THEORETICAL ANALYSIS ON A DIFFUSIVE SIR EPIDEMIC MODEL WITH NONLINEAR INCIDENCE IN A HETEROGENEOUS ENVIRONMENT

CHENGXIA LEI, FUJUN LI AND JIANG LIU

School of Mathematics and Statistics, Jiangsu Normal University
Xuzhou, 221116, Jiangsu Province, China

(Communicated by Yuan Lou)

ABSTRACT. In this paper, we deal with a diffusive SIR epidemic model with nonlinear incidence of the form $I^pS^q$ for $0 < p \leq 1$ in a heterogeneous environment. We establish the boundedness and uniform persistence of solutions to the system, and the global stability of the constant endemic equilibrium in the case of a homogeneous environment. When the spatial environment is heterogeneous, we determine the asymptotic profile of the endemic equilibrium if the diffusion rate of the susceptible or infected population is small. Our theoretical analysis shows that, different from the studies of [1, 28, 38, 44] for the SIS models, restricting the movement of the susceptible or infected population cannot lead to the extinction of infectious disease for such an SIR system.

1. Introduction. In order to model disease dynamics, in the pioneering work of Kermack and McKendrick [24], according to the principle of mass action, the bilinear incidence $\beta IS$ was used to describe the spread of an infection between susceptible and infected individuals.

Nevertheless, as explained in several research works (see, for example, [8, 9, 17, 18, 19, 20, 30, 31]), such a standard bilinear incidence rate carries some shortcomings and may require modification. As such, in certain situations a nonlinear incidence rate is used to govern the spread of infectious disease. The most commonly used nonlinear incidence takes the form $I^pS^q$, where $S$ and $I$ are the number of susceptible and infective individuals in the population, respectively, and $\beta$, $0 < p \leq 1$ and $q$ are positive constants. In recent years, epidemic models with this incidence rate have received extensive investigation; see [8, 9, 16, 17, 18, 19, 20, 30, 31, 32, 33] among many others.

In particular, Korobeinikov and Maini [25] considered the following SIR ODE model

$$
\begin{align*}
    \dot{S}(t) &= b - \beta I^pS^q - \mu S, & t > 0, \\
    \dot{I}(t) &= \beta I^pS^q - \delta I, & t > 0.
\end{align*}
$$

(1)

In (1), the population is divided into two classes: susceptible and infective, denoted by $S$ and $I$, respectively. Here, $b$, $\mu$, $\beta$, $\delta$, $0 < p \leq 1$ and $q$ are positive constants: $b$
is the birth rate, $\beta$ is the transmission rate, $\mu$ is the death rate of the susceptible, and $\delta$ is the removal rate (including the mortality rate) of the infective. Korobeinikov and Maini established the global stability of the unique endemic equilibrium (if it exists) by constructing suitable Lyapunov functions.

To take into account the inhomogeneous distribution of the population in different spatial locations within a fixed bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) at any given time, and the natural tendency of each class of population to diffuse to areas of smaller population concentration, we are led to the following PDE system of reaction-diffusion type:

$$\begin{cases}
S_t - d_S \Delta S = b(x) - \beta(x)IP^{S^q} - \mu(x)S, & x \in \Omega, \quad t > 0, \\
I_t - d_I \Delta I = \beta(x)IP^{S^q} - \delta(x)I, & x \in \Omega, \quad t > 0, \\
\partial_\nu S = \partial_\nu I = 0, & x \in \partial \Omega, \quad t > 0,
\end{cases}$$

(2)

In (2), $S$ and $I$ are the density of susceptible and infected individuals at location $x$ and time $t$, respectively, and the positive constants $d_S$ and $d_I$ are the respective movement (or diffusive) rates. The habitat $\Omega$ is a bounded domain with smooth boundary $\partial \Omega$, and the homogeneous Neumann boundary conditions biologically mean that there is no population flux across the boundary $\partial \Omega$. The functions $b$, $\beta$, $\mu$, and $\delta$ with the same interpretations as in (1) are positive and Hölder continuous on $\Omega$.

Heterogeneity of spatial environment and the movement of the individual are important factors in the theory of epidemiology. Recent studies in such a direction can be found, for instance, in [1, 3, 4, 5, 6, 7, 10, 13, 22, 27, 28, 38, 39, 41, 42, 44] and the references therein.

The steady state problem corresponding to (2) is governed by the following elliptic system:

$$\begin{cases}
-d_S \Delta S = b(x) - \beta(x)IP^{S^q} - \mu(x)S, & x \in \Omega, \\
-d_I \Delta I = \beta(x)IP^{S^q} - \delta(x)I, & x \in \Omega, \\
\partial_\nu S = \partial_\nu I = 0, & x \in \partial \Omega.
\end{cases}$$

(3)

We call a solution $(S, I) \in C^2(\Omega) \times C^2(\Omega)$ to (3) as an endemic equilibrium (EE, for short) if $(S, I)$ solves (3) pointwisely and $S \geq 0$ and $I \geq, \neq 0$ on $\Omega$. By the well-known strong maximum principle and Hopf lemma for elliptic equations (refer to [15]), one can easily show that any EE $(S, I)$ satisfies $S(x) > 0$ and $I(x) > 0$ for all $x \in \Omega$.

As in [1, 5, 6, 7, 14, 28, 38, 44], in the present work we are interested in the qualitative properties of solutions to (2) as well as its EE.

Firstly, we derive the boundedness of solutions and then the uniform persistence property in the two cases: (i) $0 < p < 1$ and (ii) $p = 1$ and the disease-free equilibrium $(\tilde{S}, 0)$ is unstable, where $\tilde{S}$ will be given below; see our Proposition 1 and Theorem 2.1. When $p = 1$, the disease-free equilibrium $(\tilde{S}, 0)$ is stable, it is shown by Proposition 2 that $(\tilde{S}, 0)$ is the global attractor. Furthermore, when the spatial environment becomes homogeneous (that is, $b$, $\beta$, $\mu$ and $\delta$ are positive constants), Theorem 2.2 below asserts that the unique constant EE is globally attractive in cases (i) and (ii).

Then, the rest of our paper is devoted to the study of the asymptotic profile of the EE (when exists) with respect to the small movement (migration) rate of the susceptible or infected population. In Theorems 4.1 and 4.2, we will show that
the densities of the susceptible and infected populations converge to the positive functions on the whole habitat $\Omega$ as $d_S \to 0$ for any fixed $0 < p \leq 1$ or as $d_I \to 0$ for any fixed $0 < p < 1$. We conjecture that, as long as the EE exists, this is also true when $d_I \to 0$ for $p = 1$. These limiting behaviors are very different from those observed in the SIS models. In the SIS models studied by [1, 28, 38, 44], it has been proved that the infected population can become extinct in the whole or part of the habitat provided that the small movement rate of the susceptible or infected population is controlled to be small. On the other hand, we want to emphasize that the qualitative behaviors of the solution to (1) and its EE do not depend on the value of $q$.

The paper is organized as follows. In section 2, we obtain the boundedness and uniform persistence of solutions to (2) and the global stability of the unique constant EE when the parameters $b, \beta, \mu, \delta$ are constants. In section 3, we establish the positive upper and lower estimates for any EE. Such a priori estimates are crucial in the analysis of the asymptotic profile of EE for the small susceptible or infected diffusion rate which is addressed in section 4.

2. The uniform persistence and the global stability of EE. In this section, we are concerned with the uniform persistence property of solutions to (2) and the global attractivity of the constant EE in the case that all the parameters in (2) are assumed to be positive constants.

By the standard theory for parabolic equations, given continuous and nonnegative initial data $(S_0, I_0)$, (2) admits a unique classical solution $(S(x, t), I(x, t))$ which exists for all positive time, and $S(x, t) > 0$ and $I(x, t) \geq 0$ for all $x \in \Omega$ and $t > 0$. Moreover, if $I_0 \geq 0$, then $I(x, t) > 0$ for all $x \in \Omega$ and $t > 0$.

