GLOBAL SPHERICALLY SYMMETRIC SOLUTIONS TO DEGENERATE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE DATA AND FAR FIELD VACUUM

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Abstract. We consider the initial-boundary value problem (IBVP) for the isentropic compressible Navier-Stokes equations (CNS) in the domain exterior to a ball in \( \mathbb{R}^d \) (\( d = 2 \) or 3). When viscosity coefficients are given as a constant multiple of the mass density \( \rho \), based on some analysis of the nonlinear structure of this system, we prove the global existence of the unique spherically symmetric classical solution for (large) initial data with spherical symmetry and far field vacuum in some inhomogeneous Sobolev spaces. Moreover, the solutions we obtained have the conserved total mass and finite total energy, \( \rho \) keeps positive in the domain considered but decays to zero in the far field, which is consistent with the facts that the total mass is conserved, and CNS is a model of non-dilute fluids where \( \rho \) is bounded away from the vacuum. To prove the existence, on the one hand, we consider a well-designed reformulated structure by introducing some new variables, which, actually, can transfer the degeneracies of the time evolution and the viscosity to the possible singularity of some special source terms. On the other hand, it is observed that, for the spherically symmetric flow, the radial projection of the so-called effective velocity \( v = U + \nabla \varphi(\rho) \) (\( U \) is the velocity of the fluid, and \( \varphi(\rho) \) is a function of \( \rho \) defined via the shear viscosity coefficient \( \mu(\rho) \): \( \varphi'(\rho) = 2\mu(\rho)/\rho^2 \)), verifies a damped transport equation which provides the possibility to obtain its upper bound. Then combined with the BD entropy estimates, one can obtain the required uniform a priori estimates of the solution. It is worth pointing out that the frame work on the well-posedness theory established here can be applied to the shallow water equations.

1. INTRODUCTION

The time evolution of the mass density \( \rho \geq 0 \) and the velocity \( U = (U^{(1)}, \ldots, U^{(d)})^\top \in \mathbb{R}^d \) of a general viscous isentropic compressible fluid occupying a spatial domain \( \Omega \subset \mathbb{R}^d \) is governed by the following isentropic CNS:

\[
\begin{align*}
\rho_t + \text{div}(\rho U) &= 0, \\
(\rho U)_t + \text{div}(\rho U \otimes U) + \nabla P &= \text{div} T.
\end{align*}
\]

(1.1)

Here, \( x = (x_1, \ldots, x_d)^\top \in \Omega, \ t \geq 0 \) are the space and time variables, respectively. For the polytropic gases, the constitutive relation is given by

\[
P = A\rho^\gamma, \quad A > 0, \quad \gamma > 1,
\]

(1.2)
where $A$ is an entropy constant and $\gamma$ is the adiabatic exponent. $T$ denotes the viscous stress tensor with the form

$$T = 2\mu(\rho)D(U) + \lambda(\rho)\text{div}U I_d,$$

where $D(U) = \frac{1}{2}(\nabla U + (\nabla U)^\top)$ is the deformation tensor, $I_d$ is the $d \times d$ identity matrix,

$$\mu(\rho) = \alpha \rho^\delta, \quad \lambda(\rho) = \beta \rho^\delta,$$

for some constant $\delta \geq 0$, $\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho) + 2d \mu(\rho)$ is the bulk viscosity coefficient, $\alpha$ and $\beta$ are both constants satisfying

$$\alpha > 0 \quad \text{and} \quad 2\alpha + d\beta \geq 0.$$

In the theory of gas dynamics, the CNS can be derived from the Boltzmann equations through the Chapman-Enskog expansion, cf. Chapman-Cowling [8] and Li-Qin [32]. Under some proper physical assumptions, one can find that the viscosity coefficients and heat conductivity coefficient $\kappa$ are not constants but functions of the absolute temperature $\theta$ such as:

$$\mu(\theta) = a_1 \theta^\frac{1}{2} F(\theta), \quad \lambda(\theta) = a_2 \theta^\frac{1}{2} F(\theta), \quad \kappa(\theta) = a_3 \theta^\frac{1}{2} F(\theta),$$

for some constants $a_i$ ($i = 1, 2, 3$) (see [8]). Actually for the cut-off inverse power force models, if the intermolecular potential varies as $\ell^{-\kappa}$, where $\ell$ is intermolecular distance and $\kappa$ is a positive constant, then in (1.6):

$$F(\theta) = \theta^b \quad \text{with} \quad b = \frac{2}{\kappa} \in [0, \infty).$$

In particular (see §10 of [8]), for ionized gas, $\kappa = 1$ and $b = 2$; for Maxwellian molecules, $\kappa = 4$ and $b = \frac{1}{2}$; while for rigid elastic spherical molecules, $\kappa = \infty$ and $b = 0$.

According to Liu-Xin-Yang [34], for isentropic and polytropic fluids, such a dependence is inherited through the laws of Boyle and Gay-Lussac:

$$P = R \rho \theta = A \rho^\gamma$$

for constant $R > 0$, i.e., $\theta = A R^{-1} \rho^{\gamma-1}$, and one can see that the viscosity coefficients are functions of $\rho$ taking the form (1.4). Actually, there do exist some physical models that satisfy the density-dependent viscosities assumption (1.4), such as Korteweg system, shallow water equations, lake equations and quantum Navier-Stokes system (see [3, 4, 5, 6, 16, 26, 37]). In particular, the viscous shallow water model in two-dimensional (2-D) space reads as

$$\begin{cases} h_t + \text{div}(hW) = 0, \\
(hW)_t + \text{div}(hW \otimes W) + \nabla h^2 = \mathcal{V}(h,W), \end{cases}$$

where $h$ denotes the height of the free surface, $W = (W^{(1)}, W^{(2)})^\top \in \mathbb{R}^2$ is the mean horizontal velocity of the fluid. $\mathcal{V}(h,W)$ is the viscous term. There are several different viscous terms imposed, such as

$$\text{div}(hD(W)), \quad \text{div}(h \nabla W), \quad h \Delta W, \quad \Delta(hW).$$

In particular, the case for $\mathcal{V} = \text{div}(h \nabla W)$ is corresponding to the well-known viscous Saint-Venant model. The derivation of Gent [16] suggests $\mathcal{V} = \text{div}(hD(W))$. A more
recent careful derivation by Marche [37] and Bresch-Noble [5, 6] suggests that
\[ V(h, W) = \text{div}(2hD(W) + 2h\text{div}W) . \]

In the current paper, let \( \Omega = \{ x \in \mathbb{R}^d \mid |x| > a \} \) (\( a > 0 \) is a constant) be an exterior domain in \( \mathbb{R}^d \) (\( d = 2 \) or 3), and \((\mu(\rho), \lambda(\rho))\) in (1.4) satisfy
\[
\mu(\rho) = \alpha \rho, \quad \lambda(\rho) = 0. \tag{1.8}
\]
We are concerned with global spherically symmetric (smooth) solutions taking the form
\[
(\rho, U)(t, x) = (\rho(t, |x|), u(t, |x|) \frac{x}{|x|}) \tag{1.9}
\]
of the equations (1.1)-(1.5) in the domain \( \Omega \) with the initial data:
\[
(\rho, U)(0, x) = (\rho_0, U_0)(x) = (\rho_0(|x|), u_0(|x|) \frac{x}{|x|}) \quad \text{for} \quad x \in \Omega, \tag{1.10}
\]
and the following boundary conditions and far field behavior:
\[
U(t, x)|_{|x|=a} = 0 \quad \text{for} \quad t \geq 0, \quad \left(\rho(t, x), U(t, x)\right) \to (0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0. \tag{1.11}
\]

There is a lot of literature on the global well-posedness of smooth solutions to the IBVP and Cauchy problem of (1.1). For constant viscous flows \((\delta = 0 \text{ in (1.4)})\), when \( \inf_x \rho_0(x) > 0 \), the global well-posedness of strong solutions with arbitrarily large data in some bounded, one-dimensional (1-D) domains has been proven by Kazhikhov-Shelukhin [29], and later, Kawashima-Nishida [28] extended this theory to the unbounded domains. In \( \mathbb{R}^3 \), Matsumura-Nishida [38] obtained a unique global classical solution for initial data close to a non-vacuum equilibrium in some Sobolev space \( H^s(\mathbb{R}^3) \) \((s > \frac{5}{2})\) (see also Danchin [12] in some critical spaces). In particular, Jiang [24] obtained the global existence of spherically symmetric smooth solutions for (large) initial data with spherical symmetry to the non-isentropic flow in the domain exterior to a ball in \( \mathbb{R}^d \) \((d = 2, 3)\). However, these approaches do not work when vacuum appears (i.e., \( \inf_x \rho_0(x) = 0 \)), which occurs when some physical requirements are imposed, such as finite total initial mass and energy in the whole space. One of the main issues in the presence of vacuum is the degeneracy of the time evolution operator, which makes it hard to understand the behavior of \( U \) near the vacuum. A remedy was suggested by Cho-Choe-Kim [9] in three-dimensional (3-D) space, where they imposed initially a compatibility condition:
\[
-\text{div} T_0 + \nabla P(\rho_0) = \sqrt{\rho_0} g \quad \text{for some} \quad g \in L^2(\mathbb{R}^3),
\]
which leads to \( \sqrt{\rho} U_t \in L^\infty([0, T_*]; L^2(\mathbb{R}^3)) \) for some time \( T_* > 0 \), and they established the local well-posedness of strong solutions with vacuum, which, recently, has been shown to be a global one with small energy by Huang-Li-Xin [23] in \( \mathbb{R}^3 \). Later, Jiu-Li-Ye [25] proved that the 1-D Cauchy problem of (1.1) admits a unique global classical solution with arbitrarily large data and vacuum. Some interesting progress on the global spherically symmetric strong solutions in annular or exterior domains of \( \mathbb{R}^d \) \((d \geq 2)\) can also be found in Choe-Kim [10], Ding-Wen-Yao-Zhu [13] and so on. We also refer Feireisl [14], Hoff [21], Hoff-Jenssen [22], Lions [33] to readers and the references therein for the existence theory of weak solutions with large data in multi-dimensional (M-D) space.
For degenerate viscous flow (\(\delta > 0\) in (1.4)), when \(\inf_x \rho_0(x) > 0\), some important progress on the global well-posedness of smooth solutions to the IBVP and Cauchy problem of (1.1) have been obtained, which include Constantin-Drivas-Nguyen-Pasqualotto [11], Haspot [19], Kang-Vasseur [27], Mellet-Vasseur [39] for 1-D flow with arbitrarily large data, and Sundbye [42], Wang-Xu [44] for initial data close to a non-vacuum equilibrium in some Sobolev spaces \(H^s(\mathbb{R}^2)\) \((s > 2)\). When vacuum appears, instead of the uniform elliptic structure in the constant viscous flow, the viscosity degenerates when density vanishes, which raises the difficulty of the problem to another level, i.e.,

\[
\rho(U_t + U \cdot \nabla U) + \nabla P = \text{div}(2\mu(\rho)D(U) + \lambda(\rho)\text{div}U I_d).
\]

(1.12)

Degenerate time evolution operator

Degenerate elliptic operator

Usually, it is very hard to control the behavior of the fluids velocity \(U\) near the vacuum. A remarkable discovery of a new mathematical entropy function was made by Bresch-Desjardins [3] for \(\lambda(\rho)\) and \(\mu(\rho)\) satisfying the relation

\[
\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)),
\]

(1.13)

which offers an estimate

\[
\mu'(\rho)\sqrt{\rho} \in L^\infty([0, T]; L^2(\mathbb{R}^d))
\]

provided that \(\mu'(\rho_0)\sqrt{\rho_0} \in L^2(\mathbb{R}^d)\) for any \(d \geq 1\). This observation plays an important role in the development of the global existence of weak solutions with vacuum for system (1.1) and some related models, see Bresch-Desjardins [3], Bresch-Vasseur-Yu [7], Guo-Jiu-Xin [17], Guo-Li-Xin [18], Li-Xin [31], Vasseur-Yu [43] and so on. However, the regularities and uniqueness of such weak solutions with vacuum remain open. Recently, for the cases \(\delta > 1\) in (1.4), the global well-posedness of regular solutions with vacuum for a class of smooth initial data that are of small density but possibly large velocities in some homogeneous Sobolev spaces has been established by Xin-Zhu [45]. Some other interesting results can also be seen in Haspot [20], Luo-Xin-Zeng [35], Yang-Zhao [46] and so on.

In the current paper, we establish the global existence and uniqueness of spherically symmetric smooth solutions for (large) initial data with spherical symmetry and far field vacuum in some inhomogeneous Sobolev spaces to the IBVP (1.1)-(1.5) with (1.10)-(1.11) in the domain \((\Omega = \{x \in \mathbb{R}^d | |x| > a\})\) exterior to a ball in \(\mathbb{R}^d\) \((d = 2\) or 3\)). Notice that, under the assumption (1.8), if \(\rho > 0\), (1.1) \(_2\) can be formally rewritten as

\[
U_t + U \cdot \nabla U + \frac{A_\gamma}{\gamma - 1} \nabla \rho^{\gamma - 1} + LU = \nabla \ln \rho \cdot Q(U),
\]

(1.14)

where the quantities \(LU\) and \(Q(U)\) are given by

\[
LU = -\alpha \Delta U - \alpha \nabla \text{div}U, \quad Q(U) = \alpha(\nabla U + (\nabla U)^\top).
\]

(1.15)

Therefore, the two quantities

\[
(\rho^{\gamma - 1}, \nabla \ln \rho)
\]

will play significant roles in our analysis on the high order regularities of the fluid velocity \(U\). Due to this observation, we first introduce a proper class of solutions called regular solutions to the IBVP (1.1)-(1.5) with (1.10)-(1.11).
Theorem 1.1. Assume the physical parameters $(\delta, \gamma, \alpha, \beta)$ satisfy
\begin{equation}
\delta = 1, \quad \gamma > \frac{3}{2}, \quad \alpha > 0 \quad \text{and} \quad \beta = 0.
\end{equation}
Let $q \in (3,6]$ be some constant, and the initial data $(\rho_0(x), U_0(x)) = (\rho_0(|x|), u_0(|x|)x/|x|)$ be spherically symmetric and satisfy
\begin{align}
0 < \rho_0(x) &\in L^1(\Omega), \quad (\rho_0^{-1}(x), U_0(x)) \in H^2(\Omega), \\
\nabla \ln \rho_0(x) &\in L^2(\Omega) \cap L^\infty(\Omega) \cap D^1(\Omega).
\end{align}
Then the IBVP (1.1)-(1.5) with (1.10)-(1.11) admits a unique global classical solution $(\rho(t,x), U(t,x))$ in $[0,\infty) \times \Omega$, and for any $0 < T < \infty$, $(\rho(t,x), U(t,x))$ is a regular one in $[0,T] \times \Omega$ as defined in Definition 1.1, and
\begin{align}
\rho &\in C([0,T]; L^1(\Omega)), \quad (\rho^{-1})_t \in C([0,T]; H^1(\Omega)), \\
(\nabla \ln \rho)_t &\in C([0,T]; L^2(\Omega)), \quad \nabla \ln \rho \in L^\infty([0,T] \times \Omega), \\
t^{\frac{\gamma}{2}} U_t &\in L^\infty([0,T]; D^1(\Omega)) \cap L^2([0,T]; D^2(\Omega)), \quad t^{\frac{\gamma}{2}} U_{tt} \in L^2([0,T]; L^2(\Omega)).
\end{align}
Moreover, $(\rho(t,x), U(t,x))$ is also a spherically symmetric one taking the form (1.9).

Remark 1.1. First, observe that we do not need any smallness assumption on $(\rho_0, U_0)$.
Second, one can find the following class of spherically symmetric initial data $(\rho_0, U_0)$ satisfying the condition (1.17):
\begin{equation}
\rho_0(x) = \frac{1}{1 + |x|^{2\gamma}}, \quad U_0(x) = u_0(|x|)\frac{x}{|x|} \in C^2_0(\Omega),
\end{equation}
for some $\sigma > \max \left\{ \frac{d}{2}, \frac{\delta}{\delta - 1} \right\} = \frac{d}{2}$, and $u_0(r) \in C^2_0([a,\infty))$ with $u_0(a) = 0$.

Another purpose of this article is to establish the well-posedness with large data applicable to most of physically relevant models in shallow water theory. Our framework for system (1.7) is applicable to the viscous terms of the forms $\text{div}(hD(W))$, $\text{div}(2hD(W))$ and $\text{div}(h\nabla W)$. Actually, when $\nu = \text{div}(hD(W))$, this is a special case of system (1.1) with $\alpha = \frac{1}{2}$, $\beta = 0$, $\delta = 1$ and $\gamma = 2$; when $\nu = \text{div}(2hD(W))$, this is also a special case of system (1.1) with $\alpha = 1$, $\beta = 0$, $\delta = 1$ and $\gamma = 2$. Therefore, one simply replaces $(\rho, U)$
by \((h, W)\) in Theorem 1.1 to obtain the same conclusion for these two classes of shallow water models, without further modifications. For the third case: \(V = \text{div}(h\nabla W)\), since the key information: BD entropy estimates (see Lemma 3.3) in the proof of Theorem 1.1 also holds, then one can still obtain the desired global-in-time well-posedness from our framework with minor changes. More precisely, considering the IBVP of system (1.7) with the initial data:

\[
(h, W)(0, x) = (h_0, W_0)(x) = (h_0(|x|), w_0(|x|) \frac{x}{|x|}) \quad \text{for} \quad x \in \Omega, \tag{1.19}
\]

and the following boundary conditions and far field behavior:

\[
W(t, x)|_{x=a} = 0 \quad \text{for} \quad t \geq 0, \quad (h(t, x), W(t, x)) \to (0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0,
\]

one has the following theorem.

**Theorem 1.2.** Assume \(V = \text{div}(hD(W)), \text{div}(2hD(W)), \) or \(\text{div}(h\nabla W)\). Let the initial data \((h_0(x), W_0(x)) = (h_0(|x|), w_0(|x|)x/|x|)\) be spherically symmetric and satisfy

\[
0 < h_0(x) \in L^1(\Omega), \quad (h_0(x), W_0(x)) \in H^2(\Omega), \quad \nabla \ln h_0(x) \in L^q(\Omega) \cap L^\infty(\Omega) \cap D^1(\Omega),
\]

for some constant \(q \in (3, 6]\). Then the IBVP (1.7) with (1.19)-(1.20) admits a unique global classical solution \((h(t, x), W(t, x))\) in \((0, \infty) \times \Omega\) satisfying for any \(0 < T < \infty\),

\[
\inf_{x \in \Omega} h(t, x) = 0 \quad \text{for} \quad 0 \leq t \leq T, \quad 0 < h \in C([0, T]; L^1(\Omega) \cap H^2(\Omega)),
\]

\[
\nabla \ln h \in C([0, T]; L^q(\Omega) \cap D^1(\Omega)) \cap L^\infty([0, T] \times \Omega),
\]

\[
W \in C([0, T]; H^2(\Omega)) \cap L^2([0, T]; D^1(\Omega)), \quad h_t \in C([0, T]; H^1(\Omega)),
\]

\[
(\nabla \ln h)_t \in C([0, T]; L^2(\Omega)), \quad W_t \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; D^1(\Omega)), \quad t^2 W_{tt} \in L^2([0, T]; L^2(\Omega)).
\]

Moreover, \((h(t, x), W(t, x))\) is also a spherically symmetric one taking the form:

\[
(h, W)(t, x) = (h(t, |x|), w(t, |x|) \frac{x}{|x|}).
\]

The rest of the paper is organized as follows: §2 is devoted to establishing the local-in-time well-posedness of regular solutions in the M-D Eulerian coordinate to the IBVP (1.1)-(1.5) with (1.10)-(1.11). Here we need to consider a well-designed reformulated structure by introducing some new variables, which, actually, can transfer the degeneracies of the time evolution and the viscosity shown in (1.12) to the possible singularity of some special source terms. In §3, we prove the global-in-time well-posedness of regular solutions in the spherically symmetric Eulerian coordinate to the reformulated IBVP (3.2) (see Theorem 3.1) by deriving some a priori estimates globally in time. Here, we have employed some arguments motivated by Bresch-Desjardins [3] and Bresch-Desjardins-Lin [4] to deal with the strong degeneracy of the density-dependent viscosity \(\mu(\rho) = \alpha \rho^\delta\), which include the well-known BD entropy estimates (see Lemma 3.3) and the effective velocity (see Lemma 3.8). In §4, we see that the global well-posedness asserted in Theorem 1.1 can be obtained by means of Theorem 3.1. We also give a brief proof for
Theorem 2.1. Finally, we give two appendixes to list some lemmas that are frequently used in our proof, and the conversion of some Sobolev spaces between the pure M-D coordinate and the spherically symmetric coordinate.

