Nonlocal symmetries of integrable two-field divergent evolutionary systems

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Abstract

Nonlocal symmetries for exactly integrable two-field evolutionary systems of the third order have been computed. Differentiation of the nonlocal symmetries with respect to spatial variable gives a few nonevolutionary systems for each evolutionary system. Zero curvature representations for some new nonevolution systems are presented.

Keywords: conserved density, nonlocal variable, nonlocal symmetry, negative flow, exact integrability.

MSC: 37K20, 37K10, 35Q58

1 Introduction

This paper is devoted to nonlocal symmetries for the systems obtained in [1] via a symmetry classification. All calculations are simple and do not require any knowledge of any zero curvature representation or Lax representation. The investigation gave several new integrable nonevolution systems besides the known Toda lattices.

Kumei’s article [2] was probably a pioneering work on generalized symmetries for the sine-Gordon equation

\[ v_{tx} = \sin v. \]  

(1)

It was discovered there that one of the symmetries coincides with the modified Korteweg-de Vries equation (mKdV)

\[ v_t = v_{xxx} + \frac{1}{2} v^3_x \]  

(2)

Rewriting the sine-Gordon equation in the evolution form \( u_t = \partial_x^{-1} \sin u \), one can say that this equation is a nonlocal symmetry of (2). Let us consider this problem in detail.

To simplify all formulas we introduce the function \( u = iv_x \), that satisfies the following equation:

\[ u_t = u_{xxx} - \frac{3}{2} u^2 u_x. \]  

(3)

It is obvious that \( u \) is the conserved density for (3) and one can introduce the nonlocal variable \( w = D_x^{-1}u \). It can be easily verified that \( e^w \) and \( e^{-w} \) are the conserved densities for

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too. This allows us to introduce two more nonlocal variables:

\[ w_1 = D^{-1}e^w, \quad w_2 = D^{-1}e^{-w}. \]

Equation (3) possesses the following nonlocal symmetry

\[ u_\tau = c_1 e^w + c_2 e^{-w} + c_3 (w_1 e^{-w} + w_2 e^w), \]

where \( c_i \) are arbitrary constants (see [3], for example).

If \( c_3 = 0 \), then adopting \( w \) in (4) as a new unknown function and setting \( u = w_x \), we obtain the following well known integrable equation

\[ w_{\tau x} = c_1 e^w + c_2 e^{-w}. \]

Differentiation of equation (4) where \( c_3 \neq 0 \), gives \( u_{\tau x} = u(c_1 e^w - c_2 e^{-w}) \). Excluding here \( w \) with the help of the initial equation, we obtain another integrable equation

\[ u_{\tau x} = u \sqrt{u_x^2 - 4c_1c_2}. \]

Obviously, the relation \( u = w_x \) connects equations (5) and (6).

Next, let \( c_3 \neq 0 \). Differentiating equation (4) and combining the result with the initial equation, one can obtain \((u^{-1}u_{\tau x})_x = uu_\tau - 2c_3u^{-2}u_x\). Using a dilatation of \( \tau \), one can adopt \( c_3 = -1/2 \) and obtain

\[ u_{\tau xx} = u^{-1}u_x(u_{\tau x} + 1) + u^2u_\tau. \]

If one sets \( u_{\tau x} = uz_\tau \), then the hyperbolic system follows:

\[ z_{\tau x} = uu_\tau + u^{-2}u_x, \quad u_{\tau x} = uz_\tau. \]

It can be shown that nonlocal equation (4) possesses a Lax representation. Hence, all differential consequences of (4) have Lax representations too. So, one integrable evolution equation (3) generates a set of integrable nonevolution equations (5) – (8).

The canonical approach to obtaining integrable hierarchies is to fix a Lax operator \( L \) and consider various operators \( A \). Usually, the operator \( A \) is presented as a polynomial with respect to positive or negative degrees of the spectral parameter. This gives positive and negative flows of the Lax equations. There is a remarkable paper on this theme [4], where a general construction of the Lax representations for the KdV-like equations has been presented in terms of the affine algebras. Later in [5], this construction was described in detail with proofs and examples provided.

The method that is used here is direct, because it deals with the evolutionary system only. But to prove integrability of a nonlocal symmetry one must construct the zero curvature representation.
2 Basic notion and notation

Consider an evolution system with two independent variables \( t, x \) and \( m \) dependent variables \( u^\alpha \)

\[
    u_t = K(t, x, u, u_x, \ldots, u_n),
\]

where \( K = \{K^\alpha\} \) and \( u = \{u^\alpha\}, \alpha = 1, \ldots, m \) are infinitely differentiable functions, \( u^\alpha = u^\alpha_0, u_x^\alpha = u^\alpha_1, u_k^\alpha = \partial^k u^\alpha / \partial x^k \). The set of whole dependent variables \( u^\alpha \) is denoted as \( u \) for brevity.

**Definition 1.** (see [6], [7]). If the vector function \( \sigma(t, x, u) \) satisfies the equation

\[
    (D_t - K_*)\sigma = 0,
\]

where

\[
    (K_*)^\alpha_\beta = \sum_{k \geq 0} \frac{\partial K^\alpha}{\partial u^\beta_k} D^k_x,
\]

\[
    D_x = \frac{\partial}{\partial x} + \sum_{\alpha, k \geq 0} u^\alpha_{k+1} \frac{\partial}{\partial u^\alpha_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_{\alpha, k \geq 0} (D^k_x K^\alpha) \frac{\partial}{\partial u^\alpha_k},
\]

then it is said to be the generalized symmetry of system (9).

Here \( D_x \) is called the total differentiation operator with respect to \( x \), \( D_t \) is called the operator of evolutionary differentiation.

The order of the differential operator \( f_* \) is called the order of the (vector-)function \( f \).

Generalized symmetries are often written as the evolution systems

\[
    u_\tau = \sigma(t, x, u),
\]

where \( \tau \) is a new evolution parameter. It is clear that to obtain local integrable equations from (12) one must find the symmetries \( \sigma \) that do not depend on \( t \) explicitly.

**Definition 2.** (see [6], [7]). If for some differentiable functions \( \rho \) and \( \theta \) the following equation

\[
    D_t \rho(t, x, u) = D_x \theta(t, x, u)
\]

is satisfied identically for any solution \( u \) of system (9), then relation (13) is called the local conservation law of system (9). The function \( \rho \) is said to be the conserved density and \( \theta \) is said to be the density of current. The pair \( (\rho, \theta) \) is said to be the conserved current.

