Critical Behavior of Light

Rafi Weill, Amir Rosen, Ariel Gordon, Omri Gat and Baruch Fischer
Department of Electrical Engineering, Technion, Haifa 32000, Israel
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Light is shown to exhibit critical and tricritical behavior in passive mode-locked lasers with externally injected pulses. It is a first and unique example of critical phenomena in a one-dimensional many body light-mode system. The phase diagrams consist of regimes with continuous wave, driven para-pulses, spontaneous pulses via mode condensation, and heterogeneous pulses, separated by phase transition lines which terminate with critical or tricritical points. Enhanced nongaussian fluctuations and collective dynamics are observed at the critical and tricritical points, showing a mode system analog of the critical opalescence phenomenon. The critical exponents are calculated and shown to comply with the mean field theory, which is rigorous in the light system.

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Statistical light-mode dynamics (SLD) [1, 2] that is based on statistical mechanics, provides a powerful approach for the study of complex nonlinear light systems, while serving as a new statistical mechanics paradigm. It was developed to treat and solve long standing questions in laser physics. A prime example is the laser pulsation threshold, a phenomenon in passive mode locking that attracted much attention [3]. Passive mode-locking occurs when a saturable absorber element is placed in the cavity, driving the laser to generate extremely short light pulses. The transition mechanism from a continuous wave (cw) to pulsation of such lasers was shown via SLD [1, 2] to be an intrinsic property of the many interacting mode system, ruled by the balance between the nonlinear interaction induced by the saturable absorber and the randomizing effect of noise. The theory, verified by experimental study, demonstrated [4] the existence of first order phase transitions between disordered (cw) and ordered (locked) mode phases, as the noise (“temperature”) or the laser power were varied. Thus noise alone stabilizes the cw state, showing a noise induced phase transition [5].

In the present Letter we report on findings of critical and tricritical phenomena in the many light-mode system. It is a first example of such behavior with light, that also provides a physical realization of a strict one-dimensional many body system. For criticality to appear in the laser mode system we add to it an external driving field which is a counterpart of the external magnetic field in magnets and the pressure in gas-liquid-solid systems [6]. It is achieved by injecting the laser with pulses from an external source, which in the simple case matches the repetition rate of the laser. When the injection is weak, the ordering phase transition persists, shifted to higher “temperature”, with a “para-pulses” phase (pulses induced and driven by the external injection). However, the phase transition line terminates in a critical point, where the distinction between para-pulses and spontaneous pulses disappears, similarly to the vapor-liquid critical point. Thus, by increasing the external injection, it is possible to obtain mode locking smoothly from cw. Near the critical point the system exhibits the familiar critical phenomena, including divergence of response coefficients, characterized by universal critical exponents, and nongaussian critical fluctuations, enhanced by a factor of \( N^{1/3} \), where \( N \) is the number of active modes, compared to normal fluctuations. The latter could mean a two orders of magnitude fluctuations enhancement in practical systems. It is a light mode system analog of the critical opalescence phenomenon.

When the injection repetition rate is higher than that of the laser, the high- and low-temperature phases are characterized by equal and unequal pulse powers, respectively, which cannot be smoothly connected. However, beyond a threshold injection level the transition becomes continuous rather than first order. The two phase transition lines meet at a tricritical point, often found in systems with interaction competing with external driving [7], around which tricritical behavior is observed, with its distinct set of universal exponents, and the tricritical fluctuations are enhanced by a factor of \( N^{1/3} \).

The inclusion of external injection has other important consequences. For example, as in the vapor-liquid case, field induced “condensation”, that is in our case mode ordering and pulsation, can be sustained to some extent when the field is removed, only below the critical point. Another practical aspect of external injection is the possibility to overcome basic deficiencies in passively mode locked lasers, their low repetition rate and timing jitter [8]. Our theory provides a basic and quantitative understanding of the way that an external field locks the laser pulsation, and the threshold power needed (or maximum noise allowed) for various operation scenarios to happen.

