Unified Hamiltonian model for mesons and baryons

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A new Hamiltonian model is introduced to study the spectrum of light hadrons. It combines relativistic field theory with elements of the constituent quark model. In addition to the standard linear confining and pseudoscalar meson exchange interactions with predetermined parameters, an additional interaction with different covariant spin structures is examined. Using a large scale Monte Carlo variational procedure, the resulting model Hamiltonian provides a very good, unified description of the light quark baryon (both octet and decuplet) and meson spectra.

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I. INTRODUCTION

Although QCD is widely accepted as the fundamental theory for the strong interaction, it is extremely challenging to calculate the observed hadron spectrum directly from the QCD Lagrangian. Due to non-Abelian and nonperturbative aspects, one method is lattice QCD which provides reasonably well described ground states \cite{1,2} and some success for excited states \cite{3,4}. However for light hadrons such as the pion and even the proton, accurate predictions are still not possible. Consequently there are many phenomenological models and effective theories, such as QCD sum rules \cite{4,5}, NRQCD \cite{6,7}, chiral perturbation theory \cite{8,9} along with potential models \cite{10,11}. Typical potential models utilize a Cornell type interaction having linear confinement supplemented with the usual Coulomb potential governing short-distance behavior. A representative example is the constituent model detailed in Ref. \cite{12} which, with an additional spin dependent interaction, obtained a good description of the light meson spectrum. This model has also been applied to baryons \cite{13} with similar success but required different model parameters. To describe the interaction between color singlet objects potential models have been extended by including a meson-exchange interaction, such as the chiral SU(3) quark potential model \cite{14,15} that gave a good description of the baryon interaction.

A more theoretical and less phenomenological approach is the Coulomb gauge model which has been successfully applied to mesons \cite{16,17}, glueballs \cite{18,19}, hybrids \cite{20,21} and tetraquark states \cite{22,23}. The predicted results are consistent with both lattice simulations and experimental data. Different from the above potential models, the Coulomb gauge approach entails relativistic field theory and is formulated in the same mathematical framework as the exact QCD Hamiltonian in the Coulomb gauge. Further, it contains no free model parameters as it only utilizes the known current quark masses and two dynamical constants, the string tension $\sigma$ and the QCD coupling constant $\alpha_s$, that are predetermined from the literature. While this model provides a reasonable hadron description it does not simultaneously reproduce both the meson and baryon spectrums with the same overall accuracy as the multi-parameter purely meson or baryon models mentioned above.

The purpose of the current work is to provide a unified model that can accurately reproduce both meson and baryon masses with the same set of model parameters which to date has not been achieved. The motivation is to develop a robust framework for reliably predicting and understanding more exotic systems such as light and heavy tetraquark states which are of intense interest. Building on the attractive theoretical features of the Coulomb gauge model and phenomenological successes of constituent quark models, a unified Hamiltonian approach has been developed that combines relativity, field theory and elements of the constituent quark model with, most significantly, a single set of parameters that can simultaneously describe both meson and baryon masses.

This paper is organized into six sections. In section \textsection II the unified Hamiltonion is described and then the meson and baryon model wavefunctions are detailed in sections \textsection III and \textsection IV respectively. Section \textsection V presents numerical results and highlights the accurate hadron descriptions. Finally, conclusions are summarized in section \textsection VI.

II. MODEL HAMILTONIAN

The model Hamiltonian is

$$H_t = H_{\text{kin}} + H_{f0} + H_{I1} + H_{ch},$$  \hspace{1cm} (1)
where $H_{\text{kine}}$ is the relativistic kinetic energy

$$H_{\text{kine}} = \int dx \Psi^\dagger(x)(-i\alpha \cdot \nabla + \beta m)\Psi(x),$$  \hspace{1cm} (2)

and $H_{I0}$, $H_{I1}$, and $H_{ch}$ are the interactions detailed below. Similar to the Coulomb gauge model $^{29,31}$, $H_{I0}$ is the confining interaction

$$H_{I0} = -\frac{1}{2} \int dx dy \bar{\Psi}(x)\gamma^0 T^a \Psi(x)\left[V_0(|x-y|)\bar{\Psi}(y)\gamma^0 T^a \Psi(y)\right],$$  \hspace{1cm} (3)

where $T^a = \frac{\lambda^a}{2}$ are the color $SU(3)$ group generators and $V_0(|x-y|)$ is a Cornell type potential

$$V_0(|x-y|) = \frac{C}{(2\pi)^3} \frac{\alpha_s}{|x-y|} + \sigma|x-y|. \hspace{1cm} (4)$$

