On Euler number of symplectic hyperbolic manifold

Teng Huang

Abstract

In this article, we introduce a class of closed $2n$-dimensional almost Kähler manifold $X$ which called the special symplectic hyperbolic manifold. Those manifolds include Kähler hyperbolic manifolds. We study the spaces of $L^2$-harmonic forms on the universal covering space of $X$. We then prove the Singer conjecture on special symplectic hyperbolic case. As an application, we can show that the Euler number of a special symplectic manifold satisfies the inequality $(-1)^n \chi(X) > 0$.

Keywords. Hopf conjecture, Singer conjecture, almost Kähler manifold, Euler number

1 Introduction

Let us start the article by recalling two well-known conjectures related to the negativity of Riemannian sectional curvature. The first one, usually attributed to Hopf, is

Conjecture 1. (Hopf Conjecture) Let $X$ be a closed $2n$-dimensional Riemannian manifold with sectional curvature $\sec$. Then

$$\begin{cases} (-1)^n \chi(X) > 0, & \text{if } \sec < 0 \\ (-1)^n \chi(X) \geq 0, & \text{if } \sec \leq 0. \end{cases}$$

This is true for $n = 1$ and 2 as the Gauss–Bonnet integrands in these two low dimensional cases have the desired sign [6]. However, in higher dimensions, it is known that the sign of the sectional curvature does not determine the sign of the Gauss-Bonnet-Chern integrand [11]. A vanishing theorem in [19] which stated that the space of $L^2$ $k$-forms is trivial for $k$ in a certain range which depends on pinching constants for the curvature. For good pinching constants the question of Hopf can thus be answered.

We denote by $h^k_{(2)}(X)$ the $k$-th $L^2$-Betti number of Riemannian manifold $X$. The second conjecture which proposed by Singer ([27] and also [9 Conjecture 2]) is

Conjecture 2. (Singer Conjecture) Let $X$ be a closed $2n$-dimensional Riemannian manifold with negative sectional curvature. Then

$$\begin{cases} h^k_{(2)}(X) = 0, & k \neq n \\ h^n_{(2)}(X) > 0. \end{cases}$$
Because of the Euler-Poincaré formula
\[
\chi(X) = \sum_{k \geq 0} (-1)^k h^k_{(2)}(X)
\]
the Singer conjecture implies the Hopf conjecture in the case where \(X\) has negative sectional curvature.

The program outlined above was carried out by Gromov [12] when the manifold in question is Kähler and is homotopy equivalent to a closed manifold with strictly negative sectional curvatures.

A differential form \(\alpha\) in a Riemannian manifold \((X, g)\) is called bounded with respect to the metric \(g\) if the \(L^\infty\)-norm of \(\alpha\) is finite, namely,
\[
\|\alpha\|_{L^\infty(X)} = \sup_{x \in X} |\alpha(x)| < \infty.
\]
By definition, a \(k\)-form \(\alpha\) is said to be \(d\)(bounded) if \(\alpha = d\beta\), where \(\beta\) is a bounded \((k-1)\)-form. It is obvious that if \(X\) is compact, then every exact form is \(d\)(bounded). However, when \(X\) is not compact, there exist smooth differential forms which are exact but not \(d\)(bounded). For instance, on \(\mathbb{R}^n\), \(\alpha = dx^1 \wedge \cdots \wedge dx^n\) is exact, but it is not \(d\)(bounded). Let’s recall some concepts introduced in [5, 20]. A differential form \(\alpha\) on a complete non-compact Riemannian manifold \((X, g)\) is called \(d\)(sublinear) if there exist a differential form \(\beta\) and a number \(c > 0\) such that \(\alpha = d\beta\) and
\[
|\beta(x)|_g \leq c(1 + \rho_g(x, x_0)),
\]
where \(\rho_g(x, x_0)\) stands for the Riemannian distance between \(x\) and a base point \(x_0\) with respect to \(g\).

Let \((X, g)\) be a closed Riemannian manifold and \(\pi : (\tilde{X}, \tilde{g}) \to (X, g)\) be the universal covering with \(\tilde{g} = \pi^*g\). A form \(\alpha\) on \(X\) is called \(\tilde{d}\)(bounded) (resp. \(\tilde{d}\)(sublinear)) if \(\pi^*\alpha\) is a \(d\)(bounded) (resp. \(d\)(sublinear)) form on \((\tilde{X}, \tilde{g})\). In geometry, various notions of hyperbolicity have been introduced, and the typical examples are manifolds with negative curvature in suitable sense [7]. The starting point for the present investigation is Gromov’s notion of Kähler hyperbolicity [12]. Gromov [12] pointed out that if the Riemannian manifold \((X, g)\) is a complete simply-connected manifold and has strictly negative sectional curvatures, then every smooth bounded closed form of degree \(k \geq 2\) is \(d\)(bounded). Then he proved the Hopf conjecture in the Kähler case. Gromov [12] also gave a lower bound on the spectra of the Laplace operator \(\Delta_d := dd^* + d^*d\) on \(L^2\)-forms in \(\Omega^{p,q}(X)\) for \(p + q \neq n\) to sharpen the Lefschetz vanishing theorem. In order to attack Hopf conjecture in the Kähler manifold with \(\sec \leq 0\), by extending Gromov’s idea, Cao-Xavier [5] and Jost-Zuo [20] independently introduced the concept of Kähler parabolicity, which includes nonpositively curved closed Kähler manifolds, and showed that their Euler numbers have the desired property. In [14], the author proved the Hopf conjecture in some locally conformally Kähler manifolds case.

Let \((X, J, \omega)\) be a closed \(2n\)-dimensional symplectic manifold. Let \(J\) be an \(\omega\)-compatible almost complex structure, i.e., \(J^2 = -id\), \(\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)\), and \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\) is a Riemannian metric on \(X\). The triple \((\omega, J, g)\) is called an almost Kähler structure on \(X\). Notice that any one of the pairs \((\omega, J)\), \((J, g)\) or \((g, \omega)\) determines the other two. An almost-Kähler metric \(g\) is Kähler if and only if \(J\) is integrable. For symplectic case, inspired by Kähler geometry, Tan-Wang-Zhou [29] gave the definition of
symplectic parabolic manifold. A closed almost Kähler manifold \((X, J, \omega)\) is called symplectic hyperbolic (resp. parabolic) if the lift \(\tilde{\omega}\) of \(\omega\) to the universal covering \((\tilde{X}, \tilde{J}, \tilde{\omega}) \to (X, J, \omega)\) is \(d\) (bounded) (resp. \(d\) (sublinear)) on \((\tilde{X}, \tilde{J}, \tilde{\omega})\).

Noticing that the proof of the vanishing theorem on Kählerian case is based on the identity \([L, \Delta_d] = 0\) due to the Kähler identities. But, in general, the complete almost Kähler manifold
\[d(\text{resp. parabolic})\] if the lift \(\tilde{\omega}\) of \(\omega\) to the universal covering \((\tilde{X}, \tilde{J}, \tilde{\omega}) \to (X, J, \omega)\) is \(d\) (bounded) (resp. \(d\) (sublinear)) on \((\tilde{X}, \tilde{J}, \tilde{\omega})\).

In this article, we observe that the operator \(\Delta_{d'} = \frac{1}{4} (\Delta_d + \Delta_{d^*})\) commutes to \(L\), where the operator \(d'\) is defined in Section 3.1. We also observe that if \(\dim \mathcal{H}_{d}^k(X) = \dim \mathcal{H}_{d}^k(X)\) holds for any \(0 \leq k < n\) over a closed almost Kähler manifold \(X\), then \(X\) satisfies Hard Lefschetz Condition (see Corollary 3.7). For non-compact case, we use Gromov’s method to study the operator \(d' + d'^* : \Omega^{even} \to \Omega^{odd}\) on a complete almost Kähler \(2n\)-manifold \(X\) with \(d\) (bounded) symplectic \(\omega\). We then give a lower bound on the spectra of the operator \(\Delta_{d'}\) on the space of \(L^2\)-forms \(\Omega^k(X)\) for \(k \neq n\) (see Theorem 1.3). In [12], Gromov proved that if \(X\) is a closed Kähler hyperbolic manifold, then \((-1)^n\chi(X) > 0\). Gromov’s method amounted to construct arbitrarily small perturbations of \(d + d^*\) on \(\Omega^{(2)}(X)\) with a non-trivial \(L^2\)-kernel. For this, he applied the \(L^2\)-index theorem to a twisted \(d + d^*\), i.e., to vector valued differential forms. We observe that the index of \(d' + d'^* : \Omega^{even} \to \Omega^{odd}\) is equal to the Euler number of \(X\), that is, \(\text{Index}(d' + d'^*) = \chi(X)\) (see Corollary 8.10). By the Atiyah’s \(L^2\) index, the \(L^2\)-index of \(d' + d'^*\) is equal to the index of \(d' + d'^*\). We construct arbitrarily small perturbations of \(d' + d'^*\) on \(\Omega^k(\tilde{X})\) with a non trivial \(L^2\)-kernel.

Let \((X, J, \omega)\) be an almost Kähler manifold with a symplectic form \(\omega\). The symplectic form \(\omega\) is closed on \(X\), this is equivalent to \(g((\nabla_X J)Y, Z) + g((\nabla_Z J)X, Y) + g((\nabla_Y J)Z, X) = 0\). We denote by \(\tau^*, \tau\) the \(*\)-curvature and scalar curvature of \(X\), respectively. There is a known identity \(|\nabla J|^2 = 2(\tau^* - \tau)\) [25] [26]. We introduce a class of closed almost Kähler manifolds as follows.

**Definition 1.1.** A closed almost Kähler manifold \((X, J, \omega)\) is called special symplectic hyperbolic if there is a bounded 1-form \(\theta\) on \(\tilde{X}\) such that \(\tilde{\omega} = d\theta\) and

\[
\|\theta\|^2_{L^\infty(\tilde{X})} \geq C \max_{x \in X}(\tau^* - \tau)(x),
\]

where \(C\) is an uniformly positive constant only depend on \(n\).

Following the identity \(g(N_J(X, Y), Z) = 2g(J(\nabla_Z J)X, Y)\), the Equation (1.1) is equivalent to

\[
\|\theta\|^2_{L^\infty(\tilde{X})} \geq \frac{C}{4}\|N_J\|^2_{L^\infty(X)},
\]

(1.2)
Remark 1.2. (1) A Kähler hyperbolic manifold is a special symplectic hyperbolic manifold since the almost complex structure $J$ is integrable, i.e., the Nijenhuis tensor is zero.

(2) Let $(X, J, \omega)$ be a closed almost Kähler manifold with sectional curvature bounded from above by a negative constant, i.e., $\text{sec} \leq -K$ for some $K > 0$. We denote by $(\tilde{X}, \tilde{J}, \tilde{\omega})$ the universal covering space of $(X, J, \omega)$. Since $\pi$ is local isometry, the sectional curvature of $\tilde{X}$ also bounded from above by the negative constant $-K$. By [7, Lemma 3.2], there exists 1-form $\theta$ on $\tilde{X}$ such that

$$\tilde{\omega} = d\theta \text{ and } \|\theta\|_{L^\infty(\tilde{X})} \leq \sqrt{nK^{-\frac{1}{2}}}.$$ 

If the sectional curvature $K$ of $X$ large enough to ensure that

$$K \geq nC \max_{x \in X} (\tau^* - \tau)(x),$$

then $X$ is a special symplectic hyperbolic manifold.

