓ, a} ∈ V_1) ≥ \frac{1}{2}$. ■

D. Proof of Lemma 3

Proof: Since Adv$_2$ randomly selects message $m ∈ M$, we have $Pr(M = m) = |M|$, and so

$$Pr(M ∈ M | C^{ℓ, a}) = \frac{\sum_{m ∈ M | C^{ℓ, a}} Pr(M = m)}{|M|}$$

From $\log |M| = H(M)$, and $\log |M | C^{ℓ, a}) \geq H(M | C^{ℓ, a})$, we have,

$$Pr(M ∈ M | C^{ℓ, a}) = \frac{|M | C^{ℓ, a})}{|M|} ≥ 2^{H(M | C^{ℓ, a}) - H(M)} ≥ 2^{-2ε'} ≥ 1 - 2ε'$$

E. Proof of Section 6

Proof: We only consider the case that $ρ_1 = 1 - ρ_2$. The case $ρ_2 > 1 - ρ_1$ can be proved, similarly. Let a pair of adversary Adv$_2$, Adv$_2 ∈ A_2$.

From $ρ_1 = 1 - ρ_2$, we have $|S^{ℓ, b}| = |S^{ℓ, d}|$. The proof has five steps:

1) Let exec denote an execution of AWTP$_{PD}$ protocol where Alice sends message $m$, Alice and Bob generate protocol messages $\{ε, d^i\}$, Adv$_2$ reads on set $S^{ℓ, r} = \{S^{ℓ, a}, S^{ℓ, b}\}$ and PD channel, and adds error on set $S^{ℓ, w} = \{S^{ℓ, b}, S^{ℓ, c}\}$. That is,

- exec: In round $i ∈ \{1 \cdots ℓ\}$, Alice encodes $m$ using randomness $r_A$ into,

$$e_i = \{e_i^a, e_i^b, e_i^c, e_i^d\}$$

$$= AWTP_{PD}(m, r_A, i, v_B, PD)$$

Adv$_2$ chooses to read the set $S^{ℓ}_r = \{S^{ℓ}_r, S^{ℓ}_c\}$, and write on set $S^{ℓ}_r = \{S^{ℓ}_a, S^{ℓ}_b\}$. Adv$_2$ randomly selects $m ∈ M$ and encodes it using randomness $r_F$ that is chosen such that, $e_i^b = \{e_i^a, e_i^b, e_i^c^i, e_i^d^i\}$.

Adv$_2$ generates error $e_i = \{e_i^a, e_i^b, e_i^c, e_i^d\}$ where, $e_i^b = e_i^d = 0$, $e_i^c = e_i^c_i - e_i^c$, and $e_i^d$ is randomly chosen from the set $Σ^{S^{ℓ}_r}$. Adv$_2$ adds error to $c_i$ and forms, $y_i = c_i + e_i$.

After receiving $y_i$, Bob replies

$$d_i = AWTP_{PD}(r_B, i, v_B, PD)$$

2) Let exec denote a second execution of AWTP$_{PD}$ in which Alice sends message $m$, Alice and Bob generate protocol messages $\{ε^i, d^i\}$, Adv$_2$ reads on set $S^{ℓ, r} = \{S^{ℓ, a}, S^{ℓ, d}\}$ and PD channel, and adds error on set $S^{ℓ, w} = \{S^{ℓ, c}, S^{ℓ, d}\}$. That is,