Following constituent quark models a constant energy $C$ is introduced and $\sigma$, $\alpha_s$ are the same as in the Coulomb gauge model. This is a “charge-charge” color interaction. Performing a Fourier transformation, the potential in momentum space is

$$V_0(|q|) = C\delta^3(q) - \frac{4\pi\alpha_s}{q^2} - \frac{8\pi\sigma}{|q|} + \delta^3(q) \int dq \frac{8\pi\sigma}{q^4}. \hspace{1cm} (5)$$

The last term is to satisfy the condition that at $r = 0$, the confining potential is zero $^{40}$. It is also important to deal with the divergence of the integral with linear confining potential at zero momentum transfer.

The interaction between two colored objects can have other forms, for example, “current-current” interaction, “spin-spin” interaction, etc. In particular to account for hadron spin splittings a hyperfine type interaction $H_{I1}$ is included with structure

$$H_{I1} = -\frac{1}{2} \int dx dy \bar{\Psi}(x)\Gamma T^a \Psi(x)\left[V_1(|x-y|)\bar{\Psi}(y)\Gamma T^a \Psi(y)\right].$$  \hspace{1cm} (6)

The $\Gamma$ matrix can be $1, \gamma, \gamma_5, \gamma_\gamma_0, \gamma_0 \gamma_0, \gamma_5 \gamma_5$. The potential $V_1(|x-y|)$ is taken to be similar to $V_0$ with linear and Coulomb terms

$$V_1(|x-y|) = -\frac{\alpha_1}{|x-y|} + \sigma|x-y|. \hspace{1cm} (7)$$

$$V_0(|x-y|) = \frac{\beta_1}{|x-y|} + \sigma|x-y|. \hspace{1cm} (8)$$

This interaction will be used to reproduce the $\pi \rho$ splitting which is large due to the small $\pi$ mass governed by chiral symmetry as documented in Ref. $^{29}$ which uses a Random Phase Approximation diagonalization to obtain a light chiral pion. Here a light pion mass is obtained entirely via spin-splitting similar to the constituent treatment of Ref. $^{29}$.

The above interaction is between two colored objects. To describe interacting color singlet hadrons a pseudoscalar meson exchange interaction $H_{ch}$ is also included using the quark-meson Lagrangian

$$\mathcal{L}_{ch} = -g_{ch}\bar{\psi}(i\gamma_5 \sum_{a=1}^8 \lambda_a \pi_a)\psi. \hspace{1cm} (9)$$

Here $\lambda_a$ are the $SU_f(3)$ generators and $\pi_a$ are the pseudoscalar meson fields. The coupling constant $g_{ch}$ is determined from the $NN\pi$ interaction $^{27}$

$$g_{ch}^2 = \frac{9}{4\pi} \frac{m_N^2}{m_N} \frac{g_{NN}^2}{4\pi}, \hspace{1cm} (10)$$

where $\frac{g_{NN}^2}{4\pi} = 13.67$ $^{41}$. The constituent quark mass $m_u$ is chosen to be 220 MeV $^{23,24}$. The Goldstone field is

$$\sum_{a=1}^8 \lambda_a \pi_a = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}K^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \hspace{1cm} (11)$$
From the quark-meson interaction the one-meson exchange potential can be extracted. For example, the one-pion exchange potential between two color singlets is

\[ H^{\pi}_{ch} = \frac{1}{2} \int d\mathbf{x} d\mathbf{y} [\bar{\Psi}(\mathbf{x}) \gamma^5 \Psi(\mathbf{x})] V^\pi_{ch}(|\mathbf{x} - \mathbf{y}|)[\bar{\Psi}(\mathbf{y}) \gamma^5 \Psi(\mathbf{y})], \]  

where \( V^\pi_{ch}(|\mathbf{x} - \mathbf{y}|) \) is the Fourier transformation of \( \tilde{V}^\pi_{ch}(\mathbf{q}) \)

\[ \tilde{V}^\pi_{ch}(\mathbf{q}) = \frac{g^2_{\pi}}{q^2 + m^2_{\pi}}. \]  