At first, we can give a lower bound on the spectra of the Laplace operator $\Delta_{d'} := \frac{1}{4}(\Delta_d + \Delta_{d^\Lambda})$ on $L^2$ $k$-forms $\Omega^k_{(2)}(X)$ for $k \neq n$. The main idea is we use the identity $[\Delta_{d} + \Delta_{d^\Lambda}, L] = 0$.

Theorem 1.3. Let $(X, J, \omega)$ be a complete $2n$-dimensional almost Kähler manifold with a $d$ (bounded) symplectic form $\omega$ i.e., there exists a bounded 1-form $\theta$ such that $\omega = d\theta$. Then every $L^2$ $k$-form $\alpha$ on $X$ of degree $k \neq n$ satisfies the inequality

$$\langle (\Delta_d + \Delta_{d^\Lambda})\alpha, \alpha \rangle \geq \lambda^2_0 \langle \alpha, \alpha \rangle,$$

where $\lambda_0$ is a strictly positive constant which depends only on $n$ and the bounded of $\theta$,

$$\lambda_0 \geq \text{const}_n \|\theta\|_{L^\infty(X)}^{-1}.$$ 

In particular,

$$\mathcal{H}^k_{(2),d'}(X) = 0,$$

unless $k \neq n$.

We then extend Gromov’s idea to the special symplectic hyperbolic manifold case. We can prove the Hopf conjecture in our case.

Theorem 1.4. (=Theorem 4.8) Let $(X, J, \omega)$ be a closed $2n$-dimensional special symplectic hyperbolic manifold. Then the Euler number of $X$ satisfies

$$(-1)^n \chi(X) > 0.$$ 

As a corollary of Theorem 1.4 we have the following result.
Corollary 1.5. Let \((X, J, \omega)\) be a closed \(2n\)-dimensional almost Kähler manifold. If the curvature of \(X\) satisfies
\[
\sec \leq -K < 0 \text{ and } K \geq nC \max_{x \in X} (\tau^* - \tau)(x),
\]
then the Euler number of \(X\) satisfies
\[
(-1)^n \chi(X) > 0,
\]
where \(C\) is an uniformly positive constant only depend on \(n\).

Suppose that the Nijenhuis tensor \(N_J\) on a complete almost Kähler manifold \((X, J, \omega)\) with a bounded symplectic form satisfies
\[
C \|N_J\|_{L^\infty(X)} \leq \|\theta\|_{L^\infty(X)}^{-1}.
\]
We then give a lower bound on the spectra of the Laplace operator \(\Delta_d := dd^* + d^* d\) on the space of \(L^2\) \(k\)-forms \(\Omega^k_{(2)}(X)\) for \(k \neq n\) (see Theorem 4.9). For the non-vanishing result, we use the Gromov’s idea in [12]. Therefore we can prove the Singer conjecture under the special symplectic hyperbolic case.

Theorem 1.6. Let \((X, J, \omega)\) be a closed \(2n\)-dimensional special symplectic hyperbolic manifold, \(\pi : (\tilde{X}, \tilde{J}, \tilde{\omega}) \to (X, J, \omega)\) the universal covering map for \(X\). Then the spaces of \(L^2\)-harmonic \(k\)-forms on its universal covering space \(\tilde{X}\) satisfy
\[
\begin{align*}
\mathcal{H}^k_{(2)}(\tilde{X}) &= \{0\}, k \neq n \\
\mathcal{H}^n_{(2)}(\tilde{X}) &\neq \{0\},
\end{align*}
\]
is equivalent to
\[
\begin{align*}
h^k_{(2)}(X) &= 0, k \neq n \\
h^n_{(2)}(X) &> 0.
\end{align*}
\]
In particular,
\[
(-1)^n \chi(X) > 0.
\]

2 Preliminaries

2.1 Differential forms on almost Kähler manifold

We recall some definitions and results on the differential forms for almost complex and almost Hermitian manifolds. Let \(X\) be a \(2n\)-dimensional manifold (without boundary) and \(J\) be a smooth almost complex structure on \(X\). There is a natural action of \(J\) on the space \(\Omega^k(X, \mathbb{C}) := \Omega^k(X) \otimes \mathbb{C}\), which induces a topological type decomposition
\[
\Omega^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^p_J(X, \mathbb{C}),
\]
where \(\Omega^p_J(X, \mathbb{C})\) denotes the space of complex forms of type \((p, q)\) with respect to \(J\). We have that
\[
d : \Omega^p_J \to \Omega^{p+2,q-1}_J \oplus \Omega^{p+1,q}_J \oplus \Omega^{p,q+1}_J \oplus \Omega^{p-1,q+2}_J,
\]
and so the operator \(d\) splits according as
\[
d = A_J + \partial_J + \bar{\partial}_J + A_J,
\]
where all the pieces are graded algebra derivations, $A_J$, $\bar{A}_J$ are 0-order differential operators. Note that each component of $d$ is a derivation, with bi-degrees given by

$$|A_J| = (2, -1), \quad |\partial_J| = (1, 0), \quad |\bar{\partial}_J| = (0, 1), \quad |\bar{A}_J| = (-1, 2).$$

The integrability theorem of Newlander and Nirenberg states that the almost complex structure $J$ is integrable if and only if $N_J = 0$, where

$$N_J : TX \otimes TX \to TX,$$

denotes the Nijenhuis tensor

$$N_J(X, Y) := [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY].$$

For $\alpha \in \Omega^1(X)$ we have

$$(A_J(\alpha) + \bar{A}_J(\alpha))(X, Y) = \frac{1}{4} \alpha(N_J(X, Y)).$$

In particular, $J$ is integrable if only if $N_J = 0$, i.e, $A_J = 0$. Expanding the equation $d^2 = 0$ we obtain the following set of equations:

$$A_J^2 = 0,$$

$$A_J \partial_J + \partial_J A_J = 0,$$

$$\partial_J^2 + A_J \bar{\partial}_J + \bar{\partial}_J A_J = 0,$$

$$\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J + A_J \bar{A}_J + \bar{A}_J A_J = 0,$$

$$\bar{A}_J \partial_J + \bar{\partial}_J \bar{A}_J = 0,$$

$$\bar{A}_J^2 = 0.$$

**Definition 2.1.** An almost Kähler structure on $2n$-dimensional manifold $X$ is a pair $(\omega, J)$ where $\omega$ is a symplectic form and $J$ is an almost complex structure calibrated by $\omega$.

If $(X, \omega, J)$ is an almost Kähler manifold, then

$$g(X, Y) = \omega(X, JY)$$

is a $J$-Hermitian metric, i.e., $g(JX, JY) = g(X, Y)$ for any $X, Y$. For any almost Kähler manifold $(X, J, \omega)$ there is an associated Hodge-star operator $[18] * : \Omega^p_J \to \Omega^{n-q,n-p}_J$ defined by

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \frac{\omega^n}{n!}.$$

The Lefschetz operator $L : \Omega^p_J \to \Omega^{p+1,q+1}_J$ defined by

$$L(\alpha) = \omega \wedge \alpha.$$
It has adjoint $\Lambda = *^{-1}L*$. There is a Lefschetz decomposition on complex $k$-forms

$$\Omega^k(X, \mathbb{C}) = \bigoplus_{r \geq 0} L^r P^k_{\mathbb{C}} - 2r,$$

where $P^k_{\mathbb{C}} = \ker \Lambda \cap \Omega^k(X, \mathbb{C})$.

We recall some definitions on almost Hermitian manifold [22, 25]. We denote by $\nabla$, $R$, $\rho$, $\tau$ the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of $X$, respectively. Here, we assume that the curvature $R$ is defined by

$$R(X, Y, Z) = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(X)$. We denote by $\{X_1, \cdots, X_{2n}\}$ a local orthonormal frame field of $X$. We have $R_{ijkl} = g(R(X_i, X_j)X_k, X_l)$, $J_{ij} = g(JX_i, X_j)$ and $\nabla_i J_{jk} = g((\nabla_X J)X_j, X_k)$. We denote by $\rho^*$ and $\tau^*$ the Ricci $*$-tensor and $*$-scalar curvature defined respectively by

$$\rho^*(X, Y) = g(Q^* X, Y) = \text{trace}(Z \mapsto R(X, JZ)JX),$$

$$\tau^* = \text{trace} Q^*,$$

where $X, Y, Z \in \mathfrak{X}(X)$. By using the first Bianchi identity, we have

$$\rho^*(X, Y) = -\frac{1}{2} \sum_{i=1}^{2n} R(X, JY, X_i, JX_i),$$

and hence

$$\tau^* = -\frac{1}{2} \sum_{a, b, i, j=1}^{2n} J_{ab} R_{abij} J_{ij}.$$

The following equality is known

**Proposition 2.2.** ([25 Lemma 2.4] or [26 Equation (3.1)–(3.3)]) Let $(X, J, \omega)$ be a closed almost Kähler manifold. Then

$$|\nabla J|^2 = 2(\tau^* - \tau),$$

$$\nabla_X J + (\nabla_{JX} J) Y = 0,$$

$$g(N_j(X, Y), Z) = 2g(J(\nabla_X J)X, Y),$$

for $X, Y, Z \in \mathfrak{X}(X)$.

**Example 2.3.** Let $(X, J, \omega)$ be a closed almost Kähler manifold. We denote by $\{Z_i\}$ the orthonormal basis of $T^1_{p,0}$, $p \in X$. We extend the curvature operator $R : \wedge^2 T^*_p X \to \wedge^2 T^*_p X$ to a complex linear transformation $R^C : \wedge^2 T^*_p X \otimes \mathbb{C} \to \wedge^2 T^*_p X \otimes \mathbb{C}$ [16]. Given a nonzero decomposition $\Pi \in \Lambda^2 T^*_p X \otimes \mathbb{C}$, its complex sectional curvature is the real number

$$K^C(\Pi) = \frac{(R^C(\Pi), \Pi)}{(\Pi, \Pi)}.$$
Then following [17, Lemma 3.3], we have
\[\tau^* - \tau = -4 \sum_{i,j=1}^n (R^C(Z_i \wedge Z_j), \overline{Z_i} \wedge \overline{Z_j}).\]

If the curvature of \(X\) satisfies
\[K^C \geq -\frac{K}{4n^3C},\]
then
\[Cn(\tau^* - \tau) \leq K.\]

It implies that \(X\) is a special symplectic hyperbolic.

### 2.2 Almost Kähler identities

In [3, 4], the authors extended the Kähler identities to the non-integrable setting and deduced several geometric and topologies consequences. In this section, we will recall almost Kähler identities constructed by Cirici-Wilson [3].

The operators \(\delta = A_J, \partial_J, \bar{\partial}_J, \bar{A}_J\) and \(\delta\) have \(L^2\)-adjoint operators \(\delta^*\) when \(X\) is closed, and one can check that
\[\bar{A}_J^* = - * A_J * \quad \text{and} \quad \bar{\partial}_J^* = - * \partial_J *.\]

For any almost Kähler manifold, there is a \(\mathbb{Z}_2\)-graded Lie algebra of operators action on the \((p, q)\)-forms, generated by eight odd operators

\[\bar{\partial}_J, \partial_J, \bar{A}_J, A_J, \bar{\partial}_J^*, \partial_J^*, \bar{A}_J^*, A_J^*\]

and even degree operators \(L, \Lambda, H\) [3, Section 3].

