Degenerate Turán densities of sparse hypergraphs

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Abstract

For fixed integers \( r > k \geq 2, e \geq 3 \), let \( f_r(n, er - (e - 1)k, e) \) be the maximum number of edges in an \( r \)-uniform hypergraph in which the union of any \( e \) distinct edges contains at least \( er - (e - 1)k + 1 \) vertices. A classical result of Brown, Erdős and Sós in 1973 showed that \( f_r(n, er - (e - 1)k, e) = \Theta(n^k) \). The degenerate Turán density is defined to be the limit (if it exists)

\[
\pi(r, k, e) := \lim_{n \to \infty} \frac{f_r(n, er - (e - 1)k, e)}{n^k}.
\]

Extending a recent result of Glock for the special case of \( r = 3, k = 2, e = 3 \), we show that

\[
\pi(r, 2, 3) := \lim_{n \to \infty} \frac{f_r(n, 3r - 4, 3)}{n^2} = \frac{1}{r^2 - r - 1}
\]

for arbitrary fixed \( r \geq 4 \). For the more general cases \( r > k \geq 3 \), we manage to show

\[
\frac{1}{r^k - r} \leq \liminf_{n \to \infty} \frac{f_r(n, 3r - 2k, 3)}{n^k} \leq \limsup_{n \to \infty} \frac{f_r(n, 3r - 2k, 3)}{n^k} \leq \frac{1}{k!(\binom{r}{k}) - \frac{k!}{2}},
\]

where the gap between the upper and lower bounds are small for \( r \gg k \).

The main difficulties in proving these results are the constructions establishing the lower bounds. The first construction is recursive and purely combinatorial, and is based on a (carefully designed) approximate induced decomposition of the complete graph, whereas the second construction is algebraic, and is proved by a newly defined matrix property which we call strongly 3-perfect hashing.

1 Introduction

Turán-type problems have been playing a central role in the field of extremal graph theory since Turán [10] determined in 1941 the Turán number of complete graphs. In this work we focus on a classical hypergraph Turán-type problem introduced by Brown, Erdős and Sós [10] in 1973.

For an integer \( r \geq 2 \), an \( r \)-uniform hypergraph \( \mathcal{H} \) (or \( r \)-graph, for short) on the vertex set \( V(\mathcal{H}) \), is a family of \( r \)-element subsets of \( V(\mathcal{H}) \), called the edges of \( \mathcal{H} \). An \( r \)-graph is said to contain a copy of \( \mathcal{H} \) if it contains \( \mathcal{H} \) as a subhypergraph. Furthermore, given a family \( \mathcal{H} \) of \( r \)-graphs, an \( r \)-graph is said to be \( \mathcal{H} \)-free if it contains no copy of any member of \( \mathcal{H} \). The Turán number \( \text{ex}_r(n, \mathcal{H}) \), is the maximum number of edges in an \( \mathcal{H} \)-free \( r \)-graph on \( n \) vertices. It can be easily shown that the sequence \( \{(\binom{n}{r})^{-1} \cdot \text{ex}_r(n, \mathcal{H})\}_{n=r}^\infty \) is bounded.

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and non-increasing, and therefore converges [28]. Hence, the Turán density \( \pi(\mathcal{H}) \) of \( \mathcal{H} \) is defined to be

\[
\pi(\mathcal{H}) := \lim_{n \to \infty} \frac{\text{ex}_r(n, \mathcal{H})}{\binom{n}{r}}.
\]

If \( \pi(\mathcal{H}) = 0 \) then \( \mathcal{H} \) is called \textit{degenerate}. It is well-known (see, e.g. [13, 29, 31]) that \( \mathcal{H} \) is degenerate if and only if it contains an \( r \)-partite \( r \)-graph, where an \( r \)-graph is called \textit{\( r \)-partite} if its vertex set admits a partition into \( r \) disjoint parts \( V_1, \ldots, V_r \), such that every edge of it contains exactly one vertex from each vertex part \( V_i \). If \( \mathcal{H} \) is degenerate and there exists a real number \( \alpha \in (0, r) \) such that \( \text{ex}_r(n, \mathcal{H}) = \Theta(n^\alpha) \), then the \textit{degenerate Turán density} \( \pi_d(\mathcal{H}) \) of \( \mathcal{H} \) is defined to be the limit (if it exists)

\[
\pi_d(\mathcal{H}) := \lim_{n \to \infty} \frac{\text{ex}_r(n, \mathcal{H})}{n^\alpha},
\]

where \( \alpha \) is called the \textit{Turán exponent} of \( \mathcal{H} \). For example, it is known (see, e.g. [23]) that \( \pi_d(C_4) = \lim_{n \to \infty} \frac{\text{ex}_2(n, C_4)}{n^{3/2}} = \frac{1}{2} \), where \( C_4 \) is the cycle of length 4.

For a positive integer \( n \) let \([n] := \{1, \ldots, n\}\), and for any \( X \subseteq [n] \) let \( \binom{X}{r} \) be the family of \( \binom{|X|}{r} \) distinct \( r \)-subsets of \( X \). For fixed integers \( r \geq 2, e \geq 2, v \geq r + 1 \), let \( \mathcal{G}_r(v, e) \) be the family of all \( r \)-graphs formed by \( e \) edges and at most \( v \) vertices; that is,

\[
\mathcal{G}_r(v, e) = \left\{ \mathcal{H} \subseteq \binom{[v]}{r} : |\mathcal{H}| = e, |V(\mathcal{H})| \leq v \right\}.
\]

Thus an \( r \)-graph is \( \mathcal{G}_r(v, e) \)-free if and only if the union of any \( e \) distinct edges contains at least \( v + 1 \) vertices. Since such \( r \)-graphs do not contain many edges (see (1) below), they are also termed \textit{sparse hypergraphs} [22]. Following previous papers on this topic (see, e.g. [1]) we will use the notation

\[
f_r(n, v, e) := \text{ex}_r(n, \mathcal{G}_r(v, e))
\]

to denote the maximum number of edges in a \( \mathcal{G}_r(v, e) \)-free \( r \)-graph.

In 1973, Brown, Erdős and Sós [10] initiated the study of the function \( f_r(n, v, e) \), which has attracted considerable attention throughout the years. More concretely, they showed that

\[
\Omega(n^{\frac{r}{r-1} - \frac{1}{r-1}}) = f_r(n, v, e) = O(n^{\frac{r}{r-1} - 1}).
\]

(1)

The lower bound was proved by a standard probabilistic argument (now known as the alteration method, see, e.g. Chapter 3 of [3]), and the upper bound follows from a double counting argument, which uses the simple fact that in a \( \mathcal{G}_r(v, e) \)-free \( r \)-graph, any set of \( \left\lceil \frac{r}{r-1} - 1 \right\rceil \) vertices can be contained in at most \( e - 1 \) distinct edges. Improvements on (1) for less general parameters were obtained in a series of works, see, e.g. [1, 9, 10, 12, 17, 14, 24, 25, 32, 33, 34, 35, 36, 38].

In this paper we are interested in the special case where \( k := \frac{e}{r-1} - 1 \) is an integer greater than one. In such a case the order of \( f_r(n, v, e) \) is determined by (1), i.e.,

\[
f_r(n, ev - (e - 1)k, e) = \Theta(n^k),
\]

(2)

We remark that in a “reverse” direction, a recent breakthrough of Bukh and Conlon [11] proved that for any rational number \( \alpha \in [1, 2] \), there exists a finite family of 2-graphs \( \mathcal{H} \) such that \( \text{ex}_2(n, \mathcal{H}) = \Theta(n^\alpha) \), which resolves a well-known conjecture of Erdős and Simonovits [15]. Similar results on \( r \)-graphs for all \( r \geq 3 \) were recently obtained by Fitch [13]. However, another conjecture of Erdős and Simonovits [16], also known as the rational exponents conjecture, which claims that in the statement above, it suffices to pick a simple graph rather than a finite family, is still widely open.
where \( v = er - (e-1)k \) and \( 2 \leq k \leq r - 1 \). Thus for fixed integers \( e \geq 2, r > k \geq 2 \), it is natural to ask whether the limit

\[
\pi_d(r, k, e) := \pi_d(G_r(ce - (e-1)k), e) = \lim_{n \to \infty} \frac{f_r(n, er - (e-1)k, e)}{n^k}
\]
exists, where we call \( \pi_d(r, k, e) \) the degenerate Turán density of sparse hypergraphs.

For \( e = 2 \) this question is already resolved, since an \( r \)-graph is \( G_r(2r - k, 2) \)-free if and only if any pair of its edges share at most \( k - 1 \) vertices, therefore \( f_r(n, 2r - k, 2) \) is equal to the maximum size of an \((n, r, k)\)-packing, where an \((n, r, k)\)-packing is a family of \( r \)-subsets of \([n]\) such that any \( k \)-subset of \([n]\) is contained in at most one member of this family. Clearly, the largest size of an \((n, r, k)\)-packing cannot exceed \( \binom{n}{k}/(\binom{r}{k}) \). Moreover, it was shown by Rödl [33] (see [26, 30] for the current state-of-the-art) that for fixed \( r, k \) and sufficiently large \( n \), this bound is essentially tight, up to a \( 1 - o(1) \) factor (where \( o(1) \to 0 \) as \( n \to \infty \)). This implies that

\[
\pi_d(r, 2, 2) = \lim_{n \to \infty} \frac{(1 - o(1))\binom{n}{k}/(\binom{r}{k})}{n^k} = \frac{1}{r \cdots (r - k + 1)}.
\]

For \( e \geq 3 \) not much is known, and only recently the existence of \( \pi_d(3, 2, 3) \) was resolved. Brown, Erdös and Sós [10] posed the following conjecture (see also [9]).

