Compacitification and Positive Mass Theorem for Fibered Euclidean End

Xianzhe Dai · Yukai Sun

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Abstract
We consider the Positive Mass Theorem for Riemannian manifolds \((M^n, g)\) with the asymptotic end \((\mathbb{R}^k \times X^{n-k}, g_{\mathbb{R}^k} + g_X)\) \((k \geq 3)\) by studying the corresponding compactification problem. Here \((X, g_X)\) is a compact scalar flat manifold. We show that the Positive Mass Theorem holds if certain generalized connected sum admits no metric of positive scalar curvature. Moreover we establish the rigidity result, namely, the mass is zero iff \(M\) is isometric to \(\mathbb{R}^k \times X^{n-k}\).

Keywords Positive mass theorem · Conformal deformation · Scalar curvature · Green function · Harmonic function

Mathematics Subject Classification 53C21 · 53C27 · 58Jxx

1 Introduction

The famous Positive Mass Theorem [12, 15] states that an asymptotically Euclidean manifold with nonnegative scalar curvature must have nonnegative ADM mass (if the dimension of the manifold is between 3 and 7 or if the manifold is spin; for the recent progress about the higher dimensional non-spin manifolds, see [13]). Furthermore, the mass is zero iff the manifold is the Euclidean space.

A crucial ingredient in Schoen–Yau’s approach (after Lohkamp’s observation) is the idea of compactification. In the end, the Positive Mass Theorem for \(M\) is reduced to showing that, for the one-point compactification \(M_1\) of \(M\), the connected sum \(M_1^n \# T^n\) admits no positive scalar curvature metric. Roughly speaking, one compactifies the asymptotically Euclidean manifold \(M^n\) to \(M_1^n \# T^n\).
In [4], motivated by string theory, the first author proves a Positive Mass Theorem for manifolds which are asymptotically approaching the product of a Euclidean space and a compact Calabi–Yau (or any exceptional holonomy) manifolds. We will be referring to the compact manifold in such a product of a Euclidean space with a compact manifold as a compact factor. The approach there is along that of Witten, using Dirac operators, and hence our manifolds are assumed to be spin. More crucially, one needs nonzero parallel spinors for the asymptotic infinity which limits the geometry at infinity. For technical reasons, we restrict to simply connected compact factors, although the method extends to non-simply connected case with some additional assumptions on the spin structure. This was later considered by [10] who treated the case of $S^1$ factors (Minerbe is motivated by the study of gravitational instantons; he also allows non-trivial circle fibrations). More recently, Positive Mass Theorems for $S^1$ or $T^2$ factors have been considered in [8], and for more general compact flat factors in [2].

More precisely, a complete Riemannian manifolds $(M^n, g)$ is said to have asymptotically fibered Euclidean end if $M^n = M_0 \cup M_{\infty}$ with $M_0$ a compact manifold with boundary and $M_{\infty} \simeq (\mathbb{R}^k \setminus B_R(0)) \times X$ for some $R > 0$ and $X$ is a closed $(n - k)$-dimensional manifold. Moreover, the metric on $M_{\infty}$ satisfies

$$g = \hat{g} + h, \quad \hat{g} = g_{\mathbb{R}^k} + g_X,$$

$$h = O(r^{-\tau}), \quad \hat{\nabla} h = O(r^{-\tau - 1}), \quad \hat{\nabla} \hat{\nabla} h = O(r^{-\tau - 2}), \quad \hat{\nabla} \hat{\nabla} \hat{\nabla} h = O(r^{-\tau - 3})$$

where $\hat{\nabla}$ is the Levi-Civita connection of $\hat{g}$, $\tau > \frac{k - 2}{2}$ is the asymptotical order and $r$ is the distance to a fixed point. (Note that it makes no difference whether the distance is induced by $g$ or $\hat{g}$. For that matter, one may also choose $r$ to be the Euclidean distance.) When we need to specify the end structure more explicitly, we will also say that $M^n$ is a manifold with the asymptotic end $\mathbb{R}^k \times X$.

The mass for such a space is then defined by [4]

$$m(g) = \lim_{R \to \infty} \frac{1}{\omega_k \text{Vol}(X)} \int_{S_R \times X} (\hat{\nabla}_{e_a} \hat{g}_{ja} - \hat{\nabla}_j \hat{g}_{aa}) \ast dx_j d\text{vol}_{g_X}.$$ 

where $\{e_a\} = \{\frac{\partial}{\partial x_i}, f_a\}$ is an orthonormal basis of $\hat{g}$ and the $\ast$ operator is the one on the Euclidean factor, the indices $i, j$ run over the Euclidean factor and the index $\alpha$ runs over $X$ while the index $a$ runs over the full index of the manifold $M^n$. (There is an additional factor of $\frac{1}{4}$ in [4].)

Strictly speaking, the existence of the limit in the definition of the mass is only guaranteed under additional assumptions such as the integrability of the scalar curvature [1]. (More generally, one replaces the limit by lim sup.) For this reason (and some others) we now assume that the metric $g_X$ on $X$ has zero scalar curvature.

On the other hand, if $M^n_1$ is a closed $n$-dimensional manifold and $X$ embeds in $M^n_1$ with trivial normal bundle, then one can construct the connected sum of $M^n_1$ and $T^k \times X$ along $X$, $M^n_1 \#_X (T^k \times X)$; see §2. Called the generalized connected sum in [8], it is first considered there in this context for $X = S^1, T^2$. Our main result is

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Theorem 1  If for any closed manifold $M^n$ admitting an embedding of $X$ with trivial normal bundle, the connected sum along $X$

$$M^n \#_X (T^k \times X)$$

has no metrics of positive scalar curvature, then the mass of any manifold $M^n$ with the asymptotic end $\mathbb{R}^k \times X$ and nonnegative scalar curvature is nonnegative. Moreover, $(M, g) \equiv (\mathbb{R}^k \times X, g_{\mathbb{R}^k} + g_X)$ if the mass of $(M^n, g)$ is zero.

Remark 2  As we mentioned, the cases of $X = S^1, T^2$ have been dealt with by Liu-Shi-Zhu in [8] (though they restrict to asymptotically Schwarzschild type metrics), and more generally, for $X = F$ a closed flat manifold, Chen-Liu-Shi-Zhu proved a similar result in [2]; see Theorem 1.8 there. Except for $X = S^1$, the rigidity statement is weaker in [2, 8]. In [2] a lifting argument is employed as $F$ is covered by the Euclidean space while here we work directly with the Green’s function on $M$. Nevertheless we have made essential use of some of the ideas in [2] such as the cut-off construction, as well as [2, Proposition 4.10], adapted to our situation, which serves as the important starting point of our proof of the rigidity statement.

As in [11], the first part of Theorem 1 is a consequence of the following result. Recall that if $M^n$ is a manifold with the asymptotic end $\mathbb{R}^k \times X$, then $M^n$ decomposes as $M_0 \cup M_\infty$ with $M_0$ a compact manifold with boundary and $M_\infty \simeq (\mathbb{R}^k \setminus B_R(0)) \times X$ for some $R > 0$. In particular $\partial M_0 \simeq S^{k-1} \times X$. Then $M^n$ has a natural compactification $M^n_1$ obtained from $M_0$ by attaching $B^k \times X$, where $B^k$ is the Euclidean $k$-ball. Note that $X$ embeds into the closed manifold $M^n_1$ as $\{0\} \times X \hookrightarrow B^k \times X$, and this embedding clearly has trivial normal bundle.

Theorem 3  Let $(M^n, g)$ be a manifold with the asymptotic end $\mathbb{R}^k \times X$ and $M^n_1$ its natural compactification introduced above. If the scalar curvature of $M^n$ is nonnegative but its mass is negative, then the connected sum along $X$

$$M^n_1 \#_X (T^k \times X)$$

admits a metrics of positive scalar curvature.

As application of Theorem 1, we have

Corollary 4  Assume that either $X$ is enlargeable or $\hat{A}(X)$ is nonzero. Then the Positive Mass Theorem holds for any spin manifolds with the asymptotic end $\mathbb{R}^k \times X$.

For definitions of the enlargeability and the $\hat{A}$-genus $\hat{A}(X)$ see [6, Sect. 5]. Here we just note that tori are the typical examples of enlargeable manifolds. On the other hand, a $K3$ surface has nonzero $\hat{A}$-genus.

As we assume $(X, g_X)$ is scalar flat, if $X$ is further assumed to be simply connected and $\dim X \geq 5$, then in fact $(X, g_X)$ is the product of Calabi–Yau manifolds or $Spin(7)$-manifolds by Theorem 1 in [5]. Thus, we give another proof of the result in [4], except for the $G_2$-manifold factors. Simply connected $G_2$-manifolds always admit positive scalar curvature metrics by [14]. Requiring a parallel spinor at infinity restricts
the geometry at infinity, and there lies the difference between the approach in [4] and
the current approach. We emphasize that our result for the non-simply connected case
is new.

The proof of the compacification theorem, Theorem 3, consists of two steps. By
solving a conformal Laplacian equation, one can deform the metric \( g \) outside a com-
pact set to a conformal metric of the product metric \( g_{\mathbb{R}^k} + g_X \), while still maintaining
negative mass and good scalar curvature control. The crucial thing here is the asymp-
totic behavior of the solution at infinity, which is obtained by studying the asymptotics
of the Green’s function of \( \Delta_g \). In the second step, which is due to an observation of
Lohkamp, one modifies the conformal factor by a subharmonic function (again outside
a compact set) which is constant near infinity. Again, the asymptotic behavior of the
conformal factor is critical.

The rigidity part of the Positive Mass Theorem makes use of two mass notions
and a result in [2], generalized to our situation. The so called Gauss-Bonnet mass
is introduced in [10] for the case \( S^1 \) factor and it, in general, is different from the
(ADM) mass discussed above. But as in [2, Proposition 4.10], under certain geometric
condition which is satisfied when the (ADM) mass is zero, the two masses differ by a
nonzero multiple. This is crucial in showing that the harmonic functions coming from
the asymptotic coordinates have parallel differentials. We then use these functions
with parallel differential to directly construct the required isometry. In [2] the authors
also obtain rigidity for the \( S^1 \) factor and more generally flat factors, although the
conclusion is weaker for the general flat factors.