Cirici-Wilson [3, 4] defined the \(\delta\)-Laplacian by letting
\[\Delta_{\delta} := \delta \delta^* + \delta^* \delta.\]

For all \(p, q\), we will denote by
\[\mathcal{H}_\delta^{p,q} := \ker \Delta_{\delta} \cap \Omega_j^{p,q} = \ker \delta \cap \ker \delta^* \cap \Omega_j^{p,q}\]
the space of \(\delta\)-harmonic forms in bi-degree \((p, q)\).

By introducing a symplectic Hodge star operator \(*_s\), Brylinski [2] proposed a Hodge theory of closed symplectic manifolds. The space of symplectic harmonic \(k\)-forms is \(\mathcal{H}_{sym}^k := \ker d \cap \ker d^\Lambda \cap \Omega^k\), where
\[d^\Lambda := [d, \Lambda] = (-1)^{k+1} *_s d *_s.\]

Given any compatible triple \((X, J, \omega)\) on a symplectic manifold, the differential operator \(d^\Lambda = [d, \Lambda]\) and the \(d^c\) operator
\[d^c := J^{-1} d J,\]
are related via the Hodge star operator defined with respect to the compatible metric $g$ by the relation

$$d^\Lambda = d^{\ast} := -\ast d^{\ast}.$$ 

Brylinski also showed that in almost Kähler manifold, a form of pure $(p, q)$ is in $\mathcal{H}^{p+q}_{\text{sym}}$ if only if it is in $\mathcal{H}^{p+q}_d$. This gives an inclusion $\bigoplus_{p+q=k} \mathcal{H}^{p,q}_d \hookrightarrow \mathcal{H}^{p+q}_{\text{sym}}$. In general, this is strict. Indeed, Yan [33] showed that $k = 0, 1, 2$, every cohomology class has a symplectic harmonic representative.

We define the graded commutator of operators $A$ and $B$ by

$$[A, B] = AB - (-1)^{\text{deg}(A)\text{deg}(B)} BA$$

where $\text{deg}(A)$ denotes the total degree of $A$.

**Proposition 2.4. ([3] Proposition 3.1)** For any almost Kähler manifold the following identities hold:

1. $[L, \bar{A}_J] = [L, A_J] = 0$ and $[\Lambda, \bar{A}_J^\ast] = [\Lambda, A_J^\ast] = 0$.
2. $[\bar{L}, \partial_J] = [L, \partial_J] = 0$ and $[\Lambda, \partial_J^\ast] = [\Lambda, \partial_J^\ast] = 0$.
3. $[\bar{L}, \bar{A}_J^\ast] = \sqrt{-1} \Lambda_J, [L, A_J^\ast] = -\sqrt{-1} \bar{A}_J$ and $[\Lambda, \bar{A}_J] = \sqrt{-1} A_J^\ast, [\Lambda, A_J] = -\sqrt{-1} \bar{A}_J^\ast$.
4. $[L, \partial_J^\ast] = -\sqrt{-1} \partial_J, [L, \bar{\partial}_J^\ast] = \sqrt{-1} \bar{\partial}_J$ and $[\Lambda, \partial_J] = -\sqrt{-1} \partial_J^\ast, [\Lambda, \bar{\partial}_J] = \sqrt{-1} \bar{\partial}_J^\ast$.

If $C$ is another operator of degree $\text{deg}(C)$, the following Jacobi identity is easy to check

$$( -1)^{\text{deg}(C)\text{deg}(A)} [A, [B, C]] + ( -1)^{\text{deg}(A)\text{deg}(B)} [B, [C, A]] + ( -1)^{\text{deg}(B)\text{deg}(C)} [C, [A, B]] = 0.$$ 

**Proposition 2.5. ([3] Proposition 3.3)** For any almost Kähler manifold the following identities hold:

1. $[\bar{A}_J, A_J^\ast] = [A_J, \bar{A}_J^\ast] = 0$.
2. $[\bar{A}_J, \partial_J] = [\partial_J, A_J^\ast]$ and $[A_J, \partial_J^\ast] = [\partial_J, \bar{A}_J^\ast]$.
3. $[\partial_J, \partial_J^\ast] = [A_J^\ast, \partial_J] + [A_J, \partial_J^\ast] + [\partial_J, \partial_J^\ast] = [\bar{A}_J^\ast, \partial_J] + [\bar{A}_J, \partial_J^\ast] + [\partial_J, \partial_J^\ast]$.

In [3], they also gave several relations concerning various Laplacians.

**Proposition 2.6. ([3] Proposition 3.4)** For any almost Kähler manifold the following identities hold:

1. $\Delta_{\bar{A}_J + A_J} = \Delta_{\bar{A}_J} + \Delta_{A_J}$.
2. $\Delta_{\partial_J} + \Delta_{A_J} = \Delta_{\partial_J^\ast} + \Delta_{\bar{A}_J}$.
3. $\Delta_d = 2(\Delta_{\partial_J} + \Delta_{A_J} + [\bar{A}_J, \partial_J^\ast] + [A_J, \partial_J^\ast] + [\partial_J, \partial_J^\ast] + [\partial_J, \partial_J^\ast])$.

## 3 Harmonic forms on symplectic manifold

### 3.1 $\Delta_{d^s}$-harmonic forms

In this section, we discuss the harmonic forms that can be constructed from the two differential operators $d'$ and $d''$. We begin with the known cohomologies with $d$ (for de Rham $H_d$) and $d^\Lambda$ (for $H_{\text{sym}}$).

We define the operators $d' = \partial_J + \bar{A}_J$ and $d'' = \bar{\partial}_J + A_J$. Those have adjoint $d'^\ast = \partial_J^\ast + \bar{A}_J^\ast$ and $d''^\ast = \partial_J^\ast + A_J^\ast$. Hence

$$d = d' + d'', \ d^s = d'^s + d''^s.$$
The operators $d'$ and $d''$ were also previously introduced by de Bartolomeis and Tomassini [8]. They also established some almost Kähler identities which obtained by Cirici-Wilson.

With the exterior derivative $d$, there is of course the de Rham cohomology

$$H^k_{dR}(X) = \frac{\ker d \cap \Omega^k(X)}{\text{Im}d \cap \Omega^k(X)},$$

that is present on all Riemannian manifolds. Since $d^\Lambda d^\Lambda = 0$, there is also a natural cohomology

$$H^k_{d^\Lambda}(X) = \frac{\ker d^\Lambda \cap \Omega^k(X)}{\text{Im}d^\Lambda \cap \Omega^k(X)}.$$ 

This cohomology has been discussed in [21, 30, 31, 32, 33].

We utilize the compatible triple $(X, J, g)$ on $X$ to write the Laplacian associated with the $d^\Lambda$-cohomology:

$$\Delta_{d^\Lambda} = [d^\Lambda, d^\Lambda^*].$$

The self-adjoint Laplacian naturally defines a harmonic form.

We consider an self-adjoint Laplacian

$$\Delta_d + \Delta_{d^\Lambda} = [d, d^*] + [d^\Lambda, d^\Lambda^*].$$

Suppose that $J$ is integrable, i.e., $N_J = 0$, then

$$\Delta_d = \Delta_{d^\Lambda} = 2\Delta_d = 2\Delta_{\bar{d}}.$$ 

In symplectic case, we introduce two self-adjoint operators

$$\Delta_{d'} := [d', d'^*] \quad (\text{resp.} \quad \Delta_{d''} := [d'', d''^*]).$$

By the inner product

$$\langle \alpha, \Delta_{d^*}\alpha \rangle_{L^2(X)} = \|d^*\alpha\|^2 + \|d^{**}\alpha\|^2$$

we are led to the following definition.

**Definition 3.1.** A differential form $\alpha \in \Omega^*(X)$ is call $d^*$-harmonic if $\Delta_{d^*}\alpha = 0$, or equivalently, $d^*\alpha = d^{**}\alpha = 0$. We denote the space of $d^*$-harmonic $k$-forms by $\mathcal{H}^k_{d^*}(X)$.

We will show that

$$\ker \Delta_{d'} \cap \Omega^k(X) = \ker \Delta_{d''} \cap \Omega^k(X) = \ker(\Delta_d + \Delta_{d^\Lambda}) \cap \Omega^k(X).$$
Proposition 3.2. For any almost Kähler manifold the following identities hold:
(1) $\Delta_d = [d, d^*] = [d', d^{**}] + [d', d'^*] + [d'', d'^*].$
(2) $\Delta_{d^*} = [d^*, d^*] = [d', d^*] - [d', d'^*] - [d'', d^*] + [d'', d'^*].$

In [8, Lemma 3.8], the authors gave a relation between $\ker \Delta_d$ and $\ker \Delta_{d^*}$ as follows. If $\alpha \in \Omega^k(X)$ is a $\Delta_d$-harmonic form in a closed almost Kähler manifold $(X, J, \omega)$, then

$$\|d^\Lambda \alpha\|^2 + \|d^\Lambda^* \alpha\|^2 = 4 \text{Re}(J^{-1}(A_J + \bar{A}_J)J\alpha, d^\Lambda \alpha) + 4 \text{Re}(J^{-1}(A^*_J + \bar{A}^*_J)J\alpha, d^\Lambda \alpha).$$

Lemma 3.3. (c.f. [8 Lemma 3.6]) Let $(X, J, \omega)$ be a closed almost Kähler manifold. Then

$$\Delta_{d'} = \Delta_{d''} = \frac{1}{4}(\Delta_d + \Delta_{d^*}).$$

In particular,

$$\ker \Delta_d \cap \ker \Delta_{d^*} \cap \Omega^k(X) = \ker d' \cap \ker d'^* \cap \Omega^k(X) = \ker d'' \cap \ker d'^* \cap \Omega^k(X).$$

Proof. Following Proposition 3.2 we get

$$\Delta_d + \Delta_{d^*} = 2[d', d^{**}] + 2[d'', d'^*] = 2\Delta_{d'} + 2\Delta_{d''}.$$

Noting that

$$[d', d'^*] = [\partial_J, \partial^*_J] + [\partial_J, \bar{A}^*_J] + [A_J, \partial^*_J] + [\bar{A}_J, \partial_J],$$

$$\Delta_{\partial_J} + \Delta_{\bar{A}_J} + [\partial_J, \bar{A}_J] + [A_J, \partial_J] = [\partial_J + A_J, \partial^*_J + A^*_J] + [\bar{A}_J, \partial_J],$$

Here we use the identities $\Delta_{\partial_J} + \Delta_{\bar{A}_J} = \Delta_{\partial_J} + \Delta_{\bar{A}_J}, [\partial_J, \bar{A}_J] = [A_J, \partial^*_J]$, and $[\bar{A}_J, \partial_J] = [\partial_J, A^*_J]$ (see Proposition 2.5 and 2.6). Therefore, we get $\Delta_{d'} = \Delta_{d''} = \frac{1}{4}(\Delta_d + \Delta_{d^*}).$ □

3.2 Hard Lefschetz Condition

From Lemma 3.3, we know that $\mathcal{H}^k_{d'}(X) = \mathcal{H}^k_{d''}(X) \subset \mathcal{H}^k_d(X)$ and

$$\dim \mathcal{H}^k_{d'}(X) = \dim \mathcal{H}^k_{d''}(X) \leq \dim \mathcal{H}^k_d(X),$$

it implies that $\mathcal{H}^k_{d'}(X)$ is finite dimensional. We denote the Betti numbers of $X$ by $b_i(X) := \dim \mathcal{H}^i_d(X)$. The Euler number of $X$ is given by

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim \mathcal{H}^i_d(X) = \sum_{i \geq 0} (-1)^i b_i.$$

We denote the $i$-th almost Kähler Betti numbers by $b_i^{AK} := \dim \mathcal{H}^i_{d'}(X)$. Therefore we define the almost Kähler Euler number as follows

$$\chi^{AK}(X) = \sum_{i \geq 0} (-1)^i \dim \mathcal{H}^i_{d'}(X) = \sum_{i \geq 0} (-1)^i b_i^{AK}.$$
Remark 3.4. Suppose the almost complex structure $J$ on $X$ is integrable, i.e., $(X, J, \omega)$ is Kählerian. Hence $A_J = \bar{A}_J = 0$. In this case, $\mathcal{H}_{d}^{k}(X) = \ker \Delta_{\theta} \cap \Omega^{k}$ and $\mathcal{H}_{d}^{k}(X) = \ker \Delta_{\bar{\theta}} \cap \Omega^{k}$. By the identity $\Delta_{\theta} = \frac{1}{2} \Delta_{d}$, one knows that

$$\mathcal{H}_{d}^{k}(X) = \mathcal{H}_{d}^{k}(X) = \mathcal{H}_{dR}^{k}(X).$$

Hence the almost Kähler Euler numbers $\chi^{AK}(X)$ is the Euler number of $X$.

