LOCUS CONFIGURATIONS AND ∨-SYSTEMS

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Abstract. We present a new family of the locus configurations which is not related to ∨-systems thus giving the answer to one of the questions raised in [1] about the relation between the generalised quantum Calogero-Moser systems and special solutions of the generalised WDVV equations. As a by-product we have new examples of the hyperbolic equations satisfying the Huygens’ principle in the narrow Hadamard’s sense. Another result is new multiparameter families of ∨-systems which gives new solutions of the generalised WDVV equation.

1. Introduction

In the paper [1] (see also [2]) a mysterious relation between the configurations of hyperplanes appeared in the theory of generalised quantum Calogero-Moser systems (locus configurations [3]) and the so-called ∨-systems describing the special solutions of the generalised WDVV equations has been observed. In this paper we investigate this relation further.

Our first result is a new family of the locus configurations, which are not related to the WDVV equations (at least, in the way described in [1]). This shows that the relation between the locus configurations and ∨-systems is not general and is applied only to a special subclass of the locus configurations, thus answering one of the questions raised in [1].

Another interesting feature of the new family is that it gives the first examples of the locus configurations in dimension more than two for which corresponding Baker-Akhiezer functions do not satisfy the so-called ”old axiomatics” (see [3] for details). It is plausible that the subclass of the locus configurations related to ∨-systems is the one with old axiomatics.

As a by-product we have new examples of hyperbolic equations satisfying the Huygens’ principle (in Hadamard’s narrow sense). The general relation between locus configurations and Huygens’ principle has been established in [3]. We should mention that as a particular two-dimensional case our family contains the configuration first discovered by Yu.Berest and I.Lutsenko in the relation with Huygens’ principle [4].

Another our result is two new multiparameter families of the ∨-systems in dimension n. As a corollary we obtain new solutions of the generalised WDVV equation.

2. Generalised quantum CMS problem and locus configurations

The famous Calogero-Moser-Sutherland integrable model describes a pairwise interaction of N particles on the line with the potential which in trigonometric case

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U = g \sum_{i<j} \frac{\omega^2}{\sin^2 \omega(x_i - x_j)}.

Olshanetsky and Perelomov [5] were the first to consider the generalisations of this problem related to any root system \( R \in \mathbb{R}^n \). Corresponding operator has the form

\[
L = -\Delta + \sum_{\alpha \in \mathcal{A}} \frac{g_\alpha \omega^2}{\sin^2 \omega(\alpha, x)}
\]  

(1)

where \( \mathcal{A} \) is a set of positive roots and \( g_\alpha \) are some parameters prescribed to the roots in a way invariant under the action of the corresponding Weyl group \( W \). Its quantum integrability in the sense of existence of \( n \) pairwise commuting quantum integrals has been shown in full generality first by Heckman and Opdam (see [6] and references therein).

In the papers [7], [8] we have shown that if the parameters have a special form \( g_\alpha = m_\alpha (m_\alpha + 1)(\alpha, \alpha) \) with integer \( m_\alpha \), then the operator (1) has more than \( n \) quantum integrals, which by definition means the algebraic integrability (see [8] for precise formulations). In that case the operator (1) can be intertwinied with \( L_0 = -\Delta \): there exists a differential operator \( D \) such that

\[
L D = D L_0.
\]

Surprisingly enough it turned out [10] (see also [3]) that the same properties are valid for the operators (1) for other finite configurations \( \mathcal{A} \). The first examples of such non-root configurations have been found in [10]. They consist of the following vectors in \( \mathbb{R}^{n+1} \):

\[
A_{n-1,1}(m) = \begin{cases} 
e e_i - e_j, & 1 \leq i < j \leq n, \text{ with multiplicity } m^*, \\
e e_i - \sqrt{m} e_{n+1}, & i = 1, \ldots, n \text{ with multiplicity } 1, 
\end{cases}
\]

Here \( m \) is an integer parameter, and the multiplicity \( m^* \) is the maximum of \( m \) and \( -1 - m \). We should mention that in [3, 10] we have denoted this system as \( A_n(m) \). The reason for the change of notation will be clear from section 3 below.

The question of quantum algebraic integrability for the operators

\[
L = -\Delta + \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha) \omega^2}{\sin^2 \omega(\alpha, x)}
\]  

(2)

and their rational limits

\[
L = -\Delta + \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}
\]  

(3)

for a general finite set of noncollinear vectors \( \mathcal{A} \) with prescribed multiplicities led to the notion of the locus conditions and locus configurations [3].

