Five dimensional charged rotating minimally gauged supergravity black hole cannot be over-spun and/or over-charged in non-linear accretion

Sanjar Shaymatov,1,2,* Naresh Dadhich,3,† Bobomurat Ahmedov,1,4,‡ and Mubasher Jamil5,6,§

1Ulugh Beg Astronomical Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan
2Institute of Nuclear Physics, Ulughbek, Tashkent 100214, Uzbekistan
3Inter University Centre for Astronomy & Astrophysics, Post Bag 4, Pune 411007, India
4National University of Uzbekistan, Tashkent 100174, Uzbekistan
5Institute for Theoretical Physics and Cosmology, Zhejiang University of Technology, Hangzhou, China
6Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), H-12, Islamabad, Pakistan

(Dated: August 6, 2019)

We investigate the Cosmic Censorship Conjecture (CCC) by overspinning and overcharging process through Gedanken experiments for five dimensional rotating charged minimally gauged supergravity black hole. Generally black hole could be over charged/spun violating CCC for linear order accretion while the same is always restored for non-linear accretion. The only exception however is that of a five dimensional rotating black hole with single rotation where CCC is obeyed at the linear order as well. For the black hole under study, we obtain the expected results with CCC being always obeyed for non-linear accretion. However in the case of single rotation CCC is respected for linear accretion when angular momentum dominates over charge of black hole, and it is violated when opposite is the case.

PACS numbers: 04.50.+h, 04.20.Dw

I. INTRODUCTION

Black holes have always very exciting and interesting objects both gravitationally and geometrically but they have taken center-stage after the discovery of gravitational waves produced by merger of two stellar mass black holes in the LIGO-VIRGO detection experiment [1, 2]. The hidden properties of black holes could be probed by analyzing gravitational waves in the near future. One of the most fundamental questions in general relativity (GR) is testing of CCC, which has so far remained unproven [3]. The physical possibility of violating CCC under test particle/field accretion has of late been a very active area of research.

A gedanken experiment was envisaged in which overcharged/rotating test particles were bombarded into a black hole to see whether a non-extremal black hole could be turned into extremal black hole [4]? The answer turned out to be negative because extremality is approached, the allowed window of parameter space of particle to reach the horizon pinches off [5] and thus extremality or zero black hole temperature can never be attained. However the interest in this question got revived when it was argued that a non-extremal black hole cannot be converted into extremal but extremality could be jumped over, and a black hole could be overcharged [6] or overspun [7]. Thus a naked singularity could be created defying CCC.

This led to a spurt in activity where various authors studied overcharging/spinning of black holes in different settings and thereby violating CCC, [see,e.g. 8–19]. In all these works, it was assumed that the test particle follows a geodesic (or Lorentz force when charged) motion while back and radiation reaction and self force effects were not considered. It is though expected that when these effects will be taken into account, there would be no overcharging/spinning and destruction of black hole horizon [20, 21, 21–25]. What happens is that particles that could cause over extremal state would not be able to reach black hole horizon. This was precisely the case, why extremality was not attainable [4, 5]. Note that in test particle accretion black hole is perturbed linearly while realistic accretion process like fluid flow would involve non-linear perturbation which could alter the situation completely. This is what has recently been done.

An extensive analysis of non-linear accretion/perturbations has been carried out [26] leading to the expected result that black hole horizon cannot indeed be destroyed, establishing validation of CCC. The same conclusion was also shown for Kerr-AdS black hole [27]. Following [26], a number of works have been done of non-linear perturbations [28, 29] – black hole horizon cannot be destroyed. Also the same analysis has been done in higher dimension [30] as well, showing that five dimensional Myers-Perry rotating black hole [31] though could be overspun at linear order, however when second order perturbations are taken into account the situation reverses — no overspinning and CCC is restored. In this case there is yet another subtle point where a black
hole with single rotation cannot be overspun even at the linear order while it could, like all other cases, however be overspun when both rotations are present [32]. A charged black hole in higher dimensions could always be over charged at linear order [33].

In this paper we would like to examine this question of linear and non-linear accretion for a charged rotating black hole in five dimension. In four dimension, it was straightforward to add charge parameter in the $\Delta$ function of rotating solution; i.e. $\Delta = r^2 - 2Mr + a^2 + Q^2$. Unfortunately this does not work in five dimension, and in fact an analogue of Kerr-Newman black hole has not yet been found. There exists a solution in slow rotation limit [34–36], and some solutions in supergravity and string theory [37–43]. The closest that comes to Kerr-Newman black hole is the one describing minimally gauged supergravity black hole [44]. Black hole energetics in terms of ergosphere and energy extraction of this solution has been investigated [45]. We shall take this solution of minimally gauged supergravity black hole for a charged and rotating black hole in five dimension and examine linear and non-linear accretion for testing CCC.

In particular it would be interesting to examine the case of single rotation for linear accretion where black hole cannot be overspun [32] but could be overcharged [33]. It turns out that the ultimate behavior would be determined by whether angular momentum is dominant over charge or the other way round. In the former case black hole cannot be over extremalized while for the latter it could be.

The paper is organized as follows: In Secs II and III, we describe the black hole metric and its properties and build up background for studying linear and non-linear accretion for over extremalizing black hole in the Sec. IV. Finally we conclude with a discussion in the Sec. V. We shall use the natural units, $G = c = 1$ throughout.

