Bose-Einstein condensation of magnons in spin pumping systems

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Abstract

We clarify the condition for the occurrence of magnon Bose-Einstein condensation (BEC) in spin pumping systems without using external pumping magnetic fields. The Goldstone model is generalized and the stability of the vacuum is closely investigated. By applying the generalized Goldstone model to spin pumping systems, the condition for the experimental realization of the stable magnon BEC in spin pumping systems is theoretically proposed.

1 Introduction

The experimental observation of BEC\cite{1} in variety kinds of systems as well as trapped ultracold atoms and molecules have recently been reported; photons in an optical microcavity,\cite{2} semiconductor microcavity exciton polaritons,\cite{3} and microcavity polaritons in a trap\cite{4} et al.\cite{5} This fact implies the universal aspects of this phenomenon. BEC is, in principle, the phenomenon that a macroscopic number of particles occupies a single-particle state.\cite{1, 6} Thus quasiparticles also undergo BEC and in particular, magnon BEC\cite{7, 8} has now become one of the most attractive subjects in condensed matter physics.

In this paper, we go after the possibility for the occurrence of magnon BEC\cite{9, 8} without using external pumping magnetic fields (i.e. quantum fluctuations)\cite{10, 11} in spin pumping systems (Fig. 1). At the interface of a ferromagnetic insulator and non-magnetic metal junction, conduction electrons interact with ferromagnetic localized spins $S; V_{ex} = -S \cdot s$. The degree of freedom of ferromagnetic localized spins are reduced to that of magnons $S$ via the Holstein-Primakoff transformation. Therefore we regard the interface of spin pumping systems as the effective area where magnons interact with conduction electrons;\cite{10, 12} the
interface can be regarded as a ferromagnetic metal. The exchange interaction $V_{\text{ex}}$ at the interface is essential to spin pumping and hence, we identify the system characterized $V_{\text{ex}}$ with the spin pumping system. From now on, we exclusively focus on the dynamics at the interface (Fig. 1). To clarify the condition for the experimental realization of the stable magnon BEC state at the interface of the spin pumping system is the final goal of this paper.

Originally, spin pumping systems have been attaching special attention from the viewpoint of spintronics, which is a rapidly developing new branch of physics. The central theme is the active manipulation of spin degrees of freedom as well as charge ones of electrons. Thus by going after the possibility for the occurrence of magnon BEC in spin pumping systems, we build a bridge between the research on spintronics\textsuperscript{15, 16} and magnon BEC.\textsuperscript{17, 18}

![Figure 1: (Color online). The schematic picture of spin pumping systems; spheres represent magnons and those with arrows are conduction electrons. The interface is characterized by the exchange interaction between conduction electrons and the ferromagnet $V_{\text{ex}}$. Thus, the interface is defined as an effective area where the Fermi gas (i.e. conduction electrons) and the Bose gas (i.e. magnons) coexist to interact. Conduction electrons cannot enter the ferromagnet, which is an insulator. Clear pictures are available at the following URL: https://dl.dropbox.com/u/5407955/MagnonBECinSP.pdf](https://dl.dropbox.com/u/5407955/MagnonBECinSP.pdf)

In this paper, we employ a non-perturbative theory to go beyond the perturbative analysis\textsuperscript{12} and investigate the possibility for the occurrence of magnon BEC in spin pumping systems without using external pumping magnetic field.\textsuperscript{7, 10} For the purpose, we generalize the Goldstone model in sec. 2 and apply the generalized Goldstone model to spin pumping systems in sec. 3. To provide details, this paper is structured as follows; first, by adopting the usual Goldstone model\textsuperscript{19, 20} as an example, we quickly review the idea of spontaneous symmetry breaking (SSB) in the classical field theory in sec. 2.1. Second, in sec. 2.2 by introducing a new complex scalar field with keeping the $U(1)$-symmetry of the system, we minimally generalize the above standard Goldstone model so
as to include the effects of other degrees of freedoms (e.g. spins carried by conduction electrons). The condition for the occurrence of the $U(1)$-SSB is clarified in the minimally generalized model. On top of this, the stability of the vacuum is closely investigated. Last, by applying the above minimally generalized Goldstone model to anisotropic spin pumping systems, we go after the possibility for the occurrence of magnon BEC in sec. 3.1. By further extending the Goldstone model, the condition for the experimental realization of the stable magnon BEC state in spin pumping systems is theoretically proposed in sec. 3.2. This is the main aim of this paper.

2 BEC and SSB

The experimental realizations of magnon BEC in variety kinds of materials have been reported; TlCuCl$_3$, Cs$_2$CuCl$_4$, Yttrium-iron-garnet (YIG), and BaCuSi$_2$O$_6$ et al. In the present study, we identify the expectation value of the bosonic annihilation operator $\langle \Psi \rangle$ with the macroscopic condensate order parameter and adopt as the criterion for the occurrence of BEC.

That is, a non-zero value of the order parameter $\langle \Psi \rangle \neq 0$ under the $U(1)$-symmetric Hamiltonian does mean the occurrence of BEC, which is accompanied by $U(1)$-SSB. This definition of BEC has now been very commonly used in the literature.