For later purpose, let us denote by $\tilde{S}$ the unique positive solution of

$$
\begin{cases}
-d_S \Delta S = b(x) - \mu(x)S, & x \in \Omega, \\
\partial_n S = 0, & x \in \partial \Omega.
\end{cases}
$$

Let $\lambda_1$ be the principal eigenvalue of the eigenvalue problem:

$$
\begin{cases}
-d_I \Delta \varphi + [\delta(x) - \beta(x)(\tilde{S}(x))^q] \varphi = \lambda \varphi, & x \in \Omega, \\
\partial_n \varphi = 0, & x \in \partial \Omega.
\end{cases}
$$

By [40, Lemma 3.2], we know that

$$\tilde{S}(x) \to \frac{b(x)}{\mu(x)} \quad \text{uniformly on } \Omega, \text{ as } d_S \to 0.$$

Thus, as $d_S \to 0$,

$$\lambda_1 \to \lambda_1^*, \quad (4)$$

where $\lambda_1^*$ is the principal eigenvalue of the eigenvalue problem

$$
\begin{cases}
-d_I \Delta \varphi + \left\{ \delta(x) - \beta(x) \left[ \frac{b(x)}{\mu(x)} \right]^q \right\} \varphi = \lambda \varphi, & x \in \Omega, \\
\partial_n \varphi = 0, & x \in \partial \Omega.
\end{cases}
$$

(5)

On the other hand, by a folklore fact (see, for instance, [1, Lemma 2.2]), we also know that

$$\lambda_1 \to \min_{\Omega} (\delta(x) - \beta(x)(\tilde{S}(x))^q), \quad \text{as } d_I \to 0.$$
2.1. The uniform persistence. We start with the boundedness of solutions to (2). Indeed, we can state the following result.

Proposition 1. There exists a positive constant $C_1$ depending on initial data such that the solution $(S, I)$ of (2) satisfies

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \forall t \geq 0. \quad (6)$$

Furthermore, there exists a positive constant $C_2$ independent of initial data such that

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2, \forall t \geq T, \quad (7)$$

for some large time $T > 0$.

Proof. By the first equation in (2), we see that

$$\begin{cases}
S_t - d_S \Delta S \leq b(x) - \mu(x)S, & x \in \Omega, \ t > 0, \\
\partial_\nu S = 0, & x \in \partial \Omega, \ t > 0, \\
S(x, 0) = S_0(x) \geq 0, & x \in \Omega.
\end{cases}$$

Consider the following problem

$$\begin{cases}
w_t - d_S \Delta w = b(x) - \mu(x)w, & x \in \Omega, \ t > 0, \\
\partial_\nu w = 0, & x \in \partial \Omega, \ t > 0, \\
w(x, 0) = S_0(x) \geq 0, & x \in \Omega.
\end{cases} \quad (8)$$

Denote by $w$ the unique solution to (8). The standard comparison principle for parabolic equations gives

$$S(x, t) \leq w(x, t) \leq \max \left\{ \max_{x \in \Omega} b(x), \max_{x \in \Omega} S_0(x) \right\}, \forall x \in \Omega, \ t \geq 0. \quad (9)$$

Moreover, it is also known that

$$\limsup_{t \to \infty} w(x, t) \leq \tilde{S}(x) \leq \max_{x \in \Omega} \frac{b(x)}{\min_{x \in \Omega} \mu(x)}, \ \text{uniformly for all} \ x \in \overline{\Omega}. \quad (10)$$

Thus, there is a large time $t_0 > 0$ such that

$$S(x, t) \leq 1 + \frac{\max_{x \in \Omega} b(x)}{\min_{x \in \Omega} \mu(x)}, \forall x \in \overline{\Omega}, \ t \geq t_0. \quad (11)$$

We next set

$$m = \min_{x \in \overline{\Omega}} \{ \min \mu(x), \min \delta(x) \}$$

and define

$$U(t) = \int_{\Omega} [S(x, t) + I(x, t)] \, dx, \forall t \geq 0.$$ 

From (2), it immediately follows that

$$\frac{dU(t)}{dt} = \int_{\Omega} bdx - \int_{\Omega} \mu Sdx - \int_{\Omega} \delta Idx$$

$$\leq \int_{\Omega} bdx - \min_{x \in \overline{\Omega}} \mu(x) \int_{\Omega} Sdx - \min_{x \in \overline{\Omega}} \delta(x) \int_{\Omega} Idx$$

$$\leq \int_{\Omega} bdx - mU(t).$$
Thus,
\[
\frac{dU(t)}{dt} + mU(t) \leq \int_{\Omega} b(x) dx \leq |\Omega| \max_{x \in \Omega} b(x),
\]
and so
\[
U(t) \leq U(0)e^{-mt} + \frac{|\Omega| \max_{x \in \Omega} b(x)}{m} (1 - e^{-mt}), \quad \forall t \geq 0.
\]
That is,
\[
\int_{\Omega} \left[ S(x,t) + I(x,t) \right] dx \leq e^{-mt} \int_{\Omega} \left[ S_0(x) + I_0(x) \right] dx + \frac{|\Omega| \max_{x \in \Omega} b(x)}{m} (1 - e^{-mt}), \quad \forall t \geq 0.
\] (12)
Together with (9), we then apply [12, Lemma 2.1] (due to [34] with \( \sigma = p_0 = 1 \)) to the second equation in (2) to conclude that (6) holds. In addition, by (12) one has
\[
\limsup_{t \to \infty} \int_{\Omega} I(x,t) dx \leq \limsup_{t \to \infty} \int_{\Omega} \left[ S(x,t) + I(x,t) \right] dx \leq \frac{|\Omega| \max_{x \in \Omega} b(x)}{m}.
\]
Resorting to [12, Lemma 2.1] to the second equation in (2) again, we are able to assert that there exists a positive constant \( C_2 \) independent of initial data such that
\[
I(x,t) \leq C_2, \quad \forall x \in \Omega, \ t \geq T_0,
\]
for some large time \( T_0 > 0 \). This and (11) imply (7).

**Proposition 2.** If \( p = 1 \), \( q > 0 \) and \( \tilde{S}(x) < \left( \frac{\delta(x)}{\beta(x)} \right)^{\frac{1}{q}} \) for all \( x \in \overline{\Omega} \), then any solution \( (S,I) \) of (2) satisfies
\[
(S,I) \to (\tilde{S},0) \text{ uniformly on } \overline{\Omega}, \text{ as } t \to \infty.
\]
Hence, (2) has no EE in this case.

**Proof.** In light of (10), for any small \( \epsilon > 0 \), there exists a large time \( t_* > 0 \), such that
\[
S(x,t) \leq \tilde{S}(x) + \epsilon, \quad \forall x \in \overline{\Omega}, \ t \geq t_*.
\]
Hence, \( I \) satisfies
\[
\begin{cases}
I_t - d_I \Delta I \leq \{ \beta(x) [\epsilon + \tilde{S}(x)]^q - \delta(x) \} I, & x \in \Omega, \ t > t_*, \\
\partial_n I = 0, & x \in \partial \Omega, \ t > t_*, \\
I(x,t_*) = I(x,t_*), & x \in \Omega.
\end{cases}
\]
Since \( \tilde{S}(x) < \left( \frac{\delta(x)}{\beta(x)} \right)^{\frac{1}{q}} \) for all \( x \in \overline{\Omega} \), we can take \( \epsilon > 0 \) to be smaller so that \( \beta(x) [\epsilon + \tilde{S}(x)]^q - \delta(x) < 0 \) for all \( x \in \overline{\Omega} \). Thus, a simple comparison argument gives
\[
I(x,t) \to 0 \text{ uniformly on } \overline{\Omega}, \text{ as } t \to \infty.
\]
Using this fact, it is easily seen from the first equation in (2) that
\[
\liminf_{t \to \infty} S(x,t) \geq \tilde{S}(x) \text{ uniformly for all } x \in \overline{\Omega},
\]
which and (10) yield
\[
\lim_{t \to \infty} S(x,t) = \tilde{S}(x) \text{ uniformly for all } x \in \overline{\Omega}.
\]
In particular, this implies that (2) admits no EE. \( \square \)
Remark 1. In fact, it is easily seen that \( \tilde{S}(x) \leq \max_{x \in \bar{\Omega}} \frac{b(x)}{\mu(x)} \) on \( \bar{\Omega} \) (by Lemma 3.1(i) below). Thus, the assertion of Proposition 2 holds provided that \( b(x) < \mu(x) \left( \frac{\delta(x)}{\beta(x)} \right)^{\frac{1}{q}} \) on \( \bar{\Omega} \).

