Particle acceleration in supernova remnants, the Bell-Lucek hypothesis and the cosmic ray “knee”.

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Abstract. Young supernova remnants are thought to be the sites where cosmic ray acceleration occurs by the mechanism of diffusive shock acceleration. The maximum energy gained in this process is conventionally estimated to have a value close to, but distinctly below, the “knee” ($\sim 10^{15}$ eV) of the cosmic-ray spectrum. Bell & Lucek (2001) have suggested that the generated cosmic rays simultaneously amplify the magnetic field around the supernova remnant shock to many times its pre-shock value. In this case the acceleration rate may be significantly increased and protons can easily reach energies up to $10^{17}$ eV. We use a “simplified” box model incorporating the magnetic field amplification suggested by Bell & Lucek to investigate the resulting modifications of the cosmic-ray spectrum. The model predicts a spectral break at high energies, close to the “knee” region, and in good accordance with observations.

1. Introduction

Bell & Lucek (2001) and Lucek & Bell (2000) have presented numerical simulations suggesting that the conventional process of particle acceleration in shocks, generally called diffusive shock acceleration, may also result in substantial amplification of the highly tangled magnetic field around the shock. The unusual strong magnetic field inferred in Cas A (e.g. Vink & Lamling 2003) seems to confirm this prediction. We note in passing that this hypothesis provides a concrete physical mechanism for one of the many ideas of the late F Hoyle who speculated (Hoyle 1960) that strong interstellar shocks might convert mechanical energy into roughly equal amounts of magnetic field energy, cosmic ray energy and thermal energy. It is also closely related to the “plastic deformation” of the magnetic field discussed qualitatively by Völk and McKenzie (1981). Bell & Lucek (2001) suggest that this may enable strong supernova driven shocks in the interstellar medium to accelerate protons to energies well beyond what are conventionally held to be the maximum realistically attainable energies of a few times $10^{14}$ eV (e.g. Lagage and Cesarsky, 1983).

The Bell-Lucek hypothesis is of great interest because it is one of the few suggestions as to how the cosmic ray particles of energies at and above the “knee”, located at a few times $10^{15}$ eV, could be made in the Galaxy. It is therefore important to determine what the expected form of the source energy spectrum would be and whether the slight break in the observed spectrum at the “knee” can be accounted for in this way. There are also interesting implications for gamma ray observations of SNRs with the next generation of imaging atmospheric Cherenkov telescopes such as HESS. For an initial examination of this problem the so-called “box” models of particle acceleration (Drury et al. 1999) offer enough accuracy and have the great advantage of computational simplicity.

2. The Bell-Lucek hypothesis

Bell & Lucek point out that in the conventional diffusive shock acceleration picture the energy density in the accelerated particles at and near the shock front is extremely large, of order the total ram pressure of the incoming plasma, and thus much larger than the energy density of the magnetic field (the ratio of particle energy to field energy is of order the square of the shock Alfvén Mach number). The standard treatment of resonant wave excitation, in which the perturbations of the field are treated as Alfvén waves to lowest order and the interaction with the particles as a small perturbation, is thus very questionable. With the support of numerical simulations and simplified analytic models they suggest that in reality the field can be highly distorted by the particle pressure and wound up to the point where approximate equipartition holds.

If this amplified and distorted field is then used to estimate the particle diffusion in the shock neighbourhood, acceleration to substantially higher energies than in the conventional picture is obviously possible. Detailed estimates and simple dimensional analysis agree that the maximum particle rigidity is given, to order of magnitude, by the product of the field strength $B$, the shock radius $R$ and the shock velocity $U$. Thus, other things being equal, the increase is directly proportional to the increase in the field strength, which by the above argument is of order the Alfvén Mach number of the shock $M_{sh}$. This can
easily be $M_{th} \approx 10^3$ for a young supernova remnant so that the effect is potentially very significant; acceleration to rigidities of a few $10^{17}$ V, rather than the $10^{14}$ V normally estimated, is easily possible.