In the rational case we say that the potential

\[
u(x) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}
\]

satisfies the locus conditions if

\[
u(x) - \nu(s_\alpha(x)) = O((\alpha, x)^{2m_\alpha})
\]  

(4)

near every hyperplane \( P_\alpha : (\alpha, x) = 0 \). Here \( s_\alpha \) stands for the reflection with respect to this hyperplane. In the trigonometric case we assume additionally that the set
\(\mathcal{A} \subset \mathbb{C}^n\) generates a \(\mathbb{Z}\)-lattice of rank \(\leq n\) and demand the locus conditions (1) to be valid for any hyperplane \((\alpha, x) = \pi l, l \in \mathbb{Z}\), where the potential has a pole. As it was shown in (1) these conditions are sufficient for the existence of the intertwining operator \(\mathcal{D}\) (and thus for the algebraic integrability).

We will call a finite set \(\mathcal{A}\) of noncollinear vectors in a (complex) Euclidean space a **locus configuration** if the corresponding potential of the operator (2) (and thus of the operator (3)) satisfies the locus conditions (4).

The list of all known so far locus configurations in dimension more than 2 is presented in (3). Besides the root systems and the configuration \(A_{n,1}(m)\) mentioned above it contains the following family

\[
C_{n,1}(m,l) = \begin{cases} 
  e_i \pm e_j, & 1 \leq i < j \leq n, \text{ with multiplicity } k^*, \\
  2e_i, & i = 1, \ldots, n, \text{ with multiplicity } m^*, \\
  e_i \pm \sqrt{k}e_{n+1}, & i = 1, \ldots, n, \text{ with multiplicity } 1, \\
  2\sqrt{k}e_{n+1} & \text{with multiplicity } l^*,
\end{cases}
\]

Here \(k, l, m\) are integer parameters related as \(k = \frac{2m+1}{2l+1}\), and \(k^*, l^*, m^*\) have the same meaning as in \(A_{n,1}(m)\) case.

For all these configurations the corresponding Baker-Akhiezer function satisfies the so-called "old axiomatics" introduced in (3) which implies the new axiomatics (3) valid for any locus configuration. In dimension two there are examples of the locus configurations first discovered by Berest and Lutsenko for which old axiomatics is not valid. The question whether it is true or not in dimension more than two was open until now. In the next section we will give the answer to this question by presenting a new family of the locus configurations which do not satisfy the old axiomatics. Actually we will classify all the locus configurations of a certain type and show that besides the known cases the list contains one new interesting family.

### 3. Locus configurations of \(A\) type

Let us consider a system \(\mathcal{A} \subset \mathbb{C}^n\) which consists of the vectors

\[\alpha = \mu_i e_i - \mu_j e_j \quad (i < j) \quad \text{with} \quad m_\alpha = m_{ij} \in \mathbb{Z}_+ \quad \text{(5)}\]

Here \(e_1, \ldots, e_n\) is a standard basis in \(\mathbb{C}^n\) and \(\mu_i \in \mathbb{C}^\times\) are some prescribed parameters. We suppose that all the vectors \(\alpha \in \mathcal{A}\) are non-isotropic, i.e. \(\mu_i^2 + \mu_j^2 \neq 0\) for all \(i, j\). Altogether we have \(n(n-1)/2+n\) parameters \(m_{ij}, \mu_i\). The question we address here is when such a system \(\mathcal{A}\) is a locus configuration, i.e. when the potential in (3) satisfies the locus conditions (4). Note that the complex orthogonal group acts naturally on the set of all locus configurations. In particular, the class of configurations we consider is invariant under the action of the group \(W = S_n \ltimes (\mathbb{Z}_2)^n\) generated by permutations of the coordinates and sign flips. In our classification below we will not differ between configurations equivalent modulo \(W\).

To start with, we recall that according to (3) \(\mathcal{A}\) is a locus configuration if and only if all its two-dimensional subsystems satisfy the locus conditions (see theorem 4.1 in (3)). So, let us start from considering a system \(\mathcal{A}^0\) consisting of three vectors

\[\alpha = ae_1 - be_2, \quad \beta = be_2 - ce_3, \quad \gamma = ae_1 - ce_3\]

with multiplicities \((m_\alpha, m_\beta, m_\gamma) = (m, l, k)\).
Proposition 1. The system $\mathcal{A}^0 \subset \mathbb{C}^3$ as above satisfies the locus conditions in the following cases only:

1. $a = b = c$ and $m = l = k$ (Coxeter case);  
2. $a = b$, $k = l = 1$, $m > 1$, $c = a \sqrt{m}$;  
3. $a = b$, $k = l = 1$, $m \geq 1$, $c = a \sqrt{-1 - m}$;  
4. $m = l = k = 1$, $a^2 + b^2 + c^2 = 0$.  

