Abstract

We deal with a graph colouring problem that arises in quantum information theory. Alice and Bob are each given a $\pm 1$-vector of length $k$, and are to respond with $k$ bits. Their responses must be equal if they are given equal inputs, and distinct if they are given orthogonal inputs; however, they are not allowed to communicate any information about their inputs. They can always succeed using quantum entanglement, but their ability to succeed using only classical physics is equivalent to a graph colouring problem. We resolve the graph colouring problem, thus determining that they can succeed without entanglement exactly when $k \leq 3$.

1 Introduction and Background

We are concerned here with a graph colouring problem that arises in quantum information theory.

The graph $\Omega_n$ has vertex set the set of $\pm 1$-vectors of length $n$; two vertices are adjacent if they are orthogonal. Our main result is that the chromatic number of this graph is equal to $n$ if and only if $n = 2^k$ with $k \leq 3$.

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This problem arises in the following scenario, introduced in [3, 2]. Alice and Bob are each given a 01-vector of length $2^k$, and they are each to respond with a 01-vector of length $k$. If their input vectors are equal, then their output vectors must also be equal; if their input vectors differ in exactly $2^{k-1}$ positions, then their output vectors must be distinct. Also, they are not allowed to communicate any information about their input vectors to each other.

Their cause is hopeless without some shared resource. Of course this resource must not allow them to share information about their inputs. We consider two possibilities, which correspond roughly to classical physics and quantum physics.

If they are allowed to share prior information then they could agree beforehand on a proper colouring of $\Omega_{2^k}$. They would then each interpret their input as a vertex of $\Omega_{2^k}$, and respond with the colour of that vertex. Since they are only allowed to output $k$ bits, this only works if $\chi(\Omega_{2^k}) \leq 2^k$.

Now consider that Alice and Bob have some strategy involving some prior shared information, and that their strategy is guaranteed to succeed. Alice and Bob are then given their respective inputs. Now before they actually answer, each writes down a list of the response they would have given to all possible inputs. If they have a winning strategy then they are able to do this. Let $a(x)$ be the entry in Alice’s list corresponding to $x$, and likewise $b(x)$ for Bob. Since Alice and Bob’s strategy is guaranteed to succeed for any pair of inputs, we have $a(x) = b(x)$ for all $x$, and $a(x) \neq b(y)$ whenever $x$ and $y$ correspond to adjacent vertices. Their answers are restricted to $k$ bits, so their lists contain at most $2^k$ distinct entries. Thus they have a proper colouring with at most $2^k$ colours. (Note that we have not shown that if Alice is given the same input on different occasions that she must respond in the same way. Rather, at each round, her copy of the shared information amounts to a proper colouring.)

In other words, any strategy based on prior shared information is equivalent to colouring $\Omega_{2^k}$ with at most $2^k$ colours.

If, instead of information, they are allowed a shared resource of quantum entanglement, then they can always succeed when $n = 2^k$ for all $k$. This was first observed by Buhr, Cleve, and Wigderson [3] (see also and Brassard, Cleve, and Tapp [2]).

We do not assume any familiarity with quantum entanglement, qubits and quantum information theory; however, the interested reader will find a good introduction to these areas in [13]. We summarize
briefly the quantum algorithm of [3, 2]. Alice and Bob share between them an entangled quantum state consisting of $k$ EPR pairs of qubits. The $\pm 1$-vectors of length $2^k$ can be interpreted as indexing a particular family of quantum operations. So given their inputs, they each apply the corresponding operation to their qubits, and then measure their qubits. They answer the result of their measurements. This turns out to be a winning strategy. See [3, 2] for a more formal description. Alternatively, see [12] for a version that does not assume any previous background in quantum information theory.

In [3] it is also shown that for sufficiently large $k$, Alice and Bob cannot succeed by sharing only prior information. This follows directly from a deep result of Frankl and Rödl [7] who show that for large enough $n = 4m$ the size of an independent set in $\Omega_n$ is at most $(2 - \epsilon)^n$ for some $\epsilon > 0$. It follows that the chromatic number must eventually be greater than $n$. We note that the motivation of [7] has nothing to do with any quantum scenario; furthermore, their result is stronger than what we state here.

