Modelling the longitudinal intensity pattern of diffraction resistant beams in stratified media

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Abstract

In this paper, we study the propagation of the Frozen Wave type beams through non-absorbing stratified media and develop a theoretical method capable to provide the desired spatially shaped diffraction resistant beam in the last material medium. In this context, we also develop a matrix method to deal with stratified media with large number of layers. Additionally, we undertake some discussion about minimizing reflection of the incident FW beam on the first material interface by using thin films. Our results show that it is indeed possible to obtain the control, on demand, of the longitudinal intensity pattern of a diffraction resistant beam even after it undergoes multiple reflections and transmissions at the layer interfaces. Remote sensing, medical and military applications, noninvasive optical measurements, etc., are some fields that can be benefited by the method here proposed.

Keywords: non-diffractive beams, stratified media, optics

1. Introduction

The propagation of electromagnetic waves through inhomogeneous media has been widely studied since a long, mainly because it is a more realistic configuration of what occurs in nature. In general, the theoretical models dealing with this subject are developed for plane waves [1][2][3][4][5] and, although researches on Localized Waves (LWs) [6][7][8] have proved to be quite advanced, few works have been addressed on the propagation of such beams along stratified media, often considering, when it is the case, just a single Bessel beam [9].

In the context of the LWs, a very interesting kind of diffraction-attenuation resistant wave is the so called Frozen Wave (FW), which possesses great potential for applications wherever one wishes to shape, on demand, the longitudinal intensity pattern of electromagnetic wave beams.

The theory of FWs was developed in [10][11] and the experimental generation of such beams was reported in [13][12][14]. Such waves are constructed by discrete superposition of co-propagating and equal frequency Bessel beams with different complex amplitudes and longitudinal wavenumbers, possessing, as the main characteristic, the possibility of assuming (approximately) any desired longitudinal intensity pattern within a spatial interval on the propagation axis, while maintaining the diffraction resistant properties of the ordinary Bessel beams [11][13][16].
In this paper, we study the behavior of a FW-type-beam when propagating through non-absorbing stratified media along $z$ direction, showing how the desired beam pattern in the last medium is affected by the inhomogeneities faced by the beam along its propagation. To overcome this issue, we develop a novel method for constructing an incident FW beam capable of compensating such inhomogeneity effects, so rendering the desired spatially-shaped-diffraction-resistant-beam in the last material medium. Additionally, we make some considerations about minimizing the reflection of the incident FW beam by using thin films. Finally, in the last section, we develop a matrix-transfer method to deal with stratified media with a large number of layers.

Our theoretical method works under a scalar model, thus, we are assuming the paraxial regime, with a linearly polarized wave beam.

2. A brief overview about the FW method for homogeneous media

A FW beam is given by a superposition of $2N + 1$ co-propagating and equal frequency Bessel beams (the time harmonic term, $e^{-i\omega t}$, is omitted along the entire paper):

$$
\Psi(\rho, \phi, z) = N_\nu e^{i\nu\phi} \sum_{q=-N}^{N} A_q J_\nu(h_q \rho) e^{i\beta_q z}
$$

where

$$
h^2_q = n^2 \omega^2 - \beta^2_q ,
$$

$n$ is the refractive index of the medium, $\omega$ the angular frequency, $c$ the light velocity and $N_\nu = 1/[J_\nu(.)]_{\text{max}}$, with $[J_\nu(.)]_{\text{max}}$ denoting the maximum value of the $\nu$-order Bessel function of first kind. The constant coefficients, $A_q$, provide the complex amplitudes for each Bessel beam in the superposition, while $h_q$ and $\beta_q$ are the Bessel beams transverse and longitudinal wavenumbers, respectively.

The method’s main goal is to find out the values of $\beta_q$, and $A_q$ in Eq.(1), in order to reproduce, approximately, within $-L/2 \leq z \leq L/2$, and over a cylindrical surface of radius $\rho_0$, a desired longitudinal intensity pattern, $|F(z)|^2$, chosen a priori. That is, we wish $|\Psi(\rho = \rho_0, \phi, z)|^2 \approx |F(z)|^2$. To obtain such result, we make the following choices [10, 11, 17]:

$$
\beta_q = Q + \frac{2\pi q}{L} , \quad \text{with } q = -N, -N + 1, ..., N - 1, N
$$

and

$$
A_q = \frac{1}{L} \int_{-L/2}^{L/2} F(z) e^{-i\frac{2\pi q}{L} z} dz ,
$$

with $0 \leq \beta_q \leq n\omega/c$ and where the value of $Q = an\omega/c$ ($0 < a < 1$) can be freely chosen, since relation (2) is obeyed.

It is also possible to get some control on the transverse beam profile, more specifically, we can choose the desired spot radius, $r_0$, of the resulting beam from the parameter $Q$, via the relation $r_0 \approx 2.4/h_0$, for
superpositions of Bessel beam of zeroth order (\(\nu = 0\)). For \(\nu > 0\), i.e. higher order FWs, the longitudinal intensity pattern will be concentrated over a cylindrical surface of radius \(\rho_\nu\), whose value corresponds the first positive root of \([[d/d\rho]J_\nu(\rho h_0)] |_{\rho=\rho_\nu} = 0\).