On the basis of this definition of BEC, we investigate the possibility for the occurrence of the $U(1)$-SSB of the vacuum in spin pumping systems, which is accompanied by a non-zero value of the order parameter under the $U(1)$-symmetric Hamiltonian. In order to go beyond the perturbative analysis by the Schwinger-Keldysh formalism, we employ a powerful theoretical technique ‘non-perturbative theory’, which does not rely on the assumption called the adiabatic theorem (i.e. the well-known Gell-Mann and Low theorem).

Therefore we can analyze beyond a perturbative theory.

2.1 Goldstone model

Before going on to the main subject, let us briefly review the idea of the SSB of the vacuum in classical field theory. As an example, we employ the (so called) ‘Goldstone model’ whose potential term is given as (see also Fig. 2)

\[
V_{\text{Goldstone}}(\varphi) := -\mu \varphi \varphi^* + \mathcal{J}(\varphi \varphi^*)^2
\]

\[
= -\mu |\varphi|^2 + \mathcal{J} |\varphi|^4,
\]

in which the variable $\varphi(\in \mathbb{C})$ denotes a complex scalar field. The parameter $\mu(\in \mathbb{R})$ represents the dimensionless chemical potential and $\mathcal{J}(\in \mathbb{R})$ does a dimensionless coupling constant. It is clear that the Goldstone model possesses the global $U(1)$-symmetry: $\varphi \mapsto e^{i\theta} \varphi$ with $\theta \equiv (\text{const.}) \in \mathbb{R}$. For the stability of the system or the vacuum (i.e. the ground state), there should be a lower bound on the energy level of the system. Thus the condition is required (see...
Figure 2: (Color online). Schematic pictures of the $U(1)$-SSB of the vacuum in the Goldstone model $V_{\text{Goldstone}}(|\varphi|)$; eq. (5). When the chemical potential becomes positive ($0 < \mu$), the Goldstone model forms ‘the Mexican-hat potential’ (b) and the $U(1)$-symmetry of the vacuum is spontaneously broken; eq. (6). As an example, each parameter is set as follows: (a) $J = +10$, $\mu = -1$ and (b) $J = +10$, $\mu = +1$. It will be useful to see also Fig. 5 (a).

From here on, we will assume that the possible vacuum states are invariant under translations and they are time-independent. Thus, the candidate of the stable vacuum of the system is given as the stationary point of the effective potential $V_{\text{Goldstone}}$: \[ \frac{\partial V_{\text{Goldstone}}}{\partial \varphi} = 0. \] (3)

In addition, within the classical theory in the sense that we omit the quantum effects (i.e. loop corrections) and discuss within the tree-level, the effective potential $V_{\text{Goldstone}}$ is reduced to the usual one $V_{\text{Goldstone}}$: \[ V_{\text{Goldstone}}(|\varphi|) = -\mu |\varphi|^2 + J |\varphi|^4 + \mathcal{O}(\hbar). \] (4a)
\[ = V_{\text{Goldstone}}(|\varphi|) + \mathcal{O}(\hbar). \] (4b)

Thus the minimum-energy classical configuration is a uniform field $\varphi = \varphi_0$ with $\varphi_0$ chosen to minimize the potential $V_{\text{Goldstone}}$: \[ V_{\text{Goldstone}}(|\varphi|) = J \left[ |\varphi|^2 - \frac{\mu^2}{2J} \right]^2 - \frac{\mu^2}{4J}. \] (5)

Consequently when the chemical potential is positive ($0 < \mu$), the vacuum expectation value of the field $\varphi_0$ reads \[ |\varphi_0| = \sqrt{\frac{\mu}{2J}} \neq 0. \] (6)
As the result, the $U(1)$-SSB of the vacuum does occur when the chemical potential is positive ($0 < \mu$, Fig. 2(b)); otherwise not (i.e. $\mu \leq 0$, Fig. 2(a)).

### 2.2 Minimally generalized Goldstone model

We have seen that the $U(1)$-symmetry of the vacuum in the Goldstone model where only one complex scalar field $\varphi$ acts is spontaneously broken when the chemical potential $\mu$ is properly adjusted; this is the rigorous theoretical result based on the non-perturbative analysis. The Goldstone model has been used to describe a dilute Bose gas in the classical limit at $T = 0$ as a (phenomenological) standard model; e.g. magnons, regardless of ferromagnets or antiferromagnets. On the other hand, in real materials and experiments, there does exist variety kinds of freedoms besides the one on which we focus, such as magnetic impurities, phonons, and photons et al. (see also sec. 3.1). Therefore it is desirable to extend the Goldstone model so as to include the effects of such degrees of freedoms by introducing a new complex scalar field $\psi(\in \mathbb{C})$ which couples with the usual field $\varphi$.

Now, our strategy of the generalization of the Goldstone model reads as follows; for clearness, we exclusively focus on when ‘the usual chemical potential $\mu$’ is negative ($\mu \leq 0$). In that case, the model reads

\[ V_{\text{Goldstone}}(\varphi) = B |\varphi|^2 + J |\varphi|^4, \]  

in which we have denoted as, $-\mu =: B (\geq 0)$, for convenience (see also sec. 3.1). In this case, it is apparent that the vacuum expectation value of the field becomes zero ($\varphi_0 = 0$), which is not accompanied by the $U(1)$-SSB of the vacuum (Fig. 2(a)).

