Research article

Pullback random attractors of stochastic strongly damped wave equations with variable delays on unbounded domains

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Abstract: In this paper, we consider the asymptotic behavior of solutions to stochastic strongly damped wave equations with variable delays on unbounded domains, which is driven by both additive noise and deterministic non-autonomous forcing. We first establish a continuous cocycle for the equations. Then we prove asymptotic compactness of the cocycle by tail-estimates and a decomposition technique of solutions. Finally, we obtain the existence of a tempered pullback random attractor.

Keywords: strongly damped wave equation; variable delays; random attractor; asymptotical compactness; tail-estimates

Mathematics Subject Classification: 37L55, 35B41, 35B40

1. Introduction

The aim of this paper is to establish the existence of pullback random attractors of the following stochastic non-autonomous strongly damped wave equation with variable delays and with additive noise in $\mathbb{R}^d$:

$$u_{tt} - \alpha \Delta u_t - \Delta u + u_t + \lambda u = f(x, u(t - \rho(t))) + g(t, x) + \sum_{j=1}^{m} h_j(x) dW_j, \quad (1.1)$$

with initial conditions

$$u(s + \tau, x) = \phi(s, x), \quad u_t(s + \tau, x) = \psi(s, x), \quad s \in [-h, 0] \quad (1.2)$$

where $x \in \mathbb{R}^d$, $t \geq \tau, \tau \in \mathbb{R}$; $\alpha > 0$ is the strong damping coefficient, $\lambda$ is a positive constant; $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^d))$ and $h_j \in H^2(\mathbb{R}^d)$; $f$ is a nonlinear function satisfying some conditions, $\rho$ is a given delay...
function; \( \{W_j\}_{j=1}^m \) are independent real-valued two-sided Wiener process on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), which will be specified later.

As we know, the concept of random attractor, as an extension of the global attractor for the deterministic systems, was first introduced in [8], which has been studied in many papers, see [2, 3, 9–12, 20, 21, 23, 28, 30, 31, 39] and references therein.

Time delay differential equations arise from some evolution phenomena in physics, biology and life science, which depend not only on the current states but also on their past history. There have been many works on the asymptotic behavior of delay differential equations, see [6, 13, 19] in the deterministic case and [4, 7, 16, 29, 35, 36] in the stochastic case and references therein.

The asymptotic behavior of solutions of stochastic wave equation have been studied extensively in [15, 22, 25, 37] in the autonomous case. For the non-autonomous stochastic wave equation, the existence of random attractors was obtained in [33] on bounded domains and in [5, 18, 27, 32, 34] on unbounded domains. Wave equations with delays are widely used in biology, physics, engineering and chemistry. Therefore, it is important for us to study the asymptotic behavior of stochastic delay wave equation. In [14, 38], stochastic wave equations with delays on bounded domains are considered. However, the results for the stochastic delay wave equation on unbounded domains are very few.

In this work, we study the pullback random attractors of stochastic non-autonomous strongly damped wave equations with variable delays on unbounded domains. To prove the existence of pullback random attractors, we need to derive some kind compactness. The main difficulty in this paper is to establish the asymptotic compactness since the Sobolev embedding is no longer compact on unbounded domains. We here overcome the difficulty by showing that the uniform tail-estimates of solutions are sufficiently small (see Lemma 2.3). On bounded domains, we decompose the solutions into a sum of two parts. One part decays exponentially and the other part has higher regularity. For the higher regularity part, we first give some uniform estimates (see Lemma 2.4) and obtain the Hölder continuity of solutions in time (see Lemma 2.5). Then we apply Arzela-Ascoli theorem to prove the precompactness (see Lemma 3.2) and hence establish our main result (see Theorem 3.3). In addition, the strongly damped term \( \alpha \Delta u_t \) and the delay term \( f(x, u(t - \rho(t))) \) introduce an additional difficulty in deriving the uniform estimates, which needs some nontrivial arguments.

The paper is organized as follows. In the next section, we establish a continuous cocycle for problem (1.1) and (1.2) and some uniform estimates of solutions are derived. Then we prove the existence and uniqueness of the tempered pullback attractors for (1.1) and (1.2) in Section 3. In Section 4, we make conclusion as well as some comments on our results.

**Notations:** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be the standard probability space with \(\Omega = \{\omega = (\omega_1, \omega_2, \cdots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\}\), \(\mathcal{F}\) is the Borel-algebra generated by the compact open topology of \(\Omega\) and \(\mathbb{P}\) is the Wiener measure on \((\Omega, \mathcal{F})\). Define the shift operator \(\{\theta_t\}_{t \in \mathbb{R}}\) by \(\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R}, \omega \in \Omega\). Then \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\) is a metric dynamical system.

Throughout this paper, we use \((\cdot, \cdot)\) and \(\|\cdot\|\) to denote the inner product and norm of \(L^2(\mathbb{R}^d)\), respectively, and use \(\|\cdot\|_X\) to denote the norm of a general Banach space \(X\). For \(h > 0\), let \(C_h\) be the Banach space \(C([-h, 0]; L^2(\mathbb{R}^d))\) endowed with the norm \(\|\varphi\|_{C_h} = \sup_{s \in [-h, 0]} \|\varphi(s)\|\) and \(u'\) be the function defined by \(u' = u(t + s), s \in [-h, 0]\). Let \(C\) be a positive constant whose value may be different from line to line or even in the same line.
Let $E = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, endowed with the following norm

$$
\|(u,v)\|_E^2 = (\sigma^2 + \lambda - \sigma)\|u\|^2 + (1 - \alpha \sigma)\|\nabla u\|^2 + \|v\|^2, \quad (u,v) \in E
$$

where $\sigma > 0$ is a fixed constant such that

$$
1 - \sigma > 0,
1 - \alpha \sigma > 0,
\sigma^2 + \lambda - \sigma > 0.
$$

Let $E = \{(u,v) : u \in C_h, \|\nabla u\| \in C_h, v \in C_h\}$, with the norm

$$
\|(u,v)\|_E^2 = (\sigma^2 + \lambda - \sigma)\|u\|^2_{C_h} + (1 - \alpha \sigma)\|\nabla u\|^2_{C_h} + \|v\|^2_{C_h}.
$$

### 2. Materials and methods

#### 2.1. Preliminaries and cocycles

In this subsection, we first show that the system (1.1) and (1.2) generates a continuous cocycle. Then, we recall some results for the existence of pullback random attractors for non-autonomous random dynamical systems.

For our purpose, we transform the system (1.1) and (1.2) into a deterministic system with random parameters but without white noise, and then show that it generates a continuous cocycle on $E$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

For $j = 1, 2, \cdots, m$, consider the one-dimensional Ornstein-Uhlenbeck equation:

$$
dz_j + z_jdt = dW_j,
$$

whose solution is given by

$$
z_j(t) = z_j(\theta_t \omega_j) = -\int_{-\infty}^{0} e^{s}(\theta_t \omega_j)(s) ds, \quad t \in \mathbb{R}.
$$

It is known that the random variable $|z_j(\omega_j)|$ is tempered and there exists a $\theta_t$–invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that $z_j(\theta_t \omega_j)$ is continuous in $t$ for each $\omega \in \tilde{\Omega}$. From now on, we will not distinguish $\tilde{\Omega}$ and $\Omega$, and write the space $\tilde{\Omega}$ as $\Omega$.

Set

$$
z(\theta_t \omega) = \sum_{j=1}^{m} h_j(x)z_j(\theta_t \omega_j),
$$

then from (2.1) we have that

$$
dz + zdt = \sum_{j=1}^{m} h_jdW_j.
$$
It follows from [1, Proposition 4.3.3] that, there exists a tempered function \( r(\omega) > 0 \) such that

\[
\sum_{j=1}^{m} |z_j(\omega_j)|^2 \leq r(\omega), \tag{2.4}
\]

where \( r(\omega) > 0 \) satisfies for each \( \omega \in \Omega \),

\[
r(\theta_t \omega) \leq e^{\sigma' |t|} r(\omega), \quad t \in \mathbb{R}, \tag{2.5}
\]

here \( \sigma' \) is a positive constant which will be fixed later. Then by (2.4) and (2.5), we obtain, for each \( \omega \in \Omega \),

\[
\sum_{j=1}^{m} |z_j(\theta_t \omega_j)|^2 \leq e^{\sigma' |t|} r(\omega), \quad t \in \mathbb{R}. \tag{2.6}
\]

In the rest of this subsection, we show that there is a continuous cocycle generated by the system (1.1) and (1.2). Firstly we give the following assumptions on \( f \) and \( g \):

(A1) There exist a function \( k_1(x) \in L^2(\mathbb{R}^d) \) and a positive constant \( k_2 \) such that the functions \( f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}), \rho \in C(\mathbb{R}, [0, h]) \) satisfy

\[
|f(x, u)|^2 \leq |k_1(x)|^2 + k_2^2 |u|^2, \quad \forall x \in \mathbb{R}^d, u \in \mathbb{R};
\]

and

\[
|\rho'(t)| \leq \rho^* < 1, \quad \forall t \in \mathbb{R};
\]

(A2) There exists a constant \( L > 0 \), such that

\[
|f(x, u) - f(x, v)| \leq L|u - v|, \quad \forall x \in \mathbb{R}^d, u, v, \in \mathbb{R};
\]

(A3) The deterministic forcing \( g(t, x) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^d)) \), and

\[
\int_{-\infty}^{\tau} e^{\lambda r} \|g(r, \cdot)\|^2 dr < \infty, \quad \forall \tau \in \mathbb{R},
\]

which implies

\[
\lim_{k \to \infty} \int_{-\infty}^{\tau} \int_{|y| < k} e^{\lambda r} |g(r, \cdot)|^2 dx dr = 0, \quad \forall \tau \in \mathbb{R}.
\]

Let \( v = u_t + \sigma u - z(\theta_t \omega) \), where \( \sigma \) is given in (1.3), then (1.1) and (1.2) can be rewritten as the following equivalent form:

\[
\begin{align*}
\frac{du}{dt} &= v - \sigma u + z(\theta_t \omega), \\
\frac{dv}{dt} &= \alpha \Delta v - (1 - \sigma) v + (1 - \alpha \sigma) \Delta u - (\sigma^2 + \lambda - \sigma) u \\
&\quad + f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta_t \omega) + \alpha \Delta z(\theta_t \omega),
\end{align*}
\tag{2.7}
\]
with the initial conditions
\[ u(t + s, x) = u^t(x) \equiv \phi(s, x), \quad v(t + s, x) = v^t(x), \quad s \in [-\gamma, 0], \quad x \in \mathbb{R}^d, \] (2.8)
where \( v^t(x) \equiv \psi(s, x) + \sigma \Phi(s, x) - z(t_+, \omega) \). Put \( \varphi(t + t, \tau, \theta_t, \varphi^t) = (u(t + t, \tau, \theta_t, u^t), v(t + t, \tau, \theta_t, v^t))^\top, \) where \( \varphi^t = (u^t, v^t)^\top \). By the classical theory in [24], we may show the following existence results of solutions of (2.7) and (2.8).

Assume that \( g \in L^2_{T,loc}(\mathbb{R}, L^2(\mathbb{R}^d)) \) and the assumption (A1) holds. Then for each \( \omega \in \Omega, \tau \in \mathbb{R}, \varphi^t \in \mathcal{E}, \) there exists a solution \( \varphi(\cdot, \tau, \omega, \varphi^t) \) to the problem (2.7) and (2.8), which satisfies \( \varphi(\cdot, \tau, \omega, \varphi^t) \in C([-\gamma, T]; \mathcal{E}) \), for any \( T > \gamma \), and for any \( t \in [\tau, T] \), \( \varphi'(\cdot, \tau, \omega, \varphi^t) \in C([-\gamma, T]; \mathcal{E}) \).

Assume moreover (A2) holds. Then the solutions to the problem (2.7) and (2.8) are unique, and the solutions depend continuously on the initial data \( \varphi^t \in \mathcal{E} \), for any \( \omega \in \Omega, t \geq \gamma \).

Now, we define a mapping: \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{E} \rightarrow \mathcal{E} \) by
\[ \Phi(t, \tau, \omega, \varphi^t) = \varphi^{t+r}(\cdot, \tau, \theta_{-r}, \omega, \varphi^t), \]
where \( \varphi^{t+r}(s, \tau, \theta_{-r}, \omega, \varphi^t) = \varphi(t + r + s, \tau, \theta_{-r}, \omega, \varphi^t) \) for \( s \in [-\gamma, 0] \). Then \( \Phi \) is a continuous cocycle on \( \mathcal{E} \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \).

