Fractional Spinor Bose Einstein condensates

MENG LI\(^1\), JINCHUN HE\(^1\), HAOYUAN XU\(^1\) and MEIHUA YANG\(^1\)*

1. School of Mathematics and Statistics, Huazhong University of Science and Technology
Wuhan, 430074, China

Abstract

We consider the system of coupled fractional Laplacian equations for spin-1 BEC

\[
\begin{align*}
(-\Delta)^su - \lambda_1 u &= \mu_1 |u|^{2p-2}u + \beta |v|^p|u|^{p-2}u \\
(-\Delta)^sv - \lambda_2 v &= \mu_2 |v|^{2p-2}v + \beta |u|^p|v|^{p-2}v
\end{align*}
\]

in \(\mathbb{R}^N\),

and study the existence of positive solution under the following constraint

\[
\int_{\mathbb{R}^N} |u|^2 = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2 = a_2^2.
\]

Assuming that the parameters \(\mu_1, \mu_2, a_1, a_2\) are fixed quantities, we prove the existence of ground state for different ranges of the coupling parameter \(\beta > 0\).

Keywords: Fractional Laplacian, Spin-1 BEC, Ground States

1 Introduction

The phenomenon of Bose-Einstein condensate (BEC) is that bosons at low temperature could occupy the same lowest-energy state, which was predicted by Einstein in 1925. In 1995, using laser cooling technique for several alkali atomic dilute gas such as Rb\(^1\), this was realized in lab by E. Cornell, W. Ketterle and C. Wieman. These Bose-Einstein condensates display various interesting quantum phenomena such as the appearance of quantized vortices in rotating traps, the effective lower dimensional behavior in strongly elongated traps, etc. The force between the atoms in the condensates can be attractive or repulsive. The dynamic of the condensate at zero temperature is generally described with the Gross-Pitaevskii equation which is effectively a mean-field approximation for the interparticle interactions. In the spin-\(f\) BEC system, the mean-field state can be described with \(2f + 1\) hyperfine states.

For the Laplacian case, Cao\(^15\) proved the existence of the ground state for the spin-1 BEC in one-dimensional case by using functional methods. Thomas et al.\(^16\) proved the existence of positive solutions for the system with any arbitrary number of components in three-dimensional space. In \(17, 17\), the authors considered the problem in one-dimensional case, and those in \(18, 19, 20\), dealing with the higher dimensional case. In \(21, 22, 23, 24\), Guo proved the existence of the ground state for the GP functional under some different trapping potentials. In \(25\), the authors proposed some efficient and robust numerical methods to compute the ground states and dynamics of fractional Schrödinger equation with a rotation term and nonlocal nonlinear interactions.

* Corresponding author.

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E-mails: yangmeih@hust.edu.cn
In this paper, we consider the following fractional spin-1 BEC with $1 + \frac{2s}{N} < p < \frac{N}{N-2s}$ and $N > 2s$, which is described by the following fractional GP system in component form
\begin{align}
\begin{cases}
i\partial_t \psi_1 = (-\Delta)^s \psi_1 + V(x)\psi_1 - \mu_1|\psi_1|^{2p-2}\psi_1 - \beta|\psi_2|^p |\psi_1|^{p-2}\psi_1, \\
i\partial_t \psi_2 = (-\Delta)^s \psi_2 + V(x)\psi_2 - \mu_2|\psi_2|^{2p-2}\psi_2 - \beta|\psi_1|^p |\psi_2|^{p-2}\psi_2.
\end{cases}
\end{align}
(1.1)

Here,
\[\int_{\mathbb{R}^N} |\psi_1|^2 = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^N} |\psi_2|^2 = a_2^2.\]

The parameters $\mu_1$, $\mu_2$ and $\beta$ can be positive or negative. In the case of Laplacian, for $\mu_1$, $\mu_2$ and $\beta$ are negative (resp. positive), the system is attractive (resp. repulsive).

The fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ of a function $f : \mathbb{R}^N \to \mathbb{R}$ is expressed by the formula
\[(-\Delta)^s f(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{f(x) - f(z)}{|x-z|^{N+2s}} dz,\]
P.V. stands for the Cauchy principal value, and $C_{N,s}$ is a normalization constant.

It can also be defined as a pseudo-differential operator
\[\mathcal{F}((-\Delta)^s f)(\xi) = |\xi|^{2s} \mathcal{F}(f)(\xi) = |\xi|^{2s} \hat{f}(\xi),\]
where $\mathcal{F}$ is the Fourier transform. For more details about the fractional Laplacian we refer to [5, 7, 8, 10, 11] and the references therein. The nature function space associated with $(-\Delta)^s$ in $N$ dimension is
\[H^s(\mathbb{R}^N) := \left\{ u \left| \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x-z|^{N+2s}} dz dx < +\infty \text{ and } \int_{\mathbb{R}^N} |u(x)|^2 dx < +\infty \right. \right\},\]
equipped with norm
\[\|u\|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \frac{|(-\Delta)^{\frac{s}{2}} u|^2}{2} + \int_{\mathbb{R}^N} u^2 \right)^\frac{1}{2},\]
where, by Fourier transform
\[\int_{\mathbb{R}^N} \frac{|(-\Delta)^{\frac{s}{2}} u|^2}{2} = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi)^2 = \int_{\mathbb{R}^N} (-\Delta)^s u \cdot u = C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(z))u(x)}{|x-z|^{N+2s}} dz dx = C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x-z|^{N+2s}} dz dx,\]

The main aim of the present paper is to analyze the existence of solution, moreover the existence of the ground state of the system (1.1) for the case that the intraspecies interaction and the interspecies interaction are both attractive, i.e. $\mu_1 > 0$, $\mu_2 > 0$ and $\beta > 0$.

In this paper, we consider the simplest case when $V(x)$ $\equiv 0$ and all $\psi_i$ ($i = 1, 2$) are real.

We rename $\psi_1$ by $u$ and rename $\psi_2$ by $v$. The energy functional associated with (1.1) is
\[E(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|(-\Delta)^{\frac{s}{2}} u|^2}{2} + \frac{|(-\Delta)^{\frac{s}{2}} v|^2}{2} - \frac{1}{2p} \int_{\mathbb{R}^N} \mu_1 u^{2p} + 2\beta u^p v^p + \mu_2 v^{2p}\]
(1.2)
on the constraint $H_{a_1} \times H_{a_2}$, where for $a \in \mathbb{R}$, we define

$$H_a := \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = a^2\}.$$ 

For $1 + \frac{2s}{N} < p < \frac{N}{N - 2s}$ and $N > 2s$, the main theorems are listed in the following:

**Theorem 1.1.** Let $a_1, a_2, \mu_1,$ and $\mu_2 > 0$ be fixed, and let $\beta_1 > 0$ be defined by

$$\max \left\{ \frac{1}{a_1^{(p-1)N-2s}}, \frac{1}{a_2^{(p-1)N-2s}}, \frac{1}{\mu_1^{(p-1)N-2s}}, \frac{1}{\mu_2^{(p-1)N-2s}} \right\} = \frac{1}{a_1^{(p-1)N-2s}} + \frac{1}{\mu_1^{(p-1)N-2s}} - \frac{(p-1)N}{2ps} \int_{\mathbb{R}^N} \mu_1 u^1_{2p} + 2\beta u^1_{2p} + \mu_2 u^2_{2p}. \tag{1.3}$$

If $0 < \beta < \beta_1$, then the following equation

$$\begin{cases}
(\Delta)^s u - \lambda_1 u = \mu_1 |u|^{2p-2}u + \beta |v|^p |u|^{p-2}u, \\
(\Delta)^s v - \lambda_2 v = \mu_2 |v|^{2p-2}v + \beta |u|^p |v|^{p-2}v,
\end{cases} \text{ in } \mathbb{R}^N \tag{1.4}$$

on the constraint $H_{a_1} \times H_{a_2}$ has a solution $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{u}, \tilde{v})$ such that $\tilde{\lambda}_1, \tilde{\lambda}_2 < 0$ and $\tilde{u}$ and $\tilde{v}$ are both positive and radial.

For the next conclusion we introduce a Pohozaev-type constraint as follows:

$$F := \{(u, v) \in H_{a_1} \times H_{a_2} : G(u, v) = 0\}, \tag{1.5}$$

where

$$G(u, v) = \int_{\mathbb{R}^N} \left( |(\Delta)^{\frac{s}{2}} u|^2 + |(\Delta)^{\frac{s}{2}} v|^2 \right) - \frac{(p-1)N}{2ps} \int_{\mathbb{R}^N} \mu_1 u^1_{2p} + 2\beta u^1_{2p} + \mu_2 u^2_{2p}. \tag{1.6}$$

We shall define a Rayleigh-type quotient as

$$R(u, v) := \frac{1}{2} \left( \frac{s}{(p-1)N} \right) \left( \frac{2ps}{(p-1)N} \right)^{2s} \left( \int_{\mathbb{R}^N} \left| (\Delta)^{\frac{s}{2}} u \right|^2 + \left| (\Delta)^{\frac{s}{2}} v \right|^2 \right)^{\frac{(p-1)N}{2ps}} \left( \int_{\mathbb{R}^N} \mu_1 |u|^{2p} + 2\beta |u|^p |v|^p + \mu_2 |v|^{2p} \right)^{\frac{(p-1)N}{2ps}}. \tag{1.7}$$

**Theorem 1.2.** Let $a_1, a_2, \mu_1,$ and $\mu_2 > 0$ be fixed, and let $\beta_2 > 0$ be defined by

$$\min \left\{ \frac{1}{a_1^{(p-1)N-2s}}, \frac{1}{a_2^{(p-1)N-2s}}, \frac{1}{\mu_1^{(p-1)N-2s}}, \frac{1}{\mu_2^{(p-1)N-2s}} \right\} = \frac{(a_1^2 + a_2^2)^{\frac{(p-1)N}{2ps}}}{(\mu_1 a_1^{p-1} + 2\beta a_1^{p} a_2 + \mu_2 a_2^{2p})^{\frac{(p-1)N}{2ps}}} \tag{1.8}$$

If $\beta > \beta_2$, then (1.4) on the constraint $H_{a_1} \times H_{a_2}$ has a solution $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{u}, \tilde{v})$ such that $\tilde{\lambda}_1, \tilde{\lambda}_2 < 0$, and $\tilde{u}$ and $\tilde{v}$ are both positive and radial. Moreover, $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{u}, \tilde{v})$ is a ground state solution in the sense that

$$E(\tilde{u}, \tilde{v}) = \inf \{ E(u, v) : (u, v) \in F \} = \inf \{ E(u, v) : (u, v) \text{ is a solution of (1.4) for some } \lambda_1, \lambda_2 \},$$

holds.
There are some difficulties in establishing the previous theorems. Firstly, in the proof of Theorem 1.1, we need to show the Liouville-type result. To deal with the difficulty, we should use the maximum principle for the fractional Laplacian equation. Secondly, through the whole process of the proof, we should give a suitable path to ensure that we can use minimax argument.

In the remainder of this paper, we shall give some important lemmas for the single equation in Section 2, and in Section 3, we shall show the existence of solution by Ekeland’s variational principle and use minimax argument to show the existence of the mountain solution for energy functional (1.2) under constraint.

2 Preliminaries

In this section, we will set out some facts with the fractional NLS equation, which are used in later. Firstly, we will need the important inequality as follows:

Gagliardo-Nirenberg-Sobolev inequality

\[
\int_{\mathbb{R}^N} |u|^{\alpha+2} \, dx \leq C_{\text{opt}} \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \right)^{\frac{N}{4s}} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{\alpha(2s-N)}{4s}+1}.
\]  

(2.1)

Here \( C_{\text{opt}} > 0 \) denotes the optimal constant depending only on \( \alpha, N \) and \( s \).

