Optimal singularities of initial data for solvability of the Hardy parabolic equation

Kotaro Hisa\textsuperscript{a},*, Jin Takahashi\textsuperscript{b}

\textsuperscript{a} Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
\textsuperscript{b} Department of Mathematical and Computing Science, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552, Japan

Abstract

We consider the Cauchy problem for the Hardy parabolic equation \( \partial_t u - \Delta u = |x|^{-\gamma} u^p \) with initial data \( u_0 \) singular at some point \( z \). Our main results show that, if \( z \neq 0 \), then the optimal strength of the singularity of \( u_0 \) at \( z \) for the solvability of the equation is the same as that of the Fujita equation \( \partial_t u - \Delta u = u^p \). Moreover, if \( z = 0 \), then the optimal singularity for the Hardy parabolic equation is weaker than that of the Fujita equation. We also obtain analogous results for a fractional case \( \partial_t u + (-\Delta)^{\theta/2} u = |x|^{-\gamma} u^p \) with \( 0 < \theta < 2 \).

Keywords: Hardy parabolic equation, solvability, optimal singularity

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1. Introduction

We consider the Cauchy problem for the Hardy parabolic equation

\[
\begin{align*}
\partial_t u - \Delta u &= |x|^{-\gamma} u^p \quad \text{in } \mathbb{R}^N \times (0, T), \\
u(\cdot, 0) &= u_0 \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where \( N \geq 1 \), \( p > 1 \), \( 0 < \gamma < \min\{2, N\} \) and \( 0 < T \leq \infty \). We assume that \( u_0 \) is nonnegative and has a singularity at some point \( z \in \mathbb{R}^N \). The aim of this paper is to identify the optimal strength of the singularity at \( z \) for the local-in-time solvability of (1.1). Intuitively, such an optimal singularity is determined by the diffusion effect and the growth rate of the nonlinear term near \( z \). Hence, if \( z \neq 0 \), it is expected that the optimal singularity of (1.1) is the same as that of the Fujita equation \( \partial_t u - \Delta u = u^p \). On the other hand, if \( z = 0 \), the optimal singularity should be weaker than that of the Fujita equation. In this paper, we show that these expectations are indeed correct. Furthermore, we also give analogous results for a fractional case \( \partial_t u + (-\Delta)^{\theta/2} u = |x|^{-\gamma} u^p \) with \( 0 < \theta < 2 \), see Section 7 below.

We recall some results for the problem (1.1). In what follows, set

\[
p_F := 1 + \frac{2}{N}, \quad p_\gamma := 1 + \frac{2 - \gamma}{N},
\]

\*Corresponding author

Email addresses: khisa@ms.u-tokyo.ac.jp (Kotaro Hisa), takahashi@c.titech.ac.jp (Jin Takahashi)

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Ben Slimene-Tayachi-Weissler [4] obtained sufficient conditions for the solvability of (1.1) in $L^q$ spaces. They also obtained a global-in-time solution under $u_0(x) < c|x|^{-(2-γ)/(p-1)}$ if $p > p_γ$ and $c > 0$ small. For related results, see [1, 3, 7, 8, 9, 10, 13, 14, 15, 18, 19]. Subsequently, the first author and Sierżega [12] examined necessary conditions of initial data for the existence of solutions including a fractional case. Their results imply that (1.1) possesses no local-in-time solutions if $u_0$ satisfies either

$$u_0(x) ≥ \begin{cases} C_1|x|^{-N} \left(\log(e + |x|^{-1})\right)^{-\frac{N}{2} - 1} & \text{if } p = p_γ, \\ C_1|x|^{-\frac{p}{p-1}} & \text{if } p > p_γ \end{cases}$$ \hspace{1cm} (1.2)$$

near $x = 0$, or

$$u_0(x) ≥ \begin{cases} C_1|x|^{-\frac{2}{p-1}}|x - z|^{-N} \left(\log(e + |x - z|^{-1})\right)^{-\frac{N}{2} - 1} & \text{if } p = p_F, \\ C_1|x|^{-\frac{2}{p-1}}|x - z|^{-\frac{2}{p-1}} & \text{if } p > p_F \end{cases}$$ \hspace{1cm} (1.3)$$

near $x = z$ ($z \neq 0$), where $C_1 > 0$ is sufficiently large. We note that $|x - z|^{-N}(\log(e + |x - z|^{-1}))^{-\frac{N}{2} - 1}$ and $|x - z|^{-\frac{2}{p-1}}$ are the same as the optimal singularities of the Fujita equation for each case, see [2, 11]. In addition, when $u_0 = δ_z$ with $z ∈ \mathbb{R}^N$, they [12] also show that the problem (1.1) possesses no local-in-time solutions if either $p_γ ≤ p < p_F$ and $z = 0$, or $p ≥ p_F$ and $z ∈ \mathbb{R}^N$. Here $δ_z$ is the Dirac measure on $\mathbb{R}^N$ concentrated at $z ∈ \mathbb{R}^N$.

The above results in [4, 12] imply that $|x|^{-(2-γ)/(p-1)}$ is the optimal singularity of the Hardy parabolic equation (1.1) in the case where $p > p_γ$ and the singularity of $u_0$ is located at the origin. However, the optimality of the other cases is still open. In this paper, we derive sufficient conditions for the existence of solutions corresponding to (1.2) and (1.3) and specify the optimal singularities of (1.1).

In order to state our results, we introduce some notation. Set

$$ψ(z) := \begin{cases} |z|^{-\frac{2}{p-1}}(1 + |z|)^{-\frac{2}{p-1}} & \text{if } p < p_F \text{ with } N ≥ 1 \text{ or } p = p_F \text{ with } N ≤ 2, \\ |z|^{-\frac{2}{p-1}} & \text{if } p = p_F \text{ with } N > 2 \text{ or } p > p_F \text{ with } N ≥ 1. \end{cases}$$ \hspace{1cm} (1.4)$$

We regard a nonnegative Radon measure $μ = Cδ_z + ϕ$ with $ϕ ∈ L^∞(\mathbb{R}^N)$ as $dμ(x) = Cδδ_z(x) + ϕdx$. We denote by $χ_1$ the characteristic function on the interval $[0, 1]$. Each of the solutions in our results is the so-called integral solution, see Definition 2.1 below.

Our first result shows that if the singularity of $u_0$ is not located at the origin, then the optimal singularity of (1.1) is the same as that of the Fujita equation.

**Theorem 1.1.** Fix $N ≥ 1$, $p > 1$, $0 < γ < \min\{2, N\}$, $0 < T < ∞$ and $z ∈ \mathbb{R}^N \setminus \{0\}$. Let $ψ$ be as in (1.4). Assume either $u_0$ is a nonnegative Radon measure satisfying

$$u_0 = cψ(z)δ_z + ϕ \quad \text{if } p < p_F$$

for a nonnegative function $ϕ ∈ L^∞(\mathbb{R}^N)$ with $\|ϕ\|_{L^∞(\mathbb{R}^N)} ≤ C_0$, or $u_0$ is a nonnegative measurable function satisfying

$$u_0(x) ≤ \begin{cases} cψ(z)|x - z|^{-N} \left(\log(e + |x - z|^{-1})\right)^{-\frac{N}{2} - 1} χ_1(|x - z|) + C_0 & \text{if } p = p_F, \\ cψ(z)|x - z|^{-\frac{2}{p-1}} + C_0 & \text{if } p > p_F \end{cases}$$

for $x ≤ z$.
for any \( x \in \mathbb{R}^N \setminus \{z\} \). Here \( c > 0 \) and \( C_0 \geq 0 \) are constants. Then there exist positive constants \( c_* \) and \( C_* \) depending on \( N, p \) and \( \gamma \) but not on \( T \) and \( z \) such that the following statements hold. If the constants \( c \) and \( C_0 \) satisfy

\[
  c \leq \begin{cases} 
    c_* T^{\frac{N(p-1)}{2(p-1)} - \frac{1}{p-1}} & \text{if } p < p_F, \\
    c_* (1 + T^{\frac{1}{2}} + T^{\frac{1}{p-1}})^{-\frac{1}{p-1}} & \text{if } p = p_F, \\
    c_* & \text{if } p > p_F,
  \end{cases} \\
  C_0 \leq C_* T^{-\frac{2-\gamma}{2(p-1)}},
\]

respectively, then (1.1) possesses a solution on \( \mathbb{R}^N \times [0, T) \). In addition, if \( p > p_F \), \( c \leq c_* \) and \( C_0 = 0 \), then (1.1) possesses a solution on \( \mathbb{R}^N \times [0, \infty) \).