**Conjecture 1** (Brown, Erdös and Sós [10]). The degenerate Turán density

\[
\pi_d(3, 2, e) = \lim_{n \to \infty} \frac{f_3(n, e + 2, e)}{n^2}
\]
exists for every fixed \( e \geq 3 \).

For the first case \( e = 3 \), they were able to show that \( 1/6 \leq \pi_d(3, 2, 3) \leq 2/9 \). To the best of our knowledge, for more than forty years no significant improvement was made until recently Glock [25] closed the gap by showing that

\[
\pi_d(3, 2, 3) = \lim_{n \to \infty} \frac{f_3(n, 5, 3)}{n^2} = \frac{1}{5}.
\]

In this paper we continue this line of research, and in the spirit of [2] and Conjecture 1 we consider the following question.

**Question 2.** For fixed integers \( r > k \geq 2, e \geq 3 \), does the limit

\[
\pi_d(r, k, e) = \lim_{n \to \infty} \frac{f_r(n, er - (e-1)k, e)}{n^k}
\]
exist? If so, what is the value of \( \pi_d(r, k, e) \)?

In general this question is widely open. The authors of [10] who established [2] did not try to optimize the coefficient of \( n^k \), however a careful analysis of their lower bound yields to

\[
\sqrt{\frac{(er - (e-1)k)!}{2(\binom{er - (e-1)k}{r}) \cdot (\binom{er - (e-1)k}{e}) \cdot (r!)^e}} \leq \pi_d(r, k, e) \leq \frac{e - 1}{r \cdots (r - k + 1)},
\]

where the upper bound follows from the observation that any \( k \)-subset of \([n]\) is contained in at most \( e - 1 \) edges of a \( G_r(ce - (e-1)k, e) \)-free \( r \)-graph, implying that \( f_r(n, er - (e-1)k, e) \leq (e - 1)(\binom{n}{k}/(\binom{r}{k})) \). Note that [4] states in fact lower and upper bounds on \( \lim_{n \to \infty} \frac{f_r(n, er - (e-1)k, e)}{n^k} \) and \( \limsup_{n \to \infty} \frac{f_r(n, er - (e-1)k, e)}{n^k} \) respectively, since it is not known whether \( \pi_d(r, k, e) \) exists.
However, to simplify the notations we keep \( \lfloor \frac{n}{2} \rfloor \) in its current form, and in the sequel we will frequently use abbreviations of this type.

Our main results are introduced in the next two subsections, and they include the determination of \( \pi_d(r, 2, 3) \) for any fixed \( r \geq 4 \), and new lower and upper bounds for \( \pi_d(r, k, 3) \) for any fixed \( r > k \geq 3 \).

**Notations.** We use standard asymptotic notations \( \Omega(\cdot), \Theta(\cdot), O(\cdot) \) and \( o(\cdot) \) as \( n \to \infty \), where for functions \( f = f(n) \) and \( g = g(n) \), we write \( f = O(g) \) if there is a constant \( c_1 \) such that \( |f| \leq c_1|g| \); we write \( f = \Omega(g) \) if there is a constant \( c_2 \) such that \( |f| \geq c_2|g| \); we write \( f = \Theta(g) \) if \( f = O(g) \) and \( f = \Omega(g) \) hold simultaneously; finally, we write \( f = o(g) \) if \( \lim_{n \to \infty} (f/g) = 0 \).

### 1.1 The exact value of \( \pi_d(r, 2, 3) \)

In an \((n, r, 2)\)-packing, any member of \( \big[ \frac{n}{2} \big] \) is contained in at most one member of \( \big[ n \big] \), therefore one can easily verify that such a packing is also \( \mathcal{G}_r(3r - 4, 3) \)-free. This implies that for all fixed \( r \geq 4 \) the result of Rödl [33], written in the above notation, is

\[
\pi_d(r, 2, 3) \geq \frac{1}{r^2 - r}.
\]

(5)

We will give a tighter bound than (5) by showing that approximately, a \( \left( \frac{1}{r^2 - r - 1} \right) \)-fraction of the 2-subsets in \([n]\) can be contained in two \( r \)-subsets, while the resulting hypergraph still has the \( \mathcal{G}_r(3r - 4, 3) \)-free property (see Remark 19). As a consequence, we obtain the following improvement on the above lower bound.

**Theorem 3.** For any fixed integer \( r \geq 4 \),

\[
\pi(r, 2, 3) = \lim_{n \to \infty} \frac{f_r(n, 3r - 4, 3)}{n^2} = \frac{1}{r^2 - r - 1}.
\]

Note that Theorem 3 extends 3 from \( r = 3 \) to arbitrary fixed \( r \geq 4 \). To prove this theorem it suffices to show that \( \limsup_{n \to \infty} f_r(n, 3r - 4, 3) \leq \frac{1}{r^2 - r - 1} \) and \( \liminf_{n \to \infty} f_r(n, 3r - 4, 3) \geq \frac{1}{r^2 - r - 1} \). The upper bound is a special case of the upper bound stated in Theorem 2 below, which will be discussed later. The main difficulty in proving Theorem 3 is the construction which establishes the lower bound. In what follows we briefly review the main ideas behind it.

Generally speaking, the lower bound is obtained by a recursive construction (recursion on the uniformity \( r \)) and a carefully designed approximate induced decomposition of \( K_n \), the complete graph on \( n \) vertices. Given a finite graph \( G \), a \( G \)-packing in \( K_n \) is simply a family of edge disjoint copies of \( G \) in \( K_n \). We will make use of the following lemma, which was proved to be very useful in many other combinatorial constructions (see, e.g. [2, 4, 19, 21, 25]).

**Lemma 4** (Graph packing lemma, see Theorem 2.2 [19] or Theorem 3.2 [5]). Let \( G \) be any fixed graph with \( e \) edges and \( \epsilon > 0 \) be any small constant. Then there is an integer \( n_0 \) such that for any \( n > n_0 \), there exists a \( G \)-packing \( \mathcal{F} = \{ G^1, \ldots, G^{l} \} \) in \( K_n \) with

\[
l \geq (1 - \epsilon)\frac{n^2}{2e}.
\]

edge disjoint copies of \( G \) such that

(i) any two distinct copies of \( G \) share at most two vertices, i.e., \( |V(G^i) \cap V(G^j)| \leq 2 \) for any \( 1 \leq i \neq j \leq l \);
(ii) if two distinct copies \(G^i, G^j\) share two vertices \(a, b\), then \(\{a, b\}\) is neither an edge of \(G^i\), nor \(G^j\).

A \(G\)-packing satisfying (ii) is called an induced \(G\)-packing (see, e.g. [19]). Note that a weaker version of the above lemma, which only considered the existence of a large \(G\)-packing, regardless of the additional properties (i) and (ii), was used in [25] (see Theorem 5 of [25]) to prove the lower bound of (3). It is easy to see that Lemma 4 is nearly-optimal in the sense that the maximum size of any \(G\)-packing in \(K_n\) cannot exceed \(\binom{n}{2}/e\).

We call the graph \(G\) in Lemma 4 the component graph, as it forms the basic component in the approximate decomposition. Following Theorem 3 it is natural to call a \(G, (3r - 4, 3)\)-free \(r\)-graph \(H \subseteq \binom{[n]}{r}\) optimal if it has roughly \((\frac{1}{r-1} + o(1))n^2\) edges as \(n \to \infty\).

The following construction summarizes the main steps taken to prove the lower bound in Theorem 3.

Construction 5. Given \(H\), an optimal \(G_r, (3r - 4, 3)\)-free \(r\)-graph, we construct an optimal \(G_{r+1}(3(r+1) - 4, 3)\)-free \((r + 1)\)-graph by performing the following three steps.

1. By applying Lemma 4 with a carefully designed component graph \(G_t\) (see Subsection 4.7), we approximately decompose the complete graph \(K_n\) to \(l = (1 - \epsilon)n^2/2|G_t|\) edge disjoint copies of \(G_t\), say, \(G_1^t, G_2^t, \ldots, G_l^t\);

2. For \(1 \leq i \leq l\), by embedding in \(V(G_i^t)\) many copies of \(H\) in a suitable way (see Subsection 4.3), we get an \((r + 1)\)-graph \(G_i^t(\mathcal{H})\) (see Lemma 16);

3. Output the \((r + 1)\)-graph \(\mathcal{F} := \bigcup_{i=1}^l G_i^t(\mathcal{H})\), the edge disjoint union of the \(G_i^t(\mathcal{H})\)'s (see Subsection 4.3).

The base case, i.e., the optimal \(G_r, (3r - 4, 3)\)-free \(r\)-graph for \(r = 3\) was given by Glock [25]. Then, by applying Construction 5 iteratively, one can construct optimal \(G_r, (3r - 4, 3)\)-free \(r\)-graphs for all \(r \geq 3\). The reader is referred to Section 4 for more details.

1.2 New lower and upper bounds for \(\pi_d(r, k, 3)\)

In the beginning of the last subsection it was mentioned that an \((n, r, 2)\)-packing is also a \(G_r, (3r - 4, 3)\)-free \(r\)-graph. However, this is not true in general, namely for \(r > k \geq 3\), an \((n, r, k)\)-packing is not necessarily a \(G_r, (3r - 2k, 3)\)-free \(r\)-graph, as \(3(k-1) < 2k\) if and only if \(k < 3\). Our next result provides new lower and upper bounds for \(\pi(r, k, 3)\) for any fixed \(r > k \geq 2\).

