Duality in Gerstenhaber algebras

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Abstract

Let $C$ be a differential graded coalgebra, $\Omega C$ the Adams cobar construction and $C^\vee$ the dual algebra. We prove that for a large class of coalgebras $C$ there is a natural isomorphism of Gerstenhaber algebras between the Hochschild cohomologies $\text{HH}^*(C^\vee;C^\vee)$ and $\text{HH}^*(\Omega C;\Omega C)$. This result permits to describe a Hodge decomposition of the loop space homology of a closed oriented manifold, in the sense of Chas-Sullivan, when the field of coefficients is of characteristic zero.

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1 Introduction

In the last two decades there has been a great deal of interest in Gerstenhaber algebras since they arise, as BV-algebras, in the BRST theory of topological field theory and in operad setting as well as in string theory. Let us denote by $C^*(A;A)$ (resp. $\text{HH}^*(A;A)$) the Hochschild complex (resp. cohomology) of the differential graded algebra $A$ with coefficients in $A$. It is well known that $\text{HH}^*(A;A)$ is a Gerstenhaber algebra. A new geometrical impact to Hochschild cohomology has been given recently by the work of Chas and Sullivan [6].

Let $\mathbb{k}$ be an integral domain and let $V^\vee$ denotes the graded dual of the graded module $V$. For a supplemented graded coalgebra $C = \mathbb{k} \oplus C$, $C^\vee$ is a supplemented graded algebra (without any finiteness restriction), and the reduced coproduct $\Delta : C \to C \otimes C$ can be iterated unambiguously to produce $\Delta^{(k)}$. If for each $x \in C$ and some $k$, $\Delta^{(k)}x = 0$, the coalgebra $C$ is called locally conilpotent.

Our first result reads

**Theorem 1.** Let $(C,d)$ be a $\mathbb{k}$-free locally conilpotent differential graded coalgebra and $\Omega C$ the normalized cobar construction on $C$. There exists a homomorphism of differential graded algebras

$$\mathcal{D} : C^*(\Omega C;\Omega C) \to C^*(C^\vee;C^\vee)$$

which induces a homomorphism of Gerstenhaber algebras

$$H(\mathcal{D}) : \text{HH}^*(\Omega C;\Omega C) \to \text{HH}^*(C^\vee;C^\vee).$$

a) The homomorphism $H(\mathcal{D})$ is an isomorphism whenever $C$ is finitely generated in each degree, and either i) $C = C_{\geq 0}$ or else ii) $C = C_{\geq 2}$. 

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b) If \( k \) is a field then \( H(D) \) is natural with respect to quasi-isomorphisms of differential graded coalgebras and is an isomorphism whenever \( H(C) \) is finitely generated in each degree, and either i) \( C = C^{>0} \) and \( H^0(C, d) = k \), or else ii) \( \overline{C} = \overline{C}_{>2} \).

The proof of Theorem 1 relies heavily on properties of free models and on differential graded Lie algebras of derivations. In order to minimize the prerequisites, we do not introduce the twisting cochain formalism. Nonetheless it underlies most of the computational peculiarities we perform by hands.

Let us precise that a free model is a tensor algebra \( TV \) together with a differential \( d \) such that \( V \) is the union \( V = \bigcup_{k=0}^{\infty} V(k) \) of an increasing family \( V(0) \subset V(1) \subset \ldots \) such that \( d(V(0)) = 0 \) and \( d(V(k)) \subset T(V(k-1)) \). Observe (Proposition 3.1) that for any differential graded algebra \( A \) there is a quasi-isomorphism of differential graded algebras \( (T(V), d) \xrightarrow{\sim} A \). If \( C \) is a locally conilpotent coalgebra, the normalized cobar construction on \( C \) is a free model (5.2).

We denote by \( \text{Der}A \) the differential graded Lie algebra of derivations on \( A \) with the commutator bracket \([ -,-] \) and differential \( D = [d, -] \). We introduce the following extension, denoted \( \widetilde{\text{Der}}A \), of this differential graded Lie algebra:

\[
\widetilde{\text{Der}}A = \text{Der}A \oplus sA,
\]

where \((sA)_i = A_{i-1}\). The next result is a corner stone in the proof of Theorem 1.

**Theorem 2.** Let \( A = (TV, d) \) be a free model. There exist two injective quasi-isomorphisms of differential graded Lie algebras

\[
sC^*(A; A) \xleftarrow{\sim} \widetilde{\text{Der}}A \xrightarrow{\sim} \text{Der} \overline{A}
\]

where \( \overline{A} \) is the differential graded algebra \( \overline{A} = (T(V \oplus k\epsilon), \tilde{d}) \) with \( \tilde{d}\epsilon = e^2 \) and \( \tilde{d}\nu = dv + \epsilon v = (1-\nu)v, v \in V \).

If we introduce, following Getzler and Jones [13, 14], the Hochschild complex, \( C^\infty_* (E; E) \) of an \( A_\infty \)-coalgebra \( E \), Theorem 2 can be read: There exist injective quasi-isomorphisms of differential graded Lie algebras

\[
sC^*(A; A) \xleftarrow{\sim} \widetilde{\text{Der}}A \xrightarrow{\sim} C^\infty_* (k \oplus sV; k \oplus sV).
\]

Since the graded Lie algebra \( sHH^* (A; A) \) is natural with respect to quasi-isomorphisms (3.4) we deduce from Theorems 1 and 2:

**Corollary 1.** Consider \( C \) as in Theorem 1. Let \( \varphi : (TV, d) \xrightarrow{\sim} C^\vee \) and \( \psi : (TW, d) \xrightarrow{\sim} \overline{\Omega} C \) be free models, then there exists isomorphisms of graded Lie algebras:

\[
H \left( \widetilde{\text{Der}}(TW, d) \right) \cong sHH^* (C^\vee; C^\vee) \cong H \left( \widetilde{\text{Der}}(TV, d) \right).
\]

The geometric meaning of Theorem 1 is in terms of loop homology. Recall that the loop homology of a closed orientable manifold \( M \) of dimension \( d \) is the ordinary homology of the free loop space \( M^{S^1} \) with degrees shifted by \( d \), i.e. \( H_* (M^{S^1}) = H_{*+d}(M^{S^1}) \). In
The Adams-Hilton model of a space is completely determined with coefficients in $\mathbb{F}$, as stated in Theorem 2. If $X$ is a simply-connected closed oriented manifold, there is a natural equivalence $\mathbb{H}_*(\Omega X) \cong \mathbb{C}^*(\Omega X)$, as shown in Corollary 1. Then, $\mathbb{H}_*(M^{S^1}) \cong \mathbb{H}^*\left(C_*(\Omega M); C_*(\Omega M)\right)$ (Corollary 1) and $\mathbb{H}^*(U(L,d); U(L,d)) \cong \mathbb{H}^*(U(L,d); U(L,d))$ (by naturality). Indeed for any pointed 1-connected space $X$ there is a natural equivalence $\mathbb{H}^*\left(C_*(\Omega X); C_*(\Omega X)\right) \cong \mathbb{C}^*(\Omega X)$, as stated in Corollary 1.

The remaining of the paper is organized as follows:

3) Hochschild cohomology of a differential graded algebra.

4) Proof of Theorem 2.

5) The Hochschild cochain complex of a differential graded coalgebra.

6) Proof of Propositions B and C (see below).
2 Sketch of the Proof of Theorem 1.

2.1. Notation. In the rest of the paper, except in 2.6 and 2.7, $\ell k$ will be a principal ideal domain. If $V = \{V_i\}_{i \in \mathbb{Z}}$ is a (lower) graded $\ell k$-module (when we need upper graded $\ell k$-module we put $V_i = V^{-i}$ as usual) then:

a) $(sV)_n = V_{n-1}$, $(sV)^n = V^{n+1}$,

b) $TV$ denotes the tensor algebra on $V$, while we denote by $TC(V)$ the free supplemented coalgebra generated by $V$.