II. THE BLACK HOLE METRIC AND ITS PROPERTIES

The metric of the five dimensional charged rotating minimally gauged supergravity black hole solution [44] in the Boyer-Lindquist coordinates $(t, r, \theta, \phi, \psi)$ takes the form

$$ds^2 = - \left( dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right) \times \left[ f \left( dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right) + \frac{2q}{\Sigma} \left( b \sin^2 \theta d\phi + a \cos^2 \theta d\psi \right) \right]$$

$$+ \Sigma \left( \frac{r^2 d\theta^2}{\Delta} + d\theta^2 \right) + \sin^2 \theta \left[ a dt - (r^2 + a^2) d\phi \right]^2$$

$$+ \frac{\Sigma}{\Delta} \left[ b dt - (r^2 + b^2) d\psi \right]^2$$

$$+ \frac{1}{r^2 \Sigma} \left[ a b dt - b (r^2 + a^2) \sin^2 \theta d\phi - a (r^2 + b^2) \cos^2 \theta d\psi \right]^2 . \tag{1}$$

Here the metric coefficients are specified by

$$f(r, \theta) = \frac{(r^2 + a^2)(r^2 + b^2) - \mu \Sigma}{r^2 \Sigma}, \quad \Sigma(r, \theta) = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta,$$

$$\Delta(r) = (r^2 + a^2)(r^2 + b^2) + 2abq + q^2 - \mu r^2, \tag{2}$$

where $a$ and $b$ are the angular momentum per unit mass parameter and related to the specific angular momenta as

$$a + b = \frac{4 J_\phi + J_\psi}{\pi \mu + q}, \tag{3}$$

with mass parameter $\mu = \frac{8M}{3\pi}$ and charge parameter $q = \frac{4Q}{\sqrt{\pi}}$ of the black hole. The electromagnetic potential is given by

$$\mathbf{A} = \frac{-\sqrt{3} q}{2 \Sigma} (dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi) . \tag{4}$$

The horizon of the black hole follows from the relation $\Delta = 0$, i.e.

$$r_\pm = \pm \sqrt{\mu - 2q - (a + b)^2} \pm \sqrt{\mu + 2q - (a - b)^2}, \tag{5}$$

from the above expression it is evident that the horizon does not exist unless the following inequalities: $a^2 + b^2 + 2|a||b| \leq \mu - 2q$ and $a^2 + b^2 - 2|a||b| \leq \mu + 2q$ are satisfied. Let’s rewrite the black hole horizon given in Eq. (5) by mass, charge and angular momenta of the black hole as

$$r_+ = \frac{1}{4 \sqrt{3\pi} \left( M + \frac{\sqrt{3} Q}{2} \right)} \left[ \alpha \right]$$

$$+ \left[ \alpha^2 + 108\pi J_\phi J_\psi + 64 \sqrt{3} Q \left( M + \frac{\sqrt{3} Q}{2} \right) \right]^2 , \tag{6}$$
where $\alpha$ is given by

$$\alpha = \left(32M^3 - 27\pi (J_\alpha + J_\psi)^2 - 72MQ^2 - 24\sqrt{3} Q^3\right)^{1/2}. \quad (7)$$

Note that black hole horizon exists if and only if $\alpha^2 > 0$, and if the opposite is true, the resulting object is des-

$$g_{\alpha\beta} = \begin{pmatrix} \Sigma & 0 \\ 0 & \left(r^2 + a^2 + \frac{\alpha[a(\mu\Sigma-q^2)+2bq\Sigma]}{\Sigma}\sin^2 \theta\right) \sin^2 \theta \\ 0 & \left(\frac{\alpha[b(\mu\Sigma-q^2)+(a^2+b^2)q\Sigma]}{2\Sigma}\right) \sin^2 2\theta \end{pmatrix}. \quad (8)$$

The area of the horizon can be then determined as

$$A = \int_{\Xi_3} \sqrt{\text{det}|g_{\alpha\beta}|} d\theta d\phi d\psi = \frac{2\pi^2}{r_+} (\mu r_+^2 - abq - q^2), \quad (9)$$

which is of considerable importance owing to the area theorem, stating that it does not decrease in any physical process.

The angular speeds along $\phi$ and $\psi$ directions at the horizon $r = r_+$ are given by

$$\Omega_+^{(\phi)} = \frac{a(r_+^2 + b^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \quad (10)$$

$$\Omega_+^{(\psi)} = \frac{b(r_+^2 + a^2) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \quad (11)$$

for which the Killing field yields

$$\chi = \chi^{(i)} + \Omega_+^{(\phi)} \chi^{(\phi)} + \Omega_+^{(\psi)} \chi^{(\psi)}, \quad (12)$$

where $\chi^{(i)} = \partial_i$. Note that the Killing field is defined by $\chi = \chi^\alpha \partial_{\alpha}$.

Based on the Killing field given in (12), the surface gravity can be defined by

$$2k\chi_\alpha = \nabla_\alpha \left(-\chi^\beta \chi_\beta\right) |_{r=r_+}, \quad (13)$$

or by

$$k^2 = -\frac{1}{2} (\nabla_\alpha \chi_\beta) (\nabla^\alpha \chi^\beta) |_{r=r_+}. \quad (14)$$

The surface gravity and electromagnetic potential at the horizon of the black hole then take forms, respectively

$$k = \frac{(2r_+^2 + a^2 + b^2 - \mu) r_+}{\mu r_+^2 - abq - q^2}, \quad (15)$$

scribed by a naked singularity. Meanwhile, $\alpha = 0$ corresponds to the extremal charged rotating black hole. The area of the event horizon of the five dimensional charged rotating minimally gauged supergravity black hole can be evaluated by setting $dr = dt = 0$ and $r = r_+$ in the line element of the metric (1). The metric tensor for black hole horizon is then given by

$$\Phi = -\chi^\alpha \mathcal{A}_\alpha |_{r=r_+} = \frac{\sqrt{3}qr_+^2}{\mu r_+^2 - abq - q^2}. \quad (16)$$

### III. VARIATIONAL IDENTITIES AND PERTURBATION INEQUALITIES

It is well known that Lagrangian $L$ for a diffeomorphism covariant theory with $n$-dimensional manifold $M$ can be described by metric $g_{\alpha\beta}$ and curvature tensors and symmetrized covariant derivatives of its curvature tensor, and it can be also given by other fields $\psi$ [46]. The variation of Lagrangian is then given by

$$\delta L = E \delta \phi + d\Theta(\phi, \delta \phi), \quad (17)$$

where the equations of motion can be described by $E = 0$ and $\Theta$ is referred to as symplectic potential $(n-1)$-form. The $(n-1)$-form yields as

$$\omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi). \quad (18)$$

The Noether current $5$-form along with a vector field $\zeta^\alpha$ is defined by

$$J_\zeta = \Theta(\phi, L\zeta \phi) - \zeta \cdot L, \quad (19)$$

for which $dJ_\zeta = 0$ verifies the equation of motion to be satisfied. According to the [47], one can define the Noether current in the following form

$$J_\zeta = dQ_\zeta + C_\zeta, \quad (20)$$

where $Q_\zeta$ is referred to as the Noether charge, while the constraint corresponds to $C_\zeta = \zeta^\alpha C_\alpha$. 
Eqs. (19) and (20) here plays an important role in getting linear variational identity on a surface $\Xi$

$$\int_{\Xi} \delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi) = \int_{\Xi} \omega(\phi, \delta \phi, L_\zeta \phi) - \int_{\Xi} \zeta \cdot E \delta \phi - \int_{\Xi} \delta C_\zeta, \quad (21)$$

where the first term on the right side is defined by

$$\delta H_\zeta = \int_{\Xi} \omega(\phi, \delta \phi, L_\zeta \phi). \quad (22)$$