When \( N > 2s \),

\[
H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \text{ for all } 2 \leq p \leq \frac{2N}{N-2s}.
\]  

(2.2)

In the following, we would introduce the Pohozaev identity for fractional Laplacian equation.

**Theorem 2.1.** [2] Let \( F(u) \in L^1(\mathbb{R}^N) \) and \( u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) be a positive solution of

\[
(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^N,
\]  

(2.3)

where \( u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and \( f \in C^2 \). Then,

\[
(N-2s) \int_{\mathbb{R}^N} uf(u) \, dx = 2N \int_{\mathbb{R}^N} F(u) \, dx,
\]  

(2.4)

where \( F(u) = \int_0^u f(t) \, dt \).

Let us consider the scalar problem

\[
\begin{align*}
(-\Delta)^s w + w &= |w|^{2p-2}w \quad \text{in } \mathbb{R}^N, \\
w &> 0 \quad \text{in } \mathbb{R}^N, \\
w(0) &= \max w \text{ and } w \in H^s(\mathbb{R}^N).
\end{align*}
\]  

(2.5)

The problem (2.5) has a unique radial solution for \( 1 < p < \frac{N}{N-2s}, \ N > 2s \), denoted by \( w_0 \), see Proposition 1.1 in [7]. We set

\[
C_0 := \int_{\mathbb{R}^N} w_0^2 \quad \text{and} \quad C_1 := \int_{\mathbb{R}^N} w_0^{2p}.
\]  

(2.6)
Lemma 2.2. [7] If \( w \in L^2(\mathbb{R}^N) \cap L^{p+2}(\mathbb{R}^N) \) solves
\[
(-\Delta)^s w - \lambda w = |w|^{2p-2} w, \tag{2.7}
\]
where \( 1 < p < \frac{N}{N-2s} \), \( N > 2s \) and \( \lambda < 0 \), then, \( w \in H^s(\mathbb{R}^N) \).

For \( a, \mu \in \mathbb{R} \), let us search for \((\lambda, w) \in \mathbb{R} \times H^s(\mathbb{R}^N)\), with \( \lambda < 0 \) in \( \mathbb{R} \), solving
\[
\begin{cases}
(-\Delta)^s w - \lambda w = \mu |w|^{2p-2} w, & \text{in } \mathbb{R}^N \\
w(0) = \max w \quad \text{and} \quad \int_{\mathbb{R}^N} w^2 = a^2. \tag{2.8}
\end{cases}
\]
Solution \( w \) of (2.8) can be found as critical points of \( I_\mu : H^s(\mathbb{R}^N) \to \mathbb{R} \), defined by
\[
I_\mu(w) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |(-\Delta)^{s/2} w|^2 - \frac{\mu}{2p} |w|^{2p} \right) \, dx,
\]
constrained on the \( L^2 \)-sphere \( H_a := \{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = a^2 \} \), and \( \lambda \) appears as Lagrange multiplier. It is well known that they can be obtained by the solution of (2.5) by scaling.

Let us introduce the set
\[
\mathcal{P}(a, \mu) := \left\{ w \in H_a : \int_{\mathbb{R}^N} |(-\Delta)^{s/2} w|^2 = \frac{(p-1)N\mu}{2ps} \int_{\mathbb{R}^N} w^{2p} \right\}. \tag{2.10}
\]
In the following, we always assume \( \mu > 0 \).

Lemma 2.3. If \( w \) is a solution of (2.8), then \( w \in \mathcal{P}(a, \mu) \). In addition the positive solution of (2.8) minimizes \( I_\mu \) on \( \mathcal{P}(a, \mu) \).

Before proving this lemma, we need to give some claims for \( 1 + \frac{2s}{N} < p < \frac{N}{N-2s} \) at firstly.

Claim 1: Let \( u \in H_a \) be arbitrary but fixed. Then we have
(i) \( |(-\Delta)^{s/2} (l \ast u)|_{L^2} \to 0 \) and \( I_\mu(l \ast u) \to 0 \) as \( l \to -\infty \),
(ii) \( |(-\Delta)^{s/2} (l \ast u)|_{L^2} \to +\infty \) and \( I_\mu(l \ast u) \to -\infty \) as \( l \to +\infty \),
where \( (l \ast u)(x) := e^{\frac{Nx}{l}} u(e^{sl} x) \).

Proof of Claim 1: At first, by direct calculation we have
\[
|l \ast u|_{L^2} = a \quad \text{and} \quad |(-\Delta)^{s/2} (l \ast u)|_{L^2} = e^{s^2 l} |(-\Delta)^{s/2} u|_{L^2}, \tag{2.11}
\]
then, \( |(-\Delta)^{s/2} (l \ast u)|_{L^2} \to 0 \) as \( l \to -\infty \), and \( |(-\Delta)^{s/2} (l \ast u)|_{L^2} \to +\infty \) as \( l \to +\infty \).

And for \( l < 0 \), we have
\[
|I_\mu(l \ast u)| = \int_{\mathbb{R}^N} \left( \frac{1}{2} |(-\Delta)^{s/2} (l \ast u)|^2 - \frac{\mu}{2p} |l \ast u|^{2p} \right) \, dx \leq \frac{e^{2sl}}{2} |(-\Delta)^{s/2} u|_{L^2}^2 + \frac{e^{(p-1)Ns}l}{2p} \mu |u|_{L^{2p}}^{2p}, \tag{2.12}
\]
thus, $I_\mu(l \star u) \to 0$ as $l \to -\infty$.

For $l > 0$, we have

$$I_\mu(l \star u) = \int_{\mathbb{R}^N} \frac{1}{2} |(-\Delta)\frac{s}{2} (l \star u)|^2 dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} |l \star u|^{2p} dx$$

$$= e^{2s^2 l} |(-\Delta)\frac{s}{2} u|^2_{L^2} - e^{(p-1)Ns} \frac{\mu}{2p} |u|^{2p}_{L^{2p}}$$

(2.13)
due to $p > 1 + \frac{2s}{N}$, then we have $I_\mu(l \star u) \to -\infty$ as $l \to +\infty$. \hfill \Box

**Claim 2:** There exists $K(a, s, \mu, N) > 0$ such that

$$0 < \sup_{u \in A} I_\mu(u) < \inf_{u \in B} I_\mu(u),$$

(2.14)

where

$$\begin{align*}
A &= \{ u \in H_a, |(-\Delta)\frac{s}{2} u|^2_{L^2} \leq K(a, s, \mu, N) \}, \\
B &= \{ u \in H_a, |(-\Delta)\frac{s}{2} u|^2_{L^2} = 2K(a, s, \mu, N) \}.
\end{align*}$$

**Proof of Claim 2:** Using the Gagliardo-Nirenberg-Sobolev inequality (2.1) and taking into account that $|u|_{L^2} = a$, we have that

$$\int_{\mathbb{R}^N} |u|^{2p} dx \leq C_{opt} \left( \int_{\mathbb{R}^N} |(-\Delta)\frac{s}{2} u|^2 dx \right)^{\frac{(p-1)N}{2s}}.$$

(2.15)

Now let $K > 0$ be arbitrary but fixed and suppose that $u, v \in H_a$ are such that $|(-\Delta)\frac{s}{2} u|^2_{L^2} \leq K$ and $|(-\Delta)\frac{s}{2} v|^2_{L^2} = 2K$. Then we have

$$I_\mu(v) - I_\mu(u) = \int_{\mathbb{R}^N} \frac{1}{2} |(-\Delta)\frac{s}{2} v|^2 dx - \int_{\mathbb{R}^N} \frac{1}{2} |(-\Delta)\frac{s}{2} u|^2 dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} |v|^{2p} dx + \frac{\mu}{2p} \int_{\mathbb{R}^N} |u|^{2p} dx$$

$$\geq \frac{K}{2} - C \frac{\mu}{2p} K^{\frac{(p-1)N}{2s}} > 0,$$

(2.16)

provided $K > 0$ sufficiently small. Furthermore making $K$ smaller if necessary, we have also for every $u \in A$,

$$I_\mu(u) = \int_{\mathbb{R}^N} \frac{1}{2} |(-\Delta)\frac{s}{2} u|^2 dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} |u|^{2p} dx$$

$$\geq \int_{\mathbb{R}^N} \frac{1}{2} |(-\Delta)\frac{s}{2} u|^2 dx - C \frac{\mu}{2p} \left( \int_{\mathbb{R}^N} |(-\Delta)\frac{s}{2} u|^2 dx \right)^{\frac{(p-1)N}{2s}} > 0.$$ 

(2.17)

Hence (2.14) holds. \hfill \Box

**Claim 3:** Let $u \in H_a$ be arbitrary but fixed. Then the function $f_a : \mathbb{R} \to \mathbb{R}$ defined by:

$$f_a(l) = I_\mu(l \star u),$$

(2.18)

reaches its unique maximum at a point $l(u) \in \mathbb{R}$ such that $l(u) \star u \in \mathcal{P}(a, \mu)$. 

6
Proof of Claim 3: By (2.13), we have
\[
f_u(l) = \frac{e^{2s^2l}}{2} |(-\Delta)\frac{s}{2} u|^2_{L^2} - \frac{e^{(p-1)Ns}}{2p} \mu |u|^{2p},
\]
then,
\[
f_u'(l) = s^2 e^{2s^2l} |(-\Delta)\frac{s}{2} u|^2_{L^2} - \frac{(p-1)Ns \mu}{2p} e^{(p-1)Ns} \int_{\mathbb{R}^N} |u|^{2p}.
\] (2.19)

From Claim 1 and Claim 2, we know that there exits a \(l_0 \in \mathbb{R}\) such that \(f_u'(l)|_{l=l_0} = 0\), and \(l_0 * u \in \mathcal{P}(a, \mu)\). Then,
\[
f_u''(l)|_{l=l_0} = \left(2s^4 e^{2s^2l} |(-\Delta)\frac{s}{2} u|^2_{L^2} - \frac{(p-1)^2 N^2 s^2 \mu}{2p} e^{(p-1)Ns} \int_{\mathbb{R}^N} |u|^{2p}\right)|_{l=l_0}
= \left(2s - (p-1)N\right) \frac{e^{(p-1)Nsl_0}}{2p} s^2 (p-1)N \mu \int_{\mathbb{R}^N} |u|^{2p}
< 0.
\] (2.20)

Note that,
\[
f_u'(l) = \begin{cases} > 0 & \text{if } l < l_0 \\ = 0 & \text{if } l = l_0 \\ < 0 & \text{if } l > l_0 \end{cases}
\] (2.21)

which implies the unicity of \(l_0\). Furthermore, from
\[
e^{s[(p-1)N-2s]l_0} = \frac{|(-\Delta)\frac{s}{2} u|^2_{L^2}}{\int_{\mathbb{R}^N} |u|^{2p}},
\]
we know that if \(u \in \mathcal{P}(a, \mu)\), then \(l_0 = 0\). \(\square\)

Now, we use above claims to proof the Lemma 2.3.

