ROTA-BAXTER 3-LIE ALGEBRAS

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ABSTRACT. In this paper we introduce the concepts of a Rota-Baxter operator and a differential operator with weights on an $n$-algebra. We then focus on Rota-Baxter 3-Lie algebras and show that they can be derived from Rota-Baxter Lie algebras and pre-Lie algebras and from Rota-Baxter commutative associative algebras with derivations. We also establish the inheritance property of Rota-Baxter 3-Lie algebras.

1. Introduction

$n$-Lie algebras [19] are a type of multiple algebraic systems appearing in many fields of mathematics and mathematical physics [33, 41, 23, 27, 26, 44]. For example, the structure of 3-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes; the Bagger-Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra; the identity in Eq. (1) for a 3-Lie algebra is essential to define the action with $N = 8$ supersymmetry; the $n$-Jaciobi identity can be regarded as a generalized Plucker relation in the physics literature, and so on. The theory of $n$-Lie algebras has been widely studied [35, 36, 11, 12, 9, 10, 3].

P. Ho, Y. Imamura and Y. Matsuo in paper [29] studied two derivations of the multiple D2 action from the multiple M2-brane model proposed by Bagger-Lambert and Gustavsson. The first one is to start from 3-Lie algebras given by arbitrary Lie algebras through 2-dimensional extensions. The first author and collaborators [9] realized 3-Lie algebras by Lie algebras and several linear functions. Filippov [19] constructed $(n - 1)$-Lie algebras from $n$-Lie algebras by fixing some non-zero vector in the multiplication. We are motivated by this to construct $n$-Lie algebras. However, it is not easy here due to the $n$-ary operation. In order to avoid the complicated equations involving the structural constants like in the classification [12], it is natural to consider to use Lie algebras and other better studied algebras to obtain 3-Lie and $n$-Lie algebras.

In recent years, Rota-Baxter (associative) algebras, originated from the work of G. Baxter [13] in probability and populated by the work of Cartier and Rota [14, 38, 39], have also been studied in connection with many areas of mathematics and physics, including combinatorics, number theory, operads and quantum field theory [1, 6, 17, 20, 21, 24, 25, 38, 39]. In particular Rota-Baxter algebras have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory of Connes and Kreimer [13, 17, 18], as well as in the application of the renormalization method in solving divergent problems in...
number theory [25, 22]. Furthermore, Rota-Baxter operators on a Lie algebra are an operator form of the classical Yang-Baxter equations and contribute to the study of integrable systems [3, 4, 8, 24]. Further Rota-Baxter Lie algebras are closely related to pre-Lie algebras and PostLie algebras.

Thus it is time to study $n$-Lie algebras and Rota-Baxter algebras together to get a suitable definition of Rota-Baxter $n$-Lie algebras. In this paper we investigate Rota-Baxter $n$-algebras in the context of associative and Lie algebras with focus on Rota-Baxter 3-Lie algebras. We establish a close relationship of our definition of Rota-Baxter 3-Lie algebras with well-known concepts of Rota-Baxter associative, commutative or Lie algebras. This on one hand justifies the definition of Rota-Baxter 3-Lie algebras and on the other hand provides a rich source of examples for Rota-Baxter 3-Lie algebras. The concepts of differential operators and Rota-Baxter operators with weights for general (non-associative) algebras are introduced in Section 2. The duality of the two concepts are established. In Section 3 we extend the connections [4, 8, 23] from Lie algebras and pre-Lie algebras to 3-Lie algebras to the context of Rota-Baxter 3-Lie algebras. In Section 4 we construct Rota-Baxter 3-Lie algebras from commutative (associative) Rota-Baxter algebras together with commuting derivations and suitable linear forms. In Section 5 we study the inheritance property of Rota-Baxter 3-Lie algebras as in the case of Rota-Baxter Lie algebras. We also consider the refined case of Lie triple systems.

2. Differential $n$-algebras and Rota-Baxter $n$-Lie algebras

An $n$-Lie algebra is a vector space $\mathfrak{g}$ over a field $k$ endowed with an $n$-ary multi-linear skew-symmetric operation $[x_1, \cdots, x_n]$ satisfying the $n$-Jacobi identity

\begin{equation}
[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [[x_1, \cdots, [x_i, y_2, \cdots, y_n]]], \quad (1)
\end{equation}

In particular, a 3-Lie algebra is a vector space $\mathfrak{g}$ endowed with a ternary multi-linear skew-symmetric operation

\begin{equation}
[[x_1, x_2, x_3], y_2, y_3] = [[x_1, x_2, y_3], x_2, x_3] + [x_1, [x_2, y_2, y_3], x_3] + [x_1, x_2, [x_3, y_2, y_3]], \quad (2)
\end{equation}

for all $x_1, x_2, x_3, y_2, y_3 \in \mathfrak{g}$. Under the skew-symmetric condition, the equation is equivalent to

\begin{equation}
[[x_1, x_2, x_3], y_2, y_3] = [[x_1, x_2, y_3], x_2, x_3] + [[x_2, y_2, y_3], x_3, x_1] + [[x_3, y_2, y_3], x_1, x_2]. \quad (3)
\end{equation}

Let $(A, \cdot)$ be a $k$-vector space with a binary operation $\cdot$ and let $\lambda \in k$. If a linear map $P : A \to A$ satisfies

\begin{equation}
P(x) \cdot P(y) = P(P(x) \cdot y + x \cdot P(y) + \lambda x \cdot y) \quad \text{for all } x, y \in A, \quad (4)
\end{equation}

then $P$ is called a Rota-Baxter operator of weight $\lambda$ and $(A, \cdot, P)$ is called a Rota-Baxter algebra of weight $\lambda$.

If a linear map $D : A \to A$ satisfies

\begin{equation}
D(x \cdot y) = D(x) \cdot y + x \cdot D(y) \quad \text{for all } x, y \in A, \quad (5)
\end{equation}

then $D$ is called a derivation on $A$. Let $\text{Der}(A)$ denote the set of all derivations of $A$. More generally, a linear map $d : A \to A$ is called a derivation of weight $\lambda$ [23] if

\begin{equation}
d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \quad \text{for all } x, y \in A. \quad (6)
\end{equation}

We generalize the concepts of a Rota-Baxter operator and differential operator to $n$-algebras.
Definition 2.1. Let $\lambda \in k$ be fixed.

