Splitting the Kemmer-Duffin-Petiau Equations

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Abstract
We study internal structure of the Kemmer-Duffin-Petiau equations for spin-0 and spin-1 mesons. We demonstrate, that the Kemmer-Duffin-Petiau equations can be splitted into constituent equations, describing particles with definite mass and broken Lorentz symmetry. We also show that solutions of the three component constituent equations fulfill the Dirac equation.

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1 Introduction
In recent years there has been a renewed interest in the Kemmer-Duffin-Petiau (KDP) theory describing spin-0 and spin-1 mesons [1, 2, 3] due to discovery of a new conserved four-vector current with positive zeroth component [4], which can be thus interpreted as a probability density. A progress was also made in demonstrating equivalence of the KDP and the Klein-Gordon equations, especially when interactions are taken into account, c.f. [5] and references therein. The KDP equations has been also studied in the context of electromagnetic interactions [6, 7, 8], parasupersymmetric quantum mechanics [6], EPR type nonlocality [9], and Riemann-Cartan space-time [10].

It is well known that the KDP equations contain redundant components - only 2(2s + 1) components are needed to describe free spin-s particles with nonzero rest masses [11] while spin-0 and spin-1 KDP equations contain 5 and 10 components, respectively. The presence of redundant components in KDP equations leads for some interactions to nonphysical effects such as superluminal velocities [12, 13] (see also Refs. [14, 15, 16] for \( s = \frac{3}{2}, 2 \) cases). It is possible however to obtain physically acceptable equations for arbitrary spin removing redundant components with use of additional covariant condition [11]. On the other hand, presence of redundant components suggests that the KDP equations posses internal structure. Indeed, a pair of three-component equations, with solutions fulfilling the five-component spin-0 KDP equation, was found
This internal structure is imperfect in a sense, that although each of the three-component equations describes a massive particle, its Lorentz symmetry is broken (i.e. equations are covariant with respect to boost in one direction and rotation around this axis only). However, these two three-component equations considered together are Lorentz covariant.

In the present paper we initiate a systematic study of the internal structure of spin-0 and spin-1 KDP equations. We shall describe a systematic procedure of splitting (five-component) spin-0 and (ten-component) spin-1 KDP equations by means of the spinor calculus into pairs of constituents equations with lesser numbers of components, such that solutions of the latter equations fulfill the initial KDP equations. In deep inelastic scattering one is probing the hadron in the infinite momentum frame. It was first shown by Susskind that the infinite momentum frame is equivalent to a change of the standard variables \((x^0, x^1, x^2, x^3)\) into the light-cone variables \((x^0 + x^3, x^0 - x^3, x^1, x^2)\) [18, 19]. This suggests that it might be useful to rewrite the tensor KDP equations within the spinor formalism in which coordinates \((x^0 + x^3, x^0 - x^3, x^1 - ix^2, x^1 + ix^2)\), complexifying the light-cone variables, appear in natural fashion. More exactly, we shall demonstrate that there is a systematic procedure of splitting (five-component) spin-0 and (ten-component) spin-1 KDP equations by means of the spinor calculus into pairs of equations with lesser numbers of components, such that solutions of the latter equations fulfill the initial KDP equations. Since mesons are spin-0 and spin-1 quark-antiquark bound states it is tempting to recognize the resulting equations as quark equations but we shall adopt a more cautious approach and will refer to them as constituent equations. Indeed, we shall show that solutions of constituent equations fulfill the Dirac equation. Lack of the full covariance shows that separation of a meson into constituents is frame dependent (yet is possible, as we shall demonstrate, in arbitrary reference frame).

The paper is organized as follows. In Section 2 the Kemmer-Duffin-Petiau equations for spin 0 and spin 1 are described. Elements of spinor calculus are given in Section 3 [20, 21]. These two Sections contain also necessary definitions and conventions. In Section 4 splitting of the KDP equations into three-component constituent equations is achieved for \(s = 0\) (covering in a new way our previous result [14]). Main results are described in the next two Sections. In Section 5 we split the KDP equations for \(s = 1\). In Section 6 we interpret all constituent equations finding direct relation with the Dirac equation. In the last Section our results are discussed in the light of several current results and problems of quark theory.

**2 Kemmer-Duffin-Petiau equations**

In what follows tensor indices are denoted with Greek letters, \(\mu = 0, 1, 2, 3\). We shall use the following convention for the metric tensor: \(g^{\mu\nu} = \text{diag}(1, -1, -1, -1)\) and we shall always sum over repeated indices. Four-momentum operators are defined in natural units \((c = 1, \hbar = 1)\) as \(p^\mu = i \partial^\mu\).
The KDP equations for spin 0 and 1 are written as:

$$\beta_\mu p^\mu \Psi = m\Psi, \quad (1)$$

with $5 \times 5$ and $10 \times 10$ matrices $\beta^\mu$, respectively, which fulfill the following commutation relations [2]:

$$\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda = g^\lambda_\mu \beta^\nu + g^\nu_\mu \beta^\lambda. \quad (2)$$