A special class of symplectic manifolds is represented by those ones satisfying the Hard Lefschetz Condition (HLC), i.e., those closed $2n$-dimensional symplectic manifolds $(X, J, \omega)$ for which the map

$$L^{k} : H_{dR}^{n-k}(X) \to H_{dR}^{n+k}(X), \; \forall \; 0 \leq k < n$$

are isomorphisms. In particular, a classical result states if $(X, J, \omega)$ is a closed Kähler manifold, then $(X, J, \omega)$ satisfies the HLC [18]. In [28], the authors studied the Hard Lefschetz property of $\ker \Delta_{d}$ on almost Kähler manifold.

Theorem 3.5. ([28, Theorem 5.2]) Let $(X, J, \omega)$ be a closed $2n$-dimensional almost Kähler manifold. Then $\mathcal{H}_{d}^{k}(X)$ satisfies the HLC.

Proof. Notice that $4\Delta_{d'} = \Delta_{d} + \Delta_{dA}$. We only need to prove that $\ker \Delta_{d} \cap \ker \Delta_{dA}$ satisfies HLC. Let us start by showing that

$$[\Delta_{d} + \Delta_{dA}, L] = 0.$$

In fact, by the Jacobi identity, we have

$$[L, [d, L]] + [d, [L, L]] - [L, [d, L]] = 0,$$

and

$$[L, [d^{A*}, d]] + [d^{A*}, [d, L]] - [d^{A*}, [L, d]] = 0.$$ 

Since $[L, d] = [L, d^{A*}] = 0$, $[d^{*}, L] = -d^{A*}$, $[d^{A*}, L] = d$, and

$$[d, d^{A*}] = [d^{A*}, d] = d^{A*}d + dd^{A*},$$

we have

$$[L, [d, L]] + [L, [d^{A*}, d]] = 0.$$ 

which gives $[\Delta_{d} + \Delta_{dA}, L] = 0$. Thus $L$ maps $\ker \Delta_{d} \cap \ker \Delta_{dA} = \ker \Delta_{d} + \Delta_{dA}$ to itself. A similar argument gives $[\Delta_{d} + \Delta_{dA}, \Lambda] = 0$, it implies that $\Lambda(\ker \Delta_{d} \cap \ker \Delta_{dA}) \subseteq \ker \Delta_{d} \cap \ker \Delta_{dA}$. Thus there is an $sl_{2}$-structure on $\ker \Delta_{d} \cap \ker \Delta_{dA}$ and our theorem follows.

Remark 3.6. The above proof gives $[L, \Delta_{d}] = [d, d^{A*}]$. Since $[d, d^{A*}] = 0$ if and only if $J$ is integrable. We then know that $X$ is Kählerian if and only if $[L, \Delta_{d}] = 0$. 


Corollary 3.7. (c.f. [28] Theorem 5.3) Let \((X, J, \omega)\) be a closed \(2n\)-dimensional almost Kähler manifold. If \(\mathcal{H}^k_d(X) = \mathcal{H}^k_d(X)\), for all \(0 \leq k < n\), then HLC on \((\mathcal{H}^*_d, L)\) and HLC on \((\mathcal{H}^*_d, L)\).

It’s easy to see that the HLC on \((\mathcal{H}^*_d, L)\) implies the HLC on \((H^*_d, L)\). But in general, we don’t know whether they are equivalent or not. In [29], the authors studied the symplectic cohomologies and symplectic harmonic forms which introduced by Tseng-Yau. Based on this, they get if \((X, J, \omega)\) is a closed symplectic parabolic manifold which satisfies the hard Lefschetz property, then its Euler number satisfies the inequality \((−1)^n \chi(X) ≥ 0\). We then have

Corollary 3.8. (c.f. [29]) Let \((X, J, \omega)\) be a closed \(2n\)-dimensional symplectic parabolic manifold. Suppose that \(\dim \mathcal{H}^k_d(X) = \dim \mathcal{H}^k_d(X)\) for all \(0 \leq k < n\). Then the Euler number of \(X\) satisfies

\[ (−1)^n \chi(X) ≥ 0. \]

3.3 Index of a family elliptic operators

Let \(X\) be a closed Riemannian manifold of real dimension \(\dim X\). Then for each \(0 \leq k \leq \dim X\), we have the following de Rham elliptic operator \(D_{dR} = d + d^*: \)

\[ D_{dR} : \bigoplus_{k=even} \Omega^k(X) \rightarrow \bigoplus_{k=odd} \Omega^k(X), \]

whose index is the Euler number of \(X\). In fact,

\[
\text{Index}(D_{dR}) = \dim \ker D_{dR} - \dim(\coker D_{dR}) = \dim \bigoplus_{k=even} \mathcal{H}^k_d(X) - \dim \bigoplus_{k=odd} \mathcal{H}^k_d(X)
\]

\[ = \sum_{k=0}^{\dim X} (-1)^k b_k = \chi(X). \]

For the almost Kähler manifold \((X, J, \omega)\), we construct a family elliptic operator

\[
D(t) = \sqrt{\frac{2}{1 + t^2}} \left( (d' + td'') + (d'^* + td'^*) \right) : \bigoplus_{k=even/odd} \Omega^k(X) \rightarrow \bigoplus_{k=even/odd} \Omega^k(X). \tag{3.1}
\]

Hence \(D(0) = d' + d'^*\) and \(D(1) = D_{dR}\).

Proposition 3.9. ([30] Proposition 3.3) For any \(t \in \mathbb{R}^{>0}\),

\[
D^2(t) : \bigoplus_{k=even/odd} \Omega^k(X) \rightarrow \bigoplus_{k=even/odd} \Omega^k(X)
\]

is an elliptic differential operator.
Proof. To calculate the symbol of $D^2(t)$, we will work a local unitary frame of $T^*X$ and choose a basis $\{e^1, \cdots, e^n\}$ such that the metric is written as

$$g = e^i \otimes \bar{e}^i + \bar{e}^i \otimes e^i,$$

with $i = 1, \cdots, n$. With an almost complex structure $J$, any $k$-form can be decomposed into a sum of $(p, q)$-forms with $p + q = k$. We can write a $(p, q)$-form in the local moving-frame coordinates

$$A_{p,q} = A_{i_1 \cdots i_p j_1 \cdots j_q} e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \bar{e}^{j_q}.$$

The exterior derivative then acts as

$$dA_{p,q} = (\partial A_{p,q})_{p+1,q} + (\bar{\partial} A_{p,q})_{p,q+1} + A_{i_1 \cdots i_p j_1 \cdots j_q} d(e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \bar{e}^{j_q}), \quad (3.2)$$

where

$$(\partial A_{p,q})_{p+1,q} = \partial_{i_{p+1}} A_{i_1 \cdots i_p j_1 \cdots j_q} e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \bar{e}^{j_q}$$

$$(\bar{\partial} A_{p,q})_{p,q+1} = \bar{\partial}_{j_{q+1}} A_{i_1 \cdots i_p j_1 \cdots j_q} e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \bar{e}^{j_1} \wedge \cdots \bar{e}^{j_q}.$$

In calculating the symbol, we are only interested in the highest-order differential acting on $A_{i_1 \cdots i_p j_1 \cdots j_q}$. Therefore, only the first two terms of (3.2) are relevant for the calculation. In dropping the last term, we are effectively working in $\mathbb{C}^n$ and can make use of all the Kähler identities involving derivative operators. So, effectively, we have (using $\simeq$ to denote equivalence under symbol calculation)

$$d' + td'' \simeq \partial_J + t\bar{\partial}_J,$$

$$d'^* + td''^* \simeq \partial^*_J + t\bar{\partial}^*_J.$$

Noting that the highest-order of the operators $[\partial_J, \partial^*_J]$, $[\bar{\partial}_J, \bar{\partial}^*_J]$ differential acting on $A_{i_1 \cdots i_p j_1 \cdots j_q}$ are the same, since $\Delta_{\partial_J} + \Delta_{\bar{\partial}_J} = \Delta_{\partial} + \Delta_{\bar{\partial}}$, We thus have

$$D^2(t) \simeq \frac{2}{1 + t^2} ([\partial_J, \partial^*_J] + t^2 [\bar{\partial}_J, \bar{\partial}^*_J])$$

$$\simeq 2[\partial_J, \partial^*_J]$$

$$\simeq ([\partial_J, \partial^*_J] + [\bar{\partial}_J, \bar{\partial}^*_J])$$

$$\simeq \Delta_d.$$

\[ \square \]

**Corollary 3.10.** Let $(X, J, \omega)$ be a closed $2n$-dimensional almost Kähler manifold. Then

$$\chi(X) = \text{Index}(D(0)).$$

**Proof.** The operator $D(t)$ is self-adjoint. Following Proposition 3.9 we know that $D^2(t)$ is a generalized Laplacian. Hence $D(t)$ is a Dirac type operator in the sense of [23, Definition 2.1.17]. Naturally, the operator $D(t)$ is elliptic. By Theorem [23, Theorem 2.1.32], for any $t \in [0, 1]$, we have

$$\text{Index}(D(t)) = \text{Index}(D(1)).$$

Noticing that $\text{Index}(D(1)) = \chi(X)$. Thus we have $\chi(X) = \text{Index}(D(0))$. \[ \square \]
We recall the Atiyah’s $L^2$ index [1, 24].

