A dichotomy phenomenon for bad minus normed Dirichlet

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Abstract
Given a norm $\nu$ on $\mathbb{R}^2$, the set of $\nu$-Dirichlet improvable numbers $\mathcal{D}_\nu$ was defined and studied in the papers (Andersen and Duke, \textit{Acta Arith}. 198 (2021) 37–75 and Kleinbock and Rao, \textit{Internat. Math. Res. Notices} 2022 (2022) 5617–5657). When $\nu$ is the supremum norm, $\mathcal{D}_\nu = \mathcal{B}_\mathcal{A} \cup \mathbb{Q}$, where $\mathcal{B}_\mathcal{A}$ is the set of badly approximable numbers. Each of the sets $\mathcal{D}_\nu$, like $\mathcal{B}_\mathcal{A}$, is of measure zero and satisfies the winning property of Schmidt. Hence for every norm $\nu$, $\mathcal{B}_\mathcal{A} \cap \mathcal{D}_\nu$ is winning and thus has full Hausdorff dimension. In this article, we prove the following dichotomy phenomenon: either $\mathcal{B}_\mathcal{A} \subset \mathcal{D}_\nu$ or else $\mathcal{B}_\mathcal{A} \setminus \mathcal{D}_\nu$ has full Hausdorff dimension. We give several examples for each of the two cases. The dichotomy is based on whether the \textit{critical locus} of $\nu$ intersects a precompact $g_t$-orbit, where $\{g_t\}$ is the one-parameter diagonal subgroup of $\text{SL}_2(\mathbb{R})$ acting on the space $X$ of unimodular lattices in $\mathbb{R}^2$. Thus, the aforementioned dichotomy follows from the following dynamical statement: for a lattice $\Lambda \in X$, either $g_\mathbb{R}\Lambda$ is unbounded (and then any precompact $g_{\mathbb{R}>0}$-orbit must eventually avoid a neighborhood of $\Lambda$), or not, in which case the set of lattices in $X$ whose $g_{\mathbb{R}>0}$-trajectories are precompact and contain $\Lambda$ in their closure has full Hausdorff dimension.

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1 INTRODUCTION

Dirichlet’s theorem [19, Theorem 1A] states that for every real number \( \alpha \) and every \( T > 1 \), the set of inequalities

\[
\begin{align*}
\lfloor \alpha q \rfloor & \leq T^{-1}, \\
|q| & < T
\end{align*}
\]

has a solution \( q \in \mathbb{N} \). Here \( \lfloor . \rfloor \) denotes the distance of a real number to a nearest integer. The question of improving Dirichlet’s theorem was initiated by Davenport–Schmidt in [6]. A real number \( \alpha \) is said to be Dirichlet improvable, written \( \alpha \in DI_\infty \), if there is a \( c < 1 \) such that the system

\[
\begin{align*}
\lfloor \alpha q \rfloor & < cT^{-1}, \\
|q| & < T
\end{align*}
\]

(1.1)

can be solved in \( q \in \mathbb{N} \) for all sufficiently large \( T \). Note that [6, Theorem 1] shows that \( DI_\infty \) is exactly the set \( BA \cup \mathbb{Q} \). In fact, this was known as far back as the paper of Morimoto [17]. We recall that badly approximable numbers are those real numbers whose continued fraction coefficients (partial quotients) are bounded from above.

A normed variant of Dirichlet’s theorem (also referred to as Minkowski’s approximation theorem) was studied in [2, 10]. The starting point is the observation that the defining condition in (1.1) can be restated as

\[
\Lambda_\alpha \cap \begin{bmatrix} cT^{-1} & 0 \\ 0 & T \end{bmatrix} B_\infty(1) \neq \{0\},
\]

where \( \Lambda_\alpha \) denotes the unimodular lattice \([1 0 \alpha 1] \mathbb{Z}^2 \) and \( B_\infty(1) \) denotes the supremum norm ball in \( \mathbb{R}^2 \) centered at the origin and with radius 1.

Now, let \( X \) denote the set of unimodular lattices in \( \mathbb{R}^2 \), fix a norm \( \nu \) on \( \mathbb{R}^2 \) and define the critical radius of \( \nu \):

\[
r_\nu := \sup \left\{ r \in \mathbb{R} : \text{there exists } \Lambda \in X \text{ with } \Lambda \cap B_\nu(r) = \{0\} \right\}.
\]

(1.2)

By Minkowski’s convex body theorem, \( r_\nu \) is finite. Let \( B_\nu(r) \) denote the \( \nu \)-norm ball with radius \( r \) centered at the origin. We say that a real number \( \alpha \) is \( \nu \)-Dirichlet improvable, written \( \alpha \in DI_\nu \), if there is some \( c < 1 \) with

\[
\Lambda_\alpha \cap \begin{bmatrix} cT^{-1} & 0 \\ 0 & T \end{bmatrix} B_\nu(r_\nu) \neq \{0\}
\]

for all sufficiently large \( T \). When \( \nu \) is the supremum norm (denoted by the subscript \( \infty \)), we have that \( r_\infty = 1 \) and so recover the definition of Davenport–Schmidt. When \( \nu \) is the Euclidean norm (denoted by the subscript \( 2 \)), we see that \( \alpha \in DI_2 \) if and only if there is some \( c < 1 \) such that

\[
\left( \frac{T\langle \alpha q \rangle}{c} \right)^2 + \left( \frac{q}{T} \right)^2 < \frac{2}{\sqrt{3}}.
\]
is solvable in $q \in \mathbb{N}$ for all sufficiently large $T$. The normalizing constant $2/\sqrt{3}$ is equal to the square of the critical radius of the Euclidean norm, see, for example, [12] for a discussion, and references.

Consider the following preliminary properties:

**Theorem 1.1** [10, Theorems 3.1 and 1.3]. For each norm $\nu$ on $\mathbb{R}^2$, the set $\text{DI}_\nu$ is of measure zero but winning in the sense of Schmidt. In particular, $\text{DI}_\nu$ has full Hausdorff dimension.

See [18] and a paragraph before Proposition 2.2 for a discussion of Schmidt games and winning. The fact that $\mathcal{B}_A$ has full Hausdorff dimension goes back to Jarník in 1928 (see [3, Theorem 7.1] for an elementary proof). Thus, natural questions that come up in the study of Theorem 1.1 are as follows.

(a) Do there exist norms for which $\mathcal{B}_A \not\subset \text{DI}_\nu$?
(b) If yes, can one characterize the norms for which $\mathcal{B}_A \not\subset \text{DI}_\nu$?

The existence of norms that satisfy the above noncontainment is a consequence of the equality

$$Q = \bigcap_{\nu} \text{DI}_\nu$$

(see [11, Theorem 1.7]). In this paper, we address Question (b) by giving a convenient dynamical condition on $\nu$ that completely determines whether or not $\mathcal{B}_A$ is a subset of $\text{DI}_\nu$. In fact, our main results are completely dynamical in nature and this paper is a twofold study of limit points of bounded diagonal orbits in $X$ and criteria to detect whether certain submanifolds intersect precompact orbits.

