The Effect of Autocorrelation on the Shewhart-RZ Control Chart

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Abstract

In many industrial manufacturing processes, the quality of products depends on the relation between two main ingredients or characteristics. Often, this calls for monitoring the ratio of two normal random variables with statistical process control (SPC) techniques. A large number of studies related to designing control charts monitoring this ratio has been published. However, these studies are based purely on the assumption of independent observations. In practice, autocorrelation between observations can exist and should be modelled to protect against the false alarm rate inflation when implementing a control charts. In this paper, we tackle this problem by investigating the performance of the Shewhart control chart monitoring the ratio of two normal variables, (denoted as Shewhart-RZ), in the presence of autocorrelation between successive observations. The autocorrelation is modelled through the bivariate autoregressive model VAR(1). We also provide an example to illustrates the use of the Shewhart-RZ control chart on a quality control problem.

Keywords: Autocorrelation, multivariate autoregressive model, control charts, ratio distribution.

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1 Introduction

In industrial manufacturing practice, control charts are a powerful tool to reduce variability and achieve process stability. Many control charts have been designed in SPC literature to monitor the stability of many parameters characterizing the distribution of a quality characteristic like the Mean, the Median, and the Coefficient of Variation (CV). Recently, the ratio between two normal random variables $X$ and $Y$ has been considered as the quality characteristic to be monitored. The reason is that the ratio can play an important role in ensuring the product quality of various processes. A number of situations where monitoring the ratio of two variables is needed has been broadly discussed in Celano and Castagliola [1] and Tran et al. [10]. Other examples related to the need to monitoring the ratio can be found in Spisak [2] and Davis and Woodall [3], from an unemployment insurance quality control program, or Öksoy et al. [4] from on-line monitoring implemented in the glass industry. In particular, the statistical properties of a Shewhart-RZ control chart have been presented in Celano et al. [5] (for individual measurements, i.e by assuming the sample size $n = 1$) and Celano and Castagliola [1] (for multiple measurements, i.e when the sample size $n > 1$). Then, other advanced control charts have been proposed, such as the Synthetic-RZ control chart (Celano and Castagliola [6]), the Run Rules-RZ control chart ([7]), the EWMA-RZ (Exponentially Weighted Moving Average) control chart (Tran et al. [8]), the CUSUM-RZ (Cumulative Sum) control chart (Tran et al. [9]), the VSI EWMA-RZ (Variable Sampling Interval) control chart ([11]). Also, the effect of measurement error on the ratio control chart has been investigated in Tran et al. [12] by using a linear covariate error model. The authors showed that the performance of this control chart is significantly affected by the presence of measurement errors.

All the control charts monitoring the ratio have been designed under the assumption that the observations collected for $X$ and $Y$ are independent. This assumption overlooks the problem of autocorrelation among consecutive observations, which is very frequent when measures are collected at high rate. Modelling the consecutive observations as time series is very important when correlation exists to avoid negative effects on the performance of the control chart monitoring the process. The first study in SPC literature that deals with autocorrelation was conducted by Alwan and Roberts [13] who illustrated statistical modeling and fitting of time-series effects and the application of standard control charts. From that pioneering work, many studies concerning the performance of various control charts in the presence of autocorrelation as well as the ways to handle this autocorrelation have been proposed. Kalgonda and Kulkarni [14] showed that the autocorrelation has a serious impact on the performance of conventional control charts. The autoregressive moving average control chart was considered in Lin et al.
The first-order autoregressive model (AR(1)) was applied to describe the wandering behavior of the process mean using the $\bar{X}$ chart in Franco et al. [16]. Leoni et al. [17] studied the effect of the autocorrelation on the performance of the $T^2$ chart. A synthetic chart to control bivariate processes with autocorrelated data was designed in Leoni et al. [18]. Other studies related to control charts monitoring the processes with autocorrelated observations are in Hwarng and Wang [19], Costa and Castagliola [20], Huang et al. [21], Huang et al. [22], Franco et al. [23], Franco et al. [24], for example.

As far as we know, up to now, the effect of the autocorrelation on the Shewhart-RZ control chart has not been yet considered in the literature. Therefore, the goal of this paper is to investigate the performance of the Shewhart control chart for monitoring the ratio of two normal random variables that, similar to Leoni et al. [17], are modelled by means of the first-order bivariate VAR(1) model to describe the autocorrelation between observations from the process.

The paper is organised as follows: in Section 2 the multivariate autoregressive model is discussed for the application to the sample ratio; in Section 4 the formulas for the control limits and the performance metrics of the Shewhart-RZ control chart are discussed; in Section 5 the effect of the autocorrelation on the Shewhart-RZ control chart performance is investigated; in Section 6 an illustrative example is provided to show the implementation of the Shewhart-RZ control chart in the presence of autocorrelation; some concluding remarks and recommendations for future research are given in Section 7. The ratio distribution is also provided in the Appendix.

2 Autoregressive model for the sample ratio

Let us assume that, at time $i = 1, 2, \ldots$, the samples $\{W_{i,j} = (X_{i,j}, Y_{i,j})\}$, $j = 1, 2, \ldots, n$ of size $n$ are collected to monitor the ratio between the normal random variables $X$ and $Y$. Autocorrelation between $X$ and $Y$ is modelled by the bivariate VAR(1) autoregressive model. According to this time series model, $W_{i,j}$ depends on $W_{i,j-1}$ through the following equation

$$W_{i,j} = \mu_W + \Phi(W_{i,j-1} - \mu_W) + \epsilon_{i,j},$$

(1)

where

$$\mu_W = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$$

is the mean vector of $W_{i,j}$,

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$
is a $(2 \times 2)$ correlation matrix that accounts for both autocorrelation, $(\Phi_{i,i}, \text{ for } i = 1, 2)$, and cross-correlation, $(\Phi_{i,j}, \text{ for } i \neq j)$, between $X$ and $Y$; $\varepsilon_{i,j}$ is an independent bivariate normal random vector with mean vector $\mu_\varepsilon = 0$ and variance-covariance matrix

$$
\Sigma_\varepsilon = \begin{pmatrix}
\sigma_{\varepsilon X}^2 & \sigma_{\varepsilon XY} \\
\sigma_{\varepsilon XY} & \sigma_{\varepsilon Y}^2
\end{pmatrix}.
$$

From (1), Reinsel [25] showed that the variance-covariance matrix $\Sigma_W$ of $W_{i,j}$ is the solution of the equation $\Sigma_W = \Phi \Sigma_W \Phi^\top + \Sigma_\varepsilon$. Then, it can be shown that

$$
\text{Vec}(\Sigma_W) = (I_4 - \Phi \otimes \Phi)^{-1} \text{Vec}(\Sigma_\varepsilon),
$$

where $I_p$ is the $p \times p$ identity matrix, $\otimes$ is the Kronecker product and Vec is the operator that transforms a matrix into a one-column vector by stacking its columns. Using (2), the variance-covariance matrix of the considered VAR(1) autoregressive model in (1) can be obtained as

$$
\Sigma_W = \begin{pmatrix}
\sigma_X^2 & \sigma_{XY} \\
\sigma_{XY} & \sigma_Y^2
\end{pmatrix}
= \begin{pmatrix}
\frac{\Delta_{11}\sigma_X^2 + (\Delta_{12} + \Delta_{13})\sigma_{XY} + \Delta_{14}\sigma_Y^2}{\Delta} & \frac{\Delta_{11}\sigma_X^2 + (\Delta_{12} + \Delta_{13})\sigma_{XY} + \Delta_{14}\sigma_Y^2}{\Delta} \\
\frac{\Delta_{21}\sigma_X^2 + (\Delta_{22} + \Delta_{23})\sigma_{XY} + \Delta_{24}\sigma_Y^2}{\Delta} & \frac{\Delta_{21}\sigma_X^2 + (\Delta_{22} + \Delta_{23})\sigma_{XY} + \Delta_{24}\sigma_Y^2}{\Delta}
\end{pmatrix},
$$

where the quantities $\Delta_{ij}$ for $i, j = 1, 2, 3, 4$ are given in Appendix 2.

