LOCAL LOG-REGULAR RINGS VS TORIC RINGS

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ABSTRACT. Local log-regular rings are a certain class of Cohen-Macaulay local rings that are treated in logarithmic geometry. Our paper aims to provide purely commutative ring theoretic proof of some ring-theoretic properties of local log-regular rings such as an explicit description of a canonical module, and the finite generation of the divisor class group.

1. Introduction

In [Kat94], Kazuya Kato established the theory of toric geometry without base by using a logarithmic structure of Fontaine–Illusie. He named schemes appearing in the theory log-regular schemes and their local rings local log-regular rings. The class of local log-regular rings has similar properties to toric rings (for example, they are Cohen–Macaulay and normal). Moreover, this class has a structure theorem such as Cohen’s structure theorem (cf. Theorem 3.10). By the structure theorem, the completion of a local log-regular ring can be expressed as a complete monoid algebra.

The class of local log-regular rings is also important from the perspective of commutative ring theory in mixed characteristic. Gabber and Ramero explicitly constructed a perfectoid ring that is an algebra over a local log-regular ring ([GR23, §17.2] or [INS22, Construction 3.58]). They apply the construction to prove that a local log-regular ring is a splinter. Moreover, Cai–Lee–Ma–Schwede–Tucker recently proved that a complete local log-regular ring is BCM-regular, which is a BCM-analogue of strong $F$-regularity ([CLM+22, Proposition 5.3.5]).

In this paper, we explore ring-theoretic properties of local log-regular rings, in particular canonical modules and divisor class groups. Firstly, we explore canonical modules of local log-regular rings. The existence of a dualizing complex of a log-regular scheme is already proved by Gabber and Ramero in [GR23, Theorem 12.5.42]. We investigate the structure of canonical modules of local log-regular rings explicitly.

Main Theorem A (Theorem 4.1). Let $(R, Q, \alpha)$ be a local log-regular ring, where $Q$ is finitely generated, cancellative, reduced, and root closed (by Remark 3.16, we may assume that $Q \subseteq \mathbb{N}^I$ for some $I > 0$). Let $x_1, \ldots, x_r$ be a sequence of elements of $R$ such that $x_1, \ldots, x_r$ is a regular system of parameters for $R/I_\alpha$. Then $R$ admits a canonical module and

$$\langle (x_1 \cdots x_r)\alpha(a) \mid a \in \text{relint } Q \rangle$$

is the canonical module of $R$, where $\text{relint } Q$ is the relative interior of $Q$.

Though Gabber and Ramero’s proof is sheaf-theoretic, we show it by reducing it to the case of a semigroup ring. Also, Robinson proved the toric case in [Rob22]. We mention the relationship between his result and Main Theorem A in Remark 4.3.
As applications of Main Theorem A, we provide a criterion of the Gorenstein property of local log-regular rings (Corollary 4.5). Moreover, we determine the generator of Gorenstein local log-regular rings with two-dimensional monoids (Proposition 4.10). Also, we prove that local log-regular rings are pseudo-rational (Proposition 4.8).

Secondly, we provide the finite generation of the divisor class group of a local log-regular ring. To show it, we prove that a log structure induces the isomorphism between the divisor class group of a local log-regular ring and that of the associated monoid:

**Main Theorem B** (Theorem 5.7). Let \((R, Q, \alpha)\) be a local log-regular ring. Then \(\alpha\) induces the group homomorphism \(\text{Cl}(\alpha) : \text{Cl}(Q) \to \text{Cl}(R)\) and it is an isomorphism. In particular, \(\text{Cl}(R)\) is finitely generated.

By the combination of Main Theorem B with Chouinard’s result in [Cho81], we know that the divisor class group of a local log-regular ring is isomorphic to that of the monoid algebra over a field.

Finally, we provide an outline of this paper. In §2, we present the basic notions of monoids. In §3, we introduce the definition and certain properties of local log-regular rings. In §4, we prove Main Theorem A. We also provide examples of Gorenstein local log-regular rings. In §5, we prove Main Theorem B.

## 2. Preliminaries on monoids

### 2.1. Properties on monoids

A monoid \(Q\) is a commutative semigroup with unity. We denote the group of units (resp. the group consisting of elements in the form of \(q - p\) where \(q, p \in Q\)) by \(Q^\times\) (resp. \(Q^{gp}\)). We also denote by \(Q^+\) the set of non-unit elements of \(Q\) (i.e. \(Q^+ = Q \setminus Q^\times\)). Let us recall the terminologies of monoids.

**Definition 2.1.** Let \(Q\) be a monoid.

1. \(Q\) is called **cancellative** (or **integral**) if for \(x, x'\) and \(y \in Q\), \(x + y = x' + y\) implies \(x = x'\).
2. \(Q\) is called **reduced** (or **sharp**) if \(Q^\times = 0\).
3. \(Q\) is called **root closed** (or **saturated**) if it satisfies the following conditions:
   - \(Q\) is cancellative,
   - If \(x \in Q^{gp}\) such that \(nx \in Q\) for some \(n > 0\), then \(x \in Q\).

**Definition 2.2.** Let \(Q\) be a monoid. Then an equivalent relation \(\sim\) on \(Q\) is called **congruence** if \(a \sim b\) implies \(a + c \sim b + c\) for any \(a, b, c \in Q\).

**Example 2.3 (Associated reduced monoids).** Let \(Q\) be a monoid. Two elements \(a, b \in Q\) are called **associates** if there exists a unit \(u \in Q^\times\) such that \(a = u + b\). If \(a, b \in Q\) are associates, we denote them by \(a \simeq b\). The relation \(\simeq\) is a congruence relation and the monoid \(Q_{\text{red}} := Q/\simeq\) is called the **associated reduced monoid of** \(Q\). By definition, we have \([a] = a + Q^\times\) where \([a]\) is an element of \(Q_{\text{red}}\). This implies that if \(Q\) is reduced, we obtain \(Q = Q_{\text{red}}\).

We recall ideals, prime ideals, and the dimension of monoids.

**Definition 2.4.** Let \(Q\) be a monoid.

1. A subset \(I\) of \(Q\) is an **s-ideal** if \(a + x \in I\) for any \(a \in Q\) and any \(x \in I\).
2. An ideal \(p \subset Q\) is called **prime** if \(p \neq Q\), and for \(p, q \in Q\), \(p + q \in Q\) implies \(p \in Q\) or \(q \in Q\).
3. The set of primes ideals of \(Q\) is called the **spectrum** of \(Q\) and is denoted by \(\text{Spec}(Q)\).

The spectrum of a monoid becomes a topological space. We note that the empty set \(\emptyset\) and the set \(Q^+\) are prime ideals. Moreover, \(\emptyset\) is the unique minimal prime, and \(Q^+\) is the unique maximal ideal. We define the dimension of a monoid.
Definition 2.5. The *dimension* of a monoid $Q$ is the maximal length $d$ of the ascending chain of prime ideals

$$\emptyset = q_0 \subset q_1 \subset \cdots \subset q_d = Q^+.$$ 

We also denote it by $\dim(Q)$.

Let $\varphi : Q \to Q'$ be a monoid homomorphism and let $p$ be a prime ideal of $Q'$. Then $\varphi^{-1}(p)$ is also prime. Thus one can define the continuous map $\text{Spec}(\varphi) : \text{Spec}(Q') \to \text{Spec}(Q)$.

