Interference and X Networks With Noisy Cooperation and Feedback
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Abstract—The Gaussian $K$-user interference and $M \times K$ X channels are investigated with no instantaneous channel state information at transmitters (CSIT). First, it is assumed that the CSI is fed back to all nodes after a finite delay (delayed CSIT), and furthermore, the transmitters operate in full-duplex mode, i.e., they can transmit and receive simultaneously. Achievable results on the degrees of freedom (DoFs) of these channels under the above assumption are obtained. It is observed that, in contrast with no CSIT and full CSIT models, when CSIT is delayed, the achievable DoFs for both channels with the full-duplex transmitter cooperation are greater than the available achievable results on their DoF without transmitter cooperation. Then, $K$-user interference and $K \times K$ X channels are considered with output feedback, wherein the channel output of each receiver is causally fed back to its corresponding transmitter. Our achievable results with output feedback demonstrate strict DoF improvements over those with the full-duplex delayed CSIT when $K > 5$ in the $K$-user interference channel and $K > 2$ in the $K \times K$ X channel. Next, the combination of delayed CSIT and output feedback, known as Shannon feedback, is studied and strictly higher DoFs compared with the output feedback model are achieved in the $K$-user interference channel when $K = 5$ or $K > 6$, and in the $K \times K$ X channel when $K > 2$.

Index Terms—Interference channel, X channel, feedback, delayed CSIT, full-duplex, degrees of freedom.

I. INTRODUCTION

The crucial role of feedback in reliability, throughput, and complexity of transmission over communication networks has made it an indispensable ingredient of all modern communication systems. In spite of the first result by Shannon that shows the capacity of a memoryless point-to-point channel is not increased with feedback [2], it has been proved that feedback enlarges the capacity region of several multi-user channels. The capacity regions of the additive white Gaussian noise (AWGN) multiple-access, broadcast, and interference channels are enlarged with noiseless output feedback as shown in [3]–[5]. It was shown in [6] that even a single output feedback link from one of receivers enlarges the capacity region of the two-user AWGN broadcast channel.

In fading channels, without any feedback, and hence, without CSI at any transmitter (no CSIT), the capacity regions of single-input single-output (SISO) fading two-user broadcast and two-user $Z$-interference channels were characterized to within constant number of bits (see [7], [8] and references therein). Also, it was shown in [9] that a large class of multiple-input single-output (MISO) multi-user channels including broadcast, interference, X, and cognitive radio channels can achieve no more that one degree of freedom (DoF) with no CSIT. As a first order approximation, the DoF of a channel characterizes its sum-capacity in high signal-to-noise ratio (SNR), i.e.,

$$C(SNR) = \text{DoF} \times \log_2(SNR) + o(\log_2(SNR)),$$

where $C(SNR)$ is the sum-capacity for a given SNR and $\text{DoF}$ is the channel sum-DoF, or simply, DoF. When there is CSI feedback to transmitter(s) and the channel variations are not too fast, it is commonly assumed that the CSI obtained through feedback links is valid at least over the current channel use, and hence, the transmitter(s) have access to perfect and instantaneous CSI (full CSIT). In this case, using the interference alignment technique, the $K$-user SISO interference channel (IC) and $M \times K$ SISO X channel$^2$ were shown to have $K/2$ and $MK/(M+K-1)$ DoF, respectively, in [10] and [11].

If the feedback delay is greater than coherence time of the channel, the CSI obtained through feedback links is outdated. This makes the "full CSIT assumption" practically implausible, since the CSIT expires prior to the beginning of each channel use. Nevertheless, it was established that the outdated CSIT (known as delayed CSIT) yields DoF gain in broadcast channels [12]–[14] and interference and X channels [15]–[20]. The two-user MIMO interference channel with both delayed CSIT and output feedback was studied recently in [21] and [22].

Output feedback in multi-user channels with distributed transmitters, such as IC and X channel, naturally provides some level of transmitter cooperation. As such, there are connections between communication over these channels with feedback and that with transmitter cooperation. A common

$^2$For the sake of brevity, we omit the term "SISO" from the SISO interference and X channels throughout this paper.
cooperation setup is the operation of transmitters in full-duplex mode, i.e., to transmit and receive simultaneously in the same time-frequency resource. The two-user IC with full-duplex transmitters and with full CSIT was investigated in [23]–[27]. In [24], [25], and [27] achievable schemes are proposed based on further splitting the common and/or private information of the Han-Kobayashi scheme into two parts, namely, non-cooperative and cooperative part. The cooperative part is decoded at the other transmitter as well to be able to cooperate in delivering the information to the desired receiver. By developing an upper bound, the sum-capacity of the two-user Gaussian IC with full-duplex transmitters was obtained to within a constant number of bits in [24]. Moreover, it was shown in [28] and [29] that under the full CSIT assumption, the full-duplex cooperation and/or output feedback cannot increase DoF of the $K$-user IC and $M \times K$ X channel. In other words, the full-duplex cooperation as well as output feedback can only yield “additive” capacity increase in the aforementioned channels when full CSI is available at transmitters. Also, the full-duplex transmitter cooperation cannot help these channels to achieve more than one DoF with no CSIT, since the MISO broadcast channel has only one DoF with no CSIT [9].

In this paper, we address the problem of communication over the $K$-user IC and $M \times K$ X channel with no instantaneous CSIT, and study the impact of full-duplex transmitter cooperation and/or different types of feedback on DoF of these channels. Specifically, after describing the problem formulation in Section II, we present our main results in Section III. Then, we offer some illustrative examples for the IC in Section IV-A and X channel in Section IV-B. These examples highlight how we exploit the transmitter cooperation and/or different feedback models to achieve DoF gains in the channels with a few number of users. We provide proofs of our main results in the subsequent sections. In particular, we consider these channels with delayed CSIT and full-duplex transmitter cooperation in Section V. Regarding the CSI of the transmitter-to-transmitter channels, we assume that each transmitter has access only to its incoming CSI. We propose transmission schemes that achieve DoF values greater than the DoF values reported previously in [18] for these channels with delayed CSIT but without transmitter cooperation.

In Section VI, we consider the same channels with output feedback, wherein we assume each transmitter has a causal access to the output of its paired receiver through a feedback link. This is indeed a limited output feedback (in contrast to providing each transmitter with the outputs of more than one receiver), however, the term “limited” will be dropped henceforth for brevity. Therefore, in the X channel, we hereafter consider only $M = K$ with a one-to-one mapping between transmitters and receivers for feedback assignment. The 3-user IC and $2 \times 2$ X channel with output feedback were investigated previously in [15], for which 6/5 and 4/3 DoF were achieved respectively. While achieving the same DoFs for the 3-user IC and $2 \times 2$ X channel, our main contribution here is to propose systematic multi-phase transmission schemes for the general $K$-user settings, which achieve DoF values strictly increasing in $K$.

Next, we study the $K$-user IC and $K \times K$ X channel with delayed CSIT and output feedback in Section VII. Under this assumption, which is referred to as Shannon feedback, we propose multi-phase transmission schemes capturing both the delayed CSI and output feedback to transmit cooperatively over the channel. The achieved DoFs are greater than those we achieved with output feedback for $K = 5$ and $K > 6$ in the $K$-user IC and for $K > 2$ in the $K \times K$ X channel. The achievable results will be compared and discussed in Section VIII, and finally, the paper is concluded in Section IX.

II. PROBLEM FORMULATION

In the following, we provide the problem formulation through a number of definitions.

Definition 1 ($K$-User AWGN Interference Channel): A set of $K$ transmitters and $K$ receivers, depicted in Fig. 1, where transmitter $i$ (TX$_i$), $1 \leq i \leq K$, wishes to communicate a message $W[i] \in \{1, 2, 3, \ldots, 2^R[i]\}$ of rate $R[i]$ to receiver $i$ (RX$_i$) over a block of $\tau$ channel uses (or time slots).

In time slot $t$, $t = 1, 2, \ldots, \tau$, signal $x[i](t) \in \mathbb{C}$ is transmitted by TX$_i$, $1 \leq i \leq K$, and signal $y[j](t) \in \mathbb{C}$ is received by RX$_j$, $1 \leq j \leq K$, where

$$y[j](t) = \sum_{i=1}^{K} h[j][i](t)x[i](t) + z[j](t),$$

and $h[j][i](t) \in \mathbb{C}$ is the channel coefficient from TX$_i$ to RX$_j$, and $z[j](t) \sim \mathcal{CN}(0, 1)$ is the additive white Gaussian noise at RX$_j$. The transmitted signal $x[i](t)$, $1 \leq i \leq K$, is subject to power constraint $P$, i.e., $\mathbb{E}[|x[i](t)|^2] \leq P$. The $K \times K$ channel matrix $H(t)$ in time slot $t$ is defined as $H(t) \triangleq (h[j][i](t))_{1 \leq i,j \leq K}$. The channel coefficients are independent and identically distributed (i.i.d.) across all nodes as well as time slots. The channel coefficients are assumed to be drawn according to a finite-variance continuous distribution. Each receiver RX$_j$, $1 \leq j \leq K$, knows all its incoming channel coefficients in time slot $t$, i.e., $(h[j][i](t))_{i=1}^{K}$, perfectly and instantaneously.

Definition 2 ($M \times K$ AWGN X Channel): A set of $M$ transmitters and $K$ receivers as depicted in Fig. 2,
coefficients in time slot \( t \) are considered in this paper as defined below. The channel matrix \( H(t) \) here is a \( K \times M \) matrix defined as \( H(t) \triangleq \{ h_{j|i}(t) \}_{1 \leq i \leq M, 1 \leq j \leq K} \). Similar to the IC, each receiver RX\(_j\), \( 1 \leq j \leq K \), knows all its incoming channel coefficients in time slot \( t \), i.e., \( \{ h_{j|i}(t) \}_{i=1}^{M} \), perfectly and instantaneously.  

**Definition 3 (Feedback Models):** We assume that each receiver knows channel coefficients of the other receivers with one time slot delay. Moreover, three different feedback models are considered in this paper as defined below.  

- **Delayed CSIT:** The channel matrix \( H(t) \) will become available at all transmitters with one time slot delay via noisless feedback links.  
- **Output Feedback:** Each channel output \( y_{i}(t) \), \( 1 \leq i \leq K \), will become available at TX\(_i\) with one time slot delay via a noisless feedback link. Therefore, for the X channel, we only consider \( M = K \) under the output feedback assumption.  
- **Shannon Feedback:** The transmitters have access to both the delayed CSIT and the output feedback. Therefore, for the X channel, we only consider \( M = K \) under the Shannon feedback assumption.  

**Definition 4 (Full-Duplex Transmitter Cooperation):** The transmitters are said to operate in full-duplex mode if they can transmit and receive simultaneously. In full-duplex mode, the received signal of TX\(_i\) in time slot \( t \) is given by \( K \)-user IC:  

\[
\tilde{x}_{i}(t) = \sum_{i'=1}^{K} \tilde{h}_{i|i'}(t) x_{i'}(t) + \tilde{z}_{i}(t), \quad 1 \leq i \leq K. \tag{3}
\]

\( M \times X \) channel:  

\[
y_{i}(t) = \sum_{i=1}^{M} \tilde{h}_{i|i}(t) x_{i}(t) + \tilde{z}_{i}(t), \quad 1 \leq i \leq M. \tag{4}
\]

The noise terms and channel coefficients are assumed to be i.i.d. across all transmitters and time. No feedback link is available between the transmitters, and hence, TX\(_i\) is assumed to have only its incoming full-duplex channel coefficients, i.e., \( \{ \tilde{h}_{i|i'}(t) \}_{i'=1}^{K} \) in the IC and \( \{ h_{i|i'}(t) \}_{i'=1}^{M} \) in the X channel, perfectly and instantaneously.  

**Definition 5 (Block Code With Feedback):** A \( (2^R, \tau) \) code of block length \( \tau \) and rate \( R = \left( R_{\tau} \right)_{1 \leq \tau \leq K} \) with feedback in the \( K \)-user IC is defined as \( K \) sets of encoding functions \( \{ \psi_{i}[\tau] \}_{\tau=1}^{K}, 1 \leq i \leq K \), such that  

\[
x_{i}(t) = \psi_{i}[\tau] (W_{\tau}(t), \mathcal{I}(t)), \quad 1 \leq t \leq \tau, \tag{5}
\]

together with \( K \) decoding functions \( \psi_{i}^{r}[j], 1 \leq j \leq K \), such that \( \hat{W}_{\tau}^j = \psi_{i}^{r}[j] (\{ y_{i}(t) \}_{t=1}^{\tau}) \), where \( \mathcal{I}(t) \) is the side information available at TX\(_i\) before time slot \( t \), which will be defined later in this section. Similarly, a \( (2^R, \tau) \) code of block length \( \tau \) and rate \( R = \left( R_{\tau} \right)_{1 \leq \tau \leq K} \) with feedback in the \( M \times X \) channel is defined as \( M \) sets of encoding functions \( \{ \psi_{i}[\tau] \}_{\tau=1}^{M}, 1 \leq i \leq M \), such that  

\[
x_{i}(t) = \psi_{i}[\tau] (W_{\tau}(t), \mathcal{I}(t)), \quad 1 \leq t \leq \tau, \tag{6}
\]

together with \( K \) decoding functions \( \psi_{i}^{r}[j], 1 \leq j \leq K \), such that \( \hat{W}_{\tau}^j = \psi_{i}^{r}[j] (\{ y_{i}(t) \}_{t=1}^{\tau}) \).  

**Definition 6 (Transmitter Side Information):** Using Definitions 3 and 4, the following feedback and/or transmitter cooperation models are investigated in this paper, each of which is equivalent to a certain transmitter side information.  

(a) The \( K \)-user IC and \( M \times X \) channel with full-duplex transmitter cooperation and delayed CSIT (a.k.a. full-duplex delayed CSIT in this paper): **K-user IC:**  

\[
\mathcal{I}(t) = \left\{ \tilde{x}_{i}(t), H(t) \right\}_{i=1}^{I-1} \cup \left\{ \tilde{h}_{i|i}(t') : 1 \leq i' \leq K \right\}_{i'=1}^{I-1}, \quad 1 \leq \tau \leq K. \tag{7}
\]

\( M \times X \) channel:  

\[
\mathcal{I}(t) = \left\{ \tilde{x}_{i}(t'), H(t') \right\}_{i'=1}^{I-1} \cup \left\{ \tilde{h}_{i|i}(t') : 1 \leq i' \leq M \right\}_{i'=1}^{I-1}, \quad 1 \leq \tau \leq M. \tag{8}
\]

(b) The \( K \)-user IC and \( K \times K \) channel with output feedback:  

\[
\mathcal{I}(t) = \left\{ \tilde{x}_{i}(t') \right\}_{i'=1}^{I-1}, \quad 1 \leq \tau \leq K. \tag{9}
\]

c) The \( K \)-user IC and \( K \times K \) channel with Shannon feedback:  

\[
\mathcal{I}(t) = \left\{ \tilde{x}_{i}(t'), H(t') \right\}_{i'=1}^{I-1}, \quad 1 \leq \tau \leq K. \tag{10}
\]

**Definition 7 (Probability of Error, Achievable Rate, and Capacity Region):** Defining the probability of error of a code as the probability of decoding any of the transmitted messages incorrectly, a rate tuple \( R \) is said to be achievable if there exists a sequence \( \{(2^R, \tau) \}_{\tau=1}^{\infty} \) of codes such that their probability of error goes to zero as \( \tau \to \infty \). The closure of the set of all
achieved achievable rate tuples $\mathbf{R}$ is called the capacity region of the channel with power constraint $P$ and is denoted by $\mathcal{C}(P)$.