**Theorem 3.11.** [24 Theorem 6.1] Let $X$ be a closed Riemannian manifold, $P$ a determined elliptic operator on sections of certain bundles over $X$. Denote by $\tilde{D}$ its lift of $D$ to the universal covering space $\tilde{X}$. Let $\Gamma = \pi_1(M)$. Then the $L^2$ kernel of $\tilde{P}$ has a finite $\Gamma$-dimension and

$$L^2\text{Index}_\Gamma(\tilde{P}) = \text{Index}(P).$$

We denote by $\tilde{D}(t)$ the lifted elliptic operator of $D(t)$. We then have

**Corollary 3.12.** Let $(X, J, \omega)$ be a closed $2n$-dimensional almost Kähler manifold, $\pi: (\tilde{X}, \tilde{J}, \tilde{\omega}) \to (X, J, \omega)$ the universal covering maps for $X$. Let $\Gamma = \pi_1(X)$. Then the Euler number of $X$ satisfies

$$\chi(X) = \text{Index}( \tilde{D}(0) ) = L^2\text{Index}_\Gamma(\tilde{D}(0)).$$

**Remark 3.13.** Noticing that

$$d^* = \partial J^2 + [\partial J, \bar{A} J] = \partial J^2 - \bar{\partial} J^2.$$

The operator $d^2$ always not zero. But it’s easy to see

$$\bigoplus_{k=even/odd} \mathcal{H}^k_{(2), d'} \subset \ker (d' + d'^*) \cap \bigoplus_{k=even/odd} \Omega^k_{(2)}.$$

### 4 Euler number of the hyperbolic symplectic manifolds

#### 4.1 Vanishing theorems

We begin the proof the Theorem 1.3 by recalling some basic notions in Hodge theory and almost Kähler geometry. If $X$ is an oriented complete Riemannian manifold, let $d^*$ be the adjoint operator of $d$ acting on the space of $L^2 k$-forms. Denote by $\Omega^k_{(2)}(X)$ and $\mathcal{H}^k_{(2)}(X)$ the spaces of $L^2 k$-forms and $L^2$ harmonic $k$-forms, respectively. By elliptic regularity and completeness of the manifold, a $k$-form in $\mathcal{H}^k_{(2)}(X)$ is smooth, closed and co-closed.

Suppose that $(X, J, \omega)$ is a complete almost Kähler manifold. We denote by

$$\mathcal{H}^k_{(2), d'}(X) = \{ \alpha \in \Omega^k_{(2)}(X) : \Delta_{d'} \alpha = 0 \}$$

the space of $L^2 \Delta_{d'}$-harmonic $k$-forms on $X$.

We follow the method of Gromov’s [12] to choose a sequence of cutoff functions $\{ f_\varepsilon \}$ satisfying the following conditions:

- (i) $f_\varepsilon$ is smooth and takes values in the interval $[0, 1]$, furthermore, $f_\varepsilon$ has compact support.
- (ii) The subsets $f_\varepsilon^{-1}(t) \subset X$, i.e., of the points $x \in X$ where $f_\varepsilon(x) = 1$ exhaust $X$ as $\varepsilon \to 0$.
- (iii) The differential of $f_\varepsilon$ everywhere bounded by $\varepsilon$,

$$\|df_\varepsilon\|_{L^\infty(X)} = \sup_{x \in X} |df_\varepsilon(x)| \leq \varepsilon.$$

Thus one obtains another useful
Lemma 4.1. Let \((X, J, \omega)\) be a complete almost Kähler manifold. If an \(L^2\) k-form \(\alpha\) is \(\Delta_d\)-harmonic, then \(d'\alpha = d^*\alpha = 0\).

Proof. We want to justify the integral identity

\[
\langle \Delta_d\alpha, \alpha \rangle = \langle d'\alpha, d'\alpha \rangle + \langle d^*\alpha, d^*\alpha \rangle
\]

If \(d'\alpha\) and \(d^*\alpha\) are \(L^2\) (i.e., square integrable on \(X\)), then this follows by Lemma \([12, 1.1.\ A]\). To handle the general case we cutoff \(\alpha\) and obtain by a simple computation

\[
0 = \langle \Delta_d\alpha, f_\varepsilon^2\alpha \rangle = \langle d'\alpha, f_\varepsilon^2d'\alpha \rangle + \langle d^*\alpha, f_\varepsilon^2d^*\alpha \rangle
\]

\[
= \langle d'\alpha, f_\varepsilon^2(d'\alpha) \rangle + \langle d^*\alpha, 2f_\varepsilon\partial f_\varepsilon \wedge \alpha \rangle + \langle d^*\alpha, f_\varepsilon^2(d^*\alpha) \rangle - \langle d^*\alpha, *(2f_\varepsilon\bar{\partial}f_\varepsilon \wedge \alpha \rangle
\]

\[
= I_1(\varepsilon) + I_2(\varepsilon),
\]

where

\[
|I_1(\varepsilon)| = |\langle d'\alpha, f_\varepsilon^2d'\alpha \rangle + \langle d^*\alpha, f_\varepsilon^2d^*\alpha \rangle| = \int_X f_\varepsilon^2(|d'\alpha|^2 + |d^*\alpha|^2)
\]

and

\[
|I_2(\varepsilon)| = |\langle d'\alpha, 2f_\varepsilon\partial f_\varepsilon \wedge \alpha \rangle - \langle d^*\alpha, *(2f_\varepsilon\bar{\partial}f_\varepsilon \wedge \alpha \rangle)|
\]

\[
\leq |\langle d'\alpha, 2f_\varepsilon\partial f_\varepsilon \wedge \alpha \rangle| + |\langle d^*\alpha, *(2f_\varepsilon\bar{\partial}f_\varepsilon \wedge \alpha \rangle)|
\]

\[
\lesssim \int_X |d\varepsilon| \cdot |f_\varepsilon| \cdot |\alpha|(|d'\alpha| + |d^*\alpha|).
\]

Then we choose \(f_\varepsilon\) such that \(|d\varepsilon| < \varepsilon f_\varepsilon\) on \(X\) and estimate \(I_2\) by Schwartz inequality. Then

\[
|I_2(\varepsilon)| \lesssim \varepsilon \|f_\varepsilon\alpha\|_{L^2(X)} \left( \int_X f_\varepsilon^2(|d'\alpha|^2 + |d^*\alpha|^2) \right)^{1/2},
\]

and hence \(|I_2| \to 0\) for \(\varepsilon \to 0\). \(\Box\)

Now, we begin to give the lower bound on the spectrum of the operator \(\Delta_d\) on \(\Omega^k_{(2)}\) for \(k \neq n\).

Proof of Theorem 1.3. To simply notation we shall write \(a \lesssim b\) for \(a \leq \text{const} \cdot b\) and \(a \approx b\), for \(b \approx a \lesssim b\).

Then we recall the operator \(L^k : \Omega^p \to \Omega^{2n-p}\) for a given \(p < n\) and \(p + k = n\). By the Lefschetz theorem \(L^k\) is a bijective quasi-isometry and so every \(L^2\)-form \(\psi\) of degree \(2n - p\) is the product \(\psi = L^k\phi = \omega^k \wedge \phi\), where \(\phi \in \Omega^p(2)\) and

\[
\|\psi\|_{L^2(X)} \approx \|\phi\|_{L^2(X)}.
\]

Since \(L^k\) commutes with \(\Delta_d + \Delta_{d^\Lambda}\), we also have

\[
\langle (\Delta_d + \Delta_{d^\Lambda})\psi, \psi \rangle = (L^k(\Delta_d + \Delta_{d^\Lambda})\phi, L^k\phi)
\]

\[
\approx (\langle (\Delta_d + \Delta_{d^\Lambda})\phi, \phi \rangle).
\]
Then we write $\psi = d\eta + \psi'$, for $\eta = \theta \wedge \omega^{k-1} \wedge \phi$ and $\psi' = \theta \wedge \omega^{k-1} \wedge d\phi$, and observe that
\[
\|\eta\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty(X)} \|\phi\|_{L^2(X)} \\
\lesssim \|\theta\|_{L^\infty(X)} \|\psi\|_{L^2(X)}.
\]
Next, since
\[
\|d\phi\|_{L^2(X)}^2 \lesssim \langle \Delta_d \phi, \phi \rangle \lesssim \langle (\Delta_d + \Delta_d^\Lambda) \phi, \phi \rangle \lesssim \langle (\Delta_d + \Delta_d^\Lambda) \psi, \psi \rangle,
\]
we have
\[
\|\psi'\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty(X)} \langle (\Delta_d + \Delta_d^\Lambda) \psi, \psi \rangle^{\frac{1}{2}}.
\]
Now,
\[
\|\psi\|_{L^2(X)}^2 = \langle \psi, \psi \rangle = \langle \psi, d\eta + \psi' \rangle \lesssim |\langle \psi, d\eta \rangle| + |\langle \psi, \psi' \rangle|,
\]
where
\[
|\langle \psi, d\eta \rangle| = |\langle d^* \psi, \eta \rangle| \\
\leq \langle d^* \psi, d^* \psi \rangle^{\frac{1}{2}} \|\eta\|_{L^2(X)} \\
\leq \langle \Delta_d \psi, \psi \rangle^{\frac{1}{2}} \|\eta\|_{L^2(X)} \\
\lesssim \langle \Delta_d \psi, \psi \rangle^{\frac{1}{2}} \|\theta\|_{L^\infty(X)} \|\psi\|_{L^2(X)} \\
\lesssim \|\theta\|_{L^\infty(X)} \langle (\Delta_d + \Delta_d^\Lambda) \psi, \psi \rangle^{\frac{1}{2}} \|\psi\|_{L^2(X)}
\]
and
\[
|\langle \psi, \psi' \rangle| \leq \|\psi\|_{L^2(X)} \|\psi'\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty(X)} \langle (\Delta_d + \Delta_d^\Lambda) \psi, \psi \rangle^{\frac{1}{2}} \|\psi\|_{L^2(X)}.
\]
This yields the desired estimate
\[
\|\phi\|_{L^2(X)} \lesssim \|\psi\|_{L^2(X)} \lesssim \langle (\Delta_d + \Delta_d^\Lambda) \psi, \psi \rangle^{\frac{1}{2}} \lesssim \langle (\Delta_d + \Delta_d^\Lambda) \phi, \phi \rangle^{\frac{1}{2}}
\]
for the forms $\phi$ of degree $p < n$. The case $p > n$ follows by the Poincaré duality as the operator $* : \Omega^p \to \Omega^{2n-p}$ commutes with $\Delta_d + \Delta_d^\Lambda$ and is isometric for the $L^2$-norms.

For the $d$(sublinear) case, we prove the following result.

**Proposition 4.2.** Let $(X, J, \omega)$ be a complete $2n$-dimensional almost Kähler manifold with a $d$(sublinear) symplectic form $\omega$. Then for any $k \neq n$,
\[
\mathcal{H}^k_{(2),d'}(X) = \{0\}.
\]

**Proof.** By hypothesis, there exists a 1-form $\theta$ with $\omega = d\theta$ and
\[
\|\theta(x)\|_{L^\infty(X)} \leq c(1 + \rho(x, x_0)),
\]
where $c$ is an absolute constant. In what follows we assume that the distance function $\rho(x, x_0)$ is smooth for $x \neq x_0$. The general case follows easily by an approximation argument.
Let \( \eta : \mathbb{R} \to \mathbb{R} \) be smooth, \( 0 \leq \eta \leq 1 \),

\[
\eta(t) = \begin{cases} 
1, & t \leq 0 \\
0, & t \geq 1
\end{cases}
\]

and consider the compactly supported function

\[
f_j(x) = \eta(\rho(x_0, x) - j),
\]

where \( j \) is a positive integer.