This paper is organized as follows. In Sect. 2, we introduce manifolds with the
asymptotic end \( \mathbb{R}^k \times X \) and their topological compactifications. For later purpose
we also discuss the existence of positive scalar curvature metrics on such manifolds
(and their generalized connected sums). In Sect. 3, we study the asymptotic behavior
at infinity of the Green function of the metric \( g \). In Sect. 4, we solve the conformal
Laplace equation and use it to prove the Theorem 4. Finally in Sect. 5 we clarify the
roles of the two mass quantities and establish the rigidity statement in Theorem 1.

2 Manifolds with Asymptotically Fibered Euclidean Ends

In this section we give the precise definition of manifolds with asymptotically fibered
Euclidean ends and their masses. Denote by \( g_{\mathbb{R}^k} \) the standard Euclidean metric on \( \mathbb{R}^k \)
and \( B_R(0) \) the ball of radius \( R \) around the origin. Let \((X, g_X)\) be a compact Riemannian
manifold.

**Definition 5** A complete noncompact smooth manifold \((M^n, g)\) is said to have
asymptotically fibered Euclidean end, or more precisely, the asymptotic end \((\mathbb{R}^k \times X^{n-k}, g_{\mathbb{R}^k} + g_X)\) \((k \geq 3)\), if there exists a compact subset \( K \subset M^n \) such that
\( M^n \setminus K \simeq (\mathbb{R}^k \setminus B_R(0)) \times X \) for some \( R > 0 \) and on \( M^n \setminus K \)

\[
\begin{align*}
g &= \hat{g} + h, \\
\hat{g} &= g_{\mathbb{R}^k} + g_X, \\
h &= O(r^{-\tau}), \quad \hat{\nabla} h = O(r^{-\tau-1}), \quad \hat{\nabla} \hat{\nabla} h = O(r^{-\tau-2}), \quad \hat{\nabla} \hat{\nabla} \hat{\nabla} h = O(r^{-\tau-3})
\end{align*}
\]
where ̂∇ is the Levi-Civita connection of ̂g, τ ≫ \( \frac{k-2}{2} \) is the asymptotical order and r is the distance to a fixed point. In addition, we assume that \( Sc_\hat{g} \in L^1(M) \).

If \( X \) is a single point, this reduces to the notion of asymptotically Euclidean manifolds. The cases when \( X \) are tori or flat manifolds arise in the study of gravitational instantons, while string theoretic considerations lead to \( X \) being the Calabi–Yau or other special holonomy manifolds.

If \( M \) is a manifold with the asymptotic end \( \mathbb{R}^k \times X^{n-k} \), \( M \) can be written as \( M_0 \cup M_\infty \), with \( M_0 \) a compact manifold with boundary and \( M_\infty \simeq (\mathbb{R}^k \setminus B_R(0)) \times X \) for some (perhaps slightly larger) \( R > 0 \).

As we mentioned, the assumption \( Sc_\hat{g} \in L^1(M) \) is imposed so that the ADM mass is well defined [1]. In particular this would imply that the scalar curvature of \( (X, g_X) \) is identically zero. From now on, we make the assumption that \( Sc_\hat{g}_X \equiv 0 \). Let \( \omega_k \) be the area of the unit sphere in \( \mathbb{R}^k \).

**Definition 6** The mass for such a space is then defined by [4]

\[
m(g) = \lim_{R \to \infty} \frac{1}{\omega_k \text{Vol}(X)} \int_{S_R \times X} (\hat{\nabla}_ja g ja - \hat{\nabla}_j gaa) \ast d\text{vol}_g.
\]

where \( \{e_a\} = \{\frac{\partial}{\partial x^i}, f_\alpha\} \) is an orthonormal basis of \( \hat{g} \) and the * operator is the one on the Euclidean factor, the indices \( i, j \) run over the Euclidean factor and the index \( \alpha \) runs over \( X \) while the index \( a \) runs over the full index of the manifold \( M^n \).

The same argument as in [1] shows that \( m(g) \) is a well-defined metric invariant of \( (M, g) \). It reduces to, by the Fubini’s theorem and divergence theorem,

\[
m(g) = \lim_{R \to \infty} \frac{1}{\omega_k \text{Vol}(X)} \int_{S_R \times X} (\partial_\alpha g_{aj} - \partial_j g_{\alpha a}) \ast d\text{vol}_g
\]

More intrinsically, it can be written as [10]

\[
m(g) = -\frac{1}{\omega_k \text{Vol}_X} \lim_{R \to \infty} \int_{\partial B_R} \ast \hat{g} (\text{div}_\hat{g} g + d \text{Tr}_\hat{g} g).
\]

Here \( \ast \hat{g} \) is the Hodge star operator of the metric \( \hat{g} \), and \( \text{Tr}_\hat{g}, \text{div}_\hat{g} \) indicate taking trace respectively divergence with respect to \( \hat{g} \). For example, \( (\text{div}_\hat{g} T)_i = -\hat{g}^{jk} \partial_j T_{ik} \) in local coordinate for a \((0, 2)\)-type tensor \( T \).

**Remark 7** Without the integrability assumption \( Sc_\hat{g} \in L^1(M) \), one can still define the ADM mass as above, by replacing the limit with \( \lim \sup \) [10].

We now define another mass quantity, the so called Gauss-Bonnet mass, first introduced in [10, Theorem 1] for the circle fibration.

\[
m^{GB}(g) = -\frac{1}{\omega_k \text{Vol}_X} \lim_{R \to \infty} \sup \int_{\partial B_R} \ast \hat{g} (\text{div}_\hat{g} g + d \text{Tr}_\hat{g} g - \frac{1}{2} d(\text{Tr}_\hat{g}_X g)).
\]
Here $\text{Tr}_{gX}$ indicates taking trace with respect to $g_X$. In other words, $\text{Tr}_{gX} = \sum_{\alpha} g(f_{\alpha}, f_{\alpha})$ for an orthonormal basis $f_{\alpha}$ of $(X, g_X)$. This mass quantity will play an important part in proving the rigidity part of Theorem 1; see Sect. 5.

A manifold $(M, g)$ with the asymptotic end $\mathbb{R}^k \times X$ has a natural topological compactification. Indeed, $M^n$ decomposes as $M_0 \cup M_\infty$ with $M_0$ a compact manifold with boundary and $M_\infty \simeq (\mathbb{R}^k \setminus B_R(0)) \times X$ for some $R > 0$. In particular $\partial M_0 \simeq S^{k-1} \times X$. One can then obtain a closed manifold $M^n_1$ by attaching $B^k \times X$ to $M_0$, where $B^k$ is the Euclidean $k$-ball. This is in some sense the fiberwise version of the one-point compactification for the asymptotically Euclidean spaces.

Note that $X$ embeds into $M_1$ with trivial normal bundle, as $\{0\} \times X \hookrightarrow B^k \times X$. In general, if $M^n_1$ is a closed $n$-dimensional manifold and $X$ embeds in $M^n_1$ with trivial normal bundle, then one can construct the connected sum of $M^n_1$ and $T^k \times X$ along $X$, denoted by $M^n_1 \#_X (T^k \times X)$. In other words, $M^n_1 \#_X (T^k \times X)$ is obtained from $M^n_1$ by removing a tubular neighborhood of $X$ and gluing in $(T^k \setminus B_\epsilon (x)) \times X$, where $x$ is a fixed point on $T^k$ and $B_\epsilon (x)$ an $\epsilon$-ball around $x$.

To consider applications of Theorem 1, we now look at some examples of manifolds of the type $M^n_1 \#_X (T^k \times X^{n-k})$ admitting no positive scalar curvature. For definitions of the enlargeability and the $\hat{A}(X)$ see Sect. 5 in [6].

**Proposition 8** Let $(M, g)$ be a manifold with the asymptotic end $\mathbb{R}^k \times X$ and its natural compactification $M_1$ be spin. If either $X$ is enlargeable or $\hat{A}(X) \neq 0$, then $M^n_1 \#_X (T^k \times X^{n-k})$ admits no metrics of positive scalar curvature.

**Proof** If $X$ is enlargeable, so is $T^k \times X$. As there is a degree one map $M^n_1 \#_X (T^k \times X^{n-k}) \longrightarrow T^k \times X^{n-k}$, there are no metrics of positive scalar curvature on $M^n_1 \#_X (T^k \times X^{n-k})$ by Proposition 5.6 in [6] if $M_1$ is, in addition, a spin manifold.

On the other hand, if $\hat{A}(X)$ is nonzero and $M_1$ is spin, then one can first construct a degree one map $f_1 : M^n_1 \#_X (T^k \times X^{n-k}) \rightarrow T^k \times X^{n-k}$ as follows. One collapses the part of $M^n_1$ in the generalized connected sum to the point of $T^k$ where we remove the $\epsilon$-ball, while mapping the compliment of $2\epsilon$-ball identically to $T^k$, and the annulus region between the $2\epsilon$-ball and $\epsilon$-ball smoothly to the $2\epsilon$-ball. The boundary of the $\epsilon$-ball is mapped to the center point and the map on the $X$ factor is the identity. The map $f_1$ is followed by the projection map $f_2 : T^k \times X^{n-k} \rightarrow T^k$ which has nonzero $\hat{A}$-degree by the assumption. Then we know that $M^n_1 \#_X (T^k \times X^{n-k})$ is enlargeable in dimension $k$ by Proposition 6.5 in [6]. Thus, it is also without positive scalar curvature.

### 3 The Green’s Function

In this section we estimate the asymptotic order of the Green’s function $G_g$ of $\Delta_g$ when $r$ is sufficiently large. This will be used crucially in the next section to establish the asymptotic behavior of the solution of conformal Laplace equation. First of all, we have

**Proposition 9** Let $(M, g)$ be a manifold with the asymptotic end $\mathbb{R}^k \times X$. Then $(M, g)$ is nonparabolic. That is, there exists a positive Green’s function $G_g$ on $M$.  