**Definition 8 (DoF):** If $\mathbf{R} = (R_1, R_2, \ldots, R_N) \in \mathcal{C}(P)$ is an achievable rate tuple, then $d \triangleq \lim_{P \to \infty} \log P/\text{DoF}$ is called an achievable DoF tuple and $d_1 + d_2 + \cdots + d_N$ is called an achievable sum-DoF or simply DoF. The closure of the set of all achievable DoF tuples is called the DoF region and denoted by $\mathcal{D}$, and the channel sum-DoF, or simply DoF, is defined as \( \max_{\mathbf{d} \in \mathcal{D}} d_1 + d_2 + \cdots + d_N \).

### III. MAIN RESULTS

The main results of this paper are summarized in the following six theorems. The proof of each theorem is provided in its respective section.

#### A. Full-Duplex Transmitter Cooperation and Delayed CSIT

**Theorem 1:** The $K$-user ($K \geq 3$) Gaussian interference channel with delayed CSIT and full-duplex transmitters can achieve $\text{DoF}_{\text{ICFD}}^{\text{ICFD}}(K)$ degrees of freedom almost surely, where

$$
\text{DoF}_{\text{ICFD}}^{\text{ICFD}}(K) = 4 \left( 3 - \frac{2}{\left\lceil \frac{K}{2} \right\rceil - 1} \right) \left( \sum_{\ell=\left\lceil \frac{K}{2} \right\rceil + 1}^{K} \frac{1}{\ell} \right)^{-1}.
$$

**Proof:** See Section V-A.\hfill $\blacksquare$

**Theorem 2:** The $M \times K$ Gaussian X channel with delayed CSIT and full-duplex transmitters can achieve $\text{DoF}_{\text{ICFD}}^{\text{ICFD}}(M, K)$ degrees of freedom almost surely, where $\text{DoF}_{\text{ICFD}}^{\text{ICFD}}(M, K)$ is given at the top of the next page.\hfill $\blacksquare$

#### B. Output Feedback

**Theorem 3:** The $K$-user ($K \geq 3$) Gaussian interference channel with output feedback can achieve $\text{DoF}_{\text{XOF}}^{\text{ICOF}}(K)$ degrees of freedom almost surely, where

$$
\text{DoF}_{\text{XOF}}^{\text{ICOF}}(K) = \max_{w \in \mathcal{W}_K} \frac{w}{a(K)(w-1)^2 + (w+1)^2},
$$

with $\mathcal{W}_K \triangleq \{[w_K^+, w_K^-] \}$, and $w_K^+$ and $w_K^-$ and $a(K)$ are defined as

$$
w_K^+ \triangleq \frac{1}{3} + \frac{1}{6} \left( \frac{8a(K) + 3\sqrt{48a(K) + 81} + 27}{a(K)} \right)^{\frac{1}{3}},
$$

$$
a(K) \triangleq \frac{1}{\left\lfloor \frac{K}{2} \right\rfloor - 1} \left( \frac{1}{\left\lfloor \frac{K}{2} \right\rfloor} + \frac{1}{\left\lceil \frac{K}{2} \right\rceil} \sum_{\ell=\left\lceil \frac{K}{2} \right\rceil + 1}^{K} \frac{1}{\ell} \right).
$$

**Proof:** See Section VI-A.\hfill $\blacksquare$

**Theorem 4:** The $K \times K$ Gaussian X channel with output feedback can achieve $\text{DoF}_{\text{XOF}}^{\text{ICOF}}(K, K)$ degrees of freedom almost surely$^1$, where

$$
\text{DoF}_{\text{XOF}}^{\text{ICOF}}(K, K) = \frac{2K}{K+1}.
$$

**Proof:** See Section VI-B.\hfill $\blacksquare$

#### C. Shannon Feedback

**Theorem 5:** The $K$-user ($K \geq 3$) Gaussian interference channel with Shannon feedback can achieve $\text{DoF}_{\text{ICSF}}^{\text{ICSF}}(K)$ degrees of freedom almost surely, where

$$
\text{DoF}_{\text{ICSF}}^{\text{ICSF}}(K) = \max_{2 \leq w \leq \lceil K/2 \rceil} \left( 1 + \frac{w - 2}{\text{DoF}_{\text{ICOF}}^{\text{ICOF}}(K)} + \frac{w}{(w+1)\text{DoF}_{\text{ICSF}}^{\text{ICOF}}(K)} \right)^{-1},
$$

with $\text{DoF}_{\text{ICOF}}^{\text{ICOF}}(K)$ and $\text{DoF}_{\text{ICSF}}^{\text{ICSF}}(K)$ given by (19) and (20), respectively.

**Proof:** See Section VII-A.\hfill $\blacksquare$

**Theorem 6:** The $K \times K$ Gaussian X channel with Shannon feedback can achieve $\text{DoF}_{\text{XSF}}^{\text{ICSF}}(K, K)$ degrees of freedom almost surely, where

$$
\text{DoF}_{\text{XSF}}^{\text{ICSF}}(K, K) = K^2 \left( \frac{K^2 + 7K - 6}{2} \right)^{-1} \left( \sum_{\ell=\left\lfloor \frac{K}{2} \right\rfloor}^{\left\lceil \frac{K}{2} \right\rceil} \frac{1}{\ell} + \sum_{\ell_2=\left\lfloor \frac{K}{2} \right\rfloor + 1}^{K} \frac{2(K-1)}{\ell_2} \sum_{\ell_1=1}^{\left\lfloor \frac{K}{2} \right\rfloor} \frac{1}{\ell_1} \right).
$$

**Proof:** See Section VII-B.\hfill $\blacksquare$

#### D. Some Comments

Before proceeding with details of the proofs, we highlight some key features of our proposed transmission schemes in the following observations.

1) For each of IC and X channel and under each of the feedback/cooperation assumptions, a “multi-phase” transmission scheme is proposed.

2) During phase 1, fresh information symbols are transmitted by a subset of transmitters.

(i) Each receiver receives a number of linear combinations in terms of its own desired information symbols and possibly some interference symbols. The received linear combinations are not enough to resolve all desired symbols (possibly including some interference symbols).

(ii) Each receiver also receives some linear combinations solely in terms of undesired information symbols. However, these linear combinations are desired by some other receivers in view of observation (i). On the other hand, by the end of phase 1, each of these linear combinations will be also

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$^1$The result of this theorem has been reported simultaneously and independently in [30].
available at a subset of transmitters based on the feedback/cooperation assumption.

3) During the remaining phases, the transmitters deliver the linear combinations mentioned in observation (ii) to the receivers where they are desired.

(i) Phase \( m, m \geq 2 \), takes some linear combinations as its inputs. Each of these linear combinations is available at a subset of transmitters and is desired by a subset of cardinality \( m \) of receivers (and is at most available at each unintended receiver as well).

(ii) During phase \( m \), the input linear combinations are transmitted over the channel such that each intended receiver obtains ”part” of the information required to decode the input linear combinations. The rest of information required by each intended receiver to decode all its desired linear combinations is obtained by a subset of unintended receivers. These pieces of information will be delivered to the intended receivers during phases \( m + 1, m + 2, \ldots \).

(iii) Specifically, the mentioned pieces of information (or a mixture of them) is now desired by a subset of cardinality \( m + 1 \) of receivers, and is available at a subset of transmitters and at most one unintended receiver. These linear combinations constitute the inputs of phase \( m + 1 \).

(iv) The transmission continues until the last phase. The input of the last phase is the linear combinations which are desired by all receivers except for at most one unintended receiver where the linear combination is already available. These linear combinations are delivered to their intended receivers by a sufficient number of transmissions.

IV. ILLUSTRATIVE EXAMPLES

In this section, we illustrate our transmission schemes for the 3-user IC and \( 3 \times 3 \) X channel under each of the channel feedback/cooperation assumptions defined in Definition 6. These examples provide insights about our proposed schemes for general multi-user settings.

A. 3-User Interference Channel

In this section, we denote fresh information symbols of TX1, TX2, and TX3 in the IC (intended for their paired receivers) by \( u, v, \) and \( w \) variables, respectively. Also, let us introduce some notations that will be used only in this section and Section IV-B.

**Notation 1:** A linear combination of transmitted symbols that is received by RX1 is denoted by \( L_a(\cdot) \) in phase 1 of the scheme, and by \( L'_a(\cdot) \) in phase 2. Similarly, \( L_b(\cdot) \) and \( L_c(\cdot) \) and their primed versions denote the linear combinations available at RX2 and RX3, respectively. A linear combination that is available at a receiver but is not desired by that receiver is coloured corresponding to that receiver. In particular, blue, red, and green are assigned to RX1, RX2, and RX3, respectively.

The schemes we propose for the 3-user IC under the considered feedback/cooperation assumptions are motivated by the scheme proposed in [15] for the 3-user IC with output feedback. Indeed, the scheme proposed here for the 3-user IC with output feedback is a modified version of the scheme proposed in [15] and achieves the same DoF of 6/5. The modification is such that our scheme can be generalized systematically to larger networks. For the full-duplex delayed CSIT and Shannon feedback, our transmission schemes also achieve 6/5 DoF. Each scheme operates in 2 distinct phases. Since phase 1 is the same for all three schemes, we present phase 1 only once, and then present phase 2 under each assumption separately.

**Phase 1 (3-User IC):**

This phase takes 3 time slots, during which 6 information symbols \( [u_1, u_2], [v_1, v_2], \) and \( [w_1, w_2] \) are fed to the system respectively by TX1, TX2, and TX3 as follows.

\[ \triangleright \text{First time slot: TX1 and TX2 transmit } u_1 \text{ and } v_1, \text{ respectively, while TX3 is silent. Hence, ignoring the noise, RX1 and RX2 each receive one linear equation in terms of } u_1 \text{ and } v_1 \text{ by the end of the first time slot, i.e.,} \]

\[
RX_1 : L_a(u_1, v_1) = h^{[11]}(1)u_1 + h^{[12]}(1)v_1, \tag{21}
\]

\[
RX_2 : L_b(u_1, v_1) = h^{[21]}(1)u_1 + h^{[22]}(1)v_1. \tag{22}
\]

Therefore, if we deliver another linearly independent combination of \( u_1 \) and \( v_1 \) to RX1, it will be able to decode both transmitted symbols (the desired symbol \( u_1 \) and the interference symbol \( v_1 \)). Similarly, if we deliver a linearly independent combination of \( u_1 \) and \( v_1 \) to RX2, it can decode both \( u_1 \), which is interference, and \( v_1 \), which is desired.
Remark 1: Since the noise variance is bounded, it does not affect the DoF. Hence, we ignore the noise in our analysis throughout this paper.

Now, we observe that RX3 has also received a linear combination of $u_1$ and $v_1$, i.e.,

$$RX_3 : L_c(u_1, v_1) = h^{[31]}(1)u_1 + h^{[32]}(1)v_1.$$  \hspace{1cm} (23)

Note that this quantity does not contain any information about the information symbols of RX3 ($w$ symbols). Therefore, it is not desired by RX3. However, since the channel coefficients are i.i.d. across the nodes, $L_c(u_1, v_1)$ is linearly independent of each of $L_c(u_1, v_1)$ and $L_b(u_1, v_1)$ almost surely. Hence, if we deliver $L_c(u_1, v_1)$ to both RX1 and RX2, each of them will be able to decode its own desired information symbol together with the interference symbol. Moreover, based on the feedback/cooperation assumption, this linear combination will be available at least one transmitter after the first time slot.

In particular,

(i) Under the full-duplex delayed CSIT assumption, TX1 receives $v_1$ and TX2 receives $u_1$ during the first time slot. This along with the delayed CSIT assumption enables both TX1 and TX2 to reconstruct $L_c(u_1, v_1)$.

(ii) Under the output feedback assumption, $L_c(u_1, v_1)$ will be available at TX3 through the feedback link from RX3.

(iii) Under the Shannon feedback assumption, $L_c(u_1, v_1)$ will be available at all three transmitters as described in the following: It will be available at TX3 directly through the output feedback link from RX3. Also, TX1 can obtain $v_1$ using the output feedback link and delayed CSI and its own transmitted symbol $u_1$. Therefore, it can reconstruct $L_c(u_1, v_1)$. Similarly, TX2 can reconstruct $L_c(u_1, v_1)$ after the first time slot.

Hence, $L_c(u_1, v_1)$ is a quantity that

1) is desired by both RX1 and RX2,
2) is available, but undesired, at RX3,
3) is available at both TX1 and TX2 with the full-duplex delayed CSIT, is available at TX3 with the output feedback, and is available at all three transmitters under the Shannon feedback assumption.

This quantity will be delivered to both RX1 and RX2 during phase 2. Before proceeding with phase 2, transmission in the second and third time slots is done similar to the first time slot, except that roles of the nodes are exchanged.

- \textit{Second time slot}: TX2 and TX3 transmit $v_2$ and $w_1$, respectively, while TX1 is silent. After this time slot, the linear combination $L_b(u_2, w_1)$, which is available at RX1, is desired by both RX2 and RX3.

- \textit{Third time slot}: TX3 and TX1 transmit $w_2$ and $u_2$, respectively, while TX2 is silent. After this time slot, $L_b(u_2, w_2)$, which is available at RX2, is desired by both RX3 and RX1.

Figure 3 illustrates the transmission during phase 1. Each coloured quantity in the figure denotes the quantity that is available and undesired at the corresponding receiver and is desired by the other two receivers. It only remains to deliver these coloured symbols, i.e., $L_c(u_1, v_1)$, $L_a(v_2, w_1)$, and $L_b(u_2, w_2)$ to the pairs of receivers where they are desired as discussed above. If two linearly independent combinations of these three quantities are delivered to each receiver, then each receiver will be able to obtain its two desired quantities by removing its known, undesired quantity. This will be accomplished in two time slots during phase 2 of the scheme, which is presented in the following under each feedback/cooperation assumption separately.

Remark 2: Each receiver desires to decode 4 symbols (two information symbols and two interference symbols) out of the six transmitted symbols. However, each receiver has received only two useful linear combinations in terms of the 4 desired symbols, since the coloured linear combination at each receiver is useless for that receiver. Therefore, it requires two more linearly independent combinations in terms of the desired symbols to be able to decode all of them.

- \textit{Phase 2 (Full-Duplex 3-User IC With Delayed CSIT)}:

  Recall that $L_c(u_1, v_1)$, $L_a(v_2, w_1)$, and $L_b(u_2, w_2)$ are available at the transmitter pairs (TX1,TX2), (TX2,TX3), and (TX3,TX1), respectively. This phase takes 2 time slots as follows.