Now, one needs to calculate mean vector $\mu_{\overline{W}}$ and variance-covariance matrix $\Sigma_{\overline{W}}$ of the sample mean vector $\overline{W}_i = (\overline{X}_i, \overline{Y}_i)^\top$ at time $i = 1, 2, \ldots$. It is proven in Appendix 3 that $\mu_{\overline{W}} = \mu_W = (\mu_X, \mu_Y)^\top$ and

$$
\Sigma_{\overline{W}} = \frac{1}{n} \left[ \Sigma_W \left( I_2 + \Lambda(\Phi^\top) - \frac{1}{n} \Pi(\Phi^\top) \right) + \left( \Lambda(\Phi) - \frac{1}{n} \Pi(\Phi) \right) \Sigma_W^\top \right]
= \begin{pmatrix}
\sigma_X^2 & \sigma_{\overline{X}, \overline{Y}} \\
\sigma_{\overline{X}, \overline{Y}} & \sigma_Y^2
\end{pmatrix},
$$

where

$$
\Lambda(\Phi) = (\Phi - \Phi^n)(I_2 - \Phi)^{-1} \\
\Pi(\Phi) = (\Phi^{-1} - I_2)^{-1}((I_2 - \Phi^{n-1})(I_2 - \Phi)^{-1} - (n - 1)\Phi^{n-1}).
$$

From (1), it is straightforward to obtain the coefficients of variations $\gamma_X = \frac{\sigma_X}{\mu_X}$ and $\gamma_Y = \frac{\sigma_Y}{\mu_Y}$ of $\overline{X}_i$ and $\overline{Y}_i$. Similarly, the coefficient of correlation between $\overline{X}_i$ and $\overline{Y}_i$ and the standard-deviation ratio can be calculated as
\( \bar{\rho} = \frac{\sigma_X \gamma_Y}{\sigma_X \sigma_Y} \) and \( \bar{\omega} = \frac{\sigma_X}{\sigma_Y} \), respectively. Concerning \( \bar{\omega} \) it is important to note that, if we define \( z = \frac{\bar{\mu}_X}{\mu_Y} \), it can also be rewritten as

\[
\bar{\omega} = \frac{\gamma_X}{\gamma_Y} \times \frac{\mu_X}{\mu_Y} = \gamma_X \times z.
\] (5)

In the rest of the paper, we will assume that \( X \) and \( Y \) are not cross-correlated and the matrix \( \Phi \) is diagonal, i.e. \( \Phi_{12} = \Phi_{21} = 0 \). This assumption allows reducing the bivariate stationary conditions to the more simple univariate stationary conditions for each variable and it has already been stated by Leoni et al. [17], who did they have not found any example in the SPC literature where the observations of one variable at a time \( i, i = 2, 3, \ldots \) are dependent of the observations of the other variable at time \( i - 1 \). As proved in Leoni et al. [17], if \( \Phi_{12} = \Phi_{21} = 0 \) then, using (2), the variance-covariance matrix \( \Sigma_W \) of \( W_{i,j} \) is equal to

\[
\Sigma_W = \begin{pmatrix}
\sigma_X^2 & \rho \sigma_X \sigma_Y \\
\rho \sigma_X \sigma_Y & \sigma_Y^2 \\
(1 - \Phi_{11}^2)^{-1} \sigma_{e,X}^2 & (1 - \Phi_{11} \Phi_{22})^{-1} \sigma_{e,XY} \\
(1 - \Phi_{11} \Phi_{22})^{-1} \sigma_{e,XY} & (1 - \Phi_{22}^2)^{-1} \sigma_{e,Y}^2
\end{pmatrix},
\] (6)

where \( \sigma_X \) and \( \sigma_Y \) are the standard deviations of two components \( X \) and \( Y \) of \( W \), respectively, and \( \rho \in (-1, 1) \) is the corresponding coefficient of correlation.

At time \( i = 1, 2, \ldots \), the sample mean vector \( \overline{W}_i = (\bar{X}_i, \bar{Y}_i) \) is also a bivariate normal random vector with mean vector \( \mu_W \) and variance-covariance matrix

\[
\Sigma_{\overline{W}} = \begin{pmatrix}
\xi_{11} & \xi_{12} \\
\xi_{21} & \xi_{22}
\end{pmatrix},
\]

where

\[
\xi_{11} = \frac{\sigma_X^2}{n} \left( 1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{11}^k \right),
\]

\[
\xi_{22} = \frac{\sigma_Y^2}{n} \left( 1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{22}^k \right),
\]

\[
\xi_{12} = \frac{\sigma_{XY}}{n} \left( 1 + \frac{1}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{11}^k + \frac{1}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{22}^k \right),
\]

\[
\xi_{21} = \xi_{12}.
\]

From these quantities, it is straightforward to obtain the coefficients of vari-
\[ \gamma_X = \frac{\sqrt{\xi_{11}}}{\mu_X} \quad \text{and} \quad \gamma_Y = \frac{\sqrt{\xi_{22}}}{\mu_Y} \] of \( \bar{X}_i \) and \( \bar{Y}_i \) as

\[ \gamma_X = \frac{\sigma_X \sqrt{1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{11}^k}}{\sqrt{n} \mu_X}, \] (7)

\[ \gamma_Y = \frac{\sigma_Y \sqrt{1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{22}^k}}{\sqrt{n} \mu_Y}, \] (8)

Similarly, the coefficient of correlation \( \bar{\rho} = \frac{\xi_{12}}{\sqrt{\xi_{11} \xi_{22}}} \) between \( \bar{X}_i \) and \( \bar{Y}_i \) is defined as

\[ \bar{\rho} = \frac{\rho \left( 1 + \frac{1}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{11}^k + \frac{1}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{22}^k \right)}{\sqrt{ \left( 1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{11}^k \right) \left( 1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{22}^k \right) }}, \] (9)

and the standard-deviation ratio \( \bar{\omega} = \sqrt{\frac{\xi_{11}}{\xi_{22}}} \) is

\[ \bar{\omega} = \frac{\sigma_X}{\sigma_Y} \times \sqrt{1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \Phi_{11}^k}. \] (10)

Concerning \( \bar{\omega} \) it is important to note that, if we define \( z = \frac{\mu_X}{\mu_Y} \), it can also be rewritten as

\[ \bar{\omega} = \frac{\gamma_X}{\gamma_Y} \times \frac{\mu_X}{\mu_Y} = \frac{\gamma_X}{\mu_Y} \times \frac{\mu_X}{\gamma_Y} \times z. \] (11)

### 3 Estimating the VAR(1) model

In practice, the VAR(1) model parameters should be estimated from the phase I data. Several methods have been presented in the literature to obtain these estimated parameters such as the least squares (LS), maximum likelihood (ML), and Bayesian methods. Tsay [26] (Chapt. 2) provided a detailed discussion on the theoretical aspects of these estimation methods and studied their key properties. They showed that the Bayesian and LS methods produce close estimates and under some conditions, the ML estimates asymptotically equivalent to the LS ones. They also prepared R demonstrations for each method and included all the programs in the MTS package available in R. MTS is a general package for analyzing multivariate linear time series. Other statistical software such as SAS can also be used to perform estimation of VAR models.

Here, as an example, we use the command `VAR` in MTS package that uses LS method to estimate the model of a real dataset. Let consider the bivariate time series data of size 186 that describes the measured pressure at the front
and back end of a furnace. The data set is reported in Montgomery et al. [27] (p. 347). They showed that VAR(1) model provides an appropriate fit for the data. Based on the data set, we have

$$\mu_W = \mu_{\bar{W}} = (\bar{W}_F, \bar{W}_B) = (10.421, 20.189)^T$$

where $W_F$ and $W_B$ are the pressures at the front and back end of the furnace, respectively. The output of VAR command shows the estimated VAR(1) autoregressive model is

$$\begin{pmatrix} W_{i,F} \\ W_{i,B} \end{pmatrix} = \begin{pmatrix} 10.421 \\ 20.189 \end{pmatrix} + \begin{pmatrix} 0.733 & 0.474 \\ 0.410 & -0.561 \end{pmatrix} \left( \begin{pmatrix} W_{i-1,F} \\ W_{i-1,B} \end{pmatrix} - \begin{pmatrix} 10.421 \\ 20.189 \end{pmatrix} \right) + \begin{pmatrix} \varepsilon_{i,F} \\ \varepsilon_{i,B} \end{pmatrix}$$

where

$$\begin{pmatrix} \varepsilon_{i,F} \\ \varepsilon_{i,B} \end{pmatrix} \sim N_2 \left( 0, \begin{pmatrix} 1.232 & 0.588 \\ 0.588 & 1.072 \end{pmatrix} \right).$$

It can be seen that these estimated parameters are very close to those of SAS ARIMA procedure in Montgomery et al. [27]. From (3) and (4), we obtained

$$\Sigma_W = \begin{pmatrix} 5.887 & 1.500 \\ 1.500 & 2.002 \end{pmatrix}$$

$$\Sigma_{\bar{W}} = \begin{pmatrix} 4.724 & 1.458 \\ 1.458 & 0.542 \end{pmatrix},$$

where $\Sigma_{\bar{W}}$ is calculated for $n = 5$. Finally, one can see that

$$\gamma_X = \frac{\sigma_X}{\mu_X} = \frac{\sqrt{4.724}}{10.421} = 0.209, \quad \gamma_Y = \frac{\sigma_Y}{\mu_Y} = \frac{\sqrt{0.542}}{20.189} = 0.036,$$

$$\bar{\rho} = \frac{\sigma_{\bar{X},\bar{Y}}}{\sigma_{\bar{X}}\sigma_{\bar{Y}}} = \frac{1.458}{\sqrt{4.724}\sqrt{0.542}} = 0.911, \quad \bar{\omega} = \frac{\gamma_X}{\gamma_Y} \times z_0 = \frac{0.209}{0.036} \times \frac{10.421}{20.189} = 2.996.$$

### 4 Implementation of a Shewhart-RZ control chart for autocorrelated data

Similar to Tran et al. [7], we suggest to monitor the ratio statistic as

$$\bar{Z}_i = \frac{\bar{X}_i}{\bar{Y}_i} = \frac{\sum_{j=1}^n X_{i,j}}{\sum_{j=1}^n Y_{i,j}}$$

at each time $i = 1, 2, \ldots$. Let $\rho = \rho_0$ and $z = \frac{\mu_X}{\mu_Y} = z_0$ denote the coefficient of correlation between the two normal variables $X_{i,j}$ and $Y_{i,j}$ and the mean ratio when the process is in-control, respectively. As in the design of any control charts, the Shewhart-RZ chart for autocorrelated data is designed by defining its control limits. It is well-known that the distribution of the ratio of two normal random variables has no moment. Thus, the control limits of the Shewhart-RZ control
chart monitoring $Z_i$ should be defined as probability control limits. Denoting as $\alpha$ the desired false alarm probability for the control chart, the lower control limit ($LCL$) and the upper control limit ($UCL$) of the Shewhart-RZ are obtained as follows:

$$LCL = F^{-1}_{Z} \left( \frac{\alpha}{2} \mid \gamma_X, \gamma_Y, \bar{\omega}, \bar{\rho} \right),$$  \hspace{1cm} (13) \\
$$UCL = F^{-1}_{Z} \left( 1 - \frac{\alpha}{2} \mid \gamma_X, \gamma_Y, \bar{\omega}, \bar{\rho} \right),$$  \hspace{1cm} (14)

where $\gamma_X, \gamma_Y, \bar{\omega}, \bar{\rho}$ are computed using equations (7)–(11) with $\rho = \rho_0$ and $z = z_0$, and $F^{-1}_{Z}(\ldots)$ is the inverse cumulative distribution function for the ratio of two normal random variables. An approximation for $F^{-1}_{Z}(\ldots)$ is given in the Appendix 1.