- exec: In round $i ∈ \{1 \cdots ℓ\}$, Alice sends $m$, encoded as,

$$e_i = \{e_i^a, e_i^b, e_i^c, e_i^d\}$$

$$= AWTP_{PD}(\tilde{m}, r_A, i, v_B, AWTP)$$
\( \text{Adv}_2 \) reads the set \( \hat{S}_i^e = \{ S_i^a, S_i^d \} \), and writes on the set \( \hat{S}_i^w = \{ S_i^a, S_i^d \} \). \( \text{Adv}_2 \) chooses \( m \in M \), and encodes it to \( c_i = \{ c_i^a, c_i^d, c_i^e, c_i^f \} \). \( \text{Adv}_2 \) generates error \( \hat{e}_i = \{ \hat{e}_i^a, \hat{e}_i^d, \hat{e}_i^e, \hat{e}_i^f \} \) with \( \hat{e}_i^a = \hat{e}_i^d = 0, \hat{e}_i^e = c_i^e - \hat{e}_i^e \), \( \hat{e}_i^f = c_i^f + \hat{e}_i^f \). \( \text{Adv}_2 \) adds error to \( \hat{e}_i \) and forms \( \hat{y}_i = c_i + \hat{e}_i \).

After receiving \( \hat{y}_i \), Bob replies
\[
\hat{d}_i = \text{AWTP}_{\text{PD}}(\hat{r}_B, i, \hat{v}_B, \text{PD})
\]

For the two executions \( \text{exec}, \hat{\text{exec}} \) of the AWTP\_PD protocol, Bob receives the same information; that is \( y_i = \hat{y}_i \) in each round \( i \). So Bob’s view of the two executions is same and so, \( v_B = \hat{v}_B \). Bob will reply to Alice, with the same protocol message \( i \hat{d}_i = \hat{d}_i \).

3) If \( m \neq \hat{m} \) and \( c^e, a \in V_1 \), the probabilities of the two executions satisfy,
\[
\Pr(\text{exec}|V_1, M \neq \hat{M}) \geq \Pr(\text{exec}|V_1, M \neq \hat{M})(1 - 2\epsilon')
\]

For AWTP\_PD protocol, if \( c^e, a \in V_1 \) and \( m \neq \hat{m} \), the probability of the execution \( \text{exec} \) is,
\[
\Pr(\text{exec}|V_1, M \neq \hat{M}) = \Pr(m)\Pr(r_A)\Pr(r_B)\Pr(\epsilon') \\
\times \Pr(\hat{m}|M \neq \hat{M})\Pr(\hat{m} \in M(c^e, a))
\]

For AWTP\_PD protocol, if \( c^e, a \in V_1 \) and \( m \neq \hat{m} \), the probability of the execution \( \hat{\text{exec}} \) is,
\[
\Pr(\hat{\text{exec}}|V_1, M \neq \hat{M}) = \Pr(\hat{m})\Pr(\hat{r}_A)\Pr(\hat{r}_B)\Pr(\hat{\epsilon'})\Pr(m|M \neq \hat{M})
\]

It is obvious that \( \Pr(m) = \Pr(\hat{m}), \Pr(r_A) = \Pr(\hat{r}_A), \Pr(r_B) = \Pr(\hat{r}_B) \). Since \( \text{Adv}_2 \) and \( \text{Adv}_2 \) chooses \( \epsilon_i^a, \epsilon_i^d \) with \( \Pr(\epsilon_i^a) = \Pr(\epsilon_i^d) = 1 \), and \( \epsilon_i^e, \epsilon_i^f \) with \( \Pr(\epsilon_i^e) = \frac{1}{|M|}, \Pr(\epsilon_i^f) = \frac{1}{|M|} \), it implies \( \Pr(\epsilon_i^a) = \Pr(\epsilon_i^d) \) for \( i = 1 \cdots \ell \). Using \( \Pr(m \in M(c^e, a)) \geq 1 - 2\epsilon' \), we have,
\[
\Pr(\text{exec}|V_1, M \neq \hat{M}) \geq \Pr(\text{exec}|V_1, M \neq \hat{M})(1 - 2\epsilon')
\]

4) If \( m \neq \hat{m} \) and \( c^e, a \in V_1 \), we have,
\[
\Pr(M_R \neq M_S|V_1, M \neq \hat{M}) \geq \Pr(M_R = M_S|V_1, M \neq \hat{M})
\]

