Research Article

Weighted Central BMO Spaces and Their Applications

Huan Zhao and Zongguang Liu

Department of Mathematics, China University of Mining and Technology, Beijing 100083, China

Correspondence should be addressed to Zongguang Liu; liuzg@cumtb.edu.cn

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In this paper, the central BMO spaces with Muckenhoupt $A_p$ weight is introduced. As an application, we characterize these spaces by the boundedness of commutators of Hardy operator and its dual operator on weighted Lebesgue spaces. The boundedness of vector-valued commutators on weighted Herz spaces is also considered.

1. Introduction

For $1 < p < \infty$ and a nonnegative locally integrable function $\omega$ on $\mathbb{R}^n$, it is said that $\omega$ is in the Muckenhoupt $A_p$ class if it satisfies the condition

$$[\omega]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty.$$ (1)

A weight function $\omega$ belongs to the class $A_1$ if

$$[\omega]_{A_1} = \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left( \text{ess sup}_{x \in Q} \omega(x)^{-1} \right) < \infty.$$ (2)

A weight $\omega$ is called an $A_\infty$ weight if

$$[\omega]_{A_\infty} = \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left( \int_Q \log \omega(x)^{-1} \, dx \right) < \infty.$$ (3)

It is well-known that $A_\infty = \bigcup_{1 < p < \infty} A_p$. Let $\omega \in A_\infty$ and $p \in (0, \infty)$; we denote $L^p(\omega)$ as the space of all measurable functions $f$ such that

$$\|f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty.$$ (4)

The definition of $A_p$ weight was introduced by Muckenhoupt [1]. Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights plays an important role in the study of boundary value problems for the Laplace equation on Lipschitz domains. Other applications of weighted inequalities include vector-valued inequalities, extrapolation of operators, and applications to certain classes of integral equations and nonlinear partial differential equations. There are a number of classical results which demonstrate that the Muckenhoupt $A_p$ classes are the right collections of weights to do harmonic analysis on weighted spaces. The main results along these lines are the equivalence between the $\omega \in A_p$ condition and the $L^p(\omega)$ boundedness (or weak boundedness) of maximal operator and singular integral operators.

A well-known result of Muckenhoupt [1] showed that the Hardy-Littlewood maximal operator $M$, that is

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$ (5)

is (weak) bounded on weighted Lebesgue spaces $L^p(\omega)$ if and only if $\omega \in A_p$ for $1 < p < \infty$ (for the case $n = 1$). Hunt et al. [2] proved that the $A_p$ condition also characterizes the $L^p(\omega)$ boundedness of the Hilbert transform $H$, where
Later, Coifman and Fefferman [3] extended the $A_p$ theory to the case $n \geq 1$ and the general Calderón-Zygmund operators; they also proved that $A_p$ weights satisfy the crucial reverse Hölder condition.

On the other hand, it is well-known that $\text{BMO}(\mathbb{R}^n)$ is just the dual space of Hardy space $H^1(\mathbb{R}^n)$. Like this, the dual space of Herz-type Hardy space is the so-called central BMO space which is defined by

$$\text{CBMO}^p(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n); \| f \|_{\text{CBMO}^p(\mathbb{R}^n)} < \infty \right\},$$

with

$$\| f \|_{\text{CBMO}^p(\mathbb{R}^n)} = \sup_{r > 0} \left( \frac{1}{|B(0, r)|} \int_{B(0, r)} \left| f(x) - f_{B(0, r)} \right|^p dx \right)^{1/p},$$

where

$$f_{B(0, r)} = \frac{1}{|B(0, r)|} \int_{B(0, r)} f(x) dx.$$

The space $\text{CBMO}^p(\mathbb{R}^n)$ can be regarded as a local version of BMO$(\mathbb{R}^n)$ at the origin, that is, $\text{BMO}(\mathbb{R}^n) \subseteq \text{CBMO}^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ (see [4]). However, they have quite different properties. For example, there is no analysis of the famous John-Nirenberg inequality of BMO$(\mathbb{R}^n)$ for $\text{CBMO}^p(\mathbb{R}^n)$.

