LARGE TIME BEHAVIOR OF SOLUTIONS OF THE HEAT EQUATION WITH INVERSE SQUARE POTENTIAL

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Abstract. Let $L := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbb{R}^N)$, where $N \geq 2$ and $V$ is a radially symmetric inverse square potential. In this paper we assume either $L$ is subcritical or null-critical and we establish a method for obtaining the precise description of the large time behavior of $e^{-tL}\varphi$, where $\varphi \in L^2(\mathbb{R}^N, e|x|^2/4 \, dx)$.

1. Introduction. Let $L := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbb{R}^N)$, where $N \geq 2$ and $V$ is a radially symmetric inverse square potential, that is

$$V(r) = \lambda_1 r^{-2} + o(r^{-2+\theta}) \quad \text{as} \quad r \to 0,$$

$$V(r) = \lambda_2 r^{-2} + o(r^{-2-\theta}) \quad \text{as} \quad r \to \infty,$$

for some $\lambda_1, \lambda_2 \in [\lambda_*, \infty)$ with $\lambda_* := -(N-2)^2/4$ and $\theta > 0$. We are interested in the precise description of the large time behavior of $u = e^{-tL}\varphi$, which is a solution of

$$\begin{cases}
\partial_t u - \Delta u + V(|x|)u = 0, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = \varphi(x), & x \in \mathbb{R}^N.
\end{cases}$$

(1.1)

Nonnegative Schrödinger operators and their heat semigroups appear in various fields and have been studied intensively by many authors since the pioneering work due to Simon [34] (see e.g., [1], [2], [4], [6], [9], [10], [13]–[19], [21]–[25], [26], [27], [29]–[37] and references therein). See also the monographs of Davies [5], Grigor’yan [7] and Ouhabaz [28]. The inverse square potential is a typical one appearing in the study of the Schrödinger operators and it arises in the linearized analysis for nonlinear diffusion equations and in the asymptotic analysis for diffusion equations.
Throughout this paper we assume the following condition on the potential $V$:

\[
(V) \begin{cases} 
(i) & V = V(r) \in C^1((0, \infty)); \\
(ii) & \lim_{r \to 0^+} r^{-\theta} |r^2 V(r) - \lambda_1| = 0, \quad \lim_{r \to \infty} r^{\theta} |r^2 V(r) - \lambda_2| = 0, \\
& \text{for some } \lambda_1, \lambda_2 \in [\lambda_*, \infty) \text{ with } \lambda_* := -(N-2)^2/4 \text{ and } \theta > 0; \\
(iii) & \sup_{r \geq 1} |r^3 V'(r)| < \infty.
\end{cases}
\]

We say that $L := -\Delta + V(|x|)$ is nonnegative on $L^2(\mathbb{R}^N)$ if

\[
\int_{\mathbb{R}^N} [|
abla \phi|^2 + V(|x|)\phi^2] \, dx \geq 0, \quad \phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}).
\]

When $L$ is nonnegative, we say that

- $L$ is subcritical if, for any $W \in C_0(\mathbb{R}^N)$, $L - \epsilon W$ is nonnegative for all sufficiently small $\epsilon > 0$;
- $L$ is critical if $L$ is not subcritical.

On the other hand, $L$ is said to be supercritical if $L$ is not nonnegative.

Consider the ordinary differential equation

\[
U'' + \frac{N-1}{r} U' - V(r)U = 0 \quad \text{in} \quad (0, \infty)
\]

under condition (V). Equation (O) has two linearly independent solutions $U$ (a regular solution) and $\tilde{U}$ (a singular solution) such that

\[
U(r) \sim r^{A^+(\lambda_1)}, \quad \tilde{U}(r) \sim \begin{cases} 
 r^{A^-(\lambda_1)} & \text{if } \lambda_1 > \lambda_*, \\
 r^{-\frac{N-2}{2}} \log r & \text{if } \lambda_1 = \lambda_*,
\end{cases}
\]

as $r \to +0$, where

\[
A^\pm(\lambda) := \frac{(N-2) \pm \sqrt{(N-2)^2 + 4\lambda}}{2} \quad \text{for} \quad \lambda \geq \lambda_*.
\]

In particular, $U \in L^2_{\text{loc}}(\mathbb{R}^N)$. Assume that $L$ is nonnegative on $L^2(\mathbb{R}^N)$. Then it follows from [19, Theorem 1.1] that $U$ is positive in $(0, \infty)$ and

\[
U(r) \sim c_* v(r) \quad \text{as} \quad r \to \infty
\]

for some positive constant $c_*$, where

\[
v(r) := \begin{cases} 
 r^{A^+(\lambda_2)} & \text{if } L \text{ is subcritical and } \lambda_2 > \lambda_*, \\
 r^{-\frac{N-2}{2}} \log r & \text{if } L \text{ is subcritical and } \lambda_2 = \lambda_*, \\
 r^{A^-(\lambda_2)} & \text{if } L \text{ is critical},
\end{cases}
\]

(See also [27] for the case $\lambda_1 = 0$.) We often call $U$ a positive harmonic function for the operator $L$. When $L$ is critical, following [32], we say that $L$ is positive-critical if $U \in L^2(\mathbb{R}^N)$ and that $L$ is null-critical if $U \notin L^2(\mathbb{R}^N)$. Generally, the behavior of the fundamental solution $p = p(x, y, t)$ corresponding to $e^{-tL}$ can be classified by whether $L$ is either subcritical, null-critical or positive-critical. Indeed, in the case of $\lambda_1 = 0$, by [32, Theorem 1.2] we have:

(L1) If $L$ is subcritical, then

\[
\lim_{t \to \infty} p(x, y, t) = 0, \quad \int_0^\infty p(x, y, t) \, dt < \infty,
\]

for $x, y \in \mathbb{R}^N$ with $x \neq y$;
(L2) If $L$ is null-critical, that is $A^-(\lambda_2) \geq -N/2$, then
\[
\lim_{t \to \infty} p(x, y, t) = 0, \quad \int_0^{\infty} p(x, y, t) \, dt = \infty,
\]
for $x, y \in \mathbb{R}^N$ with $x \neq y$;

(L3) If $L$ is positive-critical, that is $A^-(\lambda_2) < -N/2$, then
\[
\lim_{t \to \infty} p(x, y, t) = \frac{U(|x|)U(|y|)}{||U||^2_{L^2(\mathbb{R}^N)}}, \quad x, y \in \mathbb{R}^N.
\]

See Corollary 1.1 for (L1) and (L2) in the case of $\lambda_1 \neq 0$.

On the other hand, under condition (V), the first author of this paper with Kabeya and Ouhabaz recently studied in [19] the Gaussian estimate of the fundamental solution $p = p(x, y, t)$ in the subcritical case and in the critical case with $A^-(\lambda_2) > -N/2$. They proved that
\[
0 < p(x, y, t) \leq C' t^{-\frac{N}{2}} \frac{U(\min\{|x|, \sqrt{t}\})U(\min\{|y|, \sqrt{t}\})}{U(\sqrt{t})^2} \exp\left(-\frac{|x-y|^2}{C't}\right) \quad (1.6)
\]
holds for all $x, y \in \mathbb{R}^N$ and $t > 0$, where $C'$ is a positive constant (see [19, Theorem 1.3]). For related results, see e.g., [1], [4], [8], [21], [22], [24], [25], [36], [37] and references therein.

Only in the subcritical case, the precise description of the large time behavior of $e^{-tL} \varphi$ with $\varphi \in L^2(\mathbb{R}^N, e^{\frac{|x|^2}{4} dx})$ has been studied in a series of papers [13]–[16] with some additional restrictions such as $V \in C^1([0, \infty))$, $\lambda_2 > \lambda_*$ and the sign of the potential. See also [17].

The purpose of this paper is to establish a method for obtaining the precise description of the large time behavior of $e^{-tL} \varphi$ with $\varphi \in L^2(\mathbb{R}^N, e^{\frac{|x|^2}{4} dx})$ in the subcritical case and in the null-critical case with $A^-(\lambda_2) > -N/2$, under condition (V). In particular, we show that the solution $u$ of (1.1) behaves as a suitable multiple of
\[
\begin{cases}
v_{\text{reg}}(x, t) & \text{if } L \text{ is subcritical and } \lambda > \lambda_* , \\
(\log t)^{-1}v_{\text{reg}}(x, t) & \text{if } L \text{ is subcritical and } \lambda = \lambda_* , \\
v_{\text{sing}}(x, t) & \text{if } L \text{ is critical and } A^-(\lambda_2) > -N/2 ,
\end{cases}
\]
as $t \to \infty$ on all parabolic cones $\{x \in \mathbb{R}^N : R^{-1}t^{1/2} \leq |x| \leq R t^{1/2}\}$ with $R > 1$. (See Theorem 1.4.) Here
\[
\begin{align*}
v_{\text{reg}}(x, t) & := t^{-\frac{N+2A^+(\lambda_2)}{4}} |x|^{A^+(\lambda_2)} \exp\left(-\frac{|x|^2}{4t}\right), \\
v_{\text{sing}}(x, t) & := t^{-\frac{N+2A^-(\lambda_2)}{4}} |x|^{A^-(\lambda_2)} \exp\left(-\frac{|x|^2}{4t}\right),
\end{align*}
\]
which are self-similar solutions of
\[
\partial_t v = \Delta v + \lambda_2 |x|^{-2} v \quad \text{in } \mathbb{R}^N \setminus \{0\} \times (0, \infty).
\]
However, due to the fact that $v_{\text{sing}}(t) \notin H^1(\mathbb{R}^N)$ for any $t > 0$, the arguments in [13]–[16] are not applicable to the critical case. In this paper we study the large time behavior of the function $|x|^{-A} e^{-tL} \varphi$, instead of $e^{-tL} \varphi$, with
\[
A := A^+(\lambda_2) \text{ if } L \text{ is subcritical}, \quad A := A^-(\lambda_2) \text{ if } L \text{ is critical}, \quad (1.7)
\]
and overcome the difficulty arising from the fact that \( v_{\text{sing}}(t) \not\in H^1(\mathbb{R}^N) \). As far as we know, this paper is the first one treating the precise large time behavior of \( e^{-tL}\varphi \) in the critical case.

1.1. Radial solutions. In this subsection we focus on radially symmetric solutions of (1.1). Divide the operator \( L \) into the following three cases:

(S) : \( L \) is subcritical and \( \lambda_2 > \lambda_* \);   

(S) : \( L \) is subcritical and \( \lambda_2 = \lambda_* \);   

(C) : \( L \) is critical and \( A^{-}(\lambda_2) > -N/2 \).

Set
\[
d := N + 2A, \quad \rho_d(\xi) := \xi^{d-1} e^{-1/2}, \quad \psi_d(\xi) := c_d e^{-\xi^2/2},
\]
\[
|S^{d-1}| := 2\pi^{d/2}/\Gamma(d/2), \quad c_d = [2^d\pi^{d/2}/|S^{d-1}|]^{-1/2} = [2^{d-1}\Gamma(d/2)]^{-1/2},
\]
where \( R_+ := (0, \infty) \) and \( \Gamma \) is the Gamma function. Then \( \psi_d \in L^2(R_+, \rho_d d\xi) = 1 \). If \( d \) is an integer such that \( d \geq 2 \), then \( |S^{d-1}| \) coincides with the volume of \((d - 1)\)-dimensional unit sphere.

Let \( \varphi \) be radially symmetric and \( \varphi \in L^2(\mathbb{R}^N, e^{x^2/4} \, dx) \). Then \( e^{-tL}\varphi \) is radially symmetric with respect to \( x \) and set
\[
u((x,t) = [e^{-tL}\varphi](x), \quad v([x,t) = |x|^{-A}u([x,t), \quad x \in \mathbb{R}^N, \, t > 0.
\]

Then \( v \) satisfies the Cauchy problem for a \( d \)-dimensional parabolic equation
\[
\begin{dcases}
\partial_t v - \frac{1}{\rho_d^{-1}} \partial_r (r^{d-1} \partial_r v) - V_\lambda (r)v, & r \in (0, \infty), \, t > 0, \\
v(r,0) = r^{-A}\varphi(r), & r \in (0, \infty),
\end{dcases}
\]
where \( V_\lambda (r) := V(r) - \lambda_2 r^{-2} \).

In the first and the second theorems we obtain the precise description of the large time behavior of the radially symmetric solutions of (1.1) in either (S) or (C).

**Theorem 1.1.** Let \( N \geq 2 \) and assume condition (V). Let \( L \) satisfy either (S) or (C). Let \( u = u([x,t) \) be a radially symmetric solution of (1.1) such that \( \varphi \in L^2(\mathbb{R}^N, e^{x^2/4} \, dx) \). Define \( w = w(\xi, s) \) by
\[
w(\xi, s) := (1 + t)^{\frac{1}{2}} r^{-A} u(r,t) \quad \text{with} \quad \xi = (1 + t)^{-\frac{1}{2}} r \geq 0, \quad s = \log(1 + t) \geq 0.
\]

Then there exists a positive constant \( C \) such that
\[
\sup_{s > 0} \|w(\xi)\|_{L^2(R_+, \rho_d d\xi)} \leq C\|w(0)\|_{L^2(R_+, \rho_d d\xi)}.
\]

Furthermore,
\[
\lim_{s \to \infty} w(\xi, s) \equiv m(\varphi)\psi_d(\xi) \quad \text{in} \quad L^2(R_+, \rho_d d\xi) \cap C^2(K),
\]
for any compact set \( K \) in \( \mathbb{R}^N \setminus \{0\} \), where
\[
m(\varphi) := c_{1A} \int_0^{\infty} \varphi(r) U(r) r^{N-1} \, dr.
\]

In particular, if \( m(\varphi) = 0 \), then
\[
\|w(s)\|_{L^2(R_+, \rho_d d\xi)} + \|w(s)\|_{C^2(K)} = O(e^{-s}) \quad \text{as} \quad s \to \infty.
\]

**Theorem 1.2.** Assume the same conditions as in Theorem 1.1. Set \( u_*(r,t) := u(r,t)/U(r) \).

(a) For any \( j \in \{0, 1, 2, \ldots\} \), \( \partial_t^j u_* \in C([0, \infty) \times (0, \infty)) \).
(b) \( \lim_{t \to \infty} t^\frac{s}{2} u_*(0, t) = \frac{C_d}{c_*} m(\varphi) \) and \( \lim_{t \to \infty} t^\frac{s}{2+1}(\partial_t u_*)(0, t) = -\frac{dC_d}{2c_*} m(\varphi) \).