As the operators \( D_t, D_x \) are commutative, then the vector \( (\rho_0 = D_x f, \theta_0 = D_t f) \) with any function \( f \) is the conserved current for any system. Such currents are called trivial. Conserved currents are always defined by modulo of trivial currents.
Let \((\rho, \theta)\) be the conserved current, then the following system

\[
\begin{align*}
w_x &= \rho(t, x, u), \\
w_t &= \theta(t, x, u)
\end{align*}
\]  

(14)
is compatible for any \(u\) satisfying equation (9). The solution of (14) is formally written in the form \(w = D_x^{-1}\rho\). One can consider \(w\) as a new dynamical variable. It is called weakly nonlocal or quasi-local (see \[8\]). We will call such variables the first order nonlocal variables.

Let \((\rho_i, \theta_i)\) be local conserved currents and \(w^{(1)}_i = D_x^{-1}\rho_i\) be the corresponding first order nonlocal variables. If there exist conserved currents depending on \(w^{(1)}_i\) and, possibly, on local variables, then one can construct new nonlocal variables \(w^{(2)}_i\) and so on.

The order of nonlocal variables is defined inductively. Let the variables \(w^{(1)}, \ldots, w^{(n)}\) be defined till the \(n\)-th order. If there exists a nontrivial conserved density \(\rho(t, x, u, w^{(1)}, \ldots, w^{(n)})\) and the \(n\)-th order variables \(w^{(n)}_i\) can not be removed by some gauge transformation \(\rho \to \rho + D_x f, \theta \to \theta + D_t f\), then the variable \(w = D_x^{-1}\rho(t, x, u, w^{(1)}, \ldots, w^{(n)})\) is called the \((n + 1)\)-th order nonlocal variable.

Operators (11) are to be prolonged on the nonlocal variables \(w_i\) in accordance with the following formulas

\[
\begin{align*}
\hat{D}_x &= D_x + \rho_i \frac{\partial}{\partial w_i}, \\
\hat{D}_t &= D_t + \theta_i \frac{\partial}{\partial w_i}
\end{align*}
\]  

(15)

where \((\rho_i, \theta_i)\) are the nonlocal conserved currents corresponding to the nonlocal variables \(w_i\).

It is proved that the operators \(\hat{D}_x\) and \(\hat{D}_t\) are commutative (see \[9\], for example). Hence, the equation for nonlocal symmetries is (10) with the prolonged operators \(\hat{D}_x\) and \(\hat{D}_t\).

If prolonged equation (10) has a solution depending on nonlocal variables, then this solution is called a nonlocal symmetry.

Differential equations that are interesting for applications have low orders. That is why to obtain interesting nonevolution integrable systems one ought to consider low order conserved densities and nonlocal symmetries. We restrict ourselves with considering the local variables \(u_0^\alpha\) and \(u_1^\alpha\) only. Moreover, the nonlocal variables are computed till the second order because the symmetries dependent on higher order nonlocal variables are very cumbersome.

If the system takes the following form

\[
u_t^\alpha = D_x K^\alpha(t, x, u, u_x, \ldots, u_{n-1}),
\]  

(16)

then it is called divergent. Setting here \(u^\alpha = U_x^\alpha\), we obtain the system

\[
U_t^\alpha = K^\alpha(t, x, U_x, \ldots, U_n)
\]  

(17)

that is usually called a potential version of system (16), as \(U\) is the potential for \(u\).

Below we consider systems of the form (17) with two functions \(u\) and \(v\). Therefore in the general formulas considered above one must change \(u^1\) and \(u^2\) to \(u\) and \(v\) respectively.
3 Nonlocal symmetries

Some of the systems found in [1] do not possess any nonlocal symmetries. Other systems possess multi-parametric nonlocal symmetries. Arbitrary constants contained in the symmetries are denoted as \(c_i\) or \(k_i\).

1. The system

\[ u_t = u_3 + \frac{3}{2} u_1 v_2 - \frac{3}{4} u_1 v_1^2 + \frac{1}{4} u_1^3, \quad v_t = -\frac{1}{2} v_3 - \frac{3}{4} (2 u_1 u_2 + u_1^2 v_1) + \frac{1}{4} v_1^3 \]  \hspace{1cm} (18)

admits of the following nonlocal symmetry

\[ u_\tau = c_2 w_2 + c_3 w_3 + c_4 (w_4 - w_1 w_2) + c_5 (w_5 - w_1 w_3) \]
\[ + c_6 (w_3 w_4 - w_2 w_5) + c_7 (2 w_7 - w_3 w_1^2) \]
\[ + c_8 (w_2 w_7 + w_3 w_6 + w_4 w_5 - w_1 w_3 w_4 - w_1 w_2 w_5) \]
\[ + c_9 (w_1 w_3 w_4 + w_1 w_2 w_5 - 2 w_3 w_6 - w_4 w_5), \]  \hspace{1cm} (19)

where

\[ w_1 = D_x^{-1} e_v, \quad w_2 = D_x^{-1} e^{u-v}, \quad w_3 = D_x^{-1} e^{-u-v}, \quad w_4 = D_x^{-1} w_2 e_v, \quad w_5 = D_x^{-1} w_3 e_v, \]
\[ w_6 = D_x^{-1} w_1 w_2 e_v, \quad w_7 = D_x^{-1} w_3 e_v, \quad w_8 = D_x^{-1} w_2 w_3 e_v. \]

We have verified that flow (19) commutes with flow (18) and with the next flow from the same hierarchy:

\[ u_t = u_5 + \frac{5}{4} v_4 u_1 + \frac{5}{4} u_3 (2 v_2 - v_1^2) + \frac{5}{4} v_3 (2 u_2 - u_1 v_1) - \frac{5}{2} u_1 u_2 v_1 - \frac{5}{8} v_1^2 u_1 \]
\[ - \frac{5}{8} u_1 v_2 (u_1^2 + v_1^2) + \frac{1}{32} u_1 (5 v_1^5 - 3 u_1^4 + 10 u_1^2 v_1^2), \]
\[ v_t = -\frac{5}{4} v_5 - \frac{5}{4} u_1 u_4 - \frac{5}{4} u_3 (u_2 + u_1 v_1) - \frac{5}{8} v_3 (u_2 - u_1^2 v_1) + \frac{5}{8} v_1^2 u_2 - \frac{5}{4} u_1 u_2 v_2 \]
\[ + \frac{5}{8} u_1 u_2 (u_1^2 + v_1^2) + \frac{1}{32} v_1 (5 u_1^4 - 3 v_1^4 + 10 u_1^2 v_1^2). \]

So, there is a reason to believe that the exact integrability of system (19) holds.