(a) An \textit{n-(nonassociative) algebra} over a field $k$ is a pair $(A, \langle \cdot, \cdot, \cdot \rangle)$ consisting of a vector space $A$ over $k$ and a multilinear multiplication

$$\langle \cdot, \cdot, \cdot \rangle : A^\otimes n \rightarrow A.$$ 

(b) A \textit{derivation of weight} $\lambda$ on an $n$-algebra $(A, \langle \cdot, \cdot, \cdot \rangle)$ is a linear map $d : A \rightarrow A$ such that,

$$d(\langle x_1, \ldots, x_n \rangle) = \sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \hat{d}(x_1), \ldots, \hat{d}(x_{|I|}), \ldots, \hat{d}(x_n) \rangle,$$

where $\hat{d}(x_i) := \hat{d}_I(x_i) := \begin{cases} d(x_i), & i \in I, \\ x_i, & i \notin I \end{cases}$ for all $x_1, \ldots, x_n \in A$. Then $A$ is called a \textit{differential $n$-algebra of weight} $\lambda$. In particular, a \textit{differential 3-algebra of weight} $\lambda$ is a 3-algebra $(A, \langle \cdot, \cdot, \cdot \rangle)$ with a linear map $d : A \rightarrow A$ such that,

$$d(\langle x_1, x_2, x_3 \rangle) = \langle d(x_1), x_2, x_3 \rangle + \langle x_1, d(x_2), x_3 \rangle + \langle x_1, x_2, d(x_3) \rangle + \lambda d(x_1, d(x_2), x_3) + \lambda d(x_1, x_2, d(x_3)) + \lambda^2 d(x_1, x_2, x_3).$$

(c) A \textit{Rota-Baxter operator of weight} $\lambda$ on $(A, \langle \cdot, \cdot, \cdot \rangle)$ is a linear map $P : A \rightarrow A$ such that

$$\langle P(x_1), \ldots, P(x_n) \rangle = P \left( \sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_{|I|}), \ldots, \hat{P}(x_n) \rangle \right),$$

where $\hat{P}(x_i) := \hat{P}_I(x_i) := \begin{cases} x_i, & i \in I, \\ P(x_i), & i \notin I \end{cases}$ for all $x_1, \ldots, x_n \in A$. Then $A$ is called a \textit{Rota-Baxter $n$-algebra of weight} $\lambda$. In particular, a \textit{Rota-Baxter 3-algebra} is a 3-algebra $(A, \langle \cdot, \cdot, \cdot \rangle)$ with a linear map $P : A \rightarrow A$ such that

$$\langle P(x_1), P(x_2), P(x_3) \rangle = P \left( \langle P(x_1), P(x_2), x_3 \rangle + \langle P(x_1), x_2, P(x_3) \rangle + \langle x_1, P(x_2), P(x_3) \rangle + \lambda \langle P(x_1), x_2, x_3 \rangle + \lambda \langle x_1, P(x_2), x_3 \rangle + \lambda^2 \langle x_1, x_2, P(x_3) \rangle \right).$$

Theorem 2.2. Let $(A, \langle \cdot, \cdot, \cdot \rangle)$ be an $n$-algebra over $k$. An invertible linear mapping $P : A \rightarrow A$ is a Rota-Baxter operator of weight $\lambda$ on $A$ if and only if $P^{-1}$ is a differential operator of weight $\lambda$ on $A$.

Proof. If an invertible linear mapping $P : A \rightarrow A$ is a Rota-Baxter operator of weight $\lambda$, then for $x_1, \ldots, x_n \in A$, set $y_i = P^{-1}(x_i)$. Then by Eq. (7) we have

$$P^{-1}(\langle x_1, \ldots, x_n \rangle) = P^{-1}(\langle P(y_1), \ldots, P(y_n) \rangle) = \sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \hat{P}^{-1}(y_1), \ldots, \hat{P}^{-1}(y_{|I|}), \ldots, \hat{P}^{-1}(y_n) \rangle = \sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1} \langle \hat{P}^{-1}(x_1), \ldots, \hat{P}^{-1}(x_{|I|}), \ldots, \hat{P}^{-1}(x_n) \rangle.$$
Therefore, \( P^{-1} \) is a derivation of weight \( \lambda \).

Conversely, let \( P \) be a derivation of weight \( \lambda \). For \( x_1, \ldots, x_n \in A \), by Eq. (7) we have
\[
P(\langle P^{-1}(x_1), \ldots, P^{-1}(x_n) \rangle) = \sum_{\emptyset \notin I \subseteq [n]} \lambda^{\Vert I \Vert - 1} \langle \hat{P} P^{-1}(x_1), \ldots, \hat{P} P^{-1}(x_i), \ldots, \hat{P} P^{-1}(x_n) \rangle
\]
\[
= \sum_{\emptyset \notin I \subseteq [n]} \lambda^{\Vert I \Vert - 1} \langle P^{-1}(x_1), \ldots, P^{-1}(x_i), \ldots, P^{-1}(x_n) \rangle.
\]
Therefore,
\[
\langle P^{-1}(x_1), \ldots, P^{-1}(x_n) \rangle = P^{-1}\left( \sum_{\emptyset \notin I \subseteq [n]} \lambda^{\Vert I \Vert - 1} \langle P^{-1}(x_1), \ldots, P^{-1}(x_i), \ldots, P^{-1}(x_n) \rangle \right).
\]
This proves the result. \( \square \)

**Proposition 2.3.** Let \((A, \circ, P)\) be an associative Rota-Baxter (resp. differential) algebra of weight \( \lambda \). Define an \( n \)-ary multiplication \( \langle \cdot, \cdot, \cdot \rangle \) on \( A \) by
\[
\langle x_1, \ldots, x_n \rangle = x_1 \circ \cdots \circ x_n \text{ for all } x_1, \ldots, x_n \in A.
\]
Then \((A, \langle \cdot, \cdot, \cdot \rangle, P)\) is a Rota-Baxter (resp. differential) \( n \)-algebra of weight \( \lambda \).