(c) Let \( T > 0 \) and \( \epsilon \) be a sufficiently small positive constant. Define
\[
G_d(r, t) := u_*(r, t) - [u_*(0, t) + (\partial_t u_*)(0, t)F_d(r)] \quad \text{for} \quad r \in [0, \infty), \ t > 0, \quad (1.13)
\]
with
\[
U_d(s) := r^{-A}U(r), \quad F_d(r) := \int_0^r s^{1-d}[U_d(s)]^{-2} \left( \int_0^r \tau^{d-1} U_d(\tau)^2 \, d\tau \right) \, ds.
\]
Then there exists a positive constant \( C \) such that
\[
|\langle \partial_t^\ell G_d \rangle(r, t)| \leq C t^{-\frac{s}{2}-2}\epsilon^\ell \|\varphi\|_{L^2(\mathbb{R}^N, e^{\epsilon r^2/4} \, dx)} \quad (1.14)
\]
for \( \ell \in \{0, 1, 2\} \), \( 0 \leq r \leq \epsilon(1+t)^{\frac{s}{2}} \) and \( t \geq T \).

In case (S), we have:

**Theorem 1.3.** Let \( N \geq 2 \) and assume condition (V). Let \( L \) satisfy (S). Let \( u = u(|x|, t) \) be a radially symmetric solution of (1.1) such that \( \varphi \in L^2(\mathbb{R}^N, e^{\epsilon |x|^2/4} \, dx) \).

(i) Let \( w \) be as in Theorem 1.1 and \( K \) a compact set in \( \mathbb{R}^N \setminus \{0\} \). Then there exists a positive constant \( C_1 \) such that
\[
\sup_{s > 0}(1+s)\|w(s)\|_{L^2(\mathbb{R}^+, \rho_2 \, d\xi)} \leq C_1 \|w(0)\|_{L^2(\mathbb{R}^+, \rho_2 \, d\xi)}.
\]
Furthermore,
\[
\lim_{s \to \infty} sw(\xi, s) = 2m(\varphi)\psi_2(\xi) \quad \text{in} \quad L^2(\mathbb{R}^+, \rho_2 \, d\xi) \cap C^2(K),
\]
where \( m(\varphi) \) is as in (1.11).

(ii) Let \( u_*, U_2, F_2 \) and \( G_2 \) be as in Theorem 1.2 with \( d = 2 \). Then
\[
\partial_t^j u_* \in C([0, \infty) \times (0, \infty)) \quad \text{for} \quad j \in \{0, 1, 2, \ldots\},
\]
\[
\lim_{t \to \infty} t(\log t)^2 u_*(0, t) = 2\sqrt{2}c_*^{-1}m(\varphi),
\]
\[
\lim_{t \to \infty} t^3(\log t)^2(\partial_t u_*)(0, t) = -2\sqrt{2}c_*^{-1}m(\varphi).
\]
Furthermore, for any \( T > 0 \) and any sufficiently small \( \epsilon > 0 \), there exists a positive constant \( C_2 \) such that
\[
|\langle \partial_t^\ell G_2 \rangle(r, t)| \leq C_2 t^{-3}[\log(2+t)]^{-2}\epsilon^\ell \|\varphi\|_{L^2(\mathbb{R}^N, e^{\epsilon r^2/4} \, dx)} \quad (1.15)
\]
for \( \ell \in \{0, 1, 2\} \), \( 0 \leq r \leq \epsilon(1+t)^{\frac{s}{2}} \) and \( t \geq T \).

The function \( w \) defined by (1.9) satisfies
\[
\partial_x w + \mathcal{L}_d w + \hat{V}(\xi, s)w = 0 \quad \text{for} \quad \xi \in [0, \infty), \ s > 0, \quad (1.16)
\]
where
\[
\mathcal{L}_d w := -\frac{1}{\rho_d(\xi)} \partial_\xi (\rho_d(\xi)\partial_\xi w) - \frac{d}{2} w, \quad \hat{V}(\xi, s) := e^* V_{\lambda_2}(e^{\frac{s}{2}} \xi).
\]
For the proofs of Theorems 1.1–1.3, we regard the operator \( \mathcal{L}_d \) as a \( d \)-dimensional elliptic operator with
\[
\begin{align*}
&\begin{cases}
  d > 2 & \text{in the case of (S)}, \\
  d = 2 & \text{in the case of } \lambda_2 = \lambda_*,
\end{cases} \\
&\begin{cases}
  0 < d < 2 & \text{in the case of (C) with } \lambda_2 > \lambda_*
\end{cases}
\end{align*}
\]
and study the large time behavior of \( w = w(\xi, s) \) by developing the arguments in a series of papers [11]–[16]. The function \( \psi_d \) defined by (1.8) is the first eigenfunction of the eigenvalue problem

\[
L_d\psi = \mu \psi \quad \text{in} \quad \mathbb{R}_+^+, \quad \psi \in H^1(\mathbb{R}_+, \rho_d(\xi) \, d\xi)
\]

and the corresponding eigenvalue is 0 (see Lemma 2.5). We show that \( w \) behaves like a suitable multiple of \( \psi_d \) as \( s \to \infty \). Furthermore, combining the radially symmetry of \( u \) with the behavior of \( w \), we prove Theorems 1.1–1.3.

The eigenfunction \( \psi_d \) corresponds to \( v_{\text{reg}} \) in the subcritical case and \( v_{\text{sing}} \) in the null-critical case, respectively. In the null-critical case, \( v_{\text{reg}} \) is transformed by (1.9) into

\[
e^{-\frac{A^+(\lambda_2)-A^-(\lambda_2)}{2}} \tilde{\psi}_d \quad \text{with} \quad \tilde{\psi}_d := \xi^{A^+(\lambda_2)-A^-(\lambda_2)} e^{-\frac{t^2}{4}}.
\]

Here \( \tilde{\psi}_d \) is the first eigenfunction of the eigenvalue problem

\[
L_d\phi = \mu \phi \quad \text{in} \quad \mathbb{R}_+^+, \quad \phi \in H^1_0(\mathbb{R}_+, \rho_d(\xi) \, d\xi)
\]

and the corresponding eigenvalue is \([A^+(\lambda_2) - A^-(\lambda_2)]/2 > 0 \). In the null-critical case with \( \lambda_2 > \lambda_s \), we see that \( 0 < d < 2 \) and \( H^1(\mathbb{R}_+, \rho_d(\xi) \, d\xi) \neq H^1(\mathbb{R}_+, \rho_d(\xi) \, d\xi) \). This justifies that the operator \( L_d \) has two positive eigenfunctions \( \psi_d \) and \( \tilde{\psi}_d \).

The case of \( d = 0 \) is on borderline where \( L_V \) is null-critical and it is not treated in this paper. Indeed, it seems difficult to apply the arguments of this paper to the case of \( d = 0 \) since \( \rho_d(\xi) \sim \xi^{-1} \) as \( \xi \to 0 \) and \( \rho_d \notin L^1(\mathbb{R}_+) \).

1.2. Nonradial solutions. We discuss the large time behavior of solutions of (1.1) without the radially symmetry of the solutions.

Let \( \Delta_{\mathbb{S}^{N-1}} \) be the Laplace-Beltrami operator on \( \mathbb{S}^{N-1} \). Let \( \{\omega_k\}_{k=0}^\infty \) be the eigenvalues of

\[
-\Delta_{\mathbb{S}^{N-1}}Q = \omega Q \quad \text{on} \quad \mathbb{S}^{N-1}, \quad Q \in L^2(\mathbb{S}^{N-1}).
\]

Then \( \omega_k = k(N+k-2) \) for \( k = 0, 1, 2, \ldots \). Let \( \{Q_{k,i}\}_{i=1}^\ell_k \) and \( \ell_k \) be the orthonormal system and the dimension of the eigenspace corresponding to \( \omega_k \), respectively. In particular, \( Q_{0,1} \equiv |\mathbb{S}^{N-1}|^{-1/2} \). For any \( \varphi \in L^2(\mathbb{R}_N, e^{\left| x \right|^2/4} \, dx) \), we can find radially symmetric functions \( \{\phi^{k,i}\} \subset L^2(\mathbb{R}_N, e^{\left| x \right|^2/4} \, dx) \) such that

\[
\varphi = \sum_{k=0}^{\infty} \sum_{i=1}^{\ell_k} \phi^{k,i} \quad \text{in} \quad L^2(\mathbb{R}_N, e^{\left| x \right|^2/4} \, dx), \quad \phi^{k,i}(x) := \phi^{k,i}(\left| x \right|) Q_{k,i} \left( \frac{x}{\left| x \right|} \right)
\]

(see [12] and [13]). Define \( L_k := -\Delta + V_k(|x|) \) and \( V_k(r) := V(r) + \omega_k r^{-2} \). Then

\[
\begin{align*}
\left[ e^{-tL - t^{1/2}k,i} \phi^{k,i} \right](x) &= \left[ e^{-tL_k \phi^{k,i}} \right](x) Q_{k,i} \left( \frac{x}{\left| x \right|} \right), \\
\left[ e^{-tL} \varphi \right](x) &= \sum_{k=0}^{\infty} \sum_{i=1}^{\ell_k} \left[ e^{-tL_k \phi^{k,i}} \right](x) Q_{k,i} \left( \frac{x}{\left| x \right|} \right) \quad \text{in} \quad L^2(\mathbb{R}_N),
\end{align*}
\]

for any \( t > 0 \). Therefore the behavior of \( e^{-tL} \varphi \) is described by a series of the radially symmetric solutions \( e^{-tL_k \phi^{k,i}} \). (See Section 5.) Furthermore, \( V_k \) satisfies condition (V) with \( \lambda_1 \) and \( \lambda_2 \) replaced by \( \lambda_1 + \omega_k \) and \( \lambda_2 + \omega_k \), respectively. In particular, \( L_k \) is subcritical if \( k \geq 1 \). Therefore, applying our results in Section 1.1, we can obtain the precise description of the large time behavior of \( e^{-tL} \varphi \).

As an application of the above argument, we obtain the following result.
Theorem 1.4. Let $N \geq 2$ and $\varphi \in L^2(\mathbb{R}^N, e^{|x|^2/4} \, dx)$. Assume condition (V). Let

$$M(\varphi) := \frac{1}{c_\star} \int_{\mathbb{R}^N} \varphi(x)U(|x|) \, dx,$$

$$\kappa := \frac{|S^{N-1}|}{c_d^2} = 2^{N+2A} \frac{\Gamma\left(\frac{N+2A}{2}\right)}{\Gamma\left(\frac{N}{2}\right)}.$$

(a) In cases (S) and (C),

$$\lim_{t \to \infty} \frac{t^{N+2A}}{2} e^{-tL}\varphi(t \frac{1}{2} y) = M(\varphi)|y|^A e^{-\frac{|y|^2}{4}}$$

in $L^2(\mathbb{R}^N, e^{|y|^2/4} \, dy)$ and in $L^\infty(K)$ for any compact set $K \subset \mathbb{R}^N \setminus \{0\}$. Furthermore,

$$\lim_{t \to \infty} \frac{t^{N+2A}}{2} \frac{e^{-tL}\varphi(x)}{U(|x|)} = c_\star^{-1} M(\varphi)$$

uniformly on $B(0,R)$ for any $R > 0$.

(b) In case (S*),

$$\lim_{t \to \infty} \frac{t^{N+2A}}{2} \left(\log t\right)^2 e^{-tL}\varphi(t \frac{1}{2} y) = 4M(\varphi)|y|^A e^{-\frac{|y|^2}{4}}$$

in $L^2(\mathbb{R}^N, e^{|y|^2/4} \, dy)$ and in $L^\infty(K)$ for any compact set $K \subset \mathbb{R}^N \setminus \{0\}$. Furthermore,

$$\lim_{t \to \infty} \frac{t^{N+2A}}{2} \left(\log t\right)^2 \frac{e^{-tL}\varphi(x)}{U(|x|)} = 4c_\star^{-1} M(\varphi)$$

uniformly on $B(0,R)$ for any $R > 0$.

As a corollary of Theorem 1.4, we have:

Corollary 1.1. Let $N \geq 2$ and assume condition (V). Let $x, y \in \mathbb{R}^N$. Then

$$\lim_{t \to \infty} \frac{t^{N+2A}}{2} \frac{p(x, y, t)}{U(|x|)U(|y|)} = (c_\star^2 \kappa)^{-1} \quad \text{in cases (S) and (C)},$$

$$\lim_{t \to \infty} \frac{t^{N+2A}}{2} \left(\log t\right)^2 \frac{p(x, y, t)}{U(|x|)U(|y|)} = 4(c_\star^2 \kappa)^{-1} \quad \text{in case (S*).}$$

Corollary 1.1 implies the same conclusion as in (L1) and (L2). For related results, see e.g., [2], [22], [26], [30] and [32].

The above argument also enables us to obtain the higher order asymptotic expansions of $e^{-tL}\varphi$. Furthermore, similarly to [11]–[16], it is useful for the study the large time behavior of the hot spots of $e^{-tL}\varphi$. (See [18].)