(Notice that the partial case $m = 1$ of (3) appears also in the family (4).)

Proof. This result is similar to the classification of all three-line locus configurations on the plane (Theorem 4.4 from [3]), and can be proven in a similar way. To begin with, notice that the conditions (4) are equivalent to the vanishing along $\Pi\alpha$ of some first odd normal derivatives of the potential $u(x)$. More explicitly, they can be reformulated as follows: for all $s = 1, \ldots, m\alpha$

$$
\sum_{\beta \in \mathcal{A}\setminus\alpha} \frac{m_\beta (m_\beta + 1)(\beta, \beta)(\alpha, \beta)^{2s-1}}{(\beta, x)^{2s+1}} \equiv 0 \quad \text{along } \Pi\alpha. 
$$

(10)

Applying this for our particular case, we arrive at the following conditions:

$$
m_\beta (m_\beta + 1)(\beta, \beta)(\alpha, \beta)^{2s-1} = m_\gamma (m_\gamma + 1)(\gamma, \gamma)(\alpha, \gamma)^{2s-1} \quad (s = 1, \ldots, m\alpha),
$$

(11)

plus similar conditions for $\beta$ and $\gamma$. Suppose that $m\alpha > 1$, then from the first two relations in (11) we deduce immediately that $(\alpha, \beta)^2 = (\alpha, \gamma)^2$, or $a^2 = b^2$, and hence $m_\beta = m_\gamma$. As a result, we see that if at least two of the multiplicities $(m\alpha, m_\beta, m_\gamma)$ are greater than 1, then they all must be the same and $a^2 = b^2 = c^2$, which up to sign flips gives us the Coxeter case (1).

Another possibility is $(m\alpha, m_\beta, m_\gamma) = (m, 1, 1)$ with $m > 1$. As we know already, in this case $a^2 = b^2$ which provides the conditions (11). The two remaining locus conditions are:

$$
m_\alpha (m_\alpha + 1)(\alpha, \alpha)(\beta, \alpha) = m_\gamma (m_\gamma + 1)(\gamma, \gamma)(\beta, \gamma),
$$

$$
m_\alpha (m_\alpha + 1)(\alpha, \alpha)(\gamma, \alpha) = m_\beta (m_\beta + 1)(\beta, \beta)(\gamma, \beta),
$$

which gives $m(m + 1)(a^2 + b^2)b^2 = 2(c^2 + a^2)c^2$ and leads to the families (2) and (3).

Finally, for the remaining case $(m\alpha, m_\beta, m_\gamma) = (1, 1, 1)$ we have three locus conditions

$$
(b^2 + c^2)b^2 = (c^2 + a^2)a^2 \\
(c^2 + a^2)c^2 = (a^2 + b^2)b^2 \\
(a^2 + b^2)a^2 = (b^2 + c^2)c^2.
$$

It is easy to see that there are only two possibilities: either $a^2 = b^2 = c^2$ or $a^2 + b^2 + c^2 = 0$. The first one gives us the Coxeter case while the second one leads to the family (4).
The main point is that they cannot all be of the type (9). Indeed, in that case we have only two possibilities: either $\mu_1 = \cdots = \mu_n, m_{ij} \equiv m$, (Coxeter case); (2) $\mu_1 = \cdots = \mu_{n-1}, m_{ij} \equiv m$ for all $i, j < n$, $m_{in} = 1$ for all $i$, $\mu_n = \mu_1 \sqrt{m}$; (3) $\mu_1 = \cdots = \mu_{n-2} = \mu$, $m_{ij} \equiv m$ for all $1 \leq i, j \leq n-2$, $m_{i,n-1} = m_{i,n} = m_{n-1,n} \equiv m_n$ for all $i \leq n-2$, $\mu_{n-1} = \mu \sqrt{m}$, $\mu_n = \mu \sqrt{m} - 1 - m$; (4) $n = 3$, $m_{ij} \equiv 1$, $\mu_1^2 + \mu_2^2 + \mu_3^2 = 0$.