Theorem 6. For any fixed integers \(r > k \geq 2\),

\[
\frac{1}{r^k - r} \leq \liminf_{n \to \infty} \frac{f_r(n, 3r - 2k, 3)}{n^k} \leq \limsup_{n \to \infty} \frac{f_r(n, 3r - 2k, 3)}{n^k} \leq \frac{1}{k!(r)^k - \frac{k!}{2}}.
\]

One can easily check that for \(r\) much larger than \(k\) the gap between the lower and upper bounds in Theorem 6 is quite small. For example, let \(r = \frac{k!}{2}\) and \(k\) be sufficiently large, then the two bounds almost match, as \(r^k \approx k!(r)^k\). On the contrary, if \(r\) is approximately \(k\), the lower bound becomes even weaker than that of 4. We omit the detailed computation.

The upper bound in Theorem 6 which includes that of Theorem 3 as a special case, follows from a weighted counting argument, and is presented in Section 3. The lower bound is proved by an algebraic construction, which relies on a new matrix property called strongly 3-perfect hashing, which is introduced below in Definition 20. The following lemma shows that in order to construct a \(G_r, (3r - 2k, 3)\)-free \(r\)-graph it is sufficient to construct a matrix with this property.
Lemma 7. Let $r > k \geq 2$, and $q$ be integers. If $\mathcal{M}$ is a strongly 3-perfect hashing $q$-ary matrix of order $r \times q^k$, then it induces a $\mathcal{G}_r(3r - 2k, 3)$-free $r$-partite $r$-graph $\mathcal{H}_\mathcal{M}$ over $n = rq$ vertices and $q^k$ edges, where the vertices can be partitioned to $r$ disjoint parts $V_1, \ldots, V_r$ of size $q$ each.

The proof of Lemma 7 is given in Subsection 5.1. Indeed, the multipartite $r$-graph constructed using Lemma 7 is optimal up to a constant, in the sense that it is easy to verify by the pigeonhole principle that any $\mathcal{G}_r(3r - 2k, 3)$-free $r$-partite $r$-graph, which has equal part size $q$, can have at most $2q^k$ edges.

The next construction outlines the main ingredients in proving the lower bound of Theorem 6.

Construction 8 (Construction proving the lower bound of Theorem 6). By induction we assume that $f_r(n, 3r - 2k, 3) \geq \frac{n^k}{r^k} - ar^{k-1}$ holds for every integer less than $n$, where $a = a(r, k)$ is some constant not depending on $n$, and we prove the statement for $n$.

1. For fixed $r, k$, let $q$ be the largest prime power satisfying $rq \leq n$. By using the algebraic construction given in Subsections 5.2 and 5.3 we obtain an $r \times q^k$ q-ary strongly 3-perfect hashing matrix $\mathcal{M}$, which by Lemma 7 induces an $r$-partite $r$-graph $\mathcal{H}_\mathcal{M}$ over $r$ vertex parts $V_1, \ldots, V_r$;

2. By the induction hypothesis construct on each vertex part $V_i$ a $\mathcal{G}_r(3r - 2k, 3)$-free $r$-graph $\mathcal{H}_i$ with at least $\frac{q^k}{r^k - r} - aq^{k-1}$ edges;

3. Output the r-graph $\mathcal{F} := (\bigcup_{i=1}^r \mathcal{H}_i) \cup \mathcal{H}_\mathcal{M}$, whose edges are the disjoint union of the edges of $\mathcal{H}_i, 1 \leq i \leq r$ and $\mathcal{H}_\mathcal{M}$.

The $r$-graph $\mathcal{F}$ has $rq$ vertices and at least

$$q^k + r \cdot \left( \frac{q^k}{r^k - r} - aq^{k-1} \right) = \frac{(rq)^k}{r^k - r} - ar^{k-1}$$

edges. In order to complete the induction step it remains to show that $\mathcal{F}$ is $\mathcal{G}_r(3r - 2k, 3)$-free, and that the number of its edges is at least $\frac{q^k}{r^k - r} - ar^{k-1}$. The detailed proof is given in Section 5.

1.3 Outline of the paper

The rest of the paper is organized as follows. In Section 2 we briefly introduce two combinatorial problems which are closely related to the study of $\pi_d(r, k, e)$. In Section 3 we present the proof of the upper bound stated in Theorem 6. In Sections 4 and 5 we present the proofs of the lower bounds stated in Theorems 3 and 6 respectively.

2 Related work

2.1 The order of $f_r(n, er - (e - 1)k + 1, e)$

In Question 2 we asked whether $f_r(n, er - (e - 1)k + 1, e) / n^k$ converges as $n$ tends to infinity. In a similar setting, Brown, Erdős and Sós [10] and Alon and Shapira [1] posed the following conjecture.
Conjecture 9 (see, e.g. [10, 11]). For fixed integers \( r > k \geq 2, e \geq 3 \), it holds that
\[
n^{k-o(1)} < f_r(n, er - (e-1)k + 1, e) = o(n^k)
\]
as \( n \to \infty \).

Note that by [11],
\[
\Omega(n^{k-\frac{1}{2}}) < f_r(n, er - (e-1)k + 1, e) = O(n^k).
\]

Conjecture 9 plays an important role in extremal graph theory. The first case of the conjecture, namely the determination of the order of \( f_3(n, 6, 3) \), was only resolved by Ruzsa and Szemerédi [34] in the (6,3)-theorem, which was an early application of the celebrated Regularity Lemma [39], while establishing a surprising connection with additive number theory [7]. Following efforts of many researchers, the upper bound part of the conjecture is now known to hold for all \( r \geq k + 1 \geq e \geq 3 \), [11, 17, 24, 32, 34], whereas the lower bound is known to hold for all \( r \geq k \geq 2, e = 3 \) [11, 17, 34] and \( r > k = 2, e \in \{4, 5, 7, 8\} \) [24]. The reader is referred to [38] for the best known general lower bound, which shows that for all fixed \( r > k \geq 2, e \geq 3 \),
\[
f_r(n, er - (e-1)k + 1, e) = \Omega(n^{k-\frac{1}{2}}(\log n)^{\frac{r-k}{r-k}}).
\]
The smallest case of Conjecture 9 which is still unresolved is the determination whether \( f_3(n, 7, 4) = o(n^2) \), which is known as the (7,4)-problem.

2.2 Locally sparse hypergraphs

Recall that an \((n, r, t)\)-packing is of size at most \( \binom{n}{t} / \binom{r}{t} \), and such a packing is called an \((n, r, t)\)-design or an \((n, r, t)\)-Steiner system if its size attains this upper bound with equality. In [15] an \((n, 3, 2)\)-design is called \( e\)-sparse if it is simultaneously \( \mathcal{G}_3(i + 2, i)\)-free for every \( 2 \leq i \leq e \). Erdős [15] posed the following conjecture on the existence of \( e\)-sparse Steiner triple systems.

Conjecture 10 ([15]). For a fixed integer \( e \geq 2 \), there exists \( n_0(n_0) \) such that one can construct \( n \)-vertex \( e\)-sparse Steiner triple systems for every \( n \geq n_0 \) with \( n \equiv 1, 3 \pmod{6} \).

Recent results attained towards resolving this conjecture were proved independently by Bohman and Warnke [8], and Glock, Kühn, Lo and Osthus [27], who showed that for fixed \( e \), there exist \( e\)-sparse \((n, 3, 2)\)-packings with size \((1 - o(1))n^2/6\), which is near-optimal. A generalization of Conjecture 10 was made by Füredi and Ruzszikó [22] (see also Conjecture 7.2 of [27] for another generalization), who conjectured the existence of \( e\)-sparse \((n, r, 2)\)-Steiner systems, where an \((n, r, 2)\)-Steiner system is called \( e\)-sparse if it is simultaneously \( \mathcal{G}_r(\{ir - 2i + 2, i\}) \)-free for every \( 2 \leq i \leq e \).

Generalizing Question 2 in the spirit of the conjectures of Erdős, and Füredi and Ruzsikó leads to the following question. For fixed integers \( r > k \geq 2, e \geq 3 \), an \( r \)-graph is called locally \((e, k)\)-sparse if it is \( \mathcal{G}_r(\{ir - (i-1)k, i\}) \)-free for every \( 2 \leq i \leq e \).

Question 11. Let \( r > k \geq 2, e \geq 3 \) be fixed integers, and \( n \) be a sufficiently large integer. Then do there exist locally \((e, k)\)-sparse \((n, r, k)\)-packings with size at least \((1 - o(1))n^3/k \), where \( o(1) \to 0 \) as \( n \to \infty \)?
3 Proof of Theorem 6, the upper bound

To prove the upper bound in Theorem 6 we need the following technical lemma. Let \( \mathcal{H} \subseteq \binom{[n]}{r} \) be an \( r \)-graph and \( T \subseteq [n] \) be a subset. The codegree of \( T \) in \( \mathcal{H} \), \( \deg_\mathcal{H}(T) \), is the number of edges in \( \mathcal{H} \) which contain \( T \) as a subset, i.e., \( \deg_\mathcal{H}(T) = |\{A \in \mathcal{H} : T \subseteq A\}| \).

**Lemma 12.** An \( r \)-graph \( \mathcal{H} \) can be made to have no \((k - 1)\)-subset of codegree one by deleting at most \( \binom{n}{k-1} \) of its edges.

**Proof.** Successively remove the edges of \( \mathcal{H} \) which contain at least one \((k - 1)\)-subset of codegree one. Let \( A_i \) be the \( i \)-th removed edge of \( \mathcal{H} \), and let \( T_i \) be some \((k - 1)\)-subset of codegree one contained in \( A_i \). Since during this process the codegree of any \((k - 1)\)-subset can only decrease, then \( T_i \neq T_j \) for \( i \neq j \). In other words, the edges \( A_i, A_j \) are removed due to distinct \((k - 1)\)-subsets of codegree one, and therefore the process terminates after at most \( \binom{n}{k-1} \) edge removals. Note that the resulting \( r \)-graph is possibly empty.