Since we work with graded differential objects, we will make a special attention to signs. Recall that if $M = \{M_i\}_{i \in \mathbb{Z}}$ and $N = \{N_i\}_{i \in \mathbb{Z}}$ are differential graded $\ell k$-modules then:

a) $M \otimes N$ is a differential graded $\ell k$-module: $(M \otimes N)_r = \oplus_{p+q=r} M_p \otimes N_q$, $d_{M \otimes N} = d_M \otimes id_N + id_M \otimes d_N$,

b) $Hom(M, N)$ is a differential graded $\ell k$-module: $Hom_n(M, N) = \prod_{k-l=n} Hom(M_l, N_k)$, $Df = d_M \circ f - (-1)^{|f|} f \circ d_N$,

c) the commutator, $[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$, gives to the differential graded $\ell k$-module $\text{End}(M) = Hom(M, M)$ a structure of differential graded Lie algebra,

d) if $C$ is a differential graded coalgebra with diagonal $\Delta$ and $A$ is a differential graded algebra with product $\mu$ then the cup product, $f \cup g = \mu \circ (f \otimes g) \circ \Delta$, gives to the differential graded $\ell k$-module $\text{Hom}(C, A)$ a structure of differential graded algebra.

2.2. A (graded) Gerstenhaber algebra is a commutative graded algebra $G = \{G\}_{i \in \mathbb{Z}}$ with a degree 1 linear map

$$G_i \otimes G_j \rightarrow G_{i+j+1}, \quad x \otimes y \mapsto \{x, y\}$$

such that:

a) the suspension of $G$ is a graded Lie algebra with bracket

$$(sG)_i \otimes (sG)_j \rightarrow (sG)_{i+j}, \quad sx \otimes sy \mapsto [sx, sy] := s\{x, y\}$$

b) the product is compatible with the bracket, $\{-, -\}$.

This last condition means that for any $a \in G_k$ the adjunction map $ad_a : G_i \rightarrow G_{i+k+1}$, $b \mapsto \{a, b\}$ is a $(k + 1)$-derivation: ie. for $a, b, c \in G$, $\{a, bc\} = \{a, b\}c + (-1)^{|b||c|+1}b\{a, c\}$.

A homomorphism of Gerstenhaber algebras $f : G \rightarrow G'$ is a homomorphism of graded algebras such that $sf : sG \rightarrow sG'$ is a homomorphism of graded Lie algebras.

For our purpose it is convenient to introduce the notion of pre-Gerstenhaber algebra. This a differential graded algebra $G = \{G_i, d\}$ together with a degree 1 homomorphism of differential graded modules $\alpha : G \rightarrow \mathcal{L}$ where $\mathcal{L} = \{\mathcal{L}_i, D\}$ denotes a differential graded Lie algebra. Observe that

a) $D \circ \alpha = -\alpha \circ d$,

b) $\alpha$ induces on $sG$ a structure of graded a Lie algebra compatible with the differential $sd$ while no compatibility condition with the product is required.

A homomorphism of pre-Gerstenhaber algebras is a commutative diagram of homomorphisms of differential graded modules

$$
\begin{array}{ccc}
  sG & \xrightarrow{\alpha} & \mathcal{L} \\
  sf & \downarrow & \downarrow g \\
  sG' & \xrightarrow{\alpha'} & \mathcal{L}
\end{array}
$$
such that $f$ is a homomorphism of differential graded algebras and $g : L \to L'$ is a homomorphism of differential graded Lie algebras. Clearly, $f$ induces a homomorphism of graded algebras $H(f) : H(G) \to H(G')$ and a homomorphism of differential graded Lie algebras $sH(f) : H(sG) \to H(sG')$.

If $H(f)$ is an isomorphism of graded modules then $f$ is called a quasi-isomorphism of pre-Gerstenhaber algebras.

If the structure of pre-Gerstenhaber algebra on $G$ (resp. $G'$) induces a structure of Gerstenhaber algebra on $H(G)$ (resp. on $H(G')$) then $H(f)$ is a homomorphism of Gerstenhaber algebras.

2.3. Let $A$ be a differential graded algebra. We consider the degree 1 isomorphism $\beta_A$ that extends a linear map to a coderivation

$$C^*(A; A) := \text{Hom}(TC(sA), A) \xrightarrow{\beta_A} \text{Coder} (\overline{B}(A))$$

where

i) $\overline{B}(A)$ is the non-unital bar construction (3.1),
ii) $\text{Coder} (C)$ is the differential graded Lie algebra of coderivations of $C$.

The map $\beta_A$ defines the Hochschild complex $C^*(A; A)$ as a pre-Gerstenhaber algebra. (It induces on $HH^*(A; A)$ the usual Gerstenhaber algebra structure [19].)

2.4. Dually, if $C$ is a supplemented differential graded coalgebra the degree 1 isomorphism $\gamma_C$ that extends a linear map to a derivation

$$C^*(C; C) := \text{Hom}(C, Ts^{-1}C) \xrightarrow{\gamma_C} \text{Der} (\overline{\Omega}C)$$

where $\overline{\Omega}C$ is the non-counital cobar construction (5.1), makes the Hochschild complex of $C$ into a pre-Gerstenhaber algebra.

We also consider the pre-Gerstenhaber algebra $\overline{C}^*(C; C)$ together with a degree 1 linear isomorphism:

$$\overline{C}^*(C; C) := \text{Hom}(C, Ts^{-1}\overline{C}) \xrightarrow{\gamma_C} \overline{\text{Der}} (\overline{\Omega}C)$$

where

i) $\overline{\Omega}C$ is the normalized cobar construction
ii) $\overline{\text{Der}} (A)$ is the differential graded Lie algebra considered in the introduction.

2.5. Intermediate results.

**Proposition A.** Let $C = \mathbb{L}k \oplus \overline{C}$ be a $\mathbb{L}k$-free locally conilpotent differential graded coalgebra. The inclusion $\overline{C} \hookrightarrow C$ induces, by naturality, a quasi-isomorphism of pre-Gerstenhaber algebras

$$\overline{C}^*(C; C) \to C^*(C; C) .$$

**Proposition B.** Let $C$ be a differential graded coalgebra. The usual linear duality induces an homomorphism of pre-Gerstenhaber algebras $C^*(C; C) \to C^*(C^\vee; C^\vee)$ which is

i) an isomorphism whenever $C = C^\geq 0$ is a free graded $\mathbb{L}k$-module of finite type,
ii) a quasi-isomorphism if $C = \mathbb{L}k \oplus C_{\geq 2}$ is a free graded $\mathbb{L}k$-module of finite type.
Proposition C. Let $C$ be a $\mathbb{k}$-free locally conilpotent differential graded coalgebra. The bar-cobar adjunction induces a quasi-isomorphism of differential graded algebras
\[ C^\ast(\bar{\Omega}C; \bar{\Omega}C) \longrightarrow C^\ast(C; C) \]
which admits a linear section $\Gamma$ such that $s\Gamma : s\bar{C}^\ast(C; C) \longrightarrow sC^\ast(\bar{\Omega}C; \bar{\Omega}C)$ is a homomorphism of differential graded Lie algebras.

