A cork is a smooth contractible compact 4-manifold $W$ together with a self-diffeomorphism $f$ of the boundary 3-manifold that cannot extend to a self-diffeomorphism of $W$; the cork is said to be strong if $f$ cannot extend to a self-diffeomorphism of any smooth integer homology ball bounded by $\partial W$. Surprising recent work of Dai, Hedden, and Mallick showed that most of the well-known corks in the literature are strong. We construct the first non-strong corks, which also give new examples of absolutely exotic Mazur manifolds.

The problem of extending a self-diffeomorphism of a 3-manifold over a 4-manifold that it bounds depends inherently on the topology of the 4-manifold in question. It is therefore surprising that many of the corks in the existing literature turn out to be strong; Lin, Ruberman, and Saveliev originally showed that the Akbulut cork is strong [9], and Dai, Hedden, and Mallick recently exhibited several families of strong corks, encompassing most of the well-known corks in the literature [5]. In light of this, Dai, Hedden, and Mallick ask if every cork is strong [5, Question 1.14], expecting a negative answer.

**Theorem 1** Not all corks are strong.

In fact, we show that it is straightforward and requires no obstructive calculations to build a non-strong cork out of any reasonably nice Mazur-type manifold you happen to already have on hand. Our proof of Theorem 1 provides a general recipe for constructing non-strong corks at home. We then present an explicit family of examples in Figure 2, which have some additional interesting properties we record as Theorem 2. Recall that a contractible smooth 4-manifold is *Mazur-type* if it can be described with a single 0-, 1-, and 2-handle; these are the simplest contractible smooth 4-manifolds other than $B^4$.

**Theorem 2** For integers $|n| \gg 0$, the 4-manifolds $W_n$ and $W_n'$ in Figure 2 are exotic Mazur manifolds, and the indicated boundary involution $f$ extends to a diffeomorphism of $W_n$ but not to a diffeomorphism of $W_n'$.
Let \( L \) be a 2-component link of unknots with linking number one. By viewing one component as a dotted circle and the other as the attaching curve of a zero-framed 2-handle, we obtain a handle diagram for a Mazur-type manifold. We say that a Mazur manifold \( C \) is \textit{reasonably nice} if it admits such a handle diagram in which the meridian of the dotted circle defines a knot \( \mu \subset \partial C \) that does not bound a smoothly embedded disk in \( C \). The Akbulut cork [1], presented on the left in Figure 1, is reasonably nice [2], as are many other Mazur manifolds in the literature.

\textbf{Proof of Theorem 1} Choose any reasonably nice Mazur manifold \( C \) with a diagram as above; label the dotted unknot \( j \) and the 2-handle curve \( h \). Draw this diagram on the left side of the page, and let \( C' \) be a second copy of \( C \) (with corresponding dotted unknot \( j' \) and 2-handle curve \( h' \), drawn on the right side of the page by rotating the diagram for \( C \) by 180° through the vertical line down the center of the page. Then modify the linking between the 2-handles of \( C \) and \( C' \) in any manner you like so long as your diagram retains this rotational symmetry. You are, however, forbidden from introducing any new geometric linking between any 1-handle and 2-handle. If desired, you may do nothing at all and continue on with \( C \# C' \). This is your first contractible 4-manifold, denoted \( W \), and it admits a smooth involution \( F \) arising from the obvious symmetry of its diagram.

Observe that \( Y = \partial W \) bounds another contractible 4-manifold \( W' \) obtained by reversing the roles of the 1- and 2-handle curves in \( C' \), i.e. putting a dot on \( h' \) and letting \( j' \) represent a zero-framed 2-handle. We claim the involution \( f = F|_{\partial W} : Y \to Y \) does not extend to a diffeomorphism of \( W' \), hence that \( (W', f) \) is a non-strong cork.

Let \( \mu \subset Y \) denote the meridian of the 1-handle curve \( j \) in \( C \subset W' \). Since \( f(\mu) \) is represented by the meridian of the 2-handle \( j' \), \( f(\mu) \) is slice in \( W' \). If \( f \) were to extend over \( W' \), then \( \mu \) would also have to be smoothly slice in \( W' \). To show that this is not the case, consider the 4-manifold \( X \) whose diagram is obtained from that of \( W' \) by erasing the dotted circle \( h' \) on the right side of the page. Note that \( W' \) is obtained from \( X \) by carving out a slice disk represented by the dotted circle \( h' \), and hence that \( W' \) embeds in \( X \). Therefore if \( \mu \) is slice in \( W' \) then \( \mu \) is slice in \( X \). Observe also that \( X \) is obtained from \( C \) by attaching a zero-framed 2-handle along \( j' \). By construction, \( j' \) is an unknot split from the handle diagram of \( C \), so any disk passing over this 2-handle can be replaced with one disjoint from it. Therefore if \( \mu \) is slice in \( X \), then \( \mu \) is slice in \( C \), a contradiction. \( \square \)
Remark  This recipe can be modified to the reader’s taste; for example, it is possible to use a more complicated base cork $C$ or to produce a higher-order boundary diffeomorphism.

