Viscous Taylor droplets in axisymmetric and planar tubes: from Bretherton’s theory to empirical models

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Abstract The aim of this study is to derive accurate models for quantities characterizing the dynamics of droplets of non-vanishing viscosity in capillaries. In particular, we propose models for the uniform-film thickness separating the droplet from the tube walls, for the droplet front and rear curvatures and pressure jumps, and for the droplet velocity in a range of capillary numbers, $Ca$, from $10^{-4}$ to 1 and inner-to-outer viscosity ratios, $\lambda$, from 0, i.e. a bubble, to high viscosity droplets. Theoretical asymptotic results obtained in the limit of small capillary number are combined with accurate numerical simulations at larger $Ca$. With these models at hand, we can compute the pressure drop induced by the droplet. The film thickness at low capillary numbers ($Ca < 10^{-3}$) agrees well with Bretherton’s scaling for bubbles as long as $\lambda < 1$. For larger viscosity ratios, the film thickness increases monotonically, before saturating for $\lambda > 10^3$ to a value $2^{2/3}$ times larger than the film thickness of a bubble. At larger capillary numbers, the film thickness follows the rational function proposed by Aussillous & Quéré [5] for bubbles, with a fitting coefficient which is viscosity-ratio dependent. This coefficient modifies the value to which the film thickness saturates at large capillary numbers. The velocity of the droplet is found to be strongly dependent on the capillary number and viscosity ratio. We also show that the normal viscous stresses at the front and rear caps of the droplets cannot be neglected when calculating the pressure drop for $Ca > 10^{-3}$.

Keywords Film thickness · Droplet velocity · Pressure drop · Lubrication theory · Numerical simulations

List of symbols

$A$ coefficient for flow profile
$B$ coefficient for flow profile
$C$ coefficient for interface profile of static meniscus
$\bar{C}$ mean curvature of droplet interface
$D$ coefficient for interface profile of static meniscus
$c_1, c_2$ coefficient for fitting law of $P, \bar{P}$
$Ca$ capillary number based on droplet velocity
$Ca_{\infty}$ capillary number based on mean outer velocity
$F$ coefficient for minimum film thickness
$\bar{F}$ averaged $F$ coefficient
$\tilde{F}$ averaged $F$ coefficient
$G$ coefficient for minimum film thickness
$\bar{G}$ averaged $G$ coefficient
$H$ thickness of film between wall and droplet
$H_{\text{min}}$ minimum film thickness
$H_{\infty}$ uniform film thickness
$H_{\text{rec}}$ critical uniform film thickness for recirculations
$K$ coefficient for linearized lubrication equation
$I$ identity tensor
$L_d$ droplet length
$M$ coefficient for pressure model
$m$ rescaled viscosity ratio
$N$ coefficient for pressure model
$n$ unit vector normal to the droplet interface
$O$ coefficient for pressure model
1 Introduction

Two-phase flows in microfluidic devices gained considerably in importance during the last two decades [55, 21]. The key for success of these microfluidic tools is the fluid compartmentalization, allowing the miniaturization and manipulation of small liquid portions at high throughput rates with a limited number of necessary controls. Reduced liquid quantities are commonly used as individual reactors in several biological and chemical applications [34], as well as in industrial processes [1] and in micro-scale heat and mass transfer equipments [40, 58, 44]. Bubbles and droplets often flow in microchannels with a round or rectangular/square cross-section [35, 32, 44].

The dynamics of a bubble in a microchannel has been the subject of several studies, since the seminal works of Fairbrother & Stubbs [15], Taylor [56] and Bretherton [9]. These long bubbles, also referred to as Taylor bubbles, flowing in a tube of radius \( R \), have been characterized by the thickness \( H_0 \) of the uniform film separating them from the tube walls, the minimum thickness \( H_{\text{min}} \) of the film, the plane curvature of the front and rear caps in the \((z, r)\) or \((x, y)\) plane, \( \kappa_f \) and \( \kappa_r \), as well as by their velocity \( U_d \). Bretherton [9] used a lubrication approach to derive the asymptotic scalings in the limit of small capillary numbers, \( Ca \), of the film, the plane curvature of the front and rear caps scales \( \kappa_f, \kappa_r \sim R^{1 + \beta_f, \beta_r} \), with \( \beta_f, \beta_r \) a different coefficient for front and rear caps. The uniform thin-film region is connected to the static cap of constant curvature at the extremities of the bubble through a dynamic meniscus [10] (see Fig. 1). The counterpart theory for a bubble in a square duct was derived by Wong et al. [59, 60]. However, these scalings agree with Taylor’s experimental results [56] only in the small \( Ca \) limit, namely when \( Ca \gtrsim 10^{-3} \). In order to understand the dynamics of confined bubbles in a broader parameter range, researchers have performed experiments [11, 5, 16, 24, 7] as well as numerical simulations [53, 48, 47, 19, 20, 26, 35, 36, 22, 3, 2, 39]. As an outcome, several correlations have been proposed for the evolutions of the relevant quantities as a function of the different parameters (see for example Ref. [24] and Ref. [39]). Among them, Aussillous
Furthermore, being able to predict the flow field in- 
for heat transfer or cleaning of microchannels applications 
to be accurately predicted as well.

Lam et al. [33] and Cherukumudi et al. [12] tried to put a theoretical 
laratory numbers up to 1. The two recent works of Klaseboer et 
ment with the experimental results of Taylor [56] for capil-

& Quéré [5] proposed an ad-hoc rational function with a fitting 
parameter for the film thickness which is in good agree-
ment with the experimental results of Taylor [56] for capil-
lar numbers up to 1. The two recent works of Klaseboer et 
large-capillary-number range, $Ca \sim O(1)$, where the lub-

In contrast to bubbles, which have experienced a vast in-
terest of the scientific community, little amount of effort has 
been made for droplets whose viscosities are comparable to 
or much larger than that of the outer fluid. Yet, droplets of ar-
bitrary viscosities are crucial for Lab-on-a-Chip applications 
[4]. A first theoretical investigation of the effect of the inner 
phase viscosity was conducted by Schwartz et al. [52], mo-
tivated by the discrepancy in the predicted and the measured 
film thicknesses of long bubbles in capillaries. They demon-
strated that the non-vanishing inner-to-outer viscosity ratio 
could thicken the film. Hodges et al. [28] further extended 
the theory and showed that the film becomes even thicker at intermediate viscosity ratios. Numerical simulations have 
been performed to investigate the droplets in capillaries [43, 
57,36].

Models predicting the characteristic quantities such as the 
uniform film thickness and the meniscus curvatures of 
droplets in capillaries over a wide range of capillary num-
bers are still missing. For example, the velocity of a droplet of 
finit viscosity flowing in a channel still remains a simple 
question yet an open challenge. Such a prediction is, 
however, of paramount importance for the correct design of 
droplet microfluidic devices. As an example, Jakiela et 
[30] performed extensive experiments for droplets in square 
ducts, showing complex dependencies of the droplet veloc-
ity on the capillary number, viscosity ratio and droplet length. 
Also, what is the pressure drop induced by the presence of a 
drop in a channel? This question is crucial and has been the 
subjects of recent works, for example Refs. [58,37]. Other 
quantities, such as the minimum film thickness $H_{\min}$, have 
to be accurately predicted as well. $H_{\min}$ becomes essential 
for heat transfer or cleaning of microchannels applications 
[42,31]. Furthermore, being able to predict the flow field in-
side and outside of the droplet is crucial if one is interested 
in the mixing capabilities of the system.

Here, we aim at bridging this gap by combining asympto-
tic derivations with accurate numerical simulations to pro-
pose physically inspired empirical models, for the charac-
teristic quantities of a droplet of arbitrary viscosity ratio 
flowing in an axisymmetric or planar capillary with a con-
tant velocity. The model coefficients are specified by fit-
ting laws. The present work provides the reader with a rig-
orous theoretical basis, which can be exploited to understand 
the dynamics of viscous droplets. The considered capillary 
numbers vary from $10^{-4}$ to 1 and the inner-to-outer viscos-
ity ratio from 0 to $10^3$. Following the work of Schwartz et 
al. [52], we extend the low-capillary-number asymptotical 
results obtained with the lubrication approach of Bretherton 
[9] for bubbles to viscous droplets. Numerical simulations 
are performed on finite element method (FEM) employing the arbi-
trary Lagrangian-Eulerian (ALE) formulation are performed 
for the uniform-film region are derived in Sec. 3.1 
and the flow patterns are presented in Sec. 3.2. The theoretical 
part starts with the asymptotic derivation of the model 
for the uniform film thickness in Sec. 4. The derivation of the 
lubrication equation is detailed in Sec. 4.1, followed by the 
film thickness model in Sec. 4.2 and its extension to larger 
capillary numbers in Sec. 4.3. With the knowledge of the 
film thickness, the droplet velocity can be computed analyt-
ically (see Sec. 5). The minimum film thickness separating 
the droplet form the channel walls is discussed in Sec. 6. To 
build a total pressure drop model, one still needs the knowl-
edge of the front and rear caps mean curvatures (see Sec. 
7.1), the front and rear pressure jumps (see Sec. 7.2 and 7.4) 
and the front and rear normal viscous stress jumps (see Sec. 
7.3). The stresses evolutions at the channel centerline and at 
the wall are presented in Sec. 8.1 and Sec. 8.2, respectively. 
Eventually, one can sum up all these contributions to build 
the total pressure drop, which is described in Sec. 8.3. We 
summarize our results in Sec. 9.

