An extremal problem for generalized Lelong numbers

Alexander Rashkovskii

Abstract

We look for pointwise bounds on a plurisubharmonic function near its singularity point, given the value of its generalized Lelong number with respect to a plurisubharmonic weight. To this end, an extremal problem is considered. In certain cases, the problem is solved explicitly.

Subject classification: 32U05, 32U25, 32U35, 13H15.

Key words: plurisubharmonic function, generalized Lelong number, pluricomplex Green function.

1 Introduction

Let \( m = \{ f \in O_0 : f(0) = 0 \} \) be the maximal ideal in the local ring \( O_0 \) of germs of analytic functions at \( 0 \in \mathbb{C}^n \). It is not hard to see that the mixed multiplicity \( e_{n-1}(J, m) \) (due to Teissier and Risler \[15\] and Bivi\`a-Ausina \[1\]) of an ideal \( J \) of \( O_0 \) and \( n-1 \) copies of \( m \) is at least \( p, p \in \mathbb{Z}_+ \), if and only if \( J \subset m^p \); this follows from the fact that \( e_{n-1}(J, m) \) equals the minimal multiplicity of functions \( f \in J \).

Let now \( I \) be an \( m \)-primary ideal of \( O_0 \) (which means that the common zero set of functions from \( I \) is \( \{0\} \)); what can be said about an ideal \( J \) if \( e_{n-1}(J, I) \geq p? \)

If the ideal \( I \) is generated by functions \( g_1, \ldots, g_l \) with \( l \geq n \) and the ideal \( J \) is generated by functions \( f_1, \ldots, f_m \), then the value \( e_{n-1}(J, I) \) equals the residual mass of the Monge-Ampère current \( dd^c \log |f| \wedge (dd^c \log |g|)^{n-1} \) at \( 0 \), which is, actually, the generalized Lelong number (due to Demailly) \( \nu(\log |f|, \log |g|) \) of the plurisubharmonic function \( \log |f| \) with respect to the plurisubharmonic weight \( \log |g| \); here \( |f|^2 = \sum |f_j|^2 \) and similarly for \( g \) and all other vector functions (mappings) below.

Now we can restate our question in the category of plurisubharmonic functions as one on asymptotic behavior of the functions with given values of their generalized Lelong numbers: Given a plurisubharmonic weight \( \varphi \), what can be said about (the asymptotic of) a plurisubharmonic function \( u \) near \( \varphi^{-1}(-\infty) \) if \( \nu(u, \varphi) \geq c? \)

When \( \varphi(z) = \log |z - x|, x \in \mathbb{C}^n \), the value \( \nu(u, \varphi) \) is the classical Lelong number \( \nu_u(x) \) of \( u \) at \( x \). In this case, we have the bound \( u(z) \leq \nu_u \log |z - x| + O(1) \) as \( z \to x \), and \( \nu_u(x) \geq c \) if and only if \( u \leq c \log |z - x| + O(1) \) near \( x \), which is a plurisubharmonic analogue to the remark above on \( e_{n-1}(J, m) \).

More generally, let \( a \in \mathbb{R}_+^n \) and

\[
\phi_{a, x}(z) = \max_k a_k^{-1} \log |z_k - x_k|.
\]

(1.1)
Then \( u(z) \leq \nu_u(x,a)\phi_{a,x}(z) + O(1) \) with \( \nu_u(x,a) \) the directional Lelong numbers due to Kiselman at \( x \) for \( a \in \mathbb{R}_+^n \). In terms of Demailly’s generalized Lelong numbers \( \nu(u,\varphi) \) with respect to plurisubharmonic weights \( \varphi \), this gives

\[
u_u(x,a)\phi_{a,x}(z) + O(1)
\]

where \( \tau = (a_1 \ldots a_n)^{-1} = (dd^c\phi_{a,x})^n(\{x\}) \) is the Monge-Ampère mass of \( \phi_{a,x} \) at \( x \).

In the general case, the relation

\[
\nu_u(x,a)\phi_{a,x}(z) + O(1)
\]

\[\tau_{\phi} \]

been the residual mass of \( (dd^c\varphi)^n \) at \( x = \varphi^{-1}(-\infty) \), need not be true for all \( u \), even for the weights \( \varphi \) that are maximal outside \( x \).

**Example 1** Let \( \varphi = \log |f| \) with \( f(z_1,z_2) = (z_1^3 - z_2^3, z_1z_2) \); we have \( (dd^c\varphi)^2 = 6\delta_0 \). The function \( u(z) = \log |z_1| \) satisfies \( \nu(u,\varphi) = 3 \), so the inequality \( (1.2) \) would take the form \( \log |z_1| \leq \frac{1}{2} \varphi + O(1) \), which is not true for \( z = (z_1,0), z_1 \rightarrow 0 \).

To answer our question for an arbitrary maximal weight \( \varphi \), we consider an extremal problem whose solution \( d_{\varphi} \) gives the best upper bound for functions \( u \) satisfying \( \nu(u,\varphi) \geq c \):

\[
u_u(x,a)\phi_{a,x}(z) + O(1)
\]

\[\tau_{\phi} \]

need not be true for all \( u \), even for the weights \( \varphi \) that are maximal outside \( x \).

\[\tau_{\phi} \]

been the residual mass of \( (dd^c\varphi)^n \) at \( x = \varphi^{-1}(-\infty) \), need not be true for all \( u \), even for the weights \( \varphi \) that are maximal outside \( x \).

**Example 1** Let \( \varphi = \log |f| \) with \( f(z_1,z_2) = (z_1^3 - z_2^3, z_1z_2) \); we have \( (dd^c\varphi)^2 = 6\delta_0 \). The function \( u(z) = \log |z_1| \) satisfies \( \nu(u,\varphi) = 3 \), so the inequality \( (1.2) \) would take the form \( \log |z_1| \leq \frac{1}{2} \varphi + O(1) \), which is not true for \( z = (z_1,0), z_1 \rightarrow 0 \).

To answer our question for an arbitrary maximal weight \( \varphi \), we consider an extremal problem whose solution \( d_{\varphi} \) gives the best upper bound for functions \( u \) satisfying \( \nu(u,\varphi) \geq c \):

\[(1 + \epsilon) c_\epsilon \log |g_\epsilon| \leq \varphi \leq (1 - \epsilon) c_\epsilon \log |g_\epsilon| \text{ near } x
\]

with \( g_\epsilon \) a holomorphic mapping, \( g_\epsilon(x) = 0 \), and \( c_\epsilon > 0 \). The class of such weights is quite large; actually, we have no example of a maximal weight that does not have asymptotically analytic singularity. Given such a weight \( \varphi \), the function \( d_{\varphi} \) can be constructed by means of plurisubharmonic functions \( \phi_i \) generating so called Rees valuations, i.e., generic multiplicities of pullbacks of analytic functions on exceptional primes \( E_i \) of log resolutions. More precisely, \( d_{\varphi} \) is presented as the (regularized) upper envelope of the family of largest plurisubharmonic
minorants for the functions \( \min_i \gamma_i \phi_i \), where \( \gamma_i > 0 \) are such that \( \sum m_i \gamma_i = 1 \) for certain \( m_i \), determined by the weight \( \varphi \). As a consequence, it is continuous (as a map to \( \mathbb{R} \cup \{-\infty\} \)) and has asymptotically analytic singularity.

When \( \varphi \) has homogeneous (in logarithmic coordinates) singularity, the asymptotic of \( d_\varphi \) can be determined explicitly; this reflects, in particular, the fact that the largest plurisubharmonic minorant for the minimum of homogeneous plurisubharmonic functions can be computed easily. The extremal function is found by means of an integral representation for the generalized Lelong numbers with homogeneous weights [4]. The asymptotic turns out to be simplicial, equivalent to \( \phi_{u,x} \) with \( a \in \mathbb{R}^n_+ \) such that \( a_k = \tau^{-1}\nu(\log |z_k-x_k|, \varphi), 1 \leq k \leq n \).

A weight \( \varphi = \log |g| \) with analytic singularity given by a mapping \( g : \mathbb{C}_0^n \to \mathbb{C}_0^n \) is equivalent to a homogeneous weight if and only if the multiplicity of \( g \) at 0 equals \( n! \) times the covolume of the Newton polyhedron of \( g \) at 0 [15], and the latter holds, by Kouchnirenko’s theorem, for generic mappings \( g \) with given Newton polyhedron. Furthermore, if all the components of \( g \) are monomials, then \( \varphi = \log |g| \) is homogeneous. Thus, the above result on homogeneous singularities solves explicitly the problem on ideals \( J \) satisfying the condition \( e_{n-1}(J, I) \geq p \) for a monomial ideal \( I \).