Now, we go after the possibility for the occurrence of the $U(1)$-SSB of the vacuum owing to the coupling with other degrees of freedom represented by a complex scalar field $\psi(\in \mathbb{C})$ such as

\[ (\varphi\psi^* + \varphi^*\psi), \quad \psi\psi^*, \] (8a)

and

\[ |\varphi|^2 |\psi|^2, \] (8b)

which do not violate the $U(1)$-symmetry of the system; $(\varphi, \psi) \mapsto e^{i\theta}(\varphi, \psi), \text{ with } \theta \equiv (\text{const.}) \in \mathbb{R}$. It is expected that these couplings bring ‘effective chemical potential’ to $\varphi (\varphi^*)$. As the result, the total chemical potential might become positive and the $U(1)$-SSB of the vacuum might be generated.

#### 2.2.1 Minimal model

We introduce the minimally generalized Goldstone model $V_{U(1)-\text{mini}}(\varphi, \psi)$ by adding couplings, $(\varphi\psi^* + \varphi^*\psi)$ and $\psi\psi^*$, into the Goldstone model;

\[ V_{U(1)-\text{mini}}(\varphi, \psi) := V_{\text{Goldstone}}(\varphi) - \gamma(\varphi\psi^* + \varphi^*\psi) + \kappa|\psi|^2 \] (9a)

\[ = B |\varphi|^2 + J |\varphi|^4 - \gamma(\varphi\psi^* + \varphi^*\psi) + \kappa |\psi|^2, \] (9b)

\[ \text{Note that the sign of } B \text{ is opposite from the one of the chemical potential } \mu. \]
where each variable, \( \gamma (\in \mathbb{R} \text{ and } \gamma > 0) \) and \( \kappa (\in \mathbb{R}) \), represents a dimensionless coupling constant. The minimally generalized Goldstone model \( V_{U(1)-\text{mini}} \) includes two kinds of fields, \( \varphi \) and \( \psi \). Therefore for the stability of the vacuum, there should be a lower bound on the energy level of the system in respect to \( \psi \) as well as \( \varphi \). In terms of \( \varphi \), the minimally generalized Goldstone model can be expressed as

\[
V_{U(1)-\text{mini}}(\varphi) = J |\varphi|^4 + B |\varphi|^2 - \gamma(|\varphi|\psi^* + \varphi^*\psi) + \mathcal{O}(\varphi^{(\ast)}0). \tag{10}
\]

Therefore the condition is required; \( 0 < J \). In addition, from the viewpoint of \( \psi \), \( V_{U(1)-\text{mini}} \) can be regarded as

\[
V_{U(1)-\text{mini}}(\psi) = \kappa |\psi|^2 - \gamma(|\varphi|\psi^* + \varphi^*\psi) + \mathcal{O}(\psi^{(\ast)}0). \tag{11}
\]

Thus the condition should be satisfied:

\[
0 < \kappa. \tag{12}
\]

Otherwise, ‘the saddle point’ (see Fig. 4(a) as an example) cannot be eliminated from the condition for the stationary point represented by eq. 3; the saddle point gives an unstable state and the situation is out of the aim of the present study.

Under these conditions (i.e. inequalities 2 and 12), through the same procedure with sec. 2.1 and within the classical theory, we seek the true stable vacuum of the minimally generalized Goldstone model. The condition for the stationary point in respect to \( \psi \) gives

\[
\frac{\partial V_{U(1)-\text{mini}}}{\partial \psi} = 0 \Rightarrow \psi^* = \frac{\gamma}{\kappa} \psi. \tag{13}
\]

On the point, \( V_{U(1)-\text{mini}} \) can be rewritten as

\[
V_{U(1)-\text{mini}}(\varphi, \psi = \frac{\gamma}{\kappa} \psi) = (B - \frac{\gamma^2}{\kappa}) |\varphi|^2 + J |\varphi|^4 \tag{14a}
\]

\[
= J \left[ |\varphi|^2 - \frac{1}{2J} \left( \frac{\gamma^2}{\kappa} - B \right) \right]^2 - \frac{1}{4J} \left( \frac{\gamma^2}{\kappa} - B \right)^2 \tag{14b}
\]

\[
= V_{U(1)-\text{mini}}(|\varphi|). \tag{14c}
\]

It is clear that the minimally generalized Goldstone model \( V_{U(1)-\text{mini}} \) is reduced to the standard one \( V_{\text{Goldstone}} \) with ‘the effective potential’ \( (\gamma^2/\kappa - B) \); as expected, the coupling with other degrees of freedoms \( \psi \) has brought ‘the effective chemical potential’ \( \gamma^2/\kappa \) (see eq. 14). As the result, the total chemical potential can become positive and hence, the \( U(1) \)-SSB of the vacuum occurs (see Fig. 3(a)) when

\[
0 < B < \frac{\gamma^2}{\kappa}. \tag{15}
\]
Under this condition, the vacuum expectation value $\varphi_0$ (see Fig. 3 (b)) reads

$$|\varphi_0| = \sqrt{\frac{1}{2J} \left( \frac{\gamma^2}{\kappa} - B \right)}.$$  \hspace{1cm} (16)