2. Local-in-time well-posedness

This section is devoted to showing the local-in-time existence of the unique regular solution to the IBVP (1.1)-(1.5) with (1.10)-(1.11) in M-D spaces. Moreover, we show that if the initial data is spherically symmetric, so is the corresponding multi-dimensional solution to the IBVP (1.1)-(1.5) with (1.10)-(1.11) in positive time. Throughout this section, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:

\[
\begin{align*}
L^p &= L^p(\Omega), & H^s &= H^s(\Omega), & D^{k,l} &= D^{k,l}(\Omega), & D^k &= D^{k,2}(\Omega), \\
W^{m,p} &= W^{m,p}(\Omega), & |f|_p &= \|f\|_{L^p(\Omega)}, & \|f\|_s &= \|f\|_{H^s(\Omega)}, & \|f\|_{m,p} &= \|f\|_{W^{m,p}(\Omega)}, \\
|f|_{D^{k,l}} &= \|\nabla^k f\|_{L^2(\Omega)}, & |f|_{D^k} &= \|\nabla^k f\|_{L^2(\Omega)}, & \|f\|_{X_1 \cap X_2} &= \|f\|_{X_1} + \|f\|_{X_2},
\end{align*}
\]

where \(\Omega = \{x \in \mathbb{R}^d | x > a\}\) is the domain exterior to a ball in \(\mathbb{R}^d (d = 2 \text{ or } 3)\).

The main theorem of this section can be stated as follows.

**Theorem 2.1.** Assume \(d = 2 \text{ or } 3\), and (1.16) holds except \(\gamma > \frac{3}{2}\). If the initial data \((\rho_0, U_0)\) satisfy

\[
0 < \rho_0 \in L^1, \quad (\rho_0^{-1}, U_0) \in H^2, \quad \nabla \ln \rho_0 \in L^q \cap D^1, \tag{2.1}
\]

for some \(q \in (3, 6)\), then there exist a time \(T_* > 0\) and a unique regular solution \((\rho, U)(t, x)\) in \([0, T_*] \times \Omega\) to the IBVP (1.1)-(1.5) with (1.10)-(1.11) which satisfies (1.18) with \(T\) replaced by \(T_*\). Moreover, if \((\rho_0, U_0)\) are spherically symmetric in the sense of (1.10), then the solution \((\rho, U)\) is also a spherically symmetric one taking the form (1.9).

Next we only give the proof of Theorem 2.1 in 3-D space in the following Sections 2.1-2.5, and the 2-D case can be dealt with via the completely same argument. We first make a reformulation for the IBVP (1.1)-(1.5) with (1.10)-(1.11).

2.1. Reformulation. Via introducing the following new variables

\[
\phi = \frac{A_\gamma}{\gamma - 1} \rho^{\gamma - 1}, \quad \psi = \frac{1}{\gamma - 1} \nabla \ln \phi = \nabla \ln \rho, \tag{2.2}
\]

the IBVP (1.1)-(1.5) with (1.10)-(1.11) can be rewritten as

\[
\begin{align*}
\phi_t + U \cdot \nabla \phi + (\gamma - 1) \phi \text{div} U &= 0, \\
U_t + U \cdot \nabla U + \nabla \phi + LU &= \psi \cdot Q(U), \\
\psi_t + \sum_{l=1}^3 A_l(U) \partial_l \psi + B(U) \psi + \nabla \text{div} U &= 0, \\
(\phi(0, x), U(0, x), \psi(0, x)) &= \left(\frac{A_\gamma}{\gamma - 1} \rho_0^{\gamma - 1}, U_0, \nabla \ln \rho_0\right) \quad \text{for} \quad x \in \Omega, \\
U(t, x)|_{x = a} &= 0 \quad \text{for} \quad t \geq 0, \\
(\phi(t, x), U(t, x), \psi(t, x)) &\to (0, 0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0,
\end{align*}
\]
where $A_l(U) = (a^{(l)}_{ij})_{3 \times 3}(i,j,l = 1,2,3)$ are symmetric with $a^{(l)}_{ij} = U^{(l)}$ for $i = j$; otherwise $a^{(l)}_{ij} = 0$, and $B(U) = (\nabla U)^\top$. In order to solve the IBVP (1.1)-(1.5) with (1.10)-(1.11) locally in time, we need to prove the following theorem.

**Theorem 2.2.** Assume $d = 3$, and (1.16) holds except $\gamma > \frac{3}{2}$. If the initial data $(\phi_0(x),U_0(x),\psi_0(x)) = (\phi(0,x),U(0,x),\psi(0,x))$ satisfy (2.1), then there exist a time $T_* > 0$ and a unique strong solution $(\phi,U,\psi)$ in $[0, T_*] \times \Omega$ to the IBVP (2.3), satisfying

\[
0 < \phi \in C([0,T_*]; L^\frac{1}{\gamma-1} \cap H^2), \quad \phi_t \in C([0,T_*]; H^1),
\]

\[
\psi \in C([0,T_*]; L^q \cap D^1), \quad \psi_t \in C([0,T_*]; L^2),
\]

\[
U \in C([0,T_*]; H^2) \cap L^2([0,T_*]; D^3), \quad U_t \in C([0,T_*]; L^2) \cap L^2([0,T_*]; D^1),
\]

\[
t^2 U_t \in L^\infty([0,T_*]; D^1) \cap L^2([0,T_*]; D^2),
\]

\[
t^3 U_{tt} \in L^2([0,T_*]; L^2).
\]

2.2. Linearization. We begin the proof of Theorem 2.2 by considering the following linearized problem of $(\phi,U,\psi)$:

\[
\begin{cases}
\phi_t + V \cdot \nabla \phi + (\gamma - 1)\phi \text{div} V = 0, \\
U_t + V \cdot \nabla V + \nabla \phi + LU = \psi \cdot Q(V), \\
\psi_t + \sum_{l=1}^{3} A_l(V) \partial_i \psi + B(V) \psi + \nabla \text{div} V = 0, \\
(\phi(0,x),U(0,x),\psi(0,x)) = \left(\frac{A_1}{\gamma - 1} \rho_0^{-1}, U_0, \nabla \ln \rho_0\right) \quad \text{for} \quad x \in \Omega, \\
U(t,x)|_{x=a} = 0 \quad \text{for} \quad t \geq 0, \\
(\phi(t,x),U(t,x),\psi(t,x)) \rightarrow (0,0,0) \quad \text{as} \quad |x| \rightarrow \infty \quad \text{for} \quad t \geq 0,
\end{cases}
\]

where $V = (V^{(1)}, V^{(2)}, V^{(3)})^\top \in \mathbb{R}^3$ is a known vector satisfying for any $T > 0$,

\[
V(t = 0, x) = U(0,x), \quad V(t,x)|_{x=a} = 0 \quad \text{for} \quad 0 \leq t \leq T,
\]

\[
V \in C([0,T]; H^2) \cap L^2([0,T]; D^3), \quad V_t \in C([0,T]; L^2) \cap L^2([0,T]; D^1),
\]

\[
t^2 V_t \in L^\infty([0,T]; D^1) \cap L^2([0,T]; D^2), \quad t^3 V_{tt} \in L^2([0,T]; L^2).
\]

Next, the following global well-posedness of strong solutions $(\phi,U,\psi)$ to (2.5) can be obtained by classical arguments shown in ([9, 30, 36]).

**Lemma 2.1.** Assume $d = 3$, (1.16) holds except $\gamma > \frac{3}{2}$ and $T > 0$ is an arbitrarily large time. If the initial data $(\phi_0(x), U_0(x), \psi_0(x)) = (\phi(0,x), U(0,x), \psi(0,x))$ satisfy (2.1), then there exists a unique strong solution $(\phi,U,\psi)$ in $[0, T] \times \Omega$ to (2.5) which satisfies the regularities in (2.4) with $T_*$ replaced by $T$.

2.3. A priori estimates. Let $(\phi,U,\psi)$ be a strong solution in $[0,T] \times \Omega$ obtained in Lemma 2.1, and we will establish the corresponding a priori estimates later. For this purpose, we first choose a positive constant $c_0$ such that

\[
1 + \|\phi_0\|_{L^\infty([0,T];H^2)} + \|U_0\|_{L^2} + \|\psi_0\|_{L^2 \cap D^1} \leq c_0.
\]
We assume there exist some time \( T^* \in (0, T) \) and constants \( c_i \) for \( i = 1, 2, 3 \) such that
\[
1 < c_0 \leq c_1 \leq c_2 \leq c_3,
\]
and
\[
\sup_{0 \leq t \leq T^*} \| V(t) \|^2 + \int_0^{T^*} \left( \| \nabla V(t) \|^2 + |V_t(t)|^2 \right) dt \leq c_1^2,
\]
\[
\sup_{0 \leq t \leq T^*} \left( |V(t)|^2 + |V_t(t)|^2 \right) + \int_0^{T^*} \left( |V(t)|^2_{D^3} + |V_t(t)|^2_{D^1} \right) dt \leq c_2^2,
\]
\[
\text{ess sup}_{0 \leq t \leq T^*} t |V_t(t)|^2_{D^1} + \int_0^{T^*} t \left( |V_t(t)|^2_{D^2} + |V_{tt}(t)|^2 \right) dt \leq c_3^2,
\]
where \( T^* \) and \( c_i \) will be determined later, and depend only on \( c_0 \) and the fixed constants \( A, \alpha, \gamma, T \). In the rest of §2, we use \( C \geq 1 \) to denote a generic constant depending only on fixed constants \( A, \alpha, \gamma, T \) and may be different from line to line.

2.3.1. The a priori estimates for \( \phi \). We first give the estimates for \( \phi \).

**Lemma 2.2.**

\[
\| \phi(t) \|_{L^{\frac{1}{2}}(\Omega)} \leq Cc_0, \quad |\phi_t(t)|_2 \leq Cc_0c_1, \quad |\phi_t(t)|_{D^1} \leq Cc_0c_2
\]
for \( 0 \leq t \leq T_1 = \min\{T^*, (1 + Cc_3)^{-2}\} \).

**Proof.** It follows from (2.8) and standard arguments for transport equation that
\[
\| \phi(t) \|_{L^{\frac{1}{2}}(\Omega)} \leq \| \phi_0 \|_{L^{\frac{1}{2}}(\Omega)} \exp \left( C \int_0^t \| V(s) \|_{L^2} ds \right) \leq Cc_0,
\]
for \( 0 \leq t \leq T_1 \), which, along with (2.5), Hölder’s inequality and (2.9), yields that
\[
|\phi_t(t)|_2 \leq Cc_0c_1, \quad |\phi_t(t)|_{D^1} \leq Cc_0c_2 \quad \text{for } 0 \leq t \leq T_1.
\]

The proof of Lemma 2.2 is complete. \( \square \)

2.3.2. The a priori estimates for \( \psi \). Now we establish the estimates for \( \psi \).

**Lemma 2.3.**

\[
\| \psi(t) \|_{L^1(\Omega)} \leq Cc_0, \quad |\psi_t(t)|_2 \leq Cc_0c_2 \quad \text{for } 0 \leq t \leq T_1.
\]

**Proof.** First, multiplying (2.5) by \( q|\psi|^{q-2}\psi \) and integrating over \( \Omega \), one has
\[
\frac{d}{dt} |\psi|_q^q \leq C \left( |\nabla V|_\infty |\psi|_q^q + |\nabla^2 V|_q |\psi|_q^{q-1} \right) \leq C \left( |\nabla V|_\infty |\psi|_q^q + ||\nabla^2 V||_1 |\psi|_q^{q-1} \right),
\]
which, along with (2.8) and the Gronwall inequality, yields that
\[
|\psi(t)|_q \leq Cc_0 \quad \text{for } 0 \leq t \leq T_1.
\]

Second, let \( \varsigma = (\varsigma_1, \varsigma_2, \varsigma_3)^T \) (\( |\varsigma| = 1 \) and \( \varsigma_i = 0, 1 \)). Applying \( \partial_\varsigma^2 \psi \) to (2.5) and then integrating over \( \Omega \), one can get
\[
\frac{d}{dt} |\nabla \psi|_2 \leq C \left( |\nabla V|_\infty |\nabla \psi|_2 + ||\nabla^2 V||_1 \right),
\]
which, along with (2.8) and the Gronwall inequality, yields that
which, along with the Gronwall inequality, yields that
\[ |\psi(t)|_{D^1} \leq Cc_0 \quad \text{for} \quad 0 \leq t \leq T_1. \] (2.12)

Finally, it follows from the equations (2.5), (2.11) and (2.12) that for \(0 \leq t \leq T_1\),
\[ |\psi_t|_2 \leq C(|\nabla V|_q |\psi_q| + |V|_\infty |\nabla \psi|_2 + |\nabla^2 V|_2) \leq Cc_0 c_2, \]
where \(q^* = \frac{2q}{q-2} \in [3,6).\)

The proof of Lemma 2.3 is complete. \[ \square \]

2.3.3. The a priori estimates for \(U\).

**Lemma 2.4.**
\[
\|U(t)\|_2^2 + |U_t(t)|_2^2 + \int_0^t (\|\nabla U(s)\|_2^2 + |U_t(s)|_2^2 + |U_{tt}(s)|_{D^1}^2) \, ds \leq Cc_0^2,
\]
\[
|U(t)|_{D^2}^2 + t|U_t(t)|_{D^1}^2 + \int_0^t (|U(s)|_{D^3}^2 + s|U_t(s)|_2^2 + s|U_{tt}(s)|_{D^2}^2) \, ds \leq Cc_1^3 c_2^4,
\]
for \(0 \leq t \leq T_2 = \min \{T^*, (1 + Cc_3) - \frac{a_0}{r^*}\} \).

**Proof.**

**Step 1:** estimate on \( |U|_2 \). Multiplying (2.5) by \( U \) and integrating over \( \Omega \), then
\[
\frac{1}{2} \frac{d}{dt} |U|_2^2 + \alpha |\nabla U|_2^2 + \alpha |\text{div} U|_2^2 = \int_\Omega (\nabla V \cdot \nabla U + \nabla \psi \cdot Q(V)) \cdot U \, dx \leq C(|V|_{\infty} |\nabla V|_2 + |\nabla \psi|_2 + |\psi_q| |\nabla V|_q) |U|_2 \leq C|U|_2^2 + Cc_4^2,
\] (2.13)
which, along with the Gronwall inequality, yields that
\[
|U(t)|_2^2 + \int_0^t |U(s)|_{D^1}^2 \, ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2. \] (2.14)

**Step 2:** estimate on \( |\nabla U|_2 \). Multiplying (2.5) by \( U_t \) and integrating over \( \Omega \), one gets
\[
\frac{d}{dt} (\alpha |\nabla U|_2^2 + \alpha |\text{div} U|_2^2) + |U_t|_2^2 = \int_\Omega (\nabla V \cdot \nabla U + \nabla \psi \cdot Q(V)) \cdot U_t \, dx \leq C(|V|_{\infty} |\nabla V|_2 + |\nabla \psi|_2 + |\psi_q| |\nabla V|_q) |U_t|_2 \leq Cc_4^2 + \frac{1}{4}|U_t|_2^2,
\] (2.15)
which, along with the Gronwall inequality, implies that
\[
|U(t)|_{D^1}^2 + \int_0^t |U_t(s)|_2^2 \, ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2. \] (2.16)

It follows from Lemma A.5 (Appendix A) that
\[
|U|_{D^2} \leq C(|U|_{D^1} - U_t \cdot V \cdot \nabla U - \nabla \phi + \psi \cdot Q(V)|_2 + |U|_{D^3}) \leq C(|U_t|_2 + |V|_{\infty} |\nabla V|_3 + |\nabla \phi|_2 + |\psi_q| |\nabla V|_q + |U|_{D^3}) \leq C(|U_t|_2 + c_1^2 + c_2^3 c_4^2),
\] (2.17)
where one has used the fact that (see Lemma A.1 (Appendix A))
\[
|f|_3 \leq |f|_{2}^{1/2} |f|_{\sigma}^{3/2} \leq C |f|_{2}^{3/4} |f|_{1}^{1/4} \leq C(|f|_2 + |f|^2 |\nabla f|_2),
\]
\[
|f|_{q^*} \leq |f|_{2}^{q^*} |f|_{\sigma}^{q^*} \leq C |f|_{2}^{q^*} |f|_{1}^{q^*} \leq C(|f|_2 + |f|^3 |\nabla f|_2). \] (2.18)
Consequently, it follows from (2.16)-(2.17) that
\[ \int_0^t |U(s)|^2_D^2 ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2. \]

**Step 3:** estimate on $|U|_{D^2}$. Differentiating (2.5)$_2$ with respect to $t$, multiplying the resulting equation by $U_t$ and integrating over $\Omega$, one has
\[
\frac{1}{2} \frac{d}{dt} |U_t|^2_2 + \alpha |\nabla U_t|^2_2 + \alpha |\text{div}U_t|^2_2 = \int_\Omega \left( -(V \cdot \nabla)V_t - \nabla \phi_t + (\psi \cdot Q(V))_t \right) \cdot U_t dx \\
\leq C \left( |V|_\infty |\nabla V_t|_2 + |V_t|_6 |\nabla V|_3 \right) |U_t|_2 + |\phi_t|_2 |\nabla U_t|_2 \\
+ |\psi_q| |\nabla V_t|_2 |U_t|_{q^*} + |\psi_t|_2 |\nabla V|_6 |U_t|_3 \right) \right) \\
\leq C \left( 1 + c_1^2 + c_2^2 \right) |U_t|^2_2 + \frac{1}{c_2^2} |V_t|^2_2 + \frac{1}{4} |\nabla U_t|^2_2 + Cc_0^2, \tag{2.19} \right.
\]
where one has used (2.18) for dealing with the terms $(|U_t|_{q^*}, |U_t|_3)$. Then integrating (2.19) over $(\tau, t) (t \in (0, t))$, one arrives at
\[
|U_t(t)|^2_2 + \int_\tau^t \left| \nabla U_t(s) \right|^2_2 ds \leq |U_t(\tau)|^2_2 + C \int_\tau^t \left( 1 + c_1^2 + c_2^2 \right) |U_t(s)|^2_2 ds + Cc_0^2 t + C. \tag{2.20} \right.
\]
It follows from the equations (2.5)$_2$, the continuity of $(\phi, U, \psi)$ and (2.6)-(2.7) that
\[
\limsup_{\tau \to 0} |U_t(\tau)|_2 \leq C \left( |U_0|_\infty |\nabla U_0|_2 + |\nabla \phi_0|_2 + |\nabla^2 U_0|_2 + |\psi_0|_q |\nabla U_0|_{q^*} \right) \leq Cc_0^2. \tag{2.21} \right.
\]
Letting $\tau \to 0$ in (2.20) and using the Gronwall inequality, one can get
\[
|U_t(t)|^2_2 + \int_0^t |U_t(s)|^2_2 ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2. \tag{2.22} \right.
\]
It thus follows from (2.17) that
\[
|U(t)|_{D^2} \leq C \left( |U_t(t)|_2 + c_1^2 + c_1^2 c_2^2 \right) \leq Cc_1^2 c_2^2 \quad \text{for} \quad 0 \leq t \leq T_2. \right.
\]
Finally, according to Lemma A.5 (Appendix A), one has
\[
|U|_{D^3} \leq C \left( |U_t|_2 + |V|_\infty |\nabla V|_1 + |\nabla \phi|_2 + |\nabla^2 V|_1 |\nabla \phi|_1 \right) \\
+ |\psi_q| |\nabla V|_{1, q^*} + |\nabla \psi_t|_2 |\nabla V|_\infty + |U|_{D^1} \right) \leq C \left( |\nabla U_t|_2 + c_1^2 + c_0 |V|_{D^3} \right), \tag{2.23} \right.
\]
which, along with (2.22), implies that
\[
\int_0^t |U(s)|^2_{D^3} ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2. \right.
\]

