COMPACT METRIZABLE GROUPS ARE ISOMETRY GROUPS OF COMPACT METRIC SPACES

JULIEN MELLERAY

Abstract. This note is devoted to proving the following result: given a compact metrizable group $G$, there is a compact metric space $K$ such that $G$ is isomorphic (as a topological group) to the isometry group of $K$.

Introduction

It is an easy fact that, if $X$ is a Polish metric space and $\text{Iso}(X)$ is endowed with the topology induced by the product topology on $X^X$, then it turns $\text{Iso}(X)$ into a Polish group. One may then wonder whether all Polish groups are of that form; this was proved by Gao and Kechris in [2]:

Theorem 1. (Gao-Kechris) Let $G$ be a Polish group. Then there exists a Polish metric space $Y$ such that $G$ is isomorphic to $\text{Iso}(Y)$.

Similarly, it is easy to see that, if $K$ is a compact metric space, then $\text{Iso}(K)$ is a compact metrizable group.

Given what we saw above, it is natural to wonder whether the converse holds; this question was mentioned to me by Alekos Kechris (private correspondence). The aim of this note is to provide a positive answer to that question:

Theorem 2. Let $G$ be a compact metrizable group. Then there exists a compact metric space $K$ such that $G$ is isomorphic to $\text{Iso}(K)$.

To obtain the proof, it is natural to try to find a simpler proof of theorem 1 than the one in [2], which is a little buried among other considerations (it is a byproduct of a proof, so no ”direct” proof is given).

It turns out that such a simpler proof exists, and it is not very hard to use a variation of it in order to prove theorem 1.

The paper is organized as follows: first we give a simple proof of theorem 1, then we show how to adapt this proof in order to obtain theorem 2.

Acknowledgements: I would like to thank Alekos Kechris, who told me about the problem studied here.

MSC: Primary 54H11, Secondary 22A05, 51F99.
Notations and Definitions

If $(X, d)$ is a complete separable metric space, we will say that it is a Polish metric space, and will often write it simply $X$.

A Polish group is a topological group whose topology is Polish; if $X$ is a separable metric space, then we will denote its isometry group by $Iso(X)$, and endow it with the pointwise convergence topology, which turns it into a second countable topological group, and into a Polish group if $X$ is Polish (see [1] for a thorough introduction to the theory of Polish groups).

Let $(X, d)$ be a metric space; we will say that $f : X \to \mathbb{R}$ is a Katětov map iff

$$
\forall x, y \in X \ |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).$

These maps correspond to one-point metric extensions of $X$. We denote by $E(X)$ the set of all Katětov maps on $X$; we endow it with the sup-metric, which turns it into a complete metric space.

If $f \in E(X)$ and $S \subseteq X$ are such that $f(x) = \inf(f(s) + d(x, s) : s \in S)$, we say that $S$ is a support of $f$; or that $S$ controls $f$.

It is useful to note here the following easy fact: if $f, g \in E(X, \omega)$ are both supported by some set $S$, then $d(f, g) = \sup_{s \in S} |f(s) - g(s)|$.

Notice that $X$ isometrically embeds in $E(X)$ via the Kuratowski embedding $x \mapsto \delta_x$, where $\delta_x(y) = d(x, y)$, and that one has, for any $f \in E(X)$, that $d(f, \delta_x) = f(x)$.

Proofs of the theorems:

As promised, we begin by proving theorem [1]

Let $G$ be a Polish group, and $d$ be a left-invariant distance on $G$. Let $X$ be the completion of $(G, d)$. Then the left translation action of $G$ on itself extends to an action by isometries of $G$ on $X$, and it is not hard to check that this provides a continuous embedding of $G$ in $Iso(X)$. We identify $G$ with the corresponding (closed) subgroup of $Iso(X)$, and make the additional assumption that $X$ is of diameter $\leq 1$.

Claim. For all $\varphi \in Iso(X) \setminus G$, there exist $x_1, \ldots, x_m \in X$ and $\varepsilon > 0$ such that $V_\varphi = \{\psi \in Iso(X) : \forall 1 \leq k \leq m \ d(\psi(x_k), \varphi(x_k)) < \varepsilon\} \subseteq Iso(X) \setminus G$, and $2m\varepsilon = \min(d(x_1, x_k))$.

Proof. Obvious.