The point of this scenario is then that Alice and Bob can always succeed using quantum physics (i.e., by sharing quantum entanglement), whereas they cannot always succeed using classical physics (i.e., by sharing prior information). So our result quantifies the difference between what can be accomplished using quantum or classical physics for this particular scenario.

The reader may easily verify that $\chi(\Omega_n) = n$ for $n = 1, 2$, and with a little more effort for $n = 4$ as well (this case follows trivially from the recursive construction of Section 6). Unpublished computations by Gordon Royle determined that $\chi(\Omega_8) = 8$ and characterized all of the proper 8-colourings. Galliard, Tapp and Wolf [9] show that the size of an independent set in $\Omega_{16}$ is at most 3912, which implies its chromatic number is at least 17.

## 2 Some Simple Cases

It is not hard to see that $\Omega_n$ is edgeless if $n$ is odd.

If $n$ is an odd multiple of two, then the vertices can be divided into the even vertices (those with an even number of $-1$'s) and the odd vertices; every edge joins an even vertex to an odd vertex, and so the graph is bipartite.

If $n = 4m$ then every edge joins an even vertex to an even vertex, or
an odd vertex to an odd vertex. In fact, it is not hard to see that these
two subgraphs are isomorphic. Furthermore, if a vertex \( x \) is adjacent
to \( y \) then it is also adjacent to \(-y\), and \( x \) is not adjacent to \(-x\). It
follows that each component of \( \Omega_n \) can be written as a lexicographic
product \( Y_n[K_2] \) for some graph \( Y \). Note that \( \chi(\Omega_n) = \chi(Y_n) \) and that
maximum independent sets in \( \Omega_n \) are exactly four copies of maximum
independent sets in \( Y_n \). It follows that \( \alpha(\Omega_n) \) is a multiple of four.
More importantly, it will simplify some of our computational work.

3 Bounding Independent Sets

One of the main tools we use in analyzing \( \Omega_n \) is the Delsarte-Hoffman
bound on independent sets (see [6, Section 3.3] or [4, Page 115]; alter-
natively [12] for more recent work).

3.1 Theorem. Let \( X \) be a \( d \)-regular connected graph on \( v \) vertices,
and \( \tau \) the least eigenvalue of its adjacency matrix. Let \( S \) be an
independent set of size \( s \) and let \( z \) be the characteristic vector of \( S \).
Then
\[
s \leq n \frac{-\tau}{d - \tau}.
\]
Furthermore, equality holds if and only if
\[
A \left( z - \frac{s}{v} \mathbf{1} \right) = \tau \left( z - \frac{s}{v} \mathbf{1} \right).
\]

The graph \( \Omega_n \) is a graph in the Hamming scheme. We will not
go into details here, but the reader is directed to [10, Chapter 12] for
background material on the Hamming scheme and association schemes
in general. The practical consequence of this is that we know a com-
plete set of eigenvectors for \( \Omega_n \). For instance, the methods of [10,
Section 12.9] can be used to establish the following.

3.2 Lemma. Let \( n \) be a multiple of four. Then the least eigenvalue
of \( \Omega_n \) is
\[
\tau = -\frac{1}{n - 1} \binom{n}{\frac{n}{2}}
\]
and the columns of \( W \) form a basis for the \( \tau \)-eigenspace where \( W \) is
the matrix with rows indexed by subsets of \([n]\) and columns indexed
by 2-subsets and \((n-2)\)-subsets, with \((A, p)\)-entry equal to \((-1)^{|A\cap p|} \).
Note that here we are thinking of the vertices of $\Omega_n$ as being subsets of $[n]$ instead of $\pm 1$-vectors; two subsets are adjacent when they are at Hamming distance $\frac{n}{2}$.

Let $\hat{W} = (W \ 1)$. Putting Theorem 3.1 and Lemma 3.2 together, we obtain the following result for our graphs.

