STIRLING PERMUTATIONS, MARKED PERMUTATIONS AND STIRLING DERANGEMENTS

GUAN-HUEI DUH, YEN-CHI ROGER LIN, SHI-MEI MA, AND YEONG-NAN YEH

Abstract. In this paper we introduce the definition of marked permutations. We first present a bijection between Stirling permutations and marked permutations. We then present an involution on Stirling derangements. Furthermore, we present a symmetric bivariate enumerative polynomials on $r$-colored marked permutations. Finally, we give an explanation of $r$-colored marked permutations by using the language of combinatorial objects.

1. Introduction

Let $S_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Let $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$. An ascent of $\pi$ is an entry $\pi_i$, $i \in \{2, 3, \ldots, n\}$, such that $\pi_i > \pi_{i-1}$. Denote by $ASC(\pi)$ the set of all ascents of $\pi$, and $\text{asc}(\pi) = |ASC(\pi)|$. For example, $\text{asc}(5312) = |\{2, 4\}| = 2$. The classical Eulerian polynomial is defined by

$$A_n(x) = \sum_{\pi \in S_n} x^{\text{asc}(\pi)}.$$  

Set $A_0(x) = 1$. The exponential generating function for $A_n(x)$ is

$$A(x, t) = \sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1 - x}{e^{t(x-1)} - x}. \quad (1)$$

Stirling permutations were introduced by Gessel and Stanley [7]. Let $[n]_2$ denote the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. A Stirling permutation of order $n$ is a permutation of $[n]_2$ such that every entry between the two occurrences of $i$ is greater than $i$ for each $i \in [n]$. Various statistics on Stirling permutations have been extensively studied in the past decades, including descents [3, 7, 8], plateaux [1, 3, 8], blocks [13], ascent plateaux [10, 11] and cycle ascent plateaux [12].

Denote by $Q_n$ the set of Stirling permutations of order $n$ and let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in Q_n$. An occurrence of an ascent plateau is an entry $\sigma_i$, $i \in \{2, 3, \ldots, 2n-1\}$, such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$ (see [10]). Let $\text{AP}(\sigma)$ be the set of all ascent plateaux of $\sigma$, and $\text{ap}(\sigma) = |\text{AP}(\sigma)|$. As an example, $\text{ap}(221133) = |\{3\}| = 1$. Let

$$N_n(x) = \sum_{\pi \in Q_n} x^{\text{ap}(\pi)}.$$  

Set $N_0(x) = 1$. From [10] Theorem 2], we have

$$\sum_{n \geq 0} N_n(x) \frac{t^n}{n!} = \sqrt{A(x, 2t)}. \quad (2)$$

2010 Mathematics Subject Classification. Primary 05A15; Secondary 26C10.

Key words and phrases. Stirling permutations; Marked permutations; Stirling derangements; Increasing trees.
Let \( w = w_1 w_2 \cdots w_n \) be a word on \([n]\). A \textit{left-to-right minimum} of \( w \) is an element \( w_i \) such that \( w_i < w_j \) for every \( j \in [i-1] \) or \( i = 1 \); a \textit{right-to-left minimum} of \( w \) is an element \( w_i \) such that \( w_i < w_j \) for every \( j \in \{i+1, i+2, \ldots, n\} \) or \( i = n \). Let \( \text{LRMIN}(w) \) and \( \text{RLMIN}(w) \) denote the set of entries of left-to-right minima and right-to-left minima of \( w \), respectively. Set \( \text{lrmin}(w) = |\text{LRMIN}(w)| \) and \( \text{rlmin}(w) = |\text{RLMIN}(w)| \). For example, \( \text{lrmin}(223311) = |\{1,2\}| = 2 \) and \( \text{rlmin}(223311) = |\{1\}| = 1 \).

Motivated by (2), we now introduce the definition of marked permutations. Given a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n \), a \textit{marked permutation} is a permutation with marks on some of its non-left-to-right minima. An element \( i \) is denoted by \( \overline{i} \) when it is marked. Let \( \mathcal{S}_n \) denote the set of marked permutations of \([n]\). For example, \( \mathcal{S}_1 = \{1\} \), \( \mathcal{S}_2 = \{12, 1\overline{2}, 21\} \) and

\[
\mathcal{S}_3 = \{123, 1\overline{3}2, 213, 2\overline{3}1, 312, 321, 1\overline{2}3, 1\overline{2}, 1\overline{3}2, 1\overline{3}\overline{2}, 21\overline{3}, 2\overline{3}1, 31\overline{2}\}.
\]

This paper is organized as follows. In Section 2, we present a bijection between Stirling permutations and marked permutations. In Section 3, we use an involution to prove an identity between Stirling derangements and perfect matchings. In Section 4, we present a symmetric bivariate enumerative polynomials on \( r \)-colored marked permutations. And in Section 5, we use the language of combinatorial objects to give an explanation of marked permutations.