2.6. Let $C$ be a $\mathbb{k}$-free locally conilpotent differential graded coalgebra. The composite
\[ D_C : C^\ast(\bar{\Omega}C; \bar{\Omega}C) \xrightarrow{\text{Prop. C}} C^\ast(C; C) \xrightarrow{\text{Prop. A}} C^\ast(C; C) \xrightarrow{\text{Prop. B}} C^\ast(C^\vee; C^\vee) \]
is a homomorphism of differential graded algebras but not a homomorphism of pre-Gerstenhaber algebra. Nonetheless,
\[ H(D_C) : HH^\ast(\bar{\Omega}C; \bar{\Omega}C) \longrightarrow HH^\ast(C^\vee; C^\vee) \]
is a homomorphism of Gerstenhaber algebras. It results from the constructions involved in Propositions A, B and C that $H(D_C)$ is natural with respect to quasi-isomorphisms of differential graded coalgebras: Given a quasi-isomorphism of locally conilpotent supplemented $\mathbb{k}$-free differential graded coalgebras $f : C \rightarrow D$ such that either i) $\bar{C} = \bar{\mathbb{C}}^\geq 0$ and $\bar{D} = \bar{\mathbb{D}}^\geq 0$ or else ii) $\bar{C} = \bar{\mathbb{C}}^\geq 2$ and $\bar{D} = \bar{\mathbb{D}}^\geq 2$, then by Remark 2.3 of [8], $\bar{\Omega}(f) : \bar{\Omega}C \xrightarrow{\sim} \bar{\Omega}D$ is a quasi-isomorphism of $\mathbb{k}$-free differential graded algebras. Moreover, if $\mathbb{k}$ is a field, $f : D^\vee \xrightarrow{\sim} C^\vee$ is also a quasi-isomorphism between $\mathbb{k}$-free differential graded algebras. By 3.4, there exist isomorphisms of Gerstenhaber algebras $HH(\bar{\Omega}(f))$ and $HH(f^\vee)$ such that the following diagram commutes
\[ \begin{array}{ccc}
HH^\ast(\bar{\Omega}C; \bar{\Omega}C) & \xrightarrow{HH(\bar{\Omega}(f))} & HH^\ast(C^\vee; C^\vee) \\
HH^\ast(\bar{\Omega}D; \bar{\Omega}D) & \xrightarrow{HH(D_C)} & HH^\ast(D^\vee; D^\vee).
\end{array} \]

2.7. The reduction process described below permits to deduce part b) of Theorem 1 from part a).

In case ii), let $C$ be a $\mathbb{k}$-free supplemented differential graded coalgebra such that $H(C)$ is finitely generated in each degree, $\bar{C} = \bar{\mathbb{C}}^\geq 2$ and $H_2(C)$ is $\mathbb{k}$-free. By Proposition 4.2 of [8], there exists a free model $TV$ and a quasi-isomorphism of augmented differential graded algebras $\varphi : TV \xrightarrow{\sim} C^\vee$, where $V = V^\geq 2$ is $\mathbb{k}$-free of finite type. We consider then the previous diagram with $f : C \xhookrightarrow{\sim} C^\vee \xrightarrow{\varphi^\vee} TV^\vee = D$. By Propositions A, B and C, the lower line in the above diagram is an isomorphism thus so is the upper line.

The case i) is similar.

Required definitions and proofs are detailed in the following sections.

3 The Hochschild complex of a differential graded algebra.

3.1. Let $(A, d)$ be a differential graded supplemented algebra, $A = \mathbb{k} \oplus \bar{\mathbb{A}}$, and $(M, d)$ (resp. $(N, d)$) be a right (resp. left) differential graded $A$-bimodule. The two-sided bar constructions, $\mathbb{B}(M; A; N)$ and $\mathbb{B}(M; A; N)$ are defined as follows:
\[ \mathbb{B}_k(M; A; N) = M \otimes T^k sA \otimes N, \quad \mathbb{B}_k(M; A; N) = M \otimes T^k s\bar{\mathbb{A}} \otimes N \]
A generic element is written $m[a_1|a_2|...|a_k]n$ with degree $|m| + |n| + \sum_{i=1}^{k}(|sa_i|)$. The differential $d = d_0 + d_1$ is defined by

$$d_0 : B_1(M; A; N) \rightarrow B_2(M; A; N), \quad d_0 : \overline{B}_1(M; A; N) \rightarrow \overline{B}_2(M; A; N),$$

$$d_1 : B_1(M; A; N) \rightarrow B_1(M; A; N), \quad d_1 : \overline{B}_1(M; A; N) \rightarrow \overline{B}_1(M; A; N),$$

with $\epsilon_i = |m| + \sum_{j<i}(|sa_j|)$ and:

$$d_0(m[a_1|a_2|...|a_k]n) = d(m)[a_1|a_2|...|a_k]n - \sum_{i=1}^{k}(-1)^{\epsilon_i}m[a_1|a_2|...|d(a_i)|...|a_k]n$$

$$+ (-1)^{\epsilon_{k+1}}m[a_1|a_2|...|a_k]d(n)$$

$$d_1(m[a_1|a_2|...|a_k]n) = (-1)^{|m|}m[a_1|a_2|...|a_k]n + \sum_{i=2}^{k}(-1)^{\epsilon_i}m[a_1|a_2|...|a_{i-1}a_i|...|a_k]n$$

$$- (-1)^{\epsilon_k}m[a_1|a_2|...|a_{k-1}]a_k n$$

Hereafter we will consider the normalized and the non-unital bar constructions on $A$:

$$B(A) = B(k; A; k) = \left(TC(sA), \overline{d}\right), \quad \overline{B}(A) = B(k \oplus A) = \left(TC(sA), \overline{d}\right).$$

(In the latter $A$ is considered as a non unital algebra ([18]-p. 142))). The differentials $\overline{d}$ and $\overline{d}$ are given by the same formula:

$$d([a_1|a_2|...|a_k]) = - \sum_{i=1}^{k}(-1)^{\epsilon_i}[a_1|a_2|...|d(a_i)|...|a_k] + \sum_{i=2}^{k}(-1)^{\epsilon_i}[a_1|a_2|...|a_{i-1}a_i|...|a_k].$$

We will also frequently use the twisting cochain of $\overline{B}(A)$

$$\tau_{\overline{B}_A} : \overline{B}_1(A) \rightarrow A, \quad [a_1|...|a_k] \mapsto \begin{cases} 0 & \text{if } k \neq 1 \\ a_1 & \text{if } k = 1 \end{cases}$$

and the following result:

**Proposition.** [1] Lemma 4.3 If $A$ is a differential graded algebra such that $A$ is a $k$-free module then the canonical map

$$B(A, A, A) \rightarrow A$$

is a semi-free resolution of $A^e$-modules (Here $A^e := A \otimes A^{op}$ denotes the enveloping algebra).

3.2. Let $A$ be a supplemented differential graded algebra and $M$ a differential graded $A$-bimodule. The canonical isomorphism of graded modules

$$\Phi_{A,M} : \text{Hom}_A(B(A; A; A), M) \rightarrow \text{Hom}(TC(sA), M), \quad f \mapsto (1_A[a_1|...|a_k]1_A \mapsto f([a_1|...|a_k]))$$

carries a differential $D_0 + D_1$ on $\text{Hom}(TC(sA), M)$. More explicitly, if $f \in \text{Hom}(TC(sA), M)$, we have:

$$D_0(f)([a_1|a_2|...|a_k]) = d_M(f([a_1|a_2|...|a_k])) - \sum_{i=1}^{k}(-1)^{\epsilon_i}f([a_1|...|d_Aa_i|...|a_k])$$
and

\[ D_1(f)([a_1|a_2|...|a_k]) = \sum_{i=2}^{k} (-1)^{|s_{a_i}|}f([a_1|a_{i-1}a_i|...|a_k]) + (-1)^{|s_{a_2}|}f([a_1|a_2|...|a_{k-1}])a_k, \]

where \( \tau_i = |f| + |s_{a_1}| + |s_{a_2}| + ... + |s_{a_{i-1}}| \).

The Hochschild cochain complex of \( A \) with coefficients in the \( A \)-bimodule \( M \) is the differential module

\[ \mathbb{C}^*(A; M) = (\text{Hom}(TC(sA), M), D_0 + D_1). \]

(It is important here to remark that the differential graded module \( \mathbb{C}^*(A; M) \) does not coincide with none of the differential graded modules \( \text{Hom}(\mathcal{B}(A), M) \) and \( \text{Hom}(\mathcal{B}(A), M) \).)

