Primordial large-scale electromagnetic fields from Gravitoelectromagnetic Inflation

1,2 Federico Agustín Membiela *, 1,2 Mauricio Bellini †
1 Departamento de Física, Facultad de Ciencias Exactas y Naturales,
Universidad Nacional de Mar del Plata, Funes 3350, (7600) Mar del Plata, Argentina.
2 Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET).

We investigate the origin and evolution of primordial electric and magnetic fields in the early universe, when the expansion is governed by a cosmological constant $\Lambda_0$. Using the gravitoelectromagnetic inflationary formalism with $\Lambda_0 = 0$, we obtain the power of spectrums for large-scale magnetic fields and the inflaton field fluctuations during inflation. A very important fact is that our formalism is naturally non-conformally invariant.

Keywords: extra dimensions, variable cosmological parameter, inflationary cosmology, large-scale magnetic fields

I. INTRODUCTION

The origin of the primordial magnetic fields has been subject of a great amount of research[1]. The existence, strength and structure of these fields in the intergalactic plane, within the Local Supercluster, has been scrutinized recently[2]. Many spiral galaxies are endowed with coherent magnetic fields of $\mu G$ (micro Gauss) strength[3,4,5,6,7,8], having approximately the same energy density as the Cosmic Microwave Background Radiation (CMBR). In particular, the field strength of our galaxy is $B \approx 3 \times 10^{-6} G$, similar to that detected in high redshift galaxies[9] and damped Lyman alpha clouds[10]. Limits imposed by the high isotropy of CMB photons, obtained from the COBE data[11] restrict the present day strength of magnetic fields on cosmological scales to $10^{-9} G$. It is very mysterious that magnetic fields in clusters of galaxies [i.e., on scales $\sim Mpc$], to be coherent[12]. There are compelling indications of existence of large-scale microgauss magnetic fields in galaxy clusters. This would indicate that the entire universe is magnetized. There are two possible classes of mechanisms to produce cosmic fields depending on when they are generated: astrophysical mechanisms acting during large-scale structure formation, and mechanisms acting in the primordial universe. The origin of these magnetic fields is not well understood yet. The seeds of these fields could be in the early inflationary expansion of the universe, when these fields were originated. The existence of primordial magnetic fields would affect both, the temperature and polarization anisotropies of the cosmic microwave background. It also provides a plausible explanation for the possible disparity between observations and theoretical fits to the CMB power-spectrum. The ACBAR[13] and CBI[14] experiments indicate continued power up to $l \sim 4000$, but WMAP data predicts a rapidly declining power spectrum in the large multipole range[15]. This discrepancy is difficult to account from a returning of cosmological parameters. Among other possible explanations, an cosmological magnetic field generated during inflation provides a plausible mechanism to produce excess power at high multipoles. Therefore, the study of its origin and evolution in this epoch should be very important to make predictions in cosmology[16]. During inflation the extension of the causally connected regions grows as the scale factor and hence faster than in the decelerated phase. This solves the horizon problem. Furthermore, during inflation the contribution of the spatial curvature becomes very small. The way inflation solves the curvature problem is by producing a very tiny spatial curvature at the onset of the radiation epoch taking place right after inflation. The spatial curvature can well grow during the decelerated phase of expansion but it will be always subleading provided inflation lasted for sufficiently long time. It is natural to look for the possibility of generating such a large-scale magnetic field during inflation. However, the FRW universe is conformal flat and the Maxwell theory is conformal invariant, so that magnetic field generated at inflation would come vanishingly small. Therefore, the conformal invariance must be broken to generate non-trivial magnetic fields. Various conformal symmetry breaking mechanisms have been proposed so far[17]. Magnetogenesis has been studied also during the electroweak phase transition[18]. Due to this fact we are interested to study a theory that for low energies, we shall assume reduces to the Maxwell one in the limit of small fields.

Gravitoelectromagnetic Inflation (GI) was developed very recently with the aim to describe, in an unified manner, the inflaton, gravitatory and electromagnetic fields during inflation[19,20]. In this formalism all the 4D sources have a geometrical origin. This formalism can explain the origin of seed magnetic fields on cosmological scales observed today. This proposal was constructed from a 5D vacuum state on a $R^A_{BCD} = 0$ globally flat metric. As in all Space Time Matter (STM) models[21], the 4D sources are geometrically induced when we take a foliation on the fifth

* E-mail address: membiela@argentina.com
† E-mail address: mbellini@mdp.edu.ar
coordinate which is spacelike and noncompact. However, in the previous works was used the Feynman gauge in order to simplify the structure of the field equations.

In this letter we shall use this formalism using $A_0 = 0$. As we shall see, the field equations become coupled, which has interesting physical consequences. We shall study the origin and evolution of the seed large-scale electric and magnetic fields in a $Λ_0$ dominated early universe, from a 5D vacuum state, where the expansion of the universe is driven by the inflaton field.

II. VECTOR FIELDS IN 5D VACUUM

We begin considering a 5D manifold $\mathcal{M}$ described by a symmetric metric $g_{AB} = g_{BA}$. This manifold $\mathcal{M}$ is mapped by coordinates $\{x^A\}$.

$$dS^2 = g_{AB}dx^Adx^B,$$

which, we shall consider as Riemann-flat $R_{ABCD} = 0$. To introduce the fields we can define an action in $\mathcal{M}$.

$$S = \int \sqrt{-g} R^{(5)} - \frac{1}{4} F_{BC} F^{BC},$$

$(5) R$ is the 5D scalar curvature. We shall consider these fields as minimally coupled to gravity. In this space the fields are free of interactions. The Faraday tensor is antisymmetric $F_{BC} = \nabla_B A_C - \nabla_C A_B$.

