The Power of Locality: An Elementary Integrality Proof of Rothblum’s Stable Matching Formulation

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Abstract

In this paper we provide a short new proof for the integrality of Rothblum’s linear description of the convex hull of incidence vectors of stable matchings in bipartite graphs. In the spirit of iterative rounding proofs, the key feature of our proof is to show that extreme points of the formulation must have a 0, 1-component.

Keywords: Stable Matching, Polytope, Extreme Points

1. Introduction

In an instance of the stable marriage problem, we are given a bipartite graph $G = (\mathcal{A} \cup \mathcal{B}, E)$ where $\mathcal{A}$ and $\mathcal{B}$ traditionally represent sets of women and men, respectively. An edge $ab \in E$ corresponds to an acceptable pair $a$ and $b$ of man and woman. In the following, we let $N(u) := \{v : uv \in E\}$ be the set of neighbours of $u$ in $G$. Each node $u \in V := \mathcal{A} \cup \mathcal{B}$ specifies a complete preference order $>_u$ over its neighbours where node $u$ prefers neighbour $v_1$ over $v_2$ iff $v_1 >_u v_2$. For ease of notation, we will think of $>_u$ as a total ordering on $N(u) \cup \{\emptyset\}$ where $\emptyset$ is the least preferred element of each node $u \in V$. In their seminal paper \cite{GS62}, Gale and Shapley introduced the above problem, and provided a constructive proof of existence of so called stable matchings. A matching is a collection $M$ of edges in $E$ such that each node is incident to at most one edge in $M$. $M$ is stable if, for every edge $uv \notin M$, $M(u) >_u v$ or $M(v) >_v u$ where $M(u)$ is the node matched to $u$ in $M$ if that exists, and $M(u) := \emptyset$, otherwise.

In this paper, we focus on polyhedral characterizations of the set of incidence vectors of stable matchings. Vande Vate first provided such a description in \cite{Vat89} for the special case where $G$ is a complete bipartite graph. Rothblum \cite{Rot92} later generalized Vande Vate’s result to incomplete preference lists and simplified the proof of integrality.

We provide an even simpler, more compact argument for the integrality of Rothblum’s formulation. Our proof is inspired by the technique of iterative rounding as outlined in Lau et al.’s monograph \cite{LRS11}. Our arguments are elementary and rely solely on some well-known results on the symmetric difference of stable matchings as well as some knowledge of the local structure of extreme points in our formulation to achieve the desired result.

Necessary background from the literature will be covered in the section to follow. The main result, a one-page proof, will be given in section 3.

2. Stable Matchings Preliminaries

We briefly review a couple of well-known facts on stable matchings. For each edge $uv$ in $E$, we let $\delta^u(v) := \{\{v,w\} : w >_v u\}$ to be the set of edges incident to $v$ and those of its neighbours that are preferred to $u$. For $v \in V$ let $N_{\text{max}}(v)$ denote its most preferred neighbour. A matching $M$ in $G$ is now easily seen to be stable if

$$M \cap (\delta^u(v) \cup \delta^v(u) \cup \{e\}) \neq \emptyset ,$$

(1)
for all $e = uv \in E$. The following lemmas study the connected components of the symmetric difference $M_1 \triangle M_2 := (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ of stable matchings $M_1$ and $M_2$. Here, $V(C)$ and $E(C)$ denote the set of nodes and edges, respectively, of a graph $C$.

**Lemma 2.1** Let $M_1$ and $M_2$ be two stable matchings in $G$ and let $C$ be a connected component in $M_1 \triangle M_2$. Then for some $(i, j) = \{1, 2\}$, we have

$$M_i(a) >_a M_j(a) \quad \text{and} \quad M_j(b) >_b M_i(b),$$

for all $a \in V(C) \cap A$ and $b \in V(C) \cap B$.

**Proof:** Since $M_1$ and $M_2$ are matchings, $C$ is a path or a cycle. Let $v \in V$ be an end node of $C$ if $C$ is a path, otherwise let $v$ be an arbitrary node of $C$. For visualization of the proof see Figure 1.

W.l.o.g. $v := a \in A$ and $b := M_1(a) >_a M_2(a)$. If $a = M_1(b) >_b M_2(b)$, the matching $M_2$ violates (1) for the edge $ab \in E$. Thus, $M_2(b) >_b M_1(b) = a$. Thus, $M_2(b) \not\in \emptyset$ and the matching $M_1$ satisfies (1) for the edge $e := bM_2(b) \in E$ only if $M_1(M_2(b)) >_{M_1(b)} b$. Continuing in this way, we obtain statement (2).

![Figure 1: Visualizing the proof of Lemma 2.1. Here, the connected component $C$ is a cycle on four nodes. The edges of $M_1$ are marked by straight blue lines, and the edges of $M_2$ by red zigzags. For each node $w \in \{a, b, a_1, b_1\}$, the arrow at $w$ points towards the most preferred node in $\{M_1(w), M_2(w)\}$ with respect to $>_w$.](attachment:image1.png)

The next Lemma is equivalent to Theorem 2.16 in [RS92]. In the interest of self-containment we provide a short elementary proof below.