The main aim of this subsection is to establish the uniform persistence of solutions to (2). Specifically, we can state that

**Theorem 2.1.** Assume that \( 0 < p < 1 \) or \( p = 1 \) and \( \lambda_1 < 0 \). Then there exists a real number \( \eta > 0 \) independent of the initial data, such that any solution \((S, I)\) of (2) satisfies

\[
\liminf_{t \to \infty} S(x, t) \geq \eta, \quad \liminf_{t \to \infty} I(x, t) \geq \eta,
\]

uniformly for \( x \in \bar{\Omega} \), and hence, the disease persists uniformly. Furthermore, (2) admits at least one EE.

**Proof.** We shall use the abstract dynamical systems theory to obtain the desired result. Let \( X = C(\bar{\Omega}, \mathbb{R}_+^2) \), \( \|\varphi\| = \max_{x \in \bar{\Omega}} |\varphi(x)| \) and

\[
X_0 = \{ \varphi = (\varphi_1, \varphi_2) \in X : \varphi_2(x) \neq 0 \},
\]

\[
\partial X_0 = X \setminus X_0 = \{ \varphi \in X : \varphi_2(x) \equiv 0 \}.
\]

Given \( \varphi \in X \), we let

\[
[\phi(t) \varphi](x) = (S(t, x, \varphi), I(t, x, \varphi))
\]

be the unique solution to (2) with \((S_0, I_0) = \varphi\).

First of all, if \( \varphi \in \partial X_0 \), the uniqueness of solutions, we notice that \( I(t, x, \varphi) \equiv 0 \) for all \( t \geq 0 \), and so \( S(t, x, \varphi) \) solves (8). Hence, the analysis of Proposition 2 implies that

\[
S(t, x, \varphi) \to \tilde{S}(x) \quad \text{uniformly on } \bar{\Omega}, \quad t \to \infty.
\]

Let \( M_0 = (\tilde{S}, 0) \), we next conclude that there is a positive constant \( \eta_0 \) such that

\[
\limsup_{t \to \infty} d(\phi(t) \varphi, M_0) = \limsup_{t \to \infty} \|\phi(t) \varphi - M_0\| \geq \eta_0, \quad \forall \varphi \in X_0.
\]  \hspace{1cm} (13)

To the end, we have to treat two cases: \( 0 < p < 1 \); \( p = 1 \) and \( \lambda_1 < 0 \). We first consider the case of \( 0 < p < 1 \). Clearly, in such a situation, one can find a sufficiently small \( \eta_0 > 0 \) such that

\[
\frac{\beta(x)(\tilde{S}(x) - \eta_0)^q}{\eta_0^{1-p}} - \delta(x) > 0, \quad \forall x \in \bar{\Omega}.
\]

Then

\[
m := \min_{x \in \bar{\Omega}} \left( \frac{\beta(x)(\tilde{S}(x) - \eta_0)^q}{\eta_0^{1-p}} - \delta(x) \right) > 0.
\]

Suppose that \( \limsup_{t \to \infty} d(\phi(t) \varphi, M_0) < \eta_0 \) for some \( \varphi \in X_0 \). Without loss of generality, we may assume that \( d(\phi(t) \varphi, M_0) < \eta_0 \), \( \forall t \geq 0 \); otherwise, we can use \( \phi(t_0) \varphi \) as a new initial data for a large \( t_0 > 0 \). Then it is easily seen that

\[
\tilde{S}(x) - \eta_0 < S(t, x) < \tilde{S}(x) + \eta_0, \quad \forall t \geq 0, \quad x \in \bar{\Omega},
\]

\[
0 < I(t, x) < \eta_0, \quad \forall t \geq 0, \quad x \in \bar{\Omega}.
\]

This in turn gives

\[
I_t \geq d_I \Delta I + \left( \frac{\beta(x)(\tilde{S}(x) - \eta_0)^q}{\eta_0^{1-p}} - \delta(x) \right) I
\]

\[
\geq d_I \Delta I + m I, \quad \forall t \geq 0, \quad x \in \bar{\Omega}.
\]
Since \( m > 0 \), a simple comparison argument deduces that \( I(t, x) \to \infty \) uniformly on \( \bar{\Omega} \) as \( t \to 0 \), leading to a contradiction with Proposition 1. Therefore, the claim (13) holds.

When \( p = 1 \) and \( \lambda_1 < 0 \), the claim (13) remains valid by using the similar analysis to that of [5, Theorem 1.1(b-i)] (or [21, Theorem 4.2(ii)]). So the details are omitted here.

Now, in [43, Theorem 3], we set \( P(\varphi) = \min_{x \in \bar{\Omega}} \varphi_2(x) \). Then, from [43, Theorem 3] it follows that

\[
\liminf_{t \to \infty} P(\phi(t)\varphi) \geq \eta_1, \quad \forall \varphi \in X_0
\]

for some positive constant \( \eta_1 > 0 \), that is,

\[
\liminf_{t \to \infty} \min_{x \in \bar{\Omega}} I(t, x, \varphi) \geq \eta_1, \quad \forall \varphi \in X_0.
\]

This implies that for all \( \varphi \in X_0 \), there exists a large \( T_1 = T_1(\varphi) > 0 \) such that

\[
I(t, x, \varphi) \geq \frac{\eta_1}{2}, \quad \forall t \geq T_1, x \in \Omega.
\]

Making use of this fact and a comparison argument, one can easily get from the first equation of (2) that

\[
S(t, x, \varphi) \geq \frac{\eta_2}{2}, \quad \forall t \geq T_2, x \in \Omega
\]

for some positive constant \( \eta_2 > 0 \) and for some \( T_2 \geq T_1 \). As a consequence, the uniform persistence holds by taking \( \eta = \min\{\eta_1/2, \eta_2/2\} \).

Furthermore, appealing to the theory developed by Magal and Zhao (see [37, Theorem 4.5] or [45]), we also see that the system (2) admits at least one EE under our assumption. The proof is complete.

\[\square\]

2.2. Global stability of the EE. This subsection is devoted to the study of the global stability of the EE of (2) when \( b, \beta, \mu, \) and \( \delta \) are positive constants.

We first solve the following equations to find the constant EE of (2):

\[
\begin{align*}
\begin{cases}
  b - \beta ISq - \mu S &= 0, \\
  \beta ISq - \delta I &= 0,
\end{cases}
\end{align*}
\]

that is,

\[
\begin{align*}
\begin{cases}
  b = \delta I + \mu S, \\
  \beta ISq = \delta I.
\end{cases}
\end{align*}
\]

(14)

If \( 0 < p < 1 \) and \( q > 0 \), according to the second equation of (14), we have

\[
I = \left( \frac{\beta}{\delta} \right)^{\frac{1}{1-p}} S^{\frac{2}{1-p}}.
\]

(15)

Inserting (15) into (14) results in the following equality

\[
b = \mu S + \delta \left( \frac{\beta}{\delta} \right)^{\frac{1}{1-p}} S^{\frac{2}{1-p}}.
\]

Next we define a function:

\[
f(z) = \mu z + \delta \left( \frac{\beta}{\delta} \right)^{\frac{1}{1-p}} z^{\frac{q}{1-p}} - b.
\]
Clearly, \( f(z) \) is increasing in \([0, \infty)\). As \( f(0) = -b < 0 \) and \( \lim_{z \to +\infty} f(z) = +\infty \), \( f(z) = 0 \) has a unique positive solution \( z = z_0 \), in turn, (2) has a unique positive solution

\[
(S^*, I^*) = \left( z_0, \left( \frac{\beta}{\delta} \right)^{\frac{1}{1-p}} z_0^{\frac{1}{1-p}} \right).
\]

If \( p = 1 \), \( q > 0 \) and \( b > \mu \left( \frac{\delta}{\beta} \right)^{\frac{1}{q}} \), it is also easily seen that (2) has a unique positive solution

\[
(S^*, I^*) = \left( \left( \frac{\delta}{\beta} \right)^{\frac{1}{q}}, \left( \frac{1}{\delta} \left[ b - \mu \left( \frac{\delta}{\beta} \right)^{\frac{1}{q}} \right] \right) \right).
\]

Our global stability result of the constant EE of (2) reads as follows.

