Characterization of Gromov-type geodesics

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Abstract

It is well known that, when endowed with the Gromov-Hausdorff distance $d_{GH}$, the collection $\mathcal{M}$ of all isometry classes of compact metric spaces is a complete and separable space. It is also known that $(\mathcal{M}, d_{GH})$ is a geodesic metric space, but there is no known structural characterization of geodesics in $\mathcal{M}$.

In this paper we provide two characterizations of geodesics in $\mathcal{M}$. We call a Gromov-Hausdorff geodesic $\gamma : [0, 1] \to \mathcal{M}$ Hausdorff-realizable if there exists a compact metric space $Z$ containing isometric copies of $\gamma(t)$ for each $t \in [0, 1]$ such that the Hausdorff distance satisfies $d^Z_H(\gamma(s), \gamma(t)) = d_{GH}(\gamma(s), \gamma(t))$ for all $s, t \in [0, 1]$. In this way, $\gamma$ is actually a geodesic in the Hausdorff hyperspace of $Z$, and we call it a Hausdorff geodesic. We prove that in fact every Gromov-Hausdorff geodesic is Hausdorff-realizable. Inspired by this characterization, we further elucidate a structural connection between Hausdorff geodesics and Wasserstein geodesics: we show that every Hausdorff geodesic is equivalent to a so-called Hausdorff displacement interpolation. This equivalence allows us to establish that every Gromov-Hausdorff geodesic is dynamic, a notion which we develop in analogy with dynamic optimal couplings in the theory of optimal transport.

Besides geodesics in $\mathcal{M}$, we also study geodesics on the collection $\mathcal{M}^w$ of isomorphism classes of compact metric measure spaces. Sturm constructed a family of Gromov-type distances on $\mathcal{M}^w$, which we denote $d^S_{GW,p}$ (for $p \in [1, \infty]$), as an analogue of $d_{GH}$, and proved that $(\mathcal{M}^w, d^S_{GW,p})$ is also a geodesic space. We define a notion of Wasserstein-realizable $d^S_{GW,p}$ geodesics in a sense similar to Hausdorff-realizable geodesics and show that the set of all Wasserstein-realizable $d^S_{GW,p}$ geodesics is dense in the set of all $d^S_{GW,p}$ geodesics. We further identify a rich class of $d^S_{GW,p}$ geodesics which are Wasserstein-realizable.

Keywords— Gromov-Hausdorff distance, metric measure space, Sturm’s Gromov-Wasserstein distance, geodesic, optimal transport, displacement interpolation

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1 Introduction

The Gromov-Hausdorff distance $d_{GH}$ was independently introduced by Edwards [Edw75] and Gromov [Gro81] in order to quantify the difference between two given metric spaces. Let $(X, d_X)$ and $(Y, d_Y)$ be two compact metric spaces. Then, the Gromov-Hausdorff distance between them is defined as follows:

$$d_{GH}(X, Y) := \inf_{\varphi \in \mathcal{M}} d_H^2(\varphi_X(X), \varphi_Y(Y)),$$

(1)

where $d_H^2$ denotes the Hausdorff distance between nonempty closed subsets of $Z$ (cf. Definition 2.7) and the infimum is taken over all metric spaces $Z$ and isometric embeddings $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$. Let $\mathcal{M}$ denote the collection of isometry classes of compact metric spaces. In this regard, it has been established that when endowed with $d_{GH}$, $(\mathcal{M}, d_{GH})$ is a complete and separable metric space [BBI01, PAR06]. The geometry of $(\mathcal{M}, d_{GH})$ has been studied extensively recently [IT17, IT19b, Kli18].

A metric measure space $X = (X, d_X, \mu_X)$ is a metric space $(X, d_X)$ endowed with a Borel probability measure $\mu_X$. Denote by $\mathcal{M}^w$ the collection of isomorphism classes (cf. Definition 2.19) of compact metric measure spaces with full support. By replacing the Hausdorff distance in Equation (1) with the $p$-transportation distance (cf. Definition 2.17) between probability measures, Sturm introduced the $L^p$-transportation distance on $\mathcal{M}^w$ as a counterpart to $d_{GH}$ [Stu06, Stu12] and in this paper, we denote by $d_{GW}^p$ (for each $p \in [1, \infty]$) his $L^p$-transportation distance. In [Mém07, Mém11] the first author introduced the Gromov-Wasserstein distance $d_{GW}^p$ (for each $p \in [1, \infty]$) on $\mathcal{M}^w$ based on an alternative representation of $d_{GH}$ (cf. Theorem 3.1). $d_{GW}^p$ and $d_{GW}^p$ are both legitimate metrics on $\mathcal{M}^w$ which generate the same topology, yet they do not coincide in general.

In recent years, the Gromov-Hausdorff and the Gromov-Wasserstein distances have found applications in shape analysis [MS04, MS05, Mém07, BBK08, BBK10], machine learning [PC19, AMJ18, VCTF19, BAMKJ19, CN20a, XLC19, VFT19, LHY19], biology [Lie18, DSS+20, CHW20], and network analysis [Hen16, CM19, Cho19]. It is worth noting that the three distances $d_{GH}$, $d_{GW}^p$ and $d_{GW}^p$ and their variants have been applied in topological data analysis [CCSG+09, CDSO14, BGMP14, MO19, CM19, BL20, RS20] to establish stability results of invariants. As such, clarifying the properties of these distances, especially with respect to the structure of their associated geodesics, may help develop suitable concomitant statistical methods (see [Stu12, CN20b]).

1.1 Geodesic properties of $(\mathcal{M}, d_{GH})$, $(\mathcal{M}^w, d_{GW}^p)$ and $(\mathcal{M}^w, d_{GW}^p)$

The fact that $(\mathcal{M}^w, d_{GW}^p)$ is a geodesic space for each $p \in [1, \infty)$ was proved by K.T. Sturm in [Stu12, Theorem 3.1]. His proof is constructive in that a special type of geodesics, which we call straight-line geodesics, was identified for any two given metric measure spaces (cf. [Stu12, Theorem 3.1]). Furthermore, Sturm provided a complete characterization of geodesics in $(\mathcal{M}^w, d_{GW}^p)$ for each $p \in (1, \infty)$ by proving that every geodesic in $(\mathcal{M}^w, d_{GW}^p)$ is a straight-line $d_{GW}^p$ geodesic.

The geodesic property of $(\mathcal{M}, d_{GH})$ was first proved in [INT16] via the so called mid-point criterion (cf. Theorem 2.5) which did not yield explicit geodesics. Later in [CM18], the fact that $(\mathcal{M}, d_{GH})$ is geodesic was reproved by identifying the straight-line $d_{GH}$ geodesics (cf. Theorem 2.14) which are analogous to the straight-line $d_{GW}^p$ geodesics constructed in [Stu12]. In [Stu20] Sturm proved that $(\mathcal{M}^w, d_{GW}^p)$ (for each $p \in [1, \infty)$) is geodesic by constructing what we call the straight-line $d_{GW}^p$ geodesic between any two

\footnote{Sturm proved the geodesic property for the collection $\mathcal{M}^w$ larger than $\mathcal{M}^w$ for each $p \in [1, \infty]$ which contains complete metric measure spaces with finite $\ell^p$-size, a generalization of the notion of diameter. However, the same technique applies to the case of $\mathcal{M}^w$ without any changes.}
metric measure spaces, which is a slight variant of the straight-line \( d_{GW,p} \) geodesic with respect to \( d_{GW,p} \) (cf. Theorem 4.25).

Unlike the case of \((\mathcal{M}^w, d_{GW,p})\) where the only geodesics are the straight-line \( d_{GW,p} \) geodesics, neither straight-line \( d_{dH} \) geodesics nor straight-line \( d_{GW,p}^S \) geodesics completely characterize geodesics in \((\mathcal{M}, d_{dH})\) and \((\mathcal{M}^w, d_{GW,p}^S)\), respectively. Indeed, the authors of [CM18] discovered deviant geodesics in \((\mathcal{M}, d_{dH})\), i.e., geodesics which are not straight-line \( d_{dH} \) geodesics. Following a strategy similar to the one used in [CM18], we discover non-straight-line \( d_{GW,p}^S \) geodesics in \((\mathcal{M}^w, d_{GW,p}^S)\) and present our construction in Appendix C of this paper. This inspires us to obtain better understanding of geodesics in \((\mathcal{M}, d_{dH})\) and \((\mathcal{M}^w, d_{GW,p}^S)\).

1.2 Our results

In this paper we elucidate characterization results for geodesics in \((\mathcal{M}, d_{dH})\) and \((\mathcal{M}^w, d_{GW,p}^S)\). These characterizations accommodate both straight-line \( d_{dH} / d_{GW,p}^S \) geodesics and the above mentioned deviant geodesics.

**Hausdorff-realizable Gromov-Hausdorff geodesics.** In this paper, the terms “Gromov-Hausdorff geodesics” and “\( d_{dH} \) geodesics” are used interchangeably when referring to geodesics in \((\mathcal{M}, d_{dH})\).

Given a metric space \( X \), its Hausdorff hyperspace \( \mathcal{H}(X) \) is the set of all nonempty bounded closed subsets of \( X \) endowed with the Hausdorff distance \( d_H^X \) (cf. Definition 2.7). Blaschke’s compactness theorem (see for example [BBB10], Theorem 7.3.8) states that \( \mathcal{H}(X) \) is compact whenever \( X \) is compact. Furthermore, it is known that if \( X \) is a compact geodesic space then so is \( \mathcal{H}(X) \) [Bry70, Ser98]. We call a geodesic in the Hausdorff hyperspace of some metric space a Hausdorff geodesic. Hausdorff geodesics are easier to study than Gromov-Hausdorff geodesics since the definition of \( d_{dH} \) relies on finding the infimum of Hausdorff distances over certain metric embeddings (cf. Definition 2.10). This inspires us to relate Gromov-Hausdorff geodesics with Hausdorff geodesics.

It was first observed and proved in [IT19a] that every straight-line \( d_{dH} \) geodesic can be realized as a Hausdorff geodesic with respect to some ambient metric space (cf. Proposition 4.6). We further show that under mild conditions, such a metric space construction is actually compact and thus the corresponding straight-line \( d_{dH} \) geodesic can be realized as a geodesic in the Hausdorff hyperspace of certain compact metric space (cf. Proposition 4.8). We call any such a Gromov-Hausdorff geodesic Hausdorff-realizable. It turns out that Hausdorff-realizability is a universal phenomenon:

**Theorem 1.** Every Gromov-Hausdorff geodesic is Hausdorff-realizable.

**Wasserstein-realizable Gromov-Wasserstein geodesics.** Given a metric space \( X \), the \( L^p \)-Wasserstein hyperspace \( \mathcal{W}_p(X) \) is the set of all Borel probability measures on \( X \) with finite moments and endowed with the \( L^p \)-Wasserstein distance \( d_{W,p} \) (cf. Definition 2.17). Sturm’s \( L^p \)-transportation distance \( d_{GW,p} \) is defined by replacing the Hausdorff distance term in the definition of \( d_{dH} \) with an \( L^p \)-Wasserstein distance term (cf. Definition 2.20). For the sake of symmetry of nomenclature, we call the \( L^p \)-transportation distance \( d_{GW,p}^S \) the Sturm’s \( L^p \)-Gromov-Wasserstein distance. Since we only focus on \( d_{GW,p}^S \) instead of \( d_{GW,p} \) in this paper, we also simply call \( d_{GW,p}^S \) the \( L^p \)-Gromov-Wasserstein distance without causing any confusion. See also Figure 1 for an illustration. Then, in this paper, the terms “\( L^p \)-Gromov-Wasserstein geodesics” and “\( d_{GW,p}^S \) geodesics” are used interchangeably when referring to geodesics in \((\mathcal{M}^w, d_{GW,p}^S)\).
Figure 1: Nomenclature. The Gromov-Hausdorff distance between two metric spaces is defined via infimizing Hausdorff distances on certain ambient spaces (cf. Definition 2.10). This procedure of infimizing some quantities over certain ambient spaces is called “gromovization” in [Mém11], e.g., the Gromov-Hausdorff distance is the gromovization of the Hausdorff distance. In this way, Sturm’s $L^p$-transportation distance is the gromovization of the Wasserstein distance and hence we call it the $\ell^p$-Gromov-Wasserstein distance in our paper. The figure illustrates the respective gromovization processes for $d_{GH}$ and $d_{GW,p}$.

Inspired by Theorem 1, we analogously define the so-called $\ell^p$-Wasserstein-realizable geodesics in $(\mathcal{M}^w, d_{GW,p}^\varepsilon)$ (cf. Definition 4.12). For $p \in [1, \infty)$, denote $\Gamma^p$ the collection of all $d_{GW,p}^\varepsilon$ geodesics. Let $d_{\infty,p}$ be the uniform metric on $\Gamma^p$, i.e.,

$$d_{\infty,p} (\gamma_1, \gamma_2) := \sup_{t \in [0,1]} d_{GW,p}^\varepsilon (\gamma_1 (t), \gamma_2 (t))$$

for any $\gamma_1, \gamma_2 \in \Gamma^p$. Let $\Gamma_{GW}^p$ denote the subset of $\Gamma^p$ consisting of all Wasserstein-realizable geodesics. Then, by applying techniques and strategies similar to those used in proving Theorem 1, we obtain that $\Gamma_{GW}^p$ is a dense subset of $\Gamma^p$ (cf. Proposition 4.15).

Though we conjecture that every $d_{GW,p}^\varepsilon$ geodesic is Wasserstein-realizable, we have not been able to establish the closedness of $\Gamma_{GW}^p$, and the conjecture still remains open. We then turn to consider geodesics satisfying certain constraints and eventually, in Theorem 2, we are able to identify a certain type of $d_{GW,p}^\varepsilon$ geodesics called Hausdorff-bounded (cf. Definition 4.16) which turn out to be Wasserstein-realizable. For any two metric measure spaces $\gamma (s) = (X_s, d_s, \mu_s)$ and $\gamma (t) = (X_t, d_t, \mu_t)$ along a given Hausdorff-bounded geodesic $\gamma$, the Gromov-Hausdorff distance $d_{GH} (X_s, X_t)$ between their underlying metric spaces is bounded above by the Gromov-Wasserstein distance $f (d_{GW,p}^\varepsilon (\gamma (s), \gamma (t)))$ through some suitable function $f$. Such control over Gromov-Hausdorff distances allows us to exploit the techniques used for proving Theorem 1 in order to also establish Wasserstein-realizability.

**Theorem 2.** Given $p \in [1, \infty)$, every Hausdorff-bounded $\ell^p$-Gromov-Wasserstein geodesic is $\ell^p$-Wasserstein-realizable.

It turns out that both the straight-line $d_{GW,p}^\varepsilon$ geodesics introduced in [Stu20] and the deviant geodesics constructed in Appendix C are Hausdorff-bounded and thus Wasserstein-realizable (cf. Proposition 4.26 and Remark 4.27). Then, Theorem 2 indeed characterizes a large class of Gromov-Wasserstein geodesics as Wasserstein geodesics.

**Dynamic (Gromov-)Hausdorff geodesics.** Since every Gromov-Hausdorff geodesic is a Hausdorff geodesic, we turn to study properties of Hausdorff geodesics. In the last part of the paper, we devote to draw connection between Hausdorff geodesics and Wasserstein geodesics. In particular, we establish a counterpart to the theory of displacement interpolation for Hausdorff geodesics.

For $p > 1$, a geodesic $\gamma : [0, 1] \to \mathcal{W}_p (X)$ of probability measures in $\mathcal{W}_p (X)$ is also called a displacement interpolation since it is characterized by a probability measure on the set of all geodesics in $X$ itself (see [Vil08, Chapter 7] and Theorem 5.2 for a precise statement). More precisely, let $\Gamma ([0,1], X)$
denote the set of all geodesics in $X$, then there exists a probability measure $\Pi \in \mathcal{P}(\Gamma([0,1],X))$ such that 
$\gamma(t) = (e_t)_\#(\Pi)$ for each $t \in [0,1]$, where $e_t : \Gamma([0,1],X) \to X$ is the evaluation map at $t$ sending any function $\gamma : [0,1] \to X$ to $\gamma(t)$.

Now, we state an analogous characterization for Hausdorff geodesics. Let $X \in \mathcal{M}$ and $A, B \subseteq X$ be two closed subsets. Let $\rho := d_H^X(A, B) > 0$. We define

$$\mathcal{L}(A, B) := \{ \gamma : [0,1] \to X : \gamma(0) \in A, \gamma(1) \in B \text{ and } \forall s, t \in [0,1], d_X(\gamma(s),\gamma(t)) \leq |s-t|\rho \}. $$

In other words, $\mathcal{L}(A, B)$ is the set of all $\rho$-Lipschitz curves (cf. Section 2.2) in $X$ starting in $A$ and ending in $B$. We call a closed subset $\mathcal{D} \subseteq \mathcal{L}(A, B)$ a Hausdorff displacement interpolation between $A$ and $B$ if $e_0(\mathcal{D}) = A$ and $e_1(\mathcal{D}) = B$, where $e_0$ and $e_1$ are evaluation maps at $t = 0$ and $t = 1$, respectively. Then, we have the following characterization of Hausdorff geodesics via the Hausdorff displacement interpolation:

**Theorem 3.** Given a compact metric space $X$, let $\gamma : [0,1] \to \mathcal{H}(X)$ be any map. We assume that $\rho := d_H^X(\gamma(0),\gamma(1)) > 0$. Then, the following two statements are equivalent:

1. $\gamma$ is a Hausdorff geodesic;
2. there exists a nonempty closed subset $\mathcal{D} \subseteq \mathcal{L}(\gamma(0),\gamma(1))$ such that $\gamma(t) = e_t(\mathcal{D})$ for all $t \in [0,1]$.

As in the theory of optimal transport where people accept that “a geodesic in the space of laws is the law of a geodesic” [Vil08, page 126], a geodesic in the space of closed subsets is the closed subset of a certain set of Lipschitz curves. It is tempting to ask whether one can require $\mathcal{D}$ to contain only geodesics instead of Lipschitz curves. There are, however, counterexamples to this; see Remark 5.9. As an application of Theorem 3, in Theorem 5.13 we prove the existence of infinitely many distinct Gromov-Hausdorff geodesics connecting any two compact metric spaces.

We further extend our characterization of Hausdorff geodesics via displacement interpolation to Gromov-Hausdorff geodesics. Though defined via Hausdorff distances, there is a dual formula for $d_G^\mathcal{H}$ which involves correspondences between sets (cf. Section 2.3). This notion is akin to the notion of coupling between probability measures which are inherent to the Wasserstein distance (see Remark 5.15 for a more detailed comparison). We extend the notion of correspondence to the so-called dynamic optimal correspondence (cf. Definition 5.14), which is a concept analogous to dynamic optimal couplings (cf. Definition 5.1) in the theory of measure displacement interpolation. We say that a geodesic $\gamma$ in $\mathcal{M}$ is dynamic if it possesses a dynamic optimal correspondence.

The above facts are put together as follows: via Theorem 1 we identify any given Gromov-Hausdorff geodesic with a Hausdorff geodesic; then we invoke Theorem 3 to generate a Hausdorff displacement interpolation, which gives rise to a dynamic optimal correspondence. In the end, we obtain the following characterization of Gromov-Hausdorff geodesics:

**Theorem 4.** Every Gromov-Hausdorff geodesic $\gamma : [0,1] \to \mathcal{M}$ is dynamic.

**Miscellaneous results.** In the course of proving the main results described above, we established various supporting results and provided novel proofs of some known results both of which are of independent interest. Here we list some of such results:

1. In Theorem 5.10 we prove that a compact metric space is geodesic if and only if its Hausdorff hyperspace is geodesic.
2. In Example 3.9 we reconstruct the Gromov-Hausdorff geodesic defined in [CM18] between $\mathbb{S}^0$ and $\mathbb{S}^n$ via a Hausdorff geodesic construction.
3. In Theorem 3.13 we prove that the Wasserstein extensor $W_p$ for each $p \in [1, \infty]$ is 1-Lipschitz.

4. In page 17 we provide a novel succinct proof of the fact that $(\mathcal{M}, d_{GH})$ is geodesic.

5. In Appendix C we construct examples of deviant and branching $d_{GW, p}$ geodesics.

1.3 Organization of the paper

In Section 2, we collect necessary background material regarding real functions, geodesics, the Gromov-Hausdorff distance and the Gromov-Wasserstein distance. In Section 3, we further examine properties of metric extensions including Hausdorff hyperspaces, Wasserstein hyperspaces and the Urysohn universal metric space. In Section 4, we study Hausdorff-realizable and Wasserstein-realizable geodesics and prove Theorem 1 and Theorem 2. In Section 5, we study the Hausdorff displacement interpolation and dynamic Gromov-Hausdorff geodesics and we prove Theorem 3 and Theorem 4. In Section 6, we discuss some open problems and in the Appendix, we provide extra proofs and constructions.

2 Background material

In this section, we first collect some notions and basic results for real functions. Then, we provide necessary background materials regarding continuous curves and geodesics in metric spaces. We also introduce definitions and certain results of both the Gromov-Hausdorff distance and the Gromov-Wasserstein distance.

2.1 Elementary properties of real functions

In this section we collect some notions and basic results of real functions which we will use in Section 4.3.

**Definition 2.1** (Proper functions). For any function $f : I \rightarrow J$ where $I$ and $J$ are intervals in $\mathbb{R} := [0, \infty]$ containing 0, we say $f$ is proper if $f(0) = 0$ and $f$ is continuous at 0.

For any increasing real function $f : [0, \infty) \rightarrow [0, \infty)$, we define its inverse $f^{-1} : [0, \infty) \rightarrow [0, \infty)$ as follows:

$$f^{-1}(y) := \inf\{x \geq 0 : f(x) \geq y\}, \quad \forall y \in [0, \infty),$$

where we adopt the convention that $\inf \emptyset = \infty$. Then, we have the following elementary properties of inverse of increasing functions:

**Proposition 2.2** (Some properties of inverse of increasing functions). Fix an increasing function $f : [0, \infty) \rightarrow [0, \infty)$. Then, we have the following:

1. $f^{-1} : [0, \infty) \rightarrow [0, \infty]$ is still an increasing function;

2. if $f$ is unbounded, i.e., $\lim_{x \to \infty} f(x) = \infty$, then $f^{-1}(y) < \infty$ for any $y \in [0, \infty)$;

3. for all $x, y \in [0, \infty)$, $x < f^{-1}(y)$ implies that $f(x) \leq y$;

4. if $f$ is strictly increasing, then for $x, y \in [0, \infty)$, $f(x) \leq y$ implies that $x \leq f^{-1}(y)$;

5. if $f$ is proper, then for any $y > 0$, $f^{-1}(y) > 0$ and $f^{-1}(0) = 0$. If $f$ is moreover strictly increasing, then $f^{-1}$ is continuous at 0 and in particular $f^{-1}$ is proper.
2.2 Curves and geodesics in metric spaces

A metric space \( (X, d_X) \) is a pair such that \( X \) is a set and \( d_X : X \times X \to \mathbb{R}_{\geq 0} \) is a function satisfying the following conditions:

1. for any \( x, x' \in X, d_X(x, x') \leq 0 \) and the equality holds if and only if \( x = x' \);
2. for any \( x, x' \in X, d_X(x, x') = d_X(x', x) \);
3. for any \( x, x', x'' \in X, d_X(x, x') \leq d_X(x, x'') + d_X(x'', x') \).

We often abbreviate \( (X, d_X) \) to \( X \) to represent a metric space.

A map \( \varphi : X \to Y \) between metric spaces is called an isometric embedding, usually denoted by \( \varphi : X \hookrightarrow Y \), if for each \( x, x' \in X \)

\[ d_Y(\varphi(x), \varphi(x')) = d_X(x, x'). \]

We say a metric space \( X \) is isometric to another metric space \( Y \) if there exists a surjective isometric embedding \( \varphi : X \hookrightarrow Y \). When \( X \) is isometric to \( Y \), we write \( X \cong Y \).

Given a metric space \( X \), a curve in \( X \) is any continuous map \( \gamma : [0, 1] \to X \). For \( C > 0 \), a \( C \)-Lipschitz curve is any curve \( \gamma \) such that \( d_X(\gamma(s), \gamma(t)) \leq C \cdot |s - t| \) for \( s, t \in [0, 1] \). One important result that we use in the sequel is the following variant of the Arzelà-Ascoli theorem (compare with the version given in [BBI01, Theorem 2.5.14]). We omit the proof here since it is essentially the same as the one for [BBI01, Theorem 2.5.14] and it is also a direct consequence of a more general statement which we prove later (cf. Theorem 4.2).

**Theorem 2.3** (Arzelà-Ascoli theorem). Let \( X \) be a compact metric space and let \( \{\gamma_i : [0, 1] \to X\}_{i=0}^\infty \) be a sequence of \( C \)-Lipschitz curves for a fixed \( C > 0 \), i.e., \( d_X(\gamma_i(s), \gamma_i(t)) \leq C \cdot |s - t| \) for any \( s, t \in [0, 1] \) and \( i = 0, 1, \ldots \). Then, there is a uniformly convergent subsequence of \( \{\gamma_i\}_{i=0}^\infty \) with a \( C \)-Lipschitz limit \( \gamma : [0, 1] \to X \).

There is one special type of Lipschitz curves called geodesics:

**Definition 2.4** (Geodesics). A curve \( \gamma : [0, 1] \to X \) is called a geodesic if for any \( s, t \in [0, 1] \) one has \( d_X(\gamma(s), \gamma(t)) \leq |t - s| \cdot d_X(\gamma(0), \gamma(1)) \), i.e., \( \gamma \) is \( d_X(\gamma(0), \gamma(1)) \)-Lipschitz.

By the triangle inequality, it is clear that \( d_X(\gamma(s), \gamma(t)) = |t-s| \cdot d_X(\gamma(0), \gamma(1)) \) for any \( s, t \in [0, 1] \).

If for any \( x_0, x_1 \in X \), there exists a geodesic \( \gamma : [0, 1] \to X \) such that \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \), we call \( X \) a geodesic space. The following is a useful criterion for checking whether a metric space is geodesic or not.

**Theorem 2.5** (Mid-point criterion, [BBI01, Theorem 2.4.16]). A complete metric space \( X \) is geodesic if and only if for any \( x, y \in X \), there exists \( z \in X \) (which we call a mid-point between \( x \) and \( y \)) such that

\[ d_X(x, z) = d_X(y, z) = \frac{1}{2} d_X(x, y). \]

Concatenation is one way of constructing new geodesics from existing ones.

**Proposition 2.6** (Geodesic concatenation). Let \( X \) be a metric space and for \( i = 1, \ldots, n \) let \( \gamma_i : [0, 1] \to X \) be a geodesic. For \( i = 1, \ldots, n \), let \( \rho_i \coloneqq d_X(\gamma_i(0), \gamma_i(1)) \). Assume that \( \gamma_1(1) = \gamma_{i+1}(0) \) for \( i =
1, \ldots, n - 1 and \rho_i > 0 for all i = 1, \ldots, n. Let \rho := \sum_{i=1}^n \rho_i. Then, the curve \gamma : [0, 1] \to X defined as follows is a \rho-Lipschitz curve:

\[
\gamma(t) := \begin{cases}
\gamma_1 \left( \frac{a_1}{\rho^2} t \right), & t \in \left[0, \frac{a_1}{\rho} \right] \\
\gamma_2 \left( \frac{a_1}{\rho^2} \left( t - \frac{a_2}{\rho} \right) \right), & t \in \left(\frac{a_1}{\rho}, \frac{a_1 + a_2}{\rho} \right) \\
\cdots, & \\
\gamma_n \left( \frac{a_1}{\rho^n} \left( t - \sum_{i=1}^{n-1} \frac{a_i}{\rho} \right) \right), & t \in \left(\sum_{i=1}^{n-1} \frac{a_i}{\rho}, 1 \right]
\end{cases}
\]

In particular, if \rho = d_X(\gamma_1(0), \gamma_n(1)), then \gamma is a geodesic.

Proof. Given any \( s, t \in [0, 1] \), there are two cases to consider.

1. There exists \( k \) such that \( s, t \in \left[ \sum_{i=1}^{k-1} \frac{a_i}{\rho}, \sum_{i=1}^{k} \frac{a_i}{\rho} \right] \). Then,

\[
d_X(\gamma(s), \gamma(t)) = d_X \left( \gamma_k \left( \frac{\rho}{\rho_k} \left( s - \sum_{i=1}^{k-1} \frac{\rho_i}{\rho} \right) \right), \gamma_k \left( \frac{\rho}{\rho_k} \left( t - \sum_{i=1}^{k-1} \frac{\rho_i}{\rho} \right) \right) \right) \\
= \left| \frac{\rho}{\rho_k} \left( s - \sum_{i=1}^{k-1} \frac{\rho_i}{\rho} \right) - \frac{\rho}{\rho_k} \left( t - \sum_{i=1}^{k-1} \frac{\rho_i}{\rho} \right) \right| \rho_k \\
= |s - t| \rho.
\]

2. There exist \( k, l > 0 \) such that \( s \in \left[ \sum_{i=1}^{k-1} \frac{a_i}{\rho}, \sum_{i=1}^{k} \frac{a_i}{\rho} \right] \) and \( t \in \left[ \sum_{i=1}^{k+l-1} \frac{a_i}{\rho}, \sum_{i=1}^{k+l} \frac{a_i}{\rho} \right] \). Then, by case 1, we have that

\[
d_X \left( \gamma(s), \gamma \left( \sum_{i=1}^{k} \frac{\rho_i}{\rho} \right) \right) = \left| s - \sum_{i=1}^{k} \frac{\rho_i}{\rho} \right| \rho,
\]

and

\[
d_X \left( \gamma(t), \gamma \left( \sum_{i=1}^{k+l} \frac{\rho_i}{\rho} \right) \right) = \left| t - \sum_{i=1}^{k+l} \frac{\rho_i}{\rho} \right| \rho,
\]

\[
d_X(\gamma(s), \gamma(t)) \leq d_X \left( \gamma(s), \gamma \left( \sum_{i=1}^{k} \frac{\rho_i}{\rho} \right) \right) + d_X \left( \gamma(t), \gamma \left( \sum_{i=1}^{k+l} \frac{\rho_i}{\rho} \right) \right) \\
+ \sum_{j=0}^{l-2} d_X \left( \gamma \left( \sum_{i=1}^{k+j} \frac{\rho_i}{\rho} \right), \gamma \left( \sum_{i=1}^{k+j+1} \frac{\rho_i}{\rho} \right) \right) \\
= \left( \left| s - \sum_{i=1}^{k} \frac{\rho_i}{\rho} \right| + \left| t - \sum_{i=1}^{k+l} \frac{\rho_i}{\rho} \right| + \sum_{j=0}^{l-2} \frac{\rho_{k+j+1}}{\rho} \right) \rho \\
= |t - s| \rho.
\]

Therefore, \( \gamma \) is a \( \rho \)-Lipschitz curve. \( \square \)
2.3 Gromov-Hausdorff distance

Given a metric space $X$, there is a well-known notion of distance between closed subsets of $X$: the Hausdorff distance.