The details of the method can be found in [10, 17].

3. Propagation of a frozen wave in a nonabsorbing stratified medium

Following a scalar model, Eq. (1) represents the transverse cartesian component of the electric field, with a negligible longitudinal component (paraxial approximation). In the case of Bessel beams, such assumption can be made when \(h \ll n\omega/c\), resulting in a transverse spot size much larger than the wavelength.

Reflection and transmission

Let us consider a stratified medium formed by \(M\) layers with refractive indexes \(n_m (m = 1, 2, ..., M)\) and with their interfaces located at the positions \(z = d_1, d_2, ..., d_{M-1}\). See Figure 1.

![Figure 1: Schematic representation of the stratified medium.](image)

When a FW (which is a superposition of Bessel beams) coming from the first medium impinges normally upon the first interface, a process of multiple reflections and transmissions takes place at each interface of the stratified medium. It is well known that for a single Bessel beam in normal incidence on a plane interface, the reflected and transmitted waves are also Bessel beams with the same order and with the same transverse wavenumber of the incident beam, and, naturally, with amplitudes that depend on the refractive indexes of both media and also on the cone angle of the incident Bessel beam [9].

Based on this, we can calculate the field in all \(M\) layers resulting from the normally incident FW beam, \(\Psi_{inc}(\rho, \phi, z)\), coming from the first medium and given by:

\[
\Psi_{inc}(\rho, \phi, z) = N_\nu e^{i\phi\nu} \sum_{q=-N}^{N} A_q J_\nu(h_q \rho) e^{i\beta_1 q z} \tag{5}
\]

with

\[
h_q = \sqrt{n_1^2 \frac{\omega^2}{c^2} - \beta_1^2} \tag{6}
\]
In the $m$th-layer there are forward Bessel beams, $N_\nu A_q J_\nu (h_q \rho) \exp(\imath \nu \phi) \exp(\imath \beta_m q z)$, and backward ones, $N_\nu A_q \Gamma_m J_\nu (h_q \rho) \exp(\imath \nu \phi) \exp(-\imath \beta_m q z)$, which correspond to the incident Bessel beams $N_\nu A_q J_\nu (h_q \rho) \exp(\imath \nu \phi) \exp(\imath \beta_q z)$, with $q = -N, -N + 1, \ldots, N - 1, N$. In these previous equations, $\Gamma_m$ and $\tau_m$ are the reflection and transmission coefficients at the $m$th and $(M-1)$th interfaces, respectively, and $\beta_m = \sqrt{n^2_m \omega^2/c^2 - h^2_q}$.

In this way, considering the incident FW, Eq.(5), the total field within each medium will be:

$$\Psi_1(\rho, \phi, z) = N_\nu e^{\imath \nu \phi} \sum_{q=-N}^{N} [A_q J_\nu (h_q \rho) e^{\imath \beta_q z} + A_q \Gamma_q J_\nu (h_q \rho) e^{-\imath \beta_q z}] \quad (z \leq d_1)$$

$$\vdots$$

$$\Psi_m(\rho, \phi, z) = N_\nu e^{\imath \nu \phi} \sum_{q=-N}^{N} [A_q \tau_m J_\nu (h_q \rho) e^{\imath \beta_m q z} + A_q \Gamma_m J_\nu (h_q \rho) e^{-\imath \beta_m q z}] \quad (d_{m-1} \leq z \leq d_m)$$

$$\vdots$$

$$\Psi_M(\rho, \phi, z) = N_\nu e^{\imath \nu \phi} \sum_{q=-N}^{N} A_q \tau_M J_\nu (h_q \rho) e^{\imath \beta_M q z} \quad (z \geq d_{M-1})$$

with

$$\beta_{mq} = \sqrt{n^2_m \omega^2/c^2 - h^2_q} \quad m = 1, 2, \ldots, M$$

In order to evaluate the reflection and transmission coefficients, $\Gamma_m$ and $\tau_m$, respectively, we must apply the boundary conditions on each interface. Such conditions (expressing the continuity of the tangential electric and magnetic fields) are given by:

$$\Psi_1(\rho, \phi, z)|_{z=d_1} = \Psi_2(\rho, \phi, z)|_{z=d_1}$$

$$\frac{\partial \Psi_1(\rho, \phi, z)}{\partial z}|_{z=d_1} = \frac{\partial \Psi_2(\rho, \phi, z)}{\partial z}|_{z=d_1}$$

$$\vdots$$

$$\Psi_{M-1}(\rho, \phi, z)|_{z=d_{M-1}} = \Psi_{\tau M}(\rho, \phi, z)|_{z=d_{M-1}}$$

$$\frac{\partial \Psi_{M-1}(\rho, \phi, z)}{\partial z}|_{z=d_{M-1}} = \frac{\partial \Psi_{\tau M}(\rho, \phi, z)}{\partial z}|_{z=d_{M-1}}$$

By using Eqs.(7) in Eqs.(9), we can find out all $\Gamma_{mq}$ and $\tau_{mq}$ and, therefore, fully characterize the total field in each layer through Eqs.(7).