The pseudoscalar meson mass in the meson exchange potential is chosen to be the experimental value.

In the above equations, the quark field operators can be expanded

\[ \Psi(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} [u_\lambda(\mathbf{k}) b_\lambda^c(\mathbf{k}) + v_\lambda(\mathbf{k}) d_\lambda^c(\mathbf{k})] e^{i\mathbf{k} \cdot \mathbf{x}} \epsilon_C, \]

\[ \bar{\Psi}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} [\bar{u}_\lambda(\mathbf{k}) b_\lambda^c(\mathbf{k}) + \bar{v}_\lambda(\mathbf{k}) d_\lambda^c(-\mathbf{k})] e^{i\mathbf{k} \cdot \mathbf{x}} \bar{\epsilon}_C, \]

where the Dirac spinors are

\[ u_\lambda(\mathbf{k}) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{\frac{\omega + m}{\omega}} \chi_\lambda \\ \sqrt{\frac{\omega - m}{\omega}} \sigma \cdot \hat{\mathbf{k}} \chi_\lambda \end{array} \right), \]

\[ v_\lambda(\mathbf{k}) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{\frac{\omega + m}{\omega}} \chi_\lambda \\ \sqrt{\frac{\omega - m}{\omega}} \sigma \cdot \hat{\mathbf{k}} \chi_\lambda \end{array} \right), \]

and \( \omega = \sqrt{m^2 + \mathbf{k}^2} \). The spinors \( \chi_\lambda \) are, fermions, \( \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and anti-fermions, \( \chi_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \chi_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

### III. MESONS

In the center of momentum the meson state \(|\bar{q}q\rangle\) is given by

\[ |\Psi^{PC}\rangle = \sum_{C_1C_2 \lambda_1 \lambda_3} \int \frac{d\mathbf{k}}{(2\pi)^3} \Psi^{PC}_{C_1C_2 \lambda_1 \lambda_3}(\mathbf{k}) b_{C_1 \lambda_1}^c(\mathbf{k}) d_{C_2 \lambda_3}^c(-\mathbf{k}) |0\rangle, \]

with convention 1 for a quark having momentum \( \mathbf{k} \) and 3 for an anti-quark having momentum \(-\mathbf{k}\). Color and spin are represented by \( C \) and \( \lambda \), respectively. The wavefunction \( \Psi^{PC}_{C_1C_2 \lambda_1 \lambda_3}(\mathbf{k}) \) has form

\[ \Psi^{PC}_{C_1C_2 \lambda_1 \lambda_3}(\mathbf{k}) = \delta_{C_1C_2} f(k) \sum_{m_L m_S} \frac{1}{2} \chi_{\lambda_1} \chi_{\lambda_3} |S m_S \rangle (L S m_L m_S | J m_J ) (-1)^{\frac{1}{2} - \lambda_3} Y^m_L(\mathbf{k}), \]

with spin and angular momentum coupling \( \hat{S}_1 + \hat{S}_3 = \hat{S}, \hat{L} + \hat{S} = \hat{J} \), \( Y^m_L(\mathbf{k}) \) is the spherical harmonic function and \( f(k) \) is the radial wavefunction with variational parameter \( \alpha \)

\[ f(k) = k^{2L} e^{-\frac{k^2}{2\alpha^2}}. \]

