Amenability via random walks

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Abstract

We answer an open question of Grigorchuk and Žuk about amenability using random walks. Our results separate the class of amenable groups from the closure of subexponentially growing groups under the operations of group extension and direct limits; these classes are separated even within the realm of finitely presented groups.

1 Introduction

The concept of amenability, introduced by von Neumann (1929), has been central to many areas of mathematics. Kesten (1959) showed that a countable group is amenable if and only if the spectral radius equals 1; in particular, if the random walk escapes at a sublinear rate. Although this connection has been deeply exploited to study the properties of random walks, it appears that it has not yet been used to prove the amenability of groups.

A group is amenable if it admits a finitely additive invariant probability measure. The simplest examples of amenable groups (AG) are

(i) finite and Abelian groups and, more generally,

(ii) groups of subexponential growth.

Amenability is preserved by taking subgroups, quotients, extensions, and direct limits. The classes of elementary amenable (EG), and subexponentially amenable (SG, see Grigorchuk 1998, and Ceccherini et al. 1999, §14) groups are the closure of (i), (ii) under

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these operations, respectively. We have
\[ \text{EG} \subseteq \text{SG} \subseteq \text{AG}, \]
and the question arises whether these inclusions are strict: Day (1957) asked this about \( \text{EG} \subseteq \text{AG} \) (see also Grigorchuk (1998)). Chou (1980) showed that there are no elementary amenable groups of intermediate growth. Thus Grigorchuk’s group separates the class \( \text{EG} \) and \( \text{SG} \), answering Day’s question.

In this paper, we show by example that the inclusion \( \text{SG} \subset \text{AG} \) is also strict.

The group \( G \) we are considering is the iterated monodromy group of the polynomial \( z^2 - 1 \). It was first studied by Grigorchuk and Žuk (2002a), who showed that \( G \) does not belong to the class \( \text{SG} \). The main goal of this note is to show, using rate of escape for random walks, that \( G \) is amenable. This answers a question of the above authors.

Let \( T \) be the rooted binary tree with vertex set \( V \) consisting of all finite binary sequences, and edge set \( E = \{(v, vi) : v \in V, i \in \{0, 1\}\} \). Let \( \varepsilon \in \text{Aut}(T) \) send \( iv \) to \( ((i + 1) \mod 2)v \). For \( g, h \in \text{Aut}(T) \) (with the notation \( g : v \mapsto v^g \)) let \( (g, h) \) denote the element of \( \text{Aut}(T) \) sending \( 0v \mapsto 0v^g \) and \( 1v \mapsto 1v^h \). The group \( G \) is generated by the following two recursively defined elements of \( \text{Aut}(T) \):
\[ a = (1, b), \quad b = (1, a)\varepsilon. \]

Then \( G \) is the iterated monodromy group of the polynomial \( z^2 - 1 \); the scaling limit of the Schreier graphs of its action on level \( n \) of \( T \) is the basilica, i.e. the Julia set of this polynomial (see the survey Bartholdi, Grigorchuk, and Nekrashevych (2003) and Figure 1). Let \( Z_n \) be a symmetric random walk on \( G \) with step distribution supported on \( a, b \) and their inverses, and let \( | \cdot | \) denote shortest-word distance.
Theorem 1 We have $\lim |Z_n|/n = 0$ with probability 1.

This implies that the heat kernel decays sub-exponentially, the group $G$ is amenable (see Kaimanovich and Woess (2002)), and $SG \neq AG$.

In the rest of the paper, we extend this result in two directions. In Section 4, Theorem 12 we give a finitely presented example $\tilde{G}$ separating $AG$ and $SG$, showing that these classes are distinct even in this realm. Grigorchuk (1998) showed that $EG \neq AG$ (more precisely, $EG \neq SG$) for finitely presented groups.

In Section 3, Corollary 10 we give a quantitative upper bound of order $n^{5/6}$ on the rate of escape. For the heat kernel, we have the following quantitative lower bound.

**Theorem 2** There exists $c > 0$ so that for all $n$ we have $P(Z_{2n} = 1) \geq e^{-cn^{2/3}}$.

Motivated by their question about amenability, Grigorchuk and Žuk (2002b) study spectral properties of $G$. Amenability of $G$ has been claimed in the preprint Bartholdi (2002), whose proof appears to contain serious gaps and is considered altogether incomplete. The present paper uses the same starting point as Bartholdi (2002), but follows a different path; we get specific heat kernel bounds for a less general family of groups.