In the next two examples, we will use the equations developed in this section to precisely describe the propagation of a FW through stratified media with few layers. More specifically, we will see very clearly

\footnote{Naturally, $\tau_{1q} = 1$ and $\Gamma_{Mq} = 0$.}
how the desired intensity pattern of a FW, initially designed for an homogeneous medium, is affected after passing through all the material layers, suffering multiple reflections and transmissions.

In all examples of this paper we will use the angular frequency $\omega = 2.9788 \times 10^{15}\text{rad/s}$, which correspond to $\lambda = 632.8\text{nm}$ in vacuum.

**First example**

Let us consider a stratified structure formed by four different media with refractive indexes $n_1 = 1$, $n_2 = 3$, $n_3 = 2.5$, $n_4 = 4$ and whose interfaces are located at $d_1 = 0$, $d_2 = 0.01\text{m}$ and $d_3 = 0.025\text{m}$.

Considering an incident beam given by Eq.(5) with $\nu = 0$, we know that the field within each layer is given by Eqs.(7), and by using Eqs.(9) we find the reflection and transmission coefficients

\[
\Gamma_{1q} = \frac{P^+ \beta_{1q} - P^- \beta_{2q}}{P^- \beta_{2q} + P^+ \beta_{1q}}
\]

\[
\Gamma_{2q} = \frac{\beta_{1q}(1 - \Gamma_{1q})}{\beta_{2q}(e^{2i\alpha_{1q}} \xi - 1)}
\]

\[
\tau_{2q} = \Gamma_{2q} e^{-2i\alpha_{1q}} \xi
\]

\[
\Gamma_{3q} = \frac{T_{2q} e^{i\alpha_{1q}} + \Gamma_{2q} e^{-i\alpha_{1q}}}{e^{-i\alpha_{2q}} - e^{i(\alpha_{2q} - 2\alpha_{3q})} \Phi}
\]

\[
\tau_{3q} = -\Gamma_{3q} e^{-2i\alpha_{3q}} \Phi
\]

\[
\tau_{4q} = \frac{\beta_{3q}(T_{3q} e^{i\alpha_{3q}} - \Gamma_{3q} e^{-i\alpha_{3q}})}{\beta_{4q} e^{2i\alpha_{3q}}}
\]

with

\[
\Phi = \frac{\beta_{4q} + \beta_{3q}}{\beta_{4q} - \beta_{3q}}
\]

\[
\xi = \frac{\beta_{2q}(e^{-i\alpha_{2q}} - e^{i(\alpha_{2q} - 2\alpha_{3q})} \Phi) - \beta_{3q}(e^{-i\alpha_{2q}} + e^{i(\alpha_{2q} - 2\alpha_{3q})} \Phi)}{\beta_{2q}(e^{-i\alpha_{2q}} - e^{i(\alpha_{2q} - 2\alpha_{3q})} \Phi) + \beta_{3q}(e^{-i\alpha_{2q}} + e^{i(\alpha_{2q} - 2\alpha_{3q})} \Phi)}
\]

and

\[
P^+ = e^{-2i\alpha_{1q}} \xi + 1 \quad P^- = e^{-2i\alpha_{1q}} \xi - 1
\]

where $\alpha_{1q} = \beta_{2q} d_2 \quad \alpha_{2q} = \beta_{3q} d_2 \quad \alpha_{3q} = \beta_{3q} d_3 \quad \alpha_{4q} = \beta_{4q} d_3$

The values of the longitudinal wavenumbers, $\beta_q$, and the coefficients $A_q$ of the incident FW are still unknown and should depend on the beam spatial structure we wish for the last medium. But, due to the fact that, for now, we do not have a method to get the desired diffraction resistant beam in the last medium (after the stratified structure), we will design the incident beam as if we were dealing with an
homogeneous medium (the first one, in this case) and, thus, we will see the effects suffered by the beam after it has crossed the stratified structure.

Now, let us suppose we wish, in the last medium, a diffraction resistant beam with a spot radius \( r_0 \approx 4.5 \mu m \) and whose desired longitudinal intensity pattern, \( |F(z)|^2 \), has a ladder shape of three steps. More specifically, within \(-L/2 \leq z \leq L/2\),

\[
F(z) = \begin{cases} 
\sqrt{3} & l_1 \leq z < l_2 \\
\sqrt{2} & l_2 \leq z < l_3 \\
1 & l_3 \leq z \leq l_4 \\
0 & \text{elsewhere}
\end{cases}
\]  

(18)

where \( L = 0.30\text{m}, \delta = 0.035\text{m}, l_1 = 0.03\text{m}, l_2 = l_1 + \delta, l_3 = l_2 + \delta \) and \( l_4 = l_3 + \delta \). The value of \( Q \) is obtained from the desired spot radius, resulting in \( Q = 0.99856n_1\omega/c \). In this case, let us adopt the value \( N = 20 \). The coefficients \( A_q \) are calculated inserting the function (18) in (4), and the longitudinal wavenumbers \( \beta_{mq} \) by (3), being the values of \( h_q \) given by Eq.(6).