When the process runs in the out-of-control state, the in-control ratio $z_0$ is shifted to $z_1 = \tau \times z_0$, where $\tau > 0$ is the shift size, and the in-control coefficient of correlation $\rho = \rho_0$ is shifted to $\rho = \rho_1$. According to the discussion in Leoni et al. [17], we can assume that the units within a sample are collected close together in time and, at the same time, the length $\bar{h}$ of the sampling interval is large enough to eliminate any dependence between successive values of $Z_1, Z_2, \ldots$. That is to say, the observations in each subgroup of size $n$ are autocorrelated but not the statistics $Z_i, i = 1, 2, \ldots$. Then, the out-of-control average run length of the Shewhart-RZ chart is computed as

$$ARL_1 = \frac{1}{1 - \beta},$$  \hspace{1cm} (16)

5 The effect of autocorrelation on the Shewhart-RZ Control chart

In this Section, we present the effect of autocorrelation on the performance of the Shewhart-RZ control evaluated by using the average run length ($ARL$). When the process is in-control, the $ARL$ will be denoted as $ARL_0$. The $ARL$ values are computed from (16) given the fixed values of parameters $n, \gamma_X = \frac{\sigma_X}{\mu_X}, \gamma_Y = \frac{\sigma_Y}{\mu_Y}, \Phi_{11}, \Phi_{22}, z_0, \rho_0, \rho_1$ and $\tau$. Without loss of generality, we assume that $z_0 = 1$. We also suppose that $X$ and $Y$ are not cross-correlated and the matrix $\Phi$ is diagonal, i.e. $\Phi_{12} = \Phi_{21} = 0.$
It allows reducing the bivariate stationary conditions to the more simple univariate stationary conditions for each variable and it has already been stated by Leoni et al. [1]. This assumption means that \(X_{i,j}(Y_{i,j})\) and \(Y_{i,j-1}(X_{i,j-1})\) are not linearly dependent. The in-control ARL is set at \(ARL_0 = 200\), corresponding to the false alarming rate \(\alpha = 0.005\). In this study, we also assume \(n \in \{2, 5, 7, 10, 15\}\), \(\gamma_X \in \{0.01, 0.2\}\), \(\gamma_Y \in \{0.01, 0.2\}\), \(\rho_0 \in \{-0.8, -0.4, 0, 0.4, 0.8\}\), and \(\Phi_{11}, \Phi_{22} \in \{0.1, 0.7\}\).

The values control limits \((LCL, UCL)\) of the Shewhart-RZ control chart for the case \(\Phi_{11} = \Phi_{22} = 0.01\) are presented in Table 1. Results have also been obtained for other parameters but they are not presented here and are available upon request from authors. Similar to the Shewhart-RZ control chart for independent observations investigated in Celano and Castagliola [1], it can be seen from this table that with the same value of sample size \(n\), the values of \(LCL\) and \(LCL\) depend on both the value of \(\rho_0\) (given \((\gamma_X, \gamma_Y)\)), and the value of \((\gamma_X, \gamma_Y)\) (given the value of \(\rho_0\)). The increase of \(\rho_0\) leads to an increase of \(LCL\) and a decrease of \(UCL\). For example, with \((\gamma_X, \gamma_Y) = (0.01, 0.01)\) and \(n = 5\), we have \(LCL = 0.9725\) and \(UCL = 1.0283\) when \(\rho = -0.8\), compared to \(LCL = 0.9840\) and \(UCL = 1.0163\) when \(\rho_0 = 0.4\). By contrast, the increase of \((\gamma_X, \gamma_Y)\) leads to the decrease of \(LCL\) and the increase of \(UCL\). For example, with \(n = 7\) and \(\rho_0 = 0.0\), we have \(LCL = 0.9823\) and \(UCL = 1.0180\) when \((\gamma_X, \gamma_Y) = (0.01, 0.01)\) compared to \(LCL = 0.6933\) and \(UCL = 1.4423\) when \((\gamma_X, \gamma_Y) = (0.2, 0.2)\).

We present the effect of the autocorrelation between observations on the Shewhart-RZ chart in Tables 2-5 assuming that the correlation coefficient \(\rho\) between the random variables \(X\) and \(Y\) does not affect by the assignable causes, i.e \(\rho_0 = \rho_1\). In particular, Tables 2-5 show the values of the \(ARL_1\) for the case \(\Phi_{11} = \Phi_{12}\) and Tables 4-6 show the values of the \(ARL_1\) when \(\Phi_{11} \neq \Phi_{12}\). In general, the obtained results show that given the values of other parameters, the increase of \(\Phi_{11}\) or \(\Phi_{22}\), or both of them leads to the increase of the \(ARL_1\); the larger the values of \((\Phi_{11}, \Phi_{12})\), the larger the values of the \(ARL_1\). For example, in Table 2 for fixed values of \(n = 5\), \((\gamma_X, \gamma_Y) = (0.01, 0.01)\), \(\rho_0 = \rho_1 = -0.8\) and \(\tau = 0.99\), we obtain \(ARL_1 = 23.1\) when \((\Phi_{11}, \Phi_{12}) = (0.1, 0.1)\) compared to \(ARL_1 = 59.7\) when \((\Phi_{11}, \Phi_{12}) = (0.7, 0.7)\). Both of these values are larger than the value \(ARL_1 = 19.1\) when the process is free of autocorrelation (i.e. \((\Phi_{11}, \Phi_{12}) = (0, 0)\)) as pointed out in Celano and Castagliola [1] (Table 2). That is to say, the autocorrelation between observations has a negative influence on the Shewhart-RZ’s performance: it reduces the ability of the Shewhart-RZ control chart in detecting the process shift or out-of-control conditions.
We also consider the situation when the occurrence of an assignable cause shifts the correlation coefficient $\rho$ from $\rho_0$ to $\rho_1$. The values of $ARL_1$ corresponding to this situation are presented in Tables 6-7. For the brevity of the paper, we only show the $ARL_1$ values for the case $\Phi_{11} = \Phi_{22}$. The negative effect of the autocorrelation on the chart’s performance can also be seen from these two tables. Moreover, it is worth noting that when both the nominal ratio and the correlation coefficient are shifted, for the process with high values of $(\gamma_X, \gamma_Y)$, the Shewhart-RZ control chart is in average inefficient in detecting shifts as the values of $ARL_1$ are too large (see the case $(\gamma_X, \gamma_Y) = (0.2, 0.2)$ in Table 6).

In several cases when there is no preference for any specific shift size, one may suggest assigning a distribution for the process shift in a predicted interval. The expected average run length $(EARL)$ is used to evaluate the statistical performance of the corresponding chart, where

$$EARL = \int_{\Omega} ARL \times f_{\tau}(\tau) d\tau,$$

(17)

$f_{\tau}(\tau)$ stands for the distribution of the shift size $\tau$ in the interval $\Omega$ and $ARL$ is as defined in (10). When there is no information about $\tau$, a uniform distribution of $\tau$ could be applied, i.e. the distribution of $f_{\tau}(\tau) = \frac{1}{b-a}$ for $\tau \in \Omega = [a, b]$.

The effect of $\Phi_{11}$ and $\Phi_{22}$ on the overall performance of the Shewhart-RZ control chart in the presence of autocorrelation is displayed in Figure 1 ($\rho_0 = \rho_1 = -0.8$) and Figure 2 ($\rho_0 = -0.4$ and $\rho_1 = -0.8$) with the $EARL$ values for two possibilities of $\Omega$, including $\Omega = \Omega_D = [0.9, 1)$ (decreasing case, denoted as (D)) and $\Omega = \Omega_I = (1, 1.1]$ (increasing case, denoted as (I)). We also consider several values of $\Phi_{11}$ and $\Phi_{22}$ as $\Phi_{11} \in \{0.1, 0.2, \ldots, 0.7\}$ and $\Phi_{22} \in \{0.1, 0.1, 0.2, \ldots, 0.7\}$. The values of others parameters are presented in the caption of each figure. In general, the obtained values of $EARL$ in these figures show a similar trend of the negative effect of $\Phi_{11}$ and $\Phi_{22}$ (representing the autocorrelation between observations) on the overall
performance of Shewhart-RZ chart as the specific shift size cases. In most cases, the larger the values of \((\Phi_{11}, \Phi_{22})\), the larger the values of EARL, namely the faster the Shewhart-RZ control chart in detecting the of the out-of-control conditions. For example, in Figure 1 with \(n = 15\), \(\Omega = [0.9, 1]\), \(\rho_0 = \rho_1 = -0.8\), \(\gamma_X = \gamma_Y = 0.01\), we have \(EARL = 1.49\) for \(\Phi_{11} = \Phi_{22} = 0.1\), \(EARL = 2.79\) for \(\Phi_{11} = \Phi_{22} = 0.5\), and \(EARL = 4.57\) for \(\Phi_{11} = \Phi_{22} = 0.7\). These values are all larger than \(EARL = 1.4\) for \(\Phi_{11} = \Phi_{22} = (\text{the process is free of autocorrelation})\) as presented in Celano and Castagliola [1] (Table 6).