Here let us emphasize that when $\gamma = 0$ or $\kappa = 0$ (see inequalities (12) and (15)), the $U(1)$-SSB of the vacuum cannot occur. That is, the $\gamma$-term as well as the $\kappa$-term in eq. (9b) is essential for the occurrence of the $U(1)$-SSB of the vacuum. Therefore we have named $V_{U(1)}$-mini, the ‘minimally’ generalized Goldstone model.

![Figure 3](image_url)  

Figure 3: (Color online). Schematic pictures of the $U(1)$-SSB of the vacuum in the minimally generalized Goldstone model $V_{U(1)}$-mini. (| $\varphi$ |); eq. (14c). As an example, each dimensionless parameter is set as follows: $J = 7$ and $\kappa = \gamma = 1$. Therefore for the occurrence of the $U(1)$-SSB, the parameter $B$ must satisfy the condition (inequality (15)); $(0 < B < 1)$. (a) Even when parameter $\mu$ is negative $(0 < B := -\mu$, see also Fig. 2 (a)), the $U(1)$-SSB of the vacuum can occur in the minimally generalized Goldstone model because ‘the effective potential’ is not $\mu$, but $(\gamma^2/\kappa - B)$ (eq. (14b)). (b) When $B = 0.5$, the $U(1)$-symmetry of the vacuum is spontaneously broken and the vacuum expectation value $\varphi_0$ (eq. (16)) becomes $|\varphi_0| \approx 0.189$. 

7
2.2.2 Stability of vacuum

It would be useful to investigate the stability of the vacuum to confirm the importance of the repulsive interaction, \( J | \varphi |^4 \) with \( 0 < J \), for the realization of the stable vacuum. For simplicity here, each coupling constant in eq. (9b) is set as follows; \( J = \kappa = B = 0 \). On this condition, \( V_{U(1)-\text{mini.}}^{J=\kappa=B=0} \) becomes

\[
V_{U(1)-\text{mini.}}^{J=\kappa=B=0}(\varphi, \psi) = -\gamma(\varphi\psi^* + \varphi^*\psi) \quad \text{(17a)}
\]

\[
= -\gamma (\varphi^* \psi^*) A \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad \text{(17b)}
\]

with

\[
A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^\dagger. \tag{18}
\]

It is clear that \( V_{U(1)-\text{mini.}}^{J=\kappa=B=0} \) takes quadratic form and the matrix \( A \) is Hermitian. Therefore \( V_{U(1)-\text{mini.}}^{J=\kappa=B=0} \) can be easily diagonalized via an unitary matrix \( U \) as follows (see also Appendix A):

\[
V_{U(1)-\text{mini.}}^{J=\kappa=B=0}(| \Phi_+ \rangle, | \Phi_- \rangle) = -\gamma(| \Phi_+ \rangle^2 - | \Phi_- \rangle^2), \quad \text{(19)}
\]

with

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} \varphi + \psi \\ \varphi - \psi \end{pmatrix} =: \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}. \tag{20}
\]

Fig. 3 (a) describes \( V_{U(1)-\text{mini.}}^{J=\kappa=B=0}(| \Phi_+ \rangle, | \Phi_- \rangle) \). It is apparent that the origin (i.e. \( | \Phi_+ \rangle = | \Phi_- \rangle = 0 \)) has become ‘a saddle point’, which is not stable or the true vacuum; there are no true stable vacuum states in \( V_{U(1)-\text{mini.}}^{J=\kappa=B=0} \).

Note that even when a non-zero value is given to each coupling constant (\( \kappa \) and \( B \)), the situation does not change as long as \( J \) is zero; for simplicity here, we take \( \gamma = 1 \). On this condition, \( V_{U(1)-\text{mini.}}^{J=0,\gamma=1} \) becomes (see also Appendix A)

\[
V_{U(1)-\text{mini.}}^{J=0,\gamma=1}(\varphi, \psi) = B | \varphi |^2 + \kappa | \psi |^2 - (\varphi\psi^* + \varphi^*\psi) \quad \text{(21a)}
\]

\[
= (\varphi^* \psi^*) A' \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad \text{(21b)}
\]

with

\[
A' := \begin{pmatrix} B & -1 \\ -1 & \kappa \end{pmatrix} = (A')^\dagger. \tag{22}
\]

Also in this case, it is clear that \( V_{U(1)-\text{mini.}}^{J=0,\gamma=1} \) takes quadratic form and the matrix \( A' \) is Hermitian. Therefore \( A' \) can be diagonalized via an unitary matrix \( U' \):

\[
U'^\dagger A' U' = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{23}
\]
Each eigenvalue, $\lambda_{\pm}$, is determined by the following characteristic equation:

$$| \lambda E - A' | = 0 \quad (24a)$$

$$\Leftrightarrow \lambda = \frac{(B + \kappa) \pm \sqrt{(B + \kappa)^2 - 4(BK - 1)}}{2} \quad (24b)$$

$$= \frac{(B + \kappa) \pm \sqrt{(B - \kappa)^2 + 4}}{2} \quad (24c)$$

$$=: \lambda_{\pm}, \quad (24d)$$

with $\lambda_- < \lambda_+$ by definition and note that $0 < \lambda_+$ \footnote{Remember that $0 < B$ and $0 < \kappa$.}. According to eq. (24b), $\lambda_-$ becomes positive when $1 < BK (\neq 0)$; otherwise negative or zero ($\lambda_- \leq 0$).