**Theorem 2.2.** Assume that either \( 0 < p < 1 \) or \( p = 1 \) and \( b > \mu \left( \frac{\delta}{\beta} \right)^{\frac{1}{q}} \), then the constant positive steady state \((S^*, I^*)\) of (2) is globally attractive.

**Proof.** We shall construct suitable Lyapunov functionals to verify the global attractivity of \((S^*, I^*)\). The Lyapunov functionals to be employed below are inspired by [25, 26].

In the sequel, we need to consider four cases: \( 0 < p < 1, q \neq 1; 0 < p < 1, q = 1; p = 1, q \neq 1 \) and \( p = 1, q = 1 \), and present the proofs separately, for the chosen Lyapunov functionals are different.

We first consider the case of \( 0 < p < 1, q \neq 1 \). Define

\[
L(S, I) = S \left[ 1 + \frac{1}{q-1} \left( \frac{S^*}{S} \right)^q \right] + I \left[ 1 + \frac{1}{p-1} \left( \frac{I^*}{I} \right)^p \right].
\]

This function \( L(S, I) \) satisfies

\[
\frac{\partial L}{\partial S} = 1 - \left( \frac{S^*}{S} \right)^q, \quad \frac{\partial L}{\partial I} = 1 - \left( \frac{I^*}{I} \right)^p.
\]

We observe that \((S^*, I^*)\) is the only extremum and the global minimum of the function in the positive of octant \( \mathbb{R}_+^2 \). Hence, the function (16) is indeed a Lyapunov function. By (14), we have

\[
b = \delta I^* + \mu S^*, \quad \beta (I^*)^p (S^*)^q = \delta I^*.
\]

For simplicity, let us denote

\[
f_1(S, I) = b - \beta I^p S^q - \mu S, \quad f_2(S, I) = \beta I^p S^q - \delta I.
\]

Some straightforward computation yields

\[
L_S(S, I) f_1(S, I) + L_I(S, I) f_2(S, I)
= \left[ 1 - \left( \frac{S^*}{S} \right)^q \right] \left( b - \beta I^p S^q - \mu S \right) + \left[ 1 - \left( \frac{I^*}{I} \right)^p \right] \left( \beta I^p S^q - \delta I \right)
= b - \mu S - b \left( \frac{S^*}{S} \right)^q + \beta I^p (S^*)^q + \mu \left( \frac{(S^*)^q}{S^{q-1}} \right) - \delta I - \beta (I^*)^p S^q + \frac{\delta (I^*)^p}{I^{p-1}}
= \mu S^* + \delta I^* - \mu S - (\mu S^* + \delta I^*) \left( \frac{(S^*)^q}{S^q} \right)
+ \frac{\delta I^* I^p}{(I^*)^p} + \mu \left( \frac{(S^*)^q}{S^{q-1}} \right) - \delta I - \frac{\delta I^* S^q}{(S^*)^q} + \frac{\delta (I^*)^p}{I^{p-1}}
\]
= \mu S^* \left[ 1 - \frac{S}{S^*} - \left( \frac{S^*}{S} \right)^q + \left( \frac{S^*}{S} \right)^{q-1} \right] \\
+ \delta I^* \left[ 1 - \frac{I}{I^*} - \left( \frac{S^*}{S} \right)^q + \left( \frac{I^*}{I} \right)^{p-1} + \left( \frac{I}{I^*} \right)^p - \left( \frac{S^*}{S} \right)^q \right] \\
= \mu S^* \left( 1 - u - u^{-q} + u^{1-q} \right) + \delta I^* \left( 1 - v - u^{-q} + v^{1-p} + v^p - u^q \right) \\
= \mu S^* (1-u)(1-u^{-q}) - \delta I^*(v^p-1)(v^{1-p}-1) + \delta I^*(2-u^{-q} - u^q), \quad (17) 

where \quad u = \frac{S}{S^*}, \quad v = \frac{I}{I^*}.

Due to 0 < p < 1, q > 0, it is clear that 

\( (v^p-1)(v^{1-p}-1) \geq 0, \quad \text{for all } v > 0, \)

and 

\( (2-u^{-q} - u^q) \leq 0, \quad (1-u)(1-u^{-q}) \leq 0, \quad \text{for all } u > 0. \)

For an arbitrary solution \((S,I)\) of (2), we now define 

\[ \mathcal{L}(t) = \int_\Omega L(S(x,t),I(x,t))dx, \quad \forall t > 0. \]

Then,

\[
\frac{d\mathcal{L}(t)}{dt} = \int_\Omega [L_S(S,I)S_t + L_I(S,I)I_t]dx \\
= \int_\Omega [d_S L_S(S,I)\Delta S + d_I L_I(S,I)\Delta I]dx \\
+ \int_\Omega [L_S(S,I)f_1(S,I) + L_I(S,I)f_2(S,I)]dx. \quad (18)
\]

Furthermore, integrating by parts, we obtain

\[
\int_\Omega d_S L_S(S,I)\Delta S dx = - \int_\Omega d_S L_{SS}(S,I) |\nabla S|^2 dx, \\
\int_\Omega d_I L_I(S,I)\Delta I dx = - \int_\Omega d_I L_{II}(S,I) |\nabla I|^2 dx.
\]

Since

\[ L_{SS} = q \frac{(S^*)^q}{S^q + 1} \quad \text{and} \quad L_{II} = p \frac{(I^*)^p}{I^p + 1}, \]

it then follows that

\[
\int_\Omega [d_S L_S(S,I)\Delta S + d_I L_I(S,I)\Delta I] dx \\
= - \int_\Omega \left\{ q d_S \frac{(S^*)^q}{S^q + 1} |\nabla S|^2 + p d_I \frac{(I^*)^p}{I^p + 1} |\nabla I|^2 \right\} dx \leq 0, \quad \forall t > 0.
\]

In view of (17) and (18), \( \mathcal{L}(t) \) is a Lyapunov functional for (2), namely, \( \frac{d\mathcal{L}(t)}{dt} < 0 \), \( \forall t > 0 \) along all trajectories except at \((S^*, I^*)\) where \( \frac{d\mathcal{L}(t)}{dt} = 0 \), \( \forall t > 0 \). Some standard arguments claim that 

\( (S(x,t), I(x,t)) \to (S^*, I^*) \) in \( [L^2(\Omega)]^2 \), as \( t \to \infty. \)
Recall that both \( \|S(\cdot, t)\|_{L^\infty(\Omega)} \) and \( \|I(\cdot, t)\|_{L^\infty(\Omega)} \) are bounded. Hence, by [2, Theorem A2], we have

\[
\|S(\cdot, t)\|_{C^2(\Omega)} + \|I(\cdot, t)\|_{C^2(\Omega)} \leq C_0, \quad \forall t \geq 1,
\]

for some positive constant \( C_0 \). Hence, the Sobolev embedding theorem allows one to assert

\[
(S(x, t), I(x, t)) \to (S^*, I^*) \text{ in } [L^\infty(\Omega)]^2, \quad \text{as } t \to \infty,
\]

that is, \((S^*, I^*)\) attracts all solutions of \((2)\).

