Computation of the $\omega$-primality and asymptotic $\omega$-primality with applications to numerical semigroups

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Abstract

We give an algorithm to compute the $\omega$-primality of finitely generated atomic monoids. Asymptotic $\omega$-primality is also studied and a formula to obtain it in finitely generated quasi-Archimedean monoids is proven. The formulation is applied to numerical semigroups, obtaining an expression of this invariant in terms of its system of generators.

Keywords: Asymptotic $\omega$-primality, commutative monoid, finitely generated monoid, numerical semigroup, quasi-Archimedean monoid, $\omega$-primality.

MSC-class: 20M14, 13A05 (Primaries), 13F15, 20M05 (Secondaries).

Introduction

All semigroups appearing in this paper are commutative. For this reason, in the sequel we will omit this adjective. Here, an atomic monoid means a commutative cancellative semigroup with identity element such that every non-unit may be expressed as a sum of finitely many atoms (irreducible elements).

Problems involving non-unique factorizations in atomic monoids and integral domains have gathered much recent attention in the mathematical literature (see for instance [10] and the references therein). Let $S$ be a monoid, the $\omega$-invariant, introduced in [8], is a well-established invariant in the theory of non-unique factorizations, and appears also in the context of direct-sum decompositions of modules [4]. This invariant essentially measures how far an element of an integral domain or a monoid is from being prime (see [1]). In [7] it is proven that the tame degree and $\omega$-primality coincide for half-factorial affine semigroups and...
in [2] and [7] the ω-primality is computed for some kinds of affine semigroups (when the semigroup is the intersection of a group and \( \mathbb{Z}^p \)). Associated with the ω-primality there is its asymptotic version, the asymptotic ω-primality or \( \omega \)-primality, which is object of study in several works. In [3], the \( \omega \)-primality is studied for numerical semigroups generated by two elements and it is given a formula for its computation, but no other studies provide methods to calculate this invariant in other types of monoids.

Our study uses that every finitely generated commutative monoid is finitely presented (see [13]). Thus, every monoid can be completely described in terms of its presentations or from a system of generators. In [2] and [7] the \( \omega \)-primality is computed for some kinds of affine semigroups (when the semigroup is the intersection of a group and \( \mathbb{Z}^p \)). The first goal of this work is to give an algorithm to compute from a presentation of a finitely generated atomic monoid the ω-primality of any of its elements. Our second goal is for finitely generated quasi-Archimedean cancellative monoids (note that every numerical semigroup belongs to this class of monoids). For them we give an explicit formulation of the asymptotic ω-primality of its elements, and a method to compute their asymptotic ω-primality.

All the theoretical results of this work are complemented with the software **OmegaPrimality** developed in *Mathematica* (see [6]). This software provides functions to compute the ω-primality of a monoid and its elements from one of its presentations or from a system of generators.

The contents of this paper are organized as follows. In Section 1, we provide some basic tools and definitions that are used in the rest of the work. In Section 2, we recall the definitions of atomic monoid and ω-primality and we give an algorithm to compute the ω-primality of an element. In Section 3, we start by studying the ω-primality and the \( \omega \)-primality in atomic monoids minimally generated by two elements. Finally, Section 4 is devoted to give a formulation of the \( \omega \)-primality in quasi-Archimedean atomic monoids. We finish obtaining the formulation of this invariant for numerical semigroups.

## 1 Preliminaries

Every finitely generated monoid \( S \) is isomorphic to a quotient of the form \( \mathbb{N}^p/\sigma \) with \( \sigma \) a congruence on \( \mathbb{N}^p \). Thus, if \( a \in S \), it can be written as \( a = [\gamma]_\sigma \) with \( \gamma \in \mathbb{N}^p \), where \( [\gamma]_\sigma \) denotes the class of equivalence of \( \gamma \).

We denote as \( \{e_1,\ldots,e_p\} \) the minimal generating set of the monoid \( \mathbb{N}^p \) and denote by \( \| (\delta_1,\ldots,\delta_p) \| = \sum_{i=1}^p \delta_i \) the length of \( (\delta_1,\ldots,\delta_p) \in \mathbb{N}^p \). Given \( \lambda = (\lambda_1,\ldots,\lambda_p), \mu = (\mu_1,\ldots,\mu_p) \in \mathbb{Z}^p \), we define the usual cartesian product order \( \leq \) on \( \mathbb{Z}^p \) as \( \lambda \leq \mu \) if and only if \( \mu - \lambda \in \mathbb{N}^p \). If \( A \subseteq \mathbb{N}^p \), define \( \text{Minimals} \subseteq A \) as the set of the minimal elements with respect to the order \( \leq \). Given two elements \( \lambda,\mu \in \mathbb{N}^p \), define \( \lambda \lor \mu = (\max\{\lambda_1,\mu_1\},\ldots,\max\{\lambda_p,\mu_p\}) \). For any \( L \) subset of \( \mathbb{R}^p \), denote by \( L_\geq \) the set \( \{(x_1,\ldots,x_p) \in L | x_i > 0, i = 1,\ldots,p \} \) and by \( L_\geq \) the set \( \{(x_1,\ldots,x_p) \in L | x_i \geq 0, i = 1,\ldots,p \} \).