In the case of $5 \times 5$ (spin-0) representation of $\beta^\mu$ matrices Eq. (1) is equivalent to the following set of equations:

$$p^\mu \psi = m\psi^\mu, \quad p_\nu \psi^\nu = m\psi \quad \{ \quad (3)$$

if we define $\Psi$ in (1) as:

$$\Psi = (\psi^\mu, \psi)^T = (\psi^0, \psi^1, \psi^2, \psi^3)^T, \quad (4)$$

where $T$ denotes transposition of a matrix. Let us note that Eq. (3) can be obtained by factorizing second-order derivatives in the Klein-Gordon equation $p_\mu p^\mu \psi = m^2 \psi$.

In the case of $10 \times 10$ (spin-1) representation of matrices $\beta^\mu$ Eq. (1) reduces to:

$$p^\mu \psi^\nu - p^\nu \psi^\mu = m\psi^{\mu\nu}, \quad p_\mu \psi^{\mu\nu} = m\psi^{\nu} \quad \{ \quad (5)$$

with the following definition of $\Psi$ in (1):

$$\Psi = \left( \psi^{\mu\nu}, \psi^\lambda \right)^T = (\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{12}, \psi^0, \psi^1, \psi^2, \psi^3)^T, \quad (6)$$

where $\psi^\lambda$ are real and $\psi^{\mu\nu}$ are purely imaginary (in alternative formulation we have $-\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m\psi^{\mu\nu}, \partial_\mu \psi^{\mu\nu} = m\psi^{\nu}$, where $\psi^\lambda, \psi^{\mu\nu}$ are real). Because of antisymmetry of $\psi^{\mu\nu}$ we have $p_\nu \psi^\nu = 0$ what implies spin 1 condition. The set of equations (5) was first written by Proca [22].

### 3 Elements of spinor calculus

Two component undotted spinors $\xi_A$, where $A$ numbers spinor components, transform according to representation of the group $SL(2, C)$, $\xi'_A = S_A^B \xi_B$, where $S \in SL(2, C)$ is a $2 \times 2$ complex matrix with unit determinant and $A, B = 1, 2$. Analogously, two component dotted spinors $\eta^A$ transform according to $\eta'^A = \bar{S}_A^B \eta_B$, where $\bar{S}$ is a matrix complex conjugate to $S$ and $\hat{A}, \hat{B} = 1, 2$. Let us stress again that we sum over repeated indices. Spinor indices are lowered or raised with help of metric spinor $\varepsilon_{AB} = \varepsilon_{\hat{A}\hat{B}} = \varepsilon^{\hat{A}\hat{B}} = \varepsilon^{AB} = \varepsilon^{\hat{A}\hat{B}} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. 


It follows that the scalar product of two spinors, e.g. \( \xi_A \zeta^A \) is invariant and vanishes automatically for \( \xi = \zeta \) since \( \xi_A \zeta^A = -\xi^A \zeta_A \).

A general spinor with \( j \) dotted and \( k \) undotted indices transforms as \( \omega'_{A_1 \ldots A_j \bar{B}_1 \ldots \bar{B}_k} = S_{A_1} C^1 \ldots S_{A_j} C^j \bar{S}_{\bar{B}_1} \bar{D}_1 \ldots \bar{S}_{\bar{B}_k} \bar{D}_k \omega_{C_1 \ldots C_j \bar{D}_1 \ldots \bar{D}_k} \). Dirac bispinors are defined as composed from one undotted and one dotted spinor: \( \Psi = (\xi_1, \xi_2, \eta_1, \eta_2)^T \) and thus transform according to reducible representation of \( SL(2, C) \) group.

\( SL(2, C) \) group generalizes the Lorentz group \( SO(1, 3) \) (more precisely, \( SL(2, C) \) is the simply connected doubly covering group of \( SO(1, 3) \) and hence the tensors, i.e. objects transforming according to the Lorentz group, can be embedded in the spinor algebra. We shall provide two examples of such embedding, which we shall need later.

Four-vectors \( \psi^\mu = (\psi^0, \psi) \) and spinors \( \zeta^{AB} \) are related by formula:

\[
\zeta^{AB} = (\sigma^0 \psi^0 + \sigma \cdot \psi)^{AB} = \begin{pmatrix} \zeta^{11} & \zeta^{12} \\ \zeta^{21} & \zeta^{22} \end{pmatrix} = \begin{pmatrix} \psi^0 + i\psi^3 & \psi^1 - i\psi^2 \\ \psi^1 + i\psi^2 & \psi^0 - i\psi^3 \end{pmatrix}, \tag{7}
\]

where \( A, \bar{B} \) number rows and columns, respectively, and \( \sigma^j, j = 1, 2, 3 \), are the Pauli matrices, \( \sigma^0 \) is the unit matrix.

