ON ENDOMORPHISMS OF ARRANGEMENT COMPLEMENTS

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Abstract. Let Ω be the complement of a connected, essential hyperplane arrangement. We prove that every dominant endomorphism of Ω extends to an endomorphism of the tropical compactification X of Ω associated to the Bergman fan structure on the tropical variety trop(Ω).

This generalizes a result in [13], which states that every automorphism of Drinfeld’s half-space over a finite field $F_q$ extends to an automorphism of the successive blow-up of projective space at all $F_q$-rational linear subspaces. This successive blow-up is in fact the minimal wonderful compactification by de Concini and Procesi, which coincides with $X$ by results of Feichtner and Sturmfels [4]. Whereas the proof in [13] is based on Berkovich analytic geometry over the trivially valued finite ground field, the generalization proved in the present paper relies on matroids and tropical geometry.

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1. Introduction

Let $\mathcal{A}$ be a connected, essential arrangement of hyperplanes over an arbitrary field $K$, and let $\Omega_{\mathcal{A}}$ be the complement of the arrangement $\mathcal{A}$ in projective space. By $X_{vc}(\mathcal{A})$ we denote the visible contour compactification of $\Omega_{\mathcal{A}}$. It is associated to the Bergman fan structure on the tropicalization $\text{trop}(\Omega_{\mathcal{A}})$ in the sense of Tevelev [12]. Our main result Theorem 5.1 states that every dominant endomorphism of $\Omega_{\mathcal{A}}$ extends to an endomorphism of its visible contour compactification $X_{vc}(\mathcal{A})$. As a corollary we show in Corollary 5.2 that every dominant endomorphism of $\Omega_{\mathcal{A}}$ is finite. Feichtner and Sturmfels [4] have provided conditions under which the visible contour compactification coincides with the minimal wonderful compactification of $\Omega_{\mathcal{A}}$ defined by de Concini and Procesi.

This coincidence occurs for example if $K = F_q$ is a finite field and $\mathcal{A}$ is the full arrangement of all $F_q$-rational hyperplanes in projective space. Then the complement $\Omega_{\mathcal{A}}$ is Drinfeld’s half-space over $F_q$. It was shown in [13, Theorem 1.1] that for this arrangement every automorphism of $\Omega_{\mathcal{A}}$ extends to an automorphism of the ambient projective space $P_{F_q}^d$, i.e. it is given by an element in $PGL(d, F_q)$. An important step in the proof is the extension of an automorphism of $\Omega_{\mathcal{A}}$ to an automorphism of the successive blow-up $X_{\text{wod}}(\mathcal{A})$ of $F_q^d$, at all $F_q$-rational linear subspaces, which is achieved by using Berkovich analytic geometry over the trivially valued field $F_q$. In the present paper, see Corollary 5.4, we give an alternative proof of this step without using analytic geometry. Instead we use techniques from tropical geometry.

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geometry and matroid theory. Our alternative approach can then be gen-
eralized to arbitrary essential and connected hyperplane arrangements over
any ground field. To be more precise, the proof of [13, Theorem 1.1] relies on
the fact that every automorphism of \( \Omega_A \) restricts to an automorphism of a
suitable skeleton of \( \Omega_A^{\text{an}} \). In order to show that this restriction preserves the
fan structure on the skeleton, it is proved in [13, Lemma 2.2] that distinct
maximal cones in the skeleton span distinct linear spaces. In the present
paper, we prove a tropical avatar of this result in Theorem 4.2, which states
that for a loopfree matroid \( M \) distinct maximal cones in the Bergman fan
span distinct linear spaces.

Note that the Drinfeld half-space is the only hyperplane complement over
\( \mathbb{F}_q \) with automorphism group \( \text{PGL}(d, \mathbb{F}_q) \). Therefore we cannot expect that
the second step in the proof of [13, Theorem 1.1], which is a descent from
\( X_{\text{wmd}}(\mathcal{A}) \) to projective space, can be generalized to other arrangements.

The outline of the paper is as follows. We start with some basic def-
initions and properties of hyperplane arrangements in Section 2, together
with a brief account on the compactifications of arrangement complements
introduced in [12], [2] and [7]. In Section 3 we introduce the necessary defi-
nitions from matroid theory, and we describe different fan structures on the
tropical linear space of a matroid, ranging from the coarsest (the Bergman
fan) to the finest (the fine subdivision). In Section 4 we prove that distinct
cones of the Bergman fan span distinct linear spaces. Then we prove in
Section 5 that every dominant endomorphism of a connected arrangement
complement extends to an endomorphism of its visible contour compac-
tification and hence is finite. In particular, the automorphism group of the
arrangement complement is a subgroup of the automorphism group of its
visible contour compactification.

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2. Hyperplane complements and compactifications

Hyperplane complements. Fix any ground field \( K \) and a vector space
\( V \) of dimension \( d + 1 \) over \( K \). A set \( \mathcal{A} = \{ H_0, \ldots, H_n \} \) of \( n + 1 \) linear
hyperplanes in \( V \) is called a hyperplane arrangement over \( K \). It is called essential, if the intersection \( \bigcap_{i=0}^n H_i = \{0\} \).