Ptuskin and Zirakashvili (2003) have performed a sophisticated analysis of the combined effects of field amplification and wave damping on particle acceleration in supernova remnants and conclude that indeed the upper cut-off to the accelerated spectrum can be quite strongly time-dependent. Our aim in this short paper is to carry out a first exploratory analysis of the effect this has on the overall spectrum and we therefore use the simple approach of assuming an equipartition field although this is clearly a rather crude approximation.

3. The “box” model

To get a first estimate of the effect of such a dynamically generated field on the acceleration we turn to the simplest treatment of shock acceleration, the so-called “box” model. In this the accelerated particles are assumed to be more or less uniformly distributed throughout a region extending one diffusion length each side of the shock, and to be accelerated upwards in momentum space at the shock itself with an acceleration flux

$$\Phi(p) = \frac{4\pi}{3} p^3 f(p) (U_1 - U_2),$$

per unit surface area where $U_1$ and $U_2$ are the upstream and downstream velocity and $f(p)$ is the phase space density of the accelerated particles (assumed to have an almost isotropic distribution). If the diffusion length upstream is $L_1$, and that downstream is $L_2$, then

$$L_1 \approx \frac{k_1(p)}{U_1}, \quad L_2 \approx \frac{k_2(p)}{U_2},$$

where $k_1$ and $k_2$ are the upstream and downstream diffusion coefficients. To a first approximation we assume that both $L_1$ and $L_2$ are small relative to the radius of the shock and that we can neglect effects of spherical geometry (in fact it is not too difficult to develop a spherical box model, but it unnecessarily complicates the argument) so that the box volume is simply $A(L_1 + L_2)$ where $A$ is the surface area of the shock. The basic “box” model equation is then simply a conservation equation for the particles in the box; the rate at which the number in the box changes is given by the divergence of the acceleration flux in momentum space plus gains from injection and advection and minus advective losses to the downstream region.

$$\frac{\partial}{\partial t} \left[ A(L_1 + L_2) 4\pi p^2 f(p) \right] + A \frac{\partial \Phi}{\partial p} = AQ(p) + AF_1(p) - AF_2(p),$$

where $Q(p)$ is a source function representing injection at the shock (only important at very low energies), $F_1$ is a flux function representing advection of pre-existing particles into the system from upstream (normally neglected) and $F_2$ is the flux of particles advected out of the system and carried away downstream. The only complication we have to consider is that the box is time-dependent, with flow speeds, shock area and diffusion lengths all changing.

The escaping flux is determined simply by the advection across the downstream edge of the box, that is

$$F_2(p) = 4\pi p^2 f(p) \left( U_2 - \frac{\partial L_2}{\partial t} \right),$$

where we have to explicitly allow for the time varying size of the downstream region. Substituting this expression for $F_2(p)$ and neglecting the advection of pre-existing particles (the $F_1(p)$ term) the box equation simplifies to:

$$\frac{1}{A} \frac{\partial A}{\partial t} (L_1 + L_2) f + \frac{\partial L_1}{\partial t} f + (L_1 + L_2) \frac{\partial f}{\partial t} + U_1 f + (U_1 - U_2) \frac{p \partial f}{\partial p} = \frac{Q}{4\pi p^2}.$$  

4. Incorporating the Bell-Lucek effect in the “box” models

Partial differential equations of this form always reduce, by the method of characteristics, to the integration of two ordinary equations, one for the characteristic curve in the $(p, t)$ plane

$$\frac{dp}{dt} = \frac{U_1 - U_2}{L_1 + L_2} p,$$

and one for the variation of $f$ along this curve

$$(L_1 + L_2) \frac{df}{dt} = f \left[ (L_1 + L_2) \frac{1}{A} \frac{\partial A}{\partial t} + \frac{\partial L_1}{\partial t} + U_1 \right] = \frac{Q}{4\pi p^2}.$$  

