Supporting Lemmas for RISE-based Control Methods

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Abstract

A class of continuous controllers termed Robust Integral of the Signum of the Error (RISE) have been published over the last decade as a means to yield asymptotic convergence of the tracking error for classes of nonlinear systems that are subject to exogenous disturbances and/or modeling uncertainties. The development of this class of controllers relies on a property related to the integral of the signum of an error signal. A proof for this property is not available in previous literature. The stability of some RISE controllers is analyzed using differential inclusions. Such results rely on the hypothesis that a set of points is Lebesgue negligible. This paper states and proves two lemmas related to the properties.

I. INTRODUCTION

A class of continuous controllers termed Robust Integral of the Signum of the Error (RISE) have been published over the last decade as a means to yield asymptotic convergence of the tracking error for classes of nonlinear systems that are subject to exogenous disturbances and/or modeling uncertainties. RISE-based controllers all exploit a property that is instrumental for yielding an asymptotic result in the presence of disturbances. Specifically, all RISE controllers exploit the fact that the integral

\[
\int_0^x f'(y) \text{sgn}(f(y)) \, dy \approx |f(x)| - |f(0)|
\]

as a means to prove the candidate Lyapunov function is positive definite (cf. [1]–[14] and the references therein). However, no accessible proof of this fact is available. Lemmas 1 in this paper provides a proof for the property.

Motivated by robustness to measurement noise, the analysis of recent RISE-based control designs is performed using non-smooth analysis techniques (cf. [14]–[16]). To facilitate Lyapunov-based stability analysis, a majority of RISE controllers use the Mean Value Theorem to compute a proof for the property. Lemmas 2–4 in this paper provide proofs that validate the fact and further generalizations. Throughout the paper, the notation \(A^c\) is used to denote the complement of the set \(A\).

To facilitate Lyapunov-based stability analysis, a majority of RISE controllers use the Mean Value Theorem to compute a strictly increasing function that bounds the unknown functions in the system dynamics. Lemma 5 in this paper provides a constructive proof of existence of a strictly increasing bound.

II. MAIN RESULTS

**Lemma 1.** Let \(f : \mathbb{R}_+ \to \mathbb{R}\) be locally absolutely continuous. Then, \(\int_0^x f'(y) \text{sgn}(f(y)) \, dy = |f(x)| - |f(0)|\).

**Proof:** Using the fundamental theorem of calculus, local absolute continuity of \(f\) implies that \(f'\) exists almost everywhere and that \(f'\) is locally integrable. Since \(\text{sgn}(f)\) is bounded, \(f'\text{sgn}(f)\) is locally integrable. Thus, for each \(x\), \(\int_0^x f'(y) \text{sgn}(f(y)) \, dy < \infty\). Since \(f\) is continuous, \(f^{-1}(\{0\})\) is closed which means that \(f \neq 0\) only on an open subset \(O \subset [0, x]\). The open subset \(O\) can be written as an at-most countable union of mutually disjoint intervals. On some of these intervals \(\text{sgn}(f) = 1\) and on the rest, \(\text{sgn}(f) = -1\). Define a sequence of functions \((g_n)_{n=1}^\infty : \mathbb{R}_+ \to \mathbb{R}\) as

\[
g_n(y) \triangleq \begin{cases} 
\sum_{j=1}^n I_j(y) - \sum_{k=1}^n I_k(y) & \text{if } y \in O, \\
0 & \text{otherwise},
\end{cases}
\]

where \(I_j = (a_j, b_j)\) and \(I_k = (c_k, d_k)\) are the (disjoint) intervals where \(\text{sgn}(f) = +1\) or \(-1\), respectively, arranged such that \(a_j > b_{j-1}\) for all \(j > 1\) and \(c_k > d_{k-1}\) for all \(k > 1\), and \(1\) denotes the indicator function defined as \(1_I(x) \triangleq \begin{cases} 
1, & \text{if } x \in I \\
0, & \text{otherwise}.
\end{cases}\)