The rest of this paper is organized as follows. In Section 2 we formulate the definition of the solution of (1.1) and prove some preliminary lemmas. In Section 3 we obtain a priori estimates of radially symmetric solutions of (1.1) by using the comparison principle. In Section 4 we obtain the precise description of the large time behavior of radially symmetric solutions of (1.1) and complete the proofs of Theorems 1.1–1.3. In Section 5, by the argument in Section 1.2 we apply Theorems 1.1–1.3 to prove Theorem 1.4 and Corollary 1.1.
2. Preliminaries. We formulate the definition of the solution of (1.1) and obtain some properties related to the operator $L$. For positive functions $f$ and $g$ defined in $(0, R)$ for some $R > 0$, we write

$$f(r) \sim g(r) \quad \text{as} \quad r \to 0 \quad \text{if} \quad \lim_{r \to 0} \frac{f(r)}{g(r)} = 1.$$ 

Similarly, for positive functions $f$ and $g$ defined in $(R, \infty)$ for some $R > 0$, we write

$$f(r) \sim g(r) \quad \text{as} \quad r \to \infty \quad \text{if} \quad \lim_{r \to \infty} \frac{f(r)}{g(r)} = 1.$$ 

By the letter $C$ we denote generic positive constants and they may have different values also within the same line.

2.1. Definition of the solution. Assume condition $(V)$ and let $L := -\Delta + V$ be nonnegative. In this subsection we consider the Cauchy problem

$$\begin{cases}
\partial_t u_* + Lu_* = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\
u_* (x, 0) = \varphi_* (x) & \text{in } \mathbb{R}^N,
\end{cases}$$

(P)

where

$$L_* u_* := -\frac{1}{\nu} \text{div}(\nu \nabla u_*), \quad \nu := U^2 \in L^1_{\text{loc}}(\mathbb{R}^N), \quad \varphi_* \in L^2(\mathbb{R}^N, \nu \, dx).$$

**Definition 2.1.** Let $\varphi_* \in L^2(\mathbb{R}^N, \nu \, dx)$. We say that $u_*$ is a solution of (P) if $u_* \in C([0, \infty) : L^2(\mathbb{R}^N, \nu \, dx)) \cap L^2((0, \infty) : H^1(\mathbb{R}^N, \nu \, dx)),$

$$\int_0^\infty \int_{\mathbb{R}^N} [-u_* \partial \nu h + \nabla u_* \nabla h] \nu \, dx \, d\tau = 0 \quad \text{for any } h \in C_0^\infty(\mathbb{R}^N \times (0, \infty)),$$

$$\lim_{t \to 0} \|u_*(t) - \varphi_*\|_{L^2(\mathbb{R}^N, \nu \, dx)} = 0.$$ 

Problem (P) possesses a unique solution $u_*$ such that

$$\|u_*(t)\|_{L^2(\mathbb{R}^N, \nu \, dx)} \leq \|\varphi_*\|_{L^2(\mathbb{R}^N, \nu \, dx)}, \quad t > 0,$$

and we often denote by $e^{-tL_*} \varphi_*$ the unique solution $u_*$. Since $U \in C^2(\mathbb{R}^N \setminus \{0\})$ and $U > 0$ in $\mathbb{R}^N \setminus \{0\}$, applying the parabolic regularity theorems (see e.g., [20, Chapter IV]) to (P), we see that

$$\partial_t^j u_* \in C^{2,1}(\mathbb{R}^N \setminus \{0\} \times (0, \infty)), \quad j = 0, 1, 2, \ldots. \tag{2.1}$$

**Lemma 2.1.** Assume condition $(V)$ and that $L$ is nonnegative. Let $u_* := e^{-tL_*} \varphi_*$ with $\varphi_* \in L^2(\mathbb{R}^N, \nu \, dx)$.

(i) For any $j \in \{1, 2, \ldots\}$, there exists $C > 0$ such that

$$\|\partial_t^j u_* (t)\|_{L^2(\mathbb{R}^N, \nu \, dx)} \leq C t^{-j} \|\varphi_*\|_{L^2(\mathbb{R}^N, \nu \, dx)}, \quad t > 0.$$

(ii) If $\varphi_* \in L^2(\mathbb{R}^N, e^{\nu \nu} \, \nu \, dx)$, then

$$\sup_{t > 0} \|u_*(t)\|_{L^2(\mathbb{R}^N, e^{\nu \nu} \, \nu \, dx)} \leq \|\varphi_*\|_{L^2(\mathbb{R}^N, e^{\nu \nu} \, \nu \, dx)}.$$ 

**Proof.** By the same argument as in the proof of [12, Lemma 2.1] we obtain assertion (i). We prove assertion (ii). It follows that
formal, however it is justified by use of approximate solutions.)

Assume condition Lemma 2.2.

and symmetric solution of \((P)\) we see that \(\tilde{v}\) follows from (1.2) that \(\tilde{v}\) and \(\tilde{v}\) satisfies

\[
\begin{align*}
\partial_t \tilde{v}_j &= \frac{1}{\nu} \text{div}_{N+k} (\tilde{v} \nabla_{N+k} \tilde{v}_j) \quad \text{in} \quad \mathbb{R}^{N+k} \times (0, \infty), \\
\|\tilde{v}_j(t)\|_{L^2(\mathbb{R}^{N+k}, \tilde{v} \, dx)} &= \frac{|S^{N+k-1}|^{\frac{1}{2}}}{|S^{N-1}|^{\frac{1}{2}}} \|v_j(t)\|_{L^2(\mathbb{R}^N, \nu \, dx)} \leq C \tau^{-j} \|\varphi\|_{L^2(\mathbb{R}^N, \nu \, dx)},
\end{align*}
\]

Then, multiplying (P) by \(u_* e^{\frac{|x|^2}{4(1+\tau)}\nu} \) and integrating it in \(\mathbb{R}^N \times (0, \infty)\), we obtain

\[
\int_{\mathbb{R}^N} u_*(x, t) e^{\frac{|x|^2}{4(1+\tau)}\nu} \, dx \leq \int_{\mathbb{R}^N} \varphi_*(x) e^{\frac{|x|^2}{4(1+\tau)}\nu} \, dx
\]

for \(t > 0\). Thus assertion (ii) follows. (The proof of assertion (ii) is somewhat formal, however it is justified by use of approximate solutions.)

Furthermore, we have:

**Lemma 2.2.** Assume condition \((V)\) and that \(L\) is nonnegative. Let \(u_*\) be a radially symmetric solution of \((P)\). Then \(\partial_t^j u_*\) is continuous in \(\mathbb{R}^N \times (0, \infty)\), where \(j \in \{0, 1, 2, \ldots\}\).

**Proof.** Let \(j \in \{0, 1, 2, \ldots\}\) and set \(v_j = \partial_t^j u_*\). By (2.1) it suffices to prove the continuity of \(v_j\) at \((0, t) \in \mathbb{R}^N \times (0, \infty)\). Since \(v_j\) is radially symmetric, \(v_j\) satisfies

\[
\begin{align*}
\partial_t v_j &= \frac{1}{r^{N-1}\nu(r)} \partial_r (r^{N-k}\nu(r) \partial_r v_j) \\
&= \frac{1}{r^{N+k-1-r^{-k}\nu(r)}} \partial_r (r^{N+k-1-r^{-k}\nu(r)} \partial_r v_j), \quad r > 0, \; t > 0,
\end{align*}
\]

for any \(k \in \mathbb{R}\). Since \(A^+(\lambda_1) > -N/2\), we can find \(k \in \{1, 2, \ldots\}\) such that

\[
-N - k < 2A^+(\lambda_1) - k < N + k.
\]

Set \(\tilde{v}_j(x, t) := v_j(|x|, t)\) and \(\tilde{v}(x) := |x|^{-k}\nu(|x|)\) for \(x \in \mathbb{R}^{N+k}\) and \(t > 0\). By Definition 2.1, Lemma 2.1 (i) and (2.2) we see that \(\tilde{v}_j\) satisfies

\[
\begin{align*}
\partial_t \tilde{v}_j &= \frac{1}{\nu} \text{div}_{N+k} (\tilde{v} \nabla_{N+k} \tilde{v}_j) \quad \text{in} \quad \mathbb{R}^{N+k} \times (0, \infty), \\
\|\tilde{v}_j(t)\|_{L^2(\mathbb{R}^{N+k}, \tilde{v} \, dx)} &= \frac{|S^{N+k-1}|^{\frac{1}{2}}}{|S^{N-1}|^{\frac{1}{2}}} \|v_j(t)\|_{L^2(\mathbb{R}^N, \nu \, dx)} \leq C \tau^{-j} \|\varphi\|_{L^2(\mathbb{R}^N, \nu \, dx)},
\end{align*}
\]

where \(\text{div}_{N+k}\) is the \((N + k)\)-dimensional divergence operator. Furthermore, it follows from (1.2) that \(\tilde{v}(x) \sim |x|^{2A^+(\lambda_1)-k}\) as \(|x| \to 0\). This together with (2.3) implies that \(\tilde{v}\) is an \(A_2\) weight in a neighborhood of \(0 \in \mathbb{R}^{N+k}\). By Lemma 2.1 (i), applying the regularity theorems for parabolic equations with \(A_2\) weight (see e.g., [3] and [11]), we see that \(\tilde{v}_j\) is continuous at \((0, t) \in \mathbb{R}^{N+k} \times (0, \infty)\). This means that \(\partial_t^j u_*\) is continuous at \((0, t) \in \mathbb{R}^N \times (0, \infty)\). Thus Lemma 2.2 follows.

We formulate the definition of the solution of (1.1). See also [24] and [25].
Definition 2.2. Let $u$ be a measurable function in $\mathbb{R}^N \times (0, \infty)$ and $\varphi \in L^2(\mathbb{R}^N)$. Define

$$u_*(x, t) := \frac{u(x, t)}{U(|x|)}, \quad \varphi_*(x) := \frac{\varphi(x)}{U(|x|)}.$$ 

Then we say that $u$ is a solution of (1.1) if $u_*$ is a solution of (P).

In the case where $\lambda_1, \lambda_2 > \lambda_*$, we can deduce from (1.2) and (1.3) that $U \in H^1(\mathbb{R}^N)$ and that a solution $u$ of (1.1) satisfies

$$u \in C([0, \infty) : L^2(\mathbb{R}^N)) \cap L^2((0, \infty) : H^1(\mathbb{R}^N)).$$

We remark that $\varphi \in L^2(\mathbb{R}^N)$ if and only if $\varphi_* \in L^2(\mathbb{R}^N, \nu \, dx)$. Furthermore, by (1.6) we have the following lemma (see also [9, Theorem 1.2] and [10, Theorem 1.1]).

Lemma 2.3. Let $u$ be a solution of (1.1) under condition (V). Assume either $L$ is subcritical or $L$ is critical with $A^-(\lambda) > -N/2$. Then, for any $T > 0$, there exists $C > 0$ such that

$$\frac{|u(x, t)|}{U(\min\{|x|, \sqrt{t}\})} \leq Ct^{-\frac{N}{2}}U(\sqrt{t})^{-1}\|\varphi\|_{L^2(\mathbb{R}^N)}, \quad x \in \mathbb{R}^N, \ t \geq T. \quad (2.4)$$

Proof. It follows from (1.6) that

$$\frac{|u(x, t)|}{U(\min\{|x|, \sqrt{t}\})} \leq \frac{1}{U(\min\{|x|, \sqrt{t}\})} \left( \int_{\{y \leq \sqrt{t}\}} + \int_{\{y > \sqrt{t}\}} \right) p(x, y, t)|\varphi(y)| \, dy \leq Ct^{-\frac{N}{2}}U(\sqrt{t})^{-1}\int_{\{y \leq \sqrt{t}\}} |\varphi(y)|U(|y|) \, dy + Ct^{-\frac{N}{2}}U(\sqrt{t})^{-1}\int_{\{y > \sqrt{t}\}} e^{-\frac{|x-y|^2}{cr^2}}|\varphi(y)| \, dy \leq Ct^{-\frac{N}{2}}U(\sqrt{t})^{-2}\|u\|_{L^2(\{y \leq \sqrt{t}\})}\|\varphi\|_{L^2(\mathbb{R}^N)} + Ct^{-\frac{N}{2}}U(\sqrt{t})^{-1}\|\varphi\|_{L^2(\mathbb{R}^N)}$$

for $x \in \mathbb{R}^N$ and $t > 0$. On the other hand, by (1.4) and (1.5) we have

$$\|u\|_{L^2(\{y \leq \sqrt{t}\})} \leq Ct^{\frac{N}{2}}U(\sqrt{t})$$

for $t \geq T$ (see also (3.7)). These imply (2.4) and Lemma 2.3 follows. \[ \square \]

2.2. Preliminary lemmas. We prove a lemma on the decay of $U'$ as $r \to \infty$.

Lemma 2.4. Let $N \geq 2$. Assume condition (V) and that $L = -\Delta + V(|x|)$ is nonnegative. Let $U$ and $v$ be as in (1.2) and (1.5), respectively. In cases (S) and (C) there exists $\delta > 0$ such that

$$[v(r)^{-1}U(r)]' = O(r^{1-\delta}) \quad \text{as} \quad r \to \infty. \quad (2.5)$$

Proof. Let $V_\lambda(r) := V(r) - \lambda_2 r^{-2}$. Set

$$v^+(r) := \begin{cases} r^{-\frac{N+2}{2}} \log r & \text{if } L \text{ is subcritical and } \lambda = \lambda_*, \\ r^{A^-(\lambda_2)} & \text{otherwise}, \end{cases} \quad v^-(r) := r^{A^-(\lambda_2)}. \quad (2.6)$$

It follows from (1.4) and (V) (ii) that

$$\tau^{-N-1}v^-(\tau)V_\lambda(\tau)U(\tau) = O(\tau^{N-3+\lambda}A^-(\lambda_2)\nu(\tau))$$

$$= \begin{cases} O(\tau^{1-\delta}) & \text{if } L \text{ is subcritical and } \lambda_2 > \lambda_*, \\ O(\tau^{1-\delta-\sqrt{\tau}}) & \text{if } L \text{ is critical and } \lambda_2 > \lambda_*, \\ O(\tau^{1-\delta}) & \text{if } L \text{ is critical and } \lambda_2 = \lambda_*. \end{cases}$$
Lemma 2.5. Let \( \{\mu_i\}_{i=0}^\infty \) be the eigenvalues of (E) such that \( \mu_0 \leq \mu_1 \leq \mu_2 \leq \ldots \). Then, for any \( i \in \{0, 1, 2, \ldots \} \), \( \mu_i = i \) and \( \mu_i \) is simple. Furthermore, \( \psi_d \) given in (1.8) is the first eigenfunction of (E).
Proof. We leave the proof to the reader since it is proved by the same argument as in [23, Lemma 2.1].