Example 2.1. One important family of essential hyperplane arrangements
is given by the (essential) braid arrangements \( A_n \) for \( n > 1 \). Here we con-
side the hyperplanes \( H_i = V(x_i) \) for \( i \in \{0, \ldots, n-1\} \) and \( H_{ij} = V(x_i - x_j) \)
for \( i, j \in \{0, \ldots, n-1\} \) and \( i < j \) in \( n \)-space. This arrangement is in fact
the quotient of the finite reflection arrangement associated to a type \( A \) root
system after dividing by the lineality space. The complement of \( A_n \) in \( \mathbb{P}^{n-1}_\mathbb{C} \)
is isomorphic to the moduli space \( M_{0,n+2} \) of \( n+2 \) pointed curves of genus 0. ◦
Tropicalizations. From now on we assume that \( \mathcal{A} = \{H_0, \ldots, H_n\} \) is an essential arrangement of hyperplanes in \( V \). We denote by \( \Omega_{\mathcal{A}} \) the complement \( \mathbb{P}(V) - \mathcal{A} \) endowed with the reduced induced structure, where \( \mathbb{P}(V) = \text{Proj } \text{Sym} V^* \) is the projective space of lines in \( V \). Then \( \Omega_{\mathcal{A}} \) is an integral affine \( K \)-scheme. Furthermore, let \( T \) be the standard torus in \( \mathbb{P}^n_K \), which is the complement of all coordinate hyperplanes. Let \( l_i \) be an element in the dual space \( V^* \) such that \( H_i \) is the kernel of \( l_i \). Then the morphism

\[
j : \Omega_{\mathcal{A}} \to T, \quad x \mapsto [l_0(x) : l_1(x) : \ldots : l_n(x)]
\]

is a closed immersion.

Moreover, the multiplicative group \( \mathcal{O}(\Omega_{\mathcal{A}})^*/K^* \) is generated by the classes \( \frac{l_i}{l_0} \) of elements \( \frac{l_i}{l_0} \) of \( \mathcal{O}(\Omega_{\mathcal{A}})^* \), for \( i \in \{1, \ldots, n\} \). Hence \( T \) can be identified with the intrinsic torus of the very affine variety \( \Omega_{\mathcal{A}} \). By trop(\( \Omega_{\mathcal{A}} \)) we denote the associated tropicalization where the ground field \( K \) is taken with the trivial absolute value, see [9, Section 4.1]. We denote by \( N \) the cocharacter group of \( T \).

The hyperplane arrangement \( \mathcal{A} \) gives rise to a matroid \( M_{\mathcal{A}} \) on the ground set \( \{0, \ldots, n\} \), whose independent sets correspond to the linear independent subsets of \( \{l_0, \ldots, l_n\} \) in \( V^* \). A good introduction to matroids can be found in [10].

We call the arrangement \( \mathcal{A} \) connected if its associated matroid \( M_{\mathcal{A}} \) is connected. The lattice of flats of the matroid \( \mathcal{L}(M_{\mathcal{A}}) \) of \( M_{\mathcal{A}} \) is just the intersection lattice of \( \mathcal{A} \) and the rank function \( r \) on \( M_{\mathcal{A}} \) is given by the codimension of the corresponding intersection. The lattice \( \mathcal{L}(M_{\mathcal{A}}) \) is partially ordered by reverse inclusion. In fact, \( \mathcal{L}(M_{\mathcal{A}}) \) is a geometric lattice with minimal element \( 0 \), which corresponds to the empty intersection, hence to the ambient space of the arrangement.

In general, a loopfree matroid \( M \) on a finite set \( E(M) = \{0, 1, \ldots, n\} \) gives rise to a tropicalization in the following way. Write \( \mathbb{R}^{n+1}/\mathbb{R} \cdot 1 \) for the quotient space \( \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, \ldots, 1) \). Then the tropical linear space trop(\( M \)) of a loopfree matroid \( M \) is the set of vectors \( v = (v_0, v_1, \ldots, v_n) \in \mathbb{R}^{n+1} \) such that, for every circuit \( C \) of \( M \), the minimum of the numbers \( v_i \) is attained at least twice as \( i \) ranges over \( C \). If \( v \in \text{trop}(M) \) then \( v + \lambda \cdot 1 \in \text{trop}(M) \) for any \( \lambda \in \mathbb{R} \), so we regard it as a subset of \( \mathbb{R}^{n+1}/\mathbb{R} \cdot 1 \).