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Proof By [7, Corollary 20.8], which is attributed to Royden, nonparabolicity is a quasi-isometry invariant. Therefore we can assume that $g = g_{R^k} + g_X$ outside a compact region. Now for $p = (x_0, y_0), x_0 \in \mathbb{R}^k, y_0 \in X$, outside this compact region, let $r((x, y)) = d_{\mathbb{R}^k}(x, x_0)$ denote the Euclidean distance function from $x \in \mathbb{R}^k$ to $x_0 \in \mathbb{R}^k$. Then $r^{2-k}$ is a harmonic function away from the compact region whose infimum is achieved at infinity. Therefore by [7, Theorem 17.3], positive Green’s function exists on $M$.

Our main result in this section is the asymptotic behavior of $G_g$ as one approaches infinity. The following lemma gives a rough bound on this asymptotic behavior.

For each fixed $p = (x_0, y_0) = (x_0, y_0) \in M \setminus K \simeq (\mathbb{R}^k \setminus B_1(0)) \times X$ and any $(x, y) \in M \setminus K$, let $r((x, y)) = d_{\mathbb{R}^k}(x, x_0)$ as above.

Lemma 10 For any $\epsilon \in (0, 1)$, there exist a larger compact set $K' \supset K$ containing $p$ and constants $C_1 = C_1(\epsilon, K') > 0, C_2 = C_2(\epsilon, K') > 0$ such that, for $(x, y)$ outside $K'$,

$$C_1 r^{2-k-\epsilon}((x, y)) \leq G_g((x, y), p) \leq C_2 r^{2-k+\epsilon}((x, y)). \quad (3)$$

Proof For any $\epsilon > 0$, note that

$$\Delta_g r^{2-k+\epsilon}((x, y)) = \Delta_g r^{2-k+\epsilon} + (\Delta_g - \Delta_{\hat{g}}) r^{2-k+\epsilon}$$

$$= -\epsilon (k - 2 - \epsilon) r^{-k+\epsilon} + O(r^{-k-\tau+\epsilon}),$$

It follows that there exist a larger compact set $K' \supset K$ such that when $(x, y)$ is outside $K'$,

$$\Delta_g r^{2-k+\epsilon}((x, y)) \leq 0.$$

Then we can choose $C_2 = C_2(\epsilon, K') > 0$ such that

$$C_2 r^{2-k+\epsilon} |_{\partial K'} \geq G_g((x, y), p)|_{\partial K'}.$$

By the maximal principle,

$$G_g((x, y), p) \leq C_2 r^{2-k+\epsilon}((x, y))$$

outside $K'$.

The other inequality can be proven similarly.

In what follows, for $x, x_0 \in \mathbb{R}^k$, we denote by $\|x - x_0\|$ their Euclidean distance.

Proposition 11 Let $(x_0, y_0) \in M \setminus K$. The Green’s function $G_g$ has the following expansion, when $\|x - x_0\|$ is large,

$$G_g((x, y), (x_0, y_0)) = \frac{C}{\|x - x_0\|^{k-2}} + O\left(\frac{1}{\|x - x_0\|^{\min\{k-1, k-2+\tau\}}}\right)$$
for $C > 0$. Moreover

$$\|x - x_0\|^k \cdot \|\nabla G_g((x, y), (x_0, y_0))\| \to 0 \text{ as } \|x - x_0\| \to \infty.$$ 

Before we give the proof of the proposition, we recall some definitions; see [10, p.930]. Let $K$ be compact set such that $M \setminus K$ is diffeomorphic to $(\mathbb{R}^k \setminus B_R(0)) \times X$ and $r = \|x - x_0\|$ as before. Then for $\delta \in \mathbb{R}$, the weighted $L^2$-space is

$$L^2_\delta(\Omega) = \left\{ u \in L^2_{loc}(\Omega) \left| \int_{\Omega \setminus K} u^2 r^{-2\delta} \, d\nu < \infty \right. \right\}$$

with the norm

$$\|u\|_{L^2_\delta(\Omega)} = \left( \int_{\Omega \cap K} u^2 \, d\nu + \int_{\Omega \setminus K} u^2 r^{-2\delta} \, d\nu \right)^{\frac{1}{2}}.$$ 

Any $u \in L^2_{loc}(M \setminus K)$ can be written as $u = \Pi_0 u + \Pi_\perp u$, where

$$(\Pi_0 u)(x) = \frac{1}{\text{Vol}_X} \int_X u(x, y) \, d\nu_y, \quad \Pi_\perp u = u - \Pi_0 u.$$ 

Then, for any $\delta, \epsilon \in \mathbb{R}$,

$$L^2_{\delta, \epsilon}(\Omega) = \left\{ u \in L^2_{loc}(\Omega \setminus K) \left| \|\Pi_0 u\|_{L^2_\delta(\Omega \setminus K)} < \infty \text{ and } \|\Pi_\perp u\|_{L^2_\epsilon(\Omega \setminus K)} < \infty \right. \right\}$$

with the norm

$$\|u\|_{L^2_{\delta, \epsilon}(\Omega)} = \left( \|u\|^2_{L^2(K \cap \Omega)} + \|\Pi_0 u\|^2_{L^2_\delta(\Omega \setminus K)} + \|\Pi_\perp u\|^2_{L^2_\epsilon(\Omega \setminus K)} \right)^{\frac{1}{2}}.$$ 

Finally, we define the Sobolev space

$$H^2_\delta = \left\{ u \in H^2_{loc}(\Omega) \left| \|\nabla^8 d\Pi_0 u\|_{L^2_{\delta - 2}(K^c)} + \|d\Pi_0 u\|_{L^2_{\delta - 1}(K^c)} + \|\Pi_0 u\|_{L^2_\delta(K^c)} < \infty \right. \right\}.$$ 

and

$$(\Pi_\perp u)(x) = \frac{1}{\text{Vol}_X} \int_X u(x, y) \, d\nu_y, \quad \Pi_\perp u = u - \Pi_0 u.$$ 

Before we go into the proof of Proposition 11, we also need the following proposition from [10] which we quote here for convenience. The proposition gives asymptotic behavior of solutions to $\Delta_1 gu = f$ in terms of that of $f$ as well as those of harmonic functions on $\mathbb{R}^k$ expressed in the spectral decomposition of the Laplace operator $\Delta_1$ on the unit sphere $S^{k-1}$.

Recall that the eigenvalues of $\Delta_S$ are $\lambda_j = j(k - 2 + j)$, $j \in \mathbb{N}$. Let $E_j$ be the eigenspace of $\Delta_S$ with eigenvalues $\lambda_j = j(k - 2 + j)$. Let $\delta_j = \frac{k}{2} + j$, $j \in \mathbb{N}$ and we call $\delta \in \mathbb{R}$ noncritical if $\delta \neq \delta_j$ and $\delta \neq 2 - \delta_j$, for any $j \in \mathbb{N}$.
Proposition 12 (Proposition 4 in [10]) Suppose $\Delta_g u = f$ with $u$ in $L^2_{\delta}(K^c)$ and $f$ in $L^2_{\delta'-2}(K^c)$ for $K$ compact and noncritical exponents $\delta > \delta'$. Then, up to enlarging $K$, there is an element $v$ of $L^2_{\delta',\delta'-2}(K^c)$ such that $u - v$ is a linear combination of the following function:

1. $\mathcal{N}_{j,\phi_j}^+$ with $\phi_j$ in $E_j$, if $\delta' < \delta_j < \delta$;
2. $\mathcal{N}_{j,\phi_j}^-$ with $\phi_j$ in $E_j$, if $\delta' < 2 - \delta_j < \delta$.

where $\mathcal{N}_{j,\phi_j}^\pm = \Phi(r^{v_j^\pm} \phi_j + v_j^\pm)$ with $v_j^+ = j$, $v_j^- = 2 - k - j$. $\Phi$ is a smooth cut-off function which is 0 on a compact set and 1 on $K^c$. Moreover, $\Delta_g v_j^\pm = -\Delta_g (r^{v_j^\pm} \phi_j)$, $v_j^+ \in H^2_\eta$ for any $\eta > \delta_j - \tau$ and $v_j^- \in H^2_\eta$ for any $\eta > 2 - \delta_j - \tau$.

Remark 13 In [10], the author deals explicitly with the case when $X = S^1$. But the method and result in [10] generalize to the general $X$ without change.

Proof of Proposition 11 For fixed $y \in M$, consider the equation,

$$\Delta_g [\chi(x)G_g] = 2(\nabla G_g, \nabla_g \chi(x)) + G_g \Delta_g \chi(x) \tag{4}$$

where $\chi(x)$ is a cut-off function which is 0 in $B_\gamma(r_1)$ and 1 in $B_\gamma^c(r_2)$ for $r_2 >> r_1$. The order of the right hand term of equation (4) is arbitrary, hence it lies in $L^2_{\delta'-2}(B_{R_0}^c)$ especially for any $\delta' > -\frac{k}{2}$. Also $\chi(x)G_g \in L^2_{\delta}(B_{R_0}^c)$ for any $\delta > -\frac{k}{2} + 2 + \epsilon$ by (3).

Thus we can choose $\delta, \delta'$ such that only $\delta' < 2 - \delta_j < \delta$ is possible for some $j$. Thus, by Proposition 12, we have,

$$\chi(x)G_g = v + \sum_j \mathcal{N}_{j,\phi_j}^-,$$

where $v \in L^2_{\delta',\delta'-2}(B_{R_0}^c)$, the sum runs over $j \in \mathbb{N}$ such that $\delta' < 2 - \delta_j < \delta$, and

$$\Delta_g v = 2(\nabla G_g, \nabla_g \chi(x)) + G_g \Delta_g \chi(x).$$

Using Moser iteration as in the proof of Lemma 6 in [10], we have $v = O(r^{-k+\delta'})$ for (possibly different but still arbitrary) $\delta' > -\frac{k}{2}$. Since $\delta > -\frac{k}{2} + 2 + \epsilon$ and $\delta' > -\frac{k}{2}$ are arbitrary, we choose $\epsilon' > 0$ such that $\delta' = -\frac{k}{2} + \epsilon'$ and $\delta = -\frac{k}{2} + 2 + \epsilon + \epsilon'$ and $1 > \epsilon + \epsilon' > 0$. Then the only $j \in \mathbb{N}$ satisfying

$$-\frac{k}{2} + \epsilon' < 2 - \frac{k}{2} - j < -\frac{k}{2} + 2 + \epsilon + \epsilon'.$$

are $j = 0, 1$. Thus $\mathcal{N}_{j,\phi_j}^- = \Phi(r^{2-k-j} \phi_j + v_j^-)$, where $\Delta_g v_j^- = -\Delta_g (r^{v_j^-} \phi_j)$, $\Delta_g (r^{v_j^-} \phi_j) \in L^2_{\eta-\tau}$ with $\eta > \delta_j$. Then we can find $v_j^- \in H^2_{\eta-\tau}$ again as in the proof of Lemma 6 in [10]. Therefore we have $v_j^- = O(R^{2-k-\tau})$. Hence,

$$\chi(x)G_g = O(\frac{1}{r_{\min} [k-\eta+k+\tau-2\eta]} + \frac{C\Phi}{r^{2-k}}). \tag{5}$$
Since we can differentiate the both sides of (4) and use Proposition 12 again and repeat the above process, we also get $\|\nabla G_g\| = O\left(\frac{1}{r^{2-\tau}}\right)$ for $r >> 1$. By the positivity of $G_g$ and lemma 10,

$$G_g = \frac{C}{r^{k-2}} + O\left(\frac{1}{r^{\min\{k-1,k-2+\tau\}}}\right)$$

for $C > 0$ and $r >> 1$.