**Definition 2.7** (Hausdorff distance). For nonempty closed subsets $A, B \subseteq X$, the Hausdorff distance $d^X_H$ between them is defined by

$$d^X_H(A, B) := \inf \{ r : A \subseteq B^r, B \subseteq A^r \},$$

where $A^r := \{ x \in X : d_X(x, A) \leq r \}$ is called the $r$-thickening of $A$.

**Remark 2.8.** It is easy to see that $d^X_H(A, B) = \max (\sup_{x \in A} \inf_{y \in B} d_X(x, y), \sup_{y \in B} \inf_{x \in A} d_X(x, y))$ (cf. [BBI01, Exercise 7.3.2]). This formula is also sometimes given as the definition of the Hausdorff distance.

**Lemma 2.9** (Hausdorff distance under isometric embedding). Let $\varphi : X \hookrightarrow Y$ be an isometric embedding of two compact metric spaces. Let $A, B$ be nonempty closed subsets of $X$. Then, we have that

$$d^X_H(A, B) = d^Y_H(\varphi(A), \varphi(B)).$$

**Proof.** Since $\varphi$ is an isometric embedding, by Remark 2.8 we have that

$$d^X_H(A, B) = \max \left( \sup_{x \in A} \inf_{y \in B} d_X(x, y), \sup_{y \in B} \inf_{x \in A} d_X(x, y) \right)$$

$$= \max \left( \sup_{x \in A} \inf_{y \in B} d_Y(\varphi(x), \varphi(y)), \sup_{y \in B} \inf_{x \in A} d_Y(\varphi(x), \varphi(y)) \right)$$

$$= \max \left( \sup_{x \in \varphi(A)} \inf_{y \in \varphi(B)} d_Y(x, y), \sup_{y \in \varphi(B)} \inf_{x \in \varphi(A)} d_Y(x, y) \right)$$

$$= d^Y_H(\varphi(A), \varphi(B)).$$

Now we recall the definition of the Gromov-Hausdorff distance defined in Equation (1) as follows:

**Definition 2.10** (Gromov-Hausdorff distance). Given two metric spaces $X$ and $Y$, the Gromov-Hausdorff distance between them is defined by

$$d_{GH}(X, Y) := \inf_Z d^Z_H(\varphi_X(X), \varphi_Y(Y)),$$

where the infimum is taken over all metric spaces $Z$ and isometric embeddings $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$.

If $X$ and $Y$ are compact, then the infimum can be restricted to only compact metric spaces $Z$.

For two compact metric spaces $X$ and $Y$, $d_{GH}(X, Y) = 0$ if and only if $X \cong Y$. Recall that $\mathcal{M}$ denote the set of all isometry classes of compact metric spaces. Then, $(\mathcal{M}, d_{GH})$ is a metric space. Moreover, $(\mathcal{M}, d_{GH})$ is a Polish space; see [BBI01, Theorem 7.3.30] and [PAR06, Proposition 42 and 43] for more details.

---

1 A metric space is Polish if it is complete and separable.
Note: although $\mathcal{M}$ is the set of isometry classes, by a slight abuse of notation, we write $X \in \mathcal{M}$ to refer to an individual compact metric space $X$ instead of its isometry class.

One important description of $d_{GH}$ is the following duality formula (cf. Theorem 2.11) via correspondences between sets [BBI01, Chapter 7]. Given $(X,d_X),(Y,d_Y) \in \mathcal{M}$, define $\mathcal{R}(X,Y)$ as the set of all $R \subseteq X \times Y$ such that $\pi_X(R) = X$ and $\pi_Y(R) = Y$ where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are canonical projections. We call each $R \in \mathcal{R}(X,Y)$ a correspondence between $X$ and $Y$. For a correspondence $R$, we define its distortion with respect to $d_X$ and $d_Y$ by

$$\text{dis}(R) := \inf_{(x,y),(x',y') \in R} |d_X(x,x') - d_Y(y,y')|.$$  

Theorem 2.11. For any $X,Y \in \mathcal{M}$, we have

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{R \in \mathcal{R}(X,Y)} \text{dis}(R).$$

We let $\mathcal{R}^{opt}(X,Y)$ denote the set of all correspondences such that the equality in Theorem 2.11 holds. It is proved in [CM18, Proposition 1.1] that $\mathcal{R}^{opt}(X,Y) \neq \emptyset$ and there exists an $R \in \mathcal{R}^{opt}(X,Y)$ which is a compact subset of $(X \times Y, \max(d_X,d_Y))$. A direct consequence of this fact is the following result:

Lemma 2.12. If $X$ and $Y$ are compact, then there exists a compact metric space $Z$ and isometric embeddings $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$ such that

$$d_{GH}(X,Y) = d_{GH}^{\mathcal{Z}}(\varphi_X(X),\varphi_Y(Y)).$$

Proof. Let $R \in \mathcal{R}^{opt}(X,Y)$ and $Z := X \cup Y$. Let $d_Z : Z \times Z \to \mathbb{R}$ be such that $d_Z|_{X \times X} = d_X$, $d_Z|_{Y \times Y} = d_Y$ and for $x \in X$ and $y \in Y$

$$d_Z(x,y) := \inf_{(x',y') \in R} \left(d_X(x,x') + d_Y(y,y') + \frac{1}{2}\text{dis}(R)\right).$$

It is proved in [MSS18, Lemma 2.8] that $(Z,d_Z)$ is a metric space and $d_{GH}^{\mathcal{Z}}(X,Y) = d_{GH}(X,Y) \cdot (Z,d_Z)$ is obviously compact since any sequence $\{z_i\}_{i=0}^{\infty} \subseteq Z$ must contain either a convergent subsequence in $X$ or a convergent subsequence in $Y$. \qed

Geodesics. The following result is proved in [INT16, Theorem 1] using the mid-point criterion (cf. Theorem 2.5):

Theorem 2.13. $(\mathcal{M}, d_{GH})$ is a geodesic metric space.

In [CM18, Theorem 1.2], the authors proved the existence of optimal correspondences which they used to give an explicit construction of Gromov-Hausdorff geodesics and, as a consequence provided, an alternative proof of Theorem 2.13:

Theorem 2.14 (Straight-line $d_{GH}$ geodesic [CM18]). For $X,Y \in \mathcal{M}$ and any $R \in \mathcal{R}^{opt}(X,Y)$, the curve $\gamma_R : [0,1) \to \mathcal{M}$ defined as follows is a geodesic:

$$\gamma_R(0) = (X,d_X), \gamma_R(1) = (Y,d_Y) \text{ and } \gamma_R(t) = (R,d_R) \text{ for } t \in (0,1),$$

where $d_{R_t} ((x,y),(x',y')) := (1-t) \ d_X(x,x') + t \ d_Y(y,y')$.

We will henceforth use the notation: $R_t := \gamma_R(t)$ for $t \in [0,1]$.  

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Convergence. Recall that for $\varepsilon > 0$ and $X \in \mathcal{M}$, the covering number $\text{cov}_\varepsilon (X)$ is the least number of $\varepsilon$-balls\(^3\) required to cover the whole space $X$.

**Definition 2.15** (Uniformly totally bounded class). We say a class $\mathcal{K}$ of compact metric spaces is uniformly totally bounded, if there exist a bounded function $Q : (0, \infty) \to \mathbb{N}$ and $D > 0$ such that each $X \in \mathcal{K}$ satisfies the following:

1. $\text{diam} (X) \leq D,$
2. for any $\varepsilon > 0$, $\text{cov}_\varepsilon (X) \leq Q (\varepsilon)$.

We denote by $\mathcal{K} (Q,D)$ the uniformly totally bounded class consisting of all $X \in \mathcal{M}$ satisfying the conditions above.

**Theorem 2.16** (Gromov’s pre-compactness theorem). For any given bounded function $Q : (0, \infty) \to \mathbb{N}$ and $D > 0$, the class $\mathcal{K} (Q,D)$ is pre-compact in $(\mathcal{M}, d_{GH})$, i.e., any sequence in $\mathcal{K} (Q,D)$ has a convergent subsequence.

Interested readers are referred to [BBI01, Section 7.4.2] for a proof.

**2.4 Sturm’s Gromov-Wasserstein distance**

**Wasserstein distance.** Given a metric space $X$ and any $p \in [1, \infty]$, there exists a natural distance $d_{W,p}^X$, the $\ell^p$-Wasserstein distance, comparing certain Borel probability measures on $X$.

**Definition 2.17** ($\ell^p$-Wasserstein distance). For a metric space $X$ (not necessarily compact) and $p \in [1, \infty)$, let $\mathcal{P}_p (X)$ denote the collection of all Borel probability measures $\alpha$ on $X$ such that

$$\forall x_0 \in X, \quad \int_X d_X (x, x_0) \, d\alpha (x) < \infty.$$ 

For $\alpha, \beta \in \mathcal{P}_p (X)$, the $\ell^p$-Wasserstein distance between $\alpha$ and $\beta$ is defined as follows:

$$d_{W,p}^X (\alpha, \beta) := \left( \inf_{\mu \in \mathcal{C} (\alpha, \beta)} \left( \int_{X \times X} d_X^p (x_1, x_2) \, d\mu (x_1, x_2) \right)^{\frac{1}{p}} \right),$$

where $\mathcal{C} (\alpha, \beta)$ denotes the set of measure couplings between $\alpha$ and $\beta$.

For $p = \infty$, let $\mathcal{P}_\infty (X)$ denote the collection of all Borel probability measures on $X$ with bounded support. We define the $\ell^\infty$-Wasserstein distance between $\alpha, \beta \in \mathcal{P}_\infty (X)$ by

$$d_{W,\infty}^X (\alpha, \beta) := \inf_{\mu \in \mathcal{C} (\alpha, \beta)} \sup_{(x_1, x_2) \in \text{supp} (\mu)} d_X (x_1, x_2).$$

**Lemma 2.18** ([Vil08, Theorem 4.1]). Fix $p \in [1, \infty)$. For a compact metric space $X$ and $\alpha, \beta \in \mathcal{P}_p (X)$, there exists $\mu \in \mathcal{C} (\alpha, \beta)$ such that

$$d_{W,p}^X (\alpha, \beta) = \left( \int_{X \times X} d_X^p (x_1, x_2) \, d\mu (x_1, x_2) \right)^{\frac{1}{p}}.$$

We call such $\mu$ an optimal transference plan between $\alpha$ and $\beta$ (with respect to $d_{W,p}^X$) and denote by $\mathcal{C}_p^{\text{opt}} (\alpha, \beta)$ the collection of all optimal transference plans.

\(^3\)An $\varepsilon$-ball is a closed ball in $X$ with radius $\varepsilon$. 

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**Sturm’s Gromov-Wasserstein distance.** A metric measure space is a triple \( \mathcal{X} = (X, d_X, \mu_X) \) where \((X, d_X)\) is a metric space and \(\mu_X\) is a Borel probability measure on \((X, d_X)\). We use script letters such as \(\mathcal{X}\) to denote a metric measure space \(\mathcal{X} = (X, d_X, \mu_X)\).

**Definition 2.19 (Isomorphism of metric measure spaces).** Given two metric measure spaces \(\mathcal{X}\) and \(\mathcal{Y}\), we say that they are \(\varphi\)-isomorphic, if there exists an isometry \(\varphi : \mathcal{X} \rightarrow \mathcal{Y}\) such that \(\mu_Y = \varphi^*\mu_X\), where \(\varphi^*\) denotes the pushforward map under \(\varphi\). Whenever \(\mathcal{X}\) is isomorphic to \(\mathcal{Y}\), we write \(\mathcal{X} \cong \mathcal{Y}\).

Now, we provide the definition of the Gromov-Wasserstein distance given by Sturm in [Stu06, Stu12].

**Definition 2.20 (Gromov-Wasserstein distance).** Let \(p \in [1, \infty]\) and let \(\mathcal{X} = (X, d_X, \mu_X)\) and \(\mathcal{Y} = (Y, d_Y, \mu_Y)\) be two compact metric measure spaces with full support. The \(p\)-Gromov-Wasserstein distance \(d_{GW}^p\) between \(\mathcal{X}\) and \(\mathcal{Y}\) is defined by

\[
d_{GW}^p(\mathcal{X}, \mathcal{Y}) := \inf_Z d_{W,p}^Z((\varphi_X)^*\mu_X, (\varphi_Y)^*\mu_Y),
\]

where the infimum is taken over all metric spaces \(Z\) and isometric embeddings \(\varphi_X : \mathcal{X} \hookrightarrow Z\) and \(\varphi_Y : \mathcal{Y} \hookrightarrow Z\).

For notational simplicity, we sometimes identify \((\varphi_X)^*\mu_X\) with \(\mu_X\) and simply write \(d_{GW}^p(\mu_X, \mu_Y)\) to avoid carrying heavy notations of pushforward maps from isometric embeddings.

Let \(p \in [1, \infty]\) and let \(\mathcal{X} = (X, d_X, \mu_X)\) and \(\mathcal{Y} = (Y, d_Y, \mu_Y)\) be two compact metric measure spaces with full support. Then, \(d_{GW}^p(\mathcal{X}, \mathcal{Y}) = 0\) if and only if \(\mathcal{X}\) and \(\mathcal{Y}\) are isomorphic to each other. The case when \(p = 1\) was mentioned in [Stu12, Proposition 2.4] whereas the case \(p = \infty\) can be obviously derived from [Mém11, Theorem 5.1 (a) and (g)]. Let \(\mathcal{M}^w\) denote the collection of all isomorphism classes of compact metric measure spaces with full support. Then, for each \(p \in [1, \infty]\), \((\mathcal{M}^w, d_{GW}^p)\) is a metric space.

**Note:** although \(\mathcal{M}^w\) is a set of isomorphism classes, by a slight abuse of notation, we write \(\mathcal{X} \in \mathcal{M}^w\) to refer to an individual metric measure space \(\mathcal{X}\) instead of its isomorphism class.

The following is a useful alternative formulation of the Gromov-Wasserstein distance:

**Remark 2.21 (Metric coupling formulation).** Let \(\mathcal{D}(d_X, d_Y)\) denote the set of all metrics \(d : X \sqcup Y \times X \sqcup Y \rightarrow \mathbb{R}_{\geq 0}\) such that \(d|_{X \times X} = d_X\) and \(d|_{Y \times Y} = d_Y\). We call each element in \(\mathcal{D}(d_X, d_Y)\) a metric coupling between \(d_X\) and \(d_Y\). Then, it is easy to check that

\[
d_{GW}^p(\mathcal{X}, \mathcal{Y}) = \inf_{d \in \mathcal{D}(d_X, d_Y)} d_{GW,p}^d(\mu_X, \mu_Y).
\]

The following result is analogous to Lemma 2.12 for the Gromov-Hausdorff distance.

**Lemma 2.22.** Let \(\mathcal{X} = (X, \mu_X), \mathcal{Y} = (Y, \mu_Y) \in \mathcal{M}^w\) and \(p \in [1, \infty]\). Then, there exists a compact metric space \(Z\) and isometric embeddings \(\varphi_X : X \hookrightarrow Z\) and \(\varphi_Y : Y \hookrightarrow Z\) such that

\[
d_{GW}^p(\mathcal{X}, \mathcal{Y}) = d_{GW}^Z((\varphi_X)^*\mu_X, (\varphi_Y)^*\mu_Y).
\]

**Proof.** It is proved in [Stu12, Proposition 2.4] that there exists a metric space \(\hat{Z}\) and isometric embeddings \(\varphi_X : X \hookrightarrow \hat{Z}\) and \(\varphi_Y : Y \hookrightarrow \hat{Z}\) such that \(d_{GW}^p(\mathcal{X}, \mathcal{Y}) = d_{GW}^Z((\varphi_X)^*\mu_X, (\varphi_Y)^*\mu_Y)\). Now let \(Z := \varphi_X(X) \cup \varphi_Y(Y)\). Then, \(Z\) is compact. Since \(\text{im}(\varphi_X), \text{im}(\varphi_Y) \subseteq Z\), both \(\varphi_X\) and \(\varphi_Y\) are actually isometric embeddings \(\varphi_X : X \hookrightarrow Z\) and \(\varphi_Y : Y \hookrightarrow Z\), respectively. Then, it is easy to see that

\[
d_{GW}^p(\mathcal{X}, \mathcal{Y}) = d_{GW}^Z((\varphi_X)^*\mu_X, (\varphi_Y)^*\mu_Y) = d_{GW}^{\hat{Z}}((\varphi_X)^*\mu_X, (\varphi_Y)^*\mu_Y).
\]

\(\square\)
3 Metric extensions

For any two metric spaces $X$ and $Y$, if there exists an isometric embedding $X \hookrightarrow Y$, then we call $Y$ a metric extension of $X$. A metric extensor is any map $\mathcal{F}$ taking a compact metric space $X$ to another metric space $\mathcal{F}(X)$ such that $\mathcal{F}(X)$ is a metric extension of $X$. In this section, we examine three standard models of metric extensions, namely, the Hausdorff hyperspace, the Wasserstein hyperspace and the Urysohn universal metric space. Properties of these metric extensions and their corresponding metric extenders are essential for proving our main results.

3.1 Hausdorff hyperspaces

Given a metric space $X$, the Hausdorff hyperspace $\mathcal{H}(X)$ of $X$ is composed of all nonempty bounded closed subsets of $X$ and is endowed with the Hausdorff distance $d^X_{\mathcal{H}}$ as its metric.

Theorem 3.1. If $X$ is a complete metric space, then $(\mathcal{H}(X), d^X_{\mathcal{H}})$ is also complete.

Theorem 3.2 (Blaschke’s theorem). If $X$ is a compact metric space, then $(\mathcal{H}(X), d^X_{\mathcal{H}})$ is also compact.

See [BBI01, Section 7.3] for proofs of the above two results. Note that $\mathcal{H}$ mapping $X$ to $\mathcal{H}(X)$ is then a map from $\mathcal{M}$ to $\mathcal{M}$. The map sending $x \in X$ to the singleton $\{x\} \in \mathcal{H}(X)$ for each $x \in X$ is an isometric embedding from $X$ to $\mathcal{H}(X)$. This implies that $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ is a metric extensor, which we call the Hausdorff extensor. One interesting aspect of $\mathcal{H}$ as a map is the stability. In fact, it is proved in [Mik18] that $\mathcal{H}$ is a 1-Lipschitz map:

Theorem 3.3 ([Mik18, Theorem 2]). For any $X, Y \in \mathcal{M}$, we have

$$d_{\mathcal{G}\mathcal{H}}(\mathcal{H}(X), \mathcal{H}(Y)) \leq d_{\mathcal{G}\mathcal{H}}(X, Y).$$

Given an isometric embedding $\varphi : X \hookrightarrow Z$, for any closed subset $A \subseteq X$, the image $\varphi(A)$ is a closed subset of $Z$. This induces an isometric embedding $\varphi_* : (\mathcal{H}(X), d^X_{\mathcal{H}}) \hookrightarrow (\mathcal{H}(Z), d^Z_{\mathcal{H}})$ mapping $A \in \mathcal{H}(X)$ to $\varphi(A) \in \mathcal{H}(Z)$. Then, Theorem 3.3 is a direct consequence of the following interesting result:

Theorem 3.4 ([Mik18, Theorem 1]). Given two compact metric spaces $(X, d_X), (Y, d_Y)$, suppose there exist a metric space $Z$ (not necessarily compact) and isometric embeddings $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$. Then, we have that

$$d^Z_{\mathcal{H}}((\varphi_X)_*(\mathcal{H}(X)), (\varphi_Y)_*(\mathcal{H}(Y))) = d^Z_{\mathcal{H}}(\varphi_X(X), \varphi_Y(Y)).$$

Geodesics in Hausdorff hyperspaces. One interesting fact about $\mathcal{H}(X)$ is that it preserves the geodesic property of $X$:

Theorem 3.5. Given $X \in \mathcal{M}$, if $X$ is geodesic, then so is $\mathcal{H}(X)$.

The above theorem was first proved in [Bry70] using the mid-point criterion (cf. Theorem 2.5) and was later reproved in [Ser98] via the following explicit construction:

Theorem 3.6. Let $X \in \mathcal{M}$ be a geodesic space. Let $A, B$ be two closed subsets of $X$. Let $\rho := d^X_{\mathcal{H}}(A, B)$. Then, $\gamma : [0, 1] \to \mathcal{H}(X)$ defined by $\gamma(t) := A^t \rho \cap B^{(1-t)} \rho$ is a Hausdorff geodesic connecting $A$ and $B$, where for any $r \geq 0$, $A^r := \{x \in X : \exists a \in A \text{ such that } d_X(a, x) \leq r\}$.

Though the construction is correct, the proof of Theorem 3.6 given in [Ser98] is based on the following false claim:
Claim 3.7 (False claim in the proof of [Ser98, Theorem 1]). Given $X \in \mathcal{M}$ and a map $\gamma : [0, 1] \to X$, if $d_X(\gamma(0), \gamma(t)) = t \cdot d_X(\gamma(0), \gamma(1))$ and $d_X(\gamma(t), \gamma(1)) = (1 - t) \cdot d_X(\gamma(0), \gamma(1))$ hold for all $t \in [0, 1]$, then $\gamma$ is a geodesic.

A simple counterexample goes as follows: let $Y := [0, 3] \times [0, 1] \subseteq \mathbb{R}^2$ endowed with the usual Euclidean metric and let $X$ be the quotient space of $Y$ obtained by collapsing both $\{0\} \times [0, 1]$ and $\{3\} \times [0, 1]$ to points, respectively. Then, any “reasonable” curve connecting these two points will satisfy the condition in the claim while not necessary being a geodesic. See Appendix A for details and a correct proof of Theorem 3.6 which still follows the main idea in [Ser98].

It is worth noting that based on a new technique which we introduce later, i.e., the Hausdorff displacement interpolation, we are able to provide efficient alternative proofs for both Theorem 3.5 and Theorem 3.6 in Section 5.1.1. In particular, regarding Theorem 3.5 we prove a stronger result which provides both necessary and sufficient conditions instead of just a one way implication.

The following is a simple observation which we will use heavily in the sequel for transforming a Hausdorff geodesic to a Gromov-Hausdorff geodesic.

Lemma 3.8. Let $X, Y, Z \in \mathcal{M}$. Suppose that $Z$ is geodesic and there exist $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$ such that $d^Z_H(\varphi_X(X), \varphi_Y(Y)) = d^Z_{GH}(X, Y)$. Then, any Hausdorff geodesic $\gamma : [0, 1] \to \mathcal{H}(Z)$ such that $\gamma(0) = X$ and $\gamma(1) = Y$ is actually a Gromov-Hausdorff geodesic, i.e., for any $s, t \in [0, 1]$,

$$d^Z_{GH}(\gamma(s), \gamma(t)) = |s - t| \cdot d^Z_{GH}(X, Y).$$

Proof. We only need to show that $d^Z_{GH}(\gamma(s), \gamma(t)) = d^Z_{GH}(\gamma(s), \gamma(t))$ for any $s, t \in [0, 1]$. By Definition 2.10, we have $d^Z_{GH}(\gamma(s), \gamma(t)) \geq d^Z_{GH}(\gamma(s), \gamma(t))$. Without loss of generality, we assume that $s \leq t$. Since $\gamma$ is a Hausdorff geodesic, we have

$$d^Z_{GH}(X, Y) = d^Z_{GH}(\gamma(0), \gamma(1))$$

$$= d^Z_{GH}(\gamma(0), \gamma(s)) + d^Z_{GH}(\gamma(s), \gamma(t)) + d^Z_{GH}(\gamma(t), \gamma(1))$$

$$\geq d^Z_{GH}(\gamma(0), \gamma(s)) + d^Z_{GH}(\gamma(s), \gamma(t)) + d^Z_{GH}(\gamma(t), \gamma(1))$$

$$\geq d^Z_{GH}(\gamma(0), \gamma(1))$$

$$= d^Z_{GH}(X, Y).$$

Therefore, every equality holds. In particular, $d^Z_{GH}(\gamma(s), \gamma(t)) = d^Z_{GH}(\gamma(s), \gamma(t))$. \qed

Example 3.9 ($d^Z_{GH}$ geodesic connecting $S^0$ and $S^n$). In [CM18] the authors constructed explicit Gromov-Hausdorff geodesics between the spheres $S^0$ and $S^n$ with the canonical geodesic distance, for each $n \in \mathbb{N}$. We recover their construction via the techniques introduced in this section as follows. Note that if we identify $S^0$ with any pair of antipodal points, e.g., the north and south poles, in $S^n$, then $d^S_H(S^0, S^n) = \frac{n}{2}$. By [CM18, Proposition 1.2, d^Z_{GH}(S^0, S^n) = \frac{n}{2} = d^S_H(S^0, S^n). Then, by Theorem 3.6 and Lemma 3.8, the Hausdorff geodesic $\gamma : [0, 1] \to \mathcal{H}(S^n)$ defined by $t \mapsto (S^0)^t \frac{x}{n} \cap (S^n)^{(1 - t)} \frac{x}{n} = (S^0)^t \frac{x}{n}$ for $t \in [0, 1]$ is a Gromov Hausdorff geodesic connecting $S^0$ and $S^n$. Note that $\gamma$ is exactly the same geodesic connecting $S^0$ and $S^n$ constructed in [CM18, Proposition 1.3]. See also Figure 2 for an illustrative representation of $\gamma$.

3.2 Wasserstein hyperspaces

Given a metric space $X$, let $\mathcal{W}_p(X) := (\mathcal{P}(X), d_{\mathcal{W}_p})$ (cf. Definition 2.17). We call $\mathcal{W}_p(X)$ the $\ell^p$-Wasserstein hyperspace of $X$. Note that when $X$ is compact, $\mathcal{P}(X) = \mathcal{P}(X)$ for any $p \in [1, \infty]$, where $\mathcal{P}(X)$ denotes the collection of all Borel probability measures on $X$. The following two theorems are standard results about Wasserstein hyperspaces and see for example [Vil08, Section 6] for proofs.
Figure 2: Illustration of Example 3.9. In the figure we identify $S^0$ with the north and south poles of $S^1$ and illustrate $\gamma(t)$ for some $t \in (0, 1)$ as the thickened subset of $S^1$.

**Theorem 3.10.** For $p \in [1, \infty)$, if $X$ is Polish, then $W_p(X)$ is also Polish.

**Theorem 3.11.** For $p \in [1, \infty)$, if $X$ is compact, then $W_p(X)$ is also compact.

Note that $W_p$ sending $X \in \mathcal{M}$ to $W_p(X) \in \mathcal{M}$ then defines a map from $\mathcal{M}$ to $\mathcal{M}$ analogously to the case of $H$. Moreover, the map sending $x$ to the Dirac measure $\delta_x \in \mathcal{P}(X)$ is an isometric embedding from $X$ into $\mathcal{P}(X)$. Therefore, $W_p : \mathcal{M} \to \mathcal{M}$ is a metric extensor, which we call the ($\ell^p$)-Wasserstein extensor in the sequel.

Inspired by Theorem 3.4, we establish the following result:

**Theorem 3.12.** Given two compact metric spaces $(X, d_X)$ and $(Y, d_Y)$, suppose there exist a (not necessarily compact) metric space $(Z, d_Z)$ and isometric embeddings $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$. Then, for $p \in [1, \infty]$, we have that both $(\varphi_X)_# : W_p(X) \to W_p(Z)$ and $(\varphi_Y)_# : W_p(Y) \to W_p(Z)$ are isometric embeddings. Moreover, we have that

\[
d_H^{W_p}(\varphi_X)_#(W_p(X)), (\varphi_Y)_#(W_p(Y)) = d_H^Z(\varphi_X(X), \varphi_Y(Y)).
\]

See Appendix B for a proof. As a direct yet unexpected consequence, we obtain the following 1-Lipschitz property of the Wasserstein extensor $W_p : \mathcal{M} \to \mathcal{M}$:

**Theorem 3.13.** Given $X, Y \in \mathcal{M}$ and any $p \in [1, \infty]$, we have

\[
d_{GH}(W_p(X), W_p(Y)) \leq d_{GH}(X, Y).
\]

**Remark 3.14** (Comparison with related results). In the literature, there are studies about the stability of $W_2$ on $\mathcal{M}$. For example, one can derive from [LV09, Proposition 4.1] that

\[
d_{GH}(W_2(X), W_2(Y)) \leq f_{XY}(d_{GH}(X, Y))
\]

for all $X, Y \in \mathcal{M}$ and for some function $f_{XY}$ depending on the diameters of $X$ and $Y$. Theorem 3.13 is novel in that we not only proved stability of the Wasserstein extensor $W_p$ for all $p \in [1, \infty]$, but we also obtained the stronger result that $W_p$ is 1-Lipschitz.
Geodesics in $\mathcal{W}_p (X)$. We now state some results regarding geodesics in Wasserstein hyperspaces.

**Theorem 3.15** ([Vil08, Corollary 7.22]). If $X$ is a geodesic metric space, then $\mathcal{W}_p (X)$ is a geodesic metric space for all $p \in [1, \infty)$.

The case when $p = 1$ is special in that $\mathcal{W}_1 (X)$ is geodesic regardless of whether $X$ is itself geodesic:

**Theorem 3.16.** For any Polish metric space $X$, $\mathcal{W}_1 (X)$ is geodesic.

In fact, this theorem follows directly from the following explicit construction:

**Lemma 3.17** ([BALPO18, Theorem 5.1]). Let $X$ be a Polish metric space and let $\alpha, \beta \in \mathcal{P}_1 (X)$. We define $\gamma : [0, 1] \to \mathcal{P}(X)$ as follows: for each $t \in [0, 1]$, let $\gamma(t) := (1 - t) \alpha + t \beta$. Then, for each $t \in [0, 1]$, $\gamma(t) \in \mathcal{P}_1 (X)$ and $\gamma$ is an $\ell^1$-Wasserstein geodesic connecting $\alpha$ and $\beta$. We call $\gamma$ the linear interpolation between $\alpha$ and $\beta$.

Lemma 3.17 was first mentioned and proved in [BALPO18] via Kantorovich duality. In Appendix B, we provide an alternative proof which proceeds by calculating $d_{\mathcal{W}_1}^X (\gamma(s), \gamma(t))$ for all $s, t \in [0, 1]$ via an explicit construction of an optimal coupling between $\gamma(s)$ and $\gamma(t)$.

**Remark 3.18.** For $p \neq 1$, a statement similar to that of Lemma 3.17 for $\mathcal{W}_p (X)$ is not necessarily true. For example, consider the two point space $X = \{0, 1\}$ with interpoint distance 1. Then, for $p \in [1, \infty)$, one can easily verify that $\mathcal{W}_p (X) \cong ([0, 1], d_{\mathbb{R}})$ where $d$ denotes the Euclidean metric on $[0, 1]$. In this case, $\mathcal{W}_p (X)$ is geodesic if and only if $p = 1$.

The next lemma is a counterpart to Lemma 3.8 in the setting of metric measure spaces.