The meson mass is given by

\[ M = \frac{\langle \Psi^{PC} | H_{\text{kine}} | \Psi^{PC} \rangle}{\langle \Psi^{PC} | \Psi^{PC} \rangle} = \langle \Psi^{PC} | H_{\text{kine}} | \Psi^{PC} \rangle + \langle \Psi^{PC} | H_1 | \Psi^{PC} \rangle + \langle \Psi^{PC} | H_{ch} | \Psi^{PC} \rangle \]

\[ = M_{\text{kine}} + M_0 + M_1 + M_{ch}, \]
where

\[ M_{\text{kin}} = \int \frac{d\mathbf{k}}{(2\pi)^3} \left( \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2} \right) f(k) \]

\[ \times \sum_{\lambda_1, \lambda_2, m_L, m_S} \left[ \frac{1}{\sqrt{2}} \lambda_1 \lambda_3 |S\rangle \langle L \rangle m_L m_S \right] \left( J_{\text{M}} \right)^2 Y_{L}^{m_L}(k) Y_{L}^{m_L}(k), \quad (22) \]

\[ M_0 = \frac{4}{3} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\mathbf{k}'}{(2\pi)^3} \tilde{V}_0(|\mathbf{k} - \mathbf{k}'|) f(k) f(k') \]

\[ \times \sum_{\lambda_1, \lambda_2, m_L, m_S} \sum_{\lambda'_1, \lambda'_2, m'_L, m'_S} \left( \frac{1}{\sqrt{2}} \lambda_1 \lambda_3 |S\rangle \langle L \rangle m_L m_S \right) \left( J_{\text{M}} \right)^2 Y_{L}^{m_L}(k) Y_{L}^{m_L}(k) \]

\[ \left( -1 \right)^{\frac{1}{2} - \lambda_3 + \frac{1}{2} - \lambda_5} \left[ \bar{u}_{\lambda_1}(k') \gamma^0 u_{\lambda_1}(k) \right] \left[ \bar{v}_{\lambda_3}(-k') v_{\lambda_3}(-k) \right], \quad (23) \]

\[ M_1 = \frac{4}{3} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\mathbf{k}'}{(2\pi)^3} \tilde{V}_1(|\mathbf{k} - \mathbf{k}'|) f(k) f(k') \]

\[ \times \sum_{\lambda_1, \lambda_2, m_L, m_S} \sum_{\lambda'_1, \lambda'_2, m'_L, m'_S} \left( \frac{1}{\sqrt{2}} \lambda_1 \lambda_3 |S\rangle \langle L \rangle m_L m_S \right) \left( J_{\text{M}} \right)^2 Y_{L}^{m_L}(k) Y_{L}^{m_L}(k) \]

\[ \left( -1 \right)^{\frac{1}{2} - \lambda_3 + \frac{1}{2} - \lambda_5} \left[ \bar{u}_{\lambda_1}(k') \gamma^5 u_{\lambda_1}(k) \right] \left[ \bar{v}_{\lambda_3}(-k') \gamma^5 v_{\lambda_3}(-k) \right], \quad (25) \]

IV. BARYONS

The baryon state can be constructed using quark creation operators acting on the vacuum state

\[ |qqq, J^p\rangle = \sum_{C_1, C_2, C_3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{d\mathbf{k}_2}{(2\pi)^3} \frac{d\mathbf{k}_3}{(2\pi)^3} \Psi_{C_1C_2C_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) b_{C_1, C_2, C_3}^\dagger (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) |0\rangle, \quad (26) \]

here \( C, \lambda, f \) represent color, spin, and flavor, respectively. The baryon wave function, \( \Psi_{C_1C_2C_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \), can be written as the product of momentum, flavor-spin and color wave functions

\[ \Psi_{C_1C_2C_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \Psi_{\text{flavor}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times \Psi_{\text{color}}(C_1, C_2, C_3), \quad (27) \]

here \( \Psi_{\text{flavor}} \) is the flavor-spin wave function and \( \Psi_{\text{color}} \) is the color wave function. Fermi-Dirac statistics requires the total baryon wave function must be antisymmetric under the exchange of quarks. The baryon color state is a singlet and is antisymmetric

\[ \psi_{\text{color}}(C_1, C_2, C_3) = \varepsilon_{C_1C_2C_3}. \quad (28) \]