**Proposition 2.6.** Let $Q$ be a finitely generated cancellative monoid. Then $\text{Spec}(Q)$ is a finite set.

**Proof.** This is [Ogu18, Chapter I, Proposition 1.4.7 (1)]. □

This is obtained by using a discussion of convex polyhedral cones.

**Lemma 2.7.** Let $Q$ be a finitely generated cancellative and reduced monoid.

1. The equality $\dim(Q) = \text{rank}(Q^{\text{gp}})$ holds.
2. Assume that $Q^{\text{gp}}$ is a torsion-free abelian group of rank $r$. Then there is an injective monoid homomorphism $Q \hookrightarrow \mathbb{N}^r$.

**Proof.** The assertion (1) is [GR23, Corollary 6.4.12 (i)] and the assertion (2) is [GR23, Corollary 6.4.12 (iv)]. □

**Lemma 2.8.** Let $Q$ be a cancellative monoid such that $Q$ is finitely generated and cancellative. Then there exists an isomorphism of monoids

$$Q \cong \overline{Q} \times Q^\times.$$ 

**Proof.** This is [GR23, Lemma 6.2.10]. □

### 2.2. Krull monoids and their divisor class groups

In this subsection, we give an easy review on divisor class groups of Krull monoids. Krull monoids have a long history in factorization theory and they are related to many mathematical fields, such as algebraic number theory, analytic number theory, combinatorial theory, and commutative ring theory. For details, we refer the reader to [BG00], [GR23], [GHK06], or [GZ20]. First, we define fractional ideals of monoids.

**Definition 2.9** (Fractional ideals of monoids). Let $Q$ be a cancellative monoid. Then a fractional ideal of $Q$ is a $Q$-submodule $I \subseteq Q^{\text{gp}}$ such that $I \neq \emptyset$ and $xI := \{x + a \mid a \in I\} \subset Q$ for some $x \in Q$.

**Lemma 2.10.** Let $Q$ be a cancellative monoid. Then the following hold.

1. If $I_1, \ldots, I_n$ are fractional ideals of $Q$, then $\bigcap_{i=1}^n I_i$ is fractional.
2. If $I_1, I_2$ are fractional ideals of $Q$, then $I_1I_2 := \{x + y \mid x \in I_1, y \in I_2\}$.

**Proof.** (1): Since $I_i$ is a fractional ideal, there exists an element $a_i \in Q$ such that $a_iI_i \subseteq Q$. Then $a_iJ \subset a_iI_i \subset Q$.

(2): Pick elements $a_1, a_2 \in Q$ such that $a_1I_1 \subset Q$ and $a_2I_2 \subset Q$. Then, since $a_1a_2(I_1I_2) \subset P$, we can simply set $x = a_1a_2$. □

We say that a fractional ideal $I$ is *finitely generated* if it is finitely generated as a $Q$-module. For any two fractional ideals $I_1$ and $I_2$, we define $(I_1 : I_2) := \{x \in Q^{\text{gp}} \mid xI_2 \subseteq I_1\}$. 


Lemma 2.11. Let \( Q \) be a cancellative monoid and let \( I_1 \) and \( I_2 \) be fractional ideals. Then \( (I_1 : I_2) \) is also a fractional ideal.

Proof. Let \( a_1 \in Q^{gp} \) such that \( a_1 I_1 \subseteq Q \). Pick an element \( a \in I_2 \). For any \( z \in (I_1 : I_2) \), \( az \in I_1 \). Thus \( a_1 az \in a_1 I_1 \subseteq Q \). This implies \( a_1 a(I_1 : I_2) \subseteq Q \), as desired. \( \Box \)

For a fractional ideal \( I \) of \( Q \), we set \( I^{-1} := (P : I) \) and \( I^* := (I^{-1})^{-1} \). We say that a fractional ideal \( I \) is divisorial (or \( v \)-ideal) if \( I^* = I \) holds.

Lemma 2.12. Let \( Q \) be a cancellative monoid and \( I \) and \( J \) be fractional ideals. Then the following hold.

1. If \( I \subseteq J \), then \( J^{-1} \subseteq I^{-1} \) and \( I^* \subseteq J^* \) hold.
2. \( I \subseteq J^* \) holds.
3. \( I^* \) is divisorial. Especially, \( I^* \) is the smallest divisorial ideal containing \( I \).
4. For any \( a \in Q^{gp} \), \( aI^{-1} = (a^{-1}I)^{-1} \) and \( aI^* = (aI)^* \) hold.
5. \( (IJ)^* = (I^*J^*)^* \) holds.

Proof. (1): Let \( a \in Q^{gp} \) such that \( aJ \subseteq Q \). Since \( I \subseteq J \), we have \( aI \subseteq aJ \subseteq Q \), as desired. The latter assertion follows from the former assertion.

(2): Pick \( a \in I \). For any \( z \in (Q : I) \), \( zI \subseteq Q \), in particular \( za \in Q \). Thus \( a \in (Q : (Q : I)) \).

(3): This is the same proof as in [SMG4] Lemma 1.2 (1)].

(4): The inclusion \( aI^{-1} \subseteq (a^{-1}I)^{-1} \) obviously holds. Conversely, pick an element \( z \in (a^{-1}I)^{-1} \). Then we have \( (a^{-1}z)I = z(a^{-1}I) \subseteq Q \). This implies that \( z \subseteq aI^{-1} \), as desired. Next, by the former equality, we obtain \( aI^* = (a^{-1}I)^{-1} = (((a^{-1}I)^{-1})^{-1})^{-1} = (aI)^* \).

(5): Pick \( a \in I^* \). Then \( (aJ^*)^* = aJ^* = aJ^* \). This implies that \( I^*J^* = I^*J^* \). Pick \( b \in J^* \). Then \( J^*b = (Ib)^* \). This implies \( I^*J^* = (IJ)^* \). Finally, pick \( c \in I \). Then \( (cJ)^* = (cI)^* = (cJ)^* \). This implies that \( (IJ)^* = (IJ)^* \). To summarize these, we obtain \( (I^*J^*)^* = I^*J^* = IJ^* = (IJ)^* \), as desired. \( \Box \)

Definition 2.13. Let \( Q \) be a cancellative monoid. We denote by \( \text{Div}(Q) \) the set of all divisorial ideals of \( Q \). We define a binary operation on \( \text{Div}(Q) \) by

\[ I \star J := (IJ)^*. \]

Note that a monoid \( Q \) is a divisorial ideal. Moreover, for a divisorial ideal \( I \), we have \( Q \star I = I \star Q = I \). Hence \( (\text{Div}(Q), \star) \) is a monoid. To discuss when \( \text{Div}(Q) \) becomes a group, we define the completely integrally closedness of a monoid.

Definition 2.14. Let \( Q \) be a cancellative monoid.

1. An element \( x \in Q^{gp} \) is called almost integral over \( Q \) if there exists \( c \in Q \) such that \( c + nx \in Q \) for any \( n \in \mathbb{Z}_{>0} \).
2. \( Q \) is called completely integrally closed if all almost integral elements over \( Q \) lie in \( Q \).

The set of elements of \( Q^{gp} \) which are almost integral over \( Q \) is a monoid. Indeed, for an almost integral element of \( x, y \in Q^{gp} \), there exist elements \( a, b \in Q \) such that \( a + nx, b + ny \in Q \) for any \( n \in \mathbb{Z}_{>0} \). Since we have \( (a + b) + n(x + y) = (a + nx) + (b + ny) \in Q \), \( x + y \) is also almost integral over \( Q \).