## 2. A bijection between Stirling permutations and marked permutations

Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in \mathcal{Q}_n \) be a Stirling permutation of order \( n \). An entry \( k \) of \( \sigma \) is called an \textit{even indexed entry} (resp. \textit{odd indexed entry}) if the first appearance of \( k \) occurs at an even (resp. odd) position of \( \sigma \). Let \( \text{EVEN}(\sigma) \) (resp. \( \text{ODD}(\sigma) \)) denote the set of even (resp. odd) indexed entries of \( \sigma \), \( \text{even}(\sigma) = |\text{EVEN}(\sigma)| \), and \( \text{odd}(\sigma) = |\text{ODD}(\sigma)| \). For example, \( \text{even}(221331) = |\{3\}| = 1 \) and \( \text{odd}(221331) = |\{1,2\}| = 2 \).

Given \( \pi \in \mathcal{S}_n \), let \( \text{MARK}(\pi) \) denote the set of marked entries of \( \pi \), and \( \text{mark}(\pi) = |\text{MARK}(\pi)| \). The statistics \( \text{ASC}(\pi), \text{asc}(\pi), \text{LRMIN}(\pi) \) and \( \text{lrmin}(\pi) \) are defined by forgetting the marks of \( \pi \). For example, \( \text{asc}(3521467) = 4 \), \( \text{mark}(3521467) = 2 \) and \( \text{lrmin}(3521467) = 3 \).

A block in an element of \( \mathcal{Q}_n \) or \( \mathcal{S}_n \) is a substring which begins with a left-to-right minimum, and contains exactly this one left-to-right minimum; moreover, the substring is maximal, i.e. not contained in any larger such substring. It is easily derived by induction that any Stirling permutation or permutation has a unique decomposition as a sequence of blocks.

**Example 1.** The block decompositions of 34664325527711 and 3462571 are respectively given by [346643][255277][11] and [346][257][1].

We now present the first result of this paper.

**Theorem 2.** For \( n \geq 1 \), we have

\[
\sum_{\sigma \in \mathcal{Q}_n} q^{\text{lrmin}(\sigma)} x^{\text{ap}(\sigma)} y^{\text{even}(\sigma)} = \sum_{\pi \in \mathcal{S}_n} q^{\text{lrmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)}.
\]
Proof. We prove a stronger result: there is a one-to-one correspondence between the set statistics (LRMIN, AP, EVEN) on Stirling permutations and (LRMIN, ASC, MARK) on marked permutations.

Define

\[ Q(n; X, Y, Z) = \{ \sigma \in Q_n \mid \text{LRMIN}(\sigma) = X, \text{AP}(\sigma) = Y, \text{EVEN}(\sigma) = Z \}, \]
\[ S(n; X, Y, Z) = \{ \pi \in \overline{S}_n \mid \text{LRMIN}(\pi) = X, \text{ASC}(\pi) = Y, \text{MARK}(\pi) = Z \}, \]

where \( X, Y, Z \subseteq [n] \). Clearly, they form partitions of all Stirling permutations and marked permutations respectively.

Now we start to construct a bijection, denoted by \( \Phi \), between Stirling permutations and marked permutations. In addition, it maps each \( Q(n; X, Y, Z) \) onto \( S(n; X, Y, Z) \). When \( n = 1 \), it is clear that \( 11 \in Q(1; \{1\}, \emptyset, \emptyset) \) and \( 1 \in S(1; \{1\}, \emptyset, \emptyset) \). Hence setting \( \Phi(11) = 1 \) satisfies the requirement.

Fix \( m \geq 2 \), and assume that \( \Phi \) is a bijection between \( Q(m-1; X, Y, Z) \) and \( S(m-1; X, Y, Z) \) for all possible \( X, Y, Z \). Let \( \sigma' \in Q_m \) be obtained from some \( \sigma \in Q(m-1; X, Y, Z) \) by inserting the substring \( mm \) into \( \sigma \). By the assumption, \( \Phi(\sigma) = \pi \in S(m-1; X, Y, Z) \).

If \( mm \) is placed at the front of \( \sigma \), that is, \( \sigma' = mm\sigma \), then we let \( \Phi(\sigma') = m\pi \). In this case, we have \( \sigma' \in Q(m; X \cup \{m\}, Y, Z) \) and \( \Phi(\sigma') \in S(m; X \cup \{m\}, Y, Z) \). Notice that this is the only way to produce a new block.

Otherwise, suppose \( \sigma' \) is obtained from \( \sigma \) by inserting \( mm \) into the \( p^{th} \) block. Let \( r \) be the left-to-right minimum contained in the \( p^{th} \) block of \( \sigma \). There are three possible cases:

(i) If \( mm \) is inserted immediately before the second \( r \), then \( \Phi(\sigma') = \pi' \) is obtained by inserting a marked \( m \) at the end of the \( p^{th} \) block of \( \pi \). Note that \( m \) is an additional even indexed entry, as well an ascent plateau, of \( \sigma' \) after inserting \( mm \) into \( \sigma \). Meanwhile, \( m \) is a marked element, as well an ascent, of \( \pi' \). Hence \( \sigma' \in Q(m; X, Y \cup \{m\}, Z \cup \{m\}) \) and \( \Phi(\sigma') \in S(m; X, Y \cup \{m\}, Z \cup \{m\}) \).