In this paper, we will introduce the space of central BMO with Muckenhoupt $A_p$ weight and characterize these spaces by the boundedness of commutator of the Hardy operator and its dual operator on weighted Lebesgue spaces. The boundedness of vector-valued commutators on weighted Herz spaces is also considered.

Throughout this paper, the letter $C$ denotes constants which are independent of the main variables and may change from one occurrence to another. Denote $B_k = \{ x \in \mathbb{R}^n; |x| \leq 2^k \}$ and $C_k = B_k \setminus B_{k-1}$, and $\chi_k$ is the characteristic function for $k \in \mathbb{Z}$.

2. Weighted Central BMO Spaces

In this section, we will introduce the definition of weighted central BMO spaces and give some properties of $\text{CBMO}^p(\omega)$.

Let $1 \leq p < \infty$, and $\omega$ is a nonnegative locally integrable function. A function $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the weighted central BMO spaces, if

$$\| f \|_{\text{CBMO}^p(\omega)} = \sup_{r > 0} \left( \frac{1}{|B(0, r)|} \int_{B(0, r)} \left| f(x) - f_{B(0, r)} \right|^p \omega(x) dx \right)^{1/p} < \infty,$$

where

$$f_{B(0, r)} = \frac{1}{|B(0, r)|} \int_{B(0, r)} f(x) \omega(x) dx.$$

When $\omega \equiv 1$ is a constant, $\text{CBMO}^p(\omega)$ is just $\text{CBMO}^p(\mathbb{R}^n)$.

We recall some properties of the weighted Lebesgue spaces. Let $\mathcal{Y}^p$ denote the set of all families of disjoint and open cubes in $\mathbb{R}^n$. In [12], Diening et al. obtained the following lemma in the general case on Musielak-Orlicz spaces. But we only describe the special case on the weighted Lebesgue spaces now.

Lemma 1. If $\omega \in A_{p_0}$, then there exist $0 < \delta < 1$ and $C > 0$ which only depend on the $A_{p_0}$-constant of $\omega$ such that

$$\left( \sum_{Q \in \mathcal{Q}} \frac{1}{t_Q} \int_{t_Q} f \chi_Q \right)^{1/p} \lesssim C \left( \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right)^{1/p},$$

for all $\mathcal{Q} \in \mathcal{Y}^p$; all $\{ t_Q \}_{Q \in \mathcal{Q}}$, $t_Q \geq 0$; and all $f \in L^1(\mathbb{R}^n)$ with $f_Q \neq 0$, $Q \in \mathcal{Q}$.

Lemma 2 (see [11]). Let $\omega \in A_{p_0}$, $1 \leq p < \infty$; then, there exist constants $C_1, C_2, \delta > 0$ such that for all balls $B$ in $\mathbb{R}^n$ and all measurable subsets $E \subset B$,

$$C_1 \left( \frac{|E|}{|B|} \right)^p \lesssim \frac{\omega(E)}{\omega(B)} \lesssim C_2 \left( \frac{|E|}{|B|} \right)^\delta.$$

In fact, the first inequality of Lemma 2 can be improved as follows.

Lemma 3. Let $\omega \in A_{p_0}$, $1 < p < \infty$. Then, there exist $p_0$ with $1 < p_0 < p$ and $C > 0$ such that for all balls $B$ in $\mathbb{R}^n$ and all measurable subsets $E \subset B$,

$$\frac{\| X_B \|_{L^p(\omega)}}{\| X_E \|_{L^p(\omega)}} \lesssim C \left( \frac{|B|}{|E|} \right)^{1/p_0}.$$

Proof. By the fact that $A_p = \bigcup_{q < p} A_q$ and $\omega \in A_{p_0}$ (see [13]), there exist $1 < p_0 < p$ such that $\omega \in A_{p_0}$. Applying Lemma 2, there exists a constant $C > 0$ such that for any ball $B$ and any measurable set $E \subset B$,

$$\left( \frac{|E|}{|B|} \right)^{p_0} \lesssim C \frac{\omega(E)}{\omega(B)}.$$
That is,
\[ \|X_B\|_{L^p(\omega)} \leq C \left( \frac{|B|}{|E|} \right)^{1/p_0}. \] (15)

Therefore we have proved Lemma 3.