3.3 Corollary. The size of an independent set in $\Omega_n$ is bounded by

$$\alpha(\Omega_n) \leq \frac{2^n}{n}.$$ 

Furthermore, if equality holds then the characteristic vector of a maximum independent set lies in the column space of $\hat{W}$. \hfill \Box

Note that if $\Omega_n$ is $n$-colourable, then this bound must hold with equality. In other words, we have shown that $\chi(\Omega_n) > n$ whenever $n = 4m$ is not a power of two. It is noteworthy that the question “When is $\chi(\Omega_n) \leq n$?” becomes trivial when $n$ is not a power of two: the mathematical analysis is simplest for the cases that are physically uninteresting. Of more immediate use is the fact that if the bound does not hold with equality, then $\Omega_n$ is not $n$-colourable. One way to show that the bound is not tight is to use the equality condition of Theorem 3.1: it suffices to show that there are no suitable vectors in the $\tau$-eigenspace.

4 Finding Maximum Independent Sets

Since we know all of the eigenspaces of $\Omega_n$, it is not hard to see that $\Omega_n$, the even component of $\Omega_n$, and $Y_n$ all have the same least eigenvalue $\tau$. If we take only those columns of $W$ that correspond to the 2-subsets, and only those rows that correspond to the even vertices, the resulting column space gives the $\tau$-eigenspace of the even component (this amounts to taking only one of the two eigenspaces of the Hamming scheme that give the $\tau$-eigenspace on $\Omega_n$). If we further reduce by taking only one vertex (i.e., row) from each pair $\{x, -x\}$, then we obtain a matrix whose columns form a basis for the $\tau$-eigenspace for $Y_n$. We will denote this matrix by $H$. Furthermore, let $\hat{H} = (H \ 1)$.

4.1 Corollary. The size of an independent set in $Y_n$ is bounded by

$$\alpha(Y_n) \leq \frac{12^n}{4n}.$$
Furthermore, if equality holds then the characteristic vector of a maximum independent set lies in the column space of $\hat{H}$. 

Let $z$ be the characteristic vector of an independent set $S$ that meets the bound of Corollary 4.1. Then $z = \hat{H}y$ for some vector $y$. Since $Y_n$ is vertex-transitive, we are free to assume that $S$ contains any particular vertex. As $S$ is maximum, this is equivalent to assuming that $S$ is disjoint from the neighbourhood of a vertex. Let $\hat{N}$ be the submatrix of $\hat{H}$ with rows corresponding to a neighbourhood; then we may assume that $z$ takes the value 0 at the corresponding positions, meaning that $y$ is in the kernel of $\hat{N}$. Thus we are lead to the following result.

4.2 Lemma. The kernel of $\hat{N}$ is given by the row space of $\hat{B} = (B \ 1)$, where $B$ is the incidence matrix of $K_n$.

Proof. A direct computation shows that 

$$NB^T = -1.$$ 

This means that 

$$\hat{N}\hat{B}^T = 0.$$ 

We will show that this is the whole kernel by a rank argument.

We note that 

$$BB^T = (n-1)I + J.$$ 

Thus the eigenvalues of $BB^T$ are $2n - 1$ and $n - 1$, so $B$ has full rank, and $\text{rk}(B) = \text{rk}(BB^T) = n$. As $B1 = (n-1)1$, we see that $\text{rk}(\hat{B}) = n$.

Essentially the same argument determines the rank of $\hat{N}$ as well. The rows and columns of $N^TN$ are indexed by 2-subsets, and the $(a,b)$-entry depends only on $|a \cap b|$. Let $L$ and $\overline{L}$ be the incidence matrices of the line graph of $K_n$ and its complement, respectively. It follows that 

$$N^TN = c_0I + c_1L + c_2\overline{L},$$ 

where 

$$c_0 = \binom{n}{2};$$

$$c_1 = \binom{n}{2} - 8\binom{n-3}{2-1};$$

$$c_2 = \binom{n}{2} - 16\binom{n-4}{2-1}.$$
The matrices $I, L, \overline{L}$ are simultaneously diagonalizable with known eigenvectors (more precisely: the line graph of $K_n$ is strongly regular). It follows that the eigenvalues of $N^T N$ are

$$\frac{n}{2(n-1)} \binom{n}{2}, \quad \frac{n(n-2)}{(n-1)(n-3)} \binom{n}{2}, \quad 0,$$

with respective multiplicities $1$, $\binom{n}{2} - n$, and $n - 1$. As $N1 = \frac{n}{2}1$, we see that $\text{rk}(\hat{N}) = \text{rk}(N)$, and so $\dim(\text{ker}(\hat{N})) = \text{rk}(\hat{B})$. The result follows.