On the basis of linear variational identity, the non-linear one on the same surface is defined by

$$\int_{\Xi} \delta^2 Q_\zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi) = \int_{\Xi} \omega(\phi, \delta \phi, L_\zeta \delta \phi) - \int_{\Xi} \zeta \cdot E \delta \phi - \int_{\Xi} \delta^2 C_\zeta. \quad (23)$$

Taking into account Eq. (22) one can define linear order variational identity (21) as

$$\delta M - \Omega(\delta J_\phi + \delta J_\psi) = \int_{\Sigma} [\delta^2 Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi)] - \int_{\Xi} \delta C_\zeta, \quad (24)$$

for given surface $\Sigma$ with a bifurcation surface $B$ in the case in which the equation of motion is satisfied.

Meanwhile non-linear variational identity (23) is then given by

$$\delta^2 M \ - \ \Omega(\delta^2 J_\phi + \delta^2 J_\psi) = \int_{\Sigma} [\delta^2 Q_\zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi)] - \int_{\Xi} \zeta \cdot E \delta \phi - \int_{\Xi} \delta^2 C_\zeta + E_\Xi(\phi, \delta \phi), \quad (25)$$

where $E_\Xi(\phi, \delta \phi)$ is used for canonical energy on the surface $\Xi$ as a non-linear correction $\delta \phi$. For Eqs. (24) and (25), symplectic potential 4-form is defined by

$$\Theta(\phi, \delta \phi) = \frac{1}{16\pi} \epsilon_{ijk\alpha} g^{\alpha\beta} g^{\gamma\eta}(\nabla_\eta \delta g_{\beta\gamma} - \nabla_\beta \delta g_{\gamma\eta}) - \frac{1}{4\pi} \epsilon_{ijk\alpha} F^{\alpha\beta} \delta A_\beta, \quad (26)$$

where the first term on the right side is responsible for GR part while the second = electromagnetic part because Lagrangian has form as

$$L = \frac{\epsilon}{16\pi} (R - F^{\alpha\beta} F_{\alpha\beta}), \quad (27)$$

Hence we have

$$E(\phi) \delta \phi = -\epsilon (\frac{1}{2} F^{\alpha\beta} \delta g_{\alpha\beta} + j^\alpha \delta A_\alpha), \quad (28)$$

where $j^\alpha = \frac{1}{4\pi} \nabla_\beta F^{\alpha\beta}$. From Eq. (26), the corresponding symplectic current yields

$$\omega_{ijkh} = \frac{1}{4\pi} \left[ \delta_2(\epsilon_{ijkh} a^{\alpha\beta}) \delta_1 A_\beta - \delta_1(\epsilon_{ijkh} a^{\alpha\beta}) \delta_2 A_\beta \right] + \frac{1}{16\pi} \epsilon_{ijkh} w^\alpha, \quad (29)$$

with

$$w^\alpha = P^{\alpha\beta\gamma\mu\nu}(\delta g_{\beta\gamma} \nabla_\eta \delta g_{\mu\nu} - \delta g_{\beta\gamma} \nabla_\eta \delta g_{\mu\nu}) \Omega^{\alpha\beta\gamma} g^{\nu\mu} \delta g_{\kappa\lambda} \nabla_\mu g^{\eta\kappa} \nabla_\lambda g^{\rho\sigma} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\mu} g^{\nu\rho} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\mu} g^{\nu\rho}. \quad (30)$$

Taking into account $L_\zeta g_{\alpha\beta} = \nabla_\alpha \zeta_\beta + \nabla_\beta \zeta_\alpha$ and $\nabla_\alpha A_\beta = F_{\alpha\beta} + \nabla_\beta A_\alpha$, the Noether current 4-form would be defined by

$$(J_\zeta)_{ijkh} = \frac{1}{8\pi} \epsilon_{ijk\alpha} \nabla_\beta(\nabla^\beta \chi^\alpha) + \epsilon_{ijkh} \delta_2 A_\alpha \epsilon^\beta + \frac{1}{4\pi} \epsilon_{ijk\alpha} \nabla_\gamma(F^{\gamma\alpha} A_\beta \zeta^\beta) + \epsilon_{ijkh} A_\beta j^\gamma \epsilon^\beta, \quad (31)$$

and as well as the Noether charge $Q_\zeta$ and the constraint $C_\zeta$ are given by

$$(Q_\zeta)_{ijk} = \frac{1}{16\pi} \epsilon_{ijk\alpha} \nabla_\alpha \zeta^\beta - \frac{1}{8\pi} \epsilon_{ijk\alpha} F^{\alpha\beta} A_\gamma \zeta^\gamma.$$  

$$(C_\zeta)_{ijkh} = \epsilon_{ijkh}(T^\alpha_\gamma + A_\gamma j^\alpha). \quad (32)$$