In addition, the homogeneous case indicates that the functions \( d_\varphi \) form rather a small subclass of Green-like plurisubharmonic functions, containing, of course, the class of weights \( \varphi \) for which (1.2) holds true for every plurisubharmonic function \( u \) (and thus \( d_\varphi = \varphi + O(1) \)); we call such functions flat. In particular, a homogeneous weight is shown to be flat if and only if it has simplicial asymptotic. So, for homogeneous weights, the functions \( d_\varphi \) are precisely flat weight. It is plausible to conjecture that this holds in general situation as well, however we were not able to prove it.

We characterize flat weights \( \varphi \) by the relation
\[
\nu(\max_j u_j, \varphi) = \min_j \nu(u_j, \varphi)
\]
for all plurisubharmonic functions \( u_j \). The proof is based on the following result that has independent interest: If a plurisubharmonic function \( u \) dominates maximal weight \( \varphi \) and \( \nu(u, \varphi) = \nu(\varphi, \varphi) \), then \( u = \varphi + O(1) \). A sufficient "inner" condition for a weight to be flat is also given. In addition, flat weights are used to get upper bounds for \( d_\varphi \) in the general case.

The paper is organized as follows. The next section contains basic facts on generalized Lelong numbers and Green functions. In Section 3 we construct the extremal function \( d_\varphi \) as the upper envelope of negative plurisubharmonic functions \( u \) with given value of \( \nu(u, \varphi) \). This function is computed in Section 4 for the case of homogeneous singularity, and in Section 5 it is described for weights with asymptotically analytic singularities. In Section 6 we study the class of flat weights. Finally, in the last section we address a few open questions.
2 Preliminaries

Given a domain $\Omega \subset \mathbb{C}^n$, let $PSH(\Omega)$ denote the class of all plurisubharmonic functions on $\Omega$ and $PSH^-(\Omega)$ be its subclass consisting of the nonpositive functions. If $x \in \mathbb{C}^n$, $PSH_x$ will mean the collection of all germs of plurisubharmonic functions near $x$.

We recall that a function $u \in PSH_x$ is called maximal on $\Omega$ if for any $v \in PSH_x$ the relation $\{v > u\} \subset \Omega$ implies $v \leq u$ on $\Omega$. A locally bounded $u$ is maximal on $\Omega$ if and only if $(dd^c u)^n \equiv 0$ there.

2.1 Lelong numbers and generalizations

The Lelong number of $u \in PSH_x$ at the point $x \in \mathbb{C}^n$ is

$$\nu_u(x) = \lim_{r \to 0} \int_{|z-x|<r} dd^c u \wedge (dd^c \log |z-x|)^{n-1};$$

here $d = \partial + \bar{\partial}$, $d^c = (\partial - \bar{\partial})/2\pi i$. It can also be calculated as

$$\nu_u(x) = \liminf_{z \to x} \frac{u(z)}{\log |z-x|} \tag{2.1}$$

and

$$\nu_u(x) = \lim_{r \to -\infty} r^{-1} \lambda_u(x,r), \tag{2.2}$$

where $\lambda_u(x,r)$ is the mean value of $u$ over the sphere $|z-x| = e^r$.

A more detailed information on the behavior of $u$ near $x$ can be obtained by means of the directional Lelong numbers due to Kiselman [11]

$$\nu_u(x,a) = \liminf_{z \to x} \frac{u(z)}{\phi_{a,x}(z)} = \lim_{r \to -\infty} r^{-1} \lambda_u(x,ra),$$

where $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$, the function $\phi_{a,x}$ is defined by (1.1), and $\lambda_u(x,ra)$ is the mean value of $u$ over the set $\{z : |z_k - x_k| = e^{ra_k}, 1 \leq k \leq n\}$. In particular, $\nu_u(x) = \nu_u(x, (1, \ldots, 1))$.

A general notion of the Lelong number with respect to a plurisubharmonic weight was introduced and studied by J.-P. Demailly [4], [7]. Let $\varphi$ be a continuous plurisubharmonic function near $x \in \mathbb{C}^n$, locally bounded outside $x$, and $\varphi^{-1}(-\infty) = \{x\}$; we can assume $\varphi \in PSH(\mathbb{C}^n) \cap L_\text{loc}^\infty(\mathbb{C}^n \setminus \{x\})$. Such functions are called weights (centered at $x$), and the collection of the weights will be denoted by $W_x$. A weight $\varphi \in W_x$ is be called maximal if it is a maximal plurisubharmonic function on a punctured neighborhood of $x$ (i.e., satisfies $(dd^c \varphi)^n = 0$ outside $x$). We denote the class of all maximal weights centered at $x$ by $MW_x$; when we want to specify that $\varphi$ is maximal on $\omega \setminus \{x\}$, we write $\varphi \in MW_x(\omega)$. 

4
Denote $B_r^\varphi = \{ z : \varphi(z) < r \}$. The value
\[ \nu(u, \varphi) = \lim_{r \to -\infty} \int_{B_r^\varphi} dd^c u \wedge (dd^c \varphi)^{n-1} = \int_{B_r^\varphi} dd^c u \wedge (dd^c \varphi)^{n-1}(\{ x \}) \] (2.3)
is called the generalized Lelong number, or the Lelong–Demailly number, of $u$ with respect to the weight $\varphi$.

We list here some basic tools for dealing with the generalized Lelong numbers with respect to weights $\varphi \in W_x$, see details in [7].

**Theorem 1** If $u_k \to u$ in $L^1_{\mathrm{loc}}$, then $\limsup_{k \to \infty} \nu(u_k, \varphi) \leq \nu(u, \varphi)$.

**Theorem 2** If $\limsup \frac{\psi(z)}{\psi_0(z)} = l < \infty$ as $z \to x$, then $\nu(\psi, \varphi) \leq ln^{-1} \nu(u, \varphi)$.

**Theorem 3** If $\limsup \varphi_1(z) \varphi_2(z) = l < \infty$ as $z \to x$, then $\nu(u, \varphi_1) \leq ln^{-1} \nu(u, \varphi_2)$.

Denote $\varphi_r = \max(\varphi, r)$; the measure $\mu_{\varphi_r} = (dd^c \varphi_r)^n - \chi_r(dd^c \varphi)^n$ on the pseudosphere $S_{\varphi_r} = \{ z : \varphi(z) = r \}$ is called the swept out Monge-Ampère measure for $\varphi$; here $\chi_r$ is the characteristic function of the set $\mathbb{C}^n \setminus B_r^\varphi$. Any $u \in PSH(B_R^\varphi)$ is $\mu_{\varphi_r}$-integrable for $r < R$, and satisfies the following relation, which we will call the Lelong–Jensen–Demailly formula,
\[ \mu_{\varphi_r}(u) - \int_{B_r^\varphi} u(dd^c \varphi)^n = \int_{-\infty}^r \int_{B_r^\varphi} dd^c u \wedge (dd^c \varphi)^{n-1}. \] (2.4)

If $\varphi \in MW_x(B_R^\varphi \setminus \{ x \})$, then the function $r \mapsto \mu_{\varphi_r}(u)$ is convex on $(-\infty, R)$ and
\[ \nu(u, \varphi) = \lim_{r \to -\infty} r^{-1} \mu_{\varphi_r}(u), \] (2.5)
which is an analogue to formula (2.2). In particular, $\nu_u(x) = \nu(u, \log | \cdot - x |)$ and
\[ \nu_u(x, a) = a_1 \ldots a_n \nu(u, \phi_{a,x}) \] (2.6)
with the weights $\phi_{a,x}$ defined by (1.1).

Yet another generalization of the notion of Lelong number, developing its presentation (2.1), are relative types introduced in [15]. For any function $u \in PSH_x$, we denote its type relative to a weight $\varphi \in MW_x$ as
\[ \sigma(u, \varphi) = \liminf_{z \to x} \frac{u(z)}{\varphi(z)}. \] (2.7)
Maximality of $\varphi$ implies the bound
\[ u \leq \sigma(u, \varphi) \varphi + O(1). \] (2.8)

The types have properties similar to those given in Theorems 1, 2 and are related to the Lelong-Demailly numbers by the inequality
\[ \nu(u, \varphi) \geq \tau_\varphi \sigma(u, \varphi), \] (2.9)
where $\tau_\varphi = \nu(\varphi, \varphi)$, which follows from Theorem 2.
2.2 Almost homogeneous weights

Let a nonpositive plurisubharmonic function $\Phi$ in the unit polydisk $D^n$ satisfy the relation

$$\Phi(z) = \Phi(|z_1|, \ldots, |z_n|) = c^{-1}\Phi(|z_1|^c, \ldots, |z_n|^c) \quad \forall c > 0. \quad (2.10)$$

It is a continuous function in $D^n$, and $(dd^c\Phi)^n = 0$ on $\{\Phi > -\infty\}$. Such functions arise as plurisubharmonic characteristics for local behavior of plurisubharmonic functions near their singularity points. Namely, given a plurisubharmonic function $v$, its (local) indicator at a point $x$ is a plurisubharmonic function $\Psi_{v,x}$ in $D^n$ such that for any $y \in D^n$ with $y_1 \cdots y_n \neq 0$,

$$\Psi_{v,x}(y) = -\nu_v(x, a), \quad a = -(\log |y_1|, \ldots, \log |y_n|) \in \mathbb{R}^n. \quad (2.11)$$

This function satisfies (2.10) and is the largest nonpositive plurisubharmonic function in $D^n$ whose directional Lelong numbers at 0 coincide with those of $v$ at $x$, so

$$v(z) \leq \Psi_{v,x}(z - x) + O(1) \quad (2.12)$$
near $x$, see the details in [12], [13].

