A Bouncing Fab Four Quantum Cosmological Model with Bohm-de Broglie Interpretation

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Abstract: We present here a quantum cosmological model with Bohm-de Broglie interpretation of the theory described by a combination of two terms of the Fab Four cosmological theory. The first term is the John Lagrangian and the second is a potential representing matter content to avoid classical trivial solutions. This model has two free functions that provide an adjustment mechanism known classically as self-tuning. The self-tuning is a way to address the cosmological constant problem by allowing a partial break of symmetry in the scalar field sector. The Fab Four is the most general set of self-tuning scalar-tensor gravitational theories in four dimensions. The minisuperspace Hamiltonian thus obtained from this combination of Fab Four terms has fractional powers in the momenta, leading to a problem in applying canonical quantization. We have solved this problem by generalizing the canonical quantization rule using the so-called conformable fractional derivative. We show that this analysis leads to both singular and bouncing (non-singular) solutions, depending on the initial conditions over the scale factor and the homogeneous scalar field, and also depending on the free functions mentioned. This provides an adjustment mechanism in analogy with the classical self-tuning of the Fab Four, but with another interpretation.

Keywords: Models of Quantum Gravity, Spacetime Singularities, Cosmology of Theories beyond the SM
1 Introduction

In 1974, Horndeski presented the most general scalar-tensor gravitational theory in four dimensions that leads to second-order equations of motion [1]. In the notation of [2], the Horndeski action is written as
\[ S_H = \int d^4x \sqrt{-g}(L_2 + L_3 + L_4 + L_5), \] (1.1)
where
\[ L_2 = K(\phi, X), \] (1.2)
\[ L_3 = -G_3(\phi, X)\Box\phi, \] (1.3)
\[ L_4 = G_4(\phi, X)R + G_{4,X}(\phi, X)[(\Box\phi)^2 - \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi], \] (1.4)
\[ L_5 = G_5(\phi, X)\mathcal{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{6} G_{5,X}(\phi, X)[(\Box\phi)^3 \]
\[ - 3\Box\phi\nabla^\mu\phi\nabla^\nu\phi + 2\nabla_\mu\nabla_\nu\phi\nabla_\lambda\nabla^\mu\phi\nabla^\nu\nabla^\lambda\phi]. \] (1.5)
The functions \( K \) and \( G_i \) are generic differentiable functions of the scalar field \( \phi \) and of the kinetic term \( X \equiv -\frac{1}{2} \nabla^\mu\phi\nabla_\mu\phi \). The notation \( G_i,X \) denotes the derivative of \( G_i \) with respect to \( X \). The greek indices here run from 0 to 3 and the latin ones run from 1 to 3.

In 2012, Charmousis, Copeland et al. [3–5] have shown that, starting from the Horndeski action (1.1), only the following Lagrangians are able to produce the most general scalar-tensor theory in four dimensions satisfying the self-tuning filter:
\[ L_j = \sqrt{-g}V_j(\phi)\mathcal{G}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi, \] (1.6)
\[ L_p = \sqrt{-g}V_p(\phi)P^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\phi\nabla_\beta\phi, \] (1.7)
\[ L_g = \sqrt{-g}V_g(\phi)\mathcal{G}, \] (1.8)
\[ L_r = \sqrt{-g}V_r(\phi)\mathcal{G}. \] (1.9)
plus a matter action, where $R$ is the Ricci scalar, $G^{\mu\nu}$ is the Einstein tensor, $P^{\mu\nu\alpha\beta}$ is the double dual of the Riemann tensor, and $\hat{G} = R^{\mu\alpha\beta}R_{\mu\nu\alpha\beta} - 4R^{\mu\nu}R_{\mu\nu} + R^2$ is the Gauss-Bonnet combination. The self-tuning filter consists of a mechanism to obtain the most general scalar-tensor theory able to solve the fine-tuning problem of the cosmological constant by breaking the Poincar invariance in the self-adjusting scalar field, but not in the spacetime curvature. The Lagrangians (1.6) to (1.9) can produce self-tuning only if $V_j \neq 0$, $V_p \neq 0$ and $V_g \neq \text{const}$. This means that General Relativity is not able to produce self-tuning and motivates the study of alternative gravitational theories, as (1.6).