**Step 4:** time-weighted estimate on $U$. First, differentiating (2.5)$_2$ with respect to $t$, multiplying the resulting equations by $U_{tt}$ and integrating over $\Omega$, one has
\[
\frac{\alpha}{2} \frac{d}{dt} \left( |\nabla U_t|^2_2 + |\text{div}U_t|^2_2 \right) + |U_{tt}|^2_2 = - \int_\Omega \left( (V \cdot \nabla)V_t + \nabla \phi_t - (\psi \cdot Q(V))_t \right) U_{tt} dx \\
\leq C |U_t|^2_2 \left( |V_t|_2 |\nabla V|_\infty + |V|_\infty |\nabla V_t|_2 + |\nabla \phi_t|_2 + |\psi_t|_2 |\nabla V|_\infty + |\psi_q| |\nabla V|_{q^*} \right), \right.
\]
which, along with Young’s inequality and (2.18), yields that
\[
\frac{d}{dt} (|\nabla U_t|^2 + |	ext{div} U_t|^2) + |U_t|^2 \leq C(c_2^2||V||_3^2 + c_2^2|\nabla V_t|^2 + c_2^3 \|\nabla V_t\|_1^{2q\rightarrow 3}) \|
\] (2.24)

Second, multiplying (2.24) by \(s\) and integrating over \((\tau, t)\), one can get
\[
\int_\tau^t s|U_t|^2 \, ds \leq C\left( \tau |\nabla U_t(\tau)|_2^2 + c_3^3 + c_3^3 t + c_3^3 \frac{2q-3}{q} \right). \] (2.25)

Due to (2.4) and Lemma A.3 (Appendix A) that, there exists a sequence \(\{s_k\}\) such that
\[
s_k \rightarrow 0 \quad \text{and} \quad s_k |\nabla U_t(s_k, \cdot)|_2^2 \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \] (2.26)

After choosing \(\tau = s_k \rightarrow 0\) in (2.25), one obtains
\[
t|U_t(t)|_{D^1}^2 + \int_0^t s|U_t(s)|_2^2 \, ds \leq Cc_0^2 \quad \text{for} \quad 0 \leq t \leq T_2. \] (2.27)

Finally, according to Lemma A.5 (Appendix A), one has
\[
|U_t|_{D^2} \leq C(|U_t|_2 + |V \cdot \nabla V|) + |\nabla \phi| + |(\psi \cdot Q(V))|_2 + |U_t|_{D^1}) \\
\leq C(|U_t|_2 + |V| |\nabla V|_2 + |V_t|_2 |\nabla V|_\infty + |\nabla \phi|_2 + |\psi|_2 |\nabla V|_t, \|\nabla U_t|_2),
\]
which, along with (2.18) and (2.27), yields that for \(0 \leq t \leq T_2\),
\[
\int_0^t s|U_t(s)|_{L^2}^2 \, ds \leq C(c_2^3 + c_3^3 t + c_3^3 t^\frac{2q-3}{q}) \leq Cc_0^2.
\]

The proof of Lemma 2.4 is complete. \(\square\)

Finally, define the time \(T^* = \min \{ T, (1 + Cc_3)^{-\frac{6q}{6q-3}} \}\), and constants \(c_i (i = 1, 2, 3)\):
\[
c_1 = C^{\frac{2}{3}}c_0, \quad c_2 = c_3 = C^{\frac{3}{2}q}c_1^{\frac{6q}{2}q-3}) = C^{\frac{3}{2}q}c_0^{\frac{6q}{2}q-3})
\]
It follows from Lemmas 2.2-2.4 that for \(0 \leq t \leq T^*\),
\[
|U(t)|_{D^2}^2 + \int_0^t \left( |\nabla U(t)|_1^2 + |U(t)|_2^2 \right) \, ds \leq c_1^2,
\]
\[
|U(t)|_{D^2}^2 + \int_0^t \left( |U_t(t)|_{D^1}^2 + |U(t)|_2^2 \right) \, ds \leq c_2^2,
\]
\[
t|U_t(t)|_{D^1}^2 + \int_0^t \left( s|U_t(t)|_2^2 + s|U(t)|_{L^2}^2 \right) \, ds \leq c_3^2,
\]
\[
\|
\] (2.28)
2.4. Proof of Theorem 2.2. Our proof is based on the classical iteration scheme and conclusions obtained in Sections 2.2-2.3. Let us denote as in Section 2.3 that

\[ 1 + \|{\phi}_0\|_{L^\infty(T^* \cap H^2)} + \|U_0\|_2 + \|{\psi}_0\|_{L^\infty(D)} \leq c_0. \]

Next, let \( U^0 \in C([0, T^*]; H^2) \cap L^2([0, T^*]; H^2) \) satisfy the following problem:

- \( \zeta - \Delta \zeta = 0 \) in \( (0, \infty) \times \Omega; \)
- \( \zeta(0, x) = U_0 \) for \( x \in \Omega; \)
- \( \zeta(t, x)|_{x=a} = 0 \) and \( \zeta(t, x) \to 0 \) as \( |x| \to \infty \) for \( t \geq 0. \)

Choosing a time \( T_{**} \in (0, T^*] \) small enough such that

\[
\begin{align*}
\sup_{0 \leq t \leq T_{**}} \|U^0(t)\|_1^2 &+ \int_0^{T_{**}} (\|\nabla U^0(t)\|_1^2 + \|U^0_t(t)\|_2^2) \, dt \leq c_1^2, \\
\sup_{0 \leq t \leq T_{**}} (\|U^0(t)\|_{D^2}^2 + \|U^0_t(t)\|_2^2) &+ \int_0^{T_{**}} (\|U^0(t)\|_{D^3}^2 + \|U^0_t(t)\|_{D^1}^2) \, dt \leq c_2^2, \\
\text{ess sup}_{0 \leq t \leq T_{**}} t\|U^0(t)\|_{D^2}^2 &+ \int_0^{T_{**}} (t\|U^0_t(t)\|_2^2 + t\|U^0(t)\|_{D^2}^2) \, dt \leq c_3^2.
\end{align*}
\]

**Proof. Step 1: existence.** Let the beginning step of our iteration be \( V = U^0. \) Then one can get a strong solution \(({\phi}^1, U^1, \psi^1)\) of problem (2.5). Inductively, one constructs approximate sequences \((\phi^{k+1}, U^{k+1}, \psi^{k+1})\) as follows: given \((\phi^k, U^k, \psi^k)\) for \( k \geq 1, \) define \((\phi^{k+1}, U^{k+1}, \psi^{k+1})\) by solving the following problem

\[
\begin{cases}
\phi_t^{k+1} + U^k \cdot \nabla \phi^{k+1} + (\gamma - 1)\phi^{k+1} \text{div} U^k = 0, \\
U_t^{k+1} + \gamma \cdot \nabla U^k + \nabla \phi^{k+1} + LU^{k+1} = \psi^{k+1} \cdot Q(U^k), \\
\psi_t^{k+1} + \sum_{l=1}^3 A_l(U^k) \partial_t \psi^{k+1} + B(U^k) \psi^{k+1} + \nabla \text{div} U^k = 0, \\
(\phi^{k+1}, U^{k+1}, \psi^{k+1})|_{t=0} = (\phi_0, U_0, \psi_0) \quad \text{for} \quad x \in \Omega, \\
U^{k+1}|_{x=a} = 0 \quad \text{for} \quad t \geq 0, \\
(\phi^{k+1}, U^{k+1}, \psi^{k+1}) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0.
\end{cases}
\]

By replacing \( V \) with \( U^k \) in (2.5), and \((\phi^k, U^k, \psi^k)\) satisfies the uniform estimates (2.28), we can solve the problem (2.30).

Next we are going to prove that the whole sequence \((\phi^k, U^k, \psi^k)\) converges strongly to a limit \((\phi, U, \psi)\). Let

\[
\begin{align*}
\overline{\phi}^{k+1} &= \phi^{k+1} - \phi^k, \\
\overline{U}^{k+1} &= U^{k+1} - U^k, \\
\overline{\psi}^{k+1} &= \psi^{k+1} - \psi^k.
\end{align*}
\]

Then from (2.30), one can deduce that

\[
\begin{align*}
\overline{\phi}^{k+1} + U^k \cdot \nabla \overline{\phi}^{k+1} + U^k \cdot \nabla \phi^k + (\gamma - 1)(\overline{\phi}^{k+1} \text{div} U^k + \phi^k \text{div} \overline{U}^k) &= 0, \\
\overline{U}^{k+1} + U^k \cdot \nabla \overline{U}^k + U^k \cdot \nabla U^k - \nabla \overline{\phi}^{k+1} + LU^{k+1} &= \overline{\psi}^{k+1} \cdot Q(U^k), \\
\overline{\psi}^{k+1} + \sum_{l=1}^3 A_l(U^k) \partial_t \overline{\psi}^{k+1} + B(U^k) \overline{\psi}^{k+1} + \nabla \text{div} \overline{U}^k &= Y_1^k + Y_2^k.
\end{align*}
\]
where $\Upsilon^k_1$ and $\Upsilon^k_2$ are defined by

$$\Upsilon^k_1 = -\sum_{l=1}^{3}(A_l(U^k)\partial_l\psi^k - A_l(U^{k-1})\partial_l\psi^k), \quad \Upsilon^k_2 = -(B(U^k)\psi^k - B(U^{k-1})\psi^k).$$

We first consider $\overline{\psi}^{k+1}$. Actually, from Remark 2.1 at the end of this section, one has

**Lemma 2.5.**

$$\overline{\psi}^{k+1} \in L^\infty([0,T_\ast]; H^1) \quad \text{for} \quad k = 1, 2, \ldots.$$ 

Then multiplying (2.31)_3 by $2\overline{\psi}^{k+1}$ and integrating over $\Omega$, one arrives at

$$\frac{d}{dt}|\overline{\psi}^{k+1}|^2 \leq C|\overline{\psi}^{k+1}|^2(\|\nabla U^k\|_\infty|\overline{\psi}^{k+1}|_2 + |\nabla^2 U^k|_2 + |\Upsilon^k_1|_2 + |\Upsilon^k_2|_2)$$

$$\leq C|\overline{\psi}^{k+1}|^2(\|\nabla U^k\|_2 + |\nabla^2 U^k|_2 + |\nabla \psi^k|_2|U^k|_\infty + \|\psi^k\|_q|\nabla U^k|_q^*)$$

$$\leq C(\|\nabla U^k\|_2 + \epsilon^{-1}(1 + |\nabla \psi^k|_2^2 + |\psi^k|_q^2))|\overline{\psi}^{k+1}|_2 + \epsilon|\nabla U^k|_2^2,$$ 

for $t \in [0,T_\ast]$, where $\epsilon \in (0,1)$ is a constant to be determined later.

Second, for $\overline{\phi}^{k+1}$, multiplying (2.31)_1 by $2\overline{\phi}^{k+1}$ and integrating over $\Omega$, one has

$$\frac{d}{dt}|\overline{\phi}^{k+1}|^2 \leq C(\|\nabla U^k\|_\infty|\overline{\phi}^{k+1}|_2 + |\overline{U}^k|_6|\nabla \phi^k|_3 + |\nabla \psi^k|_2|\phi^k|_\infty)|\overline{\phi}^{k+1}|_2. \quad (2.32)$$

Applying $\partial^\ast_\varepsilon(|\cdot| = 1)$ to (2.31)_1, multiplying by $2\partial^\ast_\varepsilon\overline{\phi}^{k+1}$ and integrating over $\Omega$, then

$$\frac{d}{dt}|\partial^\ast_\varepsilon \overline{\phi}^{k+1}|^2 \leq C\left(|\nabla U^k|_\infty|\nabla \phi^k|_2^2 + |\nabla \phi^k|_6|\nabla \overline{U}^k|_3 + |\overline{U}^k|_\infty|\nabla^2 \phi^k|_2 \right.\right.$$ 

$$\left. + |\nabla^2 U^k|_3|\nabla \phi^k|_6 + |\nabla \psi^k|_2|\nabla \phi^k|_6 + |\phi^k|_\infty|\nabla \nabla \overline{U}^k|_2 \right)|\nabla \overline{\phi}^{k+1}|_2,$$ 

which, along with (2.32) and Young’s inequality, yields that for $t \in [0,T_\ast]$,

$$\frac{d}{dt}\|\overline{\phi}^{k+1}\|^2 \leq C(\|\nabla U^k\|_3 + \epsilon^{-1}(1 + \|\phi^k\|_2^2))\|\overline{\phi}^{k+1}\|^2_1 + \epsilon\|\nabla U^k\|^2. \quad (2.33)$$

For $\overline{U}^{k+1}$, multiplying (2.31)_2 by $2\overline{U}^{k+1}$ and integrating over $\Omega$, one gets

$$\frac{d}{dt}|\overline{U}^{k+1}|^2 \leq C\left(|U|_\infty|\nabla \overline{U}^k|_2 + |U|_6|\nabla \overline{U}^{k-1}|_3 + |\psi^k|_q|\nabla \overline{U}^k|_q \right.$$ 

$$\left. + |\overline{\psi}^{k+1}|_2|\nabla \overline{U}^{k-1}|_\infty)\|\overline{U}^{k+1}\|^2_2 + |\nabla \overline{U}^{k+1}|_2|\overline{\phi}^{k+1}|_2\right). \quad (2.34)$$
Applying \( \partial T \) to (2.31) \((|\zeta| = 1)\), multiplying by \(2\partial T U^{k+1}_t\) and integrating over \(\Omega\), then

\[
\begin{align*}
\frac{d}{dt} |\partial T U^{k+1}_t|^2 &+ 2\alpha |\nabla \partial T U^{k+1}_t|^2 + 2\alpha |\partial T \nabla U^{k+1}_t|^2 \\
&= -2 \int_{\Omega} \partial \psi^1 \left( U^k \cdot \nabla U^k + U^k \cdot \nabla U^{k-1} + \nabla \phi^k - \psi^k \cdot Q(U^k) \right) \\
&\quad - \frac{\psi^k}{k} (U(k+1)) \cdot \partial T U^{k+1}_t \, dx \\
&\leq C \left( |\nabla U^{k+1}_t|^2 |\nabla U^k|^2 + |U^k|^2 |\nabla U^{k-1}_t|^2 + |\nabla U^k|^2 |\nabla U^{k-1}||\nabla U^k| \right) \\
&\quad + |U^k|_q |\nabla U^{k-1}|_q |\nabla U^{k-1}_t|_q \\
&\quad + |\nabla \psi^{k+1}_t|_q |\nabla U^k|_q \left( |\nabla U^{k+1}_t|^2 + |\nabla U^{k+1}|_q^2 |\nabla U^{k+1}_t|_q^2 \right) \\
&\quad + |\nabla \phi^{k+1}_t|_q |\nabla U^{k-1}_t|_q |\nabla U^{k-1}||\nabla U^{k+1}|_q \\
&\leq C(\varepsilon)^{-1} \left( 1 + |U^k|^2 + |U^{k-1}|_q^2 \right) + \varepsilon^{-\frac{q}{2}} |U^{k+1}_t|^2 \\
&\quad + C (\varepsilon)^{-1} \left( 1 + |U^k|^2 + |U^{k+1}_t|^2 \right) + \varepsilon |U^{k+1}_t|^2.
\end{align*}
\]

which, along with (2.18), (2.35) and Young’s inequality, yields that for \(t \in [0, T_*]\),

\[
\frac{d}{dt} \|U^{k+1}_t\|^2 + \alpha \|\nabla U^{k+1}_t\|^2 \leq C(\varepsilon)^{-1} \left( 1 + |U^k|^2 + |U^{k-1}|_q^2 \right) + \varepsilon^{-\frac{q}{2}} |U^{k+1}_t|^2 + C \left( |U^{k+1}_t|^2 + |U^{k+1}_t|_q^2 \right) \exp \left( C + C \varepsilon \right).
\]

Finally, let

\[\Gamma^{k+1}(t) = \sup_{0 \leq s \leq t} \|\phi^{k+1}(s)\|^2 + \sup_{0 \leq s \leq t} \|\psi^{k+1}(s)\|^2 + \sup_{0 \leq s \leq t} \|U^{k+1}(s)\|^2.\]

According to the above estimates and the Gronwall inequality, one concludes that

\[
\Gamma^{k+1}(t) + \alpha \int_0^t \|\nabla U^{k+1}(s)\|^2 \, ds \leq \left( C \varepsilon \int_0^t \|\nabla U^k(s)\|^2 \, ds + C \varepsilon t \sup_{0 \leq s \leq t} \|U^k(s)\|^2 \right) \exp \left( C + C \varepsilon \right).
\]

Choose \(\varepsilon > 0\) and \(T_* \in (0, \min \{1, T_*\})\) small enough such that

\[C \varepsilon \exp C \leq \min \left\{ \frac{1}{4}, \frac{\alpha}{4} \right\} \quad \text{and} \quad \exp(C \varepsilon T_*) \leq 2.\]

Then one gets easily

\[
\sum_{k=1}^{\infty} \left( \Gamma^{k+1}(T_*) + \alpha \int_0^{T_*} \|\nabla U^{k+1}(s)\|^2 \, ds \right) \leq C < \infty,
\]

which means that the whole sequence \((\phi^k, U^k, \psi^k)\) converges to a limit \((\phi, U, \psi)\) in the following strong sense:

\[
\phi^k \to \phi \quad \text{in} \quad L^\infty([0, T_*]; H^1(\Omega)), \quad U^k \to U \quad \text{in} \quad L^\infty([0, T_*]; H^1(\Omega)), \\
\psi^k \to \psi \quad \text{in} \quad L^\infty([0, T_*]; L^2(\Omega_R)),
\]

where \(\Omega_R = \{ x \in \mathbb{R}^3 | a < |x| \leq R \} \) for any \(R > a\).

