RATIONAL POINTS ON AN INTERSECTION OF DIAGONAL FORMS

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Abstract. We consider intersections of $n$ diagonal forms of degrees $k_1 < \cdots < k_n$, and we prove an asymptotic formula for the number of rational points of bounded height on these varieties. The proof uses the Hardy-Littlewood method and recent breakthroughs on the Vinogradov system. We also give a sharper result for one specific value of $(k_1, \ldots, k_n)$, using a technique due to Wooley and an estimate on exponential sums derived from a recent approach in the van der Corput's method.

1. Introduction

Let $s, n \geq 1$ be integers. Let $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ such that

\begin{equation}
1 \leq k_1 < k_2 < \cdots < k_n.
\end{equation}

Let $F : \mathbb{R}^s \to \mathbb{R}^n$ where $F = (F_1, F_2, \ldots, F_n)$ and $F_1, F_2, \ldots, F_n \in \mathbb{Z}[t_1, t_2, \ldots, t_s]$ are diagonal forms that satisfy

\begin{equation}
F_i(t) = \sum_{j=1}^s u_{i,j} t_j^{k_i}, \quad u_{i,j} \in \mathbb{Z} \setminus \{0\} \quad (1 \leq i \leq n, 1 \leq j \leq s).
\end{equation}

We are interested in the asymptotic behaviour of the number of solutions of the following diophantine system

\begin{equation}
F_1(x) = F_2(x) = \cdots = F_n(x) = 0
\end{equation}

with $x \in \mathbb{Z}^s \cap [-X, X]^s$, as $X \to +\infty$. In the sequel, this number will be written

\begin{equation}
N_F(X) := \{x \in \mathbb{Z}^s \cap [-X, X]^s : F(x) = 0\} \quad (X \geq 1),
\end{equation}

and for $k$ as in (1.1) and $s \geq 1$, we set

\begin{equation}
D(k, s) = \left\{ F = (F_1, F_2, \ldots, F_n) \text{ that satisfy (1.2)} \right\}.
\end{equation}

The interest in particular cases of systems of the form (1.3) has widely increased in the last decades, considering in particular for $k$ as in (1.1) some variations of the historic case of the Vinogradov system, namely systems of the form

\begin{equation}
\sum_{j=1}^b (x_j^{k_i} - x_{b+j}^{k_i}) \quad (1 \leq i \leq n).
\end{equation}

The original Vinogradov system itself, the case where $k_i = i$ has been the subject of extensive studies with its culminating point in the last decade with the Vinogradov Mean Value Theorem, through the efficient congruencing techniques due to Wooley.
and the decoupling techniques due to Bourgain, Demeter and Guth (See also [8] for a remarkable survey on the Vinogradov system and these two methods). Namely, writing

\[
J_{b,k}(X) := \int_{[0,1]^b} \left| \sum_{1 \leq x \leq X} e \left( \sum_{1 \leq j \leq k} \beta_j x^j \right) \right|^{2b} \, d\beta \quad (b, k \geq 1),
\]

the Vinogradov Mean Value Theorem asserts that for any fixed \( \varepsilon > 0 \) and any \( b \geq 1 \) on has

\[
J_{b,k}(X) \ll_{b,\varepsilon} X^\varepsilon \left( X^b + X^{2b-\frac{2(k+1)}{\varepsilon}} \right) \quad (X \geq 1).
\]

Long before this Theorem has been proved, it was common knowledge in the fields of the Circle Method that whenever (1.8) is verified for some \( b \geq k(k+1)/2 \), then one may derive an asymptotic for \( J_{b+1,k}(X) \) as \( X \to \infty \) (See for example [13], [15]). As a consequence, the Vinogradov Mean Value Theorem now implies an asymptotic for \( J_{b,k}(X) \) as soon as \( b \geq 1 + k(k+1)/2 \) (See §3.4 of [8] and see [17]).

In [15], Wooley also states that the result extends to (1.3) when \( F \) satisfies (1.2), in the particular case \( k_i = i \) (\( 1 \leq i \leq n \)). Namely, if \( s \geq 2n(n+1) + 1 \), there exists a constant \( c > 0 \) such that

\[
N_{F}(X) \sim cX^{s-n(n+1)/2} \quad (X \to \infty),
\]

provided that the system (1.3) has a nonsingular solution over \( \mathbb{R} \) and over all the \( p \)-adic \( \mathbb{Q}_p \). One should point out that the condition \( s \geq 2n(n+1) + 1 \) above is not a limitation of the Hardy-Littlewood method: it merely corresponds to the value \( b \geq k(k+1) \) for which (1.3) was known at the time of [15]. Since then, [17] provides an updated version from the Vinogradov Mean value Theorem, with the new condition \( s \geq n(n+1) + 1 \) for (1.9).

The aim of our paper is to derive an asymptotic for \( N_{F}(X) \) for more general \( k \) as in (1.1), and for \( F \in \mathcal{D}(k, s) \) (with the notation (1.3)), when \( s \) is sufficiently large, still provided that the system (1.3) has a nonsingular solution over \( \mathbb{R} \) and over all the \( p \)-adic \( \mathbb{Q}_p \). Following the lines of [13] and [15], we use the Hardy-Littlewood method. Namely, the classical starting point is the identity

\[
N_{F}(X) = \int_{[0,1]^n} \left( \sum_{\mathbf{x} \in I_s(X)} e(\mathbf{\alpha} \cdot F(\mathbf{x})) \right) \, d\mathbf{\alpha},
\]

where \( I_s(X) = \mathbb{Z}^* \cap [-X, X]^s \), and where here and in the sequel, \( \mathbf{\alpha} \cdot \mathbf{\beta} \) denotes the usual scalar product over \( \mathbb{R}^n \). For \( k \in \mathbb{N}^n \) fixed as in (1.1), the number of solutions of the system (1.3) that satisfy \( |x_j| \leq X \) for each \( j \) is is equal to \( \int_{[0,1]^n} |f_k(\mathbf{\alpha}; X)|^{2b} \, d\mathbf{\alpha} \)

where we have set

\[
f_k(\mathbf{\alpha}; X) := \sum_{|x| \leq X} e \left( \sum_{i=1}^n \alpha_i x^{k_i} \right).
\]

Generalising the heuristic argument for the Vinogradov system, it is conjectured that for any \( \varepsilon > 0 \) one has

\[
\int_{[0,1]^n} |f_k(\mathbf{\alpha}; X)|^{2b} \, d\mathbf{\alpha} \ll_{\varepsilon} X^\varepsilon \left( X^b + X^{2b-\sigma(k)} \right) \quad (X \geq 1)
\]
(1.13) \( \sigma(k) = \sum_{i=1}^{n} k_i \). 

In the current state of knowledge, this conjecture is verified for large and small values of \( b \). More precisely, for large values, Fourier orthogonality yields the classical bound

\[
(1.14) \quad \int_{[0,1]^n} |f_k(\alpha; X)|^{2b} \, d\alpha \ll X^{\frac{k_n(k_n+1)}{2} - \sigma(k)} J_{b,k} n \left( 2X + 1 \right)
\]

and (1.8) implies that (1.12) is satisfied for \( b \geq \frac{k_n(k_n+1)}{2} \). In another direction, Corollary 1.2 of [17] implies that (1.12) is satisfied for \( b \leq \frac{n(n+1)}{2} \), which corresponds to the so-called quasidiagonal behaviour.

We introduce two more classical objects from the Circle Method, with a direct link to (1.10): the singular integral

\[
(1.15) \quad J(F) := \int_{\mathbb{R}^n} \left( \int_{[-1,1]^n} e(\beta \cdot F(t)) \, dt \right) \, d\beta
\]

which measures the real density of the solutions of (1.3) in the box \([-1,1]^n\), and the singular series

\[
(1.16) \quad \mathfrak{S}(F) := \sum_{q \geq 1} \frac{1}{q^s} \sum_{\mathbb{A}(q)} \sum_{r \leq q} e \left( \frac{\mathbf{a} \cdot F(r)}{q} \right),
\]

with

\[
(1.17) \quad \mathbb{A}(q) = \{ \mathbf{a} \in [1, q]^n : (a_1; a_2; \ldots; a_n; q) = 1 \},
\]

related to the \( p \)-adic densities of the solutions of (1.3). In the particular case of (1.9), the constant \( c \) is \( J(F) \mathfrak{S}(F) \), and the hypothesis about nonsingular solutions over \( \mathbb{R} \) and over the \( p \)-adic implies \( c > 0 \).

We are now ready to state our first result, an analogue of (1.9) for more general values of \( k \).

**Theorem 1.** Let \( n \geq 2 \) and \( k \) as in (1.11). Let \( s \geq 1 + k_n(1 + k_n) \) and \( F \in D(k, s) \). Then with the notation (1.15) and (1.16), both \( J(F) \) and \( \mathfrak{S}(F) \) are convergent, and for any \( \varepsilon > 0 \) one has

\[
N_F(X) = J(F) \mathfrak{S}(F) X^{s - \sigma(k)} + O(X^{s - \sigma(k) - \eta_0 + \varepsilon}) \quad (X \geq 1)
\]

where \( \sigma(k) \) has been defined in (1.13), and where we have set \( \eta_0 = \frac{1}{n k_n^2} \). If moreover the system (1.3) has a nonsingular solution over \( \mathbb{R} \) and over \( \mathbb{Q}_p \) for all \( p \), then \( J(F) \mathfrak{S}(F) > 0 \).

This result calls for several comments. First, Theorem 1 implies that the system (1.3) satisfies the Hasse Principle. One should also mention that the constraint \( s \geq 1 + k_n(1 + k_n) \) is directly related to (1.8) with \( b = k(k+1)/2 \), \( k = k_n \) and its impact on (1.14). Furthermore, in the special case \( k_i = i \) (\( 1 \leq i \leq n \)), we indeed recover Wooley’s result (1.19). We now comment on some other cases: when \( k = (k, 1) \), Theorem 1 improves Theorem 1 of [4] for \( k \geq 5 \). However, our result does not improve the cases \( k = 3 \) and \( k = 4 \), for which the sharpest current results
in the line of our theorem remain Theorem 1.5 of [16] for \( k = 3 \), and Theorem 1 of [1] for \( k = 4 \).

Next, we focus one particular case \( n = 3 \) and \((k_1, k_2, k_3) = (1, 3, 5)\). The corresponding Vinogradov-type system \((1.0)\) has already been considered, in the frame of paucity results (cf [3]). Theorem 1 applied to the corresponding Vinogradov-type system \((1.6)\) has already been considered, in the frame of paucity results (cf [3]). Theorem 1 applied to the corresponding Vinogradov-type system \((1.6)\) has already been considered, in the frame of paucity results (cf [3]).

**Theorem 2.** Let \( s = 30, k = (1, 3, 5) \) and \( F \in \mathcal{D}(k, s) \). Then with the notation \((1.15)\) and \((1.16)\), both \( \mathcal{S}(F) \) and \( \mathcal{G}(F) \) are convergent, and for any fixed \( \varepsilon > 0 \), one has

\[
N_F(X) = \mathcal{S}(F)\mathcal{G}(F)X^{21} + O(X^{21 - \frac{1}{5} + \varepsilon}) \quad (X \geq 1).
\]

If moreover the system \((1.3)\) has a nonsingular solution over \( \mathbb{R} \) and over \( \mathbb{Q}_p \) for all \( p \), then \( \mathcal{S}(F)\mathcal{G}(F) > 0 \).

The base of the proof of Theorem 2 is still the Circle Method, and our treatment of the major arcs is identical to that of Theorem 1. The main distinction comes from the approach of the minor arcs, and makes a crucial use of the structure of \((k_1, \ldots, k_n) = (1, 3, 5)\), namely the gap \( k_n - k_{n-1} \geq 2 \) between the two highest degrees. More precisely, writing \( f(\alpha_1, \alpha_2, \alpha_3) \) for the sum in \((1.11)\) for \( k = (1, 3, 5) \), and \( m \subset [0, 1]^3 \) for the minor arcs, our aim is to obtain an upper bound of the form

\[
\int_m |f(\alpha)|^{30} d\alpha \ll X^{21 - \delta_0} \quad (X \geq 1).
\]

Our proof proceeds essentially as follows: we construct two suitable sets \( \mathfrak{M}_2, \mathfrak{M}_3 \subset [0, 1] \) that resemble unions of one-dimensional major arcs.