In the following, let \( \mathcal{D}(X) \) be the collection of all tempered families of nonempty bounded subsets of \( X \). Recall that \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}(X) \) is said to be tempered in \( X \) if for each \( \gamma > 0 \),
\[ \lim_{t \rightarrow -\infty} e^{\gamma t} \|D(\tau + t, \theta_t(\omega))\|_X = 0, \] (2.9)
where \( \|D\|_X = \sup_{\omega \in \Omega} \|x\|_X \). The cocycle \( \Phi \) is said to be \( \mathcal{D}(X) \)-pullback asymptotically compact in \( X \) if for all \( \tau \in \mathbb{R}, \omega \in \Omega, \) the sequence
\[ \{\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n)\}_{n=1}^\infty \] has a convergent subsequence in \( X \), (2.10)
whenever \( t_n \rightarrow \infty \), and \( x_n \in D(\tau - t_n, \theta_{-t_n} \omega) \) with \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}(X) \).

Next, we provide the following result for non-autonomous random dynamical systems from [26].

**Proposition 2.1.** Let \( \Phi \) be a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \). Suppose \( \Phi \) is \( \mathcal{D}(X) \)-pullback asymptotically compact in \( X \) and has a closed measurable \( \mathcal{D}(X) \)-pullback absorbing set \( K \) in \( \mathcal{D}(X) \). Then \( \Phi \) has an unique \( \mathcal{D}(X) \) pullback attractor \( \mathcal{X} \) in \( \mathcal{D}(X) \). For each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), \( \mathcal{X} \) is given by,
\[ \mathcal{X}(\tau, \omega) = \bigcap_{\tau \geq 0} \bigcup_{t \in \Omega} \Phi(\tau, \tau - t, \theta_{-t} \omega, K(\tau - t, \theta_t(\omega))). \]

In the rest of this paper, we will use Proposition 2.1 to prove the existence and uniqueness of a pullback random attractor for the continuous cocycle \( \Phi \) in \( \mathcal{E} \).

### 2.2. Uniform tail-estimates

In this subsection, we derive some uniform tail-estimates of solutions of problem (2.7) and (2.8). Hereafter we suppose that \( \mathcal{D} \) is the collection of all tempered families of nonempty bounded subsets of \( X \).
Lemma 2.2. In addition to the assumptions (A1)–(A3), suppose that there exists $\sigma' > 0$ such that

$$1 - \sigma > \sigma', \quad \sigma > \sigma',$$

and

$$(2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - \frac{3k^2e^{\sigma'\theta}}{(1 - \sigma)(1 - \rho^\theta)} > 0.$$  \hfill (2.13)

Then for each $\tau \in \mathbb{R}, \omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D) > 0$, such that for all $t \geq T$, $-\hbar \leq s \leq 0$, the solution $\varphi$ of (2.7) and (2.8) satisfies

$$\|\varphi^s(s, \tau - t, \theta_{-t}\omega, \varphi^{\tau-t})\|_{L^2(\mathbb{R}^d)} + C \int_{\tau-t}^\tau e^{\sigma'\theta}||\varphi^s(s, \tau - t, \theta_{-t}\omega, \varphi^{\tau-t})||_{L^2(\mathbb{R}^d)}^2 dr + 2\alpha \int_{\tau-t}^\tau e^{\sigma'(t-r)}\|\nabla v(r, \tau - t, \theta_{-t}\omega, v^{\tau-t})\|^2 dr \leq r_1(\tau, \omega),$$

where

$$r_1(\tau, \omega) = C + Cr(\omega) + Ce^{-\sigma'\theta} \int_{-\infty}^{\tau} e^{\sigma'\theta}\|g(r, x)\|^2 dr,$$  \hfill (2.15)

$$\varphi^{\tau-t} = (u^{\tau-t}, v^{\tau-t}) \in D(\tau - t, \theta_{-t}\omega), r(\omega)$$

is the tempered function satisfying (2.4), $C$ is a constant independent of $\tau, \omega$ and $D$.

Proof. Taking the inner product of the second equation of (2.7) by $v$ in $L^2(\mathbb{R}^d)$, we have

$$\frac{d}{dt}||v||^2 = -2\alpha||\nabla v||^2 - 2(1 - \sigma)||v||^2 + 2(1 - \alpha\sigma)(\Delta u, v) - 2(\sigma^2 + \lambda - \sigma)(u, v) + 2(f(x, u(t - \rho(t))), v) + 2(g(t, x), v) + 2\sigma(z(\theta_t\omega), v) + 2\alpha(\Delta z(\theta_t\omega), v).$$

(2.16)

Note that

$$\frac{du}{dt} = v - \sigma u + z(\theta_t\omega).$$

(2.17)

Then by (2.17), we derive that

$$(\Delta u, v) = (\Delta u, \frac{du}{dt} + \sigma u - z(\theta_t\omega)) = -\frac{1}{2} \frac{d}{dt}||\nabla u||^2 - \sigma||\nabla u||^2 + (\Delta z(\theta_t\omega), u)$$

(2.18)

and

$$(u, v) = (u, \frac{du}{dt} + \sigma u - z(\theta_t\omega)) = \frac{1}{2} \frac{d}{dt}||u||^2 + \sigma||u||^2 - (z(\theta_t\omega), u).$$

(2.19)

It follows from (2.16)–(2.19) that

$$\frac{d}{dt}(||v||^2 + (1 - \alpha\sigma)||\nabla u||^2 + (\sigma^2 + \lambda - \sigma)||u||^2) + 2\alpha||\nabla v||^2$$
Now we estimate the terms on the right-hand of (2.20). By Young’s inequality, Cauchy-Schwarz inequality and (A1), we have

\[ \mathcal{J}_1 \leq \frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} \|z(\theta_t \omega)\|^2 + \frac{(1 - \alpha \sigma)^2}{\varepsilon_1} \|\Delta z(\theta_t \omega)\|^2 + 2\varepsilon_1 \|u\|^2, \]

\[ \mathcal{J}_2 \leq \frac{\sigma^2}{\varepsilon_2} \|z(\theta_t \omega)\|^2 + \frac{\alpha^2}{\varepsilon_2} \|\Delta z(\theta_t \omega)\|^2 + 2\varepsilon_2 \|v\|^2, \]

\[ \mathcal{J}_3 \leq \frac{1}{2\varepsilon_3} \|f(x, u(t - \rho(t)))\|^2 + 2\varepsilon_3 \|v\|^2 \leq \frac{1}{2\varepsilon_3} \|k_1\|^2 + \frac{k_2^2}{2\varepsilon_3} \|u(t - \rho(t))\|^2 + 2\varepsilon_3 \|v\|^2 \]

and

\[ \mathcal{J}_4 \leq 2\|g(t, x)\|^2 \|v\|^2 \leq \frac{1}{2\varepsilon_4} \|g(t, x)\|^2 + 2\varepsilon_4 \|v\|^2, \]

where \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) are fixed positive constants which will be chosen later. Combining these estimates with (2.20), we have

\[
\begin{aligned}
&\frac{d}{dt} \left( \|v\|^2 + (1 - \alpha \sigma)\|\nabla u\|^2 + (\sigma^2 + \lambda - \sigma)\|u\|^2 \right) + 2\alpha \|\nabla v\|^2 \\
\leq & -2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\|v\|^2 - 2\sigma(1 - \alpha \sigma)\|\nabla u\|^2 - 2(\sigma(\sigma^2 + \lambda - \sigma) - \varepsilon_1)\|u\|^2 \\
&+ \frac{1}{2\varepsilon_3} \|k_1\|^2 + \frac{k_2^2}{2\varepsilon_3} \|u(t - \rho(t))\|^2 + \frac{1}{2\varepsilon_4} \|g(t, x)\|^2 \\
&+ \left( \frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2} \right) \|z(\theta_t \omega)\|^2 + \left( \frac{(1 - \alpha \sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2} \right) \|\Delta z(\theta_t \omega)\|^2.
\end{aligned}
\]

Recalling the definition of norm \( \| \cdot \|_E \), from (2.21), we get

\[
\begin{aligned}
&\frac{d}{dt} \|e^{\sigma t}(u, v)\|^2 + 2\alpha \|e^{\sigma t} \|\nabla v\|^2 \\
\leq & -2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\|e^{\sigma t}v\|^2 - (2\sigma - \sigma')(1 - \alpha \sigma)\|\nabla u\|^2 e^{\sigma t} \\
&- ((2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - 2\varepsilon_1)\|u\|^2 e^{\sigma t} \\
&+ \frac{1}{2\varepsilon_3} \|k_1\|^2 e^{\sigma t} + \frac{k_2^2}{2\varepsilon_3} \|u(t - \rho(t))\|^2 e^{\sigma t} + \frac{1}{2\varepsilon_4} \|g(t, x)\|^2 e^{\sigma t}.
\end{aligned}
\]
It follows from (2.23) and (2.24) that
\[
\int^{t+s}_t e^{\sigma'(t)} \| \varphi(t + s, t - s, \varphi) \|^2_E + 2\alpha \int^{t+s}_t e^{\sigma'} \| \nabla v(r, t - t, \omega, v) \|^2 dr \\
\leq\int^{t+s}_t e^{\sigma'} \| \varphi(t - t, \varphi) \|^2_E \\
- 2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \int^{t+s}_t e^{\sigma'} \| v(t - t, \varphi) \|^2 dr \\
- (2\sigma - \sigma')(1 - \alpha\sigma) \int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr \\
\leq \frac{1}{1-\rho(r)} \leq \frac{1}{1-\rho} \quad \text{for all } t \in \mathbb{R}
\]
and the fact \( \frac{1}{1-\rho} = \frac{1}{1-\rho} \) for all \( t \in \mathbb{R} \), we infer that
\[
\int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr \\
\leq \frac{k^2_2 e^{\sigma'} h}{2 \varepsilon_3 (1 - \rho^2)} \int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr \\
= \frac{k^2_2 e^{\sigma'} h}{2 \varepsilon_3 (1 - \rho^2)} \left( \int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr + \int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr \right) \\
\leq \frac{k^2_2 e^{\sigma'} h}{2 \varepsilon_3 (1 - \rho^2)} u(t - t, \varphi) + \frac{k^2_2 e^{\sigma'} h}{2 \varepsilon_3 (1 - \rho^2)} \int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr.
\]
It follows from (2.23) and (2.24) that
\[
\int^{t+s}_t e^{\sigma'} \| \varphi(t + s, t - t, \varphi) \|_E + 2\alpha \int^{t+s}_t e^{\sigma'} \| \nabla v(t - t, \varphi) \|^2 dr \\
\leq\int^{t+s}_t e^{\sigma'} \| \varphi(t - t, \varphi) \|_E \\
- 2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \int^{t+s}_t e^{\sigma'} \| v(t - t, \varphi) \|^2 dr \\
- (2\sigma - \sigma')(1 - \alpha\sigma) \int^{t+s}_t e^{\sigma'} \| u(t - t, \varphi) \|^2 dr
\]
Replacing \( \epsilon \) with \( z \), we get

\[
- ((2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - \frac{k_2^2 e^{\sigma' t}}{2\epsilon_3(1 - \rho^*)} - 2\epsilon_1) \int_{t-\tau}^{t+s} e^{\sigma' r} |u(r, \tau - t, \omega, u^{\tau - t})|^2 dr \\
\text{and by (2.11)-(2.13), we can choose \( \epsilon_1 \) small enough such that} \\
1 - \sigma - \sigma' > 0, \quad (2.26) \\
2\sigma - \sigma' > 0 \quad (2.27)
\]

and

\[
(2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - \frac{3k_2^2 e^{\sigma' t}}{(1 - \sigma)(1 - \rho^*)} - 2\epsilon_1 > 0 \quad (2.28)
\]