A. The 5D Riemann-flat metric with decaying parameter

In particular, in this letter we are interested to deal with the following Riemann-flat metric

$$dS^2 = \psi^2 \frac{\Lambda(t)}{3} dt^2 - \psi^2 e^{2\int_0^t d\tau \sqrt{\Lambda(\tau)/3}} dv^2 - dv^2,$$

where $dv^2 = dx^i \delta_{ij} dx^j$ is the euclidean line element in cartesian coordinates and $\psi$ is the space-like extra dimension. Adopting natural units ($\hbar = c = 1$) the cosmological parameter $Λ(t)$ (with $Λ < 0$), has units of $(\text{length})^{-2}$. The metric is very interesting to study the evolution of the gravitoelectromagnetic (vectorial) field, because is Riemann-flat, but has some connections $\Gamma^C_{DE} \neq 0$. This fact is very important when we consider the covariant derivative of $A^F$.

The equations of motion for the components of the vectorial field $A_i$ are

$$\frac{\partial^2 A_i}{\partial t^2} + \left[ 3 \sqrt{\frac{\Lambda}{3}} - \frac{\dot{\Lambda}}{2A} \right] \frac{\partial A_i}{\partial t} - \frac{\Lambda}{3} e^{-2\int_0^t d\tau \sqrt{\Lambda(\tau)/3}} \frac{\partial}{\partial \psi} \left( \nabla \cdot \vec{A} \right) = 0,$$

$$\frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} \right) + \psi^2 e^{2\int_0^t d\tau \sqrt{\Lambda(\tau)/3}} \frac{\partial A_i}{\partial t} + 2 \psi e^{2\int_0^t d\tau \sqrt{\Lambda(\tau)/3}} \frac{\partial A_i}{\partial \psi} = 0,$$

$$\frac{\partial^2 A_i}{\partial t^2} + \left[ \frac{\Lambda}{3} - \frac{\dot{\Lambda}}{2A} \right] \frac{\partial A_i}{\partial t} - \frac{\Lambda}{3} e^{-2\int_0^t d\tau \sqrt{\Lambda(\tau)/3}} \frac{\partial A_i}{\partial t} - \frac{\Lambda}{3} \psi^2 \frac{\partial^2 A_i}{\partial \psi^2} + \frac{2}{\psi} \frac{\partial A_i}{\partial \psi} + \frac{2\Lambda}{3} \frac{\partial A_i}{\partial x^i}$$

$$+ \frac{\Lambda}{3} e^{-2\int_0^t d\tau \sqrt{\Lambda(\tau)/3}} \frac{\partial}{\partial x^i} \left( \nabla \cdot \vec{A} \right) + \frac{\Lambda}{3} \psi^2 \frac{\partial}{\partial x^i} \left( \frac{\partial A_i}{\partial \psi} \right) = 0.$$  

These are our equations of motion on the metric, once we consider the gauge $A^0 = 0$. To solve these equations we can begin considering $\nabla \cdot \vec{A} \equiv f(t, \vec{x}, \psi)$, and $A_4 \equiv \varphi(t, \vec{x}, \psi)$. Next, we make Fourier transforma in eqs. (4) and (5), and use separation of variables for both, $f_k(t, \psi) \sim F_1(t) F_2(\psi)$ and $\varphi_k(t, \psi) \sim \alpha(t) \beta(\psi)$ where we drop de subindex $k$ for the transformed variables $F_1, F_2, \alpha, \beta$. Working out (4) and (5), we arrive to

---

1 In our conventions capital Latin indices run from 0 to 4, greek indices run from 0 to 3 and latin indices run from 1 to 3.
\[ b^2 \dot{\alpha} = \frac{1}{\lambda_1} \dot{F}_1, \]  
(7)

\[ \ddot{\alpha} - \Gamma \dot{\alpha} + k^2 \frac{\Lambda}{3b^2} \alpha = \lambda_2 \frac{\Lambda}{3b^2} F_1, \]  
(8)

\[ \psi'^2 \beta' + 2\psi \beta = \lambda_1 F_2, \]  
(9)

\[ -\beta = \frac{1}{\lambda_2} F'_2, \]  
(10)

where primes and dots denote respectively the derivatives with respect to \( \psi \) and \( t \). Furthermore, we define \( b(t) \equiv e^{\int dt \sqrt{3\Lambda}} \) and \( \Gamma(t) \equiv \frac{d}{dt} \sqrt{3\Lambda} \). The constants \( \lambda_1, \lambda_2 \) come from the separations of variables procedure. To obtain \( \alpha \) and \( F_1 \), we work with the equations (7) and (8). We introduce \( \sigma \equiv \dot{\alpha} \) and we have

\[
\left[ \frac{3b^2}{\Lambda} \right] \ddot{\sigma} + \left[ \frac{d}{dt} \left( \frac{3b^2}{\Lambda} \right) - \frac{3b^2}{\Lambda} \Gamma \right] \dot{\sigma} + \left[ k^2 - \frac{d}{dt} \left( \frac{3b^2}{\Lambda} \Gamma - \lambda_1 \lambda_2 b^2 \right) \right] \sigma = 0,
\]  
(11)

where the solutions are given by the primitives

\[ \alpha(t) = \int \sigma(t) dt, \]  
(12)

\[ F_1(t) = \lambda_1 \int \sigma(t) b(t)^2 dt. \]  
(13)