The first step is to bound the contribution of the \( \alpha \in \mathfrak{m} \) such that \( \alpha_3 \in \mathfrak{m}_3 := [0, 1] \setminus \mathfrak{M}_3 \). Using a technique due to Wooley, for which the condition \( k_n - k_{n-1} \geq 2 \) is essential, the integral of \( |f|^{30} \) over \([0, 1]^2 \times \mathfrak{m}_3 \) gives an admissible upper bound.

For the next step, which is the main novelty in this paper, we give a more detailed sketch of the argument: writing \( \mathfrak{m}_2 := [0, 1] \setminus \mathfrak{M}_2 \), our aim is to bound the contribution of the \( \alpha \in [0, 1] \times \mathfrak{m}_2 \times \mathfrak{M}_3 \). For any interval \([z - \eta, z + \eta] \) counted in \( \mathfrak{M}_3 \), we have

\[
\int_{[0,1] \times \mathfrak{m}_2 \times [z - \eta, z + \eta]} |f(\alpha)|^{30} d\alpha \ll \sup_{\alpha_1, \alpha_2} \sup_{\alpha \in \mathfrak{m}_2} |f(\alpha)|^{10} \int_{[0,1]^2 \times [z - \eta, z + \eta]} |f(\alpha)|^{20} d\alpha.
\]

The classical minor arc technique, updated by the Vinogradov Mean Value Theorem, gives a suitable bound for the supremum, namely a saving that compensates the forthcoming summation over all intervals \([z - \eta, z + \eta] \). Hence, for the right hand side integral, it is now sufficient to have a saving close to \( X^{-9} \). Using the Beurling-Selberg function, we have

\[
\int_{[0,1]^2 \times [z - \eta, z + \eta]} |f(\alpha)|^{20} d\alpha \ll \int_{[0,1]^2 \times [-\eta, \eta]} |f(\alpha)|^{20} d\alpha.
\]

Next, we produce an upper bound of the form \( |f(\alpha)| \ll \frac{X}{(1 + |\alpha_3| X^{5})^{1/2}} \), \((|\alpha_3| \leq \eta)\) by using a new formulation of van der Corput’s upper bounds for exponential
sums introduced by the second author in a recent work. Again, the condition $k_n - k_{n-1} \geq 2$ is essential here. This yields

$$\int_{[0,1]^2 \times [-\eta, \eta]} |f(\alpha)|^{20} \, d\alpha \ll \int_{-\eta}^{\eta} X^{10} \left( \int_{[0,1]^2} |f(\alpha)|^{10} \, d\alpha_3 d\alpha_2 \right) \, d\alpha_3.$$  

Using again the Beurling-Selberg function, we remove the dependency in $\alpha_3$ in the inner integral, and we are reduced to bounding the tenth moment for the linear and cubic, for which Hua’s classical result is sufficient. Combined with a simple integration over $\alpha_3$ for the remaining term, this gives the expected saving.

Finally, for the last step, as classical trick in the Circle Method, we use some pruning techniques to fill the gap between the complementary set and the actual minor arcs.

The structure of our paper is as follows. In section 3, we give an asymptotic for Weyl sums, for the classical Weyl sum as well as for the multidimensional version, mainly in view of the major arcs and a simple estimate for the minor arcs. However, the range for these estimates goes slightly beyond what is required for this paper, and may be of independent interest. In section 4, we study variations of Vinogradov’s integrals using a technique developed in [14], and also an estimate of exponential sums in the style of van der Corput’s method, in view of the proof of Theorem 2. In section 5, we study the singular integrals and the singular series that occur in Theorems 1 and 2 following essentially Parsell’s and Schmidt’s approach. Section 6 and 7 are devoted to the contribution of major arcs in in both theorems, as well as the simplest estimate on minor arcs. In section 8, we establish Theorem 1 by proving a more general result that does not depend directly on the Vinogradov’s Mean Value Theorem. At last, Section 9 is devoted to a refined estimate on minor arcs leading to the proof of Theorem 2.

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2. Notation

For any integers $a_1, a_2, \ldots, a_n$, we set $(a_1; a_2; \ldots; a_n) = \gcd(a_1, a_2, \ldots, a_n)$. Similarly, whenever $\mathbf{a} \in \mathbb{Z}^n$ and $q \geq 1$, we write $(\mathbf{a}; q)$ for the gcd $(a_1; a_2; \ldots; a_n; q)$. For any $k \in \mathbb{N}$ and any $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$, the product $\alpha \cdot \beta$ denotes the usual scalar product, whereas $\alpha \otimes \beta$ denotes the tensor product $(\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_k \beta_k)$. Through the paper, the small letter $p$ (with or without index) represents a prime number.
3. Exponential sums and oscillating integrals

3.1. A truncated Poisson Formula.

Lemma 3.1 (Lemma 2.3 of [2]). Suppose that $\varphi$ is a twice continuously differentiable function on an interval $I$ and let $H > 2$ be a number such that $|\varphi'(x)| \leq H$ for all $x \in I$. Suppose further that $\varphi''$ has at most finitely many zeros in the interval $I$. Then

$$\sum_{n \in I} e(\varphi(n)) = \sum_{|h| \leq H} \int_I e(\varphi(t) - ht) \, dt + O(\log H).$$

In the sequel, we set

$$P_k(\beta; t) = \sum_{i=1}^k \beta_i t^i \quad (k \geq 1, \beta \in \mathbb{R}^k, t \in \mathbb{R}),$$

3.2. Estimates on complete sums.

Lemma 3.2. Let $k \geq 2$ and $\varepsilon > 0$ fixed. With the notation (3.1), For any $q \in \mathbb{N}$ and any $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, we have the following estimates:

(i) One has

$$\sum_{r=1}^q e\left(\frac{P_k(a; r)}{q}\right) \ll_{\varepsilon,k} (a; q)^{1/k} q^{1-\frac{1}{k}+\varepsilon}$$

(ii) If moreover $(a; q) = 1$ and $H \gg q$, then

$$\sum_{|h| \leq H} \left| \sum_{r=1}^q e\left(\frac{P_k(a; r) + hr}{q}\right) \right| \ll_{\varepsilon,k} H q^{1-\frac{1}{k}+\varepsilon}$$

and

$$\sum_{|h| \leq H} \frac{1}{h} \sum_{r=1}^q e\left(\frac{P_k(a; r) + hr}{q}\right) \ll_{\varepsilon,k} q^{1-\frac{1}{k}+\varepsilon} \log(2 + H)$$

(iii) If $(a; q) = 1$ and $w \in \mathbb{Z}^k$ with $w_j \neq 0$ for each $j$, then

$$\sum_{r=1}^q e\left(\frac{P_k(a \otimes w; r)}{q}\right) \ll_{\varepsilon,k} \left(\prod_{j=1}^k |w_j|\right)^{1/k} q^{1-\frac{1}{k}+\varepsilon}$$

Proof. The bound (i) is essentially a reformulation of Theorem 7.1 of [13]. For the first bound of (ii), we use (i) for the inner sum, and we now have to bound

$$\sum_{|h| \leq H} (a_1 + h; a_2; a_3; \ldots; a_n)^{1/k}. \quad \text{for a fixed } d \mid q, \text{ the contribution in this sum of the } h \text{ such that } (a_1 + h; a_2; a_3; \ldots; a_n) = d \text{ is } \ll d^{1/k}(\frac{H}{d} + 1) \text{ since, } d \mid h + a_1. \text{ Summing over the } O(q^2) \text{ choices for } d \text{ gives the expected result. The proof for the second bound of (ii) is quite similar, we omit the details. Finally, the bound (iii) is a consequence of (i), by noticing that } (a \otimes w; q) \text{ divides } |w_1 w_2 \ldots w_k|.$
3.3. Estimates on integrals.

Lemma 3.3. Let \(c_0, C_0 > 0\) such that \(c_0 < 1\). For any \(X > 0\), \(A > 0\) and any \(C^2\) function \(\varphi : \mathbb{R} \to \mathbb{R}\) such that \(|\varphi'(t)| \leq c_0 A\), \(|\varphi''(t)| \leq C_0 A/X\) \((-X \leq t \leq X)\), one has

\[
\left| \int_{-X}^{X} \left(\varphi(t) - At\right) dt \right| \leq \frac{C_0 + 1 - c_0}{(1 - c_0)^2 \pi A}.
\]

Proof. Writing

\[
\int_{-X}^{X} e\left(\varphi(t) - At\right) dt = \int_{-X}^{X} \frac{e\left(\varphi(t) - At\right)'}{2i\pi (\varphi'(t) - A)} dt,
\]

this is merely an integration by parts, using the fact that \(|\varphi'(t) - A| \geq (1 - c_0) A\) for \(-X \leq t \leq X\).

Lemma 3.4. Let \(k \geq 2\). With the notation \((3.1)\), one has

\[
\int_{-1}^{1} e\left(P_k(\beta; t)\right) dt \ll k\left(1 + \sum_{1 \leq j \leq k} |\beta_j|\right)^{-1/k} \quad (\beta \in \mathbb{R}^k)
\]

Proof. This is a direct consequence of Theorem 7.3 of [13].

Lemma 3.5. We have the following estimates:

(i) For any \(n \geq 1\), \(\sigma \in \mathbb{R}\), \(U > 0\), we have

\[
\int_{[-U,U]^n} \left(1 + \sum_{i=1}^{n} |\beta_i|\right)^{-\sigma} d\beta \leq \frac{2^n}{(n-1)!} \left(1 + (1 + nU)^{n-\sigma}\right) \log(1 + U).
\]

(ii) We have

\[
\int_{\mathbb{R}^n \setminus [-U,U]^n} \left(1 + \sum_{i=1}^{n} |\beta_i|\right)^{-\sigma} d\beta \ll_{\sigma, n} U^{n-\sigma} \quad (n \geq 1, \sigma > n, U \geq 1).
\]

(iii) We have

\[
\int_{H_n(U)} \left(1 + \sum_{i=1}^{n} |\beta_i|\right)^{-\sigma} d\beta \ll_{\sigma, n} U^{n-\sigma} \quad (n \geq 1, \sigma > n, U \geq 1).
\]

where we have set \(H_n(U) = \left\{\beta \in \mathbb{R}^n : \sum_{i=1}^{n} |\beta_i| \geq U\right\}\).

Proof. For (i), writing \(I(\sigma, U)\) for the integral on the left hand side, we start with the case \(\sigma = n\), and a direct computation gives \(I(n, U) \leq \frac{2^n}{(n-1)!} \log(1 + U)\). For \(\sigma < n\), we have \(I(\sigma, U) \leq (1 + nU)^{n-\sigma} I(n, U)\) which implies the conclusion. Finally for \(\sigma > n\), we have \(I(\sigma, U) \leq I(n, U)\) which yields again the expected result. For (ii), by symmetry, one may assume that \(|\beta_n| > U\), and a direct computation gives the result. Finally, (iii) is a consequence of (ii).
3.4. Estimates on Weyl sums. Our main result in this section is an estimate for the generating function in the Vinogradov system, crucial in the treatment of the major arcs. As usual, the main term involves the complete sum and the integral associated.

**Theorem 3.** With the previous notation, for any fixed $\varepsilon > 0$ and $k \geq 1$, one has

$$
\sum_{|x| \leq X} e(P_k(\frac{a}{q} + \beta; x)) = \frac{X}{q} \left( \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r)}{q} \right) \right) \int_{-1}^{1} e\left( P_k(\beta; X t) \right) dt + O\left( \frac{1}{q^{1-\frac{1}{q} + \varepsilon}} \right)
$$

uniformly for $a \in \mathbb{Z}$, $q \geq 1$ and $\beta \in \mathbb{R}$ such that $|\beta_1| \leq \frac{1}{2q}$, $\sum_{2 \leq j \leq k} j|\beta_j| X^{j-1} \leq \frac{1}{4q}$.

