Flatness Analysis for the Sampled-data Model of a Single Mast Stacke r Crane *

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Abstract: We show that the Euler-discretization of the nonlinear continuous-time model of a single mast stacker crane is flat. The construction of the flat output is based on a transformation of a subsystem into the linear time-variant discrete-time controller canonical form. Based on the derived flat output, which is also a function of backward-shifts of the system variables, we discuss the planning of trajectories to achieve a transition between two rest positions and compute the corresponding discrete-time feedforward control.

Keywords: discrete-time flatness; discretization; sampled-data control; stacker crane

1. INTRODUCTION

In the 1990s, the concept of flatness has been introduced by Fliess, Lévine, Martin and Rouchon for nonlinear continuous-time systems (see e.g. Fliess et al. (1995) and Fliess et al. (1999)). A nonlinear continuous-time system

\[ \dot{x} = f(x,u) \]  

with \( \dim(x) = n \) states, \( \dim(u) = m \) inputs and smooth functions \( f \) is flat, if there exists a one-to-one correspondence between solutions \( (x(t), u(t)) \) of (1) and solutions \( y(t) \) of a trivial system (sufficiently smooth but otherwise arbitrary trajectories). This one-to-one correspondence implies that for flat systems there exist flat outputs

\[ y = \varphi(x,u,\dot{u},\ldots,u^{(n)}) \]  

such that all states \( x \) and inputs \( u \) can be parameterized by \( y \) and its time derivatives, i.e.,

\[ (x,u) = F(y,\dot{y},\ldots,y^{(n)}) \]  

We call \( F \) the parameterization w.r.t. the flat output \( y \).

In the discrete-time framework one could define flatness simply by replacing the time-derivatives in (2) and (3) by forward-shifts, see e.g. Sira-Ramirez and Agrawal (2004) and Kaldmäe and Kotta (2013). However, it has recently been pointed out in Guillot and Millérioux (2020) that this definition covers only a subclass of flat systems, which we also refer to as forward-flat systems in the sequel. Accordingly, in Diwold et al. (2022a) we have derived a concise geometric definition of flatness, which is (like in the continuous-time case) solely based on the existence of a one-to-one correspondence between solutions of a flat system and solutions of a trivial system. The proposed approach leads to a definition of flatness which includes also backward-shifts of system variables in the flat output.

Both continuous-time and discrete-time flatness allow an elegant solution to motion planning problems as well as the systematic design of tracking controllers. For the practical implementation of a continuous-time controller, however, it is necessary to evaluate the control law at a sufficiently high sampling rate. If, nevertheless, the control loop is limited to lower sampling rates (possibly also due to an online-optimization), a discrete-time control law based on a suitable discretization might be preferable, see e.g. Diwold et al. (2022b). W.r.t. the discrete-time flatness-based controller design, the main challenge is to find a flat discretization as well as the corresponding flat output. So far, computationally efficient methods for the systematic construction of flat outputs have only been derived for forward-flat systems, see Kolar et al. (2021) and Kolar et al. (2022a). However, for the considered sampled-data model of a single mast stacker crane (obtained by an explicit Euler-discretization) a forward-flat output does not exist. Nevertheless, the considered discrete-time system is flat in the extended sense including backward-shifts, as we show in this contribution.

We consider the finite-dimensional, continuous-time model of a single mast stacker crane discussed in Staudecker et al. (2008) and Rams et al. (2018). While in general the construction of flat outputs is a difficult task, two flat outputs are known for the continuous-time system. The first is a configuration-flat output, which can be derived using the methods of Rathinam and Murray (1996) or Gstöttner et al. (2021). However, since the corresponding parameterizing map (3) has singularities at stationary points, the configuration-flat output is not suitable for practical purposes. Rather suitable is the second known flat output, whose first component corresponds to the vertical position of the lifting unit. As shown in Staudecker et al. (2008), the second component is of the form \( y^2 = \varphi^2(x,u,\dot{u}) \) and can be constructed systematically by transforming
In Fig. 1 the setup of a single mast stacker crane is illustrated. With \( q^4 \) we denote the position of the rigid driving unit with mass \( m_{w} \), while \( q^3 \) describes the vertical position of the lifting unit with mass \( m_h \). The mast with length \( L \) is modeled as an Euler-Bernoulli beam with the mass density \( \rho A \) and the flexural rigidity \( EI \). Like in Staudecker et al. (2008) we approximate the deformation of the beam by a first-order Rayleigh-Ritz ansatz

\[
 w(z,t) = \Phi(z)q^4(t) .
\]