4 The Compactification

Let $(M, g)$ be a manifold asymptotic to $\mathbb{R}^k \times X$ whose scalar curvature is nonnegative but its mass is negative. Following the general strategy of [11], we compactify $(M, g)$ by cutting $M$ off a large compact set $K$ such that $\partial K = (\partial[0, 1]^k) \times X$ and gluing the opposite faces of $[0, 1]^k$, with the resulting manifold $(M'_1 \#_X (T^k \times X), g)$. In order for $(M'_1 \#_X (T^k \times X), g)$ to still have a metric with positive scalar curvature we deform $g$ so that it is the product of Euclidean metric with $g_X$ outside a compact set while still maintains nonnegative scalar curvature. 

There are two key steps.

- **Step 1:** If $(M, g)$ is asymptotically $(\mathbb{R}^k \times X, g_{\mathbb{R}^k} + g_X)$ and $Sc_g \geq 0$ but $m(g) < 0$, then there is a metric $\tilde{g} = \tilde{u} g_{\mathbb{R}^k} + g_X$ with $\tilde{u} = 1 + \frac{\tilde{m}}{r^{k-2}} + O\left(\frac{1}{r^{1-k}}\right)(\tilde{m} < 0)$ and $Sc_{\tilde{g}} \geq 0$, $Sc_g = 0$ outside a large compact set.

- **Step 2:** This is an observation due to J.Lohkamp. If $(M, g)$ with $g = u^{\frac{4}{n-2}}(g_{\mathbb{R}^k} + g_X)$ and $u = 1 + \frac{m}{r^{k-2}} + O\left(\frac{1}{r^{1-k}}\right)(m < 0)$ and $Sc_g \geq 0$, $Sc_g = 0$ outside a large compact set, then there exists a metric $\tilde{g}$ with $Sc(\tilde{g}) \geq 0$ and $\tilde{g} = g_{\mathbb{R}^k} + g_X$ near $\infty$.

We first prove the Sobolev inequality on $(M, g)$ which is used in the rest part of the paper (as well as in the previous part where Moser iteration is involved). Recall that $M \setminus K \simeq (\mathbb{R}^k \setminus B_R(0)) \times X$. For $r > R$, let $B_r = K \cup (B_r(0) \setminus B_R(0)) \times X$.

**Lemma 14** There is a Sobolev constant $c > 0$ not depending on $B_r$ such that, for $f \in C_0^\infty(B_r)$,

$$\left(\int_{B_r} f \frac{2a}{n-2} d\text{vol}_g\right)^{\frac{n-2}{2a}} \leq c \left(\int_{B_r} |\nabla f|^2 d\text{vol}_g\right)^{\frac{1}{2}}. \quad (6)$$

**Proof** Write $f = \Pi_0 f + \Pi_\perp f$ in $B_r \setminus K$ where $\Delta_{g_X} \Pi_0 f = 0$, $\Pi_0 f = \frac{1}{\text{Vol}(X)} \int_X f d\text{vol}_{g_X}$ and $\Pi_\perp f = f - \Pi_0 f$. Then

$$\nabla_{g_{\mathbb{R}^k}} \Pi_0 f = \nabla_{g_{\mathbb{R}^k}} \left(\frac{1}{\text{Vol}(X)} \int_X f d\text{vol}_{g_X}\right) = \frac{1}{\text{Vol}(X)} \int_X (\nabla_{g_{\mathbb{R}^k}} f) d\text{vol}_{g_X}$$

$$\nabla_{g_X} \Pi_0 f = 0,$$
and

\[ |\nabla_{g_{rk}} \Pi_0 f|^2 = \left| \frac{1}{\text{Vol}(X)} \int_X \left( \nabla_{g_{rk}} f \right) d\text{vol}_{g_X} \right|^2 \]

\[ \leq \frac{1}{\text{Vol}(X)} \int_X \left| \nabla_{g_{rk}} f \right|^2 d\text{vol}_{g_X} \]

We only need to consider the metric \( \hat{g} \) since it is equivalent to \( g \) in \( B_r \setminus K \). Then, for some \( c_1, c_2 > 0 \)

\[ \left( \int_{B_r \setminus K} (\Pi_0 f)^{\frac{2n}{n-2}} d\text{vol}_{\hat{g}} \right)^{\frac{n-2}{2n}} \leq c_1 \left( \int_{B_r \setminus K} |\nabla_{g_{rk}} \Pi_0 f|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ \leq c_1 \left( \int_{B_r \setminus K} \left[ \frac{1}{\text{Vol}(X)} \int_X |\nabla_{g_{rk}} f|^2 d\text{vol}_{g_X} \right] d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ \leq c_1 \left( \int_{B_r \setminus K} |\nabla_{g_{rk}} f|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ \left( \int_{B_r \setminus K} (\Pi_\perp f)^{\frac{2n}{n-2}} d\text{vol}_{\hat{g}} \right)^{\frac{n-2}{2n}} \leq c_2 \left( \int_{B_r \setminus K} \left| \nabla_{g_X} \Pi_\perp f \right|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ = c_2 \left( \int_{B_r \setminus K} \left| \nabla_{g_X} f \right|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ \leq c_2 \left( \int_{B_r \setminus K} \left| \nabla_{\hat{g}} f \right|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

Let \( \chi \) be a cut-off function which is 1 on \( K \) and is 0 outside a larger compact set \( K' \), such that \( K \subset K' = B_{r_1} \subset B_r \). Then

\[ \left( \int_K f^{\frac{2n}{n-2}} d\text{vol}_{\hat{g}} \right)^{\frac{n-2}{2n}} \leq \left( \int_{K'} (\chi f)^{\frac{2n}{n-2}} d\text{vol}_{\hat{g}} \right)^{\frac{n-2}{2n}} \]

\[ \leq C_{K'} \left( \int_{K'} \| \nabla (\chi f) \|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ \leq C_{K'} \left( \int_{K'} \| \nabla f \|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} + C_{K'} \left( \int_{K'} \| f \nabla \chi \|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ \leq C_{K'} \left( \int_{K'} \| \nabla f \|^2 d\text{vol}_{\hat{g}} \right)^{\frac{1}{2}} \]

\[ + C_{K'} \left( \int_{K' \setminus K} \| f \|^{\frac{2n}{n-2}} d\text{vol}_{\hat{g}} \right)^{\frac{n-2}{2n}} \left( \int_{K' \setminus K} \| \nabla \chi \|^n d\text{vol}_{\hat{g}} \right)^{\frac{1}{n}}. \]
Using the first part we get the result.

**Proposition 15** Suppose \((M^n, g)\) has the asymptotic end \(\mathbb{R}^k \times X\). Then there exists a constant \(\epsilon_0 = \epsilon_0(g)\), such that if \(f\) is a smooth function with compact support and \(\|f-\|_{L^2} < \epsilon_0\), then the equation

\[
\begin{aligned}
\Delta_g u - fu &= 0 \text{ on } M \\
u &\to 1 \text{ as } r \to \infty
\end{aligned}
\] (7)

has a unique positive solution. Moreover, near infinity \(u\) has asymptotics

\[u = 1 - \frac{A}{r^{k-2}} + O(r^{-k+1})\]

where \(A = C \int_M fud\text{vol}_g\) for some \(C > 0\).

**Proof** Let \(v = 1 - u\). Then (7) becomes

\[
\begin{aligned}
\Delta_g v - fv &= -f \text{ on } M \\
v &\to 0 \text{ as } r \to \infty
\end{aligned}
\] (8)

On a compact subset \(B_r\), consider the Dirichlet problem

\[
\begin{aligned}
\Delta_g v_r - f v_r &= -f \text{ in } B_r \\
v_r &= 0 \text{ on } \partial B_r
\end{aligned}
\] (9)

By Fredholm alternative, if the homogeneous equation

\[
\begin{aligned}
\Delta_g v_r - f v_r &= 0 \text{ in } B_r \\
v_r &= 0 \text{ on } \partial B_r
\end{aligned}
\] (10)

has only zero solution, then equation (9) has a unique solution. Suppose \(\omega\) is a solution of equation (10). Multiplying \(\omega\) to both sides of (10) and integrating by parts, by the H"older inequality with \(p = \frac{n}{2}, q = \frac{n}{n-2}\) and the Sobolev inequality with \(p = 2, p^* = \frac{2n}{n-2}\), we have

\[
\int_{B_r} |\nabla \omega|^2\text{dvol}_g = -\int_{B_r} f\omega^2\text{dvol}_g \leq \int_{B_r} f-\omega^2\text{dvol}_g
\]

\[
\leq \left(\int_{B_r} f^2\text{dvol}_g\right)^{\frac{n}{2}} \left(\int_{B_r} \omega^{\frac{2n}{n-2}}\text{dvol}_g\right)^{\frac{n-2}{n}}
\]

\[
\leq c_1 \left(\int_{B_r} f^2\text{dvol}_g\right)^{\frac{2}{n}} \left(\int_{B_r} |\nabla \omega|^2\text{dvol}_g\right)
\]
Thus if $\|f-\| L^2_n < \frac{1}{c_1}$, then $\omega = 0$. Therefore, equation (9) has a unique solution $v_r$. Multiplying $v_r$ to both sides of (9), using Holder inequality and Sobolev inequality again,

$$\int_{B_r} |\nabla v_r|^2 d\text{vol}_g \leq \int_{B_r} f^2 v_r^2 d\text{vol}_g + \int_{B_r} f v_r d\text{vol}_g$$

$$\leq c_1 \left( \int_{B_r} f^\frac{n}{n-2} d\text{vol}_g \right)^\frac{2}{n} \left( \int_{B_r} |\nabla v_r|^2 d\text{vol}_g \right)^\frac{n-2}{2n} + c_1 \left( \int_{B_r} f^\frac{2n}{n-2} d\text{vol}_g \right)^\frac{n-2}{2n} \left( \int_{B_r} |\nabla v_r|^2 d\text{vol}_g \right)^\frac{n-2}{2n}$$

Then there is a constant $c_2$ depending on $(M, g, f)$ such that $\|v_r\| L^{\frac{2n}{n-2}} < c_2$ and $\|\nabla v_r\| L^2 < c_2$. The standard theory of elliptic equations concludes that $v_r$ has uniformly bounded $C^{2,\alpha}$ norm. By Arzela-Ascoli we may pass to a limit and conclude that equation (8) has a solution.