For all systems considered in the paper we have also verified commutativity of the nonlocal flows and of the higher members of the corresponding hierarchies. We do not mention it below and we do not write out the higher members of hierarchies for brevity.

1.a. Setting in (19) \(c_i = 0, i > 3\), we obtain the Toda lattice:

\[ u_{xx} = c_2 e^{u-v} + c_3 e^{-u-v}, \quad v_{xx} = c_1 e^v - c_2 e^{u-v} + c_3 e^{-u-v}. \]  \hspace{1cm} (20)
In notation of paper [5] consider the system $u_{i,tx} = \exp \left( \sum_j u_j A_{ji} \right)$, where $A_{ji}$, $i, j = 1, 2, 3$ is the Cartan matrix of the affine algebra $D_3^{(2)}$ with the following Dynkin diagram $\circ \Leftarrow \Rightarrow \circ$. One has explicitly:

$$u_{1,tx} = \exp(2u_1 - 2u_3), \quad u_{2,tx} = \exp(2u_2 - 2u_3), \quad u_{3,tx} = \exp(2u_3 - u_1 - u_2).$$

It is obvious that the functions $p = u_1 - u_3$, $q = u_2 - u_3$ satisfy the system $p_{tx} = e^{2p} - e^{-p-q}$, $q_{tx} = e^{2q} - e^{-p-q}$. If $c_i \neq 0$, then the same system is obtained from (20) by the substitution $u - v = 2p$, $u + v = -2q$. The constants $c_i$ may vanish in (20). In particular, if $c_1 = 0$, then the system decomposes into a pair of the Liouville equations.

System (20) can be represented in several forms. For example, choosing $p = w_1$ and $q = w_2$ as the new unknown functions, we obtain:

$$p_{rx} = p_x(c_1p - c_2q + c_3w), \quad q_{rx} = q_x(2c_2q - c_1p), \quad w_x = p_x^{-2}q_x^{-1}.$$

1.b. If in (19) $c_4 = 1$ and $c_i = 0$, $i > 4$, then there are several possibilities. Consider the following examples.

(1) Adopting $p = w_1$ and $q = \ln w_2$ as the new unknown function and setting $c_1 = c_3 = 0$ we obtain:

$$p_{rx} = e^qpp_x, \quad q_{rx} = -e^qpq_x + f(\tau)q_x,$$

where $f(\tau)$ is an integration “constant”.

(2) Substitution

$$u = \ln \left( U_x(V_x/U_x)_x \right), \quad v = \ln U_x, \quad w_1 = U, \quad w_2 = V_x/U_x, \quad w_4 = V$$

results in another system under condition $c_3 = 0$:

$$U_{rx} = c_1UU_x - c_2V_x + UV_x, \quad V_{rx} = c_1VV_x + VV_x + f(\tau)U_x.$$

If one considers in this point $c_3 \neq 0$, then the result is the third order cumbersome system. Notice that if $c_4 = 0$ and $c_5 \neq 0$ in (19), then a slightly different substitution results in (22) again.

(3) Double differentiation of system (19) gives the third order local system

$$u_{rxx} = (2u_x + z_x)u_{rx} + cu_x e^z - e^u, \quad z_{rxx} = -(z_x + u_x)z_{rx} + c(u_x + 2z_x)e^z - e^u$$

for any $c_1, c_2, c_3$. Here $z = -u - v$, $c = -2c_3$.

1.c. Adopting $c_7 = -1$, $c_i = 0$, $i > 3$ in (19) one can obtain the following system:

$$u_{txx} = -u_{tx}(p_x + 2u_x) + 2c_2 u_x e^{-p} + 2\sqrt{u_{tx}e^p - c_2 - c_3 e^{-2u}},$$

$$p_{txx} = p_{tx}(u_x + p_x) + 2c_2 (u_x + 2p_x)e^{-p} + 2\sqrt{u_{tx}e^p - c_2 - c_3 e^{-2u}},$$

(23)
where \( p = v - u \).

2. The system

\[
\begin{align*}
    u_t &= u_3 - 3v_3 + 3v_2(v_1 - 2u_1) + 3u_1v_1^2 - 2u_1^3, \\
    v_t &= -3u_3 + 4v_3 - 3u_2(v_1 - 2u_1) + 3v_1u_1^2 - 2v_1^3,
\end{align*}
\]

(24)

admits of the following nonlocal symmetry

\[
\begin{align*}
    u_r &= c_2w_2 + c_3w_3 + c_4w_4 + c_5(2w_2w_3 - w_5) + c_6w_6 \\
    &\quad + c_7(2w_3w_4 - w_7) + c_8(2w_3w_6 - w_9) + c_9w_2(w_2w_3 - w_5) \\
    &\quad - c_{10}(w_5w_6 + w_2w_9 - 2w_4w_7 + 2w_3w_4^2 - 2w_2w_3w_6), \\
    v_r &= c_1w_1 + c_2w_2 + c_4w_1w_2 + c_5w_5 + c_6(2w_1w_4 - w_6) \\
    &\quad + c_7w_1w_5 + c_8(2w_1w_7 - w_9) + c_9(2w_8 - w_2w_5) \\
    &\quad + c_{10}(w_5w_6 - w_2w_9 - 2w_1w_4w_5 + 2w_1w_2w_7),
\end{align*}
\]

(25)

where

\[
\begin{align*}
    w_1 &= D_x^{-1}e^v, & w_2 &= D_x^{-1}e^{-u-v}, & w_3 &= D_x^{-1}e^{2u}, \\
    w_4 &= D_x^{-1}w_1e^{-u-v}, & w_5 &= D_x^{-1}w_3e^{-u-v}, & w_6 &= D_x^{-1}w_1^2e^{-u-v}, \\
    w_7 &= D_x^{-1}w_1w_3e^{-u-v}, & w_8 &= D_x^{-1}w_2w_3e^{-u-v}, & w_9 &= D_x^{-1}w_1^2w_3e^{-u-v}.
\end{align*}
\]

Let us present some simple local systems that follow from (24).