Next, we consider the case of \( p = 1, q = 1 \). In such a situation, for any solution \((S, I)\) of \((2)\), we choose

\[
\mathcal{L}(t) = \int_\Omega L(S(x, t), I(x, t)) dx
\]

with

\[
L(S, I) = (S - S^* \ln S) + (I - I^* \ln I).
\]

Then, for all \( t > 0 \),

\[
\frac{d\mathcal{L}(t)}{dt} = \int_\Omega [L_S(S, I)S_t + L_I(S, I)I_t] dx
\]

\[
= \int_\Omega \left[ \left( 1 - \frac{S^*}{S} \right)(dS\Delta S) + \left( 1 - \frac{I^*}{I} \right)(dI\Delta I) \right] dx
\]

\[
+ \int_\Omega \left[ \left( 1 - \frac{S^*}{S} \right)(b - \beta IS - \mu S) + \left( 1 - \frac{I^*}{I} \right)(\beta IS - \delta I) \right] dx
\]

\[
= -\int_\Omega \left( \frac{d}{2} \frac{S^*}{S^2} \|\nabla S\|^2 + \frac{dI^*}{2} \|\nabla I\|^2 \right) dx
\]

\[
+ \int_\Omega \left( b - \mu S - b \frac{S^*}{S} + \mu S^* - \delta I - \beta I^* S + \delta I^* + \beta S^* I \right) dx
\]

\[
= -\int_\Omega \left( \frac{d}{2} \frac{S^*}{S^2} \|\nabla S\|^2 + \frac{dI^*}{2} \|\nabla I\|^2 \right) dx
\]

\[
+ \int_\Omega \left[ \mu S^* + \delta I^* - \mu S - (\mu S^* + \delta I^*) \frac{S^*}{S} \right.
\]

\[
+ \mu S^* - \delta I - \beta I^* S + \delta I^* + \beta S^* I \] dx
\]

\[
= -\int_\Omega \left( \frac{d}{2} \frac{S^*}{S^2} \|\nabla S\|^2 + \frac{dI^*}{2} \|\nabla I\|^2 \right) dx
\]

\[
+ \int_\Omega \left[ \mu S^* - \mu S - \mu S^* \frac{S^*}{S} + \mu S^* + \delta I^* - \delta I^* \frac{S^*}{S} - \delta I^* \frac{S}{S^*} + \delta I^* \right]
\]

\[
\leq 0.
\]
Here $u$ is given as before.

When $p = 1$ and $q \neq 1$, we then take
\[ L(S, I) = S \left[ 1 + \frac{1}{q-1} \left( \frac{S^*}{S} \right)^q \right] + (I - I^* \ln I) \]
and when $0 < p < 1$ and $q = 1$ we take
\[ L(S, I) = (S - S^* \ln S) + I \left[ 1 + \frac{1}{p-1} \left( \frac{I^*}{I} \right)^p \right]. \]

In these cases, the same computation as above shows that $\frac{dC(t)}{dt} < 0$, $\forall t > 0$ along all trajectories except at $(S^*, I^*)$ where $\frac{dC(t)}{dt} = 0$, $\forall t > 0$. Thus, we can assert that $(S^*, I^*)$ is globally attractive.

**3. A priori estimates for EE of (2).** In this section, we establish a priori upper and lower bounds for solutions to (3). As it will be seen in the forthcoming section, such results play a fundamental role in obtaining the asymptotic behavior of EE.

We begin by recalling a known fact, which concerns a basic convexity property of a $C^2$ function on $\overline{\Omega}$ at its local extrema; see, for instance, [23].

**Lemma 3.1.** Assume that $w \in C^2(\overline{\Omega})$ and $\frac{\partial w}{\partial \nu} = 0$ on $\partial \Omega$, then the following statements hold.

(i) If $w$ has a local maximum at $x_1 \in \overline{\Omega}$, then $\nabla w(x_1) = 0$ and $\Delta w(x_1) \leq 0$.

(ii) If $w$ has a local minimum at $x_2 \in \overline{\Omega}$, then $\nabla w(x_2) = 0$ and $\Delta w(x_2) \geq 0$.

**Theorem 3.2.** Assume that $0 < p < 1$ and let $(S, I)$ be any given positive solution of (3). Then there exists a positive constant $C$, independent of $d_S, d_I$, such that
\[ \frac{1}{C} < S(x) < C, \quad \frac{1}{C} < I(x) < C, \quad \forall x \in \overline{\Omega}. \]

**Proof.** Let $S(x_1) = \max_{x \in \overline{\Omega}} S(x)$ for some $x_1 \in \overline{\Omega}$. By Lemma 3.1 (or [36, Proposition 2.2]), we have from the first equation in (3) that
\[ b(x_1) \geq \beta(x_1)p(x_1)S^p(x_1) + \mu(x_1)S(x_1) > \mu(x_1)S(x_1). \]

Thus, we get
\[ S(x) \leq S(x_1) \leq \frac{\max_{x \in \overline{\Omega}} b(x)}{\min_{x \in \overline{\Omega}} \mu(x)} := C_1, \quad \forall x \in \overline{\Omega}. \tag{19} \]

We now set $I(x_2) = \max_{x \in \overline{\Omega}} I(x)$ for some $x_2 \in \overline{\Omega}$. Using Lemma 3.1 again, together with (19), one deduces from the second equation in (3) that
\[ C_1 \max_{x \in \overline{\Omega}} \beta(x)p(x_2) \geq \beta(x_2)p(x_2)S^p(x_2) \geq \delta(x_2)I(x_2) \geq \min_{x \in \overline{\Omega}} \delta(x)I(x_2). \]

Hence, it holds
\[ I(x) \leq I(x_2) \leq \left[ \frac{C_1 \max_{x \in \overline{\Omega}} \beta(x)}{\min_{x \in \overline{\Omega}} \delta(x)} \right]^{1/p} := C_2, \quad \forall x \in \overline{\Omega}. \tag{20} \]

Let $S(y_1) = \min_{x \in \overline{\Omega}} S(x)$ for some $y_1 \in \overline{\Omega}$. Similarly as before, combined with (20), we have
\[ \min_{x \in \overline{\Omega}} b(x) \leq b(y_1) \leq \beta(y_1)p(y_1)S^p(y_1) + \mu(y_1)S(y_1) \leq C_2 \max_{x \in \overline{\Omega}} \beta(x)S^p(y_1) + \max_{x \in \overline{\Omega}} \mu(x)S(y_1). \]
Clearly, one can find a positive constant $C_3$, independent of $d_S, d_I$, such that
\[ S(x) \geq S(y_1) \geq C_3, \quad \forall x \in \Omega. \] (21)

Finally, by taking $I(y_2) = \min_{x \in \Omega} I(x)$ for some $y_2 \in \Omega$, it follows from Lemma 3.1 and (21) that
\[ C_3 \min_{x \in \Omega} \beta(x) I_p(y_2) \leq \beta(y_2) I_p(y_2) S^\alpha(y_2) \leq \delta(y_2) I(y_2) \leq \max_{x \in \Omega} \delta(x) I(y_2), \]
which in turn yields
\[ I(x) \geq I(y_2) \geq \left[ \frac{C_3 \min_{x \in \Omega} \beta(x)}{\max_{x \in \Omega} \delta(x)} \right] \frac{1}{p}, \quad \forall x \in \Omega. \]

Next we will derive the upper and lower bounds for any EE of (3) if $p = 1$ and $d_S$ is small. Such kind of results will be used in Section 4 in the analysis of the asymptotic behavior of EE as $d_S$ goes to zero. In view of (4), Proposition 2 and Theorem 2.1, we have to assume that $\lambda_1^* < 0$ so that EE of (3) exists when $p = 1$ and $d_S$ is small.

For our purpose, the following Harnack-type inequality (see [29]) will be used below.

**Lemma 3.3.** Let $w \in C^2(\Omega) \cap C^1(\Omega)$ be a positive solution to the elliptic equation
\[ \Delta w(x) + c(x) w(x) = 0, \]
where $c \in C(\Omega)$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant $C$ which depends only on $\alpha$ where $\|c\|_\infty \leq \alpha$ such that
\[ \max_{\Omega} w \leq C \min_{\Omega} w. \]

**Theorem 3.4.** Assume $p = 1$ and $\lambda_1^* < 0$, and let $(S, I)$ be any given positive solution of (3). Then there exists a positive constant $C$, independent of all $0 < d_S \leq 1$, such that
\[ 1/C < S(x) < C, \quad 1/C < I(x) < C, \quad \forall x \in \Omega. \]

**Proof.** From the proof of Theorem 3.2, we see that
\[ S(x) \leq \frac{\max_{x \in \Omega} b(x)}{\min_{x \in \Omega} \mu(x)}, \quad \forall x \in \Omega. \] (22)

Our next aim is to establish the desired upper bound for the component $I$. To the end, let $W = d_S S + d_I I$. According to (3), we have
\[ -\Delta W = b(x) - \mu(x) S - \delta(x) I \text{ in } \Omega, \quad \partial_\nu W = 0 \text{ on } \partial \Omega. \]
Take $W(x_0) = \max_{x \in \Omega} W(x)$ for some $x_0 \in \Omega$. Then, an application of Lemma 3.1 implies that $\Delta W(x_0) \leq 0$ and so
\[ b(x_0) \geq \mu(x_0) S(x_0) + \delta(x_0) I(x_0), \]
which gives
\[ S(x_0) + I(x_0) \leq \frac{\max_{x \in \Omega} b(x)}{\min \{ \min_{x \in \Omega} \mu(x), \min_{x \in \Omega} \delta(x) \}} := \tilde{C}_1. \]
Henceforth, it holds
\[ \max_{x \in \Omega} W(x) = W(x_0) \leq \max\{d_S, d_I\} \{S(x_0) + I(x_0)\} \leq \max\{d_S, d_I\} \tilde{C}_1. \]

As a result, we obtain
\[ \max_{x \in \Omega} I(x) \leq \frac{\max_{x \in \Omega} W(x)}{d_I} \leq \frac{\max\{d_S, d_I\}}{d_I} \tilde{C}_1. \]

This is what we wanted.