6 Illustrative example

We consider a situation of monitoring the ratio of two normal variables in the food industry. In a mixture of various ingredients, pumpkin seeds and flaxseeds with added spices are two main components of a muesli brand recipe of a food company. Keeping equal weights of these two components in the recipe plays an important role in satisfying a requirement of food’s nutrient composition. Also, the company has to produce brand boxes in different sizes to meet the market’s needs. In the manufacturing process, the company would like to monitor on-line at regular intervals to check deviations from the in-control ratio for every size of boxes. This situation has been presented in Celano and Castagliola [1] and then applied in a number of other studies such as Tran et al. [8], Tran et al. [9]) and Nguyen et al. [11]. However, in these studies, the data was simulated based on an independent assumption between samples. In this section, the autocorrected data will be considered.

According to Celano and Castagliola [1], the in-control ratio \(z_0 = \frac{\mu_{p,i}}{\mu_{f,i}} = 1\) is expected for all the box sizes, where \(\mu_{p,i}\) and \(\mu_{f,i}\) stand for the mean weights of “pumpkin seeds” and “flaxseeds”. To monitor the ratio between these two components, suppose that a sample of \(n = 5\) boxes is collected. The sample average weights \(\overline{W}_{p,i} = \frac{1}{n} \sum_{j=1}^{n} W_{p,i,j}\) and \(\overline{W}_{f,i} = \frac{1}{n} \sum_{j=1}^{n} W_{f,i,j}\) and then the ratio \(\hat{Z}_i = \frac{\overline{W}_{p,i}}{\overline{W}_{f,i}}\) are calculated where \(X = W_{p,i,j}\) and \(Y = W_{f,i,j}\) denote the weight of “pumpkin seeds” and “flaxseeds”. The statistic \(Z_i\) are plotted in the Shewhart-RZ chart to detect the out-of-control condition. We suppose that

- \((W_{p,i,j}, W_{f,i,j})\) can be well approximated by a bivariate normal variable.
The observations are autocorrelated and follow the VAR(1) model with parameters of the VAR(1) model are

\[
\Phi = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \quad \text{and} \quad \Sigma_\varepsilon = \begin{pmatrix} 0.0625 & 0.01 \\ 0.01 & 0.0625 \end{pmatrix}
\]  

In practice, these parameters of the VAR(1) model should be estimated from phase I with a sufficiently enough collected data. In this example, we suppose that the in-control ratio \( z_0 = 1 \) (i.e. \( \mu_X = \mu_Y = 25 \text{ (gr)} \)). Given the input parameters of the VAR(1) model in (18), based on (6), it is straightforward to obtain the standard deviation \( \sigma_X, \sigma_Y \) of the weight of “pumpkin seeds” and “flaxseeds”. Then, the parameters \( \gamma_{\bar{X}}, \gamma_{\bar{Y}}, \hat{\rho} \) and \( \bar{\omega} \) of the distribution of the sample average weights \( \bar{W}_{p,i} \) can be calculated from the formulas (7)–(11). Eventually, from (13–14), the control limits of the Shewhart-RZ chart in this case are calculated as \( LCL = 0.9723582 \) and \( UCL = 1.0284276 \).

The simulated data collected from the process are presented in Table 8, where the values in the rightmost column are the statistics \( \hat{Z}_i \). Similar to [1], we also assume that the process runs in-control condition up to sample #10 and the occurrence of an assignable cause is generated between samples #10 and #11 to shift the in-control ratio \( z_0 = 1 \) to \( z_1 = 1.02 \) (i.e \( \mu_X = 25.05 \text{ (gr)}, \mu_Y = 24.55 \text{ (gr)} \)), corresponding to a ratio percentage increase of 2%. The statistics \( \hat{Z}_i \) are plotted on the Shewhart-RZ chart in Figure 3. As can be seen from this figure, the Shewhart-RZ chart signals the out-of-control conditions by plotting samples #14 and #15 above the upper control limit. The Shewhart-RZ chart without considering the autocorrelation can detect the occurrence of the assignable cause from sample #12 (corresponding to the upper control limit \( UCL = 0.018 \)).

7 Conclusions

In this paper, we have investigated the effects of autocorrelation on the performance of Shewhart-RZ control chart using an autoregressive model for the sample ratio. Both \( ARL \) and \( EARL \) metrics are used to evaluate the performance of the Shewhart-RZ chart for the specific shift size (using \( ARL \)) or the overall performance (using \( EARL \)). The obtained results show that the autocorrelation between observations has a negative impact on the Shewhart-RZ chart’s performance. It reduces the ability of the chart in detecting process shifts compared to no autocorrelation case. The presented results also show that the Shewhart control chart is not efficient in detecting small shifts for the process with high values of coefficient of variation (i.e. \( \gamma_X, \gamma_Y \)). Designing other advanced control charts to reduce the negative
impact of the autocorrelation on the chart’s performance or investigating the effect of autocorrelation on the performance of the EWMA-RZ control chart along similar lines, Phase I data implementation of RZ-type control charts could be interesting topics for further research.

Appendices

Appendix 1

Denote \( W = (X, Y)^T \) a bivariate normal random vector with mean vector \( \mu_W \) and variance-covariance matrix \( \Sigma_W \) where

\[
\mu_W = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix},
\]

and

\[
\Sigma_W = \begin{pmatrix} \sigma_X^2 & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma_Y^2 \end{pmatrix},
\]

The ratio of two components of \( W, X \) and \( Y \), represents the ratio of two normal variables and is defined as \( Z = \frac{X}{Y} \). Several studies on the distribution of this ratio has been published in the literature. In this study, we apply approximations for the inverse density function (i.d.f) \( F_Z^{-1}(p|\gamma_X, \gamma_Y, \omega, \rho) \) of \( Z \) proposed by Celano and Castagliola [1] as:

\[
F_Z^{-1}(p|\gamma_X, \gamma_Y, \omega, \rho) \approx \begin{cases} 
\frac{-C_2 - \sqrt{C_2^2 - 4C_1 C_3}}{2C_1} & \text{if } p \in (0, 0.5], \\
\frac{-C_2 + \sqrt{C_2^2 - 4C_1 C_3}}{2C_1} & \text{if } p \in [0.5, 1), 
\end{cases}
\]

where \( C_1, C_2 \) and \( C_3 \) are functions of \( p \), \( \gamma_X = \frac{\sigma_X}{\mu_X} \), \( \gamma_Y = \frac{\sigma_Y}{\mu_Y} \), \( \omega = \frac{\sigma_X}{\sigma_Y} \) and \( \rho \), i.e.

\[
C_1 = \frac{1}{\gamma_Y^2} - (\Phi^{-1}(p))^2, \\
C_2 = 2\omega \left( \rho \Phi^{-1}(p) - \frac{1}{\gamma_X \gamma_Y} \right), \\
C_3 = \omega^2 \left( \frac{1}{\gamma_X^2} - (\Phi^{-1}(p))^2 \right),
\]

In this paper, like in Celano and Castagliola [1] we will assume that the coefficients of variations \( \gamma_X \) and \( \gamma_Y \) are typically in the range \( (0, 0.2] \). In this range, this approximation has been found to be accurate.
Appendix 2

To calculate $\text{Vec}(\Sigma W)$ in (2), we have,

$$(I_4 - \Phi \otimes \Phi) = \begin{pmatrix}
1 - \Phi_{11}^2 & -\Phi_{11} \Phi_{12} & -\Phi_{12} \Phi_{11} & -\Phi_{12}^2 \\
-\Phi_{11} \Phi_{21} & 1 - \Phi_{11} \Phi_{22} & -\Phi_{12} \Phi_{21} & -\Phi_{12} \Phi_{22} \\
-\Phi_{21} \Phi_{11} & -\Phi_{21} \Phi_{12} & 1 - \Phi_{22} \Phi_{11} & -\Phi_{22} \Phi_{12} \\
-\Phi_{21}^2 & -\Phi_{21} \Phi_{22} & -\Phi_{22} \Phi_{21} & 1 - \Phi_{22}^2
\end{pmatrix}$$