By knowing the values of \( h_q \), we can determine the sets of longitudinal wavenumbers in the second, third and fourth medium through \( \beta_{mq} = \sqrt{n_m^2\omega^2/c^2 - h_q^2}, m = 2, 3, 4 \). Thus, we can write the transmitted field in the medium 4 as

\[
\Psi_4(\rho,\phi,z) = N_\nu \sum_{q=-N}^{N} A_q \tau_{4q} J_0(h_q\rho) e^{i\beta_{4q}z} \quad (z \geq d_3)
\]  

(19)

Figure 2: (a) Comparison between the on-axis longitudinal intensity of the FW (with \( \nu = 0 \)) and desired intensity pattern, \( |F(z)|^2 \); (b) the 3D field intensity of the resulting beam in medium 4, as well as its orthogonal projection in the detail.

Figure 2(a) shows the desired longitudinal intensity pattern, \( |F(z)|^2 \), and the on-axis intensity of the resulting field through the four media. We can see that the characteristics of the FW beam are very much
affected when compared to the desired pattern. Figure 2(b) show the 3D field intensity of the resulting wave, again in all media, with its orthogonal projection in the detail.

**Second example**

In this example we consider a stratified medium of four layers, with refractive indexes (as before) \( n_1 = 1, n_2 = 3, n_3 = 2.5 \) and \( n_4 = 4 \), whose interfaces are located at \( d_1 = 0, d_2 = 5 \text{mm} \) and \( d_3 = 10 \text{mm} \). At this time we are going to deal with a hollow FW so, in the incident beam solution, Eq. (5), we choose \( \nu = 4 \). The resulting field within each layer is given by Eq. (7). The reflection and transmission coefficients can be found through Eqs. (7,9), resulting in Eqs. (10)-(15).

Here, we wish, in the last medium, a diffraction resistant beam concentrated on a cylindrical surface of radius \( \rho_4 \approx 3 \mu \text{m} \) and with a longitudinal intensity pattern, \(|F(z)|^2\), of a ladder shape of three steps, according to Eq. (18), but now with \( L = 50 \text{ mm}, \delta = L/100, l_1 = (L/4 + 4\delta), l_2 = (l_1 + 4\delta), l_3 = (l_2 + 4\delta) \) and \( l_4 = (l_3 + 4\delta) \). Again, as we do not have, for now, a method to get the desired diffraction resistant beam in the last medium, we design the incident beam as if we were dealing with an homogeneous medium (the first one). So, from the desired cylindrical surface radius we obtain \( Q = 0.9839 n_1 \omega/c \) and from Eq. (4) we calculate the coefficients \( A_q \). The longitudinal wavenumbers are obtained from Eq. (3).

The field within each layer is given by Eqs. (7), from where we get for the last (fourth) medium:

\[
\Psi_4(\rho, \phi, z) = N_\nu e^{i4\phi} \sum_{q=-N}^{N} A_q \tau_{4q} J_4(h_q \rho) e^{i3h_q z} \quad (z \geq d_3)
\]

Figure 3(a) shows the desired longitudinal intensity pattern, \(|F(z)|^2\), to occur on the cylindrical surface of radius \( \rho_4 = 3 \mu \text{m} \), and the corresponding intensity pattern of the resulting beam occurring through the four material media. We can see how the characteristics of the FW beam are affected when compared to the desired pattern. Figure 3(b) show the 3D field intensity of the resulting wave, again in the four media, with its orthogonal projection in the detail.

In general, such deformations inflicted to the FW beam by the stratified structure will occur with more or less intensity, depending on the refractive indexes and thickness of the slabs between the first and the last media.

**4. The compensation method**

Our main objective in this work is to develop a method to obtain a FW beam in the last material medium of a stratified structure, that is, a beam resistant to diffraction effects and whose longitudinal intensity pattern can be chosen on demand. As we have seen in the previous sections, a FW designed for an homogeneous medium can undergo significant changes on its spatial structure (previously chosen) when crossing a layered medium before reaching the desired spatial region, which we consider here as being in the last material medium.

In this section, we will develop what we call the compensation method, which consists in designing a beam in the first material medium such that, when crossing the stratified structure ahead, the changes
Figure 3: (a) Comparison between the longitudinal intensity of the FW (with $\nu = 4$) over a cylindrical surface of radius $\rho_{4} \approx 3 \mu m$ and desired intensity pattern $|F(z)|^{2}$; (b) the 3D field intensity of the resulting beam in medium 4, as well as its orthogonal projection in the detail.

of amplitude and phase undergone by its constituent Bessel beams will eventually transform it into the desired FW, i.e., into the diffraction resistant beam endowed with the desired longitudinal intensity pattern.