2.a. Setting \( c_i = 0 \) for \( i > 3 \) in (25) we obtain the Toda lattice:

\[
\begin{align*}
    u_{rx} &= c_2e^{-u-v} + c_3e^{2u}, & v_{rx} &= c_1e^v + c_2e^{-u-v}.
\end{align*}
\]

(26)

Let us write the system \( u_{ix}u_{x} = \exp \left( \sum_j u_jA_{ji} \right) \), where \( A_{ji} \) is the Cartan matrix for the affine algebra \( A_4^{(2)} \) with the following Dynkin diagram \( \circ \Rightarrow \circ \Rightarrow \circ \Rightarrow \circ \). Substitution \( u = 2u_1 - u_2, v = 2u_3 - u_2 \) results in system (26) with \( c_1 = 2, c_2 = -1, c_3 = 1 \). System (26) can be rewritten in several different forms. For example, the functions \( p = w_1, q = w_2 \) satisfy the system:

\[
\begin{align*}
    p_{rx} &= p_x(c_1p + c_2q), & q_{rx} &= -q_x(c_1p + 2c_2q + c_3w), & w_x &= (p_xq_x)^{-2}.
\end{align*}
\]

2.b. If \( c_i = 0, i > 4, c_4 = 1 \), then double differentiation of system (25) gives the following local system:

\[
\begin{align*}
    u_{rxx} &= -u_{rx}(u_x + v_x) + c(3u_x + q_x)e^{2u} + e^{-u}, & (c = c_3), \\
    v_{rxx} &= v_xv_{rx} - u_{rx}(u_x + 2v_x) + c(u_x + 2v_x)e^{2u} + 2e^{-u}.
\end{align*}
\]

(27)

It is obvious that the order of the second equation can be decreased by the substitution \( v_x \rightarrow v \). If, under the previous conditions, one chooses \( p = w_1 \) and \( q = w_2 \) as new unknown functions, then another system follows:

\[
\begin{align*}
    p_{rx} &= p_x(c_1p + c_2q + pq), & (\ln q_x)_{rx} &= p_x(c_1 + q) - c_3p_x^{-2}q_x^{-2}.
\end{align*}
\]
2.c. If \( c_4 = 0, c_5 = 1, c_i = 0, i > 5 \), then double differentiation of system (25) gives the following local system:

\[
\begin{align*}
    u_{txx} &= 2u_x u_{tx} - v_{tx}(3u_x + v_x) + c(3u_x + v_x)e^v + 3e^{u-v}, \\
    v_{txx} &= -v_{tx}(u_x + v_x) + c(u_x + 2v_x)e^v + e^{u-v}, \quad (c = c_1).
\end{align*}
\]  

(28)

2.d. If \( c_6 = 1 \) and \( c_i = 0, i > 4 \), then the following local system follows

\[
\begin{align*}
    u_{txx} &= -2u_{tx}(u_x + q_x) + 2c(2u_x + q_x)e^{2u} + 2\sqrt{u_{tx} e^{2q} + be^{-2u} - cc^2(u+q)}, \\
    q_{txx} &= q_{tx}(u_x + 2q_x) + \frac{1}{2}c(2q_x - u_x)e^{2u} + \sqrt{u_{tx} e^{2q} + be^{-2u} - cc^2(u+q)},
\end{align*}
\]  

(29)

where \( b = c_4^2/4 - c_2, c = c_3, q = (v - u)/2 \). Notice that in the case \( b = c = 0 \) the order of the first equation can be decreased by the substitution \( u_x \to u \).

3. The next system

\[
    u_t = u_3 + u_1 v_2 - u_1 v_1^2, \quad v_t = (u_2 u_1 + u_1^2 v_1),
\]  

(30)

admits of the following nonlocal symmetry

\[
\begin{align*}
    u_r &= c_1 u_1 + c_2 u_2 + c_4 (w_4 - w_1 w_3) + c_5 (w_5 - w_1 w_4) \\
    &\quad + c_6 (w_6 - w_2 w_3) + c_7 (w_1 w_6 - w_2 w_4) + c_8 (w_8 - w_2 w_6), \\
    v_r &= -c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 (w_4 + w_1 w_3) + c_5 w_1 w_4 \\
    &\quad - c_6 (w_6 + w_2 w_3) - c_7 (w_1 w_6 + w_2 w_4) - c_8 w_2 w_6,
\end{align*}
\]  

(31)

where

\[
\begin{align*}
    w_1 &= D_x^{-1} e^{u-v}, \quad w_2 = D_x^{-1} e^{u-v}, \quad w_3 = D_x^{-1} e^{2v}, \quad w_4 = D_x^{-1} w_1 e^{2v}, \\
    w_5 &= D_x^{-1} w_1^2 e^{2v}, \quad w_6 = D_x^{-1} w_2 e^{2v}, \quad w_7 = D_x^{-1} w_1 w_2 e^{2v}, \quad w_8 = D_x^{-1} w_2^2 e^{2v}.
\end{align*}
\]

3.a. In the case \( c_i = 0, i > 3 \) the following Toda lattice is obtained:

\[
    u_{tx} = c_1 e^{u-v} + c_2 e^{-u-v}, \quad v_{tx} = -c_1 e^{u-v} + c_2 e^{-u-v} + c_3 e^{2v}.
\]  

(32)

In the new variables \( p = u - v, q = -u - v \) this system takes the form \( p_{tx} = 2c_1 e^p - 2c_2 e^{-p-q}, q_{tx} = -2c_2 e^q - 2c_3 e^{-p-q} \). This allows to connect system (32) with the affine algebra \( C_2^{(1)} \) having the following Dynkin diagram \( \circ \Rightarrow \circ \Leftarrow \circ \). In the case of \( c_3 = 0 \) this system decomposes into a pair of the Liouville equations obviously.

3.b. If \( c_4 \neq 0, c_i = 0, i > 4 \), then in the terms of new variables \( p = w_1, q = w_3 \) system (31) takes the following form:

\[
    p_{tx} = p_x (2c_1 p - c_3 q - 2c_4 p q), \quad (q_x^{-1} q_{tx})_x = 2c_3 q_x - 2c_1 p_x + 4c_4 p q_x + 2c_4 p q_x + 2c_2 p_x^{-1} q_x^{-1}.
\]

If one simply doubly differentiates system (31), then the result is

\[
    u_{txx} = u_{txx}(2u_x - p_x) - 2c_2 u_x e^{-p} - c_4 e^p, \quad p_{txx} = 2p_{tx}(p_x - u_x) + 2c_2 e^{-p}(2u_x - 3p_x) + 2c_2 e^p,
\]

and one obtains another pair of Liouville equations.
where \( p = u + v \). Here the order of the first equation can be decreased by the substitution \( u_x \to u \).