Next we present the proof of the upper bound in Theorem 6.

**Proof of Theorem 6, the upper bound.** Let \( \mathcal{H} \subseteq \binom{[n]}{r} \) be a \( \mathcal{G}_r(3r - 2k; 3) \)-free \( r \)-graph, and let \( \mathcal{F} \) be the resulting \( r \)-graph from Lemma 12 by removing at most \( \binom{n}{k-1} \) edges from \( \mathcal{H} \), therefore \( |\mathcal{H}| \leq |\mathcal{F}| + O(n^{k-1}) \). The upper bound stated in Theorem 6 will follow by showing that

\[
|\mathcal{F}| \leq \frac{2\binom{r}{k}}{2\binom{r}{k} - 1} \binom{n}{k}.
\]

By the \( \mathcal{G}_r(3r - 2k; 3) \)-freeness of \( \mathcal{F} \), it is clear that any \( k \)-subset of \([n]\) is contained in at most two edges of \( \mathcal{F} \). For \( i \in \{1, 2\} \), let \( \mathcal{K}_i \subseteq \binom{[n]}{k} \) be the family of \( k \)-subsets of \([n]\) with codegree \( i \) in \( \mathcal{F} \), i.e.,

\[
\mathcal{K}_i = \{K \in \binom{[n]}{k} : \deg_\mathcal{F}(K) = i\}.
\]

Then, for any \( A \in \mathcal{F} \) and \( K \in \binom{A}{k} \), either \( K \in \mathcal{K}_1 \) or \( K \in \mathcal{K}_2 \), and by counting the number of \( k \)-subsets contained in the edges of \( \mathcal{F} \) it follows that

\[
\binom{r}{k} |\mathcal{F}| = |\mathcal{K}_1| + 2|\mathcal{K}_2|.
\]  

(6)

For \( K = \{x_1, \ldots, x_k\} \in \mathcal{K}_2 \), let \( A, B \in \mathcal{F} \) be the two edges that contain it, hence \(|A \cap B| \geq k\). We claim that in fact \(|A \cap B| = k\). Indeed, let \( a \in A \setminus B \) and consider the \((k - 1)\)-subset \( \{x_1, \ldots, x_{k-2}, a\} \subseteq A \). As \( \mathcal{F} \) contains no \((k - 1)\)-subset of codegree one, \( \deg_\mathcal{F}(\{x_1, \ldots, x_{k-2}, a\}) \geq 2 \), which implies that there exists at least one edge \( C \in \mathcal{F} \setminus \{A, B\} \) such that \( \{x_1, \ldots, x_{k-2}, a\} \subseteq C \). If \(|A \cap B| \geq k + 1\), then

\[
|A \cup B \cup C| \leq 3r - |A \cap B| - |A \cap C| \leq 3r - (k + 1) - (k - 1) = 3r - 2k,
\]

a contradiction.

Next we define for a \( k \)-subset \( K \in \mathcal{K}_2 \), and the two distinct \( r \)-subsets \( A, B \in \mathcal{F} \) containing it, the family of \( k \)-subsets \( \Phi_K := \binom{A}{k} \cup \binom{B}{k} \setminus \{K\} \). Since \(|A \cap B| = k\) we have that

\[
|\Phi_K| = 2\binom{r}{k} - 2.
\]

(8)

Furthermore, by a similar calculation to (7) one can verify that

\[
\Phi_K \subseteq \mathcal{K}_1.
\]

(9)

We have the following claim.
Claim 13. \( \Phi_K \cap \Phi_{K'} = \emptyset \) for distinct \( K, K' \in \mathcal{K}_2 \).

Assuming the correctness of the claim, together with (8), (9) it follows that

\[
|\mathcal{K}_2| (2\binom{r}{k} - 2) \leq |\mathcal{K}_1|.
\]

(10)

It is also easy to see that

\[
|\mathcal{K}_1| + |\mathcal{K}_2| \leq \binom{n}{k}.
\]

(11)

Combining (6), (10), and (11), we conclude that

\[
\binom{r}{k}|F| \left(2\binom{r}{k} - 2\right) = |\mathcal{K}_1| + 2|\mathcal{K}_2| = 2\binom{r}{k}^2 - 2 - \left(2\binom{r}{k} - 2\right)|\mathcal{K}_2| - |\mathcal{K}_1|
\]

\[
\leq \frac{2\binom{r}{k}}{2\binom{r}{k} - 1}(2|\mathcal{K}_2| + |\mathcal{K}_1|) \leq \frac{2\binom{r}{k}}{2\binom{r}{k} - 1}\binom{n}{k},
\]

as needed.

It remains to prove Claim 13. For the sake of contradiction, assume that there exist two distinct \( k \)-subsets \( K, K' \in \mathcal{K}_2 \) with \( \Phi_K \cap \Phi_{K'} \neq \emptyset \), then there is an edge \( A \in \mathcal{F} \) with \( K, K' \subseteq A \), otherwise this would contradict the fact that \( \Phi_K \cap \Phi_{K'} \subseteq \mathcal{K}_1 \) which follows by (9). Let \( B, C \in \mathcal{F} \) be the edges such that \( A \cap B = K \) and \( A \cap C = K' \), then

\[
|A \cup B \cup C| \leq 3r - |A \cap B| - |A \cap C| = 3r - 2k,
\]

and we arrive at a contradiction, completing the proof of the claim. \( \square \)

4 Proof of Theorem 3

In this section we prove Theorem 3. By plugging \( k = 2 \) in the upper bound of Theorem 6 we get that \( \limsup_{n \to \infty} \frac{f_r(n, 3r-4)}{n^2} \leq \frac{1}{r^2-r-1} \), hence it remains to prove the other direction, i.e.,

\[
\liminf_{n \to \infty} \frac{f_r(n, 3r-4)}{n^2} \geq \frac{1}{r^2-r-1}.
\]

4.1 The graph \( G_t \)

In this subsection we define the graph \( G_t \), which is used in step (1) of Construction 5 as the component graph of Lemma 4 but first we will need the following definition.

Definition 14 (\( \mathcal{H} \)-embedding). For integers \( r \leq s, 1 \leq m \leq \binom{s}{r} \), let \( \mathcal{H} = \{A_1, \ldots, A_m\} \subseteq \binom{[s]}{r} \) be an \( r \)-graph with \( m \) edges and \( S \) be a set of \( s \) elements. An \( \mathcal{H} \)-embedding from \( [s] \) to \( S \) is a bijection \( \Psi_S : [s] \longrightarrow S \) that acts naturally on the edges of \( \mathcal{H} \) as follows

\[
\Psi_S : \mathcal{H} \rightarrow \binom{S}{r}
\]

\[
A \in \mathcal{H} \mapsto \{\Psi_S(a) : a \in A\} \in \binom{S}{r}.
\]

Clearly, the image of the embedding \( \Psi_S(\mathcal{H}) := \{\Psi_S(A_1), \ldots, \Psi_S(A_m)\} \) is an \( r \)-graph on the vertex set \( S \) which forms a copy of \( \mathcal{H} \).
Let $X = \{x_1, ..., x_m\}$ be a set of $m$ elements, and let $S_1, \ldots, S_t$ be $t$ disjoint sets of size $s$ each, which are also disjoint from the set $X$. Finally, for $1 \leq i \leq t$, let
\[
\Psi_{S_i}(\mathcal{H}) = \{\Psi_{S_i}(A_1), \ldots, \Psi_{S_i}(A_m)\} \subseteq \binom{S_i}{r},
\]
be an $\mathcal{H}$-embedding in $S_i$.

**Definition 15** (The definition of $G_t$). The graph $G_t$, defined on the vertex set $V(G_t) := (\cup_{i=1}^{t}S_i) \cup X$ of size $ts + m$, is constructed by taking the union of the following three edge sets:

(i) $\mathcal{E}_1 = \{\text{edges connecting any two distinct vertices of } S_i, 1 \leq i \leq t\}$;

(ii) $\mathcal{E}_2 = \{\text{edges connecting any two distinct vertices of } X\}$;

(iii) $\mathcal{E}_3 = \{\text{edges connecting } x_j \text{ and each vertex of } \Psi_{S_i}(A_j), 1 \leq j \leq m, 1 \leq i \leq t\}$.

The following two simple observations are crucial for our construction.

- The induced subgraph of $G_t$ on each of the $t+1$ subsets $S_1, \ldots, S_t$ and $X$ is the complete graph.
- The sets of edges $\mathcal{E}_i, \mathcal{E}_j$ are disjoint for $i \neq j$, therefore
\[
|G_t| = \sum_{i=1}^{3} |\mathcal{E}_i|, \text{ where } |\mathcal{E}_1| = t \binom{s}{2}, |\mathcal{E}_2| = \binom{m}{2}, \text{ and } \mathcal{E}_3 = \text{rmt.} \tag{12}
\]

### 4.2 Lifting the $\mathcal{H}$-embeddings to an $(r + 1)$-graph

In this subsection, according to step (2) of Construction 5 we lift the $t$ $r$-graphs $\Psi_{S_1}(\mathcal{H}), \ldots, \Psi_{S_t}(\mathcal{H})$ that were introduced above, to an $(r + 1)$-graphs $G_t(\mathcal{H})$ on $(\cup_{i=1}^{t}S_i) \cup X$, the vertex set $V(G_t)$.