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**Fig. 1** Sketch of the front meniscus of the bubble advancing at velocity $U_d$ in a capillary of radius $R$ with indication of the uniform thin-film region of thickness $H_\infty$, the dynamic meniscus region and the static meniscus region. The plane curvature $\kappa_f$ of the front static cap in the $(z,r)$ or $(x,y)$ plane is also highlighted.
2 Governing equations and numerical methods

2.1 Problem setup

We consider an immiscible droplet of volume $\Omega$ and dynamic viscosity $\mu_i$, translating at a steady velocity $U_d$ in a planar channel/axisymmetric tube of width/diameter $2R$ filled with an outer fluid of dynamic viscosity $\mu_o$. The volume flux of the outer fluid is $q_o$, resulting in an average velocity $U_\infty = q_o/(2R)$ and $U_\infty = q_o/(\pi R^2)$ for the planar and axisymmetric configuration, respectively (see Fig. 2). Given the small droplet velocity and size, the Reynolds number $Re$ is small and inertial effects can be neglected. Buoyancy is also neglected. The relevant dimensionless numbers include the droplet capillary number $Ca = \mu_i U_d/\gamma$ between the droplet and the outer fluid. The capillary number based on the mean flow velocity is $Ca_\infty = \mu_i U_\infty/\gamma$. For the numerical simulations, the droplet capillary number has been varied within $10^{-4} \leq Ca \leq 1$ to guarantee that the lubrication film is only influenced by the hydrodynamic forces and the dynamics is steady. For smaller capillary numbers, non-hydrodynamic forces such as disjoining pressures due to intermolecular forces might come into play as reported by the recent experiments [29]; while for larger capillary numbers, instability and unsteadiness might arise, where the droplet might form a re-entrant cavity at its rear [57], which eventually breaks up into satellite droplets. The viscosity ratios investigated numerically are from the well-known bubble limit of $\lambda = 0$ [9] to highly-viscous droplets of $\lambda = 100$ that has been scarcely investigated.

We consider both a three-dimensional axisymmetric tube, and a two-dimensional planar channel featured by the spanwise invariance. Note that the latter configuration does not correspond to the Hele-Shaw-cell-like microfluidic chips, where Darcy or Brinkman equations are more appropriate to describe the flow [8,45].

It is worth noting that the confined droplet has to be long enough to develop a region of uniformly thick film at its center [10] (see Fig. 2). However, a long axisymmetric droplet is likely to become unstable to the Rayleigh-Plateau instability. The uniform film region would resemble to a coaxial jet, which is known to be unstable to perturbations with a wavelength longer than $2\pi(R - H_{\infty})$ [14]. Within this range of droplet lengths, we found that the effect of droplet volume $\Omega$ is insignificant and hence it is fixed to $\Omega/R^2 = 12.9$ for the axisymmetric geometry and $\Omega/R^2 = 9.3$ for the planar case.

2.2 Governing equations

The governing equations are the incompressible Stokes equations for the velocity $\mathbf{u} = (u, v)$ and pressure $p$:

\[
\nabla \cdot \mathbf{u} = 0 \quad (1)
\]

\[
\nabla \cdot \sigma = 0 \quad (2)
\]

where $\sigma = -p I + \mu \left( (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \right)$ denotes the total stress tensor and $\mu$ the dynamic viscosity as $\mu_i$ (resp. $\mu_o$) inside (resp. outside) the droplet.

The imposed dynamic boundary conditions at the interface are the continuity of tangential stresses

\[
\Delta \left[ (1 - \mathbf{n}) \cdot (\sigma \cdot \mathbf{n}) \right] = 0, \quad (3)
\]

and the discontinuity of normal stresses due to the Laplace pressure jump

\[
\Delta (\sigma \cdot \mathbf{n}) = -\gamma \delta \mathbf{n}. \quad (4)
\]

$\Delta$ denotes the difference between the inner and outer quantities, $\mathbf{n}$ the unit normal vector on the interface towards the outer fluid, and $\gamma = \nabla_s \cdot \mathbf{n}$ the interfacial mean curvature ($\nabla_s$ is the surface gradient). The plane curvature of interface in the $(z, r)$ or $(x, y)$ plane is denoted as $\kappa$ and its value at the front and rear droplet extremities is given by $\kappa_f$ and $\kappa_r$, respectively. The mean curvature at the front and rear droplet extremities, which lie on the symmetry axis, is therefore given by $\kappa_{f,r} = \chi \kappa_{f,r}$, where $\chi = 1$ or 2 for a planar or axisymmetric configuration, respectively. At any other point on the droplet interface, the mean curvature is given by the sum of the two principal curvatures.

2.3 Numerical methods and implementations

Equations (1)-(2) with boundary conditions (3)-(4) are solved by the commercial FEM package COMSOL Multiphysics and the interface is well represented by the arbitrary Lagrangian-Eulerian (ALE) technique. Compared to the commonly known

![Fig. 2 Sketch of the axisymmetric $(z, r)$ and planar $(x, y)$ configurations in the frame of reference moving with the droplet. The flow profiles in the uniform film region are shown in Fig. 7.](image-url)
diffuse interface methods such as volume-of-fluid, phase-field, level-set and front-tracking all relying on a fixed Eulerian grid, the ALE approach captures the interface more accurately. Since the interface is always explicitly represented by the discretization points (see Fig. 3), the fluid quantities (viscosity, density, etc.), pressure and normal viscous stresses, are discontinuous across the interface. This technique has been used to simulate three-dimensional bubbles in complex microchannels [3], liquid films coating the interior of cylinders [25], two-phase flows with surfactants [17, 18] and head-on binary droplet collisions [41], to name a few. The Moving Mesh module of COMSOL Multiphysics has been recently employed in Refs. [49, 23] to investigate the inertial and capillary migration of bubbles in round and rectangular microchannels.

Despite the high fidelity in interface capturing, it is commonly more challenging to develop in-house ALE implementations compared to the diffuse interface counterparts. Additional difficulty arises in the case of large interfacial deformations, when the computational mesh might become highly nonuniform and skewed. It is therefore necessary to remesh the computational domain and to obtain all the quantities on the new mesh via interpolation. Hence, special expertise in scientific computing and tremendous amount of development effort is required to implement an in-house ALE-based multi-phase flow solvers, which have prevented large portion of the research community from enjoying the high fidelity and elegance of the ALE methods.

Hereby, large mesh deformations can be avoided not only thanks to the convenient moving mesh module of COMSOL, but also by solving the problem in the moving frame of droplet. To achieve so, we impose a laminar Poiseuille in-flow of mean velocity $U_\infty - U_d$ at the inlet of the channel and the velocity $-U_d$ at the walls, where the unknown droplet velocity $U_d$ is obtained as part of the solution together with that of the flow field, at each time step, by applying an extra constraint of zero volume-integrated velocity inside the droplet. Such constraints with additional unknowns are imposed in COMSOL Multiphysics by utilizing its so-called 'Global Equations'. This strategy ensures that the deforming droplet barely translates in the streamwise direction, staying approximately at its initial position (say in the center of the domain). Hence, the mesh quality and the robustness of the ALE formulation is appropriately guaranteed.

In this work, we are only concerned with the steady dynamics of the droplet reaching its equilibrium shape. We do not solve the steady Stokes Eq. (2) strictly but maintain a negligible time-derivative term $Re \frac{\partial u}{\partial t}$ for time marching. This procedure can be seen as an iterative scheme to find steady solutions of the Stokes equations. Since, indeed, the time-derivative term vanishes when the equilibrium state is reached, the solutions to the steady Stokes equations are eventually obtained.

It has to be stressed the computed transient dynamics is not physical and only the final steady solution should be considered. Theoretically, the stability of the latter might be affected by the time-derivative term. However, in practice, no unstable phenomena or multiple-branch solutions were observed in our study. We monitored, during each simulation, the temporal evolution of the velocity, the mean and minimum film thickness as well as the front and rear plane curvatures of the droplet, which exhibited precise time-invariance without exception when the equilibrium state was reached. Our method therefore converges to valid stationary solutions, which will be seen in Sec. 2.4 to compare well with asymptotic estimates and numerical solutions from the literature.

To reduce the computational cost, half of the channel is considered and axisymmetric or symmetric boundary conditions are imposed at the channel centerline for the axisymmetric and planar configurations, respectively. The setup in COMSOL Multiphysics is intrinsically parallel and the computing time required for an individual case needs no more than one hour based on a standard desktop computer.

A typical mesh is shown in Fig. 3. Triangular/quadrilateral elements are used to discretize the domain inside/outside the droplet. Furthermore, a mesh refinement is performed to best resolve the thin lubrication film (see inset of Fig. 3). It is worth-noting that quadrilateral elements have to be used to discretize the thin film because this region might undergo large radial deformation resulting in highly distorted and skewed triangular elements if used.