Every antisymmetric tensor can be decomposed into selfdual and antiselfdual parts: \( F_{\mu \nu} = -F_{\nu \mu} = F^S_{\mu \nu} + F^A_{\mu \nu} \), where \( \hat{F}^S_{\mu \nu} = F^S_{\mu \nu}, \hat{F}^A_{\mu \nu} = -F^A_{\mu \nu} \). In these formulae a definition of a dual tensor was used: \( \hat{F}_{\mu \nu} \equiv \frac{1}{2} \varepsilon_{\mu \nu \kappa \lambda} F^{\kappa \lambda} \). Selfdual and antiselfdual tensors can be expressed by symmetric spinors \( \xi^{AB} = \xi_{BA} \) and \( \eta^{AB} = \eta_{BA} \), respectively. Namely, we have \( F^S_{\mu \nu} = \Sigma_{\mu \nu}^{AB} \xi^{AB}, \quad F^A_{\mu \nu} = \Sigma_{\mu \nu}^{AB} \eta^{AB} \), where \( \Sigma_{\mu \nu}^{AB}, \Sigma_{\mu \nu}^{BA} \) are appropriate spin tensors. In explicit form we have:

\[
\begin{align*}
(F^S_{01}, F^S_{02}, F^S_{03}) &= i (F^S_{23}, F^S_{31}, F^S_{12}) = (-\xi^{11} + \xi^{22}, i\xi^{11} + \xi^{22}, 2\xi^{12}), \\
(F^A_{01}, F^A_{02}, F^A_{03}) &= -i (F^A_{23}, F^A_{31}, F^A_{12}) = (-\eta^{11} + \eta^{22}, -i\eta^{11} - \eta^{22}, 2\eta^{12}). \tag{8}
\end{align*}
\]

Spinor calculus abounds in identities. We provide for further convenience several examples of identities involving spinor \( p^{AB} \):

\[
\begin{align*}
p_{11} p^{11} + p_{21} p^{21} &= p_{\mu} p^{\mu}, & p_{12} p^{12} + p_{22} p^{22} &= p_{\mu} p^{\mu}, \tag{10}
p^{C}_{\bar{B}} p^{B}_{A} &= -\delta^{C}_{A} p_{\mu} p^{\mu}, & p^{D}_{\bar{A}} p^{A}_{\bar{B}} &= -\delta^{D}_{\bar{B}} p_{\mu} p^{\mu}, \tag{11}
p_{1} p^{1} + p_{2} p^{2} &= 0, & p_{1} p^{1} + p_{2} p^{2} &= 0, \tag{12}
\end{align*}
\]

which can be verified directly with help of \( \mathbf{F} \). For example, both identities \( \mathbf{F} \) are equivalent to the identity \( (p_{0} - p_{3})(p_{0} + p_{3}) + (-p_{1} + ip_{2})(p_{1} + ip_{2}) = (p_{0})^{2} - (p_{1})^{2} - (p_{2})^{2} - (p_{3})^{2} \).
4 Splitting the spin-0 Kemmer-Duffin-Petiau equations

Equations (13) can be written within spinor formalism as:

\[
\begin{align*}
p^{\dot{A}\dot{B}} \psi &= m \psi^{\dot{A}\dot{B}} \\
p_{\dot{A}\dot{B}} \psi^{\dot{A}\dot{B}} &= 2m \psi.
\end{align*}
\] (13)

Splitting the last of equations (13), \( p^{\dot{A}\dot{B}} \psi^{\dot{A}\dot{B}} = p_{\dot{1}\dot{1}} \psi^{\dot{1}\dot{1}} + p_{\dot{2}\dot{1}} \psi^{\dot{2}\dot{1}} + p_{\dot{1}\dot{2}} \psi^{\dot{1}\dot{2}} + p_{\dot{2}\dot{2}} \psi^{\dot{2}\dot{2}} = 2m \psi \), we obtain two sets of equations involving components \( \psi^{\dot{1}\dot{1}}, \psi^{\dot{2}\dot{1}} \) and \( \psi^{\dot{1}\dot{2}}, \psi^{\dot{2}\dot{2}} \), respectively:

\[
\begin{align*}
p^{\dot{1}\dot{1}} \psi &= m \psi^{\dot{1}\dot{1}} \\
p^{\dot{2}\dot{1}} \psi &= m \psi^{\dot{2}\dot{1}} \\
p_{\dot{1}\dot{1}} \psi^{\dot{1}\dot{1}} + p_{\dot{2}\dot{1}} \psi^{\dot{2}\dot{1}} &= m \psi
\end{align*}
\] (14)

\[
\begin{align*}
p^{\dot{1}\dot{2}} \psi &= m \psi^{\dot{1}\dot{2}} \\
p^{\dot{2}\dot{2}} \psi &= m \psi^{\dot{2}\dot{2}} \\
p_{\dot{1}\dot{2}} \psi^{\dot{1}\dot{2}} + p_{\dot{2}\dot{2}} \psi^{\dot{2}\dot{2}} &= m \psi
\end{align*}
\] (15)