Let \( a,b \in S \). We say that \( a \) divides \( b \) when there exists \( c \in S \) such that \( a + c = b \), we denote it by \( a|b \). The elements \( a,b \in S \) are associated if \( a|b \) and \( b|a \). When an element \( a \in S \) verifies that there exists \( b \in S \) such that \( a + b = 0 \), then it is called a unit; the set of units of \( S \) is denoted by \( S^\times \). An element \( x \in S \) is an atom if it fulfills that \( x \not\in S^\times \) and if \( a|x \), then either \( a \in S^\times \) or \( a \) and \( x \) are associated. If the semigroup \( S \setminus S^\times \) is generated by its set of atoms \( \mathcal{A}(S) \), the
monoid $S$ is called an atomic monoid. It is known that every non-group finitely
generated cancellative monoid is atomic (see \cite{14} Corollary 16).

A subset $I$ of a monoid $S$ is an ideal if $I+S \subseteq I$. It is straightforward to
prove that for every $a \in S$ the set $a+S = \{a+c \mid c \in S\} = \{s \in S \mid a$ divides $s\}$
is an ideal of $S$.

2 Computing the $\omega$-primality in atomic monoids

In this section we show an algorithm to compute the $\omega$-primality of an element in
a finitely generated atomic monoid from one of its presentations. The knowledge
of the computation of $\omega(a)$ for every atom $a$ can be directly used to obtain $\omega(S)$
(see \cite{1}, Definition 1.1). There exist algorithms to compute $\omega$ in some kinds of
monoids, for instance in numerical monoids (see \cite{3} and \cite{2} Remark 5.9.1), in
half-factorial affine semigroups (see \cite{7}), in saturated affine semigroups (see \cite{2},
Corollary 3.5), but there is not a general method for its computation in more
general situations.

Definition 1. (See Definition 1.1 in \cite{3}.) Let $S$ be an atomic monoid with set of
units $S^*$ and set of irreducibles $A(S)$. For $s \in S \setminus S^*$, we define $\omega(x) = n$ if $n$ is
the smallest positive integer with the property that whenever $x|a_1+\cdots+a_t$, where
each $a_i \in A(S)$, there is a $T \subseteq \{1, 2, \ldots, t\}$ with $|T| \leq n$ such that $x|\sum_{k \in T} a_k$.
If no such $n$ exists, then $\omega(s) = \infty$. For $x \in S^*$, we define $\omega(x) = 0$.

Recall that every finitely generated monoid $S$ is isomorphic to $\mathbb{N}^p/\sigma$, for
some congruence $\sigma$ on $\mathbb{N}^p$ and some positive integer $p$. For sake of simplicity,
we will identify $S$ with $\mathbb{N}^p/\sigma$. Denote by $\varphi : \mathbb{N}^p \to \mathbb{N}^p/\sigma$ the projection map.
For every $A \subset \mathbb{N}^p/\sigma$ denote by $E(A)$ the set $\varphi^{-1}(A)$. Note that for every $a \in S$
the set $E(a+S)$ is an ideal of $\mathbb{N}^p$.

The following result is proven in \cite{2} Proposition 3.3] for atomic monoids.

Proposition 2. Let $S = \mathbb{N}^p/\sigma$ be a finitely generated atomic monoid and $a \in S$.
Then $\omega(a)$ is equal to max\{\|\delta\| : \delta \in \text{Minimals}_{\leq} (E(a+S))\}.

The set $a+S$ collects all the multiples of $a$, and therefore in many cases
it is not a finite set. The problem now is to compute its minimal elements
with respect $\leq$ the usual cartesian product order on $\mathbb{N}^p$. A solution is given
by Algorithm 16 of \cite{16} which computes the set $\text{Minimals}_{\leq} E(I)$ for every ideal
$I$ of $S$. The following algorithm computes the $\omega$-primality of an element of an
atomic monoid.

Algorithm 3. Input: A finite presentation of $S = \mathbb{N}^p/\sigma$ and $\gamma$ an element of $\mathbb{N}^p$
verifying that $a = [\gamma]_{\sigma}$.
Output: $\omega(a)$.

(Step 1) Compute the set $\Delta = \text{Minimals}_{\leq} (E([\gamma]_{\sigma} + S))$ using \cite{16} Algorithm
16.

(Step 2) Set $\Psi = \{\|\mu\| : \mu \in \Delta\}$.

(Step 3) Return max $\Psi$.  

We illustrate now Algorithm 3 with some examples (all the computations were done in an Intel Core i7 with 16 GB of main memory). Note that the inputs in each example are the minimal generator set of the semigroup and an element of the semigroup. The presentations and the expressions of the elements in terms of the atoms are computed internally by the program.

**Example 4.** Let $S$ be the affine semigroup generated by $\{(5,3), (5,11), (2,7), (11,4)\}$. To compute the $\omega$-primality we use the package **OmegaPrimality** developed for this work, which is available in [6]. For the element $\langle 154, 118 \rangle$ we obtained the output

\[
\text{In[1]} := \text{OmegaPrimalityOfElemAffSG}\{\{154, 118\}, \\
\{\{5,3\}, \{5,11\}, \{2,7\}, \{11,4\}\}\}
\]

The expression of the element is $\{3,5,2,10\}$

Length of output of Alg16=40

Out[1]= 68

In this case, an expression of $\langle 154, 118 \rangle$ is $3(5,3) + 5(5,11) + 2(2,7) + 10(11,4)$, the size of Minimals $\leq (E(\gamma) + S)$ (the output of [16, Algorithm 16]) is equal to 40 and the $\omega$-primality of the element is 68.

In order to compare the timings obtained using Algorithm 3 and the numericalsgps GAP package [12], which uses the method described in [2], we consider the following examples. Note that the implementation of $\omega$-primality in the numericalsgps relies on the construction of Apéry set while our implementation is based on Gröbner basis calculations. In any case, the procedures in the numericalsgps package only apply to numerical semigroups, while the algorithm presented here applies also for any affine semigroup.