The Hochschild cohomology of \( A \) with coefficients in \( M \) is

\[ HH^*(A; M) = H(\mathbb{C}^*(A; M)) = H((\text{Hom}(TC(A), M), D_0 + D_1)). \]

Let \( \varphi : A \to A' \) be a homomorphism of differential graded algebras. Then \( A' \) is a differential graded bimodule via \( \varphi \) and \( \mathbb{C}^*(A; A') \) is a differential graded algebra.

**3.3** Let \( \psi : C \to C' \) be a homomorphism of differential graded coalgebras. We denote by \( \text{Coder}_\psi(C, C') \) the subcomplex of \( \text{Hom}(C, C') \) consisting of \( \psi \)-coderivations and by \( \beta_{C,V} : \text{Hom}(C, V) \to \text{Coder}_\varphi(C, TC(V)) \) the linear isomorphism extending each linear map to a coderivation.

**Proposition.** (11, 13, 13)

a) The degree 1 linear isomorphism \( \beta_A \):

\[ \mathbb{C}^*(A; A) = \text{Hom}(TC(sA), A) \xrightarrow{\text{Hom}(TC(sA), sA)} \text{Hom}(TC'(sA), sA) \xrightarrow{\beta_{TC(sA), sA}} \text{Coder}(\mathcal{B}A) \]

satisfies \( (D_0 + D_1)\beta_A = -D\beta_A \).

b) the structure of pre-Gerstenhaber algebra defined by \( \beta_A \) induces the usual structure of Gerstenhaber algebra on the Hochschild cohomology \( HH^*(A; A) \).

Observe that \( \beta_A^{-1} = \text{Hom}(TC(sA), \tau_{\mathcal{B}A}) \) and that for \( g \in \text{Hom}(TC(sA), A) \),

\[ \beta_A(g)[a_1|...|a_n] = \sum_{0 \leq i \leq j \leq n} (-1)^{|s_{a_i}|(|s_{a_1}|+...+|s_{a_i}|)}[a_1|...|a_i]g([a_{i+1}|...|a_j])[a_{j+1}|...|a_n]. \]

**3.4. Naturality.** Let \( \varphi : A \to A' \) and \( \psi' : A' \to B \) be homomorphisms of differential graded algebras and \( \psi := \psi' \circ \varphi \). Then the two natural maps

\[ \mathbb{C}^*(A; A') \xrightarrow{\Phi_{A,A'}} \mathbb{C}^*(A'; B) \xleftarrow{\Phi_{A,B}} \mathbb{C}^*(A; B) \]

are homomorphisms of differential graded algebras. Let \( \psi' \) and \( \varphi_1 \) be the obvious maps which make commutative the following diagram of homomorphisms of differential graded modules:

\[ \begin{array}{ccc} \mathbb{C}^*(A; A') & \xrightarrow{\Phi_{A,A'}} & \mathbb{C}^*(A; B) \\
\Phi_{A,A'} \uparrow \cong & & \Phi_{A,B} \downarrow \cong \\
\text{Hom}_A(\mathcal{B}; A) \xrightarrow{\psi'} & \xrightarrow{\Phi_{A,B}} & \text{Hom}_A(\mathcal{B}; A) \\
\Phi_{A,B} \uparrow \cong & & \Phi_{A,B} \downarrow \cong \\
\text{Hom}_A(\mathcal{B}; A; A') & \xrightarrow{\varphi_1} & \text{Hom}_A(\mathcal{B}; A; A') \\
\end{array} \]
It follows from (9)-Proposition 2.3 that if $A$, $A'$, $B$ are $k$-free modules and if $\varphi$ and $\psi$ are quasi-isomorphisms then $\psi_1 = \text{Hom}_{A'}(IB(A; A'), \psi')$ and $\varphi_1 = \text{Hom}_{A''}(\varphi, B)$ are quasi-isomorphisms. Thus, so are $C^*(A; \psi')$ and $C^*(\varphi; B)$.

**Proposition.** If $f : A \to B$ is a quasi-isomorphism of differential graded algebras and if $A$ and $B$ are $k$-free modules then the composite, denoted $\text{HH}^*(f)$

$$\text{HH}^*(A, A) \xrightarrow{\text{HH}^*(f; A)} \text{HH}^*(A; B) \xrightarrow{(\text{HH}^*(f; A))^{-1}} \text{HH}^*(B; B)$$

is an isomorphism of Gerstenhaber algebras.

**Proof.** As observed above the maps

$$C^*(A; A) \xrightarrow{C^*(f; A)} C^*(A; B) \xleftarrow{C^*(f; B)} C^*(B; B)$$

are quasi-isomorphisms of differential graded algebras. Let $f_1$ and $f_2$ be the natural maps which make commutative the following diagram of homomorphisms of differential graded modules:

\[
\begin{array}{ccc}
\beta_A \downarrow & \cong & \beta_{A,B} \downarrow \cong & \beta_B \downarrow \cong \\
\text{Coder}(\widetilde{IB}A) & \xrightarrow{f_1} & \text{Coder}(\widetilde{IB}(f)(\widetilde{IB}A, \widetilde{IB}B)) & \xleftarrow{f_2} & \text{Coder}(\widetilde{IB}B),
\end{array}
\]

We have to prove that the composite $H(f_2)^{-1} \circ H(f_1)$ is a homomorphism of graded Lie algebras. By (9), Proposition 3.1) the quasi-isomorphism $f$ factors as the composite of two quasi-isomorphisms of differential graded algebras

$$A \xrightarrow{\simeq} A \prod T(V) \xrightarrow{\simeq} pB$$

where $i : A \hookrightarrow A \prod T(V)$ admits a $k$-linear retraction $r$, and $p : A \prod T(V) \to B$ admits a $k$-linear section $s$. By functoriality, we have $\text{HH}^*(f) = \text{HH}^*(p) \circ \text{HH}^*(i)$. It suffices therefore to prove that $\text{HH}^*(p)$ and $\text{HH}^*(i)$ are homomorphisms of graded Lie algebras.

In the case $f = p$, $f_1$ admits the $k$-linear section $\text{Hom}(TC(sA), r)$ and so is surjective. Let $x_i$, $i = 1, 2$, be cycles in $\text{Coder}(\widetilde{IB}(B))$. Since $f_1$ is a surjective quasi-isomorphism of complexes, there exists cycles, $y_i$, in $\text{Coder}(\widetilde{IB}(A))$ such that

$$\widetilde{IB}(f) \circ y_i = f_1(y_i) = f_2(x_i) = x_i \circ \widetilde{IB}(f).$$

Thus

$$f_1([y_1, y_2]) = \widetilde{IB}(f) \circ y_1 \circ y_2 - (-1)^{|y_1||y_2|}\widetilde{IB}(f) \circ y_2 \circ y_1 = x_1 \circ x_2 \circ \widetilde{IB}(f) - (-1)^{|x_1||x_2|}x_2 \circ x_1 \circ \widetilde{IB}(f) = f_2([x_1, x_2]),$$

and if $a_i$ (resp. $b_i$) denotes the class of $x_i$ (resp. $y_i$), $i = 1, 2$ then $s\text{HH}^*(f)[b_1, b_2] = H(f_2)^{-1} \circ H(f_1)([b_1, b_2]) = [a_1, a_2]$.

In the case $f = i$, $f_2$ is surjective and the same argument works, mutatis mutandis. □
4 Proof of Theorem 2.

In this section $A$ will denote a differential graded algebra $(TV, d)$ (not necessarily a free model). The proof of Theorem 2 is a direct consequence of Lemmas 4.2, 4.3 and 4.7 below.

