Self organization in a minority game: the rôle of memory and a probabilistic approach

E. Burgos and Horacio Ceva

Departamento de Física, Comisión Nacional de Energía Atómica,
Avda. del Libertador 8250, 1429 Buenos Aires, Argentina
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Abstract

A minority game whose strategies are given by probabilities $p$, is replaced by a ‘simplified’ version that makes no use of memories at all. Numerical results show that the corresponding distribution functions are indistinguishable. A related approach, using a random walk formulation, allows us to identify the origin of correlations and self organization in the model, and to understand their disappearance for a different strategy’s update rule, as pointed out in a previous work.

Keywords: Minority game, Organization, Evolution

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The minority game (MG), introduced by Challet and Zhang [1], addresses the problem of self organization of a population without direct interactions between its members, but with a feedback mechanism related with its collective behavior. Each person (dubbed ‘agent’, because of the use of the model in economic problems) has to choose from a simple alternative, without knowing what the other agents will do. At the end of the game, there are two groups, one for each alternative: agents belonging to the smaller group (the ‘minority’) will be the winners. Feedback is established by a reward system for winners and losers.

Different methods to choose one or the other alternative give rise to different versions of the model. It is usual to refer to these methods as ‘strategies’. In the following, we will use the model proposed by Johnson et.al. [2]. In this version, each agent knows beforehand the previous $m$ outcomes (a ‘history’), of the game, as well as the ‘next move’ of the most recent occurrences of all $2^m$ possibilities. What is distinctive of Johnson’s et.al. formulation is the assignement to each of the $N$ agents of a single number as its strategy, $p_j$ ($0 \leq p_j \leq 1$): given a history, each agent will either choose the same outcome as that stored in the memory, with probability $p_j$, or will choose the opposite with probability $q_j = 1 - p_j$ ($1 \leq j \leq N$).

Winners, i.e. those in the minority group, will gain $G$ points ($G \equiv 1$ in Ref. [2]); those in the majority group lose a point. Strategies can be modified, following the evolution of the game: if the number of points of an agent is below a threshold value $d \leq 0$, his ‘account’ is reset to zero, and he/she gets a new strategy, whose value $p' \in [p - R/2, p + R/2)$, where $|R| \leq 2$; in what follows we will use the simpler notation $p \rightarrow p' = p \pm \Delta p$. Whenever necessary, we used reflective boundary conditions.

Johnson et.al. have shown that, as a result of correlations, the system self organizes mainly in two distinct groups of agents, with extreme values of their strategies: $p \approx 0$ or $p \approx 1$. The frequency distribution $P(p)$ is shown in Fig.1 [3]. Moreover, they also found that their results did not change if memories are not updated, or even if they are randomly chosen; a similar result was obtained by Cavagna [4] in relation with the work of Challet and Zhang [1].

In this work, we show that a ‘simplified’ version of the model, making no use of the memories, also displays self organization, and the resulting distribution $P(p)$ is indistinguishable from the original model, for all odd $N \geq 3$. Another approach (called ‘probabilistic’ in what follows) serves to present a rather detailed interpretation of this MG, explaining the mechanisms establishing correlations between the agents, and their relations with the rules of the game. This also allows us to explain the smallness of correlations found in a previous publication, where a different update rule of the strategies was used [5].

In our ‘simplified’ version of the model, agent $j$ chooses one of the two options (option “1”, say) with probability $p_j$, or the other (option “0”) with probability $q_j = 1 - p_j$, without making recourse to any history at all. All other rules, like the determination of the minority, the reward system, the upgrade of strategies $p$, etc. are the same as before. In Fig. 2 we compare results obtained with this version and with the original formulation, for $N = 101$, $d = -4$, $R = 0.2$. A single realization of the game involved $n_t = 10^5$ time steps, and the distribution was averaged over $n_s = 10^4$ samples. As it was already mentioned, both results are indistinguishable.

Even if one lets the memories totally outside of the game, there still are many parameters in the model (the variables $G, d, N, R$), and it is worth to see how it depends on these
variables. All our simulations were made in such a way that we can assure that the expected numerical fluctuations for the initial, uniform distribution, were small. To this end, we requested that the standard deviation \(1/\sqrt{Nn_s/c}\) be no greater than 0.02. We always used \(c=100\) channels in \(p\) (each of width 0.01). Moreover, to be able to compare results for different values of \(N\), we have normalized all our data so that \(\int P(p)dp = 1\).

We collected data for \(N = 3, 5, 7, 9, 11, 21, 51\) and 101, \(d = 0, -2, -4\) and \(-5\), for fixed values of \(R = 0.2\), and \(G = 1\). Figure 2 shows the density values for both extrema, \(P(0), P(1)\) vs \(1/N\). It is rather clear that \(N = 101\) is already near the ‘thermodynamic’ limit \((N \to \infty)\). Also, there is only a moderated dependence on the value of the threshold \(d\).

Results for the ‘simplified’ version of the model show that we need not to consider memories, and constitute the starting point for the following development.