Now, we show the relationship between \(CBMO(\omega)\) and central \(BMO\) spaces.

**Proposition 4.** If \(\omega \in A_p\), and \(1 < p < \infty\), then \(CBMO(\omega) \subset CBMO^*(\omega) \subset CBMO(\R^n)\).

**Proof.** Let \(f \in CBMO^*(\omega)\). For any \(B = B(0, r)\), by Hölder’s inequality, we have
\[ \left( \frac{1}{|B|} \int_B |f(x) - f_B|^p \omega(x) \, dx \right)^{1/p} \leq \left( \frac{1}{|B|} \int_B \omega(x)^{1/p} \, dx \right)^{1/p} \leq C \|f\|_{CBMO^*(\omega)}. \] (16)

On the other hand, Let \(f \in CBMO^*(\omega)\). For any \(B = B(0, r)\), by Hölder’s inequality and the condition \(\omega \in A_p\), we have
\[ \frac{1}{|B|} \int_B |f(x) - f_B| \, dx \leq \left( \frac{1}{|B|} \int_B |f(x) - f_B| \omega(x) \, dx \right)^{1/p} \left( \frac{1}{|B|} \int_B \omega(x)^{-p'} \, dx \right)^{1/p'} \leq C \|f\|_{CBMO^*(\omega)} \left( \frac{\omega(B)}{|B|} \right)^{1/p} \left( \frac{1}{|B|} \int_B \omega(x) \, dx \right)^{-1/p} \leq C \|f\|_{CBMO^*(\omega)}. \] (17)

Therefore, we only need to prove that there exists a function \(f\) such that \(f \in CBMO(\R^n) \setminus CBMO^*(\omega)\). Without loss of generality, we may assume that \(n = 1\).

Let \(A_k = \{ x \in \R : 2^k < |x| \leq 2^{k+1} \}, k \in \mathbb{Z}_+ \). Taking \(f(x) = \sum_{k=0}^{\infty} 2^k \chi_{A_k}(x) \) \(\text{sgn}(x)\), then for any \(B = B(0, r)\),
\[ f_B = \frac{1}{|B|} \int_B f(x) \, dx = 0. \] (18)

When \(r \leq 1\), we have \(f(x) \equiv 0\) and
\[ \sup_{0<r<1} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx = 0. \] (19)

When \(r > 1\), there exists \(k_0 \in \mathbb{Z}_+\) such that \(2^{k_0} < r \leq 2^{k_0+1}\); then,
\[ \sup_{r>1} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx \leq \sup_{r>1} C 2^{-k_0} \sum_{k=0}^{k_0+1} \int_{A_k} 2^k \, dx \leq C. \] (20)

From (19) and (20), it follows that \(f \in CBMO(\R^n)\).

When \(r > 4\), there exists \(k_0 \in \mathbb{Z}_+\) with \(k_0 \geq 2\) such that \(2^{k_0} < r \leq 2^{k_0+1}\); then,
\[ |f(x) - f_B| \chi_{A_k}(x) \geq \sum_{k=0}^{k_0-1} 2^k \chi_{A_k}(x) \geq 2^{k_0-1} \chi_{A_{k_0}}(x) \geq C r \chi_{A_{k_0}}(x), \] (21)

which implies that
\[ \sup_{r>4} \frac{\|f - f_B\|_{L^p(\omega)}}{\|X_B\|_{L^p(\omega)}} \geq \sup_{r>4} \frac{\|f - f_B\|_{L^p(\omega)}}{\|X_B\|_{L^p(\omega)}} \geq \sup_{r>4} \frac{\|X_{A_{k_0}}\|_{L^p(\omega)}}{\|X_B\|_{L^p(\omega)}}. \] (22)

Since \(A_{k_0} \subset B\), by Lemma 3, there exists \(p_0\) with \(1 < p_0 < p\); we have
\[ \sup_{r>4} \frac{\|f - f_B\|_{L^p(\omega)}}{\|X_B\|_{L^p(\omega)}} \geq \sup_{r>4} \frac{\|f - f_B\|_{L^p(\omega)}}{\|A_{k_0}\|} = \sup_{r>4} \frac{1}{\|A_{k_0}\|} = \infty. \] (23)

Therefore, \(f \not\in CBMO^*(\omega)\).

