TIME SERIES ANALYSIS OF THE RESPONSE OF MEASUREMENT INSTRUMENTS

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Abstract

In this work the significance of treating a set of measurements as a time series is being explored. Time Series Analysis (TSA) techniques, part of the Exploratory Data Analysis (EDA) approach, can provide much insight regarding the stochastic correlations that are induced on the outcome of an experiment by the measurement system and can provide criteria for the limited use of the classical variance in metrology. Specifically, techniques such as the Lag Plots, Autocorrelation Function, Power Spectral Density and Allan Variance are used to analyze series of sequential measurements, collected at equal time intervals from an electromechanical transducer. These techniques are used in conjunction with power law models of stochastic noise in order to characterize time or frequency regimes for which the usually assumed white noise model is adequate for the description of the measurement system response. However, through the detection of colored noise, usually referred to as flicker noise, which is expected to appear in almost all electronic devices, a lower threshold of measurement uncertainty for this particular system is obtained and the white noise model is no longer accurate.

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1 Introduction

Exploratory Data Analysis (EDA) is an approach, contrast to classical approach and Bayes theory, which allows the data itself to reveal its underlying structure and model [1]. In particular, Time Series Analysis (TSA), treats every experimental data set as a group of subsequent observations in time \{X_t\} of a random variable X (measurand).

This technique has been recently applied to the analysis of experimental measurements collected in metrological laboratories in an attempt to clarify the role of serially correlated data with stochastic characteristics with regard to more realistic measurement uncertainty estimation (see papers from Witt T.J. et al, Zhang N.F. et al). Via these analysis methods it has been shown that if measurements are autocorrelated then the usual “classical variance” as expressed through the standard deviation of the mean of a set of repeated measurements can lead to an underestimation of the measurement uncertainty [2,3]. An alternative approach for the variance calculation of autocorrelated measurements has been proposed by Allan D.W. [4] and used thoroughly by Witt T.J. in the case of voltage measurements [5,6,7].

In this work, a variety of TSA methods are being introduced and used [8] in order to best analyze univariate time series of observations collected in equally spaced time intervals. These data were obtained in an attempt to characterize the performance of a newly commissioned mass comparator for weighing in air. Eventually it will be used for the determination of the mass of density artifacts such as Si spheres.
The applied methods include both time and frequency domain techniques such as the construction of Lag Plots, the use of the Auto-Correlation function (ACF), the study of Power Spectral Density (PSD) [1,9] and the two-sample or Allan Variance [10]. More specifically, Lag Plots can reveal possible underlying data structures depending on the observed data pattern. The analysis of the PSD can identify possible system resonances and by employing noise power law models [9] the type of noise present in the measured signal can, at least qualitatively, be determined. The Allan Variance, which is an alternate but equivalent time domain method, is more efficient in characterizing the variance of autocorrelated measurements and the type of noise present. Finally, the Autocorrelation Function gives the correlation of consecutive measurements and the “memory span” of the system.

The analysis applied aims at finding autocorrelations in the time series observations, characterizing the noise in the signal output and providing a realistic estimate of the uncertainty of correlated measurements [11]. The results from the above methods will be compared to the power law models of three well-known types of noise that exist in nature: flicker noise, white noise and random walk noise. Even though the application of all the computational procedures described, is performed on a series of measurements that comes from a precision balance system, they can be applied to any electromechanical transducer system equally well with similar results.

2 Time Series Analysis Methods

Time Series are an example of a stochastic or random process. Mathematically, a stochastic process is an indexed collection of random variables \( \{X_t: t \in T\} \). When such a process is treated as a time series, then the process is indexed by time, either discrete or continuous [8]. We will specifically focus on discrete time univariate time series analysis, by using the methods presented below:

2.1 Stationary Processes

For most of the time series encountered in several scientific fields, the correlations between successive measurements have a deterministic component (linear trend, periodic characteristics etc) and a stochastic one. The factor that is in most cases responsible for a linear trend in a set of measurements is the ambient temperature. Any deterministic trend of that kind leads to non-stationary processes (non-stationary mean or non-stationary variance) where the corresponding statistical properties do not converge to any particular value.