![Fig. 3 Computationa](image)

**Fig. 3** Computational mesh. Inset: mesh refinement in the thin-film region. The triangular inner-phase (blue) and quadrilateral outer-phase meshes (red) are separated by the explicitly discretized interface (dashed green).
2.4 Validation

Our numerical results are first validated for a bubble ($\lambda = 0$) comparing the film thickness with the classical asymptotic theory $H_\infty/R \sim 0.643(3Ca)^{2/3}$ of Bretherton in the low-$Ca$ limit [9] (see Fig. 4). Excellent agreement is revealed even when the capillary number is $10^{-4}$; the discrepancy at larger $Ca$ is mostly because of the asymptotic nature of the model that becomes less accurate for increasing $Ca$. At larger capillary numbers, we compare the uniform film thickness with the FEM-based numerical results of Ref. [19] for a bubble, showing perfect agreement in Fig. 5; agreements for the front and rear plane curvatures are also observed and are not reported here.

For viscosity ratios $\lambda > 0$, we have validated our setup against the results from an axisymmetric boundary integral method (BIM) solver [36] for a droplet with $Ca_\infty = 0.05$ of viscosity ratios $\lambda = 0.1$ and 10, again exhibiting perfect agreement as displayed in Fig. 6.

Based on the carefully performed validations against the theory, numerical results from FEM and BIM solvers, we are confident that the developed COMSOL implementation can be used to carry out high-fidelity two-phase simulations efficiently, at least for the 2D and 3D-axisymmetric configurations.

3 Flow field

3.1 Velocity profiles in the thin-film region

For a sufficiently long droplet/bubble, a certain portion of the lubrication film is of uniform thickness $H_\infty$ [9] (see Fig. 2 and Fig. 8). Within this portion, the velocity field both inside and outside the droplet is invariant in the streamwise direction and resembles the well known bi-Poiseuille profile that typically arises in several interfacial flows, for example a coaxial jet [27] (see Fig. 7). For $\lambda \ll 1$, the velocity profile in the film is almost linear, whereas for $\lambda \gg 1$, the velocity inside of the droplet is almost constant (plug-like profile). Nevertheless, the parabolic component of these profiles is crucial for the accurate prediction of the droplet velocity (see Sec. 5).

Assuming the bi-Poiseuille velocity profile, we describe the streamwise velocity $u_i(r)$ inside and $u_o(r)$ outside the droplet as a function of the off-centerline distance $r$ as:

$$u_i(r) = \frac{1}{4\mu_i} \frac{dp_i}{dz} r^2 + A_i \ln r + B_i,$$

$$u_o(r) = \frac{1}{4\mu_o} \frac{dp_o}{dz} r^2 + A_o \ln r + B_o,$$

where $p_i$ and $p_o$ are the inner, respectively outer, pressures, and $A_i, B_i, A_o$ and $B_o$ are undetermined constants. Given the
the droplet, in its moving reference frame, is given by

$$u_\infty \text{ at } r = 0, \text{ we have } A_0 = 0. \text{ By satisfying the no-slip boundary condition on the channel walls } u_\infty (R) = -U_d, \text{ the continuity of velocities and tangential stresses on the interface } r = \bar{r} = R - H, \text{ namely, } u_i(\bar{r}) = u_o(\bar{r}) \text{ and}

$$

$$\frac{\mu_i du_i}{dz} \bigg|_{r=\bar{r}} = \frac{\mu_o du_o}{dz} \bigg|_{r=\bar{r}},$$

(7)

we obtain the remaining constants

$$A_o = \frac{1}{2\mu_o} \left( \frac{dp_i}{dz} - \frac{dp_o}{dz} \right) R^2,$$

(8)

$$B_i = -\frac{1}{4\mu_i \mu_o} \left( \frac{dp_i}{dz} (R^2 - r^2) \right) \mu_i + \frac{dp_i}{dz} \left( \frac{1}{\bar{r}} \right) - U_d,$$

(9)

$$B_o = -\frac{1}{4\mu_o} \left( \frac{dp_o}{dz} R^2 + 2 \left( \frac{dp_i}{dz} - \frac{dp_o}{dz} \right) R^2 \right) \ln \left( \frac{R}{\bar{r}} \right) - U_d. \quad (10)$$

Under the assumption of a slowly evolving film thickness, this velocity profile also holds in the nearby regions, where the thickness is $H$ rather than $H_\infty$. The derivation for the planar geometry is given in Appendix A.

3.2 Recirculating flow patterns

The axisymmetric velocity profile in the channel away from the droplet, in its moving reference frame, is given by

$$u_\infty (r) = 2U_\infty \left[ 1 - \left( \frac{r}{R} \right) \right] - U_d. \quad (11)$$

As will become clear in Sec. 5, the droplet velocity can be obtained by imposing mass conservation. For the particular case of a bubble with $\lambda = 0$, mass conservation reads $$(U_\infty - U_d) \pi R^2 = -\pi [R^2 - (R - H_\infty)^2] U_d \quad [54],$$ yielding $U_\infty / U_d = (1 - H_\infty / R)^2$. The velocity profile (11) can therefore be expressed as a function of $H_\infty / R$. As pointed out by Giavedoni & Saita [19], when $\lambda = 0$, the velocity at the centerline $u_\infty (r = 0)$ in the bubble frame changes sign when $H_\infty = H_\infty^* = (1 - 1/\sqrt{2}) R$. As a consequence, when the uniform film thicknesses is below $H_\infty^*$, $u_\infty (0) > 0$, an external recirculating flow pattern forms ahead of and behind the bubble. For the planar configuration, the critical film thickness for the appearance of the flow recirculation is $H_\infty^* = R/3$.

Based on the flow profiles derived in Sec. 3 and mass conservation (see Sec. 5), we can generalize the critical thickness $H_\infty^*$ to non-vanishing viscosity ratios ($\lambda > 0$) as (see Appendix D for the derivations):

$$H_\infty^* = \frac{1}{\lambda - 1}$$

(12)

for the axisymmetric case and

$$H_\infty^* = \frac{1}{3} \frac{1}{\lambda - 1}$$

(13)

for the planar case. $H_\infty^*/R$ reaches the value of 1 when $\lambda = 1/2$ or $\lambda = 2/3$, for the axisymmetric and planar configuration, respectively. Nevertheless, the uniform film thickness is always much smaller than the channel half-width, $H_\infty \ll R$. For a fixed volume of fluid, large film thicknesses would correspond to long droplets, which might be unstable to the Rayleigh-Plateau instability as discussed in Sec. 2.1.

For viscosity ratios $\lambda \geq 1/2$ ($\lambda \geq 2/3$) for the axisymmetric (planar) configuration, there is no critical film thickness above which the recirculation zones disappear, meaning that a recirculation region always exists for any capillary number. At low capillary numbers, when the film thickness is below $H_\infty^*$, the external recirculating flows are strong enough to induce the recirculation inside the droplet as well. Consequently, besides the two droplet vertices as permanent stagnation points (blue circles in Fig. 8), two stagnation rings emerge on the front and rear part of the axisymmetric interface (red stars in Fig. 8); likewise, four stagnation points arise in the planar case. The stagnation rings/points on the dynamic meniscus (red stars in Fig. 8) move outwards to the droplet vertices when $Ca$ increases. When $H_\infty > H_\infty^*$, these stagnation rings/points disappear, taking away with the recirculation regions accordingly (see Fig. 8(b)). Only the stagnation points at the droplet vertices remain.

However, since the stagnation rings/points at the droplet interface move outwards to the front and rear extremities when the film thickness increases with the capillary number, the recirculation regions might eventually detach from the interface before the critical film thickness $H_\infty^*$ is reached. Nevertheless, since recirculation regions exist far away from the droplet as long as $H_\infty < H_\infty^*$, another flow pattern must...
Fig. 8 Streamlines and recirculation patterns for an axisymmetric droplet with different capillary numbers \( Ca_\infty \) and viscosity ratios \( \lambda \) in a frame of reference moving with the droplet. The results are obtained from FEM-ALE numerical simulations. The permanent stagnation points at the droplet vertices are highlighted by blue circles, whereas the \((Ca_\infty, \lambda)\)-dependent stagnation rings/points are highlighted by red stars. Insets: detailed flow pattern at the droplet front and rear for \( \lambda = 0.3 \) and \( Ca_\infty = 0.6 \).

The phase diagram with the main different types of flow patterns as a function of the viscosity ratio \( \lambda \) and film thickness \( H_\infty / R \) is shown in Fig. 9. Note that for viscosity ratios \( \lambda \geq 1/2 \) \((\lambda \geq 2/3)\), the recirculation regions will be attached or detached from the droplet interface depending on the uniform film thickness. Other very peculiar flow fields, as a detached finite recirculation region at the rear or a detached recirculation region at the front as observed by Giavedoni & Saita [19, 20], can be obtained for some parameter combinations. However, since the flow field structures are not the main aim of this work, an extended parametric study to detect all possible patterns has not been performed. Nevertheless, our current results do not validate all the flow patterns predicted in Ref. [28] based on asymptotic arguments, which indeed have not been verified neither experimentally nor numerically.

The flow fields for the planar configuration are not presented here as they are similar to those for the axisymmetric configuration.