To be precise, this so-called self-tuning filter is the following set of properties relative to gravitational scalar-tensor theories (Charmousis, Copeland et al., op. cit.):

(i) The theory must admit a Minkowski vacuum for any value of the net cosmological constant;

(ii) This remains true before and after any phase transition in which the cosmological constant jumps instantaneously by a finite amount;

(iii) The theory must admit a non-trivial cosmology.

The study of the classical cosmological behaviour of the Fab Four was already presented in the above cited papers.

As observed in [6], due to the experimental agreement between the observations and the expanding universe [7] and also due to the fact that quantum mechanics is a fundamental theory, we should be able to describe the quantum primordial universe. This is the reason why we need a quantum theory of gravitation and in particular a quantum theory of cosmology. Our purpose here is to study the quantum cosmology of the John (1.6) term of the Fab Four cosmological theory. The usual way to face this problem is to apply the orthodox interpretation of quantum mechanics to a hamiltonian gravitational theory. However, there are strong arguments against this approach [8]. The orthodox interpretation assumes the existence of a classical exterior domain [9], and this is clearly impossible when the system under consideration is the whole universe. One way to solve this problem is to apply a different interpretation of quantum mechanics, as the Bohm-de Broglie interpretation [10–12], which is a deterministic interpretation alternative to the orthodox probabilistic interpretation of quantum mechanics [13].

In this work we describe the Bohm-de Broglie quantum cosmology of the John Lagrangian of the Fab Four

$$L = \sqrt{-g} \left[ V_j(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) \right],$$

(1.10)

where $\phi = \phi(t)$ is a homogeneous scalar field, $V_j(\phi)$ is an arbitrary differentiable function, and $V(\phi)$ is a potential to be specified, with the FLRW metric

$$ds^2 = N^2dt^2 - a^2\delta_{ij}dx^idx^j,$$

(1.11)

where $N(t)$ is the lapse function [14] and $a(t)$ is the scale factor. The quantum analysis will take place writing (1.10) in minisuperspace, i.e., we shall write $L$ as a function of $a$, $\phi$, 

\[\text{– 2 –}\]
N, and their first-order time derivatives. As stated in [15], this process is equivalent to a 3 + 1 split, since the metric (1.11) is Bianchi class A. Observe that (1.10) does not depend on the metric signature.

In section 2 we write (1.10) in minisuperspace and then apply a Hamiltonian canonical formalism to quantize the model, thus leading to the so-called Wheeler-DeWitt equation $\hat{H}\psi = 0$ of the model using the constraint $H \approx 0$, obtained from $H = HN$, where $H$ is the Hamiltonian of the theory and $\psi$ represents the wave function of the universe, from which we will obtain all quantum cosmological results. However, this is not a straightforward task because the powers of the momenta in the Hamiltonian are equal to 2/3, as we will see. Nevertheless, there are two basic ways to solve this problem: canonical transformations and fractional derivatives. We explore here the second option, because it has two basic advantages: first, no change of variables is needed; second, it directly transforms our fractional differential equation into a second-order partial differential equation. We apply this method in section 3 using the conformable fractional derivative of R. Khalil et al. [16]. We have chosen this particular fractional derivative for its numerous technical advantages and for its intuitive and very simple meaning.

Thus, in section 4 we interpret the wave function with the deterministic Bohm-de Broglie formalism for the reasons cited above involving the observer problem, but also for other reasons. In [17] is noted that the Bohmian interpretation of quantum mechanics can solve the problem of time [18] in quantum gravity and quantum cosmology. This interpretation has also been successfully applied very recently in other quantum cosmological models [19], and also to describe gravitational waves [20] and particle creation [21].

Until the determination of solutions of the Wheeler-DeWitt equation of our model, the potentials $V_j(\phi)$ and $V(\phi)$ remain absolutely free, because we have solved the Wheeler-DeWitt keeping this generality for any $V_j(\phi)$ and $V(\phi)$. However, to look to the dynamics of the system we must particularize these functions. We then study two cases. In section 5 we study the simplest non-trivial choice $V_j(\phi) \equiv$ const. with several potentials leading to singular cosmological solutions. In section 6, we study a more physically useful case: $V_j(\phi) = \phi^{-3}(\phi - \phi_0)^{-2}$ and $V(\phi) = \lambda \phi^4$, that leads to bouncing solutions, avoiding the singularity $a = 0$. Finally, in the last section of the paper we make some conclusion remarks.