In this paper, the effects of these trends have been removed by the application of the differencing method, transforming the non-stationary time series to stationary ones. Considering the first differences of the series \( \{X_t\} \) yields the same result as applying a high-pass filter to the output signal:

\[ \nabla X_t = X_t - X_{t-1} \]  

(1)

Moreover, the application of the second differences filter, eliminates any periodic characteristics of the time series:

\[ \nabla^2 X_t = X_t - 2X_{t-1} + X_{t-2} \]  

(2)

The purpose of the above transformation is the removal of any deterministic attributes that would screen possible stochastic characteristics of the system that produces the time series. The transformed series is thereafter examined with time domain as well as frequency domain techniques in order to reveal its stochastic character.

2.2 Lag Plots

The simplest test of extracting the correlation of two variables is the scatter plot of the dependent variable as a function of the independent one. The existence or not of a trend in such a scatter plot indicates the correlation or not between the plotted variables.
The same method is applicable in a time series of events where these scatter plots are called lag plots [1]. The term “lag” defines a specific step in time. Therefore, a lag plot is a plot of the observation \( X_t \) versus the observation \( X_{t-\tau} \), \( \tau \) = lag interval or time, that provides evidence of the possible autocorrelations in a time series.

The interpretation of the lag plot can answer to the questions of data randomness, serial correlations, identification of outliers and suggest a proper model that best describes the data set. According to the underlying structure of the data set, the lag plot can identify 4 different case studies: White Noise, strong positive correlations (Random Walk Noise), weak positive correlations (Pink Noise) and negative correlations (Phase Noise).

2.3 AutoCorrelation Function

In the case of any two random variables \( X, Y \) the equation for the variance of their sum is given by equation (3):

\[
var[X + Y] = var[X] + var[Y] + 2cov[X, Y]
\]  

Generalizing equation (3) for the variance of the mean of \( N \) observations, we get [3]

\[
var[\bar{X}] = \frac{1}{N^2} \sum_{i=1}^{N} var(x_i) + \frac{2}{N^2} \sum_{j=1}^{N} \sum_{k>j}^{N} R(x_j, x_k)
\]  

where the second term of the equation represents the autocovariance between adjacent measurements of the univariate time series of length \( N \).

Alternatively we have

\[
var[\bar{X}] = \frac{1}{N} var[X] \cdot \left( 1 + \frac{2}{N} \sum_{j=1}^{N} \sum_{k>j}^{N} \rho(x_j, x_k) \right)
\]  

where \( \rho(x_j, x_k) \) represents the fraction of autocovariance over variance and it is called autocorrelation. Considering that autocovariance and thus autocorrelation of any two observations depends only on their in-between time lag \( \tau \) and not on the ordering \( j, k \) we can simply derive the autocorrelation function (ACF) \( \rho(\tau) \) and its estimator \( r(\tau) \)

\[
r(\tau) = \frac{\sum_{i=1}^{N-\tau} (x(t_i) - \bar{x})(x(t_i + \tau) - \bar{x})}{\sum_{i=1}^{N} (x(t_i) - \bar{x})^2}
\]

According to the shape of the autocorrelation function [12], the following cases can be recognized:

(i) No specific shape, where the white noise model can be applied and data fall within 95% confidence bands of the ACF plot (or correlogram). (ii) Exponential or alternating positive and negative values and then decaying to zero, which is indicative of an Autoregressive (AR) model. (iii) One or more spikes and the rest zero, where a Moving Average (MA) model is appropriate. (iv) Decay starting after a few lags, where a mixed ARMA model is recommended. (v) No decay to zero, where the process is non-stationary.

