DERIVED EQUIVALENCES FOR SYMPLECTIC REFLECTION ALGEBRAS

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ABSTRACT. In this paper we study derived equivalences for Symplectic reflection algebras. We establish a version of the derived localization theorem between categories of modules over Symplectic reflection algebras and categories of coherent sheaves over quantizations of \(\mathbb{Q}\)-factorial terminalizations of the symplectic quotient singularities. To do this we construct a Procesi sheaf on the terminalization and show that the quantizations of the terminalization are simple sheaves of algebras. We will also sketch some applications: to the generalized Bernstein inequality and to perversity of wall crossing functors.

1. INTRODUCTION

The goal of this paper is to investigate derived equivalences between categories of modules over Symplectic reflection algebras (introduced by Etingof and Ginzburg in [EGI]) and give some applications.

Let us briefly recall what these algebras are. Let \(V\) be a finite dimensional symplectic vector space over \(\mathbb{C}\). Let \(\Gamma\) be a finite supgroup of \(\text{Sp}(V)\). Then we can form the smash-product algebra \(\mathbb{C}[V]#\Gamma\) that carries a natural grading. A Symplectic reflection algebra \(H_c\) is a filtered deformation of \(\mathbb{C}[V]#\Gamma\) (a note for experts: in this paper we only consider deformations with \(t = 1\) that should be thought as “quantizations” of \(\mathbb{C}[V]#\Gamma\)). Here \(c\) is a deformation parameter that is a \(\Gamma\)-invariant \(\mathbb{C}\)-valued function on the set \(S\) of symplectic reflections in \(\Gamma\). Let \(p\) denote the space of such functions. By a symplectic reflection we mean an element \(s \in \Gamma\) such that \(\text{rk}(s - 1) = 2\). We will recall the definition of \(H_c\) later.

Consider the subalgebra \(\mathbb{C}\Gamma \subset H_c\) and the averaging idempotent \(e \in \mathbb{C}\Gamma\). The spherical subalgebra \(eH_ce\) is a quantization of \(\mathbb{C}[V]^{\Gamma}\). The variety \(V/\Gamma\) is a conical symplectic singularity. Consider its \(\mathbb{Q}\)-factorial terminalization \(X\). We can talk about filtered quantizations of \(\mathcal{O}_X\), see [BPW], Section 3, these are sheaves of filtered algebras on \(X\) (in the so called conical topology). The filtered quantizations of \(X\) are parameterized by \(H^2(X^{\text{reg}}, \mathbb{C})\), see [BPW], Section 3, for \(\lambda \in H^2(X^{\text{reg}}, \mathbb{C})\) we write \(\mathcal{D}_{\lambda}\) for the corresponding quantization. Moreover, in [L8, Section 3.7], we have established an affine isomorphism \(p \rightarrow H^2(X^{\text{reg}}, \mathbb{C}), c \leftrightarrow \lambda\), such that \(eH_ce \cong \Gamma(\mathcal{D}_{\lambda})\).

Here is the first main result of this paper (that would be standard if \(X\) were smooth but, for most \(\Gamma\), the variety \(X\) is not smooth).

Theorem 1.1. The sheaf of algebras \(\mathcal{D}_{\lambda}\) is simple for any \(\lambda \in H^2(X^{\text{reg}}, \mathbb{C})\).

Our next result in this paper proves the conjecture from [L4, Section 7.1]. We say that parameters \(c, c' \in p\) have integral difference if their images in \(H^2(X^{\text{reg}}, \mathbb{C})\) lie in the image of \(\text{Pic}(X^{\text{reg}})\).

MSC 2010: 16E99, 16G99.
**Theorem 1.2.** Let \( c, c' \in p \) have integral difference. Then there is a derived equivalence \( D^b(H_c\text{-mod}) \sim D^b(H_{c'}\text{-mod}) \).

This theorem is proved using a strategy used in [GL, Section 5] in the special case of wreath-product groups. Namely, we construct a certain sheaf \( P \) on \( X \) that we call a Procesi sheaf that generalizes the notion of a Procesi bundle in the case when \( X \) is smooth. Then we quantize it to a right \( D_c \)-module denoted by \( P_c \) (where we write \( D_c \) for \( D_\lambda \), where \( \lambda \in H^2(X_{reg}, \mathbb{C}) \) corresponds to \( c \in p \)). One can show that for a suitable choice of \( P \), we have \( \text{End}_{D^{\text{op}} c}(P_c) \sim D^b(H_c\text{-mod}) \), see Proposition 5.1 below.

**Theorem 1.3.** The following is true:

1. When \( c, c' \) have integral difference, the categories \( \text{Coh}(D_c), \text{Coh}(D_{c'}) \) of coherent \( D_c \)- and \( D_{c'} \)-modules are equivalent.
2. The functor \( \Gamma(P_c \otimes_{D_c} \bullet) : D^b(\text{Coh}(D_c)) \sim D^b(H_c\text{-mod}) \) is an equivalence.

Let us describe some applications and consequences of our constructions. Every pair \((X, P)\) of a \( Q \)-factorial terminalization \( X \) of \( V/\Gamma \) and a Procesi sheaf \( P \) on \( X \) gives rise to a \( t \)-structure on \( D^b(H_c\text{-mod}) \). Below, Section 6.2, we will explain that [L7, Theorem 3.1] generalizes to our situation (and to a more general one): some of the \( t \)-structures we consider are perverse to each other.

We will also establish the following result ((2) was conjectured by Etingof and Ginzburg in [EG2]):

**Theorem 1.4.** The following is true for all \( c \in p \).

1. The regular \( H_c \)-bimodule \( H_c \) has finite length.
2. Generalized Bernstein inequality holds for \( H_c \): \( \text{GK-dim}(M) \geq \text{GK-dim} H_c/\text{Ann}(M) \).

Reduction of (2) to (1) was done in [L5]. We will explain necessary modifications of arguments from [L5] that prove (1).

**Acknowledgements.** I would like to thank Roman Bezrukavnikov, Yoshinori Namikawa and Ben Webster for stimulating discussions. This work has been funded by the Russian Academic Excellence Project '5-100' This work was also partially supported by the NSF under grant DMS-1501558.

## 2. Preliminaries

### 2.1. Symplectic reflection algebras

Let us start by recalling the definition of a symplectic reflection algebra \( H_c \) and its spherical subalgebra, due to Etingof and Ginzburg, [EG1].

Let, as before, \( V \) be a symplectic vector space (with form \( \omega \)) and \( \Gamma \subset \text{Sp}(V) \) be a finite subgroup. Let \( S \subset \Gamma \) denote the set of symplectic reflections in \( \Gamma \), it is the union of \( \Gamma \)-conjugacy classes. Let \( p \) denote the space of \( \Gamma \)-invariant maps \( S \to \mathbb{C} \). For \( s \in S \), let \( \omega_s \) denote the form whose kernel is \( \ker(s-1) \) and whose restriction to \( \text{im}(s-1) \) coincides with that of \( \omega \). For \( c \in S \) we set

\[
H_c = T(V)^\# \Gamma / (u \otimes v - v \otimes u - \omega(u, v) - \sum_{s \in S} c_s \omega_s(u, v)s).
\]

This algebra comes with a filtration inherited from \( T(V)^\# \Gamma \) The PBW property established in [EG1] says that \( \text{gr} \ H_c = \mathbb{C}[V]^\# \Gamma \).
Also we can consider the universal version $H_p$, a $\mathbb{C}[p]$-algebra that specializes to $H_c$ for $c \in p$. Its associated graded algebra $\text{gr} H_p$ is a flat graded deformation of $\mathbb{C}[V]\#\Gamma$ over $\mathbb{C}[p]$.

Now take the averaging idempotent $e \in \mathbb{C}\Gamma$. We can consider the subalgebra $eH_ce \subset H_c$ with unit $e$. This is a filtered algebra with $\text{gr} eH_ce = \mathbb{C}[V]^F$.

2.2. $\mathbb{Q}$-factorial terminalizations. The variety $Y = V/\Gamma$ has symplectic singularities. A general result of Namikawa states that there is a $\mathbb{Q}$-factorial terminalization $X$ of $Y$ meaning that $X$ is $\mathbb{Q}$-factorial, has terminal singularities and there is a crepant birational projective morphism $\rho : X \to Y$. Let us point out that the $\mathbb{C}^\times$-action lifts to $X$ making $\rho$ equivariant.

The variety $X$ has the following properties (all of them are consequences of $\rho$ being crepant).

**Lemma 2.1.** The following is true:

1. $X$ is a Cohen-Macaulay and Gorenstein variety with trivial canonical sheaf.
2. $\mathbb{C}[X] = \mathbb{C}[Y]$ and $H^i(X, O_X) = 0$ for $i > 0$.

We note that $\rho : X \to Y$ is defined over some finite integral extension $R$ of a finite localization of $Z$. So for each prime $p$ which is large enough we can consider a reduction $\rho_{\mathbb{F}} : X_{\mathbb{F}} \to Y_{\mathbb{F}} \mod p$, where $\mathbb{F}$ is an algebraically closed field of characteristic $p$. The triple $(X_{\mathbb{F}}, Y_{\mathbb{F}}, \rho_{\mathbb{F}})$ is defined over a finite field $\mathbb{F}_q$. Lemma [2.1] still holds.

Let us now recall the Namikawa Weyl group, [N2]. Let $Y$ be a conical symplectic singularity. The Namikawa-Weyl group is defined as follows. Let $L_1, \ldots, L_k$ be the codimension 2 symplectic leaves. The formal slice to each $L_i$ is the Kleinian singularity and so gives rise to the Cartan space $\tilde{h}_i$ and the Weyl group $\tilde{W}_i$ of the corresponding ADE type. The fundamental group $\pi_1(L_i)$ acts on $\tilde{h}_i, \tilde{W}_i$ by monodromy. Set $h_i = h_i^{\pi_1(L_i)}, W_i = W_i^{\pi_1(L_i)}$. By the Namikawa-Weyl group $W_Y$ we mean the product $\prod_{i=1}^k W_i$, it acts on $h := H^2(X^{\text{reg}}, \mathbb{C}) = H^2(Y^{\text{reg}}, \mathbb{C}) \oplus \bigoplus_{i=1}^k h_i$ as a crystallographic reflection group.