2 Hamiltonian in Minisuperspace

Using the usual connection satisfying $\nabla_{\alpha}g_{\mu\nu} = 0$ [22] for the FLRW metric (1.11), we obtain the components of the Ricci tensor

$$R_{00} = 3\ddot{a} \frac{\dot{N}}{aN} - 3\ddot{a}, \quad R_{0i} = 0, \quad \text{and} \quad R_{ij} = \delta_{ij} \frac{a^2}{N^2} \left(2\frac{\ddot{a}^2}{a^2} + \frac{\ddot{a}}{a} - \frac{\dot{a}}{a} \frac{\dot{N}}{aN}\right),$$

from which follows that the Lagrangian (1.10) can be rewritten in minisuperspace as

$$L = 3aV_j(\phi) \frac{a^2\dot{\phi}^2}{N^3} - Na^3V(\phi), \quad (2.2)$$
after an integration by parts to eliminate a total time derivative that do not interfere in the
dynamics. To simplify the classical equations of motion, we define the variable $\alpha \equiv \ln a$, 
thus rewriting the Lagrangian as

$$
L = e^{3\alpha} \left[ 3V_j(\phi)\frac{\dot{\phi}^2}{N^3} - NV(\phi) \right].
$$

(2.3)

With the gauge choice $N = 1$, the corresponding Euler-Lagrange equations

$$
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0
$$

(2.4)

are found to be

$$
\ddot{\alpha}^2 \dot{\phi}^2 + \frac{V}{9V_j} = 0,
$$

(2.5)

$$
e^{3\alpha} \left( 3V_j \dot{\alpha}^2 \dot{\phi}^2 - V \right) - 2\frac{d}{dt} \left( e^{3\alpha} V_j \dot{\alpha} \dot{\phi}^2 \right) = 0,
$$

(2.6)

$$
e^{3\alpha} \left( 3V'_j \dot{\alpha}^2 \dot{\phi}^2 - V' \right) - 6\frac{d}{dt} \left( e^{3\alpha} V_j \dot{\alpha} \dot{\phi} \dot{\phi} \right) = 0.
$$

(2.7)

Rearranging (2.6) and (2.7) and substituting (2.5) into them, we obtain

$$
\ddot{\alpha} + 3\dot{\alpha}^2 - \frac{V'}{V} \dot{\alpha} \dot{\phi} = 0,
$$

(2.8)

$$
\ddot{\phi} - 3\dot{\alpha} \dot{\phi} + \frac{\dot{\phi}^2}{2} \left( 3 + \frac{V_j'}{V_j} - 2\frac{V'}{V} \right) = 0.
$$

(2.9)

These are the classical equations of motion for the Lagrangian (2.3). In section 6 we will see that for some given initial conditions we can choose $V_j$ and $V$ such that the above equations give singular solutions but the quantum system obtained from fractional quantization and Bohm-de Broglie interpretation give non-singular solutions for the same initial conditions. More about the classical cosmology of the various Fab Four terms can be found in the references cited above. Here, we will restrict ourselves to compare the classical and quantum trajectories to see whose ones are able to avoid the singularity $a = 0$ and whose ones are not.

To obtain the corresponding Hamiltonian, we perform the usual Legendre transformation $H(q, \pi) = \sum \dot{q}_i(q, \pi)\pi_i - L(q, \pi)$, where $q_i = N, a, \phi$ are the generalized coordinates and $\pi_i = \pi_N, \pi_a, \pi_\phi$ are the associated canonical momenta, given by

$$
\pi_a = \frac{\partial L}{\partial \dot{a}} = 6aV_j \dot{\phi}^2 \frac{\dot{a}^2}{N^3}, \quad \pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = 6aV_j \dot{\phi}^2 \frac{\dot{a}^2}{N^3}, \quad \text{and} \quad \pi_N = \frac{\partial L}{\partial \dot{N}} = 0.
$$