3. A priori estimates of radial solutions. Let $T > 0$ and $\epsilon > 0$. Define

$$D_\epsilon(T) := \{(x, t) \in \mathbb{R}^N \times (T, \infty) : |x| < \epsilon t^2\}.$$ 

In this section we prove the following proposition.

**Proposition 3.1.** Assume condition (V). Let $L$ satisfy either (S), $(S_*)$ or (C). Let $u_* = u_*([x], t)$ be a radially symmetric solution of (P) such that $\|\varphi_*\|_{L^2(\mathbb{R}^N, \nu \, dx)} = 1$. Assume that

$$\sup_{t > 0} t^D |\log(2 + t)|^{D'} \|u_*(t)\|_{L^2(\mathbb{R}^N, \nu \, dx)} < \infty \quad \text{for some } D \geq 0 \text{ and } D' \geq 0. \quad (3.1)$$

Let $j \in \{0, 1, 2, \ldots \}$. Then the following holds for any $T > 0$ and any sufficiently small $\epsilon > 0$.

(i) There exists $C_1 > 0$ such that

$$|\langle \partial_t^j u_* \rangle([x], t)| \leq C_1 \Gamma_{D, D', j}(t)$$

for $(x, t) \in D_\epsilon(T)$, where

$$\Gamma_{D, D', j}(t) := \begin{cases} t^{-D - \frac{4}{j} - j |\log(2 + t)| - D'} & \text{in the case of (S)}, \\ t^{-D - \frac{4}{j} - j |\log(2 + t)| - D' - 1} & \text{in the case of $(S_*)$}, \\ t^{-D - \frac{4}{j} - j |\log(2 + t)| - D'} & \text{in the case of (C)}. \end{cases} \quad (3.2)$$

(ii) Let

$$F_N^j(r, t) := \int_0^r s^{1-N} \nu(s)^{-1} \left( \int_0^s \tau^{N-1} \nu(\tau) (\partial_t^{j+1} u_*)(\tau, t) \, d\tau \right) \, ds.$$ 

Then

$$\langle \partial_t^j u_* \rangle([x], t) = \langle \partial_t^j u_* \rangle(0, t) + F_N^j([x], t) \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

Furthermore, there exists $C_2 > 0$ such that

$$|F_N^j([x], t)| \leq C_2 \Gamma_{D, D', j+1}(t)|x|^2, \quad |\langle \partial_t F_N^j \rangle([x], t)| \leq C \Gamma_{D, D', j+1}(t)|x|,$$

for $(x, t) \in D_\epsilon(T)$.

For the proof, we construct supersolutions of problem (P) in $D_\epsilon(T)$.

**Lemma 3.1.** Assume condition (V). Let $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. Set

$$\zeta(t) := t^{-\gamma_1} |\log(2 + t)|^{-\gamma_2}.$$ 

Then, for any $T > 0$ and any sufficiently small $\epsilon > 0$, there exists a function $W_* = W_*(x, t)$ such that

$$\partial_t W_* + L_* W_* \leq 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad (3.3)$$

$$\zeta(t) \leq W_*(x, t) \leq 2 \zeta(t) \quad \text{in } D_\epsilon(T). \quad (3.4)$$

**Proof.** Let $T > 0$ and $\epsilon > 0$. Let $\kappa$ be a positive constant such that

$$|\zeta'(t)| \leq \kappa t^{-1} \zeta(t), \quad t > 0. \quad (3.5)$$

Let

$$F(x) := \int_0^{|x|} s^{1-N} \nu(s)^{-1} \left( \int_0^s \tau^{N-1} \nu(\tau) \, d\tau \right) \, ds,$$
which satisfies \(-L_s F = 1\) in \(\mathbb{R}^N\). Set
\[
W_s(x,t) := 2\zeta(t) \left[1 - \kappa t^{-1} F(x)\right].
\]

Since \(\zeta\) is monotone decreasing, by (3.5) we have
\[
\partial_t W_s + L_s W_s \\
\geq 2\zeta'(t) \left[1 - \kappa t^{-1} F(x)\right] + 2\kappa \zeta(t) t^{-2} F(x) + 2\kappa t^{-1} \zeta(t) \\
\geq 2\zeta'(t) + 2\kappa t^{-1} \zeta(t) \geq 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty).
\]

This implies (3.3). On the other hand, by (1.2), (1.4) and (1.5) we have
\[
\int_0^s \tau^{N-1} \nu(\tau) \, d\tau \leq C s^{2A^+} (\lambda_1 + N) \quad \text{for} \quad 0 < s \leq 1,
\]
\[
\int_0^s \tau^{N-1} \nu(\tau) \, d\tau \leq \begin{cases} s^{2A^+} (\lambda_2 + N) & \text{in the case of (S)}, \\
2^s [\log(2 + s)]^2 & \text{in the case of (S*)}, \\
s^{2A^-} (\lambda_2 + N) & \text{in the cases of (C)}
\end{cases} \quad \text{for} \quad s > 1.
\]

These imply that
\[
\int_0^s \tau^{N-1} \nu(\tau) \, d\tau \leq C s^N \nu(s), \quad s \geq 0.
\]

Then it follows that \(0 \leq F(x) \leq C|x|^2\) for \(x \in \mathbb{R}^N\). Taking a sufficiently small \(\epsilon > 0\) if necessary, we obtain
\[
0 \leq \kappa t^{-1} F(x) \leq C \epsilon^2 \kappa \leq \frac{1}{2}, \quad (x, t) \in D_\epsilon(T).
\]

This together with (3.6) implies (3.4). Thus Lemma 3.1 follows. \(\square\)

Applying the same argument as in [16, Lemma 3.2], we have:

**Lemma 3.2.** Assume the same conditions as in Proposition 3.1. Furthermore, assume (3.1) for some \(D \geq 0\) and \(D' \geq 0\). Let \(T > 0\) and let \(\epsilon\) be a sufficiently small positive constant. Then, for any \(j \in \{0, 1, 2, \ldots\}\), there exists \(C > 0\) such that
\[
|\partial_t^j u_\epsilon(|x|, t)| \leq C T^{D, D', j} \quad \text{in} \quad D_\epsilon(T).
\]

**Proof.** Let \(j \in \{0, 1, 2, \ldots\}\). Set \(v_j := \partial_t^j u_\epsilon\) and \(u_j := U(|x|) v_j(x, t)\). Since
\[
v_j(\cdot, t) = \partial_t^j \left[ e^{-(t/2) L_s} u_\epsilon(t/2) \right], \quad t > 0,
\]
Lemma 2.1 together with (3.1) implies that
\[
\sup_{t > 0} t^{D+j} [\log(2+t)]^{D'} \|u_j(t)\|_{L^2(\mathbb{R}^N)} = \sup_{t > 0} t^{D+j} [\log(2+t)]^{D'} \|v_j(t)\|_{L^2(\mathbb{R}^N, \nu dx)} < \infty.
\]

Let \(T > 0\) and let \(\epsilon\) be a sufficiently small positive constant. Since \(u_j\) satisfies
\[
\partial_t u_j = \Delta u_j - V(|x|) u_j \quad \text{in} \quad \mathbb{R}^N \times (0, \infty),
\]
by Lemma 2.3 we have
\[
|u_j(x, t)| \leq C t^{-\frac{N}{2}} \|u_j(t/2)\|_{L^2(\mathbb{R}^N)} \leq C t^{-D - \frac{N}{2} - j} [\log(2 + t)]^{-D'}
\]
for all \(x \in \mathbb{R}^N\) and \(t > T\) with \(|x| \geq \epsilon(1 + t)^{1/2}\). This together with (1.4), (1.5), (1.7), (1.8) and (3.2) implies that
\[
|v_j(x, t)| \leq \frac{C u_\epsilon(|x|, t)}{U(\epsilon(1 + t)^{1/2})} \leq C T^{D, D', j}(t)
\]
for all \((x,t) \in \mathbb{R}^N \times [T, \infty)\) with \(|x| = \epsilon(1 + t)^{\frac{1}{2}}\). On the other hand, it follows from Lemma 2.2 that

\[
|v_j(|x|,T)| \leq C \quad \text{for } x \in \mathbb{R}^N \text{ with } |x| \leq \epsilon(1 + T)^{\frac{1}{2}}.
\] (3.10)

Let \(W_*\) be as in Lemma 3.1 with \(\zeta\) replaced by \(\Gamma_{D,D'}\). Then, by Lemma 3.1, (3.9) and (3.10) we apply the comparison principle to obtain

\[
|v_j(|x|,t)| \leq CW_*(x,t) \leq 2\Gamma_{D,D',j}(t) \quad \text{in } D_\epsilon(T).
\]

This implies (3.8), and the proof is complete.

Now we are ready to complete the proof of Proposition 3.1.

**Proof of Proposition 3.1.** By Lemma 3.2 it suffices to prove assertion (ii). Let \(T > 0\) and let \(\epsilon\) be a sufficiently small positive constant. By (3.7) and (3.8) we obtain

\[
|F_N^j(|x|,t)| \leq C \Gamma_{D,D',j+1}(t) \int_0^{|x|} s^{1-N} [\nu(s)]^{-1} \left( \int_0^s r^{N-1} \nu(r) \, dr \right) \, ds
\]

\[
\leq C \Gamma_{D,D',j+1}(t)|x|^2,
\]

\[
|(\partial_r F_N^j)(|x|,t)| \leq C \Gamma_{D,D',j+1}(t)|x|,
\]

for \((x,t) \in D_\epsilon(T)\). Set

\[
v_j(|x|,t) := (\partial_t^j u_*)(|x|,t) - F_N^j(|x|,t), \quad \hat{u}_j(|x|,t) := U(|x|)v_j(|x|,t).
\]

Since \(F_N^j\) satisfies

\[
\frac{1}{\nu(r)r^{N-1}} \partial_r (\nu(r)r^{N-1} \partial_r F_N^j) = (\partial_t^{j+1} u_*)(r,t) \quad \text{for } r > 0 \text{ and } t > 0,
\]

by Lemma 2.2 and (2.2) we have

\[
\frac{1}{\nu(r)r^{N-1}} \partial_r (\nu(r)r^{N-1} \partial_r \hat{v}_j) = 0 \quad \text{for } r > 0 \text{ and } t > 0, \quad (3.11)
\]

\[
\limsup_{r \to 0} |\hat{v}_j(r,t)| < \infty \quad \text{for any } t > 0. \quad (3.12)
\]

It follows from (3.11) that \(\hat{u}_j\) satisfies (O) for any fixed \(t > 0\). On the other hand, since \(U\) and \(\tilde{U}\) are linearly independent solutions of (O), for any \(t > 0\), we can find constants \(c_j(t)\) and \(\check{c}_j(t)\) such that

\[
\hat{u}_j(r,t) = c_j(t)U(r) + \check{c}_j(t)\tilde{U}(r) \quad \text{for } r > 0.
\]

This implies that

\[
v_j(r,t) = U(r)^{-1}\hat{u}_j(r,t) = c_j(t) + \check{c}_j(t)U(r)^{-1}\tilde{U}(r) \quad \text{for } r > 0.
\]

Then, by (1.2) and (3.12) we have \(\check{c}_j(t) = 0\) and see that \(v_j(r,t) \equiv c_j(t)\) for \(r \geq 0\). Therefore we have

\[
(\partial_t^j u_*)(|x|,t) = c_j(t) + F_N^j(|x|,t) \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad c_j(t) = (\partial_t^j u_*)(0,t).
\]

Thus assertion (ii) follows, and the proof of Proposition 3.1 is complete. \qed
Large time behavior of radially symmetric solutions. In this section, under condition (V), we study the large time behavior of radially symmetric solution $u = u(|x|, t)$ of (1.1) and prove Theorems 1.1–1.3.

Let $A$, $d$ and $w$ be as in Section 1.1. Set $U_d(r) := r^{-A} U(r)$ and $\nu_d := \nu_d^2$. By (1.2), (1.4) and Lemma 2.4 we have:

$$
\frac{1}{r^{d-1}} \partial_r (r^{d-1} \partial_r U_d) - \nu_d U_d = 0 \quad \text{in} \quad (0, \infty);
$$

$$
U_d(r) \sim r^{A^{+}(\lambda_1) - A} = r^{A^+}(\lambda) \quad \text{as} \quad r \to 0;
$$

$$
U_d(r) \sim c_* \quad \text{as} \quad r \to \infty \quad \text{in cases (S) and (C)}. \quad (4.1)
$$

Here $c_*$ is as in (1.4), $\lambda := \lambda_1 - \lambda_2$ and

$$
A^+ = \frac{-(d - 2) + \sqrt{(d - 2)^2 + 4\lambda}}{2}.
$$

Furthermore, similarly to Lemma 2.4, we see that

$$
U_d(r) \sim c_* \log r, \quad U_d'(r) = O(r^{-1}) \quad \text{as} \quad r \to \infty \quad \text{in case of (S*)}. \quad (4.2)
$$

Then the function $F_N^j$ given in Proposition 3.1 satisfies

$$
F_N^j(r, t) = F_d^j(r, t) := \int_0^r s^{-d} |\nu_d(s)|^{-1} \left( \int_0^s \tau^{d-1} \nu_d(\tau) (\partial^{j+1}_t u_*)(\tau, t) \, d\tau \right) \, ds, \quad (4.3)
$$

where $j \in \{0, 1, 2, \ldots \}$. Furthermore, it follows from (3.7) that

$$
\int_0^s \tau^{d-1} \nu_d(\tau) \, d\tau \leq C s^d, \quad s \geq 0. \quad (4.4)
$$

Assume the same conditions as in Theorem 1.1. Let $\theta$ be the constant given in condition (V) and set

$$
\theta_* = \frac{\theta}{4(2 + \theta)} \in \left(0, \frac{\theta}{8}\right).
$$

Since $\tilde{V}(\xi, s) = e^s V_{\lambda_2} (e^{\frac{s}{2}} \xi)$, it follows from (V) (ii) that

$$
|\tilde{V}(\xi, s)| \leq C \xi^{-2} |e^{\frac{s}{2}} \xi|^{-\theta} \leq C \exp \left[ -\frac{\theta}{2} s + (2 + \theta) \theta_* s \right] = Ce^{-\frac{s}{2} \theta_*} \quad (4.5)
$$

for $\xi \in (e^{-\theta_* s}, \infty)$ and $s > 0$. Let $\delta$ be as in Lemma 2.4. Then, taking a sufficiently small $\theta > 0$ if necessary, we have

$$
0 < \theta < \min \{1, d, d^{-1}\}, \quad \sigma := \left(\frac{1}{2} - \theta_*\right)(1 + \delta) - \frac{1}{2} > \theta_* > 0. \quad (4.6)
$$

We prepare some lemmas on estimates of $w$.