**Lemma 3.19.** Let $\mathcal{X} = (X, d_X, \mu_X), \mathcal{Y} = (Y, d_Y, \mu_Y) \in \mathcal{M}^w$ and let $Z \in \mathcal{M}$. Fix $p \in [1, \infty)$. Suppose there exist $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$ such that

$$d_{\mathcal{W}_p}^Z ((\varphi_X)_# \mu_X, (\varphi_Y)_# \mu_Y) = d_{\mathcal{GW}_p}^Z (\mathcal{X}, \mathcal{Y}).$$

Then, any $\ell^p$-Wasserstein geodesic $\gamma : [0, 1] \to \mathcal{W}_p (Z)$ such that $\gamma(0) = (\varphi_X)_# \mu_X$ and $\gamma(1) = (\varphi_Y)_# \mu_Y$ is actually an $\ell^p$-Gromov-Wasserstein geodesic. More precisely, if for each $t \in [0, 1]$ we let $X_t := \text{supp}(\gamma(t)) \subseteq Z$ and denote by $\tilde{\gamma}(t)$ the metric measure space $(X_t, d_Z |_{X_t \times X_t}, \gamma(t))$, then $\tilde{\gamma} : [0, 1] \to \mathcal{M}^w$ is a geodesic, i.e.,

$$d_{\mathcal{GW}_p}^Z (\tilde{\gamma}(s), \tilde{\gamma}(t)) = |s - t| \cdot d_{\mathcal{GW}_p}^Z (\mathcal{X}, \mathcal{Y}) \quad \forall s, t \in [0, 1].$$

The proof is essentially the same as the one for Lemma 3.8 and we omit details.

**A new proof of Theorem 2.13.** As mentioned in the introduction, $(\mathcal{M}, d_{GH})$ is a geodesic metric space. This was proved in [INT16, CM18] respectively via the mid-point criterion and via an explicit construction of geodesics. Now, we end this section by presenting a novel and succinct proof of this fact based on geodesic properties of $\mathcal{W}_1$ and $\mathcal{H}$.

**Theorem 2.13.** $(\mathcal{M}, d_{GH})$ is a geodesic metric space.

**Proof.** Given two $X, Y \in \mathcal{M}$, let $\eta := d_{GH} (X, Y)$. Then, there exists $Z \in \mathcal{M}$ and isometric embeddings $\varphi_X : X \hookrightarrow Z$ and $\varphi_Y : Y \hookrightarrow Z$ such that $d_{\mathcal{H}}^Z (\varphi_X (X), \varphi_Y (Y)) = \eta$ (cf. Lemma 2.12). Without loss of generality, we assume that $Z$ is geodesic (otherwise we replace $Z$ with one of its extensions $\mathcal{W}_1 (Z)$, which is geodesic by Theorem 3.16). Then, $\mathcal{H} (Z)$ is geodesic by Theorem 3.5. Consequently, there exists a Hausdorff geodesic $\gamma : [0, 1] \to \mathcal{H} (Z)$ such that $\gamma(0) = X$ and $\gamma(1) = Y$ (we regard $X$ and $Y$ as subsets of $Z$). Then, by Lemma 3.8, one concludes that $\gamma$ is a geodesic in $\mathcal{M}$ connecting $X$ and $Y$. Therefore, $\mathcal{M}$ is itself geodesic. 

□
3.3 Urysohn universal metric space

In this section we introduce the Urysohn universal metric space which is a remarkable construction by Urysohn [Ury27]. This metric space can be regarded as an ambient space inside which one can isometrically embed every Polish metric space. It has some natural connections with the Gromov-Hausdorff distance and the Gromov-Wasserstein distance which we will elucidate. These connections will serve an important role in the proof of our main theorems.

Theorem 3.20 (Urysohn universal metric space). There exists a unique (up to isometry) Polish space $(U, d_U)$, which we call the Urysohn universal metric space, satisfying the following two properties:

1. (Universality) For any separable metric space $X$, there exists an isometric embedding $X \hookrightarrow U$.
2. (Homogeneity) For any isometry $\varphi$ between two finite subsets $A, A' \subseteq U$, there exists an isometry $\tilde{\varphi} : U \to U$ such that $\tilde{\varphi}|_A = \varphi$.

The universality property of $U$ makes the constant map taking each compact metric space $X$ to $U$ a metric extensor.

The homogeneity property in Theorem 3.20 above can be generalized to compact metric spaces:

Theorem 3.21 ([Huh55]). Given two compact subsets $A, A' \subseteq U$ and an isometry $\varphi : A \to A'$, there exists an isometry $\tilde{\varphi} : U \to U$ such that $\tilde{\varphi}|_A = \varphi$.

Connection with the Gromov-Hausdorff distance. For any $X, Y \in M$, due to universality, $U$ is a common ambient space. This allows us to consider the Hausdorff distance $d_U^H$ between $X$ and $Y$ which implies a connection between $U$ and $d_{GH}$. The following relation between the Gromov-Hausdorff distance and the Urysohn universal metric space was pointed out in [BV92, Ant20]:

Proposition 3.22. For any $X, Y \in M$, we have

$$d_{GH} (X, Y) = \inf_{\varphi_X, \varphi_Y} d_U^H (\varphi_X (X), \varphi_Y (Y)),$$

where the infimum is taken over all isometric embeddings $\varphi_X : X \hookrightarrow U$ and $\varphi_Y : Y \hookrightarrow U$.

We further improve this proposition through the following lemma:

Lemma 3.23. For any $X, Y \in M$, there exist isometric embeddings $\varphi_X : X \hookrightarrow U$ and $\varphi_Y : Y \hookrightarrow U$ such that

$$d_{GH} (X, Y) = d_U^H (\varphi_X (X), \varphi_Y (Y)).$$

Proof. By Lemma 2.12, there exist $Z \in M$ and isometric embeddings $\psi_X : X \hookrightarrow Z$ and $\psi_Y : Y \hookrightarrow Z$ such that $d_Z^H (\psi_X (X), \psi_Y (Y)) = d_{GH} (X, Y)$. Then, since $Z$ is compact, there exists an isometric embedding $\varphi : Z \hookrightarrow U$ (cf. Theorem 3.20). Let $\varphi_X = \varphi|_{\psi_X (X)} \circ \psi_X$ and $\varphi_Y = \varphi|_{\psi_Y (Y)} \circ \psi_Y$. Then, by Lemma 2.9 we have that

$$d_{GH} (X, Y) = d_Z^H (\varphi_X (X), \varphi_Y (Y)) = d_H^Z (\varphi_X (X), \varphi_Y (Y)) = d_H^U (\varphi_X (X), \varphi_Y (Y)).$$

Then, combining with Theorem 3.21, we derive the following lemma:
Lemma 3.24. For any $X, Y \in \mathcal{M}$, let $\varphi_X : X \hookrightarrow \mathbb{U}$ be an isometric embedding. Then, there exists an isometric embedding $\varphi_Y : Y \hookrightarrow \mathbb{U}$ such that

$$d_{GH} (X, Y) = d_{GH} (\varphi_X (X), \varphi_Y (Y)).$$

Proof. By Lemma 3.23, there exist isometric embeddings $\psi_X : X \hookrightarrow \mathbb{U}$ and $\psi_Y : Y \hookrightarrow \mathbb{U}$ such that $d_{GH} (X, Y) = d_{GH} (\psi_X (X), \psi_Y (Y))$. Now, both $\varphi_X (X)$ and $\psi_X (X)$ are compact subsets of $\mathbb{U}$ and $\tau := \varphi_X \circ \psi_X^{-1} : \psi_X (X) \rightarrow \varphi_X (X)$ is an isometry. By Theorem 3.21, there exists an isometry $\tilde{\tau} : \mathbb{U} \rightarrow \mathbb{U}$ such that $\tilde{\tau} |_{\psi_X (X)} = \tau$. Let $\varphi_Y := \tilde{\tau} |_{\psi_Y (Y)} \circ \psi_Y : Y \rightarrow \mathbb{U}$. It is clear that $\varphi_Y$ is an isometric embedding and thus

$$d_{GH} (\varphi_X (X), \varphi_Y (Y)) = d_{GH} (\tilde{\tau}^{-1} \circ \varphi_X (X), \tilde{\tau}^{-1} \circ \varphi_Y (Y)) = d_{GH} (\psi_X (X), \psi_Y (Y)) = d_{GH} (X, Y).$$

See Figure 3 for an illustration of the proof for Lemma 3.24.

Figure 3: Illustration of the proof of Lemma 3.24

Lemma 3.24 then leads us to the following key observation which is instrumental in proving Theorem 1.

Lemma 3.25. For any Gromov-Hausdorff geodesic $\gamma : [0, 1] \rightarrow \mathcal{M}$, let $\rho := d_{GH} (\gamma (0), \gamma (1))$. Then, for any finite sequence $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$, there exist for all $i = 0, \ldots, n$ isometric embeddings $\varphi_i : \gamma (t_i) \hookrightarrow \mathbb{U}$ such that

$$d_{GH} (\varphi_i (\gamma (t_i)), \varphi_j (\gamma (t_j))) = |t_i - t_j| \rho, \quad \forall 0 \leq i, j \leq n.$$

Proof. We prove the lemma by induction on $n$. When $n = 0$, the statement holds true trivially. Now, suppose the statement holds for $n \geq 0$. Consider a sequence $0 \leq t_0 < t_1 < \ldots < t_n < t_{n+1} \leq 1$. By the induction assumption, there exist $\varphi_i : \gamma (t_i) \hookrightarrow \mathbb{U}$ for $0 \leq i \leq n$ such that

$$d_{GH} (\varphi_i (\gamma (t_i)), \varphi_j (\gamma (t_j))) = |t_i - t_j| \rho, \forall 0 \leq i, j \leq n.$$
By Lemma 3.24, there exists an isometric embedding $\varphi_{n+1} : \gamma(t_{n+1}) \hookrightarrow \mathbb{U}$ such that

$$d^H_{n}(\varphi_{n}(\gamma(t_n)), \varphi_{n+1}(\gamma(t_{n+1}))) = |t_n - t_{n+1}| \rho.$$

Then, for any $i < n$, we have

$$d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_{n+1}(\gamma(t_{n+1}))) \leq d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_{n}(\gamma(t_n))) + d^H_{n}(\varphi_{n}(\gamma(t_n)), \varphi_{n+1}(\gamma(t_{n+1})))$$

$$\leq (t_n - t_i) \rho + (t_{n+1} - t_n) \rho$$

$$= (t_{n+1} - t_i) \rho.$$

Since $d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_{n+1}(\gamma(t_{n+1}))) \geq d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_{n+1}(\gamma(t_{n+1}))) = (t_{n+1} - t_i) \rho$, we have that $d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_{n+1}(\gamma(t_{n+1}))) = (t_{n+1} - t_i) \rho$ for all $i < n$. This concludes the induction step. $\blacksquare$

**Corollary 3.26.** Given any Gromov-Hausdorff geodesic $\gamma : [0, 1] \to M$, we let $\rho := d_{GH}(\gamma(0), \gamma(1))$. Then, for any finite sequence $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$, there exist $X \in M$ and isometric embeddings $\varphi_i : \gamma(t_i) \hookrightarrow X$ for $i = 0, \ldots, n$ such that

$$d^X_{n}(\varphi_i(\gamma(t_i)), \varphi_j(\gamma(t_j))) = |t_i - t_j| \rho, \quad \forall 0 \leq i, j \leq n.$$

**Proof.** By the previous lemma, there exist isometric embeddings $\varphi_i : \gamma(t_i) \hookrightarrow \mathbb{U}$ such that

$$d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_j(\gamma(t_j))) = |t_i - t_j| \rho.$$

Let $X := \bigcup_{i=0}^{n} \varphi_i(\gamma(t_i)) \subseteq \mathbb{U}$. Then, since each $\varphi_i(\gamma(t_i))$ is compact, we have that $X$ is compact and thus

$$d^X_{n}(\varphi_i(\gamma(t_i)), \varphi_j(\gamma(t_j))) = d^H_{n}(\varphi_i(\gamma(t_i)), \varphi_j(\gamma(t_j))) = |t_i - t_j| \rho, \quad \forall 0 \leq i, j \leq n.$$

$\blacksquare$

**Gromov-Wasserstein counterparts.** All the previous results in this section have counterparts for the Gromov-Wasserstein distance. We will list the most useful ones for later use and we delay the proofs to the end of this section. Recall from Section 2.4 that we use script letters such as $\mathcal{X}$ to denote metric measure spaces $\mathcal{X} = (X, d_X, \mu_X)$.

**Lemma 3.27.** For any $p \in [1, \infty)$ and any $\mathcal{X}, \mathcal{Y} \in \mathcal{M}^w$, there exist isometric embeddings $\varphi_X : X \hookrightarrow \mathbb{U}$ and $\varphi_Y : Y \hookrightarrow \mathbb{U}$ such that

$$d^G_{\mathcal{X}, \mathcal{Y}}(\varphi_X, \varphi_Y) = d^H_{\mathcal{X}, \mathcal{Y}}((\varphi_X)\#_X (\varphi_Y)\#_Y).$$

**Lemma 3.28.** For any $p \in [1, \infty)$ and any $\mathcal{X}, \mathcal{Y} \in \mathcal{M}^w$, let $\varphi_X : X \hookrightarrow \mathbb{U}$ be an isometric embedding. Then, there exists an isometric embedding $\varphi_Y : Y \hookrightarrow \mathbb{U}$ such that

$$d^G_{\mathcal{X}, \mathcal{Y}}(\varphi_X, \varphi_Y) = d^H_{\mathcal{X}, \mathcal{Y}}((\varphi_X)\#_X (\varphi_Y)\#_Y).$$

**Lemma 3.29.** For any $p \in [1, \infty)$, let $\gamma : [0, 1] \to \mathcal{M}^w$ be an $\ell^p$-Gromov-Wasserstein geodesic. Let $\rho := d^G_{\mathcal{X}, \mathcal{Y}}(\gamma(0), \gamma(1))$ and write $\gamma(t) := (X_t, d_t, \mu_t)$ for each $t \in [0, 1]$. Then, for any finite sequence $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$, there exist isometric embeddings $\varphi_i : X_{t_i} \hookrightarrow \mathbb{U}$ such that

$$d^G_{\mathcal{X}, \mathcal{Y}}((\varphi_i)\#_X (\mu_{t_i})_{(\varphi_j)\#_Y (\mu_{t_j})}) = |t_i - t_j| \rho, \quad \forall 0 \leq i, j \leq n.$$

**Corollary 3.30.** Assume the same conditions as in Lemma 3.29. Then, for any finite sequence $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$, there exist $X \in M$ and isometric embeddings $\varphi_i : X_{t_i} \hookrightarrow X$ for $i = 0, \ldots, n$ such that

$$d^G_{X, \mathcal{Y}}((\varphi_i)\#_{X_{t_i}} (\varphi_j)\#_{X_{t_j}}) = |t_i - t_j| \rho, \quad \forall 0 \leq i, j \leq n.$$
Relegated proofs.

**Proof of Lemma 3.27.** By Lemma 3.22, there exist $Z \in \mathcal{M}$ and isometric embeddings $\psi_X : X \hookrightarrow Z$ and $\psi_Y : Y \hookrightarrow Z$ such that $d_{GW,p}^{k}((\psi_X)_# \mu_X, (\psi_Y)_# \mu_Y) = d_{GW,p}^{k}(X,Y)$. Then, since $Z$ is compact, there exists an isometric embedding $\varphi : Z \hookrightarrow \mathbb{U}$ (cf. Theorem 3.20). Let $\varphi_X = \varphi|_{\psi_X(X)} \circ \psi_X$ and $\varphi_Y = \varphi|_{\psi_Y(Y)} \circ \psi_Y$. Then,

$$d_{GW,p}^{k}(X,Y) = d_{GW,p}^{k}((\psi_X)_# \mu_X, (\psi_Y)_# \mu_Y)$$
$$= d_{GW,p}^{k}(\psi_Z)_# ((\varphi_X)_# \mu_X, (\varphi_Y)_# \mu_Y)$$
$$= d_{GW,p}^{k}((\varphi_X)_# \mu_X, (\varphi_Y)_# \mu_Y).$$

\[\square\]

**Proof of Lemma 3.28.** By Lemma 3.27, there exist isometric embeddings $\psi_X : X \hookrightarrow \mathbb{U}$ and $\psi_Y : Y \hookrightarrow \mathbb{U}$ such that $d_{GW,p}^{k}(X,Y) = d_{GW,p}^{k}((\psi_X)_# \mu_X, (\psi_Y)_# \mu_Y)$. Now, both $\varphi_X(X)$ and $\psi_X(X)$ are compact subsets of $\mathbb{U}$ and $\tau := \varphi_X \circ \psi_X^{-1} : \psi_X(X) \to \varphi_X(X)$ is an isometry. By Theorem 3.21, there exists an isometry $\tilde{\tau} : \mathbb{U} \to \mathbb{U}$ such that $\tilde{\tau}|_{\psi_X(X)} = \tau$. Let $\varphi_Y := \tilde{\tau} \circ \psi_Y : Y \to \mathbb{U}$. It is clear that $\varphi_Y$ is an isometric embedding and thus

$$d_{GW,p}^{k}((\varphi_X)_# \mu_X, (\varphi_Y)_# \mu_Y) = d_{GW,p}^{k}((\tilde{\tau}^{-1})_# \circ (\varphi_X)_# \mu_X, (\tilde{\tau}^{-1})_# \circ (\varphi_Y)_# \mu_Y)$$
$$= d_{GW,p}^{k}((\psi_X)_# \mu_X, (\psi_Y)_# \mu_Y)$$
$$= d_{GW,p}^{k}(X,Y).$$

\[\square\]

**Proof of Lemma 3.29.** We prove the lemma by induction on $n$. When $n = 0$, the statement holds true trivially. Now, suppose the statement holds for $n \geq 0$. Consider a sequence $0 \leq t_0 < t_1 < \ldots < t_n < t_{n+1} \leq 1$. By the induction assumption, there exist isometric embeddings $\varphi_i : X_{t_i} \hookrightarrow \mathbb{U}$ for $0 \leq i \leq n$ such that $d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_j)_# \mu_{t_j}) = |t_i - t_j|\rho$, $\forall 0 \leq i, j \leq n$. By Lemma 3.28, there exists an isometric embedding $\varphi_{n+1} : X_{t_{n+1}} \hookrightarrow \mathbb{U}$ such that $d_{GW,p}^{k}((\varphi_n)_# \mu_n, (\varphi_{n+1})_# \mu_{t_{n+1}}) = |t_n - t_{n+1}|\rho$. Then, for any $i < n$, we have

$$d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_{n+1})_# \mu_{t_{n+1}})$$
$$\leq d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_n)_# \mu_n) + d_{GW,p}^{k}((\varphi_n)_# \mu_n, (\varphi_{n+1})_# \mu_{t_{n+1}})$$
$$\leq (t_n - t_i) \rho + (t_{n+1} - t_n) \rho$$
$$= (t_{n+1} - t_i) \rho.$$

Since $d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_{n+1})_# \mu_{t_{n+1}}) \geq d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_n)_# \mu_n) = (t_{n+1} - t_i) \rho$, we have that $d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_{n+1})_# \mu_{t_{n+1}}) = (t_{n+1} - t_i) \rho$ for all $i < n$. This concludes the induction step. \[\square\]

**Proof of Corollary 3.30.** By Lemma 3.29, there exist isometric embeddings $\varphi_i : X_{t_i} \hookrightarrow \mathbb{U}$ for $i = 0, \ldots, n$ such that $d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_j)_# \mu_{t_j}) = |t_i - t_j|\rho$. Let $X := \cup_{i=0}^{n} \varphi_i(X_{t_i}) \subseteq \mathbb{U}$. Then, since each $\varphi_i(X_{t_i})$ is compact, we have that $X$ is compact and thus for all $0 \leq i, j \leq n$ we have

$$d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_j)_# \mu_{t_j}) = d_{GW,p}^{k}((\varphi_i)_# \mu_{t_i}, (\varphi_j)_# \mu_{t_j}) = |t_i - t_j|\rho.$$
4 Hausdorff and Wasserstein-realizable geodesics

In this section, we study both Hausdorff and Wasserstein-realizable geodesics and prove Theorem 1 and Theorem 2. Both proofs rely on convergence results of Lipschitz curves under certain metric extendors and we first study such convergence results in Section 4.1.

4.1 Convergence

Given a metric space $X$ and a Hausdorff convergent sequence\(^4\) of subsets $\{A_i\}_{i=0}^\infty$ with limit $A \subseteq X$, then $\lim_{i \to \infty} d_{GH}(A_i, A) = 0$, since $d_{GH}(A_i, A) \leq d_{GH}^X(A_i, A)$ for $i = 0, \ldots$. Conversely, a Gromov-Hausdorff convergent sequence\(^5\) of compact metric spaces $\{X_i\}_{i=0}^\infty$ with limit $X \in \mathcal{M}$ can be realized as a Hausdorff convergent sequence in some ambient space. A similar statement was mentioned in [PAR06, Chapter 10] whose proof operates by passing to a subsequence. We provide a proof for our statement which involves the Urysohn universal metric space (cf. Section 3.3).

**Lemma 4.1.** Let $\{X_i\}_{i=0}^\infty$ be a convergent sequence in $(\mathcal{M}, d_{GH})$ with limit $X \in \mathcal{M}$. Then, there exist a Polish metric space $Z$ and isometric embeddings $\varphi: X \hookrightarrow Z$ and $\varphi_i: X_i \hookrightarrow Z$ for $i = 0, \ldots$ such that $\lim_{i \to \infty} d_{GH}^Z(\varphi_i(X_i), \varphi(X)) = 0$.

**Proof.** Let $Z = \cup$, the Urysohn universal metric space. Then, $Z$ is Polish. By universality (cf. Theorem 3.20), there exists an isometric embedding $\varphi: X \hookrightarrow \cup$. By Lemma 3.24, there exist isometric embeddings $\varphi_i: X_i \hookrightarrow \cup$ such that $d_{GH}^Z(\varphi(X), \varphi_i(X_i)) = d_{GH}(X, X_i)$ for $i = 0, \ldots$. Therefore, $\lim_{i \to \infty} d_{GH}^Z(\varphi_i(X_i), \varphi(X)) = 0$. \qed

**A generalized Arzelà-Ascoli theorem.** The version of Arzelà-Ascoli theorem in Theorem 2.3 requires a fixed range $X$ for all curves $\gamma_i: [0, 1] \to X$. This can be generalized to curves $\gamma_i: [0, 1] \to X_i$ with convergent ranges, i.e., $X_i$ converges (in a suitable sense) to $X \in \mathcal{M}$ as $i$ approaches $\infty$.

**Theorem 4.2** (Generalized Arzelà-Ascoli theorem). Let $(Z, d_Z)$ be a complete metric space and let $\{X_i\}_{i=0}^\infty$ be a Hausdorff convergent sequence of compact subsets of $Z$. Let $X \in \mathcal{H}(Z)$ be the limit of $\{X_i\}_{i=0}^\infty$ under $d_{GH}^Z$. Let $\gamma_i: [0, 1] \to X_i$ be a sequence of $C$-Lipschitz curves for some $C > 0$ fixed. Then, there is a uniformly convergent (in the sense of $d_Z$) subsequence of $\{\gamma_i\}_{i=0}^\infty$ with a $C$-Lipschitz limit $\gamma: [0, 1] \to X$.

**Proof.** Let $T := \{t_n\}_{n=0}^\infty$ be a countable dense subset of $[0, 1]$. Let $\rho_i := d_{GH}^Z(X_i, X)$ for $i = 0, \ldots$. Then, for each $\gamma_i(t_0) \in X_i$, there exists $x^0_i \in X$ such that $d_Z(\gamma_i(t_0), x^0_i) \leq \rho_i$. Since $X$ is compact, there exists a subsequence of $\{x^0_i\}_{i=0}^\infty$, still denoted by $\{x^0_i\}_{i=0}^\infty$, converging to a point $x^0 \in X$. Then, since $\rho_i \to 0$ as $i \to \infty$, we have

$$0 \leq \lim_{i \to \infty} d_Z(\gamma_i(t_0), x^0_i) \leq \lim_{i \to \infty} d_Z(\gamma_i(t_0), x^0) + \lim_{i \to \infty} d_Z(x^0_i, x^0) = 0,$$

and consequently, $\lim_{i \to \infty} d_Z(\gamma_i(t_0), x^0) = 0$. We similarly consider $t_1, t_2$ and so on and construct $x^n \in X$ for $n = 1, 2, \ldots$ in a manner similar to the construction of $x^0$. Then, by a standard diagonal argument, there exist a subsequence of $\{X_i\}_{i=0}^\infty$ (still denoted by $\{X_i\}_{i=0}^\infty$) and points $x^n \in X$ for $n = 0, \ldots$ such that $\lim_{i \to \infty} d_Z(\gamma_i(t_n), x^n) = 0$ for $n = 0, \ldots$. Then, for $m, n \in \mathbb{N}$, we have

$$d_Z(x^m, x^n) = \lim_{i \to \infty} d_Z(\gamma_i(t_m), \gamma_i(t_n)) \leq C \cdot |t_m - t_n|.$$ \((2)\)

\(^4\)A Hausdorff convergent sequence in a metric space $X$ is a sequence of compact subsets of $X$ converging under the Hausdorff distance $d_{GH}^X$.

\(^5\)A Gromov-Hausdorff convergent sequence is a sequence of compact metric spaces converging under the Gromov-Hausdorff distance $d_{GH}$. 
Now, we define $\gamma : [0, 1] \to X$ as follows:

$$\gamma(t) := \begin{cases} x^n, & t = t_n \in T \\ \lim_{j \to \infty} x^{n_j}, & t \in [0, 1] \setminus [t_n, a], \end{cases}$$

The existence of the limit $\lim_{j \to \infty} x^{n_j}$ is due to completeness of $Z$ and Equation (2). It is obvious that $\gamma(t)$ is well-defined, i.e., its image is independent of the choice of $\{t_n\}_{j=0}^\infty$. It is easy to check that $\gamma$ is also $C$-Lipschitz. Now, it remains to prove that $\{\gamma_i\}_{i=0}^\infty$ uniformly converges to $\gamma$. For any $\varepsilon > 0$, pick a finite subsequence $T_N = \{t_0, t_1, \ldots, t_N\}$ of $T$ (possibly relabeled) such that $T_N$ is an $\frac{\varepsilon}{\delta_0}$-net of $[0, 1]$. By definition of $\gamma$ on $T$, there exists $M > 0$ such that for all $i > M$ and for all $n = 0, \ldots, N$, we have $d_Z(\gamma(t_n), \gamma_i(t_n)) \leq \frac{\varepsilon}{3}$. Then, for any $t \in [0, 1]$, there exists $t_n \in T_N$ such that $|t - t_n| \leq \frac{\varepsilon}{3}$ and that for $i > M$

$$d_Z(\gamma(t), \gamma_i(t)) \leq d_Z(\gamma(t), \gamma(t_n)) + d_Z(\gamma(t_n), \gamma_i(t_n)) + d_Z(\gamma_i(t_n), \gamma_i(t))$$

$$\leq C \cdot |t - t_n| + \varepsilon + C \cdot |t - t_n|$$

$$\leq \varepsilon.$$

This implies that $\{\gamma_i\}_{i=0}^\infty$ converges to $\gamma$ uniformly. \hfill $\Box$

### 4.2 Hausdorff-realizable geodesics

We first define Hausdorff-realizable geodesics as follows:

**Definition 4.3** (Hausdorff-realizable geodesic). A geodesic $\gamma : [0, 1] \to M$ is called **Hausdorff-realizable**, if there exist $X \in \mathcal{M}$ and for each $t \in [0, 1]$ isometric embedding $\varphi_t : \gamma(t) \hookrightarrow X$ such that

$$d^X_{\varphi_s}(\varphi_s(\gamma(s)), \varphi_t(\gamma(t))) = d_{\mathcal{G}H}(\gamma(s), \gamma(t)), \quad \forall s, t \in [0, 1].$$

In this case, we say that $\gamma$ is $X$-$\text{Hausdorff-realizable}$.

**Remark 4.4** (Hausdorff-realizable geodesics are Hausdorff geodesics). Suppose that $\gamma : [0, 1] \to \mathcal{M}$ is a $X$-$\text{Hausdorff-realizable}$ Gromov-Hausdorff geodesic via the family of isometric embeddings

$$\{\varphi_t : \gamma(t) \hookrightarrow X\}_{t \in [0, 1]}.$$

Then, obviously the curve defined by $t \mapsto \varphi_t(\gamma(t))$ for $t \in [0, 1]$ is a geodesic in the Hausdorff hyperspace $\mathcal{H}(X)$ of $X$. This is the converse of Lemma 3.8. In short, we emphasize that a Hausdorff-realizable Gromov-Hausdorff geodesic is the same as a Hausdorff geodesic with respect to some underlying ambient space.

**Example 4.5** (Trivial Gromov-Hausdorff geodesics are Hausdorff-realizable). Let $\gamma : [0, 1] \to \mathcal{M}$ be a “trivial” Gromov-Hausdorff geodesic, i.e., there exists $X \in \mathcal{M}$ such that $\gamma(t) \cong X$ for all $t \in [0, 1]$. Then, it is obvious that $\gamma$ is $X$-$\text{Hausdorff-realizable}$.

In [IT19a], the authors show that any straight-line $d_{\mathcal{G}H}$ geodesic can be Hausdorff-realized in a metric space:

**Proposition 4.6** ([IT19a, Corollary 3.1]). Let $X, Y \in \mathcal{M}$ and $R \in \mathcal{R}^{opt}(X, Y)$ and let $\rho := d_{\mathcal{G}H}(X, Y)$. Let $\gamma_R$ be the straight-line $d_{\mathcal{G}H}$ geodesic connecting $X$ and $Y$ based on $R$ (cf. Theorem 2.14). Let $Z := R \times [0, 1]$ and define $d_Z : Z \times Z \to \mathbb{R}_+$ by

$$d_Z((x, y), (x', y'), (x'', y'')) := \inf_{(x', y') \in \mathcal{R}} (d_{R_{t'}}((x, y), (x'', y'')) + d_{R_{t'}}((x', y'), (x'', y''))) + \rho |t - t'|,$$

for any $(x, y), (x', y') \in R$ and $t, t' \in [0, 1]$. Then, by canonically identifying $R_t$ with $R \times \{t\} \subseteq Z$, we have $d^Z_{\mathcal{H}}(R_s, R_t) = d_{\mathcal{G}H}(R_s, R_t)$.
Remark 4.7. In fact, \( Z \) is a pseudo-metric space since for any \((x, y), (x, y') \in R\), we have at \( t = 0 \) that \( d_Z((x, y), 0), (x, y'), 0) = d_X(x, x) = 0 \). A similar result holds for \( t = 1 \). By identifying points at zero distance, we obtain a new metric space \( \hat{Z} \). It is obvious that the result in Proposition 4.6 still holds by replacing \( Z \) with \( \hat{Z} \).