**Remark 1.1.** We can improve \( \psi \) to \( |z|^\gamma/(p-1) \) for each case if \( T \) and \( z \neq 0 \) satisfy

\[
  0 < T \leq |z|^2. 
\]

The improved statement is optimal up to \( c_* \) compared with (1.3). Moreover, the assumption (1.3) is natural and is not so restrictive, since we can take \( T \) arbitrarily small when we only need to consider local-in-time solvability. For more details, see Remarks 3.1 and 4.1 below.

**Remark 1.2.** In [4], it was shown that \( L^q_c(\mathbb{R}^N) \) is the scaling invariant critical space for the well-posedness of (1.1), where \( q_c := N(p-1)/(2-\gamma) \) with \( p \geq p_\gamma \). For \( p > p_\gamma \), they also obtained a solution with initial data under \( c| \cdot |^{-(2-\gamma)/(p-1)} \in L^{q_c,\infty}(\mathbb{R}^N) \), where \( c > 0 \) is small and \( L^{q_c,\infty}(\mathbb{R}^N) \) is the Lorentz space. Theorem 1.1 implies that, for \( p \geq p_\gamma \), there exists \( u_0 \) such that \( u_0 \) does not belong to \( L^{q_c,\infty}(\mathbb{R}^N) \) but (1.1) possesses a solution.

We next consider the case where there is a possibility that the singularity of \( u_0 \) is located at the origin. In this case, the optimal singularity of \( u_0 \) is weaker than that of the Fujita equation.

**Theorem 1.2.** Fix \( N \geq 1, p > 1, 0 < \gamma < \min\{2, N\}, 0 < T < \infty \) and \( z \in \mathbb{R}^N \). Assume either \( u_0 \) is a nonnegative Radon measure satisfying

\[
  u_0 = c\delta_z + \phi \quad \text{if } p < p_\gamma,
\]

for a nonnegative function \( \phi \in L^\infty(\mathbb{R}^N) \) with \( \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq C_0 \), or \( u_0 \) is a nonnegative measurable function satisfying

\[
  u_0(x) \leq \begin{cases} 
    c|x-z|^{-N} \left( \log(e + |x-z|^{-1}) \right)^{-\frac{N}{(2-\gamma)/(p-1)}} \chi_1(|x-z|) + C_0 & \text{if } p = p_\gamma, \\
    c|x-z|^{-\frac{2-\gamma}{p-1}} + C_0 & \text{if } p > p_\gamma
  \end{cases}
\]

for any \( x \in \mathbb{R}^N \setminus \{z\} \). Here \( c > 0 \) and \( C_0 \geq 0 \) are constants. Then there exist positive constants \( c_* \) and \( C_* \) depending on \( N, p \) and \( \gamma \) but not on \( T \) and \( z \) such that the following statements hold. If the constants \( c \) and \( C_0 \) satisfy

\[
  c \leq \begin{cases} 
    c_* T^{\frac{N(p-1)}{2(p-1)} - \frac{1}{p-1}} & \text{if } p < p_\gamma, \\
    c_* (1 + T^{\frac{1}{2}} + T^{\frac{1}{p-1}})^{-\frac{1}{p-1}} & \text{if } p = p_\gamma, \\
    c_* & \text{if } p > p_\gamma,
  \end{cases} \\
  C_0 \leq C_* T^{-\frac{2-\gamma}{2(p-1)}},
\]

respectively, then (1.1) possesses a solution on \( \mathbb{R}^N \times [0, T) \). In addition, if \( p > p_\gamma \), \( c \leq c_* \) and \( C_0 = 0 \), then (1.1) possesses a solution on \( \mathbb{R}^N \times [0, \infty) \).
Remark 1.3. Theorem 1.2 with \( z = 0 \) is optimal up to \( c_* \) compared with (1.2).

Remark 1.4. In the case of \( u_0 = \delta_z \) with \( z \in \mathbb{R}^N \). Theorems 1.1 and 1.2 together with (1.2) imply that (1.1) possesses a local-in-time solution if and only if either \( 1 < p < p_\gamma \), or \( p_\gamma \leq p < p_F \) and \( z \neq 0 \).

Remark 1.5. The main novelty of Theorem 1.2 is the critical case of \( p = p_\gamma \), since the case of \( p < p_\gamma \) is not difficult and the case of \( p > p_\gamma \) follows from [4]. However, we give a short proof for \( p < p_\gamma \) and a unified proof for \( p \geq p_\gamma \).

The proofs of our main results are based on showing the existence of supersolutions for the corresponding integral equation to (1.1). Each of the supersolutions is constructed in the same spirit of Robinson and Sierżega [16], and has the form

\[
    u^+(x, t) := H^{-1} \left( \int_{\mathbb{R}^N} G(x - y, mt) H(u_0(y)) dy \right),
\]

where \( m \in \mathbb{N} \) and \( H : [0, \infty) \to [0, \infty) \) is a strictly increasing and convex function. However, the way to verify that \( u^+ \) is indeed a supersolution is totally different from [16]. They used uniform estimates with respect to \( x \), but we estimate \( u^+ \) uniformly with respect to \( t \) and analyze its singularity in detail.

The rest of this paper is organized as follows. In Section 2, we give the definition of solutions and collect estimates on the heat kernel. In Sections 3 and 4, we prove Theorem 1.1 for \( p \geq p_F \) and \( p < p_F \), respectively. In Sections 5 and 6, we prove Theorem 1.2 for \( p \geq p_\gamma \) and \( p < p_\gamma \), respectively. In Section 7, we state results for the fractional Hardy parabolic equation.

2. Preliminaries

In Subsection 2.1, we introduce the definition of solutions in this paper and quote a lemma on the existence of solutions. In Subsection 2.2, we give estimates concerning the heat kernel.

2.1. Integral equation

Let \( u_0 \) be a nonnegative measurable function on \( \mathbb{R}^N \) or a nonnegative Radon measure on \( \mathbb{R}^N \). In what follows, we always consider nonnegative solutions of the following integral equation corresponding to (1.1).

\[
    u(x, t) = \Phi[u](x, t),
    \Phi[u](x, t) := \int_{\mathbb{R}^N} G(x - y, t) du_0(y) + \int_t^T \int_{\mathbb{R}^N} G(x - y, t - s) |y|^{-\gamma} u(y, s)^p dy ds,
\]

where \( G(x, t) := (4\pi t)^{-N/2} e^{-|x|^2/(4t)} \) is the Gaussian heat kernel in \( \mathbb{R}^N \). Remark that, in the case where \( u_0 \) is a measurable function, we regard \( du_0(y) \) as \( u_0 dy \). Solutions and supersolutions are defined as follows.

**Definition 2.1.** Let \( 0 < T \leq \infty \). A nonnegative measurable function \( u \) on \( \mathbb{R}^N \times (0, T) \) is called a solution of (1.1) on \( \mathbb{R}^N \times [0, T) \) if \( u \) satisfies \( 0 \leq u < \infty \) and \( u = \Phi[u] \) a.e. in \( \mathbb{R}^N \times (0, T) \). In addition, a nonnegative measurable function \( \overline{u} \) on \( \mathbb{R}^N \times (0, T) \) is called a supersolution of (1.1) on \( \mathbb{R}^N \times [0, T) \) if \( \overline{u} \) satisfies \( 0 \leq \overline{u} < \infty \) and \( \overline{u} \geq \Phi[\overline{u}] \) a.e. in \( \mathbb{R}^N \times (0, T) \).
The method to construct solutions of (1.1) is based on the following lemma.

**Lemma 2.1.** Let $0 < T < \infty$. Assume that there exists a supersolution $\pi$ of (1.1) on $\mathbb{R}^N \times [0, T)$. Then there exists a solution on (1.1) on $\mathbb{R}^N \times [0, T)$.

**Proof.** This lemma was proved by [12, Lemma 2.2], see also [11, 16]. \hfill \Box

### 2.2. Estimates on the heat kernel

We first give an equality and an estimate for the heat kernel, and then we list estimates on some integrals.