Replacing \( \omega \) with \( \theta_{-\tau} \omega \) and by (2.25)–(2.28), we get

\[
\|\varphi(\tau + s, \tau - t, \theta_{-\tau} \omega, \varphi^{\tau - t})\|_{E}^2 + Ce^{-\sigma'(t+s)} \int_{t-\tau}^{t+s} e^{\sigma' r} \|\varphi(r, \tau - t, \theta_{-\tau} \omega, \varphi^{\tau - t})\|_{E}^2 dr \\
+ 2\alpha e^{-\sigma'(t+s)} \int_{t-\tau}^{t+s} e^{\sigma' r} \|\nabla v(r, \tau - t, \theta_{-\tau} \omega, v^{\tau - t})\|_{E}^2 dr \\
\leq e^{\sigma' h} e^{-\sigma' t} \|\varphi^{\tau - t}\|_{E}^2 + C|u^{\tau - t}|^2_{C_h} + C + Ce^{-\sigma'} \int_{t-\tau}^{t+s} e^{\sigma' r} \|g(r, x)\|_{E}^2 dr \\
+ Ce^{-\sigma'} \int_{t-\tau}^{t+s} e^{\sigma' r} \|z(\theta_{-\tau} \omega)\|_{E}^2 dr + Ce^{-\sigma'} \int_{t-\tau}^{t+s} e^{\sigma' r} \|\Delta z(\theta_{-\tau} \omega)\|_{E}^2 dr \\
\leq Ce^{-\sigma'} \|\varphi^{\tau - t}\|_{E}^2 + C + Ce^{-\sigma'} \int_{-\infty}^{\tau} e^{\sigma' r} \|g(r, x)\|_{E}^2 dr \\
+ Ce^{-\sigma'} \int_{t-\tau}^{\tau} e^{\sigma' r} \|z(\theta_{-\tau} \omega)\|_{E}^2 dr + Ce^{-\sigma'} \int_{t-\tau}^{\tau} e^{\sigma' r} \|\Delta z(\theta_{-\tau} \omega)\|_{E}^2 dr.
\]

Note that \( z(\theta_{j} \omega) = \sum_{j=1}^{m} h_j z(\theta_{j}, \omega) \) and \( h_j \in H^2(\mathbb{R}^d) \), we deduce that for each \( \omega \in \Omega \),

\[
e^{-\sigma' t} \int_{t-\tau}^{t} e^{\sigma' r} (\|z(\theta_{-\tau} \omega)\|_{E}^2 + \|\Delta z(\theta_{-\tau} \omega)\|_{E}^2) dr \\
= e^{-\sigma' t} \int_{t-\tau}^{0} e^{\sigma' r} (\|z(\theta_{\omega})\|_{E}^2 + \|\Delta z(\theta_{\omega})\|_{E}^2) dr \\
\leq e^{-\sigma' t} \int_{t-\tau}^{0} e^{\sigma' r} \left( \sum_{j=1}^{m} |h_j z(\theta_{j}, \omega)|^2 + \sum_{j=1}^{m} |\Delta h_j z(\theta_{j}, \omega)|^2 \right) dr
\]
\[ e^{-\alpha r} \int_{-\infty}^{0} e^{\alpha r} \sum_{j=1}^{m} |z(\theta_j, \omega)|^2 dr \leq Cr(\omega). \]  (2.30)

Since \( \varphi^{t} = (u^{t}, v^{t}) \in D(\tau - t, \theta_{t} - \omega) \), then
\[ \limsup_{t \to \infty} Ce^{-\alpha t} \| \varphi^{t} \|^2_{\Omega} \leq \limsup_{t \to \infty} Ce^{-\alpha t} \| D(\tau - t, \theta_{t} - \omega) \|^2_{\Omega} = 0. \]  (2.31)

By (2.31), there exists \( T = T(\tau, \omega, D) > 0 \), such that for all \( t \geq T \),
\[ Ce^{-\alpha t} \| \varphi^{t} \|^2_{\Omega} \leq 1. \]  (2.32)

Combining (2.32), (2.30) and (2.29), we obtain for all \( t \geq T \)
\[ \| \varphi(\tau + s, \tau - t, \theta_{t} - \omega, \varphi^{t}) \|^2_{\Omega} + C \int_{\tau-t}^{\tau} e^{\alpha \tau} \| \varphi(s, \tau - t, \theta_{t} - \omega, \varphi^{t}) \|^2_{\Omega} dr 
\]
\[ + 2\alpha \int_{\tau-t}^{\tau} e^{\alpha (\tau - t)} \| \nabla v(r, \tau - t, \theta_{t} - \omega, \varphi^{t}) \|^2_{\Omega} dr 
\]
\[ \leq C + Cr(\omega) + Ce^{-\alpha t} \int_{-\infty}^{\tau} e^{\alpha r} \| g(r, x) \|^2_{\Omega} dr \equiv r_{1}(\tau, \omega). \]  (2.33)

Thus, we complete the proof of Lemma 2.2. \( \square \)

Now we establish the following estimates on the exterior of a ball.

**Lemma 2.3.** Suppose the hypotheses in Lemma 2.2 hold, and let \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D \in D \). Then for each \( \epsilon > 0 \), there exists \( T = T(\tau, \omega, \epsilon, D) > 0 \), and \( K = K(\tau, \omega, \epsilon) \geq 1 \), such that for all \( t \geq T, k \geq K \) and \( -h \leq s \leq 0 \),
\[ \| \varphi(s, \tau - t, \theta_{t} - \omega, \varphi^{t}) \|^2_{\Omega_{k}(\mathbb{R}^{d}, \Omega_{k})} \leq \epsilon, \]
where \( \Omega_{k} = \{ x \in \mathbb{R}^{d} : |x| \leq k \} \) and \( \varphi^{t} = (u^{t}, v^{t}) \in D(\tau - t, \theta_{t} - \omega) \).

**Proof.** Choose a smooth function \( \xi(\cdot) \) as follows:
\[ \xi(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2, \end{cases} \]  (2.34)

where \( 0 \leq \xi(s) \leq 1 \), \( s \in \mathbb{R}^{+} \), and with the constants \( \mu_{1}, \mu_{2} \) satisfying \( |\xi'(s)| \leq \mu_{1}, |\xi''(s)| \leq \mu_{2} \) for \( s \in \mathbb{R}^{+} \). Taking the inner product of the second equation of (2.7) with \( \xi^2 \left( \frac{|x|^2}{k^2} \right) v \) in \( L^{2}(\mathbb{R}^{d}) \), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^{d}} \xi^2 \left( \frac{|x|^2}{k^2} \right) |v|^2 dx 
\]
\[ = -2(1 - \sigma) \int_{\mathbb{R}^{d}} \xi^2 \left( \frac{|x|^2}{k^2} \right) |v|^2 dx + 2\alpha \int_{\mathbb{R}^{d}} (\Delta v) \xi^2 \left( \frac{|x|^2}{k^2} \right) v dx 
\]
\[ + 2(1 - \alpha \sigma) \int_{\mathbb{R}^{d}} (\Delta u) \xi^2 \left( \frac{|x|^2}{k^2} \right) v dx - 2(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^{d}} u \xi^2 \left( \frac{|x|^2}{k^2} \right) v dx 
\]
\[ + 2 \int_{\mathbb{R}^{d}} f(x, u(t - \rho(t))) \xi^2 \left( \frac{|x|^2}{k^2} \right) v dx + 2 \int_{\mathbb{R}^{d}} g(x) \xi^2 \left( \frac{|x|^2}{k^2} \right) v dx 
\]

\( \square \)
\[ + 2 \int_{\mathbb{R}^d} (\sigma z(\theta, \omega) + \alpha \Delta z(\theta, \omega)) \xi^2 \left( \frac{|x|^2}{k^2} \right) vdx = -2(1 - \sigma) \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |v|^2 dx + \sum_{i=1}^{6} J_i. \quad (2.35) \]

Now we estimate each term on the right hand of (2.35). By Young’s inequality, Cauchy-Schwarz inequality and (A1), we infer that

\[ J_1 = -\frac{8\alpha}{k^2} \int_{\mathbb{R}^d} (\nabla v) \xi^2 \left( \frac{|x|^2}{k^2} \right) \xi^2 \left( \frac{|x|^2}{k^2} \right) xvdx - 2\alpha \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\nabla v|^2 dx \]
\[ \leq 8 \frac{\sqrt{2}\alpha\mu_1}{k} \int_{|k| \leq |x| \leq \sqrt{k}} |\nabla v||v| dx - 2\alpha \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\nabla v|^2 dx \]
\[ \leq 4 \frac{\sqrt{2}(1 - \sigma)\mu_1}{k} (||\nabla u||^2 + ||v||^2) \]
\[ \leq 4 \frac{\sqrt{2}(1 - \sigma)\mu_1}{k} (||\nabla u||^2 + ||v||^2), \quad (2.36) \]

where

\[ J_2 = -\frac{8(1 - \alpha \sigma)}{k^2} \int_{\mathbb{R}^d} (\nabla u) \xi^2 \left( \frac{|x|^2}{k^2} \right) \xi^2 \left( \frac{|x|^2}{k^2} \right) xvdx - 2(1 - \alpha \sigma) \int_{\mathbb{R}^d} (\nabla u) \xi^2 \left( \frac{|x|^2}{k^2} \right) \nabla vdx, \quad (2.37) \]

and

\[ -2(1 - \alpha \sigma) \int_{\mathbb{R}^d} (\nabla u) \xi^2 \left( \frac{|x|^2}{k^2} \right) \nabla vdx \]
\[ = -2(1 - \alpha \sigma) \int_{\mathbb{R}^d} (\nabla u) \xi^2 \left( \frac{|x|^2}{k^2} \right) \nabla (\frac{du}{dt} + \sigma u - z(\theta, \omega)) dx \]
\[ = -(1 - \alpha \sigma) \frac{d}{dt} \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\nabla u|^2 dx - 2\sigma(1 - \alpha \sigma) \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\nabla u|^2 dx \]
\[ - 2(1 - \alpha \sigma) \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |u||\Delta z(\theta, \omega)| dx \]
\[ \leq -(1 - \alpha \sigma) \frac{d}{dt} \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\nabla u|^2 dx - 2\sigma(1 - \alpha \sigma) \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\nabla u|^2 dx \]
\[ + \frac{(1 - \alpha \sigma)^2}{\varepsilon_1} \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |\Delta z(\theta, \omega)|^2 dx + \varepsilon_1 \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |u|^2 dx. \quad (2.39) \]

\[ J_3 = -2(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} u^2 \xi^2 \left( \frac{|x|^2}{k^2} \right) (\frac{du}{dt} + \sigma u - z(\theta, \omega)) dx \]
\[ = -(\sigma^2 + \lambda - \sigma) \frac{d}{dt} \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |u|^2 dx - 2\sigma(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) |u|^2 dx \]
+ 2(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u||z(\theta, \omega)| dx
\leq - (\sigma^2 + \lambda - \sigma) \frac{d}{dt} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u|^2 dx - 2\sigma(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u|^2 dx
+ \frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |z(\theta, \omega)|^2 dx + \varepsilon_1 \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u|^2 dx,
\tag{2.40}
\end{align*}

\begin{align*}
\mathcal{J}_4 \leq \frac{1}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |k_1(x)|^2 dx + \frac{k_2^2}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u(t - \rho(t))|^2 dx
+ 2\varepsilon_3 \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |v|^2 dx,
\tag{2.41}
\end{align*}

\begin{align*}
\mathcal{J}_5 \leq \frac{1}{2\varepsilon_4} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |g(t, x)|^2 dx + 2\varepsilon_4 \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |v|^2 dx,
\tag{2.42}
\end{align*}

\begin{align*}
\mathcal{J}_6 \leq \frac{\alpha^2}{\varepsilon_2} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |z(\theta, \omega)|^2 dx + \frac{\alpha^2}{\varepsilon_2} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |\Delta z(\theta, \omega)|^2 dx + 2\varepsilon_2 \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |v|^2 dx,
\tag{2.43}
\end{align*}

where \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\) are positive constants which will be given later. It follows from (2.35)–(2.42) that
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} (|v|^2 + (1 - \alpha\sigma)|\nabla u|^2 + (\sigma^2 + \lambda - \sigma)|u|^2) dx
&+ 2\alpha \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |\nabla v|^2 dx
\leq -2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |v|^2 dx
- (2\sigma(\sigma^2 + \lambda - \sigma) - 2\varepsilon_1) \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u|^2 dx
- 2\sigma(1 - \alpha\sigma) \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |\nabla u|^2 dx
+ \frac{4\sqrt{2}\alpha\mu_1}{k} (||\nabla v||^2 + ||v||^2) + \frac{4\sqrt{2}(1 - \alpha\sigma)\mu_1}{k} (||\nabla u||^2 + ||v||^2)
+ \frac{1}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |k_1(x)|^2 dx + \frac{k_2^2}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |u(t - \rho(t))|^2 dx
+ \frac{1}{2\varepsilon_4} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |g(t, x)|^2 dx + \frac{(1 - \alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |\Delta z(\theta, \omega)|^2 dx
+ \frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} |z(\theta, \omega)|^2 dx.
\tag{2.44}
\end{align*}