**Lemma 4.1.** Let $\|\varphi_*\|_{L^2(\mathbb{R}^N, \nu_d^{1/2} / d\xi)} = 1$. Assume the same conditions as in Theorem 1.1. Then

(i) $\sup_{s > 0} e^{-\frac{s}{2} \theta_*} \|w(s)\|_{L^2(\mathbb{R}^{+}, d\xi)} < \infty$;

(ii) Assume that

$$
\sup_{s > 0} e^{\gamma s} \|w(s)\|_{L^2(\mathbb{R}^{+}, d\xi)} < \infty \quad (4.7)
$$
for some $\gamma \geq -d/4$. Then
\begin{align}
  w(e^{-\theta_0s}, s) &= O(e^{-\gamma s}), \\
  (\partial_x w)(e^{-\theta_0s}, s) &= O(e^{-\gamma s-\theta_0s}), \\
  \int_0^\infty |w(\xi, s)|^2 \rho_d d\xi &= O(e^{-2\gamma s-\theta_0s}),
\end{align}
for all sufficiently large $s > 0$.

**Proof.** Since
\[ w(\xi, s) = (1 + t)^{\frac{d}{2}} r^{-A} u(r, t) = (1 + t)^{\frac{d}{2}} r^{-A} U(r) u_*(r, t) \]
\[ = ((1 + t)^{\frac{d}{2}} U_d(r) u_*(r, t) \text{ with } \xi = (1 + t)^{-\frac{d}{2}} r \text{ and } s = \log(1 + t), \]
it follows from Lemma 2.1 (ii) that
\begin{align}
  ||w(s)||_{L^2(R^+, \rho_d d\xi)} &= (1 + t)^{\frac{d}{2}} \int_0^\infty |u_*(r, t)|^2 U(r)^2 r^{N-1} e^{-\frac{2}{\gamma + \frac{d}{2}}} dr \\
  &= (1 + t)^{\frac{d}{2}} |S^{N-1}|^{-1/2} ||u_*(t)||_{L^2(R^+, e^{\frac{|x|^2}{2} + \gamma s} \nu dx)}^2 \\
  &\leq (1 + t)^{\frac{d}{2}} |S^{N-1}|^{-1/2} ||\varphi_*||_{L^2(R^+, e^{\frac{|x|^2}{2} \nu dx})}^2 \\
  &= (1 + t)^{\frac{d}{2}} |S^{N-1}|^{-1/2} ||\varphi_*||_{L^2(R^+, e^{\frac{|x|^2}{2} \nu dx})}^2 < \infty
\end{align}
for $s > 0$ and $t > 0$ with $s = \log(1 + t)$, where $|S^{N-1}|$ is the volume of $(N - 1)$-dimensional unit sphere, that is $|S^{N-1}| = 2\pi^{\frac{N}{2}} / \Gamma(N/2)$. Thus assertion (i) follows.

We prove assertion (ii). It follows from (4.12) that
\begin{align}
  ||w(s)||_{L^2(R^+, \rho_d d\xi)} &= (1 + t)^{\frac{d}{2}} |S^{N-1}|^{-1/2} ||u_*(t)||_{L^2(R^+, e^{\frac{|x|^2}{2} + \gamma s} \nu dx)}^2 \\
  &\geq (1 + t)^{\frac{d}{2}} |S^{N-1}|^{-1/2} ||u_*(t)||_{L^2(R^+, \nu dx)}^2 \\
  &\geq (1 + t)^{\frac{d}{2}} |S^{N-1}|^{-1/2} ||u_*(t)||_{L^2(R^+, \nu dx)}^2 < \infty
\end{align}
for $s > 0$ and $t > 0$ with $s = \log(1 + t)$. Assume (4.7) for some $\gamma \geq -d/4$. Then
\[ \sup_{t>0} (1 + t)^{\gamma + \frac{d}{2}} ||u_*(t)||_{L^2(R^+, \nu dx)} < \infty. \]

Applying Proposition 3.1 with $D = \gamma + d/4$ and $D' = 0$, we obtain
\begin{align}
  u_*(|x|, t) &= u_*(0, t) + F^0_N(|x|, t) \text{ in } R^N \times (0, \infty). \tag{4.14}
\end{align}
Furthermore, for any $T > 0$ and any sufficiently small $\epsilon > 0$,
\begin{align}
  |u_*(|x|, t)| &\leq Ct^{-\gamma - \frac{d}{2}}, \\
  |F^0_N(|x|, t)| &\leq Ct^{-\gamma - \frac{d}{2} + 1}|x|^2 \leq Ct^2 t^{-\gamma - \frac{d}{2}}, \\
  |(\partial_\nu F^0_N)(|x|, t)| &\leq Ct^{-\gamma - \frac{d}{2} + 1}|x| \leq Ct^{-\gamma - \frac{d}{2} - \frac{1}{2}}.
\end{align}
for $(x, t) \in D_\epsilon(T)$. Then, by (4.1), (4.11) and (4.15) we have $w(e^{-\theta_0s}, s) = O(e^{-\gamma s})$ for all sufficiently large $s > 0$. Furthermore,
\begin{align}
  (\partial_x w)(e^{-\theta_0s}, s) &= (1 + t)^{\frac{d + 1}{2}} [U_d(r) u_*(r, t) + U_d(r)(\partial_r u_*) (r, t)] \\
  &= (1 + t)^{\frac{d + 1}{2}} [O(r^{-1-\delta}) u_*(r, t) + (c_\epsilon + o(1))(\partial_r F^0_N)(r, t)] \\
  &= (1 + t)^{-\gamma - \frac{d}{2} + 1} O(r^{-1-\delta}) + (1 + t)^{-\gamma - \frac{d}{2}} O(r) \\
  &= O(e^{-\gamma s - \sigma}) + O(e^{-\gamma s - \theta_0 s}) = O(e^{-\gamma s - \theta_0 s})
\end{align}
for all sufficiently large \( s > 0 \), where \( r = e^{\frac{1}{2}s-\theta_0 s} \), \( s = \log(1+t) \) and \( \sigma \) is as in (4.6). So we have (4.8) and (4.9).

On the other hand, by (4.1), (4.4), (4.11), (4.14) and (4.15) we have

\[
\int_0^t e^{-\theta_0 s} \| \psi'(s) \|_{L^2}^2 \, ds = 2 \int_0^t \psi'(s) \cdot \psi''(s) \, ds + \frac{1}{2} \int_0^t (\psi'(s))^2 \, ds,
\]

and

\[
\int_0^t \psi'(s) \cdot \psi''(s) \, ds \leq C t^{-2} \int_0^t \nu_d(r) r^{d-1} \, dr
\]

for all sufficiently large \( s > 0 \) and \( t > 0 \) with \( s = \log(1+t) \). This implies (4.10).

Thus assertion (ii) follows, and the proof is complete. \( \square \)

**Lemma 4.2.** Assume the same conditions as in Lemma 4.1. Then

\[
\sup_{s > 0} \| w(s) \|_{L^2(\mathbb{R}^d, \rho_d \, dx)} < \infty.
\]  

(4.16)

**Proof.** Assume that (4.7) holds for some \( \gamma \geq -d/4 \). Let \( I(s) := (e^{-\theta_0 s}, \infty) \). It follows from (1.16) that

\[
\frac{d}{ds} \int_{I(s)} |w(\xi, s)|^2 \rho_d \, d\xi = 2 \int_{I(s)} \partial_\xi (\rho_d w) \, d\xi + \theta_0 e^{-\theta_0 s} |w(e^{-\theta_0 s}, s)|^2 \rho_d(e^{-\theta_0 s})
\]

\[
= 2 \int_{I(s)} \partial_\xi (\rho_d \partial_\xi w) \, d\xi + d \int_{I(s)} |w(\xi, s)|^2 \rho_d \, d\xi
\]

\[
- 2 \int_{I(s)} \tilde{V} w^2 \rho_d \, d\xi + \theta_0 e^{-\theta_0 s} |w(e^{-\theta_0 s}, s)|^2 \rho_d(e^{-\theta_0 s})
\]

\[
= -2 w(e^{-\theta_0 s}, s) \rho_d(e^{-\theta_0 s}) (\partial_\xi w)(e^{-\theta_0 s}, s)
\]

for \( s > 0 \). This together with Lemma 4.1 and (4.5) implies that

\[
\frac{d}{ds} \int_{I(s)} |w(\xi, s)|^2 \rho_d \, d\xi \leq -2 \int_{I(s)} |\partial_\xi w|^2 \rho_d \, d\xi + d \int_{I(s)} |w(\xi, s)|^2 \rho_d \, d\xi
\]

\[
+ C e^{-\frac{\gamma}{2} s} \int_{I(s)} |w(\xi, s)|^2 \rho_d \, d\xi + O(e^{-2\gamma s} e^{-d\theta_0 s})
\]  

(4.17)

for all sufficiently large \( s > 0 \).

Set

\[
\hat{w}(\xi, s) := \begin{cases} 
  w(\xi, s) & \text{if } \xi \geq e^{-\theta_0 s}, \\
  w(e^{-\theta_0 s}, s) & \text{if } 0 \leq \xi < e^{-\theta_0 s}.
\end{cases}
\]  

(4.18)

It follows from Lemmas 2.5 and 4.1 that

\[
-2 \int_{I(s)} |\partial_\xi (\hat{w})(\xi, s)|^2 \rho_d \, d\xi + d \int_{I(s)} |\hat{w}(\xi, s)|^2 \rho_d \, d\xi
\]

\[
= -2 \int_0^\infty |\partial_\xi \hat{w}(\xi, s)|^2 \rho_d \, d\xi + d \int_0^\infty |\hat{w}(\xi, s)|^2 \rho_d \, d\xi
\]
Lemma 4.3. Assume the same conditions as in Lemma 4.1. Let \( \hat{w} \) be as in (4.18). Then
\[
\|w(e^{-\theta_* s}, s)\|_{L^2(\mathbb{R}^N, \nu_\theta \circ e^{(d/2) s} d\xi)} < \infty, \quad \|\hat{w}(e^{-\theta_* s}, s)\|_{L^2(\mathbb{R}^N, \nu_\theta \circ e^{(d/2) s} d\xi)} < \infty,
\]
\[
\|\hat{w}(e^{-\theta_* s}, s)\|_{L^2(\mathbb{R}^N, \nu_\theta \circ e^{(d/2) s} d\xi)} < \infty, \quad \|\hat{w}(e^{-\theta_* s}, s)\|_{L^2(\mathbb{R}^N, \nu_\theta \circ e^{(d/2) s} d\xi)} < \infty.
\]

Next we study the large time behavior of \( \hat{w} \) and prove the following proposition.

Proposition 4.1. Let \( \|\varphi\|_{L^2(\mathbb{R}^N, \nu_{e^{(d/2) s}} d\xi)} = 1 \). Assume the same conditions as in Theorem 1.1. Let \( \hat{w} \) be as in (4.18). Set
\[
a(s) := \int_0^\infty \hat{w}(e^{-\theta_* s}, s) d\xi = c_d \int_0^\infty \hat{w}(\xi, s) \xi^{d-1} d\xi.
\]
Then
\[
\|\hat{w} - a(s)\psi_d\|_{L^2(\mathbb{R}^N, \nu_\theta \circ e^{(d/2) s} d\xi)} = O(e^{-\theta' s})
\]
as \( s \to \infty \), where \( \theta' := \min\{d\theta_*/2, \theta/8\} \).