2 A fractal distance

For $g \in G$ and $v \in T$ let $g[v] \in \text{Aut}(T)$ denote the action of $g$ on the descendant subtree of $v$, and let $g(v) \in C_2$ denote the action on the two children of $v$. Let $S$ be a finite binary subtree of $T$ containing the root (i.e. each vertex in $S$ has zero or two descendants). Let $\partial S$ denote its set of leaves, and $E = \#\partial S - 1$ the number of edges. Let

$$\nu_S(g) = E + \sum_{v \in \partial S} |g[v]|,$$

$$\nu(g) = \min_S \nu_S(g).$$

The quantity $\nu$ has the alternative recursive definition; for $g = (g_1, g_2) \in \ast$, let

$$\nu(g) = \min(|g|, 1 + \nu(g_1) + \nu(g_2)).$$

**Lemma 3** The function $\nu$ is a norm on $G$. Moreover, $\nu$-balls have exponential growth.

**Proof.** First note that since multiplying $g$ by a increases $|g_1| + |g_2|$ by at most 1, we get $|g| \geq |g_1| + |g_2|$. This implies that if $\nu(g) = |g|$ then $\nu(g) \geq \nu(g_1) + \nu(g_2)$. So in general, we have

$$\nu(g_1) + \nu(g_2) \leq \nu(g) \leq \nu(g_1) + \nu(g_2) + 1. \quad (1)$$
We now check that \( \nu \) satisfies the triangle inequality; this is clear if \( \nu(g) = |g|, \nu(h) = |h| \), otherwise we may assume that \( \nu(g) = \nu(g_1) + \nu(g_2) - 1 \). Then we get

\[
\nu(gh) \leq 1 + \nu((gh)_1) + \nu((gh)_2) \leq 1 + \nu(g_1) + \nu(g_2) + \nu(h_1) + \nu(h_2) \leq \nu(g) + \nu(h),
\]

where the first inequality holds by induction (some care is needed to show that the induction can be started).

We now claim that the balls \( B_n = \{ g : \nu(g) \leq n \} \) grow at most exponentially, more precisely, we have

\[
\#B_n \leq 40^n \quad \text{for all } n.
\] (2)

Indeed, there are at most \( 4^n \) such subtrees \( S \) with at most \( n \) edges. Given the subtree \( S \), the element \( g \in B_n \) is defined by its action \( g(v) \in C_2 \) at the vertices of \( S \) that are not leaves (at most \( 2^n \) possibilities), as well as the words \( g[v] \) at the vertices \( v \) that are leaves (these can be described with \( n \) symbols from the alphabet \( a, a^{-1}, b, b^{-1} \) and comma). Thus we have \( \#B_n \leq (4 \cdot 2 \cdot 5)^n \).

For the other direction, note that \( \nu \)-balls contain the word-distance balls of the same radius and \( G \) has exponential growth (see Grigorchuk and Žuk (2002a)). \( \square \)

### 3 Self-similarity of random walks on \( G \)

Fix \( r > 0 \), and consider the random walk \( Z_n \) on the free group \( F_2 \) where each step is chosen from \( (a, a^{-1}, b, b^{-1}) \) according to weights \((1, 1, r, r)\), respectively. This walk projects to \( F_2 \wr C_2 \) via the substitution \( a \mapsto (1, b) \) and \( b \mapsto (1, a) \). Let \((Y_n, X_n)\) be the projection of \( Z_n \). Define the stopping times

\[
\begin{align*}
\sigma(0) &= 0, \\
\sigma(m + 1) &= \min\{n > \sigma(m) : \varepsilon_n = 1, X_n \neq X_{\sigma(m)}\}, \quad m \geq 0, \\
\tau(0) &= \min\{n > 0 : \varepsilon_n = \varepsilon\}, \\
\tau(m + 1) &= \min\{n > \tau(m) : \varepsilon_n = \varepsilon, Y_n \neq Y_{\tau(m)}\}, \quad m \geq 0.
\end{align*}
\]

**Lemma 4** \( X_{\sigma(m)}, Y_{\tau(m)} \) are simple random walks on \( F_2 \) with step distribution given by the weights \((r/2, r/2, 1, 1)\) on \((a, a^{-1}, b, b^{-1})\), respectively.

**Proof.** The process \((X_n, \varepsilon_n)\) (i.e. we ignore the rest of the information in \( Z_n \)) is a random walk on a weighted graph, as shown on the left of Figure 2; there the value of \( \varepsilon_n \) is
The hitting distribution is given by effective conductances, and using the series law we get the picture on the right hand side of Figure 2. This process is a symmetric random walk with weights as claimed.