Then, \(g_n \to \text{sgn}(f)\) point-wise on \([0, x]\) as \(n \to \infty\). Since \(f'\) is locally integrable and \([0, x]\) is compact, \(f'\) is integrable, and hence, essentially bounded on \([0, x]\). Thus, \(f'g_n \to f'\text{sgn}(f)\) point-wise a.e. on \([0, x]\). Let \(M = \text{ess sup}_{y \in [0, x]} f'(y)\). Then,
Furthermore, since $f(y) \leq M$ for almost all $y \in [0, x]$, and hence, by the Dominated convergence theorem [18],

$$
\int_0^x f'(y) \text{sgn} (f(y)) \, dy = \lim_{n \to \infty} \int_0^x f'(y) g_n(y) \, dy = \lim_{n \to \infty} \int_0^x f'(y) \left( \sum_{j=1}^n 1_{I_j}(y) - \sum_{k=1}^n 1_{I_k}(y) \right) \, dy
$$

$$
= \lim_{n \to \infty} \int_0^x \left( \sum_{j=1}^n f'(y) 1_{I_j}(y) - \sum_{k=1}^n f'(y) 1_{I_k}(y) \right) \, dy = \lim_{n \to \infty} \left( \sum_{j=1}^n \int_0^x f'(y) 1_{I_j}(y) \, dy - \sum_{k=1}^n \int_0^x f'(y) 1_{I_k}(y) \, dy \right).
$$

Using the fundamental theorem of calculus, local absolute continuity of $f$ implies that \( \int_0^x f'(y) 1_{I_j}(y) \, dy = f(b_j) - f(a_j) \) and \( \int_0^x f'(y) 1_{I_k}(y) \, dy = f(d_k) - f(c_k) \). Thus

$$
\int_0^x f'(y) \text{sgn} (f(y)) \, dy = \lim_{n \to \infty} \left( \sum_{j=1}^n (f(b_j) - f(a_j)) - \sum_{k=1}^n (f(d_k) - f(c_k)) \right).
$$

Since $f = 0$ outside the open intervals $I_j$ and $I_k$, we get $f(b_j) = f(d_k) = 0$ and $f(a_j) = f(b_j) = f(c_k) = f(d_k) = 0$ for all $2 \leq j, k < \infty$. Furthermore,

$$
\lim_{j \to \infty} f(a_j) = \lim_{k \to \infty} f(c_k) = 0, \quad (1)
$$

and

$$
\int_0^x f'(y) \text{sgn} (f(y)) \, dy = \lim_{j \to \infty} (f(b_j) - f(a_j)) - \lim_{k \to \infty} (f(d_k) - f(c_k)) - (f(a_1) - f(c_1)). \quad (2)
$$

To evaluate $T_2$, consider the following cases:

**Case 1:** $f(0) = 0$. In this case, since $f = 0$ outside the open intervals $I_j$ and $I_k$, we get $f(a_1) = f(c_1) = 0$.

**Case 2:** $f(0) > 0$. In this case, $a_1 = 0$, and hence, $f(a_1) = f(0)$. Since $f = 0$ outside the open intervals $I_k$, $f(c_1) = 0$. Thus, $f(0) > 0 \implies (f(a_1) - f(c_1)) = f(0)$.

**Case 3:** $f(0) < 0$. In this case, $c_1 = 0$, and hence, $f(c_1) = f(0)$. Since $f = 0$ outside the open intervals $I_j$, $f(a_1) = 0$. Thus, $f(0) < 0 \implies (f(a_1) - f(c_1)) = -f(0)$.

Thus,

$$
(f(a_1) - f(c_1)) = |f(0)|. \quad (3)
$$

To evaluate $T_1$, consider the following cases:

**Case 1:** $f(x) = 0$. In this case, since $f = 0$ outside the open intervals $I_j$ and $I_k$, we get $\lim_{j \to \infty} f(b_j) = \lim_{k \to \infty} f(d_k) = 0$, which from $1$ implies $T_1 = 0$.

**Case 2:** $f(x) > 0$. In this case, $\lim_{j \to \infty} b_j = x$, which from continuity of $f$ implies that $\lim_{j \to \infty} f(b_j) = f(x)$. Furthermore, since $f = 0$ outside the open intervals $I_k$, we get $\lim_{k \to \infty} f(d_k) = 0$. Thus, $T_1 = f(x)$.

**Case 3:** $f(x) < 0$. In this case, $\lim_{k \to \infty} d_k = x$, which from continuity of $f$ implies that $\lim_{k \to \infty} f(d_k) = f(x)$. Furthermore, since $f = 0$ outside the open intervals $I_j$, we get $\lim_{j \to \infty} f(b_j) = 0$. Thus, $T_1 = -f(x)$.

Thus,

$$
\left| \lim_{j \to \infty} (f(b_j) - f(a_j)) - \lim_{k \to \infty} (f(d_k) - f(c_k)) \right| = |f(x)|. \quad (4)
$$

From 2, 3 and 4 the required result is follows.

**Lemma 2.** Let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function. Then,

$$
\mu \left\{ x \mid f(x) = 0 \wedge f'(x) \neq 0 \right\} = 0,
$$

where $\mu$ denotes the Lebesgue measure on $[0, \infty)$.