**Proof.** We start with the particular case $(a; q) = 1$. The first lines of our proof follow the standard approach used in Theorem 3 of [4] and Theorem 4.1 of [3]. One has

$$
S := \sum_{|x| \leq X} e(P_k(\frac{a}{q} + \beta; x)) = \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r)}{q} \right) \sum_{x \equiv r \mod q} e(P_k(\beta; x))
$$

$$
= \frac{1}{q} \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r)}{q} \right) \sum_{|x| \leq X} e(P_k(\beta; x)) \sum_{-\frac{b}{2} < b \leq \frac{q}{2}} e\left( \frac{b(r-x)}{q} \right)
$$

$$
= \frac{1}{q} \sum_{-\frac{b}{2} < b \leq \frac{q}{2}} \left( \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r) + br}{q} \right) \right) \left( \sum_{|x| \leq X} e(P_k(\beta; x) - \frac{br}{q}) \right)
$$

For each $b, q, \beta$, the inner sum over $x$ meets the requirements of Lemma 3.1 with $H = 3$ so that $S = S_1 + O(S_2)$ with

$$
S_1 = \frac{1}{q} \sum_{-\frac{b}{2} < b \leq \frac{q}{2}} \left( \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r) + br}{q} \right) \right) \left( \sum_{|h| \leq 3} \int_{-X}^{X} e(P_k(\beta; t) - \frac{bt}{q} - ht) dt \right)
$$

and

$$
S_2 = \frac{1}{q} \sum_{-\frac{b}{2} < b \leq \frac{q}{2}} \left| \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r) + br}{q} \right) \right|.
$$

From (ii) of Lemma 3.3 we have $S_2 \ll q^{1-\frac{1}{q} + \varepsilon}$. Now, writing $m = qh + b$ with $|h| \leq 3$ and $-\frac{q}{2} < b \leq \frac{q}{2}$, one has

$$
S_1 = \frac{1}{q} \sum_{-\frac{q}{2} < m \leq \frac{q}{2}} \left( \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r) + mr}{q} \right) \right) \int_{-X}^{X} e(P_k(\beta; t) - \frac{mt}{q}) dt.
$$

At this point we differ from the argument used in the Theorem 3 of [4]: for each $m \neq 0$, Lemma 3.3 yields

(3.2) \[ \int_{-X}^{X} e(P_k(\beta; t) - \frac{mt}{q}) dt \ll \frac{q}{|m|} \]

so that

$$
S_1 = \frac{1}{q} \left( \sum_{r=1}^{q} e\left( \frac{P_k(\frac{a}{q}; r)}{q} \right) \right) \int_{-X}^{X} e(P_k(\beta; t)) dt + O(S_3)
$$
with
\[ S_3 = \sum_{1 \leq |m| \leq 7q/2} \frac{1}{m} \left| \sum_{r=1}^{q} e\left(\frac{P_k(a;r) + mr}{q}\right) \right|. \]

Now, from (ii) of Lemma 3.2, we have \( S_3 \ll \varepsilon q^{1-\frac{1}{k}+\varepsilon} \) : this completes the proof in the case \((a;q) = 1\) after an obvious linear change of variable in the inner integral.

For the general case, writing \( a' = a/(a,q) \) and \( a'' = a/(a',q) \), we apply the previous estimate with \( q' \) and \( a'' \), and we conclude by observing that
\[ \frac{1}{q'} \sum_{r=1}^{q} e\left(\frac{P_k(a';r)}{q'}\right) = \frac{1}{q} \sum_{r=1}^{q} e\left(\frac{P_k(a;r)}{q}\right). \]

\[ \square \]

**Remark 3.1.** In the case of particular polynomial phases, results sharper than our Theorem 3 may be obtained: we have already mentioned Theorem 4.1 of [13] and Theorem 3 of [4] for the case where the phase is either a monomial or a monomial and a linear term. In that case the range of validity for \( \beta \) is significantly wider. For such particular phase, the sharpest current result is Theorem 1.1 of [2], with a new main term and a sharper error term. In the case of our Theorem 3, the polynomial phase is more general, and in particular \( t \mapsto P_k(a;t) \) does not necessarily have a monotonic first derivative, which was a crucial aspect in Theorem 3 of [4]. Hence, the bound (3.2) is not a consequence of van der Corput’s result for the first derivative; instead, we use Lemma 3.3, which induces a constraint on \( \beta \).

In the sequel, our aim is to apply Theorem 3 to the Weyl sums related to (1.11), including a multidimensional version. Considering \( \varphi: \mathbb{R} \to \mathbb{R}^n \) defined by
\[ \varphi(t) = (t^{k_1}, \ldots, t^{k_n}) \quad (t \in \mathbb{R}), \]
the sum in (1.11) may be written
\[ f_k(\alpha;X) := \sum_{|x| \leq X} e(\alpha \cdot \varphi(x)) \quad (\alpha \in \mathbb{R}^n). \]

In order to introduce the multidimensional sums, we consider the corresponding complete sum
\[ S_k(q; a) = \sum_{r=1}^{q} e\left(\frac{\alpha \cdot \varphi(r)}{q}\right) \]
and the corresponding integral
\[ v_k(\beta; X) = \int_{-1}^{1} e(\beta \cdot \varphi(Xt))dt. \]

For a fixed \( F \in D(k, s) \), we now derive a similar estimate for the associated multidimensional Weyl sum. With \( u_{i,j} \) defined by (1.2), we set
\[ u_j = (u_{1,j}, u_{2,j}, \ldots, u_{n,j}) \quad (1 \leq j \leq s). \]
and
\[ \|F\|_\infty := \max_{i,j} |u_{i,j}|. \]
The multidimensional analogues now read

\begin{equation}
(3.8) \quad f[F](\alpha; X) := \sum_{\mathbf{x} \in I_\epsilon(X)} e(\alpha \cdot F(x)) = \prod_{j=1}^s f_k(u_j \otimes \alpha; X),
\end{equation}

\begin{equation}
(3.9) \quad S[F](q; a) := \sum_{r \in [1,q]^n} e\left(\frac{a \cdot F(r)}{q}\right) = \prod_{j=1}^s S_k(q; u_j \otimes a),
\end{equation}

\begin{equation}
(3.10) \quad v[F](\beta; X) := \int_{[-1,1]^n} e(\beta \cdot F(Xt)) \, dt = \prod_{j=1}^s v_k(u_j \otimes \beta; X).
\end{equation}

Moreover, for \( q \geq 1 \) and \( X \geq 1 \), we introduce a condition on \( \gamma \in \mathbb{R}^n \) to have suitable coordinates as follows:

\begin{equation}
(3.11) \quad \begin{cases}
|\gamma_1| \leq \frac{1}{2q} \text{ and } \sum_{i=2}^n k_i |\gamma_i| X^{k_i-1} \leq \frac{1}{4q} \quad & \text{if } k_1 = 1 \\
\sum_{i=1}^n k_i |\gamma_i| X^{k_i-1} \leq \frac{1}{4q} \quad & \text{if } k_1 > 1.
\end{cases}
\end{equation}

Finally, we define

\begin{equation}
(3.12) \quad \xi_k(\beta, X) := 1 + \sum_{i=1}^n |\beta_i| X^{k_i} \quad (k \in \mathbb{N}^n, \beta \in \mathbb{R}^n).
\end{equation}

**Lemma 3.6.** Let \( s \geq 1, \ k \) as in (3.1), and \( F \in \mathcal{D}(k, s) \). Then with the notation (3.8) to (3.12), uniformly for \( q \geq 1, \ a \in A_n(q) \) and \( \beta \in \mathbb{R}^n \) such that \( \gamma := \|F\|_\infty \beta \) satisfies (3.11), we have

\[
f[F]\left(\frac{a}{q} + \beta; X\right) = \frac{X^s}{q^s} (S[F](q; a)) (v[F](\beta; X)) + O(E),
\]

where we have set \( E = X^{s-1} q^{1 - \frac{s-1}{k_1} + \varepsilon} \xi_k(\beta, X)^{-(s-1)/k_1} + q^{s-\frac{s-1}{k_1} + \varepsilon} \).

**Proof.** Our proof uses the following: for any \( s \geq 1 \) and \( z_1, z_2, \ldots, z_s, \delta_1, \ldots, \delta_s \in \mathbb{C} \) such that \( |z_j| \leq Z \) and \( |\delta_j| \leq \Delta \) for any \( j \), one has

\begin{equation}
(3.13) \quad \left| \prod_{j=1}^s (z_j + \delta_j) - \prod_{j=1}^s z_j \right| \leq s 2^{s-2} (\Delta Z^{s-1} + \Delta^s).
\end{equation}

We now use the notation introduced before our lemma. For any choice \( w = u_j \), we have \( |w_i \beta_i| \leq \|F\|_\infty |\beta_i| \) for each \( i \). Hence \( w \otimes \beta \) satisfies (3.11) and thus meets the requirements of Theorem 3. Hence we have

\begin{equation}
(3.14) \quad f_k(w \otimes \alpha; X) = X^s \frac{S_k(q; w \otimes a)}{q} v_k(w \otimes \beta; X) + O_{k, w, \varepsilon}(q^{1 - \frac{s}{k_1} + \varepsilon}).
\end{equation}

From this estimate, (iii) of Lemma 3.2 with Lemma 3.4 give the upper bound

\begin{equation}
(3.15) \quad f_k(w \otimes \alpha; X) \ll_{k, w, \varepsilon} X q^c \left( q^{s \xi_k(\beta, X)} \right)^{-1} + q^{1 - \frac{s-1}{k_1} + \varepsilon}.
\end{equation}

Finally, using (3.13), we deduce

\[
\prod_{j=1}^s f_k\left(\frac{a}{q} + \beta; X\right) = \prod_{j=1}^s \left(\frac{X}{q} S_k(q; u_j \otimes a) v_k(u_j \otimes \beta; X)\right).
\]
where $P$ is an argument in the proof of Theorem 2.1 of \cite{14}.

A technique due to Wooley related to partial minor arcs.

4.2. Proof. This is essentially Lemma 2.1 of \cite{10} followed by Hölder’s inequality.

Lemma 4.1. Let $\Omega$ be a finite set, and $\Phi : \Omega \rightarrow \mathbb{C}$. For any $m \in \mathbb{N}$ and $G : \Omega \rightarrow \mathbb{Z}^m$, consider the set $H = \{G(\omega) - G(\omega') : (\omega, \omega') \in \Omega^2 \} \subset \mathbb{Z}^m$. Then

\[
\left|\sum_{\omega \in \Omega} \Phi(\omega)\right|^2 \leq \left(\#H\right) \int_{[0,1]^m} \left|\sum_{\omega \in \Omega} \Phi(\omega) e(\beta \cdot G(\omega))\right|^2 d\beta.
\]

Proof. We have

\[
\left|\sum_{\omega \in \Omega} \Phi(\omega)\right|^2 = \sum_{(\omega, \omega') \in \Omega^2} \Phi(\omega) \overline{\Phi(\omega')} = \sum_{h \in H} \sum_{(\omega, \omega') \in \Omega^2} \Phi(\omega) \overline{\Phi(\omega')}
\]

Now, by Fourier orthogonality, for any $h \in H$, we have

\[
A(h) := \sum_{(\omega, \omega') \in \Omega^2} \Phi(\omega) \overline{\Phi(\omega')} = \int_{[0,1]^m} \left|\sum_{\omega \in \Omega} \Phi(\omega) e(\beta \cdot G(\omega))\right|^2 e(-\beta \cdot h) d\beta
\]

which gives the expected result by using the bound $A(h) \leq A(0)$. \hfill \Box

Lemma 4.2. Let $r \in \mathbb{N}$ and $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$. For any $a, b, c, d \in \mathbb{Z}$ such that $c \leq a < b \leq d$ one has

\[
\left|\sum_{a \leq x \leq b} \varphi(x)\right|^r \leq (1 + \log(d-c+1))^{r-1} \int_{-1/2}^{1/2} \sum_{c \leq x \leq d} \varphi(x) e(\gamma x)\left|\min_{-1/2 \leq \gamma \leq 1/2} \varphi(x)\right|^r d\gamma.
\]

Proof. This is essentially Lemma 2.1 of \cite{11} followed by Hölder’s inequality. \hfill \Box

4.2. A technique due to Wooley related to partial minor arcs. Let $k \geq 3$ fixed. For any $X \geq 1$ and any $\theta \in \mathbb{R}$, we set

\[
(4.1) \quad \psi_k(\theta, \mu; X) := \frac{1}{X} \sum_{1 \leq y \leq X} \min \left(\frac{1}{X^{k-1}}, \frac{1}{\|k\theta y + \mu\|}\right)
\]

\[
(4.2) \quad \psi_k^*(\theta; X) := \sup_{\mu \in \mathbb{R}} \psi_k(\theta, \mu; X)
\]

\[
(4.3) \quad g_k(\alpha, \theta; X) = \sum_{|x| \leq X} e\left(P_{k-2}(\alpha; x) + \theta x^k\right) \quad ((\alpha, \theta) \in \mathbb{R}^{k-2} \times \mathbb{R}).
\]

where $P_k(\alpha; x)$ has been defined in \cite{3.1}.