**Proposition 5.** If \(\omega \in A_p\), and \(1 < p < \infty\), then there exists a constant \(q > p\) such that \(CBMO(\R^n) \subset CBMO^*(\omega)\).

**Proof.** We can take a cube \(Q_B\) so that \(B \subset Q_B \subset \sqrt{n}B\). By Lemma 1, there exists a constant \(0 < \delta < 1\) independent of \(B\) such that for all \(f \in L^1_{\omega, \infty}(\R^n)\),
\[ \|f\|_{X_{Q_B}} \leq C \left[ \frac{|Q_B|}{|B|} \right]^{-\delta} \|X_B\|_{L^p(\omega)}. \] (24)

Let \(q = 1/\delta\) and \(f = (b - b_B)^{\delta} \chi_{Q_B}\). We conclude that
\[ \|f\|_{Q_B} \leq \frac{1}{|Q_B|} \int_B |f| \, dx \leq C \|b\|_{CBMO(\R^n)}. \] (25)

By Lemma 3, there exists a constant \(1 < p_0 < p\) such that
\[ \|X_{Q_B}\|_{L^p(\omega)} \leq C \|X_B\|_{L^p(\omega)} \leq C \left( \frac{\sqrt{n}B}{|B|} \right)^{1/p_0} \|X_B\|_{L^p(\omega)}. \] (26)

This gives us
\[ \|b\|_{CBMO(\omega)} \leq C \|b\|_{CBMO(\R^n)}. \] (27)

Hence, the proof of Proposition 5 is completed.

**Proposition 6.** If \(\omega \in A_p\), and \(1 < p < \infty\), then \(f \in CBMO^*(\omega)\) if and only if there exist a collection of numbers \(\{c_B(\delta)\}_{\delta>0}\)
(i.e., for each ball \(B(0, r)\), there exists \(c_{B(0, r)} \in \mathbb{R}\) such that

\[
\sup_{r > 0} \|X_{B(0,r)}\|_{L^p(\omega)}^{-1} \| (f - c_{B(0,r)})X_{B(0,r)} \|_{L^p(\omega)} < \infty.
\]  

(28)

Proof. We set \(c_{B(0, r)} = f_{B(0, r)}\) for all balls \(B(0, r)\); the necessity of the condition in Proposition 6 holds. Let us check the sufficiency of Proposition 6.

A similar argument as Proposition 4, we have, for any \(B := B(0, r)\),

\[
\frac{1}{|B|} \int_{B} (f(x) - c_{B})dx \leq C \|X_{B(0,r)}\|_{L^p(\omega)}^{-1} \| (f - c_{B})X_{B(0,r)} \|_{L^p(\omega)}.
\]

(29)

Thus,

\[
\| (f - f_{B})X_{B(0,r)} \|_{L^p(\omega)} \leq \| (f - c_{B})X_{B(0,r)} \|_{L^p(\omega)} + \| (c_{B} - f_{B})X_{B(0,r)} \|_{L^p(\omega)}
\]

\[
\leq C + |c_{B} - f_{B}| \leq C.
\]

(30)

Therefore, \(f \in \text{CBMO}^p(\omega)\); the proof of Proposition 6 is completed. \( \Box \)

**Proposition 7.** If \(f \in \text{A}_p\), and \(1 < p < \infty\), then \(f \in \text{CBMO}^p(\omega)\) if and only if

\[
\|f\|_{\text{CBMO}^p(\omega)} := \sup_{r > 0} \sup_{c \in \mathbb{C}} \|X_{B(0,r)}\|_{L^p(\omega)}^{-1} \| (f - c)X_{B(0,r)} \|_{L^p(\omega)} < \infty.
\]