4 Film thickness

4.1 Asymptotic result in the \( \text{low-}Ca \) limit

By following the work of Schwartz et al. [52], we derive an implicit expression predicting the thickness \( H_\infty \) of the uniformly-thick lubrication film in the \( \text{low-}Ca \) limit when \( H/R \ll 1 \) satisfies. The derivation of the axisymmetric case is presented below, see Appendix B for the planar case.
The flow rates at any axial location where the external film thickness is $H$ are:

$$q_i = 2\pi \int_0^R u_i(r) \, rdr$$

$$= -\pi (R-H)^2 \left\{ U_d + \frac{1}{8\mu_i \mu_o} \left[ 2 \frac{d \rho_o}{d z} (2R-H) \mu_i \right] + 4 \frac{d \rho_i}{d z} (R-H)^2 \mu_o \right. + 4 \left( \frac{d \rho_i}{d z} - \frac{d \rho_o}{d z} \right) (R-H)^2 \mu_i \ln \left( \frac{R}{R-H} \right) \right\},$$

$$q_o = 2\pi \int_0^R u_o(r) \, rdr$$

$$= -\pi \frac{\mu_o}{8\mu_i} \left\{ H(2R-H) \left[ H^2 \left( 2 \frac{d \rho_i}{d z} - 3 \frac{d \rho_o}{d z} \right) + 2 \left( \frac{d \rho_i}{d z} - \frac{d \rho_o}{d z} \right) R^2 - 4 \left( \frac{d \rho_i}{d z} - 6 \frac{d \rho_o}{d z} \right) R \right] + 4 \frac{d \rho_i}{d z} - \frac{d \rho_o}{d z} \right\} (R-H)^4 \ln \left( \frac{R}{R-H} \right) - \pi H(2R-H) U_d.$$  

Assuming that $H/R < 1$, the volumetric fluxes up to the second order are

$$q_i \approx -\pi R^2 \left( U_d + \frac{1}{8\mu_i} \frac{d \rho_i}{d z} R^2 + \frac{1}{2\mu_o} \frac{d \rho_o}{d z} RH + \frac{1}{2\mu_o} \frac{d \rho_o}{d z} R^2 \right),$$

$$q_o \approx -2\pi RH \left( U_d + \frac{1}{4\mu_o} \frac{d \rho_i}{d z} HR + \frac{1}{3\mu_o} \frac{d \rho_o}{d z} H^2 \right).$$

In the droplet frame, the inner flow rate is $q_i = 0$. Furthermore, in the region with a uniformly-thick film, $H = H_o$: the inner and outer pressure gradient balances, $rac{d \rho_i}{d z} = \frac{d \rho_o}{d z}$, Using these two conditions one can obtain the pressure gradient in the uniform film region

$$\left. \frac{d \rho}{d z} \right|_{r=R-H_o} \approx -\frac{8\mu_i U_d}{R^2 + 4\lambda H_o R + 4\lambda H_o^2}$$

and the outer flow rate in the $H_o/R < 1$ limit is

$$q_o \approx -2\pi RH_o \left[ \frac{3R^2 + 6\lambda H_o R + 4\lambda H_o^2}{3(R^2 + 4\lambda H_o R + 4\lambda H_o^2)} \right] U_d$$

$$\approx -2\pi RH_o \frac{R + 2\lambda H_o}{R + 4\lambda H_o} U_d.$$  

In the dynamic meniscus regions, the inner and outer pressure gradients are not equal and their difference is proportional to the mean curvature of the interface at $r = R - H$, through Laplace’s law. By assuming a quasi-parallel flow whose radial velocity is weaker than the axial velocity by one order of magnitude and a small cavillarity number, the normal stress condition at the interface is not affected by viscous stresses. Hence, the axial gradient of the normal stress condition reads

$$\frac{d \rho_i}{d z} - \frac{d \rho_o}{d z} = \frac{2}{\nu} \frac{d^3 H}{d z^3},$$

where the axial gradient of the curvature in the azimuthal direction is neglected as it is an order smaller. The pressure gradients $d \rho_i/dz$ and $d \rho_o/dz$ as functions of $H$ can be obtained from Eqs. (16), (17) imposing mass conservation, i.e. imposing $q_i = 0$ and $q_o$ given by Eq. (19):

$$\frac{d \rho_i}{d z} \approx \frac{4\lambda (-6H_o^2 \lambda + 4H_o H \lambda - 3H_o R + HR) \mu_i U_d}{H(4H_o \lambda R + R) (H \lambda + R)}$$

$$\frac{d \rho_o}{d z} \approx \frac{3(H_o - R) [8H_o \lambda R + 2(\lambda H_o + H) \lambda R + R^2] \mu_o U_d}{H^3 (4H_o \lambda R + R) (H \lambda + R)}.$$  

By plugging Eqs. (21), (22) into Eq. (20) and adopting the change of variables $H = H_o R$ and $z = H_o (3Ca)^{-1/3} \xi$ in the spirit of Bretherton [9], we obtain an universal governing equation for the scaled film thickness $\eta$ when taking the limit $H_o/R \to 0$:

$$\frac{d^3 \eta}{d \xi^3} = \eta - 1 - \frac{1}{\eta^3} \left[ \frac{1 + 2m(1 + \eta + 4m \eta)}{(1 + 4m)(1 + m \eta)} \right],$$

where $m = \lambda H_o R$ denotes the rescaled viscosity ratio. The corresponding planar counterpart reads (see derivation in Appendix B)

$$\frac{d^3 \eta}{d \xi^3} = 2 \frac{\eta - 1}{\eta^3} \left[ \frac{2 + 3m(1 + \eta + 3m \eta)}{(1 + 3m)(4 + 3m \eta)} \right].$$

If the limit of vanishing uniform film thickness is not considered, the resulting equations for $\eta$ would depend on $H_o/R$ [50]. In the limit of $m \to 0$, the classical Landau-Levich-Derjaguin equation [13, 38] is retrieved for both configurations. Following Bretherton [9], Eqs. (23) and (24) can be integrated to find the uniform film thickness $H_o/R$ (see also Cantat [10] for more details).

First, the equations can be linearized in the uniform film region around $\eta \approx 1$, giving

$$\frac{d^3 \eta}{d \xi^3} = K(\eta - 1),$$

where $K$ is a constant depending on the geometrical configurations and the viscosity ratio $m$. Equation (25) has a monotonically increasing solution with respect to $\xi$ for the front dynamic meniscus, $\xi \to \infty$, and an oscillatorily increasing solution for the rear, $\xi \to -\infty$, as derived in Sec. 6. The solution of the front meniscus is $\eta(\xi) = 1 + \alpha \exp(K^{1/3} \xi)$, where $\alpha$ is a small parameter, typically $10^{-6}$. Second, the
nonlinear equations (23) and (24) can be integrated numerically as an initial value problem with a fourth-order Runge-Kutta scheme, starting from the linear solution until the plane curvature of the interface profile becomes constant. A region of constant plane curvature, called static meniscus region (see Fig. 1) exists as $d^3\eta/d\xi^3 \approx 0$ for $\eta \gg 1$ (see red line on Fig. 11). In the static meniscus region, the interface profile is a parabola: $\eta = P(3C)^{2/3}(\xi^2 + C\xi + D)$, or, in terms of film thickness, $H = P(3C)^{2/3}(\xi^2 + C(3C)^{1/3}\xi + DH_\infty)$, where $P$, $C$ and $D$ are real-valued constants. Thus, $P$ is set by the constant plane curvature obtained by the integration of the nonlinear equation.

The procedure can be repeated for any rescaled viscosity ratio $m$ and the obtained results for the coefficient $P$ can well described by the fitting law [52]:

$$P(m) = \frac{0.643}{2} \left( 1 + 2^{2/3} + (2^{2/3} - 1) \tanh [1.2\log_{10}m + c_1] \right)$$

(26)

where the constant $c_1 = 0.1657$ for the axisymmetric configuration and $c_1 = 0.0159$ for the planar configuration (see Fig. 10). The well known limits for a bubble $P(0) = 0.643$ [9] and a very viscous droplet $P(m \to \infty) = 2^{2/3}P(0)$ [10] are recovered.

![Fig. 10](image)

Fig. 10 Film-thickness coefficient $P$ obtained for discrete $m$ values (symbols) and fitting law (26) (solid lines) as a function of the rescaled viscosity ratio $m$.

To obtain the uniform film thickness, the matching principle proposed by Bretherton [9] is employed. The plane curvature $\kappa = d^2H/dz^2 = P(3C)^{2/3}/H_\infty$ in the static region has to match that of the front hemispherical cap of radius $R$, which exists for small capillary numbers (see red dashed line in Fig. 11). A rigorous asymptotic matching can be found in Park & Homay [46] for a bubble with $m = 0$. When $m \neq 0$, the coefficient $P(m)$ depends implicitly on $H_\infty$ and thus on $Ca$, through $m$, leading to an implicit asymptotic relation for $H_\infty/R$ as:

$$H_\infty/R = P(m)(3C)^{2/3}.$$  

(27)

Strictly speaking, the uniform film thickness of viscous droplets ($\lambda \neq 0$) in the low $Ca$ limit does not scale with $Ca^{2/3}$ as for a bubble ($\lambda = 0$).