On the other hand, by virtue of the uniform (with respect to \(k\)) estimates (2.28), there exists a subsequence (still denoted by \((\phi^k, U^k, \psi^k)\)) converging to the limit \((\phi, U, \psi)\) in
the weak or weak* sense. According to the lower semi-continuity of norms, the corresponding estimates in (2.28) for \((\phi, U, \psi)\) still hold. Therefore, it is easy to show that \((\phi, U, \psi)\) is a weak solution in the sense of distributions to the problem (2.3) and satisfy the following regularities:

\[
\begin{aligned}
\phi &> 0, \quad \phi \in L^\infty([0, T_*]; L^2 \cap H^2), \quad \phi_t \in L^\infty([0, T_*]; H^1), \\
\psi &\in L^\infty([0, T_*]; L^q \cap D^1), \quad \psi_t \in L^\infty([0, T_*]; L^2), \\
U &\in L^\infty([0, T_*]; H^2) \cap L^2([0, T_*]; D^1), \quad U_t \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^1), \\
t^2 U_t &\in L^\infty([0, T_*]; D^1) \cap L^2([0, T_*]; D^2),
\end{aligned}
\] (2.38)

**Step 2:** uniqueness. Let \((\phi_1, U_1, \psi_1)\) and \((\phi_2, U_2, \psi_2)\) be two regular solutions to the problem (2.3) satisfying the uniform estimates in (2.28). Set

\[
\tilde{\phi} = \phi_1 - \phi_2, \quad \tilde{\psi} = \psi_1 - \psi_2.
\]

Then \((\tilde{\phi}, \tilde{\psi}, \tilde{U})\) satisfies the system

\[
\begin{aligned}
\tilde{\phi}_t + U_1 \cdot \nabla \tilde{\phi} + \tilde{U} \cdot \nabla \phi_2 + (\gamma - 1)(\tilde{\psi} \text{div} U_2 + \phi_1 \text{div} \tilde{U}) &= 0, \\
\tilde{U}_t + U_1 \cdot \nabla \tilde{U} + \tilde{U} \cdot \nabla U_2 + \tilde{\psi} \cdot \nabla \phi + L \tilde{U} &= \psi_1 \cdot Q(U) + \tilde{\psi} \cdot Q(U_2), \\
\tilde{\psi}_t + \sum_{l=1}^{3} A_l(U_1) \partial_l \tilde{\psi} + B(U_1) \tilde{\psi} + \nabla \text{div} \tilde{U} &= \tilde{\Upsilon}_1 + \tilde{\Upsilon}_2,
\end{aligned}
\] (2.39)

where \(\tilde{\Upsilon}_1\) and \(\tilde{\Upsilon}_2\) are defined by

\[
\tilde{\Upsilon}_1 = -\sum_{l=1}^{3} (A_l(U_1) \partial_l \psi_2 - A_l(U_2) \partial_l \psi_2), \quad \tilde{\Upsilon}_2 = -(B(U_1) \psi_2 - B(U_2) \psi_2).
\]

Let

\[
\Phi(t) = \|\tilde{\phi}(t)\|^2 + \|\tilde{\psi}(t)\|^2 + \|\tilde{U}(t)\|^2.
\]

Similarly to the derivation of (2.32)-(2.36), one can also show that

\[
\frac{d}{dt} \Phi(t) + C \|\nabla \tilde{U}(t)\|^2 \leq G(t) \Phi(t),
\] (2.40)

where \(\int_0^t G(s)ds \leq C\), for \(0 \leq t \leq T_*\). From the Gronwall inequality, one concludes that \(\tilde{\phi} = \tilde{\psi} = \tilde{U} = 0\), then the uniqueness is obtained.

**Step 3:** The time-continuity follows easily from the same procedure as in Lemma 2.1. Finally, Theorem 2.2 is proved. \(\square\)

**Remark 2.1.** Now we give the proof of Lemma 2.5.

**Proof.** Let \(X(x) \in C_0^\infty(\mathbb{R}^3)\) be a truncation function satisfying

\[
0 \leq X(x) \leq 1 \quad \text{and} \quad X(x) = \begin{cases} 
1 & \text{if } 0 \leq |x| \leq 1, \\
0 & \text{if } |x| \geq 2.
\end{cases}
\] (2.41)
Define, for any $R > 0$, $X_R(x) = X(\frac{|x|\rho}{R})$, $\overline{\psi}^{k+1,R} = \overline{\psi}^{k+1}X_R$. Then from (2.31),

$$
\overline{\psi}^{k+1,R}_t + \sum_{l=1}^{3} A_l(U^k)\partial_l \overline{\psi}^{k+1,R} + B(U^k)\overline{\psi}^{k+1,R} + \nabla \text{div} U^k X_R
$$

\[\text{(2.42)}\]

Multiplying (2.42) by $2\overline{\psi}^{k+1,R}$ and integrating over $\Omega$, one can obtain

$$
\frac{d}{dt}|\overline{\psi}^{k+1,R}|_2 \leq C(|\nabla U^k|_{\infty} |\overline{\psi}^{k+1,R}|_2 + |\nabla \text{div} U^k|_2 + |U^k|_{q^*} |\overline{\psi}^{k+1,R}|_q
$$

\[+ |\nabla \psi^k|_2 (|U^k|_{\infty} + |U^k-1|_{\infty}) + |\psi^k|_q (|\nabla U^k|_{q^*} + |\nabla U^{k-1}|_{q^*})]\]

\[\text{(2.43)}\]

where $\tilde{C} > 0$ is a generic constant depending on $C,q$ but independent of $R$. Then applying the Gronwall inequality to (2.43), one gets

$$
|\overline{\psi}^{k+1,R}(t)|_2 \leq \tilde{C}T_{**} \exp (\tilde{C}T_{**}) \text{ for } (t,R) \in [0, T_{**}] \times [0, \infty),
$$

which, along with Fatou’s lemma (see Lemma A.2 (Appendix A)), yields that

$$
\overline{\psi}^{k+1} \in L^\infty([0, T_{**}]; L^2).
$$

\[\text{(2.44)}\]

Finally, the desired conclusion can be obtained from (2.44) and $\nabla \overline{\psi}^{k+1} \in L^\infty([0, T_{**}]; L^2)$.

2.5. Proof of Theorem 2.1.

2.5.1. The well-posedness theory in Theorem 2.1. First, it follows from the assumption (2.1) and Theorem 2.2 that there exist a time $T_0 > 0$ and a unique regular solution $(\phi, U, \psi)$ in $[0, T_0] \times \Omega$ to (2.3) satisfying (2.4).

Second, from the transformation in (2.2), one has

$$
\rho(t,x) = \left(\frac{\gamma - 1}{A\gamma}\right)^{\frac{1}{\gamma - 1}}(t,x) \quad \text{and} \quad \frac{\partial \rho}{\partial \phi}(t,x) = \frac{1}{\gamma - 1} \left(\frac{\gamma - 1}{A\gamma}\right)^{\frac{2}{\gamma - 1}} \phi(t,x).
$$

Then multiplying (2.3), by $\frac{\partial \rho}{\partial \phi}(t,x)$ yields the continuity equation (1.1)$_1$; and multiplying (2.3)$_2$ by $\rho(t,x)$ gives the momentum equations (1.1)$_2$.

Thus we have shown that $(\rho, U)$ satisfies the IBVP (1.1)-(1.5) with (1.10)-(1.11) in the sense of distributions and the regularities in Definition 1.1. Moreover, it follows from the continuity equation that $\rho(t,x) > 0$ for $[0, T_0] \times \Omega$. In summary, the IBVP (1.1)-(1.5) with (1.10)-(1.11) has a unique regular solution $(\rho, U)$.

2.5.2. Spherically symmetric property in Theorem 2.1.

Lemma 2.6. If $(\rho_0, U_0)$ are spherically symmetric in the sense of (1.10), then the regular solution $(\rho, U)(t,x)$ to the IBVP (1.1)-(1.5) with (1.10)-(1.11) is also a spherically symmetric one taking the form (1.9).
Proof. First, for any orthogonal real matrix $H = (h_{kl})_{3 \times 3}$, it follows from (1.10) that
\[ \rho_0(x) = \rho_0(Hx) \quad \text{and} \quad u_0(|x|) = u_0(|Hx|). \]

Second, denote
\[ \tilde{\rho}(t,x) = \rho(t,Hx) \quad \text{and} \quad \tilde{U}(t,x) = H^\top U(t,Hx) = ((H^\top U)^{(1)}, \ldots, (H^\top U)^{(d)})^\top. \]

It follows from (2.45) that $(\tilde{\rho}(0,x), \tilde{U}(0,x)) = (\rho_0(x), U_0(x))$. Moreover, it follows from (1.11) and the fact $|Hx| = |x|$ that
\[ \tilde{U}(t,x)|_{|x|=a} = H^\top U(t,Hx)|_{|x|=a} = 0 \quad \text{for} \quad t \geq 0, \]
\[ (\tilde{\rho}(t,x), \tilde{U}(t,x)) \to (0,0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0. \]

Next we need to show that $(\tilde{\rho}(t,x), \tilde{U}(t,x))$ is also a solution to the system (1.1)-(1.5). The proof is divided into two steps.

**Step 1:** $(\tilde{\rho}(t,x), \tilde{U}(t,x))$ satisfies $(1.1)_1$. In order to facilitate the discussion, in the rest of the proof of Lemma 2.6, we adopt the Einstein summation convention: an index that appears exactly twice in a term is implicitly summed over. For any orthogonal real matrix $H = (h_{kl})_{d \times d}$ and vectors $x = (x_1, \ldots, x_d)^\top$, $\xi = (\xi_1, \ldots, \xi_d)^\top$, we denote
\[ \xi = Hx \quad \text{with} \quad \xi_k = h_{kl}x_l, \quad h_{kl}h_{lk} = 1, \quad h_{ij}h_{nj} = \delta_{ln}, \]
where $\delta_{ln}$ is the Kronecker symbol satisfying $\delta_{ln} = 1$, when $l = n$; and $\delta_{ln} = 0$, otherwise. Direct calculation gives

- For $\tilde{\rho}$, $\tilde{\rho}(t,x) = \rho(t,Hx)$;
- For $\text{div}(\tilde{\rho}\tilde{U})$,
\[
\text{div}(\tilde{\rho}\tilde{U}) = \frac{\partial}{\partial x_i}(\tilde{\rho}(H^\top U)^i) = \frac{\partial \rho}{\partial \xi_k}(H^\top U)^i + \rho \frac{\partial}{\partial \xi_k}(H^\top U)^i \frac{\partial \xi_k}{\partial x_i} H_{kl}h_{il} \xi_l
\]
\[
= \frac{\partial \rho}{\partial \xi_k} h_{kl} h_{ik} U^k + \rho h_{ik} \frac{\partial U^k}{\partial \xi_k} = \nabla \rho \cdot U + \rho \text{div} U = \text{div}(\rho U),
\]

which, along with $(1.1)_1$, implies that $(\tilde{\rho}(t,x), \tilde{U}(t,x))$ satisfies $(1.1)_1$.

**Step 2:** $(\tilde{\rho}(t,x), \tilde{U}(t,x))$ satisfies $(1.1)_2$. Direct calculation gives

- For $(\tilde{\rho}\tilde{U})_t$,
\[
(\tilde{\rho}\tilde{U})_t = (\rho(H^\top U)^t) = (\rho_{ik}U^k)_t h_{ik} = H^\top (\rho U)_t;
\]
- For $\text{div}(\tilde{\rho}\tilde{U} \otimes \tilde{U})$,
\[
\text{div}(\tilde{\rho}\tilde{U} \otimes \tilde{U}) = \frac{\partial}{\partial x_j}((\rho(H^\top U)^j) \rho_{ik}^j) h_{ik} \frac{\partial U^j}{\partial x_j} h_{il} \xi_l
\]
\[
= \frac{\partial}{\partial \xi_n} (\rho U^k U^l) h_{ik} \delta_{nl} = \frac{\partial}{\partial \xi_l} (\rho U^k U^l) h_{ik} = H^\top \text{div}(\rho U \otimes U);
\]
- For $\nabla P(\tilde{\rho})$,
\[
\nabla P(\tilde{\rho}) = \frac{\partial P(\tilde{\rho})}{\partial x_i} = \frac{\partial P(\rho)}{\partial \xi_k} \frac{\partial \rho}{\partial x_i} (h_{ki} x_l) = \frac{\partial P(\rho)}{\partial \xi_k} h_{ki} = H^\top \nabla P(\rho);
\]
For \( \text{div}(\mu(\rho)(\nabla U + (\nabla U)\top)) \)

\[
\text{div}(\mu(\rho)(\nabla U + (\nabla U)\top)) = \frac{\partial}{\partial x_j}(\mu(\rho)\left(h_{jk}U^k\right)h_{ji} + \frac{\partial}{\partial \xi_k}(h_{il}U^l)h_{kj})
\]

\[
= \frac{\partial}{\partial \xi_n}(\mu(\rho)\left(h_{jk}h_{li} + \frac{\partial U^l}{\partial \xi_k}h_{il}\right))\frac{\partial}{\partial x_j}(h_{ns}x_s)
\]

\[
= \frac{\partial}{\partial \xi_n}(\mu(\rho)\left(h_{jk}U^k\right)h_{ji} + \frac{\partial U^l}{\partial \xi_k}h_{il}) = H^\top \text{div}(\mu(\rho)(\nabla U + (\nabla U)\top))
\]

which, along with (1.1.2), yields that that \((\tilde{\rho}(t,x),\tilde{U}(t,x))\) also satisfies (1.1.2). Then \((\tilde{\rho}(t,x),\tilde{U}(t,x))\) is also a regular solution to the IBVP (1.1)-(1.5) with (1.10)-(1.11), which along with the uniqueness obtained in Theorem 2.1, yields that

\[
\rho(t,Hx) = \rho(t,x), \quad H^\top U(t,Hx) = U(t,x).
\]  

(2.48)

Then one has \(\rho(t,x) = \rho(t,|x|)\). It remains to show \(U(t,x) = u(t,|x|)\frac{x}{|x|}\) for some \(u\).

Let \(x \in \mathbb{R}^3\) be any arbitrary displacement vector, and \(H\) be the matrix that performs a 180-degree rotation about the axis parallel to \(x\). Then one has

\[
Hx = x \quad \text{and} \quad U(t,Hx) = U(t,x).
\]

Next let \(\hat{x} = \frac{x}{|x|}\), i.e., a unit vector parallel to \(x\). Let \((\hat{x},\hat{y},\hat{z})\) be an orthonormal basis. Then, for any fixed time \(t\), there exist some real constants \(x_1,y_1,z_1\) such that

\[
U(t,x) = x_1\hat{x} + y_1\hat{y} + z_1\hat{z},
\]

(2.49)

which, along with the fact that \(H\) is a 180-degree rotation about \(\hat{x}\), yields that

\[
HU(t,x) = x_1\hat{x} - y_1\hat{y} - z_1\hat{z}.
\]

(2.50)

It thus follows from (2.49)-(2.50) that

\[
x_1\hat{x} + y_1\hat{y} + z_1\hat{z} = x_1\hat{x} - y_1\hat{y} - z_1\hat{z},
\]

which means that \(y_1 = z_1 = 0\), and for any fixed time \(t\),

\[
U(t,x) = x_1\hat{x} = x_1\frac{x}{|x|},
\]

which, along with the arbitrary choice of \(x\) and \(t\), implies that \(U(t,x)\) is a radial vector.

The proof of Lemma 2.6 is complete. 

Thus the proof of Theorem 2.1 is complete.

\[\square\]
3. Global-in-time spherically symmetric estimates

The purpose of this section is to establish the global-in-time energy estimates in the spherically symmetric Eulerian coordinate. To this end, we first consider the following reformulation of the IBVP (1.1)-(1.5) with (1.10)-(1.11). Throughout this section, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces: for \( I_a = [a, \infty) \) with some \( a > 0 \),

\[
L^p = L^p(I_a), \quad H^s = H^s(I_a), \quad D^{k,l} = D^{k,l}(I_a),
\]

\[
D^k = D^{k,2}(I_a), \quad W^{m,p} = W^{m,p}(I_a), \quad |f|_p = \|f\|_{L^p(I_a)},
\]

\[
\|f\|_s = \|f\|_{H^s(I_a)}, \quad \|f\|_{m,p} = \|f\|_{W^{m,p}(I_a)}, \quad |f|_{D^{k,l}} = \|\nabla^k f\|_{L^l(I_a)},
\]

\[
|f|_{D^k} = \|\nabla^k f\|_{L^2(I_a)}, \quad \|f\|_{X_1 \cap X_2} = \|f\|_{X_1} + \|f\|_{X_2}, \quad \int f \, dr = \int_{I_a} f \, dr.
\]

3.1. Reformulation in the spherically symmetric Eulerian coordinate. Let \( (\rho, U)(t, x) \) in \([0, T_*] \times \Omega\) be the unique regular solution to the IBVP (1.1)-(1.5) with (1.10)-(1.11) obtained in Theorem 2.1, which has the following form

\[
\rho(t, x) = \rho(t, r), \quad U(t, x) = u(t, r) \frac{x}{r}, \quad r = |x|.
\] (3.1)

Denote \( m = d - 1 \) \((d = 2, 3)\). Then the IBVP (1.1)-(1.5) with (1.10)-(1.11) can be rewritten into

\[
\begin{cases}
\rho_t + (\rho u)_r + \frac{m \rho u}{r} = 0, \\
(\rho u)_t + (\rho u^2)_r + \rho_r - 2a \left( \rho \left( u_r + \frac{m u}{r} \right) \right)_r + \frac{2a m \rho u}{r} + \frac{m \rho u^2}{r} = 0, \\
(\rho(0, r), u(0, r)) = (\rho_0(r), u_0(r)) \quad \text{for} \quad r \in I_a, \\
u(t, r)|_{r=0} = 0 \quad \text{for} \quad t \geq 0, \\
(\rho(t, r), u(t, r)) \rightarrow (0, 0) \quad \text{as} \quad r \rightarrow \infty \quad \text{for} \quad t \geq 0.
\end{cases}
\] (3.2)

Then by Theorem 2.1 and Remark B.1 (Appendix B), one has

**Lemma 3.1.** Let (1.16) hold except \( \gamma > \frac{3}{2} \). Assume the initial data \((\rho_0(r), u_0(r))\) satisfy

\[
0 < r^m \rho_0 \in L^1, \quad \left( r^{-\frac{m}{2}} \rho_0^{-\frac{\gamma - 1}{2}}, r^{-\frac{m}{2}} u_0 \right) \in H^2, \quad r^\gamma (\ln \rho_0)_r \in L^q,
\]

\[
(\ln \rho_0)_r \in L^\infty, \quad r^{\frac{m}{2}} \left( (\ln \rho_0)_r, (\ln \rho_0)_{rr} \right) \in L^2,
\] (3.3)

for some \( q \in (3, 6] \). Then there exist a positive time \( T_* > 0 \) and a unique smooth solution \((\rho(t, r), u(t, r))\) in \([0, T_*] \times I_a\) to the problem (3.2) satisfying
\[0 < r^m \rho \in C([0, T_*]; L^1), \quad r^{\frac{m}{2}} \rho^{\gamma - 1} \in C([0, T_*]; H^2),\]
\[r^{\frac{m}{2}}(\rho^{\gamma - 1})_t \in C([0, T_*]; H^1), \quad r^{\frac{m}{2}}(\ln \rho)_r \in C([0, T_*]; L^q),\]
\[r^{\frac{m}{2}}(r^{-1}(\ln \rho)_r, (\ln \rho)_{rr}) \in C([0, T_*]; L^2), \quad r^{\frac{m}{2}}(\ln \rho)_{tr} \in C([0, T_*]; L^2),\]
\[r^{\frac{m}{2}} u \in C([0, T_*]; H^2(I_0)) \cap L^2([0, T_*]; D^3), \quad (\ln \rho)_r \in L^\infty([0, T_*] \times I_0),\]
\[r^{\frac{m}{2}} u_t \in C([0, T_*]; L^2) \cap L^2([0, T_*]; D^1),\]
\[t^{\frac{1}{2}} r^{\frac{m}{2}} u_{tr} \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^1).\]  

\[\text{Proof.} \quad \text{The desired well-posedness to the problem (3.2) is just a corollary of Theorem 2.1, Lemma B.1 and Remark B.1 (in Appendix B). We only need to show (ln } \rho_r \in L^\infty([0, T_*] \times I_0) \text{ if (ln } \rho_0)_r \in L^\infty.\]

Now we need to introduce the so-called effective velocity:
\[v = u + \varphi(\rho)_r = u + 2\alpha \rho^{-1} \rho_r,\]  
\[\text{where } \varphi(\rho) \text{ is a function of } \rho \text{ defined by } \varphi'(\rho) = 2\mu(\rho)/\rho^2. \text{ Then it follows from the initial assumption (3.3) that } \]
\[v_0(r) = v_0 = u_0 + \varphi(\rho_0)_r = u_0 + 2\alpha \rho^{-1}_0(\rho_0)_r \in L^\infty. \]  

Moreover, it follows from the equations (3.2)_1-(3.2)_2 and the definition of \(v\) that
\[v_t + uv_r + \frac{A_\gamma}{2\alpha} \rho^{\gamma - 1} v - \frac{A_\gamma}{2\alpha} \rho^{\gamma - 1} u = 0. \]

Via the standard characteristic method, one can obtain
\[v = \left( v_0 + \int_0^t \frac{A_\gamma}{2\alpha} \rho^{\gamma - 1} u \exp \left( \int_0^s \frac{A_\gamma}{2\alpha} \rho^{\gamma - 1} d\tau \right) ds \right) \exp \left( - \int_0^t \frac{A_\gamma}{2\alpha} \rho^{\gamma - 1} ds \right), \]
which, along with the regularities of \((\rho^{\gamma - 1}, u)\) and (3.5)-(3.6), yields that \(v, (\ln \rho)_r \in L^\infty([0, T_*] \times I_0).\)

The proof of Lemma 3.8 is complete.  