Hence the remaining wave function must be symmetric. Since the ground state momentum wave function \( f(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \) is symmetric, the flavor-spin wave function \( \psi_{\text{flavor}} \) must also be symmetric. For example the proton and \( \Delta^+ \) flavor-spin wave functions \( \psi_{\text{flavor}} \) are

\[ \psi_{\text{flavor}}(\text{proton}, \frac{1}{2}) = \frac{1}{\sqrt{2}} (\uparrow \uparrow \uparrow - \downarrow \uparrow \uparrow)(ud \uparrow - du \uparrow) + \frac{1}{\sqrt{2}} (\uparrow \uparrow \downarrow - \downarrow \uparrow \downarrow)(ud \downarrow - u \downarrow d \uparrow), \]

\[ \psi_{\text{flavor}}(\Delta^+, \frac{3}{2}) = \frac{1}{\sqrt{3}} [u(\uparrow)u(\uparrow)d(\uparrow) + u(\uparrow)d(\uparrow)u(\uparrow) + d(\uparrow)u(\uparrow)u(\uparrow)]. \]
The proper symmetric variational wave function can then be written as

\[ \psi_{\text{proton}}(fs) = C_1 \sum \rho_{12} \alpha_1 \lambda_1 + \rho_{23} \alpha_2 \lambda_2 + \rho_{31} \alpha_3 \lambda_3. \]  

where the kinetic energy has the form

\[ M_{\text{kine}} = \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} (\sqrt{m^2_{\text{p}} + k_1^2} + \sqrt{m^2_{\text{p}} + k_2^2} + \sqrt{m^2_{\text{p}} + k_3^2}) f^2(k_1, k_2, k_3). \]

To construct a completely symmetric momentum space wave function the momentum Jacobi coordinates are utilized

\[ \rho_{12} = \frac{1}{\sqrt{2}}(k_1 - k_2), \lambda_{12} = \frac{1}{\sqrt{6}}(k_1 + k_2 - 2k_3), \]  

\[ \rho_{23} = \frac{1}{\sqrt{2}}(k_2 - k_3), \lambda_{23} = \frac{1}{\sqrt{6}}(k_2 + k_3 - 2k_1), \]  

\[ \rho_{31} = \frac{1}{\sqrt{2}}(k_3 - k_1), \lambda_{31} = \frac{1}{\sqrt{6}}(k_3 + k_1 - 2k_2). \]  

The proper symmetric variational wave function can then be written as

\[ f(k_1, k_2, k_3) = e^{-\frac{\rho_{12}^2}{\alpha_1^2}} - \frac{\lambda_{12}^2}{\alpha_2^2} + e^{-\frac{\rho_{23}^2}{\alpha_1^2}} - \frac{\lambda_{23}^2}{\alpha_2^2} + e^{-\frac{\rho_{31}^2}{\alpha_1^2}} - \frac{\lambda_{31}^2}{\alpha_2^2}, \]  

where \( \alpha_1 \) and \( \alpha_2 \) are determined by the variational method.

The proton is taken as an example for calculating the baryon octet mass. According to the discussion above, the proton state can be expressed as

\[ |\text{proton}, \frac{1}{2} \rangle = \sum_{c_1c_2c_3} \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} f(k_1, k_2, k_3) \times \psi_{\text{fs}}(\text{proton}, \frac{1}{2}) \times \varepsilon_{c_1c_2c_3} b^\dagger_{c_1\lambda_1} (k_1) b^\dagger_{c_2\lambda_2} (k_2) b^\dagger_{c_3\lambda_3} (k_3) |0\rangle, \]  

where

\[ \psi_{\text{fs}}(\text{proton}, \frac{1}{2}) = \frac{1}{3\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(udd - dud) + \frac{1}{\beta\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)(uud - udu) + \frac{1}{3\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)(uud - dud) \]

\[ = \frac{2}{3\sqrt{2}} u(\uparrow) u(\downarrow) d(\uparrow) - \frac{1}{3\sqrt{2}} u(\uparrow) u(\downarrow) d(\uparrow) - \frac{1}{3\sqrt{2}} u(\uparrow) u(\downarrow) d(\uparrow) + \frac{2}{3\sqrt{2}} d(\downarrow) u(\uparrow) u(\uparrow) - \frac{1}{3\sqrt{2}} d(\downarrow) u(\uparrow) u(\uparrow) - \frac{1}{3\sqrt{2}} d(\downarrow) u(\uparrow) u(\uparrow). \]  