We briefly explain how dynamics comes into play; the idea essentially comes from the work of Davenport and Schmidt [6]. Each norm $\nu$ gives rise to the following compact subset of $X$:

$$\mathcal{L}_\nu := \{ \Lambda \in X : \Lambda \cap B_\nu(r_\nu) = \{0\} \}.$$ 

$L_\nu$ is referred to as the critical locus of the norm. It follows from the definition of $r_\nu$ and from Mahler’s compactness criterion that $\mathcal{L}_\nu$ is a nonempty compact subset of $X$. Equivalently, lattices in $\mathcal{L}_\nu$ give the densest lattice packings of $\mathbb{R}^2$ by the domain $B_\nu(\frac{r_\nu}{2})$. We have the following notation for present and future use:

$$g_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \text{ where } t \in \mathbb{R}, \text{ and } g_E := \{ g_t : t \in E \} \text{ for a subset } E \text{ of } \mathbb{R}. \quad (1.3)$$

**Proposition 1.2** [5, Theorem 2.20 and 10, Theorem 2.1]. We have the following two equivalences.

(i) A real number $\alpha$ belongs to $\mathcal{B}_A$ if and only if

$$g_{\mathbb{R}_{>0}} \Lambda_\alpha \text{ is precompact in } X.$$ 

(ii) Given a norm $\nu$ on $\mathbb{R}^2$, we have that $\alpha \in \text{DI}_\nu$ if and only if there exists $t_0 > 0$ and a neighborhood $U \supset \mathcal{L}_\nu$ such that

$$g_{\mathbb{R}_{>0}} \Lambda_\alpha \cap U = \emptyset.$$
With this context, we state our main theorem:

**Theorem 1.3.** Fix a lattice \( \Lambda \in X \). If \( g_{\mathbb{R}} \Lambda \) is precompact in \( X \), then the set

\[
S_\Lambda := \left\{ \alpha \in BA : \Lambda \in g_{\mathbb{R} > 0} \Lambda_\alpha \right\}
\]

has Hausdorff dimension 1. Conversely, if \( g_{\mathbb{R}} \Lambda \) is not precompact in \( X \), then for every \( \alpha \in BA \) there exists \( t_0 > 0 \) and a neighborhood \( U \supset \Lambda \) such that

\[g_{\mathbb{R} > t_0} \Lambda_\alpha \cap U = \emptyset.\]

**Remark 1.4.** The converse is elementary and left to the reader as an exercise, see [11, Proposition 4.1]. Thus the main content of the theorem is the first part. Its proof below can be easily modified to show that the intersection of \( S_\Lambda \) with any open interval has Hausdorff dimension 1. However, \( S_\Lambda \) is not winning as follows from [1, Theorem 2.8]. (This last theorem was used to prove Theorem 1.1.)

As a corollary of Proposition 1.2 and Theorem 1.3, we have our dichotomy phenomenon:

**Theorem 1.5.** Fix a norm \( \nu \) on \( \mathbb{R}^2 \). If \( \mathcal{L}_\nu \) contains a lattice \( \Lambda \) such that \( g_{\mathbb{R}} \Lambda \) is precompact in \( X \), then \( BA \setminus DI_\nu \) has full Hausdorff dimension. Conversely, if no such lattice exists, then \( BA \subset DI_\nu \).

**Remark 1.6.** The converse has already been proved in the more general multidimensional setting of weighted approximation of systems of linear forms, see [11, Proposition 4.1].

It can be shown that the Euclidean critical locus \( \mathcal{L}_2 \) intersects a precompact \( \mathbb{R} \)-orbit (in fact, quite a few of them). Moreover, there is a very large class of norms that also satisfy this condition. Let us say that a norm \( \nu \) on \( \mathbb{R}^2 \) is *irreducible* if whenever \( \eta \neq \nu \) is a norm on \( \mathbb{R}^2 \) with \( \eta(v) \geq \nu(v) \) for any \( v \in \mathbb{R}^2 \), we have \( r_\eta > r_\nu \). Mahler introduced this distinguished set of norms (or rather, in his terminology, of convex bounded symmetric domains) in [14], and they turned out to be central to proving Theorem 1.1 among other results. Examples of irreducible norms are those whose unit balls are ellipses, parallelograms and Reinhart’s curvilinear octagon (see [12, Examples 2.2, 2.3, 3.3]). The critical locus \( \mathcal{L}_\nu \) of an irreducible norm \( \nu \) that does not come from a parallelogram is necessarily a one-dimensional \( C^1 \)-submanifold of \( X \) (see [15, Theorem 3]). Considerations in the tangent bundle of \( X \) then allow us to conclude that \( \mathcal{L}_\nu \) must intersect some precompact orbit \( g_{\mathbb{R}} \Lambda \) (see Proposition 2.5). Thus, we obtain as a corollary:

**Theorem 1.7.** If \( \nu \) is an irreducible norm on \( \mathbb{R}^2 \) whose unit ball is not a parallelogram, then \( BA \setminus DI_\nu \) has full Hausdorff dimension. In particular, \( BA \setminus DI_2 \) has full Hausdorff dimension.

We also mention that, as the critical locus of a hexagonal norm is a singleton (see [12, Example 2.4]), one can easily construct norms \( \nu \) that are not irreducible and for which \( BA \setminus DI_\nu \) has full dimension. For example, if \( \nu \) is a norm with \( \mathcal{L}_\nu = \{ \Lambda_0 \} \), one can choose an element \( g \in SL_2(\mathbb{R}) \) that makes \( g_{\mathbb{R}} g \Lambda_0 \) precompact, and then notice that \( \mathcal{L}_{\nu \circ g^{-1}} = g \mathcal{L}_\nu \).

We organize the paper as follows. In Section 2, we explore the structure of \( X \) more carefully to establish, under the assumption that \( \nu \) is a nonparallelogram irreducible norm, the existence of lattices in \( \mathcal{L}_\nu \) with precompact \( \mathbb{R} \)-orbits. With this out of the way, all that remains is to prove
Theorem 1.3 in the case when the mentioned lattice $\Lambda$ has a precompact $\mathbb{R}$-orbit. To prove it we use a symbolic representation of the diagonal flow. In Section 3, we study a specific Poincare section for $X$ that arises in the exposition of [4]. From this it becomes apparent that the set $S_\Lambda$ from Theorem 1.3 consists of badly approximable numbers having certain predetermined blocks of coefficients appearing in their continued fraction expansion. We then apply a dimension estimate from [20] in Section 4 to show this set has full Hausdorff dimension. Some remarks on possible generalizations of the set-up of this paper are made in the last section.

2. CRITICAL LOCI WHICH INTERSECT PRECOMPACT ORBITS

We denote $G := \text{SL}_2(\mathbb{R})$ and $\Gamma := \text{SL}_2(\mathbb{Z})$. Identify $X$ with $G/\Gamma$ via the map $g \mapsto g\mathbb{Z}^2$. In addition to the notation of (1.3), for $x, y, \in \mathbb{R}$ denote

$$u_x := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ and } v_y := \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$ 

We next have the stability of precompact orbits under the action of certain subgroups of $G$:

**Proposition 2.1.** For any $\Lambda \in X$ there exists $\varepsilon > 0$ such that the maps

$$(\varepsilon, \varepsilon)^3 \rightarrow X; \ (x, t, y) \mapsto v_y g_t u_x \Lambda \text{ and } (x, t, y) \mapsto u_x g_t v_y \Lambda$$

are diffeomorphisms. Moreover, for any fixed $(x, t, y) \in \mathbb{R}^3$, we have the equivalences:

$g_{\varepsilon>0} (v_y g_t u_x \Lambda) \text{ is precompact in } X \iff g_{\varepsilon>0} (u_x \Lambda) \text{ is precompact in } X$ (2.2)

and

$g_{\varepsilon<0} (u_x g_t v_y \Lambda) \text{ is precompact in } X \iff g_{\varepsilon<0} (v_y \Lambda) \text{ is precompact in } X.$ (2.3)

**Proof.** The first assertion follows from [21, Theorem 2.10.1]. For the remaining part of the proposition, note that for any $s \in \mathbb{R}$

$$g_s v_y g_t u_x \Lambda = v_y e^{-2t} g_s + t u_x \Lambda \text{ and } g_s u_x g_t v_y \Lambda = u_x e^{2t} g_s + t v_y \Lambda.$$ (2.4)

Alternatively, see [5, Proposition 2.12].