Theorem 3.1. Let (1.16) hold. Assume that the initial data \((\rho_0(r), u_0(r))\) satisfy (3.3), then (3.2) admits a unique global classical solution \((\rho(t, r), u(t, r))\) in \((0, \infty) \times I_0\) satisfying the regularities in (3.4) with \(T_*\) replaced by arbitrarily large \(0 < T < \infty.\)

Let \(T > 0\) be some time and \((\rho(t, r), u(t, r))\) be regular solutions to the problem (3.2) in \([0, T] \times I_0\) obtained in Lemma 3.1. The main aim in the rest of this section is to establish the global-in-time a priori estimates for these solutions. Hereinafter, we denote \(C_0\) (resp. \(C_0^*\)) a generic positive constant depending only on \((\rho_0, u_0, A, \gamma, \alpha)\) (resp. \((C_0, a)\)); \(C\) (resp. \(C^a\)) a generic positive constant depending only on \((C_0, T)\) (resp. \((C, a)\)), which may be different from line to line.
3.2. The $L^\infty$ estimate of $\rho$. We consider the upper bound of the mass density $\rho$ in $[0, T] \times I_a$. First, the standard energy estimates yield that

**Lemma 3.2.** For any $T > 0$, it holds that, for $0 \leq t \leq T$,

$$\int r^m \left( \frac{1}{2} \rho u^2 + \frac{A}{\gamma - 1} \rho^\gamma \right) (t, \cdot) \, dr + \int_0^t \int (2\alpha \rho m \rho u_t^2 + 2\alpha m \rho r_{\gamma-1}^2 \rho^2) \, dr \, ds \leq C_0.$$  

**Proof.** First, multiplying (3.2) by $r^m u$ and using (3.2), one has

$$\left( \frac{r^m}{2} \rho u_t^2 \right)_t + \left( \frac{r^m}{2} \rho u^3 + 2\alpha m \rho^2 \right)_r + r^m P_{\rho u} = -2\alpha \left( r^m \rho u (u_r + \frac{m}{r} u) \right)_r + 2\alpha m \rho u + 2\alpha m \rho u^2 = 0. \quad (3.9)$$

Second, integrating (3.9) over $I_a$, one gets

$$\frac{d}{dt} \int \frac{r^m}{2} \rho u^2 \, dr + 2\alpha \int r^m \rho u^2 \, dr + 2\alpha m \int r^m \rho \, dr = -\int r^m P_{\rho u} \, dr$$

$$= -A\gamma \int r^m \rho^{-2} \rho_r u \, dr - \frac{A\gamma}{\gamma - 1} \int r^m (\rho^{\gamma-1})_r u \, dr$$

$$= -\frac{A\gamma}{\gamma - 1} \int (r^m \rho^{\gamma-1} u)_r \, dr + \frac{A\gamma}{\gamma - 1} \int \rho^{\gamma-1} (r^m \rho u)_r \, dr$$

$$= -\frac{A\gamma}{\gamma - 1} \int \rho^{\gamma-1} (r^m \rho) \, dr - \frac{A}{\gamma - 1} \frac{d}{dr} \int r^m \rho^{\gamma} \, dr,$$

where one has used the equation (3.2) and (3.2) of (3.2). Then the desired conclusion can be achieved by an integration over $[0, t]$. The proof of Lemma 3.2 is complete. □

Second, we give the well-known BD entropy estimates.

**Lemma 3.3 ([4]).** For any $T > 0$, it holds that

$$\int r^m \left( \frac{1}{2} \rho |u + \varphi(\rho) r|^2 + \frac{A}{\gamma - 1} \rho^\gamma \right) (t, \cdot) \, dr + 2A\alpha \gamma \int_0^t \int r^m \rho \varphi(\rho) \, dr \, ds \leq C_0$$

for $0 \leq t \leq T$, where $\varphi(\rho) = \frac{2\mu(\rho)}{\rho^2}$.

**Proof.** According to (3.2) and (3.2), one has

$$\varphi(\rho) u_r + (\varphi(\rho) r)_r + (\rho \varphi'(\rho) u_r)_r + \left( \frac{m}{r} \rho \varphi'(\rho) \right)_r = 0, \quad (3.10)$$

$$\rho(u_t + u u_r) + P_r - 2\alpha \left( \rho (u_r + \frac{m}{r} u) \right)_r + \frac{2\alpha m \rho_r u}{r} = 0. \quad (3.11)$$

Multiplying (3.10) by $\rho$, one has

$$\rho \varphi(\rho) u_r + \rho \varphi'(\rho) r r_r + (\rho \varphi'(\rho) u)_r + \left( \frac{m}{r} \rho^2 \varphi'(\rho) u \right)_r - \frac{m}{r} \rho \varphi'(\rho) r u = 0. \quad (3.12)$$

Then adding (3.12) to (3.11), by the definition of $v = u + \varphi(\rho)_r$, one can obtain

$$\rho(v_t + u v_r) + P_r + \left( \left( \rho \varphi'(\rho) (2\alpha - \rho \varphi'(\rho)) \frac{m}{r} u \right)_r + \frac{m \rho u}{r} (2\alpha - \rho \varphi'(\rho)) \right) = 0, \quad (3.13)$$
which, along with \( \varphi'(\rho) = \frac{2\mu(\rho)}{\rho^2} \) in (3.13), yields that
\[
\rho(v_t + w_r) + P_r = 0. \tag{3.14}
\]
Multiplying (3.14) by \( r^m v \) and integrating over \([0, t] \times I_a\), one gets
\[
\int r^m \left( \frac{1}{2} \rho v^2 + \frac{A}{\gamma - 1} \rho^\gamma \right) (t, \cdot) \, dr + 2 A \alpha \gamma \int_0^t \int r^m \rho^{\gamma-2} \rho_r^2 \, dr \, ds \leq C_0,
\]
where one has used the equation (3.2)\(_1\) and (3.2)\(_4\)-(3.2)\(_5\).

The proof of Lemma 3.3 is complete. \(\square\)

Next we show the regular solution \((\rho(t, r), u(t, r))\) keeps the conservation of total mass.

**Lemma 3.4.** For any \( T > 0 \), it holds that
\[
\int r^m \rho(t, \cdot) \, dr = \int r^m \rho_0 \, dr \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** It follows from (3.2)\(_1\) that
\[
\frac{d}{dt} \int r^m \rho(t, \cdot) \, dr = - \int (r^m \rho u)_r (t, \cdot) \, dr = 0,
\]
where one has used (3.2)\(_4\) and \( r^m \rho u(t, \cdot) \in W^{1,1} \).

The proof of Lemma 3.4 is complete. \(\square\)

Now we are ready to give the uniform upper bound of the density.

**Lemma 3.5.** For any \( T > 0 \), it holds that
\[
|\rho(t, \cdot)|_\infty \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** First, according to Lemmas 3.2-3.3, one gets
\[
\| (\sqrt{\rho})_r \|_{L^\infty([0, T]; L^2)} \leq C^a. \tag{3.15}
\]
Second, according to Lemma 3.4, one can obtain
\[
\| \sqrt{\rho} \|_{L^\infty([0, T]; L^2)} \leq C^a. \tag{3.16}
\]
It thus follows from (3.15), (3.16) and Sobolev embedding theorem (in Appendix A) that
\[
|\sqrt{\rho}|_\infty \leq C \| \sqrt{\rho} \|_1 \leq C^a,
\]
which implies that
\[
|\rho(t, \cdot)|_\infty \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.5 is complete. \(\square\)
3.3. **The $L^2$ estimate of $r^\frac{\alpha}{2} u$.** We consider the $L^2$ estimate of $r^\frac{\alpha}{2} u$. For this purpose, one first needs to show the following several auxiliary lemmas. The first one is on the $L^{\tilde{q}}$ estimate of $(r^m \rho)^{\frac{1}{2}} u$ for any $2 \leq \tilde{q} < \infty$.

**Lemma 3.6.** Assume $2 \leq \tilde{q} < \infty$. Then for any $T > 0$, it holds that

$$\| (r^m \rho)^{\frac{1}{2}} u(t, \cdot) \|_{\tilde{q}} \leq C^a$$

for $0 \leq t \leq T$,

where the constant $C^a$ depends on $\tilde{q}$.

**Proof.** First, multiplying (3.2) by $r^m |u|^p u$ ($p \geq 2$) and integrating the resulting equation over $I_a$, one arrives at

$$
\frac{1}{p+2} \frac{d}{dt} \| (r^m \rho)^{\frac{1}{2}} u \|_{p+2}^2 + 2 \alpha(p + 1) \| (r^m \rho)^{\frac{1}{2}} u \|_{p+2}^2 \\
+ 2 \alpha m \| (r^m - \rho)^{\frac{1}{2}} u \|_{p+2}^2 = A(p + 1) \int r^m \rho^{\gamma-1} |u|^p u_r \, dr + A \int r^{m-1} \rho^{\gamma-1} |u|^p u \, dr \overset{2}{=} \sum_{i=1} \mathcal{J}_i,
$$

where one has used the equation (3.2) and Young’s inequality, one can obtain that

$$
\mathcal{J}_1 \leq C \| (r^m \rho)^{\frac{1}{2}} u \|_{p+2} |(r^m \rho)^{\frac{1}{2}} u|_2 |(r^m \rho)^{\frac{1}{2}} u|_1^2 \\
\leq \frac{\alpha(p + 1)}{32} \| (r^m \rho)^{\frac{1}{2}} u \|_{p+2}^2 + C \int r^m \rho^{2\gamma-1} |u|^p \, dr.
$$

Then we need to estimate the last term on the right-hand side of (3.18). According to Lemmas 3.2 and 3.4-3.5, one has

$$
\int r^m \rho^{2\gamma-1} |u|^p \, dr \leq C \| (r^m \rho)^{\frac{1}{2}} u \|_4 |(r^m \rho)^{\frac{1}{2}} u|_2 |(r^m \rho)^{\frac{1}{2}} u|_1^2 \\
\leq C^a \| (r^m \rho)^{\frac{1}{2}} u \|_4 |(r^m \rho)^{\frac{1}{2}} u|_2 |(r^m \rho)^{\frac{1}{2}} u|_1^2
$$

for $p = 2$,

$$
\int r^m \rho^{2\gamma-1} |u|^p \, dr = \int r^m \rho^{2(\gamma-1)} \rho^{\frac{p-2}{p+2}} |u|^{\frac{p^2}{p+2}} |(r^m \rho)^{\frac{2}{p}} u^{\frac{p}{2}} \|_p \\
\leq C \| \rho^{2(\gamma-1)} |(r^m \rho)^{\frac{p-2}{p+2}} u^{\frac{p-2(p+2)}{p}} |(r^m \rho)^{\frac{2}{p}} u^{\frac{p}{2}} \|_p \\
\leq C^a \| (r^m \rho)^{\frac{p-2}{p+2}} u^{\frac{p-2(p+2)}{p}} |r^m \rho u_1^{\frac{p}{2}} |_1 \\
\leq C^a \| (r^m \rho)^{\frac{p-2}{p+2}} u^{\frac{p+2}{p+2}} \|_{p+2}
$$

for $p > 2$.

Consequently, according to (3.18)-(3.19), one can obtain that for any $p \geq 2$,

$$
\mathcal{J}_1 \leq \frac{\alpha(p + 1)}{32} \| (r^m \rho)^{\frac{1}{2}} u \|_2^2 + C^a \| (r^m \rho)^{\frac{p+2}{p+2}} u \|_{p+2}^2.
$$
For the term $J_2$, one can similarly get

$$J_2 = A\int r^{m-1}\rho^{\frac{1}{p'}}|u|^p u dr$$

$$\leq C\left( \int (r^{m}\rho)^{\frac{1}{p'+2}}|u|^p|u|_2 \right)^{p'/p+1} \int (r^{m}\rho)^{\frac{1}{p'+2}}|u|^p|u|_\infty \leq C (1 + \int (r^{m}\rho)^{\frac{1}{p'+2}}|u|^p|u|_2) \leq C (1 + \int (r^{m}\rho)^{\frac{1}{p'+2}}|u|^p|u|_2) \left( \frac{1}{p'+2} \right).$$

Substituting the estimates for $J_i$ ($i = 1, 2$) into (3.17), one has

$$\frac{d}{dt} \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} - 2 \left( \int \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} + \left| (r^{m-2}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} ds \right) \leq C \left( t + \int_0^t \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} ds \right),$$

(3.20)

Second, integrating (3.20) over $[0, t]$, one can obtain that

$$\left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} + \int_0^t \left( \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} + \left| (r^{m-2}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} ds \right) \leq C \left( t + \int_0^t \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} ds \right),$$

which, along with the Gronwall inequality and (3.3), yields that

$$\left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} + \int_0^t \left( \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} + \left| (r^{m-2}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} ds \right) \leq C \left( t + \int_0^t \left| (r^{m}\rho)^{\frac{1}{p'+2}}u \right|_{p'+2} ds \right),$$

(3.21)

Finally, the desired conclusion can be achieved from (3.21) and Lemma 3.6. The proof of Lemma 3.6 is complete.

The following lemma gives the estimate of $\int_0^t |\rho' u(s, \cdot)|_\infty ds$ ($\frac{1}{2} < \eta < 1$), which plays a crucial role in obtaining the $L^\infty$ estimate of the effective velocity $v$ (see (3.24)).

**Lemma 3.7.** For any $T > 0$, it holds that

$$\int_0^t |\rho' u(s, \cdot)|_\infty ds \leq C_1 \quad \text{for} \quad 0 \leq t \leq T.$$  

**Proof.** First, it follows from direct calculations that

$$\rho' u = \rho^{-\frac{1}{2}}\rho^{-\frac{1}{2}}\rho^{-\frac{1}{2}} u + \rho^{-\frac{1}{2}}\rho^{-\frac{1}{2}} u.$$  

According to Lemmas 3.2-3.4, 3.6 and $\frac{1}{2} < \eta < 1$, one gets for $0 \leq t \leq T$,

$$\rho^{-\frac{1}{2}}\rho^{-\frac{1}{2}} u(s, \cdot) \leq \rho^{-\frac{1}{2}}\rho^{-\frac{1}{2}} u(s, \cdot) \leq C_1 \quad \text{for} \quad 0 \leq t \leq T.$$  

(3.22)
Second, notice that \( \frac{1}{2} \in (1, 2) \) and \( \rho' u \) decays to zero at infinity, then by using Newton-Leibniz formula and Hölder’s inequality, one obtains

\[
|\rho' u|_\infty \leq C|\rho' u|_2^{1-\Xi}|(\rho' u)|_\infty^{\Xi} \quad \text{with} \quad \Xi = \frac{1}{3 - 2\xi} \in (0, 1). \tag{3.23}
\]

Thus it follows from (3.23), Lemmas 3.2, 3.5 and (3.22) that

\[
\int_0^t |\rho' u(s, \cdot)|_\infty \, ds \leq C \int_0^t \left(1 + |(\rho' u)|_2^2\right)(s, \cdot) \, ds \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.7 is complete. \( \square \)

Next we show the \( L^\infty \) estimate of the effective velocity:

\[
v = u + \varphi(\rho)_r = u + 2\alpha \rho^{-1} \rho_r. \tag{3.24}
\]

**Lemma 3.8.** Assume \( \gamma > \frac{3}{2} \) additionally. Then for any \( T > 0 \), it holds that

\[
|v(t, \cdot)|_\infty \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** According to the proof of Lemma 3.1, one has that \( v \) satisfies the damped transport equation (3.7). Then via the standard characteristic method, one can obtain

\[
v = \left(v_0 + \int_0^t \frac{A\gamma}{2\alpha} \rho^{\gamma - 1} u \exp\left(\int_0^s \frac{A\gamma}{2\alpha} \rho^{\gamma - 1} \, dr\right) \, ds\right) \exp\left(-\int_0^t \frac{A\gamma}{2\alpha} \rho^{\gamma - 1} \, ds\right). \tag{3.25}
\]

According to (3.25) and Lemma 3.5, one deduces that

\[
|v|_\infty \leq C^a \left(|v_0|_\infty + \int_0^t |\rho^{\gamma - 1} u|_\infty \, ds\right) \leq C^a \left(|v_0|_\infty + \int_0^t |\rho|_\infty^{\gamma - 1 - \iota} |\rho' u|_\infty \, ds\right), \tag{3.26}
\]

which, along with Lemmas 3.5 and 3.7, yields that for any \( \gamma \geq 1 + \iota > \frac{3}{2} \),

\[
|v(t, \cdot)|_\infty \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.8 is complete. \( \square \)

For simplicity of the following calculations, one introduces the following variables

\[
\phi = \frac{A\gamma}{\gamma - 1} \rho^{\gamma - 1}, \quad \psi = \frac{1}{\gamma - 1} (\ln \phi)_r = (\ln \rho)_r, \tag{3.27}
\]

then the problem (3.2) can be rewritten into

\[
\begin{cases}
\phi_t + u\phi_r + (\gamma - 1)\phi u_r + m(\gamma - 1) r^{-1} \phi u = 0, \\
u_t + uu_r + \phi_r - 2\alpha u_{rr} = 2\alpha \psi u_r + 2\alpha m(r^{-1} u)_r, \\
\psi_t + (\psi u)_r + u_{rr} + m(r^{-1} u)_r = 0, \\
(\phi(0, r), u(0, r), \psi(0, r)) = (\phi_0(r), u_0(r), \psi_0(r)) \\
eq \left(\frac{A\gamma}{\gamma - 1} \rho^{\gamma - 1}_0(r), u_0(r), (\ln \rho_0(r))_r\right) \quad \text{for} \quad r \in I_a, \\
u(t, r)|_{r=a} = 0 \quad \text{for} \quad t \geq 0, \\
(\phi(t, r), u(t, r), \psi(t, r)) \to (0, 0, 0) \quad \text{as} \quad r \to \infty \quad \text{for} \quad t \geq 0.
\end{cases} \tag{3.28}
\]
The following lemma shows the boundedness of \( \int_0^t |u(s, \cdot)|_2^2 \, ds \), which will be used to obtain the \( L^2 \) estimate of \( r \frac{m}{2} u \).