Proof. As we mentioned already, all we have to do is to check the locus conditions for every two-dimensional subsystem in $\mathcal{A}$. So, let us take any two-dimensional plane $\Pi$ in $\mathbb{C}^{n}$ and consider the set $\mathcal{A}_0 = \mathcal{A} \cap \Pi$ assuming that it is nonempty. Then there are three possibilities:

1) $\mathcal{A}_0$ consists of one vector;
2) $\mathcal{A}_0$ consists of two perpendicular vectors;
3) $\mathcal{A}_0$ consists of three vectors $\mu_i e_i - \mu_j e_j, \mu_j e_j - \mu_k e_k, \mu_i e_i - \mu_k e_k$ for some $i < j < k$.

In the first two cases $\mathcal{A}_0$ is a locus system for trivial reasons. So, essentially, the locus conditions for the system $\mathcal{A}$ reduce to the requirement that for all $i < j < k$ the subsystem $\mathcal{A}_0$ as in 3) must be one of those listed in Proposition 1.

Let us consider first the case $n = 4$. We have four different subsystems as in 3). The main point is that they cannot all be of the type (4). Indeed, in that case we would have that

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = \cdots = \mu_2^2 + \mu_3^2 + \mu_4^2 = 0$$

which would imply that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. Thus, either all of these subsystems are of the Coxeter type (4), or at least one of them is of type (5)–(8). In the latter case, if we suppose that $\mu_2 = \mu_3 = \mu$ and $\mu_1 = \sqrt{m} \mu$ then for $\mu_4$ we have only two possibilities: either $\mu_4 = \mu$, or $\mu_4 = \sqrt{1 - m}$. Therefore, all possible locus configurations for $n = 4$ are listed in the theorem. The general $n > 4$ case follows in a similar manner.

Notice that we classified all the systems (4) for which the corresponding rational potential (2) satisfies the locus conditions. As a matter of fact, the trigonometric version (3) of the resulting potentials will also satisfy the locus conditions. This is enough to check for all two-dimensional subsystems, which is not difficult. An explanation for this phenomenon (that the weaker rational locus conditions imply much stronger trigonometric ones) is due to the fact that from the very beginning the system (4) was affine-equivalent to the standard root system of type $A$.

As a result, the classification of all systems (2) satisfying trigonometric locus conditions leads to the same list from Theorem 1.

The families (2), (3) in Theorem $[\mathcal{F}]$ were found in $[\mathcal{G}]$–$[\mathcal{I}]$, it is the family $A_{n,1}(m)$ from the previous section. The family (4) is new. We will denote as $A_{n-1,2}(m)$ the corresponding system in $\mathbb{C}^{n+2}$:

$$A_{n-1,2}(m) = \begin{cases} e_i - e_j, & 1 \leq i < j \leq n, \\ e_i - \sqrt{me_{n+1}}, & i = 1, \ldots, n \\ e_i - \sqrt{1 - me_{n+2}}, & i = 1, \ldots, n \\ \sqrt{me_{n+1}} - \sqrt{1 - me_{n+2}} & \end{cases}$$

with multiplicity $m$, with multiplicity $1$, with multiplicity $1$, with multiplicity $1$. 


Notice that for $m = 1$ this system coincides with the system $A_{n,1}(-2)$ from the previous section. Notice also that $A_{n-1,2}(m)$ is contained in the hyperplane

$$x_1 + \cdots + x_n + \frac{1}{\sqrt{m}} x_{n+1} + \frac{1}{\sqrt{-1-m}} x_{n+2} = 0,$$

which leads to a locus configuration in $\mathbb{C}^{n+1}$. In particular, for $n = 2$ we obtain in this way a new locus configuration in dimension 3.

4. Generalised WDVV equations and $\vee$-systems.

The generalised WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations are the following overdetermined system of nonlinear partial differential equations:

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad i, j, k = 1, \ldots, n,$$

where $F$ is the $n \times n$ matrix constructed from the third partial derivatives of the unknown function $F = F(x^1, \ldots, x^n)$:

$$(F_m)_{pq} = \frac{\partial^3 F}{\partial x^m \partial x^p \partial x^q}.$$  \hfill (13)

In this form these equations have been presented by A.Marshakov, A.Mironov and A.Morozov, who showed that the Seiberg-Witten prepotential in $N = 2$ four-dimensional supersymmetric gauge theories satisfies this system [11] (for more recent developments see [12]). Originally these equations have appeared in topological field theory, their deep geometry and relations with integrable systems were first investigated by B.A.Dubrovin in [13].