Our second, probabilistic, approach originated in the observation that the change of strategies, \(p \to p'\), can be thought of as the movement of points in \(p\) space, suggesting the formulation of the model as some kind of generalized random walk (RW). It is of some interest to mention that there is a formal similarity of \(P(p)\) with a property of a RW; specifically, this is the case of the expression for the probability of the last visit to the origin, \(x = 0\), for such a system \(\exists\). In the simplest version of a RW, a point moves regularly with a constant step \(S\), randomness being only related with the sign of \(S\). Applied to the case of \(N\) points randomly distributed in a one dimensional (1D) box, it produces a uniform distribution in space, \(P(x) = P_0\). In a more general case, the probability \(m(x)\) to move a point will be, in general, a function of position \(x\). This would be the case, for instance, of a system with absorbing walls, or a gas with a thermal gradient. Moreover, there are situations where the movement needs not to be regular (in time). The stationary state would be established when \(m(x)P(x) = \text{constant}\).

We will consider the movement of \(N\) agents in a 1D space of probabilities, \(p\), with a variable step \(\Delta p\), as it was already made in Johnson’s formulation of the model. On the other hand, we will use the set \(\{p_i, G, d\}\) to decide if an agent moves or not. It is possible to write \(\mu_i\), the probability of agent \(i\) to be in the majority, in terms of \(\{p_j\}\). If \(G = d = 0\), then it is simple to see that \(\mu_i\) is equal to the probability \(m(p_i)\) to move the agent, i.e. \(p_i \to p'_i\).

In the following we will refer to \(m\) as the mobility. Note that, in general, the actual value of \(m\) will depend on all the \(p_j\); we choose to write \(m(p_i)\) for agent \(i\), to emphasize that its value changes with the position of the agent in \(p\)-space. We cannot find a closed expression for the mobility if \(G\) and \(d\) are \(\neq 0\); nevertheless, as the mobility must follow in general the behavior of \(\mu\), we still can use it to describe the system’s behavior.

Let us consider the case \(N = 3\), for simplicity. The system is characterized by the strategies \(p_1, p_2, p_3\) (and \(q_i = 1 - p_i\)). The probabilities for every agent to be in the minority are

\[
\begin{align*}
\lambda_1 &= p_1q_2q_3 + q_1p_2p_3, \\
\lambda_2 &= q_1p_2q_3 + p_1q_2p_3, \\
\lambda_3 &= q_1q_2p_3 + p_1p_2q_3
\end{align*}
\]

and the corresponding probabilities of being in the majority are \(\mu_i = 1 - \lambda_i\).

These expressions can be easily generalized to the case of \(N = 2n + 1\) agents.

Introducing \(x_i = p_i/q_i\), \(y_i = 1/x_i\), \(U = \prod_{j=1}^{j=N} p_i\), \(Q = \prod_{j=1}^{j=N} q_i\) it is \([\exists]\)
\[ \lambda_i = x_i Q(1 + \sum_{k_2} x_{k_2} + \sum_{k_2, k_3} x_{k_2} x_{k_3} + \ldots + \sum_{k_2, k_3, \ldots, k_n} x_{k_2} x_{k_3} \cdots x_{k_n}) \\
+ y_i U(1 + \sum_{k_2} y_{k_2} + \sum_{k_2, k_3} y_{k_2} y_{k_3} + \ldots + \sum_{k_2, k_3, \ldots, k_n} y_{k_2} y_{k_3} \cdots y_{k_n}) \] (2)

Equations (1)-(2) show that \( \mu_i \) depends on all the strategies, \( \{p_j\} \). In other words, it illustrates that the origin of correlations in the distribution \( P(p) \) can be traced back to the rules defining the minority game.

We have made numerical simulations based on Eqs. (1)-(2), for \( N = 3, 5, 7, 9 \) and 11: at every step of the game, an agent gains (loses) one point with probability \( \lambda(\mu) \). This procedure makes explicit how agents relate their behaviors through the strategies. Figure 3 shows results for \( N = 11 \), together with the corresponding results from our ‘simplified’ version. The similarity between the results obtained with both methods is remarkable; the small differences seen in the very narrow region near both extrema \( (p \approx 0, p \approx 1) \), are probably due to the noise attributable to our simulations. This is not to say that this approach is identical to the original model. In fact, we can only expect the probabilistic approach to be equivalent to the ‘simplified’ version in a statistical sense, but it is not possible to compare both methods at each time step. To illustrate this difference, notice that in this formulation one will accept some outcomes which are not allowed in the original MG; thus, for instance, as agents win a point with probability \( \lambda_i \), there exists a finite probability, \( W_v(N) \neq 0 \), that the majority of the agents can win a point, in an apparent violation of the basic rule of the game (indeed, it can even happen that all agents are simultaneously winners). If \( N = 3 \),

\[ W_v(3) = \lambda_1 \lambda_2 \lambda_3 + \mu_1 \lambda_2 \lambda_3 + \lambda_1 \mu_2 \lambda_3 + \lambda_1 \lambda_2 \mu_3 \] (3)

and similar relations for all \( N \).