(31)

Proof. The proof of Proposition 7 is similar as that of Proposition 6; we omit the details. \( \Box \)

### 3. Characterization of CBMO^p(\omega) Spaces via Commutators

We first review the definitions of the \(n\)-dimensional Hardy operator and its dual operator. For a locally integrable function \(f\) in \(\mathbb{R}^n\), the \(n\)-dimensional Hardy operator \(H\) is defined by

\[
Hf(x) = \frac{1}{|x|^n} \int_{|y| < |x|} f(y)dy, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

(32)

The dual Hardy operator \(H^*\) is defined by

\[
H^*f(x) = \int_{|y| > |x|} \frac{f(y)}{|y|^n} dy, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

(33)

Let \(b\) be a locally integrable function on \(\mathbb{R}^n\). The commutators of \(H\) and \(H^*\) are defined by

\[
[b, H](f) = b(Hf) - H(bf)
\]

and

\[
[b, H^*](f) = b(H^*f) - H^*(bf).
\]

(35)

The study of the Hardy operator has a very long history, and a number of papers involved its generalizations, variants, and applications. For the earlier development of this kind of integral and many important applications, we refer the interested reader to the masterpiece [14]. We are interested in the characterization of commutator of the Hardy operator.

Now, we give a remarkable result about the commutator of the Hardy operator; that is, Fu et al. [11] showed the following.

**Theorem 8.** Let \(1 < p < \infty\), \(1/p + 1/p' = 1\), and \(b \in \text{CBMO}^p(\mathbb{R}^n)\). Then, both \([b, H]\) and \([b, H^*]\) are bounded operators on \(L^p(\mathbb{R}^n)\). Conversely,

(i) if \([b, H]\) is a bounded operator on \(L^p(\mathbb{R}^n)\), then \(b \in \text{CBMO}^p(\mathbb{R}^n)\)

(ii) if \([b, H^*]\) is a bounded operator on \(L^p(\mathbb{R}^n)\), then \(b \in \text{CBMO}^p(\mathbb{R}^n)\)

The following consequence improves Theorem 8.

**Theorem 9.** If \(\omega \in \text{A}_p\), \(1 < p < \infty\) and \(\mu = \omega^{1/p'}\). Then, the following statements are equivalent:

(i) \(b \in \text{CBMO}^p(\omega) \cap \text{CBMO}^p(\mu)\)

(ii) \([b, H]\) and \([b, H^*]\) are bounded from \(L^p(\omega)\) to \(L^p(\omega)\)

Proof. (i) \(\Rightarrow\) (ii). We focus on the proof of the boundedness of \([b, H]\), since the arguments of \([b, H^*]\) are similar with necessary modifications.

For \(f \in L^p(\omega)\), we have

\[
\|[b, H](f)\|_{L^p(\omega)} = \sum_{k = -\infty}^{\infty} \|X_k[b, H](f)\|_{L^p(\omega)}
\]

\[
= \sum_{k = -\infty}^{\infty} \|X_k[f] \int_{|y| < |x|} (b(y) - b(y))f(y)dy\|_{L^p(\omega)}
\]

\[
\leq \sum_{k = -\infty}^{\infty} \|X_k[f] \int_{|y| < |x|} |b(y) - b(y)||f(y)dy|\|_{L^p(\omega)}.
\]

(36)

It is easy to see that

\[
\int_{C_y} |b(x) - b(y)||f(y)dy| \leq \int_{C_y} |b(x) - b_k||f(y)dy| + \int_{C_y} |b_k - b(x)||f(y)dy.
\]