**Lemma 3.9.** Assume \( \gamma > \frac{3}{2} \) additionally. Then for any \( T > 0 \), it holds that
\[
|u(t, \cdot)|_2^2 + \int_0^t \left( |u_r|^2_2 + |r^{-1} u|^2_2 + |u|^2_\infty \right) (s, \cdot) \, ds \leq C^a \quad \text{for } 0 \leq t \leq T.
\]

**Proof.** First, multiplying (3.28) by \( u \) and integrating over \([0, t] \), one has
\[
\frac{1}{2} \frac{d}{dt} |u|^2_2 + 2\alpha |u_r|^2_2 + \alpha m |r^{-1} u|^2_2 = - \int (u u_r + \phi_r - 2\alpha \phi u_r) u \, dr = \sum_{i=3}^5 J_i. \tag{3.29}
\]
According to Hölder’s inequality, Lemmas 3.4-3.5 and 3.8, one can obtain that
\[
J_3 = - \int u^2 u_r \, dr = 0,
\]
\[
J_4 = - A\gamma \int \rho^{\gamma-2} \rho_r u \, dr = - \frac{A\gamma}{2\alpha} \int \rho^{\gamma-1} (v - u) u \, dr 
\leq C \left( (|r^m \rho|_2^2 |u|_2 |v|_\infty |\rho|_\infty^{\gamma-1} |r^{-m} |_\infty + |\rho|_\infty^{\gamma-1} |u|^2_2) \right) 
\leq C^a (1 + |u|^2),
\]
\[
J_5 = 2\alpha \int \rho^{-1} \rho_r u \, dr = \int (v - u) u u_r \, dr 
\leq C |v|_\infty |u|_2 |u_r|_2 \leq \frac{\alpha}{4} |u_r|^2 + C^a |u|^2.
\]
Substituting the estimates for \( J_i \) \((i = 3, 4, 5)\) into (3.29) and integrating the resulting equation over \([0, t] \), one arrives that
\[
|u|^2_2 + \int_0^t \left( |u_r|^2_2 + |r^{-1} u|^2_2 \right) \, ds \leq C^a \left( t + \int_0^t |u|^2_2 \, ds \right),
\]
which, along with the Gronwall inequality, yields that
\[
|u(t, \cdot)|_2^2 + \int_0^t \left( |u_r|^2_2 + |r^{-1} u|^2_2 \right) (s, \cdot) \, ds \leq C^a \quad \text{for } 0 \leq t \leq T. \tag{3.30}
\]
It thus follows from (3.30) that
\[
\int_0^t |u(s, \cdot)|_\infty^2 \, ds \leq C \int_0^t \left( |u|^2_2 + |u_r|^2_2 \right) (s, \cdot) \, ds \leq C^a.
\]
The proof of Lemma 3.9 is complete. \( \square \)

Now we give the \( L^2 \) estimate of \( r \frac{m}{2} u \).

**Lemma 3.10.** Assume \( \gamma > \frac{3}{2} \) additionally. Then for any \( T > 0 \), it holds that
\[
|r \frac{m}{2} u(t, \cdot)|_2^2 + \int_0^t \left( |r \frac{m}{2} u_r|^2_2 + |r \frac{m-2}{2} u|^2_2 \right) (s, \cdot) \, ds \leq C^a \quad \text{for } 0 \leq t \leq T.
\]
Proof. Multiplying (3.28) by $r^m u$ and integrating over $I_a$, one has
\[
\frac{1}{2} \frac{d}{dt} |r^m u|^2 + 2\alpha |r^m u_r|^2 + 2\alpha m |r^{m-2} u_r|^2 = - \int r^m (uu_r + \phi_r - 2\alpha \psi u_r) u dr \triangleq \sum_{i=6}^{8} J_i.
\] (3.31)

Using Hölder’s inequality, Young’s inequality, Lemmas 3.4-3.5 and 3.8, we estimate $J_i$ ($i = 6, 7, 8$) as follows:

\[
J_6 = - \int r^m u_r^2 u_r dr \leq C \int r^{m-1} |u|^3 dr \leq C u_\infty |r^m u|^2,
\]

\[
J_7 = - A\gamma \int r^m \rho^{-2} \rho_r u dr = - \frac{A\gamma}{2\alpha} \int r^m \rho^{\gamma-1} (v - u) u dr \leq C \left((r^m \rho)^{\frac{2}{3}} |r^m u|_2^2 + |\rho|_{\infty}^{\gamma} + |\rho|_{\infty}^{\gamma-1} |r^m u|^2_2\right) \leq C_a(1 + |u|_\infty) |r^m u|^2_2,
\]

\[
J_8 = 2\alpha \int r^m \rho^{-1} \rho_r u_r u dr = \int r^m (v - u) u_r u dr \leq C_a \left(|u|_{\infty}|r^m u_r|^2 + |u|_{\infty} |r^m u|^2_2 \right) \leq \frac{1}{4} |r^m u_r|^2 + C_a(1 + |u|_\infty) |r^m u|^2_2.
\]

Substituting the estimates for $J_i$ ($i = 6, 7, 8$) into (3.31), using the Gronwall inequality and Lemma 3.9, one has

\[
|r^m u(t, \cdot)|^2_2 + \int_0^t \left(|r^m u|^2_2 + |r^{m-2} u|^2_2\right)(s, \cdot) ds \leq C_a \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.10 is complete. \qed

3.4. The first order estimates of $r^m u$. Now we consider the estimates of $|r^m u_r|^2$.

Lemma 3.11. Assume $\gamma > \frac{3}{2}$ additionally. Then for any $T > 0$, it holds that

\[
(|r^m u_r|^2 + |\psi|_{\infty})(t, \cdot) + \int_0^t \left(|r^m u|^2_2 + |r^m u_r|^2 + |r^m u_r|_{\infty}\right)(s, \cdot) ds \leq C_a \quad \text{for} \quad 0 \leq t \leq T.
\]

Proof. First, multiplying (3.28) by $r^m u_t$ and integrating over $I_a$, one arrives at

\[
\alpha \frac{d}{dt} |r^m u_r|^2 + |r^m u_t|^2 = - \int r^m (uu_r + \phi_r - 2\alpha \psi u_r + 2\alpha m r^{-2} u) u_t dr \triangleq \sum_{i=9}^{12} J_i.
\] (3.32)
According to Hölder’s inequality, Lemmas 3.4-3.5, 3.8 and 3.10, one can obtain that

\[ J_9 = - \int r^m u u_r u_t dr \leq C |u|_\infty |r^{\frac{m}{2}} u_r|_2 |r^{\frac{m}{2}} u_t|_2 \]

\[ \leq C |u|_\infty |r^{\frac{m}{2}} u_r|_2^2 + \frac{1}{8} |r^{\frac{m}{2}} u_t|_2^2, \]

\[ J_{10} = - A \gamma \int r^m \rho^{\gamma - 2} \rho_r u_t dr = - \frac{A \gamma}{2 \alpha} \int r^m \rho^{\gamma - 1} (v - u) u_t dr \]

\[ \leq C \left( |(r^m \rho)^{\frac{1}{2}}|_2 |v|_\infty |\rho|^{\frac{\gamma - 3}{2}} + |\rho|^{\frac{\gamma - 1}{2}} |r^{\frac{m}{2}} u_2|_2 \right) |r^{\frac{m}{2}} u_t|_2 \]

\[ \leq C^n + \frac{1}{8} |r^{\frac{m}{2}} u_t|_2^2, \]

\[ J_{11} = 2 \alpha \int r^m \rho^{-1} \rho_r u_t dr = \int r^m (v - u) u_r u_t dr \]

\[ \leq C \left( |v|_\infty + |u|_\infty \right) |r^{\frac{m}{2}} u_r|_2 |r^{\frac{m}{2}} u_t|_2 \]

\[ \leq C^n (1 + |u|_\infty^2)|r^{\frac{m}{2}} u_r|_2^2 + \frac{1}{8} |r^{\frac{m}{2}} u_t|_2^2, \]

\[ J_{12} = - 2 \alpha m \int r^{m-2} u u_t dr \]

\[ \leq C |r^{\frac{m}{2}} u_2|_2 |r^{-2}|_\infty |r^{\frac{m}{2}} u_t|_2 \leq C^n + \frac{1}{8} |r^{\frac{m}{2}} u_t|_2^2. \]

Substituting the estimates for \( J_i \) (\( i = 9, \cdots, 12 \)) into (3.32), one gets

\[ \frac{d}{dt} |r^{\frac{m}{2}} u_r|_2^2 + |r^{\frac{m}{2}} u_t|_2^2 \leq C^n (1 + |u|_\infty^2)|r^{\frac{m}{2}} u_r|_2^2 + C^n, \quad (3.33) \]

which, along with the Gronwall inequality and Lemma 3.9, yields that

\[ |r^{\frac{m}{2}} u_r(t, \cdot)|_2^2 + \int_0^t |r^{\frac{m}{2}} u_t(s, \cdot)|_2^2 ds \leq C^n \quad \text{for} \quad 0 \leq t \leq T. \quad (3.34) \]

Consequently, it follows from (3.34) and Lemmas 3.8-3.9 that

\[ |\psi(t, \cdot)|_\infty \leq C \left( |v(t, \cdot)|_\infty + |u(t, \cdot)|_\infty \right) \leq C^n (1 + \|u(t, \cdot)\|_1) \leq C^n \quad \text{for} \quad 0 \leq t \leq T. \quad (3.35) \]

Second, according to (28.2)\(_2\), Lemmas 3.4-3.5, 3.8 and (3.34)-(3.35), one has

\[ |r^{\frac{m}{2}} u_{rr}|_2 \]

\[ \leq C \left( |r^{\frac{m}{2}} u_t|_2 + |r^{\frac{m}{2}} u_{rr}|_2 + |r^{\frac{m}{2}} \phi_r|_2 + |r^{\frac{m}{2}} \psi u_r|_2 + |r^{\frac{m}{2}} \psi u_t|_2 + |r^{\frac{m}{2}} u_2|_2 \right) \]

\[ \leq C^n \left( |r^{\frac{m}{2}} u_t|_2 + |r^{\frac{m}{2}} u_{rr}|_2 + |r^{\frac{m}{2}} u_r|_2 + \left| (r^m \rho)^{\frac{1}{2}} |v|_\infty |\rho|^{\frac{\gamma - 3}{2}} + |r^{\frac{m}{2}} u_2|_2 |\rho|^{\frac{\gamma - 1}{2}} \right|_{\infty} \right) \]

\[ + |r^{\frac{m}{2}} u_r|_2 |\psi|_\infty + \|u_1\|_1 \leq C^n (1 + |r^{\frac{m}{2}} u_t|_2), \quad (3.36) \]
which, along with (3.34), yields that

\[
\int_0^t |r^{\frac{m}{a}} u_{rr}(s, \cdot)|^2 ds \leq C_0 \int_0^t (1 + |r^{\frac{m}{a}} u_t(s, \cdot)|^2) ds \leq C^a,
\]

\[
\int_0^t |r^{\frac{m}{a}} u_r(s, \cdot)|_\infty ds \leq C \int_0^t \|r^{\frac{m}{a}} u_r(s, \cdot)\|_1 ds
\]

\[
\leq C^a \int_0^t (1 + |r^{\frac{m}{a}} u_{rr}(s, \cdot)|^2) ds \leq C^a.
\]

The proof of Lemma 3.11 is complete. \( \square \)

3.5. The lower order estimates of \( \psi \). Now we show the estimates on \( |r^{\frac{m-2}{2}} \psi|_2 \) and \( |r^{\frac{m}{a}} \psi|_q \) for some \( q \in (3, 6] \).

Lemma 3.12. Assume \( \gamma > \frac{3}{2} \) additionally. Then for any \( T > 0 \), it holds that

\[
|r^{\frac{m-2}{2}} \psi(t, \cdot)|_2 + |r^{\frac{m}{a}} \psi(t, \cdot)|_q \leq C^a \quad \text{for } 0 \leq t \leq T.
\]

Proof. First, multiplying (3.7) by \( r^m |v|^{q-2} v \) and integrating the resulting equation over \( I_a \), it follows from Lemma 3.5 and (3.2)\(_4\)-(3.2)\(_5\) that

\[
\frac{d}{dt} |r^{\frac{m}{a}} v|_q \leq C \left( |u_r|_\infty + |r^{-1}|_\infty |u|_\infty \right) |r^{\frac{m}{a}} v|_q + |\rho|^{-1}_\infty |r^{\frac{m}{a}} v|_q + |\rho|^{-1}_\infty |r^{\frac{m}{a}} u|_q
\]

\[
\leq C^a (1 + |u_r|_\infty + |u|_\infty) |r^{\frac{m}{a}} v|_q + C^a,
\]

which, along with Lemmas 3.10-3.11 and the Gronwall inequality, yields that

\[
|r^{\frac{m}{a}} v(t, \cdot)|_q \leq C^a \quad \text{for } 0 \leq t \leq T.
\]

It thus follows from the definitions of \( (v, \psi) \) in (3.5), (3.27), (3.37) and Lemmas 3.10-3.11 that

\[
|r^{\frac{m}{a}} \psi(t, \cdot)|_q \leq C \left( |r^{\frac{m}{a}} v(t, \cdot)|_q + |r^{\frac{m}{a}} u(t, \cdot)|_q \right) \leq C^a \quad \text{for } 0 \leq t \leq T.
\]

Second, repeating the above procedure again, one can similarly get

\[
|r^{\frac{m-2}{2}} \psi|_2 \leq C^a |r^{\frac{m}{a}} \psi|_2 \leq C^a \left( |r^{\frac{m}{a}} v(t, \cdot)|_2 + |r^{\frac{m}{a}} u(t, \cdot)|_2 \right) \leq C^a \quad \text{for } 0 \leq t \leq T.
\]

The proof of Lemma 3.12 is complete. \( \square \)

3.6. The second order estimates of \( r^{\frac{m}{a}} u \). Now we show the estimate of \( |r^{\frac{m}{a}} u_{rr}|_2 \).

Lemma 3.13. Assume \( \gamma > \frac{3}{2} \) additionally. Then for any \( T > 0 \), it holds that

\[
(|r^{\frac{m}{a}} u_t|^2_2 + |u_{rr}|^2_2 + |r^{\frac{m}{a}} u_{rrr}|^2_2) (t, \cdot) + \int_0^t (|r^{\frac{m}{a}} u_t|^2_2 + |r^{\frac{m-2}{2}} u_{rr}|^2_2) (s, \cdot) ds \leq C^a
\]

for \( 0 \leq t \leq T \).

Proof. First, according to Lemmas 3.4-3.5, 3.8 and 3.10, one gets

\[
|r^{\frac{m}{a}} \phi_r|_2 = |A \gamma r^{\frac{m}{a}} \rho^{-1} \rho^{-1} \rho_r|_2
\]

\[
\leq C (|\rho|^{-\frac{3}{2}} (r^m \rho)^{\frac{1}{2}} |v|_\infty + |r^{\frac{m}{a}} u|_2 |\rho|^{-1}) \leq C^a.
\]
Then it follows from (3.28) that

\[ |r^2 \phi_t| \leq C(|r^m u_\phi|_2 + |r^m \phi u_r|_2 + |r^{m-2} \phi u_r|_2) \]
\[ \leq C^a \left( |u_\infty| |r^m \phi_r|_2 + |\phi_\infty| |r^m u_r|_2 + |\phi_\infty| |u|_2 \right) \leq C^a. \]

According to (3.36), one has

\[ |u_{rr}| + |r^m u_{rr}|_2 \leq C^a \left( 1 + |r^m u_t|_2 \right). \tag{3.38} \]

Second, differentiating (3.28) with respect to \( t \), multiplying the resulting equation by \( r^m u_t \) and integrating over \( I_a \), one has

\[ \frac{1}{2} \frac{d}{dt} \left( |r^m u_t|^2_2 \right) + 2\alpha |r^m u |^2_2 + 2\alpha m |r^{m-2} u_t |^2_2 \]
\[ = - \int r^m (uu_t)_t + \phi u - 2\alpha (\psi u_t)_t u_t \, dt \triangleq \sum_{i=13}^{15} J_i. \tag{3.39} \]

Using the equations (3.28) \(_1\), (3.28) \(_3\), Hölder’s inequality, Young’s inequality, Lemmas 3.5, 3.10-3.12 and (3.38), we estimate \( J_i \) \((i = 13, 14, 15)\) as follows:

\[ J_{13} = - \int r^m (uu_t)_t u_t \, dt = - \int r^m (uu_{rr} + u^2 u_{rr}) \, dt \]
\[ \leq C \left( |u_\infty| |r^m u_{rr}|_2 |r^m u_t|_2 + |u_r|_\infty |r^m u_t|_2 \right) \]
\[ \leq C^a \left( 1 + |u|_2 \right) |r^m u_t|_2 + \frac{\alpha m}{8} |r^{m-2} u_t|_2 \]
\[ \leq C^a \left( 1 + |r^m u_t|_2 \right) |r^m u_t|_2 + \frac{\alpha m}{8} |r^{m-2} u_t|_2 , \]

\[ J_{14} = - \int r^m \phi u_t \, dt = \int \phi t (mr^{-1} u_t + r^m u_{rr}) \, dt \]
\[ \leq C \left( |r^m \phi_t|_2 |r^m u_t|_2 |r^{-1}_\infty + |r^m u_{rr}|_2 |r^m \phi_t|_2 \right) \]
\[ \leq C^a \left( 1 + |r^m u_t|_2 \right) + \frac{\alpha m}{8} |r^{m-2} u_{rr}|_2 , \]

\[ J_{15} = 2\alpha \int r^m (\psi u_t)_t u_t \, dt \]
\[ = 2\alpha \int r^m \left( \psi u_{rr} + (uu)_t + (mr^{-1} u)_t \right) u_t \, dt \]
\[ \leq C^a \left( |\psi|_\infty |r^m u_{rr}|_2 |r^m u_t|_2 + (1 + \psi|_\infty |u_\infty (|r^m u_{rr}|_2 |r^m u_t|_2 \right.
\[ + |r^m u_{rr}|_2 |r^m u_t|_2 + |r^m u_{rr}|_2 |r^m u_{rr}|_2 \right) + |r^m u_t|_2 |u_r|_\infty |r^m u_{rr}|_2 \]
\[ \leq C^a \left( 1 + |r^m u_t|_2 \right) |r^m u_t|_2 + C^a + \frac{\alpha m}{8} |r^{m-2} u_t|_2 . \]

Substituting the estimates for \( J_i \) \((i = 13, 14, 15)\) into (3.39), one can obtain

\[ \frac{d}{dt} \left( |r^m u_t|^2_2 + |r^m u_{rr}|^2_2 + |r^{m-2} u_{rr}|_2^2 \right) \leq C^a \left( 1 + (1 + |r^m u_t|_2^2 |r^m u_{rr}|_2^2 \right). \tag{3.40} \]
Integrating (3.40) over \((\tau, t)(\tau \in (0, t))\), one has
\[
|r^{\pm} u_t(t, \cdot)|_2^2 + \int_{\tau}^{t} \left(|r^{\pm} u_{tr}|_2^2 + |r^{\pm} u_r|_2^2\right)(s, \cdot)ds \\
\leq |r^{\pm} u_t(\tau, \cdot)|_2^2 + C^a \int_{\tau}^{t} (1 + |r^{\pm} u_r|_2^2) |r^{\pm} u_t|_2^2(s, \cdot)ds.  \tag{3.41}
\]

It follows from (3.28) that
\[
|r^m u_t(\tau, \cdot)|_2 \\
\leq C^a (|u|_\infty |r^m u_r|_2 + |r^m \phi_r|_2 + |r^m u_{rr}|_2 + |\psi|_\infty |r^m u_r|_2 + \|u\|_1)(\tau), \tag{3.42}
\]
which, together with the time continuity of \((\rho, u)\) and (3.3), yields that
\[
\limsup_{\tau \to 0} |r^m u_t(\tau, \cdot)|_2 \\
\leq C^a (|u_0|_\infty ||r^m u_0||_1 + |r^m (\phi_0)_r|_2 + |\psi_0|_\infty |r^m (u_0)_r|_2 + \|r^m u_0\|_2) \leq C^0_0.
\]

Letting \(\tau \to 0\) in (3.41) and using the Gronwall inequality, one can obtain
\[
|r^m u_t(t, \cdot)|_2^2 + \int_0^t \left(|r^m u_{tr}|_2^2 + |r^{\pm} u_r|_2^2\right)(s, \cdot)ds \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

It thus follows from (3.38) that
\[
|u_{rr}(t, \cdot)|_2 + |r^m u_{rr}(t, \cdot)|_2 \leq C^a (1 + |r^m u_t(t, \cdot)|_2) \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.13 is complete. \(\square\)

3.7. The high order estimates of \(r^m \phi\) and \(r^m \psi\). The following lemma provides the high order estimates of \(r^m \phi\) and \(r^m \psi\).