Now, we show that the space constructed in Proposition 4.6 (or more precisely the quotient space discussed in the remark above) is compact for compact correspondences and thus show that straight-line \( d_{\mathcal{H}} \) geodesics corresponding to compact correspondences are Hausdorff-realizable.

Proposition 4.8. Assuming the same notation as in Proposition 4.6, if \( R \) is compact in the product space \((X \times Y, d_{X \times Y} := \max(d_X, d_Y))\), then \( \gamma_R \) is Hausdorff-realizable.

Proof. We only need to show that the construction \( Z \) in Proposition 4.6 is sequentially compact. Then, the metric space \( \hat{Z} \) in Remark 4.7 is also sequentially compact and thus compact.

For any sequence \( \{(x_i, y_i), t_i\}_{i=1}^{\infty} \) in \( Z \), by compactness of \([0,1]\) and \( R \), there exists a subsequence, still denoted by \( \{(x_i, y_i), t_i\}_{i=1}^{\infty} \), such that \( \{t_i\}_{i=1}^{\infty} \) converges to some \( t \in [0,1] \) and that \( \{(x_i, y_i)\}_{i=1}^{\infty} \) converges to some \((x, y) \in R\) under \( d_{X \times Y} := \max(d_X, d_Y) \).

Now, we show that \( \lim_{i \to \infty} d_Z((x_i, y_i), t_i), ((x, y), t) = 0 \). Indeed,

\[
0 \leq d_Z((x, y), t), ((x_i, y_i), t_i) = \inf_{(x', y') \in R} \left( d_{R_1}((x, y), (x', y')) + d_{R_2}((x', y'), (x_i, y_i)) \right) + \rho |t - t_i|
\]

\[
\leq d_{R_1}((x, y), (x_i, y_i)) + \rho |t - t_i|
\]

\[
= (1-t) d_X(x_i, x) + t d_Y(y, y_i) + \rho |t - t_i|
\]

\[
\leq d_{X \times Y}((x, y), (x, y_i)) + \rho |t - t_i|
\]

Then, by assumptions on the sequence \( \{(x_i, y_i), t_i\}_{i=1}^{\infty} \),

\[
\lim_{i \to \infty} d_Z((x_i, y_i), t_i), ((x, y), t) = 0.
\]

As a result, \( Z \) is sequentially compact and hence we conclude the proof.

The following lemma provides an interesting description of Hausdorff-realizable geodesics: for any \( X \)-Hausdorff-realizable geodesic \( \gamma \), there is a smallest closed subset \( G_X \subseteq X \) which Hausdorff-realizes \( \gamma \).

Lemma 4.9. Given a compact metric space \( X \) and a Hausdorff geodesic \( \gamma : [0,1] \to \mathcal{H}(X) \), the union \( G_X := \bigcup_{t \in [0,1]} \gamma(t) \) is a closed (and thus compact) subset of \( X \).

Proof. Let \( \rho := d^X_{\mathcal{H}}(\gamma(0), \gamma(1)) \). If \( \rho = 0 \), then \( \gamma(t) = \gamma(0) \) for all \( t \in [0,1] \). Thus \( G_X = \gamma(0) \) is closed.

Now, we assume that \( \rho > 0 \). Fix an arbitrary \( x \in X \). Define \( f_x : [0,1] \to \mathbb{R} \) by taking \( t \in [0,1] \) to \( d_X(x, \gamma(t)) := \inf \{d_X(x, x_t) : x_t \in \gamma(t)\} \). We first show that \( f_x \) is continuous. Fix \( t_0 \in [0,1] \). Since \( \gamma(t_0) \) is compact, there exists \( x_{t_0} \in \gamma(t_0) \) such that \( f_x(t_0) = d_X(x, x_{t_0}) \). For each \( \varepsilon > 0 \), let \( \delta = \frac{\varepsilon}{\rho} > 0 \). For any \( t \in [0,1] \) such that \( |t - t_0| < \delta \), there exists \( x_t \in \gamma(t) \) such that

\[
d_X(x_t, x_{t_0}) \leq d^X_{\mathcal{H}}(\gamma(t), \gamma(t_0)) = |t - t_0| \rho < \delta \rho = \varepsilon.
\]

Then,

\[
f_x(t) \leq d_X(x, x_t) \leq d_X(x, x_{t_0}) + d_X(x_{t_0}, x_t) < f_x(t_0) + \varepsilon.
\]

Now, assume \( x'_t \in \gamma(t) \) is such that \( f_x(t) = d_X(x, x'_t) \). Let \( x'_{t_0} \in \gamma(t_0) \) be such that

\[
d_X(x'_{t_0}, x'_t) \leq d^X_{\mathcal{H}}(\gamma(t_0), \gamma(t)) < \varepsilon.
\]
Then, 
\[ f_x(t) = d_X(x, x'_t) \geq d_X(x, x'_t_0) - d_X(x'_t_0, x'_t) > f_x(t_0) - \varepsilon. \]
Therefore, \( |f_x(t) - f_x(t_0)| < \varepsilon \) for any \( |t - t_0| < \delta \). This implies the continuity of \( f_x \).

Let \( \{x_i\}_{i=0}^\infty \) be a convergent sequence in \( X \) such that \( x_i \in \gamma(t_i) \) for some \( t_i \in [0, 1] \) and \( i = 0, \ldots, n \). Suppose \( x \in X \) is its limit. Assume that \( x \not\in G_X \) and thus \( f_x(t) > 0 \) for each \( t \in [0, 1] \). Then, by continuity of \( f_x \), there exists a constant \( c > 0 \), such that \( f > c \) on \([0, 1]\). But \( f_x(t_i) \leq d_X(x_i, x) \) and the right hand side approaches \( 0 \) as \( i \to \infty \), a contradiction. Hence, there exists \( t \in [0, 1] \) such that \( f(t) = 0 \) and thus \( x \in \gamma(t) \subseteq G_X \). This proves that \( G_X \) is closed in \( X \).

Let \( \Gamma \) be the collection of all Gromov-Hausdorff geodesics \( \gamma : [0, 1] \to \mathcal{M} \). Let \( d_{\infty} \) be the uniform metric on \( \Gamma \), i.e., \( d_{\infty}(\gamma_1, \gamma_2) := \sup_{t \in [0,1]} d_{GH}(\gamma_1(t), \gamma_2(t)) \) for any \( \gamma_1, \gamma_2 \in \Gamma \). Let \( \Gamma_H \) denote the subset of \( \Gamma \) consisting of all Hausdorff-realizable geodesics in \( \mathcal{M} \). Then, Theorem 1 is equivalent to saying that \( \Gamma_H = \Gamma \). Before proving Theorem 1, we apply the properties developed in Section 3.2 towards proving the following preliminary result:

**Proposition 4.10.** \( \Gamma_H \) is a dense subset of \( \Gamma \).

**Proof.** Fix any Gromov-Hausdorff geodesic \( \gamma : [0, 1] \to \mathcal{M} \) with \( \rho := d_{GH}(\gamma(0), \gamma(1)) > 0 \) and \( \varepsilon > 0 \). Let \( 0 = t_0 < \ldots < t_n = 1 \) be a sequence such that \( t_{i+1} - t_i < \frac{\varepsilon}{2\rho} \) for \( i = 0, \ldots, n-1 \).

Then, by Corollary 3.26, there exist \( X \in \mathcal{M} \) and isometric embeddings \( \varphi_i : \gamma(t_i) \hookrightarrow X \) such that \( d_H^X(\varphi_i(\gamma(t_i)), \varphi_j(\gamma(t_j))) = |t_i - t_j| \rho \). Let \( Z := \mathcal{W}_1(X) \) and still denote by \( \varphi_i \) the composition of \( \varphi_i : \gamma(t_i) \hookrightarrow X \) and the canonical embedding \( X \hookrightarrow \mathcal{W}_1(X) = Z \) for \( i = 0, \ldots, n \). Then, we still have \( d_Z^H(\varphi_i(\gamma(t_i)), \varphi_j(\gamma(t_j))) = |t_i - t_j| \rho \).

Since \( Z \) is compact and geodesic (cf. Theorem 3.11 and Theorem 3.16), \( \mathcal{H}(Z) \) is geodesic (cf. Theorem 3.5). Hence, there exist Hausdorff geodesics \( \gamma_i : [0, 1] \to \mathcal{H}(Z) \) such that \( \gamma_i(0) = \varphi_i(\gamma(t_i)) \) and \( \gamma_i(1) = \varphi_{i+1}(\gamma(t_{i+1})) \) for \( i = 0, \ldots, n-1 \). Then, \( \gamma_i(1) = \gamma_{i+1}(0) \) for \( i = 0, \ldots, n-1 \) and
\[
d_Z^H(\gamma_0(0), \gamma_{n-1}(1)) = d_Z^H(\varphi_0(\gamma(t_0)), \varphi_n(\gamma(t_n)))
= \sum_{i=0}^{n-1} d_Z^H(\varphi_i(\gamma(t_i)), \varphi_{i+1}(\gamma(t_{i+1})))
= \sum_{i=0}^{n-1} d_Z^H(\gamma_i(0), \gamma_i(1)).
\]

Therefore, we can concatenate (cf. Proposition 2.6) all the \( \gamma_i \)s to obtain a new geodesic \( \tilde{\gamma} : [0, 1] \to \mathcal{H}(Z) \) such that \( \tilde{\gamma}(t_i) = \varphi_i(\gamma(t_i)) \) for each \( i = 0, \ldots, n \). By Lemma 3.8, \( \tilde{\gamma} \) is a Gromov-Hausdorff geodesic and by construction, \( \tilde{\gamma} \in \Gamma_H \). Now, for any \( t \in [0, 1] \), suppose \( t \in [t_i, t_{i+1}] \) for some \( i \in \{0, \ldots, n-1\} \). Then,
\[
d_{GH}(\gamma(t), \tilde{\gamma}(t)) \leq d_{GH}(\gamma(t), \gamma(t_i)) + d_{GH}(\gamma(t_i), \tilde{\gamma}(t_i)) + d_{GH}(\tilde{\gamma}(t_i), \tilde{\gamma}(t))
= |t - t_i| \rho + 0 + |t - t_i| \rho \leq 2 \cdot \frac{\varepsilon}{2\rho} \cdot \rho = \varepsilon.
\]
So, \( d_{\infty}(\gamma, \tilde{\gamma}) \leq \varepsilon \). Therefore, we conclude that \( \Gamma_H \) is dense in \( \Gamma \).

**Proof of Theorem 1.** Now, we obtain a proof of Theorem 1 by showing that \( \Gamma_H \) is closed in \( \Gamma \). The proof is an intricate application of the generalized Arzelà-Ascoli theorem (Theorem 4.2). In order to meet the conditions in Theorem 4.2, one needs to exploit the stability of the Hausdorff extensor (Theorem 3.4) and carefully leverage Gromov’s pre-compactness theorem (Theorem 2.16).

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Theorem 1. Every Gromov-Hausdorff geodesic is Hausdorff-realizable.

Proof. By Proposition 4.10, we only need to show that $\Gamma_H$ is closed in $\Gamma$. Let $\{\gamma_i : [0, 1] \to M\}_{i=0}^\infty$ be a Cauchy sequence in $\Gamma_H$ with a limit $\gamma : [0, 1] \to M$ in $\Gamma$, i.e., $\lim_{i \to \infty} d_H(\gamma_i, \gamma) = 0$. Moreover, we let $\rho := d_{GH}(\gamma(0), \gamma(1))$.

Claim 4.11. There exist $X_i \in M$ such that $\gamma_i$ is $X_i$-Hausdorff-realizable for $i = 0, \ldots$ and moreover $\{X_i\}_{i=0}^\infty$ has a $d_{GH}$-convergent subsequence.

Assuming the claim for now, suppose $X \in M$ is such that $\lim_{i \to \infty} d_{GH}(X_i, X) = 0$ (after possibly passing to a subsequence), then by Lemma 4.1, there exist a Polish metric space $Z$ and isometric embeddings $\varphi : X \hookrightarrow Z$ and $\varphi_i : X_i \hookrightarrow Z$ for $i = 0, \ldots$ such that $\lim_{i \to \infty} d_Z(\varphi_i(X_i), \varphi(X)) = 0$. By stability of the Hausdorff extensor $H$ (cf. Theorem 3.4),

$$\lim_{i \to \infty} d_H^Z((\varphi_i)_*(H(X_i)), \varphi_*(H(X))) = \lim_{i \to \infty} d_H^Z(\varphi_i(X_i), \varphi(X)) = 0.$$  

For any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that $\gamma_i$ is $(\rho + \varepsilon)$-Lipschitz for $i \geq K(\varepsilon)$, since $\gamma_i$ is $d_{GH}^1(\gamma_i(0), \gamma_i(1))$-Lipschitz and $d_{GH}(\gamma_i(0), \gamma_i(1)) \to \rho$ as $i \to \infty$. For each $i = 0, \ldots$, since $\gamma_i$ is $X_i$-Hausdorff-realizable, we can view $\gamma_i : [0, 1] \to M$ as a Hausdorff geodesic $\gamma_i : [0, 1] \to \mathcal{H}(X_i)$. Moreover, by Theorem 3.1, $\mathcal{H}(Z)$ is complete. Then, by the generalized Arzelà-Ascoli theorem (Theorem 4.2), the sequence $\{\gamma_i : [0, 1] \to \mathcal{H}(X_i)\}_{i=K(\varepsilon)}^\infty$ uniformly converges to a $(\rho + \varepsilon)$-Lipschitz curve $\tilde{\gamma} : [0, 1] \to \mathcal{H}(X)$, where we identify $\mathcal{H}(X_i)$ with $\varphi_i(\mathcal{H}(X_i))$ and $\mathcal{H}(X)$ with $\varphi(\mathcal{H}(X))$. Obviously, for $0 < \varepsilon' < \varepsilon$, the subsequence $\{\gamma_i\}_{i=K(\varepsilon')}^\infty$ uniformly converges to the same curve $\tilde{\gamma}$ and thus $\tilde{\gamma}$ is $(\rho + \varepsilon')$-Lipschitz. Since $\varepsilon'$ is arbitrary, $\tilde{\gamma}$ is $\rho$-Lipschitz. By uniform convergence, we have that for each $t \in [0, 1]$, $\lim_{i \to \infty} d_Z^X(\tilde{\gamma}(t), \gamma_i(t)) = 0$. We know that $\gamma(t)$ is the Gromov-Hausdorff limit of $\{\gamma_i(t)\}_{i=0}^\infty$, thus $\tilde{\gamma}(t) \equiv \gamma(t)$ for all $t \in [0, 1]$. Since $\tilde{\gamma}$ is $\rho$-Lipschitz, we have that for each $s, t \in [0, 1]$,

$$d_X^X(\tilde{\gamma}(s), \tilde{\gamma}(t)) \leq |s - t|\rho = d_H(\gamma(s), \gamma(t)) \leq d_H^X(\gamma(s), \gamma(t)).$$

Therefore, $d_X^X(\tilde{\gamma}(s), \tilde{\gamma}(t)) = d_{GH}^X(\gamma(s), \gamma(t))$ for $s, t \in [0, 1]$ and thus $\gamma \in \Gamma_H$. The structure of the argument above is also captured in Figure 4.

Now, we finish by proving Claim 4.11.

Proof of Claim 4.11. Since $\gamma_i \in \Gamma_H$, there exists $Y_i \in M$ such that $\gamma_i$ is $Y_i$-Hausdorff-realizable. Then, let $X_i := \mathcal{G}_{Y_i} = \cup_{t \in [0, 1]} \gamma_i(t)$ as in Lemma 4.9. Here $\gamma_i(t)$ also denotes the isometric copy of itself into $Y_i$ and thus one can view $\gamma_i(t)$ as an element of $\mathcal{H}(Y_i)$. It is obvious that $\gamma_i$ is also $X_i$-Hausdorff-realizable. Now, we prove that $\{X_i\}_{i=0}^\infty$ has a convergent subsequence via Gromov’s pre-compactness theorem (cf. Theorem 2.16). Let $\rho_i := d_{GH}(\gamma_i(0), \gamma_i(1))$ for $i = 0, 1, \ldots$.

1. Fix $i \in \mathbb{N}$ and $t \in [0, 1]$. Then,

$$d_H^X(\gamma_i(t), \gamma_i(0)) = d_{GH}(\gamma_i(t), \gamma_i(0)) = t \rho_i \leq \rho_i.$$

Therefore, for any $x_t \in \gamma_i(t)$, there exists $x_0 \in \gamma_i(0)$ such that $d_Y(x_t, x_0) \leq \rho_i$. This implies that $\gamma_i(t) \subseteq (\gamma_i(0))^{\rho_i} \subseteq Y_i$. Since $t$ is arbitrary, we have that $X_i = \mathcal{G}_{Y_i} = \cup_{t \in [0, 1]} \gamma_i(t) \subseteq (\gamma_i(0))^{\rho_i}$.

Therefore, $\text{diam}(X_i) \leq 2\rho_i + \text{diam}(\gamma_i(0))$ for any $i = 0, \ldots$. Since $\{\text{diam}(\gamma_i(0))\}_{i=0}^\infty$ approaches $\text{diam}(\gamma(0))$ and $\{\rho_i\}_{i=0}^\infty$ approaches $\rho$ as $i \to \infty$, there exists $\delta > 0$ such that $\rho_i \leq \delta$ and $\text{diam}(\gamma_i(0)) \leq \delta$ for all $i = 0, \ldots$. Therefore, $\{X_i\}_{i=0}^\infty$ is uniformly bounded by $3\delta$. 

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2. For any $\varepsilon > 0$, pick $0 = t_0 < t_1 < \ldots < t_N = 1$ such that $t_{n+1} - t_n < \frac{\varepsilon}{2\delta}$ for $n = 0, \ldots, N - 1$. Let $S_n := \{s_n(k) : k = 0, \ldots, k_n\}$ be an $\varepsilon/2$-net of $\gamma(t_n)$ for $n = 0, \ldots, N$. Let $M > 0$ be a positive integer such that $d_\infty(\gamma_i, \gamma) \leq \frac{\varepsilon}{2}$ for all $i > M$. For $n \in \{0, \ldots, N\}$ and $i > M$, let $\gamma_i \in \gamma_i(t_n)$ be an optimal correspondence. Then,

$$\text{dis}(R^i_n) = 2d_{\mathcal{H}}(\gamma_i(t_n), \gamma(t_n)) \leq 2d_\infty(\gamma_i, \gamma) \leq \frac{\varepsilon}{4}.$$ 

For each $s_n(k) \in S_n$, choose $s^i_n(k) \in \gamma_i(t_n)$ such that $(s^i_n(k), s_n(k)) \in R^i_n$. Then, we have that $S^i_n := \{s^i_n(k)\}_{k=0}^{k_n}$ is an $\varepsilon/2$-net of $\gamma_i(t_n)$. Indeed, for any $x^i_n \in \gamma_i(t_n)$, there exists $x_n \in \gamma(t_n)$ such that $(x^i_n, x_n) \in R^i_n$. Let $s_n(k) \in S_n$ be such that $d_{\gamma(t_n)}(x_n, s_n(k)) \leq \frac{\varepsilon}{4}$. Then,

$$d_{X_i}(x^i_n, s^i_n(k)) = d_{\gamma(t_n)}(x^i_n, s^i_n(k)) \leq \text{dis}(R^i_n) + d_{\gamma(t_n)}(x_n, s_n(k)) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$ 

Furthermore, we prove that $\cup_{n=0}^{N} S^i_n$ is an $\varepsilon$-net of $X_i$. For each $t \in [0, 1]$, suppose $t \in [t_n, t_{n+1}]$ for some $n \in \{0, \ldots, N - 1\}$. For any $x^i_n \in \gamma_i(t)$, since

$$d_{\mathcal{H}}(\gamma_i(t), \gamma_i(t_n)) \leq |t - t_n| \rho_t \leq \frac{\varepsilon}{2\delta} \cdot \delta = \frac{\varepsilon}{2},$$

there exists $x^i_n \in \gamma_i(t_n)$ such that $d_{X_i}(x^i_n, x^i_i) \leq \frac{\varepsilon}{2}$. Then, there exists $s^i_n(k) \in S^i_n$ such that $d_{X_i}(x^i_n, s^i_n(k)) \leq \frac{\varepsilon}{4}$ and thus

$$d_{X_i}(x^i_n, s^i_n(k)) \leq d_{X_i}(x^i_n, x^i_n) + d_{X_i}(x^i_n, s^i_n(k)) \leq \varepsilon.$$

Now, note that $|\cup_{n=0}^{N} S^i_n| \leq \sum_{n=0}^{N} k_n$ for each $i > M$. Let

$$Q(\varepsilon) := \max \left( \max \{\text{cov}_\varepsilon(X_i) : i = 1, \ldots, M\}, \sum_{n=0}^{N} k_n \right),$$

then we have $\text{cov}_{\varepsilon}(X_i) \leq Q(\varepsilon)$ for all $i = 0, \ldots$. Therefore, $\{X_i\}_{i=0}^{\infty} \subseteq K(\mathcal{K}, 3\delta)$ (cf. Definition 2.15). By Gromov’s pre-compactness theorem (cf. Theorem 2.16), $\{X_i\}_{i=0}^{\infty}$ has a convergent subsequence. \hfill \Box

### 4.3 Wasserstein-realizable geodesics

We first specify the definition of Wasserstein-realizable geodesics as follows.

**Definition 4.12** (Wasserstein-realizable geodesic). For $p \in [1, \infty)$, an $\ell^p$-Gromov-Wasserstein geodesic $\gamma : [0, 1] \to \left( \mathcal{M}^w, d_{\mathcal{W},p}^\gamma \right)$ where $\gamma(t) := (X_t, d_t, \mu_t)$ for $t \in [0, 1]$ is called $\ell^p$-Wasserstein-realizable (or simply Wasserstein-realizable), if there exist $X \in \mathcal{M}$ and for each $t \in [0, 1]$ isometric embedding $\varphi_t : X_t \to X$ such that

$$d_{\mathcal{W},p}^\gamma \left( (\varphi_s)_\# \mu_s, (\varphi_t)_\# \mu_t \right) = d_{\mathcal{W},p}^\gamma \left( \gamma(s), \gamma(t) \right), \quad \forall s, t \in [0, 1].$$

In this case, we say that $\gamma$ is $X$-Wasserstein-realizable.
Figure 4: Illustration of the proof of Theorem 1. In this figure, we identify $X$ with $\varphi(X)$ and $\mathcal{H}(X)$ with $\varphi_*(\mathcal{H}(X))$ and similarly for $X_i$ and $\mathcal{H}(X_i)$. The figure illustrates our main strategy for proving Theorem 1 as follows: we transform the $d_H$ convergent sequence $\{X_i\}_{i=0}^\infty$ to a $d_{\mathcal{H}}(Z)$ convergent sequence $\{\mathcal{H}(X_i)\}_{i=0}^\infty$; then we use the generalized Arzelà-Ascoli theorem to establish a limit $\tilde{\gamma} : [0, 1] \to \mathcal{H}(X)$ for the Lipschitz curves $\{\gamma_i : [0, 1] \to \mathcal{H}(X_i)\}_{i=0}^\infty$; finally, we show that $\tilde{\gamma}$ coincides with $\gamma$.

Remark 4.13 (Wasserstein-realizable geodesics are Wasserstein geodesics). Suppose that an $\ell^p$-Gromov-Wasserstein geodesic $\gamma : [0, 1] \to (\mathcal{M}^w, d_{GW,p}^w)$ is $X$-Wasserstein-realizable via the following family of isometric embeddings

$$\{\varphi_t : \gamma(t) \hookrightarrow X\}_{t \in [0, 1]}.$$  

Denote $\gamma(t) = (X_t, d_t, \mu_t)$ for each $t \in [0, 1]$. Then, obviously $t \mapsto (\varphi_t)_{#} \mu_t$ for $t \in [0, 1]$ is a geodesic in the Wasserstein hyperspace $\mathcal{W}_p(X)$ of $X$. This is the converse of Lemma 3.19. In words, a Wasserstein-realizable Gromov-Wasserstein geodesic is a Wasserstein geodesic.

Example 4.14 (Trivial Gromov-Wasserstein geodesics are Wasserstein-realizable). Let $\gamma : [0, 1] \to \mathcal{M}^w$ be a “trivial” Gromov-Wasserstein geodesic, i.e., there exists $\mathcal{X} = (X, d_X, \mu_X) \in \mathcal{M}^w$ such that $\gamma(t) \equiv_w \mathcal{X}$ for all $t \in [0, 1]$. Then, it is obvious that $\gamma$ is $X$-Wasserstein-realizable.

Now, we recall some notation. Let $\Gamma^p$ be the collection of all $\ell^p$-Gromov-Wasserstein geodesics. Let $d_{\infty, p}$ be the uniform metric on $\Gamma^p$, i.e., for any $\gamma_1, \gamma_2 \in \Gamma^p$

$$d_{\infty, p}(\gamma_1, \gamma_2) := \sup_{t \in [0, 1]} d_{GW,p}^w(\gamma_1(t), \gamma_2(t)).$$

Let $\Gamma^p_W$ denote the subset of $\Gamma^p$ consisting of all Wasserstein-realizable geodesics in $\mathcal{M}^w$.

Proposition 4.15. For $p \in [1, \infty)$, $\Gamma^p_W$ is a dense subset of $\Gamma^p$.

Proof. Fix any $\ell^p$-Gromov-Wasserstein geodesic $\gamma : [0, 1] \to \mathcal{M}^w$. Let $\rho := d_{GW,p}^w(\gamma(0), \gamma(1))$ and assume that $\rho > 0$. For any $\varepsilon > 0$, let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a sequence such that $t_{i+1} - t_i < \frac{\varepsilon}{2^p}$
for \( i = 0, \ldots, n - 1 \). Then, by Corollary 3.30, there exist \( X \in \mathcal{M} \) and isometric embeddings \( \varphi_i : X_{t_i} \to X \) such that \( d_{\mathcal{W}_p, \mathcal{W}_p}^X ((\varphi_i)_{#} \mu_{t_i}, (\varphi_j)_{#} \mu_{t_j}) = |t_i - t_j| \rho \). Let \( Z := \mathcal{W}_1 (X) \) and for each \( i = 0, \ldots, n \) still denote by \( \varphi_i \) the composition of \( \varphi_i \) : \( X_{t_i} \to X \) and the canonical embedding \( X \to \mathcal{W}_1 (X) = Z \). Then, we still have \( d_{\mathcal{W}_p, \mathcal{W}_p}^Z ((\varphi_i)_{#} \mu_{t_i}, (\varphi_j)_{#} \mu_{t_j}) = |t_i - t_j| \rho \). Since \( Z \) is compact and geodesic (cf. Theorem 3.11 and Theorem 3.16), \( \mathcal{W}_p (Z) \) is geodesic (cf. Theorem 3.15). Hence, for each \( i = 0, \ldots, n - 1 \) there exists an \( \ell^p \)-Wasserstein geodesic \( \gamma_i : [0, 1] \to \mathcal{W}_p (Z) \) such that \( \gamma_i (0) = (\varphi_i)_{#} \mu_{t_i} \) and \( \gamma_i (1) = (\varphi_{i+1})_{#} \mu_{t_{i+1}} \). Then, \( \gamma_i (1) = \gamma_{i+1} (0) \) for \( i = 0, \ldots, n - 1 \) and

\[
\begin{align*}
&d_{\mathcal{W}_p, \mathcal{W}_p}^Z (\gamma_0 (0), \gamma_{n-1} (1)) = d_{\mathcal{W}_p, \mathcal{W}_p}^Z ((\varphi_0)_{#} \mu_{t_0}, (\varphi_n)_{#} \mu_{t_n}) \\
&= \sum_{i=0}^{n-1} d_{\mathcal{W}_p, \mathcal{W}_p}^Z ((\varphi_i)_{#} \mu_{t_i}, (\varphi_{i+1})_{#} \mu_{t_{i+1}}) \\
&= \sum_{i=0}^{n-1} d_{\mathcal{W}_p, \mathcal{W}_p}^Z (\gamma_i (0), \gamma_i (1)).
\end{align*}
\]

Therefore, we can concatenate all the \( \gamma_i \)'s via Proposition 2.6 to obtain a new \( \ell^p \)-Wasserstein geodesic \( \tilde{\gamma} : [0, 1] \to \mathcal{W}_p (Z) \) such that \( \tilde{\gamma} (t_i) = (\varphi_i)_{#} \mu_{t_i} \) for each \( i = 0, \ldots, n \). Then, by Lemma 3.19, \( \tilde{\gamma} \) is a Gromov-Wasserstein geodesic and thus \( \tilde{\gamma} \in \Gamma^p_{\mathcal{W}} \). Now, for any \( t \in [0, 1] \), suppose \( t \in [t_i, t_{i+1}] \) for some \( i \in \{0, \ldots, n - 1\} \). Then,

\[
\begin{align*}
d_{\mathcal{W}_p, \mathcal{W}_p}^Z (\gamma (t), \tilde{\gamma} (t)) &\leq d_{\mathcal{W}_p, \mathcal{W}_p}^Z (\gamma (t_i), \tilde{\gamma} (t_i)) + d_{\mathcal{W}_p, \mathcal{W}_p}^Z (\tilde{\gamma} (t_i), \tilde{\gamma} (t)) \\
&= |t - t_i| \rho + |t - t_i| \rho \leq 2 \cdot \frac{\varepsilon}{2 \rho} = \varepsilon.
\end{align*}
\]

So, \( d_{\infty, \ell^p} (\gamma, \tilde{\gamma}) \leq \varepsilon \). Therefore, we conclude that \( \Gamma^p_{\mathcal{W}} \) is dense in \( \Gamma^p \).

**Hausdorff-boundedness.** Now, we introduce the Hausdorff-boundedness condition mentioned in the introduction with the purpose of identifying a certain family of Wasserstein-realizable geodesics.

First recall from Section 2.1 that for any function \( f : I \to J \) where \( I \) and \( J \) are intervals in \( \mathbb{R} := [0, \infty] \) containing 0, we say that \( f \) is proper if both \( f (0) = 0 \) and \( f \) is continuous at 0 (cf. Definition 2.1).

**Definition 4.16** (Hausdorff-bounded families). Given \( p \in [1, \infty] \) and a family \( \mathcal{F} \) of metric measure spaces, we say that \( \mathcal{F} \) is \((\ell^p, \ell^p)\)-Hausdorff-bounded, if there exists an increasing and proper function \( f : [0, \infty) \to [0, \infty) \) such that for any \( \mathcal{X} = (X, d_X, \mu_X), \mathcal{Y} = (Y, d_Y, \mu_Y) \in \mathcal{F} \) and isometric embeddings \( \varphi_X : X \hookrightarrow Z \) and \( \varphi_Y : Y \hookrightarrow Z \),

\[
d_H^Z (\mathcal{X}, \mathcal{Y}) \leq f (d_{\mathcal{W}_p, \mathcal{W}_p}^Z ((\varphi_X)_{#} \mu_X, (\varphi_Y)_{#} \mu_Y))
\]

After specifying such an \( f \), we say that \( \mathcal{F} \) is \( f \)-Hausdorff-bounded.