We call the $t$ $s$-subsets $S_1, \ldots, S_t$ petals and the $m$-subset $X$ the core. An $(r + 1)$-subset $F \subseteq V(G_t)$ is called $S_i$-rooted if it contains $r$ vertices of the petal $S_i$, and one vertex of $X$. In such a case, $r(F) := F \cap S_i$ and $c(F) := F \cap X$ are called the root and the core of $F$, respectively. Let $G_t(\mathcal{H}, S_i)$ be the $(r + 1)$-graph on the vertex set $S_i \cup X$, with the following $m$ $S_i$-rooted edges
\[
G_t(\mathcal{H}, S_i) = \{\Psi_{S_i}(A_1) \cup \{x_1\}, \ldots, \Psi_{S_i}(A_m) \cup \{x_m\}\}, \tag{13}
\]
where it is easy to verify that $\{r(F) : F \in G_t(\mathcal{H}, S_i)\}$ forms a copy of $\mathcal{H}$. Next we define
\[
G_t(\mathcal{H}) = \bigcup_{i=1}^{t}G_t(\mathcal{H}, S_i), \tag{14}
\]
to be the edge disjoint union of the $t$ $(r + 1)$-graphs $G_t(\mathcal{H}, S_i)$.

The following lemma shows that the $(r + 1)$-graph $G_t(\mathcal{H})$ inherits the freeness property from the $r$-graph $\mathcal{H}$. More precisely, if $\mathcal{H}$ is $\gamma_r(3r-4, 3)$-free then $G_t(\mathcal{H})$ is $\gamma_{r+1}(3(r+1)-4, 3)$-free. Note that an $r$-graph is called almost linear if any two distinct edges of it intersect in at most two vertices. Moreover, it is easy to see that any edge $F \in G_t(\mathcal{H})$ is $S_i$-rooted for some $1 \leq i \leq t$, and therefore for any (not necessarily distinct) $F_1, F_2 \in G_t(\mathcal{H})$,
\[
r(F_1) \cap c(F_2) = \emptyset. \tag{15}
\]

---

2 Since the $S_i$'s are pairwise vertex disjoint, the $G_t(\mathcal{H}, S_i)$'s are pairwise edge disjoint.
Lemma 16. The \((r + 1)\)-graph \(G_t(\mathcal{H})\) defined in \((\ref{def:gt})\) satisfies the following properties:

(i) Any two distinct edges of \(G_t(\mathcal{H})\) that are rooted in the same petal have distinct cores;

(ii) Any two distinct edges of \(G_t(\mathcal{H})\) that are rooted in different petals have disjoint roots;

(iii) \(G_t(\mathcal{H})\) has \(m_t\) \((r + 1)\)-edges;

(iv) The vertex set of any \((r + 1)\)-edge of \(G_t(\mathcal{H})\) induces a complete subgraph in the graph \(G_t\);

(v) If \(\mathcal{H}\) is almost linear, then \(G_t(\mathcal{H})\) is also almost linear; moreover, if \(\mathcal{H}\) is also \(\mathcal{G}_r(3r - 4, 3)\)-free, then \(G_t(\mathcal{H})\) is \(\mathcal{G}_{r+1}(3r - 1, 3)\)-free.

Proof. The first four statements follow easily from the definitions of \(G_t, G_t(\mathcal{H}, S_i)\) and \(G_t(\mathcal{H})\). To prove the first part of (v) we show that \(|F_1 \cap F_2| \leq 2\) for distinct edges \(F_1, F_2 \in G_t(\mathcal{H})\). If \(F_1\) and \(F_2\) are rooted in the same petal \(S_i\), then since \(\{r(F) : F \in G_t(\mathcal{H}, S_i)\}\) forms a copy of \(\mathcal{H}\), which is almost linear, together with \((\ref{def:gt})\) we conclude that

\[|F_1 \cap F_2| = |(r(F_1) \cup c(F_1)) \cap (r(F_2) \cup c(F_2))| = |r(F_1) \cap r(F_2)| \leq 2,\]

as needed. On the other hand, if \(F_1\) and \(F_2\) are rooted in different petals, then by \((\ref{def:gt})\) and (ii) it follows that \(F_1 \cap F_2 = c(F_1) \cap c(F_2)\), implying that \(|F_1 \cap F_2| \leq 1\).

To prove the second part of (v) consider three distinct \((r + 1)\)-edges \(F_1, F_2, F_3 \in G_t(\mathcal{H})\). We have the following three cases:

(a) If \(F_1, F_2, F_3\) are rooted in three distinct petals, then by (ii) \(r(F_1), r(F_2), r(F_3)\) are pairwise disjoint, hence

\[|F_1 \cup F_2 \cup F_3| = |r(F_1) \cup r(F_2) \cup r(F_3)| + |c(F_1) \cup c(F_2) \cup c(F_3)| \geq 3r + 1;\]

(b) If \(F_1, F_2, F_3\) are rooted in two distinct petals, say, \(F_1, F_2\) are \(S_i\)-rooted and \(F_3\) is \(S_j\)-rooted for \(i \neq j\), then

\[|F_1 \cup F_2 \cup F_3| \geq |F_1 \cup F_2| + |r(F_3)| \geq 3r,\]

as \(|F_1 \cup F_2| \geq 2r\) by the first part of (v), and by \((\ref{def:gt})\) and (ii), \(r(F_3)\) is disjoint from \(F_1 \cup F_2\);

(c) If \(F_1, F_2, F_3\) are rooted in the same petal, say \(S_i\), then since \(\{r(F) : F \in G_t(\mathcal{H}, S_i)\}\) is a copy of \(\mathcal{H}\) in \(S_i\), then it follows from \((\ref{def:gt})\), (i) and the \(\mathcal{G}_r(3r - 4, 3)\)-freeness of \(G_t(\mathcal{H}, S_i)\) that

\[|F_1 \cup F_2 \cup F_3| = |r(F_1) \cup r(F_2) \cup r(F_3)| + |c(F_1) \cup c(F_2) \cup c(F_3)| \geq 3r;\]

as needed. \(\square\)
4.3 Constructing $G_t$-packings in a large complete graph

Following step (3) of Construction 5 we introduce below the key idea of the recursive construction.

By applying Lemma 4 with the graph $G_t$ and large enough $n$, one obtains a $G_t$-packing in $K_n$, denoted by $\mathcal{G} = \{G_t^1, \ldots, G_t^l\}$, which contains roughly $l = (1 - o(1))n^2/|G_t|$ edge disjoint copies of $G_t$. For each $1 \leq i \leq l$, construct on $V(G_t^i)$ the $(r + 1)$-graph $G_t^i(\mathcal{H})$, as defined in (14). Let

$$\mathcal{F} := \bigcup_{i=1}^l G_t^i(\mathcal{H})$$

be the union of those $l$ $(r + 1)$-graphs. Recall that Lemma 16 implies that if $\mathcal{H}$ has the freeness property then so has $G_t^i(\mathcal{H})$ for each $1 \leq i \leq l$. The next lemma shows that $\mathcal{F}$ preserves the $\mathcal{G}_{r+1}(3r - 1, 3)$-freeness of the $G_t^i(\mathcal{H})$’s.

**Lemma 17.** Let $r \geq 3$ be an integer, and $\mathcal{H} \subseteq \binom{[n]}{r}$ be an almost linear $\mathcal{G}_r(3r - 4, 3)$-free $r$-graph with $m$ edges. Then $\mathcal{F}$ defined in (16) is almost linear, $\mathcal{G}_{r+1}(3r - 1, 3)$-free, with $m t l$ edges.

**Proof.** We begin by showing that $|\mathcal{F}| = m t l$. It is enough to prove that $\mathcal{F}$ is an edge disjoint union of the $G_t^i(\mathcal{H})$’s, $1 \leq i \leq l$. Indeed, by Lemma 16 (iv), each $(r + 1)$-edge in $G_t^i(\mathcal{H})$ induces a complete subgraph in $G_t^i$, implying that for any $1 \leq i \neq j \leq l$, $G_t^i(\mathcal{H})$ and $G_t^j(\mathcal{H})$ cannot have any common $(r + 1)$-edge, since otherwise $G_t^i$ and $G_t^j$ would have a common 2-edge, contradicting the definition of a $G_t$-packing.

For the almost linearity we prove the following stronger claim.

**Claim 18.** For any two distinct $(r + 1)$-edges $F_1, F_2 \in \mathcal{F}$, if there exists an $1 \leq i \leq l$ such that $\{F_1, F_2\} \subseteq G_t^i(\mathcal{H})$, then $|F_1 \cap F_2| \leq 2$; otherwise $|F_1 \cap F_2| \leq 1$.

The first case of the claim follows easily from the almost linearity of $\mathcal{H}$ and Lemma 16 (v). To prove the second case, suppose there exist $1 \leq i \neq j \leq l$ such that $F_1 \in G_t^i(\mathcal{H})$ and $F_2 \in G_t^j(\mathcal{H})$. By Lemma 16 (iv) $F_1$ (resp. $F_2$) induces a complete graph on $V(G_t^i)$ (resp. $V(G_t^j)$). On the other hand, by construction $G_t^i$ and $G_t^j$ are edge disjoint, hence clearly $|F_1 \cap F_2| \leq 1$.

Next we show that $\mathcal{F}$ is $\mathcal{G}_{r+1}(3r - 1, 3)$-free. Assume to the contrary that there exist three distinct $(r + 1)$-edges $F_1, F_2, F_3 \in \mathcal{F}$ such that $|F_1 \cup F_2 \cup F_3| \leq 3r - 1$. Hence, in such a case there exist $1 \leq i \neq j \leq 3$ such that $|F_1 \cap F_j| \geq 2$, since otherwise $|F_1 \cup F_2 \cup F_3| \geq 3r$. Without loss of generality, assume that $|F_1 \cap F_2| \geq 2$. Thus it follows from Claim 18 that there exists $1 \leq i \leq l$ such that $F_1, F_2 \in G_t^i(\mathcal{H})$, and we actually have $|F_1 \cap F_2| = 2$, as $G_t^i(\mathcal{H})$ is almost linear. We claim that $F_3$ also belongs to $G_t^i(\mathcal{H})$. Then given the $\mathcal{G}_r(3r - 4, 3)$-freeness of $\mathcal{H}$, we arrive at a contradiction by Lemma 16 (v).