**Lemma 2.2.** For any $x, y, \eta \in \mathbb{R}^N$ and $0 < s < t$,

$$G(x - y, t - s)G(y - \eta, s) = G(x - \eta, t)G \left( y - \frac{s}{t}x - \frac{t - s}{t}\eta, \frac{s(t - s)}{t} \right).$$

**Proof.** This can be checked by the straightforward computations. \hfill \Box

**Lemma 2.3.** There exists a constant $C > 0$ depending only on $N$ such that, for any $x, y, \eta \in \mathbb{R}^N$ and $0 < s < t$,

$$G \left( y - \frac{s}{t}x - \frac{t - s}{t}\eta, 2\frac{s(t - s)}{t} \right) e^{-|x-y|^2/8(t-s)} \leq C(G(y - \eta, 35s) + G(y - x, 35(t - s))).$$

**Proof.** In this proof, we denote by $\tilde{G}$ the left hand side of the desired inequality. Let us first consider the case of $0 < s < t/2$. By using $|a + b|^2 \geq \delta |a|^2 - \delta (1 - \delta)^{-1}|b|^2$ for $a, b \in \mathbb{R}^N$ and $0 < \delta < 1$, we have

$$\left| y - \frac{s}{t}x - \frac{t - s}{t}\eta \right|^2 = \left| y - \eta - \frac{s}{t}(x - \eta) \right|^2 \geq \frac{1}{5}|y - \eta|^2 - \frac{1}{4} \left( \frac{s}{t} \right)^2 |x - \eta|^2,$$

and so

$$\tilde{G} \leq C \left( \frac{2(s(t - s))}{t} \right)^{-\frac{N}{2}} \exp \left( -\frac{5^{-1}|y - \eta|^2}{8s(t - s)t^{-1}} \right) \exp \left( \frac{4^{-1}s^2 t^{-2}|x - \eta|^2}{8s(t - s)t^{-1}} \right)$$

$$\times \exp \left( -\frac{|y - \eta|^2}{64(t - s)} \right) \exp \left( \frac{|y - \eta|^2}{56(t - s)} \right)$$

$$\leq Cs^{-\frac{N}{2}} \exp \left( -\frac{|y - \eta|^2}{40s} \right) \exp \left( \frac{|x - \eta|^2}{64(t - s)} \right) \exp \left( -\frac{|y - \eta|^2}{64(t - s)} \right) \exp \left( \frac{|y - \eta|^2}{56s} \right)$$

$$= Cs^{-\frac{N}{2}} \exp \left( -\frac{|y - \eta|^2}{140s} \right) = CG(y - \eta, 35s)$$

for any $0 < s < t/2$, where $C$ is a constant depending only on $N$.

We next consider the case of $t/2 < s < t$. By

$$\left| y - \frac{s}{t}x - \frac{t - s}{t}\eta \right|^2 = \left| y - x + \frac{t - s}{t}(x - \eta) \right|^2 \geq \frac{1}{5} \left( \frac{t - s}{t} \right)^2 |x - \eta|^2 - \frac{1}{4}|x - y|^2,$$
we have
\[ \tilde{G} \leq C \left( \frac{2s(t-s)}{t} \right)^{-\frac{N}{2}} \exp \left( -\frac{5^{-1}(t-s)^2t^{-2}|x-\eta|^2}{8s(t-s)t^{-1}} \right) \exp \left( \frac{|x-y|^2}{32s(t-s)t^{-1}} \right) \exp \left( -\frac{|x-y|^2}{8(t-s)} \right) \]
\[ \leq C(t-s)^{-\frac{N}{2}} \exp \left( \frac{|x-y|^2}{16(t-s)} \right) \exp \left( -\frac{|x-y|^2}{8(t-s)} \right) \]
\[ = C(t-s)^{-\frac{N}{2}} \exp \left( -\frac{|y-x|^2}{16(t-s)} \right) \leq CG(y-x, 35(t-s)) \]
for any \( t/2 < s < t \) with a constant \( C \) depending only on \( N \). Hence the lemma follows. \( \square \)

**Lemma 2.4.** Let \( 0 < k < N \). Then there exists \( C > 0 \) depending only on \( N \) and \( k \) such that, for any \( x \in \mathbb{R}^N \) and \( t > 0 \),
\[ \int_{\mathbb{R}^N} G(x-y,t)|y|^{-k}dy \leq C|x|^{-k}. \]

**Proof.** Set \( \Omega_1 := \{ y \in \mathbb{R}^N; |y| \leq |x|/2 \} \) and \( \Omega_2 := \{ y \in \mathbb{R}^N; |y| \geq |x|/2 \} \). For \( y \in \Omega_1 \), we have \( |x-y| \geq |x| - |y| \geq |x|/2 \). Thus,
\[ \int_{\Omega_1} G(x-y,t)|y|^{-k}dy \leq C t^{-\frac{N}{2}} e^{-\frac{|y|^2}{16t}} \int_{\Omega_1} |y|^{-k}dy \leq C t^{-\frac{N}{2}} e^{-\frac{|y|^2}{16t}} |x|^{-k} \leq C|x|^{-k}, \]
since \( \sup_{t>0} t^{-N/2}e^{-|x|^2/(16t)} \leq C|x|^{-N} \). On the other hand, from \( \int_{\mathbb{R}^N} e^{-|x-y|^2/(4t)}dy = C t^{N/2} \), it follows that
\[ \int_{\Omega_2} G(x-y,t)|y|^{-k}dy \leq C t^{-\frac{N}{2}} |x|^{-k} \int_{\Omega_2} e^{-\frac{|x-y|^2}{4t}}dy \leq C t^{-\frac{N}{2}} |x|^{-k} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}}dy = C|x|^{-k}. \]
Hence the lemma follows. \( \square \)

**Lemma 2.5.** Let \( k > 0 \). Then there exists \( C > 0 \) depending only on \( N \) and \( k \) such that
\[ \int_{\mathbb{R}^N} G(x-y,t)|y|^{-N} (\log(e + |y|^{-1}))^{-k-1} \chi_1(|y|)dy \leq C|x|^{-N} (\log(e + |x|^{-1}))^{-k} \]
for any \( x \in \mathbb{R}^N \) and \( t > 0 \).

**Proof.** Set \( \Omega_1 \) and \( \Omega_2 \) as in Lemma 2.4. Using \( |x-y| \geq |x|/2 \) for \( y \in \Omega_1 \) gives
\[ \int_{\Omega_1} G(x-y,t)|y|^{-N} (\log(e + |y|^{-1}))^{-k-1} \chi_1(|y|)dy \]
\[ \leq C t^{-\frac{N}{2}} e^{-\frac{|y|^2}{16t}} \int_0^{\min\{|x|/2,1\}} (er + 1) \frac{r^{-1}}{er + 1} (\log(e + r^{-1}))^{-k-1} dr \]
\[ \leq C|x|^{-N} \int_0^{\min\{|x|/2,1\}} \frac{r^{-1}}{er + 1} (\log(e + r^{-1}))^{-k-1} dr \]
\[ \leq C|x|^{-N} (\log(e + 2|x|^{-1}))^{-k} \leq C|x|^{-N} (\log(e + |x|^{-1}))^{-k}. \]
We fix a constant $K > 0$ so large that $X \mapsto X^{-N} (\log(K + X^{-1}))^{-k-1}$ is decreasing. Then,
\[
\int_{\Omega_2} G(x - y, t)|y|^{-N} \left( \log(e + |y|^{-1}) \right)^{-k-1} dy \\
\leq C \int_{\Omega_2} G(x - y, t)|y|^{-N} \left( \log(K + |y|^{-1}) \right)^{-k-1} dy \\
\leq C t^{-N} |x|^{-N} \left( \log(K + 2|x|^{-1}) \right)^{-k-1} \int_{\Omega_2} e^{-\frac{|x-y|^2}{4t}} dy \\
\leq C |x|^{-N} \left( \log(K + 2|x|^{-1}) \right)^{-k-1} \leq C |x|^{-N} \left( \log(e + |x|^{-1}) \right)^{-k},
\]
and the lemma is proved. \hfill \Box

**Lemma 2.6.** Let $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ be a nonnegative function. Then there exists $C > 0$ depending only on $N$ such that, for any $x \in \mathbb{R}^N$ and $t > 0$,
\[
\int_{\mathbb{R}^N} G(x - y, t) \varphi(y) dy \leq C t^{-N} \sup_{\zeta \in \mathbb{R}^N} \int_{B(\zeta, \frac{t}{2})} \varphi(y) dy.
\]

**Proof.** This follows from [11, Lemma 2.1] with $\theta = 2$. \hfill \Box

### 3. Proof of Theorem 1.1 for $p \geq p_F$

In this section, we prove Theorem 1.1 for $p \geq p_F$. We prepare an auxiliary function $H$, and then we define a candidate $\overline{u}$ of a supersolution. For $1 < \alpha < \min\{p, N(p - 1)/2\}$, $0 < \beta < N/2$ and $A \geq e$, set
\[
H(X) := \begin{cases} 
X^\alpha & \text{if } p > p_F, \\
X (\log(A + X))^\beta & \text{if } p = p_F.
\end{cases}
\] (3.1)

We fix $A$ so large that
\[
X \mapsto H(X), \ X \mapsto X^p/H(X) \text{ and } X \mapsto H(X)/X \text{ are strictly increasing and}
\]
\[
X \mapsto X^{-(2-\gamma)} (\log(A + X^{-1}))^{1-\beta} \text{ is strictly decreasing.}
\] (3.2)