Next, by letting \( S(y_0) = \min_{x \in \Omega} S(x) \) for some \( y_0 \in \overline{\Omega} \), we have \( \Delta S(y_0) \geq 0 \) due to Lemma 3.1. Thus,
\[ b(y_0) \leq \beta(y_0)I(y_0)S^q(y_0) + \mu(y_0)S(y_0) \]
\[ \leq \frac{\max\{d_S, d_I\}}{d_I} \tilde{C}_1 \max_{x \in \Omega} \beta(x)S^q(y_0) + \max_{x \in \Omega} \mu(x)S(y_0). \]

This clearly indicates that
\[ S(x) \geq S(y_0) \geq \tilde{C}_2, \quad \forall x \in \overline{\Omega} \]
for some positive constant \( \tilde{C}_2 \), which is independent of all \( 0 < d_S \leq 1 \).

In what follows, we are going to derive the positive lower bound for the component \( I \). First of all, from (22) and the equation of \( I \):
\[ \begin{cases} -d_I \Delta I &= \beta(x)S^q \delta(x)I, \quad x \in \Omega, \\ \partial_\nu I &= 0, \quad x \in \partial \Omega, \end{cases} \quad (23) \]
we can use the Harnack inequality (Lemma 3.3) to obtain
\[ \max_{x \in \overline{\Omega}} I \leq C \min_{x \in \overline{\Omega}} I, \quad (24) \]
where the positive constant \( C > 0 \) does not depend on \( d_S \) with \( 0 < d_S \leq 1 \).

We proceed by contradiction and suppose that there is a sequence \( \{S_i, I_i\} \) satisfying \( d_{S_i} \to 0 \) as \( i \to \infty \) and the corresponding positive solution sequence \( \{(S_i, I_i)\} \) of (3) with \( d_S = d_{S_i} \), such that \( \min_{x \in \overline{\Omega}} I_i \to 0 \) as \( i \to \infty \). By (24), there holds
\[ I_i \to 0 \quad \text{uniformly on} \quad \overline{\Omega}, \quad \text{as} \quad i \to \infty. \]

Arguing as in the proof of [11, Lemma 2.4] (or refer to [35], [40, Lemma 3.2]), one can easily obtain from the first equation of (3) that
\[ S_i \to \frac{b(x)}{\mu(x)} \quad \text{uniformly on} \quad \overline{\Omega}, \quad \text{as} \quad i \to \infty. \quad (25) \]

On the other hand, for any \( i \geq 1 \), if denoting \( \lambda_{1,i} \) by the principal eigenvalue of the eigenvalue problem:
\[ \begin{cases} -d_I \Delta \varphi + [\delta(x) - \beta(x)(S_i(x))^q] \varphi = \lambda \varphi, \quad x \in \Omega, \\ \partial_\nu \varphi = 0, \quad x \in \partial \Omega, \end{cases} \]
then (23) tells us that \( \lambda_{1,i} = 0 \) for all \( i \geq 1 \). However, in view of the continuous dependence of the principal eigenvalue of the weight function, (25), combined with our assumption asserts that \( \lambda_{1,i} \to \lambda_1^* < 0 \) as \( i \to \infty \), leading to a contradiction. This shows the existence of positive lower bound for \( I \), which is independent of \( d_S \) with \( 0 < d_S \leq 1 \).

\[ \square \]

4. Asymptotic profiles of the EE. In this section, we are going to determine the asymptotic behavior of the EE of (2). We will treat two cases: \( d_S \to 0 \) and \( d_I \to 0 \).
4.1. The case of $d_S \to 0$. Our result in this case can be stated as follows.

**Theorem 4.1.** Fix $d_I > 0$, and assume that either $0 < p < 1$ or $p = 1$ and $\lambda_1^* < 0$. Let $d_S \to 0$, then any positive solution $(S_{d_S}, I_{d_S})$ of (3) satisfies (up to a subsequence of $d_S \to 0$)

$$(S_{d_S}, I_{d_S}) \to (\overline{S}, \overline{I}) \text{ uniformly on } \overline{\Omega},$$

where $(\overline{S}, \overline{I})$ is a positive solution of

$$\begin{cases} 
-d_I \Delta \overline{I} = b(x) - \mu(x) \overline{S} - \delta(x) \overline{I}, & x \in \Omega, \\
\partial_\nu \overline{I} = 0, & x \in \partial \Omega, \\
b(x) = \beta(x)(\overline{I})^p(\overline{S})^q + \mu(x) \overline{S}, & x \in \Omega.
\end{cases}$$

**Proof.** In view of Theorem 3.2 and Theorem 3.4, under our assumption, one can find two constants $C_1, C_2 > 0$, independent of all small $d_S > 0$, such that any EE $(S_{d_S}, I_{d_S})$ of (3) satisfies

$$C_1 < S_{d_S}, \quad I_{d_S} < C_2, \quad \forall x \in \overline{\Omega}.$$

By the well-known $L^p$-theory for elliptic equations, we see that

$$\|I_{d_S}\|_{W^{2,p}(\Omega)} \leq C \quad \text{for any given } r \geq 1,$$

where $C$ is allowed to vary from place to place but does not depend on any small $d_S > 0$. Choosing $r$ to be sufficiently large, it follows from the embedding theorem that

$$\|I_{d_S}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \quad \text{for some } 0 < \alpha < 1.$$

Therefore, there exists a subsequence of $d_S \to 0$, say $d_i := d_{S,i}$, satisfying $d_i \to 0$ as $i \to \infty$, and a corresponding positive solution $(S_i, I_i) := (S_{d_{S,i}}, I_{d_{S,i}})$ of (3) with $d_S = d_i$, such that

$$I_i \to \overline{I} \quad \text{in } C^1(\overline{\Omega}), \quad \text{as } i \to \infty, \quad \text{(26)}$$

where $\overline{I} \in C^1(\overline{\Omega})$ and $\overline{I} > 0$ on $\overline{\Omega}$. Then for any given small $\epsilon > 0$, it holds

$$0 < \overline{I} - \epsilon \leq I_i \leq \overline{I} + \epsilon \quad \text{on } \overline{\Omega}, \quad \text{for all large } i.$$

Clearly for any fixed large $i$, $S_i$ is a subsolution of

$$\begin{cases} 
-d_i \Delta w = b(x) - \beta(x)(\overline{I} - \epsilon)w^q - \mu(x)w, & x \in \Omega, \\
\partial_\nu w = 0, & x \in \partial \Omega, \quad \text{(27)}
\end{cases}$$

and a supersolution of

$$\begin{cases} 
-d_i \Delta w = b(x) - \beta(x)(\overline{I} + \epsilon)w^q - \mu(x)w, & x \in \Omega, \\
\partial_\nu w = 0, & x \in \partial \Omega. \quad \text{(28)}
\end{cases}$$

It is easy to see that any large positive constant is a supersolution to (27), and any small positive constant is a subsolution to (28). Therefore, both (27) and (28) admit at least one positive solution.