Proposition 2.15. Let \( Q \) be a cancellative monoid. Then the following assertions hold.

1. \( (\text{Div}(Q), \star) \) is an abelian group if and only if \( Q \) is completely integrally closed.
2. If \( Q \) is finitely generated, cancellative and root closed, then \( Q \) is completely integrally closed.

Proof. These assertions are [GR23] Proposition 6.4.42 (i), (ii)]. \( \Box \)
Next, we define Krull monoids.

**Definition 2.16 (Krull monoids).** Let \( \mathcal{Q} \) be a cancellative monoid. Then \( \mathcal{Q} \) a **Krull monoid** if the following two condition hold:

1. The set of divisorial ideals of \( \mathcal{Q} \) contained in \( \mathcal{Q} \) satisfies the ascending chain condition, that is, for any sequence \( I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \) of divisorial ideals, there exists a number \( n \geq 0 \) such that \( I_m = I_{m+1} \) for any \( m \geq n \).
2. \( \mathcal{Q} \) is completely integrally closed.

**Lemma 2.17.** Let \( \mathcal{Q} \) be a cancellative monoid. Then the following assertions hold.

1. \( \mathcal{Q} \) is completely integrally closed if and only if \( \mathcal{Q}_{\text{red}} \) is completely integrally closed.
2. \( \mathcal{Q} \) is a Krull monoid if and only if \( \mathcal{Q}_{\text{red}} \) is a Krull monoid.

**Proof.** These are [GHK06, Corollary 2.3.6]. □

Krull monoids possess many properties that Krull rings have. In particular, the following Proposition 2.18 is important to give concrete computations of divisor class groups.

**Proposition 2.18.** Let \( \mathcal{Q} \) be a cancellative monoid and let \( D \subseteq \text{Spec}(\mathcal{Q}) \) be the subset of all prime ideals of height one. Then \( \mathcal{Q} \) is Krull if and only if there is an isomorphism \( \mathbb{Z}^{\oplus D} \cong \text{Div}(\mathcal{Q}) \) as an abelian group.

**Proof.** The proof is the same as in [SM64, Theorem 3.1]. □

Krull monoid has many other characterizations besides this one. See [GHK06, Theorem 2.3.11 and Theorem 2.4.8] for details.

Keep the notation as in Proposition 2.18. Let us denote \( (n_p)_{p \in D} \in \mathbb{Z}^{\oplus D} \) by \( \sum_{p \in D} n_p p \). Also let us denote \( \text{div} : \mathbb{Z}^{\oplus D} \xrightarrow{\sim} \text{Div}(\mathcal{Q}) \).

**Definition 2.19.** Let \( \mathcal{Q} \) be a cancellative monoid and let \( a \in \mathcal{Q}^{\text{gp}} \) be an element. Then we define a **principal fractional ideal** as \( \{a + q \mid q \in \mathcal{Q}\} \). Moreover, we denote the set of principal fractional ideals by \( \text{Prin}(\mathcal{Q}) \).

Let \( \mathcal{Q} \) be a cancellative monoid and let \( I, J \) be fractional ideals of \( \mathcal{Q} \). Here we define \( I \sim J \) if there exists an element \( a \in \mathcal{Q}^{\text{gp}} \) such that \( I = aJ \). Then \( \sim \) is an equation relation.

**Definition 2.20 (The divisor class groups of monoids).** Let \( \mathcal{Q} \) be a cancellative monoid. Then we define the divisor class group of \( \mathcal{Q} \) as \( \text{Div}(\mathcal{Q})/\sim \) and denote this by \( \text{Cl}(\mathcal{Q}) \).

For a cancellative monoid \( \mathcal{Q} \), \( \text{Cl}(\mathcal{Q}) \) is a monoid (its binary operation is induced from that of \( \text{Div}(\mathcal{Q}) \)). Furthermore, if \( \mathcal{Q} \) is completely completely integrally closed, then \( \text{Cl}(\mathcal{Q}) \) is an abelian group.

Here assume that \( \mathcal{Q} \) is a Krull monoid. Let \( p \in \text{Spec}(\mathcal{Q}) \) be a height one prime ideal of \( \mathcal{Q} \). If \( p \) is a principal ideal, then \( \text{div}(p) \) is contained in a principal fractional ideal of \( \mathcal{Q} \) by Proposition 2.18.

Hence we obtain \( \text{div}^{-1}(\text{Prin}(\mathcal{Q})) = \{\sum_{p \in D} n_p p \in D \mid p \text{ is principal}\} \) and

\[
\text{div} : Z^D/\text{div}^{-1}(\text{Prin}(\mathcal{Q})) \xrightarrow{\sim} \text{Cl}(\mathcal{Q}).
\]

By this isomorphism, we obtain the following result.

**Corollary 2.21.** Let \( \mathcal{Q} \) be a Krull monoid. Then the following assertions are equivalent.

1. \( \text{Cl}(\mathcal{Q}) = 0 \).
2. Any height one prime ideal of \( \mathcal{Q} \) is principal.
3. LOCAL LOG-REGULAR RINGS

In this section, we review on the definition and basic properties of log-regularity of commutative rings. First, we provide the log structure of commutative rings.

Definition 3.1. Let $R$ be a ring, let $Q$ be a monoid, and let $\alpha : Q \to R$ be a monoid homomorphism.

1. The triple $(R, Q, \alpha)$ is called a log ring.
2. A log ring $(R, Q, \alpha)$ is a local log ring if $R$ is local and $\alpha^{-1}(R^\times) = Q^\times$, where $R^\times$ is the group of units of $R$.

Here, we define the log-regularity of commutative rings.

Definition 3.2 (cf. [Ogu18, Chapter III, Section 1.11]). Let $(R, Q, \alpha)$ be a local log ring, where $R$ is Noetherian and $Q := \overline{Q}/Q^\times$ is finitely generated, cancellative and root closed. Let $I_{\alpha}$ be the ideal of $R$ generated by $\alpha(Q^+)$. Then $(R, Q, \alpha)$ is called a local log-regular ring if the following conditions are satisfied:

1. $R/I_{\alpha}$ is a regular local ring.
2. The equality $\dim(R) = \dim(R/I_{\alpha}) + \dim(Q)$ holds.

Remark 3.3. We note that a monoid $Q$ appearing in Definition 3.2 has a decomposition $Q \cong Q \times Q$ by Lemma 2.8. This implies that the natural projection $\pi : Q \to Q$ splits as a monoid homomorphism. Thus $\alpha$ extends to the homomorphism $\pi : Q \to R$ along $\pi$. This implies that we obtain another log structure $(R, Q, \pi)$, which becomes a local log-regular ring with a finitely generated, cancellative, reduced, and root closed monoid.

The second condition in Definition 3.2 can be paraphrased under the first condition as follows.

Theorem 3.4 (cf. [Ogu18, Chapter III, Theorem 1.11.1]). Assume that $R/I_{\alpha}$ is a regular local ring. Then the equality $\dim(R) = \dim(R/I_{\alpha}) + \dim(Q)$ holds if and only if for every prime ideal $q$ of $Q$, the ideal $qR$ of $R$ generated by $\alpha(q)$ is a prime ideal of $R$ such that $\alpha^{-1}(qR) = q$.  