**Proof:** Let $A \triangleq \{ x \mid f(x) = 0 \wedge f'(x) \neq 0 \} \subseteq [0, \infty)$. Note that $A = \left\{ f^{-1} (\{0\}) \right\} \cap \left\{ f^{-1} (\{0\}) \right\}^c$, and hence, $A$ is measurable. The first step is to prove that all the points in the set $A$ are isolated. That is,

$$
(\forall a \in A) \left( \exists \epsilon > 0 \mid ((a - \epsilon, a + \epsilon) \cap A) \setminus \{a\} = \emptyset \right). \quad (6)
$$

The negation of (6) is

$$
(\exists a \in A) \mid (\forall \epsilon > 0) \left( ((a - \epsilon, a + \epsilon) \cap A) \setminus \{a\} \neq \emptyset \right). \quad (7)
$$
For the sake of contradiction, assume that (3) is true. Thus, there exists a \( b \in ((a - \epsilon, a + \epsilon) \cap A) \setminus \{a\} \). Without loss of generality, let \( b > a \) and \( f'(a) > 0 \). As \( f \) is differentiable and \( f(a) = f(b) = 0 \), by the Mean Value Theorem, \( \exists c \in (a, b) \) such that

\[
f'(c) = 0.
\]  

(8)

By continuity of \( f' \) at \( a \),

\[(\forall \epsilon_a > 0) \ (\exists \delta_a > 0) \ | x - a | \ | f'(a) - \epsilon_a < f'(x) < f'(a) + \epsilon_a \].

In particular, pick \( \epsilon_a = f'(a) \). Then,

\[(\exists \delta_a > 0) \ (\forall x \in [0, \infty)) \ | x - a | < \delta_a \implies f'(x) > 0 \].

(9)

Now, pick \( \epsilon = \delta_a \) in (7). Thus, \( b \in ((a - \delta_a, a + \delta_a) \cap A) \setminus \{a\} \). Since \( |b - a| < \delta_a \), and \( c \in (a, b) \), it can be concluded that \( |c - a| < \delta_a \). Thus, from (9), \( f'(c) > 0 \), which contradicts (3). Thus, all the points in the set \( A \) are isolated, and hence, \( A \) is a discrete set. Using the fact that any discrete subset of \( \mathbb{R} \) is countable, (3) follows.

The following two lemmas generalize the above result.

**Lemma 3.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an everywhere differentiable function. The set \( E = \{ a \in \mathbb{R} : f(a) = 0 \text{ and } f'(a) \neq 0 \} \) is countable.

**Proof:** If \( E \) is empty, then it is countable. Now suppose that \( E \) is nonempty. We will show that \( E \) is composed of only isolated points. Let \( a \in E \), and consider the first order Taylor expansion of \( f \):

\[ f(x) = f(a) + f'(a)(x - a) + \epsilon(x) \]

First note that:

\[
\frac{\epsilon(x)}{x - a} = \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = \frac{f(x) - f(a)}{x - a} - f'(a) \to 0
\]

as \( x \to a \).

Now pick a \( \delta > 0 \) such that \( |\epsilon(x)/(x - a)| < |f'(a)| \) for \( x \in (a - \delta, a + \delta) \). For this neighborhood we have (with \( x \neq a \)):

\[ |f(x)| = \left| f'(a)(x - a) + \frac{\epsilon(x)}{x - a} \right| \geq |x - a| \left| f'(a) \right| - |\epsilon(x)/x - a| > 0 \]

Therefore we have \( f(x) \neq 0 \) in the neighborhood \( (a - \delta, a + \delta) \) unless \( x = a \). Hence each point in \( E \) is isolated, and therefore \( E \) is countable. By the proof of this theorem we can also weaken the everywhere differentiability and find that the set:

\[ E = \{ a \in \mathbb{R} : f \text{ is differentiable at } a, f(a) = 0, f'(a) \neq 0 \} \]

is countable.

**Lemma 4.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Consider the set

\[ E = \left\{ a \in \mathbb{R} : \liminf_{x \to a} \frac{f(x) - f(a)}{x - a} > 0 \text{ or } \limsup_{x \to a} \frac{f(x) - f(a)}{x - a} < 0, f(a) = 0 \right\}.
\]

This set is countable.

**Proof:** Suppose that \( E \) has some accumulation point \( a \in \mathbb{R} \). This means there is a sequence of points \( \{a_n\} \subset E \) such that \( \lim a_n = a \). Without loss of generality we may assume that

\[
\liminf_{n \to a} \frac{f(x) - f(a)}{x - a} > 0.
\]

This means for any sequence \( x_n \to a \) for which the sequence \( \frac{f(x_n) - f(a)}{x_n - a} \) converges, the limit of that convergent sequence is greater than zero.