Note that $H$ is convex and strictly increasing. In particular, the inverse function $H^{-1}$ exists and is strictly increasing. We can check that $H^{-1}$ satisfies
\[
H^{-1}(X) \begin{cases} 
X^{1/\alpha} & \text{if } p > p_F, \\
C X (\log(A + X))^{-\beta} & \text{if } p = p_F
\end{cases}
\] (3.3)

where $C > 0$ depends only on $A$ and $\beta$. Let $z \in \mathbb{R}^N \setminus \{0\}$. Then, for $c > 0$ and $C_0 \geq 0$, we define
\[
\overline{u}(x, t) := 2^{N+1}c\psi(z)U(x, t) + 2C_0.
\]

Here $\psi$ is defined by (1.4) and
\[
U(x, t) := H^{-1}\left( \int_{\mathbb{R}^N} G(x - y, 2t)H(f(y)) dy \right),
\]
\[
f(x) := \begin{cases} 
|x - z|^{-\frac{2}{p-1}} & \text{if } p > p_F, \\
|x - z|^{-N} \left( \log(e + |x - z|^{-1}) \right)^{-\frac{N}{2} - 1} \chi_1(|x - z|) & \text{if } p = p_F.
\end{cases}
\]

7
We also prepare some estimates of $U$. Note that, for $p = p_F$, we can check that $H(f(y+z)) \leq C|y|^{-(N-1)(\log(e + |y|^{-1}))^{-(N/2)-1+\beta} \chi_1(|y|)}$ with a constant $C$ depending on $A$. Thus,

$$H(U(x,t)) = \int_{\mathbb{R}^N} G(x-z-y,2t)H(f(y+z))dy$$

$$\leq \begin{cases} 
\int_{\mathbb{R}^N} G(x-z-y,2t)|y|^{-\frac{2\alpha}{p-1}}dy & \text{if } p > p_F, \\
\frac{C}{2} \int_{\mathbb{R}^N} G(x-z-y,2t)|y|^{-N} \left(\log(e + |x-z|^{-1})\right)^{-\frac{N}{2}+\beta} \chi_1(|y|)dy & \text{if } p = p_F.
\end{cases}$$

By Lemmas 2.4 and 2.5, we have

$$H(U(x,t)) \leq \begin{cases} 
\frac{C|x-z|^{-\frac{2}{p-1}}}{2} & \text{if } p > p_F, \\
\frac{C|x-z|^{-N} \left(\log(e + |x-z|^{-1})\right)^{-\frac{N}{2}+\beta}}{2} & \text{if } p = p_F.
\end{cases}$$

Then the monotonicity of $H^{-1}$ together with (3.3) implies that

$$U(x,t) \leq \begin{cases} 
\frac{C|x-z|^{-\frac{2}{p-1}}}{2} & \text{if } p > p_F, \\
\frac{C|x-z|^{-N} \left(\log(e + |x-z|^{-1})\right)^{-\frac{N}{2}+\beta}}{2} & \text{if } p = p_F.
\end{cases}$$

On the other hand, Lemma 2.6 yields

$$H(U(x,t)) \leq \begin{cases} 
\frac{Ct^{-\frac{N}{2}} \sup_{\zeta \in \mathbb{R}^N} \int_{B(\zeta,(2t)^{\frac{1}{2}})} |y|^{-\frac{2\alpha}{p-1}}dy}{2} & \text{if } p > p_F, \\
\frac{Ct^{-\frac{N}{2}} \sup_{\zeta \in \mathbb{R}^N} \int_{B(\zeta,(2t)^{\frac{1}{2}})} |y|^{-N} \left(\log(e + |y|^{-1})\right)^{-\frac{N}{2}+\beta} \chi_1(|y|)dy}{2} & \text{if } p = p_F.
\end{cases}$$

Then the monotonicity of $H^{-1}$ together with (3.3) implies that

$$U(x,t) \leq \begin{cases} 
\frac{Ct^{-\frac{1}{p-1}}}{2} & \text{if } p > p_F, \\
\frac{Ct^{-\frac{N}{2}} \left(\log(e + t^{-\frac{1}{2}})\right)^{-\frac{N}{2}+\beta}}{2} & \text{if } p = p_F.
\end{cases}$$

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1 for $p \geq p_F$.** By Lemma 2.11 it suffices to prove that $\Phi$ is a supersolution if $c$ is sufficiently small. First, we consider the case of $C_0 = 0$ and estimate $\Phi$. Remark that
\[ u_0 \leq c|\psi(z)|f \] with \( c > 0 \). This together with Jensen’s inequality gives

\[
\int_{\mathbb{R}^N} G(x, y, t) u_0(y) dy \leq c \frac{N}{2} |\psi(z)| \int_{\mathbb{R}^N} G(x, y, 2t) f(y) dy
\]

\[
= c \frac{N}{2} |\psi(z)| H^{-1} \circ H \left( \frac{\int_{\mathbb{R}^N} G(x, y, 2t) f(y) dy}{\int_{\mathbb{R}^N} G(x, y, 2t) dy} \right) \tag{3.6}
\]

\[
\leq c \frac{N}{2} |\psi(z)| U(x, t) = \frac{1}{2} \pi(x, t).
\]

By Fubini’s theorem and Lemma 2.2, we have

\[
\int_0^t \int_{\mathbb{R}^N} G(x, y, t - s) |y|^{-\gamma} \pi(y, s)^p dy ds
\]

\[
= c^p \frac{N}{2} (p+1) + p |\psi(z)|^p \int_0^t \int_{\mathbb{R}^N} G(x, y, 2(t - s)) e^{-\frac{|x-y|^2}{8(t-s)}} |y|^{-\gamma} \frac{U(y, 2s)^p}{H(U(y, 2s))} \tag{3.7}
\]

\[
\times \int_{\mathbb{R}^N} G(y, \eta, 2s) H(f(\eta)) d\eta dy ds
\]

\[
= c^p \frac{N}{2} (p+1) + p |\psi(z)|^p \int_{\mathbb{R}^N} G(x, \eta, 2t) H(f(\eta)) J(x, \eta, t) d\eta,
\]

where

\[
J(x, \eta, t) := \int_0^t \int_{\mathbb{R}^N} G \left( y - \frac{s}{t} x - \frac{t-s}{t} \eta, 2 \frac{t-s}{t} \right) e^{-\frac{|s-y|^2}{8(s-t)}} |y|^{-\gamma} \frac{U(y, 2s)^p}{H(U(y, 2s))} dy ds.
\]

We claim that

\[
J(x, \eta, t) \leq \begin{cases} 
C |\psi(z)|^{-(p-1)} t \frac{p+1}{p} & \text{if } p > p_F, \\
C |\psi(z)|^{-(p-1)} (1 + t^{\frac{1}{2}} + t^{1-\frac{3}{2}}) \left( \log(e + t^{\frac{1}{2}}) \right)^{-\beta} & \text{if } p = p_F
\end{cases} \tag{3.8}
\]

for any \( t > 0 \) with a constant \( C > 0 \) independent of \( z \). Set \( \Omega_1 := \{ y \in \mathbb{R}^N ; |y| \leq |z|/2 \} \) and \( \Omega_2 := \{ y \in \mathbb{R}^N ; |y| \geq |z|/2 \} \). Lemma 2.3 shows that

\[
J \leq C \int_0^t \int_{\mathbb{R}^N} (G(y - \eta, 35s) + G(y - x, 35(t - s))) |y|^{-\gamma} \frac{U(y, 2s)^p}{H(U(y, 2s))} dy ds
\]

\[
= C \int_0^t \int_{\Omega_1} + C \int_0^t \int_{\Omega_2} =: CJ_1 + CJ_2.
\]

In what follows, we write \( \|U(\cdot, t)\|_{L^\infty} := \|U(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \) and \( |U(x, \cdot)|_{L^\infty} := |U(x, \cdot)|_{L^\infty(0, \infty)} \).