The proof for \( Y \) is identical. Because \( \tau(0) \) has a different definition, the process \( Y_{\tau(m)} \) does not start at 0, rather at \( a \) or \( a^{-1} \). Note that the processes \( X_{\sigma(m)} \) and \( Y_{\tau(m)} \) are not independent.

Lemma 5 With probability 1, we have \( \lim\frac{m}{\sigma(m)} = \frac{2+r}{4+4r} =: f(r) \), and the same holds for \( \tau \).

Proof. The increments of the process \( \sigma(m) \) are the time the random walk in Figure 2 spends between hitting two different circles. These increments are independent and identically distributed. Let \( t_o, t_s \) denote the expected times starting from a circle or a neighboring square to hit a different circle. Conditioning on the first step of the walk gives the equations

\[
\begin{align*}
t_o &= 1 + \frac{r}{(r+1)}t_s \\
t_s &= 1 + \frac{r}{(2(r+1))}t_o + \frac{1}{(r+1)}t_s
\end{align*}
\]

And the solution is \( t_o = \frac{4(1+r)}{2+r} \). The claim now follows from the strong law of large numbers.

5
If $\mathbf{Z}_n$ denotes the image of $Z_n$ in $G$, then by construction we have $\mathbf{Z}_n = (\mathbf{X}_n, \mathbf{Y}_n)\mathbf{e}_n$. In the rest of this section we will simply (ab)use the notation $Z_n, X_n, Y_n$ for the images in $G$ of the corresponding random words.

**Proposition 6** We have $\lim \nu(Z_n)/n = 0$ a.s.

**Proof.** By Kingman’s subadditive ergodic theorem, the random limit

$$s(r) = \lim \nu(Z_n)/n$$

exists and equals a constant with probability 1. By (1) we also have

$$\nu(Z_n) \leq \nu(X_n) + \nu(Y_n) + 1$$

and therefore

$$s(r) \leq \lim \sup \nu(X_n)/n + \lim \sup \nu(Y_n)/n$$

$$= \lim \sup \nu(X_{\sigma(m)})/\sigma(m) + \lim \sup \nu(Y_{\tau(m)})/\tau(m)$$

$$= \left(\lim \nu(X_{\sigma(m)})/m\right) \left(\lim m/\sigma(m)\right) + \left(\lim \nu(Y_{\tau(m)})/m\right) \left(\lim m/\tau(m)\right)$$

$$= 2s(2/r)f(r).$$

In the last equality we used Lemmas 4 and 5. Iterating this inequality and we get

$$s(r) \leq 4s(r)f(r)f(2/r) = 4s(r)/8$$

and since $s$ is a finite constant we get $s = 0$ with probability 1. \qed

Proposition 6 implies that the walk has asymptotic entropy 0 (see the proof of Theorem 5.3 in KW (2002)). This, by Theorem 5.13 ibid. implies the following.

**Corollary 7** The random walk on $G$ satisfies $\lim |Z_n|/n = 0$ a.s.

A more technical version of Proposition 6 gives a better bound on the $\nu$-rate of escape.

**Proposition 8** There exists $c > 0$ depending on $r$ so that for all $n \geq 1$

$$u_r(n) := E \max_{i \leq n} \nu(Z_i) \leq cn^{2/3}.$$
Proof. Let $L(n)$ be the largest so that $\sigma(L) \leq n$, and let $M(n)$ be the largest so that $\tau(M) \leq n$. Following the argument of Proposition 6, we get

$$u_r(n) \leq \mathbb{E} \max_{i \leq n} \nu(X_i) + \mathbb{E} \max_{i \leq n} \nu(Y_i) + 1$$

$$= \mathbb{E} \max_{i \leq L(n)} \nu(X_{\sigma(i)}) + \mathbb{E} \max_{i \leq M(n)} \nu(Y_{\tau(i)}) + 1$$

$$\leq 2 + 2u_{2/r}(f(r)n + k_n) + 2n\mathbb{P}(L(n) > f(r)n + k_n)$$

where we can choose the constants $k_n = n^\alpha$ for some $\alpha > 1/2$ close to $1/2$. By the large deviation principle and the fact that the distribution of $\sigma(1)$ has an exponential tail, the last term can be bounded above by $c_1ne^{-c_2n^{2\alpha-1}} < c_3$. Thus for $n \geq 1$ we get

$$u_r(n) \leq u_{2/r}(f(r)n + k_n) + c_4 \leq u_{2/r}(f(r)n) + (c_4 + 1)n^\alpha.$$ 

Applying this to $u_{2/r}$ as well and using the fact that $f(r)f(2/r) = 1/8$, we easily get

$$u_r(8n) \leq 4u_r(n) + c_6n^\alpha$$

with $u_r(1) \leq 1$. Iteration at values of $n$ that are powers of 8 gives that for such values

$$u_r(n) \leq c_7n^{2/3}.$$

Since $u_r(n)$ is monotone in $n$, the claim follows.