Now let us discuss deformations of $X, Y$, see [N1], [N2], [N4]. First, there is a universal Poisson deformation $X_h$ of $X$ over $h$, it comes with a contracting $\mathbb{C}^\times$-action that restricts to the contracting $\mathbb{C}^\times$-action on $Y$ and induces a scaling action on $h$. The affinization of $Y_h$ of $X_h$ is a deformation of $Y$ over $h$. Let us write $X_{\lambda}, Y_{\lambda}$ for the fibers of $X_h, Y_h$ over $\lambda \in h$. We have $X_{\lambda} \to Y_{\lambda}$. The locus, where $X_{\lambda} \to Y_{\lambda}$ is not an isomorphism, can be shown to be a union of hyperplanes, we denote it by $h_{\text{sing}}$.

**Remark 2.2.** The classification of the deformations of $X$ carries over to formal schemes. In particular, we will use below that all Poisson deformations of a formal neighborhood of a point in a $\mathbb{Q}$-factorial terminal variety are trivial.

As Namikawa proved, the action of $W$ on $h$ lifts to $Y_h$ and $Y_h/W$ is a universal Poisson deformation of $Y$. Moreover, $h_{\text{sing}}$ is $W$-stable. The Weyl chamber for $W$ that contains the ample cone of $X$ is the movable cone of $X$. The ample cones of all $\mathbb{Q}$-factorial terminalizations of $Y$ are precisely the chambers for $h_{\text{sing}}$.

Now let us discuss the Pickard group of $X^{\text{reg}}$. It is known that the Chern character map $\text{Pic}(X^{\text{reg}}) \otimes_{\mathbb{Z}} \mathbb{C} \to h$ is an isomorphism, compare to the beginning of [BPW] Section 2.3. Let $h_{\mathbb{Z}} \subset h$ denote the image of $\text{Pic}(X^{\text{reg}})$. In the case when $H^2(Y^{\text{reg}}, \mathbb{C}) = 0$ (which is the case of interest for us) we can describe $h_{\mathbb{Z}}$ as follows. In each $h_i$ we have the coweight lattice and the lattice $h_{\mathbb{Z}}$ is the direct sum of those.
In our main example, $Y = V/\Gamma$, the codimension 2 symplectic leaves are in bijection with $\Gamma$-conjugacy classes of subgroups $\Gamma' \subset \Gamma$ such that $\dim V^{\Gamma'} = \dim V - 2$. The Kleinian group corresponding to this leaf is $\Gamma'$ and the fundamental group is $N_{\Gamma}(\Gamma')/\Gamma'$. One can show, see, e.g., [Bel] Section 3.2, Proposition 2.6 or [LS] Section 3.7 that $\mathfrak{p}$ is naturally identified with $p$ as a vector space. Namely, $H^2(Y^{reg}, \mathbb{C}) = 0$ and, for $i > 0$, the space $\mathfrak{h}_i$ is identified with the space $\mathfrak{p}_i$ of $\Gamma_i$-invariant functions on $\Gamma_i \setminus \{1\}$. It was checked by Bellamy, [Bel Theorem 1.4], that the universal deformation $Y_\mathfrak{p}$ is $\text{Spec}(e \text{ gr } H_\mathfrak{p} e)$.

2.3. Quantizations of Q-factorial terminalizations. Let again $Y$ be a general conical symplectic singularity. Let us discuss the quantizations of $X$ and $Y$ following [BPW],[LS]. Let us start with $X$. By the conical topology on $X$ we mean the topology where “open” means Zariski open and $\mathbb{C}^*$-stable. By a quantization of $X$ we mean a sheaf $\mathcal{D}$ of algebras in the conical topology on $X$ coming with

- Complete and separated, ascending exhaustive $\mathbb{Z}$-filtration,
- a graded Poisson algebra isomorphism $\text{gr} \mathcal{D} \sim \mathcal{O}_X$.

It was shown in [BPW] Sections 3.1,3.2] that the filtered quantizations of $X$ are in a canonical bijection with $\mathfrak{h}$. Let us denote the quantization corresponding to $\lambda \in \mathfrak{h}$ by $\mathcal{D}_\lambda$. We also have the universal quantization, $\mathcal{D}_0$, a sheaf (in the conical) topology of filtered algebras on $X$ whose specialization to $\lambda$ coincides with $\mathcal{D}_\lambda$ and that quantizes the universal deformation $X_\mathfrak{h}$ of $X$ over $\mathfrak{h}$. We also note, [L2 Section 2.3], that

\begin{equation}
\mathcal{D}_{-\lambda} \cong \mathcal{D}_{\lambda}^{opp}.
\end{equation}

Let us proceed to quantizations of $Y$ (or, equivalently, of the filtered algebra $\mathbb{C}[Y] = \mathbb{C}[X]$). Since $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, we see that $A_{\lambda} = \Gamma(\mathcal{D}_\lambda)$ is a quantization of $\mathbb{C}[Y]$ that coincides with the specialization of $A_\mathfrak{h}$ to $\lambda$. It was shown in [BPW] that the group $W = W_Y$ acts on $A_\mathfrak{h}$ by filtered algebra automorphisms so that the induced action on $\mathfrak{h}$ coincides with the original one.

Moreover, it was shown in [LS Proposition 3.5] that, under some mild additional assumptions, $A_\mathfrak{h}^W$ is a universal filtered quantization of $\mathbb{C}[Y]$ in the following sense. Let $B$ be a finitely generated graded commutative $\mathbb{C}$-algebra. Let $A_B$ be a filtered algebra such that $\text{gr} A_B$ is a graded Poisson deformation of $\mathbb{C}[Y]$. Then there is a unique filtered algebra homomorphism $\mathbb{C}[\mathfrak{h}]^W \rightarrow B$ and a unique filtered $B$-algebra isomorphism $B \otimes_{\mathbb{C}[\mathfrak{h}]^W} A_\mathfrak{h}^W \rightarrow A_B$ such that the associated graded homomorphism gives the identity automorphism of $\mathbb{C}[Y]$. The assumption that we need is that $\mathbb{C}[Y]_i = 0$ for $0 < i < d$, it holds for $Y = V/\Gamma$ assuming $V^{\Gamma} = \{0\}$.

Remark 2.3. The classification of quantizations carries over to the case of formal schemes as well, similarly to Remark 2.3. Here we need to deal with formal quantizations, that are (sheaves of) algebras over $\mathbb{C}[[\mathfrak{h}]]$. In particular, any family (over the spectrum of a complete local algebra) of formal quantizations of a formal neighborhood of a point in Q-factorial terminal variety is trivial.

Let us compare the quantizations $A_\lambda$ and $eH_\mathfrak{p}e$ of $\mathbb{C}[V]^{\Gamma}$, [LS] Section 3.7]. There is an affine identification $\lambda \mapsto c(\lambda) : \mathfrak{h} \rightarrow p$ (whose linear part was mentioned above) such that $eH_\mathfrak{p}e \cong A_\mathfrak{h}$ (an isomorphism of filtered algebras compatible with the isomorphism $\lambda \mapsto c(\lambda)$). In particular, $A_\lambda \cong eH_{c(\lambda)}e$ (an isomorphism of filtered quantizations of $\mathbb{C}[V]^{\Gamma}$).
2.4. Harish-Chandra bimodules over quantizations of $\mathbb{C}[Y]$. Let us discuss HC bimodules over quantizations of $Y$. By definition, a HC $A_\gamma$-bimodule (for $\gamma \in \mathfrak{h}$) is a bimodule $\mathcal{B}$ that can be equipped with a bounded below filtration $\mathcal{B} = \bigcup_i \mathcal{B}_{\leq i}$ such that

(i) $[a, b] = \langle \gamma, a \rangle b$ for $a \in \mathfrak{h}^* \subset A_\mathfrak{h}$,
(ii) $[A_{\mathfrak{h}, \leq i}, B_{\leq j}] \subset B_{\leq i+j-d}$,
(iii) $\text{gr} \mathcal{B}$ is a finitely generated $\mathbb{C}[Y]$-module.

Similarly, we have the notion of a HC $A_\lambda$-$A_\lambda$-bimodule.

Under some additional conditions, we can define restriction functors between categories of HC bimodules. Namely, assume that the formal slices to all symplectic leaves in $Y$ are conical, i.e., they come with contracting $\mathbb{C}^\times$-actions that rescale the Poisson bracket by $t \mapsto t^{-d}$. We can assume that $d$ is even.

Let $Y$ be the conical symplectic singularity such that its formal neighborhood at 0 is the formal slice to $\mathcal{L}$ at some point $y \in \mathcal{L}$. Arguing similarly to [L7, Lemma 3.3], we see that the completion $R_\mathfrak{h}(A_\lambda)^{\wedge}$ of the Rees algebra of $A_\lambda$ splits into the completed tensor product $R_\mathfrak{h}(A_\lambda)^{\wedge} \otimes_{\mathbb{C}[\mathfrak{h}]} A_\mathfrak{h}(T_x \mathcal{L})^{\wedge_0}$. Here we write $A_\mathfrak{h}(T_x \mathcal{L})$ for the homogenized Weyl algebra of the symplectic vector space $\mathcal{L}$: $A_\mathfrak{h}(T_x \mathcal{L}) = T(T_x \mathcal{L})[\mathfrak{h}]/(u \otimes v - v \otimes u - \hbar^{d/2}\omega(u, v))$. Further, we write $A_\gamma$ for the filtered quantization of $\mathbb{C}[Y]$ that corresponds to the restriction of $\lambda \in H^2(X^{reg}, \mathbb{C})$ to $X^{reg}$ (where $X$ is the $\mathbb{Q}$-factorial terminalization of $Y$ that is characterized by the property that $Y^{\mathfrak{h}} \times_\gamma X$ is the preimage of the formal slice to $\mathcal{L}$ in $X$, compare with [L7, Section 3.2]).