Let \( \alpha \) be a \((\Delta_d + \Delta_{d\Lambda})\)-harmonic \( k \)-form in \( L^2 \), \( k < n \), and consider the form \( \Phi = \alpha \wedge \theta \). Observing that \( d^* (\alpha \wedge \omega) = d^{k+2} (\alpha \wedge \omega) = 0 \) since \( \omega \wedge \alpha \) is a \((\Delta_d + \Delta_{d\Lambda})\)-harmonic \((k+2)\)-form in \( L^2 \), and noticing that \( f_j \Phi \) has compact support, one has

\[
0 = \langle d^* (\omega \wedge \alpha), f_j \Phi \rangle = \langle \omega \wedge \alpha, d(f_j \Phi) \rangle. \tag{4.1}
\]

We further note that, since \( \omega = d\theta \) and \( d\alpha = 0 \),

\[
0 = \langle \omega \wedge \alpha, d(f_j \Phi) \rangle = \langle \omega \wedge \alpha, f_j d\Phi \rangle + \langle \omega \wedge \alpha, df_j \wedge \Phi \rangle \tag{4.2}
\]

Since \( 0 \leq f_j \leq 1 \) and \( \lim_{j \to \infty} f_j(x)(\ast \alpha)(x) = \ast \alpha(x) \), it follows from the dominated convergence theorem that

\[
\lim_{j \to \infty} \langle \omega \wedge \alpha, f_j \omega \wedge \alpha \rangle_{L^2(X)} = \| \omega \wedge \alpha \|_{L^2(X)}^2. \tag{4.3}
\]

Since \( \omega \) is bounded, \( \text{supp}(df_j) \subset B_{j+1} \setminus B_j \) and \( \| \theta(x) \|_{L^\infty} = O(\rho(x_0, x)) \), one obtains that

\[
\| \langle \omega \wedge \alpha, df_j \wedge \theta \wedge \alpha \rangle \| \leq (j + 1)C \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx, \tag{4.4}
\]

where \( C \) is a constant independent of \( j \).

We claim that there exists a subsequence \( \{ j_i \}_{i \geq 1} \) such that

\[
\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 dx = 0. \tag{4.5}
\]

If not, there exists a positive constant \( a \) such that

\[
\lim_{j \to \infty} (j + 1) \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \geq a > 0.
\]

This inequality implies

\[
\int_X |\alpha(x)|^2 dx = \sum_{j=0}^{\infty} \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \geq a \sum_{j=0}^{\infty} \frac{1}{j + 1} = +\infty
\]
Theorem 4.3. Let \((X, J, \omega)\) be a complete \(2n\)-dimensional almost Kähler manifold with \(d\)(bounded) symplectic form \(\omega\), i.e., there exists a bounded 1-form \(\theta\) such that \(\omega = d\theta\). There is a uniform positive constant \(C\) only depends on \(n\) with following significance. If the Nijenhuis tensor \(N_J\) satisfies

\[ C\|N_J\|_{L^\infty(X)} \leq \|\theta\|_{L^\infty(X)}^{-1}, \]

then when \(n = \text{odd/even}\),

\[ \ker(d' + d^{*}) \cap \bigoplus_{k=\text{even/odd}} \Omega_{(2)}^{k} = \{0\}. \]

Proof. We only prove the case of \(n = \text{odd}\). Let \(\alpha = \sum_{k=0}^{n} \alpha_{2k} \in \Omega_{(2)}^{\text{even}}(X)\) be an \(L^2\)-form on \(X\), where \(\alpha_{2k} \in \Omega_{(2)}^{2k}(X)\). Noting that

\[ (d' + d^{*})^2 = [d', d^{*}] + (d')^2 + (d^{*})^2 = [d', d^{*}] + ([\partial_j + \bar{\partial}_j, A_j]) + ([\partial_j^*, \bar{\partial}_j^*]) = \frac{1}{4}(\Delta_d + \Delta_{d^{*}}) + ([\partial_j, \bar{\partial}_j] - \langle \partial_j, A_j \rangle) + ([\partial_j^*, \bar{\partial}_j^*] - \langle \partial_j^*, A_j^* \rangle). \]

Here we use the identities \([\partial_j + \bar{\partial}_j, A_j] = 0\) and \(\langle \partial_j^*, \bar{\partial}_j^* \rangle = 0\). By the inner product

\[ I = \langle ([\partial_j, \bar{\partial}_j] - \langle \partial_j, A_j \rangle) + ([\partial_j^*, \bar{\partial}_j^*] - \langle \partial_j^*, A_j^* \rangle) \rangle = 2\text{Re}(\langle d^{*}\alpha, \bar{\partial}_j^* \alpha \rangle - \langle A_j \alpha, d^{*}\alpha \rangle). \]

Here we use the identities \(A_j^2 = \bar{A}_j^2 = (A_j^*)^2 = (\bar{A}_j^*)^2 = 0\). Therefore, we get

\[ |I| \leq 2\|d^{*}\alpha\| \cdot \|\bar{\partial}_j^* \alpha\| + \|d'\alpha\| \cdot \|\bar{\partial}_j \alpha\| + \|A_j \alpha\| \cdot \|d^{*}\alpha\| + \|d''\alpha\| \cdot \|A_j^* \alpha\| \leq C(\|d^{*}\alpha\| + \|d'\alpha\| + \|d''\alpha\| + \|d^{*}\alpha\| + \|A_j \alpha\|) \leq C\varepsilon(\|d^{*}\alpha\|^2 + \|d'\alpha\|^2 + \|d''\alpha\|^2 + \|d^{*}\alpha\|^2 + \|A_j \alpha\|) \leq C\varepsilon \left(\frac{1}{2}\|\Delta_d + \Delta_{d^{*}}\| \cdot \|\alpha\| + \frac{1}{\varepsilon}\|N_J\|_{L^\infty(X)} \|\alpha\|^2 \right), \]

where \(C\) is a positive constant and where we have used the inequality

\[ 2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \]
for any \( \varepsilon > 0 \) and any real numbers \( a \) and \( b \). Noting that
\[
\langle (\Delta_d + \Delta_d^\Lambda) \alpha, \alpha \rangle = \sum_{k=0}^{n} (\Delta_d + \Delta_d^\Lambda) \alpha_{2k}, \alpha_{2k} \nexists \sum_{k=0}^{n} (\Delta_d + \Delta_d^\Lambda) \alpha_{2k}, \alpha_{2k} \rangle.
\]
Following Theorem \[1.3\] we then have
\[
\langle (\Delta_d + \Delta_d^\Lambda) \alpha, \alpha \rangle \geq \sum_{k=0}^{n} \lambda_0^2 \langle \alpha_{2k}, \alpha_{2k} \rangle = \lambda_0^2 \| \alpha \|^2.
\]
Therefore, we get
\[
\langle (d' + d'^*)^2 \alpha, \alpha \rangle \geq \frac{1}{4} \langle (\Delta_d + \Delta_d^\Lambda) \alpha, \alpha \rangle - |I|
\]
\[
\geq \left( \frac{1}{4} - \frac{1}{2} C \varepsilon \right) \langle (\Delta_d + \Delta_d^\Lambda) \alpha, \alpha \rangle - \frac{1}{2 \varepsilon} \| N_J \|_{L^\infty(X)} \| \alpha \|^2
\]
\[
\geq \left( \frac{1}{4} - \frac{1}{2} C \varepsilon \right) \lambda_0^2 - \frac{1}{2 \varepsilon} \| N_J \|_{L^\infty(X)} \| \alpha \|^2,
\]
We take \( \varepsilon = \frac{1}{4C} \) and \( 2C \| N_J \|_{L^\infty(X)} \leq \frac{\lambda_0^2}{16} \), hence
\[
\langle (d' + d'^*)^2 \alpha, \alpha \rangle \geq \frac{\lambda_0^2}{16} \| \alpha \|^2.
\]
Therefore, \( \ker (d' + d'^*) \cap \bigoplus_{k=even} \Lambda^k_{(2)} = \{0\} \). \( \square \)

### 4.2 Non-vanishing theorems

Let \( E \) and \( E' \) be \( C^\infty \)-vector bundles over a smooth manifold \( X \), and \( D : C^\infty(E) \to C^\infty(E') \) be a differential operator between \( C^\infty \)-sections of these bundle. We also suppose that \( X \) is a Riemannian manifold and \( \Gamma \) is a discrete group of isometrics of \( X \), such that the differential operator \( D \) commutes with the action of \( \Gamma \). We consider a \( \Gamma \)-invariant Hermitian line bundle \( (L, \nabla) \) on \( X \) we assume \( X/\Gamma \) is compact, and we state Atiyah’s \( L^2 \)-index theorem for \( D \otimes \nabla \).

**Theorem 4.4.** [\[12\] Theorem 2.3.A] Let \( D \) be a first-order elliptic operator. Then there exists a closed nonhomogeneous form
\[
I_D = I^0 + I^1 + \cdots + I^n \in \Omega^*(X) = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^n, \ n = \text{dim } X,
\]
invariant under \( \Gamma \), such that the \( L^2 \)-index of the twisted operator \( D \otimes \nabla \) satisfies
\[
L^2 \text{Index}_\Gamma (D \otimes \nabla) = \int_{X/\Gamma} I_D \wedge \exp [\omega],
\]
where \( [\omega] \) is the Chern form of \( \nabla \), and
\[
\exp [\omega] = 1 + [\omega] + \frac{[\omega] \wedge [\omega]}{2!} + \frac{[\omega] \wedge [\omega] \wedge [\omega]}{3!} + \cdots.
\]
Remark 4.5. (1) $L^2 \text{Index}_G(\mathcal{D} \otimes \nabla) \neq 0$ implies that either $\mathcal{D} \otimes \nabla$ or its adjoint has a non-trivial $L^2$-kernel.
(2) The operators $\mathcal{D}$ used in the present paper are the operators $d + d^*$ and $d' + d'^*$. In these cases the $I_0$-component of $I_\mathcal{D}$ is non-zero. Hence $\int_{X/\Gamma} I_\mathcal{D} \wedge \exp \alpha[\omega] \neq 0$, for almost all $\alpha$, provided the curvature form $[\omega]$ is “homologically nonsingular” $\int_{X/\Gamma} [\omega]^n \neq 0$.

Gromov defined the lower spectral bound $\lambda_0 = \lambda_0(\mathcal{D}) \geq 0$ as the upper bound of the negative numbers $\lambda$, such that $\|D_e\|_{L^2} \geq \lambda\|e\|_{L^2}$ for those sections $e$ of $E$ where $\mathcal{D}e$ in $L^2$. Let $\mathcal{D}$ be a $\Gamma$-invariant elliptic operator on $X$ of the first order, and let $I_\mathcal{D} = I_0^0 + I_1^1 + \cdots + I^n_n \in \Omega^*(X)$ be the corresponding index form on $X$. Let $\omega$ be a closed $\Gamma$-invariant 2-form on $X$ and denote by $I^n_\alpha$ the top component of product $I_\mathcal{D} \wedge \exp \alpha \omega$, for $\alpha \in \mathbb{R}$. Hence $I^n_\alpha$ is an $\Gamma$-invariant $n$-form on $X$, $\dim X = n$ depending on parameter $\alpha$.

Theorem 4.6. ([22] 2.4A. Theorem) Let $H^1_{dR}(X) = 0$ and let $X/\Gamma$ be compact and $\int_{X/\Gamma} I^n_\alpha \neq 0$, for some $\alpha \in \mathbb{R}$. If the form $\omega$ is $d$ (bounded), then either $\lambda_0(\mathcal{D}) = 0$ or $\lambda_0(\mathcal{D}^*) = 0$, where $\mathcal{D}^*$ is the adjoint operator.