4.2 Empirical model in the low-Ca limit

Equation (27) holds for capillary numbers as low as below $10^{-3}$ [9]. We solve Eq. (27) numerically and present the coefficient $P$ and the uniform film thickness $H_\infty/R$ versus $Ca$ in Fig. 12 for the axisymmetric case. In order to derive an explicit formulation to predict the film thickness in this $Ca$ regime, we define $\bar{P}$ as a $Ca$-averaged value of $P$ and define the empirical model

$$\frac{H_\infty}{R} = \bar{P}(\lambda)(3C)^{2/3},$$

(28)

where $\bar{P}(\lambda)$ is independent of $Ca$ (see dashed lines in Fig. 12(a)) and can be approximated by the fitting law (see Fig. 13):

$$\bar{P}(\lambda) = \frac{0.643}{2} \left( 1 + 2^{2/3} + (2^{2/3} - 1) \tanh [1.28\log_{10}\lambda + c_2] \right).$$

(29)

where the constant $c_2 = -2.36$ for the axisymmetric case and $c_2 = -2.52$ for the planar case are obtained by fitting. For $\lambda = 0$, $\bar{P} = 0.643$ is recovered and $H_\infty/R$ indeed scales with $Ca^{2/3}$, at least when $Ca < 10^{-3}$. Figure 12(b) also shows...
that the empirically obtained film thickness (dashed lines) Eq. (28) agrees reasonably well with the FEM-ALE simulation results (symbols), whereas the implicit law (solid lines) Eq. (27) slightly underestimates them at very low \( \text{Ca} \). To cure this mismatch, Hodges et al. [28] proposed a modified interface condition, which however is found to overestimate the thickness more than that underestimated by the original implicit law.

Despite the explicit law for the uniform-film thickness prediction with \( \bar{P} \) proved to be satisfactory, its validity range is restricted to low capillary numbers. As known since the experiments of Taylor [56], the film thickness of a bubble saturates for increasing \( \text{Ca} \). Aussillous and Quéré [5] proposed a model for \( \lambda = 0 \), which agrees well with the experimental data of [56], further inspiring the two very recent works of Refs. [33,12]. In the same vain, we propose an empirical model for the film thickness \( H_{\infty} \) as a function of both \( \text{Ca} \) and \( \lambda \)

\[
\frac{H_{\infty}}{R} = \frac{\bar{P}(\lambda)(3\text{Ca})^{2/3}}{1 + \bar{P}(\lambda)Q(\lambda)(3\text{Ca})^{2/3}}.
\]

where the coefficient \( Q \) is obtained by fitting Eq. (30) to the database constructed from our extensive FEM-ALE simulations over a broad range of \( \text{Ca} \) for different \( \lambda \). The proposed function of \( Q(\lambda) \) is given in Appendix E and plotted in Fig. 14. For an axisymmetric bubble, we find \( Q = 2.48 \), in accordance with the estimation \( Q = 2.5 \) of Ref. [5]. We now present in Fig. 15 the numerical film thickness (symbols) and the empirical model (lines) for \( \lambda = 1 \). For the sake of clarity, the results for \( \lambda = 0 \) and 100 are shown in the appendix F on Fig. 25. For \( \lambda = 1 \), the thickness of the two configurations coincide. However, when \( \text{Ca} \sim \mathcal{O}(1) \), the film is thicker in the planar configuration than in the axisymmetric one for a bubble \( (\lambda = 0) \); the trend reverses for a highly-viscous droplet \( (\lambda = 100) \). This \( \lambda \)-dependence of the film thickness is indeed implied by the crossover of the two fitting functions \( Q(\lambda) \) at \( \lambda = 1 \) shown in Fig. 14.

It has to be noted that when the capillary number is increased, the regions of constant plane curvature in the static front and rear caps reduce in size and eventually disappear (see Fig. 11), and this for all viscosity ratios. The matching to a region of constant plane curvature for large capillary numbers as proposed by Refs. [33,12] might be questionable for this \( \text{Ca} \)-range.

The uniform film thickness of droplets with 37% and 82% larger volume, resulting in longer droplets, are com-
expressions for the pressure gradient
\[ dp \Big|_{r=R-H_w} = - \frac{8R^2U_w \mu_i}{(R-H_w)^4 + H_w(2R-H_w)(2R^2-2H_wR+H_w^2)\lambda}, \]
and the droplet velocity
\[ U_d = \frac{R^2[(R-H_w)^2 + 2H_w(2R-H_w)\lambda]}{(R-H_w)^4 + H_w(2R-H_w)(2R^2-2H_wR+H_w^2)\lambda} U_w. \]

The relative velocity of the axisymmetric droplet with respect to the underlying velocity reads
\[ \frac{U_d - U_w}{U_d} = \frac{(2 - \frac{H_w}{R}) \frac{H_w}{R} [1 + (2 - \frac{H_w}{R}) \frac{H_w}{R}(\lambda - 1) \right]} {1 + (2 - \frac{H_w}{R}) \frac{H_w}{R}(\lambda - 1)}. \]

An analogous derivation for the planar configuration yields (see Appendix C):
\[ \frac{U_d - U_w}{U_d} = \frac{\frac{H_w}{R} \left( 2 - \frac{H_w}{R} \left[ 4 + 2\frac{H_w}{R} (\lambda - 1) - 3\lambda \right] \right)} {2 + \left( 2 - \frac{H_w}{R} \right) \frac{H_w}{R}(3\lambda - 2)}. \]

Eqs. (30) and (32) form a system of the two unknowns, namely the droplet capillary number \( Ca \) and the uniform film thickness \( H_w/R \). It is important to remind that the former is related to the droplet velocity via \( Ca = Ca_{w} U_{d}/U_{w} \).

For a given combination of inflow capillary number \( Ca_{w} \) and viscosity ratio \( \lambda \) as the input, the system can be solved numerically (see Matlab file filmThicknessAndVelocity.m in the Supplementary Material) outputting \( Ca \) and \( H_w/R \). The predicted relative velocity \( (U_d - U_w)/U_d \) (lines) agrees well the FEM-ALE simulation results (symbols) as shown in Fig. 16.

In the limit of \( H_w/R \to 0 \), the relative velocity can be approximated asymptotically as
\[ \frac{U_d - U_w}{U_d} = 2 \left( \frac{H_w}{R} \right) - (1 + 4\lambda) \left( \frac{H_w}{R} \right)^2 + O \left( \frac{H_w}{R} \right)^3 \]
for the axisymmetric case, and
\[ \frac{U_d - U_w}{U_d} = \left( \frac{H_w}{R} \right) - \frac{3\lambda}{2} \left( \frac{H_w}{R} \right)^2 + O \left( \frac{H_w}{R} \right)^3 \]
for the planar geometry. For very low capillary numbers, the asymptotic estimates predict that the relative droplet velocity scales with \( H_w/R \), and hence with \( Ca^{2/3} \) [54]. The viscosity ratio \( \lambda \) only enters at second order of \( H_w/R \), which however influences the validity range of the asymptotic estimates (35) and (36) considerably. The asymptotic estimates are exact for \( \lambda = 0 \). In this case, Eqs. (35) and (36) reduce to the well known predictions for bubbles \( (2-H_w/R)H_w/R \) and \( H_w/R \) [9,54,39], respectively (see Fig. 16(a)). For non-vanishing \( \lambda \), the complete expressions (33) and (34) should be employed (see Fig. 16(b)). For example, the asymptotic estimate for \( \lambda = 100 \) is only valid when \( Ca_{w} < 10^{-4} \) (see Fig. 16(c)).


6 Minimum film thickness

At low capillary numbers $Ca$, the droplet interface exhibits an oscillatory profile between the uniform thin film and the rear static cap (see Fig. 11). The minimum film thickness in the low $Ca$ limit can be computed by integrating the lubrication equation (23) or (24) for $\xi \to -\infty$. The initial condition for this initial value problem is given by the solution of the linear equation (25) for negative $\xi$: $\eta = 1 + \alpha \exp(-K^{1/3} \xi/2) \cos(\sqrt{3K^{1/3}} \xi/2 + \phi)$, where $\alpha$ is a small parameter of order $10^{-6}$ and $\phi$ is a parameter taken such that the constant plane curvature of the nonlinear integrated solution at $\xi \to -\infty$ is equal to the one of the front static cap [9,10] as discussed in Sec. 4.1. Note that the linear solution for the rear dynamic meniscus presents oscillations.

The minimum film thickness of the obtained profile is found to follow the empirical model [9]

$$H_{min} = F(m) \bar{P}(m)(3Ca)^{2/3} \quad \text{with} \quad m = \lambda \frac{H_{min}}{R},$$

where $F(m)$ is a coefficient obtained through fitting Eq. (37) to our numerical database (see Fig. 17(a)). Similar to the mean coefficient $\bar{P}$ adopted in Sec. 4, a $Ca$-averaged $F(m)$ can be introduced as $\bar{F}$, that is further assumed as $0.716$ in view of its very weak dependence on $\lambda$, as shown in Fig. 17(b).

The minimum film thickness is bounded by the thickness of the uniform film and hence will saturate at large capillary numbers. Thus, for sufficiently large $Ca$ values, the oscillations at the rear interface would disappear and $H_{min} \to H_{\infty}$. It is therefore natural to propose a rational function model of $H_{min}$ for a broader $Ca$-range as the one for $H_{\infty}$:

$$H_{min} = \frac{\bar{P}(\lambda) \bar{F}(3Ca)^{2/3}}{1 + \bar{P}(\lambda) \bar{F}G(\lambda)(3Ca)^{2/3}}.$$

The above minimum film thickness model (38) together with the coefficient $G$ is in good agreement with the results of the numerical simulations (see Fig. 18 and Fig. 26). The proposed fitting of the coefficient $G$ as a function of the viscosity ratio (see Fig. 19) is given in Appendix E.