**Proof.** We prove the case of Rota-Baxter algebras by induction on \( n \geq 2 \). The case of differential algebras is the same. There is nothing to prove when \( n = 2 \). Suppose it is true for the case \( n - 1 \geq 2 \). Then
\[
\langle P(x_1), \ldots, P(x_n) \rangle = (P(x_1) \circ \cdots \circ P(x_{n-1})) \circ P(x_n)
\]
\[
= P( \sum_{I \subseteq [n-1]} \lambda^{\Vert I \Vert - 1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_i), \ldots, \hat{P}(x_{n-1}) \rangle \circ P(x_n)
\]
\[
= P\left(P( \sum_{I \subseteq [n-1]} \lambda^{\Vert I \Vert - 1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_i), \ldots, \hat{P}(x_{n-1}) \rangle \right) \circ x_n
\]
\[
+ ( \sum_{I \subseteq [n-1]} \lambda^{\Vert I \Vert - 1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_i), \ldots, \hat{P}(x_{n-1}) \rangle \right) \circ x_n
\]
\[
+ \lambda( \sum_{I \subseteq [n-1]} \lambda^{\Vert I \Vert - 1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_i), \ldots, \hat{P}(x_{n-1}) \rangle \right) \circ x_n
\]
The first sum inside \( P \) gives \( \langle P(x_1), \ldots, P(x_{n-1}), x_n \rangle \). Together with the third sum inside \( P \), we obtain
\[
\sum_{\emptyset \notin I \subseteq [n], n \notin I} \lambda^{\Vert I \Vert - 1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_i) \rangle.
\]
The second sum inside \( P \) is
\[
\sum_{\emptyset \notin I \subseteq [n], n \notin I} \lambda^{\Vert I \Vert - 1} \langle \hat{P}(x_1), \ldots, \hat{P}(x_i) \rangle.
\]
Therefore, \( P \) is a Rota-Baxter operator of weight \( \lambda \) on the \( n \)-algebra \((A, \langle \cdot, \cdot, \cdot \rangle)\). \( \square \)
3. Rota-Baxter 3-Lie algebras from Rota-Baxter Lie algebras and pre-Lie algebras

In this section, we study the realizations of Rota-Baxter 3-Lie algebras from Rota-Baxter Lie algebras in Section 3.1 and from Rota-Baxter pre-Lie algebras in Section 3.2.

3.1. Rota-Baxter 3-Lie algebras from Rota-Baxter Lie algebras. By Definition 2.1 suppose \((L, [, ,], P)\) is a Rota-Baxter 3-Lie algebra of weight \(\lambda\). Then the linear map \(P : L \rightarrow L\) satisfies

\[
[P(x_1), P(x_2), P(x_3)] = P\left([P(x_1), P(x_2), x_3] + [P(x_1), x_2, P(x_3)] + [x_1, P(x_2), P(x_3)]\right) + \lambda[P(x_1), x_2, x_3] + \lambda^2[x_1, x_2, x_3]
\]

for all \(x_1, x_2, x_3 \in L\).

We recall the following result from Lie algebras to 3-Lie algebras.

**Lemma 3.1.** \([\text{[3]}]\) Let \((L, [, ,])\) be a Lie algebra, and let \(f \in L^* := \text{Hom}(L, k)\) satisfy \(f([x, y]) = 0\) for \(x, y \in L\). Then \(L\) is a 3-Lie algebra with the multiplication

\[
[x, y, z]_f := f(x)[y, z] + f(y)[z, x] + f(z)[x, y] \text{ for all } x, y, z \in L.
\]

We now combine this result with Rota-Baxter operators.

**Theorem 3.2.** Let \((L, [, ,], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\), \(f \in L^*\) satisfying \(f([x, y]) = 0\) for all \(x, y \in L\). Then \(P\) is a Rota-Baxter operator on the 3-Lie algebra \((L, [, ,], f)\) defined in Eq. (12) if and only if \(P\) satisfies

\[
f(x)[P(y), P(z)] + f(y)[P(z), P(x)] + f(z)[P(x), P(y)] \in \text{Ker}(P + \lambda Id_L) \text{ for all } x, y, z \in L,
\]

where \(Id_L : L \rightarrow L\) is the identity map.

**Proof.** By Lemma 3.1, \(L\) is a 3-Lie algebra with the multiplication \([, ,]_f\) defined in Eq. (12). Now for arbitrary \(x, y, z \in L\),

\[
[P(x), P(y), P(z)]_f = f(P(x))[P(y), P(z)] + f(P(y))[P(z), P(x)] + f(P(z))[P(x), P(y)]
\]

\[
= P\left(f(P(x))(P(y), z) + [y, P(z)] + f(P(y))(P(z), x) + [z, P(x)]\right) + f(P(z))(P(x), y) + [x, P(y)] + \lambda(f(P(x))[y, z] + f(P(y))[z, x] + f(P(z))[x, y])
\]

Applying Eqs. (12), (12) and regrouping, we obtain

\[
P\left([P(x), P(y), z]_f + [P(x), y, P(z)]_f + [x, P(y), P(z)]_f\right) + \lambda([P(x), y, z]_f + [x, P(y), z]_f + [x, y, P(z)]_f) + \lambda^2[x, y, z]_f
\]

\[
= P\left(f(P(x))(P(y), z) + [y, P(z)] + f(P(y))(P(z), x) + [z, P(x)]\right) + f(P(z))(P(x), y) + [x, P(y)] + \lambda(f(P(x))[y, z] + f(P(y))[z, x] + f(P(z))[x, y])
\]

\[
+ \lambda(f(x)(P(y), z) + [z, P(y)] + \lambda([y, z] + f(y)(P(z), x) + [P(z), x]) + \lambda(z, x))
\]

\[
+ f(z)(P(x), y) + [x, P(y)] + \lambda([x, y] + f(P(y), P(z)] + f(y)(P(z), P(x)) + f(z)(P(x), P(y)])
\]

\[
= [P(x), P(y), P(z)]_f + (P + \lambda Id_L)(f(x)[P(y), P(z)] + f(y)[P(z), P(x)] + f(z)[P(x), P(y)]).
\]

Then the theorem follows. \(\square\)
Corollary 3.3. Let \((L, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra of weight zero, \(f \in L^*\) satisfy 
\[ f([x, y]) = 0 \text{ for all } x, y \in L. \]
Then \(P\) is a Rota-Baxter operator on the 3-Lie algebra \((L, [\cdot, \cdot], f)\) if and only if \(P\) satisfies
\[
(f(x)P(y) - f(y)P(x), z) + [f(y)P(z) - f(z)P(y), x] + [f(z)P(x) - f(x)P(z), y] \in \ker P^2
\]
for \(x, y, z \in L\), where, \([\cdot, \cdot, \cdot]_f\) is defined in Eq. (12).
In particular, if \(P^2 = 0\), then for every \(f \in L^*\) satisfying \(f([x, y]) = 0\), \(P\) is a Rota-Baxter 
operator on the 3-Lie algebra \((L, [\cdot, \cdot], f)\).