4.1. Let $\tilde{\text{Der}}A = \text{Der}A \oplus sA$ be as defined in the introduction. We define the injective degree $-1$ linear map

$$i_A : \tilde{\text{Der}}A \hookrightarrow \text{Hom}(TC(sA), A)$$

as follow. Consider the counity $\varepsilon_{TC(sA)} : TC(sA) \to \mathbb{k}$ and the twisting cochain $\tau_{BA} : TC(sA) \to A$. Given $a \in A$, we denote by $\bar{a} : \mathbb{k} \to A$ the $\mathbb{k}$-linear map such that $\bar{a}(1) = a$.

We put

$$\begin{align*}
i_A(sa) &= \bar{a} \circ \varepsilon_{TC(sA)} & \text{if } a \in A \\
i_A(x) &= (-1)^{|x|}x \circ \tau_{BA} & \text{if } x \in \text{Der}A.
\end{align*}$$

Let $\tilde{A} = (T(V \oplus \mathbb{k} \varepsilon), \tilde{d})$ be the differential graded algebra where the differential $\tilde{d}$ is related to $d$ by

$$\begin{align*}
\tilde{d} \varepsilon &= \varepsilon^2, \\
\tilde{d}v &= dv + [\varepsilon, v] & \text{if } v \in V.
\end{align*}$$

We define

$$j_A : \tilde{\text{Der}}A \to \text{Der}A, \quad \theta + sa \mapsto j_A(\theta + sa)$$

where $j_A(\theta + sa)$ is the unique derivation of $\tilde{A}$ defined by

$$\begin{align*}
j_A(\theta + sa)(v) &= \theta(v) & \text{for } v \in A \\
j_A(\theta + sa)(\varepsilon) &= (-1)^{|a|+1}a.
\end{align*}$$

Lemma 4.2. The linear maps

$$sC^*(A; A) \xleftarrow{s\circ i_A} \tilde{\text{Der}}A \xrightarrow{j_A} \text{Der}A$$

are injective homomorphisms of graded Lie algebras.

Proof. Precise that the isomorphism $\beta_A \circ s^{-1} : sC^*(A; A) \to \text{Coder}(\tilde{B}A)$ defines on $sC^*(A; A)$ the Gerstenhaber bracket $[-, -]$ explicitly defined by:

$$\begin{align*}
(sf)\overline{\sigma}(sg) := s(f \circ [\beta_A(g)]), \\
[sf, sg] := (sf)\overline{\sigma}(sg) - (-1)^{|sf||sg|}(sg)\overline{\sigma}(sf) & \quad f, g \in \text{Hom}(TC(sA), A).
\end{align*}$$

From this formula, we deduce now that the inclusion $s \circ i_A : \tilde{\text{Der}}A \to sC^*(A; A)$ is a morphism of graded Lie algebras. One check directly that $j_A$ is a homomorphism of graded Lie algebras.

Lemma 4.3. If $A$ is a free model then

$$s \circ i_A : \tilde{\text{Der}}(A) \xrightarrow{s\circ i_A} sC^*(A; A)$$

is a quasi-isomorphism of differential graded modules.

Proof. A straightforward computation shows that the inclusion $s \circ i_A : \tilde{\text{Der}}A \to sC(A; A)$ is a morphism of differential graded modules. In order to prove that $s \circ i_A$ is a quasi-isomorphism, we introduce a quasi-isomorphism between semifree $A^e$-modules

$$\Pi : \tilde{B}(A; A) \xrightarrow{s\circ i_A} \tilde{K}_A$$

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and a commutative diagram of differential graded modules

\[
\begin{array}{ccc}
\text{Hom}_{A^e} \left( \tilde{K}_A, A \right) & \overset{\overline{\alpha}_A}{\cong} & \overline{\text{Der}}(A) \\
\text{Hom}_{A^e} \left( \Pi, A \right) \downarrow & \cong & \downarrow i_A \\
\text{Hom}_{A^e} \left( IB(A; A; A), A \right) & \overset{\cong}{\sim} & C^*(A; A)
\end{array}
\] (4.4)

where \( \overline{\alpha}_A \) is a degree 1 isomorphism and \( \Phi_A \) is the isomorphism defined in 3.2. Then, by [8, Proposition 2.3], \( \text{Hom}(\Pi, A) \) is a quasi-isomorphism and thus so is \( i_A \). In the remaining of the proof, we precise the definitions of \( \tilde{K}_A, \Pi \) and \( \overline{\alpha}_A \).

Let \( S_V \) be the universal derivation of \( A \)-bimodules:

\[
S_V : A \to A \otimes V \otimes A, \quad v_1 \ldots v_n \mapsto \sum_{i=1}^{n} v_1 \ldots v_{i-1} \otimes v_i \otimes v_{i+1} \ldots v_n.
\]

Denote by \( \overline{S}_V := (A \otimes s \otimes A) \circ S_V : A \to A \otimes sV \otimes A \). (4.5)

We put \( \tilde{K}_A := (A \otimes (lk \otimes sV) \otimes A, \tilde{\delta}) \) with \( \tilde{\delta} \) defined by

\[
\begin{cases}
\tilde{\delta}(1 \otimes \lambda \otimes 1) = 0 & \text{if } \lambda \in lk \\
\tilde{\delta}(1 \otimes sv \otimes 1) = v \otimes 1 \otimes 1 - \overline{S}_V(dv) - 1 \otimes 1 \otimes v & \text{if } v \in V.
\end{cases}
\]

The map \( \Pi : IB(A; A; A) \to \tilde{K}_A \) is defined by \( \Pi([sa_1] \cdots [sa_n]) = \begin{cases} \overline{S}_V(a_1) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases} \)

By [21, Theorem 1.4] and by [17, p. 205 and p. 207], \( \Pi \) is a quasi-isomorphism between semifree \( A \)-bimodules. We define the degree 1 linear isomorphism

\[
\overline{\alpha}_A : \text{Hom}_{A^e}(\tilde{K}_A, A) \to \overline{\text{Der}}(A) = \text{Der}(A) \oplus s(A)
\] (4.6)

by

\[
\begin{cases}
\overline{\alpha}_A(x) = (-1)^{|x|} x \circ \overline{S}_V & \text{if } x \in \text{Hom}_{A^e}(A \otimes sV \otimes A, A), \\
\overline{\alpha}_A(y) = -s(y(1 \otimes 1 \otimes 1)) & \text{if } y \in \text{Hom}_{A^e}(A \otimes lk \otimes A, A).
\end{cases}
\]

A straightforward computation shows that the diagram (4.4) commutes.

\[\square\]

**Lemma 4.7.** If \( A \) is a free model then the injective linear map

\[
j_A : \overline{\text{Der}}(A) \to \text{Der}(A)
\]

is a quasi-isomorphism of differential graded modules.

We begin the proof of Lemma 4.7 with a general construction. Consider the degree 1 ”universal derivation” \( \overline{S}_V : A = TV \to A \otimes sV \otimes A \) defined in (4.5).

**Sub-lemma 4.8.**

i) \( K_A := (A \otimes sV \otimes A, \delta) \) is a differential \( A^e \)-module with \( \delta : A \otimes sV \otimes A \to A \otimes sV \otimes A, \quad 1 \otimes sv \otimes 1 \mapsto -\overline{S}_V(dv), \quad v \in V. \)

ii) The map

\[
\alpha_A : \text{Hom}_{A^e}(K_A, A) \to \text{Der}(A), \quad f \mapsto (-1)^{|f|} f \circ \overline{S}_V
\]

is a degree 1 isomorphism of differential graded modules.

iii) Denote by \( \gamma_{sV} \) the composite

\[
\text{Hom}(sV, A) \overset{\text{Hom}(s^{-1}, A)}{\to} \text{Hom}(V, A) \to \text{Der}(A)
\]
where the map \( \text{Hom}(V, A) \to \text{Der}(A) \) is the canonical isomorphism which extends a linear map into a derivation of \( A \). The following diagram of graded modules commutes

\[
\begin{array}{ccc}
\text{Hom}(sV, A) & \xrightarrow{\gamma \cdot V} & \text{Der}(A) \\
\Phi_A & \downarrow & \\
\text{Hom}_{A^e}(K_A, A) & \xrightarrow{\alpha_A} & 
\end{array}
\]

where \( \Phi_A \) is the canonical isomorphism which restricts an \( A^e \)-linear map into a \( \mathbb{k} \)-linear map (cf 3.2).