Let $X$ be a closed almost Kähler manifold, with exact symplectic form $\omega = d\theta$ on $\tilde{X}$. Let $\Gamma = \pi_1(X)$.

For each $\varepsilon$, $\nabla_\varepsilon = d + \sqrt{-1} \varepsilon \theta$ is a unitary connection on the trivial line bundle $L = \tilde{X} \times \mathbb{C}$. One can try make it $\Gamma$-invariant by changing to a non-trivial action of $\Gamma$ on $\tilde{X} \times \mathbb{C}$, i.e., setting, for $\gamma \in \Gamma$,

$$\gamma^* (\tilde{x}, z) = (\gamma \tilde{x}, \exp \sqrt{-1} \varepsilon \theta \cdot z).$$

We want $\gamma^* \nabla_\varepsilon = \nabla_\varepsilon$, i.e., $du = - (\gamma^* \theta - \theta)$. Since $d(\gamma^* \theta - \theta) = \gamma^* \omega - \omega = 0$, there always exists a solution $u(\gamma, \cdot)$, well defined up a constant.

However, one cannot adjust the constant $\varepsilon \omega$ to obtain an action (if so, one would get a line bundle on $X$ with curvature $\varepsilon \omega$ and first Chern class $\frac{\varepsilon}{2\pi} [\omega]$). This means that the action only defined on a central extension, we call this projective representation (see [24] Chap 9).

Definition 4.7. ([24] Definition 9.2]) Let $G_\varepsilon$ be the subgroup of $\text{Diff}(\tilde{X} \times \mathbb{C})$ formed by maps $g$ which are linear unitary on fibers, preserve the connection $\nabla_\varepsilon$ and cover an element of $\Gamma$.

By construction we have an exact sequence

$$1 \to U(1) \to G_\varepsilon \to \Gamma \to 1.$$ 

Since sections of the line bundle $\tilde{X} \times \mathbb{C} \to \tilde{X}$ can be viewed as $U(1)$ equivalent functions on $\tilde{X} \times U(1)$, the operator

$$\tilde{P}_\varepsilon = \frac{\sqrt{2}}{2} (d \nabla_\varepsilon + \sqrt{-1} [L, (d') \nabla_\varepsilon]) + \frac{\sqrt{2}}{2} (I^n_\varepsilon - \sqrt{-1} [d \nabla_\varepsilon, \Lambda])$$

can be view as a $G_\varepsilon$ invariant operator on the Hilbert space $H$ of $U(1)$ equivariant basis $L^2$ differential forms on $\tilde{X} \times U(1)$. Following Theorem 4.4, the $L^2$-index of the operator $\tilde{P}_\varepsilon$ satisfies

$$L^2 \text{Index}_{G_\varepsilon}(\tilde{P}_\varepsilon) = \int_X I_{\tilde{P}_\varepsilon} \wedge \exp \left(\frac{\varepsilon}{2\pi} [\omega]\right),$$
where $I_P := I^0 + I^1 + \cdots$ denotes the Atiyah-Bott-Patodi index form of $\tilde{P} := \sqrt{2}(d' + d^*)$. Therefore $L^2\text{Index}_{G^\chi}(\tilde{P}_\varepsilon)$ is a polynomial in $\varepsilon$ whose highest degree term is $\int_X (\frac{\omega}{2\pi})^n \neq 0$.

**Theorem 4.8.** Let $(X, J, \omega)$ be a closed $2n$-dimensional special symplectic hyperbolic manifold, $\pi : (\tilde{X}, \tilde{J}, \tilde{\omega}) \to (X, J, \omega)$ the universal covering maps for $X$. Let $\Gamma = \pi_1(X)$. Then when $n = \text{even/odd}$, we have

$$\ker \tilde{P} \cap \bigoplus_{k=\text{odd/even}} \Omega^k_{(2)}(\tilde{X}) = \{0\},$$

$$\ker \tilde{P} \cap \bigoplus_{k=\text{even/odd}} \Omega^k_{(2)}(\tilde{X}) \neq \{0\}.$$  

In particular, 

$$(-1)^n \chi(X) > 0.$$  

**Proof.** The universal covering space $\tilde{X}$ is simply-connected and the lifted symplectic form $\tilde{\omega}$ is $d$(bounded). For $\varepsilon$ small enough, by Theorem 4.6 either $\ker \tilde{P} \cap \bigoplus_{k=\text{even/odd}} \Omega^k_{(2)} \neq 0$ or $\ker \tilde{P} \cap \bigoplus_{k=\text{odd/even}} \Omega^k_{(2)} \neq 0$ since $\int_X [\omega]^n \neq 0$. By the hypothesis $N_J$ is bounded from above by $\|\theta\|_{L^\infty(X)}$, we get $\pi^*(N_J)$ is also bounded from above by $\|\theta\|_{L^\infty(X)}$ since $X$ is closed and $\pi$ is a local isometry. When $n = \text{even/odd}$, according to Theorem 4.3, the spectrum of $\tilde{P}$ lies way apart from zero from possible the forms which belong to $\ker \tilde{P} \cap \bigoplus_{k=\text{even/odd}} \Omega^k_{(2)}$. Therefore, $\ker \tilde{P} \cap \bigoplus_{k=\text{even/odd}} \Omega^k_{(2)} \neq 0$. By the Positivity of the Von-Neumann Dimension [24 Section 2.1], we get 

$$\dim_{\Gamma}(\ker \tilde{P} \cap \bigoplus_{k=\text{even/odd}} \Omega^k_{(2)}) > 0, \quad \dim_{\Gamma}(\ker \tilde{P} \cap \bigoplus_{k=\text{odd/even}} \Omega^k_{(2)}) = 0.$$  

Therefore following Corollary 3.12, we have 

$$(-1)^n \chi(X) = (-1)^n L^2\text{Index}_\Gamma \tilde{P} = \dim_{\Gamma}(\ker \tilde{P} \cap \bigoplus_{k=\text{even/odd}} \Omega^k_{(2)}) > 0.$$  

\[\square\]

### 4.3 $L^2$-Hodge number

We assume throughout this subsection that $(X, g, J)$ is a closed almost Kähler $2n$-dimensional manifold with a Hermitian metric $g$, and $\pi : (\tilde{X}, \tilde{g}, \tilde{J}) \to (X, g, J)$ its universal covering with $\Gamma$ as an isometric group of deck transformations. Denote by $\mathcal{H}^k_{(2)}(\tilde{X})$ the spaces of $L^2$-harmonic $k$-forms on $\Omega^k_{(2)}(\tilde{X})$, where $\Omega^k_{(2)}(\tilde{X})$ is space of the squared integrable $k$-forms on $(\tilde{X}, \tilde{g}, \tilde{J})$, and denote by $\dim_{\Gamma} \mathcal{H}^k_{(2)}(\tilde{X})$ the Von Neumann dimension of $\mathcal{H}^k_{(2)}(\tilde{X})$ with respect to $\Gamma$ [1]. We denote by $h^k_{(2)}(X)$ the $L^2$-Hodge numbers of $X$, which are defined to be 

$$h^k_{(2)}(X) := \dim_{\Gamma} \mathcal{H}^k_{(2)}(\tilde{X}), \quad (0 \leq k \leq 2n).$$
On Euler number of symplectic hyperbolic manifold

It turns out that $h_{(2)}^k(X)$ are independent of the Hermitian metric $g$ and depend only on $X$ and $J$. By the $L^2$-index theorem of Atiyah [1], we have the following crucial identities between $\chi(X)$ and the $L^2$-Hodge numbers $h_{(2)}^k(X)$:

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k h_{(2)}^k(X).$$

Now, we give a lower bound on the spectra of the Laplace operator $\Delta_d := dd^* + d^*d$ on $L^2$-forms $\Omega^k(X)$ for $k \neq n$.

**Theorem 4.9.** Let $(X, J, \omega)$ be a complete $2n$-dimensional almost Kähler manifold with $d$(bounded) symplectic form $\omega$, i.e., there exists a bounded 1-form $\theta$ such that $\omega = d\theta$. There is a uniform positive constant $C$ only depends on $n$ with following significance. If the Nijenhuis tensor $N_J$ satisfies

$$C\|N_J\|_{L^\infty(X)} \leq \|\theta\|_{L^{\infty}(X)},$$

then every $L^2$ $k$-form $\alpha$ on $X$ of degree $k \neq n$ satisfies the inequality

$$\langle \Delta_d \alpha, \alpha \rangle \geq \frac{1}{8} \lambda_0^2 \langle \alpha, \alpha \rangle,$$

where $\lambda_0$ is positive constant in Theorem [1.3]. In particular,

$$\mathcal{H}_d^k(X) = \{0\}$$

unless $k \neq n$.

**Proof.** Following the identity on Proposition 3.2, we get

$$\begin{align*}
\Delta_d & = \frac{1}{2}(\Delta_d + \Delta_d^*) + 2[\partial_J, \bar{\partial}_J^*] + 2[\bar{\partial}_J, \partial_J^*] \\
& = \frac{1}{2}(\Delta_d + \Delta_d^*) + 2[\bar{A}_J^*, \partial_J] + 2[A_J, \partial_J^*] + 2[A_J^*, \partial_J] + 2[\bar{A}_J, \bar{\partial}_J] \\
& = \frac{1}{2}(\Delta_d + \Delta_d^*) + 2(I_1 + I_2 + I_3 + I_4)
\end{align*}$$

Here we use the identities (3) in Proposition 2.5. By the inner product

$$\begin{align*}
\langle I_1 \alpha, \alpha \rangle &= \langle \bar{\partial}_J \alpha, \bar{A}_J \alpha \rangle + \langle \bar{A}_J^* \alpha, \partial_J^* \alpha \rangle \\
& = \langle (\bar{\partial}_J + A_J) \alpha, \bar{A}_J \alpha \rangle - \langle A_J \alpha, \bar{A}_J \alpha \rangle + \langle \bar{A}_J^* \alpha, (\bar{\partial}_J^* + A_J^*) \alpha \rangle - \langle \bar{A}_J^* \alpha, A_J^* \alpha \rangle \\
& = \langle d'' \alpha, \bar{A}_J \alpha \rangle - \langle A_J \alpha, \bar{A}_J \alpha \rangle + \langle \bar{A}_J^* \alpha, d'' \alpha \rangle - \langle \bar{A}_J^* \alpha, A_J^* \alpha \rangle \\
& = \langle d'' \alpha, \bar{A}_J \alpha \rangle + \langle A_J^* \alpha, d'' \alpha \rangle - \langle [A_J^*, A_J] \alpha, \alpha \rangle \\
& = \langle d'' \alpha, \bar{A}_J \alpha \rangle + \langle A_J^* \alpha, d'' \alpha \rangle.
\end{align*}$$