For the proof of Proposition 4.1, we prepare the following lemma.
Lemma 4.4. Assume the same conditions as in Proposition 4.1. Then
\[
\sup_{s \geq 1} |a(s)| < \infty, \tag{4.23}
\]
\[
\sup_{s \geq 1} e^{2\theta s} |a'(s)| < \infty, \tag{4.24}
\]
\[
\sup_{s \geq 1} \left| \frac{d}{ds} w(e^{-\theta, s}, s) \right| < \infty. \tag{4.25}
\]

Proof. It follows from Lemma 4.3 that
\[
\sup_{s \geq 1} |a(s)| \leq \sup_{s \geq 1} \| \hat{\omega} \|_{L^2(R_+, \rho d d \xi)} \| \psi_d \|_{L^2(R_+, \rho d d \xi)} < \infty.
\]
So we have (4.23). By Proposition 3.1 (ii) and (4.11) we have
\[
w(e^{-\theta, s}, s) = e^{\frac{d}{2} s} U_d(r(s)) u_*(r(s), t(s))
\]
\[
= e^{\frac{d}{2} s} U_d(r(s))[u_*(0, t(s)) + F_N^0(r(s), t(s))]
\]
for \( s > 0 \), where \( r(s) = e^{\frac{d}{2} s - \theta, s} \) and \( t(s) = e^s - 1 \). Then
\[
\frac{d}{ds} w(e^{-\theta, s}, s)
\]
\[
= \frac{d}{2} w(e^{-\theta, s}, s) + \frac{U'_d(r(s))}{U_d(r(s))} r'(s) w(e^{-\theta, s}, s)
\]
\[
+ e^{\frac{d}{2} s} U_d(r(s)) (\partial_t u_*) (0, t(s)) t'(s)
\]
\[
+ e^{\frac{d}{2} s} U_d(r(s)) [(\partial_t F_N^0)(r(s), t(s)) r'(s) + (\partial_t F_N^0)(r(s), t(s)) t'(s)]
\]
for \( s > 0 \). It follows from (4.1) that
\[
U_d(r(s)) \sim c_+,
\]
\[
U'_d(r(s)) r'(s) = O(r(s)^{-1 - \delta} r'(s)) = O(e^{-\delta (\frac{d}{2} - \theta, s)}),
\]
for all sufficiently large \( s > 0 \). On the other hand, by Lemma 4.2 and (4.13) we have
\[
\sup_{t > 0} (1 + t)^{\frac{d}{4}} \| u_*(t) \|_{L^2(R^N, \nu d x)} < \infty. \tag{4.28}
\]
Then we apply Proposition 3.1 with \( D = d/4 \) and \( D' = 0 \) to obtain
\[
(\partial_t u_*)(0, t(s)) = O(e^{-\frac{d}{4} s - s}),
\]
\[
(\partial_t F_N^0)(r(s), t(s)) = O(e^{-\frac{d}{4} s - s} r(s)),
\]
\[
(\partial_t F_N^0)(r(s), t(s)) = F_N^0(r(s), t(s)) = O(e^{-\frac{d}{4} s - 2 s} r(s)^2),
\]
for all sufficiently large \( s > 0 \). By Lemma 4.3, (4.26), (4.27) and (4.29) we have (4.25). Furthermore, by Lemma 2.5, Lemma 4.3, (4.5), (4.16) and (4.25) we obtain
\[
a'(s) = \frac{c_d}{d} e^{-\frac{d}{4} s} [w(e^{-\theta, s}, s)]' - c_d \theta_* e^{-\frac{d}{4} s} w(e^{-\theta, s}, s)
\]
\[
+ c_d \theta_* e^{-\frac{d}{4} s} w(e^{-\theta, s}, s) + \int_{I(s)} \partial_s w \psi_d \rho d d \xi
\]
for all sufficiently large $s > 0$. This implies (4.24). Thus Lemma 4.4 follows. \hfill \Box

Proof of Proposition 4.1. Set

$$\tilde{w}(\xi, s) := \tilde{w}(\xi, s) - a(s)\psi_d(\xi).$$

It follows from Lemma 2.5 and (1.16) that

$$\partial_s \tilde{w} = \partial_s \tilde{w} - a'(s)\psi_d = -\mathcal{L}_d \tilde{w} - \tilde{V} \tilde{w} - a'(s)\psi_d = -\mathcal{L}_d \tilde{w} - \tilde{V} \tilde{w} - a'(s)\psi_d$$

for $\xi \in I(s)$ and $s > 0$. By Lemma 4.3, Lemma 4.4 and (4.5) we have

$$\frac{d}{ds} \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi$$

$$= 2 \int_{I(s)} \tilde{w}(\partial_s \tilde{w}) \rho_d d\xi + \theta \psi_s \tilde{w}(\xi, s)^2 \rho_d(e^{-\theta s})$$

$$= 2 \int_{I(s)} \left[ \tilde{w} \partial_s (\partial_s \tilde{w}) + \frac{d}{2} \tilde{w}^2 \rho_d - \tilde{V} \tilde{w} \rho_d - a'(s)\psi_d \tilde{w} \rho_d \right] d\xi + O(e^{-\theta s})$$

$$= -2 \tilde{w}(\xi, s) \rho_d(e^{-\theta s}) (\partial_s \tilde{w})(\xi, s)$$

$$- 2 \int_{I(s)} |(\partial_s \tilde{w})(\xi, s)|^2 \rho_d d\xi + d \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi$$

$$+ 2 \tilde{w} \tilde{w} \rho_d d\xi - 2a'(s) \int_{I(s)} \tilde{w} \psi_d \rho_d d\xi + O(e^{-\theta s})$$

$$= -2 \int_{I(s)} |(\partial_s \tilde{w})(\xi, s)|^2 \rho_d d\xi + d \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi$$

$$+ O(e^{-\theta s}) + O(e^{-\frac{\xi}{s}})$$

for all sufficiently large $s > 0$. Furthermore, similarly to (4.19), by Lemmas 2.5, 4.3 and 4.4 we obtain

$$\int_{I(s)} |(\partial_s \tilde{w})(\xi, s)|^2 \rho_d d\xi - \frac{d}{2} \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi$$

$$\geq \int_0^\infty |(\partial_s \tilde{w})(\xi, s)|^2 \rho_d d\xi - \frac{d}{2} \int_0^\infty |\tilde{w}(\xi, s)|^2 \rho_d d\xi$$

$$- a(s)^2 \int_0^\infty |\partial_s \psi_d|^2 \rho_d d\xi$$

$$= \int_0^\infty |(\partial_s \tilde{w})(\xi, s)|^2 \rho_d d\xi - \frac{d}{2} \int_0^\infty |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-(d+2)\theta s})$$

$$\geq \int_0^\infty |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-(d+2)\theta s})$$

$$\geq \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-(d+2)\theta s})$$
for all sufficiently large $s > 0$. Therefore we deduce from (4.30) and (4.31) that
\[
\frac{d}{ds} \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi \leq -2 \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-d\theta_s}) + O(e^{-\frac{s}{2}})
\]
\[
= -2 \int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-2\theta_s})
\]  
(4.32)
for all sufficiently large $s > 0$. Since $d\theta_s < d\theta < 1$ (see (4.6)), by (4.32) we have
\[
\int_{I(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi = O(e^{-2\theta s})
\]
(4.33)
for all sufficiently large $s > 0$. Combining (4.33) with Lemmas 4.3 and 4.4, we obtain
\[
\int_{0}^{\infty} |\tilde{w}(\xi, s)|^2 \rho_d d\xi = O(e^{-2\theta s})
\]
for all sufficiently large $s > 0$. Thus Proposition 4.1 follows. \(\square\)

**Proposition 4.2.** Let \(\|\varphi\|_{L^2(\mathbb{R}^n, \nu_d)} = 1\). Assume the same conditions as in Theorem 1.1. Then
\[
|a(s) - m(\varphi)| = O(e^{-2\theta s}),
\]
\[
|\tilde{w}(s) - m(\varphi)\psi_d|_{L^2(\mathbb{R}^n, \rho_d d\xi)} = O(e^{-\theta s}),
\]
for all sufficiently large $s > 0$. Furthermore, if $m(\varphi) = 0$, then
\[
\|\tilde{w}(s)\|_{L^2(\mathbb{R}^n, \rho_d d\xi)} = O(e^{-s}), \quad \|w(s)\|_{L^2(\mathbb{R}^n, \rho_d d\xi)} = O(e^{-s}),
\]
(4.36)
for all sufficiently large $s > 0$.

**Proof.** By (4.24) we can find a constant $a_\infty$ such that
\[
|a(s) - a_\infty| = O(e^{-2\theta s}) \quad \text{as} \quad s \to \infty.
\]
(4.37)
On the other hand, by Lemma 4.3 we have
\[
\int |\tilde{w}(\xi, s)|^2 \rho_d d\xi \leq \left( \int |\tilde{w}(\xi, s)|^2 \rho_d d\xi \right)^{\frac{1}{2}} \left( \int \xi^{d-1} e^{-\frac{\theta_s}{2}} d\xi \right)^{\frac{1}{2}} = O(e^{-d\theta_s})
\]
(4.38)
for all sufficiently large $s > 0$, where $I(s)^c := \mathbb{R}^n \setminus I(s)$. By Lemma 4.3, (4.1), (4.11) and (4.38) we obtain
\[
a(s) = c_d \int_{I(s)} w \xi^{d-1} d\xi + O(e^{-d\theta_s})
\]
\[
= c_d \int_{(1+t)^{\frac{1}{2}-s}}^{\infty} u_*(r, t) U_d(r) r^{d-1} dr + O(e^{-d\theta_s})
\]
\[
= \frac{c_d}{c_s} \int_{(1+t)^{\frac{1}{2}-s}}^{\infty} u_*(r, t) v_d(r) r^{d-1} dr + o(1)
\]
(4.39)
for all sufficiently large $s > 0$ and $t > 0$ with $s = \log(1 + t)$.

On the other hand, by (4.28) we apply Proposition 3.1 with $D = d/4$ and $D' = 0$ to obtain
\[
\sup_{0 \leq r \leq (1+t)^{\frac{d}{2}}} |u_*(r, t)| = O(t^{-\frac{d}{2}})
\]
(4.40)
for all sufficiently large \( t > 0 \). Combining (4.40) with (4.4), we see that
\[
\int_0^1 (1+ \frac{1}{t}) u_*(r, t) \nu_d(r)r^{d-1} dr = O(t^{-\frac{d}{2}}) \int_0^1 \nu_d(r)r^{d-1} dr = O(t^{-\frac{d}{2}})O(t^{-\frac{d}{2}}) = O(t^{-\frac{d}{2}})
\]
for all sufficiently large \( t > 0 \). Therefore, by (4.39) and (4.41) we obtain
\[
a_\infty = \lim_{s \to \infty} a(s) = \lim_{t \to \infty} \frac{c_d}{c_*} \int_0^\infty u_*(r, t) \nu_d(r)r^{d-1} dr.
\]
(4.42)
On the other hand, since \( u_* \) is a radial solution of problem \( (P) \), we have
\[
\int_0^\infty u_*(r, t) \nu_d(r)r^{d-1} dr = \int_0^\infty u_*(r, t) \nu(r)r^{N-1} dr
\]
(4.43)
We deduce from (4.42) and (4.43) that \( a_\infty = m(\varphi) \). This together with (4.37) implies (4.34). Furthermore, by Proposition 4.1 and (4.34) we have (4.35).

It remains to prove (4.36). Assume that \( m(\varphi) = 0 \). Then it follows from (4.35) and Lemma 4.3 that
\[
||\tilde{w}(s)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = O(e^{-\theta s}), \quad ||w(s)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = O(e^{-\theta s}),
\]
(4.44)
for all sufficiently large \( s > 0 \). Applying the same argument as in the proof of (4.32), we see that
\[
\frac{d}{ds} \int_{f(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi \leq - \int_{f(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-4\theta s})
\]
for all sufficiently large \( s > 0 \). Furthermore, similarly to (4.44), we have
\[
||\tilde{w}(s)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = O(e^{-2\theta s}), \quad ||w(s)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = O(e^{-2\theta s}),
\]
(4.45)
for all sufficiently large \( s > 0 \). Repeating this argument, we can find \( \tilde{\theta} > 1 \) such that
\[
\frac{d}{ds} \int_{f(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi \leq - \int_{f(s)} |\tilde{w}(\xi, s)|^2 \rho_d d\xi + O(e^{-\tilde{\theta} s})
\]
for all sufficiently large \( s > 0 \), instead of (4.32). This implies that
\[
||\tilde{w}(s)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = O(e^{-s}), \quad ||w(s)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = O(e^{-s}),
\]
for all sufficiently large \( s > 0 \). Thus (4.36) holds. Therefore the proof of Proposition 4.2 is complete. \( \square \)

We are ready to complete the proof of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** By the linearity of the operator \( L \) it suffices to consider only the case
\[
1 = ||\varphi||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = ||\varphi_*||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)} = |S^{N-1}|^{1/2} ||w(0)||_{L^2(\mathbb{R}_{+}\cup \mathbb{R}_d \cup \mathbb{D}_3)}.
\]
(4.45)
Let $R > 1$. By Lemma 4.2 we apply the parabolic regularity theorems (see e.g., [20]) to (1.16). Then we can find $\alpha \in (0, 1)$ such that

$$\|w\|_{C^{2,\alpha}_{\max}(I_R \times (S, \infty))} := \sum_{0 \leq \ell + 2j \leq 2} \sup_{\xi \in I_R, s \in (0, S)} |(\partial^\ell \partial^j_x w)(\xi, s)|$$

$$+ \sum_{\ell + 2j = 2} \sup_{(\xi_1, s_1), (\xi_2, s_2) \in I_R \times (0, S), (\xi_1, s_1) \neq (\xi_2, s_2)} \frac{|(\partial^\ell \partial^j_x w)(\xi_1, s_1) - (\partial^\ell \partial^j_x w)(\xi_2, s_2)|}{|\xi_1 - \xi_2|^{\alpha} + |s_1 - s_2|^{\alpha/2}} < \infty$$

(4.46)

for any $R > 1$ and $S > 0$, where $I_R := [R^{-1}, R]$. Therefore, for any sequence $\{s_i\} \subset (0, \infty)$ with $\lim_{i \to \infty} s_i = \infty$, by Proposition 4.2 and (1.16) we apply the Ascoli-Arzelà theorem and the diagonal argument to find a subsequence $\{s_i'\} \subset \{s_i\}$ such that

$$\lim_{i \to \infty} \|w(s_i') - m(\varphi)\psi_d\|_{C^2(I_R)} = 0, \quad \lim_{i \to \infty} \|\partial_s w(s_i')\|_{C^2(I_R)} = 0,$$

for any $R > 1$. Since $m(\varphi)\psi_d$ is independent of the choice of $\{s_i'\}$, we see that

$$\lim_{s \to \infty} \|w(s) - m(\varphi)\psi_d\|_{C^2(I_R)} = 0, \quad \lim_{s \to \infty} \|\partial_s w(s)\|_{C^2(I_R)} = 0,$$

(4.47)

for any $R > 1$. Furthermore, if $a_\infty = m(\varphi) = 0$, then, similarly to (4.47), by (4.36) we have

$$\sup \left\{ \|(\partial^\ell_x w)(\xi, s)\| : \xi \in I_R, s \geq S \right\} = O(e^{-s}) \quad \text{as} \quad s \to \infty$$

for any $R > 1$, where $\ell = 0, 1, 2$. These together with Proposition 4.2 imply (1.10) and (1.12). Thus Theorem 1.1 follows.