Proof. We apply Theorem 2.1 with \(f(w) = \lambda w + \mu |w|^{2p-2} w\). Let \((w, \lambda) \in H_0 \times \mathbb{R}\) be a weak solution of (2.8). Thus we have
\[
(N-2s) \int_{\mathbb{R}^N} |(-\Delta)\frac{s}{2} w|^2 = 2N \int_{\mathbb{R}^N} \int_0^w f(s) ds dx
= 2N \left(\frac{\lambda}{2} \int_{\mathbb{R}^N} |w|^2 + \frac{\mu}{2p} \int_{\mathbb{R}^N} |w|^{2p}\right),
\] (2.22)

which combining with
\[
\lambda = \frac{1}{\int_{\mathbb{R}^N} |w|^2} \left(\int_{\mathbb{R}^N} |(-\Delta)\frac{s}{2} w|^2 - \mu \int_{\mathbb{R}^N} |w|^{2p}\right),
\] (2.23)
we get that
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w|^2 = \frac{(p-1)N\mu}{2^{ps}} \int_{\mathbb{R}^N} |w|^{2p}.
\] (2.24)

In the following, we prove that the positive solution of (2.8) minimizes \( I_\mu \) on \( \mathcal{P}(a, \mu) \). Now we can define
\[
\gamma(a) := \inf_{h \in \Gamma(a)} \max_{t \in [0, 1]} I_\mu(h(t)),
\] (2.25)
where
\[
\Gamma(a) := \{ h \in C([0, 1], H_a), h(0) = u_1, h(1) = u_2 \},
\] (2.26)
with \( u_1, u_2 \in H_a \) satisfying that
\[
|(-\Delta)^{\frac{s}{2}} u_1|^2_{L^2} \leq K(a, s, \mu, N),
\]
\[
|(-\Delta)^{\frac{s}{2}} u_2|^2_{L^2} \geq 3K(a, s, \mu, N),
\]
\[
I_\mu(u_1) > 0 \geq I_\mu(u_2),
\] (2.27)
and \( K(a, s, \mu, N) \) given in Claim 2. Note that, by Claim 1 and Claim 2 we know \( u_1, u_2 \) exist. Then, we prove \( \gamma(a) = \inf_{u \in \mathcal{P}(a, \mu)} I_\mu(u) \). We use a contradiction argument. We suppose that there exists \( v \in \mathcal{P}(a, \mu) \) such that \( I_\mu(v) < \gamma(a) \) and define the map \( T_v : \mathbb{R} \to H_a \) with
\[
T_v(l) = l \star v.
\]
From Claim 1, we know that for \( K(a, s, \mu, N) \) be given in Claim 2, there exists \( l_0 > 0 \) such that \( T_v(-l_0) \in A \) and \( T_v(l_0) \in C \), where \( C = \{ u \in H_a, |(-\Delta)^{\frac{s}{2}} u|^2_{L^2} \geq 3K(a, s, \mu, N) \} \) and \( I_\mu(u) \leq 0 \}. \) Now, let \( \tilde{T}_v : [0, 1] \to H_a \) be a path defined by
\[
\tilde{T}_v(t) = e^{\frac{Ns_0(2t-1)}{2}} v(e^{s_0(2t-1)} x),
\] (2.28)
and then, \( \tilde{T}_v(0) = T_v(-l_0) \) and \( \tilde{T}_v(1) = T_v(l_0) \). Moreover by Claim 3
\[
\gamma(a) \leq \max_{t \in [0, 1]} I_\mu(\tilde{T}_v(t)) = I_\mu(v),
\] (2.29)
which is a contradiction. \( \square \)

Lemma 2.4. Equation (2.8) has a unique positive solution \((\lambda_{a, \mu}, w_{a, \mu})\) defined by
\[
\lambda_{a, \mu} := -\left[ \frac{1}{\mu} \left( \frac{C_0}{a^2} \right)^{p-1} \right]^{\frac{2^*-1}{p-1-N-2s}}, \quad w_{a, \mu} := \left( \frac{C_0}{\mu^N a^{4s}} \right)^{\frac{1}{2(p-1)} - \frac{N-2s}{p-1}} w_0 \left[ \left( \frac{C_0}{\mu^N a^{4s}} \right)^{p-1} \right]^{\frac{1}{p-1-N-2s}} x.
\] (2.30)
Furthermore, \( w_{a,\mu} \) satisfies
\[
\int_{\mathbb{R}^N} |(-\Delta)^{s} w_{a,\mu}|^2 = \frac{(p-1)N}{2ps} \frac{C_1 C_0^{\frac{2p_s-(p-1)N}{(p-1)N-2s}}}{\mu^{\frac{2s}{(p-1)N-2s}} a^{\frac{1}{2}} C_0^{\frac{4p_s-2(p-1)N}{(p-1)N-2s}}}, 
\]  
(2.31)
\[
\int_{\mathbb{R}^N} |w_{a,\mu}|^{2p} = \frac{C_1 C_0^{\frac{2p_s-(p-1)N}{(p-1)N-2s}}}{\mu^{\frac{2s}{(p-1)N-2s}} a^{\frac{1}{2}} C_0^{\frac{4p_s-2(p-1)N}{(p-1)N-2s}}}, 
\]  
(2.32)
\[
I_\mu(w_{a,\mu}) = \frac{(p-1)N-2s}{4ps} \frac{C_1 C_0^{\frac{2p_s-(p-1)N}{(p-1)N-2s}}}{\mu^{\frac{2s}{(p-1)N-2s}} a^{\frac{1}{2}} C_0^{\frac{4p_s-2(p-1)N}{(p-1)N-2s}}}. 
\]  
(2.33)

**Proof.** We can directly check that \((\lambda_{a,\mu}, w_{a,\mu})\) satisfies the equation (2.8) which is the unique positive solution by [7]. Firstly, we have
\[
\int_{\mathbb{R}^N} |w_{a,\mu}|^{2p} = \frac{C_1 C_0^{\frac{2p_s-(p-1)N}{(p-1)N-2s}}}{\mu^{\frac{2s}{(p-1)N-2s}} a^{\frac{1}{2}} C_0^{\frac{4p_s-2(p-1)N}{(p-1)N-2s}} \int_{\mathbb{R}^N} |w_0|^{2p}, 
\]  
and then, we get (2.32). Secondly, we have
\[
\int_{\mathbb{R}^N} |(-\Delta)^{s} w_{a,\mu}|^2 = \frac{C_1 C_0^{\frac{2p_s-(p-1)N}{(p-1)N-2s}}}{\mu^{\frac{2s}{(p-1)N-2s}} a^{\frac{1}{2}} C_0^{\frac{4p_s-2(p-1)N}{(p-1)N-2s}} \int_{\mathbb{R}^N} |(-\Delta)^{s} w_0|^2. 
\]  
By Lemma 2.3 with \( a^2 = C_0 \) and \( \mu = 1 \), we know
\[
\int_{\mathbb{R}^N} |(-\Delta)^{s} w_0|^2 = \frac{(p-1)N}{2ps} \int_{\mathbb{R}^N} |w_0|^{2p}, 
\]  
and then, we get (2.31). Finally, combining with (2.31) and (2.32), we obtained (2.33). \( \square \)

In order to characterize of the best constant of (2.1), we should work with several components in the system, like \( C_0 \) and \( C_1 \). To obtain it, we should consider the unique positive solution of the following system:
\[
\begin{align*}
(-\Delta)^{s} w + w &= (C_0/a^2)^{p-1} w^{2p-1} & \text{in } \mathbb{R}^N, \\
w(0) &= \max w \text{ and } \int_{\mathbb{R}^N} w^2 = a^2,
\end{align*}
\]  
and it is a minimizer of \( I_{a,(C_0/a^2)^{p-1}} \) on \( \mathcal{P}(a,(C_0/a^2)^{p-1}) \).

**Lemma 2.5.**
\[
\inf_{u \in \mathcal{P}(a,(C_0/a^2)^{p-1})} I_{a,(C_0/a^2)^{p-1}}(u) = \inf_{u \in H_a} \mathcal{R}(u), 
\]  
(2.34)
where
\[
\mathcal{R}(u) := \frac{1}{2} - \frac{s}{(p-1)N} \left( \frac{2ps}{(p-1)N} \right)^{\frac{2s}{(p-1)N-2s}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s} u|^2}{((C_0/a^2)^{p-1} \int_{\mathbb{R}^N} |u|^{2p})^{\frac{1}{2}}}. 
\]
Proof. If $u \in \mathcal{P}(a, \mu)$, then
\[
2ps \int_{\mathbb{R}^N} |(\Delta)^{\frac{r}{2}} u|^2 \frac{1}{(p-1)N \mu \int_{\mathbb{R}^N} |u|^{2p}} = 1 \quad \text{and} \quad I_\mu(u) = \left( \frac{1}{2} - \frac{s}{(p-1)N} \right) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{r}{2}} u|^2.
\]
Therefore,
\[
I_\mu(u) = \left( \frac{1}{2} - \frac{s}{(p-1)N} \right) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{r}{2}} u|^2 \left( \frac{2ps \int_{\mathbb{R}^N} |(-\Delta)^{\frac{r}{2}} u|^2}{(p-1)N \mu \int_{\mathbb{R}^N} |u|^{2p}} \right)^{\frac{p-2}{2}} dx,
\]
which proves that
\[
\inf_{u \in \mathcal{P}(a, (C_0/a^2)^{p-1})} I_{a,(C_0/a^2)^{p-1}}(u) \geq \inf_{u \in H_a} \mathcal{R}(u).
\]
On the other hand, for all $l \in \mathbb{R}$, $u \in H_a$, we have
\[
\mathcal{R}(u) = \mathcal{R}(l \ast u).
\]
By Claim 3, we know that for $u \in H_a$ be arbitrary but fixed, there exists a unique $l(u) \in \mathbb{R}$ such that $l(u) \ast u \in \mathcal{P}(a, \mu)$, and $I_\mu(l(u) \ast u)$ reaches its unique maximum. Hence, for every $u \in H_a$, we have
\[
\mathcal{R}(u) = \mathcal{R}(l(u) \ast u) = I_\mu(l(u) \ast u) = \max_l I_\mu(l(u) \ast u) \geq \inf_{u \in \mathcal{P}(a, \mu)} I_\mu(u),
\]
which proves that
\[
\inf_{u \in \mathcal{P}(a, (C_0/a^2)^{p-1})} I_{a,(C_0/a^2)^{p-1}}(u) \leq \inf_{u \in H_a} \mathcal{R}(u).
\]
\[\square\]

For the constant $C_{opt}$ in Gagliardo-Nirenberg-Sobolev inequality (2.1) with $\alpha = 2p-2$, its optimal value can be found as
\[
\frac{1}{C_{opt}^{(p-1)N-2\alpha}} = \inf_{u \in H^\alpha \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{r}{2}} u|^2 dx}{(p-1)N \mathcal{R}(u) \left( \int_{\mathbb{R}^N} |u|^{2p} dx \right)^\frac{p-2}{2}} \right)^\frac{2ps}{(p-1)N-2\alpha}
\]
\[
= \inf_{u \in H_a} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{r}{2}} u|^2 dx}{(p-1)N \mathcal{R}(u) \left( \int_{\mathbb{R}^N} |u|^{2p} dx \right)^\frac{p-2}{2}}
\]
\[
= \frac{a^{-2}C_0^{(p-1)N-2\alpha}}{\frac{2ps}{(p-1)N}} \inf_{u \in H_a} \mathcal{R}(u)
\]
\[
= \frac{a^{-2}C_0^{(p-1)N-2\alpha}}{\frac{2ps}{(p-1)N}} \inf_{u \in \mathcal{P}(a, (C_0/a^2)^{p-1})} I_{a,(C_0/a^2)^{p-1}}(u)
\]
\[
= \frac{a^{-2}C_0^{(p-1)N-2\alpha}}{\frac{2ps}{(p-1)N}} \frac{C_1}{C_0^{(p-1)N-2\alpha}}
\]
\[
= \frac{2ps}{(p-1)N} \left( \frac{C_1}{C_0^{(p-1)N-2\alpha}} \right).
\]
which implies that

$$C_{opt} = \frac{\left(\frac{2ps}{(p-1)N}\right) \left(\frac{(p-1)N}{2s}\right)_{\frac{2s}{p}}}{C_0 \left(\frac{2ps}{(p-1)N}\right) \left(\frac{(p-1)N-2s}{2s}\right)_{\frac{2s}{p}}}.$$  \hfill (2.36)

Next, we will prove the Liouville-type results by Kelvin transform that will be used in later.