We perform our theoretical study in the framework of the coarse-grained model of SLD, representing the cavity electric field envelope \( \psi \) by a single variable in an interval whose length is of the order of the pulse width. The derivation of the model from the passive mode locking master equation [10] has been discussed before [1, 2, 4], and it shows good quantitative agreement with theoreti-
The exponent in the integrand in Eq. (7) is proportional to the large parameter \( N \), and therefore the integration is concentrated near the global minimum of the integrand. It is straightforward to verify that the minimum is always obtained when \( \psi = 0 \), which then implies that 

\[
F = N \min f_n.
\]

Moments of the pulse strength can be obtained in the standard manner by taking derivatives of the free energy. Alternatively we may obtain directly the probability distribution function of the pulse amplitudes by integrating out the \( N - n \) unforced variables in Eq. (8) getting

\[
P(\psi_1, \ldots, \psi_n) \sim e^{N f_n(\psi_1, \ldots, \psi_n)}.
\]

The width of the distribution tends to zero in the thermodynamic limit \( N \to \infty \), which shows that the pulse amplitudes are thermodynamic observables.

We henceforth specialize to the case that all nonzero injection values are of the same magnitude \( h \), that is appropriate for injection of pulses from a source with a repetition rate \( n \) times faster than that of the laser. For the minimization problem there is no loss of generality in assuming that the injection values are all real, since the minimizing \( \psi \) values have the same phase as the corresponding \( \bar{h} \) values. By a simple rescaling of the variables the free energy \( f_n \) can be reduced to

\[
f = -\sum_{m=1}^{n} \left( \frac{1}{2} x_m^4 + h x_m \right) - T \log \left( 1 - \sum_{m=1}^{n} x_m^2 \right),
\]

where \( x_m \)'s are real and positive. It follows from Eq. (9) that thermodynamics depends on only two dimensionless parameters, the reduced temperature \( T = \frac{h}{\gamma P x} \) (the inverse of of the interaction strength \( h \)) and reduced driving \( \bar{h} = \frac{2 h}{\gamma P x} \). \( \bar{x}_m \), the minimizers of \( f \), are related to the expectation values (in the invariant measure) of the pulse powers by \( \bar{x}_m^2 = \langle |\psi_m|^2 \rangle / P \).

The study of thermodynamics and critical behavior is now reduced to the analysis of the function \( f \) and its minima. We note that for any values of the parameters, minima can occur only at configurations where at least \( n - 1 \) of the \( \bar{x}_m \)'s are equal, with value \( \bar{y} \), and the other minimizer, which we denote by \( \bar{x} \) is greater than or equal to \( \bar{y} \). Accordingly the free energy is obtained from

\[
f(x, y) = -\frac{1}{2} \left[ x^4 - (n - 1) y^4 \right] - h [x + (n - 1)y] - T \log[1 - x^2 - (n - 1) y^2].
\]

It is instructive to consider the above results in the \( \mathbf{k} \) or mode space \( a_k \), the discrete Fourier transform of \( \psi_n \). The Hamiltonian reads

\[
H = -\frac{\gamma}{2} \sum_{j-k+1 - m=0} a_j^* a_k a_m^* - 2 \text{Re} \sum_k \tilde{h}_k a_k^*,
\]

where \( \tilde{h}_k \) takes nonzero value for \( k \) being integer multiples of \( n \). Therefore, when \( n > 1 \), the set of modes consists...
of two types, with and without the presence of the external field. The Hamiltonian in Eq. (11) is analogous to the one of an antiferromagnet placed in an external homogeneous magnetic field \( \mathbf{h} \), but with the roles of the interaction term and the driving term reversed. Namely, the driving acts on a subset of the modes, but the interaction tends to align all modes in the same amplitude and phase. The consequences of this competition are derived below. The free energy can be expressed in the mode representation using the mode amplitude expectation values \( \langle a_f \rangle \) and \( \langle a_u \rangle \) of those with (forced) and without electric field (unforced), respectively. The relations are \( x = \frac{1}{n} \langle a_f \rangle + (n-1) \langle a_u \rangle \) and \( y = \frac{1}{n} \langle a_f \rangle - 1 \langle a_u \rangle \) and the free energy is

\[
f = -\frac{1}{2n^2} \left[ (\langle a_f \rangle)^2 + (n-1)(n^2 - 3n + 3) \langle a_u \rangle^2 \right. \\
+ 6(n-1) \langle a_f \rangle^2 \langle a_u \rangle^2 + 4(n-1)(n-2) \langle a_f \rangle^3 \langle a_u \rangle^2 \left. \right] \\
- h \langle a_f \rangle - \mathcal{T} \log \left( 1 - \frac{1}{n} \langle a_f \rangle^2 - \frac{n-1}{n} \langle a_u \rangle^2 \right) \\
(12)
\]