each of which describes particle with mass \( m \) (we check this substituting e.g. \( \psi^{\dot{1}\dot{1}}, \psi^{\dot{2}\dot{1}} \) or \( \psi^{\dot{1}\dot{2}}, \psi^{\dot{2}\dot{2}} \) into the third equations). The splitting preserving \( p_\mu p^\mu \psi = m^2 \psi \) is possible due to spinor identities (10). Thus solutions of Eqs. (14), (15) fulfill the KDP equations (13). We described these equations in [17]. From each of equations (14), (15) an identity follows:

\[
\begin{align*}
p_{\dot{2}\dot{1}} \psi^{\dot{1}\dot{1}} &= p_{\dot{1}\dot{1}} \psi^{\dot{2}\dot{1}}, \\
p_{\dot{2}\dot{2}} \psi^{\dot{1}\dot{2}} &= p_{\dot{1}\dot{2}} \psi^{\dot{2}\dot{2}}.
\end{align*}
\] (16a, 16b)

Equations (14), (15) can be written in matrix form:

\[
\rho_\mu p^\mu \Phi = m \Phi,
\] (17)

where \( \Phi = (\psi^{\dot{1}\dot{1}}, \psi^{\dot{2}\dot{1}}, \psi^{\dot{1}\dot{2}}, \psi^{\dot{2}\dot{2}})^T \),

\[
\begin{align*}
\rho^0 &= \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \end{pmatrix}, \\
\rho^1 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \end{pmatrix}, \\
\rho^2 &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -i \\
0 & -i & 0 \end{pmatrix}, \\
\rho^3 &= \begin{pmatrix} 0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0 \end{pmatrix},
\end{align*}
\] (18)

and

\[
\tilde{\rho}_\mu p^\mu \tilde{\Phi} = m \tilde{\Phi},
\] (19)
where $\tilde{\Phi} = \left(\psi^{12}, \psi^{22}, \psi\right)^T$,

\[
\begin{align*}
\tilde{\rho}^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\tilde{\rho}^1 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\tilde{\rho}^2 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\tilde{\rho}^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\end{align*}
\]

Equations (17), (19) considered together:

\[
\begin{pmatrix} \rho_{\mu}p^{\mu} & 0 \\ 0 & \tilde{\rho}_{\mu}p^{\mu} \end{pmatrix} \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix} = m \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix},
\]

are Lorentz covariant since involve all components of the spinor $\psi^{AB}$. Obviously, all solutions of Eq. (21) satisfy Eq. (13) but the reverse is not true.

### 5 Splitting the spin-1 Kemmer-Duffin-Petiau equations

KDP equations (5) can be written in spinor form as (20):

\[
\begin{align*}
p_A B C D \zeta_{CB} + p_B C D \zeta_{AB} &= 2m\eta_{AC} \\
p_A B \zeta_{AD} + p_B D \zeta_{AB} &= 2m\chi_{BD} \\
p_A \chi_{BD} + p_B \eta_{AC} &= -2m\zeta_{AB}
\end{align*}
\]

It is possible to split the spinor form of the KDP equations to get two equations for spinors $\chi_{BD}, \zeta_{AB}$ and $\eta_{AC}, \zeta_{AB}$:

\[
\begin{align*}
p_A B \zeta_{CB} &= m\eta_{AC}, \\
p_B C \eta_{AC} &= -m\zeta_{AB}
\end{align*}
\]

\[
\begin{align*}
p_A B \zeta_{AD} &= m\chi_{BD}, \\
p_B D \chi_{BD} &= -m\zeta_{AB}
\end{align*}
\]

respectively. The splitting possible due to spinor identities (11). Thus solutions of Eqs. (23), (24) fulfill the KDP equations (22).