For the sake of contradiction, assume that $F_3 \in G_t^j(\mathcal{H})$ for $j \neq i$. By the inclusion-exclusion principle, it is easy to check that $|F_3 \cap (F_1 \cup F_2)| \geq 2$, which implies that $|V(G_t^i) \cap V(G_t^j)| \geq 2$. By Lemma 4 (i) it follows that $|V(G_t^i) \cap V(G_t^j)| = 2$. Let $A = V(G_t^i) \cap V(G_t^j)$, then clearly $A \subseteq F_3$, and by Lemma 16 (iv) $A$ forms an edge in $G_t^j$, which contradicts Lemma 4 (ii).

4.4 Establishing the lower bound of $\pi_d(r, 2, 3)$

To prove the lower bound $\liminf_{n \to \infty} \frac{f_{\epsilon}(n, 3r - 4, 3)}{n^2} \geq \frac{1}{r^2 r - 1}$ it suffices to show that for any $\epsilon > 0$, there exists an integer $n(\epsilon) > 0$ such that for any $n > n(\epsilon)$, there exists an almost linear $\mathcal{G}_r(3r - 4, 3)$-free $r$-graph on $n$ vertices with at least $\frac{1}{r^2 r - 1} - \epsilon)n^2$ $r$-edges. We will prove this statement by induction on the uniformity parameter $r \geq 3$. 

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The base case \( r = 3 \) follows directly from the work of Glock \([25]\). Next, assume that the statement holds for \( r \geq 3 \), and we prove it for \( r + 1 \). Given \( \epsilon > 0 \), let \( \delta = \delta(\epsilon) > 0 \) be a sufficiently small constant. By the induction hypothesis, given \( \delta \) there exists an integer \( s(\delta) > 0 \) such that for any \( s > s(\delta) \), there exists an almost linear \( G_r(3r - 4, 3) \)-free \( r \)-graph \( H \subseteq \binom{[n]}{r} \) with \( m \) edges, where

\[
m \geq \frac{1}{r^2 - r - 1} - \delta s^2.
\]

Let \( t \) be a sufficiently large integer satisfying

\[
\frac{m}{t} < \delta.
\]

By applying Lemmas 4 and 17 with the component graph \( G_t \), one can construct an almost linear \( G_{r+1}(3r - 1, 3) \)-free \( (r + 1) \)-graph \( F = \bigcup_{i=1}^{t} G^i_{si}(H) \) on \( n > n_0^3 \) vertices, with \(|F| = tm\) edges, where

\[
l \geq \frac{(1 - \delta)n^2}{2(t\binom{s}{2} + \binom{n}{2} + rmt)}.
\]

Hence,

\[
|F| \geq tm \cdot \frac{(1 - \delta)n^2}{2(t\binom{s}{2} + \binom{n}{2} + rmt)} \geq \frac{(1 - \delta)n^2}{s^2/m + m/t + 2r} \geq \frac{(1 - \delta)n^2}{1 - \frac{r^2 - r - 1}{t(r^2 - r - 1)^2} + \delta + 2r},
\]

where the last inequality follows by (17) and (18). For \( \delta = 0 \) the right hand side of (20) is greater than

\[
\frac{(1 - \epsilon)n^2}{r^2 + r - 1} = \frac{(1 - \epsilon)n^2}{(r + 1)^2 - (r + 1) - 1}.
\]

Then by continuity there exists \( \delta > 0 \) for which \( F \) has at least \( \frac{(1 - \epsilon)n^2}{(r + 1)^2 - (r + 1) - 1} \) edges, completing the induction. \( \square \)

**Remark 19.** Let \( H \subseteq \binom{[n]}{r} \) be a \( G_r(3r - 4, r) \)-free \( r \)-graph with \( \frac{(1 - o(1))n^2}{r^2 - r - 1} \) edges, where \( o(1) \to 0 \) as \( n \to \infty \). It is clear by definition that \( H \) contains no 2-subsets with codegree larger than 2. In fact, it follows from (10) and (11) (with \( k = 2 \)) that the proportions of 2-subsets in \([n]\) with codegree 0, 1, 2 must be (approximately, as \( n \to \infty \)) \( 0, 1 - \frac{1}{r^2 - r - 1}, \frac{1}{r^2 - r - 1} \), respectively.

## 5 Proof of Theorem 6, the lower bound

### 5.1 From strongly perfect hashing matrices to sparse hypergraphs

In this subsection we define the notion of a strongly 3-perfect hashing matrix, and show that any such matrix gives rise to a sparse hypergraph with relatively many edges (see Lemma 7). We begin with some notations. Let \( M \) be an \( r \times m \) matrix over \( Q \), an alphabet of size \( q \), and for \( 1 \leq j \leq m \) let

\[
\vec{c}_j = (c_{1,j}, \ldots, c_{r,j})^T \in Q^r,
\]

be the \( j \)-th column of \( M \). We say that the \( i \)-th row of \( M \) separates a subset of columns \( T \), if the entries of row \( i \) restricted to columns in \( T \) are all distinct, i.e., \( \{c_{i,j} : \vec{c}_j \in T\} \) is a set of \( |T| \) distinct elements of \( Q \).

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*Here \( n_0 \) is a constant given by applying Lemma 4 with \( G := G_\epsilon \) and \( \epsilon := \delta \).*
A matrix is called 3-perfect hashing (see, e.g. [20]) if any three distinct columns of it are separated by at least one row. In the literature, matrices with different perfect hashing properties have been studied extensively. The reader is referred to [37] and the references therein for a detailed introduction to this topic. Here we introduce a slightly stronger notion which we term strongly 3-perfect hashing, but first we will need the following notation. For \( t \) columns \( \vec{c}_1, \ldots, \vec{c}_t \) of \( \mathcal{M} \), let \( I(\vec{c}_1, \ldots, \vec{c}_t) \subseteq [r] \) denote the collection of row indices for which \( \vec{c}_1, \ldots, \vec{c}_t \) have equal entries, i.e., \( i \in I(\vec{c}_1, \ldots, \vec{c}_t) \) if and only if \( c_{i,1} = \cdots = c_{i,t} \).

**Definition 20.** An \( r \times q^k \) matrix \( \mathcal{M} \) over \( Q \) is called strongly 3-perfect hashing if any three distinct columns \( \vec{c}_1, \vec{c}_2, \vec{c}_3 \) of \( \mathcal{M} \) are separated by more than \( r - 2k + |I(\vec{c}_1, \vec{c}_2, \vec{c}_3)| \) rows.

Clearly, this definition holds trivially if \( r - 2k + |I(\vec{c}_1, \vec{c}_2, \vec{c}_3)| < 0 \). However, if for any three distinct columns \( \vec{c}_1, \vec{c}_2, \vec{c}_3 \), \( r - 2k + |I(\vec{c}_1, \vec{c}_2, \vec{c}_3)| \geq 0 \), then a strongly 3-perfect hashing matrix is also a 3-perfect hashing matrix, justifying the name of this property.

**Observation 21.** Any \( r \times m \) matrix \( \mathcal{M} \) over \( Q \) defines an \( r \)-partite \( r \)-graph \( \mathcal{H}_\mathcal{M} \) with \( rq \) vertices and \( m \) edges as follows. The vertex set admits a partition \( V(\mathcal{H}_\mathcal{M}) = \cup_{i=1}^r V_i \), where

\[
V_i = \{(i, \alpha) : \alpha \in Q\}
\]

is the \( i \)-th vertex part of size \( q \). The \( m \) edges of \( \mathcal{H}_\mathcal{M} \) are defined by the \( m \) columns of \( \mathcal{M} \) as follows

\[
A_j := \{(1, c_{1,j}), \ldots, (r, c_{r,j})\} \subseteq (\cup_{i=1}^r V_i), 1 \leq j \leq m.
\]

It is easy to verify that for any edge and any vertex part we have \( |A_j| = r, |V_i| = q \) and \( |A_j \cap V_i| = 1 \).

In the remaining part of this section we view matrices (resp. columns of the matrices) and multipartite hypergraphs (resp. edges of the hypergraphs) as equivalent objects.

Next we present the proof of Lemma 7, but first note that for any \( 1 \leq i, j, l \leq m \),

\[
|I(\vec{c}_i, \vec{c}_j, \vec{c}_l)| = |A_i \cap A_j \cap A_l| \quad \text{and} \quad |I(\vec{c}_i, \vec{c}_j)| = |A_i \cap A_j|.
\]

**Proof of Lemma 7.** Given an \( r \times q^k \) strongly 3-perfect hashing matrix \( \mathcal{M} \) over \( Q \), let \( \mathcal{H}_\mathcal{M} = \{A_1, \ldots, A_q^k\} \) be the corresponding \( r \)-partite \( r \)-graph with \( rq \) vertices and \( q^k \) edges, given by Observation 21. We claim that \( \mathcal{H}_\mathcal{M} \) is \( \mathcal{G}(3r - 2k, 3) \)-free, i.e., for any three distinct edges \( A_i, A_j, A_l \in \mathcal{H}_\mathcal{M} \),

\[
|A_i \cup A_j \cup A_l| > 3r - 2k.
\]

Since \( \mathcal{M} \) is strongly 3-perfect hashing, the columns \( \vec{c}_i, \vec{c}_j, \vec{c}_l \) are separated by more than \( r - 2k + |I(\vec{c}_i, \vec{c}_j, \vec{c}_l)| \) rows. Equivalently, \( \vec{c}_i, \vec{c}_j, \vec{c}_l \) are not separated by less than \( 2k - |I(\vec{c}_i, \vec{c}_j, \vec{c}_l)| \) rows. Hence,

\[
|A_i \cup A_j \cup A_l| \geq 3(r - 2k + |I(\vec{c}_i, \vec{c}_j, \vec{c}_l)| + 1) + 2(2k - |I(\vec{c}_i, \vec{c}_j, \vec{c}_l)| - 1) - |I(\vec{c}_i, \vec{c}_j, \vec{c}_l)|
= 3r - 2k + 1,
\]

as desired. □

In order to construct matrices satisfying this useful property we introduce next a technical lemma of Füredi [21].
5.2 A technical lemma of Füredi

In this subsection, we introduce a lemma of Füredi [21] on a generalized linear independence property of polynomials. Let us begin with some necessary terminology.