For the analysis below we use the real space formulation. We consider first the case of \( n = 1 \), where there is a single pulse (the external field is applied on all modes), and \( f \) (Eq. 10) depends on the single variable \( x \). For zero and small values of \( h \), \( f \) has a minimum \( x_1 \) near zero and, for small enough \( \mathcal{T} \) another minimum \( x_2 > x_1 \) below 1. For such \( h \) there is a threshold temperature \( \mathcal{T}_1(h) \) where \( f(x_1) = f(x_2) \). As \( \mathcal{T} \) is decreased through this line \( \dot{x} \) jumps from \( x_1 \) to \( x_2 \) in a first order phase transition. When \( h > 0 \) the jump is between two pulsed states, but the high-temperature phase pulses are driven pulses whose power decreases smoothly to zero when \( h \to 0 \). We term this phase “para-pulse” because of its resemblance with the paramagnetic phase of magnets above the Curie point. For sufficiently strong \( h \), on the other hand, \( f \) always has a single minimum \( \bar{x} \) between zero and one, which decreases smoothly from one to zero as \( \mathcal{T} \) is increased. Thus, the coexistence line \( \mathcal{T}_c(h) \) terminates at a critical point \((h_c, \mathcal{T}_c)\). The phase diagram is shown in the upper part of Fig. 1.

The lower part of Fig. 1 shows \( \bar{x} \) as a function of \( \mathcal{T} \) for several \( h \) values. \( \bar{x} \) undergoes a jump for \( h < h_c \), and displays an infinite slope at the critical point—a manifestation of the critical divergence of the susceptibility. The critical point itself is characterized by the vanishing of the first three derivatives of \( f \), which gives three polynomial equations for the three unknowns \( \bar{x}_c, h_c \), and \( \mathcal{T}_c \). The equations can be solved explicitly by radicals giving \( \bar{x}_c \approx 0.53 \), \( h_c \approx 0.2035 \), \( \mathcal{T}_c \approx 0.34 \).

One may define the usual critical exponents \( \beta, \gamma, \delta \) by \( \langle |x - \bar{x}_c|/c_{\text{coexistence}} \rangle \sim (\mathcal{T} - \mathcal{T}_c)^\beta, \chi = \partial^2 \mathcal{F} / \partial h^2 \sim |\mathcal{T} - \mathcal{T}_c|^{\delta/\gamma}, \) and \( (\mathcal{T} - \mathcal{T}_c)|_{\mathcal{T}=\mathcal{T}_c} \sim (h - h_c)^{1/\delta} \). The exponents, as well as the nonuniversal amplitudes can be calculated by the standard procedure of expanding \( f \) up to third order near the critical point, which yields the classical mean field exponents \( \beta = 1/2, \gamma = 1, \delta = 3 \), expectedly, since mean field theory applies to our system.

The fluctuation-dissipation relations naturally hold in the laser mode system; it follows from Eq. (4) that

\[
\text{Var}(|\psi|) = \frac{2}{N} \frac{\partial}{\partial h} \langle |\psi| \rangle = \frac{2}{N} \chi, \\
(13)
\]

that is, the critical exponent \( \gamma \) also describes the divergence of pulse power fluctuations near the critical point. To study the fluctuations at the critical point we turn back to Eq. (3), which in the present context is a probability distribution for the single pulse amplitude. Letting \( x = P^{-1/2} |\psi| \), the criticality condition implies that for \( T = T_c, \ h = h_c \)

\[
P(x) \sim e^{N(a(x-x_c) + O(x-x_c)^3)}, \\
(14)
\]

where \( a \) is \( O(1) \). The fluctuations distribution is non-gaussian, and the scale of the critical fluctuations is \( O(N^{-1/4}) \), stronger by a factor of \( N^{1/4} \) than normal fluctuations. The critical fluctuations are much larger than the typical amplitude of the continuum background, \( O(N^{-1/2}) \). It follows that the fluctuations of the continuum background are correlated, being the SLD analog of the critical opalescence phenomenon.