By using these eigenvalues $\lambda_{\pm}$, $V_{U(1)\text{-mini}}^{J=0,\gamma=1}$ can be diagonalized as

$$V_{U(1)\text{-mini}}^{J=0,\gamma=1}(\Phi'_+^*, \Phi'_-) = \lambda_+ | \Phi'_+ |^2 + \lambda_- | \Phi'_- |^2, \quad (25)$$

in which the newly introduced complex scalar fields $\Phi'_\pm$ are represented by using an unitary matrix $U'$ as $(\varphi^* \psi^*) U'^\dagger =: ((\Phi'_+^*)^* (\Phi'_-^*)^*)$. Fig. 4 (b) describes $V_{U(1)\text{-mini}}^{J=0,\gamma=1}$, when $0 \leq \lambda_-$. In this case, though the origin (i.e. $| \Phi'_+ | = | \Phi'_- | = 0$) is the stable vacuum state, it is not generated by the $U(1)$-SSB; it is simply the original vacuum and it in fact gives $\varphi_0 = 0$. Let us remark that this can be easily confirmed also by the same procedure with sec. 2.2.1 (i.e. eq. (14a)); $V_{U(1)\text{-mini}}^{J=0,\gamma=1}(\varphi, \psi = \varphi/\kappa) = (B - 1/\kappa) | \varphi |^2$. On the other hand, when $\lambda_- < 0$, the situation is the same with $V_{U(1)\text{-mini}}^{J=\kappa = B=0}$, Fig. 4 (a).

Therefore, we conclude that the true stable vacuum state accompanied by the $U(1)$-SSB does not exist\footnote{25} without the repulsive interaction; $J | \varphi |^4$ with $0 < J$.

### 3 Theoretical proposal; magnon BEC in spin pumping system

We consider the application of the minimally generalized Goldstone model $V_{U(1)\text{-mini}}$ to spin pumping systems; the minimally generalized Goldstone model can be regarded to describe the dynamics of magnons interacting with spins carried by conduction electrons at $T = 0$.

#### 3.1 Anisotropic exchange interaction

In our previous work\cite{12} based on the Schwinger-Keldysh formalism (i.e. a perturbative theory), we have studied ‘thermal spin pumping’\cite{15} mediated by magnons in a ferromagnetic insulator and non-magnetic metal junction (Fig.}
Figure 4: (Color online). (a) Plot of $V_{J=\kappa=B=0}^{U(1)\text{-mini}}$ (eq. (19)) with $\gamma = 1$. The origin (i.e. $|\Phi_+| = |\Phi_-| = 0$) is a saddle point, which is not stable or the true vacuum; there are no true stable vacuum states in $V_{U(1)\text{-mini}}^{J=\kappa=B=0}\text{.}$ (b) Plot of $V_{U(1)\text{-mini}}^{J=0,\gamma=1}$ (eq. (25)) with $\lambda_+ = \lambda_- = +1$. The origin (i.e. $|\Phi_\prime_+| = |\Phi_\prime_-| = 0$) is the original vacuum; the state is not generated by the $U(1)$-SSB and it in fact gives $\varphi_0 = 0$. (a) (b) Note that although the variables are restricted to $0 \leq |\Phi_\prime_{\pm}|$ by definition, we also have plotted the region; $-1 \leq |\Phi_\prime_{\pm}|$ for clarity.

\[ V_{\text{iso}} := -2\gamma S \cdot s = -2\gamma(S^x s^x + S^y s^y + S^z s^z). \]  

(26)

The variable $\gamma'(>0)$ represents the magnitude of the exchange interaction. As far as our ‘perturbative analysis’, [12] the macroscopic condensate order parameter becomes zero and magnon BEC cannot occur. Now, by considering the correspondence of the spin pumping system (i.e. magnon-electron system) with the minimally generalized Goldstone model $V_{U(1)\text{-mini}}$, we go after the possibility for the occurrence of the stable magnon BEC state in spin pumping systems on the basis of a ‘non-perturbative theory’.