The remark below provides a first glimpse into the motivation for the definition of Hausdorff-bounded families.

**Remark 4.17.** Given an \( f \)-Hausdorff-bounded family \( \mathcal{F} \), for any \( \mathcal{X}, \mathcal{Y} \in \mathcal{F} \), we have that

\[
d_H (\mathcal{X}, \mathcal{Y}) \leq f (d_{\mathcal{W}_p}^S (\mathcal{X}, \mathcal{Y}))
\]

The definition of Hausdorff-boundedness is not superfluous. For example, the whole class \( \mathcal{M}^w \) is not a Hausdorff-bounded family for any increasing and proper function \( f \) (see also Example 4.29).
The Hausdorff-boundedness property is not easy to verify directly. We therefore seek conditions which imply it. To this end, we introduce the notion of \textit{h-boundedness} for metric measure spaces which will turn out to imply Hausdorff-boundedness (cf. Proposition 4.23).

**Note:** given a metric space \(X\) and \(\varepsilon \geq 0\), we will henceforth use the symbol \(B_\varepsilon^X(x)\) to denote the closed ball \(B_\varepsilon^X(x) := \{x' \in X : d_X(x, x') \leq \varepsilon\}\) centered at \(x\) with radius \(\varepsilon\). We abbreviate \(B_\varepsilon^X(x)\) to \(B_\varepsilon(x)\) whenever the underlying space \(X\) is clear from the context.

**Definition 4.18 (h-bounded metric measure spaces).** Let \(h : [0, \infty) \to [0, 1]\) be a strictly increasing and proper function. For any given \(X = (X, d_X, \mu_X) \in \mathcal{M}^w\), we say \(X\) is \(h\)-bounded, if for any \(x \in X\) and \(\varepsilon \geq 0\), \(\mu_X(B_\varepsilon^X(x)) \geq h(\varepsilon)\).

**Remark 4.19** (Influence of diameter on \(h\)-boundedness). Since \(\mu_X(B_0(x)) \geq 0 = h(0)\) and \(\mu_X(B_D(x)) = 1 \geq h(D)\) for any \(D \geq \text{diam}(X)\) always hold, in the above definition one can restrict \(\varepsilon\) to the interval \((0, \text{diam}(X))\).

Below we show that the doubling condition essentially implies \(h\)-boundedness.

**Example 4.20** (Examples of \(h\)-bounded metric measure spaces). In this example, we present some common types of \(h\)-bounded metric measure spaces together with explicit constructions of \(h\). Fix a metric measure space \(X = (X, d_X, \mu_X)\) with diam \((X) \leq D\).

- We say \(X\) is \(C\)-doubling for a constant \(C > 1\) if for any \(x \in X\) and \(\varepsilon \geq 0\), we have
  \[
  \mu_X(B_{2\varepsilon}(x)) \leq C \cdot \mu_X(B_\varepsilon(x)).
  \]

  Then, for any \(x \in X\) and \(0 \leq \varepsilon \leq D\),
  \[
  \mu_X(B_\varepsilon(x)) \geq C^{-1} \mu_X(B_{2\varepsilon}(x)) \geq \ldots \geq C^{-\log_2\left(\frac{D}{\varepsilon}\right)-1} \mu_X\left(B_{2^\log_2\left(\frac{D}{\varepsilon}\right)+1}(x)\right) = C^{-\log_2\left(\frac{D}{\varepsilon}\right)-1}.
  \]

  The function \(C^{-\log_2\left(\frac{D}{\varepsilon}\right)-1} : [0, D] \to [0, C^{-1}]\) is strictly increasing and proper. Since \(C > 1\), \(C^{-\log_2\left(\frac{D}{\varepsilon}\right)-1}\) can be extended to a strictly increasing and proper function \(h_X : [0, \infty) \to [0, 1]\).

  Then, \(X\) is \(h_X\)-bounded.

- Suppose \(X\) is a finite set and let \(\delta_X := \min\{\mu_X(x) : x \in X\} > 0\). Let \(h_X : [0, \infty) \to [0, 1]\) be any strictly increasing and proper function such that \(h_X(\varepsilon) \leq \delta_X\) for all \(\varepsilon \in [0, D]\). Then, \(X\) is \(h_X\)-bounded.

Given a compact metric measure space \(X\), if we define \(h_X^{\inf} : [0, \infty) \to [0, 1]\) by \(\varepsilon \mapsto \inf_{x \in X} \mu_X(B_\varepsilon(x))\) for each \(\varepsilon \in [0, \infty)\), then obviously we have for any \(x \in X\) and \(\varepsilon \geq 0\) that \(\mu_X(B_\varepsilon(x)) \geq h_X^{\inf}(\varepsilon)\). We of course cannot conclude directly that \(X\) is \(h_X^{\inf}\)-bounded since \(h_X^{\inf}\) is not necessarily strictly increasing and proper. However, it turns out that a slightly modification of the construction of \(h_X^{\inf}\) gives rise to the following result:

**Lemma 4.21.** For any \(X \in \mathcal{M}^w\), there exists a strictly increasing and proper function \(h_X : [0, \infty) \to [0, 1]\) such that \(X\) is \(h_X\)-bounded.

**Proof.** Without loss of generality, we assume that diam \((X) = 1\). For each \(n \in \mathbb{N}\), let \(X_n\) be a finite \(\frac{1}{2n}\)-net of \(X\) such that \(X_1 \subseteq X_2 \subseteq \ldots\). Let \(\eta_n := \inf_{x_n \in X_n} \mu_X\left(B_{\frac{1}{2n}}(x_n)\right)\). Since \(X_n\) is finite, the infimum is obtained and \(\eta_n > 0\) for \(n = 1, \ldots\). Obviously, \(1 \geq \eta_1 \geq \eta_2 \geq \ldots > 0\). Then, we choose any strictly decreasing positive sequence \(1 \geq \zeta_1 > \zeta_2 > \ldots\) such that \(\lim_{n \to \infty} \zeta_n = 0\) and \(\zeta_n \leq \eta_n\). Such a sequence obviously exists.
Define a function \( f : \{ \frac{1}{n} : n = 1, 2 \ldots \} \to \mathbb{R} \) by mapping \( \frac{1}{n} \) to \( \zeta_{n+1} \) for \( n = 1, \ldots \). We extend \( f \) to a new function \( g : [0, 1] \to [0, \zeta_2] \) by linearly interpolating \( f \) inside the intervals \( \left[ \frac{1}{n+1}, \frac{1}{n} \right] \) for \( n \in \mathbb{N} \) and by letting \( g(0) := 0 \). Now, let \( \tilde{g} : [1, \infty) \to [\zeta_2, 1] \) be any strictly increasing function. Then, we extend \( g \) to \( h_X : [0, \infty) \to [0, 1] \) as follows:

\[
 h_X(\varepsilon) = \begin{cases} 
 g(\varepsilon), & \varepsilon \in [0, 1] \\
 \tilde{g}(\varepsilon), & \varepsilon \in (1, \infty) 
\end{cases}.
\]

Then, it is easy to check that \( h_X \) is strictly increasing and proper. Moreover,

\[
 h_X(\varepsilon) \leq \zeta_{n+1} \quad \forall n \in \mathbb{N}, \forall \varepsilon \in \left( \frac{1}{n+1}, \frac{1}{n} \right].
\]

Now, for any \( x \in X \), there exists \( x_n \in X_n \) such that \( d_X(x, x_n) \leq \frac{1}{n} \). Then, \( B_{\frac{1}{n}}(x_n) \subseteq B_{\frac{1}{n}}(x) \). Thus, \( \mu_X \left( B_{\frac{1}{n}}(x) \right) \geq \mu_X \left( B_{\frac{1}{n}}(x_n) \right) \geq \zeta_n \). Since for any \( \varepsilon \in (0, 1] \), there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n+1} < \varepsilon \leq \frac{1}{n} \), we obtain

\[
 \mu_X (B_{\varepsilon}(x)) \geq \mu_X \left( B_{\frac{1}{n+1}}(x) \right) \geq \zeta_{n+1} \geq h_X(\varepsilon).
\]

Therefore, \( X \) is \( h_X \)-bounded. \( \square \)

We say that a family \( \mathcal{F} \subseteq \mathcal{M}^w \) of compact metric measure spaces is \textit{uniformly \( h \)-bounded} for some strictly increasing and proper \( h : [0, \infty) \to [0, 1] \) if every \( X \in \mathcal{F} \) is \( h \)-bounded.

Since each \( X \in \mathcal{M}^w \) is \( h_X \)-bounded for some strictly increasing and proper \( h_X \), the one-element family \( \mathcal{F} := \{ X \} \) is obviously uniformly \( h_X \)-bounded. Moreover, any finite family \( \mathcal{F} \) is uniformly \( h \)-bounded where \( h := \min_{X \in \mathcal{F}} h_X \). However, for an infinite family \( \mathcal{F} \), it may not be true that one can find a uniform strictly increasing and proper \( h \) for \( \mathcal{F} \); see the example below:

**Example 4.22** (An example of non-uniformly \( h \)-bounded family). For \( n \in \mathbb{N} \), denote by \( \Delta_n \) the \( n \)-point space with interpoint distance 1. Endow \( \Delta_n \) with uniform probability measure (denoted by \( \mu_n \)) and denote the corresponding metric measure space by \( \Delta_n = (\Delta_n, d_n, \mu_n) \). Let \( \mathcal{F} = \{ \Delta_n : n \in \mathbb{N} \} \). Then, there is no strictly increasing and proper function \( h \) such that \( \mathcal{F} \) is uniformly \( h \)-bounded. Indeed, we otherwise assume that \( \mathcal{F} \) is uniformly \( h \)-bounded for some strictly increasing and proper function \( h \). Then, for each \( \Delta_n \), we pick \( x_n \in \Delta_n \) and \( \varepsilon = \frac{1}{2} \). Since \( \Delta_n \) is \( h \)-bounded, we have that

\[
 \mu_n \left( B_{\frac{1}{2}}(x_n) \right) = \mu_n(\{x_n\}) = \frac{1}{n} \geq h \left( \frac{1}{2} \right) > 0.
\]

Then, this implies that \( \frac{1}{n} > h \left( \frac{1}{2} \right) > 0 \) holds for all \( n \in \mathbb{N} \), which is impossible.

The following result reveals a connection between uniform \( h \)-boundedness and Hausdorff-boundedness.

**Proposition 4.23.** Fix \( p \in [1, \infty) \). Let \( h : [0, \infty) \to [0, 1] \) be a strictly increasing and proper function and let \( \mathcal{F} \) be a family of \( h \)-bounded metric measure spaces. Then, \( \mathcal{F} \) is \( h^{-1} \)-Hausdorff-bounded, where \( h^{-1} : [0, \infty) \to [0, 1] \) is defined by \( t \mapsto \frac{1}{h^{-1}} \cdot h^{-1} \left( \frac{1}{2} \right) \) for each \( t \in [0, \infty) \).

**Proof.** Let \( X, Y \in \mathcal{F} \). Suppose that \( Z \in \mathcal{M} \) and that there exist isometric embeddings \( X \hookrightarrow Z \) and \( Y \hookrightarrow Z \). For notational simplicity, we still denote by \( \mu_X \) and \( \mu_Y \) their respective pushforwards under the respective isometric embeddings. Let \( \rho := d_{\mathcal{V}, p}^Z(\mu_X, \mu_Y) \) and \( \eta := d_H^Z(X, Y) \). Assume that \( \eta > 0 \) since the case \( \eta = 0 \) is trivial. Then, by compactness of \( X \) and \( Y \), there exists \( x_0 \in X \) and \( y_0 \in Y \) such that \( d_Z(x_0, y_0) = d_H^Z(X, Y) = \eta \). Without loss of generality, we assume that

\[
 d_Z(x_0, y_0) = d_Z(x_0, Y) := \inf \{ d_Z(x_0, y) : y \in Y \}.
\]
Then, consider the closed ball $B^X_{\frac{\eta}{2}}(x_0)$ in $X$, and let $x \in B^X_{\frac{\eta}{2}}(x_0)$. By the triangle inequality, for any $y \in Y$ we have that
\[
d_Z(x, y) \geq d_Z(x_0, y) - d_Z(x_0, x) \geq d_Z(x_0, y_0) - d_X(x_0, x) \geq \eta - \frac{\eta}{2} = \frac{\eta}{2}.
\]
Let $\mu \in C^\text{opt}_p(\mu_X, \mu_Y)$ be an optimal coupling, then we have that
\[
\delta = d^Z_{W_p}(\mu_X, \mu_Y)
\]
\[
= \left( \int_{X \times Y} (d_Z(x, y))^p \, d\mu(x, y) \right)^{\frac{1}{p}}
\]
\[
\geq \left( \int_{B^X_{\frac{\eta}{2}}(x_0) \times Y} (d_Z(x, y))^p \, d\mu(x, y) \right)^{\frac{1}{p}}
\]
\[
\geq \left( \int_{B^X_{\frac{\eta}{2}}(x_0) \times Y} \left( \frac{\eta}{2} \right)^p \, d\mu(x, y) \right)^{\frac{1}{p}}
\]
\[
= \frac{\eta}{2} \cdot \left( \mu \left( B^X_{\frac{\eta}{2}}(x_0) \times Y \right) \right)^{\frac{1}{p}}
\]
\[
= \frac{\eta}{2} \cdot \left( \mu_X \left( B^X_{\frac{\eta}{2}}(x_0) \right) \right)^{\frac{1}{p}}
\]
\[
\geq \frac{\eta}{2} \cdot h^{\frac{1}{p}} \left( \frac{\eta}{2} \right)
\]
\[
= h(\eta)
\]

Obviously, the function $\hat{h} : [0, \infty) \to [0, \infty)$ defined by $t \mapsto \frac{t}{2} \cdot h^{\frac{1}{p}} \left( \frac{t}{2} \right)$ is strictly increasing and proper. By item 4 of Proposition 2.2, we have that $d^Z_{H}(X, Y) = \eta \leq \hat{h}^{-1}(\delta) = \hat{h}^{-1} \left( d^Z_{W_p}(\mu_X, \mu_Y) \right)$.

By items 1 and 5 of Proposition 2.2 we have that $\hat{h}^{-1}$ is increasing and proper. Moreover, it is easy to see that $\lim_{t \to \infty} \hat{h}(t) = \infty$. This implies that $\hat{h}^{-1}$ is a finite function $\hat{h}^{-1} : [0, \infty) \to [0, \infty)$ (cf. item 2 of Proposition 2.2). Thus, $\mathcal{F}$ is $\hat{h}^{-1}$-Hausdorff-bounded.

For an $\ell^p$-Gromov-Wasserstein geodesic $\gamma : [0, 1] \to \mathcal{M}^w$, we say that $\gamma$ is Hausdorff-bounded if the family $\{ \gamma(t) \}_{t \in [0, 1]}$ is Hausdorff-bounded. The family of Hausdorff-bounded Gromov-Wasserstein geodesics is rich and we present two examples as follows.

**Proposition 4.24** (Geodesics consisting of finite spaces). Fix $p \in [1, \infty)$. Let $\gamma$ be an $\ell^p$-Gromov-Wasserstein geodesic. Assume that there exist a positive integer $N$, a constant $C > 0$ and a constant $D > 0$ such that for each $t \in [0, 1]$,

1. the cardinality of $\gamma(t)$ is bounded above by $N$;
2. for any $x \in \gamma(t)$, $\mu_t(\{ x \}) \geq C$;
3. $\text{diam}(\gamma(t)) \leq D$.

Then, $\gamma$ is Hausdorff-bounded.
Proof. As mentioned earlier in Example 4.20, there exists a strictly increasing and proper function $h : [0, \infty) \to [0, 1]$ such that for each $t \in [0, 1]$, $\gamma(t)$ is $h$-bounded. Then, by Proposition 4.23, $\{\gamma(t)\}_{t \in [0, 1]}$ is Hausdorff-bounded.

For any two given compact metric measure spaces, Sturm constructed in [Stu20] an $\ell^p$-Gromov-Wasserstein geodesic connecting them, which we call a straight-line $d_{GW, p}^S$ geodesic:

**Theorem 4.25 (Straight-line $d_{GW, p}^S$ geodesic [Stu20]).** Fix $p \in [1, \infty)$ and let $X, Y \in \mathcal{M}^w$. Let $Z \in \mathcal{M}$ be such that there exist isometric embeddings $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ such that

$$d_{GW, p}^S(X, Y) = d_{GW, p}^Z(\mu_X, \mu_Y),$$

whose existence is guaranteed by Lemma 2.22. Let $\mu \in C_{\text{opt}}^p(\mu_X, \mu_Y)$ be any optimal coupling between $\mu_X$ and $\mu_Y$ with respect to $d_{GW, p}^Z$. Then, the curve $\gamma_\mu : [0, 1] \to \mathcal{M}^w$ defined as follows is an $\ell^p$-Gromov-Wasserstein geodesic:

$$\gamma_\mu(t) := \begin{cases} X, & t = 0 \\ (S, d_t, \mu), & t \in (0, 1) \\ Y, & t = 1 \end{cases}$$

where $S := \operatorname{supp}(\mu) \subseteq X \times Y$ and $d_t((x, y), (x', y')) := (1 - t) d_X(x, x') + t d_Y(y, y')$ for any $(x, y), (x', y') \in S$. We call $\gamma_\mu$ a straight-line $d_{GW, p}^S$ geodesic.

Now, we use Proposition 4.23 to show the following:

**Proposition 4.26.** Given $p \in [1, \infty)$, any straight-line $d_{GW, p}^S$ geodesic is Hausdorff-bounded.

Proof. Let $X, Y \in \mathcal{M}^w$ and let $Z \in \mathcal{M}$ be the ambient space required in Theorem 4.25. Let $\mu \in C_{\text{opt}}^p(\mu_X, \mu_Y)$ be an optimal coupling for $d_{GW, p}^Z$. By Lemma 4.21, there exist $h_X$ and $h_Y$ such that $X$ is $h_X$-bounded and $Y$ is $h_Y$-bounded. Now, consider $S := (S, d_S := \max(d_X, d_Y), \mu)$, where $S := \operatorname{supp}(\mu)$. $S$ is a compact metric measure space and $\mu$ is fully supported. By Lemma 4.21 again, there exists $h_S$ such that $S$ is $h_S$-bounded.

Now, pick any $z_0 = (x_0, y_0) \in S = \operatorname{supp}(\mu)$. Then, for any $t \in (0, 1)$ and for any $\varepsilon > 0$, we have that

$$B_{\varepsilon}^{\gamma_\mu(t)}(z_0) := \{(x, y) \in S : d_t((x, y), (x_0, y_0)) \leq \varepsilon\} \supseteq (B_{\varepsilon}^X(x_0) \times B_{\varepsilon}^Y(y_0)) \cap S = B_{\varepsilon}^{(S, d_S)}(z_0)$$

Therefore,

$$\mu\left(B_{\varepsilon}^{\gamma_\mu(t)}(z_0)\right) \geq \mu\left(B_{\varepsilon}^{(S, d_S)}(z_0)\right) \geq h_S(\varepsilon).$$

Let $h := \min(h_X, h_Y, h_S)$, then we have that $\gamma_\mu(t)$ is $h$-bounded for any $t \in [0, 1]$. Then, by Proposition 4.23, we have that $\gamma_\mu$ is Hausdorff-bounded.

**Remark 4.27 (Deviant and branching GW geodesics).** Analogously to the case of Gromov-Hausdorff geodesics (cf. [CM18, Section 1.1]), for each $p \in [1, \infty)$ there exist deviant geodesics in $\Gamma_{GW}^p$; that is, geodesics which are not straight-line $d_{GW, p}^S$ geodesics. Furthermore, there also exist branching geodesics. We provide such constructions in Appendix C where we also show that these exotic geodesics are Hausdorff-bounded. In analogy with the case of $(\mathcal{M}, d_{GH})$.

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6Our definition is slightly different from the one given in [Stu20]: under our notation, for each $t \in [0, 1]$, [Stu20] defines a metric measure space $\gamma_\mu(t) := (Z \times Z, d'_t, \mu)$ where $d'_t$ is defined by $d'_t((z_1, z_2), (z'_1, z'_2)) := (1 - t) d_Z(z_1, z'_1) + t d_Z(z_2, z'_2)$ for any $z_i, z'_i \in Z$ where $i = 1, 2$. Note that our geodesic $\gamma_\mu(t)$ in Theorem 4.25 can then be obtained by simply restricting $\gamma_\mu(t)$ to the support of $\mu$. 

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1. the existence of branching geodesics shows that \((\mathcal{M}^w, d^W_{\mathcal{G}, p})\) is not an Alexandrov space with curvature bounded below [BBI01, Chapter 10];

2. the existence of deviant (non-unique) geodesics shows that \((\mathcal{M}^w, d^W_{\mathcal{G}, p})\) is also not a CAT(\(\kappa\)) space with curvature bounded above by some \(\kappa \in \mathbb{R}\) [BH13, Chapter 2.1].

**Proof of Theorem 2.** We now deal with the proof of Theorem 2. The proof consists of two parts: first we carefully approximate a given Hausdorff bounded geodesic \(\gamma\) by a convergent sequence of Wasserstein realizable geodesics; then, this carefully chosen convergent sequence allows us to utilize the Hausdorff-boundedness condition on \(\gamma\) so that we can follow a strategy analogous to the one used for proving Theorem 1 to now show that this convergent sequence has a Wasserstein realizable limit, which must agree with \(\gamma\).

**Theorem 2.** Given \(p \in [1, \infty)\), every Hausdorff-bounded \(\ell^p\)-Gromov-Wasserstein geodesic is \(\ell^p\)-Wasserstein-realizable.

**Proof.** Let \(\gamma : [0, 1] \rightarrow \mathcal{M}^w\) be a Hausdorff-bounded \(\ell^p\)-Gromov-Wasserstein geodesic. For each \(t \in [0, 1]\), we write \(\gamma(t) = (X_t, d_t, \mu_t)\). Assume that \(p := d^W_{\mathcal{G}, 1}(\gamma(0), \gamma(1)) > 0\). Let \(T^n = \{t^n_i := i \cdot 2^{-n} : i = 0, \ldots, 2^n\} \) for \(n = 0, 1, \ldots\). Then, by Corollary 3.30, there exist \(Z^n \in \mathcal{M}\) and isometric embeddings \(\varphi^n_i : X^n_{t^n_i} \hookrightarrow Z^n\) for \(i = 1, \ldots, 2^n\) such that \(d^W_{\mathcal{G}, p}(\varphi^n_i(\varphi^n_i) \# \mu^n_{t^n_i}, \varphi^n_i(\varphi^n_{t^n_{i+1}}) \# \mu^n_{t^n_{i+1}}) = |t^n_i - t^n_{i+1}|\rho\).

For \(p = 1\), by Lemma 3.17, we know that \(\gamma^n_i : [0, 1] \rightarrow \mathcal{W}_1(Z^n)\) defined by

\[
t \mapsto (1 - t) (\varphi^n_i) \# \mu^n_{t^n_i} + t (\varphi^n_{i+1}) \# \mu^n_{t^n_{i+1}}
\]

is an \(\ell^1\)-Wasserstein geodesic for each \(n = 0, \ldots\) and each \(i = 0, \ldots, 2^n - 1\). Then,

\[
d^W_{\mathcal{G}, 1}\left(\gamma^n_0(0), \gamma^n_{2^n-1}(1)\right) = d^W_{\mathcal{G}, 1}\left(\varphi^n_0(0) \# \mu^n_0, \varphi^n_{2^n-1}(1) \# \mu^n_{2^n}\right)
\]

\[
= \sum_{i=0}^{2^n-1} d^W_{\mathcal{G}, 1}\left(\varphi^n_i(\varphi^n_i) \# \mu^n_{t^n_i}, \varphi^n_i(\varphi^n_{t^n_{i+1}}) \# \mu^n_{t^n_{i+1}}\right)
\]

\[
= \sum_{i=0}^{2^n-1} d^W_{\mathcal{G}, 1}(\gamma^n_i(0), \gamma^n_i(1)).
\]

Therefore, we can concatenate all the \(\gamma^n_i\)s for \(i = 0, \ldots, 2^n - 1\) via Proposition 2.6 to obtain an \(\ell^1\)-Wasserstein geodesic \(\gamma^n : [0, 1] \rightarrow \mathcal{W}_1(Z^n)\) such that \(\gamma^n(t^n_i) = (\varphi^n_i) \# \mu^n_{t^n_i}\) for \(i = 0, \ldots, 2^n\). Since \(d^W_{\mathcal{G}, 1}(\gamma^n(0), \gamma^n(1)) = p = d^W_{\mathcal{G}, 1}(\gamma(0), \gamma(1))\), by Lemma 3.19 we have that \(\gamma^n\) is actually an \(\ell^1\)-Gromov-Wasserstein geodesic. We follow the notation from Lemma 3.19 and denote by \(\tilde{\gamma}^n : [0, 1] \rightarrow \mathcal{M}^w\) the \(\ell^1\)-Gromov-Wasserstein geodesic corresponding to \(\gamma^n\). Then, it is easy to check that \(d_{\infty, 1}(\gamma, \tilde{\gamma}^n) \leq 2 \cdot \frac{1}{2^n} \cdot p = 2^{1-n}p\) via an argument similar to the one used in the proof of Proposition 4.15. For each \(n = 0, \ldots, i = 0, \ldots, 2^n\) and \(t \in [0, 1]\), we have \(\text{supp}(\gamma^n(t)) \subseteq \varphi^n_i(X^n_{t^n_i}) \cup \varphi^n_{i+1}(X^n_{t^n_{i+1}})\). Therefore, \(\cup_{t \in [0, 1]} \text{supp}(\gamma^n(t)) = \cup_{i=0}^{2^n} \varphi^n_i(X^n_{t^n_i}) = Y^n\), and thus \(\gamma^n\) is actually a geodesic in \(\mathcal{W}_1(Y^n) \subseteq \mathcal{W}_1(Z^n)\).

For \(p > 1\), since \(W^n := \mathcal{W}_p(Y^n)\) is geodesic (cf. Theorem 3.16), we have that \(\mathcal{W}_p(W^n)\) is geodesic (cf. Theorem 3.15). We still denote \(\varphi^n_i\) the composition of \(\varphi^n_i : X^n_{t^n_i} \rightarrow Z^n(\text{or } Y^n)\) and the canonical embedding \(Y^n \hookrightarrow W^n\). Then, there exists a geodesic \(\gamma^n_i\) connecting \((\varphi^n_i) \# \mu^n_{t^n_i}\) and \((\varphi^n_{i+1}) \# \mu^n_{t^n_{i+1}}\). Similarly to the case when \(p = 1\), we concatenate \(\gamma^n_i\)s to obtain an \(\ell^p\)-Wasserstein geodesic \(\tilde{\gamma}^n : [0, 1] \rightarrow \mathcal{W}_p(W^n)\) and thus an \(\ell^p\)-Gromov-Wasserstein geodesic \(\tilde{\gamma}^n\) such that \(d_{\infty, p}(\gamma, \tilde{\gamma}^n) \leq 2 \cdot \frac{1}{2^n} \cdot p = 2^{1-n}p\). Therefore, for each \(t \in [0, 1]\), \(\gamma(t)\) is the \(\ell^p\)-Gromov-Wasserstein limit of \(\{\tilde{\gamma}^n(t)\}_{n=0}^\infty\).
Claim 4.28. There exists a $d_{GH}$-convergent subsequence of $\{Y^n\}_{n=0}^\infty$.

Assume the claim for now and consider first the case when $p = 1$. Suppose $X \in \mathcal{M}$ is such that $\lim_{n \to \infty} d_{GH}(Y^n, X) = 0$ (after possibly passing to a subsequence), then by Lemma 4.1, there exist a Polish metric space $Z$ and isometric embeddings $\varphi : X \leftrightarrow Z$ and $\varphi^n : Y^n \leftrightarrow Z$ for $n = 0, \ldots$ such that $\lim_{n \to \infty} d_{GH}^{W_1}(\varphi^n(Y^n), \varphi(X)) = 0$. By Theorem 3.12, 
\[
\lim_{i \to \infty} d_{GH}^{W_1}(\{(\varphi^n)_#(\mathcal{W}_1(Y^n)), \varphi_#(\mathcal{W}_1(X))\}) = 0.
\]
For each $n = 0, \ldots$, we have that $\gamma^n : [0,1] \to \mathcal{W}_1(Y^n)$ is $\rho$-Lipschitz. Then, since $(\varphi^n)_#$ is an isometric embedding, $(\varphi^n)_# \circ \gamma^n : [0,1] \to (\varphi^n)_#(\mathcal{W}_1(Y^n))$ is also $\rho$-Lipschitz. Moreover, by Theorem 3.10, $\mathcal{W}_1(Z)$ is complete. Then, by the generalized Arzelà-Ascoli theorem (Theorem 4.2), the sequence 
\[
\{ (\varphi^n)_# \circ \gamma^n : [0,1] \to (\varphi^n)_#(\mathcal{W}_1(Y^n)) \}_{n=0}^\infty
\]
uniformly converges to a $\rho$-Lipschitz curve $\tilde{\gamma} : [0,1] \to (\varphi)_#(\mathcal{W}_1(X))$ in the space $\mathcal{W}_1(Z)$.

By uniform convergence, for each $t \in [0,1]$ we have that 
\[
\lim_{n \to \infty} d_{W_1,1}^{Z}(\tilde{\gamma}(t), (\varphi^n)_# \circ \gamma^n(t)) = 0.
\]
Let $\tilde{X}_t := \text{supp}(\tilde{\gamma}(t))$ and let $\check{\gamma}(t) := \left(\tilde{X}_t, d_Z|_{\tilde{X}_t \times \tilde{X}_t}, \tilde{\gamma}(t)\right)$. Then, we have that 
\[
0 \leq \lim_{n \to \infty} d_{W_1,1}^{S}(\tilde{\gamma}(t), \check{\gamma}(t)) \leq \lim_{n \to \infty} d_{W_1,1}^{Z}(\tilde{\gamma}(t), (\varphi^n)_# \circ \gamma^n(t)) = 0.
\]
We know that $\gamma(t)$ is the Gromov-Wasserstein limit of $\{\gamma^n(t)\}_{n=0}^\infty$, thus $\gamma(t) \approx_{w} \gamma(t)$ for $t \in [0,1]$. Then, since $\gamma$ is $X$-Wasserstein-realizable, we have that $\gamma \in \Gamma^1_W$.

When $p > 1$, by stability of $\mathcal{W}_1$ (cf. Theorem 3.13), $\{W^n = \mathcal{W}_1(Y^n)\}_{n=0}^\infty$ also has a convergent subsequence. Then, via an argument similar to the one used in the case of $p = 1$, one concludes that $\gamma \in \Gamma^1_W$.

Now, we finish by proving Claim 4.28:

Proof of Claim 4.28. We assume that $\gamma$ is $f$-Hausdorff-bounded for an increasing and proper function $f : [0, \infty) \to [0, \infty)$, i.e., $f(0) = 0$ and $f$ is continuous at 0. We prove the claim by suitably applying Gromov’s pre-compactness theorem (cf. Theorem 2.16).