(2.10)

Thus, after the determination of the generalized velocities as functions of $(q, \pi)$ and putting them into the Legendre transformation, we find the Hamiltonian corresponding to (1.10) to be

$$
H = N \left( \frac{3}{2\sqrt{6aV_j(\phi)}} \pi_a^{2/3} \pi_\phi^{2/3} + a^3 V(\phi) \right) \equiv NH.
$$

(2.11)
Hence we obtain the scalar constraint $\mathcal{H} \approx 0$ that leads to the Wheeler-DeWitt equation $\hat{\mathcal{H}} \psi(a, \phi) = 0$ after the canonical quantization transforming $q_i$ in the operator $\hat{q}_i \psi = q_i \psi$, and $\pi_i$ in the operator $\hat{\pi}_i \psi = -i\partial_i \psi/\partial q_i$, taking units such that $c = \hbar = 1$. This last quantization rule is precisely what leads to the problem of the fractional derivative, since the power of the momenta in the Hamiltonian (2.11) is the non-integer value $2/3$.

3 Fractional Wheeler-DeWitt Equation

To solve the problem cited above, we have to generalize the quantization rule

$$\pi^n_j \longrightarrow \hat{\pi}^n_j = (-i\partial_j)^n,$$  \hspace{1cm} (3.1)

for non-integer values of $n$. This can be done by introducing some fractional derivative [23], according to the physical conditions of the problem. There are several interesting examples of applications of fractional derivatives and fractional differential equations in physics [24]. For example, they are applied in anomalous diffusion dynamics [25], wave propagation in viscoelastic media [26], lossy partial differential acoustic wave equations [27], fractional quantum mechanics [28–30], and to write a generalization of the conservation of mass to represent non-linear flux in a control volume [31], just to mention a few.

However, the fractional derivatives generally used are defined by integrals and these integral fractional derivatives have a series of limitations and a counter-intuitive behaviour. For instance, the Leibniz rule is not generally true and the fractional derivative of a constant function may not be zero for them. For all these reasons, we shall use the so-called conformable fractional derivative, introduced by R. Khalil et al. (op. cit.), because this definition satisfies the basic properties a derivative is supposed to. It satisfies the Leibniz rule, linearity, the constant function has a vanishing conformable fractional derivative and it also reduces to the ordinary derivative for integer orders. It is a very simple and direct definition in general and potential applications to physics were discussed in [32]. This derivative has been applied in several physical problems and theories. In [33] a generalized version of classical mechanics known as fractional classical mechanics was construct using the conformable fractional derivative. In [34] this derivative was applied to solve a system of fractional coupled nonlinear Schrödinger equations. It has also been applied in Optics [35]. Applying the generalizations [36, 37] we can say that the $i$-th partial conformable fractional derivative of order $\alpha$, where $0 < \alpha \leq 1$, of the function $f : X \longrightarrow \mathbb{C}$, where $X = [a_1, +\infty) \times \cdots \times [a_n, +\infty) \subset \mathbb{R}^n$ is given by the limit

$$\frac{\partial^\alpha f}{\partial x_i^\alpha} = \lim_{h \to 0} \frac{f(x + h(x_i - a_i)^{1-\alpha} e_i) - f(x)}{h},$$  \hspace{1cm} (3.2)

where $x = \sum x_i e_i \in X$, $\{e_1, \ldots, e_n\}$ is the canonical basis of euclidean space $\mathbb{R}^n$ and the $a_i$ are real constants. This definition directly leads to the relation

$$\frac{\partial^\alpha f}{\partial x_i^\alpha} = (x_i - a_i)^{1-\alpha} \frac{\partial f}{\partial x_i}.$$  \hspace{1cm} (3.3)
Using this derivative we can generalize the canonical quantization rule for the momenta to any $0 < \alpha \leq 1$:

$$\pi_j^\alpha \rightarrow \hat{\pi}_j^\alpha = (-i)^\alpha \frac{\partial}{\partial x^\alpha}.$$  

(3.4)

Then, it follows from (3.3) and (3.4) that we can write

$$\hat{\pi}_a^{2/3} \hat{\pi}_\phi^{2/3} \psi = a^{1/3} (\phi - \phi_0)^{1/3} \frac{\partial^2 \psi}{\partial a \partial \phi},$$  