7 Front and rear total stress jumps

The dynamics of a translating bubble in a capillary tube has been characterized since the seminal work of Bretherton [9] not only by the mean and minimum film thickness, the relative velocity compared to the mean velocity, but also by the mean curvature of the front and rear static menisci. In fact, for $Ca \to 0$, the pressure drop across the interface is directly related to the expression of its mean curvature via the Laplace law. Having generalized the film thickness and droplet velocity models for non-vanishing viscosity ratios, we are hereby focusing on the evolution of the plane
curvature of the front and rear static caps versus the capillary number and viscosity ratio. The mean curvature at the droplet extremities is equivalent to the corresponding plane curvature for the planar configuration or to its double for the axisymmetric configuration: $\kappa_f = \chi \kappa_r$, with $\chi = 2$ (resp. $\chi = 1$) for the axisymmetric (resp. planar) configuration (see Sec. 2.2). As it will be shown, given the rather broad range of capillary numbers considered (approaching $O(1)$), it is insufficient to consider the interface mean curvature alone to provide an accurate prediction of the pressure drop, but the jump in the normal viscous stress has to be accounted for.

For the incompressible Newtonian fluids considered, the viscous stress tensor is $\tau = \mu \left[ \nabla u + (\nabla u)^T \right]$, and hence the $z$-direction normal total stress $\sigma_{zz}$ is given by

$$\sigma_{zz} = -p + \tau_{zz} = -p + 2\mu \frac{\partial u}{\partial z}. \tag{39}$$

Applying the difference (between inner and outer phases) operator $\Delta$ to Eq. (39) and based on the dynamic boundary condition in the normal direction (4) at the droplet front and rear extremities, we get

$$\Delta \sigma_{zz} = -\Delta p + \Delta \tau_{zz} = -\gamma \chi \kappa_f, \tag{40}$$

which indicates that the total stress jump at the front/rear extremities scales with the local interface mean curvature and is the sum of the pressure jump and the normal viscous stress jump. These quantities will be modeled separately in the following sections.

7.1 Front and rear plane curvatures

In the spirit of the empirical film thickness model, the plane curvature $\kappa_f$ of the front meniscus and that of the rear, $\kappa_r$,
are approximated by the rational function model

\[ \kappa_{fr} R = \frac{1 + T_{fr}(\lambda)(3Ca)^{2/3}}{1 + Z_{fr}(\lambda)(3Ca)^{2/3}}, \quad (41) \]

where \( T_{fr} \) and \( Z_{fr} \) as \( \lambda \)-dependent constants are obtained by fitting Eq. (41) to the FEM-ALE data (see Appendix E). It is worth-noting that the asymptotic series of the proposed expression,

\[ \kappa_{fr} R \sim 1 + (T_{fr} - Z_{fr})(3Ca)^{2/3} + O(Ca^{4/3}), \quad (42) \]

is in line with the law proposed by Bretherton [9], namely \( 1 + \beta_{fr}(3Ca)^{2/3} + O(Ca^{4/3}) \). Thus, the empirical model (41), which is in excellent agreement with the numerical results (see Fig. 20 and 21 as well as Fig. 27 and 28), can be regarded as an empirical extension of Bretherton’s law to a broader capillary numbers range up to 1. The mean curvature at the droplet extremities is given by \( \kappa_{fr} = \chi \kappa_{fr} \).

Fig. 20 Curvature \( \kappa_f \) of the front meniscus predicted by the model Eq. (41) (lines) and FEM-ALE data (symbols) versus \( Ca \) for both axisymmetric (blue line, full symbols) and planar (red dashed line, empty symbols) geometries, where the viscosity ratio \( \lambda = 1 \).

7.2 Front and rear pressure jumps – classical model

Following the literature [9, 12], the dimensionless pressure jump \( \Delta p_{fr} R/\gamma = (p_{fr} - p_{fr}^o)R/\gamma \) at the front and rear of the droplet is described by the empirical model

\[ \frac{\Delta p_{fr} R}{\gamma} = \chi \left[ 1 + S_{fr}(\lambda)(3Ca)^{2/3} \right], \quad (43) \]

where \( \chi = 2 \) (resp. \( \chi = 1 \)) for the axisymmetric (resp. planar), and \( S_{fr} \) is a \( \lambda \)-dependent coefficient. Equation (43) is in fact inspired by the curvature model proposed by Bretherton [9] exploiting the Laplace law [12], reason why we call it classical model. The coefficient \( S_{fr} \) could be derived from the integration of the lubrication equation. (23) or (24), which is valid in the low-\( Ca \) limit when the viscous stresses and their jumps are negligible. To broaden the \( Ca \) range of the model, we obtain \( S_{fr} \) through fitting to the FEM-ALE data. Nevertheless, as visible in Fig. 22 and Fig. 23, the model fails to precisely describe the numerical data, particularly for the rear pressure jump at high \( Ca \) values (see Fig. 23).

After explaining our model for the normal viscous stress jump in Sec. 7.3, we will show in Sec. 7.4 that the pressure jump can be better approximated by summing up the two contributions from the interface mean curvature and the normal viscous stress jump, which are modeled separately. The importance of the normal viscous stress jump for the pressure jump is already noticeable when comparing the evolutions of the plane curvature \( \kappa_{fr} \) and the one of the pressure jump \( \Delta p_{fr} R/\gamma \) in Figs. 20 and 22 or in Figs. 21 and 23.

7.3 Front and rear normal viscous stress jumps

The dimensionless normal viscous stress jump \( \Delta \tau_{zz} R/\gamma = (\tau_{zz} - \tau_{zz}^o)R/\gamma \) at the front and rear of the droplet is approximated by the following model

\[ \frac{\Delta \tau_{zz} R}{\gamma} = \frac{M_{fr}(\lambda)(3Ca) + N_{fr}(\lambda)(3Ca)^{4/3}}{1 + O_{fr}(\lambda)(3Ca)}, \quad (44) \]

where \( M_{fr}, N_{fr} \) and \( O_{fr} \) are viscosity ratio dependent coefficients found by fitting Eq. (44) to the FEM-ALE data. The normal viscous stress jumps indeed scale with \( Ca \) for small capillary numbers, as found by Bretherton [9]. The comparison between the model and the numerical results is shown in the insets of Figs. 22 and 23, where the results for \( \lambda = 0 \)
Fig. 22 Front pressure jump $\Delta p_f$ given by Eq. (43) (solid lines) and front normal viscous stress jump $\Delta \tau_{zz}$, by Eq. (44) (inset, solid lines) and FEM-ALE data (symbols) versus $Ca$ for both axisymmetric (blue line, full symbols) and planar (red line, empty symbols) geometries, where the viscosity ratio $\lambda = 0$. The dashed lines correspond to the improved pressure jump model Eq. (46). Note the different scale in the insets.

Fig. 23 The rear counterpart, pressure jump $\Delta p_r$ and normal viscous stress jump $\Delta \tau_{zz}$, of Fig. 22.

are shown. The results for $\lambda = 1$ and 100 can be found in Figs. 29 and 30. The stress jump $\Delta \tau_{zz}$ is found to be small in the case of $\lambda = 1$ and it varies with $Ca$ non-monotonically for the other viscosities.

7.4 Front and rear pressure jumps – improved model

Using the dynamic boundary condition in the normal direction evaluated at the front and rear caps of the droplet, Eq. (40), the pressure jump at the front and rear caps can also be computed as

$$\Delta p_{f,r} = \gamma \chi f_{f,r} + \Delta \tau_{zz,f,r}. \quad (45)$$

Thus, with the proposed models (41) and (44) for the interface curvatures and normal viscous stress jumps at hand, the pressure jump model reads

$$\Delta p_{f,r} = \frac{M_{f,r}(\lambda)(3Ca) + N_{f,r}(\lambda)(3Ca)^{4/3}}{1 + O_{f,r}(\lambda)(3Ca) + X_{f,r}(\lambda)(3Ca)^{2/3}}$$

which agrees with the FEM-ALE data better than Eq. (43) does (see dashed lines on Figs. 22 and 23 or Figs. 29 and 30). Therefore, the jump in normal viscous stresses has to be taken into account for $Ca > 10^{-3}$.

8 Stresses distribution and total pressure drop

8.1 Stresses distribution along the channel centerline

The flow field can be classified into parallel and non-parallel regions (see Fig. 8). The parallel regions compose of the region sufficiently far away from the droplet and that encompassing the uniform lubrication film of constant thickness, where the flow is streamwise invariant. The profile of the streamwise velocity $u(r)$ is parabolic and the radial velocity $v \approx 0$. On the contrary, the flow is not parallel near the droplet extremities (see Fig. 8) where the flow stagnates. Therefore the nearby streamwise velocity vary significantly, leading to a non-zero radial velocity owing to the divergence-free condition.

We show in Fig. 24 the distribution of the total stress component $\sigma_z = -p + \tau_{zz}$, of the pressure $p$ and of the viscous stress component $\tau_{zz} = 2\mu\partial u/\partial z$ along the centerline of the channel. $\tau_{zz}$ vanishes where the flow is approximately parallel. As seen in Sec. 7.3, $\tau_{zz}$ is negligible at small Ca, typically below $10^{-3}$.