A function $\varphi \in PSH(\Omega)$ with $\varphi^{-1}(-\infty) = x \in \Omega$ is said to be almost homogeneous at $x$ if it is asymptotically equivalent to its indicator $\Psi_{\varphi,x}$ [14], that is,

$$\exists \lim_{z \to x} \frac{\varphi(z)}{\Psi_{\varphi,x}(z - x)} = 1. \quad (2.13)$$

A weight $\varphi = \log |g|$ generated by a holomorphic mapping $g$ with isolated zero at $x$ was proved in [15] to be almost homogeneous if and only if the multiplicity of $g$ at $x$ equals the Monge-Ampère mass of $(dd^c\Psi_{\log |g|,x})^n$ of the indicator of $\log |g|$.

Theorem 3 reduces computation of the generalized Lelong–Demailly numbers with respect to almost homogeneous weights $\varphi$ to those with respect to the homogeneous weights $\Psi_{\varphi,x}$; moreover, as shown in [14], $\nu(u, \varphi) = \nu(\Psi_{u,x}, \Psi_{\varphi,0})$.

The swept out Monge-Ampère measure for homogeneous weights can be determined by the following procedure, see [14]. A function $\Phi \in PSH^-(\mathbb{D}^n)$ satisfying (2.10) generates the function $f(t) := \Phi(e^{t_1}, \ldots, e^{t_n})$, convex and positive homogeneous in $\mathbb{R}^n_+$. Given a subset $F$ of the convex set

$$L^\Phi = \{t \in \mathbb{R}^n_+ : f(t) \leq -1\}, \quad (2.14)$$

we put

$$\Theta^\Phi_F = \{\lambda b : 0 \leq \lambda \leq 1, \ b \in \mathbb{R}^n_+, \ \sup_{t \in F} \langle b, t \rangle = \sup_{t \in L^\Phi} \langle b, t \rangle = -1\}. \quad (2.15)$$

A measure $\gamma^\Phi$ on $L^\Phi$ is defined as

$$\gamma^\Phi(F) = n! \text{Vol } \Theta^\Phi_F, \quad F \subset L^\Phi. \quad (2.16)$$
Theorem 4 \[14\] Let \( \varphi \) be an almost homogeneous weight centered at \( x \). Then for any \( u \in \text{PSH}_x \),

\[
\nu(u, \varphi) = \int_{E^\Phi} \nu_u(x, -t) \, d\gamma^\Phi(t)
\] (2.17)

where the measure \( \gamma^\Phi \) on the set \( E^\Phi \) of extreme points of the set \( L^\Phi \) (2.14) is defined by (2.16) and (2.15) with \( \Phi = \Psi_{\varphi,x} \).

2.3 Asymptotically analytic weights

Let \( \phi \in \text{W}_x \) have analytic singularity, i.e., \( \phi = c \log |F| + O(1) \) near \( x \), where \( c > 0 \) and \( F \) is a holomorphic mapping of a neighbourhood of \( x \) to \( \mathbb{C}^N \) with isolated zero at \( x \). As is known, the integral closure of the ideal generated by the components \( F_j \) of \( F \) has precisely \( n \) generators \( \xi_k = \sum a_{k,j} F_j \) (generic linear combinations of \( F_j \), \( k = 1, \ldots, n \), so \( \phi = c \log |\xi| + O(1) \). By Theorem 3, we can then assume \( \phi = c \log |\xi| \) and consider \( F \) to be equidimensional and so, \( \phi \in \text{MW}_x \), which we will tacitly do in the sequel. The collection of all weights with analytic singularities at \( x \) will be denoted by \( \text{AW}_x \).

As follows from Demailly’s approximation theorem [6], any \( \phi \in \text{W}_x \) can be approximated by weights \( \phi_k \in \text{AW}_x \) in a neighborhood \( D \) of \( x \) such that

\[
\phi(z) - \frac{C}{k} \leq \phi_k(z) \leq \sup_{|\zeta - z| < r} \phi(\zeta) + \frac{1}{k} \log \frac{C}{r^n}, \quad z \in U. \tag{2.18}
\]

Specifically,

\[
\phi_k = \frac{1}{k} \sup \{ \log |f| : f \in \mathcal{O}(D), \int_D |f|^2 e^{-2k\phi} \, dV < 1 \} = \frac{2}{k} \log \sum_i |f_{k,i}|^2,
\]

where \( \{f_{k,i}\}_i \) is an orthonormal basis for the Hilbert space

\[
\mathcal{H}(k\phi) = \{ \log |f| : f \in \mathcal{O}(D), |f| e^{-k\phi} \in L^2(D) \}. \tag{2.19}
\]

A weight \( \psi \in \text{W}_x \) will be called asymptotically analytic (\( \psi \in \text{AAW}_x \)) if for every \( \epsilon > 0 \) there exists a weight \( \phi_\epsilon \in \text{AW}_x \) such that

\[
(1 + \epsilon)\phi_\epsilon + O(1) \leq \psi \leq (1 - \epsilon)\phi_\epsilon + O(1); \tag{2.20}
\]

by Theorem 3 we get then

\[
(1 - \epsilon)^{n-1} \nu(u, \phi_\epsilon) \leq \nu(u, \psi) \leq (1 + \epsilon)^{n-1} \nu(u, \phi_\epsilon). \tag{2.21}
\]

According to [2], a weight \( \varphi \in \text{W}_x \) is called tame if there exists a constant \( C > 0 \) such that for every \( t > C \) and every analytic germ \( f \) from the multiplier ideal \( \mathcal{J}(t\varphi) \) of \( t\varphi \) at \( x \) (that is, the function \( fe^{-t\varphi} \) is \( L^2 \)-integrable near \( x \), one has \( \log |f| \leq (t - C)\varphi + O(1) \).
maximal weights $\varphi$, the latter can be written as $\sigma(\log |f|, \varphi) \geq t - C$, where $\sigma$ is the relative type (2.7). Let $\varphi$ be tame, and let $\varphi_k$ be Demailly’s approximations of $\varphi$. By [2] Lemma 5.6, they satisfy

$$\varphi + O(1) \leq \varphi_k \leq (1 - C_{\varphi}/k)\varphi + O(1)$$

(2.22)

near $x$ and therefore $\varphi$ has asymptotically analytic singularity. Moreover, conditions (2.22) characterize tame weights.

One of the main results of [2] is the following integral representation for the Lelong–Demailly numbers with respect to tame weights, which is an extension of the representation (2.17):

$$\nu(u, \varphi) = -\int_{\mathcal{V}} g_u M\Lambda(g_{\varphi})$$

(2.23)

where $g_u$ and $g_{\varphi}$ are certain formal plurisubharmonic functions on the space $\mathcal{V}$ of all centered normalized valuations and $M\Lambda(g_{\varphi})$ is a positive measure on $\mathcal{V}$. We refer to [2] for precise definitions.

We will also need the following simple result on tame weights.

**Proposition 1** If $\phi$ and $\psi$ are tame maximal weights and $\alpha, \beta \in \mathbb{R}_+$, then the greatest plurisubharmonic minorant $\varphi$ of the function $h = \min\{\alpha\phi, \beta\psi\}$ is a tame maximal weight and the constant $C_{\varphi}$ in (2.22) is independent of $\alpha$ and $\beta$.