Let us first estimate \( J_1 \). The monotonicity of \( X \mapsto X^p/H(X) \) in (3.2) yields

\[
J_1 \leq C \int_0^t \int_{\Omega_1} (G(y - \eta, 35s) + G(y - x, 35(t - s))) |y|^{-\gamma} \frac{|U(y, \eta)|_{L^\infty}^p}{H(|U(y, \eta)|_{L^\infty})} dy ds.
\]

From (3.4), \( |y - z| \geq |y| \) for \( y \in \Omega_1 \), \( N(p_F - 1) = 2 \) and the monotonicity of \( X \mapsto X^{-(2-\gamma)} (\log(A+ \cdots

9
for any $t > 0$, it follows that

$$\frac{|U(y, \cdot)|_p}{H(|U(y, \cdot)|_\infty)} \leq \begin{cases} C|y - z|^\frac{2(p - \alpha)}{p - 1} & (p > p_F) \\ C|y - z|^{-2} \left(\log(A + |y - z|^{-1})\right)^{-1 - \beta} & (p = p_F) \\ C|z|^{-\gamma} |y|^{-\frac{2(p - \alpha)}{p - 1}} & (p > p_F) \\ C|z|^{-\gamma} |y|^{-2 - \gamma} \left(\log(e + |y|^{-1})\right)^{-1 - \beta} & (p = p_F, N > 2) \\ C|z|^{-2} & (p = p_F, N \leq 2) \end{cases}$$

for $y \in \Omega_1$, and so

$$J_1 \leq \begin{cases} C|z|^{-\gamma} \sup_{\xi \in \mathbb{R}^N} \int_0^t \int_{\Omega_1} G(y - \xi, 35s) |y|^{-\frac{2(p - \alpha)}{p - 1}} dy ds & \text{if } p > p_F, \\ C|z|^{-\gamma} \sup_{\xi \in \mathbb{R}^N} \int_0^t \int_{\Omega_1} G(y - \xi, 35s) |y|^{-2} \left(\log(e + |y|^{-1})\right)^{1 + \beta} dy ds & \text{if } p = p_F, N > 2, \\ C|z|^{-2} \sup_{\xi \in \mathbb{R}^N} \int_0^t \int_{\Omega_1} G(y - \xi, 35s) |y|^{-\gamma} dy ds & \text{if } p = p_F, N \leq 2. \end{cases} \tag{3.9}$$

Then, Lemma 2.6 and straightforward computations show that, for any $t > 0$,

$$J_1 \leq \begin{cases} C|z|^{-\gamma} \int_0^t s^{-\frac{p - \alpha}{p - 1}} ds & (p > p_F) \\ C|z|^{-\gamma} \int_0^t \left(\frac{es^{\frac{1}{2}} + 1}{es^{1/2} + 1}\right) \left(\log(e + s^{-\frac{1}{2}})\right)^{-1 - \beta} ds & (p = p_F, N > 2) \\ C|z|^{-2} \int_0^t s^{-\frac{1}{2}} ds & (p = p_F, N \leq 2) \\ C|z|^{-\gamma} t^{-\frac{\alpha - 1}{p - 1}} ds & (p > p_F) \\ C|z|^{-\gamma} (1 + t^\beta) \left(\log(e + t^{-\frac{1}{2}})\right)^{-\beta} & (p = p_F, N \geq 3) \\ C|z|^{-2} t^{1 - \beta} & (p = p_F, N = 1, 2) \end{cases}$$

Note that

$$|z|^{-2} t^{1 - \beta} \leq C|z|^{-2}(1 + |z|)^{2 - \gamma}(1 + t^\frac{1}{2} + t^{1 - \beta}) \left(\log(e + t^{-\frac{1}{2}})\right)^{-\beta}$$

for any $t > 0$, where this constant $C > 0$ is independent of $z$.

We next examine $J_2$. By using (3.5), we can check that

$$\frac{\|U(\cdot, 2s)\|_\infty}{H(\|U(\cdot, 2s)\|_\infty)} \leq \begin{cases} C s^{-\frac{p - \alpha}{p - 1}} & \text{if } p > p_F, \\ C s^{-1} \left(\log(e + s^{-\frac{1}{2}})\right)^{-1 - \beta} & \text{if } p = p_F. \end{cases}$$
This together with \( \int_{\mathbb{R}^N} Gdy = 1 \) yields
\[
J_2 \leq \begin{cases} 
C|z|^{-\gamma} \int_0^t s^{-\frac{p-\alpha}{p-1}} ds & (p > p_F) \\
C|z|^{-\gamma} \int_0^t (es^{1/2} + 1) \frac{s^{-1}}{es^{1/2} + 1} \left( \log(e + s^{-\frac{1}{2}}) \right)^{-1-\beta} ds & (p = p_F) 
\end{cases}
\]
\[
\leq \begin{cases} 
C|z|^{-\gamma} \cdot \frac{1}{t^{\frac{\alpha-1}{p-1}}} & (p > p_F) \\
C|z|^{-\gamma}(1 + t^\frac{1}{2}) \left( \log(e + t^{-\frac{1}{2}}) \right)^{-\beta} & (p = p_F). 
\end{cases}
\]

Note that
\[
|z|^{-\gamma}(1 + t^\frac{1}{2}) \left( \log(e + t^{-\frac{1}{2}}) \right)^{-\beta} \leq C|z|^{-2}(1 + |z|)^{-\gamma}(1 + t^{-\frac{1}{2}} + t^{1-\frac{2}{\gamma}}) \left( \log(e + t^{-\frac{1}{2}}) \right)^{-\beta}
\]
for any \( t > 0 \) with a constant \( C > 0 \) independent of \( z \). Therefore, the above estimates show the desired inequality (3.8).

The equality (3.7) and the estimate (3.8) imply that
\[
\int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)|y|^{-\gamma} \bar{u}(y, s)^p dy ds 
\leq \begin{cases} 
Ce^p \psi(z) t^{-\frac{\alpha-1}{p-1}} \int_{\mathbb{R}^N} G(x - \eta, 2t) H(f(\eta)) d\eta & \text{if } p > p_F, \\
Ce^p \psi(z)(1 + t^\frac{1}{2} + t^{1-\frac{2}{\gamma}}) \left( \log(e + t^{-\frac{1}{2}}) \right)^{-\beta} \int_{\mathbb{R}^N} G(x - \eta, 2t) H(f(\eta)) d\eta & \text{if } p = p_F.
\end{cases}
\]

By the monotonicity of \( X \mapsto H(X)/X \) in (3.2), we have
\[
\int_{\mathbb{R}^N} G(x - \eta, 2t) H(f(\eta)) d\eta = H(U(x, t)) \leq \frac{H(\|U(\cdot, t)\|_\infty)}{\|U(\cdot, t)\|_\infty} U(x, t).
\]
Then (3.5) gives
\[
\int_{\mathbb{R}^N} G(x - \eta, 2t) H(f(\eta)) d\eta \leq \begin{cases} 
C\|U(\cdot, t)\|^{-\alpha}_{\infty} U(x, t) & (p > p_F) \\
C (\log(A + \|U(\cdot, t)\|_\infty))^{\beta} U(x, t) & (p = p_F) 
\end{cases}
\]
\[
\leq \begin{cases} 
C t^{-\frac{\alpha-1}{p-1}} U(x, t) & (p > p_F) \\
C \left( \log(e + t^{-\frac{1}{2}}) \right)^{\beta} U(x, t) & (p = p_F),
\end{cases}
\]
and so
\[
\int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)|y|^{-\gamma} \bar{u}(y, s)^p dy ds \leq \begin{cases} 
Cc^p \psi(z) U(x, t) & (p > p_F) \\
Cc^p \psi(z)(1 + t^\frac{1}{2} + t^{1-\frac{2}{\gamma}}) U(x, t) & (p = p_F) 
\end{cases}
\]
\[
= \begin{cases} 
Cc^{p-1} \bar{u}(x, t) & (p > p_F) \\
Cc^{p-1}(1 + t^\frac{1}{2} + t^{1-\frac{2}{\gamma}}) \bar{u}(x, t) & (p = p_F).
\end{cases}
\]

Then, in the case of \( C_0 = 0 \), the above computations imply that
\[
\Phi[\bar{u}] \leq \begin{cases} 
(2^{-1} + Ce^{p-1}) \bar{u} & \text{if } p > p_F, \\
(2^{-1} + Ce^{p-1}(1 + t^\frac{1}{2} + t^{1-\frac{2}{\gamma}})) \bar{u} & \text{if } p = p_F
\end{cases}
\]
for any $t > 0$, where $C > 0$ is a constant depending on $N$, $p$ and $\gamma$ but not on $t$ and $z$. In particular, for $p > p_F$, $\overline{u}$ is a supersolution on $\mathbb{R}^N \times [0, \infty)$ if $c$ is sufficiently small depending only on $N$, $p$ and $\gamma$. This together with Lemma 2.1 shows that (1.1) possesses a solution on $\mathbb{R}^N \times [0, \infty)$ provided that $p > p_F$, $C_0 = 0$ and $c$ is small enough.