We next show the uniqueness of positive solution to both (27) and (28). We only consider (27) since (28) can be treated in the same way. Suppose that (27) have two positive solutions $u_*$ and $u_{**}$. Denote $\Omega_+ = \{x \in \Omega \mid u_*(x) > u_{**}(x)\}$. We will show that $\Omega_+ = \emptyset$. Otherwise, $\Omega_+$ is a nonempty open set. To produce a contradiction, we define $U = u_* - u_{**}$ and choose a component of $\Omega_+$, say $\Omega^1_+$. Then, for any fixed large $i$, elementary calculation gives that

$$-d_i \Delta U < 0 \quad \text{in } \Omega^1_+, \quad \frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial \Omega \cap \partial \Omega^1_+, \quad U = 0 \quad \text{on } \partial \Omega^1_+ \setminus \partial \Omega.$$
The maximum principle for elliptic equations ensures that $U$ must attain its maximum at some point $x_+ \in \partial \Omega_1$. Thus, $x_+ \in \partial \Omega \cap \partial \Omega_1$. But the Hopf boundary lemma implies $\frac{\partial U}{\partial n}(x_+) > 0$, a contradiction. Hence, it is necessary that $u_* \leq u_{**}$ on $\overline{\Omega}$. Similarly, one can conclude that $u_* \geq u_{**}$ on $\Omega$. Thus, $u_* \equiv u_{**}$. This verifies the uniqueness.

Denote by $u_i$ and $v_i$, respectively, by the unique positive solution to (27) and (28). Since $S_i$ is a subsolution and a large constant $M$ with $M > \max_{x \in \Omega} S_i$ is a supersolution of (27), together with the uniqueness, we know $S_i \leq u_i$ on $\Omega$, for all large $i$. Similar analysis shows that $v_i \leq S_i$. Therefore $v_i \leq S_i \leq u_i$ on $\overline{\Omega}$, for all large $i$.

By the similar argument to that [11, Lemma 2.4], one has $u_i \rightarrow \hat{u}_\epsilon$, $v_i \rightarrow \hat{v}_\epsilon$ uniformly on $\overline{\Omega}$, as $i \rightarrow \infty$, where $\hat{u}_\epsilon$ is the unique solution of

$$b(x) = \beta(x)\left[I(x) - \epsilon\right]^{p} \hat{u}_\epsilon + \mu(x)\hat{u}_\epsilon,$$

and $\hat{v}_\epsilon$ is the unique solution of

$$b(x) = \beta(x)\left[I(x) + \epsilon\right]^{p} \hat{v}_\epsilon + \mu(x)\hat{v}_\epsilon.$$

Thus, due to the arbitrariness of $\epsilon$, it is clear that $S_i \rightarrow \overline{S}$ uniformly on $\overline{\Omega}$, as $i \rightarrow \infty$, where $\overline{S}$ is the unique solution of

$$b(x) = \beta(x)\left[I(x)\overline{S} - \delta(x)\right]^{p} \overline{S} \text{ in } \Omega. \tag{29}$$

By virtue of (26) and (29), it is not hard to see that $\overline{I}$ is a positive solution to the following elliptic problem

$$\left\{ \begin{array}{l}
-d_I \Delta \overline{I} = b(x) - \mu(x)\overline{S} - \delta(x)\overline{I}, & x \in \Omega, \\
\partial_n \overline{I} = 0, & x \in \partial \Omega.
\end{array} \right.$$ 

Thus, we complete the proof. \qed

4.2. The case of $d_I \rightarrow 0$. Our result in this case reads as follows.

**Theorem 4.2.** Fix $d_S > 0$ and assume that $0 < p < 1$. Let $d_I \rightarrow 0$, then any positive solution $(S_{d_I}, I_{d_I})$ of (3) satisfy

$$(S_{d_I}, I_{d_I}) \rightarrow (\overline{S}, \overline{I}) \text{ uniformly on } \overline{\Omega},$$

where

$$\overline{I}(x) = \left\{ \frac{\beta(x)\left[S(x)\right]^q}{\delta(x)} \right\}^{\frac{1}{p}} \text{ in } \Omega,$$

and $\overline{S}$ is the unique positive solution to the following elliptic equation

$$\left\{ \begin{array}{l}
-d_S \Delta \overline{S} = b(x) - \beta(x)\left[\frac{\beta(x)\left[S(x)\right]^q}{\delta(x)}\right]^{\frac{1}{p}} \left(\overline{S}\right)^{\frac{q}{p}} - \mu(x)\overline{S}, & x \in \Omega, \\
\partial_n \overline{S} = 0, & x \in \partial \Omega. \tag{30}
\end{array} \right.$$
Proof. By Theorem 3.2, there exist two positive constants $C_1$, $C_2$, independent of $d_I > 0$, such that any EE $(S_{d_I}, I_{d_I})$ of (3) satisfies

$$C_1 < S_{d_I}, \quad I_{d_I} < C_2, \quad \forall x \in \overline{\Omega}.$$  

So the standard $L^p$-theory for elliptic equations guarantees that

$$\|S\|_{W^{2,r}(\Omega)} \leq C$$  

for any given $r \geq 1$, where $C$ does not depend on $d_I > 0$. Taking $r$ to be properly large, we see from the embedding theorem that

$$\|S_{d_I}\|_{C^{1+\alpha}(\overline{\Omega})} \leq C$$  

with $0 < \alpha < 1$.

Hence, one can find a subsequence of $d_I \to 0$, say $d_i := d_{d_i}$, satisfying $d_i \to 0$ as $i \to \infty$, and a corresponding positive solution sequence $(S_i, I_i) := (S_{d_i}, I_{d_i})$ of (3) with $d_I = d_i$, such that

$$S_i \to \underline{S} \quad \text{in} \quad C^1(\overline{\Omega}), \quad \text{as} \quad i \to \infty,$$  

where $\underline{S} \in C^1(\overline{\Omega})$ and $\underline{S} > 0$ on $\overline{\Omega}$. Then, given any small $\epsilon > 0$, for all large $i$, we have

$$0 < \underline{S} - \epsilon \leq S_i \leq \underline{S} + \epsilon, \quad \text{on} \quad \overline{\Omega}.$$  

It is easily observed that, for fixed large $i$, $I_i$ is a subsolution of

$$\begin{cases}  
-d_i \Delta w = \beta(x)w^{p(\underline{S} + \epsilon)^q - \delta(x)u}, & x \in \Omega, \\
\partial_\nu w = 0, & x \in \partial\Omega. 
\end{cases}$$  

and is a supersolution of

$$\begin{cases}  
-d_i \Delta w = \beta(x)w^{p(\underline{S} - \epsilon)^q - \delta(x)u}, & x \in \Omega, \\
\partial_\nu w = 0, & x \in \partial\Omega. 
\end{cases}$$  

As $0 < p < 1$, it is further seen that any large positive constant is a supersolution to (32), and any small positive constant is a subsolution to (33), and hence (32) and (33) admit at least one positive solution.

Next we will show the uniqueness of positive solution of (32). Suppose that there are two positive solutions $\overline{u}$ and $\underline{u}$. We may choose a constant $M_0 > 1$ such that

$$M_0^{-1}\overline{u} < \underline{S}, \quad \underline{S} < M_0\overline{u} \quad \text{on} \quad \overline{\Omega}.$$  

Thanks to $0 < p < 1$, clearly $M_0\overline{u}$ is a supersolution of (32) and $M_0^{-1}\underline{S}$ is a subsolution. Hence, the iteration argument of sub-supersolutions implies that there exist a minimal and a maximal solution in the order interval $[M_0^{-1}\underline{S}, M_0\overline{u}]$, which we denote by $u_{\min}$ and $u_{\max}$, respectively. Thus

$$u_{\min} \leq \overline{u}, \quad \underline{S} \leq u_{\max} \quad \text{on} \quad \overline{\Omega}.$$  

Thus it suffices to show that $u_{\min} \equiv u_{\max}$.