For the purpose, first, let us consider the same situation [12] except the point that ferromagnetic localized spins $S$ anisotropically interact with conduction electrons $s$ at the interface;
electrons \( s \);

\[
V_{\text{aniso}}. := -2\gamma'(S^x s^x + S^y s^y + \Delta S^z s^z) \quad (27a)
\]

\[
= -2\gamma'\left(\frac{S^+ s^- + S^- s^+}{2} + \Delta S^z s^z\right), \quad (27b)
\]

in which \( \Delta \) represents the magnitude of the anisotropic; \( 0 \leq \Delta \leq 1 \). Reflecting the fact that the minimally generalized Goldstone model (eq. (9b)) has not included \( |\varphi|^2 |\psi|^2 \), we here focus on the strong anisotropic limit, \( \Delta \to 0 \);

\[
\begin{align*}
V_{\text{aniso}}. &\xrightarrow{\Delta \to 0} -\gamma'(S^+ s^- + S^- s^+) \\
&=: V_{\text{aniso}.(\Delta=0)}. \quad (28a)
\end{align*}
\]

Via the Holstein-Primakoff transformation; \( S^+ = \sqrt{2S}a + O(1/\sqrt{S}) \), \( S^- = \sqrt{2S}a^d + O(1/\sqrt{S}) \), \( S^z = S - a^d a \), \( V_{\text{aniso}.(\Delta=0)} \) can be expressed in terms of magnon creation/annihilation operators as follows;

\[
V_{\text{aniso}.(\Delta=0)} = -\sqrt{2S}\gamma'(as^- + a^d s^+). \quad (29)
\]

We take the classical limit\[29] and each operator is replaced with a commutative complex scalar field (i.e. c-number): \( a^{\text{classical}} \mapsto \varphi \in \mathbb{C} \), \( s^+ \mapsto \psi \in \mathbb{C} \). As the result, in the classical limit, the strong anisotropic exchange interaction between magnons (i.e. spin waves) and conduction electrons \( V^{\text{cla}}_{\text{aniso}.(\Delta=0)} \) is rewritten as follows;

\[
V^{\text{cla}}_{\text{aniso}.(\Delta=0)} = -\sqrt{2S}\gamma' (\varphi \psi^* + \varphi^* \psi). \quad (30)
\]

It is clear that this term corresponds to the \( \gamma \)-term in the minimally generalized Goldstone model \( V^{U(1)-\text{mini.}}_{U(1)} \); in the language of the spin pumping system, the \( \mathcal{B} \)-term describes the couplings with the effective magnetic field along the z-axis for magnons, the \( \kappa \)-term represents the interaction between up-spins and down-spins of conduction electrons, and the \( \mathcal{J} \)-term corresponds to the magnon-magnon interaction. Therefore, if these quantities satisfy the condition for the occurrence of the \( U(1) \)-SSB shown in sec. 2.2.1 (inequality \[15\]); \( 0 < \mathcal{B} < (\gamma^2 / \kappa) \), the stable magnon BEC state can be realized even under the interaction with conduction electrons. The vacuum expectation value \( \varphi_0 \), which corresponds to the macroscopic condensate order parameter of magnons (i.e. spin waves), becomes a non-zero value (see Fig. 3); \( |\varphi_0| = \sqrt{(\gamma^2 / \kappa - \mathcal{B})/2\mathcal{J}} \).

Here let us again remark that, as stressed in sec. 2.2.1 for the occurrence of the stable magnon BEC state in the spin pumping system, the repulsive interaction between up-spins and down-spins of conduction electrons (i.e. \( 0 < \kappa \)) is essential as well as the repulsive magnon-magnon interaction (i.e. \( 0 < \mathcal{J} \)), which can be realized, as an example, owing to the dipolar interaction.\[31\] This is the rigorous theoretical result based on the non-perturbative theory beyond a perturbative one.\[12\]
Last, it might be useful to mention that in sec. 2.2.2, we have closely investigated the stability of the vacuum in the minimally generalized Goldstone model $V_{U(1)\text{-mini}}$. There, we have concluded that the true stable vacuum state accompanied by the $U(1)$-SSB does not exist without the repulsive interaction (see Fig. 4); $J | \varphi |^4$ with $0 < J$. This means, in the language of the above spin pumping system, that the true stable magnon BEC state cannot exist without the repulsive magnon-magnon interaction; although the minimally generalized Goldstone model $V_{U(1)\text{-mini}} = 0$ can possess the stable vacuum state when $0 \leq \lambda$ (see Fig. 4 (b)), it is not generated by the $U(1)$-SSB and it in fact gives $\varphi_0 = 0$.

Therefore we suspect that magnon BEC cannot occur in the spin pumping system described by the minimally generalized Goldstone model with $J = 0$ (i.e. $V_{U(1)\text{-mini}} = 0$). Here, let us point out that BEC should not be identified with superfluid and hence there might exist a superfluid phase in that case (i.e. $J = 0$).

### 3.2 Generalized Goldstone model

The next focus lies on whether the stable magnon BEC state could exist under a finite (i.e. non-zero) $\Delta$ regime in the above spin pumping system. For the purpose, we include the term $| \varphi |^2 | \psi |^2$, which arises from the $\Delta$-term in $V_{\text{aniso}}$ (eq. (27b));

$$V_{U(1)}(\varphi, \psi) := V_{U(1)\text{-mini}}(\varphi, \psi) - \alpha | \varphi |^2 | \psi |^2$$

$$= B | \varphi |^2 + J | \varphi |^4 - \alpha | \varphi |^2 | \psi |^2$$

$$- \gamma (\varphi \psi^* + \varphi^* \psi) + \kappa | \psi |^2,$$ \hspace{1cm} (31a)

where $\alpha (\in \mathbb{R})$ is the corresponding dimensionless coupling constant. From the viewpoint of the correspondence with the above spin pumping system described by $V_{\text{aniso}}$, we restrict $\alpha$ to a positive value; $0 < \alpha$.