Proof of Theorem 1.2. Similarly to the proof of Theorem 1.1, we can assume (4.45) without loss of generality. Let $T > 0$ and let $\epsilon$ be any sufficiently small positive constant. By Lemma 4.2 and (4.3), applying Proposition 3.1 with $D = d/4$ and $D' = 0$, we obtain

$$(\partial_t^j u_*)(|x|, t) = (\partial_t^j u_*)(0, t) + F_d^j(|x|, t) \quad \text{in} \quad \mathbb{R}^N \times (0, \infty),$$

(4.48)

where $j \in \{0, 1, 2, \ldots\}$. Furthermore,

$$|F_d^j(r, t)| \leq C t^{-\frac{d}{2} - j - 1} r^2, \quad |(\partial_r F_d^j)(r, t)| \leq C t^{-\frac{d}{2} - j - 1} r,$$

(4.49)

for $0 \leq r \leq \epsilon(1 + t)^{1/2}$ and $t \geq T$. Then it follows from (4.48) and (4.49) that

$$|(\partial_r u_*)(r, t)| \leq C t^{-\frac{d}{2} - 1} r$$

(4.50)

for $0 \leq r \leq \epsilon(1 + t)^{1/2}$ and $t \geq T$. Furthermore, by (4.3) and (4.48) we have

$$F_d^0(r, t) = \int_0^r s^{1 - d}[\nu_d(s)]^{-1} \left( \int_0^s \tau^{d - 1} \nu_d(\tau) (\partial_t u_*)(\tau, t) d\tau \right) ds$$

$$= \int_0^r s^{1 - d}[\nu_d(s)]^{-1} \left( \int_0^s \tau^{d - 1} \nu_d(\tau) \left[ (\partial_t u_*)(0, t) + F_d^1(\tau, t) \right] d\tau \right) ds$$

(4.51)

$$= (\partial_t u_*)(0, t) F_d(r) + G_d(r, t)$$

for $r \geq 0$ and $t > 0$, where $F_d$ is given in Theorem 1.2 and

$$G_d(r, t) = \int_0^r s^{1 - d}[\nu_d(s)]^{-1} \left( \int_0^s \tau^{d - 1} \nu_d(\tau) F_d^1(\tau, t) d\tau \right) ds.$$  

(4.52)
Then (1.13) holds. In addition, by (4.4), (4.49) and (4.52) we have
\[
|G_d(r, t)| \leq C t^{-\frac{d}{2} - 2} \int_0^r s^{1-d} [\nu_d(s)]^{-1} \left( \int_0^s \tau^{d+1} \nu_d(\tau) d\tau \right) ds \\
\leq C t^{-\frac{d}{2} - 2} \int_0^r s^{1-d} [\nu_d(s)]^{-1} \cdot s^{d+2} \nu_d(s) ds \leq C t^{-\frac{d}{2} - 2} r^4
\] 
(4.53)
for \(0 \leq r \leq \epsilon (1 + t)^{1/2}\) and \(t \geq T\). A similar argument with (4.1) implies that
\[
|\partial_t^\ell G_d(r, t)| \leq C t^{-\frac{d}{2} - 2} r^{4-\ell}
\]
for \(0 \leq r \leq \epsilon (1 + t)^{1/2}\) and \(t \geq T\), where \(\ell \in \{0, 1, 2\}\). Thus (1.14) holds for \(\ell \in \{0, 1, 2\}\).

It remains to prove assertion (b). By (4.11) and (4.48) we have
\[
w(\xi, s) = (1 + t)^{\frac{d}{2}} U_d(r) u_*(r, t) = (1 + t)^{\frac{d}{2}} U_d(r) \left[ u_*(0, t) + F_d^0(r, t) \right]
\] 
(4.54)
for \(\xi \in (0, \infty)\) and \(s > 0\) with \(\xi = (1 + t)^{-1/2} r\) and \(s = \log(1 + t)\). Let \(0 < \xi < \epsilon\). By (4.1), (4.49) and (4.52) we obtain
\[
|w(\xi, s) - (1 + t)^{\frac{d}{2}} (c_* + o(1)) u_*(0, t)| \leq C \xi^2
\]
(4.55)
for all sufficiently large \(s > 0\) and \(t > 0\) with \(s = \log(1 + t)\) and \(0 < \xi < \epsilon\). On the other hand, it follows from (4.47) that
\[
\lim_{s \to \infty} w(\xi, s) = c_d m(\varphi) e^{-\frac{\xi^2}{2}}.
\] 
(4.56)
Then we deduce from (4.55) and (4.56) that
\[
\lim_{t \to \infty} t^\frac{d}{2} u_*(0, t) = \frac{c_d}{c_*} m(\varphi).
\] 
(4.57)
Furthermore, it follows from (4.11) that
\[
(\partial_s w)(\xi, s) = \frac{d}{2} w(\xi, s) + e^{\frac{d+1}{2} \tau} U'_d(\xi \tau) \xi \frac{1}{2} u_*(\xi \tau, \tau) \]
\[
+ e^{\frac{d+1}{2} \tau} U_d(\xi \tau) \xi \frac{1}{2} (\partial_t u_*)(\xi \tau, \tau) + e^{\frac{d+1}{2} \tau} U_d(\xi \tau) (\partial_t u_*)(\xi \tau, \tau, e^s - 1).
\] 
This together with (4.1), (4.50) and (4.56) implies that
\[
(\partial_s w)(\xi, s) = \frac{d}{2} w(\xi, s) + e^{\frac{d+1}{2} \tau} \frac{U'_d(\xi \tau)}{U_d(\xi \tau)} \frac{1}{2} w(\xi, s) \\
+ O(\xi^2) + e^{\frac{d+1}{2} \tau} (c_* + o(1)) (\partial_t u_*)(\xi \tau, e^s - 1) \\
= \frac{d}{2} m(\varphi) c_d e^{-\frac{\xi^2}{2}} + o(1) + O((e^{\frac{d+1}{2} \tau})^{-\delta}) \\
+ O(\xi^2) + e^{\frac{d+1}{2} \tau} (c_* + o(1)) (\partial_t u_*)(e^{\xi \tau}, e^s - 1)
\] 
(4.58)
for all sufficiently large \(s > 0\). On the other hand, by (4.48) and (4.49) we have
\[
e^{\frac{d+1}{2} \tau} (\partial_t u_*)(e^{\xi \tau}, e^s - 1) = e^{\frac{d+1}{2} \tau} (\partial_t u_*)(0, e^s - 1) + O(\xi^2)
\] 
(4.59)
for all sufficiently large \(s > 0\). Therefore, by (4.47), (4.58) and (4.59) we obtain
\[
\limsup_{s \to \infty} \left| (c_* + o(1)) e^{\frac{d+1}{2} \tau} (\partial_t u_*)(0, e^s - 1) + \frac{d}{2} c_d m(\varphi) \right| \leq C \xi^2.
\]
Since $0 < \xi < \epsilon$, we deduce that
\[
\limsup_{s \to \infty} \left| \int_{T}^{s} (\partial_{t}u_{s})(0, \epsilon^{s} - 1) + \frac{dc_{s}}{2\epsilon_{s}} m(\varphi) \right| = 0.
\]
This together with (4.57) implies assertion (b). Thus Theorem 1.2 follows.

**Proof of Theorem 1.3.** Similarly to the proof of Theorem 1.1, we can assume (4.45) without loss of generality. Assume the same conditions as in Theorem 1.3. Then $d = 2$ and $V_{\lambda_{2}}$ satisfies condition (V) with $\lambda_{1}$ and $\lambda_{2}$ replaced by $\lambda_{1} - \lambda_{2} (\geq 0)$ and 0, respectively. Applying a similar argument as in the proof of argument as in [15, Proposition 3.1], we have
\[
\lim_{s \to \infty} sw(\xi, s) = \frac{1}{\epsilon_{s}} \left[ \int_{0}^{\infty} w(r, 0)U_{d}(r) r dr \right] e^{-\frac{\epsilon^{2}}{4}} = 2m(\varphi)\psi_{d}(\xi) \tag{4.60}
\]
in $L^{2}(\mathbb{R}_{+}, \rho_{2} d\xi) \cap C^{2}(K)$, for any compact set $K$ in $\mathbb{R}^{2} \setminus \{0\}$. Furthermore,
\[
\lim_{t \to \infty} t(\log t)^{2}u_{s}(0, t) = 2\sqrt{2}\epsilon_{s}^{-1}m(\varphi), \quad \lim_{t \to \infty} t^{2}(\log t)^{2}(\partial_{t}u_{s})(0, t) = -2\sqrt{2}\epsilon_{s}^{-1}m(\varphi).
\]

On the other hand, similarly to (4.48), we have
\[
(\partial^{2}_{t}u_{s})(|x|, t) = (\partial^{2}_{t}u_{s})(0, t) + F_{2}^{j}(|x|, t) \quad \text{in} \quad \mathbb{R}^{N} \times (0, \infty),
\]
where $j \in \{0, 1, 2, \ldots\}$. It follows from (4.60) that
\[
\sup_{t > 0} (1 + t)^{\frac{d}{2}} \log(2 + t) \|u_{s}(t)\|_{L^{2}(\mathbb{R}^{N}, r^{\frac{d}{2}} \log(2 + t)^{-2} dx)} < \infty.
\]

Let $T > 0$ and $\epsilon$ be a sufficiently small positive constant. Then, by (4.3) we apply Proposition 3.1 with $D = d/4$ and $D' = 1$ to obtain
\[
\left| F_{2}^{j}(r, t) \right| \leq C\epsilon^{-j-1} \log(2 + t)^{-2} r^{2}
\]
for $0 \leq r \leq \epsilon(1 + t)^{1/2}$ and $t \geq T$. Similarly to (4.51) and (4.52), we have
\[
F_{2}^{0}(r, t) = (\partial_{t}u_{s})(0, t)F_{2}(r, t) + G_{2}(r, t),
\]
\[
G_{2}(r, t) = \int_{0}^{r} s^{-1}[\nu_{2}(s)]^{-1} \left( \int_{0}^{s} \tau \nu_{2}(\tau)F_{2}^{1}(\tau, t) d\tau \right) ds,
\]
for $r \geq 0$ and $t > 0$. Furthermore, similarly to (4.53), we obtain
\[
|G_{2}(r, t)| \leq C\epsilon^{-3} \log(2 + t)^{-2} \int_{0}^{r} s^{-1}[\nu_{2}(s)]^{-1} \left( \int_{0}^{s} \tau^{3} \nu_{2}(\tau) d\tau \right) ds
\]
\[
\leq C\epsilon^{-3} \log(2 + t)^{-2} r^{4}
\]
for $0 \leq r \leq \epsilon(1 + t)^{1/2}$ and $t \geq T$. A similar argument with (4.2) implies that
\[
| (\partial^{\ell}_{t}G_{2})(r, t) | \leq C\epsilon^{-3} \log(2 + t)^{-2} r^{4-\ell}, \quad \ell = 1, 2,
\]
for $0 \leq r \leq \epsilon(1 + t)^{1/2}$ and $t \geq T$. So we see that (1.15) holds for $\ell \in \{0, 1, 2\}$. Thus Theorem 1.3 follows. \(\square\)
5. Proof of Theorem 1.4. We use the same notation as in Section 1.2. Let
\( m \in \{1, 2, \ldots\} \). Then
\[
L_m := -\Delta + V(|x|) + \frac{\omega_m}{|x|^2}
\]
is subcritical and problem (O) corresponding to \( L_m \) possesses a positive solution \( U_m \) satisfying
\[
U_m(r) \sim r^{A^+(\lambda_1 + \omega_m)} \quad \text{as} \quad r \to +0,
\]
\[
U_m(r) \sim c_m r^{A^+(\lambda_2 + \omega_m)} \quad \text{as} \quad r \to \infty,
\]
for some positive constant \( c_m \). Set
\[
u(x, t) := e^{-tL} \varphi, \quad u_m(x, t) := \nu(x, t) - \sum_{k=0}^{m-1} \sum_{i=1}^{\ell_k} e^{-tL} \varphi^{k,i}.
\]

**Lemma 5.1.** Let \( m \in \{1, 2, \ldots\} \). Then there exists \( C_1 > 0 \) such that
\[
\|u_m(t)\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \leq C_1 t^{-\frac{d-2}{2}} \|u_m(0)\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \quad (5.2)
\]
for \( t > 0 \), where \( d_m := N + 2A^+(\lambda_2 + \omega_m) \). Furthermore, there exists \( C_2 > 0 \) such that
\[
\frac{\|u_m(x, t)\|}{U(\min\{|x|, \sqrt{t}\})} \leq C_2 t^{-\frac{N-d_m}{2}} U(\sqrt{t})^{-1} \|u_m(0)\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \quad (5.3)
\]
for \( x \in \mathbb{R}^N \) and \( t > 0 \).