Let \( E_{\text{succ}} \) be the set of execution with correct output, and \( E_{\text{fail}} \) be the set of executions with incorrect output. If \( \text{exec} \in E_{\text{succ}} \), Bob will output the correct message \( m = \text{Dec}(y^f, d^f) \). Bob outputs the same message \( m = \text{Dec}(y^f, d^f) \) in execution \( \hat{\text{exec}} \), since \( \text{Bob} \) receives the same information \( y^f = y^f, d^f = d^f \). Since \( m \neq \hat{m} \) in \( \text{exec} \), Bob outputs \( m \neq \hat{m} \) and execution \( \text{exec} \in E_{\text{fail}} \), and so,
\[
\Pr(M_R \neq M_S|V_1, M \neq \hat{M}) \geq \Pr(E_{\text{fail}}|V_1, M \neq \hat{M})
\]

\[
\geq \sum_{\text{exec} \in E_{\text{succ}}} \Pr(\text{exec}|V_1, M \neq \hat{M})(1 - 2\epsilon')
\]

\[
\geq \Pr(M_R = M_S|V_1, M \neq \hat{M})(1 - 2\epsilon')
\]

5) To show that if the public discussion is only invoked by \( Bob \), we have \( 2\epsilon' + 4\delta \geq 1 - \frac{1}{|M|} \), we note that,
\[
\Pr(M_R \neq M_S|V_1, M \neq \hat{M}) \geq \Pr(M_R = M_S|V_1, M \neq \hat{M})(1 - 2\epsilon')
\]

\[
\geq 1 - 2\epsilon' - (1 - 2\epsilon')\Pr(M_R \neq M_S|V_1, M \neq \hat{M})
\]

which implies that,
\[
\Pr(M_R \neq M_S|V_1, M \neq \hat{M}) \geq \frac{1 - 2\epsilon'}{2} \geq \frac{1}{2} - \epsilon'
\]

We have,
\[
\delta = \Pr(M_R \neq M_S) \geq \frac{1}{2}(1 - \frac{1}{|M|})\Pr(M_R \neq M_S|V_1, M \neq \hat{M})
\]

\[
\geq \frac{1}{2}(1 - \frac{1}{|M|})\Pr(M_R \neq M_S|V_1, M \neq \hat{M})
\]

\[
\geq \frac{1}{2}(1 - \frac{1}{|M|})\frac{1}{2} - \epsilon'
\]

where (1) is from Lemma \([4]\). This implies \( 4\delta + 2\epsilon' \geq 1 - \frac{1}{|M|} \).

\[\blacksquare\]

**F. Proof of Lemma \([7]\)**

**Proof:** We show that it is impossible to have \((\epsilon, \delta)\)-AWTP\_PD protocol over AWTP channel if public discussion channel is invoked by Alice, only. The proof is by constructing a group of adversaries \( A_3 \) which are specific type of adversaries over AWTP\_PD channel. For any AWTP\_PD protocol with PD channel invoked by Alice, there exists a pair of adversary \{Adv\_3,Adv\_3\} \( \in A_3 \) such that for the transmission over AWTP\_PD channel, the error-free fraction Bob received under Adv\_3 attacking, is same as the reading fraction of adversary under Adv\_3 attacking. Since AWTP\_PD protocol should satisfy both the secrecy and reliability for all the adversaries, including adversaries in \( A_3 \), if AWTP\_PD protocol under Adv\_3 meets the reliability requirement, the secrecy requirement of AWTP\_PD protocol under Adv\_3 will be compromised.