1. For any $t \in [0,1]$, by Remark 4.17 we have that $d_{GH}(X_0, X_t) \leq f\left(d_{GH}^{W_1}(\gamma(0), \gamma(t))\right) \leq f(\rho)$. Since $d_{GH}(X_0, X_t) \geq \frac{1}{2}|\text{diam}(X_0) - \text{diam}(X_t)|$ (cf. [Mém12, Theorem 3.4]), we have that 
\[
\text{diam}(X_t) \leq 2f(\rho) + \text{diam}(X_0).
\]

Now, fix $n \in \mathbb{N}$ and $x_0 \in \varphi_0^n(X_{t_0^n}) \subseteq Y^n$. Then, for any $t_i^n \in T^n$, there exist $x_0' \in \varphi_0^n(X_{t_0^n})$ and $x_i^n \in \varphi_i^n(X_{t_i^n})$ such that $d_Y^n(x_0', x_i^n) = d_Y^n(\varphi_0^n(X_{t_0^n}), \varphi_i^n(X_{t_i^n}))$ by compactness. Obviously, we have that 
\[
d_Y^n(x_0', x_i^n) \leq d_{W_1,1}^{S}((\varphi_0^n)_#\mu_{t_0^n}, (\varphi_i^n)_#\mu_{t_i^n}) = |t_0^n - t_i^n| \rho \leq \rho.
\]

Now, for any point $x_i \in \varphi_i^n(X_{t_i^n})$, we have that 
\[
d_Y^n(x_0, x_i) \leq d_Y^n(x_0, x_0') + d_Y^n(x_0', x_i') + d_Y^n(x_i', x_i) \leq \text{diam} (\varphi_0^n(X_{t_0^n})) + \rho + \text{diam} (\varphi_i^n(X_{t_i^n})) = \text{diam} (X_{t_0^n}) + \rho + \text{diam} (X_{t_i^n}) \leq \rho + 2f(\rho) + 2\text{diam}(X_0).
\]
The inequality holds for any $i = 0, \ldots, 2^n$. So $\text{diam}(Y^n) \leq 2(\rho + 2f(\rho) + 2\text{diam}(X_0))$ for all $n \in \mathbb{N}$. 

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2. For any $\varepsilon > 0$, there exists $M > 0$ such that $t_{j+1}^M - t_j^M < f^{-1}(\varepsilon) \cdot \rho^{-1}$. Here $f^{-1}(\varepsilon) > 0$ follows from item 5 of Proposition 2.2. Let $S_j^M$ be an $\varepsilon$-net of $\gamma(t_j^M)$ for all $j = 0, \ldots, 2^M$. For any $n > M$, $\{t_j^M\}_{j=0}^{2^n}$ is a subsequence of $\{t_n^i\}_{i=0}^n$. Indeed, $t_j^M = t_j^{2n-M}$ for $j = 0, \ldots, 2^M$. For any $t_n^j$, there exists $j$ such that $t_j^M < t_{j+1}^M$. We know by construction of $Y^n$ that

$$d_{\mathcal{W},1}^n \left( \left( \varphi_j^{2n-M} \right)_\#, \mu_j^M, \left( \varphi_i^n \right)_\# \mu_i^n \right) = |t_i^n - t_j^M| \rho < |t_{j+1}^M - t_j^M| \rho < f^{-1}(\varepsilon).$$

Therefore,

$$d_{\mathcal{W},1}^n \left( \varphi_j^{2n-M} \left( X_t^M \right), \varphi_i^n \left( X_t^n \right) \right) \leq f \left( d_{\mathcal{W},1}^n \left( \left( \varphi_j^{2n-M} \right)_\#, \mu_j^M, \left( \varphi_i^n \right)_\# \mu_i^n \right) \right) \leq \frac{\varepsilon}{2},$$

where we used item 3 of Proposition 2.2 in the second inequality.

Therefore, for each point $x_i^n \in \varphi_i^n \left( X_t^n \right)$, there exists a point $x_j^M \in \varphi_j^{2n-M} \left( X_t^M \right)$ such that $d_Y^n \left( x_i^n, x_j^M \right) \leq \frac{\varepsilon}{2}$. Note that $\varphi_j^{n-2n-M} \left( S_j^M \right)$ is an $\frac{\varepsilon}{2}$-net of $\varphi_j^{2n-M} \left( X_t^M \right)$. Then, there exists a point $x_j^M \in \varphi_j^{n-2n-M} \left( S_j^M \right)$ such that $d_Y^n \left( x_j^M, x_j^M \right) \leq \frac{\varepsilon}{2}$. So, $d_Y^n \left( x_i^n, x_j^M \right) \leq \varepsilon$. Therefore, we have that $Y^n \subseteq \bigcup_{j=0}^{2^M} \left( \varphi_j^{2n-M} \left( S_j^M \right) \right)$. Let

$$Q(\varepsilon) := \max \left( \max \left\{ \text{cov}_\varepsilon \left( Y^n \right) : n = 1, \ldots, M - 1 \right\}, \sum_{j=0}^{2^M} |S_j^M| \right),$$

then we have that $\text{cov}_\varepsilon \left( Y^n \right) \leq Q(\varepsilon)$ for all $n = 0, \ldots$.

Therefore, $\{Y^n\}_{n=0}^{\infty} \subseteq \mathcal{K}(Q, 2(\rho + 2f(\rho) + 2\text{diam}(X_0)))$ (cf. Definition 2.15). By Gromov’s pre-compactness theorem (cf. Theorem 2.16), $\{Y^n\}_{n=0}^{\infty}$ has a convergent subsequence.

There exist examples of non-Hausdorff-bounded Gromov-Wasserstein geodesics.

**Example 4.29** (An example of non-Hausdorff-bounded Gromov-Wasserstein geodesic). Let $\mathcal{X} = (X, d_X, \mu_X)$ be the one point metric space measure and $\mathcal{Y} = (Y, d_Y, \mu_Y)$ be a two-point metric measure space with unit distance and uniform probability measure. Assume that $X = \{x\}$ and $Y = \{y_1, y_2\}$. Then, under the map $\varphi_X : X \to Y$ defined by $x \mapsto y_1$ and the identity map $\varphi_Y := \text{Id} : Y \to Y$, it is easy to check that

$$d_{\mathcal{G},1}^\mathcal{W} \left( \mathcal{X}, \mathcal{Y} \right) = d_{\mathcal{W},1}^\mathcal{Y} \left( \left( \varphi_X \right)_\# \mu_X, \left( \varphi_Y \right)_\# \mu_Y \right).$$

Define $\gamma : [0, 1] \to \mathcal{W}_1(Y)$ by mapping each $t \in [0, 1]$ to $(1 - t) \left( \varphi_X \right)_\# \mu_X + t \left( \varphi_Y \right)_\# \mu_Y$. It is easy to describe $\gamma(t)$ explicitly as follows: for each $t \in (0, 1)$, $\gamma(t)(\{y_1\}) = 1 - \frac{t}{2}$ and $\gamma(t)(\{y_2\}) = \frac{t}{2}$. By Lemma 3.17, $\gamma$ is an $\ell^1$-Wasserstein geodesic. Thus $\gamma$ corresponds to an $\ell^1$-Wasserstein-realizable Gromov-Wasserstein geodesic $\tilde{\gamma}$ connecting $X$ and $Y$ (cf. Lemma 3.19).

Note that $d_{\mathcal{G},1}^\mathcal{X} \left( \mathcal{X}, \tilde{\gamma} \left( t \right) \right) = d_{\mathcal{G},1}^\mathcal{Y} \left( \mathcal{X}, \mathcal{Y} \right) = \frac{1}{2}$ for all $t \in (0, 1]$. However, $\lim_{t \to 0} d_{\mathcal{G},1}^\mathcal{W} \left( \mathcal{X}, \tilde{\gamma} (t) \right) = 0$ which precludes the existence of any increasing and proper function $f : [0, \infty) \to [0, \infty)$ such that

$$\frac{1}{2} = d_{\mathcal{G},1}^\mathcal{X} \left( \mathcal{X}, \tilde{\gamma} (t) \right) \leq f \left( d_{\mathcal{G},1}^\mathcal{W} \left( \mathcal{X}, \tilde{\gamma} (t) \right) \right).$$

Therefore, the geodesic $\tilde{\gamma}$ is not Hausdorff-bounded.
Note that the example constructed above is highly dependent on the special linear interpolation geodesic corresponding to \( d_{W_p} \) (cf. Lemma 3.17). We are not aware of any example of an \( \ell^p \)-Gromov-Wasserstein geodesic for \( p > 1 \) which is not Hausdorff-bounded.

**Conjecture 1.** For every \( p \in (1, \infty) \), any \( \ell^p \)-Gromov-Wasserstein geodesic is Hausdorff-bounded. Also, for every \( p \in [1, \infty) \), any \( \ell^p \)-Gromov-Wasserstein geodesic is Wasserstein-realizable.

## 5 Dynamic geodesics

In this section, we first carefully study the properties of the Hausdorff displacement interpolation and prove Theorem 3, then extend our results to study dynamic Gromov-Hausdorff geodesics and prove Theorem 4.

### 5.1 Displacement interpolation

**Geodesics in \( \mathcal{W}_p (X) \) for \( p \in (1, \infty) \).** Given a metric space \( X \), it is known that if \( X \) is a geodesic space, then \( \mathcal{W}_p (X) \) is a geodesic space (cf. Theorem 3.15). Geodesics in \( \mathcal{W}_p (X) \) are also called displacement interpolation, due to a refined characterization of geodesics in \( \mathcal{W}_p (X) \) which we now explain.

Let \( C([0,1],X) \) denote the set of all continuous curves \( \gamma : [0,1] \to X \) with the uniform metric \( d^X_\infty (\gamma_1, \gamma_2) := \sup_{t \in [0,1]} d_X (\gamma_1(t), \gamma_2(t)) \). Let \( \Gamma([0,1],X) \) denote the subset of \( C([0,1],X) \) consisting of all geodesics in \( X \). For \( t \in [0,1] \), let \( e_t : C([0,1],X) \to X \) be the evaluation map taking \( \gamma \in C([0,1],X) \) to \( \gamma(t) \in X \).

**Definition 5.1** (Dynamic (optimal) coupling). Let \( X \) be a metric space and let \( \alpha, \beta \in \mathcal{P}(X) \). We call \( \Pi \in \mathcal{P}(C([0,1],X)) \) (the space of probability measures on \( C([0,1],X) \)) a dynamic coupling between \( \alpha \) and \( \beta \), if \( (e_0)_\# \Pi = \alpha \) and \( (e_1)_\# \Pi = \beta \), where \( (e_t)_\# \) represents the pushforward map under \( e_t \). We call \( \Pi \) a \( \ell^p \)-dynamic optimal coupling between \( \alpha \) and \( \beta \), if \( \text{supp}(\Pi) \subseteq \Gamma([0,1],X) \) and \( (e_0, e_1)_\# \Pi \in C^\text{opt}_p (\alpha, \beta) \) is an optimal transference plan with respect to \( d_{\mathcal{W}_p} \) for a given \( p \in (1, \infty) \).

Based on the notion of dynamic optimal coupling, a probability measure on the space of geodesics, there is the following characterization of geodesics in \( \ell^p \)-Wasserstein hyperspaces. For background materials and proofs, interested readers are referred to [AG13, Section 3.2] or [Vil08, Section 7].

**Theorem 5.2** (Displacement interpolation). Let \( X \) be a Polish geodesic space and let \( p \in (1, \infty) \). Given \( \alpha, \beta \in \mathcal{P}_p (X) \), and a continuous curve \( \gamma : [0,1] \to \mathcal{W}_p (X) \), the following properties are equivalent:

1. \( \gamma \) is a geodesic in \( \mathcal{W}_p (X) \);
2. there exists an \( \ell^p \)-dynamic optimal coupling \( \Pi \) between \( \alpha \) and \( \beta \) such that \( (e_t)_\# \Pi = \gamma(t) \) for each \( t \in [0,1] \).

The notion of dynamic optimal coupling is the main inspiration for our definitions of Hausdorff displacement interpolation (cf. Definition 5.6) and dynamic optimal correspondences (cf. Definition 5.14), whereas Theorem 5.2 serves as the motivation for one of our main results Theorem 3.

### 5.1.1 Hausdorff displacement interpolation

Given a compact metric space \( X \), the following is a direct consequence of definition of the uniform metric \( d^X_\infty \) on \( C([0,1],X) \).

**Lemma 5.3.** For any \( t \in [0,1] \), the evaluation \( e_t : C([0,1],X) \to X \) taking \( \gamma \) to \( \gamma(t) \) is a continuous map.
For any closed subsets $A, B \subseteq X$ with $\rho := d_X^X(A, B) > 0$, recall the definition of $\mathcal{L}(A, B)$:

$$\mathcal{L}(A, B) := \{\gamma : [0, 1] \to X : \gamma(0) \in A, \gamma(1) \in B \text{ and } \forall s, t \in [0, 1], \ d_X(\gamma(s), \gamma(t)) \leq |s - t|\rho\}$$

$$= \{\gamma \in C([0, 1], X) : \gamma(0) \in A, \gamma(1) \in B \text{ and } \gamma \text{ is } \rho\text{-Lipschitz}\}.$$ 

We have the following basic property of $\mathcal{L}(A, B)$.

**Proposition 5.4.** $\mathcal{L}(A, B)$ is a compact subset of $C([0, 1], X)$.

**Proof.** Let $\{\gamma_i : [0, 1] \to X\}_{i=0}^{\infty}$ be a sequence in $\mathcal{L}(A, B)$. By definition of $\mathcal{L}(A, B)$, each $\gamma_i$ is $\rho$-Lipschitz. Since $X$ is also compact, by Arzelà-Ascoli theorem (Theorem 2.3), there exists a subsequence of the sequence $\{\gamma_i\}_{i=0}^{\infty}$ uniformly converging to a $\rho$-Lipschitz curve $\gamma : [0, 1] \to X$. Since $A$ is compact, $\gamma(0) = \lim_{i \to \infty} \gamma_i(0) \in A$. Similarly, $\gamma(1) \in B$ and as a result, $\gamma \in \mathcal{L}(A, B)$. This implies sequential compactness and thus compactness of $\mathcal{L}(A, B)$.

**Remark 5.5.** In particular, $e_t : \mathcal{L}(A, B) \to X$ is a closed map, i.e., for any closed subset $\mathcal{D} \subseteq \mathcal{L}(A, B)$, the image $e_t(\mathcal{D})$ is closed in $X$. Indeed, $\mathcal{D}$ is then compact since $\mathcal{L}(A, B)$ is compact. Therefore, $e_t(\mathcal{D})$ is compact and thus closed since $e_t$ is continuous.

**Definition 5.6** (Hausdorff displacement interpolation). We call a closed subset $\mathcal{D} \subseteq \mathcal{L}(A, B)$ a Hausdorff displacement interpolation between $A$ and $B$ if $e_0(\mathcal{D}) = A$ and $e_1(\mathcal{D}) = B$.

**Proof of Theorem 3.** Recall that one of our main results is the following:

**Theorem 3.** Given a compact metric space $X$, let $\gamma : [0, 1] \to \mathcal{H}(X)$ be any map. We assume that $\rho := d^X_{\mathcal{H}}(\gamma(0), \gamma(1)) > 0$. Then, the following two statements are equivalent:

1. $\gamma$ is a Hausdorff geodesic;

2. there exists a nonempty closed subset $\mathcal{D} \subseteq \mathcal{L}(\gamma(0), \gamma(1))$ such that $\gamma(t) = e_t(\mathcal{D})$ for all $t \in [0, 1]$.

Note that in item 2 of the theorem, that $\gamma(t) \in \mathcal{H}(X)$ follows from Remark 5.5. The proof of the implication $1 \Rightarrow 2$ is based on the following observations.

**Lemma 5.7.** Let $X \in \mathcal{M}$ and let $\gamma : [0, 1] \to \mathcal{H}(X)$ be a Hausdorff geodesic. Let $\rho := d^X_{\mathcal{H}}(\gamma(0), \gamma(1))$ and assume that $\rho > 0$. Then, for any $t_0 \in [0, 1]$ and any $x_0 \in \gamma(t_0)$, there exists a $\rho$-Lipschitz curve $\zeta : [0, 1] \to X$ such that $\zeta(t) \in \gamma(t)$ for $t \in [0, 1]$ and $\zeta(t_0) = x_0$.

The proof of Lemma 5.7 is technical and we postpone it to the end of this section. One immediate consequence of the lemma is the following result.

**Corollary 5.8.** Let $X \in \mathcal{M}$ and let $\gamma : [0, 1] \to \mathcal{H}(X)$ be a Hausdorff geodesic. Let $\rho := d^X_{\mathcal{H}}(\gamma(0), \gamma(1))$ and assume that $\rho > 0$. Let

$$\mathcal{D} := \{\zeta \in C([0, 1], X) : \forall t \in [0, 1], \zeta(t) \in \gamma(t), \text{ and } \zeta \text{ is } \rho\text{-Lipschitz}\}.$$ 

Then, $\mathcal{D}$ is a nonempty closed subset of $\mathcal{L}(\gamma(0), \gamma(1))$ such that $e_t(\mathcal{D}) = \gamma(t)$ for each $t \in [0, 1]$.

**Proof.** By Lemma 5.7, $\mathcal{D}$ is obviously nonempty and $e_t(\mathcal{D}) = \gamma(t)$ for each $t \in [0, 1]$. Now, it remains to prove the closedness of $\mathcal{D}$. Let $\{\zeta_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{D}$ converging to some $\zeta \in C([0, 1], X)$ with respect to the metric $d^X_{\mathcal{H}}$. Then, by Theorem 2.3, $\zeta$ is $\rho$-Lipschitz. Moreover, for each $t \in [0, 1]$, since $\zeta_i(t) \in \gamma(t)$ for all $i = 1, \ldots$ and $\lim_{i \to \infty} \zeta_i(t) = \zeta(t)$, by the closedness of $\gamma(t)$ we have that $\zeta(t) \in \gamma(t)$. Therefore, $\zeta \in \mathcal{D}$ and thus $\mathcal{D}$ is closed.
This corollary establishes the direction $1 \Rightarrow 2$ of Theorem 3. We prove the opposite direction as follows:

**Proof of $2 \Rightarrow 1$ in Theorem 3.** Suppose that there exists a closed subset $\mathcal{D} \subseteq \Sigma (\gamma (0), \gamma (1))$ such that $\gamma (t) = e_t (\mathcal{D})$ for all $t \in [0, 1]$. For any $t, t' \in [0, 1]$, choose an arbitrary $x_t \in e_t (\mathcal{D})$. Let $\zeta \in \mathcal{D}$ be such that $\gamma (t') = \zeta (t')$. Then, $d_X (x_t, x_{t'}) \leq |t - t'| \rho$. So $\gamma (t) \subseteq (\gamma (t')) |t - t'| \rho$.

Similarly, $\gamma (t') \subseteq (\gamma (t)) |t - t'| \rho$ and thus

$$d^X_H (\gamma (t), \gamma (t')) \leq |t - t'| \cdot d^X_H (\gamma (0), \gamma (1)).$$

Hence, $\gamma$ is a Hausdorff geodesic.

**Interpretations and consequences of Theorem 3.** The following remark discusses a difference between the dynamic optimal coupling and the Hausdorff displacement interpolation.

**Remark 5.9** (A difference between dynamic optimal coupling and Hausdorff displacement interpolation). Given a compact metric space $X$ and $\alpha, \beta \in \mathcal{P}(X)$, suppose $A = \text{supp} (\alpha)$ and $B = \text{supp} (\beta)$. Note in fact that any dynamic optimal coupling $\Pi$ between $\alpha$ and $\beta$ is supported on $\Gamma (A, B)$, the set of geodesics $\gamma$ with $\gamma (0) \in A$ and $\gamma (1) \in B$ (cf. [Vil08, Corollary 7.22]). It is tempting to ask whether $\Sigma (A, B)$ can be replaced by $\Sigma (A, B) \cap \Gamma (A, B)$ in Theorem 3. This is not necessarily true. Indeed, consider the following case where $X \subseteq \mathbb{R}^2$ is a large enough compact disk around the origin. Let $A$ be the square $[-3, -1] \times [-1, 1]$ and let $B$ be the square $[1, 3] \times [-1, 1]$. Then, $d^X_H (A, B) = 4$. Now, let $\gamma : [0, 1] \rightarrow \mathcal{H}(X)$ be the Hausdorff geodesic constructed in Theorem 3.6, i.e., $\gamma (t) = A^{(1-t)} H^X_H (A, B) \cap B^{td^X_H (A, B)}$ for each $t \in [0, 1]$. Then, $\gamma (\frac{1}{2}) = A^2 \cap B^2$. It is easy to see that $(0, 2) \in \gamma (\frac{1}{2})$ but there is no geodesic passing through $(0, 2)$ starting in $A$ and ending in $B$. See Figure 5 for an illustration.

![Figure 5: Illustration of the example in Remark 5.9. There is no line segment (geodesic) passing through $(0, 2)$ starting $A$ and ending in $B$. (color figure)](image)

Theorem 3 is a powerful tool which allows us to easily reprove the geodesic property of the Hausdorff hyperspace of a geodesic space (cf. Theorem 3.5). In fact, we apply Theorem 3 to prove a stronger result:

**Theorem 5.10.** Suppose $X$ is a compact metric space. Then, $\mathcal{H}(X)$ is geodesic if and only if $X$ is geodesic.
Proof. We first assume that $X$ is geodesic. Let $A, B \in \mathcal{H}(X)$ be two distinct subsets. Then, we have the following observation:

**Claim 5.11.** $\mathcal{L}(A, B) \neq \emptyset$.

**Proof of Claim 5.11.** Let $\rho := d_{\mathcal{H}}^X(A, B)$. Since $X$ is compact and $A, B$ are closed, there exists $a \in A$ and $b \in B$ such that $d_X(a, b) = \rho$. Then, there exists a geodesic $\gamma : [0, 1] \to X$ such that $\gamma(0) = a \in A$, $\gamma(1) = b \in B$ and $\gamma$ is $\rho$-Lipschitz. This implies that $\gamma \in \mathcal{L}(A, B)$ and thus $\mathcal{L}(A, B) \neq \emptyset$. □

Then by Theorem 3, any nonempty closed subset of $\mathcal{L}(A, B)$ (e.g., $\mathcal{L}(A, B)$ itself) gives rise to a Hausdorff geodesic connecting $A$ and $B$ in $\mathcal{H}(X)$. Therefore, $\mathcal{H}(X)$ is geodesic.

Now, we assume that $\mathcal{H}(X)$ is geodesic. Then, for any two distinct points $x, y \in X$, there exists a Hausdorff geodesic connecting $\{x\}$ and $\{y\}$. By Theorem 3, $\mathcal{L}(\{x\}, \{y\}) \neq \emptyset$. Note that $d_X(x, y) = d_{H, X}^A(\{x\}, \{y\}) =: \rho$. Then, any $\rho$-Lipschitz curve in $\mathcal{L}(\{x\}, \{y\})$ is automatically a geodesic connecting $x$ and $y$ which implies that $X$ is geodesic. □

Let $X$ be a compact geodesic metric space. Then, for any two distinct sets $A, B \in \mathcal{H}(X)$, we know by Claim 5.11 that $\mathcal{L}(A, B) \neq \emptyset$. Then by letting $\mathcal{D} := \mathcal{L}(A, B)$ and applying Theorem 3, we have that the curve $\gamma_{A,B}^X : [0, 1] \to \mathcal{H}(X)$ defined by $t \mapsto e_t(\mathcal{L}(A, B))$ for all $t \in [0, 1]$ is a Hausdorff geodesic. In fact, $\gamma_{A,B}^X$ coincides with the curve constructed in Theorem 3.6, which provides an alternative proof of Theorem 3.6:

**Proposition 5.12.** Let $X$ be a compact geodesic metric space. For any two distinct sets $A, B \in \mathcal{H}(X)$, we define $\gamma_{A,B}^X : [0, 1] \to \mathcal{H}(X)$ by $t \mapsto e_t(\mathcal{L}(A, B))$ for all $t \in [0, 1]$. Then, $\gamma_{A,B}^X$ is a Hausdorff geodesic. Moreover, if we let $\rho := d_{\mathcal{H}}^X(A, B)$, then for any $t \in [0, 1]$, we have that

$$e_t(\mathcal{L}(A, B)) = A^{t\rho} \cap B^{(1-t)\rho}.$$  

**Proof.** That $\gamma_{A,B}^X$ is a Hausdorff geodesic has already been proved above.

Since $\gamma_{A,B}^X$ is a Hausdorff geodesic, we have that

$$d_{\mathcal{H}}^X(A, \gamma_{A,B}^X(t)) = t\rho \quad \text{and} \quad d_{\mathcal{H}}^X(\gamma_{A,B}^X(t), B) = (1-t)\rho.$$  

Then, $\gamma_{A,B}^X(t) \subseteq A^{t\rho}$ and $\gamma_{A,B}^X(t) \subseteq B^{(1-t)\rho}$. Therefore,

$$e_t(\mathcal{L}(A, B)) = \gamma_{A,B}^X(t) \subseteq A^{t\rho} \cap B^{(1-t)\rho}. \quad (3)$$

For the other direction, we first observe that Equation (3) immediately implies that $A^{t\rho} \cap B^{(1-t)\rho} \neq \emptyset$ and thus $A^{t\rho} \cap B^{(1-t)\rho} \in \mathcal{H}(X)$. Since $\mathcal{H}(X)$ is geodesic, there exist geodesics $\gamma_A, \gamma_B : [0, 1] \to \mathcal{H}(X)$ such that $\gamma_A(0) = A$, $\gamma_A(1) = \gamma_B(0) = A^{t\rho} \cap B^{(1-t)\rho}$ and $\gamma_B(1) = B$. Note that $A^{t\rho} \cap B^{(1-t)\rho} \subseteq A^{t\rho}$ and $A \subseteq \left(\gamma_{A,B}^X(t)^{t\rho} \subseteq (A^{t\rho} \cap B^{(1-t)\rho})^{t\rho}\right)$, where $A \subseteq \left(\gamma_{A,B}^X(t)^{t\rho}\right)$ follows from the fact that $d_{\mathcal{H}}^X(A, \gamma_{A,B}^X(t)) = t\rho$. This implies that $d_{\mathcal{H}}^X(A, A^{t\rho} \cap B^{(1-t)\rho}) \leq t\rho$. Similarly, $d_{\mathcal{H}}^X(B, A^{t\rho} \cap B^{(1-t)\rho}) \leq (1-t)\rho$. By the triangle inequality,

$$\rho = d_{\mathcal{H}}^X(A, B) \leq d_{\mathcal{H}}^X(A, A^{t\rho} \cap B^{(1-t)\rho}) + d_{\mathcal{H}}^X(B, A^{t\rho} \cap B^{(1-t)\rho}) \leq t\rho + (1-t)\rho = \rho.$$  

Therefore, all equalities must hold. Then by Proposition 2.6, we concatenate $\gamma_A$ and $\gamma_B$ to obtain a geodesic $\gamma : [0, 1] \to \mathcal{H}(X)$ such that $\gamma(0) = A$, $\gamma(1) = B$ and $\gamma(t) = A^{t\rho} \cap B^{(1-t)\rho}$. By Theorem 3, there exists a closed subset $\mathcal{D} \subseteq \mathcal{L}(A, B)$ such that $\gamma(t) = e_t(\mathcal{D})$. Then,

$$A^{t\rho} \cap B^{(1-t)\rho} = \gamma(t) = e_t(\mathcal{D}) \subseteq e_t(\mathcal{L}(A, B)).$$

This concludes the proof. □
The special geodesic $\gamma^X_{1,R}$ constructed above turns out to be instrumental for proving the following interesting property of Gromov-Hausdorff geodesics:

**Theorem 5.13** (Existence of infinitely many Gromov-Hausdorff geodesics). For any $X, Y \in \mathcal{M}$ such that $X \not\equiv Y$, there exist infinitely many distinct Gromov-Hausdorff geodesics connecting $X$ and $Y$.

In [CM18] it is shown that for any $n \in \mathbb{N}$ there exist infinitely many Gromov-Hausdorff geodesics between the one point space $\Delta_1$ and the $n$-point space $\Delta_n$. Then, Theorem 5.13 is a generalization of this result to the case of two arbitrary compact metric spaces.

**Proof of Theorem 5.13.** By Lemma 2.12, there exists $Z_0 \in \mathcal{M}$ and isometric embeddings $\varphi_X^{(0)} : X \hookrightarrow Z$ and $\varphi_Y^{(0)} : Y \hookrightarrow Z$ such that

$$d_H^Z \left( \varphi_X^{(0)}(X), \varphi_Y^{(0)}(Y) \right) = d_H(X,Y).$$

Without loss of generality, we assume that $Z_0$ is geodesic (otherwise we replace $Z_0$ with one of its extension $\mathcal{W}_1(Z_0)$, which is geodesic by Theorem 3.16). Consider the following chain of isometric embeddings:

$$Z_0 \hookrightarrow Z_1 \hookrightarrow Z_2 \hookrightarrow \cdots \hookrightarrow Z_n \hookrightarrow \cdots \ (4)$$

where for each $n \geq 1$, $Z_n := \mathcal{W}_1(Z_{n-1})$ and the map $Z_{n-1} \hookrightarrow Z_n = \mathcal{W}_1(Z_{n-1})$ is the canonical embedding sending $z_{n-1} \in Z_{n-1}$ to the Dirac delta measure $\delta_{z_{n-1}} \in \mathcal{P}(Z_{n-1})$. Let $\varphi_X^{(n)} : X \hookrightarrow Z_n$ denote the composition of the map $\varphi_X^{(0)} : X \hookrightarrow Z_0$ with the following composition of canonical isometric embeddings:

$$Z_0 \hookrightarrow Z_1 \hookrightarrow Z_2 \hookrightarrow \cdots \hookrightarrow Z_n.$$  

We similarly define $\varphi_Y^{(n)} : Y \hookrightarrow Z_n$. Then, by Lemma 2.9 we have that

$$d_H^Z \left( \varphi_X^{(n)}(X), \varphi_Y^{(n)}(Y) \right) = d_H^Z \left( \varphi_X^{(0)}(X), \varphi_Y^{(0)}(Y) \right) = d_H(X,Y), \ \forall n \in \mathbb{N}. \ (5)$$

Following Equation (4), we have the following chain of isometric embeddings:

$$\mathcal{L} \left( \varphi_X^{(0)}(X), \varphi_Y^{(0)}(Y) \right) \hookrightarrow \mathcal{L} \left( \varphi_X^{(1)}(X), \varphi_Y^{(1)}(Y) \right) \hookrightarrow \cdots \ (6)$$

By Proposition 5.12 and Lemma 3.8, for each $n \in \mathbb{N}$, the curve $\gamma_{\varphi_X^{(n)},\varphi_Y^{(n)}}^Z : [0, 1] \rightarrow \mathcal{H}(Z_n)$ defined by

$$\gamma_{\varphi_X^{(n)},\varphi_Y^{(n)}}^Z(t) := \mathcal{L} \left( \varphi_X^{(n)}(X), \varphi_Y^{(n)}(Y) \right)$$

for $t \in [0, 1]$ is a Gromov-Hausdorff geodesic connecting $\varphi_X^{(n)}(X) \cong X$ and $\varphi_Y^{(n)}(Y) \cong Y$. Now, to conclude the proof, we show that for each $m \neq n \in \mathbb{N}$, $\gamma_{\varphi_X^{(m)},\varphi_Y^{(m)}}^Z(X, Y) \neq \gamma_{\varphi_X^{(n)},\varphi_Y^{(n)}}^Z(X, Y)$.