(3.5)

where we have chosen $a$ starting from 0 because it represents a non-negative quantity and $\phi_0$ is an arbitrary real constant we can chose later without loss of generality. It thus follow that the Wheeler-DeWitt equation $\hat{H} \psi = 0$ corresponding to the Lagrangian (1.10) is:

$$\frac{\partial^2 \psi}{\partial a \partial \phi} = -a^3 f(\phi) V(\phi) \psi, \quad \text{where} \quad f(\phi) = \left[ \frac{16 V_j}{9(\phi - \phi_0)} \right]^{1/3}.$$  

(3.6)

To separate variables we define

$$x = \frac{a^4}{4} - \int f(\phi) V(\phi) d\phi \quad \text{and} \quad \tau = \frac{a^4}{4} + \int f(\phi) V(\phi) d\phi,$$

(3.7)

rewriting the Wheeler-DeWitt equation as

$$\frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial x^2} + \psi = 0.$$  

(3.8)

To avoid any ambiguity, we stress that $\int f(\phi) V(\phi) d\phi$ represents the antiderivative of $f V$ with vanishing integration constant. The Wheeler-DeWitt equation (3.8) is of Klein-Gordon type. Thus the separation of variables becomes straightforward and leads to the basic plane wave solutions

$$\psi(x, y) = e^{i(kx - \omega \tau)},$$

(3.9)

where $k$ is a real separation constant and $\omega = \sqrt{1 + k^2}$.

### 4 Bohm-de Broglie Interpretation

In the last section we have found the solution of the Wheeler-DeWitt equation $\hat{H} \psi = 0$ corresponding to the problem (1.10) using the conformable fractional derivative to generalize canonical quantization, thus obtaining the plane wave solutions (3.9). We proceed now to the cosmological interpretation of these solutions. Since quantum mechanics has various interpretations, we can interpret our solution in many ways. For the reasons cited in the Introduction, we will apply the Bohm-de Broglie interpretation of quantum cosmology.

In the Bohm-de Broglie interpretation the dynamics is determined by the guidance equations:

$$\pi_j = \text{Im} \left( \frac{\partial_j \psi}{\psi} \right),$$

(4.1)

where $\pi_j$ is the classical momentum conjugated to the coordinate $q_j$ and Im means the imaginary part of a complex number or function. In this formalism we have to write
the classical momenta as functions of the coordinates and the generalized velocities to put them into the guidance equations. This procedure leads to deterministic equations of motion for $a$ and $\phi$ as if they were actual positions of particles in the canonical mechanics.

The solutions of these equations in phase space are called the *Bohmian trajectories* or the *quantum trajectories*. This is the point where the problem of time is avoided by Bohmian interpretation, because the guidance equations relate the generalized velocities $\dot{q}_j$ with the quantum wave $\psi$, thus bringing back in a quite natural way the cosmic time $t$ present as a derivative in $\dot{q}_j$. The analogous to the indefiniteness present in the probabilities in the Copenhagen interpretation is the uncertainty of the initial conditions of the guidance equations for Bohmian interpretation. Therefore, to completely determine the dynamics of a problem in Bohmian interpretation, all we need to know are the guidance equations together with their initial conditions.

In this formalism it is usual to explicitly write a quantum contribution to the classical potential known as the quantum potential. Writing $\psi$ in the polar form $\psi = R e^{iS}$ we find the equations

$$-\frac{1}{a^3 f(\phi)} \frac{\partial S}{\partial a} \frac{\partial S}{\partial \phi} + \frac{1}{a^3 f(\phi) R \partial a \partial \phi} \frac{\partial^2 R}{\partial a} + V(\phi) = 0, \tag{4.2}$$

$$\frac{1}{R} \frac{\partial R}{\partial a} \frac{\partial S}{\partial \phi} + \frac{1}{R} \frac{\partial R}{\partial \phi} \frac{\partial S}{\partial a} + \frac{\partial^2 S}{\partial a \partial \phi} = 0. \tag{4.3}$$