Proof. Let \((L, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra of weight zero, \(f \in L^*\) satisfying \(f([x, y]) = 0\) 
for \(x, y \in L\). By Eq. (11) we have
\[
f(x)[P(y), P(z)] + f(y)[P(z), P(x)] + f(z)[P(x), P(y)] = P^2([f(x)P(y) - f(y)P(x), z] + [f(y)P(z) - f(z)P(y), x] + [f(z)P(x) - f(x)P(z), y]).
\]
Then the corollary follows from Theorem 3.2. \qed

3.2. Rota-Baxter 3-Lie algebras from Rota-Baxter pre-Lie algebras. In this section, we study the 
realizations of Rota-Baxter 3-Lie algebras by Rota-Baxter associative algebras and 
Rota-Baxter pre-Lie algebras. First we recall some properties (cf. [31]).

Let \(L\) be a vector space over a field \(F\) with a bilinear product \(*\) satisfying
\[
(x * y) * z - x * (y * z) = (y * x) * z - y * (x * z) \text{ for all } x, y, z \in L.
\]
Then \((L, *)\) is called a \textit{pre-Lie algebra}. It is obvious that all associative 
algebras are pre-Lie algebras. For a pre-Lie algebra \(L\), the commutator
\[
[x, y]_c := x * y - y * x,
\]
defines a Lie algebra \(G(L) = (L, [\cdot, \cdot])\), called the \textit{sub-adjacent Lie algebra} of the pre-Lie 
algebra \(L\).

If a linear mapping \(P : L \to L\) is a Rota-Baxter operator of weight \(\lambda\) on a pre-Lie algebra 
\((L, *), \) that is, \(P\) satisfies
\[
P(x) * P(y) = P(P(x) * y + x * P(y) + \lambda x * y) \text{ for all } x, y \in L,
\]
then \(P\) is a Rota-Baxter operator of weight \(\lambda\) on its sub-adjacent Lie algebra \(G(L) = (L, [\cdot, \cdot])\).

The following facts on relationship among various Rota-Baxter algebras can be easily verified 
from definitions.

Lemma 3.4. (a) Let \((L, *, P)\) be a Rota-Baxter pre-Lie algebra of weight \(\lambda\). Then \((L, [\cdot, \cdot], P)\) 
is a Rota-Baxter Lie algebra of weight \(\lambda\), where the multiplication \([\cdot, \cdot]\), is defined in 
Eq. (7)\)
(b) If \((L, *, P)\) is a Rota-Baxter pre-Lie algebra of weight \(\lambda\). Then \((L, \cdot, P)\) is a Rota-Baxter 
pre-Lie algebra of weight \(\lambda\), where
\[
x * y := P(x) * y - y * P(x) + \lambda x * y \quad \text{for all } x, y \in L.
\]
(c) Let \((A, \cdot, P)\) be a Rota-Baxter commutative associative algebra of weight \(\lambda\), \(D\) be a derivation on the algebra \((A, \cdot)\) that satisfies \(DP = PD\). Then \((A, *, P)\) is a Rota-Baxter pre-Lie algebra of weight \(\lambda\), where

\[
(18) \quad x \ast y = D(x) \cdot y \quad \text{for all } x, y \in A.
\]

Therefore by Item \([b]\) we get a Rota-Baxter Lie algebra \((A, [, , ]_s)\), where

\[
(19) \quad [x, y], = D(x) \cdot y - D(y) \cdot x \quad \text{for all } x, y \in A.
\]

**Theorem 3.5.** Let \((L, *, P)\) be a Rota-Baxter pre-Lie algebra of weight \(\lambda\) and let \(f \in L^*\) satisfying \(f(x \ast y - y \ast x) = 0\) for \(x, y \in L\). Define

\[
(20) \quad [x, y, z]_f = f(x)(y * z - z * y) + f(y)(z * x - x * z) + f(z)(x * y - y * x) \quad \text{for all } x, y, z \in L.
\]

Then \(P\) is a Rota-Baxter operator on 3-Lie algebra \((L, [, , ]_f)\) if and only if

\[
(21) \quad f(x)(P(y) * P(z) - P(z) * P(y)) + f(y)(P(z) * P(x) - P(x) * P(z)) + f(z)(P(x) * P(y) - P(y) * P(x)) \in \text{Ker}(P + \lambda Id_L) \quad \text{for all } x, y, z \in L.
\]

**Proof.** By Lemma \([3.1]\), for \(f \in L^*\) satisfying \(f(x \ast y - y \ast x) = 0\), \((L, [, , ]_f)\) is a 3-Lie algebra with the multiplication in Eq. \(21\). By Lemma \([3.4]\) and Theorem \(3.2\), \(P\) is a Rota-Baxter operator on the 3-Lie algebra \((L, [, , ]_f)\) if and only if \(P\) satisfies Eq. \(21\). \(\square\)

**Theorem 3.6.** Let \((L, *, P)\) be a Rota-Baxter pre-Lie algebra of weight zero, \(f \in L^*\) satisfy

\[
(22) \quad f(P(x) \ast y - y \ast P(x)) = f(P(y) \ast x - x \ast P(y)) \quad \text{for all } x, y \in L.
\]

Then \((L, [, , ])\) is a 3-Lie algebra with the multiplication

\[
(23) \quad [x, y, z] := (f(x)P(y) - f(y)P(x)) \ast z - z \ast (f(x)P(y) - f(y)P(x)) + f(z)(P(x) \ast y - y \ast P(x)) + f(y)P(z) - f(z)P(y)) \ast x - x \ast (f(y)P(z) - f(z)P(y)).
\]

Further, \(P\) is a Rota-Baxter operator of weight zero on the 3-Lie algebra \((L, [, , ])\) if and only if \(P\) satisfies

\[
(24) \quad f(x)(P^2(y) \ast P^2(z) - P^2(z) \ast P^2(y)) + f(y)(P^2(z) \ast P^2(x) - P^2(x) \ast P^2(z)) + f(z)(P^2(x) \ast P^2(y) - P^2(y) \ast P^2(x)) = 0 \quad \text{for all } x, y, z \in L.
\]

**Proof.** By Lemma \([3.4, 3.5]\), \((L, \circ)\) is a pre-Lie algebra with the multiplication

\[
\circ : L \otimes L \to L, x \circ y = P(x) \ast y - y \ast P(x) \quad \text{for all } x, y \in L,
\]

and \(P\) is a Rota-Baxter operator on the pre-Lie algebra \((L, \circ)\).