For a prime power $q$ and a positive integer $k$, let $\mathbb{F}_q$ be the finite field with $q$ elements, and $\mathbb{F}_q^{<k}[x]$ be the set of polynomials of degree less than $k$, with coefficients in $\mathbb{F}_q$. Clearly $|\mathbb{F}_q^{<k}[x]| = q^k$. Let $p_1(x), p_2(x), p_3(x) \in \mathbb{F}_q^{<k}[x]$ be three arbitrary polynomials, and $k_1, k_2, k_3$ be positive integers such that

$$\sum_{i=1}^{3} k_i \leq k \quad \text{and} \quad k_i \leq k - \deg(p_i) \quad \text{for each} \quad 1 \leq i \leq 3.$$

The polynomials $p_1(x), p_2(x), p_3(x)$ are said to be $(k_1, k_2, k_3)$-independent, if for any $q_i(x) \in \mathbb{F}_q^{<k_i}[x], 1 \leq i \leq 3$, the equality

$$q_1(x)p_1(x) + q_2(x)p_2(x) + q_3(x)p_3(x) \equiv 0 \in \mathbb{F}_q^{<k}[x]$$

holds if and only if each $q_i(x)$ is the zero polynomial in $\mathbb{F}_q^{<k_i}[x]$. Equivalently, all the $q^\sum_{i=1}^{k_i}$ polynomials of the form $\sum_{i=1}^{3} q_i(x)p_i(x)$ are distinct in $\mathbb{F}_q^{<k}[x]$. Note that the case $k_1 = k_2 = k_3 = 1$ reduces to the usual $\mathbb{F}_q$-linear independence of three polynomials in $\mathbb{F}_q^{<k}[x]$.

A vector $\vec{v} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_q^r$ is called nonrepetitive if all of its entries are pairwise distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$. Given such a vector $\vec{v} \in \mathbb{F}_q^r$ with $r \geq k$, let $\mathbb{F}_q^{<k}[x, \vec{v}]$ be the $k$-dimensional subspace of $\mathbb{F}_q^r$ defined as

$$\mathbb{F}_q^{<k}[x, \vec{v}] := \left\{ \vec{c}_T = (f(\alpha_1), \ldots, f(\alpha_r))^T \in \mathbb{F}_q^r : \ f \in \mathbb{F}_q^{<k}[x] \right\}. \quad (21)$$

It is not too difficult to verify that $\mathbb{F}_q^{<k}[x, \vec{v}]$ is indeed a $k$-dimensional subspace of $\mathbb{F}_q^r$. Given a set $X \subseteq [r]$ of indices, we define the annihilator polynomial:

$$p_X(x, \vec{v}) = \prod_{i \in X} (x - \alpha_i).$$

The following lemma is crucial for Construction $S$ and it was proved by Füredi [21].

**Lemma 22** (see Lemma 10.3 and Corollary 10.4 [21]). Let $k$ be a positive integer and $q$ be a prime power. Then for all but at most $k(k-1)q^{2k-1}$ nonrepetitive vectors $\vec{v} = (\alpha_1, \ldots, \alpha_{2k}) \in \mathbb{F}_q^{2k}$, the polynomials $p_{Z_1}(x, \vec{v}), p_{Z_2}(x, \vec{v}), p_{Z_3}(x, \vec{v})$ are

$$(k - |Z_1|, k - |Z_2|, k - |Z_3|)$$-independent,

for every partition $[2k] = Z_1 \cup Z_2 \cup Z_3$, with $1 \leq |Z_i| < k$ for $i = 1, 2, 3$.

The following facts are easy to verify.

**Fact 23.** Let $\vec{v} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_q^r$ be a nonrepetitive vector and $X \subseteq [r]$. If the polynomials $f_1, f_2$ satisfy $f_1(\alpha_i) = f_2(\alpha_i)$ for each $i \in X$, then

$$p_X(x, \vec{v}) \mid f_1 - f_2.$$

**Fact 24.** Two distinct polynomials of degree less than $k$ can agree in at most $k - 1$ points in $\mathbb{F}_q$. 

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5.3 Constructing strongly 3-perfect hashing matrices

In this subsection we show the existence of strongly 3-perfect hashing matrices over large enough finite fields (see Lemma 25).

We view the k-dimensional subspace $\mathbb{F}_q^{<k}[x, \vec{v}]$ defined in (21) as an $r \times q^k$ matrix whose columns are labeled by the polynomials in $\mathbb{F}_q^{<k}[x]$, such that a column with index $f \in \mathbb{F}_q^{<k}[x]$ is the vector $\vec{c}_f$ defined in (21).

**Lemma 25.** Let $r > k \geq 2$ be fixed integers and $q > 4'k^2$ be a prime power. Then for at least $q^r - 4'k^2q^{-r}$ vectors $\vec{v} \in \mathbb{F}_q^r$, the matrix $\mathbb{F}_q^{<k}[x, \vec{v}]$ is strongly 3-perfect hashing.

**Proof.** Call a vector $\vec{v} \in \mathbb{F}_q^r$ bad if the corresponding matrix $\mathbb{F}_q^{<k}[x, \vec{v}]$ is not strongly 3-perfect hashing, and observe that there are at most $\binom{r}{2}q^{r-1}$ vectors in $\mathbb{F}_q^r$ with at least two repeated entries. To prove the lemma it suffices to show that the number of bad nonrepetitive vectors in $\mathbb{F}_q^r$ is at most $4'k(k-1)q^{-r}$, and therefore the total number of bad vectors is bounded from above by $\binom{r}{2}q^{r-1} + 4'k(k-1)q^{-r} \leq 4'k^2q^{-r}$.

We say that a vector $\vec{v} \in \mathbb{F}_q^r$ is bad for a set of row indices $I \subseteq [r]$, if there exist three distinct columns $\vec{c}_1, \vec{c}_2, \vec{c}_3$ of $\mathbb{F}_q^{<k}[x, \vec{v}]$ such that $I = I(\vec{c}_1, \vec{c}_2, \vec{c}_3)$ and they are separated by at most $r - 2k + |I| \geq 0$ rows. Clearly, a vector $\vec{v} \in \mathbb{F}_q^r$ is bad if it is bad for some set $I \subseteq [r]$. For a given subset $I \subseteq [r]$, below we give an upper bound on the number of nonrepetitive vectors that are bad for it.

Assume that $\vec{v} \in \mathbb{F}_q^r$ is a nonrepetitive vector which is bad for $I \subseteq [r]$, and let $\vec{c}_1, \vec{c}_2, \vec{c}_3 \in \mathbb{F}_q^{<k}[x, \vec{v}]$ be three distinct columns that violate the strongly 3-perfect hashing property. Let

$$X := I(\vec{c}_1, \vec{c}_2) \cup I(\vec{c}_2, \vec{c}_3) \cup I(\vec{c}_1, \vec{c}_3)$$

be the set of rows for which at least two columns attain the same value, and therefore $I \subseteq X$. Since the three columns are separated by any row whose index is not in $X$, then $r - |X| \leq r - 2k + |I|$, namely,

$$|X| \geq 2k - |I|. \quad (22)$$

By Fact 23 $|I| \leq k - 1$. If $|I| = k - 1$, then again by Fact 24 $I = X$ which contradicts (22). Therefore, we assume that $|I| < k - 1$. Note that by (22) $|X \setminus I| = |X| - |I| \geq 2k - 2|I| > 0$, and let $Y \subseteq X \setminus I$ be an arbitrary subset of size $2k - 2|I|$. Next, define the following three sets $Z_i$ that form a partition of $Y$:

$$Z_1 = \{i \in Y : i \in I(\vec{c}_1, \vec{c}_2)\}, \quad Z_2 = \{i \in Y : i \in I(\vec{c}_2, \vec{c}_3)\} \quad \text{and} \quad Z_3 = \{i \in Y : i \in I(\vec{c}_1, \vec{c}_3)\}.$$

The sets $Z_i$ satisfy the following claim.

**Claim 26.** For $i = 1, 2, 3$, $1 \leq |Z_i| < k - |I|$ and the polynomials $p_{Z_i}(x, \vec{v}), p_{Z_2}(x, \vec{v}), p_{Z_3}(x, \vec{v})$ are not

$$(k - |I| - |Z_1|, k - |I| - |Z_2|, k - |I| - |Z_3|)$$-independent.

**Proof of Claim 26.** Since $k - |I| \geq 2$ the inequalities on the sizes of the sets $Z_i$ are well-defined. The sets $I$ and $Z_1$ are disjoint and are subsets of $I(\vec{c}_1, \vec{c}_2)$, therefore $|Z_1| + |I| \leq |I(\vec{c}_1, \vec{c}_2)| < k$ which implies the upper bound. The proof for $Z_2, Z_3$ is the same. Furthermore, if one of the $Z_i$’s is the empty set, say $Z_3$, then this would imply that $2k - 2|I| = |Y| = |Z_1| + |Z_2|$, i.e., either $Z_1$ or $Z_2$ is of size at least $k - |I|$, which is a contradiction.