(ii) If \( mm \) is inserted immediately before \( s \), \( s \neq r \), then \( \Phi(\sigma') = \pi' \) is obtained by inserting \( m \) or \( \overline{m} \) into the \( p^{th} \) block of \( \pi \) such that \( m \) is immediately before \( s \). The inserted entry \( m \) is marked if and only if \( m \) is an even indexed entry of \( \sigma' \). When \( s \in Y \), let \( Y' = (Y \cup \{m\}) \setminus \{s\} \). Otherwise, let \( Y' = Y \cup \{m\} \). When \( m \) is an even indexed entry, let \( Z' = Z \cup \{m\} \). Otherwise, let \( Z' = Z \). In each possible case, we see that \( \sigma' \in Q(m; X, Y', Z') \) and \( \Phi(\sigma') \in S(m; X, Y', Z') \).

(iii) If \( mm \) is inserted at the end of the \( p^{th} \) block, then \( \Phi(\sigma') = \pi' \) is obtained by inserting an unmarked \( m \) at the end of the \( p^{th} \) block of \( \pi \). Note that \( \sigma \) does not gain any additional even indexed entry after inserting \( m \), but obtain the ascent plateau of \( m \). On the other hand, \( m \) is a new ascent of \( \pi' \) after inserting \( m \) into \( \pi \). Hence \( \sigma' \in Q(m; X, Y \cup \{m\}, Z) \) and \( \Phi(\sigma') \in S(m; X, Y \cup \{m\}, Z) \).

The above argument shows that \( \Phi(Q_n) \subseteq \overline{S}_n \), and that \( \Phi \) is one-to-one on \( Q_n \). Since the cardinality of \( Q_n \) is the same as that of \( \overline{S}_n \), \( \Phi \) must be a bijection between \( Q_n \) and \( \overline{S}_n \). By induction, we see that \( \Phi \) is the desired bijection between Stirling permutations and marked permutations. \( \square \)
Example 3. Consider $\sigma = 266255133441 \in \mathbb{Q}_6$. The correspondence between $\sigma$ and $\Phi(\sigma)$ is built up as follows:

\[
\begin{align*}
[11] & \iff [1] \\
[22][11] & \iff [2] [1] \\
[22][1331] & \iff [2] [1 3] \\
[22][134431] & \iff [2] [1 4 3] \\
[2255][134431] & \iff [2] [5 1 4 3] \\
[266255][134431] & \iff [2 5 6] [1 4 3]
\end{align*}
\]

Exploiting the bijection $\Phi$ used in the proof of Theorem 2, we also get the following result.

Theorem 4. For $n \geq 1$, we have

\[
\sum_{\sigma \in \mathbb{Q}_n} x^{\text{ap}(\sigma)} (-1)^{\text{even}(\sigma)} = \sum_{\pi \in \mathbb{S}_n} x^{\text{asc}(\pi)} (-1)^{\text{mark}(\pi)} = 1. \tag{4}
\]

Proof. From Theorem 2 we get the first equality of (4). For the last equality of (4), we will consider an involution $\iota$ on $\mathbb{S}_n$ as follows. For a marked permutation $\pi$ that has non-left-to-right minima, define $\iota(\pi)$ to be the marked permutation obtained from $\pi$ by changing the marking of the smallest non-left-to-right minimum of $\pi$. For example, when $\pi = 25314$, then $\iota(\pi) = 25314$. This map $\iota$ is clearly a sign-reversing involution because the number of marks is either plus 1 or minus 1. It is also clear that $\iota$ preserves the ascent statistic since no entry leaves its position. The only marked permutation in $\mathbb{S}_n$ which cannot be mapped by $\iota$ is the one in which every entry is a left-to-right minimum, i.e., the marked permutation $n(n-1) \cdots 21$, whose ascent is 0. This completes the proof. $\square$

Theorem 5. We have

\[
\sum_{n \geq 0} \sum_{\pi \in \mathbb{S}_n} q^{\text{lrmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)} \frac{t^n}{n!} = \frac{A(x, t(1+y))^{1+y}}{1+y},
\]

where $A(x, z)$ is the exponential generating function given by (1).

Proof. Combining [2, Proposition 7.3] and the fundamental transformation introduced by Foata and Schützenberger [6], we have

\[
\sum_{n \geq 0} \sum_{\pi \in \mathbb{S}_n} q^{\text{lrmin}(\pi)} x^{\text{asc}(\pi)} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\pi \in \mathbb{S}_n} q^{\text{cyc}(\pi)} x^{\text{exc}(\pi)} \frac{t^n}{n!} = A(x, t)^q.
\]

For a permutation $\pi$ in $\mathbb{S}_n$ with $\text{lrmin}(\pi) = \ell$, there are $n - \ell$ entries that could be either marked or not. Therefore, we have

\[
\sum_{n \geq 0} \sum_{\pi \in \mathbb{S}_n} q^{\text{lrmin}(\pi)} x^{\text{asc}(\pi)} y^{\text{mark}(\pi)} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\pi \in \mathbb{S}_n} q^{\text{lrmin}(\pi)} x^{\text{asc}(\pi)} (1 + y)^{n - \text{lrmin}(\pi)} \frac{t^n}{n!}
\]

\[
= \sum_{n \geq 0} \sum_{\pi \in \mathbb{S}_n} \left( \frac{q}{1 + y} \right)^{\text{lrmin}(\pi)} x^{\text{asc}(\pi)} \frac{(t(1+y))^n}{n!}
\]

\[
= A(x, t(1+y))^{1+y}.
\]