**Example 5.** Let $S$ be the numerical semigroup generated by $\{115, 212, 333, 571\}$. We use again the package **OmegaPrimality** obtaining for the element 10000 the output

\[
\text{In[1]} := \text{OmegaPrimalityOfElemAffSG}\{\{10000\}, \\
\{\{115\}, \{212\}, \{333\}, \{571\}\}\}
\]

The expression of the element is $\{3,2,2,15\}$

Length of output of Alg16=203

Out[1]= 109

The timing was approximately 22 seconds and the value obtained for $\omega(10000)$ was equal to 109. Using the function **OmegaPrimalityOfElementOfNumericalSemigroup** of numericalsgps GAP package we obtained

\[
gap> \text{OmegaPrimalityOfElementInNumericalSemigroup}(10000, S);
\]

109

with a timing of approximately 1389 seconds (and the same value for $\omega(10000)$).

**Example 6.** The $\omega$-primality of an atomic monoid $S$ is defined as $\omega(S) = \sup\{\omega(x) | x \text{ is irreducible}\}$. Hence, Algorithm 3 can be used to compute $\omega(S)$ when $S$ is atomic and finitely generated just computing the $\omega$-primality of its generators and taking the maximum. This is implemented in function **OmegaPrimalityOfAffSG**.
In[2]:= OmegaPrimalityOfAffSG[{{115}, {212}, {333}, {571}}]

... w-primalities of the generators: {15, 36, 36, 36}
Out[2]= 36

In this case, the \(\omega\)-primality of the generators are 15, 36, 36 and 36, respectively, and this computation took 496 milliseconds. Via the numericalsgps functions

gap> OmegaPrimalityOfNumericalSemigroup(S);
36

it took 1888 milliseconds.

Example 7. Consider the numerical semigroup \(S\) minimally generated by \(\{10, \ldots, 19\}\). The package OmegaPrimality needed approximately 3779 milliseconds for computing \(\omega(S)\),

In[1]:= OmegaPrimalityOfAffSG[{{10}, {11}, {12}, {13},
{14}, {15}, {16}, {17}, {18}, {19}}]

... w-primalities of the generators: {2, 3, 3, 3, 3, 3, 3, 3, 3, 3}
Out[1]= 3

while the numericalsgps functions took 125 milliseconds.

Example 8. Consider now \(S = \langle 101, 111, 121, 131, 141, 151, 161, 171, 181, 191 \rangle\), its \(\omega\)-primality was computed in approximately 135081 milliseconds,

In[1]:= OmegaPrimalityOfAffSG[{{101}, {111}, {121}, {131},
{141}, {151}, {161}, {171}, {181}, {191}}]

... w-primalities of the generators: {12, 23, 22, 22, 22, 22, 22, 22, 22, 22}
Out[1]= 23

and the numericalsgps functions took just 383949 milliseconds.

After some comparisons between the times required to obtain the \(\omega\)-primality using our implementation and numericalsgps, we can conclude that the larger are the elements or generators, the better performance one gets with our procedure. But, if there are many generators and small, then one should use the Apéry method.

3 Asymptotic \(\omega\)-primality in monoids generated by two elements

Let \(S\) be an atomic monoid and \(x \in S\), define \(\varpi(x) = \lim_{n \to +\infty} \omega(nx)\) the asymptotic \(\omega\)-primality of \(x\). Asymptotic \(\omega\)-primality of \(S\) is defined as \(\varpi(S) = \sup\{\varpi(x) | x \text{ is irreducible}\}\) (see \([1]\)). As in the preceding section if \(S = \langle s_1, \ldots, s_p \rangle\), then \(\varpi(S)\) is equal to \(\max\{\varpi(s_i) | i = 1, \ldots, p\}\).

In this section we focus our attention on cancellative reduced monoids minimally generated by two elements. These monoids are all isomorphic to monoids of the form \(\mathbb{N}/\sigma\), and they are atomic (see \([14]\)). The following result shows how their congruences are. Note that in particular all numerical monoids generated by two elements are monoids of this type.
Lemma 9. A non-free monoid $S$ is cancellative, reduced and minimally generated by two elements if and only if $S \cong \mathbb{N}^2 / \sigma$ with $\sigma = \langle (\alpha, 0), (0, \beta) \rangle$ and $\alpha, \beta > 1$.

**Proof.** Assume that $S$ is minimally generated by two elements. There exists a congruence $\sigma$ such that $S \cong \mathbb{N}^2 / \sigma$. From Propositions 3.6 and 3.10 in [17], there exist positive integers $a$ and $b$ verifying that if $(x, y)\sigma(x_0, y_0)$ then $a(x - x_0) + b(y - y_0) = 0$. Let $M$ be the subgroup of $\mathbb{Z}^2$ defined by the equation $ax + by = 0$. Then $M = \langle (\alpha, -\beta) \rangle$ with $\alpha = b/d$ and $\beta = a/d$, with $d = \gcd(a, b)$. Hence $(x - x_0, y - y_0) = k(\alpha, -\beta)$ for some integer $k$, and thus $(x, y) = (x_0, y_0) + k(\alpha, -\beta)$. From this easily follows that $\sigma = \langle (\alpha, 0), (0, \beta) \rangle$.

Since $S$ is not free, $\sigma$ is not trivial and therefore with $\alpha, \beta \neq 0$. Using now that $S$ is reduced and minimally generated by two elements we obtain that $\alpha, \beta > 1$.