If \(f \in L^*\) satisfies \(f(x \circ y - y \circ x) = 0\), that is, \(f\) satisfies Eq. \(22\), then by Lemma \([3.1]\), \((L, [, , ])\) is a 3-Lie algebra, where

\[
[x, y, z] = f(x)(y \circ z - z \circ y) + f(y)(z \circ x - x \circ z) + f(z)(x \circ y - y \circ x)
\]

\[
= f(x)(P(y) \ast z - z \ast P(y) - P(z) \ast y + y \ast P(z)) + f(y)(P(z) \ast x - x \ast P(z) - P(x) \ast z + z \ast P(x)) + f(z)(P(x) \ast y - y \ast P(x) - P(y) \ast x + x \ast P(y))
\]

\[
= (f(x)P(y) - f(y)P(x)) \ast z - z \ast (f(x)P(y) - f(y)P(x))
\]
Therefore, Eq. (25) holds.

By Theorem 3.3, $P$ is a Rota-Baxter operator on the 3-Lie algebra $(L, [., .])$ if and only if $P$ satisfies

$$
0 = P(f(x)(P(y) \circ P(z) - P(z) \circ P(y)) + f(y)(P(z) \circ P(x) - P(x) \circ P(z))
+ f(z)(P(x) \circ P(y) - P(y) \circ P(x)) = f(x)P^2(y) \ast P(z) - P(z) \ast P^2(y) - P^2(z) \ast P(y) + P(y) \ast P^2(z))
+ f(y)P^2(z) \ast P(x) - P(x) \ast P^2(z) - P^2(x) \ast P(z) + P(z) \ast P^2(x))
+ f(z)P^2(x) \ast P(y) - P(y) \ast P^2(x) - P^2(y) \ast P(x) + P(x) \ast P^2(y)) = f(x)(P^2(y) \ast P^2(z) \ast P^2(y) + f(y)(P^2(z) \ast P^2(x) - P^2(x) \ast P^2(z))
+ f(z)(P^2(x) \ast P^2(y) - P^2(y) \ast P^2(x))
$$

This proves the second statement. □

4. ROTA-BAXTER 3-LIE ALGEBRAS FROM ROTA-BAXTER COMMUTATIVE ASSOCIATIVE ALGEBRAS

Let $(A, \cdot)$ be a commutative associative algebra, $D$ in Der $A$, $f$ in $A^*$ satisfying $f(D(x)y) = f(xD(y))$.

Then by Lemma 3.1 and Lemma 3.4, $(A, [., .]_{f,D})$ is a 3-Lie algebra, where

$$
[x, y, z]_{f,D} := \begin{vmatrix}
f(x) & f(y) & f(z) \\
D(x) & D(y) & D(z)
\end{vmatrix}
= f(x)(D(y) \cdot z - D(z) \cdot y) + f(y)(D(z) \cdot x - D(x) \cdot z) + f(z)(D(x) \cdot y - D(y) \cdot x)
= D(f(x)y - f(y)x) \cdot z + D(f(z)x - f(x)z) \cdot y + D(f(y)z - f(z)y) \cdot x
$$

for $x, y, z \in A$.

**Theorem 4.1.** Let $(A, \cdot, P)$ be a commutative associative Rota-Baxter algebra of weight $\lambda$, $D \in$ Der $A$ satisfying $PD = DP$ and $f \in A^*$ satisfying $f(D(x)y) = f(xD(y))$. Then $P$ is a Rota-Baxter operator of weight $\lambda$ on the 3-Lie algebra $(A, [., .]_{f,D})$ if and only if $P$ satisfies

$$
\begin{vmatrix}
f(x) & f(y) & f(z) \\
DP(x) & DP(y) & DP(z)
\end{vmatrix} \in \text{Ker}(P + A\text{Id}_L), \text{ for all } x, y, z \in A.
\begin{vmatrix}
P(x) & P(y) & P(z)
\end{vmatrix}
$$

**Proof.** The result follows directly from Theorem 3.3 and Lemma 3.4. □

We prove a lemma before giving our next results on Rota-Baxter 3-Lie algebras.

**Lemma 4.2.** Let $A$ be a commutative algebra. For a $3 \times 3$-matrix $M$, we use the notation

$$
M := \begin{pmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3
\end{pmatrix}
= \begin{pmatrix}
x \ y \ z
\end{pmatrix}
$$
and the corresponding determinant, where \( \vec{x}, \vec{y} \) and \( \vec{z} \) denote the column vectors. Let \( P : A \to A \) be a Rota-Baxter operator of weight \( \lambda \) and let \( P(\vec{x}), P(\vec{y}) \) and \( P(\vec{z}) \) denote the images of the column vectors. Then we have

\[
(P(\vec{x}), P(\vec{y}), P(\vec{z})) = P \left( \sum_{\theta \in I_{\geq 3}} \lambda^{[\theta]-1} \left[ \hat{P}(\vec{x}) \quad \hat{P}(\vec{y}) \quad \hat{P}(\vec{z}) \right] \right).
\]

**Proof.** By the definition of determinants and Proposition 4.3, we have

\[
\begin{vmatrix} P(\vec{x}) & P(\vec{y}) & P(\vec{z}) \end{vmatrix} = \sum_{\sigma \in S_3} \text{sgn}(\sigma) P(x_{\sigma(1)}) P(y_{\sigma(2)}) P(z_{\sigma(3)})
\]

\[
= \sum_{\sigma \in S_3} \text{sgn}(\sigma) \left( \sum_{\theta \in I_{\geq 3}} \lambda^{[\theta]-1} \hat{P}(x_{\sigma(1)}) \hat{P}(y_{\sigma(2)}) \hat{P}(z_{\sigma(3)}) \right)
\]

\[
= P \left( \sum_{\theta \in I_{\geq 3}} \lambda^{[\theta]-1} \left[ \hat{P}(\vec{x}) \quad \hat{P}(\vec{y}) \quad \hat{P}(\vec{z}) \right] \right),
\]

as needed. \( \square \)

Let \( A \) be a commutative associative algebra, \( D_1, D_2, D_3 \) be derivations on \((A, \cdot)\) satisfying \( D_1 D_2 = D_2 D_1 \) for \( i, j = 1, 2 \). Then by [17] \( A \) is a 3-Lie algebra with the multiplication

\[
[x_1, x_2, x_3] := \begin{vmatrix} x_1 & x_2 & x_3 \\ D_1(x_1) & D_1(x_2) & D_1(x_3) \\ D_2(x_1) & D_2(x_2) & D_2(x_3) \end{vmatrix} = \begin{vmatrix} \vec{x} & D_1(\vec{x}) & D_2(\vec{x}) \end{vmatrix} \quad \text{for all } \vec{x} := [x_1, x_2, x_3]^T \in A^3.
\]

**Theorem 4.3.** Let \((A, P)\) be a Rota-Baxter commutative associative algebra of weight \( \lambda \), \( D_1, D_2 \) be derivations on \((A, \cdot)\) satisfying \( D_1 D_2 = D_2 D_1 \), \( PD_i = D_i P \) for \( i = 1, 2 \). Then \( P \) is a Rota-Baxter operator of weight \( \lambda \) on the 3-Lie algebra \((A, [, , ])\), where \([, , ]\) is defined by Eq. (28).