**Proof.** Maximality of $\varphi$ outside $x$ is evident. Let $\varphi_k, \phi_k, \psi_k$ denote Demailly’s approximating weights for $\varphi, \phi, \psi$, respectively. Then

$$\varphi + O(1) \leq \varphi_k \leq \alpha\phi_k \leq (1 - C_\phi/k)\alpha\phi + O(1)$$

and similarly with $\beta\psi$, so $\varphi_k \leq (1 - C_\phi/k)h + O(1)$ with $C_\varphi = \max\{C_\phi, C_\psi\}$, which implies

$$\varphi + O(1) \leq \varphi_k \leq (1 - C_\varphi/k)\varphi + O(1)$$

and thus the tameness of $\varphi$. \qed

### 2.4 Pluricomplex Green functions

We will use the following extremal function introduced (for the case of continuous singularity) by V. Zahariuta [17] (see also [18]); for the general case, see [15]. Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Given a plurisubharmonic function $\varphi$, locally bounded and maximal outside $x \in \Omega$, let

$$G_\varphi(z) = G_{\varphi,\Omega}(z) = \sup\{u(z) : u \in PSH^{-}(\Omega), u \leq \varphi + O(1) \text{ near } x\}.$$  

(2.24)
This function is plurisubharmonic in $Ω$, maximal in $Ω \setminus x$, $G_{ϕ,Ω} = ϕ + O(1)$ near $x$, and $G_{ϕ,Ω}(z) \to 0$ as $z \to ∂Ω$; moreover, it is a unique plurisubharmonic functions with these properties. Furthermore, if $ϕ$ is continuous (and so, $ϕ ∈ MW_x$), $G_{ϕ,Ω}$ is continuous on $Ω \setminus x$.

We will refer to this function as the Green (or Green–Zahariuta) function with singularity $ϕ$.

If $ϕ(z) = \log |z - x|$, then $G_{ϕ,Ω}$ is just the standard pluricomplex Green function $G_{x,Ω}$ of $Ω$ with pole at $x$.

When $ϕ$ is the indicator of a plurisubharmonic function $v$, the corresponding Green–Zahariuta function can be alternatively described as the upper envelope of all nonpositive plurisubharmonic functions $u$ in $Ω$ such that $ν_u(x,a) ≥ ν_v(x,a)$ for all $a ∈ R^n_+$ [12].

Since any analytic weight is equivalent to a maximal analytic weight (see Section 2.3), the Green functions $G_{ϕ_k}$ for Demailly’s approximations $ϕ_k$ of a weight $ϕ ∈ MW_x$ are well defined, too. If $ϕ$ is a tame weight, then (2.22) implies

$$G_ϕ ≤ G_{ϕ_k} ≤ (1 - C_ϕ/k)G_ϕ.$$  (2.25)

### 3 Extremal functions for Lelong–Demailly numbers

Given a function $u ∈ PSH_x$, it is convenient to consider its normalized Lelong–Demailly numbers with respect to weights $ϕ ∈ W_x$,

$$\tilde{ν}(u,ϕ) = τ_ϕ^{-1}ν(u,ϕ),$$

where

$$τ_ϕ := ν(ϕ,ϕ) = (dd^cϕ)^n(\{x\}) > 0$$

is the residual Monge-Ampère mass of $ϕ$. We have, in particular, $\tilde{ν}(ϕ,ϕ) = 1$ and $\tilde{ν}(cu,ϕ) = \tilde{ν}(u,ϕ)$ for all $c > 0$.

We will be concerned with upper bounds of functions $u$ in terms of $\tilde{ν}(u,ϕ)$. To this end, it looks reasonable to fix a bounded hyperconvex neighbourhood $Ω$ of $x$ and consider the upper envelope of the class

$$N_{ϕ,Ω}^0 = \{u ∈ PSH^-(Ω) : \tilde{ν}(u,ϕ) ≥ 1\}.$$  

Note however that it need not be closed under the operation $(u,v) ↦ \max \{u,v\}$. Indeed, as follows from Theorem [2],

$$ν(\max \{u,v\},ϕ) ≤ \min \{ν(u,ϕ),ν(v,ϕ)\},$$

and the inequality can be strict.

**Example 2** Take the weight $ϕ = \max\{3 \log |z_1|, 3 \log |z_2|, \log |z_1z_2|\}$ in $D^2$; we have then $τ_ϕ = 6$. The functions $u_j(z) = 2 \log |z_j|$ satisfy $\tilde{ν}(u_j,ϕ) = 1$, while $\tilde{ν}(\max_j u_j,ϕ) = 2τ_ϕ^{-1}ν_ϕ(0) = 2/3 < 1.$
Furthermore, we would like to work also with plurisubharmonic functions whose definition domains are proper subsets of Ω. That is why we introduce the class

\[ N_{\varphi, \Omega} = \bigcup_{k=1}^{\infty} N_{\varphi, \Omega}^k, \]

where for \( k \geq 1, \)

\[ N_{\varphi, \Omega}^k = \{ u \in PSH^-(\Omega) : u \leq \max_{1 \leq j \leq k} u_j \text{ in } \omega \ni x, u_j \in N_{\varphi, \omega}^0, \omega \subset \Omega \}. \]

Observe that the set \( N_{\varphi, \Omega} \) is convex, because if \( u_i = \max_j u_{ij}, i = 1, 2, \) then

\[ \alpha u_1 + (1 - \alpha)u_2 = \max_{j,l} \{ \alpha u_{1j} + (1 - \alpha)u_{2l} \}. \]

**Definition 1** Given a weight \( \varphi \in W_x, x \in \Omega, \) let

\[ d_{\varphi, \Omega}(z) = \limsup_{y \to z} \sup \{ u(y) : u \in N_{\varphi, \Omega} \}, \quad z \in \Omega. \quad (3.1) \]

Note that the function \( d_{\varphi, \Omega} \) need not belong to \( N_{\varphi, \Omega} \); this is the main difference with the construction of "usual" pluricomplex Green functions.

**Proposition 2** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n, \) and let \( \varphi \in W_x, x \in \Omega. \) Then

(i) \( d_{\varphi, \Omega} \in PSH^-(\Omega) \cap L_{loc}^\infty(\Omega \setminus \{x\}) \) and is maximal in \( \Omega \setminus \{x\}; \)

(ii) \( d_{\varphi, \Omega}(z) \to 0 \) as \( z \to \partial \Omega; \)

(iii) \( \nu(d_{\varphi, \Omega}, \phi) = \inf \{ \nu(u, \phi) : u \in N_{\varphi, \Omega} \} \) for any weight \( \phi \in W_x; \)

(iv) \( u \leq \tilde{\nu}(u, \varphi)d_{\varphi, \Omega} \) in \( \Omega \) for all \( u \in PSH^-(\Omega), \) and \( u \leq \tilde{\nu}(u, \varphi)d_{\varphi, \Omega} + O(1) \) near \( x \) for every \( u \in PSH_x. \)

**Proof.** By the Choquet lemma, there is a sequence \( u_j \in N_{\varphi, \Omega} \) increasing to a function \( h^* = d_{\varphi, \Omega}. \) Since \( \varphi - \sup_{\Omega} \varphi \in N_{\varphi, \Omega}, \) we can assume \( u_j \in L_{loc}^\infty(\Omega \setminus \{x\}) \) for all \( j. \) Given a ball \( B \Subset \Omega \setminus \{x\}, \) consider the functions

\[ v_j(z) = \sup \{ u(z) \in PSH^-(\Omega) : u \leq u_j \text{ in } \Omega \setminus B \}. \]

Then \( v_j \in N_{\varphi, \Omega} \) and satisfy \( (dd^c v_j)^n = 0 \) in \( B. \) Since \( v_j \geq u_j, \) the functions \( v_j \) increase a.e. to \( d_{\varphi, \Omega} \) and so, \( (dd^c d_{\varphi, \Omega})^n = 0 \) in \( B. \) This proves (i).

Assertion (ii) holds because the Green function \( G_\varphi \) belongs to \( N_{\varphi, \Omega}. \)
By Theorem 2, $\nu(d_{\phi,\Omega}, \phi) \leq \inf \{ \nu(u, \phi) : u \in N_{\phi,\Omega} \}$ for every weight $\phi \in W_x$. On the other hand, the above functions $u_j \in N_{\phi,\Omega}$ converge to $d_{\phi,\Omega}$ in $L^1_{\text{loc}}$ (the monotone convergence theorem), and Theorem 1 gives $\nu(d_{\phi,\Omega}, \phi) \geq \limsup \nu(u_j, \phi)$, which proves (iii).

The first relation in (iv) is obvious, and the second one follows from the fact that for any $u \in PSH_x$ with $\tilde{\nu}(u, \phi) \geq 1$, the function $\max \{ u, \phi \}$ can be extended from a neighbourhood of $x$ to $\Omega$ as a bounded above plurisubharmonic function. □

Corollary 1 If $\nu(u, \varphi) \geq c > 0$, then $u(z) \leq c \tau_{\varphi}^{-1} d_{\varphi,\Omega}(z) + O(1)$ as $z \to x$ for any bounded hyperconvex neighbourhood $\Omega$ of $x = \varphi^{-1}(-\infty)$.

For almost homogeneous weights $\varphi$, asymptotics of the extremal functions $d_{\varphi,\Omega}$ can be computed explicitly and they turn out to be simplicial (see the next section). In the general case, it is then likely that for the functions $d_{\varphi,\Omega}$ form rather a small subclass of Green-like functions as well; it would be nice to get a description of such functions.