Note that for any essential arrangement of hyperplanes \( \mathcal{A} \), the tropicalization trop(\( \Omega_{\mathcal{A}} \)) which we defined previously coincides with the tropical linear space trop(\( M_{\mathcal{A}} \)) by [9, Proposition 4.1.6], since the ideal of \( \text{trop}(\Omega_{\mathcal{A}}) \) as a subvariety of \( T \) is generated by a system of linear equations given by the circuits of the matroid \( M_{\mathcal{A}} \). In fact, for a linear form \( l = \sum a_i x_i \in I \) we define its support by \( \text{supp}(l) = \{i : a_i \neq 0\} \). Then the linear forms \( l_C \) of \( I \), such that \( \text{supp}(l_C) \) is a circuit of \( M_{\mathcal{A}} \), form a tropical basis.

Example 2.2. For the braid arrangement \( A_3 \) in \( \mathbb{P}^2_K \), the set of circuits of \( M_{\mathcal{A}_3} \) is just the index sets of the minimal dependent sets of column vectors of the matrix

\[
B = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & -1 \\
\end{pmatrix}.
\]
If we index the columns of $B$ from 0 to 5 then the set of circuits of $M_{A_3}$ is given by
\[ C(M_{A_3}) = \{ \{0, 1, 3\}, \{0, 2, 4\}, \{1, 2, 5\}, \{3, 4, 5\} \}. \]

The complement $\Omega \subset \mathbb{A}^5$ is identified with the very affine variety $V(I)$ in the torus $\mathbb{G}_m^6/\mathbb{G}_m$ given by the ideal
\[ I = \langle x_0 - x_1 - x_3, x_0 - x_2 - x_4, x_1 - x_2 - x_5, x_4 - x_3 - x_5 \rangle \]
in $K[x_0^\pm, \ldots, x_5^\pm]$.

**Compactifications.** Let $K$ be any field, and let $T$ be a split torus over $K$ with cocharacter group $N$. For every fan $\Sigma$ in $N_K = N \otimes \mathbb{Z}$, $\mathbb{R}$, we denote by $Y_\Sigma$ the normal toric $K$-variety associated to $\Sigma$ with dense torus $T$.

Let us recall some results by Tevelev over an algebraically closed ground field $K$. In [12, Proposition 2.3] Tevelev shows that in this case the closure $\overline{\Omega}_A$ of $\Omega_A$ in the (not necessarily complete) toric variety $Y_\Sigma$ is complete if and only if the support of $\Sigma$ contains $\text{trop}(\Omega_A)$. In particular every choice of a fan structure on $\text{trop}(\Omega_A)$ gives rise to a compactification of the arrangement complement $\Omega_A$, even if the toric variety itself is not complete.

If $K$ is an arbitrary ground field with algebraic closure $\overline{K}$, the complement $\mathbb{P}_{\overline{K}} - A$, where $A$ is regarded as an arrangement in $\mathbb{P}_{\overline{K}}$, is the base change $\Omega_A \otimes_K \overline{K} = \Omega_A \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$. The tropicalizations $\text{trop}(\Omega_A)$ and $\text{trop}(\Omega_A \otimes_K \overline{K})$ coincide. Note that for any fan structure $\Sigma$ on $\text{trop}(\Omega_A)$, the base change of the closure $\overline{\Omega}_A$ of $\Omega_A$ in the toric $K$-variety $Y_\Sigma$ coincides with the closure of $\Omega_A \otimes_K \overline{K}$ in $Y_\Sigma \otimes_K \overline{K}$ which is proper. Hence by faithfully flat descent, $\overline{\Omega}_A$ is also proper over $K$.

Compactifications of subvarieties of tori obtained in this way are called *tropical compactifications*, if the multiplication map $\mu : T \times \overline{\Omega}_A \to Y_\Sigma$ is faithfully flat. A subvariety of a torus is called *schön* if the multiplication map for one (hence for any [12, Theorem 1.4]) tropical compactification is smooth. By [12, Theorem 1.5] $\Omega_A$ is schön for any connected, essential arrangement $A$.

Note that in general there is no canonical fan structure on the tropicalization of a very affine variety, so that there are different natural tropical compactifications. Here we are mainly interested in the following two compactifications. The first one is obtained by taking the Bergman fan $\mathfrak{B}(\mathbb{M}_A)$ on the tropicalization $\text{trop}(\Omega_A)$. We denote the closure $\overline{\Omega}_A \subset Y_{\mathfrak{B}(\mathbb{M}_A)}$ by $X_{vc}(A)$ and, following [12], we call it the *visible contours compactification*. Over an algebraically closed field $K$ this is just the visible contours compactification due to Kapranov [7]. Since $X_{vc}(A)$ is a tropical compactification, $X_{vc}(A)$ is smooth if and only if the compactifying toric variety $Y_{\mathfrak{B}(\mathbb{M}_A)}$ is smooth. The Bergman fan $\mathfrak{B}(\mathbb{M}_A)$ is not necessarily simplicial, therefore $X_{vc}(A)$ is not smooth in general. A family of examples can be found in [3].