- \textit{Fourth time slot}: The symbols $L_a(u_1, v_1)$, $L_a(v_2, w_1)$, and $L_b(u_2, w_2)$ are transmitted by TX1, TX2, and TX3, respectively. Then, RX1 receives the linear combination $h^{[11]}(4)L_c(u_1, v_1) + h^{[12]}(4)L_a(v_2, w_1) + h^{[13]}(4)L_b(u_2, w_2)$ and since it already has the undesired quantity $L_a(v_2, w_1)$, it can cancel it to obtain an equation solely in terms of $L_c(u_1, v_1)$ and $L_b(u_2, w_2)$, which are both desired by RX1.

- \textit{Fifth time slot}: The symbols $L_a(u_1, v_1)$, $L_a(v_2, w_1)$, and $L_b(u_2, w_2)$ are transmitted by TX1, TX2, and TX3, respectively. Then, RX2 receives $h^{[21]}(4)L_c(u_1, v_1) + h^{[22]}(4)L_a(v_2, w_1) + h^{[23]}(4)L_b(u_2, w_2)$ and RX3 receives $h^{[31]}(4)L_c(u_1, v_1) + h^{[32]}(4)L_a(v_2, w_1) + h^{[33]}(4)L_b(u_2, w_2)$ by the end of the fourth time slot. Similarly, RX2, having the undesired quantity $L_b(u_2, w_2)$, will obtain an equation in terms of two desired quantities $L_a(v_2, w_1)$ and $L_c(u_1, v_1)$. Also, RX3 will similarly obtain an equation solely in terms of $L_a(v_2, w_1)$ and $L_b(u_2, w_2)$.

- \textit{Sixth time slot}: This time slot is an exact repetition of the fourth time slot. Since the channel coefficients are i.i.d.,
over time, each receiver obtains a linearly independent equation in terms of its own two desired quantities by the end of this time slot, and thus, can decode both desired quantities.

The transmission scheme in phase 2 is illustrated in Fig. 4a. This completes delivery of the 6 information symbols \(\{u_1, u_2, v_1, v_2, w_1, w_2\}\) to their intended receivers in 5 time slots, and thus, proves achievability of 6/5 DoF with full-duplex delayed CSIT.

- **Phase 2 (3-User IC With Output Feedback):**

  With access to output feedback, the quantities \(L_c(u_1, v_1), L_a(v_2, w_1), \) and \(L_b(u_2, w_2)\) are available at TX3, TX1, and TX2, respectively. Hence, transmission of these symbols in phase 2 can be done in two time slots using the same scheme described above, and the same DoF of 6/5 can be achieved. The only difference is that here \(L_c(v_2, w_1), L_a(u_2, w_2), \) and \(L_e(u_1, v_1)\) are transmitted by TX1, TX2, and TX3 respectively, as shown in Fig. 4b.

- **Phase 2 (3-User IC With Shannon Feedback):**

  Under the Shannon feedback assumption, \(L_c(u_1, v_1), L_a(v_2, w_1), \) and \(L_b(u_2, w_2)\) are available at all transmitters. Therefore, transmission of these quantities can be done in two time slots using the same scheme described under the full-duplex delayed CSIT assumption. Alternatively, here we can deliver two random linear combinations of these three quantities to each receiver using only one transmitter, say TX1, in two time slots \(t = 4, 5\), while the other transmitters are silent. This is depicted in Fig. 4c, where \(\{c_{a,t}, c_{b,t}, c_{c,t}\}|t = 1, 2\) are the random coefficients generated offline and revealed to all receivers before the transmission begins. Therefore, 6/5 DoF is also achieved with Shannon feedback.

**B. 3 \times 3 X Channel**

In this section, we use \(u^a, u^b, \) and \(u^c\) to denote information symbols of TX1, TX2, and TX3, respectively, all intended for RX1. Similarly, \(u^b, u^c\), and \(u^a\) denote information symbols intended for RX2, and \(u^a, u^b, \) and \(u^c\) are all intended for RX3. We also use the same notations for the linear combinations and their colouring as defined in Notation 1.

In the following, we show achievability of 24/17 DoF in the 3 \(\times\) 3 X channel with full-duplex delayed CSIT, and achievability of 3/2 DoF and 27/17 DoF in the same channel with output feedback and Shannon feedback, respectively.

1) **Full-Duplex 3 \times 3 X Channel With Delayed CSIT:**

The transmission scheme has 3 phases and delivers 24 information symbols in 17 time slots as follows.

- **Phase 1 (Full-Duplex 3 \times 3 X Channel With Delayed CSIT):**

  This phase takes 12 times slots to transmit 24 information symbols.

  \(\triangleright\) **Time slots \(t = 1, \cdots, 6\):** Only TX1 and TX2 transmit information symbols, and TX3 is silent. In particular, for each pair of receivers, TX1 and TX2 spend two time slots to transmit 4 information symbols. The first two time slots are dedicated to (RX1, RX2) and the transmission is accomplished as follows: In the first time slot, \(u_1^a\) and \(u_1^b\), both desired by RX1, are transmitted by TX1 and TX2, respectively. Ignoring the noise, RX1 receives the linear combination

  \[
  L_a(u_1^a, u_1^b) = h^{[11]}(1)u_1^a + h^{[12]}(1)u_1^b,
  \]

  in terms of the two desired information symbols, and hence, requires another linearly independent combination to resolve them. Simultaneously, RX2 receives another linear combination, namely,

  \[
  L_b(u_1^a, u_1^b) = h^{[21]}(1)u_1^a + h^{[22]}(1)u_1^b,
  \]

  in terms of \(u_1^a\) and \(u_1^b\). Since the channel coefficients are i.i.d. across the channel nodes, \(L_b(u_1^a, u_1^b)\) is linearly independent of \(L_a(u_1^a, u_1^b)\) almost surely. Therefore, if \(L_b(u_1^a, u_1^b)\) is delivered
to RX1, it will be able to decode both $u_1^a$ and $u_1^b$. On the other hand, according to full-duplex operation of the transmitters, both TX1 and TX2 will obtain both $u_1^a$ and $u_1^b$, and by the delayed CSIT assumption, they can reconstruct $L_a(u_1^a, u_1^b)$ after this time slot.

Similarly, in the second time slot, $v_1^a$ and $v_1^b$, both desired by RX2, are transmitted respectively by TX1 and TX2. Then, the linear combination

$$L_a(v_1^a, v_1^b) = h^{11}(2)v_1^a + h^{12}(2)v_1^b,$$

which is received by RX1, is desired by RX2 and available at both TX1 and TX2. Therefore, it only remains to deliver $L_b(u_2^a, u_2^b)$ and $L_a(v_1^a, v_1^b)$ to RX1 and RX2, respectively. Now, it is easy to see that if the linear combination $L_b(u_2^a, u_2^b) + L_a(v_1^a, v_1^b)$ is delivered to both RX1 and RX2, each of them can cancel its known quantity to obtain its desired quantity. On the other hand, this linear combination is available at both TX1 and TX2, and thus, can be retransmitted during the next phase of the scheme.

Time slots $t = 3, 4$ are dedicated to the pair (RX1, RX3) and time slots $t = 5, 6$ are dedicated to (RX2, RX3), as depicted in Fig. 5.

▷ Time slots $t = 7, \ldots, 12$: Similarly, another 12 information symbols are now transmitted by TX1 and TX3, while TX2 is silent.

The transmission in this phase is illustrated in Fig. 5. Each coloured linear combination in the figure denotes a quantity that is available at one receiver, desired by another receiver, and available at the two corresponding transmitters after its corresponding time slot. It only remains to deliver the six linear combinations listed in Table I to their respective pairs of receivers as discussed earlier. This will be accomplished during the remaining phases of the transmission scheme.

• Phase 2 (Full-Duplex $3 \times 3$ X Channel With Delayed CSIT):

This phase takes 3 time slots to transmit the linear combinations indicated in Table I by TX1 and TX2 as follows. TX3 is silent during this phase.

▷ Time slot $t = 13$: TX1 and TX2 transmit $L_b(u_3^a, u_3^b)$ and $L_a(v_1^a, v_1^b)$, respectively, both to be decoded by both RX1 and RX2. By the end of this time slot, each of RX1 and RX2 obtains a linear combination in terms of the transmitted quantities. Hence, they each require another linearly independent combination to decode both quantities. Now, one can easily verify that the linear combination

$$L_c' \equiv h^{31}(13)[L_b(u_3^a, u_3^b) + L_a(v_1^a, v_1^b)]$$

which is received by RX3, is linearly independent of the linear combination received by each of RX1 and RX2. Therefore, if we deliver this linear combination to both RX1 and RX2, each of them will be able to decode its both desired quantities. On the other hand, by the delayed CSIT assumption, $L_c'$ is available at TX1 as well. Note that TX1 has both transmitted quantities.

The next two time slots are similarly dedicated to the other pairs of receivers.

▷ Time slot $t = 14$: TX1 and TX2 transmit $L_c(u_4^a, u_4^b) + L_a(v_2^a, v_2^b)$ and $L_c(u_3^a, u_3^b) + L_a(v_1^a, v_1^b)$, respectively. Now, each of RX1 and RX3 receives one desired linear combination and the linear combination

$$L_b' \equiv h^{21}(14)[L_c(u_4^a, u_4^b) + L_a(v_2^a, v_2^b)]$$

which is received by RX2, is desired by both RX1 and RX3. This linear combination is also available at TX1 after this time slot.

![Fig. 5. Phase 1 of the transmission scheme for full-duplex $3 \times 3$ X channel with delayed CSIT. Each coloured linear combination is the one which is (i) available at a receiver, (ii) not desired by that receiver, and (iii) desired by one of the other receivers.](image-url)
Fig. 6. Transmission scheme for 3 × 3 X channel with output feedback. Each coloured linear combination is the one which is (i) available at a receiver, (ii) not desired by that receiver, and (iii) desired by one of the other receivers. (a) Phase 1. (b) Phase 2.

- **Time slot $t = 15$:** TX1 and TX2 transmit $L_a(v^a_2,v^c_2)$ and $L_b(v^b_2,v^c_2)$, respectively. Each of RX2 and RX3 now receives a desired linear combination $L_a(v^a_2,v^c_2)$ is received by RX1, is desired by both RX2 and RX3. This linear combination is also available at TX1 after this time slot.

In summary, each of $L'_a$, $L'_b$, and $L'_c$ is available at one receiver and desired by the other two receivers, and all of them are available at TX1. They will be delivered to their respective pairs of receivers during phase 3.

- **Phase 3 (Full-Duplex 3 × 3 Channel With Delayed CSI):**
- **Time slots $t = 16,17$:** In each time slot, a random linear combination of $L'_a$, $L'_b$, and $L'_c$ is transmitted by TX1, while the other transmitters are silent. It can be easily verified that after these two time slots, RX1, RX2, and RX3 are able to decode $(L'_a, L'_b, L'_c)$, respectively.

2) **3×3 X Channel With Output Feedback:** Our transmission scheme for this channel operates in 2 phases and delivers 9 information symbols to the receivers in 6 time slots, yielding 3/2 DoF. The scheme is illustrated in Fig. 6 and described as follows.

- **Phase 1 (3 × 3 X Channel With Output Feedback):** This phase spans 3 time slots. Each time slot is dedicated to transmission of information symbols intended for one receiver.

  - **First time slot:** The information symbols $u^a$, $u^b$, and $u^c$, all intended for RX1, are transmitted by TX1, TX2 and TX3, respectively. By the end of this time slot, RX1 receives linear combination $L_a(u^a,v^b,v^c)$ of the three desired symbols and requires two extra linearly independent combinations to resolve all three symbols. At the same time, RX2 receives linear combination $L_b(u^a,v^b,u^c)$, which is linearly independent of $L_a(u^a,v^b,u^c)$, and thus, is desired by RX1. Similarly, linear combination $L_c(u^a,v^b,u^c)$ is received by RX3 and is desired by RX1. On the other hand, $L_b(u^a,v^b,u^c)$ (resp. $L_c(u^a,v^b,u^c)$) is also available at TX2 (resp. TX3) after this time slot through the output feedback. The second and third time slots are dedicated similarly to RX2 and RX3, respectively.

  - **Second time slot:** The information symbols $v^a$, $v^b$, and $v^c$, all intended for RX2, are transmitted by TX1, TX2 and TX3, respectively. Similarly, $L_a(v^a,v^b,v^c)$ and $L_c(v^a,v^b,v^c)$ are received by RX1 and RX3, and available at TX1 and TX3, respectively, and are desired by RX2.

  - **Third time slot:** The information symbols $w^a$, $w^b$, and $w^c$, all intended for RX3, are transmitted by TX1, TX2 and TX3, respectively. Similarly, $L_a(w^a,w^b,w^c)$ and $L_b(w^a,w^b,w^c)$ are received by RX1 and RX2, and available at TX1 and TX2, respectively, and are desired by RX3.

Therefore, to deliver the transmitted information symbols to their intended receivers, it suffices to

(i) deliver $L_b(u^a,v^b,u^c)$ and $L_a(u^a,v^b,v^c)$ to RX1,
(ii) deliver $L_a(v^a,v^b,v^c)$ and $L_c(v^a,v^b,v^c)$ to RX2,
(iii) deliver $L_a(w^a,w^b,w^c)$ and $L_b(w^a,w^b,w^c)$ to RX3.

This will be done in phase 2.

- **Phase 2 (3 × 3 X Channel With Output Feedback):** This phase takes 3 time slots. Each time slot is dedicated to a pair of receivers.

  - **Fourth time slot:** In this time slot, which is dedicated to RX1 and RX2, $L_a(u^a,v^b,v^c)$ and $L_b(u^a,v^b,u^c)$ are respectively transmitted by TX1 and TX2, while TX3 is silent. After this time slot, RX1 obtains the desired linear combination $L_b(u^a,v^b,u^c)$ by cancelling the known undesired linear combination $L_a(u^a,v^b,v^c)$. Similarly, RX2 obtains its desired linear combination $L_a(v^a,v^b,v^c)$ by cancelling $L_b(u^a,u^b,u^c)$.

  - **Fifth time slot:** The quantities $L_a(w^a,w^b,w^c)$ and $L_c(w^a,w^b,w^c)$ are transmitted by TX1 and TX3, while TX2 is silent. Then, each of RX1 and RX3 similarly obtains its desired quantity.

  - **Sixth time slot:** The quantities $L_b(w^a,w^b,w^c)$ and $L_c(v^a,v^b,v^c)$ are transmitted by TX2 and TX3, while TX1 is silent. Then, each of RX2 and RX3 similarly obtains its desired quantity.

Therefore, the achieved DoF is equal to $9/(3+3) = 3/2$. 
3) 3 × 3 X Channel With Shannon Feedback: Our transmission scheme for this channel has two rounds of operation, during which 27 information symbols are delivered to the receivers in 17 time slots as follows.