**Lemma 2.6.** Let \( u \in H^s(\mathbb{R}^N) \).

(i) If \( u \) satisfies

$$\begin{cases} (-\Delta)^s u \geq 0 & \text{in } \mathbb{R}^N, \\ u \in L^q(\mathbb{R}^N), \ q \in (0, \frac{N}{N-2s}], \\ u \geq 0, \end{cases}$$

then \( u \equiv 0 \).

(ii) If \( u \) satisfies

$$\begin{cases} (-\Delta)^s u \geq u^q & \text{in } \mathbb{R}^N, \\ u \in L^q(\mathbb{R}^N), \ q \in (1, \frac{N}{N-2s}], \\ u \geq 0, \end{cases}$$

then \( u \equiv 0 \).

**Proof.** (i) At first, by Kelvin transform, let \( v(x) = \frac{1}{|x|^{N-2s}} u(\frac{x}{|x|^2}) \). Then \( v(x) \geq 0 \), and \( v(x) \) satisfies

$$(-\Delta)^s v(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s u(\frac{x}{|x|^2}) \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

so \((-\Delta)^s v \geq 0\) in distribution sense. We can prove the lemma by contraction. If \( u \neq 0 \), by maximum principle we have \( u > 0 \) in \( \mathbb{R}^N \) and at the same time \( v > 0 \) in \( \mathbb{R}^N \setminus \{0\} \). Otherwise \( u \in H^s(\mathbb{R}^N) \), so \( v \) is. Since \( H^s \subset L_{2s} \), where \( L_{2s} = \{w(x): \mathbb{R}^N \rightarrow \mathbb{R} | \int_{\mathbb{R}^N} \frac{|w(x)|}{1+|x|^{N-2s}} dx < +\infty\} \), we can see Theorem 1 in [14], there exists a constant \( C > 0 \) such that

$$\inf_{|x| < \frac{1}{4}} v(x) \geq C.$$

By Kelvin inverse transform, we obtain that

$$u(x) \geq \frac{C}{|x|^{N-2s}}, \ |x| > 1.$$  

For \( q \in (0, \frac{N}{N-2s}] \), we can compute

$$\int_{\mathbb{R}^N} u^q \geq \int_{|x| > 1} \left(\frac{C}{|x|^{N-2s}}\right)^q \geq C \int_{|x| > 1} \frac{1}{|x|^N},$$
which is a contradiction with \( u \in L^q(\mathbb{R}^N) \). So \( u \equiv 0 \).

(ii) Let \( \varphi \) be the first eigenfunction of

\[
\begin{cases}
(-\Delta)^s \varphi = \lambda_1 \varphi & \text{in } B_1(0), \\
\varphi \equiv 0 & \text{in } B_1^c(0),
\end{cases}
\]

where \( B_1(0) \) is the unit ball in \( \mathbb{R}^N \), \( \varphi > 0 \) and \( \lambda_1 \) is the first eigenvalue of \( (-\Delta)^s u \).

For any \( R \in \mathbb{R} \) but fixed, suppose that \( \varphi_R(x) = \varphi(\frac{x}{R}) \), there holds

\[
\begin{cases}
(-\Delta)^s \varphi_R = R^{-2s} \lambda_1 \varphi_R & \text{in } B_R(0), \\
\varphi_R \equiv 0 & \text{in } B_R^C(0).
\end{cases}
\]

We can compute

\[
\int_{B_R(0)} u^q \varphi_R = \int_{\mathbb{R}^N} u^q \varphi_R \leq \int_{\mathbb{R}^N} (-\Delta)^s u \varphi_R
\]

\[
= \int_{\mathbb{R}^N} (-\Delta)^s \varphi_R u = \int_{B_R(0)} u R^{-2s} \lambda_1 \varphi_R + \int_{B_R^C(0)} (-\Delta)^s \varphi_R u
\]

\[
\leq \int_{B_R(0)} u R^{-2s} \lambda_1 \varphi_R \leq R^{-2s} \lambda_1 \left( \int_{B_R(0)} u^q \varphi_R \right)^{\frac{1}{q}} \left( \int_{B_R(0)} \varphi_R \right)^{1-\frac{1}{q}}
\]

\[
\leq CR^{-2s} \lambda_1 \left( \int_{\mathbb{R}^N} u^q \varphi_R \right)^{\frac{1}{p}}.
\]

For \( q \in (1, \frac{N}{N - 2s}) \), and since the property of \( \varphi \), we have

\[
\left( \min_{B_{\frac{R}{2}}(0)} \varphi \right) \left( \int_{B_{\frac{R}{2}}(0)} u^q \right)^{-\frac{1}{q}} \leq \left( \int_{\mathbb{R}^N} u^q \varphi_R \right)^{1-\frac{1}{q}} \leq C \lambda_1 R^{-2s + N(1-\frac{1}{q})} \to 0, \text{ as } R \to \infty.
\]

So we have \( u \equiv 0 \).

For \( q = \frac{N}{N - 2s} \), we have

\[
\min_{B_{\frac{R}{2}}(0)} \varphi \int_{B_{\frac{R}{2}}(0)} u^q \leq \int_{\mathbb{R}^N} u^q \varphi_R \leq C \text{ for any } R,
\]

and from (i), we hence obtain that \( u \equiv 0 \).

\[\Box\]

3 Main Theorems

In the first part of this section, we should give the proof of Theorem 1.1. Because of compactness, we need to consider in a radial setting. So we find the critical point of the functional \( E \) constrained on \( H_{a_1}^{rad} \times H_{a_2}^{rad} \), where for any \( a \in \mathbb{R} \), we define

\[
H_a^{rad} := \{ w \in H_a : w \text{ is radial} \},
\]
and we know the fact that $H^s_{rad}(\mathbb{R}^N)$ is the subset of $H^s(\mathbb{R}^N)$ containing all the functions which are radial with respect to the origin, and when for all $2 \leq p < \frac{2N}{N-2s}$, we have $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ with compact embedding.

For $a_1, a_2, \mu_1$, and $\mu_2 > 0$, let $\beta_1 > 0$ be defined by (1.3).

**Lemma 3.1.** For $0 < \beta < \beta_1$, there holds:

$$\inf \{ E(u_1, u_2) : (u_1, u_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta) \} > \max \{ I_{\mu_1}(w_{a_1, \mu_1}), I_{\mu_2}(w_{a_2, \mu_2}) \},$$

where $I_{\mu_i}(w_{a_i, \mu_1})$, $i = 1, 2$ is defined by (2.33).

**Proof.** For $(u_1, u_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta)$, we have

$$E(u_1, u_2) = \int_{\mathbb{R}^N} \left( \frac{1}{2} \langle -\Delta \rangle^s u_1^2 + \mu_1 |u_1|^{2p} \right) dx + \int_{\mathbb{R}^N} \left( \frac{1}{2} \langle -\Delta \rangle^s u_2^2 + \mu_2 |u_2|^{2p} \right) dx - \frac{\beta}{p} \int_{\mathbb{R}^N} u_1^p u_2^p dx$$

$$\geq I_{\mu_1} + I_{\mu_2} - \frac{\beta}{2p} \int_{\mathbb{R}^N} u_1^p dx - \frac{\beta}{2p} \int_{\mathbb{R}^N} u_2^p dx$$

$$= I_{\mu_1 + \beta} + I_{\mu_2 + \beta}$$

$$\geq \inf_{u_1 \in \mathcal{P}(a_1, \mu_1 + \beta)} I_{\mu_1 + \beta} + \inf_{u_2 \in \mathcal{P}(a_2, \mu_2 + \beta)} I_{\mu_2 + \beta}$$

$$= I_{\mu_1 + \beta}(w_{a_1, \mu_1 + \beta}) + I_{\mu_2 + \beta}(w_{a_2, \mu_2 + \beta}),$$

at the same time, from (2.33) and (1.3), we know

$$\max \{ I_{\mu_1}(w_{a_1, \mu_1}), I_{\mu_2}(w_{a_2, \mu_2}) \}$$

$$= \max \left\{ \frac{(p-1)N-2s}{4ps} C_1 C_0^{\frac{2p-(p-1)N}{(p-1)N-2s}}, \frac{(p-1)N-2s}{4ps} C_1 C_0^{\frac{2p-(p-1)N}{(p-1)N-2s}} \right\}$$

$$< I_{\mu_1 + \beta}(w_{a_1, \mu_1 + \beta}) + I_{\mu_2 + \beta}(w_{a_2, \mu_2 + \beta}).$$

Then, we have

$$\inf \{ E(u_1, u_2) : (u_1, u_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta) \}$$

$$> \max \{ I_{\mu_1}(w_{a_1, \mu_1}), I_{\mu_2}(w_{a_2, \mu_2}) \}.$$

\[\square\]

Now we fix $0 < \beta < \beta_1$ and choose $\varepsilon > 0$ such that

$$\inf \{ E(u_1, u_2) : (u_1, u_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta) \}$$

$$> \max \{ I_{\mu_1}(w_{a_1, \mu_1}), I_{\mu_2}(w_{a_2, \mu_2}) \} + \varepsilon.$$  (3.1)
Introducing
\[ w_1 := w_{a_1, \mu_1 + \beta} \quad \text{and} \quad w_2 := w_{a_2, \mu_2 + \beta}, \]
and for \( i = 1, 2 \),
\[ \varphi_i(l) := I_{\mu_i}(l \ast w_i) \quad \text{and} \quad \tilde{\varphi}_i(l) := \frac{\partial}{\partial l} I_{\mu_i + \beta}(l \ast w_i). \]

**Lemma 3.2.** For \( i = 1, 2 \), there exist \( \rho_i < 0 \) and \( R_i > 0 \), depending on \( \varepsilon \) and on \( \beta \), such that

(i) \( 0 < \varphi_i(\rho_i) < \varepsilon \) and \( \varphi_i(R_i) \leq 0; \)

(ii) \( \tilde{\varphi}_i(l) > 0 \) for any \( l < 0 \) and \( \tilde{\varphi}_i(l) < 0 \) for any \( l > 0 \). In particular, \( \tilde{\varphi}_i(\rho_i) > 0 \) and \( \tilde{\varphi}_i(R_i) < 0. \)

**Proof.** For \( l < 0 \) and \( i = 1, 2 \), with Lemma 2.4 and Claim 1, we have

\[
\varphi_i(l) = \int_{\mathbb{R}^N} \left( \frac{1}{2} (-\Delta) \frac{s}{2} (l \ast w_i)^2 - \frac{\mu_i}{2p} |l \ast w_i|^{2p} \right) dx
\]

\[
= \frac{s^2 e^{2s l}}{2} |(-\Delta) \frac{s}{2} w_i|_{L^2}^2 - \frac{e^{(p-1)Ns l}}{2p} \mu_i |w_i|_{L^{2p}}^{2p}
\]

\[
= \left( \frac{(p-1)N(\mu_i + \beta)}{2ps} e^{2s l} \frac{2p}{2} \mu_i \right) |w_i|_{L^{2p}}^{2p},
\]

thus, \( I_{\mu_i}(l \ast w_i) \to 0^+ \) as \( l \to -\infty \), and \( I_{\mu_i}(l \ast w_i) \to -\infty \) as \( l \to +\infty \). Therefore, there exist \( \rho_i < 0 \) and \( R_i > 0 \) satisfying Condition (i).