Assume that $\vec{c}_1, \vec{c}_2, \vec{c}_3$ are indexed by polynomials $f_1, f_2, f_3 \in \mathbb{F}_q^{<k}[x]$, respectively. Since

$$(Z_1 \cup I) \subseteq I(\vec{c}_1, \vec{c}_2), \quad (Z_2 \cup I) \subseteq I(\vec{c}_2, \vec{c}_3) \quad \text{and} \quad (Z_3 \cup I) \subseteq I(\vec{c}_1, \vec{c}_3),$$

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then by Fact 23
\begin{align*}
p_{Z_1}(x, \vec{v})p_I(x, \vec{v}) &= p_{Z_1 \cup I}(x, \vec{v}) \mid f_1 - f_2, \\
p_{Z_2}(x, \vec{v})p_I(x, \vec{v}) &= p_{Z_2 \cup I}(x, \vec{v}) \mid f_2 - f_3, \\
p_{Z_3}(x, \vec{v})p_I(x, \vec{v}) &= p_{Z_3 \cup I}(x, \vec{v}) \mid f_3 - f_1,
\end{align*}
which implies that there exist nonzero polynomials \( q_i(x) \in F_q^{\mathbb{C}^k - |I| - |Z_i|}[x] \) for \( 1 \leq i \leq 3 \), such that
\begin{align*}
q_1(x)p_{Z_1}(x, \vec{v})p_I(x, \vec{v}) &= f_1 - f_2, \\
q_2(x)p_{Z_2}(x, \vec{v})p_I(x, \vec{v}) &= f_2 - f_3, \\
q_3(x)p_{Z_3}(x, \vec{v})p_I(x, \vec{v}) &= f_3 - f_1.
\end{align*}
By summing the left and right hand sides of (24) we conclude that
\[
(q_1(x)p_{Z_1}(x, \vec{v}) + q_2(x)p_{Z_2}(x, \vec{v}) + q_3(x)p_{Z_3}(x, \vec{v})) \cdot p_I(x, \vec{v}) = 0. \tag{25}
\]
The ring of polynomials \( F_q[x] \) is a domain, implying that
\[q_1(x)p_{Z_1}(x, \vec{v}) + q_2(x)p_{Z_2}(x, \vec{v}) + q_3(x)p_{Z_3}(x, \vec{v}) = 0,
\]
namely, \( p_{Z_1}(x, \vec{v}), p_{Z_2}(x, \vec{v}), p_{Z_3}(x, \vec{v}) \) are not \( (k - |I| - |Z_1|, k - |I| - |Z_2|, k - |I| - |Z_3|) \)-independent, completing the proof of the claim. \( \square \)

**Continuing the proof of Lemma 25**: If \( \vec{v} \in F_q^n \) is a nonrepetitive vector which is bad for \( I \), then there exists a \( (2k - 2|I|) \)-subset \( Y \subseteq [r] \setminus I \) for which the assertion of Claim 24 holds. However, Lemma 22 provides an upper bound on the number of such vectors \( \vec{v} \mid Y \in F_q^{2k-2|I|} \), where \( \vec{v} \mid Y \) is the restriction of \( \vec{v} \) to the coordinates in \( Y \). More precisely, by Lemma 22 there are at most
\[
(k - |I|)(k - |I| - 1)q^{2k-2|I|-1} \leq k(k - 1)q^{2k-2|I|-1}
\]
possible choices for \( \vec{v} \mid Y \). Thus, given the sets \( I \) and \( Y \), the number of nonrepetitive vectors \( \vec{v} \in F_q^n \) which are bad for \( I \) is at most
\[q^{|I|} \times k(k - 1)q^{2k-2|I|-1} \times q^{r - |I| - (2k-2|I|)} = k(k - 1)q^{r-1}.
\]
Indeed, there are at most \( q^{|I|} \) ways to pick \( \vec{v} \mid I \), at most \( k(k - 1)q^{2k-2|I|-1} \) ways to pick \( \vec{v} \mid Y \), and at most \( q^{r - |I| - (2k-2|I|)} \) ways to pick the remaining entries in \( \vec{v} \mid [r]\setminus(I\cup Y) \).

Since \( I \) and \( Y \) are subsets of \( [r] \), then there are at most \( (2^r)^2 \) ways to choose them. To conclude, the total number of bad nonrepetitive vectors is at most \( 4^r k(k - 1)q^{r-1} \), as desired. \( \square \)

### 5.4 Establishing the lower bound of \( \pi_d(r, k, 3) \)

Let \( r > k \geq 2 \) be fixed integers, in this subsection we will present the proof of the lower bound \( \frac{1}{r^k - r} \leq \liminf_{n \to \infty} \frac{f_r(n,3r-2k,3)}{n^k} \), finishing the comments after Construction 8. We need the following result on the distribution of primes.

**Lemma 27** (see Theorem 1 [8]). There exists a positive integer \( n_0 \) such that for any integer \( n > n_0 \), the largest prime \( q \) not exceeding \( n \) satisfies \( q \geq n - n^\delta \), where \( 0 < \delta \leq 0.525 \).

The desired lower bound is a straightforward consequence of the following claim.
Claim 28. Let $r > k \geq 2$ be fixed integers and $n_0, \delta$ be the constants given by Lemma 27. Then there exists a positive constant $a = a(r, k, n_0)$ such that for any $n \geq 1$ there exists a $\mathcal{G}_r(3r - 2k, 3)$-free $r$-graph $\mathcal{F}$ on $n$ vertices with at least

$$\frac{n^k}{r^k - r} - an^{k-1+\delta}$$

edges, such that for any distinct $A, B \in \mathcal{F}$, $|A \cap B| \leq k - 1$.

Proof. We will prove the claim by induction on $n$. Let $n^* = n^*(r, k, n_0)$ be the smallest $n$ satisfying

$$\frac{n}{r} > n_0 \quad\text{and}\quad \frac{n}{r} - \left(\frac{n}{r}\right)^\delta > 4^r k^2,$$

and pick large enough $a = a(r, k, n_0)$ such that $\frac{k^k}{r^k - r} - an^{k-1+\delta} < 0$ for $n \leq n^*$. For such a choice of $a$ the claim holds trivially for $n \leq n^*$. Let $n > n^*$ and $q$ be the largest prime not exceeding $\frac{n}{r}$, then by Lemma 27 there exists a vector $\vec{v} \in \mathbb{F}_q^r$ such that the $r \times q^k$ matrix $\mathcal{M} := \mathbb{F}_{q^k}^r[x, \vec{v}]$ defined by (21), is strongly 3-perfect hashing. By Lemma 7, $\mathcal{M}$ induces a $\mathcal{G}_r(3r - 2k, 3)$-free $r$-partite $r$-graph $\mathcal{H}_M$, with $q^k$ edges and $rq$ vertices, such that $V(\mathcal{H}_M)$ is partitioned into $r$ disjoint parts $V_1, \ldots, V_r$ with $|V_i| = \cdots = |V_r| = q$. Since the edges of $\mathcal{H}_M$ are defined by the columns of $\mathcal{M}$, and the columns of $\mathcal{M}$ are essentially defined by polynomials of degree at most $k - 1$, it follows easily from Fact 23 that for any distinct $A, B \in \mathcal{H}_M$, $|A \cap B| \leq k - 1$.

By induction hypothesis, there exists a $\mathcal{G}_r(3r - 2k, 3)$-free $r$-graph $\mathcal{H}$ on $q$ vertices with at least $\frac{q^k}{r^k - r} - aq^{k-1+\delta}$ edges, and for any distinct $A, B \in \mathcal{H}$, $|A \cap B| \leq k - 1$. For each $1 \leq i \leq r$, put a copy of $\mathcal{H}$, denoted as $\mathcal{H}_i$, into the vertex set $V_i$. Let $\mathcal{F}$ be the $r$-graph formed by the union

$$\mathcal{F} := (\bigcup_{i=1}^r \mathcal{H}_i) \cup \mathcal{H}_M.$$

It is not hard to see that for any $A, B \in \mathcal{F}$, $|A \cap B| \leq k - 1$. Moreover, since each of $\mathcal{H}_1, \ldots, \mathcal{H}_r, \mathcal{H}_M$ is $\mathcal{G}_r(3r - 2k, 3)$-free, it is routine to check that $\mathcal{F}$ is also $\mathcal{G}_r(3r - 2k, 3)$-free. We omit the details here.

It remains to prove an appropriate lower bound for $|\mathcal{F}|$. Clearly,

$$|\mathcal{F}| = |\mathcal{H}_M| + r|\mathcal{H}| \geq q^k + r \cdot \left(\frac{q^k}{r^k - r} - aq^{k-1+\delta}\right)$$

$$= \frac{(rq)^k}{r^k - r} - arq^{k-1+\delta} \geq \frac{(n - \frac{n}{r})^k}{r^k - r} - ar\left(\frac{n}{r}\right)^{k-1+\delta}$$

$$\geq \frac{n^k}{r^k - r} - n^{k-1+\delta}\left(\frac{k}{r^{k+\delta-1} - r^\delta} + \frac{a}{r^{k+\delta-2}}\right).$$

A short calculation shows that for large enough $a$, $\frac{k}{r^{k+\delta-1} - r^\delta} + \frac{a}{r^{k+\delta-2}} \leq a$. Therefore, we conclude that $|\mathcal{F}| \geq \frac{n^k}{r^k - r} - an^{k-1+\delta}$ for some appropriate $a = a(r, k, n_0)$, completing the proof of the claim. \qed

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