**Proof.** Let \( m \in \{1, 2, \ldots\} \). The comparison principle implies that
\[
\left| [e^{-tL_k} \phi^{k,i}] (x) \right| \leq \left| [e^{-tL_m} \phi^{k,i}] (x) \right| \leq \left| [e^{-tL_m} \phi^{k,i}] (x) \right| \quad \text{in} \quad \mathbb{R}^N \times (0, \infty)
\]
for \( k \in \{m, m+1, \ldots\} \) and \( i \in \{1, \ldots, \ell_k\} \). On the other hand, by Theorem 1.1 and (5.1) (see also (4.28)) we have
\[
\|e^{-tL_m} \phi^{k,i}\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \leq C(1 + t)^{-\frac{d-2}{2}} \|\phi^{k,i}\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)}, \quad t > 0,
\]
for \( k \in \{m, m+1, \ldots\} \) and \( i \in \{1, \ldots, \ell_k\} \). These together with (1.17) implies that
\[
\|e^{-tL} \varphi^{k,i}\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \leq C(1 + t)^{-\frac{d}{2}} \|\phi^{k,i}\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \leq C(1 + t)^{-\frac{d-2}{2}} \|\varphi^{k,i}\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)}
\]
for \( t > 0 \). Therefore we deduce from the orthogonality of \( \{Q_{k,i}\} \) that
\[
\|u_m(t)\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)} \leq \sum_{k=m}^{\infty} \sum_{i=1}^{\ell_k} \left\| e^{-tL} \phi^{k,i} \right\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)}^2
\]
\[
\leq C(1 + t)^{-\frac{d-2}{2}} \sum_{k=m}^{\infty} \sum_{i=1}^{\ell_k} \left\| \phi^{k,i} \right\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)}^2
\]
\[
\leq C(1 + t)^{-\frac{d-2}{2}} \|u_m(0)\|_{L^2(\mathbb{R}^N, e^{-|x|^2/(4t+1)} dx)}^2
\]
for \( t > 0 \). This implies (5.2). On the other hand, by Lemma 2.3 we have
\[
\frac{\|u_m(x, 2t)\|}{U(\min\{|x|, \sqrt{t}\})} \leq C t^{-\frac{d}{2}} U(\sqrt{t})^{-1} \|u_m(t)\|_{L^2(\mathbb{R}^N)}, \quad x \in \mathbb{R}^N, \quad t > 0.
\]
This together with (5.2) implies (5.3). Thus Lemma 5.1 follows. \( \square \)
Proof of Theorem 1.4. Let \( \varphi \in L^2(\mathbb{R}^N, e^{\|x\|^2/4} \, dx) \) and \( v := e^{-tL_0} \varphi^{0,1} \). Let \( K \) be any compact set in \( \mathbb{R}^N \setminus \{0\} \) and \( R > 0 \). In cases (S) and (C), recalling that \( d = 2N + A \), by Theorems 1.1 and 1.2 we have

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} v \left( t \frac{y}{t}, t \right) = c_d m(\varphi^{0,1}) \|y\|^A e^{-\frac{|y|^2}{4}} \quad \text{in} \quad L^2(\mathbb{R}^N, e^{\|y\|^2/4} \, dy) \cap L^\infty(K),
\]

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} \frac{v(x, t)}{U(|x|)} = c_d \frac{m(\varphi^{0,1})}{\epsilon} \quad \text{in} \quad L^\infty(B(0, R)).
\]  

(5.4)

In case (S_\ast), Theorem 1.3 implies that

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} (\log t) v \left( t \frac{y}{t}, t \right) = 2c_d m(\varphi^{0,1}) \|y\|^A e^{-\frac{|y|^2}{4}} \quad \text{in} \quad L^2(\mathbb{R}^N, e^{\|y\|^2/4} \, dy) \cap L^\infty(K),
\]

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} (\log t)^2 \frac{v(x, t)}{U(|x|)} = \frac{2\sqrt{2}}{c_\ast} m(\varphi^{0,1}) = \frac{4c_d}{c_\ast} m(\varphi^{0,1}) \quad \text{in} \quad L^\infty(B(0, R)).
\]  

(5.5)

Here

\[
c_d m(\varphi^{0,1}) = \frac{c_\ast^2}{|S|^{N-1}} \int_0^\infty \varphi^{0,1}(r) U(r) r^{N-1} \, dr
\]

\[
= \frac{c_\ast^2}{|S|^{N-1}} \int_{\mathbb{R}^N} \varphi^{0,1}(|x|) U(|x|) \, dx
\]

\[
= \frac{c_\ast^2}{|S|^{N-1}} \int_{\mathbb{R}^N} \phi(x) U(|x|) \, dx = M(\varphi).
\]  

(5.6)

Taking a sufficiently large integer \( m \), by Lemma 5.1 we have

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} u_m \left( t \frac{y}{t}, t \right) = 0 \quad \text{in} \quad L^2(\mathbb{R}^N, e^{\|y\|^2/4} \, dy) \cap L^\infty(K),
\]

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} \frac{u_m(x, t)}{U(|x|)} = 0 \quad \text{in} \quad L^\infty(B(0, R)),
\]  

(5.7)

for any compact set \( K \subset \mathbb{R}^N \setminus \{0\} \) and \( R > 0 \). On the other hand, \( L_k \) is subcritical and \( A^+(\lambda_2 + \omega_k) > A \) for \( k \in \{1, 2, \ldots, m-1\} \). Then, taking a sufficiently small \( \epsilon > 0 \) if necessary, by Theorems 1.1 and 1.2 we obtain

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} \left[ e^{-tL_k} \phi^{k,i} \right] \left( t \frac{y}{t}, t \right) = 0 \quad \text{in} \quad L^2(\mathbb{R}^N, e^{\|y\|^2/4} \, dy) \cap L^\infty(K),
\]

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} \frac{\left[ e^{-tL_k} \phi^{k,i} \right]}{U(|x|)} (x) = 0 \quad \text{in} \quad L^\infty(B(0, R)),
\]  

(5.8)

for any compact set \( K \subset \mathbb{R}^N \setminus \{0\} \) and \( R > 0 \). On the other hand, it follows from (1.2) that (5.1) that \( U_k(r)/U(r) \) is bounded on \((0, R)\) for any \( R > 0 \). This together with (5.8) implies that

\[
\lim_{t \to \infty} t^{\frac{N+A}{4}} \frac{\left[ e^{-tL_k} \phi^{k,i} \right]}{U(|x|)} (|x|) = 0 \quad \text{in} \quad L^\infty(B(0, R))
\]  

(5.9)

for any \( R > 0 \). Since

\[
[ e^{-tL} \varphi ] (x) = v(x, t) + \sum_{k=1}^{m-1} \sum_{i=1}^{k} [ e^{-tL_k} \phi^{k,i} ] (|x|) Q_{k,i} \left( \frac{x}{|x|} \right) + u_m(x, t),
\]

by (5.4)–(5.9) we obtain assertions (a) and (b). Thus the proof is complete. \( \square \)
Proof of Corollary 1.1. Let \( p = p(x,y,t) \) be the fundamental solution corresponding to \( e^{-tL} \). Let \( y \in \mathbb{R}^N \) and \( \tau > 0 \). Set \( \varphi(x) = p(x,y,\tau) \) for \( x \in \mathbb{R}^N \). Taking a sufficiently small \( \tau > 0 \) if necessary, by (1.6) we see that \( \varphi \in L^2(\mathbb{R}^N, e|x|^2/4 \, dx) \).

On the other hand, since \( \tau > 0 \), we have

\[
\int_{\mathbb{R}^N} \varphi(x) U(|x|) \, dx = \int_{\mathbb{R}^N} p(x,y,\tau) U(|x|) \, dx = \int_{\mathbb{R}^N} p(y,x,\tau) U(|x|) \, dx = U(|y|)
\]

for \( y \in \mathbb{R}^N \) and \( \tau > 0 \). Then, applying Theorem 1.4 and letting \( \tau \to +0 \), we obtain the desired results. Thus Corollary 1.1 follows.

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REFERENCES

[1] G. Barbatis, S. Filippas and A. Tertikas, Critical heat kernel estimates for Schrödinger operators via Hardy-Sobolev inequalities, J. Funct. Anal., 208 (2004), 1–30.
[2] I. Chavel and L. Karp, Large time behavior of the heat kernel: The parabolic \( \lambda \)-potential alternative, Comment. Math. Helv., 66 (1991), 541–556.
[3] F. Chiarenza and R. Serapioni, A remark on a Harnack inequality for degenerate parabolic equations, Rend. Sem. Mat. Univ. Padova 73 (1985), 179–190.
[4] D. Cruz-Uribe and C. Rios, Gaussian bounds for degenerate parabolic equations, J. Funct. Anal., 255 (2008), 283–312; Corrigendum in J. Funct. Anal., 267 (2014), 3507–3513.
[5] E. B. Davies, Heat Kernels and Spectral Theory, Cambridge Tracts in Math. 92, Cambridge Univ. Press 1989.
[6] E. B. Davies and B. Simon, \( L^p \) norms of noncritical Schrödinger semigroups, J. Funct. Anal., 102 (1991), 95–115.
[7] A. Grigor’yan, Heat Kernel and Analysis on Manifolds, AMS, Providence, RI, 2009.
[8] A. Grigor’yan and L. Saloff-Coste, Stability results for Harnack inequalities, Ann. Inst. Fourier, 55 (2005), 825–890.
[9] N. Ioku, K. Ishige and E. Yanagida, Sharp decay estimates of \( L^q \) norms of nonnegative Schrödinger heat semigroups, J. Funct. Anal., 264 (2013), 2764–2783.
[10] N. Ioku, K. Ishige and E. Yanagida, Sharp decay estimates in Lorentz spaces for nonnegative Schrödinger heat semigroups, J. Math. Pures Appl., 103 (2015), 900–923.
[11] K. Ishige, On the behavior of the solutions of degenerate parabolic equations, Nagoya Math. J., 155 (1999), 1–26.
[12] K. Ishige, Movement of hot spots on the exterior domain of a ball under the Neumann boundary condition, J. Differential Equations, 212 (2005), 394–431.
[13] K. Ishige and Y. Kabeya, Large time behaviors of hot spots for the heat equation with a potential, J. Differential Equations, 244 (2008), 2934–2962; Corrigendum in J. Differential Equations 245 (2008), 2352–2354.
[14] K. Ishige and Y. Kabeya, Hot spots for the heat equation with a rapidly decaying negative potential, Adv. Differential Equations, 14 (2009), 643–662.
[15] K. Ishige and Y. Kabeya, Hot spots for the two dimensional heat equation with a rapidly decaying negative potential, Discrete Contin. Dyn. Syst. Ser. S, 4 (2011), 833–849.
[16] K. Ishige and Y. Kabeya, \( L^p \) norms of nonnegative Schrödinger heat semigroup and the large time behavior of hot spots, J. Funct. Anal., 262 (2012), 2695–2733.
[17] K. Ishige and Y. Kabeya, Decay rate of \( L^q \) norms of critical Schrödinger heat semigroups, Geometric Properties for Parabolic and Elliptic PDE’s, 165–178, Springer INdAM Ser., 2, Springer, Milan, 2013.
[18] K. Ishige, Y. Kabeya and A. Mukai, Hot spots of solutions to the heat equation with inverse square potential, to appear in Applicable Anal., (2018), https://doi.org/10.1080/00036811.2018.1466284.
[19] K. Ishige, Y. Kabeya and E. M. Ouhabaz, The heat kernel of a Schrödinger operator with inverse square potential, Proc. Lond. Math. Soc., 115 (2017), 381–410.
[20] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva, Linear and Quasi-linear Equations of Parabolic Type, American Mathematical Society Translations, vol. 23, American Mathematical Society, Providence, RI, 1968.
[21] V. Liskevich and Z. Sobol, Estimates of integral kernels for semigroups associated with second-order elliptic operators with singular coefficients, Potential Anal., 18 (2003), 359–390.

[22] P. D. Milman and Y. A. Semenov, Global heat kernel bounds via desingularizing weights, J. Funct. Anal., 212 (2004), 373–398.

[23] N. Mizoguchi, H. Ninomiya and E. Yanagida, Critical exponent for the bipolar blowup in a semilinear parabolic equation, J. Math. Anal. Appl., 218 (1998), 495–518.

[24] L. Moschini and A. Tesei, Harnack inequality and heat kernel estimates for the Schrödinger operator with Hardy potential, Rend. Mat. Acc. Lincei, 16 (2005), 171–180.

[25] L. Moschini and A. Tesei, Parabolic Harnack inequality for the heat equation with inverse-square potential, Forum Math., 19 (2007), 407–427.

[26] M. Murata, Positive solutions and large time behaviors of Schrödinger semigroups, Simon’s problem, J. Funct. Anal., 56 (1984), 300–310.

[27] M. Murata, Structure of positive solutions to \((-\Delta + V)u = 0\) in \(\mathbb{R}^n\), Proceedings of the Conference on Spectral and Scattering Theory for Differential Operators (Fujisakura-so, 1986), (1986), 64–108.

[28] E. M. Ouhabaz, Analysis of Heat Equations on Domains, London Math. Soc. Monographs, 31, Princeton Univ. Press 2005.

[29] Y. Pinchover, On criticality and ground states of second order elliptic equations, II, J. Differential Equations, 87 (1990), 353–364.

[30] Y. Pinchover, Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators, J. Funct. Anal., 104 (1992), 54–70.

[31] Y. Pinchover, On positivity, criticality, and the spectral radius of the shuttle operator for elliptic operators, Duke Math. J., 85 (1996), 431–445.

[32] Y. Pinchover, Large time behavior of the heat kernel, J. Funct. Anal., 206 (2004), 191–209.

[33] Y. Pinchover, Some aspects of large time behavior of the heat kernel: An overview with perspectives, Mathematical Physics, Spectral Theory and Stochastic Analysis (Basel) (M. Demuth and W. Kirsch, eds.), Operator Theory: Advances and Applications, vol. 232, Springer Verlag, 2013, 299–339.

[34] B. Simon, Large time behavior of the \(L^p\) norm of Schrödinger semigroups, J. Funct. Anal., 40 (1981), 66–83.

[35] J. L. Vázquez and E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal., 173 (2000), 103–153.

[36] Q. S. Zhang, Large time behavior of Schrödinger heat kernels and applications, Comm. Math. Phys., 210 (2000), 371–398.

[37] Q. S. Zhang, Global bounds of Schrödinger heat kernels with negative potentials, J. Funct. Anal., 182 (2001), 344–370.

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