**Proof.** Let \( x_1, x_2, x_3 \in A \). Since \( PD_1 = D_1 P \) and \( PD_2 = D_2 P \), by Lemma 4.2 and Eq. (28) we have

\[
[P(x_1), P(x_2), P(x_3)] = \begin{vmatrix} P(\vec{x}) & D_1(P(\vec{x})) & D_2(P(\vec{x})) \\ P(\vec{x}) & P(D_1(\vec{x})) & P(D_2(\vec{x})) \end{vmatrix}
\]

\[
= P \left( \sum_{\theta \in I_{\geq 3}} \lambda^{[\theta]-1} \left[ \hat{P}(\vec{x}) \quad \hat{P}(D_1(\vec{x})) \quad \hat{P}(D_2(\vec{x})) \right] \right)
\]

\[
= P \left( \sum_{\theta \in I_{\geq 3}} \lambda^{[\theta]-1} \left[ \hat{P}(x_1), \hat{P}(x_2), \hat{P}(x_3) \right] \right).
\]

This is what we need. \( \square \)
Let $A$ be a commutative associative algebra, $D_1, D_2, D_3$ be derivations on $A$ satisfying $D_i D_j = D_j D_i$ for $i \neq j, i, j = 1, 2, 3$. Then by [19] $A$ is a 3-Lie algebra with the multiplicity (29)

\[
[x, y, z]_D := \begin{vmatrix} D_1(x_1) & D_1(x_2) & D_1(x_3) \\ D_2(x_1) & D_2(x_2) & D_2(x_3) \\ D_3(x_1) & D_3(x_2) & D_3(x_3) \end{vmatrix} = \begin{vmatrix} D_1(\bar{x}) & D_2(\bar{x}) & D_3(\bar{x}) \end{vmatrix}, \text{ for all } \bar{x} = [x_1, x_2, x_3]^T \in A^3.
\]

**Theorem 4.4.** Let $(A, P)$ be a Rota-Baxter commutative associative algebra of weight $\lambda$, $D_1, D_2, D_3$ be derivations of $(A, \cdot)$ satisfying $D_i D_j = D_j D_i$ and $PD_i = D_i P$ for $i, j = 1, 2, 3, i \neq j$. Then $P$ is a Rota-Baxter operator of weight $\lambda$ on the 3-Lie algebra $(A, [\cdot, \cdot], D)$, where the multiplication $[\cdot, \cdot]_D$ is defined by Eq. (29).

**Proof.** The proof follows the same argument as the proof for Theorem 4.3. $\square$

### 5. Inheritance properties of Rota-Baxter 3-Lie algebras

In this section, we study the inheritance properties of Rota-Baxter 3-Lie algebras. Such a property of Rota-Baxter Lie algebras plays an important role in their theoretical study and applications such as in integrable systems [1], [2], [4]. This is presented in Section 5.1. We also establish a similar property for Rota-Baxter Lie triple systems in Section 5.2.

#### 5.1. Rota-Baxter 3-Lie algebras constructed by Rota-Baxter 3-Lie algebras

Let $(L, [\cdot, \cdot], P)$ be a Rota-Baxter 3-Lie algebra of weight $\lambda$. Using the notation in Eq. (3), we define a ternary operation on $L$ by

\[
[x_1, x_2, x_3]_P = \sum_{\emptyset \neq I \subseteq [3]} \lambda^{[I]} \left[ \hat{P}_I(x_1), \hat{P}_I(x_2), \hat{P}_I(x_3) \right]
\]

(30) \quad = [P(x), P(y), z] + [P(x), y, P(z)] + [x, P(y), P(z)]

+ $\lambda[P(x), y, z] + \lambda[x, P(y), z] + \lambda[x, y, P(z)] + \lambda^2 [x, y, z]$ for all $x, y, z \in L$.

Then we have the following result.

**Theorem 5.1.** Let $(L, [\cdot, \cdot], P)$ be a Rota-Baxter 3-Lie algebra of weight $\lambda$. Then with $[\cdot, \cdot]$ in Eq. (30), $(L, [\cdot, \cdot]_P, P)$ is a Rota-Baxter 3-Lie algebra of weight $\lambda$.

**Proof.** First we prove that $(L, [\cdot, \cdot]_P)$ is a 3-Lie algebra. It is clear that $[\cdot, \cdot]_P$ is multi-linear and skew-symmetric.

Let $x_1, x_2, x_3, x_4, x_5 \in L$. Denote $y_1 = [x_1, x_2, x_3]_P, y_2 = x_4, y_3 = x_5$. Then by Eqs. (1), (3) and (30), we have

\[
[[x_1, x_2, x_3]_P, x_4, x_5]_P = [y_1, y_2, y_3]_P = \sum_{\emptyset \neq I \subseteq [3]} \lambda^{[I]} \left[ \hat{P}_I(y_1), \hat{P}_I(y_2), \hat{P}_I(y_3) \right]
\]

\[
= \sum_{\emptyset \neq I \subseteq [3], 1 \notin I} \lambda^{[I]} \left[ P(y_1), \hat{P}_I(y_2), \hat{P}_I(y_3) \right] + \sum_{\emptyset \neq I \subseteq [3], 1 \in I} \lambda^{[I]} \left[ y_1, \hat{P}_I(y_2), \hat{P}_I(y_3) \right]
\]

\[
= \sum_{\emptyset \neq I \subseteq [3], 1 \notin I} \lambda^{[I]} \left[ [P(x_1), P(x_2), P(x_3)], \hat{P}_I(y_2), \hat{P}_I(y_3) \right]
\]

$\square$
Thus from the above sum, we conclude that 
\[ \sum_{0 \neq I \subseteq [5]} \lambda^{[I]} \left[ \sum_{0 \neq J \subseteq [3]} \lambda^{[J]} [\hat{P}_I(x_1), \hat{P}_J(x_2), \hat{P}_J(x_3)] \right] \]

Further we have 
\[ \sum_{0 \neq K \subseteq [5], K \cap [3] \neq 0} \lambda^{[K]} \left[ \left[ \hat{P}_K(x_1), \hat{P}_K(x_2), \hat{P}_K(x_3) \right] \right] \]

Thus from the above sum, we conclude that \((L, [\ , \ ]_P)\) is a 3-Lie algebra.