However, since \( f(a_n) = 0 \) and \( f(a) = 0 \) we have

\[
\frac{f(a_n) - f(a)}{a_n - a} = 0
\]

for all \( n \). A contradiction. Thus every point is isolated and \( E \) is countable.

**Lemma 5.** Let \( D \subseteq \mathbb{R}^n \) be an open and connected set containing the origin. Let \( B_r \subset D \) denote the closed ball of radius \( r > 0 \) centered at the origin and let \( f : D \to \mathbb{R}^m = [f_1, f_2, \cdots, f_m]^T \) be a differentiable function such that \( \|x\| < \infty \implies \|f(x)\|, \|\nabla f_i(x)\| < \infty \) for all \( x \in D \). Then there exists a strictly increasing function \( \rho : [0, \infty) \to [0, \infty) \) such that \( \|f(x) - f(x_d)\| \leq \rho(\|x - x_d\|) \|x - x_d\| \) for all \( x \in D \) and \( x_d \in B_r \).
Proof: Using the Mean Value Theorem, \(\forall i = 1, \cdots, m\), and for all \(x, x_d \in D\) there exist \(0 < c_i < 1\) such that
\[
f_i (x) - f_i (x_d) = \nabla f_i |_{c_i x + (1 - c_i) x_d} \cdot (x - x_d).
\]
Using the Cauchy-Schwarz inequality,
\[
\| f(x) - f (x_d) \| = \sqrt{\sum_{i=1}^{m} \left| f_i (x) - f_i (x_d) \right|^2} \\
= \sqrt{\sum_{i=1}^{m} \left( \nabla f_i |_{c_i x + (1 - c_i) x_d} \cdot (x - x_d) \right)} \\
\leq \sqrt{\sum_{i=1}^{m} \| \nabla f_i |_{c_i (x - x_d) + x_d} \|^2} \| (x - x_d) \| \\
\leq \sqrt{\sum_{i=1}^{m} \max_i \| \nabla f_i |_{c_i (x - x_d) + x_d} \|^2} \| (x - x_d) \|.
\]
where the function \(G_1 : \mathbb{R}^n \times D \to [0, \infty)\) is defined as
\[
G_1 (x - x_d, x_d) \triangleq \sqrt{\max_{(\sigma, \omega) \in B_{xy}} \| \nabla f_i |_{c_i (x - x_d) + x_d} \|^2} \\
\text{such that } G_2 (\|x - x_d\|, \|x_d\|) \geq G_1 (x - x_d, x_d) \text{ for all } (x, x_d) \in D \times B_r. \text{ Thus,}
\]
\[
\| f(x) - f (x_d) \| \leq G_2 (\|x - x_d\|, \|x_d\|) \| (x - x_d) \|. \tag{10}
\]
To obtain a strictly increasing function, define a set \(B_{xy} \subseteq \mathbb{R} \times \mathbb{R}\) as
\[
B_{xy} \triangleq \{ (\sigma, \omega) \in \mathbb{R} \times \mathbb{R} \mid 0 < \sigma \leq \|x - x_d\|, 0 < \omega \leq \|x_d\| \}
\]
Since \(G_2 (\|x - x_d\|, \|x_d\|)\) is also bounded for all bounded \((x, x_d)\), we can define a non-decreasing function \(G_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}\) as
\[
G_3 (\|x\|, \|x_d\|) = \sup_{(\sigma, \omega) \in B_{xy}} G_2 (\sigma, \omega)
\]
such that \(G_3 (\|x - x_d\|, \|x_d\|) \geq G_2 (\|x - x_d\|, \|x_d\|)\) for all \((x, x_d) \in D \times D\). Furthermore, since \(\|x_d\| \leq r\) for all \(x_d \in B_r\),
\[
G_3 (\|x - x_d\|, r) \geq G_2 (\|x - x_d\|, \|x_d\|) \tag{11}
\]
for all \(x \in D, x_d \in B_r\). Let \(\rho : [0, \infty) \to [0, \infty)\) be defined as 
\[
\rho (\|x - x_d\|) \triangleq G_3 (\|x - x_d\|, r) + \|x - x_d\|. \text{ Then, } \rho \text{ is}
\]
strictly increasing, and using (10) and (11),
\[
\| f(x) - f (x_d) \| \leq \rho (\|x - x_d\|) \| x - x_d \|,
\]
for all \(x \in D\) and \(x_d \in B_r\).

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