$\square$
Combining Theorem 2 and Theorem 5 we have
\[ \sum_{n \geq 0} \sum_{\sigma \in Q_n} y^{\text{even}(\sigma)} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\pi \in S_n} y^{\text{mark}(\pi)} \frac{t^n}{n!} = A(1, t(1 + y)) \frac{1}{1+y}. \]

Let \([n]^{i}\) be the (signless) Stirling number of the first kind, i.e., the number of permutations of \(\mathcal{S}_n\) with \(i\) cycles. Note that \(\text{even}(\sigma) + \text{odd}(\sigma) = n\) for \(\sigma \in Q_n\). Let
\[ E_n(p, q) = \sum_{\sigma \in Q_n} p^{\text{odd}(\sigma)} q^{\text{even}(\sigma)}. \]

Now we present the following result.

**Theorem 6.** For \(n \geq 1\), we have
\[ E_n(p, q) = \sum_{k=0}^{n} \binom{n}{k} p^k (p + q)^{n-k}. \] (5)

In particular, \(E_n(1, 1) = (2n - 1)!!\), \(E_n(p, 0) = n! p^n\), \(E_n(1, -1) = 1\), \(E_n(-1, 1) = (-1)^n\) for \(n \geq 2\).

**Proof.** There are two ways in which a permutation \(\sigma' \in Q_n\) with \(\text{even}(\sigma') = k\) can be obtained from a permutation \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n-2} \in Q_{n-1}\).

(i) If \(\text{even}(\sigma) = k - 1\), then we can insert the two copies of \(n\) right after \(\sigma_{2i-1}\), where \(i \in [n-1]\).

(ii) If \(\text{even}(\sigma) = k\), then we can insert the two copies of \(n\) in the front of \(\sigma\) or right after \(\sigma_{2i}\), where \(i \in [n-1]\).

Clearly, \(E(1, 0) = 1\) corresponds to \(11 \in Q_1\). Therefore, the numbers \(E(n, k)\) satisfy the recurrence relation
\[ E(n, k) = (n - 1)E(n - 1, k - 1) + nE(n - 1, k), \] (6)

with the initial conditions \(E(1, 0) = 1\) and \(E(1, k) = 0\) for \(k \geq 1\). It follows from (6) that
\[ E_n(q) = (n + (n - 1)q)E_{n-1}(q) \]

for \(n \geq 1\), with the initial value \(E_0(q) = 1\). Thus
\[ E_n(q) = \prod_{i=1}^{n} (i + (i - 1)q). \]

Recall that
\[ \sum_{k=0}^{n} \binom{n}{k} x^k = x(x + 1) \cdots (x + n - 1). \]

Therefore,
\[ E_n(q) = \sum_{k=0}^{n} \binom{n}{k} (1 + q)^{n-k}, \]

which leads to (5), since
\[ E_n(p, q) = p^n E_n \left( \frac{q}{p} \right). \]

\[ \square \]
3. STIRLING DERANGEMENTS AND INCREASING BINARY TREES

Let \([k]^n\) denote the set of words of length \(n\) in the alphabet \([k]\). For \(\omega = \omega_1 \omega_2 \cdots \omega_n \in [k]^n\), the reduction of \(\omega\), denoted by \(\text{red}(\omega)\), is the unique word of length \(n\) obtained by replacing the \(i\)th smallest entry by \(i\). For example, \(\text{red}(33224547) = 22113435\).

Very recently, Ma and Yeh [12] introduced the definition of Stirling permutations of the second kind. A permutation \(\sigma\) of the multiset \([n]^2\) is a Stirling permutation of the second kind of order \(n\) whenever \(\sigma\) can be written as a nonempty disjoint union of its distinct cycles and \(\sigma\) has a standard cycle form that satisfies the following conditions:

(i) For each \(i \in [n]\), the two copies of \(i\) appear in exactly one cycle;
(ii) Each cycle is written with one of its smallest entry first and the cycles are written in increasing order of their smallest entries;
(iii) The reduction of the word formed by all entries of each cycle is a Stirling permutation.

In other words, if \((c_1, c_2, \ldots, c_{2k})\) is a cycle of \(\sigma\), then \(\text{red}(c_1 c_2 \cdot \cdot \cdot c_{2k}) \in Q_k\).