Since \((L, [\ , \ ])\) is a 3-Lie algebra, for any given \(\emptyset \neq I \subseteq [5]\), we have
\[
[[\hat{P}_K(x_1), \hat{P}_K(x_2), \hat{P}_K(x_3)], \hat{P}_K(x_4), \hat{P}_K(x_5)] = [[\hat{P}_K(x_1), \hat{P}_K(x_2), \hat{P}_K(x_3)], \hat{P}_K(x_4), \hat{P}_K(x_5)]
+ [[\hat{P}_K(x_2), \hat{P}_K(x_4), \hat{P}_K(x_5)], \hat{P}_K(x_3), \hat{P}_K(x_1)] + [[\hat{P}_K(x_3), \hat{P}_K(x_4), \hat{P}_K(x_5)], \hat{P}_K(x_1), \hat{P}_K(x_2)].
\]

Further we have
\[
[P(x_1), P(x_2), P(x_3)]_P = \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} [\hat{P}_I(P(x_1)), \hat{P}_I(P(x_2)), \hat{P}_I(P(x_3))]
= \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} [P(\hat{P}_I(x_1)), P(\hat{P}_I(x_2)), P(\hat{P}_I(x_3))]
= \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} P \left( [\hat{P}_I(x_1), \hat{P}_I(x_2), \hat{P}_I(x_3)]_P \right).
\]

This proves that \(P\) is a Rota-Baxter operator on \((L, [\ , \ ]_P)\).

**Theorem 5.2.** Let \((L, [\ , \ ], P)\) be a Rota-Baxter 3-Lie algebra of weight \(\lambda\). Let \(d\) be a differential operator of weight \(\lambda\) on \(L\) satisfying \(dP = Pd\). Then \(d\) is a derivation of weight \(\lambda\) on the 3-Lie algebra \((L, [\ , \ ]_P)\), where \([\ , \ ]_P\) is defined in Eq. (34).

**Proof.** Let \(x_1, x_2, x_3 \in L\). Using the notation in Eqs. (3) and (4), we have
\[
d([x_1, x_2, x_3]_P) = \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} d([\hat{P}_I(x_1), \hat{P}_I(x_2), \hat{P}_I(x_3)])
= \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} \left( \sum_{0 \neq J \subseteq [3]} \lambda^{[J]} [d_J \hat{P}_I(x_1), d_J \hat{P}_I(x_2), d_J \hat{P}_I(x_3)] \right)
= \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} \left( \sum_{0 \neq J \subseteq [3]} \lambda^{[J]} [\hat{P}_I d_J(x_1), \hat{P}_I d_J(x_2), \hat{P}_I d_J(x_3)] \right)
= \sum_{0 \neq I \subseteq [3]} \lambda^{[I]} [d_J(x_1), d_J(x_2), d_J(x_3)]_P.
\]

Therefore, \(d\) is a derivation of weight \(\lambda\) on the 3-Lie algebra \((L, [\ , \ ]_P)\). \(\square\)
Corollary 5.3. Let \((L, \cdot, \cdot)\) be a 3-Lie algebra, \(d\) be an invertible derivation of \(L\) of weight \(\lambda\). Then \((L, [x, y, z]_{d^{-1}})\) with \([\cdot, \cdot, \cdot]\) defined in Eq. \((30)\) is a 3-Lie algebra. Further
\[
[x, y, z]_{d^{-1}} = d([d^{-1}(x), d^{-1}(y), d^{-1}(z))] \text{ for all } x, y, z \in L,
\]
and \(d\) is a derivation of weight \(\lambda\) on the 3-Lie algebra \((L, [\cdot, \cdot, \cdot]_{d^{-1}})\).

**Proof.** By Theorem 2.2, \(d^{-1}\) is a Rota-Baxter operator of weight \(\lambda\) on the 3-Lie algebra \((L, [\cdot, \cdot, \cdot])\). Then by Theorem 5.4, \(d^{-1}\) is a Rota-Baxter operator on the 3-Lie algebra \(L\) equipped with the multiplication \([x, y, z]_{d^{-1}}\) defined in Eq. \((30)\). By Eq. \((3)\) we have
\[
[x, y, z]_{d^{-1}} = d([d^{-1}(x), d^{-1}(y), d^{-1}(z)]),
\]
as needed. The last statement follows from Theorem 5.2. □

Corollary 5.4. Let \((L, [\cdot, \cdot, \cdot])\) be a Rota-Baxter Lie algebra of weight \(\lambda\) and let \(f \in L^*\). Suppose \(f\) and \(P\) satisfy
\[
f([x, y]) = 0, \quad (P + \lambda Id_L)(f(x)[P(y), P(z)] + f(y)[P(z), P(x)]) + f(z)[P(x), P(y)]) = 0.
\]
Define
\[
[x, y, z]_{f,P} := f(P(x))[P(y), P(z)] + [y, P(z)] + \lambda [y, z] + f(P(y))[P(z), x] + z, P(x)] + \lambda [z, x])
\]
and \(P\) is a Rota-Baxter operator of weight \(\lambda\). Then by Theorem 5.4, the derived ternary multiplication \([\cdot, \cdot, \cdot]_{f,P}\) from \([\cdot, \cdot, \cdot]_{f}:= [\cdot, \cdot, \cdot]_{f}\) defined in Eq. \((30)\) also equips \(L\) with a 3-Lie algebra structure for which \(P\) is a Rota-Baxter operator of weight \(\lambda\). By direct checking, we see that \([\cdot, \cdot, \cdot]_{f,P}\) thus obtained agrees with the one defined in Eq. \((30)\).