**Round 1 (3 × 3 X Channel With Shannon Feedback):**

The first round consists of two phases. Phase 1 takes 3 time slots to transmit 9 information symbols \(\{u_a^1, u_b^1, u_c^1, u_a^2, u_b^2, u_c^2, u_a^3, u_b^3, u_c^3\}\) exactly as in phase 1 of the scheme proposed above for the same channel with output feedback. Before proceeding with phase 2, one notes that TX1 obtains the linear combination

\[
L_a(u_a^1, u_b^1, u_c^1) = h^{[11]}(1)u_a^1 + h^{[12]}(1)u_b^1 + h^{[13]}(1)u_c^1,
\]

after the first time slot through the output feedback. Since TX1 has access to delayed CSI according to Shannon feedback assumption, it can cancel its own transmitted symbol \(u_c^1\) to obtain

\[
h^{[12]}(1)u_b^1 + h^{[13]}(1)u_c^1.
\]

TX1 knows the coefficients \(h^{[12]}(1)\) and \(h^{[13]}(1)\) of this linear combination. Similarly, TX2 will obtain \(h^{[21]}(2)v_a^1 + h^{[22]}(2)v_b^1\) after the second time slot using Shannon feedback.

During phase 2, in one time slot, TX1 and TX2 transmit \(L_a(u_a^1, v_b^1, v_c^1)\) and

\[
L_b(u_a^1, u_b^1, u_c^1) = h^{[21]}(1)v_a^1 + h^{[22]}(1)v_b^1 + h^{[23]}(1)v_c^1,
\]

while TX3 is silent. Hence, \(L_a(v_a^1, v_b^1, v_c^1)\) and \(L_b(u_a^1, u_b^1, u_c^1)\) are delivered to RX2 and RX1, respectively, as in phase 2 of the scheme proposed with output feedback. Now, TX1 obtains \(L_a(v_a, v_b^1, v_c^1)\) since it has access to Shannon feedback and its own transmitted quantity i.e., \(L_a(v_a^1, v_b^1, v_c^1)\). Therefore, by cancelling \(u_c^1\) from \(L_b(u_a^1, v_b^1, v_c^1)\), TX1 can obtain

\[
h^{[22]}(1)v_a^1 + h^{[23]}(1)v_c^1.
\]

Hence, TX1 is now able to decode both \(u_a^1\) and \(v_c^1\) using the linear combinations (27) and (28). Therefore, having access to delayed CSI, TX1 can reconstruct \(L_a(v_a^1, v_b^1, u_c^1)\). Likewise, TX2 is able to decode both \(v_a^1\) and \(v_c^1\), and hence, reconstruct \(L_c(v_a^1, v_b^1, v_c^1)\).

In summary, after these 4 time slots, it only remains to deliver \(L_a(v_a^1, v_b^1, u_c^1)\) (available at RX3) to RX1,

(i) deliver \(L_a(u_a^1, u_b^1, w_c^1)\) (available at RX1) to RX3,

(ii) deliver \(L_a(u_a^1, u_b^1, w_c^1)\) (available at RX1) to RX3,

(iii) deliver \(L_c(u_a^1, v_b^1, v_c^1)\) (available at RX3) to RX2,

(iv) deliver \(L_b(u_a^1, w_b^1, w_c^1)\) (available at RX2) to RX3.

On the other hand, in addition to \(L_a(u_a^1, u_b^1, u_c^1)\), TX1 has also access to \(L_a(u_a^1, u_b^1, w_c^1)\) through output feedback, and thereby, it has access to \(L_c(u_a^1, u_b^1, u_c^1)\). Similarly, TX2 has access to \(L_a(v_a^1, v_b^1, v_c^1)\) and \(L_a(v_a^1, v_b^1, w_c^1)\), and thus, \(L_c(v_a^1, v_b^1, v_c^1)\) + \(L_a(u_a^1, v_b^1, w_c^1)\). Moreover, it is easy to verify that the goals (i)-(iv) above are attained if \(L_c(u_a^1, u_b^1, w_c^1)\) is delivered to both RX1 and RX3 and \(L_a(v_a^1, v_b^1, v_c^1)\) + \(L_a(u_a^1, v_b^1, w_c^1)\) is delivered to both RX2 and RX3. This will be accomplished during round 2 of the scheme.

Before proceeding to round 2, we repeat the above procedure twice more and transmit another \(2 \times 3 = 18\) fresh information symbols, namely \(\{u_a^1, u_b^1, u_c^1, v_a^1, v_b^1, v_c^1, w_a^1, w_b^1, w_c^1\}\) in another \(2 \times 4 = 8\) time slots. However, in the first repetition, \(L_a(w_a^1, w_b^1, w_c^1)\) and \(L_a(u_a^1, u_b^1, u_c^1)\) are transmitted by TX1 and TX3 in phase 2, and in the second repetition \(L_b(w_a^1, w_b^1, w_c^1)\) and \(L_c(v_a^1, v_b^1, v_c^1)\) are transmitted by TX2 and TX3 in phase 2.

Up to this point, we have spent 12 time slots and transmitted 27 information symbols. Table II summarized the quantities that must be delivered to their respective pairs of receivers in round 2.

**Round 2 (3 × 3 X Channel With Shannon Feedback):**

This round takes 5 time slots, i.e., \(t = 13, \ldots, 17\). During the first 3 time slots, the 6 linear combinations listed in Table II are transmitted over the channel. Each time slot is dedicated to a pair of receivers as follows.

**Time slot \(t = 13\):** TX1 and TX3 respectively transmit \(L_a(u_a^1, u_b^1, u_c^1) + L_a(u_a^2, u_b^2, u_c^2)\) and \(L_a(u_a^2, u_a^2, u_b^3) + L_a(u_b^1, u_b^1, u_c^1)\), both to be delivered to both RX1 and RX2 according to Table II, while TX3 is silent. Then, each of RX1 and RX2 requires another linear combination in terms of the transmitted quantities to be able to decode both quantities. Therefore, the linear combination

\[
L_r' = h^{[31]}(13)[L_a(u_a^1, u_b^1, u_c^1) + L_a(u_a^2, u_b^2, u_c^2)]
\]

+ \(h^{[32]}(13)[L_a(u_a^2, u_b^3, u_c^2) + L_a(u_b^1, u_b^1, u_c^1)]\), which is received by RX3, is desired by both RX1 and RX2. It can also be easily verified that \(L_r'\) can be reconstructed by TX1 using Shannon feedback.

**Time slot \(t = 14\):** TX1 and TX3 respectively transmit \(L_a(u_a^1, u_b^1, u_c^1)\) + \(L_a(u_a^1, u_b^1, u_c^1)\) and \(L_a(u_b^1, u_b^1, u_c^1)\) + \(L_a(u_a^2, u_a^2, u_b^3)\), both desired by RX1 and RX3, while TX2 is silent. Then, the linear combination

\[
L_r' = h^{[21]}(14)[L_a(u_a^1, u_b^1, u_c^1) + L_a(u_a^2, u_b^2, u_c^2)]
\]

+ \(h^{[23]}(14)[L_a(u_a^2, u_b^3, u_c^2) + L_a(u_a^2, u_a^2, u_c^1)]\), which is received by RX2, is desired by both RX1 and RX3 and can be reconstructed by TX1 using Shannon feedback.

**Time slot \(t = 15\):** TX2 and TX3 respectively transmit \(L_a(u_a^1, u_b^1, u_c^1) + L_a(u_a^2, u_b^2, u_c^2)\) and \(L_a(u_b^3, u_b^3, u_c^2)\), both desired by TX3 and RX3, while TX1 is silent. Then, the linear combination

\[
L_r' = h^{[12]}(15)[L_a(u_a^1, u_a^2, u_b^3) + L_a(u_a^2, u_b^2, u_c^2)]
\]

+ \(h^{[13]}(15)[L_a(u_b^3, u_b^3, u_c^2) + L_a(u_a^1, u_a^2, u_c^2)]\),
which is received by RX$_1$, is desired by both RX$_2$ and RX$_3$ and is available at TX$_1$ through Shannon feedback (output feedback).

During the last two time slots of this round, $L'_m$, and $L'_{i'}$, each of which is available at one receiver and desired by the other two, are delivered to their intended pairs of receivers.

▷ Time slots $t = 16, 17$: Two random linear combinations of $L'_m$, and $L'_{i'}$, and $L'_m$, transmitted by TX$_1$, while the other transmitters are silent. Each receiver will then be able to decode its two desired linear combinations.

The achieved DoF is therefore equal to $27/(12 + 3 + 2) = 27/17$.

V. INTERFERENCE AND X CHANNELS WITH FULL-DUPLEX TRANSMITTER COOPERATION AND DELAYED CSIT

Let us introduce some notations that are widely used throughout Sections V to VII.

Notation 2: In the $K$-user IC and $K \times K$ X channel, $S_m \subseteq \{1, 2, \ldots, K\}$ denotes a set of $m$ transmitters (or receivers), $m \leq K$, where $S_K = \{1, 2, \ldots, K\}$ is the index set of all transmitters (or receivers). In the $M \times K$ X channel, subsets of cardinality $m_1$ and $m_2$ of transmitters and receivers are denoted by $U_{m_1} \subseteq U_M$ and $V_{m_2} \subseteq V_K$, respectively, where $U_M = \{1, 2, \ldots, M\}$ and $V_K = \{1, 2, \ldots, K\}$ are respectively the index sets of all transmitters and all receivers. A symbol that is available at all transmitters TX$_1$, $i \in U_{m_1}$, and all receivers RX$_j$, $j \in V_{m_2}$, and is intended to be decoded at all receivers RX$_j$, $j \in V_{m_2}$, is denoted by $u |U_{m_1}\rangle |V_{m_2}\rangle$. If $V_{m_2} = \{\}$, the mentioned symbol is denoted by $u |U_{m_1}\rangle$ and is called an order-$m_2$ symbol.

We also define the notion of transmission graph, which is used for illustration of the transmission schemes throughout Sections V to VII.

Definition 9 (Transmission Graph): A transmission graph is a complete bipartite graph whose edges connect top nodes to bottom nodes of the graph. Each solid edge implies that the quantity transmitted by the top node is unavailable at and desired by the bottom node, whereas each dashed edge implies that the quantity transmitted by the top node is already available at the bottom node.

In this section, we prove Theorems 1 and 2 by proposing transmission schemes for the interference and X channels with full-duplex transmitter cooperation and delayed CSIT.

A. Proof of Theorem 1

Before providing the details of the transmission scheme and analyzing its achievable DoF, we highlight the transmission strategy through the following observations: The transmission scheme operates in $K - 1$ phases. During each time slot of phase $m$, $m + 1$ transmitters are active and the rest are silent. In each time slot, the group of active transmitters together with their transmitted quantities is chosen such that:

(a) Each active transmitter has access to $m$ out of the $m + 1$ to-be-transmitted quantities, and thus, obtains the last quantity during this time slot via full-duplex reception. Therefore, having access to delayed CSI, each active transmitter can reconstruct all linear combinations received by the receivers during this time slot.

(b) The quantity transmitted by each active transmitter is a linear combination of a number of independent symbols. The paired receiver of each active transmitter desires to decode all the symbols of its paired transmitter. Moreover, for any other active transmitter, the mentioned receiver either already has all the independent symbols of that transmitter (obtained during previous phases) or desires to decode all of them. Therefore, the linear combination received by the mentioned receiver is totally desired (after removing the part corresponding to the known symbols). Furthermore, it requires extra linear combinations to be able to decode all the (unknown) symbols transmitted in this time slot. In particular, it desires to obtain $K - m - 1$ linear combinations received by the paired receivers of the $K - m - 1$ silent transmitters, or a subset of them depending on $m$ and $K$.

These linear combinations are all available at the $m + 1$ active transmitters in view of observation (a), and hence, can be retransmitted during phase $m + 1$.

• Phase 1 (Full-Duplex K-User IC With Delayed CSIT): In this phase, fresh information symbols are fed to the channel. For every subset $S_1 = \{i_1, i_2, i_3\} \subseteq S_K$, spend 3 time slots to transmit 6 fresh information symbols $[u_{1[i_1]}, u_{2[i_1]}, u_{1[i_2]}, u_{2[i_2]}, u_{1[i_3]}, u_{2[i_3]}]$ by TX$_{i_1}$, TX$_{i_2}$, TX$_{i_3}$ as follows: In the first time slot, TX$_{i_1}$, TX$_{i_2}$ transmit $u_{1[i_1]}$ and $u_{i_2[i_1]}$, respectively, and the rest of transmitters are silent. Hence, RX$_{i_1}$ and RX$_{i_2}$ each receive one linear combination in terms of $u_{1[i_1]}$ and $u_{i_2[i_1]}$ by the end of the first time slot. Therefore, if we deliver a linearly independent combination in terms of $u_{1[i_1]}$ and $u_{i_2[i_1]}$ to both RX$_{i_1}$ and RX$_{i_2}$, each of them will be able to decode both transmitted symbols (desired and interference). This linearly independent combination is indeed the linear combination $h_{i_1[i_1]}(1)u_{1[i_1]} + h_{i_2[i_1]}(1)u_{i_2[i_1]}$ received by RX$_{i_1}$ during this time slot. On the other hand, according to full-duplex operation of the transmitters, both TX$_{i_1}$ and TX$_{i_2}$ will have both $u_{1[i_1]}$ and $u_{i_2[i_1]}$ by the end of the first time slot. This along with the delayed CSIT assumption enables both TX$_{i_1}$ and TX$_{i_2}$ to reconstruct $h_{i_1[i_1]}(1)u_{1[i_1]} + h_{i_2[i_1]}(1)u_{i_2[i_1]}$.

Thus, according to Notation 2, one can define

$$u_{1[i_1]i_2[i_1][i_2];i_3} \triangleq h_{i_1[i_1]}(1)u_{1[i_1]} + h_{i_2[i_1]}(1)u_{i_2[i_1]}.$$  \hspace{1cm} (29)

Similarly, the second and third time slots are described as follows:

• Second time slot: TX$_{i_3}$ and TX$_{i_1}$ transmit $u_{2[i_1]}$ and $u_{2[i_1]}$, respectively. The symbol $u_{2[i_1][i_2][i_1][i_2];i_1}$ is generated accordingly after this time slot.

• Third time slot: TX$_{i_2}$ and TX$_{i_3}$ transmit $u_{2[i_2]}$ and $u_{2[i_2]}$, respectively. The symbol $u_{2[i_1][i_2][i_3][i_2];i_1} = 25$ is generated accordingly after this time slot.