The thermodynamics with \( n > 1 \) is qualitatively different. The external injection encourages the formation of \( n \) equal pulses, clashing with the tendency of the saturable absorber to form a single strong pulse. As a result, the
phase diagram consists of unequal pulse phase for weak noise and weak injection, and an equal pulse phase for strong noise or strong injection. As the two phases are characterized by different symmetries, there can be no smooth transition between them, and they are separated by a phase transition line. However, the phase transition may be continuous or first order, depending on whether the transition order parameter \( q = (x - y)/\sqrt{x^2 + y^2} \) is continuous or jumps to a nonzero value at the transition.

We consider in detail the case \( n = 2 \) where both behaviors occur. It is more convenient to express \( f \) of Eq. (11) in terms of \( q \) and \( p = x^2 + y^2 \)

\[
f(p, q) = -\frac{p^2}{2}(1+2q^2-q^4)-\beta_4\sqrt{2p\left(1 - \frac{q^2}{2}\right)} - T\log(1-p)
\]

(15)
to be minimized over \( p \) and \( q \). The values of the minimizers \( \tilde{p} \) and \( \tilde{q} \), giving their “thermal” averages vs. \( T \), are given in Fig. 2. \( f \) is manifestly symmetric in \( q \), since the pulse amplitudes \( x \) and \( y \) play symmetric roles, wherefrom it follows that \( f \) is always stationary with respect to \( q \) when \( q = 0 \). The condition \( \partial_q f = 0 \) has another solution \( p^3 = \frac{\tilde{q}^2}{2}(q^2 - 1)^2(2 - q^2) \), and the global minimum of \( f \) is reached in one of these configurations, the first corresponding to equal and the second to unequal pulse mode locking.



Straightforward analysis shows that for large \( T \) the function \( f \) has a single minimum, which occurs at \( q = 0 \). For large \( \beta \) this situation persists as \( T \) is lowered until at \( T = T_0 \equiv \frac{2}{3}\beta^{2/3}(2 - \beta^{2/3}) \) the minimum becomes a saddle and two minima with nonzero \( q \) form, i.e., \( q \) undergoes a continuous phase transition, see Fig. 3. For small \( \beta \), on the other hand, nonzero \( q \) minima appear for \( T > T_0(h) \), and at \( T_1(h) \) exchange stability with the other pair of minima.

The intersection of the line of first order phase transition \( T_1(h) \) and the line of continuous phase transition \( T_0(h) \) can be shown to occur at \( (T_1, b_1) = (\frac{3}{4}, \frac{45}{128}) \).

(11) is a tricritical point \( T_1 \), with symmetric tricritical phenomena, and the phase diagram Fig. 3 is quite similar to that of metamagnets \( \Delta \), where tricritical behavior is known to occur. In particular we may define tricritical exponents such as \( \beta_1 \) and \( \beta_2 \) associated with nonsymmetric (e.g. \( q \)) and symmetric (e.g. \( p \)) fields respectively by \( q \sim (T - T_1)^{\beta_1} \) and \( p - p_1 \sim (T - T_1)^{\beta_2} \) near the tricritical point, see Fig. 2. As before, the exponents take the classical values \( \beta_1 = 1/4, \beta_2 = 1/2 \).

Critical and multicritical phenomena could be observed in these cases under external injection of unequal pulses.

The continuation of the first order phase transition line (in all cases) under external injection to power levels below the one required for spontaneous passive mode locking (self-starting threshold) has an interesting practical implication. A laser operating below the threshold pumping level may be mode locked by external injection. The injection may be removed and, provided pumping is strong enough, the laser would remain metastably mode locked for an exponentially long lifetime.

\[ x > y \]

\[ \mathcal{P} \]

\[ \mathcal{N} \]

\[ \mathcal{L} \]

\[ \mathcal{R} \]

\[ \mathcal{M} \]

\[ \mathcal{O} \]

\[ \mathcal{F} \]

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