The function $d_{\varphi,\Omega}$ can be estimated by means of flat weights defined as follows. Let $\sigma(u, \phi)$ denote the type of $u$ relative to $\phi$, see Section 2.1.

Definition 2 A maximal weight $\phi \in MW_x$ is flat if $\tilde{\nu}(u, \phi) = \sigma(u, \phi)$ for any function $u \in PSH_x$.

As follows from (2.6), the directional weights $\phi_{a,x}$ are flat. More properties of flat weights will be given in Section 6.

Evidently, $d_{\phi,\Omega} = G_{\phi,\Omega}$ (the Green–Zahariuta function) for any flat weight $\phi$. This gives the following simple bound.

Proposition 3 If $\varphi \in W_x$, $x \in \Omega$, then $d_{\varphi,\Omega} \leq \tau_{\varphi}^{-1} \tau_{\phi} G_{\phi,\Omega}$ and, consequently, $\tilde{\nu}(d_{\varphi,\Omega}, \varphi) \geq \tau_{\varphi}^{-1} \tau_{\phi}$ for every flat weight $\phi$ satisfying $\phi \leq \varphi + O(1)$ near $x$.

Proof. Let $\tilde{\nu}(u, \varphi) \geq 1$. By Theorem 3 we have $\nu(u, \phi) \geq \nu(u, \varphi) \geq \tau_{\phi}$ for any flat weight $\phi$ satisfying $\phi \leq \varphi + O(1)$. Therefore $\sigma(u, G_{\phi,\Omega}) = \sigma(u, \phi) = \tilde{\nu}(u, \phi) \geq \tau_{\phi}^{-1} \tau_{\varphi}$, which implies the statements in view of (2.8) and Theorem 2. □

Corollary 2 If $\varphi \in W_x$ is such that $\varphi \geq N \log |z - x|$ near $x$ for some $N > 0$, then $d_{\varphi,\Omega} \leq N^{1-n} \tau_{\varphi} G_{x,\Omega}$, where $G_{x,\Omega}$ is the pluricomplex Green function of $\Omega$ with logarithmic pole at $x$.

Remarks. 1. If $\varphi \in AW_x$, $\varphi = \log |F| + O(1)$ for a holomorphic mapping $F$ with isolated zero of multiplicity $m$ at $x$, then Corollary gives the bound $d_{\varphi,\Omega} \leq L^{1-n} m G_{x,\Omega}$, where $L > 0$
is the Lojasiewicz exponent of $F$ at $x$, i.e., the infimum of $\gamma > 0$ such that $|F(z)| \geq |z - x|^{\gamma}$ near $x$.

2. By analogy with the analytic case, we will call the value

$$L_\varphi = \limsup_{z \to x} \frac{\varphi(z)}{\log |z - x|}$$

(3.2)

the Lojasiewicz exponent of the weight $\varphi$. Corollary 2 implies $d_{\varphi, \Omega} \neq 0$ for $\varphi$ with finite $L_\varphi$. It is easy to construct weights with infinite Lojasiewicz exponent, however we have no examples of maximal weights $\varphi$ with $L_\varphi = \infty$. Moreover, we do not know if there exists weights $\varphi \in W_x$ such that $d_\varphi \equiv 0$.

3. Another upper bound for weights with finite Lojasiewicz exponent will be given in Corollary 3.

4 The case of almost homogeneous weights

Here we will show that if $\varphi \in W_x$ is an almost homogeneous weight (2.13), then the function $d_{\varphi, \Omega}$ (3.1) is flat. Moreover, it has a simplicial asymptotic $d_{\varphi, \Omega} = \phi_{a,x} + O(1)$ with $a \in \mathbb{R}^n_+$ determined explicitly.

Let us assume $x = 0$. Given a weight $\varphi \in W_0$, we set

$$a_k = \tilde{\nu}(\log |z_k|, \varphi), \quad 1 \leq k \leq n.$$  

(4.1)

**Theorem 5** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, $0 \in \Omega$, and let $\varphi$ be an almost homogeneous plurisubharmonic weight centered at 0. Then $d_{\varphi, \Omega}$ equals the Green–Zahariuta function for the singularity $\phi_{a,0}$ (1.1) in $\Omega$ with $a \in \mathbb{R}^n_+$ defined by (4.1).

**Proof.** Take any $u \in PSH_0$ with $\tilde{\nu}(u, \varphi) \geq 1$; by Theorem 4

$$\tilde{\nu}(u, \varphi) = \int_{E^\Phi} \nu_u(0, -t) \, d\gamma^\Phi(t),$$

where $\gamma^\Phi = \tau_{\varphi}^{-1} \gamma^\Phi$. By the definition of the indicator, $\nu_u(0, -t) = -\psi(t)$, where $\psi(t) = \Psi_u(e^{t_1}, \ldots, e^{t_n})$. Therefore, the condition $\tilde{\nu}(u, \varphi) \geq 1$ implies

$$\int_{E^\Phi} \psi(t) \, d\gamma^\Phi \leq -1.$$

The function $\psi$ is the restriction to $\mathbb{R}^n_+$ of the supporting function of a convex subset $S$ of $\mathbb{R}^n_+$: $\psi(t) = \sup \{ \langle b, t \rangle : b \in S \}$. This gives us

$$-1 \geq \int_{E^\Phi} \sup_{b \in S} \langle b, t \rangle \, d\gamma^\Phi \geq \sup_{b \in S} \langle b, \int_{E^\Phi} t \, d\gamma^\Phi \rangle = \sup_{b \in S} \langle b, \mu \rangle,$$

12
where the vector $\mu \in \mathbb{R}^n_-$ has the components

$$
\mu_k = \int_{E^c} t_k d\gamma, \quad k = 1, \ldots, n.
$$

Therefore the set $S$ lies in the half-space $\Pi_{\mu} = \{ b : \langle b, \mu \rangle \leq -1 \}$ and so, $\psi(t)$ is dominated by the supporting function $\psi_\mu$ of the set $\Pi_{\mu} \cap \mathbb{R}^n_+$. It is easy to see that $\psi_\mu(t) = \max_k t_k/|\mu_k|.

As follows from Theorem 4 applied to the functions $\log |z_k|$, $\mu_k = -a_k$ with $a_k$ defined by (4.1), and thus $\psi(t) \leq \max_k t_k/a_k$. This means that $\Psi_{a,0} \leq \phi_{a,0}$.

Let now $v \in N_{\phi,\Omega}$ have the form $v = \max_j u_j$ near 0 and $\tilde{\nu}(u_j, \varphi) \geq 1$. Then $\Psi_{v,x} = \max_j \Psi_{u_j,x} \leq \phi_{a,0}$. Therefore, in view of (2.12), $v$ is dominated by the Green–Zahariuta function $G$ for the singularity $\phi_{a,0}$, so $d_{\varphi,\Omega} \leq G$.

Finally, the relations $\tilde{\nu}(a_k^{-1} \log |z_k|, \varphi) = 1$ imply $d_{\varphi,\Omega} \geq \phi_{a,0} + C$ with some constant $C$. Since $G = \phi_{a,0} + O(1)$ near 0, the equality $d_{\varphi,\Omega} = G$ follows from the uniqueness property for the Green–Zahariuta functions, which completes the proof. \hfill \Box

This can be used for estimation of the functions $d_{\varphi}$ for arbitrary weights $\varphi \in W_0$ with finite Lojasiewicz exponent (3.2). It is easy to see that such weights are characterized by the property

$$
l := \limsup_{z \to 0} \frac{\varphi(z)}{\Psi_{\varphi,0}(z)} < \infty,
$$

just because the indicator of every weight has finite Lojasiewicz exponent.

**Corollary 3** If $\varphi \in W_0$ satisfies (4.2), then $d_{\varphi,\Omega} \leq l^{1-n} G_a$, where $G_a$ is the Green–Zahariuta function for the singularity $\phi_{a,0}$ in $\Omega$ with $a \in \mathbb{R}^n_+$ defined by (4.1).

**Proof.** Denote $\Psi = \Psi_{\varphi,0}$ and take any $u \in PSH_x$ with $\tilde{\nu}(u, \varphi) \geq 1$. By Theorem 3 condition (4.2) implies the inequality $\nu(u, \Psi) \geq l^{1-n} \tau_{\varphi}$. Therefore, by Theorem 5, $u \leq l^{1-n} \tau_{\varphi}^{-1} \tau_{\Psi}^{-1} \phi_{b,0} + C$ near 0, where $b_k = \tilde{\nu}(\log |z_k|, \Psi)$, $k = 1, \ldots, n$. This implies the statement because $\tau_{\varphi}^{-1} \phi_{b,0} \leq \phi_{a,0}$ with $a$ defined by (4.1). \hfill \Box

In an analytic setting, Theorem 5 gives us the following result on monomial ideals. Given an ideal $\mathcal{J}$ and an $m$-primary ideal $\mathcal{I}$ of $\mathcal{O}_0$, denote by $e(\mathcal{I})$ the Samuel multiplicity of $\mathcal{I}$ and by $e_{n-1}(\mathcal{J}, \mathcal{I})$ the mixed multiplicity, due to Teissier and Risler [16] and Bivià-Ausina [1], of $\mathcal{J}$ and $n-1$ copies of $\mathcal{I}$. Denote, furthermore, $e_k(\mathcal{I}) = e_{n-1}(m_k, \mathcal{I})$ for the principal ideal $m_k$ generated by the function $z_k$.