The method is presented in four steps described below:

1. First, once it is given the stratified medium, composed of $M$ layers, and considering the incident beam of the type given by Eqs. (5,6), where we still do not know the values of the coefficients, $A_{q}$, and the longitudinal wavenumbers, $\beta_{Mq}$, we use Eqs. (7,9) to calculate the reflection and transmission coefficients, $\Gamma_{Mq}$ and $\tau_{Mq}$, respectively, in function of the refractive indexes, $n_{M}$, the interface positions, $d_{M}$, and the longitudinal wavenumbers, $\beta_{Mq}$ (still unknown).

2. Afterwards, we choose the characteristics of the diffraction resistant beam that we wish to occur in the last medium, $z \geq d_{M-1}$, i.e., we choose the beam spot size, $r_{0}$, or the radius of the cylindrical surface, $\rho_{\nu}$ (in the case of a hollow beam), and also the desired longitudinal intensity pattern, $|F(z)|^{2}$, which has to be defined within $-L/2 \leq z \leq L/2$; such spatial range must be large enough to enclose part of the first medium, all the stratified structure and part of the last medium, which must include the region where the desired beam pattern will be located.

3. We directly construct the desired beam solution, $\Psi_{M}(\rho, \phi, z)$ in the last medium through the following equations:

$$\Psi_{M}(\rho, \phi, z) = N_{\nu} e^{i\nu\phi} \sum_{q=-N}^{N} A_{q} \tau_{Mq} J_{\nu}(h_{q}\rho) e^{i\beta_{Mq}z}$$

(21)
with

\[ \beta_{Mq} = Q + \frac{2\pi}{L} q, \quad (22) \]

\[ h_q = \sqrt{n_{Mq}^2 - \beta_{Mq}^2}, \quad (23) \]

and

\[ A_q = \frac{1}{\tau_{Mq} L} \int_{-L/2}^{L/2} F(z) e^{-i \frac{2\pi q}{L} z} \, dz, \quad (24) \]

With \( Q \) obtained from \( r_0 = 2.4/\sqrt{n_{Mq}^2 \omega^2/c^2 - Q^2} \) (with \( \nu = 0 \)) or from \( [(d/d\rho) J_\nu(\rho \sqrt{n_{Mq}^2 \omega^2/c^2 - Q^2})] \big|_{\rho=\rho_0} = 0 \) (with \( \nu \geq 1 \) of our choice), where \( |F(z)|^2, \ r_0 \) and \( \rho_0 \) have been chosen in the previous step.

Note that, now, once we have the values of \( h_q \), the reflection and transmission coefficients, analytically calculated in the first step, can be numerically evaluated.

4. This final step is directed to calculate the incident beam, \( \Psi_{\text{inc}}(\rho, \phi, z) \), i.e., the values of \( A_q \) and \( \beta_{1q} \) to be used in Eq.(5), in such way that the beam in the last medium will result to be the desired one and already calculated in step 3. Actually, this can be done through Eqs.(21-24), from which we get:

\[ \beta_{1q} = \sqrt{n_1^2 \omega^2/c^2 - h_q^2} = \sqrt{(n_1^2 - n_{Mq}^2) \omega^2/c^2 - (Q + \frac{2\pi}{L} q)^2}, \quad (25) \]

\[ A_q = \frac{1}{\tau_{Mq} L} \int_{-L/2}^{L/2} F(z) e^{-i \frac{2\pi q}{L} z} \, dz, \quad (26) \]

with \( h_q \) given by Eq.(23).

Naturally, the resulting field within the \( m \)th medium can be calculated via Eqs.(7,8).

4.1. The method applied to FWs

Now, we are going to use the compensation method to the previous examples (1) and (2) for obtaining the desired FW beam in the last medium of the multilayered structure.

First example revisited

Returning to the first example, we note that the first and second steps of the compensation method were already implemented through the Eqs.(10-15, 18) and the chosen spot radius \( r_0 = 4.5 \mu m \). Now, we implement the third step by constructing the solution for the desired beam in the fourth medium (the last one) through Eqs.(21-24), with \( M = 4 \). In this case, the value of \( Q \) results to be \( Q = 0.99991 n_4 \omega/c \).

Finally, we go through step four and calculate the incident beam via Eq.(5, 25, 26). By construction, this incident beam, after crossing the stratified structure ahead, will result, in the last medium, into the diffraction resistant beam endowed with the desired longitudinal intensity pattern (obtained in the
Figure 4: The resulting FW (with \( \nu = 0 \)) in medium 4 after applied the compensation method. (a) Comparison between the on-axis longitudinal intensity of the FW and desired pattern, \( |F(z)|^2 \); (b) the three-dimensional field intensity of the resulting FW, as well as its orthogonal projection in the detail.

previous paragraph). The resulting fields within the second and third media are calculated via Eqs.\((7,8)\), with \( m = 2, 3 \).