In the language of the spin pumping system (i.e. the Holstein-Primakoff transformation), the variable $| \varphi |^2$ represents the number of magnons obeying the parastatistics and hence, the relation; $| \varphi |^2 \ll S = O(1)$, is required by definition. Moreover because we here treat the extremely low temperature regime (i.e. $T = 0$), the variable $| \varphi |^2$ is supposed to be very small enough to satisfy the relation; $| \varphi |^2 \ll O(1)$. Therefore when we choose variables, $\kappa$ and $\alpha$, to satisfy the condition; $\kappa/\alpha = O(1)$, we are allowed to assume the relation; $| \varphi |^2 \ll \kappa/\alpha \Leftrightarrow \alpha \ll | \varphi |^2 \ll \kappa$. Also from the viewpoint of the stability of the system in respect to $\psi(\psi^*)$, $V_{U(1)}(\psi) = (\kappa - \alpha | \varphi |^2) | \psi |^2 + O(\psi(\psi^*) + O(\psi^*)^0)$, the relation is strongly required. Thus from now on, we discuss on the basis of the assumption; $\alpha \ll | \varphi |^2 \ll \kappa$. In other words, the following our analysis is adequate in the region.

Through the same procedure with the minimally generalized Goldstone model $V_{U(1)\text{-mini}}$, and the approximation; $(\kappa - \alpha | \varphi |^2)^{-1} \simeq (1 + \alpha | \varphi |^2 / \kappa)$, the generalized Goldstone model $V_{U(1)}$ on the point, $\psi = \gamma \varphi / (\kappa - \alpha | \varphi |^2)$ \hspace{1cm} (31b)
\[ \partial V_{U(1)}/(\partial \psi) = 0, \]
reads

\[ V_{U(1)}(\varphi, \psi) = \frac{\gamma}{\kappa - \alpha |\varphi|^2} \varphi = (B - \frac{\gamma^2}{\kappa})\chi + (J - \frac{\alpha \gamma^2}{\kappa^2})\chi^2 - \frac{2\alpha^2 \gamma^2}{\kappa^3} \chi \]
\[ =: V_{U(1)}(\chi), \tag{32a} \]

with \( \chi := |\varphi|^2 (\geq 0) \).

Here let us denote the solution of the equation, \( dV_{U(1)}(\chi)/d\chi = 0 \), as \( \chi_{\pm} \) by definition (see Fig. 5 (b)). The coefficient of \( \chi^3 \) in \( V_{U(1)}(\chi) \), \( 2\alpha^2 \gamma^2/(\kappa^3) \), takes a positive value. Therefore for the occurrence of the \( U(1) \)-SSB accompanied by the stable magnon BEC state, the condition is required (Fig. 5 (b)):

\[ 0 < \chi_- \quad \text{and} \quad S < \chi_0. \tag{33a} \]

That is, when

\[ \frac{\alpha \gamma^2}{\kappa^2} < J, \tag{34a} \]

\[ \frac{\gamma^2}{\kappa} - \frac{\kappa^3}{6 \alpha^2 \gamma^2} (J - \frac{\alpha \gamma^2}{\kappa^2})^2 < B < \frac{\gamma^2}{\kappa}, \tag{34b} \]

and

\[ S < \chi_0 \tag{34c} \]

with

\[ \chi_0 = \frac{\kappa^3}{6 \alpha^2 \gamma^2} \left[ (J - \frac{\alpha \gamma^2}{\kappa^2}) + 2 \sqrt{(J - \frac{\alpha \gamma^2}{\kappa^2})^2 + \frac{6 \alpha^2 \gamma^2}{\kappa^3} (B - \frac{\gamma^2}{\kappa})} \right], \tag{35} \]

the \( U(1) \)-SSB accompanied by the stable magnon BEC state (\( \chi_- \)) occurs:

\[ \chi_- = \frac{\kappa^3}{6 \alpha^2 \gamma^2} \left[ (J - \frac{\alpha \gamma^2}{\kappa^2}) - \sqrt{(J - \frac{\alpha \gamma^2}{\kappa^2})^2 + \frac{6 \alpha^2 \gamma^2}{\kappa^3} (B - \frac{\gamma^2}{\kappa})} \right]. \tag{36} \]