We only consider the case that \( \rho_r > 1 - \rho_w \). The case that \( \rho_r > 1 - \rho_w \) can be proved in similar method. Let \( A_3 \) be a group of adversaries. Each Adv\_3 \( \in A_3 \) can be one of the possible adversarial strategies that can be used against the protocol.
Definition 15: Each adversary $\text{Adv}_3 \in \mathcal{A}_3$ has the reading and writing capability:

1) $\text{Adv}_3$ selects the reading sets $S^{\ell,r}$ on all round before the AWTP$_{PD}$ protocol.
2) $\text{Adv}_3$ selects the writing sets $S^{\ell,w}$ on all round before the AWTP$_{PD}$ protocol. In each round, adversary chooses the error $e_i$ with the value $e_i^w$ randomly and uniformly over $\Sigma^{w} N_i$. That is $\text{Pr}(e_i^w) = \frac{1}{|\Sigma^w N_i|}$.

Denote $\text{Adv}_3$, $\text{Adv}_3 \in \mathcal{A}_3$ be a pair of adversaries of AWTP$_{PD}$ protocol. The proof are three steps:

1) For adversary $\text{Adv}_3$, in each round of transmission, Alice generate $c_i$ over AWTP channel, that is,

$$c_i = \text{AWTP}_{PD}(m, r_A, v_A, i, \text{AWTP})$$

or $d_i = \text{AWTP}_{PD}(m, r_A, v_A, i, \text{PD})$ over PD channel.

Adv$_3$ reads on set $\{S^{\ell,r}, S^{\ell,b}\}$ and writes on set $\{S^{\ell,w}, S^{\ell,c}\}$. In each round, the error $e_i$ is chosen with $\text{SUPP}(e_i) = S^{\ell,w}$ and the value $e_i^w$ randomly and uniformly over $\Sigma^{w} N_i$.

Bob receivers $y_i = c_i + e_i$ over AWTP channel, or $d_i$ over PD channel.

The reading capability of adversary $\text{Adv}_3$ is same as $\text{Adv}_1$. From Lemma 3 the reliability of AWTP$_{PD}$ protocol implies,

$$H(M|C^{\ell,a}C^{\ell,d}D^{\ell}) \leq H(\delta) + \delta(H(M) - 1)$$

2) For adversary $\text{Adv}_3$, in each round of transmission, Alice generate $\hat{c}_i$ over AWTP channel, that is,

$$\hat{c}_i = \text{AWTP}_{PD}(\hat{m}, r_A, v_A, i, \text{AWTP})$$

or $\hat{d}_i = \text{AWTP}_{PD}(\hat{m}, r_A, v_A, i, \text{PD})$ over PD channel.

Adv$_3$ reads on set $\{S^{\ell,r}, S^{\ell,b}\}$ with $S^{\ell,a} = S^{\ell,c}$ and $S^{\ell,b} = S^{\ell,d}$. Adv$_3$ writes on set $\{S^{\ell,w}, S^{\ell,c}\}$ with $S^{\ell,a} = S^{\ell,c}$ and $S^{\ell,b} = S^{\ell,d}$. In each round, the error $\hat{e}_i$ is chosen with $\text{SUPP}(\hat{e}_i) = S^{\ell,w}$, and the value $e_i^w$ randomly and uniformly over $\Sigma^{w} N_i$.

Bob receivers $\hat{y}_i = \hat{c}_i + \hat{e}_i$ over AWTP channel, or $\hat{d}_i$ over PD channel.

The writing capability of $\text{Adv}_3$ is same as $\text{Adv}_1$. From Lemma 2 the secrecy of AWTP$_{PD}$ protocol implies,

$$I(\hat{M}; C^{\ell,a}C^{\ell,d}D^{\ell}) \leq 2\epsilon'$$

3) Since the public discussion channel is only invoked by Alice, Bob can not reply information to Alice. So Alice’s view only depends on message $m$ and randomness $r_A$. For adversary $\text{Adv}_3$ and $\text{Adv}_1$, the distribution of message $M$ and $\hat{M}$ are same. The distribution of randomness $R_A$ and $R_A$ are also same. So the distribution of Alice’s view $V_A$ is same as $\hat{V}_A$. From (20) and (22), in each round, the transmission over AWTP channel is uniquely determined by $m, r_A$ and $v_A$. So the distribution of transmission over AWTP channel $C_i$ is same as $\hat{C}_i$. Similarly, the distribution over PD channel $D_i$ is same as $\hat{D}_i$. So the distribution of $\{C_i, D_i\}$ is same as the distribution of $\{\hat{C}_i, \hat{D}_i\}$.