Note that $\mathbb{C}[Y]^{\wedge}$ is equipped with two derivations, $D, D'$ that multiply $\cdot, \cdot$ by $-d$. Arguing as in the proof of (1) of [L7, Lemma 3.3] and using [L8, Lemma 2.15], we show that all Poisson derivations of $\mathbb{C}[Y]^{\wedge}$ are inner. Now we can argue as in [L7, Section 3.3] to construct the restriction functor $\bullet_{T_x} : \text{HC}(A_\gamma, A_\lambda) \rightarrow \text{HC}(A_\gamma, A_\lambda)$. Direct analogs of [L7, Lemma 3.5, Lemma 3.6, Lemma 3.7] hold (in the case of $Y = V/\Gamma$ these properties were established in [L1]).

Let us discuss supports of HC bimodules in $\mathfrak{h}$. Let $\mathcal{B} \in \text{HC}(A_\mathfrak{h}, \gamma)$. By the (right) support $\text{Supp}_h(\mathcal{B})$ of $\mathcal{B}$ in $\mathfrak{h}$, we mean $\{\lambda \in \mathfrak{h}| B \otimes_{\mathbb{C}[\mathfrak{h}]} C_\lambda \neq 0\}$. Similarly to [L6, Proposition 2.6], we get the following result.

**Proposition 2.4.** The subset $\text{Supp}_h(\mathcal{B})$ is closed and its asymptotic cone coincides with $\text{Supp}_h(\text{gr} \mathcal{B})$, where the associated graded is taken with respect to any good filtration on $\mathcal{B}$.

3. Simplicity of $D_\lambda$

3.1. Main conjecture and result. Let $Y$ be an arbitrary conical symplectic singularity and let $X$ be its $\mathbb{Q}$-factorial terminalization. Let $D$ be a quantization of $X$.

We conjecture the following.

**Conjecture 3.1.** The sheaf of algebras $D$ is simple.

In the case when $X$ is smooth, the conjecture is standard because $O_X$ has no Poisson ideals.

The main result of this section is a proof of this conjecture in the case when $Y = V/\Gamma$ (Theorem [L1] from the introduction).

The most essential result about $V/\Gamma$ that we use (and which is not available in general) is that the algebra $A_\lambda$ is simple for a Weil generic $\lambda$ (that follows from [L1, Theorem 4.2.1]).
3.2. Abelian localization. Let $Y$ again be arbitrary. Note that since $X$ is $\mathbb{Q}$-factorial the natural map $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{Pic}(X^{reg}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. Pick $\chi \in \text{Pic}(X)$ and let $\mathcal{O}(\chi)$ denote the corresponding line bundle on $X$. Abusing the notation we will denote its image in $\mathfrak{h}$ also by $\chi$.

Now pick $\lambda \in \mathfrak{h}$. We can consider the category $\text{Coh}(\mathcal{D}_\lambda)$ of coherent $\mathcal{D}_\lambda$-modules. There is the global section functor $\Gamma_\lambda : \text{Coh}(\mathcal{D}_\lambda) \to \mathcal{A}_\lambda$-mod that has left adjoint $\text{Loc}_\lambda := \mathcal{D}_\lambda \otimes_{\mathcal{A}_\lambda} \bullet$. We say that abelian localization holds for $(X, \lambda)$ if these functors are mutually inverse equivalences.

The following result is a direct generalization of results of [BPW, Section 5.3].

**Proposition 3.2.** Suppose $\chi$ is ample for $X$ and $H^i(X, \mathcal{O}(\chi)) = 0$ for $i > 0$. For any $\lambda \in \mathfrak{h}$, there is $n_0 \in \mathbb{Z}$ with the property that abelian localization holds for $(X, \lambda + n\chi)$ for any $n \geq n_0$.

Below we will establish a stronger version of this result.

The scheme of proof is similar to what is used in [BPW] and is based on translation bimodules.

The line bundle $\mathcal{O}(\chi)$ on $X$ quantizes to a $\mathcal{D}_{\lambda+\chi}-\mathcal{D}_\lambda$-bimodule (that is a sheaf on $X$) to be denoted by $\mathcal{D}_{\lambda,\chi}$. Note that tensoring with $\mathcal{D}_{\lambda,\chi}$ gives an equivalence $\text{Coh}(\mathcal{D}_\lambda) \sim \text{Coh}(\mathcal{D}_{\lambda+\chi})$, compare to [BPW Section 5.1]. Set $\mathcal{A}_{\lambda,\chi} := \Gamma(\mathcal{D}_{\lambda,\chi})$, this is an $\mathcal{A}_{\lambda+\chi}-\mathcal{A}_\lambda$-bimodule.

We can also consider the universal version of $\mathcal{D}_{\lambda,\chi}$, the $\mathfrak{h}$-bimodule $\mathcal{D}_{\mathfrak{h},\chi}$, so that $\mathcal{D}_{\lambda,\chi} = \mathcal{D}_{\mathfrak{h},\chi} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathcal{C}_\lambda$. Let $\mathcal{A}_{\mathfrak{h},\chi} := \Gamma(\mathcal{D}_{\mathfrak{h},\chi})$. Similarly to [BPW Proposition 6.24], see also [BL Proposition 3.3], we see that $\mathcal{A}_{\mathfrak{h},\chi}$ is independent of the choice of $X$ (the Picard groups of different $\mathbb{Q}$-factorial terminalizations of $Y$ are naturally identified). If $H^1(X, \mathcal{O}(\chi)) = 0$, then $\mathcal{A}_{\lambda,\chi}$ coincides with the specialization of $\mathcal{A}_{\mathfrak{h},\chi}$ to $\lambda$.

**Proof of Proposition 3.2.** Similarly to [BPW Proposition 5.13], abelian localization holds for $(X, \lambda)$ provided each $\mathcal{A}_{\lambda+n\chi,\lambda}$ is a Morita equivalence and for all $m > 0$ the natural map

$$\mathcal{A}_{\lambda+(m-1)\chi,\lambda} \otimes_{\mathcal{A}_{\lambda+(m-1)\chi}} \mathcal{A}_{\lambda+(m-2)\chi,\lambda} \cdots \otimes_{\mathcal{A}_{\lambda+\chi}} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_{\lambda,m\chi}$$

is an isomorphism. Similarly to the proof of [BL Lemma 4.4], we see that the latter will follow if we show that $\mathcal{A}_{\mathfrak{h},-\chi}|_{\lambda+(n+1)\chi}$ is inverse to $\mathcal{A}_{\lambda+n\chi,\lambda}$ for all $n \geq 0$.

Following [BL Section 2.2.5], we say that a Zariski open subset $U \subset \mathfrak{h}$ is asymptotically generic, if the asymptotic cone of $\mathfrak{h} \setminus U$ is contained in $\mathfrak{h}^{\text{sing}}$. Arguing as in the proof of [BL Proposition 4.5(2)], we see that the locus, where $\mathcal{A}_{\lambda,\chi}, \mathcal{A}_{\mathfrak{h},-\chi}|_{\lambda+\chi}$ are mutually inverse Morita equivalences is asymptotically generic. In particular, its intersection with the line $\{\lambda + z\chi | z \in \mathbb{C}\}$ is nonempty. This finishes the proof. □

**Corollary 3.3.** Assume that algebra $\mathcal{A}_\lambda$ is simple for a Weil generic $\lambda$. Then, for a Weil generic $\lambda \in \mathfrak{h}$, the sheaf of algebras $\mathcal{D}_\lambda$ is simple.

**Proof.** Recall, (2.1), that $\mathcal{D}_\lambda^{\text{opp}} \cong \mathcal{D}_{-\lambda}$. We can view $\mathcal{D}_\lambda \otimes_{\mathcal{D}_{-\lambda}}$ as a quantization of $X \times X$. The proof of Proposition 3.2 shows that abelian localization holds for $(X \times X, (\lambda, -\lambda))$ assuming $\lambda$ is in the intersection of integral translates of some asymptotically generic Zariski open subset, in particular, when $\lambda$ is Weil generic. Note that the global section functor sends the regular bimodule $\mathcal{D}_\lambda$ to the regular bimodule $\mathcal{A}_\lambda$. The latter is simple, so is the former. □
3.3. Leaves in $X$ and two-sided ideals in $\mathcal{D}_\lambda$. Thanks to Corollary 3.3 to prove Theorem \[L.1\] we need to show that if $\mathcal{D}_\lambda$ has a two-sided ideal for given $\lambda$, then $\mathcal{D}_X$ also has a two-sided ideal for all $\lambda$. The starting point here is to observe that the formal neighborhood of a point in a \(\mathbb{Q}\)-factorial terminal variety admits only one quantization. In this section, we will prove technical results that are analogous to several results obtained in \[L.5\] Section 3.2.

So let $Y$ be a conical symplectic singularity, $X$ be its \(\mathbb{Q}\)-terminalization, $\mathfrak{h} := H^2(X^{reg}, \mathbb{C})$. The variety $X$ has finitely many symplectic leaves. Let $\mathcal{L}$ be one of these leaves. Note that the $\mathbb{C}^\times$-action preserves $\mathcal{L}$ and the action on the closure of $\mathcal{L}$ is contracting.

Pick a point $x \in \mathcal{L}$. Consider the formal neighborhood $\mathcal{L}^{\wedge_x}$ and its algebra of functions $\mathbb{C}[\mathcal{L}^{\wedge_x}]$. This is a Poisson algebra. The action of $\mathbb{C}^\times$ on $X$ induces a derivation of $\mathbb{C}[\mathcal{L}^{\wedge_x}]$ that rescales the Poisson bracket. We call it the Euler derivation and denote it by eu. Consider the category $\mathcal{C}(\mathcal{L}^{\wedge_x})$ of all finitely generated Poisson $\mathbb{C}[\mathcal{L}^{\wedge_x}]$-modules that come equipped with an Euler derivation, compare with \[L.5\] Section 3.2. On the other hand, consider the category $\mathcal{C}(\mathcal{L})$ consisting of all finitely generated weakly $\mathbb{C}^\times$-equivariant Poisson $\mathcal{O}_\mathcal{L}$-modules. We have the functor $\bullet_{\mathfrak{i},x} : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}^{\wedge_x})$ of completing at $x$.

**Lemma 3.4.** Assume that the algebraic fundamental group of $\mathcal{L}$ is finite. The functor $\bullet_{\mathfrak{i},x} : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}^{\wedge_x})$ admits a right adjoint functor.