\[
\tilde{\varphi}_i(l) = s^2 e^{2s l} |(-\Delta) \frac{s}{2} w_i|_{L^2}^2 - \frac{e^{(p-1)Ns l}}{2p} \mu_i \left( (p-1)Nsl \frac{(p-1)Ns l}{2} \right) \int_{\mathbb{R}^N} |w_i|^{2p}
\]

\[
= \left( \frac{(p-1)N(\mu_i + \beta)}{2ps} e^{2s l} \right) \left( \frac{(p-1)Ns l}{2} - \frac{(p-1)Ns l}{2} \right) \int_{\mathbb{R}^N} |w_i|^{2p}
\]

\[
= \left( \frac{(p-1)N(\mu_i + \beta)}{2ps} e^{2s l} \right) \left( e^{(2s-(p-1)Ns l l)} - 1 \right) \int_{\mathbb{R}^N} |w_i|^{2p},
\]

then,

\[
\tilde{\varphi}_i(l) = \begin{cases} 
> 0 & \text{if } l < 0 \\
= 0 & \text{if } l = 0 \\
< 0 & \text{if } l > 0
\end{cases},
\]

which implies that Condition (ii) holds. \( \square \)
Let $Q := [\rho_1, R_1] \times [\rho_2, R_2]$, and let 
\[ \gamma_0(t_1, t_2) := (t_1 \ast w_1, t_2 \ast w_2) \in H_{a_1}^{rad} \times H_{a_2}^{rad}, \ \forall (t_1, t_2) \in \bar{Q}, \]
We introduce the minimax class
\[ \Gamma := \{ \gamma \in C(\bar{Q}, H_{a_1}^{rad} \times H_{a_2}^{rad}) : \gamma = \gamma_0 \text{ on } \partial Q \}. \]

**Lemma 3.3.** There holds
\[ \sup_{\partial Q} E(\gamma_0) \leq \max \{ I_{\mu_1}(w_{a_1, \mu_1}), I_{\mu_2}(w_{a_2, \mu_2}) \} + \varepsilon. \]

**Proof.** For every $(u_1, u_2) \in H_{a_1}^{rad} \times H_{a_2}^{rad}$, we have
\[ E(u_1, u_2) = I_{\mu_1}(u_1) + I_{\mu_2}(u_2) - \frac{\beta}{p} \int_{\mathbb{R}} u_1^p u_2^p dx \leq I_{\mu_1}(u_1) + I_{\mu_2}(u_2). \]
Then, from Lemma 3.2,
\[ E(t_1 \ast w_1, \rho_2 \ast w_2) \leq I_{\mu_1}(t_1 \ast w_1) + I_{\mu_2}(\rho_2 \ast w_2) \]
\[ \leq I_{\mu_1}(t_1 \ast w_1) + \varepsilon \]
\[ \leq \sup_{l \in \mathbb{R}} I_{\mu_1}(l \ast w_1) + \varepsilon. \]
By Lemma 2.4 we have
\[ w_{a_i, \mu_i} = \tilde{l}_i \ast w_i, \quad \text{for } e^{\tilde{l}} := \left( \frac{\mu_i + \beta}{\mu_i} \right)^{\frac{1}{p-1}}. \]
Then, with $l_1 \ast (l_2 \ast w) = (l_1 + l_2) \ast w$ for every $l_1, l_2 \in \mathbb{R}$ and $w \in H^s(\mathbb{R})$, we have
\[ \sup_{l \in \mathbb{R}} I_{\mu_1}(l \ast w_1) = \sup_{l \in \mathbb{R}} I_{\mu_1}(l \ast w_{a_1, \mu_1}). \] (3.5)
As a consequence of **Claim 3**, the supremum on the right hand side is achieved for $l = 0$, and hence
\[ E(t_1 \ast w_1, \rho_2 \ast w_2) \leq I_{\mu_1}(w_{a_1, \mu_1}) + \varepsilon, \quad \forall t_1 \in [\rho_1, R_1], \] (3.6)
and in a similar way one can show that
\[ E(\rho_1 \ast w_1, t_2 \ast w_2) \leq I_{\mu_2}(w_{a_2, \mu_2}) + \varepsilon, \quad \forall t_2 \in [\rho_2, R_2]. \] (3.7)
At the same time, by Lemma 3.2 and (3.5), we have
\[ E(t_1 \ast w_1, R_2 \ast w_2) \leq I_{\mu_1}(t_1 \ast w_1) + I_{\mu_2}(R_2 \ast w_2) \]
\[ \leq \sup_{l \in \mathbb{R}} I_{\mu_1}(l \ast w_1) \]
\[ = I_{\mu_1}(w_{a_1, \mu_1}), \quad \forall t_1 \in [\rho_1, R_1], \] (3.8)
and
\[
E(R_1 \ast w_1, t_2 \ast w_2) \leq I_{\mu_1}(R_1 \ast w_1) + I_{\mu_2}(t_2 \ast w_2)
\]
\[
\leq \sup_{t \in \mathbb{R}} I_{\mu_2}(t \ast w_2)
\]
\[
= I_{\mu_2}(w_{o_2, \mu_2}), \quad \forall t_2 \in [\rho_2, R_2]. \quad (3.9)
\]
Hence, the conclusion of Lemma 3.3 holds.

**Lemma 3.4.** For every \( \gamma \in \Gamma \), there exists \((t_1, \gamma, t_2, \gamma) \in Q\) such that \( \gamma(t_1, \gamma, t_2, \gamma) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta) \).

**Proof.** For \( \gamma \in \Gamma \), we use the notation \( \gamma(t_1, t_2) = (\gamma_1(t_1, t_2), \gamma_2(t_1, t_2)) \in H^{rad}_{a_1} \times H^{rad}_{a_2} \). Considering the map \( F_{\gamma} : Q \to \mathbb{R}^2 \) defined by
\[
F_{\gamma}(t_1, t_2) := \left( \frac{\partial}{\partial l} I_{\mu_1 + \beta}(l \ast \gamma_1(t_1, t_2))|_{l=0}, \frac{\partial}{\partial l} I_{\mu_2 + \beta}(l \ast \gamma_2(t_1, t_2))|_{l=0} \right).
\]
From
\[
\frac{\partial}{\partial l} I_{\mu_1 + \beta}(l \ast \gamma_1(t_1, t_2))|_{l=0}
\]
\[
= \frac{\partial}{\partial l} \left( \frac{e^{2s \rho_1 t}}{|(-\Delta)^{\frac{s}{2}} \gamma_1(t_1, t_2)|^2L^2} - \frac{e^{(p-1)Ns \rho_1}}{2p} (\mu_1 + \beta)^s \gamma_1(t_1, t_2) \right) \big|_{l=0}
\]
\[
= s^2 |(-\Delta)^{\frac{s}{2}} \gamma_1(t_1, t_2)|^2L^2 - \frac{(p-1)Ns}{2p} (\mu_1 + \beta) s \gamma_1(t_1, t_2) \big|_{L^2},
\]
we deduce that
\[
F_{\gamma}(t_1, t_2) = (0, 0) \quad \text{if and only if} \quad \gamma(t_1, t_2) \in \mathcal{P}(a_1, \mu_1 + \beta) \times \mathcal{P}(a_2, \mu_2 + \beta).
\]
Now, we will show that \( F_{\gamma}(t_1, t_2) = (0, 0) \) has a solution in \( Q \) for every \( \gamma \in \Gamma \). At first, for \( \gamma \in \Gamma \), we observe that \( F_{\gamma}(\partial^+ Q) = F_{\gamma_0}(\partial^+ Q) \) depends only on the choice of \( \gamma_0 \), and not on \( \gamma \). Then, we have
\[
F_{\gamma_0}(t_1, t_2) = \left( s^2 e^{2s \rho_1 t} |(-\Delta)^{\frac{s}{2}} w_1|^2L^2 - \frac{(p-1)Ns}{2p} e^{(p-1)Ns \rho_1 (\mu_1 + \beta)} |w_1|^{2p}, \right.
\]
\[
\left. s^2 e^{2s \rho_2 t} |(-\Delta)^{\frac{s}{2}} w_2|^2L^2 - \frac{(p-1)Ns}{2p} e^{(p-1)Ns \rho_2 (\mu_2 + \beta)} |w_2|^{2p} \right)
\]
\[
= (\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)).
\]
Therefore, the restriction of \( F_{\gamma_0} \) on \( \partial Q \) is completely described by Lemma 3.2-(ii). In particular, we have the topological degree
\[
\text{deg}(F_{\gamma}, Q, (0, 0)) = \iota(F_{\gamma_0}(\partial^+ Q), (0, 0)) = 1,
\]
where \( \iota(\sigma, P) \) denotes the winding number of the curve \( \sigma \) with respect to the point \( P \). Hence, there exists \((t_1, \gamma, t_2, \gamma) \in Q\) such that \( F_{\gamma}(t_1, \gamma, t_2, \gamma) = (0, 0) \). \( \square \)
Lemma 3.5. There exists a Palais-Smale sequence \((u_n,v_n)\) for \(E\) on \(H_{a_1}^{rad} \times H_{a_2}^{rad}\) at the level
\[
c := \inf_{\gamma \in \Gamma} \max_{(t_1,t_2) \in Q} E(\gamma(t_1,t_2)) \geq \max\{I_{\mu_1}(w_{a_1,\mu_1}), I_{\mu_2}(w_{a_2,\mu_2})\},
\]
satisfying the additional condition
\[
G(u_n,v_n) = o(1),
\]
where \(o(1) \to 0\) as \(n \to \infty\). Furthermore, \(u_n^-,v_n^- \to 0\) a.e. in \(\mathbb{R}^N\) as \(n \to \infty\).