Evaluating the Euler-Lagrange equations for the finite-dimensional approximation yields a finite-dimensional system representation of the form

\[
 M(q)\ddot{q} + C(q,\dot{q}) = Gu ,
\]

with \( q = [q^1, q^2, q^3]^T \) and \( u = [F_1, F_2]^T \). The symmetric matrix \( M(q) \) reads as

\[
 M(q) = \begin{bmatrix}
 m_{11} & m_{12} + m_h\Phi(q^3) & m_hq^2\partial_3\Phi(q^3) \\
 * & m_{22} + m_h\Phi(q^3)^2 & m_hq^2\partial_3\Phi(q^3) \\
 * & * & m_h + m_h(q^2\partial_3^2\Phi(q^3))^2 \\
\end{bmatrix}
\]

while \( C(q,\dot{q}) \) as well as \( G \) are given by

\[
 C(q,\dot{q}) = q^2EI \int_0^L (\partial_2^2\Phi(z))^2dz + \Phi(q^3) C_1, \quad G = \begin{bmatrix}
 1 & 0 \\
 0 & 0 \\
 0 & 1 \\
\end{bmatrix},
\]

with

\[
 C_1 = m_h(q^2)^2\partial_3^2\Phi(q^3)^2 + 2m_hq^3\partial_3\Phi(q^3)^3 .
\]

Note that we adopted the abbreviations \( m_{11}, m_{12} \) and \( m_{22} \) from Rams et al. (2018). If we introduce the time-derivatives of \( q \) as states \( v \) (velocities), then we get a state representation of the form

\[
 \begin{bmatrix}
 \dot{q} \\
 \dot{v} \\
\end{bmatrix} = \begin{bmatrix}
 v \\
 -M^{-1}(q)C(q,v) + M^{-1}(q)Gu \\
\end{bmatrix} \\
 f(x,u) \\
 (4)
\]

with \( x = [x^1, x^2, x^3, x^4, x^5]^T = [q^1, q^2, q^3, v^1, v^2, v^3]^T \) and \( u = [u^1, u^2]^T = [F_1, F_2]^T \).

### 2.2 Flatness of the Continuous-time Model

First, we want to emphasize that the system (4) is not static feedback-linearizable. This can be shown with the geometric necessary and sufficient conditions stated e.g. in Nijmeijer and van der Schaft (1990). Nevertheless, as mentioned in Rams et al. (2018), the system (4) is flat and possesses a configuration-flat output of the form

\[
 y = (x^1 + \Phi(x^3)x^2, x^1(m_{22} - m_h\Phi(x^3))) .
\]

The flat output (5) can be derived systematically using the approach of Rathinam and Murray (1996) for underactuated mechanical systems or the more recent geometric methods of Gstötzner et al. (2021). However, for practical purposes, the configuration-flat output (5) is not suitable, because the corresponding parameterizing map (3) has singularities at stationary points. Fortunately, there exists another flat output that can be constructed intuitively using the controller canonical form for linear time-variant systems as shown in Staudecker et al. (2008).

First, by means of the input transformation

\[
 \begin{bmatrix}
 \dot{u}_1 \\
 \dot{u}_2 \\
\end{bmatrix} = \begin{bmatrix}
 f^4(x,u) \\
 f^6(x,u) \\
\end{bmatrix},
\]

the system (4) can be decoupled into two subsystems

\[
 \begin{bmatrix}
 \dot{x}_1 \\
 \dot{x}_2 \\
\end{bmatrix} = \begin{bmatrix}
 A_1 & 0 \\
 0 & A_2 \\
\end{bmatrix} \begin{bmatrix}
 x_1 \\
 x_2 \\
\end{bmatrix} + \begin{bmatrix}
 b_1(x^3) \\
 b_2 \\
\end{bmatrix} \begin{bmatrix}
 \dot{u}_1 \\
 \dot{u}_2 \\
\end{bmatrix},
\]