Let us next consider the case of $C_0 \geq 0$. Fix $0 < T < \infty$. By Lemma 2.2 and the same computations as (3.6) and (3.11), we have

$$
\Phi[\overline{u}] \leq \left\{\begin{array}{ll}
2^{\frac{N}{p}+1}c\psi(z)(1 + \tilde{C}c^{p-1})U + (1 + \tilde{C}C_0^{p-1}T^{1-\gamma})C_0 & \text{if } p > p_F, \\
2^{\frac{N}{p}+1}c\psi(z)(1 + \tilde{C}c^{p-1}T^{1-\gamma})U + (1 + \tilde{C}C_0^{p-1}T^{1-\gamma})C_0 & \text{if } p = p_F
\end{array}\right.
$$

for any $0 < t < T$, where $\tilde{C} > 0$ is a constant depending on $N$, $p$ and $\gamma$ but not on $T$ and $z$. If $c$ and $C_0$ satisfy

$$
c \leq \begin{cases} (2\tilde{C})^{-\frac{1}{p-1}} & \text{if } p > p_F, \\
(2\tilde{C})^{-\frac{1}{p-1}}(1 + \tilde{C}c^{p-1}T^{1-\gamma})^{-\frac{1}{p-1}} & \text{if } p = p_F
\end{cases}, \quad C_0 \leq \tilde{C}^{-\frac{1}{p-1}}T^{-\frac{2-\gamma}{p-1}},
$$

then $\overline{u}$ is a supersolution on $\mathbb{R}^N \times [0, T)$. By Lemma 2.1, we obtain a solution on $\mathbb{R}^N \times [0, T)$. The proof is complete. 

**Remark 3.1.** We consider the case of $p = p_F$ under the additional assumption that $0 < T < \infty$ and $z \neq 0$ satisfy (1.5). For $0 < t < T$ ($\leq |z|^2$), we can improve the case of $N \leq 2$ in (3.9) to $J_1 \leq C|z|^{-\gamma}$, and then we have

$$
J(x, \eta, t) \leq C|z|^{-\gamma}(1 + t^{\frac{1}{p}})\left(\log(e + t^{\frac{1}{p}})\right)^{-\beta}
$$

for any $0 < t < T$ with a constant $C > 0$ independent of $T$ and $z$. Thus, by replacing $\psi$ with $|z|^{\gamma/(p-1)}$ for each case, we see that the function $\overline{u}^+(x, t) := 2^{(N/2)+1}c|z|^{\gamma/(p-1)}U(x, t) + 2C_0$ satisfies

$$
\Phi[\overline{u}^+] \leq (2^{1} + Ce^{p-1}(1 + T^{1/2}))\overline{u}^+ \text{ for any } 0 < t < T.
$$

This improves the condition on $\psi$ and $c$ in the statement of Theorem 1.1 for $p = p_F$. We note that the condition on $c$ is improved to $c \leq c_*(1 + T^{1/2})^{-1/(p-1)}$ for $p = p_F$.

4. **Proof of Theorem 1.1 for $p < p_F$**

Let $z \in \mathbb{R}^N \setminus \{0\}$ and $\phi \in L^\infty(\mathbb{R}^N)$. For $c > 0$, we set

$$
\overline{w}(x, t) := 2^{\frac{N}{p}+1}c\psi(z)G(x - z, 2t) + 2C_0,
$$

where $C_0 := \|\phi\|_{L^\infty(\mathbb{R}^N)}$.

**Proof of Theorem 1.1 for $p < p_F$**. We check that $\overline{w}$ is a supersolution of (1.1) if $c$ is small. The assumption on $u_0$ gives

$$
\int_{\mathbb{R}^N} G(x - y, t)du_0(y) \leq c\psi(z)G(x - z, t) + C_0 \leq \frac{1}{2}\overline{w}(x, t).
$$
By similar computations to \([3.7]\) and Lemma \([2.6]\) we have
\[
\int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)|y|^{-\gamma} \psi(y,s)pdyds \leq CC^p\psi(z)^p G(x-z,2t) + CC_0^p t^{1-\frac{2}{p}},
\]
\[
\mathcal{J}(x,t) := \int_0^t \int_{\mathbb{R}^N} G \left( y - \frac{s}{t}x - \frac{t-s}{t} z, 2 \frac{s(t-s)}{t} \right) e^{-\frac{|x-y|^2}{4t}} |y|^{-\gamma} G(y-z,2s)^{p-1} dyds.
\]

We estimate \(\mathcal{J}\). Set \(\Omega_1 := \{y \in \mathbb{R}^N; |y| \leq |z|/2\}\), \(\Omega_2 := \{y \in \mathbb{R}^N; |y| \geq |z|/2\}\) and \(\mathcal{J} = \int_0^t \int_{\Omega_1} + \int_0^t \int_{\Omega_2} =: \mathcal{J}_1 + \mathcal{J}_2\). For \(y \in \Omega_1\), we see that
\[
G(y-z,2s) = (8\pi)^{-\frac{N}{2}} s^{-\frac{N}{2}-\frac{1}{p-1}} e^{-\frac{|y-z|^2}{8s}} \leq CS s^{-\frac{1}{p-1}-\frac{N}{2}} |y-z|^{-\frac{2}{p-1}} \leq CS s^{-\frac{1}{p-1}-\frac{N}{2}} |z|^{-\frac{2}{p-1}}.
\]
This together with Lemmas \([2.3]\) and \([2.6]\) yields
\[
\mathcal{J}_1 \leq C t^{1-\frac{N}{2}(p-1)} |z|^{-2} \int_0^t \int_{\Omega_1} (G(y-z,35s) + G(y-x,35(t-s))) |y|^{-\gamma} dyds \leq C t^{1-\frac{N}{2}(p-1)+1-\frac{2}{p}} |z|^{-2} \leq C t^{\frac{N}{2}(p-pF)-(p-1)}
\]
for any \(t > 0\). On the other hand, by \(G(y-z,2s) \leq Cs^{-N/2}\) and \(\int_{\mathbb{R}^N} Gdy = 1\), we have
\[
\mathcal{J}_2 \leq \int_0^t \int_{\Omega_2} G \left( y - \frac{s}{t}x - \frac{t-s}{t} z, 2 \frac{s(t-s)}{t} \right) |y|^{-\gamma} G(y-z,2s)^{p-1} dyds \leq C |z|^{-\gamma} \int_0^t s^{-\frac{N}{2}(p-1)} \int_{\Omega_2} G \left( y - \frac{s}{t}x - \frac{t-s}{t} z, 2 \frac{s(t-s)}{t} \right) dyds \leq C |z|^{-\gamma} t^{\frac{N}{2}(pF-p)} \leq C t^{\frac{N}{2}(pF-p)-(p-1)}(1+t^{1-\frac{2}{p}})\psi(z)^{-(p-1)}
\]
for any \(t > 0\). Hence \(\mathcal{J}(x,t) \leq C t^{(N/2)(pF-p)+(1+t^{1-\gamma/2})}\psi(z)^{-(p-1)}\) for \(t > 0\).

Fix \(0 < T < \infty\). The above computations imply that
\[
\Phi[\overline{w}] \leq \frac{1}{2} \overline{w} + \tilde{C} c^{p-1} T^{\frac{N}{2}(pF-p)}(1 + T^{1-\gamma/2})c\psi(z)G(x-z,2t) + \tilde{C} C_0^p T^{1-\frac{2}{p}}
\]
for any \(0 < t < T\), where \(\tilde{C} > 0\) is a constant independent of \(T\) and \(z\). If \(c\) and \(C_0\) satisfy
\[
c \leq (2\tilde{C})^{-\frac{1}{p-1}} T^{\frac{N}{2}(pF-p)} (1 + T^{1-\gamma/2})^{-\frac{1}{p-1}}, \quad C_0 \leq (2\tilde{C})^{-\frac{1}{p-1}} T^{-\frac{2-\gamma}{p-1}},
\]
then \(\overline{w}\) is a supersolution on \(\mathbb{R}^N \times [0,T]\). By Lemma \([2.1]\) we obtain a solution on \(\mathbb{R}^N \times [0,T]\). The proof is complete. \(\square\)

**Remark 4.1.** We consider the case of \(p < p_F\) under \([1.5]\). For \(0 < t < T (\leq |z|^2)\), we can improve \([1.1]\) to \(J_1 \leq C t^{(N/2)(pF-p)}|z|^{-\gamma}\), and then we have \(\mathcal{J}(x,t) \leq C t^{(N/2)(pF-p)}|z|^{-\gamma}\) for any \(0 < t < T\) with a constant \(C > 0\) independent of \(T\) and \(z\). Thus, we see that the function \(\overline{w}^+(x,t) := 2^{(N/2)+1} c|z|^{\gamma/(p-1)} G(x-z,2t) + 2C_0\) satisfies
\[
\Phi[\overline{w}^+] \leq \frac{1}{2} \overline{w} + C c^{p-1} T^{\frac{N}{2}(pF-p)} c|z|^{\gamma/2} G(x-z,2t) + CC_0^p T^{1-\frac{2}{p}}\]
for any \(0 < t < T\). This improves the condition on \(\psi\) and \(c\) in the statement of Theorem \([1.1]\) for \(p < p_F\). The improved condition on \(c\) is \(c \leq c_s T^{-N(pF-p)/(2(p-1))}\) for \(p < p_F\).
5. Proof of Theorem 1.2 for \( p \geq p_\gamma \)