Corollary 9 There exists $c > 0$ depending on $r$ so that for all $n \geq 0$ we have

$$\mathbb{P}(Z_{2n} = 1) \geq e^{-cn^{2/3}}.$$

Proof. By Markov’s inequality, we have

$$\mathbb{P}(\nu(Z_n) \leq 2cn^{2/3}) \geq 1/2$$

and therefore there exists $g \in B_{2cn^{2/3}}$ so that

$$\mathbb{P}(Z_n = g) \geq 1/(2\#B_{2cn^{2/3}})$$

and since balls grow at most exponentially, we get

$$\mathbb{P}(Z_{2n} = 1) \geq \mathbb{P}(Z_n = g)\mathbb{P}(Z_n^{-1}Z_{2n} = g^{-1}) = \mathbb{P}(Z_n = g)^2 \geq c_1e^{-c_2n^{2/3}}$$

and the claim follows.
Let $M_n = \max(|X_1|, \ldots, |X_n|)$. We have the following bound on the rate of escape.

**Corollary 10** There exists $c > 0$ so that for all $a, n \geq 1$ we have $P(M_n > an^{5/6}) < c/a$.

**Proof.** Let $K_n = \max(\nu(X_1), \ldots, \nu(X_n))$. We have

$$P(M_n > an^{5/6}) = P(M_n > an^{5/6}, K_n > ac_1n^{2/3}) + P(M_n > an^{5/6}, K_n \leq ac_1n^{2/3}). \quad (3)$$

The first term is at most $P(M_n > ac_1n^{2/3}) < c_2/c_1a$ by Proposition 8 and Markov’s inequality. The second term is bounded above by the sum of $P(X_m = g)$ over all $m \leq n$ and all $g$ with $|g| > an^{5/6}$ and $\nu(g) \leq ac_1n^{2/3}$. By the Varopoulos-Carne bounds (see KW, 2002) the first constraint on $g$ implies

$$P(X_m = g) \leq e^{-(an^{5/6})^2/(2n)},$$

and since $\nu$-balls grow exponentially [2] the second term of (3) is bounded above by

$$ne^{-(an^{5/6})^2/(2n) + c_3c_1an^{2/3}} = ne^{(c_3c_1a - a^2/2)n^{2/3}},$$

which is at most $c_4/a$ for an appropriate choice of $c_1$. \qed

4 A finitely presented example and generalizations

Our first goal is to show that an HNN-extension of $G$ gives a finitely presented example separating $AG$ and $SG$. The following lemma is needed.

**Lemma 11** $G$ has the following presentation:

$$G = \langle a, b \mid \sigma^n[a, a^b] \quad \forall n \in \mathbb{N} \rangle,$$

where $\sigma$ is the substitution $b \mapsto a, a \mapsto b^2$.

**Proof.** By Proposition 9 of Grigorchuk and Žuk (2002a) we have

$$G = \langle a, b \mid \sigma^n[a, a^{b^{2m+1}}] \quad \forall n, m \in \mathbb{N} \rangle.$$

For odd $i$, we have $a^{b^i} = [a^{-1}, b^{-2}]a^{b^{-2}}$ using the relation $[b^{2a}, b^2] = \sigma([a^b, a])$; therefore $[a^{b^n}, a]$ follows from $[a^{b^{n-2}}, a]$ and $[[a^{-1}, b^{-2}]b, a]$, which itself is a consequence of $[a^b, a]$. So the relations $\sigma^n([a, a^{b^{2m+1}}])$ may be eliminated for all $m > 1$ as long as $\sigma^n([a, a^b])$ and $\sigma^{n+1}([a, a^b])$ are kept. \qed
Theorem 12 \( G \) embeds in the finitely presented group

\[
\tilde{G} = \langle a, t \mid a^2 = a^2, [[a, t^{-1}], a] = 1 \rangle.
\]

Furthermore, \( \tilde{G} \) is also amenable, and does not belong to the class \( \text{SG} \).

This implies that the classes \( \text{SG} \) and \( \text{AG} \) are distinct, even in the realm of finitely presented groups.