We note that the rank arguments in the above proof amount to observing that $BB^T$ and $N^T N$ both lie in the Bose-Mesner algebras of known association schemes.

Let $C$ be the reduced column echelon form of the matrix $\hat{H}\hat{N}$. Then it follows that there are vectors $x, x'$ such that

$$z = \hat{H}y = \hat{H}\hat{B}^T x = Cx'.$$

Furthermore, since $z$ is a 01-vector so is $x'$. Since $H$ and $B^T$ have full column rank, it follows that the rank of $C$ is $n$. Thus it suffices to check all $2^n$ possibilities for $x'$ in order to determine if there exist any independent sets that meet the bound. We have carried out this computation for $n = 8, 16$: for $n = 8$, we find that there are eight independent sets of the required size containing a given vertex; for $n = 16$, there are none.

On its own this computation is not particularly satisfying: we have not contradicted Royle’s result mentioned above, and we have established a weaker bound on $\alpha(\Omega_{16})$ than the one given in [9]. However, it will turn out that our computations for $n = 8, 16$ suffice to determine all values of $n$ for which $\Omega_n$ is $n$-colourable. For this purpose, we will rederive the bound of Theorem 3.1 for $\Omega_n$ twice more.

5 Colouring $\Omega_n$

It is well-known that for any vertex-transitive graph $X$ on $v$ vertices, $\alpha(X)\omega(X) \leq v$. This is the clique-coclique bound. It also holds for any graph that is a union of classes in an association scheme. $\Omega_n$ falls into both of these categories, but it is also a normal Cayley graph, for which we can extend this result. In particular, we will show that for a normal Cayley graph $X$, if $\alpha(X)\omega(X) = |V(X)|$, then $\chi(X) = \omega(X)$. We will need some preliminary results first.
Recall that the connection set of a Cayley graph for a group $G$ is the subset $D$ of $G$ such that $a \sim b$ whenever $ba^{-1} \in D$. If $S$ is a subset of $G$ then we write

$$S^{-1} = \{g^{-1} : g \in S\},$$

$$Sa = \{ga : g \in S\}.$$ 

5.1 Lemma. Let $X$ be a Cayley graph for a group $G$ with connection set $D \subseteq G$. Let $S$ be an independent set of $X$. If $a$ and $b$ are adjacent then $S^{-1}a \cap S^{-1}b = \emptyset$.

Proof. Assume $g^{-1}a = h^{-1}b$ for some $g, h \in S$. Then $hg^{-1} = ba^{-1}$. But $h \sim g$ so $hg^{-1} \notin D$, while $b \sim a$ so $ba^{-1} \in D$.

5.2 Corollary. If $X$ is a Cayley graph, then $\alpha(X) \omega(X) \leq |V(X)|$.

Proof. Let $S$ be an independent set and $C$ a clique. Then by the previous result the sets $S^{-1}c, c \in C$

are all disjoint.

We note parenthetically that this can be extended to a proof for all vertex-transitive graphs. It is an old result of Sabidussi [14] that if $X$ is a vertex-transitive graph, there is an integer $m$ such that the lexicographic product $X[K_m]$ is a Cayley graph. Since

$$\alpha(X[K_m]) = m\alpha(X), \quad \omega(X[K_m]) = \omega(X), \quad |V(X[K_m])| = m|V(X)|,$$

the result follows for all vertex transitive graphs.

Recall that a Cayley graph is normal if its connection set is closed under conjugation. Our purpose in approaching the clique-coclique bound through Cayley graphs is the following extension from [11].

5.3 Corollary. If $X$ is a normal Cayley graph and $\alpha(X) \omega(X) = v$, then $\chi(X) = \omega(X)$.

Proof. Let $S$ be an independent set and $C$ be a maximum clique. Again, the sets $S^{-1}c, c \in C$

are disjoint, and so they partition the vertex set. Since $X$ is normal they are also independent sets, and hence form a colouring.
For our purposes, Corollary 5.3 rederives the bound of Theorem 3.1 for \( \Omega_n \), but with a different equality condition. Notice that any clique in \( \Omega_n \) is a set of pairwise orthogonal vectors, hence linearly independent, and hence has size at most \( n \). Furthermore, an \( n \)-clique would correspond to a Hadamard matrix, which certainly exists if \( n = 2^k \). It follows that we can use Corollary 5.3 to rederive the bound of Theorem 3.1, but with a different equality condition.