IV. OVER EXTREMALIZING BLACK HOLE VIA GEDANKEN EXPERIMENTS

A. Extremal case

Here, we consider a particle absorption by an extremal five dimensional charged rotating black hole with mass $M$, angular momenta $J_\psi$ and $J_\phi$ and electric charge $Q$. The extremality condition for black hole reads as

$$32M^2 = 27\pi (J_\psi + J_\phi)^2 + 72MQ^2 + 24\sqrt{3} Q^3. \quad (33)$$

A particle of energy $\delta M$ and angular momenta $\delta J_\psi$ and $\delta J_\phi$ is thrown onto black hole horizon. This leads to increase in the corresponding parameters of black hole and a perturbed stationary state would be attained with parameters, $M + \delta M$, $J + \delta J_\psi$, $J + \delta J_\phi$, and $Q + \delta Q$. The condition for CCC violation would require the following inequality

$$96M^2 \delta M < 54\pi (J_\psi + J_\phi) (\delta J_\phi + \delta J_\psi) + 72Q^2 \delta M + 144MQ \delta Q + 72\sqrt{3} Q^2 \delta Q, \quad (34)$$

for the first order linear accretion. An extremal black hole will be pushed to over-extremal state if and only if
the following linear order accretion satisfies
\[
\delta M = \frac{9\pi}{4(4M^2 - 3Q^2)} (J_\phi + J_\psi) (\delta J_\phi + \delta J_\psi) - \frac{3}{(4M^2 - 3Q^2)} \delta Q < 0 .
\] (35)

We must then ensure that whether over-extremal state happens or not, satisfying Eq. (35). Let’s suppose that a black hole with initial state is interacted by the absorbed particle with small appropriate parameters described by the stress-energy tensor \( T_{\alpha\beta} \). Consequently, the mass and angular momenta of the black hole are increased by following amounts
\[
\delta M = \int_H \epsilon_{ijkh}\chi^{\gamma}_i (\delta T^\alpha_{\gamma} + A_\gamma \delta j^\alpha) , \tag{36}
\]
\[
\delta J_\phi = -\int_H \epsilon_{ijkh}\chi^{\gamma}_i (\delta T^\alpha_{\gamma} + A_\gamma \delta j^\alpha) , \tag{37}
\]
\[
\delta J_\psi = -\int_H \epsilon_{ijkh}\chi^{\gamma}_i (\delta T^\alpha_{\gamma} + A_\gamma \delta j^\alpha) , \tag{38}
\]
where the integrations are over surface element on the event horizon \( r_+ \). We assume that at the end of the process, the black hole is returned to another stationary state. Using Eqs. (24) and (36–38) with \( \Omega_+^{(\phi)} \) and \( \Omega_+^{(\psi)} \), we obtain the following equation which ensures that particle crossed the horizon eventually
\[
\delta M - \Omega_+^{(\phi)} \delta J_\phi - \Omega_+^{(\psi)} \delta J_\psi = - \int_{\equiv} \delta C_\gamma =
- \int_H \epsilon_{ijkh}\chi^{\gamma}_i \left( \chi^{\gamma}_i + \Omega_+^{(\phi)} \chi^{\phi}_i + \Omega_+^{(\psi)} \chi^{\psi}_i \right)
\times (\delta T^\alpha_{\gamma} + A_\gamma \delta j^\alpha) , \tag{39}
\]
where \( \chi^{\gamma} \) is null vector on the horizon \( r_+ \). Bearing in mind \( \Phi = -\chi^{\gamma} A_\gamma \rvert_{r_+} \) and considering \( \int_H \delta (\epsilon_{ijkh}\chi^{\gamma}_i) = \delta Q \) for the perturbed charge fallen into the black hole through the horizon +, we rewrite Eq. (39) as
\[
\delta M - \Omega_+^{(\phi)} \delta J_\phi - \Omega_+^{(\psi)} \delta J_\psi - \Phi_+ \delta Q
= - \int_H \epsilon_{ijkh}\chi^{\gamma}_i \delta T^\gamma_{\alpha\gamma} , \tag{40}
\]
where the volume element on the horizon can be written as \( \epsilon_{ijkh} = -5k_i \epsilon_{ijkh} \). Given the volume element on the horizon one may then write
\[
- \int_H \epsilon_{ijkh}\chi^{\gamma}_i \delta T^\gamma_{\alpha\gamma} = \int_H \epsilon_{ijkh}\chi^{\gamma}_i \delta T^\gamma_{\alpha\gamma} . \tag{41}
\]
This clearly shows that the right side is only positive in the case when null energy condition on the horizon is satisfied, i.e. \( \delta T^\alpha_{\beta\gamma} k^\alpha k^\beta \geq 0 \), thereby having the inequality for linear order accretion for an extremal black hole
\[
\delta M - \Omega_+ (\delta J_\phi + \delta J_\psi) - \Phi_+ \delta Q \geq 0 . \tag{42}
\]
In the extremal black hole case, the angular velocity and the electric potential will take the forms
\[
\Omega_+ = \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} , \tag{43}
\]
\[
\Phi_+ = \frac{3 (2MQ - \sqrt{3}Q^2)}{(4M^2 - 3Q^2)} . \tag{44}
\]
The inequality (42) for extremal black hole then becomes
\[
\delta M - \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} (\delta J_\phi + \delta J_\psi)
- \frac{3}{(4M^2 - 3Q^2)} \delta Q \geq 0 . \tag{45}
\]
This inequality clearly shows that (35) cannot be satisfied. The inequalities (35) and (45) are in clear conflict, hence an extremal five dimensional charged rotating minimally gauged supergravity black hole cannot be over-extremalized. Thus no violation of cosmic censorship conjecture occurs for extremal black hole.

Now we must ensure that it is indeed not possible to over-extremalize an extremal black hole. Thus, we must understand whether or not the black hole after interacting with test particle still remains an extremal black hole.