In analogy with the Bohmian Quantum Mechanics of the Schrödinger equation, we define the quantum potential of the problem we are dealing with as

$$Q = \frac{1}{a^3 f(\phi) R \partial a \partial \phi}. \tag{4.4}$$

The basic solution (3.9) leads to a null quantum potential. So to avoid this we will take a linear combination of basic solutions:

$$\psi(x, y) = (e^{ik_1 x - i\omega_1 \tau} + e^{ik_2 x - i\omega_2 \tau})/\sqrt{2}. \tag{4.5}$$

For this solution we find $R = \sqrt{\psi^* \psi} = \{1 + \cos[(k_1 - k_2)x - (\omega_1 - \omega_2)\tau]\}^{1/2}$ and the corresponding quantum potential is

$$Q = \frac{1}{2}(\omega_1 \omega_2 - k_1 k_2 - 1) V(\phi). \tag{4.6}$$

Observe that expanding $(k_1^2 - k_2^2)^2 > 0$ (because $k_1 = k_2$ means we have only one wave instead of a superposition) we can see $(\omega_1 \omega_2 - k_1 k_2 - 1) > 0$, so that the quantum and the classical potential have both the same sign and they are proportional to each other. Although the quantum potential $Q$ for superposition of plane waves is null or proportional to the classical potential $V$, it is interesting to observe that we have a quantum dynamics different from the classical one, leading to bouncing universes not allowed classically, as we shall see.

To determine the guidance equations, we invert (2.10) obtaining

$$\dot{a} = \frac{N^3}{6aV_j(\phi)} \pi_a^{-1/3} \pi_{\phi}^{2/3} \quad \text{and} \quad \dot{\phi} = \frac{N^3}{6aV_j(\phi)} \pi_a^{2/3} \pi_{\phi}^{-1/3}. \tag{4.7}$$
Thus, using (4.1) for the solution (4.5) we find

$$\pi_a = -\bar{\omega} + \bar{k}a^3$$

and

$$\pi_\phi = -(\bar{\omega} + \bar{k})f(\phi)V(\phi),$$

(4.8)

where $\bar{k} = \frac{1}{2}(k_1 + k_2)$ and $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$. Fixing the lapse function as $N = 1$ (cosmological time), defining $\alpha \equiv \ln a$, as it is conventional in cosmological dynamical systems, and using (4.8) the dynamical system of equations of motion (4.7) becomes:

$$\dot{\alpha} = -\frac{2^{5/9}3^{-7/9}}{9} \left[ \frac{(\bar{\omega} + \bar{k})}{\bar{\omega} - \bar{k}} \right]^{1/3} e^{-7\alpha/3}(\bar{\omega} - \bar{k})^{-2/9}V_j^{-1/9}(\phi)V^2/3(\phi),$$

(4.9)

$$\dot{\phi} = -2^{-7/9}3^{-1/9} \left[ \frac{(\bar{\omega} - \bar{k})}{\bar{\omega} + \bar{k}} \right]^{1/3} e^{5\alpha/3}(\phi - \phi_0)^{1/9}V_j^{-4/9}(\phi)V^{-1/3}(\phi).$$

(4.10)

It is interesting to observe that this is an autonomous dynamical system, because we have two equations of first order and the right hand side does not explicitly depends on time. This is a consequence of the structure of the Wheeler-DeWitt equation $\hat{H}\psi = 0$, that differs from the Schrödinger equation in the time dependence. The Schrödinger equation has the term $i\hbar \partial\psi/\partial t$ while the Wheeler-DeWitt equation is stationary, so then the solution of the Wheeler-DeWitt equation is also stationary and as a consequence, the dynamical system of equations is autonomous.