(37)
By Hölder’s inequality, we get
\[
\int_{C_j} |b(x) - b_{B_j}| |f(y)| \, dy 
\leq C |b(x) - b_{B_j}| \left( \int_{C_j} |f(y)|^p \omega(y) \, dy \right)^{1/p} \left( \int_{C_j} \omega(y)^{-p'} \, dy \right)^{1/p'} 
\leq C |b(x) - b_{B_j}| \|f_X\|_{L^p(\omega)} \|X_l\|_{L^{p'}(\mu)}.
\] (38)

In [11], Fu et al. showed that for \( b \in \text{CBMO}(R^n) \) and \( j, k \in \mathbb{Z}_+ \),
\[
|b(t) - b_{B_j}| \leq |b(t) - b_{B_j}| + C(j-k)|b|_{\text{CBMO}(R^n)}.
\] (39)

By Proposition 4, for \( b \in \text{CBMO}^p(\omega) \subset \text{CBMO}(R^n) \) and \( j, k \in \mathbb{Z}_+ \), we have
\[
|b(t) - b_{B_j}| \leq |b(t) - b_{B_j}| + Cj - k|b|_{\text{CBMO}^p(\omega)}.
\] (40)

This gives us
\[
\int_{C_j} |b(y) - b_{B_j}| |f(y)| \, dy 
\leq \int_{C_j} |b(y) - b_{B_j}| |f(y)| \, dy + |b_{B_j} - b_{B_j}| \int_{C_j} |f(y)| \, dy 
\leq \int_{C_j} |b(y) - b_{B_j}| |f(y)| \omega(y)^{1/2} \omega(y)^{-1/2} \, dy 
+ C(j-k)|b|_{\text{CBMO}^p(\omega)} \int_{C_j} |f(y)|\omega(y)^{1/2} \omega(y)^{-1/2} \, dy 
\leq C \left( \|b(b-B_j)\|_{L^{p'}(\mu)} \|f_X\|_{L^p(\omega)} \|X_l\|_{L^{p'}(\mu)} 
+ C(k-j)\|b\|_{\text{CBMO}^p(\omega)} \|f_X\|_{L^p(\omega)} \|X_l\|_{L^{p'}(\mu)} 
\leq C\|b\|_{\text{CBMO}^p(\omega)} \|f_X\|_{L^p(\omega)} \|X_l\|_{L^{p'}(\mu)}.
\] (41)

Combing (38) and (41), we get
\[
\left\| X_k(\cdot) \int_{C_j} (b(\cdot) - b(y)) |f(y)| \, dy \right\|_{L^p(\omega)} 
\leq C 2^{-kn} \|X_k\|_{L^p(\omega)} \|f_X\|_{L^p(\omega)} \|X_l\|_{L^{p'}(\mu)}
\] (42)

From the condition \( \omega \in A_p \) and Lemma 2, it follows that for \( k \geq j \), there exists a constant \( \delta \in (0,1) \) such that
\[
2^{-k\delta n} \|X_k\|_{L^p(\omega)} \|f_X\|_{L^p(\omega)} \|X_l\|_{L^{p'}(\mu)} \leq C \left( \frac{|b|_{L^\infty}}{|B_j|^\delta} \right)^{2^{k\delta n} p'} \leq C(\kappa)^{2^{(j-k)\delta n} p'}.
\] (43)

Therefore, generalized Minkowski’s inequality implies
\[
\|b(H')(f)\|_{L^p(\omega)} \leq C \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (k-j)^{2^{(j-k)\delta n} p'} \|f_X\|_{L^p(\omega)} 
\leq C \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (k-j)^{2^{(j-k)\delta n} p'} \|f_X\|_{L^p(\omega)} 
\leq C \|f\|_{L^p(\omega)}.
\] (44)

(ii) \( \Rightarrow \) (i). The condition \( b \in \text{CBMO}^p(\omega) \cap \text{CBMO}^p(\mu) \) turns out to be necessary for the conclusion that both \([b,H]\) and \([b,H^*]\) are bounded on \( L^p(\omega) \).