Let \(Y = |v|^2 + (1 - \alpha\sigma)|\nabla u|^2 + (\sigma^2 + \lambda - \sigma)|u|^2\). Multiplying (2.44) by \(e^{\int \tau_1} \) and integrating over \((\tau - t, \tau + s)\), where \(s \in [-h, 0]\), \(\sigma_1\) is a fixed constant chosen as in Lemma 2.2, we obtain for each \(\omega \in \Omega\)
\begin{align*}
e^{\int \tau_1} \int_{\mathbb{R}^d} \xi^2 \frac{1}{k^2} Y(\tau + s, \tau - t, \omega, \tau_1) dx
\end{align*}
Set $r' = r - \rho(r)$. By using the similar arguments in (2.24), we obtain

$$
\frac{k^2}{2\varepsilon_3} \int_{\mathbb{R}^d} e^{r'} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})||u(r - \rho(r), \tau - t, \omega, u^{\tau-})||^2 dx dr
\leq\frac{k^2}{2\varepsilon_3(1 - \rho^*)} \int_{\mathbb{R}^d} e^{r'} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})||u(r, \tau - t, \omega, u^{\tau-})||^2 dx dr
= \frac{k^2}{2\varepsilon_3(1 - \rho^*)} \int_{\mathbb{R}^d} e^{r'} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})||u(r, \tau - t, \omega, u^{\tau-})||^2 dx dr
+ \frac{k^2}{2\varepsilon_3(1 - \rho^*)} \int_{\mathbb{R}^d} e^{r'} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})||u(r, \tau - t, \omega, u^{\tau-})||^2 dx dr
\leq \frac{k^2}{2\varepsilon_3(1 - \rho^*)} ||u^{\tau-}||^2 + \frac{k^2}{2\varepsilon_3(1 - \rho^*)} \int_{\mathbb{R}^d} e^{r'} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})||u(r, \tau - t, \omega, u^{\tau-})||^2 dx dr. (2.46)
$$

Combining (2.45) and (2.46), we have for $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{1 - \sigma}{6}$

$$
\int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2}) Y(\tau + s, \tau - t, \omega, Y^{\tau-}) dx
$$
Choosing $\varepsilon_1$ sufficiently small, together with (2.26)–(2.28), and replacing $\omega$ by $\theta_{-\tau, \omega}$, we have

\[
\begin{align*}
\int_{\mathbb{R}^d} &\xi^2\left(\frac{|x|^2}{k^2}\right)Y(\tau + s, \tau - t, \theta_{-\tau, \omega}, Y_{t-\tau})dx \\
+ 2ae^{-\sigma'(t+s)} &\int_{\tau-t}^{\tau+s} e^{|r-r'|} \int_{\mathbb{R}^d} \xi^2\left(\frac{|x|^2}{k^2}\right)|\nabla v(\tau + s, \tau - t, \omega, v_{t-\tau})|^2 dx dr \\
\leq &\ C e^{-\sigma'(t)} \int_{\mathbb{R}^d} \xi^2\left(\frac{|x|^2}{k^2}\right)Y_{t-\tau} dx + C e^{-\sigma'(t)} ||u_{t-\tau}||_{C^3_h}^2 \\
+ C &\ k e^{-\sigma'(t)} \int_{\tau-t}^{\tau+s} e^{|r-r'|} (||\nabla u(\tau - t, \theta_{-\tau, \omega}, u_{t-\tau})||^2 + ||v(r, \tau - t, \theta_{-\tau, \omega}, v_{t-\tau})||^2) dr \\
+ C &\ k e^{-\sigma'(t)} \int_{\tau-t}^{\tau+s} e^{|r-r'|} ||\nabla v(\tau - t, \theta_{-\tau, \omega}, v_{t-\tau})||^2 dr \\
+ Ce^{-\sigma'(t)} &\ \int_{\tau-t}^{\tau+s} e^{|r-r'|} \int_{\mathbb{R}^d} \xi^2\left(\frac{|x|^2}{k^2}\right)|k_1(x)|^2 dx dr \\
+ Ce^{-\sigma'(t)} &\ \int_{\tau-t}^{\tau+s} e^{|r-r'|} \int_{\mathbb{R}^d} \xi^2\left(\frac{|x|^2}{k^2}\right)|g(r, x)|^2 dx dr
\end{align*}
\]
We now estimate the terms on the right-hand side of (2.48). Since \( \varphi^{r-t} = (u^{r-t}, v^{r-t}) \in D(\tau - t, \theta_{r-t}, \omega) \), given \( \epsilon > 0 \), there exists \( T_1 = T_1(\tau, \omega, D, \epsilon) > 0 \), \( K_1 = K_1(\tau, \omega, \epsilon) \geq 1 \), such that for all \( t > T_1, k > K_1, \)

\[
Ce^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} \int_{\R^d} \xi^2 \left( \frac{|\xi|^2}{k^2} \right) |\Delta z(\theta_{r-t}, \omega)|^2 dxdr + Ce^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} \int_{\R^d} \xi^2 \left( \frac{|\xi|^2}{k^2} \right) |z(\theta_{r-t}, \omega)|^2 dxdr.
\]

Due to (A3) and \( k_1 \in L^2(\R^d) \), there exists \( K_2 = K_2(\tau, \omega, \epsilon) \geq 1 \), such that for all \( k > K_2 \)

\[
Ce^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} \int_{\R^d} \xi^2 \left( \frac{|\xi|^2}{k^2} \right) |k_1(x)|^2 dxdr \leq Ce^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} \int_{|x|>k} |k_1(x)|^2 dxdr \leq Ce,
\]

and

\[
Ce^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} \int_{\R^d} \xi^2 \left( \frac{|\xi|^2}{k^2} \right) |g(r, x)|^2 dxdr \leq Ce^{-\sigma't} \int_{-\infty}^\tau e^{\sigma'r} \int_{|x|>k} |g(r, x)|^2 dxdr \leq Ce^{-\sigma't} \int_{-\infty}^\tau e^{\sigma'r} ||g(r, x)||^2 dxdr \leq Ce.
\]

Note that \( z(\theta, \omega) = \sum_{j=1}^m h_j z(\theta_{j, \omega_j}) \) and \( h_j \in H^2(\R^d) \), there exists \( K_3 = K_3(\omega, \epsilon) \) such that for all \( k \geq K_3 \) and \( j = 1, 2, \ldots, m, \)

\[
\int_{|x|>k} (|h_j(x)|^2 + |\nabla h_j(x)|^2 + |\Delta h_j(x)|^2)dx \leq \frac{\epsilon}{r(\omega)},
\]

where \( r(\omega) \) is tempered satisfying (2.4) and (2.5).

By (2.52), we have

\[
Ce^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} \int_{\R^d} \xi^2 \left( \frac{|\xi|^2}{k^2} \right) (|\Delta z(\theta_{r-t}, \omega)|^2 + |z(\theta_{r-t}, \omega)|^2) dxdr \leq C \int_{\tau-t}^\tau e^{\sigma'(r-t)} \sum_{j=1}^m \int_{|x|>k} (|\Delta h_j|^2 |z_j(\theta_{r-t}, \omega_j)|^2 + |h_j|^2 |z_j(\theta_{r-t}, \omega_j)|^2) dxdr \leq C \int_{-\infty}^0 e^{-\sigma r} \sum_{j=1}^m \int_{|x|>k} (|\Delta h_j|^2 |z_j(\theta, \omega_j)|^2 + |h_j|^2 |z_j(\theta, \omega_j)|^2) dxdr \leq \frac{\epsilon}{r(\omega)} \int_{-\infty}^0 e^{-\sigma r} \sum_{j=1}^m |z_j(\theta, \omega_j)|^2 dr \leq Ce.
\]

In view of Lemma 2.2, (2.4) and (A3), there exists \( T_2 = T_2(\tau, \omega, D, \epsilon), K_4 = K_4(\epsilon) \geq 1 \) such that

\[
\frac{C}{k} e^{-\sigma't} \int_{\tau-t}^\tau e^{\sigma'r} (||\nabla u(r, \tau - t, \theta_{r-t}, \omega, u^{r-t})||^2 + ||v(r, \tau - t, \theta_{r-t}, \omega, v^{r-t})||^2) dr
\]
\[ + \frac{C}{k} e^{-\sigma_2^2 \frac{|x|^2}{k^2}} \int_{r-t}^r e^{\sigma_2^2 \frac{|x|^2}{k^2}} dr \leq \frac{C}{k} e^{-\sigma_2^2 \frac{|x|^2}{k^2}} \int_{-\infty}^r e^{\sigma_2^2 \frac{|x|^2}{k^2}} dr + \frac{C}{k} \int_0^r \sum_{j=1}^m |z_j(\theta_i, \omega_j)|^2 dr + \frac{C}{k} \leq C \epsilon. \quad (2.54) \]

Let \( T = \max\{T_1, T_2\} > 0, K = \max\{K_1, K_2, K_3, K_4\} \geq 1. \) From (2.48)–(2.54) we have for all \( t > T, k > K, \)
\[ \int_{\mathbb{R}^d} \xi^2 \left( \frac{|x|^2}{k^2} \right) Y(\tau + s, \tau - t, \theta_\tau, \varphi^{-1}) dx \leq C \epsilon, \]
which implies
\[ \| \varphi^r (s, \tau - t, \theta_\tau, \varphi^{-1}) \|_{\partial(\mathbb{R}^d)} \leq C \epsilon. \]

Then the proof of Lemma 2.3 is finished. \( \square \)

2.3. Estimates on bounded domain

We now decompose the solutions of (2.7) and (2.8) in bounded domains and derive some uniform estimates.

Let \( \Omega_k = \{ x \in \mathbb{R}^d : |x| \leq k \} \), given \( k \geq 1 \) and set
\[ \begin{cases} \bar{u}(t, \tau, \omega, \bar{u}) = (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) u(t, \tau, \omega, u), \\ \bar{v}(t, \tau, \omega, \bar{v}) = (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) v(t, \tau, \omega, v), \end{cases} \quad (2.55) \]
where \( \xi \) is the cutoff function defined in (2.34).

Multiplying (2.7) by \( 1 - \xi^2 \left( \frac{|x|^2}{k^2} \right) \), we have
\[ \begin{align*}
\frac{d\bar{u}}{dt} &= \bar{v} - \sigma \bar{u} + (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) \zeta(\theta_\tau, \omega), \\
\frac{d\bar{v}}{dt} &= \alpha \bar{v} + 2 \alpha \nabla \bar{v} \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) + \alpha \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) - (1 - \sigma) \bar{v} + (1 - \alpha \sigma) \Delta \bar{u} \\
&\quad + 2(1 - \alpha \sigma) \nabla \bar{u} \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) + (1 - \alpha \sigma) \Delta \xi^2 \left( \frac{|x|^2}{k^2} \right) - (\sigma^2 + \lambda - \sigma) \bar{u} \\
&\quad + (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) (f(x, u(t - \rho(t))) + g(t, x) + \sigma \zeta(\theta_\tau, \omega) + \alpha \Delta \zeta(\theta_\tau, \omega)),
\end{align*} \]
with the initial conditions
\[ \bar{u}(\tau + s, x) = \bar{u}^r(x) \equiv (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) \phi(s, x), \bar{v}(\tau + s, x) = \bar{v}^r(x), \quad s \in [-h, 0], x \in \mathbb{R}^d, \quad (2.57) \]
where \( \bar{v}^r(x) \equiv (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) \psi(s, x) + \sigma (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) \phi(s, x) - (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) \zeta(\theta_{t+}, \omega). \) Now we decompose (2.56) into two parts. Set \( \bar{u} = u_1 + u_2, \bar{v} = v_1 + v_2, \) we have two new systems:
\[ \begin{align*}
\frac{du_1}{dt} &= v_2 - \sigma u_2, \\
\frac{du_2}{dt} &= \alpha \Delta v_2 - (1 - \sigma) v_2 + (1 - \alpha \sigma) \Delta u_2 - (\sigma^2 + \lambda - \sigma) u_2, \quad (2.58)
\end{align*} \]
with the initial conditions
\[ u_2(x) = \bar{u}(\tau + s, x), \quad \tau \in [-h, 0], \quad x \in \Omega_{2k}, \quad (2.59) \]
and boundary conditions
\[ u_2(t, x) = 0, \quad v_2(t, x) = 0, \quad t \in [-h, +\infty), \quad |x| = 2k, \quad (2.60) \]
and
\[
\begin{align*}
\frac{du_1}{dt} &= v_1 - \sigma u_1 + (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right)) z(\theta, \omega), \\
\frac{dv_1}{dt} &= \alpha \Delta v_1 + 2\alpha \nabla v \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) + (1 - \sigma) v_1 + (1 - \alpha \sigma) \Delta u_1 \\
&+ 2(1 - \alpha \sigma) \nabla u \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) + (1 - \alpha \sigma) u \Delta \xi^2 \left( \frac{|x|^2}{k^2} \right) - (\sigma^2 + \lambda - \sigma) u_1 \\
&+ (1 - \xi^2 \left( \frac{|x|^2}{k^2} \right))(f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta, \omega) + \alpha \Delta z(\theta, \omega)).
\end{align*}
\]
with the initial conditions
\[ u_1^0(x) = 0, \quad v_1^0(x) = 0, \quad s \in [-h, 0], \quad x \in \Omega_{2k}, \quad (2.62) \]
and boundary conditions
\[ u_1(t, x) = 0, \quad v_1(t, x) = 0, \quad t \in [-h, +\infty), \quad |x| = 2k. \quad (2.63) \]