Lemma 3.5. Let $(R, Q, \alpha)$ be a log ring, where $Q$ is a cancellative monoid. Assume that $\alpha$ is injective. Then the image of $\alpha$ is contained in $R^\star = R \setminus \{0\}$.

Proof. If $Q$ is the zero monoid, the claim holds obviously. Thus we may assume that $Q$ is a non-zero monoid. Suppose that there exists $x \in Q$ such that $\alpha(x) = 0$. Then, for a non-zero element $y \in Q$, we have the equality $\alpha(x + y) = \alpha(x)$. Since $\alpha$ is injective and $Q$ is cancellative, we obtain $y = 0$. This is a contradiction. Thus $\text{Im} \alpha \subseteq R^\star$ holds.  

In the situation of Lemma 3.5 we obtain the monoid homomorphism $\alpha^\star : Q \to R^\star$ which decomposes $\alpha$.

Definition 3.6. A monoid homomorphism $\theta : P \to Q$ is exact if the following diagram is cartesian:

\[
\begin{array}{ccc}
P^{\text{gp}} & \longrightarrow & Q^{\text{gp}} \\
\uparrow & & \uparrow \\
\downarrow & & \\
P & \longrightarrow & Q,
\end{array}
\]

that is, $Q \times Q^{\text{gp}} P^{\text{gp}} = P$.

\[\text{In [Ogu18], a monoid homomorphism } \alpha \text{ associated with a local log-regular ring } (R, Q, \alpha) \text{ which satisfies the latter condition is called very solid}.\]
By Theorem 3.10 which we introduce later, we obtain the injectivity of structure morphisms of log-regular rings. To prove Corollary 4.5, we need the exactness of $\alpha^\bullet$.

**Lemma 3.7.** Let $(R, Q, \alpha)$ be a local log-regular ring. Assume that $Q$ is finitely generated, cancellative, reduced, and root closed. Then $\alpha^\bullet$ is exact.

**Proof.** Since $Q$ is finitely generated, cancellative and root closed and $R^\bullet$ is cancellative, it suffices to show that $\text{Spec}(\alpha^\bullet)$ is surjective by [Ogu18, Chapter I, Proposition 4.2.2]. For any $q \in \text{Spec}(Q)$, $qR$ is prime of $R$ and $\alpha^{-1}(qR) = q$ by Theorem 3.4. Set $q^\bullet := q \setminus \{0\} \subseteq R^\bullet$. Since $q^\bullet$ is a prime ideal of $R^\bullet$, $\text{Spec}(\alpha^\bullet)(q^\bullet) = q$ holds. Hence $\text{Spec}(\alpha^\bullet)$ is surjective. □

Let $Q$ be a finitely generated, cancellative reduced monoid and let $R$ be a commutative ring. Then we denote by $R[[Q]]$ the set of functions $Q \rightarrow R$, viewed as an $R$-module using the usual point-wise structure and endowed with the product topology induced by the discrete topology on $R$, that is, we have the explicit description

$$R[[Q]] = \left\{ \sum_{q \in Q} a_q e_q \mid a_q \in R \right\}.$$ 

By using this description, the $R$-module $R[Q]$ admits the unique multiplication (see [Ogu18, Chapter I, Proposition 3.6.1 (2)]). Also, $R[Q]$ of $Q$ can be view as the completion of $R[Q^+]$ with respect to an ideal $R[Q^+]$ (see [Ogu18, Chapter I, Proposition 3.6.1 (3)]).

**Remark 3.8.** It is easy to miss symbolically, but $R[Q]$ is not complete and separated with respect to a maximal ideal even if $R$ is a field.

**Proposition 3.9.** Keep the notation as above. Then the following assertions hold.

1. If $Q^{gp}$ is torsion free and $R$ is also an integral domain, then $R[[Q]]$ is an integral domain.
2. If $R$ is a local ring with the maximal ideal $m$, then $R[[Q]]$ is a local ring with the maximal ideal consisting of elements of $R[[Q]]$ such that their constant term belongs to $m$.

**Proof.** These are [Ogu18, Chapter I, Proposition 3.6.1 (4) and (5)]. □

The following theorem is an analogue of Cohen’s structure theorem and it is one of the main tools to find out properties of local log-regular rings.

**Theorem 3.10 (Ogu18, Chapter III, Theorem 1.11.2 or INS22, Theorem 2.20).** Let $(R, Q, \alpha)$ be a local log ring, where $R$ is a Noetherian ring and $Q$ is finitely generated, cancellative, reduced, and root closed. Let $k$ be the residue field of $R$. Then the following assertions hold.

1. Suppose that $R$ is of equal characteristic. Then $(R, Q, \alpha)$ is log-regular if and only if there exists a commutative diagram of the form

$$\begin{array}{ccc}
Q & \xrightarrow{\alpha} & k[[Q \oplus \mathbb{N}^\bullet]] \\
\downarrow & & \downarrow \cong \\
R & \xrightarrow{\phi} & \widehat{R},
\end{array}$$

where the top arrow is the natural injection and $\widehat{R}$ is the $m$-adic completion of $R$.

2. Suppose that $R$ is of mixed characteristic. Let $C(k)$ be a Cohen ring of $k$ and let $p > 0$ be the characteristic of $k$. Then $(R, Q, \alpha)$ is log-regular if and only if there exists a commutative
Moreover, let $e_1, \ldots, e_r$ be the canonical bases on $N'$ and let $x_1, \ldots, x_r$ be a sequence of elements of $R$ such that $\bar{x}_1, \ldots, \bar{x}_r$ is a regular system of parameters for $R/I_\alpha$. If $(R, Q, \alpha)$ is a local log-regular ring, then one may assume that $\phi$ sends $e_i$ to $\bar{x}_i$ where $\bar{x}_i$ is the image of $x_i$ in $\widehat{R}$.

**Proof.** The former assertions (1) and (2) are exactly [Ogu18, Chapter III, Theorem 1.11.2]. The latter assertion is obtained in the proof of [Ogu18, Chapter III, Theorem 1.11.2].

**Definition 3.11.** Let $(R, Q, \alpha)$ be a log ring. Then $R$ is $\alpha$-flat if $\text{Tor}^1_1(\mathbb{Z}[Q]/\mathbb{Z}[I], R) = 0$ for any ideal $I \subseteq Q$.

Under the first condition in Definition 3.2, the second condition is equivalent to several conditions.

**Proposition 3.12.** Keep the notation and the assumption as in Definition 3.2. Assume that $R/I_\alpha$ is regular. Then the following are equivalent:

1. $(R, Q, \alpha)$ is a local log-regular ring.
2. For every prime ideal $q$ of $Q$, the ideal $qR$ is generated by $\alpha(q)$ is a prime ideal of $R$ such that $\alpha^{-1}(qR) = q$ (then we say that $\alpha$ is very solid).
3. $R$ is $\alpha$-flat.
4. $\text{Tor}^1_1(\mathbb{Z}[Q]/\mathbb{Z}[I], R) = 0$.
5. $\text{gr}\mathbb{Z}[Q^+]/(\mathbb{Z}[Q]) \otimes_{\mathbb{Z}} R/I_\alpha \cong \text{gr}_{I_\alpha} R$ is an isomorphism.

**Proof.** The equivalences (1) $\iff$ (2) $\iff$ (4) $\iff$ (5) are a combination of [Ogu18, Chapter III, Theorem 1.11.1] and [Ogu18, Chapter III, Proposition 1.11.5]. The equivalence (1) $\iff$ (3) is [Tho06, Proposition 52].