Choose for all $\varphi \in Iso(X) \setminus G$ such a $V_\varphi$; then, since $Iso(X)$ is Lindelöf, there are $\{\varphi_i\}_{i \geq 1}$ such that $Iso(X) \setminus G = \bigcup_{i \geq 1} V_{\varphi_i}$. We denote
$V_{\varphi_i} = \{ \psi \in Iso(X) : \forall 1 \leq k \leq m_i \ d(y^i_k, \psi(x^i_k)) < \varepsilon_i \}.$

For each $i \geq 1$ we define maps $f_i, g_i \in E(X)$:

$f_i(x) = \min \{ \min_{1 \leq k \leq m_i} (1 + d(x, x^i_k) + 2(k-1)\varepsilon_i), 1 + 2m_i\varepsilon_i) \},$ and

g_i(x) = \min \{ \min_{1 \leq k \leq m_i} (1 + d(x, y^i_k) + 2(k-1)\varepsilon_i), 1 + 2m_i\varepsilon_i) \}.

If $\varphi \in Iso(X)$, we let $\varphi^*$ denote its (unique) extension to $E(X)$, defined by $\varphi^*(f)(x) = f(\varphi^{-1}(x))$. We now have the following lemma:

**Lemma 3.** $\forall \varphi \in Iso(X) \forall i \geq 1 \ (\varphi \in V_{\varphi_i}) \iff (d(\varphi^*(f_i), g_i) < \varepsilon_i)$.

**Proof of Lemma 3.**
Let $\varphi \in V_{\varphi_i}$. Then the various inequalities involved imply that:

$\varphi^*(f_i)(y^i_k) = 1 + d(\varphi(x^i_k), y^i_k) + 2(k-1)\varepsilon_i$, so $|\varphi^*(f_i)(y^i_k) - g_i(y^i_k)| < \varepsilon_i$.

$g_i((\varphi(x^i_k)) = 1 + d(\varphi(x^i_k), y^i_k) + 2(k-1)\varepsilon_i$, so $|\varphi^*(f_i)(\varphi(x^i_k)) - g_i(\varphi(x^i_k))| < \varepsilon_i$.

Since $\varphi^*(f_i)$ and $g_i$ are both supported by the set $\{\varphi(x^i_k)\}_{k=1...m_i} \cup \{y^i_k\}_{k=1...m_i}$, the inequalities above are enough to show that $d(\varphi^*(f_i), g_i) < \varepsilon_i$.

Conversely, let $\varphi \in Iso(X)$ be such that $d(\varphi^*(f_i), g_i) < \varepsilon_i$. We prove by induction on $k = 1...m$ that $d(\varphi(x^i_k), y^i_k) < \varepsilon_i$.

To see that this is true for $k = 1$, remark that we have that $g_i(\varphi(x^i_1)) < 1 + \varepsilon_i$, so that we must have $g_i(\varphi(x^i_1)) = g_i(y^i_1) + d(y^i_1, x^i_1)$. In turn, this implies that $d(y^i_1, x^i_1) < \varepsilon_i$.

Suppose now that we have proved the result up to rank $k - 1 \leq m - 1$.

Notice that we have this time that $g_i(\varphi(x^i_k)) < 1 + \varepsilon_i + 2(k-1)\varepsilon_i$ (*). Also, we know that for all $l < k$ we have $d(x^i_k, x^i_l) > 2m_i\varepsilon_i$, and $d(x^i_k, x^i_l) < \varepsilon_i$.

Thus, $d(x^i_k, y^i_l) > (2m_i - 1)\varepsilon_i$ for all $l < k$.

It is then clear that (*) implies that $g_i(\varphi(x^i_k)) = g_i(y^i_k) + d(y^i_k, x^i_k)$, which means that $d(y^i_k, x^i_k) < \varepsilon_i$.

We now let $F_0 = X$, $F_i = \{\varphi^* f_i : g \in G\} \subset E(X)$, and $Z = \overline{F_i}$. Notice that $Z$ is a Polish metric space, since it is closed in $E(X)$, which is complete, and it admits a separable dense subset.

Also, it is important to remark that lemma 3 implies that $d(\psi^*(f_i), g_i) \geq \varepsilon_i$ for all $\psi \in G$; in other words, $d(F_i, g_i) \geq \varepsilon_i$ for all $i$.

**Lemma 4.** Any element $\varphi$ of $G$ extends (uniquely) to an isometry $\varphi^Z$ of $Z$, and

$\{\varphi^Z : \varphi \in G\} = \{\varphi \in Iso(Z) : \varphi(X) = X \text{ and } \forall i \geq 0 \varphi(F_i) = F_i\}$

**Proof of Lemma 4.**

The first assertion is easy to prove: since $\varphi^*(F_i) = F_i$ for all $\varphi \in G$, we see that $\varphi^*(Z) = Z$ for all $\varphi \in G$. The fact that this extension is unique is a classical consequence of the definition of the distance on $E(X)$, see for instance 3.