Figure 4(a) shows the desired intensity pattern, \( |F(z)|^2 \), and the intensity of the resulting field, from the first until the last medium. We can see a very good agreement between them. Figure 4(b) shows the 3D intensity of the resulting FW beam through the four media. It is very clear that the compensation method provides the desired diffraction resistant beam in the last medium. In this case, we use \( N = 20 \).

**Second example revisited**

In applying the compensation method to the second example, we note that the first and second steps were already implemented through the Eqs.\((10,15,18)\) and the chosen of the cylindrical surface’s radius \( \rho_4 = 3 \mu m \). The step three, i.e, the construction of the desired beam solution in medium four, is implemented through Eqs.\((21,24)\), with \( \nu = 4 \) and \( M = 4 \). Here, the value of \( Q \) results to be \( Q = 0.999n_4\omega/c \). The fourth step is executed by calculating the incident beam via Eq.\((5,25,26)\). The resulting fields within the second and third media are calculated via Eqs.\((7,8)\), with \( m = 2, 3 \). In this case, we use \( N = 300 \).

Figure 5 shows that the resulting FW beam has the very same spatial shape chosen \emph{a priori}, so confirming the efficiency of the method.

**4.2. Restriction over the longitudinal wavenumbers**

Let us stress that our preference here is to deal with propagating waves whenever it is possible. For this, all longitudinal wavenumbers have to be real in all layers. In the last medium, this condition is
Figure 5: The resulting FW (with $\nu = 4$) in medium 4 after applying the compensation method. (a) Comparison between the longitudinal intensity of the FW over the cylindrical surface of radius $\rho = \rho_4 = 3 \mu m$ and the desired intensity pattern, $|F(z)|^2$; (b) the 3D field intensity of the resulting FW, as well as its orthogonal projection in the detail.

guaranteed by construction, since the $\beta_{Mq}$ are chosen according to Eq. \((22)\). Now, within the $m$th medium the longitudinal wavenumbers, $\beta_{mq}$ will be real if

$$
(\beta_{mq})^2 = n_m^2 \frac{\omega^2}{c^2} - h_q^2 = (n_m^2 - n_M^2) \frac{\omega^2}{c^2} + \left( Q + \frac{2\pi}{L} q \right)^2 \geq 0 ,
$$

where we have used Eq. \((23)\).

A sufficient (but not necessary) condition for getting inequation \((27)\) satisfied for all $m = 1, 2, ..., M - 1$ and $q = -N, 0, ..., N$ is to have

$$
[(\beta_{Mq})_{min}]^2 = \left( Q - \frac{2\pi}{L} q \right)^2 \geq (n_M^2 - n_m^2) \frac{\omega^2}{c^2} \tag{27}
$$

In case inequation \((27)\) is not satisfied for one or more values of $q$ within the $m$th medium, we have to change the value of the parameter $Q$ and/or the value of $N$, depending on the case.

5. Minimizing the reflection

Anti-reflection coating is a very important application of the optics of thin films [18] [19] [20]. The technics used for minimizing reflection of ordinary beams when impinging on dielectric surfaces can also be used when the incident wave is a FW-type beam. In this section we will analyze the very simple case of determining the refractive index and the thickness of a film to be used on the separation plane surface of two dielectric media for minimizing the reflection of a normally incident FW beam.

The reader may note that we say “minimize” rather than “nullify” reflection. This is due to the fact that a FW is composed of a superposition of Bessel beams with different longitudinal wavenumbers,
which makes it impossible to extinguish the reflection coefficients of all them simultaneously. Thus, we have to choose the most important Bessel beam of the superposition and characterize the thin film that extinguishes its reflection. In doing so, we will minimize the reflection of the incident FW, since, in general, the other Bessel beams composing it differ little from the main Bessel beam with respect to the values of the longitudinal wavenumbers.

In general, the Bessel beam that most contribute to the superposition that defines a FW, like that given by Eq. (21), is the central one, which corresponds to \( q = 0 \). The anti-reflective film will be characterized considering this Bessel beam. In cases where the most important beam of the FW is not the central one (as it may occur when the function \( F(z) \), which defines the desired longitudinal pattern, oscillates considerably), one should first find which Bessel beam is the most relevant to the superposition and perform the characterization of the thin film based on it.

Before to proceed, let us remember that a Bessel beam is characterized for its order and its cone-angle \( \theta \), which determines its longitudinal and transverse wavenumbers, \( \beta = (n\omega/c) \cos \theta \) and \( h = (n\omega/c) \sin \theta \) [6]. From the compensation method we already know that the central Bessel beam of the resulting FW in the last medium (the third one, in this case) has its longitudinal wavenumber \( \beta_{30} = Q \equiv \bar{\beta}_3 \). The longitudinal wave numbers of the correspondent central Bessel beams in media 1 and 2 will be \( \beta_{10} = \sqrt{(n_1^2 - n_3^2)\omega^2/c^2 - Q^2} \equiv \bar{\beta}_1 \) and \( \beta_{20} = \sqrt{(n_2^2 - n_3^2)\omega^2/c^2 - Q^2} \equiv \bar{\beta}_2 \), respectively. The cone angle of these Bessel beams are given by \( \theta_1 = \arccos(c \bar{\beta}_1/n_1\omega) \), \( \theta_2 = \arccos(c \bar{\beta}_2/n_2\omega) \) and \( \theta_3 = \arccos(c Q/n_3\omega) \).