The second compactification of interest here is constructed by taking the minimal nested set fan $\Sigma_{\text{min}}(\mathbb{M}_A)$ as fan structure supported on $\text{trop}(\Omega_A)$. We write $X_{\text{wnd}}(A)$ for the closure $\overline{\Omega}_A \subset Y_{\Sigma_{\text{min}}(\mathbb{M}_A)}$ and call it the *wonderful compactification* of $\Omega_A$. Feichtner and Sturmfels have shown in [4] that over an algebraically closed field $X_{\text{wnd}}(A)$ coincides with the minimal wonderful model of the arrangement complement introduced by de Concini and Procesi.
in [2]. The compactification \( X_{\text{wd}}(A) \) can also be obtained by iteratively blowing up the ambient projective space of \( A \) along strict transforms of linear subspaces in increasing order of dimension. The boundary \( X_{\text{wd}}(A) \setminus \Omega_A \) is a divisor with normal crossings whose irreducible components are indexed by the elements of the so-called building set. A subset of boundary components intersect if and only if the corresponding subset of the building set forms a nested set. In Section 3 we give a formal definition of building sets, nested sets as well as of the fans \( \mathcal{B}(M_A) \) and \( \Sigma_{\text{min}}(M_A) \).

3. Matroids and fan structures

Let us begin by recalling some facts on matroids. Let \( M \) be a matroid on the finite set \( E(M) \). We denote by \( C(M) \) the set of circuits and by \( \text{cl}_M \) its closure operator.

Restriction and contraction of matroids. For a subset \( X \subset E(M) \) we define the restriction of \( M \) to \( X \) as the matroid \( M|_X \) on the ground set \( X \) for which a subset of \( X \) is independent if and only if it is independent in the original matroid \( M \).

A subset \( F \) of \( X \) is a flat of \( M|_X \) if and only if there is a flat \( \tilde{F} \) of \( M \) such that \( F = \tilde{F} \cap X \).

Now let \( B_X \) be a basis of the restriction \( M|_X \). We define the contraction of \( M \) to \( E(M) \setminus X \) as the matroid \( M/X \) on the ground set \( E(M) \setminus X \) whose independent sets are subsets \( I \) of \( E(M) \setminus X \) for which \( I \cup B_X \) is an independent set of \( M \).

A subset \( F \) of \( E(M) \setminus X \) is a flat of \( M/X \) if and only if \( F \cup X \) is a flat of \( M \).

Example 3.1. Let \( A \) be an essential arrangement of \( n+1 \) hyperplanes in a vector space \( V \), let \( M(A) \) its associated matroid on \( \{0,1,\ldots,n\} \) and \( F \) a flat of \( M(A) \). Moreover let \( L_F \) be the linear space \( \bigcap_{i \in F} H_i \) associated to \( F \).

We define

\[
A_F = \{ H_i \in A : L_F \subset H_i \} \\
A^F = \{ H_i \cap L_F \neq \emptyset : H_i \in A \setminus A_F \}.
\]

Then \( A_F \) is an arrangement in \( V \) such that \( M(A_F) = M|_F \), while \( A^F \) is an arrangement of hyperplanes in \( L_F \) such that \( M(A^F) = M/F \), provided that \( M/F \) is simple.

Building sets and nested sets. Let \( \mathcal{L}(M) \) be the lattice of flats of \( M \) with unique minimal element \( \emptyset \). A subset \( G \subset \mathcal{L}(M) \setminus \{\emptyset\} \) is called a building set if, for every \( F \in \mathcal{L}(M) \), we have an order-isomorphism

\[
[\emptyset, F] \simeq \prod_{X \in \max(G \cap F)} [\emptyset, X],
\]

where, for a set \( H \subset \mathcal{L}(M) \), the notation \( \max H \) denotes the set of maximal elements.

Example 3.2. The set \( G_{\text{min}} = \{ F \in \mathcal{L}(M) : M|_F \text{ is connected} \} \) is the unique minimal building set, while \( G_{\text{max}} = \mathcal{L}(M) \setminus \{\emptyset\} \) is the unique maximal building set of \( \mathcal{L}(M) \).
Example 3.3. For the maximal building set called the Bergman fan, to the finest, called the fine subdivision of trop(M), Feichtner and Sturmfels compare these fan structures, from the coarsest, natural polyhedral fan structures on the tropical linear space trop(M). In particular, if M is connected, then a nested set can contain incomparable elements. ♦

Fan structures on trop(M). For a loopfree matroid M there are several natural polyhedral fan structures on the tropical linear space trop(M). In [4], Feichtner and Sturmfels prove in [4, Proposition 2.4] that the dimension of the matroid polytopes. In fact, if S is a face of P_M, then we define the degeneration matroid M_S as the matroid on the ground set E(M) such that, I ⊂ E(M) is an independent set of M_S if and only if there exists a vertex e_B of the face S with I ⊂ B. Then S coincides with the matroid polytope P_M.S.