To solve the equation for \( F_2 \) we replace (10) and its derivative in (9)

\[ \psi'^2 F_2'' + 2\psi F_2' + \lambda_1 \lambda_2 = 0. \]  
(14)

The solution is

\[ F_2(\psi) = \psi^{-\frac{3}{2}} \left[ c_1 \psi^w + c_2 \psi^{-w} \right], \]  
(15)

\[ \beta(\psi) = \psi^{-\frac{3}{2}} \left[ c_1 \left( \frac{\psi^w + \frac{1}{2} w}{\lambda_2} \right) \psi^w + c_2 \left( \frac{\psi^w + \frac{1}{2} w}{\lambda_2} \right) \psi^{-w} \right], \]  
(16)

with \( w \equiv \sqrt{\frac{1}{4} - \lambda_1 \lambda_2} \). In order to illustrate the formalism we can study an example, which is interesting for the cosmological expansion of the early universe.

### III. AN EXAMPLE WITH \( \Lambda = \Lambda_0 \): DE SITTER EXPANSION

We consider the case where \( \Lambda = \Lambda_0 \). When we make the foliation \( \psi = \psi_0 = \sqrt{\frac{1}{\Lambda_0}} = H_0^{-1} \) on the \( \mathbb{R} \), this case give us a de Sitter inflationary expansion of the universe with tetra-velocities: \( u^a = (1, 0, 0, 0) \) for a comoving frame. Furthermore, the effective 4D line element is

\[ ds^2 = dt^2 - H_0^{-2} e^{2H_0 t} dr^2. \]  
(17)

In this case equation (11) yields

\[ \ddot{\sigma} + 5H_0 \dot{\sigma} + H_0^2 \left( k^2 e^{-2H_0 t} + 6 - \lambda_1 \lambda_2 \right) \sigma = 0, \]  
(18)

which has the solution \( \sigma(t) = e^{-\frac{5}{2}H_0 t} \left[ N_1 J_{-\mu} \left( k e^{-H_0 t} \right) + N_2 Y_{-\mu} \left( k e^{-H_0 t} \right) \right] \), with \( \mu \equiv \sqrt{\frac{1}{4} - \lambda_1 \lambda_2} \). To get \( \alpha(t) \) we have to integrate the Bessel functions with the exponential, this can be achieved by changing variables to \( \eta = \frac{1}{2} e^{-H_0 t} \).
\[ -\frac{\dot{\phi}}{\tau} H_0 \int dt e^{-\frac{\dot{\phi}}{\tau} H_0 t} = e^{-\frac{\dot{\phi}}{\tau} H_0 t} \]. The primitive of a first kind Bessel function gives a Regularized Hypergeometric Function (RHF), and for the second kind Bessel function we obtain the solution as a combination of two RHFs

\[
\alpha(\eta) = -\frac{1}{H_0} M_1 2^{\mu - 1} k^{-\mu} \eta^{-\frac{\mu}{2}} \Gamma\left[\frac{\mu + 1}{2}\right] F_1 \left[\left\{\frac{5 - 2\mu}{4}\right\}; \left\{1 - \mu, \frac{9 - 2\mu}{4}\right\}; -\frac{k^2}{4} \eta^{4/5}\right]
\]

\[
-\frac{1}{H_0} M_2 2^{-(\mu + 1)} k^{\mu + 1} \eta^{\frac{\mu}{2}} \Gamma\left[\frac{\mu + 1}{2}\right] F_2 \left[\left\{\frac{5 + 2\mu}{4}\right\}; \left\{1 + \mu, \frac{9 + 2\mu}{4}\right\}; -\frac{k^2}{4} \eta^{4/5}\right],
\]

where \( M_1 \equiv N_1 - \frac{N_2}{\tan(\mu \pi)} \) and \( M_2 \equiv N_2 \). To calculate \( F_1 \) we can define \( \eta' = -\frac{1}{2} H_0 \int dt e^{-\frac{\dot{\phi}}{\tau} H_0 t} = e^{-\frac{\dot{\phi}}{\tau} H_0 t} \)

\[
\frac{1}{\lambda_1} F_1(\eta') = -\frac{1}{H_0} M_1 2^{\mu - 1} k^{-\mu} \eta'^{-1+2\mu} \Gamma\left[\frac{\mu + 1}{2}\right] F_1 \left[\left\{\frac{1 - \mu, 5 - 2\mu}{4}\right\}; -\frac{k^2}{4} \eta'^{4}\right]
\]

\[
-\frac{1}{H_0} M_2 2^{-(\mu + 1)} k^{\mu+1} \eta'^{1+2\mu} \Gamma\left[\frac{\mu + 1}{2}\right] F_2 \left[\left\{\frac{1 + \mu, 5 + 2\mu}{4}\right\}; -\frac{k^2}{4} \eta'^{4}\right].
\]

The total solution \( \varphi_k(t, \psi) = [\varphi_k^{(hom)}(t) + \varphi_k(t)] \beta(\psi) \), where we have included the homogeneous solution

\[
\varphi_k^{(hom)}(t) = A_1 e^{-\frac{\dot{\phi}}{\tau} H_0 H_0^{(1)}(k e^{-\dot{\phi} H_0 t})} + A_2 e^{-\frac{\dot{\phi}}{\tau} H_0 H_0^{(2)}(k e^{-\dot{\phi} H_0 t})}.
\]

that has the typical scale invariant spectrum of a de Sitter model.