**Proof.** A straightforward computation proves i) and ii). In order to prove iii), we use the fact that \((\gamma \cdot V)^{-1} = -\text{Hom}(\tau_A, A)\) where \( \tau_A : sV \to A, sv \mapsto v, v \in V \).

\[\square\]

4.9 Remark. An explicit formula for \( \gamma \cdot V \) is:

\[\gamma \cdot V(\varphi) = (-1)^{|\varphi|} \mu_A^2 \circ (A \otimes (\varphi \circ s) \otimes A) \circ S \]

for \( \varphi \in \text{Hom}(sV, A) \). Here \( \mu_A^2 : A \otimes A \otimes A \to A \) denotes the iterated multiplication of \( A \).

The differential \( D^A \) defined on \( \text{Hom}(sV, A) \) via the degree 1 isomorphism \( \gamma \cdot V \) is explicitly given by

\[D^A(\varphi) = d \circ \varphi + \gamma \cdot V(\varphi) \circ d \mid_{sV} \circ s^{-1} = d \circ \varphi + (-1)^{|\varphi|} \mu_A^2 \circ (A \otimes (\varphi \circ s) \otimes A) \circ S \circ d \mid_{sV} \circ s^{-1}\]

for \( \varphi \in \text{Hom}(sV, A) \). In particular, the diagram above is a commutative diagram of differential graded modules.

4.10. End of proof of Lemma 4.7. Let \( \tilde{A} = (T(V \oplus \mathbb{k} \varepsilon), \tilde{d}) \) defined in 4.1 and consider the following diagram of graded modules

\[
\begin{array}{ccc}
\text{Hom}_{A^e}(K_{\tilde{A}}, \tilde{A}) & \xrightarrow{\tilde{\alpha}_A} & \text{Der}(\tilde{A}) \\
\kappa_A & \downarrow & \\
\text{Hom}_{A^e}(\tilde{K}_{\tilde{A}}, \tilde{A}) & \xrightarrow{\tilde{\alpha}_A} & \text{Der}(\tilde{A}) 
\end{array}
\]

where \( \tilde{\alpha}_A \) is the isomorphism of differential graded modules defined in (4.6) and \( \alpha_{\tilde{A}} \) is the isomorphism of graded modules defined in the sublemma 4.8 with \( \tilde{A} = (T(V \oplus \mathbb{k} \varepsilon), \tilde{d}) \) in place of \( A \). Our strategy to prove Lemma 4.7 is to complete diagram (4.11) by a quasi-isomorphism of differential graded modules \( \kappa_A : \text{Hom}_{A^e}(\tilde{K}_{\tilde{A}}, \tilde{A}) \to \text{Hom}_{A^e}(K_{\tilde{A}}, \tilde{A}) \) so that it commutes.

First observe that

\[K_{\tilde{A}} = (\tilde{A} \otimes s(V \oplus \mathbb{k} \varepsilon) \otimes \tilde{A}, \delta) = (\tilde{A} \otimes (\mathbb{k} \oplus sV) \otimes \tilde{A}, \delta) .\]

We introduce the differential graded algebra

\( \hat{\tilde{A}} = (T(V \oplus \mathbb{k} \varepsilon), \hat{d}) \) with

\[
\begin{aligned}
\hat{d} \varepsilon & = -\varepsilon^2 , \\
\hat{d} v & = dv .
\end{aligned}
\]

Then the natural inclusion of differential graded algebras

\[i : A \xrightarrow{\sim} \hat{\tilde{A}} = A \coprod (T(\varepsilon), \hat{d})\]
is a quasi-isomorphism since \( T(\varepsilon, \hat{d}) \) is acyclic. Therefore by \([1, \text{ Proposition 2.3}], \) \( \text{Hom}_{A^e}(\bar{K}_A, i) \) a quasi-isomorphism of differential graded modules, since \( \bar{K}_A \) is a semi-free \( A^e \)-module. Now we have the following diagram of graded modules

\[
\begin{array}{ccc}
\text{Hom}_{A^e}(\bar{K}_A, \hat{A}) & \xrightarrow{\alpha_{\hat{A}}} & \text{Der}(\hat{A}) \\
\downarrow \quad \text{id} \quad & & \downarrow \quad j_A \\
\text{Hom}_{\hat{A}^e}(\hat{A}^e \otimes_{A^e} \bar{K}_A, \hat{A}) & \cong & \text{Hom}_{A^e}((\hat{A}^e \otimes_{A^e} \bar{K}_A), \hat{A}) \\
\downarrow \quad k_A & & \downarrow \quad j_A \\
\text{Hom}_{A^e}(\bar{K}_A, \hat{A}) & \cong & \text{Hom}_{A^e}(\bar{K}_A, \hat{A}) \\
\end{array}
\]

where the isomorphism of differential graded modules

\[
\text{Hom}_{A^e}(\bar{K}_A, \hat{A}) \cong \text{Hom}_{A^e}(\hat{A}^e \otimes_{A^e} \bar{K}_A, \hat{A})
\]

is given by “extension of scalars”. The identity map \( \text{id} \) commutes with the differentials. Indeed, the differential \( \delta \) on \( \hat{A}^e \otimes_{A^e} \bar{K}_A \) is given by \( \delta(a) = \delta(a) - [\varepsilon, a] \) for \( a \in \hat{A} \otimes (k \oplus sV) \otimes \hat{A} \). The proof ends by checking that the above diagram commutes.

\[
\square
\]

5 The Hochschild cochain complex of a differential graded coalgebra

5.1 Let \( (C, d) \) be a supplemented differential graded coalgebra, \( C = k \oplus \overline{C} \), and \( (R, d) \) (resp. \( (L, d) \)) be a right (resp. left) \( C \)-comodule. The two-sided cobar constructions, \( \Omega(R; C; L) \) and \( \overline{\Omega}(R; C; L) \) are defined as follows:

\[
\Omega(R; C; L) = (R \otimes T(s^{-1}C) \otimes L, d_0 + d_1), \quad \overline{\Omega}(R; C; L) = (R \otimes T(s^{-1}\overline{C}) \otimes L, d_0 + d_1).
\]

A generic element is denoted \( r \langle c_1 | c_2 | \cdots | c_k \rangle \) with degree \(|r| + |l| + \sum_{i=1}^{k} |s^{-1}c_k| \). The differential \( d_0 \) is the unique derivation extending \(-sd\) and \( d_1 \) is given by the formula:

\[
d_1(r \langle c_1 | \cdots | c_{p-1} \rangle l) = -\sum_{k=1}^{p}(1)^{|c|}|c|^l k \langle r \langle c_1 | \cdots | c_{p-1} \rangle \rangle l
\]

\[
+ \sum_{j=1}^{p-1} \sum_{i} (-1)^{1+|c_j|/l} r \langle c_1 | \cdots | c_j | \cdots | c_{p-1} \rangle l + \sum_{j} (-1)^{e_j} r \langle c_1 | \cdots | c_{p-1} \rangle y_j \langle y_j \rangle ^{l_j}_{y_j}
\]

with \( \varepsilon_j = |r| + |s^{-1}c_1| + \cdots + |s^{-1}c_{j-1}| \), and \( \Delta c_j = \sum_i c_ji \otimes \overline{c}_j, \Delta r = \sum_i r_k j \otimes x_k \) and \( \Delta l = \sum_j y_j \otimes l_{y_j}^{l_j} \) denote the non reduced diagonals (resp. reduced diagonals), \( c_j, c_ji, y_j, x_j, y_i \in C \) (resp. \( \overline{C} \)), \( l, l_{y_j}^{l_j} \in L \) and \( r, r_l \in R \).

