A NOTE ON GENERATING FUNCTIONS
FOR HAUSDORFF MOMENT SEQUENCES

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Abstract. For functions $f$ whose Taylor coefficients at the origin form a Hausdorff moment sequence we study the behaviour of $w(y) := |f(\gamma + iy)|$ for $y > 0$ ($\gamma \leq 1$ fixed).

1. Introduction and statement of the results

A sequence $\{a_k\}_{k \geq 0}$ of non-negative real numbers, $a_0 = 1$, is called a Hausdorff moment sequence if there is a probability measure $\mu$ on $[0, 1]$ such that

$$a_k = \int_0^1 t^k \, d\mu(t), \quad k \geq 0,$$

or, equivalently,

$$F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz},$$

and $F$ is its generating function.

It is well known (Hausdorff [2]) that a sequence $\{a_k\}_{k \geq 0}$ with $a_0 = 1$ is a Hausdorff moment sequence if and only if it is completely monotone, i.e.

$$\Delta^n a_k := \Delta^{n-1} a_k - \Delta^{n-1} a_{k+1} \geq 0, \quad k \geq 0, \quad n \geq 1,$$

where $\Delta^0$ is the identity operator: $\Delta^0 a = a$.

Let $T$ denote the set of such generating functions $F$. They are analytic in the slit domain $\Lambda := \mathbb{C} \setminus [1, \infty)$ and also belong to the set of Pick functions $P(-\infty, 1)$ (see Donoghue [1] for more information on Pick functions).

Wirths [5] has shown that $f \in T$ implies that the function $zf(z)$ is univalent in the half-plane $\text{Re} \, z < 1$, and recently the theory of universally prestarlike mappings...
has been developed, showing a close link to $T$; see [4]. Many classical functions belong to $T$ or are closely related to it. We mention only the polylogarithms

$$Li_\alpha(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha}, \quad \alpha \geq 0,$$

where $Li_\alpha(z)/z \in T$ and which we are going to study somewhat closer in the sequel.

The main result in this paper is

**Theorem 1.1.** For $f \in T$ we have

$$\text{Re} \frac{f(\gamma + iy)}{f(\gamma + iy_2)} \geq 1, \quad \gamma \in (-\infty, 1], \; 0 < y_1 \leq y_2.$$  \hfill (1.1)

This relation does not hold, in general, for $\gamma > 1$.

Theorem 1.1 has the following immediate consequence.

**Corollary 1.2.** For $f \in T$ and $\gamma \in (-\infty, 1]$ fixed, the function $|f(\gamma + iy)|$ is monotonically decreasing with $y > 0$ increasing.

In the case $\gamma = 0$ Theorem 1.1 admits a slight generalization. It is well-known and easy to verify that $T$ is invariant under the Hadamard product: if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \in T, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k \in T,$$

then also

$$(f \ast g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k \in T.$$  

**Theorem 1.3.** For $f, g \in T$ we have

$$\text{Re} \left( \frac{f \ast g(iy)}{f(iy)} \right) \geq 1, \quad y > 0.$$  

Therefore, under the same assumption,

$$|f(iy)| \leq |(f \ast g)(iy)|, \quad y > 0. \hfill (1.2)$$

For the polylogarithms and $0 < \alpha \leq \beta$ it is clear that $Li_\beta = Li_\alpha \ast Li_{\beta - \alpha}$, so that we get

**Corollary 1.4.** For $0 \leq \alpha < \beta$

$$|Li_\alpha(iy)| \leq |Li_\beta(iy)|, \quad y > 0.$$  

This result can also be obtained and even strengthened using Corollary 1.2 and the deeper relation

$$\frac{Li_\alpha}{Li_\beta} \in T, \quad 0 \leq \alpha \leq \beta,$$

recently established in [4].

For a certain subset of $T$ we can go one step beyond Corollary 1.2 as far as the behaviour of $|f(iy)|$ for $y > 0$ is concerned.
Theorem 1.5. Let

\[ f(z) = \int_0^1 \frac{\sigma(t)dt}{1 - tz}, \]

where \( \sigma \in C^1((0,1)) \) is positive and with \( t\sigma'(t)/\sigma(t) \) decreasing. Then, for \( w(y) := |f(iy)| \), the function \( \frac{yw'(y)}{w(y)} \) decreases with \( y > 0 \) increasing.