**Proof.** Let $\pi : \mathcal{L} \to \mathcal{L}$ denote the universal algebraic cover that exists because the algebraic fundamental group is finite. The action of $\mathbb{C}^\times$ lifts to $\mathcal{L}$ possibly after replacing the given $\mathbb{C}^\times$ with a covering $\mathbb{C}^\times$. So we can consider the category $\mathcal{C}(\mathcal{L})$ that comes with adjoint functors $\pi^\ast : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L})$, $\pi^\ast : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L})$, where we write $\pi^\ast$ for the equivariant descent. We still have the functor $\bullet_{\mathfrak{i},x} : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}^{\wedge_x})$ that satisfies $\bullet_{\mathfrak{i},x} \circ \pi^\ast \cong \bullet_{\mathfrak{i},x}$. So it is enough to show that $\bullet_{\mathfrak{i},x}$ admits a right adjoint functor, say $\bullet_{\mathfrak{i},x}$. Then the right adjoint to $\bullet_{\mathfrak{i},x}$ is given by $\pi^\ast \circ \bullet_{\mathfrak{i},x}$.

Arguing as in Step 3 of the proof of \[L.5\] Lemma 3.9, we see that every object $M \in \mathcal{C}(\mathcal{L})$ is of the form $\mathcal{O}_{\mathcal{L}} \otimes V$, where $V$ is a finite dimensional rational representation of $\mathbb{C}^\times$. Similarly, any object $N \in \mathcal{C}(\mathcal{L}^{\wedge_x})$ is of the form $\mathbb{C}[\mathcal{L}^{\wedge_x}] \otimes V'$, where $V'$ is a finite dimensional vector space with a linear operator ($V'$ arises as the Poisson center of $N$, the linear operator is obtained by restricting the Euler derivation). The functor $\bullet_{\mathfrak{i},x}$ becomes $V \mapsto V'$ (where we take the linear operator coming by differentiating the $\mathbb{C}^\times$-action). This functor clearly has right adjoint (we send $V'$ the maximal subspace where the operator acts diagonally with integral eigenvalues). \[1\]

Let us give corollaries of this lemma. The first concerns two-sided ideals in quantizations. Consider the $h$-adically completed Rees sheaf $\mathcal{D}_{h,h}$ of $\mathcal{D}_h$. It makes sense to complete the sheaf $\mathcal{D}_{h,h}$ at $x \in \mathcal{L}$ getting an algebra $\mathcal{D}_{h,h}^{\wedge_x}$ that is a formal quantization of $X^{\wedge_x}$. It again comes equipped with an Euler derivation eu (that rescales $h$). Let $\mathcal{I}_h \subset \mathcal{D}_{h,h}$ be a two-sided ideal such that $\mathcal{D}_{h,h}^{\wedge_x}/\mathcal{I}_h$ is finitely generated over $\mathbb{C}[h] \otimes A^{0_0}(T_x\mathcal{L})$ (here the second factor is the completed homogenized Weyl algebra of the symplectic vector space $T_x\mathcal{L}$).

The following proposition is deduced from Lemma 3.4 in the same way as \[L.5\] Proposition 3.8] is deduced from \[L.5\] Lemma 3.9. Let $\mathcal{D}_h$ be the universal quantization of $X$. 

Proposition 3.5. Assume that the algebraic fundamental group of $\mathcal{L}$ is finite. Then there is a maximal sheaf of ideals $\mathcal{J}_h \subset \mathcal{D}_{h,h}$ whose completion at $x$ is contained in $\mathcal{I}_h$. This ideal has the following properties:

1. It is $\mathbb{C}^\times$-stable.
2. Its completion coincides with $\mathcal{I}_h$.
3. The intersection of the support of $\mathcal{D}_{h,h}/\mathcal{J}_h$ with $X$ is $\overline{\mathcal{L}}$.

Similarly (compare to the proof of [L1, Theorem 1.3.2] given in [L1, Section 3.9]) one gets the following.

Lemma 3.6. Let $\mathcal{L}$ be a symplectic leaf in $X$ with finite algebraic fundamental group. Let $\mathcal{L}_1$ be a leaf in some fiber of $X_h \to \mathfrak{h}$ such that $\overline{\mathcal{L}}$ is an irreducible component in $\mathbb{C}^\times \mathcal{L}_1 \cap X$. Then $\overline{\mathcal{L}} = X \cap \mathbb{C}^\times \mathcal{L}_1$.

Proposition 3.7. The algebraic fundamental group of every leaf $\mathcal{L}$ is finite.

Proof. Consider a point $x \in \mathcal{L}$ and its formal neighborhood $X_h^{\wedge,x}$. By Remark 2.2, $X_h^{\wedge,x} \cong X^{\wedge,x} \times h^{\wedge,x}$ (an isomorphism of formal Poisson schemes). It follows that for a generic $\lambda \in \mathfrak{h}$ there is a symplectic leaf $\mathcal{L}_1 \subset X_\lambda$ of the same dimension as $\mathcal{L}$ such that $\mathcal{L} \subset \mathbb{C}^\times \mathcal{L}_1$ (the closure is taken in $X_h$). The intersection of $\mathbb{C}^\times \mathcal{L}_1$ (the closure is taken in $Y_h$) with $Y$ has therefore the same dimension as $\mathcal{L}$. It follows that for some open leaf $\mathcal{L}'$ in the intersection of $\mathbb{C}^\times \mathcal{L}_1$ with $X$, we have $\dim \mathcal{L}' = \dim \rho(\mathcal{L}')$.

Since $\rho$ is a Poisson morphism, $\dim \mathcal{L}' = \dim \rho(\mathcal{L}')$ implies that there is an open subset $\mathcal{L}'_0 \subset \mathcal{L}'$ that is an unramified cover of an open leaf $\mathcal{L}$ in $\rho(\mathcal{L}')$. By the work of Namikawa, [N3] (the case of open leaf) and of Proudfoot and Schedler, the proof of [PS, Proposition 3.1] (the general case), the algebraic fundamental group of $\mathcal{L}$ is finite. It follows that the algebraic fundamental group of $\mathcal{L}'_0$ is finite. Since $\mathcal{L}'$ is smooth, we deduce that the algebraic fundamental group of $\mathcal{L}'$ is finite.

By Lemma 3.8, the intersection of $\mathbb{C}^\times \mathcal{L}_1$ with $X$ is irreducible, so $\mathcal{L}' = \mathcal{L}$. This finishes the proof.

Let us deduce the following useful result whose proof in the case when $X$ is due to Kaledin, [K].

Lemma 3.8. The morphism $\rho : X \to Y$ is semismall.

Proof. Arguing as in the proof of [K, Lemma 2.11], we see that the morphism $X^{\text{reg}} \to Y$ is semismall. The subvariety $X$ is.

3.4. Proof of Theorem 1.1. We start with the following proposition.

Proposition 3.9. Suppose that the sheaf of algebras $\mathcal{D}_\lambda$ is simple for a Weil generic $\lambda \in \mathfrak{h}$. Then $\mathcal{D}_\lambda$ is simple for any $\lambda \in \mathfrak{h}$.

Proof. Let $\mathcal{J} \subset \mathcal{D}_\lambda$ be a proper ideal. Let $\mathcal{L} \subset X$ be an open symplectic leaf in the support of $\mathcal{D}_\lambda/\mathcal{J}_\lambda$. Pick a point $x \in X$. Let $\mathcal{J}_{\lambda,h}$ be the two-sided ideal in $\mathcal{D}_{\lambda,h}$ corresponding to $\mathcal{J}_\lambda$. Then $\mathcal{D}_{\lambda,h}^{\wedge,x}/\mathcal{J}_{\lambda,h}^{\wedge,x}$ is finitely generated over $\mathbb{A}^{\wedge,x}_h(T_x \mathcal{L})$. On the other hand, by Remark 2.3, $\mathcal{D}_{\lambda,h}^{\wedge,x} = \mathbb{C}[[h]] \otimes \mathcal{D}_{\lambda,h}^{\wedge,x}$. Set $\mathcal{I}_h := \mathbb{C}[[h]] \otimes \mathcal{J}_{\lambda,h}^{\wedge,x}$. By Proposition 3.5 we can find a $\mathbb{C}^\times$-stable ideal $\mathcal{J}_h \subset \mathcal{D}_{h,h}$ such that $\mathcal{J}_h^{\wedge,x} = \mathcal{I}_h$. Let $\mathcal{J}_{h,\text{fin}}$ denote the $\mathbb{C}^\times$-finite part of $\mathcal{J}_h$. Since $\mathcal{D}_{h,h}^{\wedge,x}/\mathcal{I}_h$ is flat over $\mathbb{C}[[h]]$, we see that $R_h(\mathcal{D}_h)/\mathcal{J}_{h,\text{fin}}$ is torsion free over $\mathbb{C}[h]$. So for a Weil generic $\lambda \in \mathfrak{h}$ the specialization of $\mathcal{D}_{h,h,\text{fin}}/\mathcal{J}_{h,\text{fin}}$ to $\lambda/h$ is nonzero. It follows that $\mathcal{D}_\lambda$ is not simple, a contradiction.

\qed
Proof of Theorem 1.1. We know by [L1, Theorem 4.2.1] that the algebra $H_c$ is simple for a Weil generic $c \in p$. It follows that $eH_ce$ is simple for such $c$. By Corollary 3.3, $D_\lambda$ is simple for a Weil generic $\lambda$. Now we are done by Proposition 3.9. □

Also let us record the following result.

**Corollary 3.10.** Suppose $D_\lambda$ is simple. Then the support of every coherent $D_\lambda$-module intersects $X^{reg}$.

**Proof.** Let $M$ be a coherent $D_\lambda$-module whose support does not intersect $X^{reg}$. The sheaf of algebras $D_\lambda$ is left Noetherian. It follows that $M$ has an irreducible quotient, so we can assume $M$ itself is irreducible. As in the proof of [L5, Theorem 1.1], Proposition 3.5 implies that the support of $D_\lambda/\text{Ann}_{D_\lambda}(M)$ is the closure of the single leaf that is a maximal (with respect to inclusion) leaf intersected by the support of $M$. It follows that $\text{Ann}(M)$ is a proper ideal. This finishes the proof. □

### 4. Procesi sheaves

In this section we axiomatically define and construct a Procesi sheaf on $X$, a $\mathbb{Q}$-factorial terminalization of $Y := V/\Gamma$. In the case when $\Gamma$ is a so called wreath-product group, and so $X$ is smooth, the construction was carried out by Bezrukavnikov and Kaledin, [BK], and our construction follows theirs.