The Gröbner fan G(M) of a matroid is the outer normal fan of the matroid polytope P_M. There is an equivalence relation on vectors in \( \mathbb{R}^{n+1}/\mathbb{R}.1 \), where \( u \sim v \) if and only if \( u \) and \( v \) achieve their maximum value on the same face of P_M. The equivalence classes form the relative interiors of convex polyhedral cones. For a face S of P_M, we will denote by \( \sigma_S \) the cone obtained by taking the closure of the equivalence class of vectors attaining their maximum value on the face S.

Finally, the Bergman fan \( \mathfrak{B}(M) \) is the subfan of the Gröbner fan G(M) consisting of those cones \( \sigma_S \) for which the degeneration matroid M_S is loopfree, i.e. the union of its bases is the complete ground set. The support of \( \mathfrak{B}(M) \) of the Bergman fan is the tropical linear space trop(M), by [9, Corollary 4.2.11]. Let L be the lineality space of a tropical linear space trop(M) ⊂ \( \mathbb{R}^{n+1}/\mathbb{R}.1 \), i.e. L is a subspace of \( \mathbb{R}^{n+1} \) such that \( \mathfrak{B}(M) \) is invariant under translations by vectors in L. Then by [6, Lemma 3.3] \( \dim(L) \) coincides with the number of connected components of M. In particular, if M is connected then L is spanned by (1, ..., 1). More details about the fans G(M) and \( \mathfrak{B}(M) \) can be found in [8] and [11].
of flats

A particular, for a finite reflection arrangement

fan and the minimal nested set fan coincide for finite reflection arrangements.

(2) In [1, Theorem 1.2] Ardila, Reiner and Williams prove that the Bergman

while

result due to Feichtner and Sturmfels gives a combinatorial criterion for the

fans

Theorem 3.4. [4, Proposition 5.3] For a loopfree matroid the minimal

nested set fan \( \Sigma_{\min}(M) \) equals the Bergman fan \( \mathfrak{B}(M) \) if and only if the

matroid \( M[F,G] \) is connected for every pair of flats \( F \subset G \) with \( G \) connected.

Example 3.5. (1) For an arrangement \( A \) of \( n \) lines in \( \mathbb{P}^2 \), the Bergman

fan and the nested set fan coincide in all cases but the following: There

exists a line \( L \) in \( A \) and points \( a, b \in L \) such that each remaining line passes

through \( a \) or \( b \). In this case \( \mathfrak{B}(M_A) \neq \Sigma_{\min}(M_A) \). In fact, the wonderful

compactification \( X_{\text{wnd}}(A) \) is just the blow-up of \( \mathbb{P}^2 \) in the two points \( a \) and

\( b \), while \( X_{\text{vc}}(A) \) is obtained by blowing-down the strict transform of \( L \) in

\( X_{\text{wnd}}(A) \), hence \( X_{\text{vc}}(A) \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).

(2) In [1, Theorem 1.2] Ardila, Reiner and Williams prove that the Bergman

fan and the minimal nested set fan coincide for finite reflection arrangements.

In particular, for a finite reflection arrangement \( A \) the compactifications

\( X_{\text{wnd}}(A) \) and \( X_{\text{vc}}(A) \) coincide. Thus, we have \( X_{\text{vc}}(A_n) = X_{\text{wnd}}(A_n) \) for the

braid arrangement \( A_n \) defined in Example 2.1.

(3) The Deligne-Mumford-Knudsen compactification \( \overline{M}_{0,n} \) coincides with

the minimal wonderful compactification of the complement of the complex

braid arrangement \( A_{n-2} \) by [2, Section 4.3].

\diamond

4. The span of cones in the Bergman fan

In this section we prove that distinct maximal cones in the Bergman fan

of a matroid span distinct linear spaces. This will be useful in the proof of

our main theorem.

Recall that every cone \( \sigma_F \) in the fine subdivision \( \Sigma(M) \) is given by a chain

of flats

\[ F : \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_d \subsetneq F_{d+1} \subset E(M). \]
We will decompose such a chain in the following way: set \( I_1^F = F_1 \) and \( I_j^F = F_j \setminus F_{j-1} \) for \( j \) in \( \{2, \ldots, d + 1\} \). Then we can rewrite the chain \( \mathcal{F} \) as
\[
\mathcal{F} : \emptyset \subsetneq I_1^F \subsetneq I_2^F \cup I_1^F \subsetneq \cdots \subsetneq I_d^F \cup I_{d-1}^F \cup \cdots \cup I_1^F \subsetneq E(M).
\]

**Proposition 4.1.** Let \( M \) be a loopfree matroid on the ground set \( E(M) = \{0, 1, \ldots, n\} \) of rank \( r(M) = r + 1 \), and let \( \sigma_F \) and \( \sigma_G \) be two maximal cones in the fine subdivision \( \Sigma(M) \subseteq \mathbb{R}^{n+1}/\mathbb{R} \cdot 1 \) such that the linear spans \( \langle \sigma_F \rangle \) and \( \langle \sigma_G \rangle \) coincide. Then there exists a cone \( \sigma \) in the Bergman fan \( \mathcal{B}(M) \) containing both of them, i.e. \( \sigma_F \cup \sigma_G \subseteq \sigma \).