2.4 Power Spectral Density

The Power Spectrum is a very popular method used in the characterization of physical systems [9]. By simply applying a Discrete Fourier Transform to the time series data, the real \( \text{Re}(X(f)) \) and imaginary \( \text{Im}(X(f)) \) parts of the Fourier coefficients are derived. The periodogram is a plot of the magnitude squared of the coefficients \( S(f) \) versus frequency \( f \), constituting the Power Spectral Density (PSD).
Moreover, the PSD is the Fourier Transform of the Autocorrelation Function $\rho(\tau)$ and that relation, given by equation (7), makes these techniques equivalent in the analysis of time series:

$$S(f) = \int_{-\infty}^{+\infty} \rho(\tau) \cdot exp(-2\pi if\tau) d\tau \quad (7)$$

By plotting $S(f) - f$ in a log-log diagram, the different noise levels in the power spectrum can be estimated from the adjustment of equation (8) to the log-log plot. The calculation of the spectral exponent $n$, leads to the identification of white noise ($n=0$), flicker noise ($n=1$), random walk noise ($n=2$) or any intermediate case $0<n<2$ of colored noise.

$$S(f) = \frac{h_n}{f^n}, 0 \leq n \leq 2 \quad (8)$$

### 2.5 Allan Variance

The Allan Variance or Two-sample Variance was first introduced by David W. Allan for the evaluation of the stability of time and frequency standards [10]. The main idea behind this method lies in the relation of the dispersion that a set of measurements has, with the noise we expect to find in the output signal. The most common measure of dispersion is the classical variance, whose value decreases as the number of data points included in the calculations, increases.

Unfortunately this is only true in the case of truly random processes, where the variance of the mean decreases with the number $N$ of data points

$$\text{var}[\bar{X}] = \frac{\sigma^2}{N} \quad (9)$$

In the case of autocorrelated data the above equation does not apply since there exists a possibility that the estimated variance will diverge as the number of data points increases [4]. So, we create a high-pass filter by extracting each $k+1$ value from its previous one $k$ and thus we remove any possible trends, fast fluctuations or other peculiar characteristics. The Allan variance is estimated at time intervals $\tau=m\tau_0$, where $\tau_0$ is a minimum sampling time and $m$ is usually chosen to denote powers of two. From the resulting plot of $\sigma(\tau) - \tau$ one can estimate the cut-off value after which the inclusion of more data points does not lead to the decrease of the variance, as classically indicated by equation (9). In a log-log plot the Allan variance is proportional to $\tau^\mu$ and $\mu = -n-1$, where $n$ is the spectral exponent appearing in equation (8).

$$\sigma^2_y(\tau) = \frac{1}{2(N-1)} \sum_{k=1}^{N-1} (y_{k+1}(\tau) - y_k(\tau))^2 \quad (10)$$

In this paper lag plots and autocorrelation function are used for the analysis of correlations in the time series while spectral analysis and Allan variance are used to test time series for colored noise and estimate realistic uncertainties. It should be noted that a normal distribution of observations couldn’t tell between white noise and colored noise thus the mere use of histograms for the acceptance or not of equation (9) is pointless.

### 3 Experimental

Data was obtained from characterization experiments of a newly commissioned 1kg/ 10μg resolution mass comparator at the Hellenic Institute of Metrology density laboratory. The principle function of this comparator will be the measurement of mass in air of Si and ceramic density artifacts of mass ca. 1kg in the form of spheres through comparative weighing. Due to the limited space on its weighing pan as well as the weighing procedure itself it was deemed necessary to also use its under-floor weighing
facility in order to perform the measurements. Thus the object of which the mass is to be determined is placed in a ring cradle and hung from the balance weighing cell via a ca. 80cm steel wire which passes through a hole at the bottom of the balance. In order to have stable weighing conditions, a plexi-glass draft shield of approximately 0.3 \( m^3 \) volume was used with a facility for installing temperature and relative humidity sensors. A simple weight exchange mechanism was used to alternately weigh the weight standard placed on the balance pan and the test object hanging from the balance in an attempt to minimize the disturbance of the weighing environment [13]. Since weighing of non-conducting objects is prone to electrostatic charging effects it was ensured that the relative humidity within the draft shield was above 40\%. The air temperature was set at 22.5 °C. The data presented below resulted from a Si sphere of mass approximately 1kg hanging from the balance weighing cell. Sampling was performed every 20 sec of the balance indication and of the environmental conditions.