Assume now that $S \cong \mathbb{N}^2 / \sigma$ with $\sigma = \langle (\alpha, 0), (0, \beta) \rangle$ and $\alpha, \beta > 1$. Using that $\alpha, \beta > 1$ it is straightforward to prove that $S$ is minimally generated by two elements and that $S$ is reduced. We see that $S$ is cancellative. For every $\gamma \in \mathbb{N}^2$, $E(\gamma) = \{ \gamma + \lambda(\alpha, -\beta) | \lambda \in \mathbb{Z} \} \cap \mathbb{N}^2$. This implies that for all $\gamma' \in \mathbb{N}^2$, we have $\gamma \sim \gamma'$ if and only if $\gamma - \gamma'$ belongs to the subgroup $G$ of $\mathbb{Z}^2$ generated by $\langle (\alpha, -\beta) \rangle$. Hence, the congruence $\sigma$ is defined by a subgroup of $\mathbb{Z}^2$ and hence $S$ is cancellative (see [17 Proposition 1.4]). 

In order to compute the asymptotic $\omega$-primality of an element $[\gamma]_\sigma \in S$ we give explicitly Minimals$_\leq (E(\gamma) + S)$ and $\omega([\gamma]_\sigma)$. Denote by $[\frac{a}{b}]$ the integer part of $\frac{a}{b}$ with $a, b \in \mathbb{N}$ and $b \neq 0$.

**Lemma 10.** Let $S = \mathbb{N}^2 / \sigma$ with $\sigma = \langle (\alpha, 0), (0, \beta) \rangle$ and $\alpha, \beta > 1$. Then for all $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$, $E(\gamma) = \{ \gamma + \lambda(\alpha, -\beta) | \lambda \in \mathbb{Z}, -\frac{\gamma_1}{\alpha} \leq \lambda \leq \frac{\gamma_2}{\beta} \}$.

Minimals$_\leq (E(\gamma) + S)$

$= \text{Minimals}_\leq (E(\gamma) + S) \cup \{ (0, \gamma_2 + (\frac{\gamma_1}{\alpha} + 1) \beta, (\gamma_1 + (\frac{\gamma_2}{\beta} + 1) \alpha, 0) \}$ (1)

and $\omega([\gamma]_\sigma) = \max\{ \gamma_2 + (\frac{\gamma_1}{\alpha} + 1) \beta, \gamma_1 + (\frac{\gamma_2}{\beta} + 1) \alpha \}$.

**Proof.** For every element $\gamma$ the set $E(\gamma)$ is equal to $\{ \gamma + \lambda(\alpha, -\beta) | \lambda \in \mathbb{Z} \} \cap \mathbb{N}^2$. Note that $\gamma + \lambda(\alpha, -\beta) \in \mathbb{N}^2$ if and only if $-\frac{\gamma_1}{\alpha} \leq \lambda \leq \frac{\gamma_2}{\beta}$.

Using now that for every element $z$ in $[\gamma]_\sigma + S$ there exists $\epsilon \in \mathbb{N}^2$ such that $z = [\gamma + \epsilon]_\sigma$, we obtain $E([\gamma]_\sigma + S) = \{ \gamma + \lambda(\alpha, -\beta) | \lambda \in \mathbb{Z} \} \cap \mathbb{N}^2$. To describe this set we need to obtain the intersection of $E([\gamma]_\sigma$ with the axes. These are the points $(0, \gamma_2 + (\frac{\gamma_1}{\alpha} + 1) \beta)$ and $(\gamma_1 + (\frac{\gamma_2}{\beta} + 1) \alpha, 0)$. Thus $E([\gamma]_\sigma + S) = E([\gamma]_\sigma + \mathbb{N}^2) \cup ((0, \gamma_2 + (\frac{\gamma_1}{\alpha} + 1) \beta) + \mathbb{N}^2) \cup ((\gamma_1 + (\frac{\gamma_2}{\beta} + 1) \alpha, 0) + \mathbb{N}^2)$ and therefore (1) is satisfied.

The set $\{ \|x\| \in \mathbb{N}^2 | x \in E([\gamma]_\sigma) \}$ is equal to $\{ \gamma_1 + \gamma_2 + \lambda(\alpha, -\beta) | \lambda \in \mathbb{Z}, -\frac{\gamma_1}{\alpha} \leq \lambda \leq \frac{\gamma_2}{\beta} \}$. If $\alpha \geq \beta$ the maximum is achieved by the element $\gamma + [\frac{\gamma_2}{\beta}] (\alpha, -\beta)$. Using now that $\gamma_2 - [\frac{\gamma_2}{\beta}] \beta \leq \beta$, we obtain $\| \gamma + [\frac{\gamma_2}{\beta}] (\alpha, -\beta) \| \leq \| (\gamma_1 + [\frac{\gamma_2}{\beta}] + 1) \alpha, 0) \|$. Analogously, if $\alpha \leq \beta$ it can be proved that $\| \gamma - [\frac{\gamma_1}{\alpha}] (\alpha, -\beta) \| \leq \| (0, \gamma_2 + (\frac{\gamma_1}{\alpha} + 1) \beta) \|$. Hence, $\omega([\gamma]_\sigma) = \max\{ \gamma_2 + (\frac{\gamma_1}{\alpha} + 1) \beta, \gamma_1 + (\frac{\gamma_2}{\beta} + 1) \alpha \}$.
the notation we could say that

Proposition 13. Let \( \omega(\beta) = \omega([a, b])_{\sigma} = ||(a, b) + q(\alpha, -\beta) + (\alpha, -r)|| \)
when \( b = \beta q + r \) with \( 0 \leq r < \beta \).