3.c. If \( c_5 \neq 0 \) and the other constants \( c_i = 0, i > 3 \), then a double differentiation of system (31) gives

\[
\begin{align*}
 u_{txx} &= u_{tx}(2u_x + q_x) - 2c_2u_x e^q + \sqrt{aq_{rx}e^{2u} + b e^{-2q} + 2ac_2e^{2u+ q}}, \\
 q_{txx} &= -2q_{rx}(u_x + q_x) - 2c_2(2u_x + 3q_x) e^q + 2\sqrt{aq_{rx}e^{2u} + b e^{-2q} + 2ac_2e^{2u+ q}},
\end{align*}
\]

(33)

where \( a = -c_5, b = -c_3c_5, q = -u - v \).

Notice that systems (29) and (33) coincide when \( c_2 = c_3 = c_4 = 0 \). This is surprising, because the symmetries of these systems, i.e. systems (24) and (30), are entirely different. A possible explanation is as follows. The system

\[
\begin{align*}
 u_{txx} &= u_{tx}(2u_x + q_x) + e^u \sqrt{q_{tx}}, \\
 q_{txx} &= -2q_{rx}(u_x + q_x) + 2e^u \sqrt{q_{tx}}
\end{align*}
\]

is Liouvillean and possesses a double sequence of symmetries that are constructed by different integrals.

4. The next system

\[
\begin{align*}
 u_t &= u_3 + v_1v_2 - \frac{1}{2} u_1^3 + \frac{1}{2} u_1v_1^2 + c_1v_1, \\
 v_t &= u_2v_1 - \frac{1}{2} u_1^3v_1 + \frac{1}{2} v_1^3 - c_1u_1 + c_2v_1
\end{align*}
\]

(34)

contains two essential constants that affect the form and quantity of admissible symmetries. It becomes clear if one takes into account that system (34) can be obtained from the Ito system by a differential substitution depending on \( c_1 \) and \( c_2 \) (see [1]). If \( c_1 = 0 \), then the differential substitution is essentially simplified, and the differential substitution vanishes when \( c_1 = c_2 = 0 \).

4.1. If \( c_1 = 0, c_2 = 0 \), then system (34) is degenerate. In fact, the substitution \( u = \ln U_x, v = V_xU_x^{-1} \) gives

\[
\begin{align*}
 U_t &= U_{xxx} - \frac{3}{2} U_{xx}^2 + \frac{V_xV_{xx}U_{xx}}{U_x^2} + \frac{1}{2} V_x^2U_{xx}^2 + \frac{V_x^2}{2 U_x}, \\
 V_t &= \frac{V_xU_t}{U_x}.
\end{align*}
\]

Hence, \( V = F(U) \) and we have:

\[
U_t = U_{xxx} - \frac{3}{2} U_{xx}^2 + \frac{1}{2} (F')^2 U_x^3.
\]

This equation is exactly integrable iff \( F^{IV} = 0 \) (see [7]). Hence, system (34) with \( c_1 = 0, c_2 = 0 \) is not integrable in the general case.

4.2. If \( c_1 = 0, c_2 \neq 0 \), then system (34) admits of the following nonlocal 4-parametric symmetry

\[
\begin{align*}
 u_x &= k_1w_1 + k_2w_2 + k_3(w_5 + 2w_2w_3) + k_4(w_1w_5 - 4w_1w_2 + 2w_2w_4), \\
 v_x &= -2k_2e^{-u}v_x - 4k_3e^{-u}v_x(2 + w_3) - 4k_4e^{-u}v_xw_4,
\end{align*}
\]

(35)
where
\[ w_1 = D_x^{-1}e^u, \quad w_2 = D_x^{-1}e^{-u}(v_x^2 + c_2), \quad w_3 = D_x^{-1}e^uw_2, \]
\[ w_4 = D_x^{-1}e^uw_2w_1, \quad w_5 = D_x^{-1}(4v_x^2e^{-u} - e^uw_2^2). \]

4.2.a. If \( k_i = 0, i > 2 \), then differentiation of system (35) gives:
\[ u_{rx} = k_1e^u + k_2(v_x^2 + c_2)e^{-u}, \quad v_r = -2k_2vxe^{-u}. \]

4.2.b. If \( k_3 \neq 0, k_4 = 0 \), then one can obtain \( k_3 = 1/4 \) by a dilatation of \( \tau \). In this case differentiation of system (35) gives the following system:
\[ u_{rx} = \frac{1}{4}e^u \left( \left( \frac{v_r}{v_x} \right)_x + \frac{v_ru_x}{v_x} \right)^2 - \frac{1}{2}v_r(v_x + c_2v_x^{-1}) + k_1e^u - c_2e^{-u}, \]
\[ (\ln v_x)_{rx} = \left[ \frac{v_r}{v_x^2}(v_{xx} - u_xv_x) \right]_x - (v_x^2 + c_2)e^{-u}. \]

4.3. If \( c_1 \neq 0 \) in (34), then a dilatation of \( v, t \) and \( x \) gives \( c_1 = 1 \):
\[ u_t = u_3 + v_1v_2 - \frac{1}{2}u_1^2 + \frac{1}{2}u_1v_1^2 + v_1, \quad v_t = u_2v_1 - \frac{1}{2}u_1^2v_1 + \frac{1}{2}v_1^3 - u_1 + c_2v_1, \] \( (34a) \)
There are three cases for three different values of \( c_2 \).