The next theorem is a reformulation and a slight generalisation of the crucial argument in the proof of Theorem 2.1 of \cite{13}.
Theorem 4. Let $k \geq 3$ and $b \geq 1$ be fixed. Then, with the notation (3.1), (4.2), (4.3) and (4.4), one has

$$
\int_{[0,1)^{k-2} \times A} |g_k(\alpha, \theta; X)|^{2b} \, d\alpha d\theta \ll_{b,k} (\log(4X))^{2b} \left( \sup_{\theta \in A} \psi_k'(\theta; X) \right) J_{b,k}(4X + 1)
$$

uniformly for $X \geq 1$, $c \in \mathbb{R}$ and $A \subset [c, c+1]$, where $A$ is a Lebesgue-measurable set.

Proof. One has

$$g_k(\alpha, \theta; X) = \sum_{y - X \leq x \leq y + X} e \left( P_{k-2}(\alpha; x) + \theta(x - y)^k \right) \quad (1 \leq y \leq X).$$

Since $[y - X, y + X] \subset [-2X, 2X]$, Lemma (4.2) yields

$$
|g_k(\alpha, \theta; X)|^{2b} = \left| \sum_{y - X \leq x \leq y + X} e \left( P_{k-2}(\alpha; x) + \theta(x - y)^k \right) \right|^{2b}
\ll (\log(4X))^{2b-1} \int_{-1/2}^{1/2} \left| \sum_{|x| \leq 2X} e \left( \gamma x + P_{k-2}(\alpha; x) + \theta(x - y)^k \right) \right|^{2b} \min \left( 4X, \frac{1}{|\gamma|} \right) d\gamma
$$

uniformly for $1 \leq y \leq X$. Now, integrating over $\alpha$, and averaging over $y$, we have

$$
G(\theta) \ll (\log(4X))^{2b-1} \frac{1}{X} \sum_{1 \leq y \leq X} \int_{-1/2}^{1/2} I(\gamma, y; \theta) \min \left( 4X, \frac{1}{|\gamma|} \right) d\gamma,
$$

where we have set $G(\theta) = \int_{[0,1)^{k-2}} |g_k(\alpha, \theta; X)|^{2b} \, d\alpha$ and

$$
I(\gamma, y; \theta) := \int_{[0,1)^{k-2}} \left| \sum_{|x| \leq 2X} e \left( \gamma x + P_{k-2}(\alpha; x) + \theta(x - y)^k \right) \right|^{2b} \, d\alpha.
$$

Writing

$$
\sigma_j(x; y) := \sum_{m=1}^{b} \left( (x_m - y)^j - (x_{b+m} - y)^j \right), \quad \sigma_j(x) := \sigma_j(x; 0) \quad (x \in \mathbb{Z}^s),
$$

we have $I(\gamma, y; \theta) = \sum_{x \in J_1(y)} e(\gamma \sigma_1(x) + \theta \sigma_k(x; y))$ where $J_1(y)$ is the set of solutions of the system

$$
\sigma_j(x; y) = 0 \quad (1 \leq j \leq k - 2) \quad (x \in [-2X, 2X]^{2b}).
$$

By translation invariance of $J_1(y)$, we have $J_1(y) = J_1(0)$, and $\gamma \sigma_1(x) + \theta \sigma_k(x; y) = \theta \sigma_k(x) - ky \theta \sigma_{k-1}(x)$ for $x \in J_1(0)$. Hence

$$
I(\gamma, y; \theta) = \sum_{x \in J_1(0)} e(\theta \sigma_k(x) - ky \theta \sigma_{k-1}(x)).
$$
Now in this sum, by Fourier orthogonality, the contribution of the \( x \) such that \( \sigma_{k-1}(x) = h \) is

\[
\int_{[0,1]^{k-1}} |\Phi(\alpha, \theta; 2X)|^{2b} e\left(-\alpha_{k-1} h - k y \theta h\right) d\alpha
\]

where we have set

\[
\Phi(\alpha, \theta; X) := \sum_{|x| \leq X} e\left(P_{k-1}(\alpha; x) + \theta x^k\right).
\]

Due to the size of \( x \), we necessarily have \( |h| \ll X^{k-1} \). Summing up over \( h \), we have

\[
I(\gamma, y; \theta) = \int_{[0,1]^{k-1}} |\Phi(\alpha, \theta; 2X)|^{2b} \min\left(X^{k-1}, \frac{1}{||k \theta y + \alpha_{k-1}||}\right) d\alpha.
\]

Now inserting this estimate in (4.4), we have

\[
G(\theta) \ll (\log(4X))^{2b} \int_{[0,1]^{k-1}} |\Phi(\alpha, \theta; 2X)|^{2b} \psi_k(\theta, \alpha_{k-1}; X) d\alpha d\theta
\]

Integrating over \( \theta \), we have

\[
\int_{A} G(\theta) d\theta \ll (\log(4X))^{2b} \int_{[0,1]^{k-1} \times A} |\Phi(\alpha, \theta; 2X)|^{2b} \psi_k(\theta, \alpha_{k-1}; X) d\alpha d\theta
\]

and this last integral is \( J_{b,k}(4X + 1) \) by translation invariance of the Vinogradov system, which gives concludes the proof.

In order to estimate \( \psi_k^*(\theta; X) \), we recall the following classical result:

**Lemma 4.3.** Let \( \alpha, \mu \in \mathbb{R} \) such that \( |\alpha - \frac{a}{q}| \leq \frac{1}{q^2} \), and let \( Y, \Delta > 0 \). Then

\[
\sum_{y=1}^{X} \min\left(Y, \frac{1}{||\alpha y + \mu||}\right) \ll Y(1 + \frac{X}{q}) + (X + q) \log Y
\]

**Proof.** Under the assumption made on \( \alpha \) and \( \mu \), Lemma 6 of [6] gives

\[
\#\{1 \leq y \leq X : ||\alpha y + \mu|| \leq \Delta\} \ll 1 + X \Delta + \frac{X}{q} + q \Delta.
\]

The announced result then follows from a dyadic summation according to the size of \( ||\alpha y + \mu|| \) (see also equation (2.13) of [34]).
4.3. Applications of the Beurling-Selberg function.

**Lemma 4.4.** Let $k \geq 1$. Let $E$ be a finite set, and consider $\varphi : E \to \mathbb{R}^k$. Let $(T_j)_{1 \leq j \leq k}$ and $(\delta_j)_{1 \leq j \leq k}$ be sequences of positive real numbers. Write $P_k = \prod_{j=1}^{k} [-T_j, T_j]$, $P'_k = \prod_{j=1}^{k} [-T'_j, T'_j]$ and $\Delta_k = \prod_{j=1}^{k} [-\delta_j, \delta_j]$.

(i) For any sequence $(a(z))_{z \in E}$ of complex numbers of modulus at most one, one has

$$\int_{P_k} \left| \sum_{z \in E} a(z) e(\alpha \cdot \varphi(z)) \right|^2 \, d\alpha \leq \left( \prod_{j=1}^{k} (2T_j \cdot \frac{1}{\delta_j}) \right) \# \{(z, w) \in E^2 : \varphi(z) - \varphi(w) \in \Delta_k \}.$$

(ii) One has

$$\# \{(z, w) \in E^2 : \varphi(z) - \varphi(w) \in \Delta_k \} \leq \left( \prod_{j=1}^{k} (2\delta_j + \frac{1}{T_j}) \right) \int_{P_k} \left| \sum_{z \in E} e(\alpha \cdot \varphi(z)) \right|^2 \, d\alpha.$$

(iii) One has

$$\int_{P_k} \left| \sum_{z \in E} a(z) e(\alpha \cdot \varphi(z)) \right|^2 \, d\alpha \leq 8^k \left( \prod_{j=1}^{k} \frac{T_j}{T'_j} \right) \int_{P'_k} \left| \sum_{z \in E} e(\alpha \cdot \varphi(z)) \right|^2 \, d\alpha.$$

**Proof.** The proof relies on properties of the Beurling-Selberg function: writing $B_0 := \prod_{j=1}^{k} (2T_j + \frac{1}{\delta_j})$, there exists a function $f \in L^1(\mathbb{R}^k)$ such that

$$1_{P_k} \leq f \quad \text{and} \quad \hat{f} \leq B_0 1_{\Delta_k},$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^k} f(\alpha) e(-\xi \cdot \alpha) \, d\alpha \quad (\xi \in \mathbb{R}^k)$$

(see [12]). Assertion (i) is essentially Lemma 7.4 of [3]. The second assertion may be derived using the same argument (permuting the $T_j$'s and the $\delta_j$'s), and finally, (iii) is obtained using the two first inequalities with a straightforward optimisation over the $\delta_j$'s.

4.4. An application on van der Corput’s method for a polynomial phase.

The following result is a consequence of a van der Corput estimate proved in [10], with a new approach (see [9]).

**Lemma 4.5.** One has

$$\sum_{|x| \leq X} e\left(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5\right) \ll \frac{X}{(1 + X^5|\alpha_3|)^{1/10}}$$

uniformly for $X \geq 1$ and $\alpha \in \mathbb{R}^3$ such that $|\alpha_3| \leq X^{-10/3}$.

**Proof.** We first observe that in the case $|\alpha_3| \leq X^{-5}$, the trivial bound gives the expected result. Therefore, in the sequel, we assume that $X^{-5} < |\alpha_3| \leq X^{-10/3}$. We start with the upper bound

$$\left| \sum_{|x| \leq X} e\left(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5\right) \right| \ll 1 + \sum_{1 < x \leq X} e\left(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5\right).$$
In the terminology of Lemma 5 of [9], we establish, using van der Corput’s A-process and Corollaire 4.2 of [10], that \(\alpha\) is a van der Corput 4-couple. Using Lemma 5 (ii) of [9], this implies that for \(\varphi : [U, 2U] \to \mathbb{R}\) defined by \(\varphi(x) = \alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5\) for \(x \in [U, 2U]\), since \(|\varphi^{(4)}(x)| \gg U|\alpha_3|\) for \(x \in [U, 2U]\), one has

\[
\sum_{U < x \leq 2U} e(\varphi(x)) \ll U^{3/5}(U|\alpha_3|)^{-1/10}
\]

uniformly for \(U \leq (U|\alpha_3|)^{3/7}\). Hence, uniformly for \(X^{-5} < |\alpha_3| \leq X^{-10/3}\) and \(1 \leq U \leq X\), one has

\[
\sum_{U < x \leq 2U} e(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5) \ll U^{1/2}|\alpha_3|^{-1/10}.
\]

Now, using the dyadic sums

\[
\sum_{1 < x \leq X} \sum_{2^k \leq x} \sum_{x^{2-k-1} < x_2 \leq x^{2-k}}
\]

and applying the previous bound on the inner sums with the choice \(U = X^{2-k-1}\), one has

\[
\sum_{0 \leq x \leq X} e(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5) \ll 1 + X^{1/2}|\alpha_3|^{-1/10} \ll \frac{X}{(1 + X^5|\alpha_3|)^{1/10}}
\]

which concludes the proof. \(\square\)

**Lemma 4.6.** Let \(k = (1, 3, 5)\). Then

\[
\int_{[0,1]^2 \times [-T, c+T]} \sum_{|x| \leq X} e(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5)^{20} d\alpha_1 d\alpha_2 d\alpha_3 \ll \varepsilon X^{11 + \varepsilon}
\]

uniformly for \(c \in [0, 1]\) and \(0 \leq T \leq X^{-10/3}\).