Notice that

$$-d_i \Delta u_{\min} = \beta(x)u_{\min}^{p(\overline{S} + \epsilon)^q - \delta(x)u_{\min}}$$
$$-d_i \Delta u_{\max} = \beta(x)u_{\max}^{p(\overline{S} + \epsilon)^q - \delta(x)u_{\max}}.$$  

We multiply the two equations by $u_{\max}$ and $u_{\min}$, respectively, and then integrate by parts to obtain

$$\int_{\Omega} \beta(x)(\overline{S} + \epsilon)^q [u_{\min}^{p}u_{\max} - u_{\min}^{p}u_{\max}^p] = 0.$$
That is,
\[
\int_{\Omega} \beta(x)(S + \epsilon)^q u_{\min} u_{\max} [u_{\min}^{p-1} - u_{\max}^{p-1}] = 0. \tag{34}
\]
As \( u_{\min} \leq u_{\max} \) on \( \overline{\Omega} \), \( u_{\min}^{p-1} \geq u_{\max}^{p-1} \) on \( \overline{\Omega} \). The equality (34) implies that \( u_{\min} \equiv u_{\max} \), and in turn \( \overline{\pi} \equiv \bar{u} \) on \( \overline{\Omega} \). Denote by \( u_i^\epsilon \) the unique positive solution of (32).

By a similar argument, we see that (33) has a unique positive solution, denoted by \( v_i^\epsilon \). A simple sub-supersolution analysis, together with the uniqueness, guarantees that
\[
v_i^\epsilon \leq I_i \leq u_i^\epsilon \quad \text{on} \quad \Omega, \quad \text{for all large} \quad i. \tag{35}
\]
The proof of [11, Lemma 2.4] with some obvious modifications can be used to conclude that
\[
u_i^\epsilon \to \left[ \frac{\beta(x)(S + \epsilon)^q}{\delta(x)} \right]^{\frac{1}{p-q}} \quad \text{uniformly on} \quad \Omega, \quad \text{as} \quad i \to \infty
\]
and
\[
v_i^\epsilon \to \left[ \frac{\beta(x)(S - \epsilon)^q}{\delta(x)} \right]^{\frac{1}{p-q}} \quad \text{uniformly on} \quad \Omega, \quad \text{as} \quad i \to \infty.
\]
Thus, letting \( i \to \infty \) in (35) gives
\[
\left[ \frac{\beta(x)(S - \epsilon)^q}{\delta(x)} \right]^{\frac{1}{p-q}} \leq \liminf_{i \to \infty} I_i(x) \leq \limsup_{i \to \infty} I_i(x) \leq \left[ \frac{\beta(x)(S + \epsilon)^q}{\delta(x)} \right]^{\frac{1}{p-q}}. \tag{36}
\]
Due to the arbitrary choice of small \( \epsilon \) in (36), one immediately gets
\[
I_i \to \left[ \frac{\beta(x)(S)^q}{\delta(x)} \right]^{\frac{1}{p-q}} \quad \text{uniformly on} \quad \Omega, \quad \text{as} \quad i \to \infty.
\]
Combined with this and (31), one can easily see from the equation satisfied by \( S_i \) that \( S \) solves (30). The same analysis as in the proof of uniqueness of positive solutions to (27) shows that (30) has a unique positive solution. The proof is thus complete.

\textbf{Acknowledgments.} We would like to thank Professor Rui Peng for his helpful discussions during the preparation of this work.

\textbf{REFERENCES}

[1] L. J. S. Allen, B. M. Bolker, Y. Lou and A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, *Discrete Contin. Dyn. Syst.*, 21 (2008), 1–20.

[2] K. J. Brown, P. C. Dunne and R. A. Gardner, A semilinear parabolic system arising in the theory of superconductivity, *J. Differential Equations*, 40 (1981), 232–252.

[3] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-diffusion Equations*, John Wiley and Sons Ltd., Chichester, UK, 2003.

[4] J. Cui, X. Tao and H. Zhu, An SIS infection model incorporating media coverage, *Rocky Mount. J. Math.*, 38 (2008), 1323–1334.

[5] R. Cui, K.-Y. Lam and Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, *J. Differential Equations*, 263 (2017), 2343–2373.

[6] R. Cui and Y. Lou, A spatial SIS model in advective heterogeneous environments, *J. Differential Equations*, 261 (2016), 3305–3343.

[7] K. Deng and Y. Wu, Dynamics of an SIS epidemic reaction-diffusion model, *Proc. Roy. Soc. Edinburgh Sect. A*, 146 (2016), 929–946.

[8] W. R. Derrick and P. van den Driessche, A disease transmission model in a nonconstant population, *J. Math. Biol.*, 31 (1993), 495–512.
[9] W. R. Derrick and P. van den Driessche, Homoclinic orbits in a disease transmission model with nonlinear incidence and nonconstant population, *Discrete Contin. Dyn. Syst. Ser. B*, 3 (2003), 299–309.

[10] W. Ding, W. Huang and S. Kansakar, Traveling wave solutions for a diffusive SIS epidemic model, *Discrete Contin. Dyn. Syst. Ser. B*, 18 (2013), 1291–1304.

[11] Y. Du, R. Peng and M. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, *J. Differential Equations*, 246 (2009), 3932–3956.

[12] Z. Du and R. Peng, A priori $L^\infty$ estimates for solutions of a class of reaction-diffusion systems, *J. Math. Biol.*, 72 (2016), 1429–1439.

[13] D. Gao and S. Ruan, An SIS patch model with variable transmission coefficients, *Math. Biosci.*, 232 (2011), 110–115.

[14] J. Ge, K. I. Kim, Z. Lin and H. Zhu, A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, *J. Differential Equations*, 259 (2015), 5486–5509.

[15] A. Korobeinikov and P. K. Maini, A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence, *Math. Biosci. Eng.*, 1 (2004), 57–60.

[16] A. Korobeinikov and G. C. Wake, A Lyapunov function and global stability for SIR, SEIR and SIS epidemiological models, *Appl. Math. Lett.*, 15 (2002), 955–960.

[17] K. Kuto, H. Matsuzawa and R. Peng, Concentration profile of the endemic equilibria of a reaction-diffusion-advection SIS epidemic model, *Calc. Var. Partial Differential Equations*, 56 (2017), Art. 112, 28 pp.

[18] W. Li, R. Peng and F.-B. Wang, Vary total population enhances disease persistence: Qualitative analysis on a diffusive SIS epidemic model, *J. Differential Equations*, 262 (2017), 885–913.

[19] C. Lin, W.-M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, *J. Differential Equations*, 72 (1988), 1–27.

[20] W. M. Liu, H. W. Hethcote and S. A. Levin, Dynamical behavior of epidemiological models with nonlinear incidence rates, *J. Math. Biol.*, 25 (1987), 359–380.

[21] W. M. Liu, S. A. Levin and Y. Isawa, Influence of nonlinear incidence rates upon the behaviour of SIRS epidemiological models, *J. Math. Biol.*, 23 (1986), 187–204.

[22] J. Liu, B. Peng and T. Zhang, Effect of discretization on dynamical behavior of SEIR and SIR models with nonlinear incidence, *Appl. Math. Lett.*, 39 (2015), 60–66.

[23] D. Le, Dissipativity and global attractors for a class of quasilinear parabolic systems, *Comm. Partial Differential Equations*, 22 (1997), 413–433.

[24] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differential Equations*, 223 (2006), 400–426.

[25] Y. Lou and W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, 131 (1996), 79–131.
[37] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM. J. Math. Anal.*, 37 (2005), 251–275.

[38] R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model. Part I, *J. Differential Equations*, 247 (2009), 1096–1119.

[39] R. Peng and S. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, *Nonlinear Anal.*, 71 (2009), 239–247.

[40] R. Peng, J. Shi and M. Wang, On stationary patterns of a reaction-diffusion model with autocatalysis and saturation law, *Nonlinearity*, 21 (2008), 1471–1488.

[41] R. Peng and F.-Q. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: Effects of epidemic risk and population movement, *Phys. D*, 259 (2013), 8–25.

[42] R. Peng and X.-Q. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, 25 (2012), 1451–1471.

[43] H. L. Smith and X.-Q. Zhao, Robust persistence for semidynamical systems, *Nonlinear Anal.*, 47 (2001), 6169–6179.

[44] Y. Wu and X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, *J. Differential Equations*, 261 (2016), 4424–4447.

[45] X.-Q. Zhao, *Dynamical Systems in Population Biology*, 2nd edition, Springer, New York, 2017.

Received August 2017; revised January 2018.

E-mail address: leichengxia001@163.com
E-mail address: fujunlimath@163.com
E-mail address: jiangliu@jsnu.edu.cn