Example 11. Let us consider the congruence \( \sigma = \langle (7, 0), (0, 5) \rangle \) and take \( \gamma = (6, 7) \). In this case, the ideal \( E(\gamma_{\sigma} + N^2/\sigma) \) is showed in Figure [1] and it is generated by the elements \( \{0, 12, 6, 7, 13, 2, 20, 0\} \). The \( \omega \)-primality of \( (6, 7)_{\sigma} \) is

![Figure 1: E((6, 7)_{\sigma} + N^2/\sigma).](image)

equal to \( \max\{0 + 12, 6 + 7, 13 + 2, 20 + 0\} = 20 \).

Lemma 12. For all \( a, b \in \mathbb{N} \) with \( b \neq 0 \), it holds that \( \lim_{n \to \infty} \left\lfloor \frac{na}{b} \right\rfloor = \frac{a}{b} \).

Proof. For every \( n \in \mathbb{N} \), \( na \) can be expressed as \( na = bm + t \) with \( m \in \mathbb{N} \) and \( 0 \leq t < b \). Hence, \( \left\lfloor \frac{na}{b} \right\rfloor = \left\lfloor \frac{bm + t}{b} \right\rfloor = \left\lfloor \frac{m}{b} \right\rfloor \frac{a}{b} = m \frac{a}{b} + \frac{t}{b} \). Using now Lemma 12, we have \( \omega([\gamma]_{\sigma}) = \gamma_1 + \frac{a}{b} \gamma_2 \).

The following result gives us an explicit formulation of the \( \omega \)-primality of the elements of the monoids studied in this section.

Proposition 13. Let \( S = N^2/\sigma \) with \( \sigma = \langle (\alpha, 0), (0, \beta) \rangle \) and \( \alpha, \beta > 1 \). If \( \alpha \geq \beta \), then \( \omega(\gamma_{\sigma}) = \gamma_1 + \frac{\alpha}{\beta} \gamma_2 \) and if \( \alpha < \beta \), then \( \omega(\gamma_{\sigma}) = \frac{\beta}{\alpha} \gamma_1 + \gamma_2 \).

Proof. Assume that \( \alpha \geq \beta \) (if \( \alpha < \beta \) proceed similarly). By Lemma 10, \( \omega(\gamma_{\sigma}) = \gamma_1 + (\left\lfloor \frac{\alpha}{\beta} \right\rfloor + 1) \alpha \). Thus, \( \omega(\gamma_{\sigma}) = \lim_{n \to \infty} \frac{\omega(n [\gamma_{\sigma}])}{n} = \lim_{n \to \infty} (n \gamma_1 + (\left\lfloor \frac{n \alpha}{\beta} \right\rfloor + 1)\alpha) / n \). Using now Lemma 12, we have \( \omega([\gamma]_{\sigma}) = \gamma_1 + \frac{\alpha}{\beta} \gamma_2 \).

If \( \alpha \geq \beta \), since \( (\alpha, 0), (0, \beta) \in \sigma \) means that \( \alpha e_1_{\sigma} = \beta e_2_{\sigma} \), abusing of the notation we could say that \( \omega([\gamma]_{\sigma}) = \frac{\gamma_1}{e_1_{\sigma}} = \frac{\gamma_1 e_1_{\sigma} + \gamma_2 e_2_{\sigma}}{e_1_{\sigma}} = \gamma_1 + \gamma_2 \frac{e_2_{\sigma}}{e_1_{\sigma}} = \gamma_1 + \gamma_2 \frac{\beta}{\alpha} \).

Corollary 14. Let \( S = N^2/\sigma \) be an reduced atomic monoid finitely generated by two different elements and let \( \sigma = \langle (\alpha, 0), (0, \beta) \rangle \) a non-trivial congruence. Then:

- If \( \alpha \geq \beta \) then \( \omega(e_1_{\sigma}) = 1 \) and \( \omega(S) = \omega(e_2_{\sigma}) = \frac{\beta}{\alpha} \).

- If \( \alpha < \beta \) then \( \omega(e_2_{\sigma}) = 1 \) and \( \omega(S) = \omega(e_1_{\sigma}) = \frac{\alpha}{\beta} \).
Note that if \( \varpi([e_1]_\sigma) \neq 1 \), the elasticity of \( S \) is equal to \( \varpi([e_1]_\sigma) \) (see Lemma 16).

The monoid \( S \) is a numerical monoid if and only if \( \alpha \) and \( \beta \) are coprime. In such case \( S \cong \langle \alpha, \beta \rangle \).

In [3] the asymptotic \( \omega \)-primality and \( \omega \)-primality of numerical monoids generated by two elements is computed. In that paper it is also given the formulation of the \( \omega \)-primality for two other families of numerical monoids (monoids generated by \( \{n, n+1, \ldots, 2n-1\} \) with \( n \geq 3 \) and by \( \{n, n+1, \ldots, 2n-2\} \) with \( n \geq 4 \)). The approach proposed in this work allows us to obtain the same formulation for the \( \omega \)-primality of an element, but it can be used in a more general scope. For example, consider \( S \) to be the non-torsion free monoid given by the presentation \( \{(4,0),(0,2)\} \). Since 4 and 2 are not coprime, \( S \) is not a numerical monoid. In this case, we have \( \varpi([e_1]_\sigma) = 1 \) and \( \varpi(S) = \varpi([e_2]_\sigma) = \frac{4}{2} = 2 \).