Apart from at the injection momentum $Q = 0$ and we can write the above equation as

$$\frac{d \ln f}{dt} = \frac{1}{A} \frac{\partial A}{\partial t} - \frac{1}{L_1 + L_2} \frac{\partial L_1}{\partial t} \frac{U_1}{L_1 + L_2}.$$  

But the shock area $A$ is a function only of time so that

$$\frac{\partial A}{\partial t} = \frac{dA}{dt},$$

and, although the upstream diffusion length does depend on both time and momentum, if we assume Bohm scaling for the two lengths so that

$$L \propto \frac{k}{U} \propto \frac{pv}{UB},$$

(where $v$ is the particle velocity) we can write

$$\frac{1}{L_1 + L_2} \frac{\partial L_1}{\partial t} = -\vartheta \frac{d \ln (U_1 B_1)}{dt},$$

where

$$\vartheta = \frac{L_1}{L_1 + L_2},$$

(0 < $\vartheta$ < 1). Finally, noting that

$$\frac{U_1}{L_1 + L_2} = \frac{3U_1}{U_1 - U_2} \frac{d \ln p}{dt}. $$
and assuming that the shock remains strong, which yields a fixed compression ratio, we can simplify equation (13) to
\[
\frac{d \ln f}{dt} = -\frac{d \ln A}{dt} + \theta \left( \frac{d \ln (U_1 B_1)}{dt} - \frac{3U_1}{U_1 - U_2} \frac{d \ln p}{dt} \right),
\]
which integrates trivially to relate the value of \( f \) at the end of one of the characteristic curves, say at the point \((p_1, t_1)\), to the value at the start, say at \((t_0, p_0)\), as follows;
\[
\frac{f(t_1, p_1)}{f(t_0, p_0)} = \left( \frac{A(t_1)}{A(t_0)} \right)^{-1} \left( \frac{U_1(p_1)B_1(t_1)}{U_1(p_0)B_1(t_0)} \right)^{\theta} \left( \frac{p_1}{p_0} \right)^{-s}
\]
where
\[
s = \frac{3U_1}{U_1 - U_2}
\]
is the standard exponent of the steady-state power-law spectrum associated with shock acceleration.

This rather beautiful result shows how the standard test particle power-law is modified by a combination of effects as the box volume changes. As one would expect the amplitude varies inversely as the shock area and also decreases if the upstream diffusion length (at fixed energy) increases, but with an exponent between zero and one determined by the ratio of the upstream diffusion length to the total width of the diffusion region. It is very interesting that the result is not simply a variation inversely as the box volume, which one would naively expect from geometrical dilution. This reflects the fundamental asymmetry between the upstream and downstream regions, that upstream is empty outside the diffusion region whereas the entire downstream region is filled with accelerated particles.

If we assume pure Bohm scaling the other differential equation is also integrable so that the problem is reduced entirely to quadratures (of course only within the various approximations we are making; but still a remarkable result). Bohm scaling implies that the mean free path is of order and proportional to the particle gyroradius, so that if the particle charge is \( e \)
\[
L_1 + L_2 \approx \frac{pv}{eB_1 U_1},
\]
where \( \alpha \) is a dimensionless parameter (probably of order ten). Substituting in the equation of the characteristic (equation (6)) we get
\[
\frac{dp}{dt} = 1 - \frac{1}{\alpha} (U_1 - U_2)U_1eB_1,
\]
and noting the relativistic identity between kinetic energy \( T \), momentum \( p \) and velocity \( v \),
\[
v = \frac{dT}{dp},
\]
we can integrate this as
\[
T_1 - T_0 = \frac{e}{\alpha} \int_{t_0}^{t_1} (U_1 - U_2)U_1B_1 \, dt.
\]
For relativistic particles the kinetic energy and the momentum are essentially interchangeable with \( T = c \sqrt{p^2 + m^2c^2} - mc^2 \approx c p \).