with \( x_1 = [x^1, x^2, x^4, x^5]^T \), \( x_2 = [x^3, x^6]^T \) and

\[
 A_1 = \begin{bmatrix}
 0 & 1 \\
 0 & 0 \\
\end{bmatrix}, \quad b_1 = \begin{bmatrix}
 0 \\
 1 \\
\end{bmatrix}
\]

and

\[
 A_2 = \begin{bmatrix}
 0 & 0 \\
 0 & 1 \\
\end{bmatrix}, \quad b_2 = \begin{bmatrix}
 0 \\
 0 \\
\end{bmatrix}
\]
Since the subsystem related to $x_2$ is in controller canonical form, the state $x^3$ (vertical position of the lifting unit) is a candidate for the first component of the flat output $y$. With the choice $y^1 = x^3$ we can already parameterize the states $x_2$ and the input $\ddot{u}^2$ of the second subsystem by the flat output and its time-derivatives, i.e.,

$$x^3 = y^1, \quad x^0 = y^1, \quad \ddot{u}^2 = \ddot{y}^1.$$  

In order to compute the second component of the flat output, let us fix in an intermediate step a trajectory $y^1(t)$ for the first component of the flat output. The corresponding trajectories for $x^3(t)$, $x^0(t)$ and $\ddot{u}^2(t)$ are then fixed via the relations (8). If we insert $x^3(t), x^0(t)$ and $\ddot{u}^2(t)$ into the first subsystem, we get a linear time-variant system

$$\dot{x}_1 = A_1(x^3(t), x^0(t), \ddot{u}^2(t)) x_1 + b_1(x^3(t)) \ddot{u}^1.$$

For linear time-variant flat systems, however, a flat output can be constructed systematically using the state transformation

$$\dot{x}_1 = T(x^3(t), x^0(t), \ddot{u}^2(t), \ldots, \ddot{u}^{2(4)}(t)) x_1 \quad (9)$$

into the time-variant controller-canonical form, see e.g. Ilchmann and Mueller (2007). In such coordinates, the first component of $x_1$ corresponds to the second component of the flat output. With the inverse transformation of (6) and (9), the second component of the flat output can be expressed in terms of the original coordinates and is of the form $y^2 = \varphi^2(x, u, \dot{u})$. In contrast to (5), the corresponding parameterizing map for the flat output

$$y = (x^3, \varphi^2(x, u, \dot{u}))$$

has no singularities at stationary points and can therefore be used for practical purposes.

### 2.3 Motivation and Problem Statement

The concept of continuous-time flatness allows the systematic design of tracking controllers. Nevertheless, for the practical implementation it is necessary to evaluate the continuous-time control law at a sufficiently high sampling rate. For lower sampling rates, a discrete evaluation of such a continuous-time control law could lead to unsatisfactory results, as we have shown e.g. in Diwold et al. (2022b) by means of the laboratory setup of a gantry crane. An alternative proposition is to design the controller directly for a suitable discretization. The main challenge in the design of discrete-time flatness-based control laws, however, is to find a flat discretization, since in general flatness is not preserved by an approximate or exact discretization of a continuous-time system. For this reason, in Diwold et al. (2022b) we proposed a method that combines the explicit Euler-discretization and a prior state transformation in such a way that not only the flatness but also the flat output is preserved. Although this method is restricted to a subclass of continuous-time $x$-flat systems, it can be applied to many practical examples such as the induction motor, the VTOL aircraft, or the unmanned aerial vehicle discussed in Greff et al. (2022). Even though the model of a stacker crane (4) together with the flat output (5) also belongs to this class, in this case our method is not useful for practical applications. The reason is that not only the flat output (5) would be preserved but also the corresponding singularities in the parameterizing map. Thus, the main objective of this paper is to construct a flat output for the explicit Euler-discretization of (4) in a similar way as it is shown in Section 2.2 for the continuous-time model. For this purpose, we subsequently extend the definition of flatness as proposed in Diwold et al. (2022a) to time-variant systems and briefly discuss the controller canonical form for linear time-variant discrete-time systems. Based on these results, in Section 4 we assume again that the vertical position of the lifting unit corresponds to the first component of the flat output, and construct the second component by transforming a linear time-variant discrete-time subsystem into the controller canonical form. As we will see below, the constructed flat output depends also on backward-shifts of system variables.