Therefore, $6(\binom{K}{3})$ information symbols are transmitted in $3(\binom{K}{3})$ time slots and $3(\binom{K}{3})$ symbols of type $u_{[S_j|S_j; j]}$, $j \in S_K \setminus S_m$, are generated by the end of phase 1. We denote by $\text{DoF(CFD)}(K)$, $2 \leq m \leq K - 1$, our achievable DoF for transmission of symbols of type $u_{[S_m|S_m; j]}$, $j \in S_K \setminus S_m,$
over the $K$-user IC with full-duplex delayed CSIT. Then, the $3^{(K)}$ symbols of type $u_{S_2 \setminus S_1}$ can be delivered to their respective pairs of receivers in $3^{(K)}/3^{ICFD}(K)$ time slots. Therefore, the total duration of the scheme is $3^{(K)} + 3^{(K)}/3^{ICFD}(K)$ time slots and the achievable DoF is equal to

$$\text{DoF}^{ICFD}(K) = \frac{6^{(K)}}{3^{(K)} + 3^{(K)}/3^{ICFD}(K)} = \frac{2}{1 + 1/\text{DoF}^{ICFD}(K)},$$

(30)

- **Phase m, $2 \leq m \leq K - 2$ (Full-Duplex K-User IC With Delayed CSIT):** For $m, n \in \mathbb{Z}$, define

$$L_m(n) \triangleq \text{lcm}[n - m, m],$$

$$Q_m(n) \triangleq \text{min}[n - m, m],$$

(31)

(32)

where $\text{lcm}(x, y), x, y \in \mathbb{Z}$, is the least common multiplier of $x$ and $y$. This phase takes $m+1$ symbols $u_{S_m \setminus S_1}$, $j \in S_K \setminus S_m$, transmits them over the channel in $a_m(K)/Q_m(n)$ time slots, and generates $a_{m+1}(K)/Q_m(n)$ symbols of type $u_{S_{m+1} \setminus S_m}$, $j \in S_K \setminus S_{m+1}$, where $a_m(K)$ is defined as

$$a_m(K) \triangleq \left( \frac{K}{m+1} \right) \left( \frac{K-m-1}{Q_m(K)} - 1 \right) L_m(K).$$

(33)

Fix a subset $S_{m+1} = \{i_1, i_2, \cdots, i_{m+1}\} \subset S_K$, and a subset $S_{m+1} \setminus S_{m+1} \subset S_K \setminus S_{m+1}$. During $L_m(K)/Q_m(n)$ time slots, each TX$_i$, $1 \leq n \leq m + 1$, transmits a random linear combination of $u_{S_m \setminus S_1}$ in each time slot, as illustrated in Fig. 7. Therefore, a total of $(m+1) L_m(K)/Q_m(n)$ symbols are transmitted in $L_m(K)/Q_m(n)$ time slots. We note that the random coefficients of these linear combinations are generated offline and shared with all nodes. Now, the following observations are important.

(i) RX$_j$, $j \in S_{m+1}$, wishes to decode the $L_m(K)$ symbols $\{u_{S_{m+1} \setminus S_1}^{[j]}\}_{k=1}^{L_m(K)/m}$, $j' \in S_{m+1} \setminus \{j\}$. Since it has all the symbols $\{u_{S_{m+1} \setminus S_1}^{[j]}\}_{k=1}^{L_m(K)/m}$, by cancelling them, it can obtain $L_m(K)/Q_m(n)$ equations out of its received equations, solely in terms of its desired symbols.

(ii) TX$_i$, $i \in S_{m+1}$, has all the transmitted symbols except for $\{u_{S_{m+1} \setminus S_1}^{i} \in \{[j] \in S_{m+1} \setminus \{i\}, l_m(K)/m\}}_{k=1}^{L_m(K)/m}$. According to the full-duplex operation, it will obtain $L_m(K)/Q_m(n)$ random linear combinations of these symbols after cancelling its known symbols, and since $L_m(K)/Q_m(n) \geq L_m(K)/m$, it can decode all of them.

(iii) RX$_j'$, $j' \in S_{Q_m(K)-1}$, receives $L_m(K)/Q_m(n)$ linear combinations in terms of all transmitted symbols. If we deliver these linear combinations to RX$_j'$, $j \in S_{m+1}$, it will be able to cancel its undesired part as argued in observation (i) and obtain $L_m(K)/Q_m(n)$ linear combinations solely in terms of its desired symbols. On the other hand, in view of observation (ii) and according to the delayed CSIT assumption, TX$_i$, $i \in S_{m+1}$, can reconstitute all these linear combinations by the end of the $L_m(K)/Q_m(n)$ time slots. Thus, the $L_m(K)/Q_m(n)$ linear combinations received by RX$_j'$, $j \in S_{m+1}$, is denoted by $\{u_{S_{m+1} \setminus S_1}^{i} \in \{[j] \in S_{m+1} \setminus \{j\}, l_m(K)/m\}}_{k=1}^{L_m(K)/m}$. After delivering these $(Q_m(K) - 1) \times L_m(K)/Q_m(n)$ symbols to RX$_j$, $j \in S_{m+1}$, it is provided with a total of $L_m(K)$ linear combinations in terms of its $L_m(K)$ desired symbols. Also, it is easy to show that these linear combinations are linearly independent almost surely, and hence, can be solved for the desired symbols.

Since there are $K_{m+1}$ choices of $S_{m+1}$ and $(K-m-1)$ choices of $S_{Q_m(K)-1}$ for each $S_{m+1}$, the achievable DoF for $2 \leq m \leq K - 2$ equals

$$\text{DoF}_{m}^{ICFD}(K) = \frac{(m+1)a_m(K)/m}{Q_m(K)} + \frac{(Q_m(K)-1)a_m(K)/Q_m(K)}{\text{DoF}_{m}^{ICFD}(K)} = \frac{m+1}{m} \times Q_m(K),$$

(34)

- **Phase K - 1 (Full-Duplex K-User IC With Delayed CSIT):** During $K - 1$ consecutive time slots, TX$_i$, $i \in S_K$, repeats transmission of the symbol $u_{S_K \setminus \{i\} \setminus \{l\}}$, $1 \leq k \leq L_m(K)/m$, (with $u_{S_K \setminus \{0\} \setminus \{0\}} \triangleq u_{S_K \setminus \{0\}}$) for each receiver. It is easily verified that each receiver obtains a linear combination of its $K - 1$ desired symbols in each time slot. Hence, each receiver is able to decode all its $K - 1$ desired symbols after $K - 1$ time slots. Then, one can write

$$\text{DoF}_{K-1}^{ICFD}(K) = \frac{K}{K-1}.$$ 

(35)

It is shown in Appendix A that (11) is indeed the closed form solution to the recursive Eqs. (30) and (34) with initial condition (35).

B. Proof of Theorem 2

Let us highlight our transmission strategy for the full-duplex $M \times K$ X channel with delayed CSIT through the following observations before elaborating on the details of the scheme:

The transmission scheme operates in $K$ phases. During each time slot of phase $m$,

(a) Order-$m$ symbols, which are all desired by the same set of $m$ receivers, are transmitted by a number of
active transmitters. Depending on the triple \((m, M, K)\), each active transmitter already knows either all of the transmitted order-\(m\) symbols or all but one of them. In the latter case, the active transmitters use their full-duplex reception capability to obtain all the symbols. In either case, having access to delayed CSI, all active transmitters are able to reconstruct all linear combinations received by the receivers.

(b) Each of the intended \(m\) receivers receives one linear combination of the (desired) transmitted order-\(m\) symbols, and thus, requires extra linearly independent combinations to decode all of them. In particular, it desires to obtain \(K - m\) linear combinations received by the \(K - m\) unintended receivers, or a subset of them depending on \((m, M, K)\). These linear combinations are all available at the active transmitters in view of observation (a), and thus, after being combined appropriately to construct order-\((m + 1)\) symbols, are retransmitted during phase \(m + 1\).

• Phase 1 (Full-Duplex \(M \times K\) Channel With Delayed CSIT):

Fix \(i_1, i_2 \in \mathcal{U}_M\). For any \(\{j_1, j_2\} \in \mathcal{V}_K\), \(\text{TX}_{j_1}\) and \(\text{TX}_{j_2}\) transmit 4 fresh information symbols \(u_{[i_1][j_1]}, u_{[i_2][j_1]}, u_{[i_1][j_2]}, u_{[i_2][j_2]}\) in two time slots as follows (we have ignored the indices of symbols for ease of notations): In the first time slot, \(\text{TX}_{i_1}\) and \(\text{TX}_{i_2}\) transmit \(u_{[i_1][j_1]}, u_{[i_2][j_1]}\), respectively, both intended for \(\text{RX}_{j_1}\). After this time slot, the linear combination \(h_{[j_1][i_1]}u_{[i_1][j_1]} + h_{[j_2][i_2]}u_{[i_2][j_1]}\), which was received by \(\text{RX}_{j_2}\), is available at \(\text{TX}_{i_1}\) and \(\text{TX}_{i_2}\) due to full-duplex operation of the transmitters and delayed CSIT, and is desired by \(\text{RX}_{j_2}\) to be able to decode \(u_{[i_1][j_1]}\) and \(u_{[i_2][j_1]}\). Hence, it is denoted as \(u_{[i_1][j_1:j_2]}\).

Similarly, in the second time slot, \(\text{TX}_{i_1}\) and \(\text{TX}_{i_2}\) transmit \(u_{[i_1][j_2]}, u_{[i_2][j_2]}\), respectively, both intended now for \(\text{RX}_{j_2}\), and the symbol \(u_{[i_1][j_1:j_2]}\) is generated. It is easily verified that \(u_{[i_1][j_1:j_2]} + u_{[i_1][j_1:j_2]}\) is desired by both \(\text{RX}_{j_1}\) and \(\text{RX}_{j_2}\). Hence, one can define the order-2 symbol

\[
u_{[i_1][j_1:j_2]} = u_{[i_1][j_1:j_2]} + u_{[i_1][j_1:j_2]},
\]

(36)

By the end of this phase, \(4\binom{M}{2}\binom{K}{2}\) fresh information symbols are transmitted in \(2\binom{M}{2}\binom{K}{2}\) time slots and \(4\binom{M}{2}\binom{K}{2}\) order-2 symbols are generated, which will be delivered to their corresponding pairs of receivers during the rest of the transmission scheme. Then, the achievable DoF is given by

\[
\text{DoF}^{\text{FDD}}_{m}(M, K) = \frac{4\binom{M}{2}\binom{K}{2}}{\binom{M}{2}\binom{K}{2} + \binom{M}{2}\binom{K}{2}}
= \frac{2 + \frac{1}{\text{DoF}^{\text{FDD}}_{m}(M, K)}}{4},
\]

(37)

where \(\text{DoF}^{\text{FDD}}_{m}(M, K)\) denotes our achievable DoF for transmission of order-2 symbols of type \(u_{[i][j:j]}\) over the full-duplex \(M \times K\) channel with delayed CSIT.

• Phase 2, \(2 < m < K - 1\) (Full-Duplex \(M \times K\) Channel With Delayed CSIT): Consider the following mutually exclusive cases.

(i) \(M > \frac{K}{2}\), \(2 < m < \frac{K}{2}\): In this case, order-\(m\) symbols of type \(u_{[i][j]}\) are transmitted over the channel. Fix a subset \(\mathcal{V}_m \subseteq \mathcal{V}_K\), and a subset \(\mathcal{U}_{m+1} = \{i_1, i_2, \ldots, i_{m+1}\} \subseteq \mathcal{U}_M\). Note that since \(m \leq K/2 < M\), both subsets exist. All transmitters \(\text{TX}_j, j \in \mathcal{U}_M \setminus \mathcal{U}_{m+1}\), are silent, while the transmitters \(\text{TX}_{i_n}, 1 \leq n \leq m+1\), transmit as follows: For every subset \(\mathcal{V}_m \subseteq \mathcal{V}_m\), spend one time slot to transmit \(u_{[i_n][i_{n+1}]}|_{\mathcal{V}_m} \) by \(\text{TX}_{i_n}, n = 1, \ldots, m + 1\), where \(i_k \triangleq i_{k-m-1}\) for \(m + 1 < k \leq 2m\), as illustrated in Figs. 8a and 8b. Every \(\text{RX}_j, j \in \mathcal{V}_m\), receives one linear combination in terms of \(m + 1\) desired symbols, and thus, requires \(m\) extra linearly independent combinations to resolve all the \(m + 1\) symbols. It is easy to see that the linear combination received by \(\text{RX}_j, j \in \mathcal{V}_m\), is linearly independent of the one received by any \(\text{RX}_j, j \in \mathcal{V}_m\), and hence, is desired by all of them. On the other hand, every \(\text{TX}_{i_n}, 1 \leq n \leq m + 1\), has access to exactly \(m\) out of the \(m + 1\) transmitted symbols, and thus, obtains the last one using the full-duplex operation by the end of this time slot. Hence, \(\text{TX}_{i_n}, 1 \leq n \leq m + 1\), having access to delayed CSI, can reconstruct the linear combinations received by all receivers. In particular, one can denote the linear combination received by \(\text{RX}_j, j \in \mathcal{V}_m \setminus \mathcal{V}_m\), as \(u_{[i][j]}|_{\mathcal{V}_m \setminus \mathcal{V}_m}\).

Now, we have the following observation: For any subset \(\mathcal{V}_{m+1} \subseteq \mathcal{V}_m\), consider the \(m + 1\) symbols \(u_{[i][j]}|_{\mathcal{V}_m \setminus \mathcal{V}_m}\), \(j \in \mathcal{V}_{m+1}\), as defined above. Each receiver \(\text{RX}_j, j \in \mathcal{V}_{m+1}\), has exactly one of these symbols and requires the other \(m\). Therefore, if we deliver \(m\) random linear combinations of these \(m + 1\) symbols to all receivers \(\text{RX}_j, j \in \mathcal{V}_{m+1}\), each of them is provided with \(m\) random linear combinations of \(m\) desired unknowns, and thus, can resolve all of them. Hence, these \(m\) random linear combinations can be denoted as \(u_{[i][j]}|_{\mathcal{V}_m \setminus \mathcal{V}_m}\), \(i \in \mathcal{V}_{m+1}\), \(j \in \mathcal{V}_{m+1}\). These order-(\(m + 1\)) symbols will be delivered to their corresponding receivers during the rest of the transmission scheme. We denote by \(\text{DoF}^{\text{FDD}}_m(M, K)\), \(2 \leq m \leq K/2 < M\), our achievable DoF for transmission of order-\(m\) symbols of type \(u_{[i][j]}|_{\mathcal{V}_m}\) over the full-duplex \(M \times K\) channel with delayed CSIT. Since there are \(\binom{M}{m}\) choices for \(\mathcal{V}_m\), \(\binom{M}{m+1}\) choices for \(\mathcal{U}_{m+1}\), and \(\binom{M}{m}\) choices for \(\mathcal{V}_m\), the achievable DoF for \(M > \frac{K}{2}\), \(2 \leq m \leq \frac{K}{2}\) is obtained by

\[
\text{DoF}^{\text{FDD}}_m(M, K) = \frac{\binom{M}{m+1}\binom{K}{m+1}(m+1)}{\binom{M}{m+1}\binom{K}{m+1} + \binom{M}{m}\binom{K}{m+1} + \binom{M}{m}\binom{K}{m+1}}
= \frac{(m+1)^2}{m+1 + \frac{1}{\text{DoF}^{\text{FDD}}_m(M, K)}}.
\]

(38)

(ii) \(M > \frac{K}{2}, \frac{K}{2} < m \leq K - 1\): In this case, order-\(m\) symbols of type \(u_{[i][j]}|_{\mathcal{V}_m}\) are transmitted over the channel. Since \(K/2 < M\) and \(K/2 < m\), we have \(K - m + 1 \leq |K/2| + 1 \leq M\), and thus, these symbols can be optimally transmitted over the channel using the scheme proposed in [12] for transmission of order-\(m\) symbols over an
order-m symbols (see Figures 8c and 8d for the transmission graphs of this case). Here, we denote by $\text{DoF}_m^{\text{XFD}}(M, K)$, $0 \leq K - m < K/2 < M$, the achievable DoF for transmission of order-m symbols of type $u^{\{U_{m+1}\}V_{m}}$ over the full-duplex $M \times K$ X channel with delayed CSIT. Hence, the following recursion holds for $M > \frac{K}{2}$, $\frac{K}{2} < m \leq K - 1$ (see [12, eq. (27)]).

$$\text{DoF}_m^{\text{XFD}}(M, K) = \frac{(m+1)(M-1,K-m)+1}{m+1 + m \times \min \{M-1,K-m\}}.$$ (39)

(iii) $2 \leq M \leq \frac{K}{2}$, $2 \leq m < M$: In this case, order-m symbols of type $u^{\{U_{m}\}V_{m}}$ are transmitted over the channel. Since $m < M < K/2$, the scheme proposed for case (i) works equally well here and the achievable DoF is given by (38).