**Lemma 2.2** Let $M_1$ and $M_2$ be two stable matchings in $G$. Let $J_i$ be those connected components of $M_1 \triangle M_2$ that satisfy (2) for $i = 1$ and $j = 2$ (i.e., $A$ nodes prefer $M_1$ edges); let $J_2$ be all remaining connected components of $M_1 \triangle M_2$. Then both $M_1’ = M_1 \triangle J_1$ and $M_2’ = M_2 \triangle J_2$ are stable matchings in $G$.

**Proof:** For contradiction assume that one of the matchings $M_1’$ and $M_2’$ is not stable; w.l.o.g. assume that $M_1’$ does not satisfy (1) for some edge $ab \in E$ with $a \in A$ and $b \in B$. For visualization of the proof see Figure 2.

Since $M_1$ and $M_2$ are stable, $a$ and $b$ cannot both lie in the same connected component of $J_1 \cup J_2$. Suppose first that $a \in V(J_1)$, and $b \in V(J_2)$. In this case $M_1’(a) = M_2(a)$ and $M_2’(b) = M_1(b) >_b M_2(b)$. Thus $M_2$ violates (1) for edge $ab$.

If $a \in V(J_2)$ and $b \in V(J_1)$, then $M_1’(a) = M_1(a)$ and $M_2’(b) >_b M_1(b)$, and hence $M_1$ violates (1) for edge $ab$, contradiction.

![Figure 2: Visualizing the proof of Lemma 2.2. Here, both $J_1$ and $J_2$ consist of a cycle on four nodes. The edges of $M_1$ are marked by blue straight lines, and the edges of $M_2$ by red zigzags; the edges of $M_1 \triangle J_1$ are the highlighted edges of $M_1$ and $M_2$. For each node $w$, the arrow at $w$ points towards the most preferred node in $\{M_1(w), M_2(w)\}$ with respect to $>_w$. The figure illustrates the case, when the matching $M_1 \triangle J_1$ violates (1) for the edge $(a, b)$ with $a \in V(J_2)$ and $b \in V(J_1)$. In this case, $M_1$ violates (1) for the edge $(a, b)$ as well, contradiction.](attachment:image2.png)
Definition 2.3 Let us define the stable matching polytope $P(G) \subseteq \mathbb{R}^E$ for graph $G$ as follows

$$P(G) := \text{conv}(\chi(M) \in \mathbb{R}^E : M \text{ is a stable matching in } G).$$

By [GS62], $P(G)$ is a nonempty polytope because every graph $G$ has a stable matching.

Clearly, the vertices of $P(G)$ are in one-to-one correspondence with stable matchings in $G$. Moreover, Lemma 2.2 helps to understand what pairs of stable matchings in $G$ do not correspond to edges of $P(G)$.

Lemma 2.4 Let $M_1$ and $M_2$ be two stable matchings in $G$ which define an edge of the polytope $P(G)$. Then all connected components in $M_1 \triangle M_2$ satisfy 1 for unique choice of $i$ and $j$.

Proof: Suppose for contradiction that the statement of the lemma does not hold. Hence the sets $J_1$ and $J_2$ are both nonempty in Lemma 2.2 and we obtain stable matchings $M_1 \triangle J_1$ and $M_1 \triangle J_2$ that are different from $M_1$, $M_2$. We also have

$$\frac{1}{2} \chi(M_1 \triangle J_1) + \frac{1}{2} \chi(M_1 \triangle J_2) = \frac{1}{2} \chi(M_1) + \frac{1}{2} \chi(M_2),$$

and hence there are two distinct convex combinations of the midpoint of the edge between $M_1$ and $M_2$; a contradiction. ■

The next Corollary can be obtained from Ratier’s characterization of edges of the stable matching polytope [Rat96].

Corollary 2.5 Let $M_1$ and $M_2$ be two stable matchings in $G$ such that

$$M_1 \cap \delta^a(b) \neq \emptyset, \ M_1 \cap \delta^b(a) \neq \emptyset \ \text{and} \ M_2 \cap (\delta^a(b) \cup \delta^b(a)) = \emptyset$$

for some $a \in A$ and $b \in B$. Then, $M_1$ and $M_2$ do not define an edge of the polytope $P(G)$.

Proof: Condition (3) implies that both $a$ and $b$ prefer $M_1$ over $M_2$. Hence, both sets $J_1$ and $J_2$ as given in Lemma 2.2 must be non-empty. An application of Lemma 2.4 completes the proof of the corollary. ■

3. Linear Description

Let us define $Q(G) \subseteq \mathbb{R}^E$ to be the polytope described by the following linear constraints

$$x(\delta(v)) \leq 1, \ \forall v \in V \quad \text{and} \quad x_e \geq 0, \ \forall e \in E,$$

(4)

$$x(\delta^a(b)) + x(\delta^b(a)) + x_{ab} \geq 1, \ \forall ab \in E$$

(5)

where $x(J) := \sum_{e \in J} x_e$ for any $J \subseteq E$.