These two integrals (equations (15) & (20)) together reduce the problem of calculating the final spectrum to that of determining the initial amplitude \( f(t_0, p_0) \) which in turn depends on the injection rate and its time dependence.

5. The injection rate

There are two main approaches to the injection rate. The simplest, which is perhaps more consistent with the test particle approach, is to simply parametrise it by assuming that some fraction of the incoming thermal particles are "injected" as non-thermal particles at some suitably chosen "injection momentum" which separates the thermal particle population from the non-thermal. In other words one writes
\[
Q(p, t) = \eta(t)n_1U_1\delta(p - p_{mj}(t)),
\]
where \( n_1 \) is the upstream thermal particle number density, \( \eta \ll 1 \) is the injection fraction, \( p_{mj} \) is the injection momentum and as usual \( \delta \) is Dirac's delta distribution. It should be clear that this is a parametrisation rather than a true injection model, however it, or equivalent parametrisations, have been very widely used, typically with \( \eta \) taken to be a constant of order \( 10^{-5} \) to \( 10^{-4} \) for protons and \( p_{mj} \approx 10m_pU_1 \) where \( m_p \) is the proton mass. However there is no real justification for this apart from the fact that it seems to yield reasonable results in many cases.

With the above parametrisation the distribution function just above the injection energy can be simply determined by equating the acceleration flux to the injection flux,
\[
\frac{4\pi p^3_{mj}}{3}(U_1 - U_2) f(p_{mj}) = \eta n_1 U_1,
\]
giving
\[
f(p_{mj}) = \frac{3}{4\pi p^3_{mj}} \frac{U_1}{U_1 - U_2} n_1 \eta.
\]

The second approach adopts the idea, which can be traced back to the early work of Eichler, that the injection process is inherently extremely efficient but that various feedback processes operate to reduce it to the point where the accelerated particles carry a significant part of the energy dissipated in the shock. Probably the most sophisticated modern version of this idea is to be found in the papers by Malkov (eg Malkov 1998, Malkov et al 2000; see also Kang et al 2002). This, or something similar, is in fact required for the Bell-Lucek hypothesis to operate because it requires the accelerated particle energy density to be substantial and of order the ram pressure of the upstream flow. For a standard spectrum close to \( p^{-4} \) the energy is almost uniformly distributed per logarithmic interval over the relativistic part of the spectrum. This suggests taking a reference momentum in the mildly relativistic region, \( p_0 \approx mc \), and determining \( f \) by a relation of the form
\[
\frac{4\pi p_0^3}{3} f(p_0)mc^2 \approx \beta \rho U_1(U_1 - U_2)
\]
where \( \beta \) is a number which depends logarithmically on the upper cut-off and which for supernova remnants is probably somewhere between \( 10^{-1} \) and \( 10^{-2} \).

It is important to note that both injection models are models for proton injection, the protons being the dynamically dominant species. Unfortunately very little is known about the factors controlling the injection of electrons and other minor species despite their importance for diagnostic tests. It is also
very probable that the injection is nonuniform over the shock surface with a strong dependence on the angle between the mean background field and the shock normal (Völk et al, 2003).

6. Application to the Sedov solution

Let us now apply these ideas to the Sedov solution (also studied by Taylor and von Neumann) for a strong spherical explosion in a cold gas where the shock radius expands as \( R \propto t^{2/3} \) and the shock velocity decreases as \( U \propto t^{-3/5} \). On the Bell-Lucek hypothesis the magnetic field also scales as the shock velocity, \( B \propto t^{-3/5} \) and thus the characteristic acceleration curves (equation (20)) are given by

\[
T_1 - T_0 \propto \int_{t_0}^{t_1} (U_1 - U_2) U_1 B dt \\
\propto \int_{t_0}^{t_1} t^{-9/5} dt \\
\propto t^{-4/5} - t_1^{-4/5}.
\]

These curves, illustrated schematically in Fig. 1, all rise extremely steeply, representing an initial phase of rapid acceleration, turn over and then become asymptotically flat. Physically it is clear that, as the shock slows and the field drops, the high energy particles cease to be significantly accelerated and simply diffuse further and further in front of the shock. In fact in reality they should probably be thought of as decoupling from the shock and forming part of the general interstellar population at this point, but within the box model they simply fill a steadily growing upstream region. We will return to this point later.