### 3. Flatness of Time-Variant Discrete-Time Systems

In the following we recall and extend the definition of flatness as proposed in Diwold et al. (2022a) to time-variant discrete-time systems. Note that in this paper we use a slightly different notation than in previous contributions (e.g. Kolar et al. (2021), Diwold et al. (2022a) and Diwold et al. (2022b)), since we focused only on time-variant discrete-time systems so far.

#### 3.1 Flatness of Nonlinear Time-variant Systems

We consider time-variant discrete-time systems in state representation

$$x_{k+1} = f_k(x_k, u_k) \quad (10)$$

with $\dim(x_k) = n$, $\dim(u_k) = m$ and smooth functions $f_k(x_k, u_k)$. Furthermore, we assume that the Jacobian matrix of $f_k$ w.r.t. $(x_k, u_k)$ meets

$$\text{rank}(\partial(x_k, u_k)f_k) = n \quad (11)$$

for all $k$ (submersivity). As stated in Diwold et al. (2022a), flatness for discrete-time systems can be characterized by a one-to-one correspondence between the system trajectories $(x_k, u_k)$ and trajectories $y_k$ of a trivial system, i.e., arbitrary trajectories that need not satisfy any difference equation. In contrast to the time-invariant systems considered in Diwold et al. (2022a), for time-variant systems (10) the maps

$$(x_k, u_k) = F(k, y_{k-r_1}, \ldots, y_k, \ldots, y_{k+r_2}) \quad (12)$$

and

$$y_k = \varphi(k, x_{k-q_1}, u_{k-q_1}, \ldots, x_k, u_k, \ldots, x_{k+q_2}, u_{k+q_2}) \quad (13)$$

describing this one-to-one correspondence may also depend explicitly on $k$. As explained in Diwold et al. (2022a), the composition of (12) with the occurring shifts of (13) must yield the identity map, and vice versa. Furthermore, since the trajectories $y_k$ of a trivial system are arbitrary, substituting (12) into (10) must also yield an identity. The map (13) can be further simplified by taking into account that every trajectory $\ldots, x_{k-1}, u_{k-1}, x_k, u_k, x_{k+1}, u_{k+1}, \ldots$ satisfies (10). Hence, the forward-shifts of $x_k$ can be expressed as functions of $k, x_k, u_k, u_{k+1}, \ldots$ via the successive compositions

$$x_{k+1} = f_k(x_k, u_k)$$

$$x_{k+2} = f_{k+1}(f_k(x_k, u_k), u_{k+1}) \quad (14)$$

1 Instead of $f'(k, x_k, u_k)$ we write $f'_k(x_k, u_k)$ for a better readability.
A similar argument holds for the backward-direction. Since (10) meets the submersivity condition (11), there always exist \( m \) functions \( g_k(x_k, u_k) \) such that the map
\[
x_{k+1} = f_k(x_k, u_k), \quad \zeta_k = g_k(x_k, u_k),
\]
is locally invertible. If we denote by \( (x_k, u_k) = \psi_k(x_{k+1}, \zeta_k) \) its inverse
\[
x_k = \psi_x(x_{k+1}, \zeta_k), \quad u_k = \psi_u(x_{k+1}, \zeta_k)
\]
then we can express the backward-shifts of \((x_k, u_k)\) as functions of \(k, x_k, \zeta_{k-1}, \zeta_{k-2}, \ldots \) via the successive compositions
\[
(x_{k-1}, u_{k-1}) = \psi_{k-1}(x_k, \zeta_{k-1})
\]
\[
(x_{k-2}, u_{k-2}) = \psi_{k-2}(x_{k-1}, \zeta_{k-1}), \zeta_{k-2}) \, .
\]
(17)

Thus, with (14) and (17), the map (13) can be written as
\[
y_k = \varphi(k, \zeta_{k-1}, \zeta_{k-2}, \ldots, \zeta_1, x_k, u_k, \ldots, u_{k+q_0}).
\]
(18)

The map (18) is called a flat output of the time-variant system (10), and the corresponding map (12) describes the parameterization of the system variables \((x_k, u_k)\) by this flat output.

Remark 1. The flatness of the system (10) does not depend on the choice of the functions \(g_k(x_k, u_k)\) of (15). The representation (18) of the flat output may differ, but the parameterization (12) is not affected.