Fundamental for the proof of Theorem 1.5 is the following result, which is based on a general theorem in [4].

Theorem 1.6. Let \( f \) be as in Theorem 1.5. Then, for \( x \in [0,1] \),

\[ \frac{f(z)}{f(xz)} \in T. \]

One can show that the conclusion of Theorem 1.6 is not generally valid for \( f \in T \). However, for the functions \( g_\alpha(z) := \frac{1}{z} \text{Li}_\alpha(z), \alpha > 0 \), we have

\[ g_\alpha(z) = \frac{1}{\Gamma(a)} \int_0^1 \frac{\log^{a-1}(1/t)}{1 - tz} dt, \]

for which the assumptions of Theorem 1.5 are fulfilled. Thus both Theorem 1.5 and Theorem 1.6 apply to \( g_\alpha \).

2. Proofs

We first note that the convex set \( T \) satisfies the condition of the main theorem in [3], which for the present case can be stated as follows:

Lemma 2.1. Let \( \lambda_1, \lambda_2 \) be two continuous linear functionals on \( T \) and assume that \( 0 \notin \lambda_2(T) \). Then the range of the functional

\[ \lambda(f) := \frac{\lambda_1(f)}{\lambda_2(f)} \]

over \( T \) equals the set

\[ \left\{ \lambda \left( \frac{\rho}{1 - t_1 z} + \frac{1 - \rho}{1 - t_2 z} \right) : \rho, t_1, t_2 \in [0,1] \right\}. \]

Proof of Theorem 1.1. First we note that it is enough to prove (1.1) for \( \gamma = 1 \) only. This is because \( f \in T \) implies \( f(z - \delta)/f(-\delta) \in T \) for all \( \delta > 0 \). In Lemma 2.1 we choose \( \lambda_j(f) := f(1 + iy_j), j = 1, 2 \). Since \( \text{Im} f(z) > 0 \) for \( f \in T \) and \( \text{Im} z > 0 \), it is clear that \( 0 \notin \lambda_2(T) \). Lemma 2.1 now implies that for the proof of Theorem 1.1 we only need to show that the expression

\[ \frac{\rho}{1 - t_1 - it_1 y_1} + \frac{1 - \rho}{1 - t_2 - it_2 y_1}, \rho, t_1, t_2 \in [0,1], \]

is located in the half-plane \( \{w : \text{Re} w \geq 1\} \). To simplify this expression we set \( \kappa := (1 - \rho)/\rho, \tau := y_1/y_2 \). Then our claim is

\[ \text{Re} q(\kappa, y, \tau, t_1, t_2) \geq 1, \quad \kappa \geq 0, \quad y > 0, \quad t_1, t_2, \tau \in [0,1], \]
where

\[ q(\kappa, y, \tau, t_1, t_2) = \frac{1}{1 - t_1 - i\tau y t_1} + \frac{\kappa}{1 - t_1 - i\tau y t_2}, \]

\[ \frac{1}{1 - t_1 - i\tau y t_1} + \frac{\kappa}{1 - t_2 - i\tau y t_2}. \]

Note that by symmetry we may assume that \( t_1 \leq t_2 \). For fixed \( y, \tau, t_1, t_2 \) the values of \( w(\kappa) := q(\kappa, y, \tau, t_1, t_2), \kappa \geq 0, \) form a circular arc connecting the points \( w(0) = v(t_1) \) and \( w(\infty) = v(t_2) \), where

\[ v(t) = \frac{1 - t - i\tau y t}{1 - t - i\tau y t}. \]

It is easily checked that under our assumptions for \( y \) and \( \tau \) the function \( \text{Re} v(t) \) increases with \( t \in [0, 1] \) and, in particular, \( \text{Re} v(t) \geq \text{Re} v(0) = 1. \) This implies that \( 1 \leq \text{Re} w(0) \leq \text{Re} w(\infty). \)

We will prove that \( \text{Re} w'(0) \geq 0. \) Once this is done a simple geometric consideration shows that under these circumstances the circular arc \( w(\kappa), \kappa \geq 0, \) cannot leave the half-plane \( \{w : \text{Re} w \geq 1\} \), which then completes the proof of (1.1).