As was known that the first law of black hole dynamics states that the changes in black hole parameters and horizon area can be defined by
\[
\delta M = \frac{k}{8\pi} \delta A + \Omega^{(\phi)} \delta J_\phi + \Omega^{(\psi)} \delta J_\psi + \Phi_+ \delta Q , \tag{46}
\]
which completely satisfies
\[
M = M(A, J_\phi, J_\psi, Q) , \tag{47}
\]
where the horizon area can be also defined as a function of momenta and charge of an extremal black hole \( A = A_{ext}(J_\phi, J_\psi) \). For an extremal black hole, we will consider variation in the mass
\[
\delta M_{ext} = \left( \frac{\partial M}{\partial A} \frac{\partial A_{ext}}{\partial J_\phi} + \frac{\partial M}{\partial J_\phi} \right) \delta J_\phi
+ \left( \frac{\partial M}{\partial A} \frac{\partial A_{ext}}{\partial J_\psi} + \frac{\partial M}{\partial J_\psi} \right) \delta J_\psi
+ \left( \frac{\partial M}{\partial Q} \frac{\partial A_{ext}}{\partial J_\phi} + \frac{\partial M}{\partial J_\phi} \right) \delta Q
= \frac{k}{8\pi} \delta A + \Omega^{(\phi)} \delta J_\phi + \Omega^{(\psi)} \delta J_\psi + \Phi_+ \delta Q , \tag{48}
\]
where
\[
k = \frac{\partial M}{\partial A} , \tag{49}
\]
\[
\delta A = \frac{\partial A_{ext}}{\partial J_\phi} \delta J_\phi + \frac{\partial A_{ext}}{\partial J_\psi} \delta J_\psi + \frac{\partial A_{ext}}{\partial Q} \delta Q . \tag{50}
\]
The surface gravity does go to zero \( k \to 0 \) for an extremal black hole. As a result, Eq. (48) yields
\[
\delta M_{ext} = \Omega_+ (\delta J_\phi + \delta J_\psi) + \Phi_+ \delta Q , \tag{51}
\]
which characterizes an extremal black hole $M = M_{\text{ext}}(J_\phi, J_\psi, Q)$. The black hole exists provided that $M \geq M_{\text{ext}}(J_\phi, J_\psi, Q)$, and if the opposite, $M < M_{\text{ext}}(J_\phi, J_\psi, Q)$, is true, over-extremal state could happen. If the particle with angular momenta and charge crosses the horizon of an extremal five dimensional charged rotating black hole, the black hole’s angular momenta and charge are given by $J_\phi + \delta J_\phi$, $J_\psi + \delta J_\psi$ and $Q + \delta Q$. Thus, its final mass, according to Eqs. (42) and (51) is given

$$M + \delta M \geq M + \Omega_+ (\delta J_\phi + \delta J_\psi) + \Phi_+ \delta Q$$

$$= M_{\text{ext}}(J_\phi, J_\psi, Q) + \delta M_{\text{ext}}$$

$$= M_{\text{ext}}(J_\phi + \delta J_\phi, J_\psi + \delta J_\psi, Q + \delta Q).$$ (52)

As can be seen from Eq. (52), the black hole’s final mass is not less than an extremal black hole’s mass. This is in agreement with the third law of black hole thermodynamics [5, 48, 49].

From the above approaches, it follows that an extremal five dimensional charged rotating black hole cannot be over-extremalized. At the end of this process, an extremal black hole keeps its extremality, occurring no violation of the cosmic censorship conjecture.

Next, we investigate over-extremal state for a near-extremal five dimensional charged rotating black hole for linear and non-linear perturbations through gedanken experiments.

### B. Near-extremal case

In this subsection we apply gedanken experiments to overspin/overcharge near extremal black hole. Let’s re-call the extremality condition Eq. (33),

$$32M^3 - 27\pi (J_\phi + J_\psi)^2 - 72MQ^2 - 24\sqrt{3} Q^3 = 0.$$ (53)

To test gedanken experiments in order to overspin/overcharge a nearly extremal black hole, let us introduce a one-parameter family of function $f(\lambda)$ as

$$f(\lambda) = 32M(\lambda)^3 - 27\pi [J_\phi(\lambda) + J_\psi(\lambda)]^2$$

$$- 72M(\lambda)Q(\lambda)^2 - 24\sqrt{3} Q(\lambda)^3,$$ (54)

where $f(0) = \alpha^2$, being a bit larger than zero, corresponds to the near extremal black hole, and $M(\lambda)$, $J_\phi(\lambda)$, $J_\psi(\lambda)$ and $\delta Q(\lambda)$ are defined by

$$M(\lambda) = M + \lambda \delta M,$$

$$J_\phi(\lambda) = J_\phi + \lambda \delta J_\phi,$$

$$J_\psi(\lambda) = J_\psi + \lambda \delta J_\psi,$$

$$Q(\lambda) = Q + \lambda \delta Q.$$ (55)

Let’s now follow a nearly extremal black hole. To jump from sub-extremal to over-extremal state we must obtain $f(\lambda) < 0$. To proceed, we expand $f(\lambda)$ up to second order in $\alpha$ and $\lambda$ as

$$f(\lambda) = \alpha^2 + f_1 \lambda + f_2 \lambda^2 + O(\lambda^3, \lambda^2 \alpha, \lambda \alpha^2, \alpha^3),$$ (56)

where

$$f_1 = 24 \left(4M^2 - 3Q^2\right) \left[\delta M - \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} (\delta J_\phi + \delta J_\psi) - \frac{3 \left(2MQ - \sqrt{3}Q^3\right)}{4M^2 - 3Q^2} \delta Q\right],$$ (56)

$$f_2 = \left\{12 \left(4M^2 - 3Q^2\right) \left[\delta^2 M - \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} (\delta^2 J_\phi + \delta^2 J_\psi) - \frac{3 \left(2MQ - \sqrt{3}Q^3\right)}{4M^2 - 3Q^2} \delta^2 Q\right]ight.$$ \left[\delta^2 M - \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} (\delta^2 J_\phi + \delta^2 \right.$$

$$+ \ 96M(\delta M)^2 - 27\pi (\delta J_\phi + \delta J_\psi)^2 + 72 \left(M(\delta Q)^2 + 2Q\delta M \delta Q + \sqrt{3}Q(\delta Q)^2\right)\right\}.$$ (57)