**4.1. Definition of Procesi sheaf.** Set $Y = V/\Gamma$ and let $X$ still be a $\mathbb{Q}$-factorial terminalization of $Y$.

**Definition 4.1.** A $\mathbb{C}^\times$-equivariant coherent sheaf $\mathcal{P}$ on $X$ together with an isomorphism $\text{End}(\mathcal{P}) \cong \mathbb{C}[V]\#\Gamma$ is called a Procesi sheaf if the following holds:

1. The isomorphism $\text{End}(\mathcal{P}) \cong \mathbb{C}[V]\#\Gamma$ is $\mathbb{C}^\times$-equivariant and $\mathbb{C}[Y]$-linear.
2. We have $H^i(X, \text{End}(\mathcal{P})) = 0$ for $i > 0$.
3. $\mathcal{P}_\Gamma \cong \mathcal{O}_X$, an isomorphism of $\mathbb{C}^\times$-equivariant coherent sheaves.
4. $\mathcal{E}nd(\mathcal{P})$ is a maximal Cohen-Macaulay $\mathcal{O}_X$-module.

Note a few standard consequences of these conditions. By (iii), $\mathcal{P} = \mathcal{E}nd(\mathcal{P})e$ and hence (iv) implies that $\mathcal{P}$ is a maximal Cohen-Macaulay $\mathcal{O}_X$-module. In particular, $\mathcal{P}|_{X^{reg}}$ is a vector bundle. So when $X$ is smooth we recover an axiomatic description of a Procesi bundle from [L3, Section 1.1].

Let us also note that the definition makes sense for other fields as well. For example, we can reduce $X \to Y$ modulo $p$ for $p \gg 0$ getting a $\mathbb{Q}$-factorial terminalization $X_{\mathbb{F}} \to Y_{\mathbb{F}}$ for $\mathbb{F} := \mathbb{F}_p$. We can define a Procesi sheaf $\mathcal{P}_{\mathbb{F}}$ on $X_{\mathbb{F}}$ similarly. In fact, we will need a Frobenius twisted version $\mathcal{P}_{\mathbb{F}}^{(1)}$ on $X_{\mathbb{F}}^{(1)}$, which is again defined completely analogously.

**4.2. Frobenius constant quantization.** Our construction of $\mathcal{P}$ closely follows that of [BK], see Sections 5 and 6 there. The first step is to produce a Frobenius-constant quantization of $X_{\mathbb{F}}$ (where $\mathbb{F}$ is as before) with a specified algebra of global sections. Let us start by explaining what we mean by a Frobenius constant quantization in this context.

**Definition 4.2.** A Frobenius constant quantization $\mathcal{A}$ of $X_{\mathbb{F}}$ is a coherent sheaf of algebras on $X_{\mathbb{F}}^{(1)}$ whose restriction to the conical topology is equipped with a separated ascending filtration such that $\text{gr} \mathcal{A} \cong \text{Fr}_* \mathcal{O}_{X_{\mathbb{F}}}$.

Let us deduce several corollaries of this definition.
Lemma 4.3. The following is true:

1. $\mathcal{A}|_{X_F^{(1)}_{\text{reg}}}$ is an Azumaya algebra.

2. $\mathcal{A}$ is a maximal Cohen-Macaulay $\mathcal{O}_{X_F^{(1)}}$-module.

3. $H^i(X_F^{(1)}, \mathcal{A}) = 0$ and $\text{gr} \Gamma(X_F^{(1)}, \mathcal{A}) = \mathbb{F}[V_F]^\Gamma$.

Proof. Let us prove (1). Let $\mathcal{A}_h$ denote the $h$-adic completion of the Rees sheaf of the filtered sheaf $\mathcal{A}$. Then we have a central inclusion $\mathcal{O}_{X_F^{(1)}}[[h]] \hookrightarrow \mathcal{A}_h$. Now pick $x \in X_F^{(1),\text{reg}}$ and consider the specialization $\mathcal{A}_{h,x}$ to that point. It is a formal deformation of the symplectic Frobenius neighborhood of $x$ in $X_F$. It follows that the localization $\mathcal{A}_{h,x}[h^{-1}]$ is a simple algebra.

Let us prove (2). Since $p \gg 0$ and $X$ is Cohen-Macaulay, we see that $X_F$ is Cohen-Macaulay. It follows that $\text{Fr}_* \mathcal{O}_{X_F}$ is a maximal Cohen-Macaulay $\mathcal{O}_{X_F^{(1)}}$-module. Since $\mathcal{A}$ is a filtered deformation of a maximal Cohen-Macaulay module, it is maximal Cohen-Macaulay as well.

(3) follows from $H^i(X_F, \mathcal{O}_{X_F}) = 0$ and $\text{gr} \mathcal{A} = \text{Fr}_* \mathcal{O}_{X_F}$.

Proposition 4.4. There is a Frobenius constant quantization $\mathcal{A}$ of $X_F$ such that we have an isomorphism $\Gamma(X_F^{(1)}, \mathcal{A}) = W(V_F)^\Gamma$ of filtered $\mathbb{F}[V_F^{(1)}]^\Gamma$-algebras, where we write $\mathbb{A}(V_F)$ for the Weyl algebra of $V_F$.

Proof. Let us write $V_F^{(1),sr}/\Gamma$ for the union of the open leaf and all codimension 2 symplectic leaves in $V^{(1)}/\Gamma$. The preimage $X_F^{(1),sr}$ of $V_F^{(1),sr}/\Gamma$ in $X_F^{(1)}$ consists of smooth points. Moreover, the morphism $\rho_F$ is semismall because, by Lemma 3.3, $\rho$ is. It follows that the complement of $X_F^{(1),sr}$ in $X_F^{(1)}$ has codimension at least 2.

Similarly to the proof of [BK, Proposition 5.10], we get a Frobenius constant quantization $\mathcal{A}^{sr}$ of $X_F^{sr}$ (Frobenius constant quantization of $X_F^{sr}$ are defined similarly to those of $X_F$) with $\Gamma(X_F^{(1),sr}, \mathcal{A}^{sr}) = \mathbb{A}(V_F)^\Gamma$ (an equality of filtered $\mathbb{F}[V_F^{(1)}]^\Gamma$-algebras). Similarly to the proof of [BK, Proposition 5.11], we see that $\mathcal{A}^{sr}$ uniquely extends to a Frobenius constant quantization $\mathcal{A}^{\text{reg}}$ of $X_F^{(1),\text{reg}}$. But $\text{gr} \mathcal{A}^{\text{reg}} = \text{Fr}_* \mathcal{O}_{X_F}^{\text{reg}}$ and the complement of $X_F^{(1),\text{reg}}$ in $X^{(1)}$ has codimension 4. From here one deduces that $\mathcal{A} := i_* \mathcal{A}^{\text{reg}}$ is a Frobenius constant quantization of $X_F$, compare to [BPW, Proposition 3.4]. By the construction, $\mathcal{A}$ has required properties.

4.3. Construction of Procesi sheaf in characteristic $p$. Consider the formal neighborhood $\hat{Y}_F^{(1)}$ of 0 in $V_F^{(1)}$ and the formal neighborhood $\hat{Y}_F^{(1)}$ of 0 in $Y_F^{(1)}$. Set

$\hat{X}_F^{(1)} := \hat{Y}_F^{(1)} \times_{Y_F^{(1)}} X_F^{(1)}$, $\hat{\mathcal{A}} := \mathcal{A}_{X_F^{(1)}}$, $\hat{\mathcal{A}} := \mathbb{F}[\hat{V}_F^{(1)}] \otimes_{\mathbb{F}[V_F^{(1)}]} \mathbb{A}(V_F)$.

So we have $R\Gamma(\hat{\mathcal{A}}) = \hat{\mathcal{A}}^\Gamma$.

Similarly to [BK, Section 6.3], we see that

$\hat{\mathcal{A}}^{\text{reg}} := \hat{\mathcal{A}}|_{X_F^{(1),\text{reg}} \cap \hat{X}_F^{(1)}}$

(that, by, (1) of Lemma 4.3 is an Azumaya algebra) splits, while

$\hat{\mathcal{A}}|_{\hat{Y}_F^{(1)}}$

$\Gamma$-equivariantly splits (here we write $V_F^{(1)r}$ for the locus in $V_F^{(1)}$ consisting of all points with trivial stabilizer in $\Gamma$).
Now let us define a sheaf of algebras $\hat{A}$ on $\hat{X}_F^{(1)}$ that is Morita equivalent to $\hat{A}$ and has global sections $F[\hat{Y}_F^{(1)}]#\Gamma$. Similarly to [BK, Section 6], the algebras $\hat{A}^\Gamma$ and $F[\hat{Y}_F^{(1)}]#\Gamma$ are Morita equivalent. It follows that we can find central idempotents $e_i \in \hat{A}^\Gamma$, one per an irreducible representation of $\Gamma$ and nonzero multiplicities $n_i$ such that $F[\hat{Y}_F^{(1)}]#\Gamma = \bigoplus_{i,j}(e_i \hat{A}^\Gamma e_j)^{n_i n_j}$. Set $\hat{A} := \bigoplus_{i,j}(e_i \hat{A}^\Gamma e_j)^{n_i n_j}$.

**Lemma 4.5.** The sheaf $\hat{A}$ on $\hat{X}_F^{(1)}$ has the following properties.

1. $\Gamma(\hat{X}_F^{(1)}, \hat{A}) = F[\hat{Y}_F^{(1)}]#\Gamma$.
2. $H^i(\hat{X}_F^{(1)}, \hat{A}) = 0$ for $i > 0$.
3. $\hat{A}$ is a maximal Cohen-Macaulay $O_{\hat{X}_F^{(1)}}$-module.
4. $\hat{A}^{reg} := \hat{A}|_{\hat{X}_F^{(1)} \cap \hat{X}_F^{(1)}}$ is a split Azumaya algebra.
5. Let $e$ denote the averaging idempotent in $F\Gamma$. Then $e \hat{A} e = O_{\hat{X}}$.
6. The sheaf of algebras $\hat{A}$ coincides with the endomorphism sheaf of $\hat{P} := \hat{A} e$.