**Proof.** Let us first show that \( \langle \sigma_F \rangle = \langle \sigma_G \rangle \) implies that \( \{I_j^F : j = 1, \ldots, r + 1\} = \{I_j^G : j = 1, \ldots, r + 1\} \).

Hence assume \( \langle \sigma_F \rangle = \langle \sigma_G \rangle \) for two maximal cones \( \sigma_F \) and \( \sigma_G \) in \( \Sigma(M) \) given by the chains
\[
\mathcal{F} : \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E(M)
\]
and
\[
\mathcal{G} : \emptyset \subsetneq G_1 \subsetneq \cdots \subsetneq G_r \subsetneq G_{r+1} = E(M),
\]
respectively.

Recall that the cone \( \sigma_F \) associated to a chain \( \mathcal{F} \) is defined as \( \sigma_F = \text{cone}(v_F : F \in \mathcal{F}) + \mathbb{R} \cdot 1 \). Since every maximal chain of flats contains \( E(M) = \{0, 1, \ldots, n\} \) as the maximal element, the ray generated by \( 1 = (1, \ldots, 1) \) belongs to all maximal cones of \( \Sigma(M) \). In particular, \( \langle \sigma_F \rangle = \langle \sigma_G \rangle \) if and only if \( \langle \text{cone}(v_F : F \in \mathcal{F}) \rangle = \langle \text{cone}(v_G : G \in \mathcal{G}) \rangle \).

Now, the set of vectors \( (x_0, \ldots, x_n) \in \langle \text{cone}(v_F : F \in \mathcal{F}) \rangle \) such that \( x_i \in \{0, 1\} \) for all \( i \in \{0, \ldots, n\} \) are sums of incidence vectors
\[
v_{I_j}^F = \sum_{i \in I_j^F} e_i.
\]
Hence every \( v_{I_j}^F \) is a sum of suitable \( v_{I_j^G}^F \). This sum cannot involve more than one summand since all \( v_{I_j^G}^F \) are contained in \( \langle \text{cone}(v_F : F \in \mathcal{F}) \rangle \) by assumption and \( I_j^F \subseteq I_j^G \) implies \( I_j^F = I_j^G \). Hence we find \( \{v_{I_j}^F : j\} = \{v_{I_j^G} : j\} \) which implies our claim.

To each cone \( \sigma_F \) we can now associate the set \( \mathcal{B}(\sigma_F) \) of all bases of \( M \) of the form \( \{i_1, \ldots, i_{r+1} : i_j \in I_j^F\} \). It is straightforward to show that \( \mathcal{B}(\sigma_F) \) is in fact a subset of the bases \( \mathcal{B}(M) \) of \( M \).

Our next goal is to show that points in the relative interior of \( \sigma_F \) achieve their maximum value on the face \( S = \text{conv}(e_B : B \in \mathcal{B}(\sigma_F)) \) of \( P_M \). For a point \( p = (p_0, \ldots, p_n) \in \mathbb{R}^{n+1} \) whose image in \( \mathbb{R}^{n+1}/\mathbb{R} \cdot 1 \) lies in the relative interior of \( \sigma_F \) the entries \( p_i \) and \( p_j \) coincide if and only if there exists \( k \in \{1, \ldots, r + 1\} \) such that \( i, j \in I_k^F \). In particular the point \( p \) has \( r + 1 \) distinct entries \( x_1, \ldots, x_{r+1} \) satisfying \( x_i > x_{i+1} \), and therefore \( \langle p, e_B \rangle = \sum_{i=1}^{r+1} x_i \) for all \( B \in \mathcal{B}(\sigma_F) \). Now let \( B \in \mathcal{B}(M) \) be an arbitrary basis of the matroid \( M \) and \( \lambda_j = |B \cap I_j^F| \) for \( j \in \{1, \ldots, r + 1\} \). The sum \( \sum_{j=1}^{r+1} \lambda_j \) is equal to \( r + 1 \) since \( |B| = r + 1 \). Moreover for each \( F_k \) in the
Theorem 4.2. Let $B$ be a loopfree matroid and $\mathcal{B}(M)$ its Bergman fan. Then distinct maximal cones of $\mathcal{B}(M)$ span distinct linear spaces.