A very important aspect of the curves is that they uniquely relate final energies (or equivalently momenta) to starting times. Asymptotically the relation is a simple power-law; for \( T_1 \gg T_0 \) and \( t_0 \ll t_1 \) we have simply

\[
p_1 \propto T_1 \propto t_1^{-4/5}, \quad t_0 \propto p_1^{-5/4}.
\]

Using this we can now translate the dilution factors from equation (15) to additional power-law terms in the final momentum. Explicitly, a given final momentum maps to a starting radius using a Sedov expansion-law:

\[
R(t_0) \propto t_0^{2/5} \propto p_1^{-1/2},
\]

and thus the first term on the RHS of equation (15) translates to a \( p_1^{-1} \) factor:

\[
\left( \frac{A(t_0)}{A(t_1)} \right)^{-1} \propto p_1^{-1}.
\]

The final momentum can also be mapped to a starting velocity, using again a Sedov expansion-law:

\[
U(t_0) \propto t_0^{-3/5} \propto p_1^{3/4}.
\]

The magnetic field, which on the Bell-Lucek hypothesis scales as velocity, gives an additional \( p_1^{1/4} \) factor, thus the second term on the RHS of equation (15) scales as:

\[
\left( \frac{U_1(t_1)B_1(t_1)}{U_1(t_0)B_1(t_0)} \right) \propto p_1^{-30/2} \quad \text{(30)}
\]

Furthermore we need to determine the initial amplitude of \( f \) from an injection model. First we use the \( \eta \) parametrisation as discussed in section 5:

\[
p_{\text{inj}} = p_0 \propto U(t_0) \propto p_1^{3/4},
\]

together with

\[
f_0 \propto \eta p_0^{-3}. \quad \text{(32)}
\]

Finally assuming a strong shock which, using the Rankine-Hugoniot conditions for a non-relativistic fluid, yields \( U_1/U_2 = 4 \) and \( 3U_1/(U_1 - U_2) = 4 \) and substituting the equations (28), (30), (31) and (32) into equation (15) one obtains a scaling-law for the particle distribution \( f(p_1) \) at a fixed time \( t_1 \):

\[
f(p_1) \propto \eta p_0^{-3} A(t_0) \left[ U_1(t_0)B_1(t_0) \right]^{-4} \left( \frac{p_1}{p_0} \right)^{-4}
\]

\[
\propto \eta p_1^{3/4} p_1^{-30/2} p_1^{-4}.
\]

If \( \eta \) is constant, the slope is steepened from the canonical value of 4 to

\[
4.25 + \frac{3\theta}{2} \quad \text{(34)}
\]

If we use the alternative “equipartition” argument as an injection model, meaning that \( f_0 \) should be dynamically determined in the mildly relativistic region, we have \( p_0 \approx mc \) independent of \( p_1 \) and

\[
f(p_0) \propto U_1^4 \propto r^{-3} \propto p_1^{3/2},
\]

which gives the following scaling-law for the particle distribution:

\[
f(p_1) \propto p_1^{3/2} p_1^{-4} p_1^{-30/2} p_1^{-4}.
\]

In this approach the slope is given by

\[
4 + \frac{3\theta - 1}{2}.
\]
Fig. 2. Example of $f(t_1, p_1)$ (dotted line) from equation (15), where a Sedov-like shock has been to describe the velocity and $\theta = 0.9$. The solid line is the standard $p^{-4}$ curve, the dotted line converges towards the spectral index given by equation (34). It is interesting that because of the strong injection at early times this model can even, if $\theta < 1/3$, lead to a slight flattening of the spectrum. However, especially at high energies, it is unlikely that the upstream diffusion region could be so small and a modest steepening of the spectrum is more likely.