(iv) $2 \leq M \leq \frac{K}{2}$, $M \leq m < K - 1$: In this case, order-m symbols of type $u^{\{U_{m}\}V_{m}}$ are transmitted over the channel without operating in the full-duplex mode using the scheme proposed in [12] for transmission of order-m symbols over an $M \times K$ MISO broadcast channel with delayed CSIT (see [12] Section VI-B). Figures 8e and 8f show the transmission graphs for this case. Then, the following DoF is achievable for $M \leq \frac{K}{2}$, $M \leq m < K - 1$ (see [12, eq. (39)]):

$$\text{DoF}_m^{\text{XFD}}(M, K) = \frac{(m+1)(\min \{M-1,K-m\} + 1)}{m+1 + m \times \min \{M-1,K-m\}}.$$ (40)

where $\text{DoF}_m^{\text{XFD}}(M, K)$ (resp. $\text{DoF}_m^{\text{XFD}}(M, K)$) here denotes our achievable DoF for transmission of symbols of type $u^{\{U_{m}\}V_{m}}$ (resp. $u^{\{U_{m}\}V_{m+1}}$) over the full-duplex $M \times K$ X channel with delayed CSIT.

To summarize the above cases, for $m, K \in \mathbb{Z}$, we define

$$Q_m(M, K) \triangleq \min \{M-1, K-m, m\},$$ (41)

$$\Theta_m(M, K) \triangleq \min \{M, K/2 \} + 1, m),$$ (42)

and denote by $\text{DoF}_m^{\text{XFD}}(M, K)$ our achievable DoF for transmission of order-m symbols of type $u^{\{U_{m}\}V_{m}}$ over the full-duplex $M \times K$ X channel with delayed CSIT. It is easy to see from (37) to (40) that our achievable DoF satisfies the following recursive equation for $1 \leq m \leq K - 1$.

$$\text{DoF}_m^{\text{XFD}}(M, K) = \frac{(m+1)(Q_m(M, K) + 1)}{m+1 + m \times \Theta_m(M, K)}.$$ (43)

• **Phase K (Full-Duplex $M \times K$ X Channel With Delayed CSIT):**

In this phase, the symbols of type $u^{\{U_{m}\}V_{m}}$ are delivered to all $K$ receivers via transmission of one symbol per time slot by one transmitter that has access to that symbol. Therefore,

$$\text{DoF}_K^{\text{XFD}}(M, K) = 1.$$ (44)

It is shown in Appendix B that (18) is indeed the closed form solution to the recursive Eq. (43) with the initial condition (44).
VI. INTERFERENCE AND X CHANNELS WITH OUTPUT FEEDBACK

In this section, we investigate the impact of output feedback on the DoF of the K-user IC and K × K X channel. The output feedback from each receiver to its paired transmitter provides that transmitter with a side information about other transmitters’ messages. This enables the transmitters to cooperate in their subsequent transmissions. This is exploited to propose transmission schemes for these channels with output feedback and prove Theorems 3 and 4.

A. Proof of Theorem 3

We propose a multiphase transmission scheme for this channel, in which m + 1 transmitters are active during each time slot of phase m and the rest are silent. In each time slot, the group of active transmitters and their transmitted quantities are chosen such that the quantity transmitted by each active transmitter is a linear combination of a number of independent symbols that are already available at its paired receiver. Moreover, the paired receiver desires to decode all the symbols of the rest of active transmitters. Therefore, the linear combination received by the mentioned receiver is totally desired (after removing the part corresponding to the symbols known to this receiver). Furthermore, it requires extra linear combinations to be able to decode all the (unknown) symbols transmitted in this time slot. In particular, it desires to obtain K - m - 1 linear combinations received by the paired receivers of the K - m - 1 inactive transmitters, or a subset of them depending on m and K. Each of these linear combinations is available at the corresponding (inactive) transmitter through the output feedback, and hence, can be retransmitted during phase m + 1.

In particular, the transmission scheme consists of K - μ(K) + 1 phases, where the integer μ(K), 2 ≤ μ(K) ≤ ⌈K/2⌉, will be determined later.

- Phase 1 (K-User IC With Output Feedback):

For every subset $S_{\mu(K)} \subset S_K$, and every subset $S_{\mu(K)-1} \subset S_K \setminus S_{\mu(K)}$, in one time slot, each TX$_i$, $i \in S_{\mu(K)}$, transmits a fresh information symbol $u[i]$. Then, if we deliver $\mu(K) - 1$ linearly independent combinations of the K symbols transmitted to RX$_i$, $i \in S_{\mu(K)}$, it will be able to decode all the transmitted symbols. Thus, the linear combination received by RX$_j$, $j \in S_{\mu(K)}$-1, which is available at TX$_j$ via the output feedback, is desired by all the receivers RX$_j$, $i \in S_{\mu(K)}$. Hence, they are denoted as $u[j|S_{\mu(K)};j]$, $j \in S_{\mu(K)}$-1.

Therefore, $\mu(K)(K - \mu(K))^{K-\mu(K)-1}$ information symbols are transmitted in $(K - \mu(K))^{K-\mu(K)-1}$ time slots and $(\mu(K) - 1)$ $(K - \mu(K))^{K-\mu(K)-1}$ symbols are generated by the end of phase 1. Denoting by $\text{DoF}^{\text{ICOF}}_m(K)$, $2 \leq m \leq K - 1$, our achievable DoF for transmission of symbols $u[j|S_{\mu(K)};j]$, $j \in S_K \setminus S_m$, over the K-user IC with output feedback, the achievable DoF is equal to

$$\text{DoF}^{\text{ICOF}}_m(K) = \frac{\mu(K)}{1 + \frac{\mu(K) - 1}{\text{DoF}^{\text{ICOF}}_{m-1}(K)}}. \quad (45)$$

- Phase $m$, 2 ≤ m ≤ K - 2 (K-User IC With Output Feedback):

This phase feeds $\frac{m+1}{m} \alpha_m(K)$ symbols of type $u[j|S_{m+1};j]$, $j \in S_K \setminus S_m$, to the channel in $\frac{\alpha_m(K)}{Q_m(K)}$ time slots, and generates $\frac{Q_m(K) - 1}{Q_m(K)} \alpha_m(K)$ symbols of type $u[j|S_{m+1};j]$, $j \in S_K \setminus S_{m+1}$, where $\alpha_m(K)$ and $Q_m(K)$ are defined in Eqs. (32) and (33). Specifically, for every subset $S_{m+1} \subset S_K$, and every subset $S_{Q_m(K)-1} \subset S_K \setminus S_{m+1}$, during $\frac{L_m(K)}{Q_m(K)}$ time slots, each TX$_i$, $i \in S_{m+1}$, transmits $\frac{L_m(K)}{Q_m(K)}$ random linear combinations of symbols $u[j|S_{m+1};j]$, $j \in S_{m+1}$, where $L_m(K)$ is defined in (31). The corresponding transmission graph is shown in Fig. 9. Each RX$_j$, $j \in S_{m+1}$, wishes to decode the $L_m(K)$ symbols $u[j|S_{m+1};j]$, $j \in S_{m+1}$. Also, RX$_j$, $j \in S_{m+1}$, removing $u_k[j|S_{m+1};j]$, $k \in S_{m+1}$, from its received linear combinations, obtains $\frac{L_m(K)}{Q_m(K)}$ linear combinations solely in terms of its desired symbols. If we deliver the $\frac{L_m(K)}{Q_m(K)}$ linear combinations received by RX$_j$, $j' \in S_{Q_m(K)-1}$, to RX$_j$, $j \in S_{m+1}$, it obtains another $Q_m(K) - 1 \times \frac{L_m(K)}{Q_m(K)}$ linear combinations solely in terms of its desired symbols. Since these linear combinations are available at TX$_j$, $j' \in S_{Q_m(K)-1}$, via the output feedback, they are denoted as $u[j'|S_{m+1};j]$, $j' \in S_{m+1}$. Therefore, RX$_j$, $j \in S_{m+1}$, will have $L_m(K)$ linearly independent combinations in terms of its $L_m(K)$ desired symbols, and can solve them for its desired symbols.

Finally, since the number of input symbols, spent time slots, and output symbols of this phase are equal to those of phase m in the proposed transmission scheme for the full-duplex K-user IC with delayed CSI described in proof of Theorem 1, the achievable DoF for phase m satisfies the same recursive equation, given by (34), i.e., for 2 ≤ m ≤ K - 2,

$$\text{DoF}^{\text{ICOF}}_m(K) = \frac{m + 1}{m} \times \frac{Q_m(K)}{1 + \frac{Q_m(K) - 1}{\text{DoF}^{\text{ICOF}}_{m-1}(K)}}. \quad (46)$$

- Phase $K-1$ (K-User IC With Output Feedback):

During K - 1 consecutive time slots, TX$_i$, $i \in S_K$, repeats transmission of the symbol $u[i|S_{k+1};i]$. Therefore, each receiver receives K - 1 linear combinations of its K - 1 desired symbols, and thus, is able to decode all its K - 1 desired symbols. Hence,

$$\text{DoF}^{\text{ICOF}}_{K-1} = \frac{K}{K-1}. \quad (47)$$

It is shown in Appendix A that the solution to recursive Eq. (46) with initial condition (47) is given by (19).
output feedback has 2 phases.

B. Proof of Theorem 4

Substituting (19) for $\text{DoF}_{\mu(K)}^\text{ICOF}(K)$ in (45), we get

$$\text{DoF}_{\mu(K)}^\text{ICOF}(K) = \frac{\mu(K)}{a(K)\mu(K)(\mu(K)-1)^2 + (\mu(K)+1)^2},$$

where $a(K)$ is defined by (14). Now, we choose $\mu(K)$ such that $\text{DoF}_{\mu(K)}^\text{ICOF}(K)$ given in (48) is maximized. In other words,

$$\mu(K) = \arg \max_{2 \leq w \leq \lceil K/2 \rceil} f_K^\text{ICOF}(w),$$

where $f_K^\text{ICOF}(w)$ is defined as

$$f_K^\text{ICOF}(w) = \frac{w}{a(K)w(w-1)^2 + (w+1)^2}.$$  \hspace{1cm} (50)

By taking the derivative of $f_K^\text{ICOF}(w)$ with respect to $w$, it can be shown that the solution $w^*_K$ to the maximization problem $w^*_K = \arg \max_{2 \leq w \leq \lceil K/2 \rceil} f_K^\text{ICOF}(w)$ is given by (13). Since $f_K^\text{ICOF}(w)$ is a continuous and concave function of $w$, the solution $\mu(K)$ to the maximization problem (49) is either $[w^*_K]$ or $[w^*_K]$, whichever yields a greater $f_K^\text{ICOF}(w)$, i.e.,

$$\mu(K) = \arg \max_{w \in ([w^*_K], [w^*_K])} f_K^\text{ICOF}(w),$$

which in view of Eqs. (48) and (50) completes the proof. Figure 10 shows the achievable DoF for different values of $\mu(K)$ together with the optimized achievable DoF, i.e., $\text{DoF}_{\mu(K)}^\text{ICOF}(K)$, for $3 \leq K \leq 30$.

B. Proof of Theorem 4

Our transmission scheme for the $K \times K$ X channel with output feedback has 2 phases.

• Phase 1 ($K \times K$ X Channel With Output Feedback):

For every $j \in S_K$, spend one time slot to transmit $u_i^{|j|}$, $u_i^{2|j|}$, $u_i^{3|j|}$, $u_i^{4|j|}$ by TX$_i$, TX$_{i+1}$, TX$_{i+2}$, respectively, all intended for RX$_j$. During this time slot, RX$_j$ receives one linear combination of all $K$ desired symbols.

Therefore, if the linear combinations received by RX$_j$, $j \in S_K \setminus \{j\}$, are delivered to RX$_j$, it can decode all the $K$ symbols. On the other hand, according to the output feedback, the linear combination received by RX$_j$, $j \in S_K \setminus \{j\}$, is available at TX$_{j'}$ after this time slot. Hence, it is denoted as $u_{j'|j; j'}$, $j \in S_K \setminus \{j\}$. Therefore, after $K$ time slots, $K(1 - 1)$ symbols $u_{j'|j; j'}$, $j \in S_K$, $j' \in S_K \setminus \{j\}$, are generated. These symbols will be delivered to their respective receiver during the next phase.

• Phase 2 ($K \times K$ X Channel With Output Feedback):

This phase takes $K(K - 1)/2$ time slots to deliver the $K(K - 1)$ symbols generated in phase 1 as follows: For any subset $\{j, j'\} \subseteq S_K$, spend one time slot to transmit $u_{j|j'; j'}$ and $u_{j'|j; j'}$ by TX$_j$ and TX$_{j'}$, respectively, while the other transmitters are silent. After this time slot, each of RX$_j$ and RX$_{j'}$ can decode its desired symbol by cancelling the interference symbol that it already has. Then, the achievable DoF is equal to

$$\text{DoF}_{\mu(K)}^\text{XOF}(K, K) = \frac{K^2}{K + K(K - 1)/2} = 2K/K + 1.$$  \hspace{1cm} (52)

and the proof is complete.

VII. INTERFERENCE AND X CHANNELS

WITH SHANNON FEEDBACK

The proposed transmission schemes for $K$-user IC and $K \times K$ channel with Shannon feedback have two rounds of operation. Round 1 of each scheme operates in parallel with the scheme proposed in Section VI for the same channel with output feedback. However, as the scheme proceeds in round 1, the transmitters obtain enough number of linear combinations gradually through the output feedback links. This in conjunction with delayed CSIT enables each transmitter to obtain a subset of symbols of the other transmitters, and thus, cooperate in transmission of them through round 2 of the scheme. The following two subsections offer proofs of Theorems 5 and 6.

A. Proof of Theorem 5

Consider the following two-round scheme for the $K$-user IC with Shannon feedback.

• Round 1 ($K$-User IC With Shannon Feedback):

In this round, the transmitters use only the output feedback in parallel with the scheme proposed in proof of Theorem 3. Specifically, during phase 1, for every subset $S_{(K)} \subset S_K$, every subset $S_{(K-1)} \subseteq S_K \setminus S_{(K)}$, and every $j_0 \in S_{(K-1)}$, in one time slot, each TX$_i$, $i \in S_{(K)}$, transmits a fresh information symbol $u_i^{|j_0|}$. The integer $v(K)$, $2 \leq v(K) \leq \lceil K/2 \rceil$, will be determined later. The linear combination received by RX$_j$, $j \in S_{(K)}$, which is available at TX$_j$ via the output feedback, is desired by any RX$_i$, $i \in S_{(K)}$.