### 4 Asymptotic \( \omega \)-primality in Archimedean semigroups

We start by introducing some basic concepts used in this section. An element \( x \neq 0 \) of a monoid \( S \) is Archimedean if for all \( y \in S \setminus \{0\} \) there exists a positive integer \( k \) such that \( y/kx \). We say that \( S \) is quasi-Archimedean if the zero element is not Archimedean and the rest of elements in \( S \) are Archimedean. If a monoid is finitely generated, cancellative and quasi-Archimedean, then it verifies that for all \( x, y \in S \setminus \{0\} \), there exist positive integers \( p \) and \( q \) such that \( px = qy \) (see Theorem 2.1 or [11]). All monoids in Section 3, the submonoids of \( \mathbb{N} \) and in particular numerical semigroups are quasi-Archimedean.

**Remark 15.** If \( S \) is a quasi-Archimedean cancellative monoid, for all \( i = 2, \ldots, p \) there exist \( a_i, b_i \in \mathbb{N}\setminus\{0\} \) such that \( a_i[e_1]_\sigma = b_i[e_1]_\sigma \). Consider now \( a_2 \ldots a_p[e_1]_\sigma \), applying the above equalities we obtain that \( a_2 \ldots a_p[e_1]_\sigma = b_2a_3 \ldots a_p[e_2]_\sigma = \cdots = a_2 \cdots a_p - 1b_p[e_p]_\sigma \). For instance if \( S \) is the numerical semigroup generated by 5, 7 and 11 we can take \( a_1 = 7, b_1 = 5, a_2 = 11 \) and \( b_2 = 5 \), obtaining that \( 77 \cdot 5 = 55 \cdot 7 = 35 \cdot 11 \).

Let \( S \) be a quasi-Archimedean cancellative monoid. After rearranging its minimal generators we obtain that there exist \( k_1 \geq \cdots \geq k_p \in \mathbb{N} \setminus \{0\} \) verifying that \( k_1[e_1]_\sigma = \cdots = k_p[e_p]_\sigma \). In this way some elements of \( S \) can be expressed using only the generator \( [e_1]_\sigma \). Abusing again of the notation, we say that \( \frac{[e_1]_\sigma}{[e_1]_\sigma} = k_1 \) and in the same manner that \( \frac{[e_1]_\sigma}{[e_2]_\sigma} = \sum_{i=1}^{p} \gamma_i \frac{[e_2]_\sigma}{[e_1]_\sigma} = \sum_{i=1}^{p} \gamma_i k_i \).

Theorem 18 proves that \( \varpi([\gamma]_\sigma) = \frac{[\gamma]_\sigma}{[e_1]_\sigma} \). In order to prove it, we need two lemmas.

**Lemma 16.** Let \( k_1 \geq \cdots \geq k_p \in \mathbb{N}, \gamma \in \mathbb{N}\setminus\{0\}, \Gamma \) be equal to \( \{\gamma + \sum_{i=1}^{p} \mu_i (k_1 e_i - k_i e_1) | \mu_i \in \mathbb{Z} \} \) and let \( \Theta = \{\gamma \vee (k_2 e_2 - k_1 e_1) \vee \cdots \vee (k_p e_p - k_1 e_1) | \gamma \in \Gamma \} \). The set \( I = \{x \in \mathbb{Z}^p \} \) there exists \( \gamma \in \Gamma \) such that \( x \geq \gamma \) is equal to \( \mathbb{Z}^p \setminus \{x \in \mathbb{Z}^p \} \) there exists \( \phi \in \Theta \) such that \( x_i < \phi_i \) for all \( i = 1, \ldots, p \).

**Proof.** Let \( G \) be the subgroup of \( \mathbb{Z}^p \) spanned by \( \{k_2 e_2 - k_1 e_1, \ldots, k_p e_p - k_1 e_1\} \). Then \( \Gamma = \gamma + G \) and \( I = \Gamma + \mathbb{N}^p \). An element \( x \) is in \( I \) if and only if \( x - \gamma \in G + \mathbb{N}^p \). Also \( a + (b \vee c) = (a + b) \vee (a + c) \) for all \( a, b, c \in \mathbb{Z}^p \). For this reason, we may assume without loss of generality that \( \gamma = 0 \).
Take \( x = (x_1, \ldots, x_p) \in \mathbb{Z}^p \). For every \( i \in \{2, \ldots, p\} \) let \( q_i, r_i \in \mathbb{Z} \) such that \( x_i = k_i q_i + r_i \), with \( 0 \leq r_i < k_i \) (division algorithm). Set \( y = (y_1, y_2, \ldots, y_p) \) and \( y' = (y_1 + k_1, y_2 + k_2, \ldots, y_p + k_p) \) if \( x, y \in G \) and \( y' = (y_1 + k_1 e_1, y_2 + k_2 e_2, \ldots, y_p + k_p e_p) \) if \( x \in G \) and \( y \notin G \). If \( y \notin G \), then \( y' \notin G \). Otherwise, \( y < y' \).

**Lemma 17.** Let \( S = \mathbb{N}/\sigma = \langle s_1, \ldots, s_p \rangle \) be a cancellative monoid with \( \sigma \) a congruence, let \( k_1 \geq \cdots \geq k_p \in \mathbb{N} \) such that \( k_1 s_1 = \cdots = k_p s_p \) and let \( \gamma \in \mathbb{N}^p \). Then every element \( x = (x_1, \ldots, x_p) \in \mathbb{N}^p \backslash \{0\} \) fulfilling that

\[
\sum_{i=1}^{p} \frac{k_i}{i} x_i \geq (p - 1) k_1 \cdots k_p + \sum_{i=1}^{p} \frac{k_i}{i} \gamma_i
\]

belongs to \( E(\gamma, \sigma + S) \).