**Proof.** (1) to (4) follow from the construction of $\hat{A}$ and the analogous properties of $\hat{A}$.

Let us prove (5). We have an inclusion $O_{\hat{X}_F^{(1)}} \hookrightarrow e \hat{A} e$ that is an iso over $\hat{X}_F^{(1)}reg$. Since both $O_{\hat{X}_F^{(1)}}$ and $e \hat{A} e$ are maximal Cohen-Macaulay, we see that $O_{\hat{X}_F^{(1)}} \tilde{\to} e \hat{A} e$.

Let us prove (6). By the construction, $\hat{P}$ is a maximal Cohen-Macaulay $O_{\hat{X}_F^{(1)}}$-module. So $\hat{P} = i_* i^* \hat{P}$, where we write $i$ for the inclusion $\hat{X}_F^{(1)}reg \hookrightarrow \hat{X}_F^{(1)}$. It follows that $\text{End}(\hat{P}) = i_* \text{End}(i^* \hat{P})$. Similarly, $\hat{A} = i_* i^* \hat{A}$. And since $i^* \hat{A}$ is a split Azumaya algebra, we have $i^* \hat{A} = \text{End}(i^* \hat{P})$. This implies (6). \hfill $\Box$

So the sheaf $\hat{P}$ behaves almost like a Procesi sheaf with two differences: it does not carry an action of $F^\times$ and it is defined on $\hat{X}_F^{(1)}$, while we originally wanted a sheaf on $X$. For this we first equip it with an $F^\times$-equivariant structure and extend to a sheaf $P_F^{(1)}$ on $X_F^{(1)}$. Then we lift (a twist of) $P_F^{(1)}$ to characteristic 0.

**Lemma 4.6.** There is a unique Procesi sheaf $P_F^{(1)}$ on $X_F^{(1)}$ whose restriction to $\hat{X}_F^{(1)}$ coincides with $\hat{P}$.

**Proof.** The proof is in several steps.

**Step 1.** Let us equip $\hat{P}$ with an $F^\times$-equivariant structure. Since $\hat{P} = i_* i^* \hat{P}$, it is enough to equip the vector bundle $i^* \hat{P}$ on $\hat{X}_F^{(1)}reg$ with an $F^\times$-equivariant structure. The proof follows [V] and consists of two parts. First, we need to show that the isomorphism class of $i^* \hat{P}$ is $F^\times$-invariant. Second, we need to deduce that $i^* \hat{P}$ is equivariant. The second claim follows precisely as in [V].

**Step 2.** The proof of the first claim of the previous step is a ramification of that of [V, Lemma 6.3]. Namely, although $\hat{X}_F^{(1)}reg \to \hat{Y}_F^{(1)}$ is not proper, since the codimension of the complement of $\hat{X}_F^{(1)}reg$ in $\hat{X}_F^{(1)}$ is bigger than 3, the sheaves on $\hat{Y}_F^{(1)} \times F^\times$ that are coherent in the proof of that lemma are coherent in our case as well. This produces an $F^\times$-equivariant structure on $\hat{P}$.

**Step 3.** Let us show that we can choose an $F^\times$-equivariant structure on $i^* \hat{P}$ in such a way that the isomorphism $\text{End}(i^* \hat{P}) \tilde{\to} F[[V_F^{(1)}]]#\Gamma$ is $F^\times$-equivariant. The proof follows an analogous proof in [BL, Section 10.3]. Pick a basis in each irreducible representation
of \( \Gamma \) and let \( \mathcal{B} \) denote the union of these bases. Let \( e_v \) be the primitive idempotent in \( \mathbb{F}[\Gamma] \) corresponding to \( v \in \mathcal{B} \). We can decompose \( i^* \mathcal{P} \) as \( \bigoplus_{v \in \mathcal{B}} \mathcal{P}_v \), where \( \mathcal{P}_v := e_v i^* \mathcal{P} \) is an indecomposable vector bundle on \( \hat{X}_F^{(1)} \). We can assume that the \( \mathbb{F}^x \)-equivariant \( \mathbb{F}[\Gamma] \)-module structure on \( i^* \mathcal{P} \) is chosen in such a way that each \( \mathcal{P}_v \) is stable. We note that \( \Gamma(\mathcal{P}_v) = e_v \mathbb{F}[\Gamma] \). It is easy to see that an \( \mathbb{F}^x \)-equivariant structure on \( e_v \mathbb{F}[\Gamma] \) is unique up to a twist with a character. So twisting an \( \mathbb{F}^x \)-equivariant structure on each \( \mathcal{P}_v \) we achieve that the isomorphism \( \text{End}(i^* \mathcal{P}) \cong \mathbb{F}[\Gamma] \) holds in our case for the same reason they hold in \( \mathbb{F} \). The other axioms of \( \mathbb{F}^x \) hold in our case for the same reason they hold in \( \mathbb{F} \).

\[ \text{Step 4.} \] Since the action of \( \mathbb{F}^x \) on \( X_F^{(1)} \) is contracting and \( X_F^{(1)} \) is projective over an affine scheme, we see that \( \mathcal{P}_F \) extends to a unique \( \mathbb{F}^x \)-equivariant sheaf \( \mathcal{P}_F^{(1)} \) on \( X_F^{(1)} \).

We can twist \( \mathcal{P}_F^{(1)} \) with \( \text{Fr} \) several times to get a Procesi sheaf \( \mathcal{P}_F \) on \( X_F \).

4.4. \textbf{Construction of Procesi sheaf in characteristic 0.} We can extend the Procesi sheaf to characteristic 0 basically as that was done in [BK] with some modifications. Namely, \( X_F, \mathcal{P}_F \) are defined over some finite field \( \mathbb{F}_q \), let \( X_{\mathbb{F}_q}, \mathcal{P}_{\mathbb{F}_q} \) be the corresponding reductions. Let \( R \) denote an integral extension of \( \mathbb{Z}_p \) with residue field \( \mathbb{F}_q \). We can assume that \( X \) is defined over \( R \), let \( X_R \) denote the corresponding form so that \( X_{\mathbb{F}_q} = \text{Spec}(\mathbb{F}_q) \times_{\text{Spec}(R)} X_R \).

\[ \text{Lemma 4.7.} \] \( \mathcal{P}_{\mathbb{F}_q} \) uniquely extends to a Procesi bundle \( \mathcal{P}_R \) on \( X_R \).

\[ \text{Proof.} \] Consider the open subscheme \( X_{\mathbb{F}_q}^{reg} \subset X_{\mathbb{F}_q} \), the complement still has codimension at least 4. Let \( X_R^{\circ} \) denote the formal neighborhood of \( X_{\mathbb{F}_q} \) in \( X_R \) and let \( X_R^\circ \) be the formal neighborhood of \( X_{\mathbb{F}_q}^{reg} \) in \( X_R \) so that we have an inclusion of formal schemes \( i : X_R^\circ \hookrightarrow X_R \).

So \( X_R^\circ \) is a formal deformation of \( X_{\mathbb{F}_q}^{reg} \).

Note that \( \text{Ext}^i(i^* \mathcal{P}_{\mathbb{F}_q}, i^* \mathcal{P}_{\mathbb{F}_q}) = 0 \) for \( i = 1, 2 \) because \( \text{End}(\mathcal{P}_{\mathbb{F}_q}) \) is maximal Cohen-Macaulay and the codimension of the complement to \( X_{\mathbb{F}_q}^{reg} \) in \( X_{\mathbb{F}_q} \) is at least 4. So \( i^* \mathcal{P}_{\mathbb{F}_q} \) admits a unique \( G_m \)-equivariant deformation to \( X_R^{\circ} \), let us denote it by \( \mathcal{P}_R^{\circ} \). Now the codimension condition guarantees that \( \mathcal{P}_R^\Lambda := i_* \mathcal{P}_R^{\circ} \) is an \( R \)-flat deformation of \( i_* i^* \mathcal{P}_{\mathbb{F}_q} = \mathcal{P}_{\mathbb{F}_q} \). For a similar reason, \( \text{End}(\mathcal{P}_R^\Lambda) \) is an \( R \)-flat deformation of \( \text{End}(\mathcal{P}_{\mathbb{F}_q}) \). Because of the \( G_m \)-equivariance we can extend \( \mathcal{P}_R^\Lambda \) to a unique \( G_m \)-equivariant coherent sheaf \( \mathcal{P}_R \) on \( X_R \). That \( \text{End}(\mathcal{P}_R) \) is isomorphic to \( R[\mathbb{F}_R][\# \Gamma] \) is proved similarly to [BK] Section 6.4, as the assumptions of [BK] Proposition 4.3 hold in our case for the same reason they hold in [BK]. The other axioms of the Procesi sheaves for \( \mathcal{P}_R \) are straightforward.

Then we can base change from \( R \) to \( \mathbb{C} \) and get a required Procesi sheaf on \( X \).

\[ \text{Remark 4.8.} \] One can also classify all Procesi bundles on \( X \) similarly to [L3]: they are in bijection with the Namikawa-Weyl group of \( Y \). The proof basically repeats [L3] Sections 2,3] (see, e.g., (2) of Proposition 5.1).

5. \textbf{Derived equivalences from Procesi sheaves}

In this section we are going to use the Procesi sheaf \( \mathcal{P} \) on \( X \) to prove Theorems [L2] and [L3]. We use the following notation – \( \mathcal{H}^0 = \text{End}(\mathcal{P}) \), this is a maximal Cohen-Macaulay sheaf on \( X \).
5.1. Quantizations of the Procesi sheaf. Let $\mathcal{D}_c$ be the quantization of $X$ corresponding to $c \in \mathfrak{p}$. Let us write $\mathcal{D}_c^{reg}$ for the restriction of $\mathcal{D}_c$ to $X^{reg}$ and $\mathcal{P}^{reg}$ for the restriction of $\mathcal{P}$ to $X^{reg}$. As in the proof of Lemma 4.7, we see that $\text{Ext}^i(\mathcal{P}^{reg}, \mathcal{P}^{reg}) = 0$ for $i = 1, 2$. It follows that $\mathcal{P}^{reg}$ admits a unique deformation to a locally free right $\mathcal{D}_c^{reg}$-module to be denoted by $\mathcal{P}_c^{reg}$. As usual, we set $\mathcal{P}_c := i_* \mathcal{P}_c^{reg}$, where $i$ stands for the inclusion $X^{reg} \hookrightarrow X$. We can also define the universal version $\mathcal{P}_p^{reg}$ on $X^{reg}$ and its push-forward $\mathcal{P}_p := i_* \mathcal{P}_p^{reg}$. Let us write $\mathcal{H}_c$ for $\mathcal{E}nd_{\mathcal{D}_c^{reg}}(\mathcal{P}_c)$.