5 Singular Quantum Trajectories

Thanks to the structure of the Wheeler-DeWitt equation (3.6) we have determined the autonomous dynamical system (4.9), (4.10). This system maintains absolute freedom over the potentials $V_j(\phi)$ and $V(\phi)$. Consequently the dynamics has a strong dependence on the choice of these potentials. In this section, we will see that the most obvious choices for $V_j$ and $V$ lead only to singular solutions. Choosing $V_j = 1$, which is the simplest possible choice, we can study the behaviour of various different potentials, thus obtaining Figure 1. Observe that $\alpha \to -\infty$ is equivalent to the singularity $a \to 0^+$. The dynamical system plot $\alpha \times \phi$ for the potentials $V = \text{const.}$ and $V = e^{\gamma \phi}$ above show that all the solutions go to the singularity, thus presenting a very similar behaviour and not being able to avoid the singularity. For the other dynamical systems with the usual power-law potentials $V = m\phi^2$ and $V = \lambda\phi^4$, there are three possibilities: the solutions can come from the singularity, they can evolve to the singularity or they can stop at the attractor $\phi = 0$ that represents stationary states. Therefore, in any case, we obtain singular or stationary solutions. The reason why we have made such simple choices for the constants $\gamma$, $m$ and $\lambda$ is because these choices do not really change the qualitative analysis of whose trajectories avoid the singularity and whose ones do not. In other words, all the possible cases above do not avoid the singularity, and the reason is we have to make a different choice for the potentials $V_j(\phi)$ and $V(\phi)$. 

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Figure 1. Plot of the dynamical system $\alpha \times \phi$ for $V_j = 1$ and four different potentials.

6 Non-singular Quantum Trajectories

To obtain non-singular Bohmian trajectories, we have to make very particular choices for the functions $V_j$ and $V$. In this work, we have made the choices

$$V_j(\phi) = \frac{1}{\phi^3(\phi - \phi_0)^2} \quad \text{and} \quad V(\phi) = \lambda \phi^4. \quad (6.1)$$

With these potentials, the dynamical system $(4.9)$ and $(4.10)$ becomes

$$\dot{\alpha} = -2^{5/3} - 7/3 \lambda^{2/3} \left[ \frac{(\bar{\omega} + \bar{k})^2}{\bar{\omega} - \bar{k}} \right]^{1/3} e^{-7\alpha/3} \phi^3, \quad (6.2)$$

$$\dot{\phi} = -2^{-7/3} \lambda^{-1/3} \left[ \frac{(\bar{\omega} - \bar{k})^2}{\bar{\omega} + \bar{k}} \right]^{1/3} e^{5\alpha/3} (\phi - \phi_0). \quad (6.3)$$

For this configuration, we obtain the graph on the left of Figure 2.

The flow of the dynamical system was obtained taking $\lambda = 1$, $\bar{k} = 1$, and $\phi_0 = -100$. The orientation of the flow indicates time evolution. We have numerically plotted some interesting Bohmian trajectories. In the left of Figure 2, the green curve indicates
Figure 2. In the left, we have the plot of the Bohmian quantum dynamical system $\alpha \times \phi$ for the bouncing condition (6.1) with some examples of bouncing (blue and red) and singular (yellow and green) trajectories. In this plot, we can see clearly that depending on the choice of the initial conditions for $\alpha$ and $\phi$, there is an infinity family of solutions that avoids the singularity $\alpha = -\infty$.

The initial conditions for the solutions showed here are the following. Red: $\alpha(0) = 0.02667$ and $\phi(0) = 2$; blue: $\alpha(0) = 0.08$ and $\phi(0) = 2$; green: $\alpha(0) = -0.0001$ and $\phi(0) = 2$; yellow: $\alpha(0) = -2$ and $\phi(0) = -1.1$. In the right, we show the comparison between the classical (dashed lines) and quantum (solid lines) trajectories in phase space.

a contracting universe that collapses to the singularity in a finite time. The yellow curve represents an expanding universe starting from the Big Bang singularity. The blue and the red curves represent bouncing models which means that they avoid the Big Bang singularity, with different initial conditions. In these trajectories, the universe contracts to a minimum radius and then expands on. This minimum radius depends on the initial condition over $a$ and $\phi$ and we have a relative calibration freedom over it. As we mentioned in section 4, the solutions have a strong dependence on the initial conditions, an expected behaviour for a Bohmian approach. Moreover, the only effect of changing the value of $\bar{k}$ is to shift the whole dynamical flow up (if $\bar{k}$ increases) or down (if $\bar{k}$ decreases). The graph on the right of Figure 2 shows the quantum (solid) and classical (dashed) trajectories put together with the same respective initial conditions. The classical trajectories were obtained solving the classical system of equations of motion (2.8) and (2.9) numerically using the same initial conditions of the correspondent quantum ones of same color. In Figure 2 it becomes clear that even the initial conditions that give us bouncing universes are singular in the classical regime (curves blue and red) and also that the singular quantum trajectories (yellow and green) are also singular in the classical theory, as expected. Therefore, we can conclude that our bouncing solutions are indeed quantum effects, as we claimed above. Finally, we can resume the above result by saying that with the choice (6.1) the Lagrangian