**Proof.** Writing

\[
(4.6) \quad f(\alpha_1, \alpha_2, \alpha_3; X) = \sum_{|x| \leq X} e(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5), \quad (\alpha \in \mathbb{R}^3),
\]

we have

\[
\int_{[0,1]^2 \times [-T, c+T]} |f(\alpha_1, \alpha_2, \alpha_3; X)|^{20} d\alpha_1 d\alpha_2 d\alpha_3
\]

\[
= \int_{[0,1]^2 \times [-T, T]} \sum_{|x| \leq X} e(\alpha_1 x + \alpha_2 x^3 + \alpha_3 x^5)^{20} d\alpha_1 d\alpha_2 d\alpha_3
\]

\[
\ll \int_{[0,1]^2 \times [-T, T]} |f(\alpha_1, \alpha_2, \alpha_3)|^{20} d\alpha_1 d\alpha_2 d\alpha_3
\]

by using (iii) of Lemma 4.4 with \(T = T' = T\). Using Lemma 4.5, since \(T \leq X^{-10/3}\), we have

\[
|f(\alpha_1, \alpha_2, \alpha_3; X)|^{10} \ll \frac{X^{10}}{1 + X^5|\alpha_3|}, \quad (\alpha \in [0, 1]^2 \times [-T, T], 0 \leq T \leq X^{-10/3})
\]

so that

\[
\int_{[0,1]^2 \times [-T, T]} |f(\alpha; X)|^{20} d\alpha
\]

\[
\ll \int_{-T}^{T} \frac{X^{10}}{1 + X^5|\alpha_3|} \left(\int_{[0,1]^2} |f(\alpha_1, \alpha_2, \alpha_3; X)|^{10} d\alpha_1 d\alpha_2\right) d\alpha_3
\]
\[ \ll \int_{-T}^{T} \frac{X^{10}}{1 + X^{2} |\alpha_3|} \left( \int_{[0,1]^2} |f(\alpha_1, \alpha_2, 0; X)|^{10} \, d\alpha_1 \, d\alpha_2 \right) \, d\alpha_3 \]

where once again we have used (iii) of Lemma 4.3 for the inner integral. Now, using Lemma 5 of \[\S\] , the new inner integral is \[\ll X^{6+\varepsilon}\]. We conclude with a direct computation of the remaining integral over \(\alpha_3\).

5. Singular integrals and singular series

Let \(s \geq 1\) and \(k\) be as in \([1, k]\). For any \(F \in \mathcal{D}(k, s)\), we recall that \(\mathcal{I}(F)\) and \(\mathcal{S}(F)\) have been defined in \([1.5]\) and \([1.16]\) respectively. The purpose of this section is to prove that for \(s\) sufficiently large, these constants are positive, as soon as \([1.3]\) has nonsingular solution over \(\mathbb{R}\) and the \(p\)-adic, in view of the asymptotic in Theorems \([4, 5]\) and \([2]\). For both constants, we follow quite closely the lines of \([4, 5]\) and \([1, k]\).

5.1. Singular integrals.

**Lemma 5.1.** Let \(k\) be as in \([1, k]\), \(s \geq nk_n + 1\) and \(F \in \mathcal{D}(k, s)\). Then \(\mathcal{I}(F)\) is absolutely convergent. Moreover, if the system \([1.3]\) has a nonsingular solution over \(\mathbb{R}\), then \(\mathcal{I}(F) > 0\).

**Proof.** For any \(T \geq 1\) and any \(\beta \in \mathbb{R}^n\) we set

\[ w_T(\beta) = \prod_{i=1}^{n} \sin^2 \left( \frac{\pi \beta_i / T}{\pi \beta_i / T} \right) \quad (\beta \neq 0), \quad w_T(0) = 1. \]

Classically, for any \(y \in \mathbb{R}^n\) we have

\[ \hat{w}_T(y) := \int_{\mathbb{R}^n} w_T(\beta) e(\beta \cdot y) \, d\beta = T^n \prod_{i=1}^{n} \max \left(0, 1 - T |y_i| \right) \]

Using Lemma 5.4 we have

\[ v(F; \beta) := \int_{[-1,1]^n} e(\beta \cdot F(t)) \, dt \ll_F \left( 1 + \sum_{i=1}^{n} |\beta_i| \right)^{-s/k_n} \quad (\beta \in \mathbb{R}^n), \]

hence (ii) of Lemma 5.3 implies \(\int_{\mathbb{R}^n} |v(F; \beta)| \, d\beta \ll < +\infty \) since \(s > nk_n\). Setting

\[ \mathcal{I}_T(F) = \int_{\mathbb{R}^n} w_T(\beta) v(F; \beta) \, d\beta \quad (T \geq 1) \]

it follows easily from Lebesgue’s theorem that \(\lim_{T \to +\infty} \mathcal{I}_T(F) = \mathcal{I}(F)\). Moreover, using Fubini’s theorem, one can easily deduce that

\[ \mathcal{I}_T(F) = \int_{[-1,1]^n} \hat{w}_T(F(t)) \, dt \geq 0 \quad (T \geq 0). \]

By homogeneity of \(F\), we may assume that system \([1.3]\) has a nonsingular solution \(\eta \in [\frac{1}{2}, \frac{1}{2}]^n\). Up to a renumbering of the coordinates \(\eta_1, \ldots, \eta_n\), we may assume that the matrix \(DF(\eta) = \left( \frac{\partial F_i}{\partial \eta_j}(\eta) \right)_{1 \leq i, j \leq n}\) has maximal rank. Consider the map \(\psi : \mathbb{R}^n \to \mathbb{R}^n\) defined by

\[ \psi(t) = (F_1(t), F_2(t), \ldots, F_n(t), t_{n+1}, t_{n+2}, \ldots, t_s) \quad (t \in \mathbb{R}^s). \]

Writing \(J_{\psi}(t)\) the jacobian of \(\psi\) at \(t\), one has \(\det J_{\psi}(\eta) = \det DF(\eta) \neq 0\). Hence, the Inverse Function Theorem implies that for some open neighbourhood \(U_0 \subset \)
of the coordinates we assumed that (1.3) has a nonsingular solution \( \eta \). Using the change of variables \( y = \psi(t) \), we let \( T \) tend to \(+\infty\) that
\[
[T \geq T_0).
\]
For some \( C_0 > 0 \) we have \( \frac{1}{C_0} \leq |\det J_\psi(t)| \leq C_0 \) whenever \( t \in K_{T_0} \). Now, for \( T \geq T_0 \), one has
\[
\mathcal{J}_T(F) \geq \int_{K_T} \hat{w}_T(F(t)) dt \geq \frac{1}{C_0} \int_{K_T} \hat{w}_T(F(t)) |\det J_\psi(t)| dt.
\]
Using the change of variables \( y = \psi(t) \), this last integral is equal to
\[
\int_{\psi(K_T)} \hat{w}_T(y_1, \ldots, y_n) dy = \text{Meas}(W_1) \int_{[-\frac{1}{q}, \frac{1}{q}]^n} \hat{w}_T(y_1, \ldots, y_n) dy_1 \cdots dy_n.
\]
By a simple computation, this last integral is equal to 1. Hence, one has
\[
\mathcal{J}_T(F) \geq \frac{\text{Meas}(W_1)}{C_0} \quad (T \geq T_0).
\]
Letting \( T \) tend to \(+\infty\), one has \( \mathcal{J}(F) \geq \frac{\text{Meas}(W_1)}{C_0} > 0 \), which is the expected result.

\[\square\]

5.2. Singular series.

Lemma 5.2. Let \( k \) be as in (3), \( s \geq (n + 1)k_n + 1 \) and \( F \in \mathcal{D}(k, s) \). Then \( \mathcal{S}(F) \) is absolutely convergent. Moreover, if the system (1.3) has a nonsingular solution over each \( p \)-adic \( \mathbb{Q}_p \), then \( \mathcal{S}(F) > 0 \).

Proof. We set
\[
(5.3) \quad T(q) = \frac{1}{q^s} \sum_{\mathbf{a} \in A_n(q) \times \{1, q\}^n} e\left(\frac{\mathbf{a} \cdot F(x)}{q}\right) \quad (q \geq 1).
\]
Writing the sum over \( x \) as in (4.3) and using (iii) of Lemma 5.2 we have the estimate \( T(q) \ll_{k,c} q^{s-\frac{1}{2}+c} \) \((q \geq 1)\) so that for \( s > (n + 1)k_n \) the series \( \mathcal{S}(F) \) is absolutely convergent. We now recall that \( T(q) \) is multiplicative, i.e. that \( T(qq') = T(q)T(q') \) whenever \( q \) and \( q' \) are coprime. The proof is quite similar to that of [7]. We omit the details. We now have
\[
(5.4) \quad \mathcal{S}(F) = \prod_p \left(1 + \sum_{h \geq 1} T(p^h)\right)
\]
where this product is absolutely convergent. Moreover, for each \( p \geq 2 \), one has
\[
(5.5) \quad 1 + \sum_{h \geq 1} T(p^h) = \lim_{H \to +\infty} p^{H(n-s)} M(p^H)
\]
where \( M(q) \) is the number of solutions of (1.3) in \((\mathbb{Z}/q\mathbb{Z})^n\). For \( p \geq 2 \) fixed, we assumed that (1.3) has a nonsingular solution \( \eta \in \mathbb{Z}_p^n \). Up to a renumbering of the coordinates \( \eta_1, \ldots, \eta_s \), we may assume that \( D F(\eta) = \left(\frac{\partial F_i}{\partial t_j}(\eta)\right)_{1 \leq i,j \leq n} \) has
maximal rank. We set $F_i^{(1)}(t) = \sum_{j=1}^{n} u_{i,j} t_j^{k_i}$ and $F_i^{(2)}(t) = \sum_{j=n+1}^{s} u_{i,j} t_j^{k_i}$ for $1 \leq i \leq n$ so that we have $F = F^{(1)} + F^{(2)}$ where $F^{(j)} = (F^{(j)}_1, F^{(j)}_2, \ldots, F^{(j)}_n)$ $(j = 1, 2)$. Hence, with this notation we have $F^{(1)}(\eta_1, \ldots, \eta_n) + F^{(2)}(\eta_{n+1}, \ldots, \eta_s) = 0$. We recall that $\det D F(\eta) \neq 0$ and let $\nu_p$ be its $p$-adic valuation. For $u_p := 2\nu_p + 1$, we have the following: for any fixed $(\mu_{n+1}, \ldots, \mu_s)$ such that
\begin{equation}
(\mu_{n+1}, \ldots, \mu_s) \equiv (\eta_{n+1}, \ldots, \eta_s) \mod p^{\nu_p},
\end{equation}
we have $F^{(1)}(\eta_1, \ldots, \eta_n) + F^{(2)}(\mu_{n+1}, \ldots, \mu_s) \equiv 0 \mod p^{\nu_p}$. From this, Hensel’s Lemma asserts that $(\eta_1, \ldots, \eta_n)$ lifts to a unique $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}_p^n$ such that
\begin{equation}
F^{(1)}(\mu_1, \ldots, \mu_n) + F^{(2)}(\mu_{n+1}, \ldots, \mu_s) = 0
\end{equation}
with $(\mu_1, \ldots, \mu_n) \equiv (\eta_1, \ldots, \eta_n) \mod p^{\nu_p+1}$. Finally, for any $H \geq u_p$, there are at least $p^{(H-u_p)(s-n)}$ choices of $(\mu_{n+1}, \ldots, \mu_s) \in (\mathbb{Z}/p^H \mathbb{Z})_{s-n}$ that satisfy (6.6), and each of them contributes for at least one solution of (1.3) in $(\mathbb{Z}/p^H \mathbb{Z})^s$. Hence
\begin{equation}
M(p^H) \geq p^{(H-u_p)(s-n)}.
\end{equation}

Inserting this last inequality into (5.3), the corresponding series is also $\geq p^{-u_p(s-n)}$, so that each of the factors in (5.4) is positive. Since the product (5.4) is absolutely convergent, this implies that $S(F) > 0$.