Furthermore, for a larger but still moderate $Ca$ number, it is observed in Fig. 24(b) that the pressure (red line) deviates from the linearly varying pressure, $p_{\text{linear}}$ (black line), of the unperturbed flow (without droplet) featured with a constant pressure gradient. The deviation is attributed to the non-parallel flow structure near the front and rear caps of the droplet (see Fig. 8), hence the pressure based on $p_{\text{linear}}$ need to be corrected by $\Delta p^{\text{NP}} = p - p_{\text{linear}}$. Typical values for the pressure corrections can be found in the Appendix G. These corrections are particularly large at large viscosity ratios for the region inside of the droplet. We did not succeed in providing a model to quantify this pressure correction.

Finally, in agreement with the results of Section 7, the jump in total stress or pressure at the rear of the droplet is smaller than the one at the front.
8.2 Pressure distribution along the channel wall

The pressure distribution on the channel wall is presented on Fig. 24 as well (continuous grey line). The influence of the interface mean curvature is clearly visible. The non-monotonic pressure at the wall close the droplet rear results from the variation of the plane curvature in the dynamic meniscus region, where the interface oscillates (see also Fig. 11).

8.3 Droplet-induced total pressure drop along a channel

The prediction of the total pressure drop along a channel induced by the presence of a droplet flowing with a velocity $U_2$ is of paramount importance for the design of two-phase flow pipe networks [6, 37]. This allows for a coarse-grained quantification of the complicated local effects induced by the droplet. Droplets can thus be seen as punctual perturbations in the otherwise linear pressure evolution. In this section, we will show that it is possible to predict the total pressure drop induced by a droplet with the models proposed so far.

The total pressure drop can be defined as the difference between the pressure in the outer phase ahead and behind the droplet, namely $\Delta p_{tot} = p^o_f - p^o_r$ [35]. It is given by

$$\Delta p_{tot} = \Delta p_{v,NP} + \Delta p_r - \Delta p_{v,f} + \frac{d p_i}{dz} L_d + \Delta p_{v,f} - \Delta p_{o,f} + \Delta p_{v,f}, \quad (47)$$

where $\Delta p_{v,f}$ are given by the model for the pressure jumps at interfaces, equation (46). The pressure gradient $d p_i/dz$ in the parallel region inside the droplet is given by Eq. (31) and Eq. (67) for the axisymmetric and planar geometries, respectively. Assuming the droplet of volume/area $\Omega$ (axisymmetric/planar geometry) as a composition of two hemispherical caps of radius $R - H_o$, with $H_o$ given by Eq. (30), connected by a cylinder of the same radius, the droplet length $L_d$ can be approximated at first order for low $Ca$, i.e. for $H_o/R \ll 1$, as

$$L_d = \frac{(R + 2H_o)}{\pi} \frac{\Omega}{R^3} + \frac{2}{3}(R - H_o) \quad (48)$$

for the axisymmetric case and

$$L_d = \frac{(R + H_o)}{2} \frac{\Omega}{R^2} + \frac{4}{2}(R - H_o) \quad (49)$$

for the planar case.

Equivalently, the total pressure drop can also be calculated using the models for the normal viscous stress jump, equation (44), and the front and rear plane curvatures, equation (41), yielding:

$$\Delta p_{tot} = \Delta p_{v,NP} + \Delta \tau_{zz} + \chi \gamma \kappa_f - \Delta p_{v,f} + \frac{d p_i}{dz} L_d + \Delta p_{v,f} - \Delta \tau_{zz} - \chi \gamma \kappa_f - \Delta p_{o,f} \quad (50)$$

where $\chi = 2$ for the axisymmetric configuration and $\chi = 1$ for the planar one.

If we neglect the non-parallel flows effects on the pressure, $\Delta p^{NP}$, the total pressure drop would then be:

$$\Delta p_{tot} = \Delta p_r + \frac{d p_i}{dz} L_d - \Delta p_f. \quad (51)$$

Neglecting the effects of the non-parallel flows would induce an error on the pressure drop, increasing with $Ca$. For
a single droplet of volume $\Omega = 12.9$, the error of Eq. (51) compared to the numerical results is less than 3% for $\lambda = 0$, but reaches 15% for $\lambda = 1$ and even 48% for $\lambda = 100$. It is thus important to include the corrections accounting for the non-parallel flow effects to predict the pressure drop accurately, especially when the viscosity ratios $\lambda \approx 1$. Numerical simulations are therefore crucial to achieve so.

9 Conclusions

This paper generalizes the theory of a confined bubble flowing in an axisymmetric or planar channel to droplets of non-vanishing viscosity ratios. Empirical models for the relevant quantities such as the uniform and minimal film thicknesses separating the wall and the droplet, the front and rear droplet plane curvatures, the total pressure drop in the channel and separating the wall and the droplet, the front and rear droplet quantities such as the uniform and minimal film thicknesses vanishing viscosity ratios. Empirical models for the relevant quantities. The models are inspired by the low-capillary-number predictions obtained by the lubrication approach of Bretherton [9] for bubbles to viscous droplets. Extensive accurate moving-mesh arbitrary Lagrangian-Eulerian (ALE) finite-element numerical simulations are performed for the viscosity-ratio range $\lambda \in [0-100]$ to build a numerical database, based on which we propose empirical models for the relevant quantities. The models are inspired by the low-Ca theoretical asymptotes, but their validity range reaches large capillary numbers ($Ca > 10^{-3}$), where the lubrication approach no longer holds.

We have found that the uniform film thickness for $Ca < 10^{-3}$ does not differ significantly with that of a bubble as long as $\lambda < 1$. For larger viscosity ratios, instead, the film thickness increases monotonically and saturates to a value $2^{3/2}$ times the Bretherton’s scaling for bubbles when $\lambda > 10^4$. The film thickness can be modeled by a rational function similar to that proposed by Aussillous and Quéré [5] for bubbles, where the fitting coefficient $Q$ depends on the viscosity ratio. Furthermore, the uniform film thickness saturates at large capillary numbers to a value depending on $Q$. The minimum film thickness can be predicted analogously. The velocity of a droplet can be unambiguously derived once the uniform film thickness is known. We have shown that considering the full expression of the droplet velocity is crucial as the asymptotic series for low $Ca$ has a very restricted range of validity for non-vanishing viscosity ratios.

Furthermore, we have found that the evolution of the front and rear cap curvatures as a function of the capillary number differs from the one of the pressure jumps at the front and rear droplet interfaces. This is due to the normal viscous stress jumps. The contribution of the jumps has been overlooked in the literature, though it has to be considered for $Ca > 10^{-3}$. With all these models at hand, the pressure drop across a droplet can be computed, which will be valuable for engineering practices.

We also have shown that the flow patterns inside and outside of the droplet strongly depend on the capillary number and viscosity ratio. In particular, for $\lambda < 1/2 (\lambda < 2/3)$ for the axisymmetric (planar) configuration, when the film thickness is larger than a critical value $H^*/R$, recirculating regions at the front and rear of the droplet disappear. Furthermore, the recirculation region in the outer phase detaches from the droplet’s rear interface for large film thickness yet smaller than $H^*/R$, implying the disappearance of the inner recirculating region at the rear.

The considered problem in a planar configuration could be relevant for the study of a front propagation in a Hele-Shaw cell [46,48], where the second-phase viscosity is non-vanishing. For instance, one could compute the amount of fluid left on the walls when a finger of immiscible fluid penetrates [51]. Furthermore, the problem in the planar configuration can be seen as a first step towards understanding the dynamics of pancake droplets in a Hele-Shaw cell [29,61]. Another possible outlook is the extension of the present theory to capillaries with polygonal cross sections, where the film between the droplet and the walls is not axisymmetric, but thick films known as gutters develop in the capillary corners. Three-dimensional numerical simulations are then necessary to resolve this asymmetry. A force balance will determine the droplet velocity and an equivalent pressure drop model could be proposed for these geometries.

Despite the fact that this work was motivated by the vast number of droplet-based microfluidic applications, the analytically derived equation (24) serves as a generalization of the well known Landau-Levich-Derjaguin-Bretherton equation [38,13,9] when the second fluid has a non-negligible viscosity. This equation could therefore be adapted to predict the film thickness in coating problems with two immiscible liquids.
A Derivation of the flow profiles in the thin-film region for the planar configuration

Consider an axial location in the thin-film region. The velocity profiles inside, \( u_i \), and outside, \( u_o \), of the droplet can be described by:

\[
\begin{align*}
  u_i(r) &= \frac{1}{2\mu_o} \frac{dp_i}{dz} r^2 + A_r r + B_i, \\
  u_o(r) &= \frac{1}{2\mu_o} \frac{dp_o}{dz} r^2 + A_o r + B_o,
\end{align*}
\]

(52) (53)

where \( p_i \) and \( p_o \) are the inner, respectively outer, pressures, and \( A_i, B_i, A_o, \) and \( B_o \) are real constants to be determined. Given the symmetry at \( r = 0 \) of the inner velocity, \( A_i = 0 \). The other constants are found by imposing the no-slip boundary condition at the channel walls \( u(R) = -U_d \) in the droplet reference frame, the continuity of velocities at the interface located at \( r = R - H \), \( u_i(R - H) = u_o(R - H) \), and the continuity of tangential stresses at the interface

\[
\mu_o \frac{du_i}{dz} \bigg|_{r=R-H} = \mu_i \frac{du_o}{dz} \bigg|_{r=R-H}.
\]

(54)