5.4 Corollary. Let \( n \) be a power of two.

The size of an independent set in \( \Omega_n \) is bounded as

\[
\alpha(\Omega_n) \leq \frac{2^n}{n}.
\]

Furthermore, if equality holds then \( \chi(\Omega_n) = n \). \( \square \)

So \( \chi(\Omega_n) = n \) if and only if \( \alpha(\Omega_n) = \frac{2^n}{n} \). Our determination of \( \alpha(\Omega_8) \) above is now upgraded to a proof that \( \chi(\Omega_8) = 8 \).

More generally, we see that not only is it impossible to \( n \)-colour \( \Omega_n \) if \( n \) is not a power of two, but if it is possible, then the colouring is exactly a partition of the vertex set into maximum independent sets that meet the bound of Theorem 3.1.

6 A Recursive Construction

The graph \( \Omega_n \) is an induced subgraph of \( \Omega_{2n} \): take exactly those vertices of \( \Omega_{2n} \) whose last \( n \) entries are the same as the first \( n \) entries. In fact, we can say much more than this.

For vertices \( x, r \) of \( \Omega_n \), let \( x^{(r)} \) be the vertex of \( \Omega_{2n} \) obtained by concatenating \( x \) with the entrywise product of \( x \) and \( r \) (recall that vertices are \( \pm 1 \)-vectors). Let \( \Omega_n^{(r)} \) be the subgraph induced by the vertices

\[
\{x^{(r)} : x \in V(\Omega_n)\}.
\]

Then we see that \( \Omega_n^{(r)} \) is isomorphic to \( \Omega_n \), for any \( r \). (The previous example was \( \Omega_8^{(1)} \).) The vertex set of \( \Omega_{2n} \) can be partitioned as

\[
\{V(\Omega_n^{(r)}) : r \in V(\Omega_n)\}
\]

Furthermore, every vertex of \( \Omega_n^{(r)} \) is adjacent to every vertex of \( \Omega_n^{(-r)} \).

Recall that the join of two graphs \( X_1 \) and \( X_2 \) is \( X_1 + X_2 = \overline{X_1} \cup X_2 \). We have established the following result.
6.1 Lemma. The vertex set of $\Omega_{2n}$ can be partitioned into $2^{n-1}$ copies of $\Omega_n + \Omega_n$. \hfill \Box

Note that any independent set in the join of two graphs must lie entirely within one or the other. This gives us a bound on the size of an independent set in $\Omega_{2n}$; it is at most half of the size of $2^n$ maximum independent sets in $\Omega_n$. We can use this to again rederive the bound of Theorem 3.1, with yet another equality condition.

6.2 Corollary. Let $n = 2^k$ where $k > 1$.

The size of an independent set in $\Omega_n$ is bounded as

$$\alpha(\Omega_n) \leq \frac{2^n}{n}.$$  

Furthermore, if equality holds then it also holds for $n = 2^{k-1}$. \hfill \Box

We can in fact apply the recursive construction when $n$ is not a power of two. If $n = m2^k$, where $m$ is odd, then we find that

$$\alpha(\Omega_n) \leq \frac{2^n}{2^k}. \quad (1)$$

When $k > 1$ the bound of Corollary 3.3 is better than (1) by a factor of $m$. When $k = 1$, the bound of (1) is half the number of vertices, and is tight since $\Omega_n$ is then bipartite. But for $k = 1$ the least eigenvalue of $\Omega_n$ is no longer given by Lemma 3.2, and applying Theorem 3.1 in this case again gives half the number of vertices.

In other words, this recursion does not ever give a tighter bound than Theorem 3.1; rather, it is useful because of the equality condition of Corollary 6.2.

We mention another point of view on this recursion.