Using the similar arguments in Lemma 2.2, we derive the following estimate of solutions of (2.56) and (2.57),
\[
\begin{align*}
&\| (\bar{u}(\tau + s, \tau - t, \theta, \omega, \bar{u}^{-t}), \bar{v}(\tau + s, \tau - t, \theta, \omega, \bar{v}^{-t})) \|_{\tilde{\Omega}(\Omega_{2k})}^2 \\
&+ C \int_{\tau-t}^{\tau} e^{\nu \tau'} \|[\bar{u}(r + s, \tau - t, \theta, \omega, \bar{u}^{-t}), \bar{v}(r + s, \tau - t, \theta, \omega, \bar{v}^{-t})]\|_{\tilde{\Omega}(\Omega_{2k})}^2 dr \\
&+ 2\alpha \int_{\tau-t}^{\tau} e^{\nu (\tau' - t)} |\nabla \bar{v}(r, \tau - t, \theta, \omega, \bar{v}^{-t})|^2 dr \\
&\leq C e^{-\nu \tau} \| \tilde{\varphi}^{\tau-t} \|_{\tilde{\Omega}(\Omega_{2k})}^2 + C e^{-\nu \tau} \int_{-\infty}^{\tau} e^{\nu r} \| g(r, x) \|^2 dr \leq C r(\omega) + C.
\end{align*}
\]
(2.64)

It follows from (2.64), we obtain the estimate of \((u_2, v_2)\)
\[
\begin{align*}
&\| (u_2(\tau + s, \tau - t, \theta, \omega, u_2^{-t}), v_2(\tau + s, \tau - t, \theta, \omega, v_2^{-t})) \|_{\tilde{\Omega}(\Omega_{2k})}^2 \\
&+ C \int_{\tau-t}^{\tau} e^{\nu \tau'} \|[u_2(r + s, \tau - t, \theta, \omega, u_2^{-t}), v_2(r + s, \tau - t, \theta, \omega, v_2^{-t})]\|_{\tilde{\Omega}(\Omega_{2k})}^2 dr \\
&+ 2\alpha \int_{\tau-t}^{\tau} e^{\nu (\tau' - t)} |\nabla v_2(r, \tau - t, \theta, \omega, v_2^{-t})|^2 dr \\
&\leq C e^{-\nu \tau} \| \varphi_2^{\tau-t} \|_{\tilde{\Omega}(\Omega_{2k})}^2 = C e^{-\nu \tau} \| \tilde{\varphi}^{\tau-t} \|_{\tilde{\Omega}(\Omega_{2k})}^2.
\end{align*}
\]
(2.65)
It follows from (2.64) and (2.65) that

\[
\|(u_1(t + s, \tau - t, \theta_{-t}, 0), v_1(t + s, \tau - t, \theta_{-t}, 0))\|^2_{E(\Omega_{2\kappa})} \\
+ C \int_{-T}^{\tau} e^{\sigma r} \|(u_1(r + s, \tau - t, \theta_{-t}, 0), v_1(r + s, \tau - t, \theta_{-t}, 0))\|^2_{E(\Omega_{2\kappa})} dr \\
+ 2\alpha \int_{-T}^{\tau} e^{\sigma r(s-r)} \|\nabla v_1(r, t - \theta_{-t}, 0)\|^2 dr \\
\leq 2\|(\tilde{u}(r + s, \tau - t, \theta_{-t}, \tilde{v}^{\tau-t-i}), \tilde{v}(r + s, \tau - t, \theta_{-t}, \tilde{v}^{\tau-t-i}))\|^2_{E(\Omega_{2\kappa})} \\
+ 2\|(u_2(r + s, t - \theta_{-t}, \theta_{-t}, 0), u_2^{\tau-t-i}, v_2(r + s, t - \theta_{-t}, \theta_{-t}, 0))\|^2_{E(\Omega_{2\kappa})} dr \\
+ 4\alpha \int_{-T}^{\tau} e^{\sigma r(s-r)} \|\nabla v_2(r, t - \theta_{-t}, \theta_{-t}, 0)\|^2 dr \\
+ 4\alpha \int_{-T}^{\tau} e^{\sigma r(s-r)} \|\nabla v_2(r, t - \theta_{-t}, \theta_{-t}, 0)\|^2 dr \\
\leq Ce^{-\sigma r}\|\tilde{v}_{\tau-t}^{\tau-t}\|^2_{E(\Omega_{2\kappa})} + Ce^{-\sigma r} \int_{-\infty}^{\tau} e^{\sigma r} g(r, x)\|^2 dr + Cr(\omega) + C,
\]

Since \(\tilde{v}_{\tau-t}^{\tau-t} = (\tilde{u}^{\tau-t-i}, \tilde{v}^{\tau-t-i}) \in D(\tau - t, \theta_{-t}) \in D(\delta(\Omega_{2\kappa})),\) we have

\[
\limsup_{t \to \infty} Ce^{-\sigma r}\|\tilde{v}_{\tau-t}^{\tau-t}\|^2_{E(\Omega_{2\kappa})} \leq \limsup_{t \to \infty} Ce^{-\sigma r}\|D(\tau - t, \theta_{-t})\|^2_{E(\Omega_{2\kappa})} = 0,
\]

Therefore, there exists \(T = T(\tau, \omega, D) > 0\) such that for all \(t \geq T,\)

\[
Ce^{-\sigma r}\|\tilde{v}_{\tau-t}^{\tau-t}\|^2_{E(\Omega_{2\kappa})} \leq 1.
\]

Thus, for all \(t \geq T,\) we have

\[
\|(u_1(t + s, \tau - t, \theta_{-t}, 0), v_1(t + s, \tau - t, \theta_{-t}, 0))\|^2_{E(\Omega_{2\kappa})} \\
+ C \int_{-T}^{\tau} e^{\sigma r} \|(u_1(r + s, \tau - t, \theta_{-t}, 0), v_1(r + s, \tau - t, \theta_{-t}, 0))\|^2_{E(\Omega_{2\kappa})} dr \\
+ 2\alpha \int_{-T}^{\tau} e^{\sigma r(s-r)} \|\nabla v_1(r, t - \theta_{-t}, 0)\|^2 dr \\
\leq C + Cr(\omega) + Ce^{-\sigma r} \int_{-\infty}^{\tau} e^{\sigma r} g(r, x)\|^2 dr \leq Cr_1(\tau, \omega), \tag{2.66}
\]

where \(r_1(\tau, \omega)\) is defined in Lemma 2.2.

Furthermore, we can give the uniform estimates of \((\Delta u_1, \Delta v_1),\) which imply the higher regularity of \((u_1, v_1).\)

**Lemma 2.4.** Assume the hypotheses in Lemma 2.2 hold, and let \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D \in D(\delta(\Omega_{2\kappa})).\) Then for given \(\epsilon > 0,\) there exist random variables \(r_2(\tau, \omega),\) and \(T = T(\tau, \omega, D) > 0,\) such that for all \(t \geq T, k \geq 1\) and \(-h \leq s \leq 0,\)

\[
(1 - \alpha\sigma)\|\Delta u_1(t + s, \tau - t, \theta_{-t}, 0)\|^2 + \|\nabla v_1(t + s, \tau - t, \theta_{-t}, 0)\|^2
\]
where \( r_2(\tau, \omega) = Cr_1(\tau, \omega), r_1(\tau, \omega) \) is defined as in Lemma 2.2.

**Proof.** From the first equation in (2.61), we have

\[
\frac{d}{dt} \Delta u_1 = \Delta v_1 - \sigma \Delta u_1 + (1 - \xi^2(\frac{|x|^2}{k^2}))\Delta z(\theta_1, \omega). \tag{2.67}
\]

Taking the inner product of the second equation in (2.61) and (2.67) with \( -\Delta v_1 \) and \( \Delta u_1 \) respectively, we have

\[
\begin{align*}
\frac{d}{dt} ||v_1||^2 & = -2\alpha ||v_1||^2 - 2(1 - \sigma)||\nabla v_1||^2 - 2(1 - \alpha \sigma)(\Delta u_1, \Delta v_1) + 2(2(1 - \alpha \sigma) + \lambda - \sigma)(u_1, \Delta v_1) \\
& - 4\alpha(\nabla v \nabla \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) - 2\alpha(v \Delta \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) \\
& - 4(1 - \alpha \sigma)(\nabla u \nabla \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) - 2(1 - \alpha \sigma)(u \Delta \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) \\
& - 2(1 - \xi^2(\frac{|x|^2}{k^2}))(f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta_1, \omega) + \sigma \Delta z(\theta_1, \omega), \Delta v_1), \tag{2.68}
\end{align*}
\]

and

\[
\begin{align*}
\frac{d}{dt} ||\Delta u_1||^2 & = 2(\Delta v_1, \Delta u_1) - 2\alpha ||\Delta u_1||^2 + 2(1 - \xi^2(\frac{|x|^2}{k^2}))(\Delta z(\theta_1, \omega), \Delta u_1). \tag{2.69}
\end{align*}
\]

Adding up (2.69) \( \times (1 - \alpha \sigma) \) and (2.68), we obtain

\[
\begin{align*}
\frac{d}{dt} ((1 - \alpha \sigma)||\Delta u_1||^2 + ||\nabla v_1||^2) & = -2\alpha(1 - \alpha \sigma)||\Delta u_1||^2 - 2\alpha ||\Delta v_1||^2 - 2(1 - \sigma)||\nabla v_1||^2 + 2(2(1 - \alpha \sigma) + \lambda - \sigma)(u_1, \Delta v_1) \\
& - 2(1 - \xi^2(\frac{|x|^2}{k^2}))(f(x, u(t - \rho(t))) + g(t, x), \Delta v_1) \\
& - 2(1 - \xi^2(\frac{|x|^2}{k^2}))(\sigma z(\theta_1, \omega) + \sigma \Delta z(\theta_1, \omega), \Delta v_1) + 2(1 - \alpha \sigma)(1 - \xi^2(\frac{|x|^2}{k^2}))(\Delta z(\theta_1, \omega), \Delta u_1) \\
& - 4\alpha(\nabla v \nabla \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) + 2\alpha(v \Delta \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) \\
& - 4(1 - \alpha \sigma)(\nabla u \nabla \xi^2(\frac{|x|^2}{k^2}), \Delta v_1) + 2(1 - \alpha \sigma)(u \Delta \xi^2(\frac{|x|^2}{k^2}), \Delta v_1)) \\
& = -2\alpha(1 - \alpha \sigma)||\Delta u_1||^2 - 2\alpha ||\Delta v_1||^2 - 2(1 - \sigma)||\nabla v_1||^2 + \sum_{i=1}^{6} T_i.. \tag{2.70}
\end{align*}
\]