In the papers [1], [2] the following special class of solutions to (12):

$$F^\mathfrak{A} = \sum_{\alpha \in \mathfrak{A}} (\alpha, x)^2 \log (\alpha, x)^2,$$

where $\mathfrak{A}$ be a finite set of covectors $\alpha$ in the space $V^*$ dual to a vector space $V$, has been investigated. It was shown that [14] satisfies the generalised WDVV equations if $\mathfrak{A}$ satisfies certain conditions which led to the notion of $\vee$-systems.

Let us give the definition of the $\vee$-systems following [1].

Let $V$ be a vector space (real or complex), $V^*$ be its dual, $\mathfrak{A} \subset V^*$ be a finite set of covectors which we assume to be non-collinear. Let’s introduce the following bilinear form

$$G^\mathfrak{A} = \sum_{\alpha \in \mathfrak{A}} \alpha \otimes \alpha.$$  \hfill (15)

We will assume that the form $G^\mathfrak{A}$ is non-degenerate, in the real case this equivalent to the fact that covectors $\alpha \in \mathfrak{A}$ generate $V^*$.

This means that the natural linear mapping $\varphi_{\mathfrak{A}} : V \to V^*$ defined by the formula

$$(\varphi_{\mathfrak{A}}(u), v) = G^\mathfrak{A}(u, v), \quad u, v \in V$$

is invertible. We will denote $\varphi_{\mathfrak{A}}^{-1}(\alpha), \alpha \in V^*$ as $\alpha^\vee$. By definition

$$\sum_{\alpha \in \mathfrak{A}} \alpha^\vee \otimes \alpha = Id$$

as an operator in $V^*$ or equivalently

$$\langle \alpha, v \rangle = \sum_{\beta \in \mathfrak{A}} (\alpha, \beta^\vee)(\beta, v).$$  \hfill (16)
for any $\alpha \in V^*, v \in V$.

Recall that for a pair of bilinear forms $F$ and $G$ on the vector space $V$ one can define an eigenvector $e$ as the kernel of the bilinear form $F - \lambda G$ for a proper $\lambda$:

$$(F - \lambda G)(v, e) = 0$$

for any $v \in V$. When $G$ is non-degenerate $e$ is the eigenvector of the corresponding operator $\tilde{F} = G^{-1}F$:

$$\tilde{F}(e) = G^{-1}F(e) = \lambda e.$$ 

Now let $\mathfrak{A}$ be as above any finite set of non-collinear covectors $\alpha \in V^*$, $G = G^\mathfrak{A}$ be the corresponding bilinear form (15), which is assumed to be non-degenerate, $\alpha^\vee$ are defined by (16). Define now for any two-dimensional plane $\Pi \subset V^*$ a form

$$G^\mathfrak{A}_\Pi(x, y) = \sum_{\alpha \in \Pi \cap \mathfrak{A}} (\alpha, x)(\alpha, y).$$ (17)

**Definition.** We will say that $\mathfrak{A}$ satisfies the $\vee$-conditions if for any plane $\Pi \subset V^*$ the vectors $\alpha^\vee, \alpha \in \Pi \cap \mathfrak{A}$ are the eigenvectors of the pair of the forms $G^\mathfrak{A}$ and $G^\mathfrak{A}_\Pi$. In this case we will call $\mathfrak{A}$ as $\vee$-system.

The $\vee$-conditions can be written explicitly as

$$\sum_{\beta \in \Pi \cap \mathfrak{A}} \beta(\alpha^\vee)\beta^\vee = \lambda\alpha^\vee,$$ (18)

for any $\alpha \in \Pi \cap \mathfrak{A}$ and some $\lambda$, which may depend on $\Pi$ and $\alpha$.

Geometrically we have three different cases:

1) If the plane $\Pi$ contains no more than one covector from $\mathfrak{A}$ then $\vee$-conditions are obviously satisfied (this means that these conditions should be checked only for a finite number of planes $\Pi$);

2) If the plane $\Pi$ contains only two covectors $\alpha$ and $\beta$ from $\mathfrak{A}$ then the condition (18) means that $\alpha^\vee$ and $\beta^\vee$ are orthogonal with respect to the form $G^\mathfrak{A}$:

$$\beta(\alpha^\vee) = G^\mathfrak{A}(\alpha^\vee, \beta^\vee) = 0;$$

3) If the plane $\Pi$ contains more than two covectors from $\mathfrak{A}$ this condition means that $G^\mathfrak{A}$ and $G^\mathfrak{A}_\Pi$ restricted to the plane $\Pi^\vee \subset V$ are proportional:

$$G^\mathfrak{A}_\Pi|_{\Pi^\vee} = \lambda(\Pi) \ G^\mathfrak{A}|_{\Pi^\vee}. $$ (19)

The $\vee$-conditions are known to be sufficient (in the real case - necessary and sufficient) for $F$ of the form (14) to satisfy the generalised WDVV (see [1,2]).