6. Estimates related to major arcs

We start with the definition of the major arcs and the minor arcs for our problem. Let $k$ be fixed as in (1.1), and $\tau$ fixed such that $1/k \leq \tau \leq 1$. For $X$ sufficiently large, writing
\begin{equation}
Q = \lfloor X^{\tau} \rfloor,
\end{equation}
the set of major arcs is
\begin{equation}
\mathcal{M} = \mathcal{M}(X) = \bigcup_{q \leq Q} \bigcup_{a \in A(q)} \mathcal{M}(q, a)
\end{equation}
where we have set
\begin{equation}
\mathcal{M}(q, a) = \prod_{i=1}^{n} \left[ \frac{a_i - \frac{Q}{q X^{k_i}} a_i}{q x^{k_i}} \right] \frac{Q}{q X^{k_i}}.
\end{equation}
Writing
\begin{equation}
Q_0 = 2Q,
\end{equation}
one has $\mathcal{M} \subset \left[ \frac{1}{Q_0}, 1 + \frac{1}{Q_0} \right]^n$. The set of minor arcs is
\begin{equation}
m = \left[ \frac{1}{Q_0}, 1 + \frac{1}{Q_0} \right]^n \setminus \mathcal{M}.
\end{equation}

We are now ready to state our estimate for the major arcs.

Theorem 5. Let $k$ as in (1.1), $\frac{1}{(n+1)k_n} \leq \tau \leq 1$ and $\mathcal{M}$ as in (6.2). Then for any $s \geq (n+1)k_n + 1$ and any $F \in D(k, s)$, we have
\begin{equation}
\int_{\mathcal{M}} \left( \sum_{x \in I_s(x)} e(\alpha \cdot F(x)) \right) d\alpha = 3 S(F) \mathcal{S}(F) X^{s-\sigma(k)} + O(X^{s-\sigma(k)} - \varepsilon) \quad (X \geq 1).
\end{equation}
Proof: Throughout this proof, the quantities $f(\mathbf{F}(\alpha, X))$, $S(\mathbf{F}(q, a))$, $v(\mathbf{F}(\beta, X))$, $\xi(\beta, X)$ defined in equations \( \text{(3.8)} \) to \( \text{(3.12)} \) are written more simply $f(\alpha, X)$, $S(q, a)$, $v(\beta, X)$ and $\xi(\beta, X)$. Writing $I(\mathcal{M})$ for the integral over the major arcs, using that $\mathcal{M}$ is a disjoint union, we have

$$I(\mathcal{M}) = \sum_{q \leq Q} \sum_{a \in A_n(q)} \int_{\mathcal{M}(q, a)} f(\alpha, X) d\alpha = \sum_{q \leq Q} \sum_{a \in A_n(q)} \int_{\mathcal{M}(q, 0)} f\left(\frac{a}{q} + \beta, X\right) d\beta.$$  

Now inserting the estimate from Lemma 3.6 in each of these right hand side integrals, we have $I(\mathcal{M}) = I_1(\mathcal{M}) + O(I_2(\mathcal{M}) + I_3(\mathcal{M}))$ where we have set

$$I_1(\mathcal{M}) = \sum_{q \leq Q} \sum_{a \in A_n(q)} \int_{\mathcal{M}(q, 0)} \left(\frac{X^s}{q^n} S(q, a) v(\beta, X)\right) d\beta,$$

$$I_2(\mathcal{M}) = \sum_{q \leq Q} \sum_{a \in A_n(q)} \int_{\mathcal{M}(q, 0)} \left(X^{s-1} q^{-\frac{1}{k} \tau + \varepsilon} \xi(\beta, X)^{-\frac{1}{k} + \frac{1}{k_n}}\right) d\beta,$$

$$I_3(\mathcal{M}) = \sum_{q \leq Q} \sum_{a \in A_n(q)} q^{s-\frac{1}{k_n} + \varepsilon} \int_{\mathcal{M}(q, 0)} d\beta.$$

We already have

$$I_3(\mathcal{M}) \ll \sum_{q \leq Q} \sum_{a \in A_n(q)} q^{s-\frac{1}{k} + \varepsilon} Q^{n} q^{n} X^{\sigma(k)} \ll \frac{Q^{s+n-1-\frac{1}{k_n} + \varepsilon}}{X^{\sigma(k)}} \ll \frac{X^{s-\sigma(k)-\frac{1}{k_n} + \varepsilon}}{X^{\sigma(k)}}$$

by using the bounds $Q^s \ll X^{s+\varepsilon}$ and $Q^{n+1-\frac{1}{k_n}} \ll Q^{1/k_n} = X^{-\tau/k_n}$, where for the first bound, we have used the fact that $\tau \leq 1$, and for the last bound, we have used the inequality $n + 1 - s/k_n \leq -1/k_n$. Now, using the change of variables $\gamma_i = X^{k_i} \beta_i$ in the integrals of $I_1(\mathcal{M})$ and $I_2(\mathcal{M})$, we have

$$I_1(\mathcal{M}) = X^{s-\sigma(k)} \sum_{q \leq Q} \sum_{a \in A_n(q)} S(q, a) \int_{[-\frac{q}{Q}, \frac{q}{Q})} v(\gamma, 1) d\gamma$$

and

$$I_2(\mathcal{M}) = X^{s-1-\sigma(k)} \sum_{q \leq Q} \sum_{a \in A_n(q)} q^{1-\frac{1}{k} + \varepsilon} \int_{[-\frac{q}{Q}, \frac{q}{Q})} \xi(\gamma, 1)^{-\frac{1}{k} + \frac{1}{k_n}} d\gamma.$$

Using Lemma 3.4 and (ii) of Lemma 3.5, the inner integrals in $I_1(\mathcal{M})$ satisfy

$$\mathcal{J}(F) - \int_{[-\frac{q}{Q}, \frac{q}{Q})} v(\gamma, 1) d\gamma \ll (Q/q)^{n-s/k_n} \quad (1 \leq q \leq Q).$$

Hence

$$I_1(\mathcal{M}) = X^{s-\sigma(k)} \sum_{q \leq Q} \sum_{a \in A_n(q)} S(q, a) \left(\mathcal{J}(F) + O\left((Q/q)^{n-s/k_n}\right)\right).$$

Using 3.4 and (i) of Lemma 3.2, we have

$$I_1(\mathcal{M}) = X^{s-\sigma(k)} \mathcal{J}(F) \sum_{q \leq Q} \sum_{a \in A_n(q)} S(q, a) + O(I_4)$$
where
\[
I_4 = X^{s-\sigma(k)} \sum_{q \leq Q} \frac{1}{q^n} \sum_{a \in A_n(q)} q^{s - \frac{\sigma}{\log q} + \varepsilon} (Q/q)^{n-s/k} \ll X^{s-\sigma(k)-\frac{\sigma}{\log q} + \varepsilon}
\]

by using the same inequalities as for \(I_3(M)\). Moreover, using again (3.9) and (i) of Lemma 3.2 we have
\[
\Theta(F) - \sum_{q \leq Q} \frac{1}{q^n} \sum_{a \in A_n(q)} S(q, a) \ll \sum_{q > Q} \frac{1}{q^n} \sum_{a \in A_n(q)} q^{s - \frac{\sigma}{\log q} + \varepsilon} \ll X^{-\frac{\sigma}{\log q} + \varepsilon}
\]

which gives \(I_1(M) = 3(F) \Theta(F) X^{s-\sigma(k)} + O\left(X^{s-\sigma(k)}-\frac{\sigma}{\log q} + \varepsilon\right)\). Finally, using the same inequalities, we have
\[
I_2(M) \ll X^{s-1-\sigma(k)} \sum_{q \leq Q} \sum_{a \in A_n(q)} q^{-\frac{\sigma}{\log q} + \varepsilon} \left(1 + \frac{Q}{q}\right)^{1/k} \ll X^{s-\sigma(k)-\frac{\sigma}{\log q} + \varepsilon},
\]

which completes the proof.

7. Classical minor arcs estimates

The following result is merely Theorem 5.2 of [13] applied to the sum \(f_k(\alpha; X)\) defined in (1.11).

**Proposition 1.** Let \(n \geq 2, k\) as in (1.1) and \(b \geq 1\). Let \(\alpha \in \mathbb{R}^n\). Suppose that there exist \(j, a_j, q_j\) with \(k_j \geq 2\), \(|\alpha_j - \frac{a_j}{q_j}| \leq \frac{1}{q_j}\), \((a_j; q_j) = 1\), \(q_j \leq X^{k_j}\). Then, with the notation (1.7), one has
\[
f_k(\alpha; X) \ll_{b,n,k} \left(X^{k_j(k_j-1)/2} j_{b,k_j-1}(2X)\right)^{1/(2b)} \left(\frac{q_j}{X^{k_j}} + \frac{1}{X} + \frac{1}{q_j}\right)^{1/(2b)} \log(2X)
\]

In order to treat the minor arcs for an asymptotic for the Vinogradov-type system, Proposition 111 and some analogue of our Theorem 3 are sufficient to derive a bound of the form
\[
\sup_{\alpha \in \mathbb{R}^n} |f_k(\alpha; X)| \ll X^{1-\epsilon_0} \quad (X \geq X_0)
\]

for some \(\epsilon_0 > 0\). In the case of our system (1.3), we shall require an analogue for exponential sums of the form \(f_k(w \otimes \alpha; X)\) for some fixed \(w \in \mathbb{Z}^n\) with \(w_1 \ldots w_n \neq 0\). Although it is no trouble to derive an analogue with the same tools, in order to ease our presentation and set some notation, we state a lemma that produces a suitable approximation for \(\alpha\) from an approximation of \(w \otimes \alpha\).

**Lemma 7.1.** Let \(w \in \mathbb{Z}^n\) fixed such that \(w_1 \ldots w_n \neq 0\). Set \(M_0 := |w_1 w_2 \ldots w_n|\).

(i) For any \(q \geq 1\) and any \(a \in \mathbb{Z}^n\) with \((a; q) = 1\), there exists \(c = c(q, a, w) \in \mathbb{Z}^n\) and \(h = h(q, a, w) \geq 1\) unique such that \(\frac{w \otimes a}{q} = \frac{c}{h}\) with \((c; h) = 1\). Moreover one has \(\frac{\lambda}{\lambda_{h}} \leq h(q, a, w) \leq q\).

(ii) Let \(k\) be as in (1.1), and \(\lambda \in \mathbb{R}^n\) such that \(\lambda_i > 0\) \((1 \leq i \leq n)\) and \(\sigma := \sum_{1 \leq i \leq n} \lambda_i < 1\). Suppose that for any \(i\) with \(k_i \geq 2\) there exists \(b_i \in \mathbb{Z}, q_i \geq 1\) coprime such that
\[
|w_i \alpha_i - \frac{b_i}{q_i}| \leq \frac{X^{\lambda_i}}{q_i X^{\sigma}}, \quad 1 \leq q_i \leq X^{\lambda_i}.
\]
Then, there exists $X_0 = X_0(\lambda, w)$ such that whenever $X \geq X_0$, there exists $q \in \mathbb{N}$ with \( q \leq M_0 X^\sigma \), \( a_1, a_2, \ldots, a_n \in \mathbb{Z} \) unique with \( (q; a_1; a_2; \ldots; a_n) = 1 \) such that, writing $\beta = \alpha - \frac{w_i}{q}$, we have the following:

- If $k_1 = 1$, then $|\beta_i| \leq \frac{1}{2q^{w_i} X(q,w)}$, $|\beta_i| \leq \frac{M_0 X^\sigma}{q X^\sigma}$, \( 2 \leq i \leq n \),
- If $k_1 \geq 2$, then $|\beta_i| \leq \frac{M_0 X^\sigma}{q X^\sigma}$, \( 1 \leq i \leq n \),

and such that $w \otimes \beta$ satisfies (3.11) with the choice $q = h(q, a, w)$.