Here we provide an example of non-complete local log-regular rings which is called Jungian domain. This is defined by S. Abhyankar [Abh65] (see also [Kat94, §12]). He introduced it and explored how to construct it. For example, see [Abh65, Theorem 10] or [Abh65, Theorem 14]. Here we recall the definition of Jungian domains and give an induced log-structure.

**Definition 3.13** (Abh65, P23, Definition 2). Let $(R, m)$ be a Noetherian local domain. We say that $(R, m)$ is a Jungian domain if it is a two-dimensional normal domain such that the following condition satisfies: There exist integers $m, n \in \mathbb{Z}$ with $0 \leq m \leq n$ and $\text{GCD}(m, n) = 1$ and generators $x, y, z_1, \ldots, z_{n-1}$ of $m$ such that $z_i^n = x^iy^{m_i}$ for any $i = 1, \ldots, n-1$, where $m_i$ is the unique integer such that $0 \leq m_i \leq n$ and $m_i = m_i \pmod{n}$.

**Lemma 3.14.** Let $(R, m)$ be a Jungian domain, let $\mathcal{M}$ be the multiplicative submonoid

$$\langle x_1^l y_2^l z_3^l \cdots z_{n-1}^l | l_1, \ldots, l_{n-1} \geq 0 \rangle,$$

and let $\alpha : \mathcal{M} \to R$ be the inclusion map. Then $\mathcal{M}$ is finitely generated, cancellative, reduced, and root closed. Moreover, $(R, \mathcal{M}, \alpha)$ is a local log-regular ring.
Proof. Since $R$ is an normal domain and $M$ is generated by $x, y, z_1, \ldots, z_{i-1}$, $M$ is obviously finitely generated, cancellative and root closed. Moreover, it follows from $I_n = \mathfrak{m}$ that $M$ is reduced and $R/I_n$ is regular. Finally, we can easily check that any prime ideal of $M$ forms $\mathfrak{p} \cap M$ where $\mathfrak{p}$ is a prime ideal of $R$. Hence $\dim(M) = \dim(R)$.  

It is well-known that a normal affine semigroup ring is Cohen–Macaulay and normal, which is proved by Hochster. The same assertion holds for a local log-regular ring.

**Theorem 3.15.** Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Then $R$ is Cohen-Macaulay and normal.

**Proof.** See [Kat91 (4.1) Theorem] or [GR23 Corollary 12.5.13]. □

**Remark 3.16.** If $\mathcal{Q}$ is finitely generated, cancellative, reduced, and root closed, then there is an exact injection $\mathcal{Q} \hookrightarrow \mathbb{N}^l$ for some $l \in \mathbb{N}$ (see [Ogu18 Chapter I, Proposition 1.3.5] and [Ogu18 Chapter I, Corollary 2.2.7]). Thus, in the following sections, we assume that a finitely generated, cancellative, reduced, and root closed monoid is a submodule of some $\mathbb{N}^l$.

### 4. Canonical modules of local log-regular rings

In this section, we provide an explicit structure of the canonical module of a local log-regular ring.

**Theorem 4.1.** Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring, where $\mathcal{Q}$ is finitely generated, cancellative, reduced, and root closed (by Remark 3.16, we may assume that $\mathcal{Q} \subseteq \mathbb{N}^l$ for some $l > 0$). Let $x_1, \ldots, x_r$ be a sequence of elements of $R$ such that $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ is a regular system of parameters for $R/I_\mathcal{Q}$. Then $R$ admits a canonical module and its form is

\[(4.1) \quad \langle (x_1 \cdot \ldots \cdot x_r)\alpha(a) \mid a \in \text{relint} \mathcal{Q} \rangle \]

is the canonical module of $R$, where $\text{relint} \mathcal{Q}$ is the relative interior of $\mathcal{Q}$.

**Proof.** First, assume that $R$ is $\mathfrak{m}$-adically complete and separated. If $R$ is of equal characteristic, then $R$ is isomorphic to $k[\mathcal{Q} \oplus \mathbb{N}^r]$ by Theorem 3.10. Let us check that

\[(4.2) \quad k[\text{relint} \mathcal{Q} \oplus (e + \mathbb{N}^r)] := \langle (q, e) \mid q \in \text{relint} \mathcal{Q} \rangle \subseteq k[\mathcal{Q} \oplus \mathbb{N}^r] \]

is a canonical module of $k[\mathcal{Q} \oplus \mathbb{N}^r]$, where $e = (1, 1, \ldots, 1) \in \mathbb{N}^r$. Indeed, note that we have the ring isomorphism $k[\mathcal{Q}] \otimes_k k[\mathbb{N}^r] \cong k[\mathcal{Q} \oplus \mathbb{N}^r]$. Also note that the canonical module $\omega_k[\mathcal{Q}] = k[\text{relint} \mathcal{Q}]$ and $\omega_k[\mathbb{N}^r] = k[e + \mathbb{N}^r]$ by [BH98 Theorem 6.3.5 (b)]. This induces the following isomorphisms

\[(4.3) \quad \omega_k[\mathcal{Q} \oplus \mathbb{N}^r] \cong \omega_k[\mathcal{Q}] \otimes_k \omega_k[\mathbb{N}^r] = k[\text{relint} \mathcal{Q}] \otimes_k k[e + \mathbb{N}^r] .\]

If you trace [4.3] backwards, then it turns out that $\omega_k[\mathcal{Q} \oplus \mathbb{N}^r]$ is the form of (4.2). Since $R$ is isomorphic to the completion of $k[\mathcal{Q} \oplus \mathbb{N}^r]$ along a maximal ideal $k[(\mathcal{Q} \oplus \mathbb{N}^r)^+]$, the image of (4.2) in $R$ is the canonical module of $R$.

If $R$ is of mixed characteristic, then $R$ is isomorphic to $C(k)[\mathcal{Q} \oplus \mathbb{N}^r]/(\theta)$ for some $\theta \in W(k)[\mathcal{Q} \oplus \mathbb{N}^r]$. If $C(k)[\mathcal{Q} \oplus \mathbb{N}^r]$ has a canonical module, then its image in $R$ is the canonical module of $R$. Thus it suffices to show the case where $R = C(k)[\mathcal{Q} \oplus \mathbb{N}^r]$.

Set $\omega_R := \langle (p(q_i, e) \mid q_i \in \text{relint} \mathcal{Q} \rangle \subseteq C(k)[\mathcal{Q} \oplus \mathbb{N}^r]$. Since $\omega_R/p\omega_R$ is a canonical module of $R/pR \cong k[\mathcal{Q} \oplus \mathbb{N}^r]$ and $p$ is a regular element on $R$ and $\omega_R$, $\omega_R$ is a maximal Cohen-Macaulay module of type 1. Finally, since $R$ is a domain, $\omega_R$ is faithful. Thus $\omega_R$ is a canonical module of $R$.

Next, let us consider the general case. We define the ideal $\omega_R$ as (4.1). Then, by considering the diagrams [3.1] or [3.2], the image of $\omega_R$ in the $\mathfrak{m}$-adic completion of $R$ is the canonical module. Thus, by [BH98 Theorem 3.3.14 (b)], $\omega_R$ is a canonical module of $R$. □
Remark 4.2. Set $\omega_R := (x_1 \cdots x_r)\alpha(a) | a \in \text{relint } Q$ and $\omega'_R := \langle \alpha(a) | a \in \text{relint } Q \rangle$. Then we note that the homomorphism $\omega'_R \times x_1 \cdots x_r \to \omega_R$ is isomorphism. Namely, the ideal of $R$ generated by the image of the relative interior of the associated monoid is also the canonical module of $R$.