Let us remark that magnons obey the parastatistics and hence when \( S > \chi_0 \), the state \( \chi_- \) becomes the classically metastable state \([19, 20, 37]\) and it does not give the absolute minimum. That is, the state \( \chi_- \) is not the true stable vacuum and it can decay to the true vacuum by the quantum-mechanical tunneling effect \([19]\) (see Fig. 5 (a) as an example). Of course we have noted that we have been theoretically discussing within the classical theory, but quantum effects are inevitable in real materials (i.e. experiments). Thus, the condition \( S < \chi_0 \) (eq. (34c)) is required for the experimental realization of stable magnon BEC in spin pumping systems with the non-zero \( \Delta \)-term (i.e. \( \alpha \)-term).
Figure 5: (Color online). (a) Plot of the Goldstone model $V_{\text{Goldstone}}$ (eq. 5) with $\mathcal{J} < 0$. The $U(1)$-SSB of the vacuum does not occur. The origin (i.e. $|\varphi| = 0$) is unstable ($0 \leq \mu$) or the classically metastable state ($\mu < 0$). (b) A schematic picture of the generalized Goldstone model $V_{U(1)}(\chi)$ (eq. 32). When $\chi_- \leq 0$, the situation is the same with (a). The state $\chi_0$ is defined as $V_{U(1)}(\chi = \chi_0) = V_{U(1)}(\chi = \chi_-)$.

4 Summary and discussion

In order to go after the possibility for the stable magnon BEC state in spin pumping systems, we have employed a non-perturbative theory to go beyond the perturbative analysis and have extended the standard Goldstone model. For the realization of the stable magnon BEC state, the repulsive interaction between up-spins and down-spins of conduction electrons is essential as well as the repulsive magnon-magnon interaction. By realizing the condition we have clarified in sec. 3.1 and 3.2 (it depends on materials), the true stable magnon BEC state can be experimentally observed also in spin pumping systems without using external pumping magnetic fields.

On the other hand, to extend the system at finite temperature with quantum effects is left as a future work. On top of this, we consider that to clarify the effects of the unusual energy dispersion of the lowest magnon mode in YIG, which is a relevant material to the experiment of magnon BEC[7, 22, 23] and spin pumping[11, 38] is a significant theoretical issue. In addition, as stressed by Hick et al[34] BEC of quasiparticles is not necessarily accompanied by superfluidity[29]. In other words, they should not be identified[25]. Of course it is roughly expected, owing to Bogoliubov theory[29] that superfluid of magnons is accompanied by magnon BEC in spin pumping systems, but to reveal the detailed relationship between magnon BEC and superfluid[9] of
magnons in spin pumping systems is left as an important future work.

BEC state (i.e. coherent state) is the robust macroscopic quantum state against the loss of information. Therefore we hope this work becomes a bridge between the research on spintronics and magnon BEC to lead to the green information technologies.

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A Appendix: Diagonalization of quadratic form

In this Appendix, we show the detail of the diagonalization in sec. 2.2.2. Remember that any Hermitian matrices $\mathcal{A}$ can be diagonalized via an unitary matrix $U$ as follows;

$$U^{\dagger}\mathcal{A}U = \begin{pmatrix} \lambda^\prime_+ & 0 \\ 0 & \lambda^\prime_- \end{pmatrix},$$

(37)

where $\lambda^\prime_\pm$ represent eigenvalues which are determined by the following characteristic equation; $| \lambda^\prime E - \mathcal{A} | = 0$. Here the $(2 \times 2)$ identity matrix is represented as $E$. In addition, the unitary matrix $U$ is constructed, via the eigenvector $u_\pm$ which satisfy the relation; $(\lambda^\prime_\pm E - \mathcal{A})u_\pm = 0$, as $U = (u_+ \quad u_-)$.

On the basis of the above procedure, we diagonalize $V_{U(1)\text{-mini}}^{J=\kappa=B=0}$ as an example;

$$V_{U(1)\text{-mini}}^{J=\kappa=B=0}(\varphi, \psi) = -\gamma (\varphi\psi^* + \varphi^*\psi)$$

(38a)

$$= -\gamma (\varphi^* \quad \psi^*) \mathcal{A} \begin{pmatrix} \varphi \\ \psi \end{pmatrix},$$

(38b)

with

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{A}^\dagger.$$
It is clear that $V_{U(1)\text{-mini.}}^{J=\kappa=B=0}$ takes quadratic form and the matrix $A$ is Hermitian. Therefore $V_{U(1)\text{-mini.}}^{J=\kappa=B=0}$ can be diagonalized;

$$U^\dagger A U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{with} \quad U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (40)$$

Note that the characteristic equation gives eigenvalues, $\lambda_+ \equiv +1$ and $\lambda_- \equiv -1$, and the corresponding eigenvectors read

$$u_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (41)$$

As the result, $V_{U(1)\text{-mini.}}^{J=\kappa=B=0}$ can be rewritten as

$$V_{U(1)\text{-mini.}}^{J=\kappa=B=0} = -\gamma \left( \varphi^* \psi^* \right) U U^\dagger A U U^\dagger \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad (42a)$$

$$= -\gamma \left( |\Phi_+|^2 - |\Phi_-|^2 \right), \quad (42b)$$

where the newly introduced complex scalar fields $\Phi_\pm$ are represented by using an unitary matrix $U$ as

$$U^\dagger \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi + \psi \\ \varphi - \psi \end{pmatrix} \quad (43a)$$

$$=: \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}. \quad (43b)$$

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