Let $\mathcal{I}$ be an $m$-primary monomial ideal generated by monomials $g_1, \ldots, g_l$, $l \geq n$, and let $\varphi = \max_j \log |g_j|$; observe that $\varphi$ is an indicator. Then $e(\mathcal{I}) = \tau_{\varphi}$ (because in the monomial case the both values equal $n!$ times the covolume of the Newton polyhedron of the mapping $g$); in the general case, the equality is proved, for example, in [8]). Furthermore, if an ideal $c\mathcal{J}$
is generated by functions \( f_1, \ldots, f_m \), then \( e_{n-1}(\mathcal{J}, I) = \nu(\log |f|, \varphi) \), where \( f = (f_1, \ldots, f_m) \) (this follows from multilinearity of the mixed multiplicities).

The numbers \( a_k \) from (4.1) can now be computed now as

\[
a_k = e_{n-1}(m_k, I)e^{-1}(I),
\]

where \( m_k \) is the principal ideal generated by the function \( z_k \), and Theorem 5 takes the following form.

**Theorem 6** Let \( I \) be an \( m \)-primary monomial ideal in \( O_0 \). If an ideal \( J \) satisfies \( e_{n-1}(\mathcal{J}, I) \geq p \), then \( J \) is contained in the ideal generated by the functions \( z_{p_k} \), where

\[
p_k = \min \{ q \in \mathbb{Z}_+ : p/q \leq e_{n-1}(m_k, I) \}, \quad k = 1, \ldots, n.
\]

5 The case of (asymptotically) analytic weights

Given \( \phi = c \log |F| \in AW_x \), denote by \( a \) the primary ideal generated by \( F_1, \ldots, F_n \). By [2] Prop. 4.10, there exists a simple modification \( \pi : X_\pi \to X \) above a neighbourhood \( X \setminus x \) with a normal crossing exceptional divisor \( \pi^{-1}(x) \), such that \( \pi^{-1}a \) is a principal ideal and the measure \( MA(g_\phi) \) is a finite sum of weighted Dirac measures with masses \( c_i = c^{n-1} I_i \) at the divisorial valuations (Rees valuations) \( \text{ord}_{E_i} \) over the irreducible components \( E_1, \ldots, E_N \) of \( \pi^{-1}(x) \). Therefore, (2.23) takes the form

\[
\tilde{\nu}(u, \phi) = \sum_{1 \leq i \leq N} a_i \nu_i(u), \quad u \in PSH(\Omega),
\]

where \( a_i = \tau_{\phi}^{-1} c_i \) and \( \nu_i(u) \) is the generic Lelong number of the function \( u \circ \pi \) along the set \( E_i \).

By [15] Thm. 4.3], there exist maximal weights \( \phi_i \) such that \( a_i \nu_i(u) = \sigma(u, \phi_i) \) for every \( u \), and in [2] Thm. 5.13] these weights are proved to be continuous and tame. We will call them *elementary representing weights* for the weight \( \phi \). Tameness of these weights implies \( \sigma(\phi_k, \phi_j) > 0 \) for all \( k, j \).

If \( \tilde{\nu}(u, \phi) \geq 1 \), then

\[
\sum_{1 \leq i \leq N} \sigma(u, \phi_i) \geq 1.
\]

This implies existence of positive numbers \( \beta_1, \ldots, \beta_N \) with \( \sum_i \beta_i \geq 1 \), such that \( \sigma(u, \phi_i) \geq \beta_i \) and thus, by (2.28), \( u(z) \leq \beta_i \phi_i(z) + O(1) \) near \( x \). Therefore,

\[
u(z) \leq \phi(\beta) + O(1), \quad z \to x, \tag{5.1}
\]
where $\phi_{(\beta)}$ denotes the greatest plurisubharmonic minorant of the function $\min_i \beta_i \phi_i$. Notice that $\sigma(\phi_{(\beta)}, \phi_i) \geq \beta_i \geq \inf_{k,j} \sigma(\phi_k, \phi_j) > 0$ for all $i$.

Furthermore, let $\beta^0_j$ denote the the least lower bound for $\sigma(u, \phi_j)$ over all $u$ with $\tilde{\nu}(u, \phi) \geq 1$. Then

$$\phi_{(\beta)} \leq \phi_{(\beta^0)}, \quad \beta \in \Pi_1,$$

where $\Pi_1 = \{ \beta \in \mathbb{R}^N : \sum_i \beta_i \geq 1 \}$. Note that $\beta^0$ need not belong to $\Pi_1$.

By Proposition 1, $\phi_{(\beta)}$ is a tame maximal weight and the constant $C_\phi$ in (2.22) for $\phi = \phi_{(\beta)}$ is independent of $\beta$; let us call it $C$. Then the Green–Zahariuta function $G_{(\beta)}$ of $\Omega$ for the singularity $\phi_{(\beta)}$ satisfies

$$G_{(\beta)} \leq G_{(\beta),k} \leq (1 - C/k)G_{(\beta)},$$

where $G_{(\beta),k}$ are the Green functions for Demailly’s approximations of $G_{(\beta)}$.

Any function $u \in N_{\phi,\Omega}$ satisfies $u \leq \max_j u_j$ for finitely many functions $u_j$ with $\tilde{\nu}(u_j, \phi) \geq 1$. Since $u$ is negative in $\Omega$, (5.1) applied to each $u_j$ implies the relation $u \leq \max \{ G_{(\beta)} : \beta \in B_u \}$, where $B_u$ is a finite subset of the half-space $\Pi_1$, depending on $u$. Therefore, by (5.2),

$$d_{\phi,\Omega}(z) \leq \sup \{ G_{(\beta)}(z) : \beta \in \Pi_1 \} = G_{(\beta^0)}(z).$$

Since for $\beta \in \Pi_1$,

$$\tilde{\nu}(G_{(\beta)}, \phi) = \tilde{\nu}(\phi_{(\beta)}, \phi) = \sum_{1 \leq i \leq N} \sigma(\phi_{(\beta)}, \phi_i) \geq 1$$

and thus $G_{(\beta)} \in N^0_{\phi,\Omega}$, we have actually the equality $d_{\phi,\Omega} = G_{(\beta^0)}$.

In addition, (5.3) with $\beta = \beta^0$ implies

$$d_{\phi,\Omega} \leq d_{\phi,\Omega,k} + O(1) \leq (1 - C/k)d_{\phi,\Omega}$$

and thus its continuity and tameness.

Finally, if $\psi$ is an asymptotically analytic weight, then (2.21) implies

$$(1 + \epsilon)^{n-1} d_{\varphi,\Omega} \leq d_{\psi,\Omega} \leq (1 - \epsilon)^{n-1} d_{\varphi,\Omega},$$

thus $d_{\psi,\Omega}$ is asymptotically analytic (since so are $d_{\varphi,\Omega}$) and

$$d_{\psi,\Omega} = \sup_{\epsilon > 0} (1 + \epsilon)^{n-1} d_{\varphi,\Omega}$$

which gives, in particular, its continuity.

As a result, we have got the following properties of the extremal functions for analytic and asymptotically analytic weights.

**Theorem 7** If $\psi$ is an asymptotically analytic weight and $\Omega$ is a bounded hyperconvex domain, then the function $d_{\psi,\Omega}$ is continuous and has asymptotically analytic singularity. If $\psi$ has analytic singularity, then $d_{\psi,\Omega}$ is tame.
6 Flat weights

Here we consider the situation when the function $d_{\varphi, \Omega}$ has the same asymptotic as $\varphi$. Assuming $\varphi$ to be maximal, it is easy to see that this happens if and only if the weight $\varphi$ is flat (i.e., satisfying $\tilde{\nu}(u, \varphi) = \sigma(u, \varphi)$ for all $u$, see Section 3). We denote the class of all flat maximal weights centered at $x$ by $FW_x$.

As follows from the definition of relative types, flat weights $\varphi$ satisfy the relation $\nu(\max \{u, v\}, \varphi) = \min \{\nu(u, \varphi), \nu(v, \varphi)\}$ for any plurisubharmonic functions $u$ and $v$. We will show that this is a characteristic property of the class of flat weights (see Corollary 5 below). Moreover, it suffices to check it only on $u$ with $\tilde{\nu}(u, \varphi) \geq 1$ and $v = \varphi$ (Corollary 4).