From (21) (23), and the distribution of $M, C^{\ell}, D^{\ell}$ is same as $\{M, C^{\ell}, D^{\ell}\}$, it implies,

$$H(\delta) + \delta H(M) \geq H(M|C^{\ell,a}C^{\ell,d}D^{\ell})$$

$$= H(M|C^{\ell,a}C^{\ell,d}D^{\ell})$$

$$\geq H(M) - 2\epsilon'$$

So there is,

$$2\epsilon' - \frac{H(\delta)}{1 - \delta} \geq H(M)$$

Since $0 \leq \delta < \frac{1}{2}$ and message is uniformly distributed, it implies,

$$1 - 4\epsilon' + 2H(\delta) \leq 2^{-2(4\epsilon' + 2H(\delta))} \leq 2^{-H(M)} = \frac{1}{|M|}$$

So there is,

$$4\epsilon' + 2H(\delta) \geq 1 - \frac{1}{|M|}$$

G. Proof of Lemma 8

Proof: First, assume the adversary reads the last $\rho_c N$ components of $c$, and the first $(1 - \rho_c) N$ components is the set of components that is neither read, nor written to, be the adversary. Let $v'_E = \{r_{1, \rho_c N + 1}, \ldots, r_N, \beta_{1, \rho_c N + 1}, \ldots, \beta_N, \alpha_1, \ldots, \alpha_N, l_1, \ldots, l_N, v_0, \ldots, v_N\}$ denote the view of the adversary, except for $c$.

If $\ell \leq (u - 1)(1 - \rho_c) N$, the vector of random variables, $(r_1 || \ldots || r_u)$, corresponds to a symbol-fixing source. The components that the adversary do not read are uniformly distributed and are independent from the adversary’s view $v'_E$, and the components that the adversary reads are determined and fixed. So the randomness $k$ that is generated from the extractor, is uniformly distributed and is independent of the adversarial view. That is,

$$\text{Pr}(k|v'_E) = \text{Pr}(k)$$

(24)

Second, since Alice selects the message $m \in \mathcal{M}$ independent from $k$ and $v'_E$, we have $\text{Pr}(m|k, v'_E) = \text{Pr}(m)$. For any message $m \in \mathcal{M}$, we have,

$$\text{Pr}(m) \leq \text{Pr}(m|v'_E) \leq \text{Pr}(m|k, v'_E) = \text{Pr}(m)$$

This implies,

$$\text{Pr}(m) = \text{Pr}(m|v'_E) = \text{Pr}(m|k, v'_E)$$

(25)

so and we have,

$$\text{Pr}(k|m, v'_E) = \frac{\text{Pr}(k, m, v'_E)}{\text{Pr}(m, v'_E)} = \frac{\text{Pr}(m|k, v'_E) \text{Pr}(k, v'_E)}{\text{Pr}(m|v'_E) \text{Pr}(v'_E)}$$

$$= \text{Pr}(k|v'_E)$$
Third, the adversarial view for any $m \in \mathcal{M}$ is $v_E = \{c, v'_E\}$, and so,

$$\Pr(v_E|m) = \Pr(c, v'_E|m) = \Pr(c|m, v'_E)\Pr(v'_E|m)$$

$$\leq \Pr(k|m, v'_E)\Pr(v'_E)$$

$$\leq \Pr(k)\Pr(v'_E)$$

where, (1) is from $c_i = k_i + m_i \mod q$ for $i = 1 \ldots \ell$, and (2) is from \((24)\) and \((26)\).