Our main tool in establishing the existence of precompact orbits passing through critical loci is the following proposition that constructs a winning set of precompact orbits along the $u_x$ and $v_y$ directions about any fixed lattice. It is a well-known application of Schmidt’s results in [18]; we give an elementary proof here for convenience (see also [13, Theorem 3.7] and [16, Theorem 1.3]).

For the benefit of the reader, we recall the rules of Schmidt’s game introduced in [18]. It involves two parameters $\alpha, \beta \in (0, 1)$ and is played by two players Alice and Bob on a complete metric space (which we shall take to be the set of real numbers) with a target set $S$. Bob starts the game by
choosing a closed ball $B_0 = B(x_0, r_0)$ in $\mathbb{R}$ with center $x_0$ and radius $r_0$. After Bob chooses a closed ball $B_i = B(x_i, r_i)$, Alice chooses $A_i = B(x_i', r_i') \subset B_i$ with $r_i' = \alpha r_i$, and then Bob chooses $B_{i+1} = B(x_{i+1}, r_{i+1}) \subset A_i$ with $r_{i+1} = \beta r_i'$, and so on. Alice wins the game if the unique point $\bigcap_{i=0}^{\infty} A_i = \bigcap_{i=0}^{\infty} B_i$ belongs to $S$, and Bob wins otherwise. The set $S$ is $(\alpha, \beta)$-\textit{winning} if Alice has a winning strategy, and is \textit{winning} if it is $(\alpha, \beta)$-winning for some $\alpha > 0$ and all $\beta \in (0, 1)$.

\textbf{Proposition 2.2.} For any $\Lambda \in X$, the sets

$$\{ x \in \mathbb{R} : g_{\mathbb{R} > 0} u_x \Lambda \text{ is precompact in } X \} \quad \text{and} \quad \{ y \in \mathbb{R} : g_{\mathbb{R} < 0} v_y \Lambda \text{ is precompact in } X \}$$

are \textit{winning}.

\textit{Proof.} Recall that we are given $\Lambda \in X$. Let us first consider the case when

$$\Lambda = \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2$$

(2.5)

for some $a \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$. Take $\alpha < 1/2$. After making the first move Alice can ensure that all $x$ in the ball of her choice satisfy $a + cx \neq 0$. For $\phi(x) := a^{-1} x (a + cx)^{-1}$, we have

$$u_x \Lambda = \begin{bmatrix} a + cx & 0 \\ c & (a + cx)^{-1} \end{bmatrix} \begin{bmatrix} 1 & \phi(x) + b \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2.$$

Using the equivalence of Proposition 2.1 and the characterization of Proposition 1.2(i), we see that

$$\phi(x) + b \in \mathbb{B} \equiv g_{\mathbb{R} > 0} u_x \Lambda \text{ is precompact in } X.$$

As $\phi^{-1}$ is Lipschitz, the winning property of $\mathbb{B}$ and [18, Theorem 1] establishes the winning property for the first set in the proposition.

Next we consider the case when $\Lambda$ cannot be written in the form (2.5). This happens if and only if $\Lambda = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} \mathbb{Z}^2$ for some $a, b, c \in \mathbb{R}$. A consequence is that, for any $x \neq 0$, $u_x \Lambda$ can be written in the form (2.5); thus one can replace $\Lambda$ with $u_x \Lambda$ and reduce the problem to the previously considered case. Hence in both cases we have the winning property for the first set in the proposition; the second set, in view of an elementary observation that

$$g_{\mathbb{R} > 0} u_x \mathbb{Z}^2 \text{ is precompact in } X \iff g_{\mathbb{R} < 0} v_y \mathbb{Z}^2,$$

(2.6)

can be handled similarly. \hfill \square

We have the notation for the following subgroups of $G$:

$$F := \{ g_t : t \in \mathbb{R} \}, \quad H^+ := \{ u_x : x \in \mathbb{R} \}, \quad H^- := \{ v_y : y \in \mathbb{R} \}.$$

\textbf{Remark 2.3.} We use the notation $T_m(M)$ to denote the tangent space of a ($C^1$ or smoother) manifold $M$ at a point $m$. We also use the prefix $d$ to indicate the derivative of a smooth map. $T_z(Fz)$
is understood to be the image, in $T_x(X)$, of the tangent space $T_x(F)$ via the derivative of the map $f \mapsto f z$. The same interpretation is used for the other subgroups of $G$.

**Proposition 2.4.** Let $Z \subset X$ be a one-dimensional compact $C^1$-submanifold satisfying, for some $z \in Z$,

$$T_z(X) = T_z(Z) + T_z(Fz) + T_z(H^+z)$$

(2.7)

and

$$T_z(X) = T_z(Z) + T_z(Fz) + T_z(H^-z).$$

(2.8)

Then $Z$ contains a lattice $\Lambda$ for which $g_{\mathbb{R}}\Lambda$ is precompact in $X$.

**Proof of Proposition 2.4.** Let

$$\phi : (-\varepsilon, \varepsilon) \to X$$

be a local parameterization of $Z$ with $\phi(0) = z$. Using the first set of local coordinates at $z$ in (2.1), and the tangent space decomposition of (2.8), we can write

$$\phi(s) = u_y(s) g_t(s) u_x(s) z,$$

where $x(\cdot), t(\cdot), y(\cdot)$ are now $C^1$-functions and $x'(0) \neq 0$. Thus, the inverse function $x^{-1}$ exists in a neighborhood of 0 and is Lipschitz. Using Proposition 2.2, [18, Theorem 1] and the equivalence (2.2), we see that

$$\left\{ s \in (-\varepsilon, \varepsilon) : g_{\mathbb{R}}^{-1} \phi(s) \text{ is precompact in } X \right\}$$

is winning. An entirely analogous argument using (2.7) shows that

$$\left\{ s \in (-\varepsilon, \varepsilon) : g_{\mathbb{R}}^{-1} \phi(s) \text{ is precompact in } X \right\}$$

is winning. As the intersection of winning sets is also winning [18, Theorem 2], we arrive at the desired result. \[\square]\n
**Proposition 2.5.** If $\nu$ is an irreducible norm on $\mathbb{R}^2$ whose unit ball is not a parallelogram, then $\mathcal{L}_\nu$ is a compact $C^1$-submanifold of $X$ satisfying conditions (2.7) and (2.8) at every point $z \in \mathcal{L}_\nu$. Consequently, $\mathcal{L}_\nu$ must intersect some precompact $\mathbb{R}$-orbit.