Next, we give the estimates of the terms on the right-hand of (2.70). It follows from Cauchy-Schwarz inequality and Young’s inequality that

\[
T_1 \leq \frac{\alpha}{9} ||\Delta v_1||^2 + \frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha} ||u_1||^2,
\]
Thus we have

\[ \mathcal{T}_2 \leq \frac{2\alpha}{9} \|\Delta v_1\|^2 + \frac{9}{\alpha} \|k_1\|^2 + \frac{9k_2^2}{\alpha} \|\mu'|\|_{C_0}^2 + \frac{9}{\alpha} \|g(t,x)\|^2, \]

\[ \mathcal{T}_3 \leq \frac{2\alpha}{9} \|\Delta v_1\|^2 + \frac{9\sigma^2}{\alpha} \|z(\theta,\omega)\|^2 + \frac{9}{\alpha} \|\Delta z(\theta,\omega)\|^2, \]

and

\[ \mathcal{T}_4 \leq \sigma(1 - \alpha\sigma)\|\Delta u_1\|^2 + \sigma(1 - \alpha\sigma)\|\Delta z(\theta,\omega)\|^2. \]

Using the properties of the cutoff function \( \xi \), we infer that

\[ \mathcal{T}_5 = -4\alpha \left( \frac{4x}{k^2} \nabla v \xi\left( \frac{|x|^2}{k^2} \right) \xi'(\frac{|x|^2}{k^2}), \Delta v_1 \right) \]

\[ -2\alpha \left( v \left( \frac{4x}{k^2} \xi\left( \frac{|x|^2}{k^2} \right) \xi'(\frac{|x|^2}{k^2}) + \frac{8x^2}{k^4} \xi'(\frac{|x|^2}{k^2}) \right)^2 + \frac{8x^2}{k^4} \xi(\frac{|x|^2}{k^2}) \xi''(\frac{|x|^2}{k^2}), \Delta v_1 \right), \]

where

\[ -4\alpha \left( \frac{4x}{k^2} \nabla v \xi\left( \frac{|x|^2}{k^2} \right) \xi'(\frac{|x|^2}{k^2}), \Delta v_1 \right) = -\frac{16\alpha}{k^2} \int_{\Omega_{\alpha}} x \nabla v \xi\left( \frac{|x|^2}{k^2} \right) \xi'(\frac{|x|^2}{k^2}) \Delta v_1 dx \]

\[ \leq \frac{16 \sqrt{2} \alpha \mu_1}{k} \int_{k<|x|<\sqrt{k}} |\nabla v||\Delta v_1| dx \]

\[ \leq \frac{16 \sqrt{2} \alpha \mu_1}{k} \int_{\Omega_{\alpha}} |\nabla v||\Delta v_1| dx \]

\[ \leq \frac{\alpha}{9} \|\Delta v_1\|^2 + \frac{2^{2}3^{2}\mu_{1}^{2}\alpha}{k^2} \|\nabla v\|^2, \]

and similarly,

\[ -2\alpha \left( v \left( \frac{4x}{k^2} \xi\left( \frac{|x|^2}{k^2} \right) \xi'(\frac{|x|^2}{k^2}) + \frac{8x^2}{k^4} \xi'(\frac{|x|^2}{k^2}) \right)^2 + \frac{8x^2}{k^4} \xi(\frac{|x|^2}{k^2}) \xi''(\frac{|x|^2}{k^2}), \Delta v_1 \right) \]

\[ \leq \frac{\alpha}{9} \|\Delta v_1\|^2 + \left( \frac{2^{4}3^{3}\mu_{1}^{2}\alpha}{k^4} + \frac{2^{8}3^{3}\mu_{1}^{4}\alpha}{k^4} + \frac{2^{8}3^{3}\mu_{2}^{2}\alpha}{k^4} \right) \|\nabla v\|^2. \]

Thus we have

\[ \mathcal{T}_5 \leq \frac{2\alpha}{9} \|\Delta v_1\|^2 + \frac{2^{3}3^{2}\mu_{1}^{2}\alpha}{k^2} \|\nabla v\|^2 + \left( \frac{2^{4}3^{3}\mu_{1}^{2}\alpha}{k^4} + \frac{2^{8}3^{3}\mu_{1}^{4}\alpha}{k^4} + \frac{2^{8}3^{3}\mu_{2}^{2}\alpha}{k^4} \right) \|\nabla v\|^2. \]

In the same way we obtain

\[ \mathcal{T}_6 \leq \frac{2\alpha}{9} \|\Delta v_1\|^2 + \frac{2^{3}3^{2}\mu_{2}^{2}\alpha}{k^2} \|\nabla u\|^2 + \left( \frac{2^{4}3^{3}\mu_{2}^{2}(1 - \alpha\sigma)}{k^4\alpha} + \frac{2^{8}3^{3}\mu_{2}^{4}(1 - \alpha\sigma)}{k^4\alpha} + \frac{2^{8}3^{3}\mu_{2}^{2}(1 - \alpha\sigma)}{k^4\alpha} \right) \|\nabla u\|^2. \]
Combining these estimates with (2.70), we get
\[
\frac{d}{dt}((1 - a\sigma)\|\Delta u_1\|^2 + \|\nabla v_1\|^2) + \|\Delta v_1\|^2 \\
\leq -\sigma(1 - a\sigma)\|\Delta u_1\|^2 - 2(1 - \sigma)\|\nabla v_1\|^2 + \frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha}\|u_1\|^2 \\
+ \frac{C}{k^2}(|\nabla u|^2 + |\nabla v|^2) + \frac{C}{k^4}(|u|^2 + |v|^2) \\
+ \frac{9}{\alpha}k_1|\xi|^2 + \frac{9k_2}{\alpha}u_t^2 + \frac{9}{\alpha}g(t, x)^2 \\
+ \frac{9\sigma^2}{\alpha}|z(\theta, \omega)|^2 + (\sigma(1 - a\sigma) + \frac{9}{\alpha})\|\Delta z(\theta, \omega)\|^2.
\]
(2.71)

Let \(\sigma' > 0\) be a constant satisfying (2.11)–(2.13). Multiplying \(e^{\sigma't}\) on both sides of (2.71) and integrating over \((\tau - t, \tau + s)\), where \(s \in [-h, 0]\), we obtain
\[
(1 - a\sigma)\|\Delta u_t(\tau + s, \tau - t, \omega, 0)\|^2 + \|\nabla v_1(\tau + s, \tau - t, \omega, 0)\|^2 \\
+ \alpha e^{-\sigma'(t+s)} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta u_1(r, \tau - t, \omega, 0)\|^2 dr \\
\leq -\sigma - \sigma'\sigma(1 - a\sigma)e^{-\sigma'(t+s)} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta u_1(r, \tau - t, \omega, 0)\|^2 dr \\
- (2(1 - \sigma) - \sigma')e^{-\sigma'(t+s)} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\nabla v_1(r, \tau - t, \omega, 0)\|^2 dr \\
+ \frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta v_1(r, \tau - t, \omega, 0)\|^2 dr \\
+ \frac{C}{k^2}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\nabla u(r, \tau - t, \omega, u^{\tau-t})\|^2 + \|\nabla v(r, \tau - t, \omega, v^{\tau-t})\|^2 dr \\
+ \frac{C}{k^4}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta u(r, \tau - t, \omega, u^{\tau-t})\|^2 + \|\Delta v(r, \tau - t, \omega, v^{\tau-t})\|^2 dr \\
+ C + \frac{9k_2}{\alpha}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta u'(s, \tau - t, \omega, u^{\tau-t})\|^2 dr \\
+ C e^{-\sigma't} \int_{-\infty}^{\tau} e^{\sigma't}||g(r, x)\|^2 dr + C e^{-\sigma't} \int_{-\infty}^{0} e^{\sigma't}||\Delta z(\theta, \omega)\|^2 + \|z(\theta, \omega)\|^2 dr.
\]
(2.72)

Now we replace \(\omega\) by \(\theta_{-\tau}\omega\) and estimate the terms on the right-hand of (2.72).

By Lemma 2.2, there exists \(T_1 = T_1(\tau, \omega, D) > 0\), such that for all \(t \geq T_1\),
\[
\frac{C}{k^2}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 + \|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr \\
+ \frac{C}{k^4}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 + \|\Delta v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr \\
\leq C r_1(\tau, \omega),
\]
(2.73)

and
\[
\frac{9k_2}{\alpha}e^{-\sigma't} \int_{\tau-t}^{\tau+t} e^{\sigma't}||\Delta u'(s, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 dr \leq C r_1(\tau, \omega).
\]
(2.74)
From (2.66), there exists $T_2 = T_2(\tau, \omega, D) > 0$, such that for all $t \geq T_2$,

$$
\frac{9(\sigma^2 + \lambda - \sigma^2)}{\alpha} e^{-\alpha \tau} \int_{t-T}^{t} e^{\alpha \tau} \|u_1(r, \tau - t, \theta_{-\tau}, 0)\|^2 dr \leq C_r(\tau, \omega).
$$

(2.75)

By (2.30), we obtain

$$
Ce^{-\alpha \tau} \int_{-\infty}^{T} e^{\alpha \tau} \|g(r, x)\|^2 dr + Ce^{-\alpha \tau} \int_{-\infty}^{T} e^{\alpha \tau} (\|\Delta u_1(r, \tau - t, \theta_{-\tau}, 0)\|^2 + \|\partial^2 u_1(r, \tau - t, \theta_{-\tau}, 0)\|^2) dr \leq C_r(\tau, \omega).
$$

(2.76)

Let $T = \max\{T_1, T_2\}$. It follows from (2.72)–(2.76) that for all $t \geq T$,

$$(1 - \alpha \sigma) \|\Delta u_1(\tau + s, \tau - t, \theta_{-\tau}, 0)\|^2 + \|\nabla v_1(\tau + s, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$+ \alpha \int_{-\infty}^{t} e^{\alpha \tau} \|\Delta u_1(r, \tau - t, \theta_{-\tau}, 0)\|^2 dr \leq C_r(\tau, \omega) \equiv r_2(\tau, \omega).$$

Thus we complete the proof of Lemma 2.4.

Next, we establish the Hölder continuity of $\varphi_1$ in time, which will be useful to show the equicontinuity of solutions in $C([-h, 0], E(\Omega_{2h}))$ based on the Arzela-Ascoli theorem.

**Lemma 2.5.** Assume the hypotheses in Lemma 2.2 hold, and let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}(\mathbb{R})$. Then there exist random variables $r_3(\tau, \omega)$ and $T = T(\tau, \omega, D) > 0$, such that for all $t \geq T, k \geq 1$ and $-h < \eta_1 < \eta_2 < 0$,

$$
\|\varphi^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - \varphi^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|_{E(\Omega_{2h})}^2 \leq r_3(\tau, \omega)|\eta_2 - \eta_1|^\frac{4}{\alpha}
$$

where $\varphi^\tau_1(\eta, \tau - t, \theta_{-\tau}, 0) = (u^\tau_1(\eta, \tau - t, \theta_{-\tau}, 0), v^\tau_1(\eta, \tau - t, \theta_{-\tau}, 0))$, $i = 1, 2$.