Note also that considering backward-shifts in both (12) and (18) is actually not necessary. If there exist a parameterizing map (12) and a flat output (18), then one can always define a new flat output as the \(r_1\)-th backward-shift of the original flat output. The corresponding parameterizing map is then of the form
\[
(x_k, u_k) = F(k, y_k, \ldots, y_{k+r})
\]
with \( r = r_1 + r_2 \) and does not contain backward-shifts. Systems that possess a flat output \( y_k = \varphi(k, x_k, u_k, \ldots, u_{k+q})\) which is independent of backward-shifts of \( \zeta_k \) together with a parameterizing map (19) will be denoted in the following as forward-flat.

### 3.2 Flatness of Linear Time-variant Single-input Systems

In this section, we focus on the special case of linear time-variant discrete-time systems\(^2\)
\[
x_{k+1} = A_k x_k + b_k u_k
\]
with \( \dim(x_k) = n \) and \( \dim(u_k) = 1 \). For constructing a flat output of the sampled-data model of the single mast stacker crane, like in the continuous-time case a linear time-variant transformation
\[
x_k = T_k \bar{x}_k
\]
into the controller canonical form is useful. In general, the transformation (21) yields again a linear time-variant system of the form
\[
x_{k+1} = \bar{A}_k \bar{x}_k + \bar{b}_k u_k
\]
with \( \bar{A}_k = T_{k+1}^{-1} A_k T_k \) and \( \bar{b}_k = T_{k+1}^{-1} b_k \). A system (22) is said to be in controller canonical form if the matrix \( \bar{A}_k \) and the vector \( \bar{b}_k \) read as
\[
\bar{A}_k = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{0,k} & -a_{1,k} & \cdots & -a_{n-1,k} \end{bmatrix}, \quad \bar{b}_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]
with time-variant coefficients \(a_{0,k}, \ldots, a_{n-1,k}\), see also Hwang et al. (1985). Note that \(a_{0,k}, \ldots, a_{n-1,k}\) do not necessarily correspond to the coefficients of the characteristic polynomial of \(A_k\). Obviously, the existence of a transformation into controller canonical form is a sufficient condition for flatness, since the first component of \(\bar{x}_k\) forms a flat output. In the following, we show how the transformation into controller canonical form can be constructed in a systematic way. For this purpose, we use an ansatz \(\dot{y}_k = c_k^T \bar{x}_k\) and require that all forward-shifts of \(y_k\) up to the order \(n-1\) are independent of \(u_k\), i.e.,
\[
y_{k+1} = c_{k+1}^T A_k x_k + c_{k+1}^T b_k u_k
\]
\[
y_{k+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -a_{0,k} & -a_{1,k} & \cdots & -a_{n-1,k} \end{bmatrix} \begin{bmatrix} x_k \\ \vdots \\ x_0 \end{bmatrix} + b_k u_k.
\]
(23)

For the \(n\)-th forward-shift of \(y_k\) we require
\[
y_{k+n} = c_{k+n}^T A_{k+n-1} \cdots A_k x_k + c_{k+n}^T A_{k+n-1} \cdots A_k b_k u_k.
\]
(24)

By applying suitable backward-shifts, the conditions (23) and (24) can be written as
\[
c_k^T [b_{k-1} A_{k-1} b_{k-2} \ldots A_{k-1} \cdots A_{k-n} b_{k-n}] = e_n^T
\]
(25)
\[
= M_k
\]
with the \(n\)-th unit vector \(e_n\). Obviously, there exists a solution for \(c_k^T\) if and only if the matrix \(^3\) \(M_k\) is regular for all \(k\). The transformation into controller canonical form is then given by
\[
\bar{x}_k = \begin{bmatrix} y_k \\ \vdots \\ y_{k+1} \\ \vdots \\ y_{k+n-1} \end{bmatrix} = \begin{bmatrix} c_k^T \\ c_{k+1}^T A_k \\ \vdots \\ c_{k+n-1}^T A_{k+n-2} \cdots A_k \end{bmatrix} T_k^{-1} x_k.
\]
(26)

In the controller canonical form (22), the first component of the state \(\bar{x}_k\) is a flat output. For the system (20) in original coordinates, this flat output is given by
\[
y_k = \varphi(k, x_k) = c_k^T x_k.
\]
(27)

The corresponding parameterization of \((x_k, u_k)\) follows as
\[
x_k = F_k(k, y_k, \ldots, y_{k+n-1}) = T_k \bar{x}_k
\]
and
\[
u_k = F_u(k, y_k, \ldots, y_{k+n}) = y_{k+n} + \sum_{i=0}^{n-1} a_{i,k} y_{k+i}.
\]
(29)

Since neither (27) nor (28) and (29) depend on backward-shifts, the condition
\[
\text{rank}(M_k) = n, \quad \forall k
\]
even ensures forward-flatness. In fact, it can be shown that this condition is not only sufficient but also necessary for flatness.