**Proof.** The first condition (2.7) is proved in [10, Theorem 3.11]. For the second condition, consider the linear automorphism of $\mathbb{R}^2$ given by

$$v \mapsto pv,$$

where $p := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Of course, it also induces a map on subsets of $\mathbb{R}^2$ that gives a diffeomorphism of $X$

$$\Lambda \mapsto p\Lambda.$$ 

On the level of critical loci and norms, we have that

$$p\mathcal{L}_\nu = \mathcal{L}_{\nu \circ p}.$$ 

Moreover, if $\nu$ is irreducible with unit ball not a parallelogram, then the same holds true for $\nu \circ p$. (Indeed, the irreducibility of $\nu \circ p$ follows from the definition after noting that $r_{\nu \circ p} = r_{\nu}$; on the other hand, it is clear that the image of a parallelogram under a linear transformation remains a parallelogram.) On the level of tangent spaces we have that, for $z \in X$,

$$(dp)_z(T_z(Fz)) = T_{pz}(Fpz), (dp)_z(T_z(H^+z)) = T_{pz}(H^+pz), (dp)_z(T_z(H^-z)) = T_{pz}(H^-pz).$$

The upshot now is that as $\mathcal{L}_{\nu \circ p}$ satisfies (2.7) at $pz \in p\mathcal{L}_{\nu}$, it follows that $\mathcal{L}_{\nu}$ satisfies (2.8) at $z$. This shows that Proposition 2.4 can be applied, giving the final assertion in the present proposition. □

3 | MINIMAL VECTORS AND A POINCARE SECTION FOR $X$

We study a well-known correspondence between diagonal orbits in $X$ and orbits in an invertible extension of the Gauss map. We follow the exposition of [4, section 3.1] and presume the reader is familiar with the theory of continued fractions at the level of [8].

If $a, b$ are real numbers, we define the rectangle $R(a, b)$ to be

$$R(a, b) := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq a \text{ and } |x_2| \leq b\}.$$ 

**Definition 3.1** (Minimal vectors). Let $\Lambda \in X$. A vector $r = (r_1, r_2) \in \Lambda$ is called a *minimal vector* if it is nonzero, and if

$$s = (s_1, s_2) \in R(|r_1|, |r_2|) \cap \Lambda \setminus \{0\} \Rightarrow |s_1| = |r_1| \text{ and } |s_2| = |r_2|.$$ 

**Definition 3.2** (Consecutive minimal vectors). An ordered pair of minimal vectors $r = (r_1, r_2)$ and $s = (s_1, s_2)$ in $\Lambda$ is said to be the pair of *consecutive minimal vectors* if $|r_2| < |s_2|$, and if there are no minimal vectors $w = (w_1, w_2) \in \Lambda$ with $|r_2| < |w_2| < |s_2|$.

**Lemma 3.3** [4, Lemma 2 and Proposition 3]. If $r, s$ are consecutive minimal vectors in $\Lambda$, then we have

$$\text{Interior}(R(|r_1|, |s_2|)) \cap \Lambda = \{0\}.$$ 

Moreover, $r$ and $s$ form a basis of $\Lambda$.

If $x \in \mathbb{R}$, let $[x]$ and $\{x\}$ denote the integer and fractional parts of $x$. Consider the set

$$\mathcal{U}^* = (0, 1)^2 \cup \left(\left[0, \frac{1}{2}\right] \times \{0\}\right) \cup \left(\{0\} \times \left[0, \frac{1}{2}\right]\right).$$
Consider also the maps

**Definition 3.4.** $T : (0, 1)^2 \cup ((0, \frac{1}{2}] \times \{0\}) \rightarrow (0, 1)^2 \cup ([0] \times (0, \frac{1}{2}])$ given by

$$T(x, y) = \left( \left\{ x^{-1} \right\}, \frac{1}{\left\lfloor x^{-1} \right\rfloor + y} \right)$$

(3.1)

and $S : (0, 1)^2 \cup ([0] \times (0, \frac{1}{2}]) \rightarrow (0, 1)^2 \cup ((0, \frac{1}{2}] \times \{0\})$ given by

$$S(a, b) = \left( \frac{1}{a + \left\lfloor b^{-1} \right\rfloor}, \left\{ b^{-1} \right\} \right).$$

It is then straightforward to check that

**Lemma 3.5.** The compositions $S \circ T$ and $T \circ S$ give the identity on the domains of $T$ and $S$, respectively.

We now discuss a mapping from pairs of minimal vectors of a lattice onto $U' \times \{\pm 1\}$ that, in a sense made precise below, intertwines the diagonal action on $X$ with the $T$-action on $U'$.
Proposition 3.6 [4, Proposition 6]. Let \( \Lambda \in X \) and let \( \mathbf{r}, \mathbf{s} \) be a pair of consecutive minimal vectors in \( \Lambda \) with

\[
0 \leq r_2 < s_2.
\]

If we happen to have \( r_2 = 0 \) and \( r_1 s_1 > 0 \), replace \( \mathbf{r} \) by \(-\mathbf{r}\). For such pairs of minimal vectors, define the functions

\[
x(\mathbf{r}, \mathbf{s}) := -\frac{s_1}{r_1}, \quad y(\mathbf{r}, \mathbf{s}) := \frac{r_2}{s_2}, \quad \varepsilon(\mathbf{r}, \mathbf{s}) := \frac{r_1}{|r_1|}.
\]

We then have that \((x, y) \in U'\). Moreover, if \( s_1 \neq 0 \), \( n := \lfloor \frac{1}{x} \rfloor \) and \( \mathbf{w} := \mathbf{r} + n \mathbf{s} \), then \( \mathbf{s}, \mathbf{w} \) is a pair of consecutive minimal vectors for \( \Lambda \). Further, we have that

\[
(x(s, w), y(s, w), \varepsilon(s, w)) = (T(x(\mathbf{r}, \mathbf{s}), y(\mathbf{r}, \mathbf{s})), -\varepsilon(\mathbf{r}, \mathbf{s})),
\]

(3.2)

where \( T \) is as in Definition 3.4.

Remark 3.7. Note that \( r_1 \) cannot be zero by minimal vector considerations. In subsequent notation, we often drop the dependence of \( x, y, \varepsilon \) on the minimal vectors when the context is clear. We repeat the proof from [4] in order to make use of Equation (3.4).

Proof of Proposition 3.6. We first show that \((x, y) \in U'\). It is clear that \( 0 \leq y < 1 \).

If \( y = 0 \), then we have \( r_2 = 0 \) and, by definition, either \( s_1 = 0 \) or \( r_1 s_1 < 0 \). This shows that \( 0 \leq x \).

Moreover, if \( s_1 = 0 \), then \((x, y) = (0, 0) \in U'\) as desired. So, assume we are in the case when \( s_2 = 0 \) and \( r_1 s_1 < 0 \). Consider the lattice vector \( \mathbf{r} + \mathbf{s} = (r_1 + s_1, s_2) \). As \( \mathbf{s} \) is minimal, we must have

\[
|r_1 + s_1| \geq |s_1|.
\]

(3.3)

When \( r_1 > 0 \), as \( |s_1| < r_1 \) and \( r_1 s_1 < 0 \), this implies

\[
r_1 + s_1 \geq -s_1
\]

that shows that \( x \leq 1/2 \). When \( r_1 < 0 \), Equation (3.3) leads to

\[
-(r_1 + s_1) \geq s_1,
\]

which again shows that \( x \leq 1/2 \).