Now, considering the system formed by the three parallel layers (the intermediate layer is the film with thickness \( d \) and refractive index \( n_2 \)), the (central) incident Bessel beam is totally transmitted if its reflection coefficient is equal to zero, that is:

\[
\Gamma_1 = \frac{\bar{\beta}_2(1 + e^{-2i\alpha \xi}) + \bar{\beta}_1(1 - e^{-2i\alpha \xi})}{\bar{\beta}_1(1 - e^{-2i\alpha \xi}) - \bar{\beta}_2(1 + e^{-2i\alpha \xi})} = 0
\]

where \( \alpha = \bar{\beta}_2 d \) and \( \xi = (\bar{\beta}_3 + \bar{\beta}_2)/(\bar{\beta}_3 - \bar{\beta}_2) \). From Eq. (28), we get the following system of two equations

\[
\begin{align*}
\bar{\beta}_2 + \bar{\beta}_2 \xi \cos(2\alpha) &= -\bar{\beta}_1 + \bar{\beta}_1 \xi \cos(2\alpha) \\
\bar{\beta}_2 \sin(2\alpha) &= \bar{\beta}_1 \sin(2\alpha),
\end{align*}
\]

which predicts two cases:

1. **When \( \bar{\beta}_1 = \bar{\beta}_3 \)**

   In this case, the first equation of (29) is satisfied if \( \cos(2\alpha) = 1 \), where \( \alpha = m\pi \), and \( m \) are integers (note that the second equation of (29) is also satisfied). So

\[
d = \frac{m}{\cos \theta_2} \frac{\lambda}{2n_2},
\]

where \( \lambda = 2\pi c/\omega \).

2. **When \( \bar{\beta}_1 \neq \bar{\beta}_3 \)**
The second equation of (29) is satisfied if \( \sin(2\alpha) = 0 \), so \( 2\alpha = m\pi \), with \( m \) an odd number, otherwise the first equation can not be satisfied. So, the film thickness is given by

\[
d = \frac{m\lambda}{\cos\theta_2 4n_2},
\]

where \( \lambda = 2\pi c/\omega \). With respect to the first equation of (29), by knowing that \( \cos(2\alpha) = -1 \), we have

\[
\beta_2 \sqrt{\beta_1 \beta_3}
\]

So, the refractive index of the film must be

\[
n_2 = \frac{\sqrt{n_1 \cos\theta_2 n_3 \cos\theta_3}}{\cos\theta_2}
\]

With this, we can ensure that the central Bessel beam of the incident FW will be totally transmitted, and so we can expect that the transmission of the entire FW beam will be maximized.

Now, let us test this with an example.

**Third example**

Let us assume the refractive indexes \( n_1 = 1 \) (air) and \( n_3 = 1.5 \) (glass lens) for the first and third layer, respectively. The anti-reflective film, deposited over the glass surface in contact with air, will have refractive index \( n_2 \) and thickness \( d \) (both to be calculated).

We can use the compensation method to obtain, in the third medium, a zero-order FW beam whose the desired longitudinal intensity pattern, \( |F(z)|^2 \), is given (let us suppose) by Eq.(18), with \( L = 0.3m, \delta = 0.035m, l_1 = 0.03m, l_2 = l_1 + \delta, l_3 = l_2 + \delta \) and \( l_4 = l_3 + \delta \) and a spot radius \( r_0 = 20\mu m \). With the value of \( r_0 \) we obtain \( Q = 0.99997 n_3 \omega/c \), from which we can calculate, for the anti-reflective film, \( n_2 = 1.22 \) and \( d = 0.129\mu m \) (for the first maximum transmission). With all the values of the refractive indexes and also of \( d \), we can proceed by using the compensation method to obtain the incident beam that will result into the desired FW in the last medium.

Figure 6 shows the effect of using (or not) the anti-reflective film. We can notice that the use of the film almost eliminates the reflection of the incident FW in the interface air-film, as expected. This fact implies that with the implementation of the thin film a lower power is necessary to generate the desired FW in the third medium.

**6. The transfer-matrix method**

In this section we are going to develop a transfer-matrix formulation, which will enable us to speed up the implementation of the longitudinal intensity modelling of diffraction resistant beams in the last medium of a stratified dielectric structure. More specifically, with this matrix method, once we have chosen the desired FW for the last medium, we will be able to quickly calculate the incident beam to be generated in the first medium in such a way that the resulting beam in the last one is exactly that we
Figure 6: Intensity of the reflected FW in a setting (a) without the film and (b) with the film, both normalized with a peak intensity of the incident beam. Comparison between the intensity of the desired longitudinal pattern of the transmitted FW (c) without the film and (e) with the film. The solid line represents the function $F(z)$ and the dotted one the FW.

have previously chosen. Such a formulation will be very suitable for stratified media with a large number of layers.