The natural examples of the $\vee$-systems are given by the Coxeter systems consisting of the normals to the reflection hyperplanes of some Coxeter group. It turned out that the locus configurations can be used to construct non-Coxeter examples of the $\vee$-systems. Namely for a given locus configuration $\mathcal{A}$ consisting from the vectors $\alpha$ in the Euclidean space $V$ with multiplicities $m_\alpha$ we can define a new set of vectors in $V \approx V^*$

$$\mathfrak{A} = \sqrt{m_\alpha} \alpha, \alpha \in \mathcal{A}.$$ 

A surprising fact discovered in [1] is that for all locus configurations described in the section 2 the corresponding sets $\mathfrak{A}$ satisfy the $\vee$-conditions. A natural question arose whether this is a common property of all locus configurations or not. Now we are ready to answer this question.
Proposition 2. The systems $\mathfrak{X}$ corresponding to the new family of the locus configurations $A_{n,2}(m)$ do not satisfy the $\lor$-conditions. Corresponding function (14) is a solution of the generalised WDVV equation only if $m = 1$.

To prove this let’s notice that the definition of the $\lor$-systems is affine invariant. This means that we can consider the projection of the corresponding system $\mathfrak{X}$ into the hyperplane $x_{n+2} = 0$, which has the form

$$\mathfrak{X}_{n-1,2}(m) = \begin{cases} \sqrt{m}(e_i - e_j), & 1 \leq i < j \leq n, \\ e_i - \sqrt{me_{n+1}}, & i = 1, \ldots, n \\ e_i, & i = 1, \ldots, n \\ \sqrt{me_{n+1}}. \end{cases}$$

A straightforward calculation shows that the $\lor$-conditions corresponding to the plane containing the vectors $e_i - \sqrt{me_{n+1}}$, $e_i$, $\sqrt{me_{n+1}}$ are satisfied if and only if $m = 1$ when we have the configuration of the type $A_{n,1}(-2)$. Notice that since the system $\mathfrak{X}_{n-1,2}(m)$ is real this implies that if $m \neq 1$ the corresponding function (14) does not satisfy the generalised WDVV equation according to the general result from [2].

In analogy with the previous section, it is natural to consider $\lor$-systems of $A$ type which consist of the covectors $\mu_{ij}(e_i - e_j) \in V^*$. Then the $\lor$-conditions imply some algebraic relations on the parameters $\mu_{ij}$. In case $n = 4$ we are able to give the complete solution.

Proposition 3. The system of $A_3$-type

$$\mathfrak{X} = \{\mu_{ij}(e_i - e_j), \ 1 \leq i < j \leq 4\}$$

satisfies the $\lor$-conditions if and only if

$$\mu_{12}\mu_{34} = \mu_{13}\mu_{24} = \mu_{14}\mu_{23}.$$  

The corresponding family of solutions of the generalised WDVV equation has the form

$$F = c_1c_2(x_1 - x_2)^2 \log(x_1 - x_2)^2 + c_2c_3(x_2 - x_3)^2 \log(x_2 - x_3)^2 + c_1c_3(x_1 - x_3)^2 \log(x_1 - x_3)^2 + c_1x_1^2 \log x_1^2 + c_2x_2^3 \log x_2^3 + c_3x_3^2 \log x_3^2,$$

with arbitrary $c_1, c_2, c_3$.

It is interesting that this family of $\lor$-systems can be extended to higher dimensions, though we are not sure whether or not this exhausts all possibilities.

Namely, let us consider the system $\mathfrak{X}_n(c) = \{\sqrt{c_i c_j}(e_i - e_j), \ 1 \leq i < j \leq n+1\}$ in $\mathbb{R}^{n+1}$ where $c_1, \ldots, c_{n+1}$ are arbitrary (positive) parameters. Without loss of generality, we may assume that $c_{n+1} = 1$ and restrict the system onto hyperplane $x_{n+1} = 0$. Thus we arrive at the following $n$-parametric family of configurations in $\mathbb{R}^n$:

$$\mathfrak{X}_n(c) = \begin{cases} \sqrt{c_i c_j}(e_i - e_j), & 1 \leq i < j \leq n, \\ \sqrt{c_i e_i}, & i = 1, \ldots, n. \end{cases}$$  (20)
Theorem 2. The system (20) satisfies ∨-conditions for any $c_1, \ldots, c_n$. The corresponding family of solutions of the generalised WDVV equation has the form

$$F = \sum_{i<j} c_i c_j (x_i - x_j)^2 \log(x_i - x_j)^2 + \sum_{i=1}^n c_i x_i^2 \log x_i^2.$$