It is also clear that $\{\varphi^Z : \varphi \in G\} \subseteq \{\varphi \in Iso(Z) : \varphi(X) = X\}$; to prove the
converse, take \( \varphi \in Iso(Z) \) such that \( \varphi(X) = X \) and \( \varphi(F_i) = F_i \) for all \( i \).
So \( \varphi|_X \) is an isometry of \( X \) such that \( d((\varphi|_X)^*(f_i), g_i) = d(\varphi(f_i), g_i) \geq \varepsilon_i \) for all \( i \), which means that \( \varphi|_X \notin V_{\varphi_i} \) for all \( i \), so that \( \varphi|_X \in G \) and we are done.

To conclude the proof of the theorem, notice that \( Z \) is a bounded Polish metric space, so we may assume that \( \text{diam}(Z) \leq 1 \).
Then we conclude as in [2]: for each \( i \in I \) we let \( y_i \in E(Z) \) be defined by \( y_i(z) = d(y_i, z) = (i + 2) + d(z, F_i) \) for all \( z \in Z \). Notice that \( \varphi^*(y_i) = y_i \) for all \( i \in I \) and all \( \varphi \in G \).
Let \( Y = Z \cup \{y_i\} \subset E(Z) \); \( Y \) is complete, and we claim that \( G \) is isomorphic to \( Iso(Y) \).
Indeed, any element \( \varphi \) of \( G \) has a unique isometric extension \( \varphi^Y \) to \( Y \), and the mapping \( \varphi \mapsto \varphi^Y \) is continuous. Conversely, let \( \psi \) be an isometry of \( Y \); necessarily \( \psi(Z) = Z \), and \( \psi(y_i) = y_i \) for all \( i \).
Since \( F_i = \{z \in Z : d(z, y_i) = 0\} \), we must have \( \psi(F_i) = F_i \) for all \( i \). So, we see that there is some \( \varphi \in G \) such that \( \psi|_X = \varphi^Z \), so that \( \psi = \varphi^Y \).
To conclude the proof of theorem [4] recall that any bijective, continuous morphism between two Polish groups is actually bicontinuous.

Now we will see that the ideas of this proof enable one to also prove theorem [2]

**Proof of theorem [2]**
We may again assume that \( G \) has more than two elements. Let \( d \) be an invariant metric on \( G \); the metric space \( X = (G, d) \) is compact, and \( G \) embeds topologically in \( Iso(X) \), via the mapping \( g \mapsto (x \mapsto g.x) \). We again identify \( G \) with the corresponding (closed) subgroup of \( Iso(X) \), and make the additional assumption that \( X \) is of diameter \( \leq 1 \).

We again choose for, all \( \varphi \in Iso(X) \setminus G \), a \( V_{\varphi} \) as in the claim; there are \( \{\varphi_i\}_{i \geq 1} \) such that \( Iso(X) \setminus G = \bigcup_{i \geq 1} V_{\varphi_i} \). We again denote \( V_{\varphi_i} = \{\psi \in Iso(X) : \forall 1 \leq k \leq m_i \ \ d(y_k^i, \psi(x_k^i)) < \varepsilon_i \} \).
For each \( i \geq 1 \) we define slightly different maps \( f_i, g_i \):

\[
f_i(x) = \min \left( \min_{1 \leq k \leq m_i} \left( 1 + \frac{1}{2^i} + d(x, x_k^i) + 2(k - 1)\varepsilon_i \right), 1 + \frac{1}{2^i} + 2m_i\varepsilon_i \right),
\]

\[
g_i(x) = \min \left( \min_{1 \leq k \leq m_i} \left( 1 + \frac{1}{2^i} + d(x, y_k^i) + 2(k - 1)\varepsilon_i \right), 1 + \frac{1}{2^i} + 2m_i\varepsilon_i \right).
\]

If \( \varphi \in Iso(X) \), we let \( \varphi^* \) denote its (unique) extension to \( E(X) \); we have again that

\[
\forall \varphi \in Iso(X) \forall i \geq 1 \ (\varphi \in V_{\varphi_i}) \iff (d(\varphi^*(f_i), g_i) < \varepsilon_i).
\]
Now, we let $Y$ be the set of $f \in E(X)$ such that

$$\exists x_1 \ldots x_n \ \forall x \ f(x) = \min \left( \min_{1 \leq i \leq n} (1 + 2(i - 1)\varepsilon + d(x, x_i)), 1 + 2n\varepsilon \right),$$

where $2n\varepsilon = \min(d(x_i, x_j))$.