Proof. Suppose there are distinct maximal cones $\sigma_1$ and $\sigma_2$ in the Bergman fan $\mathcal{B}(M)$ such that $\langle \sigma_1 \rangle = \langle \sigma_2 \rangle$. Since the fine subdivision refines the Bergman fan, there are maximal cones $a_1$ and $a_2$ in $\Sigma(M)$ such that $a_1 \subseteq \sigma_1$ and $a_2 \subseteq \sigma_2$. In particular $\dim a_1 = \dim \sigma_1$ and $\dim a_2 = \dim \sigma_2$ and hence...
the classes \((a_1) = (\sigma_1)\) and \((a_2) = (\sigma_2)\). Since the linear hulls of \(\sigma_1\) and \(\sigma_2\) coincide, so do the linear hulls of \(a_1\) and \(a_2\). Therefore the cones \(a_1\) and \(a_2\) are maximal cones in the fine subdivision \(\Sigma(M)\) whose linear hulls coincide. By Proposition 4.1 there exists a cone \(\sigma\) in the Bergman fan \(\mathcal{B}(M)\) which contains the union \(a_1 \cup a_2\). In particular \(a_1 \subseteq \sigma_1 \cap \sigma\) and \(a_2 \subseteq \sigma_2 \cap \sigma\). That means \(\sigma\) and \(\sigma_i\) are cones in the Bergman fan which intersect in full dimension. Hence \(\sigma_1 = \sigma = \sigma_2\).

### 5. Extending morphisms between arrangement complements

Let \(\mathcal{A}\) be a finite arrangement of hyperplanes in projective space \(\mathbb{P}(V)\), where \(V\) is a vector space of dimension \(d + 1\) over an arbitrary ground field \(K\). We will now show our main result on extension of endomorphisms of \(\Omega_\mathcal{A}\), using the notation from Section 2.

**Theorem 5.1.** (i) If the hyperplane arrangement \(\mathcal{A}\) is essential and connected, then every dominant morphism \(f : \Omega_\mathcal{A} \to \Omega_\mathcal{A}\) can be extended to a morphism on the visible contour compactification \(\overline{f} : X_{vc}(\mathcal{A}) \to X_{vc}(\mathcal{A})\).

(ii) If \(\mathcal{A}\) is essential and connected, then every automorphism of \(\Omega_\mathcal{A}\) extends to an automorphism of its visible contour compactification \(X_{vc}(\mathcal{A})\).

**Proof.** (i) Let \(f : \Omega_\mathcal{A} \to \Omega_\mathcal{A}\) be a dominant endomorphism and denote by \(j : \Omega_\mathcal{A} \to T\) the natural embedding of \(\Omega_\mathcal{A}\) into its intrinsic torus as in section 2. Let \(l_0, \ldots, l_n\) are the linear forms associated to the hyperplanes in \(\mathcal{A}\), and let \(x_0, \ldots, x_n\) be the basis of the character group \(N^*\) of \(T\) such that \(j\) is given by \(x_i \mapsto l_i/l_0\).

Since \(f\) is dominant, the associated map on coordinate rings \(f^* : \mathcal{O}(\Omega_\mathcal{A}) \to \mathcal{O}(\Omega_\mathcal{A})\) is injective. As the multiplicative group \(\mathcal{O}(\Omega_\mathcal{A})^*/K^*\) is generated by the classes \(\left[\frac{k}{l_0}\right]\) for \(k\) in \(\{1, \ldots, n\}\) we find that \(f^*\left(\frac{k}{l_0}\right) = \lambda, \prod_{j=1}^n \left(\frac{k}{l_0}\right)^{a_{ij}}\) for integers \(a_{ij}\) and \(\lambda_i \in K^*\). In terms of the coordinates \(x_1, \ldots, x_n\), the matrix \(A = (a_{ij})\) defines an endomorphism of \(N^*\). Hence it gives rise to a torus homomorphism \(h : T \to T\). Let \(\lambda\) be the \(K\)-rational point of \(T\) with coordinates \((\lambda_1, \ldots, \lambda_n)\), and write \(t_\lambda : T \to T\) for translation by \(\lambda\).

Putting \(g = t_\lambda \circ h\), we have a commutative diagram:

\[
\begin{array}{ccc}
\Omega_\mathcal{A} & \longrightarrow & T \\
\downarrow f & & \downarrow g \\
\Omega_\mathcal{A} & \longrightarrow & T \\
\end{array}
\]

We claim that \(g\) extends to a morphism of the toric variety \(Y_{\mathcal{B}(\mathcal{M}_A)}\). In fact, since the natural action of the intrinsic torus on itself extends to \(Y_{\mathcal{B}(\mathcal{M}_A)}\), so does the automorphism \(t_\lambda\). Therefore we only need show that the group homomorphism \(h\) extends as well. In order to prove this, we will show that the linear endomorphism induced by \(A\) on the cocharacter space \(N_R\) is compatible with the fan \(\mathcal{B}(\mathcal{M}_A)\) and hence gives rise to a toric morphism \(\overline{g} : Y_{\mathcal{B}(\mathcal{M}_A)} \to Y_{\mathcal{B}(\mathcal{M}_A)}\) extending \(g\).

The linear map \(a : N_R \to N_R\) maps \(\text{trop}(\Omega_\mathcal{A})\) to \(\text{trop}(\Omega_\mathcal{A})\). By [12, Proposition 3.1], this is a surjection \(a : \text{trop}(\Omega_\mathcal{A}) \to \text{trop}(\Omega_\mathcal{A})\). In order to show that \(a\) is compatible with the fan structure on \(\mathcal{B}(\mathcal{M}_A)\), we first note
that all vectors of the standard basis $e_1, \ldots, e_n$ of $N_R$ lie in $\mathrm{trop}(\Omega_A)$. Since $a$ is surjective on $\mathrm{trop}(\Omega_A)$ the linear map $a$ is an element of $\mathrm{GL}_n(R)$, and it lies in $\mathrm{GL}_n(Z)$ if and only if the dominant map $f$ is an automorphism of $\Omega_A$.