For $n = 2^k$, let the graphs $\Psi_n$ be defined by setting $\Psi_1 := \Omega_1 = K_2$, and recursively defining $\Psi_{2n}$ to be the disjoint union of $2^{n-1}$ copies of $\Psi_n + \Psi_n$. Then $\Psi_n$ is a spanning subgraph of $\Omega_n$, and so Corollary 3.3 gives a bound on independent sets in $\Psi_n$ as well. It follows that

$$\alpha(\Psi_n) = \frac{2^n}{n},$$

$$\chi(\Psi_n) = n.$$  

Furthermore we see that $\Psi_n = \Omega_n$ for $n = 1, 2, 4$, by simply observing that their degrees are the same. This is an easy way to see that $\chi(\Omega_4) = 4$. 

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There are of course other edges in $\Omega_n$ for general $n$. In fact, $\Psi_n$ is not only spanning, it is asymptotically sparse, in the following sense.

6.3 Lemma.

$$\lim_{k \to \infty} \frac{|E(\Psi_{2^k})|}{|E(\Omega_{2^k})|} = 0$$

In other words, for $n = 2^k$, Theorem 3.1 effectively only “sees” the edges of $\Psi_n$. This is expected, since one consequence of the Frankl-Rödl result is that for large enough $n$, the bound of Corollary 3.3 is exponentially too big.

7 Main Result

Recall that although we can easily define $\Omega_n$ for any positive integer $n$, it is the cases where $n = 2^k$ that are most of interest. In exactly these cases, we have three different ways of proving the same bound on independent sets: using the Delsarte-Hoffman bound, using the maximum cliques, and a recursive construction. Furthermore, each approach gives different information in the case where the bound is tight.

We now find that our main result follows directly.

7.1 Theorem. $\chi(\Omega_n) = n$ if and only if $n = 2^k$ with $k \leq 3$.

Proof. It follows from Corollary 3.3 and the comments after it that $\chi(\Omega_n) \neq n$ if $n$ is not a power of two. Furthermore, if $\chi(\Omega_n) = n$ then the bound of Corollary 3.3 holds with equality, and Corollary 5.4 tells us that it is sufficient that this bound holds with equality. Our computations of Section 4 deal with the cases $n = 2^k$ for $k = 3, 4$. We then invoke Corollary 6.2 to conclude that $\chi(\Omega_{2^k}) > 2^k$ for all $k > 4$. □

8 Further Bounds

Galliard [8] has a construction of an independent set inspired by the methods of Ahlswede and Khachatrian [1]. We state it in terms of the graph $Y_n$. It is convenient to regard the vertices as being subsets of $[n]$. 

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Let \( n = 2^k \) and \( c = \frac{n}{4} - 1 \). Then the following collection is an independent set in \( Y_n \).

\[
\mathcal{F}_n = \{ F \subseteq [n] : |F| = 2i, \ 2c \leq i \leq 2c; \ |F \cap [c]| \geq |F \setminus [c]| \}
\]

It is not hard to see that \( \mathcal{F} \) is not properly contained in any larger independent set.

It turns out that for \( k \leq 3 \) this set meets the bound of Corollary 3.3, and hence this construction is maximum. Up to automorphisms of \( \Omega_n \), this is unique (this follows both from Gordon Royle’s computations and from our work in Section 4). Galliard conjectured that \( \mathcal{F}_n \) is maximum for all \( n = 2^k \).

A recent computation of de Klerk and Pasechnik [5] using a technique of Schrijver [15] gives that \( \alpha(Y_{16}) \leq 576 \), which is exactly the size of \( \mathcal{F}_{16} \). The reader is referred to [5] for details and further results.

We do not know of any determination of \( \alpha(\Omega_{4m}) \) for \( m > 4 \).

There is another way to look at the collection \( \mathcal{F} \). If we replace each element \( F \) of \( \mathcal{F}_n \) with its symmetric difference with \( [c] \) we obtain the collection of all odd subsets of \( [n] \) of size at most \( c \). More generally, if we assume only that \( n = 4m \), we have the following construction.

\[
\mathcal{S}_n = \{ F \subseteq [n] : |F| \equiv m \pmod{2}; \ |F| < m \}
\]

We conjecture that these are in fact maximum in general. If this is true, it would imply our statement of the Frankl-Rödl result, namely that

\[
\alpha(\Omega_{4m}) \leq (2 - c)^{4m}
\]

for some \( c > 0 \), for large enough \( m \).

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