Contributions to the Hamiltonian expectation value are summarized in Fig. 1. The expectation value for the proton mass is

\[ \langle \text{proton}, \frac{1}{2} | H | \text{proton}, \frac{1}{2} \rangle = M_{\text{kine}} + M_{12} + M_{23} + M_{31}, \]  

where the kinetic energy has the form

\[ M_{\text{kine}} = \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} (\sqrt{m^2_{\text{p}} + k_1^2} + \sqrt{m^2_{\text{p}} + k_2^2} + \sqrt{m^2_{\text{p}} + k_3^2}) f^2(k_1, k_2, k_3). \]
The expressions for $E_i$ have many terms due to the complexity of proton flavor-spin wave function $\psi_f$. For example $M_{12}$ is

$$M_{12} = -\frac{2}{3} \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dq}{(2\pi)^3} f(k_1, k_2, k_3) f(k'_1, k'_2, k'_3) V(|q|) \left( \frac{4}{18} E_1 + \frac{4}{18} E_2 + \frac{2}{18} E_3 + \frac{8}{18} E_4 + \frac{2}{18} E_5 + \frac{2}{18} E_6 - \frac{4}{18} E_7 - \frac{4}{18} E_8 \right).$$

(37)

The eight contributions $E_i$ are given in Table I. They are classified by different spin configurations. The expressions for $M_{13}$ ($M_{23}$) are similar to $M_{12}$ with the replacement of $k_2$ by $k_3$ and $k_3$ by $k_2$ ($k'_1$ and $k'_1$ by $k'_3$ and $k'_3$). Due to the symmetry of the wave function, the numerical values of $M_{12}$, $M_{13}$ and $M_{23}$ are the same.

For the decuplet states the $\Delta^+$ is used as a representative example and has wavefunction given by

$$|\Delta^+, \frac{3}{2}\rangle = \sum_{c_1, c_2, c_3} \frac{1}{\sqrt{3}} [u(\uparrow) u(\uparrow) d(\uparrow) + u(\uparrow) d(\uparrow) u(\uparrow) + d(\uparrow) u(\uparrow) u(\uparrow)],$$

(38)

in which

$$\psi_f(\Delta^+, \frac{3}{2}) = \frac{1}{\sqrt{3}} [u(\uparrow) u(\uparrow) d(\uparrow) + u(\uparrow) d(\uparrow) u(\uparrow) + d(\uparrow) u(\uparrow) u(\uparrow)],$$

(39)

and the mass is given by

$$\langle \Delta^+, \frac{3}{2} | H_i | \Delta^+, \frac{3}{2} \rangle = M'_{\text{kine}} + M'_{12} + M'_{23} + M'_{31},$$

(40)

where

$$M'_{\text{kine}} = \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dq}{(2\pi)^3} \sqrt{m_1^2 + k_1^2 + m_2^2 + k_2^2 + \sqrt{m_3^2 + k_3^2}} f^2(k_1, k_2, k_3).$$

(41)

$$M'_{12} = \frac{2}{3} \int \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dq}{(2\pi)^3} f(k_1, k_2, k_3) f(k'_1, k'_2, k'_3) V(|q|) [E'_1 + E'_2],$$

(42)

The expressions for $E'_1$ and $E'_2$ are given in Table II. Again, due to the symmetry of the wave function, the numerical values for $M'_{13}$ and $M'_{23}$ are the same as $M'_{12}$.
TABLE III: Unified model parameters. The different meson \ref{23} and baryon \ref{24} parameters from Isgur et al. are also listed.