Now, TX$_i$, $i \in S_{(K)}$, using Shannon feedback and having $u_i^{|j_0|}$ obtains a linear combination in terms of the symbols $u_i^{|j'|}$, $i' \in S_{(K)} \setminus \{i\}$. We deliver the $v(K) - 2$ linear combinations available at the receivers RX$_j$, $j \in S_{(K-1)} \setminus \{j_0\}$, to every RX$_i$, $i \in S_{(K)}$, using the scheme proposed in proof of Theorem 3. Concurrently, TX$_i$ using Shannon feedback and
having \( u[i] \), will obtain another \( v(K) - 2 \) linearly independent combinations of \( u[i] \), \( i \in S_{v(K)} \), and hence, can decode all of them. Therefore, it can reconstruct the linear combination available at RX\( j \), which is still required by all receivers RX\( i \), \( i \in S_{v(K)} \). Hence, this linear combination is denoted as \( u[S_{v(K)} \cup \{j\}] S_{v(K) \cup \{j\}} \).

We note that, for every subset \( S_{v(K)} + 1 \subseteq S_{K} \), and every subset \( S_{v(K)} + 2 \subseteq S_{K} \), we have generated \( v(K) + 1 \) symbols \( u[S_{v(K)} + 1 \cup \{j\}] S_{v(K)} + j \), \( j \in S_{v(K)} + 1 \). Since each RX\( i \), \( i \in S_{v(K)} + 1 \), requires exactly \( v(K) \) out of these \( v(K) + 1 \) symbols, \( v(K) \) random linear combinations of these symbols are desired by each RX\( i \), \( i \in S_{v(K)} + 1 \), and are denoted as \( \{u[\{S_{v(K)} + 1 \cup \{j\}] S_{v(K)} + j\}] v(K) \}_{k=1}^{K} \). They will be delivered during round 2 of the transmission scheme. Therefore, the achievable DoF is given by

\[
\text{DoF}_{\text{ICSF}}(K) = \frac{v(K) \beta(K)}{\beta(K) + \frac{1}{v(K)} v(K) - v(K)} (54)
\]

where

\[
\beta(K) = \frac{K - v(K)}{v(K) - 1},\]

and \( \text{DoF}_{\text{ICSF}}(K) \) denotes our achievable DoF for transmission of the symbols of type \( u[S_{v(K)}] S_{K} \) over the K-user IC with Shannon feedback.

**Round 2 (K-User IC With Shannon Feedback):**

This round consists of \( K - v(K) \) phases.

- **Phase m, v(K) + 1 \leq m \leq K - 1 (K-User IC With Shannon Feedback):**

In this phase, symbols of type \( u[S_{v(K) + 1}\cup\{s\}] S_{K} \) are fed to the channel and symbols of type \( u[S_{v(K) + 1}\cup\{s\}] S_{K} \) are generated as follows: Fix a subset \( S_{Q_m(K+1)+m-1} \subseteq S_{K} \), where \( Q_m(n) \) is defined in (32). For any \( S_m \subseteq S_{Q_m(K+1)+m-1} \), spend one time slot to transmit \( \{u[S_{Q_m(n)} ] S_{Q_m(n)}\} Q_m(n) K + 1 \) arbitrary transmitters out of \( \{TX_j : j \in S_m\} \), as illustrated in Fig. 11. Then, RX\( j \), \( j \in S_m \), transmits \( Q_m(K+1) - 1 \) extra linear combinations to resolve all the transmitted symbols. Thus, the linear combination received by RX\( j' \), \( j' \in S_{Q_m(K+1)+m-1} \), which is available at TX\( j' \) via the output feedback, is desired by every RX\( j \), \( j \in S_m \). On the other hand, every TX\( j \), \( j \in S_m \), having access to all the transmitted symbols and delayed CSI, can reconstruct this linear combination. Therefore, it is denoted as \( u[S_{v(K) + 1}\cup\{s\}] S_{v(K) + 1} \).

Now, for any subset \( S_{m+1} \subseteq S_{Q_m(K+1)+m-1} \), consider \( m + 1 \) symbols \( u[S_{m+1}\cup\{s\} \cup \{j\}] S_{m+1} \), \( j \in S_{m+1} \). It is easy to see that \( m \) linear random combinations of these symbols are desired by each RX\( i \), \( i \in S_{m+1} \), and can be denoted as \( \{u[S_{m+1}\cup\{j\}] S_{m+1}\}_{k=1}^{K} \). The achievable DoF for \( 2 \leq m \leq K - 1 \) equals

\[
\text{DoF}_{\text{ICSF}}(K) = \frac{m+1}{m+1 - \frac{1}{m+1}} (55)
\]

It is shown in Appendix C that the solution \( \text{DoF}_{\text{ICSF}}(K) \) to the recursive Eq. (55) with initial condition (56) is given by (20). Therefore, the proof is complete in view of (53) and the fact that \( v(K) \) is chosen to maximize \( \text{DoF}_{\text{ICSF}}(K) \). The achievable DoF for different values of \( v(K) \) and the optimized achievable DoF are plotted in Fig. 12 for \( 2 \leq K \leq 30 \).

**B. Proof of Theorem 6**

For the \( K \times K \) X channel with Shannon feedback, we propose the following two-round scheme.

- **Round 1 (K \times K \) X Channel With Shannon Feedback):**

This round has 2 phases in parallel with the scheme proposed in proof of Theorem 4 for the same channel with output feedback. In particular, in phase 1, \( K^2 \) fresh information symbols \( u[i,j] \), \( 1 \leq i, j \leq K \), are transmitted over the channel during \( K \) time slots in the same way as phase 1 of the scheme proposed in Section VI-B, and \( K(K - 1) \)
symbols \(u^{[j:j']}, \{j, j'\} \subseteq S_K\), are generated correspondingly. After time slot \(j\), TX\(_j\), having access to its own transmitted symbol and Shannon feedback, obtains a linear combination of the \(K - 1\) symbols \(u^{[i]}, i \in S_K\setminus\{j\}\). Therefore, if TX\(_j\) is provided with extra \(K - 2\) linearly independent combinations of these \(K - 1\) symbols (with known coefficients), it will be able to decode all of them.

In phase 2, the symbols \(u^{[j:j']}, \{j, j'\}\) are transmitted according to phase 2 of the scheme presented in Section VI-B. However, here, each TX\(_i\) obtains more linear combinations of the symbols \(u^{[i]}, \{j, j'\} \subseteq S_K\setminus\{j\}\), using the Shannon feedback as we proceed with the transmissions. Specifically, for a fixed index \(j_0\), if \(j_0 \in S_K\), and for any \(\{j, j'\} \subseteq S_K\setminus\{j_0\}\), one can define the order-2 symbol which is available at the end of time slot \(j\) and \(j'\) respectively by TX\(_j\) and TX\(_{j'}\), while the other transmitters are silent.

By the end of this time slot, \(u^{[j:j']}, \{j, j'\}\) are delivered to RX\(_j\) and RX\(_{j'}\) respectively. Also, TX\(_j\) obtains \(u^{[j:j']}, \{j, j'\}\) through Shannon feedback, which is a linear combination of \(u^{[i]}, i \in S_K\setminus\{j\}\). Similarly, TX\(_{j'}\) obtains \(u^{[i]}\) which is a linear combination of \(u^{[i]}, i \in S_K\setminus\{j'\}\). Therefore, one can verify that, after the \((K - 2)\) time slots of this phase, (i) each RX\(_j, j \in S_K\setminus\{j_0\}\), receives all the symbols \(u^{[j:j']}, j' \in S_K\setminus\{j_0, j\}\); (ii) each RX\(_j, j \in S_K\setminus\{j_0\}\), obtains \(u^{[j:j']}, j' \in S_K\setminus\{j, j_0\}\), which are \(K - 2\) linear combinations of the symbols \(u^{[i]}, i \in S_K\setminus\{j\}\). These linear combinations together with the linear combination obtained during phase 1, constitute \(K - 1\) linearly independent combinations of \(K - 1\) unknowns, and thus, can be solved for the symbols \(u^{[i]}, i \in S_K\setminus\{j\}\).

By observation (i), it only remains to deliver the \(2(K - 1)\) symbols \(u^{[j:j_0]}, j \in S_K\setminus\{j_0\}\), to their respective receivers. On the other hand, by observation (ii), the symbol \(u^{[j:j_0]}, j \in S_K\setminus\{j_0\}\), can now be reconstructed by TX\(_j\), and thus, is denoted as \(u^{[j_0:j]}\). Consequently, one can define the following order-2 symbol which is available at TX\(_j\):

\[
u^{[j:j_0]} \triangleq u^{[j_0:j]} + u^{[j_0:j]}\] (57)

Therefore, it only remains to deliver the above \(K - 1\) order-2 symbols to their respective pairs of receivers. By \(K\) times repetition of phase 1, each time with \(K^2\) fresh information symbols and a new \(j_0\), \(1 \leq j_0 \leq K\), we generate \(K(K - 1)\) order-2 symbols \(u^{[j:j_0]}, j \in S_K, j \in S_K\setminus\{j_0\}\), as above. Then, the achievable DoF is given by

\[
\text{DoF}^{\text{XSF}}_i(K, K) = \frac{K \times K^2}{K + K \times \left(\binom{K - 1}{2}\right)} + \frac{K \times (K - 1)}{\text{DoF}^{\text{XSF}}_i(K, K)} = \frac{K^2}{K + \frac{(K - 1)(K - 2)}{2} + \frac{K - 1}{\text{DoF}^{\text{XSF}}_i(K, K)}},
\] (58)

where \(\text{DoF}^{\text{XSF}}_i(K, K)\) represents our achievable DoF for transmission of symbols \(u^{[i]}, \{i, j\} \subseteq S_K\), over the \(K \times K\) channel with Shannon feedback. These symbols will be delivered to their respective pairs of receivers during round 2 of the scheme.

**Round 2 (\(K \times K\) Channel With Shannon Feedback):**

This round has \(K - 1\) phases (i.e., phases 2 to \(K\)). If \(K = 2\), the symbols \(u^{[i]}\) and \(u^{[j]}\) are transmitted respectively by TX\(_1\) and TX\(_2\) in 2 time slots, by the end of which both receivers obtain both symbols.

If \(K > 2\), the \(K(K - 1)\) order-2 symbols of type \(u^{[i]}, \{i, j\}\) and \(u^{[j]}, \{i, j\}\), \(i, j \in S_K\), are transmitted over the channel in phase 2 as follows: For each \(S_1 = \{i_1, i_2, i_3\} \subseteq S_K\), spend three time slots to transmit \(u^{[i]}\) and \(u^{[i]}, \{i, i_1, i_2, i_3\}\), while the other transmitters are silent. Then, RX\(_1\) and RX\(_2\) each require an extra linear combination to decode both symbols. Hence, after this time slot, the linear combination \(h^{[i]}u^{[i_1]} + h^{[i]}u^{[i_2]} + h^{[i]}u^{[i_3]}\) received by RX\(_3\), which is now available at TX\(_3\) via the output feedback, is desired by both RX\(_1\) and RX\(_2\) (the time indices have been omitted for brevity).

By observation (i), it only remains to deliver the \(2(K - 1)\) symbols \(u^{[i_1:i_2]}, u^{[i_1:i_2]}, i_1, i_2 \in S_K\setminus\{i_3\}\), to their respective receivers. On the other hand, by observation (ii), the symbol \(u^{[i_1:i_2]}, i_1, i_2 \in S_K\setminus\{i_3\}\), can now be reconstructed by TX\(_j\), and thus, is denoted as \(u^{[i_1:i_2]}\). Consequently, one can define the following order-2 symbol which is available at TX\(_j\):

\[
u^{[i_1:i_2]} \triangleq a_1u^{[i_1:i_2]} + a_2u^{[i_1:i_2]} + a_3u^{[i_1:i_2]},
\] (57)

where \(a_1, a_2, a_3\) are random coefficients. The achievable DoF is given by

\[
\text{DoF}^{\text{XSF}}_i(K, K) = \frac{6K^2}{3K + \frac{2K^2}{\text{DoF}^{\text{XSF}}_i(K, K)}} = \frac{6}{3 + \frac{2K^2}{\text{DoF}^{\text{XSF}}_i(K, K)}},
\] (59)

where \(\text{DoF}^{\text{XSF}}_i(K, K)\) denotes our achievable DoF for transmission of symbols of type \(u^{[S]}\) over the \(K \times K\) channel with Shannon feedback.

Since the \(K \times K\) channel has the same input-output relationship as the \(K\)-user IC, the problem of transmission of order-3 symbols of type \(u^{[S]}\) over the \(K \times K\) channel with Shannon feedback is equivalent to that of the IC with Shannon feedback. Hence, phase \(m\) of round 2 of the scheme proposed in Section VII-A for \(3 \leq m \leq K\) can be used for transmission of the order-3 symbols and generation of higher order symbols up to order-\(K\) symbols, which are delivered to all receivers in phase \(K\). Therefore, the same recursive equation, i.e., (55), holds for \(\text{DoF}^{\text{XSF}}_i(K, K),\)
Fig. 13. Achievable DoFs for the $K$-user IC with Shannon feedback, output feedback, full-duplex delayed CSIT, and delayed CSIT.

$$3 \leq m \leq K - 1,$$ with \( \text{DoF}_{\text{XSF}}(K, K) = 1 \), and thus, \( \text{DoF}_{\text{Im}}(K, K), 3 \leq m \leq K \), is given by (20). Finally, (17) results from Eqs. (20), (58) and (59).

\( \text{VIII. COMPARISON AND DISCUSSION} \)

A. DoF Comparison

We compare our achievable DoFs with the achievable results on the DoF of both channels with delayed CSIT reported in [18]. Figure 13 plots our achievable DoF for the $K$-user IC with delayed CSIT and full-duplex transmitter cooperation, given by (11), together with our achievable DoFs for the $K$-user IC with output and Shannon feedback, respectively given by Eqs. (12) and (16), and compares them with the achievable DoF for the same channel with delayed CSIT [18] for $2 \leq K \leq 30$. It is seen from the figure that our achievable DoFs are greater than the achievable DoF for the same channel with delayed CSIT for $K \geq 3$. Also, for $K \geq 6$, we achieve greater DoF with output feedback than with full-duplex delayed CSIT. Our achievable DoF with Shannon feedback is greater than that with output feedback for $K = 5$ and $K \geq 7$. One can also verify from (11) that

$$\lim_{K \to \infty} \text{DoF}_{\text{ICOF}}(K) = \frac{4}{3}. \quad (60)$$

Using Eqs. (13) and (14) and the fact that $\mu(K)$ is either $[w/K]$ or $[w_K/K]$, one can show $\mu(K) = o(K)$, which in view of (19) yields $\lim_{K \to \infty} \text{DoF}_{\text{ICOF}}(K) = 2$. This together with (45) and the fact that $\lim_{K \to \infty} \mu(K) = \infty$, implies that

$$\lim_{K \to \infty} \text{DoF}_{\text{ICOF}}(K) = 2. \quad (61)$$

We now show that $\lim_{K \to \infty} \text{DoF}_{\text{ICOF}}(K) = 2$. To do so, it suffices to show that $\text{DoF}_{\text{ICOF}}(K) < 2$. Then, an application of the Squeeze theorem regarding (61) and the fact that $\text{DoF}_{\text{ICOF}}(K) \leq \text{DoF}_{\text{ICOF}}(K)\text{ICOF}(K)$ will yield the desired result.