Once obtained the solutions for \( \varphi_k(t, \psi) \) and \( f_k(t, \psi) \), we can try to solve the equations for the potential 3-vector \( A_j(x^a) \). We take the Fourier transform in the \( \vec{x} \)-space, as before. From eq. (10), we obtain the following equation for the modes \( \xi^{(j)}(t, \psi) \):

\[
\xi^{(j)}(k) + \frac{\partial^2}{\partial \psi^2} \xi^{(j)}(k) + \frac{2}{\psi} \frac{\partial}{\partial \psi} \xi^{(j)}(k) = -ik_j H_0^2 \left[ 2\psi \dot{\varphi}_k(t) \beta(\psi) + e^{-2\dot{\phi} H_0 t} f_k(t, \psi) + \psi^2 \varphi_k(t) \beta'(\psi) \right],
\]

where \( K(t, \psi) \) is the source term and

\[
\varphi_k(t) = \varphi_k^{(hom)}(t) + \varphi_k(t).
\]

In this letter we consider that the particle excitations for an observer in (3) appear are the Mellin transform in the extra coordinate \( \psi \). We will see that the extra terms will become massive terms for each \( m \)-mode. Thus, this extra coordinate formalism, besides it produces couplings between the effective vector and scalar components of the field (in a curved background), provide us of a contribution to the mass of vector excitations. The Mellin transform on the foliated spacetime (17), is

\[
\xi^{(j)}_{k,m}(t) = \int_0^1 \psi^{m-1} \xi^{(j)}(\tau, \psi') \tau dt, \quad \psi' = \frac{\psi}{\psi_0},
\]

and the equation (22) becomes

\[
\xi^{(j)}_{k,m} + H_0^2 \xi^{(j)}_{k,m} + H_0^2 \left[ k^2 e^{-2\dot{\phi} H_0 t} - m(m-1) \right] \xi^{(j)}_{k,m} = -ik_j H_0^2 K_{m}(t),
\]

where \( m \) is a free parameter on the metric (17) [but not on the 5D Riemann-flat metric (3)], to be experimentally determined by the spectrum of large scale magnetic fields. The total solution to this ordinary differential equation is the homogenous part plus the inhomogeneous one

\[
\xi^{(j)}_{k,m}(t) = D_1 e^{-\frac{\dot{\phi}}{\tau} H_0 t} H_0^{(1)} \left[ x(t) \right] + D_2 e^{-\frac{\dot{\phi}}{\tau} H_0 t} H_0^{(2)} \left[ x(t) \right] + \xi^{(j)}_{k,m, inh}(t),
\]

where

\[
\xi^{(j)}_{k,m, inh}(t) = i k_j H_0^2 e^{-\frac{\dot{\phi}}{\tau} H_0 t} \int d\tau K_m(\tau) e^{\frac{1}{2} \dot{\phi} H_0 \tau} \left[ Y_{\frac{1}{2} - m} \left[ x(\tau) \right] J_{\frac{1}{2} - m} \left[ x(\tau) \right] + J_{\frac{1}{2} - m} \left[ x(\tau) \right] Y_{\frac{1}{2} - m} \left[ x(\tau) \right] \right].
\]

The source term, after the Mellin transform, yields

\[
K_m(\tau) = \frac{c_1}{(m - \frac{1}{2} + w)} \left[ e^{-2\dot{\phi} H_0 \tau} F_1(\tau) + \frac{(\frac{1}{2} + w)^2}{\lambda_2} \alpha(\tau) \right] + \frac{c_2}{(m - \frac{1}{2} - w)} \left[ e^{-2\dot{\phi} H_0 \tau} F_1(\tau) + \frac{(\frac{1}{2} - w)^2}{\lambda_2} \alpha(\tau) \right].
\]
Notice that $F_1(\tau)$ and $\alpha(\tau)$ are given by hypergeometric functions and we have to integrate them with the Bessel functions. This can be done analytically by evaluating the integral for each term of the power series of the RHF. To make it, we write explicitly the RHFs in the form

$$1 \hat{F}_2 \{\{a\}; \{b_1, b_2\}; z\} = \sum_{p=0}^\infty \gamma_p \frac{z^p}{z!},$$

with $\gamma_p \equiv \frac{\Gamma(p+1)}{(b_1+p)! (b_2+1)! (p)!}$. The integrals we have to evaluate are of the form

$$\sum_{p=0}^\infty \gamma_p \left(\frac{-k^2}{t^2}\right)^p \int d\tau \left[ e^{-2pH_0\tau} K_{\frac{1}{2}-m} \left(k e^{-H_0\tau}\right) \right] e^{-\left(\pm \mu + \frac{1}{2}\right)H_0\tau},$$

where $K_{\frac{1}{2}-m}[x(t)]$ can be either the first kind or second kind Bessel function. To solve the integral again we repeat the procedure of changing variables $\eta$, make it, we write explicitly the RHFs in the form

$$\int \xi d\eta \equiv \int \frac{\xi}{\eta^{\mu-1}} d\eta,$$

and

$$\mu \equiv \frac{\Gamma(1/2 + \frac{m}{2})}{\Gamma(1 + \mu + p)} \left(\frac{1 + \frac{m}{2}}{4}\right) + p,$$

and

$$\gamma_q \equiv \frac{\Gamma(1/2 + \frac{m}{2})}{\Gamma(1 + \mu + p)} \left(\frac{1 + \frac{m}{2}}{4}\right) + q.$$
we require that $\mu$ and $w$ be real and positive, then we obtain that they are restricted to the interval $\left[ 0, \frac{2}{5} \right]$. This restricts the parameter space of $\lambda_1, \lambda_2$.