| Parameters | \(\xi = 2.1\) | \(\xi = 1.0\) | Ref. \[23\] | Ref. \[24\] |
|------------|----------------|----------------|----------------|----------------|
| \(m_u/m_d\) (MeV) | 313 | 50 | 220 | 220 |
| \(m_s\) (MeV) | 660 | 640 | 419 | 419 |
| \(\alpha_s\) | 0.40 | 0.40 | 0.60 | 0.60 |
| \(\sigma\) (GeV\(^2\)) | 0.18 | 0.18 | 0.18 | 0.15 |
| \(C\) (MeV) | -195 | -198 | -253 | -615 |
| \(\alpha_1\) | 0.762 | 0.490 | | |
| \(\sigma_1\) (GeV\(^2\)) | 0.207 | 0.0625 | | |

V. NUMERICAL RESULTS

The new spin splitting interaction Hamiltonian \(H_{11}\) was investigated by calculating the meson and baryon masses for all possible \(\Gamma\) matrices. The interaction with matrices 1 and \(\gamma_5\) invert the baryon octet and decuplet spectra, i.e., produce larger octet masses than decuplet masses. The matrices \(\gamma_5\), \(\gamma_5\gamma_0\), and \(\gamma_5\gamma_1\) produce \(P\) wave meson masses several hundred MeV lower than the experimental values. Only the interaction with the \(\Gamma = \bar{\gamma}\) could produce reasonable baryon and meson masses simultaneously. This is the same Lorenz structure as in Ref. \[30\] using an effective one gluon exchange hyperfine interaction. Then using the predetermined values \(\sigma = 0.18\) GeV\(^2\) and \(\alpha_s = A\), the remaining free model parameters, the \(u/d, s\) quark masses, the potential strengths \(\sigma_1\) and \(\alpha_1\) in \(H_{11}\) and constant \(C\) in \(H_{10}\), were determined by reproducing the light quark meson spectrum. These parameters are listed in Table III along with Godfrey and Isgur’s quark model values for comparison. The inclusion of the meson-exchange interaction does not add a free parameter.

In this model the relativistic four-component spinor \(u_\lambda\) is related to the free quark propagator by

\[
\sum_{\lambda=1,2} u_\lambda(k) \bar{u}_\lambda(k) = p + m. \tag{43}
\]

However for confined quarks lattice results obtain a different propagator so modified spinors of the form

\[
u_\lambda(k) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{\frac{\omega + \xi m}{\omega}} \chi_\lambda \\ \sqrt{\frac{\omega - \xi m}{\omega}} \sigma \cdot k \chi_\lambda \end{array} \right), \tag{44}
\]

\[
u_\lambda(k) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{\frac{\omega + \xi m}{\omega}} \chi_\lambda \\ \sqrt{\frac{\omega - \xi m}{\omega}} \sigma \cdot k \chi_\lambda \end{array} \right), \tag{45}
\]

have also been investigated. Here \(\omega = \sqrt{k^2 + \xi^2 m^2}\) with parameter \(\xi\). The free quark spinor is obtained for \(\xi = 1\) while \(\xi \to \infty\) produces the non-relativistic two component spinor \(\chi_\lambda\). The hadron spectra were studied for different values of \(\xi\) and quark masses and the results are listed in Tables III and IV. While different sets of values produce comparable hadron spectra, the value \(\xi = 2.1\) yields quark masses similar to the constituent quark model while for \(\xi = 1\) a much lighter quark mass of 50 MeV is required. Note the quark masses are purely parameters which should not necessarily be identified as constituent quark masses.

The hadron masses are obtained by variationally using the Monte Carlo method and are compared to experiment in Table IV. Only meson states suggested as \(q\bar{q}\) in the PDG review table (Table 14.2) \[12\] are addressed. For mesons the Goldstone exchange interaction is small, less than 20 MeV. However for baryons it is larger, reducing the decuplet masses by about 30 MeV and for octets between 60 and 100 MeV which is now sufficient to reproduce the observed 300 MeV \(N\Delta\) mass splitting. This is gratifying because this splitting without Goldstone exchange is only 250 MeV. It appears Goldstone exchange interactions play an important role in the baryon spectrum \[13\].