Let \(Q_n^2\) denote the set of Stirling permutations of the second kind of order \(n\). In the following discussion, we always write \(\sigma \in Q_n^2\) in standard cycle form. Let \((c_1, c_2, \ldots, c_{2k})\) be a cycle of \(\sigma\), where \(k \geq 2\). An entry \(c_i\) is called a cycle ascent plateau if \(c_i - 1 < c_i = c_{i+1}\), where \(2 \leq i \leq 2k - 1\). Denote by \(\text{cap}(\sigma)\) (resp. \(\text{cyc}(\sigma)\)) the number of cycle ascent plateaux (resp. cycles) of \(\sigma\). An entry \(k \in [n]\) be called a fixed point of \(\sigma\) if \((kk)\) is a cycle of \(\sigma\). Let \(\text{fix}(\sigma)\) denote the number of fixed points of \(\sigma\). Using the fundamental transformation of Foata and Schützenberger [6], we have

\[
\sum_{\sigma \in Q_n^2} q^{\text{lrmin}(\sigma)} x^{\text{cap}(\sigma)} y^{\text{bk}_2(\sigma)} = \sum_{\tau \in Q_n^2} q^{\text{cyc}(\tau)} x^{\text{cap}(\tau)} y^{\text{fix}(\tau)},
\]

where \(\text{bk}_2(\sigma)\) is the number of blocks of \(\sigma\) with length 2.

The exponential generating function could be obtained as follows.

**Theorem 7.**

\[
\sum_{n \geq 0} \sum_{\tau \in Q_n^2} q^{\text{cyc}(\tau)} x^{\text{cap}(\tau)} y^{\text{fix}(\tau)} \frac{t^n}{n!} = \left( A(x, t) e^{t(y-1)} \right)^q,
\]

where \(A(x, z)\) is the exponential generating function given by (1).

**Proof.** In the same spirit as Theorem 5 the generating function is equal to

\[
\sum_{n \geq 0} \sum_{\pi \in S_n} q^{\text{cyc}(\pi)} x^{\text{asc}(\pi)} y^{\text{fix}(\pi)} \frac{t^n}{n!},
\]

with the help of (7). The statistic \(\text{fix}(\pi)\) denotes the number of blocks of size 1 in \(\pi\).

We use the language of combinatorial objects as in the book of Flajolet and Sedgewick [5]. Take \(C\) to be the class of permutations in which the atom with label 1 is in the first place. Then \(\text{Set}(C)\) is exactly the class of permutations. Let the generating function of \(C\) be

\[
\text{Gen}(C; \text{asc}) := f(x, t) = t + x^2 \frac{t^2}{2!} + (x + x^2) \frac{t^3}{3!} + O(t^4),
\]
where the variable $x$ records the statistic ascent. The interpretation of $\text{Set}(C)$ tells us that $ef(x,t) = A(x,t)$. Moreover,

$$\sum_{n \geq 0} \sum_{\pi \in S_n} q^{\text{cyc}(\pi)} x^{\text{asc}(\pi)} \frac{t^n}{n!} = e^q f(x,t) = A(x,t)^q$$

is also straightforward. Now to take the statistic fix into account, the first-order term of $f(x,t)$ should be changed from $t$ to $yt$. Hence the generating function in question is

$$e^q(f(x,t) - t + yt) = (A(x,t) e^{t(y-1)})^q$$

A perfect matching of $[2n]$ is a set partition of $[2n]$ with blocks (disjoint nonempty subsets) of size exactly 2. Let $\mathcal{M}_{2n}$ be the set of matchings of $[2n]$, and let $M \in \mathcal{M}_{2n}$. The standard form of $M$ is a list of blocks $\{(i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n)\}$ such that $i_r < j_r$ for all $1 \leq r \leq n$ and $1 = i_1 < i_2 < \cdots < i_n$. Throughout this paper we always write $M$ in the standard form. If $(i_s,j_s)$ is such a block of $M$, then we say that $i_s$ (resp. $j_s$) is an ascending (resp. a descending) entry of $M$.

**Definition 8.** Let $h_k$ be the number of perfect matchings of $[4k]$ such that $2i-1$ and $2i$ are either both ascending or both descending for every $i \in [2k]$.

Let $i^2 = -1$. Following [15], we have

$$\sqrt{\sec(2iz)} = \sum_{n \geq 0} (-1)^n h_n \frac{z^{2n}}{(2n)!}.$$  

**Definition 9.** A Stirling derangement is a Stirling permutation without blocks of length 2.

Let $DQ_n$ be the set of Stirling derangements of order $n$, i.e., $DQ_n = \{\sigma \in Q_n | \text{bk}_2(\sigma) = 0\}$. The following result (in an equivalent form) has been algebraically obtained by Ma and Yeh [12, Page 15].

**Theorem 10.**

$$\sum_{\sigma \in DQ_n} (-1)^{\text{ap}(\sigma)} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^k h_k, & \text{if } n = 2k \text{ is even}. \end{cases}$$

In the rest of this section, we shall present a bijective proof of Theorem 10.

Let $\varphi$ be the bijection between permutations and increasing binary trees defined by the inorder traversal in depth-first search. The left-to-right minima of $\pi \in S_n$ will be the labels in the leftmost path of the corresponding trees. Let $T_n$ denote the set of bicolored increasing binary trees on $n$ nodes such that all the nodes in the leftmost path are white and the other nodes are white or black. Then $\varphi$ is a bijection from $\mathfrak{S}_n$ to $T_n$ if we match white nodes (resp. black nodes) with those unmarked letters (resp. marked letters.) Note that all of the left-to-right minima of a marked permutation are mapped to the nodes of the leftmost path of its increasing binary tree. These nodes will be called special nodes.
For example, the Stirling permutation $\sigma = 223315544166$ corresponds to the marked permutation $\pi = 231546$, which in turn corresponds to the bicolored increasing tree with 6 nodes in Figure 1. Note that the special nodes are labelled 1 and 2.