### 4 Results

The following figures show the raw data transformed into a time series of equal time intervals (figure 1), the variation of temperature (figure 2) and relative humidity (figure 3) over time.

For a time interval approximately equal to three hours, the time series evolves in accordance with ambient temperature, a consequence of the increase of air density and thus of buoyancy due to air. Afterwards, it shows an increasing trend most probably due to balance drift. In order to eliminate these deterministic affects, a first difference filter is applied and the resulting series is thoroughly examined for stochastic correlations and noise.

![Figure 1. Time series of hanging Si sphere.](image1)

![Figure 2. Ambient conditions, temperature.](image2)

![Figure 3. Ambient conditions, relative humidity.](image3)
The lag plot of the differenced series in figure 4 is comprised of a small number of points, due to the limited discretization available, gathered around (0,0) with no particular structure that could reveal any serial correlations.

The correlogram in figure 5 shows positive serial correlations in the first lags and the sample ACF approaches zero very slowly (after 2000 sec), suggesting the inadequacy of the white noise model and the use of classical variance. From approximately the 200th lag and after, the data become independent falling well within the 95% confidence bands of $\pm \frac{2}{\sqrt{N}}$, implying that the white noise model is now sufficient to describe the measurements. Consequently, two different noise areas are also expected in the PSD diagram. The shape of the ACF is exponentially decaying to zero so an AR model would be appropriate to model the data [5].
The periodogram of the differenced data as well as the PSD (in log-log) of the original and the differenced series are presented in figures 6 and 7. In figure 6, the presence of strong power intensity in lower frequencies indicates an AR model with $\varphi > 0$ to model the data. Figure 7 shows a comparison between the original time series before the removal of deterministic affects and the differenced series. As expected, a Random-Walk Noise Model adequately describes the non-stationary series and the slope of the regression line (red line) is close to $-2$. On the contrary, the differenced series exhibits a more complex behavior. At lower frequencies, a line of slope close to $-0.6$ (blue line) can approximate the data but at higher frequencies the white noise model of zero slope seems to be more appropriate (green line). Thus, there exists a cut-off frequency at which a marked change of slope occurs, revealing the two different noise areas of flicker or pink noise and white noise [5]. This implies that after about 500 sec the classical variance would be valid as a measure of dispersion of the data. In addition, a suitable power law model would be $S_y(f) = h_0 f^0 + h_{-1} f^{-1}$, where $h_0$ and $h_{-1}$ are the intensity coefficients.

Table I summarizes the results for the two time series.

Table I

| Frequency (sec) | Power |
|----------------|-------|
| 0.005          | 0.01  |
| 0.015          | 0.02  |

Figure 6. The periodogram of the first-differenced data, showing high intensity at low frequencies and white noise at the rest of the spectrum.

Figure 7. The PSD of the original compared to the first-differenced data. After eliminating the deterministic affects, the random walk noise turns into a combination of flicker and white noise. The fitted line is of the form $y = -n^x + \log h$, where $y=\log$(Power) and $x=\log$(Frequency).
From the above analysis, one concludes that the use of Allan Variance could be more appropriate than Classical Variance, particularly during the initial time duration [4,5]. The dependence of the Allan deviation that is shown in figure 8 is in complete agreement with the PSD results, indicating clearly the white noise and the flicker noise area. Notice that the last three points, if included in the calculation, will lead to an overestimation of the sample deviation. Indeed, the classical standard deviation is equal to $6.53 \times 10^{-6}$ where the Allan deviation gives a value of one less order of magnitude $\sigma/\tau = 7.34 \times 10^{-7}$ for $\tau=256$. The last three data points after $\tau=256$, are indicative of the flicker noise and thus are not included in the estimation of the sample variance.