Now consider \( x \). By minimality, \( |s_1| < |r_1| \) so that \(|x| < 1 \). If \( x = 0 \) so that \( s_1 = 0 \), we consider the lattice vector \( \mathbf{r} - \mathbf{s} = (r_1, r_2 - s_2) \). As \( \mathbf{r} \) is minimal, we must have

\[
|r_2 - s_2| = s_2 - r_2 \geq r_2
\]

which leads to \( y \leq 1/2 \). Taking stock so far, we have shown that \((x, y)\) always lies in

\[
[0, 1/2] \times \{0\} \cup (0, 1) \times (0, 1) \cup (-1, 0) \times (0, 1) \cup \{0\} \times [0, 1/2].
\]

Thus, we are left with proving that \( x > 0 \) when \( y > 0 \). If \( y > 0 \), consider the vector \( \mathbf{r} - \mathbf{s} \) which is equal to \((r_1 - s_1, r_2 - s_2)\). If \( r_1 \) and \( s_1 \) had the same sign, this vector would belong to the interior of the rectangle \( B(|r_1|, |s_2|) \) that contradicts Lemma 3.3. Thus, \( x > 0 \) and this concludes the proof that \((x, y) \in U'\).
Moreover, we can write \( \mathbf{r}, \mathbf{s} \) in a matrix as

\[
\begin{bmatrix}
\mathbf{r} \\
\mathbf{s}
\end{bmatrix} = \begin{bmatrix}
\varepsilon|r_1| & -\varepsilon|r_1|x \\
-s_2y & s_2
\end{bmatrix}.
\]

Or rather, noting that \( \Lambda \) is a covolume one lattice, we can solve for \( s_2 \) and write:

\[
\begin{bmatrix}
\mathbf{r} \\
\mathbf{s}
\end{bmatrix} = \begin{bmatrix}
|r_1| & 0 \\
0 & |r_1|^{-1}
\end{bmatrix} \begin{bmatrix}
\varepsilon \\
\frac{y}{1+xy} & \frac{-\varepsilon x}{1+xy}
\end{bmatrix}.
\]

(3.4)

Now we study the case when \( s_1 \neq 0 \). In this case, Minkowski’s convex body theorem shows that there is a nonzero lattice vector in the strip

\[
\left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0 \text{ and } |x_1| < |s_1| \right\}.
\]

But using the discreteness of \( \Lambda \), we get a minimal lattice vector \( \mathbf{w} = (w_1, w_2) \) in this region such that \( \mathbf{s}, \mathbf{w} \) form a consecutive pair of minimal vectors. By Lemma 3.3, we see that \( \mathbf{r}, \mathbf{s} \) and \( \mathbf{s}, \mathbf{w} \) form bases of \( \Lambda \). The orientations of each basis are distinct by what we have proved above; namely, if \( r_1 > 0 \), then \( \det(\mathbf{r}, \mathbf{s}) = 1 \), and if \( r_1 < 0 \), then \( \det(\mathbf{r}, \mathbf{s}) = -1 \). And further, we always have the condition that \( r_1s_1 < 0 \). Thus, we can write

\[
\mathbf{w} = \mathbf{r} + k\mathbf{s} = \left(\varepsilon|r_1|x \left(\frac{1}{x} - k\right), s_2(y + k)\right)
\]

for some \( k \in \mathbb{N} \).

(Note that \( x \neq 0 \) as \( s_1 \neq 0 \).) As \( \mathbf{w} \) is minimal, we must have

\[
\left|\frac{1}{x} - k\right| < 1,
\]

so that \( k = n \) or \( n + 1 \). On the other hand, the presence of the lattice vector

\[
\mathbf{r} + n\mathbf{s} = \left(\varepsilon|r_1|x \left(\frac{1}{x} - n\right), s_2(y + n)\right),
\]

and the fact that \( \mathbf{s}, \mathbf{w} \) are consecutive shows that we indeed have \( \mathbf{w} = \mathbf{r} + n\mathbf{s} \). Also we have

\[
x(\mathbf{s}, \mathbf{w}) = -\frac{w_1}{s_1} = \left\{ \frac{1}{x(\mathbf{r}, \mathbf{s})} \right\},
\]

\[
y(\mathbf{s}, \mathbf{w}) = \frac{s_2}{w_2} = \frac{1}{y(\mathbf{r}, \mathbf{s}) + n},
\]

\[
\varepsilon(\mathbf{s}, \mathbf{w}) = -\varepsilon(\mathbf{r}, \mathbf{s}),
\]

thereby completing the proof.

\[\square\]

Remark 3.8. Henceforth, whenever we refer to a minimal vector \( \mathbf{r} = (r_1, r_2) \), we always make the tacit assumption that \( r_2 \geq 0 \).

We also note the following characterization of precompact orbits:

**Proposition 3.9.** The orbit \( g_{\mathbf{r}} \Lambda \) is precompact if and only if, for any choice of consecutive minimal vectors \( \mathbf{r}, \mathbf{s} \in \Lambda \), we have that both \( x(\mathbf{r}, \mathbf{s}) \) and \( y(\mathbf{r}, \mathbf{s}) \) are badly approximable real numbers.
Proof. Say $g_{\mathbb{R}}\Lambda$ is precompact in $X$. Let $r = (r_1, r_2), s = (s_1, s_2)$ be any pair of consecutive minimal vectors. By Lemma 3.3, they form a basis of $\Lambda$ and we can write

$$\Lambda = \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} \mathbb{Z}^2.$$ 

Moreover, precompactness shows that $\Lambda$ does not contain any vector on the coordinate axes, and so we have

$$\begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ r_2 & s_2 - \frac{s_1 r_2}{r_1} \end{bmatrix} \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ r_2 & s_2 - \frac{s_1 r_2}{r_1} \end{bmatrix} u_{-x(r,s)},$$

so that, from the commutator relation (2.4) and the equivalence of Proposition 1.2(i), $-x(r,s)$ is badly approximable, which, of course, is equivalent to $x(r,s)$ being badly approximable.

Similarly, we also have the decomposition

$$\begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} = \begin{bmatrix} r_1 - \frac{s_1 r_2}{s_2} \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s_2 & 1 \end{bmatrix} = \begin{bmatrix} r_1 - \frac{s_1 r_2}{s_2} \\ 0 \end{bmatrix} v_{y(r,s)}.$$

Using (2.4) and (2.6), we see that $y(r,s)$ is badly approximable. We remark that, in the above proof, it does not matter if the matrix formed by the ordered pair $r, s$ has determinant $-1$.

For the converse, we simply note that we can reverse the order in the argument just given. □

4 \hspace{1cm} AN ABUNDANCE OF PRECOMPACT ORBITS WITH A PRESCRIBED LIMIT POINT

We set out some further notation involving continued fractions. The symbols

$$a_1, a_2, \ldots, a_n, \ldots$$

denote the countable family of measurable functions giving the continued fraction coefficients of a number in $[0,1]$. Of course, $a_n$ is not defined on certain rationals. Other letters such as $b_n, c_n, d_n$ are, as before, used to denote continued fraction coefficients of specific numbers. We also have the associated convergent functions

$$\frac{p_n}{q_n} := [0; a_1, \ldots, a_n],$$

and the remainder functions defined by

$$\alpha = [0; a_1(\alpha), \ldots, a_{n-1}(\alpha), \rho_n(\alpha)].$$

Finally, we denote the successive iterates of the Gauss map on $[0,1)$ by

$$z_n(\alpha) := \rho_n(\alpha) - a_n(\alpha) \in [0,1).$$
Given a multi-index \( k = (k_1, \ldots, k_n) \) of natural numbers, we define the \( n \)th order cylindrical interval

\[
I(k) := \{ \alpha \in [0,1] : a_i(\alpha) = k_i \text{ for } i = 1, \ldots, n \}.
\]

It is well-known that these intervals are given by

\[
I(k) = \begin{cases} 
\left[ \frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}} \right] & \text{if } n \text{ is even,} \\
\left( \frac{p_{n+1}}{q_{n+1}+q_{n-1}}, \frac{p_n}{q_n} \right] & \text{if } n \text{ is odd.}
\end{cases}
\]

(4.1)

Here, the convergents are associated to \( k \). The following estimate on conditional probability is essential for our computation of dimension. Here and hereafter, \( | \cdot | \) denotes Lebesgue measure.