**A bijective proof of Theorem 10.**

Now we consider the composition $\varphi \circ \Phi$ from $Q_n$ to $T_n$, where $\Phi$ is defined in the proof of Theorem 2. It is easily observed that $k$ is an ascent plateau of $\sigma \in Q_n$ if and only if the node $k$ in the bicolored increasing binary tree $\varphi(\theta(\sigma))$ is non-special and does not have a left child. Also the Stirling derangements are mapped to those trees each of whose special nodes has a right child.

For trees in $T_n$, there are some trees that have a non-special node with only one child. An involution can be introduced on these trees: among the non-special nodes with only one child, find the one with the minimal label, then move its whole subtree to the other branch. An example on this involution is shown in Figure 2.

Following the discussion in the previous paragraph, we immediately see that this involution is sign-reversing if we attach the sign $(-1)^{ap(\sigma)}$ to the tree $\varphi(\theta(\sigma))$. 

![](image1.png)

**Figure 1.** The bicolored tree corresponded to $231546$

![](image2.png)

**Figure 2.** Involution on trees in $T_n$; the move is made on Node 5.
For those trees to which the above involution cannot be applied, all of its nodes have either zero or two children; these trees are called complete, whose number of nodes $n$ must be even, and it has the sign $(-1)^{ap(\sigma)} = (-1)^{n/2}$. From here we conclude that when $n$ is odd, these trees are sign-balanced. Our remaining task is to give a proper count on those complete bicolored increasing binary trees on $n = 2k$ nodes.

Let $\sigma \in DQ_n$ such that $\varphi(\theta(\sigma))$ is complete. Then the intermediate marked permutation

$$\pi = \theta(\sigma) = \pi_1 \pi_2 \ldots \pi_n$$

is reverse alternating, i.e., $\pi_1 < \pi_2 > \pi_3 < \cdots$. Note that in these situations the left-to-right minima of $\pi$ always occur at odd-indexed positions. We now apply the Foata transformation on $\pi$ by making its left-to-right minima as the head of each cycle; say $\pi \mapsto C_1 C_2 \ldots C_j$. Each cycle determines some pairings on $\pm[n]$: for any $C = (c_1 c_2 \ldots c_\ell)$, we pair $\overline{c_i}$ with $c_{i+1}$ for $1 \leq i < \ell$, and pair $\overline{c_\ell}$ with $c_1$. Since the number at the beginning of a cycle must be unmarked, we have a bijection between complete bicolored increasing binary trees on $n$ nodes with those special perfect matchings on $[4k] = [2n] \cong \pm[n]$. Using definition of the numbers $h_k$, we get the desired result. $\square$

**Example 11.** We use the ordering $\overline{1} < \overline{2} < 1 < \overline{2} < \cdots < \overline{n} < n$ on $\pm[n] \cong [2n] = [4k]$. The reverse alternating marked permutation $\pi = 3 \overline{5} 1 \overline{6} 2 \overline{4}$ corresponds to the following perfect matching on $\pm[6]$:

$$3\overline{5}/5\overline{3}/1\overline{6}/6\overline{1}/24/2\overline{1}.$$ 

Because $\pi$ is reverse alternating, the numbers $i$ and $\overline{i}$ are both ascending or both descending for all $i \in [n]$ in the corresponding perfect matching.

**4. On $r$-colored marked permutations**

The definition of marked permutations can easily be extended to the $r$-colored version for any positive integer $r$. We think that a marked permutation (resp. an ordinary permutation) is in the special case $r = 2$ (resp. $r = 1$) in which the unmarked elements are painted by the first color, while the others are painted by the second color. The definition for the $r$-colored marked permutations is similar to that in the two-colored situation, i.e., every left-to-right minimum of $\pi$ must be painted by the first color, which is always referred as white in this section. We find that on the $r$-colored marked permutations the ascent statistic is equidistributed with another statistic, which is defined below.

Let $\overline{\mathcal{S}}_n^{(r)}$ be the set of $r$-colored marked permutations of $[n]$. For $\pi \in \overline{\mathcal{S}}_n^{(r)}$, we define

$$\text{desrlmin}(\pi) = \left| \{ \pi_i \mid \pi_{i-1} > \pi_i \text{ for } i > 1, \text{ or } \pi_i \text{ is a right-to-left minimum of } \pi \} \right|.$$

**Theorem 12.** Let $r$ be a positive integer. The following polynomial is symmetric in the variables $x$ and $y$:

$$F_r(x,y) = \sum_{\pi \in \overline{\mathcal{S}}_n^{(r)}} x^{\text{asc}(\pi)+1} y^{\text{desrlmin}(\pi)}.$$
Proof. We will construct an involution $\psi$ on $\mathfrak{S}_n$ first. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$. Again we divide $\pi$ into several blocks by marking each right-to-left minimum of $\pi$ as the end of a block; we will view each block as a cycle. Now we rotate each block in order to list its largest number at the end of its block; then arrange these blocks so that these largest numbers are in decreasing order; lastly take the complement of every number to get the final result $\psi(\pi)$.