Proof. We consider the augmented functional \(\tilde{E} : \mathbb{R} \times H_{a_1}^{rad} \times H_{a_2}^{rad} \to \mathbb{R}\) defined by
\[
\tilde{E}(l,u_1,u_2) := E(l \ast u_1,l \ast u_2). \text{ Let also}
\]
\[
\tilde{\gamma}(t_1,t_2) := (l(t_1,t_2),\gamma_1(t_1,t_2),\gamma_2(t_1,t_2)),
\]
\[
\tilde{\gamma}_0(t_1,t_2) := (0,\gamma_0(t_1,t_2)) = (0,t_1 \ast w_1,t_2 \ast w_2),
\]
\[
\tilde{\Gamma} := \{\tilde{\gamma} \in C(Q,\mathbb{R} \times H_{a_1}^{rad} \times H_{a_2}^{rad} : \tilde{\gamma} = \tilde{\gamma}_0 \text{ on } \partial Q)\},
\]
and
\[
\tilde{c} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{(t_1,t_2) \in Q} \tilde{E}(\tilde{\gamma}(t_1,t_2)).
\]
Since \(\Gamma \subset \tilde{\Gamma}\), we have \(\tilde{c} \leq c\). At the same time, for any \(\tilde{\gamma} \in \tilde{\Gamma}\) and \((t_1,t_2) \in Q\), we have
\[
\tilde{E}(\tilde{\gamma}(t_1,t_2)) = E(l(t_1,t_2) \ast \gamma_1(t_1,t_2),l(t_1,t_2) \ast \gamma_2(t_1,t_2)),
\]
and \((l(\cdot) \ast \gamma_1(\cdot),l(\cdot) \ast \gamma_2(\cdot)) \in \Gamma\), then, \(c \leq \tilde{c}\). Hence, \(c = \tilde{c}\). Note that, since \(\tilde{E}(\tilde{\gamma}_0) = E(\gamma_0)\) on \(\partial Q\), by Lemma 3.3 and Lemma 3.4, the assumptions of minimax principle are satisfied(Theorem 3.2 in [6]). Furthermore, from
\[
\tilde{E}(\tilde{\gamma}(t_1,t_2)) = E(l(t_1,t_2) \ast \gamma_1(t_1,t_2),l(t_1,t_2) \ast \gamma_2(t_1,t_2))
\]
\[
= \tilde{E}(0,l(t_1,t_2) \ast \gamma_1(t_1,t_2),l(t_1,t_2) \ast \gamma_2(t_1,t_2)),
\]
and \(\tilde{E}(l,u,v) = \tilde{E}(l,|u|,|v|)\), we can choose the minimizing sequence \(\tilde{\gamma}_n = (l_n,\gamma_{1,n},\gamma_{2,n})\) for \(\tilde{c}\) satisfying the additional conditions:
\[
l_n \equiv 0,
\]
\[
\gamma_{1,n}(t_1,t_2) \geq 0 \text{ a.e. in } \mathbb{R} \text{ for every } (t_1,t_2) \in Q,
\]
\[
\gamma_{2,n}(t_1,t_2) \geq 0 \text{ a.e. in } \mathbb{R} \text{ for every } (t_1,t_2) \in Q.
\]
In conclusion, Theorem 3.2 in [6] implies that there exists a Palais-Smale sequence \((\tilde{l}_n, \tilde{u}_{1,n}, \tilde{u}_{2,n})\) for \(E\) on \(\mathbb{R} \times H_{a_1}^{rad} \times H_{a_2}^{rad}\) at level \(\tilde{c}\), and such that
\[
\lim_{n \to \infty} |\tilde{l}_n| + dist_{H^r}((\tilde{u}_n, \tilde{v}_n), \tilde{\gamma}_n(Q)) = 0. \tag{3.11}
\]

To obtain a Palais-Smale sequence for \(E\) at level \(c\) satisfying (3.10), it is possible to argue as in Lemma 2.4 in [9] with minor changes. From (3.11), we know \(u_n^-, v_n^- \to 0\) as \(n \to \infty\), and by Lemma 3.3 we know the lower estimate for \(c\).

\[\square\]

**Lemma 3.6.** The sequence \(\{(u_n, v_n)\}\) is bounded in \(H^s\). Furthermore, there exists \(\bar{C} > 0\) such that
\[
\int_{\mathbb{R}} \left( |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 \right) dx \geq \bar{C} \quad \text{for all } n.
\]

**Proof.** By (3.10), we have
\[
E(u_n, v_n) = \left( \frac{1}{2} - \frac{s}{(p-1)N} \right) \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \right) - o(1),
\]
where \(o(1) \to 0\) as \(n \to \infty\). Combining with \(E(u_n, v_n) \to c > 0\), we know that the conclusion of Lemma 3.6 holds. \[\square\]

From Lemma 3.6, up to a subsequence \((u_n, v_n) \to (\tilde{u}, \tilde{v})\) weakly in \(H^s(\mathbb{R}^N)\), strongly in \(L^{2p}(\mathbb{R}^N)\) (because of the compactness of embedding \(H^s_{rad} \hookrightarrow L^{2p}(\mathbb{R}^N)\)), and a.e. in \(\mathbb{R}^N\); in particular, both \(\tilde{u}\) and \(\tilde{v}\) are nonnegative in \(\mathbb{R}^N\). As a consequence, \(dE|_{H_{a_1}^{rad} \times H_{a_2}^{rad}}(u_n, v_n) \to 0\), there exist two sequences of real number \(\{\lambda_{1,n}\}\) and \(\{\lambda_{2,n}\}\) such that
\[
\begin{align*}
\int_{\mathbb{R}^N} \left( (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} g + (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} h \right) - \mu_1 u_n^{2p-1} g \\
- \mu_2 v_n^{2p-1}h - \beta u_n^{p-1}v_n^{p-1}(u_nh + v_ng) - \int_{\mathbb{R}^N} (\lambda_{1,n} u_n g + \lambda_{2,n} v_n h) \\
= o(1)|(g, h)|_{H^s},
\end{align*}
\tag{3.12}
\]
for every \((g, h) \in H^s(\mathbb{R}^N, \mathbb{R}^2)\) with \(o(1) \to 0\), as \(n \to \infty\).

**Lemma 3.7.** Both \(\{\lambda_{1,n}\}\) and \(\{\lambda_{2,n}\}\) are bounded sequences and at least one of them is converging, up to a sequence, to a strictly negative value.

**Proof.** By using \((u_n, 0)\) and \((0, v_n)\) as test functions in (3.12), we can find the value of \(\{\lambda_{i,n}\}\):
\[
\begin{align*}
\lambda_{1,n} a_1^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 - \mu_1 u_n^{2p} - \beta u_n^p v_n^p - o(1), \\
\lambda_{2,n} a_2^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 - \mu_2 v_n^{2p} - \beta u_n^p v_n^p - o(1),
\end{align*}
\]

18
with \( o(1) \to 0 \), as \( n \to \infty \). By the boundedness of \((u_n, v_n)\) in \( H^s(\mathbb{R}^N) \) and in \( L^{2p}(\mathbb{R}^N) \). Furthermore, there holds

\[
\lambda_{1,n} a_1^2 + \lambda_{2,n} a_2^2 = \int_{\mathbb{R}^N} |(-\Delta)^s u_n|^2 + |(-\Delta)^s v_n|^2 - \mu_1 u_n^{2p} - 2\beta u_n^p v_n^p - \mu_2 v_n^{2p} - o(1) \\
= (1 - \frac{2ps}{(p-1)N}) \int_{\mathbb{R}^N} |(-\Delta)^s u_n|^2 + |(-\Delta)^s v_n|^2 + o(1) \\
\leq (1 - \frac{2ps}{(p-1)N}) C,
\]

for \( 1 + \frac{2s}{N} < p < \frac{N}{N-2s} \) and every \( n \) sufficiently large, then at least one sequence of \( \{\lambda_{1,n}\} \) is negative and bounded away from 0. \( \square \)

Next we still consider converging subsequence \( \lambda_{1,n} \to \lambda_1 \in \mathbb{R} \) and \( \lambda_{2,n} \to \lambda_2 \in \mathbb{R} \), as \( n \to \infty \). In the following argument the sign of the limit value plays an important role.

**Lemma 3.8.** If \( \lambda_1 < 0 \) (resp. \( \lambda_2 < 0 \)), then \( u_n \to \tilde{u} \) (resp. \( v_n \to \tilde{v} \)) strongly in \( H^s(\mathbb{R}^N) \).

*Proof.* Let us suppose that \( \lambda_1 < 0 \). By weak convergence in \( H^s(\mathbb{R}^N) \), strongly convergence in \( L^{2p}(\mathbb{R}^N) \), and using (3.12), we have

\[
o(1) = (dE(u_n, v_n) - dE(\tilde{u}, \tilde{v}))(u_n - \tilde{u}), 0) - \lambda_1 \int_{\mathbb{R}^N} (u_n - \tilde{u})^2 \\
= \int_{\mathbb{R}^N} |(-\Delta)^s (u_n - \tilde{u})|^2 - \lambda_1 (u_n - \tilde{u})^2 + o(1),
\]

with \( o(1) \to 0 \) and \( n \to \infty \). Since \( \lambda_1 < 0 \), this equivalents to the strong convergence in \( H^s \). The proof in the case \( \lambda_2 < 0 \) is similar. \( \square \)

**The proof of Theorem 1.1.** By the convergence of \( \{\lambda_{1,n}\} \) and \( \{\lambda_{2,n}\} \), and the weak convergence \( (u_n, v_n) \to (\tilde{u}, \tilde{v}) \), we can obtain that \((\tilde{u}, \tilde{v})\) is a solution of (1.4). It remains to prove the convergence in the sense of \( L^2 \) norm. Without loss of generality, by Lemma 3.8 we can suppose that \( \lambda_1 < 0 \) and then \( u_n \to \tilde{u} \) strongly in \( H^s(\mathbb{R}^N) \). If \( \lambda_2 < 0 \) we can get that \( v_n \to \tilde{v} \) strongly in \( H^s(\mathbb{R}^N) \), which means the proof is completed. Now we prove that by contradiction and assume that \( \lambda_2 \geq 0 \) and \( v_n \to \tilde{v} \) strongly in \( H^s(\mathbb{R}^N) \). Since both \( \tilde{u}, \tilde{v} \geq 0 \), we have that

\[
(-\Delta)^s \tilde{v} = \lambda_2 \tilde{v} + \mu_2 \tilde{v}^{2p-1} + \beta \tilde{u}^p \tilde{v}^{p-1} \geq 0 \quad \text{in} \ \mathbb{R}^N,
\]

and from Lemma 2.6 we can deduce that \( \tilde{v} \equiv 0 \). In particular, this implies that \( \tilde{u} \) solves

\[
\begin{aligned}
(-\Delta)^s \tilde{u} - \lambda_1 \tilde{u} - \mu_1 \tilde{u}^{2p-1} &= 0 \quad \text{in} \ \mathbb{R}^N, \\
\tilde{u} &> 0 \quad \text{in} \ \mathbb{R}^N, \\
\int_{\mathbb{R}^N} \tilde{u}^2 &= a_1^2,
\end{aligned}
\]

(3.13)
so that \( \tilde{u} \in \mathcal{P}(a_1, \mu_1) \). However, we can obtain
\[
c = \lim_{n \to \infty} E(u_n, v_n) = \lim_{n \to \infty} \frac{(p-1)N - 2s}{4ps} \int_{\mathbb{R}^N} (\mu_1 u_n^{2p} + 2\beta u_n^p v_n^p + \mu_2 v_n^{2p})
= \frac{(p-1)N - 2s}{4ps} \int_{\mathbb{R}^N} \mu_1 \tilde{u}^{2p} = I_{\mu_1}(\tilde{u}).
\]
This is a contradiction with Lemma 3.5.