**Proof.** By the definition of the norm $\| \cdot \|_{\mathcal{E}}$, we have

$$
\|\varphi^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - \varphi^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2_{E(\Omega_{2h})}$$

$$= (\sigma^2 + \lambda - \sigma^2) \|u^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - u^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$+ (1 - \alpha \sigma) \|\nabla u^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - \nabla u^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$+ \|v^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - v^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$\leq (\sigma^2 + \lambda - \sigma^2) \|u^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - u^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$+ (1 - \alpha \sigma) \|\nabla u^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - \nabla u^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$+ \|v^\tau_1(\eta_2, \tau - t, \theta_{-\tau}, 0) - v^\tau_1(\eta_1, \tau - t, \theta_{-\tau}, 0)\|^2$$

$$\leq (\sigma^2 + \lambda - \sigma^2) \int_{\tau+\eta_1}^{\tau+\eta_2} \frac{d}{dr} u^\tau_1(r, \tau - t, \theta_{-\tau}, 0) dr$$

$$+ (1 - \alpha \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \frac{d}{dr} \nabla u^\tau_1(r, \tau - t, \theta_{-\tau}, 0) dr$$

$$+ \int_{\tau+\eta_1}^{\tau+\eta_2} \frac{d}{dr} v^\tau_1(r, \tau - t, \theta_{-\tau}, 0) dr.
$$

(2.77)
From (2.61), we have
\[
\frac{du_1}{dt} = v_1 - \sigma u_1 + (1 - \bar{\xi}(\frac{|x|^2}{k^2}))z(\theta, \omega),
\]
together with (2.66), we know that there exists \( T_1 = T_1(\tau, \omega, D) > 0 \) such that for all \( t \geq T_1 \),
\[
\int_{\tau+\eta_1}^{\tau+\eta_2} \left\| \frac{d}{dr}u_1(r, \tau - t, \theta, \omega, 0) \right\| dr
\leq \int_{\tau+\eta_1}^{\tau+\eta_2} \|v_1(r, \tau - t, \theta, \omega, 0)\| dr + \sigma \int_{\tau+\eta_1}^{\tau+\eta_2} \|u_1(r, \tau - t, \theta, \omega, 0)\| dr
+ \int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta, \omega)\| dr
\leq \left\{ \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|v_1(r, \tau - t, \theta, \omega, 0)\|^2 dr \right)^{\frac{1}{2}} + \sigma \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|u_1(r, \tau - t, \theta, \omega, 0)\|^2 dr \right)^{\frac{1}{2}} \right\}
+ \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta, \omega)\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^\frac{1}{2}
\leq \left\{ \left( 1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda - \sigma}} \right) \left( \int_{\tau+\eta_1}^{\tau+\eta_2} r_1(\tau, \theta) dr \right)^{\frac{1}{2}} \right\}
+ \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta, \omega)\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^\frac{1}{2}
= \left\{ \left( 1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda - \sigma}} \right) \left( \int_{\eta_1}^{\eta_2} r_1(\tau, \theta) dr \right)^{\frac{1}{2}} + \left( \int_{\eta_1}^{\eta_2} \|z(\theta, \omega)\|^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^\frac{1}{2}
\leq \left\{ \left( 1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda - \sigma}} \right) h^\frac{1}{2} \left( \sup_{r \in [-h, h]} r_1(\tau, \theta) \right)^{\frac{1}{2}} + e^{\sigma h^\frac{1}{2}} h^\frac{1}{2} r_1(\tau, \omega) \right\} |\eta_2 - \eta_1|^\frac{1}{2}
\right.
\left. \right\} |\eta_2 - \eta_1|^\frac{1}{2}
(2.78)

Since
\[
e^{-\sigma h} \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v_1(r, \tau - t, \theta, \omega, 0)\|^2 dr \leq \int_{\tau+\eta_1}^{\tau+\eta_2} e^{\sigma(r-\tau-\eta_2)} \|\nabla v_1(r, \tau - t, \theta, \omega, 0)\|^2 dr
\leq \int_{\tau-t}^{\tau+\eta_2} e^{\sigma(r-\tau-\eta_2)} \|\nabla v_1(r, \tau - t, \theta, \omega, 0)\|^2 dr,
\]
by (2.66) and (2.79), we get for \( t \geq T_1 \),
\[
\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v_1(r, \tau - t, \theta, \omega, 0)\|^2 dr \leq C e^{\sigma h} r_1(\tau, \omega).
\]
Using the similar computation in (2.78), we obtain for \( t \geq T_1 \),
\[
\int_{\tau+\eta_1}^{\tau+\eta_2} \|\frac{d}{dr}\nabla u_1(r, \tau - t, \theta, \omega, 0)\| dr
\]
\[
\begin{align*}
&\leq \left( \int_{\tau_{\eta_2}}^{\tau_{\eta_1}} ||\nabla v_1(r, \tau - t, \theta_{-\tau}, 0)||dr + \sigma \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla u_1(r, \tau - t, \theta_{-\tau}, 0)||dr \\
&\quad + \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla z(\theta_{-r})||dr \\
&\quad \leq \left\{ \left( \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla v_1(r, \tau - t, \theta_{-\tau}, 0)||^2 dr \right)^{\frac{1}{2}} + \sigma \left( \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla u_1(r, \tau - t, \theta_{-\tau}, 0)||^2 dr \right)^{\frac{1}{2}} \\
&\quad + \left( \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla z(\theta_{-r})||^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \\
&\leq \left\{ C^2 \sqrt{r_1(\tau, \omega)} + \frac{C^2 \sigma}{\sqrt{1 - \alpha \sigma}} \left( \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} r_1(\tau, \theta_{-\tau}, \omega) dr \right)^{\frac{1}{2}} + \left( \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla z(\theta_{-r})||^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \\
&= \left\{ C^2 \sqrt{r_1(\tau, \omega)} + \frac{C^2 \sigma}{\sqrt{1 - \alpha \sigma}} \left( \int_{\tau_{\eta_1}}^{\eta_2} r_1(\tau, \theta_{-\tau}, \omega) dr \right)^{\frac{1}{2}} + \left( \int_{\tau_{\eta_1}}^{\eta_2} ||\nabla z(\theta_{-r})||^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \\
&\leq \left\{ C^2 \sqrt{r_1(\tau, \omega)} + \frac{C^2 \sigma}{\sqrt{1 - \alpha \sigma}} h^\frac{1}{2} \left( \sup_{\tau_{\eta_1} - \eta_2} r_1(\tau, \theta_{-\tau}, \omega) \right)^{\frac{1}{2}} + e^{\frac{\alpha \sigma}{\sigma}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \\
&\leq \left\{ C^2 \sqrt{r_1(\tau, \omega)} + \frac{C^2 \sigma}{\sqrt{1 - \alpha \sigma}} h^\frac{1}{2} \left( \sup_{\tau_{\eta_1} - \eta_2} r_1(\tau, \theta_{-\tau}, \omega) \right)^{\frac{1}{2}} + e^{\frac{\alpha \sigma}{\sigma}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \quad \text{(2.81)}
\end{align*}
\]

By (2.61), we get
\[
\int_{\tau_{\eta_1}}^{\tau_{\eta_2}} \frac{d}{dr} v_1(r, \tau - t, \theta_{-\tau}, 0) dr \\
\leq \alpha \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\Delta v_1(r, \tau - t, \theta_{-\tau}, 0)|| dr + (1 - \alpha \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\Delta u_1(r, \tau - t, \theta_{-\tau}, 0)|| dr \\
+ (1 - \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||v_1(r, \tau - t, \theta_{-\tau}, 0)|| dr + (\sigma^2 + \lambda - \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||u_1(r, \tau - t, \theta_{-\tau}, 0)|| dr \\
+ 2\alpha \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla v(r, \tau - t, \theta_{-\tau}, \omega, v_{-\tau}) \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) || dr \\
+ \alpha \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||v(r, \tau - t, \theta_{-\tau}, \omega, v_{-\tau}) \Delta \xi^2 \left( \frac{|x|^2}{k^2} \right) || dr \\
+ 2(1 - \alpha \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||\nabla u(r, \tau - t, \theta_{-\tau}, \omega, u_{-\tau}) \nabla \xi^2 \left( \frac{|x|^2}{k^2} \right) || dr \\
+ (1 - \alpha \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||u(r, \tau - t, \theta_{-\tau}, \omega, u_{-\tau}) \Delta \xi^2 \left( \frac{|x|^2}{k^2} \right) || dr \\
+ \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||f(x, u(r - \rho(r)))|| + ||g(r, x)|| dr + \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} (\sigma ||z(\theta, \omega)|| + \alpha ||\Delta z(\theta, \omega)||) dr. \quad \text{(2.82)}
\]

By (2.66), we have for \( t \geq T_1 \),
\[
(1 - \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||v_1(r, \tau - t, \theta_{-\tau}, 0)|| dr + (\sigma^2 + \lambda - \sigma) \int_{\tau_{\eta_1}}^{\tau_{\eta_2}} ||u_1(r, \tau - t, \theta_{-\tau}, 0)|| dr
\]

\[AIMS Mathematics\]
By Lemma 2.4, we know that there exists $T_2 = T_2(\tau, \omega, D)$ such that for $t > T_2$,

$$
\alpha \int_{\tau + \eta_2}^{\tau + \eta_1} \|\Delta v_1(r, \tau - t, \theta, \omega, 0)\|dr + (1 - \alpha\sigma) \int_{\tau + \eta_1}^{\tau + \eta_2} \|\Delta u_1(r, \tau - t, \theta, \omega, 0)\|dr \\
\leq \left\{ \alpha \left( \int_{\tau + \eta_1}^{\tau + \eta_2} \|\Delta v_1(r, \tau - t, \theta, \omega, 0)\|^2 dr \right) ^{\frac{1}{2}} \\
+ (1 - \alpha\sigma) \left( \int_{\tau + \eta_1}^{\tau + \eta_2} \|\Delta u_1(r, \tau - t, \theta, \omega, 0)\|^2 dr \right) ^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}
$$

From Lemma 2.2, we find that there exists $T_3 = T_3(\tau, \omega, D)$ such that for $t > T_3$,

$$
2\alpha \int_{\tau + \eta_1}^{\tau + \eta_2} \|\nabla v(r, \tau - t, \theta, \omega, v^{r-t})\|dr \\
= \frac{8\alpha}{k^2} \int_{\tau + \eta_1}^{\tau + \eta_2} \|\nabla v(r, \tau - t, \theta, \omega, v^{r-t})\|dr \\
\leq \frac{8\sqrt{2}\alpha\mu_1}{k} \left( \int_{\tau + \eta_1}^{\tau + \eta_2} \|\nabla v(r, \tau - t, \theta, \omega, v^{r-t})\|^2 dr \right) ^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}
$$

and

$$
\alpha \int_{\tau + \eta_1}^{\tau + \eta_2} \|v(r, \tau - t, \theta, \omega, v^{r-t})\|dr \\
= \frac{4\alpha}{k^2} \int_{\tau + \eta_1}^{\tau + \eta_2} \|v(r, \tau - t, \theta, \omega, v^{r-t})\|dr \\
+ \frac{4\alpha}{k^2} \int_{\tau + \eta_1}^{\tau + \eta_2} \|v(r, \tau - t, \theta, \omega, v^{r-t})\| \xi'' \left( \frac{|x|^2}{k^2} \right) \xi' \left( \frac{|x|^2}{k^2} \right) \|dr \\
\leq \frac{4\sqrt{2}k\mu_1^2 + \mu_2}{k^2} \left( \int_{\tau + \eta_1}^{\tau + \eta_2} \|v(r, \tau - t, \theta, \omega, v^{r-t})\|^2 dr \right) ^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}
$$
By (A1) and Lemma 2.2, we obtain for $t$ and $\eta$

\[
\int_{\tau+\eta_1}^{\tau+\eta_2} \|u(r, \tau - t, \theta - r, u^{r_i})\| dr \leq h^2 \left( \sup_{r \in [-h,0]} r_1(\tau, \theta, \omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.86}
\]

Similarly, we have for $t > T_3$,

\[
(1 - \alpha \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|u(r, \tau - t, \theta - r, u^{r_i})\| \eta^2 \left( \frac{|x|^2}{k^2} \right) dr \leq 4 \sqrt{2} k(1 - \alpha \sigma)(\mu_1^2 + \mu_2) + 4(1 - \alpha \sigma)\mu_1 h^2 \left( \sup_{r \in [-h,0]} r_1(\tau, \theta, \omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}, \tag{2.87}
\]

and

\[
2(1 - \alpha \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u(r, \tau - t, \theta - r, u^{r_i})\| \eta^2 \left( \frac{|x|^2}{k^2} \right) dr \leq 8 \sqrt{2} (1 - \alpha \sigma)\mu_1 \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u(r, \tau - t, \theta - r, u^{r_i})\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}} \tag{2.88}
\]

By (A1) and Lemma 2.2, we obtain for $t > T_3$,

\[
\int_{\tau+\eta_1}^{\tau+\eta_2} \|f(x, u(r - \rho(r)))\| dr \leq \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|f(x, u(r - \rho(r)))\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}} \tag{2.89}
\]

and

\[
\int_{\tau+\eta_1}^{\tau+\eta_2} \|g(x)\| dr \leq \left( \int_{\tau+\eta_1}^{\tau+\eta_2} \|g(r, x)\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}} \tag{2.90}
\]
Noting that \( h_j \in H^2(\mathbb{R}^d) \) and by (2.6), we have
\[
\int_{\tau+\eta_1}^{\tau+\eta_1} (\sigma||z(\theta, \omega)|| + \alpha||\Delta z(\theta, \omega)||)\,dr \leq (\sigma + \alpha)C^2 h^\frac{1}{2} e^{\frac{\alpha}{\sigma} r^2} (\omega)\eta_2 - \eta_1^\frac{1}{2}.
\] (2.91)