For any ball \( B = B(x,r) \) and \( x \in B \), we have
\[
|b(x) - b_{B_j}| = \frac{1}{|B|} \int_B (b(x) - b(y)) \, dy 
\leq C \frac{1}{|x|^{n-1}} \int_{|y|<|x|} (b(x) - b(y)) \omega(y) \, dy 
+ C \int_{|y|>|x|} (b(x) - b(y)) \omega(y) \, dy 
\leq C |b(H)(\omega)| + C|b,H^*(f_0)(\omega)|,
\] (45)

where \( f_0 = |x|^n |B|^{-1} \omega(B) \).

From \([b,H]\) and \([b,H^*]\) that are bounded on \( L^q(\omega) \), it follows that
\[
\|b(b-B_j)\|_{L^q(\omega)} \leq C |b(H)(\omega)| + C|b,H^*(f_0)(\omega)| 
\leq C \|X_B\|_{L^q(\omega)} + C|f_0|_{L^q(\omega)} \leq C \|X_B\|_{L^q(\omega)}.
\] (46)

Therefore, we obtain that \( b \) belongs to \( \text{CBMO}^p(\omega) \).

Note that \((L^p(\omega))^\prime = L^{p'}(\omega^{1-p'})\) (see [15]). We know that \([b,H]\) and \([b,H^*]\) are bounded on \( L^{p'}(\mu) \). Therefore, we obtain that \( b \in \text{CBMO}^p(\mu) \).

This completes the proof of Theorem 9. \( \square \)

4. Vector-Valued Inequality

In this section, we give the definition of weighted Herz spaces \([16]\). Let \( \alpha \in \mathbb{R}, \ 0 < p, q < \infty \), and \( \omega \) be weight functions on \( \mathbb{R}^n \). The homogeneous weighted Herz space \( K_p^q(\omega) \) is
defined by
\[
K_p^{\omega q}(\omega) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}); \|f\|_{K_p^{\omega q}(\omega)} < \infty \right\},
\]
where
\[
\|f\|_{K_p^{\omega q}(\omega)} := \left\{ \left( \sum_{k=-\infty}^{\infty} 2^{nk} \|f_k\|_{L^p(\omega)}^q \right)^{1/q} \right\} \quad \text{for all sequences of functions } \{f_k\}_{k=-\infty}^{\infty}.
\]

We prove the boundedness of the vector-valued commutator of the Hardy operator on weighted Herz spaces.

**Theorem 10.** Let \( \omega \in A_p, 1 < r, p < \infty, 0 < q < \infty, \) and \( b \in \text{CBMO}^r(\omega) \cap \text{CBMO}^q(\mu). \)

(i) If \( \alpha < n/lp' \), then there exists a constant \( C \) such that
\[
\| \left( \sum_{j=1}^{\infty} \left\| [b, H] \left( f_j \right) \right\|_{K_p^{\omega q}(\omega)} \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \leq C \left\{ \left( \sum_{j=1}^{\infty} \left\| f_j \right\|_{l^r} \right) \|f_j\|_{K_p^{\omega q}(\omega)} \right\} \quad \text{for all sequences of functions } \{f_j\}_{j=1}^{\infty}.
\]

(ii) If \( \alpha > -nlp \), then there exists a constant \( C \) such that
\[
\left\| \left( \sum_{j=1}^{\infty} \left\| [b, H^*] \left( f_j \right) \right\|_{K_p^{\omega q}(\omega)} \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \leq C \left\{ \left( \sum_{j=1}^{\infty} \left\| f_j \right\|_{l^r} \right) \|f_j\|_{K_p^{\omega q}(\omega)} \right\} \quad \text{for all sequences of functions } \{f_j\}_{j=1}^{\infty}.
\]

In order to prove Theorem 10, we additionally introduce the next lemma well-known as the generalized Minkowski inequality.