Remark 4.3. In Theorem 4.1, the case when $R = W(k)[\sigma^\vee \cap M]$ follows from the following Marcus Robinson’s result. Let $A := W(k)[\sigma^\vee \cap M]$, where $M$ is a lattice and $\sigma$ is the strongly convex polyhedral cone. Let $X = \text{Spec}(A)$. Then one can choose codimension one subschemes $D_1, \ldots, D_n$ of $X$ such that $K_X = -\sum D_i$ is a canonical divisor on $X$. Indeed, his result implies that the ideal $\omega_A := \bigcap p_i$ is a canonical module of $A$, where $p_i$ is the corresponding height one prime ideal to $D_i$.

As an application of Theorem 4.1, let us provide a Gorenstein criterion of local log-regular rings. In order to prove it, we need the following proposition.

Proposition 4.4. Let $(R, Q, \alpha)$ be a local log-regular ring. Let $x := x_1, \ldots, x_r$ be a sequence of elements of $R$ such that $x_1, \ldots, x_r$ is a regular system of parameters for $R/I\alpha$. Set $R_i := R/(x_1, \ldots, x_i)$ and $\alpha_i : Q \to R \to R_i$. Then $x$ is a regular sequence on $R$ and $(R_i, Q, \alpha_i)$ is also a local log-regular ring for any $1 \leq i \leq r$.

Proof. Since a local homomorphism preserves the locality of the log structure (see [INS22, Lemma 2.16]), $(R_i, Q, \alpha_i)$ is a local log ring. By the induction for $i$, it suffices to check that $i = 1$. Since $R$ is a domain, $x_1$ is a regular element. Thus we obtain the isomorphism $R_1/I\alpha_1 \cong (R/I\alpha)/x_1(R/I\alpha)$. Since the image of $x_1$ is a regular element on $R/I\alpha$ by the assumption and $R/I\alpha$ is a regular local ring, $R_1/I\alpha_1$ is regular. Moreover, the above isomorphism implies that the equality $\dim(R_1/I\alpha_1) = \dim(R_1) - \dim(Q)$ holds. Thus $(R_1, Q, \alpha_1)$ is a local log-regular ring.

Corollary 4.5. Keep the notation as in Theorem 4.1. The following assertions are equivalent:

1. $R$ is Gorenstein.
2. For a fixed field $k$, $k[Q]$ is Gorenstein.
3. There exists an element $c \in \text{relint } Q$ such that $\text{relint } Q = c + Q$.

Proof. The equivalence of (2) and (3) is well-known (for example, see [BH98, Theorem 6.3.5 (a)]). Thus it suffices to show the equivalence of (1) and (3). Since the Gorenstein property of $R$ is preserved under the completion and the quotient by a regular sequence, one can assume that $\alpha$ is injective by Theorem 4.10 and that $\dim(R) = \dim(Q)$ by Proposition 4.1. Hence $\omega_R = \langle \alpha(x) | x \in \text{relint } Q \rangle$. Now, assume that $R$ is Gorenstein. There exists an element $c \in \text{relint } Q$ such that $\omega_R = \langle \alpha(c) \rangle$. This implies that for any $a \in \text{relint } Q$, there exists $x \in R$ such that $\alpha(a) = \alpha(c)x$. Since we have $\alpha(a) = \alpha^*(a)$ and $\alpha(c) = \alpha^*(c)$ by Lemma 3.5, we obtain

$$\alpha^*(a) = \alpha^*(c)x.$$  

Hence $x = \alpha^*(a - c) \in \text{Im}(\langle \alpha^* \rangle^{gp})$. Since $\alpha^*$ is exact, we obtain $x \in \text{Im} \alpha^*$. Now, there exists $y \in Q$ such that $x = \alpha^*(y)$. By (4.4) and the injectivity of $\alpha^*$, we obtain $a = c + y \in c + Q$. Hence $\text{relint } Q \subseteq c + Q$. Since $\text{relint } Q$ is an ideal of $Q$, the converse inclusion holds. Therefore we obtain $\text{relint } Q = c + Q$.

Conversely, assume that $\text{relint } Q = c + Q$ for some $c \in \text{relint } Q$. Then we obtain the equalities $\omega_R = \alpha(c)\langle \alpha(x) | x \in R \rangle = \alpha(c)\text{relint } Q$. This implies that $R$ is Gorenstein, as desired.

3For readers who are not familiar with algebraic geometry, see [ST12, Appendix B].
If a Cohen-Macaulay local ring has a canonical module, it is a homomorphic image of a Gorenstein local ring. Namely, we obtain the following corollary (a toric ring is always a homomorphic image of a regular ring, but we don’t know whether a local log-regular ring is a homomorphic image of a regular ring or not).

**Corollary 4.6.** Let \((R, \mathcal{Q}, \alpha)\) be a local log-regular ring. Then \(R\) is a homomorphic image of a Gorenstein local ring.

Next, we prove that local log-regular rings are pseudo-rational. See [LT81] or [HM18] for the definition of pseudo-rationality.

**Theorem 4.7 ([Bou87], [HM18], [HH90]).** Let \((R, m) \to (S, n)\) be a pure map of local rings such that \((S, n)\) is regular. Then \(R\) is pseudo-rational. In particular, direct summands of regular rings are pseudo-rational.

Applying this theorem, we obtain the pseudo-rationality of a local log-regular ring. Our proof does not need the discussion using divisors.

**Proposition 4.8.** Let \((R, \mathcal{Q}, \alpha)\) be a local log-regular ring. Then \(R\) is pseudo-rational.

**Proof.** Since a local log-regular ring has the canonical module by Theorem 4.14 by applying [Mu22] Proposition 4.20, it suffices to show that a complete local log-regular ring is pseudo-rational. Thus we may assume that \(R\) is \(m\)-adically complete and separated. Namely, \(R\) is isomorphic to either \(k[Q \oplus \mathbb{N}^r]\) or \(C(k)[Q \oplus \mathbb{N}^r]/(\theta)\). Now, we prove that \(R\) is the direct summand of a regular local ring. Our approach is the same as in the proof of [LNS22] Theorem 2.27, so we give the sketch of the proof here. We refer the reader to it for details. Since the same argument is made, we will show the case \(R \cong C(k)[Q \oplus \mathbb{N}^r]/(\theta)\). An embedding \(Q \hookrightarrow \mathbb{N}^r\) given in Remark 3.16 induces a split injection \(C(k)[Q \oplus \mathbb{N}^r] \to C(k)[\mathbb{N}^d]\) for some \(d > 0\). This induces the split injection \(C(k)[Q \oplus \mathbb{N}^r] \to C(k)[\mathbb{N}^d]\). And after taking the quotient by some element \(\theta \in A[Q \oplus \mathbb{N}^r]\), we also obtain the split injection \(C(k)[Q \oplus \mathbb{N}^r]/(\theta) \to C(k)[\mathbb{N}^d]/(\theta)\).