After some time-consuming calculations, one can obtain:

$$(I_4 - \Phi \otimes \Phi)^{-1} = \frac{1}{\Delta} \begin{pmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} \\
\Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} \\
\Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} \\
\Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44}
\end{pmatrix}$$

where:

$$\Delta = (\Phi_{11} \Phi_{22} - \Phi_{12} \Phi_{21} - 1)(\Phi_{11}^2 \Phi_{22}^2 - 2\Phi_{11} \Phi_{12} \Phi_{21} \Phi_{22} + \Phi_{12}^2 \Phi_{21}^2 - \Phi_{11}^2 - 2\Phi_{12} \Phi_{21} - \Phi_{22}^2 + 1)$$

and

$$\Delta_{11} = -(\Phi_{11} \Phi_{22}^3 - \Phi_{12} \Phi_{21} \Phi_{22}^2 - \Phi_{12} \Phi_{22} - \Phi_{12} \Phi_{21} - \Phi_{22}^2 + 1)$$

$$\Delta_{12} = \Delta_{13} = \Phi_{12}(\Phi_{11} \Phi_{22}^2 - \Phi_{12} \Phi_{21} \Phi_{22} - \Phi_{11})$$

$$\Delta_{14} = -\Phi_{12}^2(\Phi_{11} \Phi_{22} - \Phi_{12} \Phi_{21} + 1)$$

$$\Delta_{21} = \Phi_{21}(\Phi_{11} \Phi_{22}^2 - \Phi_{12} \Phi_{21} \Phi_{22} - \Phi_{11})$$

$$\Delta_{22} = -(\Phi_{11}^2 \Phi_{22}^2 - \Phi_{11} \Phi_{12} \Phi_{21} \Phi_{22} - \Phi_{11}^2 - \Phi_{12} \Phi_{21}^2 - \Phi_{22}^2 + 1)$$

$$\Delta_{23} = -\Phi_{21} \Phi_{12}(\Phi_{11} \Phi_{22} - \Phi_{12} \Phi_{21} + 1)$$

$$\Delta_{24} = \Phi_{12}(\Phi_{11}^3 \Phi_{22} - \Phi_{11} \Phi_{12} \Phi_{21} - \Phi_{22})$$

$$\Delta_{31} = \Delta_{21}, \Delta_{32} = \Delta_{23}, \Delta_{33} = \Delta_{22}, \Delta_{34} = \Delta_{24}$$

$$\Delta_{41} = -\Phi_{21}^2(\Phi_{11} \Phi_{22} - \Phi_{12} \Phi_{21} + 1)$$

$$\Delta_{42} = \Phi_{21}(\Phi_{11} \Phi_{22} - \Phi_{11} \Phi_{12} \Phi_{21} - \Phi_{22})$$

$$\Delta_{43} = \Delta_{42}$$

$$\Delta_{44} = -(\Phi_{11}^3 \Phi_{22} - \Phi_{11} \Phi_{12} \Phi_{21} - \Phi_{11}^2 - \Phi_{11} \Phi_{22} - \Phi_{12} \Phi_{21} + 1)$$

Accordingly, from these calculations and the formula of $\text{Vec}(\Sigma W)$ in (2), it
is resulted that

\[ \text{Vec}(\Sigma_W) = \frac{1}{\Delta} \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} \end{pmatrix} \begin{pmatrix} \sigma_{eX}^2 \\ \sigma_{eY}^2 \end{pmatrix} \]

\[ = \frac{1}{\Delta} \begin{pmatrix} \Delta_{11}\sigma_{eX}^2 + (\Delta_{12} + \Delta_{13})\sigma_{eXY} + \Delta_{14}\sigma_{eY}^2 \\ \Delta_{21}\sigma_{eX}^2 + (\Delta_{22} + \Delta_{23})\sigma_{eXY} + \Delta_{24}\sigma_{eY}^2 \\ \Delta_{31}\sigma_{eX}^2 + (\Delta_{32} + \Delta_{33})\sigma_{eXY} + \Delta_{34}\sigma_{eY}^2 \\ \Delta_{41}\sigma_{eX}^2 + (\Delta_{42} + \Delta_{43})\sigma_{eXY} + \Delta_{44}\sigma_{eY}^2 \end{pmatrix} \]

Finally, this leads us to obtain the variance-covariance matrix \( \Sigma_W \) in (3).

Appendix 3

Let \( W_i = (W_{i,1}, W_{i,2}, \ldots, W_{i,p})' \) be a stationary multivariate time series of dimension \( p \) with mean vector \( \mu_W \) and cross-covariances matrix at lag \( k \) as

\[ \Gamma(k) = E[(W_i - \mu_W)(W_{i+k} - \mu_W)^T]_{p \times p} = [\gamma_{j,t}]_{j,t=1,2,\ldots,p} \]

where \( \gamma_{j,t} = E[(W_{i,j} - \mu_j)(W_{i+k,t} - \mu_t)] \) is the covariance between \( W_{i,j} \) and \( W_{i+k,t} \) for \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \). By definition, we have \( \Sigma_W = \Gamma(0) \). For the random sample \( W_{i,1}, W_{i,2}, \ldots, W_{i,n} \), Reinsel [25] showed that

\[ \mu_{\Sigma} = \mu_W \]

\[ \Sigma_{\Sigma} = \frac{1}{n^2} \sum_{j=1}^n \sum_{t=1}^n \Gamma(j-t) = \frac{1}{n^2} \sum_{k=-|n|}^{n-1} (n-|k|) \Gamma(k). \]

Adopting this result for VAR(1) autoregressive model, \( \Sigma_{\Sigma} \) can be obtained as

\[ \Sigma_{\Sigma} = \frac{1}{n} \Sigma_W + \frac{1}{n^2} \sum_{k=1}^{n-1} (n-k) \Gamma(k) + \frac{1}{n^2} \sum_{k=1}^{n-1} (n-k) \Gamma(-k) \]

\[ = \frac{1}{n} \Sigma_W + \frac{1}{n^2} \sum_{k=1}^{n-1} (n-k) [\Gamma(k) + \Gamma(k)^T] \quad \text{(}\Gamma(-k) = \Gamma(k)^T\text{)} \]

\[ = \frac{1}{n} \Sigma_W + \frac{1}{n^2} \sum_{k=1}^{n-1} (n-k) [\Sigma_W \Phi^k + \Phi^k \Sigma_W^T] \quad \text{(}\Gamma(k) = \Sigma_W \Phi^k\text{)} \]

\[ = \frac{1}{n} \Sigma_W + \frac{1}{n^2} \sum_{k=1}^{n-1} [\Sigma_W \Phi^k + \Phi^k \Sigma_W^T] - \frac{1}{n^2} \sum_{k=1}^{n-1} [\Sigma_W \Phi^k + \Phi^k \Sigma_W^T]. \quad (25) \]

In order to further simplify the above relation, we use proposition 1.5.38. of Hubbard and Hubbard [29] in which they proved that \( \sum_{n=0}^{\infty} \Phi^n = (I_p - \Phi)^{-1} \) for a \( p \times p \) square matrix \( \Phi \) that the absolute values of all its eigenvalues
are less than 1. Here, we know that this condition about \( \Phi \) is hold because it is the necessary and sufficient condition for the stationarity of a VAR(1) autoregressive model. Using the idea of their proof, it can be showed that
\[
\sum_{k=1}^{n-1} \Phi^k = \Lambda(\Phi) \quad \text{and} \quad \sum_{k=1}^{n-1} k \Phi^k = \Pi(\Phi)
\]
where
\[
\Lambda(\Phi) = (\Phi - \Phi^n)(I_2 - \Phi)^{-1}
\]
\[
\Pi(\Phi) = (\Phi^{-1} - I_2)^{-1} \left( (I_2 - \Phi^{n-1})(I_2 - \Phi)^{-1} - (n-1)\Phi^{n-1} \right)
\]
Rewriting (25) based on \( \Lambda(\Phi) \) and \( \Pi(\Phi) \) and some mathematical calculations result \( \Sigma_W \) in [1].

References

[1] G. Celano and P. Castagliola. Design of a phase II Control Chart for Monitoring the Ratio of two Normal Variables. Quality and Reliability Engineering International, 2014. doi: 10.1002/qre.1748. In press.

[2] A.W. Spisak. A Control Chart for Ratios. Journal of Quality Technology, 22(1):34–37, 1990.

[3] R.B. Davis and W.H. Woodall. Evaluation of Control Charts for Ratios. In 22nd Annual Pittsburgh Conference on Modeling and Simulation, Pittsburgh, PA, 1991.

[4] D. Öksoy, E. Boulos, and L.D. Pye. Statistical Process Control by the Quotient of two Correlated Normal Variables. Quality Engineering, 6(2):179–194, 1994.

[5] G. Celano, P. Castagliola, A. Faraz, and S. Fichera. Statistical Performance of a Control Chart for Individual Observations Monitoring the Ratio of two Normal Variables. Quality and Reliability Engineering International, 30(8):1361–1377, 2014.

[6] G. Celano and P. Castagliola. A Synthetic Control Chart for Monitoring the Ratio of Two Normal Variables. Quality and Reliability Engineering International, 32:681–696, 2016.

[7] K. P. Tran, P. Castagliola, and G. Celano. Monitoring the Ratio of two Normal Variables using Run Rules Type Control Charts. International Journal of Production Research, 54(6):1670–1688, 2016.