4.3.a. \( c_2 = -2\varepsilon, \varepsilon = \pm 1 \). In this case system (34a) admits the following nonlocal symmetry:
\[ u_r = k_1w_1 - k_2w_2 + k_3w_3 + k_4w_4 + k_5w_1w_3 + k_6(6w_1w_5 - 6w_3w_6 - 3w_1 + w_8), \]
\[ v_r = (k_2 - \varepsilon k_1)w_1 + \varepsilon k_2w_2 + \varepsilon(k_3 + k_5w_1)(2v_xe^{-u+\varepsilon v} - w_3) \]
\[ + k_4(w_3 + 2\varepsilon v_xe^{-u-\varepsilon v} - \varepsilon w_4 - 2v_xe^{-u-\varepsilon v}) \]
\[ + 4k_6e^{-u-\varepsilon v}(6v_xv + v_xv^3 + 3\varepsilon v_xv_w_5 - 3v_xw_5 - 6\varepsilon v_xv^2 - 3\varepsilon v_xw_0) \]
\[ + 3k_6(\varepsilon w_1 + 2w_3w_5 + w_7 - 4w_4 + 2\varepsilon w_3w_6 - 2\varepsilon w_4w_5) - k_6\varepsilon w_8. \] \( (36) \)

Here,
\[ w_1 = D_x^{-1}e^{u+\varepsilon v}, \quad w_2 = D_x^{-1}ve^{u+\varepsilon v}, \quad w_3 = D_x^{-1}e^{-u-\varepsilon v}(1 - \varepsilon v_x^2), \quad w_5 = D_x^{-1}w_3e^{u+\varepsilon v}, \]
\[ w_4 = D_x^{-1}e^{-u-\varepsilon v}(v + v_x^2(1 - \varepsilon v)), \quad w_6 = D_x^{-1}e^{u+\varepsilon v}(vw_3 - w_4), \]
\[ w_7 = D_x^{-1}(2ve^{-u-\varepsilon v}(v + v_x^2(\varepsilon v - v)) - e^{u+\varepsilon v}w_3^2), \]
\[ w_8 = D_x^{-1}(2ve^{-u-\varepsilon v}(v^2(\varepsilon - v_x^2) - 3v + 6v_x^2(\varepsilon v - 1)) + 3e^{u+\varepsilon v}w_3(vw_3 - 2w_4)). \]

If \( k_5 = k_6 = 0 \), then differentiation of system (35) gives the following system:
\[ u_{rx} = (k_1 - k_2v)e^{u+\varepsilon v} + k_3e^{-u-\varepsilon v}(1 - \varepsilon v_x^2) + k_4e^{-u-\varepsilon v}(v + v_x^2(1 - \varepsilon v)), \]
\[ v_{rx} = (k_2 - \varepsilon k_1 + \varepsilon k_2v)e^{u+\varepsilon v} - \varepsilon k_3 e^{-u-\varepsilon v}(2u_xv_x - 2v_{xx} + \varepsilon v_x^2 + 1) \]
\[ + k_4e^{-u-\varepsilon v}((2v_{xx} - 2u_xv_x - 1)(\varepsilon v - 1) + v_x^2(2\varepsilon v - v)). \]

If one set here \( k_3 = k_4 = 0 \), then the triangle system \((u + \varepsilon v)_{rx} = \varepsilon k_2e^{u+\varepsilon v}\) follows.
If \( k_5 = \varepsilon/2 \), and the other constants \( k_i = 0 \), then the following local system follows

\[
p_{tx} = \varepsilon p\sqrt{\varepsilon - \varepsilon p_x q_x},
q_{tx} = 2(q_x p_x - 1)p_x^{-1} + pp_{xx}(2 - q_x p_x)p_x^{-3} + pp_x^{-1}q_{xx} + \varepsilon q\sqrt{\varepsilon - \varepsilon p_x q_x},
\]

(37)

where \( p = w_1, q = w_3 \). Other combinations of constants in (36) give very cumbersome systems.

4.3.4. If \( c_2 = -2k, \ |k| < 1 \), then system (34) possesses the following nonlocal symmetry:

\[
u_t = k_1 w_1 - k_2 w_2 + k_3 w_3 + k_4 w_4 + k_5(w_4 w_1 + w_3 w_2) + k_6(-w_4 w_2 + w_1 w_3) \\
- k_7(-w_7 + 2c^2 w_3 w_6 - 2c^2 w_4 w_5) + k_8(2c^2 w_4 w_6 + 2c^2 w_3 w_5 + w_8),
\]

\[
v_r = k_1(c w_2 - k w_1) + k_2(k w_2 + c w_1) + k_3\left(2v_x \sin(c v)e^{-u-kv} + w_4 c - k w_3\right) \\
+ k_4\left(2v_x \cos(c v) e^{-u-kv} - w_3 c - k w_4\right) + 2v_x k_5\left(w_1 \cos(c v) + w_2 \sin(c v)\right)e^{-u-kv} \\
- k_5(k w_3 w_2 + c w_3 w_1 - c w_4 w_2 + k w_4 w_1) + 2v_x k_6\left(w_1 \sin(c v) - w_2 \cos(c v)\right)e^{-u-kv} \\
+ k_6(c w_1 w_4 + c w_2 w_3 - k w_1 w_3 + k w_2 w_4) \\
- 2v_x k_7\left(\sin(c v)(1 - 2c^2 + 2c^2 w_6) - 2c \cos(c v)(v + k - c w_5)\right)e^{-u-kv} \\
+ k_7(2kc^2(w_3 w_6 - w_4 w_5) - 2c^3(w_4 w_6 + w_3 w_5) + 2c^4 - k w_2 - c w_8) \\
+ 2k_8 v_x\left(\cos(c v)(1 - 2c^2 + 2c^2 w_6) + 2c \sin(c v)(c w_5 - k + v)\right)e^{-u-kv} \\
+ k_8\left(2c^3(w_4 w_5 - w_3 w_6) - 2kc^2(w_4 w_6 + w_3 w_5) + c w_7 + 2c w_3 + k w_8\right).
\]

Here \( c = \sqrt{1-k^2} \) and

\[
w_1 = D_x^{-1} \cos(c v)e^{u+kv}, \quad w_3 = D_x^{-1}\left(\cos(c v - \alpha) - v_x^2 \sin(c v)\right)e^{-u-kv},
\]

\[
w_2 = D_x^{-1} \sin(c v)e^{u+kv}, \quad w_4 = D_x^{-1}\left(\sin(\alpha - c v) - v_x^2 \cos(c v)\right)e^{-u-kv},
\]

\[
w_5 = D_x^{-1}(w_4 \sin(c v) - w_3 \cos(c v))e^{u+kv}, \quad w_6 = D_x^{-1}(w_4 \cos(c v) + w_3 \sin(c v))e^{u+kv},
\]

\[
w_7 = D_x^{-1}\left(c^2 e^{u+kv}(\sin(c v)(w_3^2 - w_1^2) + 2w_3 w_4 \cos(c v)) + \\
+ e^{-u-kv}(c \cos(c v)(2kv_x^2 - 2vv_x^2 + 2kv - 1) - \sin(c v)(2c^2 v_x^2 - v_x^2 + 2c^2 v))\right),
\]

\[
w_8 = D_x^{-1}\left(c^2 e^{u+kv}(\cos(c v)(w_3^2 - w_1^2) - 2w_4 w_3 \sin(c v)) + \\
+ e^{-u-kv}(c \cos(c v)(2c^2 v_x^2 - v_x^2 + 2c^2 v + k) + c \sin(c v)(2kv_x^2 - 2vv_x^2 + 2kv - 1))\right).
\]

\[
k = \sin(\alpha), \quad c = \cos(\alpha), \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.
\]