Hereafter we will use the normalized and the non-counital cobar constructions

\[
\overline{\Omega}C = \overline{\Omega}(k; C; k) = (TC(s^{-1}\overline{C}), \alpha), \quad \overline{\Omega}C = \overline{\Omega}(C \otimes k) = (TC(s^{-1}C), \alpha).
\]
and the twisting cochain \( \tau_{\tilde{\Omega}C} : C \to \tilde{\Omega}C \), \( c \mapsto \langle c \rangle \) of \( \tilde{\Omega}C \).

**Lemma 5.2.** Let \( C \) be a locally conilpotent differential graded coalgebra. If \( C \) is a free \( k \)-module then

a) \( \tilde{\Omega}C = (T(V, d)) \) is a free model

b) \( \tilde{\Omega}C = (T(V \oplus k\epsilon), d) \) with \( d\epsilon = \epsilon^2 \) and \( d\epsilon = d\epsilon + \epsilon v - (-1)^{|v|}v\epsilon, v \in V \).

**Proof**

a) Write \( C = k1_C \oplus \mathcal{C}, d = \tilde{d}_0 + \tilde{d}_1, V = s^{-1} \mathcal{C} \) and

\[
V(k) = \begin{cases} 
    s^{-1} \text{Ker} \Delta^{(n)} & \text{if } k = 2n \\
    V(2n) + s^{-1} (\text{Ker} \Delta^{(n+1)} \cap \text{Ker} \tilde{d}_0) & \text{if } k = 2n + 1
\end{cases}
\]

(Here \( \Delta^{(0)} = id_C \) and \( V_0 = 0 \). Thus \( V \) is the union of the increasing sequence \( (V(k))_{k \geq 0} \).)

Now observe that if \( c \in \text{Ker} \Delta^{(n)} \) then \( \Delta(c) \in \text{Ker} \Delta^{(n-1)} \otimes \text{Ker} \Delta^{(n-1)} \). Thus \( d(V(k)) \subset T(V(k - 1)) \).

b) Write \( \epsilon = \langle 1_c \rangle \) and \( \tilde{\Omega}C = \tilde{\Omega}(C \oplus k1_k) = (T(V \oplus k\epsilon), d) \) with \( d = \tilde{d}_0 + \tilde{d}_1 \) satisfies \( d\epsilon = d\epsilon, d\epsilon = \tilde{d}_1(1_c) = \langle 1_c | 1_c \rangle = \epsilon^2 \) and if \( v = \langle c \rangle \) and \( \Delta c = \sum c_i \otimes c'_i \) then \( \tilde{d}_1 v = (-1)^{|c|} \langle c | 1_c \rangle + \sum_i (-1)^{|c_i|} \langle c_i | c'_i \rangle + \langle 1_c | c \rangle = (-1)^{|c|} v \epsilon + \tilde{d}_1 v + v \epsilon \).

\( \square \)

5.3 Let \( C \) be a differential graded coalgebra and \( N \) a differential graded \( C \)-bicomodule. For any graded \( k \)-module \( V \), we consider the natural isomorphism of graded \( k \)-modules

\[
\Psi_{N,C} : \text{Hom}(N, V) \to \text{Hom}_{C^e}(N, C \otimes V \otimes C), \quad \Psi_{N,C}(f) = (C \otimes f \otimes C) \circ \Delta_N,
\]

where \( \Delta_N = (C \otimes \Delta_N^e) \circ \Delta_N^l, \Delta_N : N \to N \otimes C \) and \( \Delta_N^l : N \to C \otimes N \) denoting the left and right comodule maps of \( N \). Here \( \text{Hom}_C(L, R) \) denotes the graded \( k \)-module of morphisms of \( C \)-comodules from \( L \) into \( R \) and \( C^e = C \otimes C^{op} \) is the enveloping coalgebra of \( C \). The inverse of \( \Psi_{N,C} \) is simply \( \text{Hom}(N, \varepsilon_C \otimes V \otimes \varepsilon_C) \).

The isomorphism \( \Psi_{N,C} : \text{Hom}(N, T(s^{-1}C)) \to \text{Hom}_{C^e}(N, \Omega(C; C; C)) \) carries on the graded module \( \text{Hom}(N, T(s^{-1}C)) \) a differential \( D_0 + D_1 \) explicitly defined by:

\[
(D_0 + D_1)(\varphi) = d_{\tilde{\Omega}C} \circ \varphi - (-1)^{|\varphi|} \varepsilon_N \circ D_N - \left( \mu_{\tilde{\Omega}C} \circ (\tau_{\tilde{\Omega}C} \otimes \varphi) \circ \Delta_N^l - (-1)^{|\varphi|} \tau_{\tilde{\Omega}C} \circ (\varphi \otimes \tau_{\tilde{\Omega}C}) \circ \Delta_N^l \right)
\]

for \( \varphi \in \text{Hom}(N, \tilde{\Omega}C) \).

The Hochschild cochain complex of a differential graded coalgebra \( C \) with coefficients in the \( C \)-bicomodule \( N \) is the differential graded module \([12, \text{p. 57}]:

\[
C^*(N; C) = \left( \text{Hom}(N, T(s^{-1}C)), D_0 + D_1 \right).
\]

The Hochschild cohomology of the coalgebra \( C \) with coefficients in the bicomodule \( N \) is

\[
HH^*(N; C) = H(C^*(N; C)) = H \left( \text{Hom}(N, T(s^{-1}C)), D_0 + D_1 \right).
\]

If \( \psi : C^e \to C \) is a homomorphism of differential graded coalgebras, then \( C^*(C^e; C) \) is a differential graded algebra.

5.4 **Proposition 5.4.** The degree 1 linear isomorphism \( \gamma_C : \)

\[
C^*(C; C) = \text{Hom}(C, T(s^{-1}C)) \xrightarrow{\text{Hom}(\text{Id}, T(s^{-1}C))} \text{Hom}(\text{Id}, T(s^{-1}C) = \text{Der}(\tilde{\Omega}C)
\]

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satisfies $\gamma_C \circ (D_0 + D_1) = -D \circ \gamma_C$.

**Proof.** By sub-lemma 4.8 and Remark 4.9 applied to $\tilde{\Omega}C$, the degree 1 linear map $\gamma_C$:

$$\left( \text{Hom}(C, Ts^{-1}C), D\tilde{\Omega}C \right) \to \text{Der}(\tilde{\Omega}C)$$

anticommutes with the differentials. A straightforward computation shows that the differential $D\tilde{\Omega}C$ coincides with differential $D_0 + D_1$ of $C^*(C; C)$.

5.5 We now suppose that $C$ is a locally conilpotent. The normalized Hochschild cochain complex of $C$ is the differential graded module:

$$C^*(C; C) = \left( \text{Hom}(C, Ts^{-1}C), D_0 + D_1 \right)$$

(The isomorphism $\Psi_C : C^*(C; C) \cong \text{Hom}_C(C, \Omega(C; C))$ defined above being an homomorphism of differential graded modules).

**Proposition 5.6.** The inclusion $i : \Omega C \hookrightarrow \tilde{\Omega}C$ induces a quasi-iso-morphism of differential graded algebras

$$\text{Hom}(C, i) : C^*(C; C) \cong \tilde{\Omega}C^*(C; C)$$

Moreover, there exists a structure of differential graded Lie algebra on $s\tilde{\Omega}C^*(C; C)$ such that

$$s\text{Hom}(C, i) : s\tilde{\Omega}C^*(C; C) \cong sC^*(C; C)$$

is a homomorphism of differential graded Lie algebras.