The crucial step in proving the claim is the following extremal property of maximal weights.

**Lemma 1** Let $\varphi \in MW_x(B_R^\varphi)$. If $v \in PSH(B_R^\varphi)$ is such that $v \geq \varphi$ in $B_R^\varphi$, $v(x) \to R$ as $x \to \partial B_R^\varphi$, and $\nu(v, \varphi) = \tau_\varphi$, then $v \equiv \varphi$.

**Proof.** We can assume $R = 0$, then $\varphi = G_{\varphi, D}$ with $D = B_0^\varphi$.

Given $\epsilon > 0$, let $w_\epsilon = \max\{v - \epsilon, \varphi\}$. Since $w_\epsilon = \varphi$ near $\partial D$, we have

$$\int_D dd^c w_\epsilon \wedge (dd^c \varphi)^{n-1} = \int_D (dd^c \varphi)^n = \tau_\varphi. \tag{6.1}$$

The inequality $v \geq \varphi$ implies $\varphi \leq w_\epsilon \leq v$. Since $\nu(w, \varphi) = \nu(v, \varphi) = \tau_\varphi$, this gives the relation $\nu(w_\epsilon, \varphi) = \tau_\varphi$, so

$$\int_D dd^c w_\epsilon \wedge (dd^c \varphi)^{n-1} = \tau_\varphi + \int_{D \setminus \{x\}} dd^c w_\epsilon \wedge (dd^c \varphi)^{n-1}.$$

Comparing this with (6.1), we get

$$dd^c w_\epsilon \wedge (dd^c \varphi)^{n-1} = \tau_\varphi \delta_x. \tag{6.2}$$

By the Lelong–Jensen–Demailly formula (2.4),

$$\mu_\varphi^c(w_\epsilon) = \mu_{r_0}^c(w_\epsilon) + \int_{r_0}^r \int_{B_t^\varphi} dd^c w_\epsilon \wedge (dd^c \varphi)^{n-1} dt, \quad r < r_0 < 0.$$

Choose $r_0 = r_0(\epsilon) < 0$ such that $w_\epsilon = \varphi$ near $\partial B_{r_0}^\varphi$, then $\mu_{r_0}^c(w_\epsilon) = r_0 \tau_\varphi$, while the second term, by (6.2), equals $(r - r_0) \tau_\varphi$. Therefore $\mu_\varphi^c(w_\epsilon) = r \tau_\varphi$, $r < r_0$. By the construction of the function $w_\epsilon$, this means

$$\int \max\{v - \epsilon - r, 0\} d\mu_\varphi^c = 0, \quad r < r_0,$$
which implies
\[
\mu_r^\varphi(\{v - \epsilon > r\}) = 0, \quad r < r_0.
\] (6.3)
By Demailly’s maximum principle [3, Thm. 5.1], sup \{v(z) - \epsilon : \varphi(z) < r\} equals the essential supremum of \(v - \epsilon\) with respect to the measure \(\mu_r^\varphi\), so \(v - \epsilon \leq r\) in \(B_r^\varphi\) for all \(r < r_0\) and \(\sigma(v, \varphi) = \sigma(v - \epsilon, \varphi) \geq 1\). Therefore, \(v \leq G_{\varphi, D} = \varphi\).

Corollary 4
If a weight \(\varphi \in MW_x\) is such that \(\tilde{\nu}(\max \{w, \varphi\}, \varphi) = 1\) for every \(w \in PSH_x\) with \(\tilde{\nu}(w, \varphi) \geq 1\), then \(\varphi \in FW_x\).

Proof. Take any \(w \in PSH_x\) with \(\tilde{\nu}(w, \varphi) \geq 1\) and a real \(R\) such that \(w < 0\) in \(B_R^\varphi\) and \(\varphi \in MW_x(B_R^\varphi)\). Then the function \(v := \max\{w + R, \varphi\}\) satisfies the conditions of Lemma 1, so \(v \equiv \varphi\) and consequently, \(w \leq \varphi - R\) in \(B_R^\varphi\); in other words, \(\sigma(w, \varphi) \geq 1\).

Given now \(u \in PSH_x\) with \(\nu(u, \varphi) > 0\), the function \(w = [\tilde{\nu}(u, \varphi)]^{-1}u\) satisfies \(\tilde{\nu}(w, \varphi) = 1\) and so, as we have just shown, \(\sigma(w, \varphi) \geq 1\), which means \(\tilde{\nu}(u, \varphi) \leq \sigma(u, \varphi)\). In view of (2.9) this completes the proof.

Corollary 5
If \(\varphi \in MW_x\) is such that \(\nu(\max_{j} u_j, \varphi) = \min_j \nu(u_j, \varphi)\) for any plurisubharmonic functions \(u_j, j = 1, 2\), then \(\varphi \in FW_x\).

We can also give some "inner" sufficient conditions for a weight to be flat. Let us assume that there exist numbers \(K \geq 0, \tau > 0\), and a family of plurisubharmonic functions \(\{\varphi_x\}_{x \in \omega}\), where \(\omega\) is a bounded domain in \(\mathbb{C}^n\), such that
\[
\varphi_x \in MW_x(\omega),
\]
(6.4)
\[
\varphi_y(x) \leq K + \max\{\varphi_z(x), \varphi_z(y)\},
\]
(6.5)
\[
\tau_{\varphi_x} = \tau
\]
(6.6)
for all \(x, y, z \in \omega\).

For example, this is the case for a family \(\varphi_x(y) = \varphi(y - x)\) with a weight \(\varphi \in MW_0(\Omega)\) in a neighbourhood \(\Omega\) of 0 in \(\mathbb{C}^n\) that satisfies
\[
\varphi(y - x) \leq K + \max\{\varphi(x), \varphi(y)\}, \quad x, y \in \omega.
\]
(6.7)
Since \(\varphi_x(x) = -\infty\), (6.5) implies, in particular,
\[
\varphi_y(x) \leq K + \varphi_x(y);
\]
(6.8)
[together with (6.5), this means that the function \(\rho(x, y) = \exp(\varphi_y(x))\) is a non-Archimedean quasi-metric on \(\omega\), that is,
\[
\rho(x, x) = 0, \quad \rho(x, y) \leq C \rho(y, x), \quad \rho(x, y) \leq C \max\{\rho(x, z), \rho(y, z)\}.
\]
**Theorem 8** Let a family of weights \( \varphi_x, x \in \omega \), satisfy conditions \((6.4)\)–\((6.6)\). Then \( \varphi_x \in FW_x \).

**Proof.** Fix any \( x \in \omega \); for brevity, we denote \( \varphi_x \) just by \( \varphi \). We can assume \( \tau = \tau_\varphi = 1 \); then, in view of \((2.9)\), it is only the relation \( \sigma(u, \varphi) \geq \nu(u, \varphi) \) to be proved.

Let \( r < 0 \) be such that \( B_r^\varphi = \{ y : \varphi(y) < r \} \subset \omega \) and let \( t < r - 3K \). Fix any \( y \in \omega \) with \( \varphi(y) = t \). Then relation \((6.5)\) implies

\[
\varphi_y(z) \leq K + \max \{ \varphi(z), t \}, \quad z \in B_r^\varphi.
\]  

(6.9)

In particular, for all \( z \) with \( t < \varphi(z) \leq r \) we get \( \varphi_y(z) \leq K + \varphi(z) \). On the other hand, for \( z \) with \( \varphi(z) > 2K + t \) relations \((6.5)\) and \((6.8)\) give

\[
\varphi(z) \leq K + \max \{ \varphi_y(z), \varphi_y(0) \} \leq K + \max \{ \varphi_y(z), K + t \},
\]

which implies \( \varphi_y(z) \geq \varphi(z) - K \). Therefore \( \varphi(z) - K \leq \varphi_y(z) \leq \varphi(z) + K \) near the boundary of \( B_r^\varphi \); in particular,

\[
r - K \leq \varphi_y(z) \leq r + K, \quad z \in S_r^\varphi.
\]  

(6.10)

Furthermore, on a neighbourhood of \( S_t^\varphi \) we have, in view of \((6.9)\),

\[
\varphi_y(z) \leq 2K + t.
\]  

(6.11)

Let \( G \) denote the Green–Zahariuta function of \( B_r^\varphi \) with the singularity \( \varphi_y \). Then \( G(z) \leq \varphi_y(z) + O(1) \) near \( y \) and

\[
(dd^c G)^n = (dd^c \varphi_y)^n = \delta_y.
\]

Since the function \( \varphi_y \) is maximal on \( B_r^\varphi \setminus \{ y \} \), relation \((6.10)\) implies \( G \leq \varphi_y - r + K \) in \( B_r^\varphi \); on \( S_t^\varphi \) we have then, by \((6.11)\), \( G \leq 3K + t - r \). Therefore

\[
G(z) \leq M(\varphi(z) - r), \quad z \in S_t^\varphi,
\]  

(6.12)

where \( M = M(r, t) = (3K + t - r)(t - r)^{-1} \).