Simple local equations exist under conditions \( k_i = 0, i > 4 \) only:

\[
u_{tx} = (k_1 \cos(c v) + k_2 \sin(c v))e^{u+kv} + k_3 v_x^2 \sin(c v + \alpha)e^{-u-kv} \\
- k_4 v_x^2 \cos(c v + \alpha)e^{-u-kv} - (k_3 \cos(c v) + k_4 \sin(c v))e^{-u-kv},
\]

\[
v_{tx} = (k_1 \sin(c v - \alpha) - k_2 \cos(c v - \alpha))e^{u+kv} \\
+ (2v_x u_x - 2v_{xx} + 1)(k_3 \sin(c v + \alpha) - k_4 \cos(c v + \alpha))e^{-u-kv} \\
- v_x^2(k_3 \cos(c v + 2\alpha) + k_4 \sin(c v + 2\alpha))e^{-u-kv}.
\]

(39)
If \( k_3 = k_4 = 0 \), then this system decomposes into two Liouville equations in the terms of variables \( p = u + ie^{-i\theta}v, q = u - ie^{i\theta}v \).

**4.3.c.** If \( c_2 = -a - a^{-1}, \ |a| \neq 1 \), then system \((34)\) possesses the following nonlocal symmetry:

\[
\begin{align*}
 u_r &= -ak_1w_1 + k_2w_2 + ak_3w_3 + k_4w_4 + ak_5w_5 + k_6w_6w_4 \\
 &+ k_7(w_7 + 2w_4w_5(a^2 - 1)^2) + ak_8(2w_3w_6(a^2 - 1)^2 + w_8), \\
 v_r &= k_1w_1 - ak_2w_2 - k_3(w_3 - 2av_xe^{-u-v/a}) - ak_4(w_4 - 2v_xe^{-u-v}) \\
 &+ k_5w_1(-w_3 + 2ae^{-u-v/a}v_x) + k_6w_2a(2v_xe^{-u-v} - w_4) \\
 &- ak_7(w_7 + 2w_4(a^2 - 1)(w_5a^2 + 2a - w_5)) \\
 &- 4ak_7e^{-u-v}v_x(2a^2v(a^2 - 1) + 2a - w_5(a^2 - 1)^2) \\
 &+ k_8(2w_3(a^2 - 1)(w_6 - a^2w_6 + 2a^2) - w_8) \\
 &+ 4ak_8e^{-u-v/a}v_x(w_6(a^2 - 1)^2 + 2av(a^2 - 1) - 2a^4)).
\end{align*}
\]

Here,

\[
\begin{align*}
 w_1 &= D_x^{-1}e^{u+v/a}, \quad w_2 = D_x^{-1}e^{u+av}, \quad w_3 = D_x^{-1}e^{-u-v/a}(a - v_x^2), \\
 w_4 &= D_x^{-1}e^{-u-av}(1 - av_x^2), \quad w_5 = D_x^{-1}w_4e^{u+av}, \quad w_6 = D_x^{-1}w_3e^{u+av} \\
 w_7 &= D_x^{-1}(4a^2e^{-u-av}(v - a^2v - a + v_x^2(a^3v - av + 1)) - e^{u+av}w_4^2(a^2 - 1)^2), \\
 w_8 &= D_x^{-1}(4ae^{-u-v/a}(a^3v - a^2 - av + v_x^2(-a^2v + a^3 + v)) - e^{u+v/a}w_3^2(a^2 - 1)^2). 
\end{align*}
\]

If \( k_i = 0, i > 4 \), then the following local system follows:

\[
\begin{align*}
 u_{rx} &= -ak_1e^{u+v/a} + k_2e^{u+av} + ak_3e^{-u-v/a}(a - v_x^2) + k_4e^{-u-av}(1 - av_x^2), \\
 v_{rx} &= k_1e^{u+v/a} - ak_2e^{u+av} - k_3e^{-u-v/a}(2au_xv_x - 2av_{xx} + v_x^2 + a) \\
 &- ak_4e^{-u-av}(2uxv_x - 2v_{xx} + av_x^2 + 1).
\end{align*}
\]

If \( k_3 = k_4 = 0 \) then this system decomposes into two Liouville equations in the terms of variables \( p = u + av, q = u + v/a \).

If \( k_5 = a \) and the other constants \( k_i = 0 \), then the following local system follows

\[
\begin{align*}
 p_{rx} &= pqp_x(a^2 - 1) + 2ap\sqrt{a + p_xq_x}, \\
 q_{rx} &= 4a^2(a + p_xq_x)p_x^{-1} + 2a^2p_x^{-1}q_{xx} - 2ap_x^{-1}(qp_x + 2pq_xp_{xx}) \sqrt{a + p_xq_x} \\
 &- 2a^2p_{xx}(p_xq_x + 2a)p_x^{-3} + (1 - a^2)pqq_x,
\end{align*}
\]

where \( p = w_1, q = w_3 \). Other combinations of the constants in \((40)\) give more cumbersome systems.

Notice that all formulas from points 4.3.b and 4.3.c are connected with each other by the transformation \( a = k + ic, a^{-1} = k - ic, c = \sqrt{1 - k^2} \). All formulas from point 4.3.a can be obtained from corresponding formulas of point 4.3.b as the limit \( k \to \varepsilon = \pm 1, c \to 0. \)
But these calculations are very cumbersome. In particular, system (41) is reduced into (37) under the substitution $a = \varepsilon = \pm 1$, $q \rightarrow -\varepsilon q$

All remaining systems found in [1] have no nonlocal symmetries or have trivial nonlocal symmetries that lead to the Liouville equation.