**Proposition 4.1** cf. [7, Corollary 1.2.6]. If \( k = (k_1, \ldots, k_n) \) is a multi-index of natural numbers and \( x \in [0,1] \), then we have

\[
\frac{|\{\alpha \in I(k) : z_n(\alpha) \geq x\}|}{|I(k)|} = \frac{1 - x}{\sigma_n x + 1}
\]

where \( \sigma_n := q_{n-1}q_n^{-1} \) (\( \sigma_n \) is dependent on \( k \) only).

As a corollary, we obtain

**Corollary 4.2.** If \( 0 < x < y < 1 \), we have

\[
\frac{|\{\alpha \in I(k) : x \leq z_n(\alpha) \leq y\}|}{|I(k)|} = \frac{(y-x)(\sigma_n + 1)}{(\sigma_n x + 1)(\sigma_n y + 1)}.
\]

We also have the following consequence of the mass distribution principle, see [20, Lemma 2.1]. For each \( m \in \mathbb{N} \cup \{0\} \) let \( \mathcal{E}_m \subseteq 2^{[0,1]} \) be a finite collection of nondegenerate compact intervals in \([0,1]\). By abuse of notation, we write

\[
\bigcup \mathcal{E}_m := \bigcup_{I \in \mathcal{E}_m} I.
\]

Define for each \( I \in \mathcal{E}_m \),

\[
\text{density}(\mathcal{E}_{m+1}, I) := \frac{|\bigcup \mathcal{E}_{m+1} \cap I|}{|I|}.
\]

Assume we have the following properties.

(a) \( \bigcup \mathcal{E}_0 = [0,1] \).

(b) For \( I \neq J \) in \( \mathcal{E}_m \), we have \( |I \cap J| = 0 \).

(c) For every \( I \in \mathcal{E}_{m+1} \), there is a unique \( J \in \mathcal{E}_m \) with \( I \subset J \).

(d) We have that

\[
\Theta_m := \inf \{ \text{density}(\mathcal{E}_{m+1}, I) : I \in \mathcal{E}_m \} > 0.
\]
(e) If $diam_m$ is the supremum of diameter($I$) over all $I \in \mathcal{E}_m$, then we have

$$\lim_{m \to \infty} diam_m = 0.$$ 

**Theorem 4.3** [20, Theorem 2.1]. *Given the conditions above, let $E = \bigcap_{m \in \mathbb{N}} \cup \mathcal{E}_m$. Then we have that*

$$1 - \dim E \leq \limsup_{m \to \infty} \frac{\sum_{j=1}^{m-1} \log \Theta_j}{\log diam_m}.$$ 

**Proof of Theorem 1.3.** We have that $g_{\mathbb{R}^\Lambda}$ is precompact. Choose consecutive minimal vectors $r, s \in \Lambda$. By Proposition 3.9, we have that $x = x(r, s)$ and $y = y(r, s)$ are badly approximable. From Proposition 3.6, we can assume without loss of generality, that $\varepsilon = \varepsilon(r, s) = 1$. From Equation (3.4), $r, s$ have coordinates given by

$$\begin{bmatrix} r & s \end{bmatrix} = \begin{bmatrix} |r_1| & 0 \\ 0 & |r_1|^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x \\ \frac{y}{1+xy} & \frac{1}{1+xy} \end{bmatrix}.$$ 

And by Proposition 3.3, $r$ and $s$ form a basis of $\Lambda$.

Write out the continued fraction expansions

$$x = [0; b_1, \ldots, b_m, \ldots]$$

and, recalling that $x, y$ are badly approximable, let $M \in \mathbb{R}$ be a uniform upper bound for $b_m, c_m$. For each $k \in \mathbb{N}$, let $B_k$ be the block of $2k$ digits

$$B_k = (c_k, \ldots, c_1, b_1, \ldots, b_k).$$

Let $\alpha \in (0, 1)$ be any real number that has infinite continued fraction expansion

$$\alpha = [0; d_1, \ldots, d_m, \ldots].$$

Assume that, for each $k \in \mathbb{N}$, there is a digit $d_{m_k}$ with $m_k$ odd and such that

$$\left( d_{m_k-(k-1)}, \ldots, d_{m_k}, d_{m_k+1} \ldots, d_{m_k+k} \right) = B_k.$$ 

Moreover, assume that $d_1 = 2$. We make this assumption for convenience in writing out the minimal vectors of $\Lambda_\alpha$. In any case, it will lead us to a set of full dimension.

We then claim that $\Lambda \in g_{\mathbb{R}_{>0}} \Lambda_\alpha$. To prove the claim, it suffices to show that

$$g_{\log |r_1|} \Lambda = \begin{bmatrix} 1 & -x \\ \frac{y}{1+xy} & \frac{1}{1+xy} \end{bmatrix} \mathbb{Z}^2 \text{ belongs to } g_{\mathbb{R}_{>0}} \Lambda_\alpha.$$ 

(4.3)

Consider the following consecutive minimal vectors in $\Lambda_\alpha$:

$$w_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}.$$
Here we have used that the first digit of $\alpha$ is 2. According to Proposition 3.6, the associated coordinates of $w_1, w_2$ are given by

$$x(w_1, w_2) = \alpha, \ y(w_1, w_2) = 0, \text{ and } \varepsilon(w_1, w_2) = -1. \tag{4.4}$$

We claim that, by induction, it is possible to continue choosing minimal vectors and obtain a sequence $(w_n)_{n \in \mathbb{N}} \subset \Lambda_\alpha$ such that $w_n, w_{n+1}$ form consecutive pairs and that

$$x(w_n, w_{n+1}) = \left[0; d_n, d_{n+1}, \ldots \right], \ y(w_n, w_{n+1}) = \left[0; d_{n-1}, \ldots, d_1 \right], \ \varepsilon(w_n, w_{n+1}) = (-1)^n. \tag{4.5}$$

The convention for $n = 1$ is that $y(w_1, w_2) = 0$. The base case of (4.5) is nothing but (4.4). For the induction step, assume that (4.5) holds. Then note that, as $\alpha$ is irrational, $\Lambda_\alpha$ does not contain any vector on the $y$-axis. Thus, given the consecutive pair of minimal vectors $w_n, w_{n+1}$ for $\Lambda_\alpha$, the latter part of Proposition 3.6 applies. This gives us the vector $w_{n+2}$, and formula (3.2) says that

$$x(w_{n+1}, w_{n+2}) = \left[0; d_n, d_{n+1}, \ldots \right]^{-1} = \left[0; d_{n+1}, d_{n+1}, \ldots\right],$$

$$y(w_{n+1}, w_{n+2}) = \frac{1}{\left[0; d_n, d_{n+1}, \ldots \right]^{-1} + \left[0; d_{n-1}, \ldots, d_1 \right]} = \left[0; d_n, \ldots, d_1 \right]$$

and

$$\varepsilon(w_{n+1}, w_{n+2}) = -(-1)^n = (-1)^{n+1}.$$ 

Thus, formula (4.5) is proved and we write $x_n, y_n, \varepsilon_n$ for the respective coordinates. Again using (3.4) and the fact that $w_n, w_{n+1}$ form a basis of $\Lambda_\alpha$, we write, for some $t_n > 0$, that

$$\Lambda_\alpha = \begin{bmatrix} e^{-t_n} & 0 \\ 0 & e^{t_n} \end{bmatrix} \begin{bmatrix} \varepsilon_n & -\varepsilon_n x_n \\ y_n & 1 + x_n y_n \end{bmatrix} \mathbb{Z}^2.$$