Finally, applying Theorem 4.7 we obtain the desired claim. □

**Remark 4.9.** There is another way to prove equal characteristic cases. If \(R\) is \(F\)-finite complete local log-regular ring, then it is strongly \(F\)-regular ring. Since strong \(F\)-regularity implies \(F\)-rationality, \(R\) is \(F\)-rational. Hence we obtain \(R\) is pseudo-rational because an \(F\)-rational ring is pseudo-rational by [Smi97] Theorem 3.1. Also, the equal characteristic 0 case is due to [Sch08] Main Theorem A1 and the above discussion.

At the last of this section, we determine the form of Gorenstein local log-regular rings consisting of two-dimensional monoids by using Corollary 4.5.

**Proposition 4.10.** Let \((R, \mathcal{Q}, \alpha)\) be a local log-regular ring where \(\mathcal{Q}\) is finitely generated, cancellative, reduced, and root closed. Assume that \(\mathcal{Q}\) is two-dimensional. Then \(R\) is Gorenstein if and only if \(\mathcal{Q}\) is isomorphic to the submonoid of \(\mathbb{N}^2\) generated by \((n + 1, 0), (1, 1), (0, n + 1)\) for some \(n \geq 1\).

**Proof.** By Corollary 4.5 one can reduce to the case of a toric ring \(k[Q]\) where \(k\) is an algebraically closed field, and in this case, we know that there exists \(n \geq 1\) such that \(k[Q]\) is isomorphic to \(k[P]\) where \(P\) is the submonoid of \(\mathbb{N}^2\) generated by \((n + 1, 0), (1, 1), (0, n + 1)\). By applying [Gub98] Theorem 2.1 (b) (see also [BG99]), we can show that \(Q\) is isomorphic to \(P\), as desired. Conversely, assume that \(Q\) is isomorphic to the submonoid generated by \((n + 1, 0), (1, 1), (0, n + 1) \in \mathbb{N}^2\). Then \(k[Q]\) is Gorenstein for an algebraically closed field \(k\) because this is an \(A_n\)-type singularity. Thus \(R\) is also Gorenstein by Corollary 4.5 as desired. □
From the above proposition, it follows that a complete Gorenstein local log-regular ring with the two-dimensional monoid has the following form.

**Corollary 4.11.** Let \((R, Q, \alpha)\) be a local log-regular ring where \(Q\) is a two-dimensional finitely generated, cancellative, reduced, and root closed monoid. Then the following assertions hold.

1. Suppose that \(R\) is of equal characteristic. Then \(R\) is Gorenstein if and only if \(R\) is isomorphic to \(k[s^{n+1}, s^t, t^n+1, x_1, \ldots, x_r]\) for some \(n \geq 1\).
2. Suppose that \(R\) is of mixed characteristic. Then \(R\) is Gorenstein if and only if \(R\) is isomorphic to \(C(k)[s^{n+1}, s^t, t^n+1, x_1, \ldots, x_r]/(\theta)\) for some \(n \geq 2\) where \(C(k)\) is the Cohen ring of the residue field \(k\) and \(\theta\) is an element of \(C(k)[s^{n+1}, s^t, t^n+1, x_1, \ldots, x_r]\) with the constant term \(p\).

**Proof.** These follow from Proposition 4.10 and Theorem 3.10. \(\square\)

We also provide examples of non-Gorenstein local log-regular rings.

**Example 4.12.** Let \(P\) be a monoid generated by \(a_1 := (1, 0), a_2 := (1, 1), a_3 := (1, 2), a_4 := (1, 3)\).

1. Set \(R := \mathbb{Z}_p[P]/(p - a_1) \cong \mathbb{Z}_p[s, st, st^2, st^3]/(p - s)\) and set \(\alpha : P \to \mathbb{Z}_p[P] \to R\) Then \((R, Q, \alpha)\) is a local log-regular ring. By Proposition 4.10, we know that \(R\) is not Gorenstein. Moreover, \(R\) is also isomorphic to \(\mathbb{Z}_p[pt, pt^2, pt^3]\). Since the relative interior of \(Q\) is generated by \((1, 1)\) and \((1, 2)\), its canonical module is generated by \(e^{a_2}, e^{a_3}\) and it is isomorphic to the ideal of \(\mathbb{Z}_p[pt, pt^2, pt^3]\) generated by \(pt\) and \(pt^2\).

2. Set \(S := \mathbb{Z}_p[P]\). Then \((S, P, P \to S)\) is a local log-regular ring. We note that \(S\) is not complete and separated with respect to the maximal ideal \(m\) (see Remark 3.3) and its \(m\)-adic completion is isomorphic to \(\mathbb{Z}_p[s, st, st^2, st^3]/(p - x)\). Then the canonical module of \(S\) is isomorphic to the ideal of \(pe^{a_2}\) and \(pe^{a_3}\).

5. **Divisor class groups of local log-regular rings**

**Lemma 5.1.** Let \((R, Q, \alpha)\) be a local log-regular ring and let \(p\) be a height one prime ideal of \(R\). Then the following are equivalent.

1. There exists a height one prime ideal \(q\) of \(Q\) such that \(p = qR\),
2. The intersection of \(\text{Im} \alpha\) and \(\alpha^{-1}(p)\) is not empty.

**Proof.** The implication \((1) \Rightarrow (2)\) is obvious, hence let us consider the implication \((2) \Rightarrow (1)\). Note that \(\alpha^{-1}(p)\) is a height one prime ideal by assertion \((2)\). Since \(\alpha\) is very solid and any element of \(Q\) does not map to 0 by Lemma 3.3, we obtain \(\alpha^{-1}(p)R = p\). Hence assertion \((1)\) holds. \(\square\)

**Lemma 5.2.** Let \((R, Q, \alpha)\) be a log ring and let \(I, J\) be ideals of \(Q\). Assume that \(R\) is \(\alpha\)-flat. Then

\[ \alpha(I)R \cap \alpha(J)R = \alpha(I \cap J)R \]

holds.

**Proof.** Let us consider the following diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}[I \cap J] & \longrightarrow & \mathbb{Z}[I] & \longrightarrow & \mathbb{Z}[I]/\mathbb{Z}[I \cap J] & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}[J] & \longrightarrow & \mathbb{Z}[Q] & \longrightarrow & \mathbb{Z}[Q]/\mathbb{Z}[J] & \longrightarrow & 0.
\end{array}
\]
By the $\alpha$-flatness of $R$, we obtain the following diagram.
\[
0 \longrightarrow Z[I \cap J] \otimes_{\mathbb{Z}[Q]} R \longrightarrow Z[I] \otimes_{\mathbb{Z}[Q]} R \longrightarrow Z[I]/Z[I \cap J] \otimes_{\mathbb{Z}[Q]} R \longrightarrow 0
\]
This diagram is isomorphic to the following one:
\[
0 \longrightarrow \alpha(I \cap J)R \longrightarrow \alpha(I)R \longrightarrow \alpha(I)R/\alpha(I \cap J)R \longrightarrow 0
\]
Since the vertical arrows are injective, we obtain the following exact sequence by the snake lemma.
\[\text{0} \longrightarrow \alpha(J_0)R/\alpha(I \cap J_0)R \longrightarrow R/\alpha(I\alpha)R \longrightarrow \alpha(J)R/\alpha(I \cap J)R \longrightarrow 0.\]
Thus since we obtain $\alpha(J)R/\alpha(I \cap J)R \cong \text{Ker} p = \alpha(I)R + \alpha(J)R/\alpha(I\alpha)R \cong \alpha(J)R/\alpha(I\alpha)R$, the equality $\alpha(I \cap J)R = \alpha(I)R \cap \alpha(J)R$ holds. \hfill \Box

**Lemma 5.3.** Let $(R, Q, \alpha)$ be a log ring where $R$ is a domain and $Q$ is cancellative. Let $J, J'$ be a fractional ideal of $Q$. Then the equality $(J \cap J')R = JR \cap J'R$ holds.