**Lemma 3.14.** Assume \(\gamma > \frac{3}{2}\) additionally. Then for any \(T > 0\), it holds that
\[
|r^m \phi_{rr}(t, \cdot)|^2 + |r^m \phi_{tr}(t, \cdot)|^2 + |r^m \phi_r(t, \cdot)|^2 + |r^m \psi(t, \cdot)|^2 \\
+ \int_0^t \left(|u_{rrr}|^2 + |r^{m} u_{rrr}|^2 + |r^{m} \phi_{rr}|^2\right)(s, \cdot)ds \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** The proof will be divided into two steps.

**Step 1:** the estimate on \(r^m u_{rrr}\). First, it follows from (3.28)\(_2\), Hölder’s inequality, Young’s inequality, Lemmas 3.10-3.11 and 3.13 that
\[
|r^m u_{rrr}|_2 \leq C (|r^m u_{tr}|_2 + |r^m (u_{tr})_r|_2 + |r^m \phi_{rrr}|_2 + |r^m (\psi u_r)_r|_2 \\
+ |r^m (r^{-1}u_r)_r|_2 + |r^m (r^{-2}u)_r|_2) \\
\leq C^a (|r^m u_{tr}|_2 + |u|_\infty |r^m u_{rrr}|_2 + |u_r|_\infty |r^m u_r|_2 + |r^m \phi_{rrr}|_2 \\
+ |r^m \psi_r|_2 |u_r|_\infty + |r^m u_{rrr}|_2 |\psi|_\infty + \|r^m u\|_2) \tag{3.43}
\]
\[
\leq C^a (1 + |r^m u_{tr}|_2 + |r^m \psi_r|_2 + |r^m \phi_{rrr}|_2).
\]

Second, considering the last two terms on the right-hand side of (3.43), for \(|r^m \psi_r|_2\), differentiating (3.28)\(_3\) with respect to \(r\), multiplying the resulting equation by \(r^m \psi_r\) and
integrating over $I_a$, one can obtain
\[
\frac{d}{dt}[m \psi_r]_2^2 = -2 \int r^m ((\psi u)_{rr} + u_{rrr} + m(r^{-1}u)_r) \psi_r \, dr
\]
\[
\leq C^a [r^m \psi_r]_2^2 (\|u\|_2 + |\psi|_\infty |r^m u_{rrr}|_2 + |r^m u_{rrr}|_2 + \|r^m u\|_2)
\]
\[
\leq C^a (1 + |r^m \psi_r|_2^2 + |r^m u_{rrr}|_2^2)
\]
\[
\leq C^a (1 + |r^m \psi_r|_2^2 + |r^m u_{tr}|_2^2 + |r^m \phi_{rrr}|_2^2),
\]
where one has used (3.43), Lemmas 3.10-3.11, 3.13.

For $|r^m \phi_{rrr}|_2$. Applying $\phi_r$ to (3.28)$_1$, multiplying the resulting equation by $r^m \phi_{rrr}$ and integrating over $I_a$, one has
\[
\frac{d}{dt}[r^m \phi_{rrr}]_2^2 = -2 \int r^m ((u \phi_r)_{rrr} + (\gamma - 1)(\phi u_r)_{rrr} + m(\gamma - 1)(r^{-1}u)_r) \phi_{rrr} \, dr
\]
\[
\leq C^a (|u_r|_\infty |r^m \phi_{rrr}|_2^2 + |r^m u_{rrr}|_2 |r^m \phi_{rrr}|_2 |\phi_r|_\infty
\]
\[
+ |\phi|_\infty |r^m \phi_{rrr}|_2 (|r^m u_{rrr}|_2 + |r^m u_{rrr}|_2) + |u|_\infty |r^m \phi_{rrr}|_2
\]
\[
+ |r^m \phi_{rrr}|_2 |r^m u_r|_\infty |\phi_r|_1 + |r^m \phi_{rrr}|_2 |\phi|_1 |u|_1)
\]
\[
\leq C^a (1 + |r^m \phi_{rrr}|_2^2 + |r^m u_{rrr}|_2^2)
\]
\[
\leq C^a (1 + |r^m \psi_r|_2^2 + |r^m u_{tr}|_2^2 + |r^m \phi_{rrr}|_2^2),
\]
where one has used (3.43) and Lemmas 3.10-3.11, 3.13.

Combining (3.44)-(3.45), one can obtain that
\[
\frac{d}{dt}[r^m \psi_r]_2^2 + |r^m \phi_{rrr}|_2^2 \leq C^a (1 + |r^m u_{tr}|_2^2 + |r^m \psi_r|_2^2 + |r^m \phi_{rrr}|_2^2),
\]
which, along with the Gronwall inequality and Lemma 3.13, yields that
\[
|r^m \psi_r(t, \cdot)|_2 + |r^m \phi_{rrr}(t, \cdot)|_2 \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]
At last, it follows from (3.43), (3.46) and Lemma 3.13 that
\[
\int_0^t (|u_{rrr}|_2^2 + |r^m u_{rrr}|_2^2)(s, \cdot) \, ds
\]
\[
\leq C^a \int_0^t (1 + |r^m \psi_r|_2^2 + |r^m u_{tr}|_2^2 + |r^m \phi_{rrr}|_2^2)(s, \cdot) \, ds \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

**Step 2:** estimates on time derivatives of the density. First, according to (3.28)$_3$, Hölder’s inequality, Lemmas 3.10-3.11, 3.13 and (3.46), one has
\[
|r^m \psi_t| \leq C(r^m (\psi u)_r|_2 + |r^m u_{rrr}|_2 + |r^m (r^{-1}u)_r|_2)
\]
\[
\leq C^a (|r^m u_r|_2 |\psi|_\infty + |r^m \psi_r|_2 |u|_\infty + \|r^m u\|_2) \leq C^a.
\]
Lemma 3.15. Assume $r$

Proof. First, differentiating (3.48) by $\frac{m}{\gamma}$, Hölder’s inequality, Lemmas 3.10-3.11, 3.13 and (3.46), one obtains

\[
|\frac{m}{\gamma} u_{tt}| \leq C \left( |\frac{m}{\gamma} (u_{tt})| + |\frac{m}{\gamma} (u_{t})| + |\frac{m}{\gamma} (-u_{t})| \right)
\]

which, along with Young’s inequality, yields that

\[
|\frac{m}{\gamma} u_{tt}| \leq C\left( |u_{tt}| + |u_{t}| + \left| \frac{m}{\gamma} (u_{t})| + |\frac{m}{\gamma} (-u_{t})| \right) \right)
\]

which, along with Lemma 3.13, yields that

\[
\int_{0}^{t} |\frac{m}{\gamma} u_{tt}(s, \cdot)|^{2} ds \leq C \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.14 is complete.

3.8. Time-weighted energy estimates of $r \frac{m}{\gamma} u$. At last, one gives the time-weighted energy estimates of $r \frac{m}{\gamma} u$.

Lemma 3.15. Assume $\gamma > \frac{3}{2}$ additionally. Then for any $T > 0$, it holds that

\[
t |r \frac{m}{\gamma} u_{tt}(t, \cdot)|^{2} + \int_{0}^{t} s\left( |r \frac{m}{\gamma} u_{tt}|^{2} + |r \frac{m}{\gamma} u_{tt}|^{2} \right) ds \leq C \quad 0 \leq t \leq T.
\]

Proof. First, differentiating (3.28) with respect to $t$, multiplying the resulting equation by $r \frac{m}{\gamma} u_{tt}$ and integrating over $t_{a}$, it follows from Lemmas 3.10-3.11 and 3.13-3.14 that

\[
\frac{d}{dt} |r \frac{m}{\gamma} u_{tt}|^{2} + |r \frac{m}{\gamma} u_{tt}|^{2} = - \int t \left( (u_{tt})_{t} + \phi_{tt} - 2\alpha (u_{tt})_{t} + 2\alpha r^{-2} u_{tt} - 2r \frac{m}{\gamma} \phi_{tt} \right) u_{tt} dr
\]

which, along with Young’s inequality, yields that

\[
\frac{d}{dt} |r \frac{m}{\gamma} u_{tt}|^{2} + |r \frac{m}{\gamma} u_{tt}|^{2} \leq C \left( 1 + |r \frac{m}{\gamma} u_{tt}|^{2} \right).
\]

Second, multiplying (3.48) by $s$ and integrating with respect to $s$ over $[t, \tau]$ for $t \in (0, t)$, one has

\[
t |r \frac{m}{\gamma} u_{tt}(t, \cdot)|^{2} + \int_{t}^{\tau} s |r \frac{m}{\gamma} u_{tt}(s, \cdot)|^{2} ds \leq \tau |r \frac{m}{\gamma} u_{tt}(\tau, \cdot)|^{2} + C \theta.
\]

It follows from Lemma 3.13 that

\[
r \frac{m}{\gamma} u_{tt} \in L^{2}([0, T]; L^{2}).
\]
which, along with Lemma A.3 (Appendix A), implies that there exists a sequence \( \{s_k\} \) such that
\[
s_k \to 0, \quad \text{and} \quad s_k |r^m u_{tr}(s_k, \cdot)|^2_2 \to 0 \quad \text{as} \quad k \to \infty.
\]
After choosing \( \tau = s_k \to 0 \) in (3.49), one can obtain
\[
t |r^m u_{tr}(t, \cdot)|^2_2 + \int_0^t s |r^m u_{tt}(s, \cdot)|^2_2 ds \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]
(3.50)

Finally, according to (3.28)\(_2\), Lemmas 3.10-3.11 and 3.13-3.14, one has
\[
|r^m u_{tr}|_2 \leq C (|r^m u_{tt}|_2 + |r^m (u u_{tr})|_2 + |r^m \phi_{tr}|_2 + |r^m (\psi u_{tr})|_2 + |r^m u_{tr2}| + |r^m u_t|_2) 
\leq C^a (1 + |r^m u_{tt}|_2 + |r^m u_t|_2 |u_t|_\infty + |r^m u_{tr}|_2 |u_t|_\infty + |r^m \phi_{tr}|_2 + |\psi|_\infty |r^m u_{tr}|_2 
+ |r^m \psi|_2 |u_t|_\infty + |r^m u_{tr}|_2) \leq C^a (1 + |r^m u_{tt}|_2 + |r^m u_{tr}|_2),
\]
which, along with (3.50), implies that
\[
\int_0^t s |r^m u_{tr}(s, \cdot)|^2_2 ds \leq C^a \quad \text{for} \quad 0 \leq t \leq T.
\]

The proof of Lemma 3.15 is complete. \(\square\)

3.9. **Proof of Theorem 3.1.** Based on the local-in-time well-posedness obtained in Lemma 3.1 and the global-in-time a priori estimates established in Lemmas 3.2-3.15, now we are ready to give the proof of Theorem 3.1.

**Step 1:** the global well-posedness of regular solutions. First, Lemma 3.1 guarantees that there exists a local-in-time regular solution \((\rho(t, r), u(t, r))\) to the problem (3.2) in \([0, T_*] \times I_\rho\) for some \(T_* > 0\). Let \(\bar{T} > 0\) be the life span of the regular solution shown in Lemma 3.1. It is obvious that \(\bar{T} \geq T_*\). Then we claim that \(\bar{T} = \infty\). Otherwise, if \(\bar{T} < \infty\), according to the uniform a priori estimates obtained in Lemmas 3.2-3.15 and the standard weak compactness theory, one can know that for any sequence \(\{t_k\}_{k=1}^\infty\) satisfying \(0 < t_k < \bar{T}\) and
\[
t_k \to \bar{T} \quad \text{as} \quad k \to \infty,
\]
there exists a subsequence \(\{t_{k}\}_{k=1}^\infty \subset \{t_k\}_{k=1}^\infty\) and functions \((\phi, u, \psi)(\bar{T}, r)\) such that
\[
\begin{align*}
 r^m \phi(t_{k1}, r) &\to r^m \phi(\bar{T}, r) \quad \text{in} \quad H^2 \quad \text{as} \quad k \to \infty, \\
 r^m u(t_{k1}, r) &\to r^m u(\bar{T}, r) \quad \text{in} \quad H^2 \quad \text{as} \quad k \to \infty, \\
 r^m \psi(t_{k1}, r) &\to r^m \psi(\bar{T}, r) \quad \text{in} \quad L^q \quad \text{as} \quad k \to \infty, \quad (3.51)
\end{align*}
\]
\[
\begin{align*}
 \psi(t_{k1}, r) &\to \psi(\bar{T}, r) \quad \text{in} \quad L^\infty \quad \text{as} \quad k \to \infty, \\
 r^m \psi(t_{k1}, r) &\to r^m \psi(\bar{T}, r) \quad \text{in} \quad L^2 \quad \text{as} \quad k \to \infty, \\
 r^m (\psi(t_{k1}, r))_{r} &\to r^m (\psi(\bar{T}, r))_{r} \quad \text{in} \quad L^2 \quad \text{as} \quad k \to \infty.
\end{align*}
\]

Second, we want to show that functions \((\phi, u, \psi)(\bar{T}, r)\) satisfy all the initial assumptions shown in Lemma 3.1, which include (3.3) and the following relationship between \(\phi\)
and \( \psi \):
\[
\psi = \frac{1}{\gamma - 1} \left( \ln \phi \right)_r. 
\] (3.52)

It follows from (3.51) that (3.3) except \( r^m \rho \in L^1 \) still holds at the time \( t = T \).

Next for the relation in (3.52), we need to consider the following equation
\[
\phi_t + u \phi_r + (\gamma - 1) \phi u_r + m(\gamma - 1)r^{-1} \phi u = 0, 
\] (3.53)

which holds in \([0, T) \times I_a\) in the classical sense. Actually, if we regard \((\phi(T, r), u(T, r), \psi(T, r))\) and
\[
\phi_t(T, r) = -u(T, r) \phi_r(T, r) - (\gamma - 1) \phi(T, r)(u_r(T, r) + mr^{-1}u(T, r)),
\]
as the extended definitions of \((\phi(t, r), u(t, r), \psi(t, r), \phi(t, r))\) at the time \( t = T \), then one has
\[
\text{ess sup}_{0 \leq t \leq T} \left( \|\phi(t, \cdot)\|_2 + \|\phi_t(t, \cdot)\|_1 \right) \leq C_1 < \infty,
\]
which, together with the Sobolev embedding theorem (Appendix A), implies that
\[
\phi(t, r) \in C([0, T]; H^1). 
\] (3.54)

It follows from the first line in (3.51) and the consistency of weak convergence and strong convergence in \( H^1 \) space that
\[
\phi(t_{1k}, r) \to \phi(T, r) \quad \text{in} \quad H^1 \quad \text{as} \quad k \to \infty. 
\] (3.55)

Notice that for any \( R > a \), there exists a generic constant \( C(T, R) \) such that
\[
\phi(t, r) \geq C(T, R) \quad \text{for any} \quad (t, r) \in [0, T] \times (a, R], 
\] (3.56)

which, along with the last four lines in (3.51) and (3.55), implies that (3.52) holds.

It remains to show that \( r^m \rho(T, r) \in L^1 \). According to (3.54) and the Sobolev embedding theorem (Appendix A), one can obtain
\[
\phi(t, r) \in C([0, T] \times I_a),
\]
which, along with the fact \( \rho = \left( \frac{2}{A\gamma} \right) \frac{1}{\ln \phi} \), yields that
\[
r^m \rho(t, r) \in C([0, T] \times I_a). 
\] (3.57)

Moreover, one has
\[
\int r^m \rho(t, r) dr = \int r^m \rho_0(r) dr \quad \text{and} \quad 0 \leq t < T. 
\] (3.58)

It follows from (3.57)-(3.58) and Fatou’s lemma (Appendix A) that,
\[
\int r^m \rho(T, r) dr = \int \liminf_{k \to \infty} r^m \rho(t_{1k}, r) dr \leq \liminf_{k \to \infty} \int r^m \rho(t_{1k}, r) dr < \infty,
\]

which implies that \( r^m \rho(T, r) \in L^1 \). Thus we have shown that \((\rho(T, r), u(T, r))\) satisfy all the initial assumptions on the initial data of Lemma 3.1.

Finally, if we solve (3.2) with the initial time \( \bar{T} \), then Lemma 3.1 ensures that for some constant \( T_0 > 0 \), \((\rho, u)(t, r)\) is the unique regular solution in \([\bar{T}, \bar{T} + T_0] \times I_a\) to this problem. It follows from the boundedness of all required norms of the solution \((\rho, u)(t, r)\) in \([0, \bar{T} + T_0] \times I_a\) and the standard arguments for proving the time continuity of the regular solution that, \((\rho, u)(t, r)\) is actually the unique regular solution in \([0, \bar{T} + T_0] \times I_a\).
to the problem (3.2), which contradicts to the fact that $0 < \bar{T} < \infty$ is the maximal existence time.

**Remark 3.1.** According to the proof in this subsection, the definitions of $(\phi(\bar{T}, r), u(\bar{T}, r), \psi(\bar{T}, r))$ do not depend on the choice of the time sequences $\{t_k\}_{k=1}^{\infty}$.

**Step 2:** the global well-posedness of classical solutions. Now we show that the regular solution obtained above is indeed a classical one in $(0, \infty) \times I_a$.

Actually, for any $T > 0$, first due to $\phi > 0$ and $\rho = (\gamma - 1)A\gamma \phi^{\frac{1}{\gamma-1}}$, one has $(\rho, \rho_t, \rho_r, u, u_r) \in C([0, T] \times I_a)$. (3.59)

Second, for the term $u_t$, according to Lemma 3.15, one has $t^{\frac{1}{2}}u_{tt} \in L^\infty([0, T]; L^2)$, $t^{\frac{1}{2}}u_{ttt} \in L^2([0, T]; H^{-1})$, (3.60) which, along with the classical Sobolev embedding theorem:

$L^2([0, T]; H^1) \cap W^{1,2}([0, T]; H^{-1}) \hookrightarrow C([0, T]; L^2)$, (3.61)
yields that for any $0 < \tau < T$, $tu_t \in C([0, T]; H^1)$ and $u_t \in C([\tau, T] \times I_a)$. (3.62)

Finally, it remains to show that $u_{rr} \in C([\tau, T] \times I_a)$ for any $0 < \tau < T$. Actually, it follows from the fact $\rho > 0$, (3.59) and (3.62) that $u_{rr} = \frac{1}{2\alpha}(u_t + u_{rr} + A\gamma \rho^{\gamma-2} \rho_r - 2u_{rr} \rho_r u_r - 2u_{rr} (r^{-1}u)_r) \in C([\tau, T] \times I_a)$.