Or rather,

$$g_{t_n} \Lambda_\alpha = \begin{bmatrix} \varepsilon_n & -\varepsilon_n x_n \\ y_n & 1 + x_n y_n \end{bmatrix} \mathbb{Z}^2.$$

If we choose $n = m_k + 1$, the choices in (4.2) and the formulae in (4.5) show that

$$x_n = \left[0; d_{m_k+1}, \ldots \right] = \left[0; b_1, \ldots, b_k, d_{m_k+k+1}, \ldots \right],$$

$$y_n = \left[0; d_{m_k}, \ldots, d_{m_k-(k-1)}, \ldots, d_1 \right] = \left[0; c_1, \ldots, c_k, d_{m_k-k}, \ldots, d_1 \right]$$

and that

$$\varepsilon_n = (-1)^{m_k+1} = 1.$$ 

Hence, the matrices

$$\begin{bmatrix} 1 & -x \\ y & 1 + x y \end{bmatrix} \text{ and } \begin{bmatrix} \varepsilon_n & -\varepsilon_n x_n \\ y_n & 1 + x_n y_n \end{bmatrix}$$
can be made arbitrarily close, which establishes the claim in (4.3). Thus, we have shown that $S_\Lambda$ is nonempty, and it remains to estimate its Hausdorff dimension.

We will work with a convenient subset of $S_\Lambda$. What we have so far is that the lattice $\Lambda$ with precompact orbit gives sequences of digits $(b_n), (c_n)$ and a bound $M \in \mathbb{N}$ from which we get the blocks

$$B_k = (c_k, \ldots, c_1, b_1, \ldots, b_k) \text{ with } 1 \leq c_j, b_j \leq M.$$ 

The set we are concerned with is

$$\tilde{S}_\Lambda := \bigcap_k \bigcup_{m \text{ is odd}} \left\{ \alpha \in \mathbb{B}A : a_1(\alpha) = 2 \text{ and } (a_i(\alpha))_{m+k}^{m-(k-1)} = B_k \right\},$$

which we have just shown to be a subset of $S_\Lambda$. We are left to show $\dim \tilde{S}_\Lambda = 1$. Let $L > M$ be a natural number. Choose a sequence of odd numbers $(m_k)_{k \in \mathbb{N}}$ sparse enough so that

(a) for each $k > 1$, $m_k - m_{k-1} > 2k$;
(b) if we define

$$\text{density}((m_k), n) := \sum_{m_k < n} \frac{2k}{n}, \text{ then } \lim_{n \to \infty} \text{density}((m_k), n) = 0.$$ \hfill (4.6)

We define the following subset of $\tilde{S}_\Lambda$:

$$S(L, (m_k)) := \bigcap_k \left\{ \alpha \in S_\Lambda : a_n(\alpha) \leq L \text{ for each } n, \text{ and } (a_n(\alpha))_{m+k}^{m-(k-1)} = B_k \right\}.$$

This is nothing but the set of badly approximable numbers with $a_1 = 2$ having upper bound $L$ on their partial quotients and having the block $B_k$ appear at the $m_k$ position. The upper bound on the entries of $B_k$ and condition (a) above guarantee that this set is nonempty.

We now use the Cantor set structure of $S(L, (m_k))$ to estimate its Hausdorff dimension. According to the notation in Theorem 4.3, we first define

$$\mathcal{E}_0 := \{[0, 1]\} \text{ and } \mathcal{E}_1 := \left\{ \alpha \in [0, 1] : a_1(\alpha) = 2 \right\} = \{1/3, 1/2\}.$$ 

Now, assuming we have defined $\mathcal{E}_m$ as a collection of closed intervals $\{I\}$ where each $I$ is of the form (4.1) (with $n = m$), we define the family $\mathcal{E}_{m+1}$ as follows:

$$\mathcal{E}_{m+1} := \begin{cases} \bigcup_{I \in \mathcal{E}_m} \bigcup_{i \leq L} \left\{ \alpha \in I : a_{m+1}(\alpha) = i \right\} & \text{if for all } k, \ m + 1 \not\in \{m_k - (k-1), \ldots, m_k + k\}, \\ \bigcup_{I \in \mathcal{E}_m} \left\{ \alpha \in I : a_{m+1}(\alpha) = c_i \right\} & \text{if } m + 1 = m_k - i \text{ where } 0 \leq i \leq k - 1, \\ \bigcup_{I \in \mathcal{E}_m} \left\{ \alpha \in I : a_{m+1}(\alpha) = b_i \right\} & \text{if } m + 1 = m_k + i \text{ where } 1 \leq i \leq k. \end{cases}$$ \hfill (4.7)
The cylindrical intervals of (4.1) are either disjoint or satisfy a containment relation. Thus, each $I,J$ that belong to some $\mathcal{E}_m$ can intersect in at most a point. The diameter of these sets converge to 0 as $m \to \infty$. Moreover for each $I \in \mathcal{E}_m$, we have

$$| \cup \mathcal{E}_{m+1} \cap I | > 0.$$ 

Thus, the conditions for Theorem 4.3 are satisfied. We also have that

$$S(L,(m_k)) = \bigcap_m \cup \mathcal{E}_m.$$ 

We now turn to computing $\Theta_m$. Assume $\mathcal{E}_{m+1}$ is defined according to the first case in Equation (4.7). If $I \in \mathcal{E}_m$, then, using Proposition 4.1, we can write

$$| \cup \mathcal{E}_{m+1} \cap I | = \left| \{ \alpha \in I : a_{m+1}(\alpha) \leq L \} \right|$$

$$= \left| \{ \alpha \in I : z_m(\alpha) \geq (L+1)^{-1} \} \right|$$

$$= \frac{1 - (L+1)^{-1}}{\sigma_m(L+1)^{-1} + 1} \cdot |I|.$$ 

Here, $\sigma_m$ depends on the defining continued fraction coefficients for $I$. But as $\sigma_m < 1$ always, we get that

$$\Theta_m \geq \frac{L}{L + 2}. \quad (4.8)$$

Now if $\mathcal{E}_{m+1}$ is defined according to the second or third case in Equation (4.7), using Corollary 4.2 and the bound $M$, we have

$$\Theta_m \geq \frac{1}{4(M+1)^2}. \quad (4.9)$$

To differentiate these two ways in which $\mathcal{E}_{m+1}$ can be defined, we introduce, for each $m \in \mathbb{N}$, the index sets

$$I_m = \bigcap_{k=1}^{\infty} \{ j \in \mathbb{N} \leq m : j + 1 \notin \{m_k - (k-1), \ldots, m_k + k\} \}$$

and

$$J_m = \bigcup_{k=1}^{\infty} \{ j \in \mathbb{N} \leq m : j + 1 \in \{m_k - (k-1), \ldots, m_k + k\} \}.$$ 

In considering $\text{diam}_m$, we note by induction and Corollary 4.2 that

$$\text{diam}_m \leq 2^{-m}. \quad (4.10)$$
Thus, we can finally estimate, using (4.8), (4.9), (4.10),
\[
-\sum_{j=1}^{m-1} \log \Theta_j = -\log \text{diam}_m \left( -\sum_{j \in I_m} \log \Theta_j - \sum_{j \in J_m} \log \Theta_j \right)
\leq \frac{1}{m \log 2} \left( -\sum_{j \in I_m} \log \frac{L}{L+2} - \sum_{j \in J_m} \log \frac{1}{4(M+1)^2} \right)
\leq \frac{-1}{\log 2} \left( \frac{#I_m}{m} \log \frac{L}{L+2} + \frac{#J_m}{m} \log \frac{1}{4(M+1)^2} \right).
\]

Using Theorem 4.3 and (4.6) on the above estimate, we get
\[
1 - \dim S(L, (m_k)) \leq \limsup_{m \to \infty} \frac{-\sum_{j=1}^{m-1} \log \Theta_j}{-\log \text{diam}_m} \leq \frac{1}{\log 2} \cdot \log \frac{L+2}{L}.
\]

As $S(L, (m_k)) \subset S_{\Lambda}$, taking $L \to \infty$ gives the result. \hfill \qed

5 | FURTHER QUESTIONS ON DIRICHLET IMPROVABILITY AND PRECOMPACT ORBITS

Some amusing problems on $\nu$-Dirichlet numbers that we omitted are as follows.