Note that for any $n \in \mathbb{N}$ and any $t \in (0, 1)$, we have the isometric embedding (induced from Equation (6)):

$$\Psi_t^n : \gamma_{\varphi_X^{(n)},\varphi_Y^{(n)}}^Z(t) \hookrightarrow \gamma_{\varphi_X^{(n+1)},\varphi_Y^{(n+1)}}^Z(t).$$

In fact, $\Psi_t^n$ is the restriction of the canonical embedding $Z_n \hookrightarrow Z_{n+1}$ and sends each $z_n \in \gamma_{\varphi_X^{(n)},\varphi_Y^{(n)}}^Z(t)$ to $\delta_{z_n} \in \gamma_{\varphi_X^{(n+1)},\varphi_Y^{(n+1)}}^Z(t)$. Let $\rho := d_H(X,Y)$. Then, for any $x \in \varphi_X^{(n)}(X)$ there exists $y \in \varphi_Y^{(n)}(Y)$ such that $d_{Z_n}(x, y) \leq \rho$ (cf. Equation (5)). By Lemma 3.17, the curve $s \mapsto (1-s)\delta_x + s \delta_y$ is a geodesic.
in the space $Z_{n+1} = W_1(Z_n)$ and therefore it is a $\rho$-Lipschitz curve in $\mathcal{L} \left( \varphi^{(n+1)}_X(X), \varphi^{(n+1)}_Y(Y) \right)$. Then, it is easy to see that for the given $t \in (0, 1)$

$$(1 - t)\delta_x + t\delta_y \in \gamma_{Z^{n+1}}^{\varphi^{(n+1)}_X(X), \varphi^{(n+1)}_Y(Y)}(t) \setminus \Psi^n \left( \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t) \right).$$

Therefore, $\Psi^n_t$ is not surjective. By [BBI01, Theorem 1.6.14],

$$\gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t) \not\sim \gamma_{Z^{n+1}}^{\varphi^{(n+1)}_X(X), \varphi^{(n+1)}_Y(Y)}(t),$$

since both spaces are compact. Therefore

$$\gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t) \neq \gamma_{Z^{n+1}}^{\varphi^{(n+1)}_X(X), \varphi^{(n+1)}_Y(Y)}(t).$$

Now, for $m < n$ and $t \in (0, 1)$, we have an embedding $\Psi^{m,n}_t : \gamma_{Z^m}^{\varphi^{(m)}_X(X), \varphi^{(m)}_Y(Y)}(t) \hookrightarrow \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t)$ defined as the following composition of maps:

$$\Psi^{m,n}_t : \gamma_{Z^m}^{\varphi^{(m)}_X(X), \varphi^{(m)}_Y(Y)}(t) \xrightarrow{\Psi^m_t} \gamma_{Z^{m+1}}^{\varphi^{(m+1)}_X(X), \varphi^{(m+1)}_Y(Y)}(t) \xrightarrow{\Psi^{m+1}_t} \cdots \xrightarrow{\Psi^{n-1}_t} \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t),$$

Then, it follows from the above that $\Psi^{m,n}_t$ is not surjective. Hence

$$\gamma_{Z^m}^{\varphi^{(m)}_X(X), \varphi^{(m)}_Y(Y)}(t) \not\sim \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t),$$

and thus

$$\gamma_{Z^m}^{\varphi^{(m)}_X(X), \varphi^{(m)}_Y(Y)}(t) \neq \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t) \quad \forall m \neq n \in \mathbb{N}.$$ 

Therefore, $\left\{ \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t) \right\}_{n \in \mathbb{N}}$ is an infinite family of distinct Gromov-Hausdorff geodesics connecting $X$ and $Y$.

We summarize the construction of $\left\{ \gamma_{Z^n}^{\varphi^{(n)}_X(X), \varphi^{(n)}_Y(Y)}(t) \right\}_{n \in \mathbb{N}}$ above via the following diagram:

\[\begin{array}{cccccc}
[0, 1] & = & [0, 1] & = & [0, 1] & = \cdots \\
\gamma_{Z^0}^{\varphi^{(0)}_X(X), \varphi^{(0)}_Y(Y)} & \rightarrow & \gamma_{Z^1}^{\varphi^{(1)}_X(X), \varphi^{(1)}_Y(Y)} & \rightarrow & \gamma_{Z^2}^{\varphi^{(2)}_X(X), \varphi^{(2)}_Y(Y)} & \rightarrow \cdots \\
\mathcal{H}(Z^0) & \rightarrow & \mathcal{H}(Z^1) & \rightarrow & \mathcal{H}(Z^2) & \rightarrow \cdots \\
Z^0 & \xrightarrow{\mathcal{W}_1} & Z^1 & \xrightarrow{\mathcal{W}_1} & Z^2 & \xrightarrow{\mathcal{W}_1} \cdots \\
\mathcal{H} & \rightarrow & \mathcal{H} & \rightarrow & \mathcal{H} & \rightarrow \mathcal{H}
\end{array}\]

We end this section by proving Lemma 5.7:
Proof of Lemma 5.7. Without loss of generality, we assume that $0 < t_* < 1$. For $k = 0, \ldots$ and $i = 0, \ldots, 2^k + 1$, let
\[
t^k_i := \begin{cases} \frac{i}{2^k} \cdot t_*, & 0 \leq i \leq 2^k \\ t_* + \left(\frac{i}{2^k} - 1\right) (1 - t_*), & 2^k + 1 \leq i \leq 2^{k+1} \end{cases}.
\]
Let $T^k = \{t^k_i\}_{i=0}^{2^{k+1}}$. Then, $t_* = t^k_{2^k} \in T^k$, $T^k \subseteq T^{k+1}$ and $T := \cup_{k=0}^{\infty} T^k$ is dense in $[0, 1]$. Now, for any given $k \in \mathbb{N}$, let $x^k_{2^k} := x_*$. Then, there exist $x^k_{2^k+1} \in \gamma\left(t^k_{2^k+1}\right)$ and $x^k_{2^k-1} \in \gamma\left(t^k_{2^k-1}\right)$ such that
\[
dx\left(x^k_{2^k+1}, x^k_{2^k}\right) \leq |t^k_{2^k+1} - t_*| \rho \quad \text{and} \quad ndx\left(x^k_{2^k-1}, x^k_{2^k}\right) \leq |t^k_{2^k-1} - t_*| \rho
\]
and
\[
ndx\left(x^k_{2^k-2}, x^k_{2^k-1}\right) \leq |t^k_{2^k-2} - t^k_{2^k-1}| \rho \quad \text{for all } i, j = 0, \ldots, 2^{k+1}.
\]

Let $Z := \mathcal{W}_1(X)$. Since $X$ is compact, $Z$ is geodesic by Theorem 3.16. We identify $X$ with its image under the canonical embedding $X \hookrightarrow \mathcal{W}_1(X) = Z$, which then is a closed subset of $Z$. For each $k \in \mathbb{N}$, we interpolate between points $x^k_i$ and $x^k_{i+1}$ by a geodesic in $Z$ for all $i = 0, \ldots, 2^k - 1$ and concatenate all such geodesics via Proposition 2.6 to obtain a sequence of points $\{x^k_i\}_{i=0}^{2^{k+1}}$ such that $x^k_i \in \gamma\left(t^k_i\right)$ and $\sum_{i=0}^{2^{k+1}-1} dx\left(x^k_i, x^k_{i+1}\right) \leq |t^k_i - t^k_{i+1}| \rho$ for all $i, j = 0, \ldots, 2^{k+1}$. Moreover,
\[
\sum_{i=0}^{2^{k+1}-1} dx\left(x^k_i, x^k_{i+1}\right) \geq \sum_{i=0}^{2^{k+1}-1} |t^k_i - t^k_{i+1}| \rho = \rho.
\]
Then, $\zeta_k$ is $\rho$-Lipschitz. By the Arzelà-Ascoli theorem (Theorem 2.3), $\{\zeta_k\}_{k=0}^{\infty}$ has a subsequence, still denoted by $\{\zeta_k\}_{k=0}^{\infty}$, uniformly converging to a $\rho$-Lipschitz curve $\zeta : [0, 1] \to Z$.

Since $\zeta_k(t_*) = x_*$ for all $k$, we have that $\zeta(t_*) = x_*$. For each $t^k_i$, since it belongs to $T^m$ for all $m \geq k$, we have $\zeta(t^k_i) = \lim_{m \to \infty} \zeta_m(t^k_i) \in X$. Now, it remains to prove that $x_t \in \gamma(t)$ for each $t \in [0, 1] \setminus T$. Let $\{t_{n_k}\}_{k=0}^{\infty}$ be a subsequence of $T = \cup_k T^k$ converging to $t \in [0, 1] \setminus T$. Assume on the contrary that $x_t \not\in \gamma(t)$ and let $\delta := dx\left(x_t, \gamma(t)\right) > 0$. For $k$ large enough, we have $dx\left(x_{t_{n_k}}, x_t\right) < \frac{\delta}{2}$, which implies that $dx\left(x_{t_{n_k}}, \gamma(t)\right) > \frac{\delta}{2}$. This contradicts with the fact that $\lim_{k \to \infty} dx\left(\gamma(t_{n_k}), \gamma(t)\right) = 0$. This concludes the proof.

### 5.2 Dynamic Gromov-Hausdorff geodesics.

In this section we extend our results about Hausdorff geodesics in the previous section to the case of Gromov-Hausdorff geodesics.
Definition 5.14 (Dynamic correspondence). For a continuous curve $\gamma : [0, 1] \to \mathcal{M}$, we call $\mathfrak{R} \subseteq \Pi_{t \in [0,1]} \gamma (t)$ a dynamic correspondence for $\gamma$ if for any $s, t \in [0, 1]$, the image of $\mathfrak{R}$ under the evaluation $e_{st} : \Pi_{t \in [0,1]} \gamma (t) \to \gamma (s) \times \gamma (t)$ taking $(x_t)_{t \in [0,1]}$ to $(x_s, x_t)$ is a correspondence between $\gamma (s)$ and $\gamma (t)$. We call $\mathfrak{R}$ a dynamic optimal correspondence if each $e_{st} (\mathfrak{R})$ is an optimal correspondence between $\gamma (s)$ and $\gamma (t)$.

In either case, we say that $\gamma$ admits a dynamic (optimal) correspondence.

Remark 5.15 (Comparison between (dynamic) coupling and (dynamic) correspondence). The following observation from optimal transport inspires our definition of dynamic correspondence. Given a compact metric space $X$ and $\alpha, \beta \in \mathcal{P} (X)$, note that any coupling $\mu \in \mathcal{C} (\alpha, \beta)$ satisfies $\text{supp} (\mu) \in \mathcal{R} (\text{supp} (\alpha), \text{supp} (\beta))$ (cf. [Mém11, Lemma 2.2]), i.e., the support of a coupling is a correspondence between supports of measures. Now, for a dynamic coupling $\Pi$ between $\alpha$ and $\beta$, we know $(e_s, e_t) \# \Pi$ is a coupling between $(e_s) \# \Pi$ and $(e_t) \# \Pi$ for any $s, t \in [0,1]$. So

$$\text{supp} ((e_s, e_t) \# \Pi) \subseteq \mathcal{R} (\text{supp} ((e_s) \# \Pi), \text{supp} ((e_t) \# \Pi)).$$

Definition 5.16 (Dynamic geodesic). A Gromov-Hausdorff geodesic $\gamma : [0,1] \to \mathcal{M}$ is dynamic, if it admits a dynamic optimal correspondence $\mathfrak{R}$.

It is easy to check that straight-line $d_{GH}$ geodesics are dynamic.

Proposition 5.17. Any straight-line $d_{GH}$ geodesic is dynamic.

Proof. We adopt the notation from Theorem 2.14. We first define

$$\mathfrak{R} := \{ ((x, y))_{t \in [0,1]} : (x, y) \in R \} \subseteq \Pi_{t \in [0,1]} R.$$ 

Consider the quotient map $q : \Pi_{[0,1]} R \to \Pi_{[0,1]} R_t$ taking $((x, y))_{t \in [0,1]}$ to the tuple $(z_t)_{t \in [0,1]}$ such that $z_0 = x$, $z_1 = y$ and $z_t = (x, y)$ for any $t \in (0,1)$. Then, $\mathfrak{R} := q (\mathfrak{R})$ is a dynamic optimal correspondence for $\gamma_R$. \hfill \Box

Now, we proceed to proving Theorem 4. The proof combines Theorem 1 with Theorem 3 in a direct way: we transform a Gromov-Hausdorff geodesic into a Hausdorff geodesic, and its Hausdorff displacement interpolation will then generate a dynamic optimal correspondence.

Theorem 4. Every Gromov-Hausdorff geodesic $\gamma : [0,1] \to \mathcal{M}$ is dynamic.

Proof. Let $\gamma : [0,1] \to \mathcal{M}$ be a Gromov-Hausdorff geodesic and let $\rho := d_{GH} (\gamma (0), \gamma (1))$. Without loss of generality, assume $\rho > 0$. By Theorem 1, there exist $Z \in \mathcal{M}$ and isometric embeddings $\varphi_t : \gamma (t) \to Z$ for $t \in [0,1]$ such that $d^Z (\varphi_s (\gamma (s)), \varphi_t (\gamma (t))) = d_{GH} (\gamma (s), \gamma (t))$ for $s, t \in [0,1]$.

Since $t \mapsto \varphi_t (\gamma (t))$ is a Hausdorff geodesic in $Z$, by Theorem 3 there exists a Hausdorff displacement interpolation $\mathfrak{D} \subseteq \mathcal{L} (\varphi_0 (\gamma (0)), \varphi_1 (\gamma (1)))$ such that $e_t (\mathfrak{D}) = \varphi_t (\gamma (t))$ for each $t \in [0,1]$. Now, let

$$\mathfrak{R} := \{ (x_t)_{t \in [0,1]} \in \Pi_{t \in [0,1]} \gamma (t) : \exists \zeta \in \mathfrak{D} \text{ such that } \varphi_t (x_t) = \zeta (t), \forall t \in [0,1] \}.$$ 

It is obvious that $\mathfrak{R} \neq \emptyset$ and $e_t (\mathfrak{R}) = \gamma (t)$ for any $t \in [0,1]$. Since $e_t = e_t \circ e_{st}$ and $e_s = e_s \circ e_{st}$ for $s, t \in [0,1]$, we have that $e_s (e_{st} (\mathfrak{R})) = \gamma (s)$ and $e_t (e_{st} (\mathfrak{R})) = \gamma (t)$ so that $e_{st} (\mathfrak{R})$ is a correspondence between $\gamma (s)$ and $\gamma (t)$. Therefore, $\mathfrak{R}$ is a dynamic correspondence for $\gamma$. 

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Now, we show that $\mathcal{R}$ is optimal. In fact, for any $s, t \in [0, 1],
\begin{align*}
e_{st}(\mathcal{R}) &= \{(x_s, x_t) \in \gamma(s) \times \gamma(t) : \exists \zeta \in \mathcal{D} \text{ such that } \varphi_s(x_s) = \zeta(s) \text{ and } \varphi_t(x_t) = \zeta(t)\} \\
&\subseteq \{(x_s, x_t) \in \gamma(s) \times \gamma(t) : d_{Z}(\varphi_s(x_s), \varphi_t(x_t)) \leq |s - t| \rho\} =: R_{st}.
\end{align*}

For any $(x_s, x_t), (x'_s, x'_t) \in R_{st}$, by identifying $x_t$ with $\varphi_t(x_t) \in Z$, we have
\begin{align*}
|d_Z(x_s, x'_s) - d_Z(x_t, x'_t)| &\leq |d_Z(x_s, x'_s) - d_Z(x'_s, x_t)| + |d_Z(x'_s, x_t) - d_Z(x_t, x'_t)| \\
&\leq d_Z(x_s, x_t) + d_Z(x'_s, x'_t) \\
&\leq 2|s - t| \rho \\
&= 2d_{GH}(\gamma(s), \gamma(t)).
\end{align*}

Therefore, $R_{st}$ is an optimal correspondence between $\gamma(s)$ and $\gamma(t)$ and so must be $e_{st}(\mathcal{R}) \subseteq R_{st}$. Therefore, $\mathcal{R}$ is an dynamic optimal correspondence for $\gamma$. \hfill \square

6 Discussion

Gromov-Hausdorff and Gromov-Wasserstein geodesics have been studied in [INT16, CM18, IT19a, Stu06, Stu12]. However, those papers did not address the characterization of such geodesics. In this paper, we have proved that not only Gromov-Hausdorff geodesics are actually Hausdorff geodesics but also, in an analogous sense, that a large collection of Gromov-Wasserstein geodesics are Wasserstein geodesics. We further drew structural connections between Hausdorff geodesics and Wasserstein geodesics and studied the dynamic characterization of Gromov-Hausdorff geodesics.

Some open problems

Besides Conjecture 1 about Wasserstein-realizable Gromov-Wasserstein geodesics, there are other related unsolved problems which we summarize next.

Geodesic hull. Lemma 4.9 states that for any $X$-Hausdorff-realizable Gromov-Hausdorff geodesic $\gamma$, the union $G_X \coloneqq \cup_{t \in [0,1]} \gamma(t) \subseteq X$ also Hausdorff-realizes $\gamma$. Intuitively speaking, such a $G_X$ is a “minimal” ambient space containing $\gamma$ without any redundant points. It is tempting to call $G_X$ “the geodesic hull” of $\gamma$. In order to make sense of this nomenclature, we need to understand the relation between $G_X$ and $G_Y$ whenever $\gamma$ is also $Y$-Hausdorff-realizable for some $Y \in \mathcal{M}$ which is not isometric to $X$. Is it true that $G_X \cong G_Y$ for any such $Y$? Even if this is not the case, it still remains interesting to unravel commonalities between elements of the family
\[ \mathcal{M}(\gamma) := \{G_X : \gamma \text{ is } X\text{-Hausdorff-realizable}\} \]

where each $X$ Hausdorff-realizes the given Gromov-Hausdorff geodesic $\gamma$. For example: what is the Gromov-Hausdorff diameter of $\mathcal{M}(\gamma)$?

Beyond geodesics. One of the main insights behind our characterization of Gromov-Hausdorff geodesics is the observation in Corollary 3.26 that any finite collection of compact metric spaces along a given Gromov-Hausdorff geodesic is embeddable into a common ambient space in a way such that pairwise Hausdorff distances agree with the corresponding Gromov-Hausdorff distances.

A natural question is whether such an ambient space still exists for an arbitrary finite collection of compact metric spaces that do not necessarily reside along the trace of a Gromov-Hausdorff geodesic. The following conjecture pertains to the simplest unsolved case: the case involving only three spaces.
Conjecture 2. Given three compact metric spaces $X_1, X_2, X_3 \in \mathcal{M}$, there exist $Z \in \mathcal{M}$ and isometric embeddings $\varphi_i : X_i \rightarrow Z$ such that $d^2_{\mathcal{H}}(\varphi_i(X_i), \varphi_j(X_j)) = d_{\mathcal{H}}(X_i, X_j)$ for all $i, j \in \{1, 2, 3\}$.

A similar question can be posed in relation to dynamic optimal correspondences:

Conjecture 3. Given three compact metric spaces $X_1, X_2, X_3 \in \mathcal{M}$, there exists $R \subseteq X_1 \times X_2 \times X_3$ such that $e_{ij}(R)$ is an optimal correspondence between $X_i$ and $X_j$ for all $i, j \in \{1, 2, 3\}$.

It seems also interesting to elucidate the relationship between these two conjectures: note that when only dealing with two spaces, in the proof of Lemma 2.12 we used the existence of an optimal correspondence to establish the existence of an ambient space which realizes the Gromov-Hausdorff distance between the two given spaces; it is then plausible that Conjecture 3 could imply Conjecture 2.

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Declarations of interest: none

A Geodesics in Hausdorff hyperspaces

A.1 A counterexample

Example A.1 (Counterexample to Claim 3.7). Consider $X = [0, 3] \times [0, 1]$ endowed with the usual Euclidean metric $d_X$. We define $\tilde{d}_X : X \times X \rightarrow \mathbb{R}$ on $X$ as follows:

$$\tilde{d}_X ((x, y), (x', y')) := \min \left(x + x', d_X ((x, y), (x', y')), 6 - (x + x')\right).$$

It is easy to check that $\tilde{d}_X$ is a pseudo-metric on $X$ and that $\tilde{d}_X|_{[1, 2] \times [0, 1]} = d_X|_{[1, 2] \times [0, 1]}$.

Note that $\tilde{d}_X ((x, y), (x', y')) = 0$ if and only if one of the following three cases holds: $(x, y) = (x', y')$, $x = x' = 0$, or $x = x' = 3$. This induces an equivalence relation $\sim$ on $X$. In fact, $\tilde{d}_X$ is the quotient metric ([BB101, Definition 3.1.12]) on $X/\sim$ which arises by identifying all points along $\{0\} \times [0, 1]$ and all points along $\{3\} \times [0, 1]$, respectively. Let $[0]$ denote the equivalence class of all points on $\{0\} \times [0, 1]$ and by $[3]$ the equivalence class of all points on $\{3\} \times [0, 1]$. Then, it follows from definition of $\tilde{d}_X$ that for any $(x, y) \in X/\sim$, we have $\tilde{d}_X ((x, y), [0]) = x$ and $\tilde{d}_X ((x, y), [3]) = 3 - x$.

So, any continuous curve $\gamma : [0, 1] \rightarrow X/\sim$ such that $\gamma (0) = [0]$, $\gamma (1) = [3]$ and $p_1 (\gamma (t)) = 3t$ satisfies the condition in Claim 3.7, where $p_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection onto the first component. For example, the graph of any continuous function defined on $[0, 3]$ with value bounded between 0 and 1 composed with the quotient $X \rightarrow X/\sim$ generates such a continuous curve $\gamma$. However, for such a curve, the image of $\gamma|_{[1, 2]}$ lies in $[1, 2] \times [0, 1] \subseteq X/\sim$, where $\tilde{d}_X$ agrees with the Euclidean metric. As long as the image of $\gamma|_{[1, 2]}$ is not straight, $\gamma$ is not a geodesic in $X/\sim$. See Figure 6 for an illustration. For an explicit counterexample, we let $\gamma$ be the composition of the curve $t \mapsto (3t, \sin(3t \pi)) \in X$ for $t \in [0, 1]$ and the quotient $X \rightarrow X/\sim$. Then, obviously the image of $\gamma|_{[1, 2]}$ is not straight and thus $\gamma$ is not a geodesic in $X/\sim$. 
A.2 Proof of Theorem 3.6

We now provide a proof for Theorem 3.6 which is an amended version of the one given in [Ser98]. Recall that for a metric space \( X \), a subspace \( A \) and \( r \geq 0 \), \( A^r := \{x \in X : \exists a \in A \text{ such that } d_X(a, x) \leq r\} \).

**Lemma A.2.** Given a geodesic metric space \( X \) and a closed subset \( A \subseteq X \), for any \( r_1, r_2 \geq 0 \), we have \((A^r_1)^{r_2} = A^{r_1+r_2}\).

**Proof.** The case when \( r_1 = 0 \) or \( r_2 = 0 \) is trivial. Now, we assume that \( r_1 > 0 \) and \( r_2 > 0 \). That \((A^r_1)^{r_2} \subseteq A^{r_1+r_2}\) holds obviously for any metric space \( X \) (not necessarily geodesic). For the other direction, suppose \( x \in A^{r_1+r_2} \) and let \( a \in A \) such that \( d_X(x, a) \leq r_1 + r_2 \). Let \( \gamma : [0, 1] \rightarrow X \) be a geodesic such that \( \gamma(0) = a \) and \( \gamma(1) = x \). Let \( x_1 := \gamma\left(\frac{r_1}{r_1+r_2}\right) \).

\[
d_X(a, x_1) = d_X\left(\gamma(0), \gamma\left(\frac{r_1}{r_1+r_2}\right)\right) \leq \frac{r_1}{r_1+r_2} \cdot d_X(\gamma(0), \gamma(1)) \leq \frac{r_1}{r_1+r_2} \cdot (r_1 + r_2) = r_1.
\]

Similarly, \( d_X(x_1, x) \leq r_2 \). Therefore, \( x_1 \in A^{r_1} \) and thus \( x \in (A^r_1)^{r_2}\). \(\square\)

**Lemma A.3.** Given a geodesic metric space \( X \) and \( A, B \in H(X) \), suppose \( \rho := d_H^X(A, B) > 0 \). Then, for any \( r_1, r_2 \geq 0 \) such that \( r_1 + r_2 = \rho \), we have \( A^{r_1} \cap B^{r_2} \neq \emptyset \).

**Proof.** For any \( a \in A \), there exists \( b \in B \) such that \( d_X(a, b) \leq \rho \). Let \( \gamma : [0, 1] \rightarrow X \) be a geodesic such that \( \gamma(0) = a \) and \( \gamma(1) = b \). Let \( x_1 := \gamma\left(\frac{r_1}{\rho}\right) \), then it is easy to check as in the previous lemma that \( d_X(a, x_1) \leq r_1 \) and \( d_X(x_1, b) \leq r_2 \). This implies that \( x_1 \in A^{r_1} \cap B^{r_2} \) and thus \( A^{r_1} \cap B^{r_2} \neq \emptyset \). \(\square\)

**Proof of Theorem 3.6.** Given \( A, B \in H(X) \), suppose \( d_H^X(A, B) = \rho \). Define \( \gamma : [0, 1] \rightarrow H(X) \) by \( t \mapsto A^t \cap B^{(1-t)\rho} \). Obviously, \( \gamma(0) = A, \gamma(1) = B \) and \( \gamma(t) \neq \emptyset \) due to Lemma A.3. Now, for any \( s < t \in [0, 1] \), we need to show that \( d_H^X(\gamma(s), \gamma(t)) \leq |s-t|\rho \).

**Claim A.4.** For any \( E, F \in H(X) \), suppose that for a given \( r \geq 0 \) we have \( E \cap F^r \neq \emptyset \). Then, for any \( r_0 \in [0, r] \) we have \( E^{r_0} \cap F^{r-r_0} \subseteq (E \cap F^r)^{r_0} \).

**Proof of Claim A.4.** Given \( x \in E^{r_0} \cap F^{r-r_0} \), there exist \( e \in E \) and \( f \in F \) such that \( x \in B_s(e) \cap B_{r-r_0}(f) \neq \emptyset \). Hence \( d_X(e, f) \leq d_X(x, e) + d_X(x, f) \leq r_0 + r - r_0 = r \). This implies \( e \in E \cap F^r \) and thus \( x \in (E \cap F^r)^{r_0} \). \(\square\)
Now, let \( r_0 = (t-s) \rho, r = (1-s) \rho, E = A^{s \rho} \) and \( F = B \), then one has the following from Lemma A.2 and the Claim:

\[
A^{t \rho} \cap B^{(1-t) \rho} = E^{r_0} \cap F^{r - r_0} \subseteq (E \cap F)^{r_0} = (A^{s \rho} \cap B^{(1-s) \rho})^{(t-s) \rho}.
\]

Similarly, if we let \( r_0 = (t-s) \rho, r = t \rho, E = B^{(1-t) \rho}, F = A \), then we have

\[
A^{s \rho} \cap B^{(1-s) \rho} = F^{r - r_0} \cap E^{r_0} \subseteq (F \cap E)^{r_0} = (A^{t \rho} \cap B^{(1-t) \rho})^{(t-s) \rho}.
\]

Therefore \( d_{H}^{X}(\gamma(s), \gamma(t)) \leq \rho|s-t| \).

\[\square\]

**B Additional proofs regarding Wasserstein hyperspaces**

**B.1 Proof of Theorem 3.12**

We prove Theorem 3.12 via the following series of lemmas.

**Lemma B.1.** For any \( X \in M \) and a (not necessarily compact) metric space \( Y \) and for any \( p \in [1, \infty] \), if \( \varphi : X \to Y \) is an isometric embedding, then the pushforward map \( \varphi_{#} : W_{p}(X) \to W_{p}(Y) \) is also an isometric embedding.

**Proof.** Given \( \alpha, \beta \in P(X) \), let \( \mu \in C(\alpha, \beta) \) be any coupling. Consider the pushforward \( \tilde{\mu} = (\varphi \times \varphi)_{#} \mu \), where \( \varphi \times \varphi : X \times X \to Y \times Y \) takes \( (x, x') \) to \( (\varphi(x), \varphi(x')) \). It is easy to check that \( \varphi_{#}\alpha, \varphi_{#}\beta \in P_{p}(Y) \) and \( \tilde{\mu} \in C(\varphi_{#}\alpha, \varphi_{#}\beta) \). Then, for \( p \in [1, \infty) \) we have the following

\[
d_{W_{p}}^{Y}(\varphi_{#}\alpha, \varphi_{#}\beta) \leq \left( \int_{Y \times Y} d_{Y}^{p}(y_{1}, y_{2}) d\tilde{\mu}(y_{1}, y_{2}) \right)^{\frac{1}{p}}
\]

\[
= \left( \int_{X \times X} d_{Y}^{p}(\varphi(x_{1}), \varphi(x_{2})) d\mu(x_{1}, x_{2}) \right)^{\frac{1}{p}}
\]

\[
= \left( \int_{X \times X} d_{X}^{p}(x_{1}, x_{2}) d\mu(x_{1}, x_{2}) \right)^{\frac{1}{p}}.
\]

The last equality follows from the fact that \( \varphi \) is an isometric embedding.