The spinor equations (22), (24) describe spin-1 bosons where spinors $\eta_{CA}, \chi_{BD}$ correspond to selfdual or antiselfdual antisymmetric tensors $\psi^{\mu\nu}$, respectively. Each of the above equations is covariant except from space reflection but both equations taken together are fully covariant. These equations written in tensor form, $\beta_{\mu\nu} p_\mu \Psi = m \Psi$, $\Psi = [\psi_{01}, \psi_{02}, \psi_{03}, \psi_0, \psi_1, \psi_2, \psi_3]^T$ where $\psi^{\mu\nu}$ are selfdual or antiselfdual antisymmetric tensors, with $7 \times 7$ matrices $\beta_{\mu}$ fulfilling Eq. (38), are the Hagen-Hurley equations [21, 22].
We shall now split the spinor form of the Hagen-Hurley equations (23), (24) to arrive at the main result of this Section. To this end equation (23) is written in explicit form. Since 
\[ \eta_{12} = \eta_{21} = \eta, \]
the left hand side of the subequation
\[ p_1 \dot{\xi}_{12} + p_2 \dot{\xi}_{21} = \eta, \]
is decomposed into symmetric and antisymmetric parts, the latter equal zero:
\[ p_1 \dot{\xi}_{11} + p_2 \dot{\xi}_{12} = m \eta_{11}, \]  
(25a)
\[ \frac{1}{2} (p_1 \dot{\xi}_{21} + p_2 \dot{\xi}_{22} + p_1 \dot{\xi}_{11} + p_2 \dot{\xi}_{12}) = m \eta, \]  
(25b)
\[ p_1 \dot{\xi}_{21} + p_2 \dot{\xi}_{22} - p_2 \dot{\xi}_{11} - p_1 \dot{\xi}_{12} = 0, \]  
(25c)
\[ p_2 \dot{\xi}_{21} + p_2 \dot{\xi}_{22} = m \eta_{22}, \]  
(25d)
\[ p_1 \eta_{11} + p_2 \eta = -m \zeta_{11}, \]  
(25e)
\[ p_1 \eta_{11} + p_2 \eta = -m \zeta_{12}, \]  
(25f)
\[ p_1 \eta + p_1 \eta_{22} = -m \zeta_{21}, \]  
(25g)
\[ p_1 \eta + p_2 \eta_{22} = -m \zeta_{22}, \]  
(25h)
Let us note that vanishing of the antisymmetric part, i.e. the third equation, expresses spin-1 condition
\[ p_\mu \psi^\mu = 0, \]
what can be verified directly with help of Eq.(7).

We shall demonstrate that the set of eight equations above is equivalent to the following equations:
\[ p_1 \eta_{11} = -m \hat{\zeta}_{11}, \]  
(26a)
\[ p_2 \eta_{11} = -m \hat{\zeta}_{12}, \]  
(26b)
\[ p_1 \hat{\zeta}_{11} + p_2 \hat{\zeta}_{12} = m \eta_{11}, \]  
(26c)
\[ p_2 \eta_{22} = -m \hat{\zeta}_{21}, \]  
(27a)
\[ p_2 \eta_{22} = -m \hat{\zeta}_{22}, \]  
(27b)
\[ p_2 \hat{\zeta}_{21} + p_2 \hat{\zeta}_{22} = m \eta_{22}, \]  
(27c)
\[ p_\mu p^\mu \eta = m^2 \eta, \]  
(28)
where \( \hat{\zeta}_{11} \equiv \xi_{11} + \frac{2}{m} \eta, \hat{\zeta}_{12} \equiv \frac{\xi_{12}}{m} + \frac{p^1 \eta}{m}, \hat{\zeta}_{21} \equiv \frac{\xi_{21}}{m} + \frac{p^2 \eta}{m}, \frac{\xi_{22}}{m} = \xi_{22} + \frac{p^1 \eta}{m}. \)

Indeed, \( p_1 \hat{\zeta}_{11} + p_1 \hat{\zeta}_{12} = p_1 \xi_{11} + p_1 \frac{\xi_{12}}{m} \) and \( p_2 \hat{\zeta}_{21} + p_2 \hat{\zeta}_{22} = p_2 \xi_{21} + p_2 \frac{\xi_{22}}{m} \) due to spinor identities (12). Hence Eqs. (26a), (26b), (26c) are identical with equations (25e), (25f), (25a) and Eqs. (27a), (27b), (27c) are identical
with equations (25g), (25h), (25d). Furthermore, from Eqs. (26a), (26b) and Eqs. (27a), (27b) two identities follow:

\[ p_{2} \dot{\zeta}_{11} = p_{1} \dot{\zeta}_{12}, \]  
\[ p_{2} \dot{\zeta}_{21} = p_{1} \dot{\zeta}_{22}. \]  
\[ (29a) \]
\[ (29b) \]

Now, \( \left( p_{2} \dot{\zeta}_{11} - p_{1} \dot{\zeta}_{12} \right) + \left( p_{2} \dot{\zeta}_{21} - p_{1} \dot{\zeta}_{22} \right) \) reduces to \( p_{2} \dot{\zeta}_{21} - p_{1} \dot{\zeta}_{12} + p_{2} \dot{\zeta}_{21} - p_{1} \dot{\zeta}_{22} = 0 \) and implies spin-1 condition \( p_{\mu} \dot{\psi}^{\mu} = 0 \), where \( \zeta^{A \dot{B}} = (\sigma^{0} \psi^{0} + \sigma \cdot \dot{\psi})^{A \dot{B}} \); this condition is thus equivalent to Eq. (25c). On the other hand, it can be directly verified using definition of spinor \( \dot{\zeta}_{A \dot{B}} \) and rearranging indices of the spinor \( p_{A}^{\dot{B}} \) to get \( \rho_{C}^{\dot{B}} \), that \( \left( p_{1} \dot{\zeta}_{11} - p_{1} \dot{\zeta}_{12} \right) - \left( p_{2} \dot{\zeta}_{21} - p_{2} \dot{\zeta}_{22} \right) = 0 \) is equivalent to equality

\[ p_{1} \dot{\zeta}_{21} + p_{1} \dot{\zeta}_{22} + p_{1} \dot{\zeta}_{11} + p_{2} \dot{\zeta}_{12} = \frac{2\eta}{m} \]  

and becomes the equation (25b) due to (28).