For example, let us go through step by step on the permutation $\pi = 214853697 \in \mathfrak{S}_9$:

$\pi = 214853697 \rightarrow 21|4853|6|97$  
$\rightarrow 12|5348|6|79$  
$\rightarrow 79|5348|6|12$  
$\rightarrow 31|5762|4|98 \mapsto \psi(\pi) = 315762498$.

A careful examination shows that $\psi$ is an involution on $\mathfrak{S}_n$, and $\text{asc}(\pi) + 1 = \text{desrlmin}(\psi(\pi))$.

For $r > 1$, we can take a certain algorithm to ensure that every number in $\psi(\pi)$ is still colored white. For example, consider the colors are taken modulo $r$ by addition in each block; or during the block rotation process we keep the colors fixed at their respective positions. Since the above involution $\psi$ does not depend on the coloring, we still have the result in symmetry. □

The generating function

$$F_r(x, y) := \sum_{\pi \in \mathfrak{S}_n^{(r)}} x^\text{asc}(\pi)+1 y^\text{desrlmin}(\pi)$$

can be obtained by using context-free grammars, as done by Dumont [4]. For $r = 1$, the grammar is $\{a \rightarrow ab, b \rightarrow ab, c \rightarrow ab, e \rightarrow ce\}$, which had been studied by Roselle [14]. Let $A(x, y, z, t)$ be a refinement for the Eulerian polynomials:

$$A(x, y, z, t) = zt + \sum_{n \geq 2} \sum_{\pi \in \mathfrak{S}_n} x^\text{asc}(\pi)+1 y^\text{des}(\pi) \frac{t^n}{n!}.$$ 

Then the generating function for $F_1(x, y)$ is shown in [4] to be

$$G_1(x, y, t) := \sum_{n \geq 0} F_1(x, y) \frac{t^n}{n!} = \exp(A(x, y, xyt)).$$

We define $G_r(x, y, t) := \sum_{n \geq 0} F_r(x, y) \frac{t^n}{n!}$. For any $r$-colored permutation, it can also be partitioned into blocks by the method described in the proof of Theorem [12]. Then we can assign those blocks whose end entries are painted by the same color into the same group. It is clear that the number of $r$-colored marked permutations of length $n$ is the same as those $r$-colored permutations of length $n$ whose right-to-left minima are all painted by any other color. Hence we have the identity:

$$G_1(x, y, rt) = (G_r(x, y, t))^r,$$

that is,

$$G_r(x, y, t) = (G_1(x, y, rt))^{1/r}.$$
5. An interpretation of \( r \)-colored marked permutations

Suppose a class of combinatorial objects \( \mathcal{C} \) has the generating function \( \text{Gen}(\mathcal{C}) = f(x, t) \), where \( x \) is the vector \( x = (x_1, x_2, \ldots, x_s) \), and the power of \( x_i \) records certain statistic \( \text{stat}_i \) on \( \mathcal{C} \). Here the generating function can be either ordinary or exponential, as an ordinary one can be seen as an exponential one with arbitrary labelling.

Recall that
\[
g(x, t, r_1, r_2, \ldots, r_k) := f(x, (r_1 + r_2 + \cdots + r_k)t)
\]
counts a weighted version of \( \mathcal{C} \), denoted by \( \mathcal{C}^k \), where each atom in an object in \( \mathcal{C} \) is weighted by a number in \([k]\). An object in \( \mathcal{C}^k \) with \( n \) atoms could be seen as \((C, w)\), where \( C \in \mathcal{C} \) and \( w \) is a map from \([n]\) to \([k]\). Then the power of \( x_i \) in \( g \) records \( \text{stat}_i(C) \), and the power of \( r_j \) records the number of atoms in \( C \) with weight \( j \), i.e., the cardinality of the pre-image \( w^{-1}(j) \).

It is well-known that for a given non-negative integer \( k \),
\[
f_{\text{seq}, k}(x, t) := f(x, t)^k
\]
counts \( \text{SEQ}_k(\mathcal{C}) \). Each object in \( \text{SEQ}_k(\mathcal{C}) \) is of the form of a \( k \)-sequence: \((C_1, C_2, \ldots, C_k)\) with each \( C_i \in \mathcal{C} \). The power of \( x_i \) in \( h_s \) records \( \text{stat}_i(C_1) + \text{stat}_i(C_2) + \cdots + \text{stat}_i(C_k) \). On the other hand,
\[
f_{\text{set}, k}(x, t) := \frac{f(x, t)^k}{k!}
\]
counts \( \text{SET}_k(\mathcal{C}) \). This induced class contains objects of the form of a \( k \)-set \( \{C_1, C_2, \ldots, C_k\} \) with each \( C_i \in \mathcal{C} \). The power of \( x_i \) in \( h_s \) also records \( \text{stat}_i(C_1) + \text{stat}_i(C_2) + \cdots + \text{stat}_i(C_k) \).