Proof. The proof of (i) only use classical divisibility properties: we omit the details. For (ii), due to the constraints over the $\lambda$ and $k_1$ and the size of the $q_i$, it is plain that for $X$ sufficiently large, the $b_i, q_i$ in (7.1) are unique. We start with the case $k_1 = 1$. Using (7.1), there exist $q'$ and $c_i (2 \leq i \leq n)$ unique such that $\frac{b_i}{q_i w_i} = \frac{c_i}{q}$ with $(c_2; c_3; \ldots; c_n; q') = 1$. Then we have $q' \leq X^\sigma$ and

\[
|\alpha - \frac{w_i}{q'}| \leq \frac{X^\sigma}{q' X^\sigma} \quad (2 \leq i \leq n).
\]

Next, we choose $c_1$ minimal such that $|w_1 \alpha - \frac{c_1}{q'}| \leq \frac{1}{2q'}$. Writing now $c = (c_1, c_2, \ldots, c_n)$, we still have $(c; q') = 1$. Similarly, there exist $q \geq 1$ and $a \in \mathbb{Z}^n$ unique with $(a; q) = 1$ such that $c_i = \frac{a_i}{q}$ for $1 \leq i \leq n$. As previously, we have $q \leq M_0 X^\sigma$ and $|\alpha - \frac{c_i}{q}| \leq \frac{1}{2q |w_i|}, |\alpha - \frac{c_i}{q}| \leq \frac{X^\sigma M_0}{q X^\sigma}$ \( 2 \leq j \leq n \). By unicity, it is now plain that $q' = h(q, a, w)$, which gives the expected result.

The case $k_1 \geq 2$ is more straightforward: the construction leading to (7.2), initially valid for $2 \leq i \leq n$ is now also valid for $i = 1$, hence the choice $a = c$ and $q = q'$ is sufficient to have the expected result.

Finally, since $q \geq h(q, a, w)$, it is a simple observation that for $X$ sufficiently large, $w \otimes \beta$ satisfies (3.11) with the choice $q = h(q, a, w)$.

We can now state our first result for the minor arcs defined in (6.3).

Lemma 7.2. Let $n \geq 2$ and $k$ fixed as in (6.1). Let $w \in \mathbb{Z}^n$ fixed such that all $w_1 w_2 \ldots w_n \neq 0$. Set $\eta_0 = \frac{1}{n k \lambda}$. With the notation (1.11), we have

\[ f_k(w \otimes \alpha; X) \ll_{w, \varepsilon} X^{1-\eta_0 + \varepsilon} \]

uniformly for $\alpha \in m$ and $X \geq X_0(w)$.

Proof. We set $\lambda_i = \frac{k_{n-1}}{n k} \eta_0$ for $1 \leq i \leq n$. For any $i$ such that $k_i \geq 2$, there exist $b_i, q_i$ with $1 \leq q_i \leq X^{k_i - \lambda_i}$ such that $|w_i \alpha_i - \frac{b_i}{q_i}| \leq \frac{X^{k_i - \lambda_i}}{q_i X^\sigma}$. If one has $q_i > X^{k_i}$ for some of these $i$, then Proposition 11 used with $b = k_n(k_n - 1)/2$ and the bound (1.8) implies

\[ f_k(w \otimes \alpha; X) \ll_{w, \varepsilon} X^{1+\varepsilon} (X^{\lambda_i} - X^{-\lambda_i})^{1/2b} \ll X^{1-\eta_0 + \varepsilon} \]

We may now assume that for any $1 \leq i \leq n$ such that $k_i \geq 2$ we have $q_i \leq X^{\lambda_i}$. Using Lemma 7.1 and its notation, we have $\sigma = 1 - \frac{1}{k_n}, q \ll X^\sigma$, and for $X$ sufficiently large, $w \otimes \beta$ satisfies (3.11) for some $q < q$. Hence Theorem 3 yields

\[ f_k(w \otimes \alpha; X) \ll_{w, k, \varepsilon} X^{1+\varepsilon} \left( q \xi_k(\beta, X) \right)^{-1/k_n} + q^{1-\frac{1}{k_n} + \varepsilon} \]

(7.3)
where for the last term the bound \( q \ll X \) is sufficient. Moreover, with the notation \( 3.3 \) we have \( q \xi_\kappa(\beta, X) \gg X^\tau \) uniformly for \( \alpha \in m \). This combined with \( 7.3 \) yields the announced upper bound.

\[ \square \]

### 8. Proof of Theorem 1

#### 8.1. Full-saving index for \( \{k_1, \ldots, k_n\} \)

In order to establish Theorem 4, we shall prove a more general result. The starting point is to work with a case where \( 1.12 \) is verified. We shall say that a number \( A \) is a full-saving index for \( k \) if for any \( \varepsilon > 0 \) one has

\[ \int_{[0,1]^n} |f_k(\alpha; X)|^A \, d\alpha \ll \varepsilon X^{A-\sigma(k)+\varepsilon} \quad (X \geq 1). \]

**Theorem 6.** Let \( n \geq 2 \) and \( k \) as in \( 1.1 \). Let \( A \) be a full saving index for \( k \). Let \( s \geq 1 + \max \left(1 + A, (n+1)k_n\right) \) and \( \mathbf{F} \in \mathcal{D}(k, s) \). Then with the notation \( 1.10 \) and \( 1.16 \), both \( \mathfrak{I}(\mathbf{F}) \) and \( \mathfrak{S}(\mathbf{F}) \) are convergent, and one has, for any \( \varepsilon > 0 \) fixed

\[ N_\mathbf{F}(X) = \mathfrak{I}(\mathbf{F})\mathfrak{S}(\mathbf{F})X^{s-\sigma(k)} + O(X^{s-\sigma(k)-\eta_0+\varepsilon}) \quad (X \geq 1), \]

where we have set \( \eta_0 = \frac{1}{kn+1} \). If moreover the system \( 1.3 \) has a non singular solution over \( \mathbb{R} \) and over \( \mathbb{Z}_p \) for all \( p \), then \( \mathfrak{I}(\mathbf{F})\mathfrak{S}(\mathbf{F}) > 0 \).

It is clear from \( 1.14 \) that \( A = k_n(k_n+1) \) is a full-saving index for \( k \). Hence Theorem 4 follows immediately from Theorem 6.

#### 8.2. Proof of Theorem 6

We already have the suitable estimate on the major arcs with Theorem 5. We follow the classic approach described in §3.4 of §8. First, under our assumptions, Lemmas 5.1 and 7.2 take care of the singular constants. We may now estimate the contribution of the minor arcs. Using the notation in \( 3.3 \) and Hölder’s inequality, one has

\[ \int_m f^{|m|} (\alpha; X) \, d\alpha \ll \prod_{j=1}^{s} \left( \int_m |f_k(u_j \otimes \alpha; X)|^s \, d\alpha \right)^{1/s}. \]

Hence, it is sufficient to establish the upper bound

\[ \int_m |f_k(w \otimes \alpha; X)|^s \, d\alpha \ll X^{s-\sigma(k)-\eta_0+\varepsilon} \]

for any \( w \in \mathbb{Z}^n \) with \( w_i \neq 0 \) (\( 1 \leq i \leq n \)). For such a \( w \), we have

\[ \int_m |f_k(w \otimes \alpha; X)|^s \, d\alpha \leq \sup_{\alpha \in m} |f_k(w \otimes \alpha; X)| \int_{[0,1]^n} |f_k(w \otimes \alpha; X)|^{s-1} \, d\alpha \]

\[ \ll X^{1-\eta_0+\varepsilon} \int_{[0,1]^n} |f_k(\alpha; X)|^{s-1} \, d\alpha \]

by using Lemma 7.2. Moreover, since \( s - 1 \geq A \), then

\[ \int_{[0,1]^n} |f_k(\alpha; X)|^{s-1} \, d\alpha \ll X^{s-1-\sigma(k)+\varepsilon}, \]

which concludes the proof of Theorem 6.
9. Refined estimate on the minor arcs

In this section, \( k \) is fixed equal to \((1,3,5)\). We use the notation \( f(\alpha_1, \alpha_2, \alpha_3; X) \) introduced in \((4.6)\), and we consider the major arcs introduced in \((6.2)\) with the choice \( \tau = 5/8 \). To prove Theorem 2, it is clear, using \((8.2)\), that it is now sufficient to prove the following:

**Theorem 7.** Let \( w = (w_1, w_2, w_3) \in \mathbb{Z}^3 \) with \( w_1w_2w_3 \neq 0 \). With the notation \((4.0)\), we have

\[
\int_{m} |f(w_1 \alpha_1, w_2 \alpha_2, w_3 \alpha_3; X)|^{30} d\alpha_1 d\alpha_2 d\alpha_3 \ll_{w, \varepsilon} X^{21 - \frac{1}{3} + \varepsilon}.
\]

9.1. Partial minor arcs over \( \alpha_3 \). We consider the set \( m_3 \) of the \( \alpha \in \left[ \frac{q_3}{15}, 1 + \frac{1}{q_3} \right] \) such that whenever we have \( |5\alpha - \frac{b_3}{q_3}| \leq \frac{1}{q_3} \) with \((b_3; q_3) = 1\), we have \( q_3 > \frac{1}{8} X^{1/8} \).

**Lemma 9.1.** With the notation above, we have

\[
\iint_{[0,1]^2 \times m_3} |f(\alpha_1, \alpha_2, \alpha_3; X)|^{30} d\alpha_1 d\alpha_2 d\alpha_3 \ll_{\varepsilon} X^{21 - \frac{1}{3} + \varepsilon}
\]

**Proof.** First, we claim that

\[
\iint_{[0,1]^2 \times m_3} |f(\alpha_1, \alpha_2, \theta; X)|^{30} d\alpha_1 d\alpha_2 d\theta \ll X^2 \iint_{[0,1]^3 \times m_3} |g_5(\alpha, \theta; X)|^{30} d\alpha d\theta
\]

where \( g_k(\alpha, \theta; X) \) is defined in \((4.3)\). Indeed, for fixed \( \alpha_1, \alpha_2, \theta \), we consider the set \( \Omega = \mathbb{Z}^{15} \cap [-X, X]^{15} \) and for \( \omega = (x_1, x_2, \ldots, x_{15}) \in \Omega \), we consider \( \Phi(\omega) \) such that \( \sum_{\omega \in \Omega} \Phi(\omega) = (f(\alpha_1, \alpha_2, \theta; X))^{15} \). Now consider the set \( \mathcal{H} \) of the integers of the form \( \sum_{i=1}^{15} x_i^2 - \sum_{i=1}^{15} y_i^2 \) with \(|x_i|, |y_i| \leq X\). Then we apply Lemma \((4.1)\) and use the bound \( \#\mathcal{H} \ll X^2 \): we have

\[
|f(\alpha_1, \alpha_2, \theta; X)|^{30} \ll X^2 \int_0^1 |g_5(\alpha, \theta; X)|^{30} d\alpha_3,
\]

and the expected result follows by integrating over \( \alpha_1, \alpha_2, \theta \).

We are now in a position to apply Theorem \((4)\): we have

\[
\iint_{[0,1]^3 \times m_3} |g_5(\alpha, \theta; X)|^{30} d\alpha d\theta \ll (\log(4X))^{30} \left( \sup_{\theta \in m_3} \psi_5^*(\theta; X) \right) J_{15,5}(4X + 1)
\]

where \( \psi_5^*(\theta; X) \) has been defined in \((4.2)\). Now, for any \( \mu \in \mathbb{R} \) and any \( \theta \in \mathbb{R} \) there exists \( b_3 \in \mathbb{Z} \) and \( q_3 \geq 1 \) coprime such that \( |5\theta - \frac{b_3}{q_3}| \leq \frac{X^{1/8}}{q_3|\theta|}, q_3 \leq X^{5-\frac{1}{3}} \). Then Lemma \((4.3)\) implies

\[
\frac{1}{X} \sum_{y=1}^{X} \min \left( X^4, \frac{1}{\|5\theta + \mu\|} \right) \ll X^3 \left( 1 + \frac{X}{q_3} \right) + \left( 1 + \frac{X}{q_3} \right) \log X
\]

and since \( \theta \in m_3 \), this implies \( \frac{1}{5} X^{1/8} < q_3 \leq X^{5-\frac{1}{3}} \), hence

\[
\sup_{\theta \in m_3} \psi_5^*(\theta; X) \ll X^{4\varepsilon + \frac{2}{3} + \varepsilon}.
\]

We now conclude using the bound \( J_{15,5}(4X + 1) \ll X^{15 + \varepsilon} \) from \((1.8)\).  \( \square \)
9.2. Partial minor arcs over $\alpha_2$. We now consider the set
\[
\mathfrak{m}_3 = \bigcup_{q_3 \leq X^{1/8}} \bigcup_{b_3 \in A_1(q_3)} \mathfrak{m}_3(q_3, b_3)
\]
where
\[
\mathfrak{m}_3(q_3, b_3) := \left[b_3 - \frac{X^{1/8}}{q_3} - \frac{b_3}{q_3^2} + \frac{X^{1/8}}{q_3^2}, q_3 X^5\right].
\]
By construction, one has $\left[\frac{1}{q_0^3}, 1 + \frac{1}{q_0^3}\right] \setminus m_3 \subset \mathfrak{m}_3$. As previously, we consider the set $m_2$ of the $\alpha \in \left[\frac{1}{q_0^3}, 1 + \frac{1}{q_0^3}\right]$ such that whenever we have $|\alpha - \frac{b_2}{q_2}| \leq \frac{1}{q_2^2}$ with $(b_2; q_2) = 1$, this implies $q_2 > X^{3/4}$.