$$L = \sqrt{-g} \left[ \frac{1}{\phi^4(\phi - \phi_0)^2} G^\mu\nu \nabla_\mu \phi \nabla_\nu \phi - \lambda \phi^4 \right]$$  \hspace{1cm} (6.4)$$

leads to bouncing universes, thus avoiding the Big Bang singularity $a = 0$.

For the solutions showed in Figure 2, we can numerically plot the time evolution of $\alpha = \ln a$ and $\phi$, thus obtaining Figure 3. In this plot it becomes evident that the scale
factor for the bouncing (blue and red) solutions never reach the singular point \( a = 0 \) that corresponds to the asymptotic limit \( \alpha \to -\infty \). It is also easy to see that the universe near the bounce is indeed passing through a minimum point, coming from a contracting phase and going into an expanding phase. But we have to say that this bounces strongly depend on the choice of the initial conditions. Actually, we need a very specific choice for the initial condition \( \alpha(0) \) to get a bouncing universe. Otherwise, the solution goes to the singularity. And from the Figure 3 it is also clear that this choice determines the specific instant of time \( t_0 \) in which the bounce occurs and the associated minimum value of the “radius of the universe” \( e^{\alpha(t_0)} \). For instance, we can compare the blue and red curves, because both are non-singular. The red one has as its initial value \( \alpha(0) = 0.02667 \) and for the blue one \( \alpha(0) = 0.08 \). This makes the red solution reach a smaller minimum radius than the blue one, as we can see from Figure 3. So depending on the initial condition, we can fit the value of the minimum radius of the Universe to adjust it with some suitable physical condition and then make a trivial change in the initial value of time to enforce \( t_0 = 0 \), so that the bounce occurs exactly in the same time we expect the Big Bang to occur.

7 Conclusion

In this work, we have shown that the John Lagrangian of the Fab Four with a \( \lambda \phi^4 \) potential and the very specific function \( V_j(\phi) = \phi^{-3}(\phi - \phi_0)^{-2} \) can lead to a cosmological solution that never reaches the singularity, using the conformable fractional derivative and applying a Bohm-de Broglie interpretation of quantum mechanics to the result.

We have to stress the role of the conformable fractional derivative in this paper, because this fractional derivative has provided us a method to generalize the canonical quantization \( \pi_j^n = -i\partial_j^n \) for non-integer powers. This allowed us to obtain physical results by attributing a unambiguous meaning to \( \pi_a^{2/3}, \pi_\phi^{2/3} \) as a differential operator without using any change of variables, such as canonical transformations. Another advantage of this fractional derivative is the reduction of a fractional equation into a second-order differen-
tial equation, the most common physical equation of motion. This makes the choice of this fractional derivative seems physically quite natural.

Beyond this technical question, we emphasize that the Bohm-de Broglie interpretation of quantum mechanics is fundamental because it allows us to make a very precise analysis in terms of trajectories in phase space to conclude that our model in fact can tune to a bouncing model with no need of probabilities, and also not leading to a problem in the definition of time. Moreover, it is interesting to observe the crucial role of the free function $V_j$, a characteristic of the John Lagrangian of the Fab Four, and the potential $V$. The freedom to choose these functions is precisely what allowed us to tune the theory in order to obtain a bouncing universe. Therefore, we can say we obtained a kind of quantum self-tuning mechanism analog to that of the classical Fab Four theory, but with a crucial difference. The Fab Four avoids the problem of the fine-tuning of the cosmological constant. By the other hand, we have found a way to avoid the Big Bang singularity, of course, in the context of assuming a classical theory and particular interpretations. The interesting point is that both these adjustment mechanisms came from the same freedom to make the most physically suitable choices for the functions: the John coefficient $V_j$ and the potential $V$.

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