The proof of Theorem 3.1 is complete.

4. **Proof of Theorems 1.1-1.2**

With Theorems 2.1 and 3.1 at hand, now we turn to the proof of Theorems 1.1-1.2. In the rest of this section, we denote $r = |x|$.

4.1. **Proof of Theorem 1.1.**

**Proof.** **Step 1:** the existence of the global unique regular solution. First, in Theorem 1.1, one assumes that the initial data $(\rho_0(x), U_0(x)) = (\rho_0(r), u_0(r)\frac{x}{r})$ be spherically symmetric and satisfy (1.17), which, along with Lemma B.1 and Remark B.1 in Appendix B, yields that the data $(\rho_0(r), u_0(r))$ satisfy (3.3). Then it follows from Theorem 3.1 that the IBVP (3.2) admits a unique global classical solution $(\rho(t, r), u(t, r))$.
Proof. It follows from the definition of \((\rho(t, r), u(t, r))\) in (4.1), direct calculations and the equations of \((\rho(t, r), u(t, r))\) in (3.2) that \((\rho(t, x), U(t, x))\) also satisfies the equations (1.1)-(1.5) pointwisely.

Second, similarly to the proof of Lemma B.1 and Remark B.1, based on the regularities of \((\rho(t, r), u(t, r))\) in (3.4), one has that for any \(T > 0\), \((\rho(t, x), U(t, x))\) satisfies the assumptions (1.10)-(1.11). In summary, we have shown that the regular solution obtained above is indeed a classical one in \((0, \infty) \times \Omega\), which can be achieved by the following lemma.

Lemma 4.1. Since
\[
(\rho, \rho_t, \rho_r, u, u_r) \in C([0, \infty) \times I_a), \quad (u, u_{rr}) \in C((0, \infty) \times I_a),
\]
then one has
\[
(\rho, \rho_t, \nabla \rho, U, \nabla U) \in C([0, \infty) \times \Omega), \quad (U_t, \nabla^2 U) \in C((0, \infty) \times \Omega).
\]

Proof. It follows from the definition of \((\rho(t, x), U(t, x))\) in (4.1) and direct calculations that
\[
\frac{\partial \rho(t, x)}{\partial x_i} = \rho(t, r) \frac{x_i}{r}, \quad \frac{\partial \rho(t, x)}{\partial t} = \rho(t, r), \quad \frac{\partial U(t, x)}{\partial t} = u(t, r) \frac{x}{r}, \quad \frac{\partial \rho(t, x)}{\partial t} = \rho(t, r) \frac{x_i}{r},
\]
\[
\frac{\partial U(t, x)}{\partial x_i} = \frac{\partial}{\partial x_i} \left( u(t, r) \frac{x_i}{r} \right) = (u(t, r))_r \frac{x_i x_j}{r^2} + u(t, r) \frac{\delta_{ij} r^2 - x_i x_j}{r^3},
\]
\[
\frac{\partial^2 U(t, x)}{\partial x_i \partial x_j} = (u(t, r))_{rr} \frac{x_i x_j x_k}{r^3} + (u(t, r))_r \frac{\delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i - 3 x_i x_j x_k}{r^4}
\]
\[
+ u(t, r) \left( \frac{3 x_i x_j x_k}{r^5} - \frac{\delta_{jk} x_i + \delta_{ij} x_k + \delta_{ik} x_j}{r^3} \right),
\]
which, along with (4.3), yields (4.4) holds.

The proof of Lemma 4.1 is complete.

□

The proof of Theorem 1.1 is complete.

□
4.2. Proof of Theorem 1.2.

Proof. **Step 1**: the cases \( V = \text{div}(hD(W)) \), or \( \text{div}(2hD(W)) \). Actually, when \( V = \text{div}(hD(W)) \), this is a special case of system (1.1) with
\[
d = 2, \quad \alpha = \frac{1}{2}, \quad \beta = 0, \quad \delta = 1 \quad \text{and} \quad \gamma = 2;
\]
and when \( V = \text{div}(2hD(W)) \), this is also a special case of system (1.1) with
\[
d = 2, \quad \alpha = 1, \quad \beta = 0, \quad \delta = 1 \quad \text{and} \quad \gamma = 2.
\]
Therefore, one simply replaces \((\rho, U)\) by \((h, W)\) in Theorem 1.1 to obtain the same conclusion for these two classes of shallow water models, without further modifications.

**Step 2**: the case \( V = \text{div}(h\nabla W) \). We still hope to prove the desired global-in-time well-posedness based on the framework established in the proof of Theorem 1.1. Since system (1.7) with \( V = \text{div}(h\nabla W) \) is not a special case of the system (1.1)-(1.5) formally, some necessary explanations need to be given.

First, according to the proof of Theorem 2.1, one has that the corresponding local-in-time well-posedness can be obtained via the completely same argument used in §2.

Second, according to the proof of Theorem 3.1, for getting the corresponding global-in-time a priori estimates, one needs to consider a reformulated problem in the spherically symmetric Eulerian coordinate.

Concerning the spherically symmetric solutions \((h(t,x), W(t,x))\) taking the form:
\[
(h, W)(t,x) = (h(t,r), w(t,r) \frac{x}{r}),
\]
then the IBVP (1.7) with (1.19)-(1.20) can be rewritten as
\[
\begin{align*}
\left\{ \begin{array}{l}
h_t + (hw)_r + \frac{hw}{r} = 0, \\
(hw)_t + (hw^2)_r + (h^2)_r - \left(h(w_r + \frac{1}{r} w)\right)_r + \frac{h_r w}{r} + \frac{hw^2}{r} = 0, \\
(h(0,r), w(0,r)) = (h_0(r), w_0(r)) \quad \text{for} \quad r \in I_a, \\
w(t,r)|_{r=a} = 0 \quad \text{for} \quad t \geq 0, \\
(h(t,r), w(t,r)) \to (0,0) \quad \text{as} \quad r \to \infty \quad \text{for} \quad t \geq 0.
\end{array} \right.
\end{align*}
\]
(4.6)

It is obvious that the problem (4.6) is actually a special case of problem (3.2) with
\[
d = 2, \quad \alpha = \frac{1}{2} \quad \text{and} \quad \gamma = 2.
\]
Therefore, one simply replaces \((\rho, u)\) by \((h, w)\) in the proof of Theorem 3.1 to obtain the same uniform a priori estimates shown in Lemmas 3.2-3.15 for this shallow water model, without further modifications.

Then the proof of Theorem 1.2 is complete. \(\Box\)
Appendix A. Basic lemmas

This appendix is devoted to listing some useful lemmas which were used frequently in the previous sections. The first one is the Sobolev embedding theorem (see Theorem 4.12 (page 85) of [1]).

Lemma A.1 ([1]). Assume \( \mathbb{D} \) is a domain in \( \mathbb{R}^d \) which satisfies the cone condition, and \((j,m,p)\) are all constants satisfying \( j \geq 0 \), \( m^* \geq 1 \) and \( 1 \leq p < \infty \).

(i) If \( m^* p < d \), then
\[
W^{j+m^*p}(\mathbb{D}) \hookrightarrow W^{j,q}(\mathbb{D}) \quad \text{for} \quad p \leq q \leq \frac{dp}{d-m^*p};
\]
(ii) if \( m^* p = d \), then
\[
W^{m^*p}(\mathbb{D}) \hookrightarrow L^q(\mathbb{D}) \quad \text{for} \quad p \leq q < \infty;
\]
(iii) if \( m^* p > d \) or \( m^* = d \) and \( p = 1 \), then
\[
W^{j+m^*p}(\mathbb{D}) \hookrightarrow C^j_B(\mathbb{D}),
\]
where \( C^j_B(\mathbb{D}) = \{ f \in C^j(\mathbb{D}) | D^\alpha f \in L^\infty(\mathbb{D}), |\alpha| \leq j \} \). The embedding constants here depend only on \( d, m^*, p, q, j \) and the property of the cone \( C \) in the cone condition.

Remark A.1 (see Section 4.6 (page 82) of [1]). \( \mathbb{D} \) satisfies the cone condition if there exists a finite cone \( C \) such that each \( x \in \mathbb{D} \) is the vertex of a finite cone \( C_x \) contained in \( \mathbb{D} \) and congruent to \( C \). Note that \( C_x \) need not to be obtained from \( C \) by parallel transformation, but simply by rigid motion.

The second one is the well-known Fatou’s lemma which can be found in [40].

Lemma A.2 ([40]). Given a measure space \((V,F, \nu)\) and a set \( X \in F \), let \( \{f_n\} \) be a sequence of \((F, B_{\mathbb{R}^d \geq 0})\)-measurable non-negative functions \( f_n : X \to [0, \infty] \). Define the function \( f : X \to [0, \infty] \) by setting
\[
f(x) = \lim \inf_{n \to \infty} f_n(x),
\]
for every \( x \in X \). Then \( f \) is \((F, B_{\mathbb{R}^d \geq 0})\)-measurable, and
\[
\int_X f(x)d\nu \leq \lim \inf_{n \to \infty} \int_X f_n(x)d\nu.
\]

The third one is used to obtain the time-weighted estimates of the velocity.

Lemma A.3 ([2]). If \( f(t,\cdot) \in L^2([0,T]; L^2(\mathcal{O})) \) \((\mathcal{O} \text{ can be any domain in } \mathbb{R}^d)\), then there exists a sequence \( \{s_k\} \) such that
\[
s_k \to 0 \quad \text{and} \quad s_k \|f(s_k,\cdot)\|^2_{L^2(\mathcal{O})} \to 0 \quad \text{as} \quad k \to \infty.
\]
Proof. Denote \( h(s) = \|f(s,\cdot)\|^2_{L^2(\mathcal{O})} \), then one can see that \( 0 \leq h(s) \in L^1([0,T]) \). We claim that for any \( k \geq 1 \), there exists \( 0 < s_k < \frac{1}{N_k} \) such that
\[
s_k h(s_k) < \frac{1}{N_k} \to 0 \quad \text{as} \quad k \to \infty,
\]
where \( N_k = N_0 + k \) and \( N_0 \) is a positive constant satisfying \( \frac{1}{N_0} \leq \frac{T}{2} \). Indeed, assume by contradiction that there exist some \( k \geq 1 \) such that for any \( s \in (0, \frac{1}{N_k}) \),

\[
sh(s) \geq \frac{1}{N_k},
\]

which yields that

\[
\int_0^T h(s) ds \geq \int_0^{\frac{N_k}{sN_k}} \frac{1}{sN_k} ds = \infty.
\]

This contradicts with the fact \( h(s) \in L^1([0, T]) \). Thus the claim holds.

The proof of Lemma A.3 is complete. \( \Box \)

The following one is on compactness theories obtained via the Aubin-Lions Lemma.

**Lemma A.4** ([41]). Let \( X_0 \subset X \subset X_1 \) be three Banach spaces. Suppose that \( X_0 \) is compactly embedded in \( X \) and \( X \) is continuously embedded in \( X_1 \). Then the following statements hold

1. If \( J \) is bounded in \( L^p([0, T]; X_0) \) for \( 1 \leq p < \infty \), and \( \frac{\partial J}{\partial t} \) is bounded in \( L^1([0, T]; X_1) \), then \( J \) is relatively compact in \( L^p([0, T]; X) \).

2. If \( J \) is bounded in \( L^\infty([0, T]; X_0) \) and \( \frac{\partial J}{\partial t} \) is bounded in \( L^p([0, T]; X_1) \) for \( p > 1 \), then \( J \) is relatively compact in \( C([0, T]; X) \).

The last one provides the regularity estimates for the Lamé operator in the exterior domain \( \Omega \) in \( \mathbb{R}^d \) (\( d = 2 \) or \( 3 \)). We consider

\[
\begin{aligned}
&LU = -\alpha \Delta U - \alpha \nabla \text{div} U = F \quad \text{in} \quad \Omega, \\
&U|_{\partial \Omega} = 0, \quad U(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{aligned}
\]

(A.1)

**Lemma A.5** ([9]). Let \( \Omega \) be an exterior domain in \( \mathbb{R}^d \) with smooth boundary. If \( U \in D^{1,\varphi}(\Omega) \) is a weak solution to (A.1), then for any \( 1 < q < \infty \),

\[
\|U\|_{D^{k+2,\varphi}(\Omega)} \leq C\left(\|F\|_{W^{k,\varphi}(\Omega)} + \|U\|_{D^{1,\varphi}(\Omega)}\right),
\]

where \( C \) is a positive constant independent of \( (U, F) \).

**APPENDIX B. Conversion of Sobolev spaces**

In order to understand the well-posedness theories established in Theorems 1.1-1.2, 2.1 and 3.1 clearly, this appendix is devoted to showing the conversion of some Sobolev spaces between the pure M-D coordinate and the spherically symmetric coordinate.

**Lemma B.1.** Let \( m = d - 1 \). If the initial data \((\rho_0(x), U_0(x))\) have the following form:

\[
\rho_0(x) = \rho_0(r), \quad U_0(x) = u_0(r)\frac{x}{r},
\]

(B.1)
where $r = |x|$, then the following two assertions are equivalent:

(i) $\left( \rho_0^{-1}(x), U_0(x) \right) \in H^2(\Omega)$, \quad $\nabla \ln \rho_0(x) \in L^q(\Omega) \cap L^\infty(\Omega) \cap D^1(\Omega)$.

(ii) $r \frac{m}{3} \left( \rho_0^{-1}(r), (\rho_0^{-1}(r))_r, r^{-1}(\rho_0^{-1}(r))_r, (\rho_0^{-1}(r))_{rr} \right) \in L^2(I_a)$,

\begin{align*}
&\quad r \frac{m}{3} \left( r^{-2} u_0(r), r^{-1} u_0(r), (u_0(r))_r, (u_0(r))_{rr} \right) \in L^2(I_a), \\
&\quad r \frac{m}{3} \left( (\rho_0(r))_r \right) \in L^q(I_a), \quad r \frac{m}{3} \left( r^{-1}(\ln \rho_0(r))_r, (\ln \rho_0(r))_{rr} \right) \in L^2(I_a), \\
&\quad (\ln \rho_0(r))_r \in L^\infty(I_a).
\end{align*}

**Proof.** First, it follows from direct calculations that

\begin{align*}
\frac{\partial \rho_0^{-1}(x)}{\partial x_i} =& (\rho_0^{-1}(r))_r \frac{x_i}{r}, \quad \frac{\partial \ln \rho_0(x)}{\partial x_i} = (\ln \rho_0(r))_r \frac{x_i}{r}, \\
\frac{\partial^2 \rho_0^{-1}(x)}{\partial x_i \partial x_j} =& (\rho_0^{-1}(r))_{rr} \frac{x_i x_j}{r^2} + (\rho_0^{-1}(r))_r \frac{\delta_{ij} r^2 - x_i x_j}{r^3}, \\
\frac{\partial U_0(x)}{\partial x_i} =& \frac{\partial}{\partial x_i} \left( u_0(r) \frac{x_i}{r} \right) = (u_0(r))_r \frac{x_i x_j}{r^2} + u_0(r) \frac{\delta_{ij} r^2 - x_i x_j}{r^3}, \\
\frac{\partial^2 U_0(x)}{\partial x_i \partial x_j} =& (u_0(r))_{rr} \frac{x_i x_j x_k}{r^3} + (u_0(r))_r \left( \frac{\delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i}{r^2} - 3 \frac{x_i x_j x_k}{r^4} \right) \\
&\quad + u_0(r) \left( \frac{3 x_i x_j x_k}{r^5} - \frac{\delta_{jk} x_i + \delta_{ij} x_k + \delta_{ik} x_j}{r^3} \right), \\
\frac{\partial^2 \ln \rho_0(x)}{\partial x_i \partial x_j} =& (\ln \rho_0(r))_{rr} \frac{x_i x_j}{r^2} + (\ln \rho_0(r))_r \frac{\delta_{ij} r^2 - x_i x_j}{r^3}.
\end{align*}

Second, via introducing the unit spherical coordinate transformation in $\mathbb{R}^d (d = 2, 3)$

\begin{align*}
x'_1 &= \cos \varphi_1, \quad x'_2 = \sin \varphi_1 \cos \varphi_2, \\
\vdots \\
x'_{d-1} &= \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{d-2} \cos \varphi_{d-1}, \\
x'_d &= \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{d-2} \sin \varphi_{d-1}, \tag{B.3}
\end{align*}

where $x' = (x'_1, \ldots, x'_d) \in \Omega'$, $\Omega'$ is the unit sphere in $\mathbb{R}^d$, $\varphi_k \in [0, \pi]$ ($k = 1, \ldots, d - 2$) and $\varphi_{d-1} \in [0, 2\pi]$, then one can obtain that for any real integrable function $f(x)$

\begin{align*}
\int_{\Omega} f(x) dx = \int_{\Omega'} \int_{a}^{\infty} f(x') r^m dr dx'. \tag{B.4}
\end{align*}

Therefore, one can easily verify that (i) and (ii) are equivalent by using (B.1)-(B.4).

\[ \square \]

**Remark B.1.** For any fixed constant $a > 0$, one can also show that (ii) is equivalent to

\begin{align*}
&\quad r \frac{m}{3} \left( \rho_0^{-1}(r), u_0(r) \right) \in H^2(I_a), \quad r \frac{m}{3} \left( \ln \rho_0(r) \right)_r \in L^q(I_a), \\
&\quad r \frac{m}{3} \left( r^{-1}(\ln \rho_0(r))_r, (\ln \rho_0(r))_{rr} \right) \in L^2(I_a), \quad (\ln \rho_0(r))_r \in L^\infty(I_a).
\end{align*}
Indeed, according to Lemma B.1 and the following facts

\[ (r^{\frac{m}{2}} \rho_0^{-1}(r))_r = \frac{m}{2} r^{\frac{m-2}{2}} \rho_0^{-1}(r) + r^{\frac{m}{2}} (\rho_0^{-1}(r))_r, \]

\[ (r^{\frac{m}{2}} \rho_0^{-1}(r))_{rr} = \frac{m(m-2)}{4} r^{\frac{m-4}{2}} \rho_0^{-1}(r) + m r^{\frac{m-2}{2}} (\rho_0^{-1}(r))_r + r^{\frac{m}{2}} (\rho_0^{-1}(r))_{rr}, \]

\[ (r^{\frac{m}{2}} u_0(r))_r = \frac{m}{2} r^{\frac{m-2}{2}} u_0(r) + r^{\frac{m}{2}} (u_0(r))_r, \]

\[ (r^{\frac{m}{2}} u_0(r))_{rr} = \frac{m(m-2)}{4} r^{\frac{m-4}{2}} u_0(r) + m r^{\frac{m-2}{2}} (u_0(r))_r + r^{\frac{m}{2}} (u_0(r))_{rr}, \]

one can get the desired conclusion. Therefore, the initial assumption (3.3) in Lemma 3.1 is reasonable.

Acknowledgement: The research was supported in part by National Natural Science Foundation of China under the Grant 12101395. The research of Shengguo Zhu was also supported in part by The Royal Society–Newton International Fellowships Alumni AL/201021 and AL/211005.

Conflict of Interest: The authors declare that they have no conflict of interest. The authors also declare that this manuscript has not been previously published, and will not be submitted elsewhere before your decision.

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