Considering that the magnetic fields produced during this epoch are scale invariant in the cosmological level, we can define a value for the parameter $m$

$$\mathcal{B}^2_{com} = \langle B^2 \rangle = \frac{1}{2\pi^2} \int_0^{\theta k_H(t)} \frac{dk}{k^2} \xi_{k,m}^{(j)} \xi_{k,m}^{(j)*}$$

(38)

The contribution to the magnetic fields is exclusive from the homogeneous solution in (26), this should be clear from (6) because the source terms are all gradients. The k-power that come from the homogeneous solution of the modes is $m - \frac{1}{2}$. Then, to obtain a nearly scale invariant spectrum of the magnetic field we need that $5 - 1 + 2m \simeq 0$, yielding $m \simeq -2$. In the cosmological limit the inhomogeneous equation can be reduced to

$$\xi_{k,m}^{(j)} \bigg|_{inh} = - \sum_{n=1,2} \frac{i\pi M_1 c_n}{2^{2-n} k_H} \frac{k_j e^{-2H_0 t} (ke^{-H_0 t})^{-\mu}}{\sqrt{\Gamma (1+2\mu)}} \left\{ \frac{4\lambda_1}{(1 + 2\mu)} \right\}$$

$$\times \left[ \left( \frac{\epsilon}{225} (ke^{-H_0 t})^5 - \frac{2^4}{45\pi} \frac{\Gamma (2+\mu/2)}{2 + \mu/2 - 3(\mu - 1)} \right) \right],$$

(39)

where $\epsilon$ is a parameter that takes into account small deviations from scale invariance. In the last expression should be noted that we have drop the terms involving positive $\mu$-powers of the physical wavenumber $ke^{-H_0 t}$ (this because of the same idea we stayed with $p = q = 0$). The Bessel functions have been approximated to their asymptotic expressions in the infrared limit. Note that the term involving $\epsilon$ decays very strongly due to the factor $x(t)^5$.

### A. Quantization and normalization of homogeneous solutions of the vector and scalar fields

We consider the usual commutation relations for the fields and their conjugate momenta on the effective 4D metric (17). For the scalar field we obtain

$$[\varphi_m(t, \vec{x}), \varphi_m(t, \vec{x}')] = i H_0^3 e^{-3H_0 t} \delta^{(3)}(\vec{x} - \vec{x}'),$$

(40)

where $\varphi_m(t, \vec{x}) = \varphi(t, \vec{x}) \hat{\beta}_m$, and $\hat{\beta}_m$ is the Mellin transform of $\beta(\psi)$. From these relations we derive the normalization condition for the modes

$$\phi_k \phi_k^* - \hat{\phi}_k \hat{\phi}_k^* = \frac{i}{m^2 |\hat{\beta}_m|^2} H_0^3 e^{-3H_0 t},$$

(41)

where the Bunch-Davies vacuum is given by $A_1 = 0$ and then, we obtain $A_2 = i \frac{H_0 \sqrt{\pi}}{2|m|}$. We repeat the same for the vector solution, the commutation relation is in this case

$$[A_{m,j}(t, \vec{x}), A_{m',k}(t, \vec{x}')] = i \delta_{jk} \delta_{mm'} H_0 e^{-H_0 t} \delta^{(3)}(\vec{x} - \vec{x}'),$$

(42)

where $A_{m,j}(t, \vec{x})$ is the Mellin transform of the $A_j$ field. We impose the Bunch-Davies vacuum to the modes choosing $D_1 = 0$ and $D_2 = i \frac{\sqrt{\pi}}{2}$ in (26). Then, the Fourier-Mellin modes comply

$$\xi_{k,m}^{(j)}(t) \xi_{k,m}^{(j)*}(t) - \xi_{k,m}^{(j)}(t) \xi_{k,m}^{(j)*}(t) = i H_0 e^{-H_0 t}.$$

(43)

We have noted that the magnetic fields depend exclusively from the homogeneous solution, then we can compute their quadratic amplitude from (28)

$$\langle B^2 \rangle = H_0^2 e^{-3H_0 t} \mathcal{B}_{com} = \frac{\Gamma (1/2-m)}{(2\pi)^{3/2}} H_0^2 e^{-(1+m)H_0 t} \frac{(\theta k_H)^{2+m}}{\sqrt{2 + m}},$$

(44)

$$B_{phys} = H_0^2 e^{-2H_0 t} \mathcal{B}_{com} = \frac{\Gamma (1/2-m)}{(2\pi)^{3/2}} H_0^2 e^{-(m+2)H_0 t} \frac{(\theta k_H)^{2+m}}{\sqrt{2 + m}}.$$

(45)
The horizon wavenumber is found to be \( k_H(t) = \sqrt{\frac{1}{4} + m(m - 1)e^{H_0t}} \). This means that the physical magnetic field is constant for any \( m \)

\[
E_{\text{phys}} = H_0^2 e^{-2H_0t} E_{\text{com}} = \frac{\Gamma(\frac{1}{2} - m)}{(2\pi)^{3/2} 2^m H_0^2} \left[ \frac{\theta \sqrt{\frac{1}{4} + m(m - 1)}}{\sqrt{2 + m}} \right]^{2+m}.
\]  

(46)

which is divergent for \( m = -2 \). If we require that these fields are nearly invariant in cosmological scales, then \( m = -2 + \epsilon \), \( \epsilon \) being a small parameter, this yields for the amplitude of the physical magnetic field

\[
E_{\text{phys}} = \frac{3}{4} \sqrt{\frac{2}{2\pi}} H_0^2 \frac{(5\theta/2)^\epsilon}{\sqrt{\epsilon}}.
\]  

(47)