From the Eq. (56), the expression in the bracket is defined by assuming optimal choice of linear order correction
\[ \delta M = \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} (\delta J_\phi + \delta J_\psi) + \frac{3(2MQ - \sqrt{3}Q^2)}{(4M^2 - 3Q^2)} \delta Q = \]
\[ - \frac{\sqrt{27\pi J_\phi J_\psi + 4\sqrt{3}Q(2M + \sqrt{3}Q)^2}}{(27\pi J_\phi J_\psi + 4\sqrt{3}Q(2M + \sqrt{3}Q)^2)^{1/2}} \left( 9\pi (J_\phi + J_\psi)^2 + \frac{4\sqrt{3}Q}{2} Q (2M + \sqrt{3}Q)^2 \right)^2 \]
\[ \times \left[ 6\pi \left( M + \frac{\sqrt{3}Q}{2} \right) \left( 144\sqrt{3}Q \left( M + \frac{\sqrt{3}Q}{2} \right)^2 \right. \right. \]
\[ \left. \delta J_\psi J_\phi^2 + 2J_\psi J_\phi^2 \delta J_\phi \right) + \frac{2}{3} \left. \delta J_\phi J_\psi^2 \right) + 2J_\phi J_\psi^2 \delta J_\phi + \delta J_\phi J_\psi^2 \right] \]
\[ + 243\pi^2 J_\phi J_\psi (J_\phi + J_\psi)^2 (\delta J_\phi J_\phi + \delta J_\phi J_\psi) + 16Q^2 \left( 2M + \sqrt{3}Q \right)^4 \left[ J_\phi (\delta J_\phi + 2\delta J_\psi) + J_\psi (\delta J_\phi + 2\delta J_\psi) \right] \]
\[ + 256Q^2 \left( M + \frac{\sqrt{3}Q}{2} \right)^4 \left( 9\sqrt{3}J_\phi J_\psi + 4Q \left( 2M + \sqrt{3}Q \right)^2 \right) \delta Q \alpha. \]

(58)

Further, taking into account the optimal choice of linear order perturbation, we test gedanken experiments to overextremalize black hole for both linear and non-linear particle accretion.

C. With two rotations

1. Linear order accretion

In view of the above equation (58), we rewrite \( f(\lambda) \) for linear order correction as

\[ f(\lambda) = \alpha^2 - \frac{6(2M + \sqrt{3}Q)^{-1}}{(27\pi J_\phi J_\psi + 4\sqrt{3}Q(2M + \sqrt{3}Q)^2)^{1/2}} \left( 9\pi (J_\phi + J_\psi)^2 + \frac{4\sqrt{3}Q}{2} Q (2M + \sqrt{3}Q)^2 \right)^2 \]
\[ \times \left[ 6\pi \left( M + \frac{\sqrt{3}Q}{2} \right) \left( 144\sqrt{3}Q \left( M + \frac{\sqrt{3}Q}{2} \right)^2 \right. \right. \]
\[ \left. \delta J_\psi J_\phi^2 + 2J_\psi J_\phi^2 \delta J_\phi \right) + \frac{2}{3} \left. \delta J_\phi J_\psi^2 \right) + 2J_\phi J_\psi^2 \delta J_\phi + \delta J_\phi J_\psi^2 \right] \]
\[ + 243\pi^2 J_\phi J_\psi (J_\phi + J_\psi)^2 (\delta J_\phi J_\phi + \delta J_\phi J_\psi) + 16Q^2 \left( 2M + \sqrt{3}Q \right)^4 \left[ J_\phi (\delta J_\phi + 2\delta J_\psi) + J_\psi (\delta J_\phi + 2\delta J_\psi) \right] \]
\[ + 256Q^2 \left( M + \frac{\sqrt{3}Q}{2} \right)^4 \left( 9\sqrt{3}J_\phi J_\psi + 4Q \left( 2M + \sqrt{3}Q \right)^2 \right) \delta Q \alpha + O(\alpha^2), \]

(59)

from which it is certain that it is possible to obtain \( f(\lambda) < 0 \) for suitable values of given parameters. Thus black hole could be over-extremalized. To ensure this, we try to explore \( f(\lambda) \) numerically. From Eq. (5), the extremal condition \( \mu - 2q = (a + b)^2 \) yields

\[ \sqrt{\frac{32}{27\pi}} \left( M - \sqrt{3}Q \right) = \frac{J_\phi + J_\psi}{M + \frac{\sqrt{3}Q}{2}}, \]

(60)

for black hole parameters. From Eq. (60) it is clear that a
near-extremality requires \( Q < (\sqrt{3})^{-1} M \), which in turn allows us to choose \( Q = 0.5M \). For given \( Q = 0.5M \), \( f(0) = \alpha^2 \) corresponding to the near extremality defines the angular momenta numerically, \( J_\phi + J_\psi = 0.322011 \) for the given value \( \alpha = 0.01 \). For this thought experiment one can take different values of black hole parameters and even smaller values of \( \alpha \). Setting \( M = 1 \), let’s choose \( \delta J_\phi = 0.001 \ll J_\phi \), \( \delta J_\psi = 0.001 \ll J_\psi \) and \( \delta Q = 0.003 \ll Q \) in order for the test particle approximation to hold well. Now let’s then evaluate Eq. (59) numerically, thereby \( f(0.1) = -0.00045 \). That is a charged black hole with two rotations could certainly be over extremalized by linear order accretion.

2. Non-linear order accretion

We here consider the second order particle accretion \( O(\lambda^2) \) so as to understand what might happen in the case of non-linear regime. Let’s start from Eq. (57), where the non-linear terms are defined by

\[
\delta^2 M - \frac{9\pi (J_\phi + J_\psi)}{4(4M^2 - 3Q^2)} (\delta^2 J_\phi + \delta^2 J_\psi) - \frac{3(2MQ - \sqrt{3}Q^2)}{(4M^2 - 3Q^2)} \delta^2 Q \geq -\frac{k}{8\pi} \delta^2 A \geq \frac{1}{12(4M^2 - 3Q^2)} \alpha^2.
\]

\[
\times \left( N_1 (M, Q, J_\phi, J_\psi, \delta J_\phi, \delta J_\psi, \delta Q) \delta M + N_2 (M, Q, J_\phi, J_\psi) \delta M^2 + N_3 (M, Q, J_\phi, J_\psi) \delta J_\phi \delta J_\psi + N_4 (M, Q, J_\phi) \delta J_\phi^2 + N_5 (M, Q, J_\psi) \delta J_\psi^2 + N_6 (M, Q, \delta J_\phi, \delta J_\psi) \delta Q^2 \right).
\]

(61)