We set

\[
\psi(z) = \begin{cases} \varphi(z) - r, & z \in B_r^\varphi \\ \max \{ \varphi(z) - r, M^{-1}G(z) \}, & z \in B_r^\varphi \setminus B_t^\varphi. \end{cases}
\]

Due to \((6.12)\), \( \psi \in PSH^-(B_t^\varphi) \). Then it is dominated by the Green–Zahariuta function for \( B_t^\varphi \) with the singularity \( \varphi \), that is, by \( \varphi - r \), and so,

\[
M^{-1}G(z) \leq \varphi(z) - r, \quad z \in B_t^\varphi \setminus B_t^\varphi.
\]

18
By Demailly’s comparison theorem [5 Thm. 3.8] for the swept out Monge-Ampère measures, this implies the inequality $d\mu^\varphi - r \leq M^{-n}d\mu^G$ and thus, as $d\mu^\varphi - r = d\mu^\varphi_r$,

$$d\mu^\varphi_r \leq M^{-n}d\mu^G.$$  \hspace{1cm} (6.13)

The Lelong–Jensen–Demailly formula (2.4) shows that for a plurisubharmonic in a neighbourhood of $B^\varphi_r$ and every $\tau < 0$,

$$\mu^G(u) \geq \mu^G(\tau) \geq \int_{B^{\varphi G}_r} (dd^cG)^n = u(y).$$

Assuming $u$ negative in $B^\varphi_r$, inequality (6.13) gives then

$$\mu^\varphi_r(u) \geq M^{-n}u(y)$$

and, since the only condition on $y$ is that $\varphi(y) = t$,

$$\sup\{u(z) : z \in B^\varphi_t\} \leq M^n\mu^\varphi_r(u).$$

Now let $t = (1+\epsilon)r$, $\epsilon > 0$, then $M = 1+3K\epsilon^{-1}r^{-1}$ satisfies $M \geq 1-\epsilon$ for all $r < -3K\epsilon^{-2}$, so

$$\sup\{u(z) : z \in B^\varphi_{(1+\epsilon)r}\} \leq (1-\epsilon)^n\mu^\varphi_r(u)$$

and, by the definition of the relative type and (2.5),

$$\sigma(u, \varphi) \geq (1-\epsilon)^n(1+\epsilon)^{-1}\nu(u, \varphi).$$

Letting $\epsilon \to 0$ completes the proof. \hfill \Box

**Example 3** It is easy to see that the weights $\varphi = \phi_{a,0}$ (1.1) satisfy (6.7) with $K = (\min a_k)^{-1}\log 2$ and therefore are flat. In particular, this recovers Kiselman’s result that the directional Lelong number can be calculated by means of both the maximal and mean values.

**Example 4** An almost homogeneous weight $\varphi \in W_x$ is flat if and only if its indicator $\Psi_{\varphi,x}$ is simplicial, which follows from Theorem [5]. This does not hold true without the homogeneity assumption.

**Example 5** Let $f$ be an irreducible holomorphic function on a neighbourhood of $0 \in \mathbb{C}^2$ such that $f(0) = 0$ and $\{f = 0\}$ is transverse to $\{z_1 = 0\}$. By [9 Prop. 3.6 and 3.9], the weight $\varphi = \log \max\{|z_1|^s, |f|\}$ is flat for any $s \geq \text{mult}_0 f$. We do not know if this can be deduced from Theorem [8].

**Example 6** If $F$ is a biholomorphic mapping between neighbourhoods of $x$ and $y$, and if a weight $\varphi \in FW_y$, then the weight $\psi = F^*\varphi \in FW_x$.  

19
7 Open questions

Here we would like to mention a few questions that are, as we believe, quite important in understanding the nature of plurisubharmonic singularities.

1. Do there exist weights \( \varphi \) such that \( d_\varphi \equiv 0 \)?

2. As pointed out in Example 4, solutions to the extremal problem for homogeneous weights are exactly flat weight functions. Is \( d_\varphi \), for any weight \( \varphi \), flat? It would be interesting to have an answer even in the case of weights with analytic singularities.

3. Flatness means some regularity of the weight. What kind of regularity is it? Specifically, are flat weights tame? asymptotically analytic? Do they have finite Lojasiewicz exponent? All known examples of flat weights have analytic or tame singularities.

4. In Section 5 we presented a procedure for constructing the extremal functions for weights with analytic singularities. It rests on finding best plurisubharmonic minorants for certain tame weights, which makes the procedure somewhat implicit. Is it possible to describe explicitly the extremal functions for weights with analytic singularities? One more question, do such extremal functions have analytic singularities?

5. The construction of the extremal function for analytic singularities \( \phi \) is based on elementary representing weights \( \phi_i \) for divisorial (or Rees) valuations. Such valuations play central role in investigation of singularities, as well as in many other problems of algebraic geometry and commutative algebra. Are elementary representing weights analytic? How are the asymptotics of \( \phi_i \) related to the asymptotic of the weight \( \phi \)?

6. What are flat weights with analytic singularities? How can they be explicitly described? So far, the only known flat analytic weights come from Examples 3 and 5, modulo all holomorphic coordinate changes (as noticed in Example 6).

7. It was shown in Corollary 5 that the condition

\[
\nu(\max_i u_i, \varphi) = \min_i \nu(u_i, \varphi)
\]

for all \( u_i \) implies flatness of \( \varphi \). On the other hand, any flat weight \( \varphi \) satisfies

\[
\sigma\left(\sum_i u_i, \varphi\right) = \sum_i \sigma(u_i, \varphi)
\]

Is it a characteristic property for flat weights as well?

Acknowledgement. The author is grateful to the Mathematics Department of Purdue University where part of the work was done.
References

[1] C. Bivià-Ausina, Joint reductions of monomial ideals and multiplicity of complex analytic maps, Math. Res. Lett. 15 (2008), no. 2, 389–407.

[2] S. Boucksom, C. Favre and M. Jonsson, Valuations and plurisubharmonic singularities, Publ. Res. Inst. Math. Sci. 44 (2008), no. 2, 449–494.

[3] J.-P. Demailly, Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines, Mém. Soc. Math. France (N. S.) 19 (1985), 1-124.

[4] J.-P. Demailly, Nombres de Lelong généralisés, théorèmes d’intégralité et d’analyticité, Acta Math. 159 (1987), 153–169.

[5] J.-P. Demailly, Mesures de Monge-Ampère et mesures plurisousharmoniques, Math. Z. 194 (1987), 519–564.

[6] J.-P. Demailly, Regularization of closed positive currents and intersection theory, J. Algebraic Geometry 1 (1992), 361–409.

[7] J.-P. Demailly, Monge-Ampère operators, Lelong numbers and intersection theory, Complex Analysis and Geometry (Univ. Series in Math.), ed. by V. Ancona and A. Silva, Plenum Press, New York 1993, 115–193.

[8] J.-P. Demailly, Estimates on Monge-Ampère operators derived from a local algebra inequality, preprint at [http://arxiv.org/abs/0709.3524v2](http://arxiv.org/abs/0709.3524v2)

[9] C. Favre and M. Jonsson, Valuative analysis of planar plurisubharmonic functions, Invent. Math. 162 (2005), 271–311.

[10] C.O. Kiselman, Densité des fonctions plurisousharmoniques, Bull. Soc. Math. France 107 (1979), 295–304.

[11] C.O. Kiselman, Un nombre de Lelong raffiné, In: Séminaire d’Analyse Complexe et Géométrie 1985-87, Fac. Sci. Monastir Tunisie 1987, 61–70.

[12] P. Lelong and A. Rashkovskii, Local indicators for plurisubharmonic functions, J. Math. Pures Appl. 78 (1999), 233–247.

[13] A. Rashkovskii, Newton numbers and residual measures of plurisubharmonic functions, Ann. Polon. Math. 75 (2000), no. 3, 213-231.
[14] A. Rashkovskii, Lelong numbers with respect to regular plurisubharmonic weights, Results Math. 39 (2001), 320-332.

[15] A. Rashkovskii, Relative types and extremal problems for plurisubharmonic functions, Int. Math. Res. Not. 2006 (2006), Article ID 76283, 26 p.

[16] B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, Singularités à Cargèse, Asterisque 7-8 (1973), 285–362.

[17] V.P. Zahariuta, Spaces of analytic functions and maximal plurisubharmonic functions. D.Sci. Dissertation, Rostov-on-Don, 1984.

[18] V.P. Zahariuta, Spaces of analytic functions and Complex Potential Theory, Linear Topological Spaces and Complex Analysis 1 (1994), 74–146.