On the other hand the electric fields are affected by the extra terms. Then, at the end of the inflationary epoch there will be a contribution from the inhomogeneous solution to the spectrum and amplitude of the electric field. This field has components \( E^\alpha = F^{\alpha\beta} v_\beta \), where \( v_\beta \) are the components of the observer velocity. For a physical observer we have \( \vec{E}_{\text{phys}} = a^{-1}(t) \partial_t \vec{A} \). Therefore

\[
E_{\text{phys}}^2 \equiv \langle E_{\text{phys}}^2 \rangle = \frac{H_0^2 e^{-2H_0t}}{2\pi^2} \int \frac{dk}{k} k^3 |\xi^{[h]}_{k,m}|^2.
\]  

(48)

From the homogeneous solution product we obtain the amplitude

\[
\langle E_{\text{phys}}^{[h]} \rangle^2 = \frac{H_0^2 e^{-2H_0t}}{2\pi^2} \int \frac{dk}{k} k^3 |\xi^{[h]}_{k,m}|^2.
\]  

(49)

Hence, the homogeneous amplitude of the electric field, is

\[
E_{\text{phys}}^{[h]} = \frac{\sqrt{2} \Gamma \left[ \frac{1}{2} - m \right]}{(2\pi)^{3/2} 2^m} \left[ H_0^2 \right]^{m+3/2} \left[ \frac{\theta \sqrt{\frac{1}{4} + m(m - 1)}}{\sqrt{2m + 3}} \right]^{m+3/2} e^{H_0t}.
\]  

(50)

The relation between the energy density of the electric and magnetic fields, in the physical frame, is

\[
\frac{\rho_{\text{elec}}^{[h]}}{\rho_{\text{mag}}} = \left( \frac{2m^2}{\theta} \right) \left( \frac{2 + m}{3 + 2m} \right) \frac{e^{H_0t}}{\sqrt{\frac{1}{2} + m(m - 1)}}.
\]  

(51)

For a nearly scale invariant magnetic field, we obtain

\[
\frac{\rho_{\text{elec}}^{[h]}}{\rho_{\text{mag}}} = \left( \frac{8\epsilon}{5\theta} \right) e^{H_0t}.
\]  

(52)

This means that the energy of the electric field is dominant during exponential inflation on cosmological scales. For a nearly scale invariant electric field we obtain \( m = -3/2 + \epsilon \), and then

\[
E_{\text{phys}} = \sqrt{\frac{2\theta}{\pi^3}} H_0^2.
\]  

(53)

Thus, the magnetic field would be constant but bigger on smaller (astrophysical) scales, while the electric field would still dominate in larger scales

\[
\frac{\rho_{\text{elec}}^{[h]}}{\rho_{\text{mag}}} = \left( \frac{9}{8\epsilon\theta} \right) e^{H_0t}.
\]  

(54)

B. Spectrum of the inhomogeneous solutions of the electric field and the scalar

The amplitude and spectrum of the electric field have terms that involve the inhomogeneous solution, with double infinitum power series, coming from an integral of the hypergeometric functions. For convenience we only keep the
first term ($p = q = 0$), because it is of cosmological relevance. The contribution of terms with $p$ and $q \neq 0$ can be neglected on cosmological scales. The three remaining contributions are

\[
\langle E_{\text{phys}}^1 \rangle^2 = \frac{H_0^2 e^{-2H_0 t}}{2\pi^2} \int \frac{dk}{k} k^3 \xi_{k, m}^{(inh)(j)} \xi_{k, m}^{(inh)(j)*}, \tag{55}
\]

\[
\langle E_{\text{phys}}^2 \rangle^2 = \frac{H_0^2 e^{-2H_0 t}}{2\pi^2} \int \frac{dk}{k} k^3 \xi_{k, m}^{(inh)(j)} \xi_{k, m}^{(inh)(j)*}, \tag{56}
\]

\[
\langle E_{\text{phys}}^3 \rangle^2 = \frac{H_0^2 e^{-2H_0 t}}{2\pi^2} \int \frac{dk}{k} k^3 \xi_{k, m}^{(inh)(j)} \xi_{k, m}^{(inh)(j)*}. \tag{57}
\]

In order to simplify the notation, we write (55) in the compact form

\[
\xi_{k, -1 + \epsilon}^{(inh)(j)}(t) = -\frac{i e_j}{H_0} e^{-H_0 t} (k e^{-H_0 t})^{1-\mu} \sum_{n=1,2} \mathbb{D}_{\lambda_1\lambda_2}^{(n)}, \tag{58}
\]

where we used $k_j = k e_j$ and the coefficients $\mathbb{D}_{\lambda_1\lambda_2}^{(n)}$ have units of $H_0$. This yields the respective power spectrums, for (55), (56) and (57)

\[
\langle E_{\text{phys}}^1 \rangle^2 \sim \langle E_{\text{phys}}^2 \rangle^2 \sim \int \frac{dk}{k} k^{5-\mu}, \tag{59}
\]

\[
\langle E_{\text{phys}}^3 \rangle^2 \sim \int \frac{dk}{k} k^{5-2\mu}, \tag{60}
\]

where we have considered only terms with $p = q = 0$ in eqs. (52) and (53). It is important to notice that the spherical symmetry is broken because $e_j$ is an unitary vector.