**Proposition 5.1.** The following is true:

1. $\text{gr} \mathcal{H}_c = \mathcal{H}^0$ for all $c$.
2. $\Gamma(\mathcal{H}_c) = H_{w(c)}$, where $w$ is an element of the Namikawa-Weyl group depending only on $\mathcal{P}$.

**Proof.** Let us prove (1). Consider the $h$-adically completed Rees sheaf $\mathcal{P}_{c,h}$ of $\mathcal{P}_c$ so that we have an exact sequence $0 \to \mathcal{P}_{c,h} \to \mathcal{P}_{c,h} \to \mathcal{P} \to 0$. We need to prove that $\mathcal{E}nd(\mathcal{P}_{c,h})/(h) = \mathcal{H}^0$. We know from the construction that $i^* \mathcal{E}nd(\mathcal{P}_{c,h})/(h) = i^* \mathcal{H}^0$. Also we have $i_* i^* \mathcal{H}^0 = \mathcal{H}^0$. From the inclusion $\mathcal{E}nd(\mathcal{P}_{c,h})/(h) \hookrightarrow \mathcal{H}^0$ and the definition of $\mathcal{P}_{c,h}$ we conclude that the natural homomorphism $\mathcal{E}nd(\mathcal{P}_{c,h}) \to i_* i^* \mathcal{E}nd(\mathcal{P}_{c,h})$ is an isomorphism. So what we need to prove is that $R^1 i_! i^* \mathcal{E}nd(\mathcal{P}_{c,h}) = 0$. We know $R^1 i_! i^* \mathcal{H}^0 = 0$ (because $\mathcal{H}^0$ is maximal Cohen-Macaulay and the codimension of the complement of $X^{reg}$ is, at least, $4$). Similarly to the proof of [GL, Lemma 5.6.3], it follows that $R^1 i_! i^* \mathcal{E}nd(\mathcal{P}_{c,h}) = 0$.

Let us prove (2). Since $\text{Ext}^1(\mathcal{P}, \mathcal{P}) = 0$ (compare to the proof of Lemma 4.7), we see that $\text{gr} \mathcal{E}nd(\mathcal{P}_p) = \text{gr} H_p$. The unversality property of $H_p$, see [L2, Section 6.1], implies that there is an filtered algebra isomorphism $\mathcal{E}nd(\mathcal{P}_p) \cong H_p$ that induces an affine map on $\mathfrak{p}$ and induces a trivial map on $\mathbb{C}[V] \# \Gamma$. Passing to spherical subalgebras we get a filtered algebra automorphism $e H_p e \cong e H_p e$ that induces the trivial automorphism of $\mathbb{C}[V] \# \Gamma$. This isomorphism has to be given by an element of the Namikawa-Weyl group. \hfill $\square$

5.2. Mckay equivalence. In this section we will prove the following result. Consider the category $\text{Coh}(\mathcal{H}^0)$ of coherent $\mathcal{H}^0$-modules on $X$.

**Proposition 5.2.** Let $\mathcal{P}$ be a Procesi bundle on $X$. Then the derived global section functor $R\Gamma : D^b(\text{Coh}(\mathcal{H}^0)) \cong D^b(\mathbb{C}[V] \# \Gamma \text{-mod})$ is a category equivalence.

Similarly to [BK, Section 2.2], a crucial step in the proof of of Proposition 5.2 is to prove the following lemma.

**Lemma 5.3.** The following is true:

1. The category $D^b(\text{Coh}(\mathcal{H}^0))$ is indecomposable (as a triangulated category).
2. The functor $\bullet[\text{dim } X]$ is a Serre functor for $D^b(\text{Coh}(\mathcal{H}^0))$ meaning that $R\text{Hom}_{\mathbb{C}[Y]}(R\text{Hom}_{\mathcal{H}^0}(\mathcal{F}, \mathcal{G}), \mathbb{C}[Y][\text{dim } Y]) \cong R\text{Hom}_{\mathcal{H}^0}(\mathcal{G}, \mathcal{F}[\text{dim } X])$.

**Proof.** Let us note that the objects of the form $\mathcal{H}^0 \otimes \mathcal{L}$, where $\mathcal{L}$ is a line bundle on $X$, generate $\text{Coh}(\mathcal{H}^0)$.

Let us prove (1). Let $e$ be a primitive idempotent in $\mathbb{C}[\Gamma]$. Then $\text{Hom}(\mathcal{H}^0 e \otimes \mathcal{L}, \mathcal{F}) = \mathcal{L}^* \otimes e \mathcal{F}$, an equality of coherent sheaves on $X$. So $\text{Hom}(\mathcal{H}^0 e \otimes \mathcal{L}, \mathcal{F}) = \Gamma(\mathcal{L}^* \otimes e \mathcal{F})$. This space is nonzero as long as $e \mathcal{F}$ is nonzero and $\mathcal{L}$ is sufficiently anti-ample.
Note that the object $\mathcal{H}^0e \otimes \mathcal{L}$ is indecomposable. Indeed, it is enough to assume that $\mathcal{L}$ is trivial. The global sections of $\mathcal{H}^0e$ is $(\mathbb{C}[V]\#\Gamma)e$. This is an indecomposable $\mathbb{C}[V]\#\Gamma$-module because $e$ is primitive. So if $\mathcal{H}^0e$ is decomposable, then one of its summands has trivial global sections, and, in particular, proper support in $X$. The latter is impossible because $\mathcal{H}^0$ is maximal Cohen-Macaulay. So we see that $\mathcal{H}^0e \otimes \mathcal{L}$ is indecomposable. If we can decompose $D^b(\text{Coh}(\mathcal{H}^0))$ into the direct sum of two triangulated categories, $\mathcal{C}_1 \oplus \mathcal{C}_2$, then for each $e, \mathcal{L}$ the object $\mathcal{H}^0e \otimes \mathcal{L}$ lies in one summand.

Now note that $\text{Hom}(\mathcal{H}^0e, \mathcal{H}^0e') = e(\mathbb{C}[V]\#\Gamma)e' \neq 0$. So all $\mathcal{H}^0e$ lie in the same summand, say $\mathcal{C}_1$. It follows that for any sufficient anti-ample $\mathcal{L}$, we have $\mathcal{H}^0e \otimes \mathcal{L} \in \mathcal{C}_1$. So $\mathcal{C}_2 = 0$ and the proof of (1) is finished.

Let us prove (2). Recall that $K_X$ is trivial. As in [BK, Section 2.1], since the Grothendieck-Serre duality commutes with proper direct images, what we need to check is

\begin{equation}
\text{Proof of Proposition 5.2.}
\text{The proof closely follows that of [BK, Proposition 2.2]. Namely, we have the left adjoint functor $\mathcal{H}^0 \otimes_{\mathbb{C}[V]\#\Gamma}^L \bullet$. It is right inverse to $R\Gamma$ because of $R\Gamma(\mathcal{H}^0) = \mathbb{C}[V]\#\Gamma$. We need to prove that $\mathcal{H}^0 \otimes_{\mathbb{C}[V]\#\Gamma}^L \bullet$ is essentially surjective. Since $\bullet_{\mathbb{C}[V]\#\Gamma}^{[\dim X]}$ is a Serre functor for $\text{Coh}(\mathcal{H}^0)$ and the category $\text{Coh}(\mathcal{H}^0)$ is indecomposable, we can apply [BK, Lemma 2.7] and conclude that $\mathcal{H}^0 \otimes_{\mathbb{C}[V]\#\Gamma}^L \bullet$ is essentially surjective.}
\end{equation}

5.3. **Proofs of Theorems 1.2,1.3.** We prove Theorem 1.3 and then deduce Theorem 1.2 from here. Our first step is the following corollary that follows from Proposition 5.2 similarly to what was done in [GL, Section 5.5].

**Corollary 5.4.** The derived global section functor $R\Gamma : D^b(\text{Coh}(\mathcal{H}_c)) \sim D^b(\mathcal{H}_w(c)\text{-mod})$ is a category equivalence for all $c$.

Theorem 1.3 is now a consequence of the following proposition.
Proposition 5.5. The functor \( M \mapsto eM : \text{Coh}(\mathcal{H}_c) \to \text{Coh}(\mathcal{D}_c) \) is a category equivalence.

Proof. Suppose, first, that abelian localization holds for \((X, c)\) and that the parameter \(w(c)\) (where \(w \in W_Y\) is the element defined by \(\mathcal{P}\), see (2) of Proposition 5.1) is spherical. Note that the global section functors \(\Gamma^H : \text{Coh}(\mathcal{H}_c) \to H_{w(c)}\text{-mod}\) and \(\Gamma^D : \text{Coh}(\mathcal{D}_c\text{-mod}) \to eH_{w(c)}e\text{-mod}\) intertwine the multiplication by \(e\). The functors \(\Gamma^D\) and \(\mathcal{N} : H_{w(c)}\text{-mod} \to eH_{w(c)}e\text{-mod}\) are category equivalences. We conclude that the functor \(\Gamma^H\) is exact. By Corollary 5.4, the functor \(R\Gamma^H\) is an equivalence, hence \(\Gamma^H\) is an equivalence. It follows that the functor \(M \mapsto eM : \text{Coh}(\mathcal{H}_c) \to \text{Coh}(\mathcal{D}_c)\) is an equivalence under our assumption on \(c\).