Proof of Theorem 2 For all integers $n$, the manifolds $W_n$ and $W'_n$ in Figure 2 are produced via the recipe outlined above, so the obvious boundary involution $f$ extends smoothly over $W_n$ and not over $W'_n$. Thus it remains to show that $W_n$ and $W'_n$ are both Mazur-type for all $n$, and that the pair are homeomorphic but not diffeomorphic for $|n| \gg 0$.

To see that $W'_n$ is Mazur-type, observe that the 2-component link $h \cup h'$ is a Hopf link, thus the corresponding 1- and 2-handles form a canceling pair. To see that $W_n$ is Mazur-type, first perform the handle slide of $h$ indicated by the orange arrow. In the new diagram, this modified 2-handle and the 1-handle represented by $j$ form a canceling pair.

Observe that $\partial W_n$ and $\partial W'_n$ are diffeomorphic, each identified with the 3-manifold $Y_n$ obtained by performing zero-framed Dehn surgery on all four link components in the diagram of $W_n$ or $W'_n$. Since $W_n$ and $W'_n$ are contractible 4-manifolds with the same boundary, they are homeomorphic by work of Freedman [6]. For $|n| \gg 0$, we will show that there are exactly two self-diffeomorphisms of $Y_n$ up to isotopy, namely $f$ and the identity. Assuming this for the moment, we show that $W_n$ and $W'_n$ (for $|n| \gg 0$) are not diffeomorphic: To the contrary, suppose there exists a diffeomorphism $\phi$ of $Y_n$ extending to a diffeomorphism from $W'_n$ to $W_n$. Since $f$ extends over $W_n$, the composition $\phi^{-1} \circ f \circ \phi$ extends to a diffeomorphism of $W'_n$. However, $\phi^{-1} \circ f \circ \phi$ cannot be isotopic to the identity (because the isotopy class of the identity is preserved under conjugation), so it must be isotopic to $f$. This implies that $f$ extends to a diffeomorphism of $W'_n$, a contradiction.

It remains to show that, for $|n| \gg 0$, there are exactly two self-diffeomorphisms of $Y_n$ up to isotopy, namely $f$ and the identity. Since $f$ fails to extend to a diffeomorphism of $W'_n$, it is not isotopic to the identity. So it suffices to show that the mapping class group of $Y_n$ has no more than two elements. To do so, we realize $Y_n$ as $-1/2n$-surgery on the knot $\gamma \subset Y_0$ depicted in Figure 2. Using SnapPy [3] and Sage [12], we verify that $\gamma \subset Y_0$ has hyperbolic exterior; all computer calculations for this proof are documented in [7]. For large $|n|$, Thurston’s hyperbolic Dehn surgery theorem [13] ensures that the Dehn-filled 3-manifold $Y_n$ is hyperbolic and that the core $\tilde{\gamma} \subset Y_n$ of the surgered solid torus is the unique shortest
closed geodesic in $Y_n$. By Mostow rigidity [11], the mapping class group of a hyperbolic 3-manifold is isomorphic to its isometry group. As in [8, §5], we note that any isometry of $Y_n$ fixes the short geodesic $\tilde{\gamma}$ setwise, hence $\text{Isom}(Y_n)$ is isomorphic to a subgroup of $\text{Isom}(Y_n \setminus \tilde{\gamma})$. Since the hyperbolic 3-manifolds $Y_n \setminus \tilde{\gamma}$ and $Y_0 \setminus \gamma$ are naturally identified, we have $\text{Isom}(Y_n \setminus \tilde{\gamma}) \cong \text{Isom}(Y_0 \setminus \gamma)$. Using [3, 12], we calculate $\text{Isom}(Y_0 \setminus \gamma) \cong \mathbb{Z}_2$, so we conclude that $\text{Isom}(Y_n)$ contains at most (indeed, exactly) two elements.

**Remark** For $|n| \gg 0$, the exotic 4-manifolds $W_n$ and $W'_n$ have very distinct diffeomorphism groups: All diffeomorphisms of $\partial W_n$ extend to diffeomorphisms of $W_n$, whereas a diffeomorphism of $\partial W'_n$ extends over $W'_n$ only if it is isotopic to the identity.

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