Set \( T = \max\{T_1, T_2, T_3\}, \)
\[
r_3(\tau, \omega) \equiv (\sigma^2 + \lambda - \sigma) \left\{ \left( 1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda - \sigma}} \right) h^\frac{1}{2} \left( \sup_{r \in [-h, 0]} r_1(\tau, \theta, \omega) \right)^\frac{1}{2} + e^{\frac{\alpha}{\sigma} r^2} h^\frac{1}{2} (\omega) \right\}
\]
\[
+ (1 - \alpha r) \left\{ C^2 \sqrt{r_1(\tau, \omega)} + \frac{C^2 \sigma}{\sqrt{1 - \alpha \sigma}} h^\frac{1}{2} \left( \sup_{r \in [-h, 0]} r_1(\tau, \theta, \omega) \right)^\frac{1}{2} + e^{\frac{\alpha}{\sigma} r^2} h^\frac{1}{2} (\omega) \right\}
\]
\[
+ Ch^\frac{1}{2} \left( \sup_{r \in [-h, 0]} r_1(\tau, \theta, \omega) \right)^\frac{1}{2} + \left\{ \alpha \sqrt{r_2(\tau, \omega)} + (1 - \alpha \sigma) \left( \sup_{r \in [-h, 0]} r_2(\tau, \theta, \omega) \right) \right\} ^\frac{1}{2}
\]
\[
+ 8 \sqrt{2}(1 - \alpha \sigma)(\mu_1^2 + \mu_2) + 4(1 - \alpha \sigma)(\mu_1)
\]
\[
+ 8 \sqrt{2}(1 - \sigma \alpha)(\mu_1^2 + \mu_2) + 4(1 - \alpha \sigma)(\mu_1)
\]
\[
+ \left( \frac{\alpha}{\sigma} r^2 \right) + \left( \frac{\alpha}{\sigma} r^2 \right) + C^2 h^\frac{1}{2} e^{\frac{\alpha}{\sigma} r^2} (\omega).
\] (2.92)

It follows from (2.77)–(2.92) that, for \( t > T \),
\[
||\varphi^T_1(\eta_2, \tau - t, \theta, \omega, 0) - \varphi^T_1(\eta_1, \tau - t, \theta - \omega, 0)||^2_{E(\Omega_{23})} \leq r_3(\tau, \omega)\eta_2 - \eta_1^\frac{1}{2}.
\]

This completes the proof of Lemma 2.5. \( \square \)

3. Results and discussion

In this section, we aim to prove the existence of tempered pullback random attractors for the system (1.1) and (1.2) in \( \mathcal{E} \). Firstly we show the existence of the pullback absorbing set as follows.

**Lemma 3.1.** Suppose the hypotheses in Lemma 2.2 hold. Then the continuous cocycle \( \Phi \) has a closed measurable \( \mathcal{D} \)-pullback absorbing set \( K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \).

**Proof.** Set
\[
K(\tau, \omega) = \{ \varphi \in \mathcal{E} : ||\varphi||^2_{\mathcal{E}} \leq r_1(\tau, \omega) \},
\]
where \( r_1(\tau, \omega) \) is given by (2.15). It is evident that, for each \( \tau \in \mathbb{R}, r_1(\tau, \cdot) : \Omega \to \mathbb{R} \) is (\( \mathcal{F}, \mathcal{B}(\mathbb{R}) \))-measurable.

Note that
\[
r_1(\tau, \omega) = C + Cr(\omega) + Ce^{\sigma r} \int_{-\infty}^{T} e^{\sigma r} ||g(r, x)||^2 dr.
\]
By simple calculations, we have for each $\gamma > 0$,

$$\lim_{t \to -\infty} e^{\gamma t} \|K(t + t, \theta_t \omega)\|^2_{\mathcal{E}} = \lim_{t \to -\infty} e^{\gamma t} r_1(t + t, \theta_t \omega) = 0.$$ 

In addition, for each $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, by Lemma 2.2, there exists $T = T(\tau, \omega, D) > 0$, such that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t} \omega, D(t - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega),$$

that is, $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for $\Phi$. \hfill \Box

Next, we will prove that the continuous cocycle $\Phi$ is asymptotically compact in $\mathcal{E}$.

**Lemma 3.2.** Suppose the hypotheses in Lemma 2.2 hold. Then the continuous cocycle $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{E}$. That is, if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, the sequence $\{\Phi(t_m, \tau - t_m, \theta_{-t_m} \omega, x_m)\}_{m=1}^{\infty}$ has a convergent subsequence in $\mathcal{E}$.

**Proof.** Let $t_m \to \infty$, $D \in \mathcal{D}(\mathcal{E}(\Omega_{2k}))$ and $\tilde{\varphi}^{\tau - t_m} = (\tilde{u}^{\tau - t_m}, \tilde{v}^{\tau - t_m}) \in D(t - t_m, \theta_{-t_m} \omega)$. We firstly show that $\tilde{\varphi}^{\tau - t_m} = (\tilde{u}^{\tau - t_m}, \tilde{v}^{\tau - t_m})$ is precompact in $\mathcal{E}(\Omega_{2k})$, where $\tilde{\varphi}^{\tau - t_m} = (\tilde{u}^{\tau - t_m}, \tilde{v}^{\tau - t_m})$. It follows from Lemma 2.4 that, for $M_1 = M_1(\tau, \omega, D)$ large enough, $m > M_1$ and $r \in [-h, 0]$,

$$||u_1(r, \tau - t_m, \theta_{-t_m} \omega, 0)||_{H^1(\Omega_{2k})}^2 + ||v_1(r, \tau - t_m, \theta_{-t_m} \omega, 0)||_{H^1(\Omega_{2k})}^2 \leq Cr_1(\tau, \omega).$$

We know that $H^1(\Omega_{2k}) \hookrightarrow L^2(\Omega_{2k})$ and $H^2(\Omega_{2k}) \hookrightarrow H^1(\Omega_{2k})$ are compact. Therefore, for $m > M_1$ and $r \in [-h, 0]$, $\{u_1(r, \tau - t_m, \theta_{-t_m} \omega, 0), v_1(r, \tau - t_m, \theta_{-t_m} \omega, 0)\}$ is precompact in $E(\Omega_{2k})$. From Lemma 2.5, for $m > M_1$, $\{u_1(\tau - t_m, \theta_{-t_m} \omega, 0), v_1(\tau - t_m, \theta_{-t_m} \omega, 0)\}$ is equi-continuous in $C([-h, 0], E(\Omega_{2k}))$. Then by Arzela-Ascoli theorem, $\{u_1(\tau - t_m, \theta_{-t_m} \omega, 0), v_1(\tau - t_m, \theta_{-t_m} \omega, 0)\}$ is precompact in $C([-h, 0], E(\Omega_{2k}))$.

Hence, there exists a subsequence $t_{m_j}$, still denote as $t_m$, such that

$$(u_1^{t_m}(\tau - t_m, \theta_{-t_m} \omega, 0), v_1^{t_m}(\tau - t_m, \theta_{-t_m} \omega, 0)) \to (\zeta(\cdot), \xi(\cdot)), \quad \text{in } C([-h, 0], E(\Omega_{2k})).$$

In other words, for any $\varepsilon > 0$, there exists $M_2 = M_2(\tau, \omega, \varepsilon, D)$, such that for $m > M_2$ and $r \in [-h, 0]$,

$$||(u_1^{t_m}(\tau - t_m, \theta_{-t_m} \omega, 0) - \zeta(\cdot), v_1^{t_m}(\tau - t_m, \theta_{-t_m} \omega, 0) - \xi(\cdot))||_{\mathcal{E}(\Omega_{2k})} < \varepsilon. \quad (3.1)$$

For any $\{u_2^{\tau - t_m}, v_2^{\tau - t_m}\} = (\tilde{u}^{\tau - t_m}, \tilde{v}^{\tau - t_m}) \in D(t - t_m, \theta_{-t_m} \omega, 0)$, by (2.65), there exists $M_3 = M_3(\tau, \omega, \varepsilon, D)$, such that for $m > M_3$,

$$||(u_2^{\tau - t_m}(\tau - t_m, \theta_{-t_m} \omega, 0), v_2^{\tau - t_m}(\tau - t_m, \theta_{-t_m} \omega, 0))||_{\mathcal{E}(\Omega_{2k})} < \varepsilon. \quad (3.2)$$

By (3.1) and (3.2), we derive for $m > M = \max\{M_1, M_2, M_3\}$,

$$\begin{align*}
||\hat{\Phi}(\tau - t_m, \theta_{-t_m} \omega, 0) - \zeta(\cdot), \hat{v}(\tau - t_m, \theta_{-t_m} \omega, 0)||_{\mathcal{E}(\Omega_{2k})} \\
\leq 2||u_1^{t_m}(\tau - t_m, \theta_{-t_m} \omega, 0) - \zeta(\cdot), v_1^{t_m}(\tau - t_m, \theta_{-t_m} \omega, 0) - \xi(\cdot))||_{\mathcal{E}(\Omega_{2k})} \\
+ 2||u_2^{\tau - t_m}(\tau - t_m, \theta_{-t_m} \omega, 0), v_2^{\tau - t_m}(\tau - t_m, \theta_{-t_m} \omega, 0))||_{\mathcal{E}(\Omega_{2k})} \leq 4\varepsilon. \quad (3.3)
\end{align*}$$
Thus, \( \tilde{\varphi}^\tau(\cdot, \tau - t_m, \theta - \tau \omega, \tilde{\varphi}^{\tau - t_m}) = (\tilde{u}^\tau(\cdot, \tau - t_m, \theta - \tau \omega, \tilde{u}^{\tau - t_m}), \tilde{v}^\tau(\cdot, \tau - t_m, \theta - \tau \omega, \tilde{v}^{\tau - t_m})) \) is precompact in \( \mathcal{E}(\Omega_{2k}) \). By Lemma 2.3, there exist \( k_1 = k_1(\tau, \omega, \epsilon) \) and \( M_4 = M_4(\tau, \omega, \epsilon, D) \), such that for each \( m > M_4 \),

\[
\|\varphi^\tau(\cdot, \tau - t_m, \theta - \tau \omega, \varphi^{\tau - t_m})\|_{\mathcal{E}(\mathbb{R}^d \setminus \Omega_{k_1})} \leq \epsilon. \tag{3.4}
\]

By (3.3), there exists \( k_2 = k_2(\tau, \omega, \epsilon) \geq k_1 \) such that \( \tilde{\varphi}^\tau(\cdot, \tau - t_m, \theta - \tau \omega, \tilde{\varphi}^{\tau - t_m}) \) is precompact in \( E(\Omega_{2k_2}) \). Recalling (2.55) and the fact \( 1 - \xi^2 \left( \frac{|x|^2}{k_2^2} \right) = 1 \) for \( |x| \leq k_2 \), we know that \( \varphi^\tau(\cdot, \tau - t_m, \theta - \tau \omega, \varphi^{\tau - t_m}) \) is precompact in \( \mathcal{E}(\Omega_{k_2}) \). Along with (3.4), we have that the continuous cocycle \( \Phi \) is asymptotically compact in \( \mathcal{E} \).

We are now to give our main result.

**Theorem 3.3.** Suppose the hypotheses in Lemma 2.2 hold. Then the continuous cocycle \( \Phi \) has a unique \( D \)-pullback random attractor in \( \mathcal{E} \).

**Proof.** By Proposition 2.1, Lemma 3.1 and Lemma 3.2, we can obtain the existence and uniqueness of \( D \)-pullback random attractor of \( \Phi \) in \( \mathcal{E} \) immediately. \( \square \)

### 4. Conclusions

Since the Sobolev embedding is no longer compact on unbounded domains, we obtained the existence of random attractor for the problem (1.1) and (1.2) by using the uniform tail-estimates of solutions and the decomposition technique as well as the compactness argument. In addition, to derive the uniform estimates, we make some nontrivial arguments due to the presence of strongly damped term \( \alpha \Delta u_t \) and the delay term \( f(x, u(t - \rho(t))) \) in (1.1).

**Conflict of interest**

The author declares that there is no conflict of interest.

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