Theorem 1.2 with \( p > p_\gamma \) was proved in [4, 12]. However, we handle \( p > p_\gamma \) and \( p = p_\gamma \) in a unified way. Let \( H \) be as in (3.1), where \( A \) is chosen so large that (3.2) holds. Fix \( z \in \mathbb{R}^N \). For \( c > 0 \), define

\[
\overline{v}(x, t) := 2cV(x, t) + 2C_0,
\]

where

\[
V(x, t) := H^{-1}\left( \int_{\mathbb{R}^N} G(x - y, t)H(g(y))dy \right),
\]

\[
g(x) := \begin{cases} |x - z|^{-\frac{2}{p-1}} & \text{if } p > p_\gamma, \\ |x - z|^{-N} \left( \log \left( e + |x - z|^{-1} \right) \right)^{-\frac{N}{2-\gamma}} & \text{if } p = p_\gamma. \end{cases}
\]

We give estimates of \( V \). Lemma 2.6 shows that

\[
H(V(x, t)) \leq \begin{cases} Ct^{-\frac{N}{2-p}} \left( e^{-N\log(1+t)} \right) & (p > p_\gamma) \\ Ct^{-\frac{N}{2-p}} \left( e^{-N \log(1+t)} \right) & (p = p_\gamma) \end{cases}
\]

Then the monotonicity of \( H^{-1} \) together with (3.3) implies that

\[
V(x, t) \leq \begin{cases} Ct^{-\frac{2-\gamma}{2(p-1)}} & \text{if } p > p_\gamma, \\ Ct^{-\frac{N}{2-\gamma}} \left( \log(e + t^{-\frac{1}{2}}) \right)^{\frac{N}{2-\gamma}} & \text{if } p = p_\gamma. \end{cases} \tag{5.1}
\]

We are now in a position to prove Theorem 1.2

**Proof of Theorem 1.2 for \( p \geq p_\gamma \).** We only give a proof in the case of \( C_0 = 0 \), since the case of \( C_0 > 0 \) can be handled in the same way as in the last part of Section 3. We check that \( \overline{v} \) is a supersolution of (1.1) if \( c \) is small. By the same computations as (3.6) and (3.7), we have

\[
\int_{\mathbb{R}^N} G(x - y, t)u_0(y)dy \leq c \int_{\mathbb{R}^N} G(x - y, t)g(y)dy \leq cV(x, t) = \frac{1}{2}\overline{v}(x, t),
\]

\[
\int_0^t \int_{\mathbb{R}^N} G(x - y, t - s) |y|^{-\gamma \overline{v}(y, s)} dyds = 2^p c^p \int_{\mathbb{R}^N} G(x - \eta, t)H(g(\eta))J(x, \eta, t)d\eta,
\]

where

\[
J(x, \eta, t) := \int_0^t \int_{\mathbb{R}^N} G \left( y - \frac{s}{t} x - \frac{t-s}{t} \eta, \frac{s(t-s)}{t} \right) |y|^{-\gamma} \frac{V(y, s)^p}{H(V(y, s))} dyds.
\]

In what follows, we write \( \|V(\cdot, t)\|_\infty := \|V(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \). Then the monotonicity of \( X \mapsto X^p/H(X) \) in (3.2) implies that

\[
J(x, \eta, t) \leq \int_0^t \frac{\|V(\cdot, s)\|_\infty^p}{H(\|V(\cdot, s)\|_\infty)} \int_{\mathbb{R}^N} G \left( y - \frac{s}{t} x - \frac{t-s}{t} \eta, \frac{s(t-s)}{t} \right) |y|^{-\gamma} dyds.
\]

14
Note that
\[ \sup_{\xi \in \mathbb{R}^N} \int_{\mathbb{R}^N} G \left( y - \xi, \frac{s(t - s)}{t} \right) |y|^{-\gamma} dy = \int_{\mathbb{R}^N} G \left( y, \frac{s(t - s)}{t} \right) |y|^{-\gamma} dy \leq C \left( \frac{s(t - s)}{t} \right)^{-\frac{\gamma}{2}}. \] (5.3)

Moreover, by (5.1), we have
\[ \frac{\| V(\cdot, s) \|_p}{H(\| V(\cdot, s) \|_\infty)} \leq \begin{cases} C s^{-\frac{(2-\gamma)(p-\alpha)}{2(p-1)}} & \text{if } p > p_\gamma, \\ C s^{-\frac{N}{2}(p-1)} \left( \log(1 + s^{-\frac{1}{2}}) \right)^{-1-\beta} & \text{if } p = p_\gamma. \end{cases} \]

Recall \( p_\gamma = 1 + (2 - \gamma)/N \). Then,
\[ \tilde{J} \leq \begin{cases} C \int_0^{t/2} + C \int_{t/2}^t s^{-\frac{(2-\gamma)(p-\alpha)}{2(p-1)}} \left( \frac{s(t - s)}{t} \right)^{-\frac{\gamma}{2}} ds & \text{if } p > p_\gamma, \\ C \int_0^{t/2} s^{-\frac{N}{2}(p-1)} \left( \log(1 + s^{-\frac{1}{2}}) \right)^{-1-\beta} \left( \frac{s(t - s)}{t} \right)^{-\frac{\gamma}{2}} ds & \text{if } p = p_\gamma. \end{cases} \]

Hence the same computations as (3.9) yield
\[ \tilde{J}(x, t) \leq \begin{cases} Ct^{-\frac{(2-\gamma)(\alpha-1)}{2(p-1)}} & \text{if } p > p_\gamma, \\ C(1 + t^{\frac{1}{2}}) \left( \log(1 + t^{-\frac{1}{2}}) \right)^{-\beta} & \text{if } p = p_\gamma. \end{cases} \]

From this, it follows that
\[ \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s) |y|^{-\gamma} v(y, s)^p dy ds \leq \begin{cases} C e^{p} t^{-\frac{(2-\gamma)(\alpha-1)}{2(p-1)}} \int_{\mathbb{R}^N} G(x - \eta, t) H(g(\eta)) d\eta & \text{if } p > p_\gamma, \\ C e^{p} (1 + t^{\frac{1}{2}}) \left( \log(1 + t^{-\frac{1}{2}}) \right)^{-\beta} \int_{\mathbb{R}^N} G(x - \eta, t) H(g(\eta)) d\eta & \text{if } p = p_\gamma. \end{cases} \]

By (5.1) together with the same computations as (3.11), we have
\[ \int_{\mathbb{R}^N} G(x - \eta, t) H(g(\eta)) d\eta \leq \frac{H(\| V(\cdot, t) \|_\infty)}{\| V(\cdot, t) \|_\infty} V(x, t) \leq \begin{cases} C t^{-\frac{(2-\gamma)(\alpha-1)}{2(p-1)}} V & \text{if } p > p_\gamma, \\ C \left( \log(1 + t^{-\frac{1}{2}}) \right)^{\beta} V & \text{if } p > p_\gamma, \end{cases} \]
so that
\[ \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s) |y|^{-\gamma} v(y, s)^p dy ds \leq \begin{cases} C e^{p-1} v & \text{if } p > p_\gamma, \\ C e^{p-1} (1 + t^{\frac{1}{2}}) v & \text{if } p > p_\gamma. \end{cases} \]
The above computations imply that

\[ \Phi[\bar{\varphi}] \leq \begin{cases} (2^{-1} + Cc^{p-1})\bar{\varphi} & \text{if } p > p_\gamma, \\ (2^{-1} + Cc^{p-1}(1 + t^{\frac{1}{2}}))\bar{\varphi} & \text{if } p = p_\gamma, \end{cases} \]

where \( C > 0 \) is a constant depending on \( N, p, \gamma \) but not on \( c \). Then \( \bar{\varphi} \) is a supersolution of (1.1) if we restrict the range of \( t \) suitably and \( c \) is small. The proof is complete. \( \square \)

6. Proof of Theorem 1.2 for \( p < p_\gamma \)

Let \( z \in \mathbb{R}^N \) and \( \phi \in L^\infty(\mathbb{R}^N) \). For \( c > 0 \), we define

\[ \tilde{w}(x,t) := 2cG(x-z,t) + 2C_0, \]

where \( C_0 := \|\phi\|_{L^\infty(\mathbb{R}^N)} \).