This means the statistical distance between adversarial views of any two messages $m_1, m_2 \in \mathcal{M}$, is zero and the $\text{AWTP}_{PD}$ protocol is perfectly secure. That is,

$$[\text{SD} (\text{View}_E|m_1, \text{View}_E|m_2)] = \sum_{v_E \in \text{View}_E} |\Pr(v_E|m_1) - \Pr(v_E|m_2)| = 0$$

H. Proof of Lemma 2

Proof: First, we show the probability that vector $(r_{i_1}, \ldots, r_{i_N}) \neq (r'_{i_1}, \ldots, r'_{i_N})$ is no more than $\frac{uN}{q}$. This is from,

$$\Pr((r_{i_1}, \ldots, r_{i_N}) \neq (r'_{i_1}, \ldots, r'_{i_N}))$$

$$\leq \sum_{i=1}^{N} \Pr(r_i \neq r'_i)$$

$$= \sum_{i=1}^{N} \Pr(r_i \neq r'_i, v_i = 1)$$

$$\leq \sum_{i=1}^{N} \Pr(r_i \neq r'_i)$$

$$|\text{hash}_{\alpha_i}(r_i) - \text{hash}_{\alpha_i}(r'_i) = [\beta_i - \beta'_i] | \leq \frac{uN}{q}$$

Second, for the two random vectors $k = \text{Ext}(r_{i_1}, \ldots, r_{i_N})$ and $k' = \text{Ext}(r'_{i_1}, \ldots, r'_{i_N})$, we have,

$$\Pr(k \neq k') \leq \Pr((r_{i_1}, \ldots, r_{i_N}) \neq (r'_{i_1}, \ldots, r'_{i_N}))$$

(28)

Third, Bob correctly receives $d_2 = \{c, v\}$ sent by Alice and so, $m_i + k_i = m'_i + k'_i \mod q$ for $i = 1 \ldots \ell$. That is, the probability that the message $m \neq m'$ is the same as the probability $k \neq k'$.

$$\Pr(m \neq m') = \Pr(k \neq k')$$

(29)

From \((27)\), \((28)\), and \((29)\), there is $\Pr(m \neq m') = \Pr(k \neq k') \leq \frac{N}{q}$.

I. Proof of Lemma 7

Proof: We show that there is a one-to-one correspondence between $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocols and restricted $(\epsilon, \delta)$-$\text{AWTP}_{PD}$ protocols, in the sense that given one of the former, a corresponding one in the latter can be constructed, and vice versa, and (i) given one of the that the security and reliability parameters of the two protocols are the same.

1. Consider a $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol, with a fixed public numbering of wires. Recall that the in each round of the $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol, both the wires and the PD can be invoked by Alice, while in our $\text{AWTP}_{PD}$ model, only one type channel is invoked by Alice in each round. In both models Bob can invoke the PD in each round. We can convert the protocol messages in round $i$ of a $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol to the protocol messages of round $j$ and $j+1$, of a $\text{AWTP}_{PD}$ protocol. In round $i$, transmissions over wire 1 to $N$, defines a codeword of length $N$ in the $i^{th}$ round of the $\text{AWTP}$. The transmission over the PD directly defines the transmission over the PD in $\text{AWTP}_{PD}$, in the $j+1$ round. Each round of the transmission over PD, when invoked by Bob in the $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$, defines a transmission over the PD for the a $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol. The above transformation gives a $\text{AWTP}_{PD}$ from a $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$. Similarly, a $\text{AWTP}_{PD}$ protocol defines an $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol.

So a restricted $(\epsilon, \delta)$-$\text{AWTP}_{PD}$ protocol can be constructed from $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol. Similarly, a $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ protocol can also be constructed from restricted $(\epsilon, \delta)$-$\text{AWTP}_{PD}$ protocol.