4 Zero curvature representations

We present here the matrices $U$ and $V$ realizing zero curvature representations

$$U_T - V_x + [U, V] = 0$$

for some of the systems connected with (24). Spectral parameter is denoted as $k$ everywhere.

System (24) can be obtained from the Drinfeld-Sokolov system [5]

$$m_t = m_3 - 3m_3 - 3m_x(4m - 9n) + 3n_x(8m - 15n),$$
$$n_t = -3m_3 + 4n_3 + 12m_xn + 6n_x(m - 4n)$$

by the following differential substitution:

$$m = u_x^2 + \frac{1}{2}v_x^2 - u_2 - v_2, \quad n = u_x^2 - u_2.$$  \hspace{1cm} (43)

First, we write the matrices $U_0$, $V_0$ that form the zero curvature representation for system (42):

$$U_0 = \begin{pmatrix} 0 & 1 & n-m & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & m-n \\ 0 & n & 0 & 0 & k \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} h_{1,x} & h_2 & f_1 & 0 & -5k \\ 0 & h_{3,x} & -5k & -2h_3 & 0 \\ -h_1 & 0 & 0 & 5 & -f_1 \\ 5k & f_2 & 0 & -h_{3,x} & kh_2 \\ 0 & 5 & h_1 & 0 & -h_{1,x} \end{pmatrix}. \hspace{1cm} (44)$$

Here,

$$h_1 = 7n - 4m, \quad h_2 = m - 3n, \quad h_3 = 4n - 3m,$$

$$f_1 = -4m_2 + 7n_2 + 4m^2 + 7n^2 - 11mn,$$

$$f_2 = -3m_2 + 4n_2 - 8n^2 + 6mn.$$

Matrices (44) are embedded in $sl(5, \mathbb{C})$.

Performing substitution (43) in matrices (44) and excluding $u_2$ and $v_2$ from $U_0$ by a gauge transformation $U = S^{-1}(U_0S - S_x)$, $V = S^{-1}(V_0S - S_t)$, we obtain the zero curvature representation for system (24):

$$U = \begin{pmatrix} v_x & 1 & 0 & 0 & 0 \\ 0 & -v_x & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_x & k \\ 0 & 0 & 1 & 0 & -v_x \end{pmatrix}, \quad V = \begin{pmatrix} \varphi_1 & \varphi_2 & 0 & -5v_x & -5k \\ 0 & \varphi_3 & -5k & \varphi_4 & 5kv_x \\ -\varphi_5 & 5h & 0 & 5 & 0 \\ 5k & 0 & 5h & -\varphi_3 & k\varphi_2 \\ 0 & 5 & \varphi_5 & 0 & -\varphi_1 \end{pmatrix}. \hspace{1cm} (45)$$
Here,
\[
\begin{align*}
\varphi_1 &= 4v_3 - 3u_3 + 3u_2(2u_x - v_x) + 3v_xu_x^2 - 2v_x^3, \\
\varphi_2 &= 2u_x - v_x - 2u_x^2 - 2v_x^2 + 5u_xv_x, \\
\varphi_3 &= 3v_3 - u_3 + 3v_2(2u_x - v_x) - 3u_xv_x^2 + 2u_x^3, \\& h = v_x - u_x, \\
\varphi_4 &= 2u_2 - 6v_2 - 2u_x^2 + 3v_x^2, \\
\varphi_5 &= 4v_2 - 3u_2 + 3u_x^2 - 2v_x^2.
\end{align*}
\]

The system \( \Psi_x = U\Psi \), where \( U \) takes the form \((45)\), can be reduced to the following single equation
\[
(\partial_x - u_x)(\partial_x + u_x)(\partial_x - v_x)\partial_x(\partial_x - v_x)\Psi_5 + k\Psi_5 = 0.
\]

The spectral problem for this equation is obviously nontrivial.

System \((24)\) is presented in \([5]\), but in another form (see table 5, \(A^{(2)}_1\)). The zero curvature representations for this system and corresponding Toda lattice are contained in the same paper. But it was simpler for us to compute these zero curvature representations anew.

Matrix \( U \) for the Toda lattice \((26)\) is shown in \((45)\) and \( V \) takes the following form:
\[
V = \begin{pmatrix}
0 & 0 & -c_1e^v & 0 & 0 \\
-c_2e^{-u-v} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1e^v \\
0 & c_3e^{2u} & 0 & 0 & 0 \\
0 & 0 & 0 & -k^{-1}c_2e^{-u-v} & 0
\end{pmatrix}.
\]

We have assumed that systems \((27) - (29)\) belong to the same hierarchy as system \((24)\).

If this is true, the matrix \( U \) is common for all mentioned systems. The calculations have confirmed our assumption and we present below only the matrices \( V \) for the mentioned systems.

For system \((27)\):
\[
V = \begin{pmatrix}
0 & 0 & u_{tx} - v_{tx} + ce^{2u} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -k^{-1}e^{-u} & v_{tx} - u_{tx} + ce^{2u} \\
0 & 0 & ce^{2u} & 0 & 0 \\
0 & 0 & 0 & k^{-1}(ce^{2u} - u_{tx}) & 0
\end{pmatrix}.
\]

For system \((28)\):
\[
V = \begin{pmatrix}
0 & 0 & -ce^v & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ce^v \\
0 & 0 & u_{tx} - v_{tx} + ce^v & 0 & 0 \\
0 & 0 & k^{-1}e^{u-v} & 0 & k^{-1}(ce^v - v_{tx})
\end{pmatrix}.
\]

For system \((29)\):
\[
V = \begin{pmatrix}
0 & 0 & -2q_{tx} - ce^{2u} & 2k^{-1}e^{2q} & 0 \\
0 & 0 & 0 & -2k^{-1}e^{-2q} & 2e^{2q} \\
0 & 0 & ce^{2u} & 0 & 0 \\
0 & 0 & 0 & k^{-1}(ce^{2u} - u_{tx}) & 0
\end{pmatrix}.
\]
Here \( r = \sqrt{u_{xx} e^{2q} + be^{-2u} - ce^{2(u+q)}} \) and the substitution \( v = u + 2q \) must be performed in the matrix \( U \) (see (45)).

**Conclusion**

As it was mentioned above, each nonlocal symmetry presented in this paper is a symmetry for the system under consideration as well as for its higher analogue. This gives grounds to believe that all presented systems are exactly integrable. But this assumption must be proved, of course. Such proofs have been presented for systems (27) – (29). For other systems this problem should be further investigated.

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