(c) In the case of the Euclidean norm, does $\text{DI}_2 \setminus \text{BA}$ have full Hausdorff dimension?
(d) In general, for two norms $\nu_1, \nu_2$ on $\mathbb{R}^2$, what can be said about $\text{DI}_{\nu_1} \setminus \text{DI}_{\nu_2}$?

We expect (c) to have an affirmative answer, although constructing the required orbits in $X$ will require a more delicate inductive procedure than the one produced here.

Dirichlet-improvability can be studied in a variety of different settings. And each setting relates, via the Dani correspondence, to diagonal orbits on a homogeneous space avoiding a critical locus. The setting of diagonal flows on $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, described in [11], is as follows. Fix positive integers $m, n$ with $m + n = d$ and a set of weights
\[
\omega = (\alpha, \beta) \in \mathbb{R}_+^m \times \mathbb{R}_+^n \text{ with } \sum_i \alpha_i = \sum_j \beta_j = 1.
\]

Fix also a norm $\nu$ on $\mathbb{R}^d$ and define the critical radius $r_\nu$ in analogy with (1.2). A matrix $A \in M_{m,n}(\mathbb{R})$ is said to be $(\nu, \omega)$-Dirichlet improvable (written $A \in \text{DI}_{\nu, \omega}$) if there is a constant $c < 1$ such that
\[
\Lambda_A \cap \left[ \begin{array}{cc} (ct^{-1})^\alpha & 0 \\ 0 & (t)^\beta \end{array} \right] B_\nu(r_\nu) \neq \{0\}
\]
for all $t$ sufficiently large. Here, we define a positive real number $x$ raised to a vector power $a \in \mathbb{R}^k$ as the $k \times k$ matrix
\[
(x)^a := \text{diag}(x^{a_1}, \ldots, x^{a_k}),
\]
and use the notation
\[
\Lambda_A := \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} \mathbb{Z}^d.
\]

We also have the set of \textit{weighted badly approximable} matrices:
\[
BA_\omega := \left\{ A \in M_{m,n}(\mathbb{R}) : \inf_{p \in \mathbb{Z}^m, q \in \mathbb{Z}^n \setminus \{0\}} \| Aq - p \|_\omega \| q \|_\beta > 0 \right\},
\]
where we have used the quasi-norms
\[
\| x \|_\alpha := \max_i |x_i|^{1/\alpha_i} \quad \text{and} \quad \| y \|_\beta := \max_j |y_j|^{1/\beta_j}.
\]

Now one can naturally ask for a comparison study between \( \mathcal{D}_\omega \) and \( \mathcal{B}_\omega \). We predict that the same dichotomy phenomenon as in Theorem 1.5 holds for this case. (See [11] for some preliminary theorems about these sets.)

An approach to this problem via symbolic dynamics is currently unavailable. However, one can try and modify the construction (via equidistribution of expanding horocycles) in [9] to force the resulting precompact orbits to have prescribed limit points. (Of course, it is necessary that these limit points themselves have precompact \( \mathbb{R} \)-orbits.) We mention that, although this technique via equidistribution was available to us, we chose to use continued fractions in the case of \( \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z}) \) for the simplicity of their heuristics and as an exercise in understanding Poincare sections.

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\REFERENCES

1. J. An, L. Guan, and D. Kleinbock, \textit{Nondense orbits on homogeneous spaces and applications to geometry and number theory}, Ergodic Theory Dynam. Systems \textbf{42} (2022), no. 4, 1327–1372.
2. N. Andersen and W. Duke, \textit{On a theorem of Davenport and Schmidt}, Acta Arith. \textbf{198} (2021), no. 1, 37–75.
3. V. Beresnevich, F. Ramírez, and S. Velani, *Metric diophantine approximation: aspects of recent work*, In D. Badziahin, A. Gorodnik, & N. Peyerimhoff (Eds.), Dynamics and analytic number theory, London Mathematical Society Lecture Note Series, (pp. 1–95, vol. 437), 2016, Cambridge: Cambridge University Press.

4. N. Chevallier, *The natural extension of the Gauss map and Hermite best approximations*, J. Théor. Nombres Bordeaux 34 (2022), no. 2, 619–636.

5. S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. Reine Angew. Math. 359 (1985), 55–89.

6. H. Davenport and W. M. Schmidt, *Dirichlet’s theorem on diophantine approximation*, 1970 Symposia mathematica, vol. IV (INDAM, Rome, 1968/69), Academic Press, London, pp. 113–132.

7. M. Iosifescu and C. Kraaikamp, *Metrical theory of continued fractions*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2002.

8. A. Khintchine, *Continued fractions*, Translated by Peter Wynn, P. Noordhoff Ltd., Groningen, 1963.

9. D. Kleinbock and G. A. Margulis, *Bounded orbits of nonquasunipotent flows on homogeneous spaces*, Sinai’s Moscow Seminar on Dynamical Systems, Amer. Math. Soc. Transl. Ser. 2, vol. 171, Amer. Math. Soc., Providence, RI, 1996, pp. 141–172.

10. D. Kleinbock and A. Rao, *A zero-one law for uniform Diophantine approximation in Euclidean norm*, Internat. Math. Res. Notices 2022 (2022), no. 8, 5617–5657.

11. D. Kleinbock and A. Rao, *Weighted uniform Diophantine approximation of systems of linear forms*, Pure Appl. Math. Q. 18 (2022), no. 3, 1095–1112.

12. D. Kleinbock, A. Rao, and S. Sathiamurthy, *Critical loci of convex domains in the plane*, Indag. Math. (N.S.) 32 (2021), no. 3, 719–728.

13. D. Kleinbock and B. Weiss, *Values of binary quadratic forms at integer points and Schmidt games*, Recent trends in ergodic theory and dynamical systems, Contemp. Math., vol. 631, Amer. Math. Soc., Providence, RI, 2015, pp. 77–92.

14. K. Mahler, *On irreducible convex domains*, Nederl. Akad. Wetensch., Proc. 50 (1947), 98–107.

15. K. Mahler, *On the minimum determinant and the circumscribed hexagons of a convex domain*, Nederl. Akad. Wetensch., Proc. 50 (1947), 326–337.

16. C. McMullen, *Winning sets, quasiconformal maps and Diophantine approximation*, Geom. Funct. Anal. 20 (2010), 726–740.

17. S. Morimoto, *Zur Theorie der Approximation einer irrationalen Zahl durch rationale Zahlen*, Tohoku Math. J. 45 (1939), 177–187.

18. W. M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. 123 (1966), 178–199.

19. W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics, vol. 785, Springer, Berlin, 1980.

20. M. Urbanski, *The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system*, Nonlinearity 2 (1991), 385–397.

21. V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, vol. 102, Springer, New York, 1984.