**Proof.** Assume that \( x \in \mathbb{N}^p \) verifies \( 2 \). Take \( \gamma = \gamma + \Gamma, \Theta, \) and \( I \) as in Lemma 11. It is easy to prove that for every element in \( \Theta \) equality in \( 2 \) holds, and thus, by Lemma 10, \( x \in I \). This implies that there exist \( \gamma' \in \Gamma \) and \( y \in \mathbb{N}^p \) such that \( x = \gamma' + y \). Since \( \gamma' + y \in \Gamma \), there exist \( \mu_i \in \mathbb{N} \) with \( i, j \in \{1, \ldots, p\} \) such that \( \gamma = \gamma' + \sum_{i,j=1}^{p} \mu_i (k_i e_i - k_j e_j) \). Thus, \( \gamma + y = x + \sum_{i,j=1}^{p} \mu_i (k_i e_i - k_j e_j) \in \mathbb{N}^p \).

Since \( S \) is cancellative there exists \( G \) a subgroup of \( \mathbb{Z}^p \) such that for every \( a, b \in \mathbb{N}^p \), \( a \leq b \) if and only if \( a - b \in G \) (see [17] Proposition 1.4). Using this fact and that \( k_i e_i, k_j e_j \) we have that \( \sum_{i,j=1}^{p} \mu_i (k_i e_i - k_j e_j) \in \mathbb{N}^p \).

**Theorem 18.** Let \( S = \mathbb{N}/\sigma \) be a quasi-Archimedean cancellative reduced monoid. There exists a rearrange \( \{t_1, \ldots, t_p\} \) of the set \( \{1, \ldots, p\} \) such that \( \mathbb{N}(a) = \gamma_1 + \sum_{i=1}^{p} \frac{b_i}{k_i} \gamma_i \), for every \( a = (\gamma_1, \ldots, \gamma_p) \in S \).

**Proof.** By \[3], \( \omega(a+b) \leq \omega(a) + \omega(b) \) for all \( a, b \in S \). Thus, for every \( n, m \in \mathbb{N} \) and every \( a \in S \), \( \omega((n + m) a) \leq n \omega(a) + m \omega(a) \). Fekete’s Subadditive Lemma (see \[5\]) states that for every subadditive sequence \( \{z_n | n = 1, \ldots, \infty\} \), the limit \( \lim_{n \to \infty} \frac{z_n}{n} \) exists and it is equal to \( \inf \frac{z_n}{n} \) or \( -\infty \). Since \( \omega(a) \geq 0 \) for every \( a \in S \), the limit \( \mathbb{N}(a) = \lim_{n \to \infty} \frac{\omega(n a)}{n} \) always exists for all \( a \in S \).

Without loss of generality we can assume that there exist \( k_1 \cdots k_p \in \mathbb{N} \) verifying \( k_1 e_1 = \cdots = k_p e_p \). Then for all \( \gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p \), it holds that \( k_2 \cdots k_p [\gamma_1 e_1] = k_2 \cdots k_p [\gamma_2 e_2 + \gamma_3 e_3 + \cdots + k_2 \cdots k_p \gamma_p e_p e_p] = (\gamma_1 k_2 \cdots k_p + k_1 \gamma_2 k_3 \cdots k_p + \cdots + k_1 \cdots k_{p-1} k_p e_p e_p) \).

Let \( a = [\gamma] \in S \) with \( \gamma \in \mathbb{N}^p \). Denote by \( H \) the hyperplane of \( \mathbb{Q}^p \) spanned by \( \{k_i e_i - k_j e_j | i = 2, \ldots, p\} \); \( H \) is defined by the equation \( \sum_{i=1}^{p} \frac{k_i}{k_i} x_i = 0 \). It is straightforward to prove that \( E([\gamma]) \subseteq H \), where \( H_\gamma \) is the affine subvariety of \( \mathbb{Q}^p \) defined by the equation \( \sum_{i=1}^{p} \frac{k_i}{k_i} x_i = \sum_{i=1}^{p} \frac{k_i}{k_i} \gamma_i \). Using now that \( 0 \leq \frac{k_i}{k_i} \leq \cdots \leq \frac{k_1}{k_1} \) and that \( \sum_{i=1}^{p} \frac{k_i}{k_i} \gamma_i \geq 0 \) we deduce that max \( \{\sum_{i=1}^{p} x_i | (x_1, \ldots, x_p) \in H_\gamma \cap \mathbb{Q}^p \} \) is achieved by the element \( \frac{1}{k_2 \cdots k_p} (\sum_{i=1}^{p} \frac{k_i}{k_i} \gamma_i, 0, \ldots, 0) \). For the elements of the form \( nk_2 \cdots k_p \gamma_1 \) with \( n \in \mathbb{N} \backslash \{0\} \) its maximum is achieved by the element \( nk_2 \cdots k_p \frac{1}{k_2 \cdots k_p} (\sum_{i=1}^{p} \frac{k_i}{k_i} \gamma_i, 0, \ldots, 0) = n(\sum_{i=1}^{p} \frac{k_i}{k_i} \gamma_i, 0, \ldots, 0) \in \mathbb{N}^p \).
\(N^p\), which clearly verifies that
\[
nk_2 \cdots k_p[\gamma_\sigma] = [n(\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i, 0, \ldots, 0)]_\sigma.
\]