[8] K.P. Tran, P. Castagiola, and G. Celano. Monitoring the Ratio of Two Normal Variables Using EWMA Type Control Charts. Quality and Reliability Engineering International, 32(2):1853–1869, 2016.
Table 1: Values of \( LCL \) (first row) and \( UCL \) (second row) for the Shewhart-RZ control chart in the presence of autocorrelation, for \( z_0 = 1, ARL_0 = 200, \Phi_{11} = \Phi_{22} = 0.1, n \in \{2, 5, 7, 10, 15\}, \gamma_X \in \{0.01, 0.2\}, \gamma_Y \in \{0.01, 0.2\} \) and \( \rho_0 \in \{-0.8, -0.4, 0, 0.4, 0.8\} \).

| \( \gamma_X \) | \( \gamma_Y \) | \( \rho_0 \) | \( n = 1 \) | \( n = 5 \) | \( n = 7 \) | \( n = 10 \) | \( n = 15 \) |
|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.01   | 0.01   | -0.8   | 0.9596 | 0.9725 | 0.9764 | 0.9800 | 0.9835 |
|        |        |        | 1.0421 | 1.0283 | 1.0242 | 1.0204 | 1.0167 |
|        |        |        | -0.4   | 0.9643 | 0.9757 | 0.9791 | 0.9823 | 0.9855 |
|        |        |        |        | 1.0371 | 1.0249 | 1.0213 | 1.0180 | 1.0148 |
|        |        | 0.0    | 0.9697 | 0.9794 | 0.9823 | 0.9851 | 0.9877 |
|        |        |        |        | 1.0312 | 1.0210 | 1.0180 | 1.0152 | 1.0125 |
|        |        | 0.4    | 0.9765 | 0.9840 | 0.9863 | 0.9884 | 0.9905 |
|        |        |        |        | 1.0241 | 1.0163 | 1.0139 | 1.0117 | 1.0096 |
|        |        | 0.8    | 0.9863 | 0.9907 | 0.9921 | 0.9933 | 0.9945 |
|        |        |        |        | 1.0138 | 1.0094 | 1.0080 | 1.0068 | 1.0056 |
| 0.20   | 0.20   | -0.8   | 0.4119 | 0.5619 | 0.6131 | 0.6634 | 0.7148 |
|        |        |        | 2.4276 | 1.7798 | 1.6311 | 1.5075 | 1.3990 |
|        |        |        | -0.4   | 0.4550 | 0.6005 | 0.6489 | 0.6959 | 0.7435 |
|        |        |        |        | 2.1980 | 1.6652 | 1.5410 | 1.4370 | 1.3450 |
|        |        | 0.0    | 0.5115 | 0.6490 | 0.6933 | 0.7357 | 0.7782 |
|        |        |        |        | 1.9549 | 1.5408 | 1.4423 | 1.3592 | 1.2850 |
|        |        | 0.4    | 0.5927 | 0.7147 | 0.7525 | 0.7881 | 0.8233 |
|        |        |        |        | 1.6871 | 1.3992 | 1.3289 | 1.2688 | 1.2147 |
|        |        | 0.8    | 0.7377 | 0.8232 | 0.8483 | 0.8714 | 0.8937 |
|        |        |        |        | 1.3556 | 1.2147 | 1.1788 | 1.1476 | 1.1189 |
| 0.01   | 0.20   | -0.8   | 0.6846 | 0.7633 | 0.7905 | 0.8174 | 0.8450 |
|        |        |        | 1.8005 | 1.4341 | 1.3505 | 1.2813 | 1.2208 |
|        |        |        | -0.4   | 0.6904 | 0.7677 | 0.7944 | 0.8208 | 0.8479 |
|        |        |        |        | 1.7853 | 1.4259 | 1.3439 | 1.2760 | 1.2166 |
|        |        | 0.0    | 0.6964 | 0.7722 | 0.7984 | 0.8242 | 0.8508 |
|        |        |        |        | 1.7700 | 1.4176 | 1.3373 | 1.2707 | 1.2124 |
|        |        | 0.4    | 0.7025 | 0.7768 | 0.8025 | 0.8278 | 0.8538 |
|        |        |        |        | 1.7546 | 1.4092 | 1.3305 | 1.2652 | 1.2081 |
|        |        | 0.8    | 0.7088 | 0.7815 | 0.8066 | 0.8314 | 0.8569 |
|        |        |        |        | 1.7389 | 1.4007 | 1.3236 | 1.2597 | 1.2038 |
| 0.20   | 0.01   | -0.8   | 0.5554 | 0.6973 | 0.7404 | 0.7804 | 0.8192 |
|        |        |        | 1.4607 | 1.3101 | 1.2650 | 1.2234 | 1.1834 |
|        |        |        | -0.4   | 0.5601 | 0.7013 | 0.7441 | 0.7837 | 0.8220 |
|        |        |        |        | 1.4484 | 1.3026 | 1.2588 | 1.2184 | 1.1794 |
|        |        | 0.0    | 0.5650 | 0.7054 | 0.7478 | 0.7870 | 0.8248 |
|        |        |        |        | 1.4360 | 1.2950 | 1.2525 | 1.2132 | 1.1754 |
|        |        | 0.4    | 0.5699 | 0.7096 | 0.7516 | 0.7904 | 0.8277 |
|        |        |        |        | 1.4234 | 1.2874 | 1.2462 | 1.2080 | 1.1712 |
|        |        | 0.8    | 0.5751 | 0.7139 | 0.7555 | 0.7938 | 0.8307 |
|        |        |        |        | 1.4107 | 1.2796 | 1.2397 | 1.2028 | 1.1670 |
| \( n \) | (\( \gamma_X = 0.01, \gamma_Y = 0.01 \)) | (\( \gamma_X = 0.2, \gamma_Y = 0.2 \)) |
|---|---|---|
| 2 | \( \rho_0 = \rho_1 = -0.8 \) | \( \rho_0 = \rho_1 = -0.4 \) |
| 5 | \( (1.0, 1.0) \) | \( (1.0, 1.0) \) |
| 7 | \( (1.0, 1.0) \) | \( (1.0, 1.0) \) |
| 10 | \( (1.0, 1.0) \) | \( (1.0, 1.0) \) |
| 15 | \( (1.0, 1.0) \) | \( (1.0, 1.0) \) |

Table 2: The \( ARL_1 \) values of the Shewhart-RZ in the presence of autocorrelation for \( \Phi_{11} = \Phi_{22} = 0.1 \) (left side), \( \Phi_{11} = \Phi_{22} = 0.7 \) (right side), \( \gamma_X = \gamma_Y \in \{0.01, 0.2\} \), \( \rho_0 \in \{-0.8, -0.4, 0.4, 0.8\} \), \( \rho_0 = \rho_1, \ n \in \{2, 5, 7, 10, 15\} \) and \( ARL_0 = 200 \).
| $n_0$ | $n_5$ | $n_7$ | $n_{10}$ | $n_{15}$ |
|-------|-------|-------|----------|---------|
| 0.90  | 34.0  | 16.6  | 11.9    | 8.0     |
| 0.95  | 94.2  | 65.9  | 53.5    | 41.5    |
| 0.98  | 159.3 | 140.6 | 138.2   | 127.8   |
| 0.99  | 181.2 | 172.1 | 174.4   | 170.5   |
| 1.01  | 212.9 | 207.2 | 204.4   | 206.9   |
| 1.02  | 218.7 | 199.4 | 188.1   | 173.2   |
| 1.05  | 199.3 | 131.6 | 165.4   | 170.6   |
| 1.10  | 128.1 | 50.3  | 35.1    | 21.6    |

| $\gamma$ values | $\gamma_X = 0.01$, $\gamma_Y = 0.2$ | $\gamma_X = 0.2$, $\gamma_Y = 0.01$ |
|-----------------|-----------------------------|-----------------------------|
| $\rho_0 = 0.0$  | 114.8 167.9 (43.7 127.6)  | 28.1 108.3 (16.8 86.3)     |
| $\rho_0 = 0.1$  | 195.3 225.9 (125.6 203.9) | 99.4 191.0 (74.0 173.4)    |
| $\rho_0 = 0.2$  | 218.8 224.6 (198.8 220.5) | 187.1 217.8 (171.8 213.6) |
| $\rho_0 = 0.3$  | 210.3 214.4 (204.7 212.3) | 199.5 211.7 (191.8 209.0) |
| $\rho_0 = 0.4$  | 181.6 182.7 (177.4 181.9) | 174.7 181.4 (170.9 180.6) |
| $\rho_0 = 0.5$  | 160.2 163.9 (147.3 161.3) | 139.7 159.9 (129.5 156.9) |
| $\rho_0 = 0.6$  | 98.7 111.0 (60.9 101.9)    | 57.9 96.9 (45.4 90.0)      |
| $\rho_0 = 0.7$  | 40.8 525.2 (20.8 43.8)     | 15.2 29.3 (10.4 33.6)      |
| $\rho_0 = 0.8$  | 113.3 136.9 (42.1 126.2)  | 27.1 106.8 (16.1 84.7)     |
| $\rho_0 = 0.9$  | 195.2 226.6 (124.0 203.8) | 97.6 190.5 (72.2 172.5)    |
| $\rho_0 = 1.0$  | 213.5 213.4 (208.0 210.9) | 204.5 212.3 (199.5 210.2) |
| $\rho_0 = 1.1$  | 180.9 181.2 (176.6 181.3) | 173.9 180.7 (169.8 179.0) |
| $\rho_0 = 1.2$  | 158.8 162.7 (145.6 160.0) | 137.0 158.2 (127.5 155.5) |
| $\rho_0 = 1.3$  | 96.3 107.6 (67.7 99.4)     | 55.9 94.5 (43.7 86.7)      |
| $\rho_0 = 1.4$  | 39.0 50.3 (19.8 41.9)      | 14.4 37.5 (9.9 32.1)       |