**Lemma 11.** If \( 1 < r < \infty \), then there exists a constant \( C > 0 \) such that for all sequences of functions \( \{f_j\}_{j=1}^{\infty} \) satisfying \( \|f_j\|_{K_p^{\omega q}(\omega)} < \infty \),
\[
\left\{ \left( \int_{\mathbb{R}^n} \left| f_j(y) \right| dy \right)^{1/r} \right\} \leq C \left\{ \left( \int_{\mathbb{R}^n} \left| f_j(y) \right|^r dy \right) \right\}^{1/r}.
\]

**Proof of Theorem 10.** We focus on the proof of the boundedness of \([b, H]\), since the arguments of \([b, H^*]\) are similar with necessary modifications. For every \( \{f_j\}_{j=1}^{\infty} \) with \( \|f_j\|_{K_p^{\omega q}(\omega)} < \infty \), we obtain
\[
\left\{ \left( \sum_{j=1}^{\infty} \left\| [b, H] \left( f_j \right) \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \right\} \leq C \left( \sum_{j=1}^{\infty} \left\| f_j \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \right) \quad \text{for all sequences of functions } \{f_j\}_{j=1}^{\infty}.
\]

For convenience, below, we denote \( F := \|f_j\|_{l^r} \). For \( x \in C_k \), generalized Hölder’s inequality and generalized Minkowski’s inequality (51) imply
\[
\left\{ \left( \sum_{j=1}^{\infty} \left\| [b, H] \left( f_j \right) \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \right\} \leq C \left\{ \left( \sum_{j=1}^{\infty} \left\| f_j \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \right) \right\} \quad \text{for all sequences of functions } \{f_j\}_{j=1}^{\infty}.
\]

from the fact that
\[
|b(x) - b_k| \leq |b(t) - b_k| + C|l-k||b||_{\text{CBMO}^r(\omega)},
\]
which gives us
\[
\left\{ \left( \int_{\mathbb{R}^n} \left| f_j(y) \right| dy \right)^{1/r} \right\} \leq C \left\{ \left( \int_{\mathbb{R}^n} \left| f_j(y) \right|^r dy \right) \right\}^{1/r}.
\]

By Lemma 3 and \( a/n < 1/p' \), there exists a constant \( 1 < p_0 < p \) such that \( a/n < 1/p_0' \) and
\[
\left\{ \left( \sum_{j=1}^{\infty} \left\| [b, H] \left( f_j \right) \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \right\} \leq C \left\{ \left( \sum_{j=1}^{\infty} \left\| f_j \right\|_{l^r} \|f_j\|_{K_p^{\omega q}(\omega)} \right) \right\} \quad \text{for all sequences of functions } \{f_j\}_{j=1}^{\infty}.
\]
This completes the proof of Theorem 10.

\[
\|X_h\|_{L^p(w)} \leq \|X_{h0}\|_{L^p(w)} \leq C2^{n(k-l)/p_0}, \quad (56)
\]

Then,

\[
\|X_h\| \|\{b, H\}(f_jX_i)\|_{L^r(w)} \leq C(k-l)2^{n(k-l)/p_0}\|F_{X_i}\|_{L^p(w)}.
\]

This implies that

\[
\|\{b, H\}(f_j)\|_{L^p(w)} \leq C\left(\sum_{k=-\infty}^{\infty} 2^{nk}\left(\sum_{j=-\infty}^{\infty} (k-l)2^{n(k-l)/p_0}\|F_{X_i}\|_{L^p(w)}\right)^{q} \right)^{1/q}.
\]

If \(0 < q \leq 1\), then we obtain

\[
\|\{b, H\}(f_j)\|_{L^p(w)} \leq C\left(\sum_{k=-\infty}^{\infty} 2^{nk}\|F_{X_i}\|_{L^p(w)}2^{(k-l)(1/p_0'-\alpha)}(k-l)\|F\|_{L_p^\alpha(w)}\right)^{1/q}.
\]

If \(1 < q < \infty\), then we use Hölder’s inequality and obtain

\[
\|\{b, H\}(f_j)\|_{L^p(w)} \leq C\left(\sum_{k=-\infty}^{\infty} 2^{nk}\|F_{X_i}\|_{L^p(w)}2^{(k-l)(1/p_0'-\alpha)}(k-l)\|F\|_{L_p^\alpha(w)}\right)^{1/q}.
\]

This completes the proof of Theorem 10.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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