A similar argument proves that the solution of equation (7) is nonnegative everywhere. Otherwise there exists an open set $\Omega$ such that

$$\left\{ \begin{array}{l}
\Delta_g u - f u = 0 \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega.
\end{array} \right.$$ 

This contradicts with the Sobolev inequality and the choice of $\epsilon_0$ as above since $u$ is the nonzero solution. By the strong maximum principle $u$ is positive everywhere.

In the following, we use the asymptotic estimate of the Green’s function $G$ and $\nabla G$ to obtain the asymptotic behavior of the solution.

Let $\phi(r)$ be a smooth cut-off function on $M$ with $\phi(r) = 1$ on $M \setminus B_{R_1}$, $\phi(r) = 0$ on $B_{R_2}$ for $R < R_2 < R_1$. Let $dS$ be the boundary area form. Fix $(x, y) \in M \setminus B_{R_1}$ and choose $s$ sufficiently large so that $B_s$ contains $(x, y)$. Multiply $G = G(z, (x, y))$ to $\Delta_g v = f v - f$ and integrate on $B_s$ about the variable $z \in B_s$. Then

$$\int_{B_s} \phi G(f v - f) d\text{vol}_g = \int_{B_s} \phi G \Delta_g v d\text{vol}_g$$

$$= -\int_{B_s} \langle \nabla(\phi G), \nabla v \rangle d\text{vol}_g + \int_{\partial B_s} \phi G \langle v, \nabla v \rangle dS$$

$$= \int_{B_s} G \Delta_g (\phi v) d\text{vol}_g + \int_{B_s} \phi v G \Delta_g d\text{vol}_g + 2 \int_{B_s} \langle \nabla(\phi), \nabla G \rangle v d\text{vol}_g$$

$$+ \int_{\partial B_s} (\phi G) \langle v, \nabla v \rangle dS - \int_{\partial B_s} v \langle \nabla(\phi G) \rangle dS$$
Since
\[ \left| \int_{\partial B_s} (\phi G) \langle v, \nabla v \rangle dS \right| \leq G((x, y), z) \int_{\partial B_s} |\langle v, \nabla v \rangle| dS \]
\[ \leq \max_{z \in \partial B_s} G((x, y), z) \int_{\partial B_s} |\nabla v|^2 dS. \]

As \( s \to \infty \), \( \max_{z \in \partial B_s} G((x, y), z) \to 0 \). Since \( f \in C^{2,1}(M) \) and \( \int_M |\nabla v|^2 \, d\text{vol}_g < \infty \), we deduce that
\[ \int_{\partial B_s} (\phi G) \langle v, \nabla v \rangle dS \to 0, \quad \text{as} \quad s \to \infty. \]

Similarly, we have
\[ \int_{\partial B_s} v \langle \nabla (\phi G) \rangle dS \to 0, \quad \text{as} \quad s \to \infty. \]

Thus, taking \( s \to \infty \) so that \( B_s \to M \), we obtain
\[ \int_M \phi G(f v - f) \, d\text{vol}_g = v((x, y)) + \int_M G \Delta_g (\phi) v \, d\text{vol}_g + 2 \int_M \langle \nabla \phi, \nabla G \rangle v \, d\text{vol}_g. \]

Therefore
\[ \lim_{r \to \infty} r^k v((x, y)) = -2 \int_M v((x, y)) + \int_M G \Delta_g (\phi) v \, d\text{vol}_g + 2 \int_M \langle \nabla \phi, \nabla G \rangle v \, d\text{vol}_g \]
\[ + \int_M \lim_{r \to \infty} r^k G \Delta_g (1 - \phi) v \, d\text{vol}_g \]
\[ = C \left[ \int_M \phi (f v - f) \, d\text{vol}_g + \int_M \Delta_g (1 - \phi) v \, d\text{vol}_g \right] \]
\[ = C \left[ \int_M \phi (f v - f) \, d\text{vol}_g + \int_M (1 - \phi) \Delta_g v \, d\text{vol}_g \right] \]
\[ = C \int_M (f v - f) \, d\text{vol}_g \]
for \( C > 0 \).

The next lemma is standard, which relates the masses of conformally related metrics.

**Lemma 16.** For metric \( \tilde{g} = u^{4/n}((x, y)) g \) outside a large compact set on \( M^n \), and \( u = 1 + \frac{m_0}{r^{k-2}} + O(r^{-k+1}) \),

\[ m(\tilde{g}) = m(g) + \frac{4(n-1)(k-2)}{n-2} m_0 \]
Proof Let $dS_{\rho}$ be the area form for the sphere of radius $\rho$ in $\mathbb{R}^k$.

$$m(\hat{g}) = \frac{1}{\omega_k \text{Vol}(X)} \lim_{\rho \to \infty} \int_{S_\rho \times X} \sum_{i,j} \frac{(\partial_i \tilde{g}_{ij} - \partial_j \tilde{g}_{aa}) x^j}{\rho} dS_{\rho} d \text{vol}_X$$

$$= \frac{1}{\omega_k \text{Vol}(X)} \lim_{\rho \to \infty} \int_{S_\rho \times X} \sum_{i,j} \frac{\left(\partial_i (u^{4-n} g_{ij}) - \partial_j (u^{4-n} g_{aa})\right) x^j}{\rho} dS_{\rho} d \text{vol}_X$$

$$= \frac{1}{\omega_k \text{Vol}(X)} \lim_{\rho \to \infty} \int_{S_\rho \times X} \sum_{i,j} \frac{u^{4-n} \left(\partial_i g_{ij} - \partial_j g_{aa}\right) x^j}{\rho} dS_{\rho} d \text{vol}_X$$

$$+ \frac{1}{\omega_k \text{Vol}(X)} \lim_{\rho \to \infty} \int_{S_\rho \times X} \sum_{i,j} \frac{4}{n-2} u^{4-n-1} \left(\partial_i u g_{ij} - \partial_j u g_{aa}\right) x^j}{\rho} dS_{\rho} d \text{vol}_X$$

$$= \frac{1}{\omega_k \text{Vol}(X)} \lim_{\rho \to \infty} \int_{S_\rho \times X} \left(1 + \frac{m_0}{2 \rho^{k-2}}\right)$$

$$+ O(\rho^{-k+1}) \frac{u^{4-n}}{n-2} \sum_{i,j} \left(\partial_i g_{ij} - \partial_j g_{aa}\right) x^j}{\rho} dS_{\rho} d \text{vol}_X$$

$$+ \frac{1}{\omega_k \text{Vol}(X)} \lim_{\rho \to \infty} \int_{S_\rho \times X} \sum_{i,j} \frac{4}{n-2} u^{4-n-1} \left(1 + ((2 - k) m_0 \rho^{1-k}) \frac{x^j}{\rho}\right)$$

$$+ O(\rho^{-k}) \frac{g_{ij}}{x^j}{\rho} dS_{\rho} d \text{vol}_X$$

$$= m(g) + \frac{4(n-1)(k-2)}{n-2} m_0.$$ 

We are now ready to start the proof of compactification result.

Proof of Step 1 Following the proof of Proposition 4.11 in [2] and Proposition 3.2 in [17], write the metric $g$ as

$$g = (1 + \frac{m_1}{2 \rho^{k-2}}) \hat{g}^{4-n} + \tilde{g}$$

outside a large compact set with $m_1 = \frac{n-2}{4(n-1)(k-2)} m$ and

$$\lim_{\rho \to \infty} \int_{S_\rho \times X} \sum_{i,j} \left(\partial_i \tilde{g}_{ij} - \partial_j \tilde{g}_{aa}\right) x^j}{\rho} dS_{\rho} d \text{vol}_X = 0 \quad (11)$$

Let $\phi(r)$ be a cut-off function, $\phi(r) = 1$ for $r \leq 2$ and $\phi(r) = 0$ for $r \geq 3$ and $0 \leq \phi(r) \leq 1$. Define the metric $g^\sigma = (1 + \frac{m_1}{2 \rho^{k-2}}) \hat{g}^{4-n} + \phi(\frac{r}{\sigma}) \tilde{g}$. Then $Sc_{g^\sigma}$ is a
smooth function with compact support in $B_{3\sigma} \times X$. More precisely,

$$Sc_{g^\sigma} = \begin{cases} Sc_g, & \text{if } r \leq 2\sigma \\ O(\sigma^{-\tau-2}), & \text{if } 2\sigma \leq r \leq 3\sigma \\ 0, & \text{otherwise.} \end{cases}$$