Here we use the identity $[\bar{A}_J^*, A_J] = 0$. Similarly,

$$\begin{align*}
\langle I_2 \alpha, \alpha \rangle &= \langle d^* \alpha, A_J^* \alpha \rangle + \langle A_J \alpha, d' \alpha \rangle \\
\langle I_3 \alpha, \alpha \rangle &= \langle d' \alpha, A_J \alpha \rangle + \langle A_J^* \alpha, d^* \alpha \rangle \\
\langle I_4 \alpha, \alpha \rangle &= \langle d'' \alpha, A_J^* \alpha \rangle + \langle A_J \alpha, d'' \alpha \rangle
\end{align*}$$
Therefore, we obtain
\[
|\langle (I_1 + I_4)\alpha, \alpha \rangle| \leq 2\|d''\alpha\| \cdot \|\bar{A}_J\alpha\| + 2\|A^*_J\alpha\| \cdot \|d''^*\alpha\|
\]
\[
\leq \frac{1}{4}(\|d''\alpha\|_{L^2(X)}^2 + \|d''^*\alpha\|_{L^2(X)}^2) + 4\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2
\]
\[
\leq \frac{1}{4}(\|d', d''\alpha\| + 4\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2
\]
\[
\leq \frac{1}{16}(\langle (\Delta_d + \Delta_d^\alpha)\alpha, \alpha \rangle + 4\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2),
\]
and
\[
|\langle (I_2 + I_3)\alpha, \alpha \rangle| \leq 2\|d'\alpha\| \cdot \|A_J\alpha\| + 2\|A^*_J\alpha\| \cdot \|d^*\alpha\|
\]
\[
\leq \frac{1}{4}(\|d'\alpha\|_{L^2(X)}^2 + \|d^*\alpha\|_{L^2(X)}^2) + 4\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2
\]
\[
\leq \frac{1}{4}(\|d'^*\alpha\| + 4\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2
\]
\[
\leq \frac{1}{16}(\langle (\Delta_d + \Delta_d^\alpha)\alpha, \alpha \rangle + 4\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2),
\]
where \(C\) is a positive constant only depend on \(n\). Combining above inequalities, we get
\[
\langle \Delta_d\alpha, \alpha \rangle \geq \frac{1}{2}(\langle (\Delta_d + \Delta_d^\alpha)\alpha, \alpha \rangle - 2|\langle (I_1 + I_4)\alpha, \alpha \rangle| - 2|\langle (I_2 + I_3)\alpha, \alpha \rangle|
\]
\[
\geq \frac{1}{4}(\langle (\Delta_d + \Delta_d^\alpha)\alpha, \alpha \rangle - 16C\|N_J\|_{L^\infty(X)}^2 \|\alpha\|_{L^2(X)}^2
\]
\[
\geq \left(\frac{1}{4}\lambda_0^2 - 16C\|N_J\|_{L^\infty(X)}^2\right) \|\alpha\|_{L^2(X)}^2.
\]
If the Nijenhuis tensor \(N_J\) satisfies
\[
16C\|N_J\|_{L^\infty}^2 \leq \frac{1}{8}\lambda_0^2,
\]
then
\[
\langle \Delta_d\alpha, \alpha \rangle \geq \frac{\lambda_0^2}{8\|\alpha\|_{L^2(X)}^2}.
\]
\[\square\]

Let \(L \to X\) be a vector bundle equipped with a Hermitian metric and Hermitian connection \(\nabla\). Then there is an induced exterior differential \(d^\nabla\) on \(\Omega^*(X) \otimes L\). If \(D = d^\nabla + (d^*)^\nabla\), then Atiyah-Singer’s index theorem states
\[
\text{Index}(D) = \int_X \mathcal{L}_X \wedge \text{Ch}(L).
\]
Here \(\mathcal{L}_X\) is Hizebruch’s class,
\[
\mathcal{L}_X = 1 + \cdots + e(X)
\]
where \(1 \in H^0(X)\) and \(e(X) \in H^{\dim X}(X)\) is the Euler class. For each \(\varepsilon\), \(\nabla_\varepsilon = d + \sqrt{-1}\varepsilon\theta\) is a unitary connection on the trivial line bundle \(L = \tilde{X} \times \mathbb{C}\). The operator \(D_\varepsilon := d^\nabla_\varepsilon + (d^*)^\nabla_\varepsilon\) can be view as a \(G_\varepsilon\) operator on the Hilbert space \(H\) of \(U(1)\) equivalent basis \(L^2\) differential forms on \(\tilde{X} \times U(1)\) [24].
Theorem 4.10. ([24, Theorem 9.3]) The operator $\tilde{D}_\varepsilon$ has a finite projective $L^2$ index given by

$$L^2\text{Index}_{G_\varepsilon}(\tilde{D}_\varepsilon) = \int_X L_X \wedge \exp\left(\frac{\varepsilon}{2\pi}\omega\right).$$

Now, we begin to proof the Singer conjecture under the special symplectic hyperbolic case.

**Proof of Theorem 1.6** This number is a polynomial in $\varepsilon$ whose highest degree term is $\int_X \left(\frac{\omega}{2\pi}\right)^n \neq 0$ thus for $\varepsilon$ small enough, $\tilde{D}_\varepsilon$ has a non-zero $L^2$ kernel. By construction, $\tilde{D}_\varepsilon$ is an $\varepsilon$-small perturbation of $d + d^*$, so $d + d^*$ is not invertible. For any $k \neq n$, $\mathcal{H}^k_{(2)}(\tilde{X}) = \{0\}$, i.e., $h^k_{(2)}(X) = 0$. Therefore, we get $\mathcal{H}^n_{(2)}(\tilde{X}) \neq \{0\}$, i.e, $h^n_{(2)}(X) > 0$. Hence

$$(-1)^n \chi(X) = (-1)^n \sum_{k=0}^{2n} (-1)^k h^k_{(2)}(X) = h^n_{(2)}(X) > 0.$$

☐

**Acknowledgements**

We would like to thank Professor H.Y. Wang for drawing our attention to the symplectic cohomology. I would like to thank S.O. Wilson and J. Cirici for helpful comments regarding their article [3, 4]. We would also like to thank the anonymous referee for careful reading of my manuscript and helpful comments. This work is supported by the National Natural Science Foundation of China (Nos. 12271496, 11801539) and the Youth Innovation Promotion Association CAS, the Fundamental Research Funds of the Central Universities, the USTC Research Funds of the Double First-Class Initiative.

**References**

[1] Atiyah, M., *Elliptic operators, discrete group and Von Neumann algebras*. Astérisque. **32–33** (1976), 43–72.

[2] Brylinski, J.-L., *A differential complex for Poisson manifolds*. J. Differential Geom. **28** (1988), 93–114.

[3] Cirici, J., Wilson, S.O., *Topological and geometric aspects of almost Kähler manifolds via harmonic theory*. Sel. Math. New Ser. **26** (2020), 35.

[4] Cirici, J., Wilson, S.O., *Dolbeault cohomology for almost Kähler manifolds*. Adv. Math. **391** (2021), 107970.

[5] Cao, J. G., Xavier, F., *Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature*. Math. Ann. **319** (2001), 483–491.

[6] Chern, S. S., *On curvature and characteristic classes of a Riemannian manifold*. Abh. Math. Sem. Univ. Hamburg. **20** (1955), 117–126.

[7] Chen, B. L., Yang, X. K., *Compact Kähler manifolds homotopic to negatively curved Riemannian manifolds*. Math. Ann. **370** (2018), 1477–1489.

[8] de Bartolomeis, P., Tomassini, A., *On formality of some symplectic manifolds*. Int. Math. Res. Not. **24** (2001), 1287–1314.
| Reference | Text                                                                                     |
|-----------|------------------------------------------------------------------------------------------|
| [9]       | Dodziuk, J., *L²* harmonic forms on rotationally symmetric Riemannian manifolds.* Proc. Amer. Math. Soc. 77(3) (1979), 395–400. |
| [10]      | Dodziuk, J., *L²* harmonic forms on complete manifolds. In: Yau, S. T. (ed.) Seminar on Differential Geometry, Princeton University Press, Princeton. Ann. Math Studies, 102 (1982), 291–302. |
| [11]      | Geroch, R., Positive sectional curvatures does not imply positive Gauss-Bonnet integrand. Proc. Amer. Math. Soc. 54 (1976), 267–70. |
| [12]      | Gromov, M., Kähler hyperbolicity and *L²*-Hodge theory. J. Differential Geom. 33 (1991), 263–292. |
| [13]      | Hind, R., Tomassini, A., *On *L²*-cohomology of almost Hermitian manifolds.* J. Symplectic Geom. 17 (2019), 1773–1792. |
| [14]      | Huang, T., A note on Euler number of locally conformally Kähler manifolds. Math. Z. 296 (2020), 1725–1733. |
| [15]      | Huang, T., Tan, Q., *L²*-hard Lefschetz complete symplectic manifolds. Ann. Mat. Pura Appl. 200 (2021), 505–520. |
| [16]      | Hernández, L., Kähler manifolds and *L²*pinching. Duke Math. J. 62 (1991), 601–611. |
| [17]      | Hernández, L., Curvature vs. almost Hermitian structures. Geom. Dedicata 79 (2000), 205–218. |
| [18]      | Huybrechts, D., Complex geometry: an introduction. Springer Science and Business Media. (2006) |
| [19]      | Jost, J., Xin, Y. L., Vanishing theorems for *L²*-cohomology groups. J. Reine Angew. Math. 525 (2000), 95–112. |
| [20]      | Jost, J., Zuo, K., Vanishing theorems for *L²*-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. Comm. Anal. Geom. 8 (2000), 1–30. |
| [21]      | Mathieu, O., Harmonic cohomology classes of symplectic manifolds. Comm. Math. Helv. 70 (1995), 723–733. |
| [22]      | Murakoshi, N., Oguro, T., Sekigawa, K., Four–dimensional almost Kähler locally symmetric spaces. Differential Geom. Appl. 6 (1996), 237–244. |
| [23]      | Nicolaescu, Li., Notes on the Atiyah-Singer Index Theorem. Notes for a topics in topology course, University of Notre Dame (2013). |
| [24]      | Pansu, P., Introduction to *L²* Betti numbers. Riemannian geometry (Waterloo, ON, 1993) 4 (1993), 53–86. |
| [25]      | Sekigawa, K., On some compact Einstein almost Kähler manifolds. J. Math. Soc. Japan 39 (1987), 677–684. |
| [26]      | Sekigawa, K., Vanhecke, L., Four-dimensional almost Kähler Einstein manifolds. Ann. Mat. Pura Appl. 157 (1990), 149–160. |
| [27]      | Singer, I.M., Some remarks on operator theory and index theory. In K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pages 128–138. Lecture Notes in Math., Vol. 575. Springer-Verlag, Berlin, 1977. |
| [28]      | Tomassini, A., Wang, X., Some results on the Hard Lefschetz Condition. Internat. J. Math. 29 (2018), 1850095 (30 pages) |
| [29]      | Tan, Q., Wang, H. Y., Zhou, J. R., Symplectic parabolicity and *L²* symplectic harmonic forms. Q. J. Math. 70 (2019), 147–169. |
| [30]      | Tseng, L.S., Yau, S.T., Cohomology and Hodge theory on symplectic manifolds: I. J. Differential Geom. 91 (2012), 383–416. |
| [31]      | Tseng, L.S., Yau, S.T., Cohomology and Hodge theory on symplectic manifolds: II. J. Differential Geom. 91 (2012), 417–443. |
| [32]      | Tseng, L.S., Yau, S.T., Cohomology and Hodge theory on symplectic manifolds: III. J. Differential Geom. 103 (2016), 83–143. |
| [33]      | Yan, D., Hodge structure on symplectic manifolds. Adv. Math. 20 (1996), 143–154. |