2. $\text{AWTP}_{PD}$ and $(\epsilon, \delta)$-$\text{SMT}^{[1]}$-$\text{PD}$ definitions of secrecy and reliability are the same. Definition of $\epsilon$-secrecy in both primitives requires statistical distance of the adversary’s view for two messages chosen by the adversary (Compare definition 13 and definition 5), to be bounded by $\epsilon$. For $\delta$-reliability, both primitives require the probability of outputting the correct message to be at least $1 - \delta$, and the probability of outputting the wrong message to be at most $\delta$.

J. Proof of Lemma 12

Proof: Using Theorem 11 for a 1-$(\epsilon, \delta)$-$\text{SMT}$-$\text{PD}$ over $N$ wires and $t = \rho N$, there is a corresponding restricted $(\epsilon, \delta)$-$\text{AWTP}_{PD}$ protocol whose rate is upper bounded by,

$$R \leq 1 - \rho + 2\epsilon(1 + \log_{\|W\|\epsilon}) + 2\epsilon n$$

Since the transmission rate of a 1-$(\epsilon, \delta)$-$\text{SMT}$-$\text{PD}$ is the inverse of the rate of the corresponding restricted $(\epsilon, \delta)$-$\text{AWTP}_{PD}$ protocol, we have

$$TR = \frac{1}{R}$$

$$\geq 1 - 2\rho + 2\epsilon(1 + \log_{\|W\|\epsilon}) + 2\epsilon n$$

$$= \frac{N}{N - 2t + 2N\epsilon(1 + \log_{\|W\|\epsilon}) + 2\epsilon n N}$$

K. Proof of Lemma 13

We first prove a lemma on the round structure of round optimal $(\epsilon, \delta)$-$\text{AWTP}_{PD}$ protocols, and use that to show the
lower bound on the round complexity of \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocols.

**Lemma 16:** If \(\rho_r + \rho_w \geq 1\), a three-round \((\epsilon, \delta)\)-AWTP\(_{PD}\) protocol will have the following structure:
Round 1: Alice \(\xrightarrow{AWTP} Bob\); Round 2: Bob \(\xrightarrow{PD} Alice\); Round 3: Alice \(\xrightarrow{PD} Bob\).

**Proof:** In Lemma 11 we showed that an \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocol that is constructed from an AWTP\(_{PD}\) will have the same number of round and so the \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocol corresponding to a round optimal AWTP\(_{PD}\) protocol will have three rounds.

In the following we will show that no two-round \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocol can exist. We show this by considering the corresponding AWTP\(_{PD}\) protocol and use the impossibility results proven for these protocol. Firstly, note that 2-round \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocols in which PD channel is used by Alice only, cannot exist because, Lemmas 6 and 7 showed that for AWTP\(_{PD}\) protocols, it is impossible to have PD channel invoked only by Alice, or only by Bob. Next, for two-round \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocols in which the PD channel is invoked by Alice and Bob both, we will have the following two types with their corresponding round structures.
Type 1) R1: Alice \(\xrightarrow{AWTP,PD} Bob\); R2: Bob \(\xrightarrow{PD} Alice\).
Type 2) R1: Bob \(\xrightarrow{PD} Alice\); R2: Alice \(\xrightarrow{AWTP,PD} Bob\).

Type 1 corresponds to the following three-round AWTP\(_{PD}\) protocol:
R1: Alice \(\xrightarrow{PD} Bob\); R2: Alice \(\xrightarrow{AWTP} Bob\); R3: Bob \(\xrightarrow{PD} Alice\).

Lemma 16 shows that it is impossible to have this three-round AWTP\(_{PD}\) protocol and so the corresponding \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocol is impossible, also. Using a similar argument shows that Type 2 two-round \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocol is also impossible and so the minimum number of rounds for \((\epsilon, \delta)\)-SMT\(^{(1)}\)-PD protocol when \(\rho_r + \rho_w \geq 1\), is three.