Using that \(S\) is cancellative and reduced, it is easy to prove that
\[
n(\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i, 0, \ldots, 0) \in \text{Minimals}_\leq (E(nk_2 \cdots k_p[\gamma_\sigma] + S)).
\]
Thus, \(\omega(nk_2 \cdots k_p[\gamma_\sigma]) \geq \|n(\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i, 0, \ldots, 0)\| = n \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i.\) Since the limit \(\varpi(a) = \lim_{n \to \infty} \frac{\omega(n[\gamma_\sigma])}{n}\) exists and every subsequence converges to the same limit. Therefore,
\[
\varpi(a) = \lim_{n \to \infty} \frac{\omega(n[\gamma_\sigma])}{n} = \lim_{n \to \infty} \frac{\omega(nk_2 \cdots k_p[\gamma_\sigma])}{nk_2 \cdots k_p} \geq \lim_{n \to \infty} \frac{n \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i}{nk_2 \cdots k_p} = \frac{nk_1 k_2 \cdots k_p (\gamma_1/k_1 + \cdots + \gamma_p/k_p)}{nk_2 \cdots k_p} = k_1 (\gamma_1/k_1 + \cdots + \gamma_p/k_p).
\]

Obtaining in this way a lower bound for \(\varpi(a)\).

We now prove that the equality holds. By Lemma 17, every element \((x_1, \ldots, x_p) \in N^p\) verifying that \(\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} x_i = (p-1)k_1 \cdots k_p + \sum_{i=1}^{p} \frac{n k_1 \cdots k_p}{k_i} \gamma_i\) belongs to \(E(n[\gamma_\sigma] + S)\). Thus, every element fulfilling
\[
\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} (x_i - 1) \geq (p-1)k_1 \cdots k_p + \sum_{i=1}^{p} \frac{n k_1 \cdots k_p}{k_i} \gamma_i \text{ also belongs to } E(n[\gamma_\sigma] + S),
\]
but does not belong to \(\text{Minimals}_\leq (E(n[\gamma_\sigma] + S))\). Therefore, the elements of \(\text{Minimals}_\leq (E(n[\gamma_\sigma] + S))\) satisfy the inequality
\[
\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} x_i \leq \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} + (p-1)k_1 \cdots k_p + \sum_{i=1}^{p} \frac{n k_1 \cdots k_p}{k_i} \gamma_i. \quad (3)
\]

Hence, the set \(\text{Minimals}_\leq (E(n[\gamma_\sigma] + S))\) is included in \(\{ (x_1, \ldots, x_p) \in Q^p_\leq : (x_1, \ldots, x_p) \text{ satisfies } (3) \}\) and for this reason
\[
\omega(n[\gamma_\sigma]) \leq \sup \{ \|(x_1, \ldots, x_p)\| : (x_1, \ldots, x_p) \in Q^p_\leq \text{ and satisfies } (3) \}. \quad (4)
\]

Using that the coefficients of the left-hand side of (3) verify that \(0 \leq \frac{k_1 \cdots k_p}{k_i} \leq \cdots \leq \frac{k_1 \cdots k_p}{k_1}\), it is straightforward to prove that for every \(d \in Q\) the set \(\{(x_1, \ldots, x_p) \in Q^p_\leq : \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} x_i = d \}\) is nonempty if and only if \(d \geq 0\). Besides, the maximum of \(\{ \|(x_1, \ldots, x_p)\| : (x_1, \ldots, x_p) \in Q^p_\leq \text{ and } \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} x_i = d \}\) is achieved by the element \((d/(k_2 \cdots k_p), 0, \ldots, 0)\), it is equal to \(d/(k_2 \cdots k_p)\), and if we increase \(d\), then it increases. Since the right-hand side of (3) is greater than zero, we deduce that the supreme of (4) is achieved by an element of \(Q^p_\leq\) fulfilling that equality in (3) holds and having all its coordinates equal to zero but the first one. That point is equal to \(n\xi\) with
\[
\xi = \left( \frac{\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i}}{nk_2 \cdots k_p} + \frac{(p-1)k_1}{n} + \frac{\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_i} \gamma_i}{k_2 \cdots k_p}, 0, \ldots, 0 \right) \in Q^p_\leq.
\]
and it verifies that $\omega(n[\gamma]_{\sigma}) \leq \|n\xi\|$. Therefore,

$$
\varpi(n) = \lim_{n \to \infty} \frac{\omega(n[\gamma]_{\sigma})}{n} \leq \lim_{n \to \infty} \frac{\|n\xi\|}{n}
\leq \lim_{n \to \infty} \frac{\sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_2 \cdots k_p}}{n} + \frac{(p-1)k_1}{1} + n \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_2 \cdots k_p} \gamma_i
= \sum_{i=1}^{p} \frac{k_1 \cdots k_p}{k_2 \cdots k_p} \gamma_i
= k_1(\gamma_1/k_1 + \cdots + \gamma_p/k_p).
\square
$$

**Corollary 19.** Let $S = \mathbb{N}^p / \sigma$ be a quasi-Archimedean cancellative reduced monoid. There exist $k_1, \ldots, k_p \in \mathbb{N}$ such that $\varpi([e_i]_{\sigma}) = \frac{\max\{k_1, \ldots, k_p\}}{k_i}$ for all $i = 1, \ldots, p$.

As we pointed out in the abstract, we give a formula to compute the $\varpi$-primality in numerical semigroups.

**Corollary 20.** Let $S$ be a numerical monoid minimally generated by $\langle s_1 < s_2 < \cdots < s_p \rangle$. For every $s \in S$, we have that $\varpi(s) = \frac{s}{s_1}$.

**Proof.** Use that there exist $\gamma_1, \ldots, \gamma_p \in \mathbb{N}$ such that $s = \sum_{i=1}^{p} \gamma_i s_i$ and take $k_i = \frac{s_{\text{cm}(s_1, \ldots, s_p)}}{s_i}$.

\[ \square \]

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