Solve

$$\Delta_{g^\sigma} u - \frac{n-2}{4(n-1)} \varphi Sc_{g^\sigma} u = 0$$

for $u \to 1$ as $x \to \infty$

where $\varphi(r) = 1$ for $2\sigma \leq r \leq 3\sigma$ and $\varphi(r) = 0$ for $0 \leq r \leq \sigma$ and $4\sigma \leq r < \infty$, $0 \leq \varphi(r) \leq 1$. Choose $\sigma$ sufficiency large to make $(\int_M (|\varphi Sc_{g^\sigma}| - |n/2 d \text{vol}_{g^\sigma}|)^{2/n} = O(\sigma^{-\tau-2+2k/n})$ small. By Proposition 15,

$$u = 1 - \frac{A_{\sigma}}{r^{k-2}} + O(r^{1-k})$$

and

$$A_{\sigma} = C \int_M \varphi Sc_{g^\sigma} u d \text{vol}_{g^\sigma}.$$ 

Let $\tilde{g} = u^{4n-2} g^\sigma$. Then

$$Sc_{\tilde{g}} = \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} (-\Delta_{g^\sigma} u + \frac{n-2}{4(n-1)} Sc_{g^\sigma} u)$$

$$= \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} (-\Delta_{g^\sigma} u + \frac{n-2}{4(n-1)} \varphi Sc_{g^\sigma} u + \frac{n-2}{4(n-1)} (1 - \varphi) Sc_{g^\sigma} u)$$

$$\geq u^{-\frac{4}{n-2}} (1 - \varphi) Sc_{g^\sigma} \geq 0.$$ 

Moreover,

$$m(\tilde{g}) = -\frac{4(n-1)(k-2)}{n-2} A_{\sigma} + m(g^\sigma) = -\frac{4(n-1)(k-2)}{n-2} A_{\sigma} + m$$

and thus

$$|m(\tilde{g}) - m(g)| = \frac{4(n-1)(k-2)}{n-2} |A_{\sigma}|.$$

Thus if $|A_{\sigma}| < \epsilon$ can be made arbitrarily small for sufficiency large $\sigma$, then $|m(\tilde{g}) - m(g)| < \epsilon$. Since $m(g) < 0$, we can make $m(\tilde{g}) < 0$ by taking $\sigma$ sufficiency large.
Now we prove $|A_\sigma| < \epsilon$ for sufficiency large $\sigma$. Let $v = 1 - u$, then

$$|A_\sigma| = C \left| \int_M \varphi S_{g^\sigma} u d\text{vol}_{g^\sigma} \right|$$

$$\leq C \left| \int_M \varphi S_{g^\sigma} d\text{vol}_{g^\sigma} \right| + C \left| \int_M \varphi S_{g^\sigma} u d\text{vol}_{g^\sigma} \right|$$

$$\leq C \left| \int_M \varphi S_{g^\sigma} d\text{vol}_{g^\sigma} \right| + C\sigma^{-k+2} \int_{\{\sigma \leq r \leq 2\sigma\} \times X} |S_{g_\sigma}| d\text{vol}_{g}$$

$$+ C \left( \int_{\{2\sigma \leq r \leq 3\sigma\} \times X} |S_{g_\sigma}| |v| d\text{vol}_{g^\sigma} \right)$$

$$\leq C \left| \int_M \varphi S_{g^\sigma} d\text{vol}_{g^\sigma} \right| + O(\sigma^{-\tau}) + C\sigma^{-k+2} \int_{\{\sigma \leq r \leq 2\sigma\} \times X} |S_{g_\sigma}| d\text{vol}_{g}$$

$$\leq C \left| \int_M \varphi S_{g^\sigma} d\text{vol}_{g^\sigma} \right| + O(\sigma^{-\tau}) + C\sigma^{-k+2}$$

by $S_{g_\sigma} \in L^1(M)$.

We now estimate the first term, where the integrand is nonzero only when $\sigma \leq r \leq 4\sigma$. In the asymptotic coordinates,

$$S_{g^\sigma} = |g^\sigma|^{-\frac{1}{2}} \partial_a (|g^\sigma|^\frac{1}{2} (g^\sigma)^{ab} (\Gamma_b - \frac{1}{2} \partial_b (\log |g^\sigma|))$$

$$- \frac{1}{2} (g^\sigma)^{ab} \Gamma_a \partial_b (\log |g^\sigma|) + (g^\sigma)^{ab} (g^\sigma)^{cd} (g^\sigma)^{ef} \Gamma_{ace} \Gamma_{bdf},$$

where $\Gamma_{abc} = \frac{1}{2} (g^\sigma_{bc,a} + g^\sigma_{ac,b} - g^\sigma_{ab,c})$ and $\Gamma_c = (g^\sigma)^{ab} \Gamma_{abc}$. If

$$\left| \int_{\{\sigma \leq r \leq 4\sigma\} \times X} \varphi S_{g^\sigma} d\text{vol}_{g^\sigma} \right| = \int_{\{\sigma \leq r \leq 4\sigma\} \times X} \varphi S_{g^\sigma} d\text{vol}_{g^\sigma},$$

then

$$0 \leq \int_{\{\sigma \leq r \leq 2\sigma\} \times X} (1 - \varphi) S_{g^\sigma} d\text{vol}_{g^\sigma} = \int_{\{\sigma \leq r \leq 4\sigma\} \times X} (1 - \varphi) S_{g^\sigma} d\text{vol}_{g^\sigma}.$$

If

$$\left| \int_{\{\sigma \leq r \leq 4\sigma\} \times X} \varphi S_{g^\sigma} d\text{vol}_{g^\sigma} \right| = - \int_{\{\sigma \leq r \leq 4\sigma\} \times X} \varphi S_{g^\sigma} d\text{vol}_{g^\sigma},$$
Thus, by
\[ 0 \geq \int_{[\sigma \leq r \leq 4\sigma] \times X} \varphi Scg^\sigma d \text{vol}_{g^\sigma} \]
\[ = \int_{[\sigma \leq r \leq 2\sigma] \times X} (1 - \varphi)Scg^\sigma d \text{vol}_{g^\sigma} + \int_{[2\sigma \leq r \leq 3\sigma] \times X} Scg^\sigma d \text{vol}_{g^\sigma} \]
\[ = \int_{[\sigma \leq r \leq 2\sigma] \times X} (1 - \varphi)Scg d \text{vol}_{g^\sigma} + \int_{[2\sigma \leq r \leq 3\sigma] \times X} Scg^\sigma d \text{vol}_{g^\sigma} \]
\[ \geq \int_{[2\sigma \leq r \leq 3\sigma] \times X} Scg^\sigma d \text{vol}_{g^\sigma} . \]

Therefore
\[ \left| \int_M \varphi Scg^\sigma d \text{vol}_{g^\sigma} \right| \leq \max \left\{ \left| \int_{[\sigma \leq r \leq 4\sigma] \times X} Scg^\sigma d \text{vol}_{g^\sigma} \right|, \left| \int_{[2\sigma \leq r \leq 3\sigma] \times X} Scg^\sigma d \text{vol}_{g^\sigma} \right| \right\} . \]

Thus, by \(|g^\sigma|^2 (g^\sigma)^{ab} (\Gamma_b - \frac{1}{2} \partial_b (\log |g^\sigma|)) = (g^\sigma)_{ab,b} - (g^\sigma)_{bb,a} + O(r^{-2\tau - 1})\), there is
\[
\int_{[\sigma \leq r \leq 4\sigma] \times X} Scg^\sigma d \text{vol}_{g^\sigma} \\
= \int_{[r=4\sigma] \times X} \left| g^\sigma \right|^2 \left( (g^\sigma)^{ab} (\Gamma_b - \frac{1}{2} \partial_b (\log |g^\sigma|)) \nu^a dS_{4\sigma} d \text{vol}_{g^\sigma} \right) \\
- \int_{[r=\sigma] \times X} \left| g^\sigma \right|^2 \left( (g^\sigma)^{ab} (\Gamma_b - \frac{1}{2} \partial_b (\log |g^\sigma|)) \nu^a dS_{\sigma} d \text{vol}_{g^\sigma} + O(\sigma^{-2\tau - 2 + k}) \right) \\
= \int_{[r=4\sigma] \times X} \left( (g^\sigma)_{ab,b} - (g^\sigma)_{bb,a} \right) \nu^a dS_{4\sigma} d \text{vol}_{g^\sigma} \\
- \int_{[r=\sigma] \times X} \left( (g^\sigma)_{ab,b} - (g^\sigma)_{bb,a} \right) \nu^a dS_{\sigma} d \text{vol}_{g^\sigma} + O(\sigma^{-2\tau - 2 + k}) \\
= \int_{[r=4\sigma] \times X} (g_{ab,b} - g_{bb,a}) \nu^a dS_{4\sigma} d \text{vol}_{g^\sigma} \\
- \int_{[r=\sigma] \times X} (g_{ab,b} - g_{bb,a}) \nu^a dS_{\sigma} d \text{vol}_{g^\sigma} \\
+ \int_{S_{\sigma}} \sum_{i,j} \left( \partial_i \tilde{g}_{ij} - \partial_j \tilde{g}_{aa} \right) \nu^a dS_{\sigma} d \text{vol}_{g^\sigma} + O(\sigma^{-2\tau - 2 + k})
\]

where \( \nu = (\nu^a) \) is the outer normal vector. The difference of the first two terms is arbitrarily small if \( \sigma \) is sufficiently large by the existence of mass [1]. The third term is small for large \( \sigma \) by (11). The other term can be handled similarly. Therefore, we have \(|A_{\sigma}| < \epsilon \) for sufficiency large \( \sigma \).