For a maximal cone $\sigma$ in $\mathcal{B}(M_A)$ of dimension $\dim(\sigma) = \dim(\Omega_A) = d$, the image $a(\sigma)$ is again a $d$-dimensional cone in $N_R$. We need to show that $a(\sigma)$ is contained in a cone of $\mathcal{B}(M_A)$. Suppose $a(\sigma)$ intersects two different maximal cones $\sigma_1$ and $\sigma_2$ in $\mathcal{B}(M_A)$ in their relative interiors. Since $d = \dim(a(\sigma)) = \dim(\sigma_1) = \dim(\sigma_2)$ it follows that the linear hulls spanned by $\sigma_1$ and $\sigma_2$ coincide. This is a contradiction to Theorem 4.2. In particular $a$ is a homomorphism of fans and hence $h$ extends to a morphism $h : Y_{\mathcal{B}(M_A)} \rightarrow Y_{\mathcal{B}(M_A)}$. By restricting $h$ to the closure of $\Omega_A$ in the toric variety $Y_{\mathcal{B}(M_A)}$ we get the required extension.

(ii) follows by applying (i) to $f$ and to its inverse morphism. □

Note that in the proof of the previous theorem we could also have argued with the fact that the Bergman fan is the coarsest fan structure on $\mathrm{trop}(\Omega_A)$. The argument given here is more intrinsic, and we hope that Theorem 4.2 is also useful for other purposes.

**Corollary 5.2.** If the hyperplane arrangement $A$ is essential and connected, every dominant morphism $f : \Omega_A \rightarrow \Omega_A$ is finite.

**Proof.** Let $f : \Omega_A \rightarrow \Omega_A$ be a dominant endomorphism. Then by Theorem 5.1, $f$ extends to a morphism $\overline{f} : X_{vc}(A) \rightarrow X_{vc}(A)$ and this yields the following Cartesian diagram:

$$
\begin{array}{ccc}
\Omega_A & \longrightarrow & X_{vc}(A) \\
\uparrow f & & \uparrow \overline{f} \\
\Omega_A & \longrightarrow & X_{vc}(A)
\end{array}
$$

Since $\overline{f}$ is proper, so is $f$. As a proper morphism of affine varieties $f$ is indeed finite. □

In view of the previous Theorem 3.4 by Feichtner and Sturmfels, we also have the following corollary.

**Corollary 5.3.** Assume that $A$ is essential and connected, and that $M[F,G]$ is connected for every pair of flats $F$ and $G$ in $\mathcal{L}(M_A)$ with $G$ connected and $F \subset G$. Then every dominant morphism $f : \Omega_A \rightarrow \Omega_A$ extends to a morphism on the wonderful compactification $\overline{f} : X_{\text{wnd}}(A) \rightarrow X_{\text{wnd}}(A)$.

An important example where the conditions of this corollary are fulfilled is the following one.

**Corollary 5.4.** Assume that $K = \mathbb{F}_q$ is a finite field and $A$ is the arrangement consisting of all $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^n_F$, so that $\Omega_A$ is Drinfeld’s half-space over $\mathbb{F}_q$.

(i) Every dominant morphism $f : \Omega_A \rightarrow \Omega_A$ extends to a morphism $\overline{f} : X_{\text{wnd}}(A) \rightarrow X_{\text{wnd}}(A)$ on the wonderful compactification. In particular, $f$ is a finite morphism.
(ii) Every automorphism $f : \Omega_A \to \Omega_A$ extends to an automorphism $\tilde{f} : X_{\text{wind}}(A) \to X_{\text{wind}}(A)$.

Proof. Let $M_A$ be the matroid associated to $A$ and $F$ and $G$ two flats of $M$, such that $F \subset G$. Then $M[F,G]$ is the matroid associated to the full arrangement in $\mathbb{P}^{r(G)-r(F)-1}$, where $r(F)$ and $r(G)$ denotes the rank of the flats $F$ and $G$ respectively. In particular, $M[F,G]$ is connected for all pairs of flats $F \subset G$ of $M_A$. Therefore in the case of the Drinfeld’s half-space the fans $\mathcal{B}(M_A)$ and $\Sigma_{\text{min}}(M_A)$ coincide by Theorem 3.4 and, in particular, so do the visible contours and the wonderful compactification. Hence our claims are a direct consequence of Theorem 5.1 and Corollary 5.2. □

In particular, Corollary 5.4 gives an alternative proof of [13, Proposition 2.1] without using analytic geometry. Moreover, it generalizes this result to a large class of arrangement complements.

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