**Proof.** By Lemma 2.1, it suffices to construct a supersolution of (1.1). The assumption on \( u_0 \) gives

\[ \int_{\mathbb{R}^N} G(x-y,t)du_0(y) \leq cG(x-z,t) + C_0 = \frac{1}{2} \tilde{w}(x,t). \]

From the same computations as (5.2), \( G(y-z,s) \leq Cs^{-N/2} \) and (5.3), it follows that

\[ \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)|y|^{-\gamma}\tilde{w}(y,s)^pdyds \]

\[ \leq Cc^pG(x-z,t)\int_0^t \int_{\mathbb{R}^N} G\left(y - s\frac{x}{t} + t - s, \frac{s(t-s)}{t}, t\right)|y|^{-\gamma}G(y-z,s)^{p-1}dyds + CC_0^p t^{1-\frac{p}{2}} \]

\[ \leq Cc^pG(x-z,t) \int_0^t s^{-\frac{N}{2}(p-1)}\left(s\frac{(t-s)}{t}\right)^{-\frac{p}{2}}ds + CC_0^p t^{1-\frac{p}{2}}. \]

Since \( p < p_\gamma \), we have

\[ \int_0^t s^{-\frac{N}{2}(p-1)}\left(s\frac{(t-s)}{t}\right)^{-\frac{p}{2}}ds \leq C \int_0^{t/2} s^{-\frac{N}{2}(p-1)-\frac{p}{2}}ds + C \int_{t/2}^t (t-s)^{-\frac{N}{2}(p-1)-\frac{p}{2}}ds \leq Ct^{\frac{N}{2}(p_\gamma-p)}. \]

Then,

\[ \Phi[\tilde{w}] \leq \frac{1}{2}\tilde{w} + C(c^{p-1}t^{\frac{N}{2}(p_\gamma-p)})cG(x-z,t) + CC_0^p t^{1-\frac{p}{2}}, \]

where \( C > 0 \) is a constant independent of \( c \). By restricting the range of \( t \) and taking a small constant \( c > 0 \), we see that \( \tilde{w} \) is a supersolution of (1.1). The proof is complete. \( \square \)

7. Fractional Hardy parabolic equation

In this section, we consider the Cauchy problem for the fractional Hardy parabolic equation

\[ \begin{cases} \partial_t u + (-\Delta)^\theta u = |x|^{-\gamma}u^p & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (7.1) \]
where \( N \geq 1, 0 < \theta < 2, p > 1, 0 < \gamma < \min\{\theta, N\} \) and \( 0 < T \leq \infty \). Set

\[
p_{F,\theta} := 1 + \frac{\theta}{N}, \quad p_{\gamma,\theta} := 1 + \frac{\theta - \gamma}{N}
\]

and

\[
\psi_\theta(z) := \begin{cases} 
|z|^{\frac{\theta}{p-1}} (1 + |z|)^{-\frac{\theta p}{p-1}} & \text{if } p < p_{F,\theta} \text{ with } N \geq 1 \text{ or } p = p_{F,\theta} \text{ with } N \leq \theta, \\
|z|^{\frac{\theta}{p-1}} & \text{if } p = p_{F,\theta} \text{ with } N > \theta \text{ or } p > p_{F,\theta} \text{ with } N \geq 1.
\end{cases}
\] (7.2)

Our results for (7.1) are as follows.

**Theorem 7.1.** Fix \( N \geq 1, 0 < \theta < 2, p > 1, 0 < \gamma < \min\{\theta, N\}, 0 < T < \infty \) and \( z \in \mathbb{R}^N \setminus \{0\} \).

Let \( \psi_\theta \) be as in (7.2). Assume either \( u_0 \) is a nonnegative Radon measure satisfying

\[
u_0 = c\psi_\theta(z)\delta_z + \phi \quad \text{if } p < p_{F,\theta}
\]

for a nonnegative function \( \phi \in L^\infty(\mathbb{R}^N) \) with \( \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq C_0 \), or \( u_0 \) is a nonnegative measurable function satisfying

\[
u_0(x) = \begin{cases} 
c\psi_\theta(z)(x-z)^{\gamma-N} (\log(e + |x-z|^{-1}))^{-\frac{N}{p-1}} \chi_1(|x-z|) + C_0 & \text{if } p = p_{F,\theta}, \\
c\psi_\theta(z)|x-z|^{-\frac{\theta p}{p-1}} + C_0 & \text{if } p > p_{F,\theta}
\end{cases}
\]

for any \( x \in \mathbb{R}^N \setminus \{z\} \). Here \( c > 0 \) and \( C_0 \geq 0 \) are constants. Then there exist positive constants \( c_* \) and \( C_* \) depending on \( N, \theta, p \) and \( \gamma \) but not on \( T \) and \( z \) such that the following statements hold. If the constants \( c \) and \( C_0 \) satisfy

\[
c \leq \begin{cases} 
c_*(1 + T^\frac{\theta}{p} + T^{1\gamma-\frac{\theta}{p}})^{-\frac{1}{p-1}} & \text{if } p = p_{F,\theta}, \\
c_* T^{-\frac{N(p_{F,\theta}-p)}{\theta(p-1)}} (1 + T^{1\frac{\theta}{p}})^{-\frac{1}{p-1}} & \text{if } p < p_{F,\theta}, \\
c_* T^{-\frac{\theta p}{p-1}} & \text{if } p > p_{F,\theta},
\end{cases}
\]

respectively, then (7.1) possesses a solution on \( \mathbb{R}^N \times [0, T) \). In addition, if \( p > p_{F,\theta}, c \leq c_* \) and \( C_0 = 0 \), then (7.1) possesses a solution on \( \mathbb{R}^N \times [0, \infty) \).

**Remark 7.1.** We can improve \( \psi_\theta \) to \( |z|^{\gamma/p-1} \) for each case if \( T \) and \( z \neq 0 \) satisfy \( 0 < T \leq |z|^{\theta} \).

**Theorem 7.2.** Fix \( N \geq 1, 0 < \theta < 2, p > 1, 0 < \gamma < \min\{\theta, N\}, 0 < T < \infty \) and \( z \in \mathbb{R}^N \).

Assume either \( u_0 \) is a nonnegative Radon measure satisfying

\[
u_0 = c\delta_z + \phi \quad \text{if } p < p_{\gamma,\theta}
\]

for a nonnegative function \( \phi \in L^\infty(\mathbb{R}^N) \) with \( \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq C_0 \), or \( u_0 \) is a nonnegative measurable function satisfying

\[
u_0(x) = \begin{cases} 
c|x-z|^{-\gamma} (\log(e + |x-z|^{-1}))^{-\frac{N}{p-1}} \chi_1(|x-z|) + C_0 & \text{if } p = p_{\gamma,\theta}, \\
c|x-z|^{-\frac{\theta p}{p-1}} + C_0 & \text{if } p > p_{\gamma,\theta}
\end{cases}
\]
for any $x \in \mathbb{R}^N \setminus \{z\}$. Here $c > 0$ and $C_0 \geq 0$ are constants. Then there exist positive constants $c_*$ and $C_*$ depending on $N$, $\theta$, $p$ and $\gamma$ but not on $T$ and $z$ such that the following statements hold. If the constants $c$ and $C_0$ satisfy

$$c \leq \begin{cases} 
  c_* T^{\frac{-N(p,\theta) - p}{\sigma(p-1)}} & \text{if } p < p_{\gamma,\theta}, \\
  c_* (1 + T^{\frac{1}{p-1}})^{-\frac{1}{p-1}} & \text{if } p = p_{\gamma,\theta}, \\
  c_* & \text{if } p > p_{\gamma,\theta},
\end{cases} \quad \quad \quad C_0 \leq C_* T^{\frac{\theta - \gamma}{\sigma(p-1)}},$$

respectively, then (7.1) possesses a solution on $\mathbb{R}^N \times [0, T)$. In addition, if $p > p_{\gamma,\theta}$, $c \leq c_*$ and $C_0 = 0$, then (7.1) possesses a solution on $\mathbb{R}^N \times [0, \infty)$.

**Remark 7.2.** We remark that each of the singularities in Theorems 7.1 and 7.2 is optimal at least we additionally assume that $\theta$, $p$ and $\gamma$ satisfy $0 < \gamma < \theta(p - 1)$. This assumption is needed to show necessary conditions for the existence of solutions of (7.1). For more details of necessary conditions, see [12].

The key to prove the above theorems is the following Lemma.

**Lemma 7.1** ([5, Inequality (9)]). Let $G_{\theta}$ be the fundamental solution of the fractional heat equation $\partial_t u + (-\Delta)^{\theta/2} u = 0$ in $\mathbb{R}^N \times (0, \infty)$. Then there exists $C > 0$ depending only on $N$ and $\theta$ such that, for any $x, y, \eta \in \mathbb{R}^N$ and $0 < s < t$,

$$G_{\theta}(x - y, t - s) G_{\theta}(y - \eta, s) \leq CG_{\theta}(x - \eta, t) (G_{\theta}(y - \eta, s) + G_{\theta}(y - x, t - s)).$$

By applying this lemma instead of Lemmas 2.2 and 2.3 and using upper and lower estimates of $G_{\theta}$ (see for instance [6, 17]), we can prove Theorems 7.1 and 7.2 in much the same way as Theorems 1.1 and 1.2. We leave the details to the reader.

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