Proof. Let us identify $V = \mathbb{R}^n$ with its dual using the standard Euclidean structure. Then the bilinear form $G = G^\mathfrak{A}$ associated to the system (20) according to the formula (15) looks as follows:

$$G(x, y) = \sum_{i<j} c_i c_j (x_i - x_j)(y_i - y_j) + \sum_{i=1}^n c_i x_i y_i.$$

The associated matrix which we will denote by the same symbol $G$ has the form

$$G = (1 + \sum_i c_i) C - c \otimes c,$$

where $C = \text{diag}(c_1, \ldots, c_n)$ and $(e \otimes e)_{ij} = c_i c_j$. A straightforward check shows that its inverse has the form

$$G^{-1} = (1 + \sum_i c_i)^{-1} (C^{-1} - e \otimes e),$$

where $e = (1, \ldots, 1)$ and $(e \otimes e)_{ij} \equiv 1$ for all $i, j$.

To verify ∨-conditions, we should deal with two-dimensional planes $\Pi$ containing at least two of the vectors $\alpha, \beta \in \mathfrak{A}$. Altogether we have the following 4 different types of such planes:

1. $\Pi = \langle e_i, e_j, e_i - e_j \rangle$;
2. $\Pi = \langle e_i - e_j, e_j - e_k, e_i - e_k \rangle$;
3. $\Pi = \langle e_i - e_j, e_k \rangle$;
4. $\Pi = \langle e_i - e_j, e_k - e_l \rangle$.

Let us consider the first case. Let us fix a basis in $\Pi$ as $\alpha = e_i$ and $\beta = e_j$. Then the corresponding plane $\Pi^\vee$ is spanned by $\alpha^\vee = G^{-1} \alpha$ and $\beta^\vee = G^{-1} \beta$. Using the explicit formula for $G^{-1}$ one easily finds that (up to a nonessential factor)

$$\alpha^\vee = (1, \ldots, 1 + c_i^{-1}, 1, \ldots, 1)$$

(with $c_i^{-1}$ appearing in the $i$-th component) and similarly

$$\beta^\vee = (1, \ldots, 1 + c_j^{-1}, 1, \ldots, 1).$$

Now we should check that the restrictions of the forms $G$ and $G_{\Pi}$ onto $\Pi^\vee$ are proportional. Here $G_{\Pi}$ is given by the formula

$$G_{\Pi}(x, y) = c_j c_j (x_i - x_j)(y_i - y_j) + c_i x_i y_i + c_j x_j y_j.$$

After some calculations one finds that

$$G(\alpha^\vee, \alpha^\vee) = (1 + \sum_k c_k)(1 + c_i^{-1}),$$

$$G(\beta^\vee, \beta^\vee) = (1 + \sum_k c_k)(1 + c_j^{-1}),$$

$$G(\alpha^\vee, \beta^\vee) = 1 + \sum_k c_k.$$
On the other hand, evaluating $G_{\Pi}$ we obtain that

\[
G_{\Pi}(\alpha^\vee, \alpha^\vee) = (1 + c_i + c_j)(1 + c_i^{-1}), \\
G_{\Pi}(\beta^\vee, \beta^\vee) = (1 + c_i + c_j)(1 + c_j^{-1}), \\
G_{\Pi}(\alpha^\vee, \beta^\vee) = 1 + c_i + c_j.
\]

This demonstrates that $G$ and $G_{\Pi}$ are proportional and gives $\vee$-condition for the case (1).

In case (3) we take $\alpha = e_i - e_j$, $\beta = e_k$ and have only to check that the Euclidean product $(\alpha^\vee, \beta)$ is zero. The latter becomes obvious after calculating $\alpha^\vee$ which is proportional to the vector $c_i^{-1}e_i - c_j^{-1}e_j$.

Two other cases are completely analogous. As a result, we conclude that the system (20) is a $\vee$-system for any values of the parameters $c_1, \ldots, c_n$ and the corresponding function (14) is a solution of the generalised WDVV equation.

When $c_1 = \cdots = c_k$ for some $k < n$ and $c_{k+1} = \cdots = c_n = 1$ the system (20) reduces to the configuration $A_k \ast A$ discovered by Berest and Yakimov (see [2]). For general $c_i$ the constructed solutions of the generalised WDVV equation seem to be new.