As we know, the incident FW beam is given by Eq.(5). Now, let us to rewrite the Eqs.(7), which describe the resulting waves within each medium, in the following compact form:

$$
\Psi_m(\rho, \phi, z) = \sum_{q=-N}^{N} \psi_{mq}(\rho, \phi, z),
$$

with

$$
\psi_{mq}(\rho, \phi, z) = N_\nu A_q \tau_{mq} J_\nu(h_q \rho)e^{i \nu \phi} e^{i \beta_{mq} z} + N_\nu A_q \Gamma_{mq} J_\nu(h_q \rho)e^{i \nu \phi} e^{-i \beta_{mq} z},
$$

for $m = 1, 2, \ldots, M$.

From Eq.(35), it is not difficult to show the following matrix equation, for $m = 2, \ldots, M - 1$, relating
Ψ_{mq} and its derivative ∂zΨ_{mq} evaluated on the interface at z = d_m with their values on the preceding interface z = d_{m-1}:

\[
\begin{bmatrix}
Ψ_{mq}(ρ, φ, z = d_m) \\
∂zΨ_{mq}(ρ, φ, z = d_m)
\end{bmatrix}
= M_m^q
\begin{bmatrix}
Ψ_{mq}(ρ, φ, z = d_{m-1}) \\
∂zΨ_{mq}(ρ, φ, z = d_{m-1})
\end{bmatrix},
\] (36)

with

\[
M_m^q =
\begin{bmatrix}
\cos(β_{mq}Δ_m) & \frac{1}{β_{mq}} \sin(β_{mq}Δ_m) \\
-β_{mq} \sin(β_{mq}Δ_m) & \cos(β_{mq}Δ_m)
\end{bmatrix},
\] (37)

and

\[Δ_m = d_m - d_{m-1}\] (38)

Now, from Eqs. (36 - 38) and due to the boundary conditions asserting the continuity of Ψ_{mq} and ∂zΨ_{mq} through each plane interface at z = d_m (m = 1, ..., M - 1), we can write down the following equation relating Ψ_{Mq} and ∂zΨ_{Mq} with Ψ_{1q} and ∂zΨ_{1q}:

\[
\begin{bmatrix}
Ψ_{Mq}(ρ, φ, z = d_{M-1}) \\
∂zΨ_{Mq}(ρ, φ, z = d_{M-1})
\end{bmatrix}
= M^q
\begin{bmatrix}
Ψ_{1q}(ρ, φ, z = 0) \\
∂zΨ_{1q}(ρ, φ, z = 0)
\end{bmatrix},
\] (39)

with the transfer-matrix,

\[M^q = M_{M-1}^q \cdot ... \cdot M_4^q \cdot M_2^q\] (40)

From Eqs. (35 - 39 - 40) we get

\[Γ_{1q} = \frac{(M_{21}^q + β_{1q}β_{Mq}M_{12}^q) + i(β_{1q}M_{22}^q - β_{Mq}M_{11}^q)}{(-M_{21}^q + β_{1q}β_{Mq}M_{12}^q) + i(β_{1q}M_{22}^q + β_{Mq}M_{11}^q)}\] (42)

and

\[τ_{Mq} = 2iβ_{1q}e^{-iβ_{Mq}d_{M-1}} \left[ \frac{1}{-M_{21}^q + β_{1q}β_{Mq}M_{12}^q + i(β_{1q}M_{22}^q + β_{Mq}M_{11}^q)} \right] \] (43)

Now, with respect our compensation method, we can see from the receipt of four steps developed in Section 4 that Eq. (43) can be directly used in Eq. (26) to furnish the coefficients of the Bessel beam superposition (5) which defines the incident beam (in the first medium) that, after crossing the stratified structure, will result into the desired FW beam in the last medium.

Naturally, we can also use Eq. (42) to get the reflected beam in the first medium.
As we have already said, this matrix-transfer formulation can be very effective in the application of the compensation method in cases where the stratified medium possesses a large number of layers. It can be equally useful for applying the compensation method to a continuously inhomogeneous medium, with refractive index \( n = n(z) \), if we discretize it as a sequence of thin layers, each one with its (different) constant refractive index.

7. Conclusion

In this paper, we describe analytically the propagation of Frozen-Wave-type beams through non-absorbing stratified media, showing how the desired beam pattern for the last medium is affected by the inhomogeneities. More importantly, we develop a novel method for constructing an incident FW beam capable of compensating such inhomogeneity effects, so rendering the desired spatially-shaped-diffraction-resistant-beam in the last material medium. In this context, we also develop a matrix method to deal with stratified media with large number of layers.

Additionally, we undertake some discussion about minimizing reflection of the incident FW beam on the first material interface by using thin films.

The results here presented can have important applications in remote sensing, medical and military purposes, non-invasive optical measurements, etc..

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