The three component equations can be written in matrix form as \( \rho_{\mu} p^{\mu} \Psi = m \Psi \):

\[
\rho^{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \rho^{1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \rho^{2} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \rho^{3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
\]

\[ \Psi = \left( \dot{\zeta}_{11}, \dot{\zeta}_{12}, \eta_{11} \right)^{T} \]  

and \( \tilde{\rho}_{\mu} \tilde{p}^{\mu} \tilde{\Psi} = m \tilde{\Psi} \):

\[
\tilde{\rho}^{0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tilde{\rho}^{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
\]
\[ \tilde{\rho}^{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \tilde{\rho}^{3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

\[ \tilde{\Psi} = \left( \dot{\zeta}_{21}, \dot{\zeta}_{22}, \eta_{22} \right)^{T}. \]
Analogously, we split equations (24):

\[
\begin{align*}
\frac{1}{2} \left( p_1^1 \dot{\zeta}_{11} + p_2^1 \dot{\zeta}_{21} + p_1^1 \dot{\zeta}_{12} + p_2^1 \dot{\zeta}_{22} - p_1^2 \dot{\zeta}_{11} - p_2^2 \dot{\zeta}_{21} + p_1^2 \dot{\zeta}_{12} + p_2^2 \dot{\zeta}_{22} \right) &= m\chi_11, \\
p_1^1 \dot{\zeta}_{11} + p_1^2 \dot{\zeta}_{22} &= m\chi_12, \\
p_2^1 \dot{\zeta}_{11} + p_2^2 \dot{\zeta}_{22} &= m\chi_22, \\
p_1^1 \chi_{11} + p_2^1 \chi &= -m\zeta_{11}, \\
p_1^2 \chi_{11} + p_2^2 \chi &= -m\zeta_{21}, \\
p_1^1 \chi + p_1^2 \chi_{22} &= -m\zeta_{12}, \\
p_2^1 \chi + p_2^2 \chi_{22} &= -m\zeta_{22},
\end{align*}
\]

(32)

where \( \chi_{12} = \chi_{21} \equiv \chi \) (again the third equation is equivalent to spin-1 condition \( p_\mu \psi^\mu = 0 \)). We thus get:

\[
\begin{align*}
p_1^1 \chi_{11} &= -m\tilde{\zeta}_{11}, \\
p_2^1 \chi_{11} &= -m\tilde{\zeta}_{21}, \\
p_1^1 \dot{\zeta}_{11} + p_2^2 \dot{\zeta}_{22} &= m\chi_{11}, \\
p_1^1 \dot{\zeta}_{12} &= -m\tilde{\zeta}_{12}, \\
p_2^2 \dot{\zeta}_{22} &= -m\tilde{\zeta}_{22},
\end{align*}
\]

(33)

\[
\begin{align*}
p_1^2 \chi_{11} &= -m\tilde{\zeta}_{11}, \\
p_2^2 \chi_{11} &= -m\tilde{\zeta}_{21}, \\
p_1^2 \chi_{12} &= -m\tilde{\zeta}_{12}, \\
p_2^2 \chi_{22} &= -m\tilde{\zeta}_{22},
\end{align*}
\]

(34)

\[
\rho_\mu p^\mu \chi = m^2 \chi,
\]

(35)

where \( \tilde{\zeta}_{11} \equiv \zeta_{11} \), \( \tilde{\zeta}_{21} \equiv \zeta_{21} \), \( \tilde{\zeta}_{12} \equiv \zeta_{12} \), \( \tilde{\zeta}_{22} \equiv \zeta_{22} \), \( \tilde{\chi}_{11} \equiv \chi_{11} \), \( \tilde{\chi}_{21} \equiv \chi_{21} \), \( \tilde{\chi}_{12} \equiv \chi_{12} \), \( \tilde{\chi}_{22} \equiv \chi_{22} \).

Equations (33), (34) with these definitions are equivalent to (32) due to appropriate spinor identities as well as to identities:

\[
\begin{align*}
p_2^1 \tilde{\zeta}_{11} &= p_1^1 \tilde{\zeta}_{21}, \\
p_2^2 \tilde{\zeta}_{12} &= p_1^2 \tilde{\zeta}_{22},
\end{align*}
\]

(36a, 36b)

which follow from (33), (34).

The equations (33), (34) can be written in matrix form \( \rho_\mu p^\mu \Psi = m\Psi \):

\[
\begin{align*}
\rho^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \rho^1 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\rho^2 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \rho^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
\end{align*}
\]

(37)

\[\Psi = (\tilde{\zeta}_{11}, \tilde{\zeta}_{21}, \chi_{11})^T\] and \( \tilde{\rho}_\mu p^\mu \tilde{\Psi} = m\tilde{\Psi} \).
\[ \tilde{\rho}^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tilde{\rho}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \]

\[ \tilde{\rho}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ -i & 0 \end{pmatrix}, \tilde{\rho}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \]

\[ \tilde{\Psi} = (\tilde{\zeta}_{12}, \tilde{\zeta}_{23}, \chi_{22})^T. \]

Let us note that matrices (37), (38) can be obtained from matrices (30), (31) by complex conjugation.