On the other hand, the power spectrum for the homogeneous part of the scalar field is scale invariant for the solution (23), while that for the other terms in (24) we obtain two different spectrums to zero order in the hypergeometric function (29)

\[
\left\langle \phi^{[1]} \right\rangle^2 = \frac{1}{2\pi^2} \int \frac{dk}{k} k^3 \phi_k^{(hom)}(t) \alpha_k^*(t) \sim e^{-(5/2-\mu)} \int \frac{dk}{k} k^{5-\mu}, \tag{61}
\]

\[
\left\langle \phi^{[2]} \right\rangle^2 = \frac{1}{2\pi^2} \int \frac{dk}{k} k^3 [\phi_k^{(hom)}(t) \alpha_k(t) \sim e^{-(5/2-\mu)} \int \frac{dk}{k} k^{5-2\mu}, \tag{62}
\]

\[
\left\langle \phi^{[3]} \right\rangle^2 = \frac{1}{2\pi^2} \int \frac{dk}{k} k^3 \alpha_k(t) \alpha_k^*(t) \sim e^{-(5/2-\mu)} \int \frac{dk}{k} k^{3-2\mu}. \tag{63}
\]

Notice that these inhomogeneous terms are exponentially damped. The parameter $\mu$ can be fixed, with the smaller index that decays weaker, so as to yield the experimental data for the scalar spectral index (23), $n_s = 0.958$, then we write $3/2 - \mu = \epsilon'$, with $\epsilon' = n_s - 1 \sim -0.042$.

**IV. FINAL COMMENTS**

In this letter we have studied the primordial spectrum of electromagnetic fields using GI. Starting from a gauge with $A_0 = 0$, we have obtained some interesting properties. In the example here studied the spectrum of large-scale magnetic fields is nearly scale-invariant for $m \simeq -2$. The amplitude for the strength of comoving magnetic fields is dramatically increasing, but they are frozen in physical coordinates. The important result here obtained is that the modes of $A_j$ are affected by a source, which is originated in the modes of the inflaton field, so that the spectrum of the large-scale electric field during inflation depends of the modes of the inflaton field. These modes can be considered as massive photons which are gauge-invariant in a 5D sense, but once the foliation $d\psi = 0$ is done (which implies the choice of a relativistic system), these photons acquire mass because they live in an effective 4D curved spacetime. In this sense the choice of the relativistic system acts as an effective Higgs's mechanism.

But the more interesting result relies in that the spectrum of the inflaton field depends of the modes of $A_1$, because they are coupled to the modes of $A_1$. Of course, this scale invariance is significatively affected on shorten scales, so that it is nearly scale invariant on very large scales. This result disagrees with standard 4D versions of inflation, but
it agree very much with experience, because it is very known that for shorten scales the mass spectrum of matter has a positive index with a scale dependent power.

The effects of a conducting plasma in the early inflationary universe are negligible. During inflation conformal invariance is broken and the strength of comoving magnetic fields increases dramatically as $a^2$, until values of the order of $B_{\text{com}} \approx 10^{127}$ Gauss after 63 $e$-folds, so that the flatness problem is resolved in the model. After inflation, the universe enters in the so-called reheating phase, during which the energy of the inflaton is converted into ordinary matter. In this epoch, the conductivity $\sigma$ of the universe is of the order of $\sigma_e \sim T \gg H$ (with a background temperature $T \ll M_p$). Magnetic fields evolves adiabatically from the end of inflation until today, due to the high electrical conductivity of the cosmic plasma. In this epoch the universe is thermalized, so that the comoving magnetic field decreases with the expansion to take actual values of the order of $10^{-12}$ Gauss. Notice that the results here obtained depends on the gauge $A_0 = 0$. It is well known that any viable mechanism to generate seed magnetic fields during inflation must reposition on the breaking of conformal invariance of standard electrodynamics. Otherwise, the produced fields are vanishingly small. Notice that the approach here worded is not conformally invariant on the effective 4D metric. The origin of this rupture is in the fact that some connections $\Gamma^\alpha_{\beta\gamma}$ are non-zero on the 5D Riemann flat metric. This is the reason by which bosons are massive on the effective 4D spacetime, on which move the observers. Concerning electric fields, there is a damping of the longitudinal component of the field strength, corresponding to the gradual neutralization of charged particles in the primordial plasma, in the first stages of reheating. Finally, in our model, inflation occurs at a very low scale with $H \sim 10^{-9} M_p$ and with the inflaton taking values much below of the Planckian scale: $\langle \phi \rangle \sim 10^{-12} M_p$. In this sense our model evolves on scales similar to the MSSM inflationary model, where fine tuning and slow rolling problems joined with reheating were considered and the inflaton field couplings to Standard Model physics is explained from first principles. In our case the couplings between the fields $A_C$ is explained from the induced curvature of the metric. The problem of back-reaction should be considered in future works.

Acknowledgments

The authors acknowledge CONICET and UNMdP (Argentina) for financial support.

[1] T. Vachaspati, Phys. Lett. B265, 258 (1991); G. Sigl, A. V. Olinto, K. Jedamzik, Phys. Rev. D55, 4582 (1997); E. A. Calzetta, A. Kandus, F. D. Mazzitelli, Phys. Rev. D57, 7139 (1998); O. Tornkvist, Phys. Rev. D58, 043501 (1998); G. B. Field, Phys. Rev. D62, 103008 (2000); A. Kandus, E. A. Calzetta, F. D. Mazzitelli, Phys. Lett. B472, 287 (2000); N. Y. Gnedin, A. Ferrara, E. G. Zweibel, Astrophys. J. 539, 505 (2000); M. Giovannini, Phys. Rev. D62, 067301 (2000); G. Sigl, Phys. Rev. D66, 123002 (2002); A. Ashoorioon, R. B. Mann, Phys. Rev. D71, 103509 (2005); K. Bamba, M. Sasaki, JCAP 0702: 030 (2007); K. Bamba, Phys. Rev. D75: 083516 (2007); K. Bamba, JCAP 0710: 015 (2007); L. Campanelli, P. Cea, G. L. Fogli, L. Tedesco, Phys. Rev. D77: 123002 (2008); L. Campanelli, Helical Magnetic Fields from inflation, E-print: arXiv:0805.0575.