![Allan Deviation](image)

Figure 8. The Allan deviation of the first-differenced series, showing white and flicker noise.

Table II gives the results from the fitting and the estimation of exponent $\mu$, as well as the calculation of the spectral exponent from the relation $n = -\mu - 1$. White noise and flicker noise areas are clearly indicated.

|        | $\mu$  | $n$    |
|--------|--------|--------|
| Area 1 | -0.8942| -0.1058|
| Area 2 | 0.027  | -1.02  |

5 Conclusions

The main purpose of this work is to point out the importance of considering measurement results as time series and of using the time series analysis methods to test for correlations. Only in the absence of correlation the use of the expression $\sigma/\sqrt{N}$ is appropriate to characterize the random uncertainty.
Otherwise, the Allan variance is an alternate, more appropriate tool to characterize the dispersion of a set of experimental results.

Lag Plots, Correlograms and Periodograms are graphic tools of EDA analysis used to identify serial correlations in measurements and estimate the type of the underlying stochastic noise. For the case study of mass measurements we consider here, all the above techniques give consistent results.

Allan Variance and PSD analysis provide almost the same information for a set of measurements but with the strict limitation of data collected at regular time intervals. Also, the length of the time series must be of a value of $2^k$ otherwise not all points are included in the Fourier Transform and Allan Variance calculations.

Future work will mainly focus on the application of TSA methods to other measurement circumstances, the comparison of the PSD with Bayesian spectrum analysis methods, the possible use of a second difference filter and the phase noise analysis.

References

[1] Exploratory Data Analysis, Engineering Statistics Handbook Chapter 1, NIST

[2] Zhang N.F., Calculation of the uncertainty of the mean of autocorrelated measurements, Metrologia 43 (2006), p. S276-S281

[3] Witt T.J., Using the autocorrelation function to characterize time series of voltage measurements, Metrologia 44 (2007), p.201-209

[4] Allan D.W., Should the Classical Variance Be Used As a Basic Measure in Standards Metrology? , IEEE Transactions on Instrumentation and Measurement, VOL IM-36, NO 2 (1987), p. 646-654

[5] Witt T.J., Testing for Correlations in Measurements, Advanced Mathematical and Computational Tools in Metrology IV, World Scientific Publishing Company, (2000), p. 273-288

[6] Witt T.J., Using the Allan Variance and Power Spectral Density to Characterize DC Nanovoltmeters, IEEE Transactions on Instrumentation and Measurement, VOL 50, NO 2 (2001), p. 445-448

[7] Witt T.J., Allan Variances and Spectral Densities for DC Voltage Measurements with Polarity Reversals, IEEE Transactions on Instrumentation and Measurement, VOL 54, NO 2 (2005), p. 550-553

[8] Falk M, A first course in time-series analysis – examples with SAS, Chair of Statistics, University of Wurzburg (2006)

[9] Priestley M.B., Spectral Analysis and Time Series -vol 1 Univariate Series-, Department of Mathematics, University of Manchester, Academic Press London (1981)

[10] Jespersen J., Introduction to the time domain characterization of frequency standards, Time & Frequency Division, NIST

[11] Boulanger J.S., Douglas R.J., Evaluating and Communicating Uncertainty in the Presence of Non-white Noise, AMUEM (2005)

[12] Process or Product Monitoring and Control, Engineering Statistics Handbook Chapter 6, NIST

[13] Euramet 1031, Solid Density Comparison, Report Form: Apparatus and procedure for weighing in air, EIM (2009)