For \( p = \infty \), since \( \varphi \) is an isometric embedding, we have that \( (\varphi \times \varphi)(\text{supp}(\mu)) = \text{supp}(\tilde{\mu}) \). Then,

\[
d_{W_{\infty}}^{Y}(\varphi_{#}\alpha, \varphi_{#}\beta) \leq \sup_{(y_{1}, y_{2}) \in \text{supp}(\tilde{\mu})} d_{Y}(y_{1}, y_{2}) = \sup_{(y_{1}, y_{2}) \in \varphi \times \varphi(\text{supp}(\mu))} d_{Y}(y_{1}, y_{2})
\]

\[
= \sup_{(x_{1}, x_{2}) \in \text{supp}(\mu)} d_{Y}(\varphi(x_{1}), \varphi(x_{2})) = \sup_{(x_{1}, x_{2}) \in \text{supp}(\mu)} d_{X}(x_{1}, x_{2}).
\]

By taking infimum over \( \mu \in C(\alpha, \beta) \), we conclude that for \( p \in [1, \infty) \),

\[
d_{W_{p}}^{Y}(\varphi_{#}\alpha, \varphi_{#}\beta) \leq d_{W_{p}}^{X}(\alpha, \beta).
\]

Since \( \varphi \) is continuous, \( \varphi(X) \) is compact in \( Y \) and hence closed. Then, it is obvious to see that \( \text{supp}(\varphi_{#}\alpha) \subseteq \varphi(X) \) for any \( \alpha \in P(X) \). Hence we have that

\[
d_{W_{p}}^{Y}(\varphi_{#}\alpha, \varphi_{#}\beta) = d_{W_{p}}^{X}(\varphi_{#}\alpha, \varphi_{#}\beta) \leq d_{W_{p}}^{X}(\alpha, \beta).
\]
Since \( \varphi^{-1} : \varphi (X) \to X \) is also an isometric embedding, one has that
\[
d_{W,p}^X (\alpha, \beta) = d_{W,p}^X \left( \varphi^{-1} \circ \varphi_\# \alpha, \varphi^{-1} \circ \varphi_\# \beta \right) \leq d_{W,p}^Y (\varphi_\# \alpha, \varphi_\# \beta).
\]
Therefore, \( d_{W,p}^X (\alpha, \beta) = d_{W,p}^Y (\varphi_\# \alpha, \varphi_\# \beta) \) and thus \( \varphi_\# \) is an isometric embedding.

**Lemma B.2.** Given two compact metric spaces \((X, d_X)\) and \((Y, d_Y)\), suppose there exist a (not necessarily compact) metric space \((Z, d_Z)\) and isometric embeddings \(\varphi_X : X \hookrightarrow Z\) and \(\varphi_Y : Y \hookrightarrow Z\). Then, for any \(p \in [1, \infty)\), we have that
\[
d_{W,p}^Z (\varphi_X (X), \varphi_Y (Y)) \leq d_{W,p}^Z (\varphi_X (X), \varphi_Y (Y))
\]
\[
\text{Proof.} \quad \text{For notational simplicity, we identify } X \text{ with } \varphi_X (X) \text{ and } Y \text{ with } \varphi_Y (Y) \text{ with } (\varphi_X)_\# (W_p (X)) \text{ (same for } Y \text{ and } W_p (Y)).
\]
Let \(\eta := d_{W,p}^Z (X, Y)\), and consider some small \(\varepsilon > 0\).

**Claim B.3.** There exists a Borel measurable map \(\xi : X \to Y\) such that for any \(x \in X\),
\[
d_Z (x, \xi (x)) \leq \eta + \varepsilon.
\]

**Proof of Claim B.3.** Consider an \(\varepsilon\)-net \([x_k]_{k=1}^n\) of \(X\). Then, we construct the Voronoi cells \([X_k]_{k=1}^n\) of \(X\) with respect to the net, i.e., \(X_k := \{ x \in X : d_X (x, x_k) = \min_{i=1, \ldots, n} d_X (x, x_i) \}\). We adjust the Voronoi cells to make them disjoint. For example, let \(Y_1 = X_1\) and \(Y_k = X_k \cup \bigcup_{i=1}^{k-1} X_i\) for \(k \geq 1\). We still let \([X_k]_{k=1}^n\) denote the cells after adjustment. Note that these cells are Borel measurable. For each \(x_k\), we let \(y_k \in Y\) such that \(d_Z (x_k, y_k) \leq \eta\). Then, we define \(\xi : X \to Y\) by mapping \(x\) to \(y_k\) if \(x \in X_k\). The map \(\xi\) thus constructed is obviously measurable. Moreover, since \(x \in X_k\) and \([x_k]_{k=1}^n\) is an \(\varepsilon\)-net, one has that \(d_Z (x, x_k) = d_X (x, x_k) \leq \varepsilon\). Therefore
\[
d_Z (x, \xi (x)) \leq d_Z (x, x_k) + d_Z (x_k, y_k) \leq \varepsilon + \eta.
\]

Let \(\xi : X \to Y\) be the map constructed in Claim B.3. For any \(\alpha \in \mathcal{P} (X)\), let \(\beta := \xi_\# \alpha \in \mathcal{P} (Y)\). For \(p \in [1, \infty)\), we have
\[
\left( d_{W,p}^Z (\alpha, \beta) \right)^p \leq \int_X d_{W,p}^Z (x, \xi (x)) \, d\alpha (x) \leq \int_X (\eta + \varepsilon)^p \, d\alpha (x) = (\eta + \varepsilon)^p.
\]
For \(p = \infty\), define \(\hat{\xi} : X \to X \times Y\) by letting \(\hat{\xi} (x) := (x, \xi (x))\) for any \(x \in X\). Then, \(\mu := \hat{\xi}_\# \alpha\) is a coupling between \(\alpha\) and \(\beta\). Let \(C = \hat{\xi} (X)\), then \(\text{supp} (\mu) \subseteq C\). Indeed, for any \((x, y) \in (X \times Y) \setminus C\), there exists an open neighborhood \(N\) of \((x, y)\) such that \(N \cap C = \emptyset\) and therefore
\[
\mu (N) = \alpha \left( \hat{\xi}^{-1} N \right) = \alpha (\emptyset) = 0.
\]
Now, for any point \((x, y) \in C\), there exists a sequence \(\{(x_i, y_i)\}_{i=1}^\infty\) in \(\hat{\xi} (X)\) approaching \((x, y)\) as \(i \to \infty\). Then, we have that
\[
d_Z (x, y) = \lim_{i \to \infty} d_Z (x_i, y_i) = \lim_{i \to \infty} d_Z (x_i, \xi (x_i)) \leq \eta + \varepsilon.
\]
Hence we have that
\[
d_{W,\infty}^Z (\alpha, \beta) \leq \sup_{(x,y) \in \text{supp} (\mu)} d_Z (x, y) \leq \eta + \varepsilon.
\]

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Therefore, for any $p \in [1, \infty]$ we have that $\mathcal{W}_p (X) \subseteq (\mathcal{W}_p (Y))^{\eta + \varepsilon}$, and similarly we have that $\mathcal{W}_p (Y) \subseteq (\mathcal{W}_p (X))^{\eta + \varepsilon}$. This implies that

$$d_{\mathcal{H}}^{\mathcal{W}_p (Z)} (\mathcal{W}_p (X), \mathcal{W}_p (Y)) \leq \eta + \varepsilon = d_{\mathcal{H}}^{Z} (X, Y) + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we conclude that

$$d_{\mathcal{H}}^{\mathcal{W}_p (Z)} (\mathcal{W}_p (X), \mathcal{W}_p (Y)) \leq d_{\mathcal{H}}^{Z} (X, Y).$$

\[\square\]

**Lemma B.4.** Under the same assumptions as in Lemma B.2, we have that

$$d_{\mathcal{H}}^{\mathcal{W}_p (Z)} ((\varphi_X)_\# (\mathcal{W}_p (X)), (\varphi_Y)_\# (\mathcal{W}_p (Y))) \geq d_{\mathcal{H}}^{Z} (\varphi_X (X), \varphi_Y (Y)).$$

**Proof.** Again for notational simplicity, we will omit the symbols for isometric embedding maps such as $\varphi_X$ or $(\varphi_X)_\#$ in the following proof.

It follows from Remark 2.8 that $d_{\mathcal{H}}^{Z} (X, Y) = \max (\sup_{x \in X} \inf_{y \in Y} d_Z (x, y), \sup_{y \in Y} \inf_{x \in X} d_Z (x, y))$. Without loss of generality, we assume that $d_{\mathcal{H}}^{Z} (X, Y) = \sup_{y \in Y} \inf_{x \in X} d_Z (x, y)$. Suppose that the points $x_0 \in X, y_0 \in Y$ are such that $d_Z (x_0, y_0) = \sup_{y \in Y} \inf_{x \in X} d_Z (x, y)$. The existence of such $(x_0, y_0)$ follows from compactness of $X$ and $Y$. Then, we consider the Dirac delta measure $\delta_{y_0} \in \mathcal{P} (Y)$ and any $\mu_X \in \mathcal{P} (X)$. We identify $\delta_{y_0}$ with $(\varphi_Y)_\# \delta_{y_0} \in \mathcal{P} (Z)$ and $\mu_X$ with $(\varphi_Y)_\# \mu_X \in \mathcal{P} (Z)$. Then, for any $p \in [1, \infty)$,

$$d_{\mathcal{H}}^{Z} (\delta_{y_0}, \mu_X) = \left( \int_X d_Z (x, y_0)^p \, d\mu_X (x) \right)^{\frac{1}{p}} \geq d_Z (x_0, y_0).$$

For $p = \infty$, we have that

$$d_{\mathcal{H}}^{Z} (\delta_{y_0}, \mu_X) = \sup_{x \in \text{supp} (\mu_X)} d_Z (x, y_0) \geq d_Z (x_0, y_0).$$

Consequently, we have for all $p \in [1, \infty]$ that

$$d_{\mathcal{H}}^{\mathcal{W}_p (Z)} (\mathcal{W}_p (X), \mathcal{W}_p (Y)) \geq \inf_{\mu_X \in \mathcal{P} (X)} d_{\mathcal{H}}^{\mathcal{W}_p (Z)} (\delta_{y_0}, \mu_X) \geq d_Z (x_0, y_0) = d_{\mathcal{H}}^{Z} (X, Y).$$

\[\square\]

Then, Theorem 3.12 follows from Lemma B.1, Lemma B.2 and Lemma B.4.

**B.2 Proof of Lemma 3.17 (linear interpolation)**

**Proof.** For any $x_0 \in X$ and each $t \in [0, 1]$, we have that

$$\int_X d_X (x, x_0) \, d (\gamma (t)) (x) = (1 - t) \int_X d_X (x, x_0) \, d\alpha (x) + t \int_X d_X (x, x_0) \, d\beta (x) < \infty.$$ 

Therefore, for each $t \in [0, 1]$, $\gamma (t) \in \mathcal{P}_1 (X)$.

Since $X$ is Polish, there exists an optimal coupling $\mu$ between $\alpha$ and $\beta$ such that $d_{\mathcal{W}_1} (\alpha, \beta) = \int_{X \times X} d_X (x, y) \, d\mu (x, y)$ (see for example [Vil08, Chapter 4]). For $0 \leq s < t \leq 1$, let

$$\mu (s, t) := (1 - t) \cdot \iota_{\#} \alpha + s \cdot \iota_{\#} \beta + (t - s) \cdot \mu,$$
where \( \iota : X \to X \times X \) taking \( x \) to \( (x, x) \) is the diagonal map.

Then, it is easy to show that \( \mu(s, t) \in C(\gamma(s), \gamma(t)) \). Therefore, we have that

\[
d_{\mathcal{W}, 1}(\gamma(s), \gamma(t)) \leq \int_{X \times X} d_X(x, y) d(\mu(s, t))(x, y)
\]

\[
= \int_{X \times X} d_X(x, y) ((1 - t) d(\iota_#\alpha) + s d(\iota_#\beta) + (t - s) d\mu(x, y)
\]

\[
= \int_X d_X(x, x) ((1 - t) d\alpha + s d\beta)(x) + (t - s) \int_{X \times X} d_X(x, y) d\mu(x, y)
\]

\[
= (t - s) d_{\mathcal{W}, 1}(\alpha, \beta).
\]

Hence, by the triangle inequality we obtain that \( d_{\mathcal{W}, 1}(\gamma(s), \gamma(t)) = |t - s| \cdot d_{\mathcal{W}, 1}(\alpha, \beta) \) and thus \( \gamma \) is a geodesic in \( \mathcal{W}_1(X) \) connecting \( \alpha \) and \( \beta \).

\[\square\]

C Deviant and branching Gromov-Wasserstein geodesics

For \( n \in \mathbb{N} \), denote by \( \Delta_n \) the \( n \)-point space with interpoint distance 1. Now, endow \( \Delta_n \) with uniform probability measure (denoted by \( \mu_n \)) and denote the corresponding metric measure space by \( \tilde{\Delta}_n = (\Delta_n, d_n, \mu_n) \).

In [CM18], the authors constructed an infinite family of deviant Gromov-Hausdorff geodesics between \( \Delta_1 \) and \( \Delta_n \) for \( n \geq 2 \) and an infinite family of Gromov-Hausdorff geodesics branching from the straight-line \( d_{\mathcal{G}H} \) geodesic from \( \Delta_1 \) to \( \Delta_n \). In this section, we mimic their constructions and construct deviant and branching \( L^p \)-Gromov-Wasserstein geodesics for \( p \in [1, \infty) \).

Since there is only one measure coupling between \( \mu_1 \) and \( \mu_n \), there exists a unique straight-line \( d_{\mathcal{G}W, p}^\delta \) geodesic \( \gamma \) that connects \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_n \) (cf. Theorem 4.25). We first write down \( \gamma \) explicitly: \( \gamma(0) = \tilde{\Delta}_1 \) and \( \gamma(1) = \tilde{\Delta}_n \); for each \( t \in (0, 1) \), \( \gamma(t) \) has the underlying set \( X_n = \{x_1, \ldots, x_n\} \), distance function \( d_t \) such that \( d_t(x_i, x_j) = t \cdot \delta_{i \neq j} \) for \( i, j = 1, \ldots, n \) and uniform probability measure \( \nu_t \).

Next, we compute explicitly \( d_{\mathcal{G}W, p}^\delta(\gamma(0), \gamma(1)) = d_{\mathcal{G}W, p}^\delta(\tilde{\Delta}_1, \tilde{\Delta}_n) \) for later use.

**Proposition C.1.** For each \( p \in [1, \infty) \) and any positive integer \( n \in \mathbb{N} \), we have that

\[
d_{\mathcal{G}W, p}^\delta(\tilde{\Delta}_1, \tilde{\Delta}_n) = \frac{1}{2}.
\]

**Proof.** Note that there exists only one coupling \( \pi_n \) between \( \mu_1 \) and \( \mu_n \). Let \( \{p\} \) and \( X_n = \{x_1, \ldots, x_n\} \) be the underlying sets of \( \Delta_1 \) and \( \Delta_n \), respectively. Assume that \( d \in \mathcal{D}(d_1, d_n) \), i.e., \( d \) is a metric coupling between \( d_1 \) and \( d_n \) (cf. Remark 2.21). Then, we have that

\[
d_{\mathcal{G}W, p}^\delta(\tilde{\Delta}_1, \tilde{\Delta}_n) \leq \left( \frac{1}{n} \sum_{i=1}^n d^p(p, x_i) \right)^{\frac{1}{p}}.
\]

Since \( p \geq 1 \), by the generalized means inequality (see [BMV13]), we have that

\[
\left( \frac{1}{n} \sum_{i=1}^n d^p(p, x_i) \right)^{\frac{1}{p}} \geq \frac{1}{n} \sum_{i=1}^n d(p, x_i) = \frac{1}{n} \cdot \frac{1}{n - 1} \sum_{i < j} (d(x_i, p) + d(p, x_j))
\]

\[
\geq \frac{1}{n(n - 1)} \sum_{i < j} d_n(x_i, x_j) = \frac{1}{n(n - 1)} \cdot \frac{n(n - 1)}{2} \cdot 1 = \frac{1}{2}.
\]
If we let $d(p, x_i) = \frac{1}{2}$ for each $i = 1, \ldots, n$, then $d$ is a metric coupling between $d_1$ and $d_n$ satisfying

$$
\left( \frac{1}{n} \sum_{i=1}^{n} d^p(p, x_i) \right)^{\frac{1}{p}} = \frac{1}{2}.
$$

Therefore, we obtain the proposition.

**C.1 Deviant geodesics**

In this section, we construct an infinite family of Gromov-Wasserstein geodesics $\{\tilde{\gamma}_{\sigma}\}_{\sigma \in (0,1)}$ connecting $\tilde{\Delta}_1$ and $\tilde{\Delta}_n$ such that for each $\sigma \in (0,1)$, $\tilde{\gamma}_{\sigma}$ is different from the straight-line $d_{GW, p}^{\tilde{\Delta}_1, \tilde{\Delta}_n}$ geodesic $\gamma$. Since $\gamma$ is the unique straight-line $d_{GW, p}^{\tilde{\Delta}_1, \tilde{\Delta}_n}$ geodesic connecting $\tilde{\Delta}_1$ and $\tilde{\Delta}_n$, $\{\tilde{\gamma}_{\sigma}\}_{\sigma \in (0,1)}$ is an infinite family of deviant geodesics.

For any $\sigma \in (0,1)$ and $t \in [0,1]$, define

$$
f(\sigma, t) := \begin{cases} 
\sigma, & 0 \leq t \leq \frac{1}{2} \\
\sigma - t, & \frac{1}{2} < t \leq 1.
\end{cases}
$$

Let $m$ be an integer such that $1 \leq m \leq n$. For each $t \in (0,1)$, let $X_{n+m}^t := \{x_1^t, x_2^t, \ldots, x_{n+m}^t\}$ be an $(n+m)$-point set. Choose $\sigma \in (0,1)$ and define for each $t \in (0,1)$ a function $d^p_t : X_{n+m}^t \times X_{n+m}^t \to \mathbb{R}_{\geq 0}$ as follows:

$$
\forall 1 \leq i, j \leq n+m, \quad d^p_t(x_i, x_j) := \begin{cases} 
0, & i = j \\
f(\sigma, t), & |j - i| = n \\
t, & \text{otherwise}.
\end{cases}
$$

It was proved in [CM18, Section 1.1.1] that $(X_{n+m}^t, d^p_t)$ is a metric space when $0 < t < 1$. Moreover, they proved that

$$
\gamma_{\sigma}(t) := \begin{cases} 
\Delta_1, & t = 0 \\
(X_{n+m}^t, d^p_t), & t \in (0,1) \\
\Delta_n, & t = 1
\end{cases}
$$

is a Gromov-Hausdorff geodesic connecting $\Delta_1$ and $\Delta_n$. Now, we assign to $\gamma_{\sigma}(t)$ for each $t \in (0,1)$ a probability measure $\nu_t$ as follows:

$$
\nu_t(\{x_i^t\}) := \begin{cases} 
\frac{1}{2n}, & 1 \leq i \leq m \text{ or } n+1 \leq i \leq n+m \\
\frac{1}{n}, & \text{otherwise}.
\end{cases}
$$

Define a curve $\tilde{\gamma}_{\sigma}(t) : [0,1] \to \mathcal{M}^w$ of metric measure spaces as follows:

$$
\tilde{\gamma}_{\sigma}(t) := \begin{cases} 
\tilde{\Delta}_1, & t = 0 \\
(X_{n+m}^t, d^p_t, \nu_t), & t \in (0,1) \\
\tilde{\Delta}_n, & t = 1
\end{cases}
$$

**Proposition C.2.** For each $\sigma \in (0,1)$ and each $p \in [1, \infty)$, we have that

1. $\tilde{\gamma}_{\sigma}$ is an $\ell^p$-Gromov-Wasserstein geodesic;
2. for any \( t \in (0, 1) \), we have that \( \tilde{\gamma}_\sigma(t) \not\approx_w \gamma(t) \). Given any \( \sigma' \in (0, 1) \) such that \( \sigma \neq \sigma' \), we have that \( \tilde{\gamma}_\sigma(t) \not\approx_w \tilde{\gamma}_{\sigma'}(t) \).

3. \( \tilde{\gamma}_\sigma \) is Hausdorff-bounded.

**Proof.** For item 1, we need to prove for any \( 0 \leq s < t \leq 1 \) that \( d_{GW,p}^\omega(\tilde{\gamma}_\sigma(s), \tilde{\gamma}_\sigma(t)) \leq |t - s| \cdot d_{GW,p}^\omega(\tilde{\Delta}_1, \tilde{\Delta}_n) \). There are three cases: (i) \( s, t \in (0, 1) \); (ii) \( s = 0 \) and \( t \in (0, 1) \); (iii) \( s \in (0, 1) \) and \( t = 1 \). We only prove case (i) and the rest two cases follow a similar strategy. Let \( \pi_{st} \in C(\nu_s, \nu_t) \) be the identity coupling, i.e., \( \pi_{st} = \sum_{i=1}^{n+m} \nu_s(x_i^s) \delta(x_i^s, x_i^t) \). Define a function \( d_{st}^p : X_{n+m} \cup X_{n+m} \cup X_{n+m} \to \mathbb{R}_{\geq 0} \) as follows:

1. \( d_{st}^\omega(X_{n+m} \times X_{n+m}) := d_s^\omega \) and \( d_{st}^\omega(X_{n+m} \times X_{n+m}) := d_t^\omega \);

2. for any \( x_i^s \in X_{n+m} \) and \( x_j^t \in X_{n+m} \),

\[
d_{st}^p(x_i^s, x_j^t) := \begin{cases} \frac{|t - s|}{2}, & i = j \\ \frac{|t - s|}{2} + \min(f(\sigma, s), f(\sigma, t)), & |i - j| = n \\ \frac{s + t}{2}, & \text{otherwise.} \end{cases}
\]

3. for any \( x_i^s \in X_{n+m} \) and \( x_j^t \in X_{n+m} \),

\[
d_{st}^\omega(x_i^s, x_j^t) := d_{st}^\omega(x_i^s, x_i^t).
\]

Then, it is easy to check that \( d_{st}^p \in D(d_s^\omega, d_t^\omega) \) (cf. Remark 2.21). Therefore, for any \( p \in [1, \infty) \),

\[
d_{GW,p}^\omega(\tilde{\gamma}_\sigma(s), \tilde{\gamma}_\sigma(t)) \leq \left( \int_{X_{n+m} \times X_{n+m}} \left( d_{st}^p(x_i^s, x_j^t) \right)^p \, d\pi_{st}(x_i^s, x_j^t) \right)^{\frac{1}{p}}
\]

\[
= \left( \sum_{i=1}^{m} \frac{|t - s|}{2} \right)^p \cdot \frac{1}{2n} + \sum_{i=m+1}^{n} \frac{|t - s|}{2} \right)^p \cdot \frac{1}{2n} + \sum_{i=n+1}^{n+m} \frac{|t - s|}{2} \right)^p \cdot \frac{1}{2n} \right)^{\frac{1}{p}}
\]

\[
= \frac{|t - s|}{2} = |t - s| \cdot d_{GW,p}^\omega(\tilde{\Delta}_1, \tilde{\Delta}_n),
\]

where we use Proposition C.1 in the last equality. This implies that \( \tilde{\gamma}_\sigma \) is an \( \ell^p \)-Gromov-Wasserstein geodesic connecting \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_n \).

For item 2, given \( t \in (0, 1) \) and \( \sigma \in (0, 1) \), we have that \( \tilde{\gamma}_\sigma(t) \) is a \((n + m)\)-point metric measure space (with full support). Since \( \gamma \left( \frac{1}{2} \right) \) is an \( n \)-point space, then \( \tilde{\gamma}_\sigma \left( \frac{1}{2} \right) \not\approx_w \gamma \left( \frac{1}{2} \right) \). For \( \sigma \neq \sigma' \), the underlying metric spaces of \( \tilde{\gamma}_\sigma \) and \( \tilde{\gamma}_{\sigma'} \) have different sets of distance values and thus they must not be isometric, let alone isomorphic. Therefore, \( \tilde{\gamma}_\sigma(t) \not\approx_w \tilde{\gamma}_{\sigma'}(t) \).

For item 3, that \( \tilde{\gamma}_\sigma \) is Hausdorff-bounded follows directly from Proposition 4.24.

From the above proposition, we conclude that \( \gamma \) is different from \( \tilde{\gamma}_\sigma \) for all \( \sigma \in (0, 1) \). Moreover, \( \tilde{\gamma}_\sigma \neq \tilde{\gamma}_{\sigma'} \) for all \( \sigma, \sigma' \in (0, 1), \sigma' \neq \sigma \). Therefore, \( \{ \tilde{\gamma}_\sigma \}_{\sigma \in (0, 1)} \) is an infinite family of deviant Gromov-Wasserstein geodesics connecting \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_n \).

**C.2 Branching geodesics**

We construct an infinite family of \( \ell^p \)-Gromov-Wasserstein geodesics \( \{ \gamma^a \}_{a \in (0, 1)} \) branching off from the straight-line \( d_{GW,p}^\omega \) geodesic \( \gamma : [0, 1] \to \mathcal{M}^w \) from \( \tilde{\Delta}_1 \) to \( \tilde{\Delta}_n \). More precisely, for each \( a \in (0, 1) \), \( \gamma(t) = \gamma^a(t) \) for \( t \in [0, a] \) and \( \gamma(t) \not\approx_w \gamma^a(t) \) for \( t \in (a, 1] \). See Figure 7 for an illustration.
First of all, using a similar strategy as in proving Proof.

For each Proposition C.3.

(iv) and omit the proof of case (iii) since it follows a strategy similar to the one used for proving case (iv).

follows from the fact that $\gamma$

show that $\gamma$

Given $a \in (0, 1)$, we define for each $t \in (a, 1]$ a metric space as follows: let $X^t_{n+1} := \{x_1^t, \ldots, x_n^t, x^t_{n+1}\}$ and let $d^a_t : X^t_{n+1} \times X^t_{n+1} \to \mathbb{R}_{\geq 0}$ be such that $d^a_t(x_i^t, x_j^t) = t \cdot \delta_{i\neq j}$ for $i, j = 1, \ldots, n$, $d^a_t(x_i^t, x^t_{n+1}) = t$ for $i = 1, \ldots, n - 1$ and $d^a_t(x^t_{n+1}, x^t_{n+1}) = t - a$. That $(X^t_{n+1}, d^a_t)$ is a metric space follows from [CM18, Section 1.1.2]. Endow $(X^t_{n+1}, d^a_t)$ with a probability measure $\nu_t$ as follows:

$$\nu_t(\{x_i^t\}) := \begin{cases} \frac{1}{n}, & 1 \leq i \leq n - 1 \\ \frac{1}{2n}, & i = n \text{ or } i = n + 1 \end{cases}$$

Now, we define a curve $\gamma^a : [0, 1] \to \mathcal{M}^w$ as follows:

$$\gamma^a(t) := \begin{cases} \gamma(t), & t \in [0, a] \\ (X^t_{n+1}, d^a_t, \nu_t), & t \in (a, 1] \end{cases}$$

Proposition C.3. For each $a \in (0, 1)$ and each $p \in [1, \infty)$, we have that

1. $\gamma^a$ is an $\ell^p$-Gromov-Wasserstein geodesic;
2. $\gamma^a(t) \equiv_w \gamma(t)$ for $t \in [0, a]$ whereas $\gamma^a(t) \not\equiv_w \gamma(t)$ for $t \in (a, 1]$;
3. $\gamma^a$ is Hausdorff-bounded.

Proof. First of all, using a similar strategy as in proving $d^a_{\mathcal{GW}, p}(\tilde{\Delta}_1, \tilde{\Delta}_n) = \frac{1}{2}$ (cf. Proposition C.1), one can show that $d^a_{\mathcal{GW}, p}(\gamma^a(0), \gamma^a(1)) = \frac{1}{2}$ and we omit details here.

For item 1, we need to prove for any $0 \leq s < t \leq 1$ that

$$d^a_{\mathcal{GW}, p}(\gamma^a(s), \gamma^a(t)) \leq |t - s| \cdot d^a_{\mathcal{GW}, p}(\gamma^a(0), \gamma^a(1)).$$

There are four cases: (i) $s, t \in [0, a]$; (ii) $0 \leq s < a < t \leq 1$; (iii) $s = a < t \leq 1$; (iv) $s, t \in (a, 1]$. Case (i) follows from the fact that $\gamma$ is a geodesic. Case (ii) follows from case (i) and case (iii). We only prove case (iv) and omit the proof of case (iii) since it follows a strategy similar to the one used for proving case (iv).

Fix $a < s < t \leq 1$. Let $\pi_{st} \in \mathcal{C}(\nu_s, \nu_t)$ be the identity coupling, i.e., $\pi_{st} = \sum_{i=1}^{n+1} \nu_s(x_i^s) \delta_{(x_i^s, x_i^t)}$. Define a function $d^a_{st} : X^s_{n+1} \sqcup X^t_{n+1} \times X^t_{n+1} \sqcup X^s_{n+1} \to \mathbb{R}_{\geq 0}$ as follows:

1. $d^a_{st}|_{X^s_{n+1} \times X^t_{n+1}} := d^a_s$ and $d^a_{st}|_{X^t_{n+1} \times X^s_{n+1}} := d^a_t$;
2. for any $x_i^s \in X^s_{n+1}$ and $x_j^t \in X^t_{n+1}$,

$$d^a_{st}(x_i^s, x_j^t) := \begin{cases} \frac{|t-s|}{2}, & i = j \\ \frac{2+t-s}{2} - a, & (i, j) = (n, n+1) \text{ or } (n+1, n) \\ \frac{t+s}{2}, & \text{otherwise} \end{cases}$$

$$\gamma^a$$

Illustration of the family

Figure 7: Illustration of the family $\{\gamma^a\}_{a \in (0, 1)}$
3. for any $x_i^s \in X_{n+1}^s$ and $x_j^t \in X_{n+1}^t$, $d_{st}^{a}(x_i^s, x_j^t) := d_{st}^{a}(x_j^t, x_i^s)$.

Then, it is easy to check that $d_{st}^{a} \in D(d_{st}, d_{st})$ (cf. Remark 2.21). Therefore, for any $p \in [1, \infty)$,

$$d_{GW,p}^S(\gamma^a(s), \gamma^a(t)) \leq \left( \int_{X_{n+1}^s \times X_{n+1}^t} (d_{st}^{a}(x^s, x^t))^p \, d\pi_{st}(x^s, x^t) \right)^{\frac{1}{p}}$$

$$= \left( \frac{1}{n} \sum_{i=1}^{n-1} \left( \frac{|t-s|}{2} \right)^p + \frac{1}{n+1} \sum_{i=n}^{n+1} \left( \frac{|t-s|}{2} \right) \cdot \frac{1}{2n} \right)^{\frac{1}{p}}$$

$$= \frac{|t-s|}{2} \cdot d_{GW,p}^S(\gamma^a(0), \gamma^a(1)).$$

For item 2, by definition of $\gamma^a$, $\gamma^a(t) = \gamma(t)$ for $t \in [0, a]$. When $t \in (a, 1]$, $\gamma^a(t)$ is an $(n + 1)$-point space and as a result, $\gamma^a(t) \not\sim_w \gamma(t)$.

For item 3, that $\gamma^a$ is Hausdorff-bounded is a direct consequence of Proposition 4.24.

Therefore, for each $a \in (0, 1)$, $\gamma^a$ is an $\ell^p$-Gromov-Wasserstein geodesic branching off from the straight-line $d_{GW,p}^S$ geodesic $\gamma$ at $t = a$ and thus $\{\gamma^a\}_{a \in (0, 1)}$ is an infinite family of Gromov-Wasserstein geodesics branching off from the straight-line $d_{GW,p}^S$ geodesic $\gamma$.

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