Thus, \( \tilde{g} = \tilde{u} \frac{4}{d} (g_{2\sigma} + g_X) \) with \( \tilde{u} = 1 + \frac{\tilde{m}}{r^{-\tau}} + O(r^{-1-k})(\tilde{m} < 0) \) and \( Sc\tilde{g} \geq 0 \). Moreover, \( Sc\tilde{g} = 0 \) by (12) outside a large compact set \( K \), for example \( K = B_{3\sigma} \).
Proof of Step 2  From Step 1, we know that there is a metric $g = u^{4/(n-2)}(g_{\mathbb{R}^k} + g_X)$ with $u = 1 + \frac{m}{r^{k-2}} + O(r^{1-k})(m < 0)$ and $Sc_g = 0$ outside a large compact set and $Sc_g \geq 0$ on $M$. Since

$$Sc_g = \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \left( -\Delta g_{\mathbb{R}^k} + g_X u + \frac{n-2}{4(n-1)} Sc_{g_{\mathbb{R}^k} + g_X} u \right)$$

therefore, $\Delta g_{\mathbb{R}^k} + g_X u = 0$ outside a large compact set. By $u = 1 + \frac{m}{r^{k-2}} + O(r^{1-k})(m < 0)$, we can take $s_1$ large enough such that $u < 1$ on $\{s_1\} \times X$ and $s_1 > 3\sigma$. Let

$$\epsilon = 1 - \sup_{\{s_1\} \times X} u(x).$$

Then $u > 1 - \frac{\epsilon}{4}$ in $r \geq s_2$ for sufficiently large $s_2 > s_1$. Take a cutoff function $\zeta : [0, +\infty) \to [0, 1 - \frac{\epsilon}{2}]$ such that $\zeta(t) = t$ for $t \leq 1 - \frac{3\epsilon}{4}$ and $\zeta(t) = 1 - \frac{\epsilon}{2}$ for $t \geq 1 - \frac{\epsilon}{4}$ with $\zeta' \geq 0$ and $\zeta'' \leq 0$ as well as $\zeta'' < 0$ in $(1 - \frac{3\epsilon}{4}, 1 - \frac{\epsilon}{4})$. Let

$$v = \begin{cases} \zeta \circ u, & r \geq s_1; \\ u, & r \leq s_1. \end{cases}$$

$v$ is a smooth function defined on entire $M$ and $v = u$ around $\{r = s_1\} \times X$. We also have

$$\Delta g_{\mathbb{R}^k} + g_X v = \zeta'' \left( \nabla^2 g_{\mathbb{R}^k} + g_X u \right)^2 + \zeta' \Delta g_{\mathbb{R}^k} + g_X u \leq 0 \quad \text{in} \quad \{r \geq s_1\} \times X$$

and $\Delta g_{\mathbb{R}^k} + g_X v < 0$ at some point in $\{s_1 < s < s_2\} \times X$. Define

$$\tilde{g} = \left( \frac{v}{u} \right)^{4/(n-2)} g.$$

Then $Sc_{\tilde{g}} \geq 0$ with strict inequality at some point on $M$ and $\tilde{g}$ is the metric $g_{\mathbb{R}^k} + g_X$ near infinity.

Then we can cut $M$ off outside a large compact set $K$ such that $\partial K = (\partial[0, 1]^k) \times X$ and glue the opposite faces of $[0, 1]^k$, such that it becomes $(M''_1 \times X(T^k \times X), \tilde{g})$ with nonnegative scalar curvature and strictly positive scalar curvature at some point. It can then be deformed to a metric with positive scalar curvature.

5 Rigidity

This section is devoted to the proof of the rigidity part of the Positive Mass Theorem, Theorem 1. Thus let $(M, g)$ be a manifold asymptotic to $\mathbb{R}^k \times X$ whose scalar curvature is nonnegative but its mass is zero. We first show that $Sc_g = 0$, and then $\text{Ric}_g = 0$. 

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If the scalar curvature is not identically zero. Then $Sc_{g}(p) > 0$ for some $p \in M^n$. Choose compact sets $K_1, K_2$ such that $K_1$ is contained in the interior of $K_2$, $p \in K_1$, and $Sc_{g} > 0$ on $K_2$. Let $\varphi$ be a nonnegative smooth function which is 1 on $K_1$ and 0 on $M \setminus K_2$. Solve the equation

$$\begin{cases}
\Delta_g u - \frac{n-2}{4(n-1)} \varphi Sc_{g} u = 0 & \text{on } M^n, \\
u \rightarrow 1 & \text{as } r \rightarrow \infty.
\end{cases}$$

Then Proposition 15 gives a unique positive solution $u$. And the metric $\hat{g} = u^{4 \over n-2}g$ has $Sc_{\hat{g}} \geq 0$ but

$$m(\hat{g}) = \frac{4(n-1)(k-2)}{n-2} m_0 + m(g) = \frac{4(n-1)(k-2)}{n-2} m_0,$$

where $u = 1 + \frac{m_0}{r^{n-2}} + O(r^{1-k})$, and $m_0 = -C \int_M \varphi Sc_{g} u \, dvol_g < 0$. That is, $m(\hat{g}) < 0$, which is a contradiction.

Next we prove that the Ricci curvature of $M^n$ is identically zero. Let $h$ be a compactly supported $(0, 2)$ tensor and consider the deformation $g_t = g + th$. For $t$ sufficiently small, since the scalar curvature depends smoothly on $t$, we have that $\|Sc(g_t)\|_{L^2_g}$ will be small. Thus, by Proposition 15 we can again solve the equation

$$\begin{cases}
\Delta_g u_t - \frac{n-2}{4(n-1)} Sc_{g_t} u_t = 0 & \text{on } M^n, \\
u_t \rightarrow 1 & \text{as } r \rightarrow \infty
\end{cases}$$

with a unique positive solution $u_t$. Define $\hat{g}_t = u_t^{4 \over n-2}g_t$. Then $Sc_{\hat{g}_t} = 0$. Let $m(t)$ denote the mass of the metric $\hat{g}_t$. Using the asymptotic formula again we see that

$$m(t) = -C \int_M Sc_{g_t} u_t dvol_{g_t},$$

therefore $m(t)$ is $C^1$ differentiable about $t$. Taking its first derivative at $t = 0$, and use the facts that $u_0 \equiv 1, Sc_{g_0} = 0$, we have

$$\frac{d}{dt} \bigg|_{t=0} m(t) = -C \int_M \hat{Sc}(0) dvol_g$$

$$= -C \int_M \langle \nabla_i \nabla_j h_{ij} - \Delta_{\hat{g}} (g^{ij} h_{ij}) - (\text{Ric}_g, h) \rangle dvol_g$$

$$= C \int_M (\text{Ric}_g, h) dvol_g$$
If $\text{Ric}_g$ is not identically zero, then taking $h = \eta \text{Ric}_g$, with $\eta$ a cutoff function yields that

$$\frac{d}{dt} \bigg|_{t=0} m(t) < 0.$$  

This means that for some small $t$, $m(t) < 0$, again a contradiction. Then $(M^n, g)$ is Ricci flat. Since $\text{Ric}_g = \text{Ric}_g^{g_{\epsilon k}} + \text{Ric}_{g_X} + O(r^{-\tau})$ outside a compact set, $\text{Ric}_{g_X} \equiv 0$, i.e. $(X, g_X)$ is Ricci flat.

As in the proof of lemma 6 in [10] and Proposition 4.12 in [2], we can assume that $\tau \leq k - 2$. We first find smooth functions on $M$, $y^i$, $i = 1, \cdots, k$, such that $\Delta_g y^i = 0$ and $y^1, \cdots, y^k$ form an asymptotic coordinate system for the $\mathbb{R}^k$ factor. Let $x^i \in C^\infty(M\setminus K), i = 1, \cdots, k$, be an asymptotic coordinate system for the $\mathbb{R}^k$ factor. Let $x$ be a cut-off function vanishing on $B_{r_1}$ and is identically 1 on $M \setminus B_{r_2}$ for some $r_1 < r_2$. Since $\Delta_g (x^i) = O(r^{1 - \tau}) \in L_{\delta - 2}^2(M)$ for $\delta > 1 + \frac{k}{2} - \tau$, there exists $u^i \in H^2_\delta(M)$ such that $\Delta_g u^i = \Delta_g (x^i)$ by Corollary 2 in [10]. Set $y^i = x^i - u^i$. Then $\Delta_g y^i = 0$ for $1 \leq i \leq k$ on $M$. By the Moser iteration and Schauder estimate,

$$|u^i| + r|\partial u^i| + r^2|\partial^2 u^i| = O(r^{1 - \delta}), \quad \delta = 1 + \frac{k}{2} - \tau + \epsilon_1.$$  

Fix $\epsilon_1 > 0$ small enough such that if $\delta = 1 + \frac{k}{2} - \tau + \epsilon_1$, then $\delta' = \tau - \epsilon_1 > \frac{k - 2}{2}$. With a similar analysis on the derivative of $\Delta_g u^i = \Delta_g (x^i)$, one also deduces

$$r^3|\partial^3 u^i| = O(r^{1 - \delta'}).$$

The above computation is in the basis $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^k}, \{f_\alpha\}\}$.

Since $y^i = x^i - u^i$, $\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^k}, \{f_\alpha\}\}$ form a basis outside a large compact set by the above estimates. With respect to the new coordinate system $y^i = x^i - u^i$ and the basis $\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^k}, \{f_\alpha\}\}$ the metric $g$ can be written as $g = g_0 + \omega$ where $g_0 = dy^2 + g_X$ and $\omega$ satisfies

$$|\omega| + r|\partial \omega| + r^2|\partial^2 \omega| = O(r^{-\tau}), \quad \tau > \frac{k - 2}{2}.$$  

Now we compute in this new coordinate system and the derivative will be taken with respect to $y^i = x^i - u^i$ and $\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^k}, \{f_\alpha\}\}$. Let $\partial_i = \frac{\partial}{\partial y^i}, g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$ and $r = |y|$. From the Bochner formula and $\Delta_g y^i = 0$, we have

$$\Delta_g \left(\frac{1}{2}|dy^i|^2_g\right) = |\nabla_g dy^i|^2_g,$$

$$\Delta_g \left(\frac{1}{2}g^{ij}\right) = g(\nabla_g dy^i, \nabla_g dy^j).$$
Then $\Delta_g (g^{ij} - \delta_{ij}) = O(r^{-2\delta' - 2})$. From the above discussion we conclude that

$$g^{ij} - \delta_{ij} \in L^2_\mu \text{ for any } \mu > \frac{k}{2} - \delta'$$

$$\Delta_g (g^{ij} - \delta_{ij}) \in L^2_{\mu' - 2} \text{ for any } \mu' > \frac{k}{2} - 2\delta'.$$

Since $\delta' \leq \tau \leq k - 2$ and $\delta' > \frac{k - 2}{2}$, therefore we can choose $1 - \frac{k}{2} < \mu' < 2 - \frac{k}{2} < \mu \leq \frac{k}{2}$. By Proposition 12,

$$g^{ij} = \delta_{ij} - c_{ij} r^{2-k} + v^{ij}$$

where $c_{ij}$ are constants with $c_{ij} = c_{ji}$ and $v^{ij} \in H^2_\mu$. After repeating the argument for $u^i$, we see that $v^{ij}$ are higher-order error terms satisfying, for some small $\epsilon_2 > 0$,

$$|v^{ij}| + r|\partial v^{ij}| + r^2|\partial^2 v^{ij}| = O(r^{2-k-\epsilon_2}).$$