\(^2\) Again, instead of \(A(k)\) and \(b(k)\), we write \(A_k\) and \(b_k\), respectively.

\(^3\) For linear time-invariant systems the matrix \(M_k = M\) corresponds to the controllability matrix \(M = [b, Ab, \ldots, A^{n-1} b]\).
4. SAMPLED-DATA MODEL OF THE SINGLE MAST STACKER CRANE

In this section we consider the explicit Euler-discretization
\[ x_{k+1} = x_k + T_s f(x_k, u_k) \]  
(31)
of the single mast stacker crane (4) and derive a flat output similar as in the continuous-time case. Subsequently, we use the corresponding parameterizing map to compute a feedforward control which achieves a transition of the single mast stacker crane between two rest positions.

4.1 Flatness of the Sampled-data Model

In the following, we proceed with the explicit Euler-discretization (31) of the transformed system (7), i.e.
\[ x_{k+1} = x_k + T_s f(x_k, u_k). \]
This is possible since the explicit Euler-discretization and input-transformations commute, see e.g. Diwold et al. (2022b). The Euler-discretized system reads as
\[
\begin{bmatrix}
  x_{1,k+1} \\
  x_{2,k+1}
\end{bmatrix} =
\begin{bmatrix}
  A_1(x_k^2, x_k^3, x_k^4, u_k^2, u_k^3) & 0 \\
  0 & A_2
\end{bmatrix}
\begin{bmatrix}
  x_{1,k} \\
  x_{2,k}
\end{bmatrix} +
\begin{bmatrix}
  b_1(x_k^2) & 0 \\
  0 & b_2
\end{bmatrix} \tilde{u}_k
\]with
\[
x_{1,k} = [x_k^1, x_k^2, x_k^3, x_k^4]^T, \quad x_{2,k} = [x_k^5, x_k^6]^T
\]and
\[
A_2 = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ T_s \end{bmatrix}
\]Note that also the matrix \( A_1 \) and the vector \( b_1 \) differ from the ones defined in (7). Like in the continuous-time case, the vertical position of the lifting unit \( x_k^5 \) is a candidate for \( y_k \), since with (32) we can already parameterize
\[
x_k^3 = y_k^1, \quad x_k^5 = \frac{y_{k+1} - y_k}{T_s}, \quad \tilde{u}_k^2 = \frac{y_k^1 - x_k^5}{T_s}.
\]Once a trajectory \( y_k^1 \) is fixed, there remains a linear time-variant system of the form
\[
x_{1,k+1} = A_1(x_k^1, x_k^2, x_k^3, x_k^4, u_k^2)x_{1,k} + b_1(x_k^2)u_k^1
\]which can be checked w.r.t. flatness by condition (30). Since there appear backshifts of \( A_1(x_k^1, x_k^2, x_k^3, x_k^4, u_k^2) \) and \( b_1(x_k^2) \) within the matrix \( M_k \), we need to introduce the functions \( g_k(x_k, u_k) \) of (15). With the inverse (16), we can then express the backshifts of \( x_k^1 \) as functions of \( x_k \) and backshifts of \( \xi_k \), see also equation (17). With the choice
\[
\xi_k^1 = x_k^2, \quad \xi_k^2 = x_k^1,
\]we get the matrix
\[
M_k = [h(\xi_k^1, A_1(\xi_k^1, x_k^3, x_k^4, \xi_k^2)h(\xi_k^1, x_k^3, x_k^4, \xi_k^2), \ldots, A_1(\xi_k^1, x_k^3, x_k^4, \xi_k^2), A_1(\xi_k^1, x_k^3, x_k^4, \xi_k^2)]
\]which is regular for arbitrary values \( \xi_k^1, \ldots, \xi_k^1, x_k^3, x_k^4, \xi_k^2 \). This holds independently of the chosen extension \( \xi_k = g_k(x_k, u_k) \) with the corresponding controller canonical form, we can now determine the second component of the flat output. Due to relation (29), also \( \xi_k^1 \) depends on \( \xi_k^1, x_k^3, x_k^4, \xi_k^2 \) and the second component of the flat output \( y_k^2 \) has the form
\[
y_k^2 = c_1 x_k = c_1^T(\xi_k^1, x_k^3, x_k^4, \xi_k^2)x_k.
\]In order to obtain the parameterization for \( x_k \) and \( u_k^1 \), we use the matrix \( T_k \) of (26), which is
\[
T_k = T(\xi_k^1, \ldots, \xi_k^1, x_k^3, x_k^4, \xi_k^2, u_k^2, \ldots, u_k^2).
\]
4.2 Trajectory Planning and Feedforward Control