Eventually one obtains:

\[
\begin{align*}
  A_o &= \frac{1}{\mu_o} \left( \frac{dp_o}{dz} - \frac{dp_i}{dz} \right) (R - H), \\
  B_i &= \frac{1}{2\mu_o} \left[ -(R - H)^2 \frac{dp_o}{dz} \mu_o + H \left( 2H^2 \frac{dp_o}{dz} - \frac{dp_i}{dz} R \right) \right] - U_d, \\
  B_o &= \frac{1}{2\mu_o} \left( \frac{dp_o}{dz} - 2 \frac{dp_i}{dz} \right) R^2 - 2HR \left( \frac{dp_o}{dz} - \frac{dp_i}{dz} R \right) - U_d.
\end{align*}
\]

(55) (56) (57)

B Derivation of the interface profile equation for the planar configuration

The flow rates at any axial location where the external film thickness is \( H \) are:

\[
q_i = 2 \int_0^{R-H} u_i(r) dr
\]

(58)

and substituted into Eq. (62). Following Bretherton [9], the resulting equation can be put in an universal form by the substitutions \( H = H_o \eta \) and \( z = H_o (3\alpha)^{-1/3} \zeta \). In the limit of \( H_o / R \to 0 \), the governing equation for the interface profile reads:

\[
\frac{d^2 \eta}{dz^2} = \frac{2\eta - 1}{\eta^3} \left[ 2 + 3m(1 + \eta + 3m) \right] \left( 1 + 3m \right),
\]

(65)

where

\[
m = \frac{\lambda H_o}{R}
\]

(66)

is the rescaled viscosity ratio.

C Derivation of the droplet velocity model for the planar configuration

The velocity profiles in the uniform film region have been derived in Appendix A. In particular, the inner and outer volumetric fluxes are given by Eqs. (58) and (59), respectively. At the location where \( H = H_o \) the interface is flat and the pressure gradients are equal, \( dp_i/dz = dp_o/dz \). Furthermore, mass conservation imposes that \( q_o = 2R(U_o - U_d) \) and since we are in the reference frame of the droplet, \( q_i = 0 \). The system of two equations can be solved for the pressure gradient

\[
\left. \frac{dp}{dz} \right|_{r=R-H} = \frac{-3RU_o \mu_i}{(3H_o^2 + 3H_o^2(3H_o^2 - 3H_o R + H_o^2)\lambda)}
\]

(67)

and the droplet velocity

\[
U_d = \frac{R(2R - H_o^2) + 3H_o^2(2R - H_o^2)\lambda}{2(R - H_o^2)^3 + 2H_o^2(3R^2 - 3H_o^2 + H_o^2)\lambda} U_o.
\]

(68)

The relative velocity of the planar droplet reads

\[
\frac{U_d - U_o}{U_d} = \frac{H_o}{R} \left[ 2 - \frac{H_o}{R} \left( 4 + 2\frac{H_o}{R} (\lambda - 1) - 3\lambda \right) \right]
\]

(69)
Table 1 Coefficients of the fitting law for the axisymmetric configuration.

| $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-------|-------|-------|-------|
| $Q$   | 2.21  | 111.25| 33.84 | 1.37  |
| $G$   | 130.37| 186.67| -4.82 | 1.30  |
| $T_r$ | 3262.57| 1573.07| 7222.70| 9.90 |
| $T_i$ | -12031.87| -21476.98| 2820.73| 77.21 |
| $Z_r$ | 3392.32| -1773.73| 2984.79| 39.56 |
| $Z_i$ | -1842.14| -14129.53| 26169.48| 160.45 |
| $M_f$ | -4850.40| 5797.90| -507.02| 1.22 |
| $M_i$ | -6.38 | 18.59 | -10.85 | -0.82 |
| $N_f$ | -5293.51| 14808.02| -9344.15| -126.15 |
| $N_i$ | -2.93 | -17.28 | 18.94 | 1.08 |
| $O_f$ | 0.01 | -0.02 | 0.08 | -0.11 |
| $O_i$ | 32.38 | -429.86| 638.07| -5.84 |

Table 2 Coefficients of the fitting law for the planar configuration.

| $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-------|-------|-------|-------|
| $Q$   | 98.76 | 146.42| 70.42 | 1.45  |
| $G$   | 168.27| 348.60| 26.76 | 1.48  |
| $T_f$ | 0.35  | 1.17  | 5.41  | 2.43  |
| $T_i$ | -130.18| -298.84| -55.49| -0.66 |
| $Z_f$ | 1096.45| 191.51| 395.61| -0.08 |
| $Z_i$ | -0.62 | -0.66 | 0.08  | -0.21 |
| $M_f$ | -6.41 | 17.26 | -11.54| 0.80  |
| $M_i$ | 4.10  | -3.52 | 0.17  | 0.04  |
| $N_f$ | -6.12 | 17.95 | -11.90| 0.19  |
| $N_i$ | -50.40| 61.95 | -14.79| 0.51  |
| $O_f$ | 4.52  | -2.80 | -1.73 | 0.89  |
| $O_i$ | 2.10  | -6.97 | 2.79  | -0.01 |

D Derivation of the critical uniform film thickness for the appearance of the recirculation regions

The velocity profile in the channel away from the droplet is given by Eq. (11) for the axisymmetric configuration and by

$$ u_\infty(y) = \frac{3}{2} U_\infty \left[ 1 - \left( \frac{y}{R} \right)^2 \right] - U_d $$

(70)

for the planar one. The droplet velocity for the former case is given by Eq. (32), whereas for the latter it is given by Eq. (68). With the use of Eqs. (32) and (68), the velocity $u_d$ can be expressed as a function of recirculation regions, $H_2/R$. The critical uniform film thickness for the appearance of recirculation, $H_2$, is obtained by solving $u_d(0) = 0$, resulting in Eqs. (12) and (13) for the axisymmetric and planar configurations, respectively.

E Fitting laws for the model coefficients

The model coefficients $Q$ in Eq. (30), $G$ in Eq. (38), $T_r$, $T_i$, and $Z_r$ in Eq. (41) and $M_f$, $N_f$, and $O_f$ in Eq. (44) can be well approximated by the rational function

$$ a_1 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 $$

$$ \lambda^3 + b_0 \lambda^2 + b_1 \lambda + b_0 $$

(71)

where the constants $a_i$ with $i = 0, \ldots, 3$ and $b_j$ with $j = 0, \ldots, 2$ are given in tables 1 and 2 for the axisymmetric and planar geometries, respectively.

F Additional results

For seek of clarity, the results for $\lambda = 0$ and 100 are shown in the appendix rather than in the main text, except for the normal viscous stresses jump, whose results for $\lambda = 0$ are presented in the main text as for $\lambda = 1$ the normal viscous stress jumps are small.

G Pressure corrections due to non-parallel flow

Some typical total stresses corrections at the outer and inner sides of the droplet interface as a function of $Ca$ and for different viscosity ratios $\lambda$ are shown in Fig. 31 and Fig. 32, respectively.

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Fig. 25 Uniform film thickness given by Eq. (30) (lines) and FEM-ALE numerical results (symbols) as a function of the droplet capillary number for $\lambda = 0$ (a) and 100 (b) and both axisymmetric (blue solid line, full symbols) and planar (dashed red line, empty symbols) geometries.

Fig. 26 Minimum film thickness given by Eq. (38) (lines) and FEM-ALE numerical results (symbols) as a function of the droplet capillary number for $\lambda = 0$ (a) and 100 (b) and both axisymmetric (blue solid line, full symbols) and planar (dashed red line, empty symbols) geometries.
Fig. 27 Curvature $\kappa_f$ of the front meniscus predicted by the model Eq. (41) (lines) and FEM-ALE data (symbols) versus $Ca$ for both axisymmetric (blue line, full symbols) and planar (red dashed line, empty symbols) geometries, where the viscosity ratio $\lambda = 0$ (a) and 100 (b).

Fig. 28 The rear counterpart $\kappa_r$ of Fig. 27.
Viscous Taylor droplets in axisymmetric and planar tubes: from Bretherton’s theory to empirical models

Fig. 29 Front pressure jump $\Delta p_f$ given by Eq. (43) (solid lines) and front normal viscous stress jump $\Delta \tau_{zz,f}$ by Eq. (44) (inset, solid lines) and FEM-ALE data (symbols) versus $Ca$ for both axisymmetric (blue line, full symbols) and planar (red line, empty symbols) geometries, where the viscosity ratio $\lambda = 1$ (a) and 100 (b). The dashed lines correspond to the improved pressure jump model Eq. (46). Note the different scale in the insets.

Fig. 30 The rear counterpart, pressure jump $\Delta p_r$ and normal viscous stress jump $\Delta \tau_{zz,r}$, of Fig. 29.
Fig. 31 Pressure correction due to non-parallel flow effects at the rear (a) and front (b) outer sides of the interface for $\lambda = 0.04$ (blue squares), 0.12 (red crosses), 1 (yellow circles), 15 (purple stars) and 50 (green diamonds) for the axisymmetric configuration. The results are obtained from FEM-ALE numerical simulations.

Fig. 32 Pressure correction due to non-parallel flow effects at the rear (a) and front (b) inner sides of the interface for $\lambda = 0.04$ (blue squares), 0.12 (red crosses), 1 (yellow circles), 15 (purple stars) and 50 (green diamonds) for the axisymmetric configuration. The results are obtained from FEM-ALE numerical simulations.
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