In the second part of section, we devote to prove Theorem 1.2 by dividing the process into two parts. Firstly, we show the existence of a positive solution \((\bar{u}, \bar{v})\), and secondly we characterize it as a ground state. The proof of the main theorem is depended upon a mountain pass argument. We should usually consider, for \((u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}\), the function
\[
E(l \ast (u, v)) = \frac{e^{2s^2l}}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 - \frac{e^{(p-1)Ns/2}}{2p} \int_{\mathbb{R}^N} (\mu_1 u^{2p} + 2\beta u^p v^p + \mu_2 v^{2p})
\]
where \(l \ast (u, v) = (l \ast u, l \ast v)\). If \((u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}\), then \(l \ast (u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}\) for any \(l \in \mathbb{R}\). From the definition, we can immediately obtain that holds:

**Lemma 3.9.** Let \((u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}\). Then
\[
\lim_{l \to -\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} l \ast u|^2 + |(-\Delta)^{\frac{s}{2}} l \ast v|^2 = 0,
\]
\[
\lim_{l \to +\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} l \ast u|^2 + |(-\Delta)^{\frac{s}{2}} l \ast v|^2 = +\infty,
\]
\[
\lim_{l \to -\infty} E(l \ast (u, v)) = 0^+ \text{ and } \lim_{l \to +\infty} E(l \ast (u, v)) = -\infty.
\]

From preliminaries, we can find the fact the proof of the existence of normalized solutions for system is closer to the proof of the single equation. The following lemma is similar to Lemma 2.3 Claim 2:

**Lemma 3.10.** There exists \(K > 0\) sufficiently small such that
\[
\sup_{\tilde{A}} E < \inf_{\tilde{B}} E \text{ and } E(u, v) > 0 \text{ on } \tilde{A},
\]
where
\[
\tilde{A} = \{u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}, |(-\Delta)^{\frac{s}{2}} u|_{L_2}^2 + |(-\Delta)^{\frac{s}{2}} v|_{L_2}^2 \leq K\},
\]
\[
\tilde{B} = \{u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}, |(-\Delta)^{\frac{s}{2}} u|_{L_2}^2 + |(-\Delta)^{\frac{s}{2}} v|_{L_2}^2 = 2K\}.
\]

**Proof.** By Gagliardo-Nirenberg-Sobolev inequality (2.1), there holds
\[
\int_{\mathbb{R}^N} (\mu_1 u^{2p} + 2\beta u^p v^p + \mu_2 v^{2p})
\leq C \int_{\mathbb{R}^N} (u^{2p} + v^{2p})
\leq C(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2)^{(p-1)N/2s},
\]
\[
20
for every \((u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad}\), where \(C > 0\) depends on \(\mu_1, \mu_2, \beta, a_1, a_2 > 0\), but not on the choice of \((u, v)\). Now if \((u_1, v_1) \in \bar{B}\) and \((u_2, v_2) \in \bar{A}\) (with \(K\) to be determined), we have

\[
E(u_1, v_1) - E(u_2, v_2) \geq \frac{1}{2} \left\{ \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u_1|^2 + |(-\Delta)^{\frac{1}{2}} v_1|^2 - \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u_2|^2 + |(-\Delta)^{\frac{1}{2}} v_2|^2 \right) \right\}
- \frac{1}{2p} \int_{\mathbb{R}^N} \left( \mu_1 u_1^{2p} + 2\beta u_1 v_1^{p} + \mu_2 v_1^{2p} \right) \geq \frac{K}{2} - \frac{C}{2p} (2K)^{(p-1)N/2p},
\]

provided \(K > 0\) is sufficiently small. Furthermore if necessary, we can make \(K\) smaller, then there holds

\[
E(u_2, v_2) \geq \frac{1}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u_2|^2 + |(-\Delta)^{\frac{1}{2}} v_2|^2 \right)
- \frac{C}{2p} \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u_2|^2 + |(-\Delta)^{\frac{1}{2}} v_2|^2 \right)^{(p-1)N/2p}
> 0,
\]

for every \((u_2, v_2) \in \bar{A}\). \(\square\)

For the next part, we shall introduce a suitable minimax class, we can see Lemma 2.4 that \(\bar{C}\) is the unique positive radial solution of (2.8). Now we define

\[
\bar{C} := \{(u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad} : |(-\Delta)^{\frac{1}{2}} u|_{L^2}^2 + |(-\Delta)^{\frac{1}{2}} v|_{L^2}^2 \geq 3K\text{ and } E(u, v) \leq 0\}.
\]

By Lemma 3.9, there exist \(l_1 < 0\) and \(l_2 > 0\) such that

\[
\begin{align*}
&l_1 \ast (w_{a_1, C_0/a_1^2}, w_{a_2, C_0/a_2^2}) := (\bar{u}_1, \bar{v}_1) \in \bar{A}, \\
&l_2 \ast (w_{a_1, C_0/a_1^2}, w_{a_2, C_0/a_2^2}) := (\bar{u}_2, \bar{v}_2) \in \bar{C}.
\end{align*}
\]

At last, we define

\[
\bar{\Gamma} := \{\bar{\gamma} \in C([0, 1], H_{a_1}^{rad} \times H_{a_2}^{rad}) : \bar{\gamma}(0) = (\bar{u}_1, \bar{v}_1), \bar{\gamma}(1) = (\bar{u}_2, \bar{v}_2)\}.
\]

**Lemma 3.11.** There exists a Palais-Smale sequence \((u_n, v_n)\) for \(E\) on \(H_{a_1}^{rad} \times H_{a_2}^{rad}\) at the level

\[
d := \inf_{\bar{\gamma} \in \bar{\Gamma}} \max_{t \in [0, 1]} E(\bar{\gamma}(t)),
\]

satisfying the additional condition

\[
G(u_n, v_n) = o(1),
\]

with \(o(1) \to 0\) as \(n \to \infty\). Furthermore, \(u_n, v_n \to 0\) a.e. in \(\mathbb{R}^N\) as \(n \to \infty\).
Lemma 3.12. If $\beta > \beta_2$, then

$$\sup_{t \in \mathbb{R}} E\left(l \star (w_{a_1, (C_0/a^2)^{p-1}}, w_{a_2, (C_0/a^2)^{p-1}})\right) < \min \left\{ I_{\mu_1}(w_{a_1, \mu_1}), I_{\mu_2}(w_{a_2, \mu_2}) \right\}.$$  

Proof. We can obtain immediately

$$\int_{\mathbb{R}^N} \left(l \star (w_{a_1, (C_0/a^2)^{p-1}}, w_{a_2, (C_0/a^2)^{p-1}})\right)^p = \int_{\mathbb{R}^N} e^{(p-1)Ns l} \left(\frac{a_1}{C_0^p} w_0(x)\right)^p \left(\frac{a_2}{C_0^p} w_0(x)\right)^p = \frac{a_1^p a_2^p}{C_0^p} e^{(p-1)Ns l} \int_{\mathbb{R}^N} w_0^{2p} dx = \frac{a_1^p a_2^p C_1}{C_0^p} e^{(p-1)Ns l}.$$  

Using Lemma 2.4, we can explicitly conclude the maximum in $l$ of the function

$$E\left(l \star (w_{a_1, (C_0/a^2)^{p-1}}, w_{a_2, (C_0/a^2)^{p-1}})\right) = \frac{(p-1)Ns l e^{2s l}}{4ps} \frac{a_1^2 \gamma_1 + a_2^2 \gamma_2}{C_0} - \frac{e^{(p-1)Ns l}}{2p} \left(\mu_1 a_1^{2p} + 2\mu_2 a_1^{2p} + \mu_2 C_1^{2p} a_2^{2p}\right).$$  

The maximum is given by

$$\max_{l \in \mathbb{R}} E\left(l \star (w_{a_1, (C_0/a^2)^{p-1}}, w_{a_2, (C_0/a^2)^{p-1}})\right) = \frac{(p-1)Ns l e^{2s l}}{4ps} \frac{2^{2p-1} a_1^{2p} a_2^{2p} \gamma_2^{(p-1)Ns l} (a_1^{2p} + a_2^{2p})^{(p-1)Ns l}}{(\mu_1 a_1^{2p} + 2\mu_2 a_1^{2p} + \mu_2 C_1^{2p} a_2^{2p})^{2s l} - 2s l}.$$  

According to the definition of $\beta_2$, $I_{\mu_1}(w_{a_1, \mu_1})$ and $I_{\mu_2}(w_{a_2, \mu_2})$, if $\beta > \beta_2$, the proof of lemma is completed.  

Existence of a positive solution. Next we will prove the existence of positive solution at level $d$ by contradiction. As in the proof of Theorem 1.1, we should show that $(u_n, v_n) \to (\bar{u}, \bar{v})$ in $H^s$, and $(\bar{u}, \bar{v})$ is a solution of (1.4) on the constraint $H_{a_1} \times H_{a_2}$. Firstly, up to a sequence $(u_n, v_n) \to (\bar{u}, \bar{v})$ weakly in $H^s$, strongly in $L^{2p}$, a.e. in $\mathbb{R}^N$. By weak convergence and Lemma 3.8, $(\bar{u}, \bar{v})$ is a solution of (1.4) for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Moreover, we can also suppose that one of the parameters, say $\lambda_1$ is strictly negative. Thus Lemma 3.9 implies that $u_n \to \bar{u}$ strongly in $H^s$. Suppose that $v_n \to \bar{v}$ strongly in $H^s(\mathbb{R}^N)$, then $\bar{v} \equiv 0$ and $d = I_{\mu_1}(w_{a_1, \mu_1})$. We can consider the path

$$\bar{\gamma}(t) := ((1-t)l_1 + tl_2) \star (w_{a_1, \mu_1}, w_{a_2, \mu_2}).$$  

Obviously, $\bar{\gamma} \in \bar{\Gamma}$. Then by Lemma 3.12

$$d \leq \sup_{t \in [0, 1]} E(\bar{\gamma}(t)) \leq \sup_{t \in \mathbb{R}} E(l \star (w_{a_1, \mu_1}, w_{a_2, \mu_2})) < I_{\mu_1}(w_{a_1, \mu_1}),$$  

which is a contradiction.
**Variational characterization.** In the following, we will use variational characterization for $(\bar{u}, \bar{v})$. Next we shall prove that

$$E(\bar{u}, \bar{v}) = \inf\{E(u, v); (u, v) \in F\} = \inf_{(u,v) \in H_{a_1} \times H_{a_2}} \mathcal{R}(u,v)$$

$$= \inf\{E(u, v) \text{ is a solution of } (1.4) \text{ for some } \lambda_1, \lambda_2\},$$

where $F$ and $\mathcal{R}$ can be seen in the previous section. Recalling the definition of $\bar{A}$ and $\bar{C}$. Let us define

$$\bar{A}^+ := \{(u, v) \in \bar{A}, u, v \geq 0 \text{ a.e. in } \mathbb{R}^N\},$$

$$\bar{C}^+ := \{(u, v) \in \bar{C}, u, v \geq 0 \text{ a.e. in } \mathbb{R}^N\}.$$  

For any $(u_1, v_1) \in \bar{A}^+$ and $(u_2, v_2) \in \bar{C}^+$, let

$$\bar{\Gamma}(u_1, v_1, u_2, v_2) = \left\{ \bar{\gamma} \in C([0,1], H^rad_{a_1} \times H^rad_{a_2}) : \bar{\gamma}(0) = (u_1, v_1), \bar{\gamma}(1) = (u_2, v_2) \right\}.$$

**Lemma 3.13.** The sets $\bar{A}^+$ and $\bar{C}^+$ are connected by arcs, so that

$$d = \inf_{\bar{\gamma} \in \bar{\Gamma}(u_1, v_1, u_2, v_2)} \max_{t \in [0,1]} E(\bar{\gamma}(t)), \quad (3.20)$$

for every $(u_1, v_1) \in \bar{A}^+$ and $(u_2, v_2) \in \bar{C}^+$.