Using (16), we have

$$\text{DoF}_{\text{ICOF}}(K) = \max_{2 \leq w \leq \left\lfloor K/2 \right\rfloor} \frac{w}{2} + \frac{w}{\text{DoF}_{\text{ICF}}(K)} + \frac{w}{(w+1) \text{DoF}_{\text{ICF}}(K+1)}$$

$$\leq \max_{2 \leq w \leq \left\lfloor K/2 \right\rfloor} \frac{w}{2} + \frac{w}{\text{DoF}_{\text{ICF}}(K)} + \frac{w}{(w+1) \text{DoF}_{\text{ICF}}(K+1)}$$

$$\leq 2, \quad (62)$$

where (a) follows from Eqs. (14) and (19), and (b) uses the fact that the denominator is strictly increasing in $w$ for $w \geq 2$, and thus, is minimized by $w = 2$.

Figure 14 plots our achievable DoFs for the $M \times K$ X channel with delayed CSIT and full-duplex transmitter cooperation, given by (18), for $M = 2, 3$, and $M > \frac{K}{2}$, and $2 \leq K \leq 30$, and compares them with the achievable DoF reported in [18] for the $2 \times K$ X channel with delayed CSIT. For all values of $M$, our achievable DoF for the full-duplex $M \times K$ X channel with delayed CSIT is greater than that of the $2 \times K$ X channel with delayed CSIT. Also, it can be shown using (18) that for a fixed $M$,

$$\lim_{K \to \infty} \text{DoF}_{\text{ICOF}}(M, K) = \frac{1}{\ln 2}. \quad (64)$$

and

$$\lim_{K \to \infty} \text{DoF}_{\text{ICOF}}(3, K) = \frac{8}{3 \ln 3 + 2}. \quad (65)$$
as indicated in Fig. 14. Moreover, it follows from (18) and \(\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\) that, if \(M > K/2\) for sufficiently large \(K\),

\[
\lim_{K \to \infty} \text{DoF}^{XFD}(M, K) = \frac{6}{\pi^2 - 6}.
\]

Figure 15 compares our achievable DoF for the \(K \times K\) X channel with Shannon feedback (given by (17)), output feedback (which is \(2K/(K+1)\) by Theorem 4), full-duplex delayed CSIT (given by (18)), and delayed CSIT [18] for \(2 \leq K \leq 30\). It is observed that for \(K > 2\),

\[
\text{DoF}^{XFD}(K, K) < \text{DoF}^{XOF}(K, K) < \text{DoF}^{XSF}(K, K).
\]

Also, one can easily verify using (15) and (17) that

\[
\lim_{K \to \infty} \text{DoF}^{XOF}(K, K) = \lim_{K \to \infty} \text{DoF}^{XSF}(K, K) = 2.
\]

B. Comments on DoF Scaling

Although the DoF values achieved in this paper under all the considered feedback/cooperation assumptions are greater than those achieved in [18] for the same channels with delayed CSIT, they all approach limiting values not greater than 2 for asymptotically large networks. While no nontrivial upper bound is available for the DoF of any of the considered channels under the considered feedback/cooperation assumptions, it was conjectured in [18] that the DoF of the considered channels under the delayed CSIT assumption is bounded above by a constant. Here, we make a similar conjecture under the considered feedback/cooperation assumptions. Specifically, we conjecture that the DoF of the K-user IC and \(M \times K\) X channel under any of the considered feedback/cooperation models is bounded above by a constant. The intuition behind this conjecture is as follows: When the transmitters do not access the current CSI, they only rely on retransmission of the past interference to partially align the interference subspace of each receiver. However, when the transmitters are distributed, the access of each transmitter to the past interference, which is obtained through the underlying feedback/cooperation model, is very limited. As such, the number of retransmissions required to align the past interference grows significantly with the number of initial transmissions. This has limited the achievable DoF considerably to the extent that none of the proposed schemes was able to achieve more than 2 DoF for any number of users.

IX. Conclusions

The interference and X channels with arbitrary number of users were investigated in this paper, where it was assumed that the CSI is not instantaneously available at the transmitters. Achievable results were obtained on the DoF of these channels under three different assumptions, namely, full-duplex delayed CSIT (where the transmitters access the delayed CSI and can operate in full-duplex mode), output feedback (where each transmitter causally accesses the output of its paired receiver), and Shannon feedback (where each transmitter accesses both the output feedback and delayed CSI). Under each assumption, the transmitters, obtaining side information about each other's messages through full-duplex or feedback links, were able to cooperate to align the interference at the receivers in a multi-phase fashion. Despite the attained DoF gains, all achievable DoFs in this paper were bounded above by 2. The DoF characterization of both channels under each of the considered assumptions remains open in the lack of tight upper bounds.

APPENDIX A

CLOSED FORM EXPRESSION FOR THE RECURSIVE EQUATIONS (34) AND (46)

Consider the recursive equation

\[
\text{DoF}_m(K) = \frac{m+1}{m} \times \frac{Q_m(K)}{1 + \frac{Q_m(K)}{\text{DoF}_{m+1}(K)}}.
\]

for \(2 \leq m \leq K - 2\), with \(Q_m(K) = \min\{K - m, m\}\) and the initial condition \(\text{DoF}_{K-1}(K) = K/(K-1)\). We treat the following two different cases separately.

(i) \([K/2] \leq m \leq K - 1\): In this case, we have \(Q_m(K) = K - m\), and hence,

\[
\text{DoF}_m(K) = \frac{K - m}{m+1} + \frac{K - m - 1}{(m+1)\text{DoF}_{m+1}(K)}.
\]

Then, defining \(\gamma_m(K) \triangleq \frac{K - m - 1}{(m+1)\text{DoF}_{m+1}(K)}\), one can write \(\gamma_m(K) = \frac{1}{m+1} + \gamma_{m+1}(K)\), which implies that \(\gamma_m(K) = \sum_{\ell=m+1}^{K} \frac{1}{\ell}\), or equivalently, for \([K/2] \leq m \leq K - 1\),

\[
\text{DoF}_m(K) = \left(\frac{m}{K - m}\right) \sum_{\ell=m+1}^{K} \frac{1}{\ell}.
\]

(ii) \(2 \leq m < [K/2]\): In this case, we have

\[
\frac{1}{\text{DoF}_m(K)} = \frac{1}{m+1} + \left(\frac{1}{m+1}\right)\text{DoF}_{m+1}(K),
\]

which can be rewritten as

\[
\frac{2}{\text{DoF}_m(K) - 1} = \frac{m-1}{m+1} \left(\frac{2}{\text{DoF}_{m+1}(K) - 1}\right).
\]
It immediately follows that for $2 \leq m < \lceil K/2 \rceil$,

$$\frac{2}{\text{DoF}_m(K)} - 1 = \frac{m(m - 1)}{\left\lceil \frac{K}{2} \right\rceil(\left\lceil \frac{K}{2} \right\rceil - 1)} \left(\frac{2}{\text{DoF}_{\left\lceil \frac{K}{2} \right\rceil}(K)} - 1\right),$$

where (a) uses (70) with $m = \left\lceil \frac{K}{2} \right\rceil$, and the fact that $K - \left\lceil \frac{K}{2} \right\rceil = \left\lfloor \frac{K}{2} \right\rfloor$.

Finally, the closed form expression for $\text{DoF}_m(K)$ follows from Eqs. (70) and (73) and is given by (19).

**APPENDIX B**

**CLOSED FORM EXPRESSION FOR THE RECURSIVE EQ. (43)**

Consider the recursive equation

$$\frac{\text{DoF}_m(M, K)}{\text{DoF}_m(M, K)} = \frac{(m + 1)(Q_m(M, K) + 1)}{m + 1 + mQ_m(M, K)}$$

for $1 \leq m \leq K - 1$, with $Q_m(M, K) = \min\{M - 1, K - m, m\}$ and initial condition $\text{DoF}_1(M, K) = 1$. The following mutually exclusive cases can be differentiated.

(i) $M - 1 \geq \lceil K/2 \rceil$: In this case, $Q_m(M, K) = Q_m(K) = \min\{K - m, m\}$, and hence,

$$\frac{Q_m(M, K) + 1}{m\text{DoF}_m(M, K)} = \frac{1}{m + (m + 1)\text{DoF}_{m+1}(M, K)}$$

Now, if $\lceil K/2 \rceil \leq m \leq K$,

$$\text{DoF}_m(K) = \left(\frac{m}{K - m + 1}\right)^{-1} \sum_{\ell=m}^{K} \frac{1}{\ell}.$$  

(75)

If $1 \leq m < \lceil K/2 \rceil$, the recursive Eq. (74) can be rewritten as

$$\frac{m^2\text{DoF}_m(M, K)}{\text{DoF}_m(M, K)} = \frac{1}{m^2(m + 1) + (m + 1)^2\text{DoF}_{m+1}(M, K)} + \sum_{\ell=m}^{(\left\lceil \frac{K}{2} \right\rceil - 1)} \frac{1}{\ell^2}$$

$$= \left(\frac{1}{\left\lceil \frac{K}{2} \right\rceil} - \frac{1}{m} + \sum_{\ell=m}^{(\left\lceil \frac{K}{2} \right\rceil - 1)} \frac{1}{\ell^2} + \frac{1}{\left\lceil \frac{K}{2} \right\rceil\left(\frac{K}{2} + 1\right)} \sum_{\ell=m}^{(\left\lceil \frac{K}{2} \right\rceil - 1)} \frac{1}{\ell},$$

(76)

where (a) uses (75) with $m = \left\lceil \frac{K}{2} \right\rceil$, and the fact that $K - \left\lceil \frac{K}{2} \right\rceil = \left\lfloor \frac{K}{2} \right\rfloor$.

Equations (75) and (76) yield $\text{DoF}_m(M, K)$, given by (83) at the bottom of this page.

(ii) $M - 1 < \lceil K/2 \rceil$: In this case, if $K - M + 1 \leq m \leq K$, then the same expression as (75) holds for $\text{DoF}_m(K)$. Otherwise, if $M - 1 < m < M - 1$, then $Q_m(M, K) = M - 1$, and we have

$$\frac{1}{m\text{DoF}_m(M, K)} = \frac{1}{m + (M - 1)m + 1}\text{DoF}_m(K)$$

$$= \frac{1}{M} \sum_{\ell=m}^{K - M - 1} \frac{1}{\ell^2}$$

$$+ \frac{1}{M} \sum_{\ell=m}^{K - M - 1} \frac{1}{\ell^2}$$

(77)

where (a) follows from (75) with $m = K - M - 1$. Therefore, for $m = M - 1$,

$$\text{DoF}_m(M, K) = \left(\frac{m}{M} \sum_{\ell=m}^{K} \frac{1}{\ell} \left(\frac{M - 1}{M}\right)^{\min(\ell, K - M + 1) - m - 1}\right)^{-1}.$$  

(78)

Finally, if $1 \leq m < M - 1$,

$$\frac{1}{m^2\text{DoF}_m(M, K)}$$

$$= \frac{1}{m^2(m + 1)} + \frac{1}{(m + 1)^2\text{DoF}_{m+1}(M, K)}$$

$$= \frac{1}{M - 1} - \frac{1}{m} + \sum_{\ell=m}^{M - 2} \frac{1}{\ell^2} + \frac{1}{(M - 1)^2\text{DoF}_{M-1}(M, K)}$$

(79)

where (a) uses (78) with $m = M - 1$. Thus, for $1 \leq m < M - 1$

$$\text{DoF}_m(M, K)$$

$$= \left(\frac{m^2}{M - 1} - m + m^2 \sum_{\ell=m}^{M - 2} \frac{1}{\ell^2} + \frac{m^2}{M}\sum_{\ell=m}^{K} \frac{1}{\ell^2}\right)^{-1}$$

(80)

$$+ \left(\frac{m}{M}\sum_{\ell=m}^{K} \frac{1}{\ell^2} \left(\frac{M - 1}{M}\right)^{\min(\ell, K - M + 1) - M - 1}\right)^{-1}.$$  

(83)
APPENDIX C

CLOSED FORM EXPRESSION FOR THE RECURSIVE EQ. (55)

Consider the recursive equation

$$\text{DoF}_m(K) = \frac{(m+1)Q_m(K+1)}{m+1 + \frac{x(Q_m(K+1))}{\text{DoF}_{m+1}(K)}},$$

for $2 \leq m \leq K - 1$, with initial condition $\text{DoF}_2(K) = 1$. For $\lfloor \frac{K}{2} \rfloor < m \leq K$, it is easily shown that $\text{DoF}_m(K)$ is given by (75). For $2 \leq m \leq \lfloor \frac{K}{2} \rfloor$, we have

$$\text{DoF}_m(K) = \frac{1}{m + \left( \frac{m-1}{m+1} \right) \text{DoF}_{m+1}(K)}$$

$$= \frac{1}{m + \frac{m-1}{m+1} + \sum_{\ell=m}^{\frac{K}{2}-1} \frac{1}{\ell(\ell+1)}}$$

$$+ \left( \frac{m(m-1)}{\lfloor \frac{K}{2} \rfloor(\lfloor \frac{K}{2} \rfloor + 1)} \right) \text{DoF}_{\lfloor \frac{K}{2} \rfloor + 1}(K)$$

$$= \left( \frac{1}{m} + m(m-1) \sum_{\ell=m+1}^{\lfloor \frac{K}{2} \rfloor} \frac{1}{\ell} \right)$$

$$+ \frac{m(m-1)}{\lfloor \frac{K}{2} \rfloor} \sum_{\ell_1=\lfloor \frac{K}{2} \rfloor+1}^{K} \sum_{\ell_2=\ell_1}^{\lfloor \frac{K}{2} \rfloor} \frac{1}{\ell_2},$$

where (a) uses (75) with $m = \lfloor \frac{K}{2} \rfloor + 1$, and the fact that $K - \lfloor \frac{K}{2} \rfloor = \lfloor \frac{K}{2} \rfloor$. Therefore, for $2 \leq m \leq \lfloor K/2 \rfloor$,

$$\text{DoF}_m(K) = \left( \frac{1}{m} + m(m-1) \frac{1}{\lfloor \frac{K}{2} \rfloor} \right)$$

$$- \sum_{\ell_1=m+1}^{\lfloor \frac{K}{2} \rfloor} \frac{1}{\ell_1} \sum_{\ell_2=\lfloor \frac{K}{2} \rfloor+1}^{\lfloor \frac{K}{2} \rfloor} \frac{1}{\ell_2}.$$ 

(81)
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