**Proof.** Let \( \tilde{G} \) be the HNN extension of \( G \) along the endomorphism \( \sigma \) identifying \( G \) and \( \sigma(G) \): it is given by the presentation

\[
\tilde{G} = \langle a, b, t \mid a^t = \sigma(a), b^t = \sigma(b), \text{relations in } G \rangle.
\]

A simpler presentation follows by eliminating the generator \( b \).

Consider the kernel \( H \) of the map \( a \mapsto 1, t \mapsto t \) from \( \tilde{G} \) to \( \langle t \rangle \). Since the HNN extension is “ascending”, we have \( H = \bigcup_{n \in \mathbb{Z}} G^n \), an ascending union. Therefore \( H \) is amenable, and since \( \tilde{G} \) is an extension of \( H \) by \( \mathbb{Z} \), it is also amenable.

Finally, if \( \tilde{G} \) were in \( \text{SG} \), then \( G \) would also be in \( \text{SG} \), since it is the subgroup of \( \tilde{G} \) generated by \( a \) and \( a^{t^{-1}} \). However, Proposition 13 of Grigorchuk and Žuk (2002a) shows that \( G \) is not in \( \text{SG} \). \( \square \)

**Generalizations.** In what setting does the proof for amenability work? Let \( G \) be a group acting spherically transitively on a \( b \)-ary rooted tree \( (b \geq 2) \), and suppose that it is defined recursively by the set \( S \) of generators \( g_i = (g_{i,1}, \ldots, g_{i,b}) \sigma_i \), where each \( g_{i,v} \) is one of the \( g_j \). Consider the Schreier graph of the action of \( G \) on \( T_1 \), that is level 1 of the tree; we label level 1 of the tree by the integers \( 1, \ldots b \). Furthermore, we label each directed edge \( (v, g_i) \) by \( g_{i,v} \).

Fix a vertex at level 1, without loss of generality the vertex 1. Consider the set of cycles in the Schreier graph that go from 1 to 1 and may traverse edges either forwards or backwards; such a cycle is called “irreducible” if it only visits 1 at its endpoints. The label of a cycle is the product of the labels along

If \#S \geq 2 then there are infinitely many irreducible cycles. A necessary condition (1) for our proof to work is that the set of labels of irreducible cycles is finite and agrees with the set \( S \) of generators together with the identity.

Given a probability distribution \( \mu \) on the set of generators, we get a distribution on the set of irreducible loops by considering the path of a random walk on \( G \) up to the first positive
time \( \tau \) that it fixes 1 and has a cycle whose label is not 1. Call the distribution of this label \( \mu' \). The transformation \( \mu \mapsto \mu' \) is a continuous map from a convex set to itself, so it has a fixed point.

A further necessary condition is that at least one fixed point is in the interior of the convex set, i.e. assigns positive weight to each generator. For this, it is sufficient that condition (1) does not hold for any proper subset of \( S \).

Now let \( \mu_0 \) be a such a fixed point, and let \( \alpha = \log b / \log E \tau \) for the corresponding random time \( \tau \). If \( \alpha > 1/2 \), then the argument above gives a heat kernel lower bound of \( e^{-cn^\alpha} \). The argument above cannot give an exponent below \( 1/2 \) as the rate of escape cannot be slower than \( n^{1/2} \). In the proof, the large deviation bounds for \( \sigma \) break down at \( \alpha = 1/2 \).

**Example.** Consider the group acting on the binary tree generated by \( a_i = (1, a_{i+1}) \) for \( i < k \), and \( a_k = (1, a_1) \varepsilon \). The distribution \( \mu = (m_1, \ldots, m_k) \) on the generators (and symmetrically on their inverses) is then sent to \( T \mu' = (m_k/2, m_1, \ldots, m_{k-1})/(1 - m_k/2) \). A fixed point is given by \( (1, 2^{1/k}, \ldots, 2^{(k-1)/k}) \) normalized to be a probability distribution. A simple computation gives \( E \tau = 2^{1+1/k} \), and we get the heat kernel lower bound \( e^{-cn^\alpha} \) with \( \alpha = k/(k + 1) \).

In this example, it is not important to consider a fixed point. Since \( T^k = 1 \), one may iterate the decomposition process \( k \) times starting from an arbitrary \( \mu \). Then one is lead to consider \( 2^k \) processes having the same distribution as the original walk, each with time running slower by a constant factor. After massive cancellations, one finds that the constant \( \prod E \tau_i \) does not depend on \( \mu \), and equals \( 2^{k+1} \). This gives the same heat kernel bound as above.

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