The derived parameterizing map can be used efficiently for planning trajectories. We illustrate this by planning a trajectory which achieves a transition between two rest positions \( x_0 = 0 \) when \( N \) is at \( x_N = 0 \). In rest positions, the shape of the beam is straight and the velocities are zero. Hence, the corresponding state variables must \( x_0 = x_N = 0 \) for \( i = 2, 4, 5, 6 \). Furthermore, since in a rest position the shifts of the flat output are constant, a desired reference trajectory must meet
\[
y_{d,0} = \cdots = y_{d,9}, \quad y_{d,0} = \cdots = y_{d,4},
\]\[
y_{d,2N} = \cdots = y_{d,2N+4}, \quad y_{d,2N} = \cdots = y_{d,2N+4}.
\]If we insert (35) into the parameterization (34) for \( x_k \), we get
\[
x_0 = F_x(y_{d,0}, \ldots, y_{d,0}, y_{d,0}, \ldots, y_{d,0})
\]\[
x_N = F_x(y_{d,2N}, \ldots, y_{d,2N}, y_{d,2N}, \ldots, y_{d,2N}),
\]which can be solved for \( y_{d,0}, y_{d,0}, y_{d,0}, y_{d,0} \). The remaining values of \( y_{d,10}, \ldots, y_{d,20}, \ldots, y_{d,2N} \) can be chosen arbitrarily. The simplest approach is to use polynomial functions \( y_{d,k} = \sum_{l=0}^{l_k} p_k^l k^l \) of appropriate order \( l_k \) with suitable coefficients \( p_k^l \). Once the trajectories \( y_{d,0}, y_{d,0} \) and \( y_{d,0}, y_{d,0} \) are fixed, the corresponding state- and input-trajectories \( (x_{d,k}, u_{d,k}) \) are uniquely determined by the parameterizing map. In Fig. 2 the transition between two rest positions is illustrated, and the corresponding feedforward control is depicted in Fig. 3. All parameters for the model (4) as well as the Rayleigh-Ritz ansatz function \( \Phi(z) \) were adopted from Staudt et al. (2008) and Rams et al. (2018). The sampling time was set to \( T_s = 50ms \).
In this contribution, we have shown that the explicit Euler-discretization of the single mast stacker crane (4) is flat. For this purpose, we have constructed a flat output by means of the controller canonical form for linear time-variant discrete-time systems. In contrast to the sampled-data model of a gantry crane that we derived in Diwold et al. (2022b), the explicit Euler-discretization of (4) is flat but not forward-flat. However, as illustrated in Section 4.2, also the more general class of flat systems (including backward-shifts) allows for the systematic planning of trajectories and design of feedforward controls. Like in the continuous-time case, one could also formulate an optimization problem (e.g. time-optimal transition) instead of using polynomial functions for \( y_{d,k} \). While often ansatz functions (e.g. spline curves) are used for this task in the continuous-time case in order to obtain a finite number of optimization variables, the optimization problem is inherently finite-dimensional in the discrete-time case. Apart from formulating and solving an optimization problem, subject to further research could be the design of discrete-time flatness-based tracking controllers based on an exact linearization as proposed in Kolar et al. (2022b). Subsequently, the discrete-time approach could be compared to existing continuous-time (sampled) control laws in simulations as well as on the laboratory setup of the single mast stacker crane.