[2] J. Vallée, Astron. J. 124, 1322 (2001).

[3] Y. Sofue, M. Fujimoto and R. Wielebinski, Ann. Rev. Astron. Astrophys. 24, 459 (1986).

[4] E. Asseo and H. Sol, Phys. Rept. 148, 307 (1987).

[5] P. P. Kronberg, Rep. Prog. Phys. 57, 325 (1994).

[6] R. Beck, A. Brandenburg, D. Moss, A. A. Surkhurov and D. Sokloff, Annu. Rev. Astron. Astrophys. 34, 155 (1996).

[7] D. Grasso and H. R. Bubinstein, Phys. Rept. 348, 163 (2001).

[8] J. Bagchi, et al, New Astron. 7, 249 (2002).

[9] P. P. Kronberg, J. J. Perri and A. L. Zukowski, Ap. J. 33, 528 (1992).

[10] M. Wolfe, K. Lanzetta and A. L. Oren, Ap. J. 388, 17 (1992).

[11] C. A. Clarkson, A. A. Coley, R. Maartens and C. G. Tsagas, Class. Quant. Grav. 20, 1519 (2003).

[12] L. Campanelli, P. Cea, G. L. Fogli, L. Tedesco, Phys. Rev. D77: 043001 (2008).

[13] C.-L. Kuo et al., Astrophys. J. Suppl. Ser. 170, 335 (2007).

[14] A. C. S. Readhead et al., Astrophys. J. 609, 498 (2004); J. L. Sievers et al., arXiv:astro-ph/0509203.

[15] G. Yamazaki, K. Ichiki, T. Kajino, G. J. Mathews, Phys. Rev. D77, 043005 (2008).

[16] M. Giovannini, Int. J. Mod. Phys. D13, 391 (2004).

[17] M. S. Turner and L. M. Widrow, Phys. Rev. D37, 2743 (1998); F. D. Mazzitelli and F. M. Spedalieri, Phys. Rev. D52, 6694 (1995); G. Lambiase and A. R. Prasanna, Phys. Rev. D70, 063502 (2004); B. Ratna, Astrophys. J. 391, L1 (1992); M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. 75, 3796 (1995); E. A. Calzetta, A. Kandus and F. D. Mazzitelli, Phys. Rev. D57, 7139 (1998); A. C. Davis, K. Dimopoulos, T. Prokopec and O. Tornkvist, Phys. Lett. B501, 165 (2001); B. A. Bassett, G. Pollifrone, S. Tsuchikawa and F. Viniegra, Phys. Rev. D63, 103515 (2001); A. D. Dolgov,
[18] C. J. Hogan, Phys. Rev. Lett. 51, 1488 (1983); J. M. Quashnock, A. Loeb, D. N. Spergel, Astrophys. J. 344, L49 (1989); B. L. Cheng, A. V. Olinto, Phys. Rev. D50, 2421 (1994); M. Joyce, M. E. Shaposhnikov, Phys. Rev. Lett. 79, 1193 (1997); M. Giovannini, M. E. Schaposhnikov, Phys. Rev. D57, 2186 (1998); M. Giovannini, M. E. Schaposhnikov, Phys. Rev. D62, 103512; M. Christensson, M. Hindmarsh, A. Brandenburg, Astron. Nachr. 326, 393 (2005).

[19] A. Raya, J. E. Madriz Aguilar, M. Bellini, Phys. Lett. B638, 314 (2006).

[20] J. E. Madriz Aguilar, M. Bellini, Phys. Lett. B642, 302 (2006).

[21] P. Wesson, Gen. Rel. Grav. 22, 707 (1990); P. S. Wesson, Phys. Lett. B276, 299 (1992); P. S. Wesson and J. Ponce de Leon, J. Math. Phys. 33, 3883 (1992); P. Wesson, H. Liu and P. Lim, Phys. Lett. B298, 69 (1993); H. Liu and P. S. Wesson, J. Math. Phys. 33, 3888 (1992); T. Liko and P. S. Wesson, J. Math. Phys. 46, 062504 (2005); J. M. Overduin, P. S. Wesson, Phys. Rept. 283, 303 (1997).

[22] M. Bellini, Phys. Lett. B632, 610 (2006).

[23] Review of Particle Physics. Phys. Lett. B667, 103-105 (2008).

[24] C. A. Clarkson, A. A. Coley, R. Maartens, C. G. Tsagas, Class. Quantum Grav. 20, 1519 (2003).

[25] A. Díaz-Gil, J. García-Bellido, M. García-Pérez, A. González-Arroyo, PoSLAT: 242 (2006); A. Díaz-Gil, J. García-Bellido, M. García-Pérez, A. González-Arroyo, Phys. Rev. Lett. 100: 241301 (2008).

[26] A. Membiela, M. Bellini, Phys. Lett. B635, 243 (2006).

[27] R. Allahverdi, K. Enquist, J. García-Bellido, A. Jokinen, A. Mazumdar, JCAP 0706: 019 (2007).

[28] N. C. Tsamis, R. P. Wooderd, Annals Phys. 253, 1 (1997); M. Bellini, Class. Quant. Grav. 17, 145 (2000); R. H. Brandenberger, J. Martin, Phys. Rev. D71: 023504(2005).