Clearly, $P(G) \subseteq Q(G)$ because for every stable matching $M$ in $G$ the point $x := \chi(M)$ satisfies (4) and by (1) the point $x$ also satisfies (5). On the other hand, every integral point in $Q(G)$ equals $\chi(M)$ for some stable matching $M$ in $G$. In the remaining part of the paper we show that every vertex of $Q(G)$ is integral, thus proving the main theorem.

Theorem 3.1 For every graph $G$ the polytope $P(G)$ equals $Q(G)$.

Lemma 3.2 For every graph $G$ every vertex of the polytope $Q(G)$ is integral.

Proof: We first claim that every vertex $x$ of $Q(G)$ satisfies $x_e \in [0, 1]$ for at least one $e \in E$. Assume for contradiction that $0 < x_e < 1$ for all $e \in E$. Hence, $x$ is uniquely defined by $|E|$ linearly independent tight constraints describing $Q(G)$. Since $x$ has no zero coordinate, we can assume that the tight constraints are $x(\delta(v)) \leq 1$ for $v \in V$ and for $e \in E_x$, where $|V_x| + |E_x| = |E|$. Moreover, let us assume that we choose the $|E|$ tight constraints so that $|V_x|$ is as large as possible.

The constraints $x(\delta(v)) = 1, \ v \in V$ are linearly dependent, in particular $\sum_{a \in A} \chi(\delta(a)) = \sum_{b \in B} \chi(\delta(b))$. Hence, we have $V_x \subseteq V$. On the other hand if $a = N_{\max}(b)$ then $e := ab \not\in E_x$. Indeed, $a = N_{\max}(b)$ implies $\delta^a(b) = \emptyset$ then

$$1 \leq x(\delta^a(b)) + x(\delta^b(a)) + x_{ab} = x(\delta^b(a)) \leq x(\delta(a)) \leq 1,$$
showing that $\delta^{ab}(a) = \emptyset$ and hence $E_x \setminus e, V_x \cup \{a\}$ also define the vertex $x$. Analogously, we can show that if $b \in V_x$ and $a = N_{\min}(b)$ then $e := ab \notin E_x$. Moreover, notice that $N_{\min}(v) \neq N_{\max}(v)$ for $v \in V_x$ since no coordinate of $x$ equals 1. Thus,

$$|E_x| = \frac{1}{2} \sum_{v \in V} |\delta(v) \cap E_x| \leq \frac{1}{2} \sum_{v \in V_x} (|\delta(v)| - 2) + \frac{1}{2} \sum_{v \in V \setminus V_x} (|\delta(v)| - 1) = |E| - |V_x| - \frac{1}{2} |V \setminus V_x|,$$

which implies $|E_x| + |V_x| < |E|$, contradiction.

Now let us assume that $G$ is a graph with the minimum number of edges such that $Q(G)$ is not an integral polytope. Let $x$ be a non-integral vertex of $Q(G)$.

Case $x_{ab} = 0$ for some $a \in \mathcal{A}, b \in \mathcal{B}$ and $e := ab \in E$. In this case, let $P'$ and $x'$ be obtained from $P \cap \{x \in \mathbb{R}^E : x_{ab} = 0\}$ and $x$ by dropping the coordinate corresponding to $ab$. Then, $x'$ is a vertex of the polytope $P'$. Let $G'$ be the graph with $V(G') = V$ and $E(G') = E \setminus \{e\}$. Define $H'$ to be the hyperplane $\{x \in \mathbb{R}^{E(G')} : x(\delta^{ab}(b)) + x(\delta^{ba}(a)) = 1\}$. Then every vertex of $P'$ is either a vertex of $P(G')$ or the intersection of an edge of $P(G')$ with the hyperplane $H'$. Since vertices of $P(G')$ are integral by the minimal choice of graph $G$, it remains to consider vertices at the intersection of $H'$ and an edge of $P(G')$. Such an edge would be defined by stable matchings $M_1$ and $M_2$ so that w.l.o.g. $|M_1 \cap (\delta^{ab}(b) \cup \delta^{ba}(a))| = 2$ and $|M_2 \cap (\delta^{ba}(b) \cup \delta^{ab}(a))| = 0$. Therefore, $M_1$ and $M_2$ satisfy (3) for the given edge $ab$. So Corollary 2.5 readily implies that $P(G')$ cannot have an edge connecting $M_1$ and $M_2$.

Case $x_{ab} = 1$ for some $a \in \mathcal{A}, b \in \mathcal{B}$. Let $x'$ be obtained by dropping the coordinates corresponding to $\delta(a) \cup \delta(b)$, and let $G'$ be the graph with $V(G') = V \setminus \{a, b\}$ and $E(G') = E \setminus (\delta(a) \cup \delta(b))$. It is straightforward to see that $x'$ is a vertex of $P(G')$. Thus by minimality of $G$ both $x'$ and $x$ are integral, a contradiction.

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