Table 3: The ARL values of the Shewhart-R2 in the presence of autocorrelation for $\Phi_1 = \Phi_2 = 0.1$ (left side), $\Phi_1 = \Phi_2 = 0.7$ (right side), $\gamma_X \in \{0.01, 0.2\}$, $\gamma_Y \in \{0.01, 0.2\}$, $\gamma_X \neq \gamma_Y$, $\rho_0 \in \{-0.8, -0.4, 0, 0.4, 0.8\}$, $\rho_0 = \rho_1$, $n \in \{2, 5, 7, 10, 15\}$ and $ARL_0 = 200$. 
$$\gamma_X = 0.01, \gamma_Y = 0.01$$

| \( \gamma_X = 0.2, \gamma_Y = 0.2 \) |
|---|
| \( \gamma_X = 0.1, \gamma_Y = 0.1 \) |

| \( n = 2 \) | \( n = 5 \) | \( n = 7 \) | \( n = 10 \) | \( n = 15 \) |
|---|---|---|---|---|
| \( \rho_0 = \rho_1 = -0.8 \) | \( \rho_0 = \rho_1 = -0.8 \) |
| \( \rho_0 = \rho_1 = 0.4 \) | \( \rho_0 = \rho_1 = 0.4 \) |

Table 4: ARL values of the Shewhart-RZ in the presence of autocorrelation for \( \Phi_{11} = 0.1, \Phi_{22} = 0.7 \) (left side), \( \Phi_{11} = 0.7, \Phi_{22} = 0.1 \) (right side), \( \gamma_X \in \{0.01, 0.2\}, \gamma_Y \in \{0.01, 0.2\}, \gamma_X = \gamma_Y, \rho_0 \in \{-0.8, -0.4, 0, 0.4, 0.8\}, \rho_0 = \rho_1, n \in \{2, 5, 7, 10, 15\} \) and \( ARL_0 = 200 \).
| n = 4 | n = 5 | n = 7 | n = 10 | n = 15 |
|-------|-------|-------|--------|-------|
| ρ₀ = 0.9 | (414.350) | (35.5,18.1) | (31.4,13.3) | (26.5,9.2) | (20.6,5.9) |
| ρ₀ = 0.95 | (1049.956) | (95.0,69.0) | (90.6,57.7) | (81.5,45.8) | (73.3,33.4) |
| ρ₀ = 0.98 | (1627.160) | (159.5,148.8) | (157.6,142.0) | (154.8,133.2) | (150.1,119.3) |
| ρ₀ = 0.99 | (1823.181) | (181.3,178.5) | (180.7,176.3) | (179.8,172.9) | (178.3,165.7) |
| ρ₀ = 1.01 | (2145.212) | (213.6,207.2) | (213.0,204.1) | (212.0,199.6) | (210.2,192.8) |
| ρ₀ = 1.02 | (2247.218) | (220.9,199.4) | (218.2,188.8) | (214.1,175.3) | (207.3,156.0) |
| ρ₀ = 1.05 | (2276.199) | (207.0,134.0) | (194.5,109.0) | (177.5,83.9) | (153.8,59.9) |
| ρ₀ = 1.10 | (180.0,120.0) | (140.2,55.3) | (120.7,37.5) | (97.9,23.8) | (71.7,13.5) |
| ρ₀ = 0.9 | (423.329) | (34.1,16.5) | (30.4,12.0) | (25.6,8.2) | (19.9,5.2) |
| ρ₀ = 0.95 | (102.8,962) | (94.0,65.2) | (89.3,53.9) | (81.9,42.2) | (71.8,30.3) |
| ρ₀ = 0.98 | (161.6,158) | (158.5,146.0) | (156.6,138.6) | (153.8,128.7) | (149.9,114.7) |
| ρ₀ = 0.99 | (181.7,180) | (180.8,177.2) | (180.2,174.6) | (179.3,170.9) | (177.8,165.0) |
| ρ₀ = 1.01 | (214.1,212) | (214.0,207.7) | (213.3,204.3) | (213.2,199.6) | (210.4,192.2) |
| ρ₀ = 1.02 | (225.4,219) | (219.1,493) | (218.7,181.8) | (215.4,173.4) | (207.5,153.2) |
| ρ₀ = 1.05 | (228.2,198) | (207.0,131.3) | (194.3,105.6) | (177.0,80.2) | (152.5,55.3) |
| ρ₀ = 1.10 | (179.3,127) | (139.3,32.9) | (119.7,35.2) | (96.8,21.9) | (70.7,12.2) |

Table 5: ARL₁ values of the Shewhart-R2 in the presence of autocorrelation for ϕ₁ = 0.1, ϕ₂ = 0.7 (left side), ϕ₁ = 0.7, ϕ₂ = 0.1 (right side), γ ∈ {0.01, 0.2}, γ ≠ γ₀, ρ₀ ∈ {−0.8, −0.4, 0.0, 0.4, 0.8}, ρ₀ = ρ₁, n ∈ {2, 5, 7, 10, 15} and ARL₀ = 200.
| n   | $\gamma_x = 0.01, \gamma_y = 0.01$ | n   | $\gamma_x = 0.2, \gamma_y = 0.2$ |
|-----|-----------------------------------|-----|-----------------------------------|
| 2   | $(1.0,1.0)$                        | 2   | $(0.4,0.2)$                        |
| 5   | $(1.0,1.0)$                        | 5   | $(0.4,0.2)$                        |
| 7   | $(1.0,1.0)$                        | 7   | $(0.4,0.2)$                        |
| 10  | $(1.0,1.0)$                        | 10  | $(0.4,0.2)$                        |
| 15  | $(1.0,1.0)$                        | 15  | $(0.4,0.2)$                        |

| $\rho_0 = -0.4, \rho_1 = -0.2$ | $\rho_0 = -0.4, \rho_1 = -0.8$ |
|---------------------------------|---------------------------------|
| 0.90                            | 0.90                            |
| 0.95                            | 0.95                            |
| 0.98                            | 0.98                            |
| 0.99                            | 0.99                            |
| 1.01                            | 1.01                            |
| 1.02                            | 1.02                            |
| 1.05                            | 1.05                            |
| 1.10                            | 1.10                            |

Table 6: ARL$_1$ values of the Shewhart-RZ in the presence of autocorrelation for $\Phi_{11} = \Phi_{22} = 0.1$ (left side), $\Phi_{11} = \Phi_{22} = 0.7$ (right side), $\gamma_x \in \{0.01, 0.2\}$, $\gamma_x \in \{0.01, 0.2\}$, $\gamma_x = \gamma_y$, $(\rho_0, \rho_1) = \{(-0.4, -0.2), (-0.4, -0.8), (0.4, 0.2), (0.4, 0.8)\}$, $n \in \{2, 5, 7, 10, 15\}$ and ARL$_0 = 200$. 

The ARL values are calculated for different combinations of parameters, including $\gamma_x$, $\gamma_y$, and $\rho_0, \rho_1$. The table provides a comprehensive view of how these parameters affect the ARL values for specific sample sizes.
\[
\begin{array}{cccccccc}
\nu & n = 2 & n = 5 & n = 10 & n = 15 & n = 2 & n = 5 & n = 10 & n = 15 \\
\rho_0 = -0.4, \rho_1 = -0.2 & \Phi & \Phi & \Phi & \Phi & \Phi & \Phi & \Phi & \Phi \\
0.90 & (34.2, 45.5) & (16.2, 37.0) & (11.5, 32.7) & (7.8, 27.4) & (4.8, 21.2) & (118.1, 173.9) & (44.0, 131.6) & (28.0, 111.4) & (16.5, 88.3) & (8.7, 62.7) \\
0.95 & (99.3, 112.5) & (67.3, 103.0) & (54.7, 97.3) & (42.0, 89.4) & (29.4, 78.2) & (206.7, 240.0) & (130.8, 215.9) & (102.7, 201.7) & (75.8, 182.5) & (50.4, 155.7) \\
0.99 & (172.4, 176.8) & (157.0, 174.7) & (148.0, 171.6) & (136.2, 168.4) & (119.8, 163.1) & (236.4, 243.0) & (213.6, 238.4) & (200.4, 235.2) & (183.1, 230.5) & (159.5, 222.7) \\
0.999 & (197.1, 198.4) & (192.2, 197.5) & (189.1, 196.9) & (184.5, 196.0) & (177.2, 194.3) & (231.2, 232.8) & (225.3, 231.7) & (221.4, 231.0) & (215.7, 229.8) & (206.9, 227.8) \\
1.0 & (231.2, 232.7) & (225.3, 231.6) & (215.9, 229.9) & (207.3, 227.7) & (197.4, 198.6) & (192.6, 197.7) & (189.4, 197.1) & (184.9, 196.2) & (177.8, 194.6) & \\
1.02 & (236.3, 242.7) & (214.3, 238.2) & (201.6, 235.3) & (184.7, 230.7) & (161.6, 223.1) & (174.4, 177.7) & (158.4, 174.7) & (149.7, 172.6) & (138.0, 169.6) & (121.9, 164.4) \\
1.05 & (210.7, 241.9) & (137.3, 219.3) & (109.2, 205.9) & (81.1, 187.7) & (55.4, 161.8) & (104.3, 117.1) & (72.4, 107.9) & (59.4, 102.3) & (46.1, 94.6) & (32.7, 83.8) \\
1.1 & (132.5, 187.2) & (57.3, 145.9) & (35.2, 125.2) & (21.4, 101.8) & (11.6, 74.3) & (41.4, 53.8) & (20.6, 44.5) & (14.8, 39.7) & (10.1, 33.8) & (6.3, 25.6) \\
\end{array}
\]

\textbf{Table 7:} $ARL_1$ values of the Shewhart-RZ in the presence of autocorrelation for $\Phi_{11} = \Phi_{22} = 0.1$ (left side), $\Phi_{11} = \Phi_{22} = 0.7$ (right side), $\gamma_X \in \{0.01, 0.2\}$, $\gamma_Y \in \{0.01, 0.2\}$, $\gamma_X \neq \gamma_Y$, $(\rho_0, \rho_1) = \{(-0.4, -0.2), (-0.4, -0.8), (0.4, 0.2), (0.4, 0.8)\}$, $n \in \{2, 5, 7, 10, 15\}$ and $ARL_0 = 200$. 