The calculated meson masses agree quite well with PDG data, especially the \(0^{-}\) and \(1^{-}\) states. The \(\pi\rho\) splitting is close to 640 MeV and the \(KK^*\) splitting is about 400 MeV. In traditional quark models these splittings are produced by the color hyperfine interaction. In this work it is predominantly obtained from the Hamiltonian \(H_{11}\) which lowers the \(0^{-}\) masses by about 500 MeV while reducing \(1^{-}\) masses less than 100 MeV.

The model parameters were mainly determined by the \(\pi, \rho/\omega, K\) and \(K^*\) masses and then the remaining hadron masses were predicted. With the exception of three states (\(h_1, K^*_0\) and \(f_0\)) the overall meson and baryon spectra are in very good agreement with observation. This model calculation also predicts that the lightest scalar mesons have
TABLE IV: Calculated hadron spectrum in MeV. The experimental values from PDG are listed in the last column.

| $J^{PC}$ | Meson | This work($\xi = 2.1$) | This work($\xi = 1.0$) | PDG |
|----------|-------|------------------------|------------------------|-----|
| 0$^{+-}$ | $\pi$ | 141                    | 137                    | 135 |
|          | $K$   | 494                    | 498                    | 493 |
| 1$^{--}$ | $\rho$| 778                    | 779                    | 776 |
|          | $K^*$ | 891                    | 888                    | 894 |
|          | $\phi$| 1029                   | 995                    | 1020|
| 1$^{+-}$ | $b_1$ | 1195                   | 1043                   | 1235|
|          | $K_{1B}$ | 1346                  | 1277                   |     |
|          | $h_1$ | 1512                   | 1485                   | 1380|
| 0$^{++}$ | $a_0$ | 1460                   | 1352                   | 1450|
|          | $K^*_0$ | 1519                  | 1473                   | 1430|
|          | $f_0$ | 1623                   | 1609                   | 1710|
| $\frac{1}{2}^+$ | $N$   | 934                    | 941                    | 938 |
|          | $\Lambda$ | 1158                  | 1180                   | 1116|
|          | $\Sigma$ | 1204                  | 1196                   | 1189|
|          | $\Xi$  | 1360                   | 1340                   | 1314|
| $\frac{3}{2}^+$ | $\Delta$ | 1233                  | 1254                   | 1232|
|          | $\Sigma^*$ | 1385                  | 1400                   | 1385|
|          | $\Xi^*$ | 1544                   | 1543                   | 1533|
|          | $\Omega$ | 1704                  | 1681                   | 1672|

mass well above 1 GeV. This would indicate that the $a_0(980)$, $f_0(980)$ and $f_0(500)$ mesons are not pure $q\bar{q}$ states and possibly have a tetraquark structure.

VI. SUMMARY

A new, unified Hamiltonian model has been developed which combines the attractive features of phenomenologically based quark models with many of the theoretical ingredients common to QCD. A new spin interaction has also been investigated for a variety of Lorentz structures with a clear preference for $\Gamma = \vec{\gamma}$. A Goldstone exchange interaction was also included and, along with the spin interaction, found necessary to accurately reproduce the $N\Delta$ mass splitting.

The parameters are mainly determined by fitting the $\pi$, $\rho/\omega$, $K$ and $K^*$ masses. The remaining meson and baryon masses were then predicted and found to be in good agreement with observation. All scalar mesons are predicted to have mass well above 1 GeV suggesting the $a_0(980)$, $f_0(980)$ and $f_0(500)$ are not simple $q\bar{q}$ states but perhaps tetraquarks. Most significantly, a good Hamiltonian description for the meson, baryon (octet and decuplet) spectra has been obtained with a common set of parameters which has previously not been achieved.

Future work will address heavy quark systems to further test this model. If robust results are obtained applications to exotic systems will be performed.

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