A natural question is what is the analogue of the family (20) for other classical root systems. The answer is given by the following family

\[
\mathfrak{N}_n(c) = \begin{cases} 
\sqrt{c_i c_j (e_i \pm e_j)}, & 1 \leq i < j \leq n, \\
\sqrt{2c_i (c_i + c_0)} e_i, & i = 1, \ldots, n.
\end{cases}
\]

One can easily check that the $\vee$-conditions are satisfied for arbitrary values of the parameters $c_0, c_1, \ldots, c_n$. The corresponding new solution of the generalised WDVV equation has the form

\[
F = \sum_{i < j} c_i c_j (x_i + x_j)^2 \log(x_i + x_j)^2 + \sum_{i < j} c_i c_j (x_i - x_j)^2 \log(x_i - x_j)^2 + \sum_{i=1}^n 2c_i (c_i + c_0) x_i^2 \log x_i^2.
\]

5. RELATION TO HUYGENS’ PRINCIPLE.

Let us consider the second order hyperbolic equation

\[
L \phi(t, x) = 0, \quad L = \Box_{N+1} + u(x),
\]

where $\Box_{N+1}$ is the D’Alembert operator, $\Box_{N+1} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_N^2}$.

J.Hadamard raised the question when such an equation has the fundamental solution located on the characteristic cone, or equivalently, when it satisfies the Huygens’ Principle in the narrow Hadamard’s sense. For the review of the current situation with this problem we refer to [3]. In particular, the theorem 6.1 from [3] claims that if $u(x)$ is a real rational potential (3) related to a locus configuration, then the equation (22) satisfies the Huygens’ Principle for large enough odd $N$. More precisely, if $u(x) = u(x_1, \ldots, x_n)$ is a potential (3) related to a locus configuration $\mathcal{A} \subset \mathbb{C}^n$ then one should take $N \geq 2 \sum_{\alpha \in \mathcal{A}} m_\alpha + 3$. Converse statement is also true if we assume that all the Hadamard’s coefficients are rational functions (see theorem 6.2 in [3]).
Applying this first for the configuration $A_{n-1,1}(m)$, we arrive at the following potential $u$ depending on $x_1, \ldots, x_{n+1}$:

$$u = \sum_{i<j}^{n} 2m(m+1)(x_i - x_j)^{-2} + \sum_{i=1}^{n} 2(m+1)(x_i - \sqrt{mx_{n+1}})^{-2}. \quad (23)$$

For any $m \in \mathbb{Z}_+$ the corresponding equation (22) will satisfy the Huygens’ Principle if $N$ is odd and $N \geq mn(n-1) + 2n + 3$. When $m$ is negative integer, the potential (23) is no longer real-valued. However, one can make a change of coordinates and think of $\sqrt{-1}x_{n+1}$ as a $t$-variable. In this way we arrive at the time-dependent real potential $u(t, x_1, \ldots, x_n)$ as follows:

$$u = \sum_{i<j}^{n} 2m(m+1)(x_i - x_j)^{-2} + \sum_{i=1}^{n} 2(m+1)(x_i - \sqrt{-mt})^{-2}, \quad (24)$$

and the corresponding huygensian equation (22) for odd $N \geq (-1-m)n(n-1) + 2n + 3$.

In case of the configuration $A_{n-1,2}(m)$ with $m \in \mathbb{Z}_+$ one can make a similar change of coordinates and think of $\sqrt{-1}x_{n+2}$ as a $t$-variable. The corresponding potential $u(t, x)$ will be of the form

$$u = \sum_{i<j}^{n} 2m(m+1)(x_i - x_j)^{-2} + \sum_{i=1}^{n} 2(m+1)(x_i - \sqrt{mx_{n+1}})^{-2}$$

$$- \sum_{i=1}^{n} 2m(x_i - \sqrt{m+1}t)^{-2} - 2(\sqrt{mx_{n+1}} - \sqrt{m+1}t)^{-2}. \quad (25)$$

Thus, we arrive at the following result.

**Proposition 4.** The equation $(\Box_{N+1} + u(t, x))\phi = 0$ with the potential $u(t, x)$ given by (25) with a positive integer $m$ satisfies the Huygens’ Principle for odd $N \geq mn(n-1) + 4n + 5$.

This gives us the new examples of the huygensian equations, the first of which appears in dimension $N = 17$ for $n = 2$ and $m = 2$.

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