Here the function \( N_i \) is related to the black hole parameters in a complex way. When we take into account non-linear term \( O(\lambda^3) \) by using Eq. (61) and optimal choice of linear order correction, the function \( f(\lambda) \) is given by

\[
f(\lambda) > \alpha - \frac{3(2M + \sqrt{3}Q)^{-1} \lambda}{\left( 27\pi J_\phi J_\psi + 4\sqrt{3}Q (2M + \sqrt{3}Q)^2 \right)^{1/2} \left( 9\pi (J_\phi + J_\psi)^2 + \frac{4\sqrt{3}}{3} Q (2M + \sqrt{3}Q)^2 \right)}
\]

\[
\times \left[ 6\pi \left( M + \frac{\sqrt{3}Q}{2} \right) \left( 144\sqrt{3}\pi Q \left( M + \frac{\sqrt{3}Q}{2} \right)^2 \left[ \delta J_\phi J_\psi^3 + 2J_\psi J_\phi^2 (\delta J_\phi + 2\delta J_\psi) + 2J_\phi J_\psi^2 (\delta J_\phi + \delta J_\psi) + \delta J_\phi J_\psi^3 \right] \right. \right.
\]

\[\left. \quad + 243\pi^2 J_\phi J_\psi (J_\phi + J_\psi)^2 (\delta J_\phi J_\psi + \delta J_\phi J_\psi) + 16Q^2 \left( 2M + \sqrt{3}Q \right)^4 \left[ J_\phi (\delta J_\phi + 2\delta J_\psi) + J_\psi (\delta J_\phi + \delta J_\psi) \right] \right) \right]
\]

\[+ 256Q^2 \left( M + \frac{\sqrt{3}Q}{2} \right)^4 \left( 9\sqrt{3}\pi J_\phi J_\psi + 4Q (2M + \sqrt{3}Q)^2 \delta Q \right) \right)^2 + O(\alpha^3, \alpha^2 \lambda, \alpha \lambda^2, \lambda^3).
\]

(62)

This clearly shows \( f(\lambda) \geq 0 \) always. Thus, it verifies that a five dimensional charged rotating black hole cannot be over extremalized for a non-linear order accretion while the opposite is true for a linear order accretion.

D. With single rotation

1. Linear order accretion

Let’s consider a particular case of single rotation, for which Eq. (59) takes the following form

\[
f(\lambda) = \alpha^2 - \frac{48 \times 3^{3/4} Q^{3/2} (2M + \sqrt{3}Q)^3}{\left( 9\sqrt{3}J_\psi^2 + 4Q (2M + \sqrt{3}Q)^2 \right)^2}
\]

\[
\times \left( 3\pi J_\psi \delta J_\psi + 4Q (2M + \sqrt{3}Q) \delta Q \right) \alpha \lambda + O(\lambda^2).
\]

(63)
It is clear from the above equation that overspinning/charging is quite possible in general. However let’s consider various cases separately.

- $\delta Q = 0$. Note that in the limit $Q \to 0$ one can reach $f(\lambda) > 0$, for which black hole could not be overspun, thereby verifying the validity of the CCC for black hole having a single rotation. This verifies the recently obtained result Ref. [32] that CCC is obeyed for single rotation even at linear order accretion. Consider the numerical example: For $Q = 0.5$, $J_\psi = 0.322011$, $\delta J_\psi = 0.001$, and $\alpha = 0.01$ with $\lambda = 0.1$ we get $f(\lambda) = -0.00048$. With this we again verify the result of Ref. [33] that the CCC could as in four dimension be violated.

In five dimension, a single rotating black hole could be overcharged but not overspun, thereby verifying the validity of the CCC for black hole having a single rotation. This verifies the recently obtained result Ref. [32] that CCC is obeyed for single rotation even at linear order accretion. Consider the numerical example: For $Q = 0.5$, $J_\psi = 0.322011$, $\delta J_\psi = 0.001$, and $\alpha = 0.01$ with $\lambda = 0.1$ we get $f(\lambda) = -0.00048$. With this we again verify the result of Ref. [33] that the CCC could as in four dimension be violated.

- $\delta J_\psi = 0$. It is well known that a four dimensional charged black hole could be overcharged [33]. To be a bit more quantitative let’s consider the Eq. (63), for $Q = 0.5$, $J_\psi = 0.322011$, $\delta Q = 0.003$, and $\alpha = 0.01$ with $\lambda = 0.1$, we get $f(\lambda) = -0.00048$. With this we again verify the result of Ref. [33] that the CCC could as in four dimension be violated.

In five dimension, a single rotating black hole could be overcharged but not overspun, thereby verifying the validity of the CCC for black hole having a single rotation. This verifies the recently obtained result Ref. [32] that CCC is obeyed for single rotation even at linear order accretion. Consider the numerical example: For $Q = 0.5$, $J_\psi = 0.322011$, $\delta J_\psi = 0.001$, and $\alpha = 0.01$ with $\lambda = 0.1$, we get $f(\lambda) = -0.00048$. With this we again verify the result of Ref. [33] that the CCC could as in four dimension be violated.

- $\delta J_\psi > \delta Q$. Let’s consider the numerical exercise: Take a) $Q = 0.5$, $J_\psi = 0.322011$ and b) $Q = 0.353553$, $J_\psi = 0.499394$ for given $\delta Q = 0.0003$, $\delta J_\psi = 0.001$, and $\alpha = 0.01$ with $\lambda = 0.1$ and that leads to a) $f(\lambda) = 6.3832 \times 10^{-6} > 0$ and b) $f(\lambda) = 50.1196 \times 10^{-6} > 0$. It cannot be over extremalized, and the CCC continues to hold ground.

- $\delta J_\psi = \delta Q$. Let’s again consider the numerical exercise: Take a) $Q = 0.5$, $J_\psi = 0.322011$ and b) $Q = 0.353553$, $J_\psi = 0.499394$ for given $\delta Q = 0.0003$, $\delta J_\psi = 0.001$, and $\alpha = 0.01$ with $\lambda = 0.1$, we get a) $f(\lambda) = -0.000417867$ and b) $f(\lambda) = -0.000116191$. This shows that black hole could reach over-extremal state when impinging particles have angular momentum equal to charge.

What emerges from this analysis is that black hole with single rotation for linear accretion obeys the CCC so long as $\delta Q < \delta J_\psi$, and the opposite is true for $\delta Q \geq \delta J_\psi$, irrespective of relative dominance of black hole rotation and charge parameters. Interestingly in the case of equality of angular momentum and charge of impinging particles, it is charge’s interaction plays dominating role for over extremalizing process.