**Lemma 9.2.** With the notation above, we have
\[
\int\int\int_{[0, 1] \times \mathfrak{m}_2 \times \mathfrak{m}_3} |f(\alpha; X)|^{30} d\alpha \ll X^{21-\frac{4}{8}+\varepsilon}
\]

**Proof.** Writing $U_2 = [0, 1] \times \mathfrak{m}_2 \times \mathfrak{m}_3$, we have
\[
\int\int\int_{U_2} |f(\alpha; X)|^{30} d\alpha \ll \sup_{\alpha \in U_2} |f(\alpha; X)|^{10} \int\int\int_{[0, 1] \times \mathfrak{m}_3} |f(\alpha; X)|^{20} d\alpha
\]

Now, for any $\alpha_2 \in \mathbb{R}$, there exists $b_2 \in \mathbb{Z}$ and $q_2 \geq 1$ coprime such that
\[
|\alpha_2 - \frac{b_2}{q_2}| \leq X^{3/4} q_2^{-1}, \quad q_2 \leq X^{3-\frac{4}{2}}.
\]

For such an $\alpha_2$, and for any $\alpha_1, \alpha_3 \in \mathbb{R}$, Proposition $\|$ applied to $k = (1, 3, 5)$ and $b = 10$ with (1.3) implies
\[
f(\alpha_1, \alpha_2, \alpha_3; X) \ll X^{1+\varepsilon} \left(\frac{q_2}{X^3} + \frac{1}{X} + \frac{1}{q_2}\right)^{1/20}.
\]

Since $\alpha_2 \in \mathfrak{m}_2$, then $q_2 > X^{3/4}$, which implies $\sup_{\alpha \in U_2} |f(\alpha; X)|^{10} \ll X^{10-\frac{4}{2}+\varepsilon}$. Moreover,
\[
\int\int\int_{[0, 1] \times \mathfrak{m}_3} |f(\alpha; X)|^{20} d\alpha = \sum_{q_2 \leq X^{3/4}} \sum_{b_3 \in A_1(q_3)} \int\int\int_{[0, 1] \times \mathfrak{m}_3(q_3, b_3)} |f(\alpha; X)|^{20} d\alpha.
\]

Using Lemma 9.10 for the inner integrals, we obtain
\[
\int\int\int_{[0, 1] \times \mathfrak{m}_3} |f(\alpha_1, \alpha_2, \alpha_3; X)|^{20} d\alpha \ll X^{11+\frac{4}{2}+\varepsilon}.
\]

Gathering these estimates gives the result announced. \hfill $\square$

9.3. Pruning. Let $w \in \mathbb{Z}^n$ fixed with $w_1 w_2 \ldots w_n \neq 0$. Using the notation of Lemma 7.11 we set
\[
\mathcal{L} = \bigcup_{1 \leq q \leq M_0 X^{7/8}} \bigcup_{a \in A_3(q)} \mathcal{L}(q, a)
\]
where $\mathcal{L}(q, a)$ is the set of $(\alpha_1, \alpha_2, \alpha_3) \in \left[\frac{1}{q_0^3}, 1 + \frac{1}{q_0^3}\right]^{3}$ such that
\[
|\alpha_1 - \frac{a_1}{q}| \leq \frac{1}{2 |w_1| (q, q, w)}, \quad |\alpha_2 - \frac{a_2}{q}| \leq \frac{M_0 X^{7/8}}{q X^5}, \quad |\alpha_3 - \frac{a_3}{q}| \leq \frac{M_0 X^{7/8}}{q X^5}.
\]

It is plain that $\mathfrak{M} \subset \mathcal{L}$. 

Lemma 9.3. One has
\[
\int_{L \setminus M_2} |f(w_1\alpha_1, w_2\alpha_2, w_3\alpha_3)|^{30} \, d\alpha_1 d\alpha_2 d\alpha_3 \ll_{w, \varepsilon} X^{21 - 2\tau + \varepsilon}
\]

Proof. Throughout the proof we use the following notation
\[
\xi(\beta, X) = 1 + |\beta_1|X + |\beta_2|X^3 + |\beta_3|X^5 \quad (\beta \in \mathbb{R}^3).
\]
The integral we have to estimate is equal to
\[
\sum_{q \leq M_0 X^{7/8}} \sum_{a \in A_3(q)} \int_{L(q,a)} \, f(w_1\alpha_1, w_2\alpha_2, w_3\alpha_3) \, d\alpha_1 d\alpha_2 d\alpha_3.
\]
Using Lemma 7.1 and its notation, for \( \alpha \in L(q,a) \), \( w \otimes \beta \) satisfies (3.11) for some \( q \ll q \), hence using 7.3, we have
\[
f(w_1\alpha_1, w_2\alpha_2, w_3\alpha_3) \ll X q^{-\frac{1}{3} + \varepsilon} \xi(\beta, X)^{\frac{1}{5} + q^\frac{1}{2}} + \varepsilon
\]
uniformly for \( q \leq M_0 X^{7/8}, a \in A_3(q) \) and \( \alpha \in L(q,a) \). Thus,
\[
\int_{L \setminus M_2} |f(w_1\alpha_1, w_2\alpha_2, w_3\alpha_3)|^{30} \, d\alpha_1 d\alpha_2 d\alpha_3 \ll X^7 S_1 + X^{17 + \varepsilon}
\]
where we have set
\[
S_1 := \sum_{q \leq M_0 X^{7/8}} \sum_{\alpha \in A_3(q)} X^{30} q^{-6} \int_{L(q,0)} \, 1_m \left( \frac{a}{q} + \beta \right) \xi(\beta, X)^{-6} d\beta_1 d\beta_2 d\beta_3.
\]
Since
\[
\int_{L(q,0)} \, 1_m \left( \frac{a}{q} + \beta \right) \xi(\beta, X)^{-6} d\beta_1 d\beta_2 d\beta_3 \ll X^{-9} \int_{\mathbb{R}^3} \xi(\beta, 1)^{-6} d\beta_1 d\beta_2 d\beta_3 \ll X^{-9},
\]
the contribution of the \( q > \frac{1}{2} X^\tau \) in \( S_1 \) is \( \ll X^{21 - 2\tau} \), which does not exceed the expected bound. Now, since for \( \frac{a}{q} + \beta \in \mathbb{m} \), we have
\[
q + q|\beta_1|X + q|\beta_2|X^3 + q|\beta_3|X^5 > X^\tau,
\]
Hence, for \( q \leq \frac{1}{2} X^\tau \), we have \( |\beta_1|X + |\beta_2|X^3 + |\beta_3|X^5 > \frac{X^\tau}{2q} \), which implies
\[
\int_{L(q,0)} \, 1_m \left( \frac{a}{q} + \beta \right) \xi(\beta, X)^{-6} d\beta \ll X^{-9} \int_{H_3(\frac{X^\tau}{2q})} \xi(\beta, 1)^{-6} d\beta \ll X^{-9} \left( 1 + \frac{X^\tau}{q} \right)^{-3}
\]
by using (iii) of Lemma 3.5. Thus, the contribution of the \( q \leq \frac{1}{2} X^\tau \) in \( S_1 \) is
\[
\ll \sum_{q \leq \frac{1}{2} X^\tau} X^{21q^{-3}} \left( 1 + \frac{X^\tau}{q} \right)^{-3} \ll X^{21 - 2\tau}.
\]
\[\square\]

9.4. Proof of Theorem 7. We keep the notation \( m_3, m_2, \mathfrak{M}_3 \) and \( L \) introduced in the sections 9.1, 9.2 and 9.3. Similarly to \( \mathfrak{M}_3 \), we construct the set
\[
\mathfrak{M}_2 = \bigcup_{q_2 \leq X^{3/4}} \bigcup_{b_2 \in A_1(q_2)} \mathfrak{M}_2(q_2, b_2),
\]
where
\[
\mathfrak{M}_2(q_2, b_2) := \left[ b_2 \frac{X^{3/4}}{q_2} - \frac{X^{3/4} b_2}{2q_2 X^3} + \frac{X^{3/4}}{2q_2 X^3} \right].
\]
Again by construction, one has \( \left[ \frac{1}{q_2^{3/4}}, 1 + \frac{1}{q_2^{3/4}} \right] \setminus m_2 \subset \mathfrak{M}_2. \)
For a fixed $w \in \mathbb{Z}^3$ such that $w_1w_2w_3 \neq 0$, we introduce various subsets of $m$:

- We define the set $n_1$ of $\alpha \in m$ such that $w \otimes \alpha \mod 1$ belongs to $[0, 1] \times \mathfrak{M}_2 \times \mathfrak{M}_3$.
- We also define $n_2$, the set of $\alpha \in m$ such that $w \otimes \alpha \mod 1$ belongs to $[0, 1] \times m_2 \times \mathfrak{M}_3$.
- Finally $n_3$ is the set of $\alpha \in m$ such that $w \otimes \alpha \mod 1$ belongs to $[0, 1]^2 \times m_3$.

It is plain that $m = n_1 \cup n_2 \cup n_3$. Writing

$$I_j := \iiint_{n_j} |f(w_1\alpha_1, w_2\alpha_2, w_3\alpha_3; X)|^{30} \, d\alpha_1 d\alpha_2 d\alpha_3 \quad (1 \leq j \leq 3),$$

we now have

$$\iiint_m |f(w_1\alpha_1, w_2\alpha_2, w_3\alpha_3; X)|^{30} \, d\alpha_1 d\alpha_2 d\alpha_3 = I_1 + I_2 + I_3.$$

Using Lemma 7.1 we deduce

$$I_3 \ll \iiint_{[0,1]^2 \times m_3} |f(\alpha_1, \alpha_2, \alpha_3; X)|^{30} \, d\alpha_1 d\alpha_2 d\alpha_3 \ll X^{21 - \frac{1}{6} + \varepsilon}.$$

Similarly

$$I_2 \ll \iiint_{[0,1] \times m_2 \times \mathfrak{M}_3} |f(\alpha_1, \alpha_2, \alpha_3; X)|^{30} \, d\alpha_1 d\alpha_2 d\alpha_3 \ll X^{21 - \frac{1}{6} + \varepsilon}$$

by using Lemma 7.2.

Finally, we have $n_1 \subset \mathcal{L} \setminus \mathfrak{M}$. Indeed, any $\alpha \in n_1$ satisfies

$$|w_2\alpha_2 - \frac{b_1}{q_2}| \leq \frac{X^{3/4}}{q_2 X}$$

and

$$|w_3\alpha_3 - \frac{b_3}{q_3}| \leq \frac{X^{1/8}}{q_3 X}$$

for some $b_i, q_i$ with $q_2 \leq X^{3/4}$ and $q_3 \leq X^{1/8}$. Therefore, using Lemma 7.1 we have $\alpha \in \mathcal{L} \setminus \mathfrak{M}$.

We may now use Lemma 9.3: this implies $I_1 \ll X^{21 - 2\tau + \varepsilon}$, which completes the proof.

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