Using Eq. (2), and the generalization of Eq. (3), we calculate for a uniform distribution that \( W_v(3) \approx 0.14 \), while it can be estimated that \( W_v(\infty) \leq 0.25 \). Similar values are obtained for non uniform distributions with a shape analogous to that of Fig. 3.

Figure 4 has results for \( N = 3, d = 0, -1 \) and \( G = 0, 1 \). In this case, results obtained with the ‘simplified’ version (not shown here) are indistinguishable from those coming from Eq. (1). If \( G = 0 \), the ensuing self-organization is small; we have verified the same type of behavior for all \( N \leq 11 \). On the contrary, if \( G = 1 \) self-organization is very important. In both cases, \( d \) has a smaller influence on the behavior of the system. We have included results for \( N = 3, d = G = 0 \) in Fig. 4, because in this case \( m(p) = \mu(p) \), and it is possible to get a clear picture of the resulting (small) organization. Assume agents are numbered so that \( p_1 < p_2 < p_3 \), and consider the case \( p_1 < 1/2, p_3 > 1/2 \). It follows from Eq. (1) that \( m_2 > m_1, m_3 \). This describes a situation where agents near \( p \approx 0 \) and \( p \approx 1 \) have a tendency to remain in their positions, while the agent in between moves more frequently. Eventually this will change if, as a result of the movement, either \( p_2 < p_1 \) or \( p_2 > p_3 \), increasing the accumulation of agents near \( p = 0 \) and \( p = 1 \). Incidentally, we can use this picture to understand why the use of a different updating rule can destroy self-organization, as previously reported by one of us. Assume the same situation as before, i.e. an agent with \( p_2 \approx 1/2 \) and high mobility, and two agents near \( p_1 \approx 0 \) and \( p_3 \approx 1 \), with smaller mobility. If now strategies are updated using \( p' = 1 - p \pm \Delta p \), as the agent with \( p_1 \approx 0 \)
moves, he will go near $p_1 \approx 1$. It is easy to see from Eq.(1) that the corresponding mobilities are $m_1 \approx m_3 \approx 1 > m_2$. In words, correlations are broken in a single step, so that only a very tiny indication of self-organization remains. It is worthwhile to mention that this type of evolution can no longer be sensibly described as a random walk.

The probabilistic approach provides a natural starting point for an analytic formulation of this model. A sample is described by a point in an $N$ dimensional space of probabilities $p_j$; the set of all the $n_s$ points can be thought of as a non interacting gas that evolves towards a stationary distribution, driven by the rules of the game. We are presently working on the implementation of these ideas [10].

In summary, we have made mainly two contributions to the knowledge of this version of the MG: (i) we have shown the irrelevance of memory in the resulting self organization of the system; in this respect, therefore, this version of the MG behaves differently than that of [1], where it has been claimed [4] that all agents need to receive the same information, whether it be true or false, to be able to self organize; (ii) our probabilistic formulation proved to be a very good approximation to the model and, equally important, allows us to understand in detail how the game’s rules establish correlations between the agents.

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* e-mail address: burgos@cnea.gov.ar
** e-mail address: ceva@cnea.gov.ar
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[9] It can be seen that the mean value of the initial mobility for \( N = 3 \) is \( 3/4 \); in general, 
\[
\frac{2}{3} < \frac{m_1 + m_2 + m_3}{3} < 1.
\]
If all \( p' \)s are smaller (greater) than \( 1/2 \), a similar analysis shows that all agents will try to be near \( p \approx 1/2 \), with a mobility \( m \approx 1 - p + p^2 \), which is even greater than the initial value. This suggests that this configuration should be unstable.
[10] F. Parisi, E. Burgos and H. Ceva, to be published
FIGURES

FIG. 1. Distribution of strategies $P(p)$ for the original model and for our ‘simplified’ approximation. $N = 101$, $G = 1$, $d = -4$, $R = 0.2$, $n_t = 10^5$, $n_s = 10^4$. In the original model, $m = 3$

FIG. 2. Extreme strategies $P(0)$ and $P(1)$ vs $1/N$. Filled symbols refer to $P(0)$, open ones to $P(1)$

FIG. 3. Distribution of strategies $P(p)$ for our two approximations. $N = 11$, $G = 1$, $d = -1$, $R = 0.1$, $n_t = 10^5$, $n_s = 10^4$

FIG. 4. Distribution of strategies $P(p)$, in the probabilistic approximation, for $N = 3$. Open symbols refer to $d = -1$, filled symbols to $d = 0$. The inset shows an enlarged picture for $G = 0$
A plot showing the function $P(p)$ versus $p$. The graph includes data points labeled as 'probabilistic' and 'simplified'. The x-axis represents $p$, ranging from 0 to 1, and the y-axis represents $P(p)$, ranging from 0 to 6.