When \( \mathcal{C} \) has no objects of size 0, the power series
\[
f_{\text{seq}}(x, t) := \frac{1}{1 - f(x, t)}
\]
is well-defined, and it counts \( \text{SEQ}(\mathcal{C}) := \bigcup_{k=0}^{\infty} \text{SEQ}_k(\mathcal{C}) \); while the power series
\[
f_{\text{set}}(x, t) := \exp(f(x, t))
\]
counts \( \text{SET}(\mathcal{C}) := \bigcup_{k=0}^{\infty} \text{SET}_k(\mathcal{C}) \).

Suppose now that \( \mathcal{C} \) is a combinatorial objects with no objects of size 0, with generating function \( \text{Gen}(\mathcal{C}) = f(x, t) \). Observe the formal power series
\[
h(x, t) := \frac{1}{k} f(x, kt)
\]
has non-negative integer coefficients. This fact suggests that there should be some general method to induce a new class \( ^*k\mathcal{C} \) of combinatorial objects that is enumerated by \( h \). Suppose we have a rule to specify an atom from any object \( C \) in \( \mathcal{C} \); that atom will be called a distinguished atom of \( C \). For example, let the distinguished atom be the one with label 1. Then we could define \( ^*k\mathcal{C} \) to be a weighted version of \( \mathcal{C} \), where each atom except the distinguished one in an object in \( \mathcal{C} \) is weighted by a number in \([k]\). The combinatorial interpretation of the refined generating function
\[
h(x, t, r_1, r_2, \ldots, r_k) := \frac{1}{r_1 + r_2 + \cdots + r_k} f(x, (r_1 + r_2 + \cdots + r_k)t)
\]
is straightforward.
Take $\mathcal{C}$ to be the class of permutations in which the atom with label 1 is in the first place. Then $\text{Set}(\mathcal{C})$ (resp. $\text{Set}(\ast \mathcal{C})$) is exactly the class of permutations (resp. $r$-colored marked permutations) that we introduced in this article.

References

[1] M. Bóna. Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley. *SIAM J. Discrete Math.*, 23:401–406, 2008/09.
[2] F. Brenti. A class of $q$-symmetric functions arising from plethysm. *J. Combin. Theory Ser. A*, 91:137–170, 2000.
[3] W.Y.C. Chen, A.M. Fu. Context-free grammars for permutations and increasing trees, *Adv. in Appl. Math.*, 82:58–82, 2017.
[4] Dominique Dumont. Grammaires de William Chen et dérivations dans les arbres et arborescences, *Sém. Lothar. Combin.*, 37 (1996), Art. B37a–21 pp. (electronic).
[5] Philippe Flajolet and Robert Sedgewick. Analytic combinatorics. Cambridge University Press, Cambridge, 2009.
[6] D. Foata, M. Schützenberger. Théorie Géométrique des Polynômes Euleriens. *Lecture Notes in Mathematics*, vol. 138, Springer-Verlag, Berlin-New York, 1970.
[7] I. Gessel and R.P. Stanley. Stirling polynomials. *J. Combin. Theory Ser. A*, 24:25–33, 1978.
[8] J. Haglund, M. Visontai. Stable multivariate Eulerian polynomials and generalized Stirling permutations, *European J. Combin.*, 33:477–487, 2012.
[9] S. Janson, M. Kuba and A. Panholzer. Generalized Stirling permutations, families of increasing trees and urn models, *J. Combin. Theory Ser. A*, 118:94–114, 2011.
[10] S.-M. Ma, T. Mansour. The $1/k$-Eulerian polynomials and $k$-Stirling permutations. *Discrete Math.*, 338:1468–1472, 2015.
[11] S.-M. Ma, Y.-N. Yeh. Stirling permutations, cycle structure of permutations and perfect matchings. *Electron. J. Combin.*, 22(4):#P4.42, 2015.
[12] S.-M. Ma, Y.-N. Yeh. Eulerian polynomials, perfect matchings and Stirling permutations of the second kind. *arXiv:1607.01311*.
[13] J.B. Remmel, A.T. Wilson. Block patterns in Stirling permutations. *arXiv:1402.3358*.
[14] D P Roselle. Permutations by number of rises and successions, *Proc. Amer. Math. Soc.*, 19(1): 8–16, 1968.
[15] C.V. Sukumar and A. Hodges. Quantum algebras and parity-dependent spectra. *Proc. R. Soc. A* 463: 2415–2427, 2007.

Institute of Mathematics, Academia Sinica, Taipei, Taiwan

E-mail address: arthurduhl@gmail.com (G.-H. Duh)

Department of Mathematics, National Taiwan Normal University, Taipei 116, Taiwan

E-mail address:yclinpa@gmail.com (Y.-C. Lin)

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P.R. China

E-mail address: shimeimapapers@163.com (S.-M. Ma)

Institute of Mathematics, Academia Sinica, Taipei, Taiwan

E-mail address: mayeh@math.sinica.edu.tw (Y.-N. Yeh)