Taking the case when \(\lambda = 0\), we obtain

Corollary 5.5. Let \((L, [\cdot, \cdot, \cdot])\) be a Rota-Baxter Lie algebra of weight zero, and \(f \in L^*\). If \(f\) and \(P\) satisfy
\[
f([x, y]) = 0, \quad P(f(x)[P(y), P(z)] + f(y)[P(z), P(x)]) + f(z)[P(x), P(y)]) = 0,
\]
then \((L, [\cdot, \cdot, \cdot]_{P})\) is a 3-Lie algebra of weight zero, where
\[
[x, y, z]_{P} = f(P(x))[P(y), z] + [y, P(z)] + f(P(y))[P(z), x] + z, P(x)]
\]
and \(P\) is a Rota-Baxter operator of weight \(\lambda\). Then by Theorem 5.4, the derived ternary multiplication \([\cdot, \cdot, \cdot]_{P}:= [\cdot, \cdot, \cdot]_{P}\) defined in Eq. \((30)\) also equips \(L\) with a 3-Lie algebra structure for which \(P\) is a Rota-Baxter operator of weight \(\lambda\). By direct checking, we see that \([\cdot, \cdot, \cdot]_{P}\) thus obtained agrees with the one defined in Eq. \((30)\). □
Remark 5.6. For a Rota-Baxter 3-Lie algebra \( (L, [\cdot, \cdot], P) \) of weight zero, \( L \) may not be a 3-Lie algebra with the multiplication

\[
[x, y, z]_1 = [P(x), y, z] + [x, P(y), z] + [x, y, P(z)] \quad \text{for all } x, y, z \in L.
\]

For Example, let \( L \) be a 3-Lie algebra in the multiplication

\[
[x_1, x_2, x_3] = x_4, \quad [x_1, x_2, x_4] = x_3, \quad [x_1, x_3, x_4] = x_2, \quad [x_2, x_3, x_4] = x_1,
\]

where \( \{x_1, x_2, x_3, x_4\} \) is a basis of \( L \). Since \( D = \text{ad}(x_1, x_2) + \text{ad}(x_3, x_4) \) is an invertible derivation of \( L \) and \( D^{-1} = D \), \( D \) is a Rota-Baxter of weight zero on \( (L, [\cdot, \cdot]) \). And

\[
\begin{align*}
[x_1, x_2, x_3]_1 &= [D(x_1), x_2, x_3] + [x_1, D(x_2), x_3] + [x_1, x_2, D(x_3)] = x_3, \\
[x_1, x_2, x_4]_1 &= [D(x_1), x_2, x_4] + [x_1, D(x_2), x_4] + [x_1, x_2, D(x_4)] = x_4, \\
[x_1, x_3, x_4]_1 &= [D(x_1), x_3, x_4] + [x_1, D(x_3), x_4] + [x_1, x_3, D(x_4)] = x_1, \\
[x_2, x_3, x_4]_1 &= [D(x_2), x_3, x_4] + [x_2, D(x_3), x_4] + [x_2, x_3, D(x_4)] = x_2.
\end{align*}
\]

Since \( [[x_1, x_2, x_3], x_2, x_4] = -x_2 \), and

\[
[[x_1, x_2, x_3], x_2, x_4] + [x_1, [x_2, x_2, x_4]], x_3] + [x_1, x_2, [x_3, x_2, x_4])] = x_2,
\]

\( L \) is not a 3-Lie algebra in the multiplication \([\cdot, \cdot]\).}

5.2. Rota-Baxter Lie triple systems. A Lie triple system \([30]\) is a vector space \( L \) equipped with a ternary linear bracket \([\cdot, \cdot, \cdot] : L \otimes L \otimes L \to L\), satisfying

\[
[x, y, y] = 0,
\]

\[
\{x, y, z\} + \{y, z, x\} + \{z, x, y\} = 0,
\]

\[
\{\{x, y, z\}, a, b\} = \{\{x, a, b\}, y, z\} + \{\{y, a, b\}, z\} + \{\{z, a, b\}, x\} \quad \text{for all } x, y, z \in L.
\]

It is important in the study of symmetric spaces.

A Lie triple system is a 3-Lie algebra and thus it makes sense to define a Rota-Baxter Lie triple system.

Theorem 5.7. Let \((L, [\cdot, \cdot], P)\) be a Rota-Baxter Lie triple system of weight \( \lambda \). Define a ternary multiplication on \( L \) by \([\cdot, \cdot, \cdot]_P : L \otimes L \otimes L \to L\) in Eq. (34). Then \((L, [\cdot, \cdot, \cdot]_P, P)\) is a Rota-Baxter Lie triple system.

Proof. It is clear that \([x, y, y]_P = 0\) and

\[
[x, y, z]_P + [y, z, x]_P + [z, x, y]_P = 0 \quad \text{for all } x, y, z \in L.
\]

Then the theorem follows from Theorem 5.1. \(\square\)

If \((L, [\cdot, \cdot])\) is a Lie algebra, then \((L, [\cdot, \cdot])\) is a Lie triple system \([37]\), where \([\cdot, \cdot] \) is defined by

\[
[x, y, z] := [x, [y, z]] \quad \text{for all } x, y, z \in L.
\]

Theorem 5.8. Let \((L, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra of weight \( \lambda \). Then \((L, [\cdot, \cdot], P)\) is a Rota-Baxter Lie triple system of weight \( \lambda \).
Proof. By Eqs. (8) and (10), we have

\[ [P(x), [P(y), P(z)]] = [P(x), P[y, P(z)]] + [P(x), P[P(y), z]] + \lambda [P(x), P[y, z]] \]
\[ = P[P(x), [y, P(z)]] + P[x, [y, P(z)]] + \lambda P[x, [y, P(z)]] \]
\[ + P[P(x), (y, z)] + P[x, P[P(y), z]] + \lambda P[x, (y, z)] \]
\[ + \lambda P[P(x), [y, z]] + \lambda P[x, [y, z]] + \lambda^2 P[x, [y, z]]. \]

Since \( P[x, [P(y), P(z)]] = P[x, P[y, P(z)]] + P[x, P[P(y), z]] + \lambda P[x, P[y, z]], \) we have

\[ [P(x), P(y), P(z)] = [P(x), P(y), P(z)] \]
\[ = P[P(x), [P(y), P(z)]] + P[P(x), [y, P(z)]] + P[x, [P(y), P(z)]] \]
\[ + \lambda P[P(x), [y, z]] + \lambda P[x, [P(y), z]] + \lambda P[x, [y, P(z)]] + \lambda^2 P[x, [y, z]] \]
\[ = P[P(x), P(y), z] + P[P(x), y, P(z)] + P[x, P(y), P(z)] \]
\[ + \lambda P[P(x), y, z] + \lambda P[x, P(y), z] + \lambda P[x, y, P(z)] + \lambda^2 P[x, y, z] \]

for all \( x, y, z \in L. \) Hence \( P \) is a Rota-Baxter operator on the 3-Lie triple system \( (L, [ , , ]). \)

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