2. Non-linear order accretion

Let’s rewrite Eq. (62) in the case of a single rotation,

$$f(\lambda) = \alpha - \frac{48 \times 3^{3/4} Q^{3/2} (2M + \sqrt{3Q})^3 (3\pi J_\psi \delta J_\psi + 4Q (2M + \sqrt{3Q}) \delta Q)}{(9\sqrt{3\pi J_\psi^2} + 4Q (2M + \sqrt{3Q})^2)^2} \lambda^2 + \mathcal{O}(\alpha^3, \alpha^2 \lambda, \alpha \lambda^2, \lambda^3). \quad (64)$$

From Eq. (64), it is clear that black hole cannot be over extremalized when second order perturbations, $\mathcal{O}(\lambda^2)$, are taken in. For non-linear accretion the CCC continued to hold good.

V. CONCLUSIONS

It is known that there does not exist a true analogue of four dimensional Kerr-Newman rotating charged black hole in five dimension. On the other hand there exists
an analogue of Kerr rotating black hole in five or higher dimensions. Strangely electric charge cannot be injected to rotating black hole. However there exists a very close cousin of Kerr-Newman black hole in minimally gauged supergravity solution of rotating and charged black hole [44]. To this black hole we have in this paper extended the analysis of over extremalizing under linear and non-linear accretion process [30].

In general it turns out that as is the case in for all other cases, over extremalizing is possible for linear order while it gets miraculously reversed when non-linear perturbations are included. The five dimensional black hole in question thus falls in line with all other black holes that CCC could be violated at linear order but it is restored at non-linear order accretion. However there is a subtle exception for rotating black hole in five dimension which has two rotation axes permitting two rotation parameters.

Very recently, some of us [32] had demonstrated a remarkable property of a black hole with single rotation. Unlike four dimensional black hole, it cannot be overspun even at the linear order accretion while it could be overspun when both rotations are present. This property is however carried through for the five dimensional rotating charged black hole under study. A charged black hole could always be overcharged under linear accretion. In this case there are both rotation and charge present. Hence the question, when would it be over extremalized and when not? As expected it turns out that when rotation parameter of impinging particle is greater than its charge, over extremalizing is prohibited while the opposite is the case when charge is greater than or equal to rotation parameter. It is interesting that in the case of equality of rotation and charge parameters, it is the latter’s contribution that dominates.

As pointed out in [32], a black hole with single rotation in five dimension is a different entity like extremal black hole. The latter can never be over extremalized and interestingly so is the case for the former as well. It seems when black hole has the maximum number of rotations that are permitted in a given spacetime dimension, it can be overspun while it has less than the maximum allowed, it cannot be overspun. In four dimension maximum allowed parameter is one and that is why it can be overspun while in five dimensions maximum allowed are two. That is why it can perhaps only violate CCC when two rotations are present but not for single rotation.

Acknowledgments

BA and SS acknowledge the Faculty of Philosophy and Science, Silesian University in Opava, Czech Republic, Inter-University Centre for Astronomy and Astrophysics, Pune, India, and Goethe University, Frankfurt am Main, Germany, for warm hospitality. ND wishes to acknowledge visits to Albert Einstein Institute, Golm and to Astronomical Institute, Tashkent supported by the Abdus Salam International Centre for Theoretical Physics, Trieste under the Grant No. OEA-NT-01. This research is supported in part by Projects No. VA-FA-F-2-008 and No. MRB-AN-2019-29 of the Uzbekistan Ministry for Innovation Development, by the Abdus Salam International Centre for Theoretical Physics under the Grant No. OEA-NT-01 and by the Erasmus + Exchange Grant between Silesian University in Opava and National University of Uzbekistan.

[1] B. P. Abbott and et al. (VIRGO and LIGO Scientific Collaborations), Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc].
[2] B. P. Abbott and et al. (VIRGO and LIGO Scientific Collaborations), Phys. Rev. Lett. 116, 241102 (2016), arXiv:1602.03840 [gr-qc].
[3] R. Wald, Ann. Phys. (N.Y.) 82, 548 (1974).
[4] R. Penrose, Nuovo Cimento Rivista Serie 1, 252 (1969).
[5] N. Dadhich and K. Narayan, Phys. Lett. A 231, 335 (1997).
[6] V. E. Hubeny, Phys. Rev. D 59, 064013 (1999), gr-qc/9808043.
[7] T. Jacobson and T. P. Sotiriou, in Journal of Physics Conference Series, Vol. 222 (2010) p. 012041, arXiv:1006.1764 [gr-qc].
[8] A. Saa and R. Santarelli, Phys. Rev. D 84, 027501 (2011), arXiv:1105.3950 [gr-qc].
[9] M. Bouhmadi-López, V. Cardoso, A. Nerozzi, and J. V. Rocha, Phys. Rev. D 81, 084051 (2010), arXiv:1003.4295 [gr-qc].
[10] Z. Li and C. Bambi, Phys. Rev. D 87, 124022 (2013), arXiv:1304.6592 [gr-qc].
[11] J. V. Rocha and R. Santarelli, Phys. Rev. D 89, 064065 (2014), arXiv:1402.4840 [gr-qc].
[12] S. Shaymatov, M. Patil, B. Ahmedov, and P. S. Joshi, Phys. Rev. D 91, 064025 (2015), arXiv:1409.3018 [gr-qc].
[13] J. Natário, L. Queimada, and R. Vincente, Class. Quantum Grav. 33, 175002 (2016), arXiv:1601.06809 [gr-qc].
[14] Y. Song, M. Zhang, D.-C. Zou, C.-Y. Sun, and R.-H. Yue, Commun. Theor. Phys. 69, 694 (2018), arXiv:1705.01676 [gr-qc].
[15] K. Düztaş, Class. Quantum Grav. 35, 045008 (2018), arXiv:1710.06610 [gr-qc].
[16] S. Jana, R. Shaikh, and S. Sarkar, Phys. Rev. D 98, 124039 (2018), arXiv:1808.09656 [gr-qc].
[17] K. Düztaş and M. Jamil, arXiv e-prints (2018), arXiv:1808.04711 [gr-qc].
[18] K. Düztaş and M. Jamil, Mod. Phys. Lett. A 34, 1950248 (2019).