Now let us prove the claim of the proposition without restrictions on \(c\). Recall that, for \(\chi \in \text{Pic}(X)\), we have the \(\mathcal{D}_{c+\chi}\text{-D}_{c}\)-bimodule \(\mathcal{D}_{c,\chi}\) quantizing the line bundle \(\mathcal{O}(\chi)\) and also the \(\mathcal{D}_{c}\text{-D}_{c+\chi}\)-bimodule \(\mathcal{D}_{c,\chi, -\chi}\). Tensor products with these bimodules define mutually quasi-inverse equivalences between \(\text{Coh}(\mathcal{D}_c)\) and \(\text{Coh}(\mathcal{D}_{c+\chi})\). In particular, we have natural isomorphisms

\[
\mathcal{H}_c \cong \text{End}_{\mathcal{D}^{opp}}(\mathcal{P}_c \otimes_{\mathcal{D}_c} \mathcal{D}_{c+\chi,-\chi}), \mathcal{H}_{c+\chi} \cong \text{End}_{\mathcal{D}^{opp}}(\mathcal{P}_{c+\chi} \otimes_{\mathcal{D}_{c+\chi}} \mathcal{D}_{c,\chi}).
\]

So we can consider \(\mathcal{H}_{c+\chi}\text{-H}_c\)-bimodule \(\mathcal{E}_{c,\chi} := \text{Hom}_{\mathcal{D}^{opp}}(\mathcal{P}_c \otimes_{\mathcal{D}_c} \mathcal{D}_{c+\chi,-\chi}, \mathcal{P}_{c+\chi})\) and the \(\text{End}(\mathcal{P}_c)\text{-End}(\mathcal{P}_{c+\chi})\)-bimodule \(\mathcal{E}_{c+\chi,\chi} := \text{Hom}_{\mathcal{D}^{opp}}(\mathcal{P}_{c+\chi} \otimes_{\mathcal{D}_{c+\chi}} \mathcal{D}_{c,\chi}, \mathcal{P}_c)\).

We claim that the bimodules \(\mathcal{E}_{c,\chi}\) and \(\mathcal{E}_{c+\chi, -\chi}\) are mutually inverse Morita equivalences. Indeed, these bimodules come with natural filtrations and \(\text{gr} \mathcal{E}_{c,\chi} \cong \mathcal{H}^0 \otimes \mathcal{O}(\chi)\) and \(\text{gr} \mathcal{E}_{c+\chi, -\chi} \cong \mathcal{H}^0 \otimes \mathcal{O}(-\chi)\). The latter \(\mathcal{H}^0\)-bimodules are locally trivial. Tensoring over \(\mathcal{H}^0\) with these associated graded bimodules give autoequivalences of the category \(\text{Coh}(\mathcal{H}^0)\). Since the associated graded bimodules are mutually inverse Morita equivalences, so are the initial bimodules.

For any \(c\), we can choose \(\chi\) so that abelian localization holds for \((X, c + \chi)\). Note that \(\mathcal{P}_{c+\chi} \otimes_{\mathcal{D}_{c+\chi}} \bullet\) is an equivalence \(\text{Coh}(\mathcal{D}_c) \tilde\to \text{Coh}(\mathcal{H}_c)\) inverse to \(M \mapsto eM\). Note also that

\[
e(\mathcal{E}_{c+\chi, -\chi} \otimes_{\mathcal{H}_c} \mathcal{P}_{c+\chi} \otimes_{\mathcal{D}_{c+\chi}} \bullet) \cong \mathcal{D}_{c+\chi, -\chi} \otimes_{\mathcal{D}_{c+\chi}} \bullet.
\]

It follows that \(M \mapsto eM : \text{Coh}(\mathcal{H}_c) \to \text{Coh}(\mathcal{D}_c)\) is a category equivalence. \(\square\)

Proof of Theorem 1.3. Theorem 1.3 follows now from Propositions 5.5 and Corollary 5.4. \(\square\)

Proof of Theorem 1.2. Thanks to Theorem 1.3, it is enough to show that \(\text{Coh}(\mathcal{D}_c) \tilde\to \text{Coh}(\mathcal{D}_{c+\chi})\) for any \(\chi \in \text{Pic}(X^{reg})\) (so far, we know an equivalence for \(\chi \in \text{Pic}(X)\)). Let \(\mathcal{O}^{reg}(\chi)\) be a line bundle on \(X^{reg}\) corresponding to \(\chi\). Since \(H^i(X^{reg}, \mathcal{O}^{reg}) = 0\) for \(i = 1, 2\), the bundle \(\mathcal{O}^{reg}(\chi)\) admits a unique quantization to a \(\mathcal{D}_{c+\chi}\text{-D}_{c}^{reg}\)-bimodule that we denote by \(\mathcal{D}_{c,\chi}^{reg}\). Set \(\mathcal{D}_{c,\chi} := i_! \mathcal{D}_{c,\chi}^{reg}\), where \(i_!\) denotes the inclusion of \(X^{reg}\) into \(X\). We claim that \(\mathcal{D}_{c,\chi}\) are mutually dual Morita equivalences. Indeed, we have natural bimodule homomorphisms

\[
\mathcal{D}_{c,\chi} \otimes_{\mathcal{D}_c} \mathcal{D}_{c+\chi, -\chi} \to \mathcal{D}_{c+\chi}, \mathcal{D}_{c+\chi, -\chi} \otimes_{\mathcal{D}_{c+\chi}} \mathcal{D}_{c,\chi} \to \mathcal{D}_c.
\]

They becomes iso after restriction to \(X^{sing}\). So their kernels and cokernels are supported on \(X^{sing}\). By Corollary 5.10 they are zero. \(\square\)

6. Applications

Let us discuss some applications of Theorems 1.1, 1.3.
6.1. **Generalized Bernstein inequality.** Let $Y$ be a conical symplectic singularity. Suppose that $D_\lambda \otimes D_\mu$ is simple for all $\lambda, \mu$ (that is true for $Y = V/\Gamma$ because $Y \times Y = (V \oplus V)/(\Gamma \times \Gamma)$ is again a symplectic quotient singularity).

**Proposition 6.1.** Any HC $A_\lambda$-bimodule has finite length.

**Proof.** The proof repeats that from [L5, Section 4.3]: note that we can prove the finiteness of length for $D_\lambda \otimes D_\lambda^{opp}$ modules supported on $X \times_Y X^-$ (where $X^-$ is the terminalization of $X$ corresponding to the cone opposite to that of $X$) using the characteristic cycle argument because of Corollary 3.10.

**Corollary 6.2.** For any $c \in p$, the regular $H_\mu$-bimodule has finite length.

**Proof.** It was explained in [L5, Remark 4.5] that the finiteness of length will follow if we show that $H_c$ contains a minimal ideal of finite codimension. In the case, when $c$ is spherical (i.e., $H_c e H_c = H_c$) the existence of a minimal ideal of finite codimension follows from Proposition 6.1 applied to the regular $e H_c e$-bimodule. In the general case, we can find $\chi \in p^\ast$ such that $c + \chi$ is spherical and abelian localization holds for $c + \chi$ and some choice of $X$. Then we can use the bimodule $\Gamma(E_{c, \chi})$ similarly to what was done in [L5, Section 4.3].

As was noted in the introduction, Corollary 6.2 implies Theorem 1.4. An analog of this theorem also follows for the algebras $A_\lambda$, where $Y$ satisfies the additional assumption in the beginning of this section.

6.2. **Perverse equivalences.** Here we assume that $Y$ is such that the formal slices to all symplectic leaves are conical. Moreover, we assume that the quantization $D_\lambda$ is simple for all $\lambda$ (equivalently, for a Weil generic $\lambda$). Then we can generalize results of [L7, Section 3.1] to this setting.

Namely, let us choose two chambers $C, C'$ in $\mathfrak{h}$ that are opposite with respect to a common face. Let us pick parameters $\lambda, \lambda' \in \mathfrak{h}$ with $\chi := \lambda' - \lambda \in \mathfrak{h}_\mathbb{Z}$ such that abelian localization holds for $(\lambda', C')$ and $(\lambda, C)$. We can consider the wall-crossing $A_{\lambda'}$-$A_\lambda$-bimodule $A_{\lambda, \chi}$ defined as in [BPW, Section 6.3].

**Proposition 6.3.** After suitably modifying $\lambda, \lambda'$ (and hence $\chi$) by adding elements from $\mathfrak{h}_\mathbb{Z}$ so that $(\lambda, C), (\lambda', C')$ still satisfy abelian localization, the functor $A_{\lambda, \chi} \otimes A_{\lambda} \bullet$ becomes a perverse derived equivalence $D^b(A_\lambda \text{-mod}) \to D^b(A_{\lambda'} \text{-mod})$, where the filtrations by Serre subcategories on $A_\lambda \text{-mod}, A_{\lambda'} \text{-mod}$ making these equivalences perverse are introduced as in [L7, Section 3.1].

**Proof.** The proof basically repeats that of [L7, Theorem 3.1]. The case when $\lambda$ is Weil generic in an affine subspace with associated vector space $\text{Span}_C(C \cap C')$ is handled using Corollary 3.10 (that in the setting of [L7], where the case of smooth $X$ is considered, is straightforward). The rest of the proof is the same as that of [L7, Theorem 3.1].

6.3. **Abelian localization and simplicity.** Also let us mention two results that are proved as in [BL, Section 8.4] using the case of $C' = -C$ in Proposition 6.3. In that case, the filtration on $A_\lambda \text{-mod}$ is defined by $A_\lambda \text{-mod}_i := \{M \in A_\lambda \text{-mod} \mid \dim V(A_\lambda \text{/} \text{Ann}_{A_\lambda}(M)) \leq \dim X - 2i\}$. We still keep the assumptions of the previous section.

**Proposition 6.4.** There is a finite collection of hyperplanes in $\mathfrak{h}^{\text{sing}}$ with the following property: if $\lambda + \mathfrak{h}_\mathbb{Z}$ does not intersect the union of these hyperplanes, then $A_\lambda$ is simple.
The same holds for the algebras $H_c$, this strengthens [L1, Theorem 4.2.1].

**Proposition 6.5.** Let $\lambda \in p$. Then there is $\lambda_0 \in \lambda + h_{\mathbb{Z}}$ with the following property: for any $\lambda_1 \in \lambda_0 + (C \cap h_{\mathbb{Z}})$ the functor $\Gamma_{\lambda_1}$ is a category equivalence.

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