**Proof.** Equation (3.20) holds easily once we show that $\bar{A}^+$ and $\bar{C}^+$ are connected by arcs. Let $(u_1, v_1), (u_2, v_2) \in H^rad_{a_1} \times H^rad_{a_2}$ be nonnegative functions such that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\nu}{2}} u_1|^2 + |(-\Delta)^{\frac{\nu}{2}} v_1|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\nu}{2}} u_2|^2 + |(-\Delta)^{\frac{\nu}{2}} v_2|^2 dx = \alpha^2, \quad (3.21)$$

for some $\alpha > 0$. For $l \in \mathbb{R}$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, 

$$h(l, \theta) = (\cos \theta (l \ast u_1)(x) + \sin \theta (l \ast u_2)(x), \cos \theta (l \ast v_1)(x) + \sin \theta (l \ast v_2)(x)).$$

Setting $h = (h_1, h_2)$, we have that $h_1(l, \theta), h_2(l, \theta) \geq 0$ a.e. in $\mathbb{R}^N$. It is not difficult to check that

$$\int_{\mathbb{R}^N} h_1^2(l, \theta) = a_1^2 + \sin(2\theta) \int_{\mathbb{R}^N} u_1 u_2,$$

$$\int_{\mathbb{R}^N} h_2^2(l, \theta) = a_2^2 + \sin(2\theta) \int_{\mathbb{R}^N} v_1 v_2,$$

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\nu}{2}} h_1(l, \theta)|^2 + |(-\Delta)^{\frac{\nu}{2}} h_2(l, \theta)|^2$$

$$= e^{2\gamma^2} \left( \alpha^2 + \sin(2\theta) \int_{\mathbb{R}^N} (-\Delta)^{\frac{\nu}{2}} u_1 (-\Delta)^{\frac{\nu}{2}} u_2 + (-\Delta)^{\frac{\nu}{2}} v_1 (-\Delta)^{\frac{\nu}{2}} v_2 \right),$$

23
Thus we can define the function
\[
\hat{h}(l, \theta)(x) = (a_1 \frac{h_1(l, \theta)}{|h_1(l, \theta)|_{L^2}}, a_2 \frac{h_2(l, \theta)}{|h_2(l, \theta)|_{L^2}}),
\]
for \((l, \theta) \in \mathbb{R} \times [0, \frac{\pi}{2}]\). Notice that \(\hat{h}(l, \theta) \in L^r_{a_1} \times L^r_{a_2}\) for every \((l, \theta)\). We can obtain that
\[
\frac{1}{2} C e^{2s_2t} \min\{a_1^2, a_2^2\} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} h_1(l, \theta)|^2 + |(-\Delta)^{\frac{s}{2}} h_2(l, \theta)|^2 \leq 2\alpha^2 e^{2s_2t} \min\{a_1^2, a_2^2\}.
\]
(3.22)
We can check the following
\[
\int_{\mathbb{R}^N} \hat{h}_{1}^{2p}(l, \theta) \geq C e^{(p-1)Ns_t} \text{ and } \int_{\mathbb{R}^N} \hat{h}_{2}^{2p}(l, \theta) \geq C e^{(p-1)Ns_t},
\]
(3.23)
for all \((l, \theta) \in \mathbb{R} \times [0, \frac{\pi}{2}]\) and \(C\) is a smaller quantity. Let \((u_1, v_1), (u_2, v_2) \in \tilde{A}^+\), and let \(\tilde{h}\) as the previous. From (3.21), we can deduce there exists \(l_0 > 0\) such that
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} h_1(-l_0, \theta)|^2 + |(-\Delta)^{\frac{s}{2}} h_2(-l_0, \theta)|^2 \leq K,
\]
for all \(\theta \in [0, \frac{\pi}{2}]\), where \(K\) is defined in Lemma 3.10. For the choice of \(l_0\), let
\[
\sigma_1(r) := \begin{cases} 
-r \ast (u_1, v_1) = \hat{h}(-r, 0), & 0 \leq r \leq l_0, \\
\hat{h}(-l_0, r - l_0), & l_0 \leq r \leq l_0 + \frac{\pi}{2}, \\
(r - 2l_0 - \frac{\pi}{2}) \ast (u_2, v_2) = \hat{h}(r - 2l_0 - \frac{\pi}{2}, \frac{\pi}{2}), & l_0 + \frac{\pi}{2} < r \leq 2l_0 + \frac{\pi}{2}.
\end{cases}
\]
It is not difficult to check that \(\sigma_1\) is a continuous path connecting \((u_1, v_1)\) and \((u_2, v_2)\) and lying in \(\tilde{A}^+\). But we should still consider that cases when condition (3.21) is not satisfied.
Suppose for instance
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_1|^2 + |(-\Delta)^{\frac{s}{2}} v_1|^2 > \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_2|^2 + |(-\Delta)^{\frac{s}{2}} v_2|^2.
\]
Then by Lemma 3.9, there exists \(l_1 < 0\) such that
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} (l_1 \ast u_1)|^2 + |(-\Delta)^{\frac{s}{2}} (l_1 \ast v_1)|^2 > \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_2|^2 + |(-\Delta)^{\frac{s}{2}} v_2|^2.
\]
Therefore, to connect \((u_1, v_1)\) and \((u_2, v_2)\) by a path in \(\bar{A}^+\), we can at first connect \((u_1, v_1)\) with \(l_1 \ast (u_1, v_1)\) and then connect this point with \((u_2, v_2)\).

Let fix \((u_1, v_1), (u_2, v_2) \in \bar{G}^+\). Suppose that (3.21) holds. By using (3.22) and (3.23), there exists \(l_0 > 0\) such that

\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{h}_1(l_0, \theta)|^2 + |(-\Delta)^{\frac{s}{2}} \hat{h}_2(l_0, \theta)|^2 \geq 3K \quad \text{and} \quad E(\hat{h}(l_0, \theta)) \leq 0,
\]

for all \(\theta \in [0, \frac{\pi}{2}]\). For the choice of \(l_0\), we set

\[
\sigma_2(r) := \begin{cases} 
    r \ast (u_1, v_1) = \hat{h}(r, 0), & 0 \leq r \leq l_0, \\
    \hat{h}(l_0, r - l_0), & l_0 < r \leq l_0 + \frac{\pi}{2}, \\
    (2l_0 + \frac{\pi}{2} - r) \ast (u_2, v_2) = \hat{h}(2l_0 + \frac{\pi}{2} - r, \frac{\pi}{2}), & l_0 + \frac{\pi}{2} < r \leq 2l_0 + \frac{\pi}{2},
\end{cases}
\]

which is desired continuous path connecting \((u_1, v_1)\) and \((u_2, v_2)\) and lying in \(\bar{G}^+\). ☐

From the previous notation, we know

\[
F := \{(u, v) \in H_{a_1} \times H_{a_2} : G(u, v) = 0\},
\]

and its radial subset

\[
F_{rad} := \{(u, v) \in H_{a_1}^{rad} \times H_{a_2}^{rad} : G(u, v) = 0\}, \tag{3.24}
\]

where

\[
G(u, v) = \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \right) - \frac{(p-1)N}{2ps} \int_{\mathbb{R}^N} (\mu_1 u^{2p} + 2\beta u^p v^p + \mu_2 v^{2p}).
\]

**Lemma 3.14.** If \((u, v)\) is a solution of (1.4) on the constraint \(H_{a_1} \times H_{a_2}\) for some \(\lambda_1, \lambda_2 \in \mathbb{R}\), then \((u, v) \in F\).

**Proof.** From the Pohozaev identity for the system, there holds

\[
(N - 2s) \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \right) = N \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2) + \frac{N}{p} \int_{\mathbb{R}^N} (\mu_1 u^{2p} + 2\beta u^p v^p + \mu_2 v^{2p}).
\]

On the other hand, testing (1.4) with \((u, v)\), we find

\[
\lambda_1 \int_{\mathbb{R}^N} u^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \int_{\mathbb{R}^N} (\mu_1 u^{2p} + \beta u^p v^p),
\]

\[
\lambda_2 \int_{\mathbb{R}^N} v^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 - \int_{\mathbb{R}^N} (\mu_2 v^{2p} + \beta u^p v^p),
\]

which substituted into the Pohozaev identity gives the desired result. ☐
For \((u, v) \in H_{a_1} \times H_{a_2}\), let us set
\[
\Psi_{(u,v)}(l) = E(l \ast (u, v)),
\]
where \(l \ast (u, v) = (l \ast u, l \ast v)\) for short.

**Lemma 3.15.** For every \((u, v) \in H_{a_1} \times H_{a_2}\), there exists a unique \(l_{(u,v)} \in \mathbb{R}\) such that
\(l_{(u,v)} \ast (u, v) \in F\). Moreover \(l_{(u,v)}\) is the unique critical point of \(\Psi_{(u,v)}\), which is a strict maximum.

**Lemma 3.16.** There holds \(\inf_F E = \inf_{F_{rad}} E\).

**Proof.** We prove the lemma by contradiction. Suppose there exists \((u, v) \in F\),
\[
0 < E(u, v) < \inf_{F_{rad}} E.
\]
(3.25)

For \(u \in H^s(\mathbb{R}^N)\), let \(u^*\) denotes its Schwarz spherical rearrangement. According to the property of Schwarz symmetrization, we have that \(E(u^*, v^*) \leq E(u, v)\) and \(G(u^*, v^*) \leq G(u, v) = 0\). Thus there exists \(l_0 \leq 0\) such that \(G(l_0 \ast (u^*, v^*)) = 0\). Using that \(G(l_0 \ast (u^*, v^*)) = G(u, v) = 0\), we have
\[
E(l_0 \ast (u^*, v^*)) = \left(\frac{1}{2} - \frac{s}{(p-1)N}\right)e^{2s^2l_0}\{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u^*|^2 + |(-\Delta)^{\frac{s}{2}} v^*|^2\} \leq \left(\frac{1}{2} - \frac{s}{(p-1)N}\right)e^{2s^2l_0}\{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2\} = e^{2s^2l_0}E(u, v).
\]
Thus
\[
0 < E(u, v) < \inf_{F_{rad}} E \leq E(l_0 \ast (u^*, v^*)) \leq e^{2s^2l_0}E(u, v),
\]
which contradicts \(l_0 \leq 0\). \(\Box\)

**The proof Theorem 1.2.** Considering that the solution of (1.4) on the constraint \(H_{a_1} \times H_{a_2}\) stays in \(F\). To obtain the result, we claim that
\[
E(\bar{u}, \bar{v}) \leq d \leq \inf\{E(u, v) : (u, v) \in F_{rad}\}.
\]
Indeed we can choose an arbitrary \((u, v) \in F_{rad}\) that implies \((|u|, |v|) \in F_{rad}\) and \(E(u, v) = E(|u|, |v|)\), we can suppose that \(u, v \geq 0\) a.e. \(\text{in } \mathbb{R}^N\). Let us consider the function \(\Psi_{(u,v)}\). By Lemma 3.9 there exists \(l_0 \gg 1\) such that \((-l_0) \ast (u, v) \in \bar{A}^+\) and \(l_0 \ast (u, v) \in \bar{C}^+\). Therefore,
\[
\tilde{\gamma}(t) = ((2t - 1)l_0) \ast (u, v) \quad t \in [0, 1],
\]
which is the continuous path connecting \(\bar{A}^+\) with \(\bar{C}^+\), and by Lemma 3.13 and Lemma 3.15, we can deduce that
\[
d \leq \max_{t \in [0,1]} E(\tilde{\gamma}(t)) = E(u, v).
\]

26
Since this holds for all the elementary in $F_{rad}$, the inequality $d \leq \inf \{E(u, v) : (u, v) \in F_{rad}\}$ holds. The equality $E(\bar{u}, \bar{v}) = \inf_{F} E$ can be acquired immediately from Lemma 3.16. It remains to show that

$$\inf_{F} E = \inf_{H_{a_{1}} \times H_{a_{2}}} \mathcal{R}. \quad (3.26)$$

The proof of (3.26) is similar to the single equation, see Lemma 2.5.

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