\textbf{23}
Figure 1: The effects of $\Phi_{11}$ and $\Phi_{22}$ on the overall performance of the Shewhart-RZ control in the presence of autocorrelation for $n \in \{2, 15\}$, $\gamma_X \in \{0.01, 0.2\}$, $\gamma_Y \in \{0.01, 0.2\}$, $\gamma_X = \gamma_Y$, $\rho_0 = \rho_1 = -0.8$ and $ARL_0 = 200$. 
\[ \rho_0 = -0.4; \rho_1 = -0.8; \Omega_D = [0.9; 1) \]

\[ (\gamma_X = 0.01, \gamma_Y = 0.01) \] \hspace{1cm} \[ (\gamma_X = 0.2, \gamma_Y = 0.2) \]

\[ \rho_0 = -0.4; \rho_1 = -0.8; \Omega_I = (1; 1) \]

\[ (\gamma_X = 0.01, \gamma_Y = 0.01) \] \hspace{1cm} \[ (\gamma_X = 0.2, \gamma_Y = 0.2) \]

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Figure 2: The effects of \( \Phi_{11} \) and \( \Phi_{22} \) on the overall performance of the Shewhart-RZ control chart in the presence of autocorrelation for \( n \in \{2, 15\} \), \( \gamma_X \in \{0.01, 0.2\} \), \( \gamma_Y \in \{0.01, 0.2\} \), \( \gamma_X = \gamma_Y \), \( \rho_0 = -0.4 \), \( \rho_1 = -0.8 \) and \( ARL_0 = 200 \).
| Sample | \( W_{p,i,j} \) [gr] | \( W_{f,i,j} \) [gr] | \( W_{p,i} \) [gr] | \( W_{f,i} \) [gr] | \( Z_i = \frac{W_{p,i}}{W_{f,i}} \) |
|--------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1      | 24.923            | 24.784            | 24.576            | 25.154            | 24.839            |
|        | 25.086            | 25.061            | 24.419            | 24.     | 24.814            |
| 2      | 24.735            | 24.973            | 25.167            | 25.084            | 24.710            |
|        | 24.463            | 25.489            | 25.355            | 24.725            | 24.676            |
|        | 25.071            | 24.600            | 25.008            | 25.129            | 25.364            |
|        | 24.455            | 24.684            | 25.240            | 25.170            | 25.034            |
| 3      | 25.591            | 25.122            | 25.006            | 24.772            | 24.504            |
|        | 25.440            | 25.057            | 24.878            | 24.762            | 24.832            |
| 4      | 24.695            | 25.094            | 24.787            | 24.941            | 25.261            |
|        | 24.942            | 25.347            | 25.001            | 24.646            | 25.178            |
|        | 25.087            | 25.080            | 25.366            | 25.225            | 25.118            |
|        | 25.219            | 25.433            | 25.431            | 25.057            | 25.478            |
| 5      | 24.675            | 25.011            | 24.809            | 24.919            | 24.740            |
|        | 25.169            | 25.749            | 24.930            | 25.234            | 25.506            |
| 6      | 25.114            | 25.379            | v25.491            | 25.376            | 25.050            |
|        | 24.889            | 24.822            | 25.123            | 24.785            | 25.072            |
|        | 25.223            | 25.036            | 24.880            | 24.992            | 25.055            |
|        | 25.001            | 24.923            | 24.964            | 24.913            | 24.729            |
|        | 24.955            | 24.985            | 24.720            | 24.488            | 24.943            |
|        | 25.555            | 25.458            | 25.012            | 25.279            | 25.295            |
| 7      | 25.148            | 25.250            | 25.169            | 25.007            | 24.982            |
|        | 24.888            | 24.916            | 24.898            | 24.891            | 24.581            |
| 8      | 25.312            | 25.089            | 25.322            | 25.140            | 24.813            |
|        | 24.753            | 24.304            | 24.502            | 24.966            | 24.751            |
|        | 25.165            | 25.203            | 25.165            | 25.536            | 25.288            |
|        | 24.819            | 24.874            | 24.860            | 24.736            | 24.526            |
| 9      | 24.955            | 24.914            | 24.971            | 24.739            | 24.763            |
|        | 24.447            | 24.038            | 24.076            | 23.917            | 24.085            |
|        | 24.265            | 24.660            | 25.462            | 25.343            | 25.108            |
|        | 24.532            | 24.716            | 24.745            | 24.113            | 24.436            |

Table 8: The food industry example data
Figure 3: Shewhart-RZ control chart in the presence of autocorrelation for the food industry example

[9] K.P. Tran, P. Castagliola, and G. Celano. Monitoring the Ratio of Population Means of a Bivariate Normal distribution using CUSUM Type Control Charts. Statistical Papers, 2016, DOI: 10.1007/s00362-016-0769-4.

[10] K.P. Tran, P. Castagliola, G. Celano, and M.B.C. Khoo. Monitoring compositional data using multivariate exponentially weighted moving average scheme. Quality and Reliability Engineering International, 34 (3):391–402, 2018. doi: https://doi.org/10.1002/qre.2260.

[11] H. D. Nguyen, K.P. Tran, C. Heuchenne. Monitoring the ratio of two normal variables using variable sampling interval exponentially weighted moving average control charts. Quality and Reliability Engineering International, 35(1):439–46, 2019.

[12] K.P. Tran, P. Castagliola, and G. Celano. The performance of the Shewhart-RZ control chart in the presence of measurement error. International Journal of Production Research, 54: 7054–7522, 2016.

[13] L.C. Alwan and H.V. Roberts. Time-series modeling for statistical process control. Journal of Business & Economic Statistics, 6(1):87–95, 1988.

[14] A. A. Kalgonda and S. R. Kulkarni. Multivariate quality control chart
for autocorrelated processes. *Journal of Applied Statistics*, 31(3):317–327, 2004.

[15] S. N. Lin, C. Y. Chou, S. L. Wang, and H. R. Liu. Economic design of autoregressive moving average control chart using genetic algorithms. *Expert Systems with Applications*, 39(2):1793–1798, 2012.

[16] B. C. Franco, A. F. B. Costa, and M. A. G. Machado. Economic-statistical design of the X chart used to control a wandering process mean using genetic algorithm. *Expert Systems with Applications*, 39(17):12961–12967, 2012.

[17] R. C. Leoni, A. F. B. Costa, and M. A. G. Machado. The effect of the autocorrelation on the performance of the $T^2$ chart. *European Journal of Operational Research*, 247(1):155 – 165, 2015.

[18] R.C. Leoni, A. F. B. Costa, B. C. Franco, and M. A. G. Machado. The skipping strategy to reduce the effect of the autocorrelation on the $T^2$ chart’s performance. *The International Journal of Advanced Manufacturing Technology*, 80(9):1547–1559, 2015.

[19] H. B. Hwarng and Y. Wang. Shift detection and source identification in multivariate autocorrelated processes. *International Journal of Production Research*, 48(3):835–859, 2010.

[20] A. F. Costa and P. Castagliola. Effect of measurement error and autocorrelation on the x chart. *Journal of Applied Statistics*, 38(4):661–673, 2011.

[21] X. Huang, N. Xu, and S. Bisgaard. A class of markov chain models for average run length computations for autocorrelated processes. *Communications in Statistics - Simulation and Computation*, 42(7):1495–1513, 2013.

[22] X. Huang, S. Bisgaard, and N. Xu. Model-based Multivariate Monitoring Charts for Autocorrelated Processes. *Quality and Reliability Engineering International*, 30(4):527–543, 2014.

[23] B.C. Franco, G. Celano, P. Castagliola, and A. F. B. Costa. Economic design of shewhart control charts for monitoring autocorrelated data with skip sampling strategies. *International Journal of Production Economics*, 151:121 – 130, 2014.

[24] B.C. Franco, P. Castagliola, G. Celano, and A. F. B. Costa. A new sampling strategy to reduce the effect of autocorrelation on a control chart. *Journal of Applied Statistics*, 41(7):1408–1421, 2014.
[25] G.C. Reinsel. *Elements of Multivariate Time Series Analysis*. Springer Science & Business Media, 2003.

[26] R.S. Tsay. *Multivariate time series analysis: with R and financial applications*. John Wiley & Sons, 2013.

[27] D.C. Montgomery, C.L. Jennings, and M. Costa. *Introduction to Time Series Analysis and Forecasting*. John Wiley & Sons, 2008.

[28] H. Lutkepohl. *New Introduction to Multiple Time Series Analysis*. Springer-Verlag Berlin Heidelberg, 2005.

[29] J.H. Hubbard and B.B. Hubbard. *Vector calculus, linear algebra, and differential forms: a unified approach*. Matrix Editions, 2015.