Calculation yields

\[ \text{Re} w'(0) = (1 - \tau)(t_2 - t_1)y^2 \frac{Z}{N}, \]

where

\[ Z = t_1^* t_2^*(t_2 - t_1) + (t_1 t_2 + t_2^* t_1^*)t_1^* t_2^* \tau + t_1 t_2 y^2 \tau (t_1 t_2^* + t_2^* t_1^* - \tau(t_2 - t_1)), \]

\[ N = ((1 - t_1)^2 + (t_1 y \tau)^2)((1 - t_2)^2 + (t_2 y \tau)^2)((1 - t_2)^2 + (t_2 y \tau)^2), \]

and \( t_j^* := 1 - t_j. \) Here all terms are non-negative (note that \( s(\tau) := t_1 t_2^* + t_2^* t_1^* - \tau(t_2 - t_1) \) decreases with \( \tau \) and is therefore not smaller than \( s(1) = 2t_1 t_2^* \geq 0). \)

It remains to show that (1.1) does not hold, in general, for \( \gamma > 1. \) Let \( \gamma = 1 + \varepsilon, \varepsilon > 0, \) and choose

\[ f(z) := \frac{1}{1 + 2\varepsilon} + \frac{2\varepsilon}{1 + 2\varepsilon} \frac{1}{1 - z} \in T. \]

Then, using \( y_1 = \varepsilon, y_2 = 1, \)

\[ \text{Re} \frac{f(\gamma + i\varepsilon)}{f(\gamma + i)} = \frac{2\varepsilon}{1 + \varepsilon^2} < 1. \]

**Proof of Theorem 1.3.** If

\[ g(z) = \int_0^1 \frac{d\mu(t)}{1 - tz}, \]

then

\[ \frac{(f * g)(iy)}{f(iy)} = \int_0^1 f(iy) \frac{d\mu(t)}{f(iy)} \frac{d\mu(t)}{d\mu(t)}, \]

which is a convex combination of the values of \( f(iy)/f(iy). \) By Theorem 1.1 these are all in the half-plane \( \{w : \text{Re} w \geq 1\}. \)

For the proof of Theorem 1.6 we need the following result from [4].
Lemma 2.2. Let $f, g \in \mathcal{T}$ be represented by

$$f(z) = \int_0^1 \frac{\varphi(t)dt}{1-tz}, \quad g(z) = \int_0^1 \frac{\psi(t)dt}{1-tz}$$

with non-negative Borel functions $\varphi, \psi$ on $(0, 1)$. If $\varphi(t)\psi(s) \geq \varphi(s)\psi(t)$ holds for all $0 < s < t < 1$, then $f/g \in \mathcal{T}$.

Proof of Theorem 1.6. We have

$$f(xz) = \int_0^1 \frac{\sigma(t)dt}{1-txz} = \int_0^1 \frac{\sigma^*(t)dt}{1-tz},$$

with

$$\sigma^*(t) := \begin{cases} \frac{1}{x} \sigma(t/x), & 0 < t \leq x, \\ 0, & x < t < 1. \end{cases}$$

The condition

$$(2.1) \quad \sigma(t)\sigma^*(s) \geq \sigma(s)\sigma^*(t), \quad 0 < s < t < 1,$$

is immediately fulfilled if $t > x$. Otherwise we are left with

$$\sigma(t)\sigma(s/x) \geq \sigma(s)\sigma(t/x), \quad 0 < s < t \leq x.$$

This requires that $\sigma(t)/\sigma(t/x)$ increases with $t$. Taking logarithms and differentiating w.r.t. the variable $t$, we find as a necessary and sufficient condition for (2.1) that $t\sigma'(t)/\sigma(t)$ decreases for $t$ increasing. The result now follows from Lemma 2.2. □

Proof of Theorem 1.5. We apply Theorem 1.1 to the function $F$ of Theorem 1.6. Then, for $x, \tau \in (0, 1)$, we get

$$\left| \frac{\frac{f(iy\tau)f(iyx)}{f(iyx\tau)f(iy)}} \right| \geq 1, \quad y > 0.$$

Taking logarithms we obtain

$$(\log w(y) - \log w(xy)) - (\log w(\tau y) - \log w(x\tau y)) \leq 0.$$

Dividing by $1-x$ and letting $x \rightarrow 1-0$ yield

$$\frac{yw'(y)}{w(y)} \leq \frac{\tau yw'(\tau y)}{w(\tau y)},$$

which implies the assertion. □

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