(for $n = 0$ one gets $g$ defined by $g(x) = 1$ for all $x \in X$.)

**Lemma 5.** $Y$ is compact.

**Proof.** Let $(f_i)$ be a sequence of maps in $Y$, and let $x_1^i, \ldots, x_n^i$ be points witnessing the fact that $f_i \in Y$.

Then, either we can extract a sequence $f_{\varphi(i)}$ such that $n_{\varphi(i)} \to +\infty$, or $(n_i)$ is bounded.

In the first case, notice that, since $X$ is totally bounded, one must necessarily have that $\min_{1 \leq j < k \leq n_{\varphi(i)}} d(x_j^{\varphi(i)}, x_k^{\varphi(i)}) \to 0$ when $n \to +\infty$, so that the definition of $f_i$ ensures that $f_{\varphi(i)} \to g$.

In the other case, we may extract a subsequence $f_{\psi(i)}$ such that $n_{\psi(i)} = n$ for all $i$.

We assume that $\min_{1 \leq j < k \leq n} d(x_j^{\psi(i)}, x_k^{\psi(i)}) \geq \delta$ for some $\delta > 0$ (if not, we can conclude as in the first case that some subsequence of $(f_{\psi(i)})$ converges to $g$). But then, up to another extraction, we can suppose that

$x_1^{\psi(i)} \to x_1, \ldots, x_n^{\psi(i)} \to x_n$.

This implies that $\min_{1 \leq j < k \leq n} d(x_j^{\psi(i)}, x_k^{\psi(i)}) \to \min_{1 \leq j < k \leq n} d(x_j, x_k)$, and one checks easily that $f_{\psi(i)} \to f$ for some $f \in Y$.

We let, for all $i \geq 1$, $F_i = G^* \cdot \{f_i\}$ (which is a compact subset of $E(X)$), and $Z = X \cup Y \cup \bigcup F_i$.

We proceed to prove that $Z$ is compact: for that, it is enough to see that any sequence $(x_n)$ of elements of $\bigcup F_i$ admits a subsequence converging to some $z \in Z$.

We know by definition that $x_n = \varphi_n(f_{i_n})$ for some $\varphi_n \in G$ and $i_n \in \mathbb{N}$. Since $G$ is compact, we may assume that $\varphi_n \to \varphi$, so that it is enough to show that $f_{i_n}$ admits a subsequence converging to some $z' \in Z$.

We may of course assume that $i_n \to +\infty$. Notice that, by definition, $f_{i_n} = \frac{1}{2i_n} + h_n$, for some $h_n \in Y$; since $Y$ is compact, and $i_n \to +\infty$, we are done.

The end of the proof is very similar to that of theorem 1, only a bit simpler (that is why we have chosen the $f_i$ more carefully this time): we pick $k \in E(Z)$ such that $d(k, z) = \text{diam}(Z) + d(z, X)$, and let $\{k\} \cup Z = K$.

$k$ is compact, any element of $G$ extends uniquely to an isometry of $K$, and the extension morphism is continuous.

So, we only need to prove that all isometries of $K$ are extensions of elements of $G$; to that end, let $\psi \in \text{Iso}(K)$.

We see that $\psi(k) = k$, so $\psi(Z) = Z$. 

COMPACT METRIZABLE GROUPS ARE ISOMETRY GROUPS OF COMPACT METRIC SPACES 5
Also, since \( X = \{ z \in Z : d(z,k) = \text{diam}(Z) \} \), we must have \( \psi(X) = X \).
Similarly, \( F_i = \{ z \in Z : d(z, X) = 1 + \frac{1}{2^i} \} \), so \( \psi(F_i) = F_i \).
We may now conclude as above: \( \psi|_X \not\in V_{\varphi_i} \) for all \( i \), so \( \psi|_X \in G \), and we are done.

\[ \Diamond \]

References

[1] H. Becker and A. S. Kechris, The Descriptive Set Theory of Polish Group Actions, London Math. Soc. Lecture Notes Series, 232, Cambridge University Press (1996).
[2] S. Gao and A. S. Kechris, On the classification of Polish metric spaces up to isometry, Memoirs of Amer. Math. Soc., 766, Amer. Math. Soc. (2003).
[3] M. Katětov, On universal metric spaces, Proc. of the 6th Prague Topological Symposium (1986), Frolík (ed). Helderman Verlag Berlin, pp 323-330 (1988).
[4] V. V. Uspenskij, On the group of isometries of the Urysohn universal metric space, Comment. Math. Univ. Carolinae, 31(1) (1990), pp 181-182.