After a possible orthogonal transformation of $\{y^1, \ldots, y^k\}$, we can assume $c_{ij} = c_i \delta_{ij}$ without loss of generality. Finally, we have

$$g_{ij} = \delta_{ij} + c_i \delta_{ij} r^{2-k} + \omega_{ij}, \quad g_{i\alpha} = \omega_{i\alpha}, \quad g_{\alpha\alpha} = 1 + \omega_{\alpha\alpha},$$

such that

$$|\omega^{ij}| + r|\partial \omega^{ij}| + r^2|\partial^2 \omega^{ij}| = O(r^{2-k-\epsilon_2}),$$

and

$$|\omega^{\alpha\alpha}| + r|\partial \omega^{\alpha\alpha}| + r^2|\partial^2 \omega^{\alpha\alpha}| = O(r^{-\delta'}), \quad \delta' > \frac{k - 2}{2}.$$

$$|\omega^{i\alpha}| + r|\partial \omega^{i\alpha}| + r^2|\partial^2 \omega^{i\alpha}| = O(r^{-\delta'}), \quad \delta' > \frac{k - 2}{2}.$$

Note that this refined asymptotic information about the metric $g$ uses only the Ricci curvature condition $\text{Ric} = 0$, and not the mass condition $m(g) = 0$. In fact we are going to use this refined asymptotic to compute the mass $m(g)$. The mass condition $m(g) = 0$ is only used at the end to conclude that the one-forms $dy^i$’s are parallel.

Note also that, as pointed out in [2, Proposition 4.12], the refined asymptotics actually hold under some Ricci decay condition.

By the relation $y^i = \chi x^i - u^i$ and the estimate of $u^i$, we see that the mass $m(g)$ calculated with respect to $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^k}, \{f_\alpha\}\}$ is the same as that with respect to
{$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}, \{f_\alpha\}$}. Since $\Delta_g y^i = 0$, a straightforward computation yields

$$(k - 2) \left( c_i - \frac{1}{2} \left( \sum_{j=1}^k c_j \right) \right) \frac{y^i}{|y|^k} + \sum_\alpha \partial_\alpha g_{i\alpha} + \frac{1}{2} \sum_\alpha \partial_\alpha g_{\alpha\alpha} + O(r^{-\mu''}) = 0 \quad (14)$$

where $\mu'' = \min\{k - 1 + \epsilon_2, 2\delta' + 1\} > k - 1$. Integrating (14) and summing over $i$ gives

$$\lim_{\rho \to +\infty} \int_{S^k \times X} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{y^i}{\rho} dS_{\rho} d\operatorname{vol}_g = \lim_{\rho \to +\infty} \int_{S^k \times X} \frac{(k - 2)(k - 1)}{k} \left( \sum_{i=1}^k c_i \right) \rho^{1-k} dS_{\rho} d\operatorname{vol}_g = \frac{(k - 2)(k - 1)}{k} \omega_k \left( \sum_{i=1}^k c_i \right) \operatorname{Vol}(X).$$

And a direct computation yields

$$\lim_{\rho \to +\infty} \int_{S^k \times X} (\partial_j g_{ij} - \partial_i g_{aa}) \frac{y^i}{\rho} dS_{\rho} d\operatorname{vol}_g = \frac{(k - 2)(k - 1)}{k} \omega_k \left( \sum_{i=1}^k c_i \right) \operatorname{Vol}(X).$$

Thus

$$m(g) = \lim_{\rho \to +\infty} \frac{1}{\omega_k \operatorname{Vol}(X)} \int_{S^k \times X} (\partial_j g_{ij} - \partial_i g_{aa}) \frac{y^i}{\rho} dS_{\rho} d\operatorname{vol}_g = \frac{k - 2}{k} \left( \sum_{i=1}^k c_i \right). \quad (15)$$

On the other hand, since $\Delta_g dy^i = (dd^* + d^* d)dy^i = d\Delta_g y^i = 0$, by the Weitzenbock formula for 1-form $\omega$, i.e.

$$\Delta_g \omega = \nabla^* \nabla \omega + \operatorname{Ric}(\omega^\#, \cdot),$$

one has,

$$0 = \sum_i \int_M \langle \nabla^* \nabla dy^i, dy^i \rangle d\operatorname{vol}_g$$

$$= \sum_i \int_M \langle \nabla dy^i, \nabla dy^i \rangle d\operatorname{vol}_g - \sum_i \lim_{\rho \to \infty} \int_{S^{k-1} \times X} \langle \nabla_d dy^i, dy^i \rangle \nu^a dS_{\rho} d\operatorname{vol}_g$$
where $\nu = (\nu^1, \cdots, \nu^n)$ is the unit outer normal. On the other hand,

$$\nabla_a dy^i = -\Gamma^{i}_{ak} dy^k - \Gamma^{i}_{aa} f^a.$$ 

Hence,

$$\lim_{\rho \to \infty} \sum_i \int_{S_{\rho}^{k-1} \times X} (\nabla_a dy^i, dy^i) \nu^a dS_{\rho} d \text{vol}_{g_X}$$

$$= \lim_{\rho \to \infty} \int_{S_{\rho}^{k-1} \times X} (\partial_j g_{ij} - \partial_i g_{aa} + \frac{1}{2} \partial_i g_{aa}) \frac{y^i}{\rho} dS_{\rho} d \text{vol}_{g_X}$$

$$= \frac{1}{2} (k - 2) \omega_k \left( \sum_{i=1}^{k} c_i \right) \text{Vol}(X)$$

$$= \frac{k}{2} \omega_k \text{Vol}(X) m(g)$$

by (15). Combining the discussion above we arrive at the equation

$$\sum_i \int_M \langle \nabla_d y^i, \nabla d y^i \rangle d \text{vol}_g = \frac{k}{2} \omega_k \text{Vol}(X) m(g).$$

In particular $m(g) = 0$ implies that $dy^i$ is parallel.

Therefore $dy^i$ is parallel for $1 \leq i \leq k$, and as they are approaching orthonormal at infinity, they are exactly orthonormal on $M$. Consider the map $F : M^n \to \mathbb{R}^k$, $F(p) = (y^1(p), \cdots, y^k(p))$. Let $\Phi^i_t$ be the flow of $\nabla y^i$, which is a complete vector field since it has norm one. Since $\nabla y^i$ is parallel, it is a Killing vector, i.e. the flow $\Phi^i_t$ action on $M^n$ is isometric. Therefore, for $q \in \mathbb{R}^k$, $\Phi^i_q : F^{-1}(q) \to F^{-1}(q + te_i)$ is isometric and so is $\Phi^1_t \circ \cdots \circ \Phi^k_t : F^{-1}(q) \to F^{-1}(q + y)$. Since $y^i$, $1 \leq i \leq k$, form an asymptotic coordinate system for the $\mathbb{R}^k$ factor and $g$ is asymptotic to the product metric, letting $y \to \infty$, we find an isometry $F^{-1}(q) \cong X$. Thus $M^n$ is a fiber bundle over $\mathbb{R}^k$ with fibers isometric to $X$. Since $\mathbb{R}^k$ is contractible, the bundle must be trivial. On the other hand, the de Rham Decomposition Theorem says that this is a local metric product. Therefore $(M^n, g) \cong (\mathbb{R}^k \times X, g_{\mathbb{R}^k} + g_X)$.

**Remark 17** From the discussion above, for the harmonic asymptotic coordinates $y^i$, $1 \leq i \leq k$, for the Euclidean factor,

$$\sum_i \int_M \langle \nabla_d y^i, \nabla_d y^i \rangle d \text{vol}_g = \lim_{\rho \to \infty} \int_{S_{\rho}^{k-1} \times X} (\partial_j g_{ij} - \partial_i g_{aa} + \frac{1}{2} \partial_i g_{aa}) \frac{y^i}{\rho} dS_{\rho} d \text{vol}_{g_X}$$

$$= \omega_k \text{Vol}(X) m^{GB}(g)$$

which is exactly the motivation for Minerbe to introduce the Gauss-Bonnet mass [10].

From the proof of the rigidity part, as a generalization of Proposition 4.12 in [2], we have

\[ \mathcal{S} \] Springer
Proposition 18  For a complete noncompact smooth manifold \((M^n, g)\) with asymptotic end \((\mathbb{R}^k \times X^{n-k}, \hat{g} = g_{\mathbb{R}^k} + g_{X^n})\), if \(|\text{Ric}_\hat{g}| + r|\nabla \text{Ric}_\hat{g}| = O(r^{-k-\epsilon})\) for some \(\epsilon > 0\) then

\[
m^{GB}(g) = \frac{k}{2} m(g).
\]

It is very interesting that this notion of the Gauss-Bonnet mass plays a crucial role in the rigidity part of the Positive Mass Theorem. On the other hand, it should be emphasized that, in general, no such relation is known between the two mass quantities.

We end our discussion with an example. The so called Euclidean Schwarzschild space is \((\mathbb{R}^2 \times S^{n-2}, g)\), where

\[
g = \frac{dr^2}{1 - \frac{m}{r^{n-3}}} + r^2 g_{S^{n-2}} + (1 - \frac{m}{r^{n-3}})dt^2
\]

for \(r \in (0, \infty)\) and \(t \in S^1_l\), a circle of radius \(l = \frac{2m^{\frac{1}{n-3}}}{n-3}\). Here \(m \geq 0\) is a parameter. The Euclidean Schwarzschild space has asymptotic end \(\mathbb{R}^{n-1} \times S^1_l\) and \(\text{Ric}_g = 0\). Moreover, one has \(m(g) = m\) and \(m^{GB}(g) = \frac{n-1}{2} m\).

This should be contrasted with the case of asymptotic Euclidean spaces where the ADM mass must vanish under the even weaker Ricci decay condition \(\text{Ric} = O(r^{-n-\epsilon})\) for some \(\epsilon > 0\), which follows from a formula of the ADM mass via the Ricci tensor; see, for example, [9].

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