All matrices: \( \tilde{\rho}^\mu \), \( \tilde{\rho}^\mu \) discussed above, c.f. Eqs.(18), (20), (30), (31), (37), (38), fulfill the Tzou commutation relations [25, 17, 6]

\[ \rho^{(\lambda} \rho^\mu \rho^{\nu)} = g^{(\lambda} \rho^\mu), \]

more complicated than (2), where \( (\lambda \mu \nu) \) is the symmetrizer. There is however no conjugation rule for matrices \( \rho^\mu \) and \( \tilde{\rho}^\mu \), for example there is no such matrix \( S \) that \( \tilde{\rho}^\mu = S \rho^\mu S^{-1} \). We shall see in the next Section that a conjugation rule (charge conjugation) exists if \( 3 \times 3 \) matrices \( \rho^\mu \) are extended to \( 4 \times 4 \) Dirac matrices \( \gamma^\mu \).

Equations (26a), (26b), (26c) considered together are Lorentz covariant, except from space reflection, since involve all components of the spinors \( \hat{\zeta}_{AB}, \eta_{CD} \) in analogy with spin-0 case (the same applies to the set of equations (33), (34), (35)).

6 Subsolutions of the Dirac equation

Since the constituent equations (14), (15) and (26a, 26b, 26c), (27a, 27b, 27c) seem to be fundamental we shall attempt in the first place to interpret these equations. We shall first interpret equations (14), (15) and identities (16a), (16b).

Equation (14) and the identity (16a), as well as equation (15) and the identity (16b) can be written in form of the Dirac equations:

\[
\begin{pmatrix}
0 & 0 & p^0 + p^3 & p^1 - ip^2 \\
0 & 0 & p^1 + ip^2 & p^0 - p^3 \\
p^0 - p^3 & -p^1 + ip^2 & 0 & 0 \\
-p^1 - ip^2 & p^0 + p^3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi^{11} \\
\psi^{21} \\
\chi \\
0
\end{pmatrix}
= m
\begin{pmatrix}
\psi^{11} \\
\psi^{21} \\
\chi \\
0
\end{pmatrix},
\]

(40)

\[
\begin{pmatrix}
0 & 0 & p^0 - p^3 & p^1 + ip^2 \\
0 & 0 & p^1 - ip^2 & p^0 + p^3 \\
p^0 + p^3 & -p^1 - ip^2 & 0 & 0 \\
-p^1 + ip^2 & p^0 - p^3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi^{22} \\
\psi^{12} \\
\chi \\
0
\end{pmatrix}
= m
\begin{pmatrix}
\psi^{22} \\
\psi^{12} \\
\chi \\
0
\end{pmatrix},
\]

(41)
respectively, with one zero component. Equation (10) can be written as $\gamma^\mu p_\mu \Psi = m\Psi$ with spinor representation of the Dirac matrices, $\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}$, $\gamma^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix}$, $j = 1, 2, 3$, $\gamma^5 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}$, $\Psi = \begin{pmatrix} \psi^{11} \\ \psi^{2i} \\ \chi, 0 \end{pmatrix}^T$. Equation (40) can be analogously written as $(\gamma^0 p^0 - \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3) \Phi = m\Phi$, $\Phi = \begin{pmatrix} \psi^{22}, \psi^{12}, \chi, 0 \end{pmatrix}^T$.

We shall demonstrate now that equations (40) and (41) are charge conjugated one to another. Complex conjugation of Eq. (40) yields:

$$(-1) \begin{pmatrix} 0 & 0 & p^0 + p^3 & p^1 + ip^2 \\ 0 & 0 & p^1 - ip^2 & p^0 - p^3 \\ -p^1 + ip^2 & 0 & 0 & p^0 + p^3 \\ -p^0 - p^3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{11}^* \\ \psi^{2i}^* \\ \chi^* \\ 0 \end{pmatrix} = m \begin{pmatrix} \psi^{11}^* \\ \psi^{2i}^* \\ \chi^* \\ 0 \end{pmatrix}, \tag{42}$$

i.e. $(-1) (\gamma^0 p^0 - \gamma^1 p^1 + \gamma^2 p^2 - \gamma^3 p^3) \Psi^* = m\Psi^*$ where $^*$ denotes complex conjugation. Acting from the left with matrix $\gamma^3$ on Eq. (42) we obtain equation $(\gamma^0 p^0 - \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3) \gamma^3 \Psi^* = m\gamma^3 \Psi^*$ which has the same form as Eq. (10) (the charge conjugation matrix $C$ is thus defined as $C\gamma^0 \equiv \gamma^3 \tag{26}$). Hence the initial equations (10) and (11) are charge conjugated one to another in a sense that they are charge conjugated after extension to the Dirac form.