These results refer of course only to the asymptotic behaviour of the high energy part of the spectrum. As $p_1$ is decreased there comes a point where $t_0$ is no longer small relative to $t_1$. At this point all values of the final momentum map down to a small approximately constant region and the spectrum becomes simply the standard $p^{-4}$ spectrum. This break occurs at the point to which efficient acceleration is possible at that stage in the remnant evolution, and decreases as the the remnant ages. Shock acceleration will terminate when the shock is beginning to weaken and the amplified field is only a few times the ambient field, which for ambient fields of a few $\mu$Gauss and typical SNR parameters places the break exactly in the “knee” region. Figure 2 shows an example of the total spectrum for a Sedov-like shock, which was obtained by using equation (15). The injection mechanism for this example was taken according to the equations (22) and (23) and $\theta = 0.9$. The spectrum clearly shows the smooth transition from the standard $p^{-4}$ spectrum to the asymptotic spectral index given by equation (34).

7. Conclusions

We have applied the Bell-Lucek hypothesis to a Sedov-like shock, using a simplified Box model. We showed that such a model exhibits a spectral break at an energy determined by the current acceleration cut-off below which one observes the standard shock acceleration spectrum, but above which a slightly different power-law continues to higher energies. For older remnants the break is expected to be in the “knee” region at rigidities of order $10^{15}$ V. The key point is that the dynamical field amplification both increases the maximum attainable energy and makes it a relatively strongly time-dependent quantity unlike the situation with no field amplification where, as is well known, there is only a very weak dependence of the cut-off energy on the remnant age, at least during the Sedov phase (the much more sophisticated analysis by Ptuskin and Zirakashvili, 2003, is relevant here).

While the spectrum at the shock is the more relevant quantity from the point of view of gamma-ray tests, in the context of cosmic ray propagation theory what one would really like is the effective source spectrum of an individual supernova integrated over its history. This, while obviously related to the spectrum discussed here, is a somewhat different quantity and harder to evaluate. However it is clear that here also one should expect a broken power-law with a relatively small break in the exponent at a rigidity corresponding to acceleration in a mildly amplified Galactic field.

Obviously further and more detailed work is needed, but it is very encouraging that even such a simple model can produce spectra remarkably close to the inferred cosmic ray source spectrum through the “knee” region. In fact we are not aware of any other acceleration model that can naturally produce a break of the right magnitude (about 0.5 in the exponent) at the right position (modulo major uncertainties in interstellar propagation at these energies of course).

References

Bell, A. R. & Lucek, S. G. 2001, MNRAS, 321, 433
Drury, L. O’C., Duffy, P., Eichler, D., Mastichiadis, A. 1999, A&A, 347, 370
Hoyle, F. 1960, MNRAS, 120, 338
Kang, H, Jones, T. W. & Gieseler, U. D. J. 2002, Ap J 579, 337
Lagage, P. O.,& Cesarsky, C., 1983, A&A, 125, 249
Lucek, S. G. & Bell, A. R., 2000, MNRAS, 314, 65
Malkov, M. A., 1998, Phys Rev E 58, 4911
Malkov, M. A., Diamond, P. H. & Völk, H. J., 2000, Ap J 533 L171.
Ptuskin, V. S., & Zirakashvili, V. N., 2003, A&A 403, 1.
Vink, J., & Laming, J. M., 2003, ApJ, 584, 758
Völk, H. J., & McKenzie, J. F., 1981, Proc 17th ICRC (Paris) 9 246.
Völk, H. J., Berezhko, E. G. & Ksenofontov, L. T., 2003, [http://arXiv/astro-ph/0306016]