**Proof.** Choose $x \in Q$ such that $xJ, xJ' \subseteq Q$. Then it suffices to show that $x(JR \cap J'R) = xJR \cap xJ'R$, but this follows from $x(J \cap J') = xJ \cap xJ'$. \hfill \Box

**Lemma 5.4.** Let $(R, Q, \alpha)$ be a local log-regular ring and let $I, J, J' \subseteq Q^{gp}$ be fractional ideals of $Q$. Assume that $I$ is finitely generated. Then the following assertions hold.

(1) The equality $(J : I)R = (JR : IR)$ holds.

(2) $JR$ is equal to $J'R$ if and only if $J$ is equal to $J'$.

(3) $\text{Div}(\alpha) : \text{Div}(Q) \to \text{Div}(R)$ is well-defined and it is injective.

**Proof.** We express $I = a_1Q \cup \cdots \cup a_nQ$ for some $a_1, \ldots, a_n \in Q^{gp}$. Thus we obtain $(J : I) = a_1^{-1}J \cap \cdots \cap a_n^{-1}J$ and $(JR : IR) = a_1^{-1}JR \cap \cdots \cap a_n^{-1}JR$. Here, by Lemma 5.3, the equality $a_1^{-1}JR \cap \cdots \cap a_n^{-1}JR = (a_1^{-1}J \cap \cdots \cap a_n^{-1}J)R$ holds. Hence the assertion (1) holds.

Next to prove the assertion (2), we may assume that $J \subseteq J'$ after replacing $J'$ with $J' \cup J$. Assume that $JR = J'R$. Then, since the equality $Z[xJ] \otimes_{\mathbb{Z}[Q]} R = Z[xJ'] \otimes_{\mathbb{Z}[Q]} R$ holds, we obtain $Z[xJ] = Z[xJ']$ by faithfully $\alpha$-flatness, and hence $xJ = xJ'$ holds. The converse implication is obvious, the assertion (2) holds.

Finally, the first assertion of (3) follows from (1), and the second follows from (2). \hfill \Box

**Proposition 5.5.** Let $(R, Q, \alpha)$ be a local log-regular ring. Then $\text{Cl}(\alpha) : \text{Cl}(Q) \to \text{Cl}(R)$ is well-defined and it is injective.

**Proof.** This is the same as in [GR23, Proposition 6.4.55]. \hfill \Box

**Lemma 5.6.** Let $(R, Q, \alpha)$ be a complete local log-regular ring. Let $S$ be the image of $\alpha$. There exists an $R$-algebra $T$ such that $T$ is a regular local ring and $S^{-1}R \cong S^{-1}T$.

**Proof.** By replacing the monoid $Q$ with $\mathbb{Q} \oplus \mathbb{N}^{\dim(R/I_0)}$, we may assume $\dim(R/I_0) = 0$.

First, suppose that $R$ is of equal characteristic. Then $R$ is isomorphic to $k[Q]$ by Theorem 3.10 [1]. Here, by Lemma 2.7 [2], the monoid homomorphism $Q \to \mathbb{N}$ induces the injective ring homomorphism $k[Q] \hookrightarrow k[\mathbb{N}]$. Moreover, since $S^{-1}k[Q]$ is isomorphic to $S^{-1}k[\mathbb{N}]$ where $S$ is the multiplicatively closed subset of $k[Q]$ generated by an element of the relative interior of $Q$ and
$S'$ is its image, $S^{-1}k[Q]$ is a unique factorization domain. Thus $k[N']$ is the desired regular local ring.

Next, suppose that $R$ is of mixed characteristic. Then $R$ is isomorphic to $V[[Q]]/(\theta)$ by Theorem 3.10 [2]. By the same discussion of the equal characteristic case, we obtain the injection $V[[Q]]/(p-f)V[Q] \rightarrow V[N']/(p-f)V[N']$, and $S^{-1}(V[[Q]]/(p-f)V[Q])$ is isomorphic to $S'^{-1}(V[N']/(p-f)V[N'])$. This also implies that $V[N']/(p-f)V[N']$ is the desired regular local ring. \hfill \Box

We prove the second main result in this paper. See [INS22] and [CLM+22] of the torsion part of the divisor class group of a local log-regular ring

**Theorem 5.7.** Let $(R, Q, \alpha)$ be a local log-regular ring. Then $\text{Cl}(\alpha) : \text{Cl}(Q) \rightarrow \text{Cl}(R)$ is isomorphism. In particular, the divisor class group $\text{Cl}(R)$ is finitely generated.

**Proof.** Consider the composite map

$$\text{Cl}(Q) \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(\tilde{R}).$$

Note that the former group homomorphism $\text{Cl}(Q) \rightarrow \text{Cl}(R)$ is injective by Proposition 5.3 and it is well-known that the latter group homomorphism $\text{Cl}(R) \rightarrow \text{Cl}(\tilde{R})$ is injective. Since it suffices to show that $\text{Cl}(Q) \rightarrow \text{Cl}(\tilde{R})$ is surjective, we may assume that $R$ is complete. By Nagate’s theorem, we obtain the following short exact sequence:

$$0 \rightarrow H \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(S^{-1}R) \rightarrow 0,$$

where $H$ is the subgroup of $R$ generated by the isomorphic class of a height one prime ideal that meets $S$. Since $\text{Cl}(S^{-1}R)$ is trivial by Lemma 5.6 we obtain $H = \text{Cl}(R)$. Moreover, we have an isomorphism $\text{Cl}(\alpha) : \text{Cl}(Q) \xrightarrow{\cong} \text{Im}(\text{Cl}(\alpha)) = H$ by Lemma 5.1. This implies that $\text{Cl}(\alpha)$ is an isomorphism. Finally, since the set of height one prime ideals of $Q$ is finite by Lemma 2.6, $\text{Cl}(Q)$ is finitely generated. Thus so is $\text{Cl}(R)$.

By combining Theorem 5.7 with Chouinard’s Theorem [Cho81] (or see [G184] Corollary 16.8), for a local log-regular ring $(R, Q, \alpha)$, we obtain the following isomorphism

\begin{equation}
\text{Cl}(R) \cong \text{Cl}(Q) \cong \text{Cl}(k[Q]).
\end{equation}

**Example 5.8.** Let $Q \subseteq \mathbb{N}^4$ be a root closed submonoid generated by $x_1 := (1, 1, 0, 0), x_2 := (0, 0, 1, 1), x_3 := (1, 0, 0, 1),$ and $x_4 := (0, 1, 1, 0)$. Let $R := W(k)[Q]/(p - \varepsilon x)W(k) \cong W(k)[x, y, z, w]/(xy - zw, p - w) = W(k)[x, y, z]/(xy - pz)$ where $k$ is a perfect field. Then $(R, Q, \alpha)$ is a local log-regular ring by Theorem 3.10 where $\alpha : Q \rightarrow W(k)[Q] \rightarrow R$ is the composition of monoid homomorphisms. Moreover, by applying the isomorphisms (5.1), we obtain $\text{Cl}(R) \cong \text{Cl}(k[Q]) = \mathbb{Z}$.

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