Similar considerations lead to conclusion that also Eqs. (26a), (26b), (26c), (27a), (27b), (27c), and identities (29a), (29b) as well as Eqs. (33), (34) and identities (36a), (36b) can be written in Dirac form to reveal that they are charge conjugated one to another.

The observations made above can be given representation independent formulation. Let us notice that the three component equations, for instance (11), (15) as well as the identities (16a), (16b), can be obtained by projecting the Dirac equation with projection operator $P_1 = \text{diag}(1, 1, 1, 0)$. Incidentally, there are other projection operators which lead to analogous three component equations, $P_1 = \text{diag}(0, 1, 1, 1)$, $P_2 = \text{diag}(1, 0, 1, 1)$, $P_3 = \text{diag}(1, 1, 0, 1)$ but we shall need only the operator $P_4$.

In general, we can consider subsolutions, of form $P_4 \Psi$, of the Dirac equation:

$$\gamma^\mu p_\mu P_4 \Psi = mP_4 \Psi, \tag{43}$$

which is equivalent to (11) in the case of spinor representation of the Dirac matrices.

Accordingly, acting from the left on (43) with $P_4$ and $(1 - P_4)$ we obtain two equations:

$$P_4 (\gamma^\mu p_\mu) P_4 \Psi = mP_4 \Psi, \tag{44a}$$

$$P_4 (\gamma^\mu p_\mu) P_4 \Psi = 0. \tag{44b}$$

In the spinor representation of $\gamma^\mu$ matrices Eq. (44a) is equivalent to (11), while (44b) is equivalent to the identity (16a).
Now the projection operator can be written as
\[ P_4 = \frac{1}{4}(3 + \gamma^5 - \gamma^0 \gamma^3 + i \gamma^1 \gamma^2) \]
(and similar formulae can be given for other projection operators \( P_1, P_2, P_3 \)), i.e. all equations (43), (44a), (44b) are now given representation independent form. Let us also note that the projection operator \( P_4 \) commutes with two generators of Lorentz transformations \( \gamma^0 \gamma^3 \) and \( \gamma^1 \gamma^2 \) (and does not commute with other generators), i.e. is invariant under boosts in \( x^0 x^3 \) plane and rotations in \( x^1 x^2 \) plane. Accordingly, the three component equations are covariant with respect to such Lorentz transformations only. Let us note finally that all three component equations describe particles with definite mass and only one component of spin defined. Results of this Section are directly generalized for the case of interaction introduced via minimal coupling, \( p^\mu \rightarrow \pi^\mu = p^\mu - eA^\mu \).

7 Discussion

We have shown that spinor formalism discloses internal structure of KDP equations which manifests itself by presence of redundant components - there are special three-component solutions of these equations. Accordingly, the meson spin-0 and spin-1 KDP equations split into pairs of three-component constituent equations, each equation describing a particle with definite mass and only one component of spin defined (all three-component constituent equations discussed above are similar in a sense that their matrices \( \rho^\mu \) fulfill the same commutation relations (39) \[ 25, 17, 6 \]) and in the case of spin-1 KDP equations an additional wavefunction fulfilling the Klein-Gordon equation is present. Moreover, solutions of the constituent equations are subsolutions of appropriate Dirac equations and pairs of such Dirac equations, corresponding to pairs of constituent equations, are charge conjugated one to another. This last finding entitles us to conclude that Kemmer-Duffin-Petiau equations describe mesons as composed from quark-antiquark pairs. These results are consistent with quark theory of mesons \[ 27 \]. Furthermore, existence of additional constituent particle with the same mass in the case of spin-1 mesons elucidates problem of meson multiplets. It is well known that pseudoscalar mesons can be approximately arranged in \( SU(3) \) octets while in the case of vector mesons octets are strongly mixed with \( SU(3) \) singlets to form nonets \[ 25 \]. Since we get two constituent, charge conjugated equations, plus a single equation it might be inferred that this picture is consistent with octet-singlet mixing if we interpret constituents in three-component equations as quarks and an additional constituent as a \( SU(3) \) singlet vector meson. The mixing is indeed present since wavefunction fulfilling (25) is dynamically coupled to wavefunctions in three-component equations \[ 25, 26, 26a \).

Let us stress that the separation of a meson into constituents is imperfect, since although each of the constituent equations describes a massive particle, its Lorentz symmetry is broken. This offers explanation of quark confinement different than in quantum chromodynamics where linearly increasing potential energy between a quark and other quarks in a hadron is responsible for confinement \[ 29 \]. We hope that our results can also cast light on the problem of spin...
crisis \[30\] since meson constituents in our theory have one component of spins defined only. Let us assume that proton constituents (quarks) have the same nature as the meson constituents of our theory. It follows that the proton spin cannot be obtained as a sum of spins of its constituents since the constituents have not fully defined spins.

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