**Proof.** By diagram (4.11) applied when $TV = \tilde{\Omega}C$, we obtain the following diagram of differential graded modules

$$\begin{array}{cccc}
\text{Hom}_{\tilde{\Omega}C^e}(K_{\tilde{\Omega}C}, \tilde{\Omega}C) & \text{Hom}(\tilde{\Omega}C, \tilde{\Omega}C) & \text{Der}(\tilde{\Omega}C) \\
\uparrow s\text{Hom}_{\tilde{\Omega}C^e}(K_{\tilde{\Omega}C}, \tilde{\Omega}C) & \uparrow \alpha_{\tilde{\Omega}C} \circ s^{-1} & \uparrow \tilde{\Omega}C \\
\text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(i) & \text{Hom}_C(C, \Omega(C; C)) \\
\uparrow \Psi_C & \uparrow \Phi_{\tilde{\Omega}C} & \uparrow \Phi_{\tilde{\Omega}C} \\
\text{Hom}_C(C, C \otimes i \otimes C) & \text{Hom}_C(i) & \text{Hom}_C(C, C \otimes i \otimes C) \\
\uparrow & \uparrow & \uparrow \\
\text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(C, \Omega(C; C))
\end{array}$$

On the other hand, the following diagram of graded modules commutes obviously

$$\begin{array}{cccc}
\text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(C, \Omega(C; C)) \\
\uparrow \Psi_C & \uparrow \Phi_{\tilde{\Omega}C} & \uparrow \Phi_{\tilde{\Omega}C} \\
\text{Hom}_C(C, C \otimes i \otimes C) & \text{Hom}_C(i) & \text{Hom}_C(C, C \otimes i \otimes C) \\
\uparrow & \uparrow & \uparrow \\
\text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(C, \Omega(C; C)) & \text{Hom}_C(C, \Omega(C; C))
\end{array}$$

Since the maps $\Psi_C$’s, $\Phi_{\tilde{\Omega}C}$, $\text{Hom}_C(C, C \otimes i \otimes C)$ and $\text{Hom}_C(C, \Omega(C; C))$ are homomorphisms of differential graded modules, then $\Phi_{\tilde{\Omega}C}$ and $\text{Hom}(C, i)$ are also homomorphisms of differential graded modules. Define $\gamma_C := \alpha_{\tilde{\Omega}C} \circ \Phi^{-1}_{\tilde{\Omega}C}$. The remaining of the statement follows also from the commutativity of above diagrams.

$\square$
6 Proof of Propositions B and C

6.1 Proof of Proposition C. Let $C$ be a supplemented conilpotent differential graded coalgebra. We denote by $\sigma_C : C \xrightarrow{\Phi} \Omega \Omega C \xrightarrow{\Phi} \mathcal{B}(k; \Omega C; k)$ the counity of the bar-cobar adjunction,

$$\sigma_C(c) = [c >] + \sum_{i \geq 1} \sum_{j} [c_{i,j} > | \cdots | < c_{i+1,j}],$$

where $\sum c = \sum_{j} c_{i,j} \otimes \cdots \otimes c_{i+1,j}, c \in C$. Define $D_2 = \text{Hom}(\sigma_C; \Omega \Omega C) : C^\ast(\Omega \Omega C; \Omega C) \rightarrow \mathcal{C}^\ast(C; C)$. We will prove that

a) $D_2$ is a quasi-isomorphism of differential graded algebras which admits a section $\Gamma$.

b) The map $s\Gamma : sC^\ast(C; C) \rightarrow sC^\ast(\Omega \Omega C; \Omega C)$ is a homomorphism of differential graded Lie algebras.

a) The quasi-isomorphism $\Pi : \mathcal{B}(\Omega \Omega C; \Omega C) \rightarrow \tilde{\Lambda}_{\Omega C}$ of semifree $\Omega C^\ast$-modules described in the proof of Lemma 4.3 with $A = \Omega C$, admits the section $\nabla_\infty := \Omega C \otimes \sigma_C \otimes \Omega C$ which is a homomorphism of differential graded modules [17, p. 209] and we have the following commutative diagram of graded modules

$$\begin{array}{ccc}
\text{Hom}(C, \Omega \Omega C), D_0 + D_1 & \xrightarrow{\Phi_{\Omega \Omega C}} & \text{Hom}_{\Omega \Omega C^\ast}(\tilde{\Lambda}_{\Omega C}, \Omega C) \\
\text{Hom}(\sigma_C, \Omega \Omega C) \uparrow & & \uparrow \text{Hom}_{\Omega \Omega C^\ast}(\nabla_\infty, \Omega C) \\
(\text{Hom}(\mathcal{B} \Omega \Omega C, \Omega \Omega C), D_0 + D_1) & \xrightarrow{\Phi_{\Omega \Omega C}} & \text{Hom}_{\Omega \Omega C^\ast}(B(\Omega \Omega C; \Omega \Omega C), \Omega \Omega C)
\end{array}$$

where the upper homomorphism is defined as the lower homomorphism has been defined in 3.2. Since the right vertical map is a quasi-isomorphism of differential graded modules and since the horizontal maps are isomorphisms of differential graded modules, the homomorphism $D_2 := \text{Hom}(\sigma_C, \Omega \Omega C)$ is also a quasi-isomorphism of differential graded modules. The section $\Gamma$ is then defined by

$$\Gamma := \Phi_{\Omega \Omega C} \circ \text{Hom}_{\Omega \Omega C^\ast}(\Pi, \Omega \Omega C) \circ \Phi_{\Omega \Omega C}^{-1}.$$

b) By Lemma 4.2,

$$s \circ i_{\Omega \Omega C} : \hat{\text{Der}}(\Omega \Omega C) \hookrightarrow sC^\ast(\Omega \Omega C; \Omega \Omega C)$$

is a homomorphism of differential graded Lie algebras and diagram (4.4) becomes

$$\begin{array}{ccc}
\text{sHom}_{\Omega \Omega C^\ast}(\tilde{\Lambda}_{\Omega C}, \Omega \Omega C) & \xrightarrow{\Phi_{\Omega \Omega C} \circ s^{-1}} & \text{Der}(\Omega \Omega C) \\
\text{sHom}_{\Omega \Omega C^\ast}(\Pi, \Omega \Omega C) \downarrow & & \downarrow s \circ i_{\Omega \Omega C} \\
\text{sHom}_{\Omega \Omega C^\ast}(B(\Omega \Omega C; \Omega \Omega C), \Omega \Omega C) & \xrightarrow{\Phi_{\Omega \Omega C}} & sC^\ast(\Omega \Omega C; \Omega \Omega C)
\end{array}$$

Therefore $s\Gamma$ is a homomorphism of differential graded Lie algebras.

\[ \square \]

6.2 Proof of Proposition B.

Let $C$ be a differential graded coalgebra and consider the linear map

$$D_1 : C^\ast(C; C) \rightarrow C^\ast(C^\vee; C^\vee), \quad f \mapsto f^\vee \circ \Theta$$

where $\Theta : TC(sA) \rightarrow (T(s^{-1}C))^\vee$ is defined by:

$$\Theta([f_1] \cdots [f_n])([c_1] \cdots [c_k]) = \begin{cases} 0 & \text{if } k \neq n, \\ (-1)^n \varepsilon c_1 f_1(c_1) \cdots f_n(c_n) & \text{if } k = n. \end{cases}$$
Here $\varepsilon_\phi$ is the graded signature obtained by the strict application of the Koszul rule to the graded permutation $s, f_1, s, f_2, \ldots, s, f_n, s^{-1}, c_1, s^{-1}, c_2, \ldots, s^{-1}, c_n \mapsto f_1, s, s^{-1}, c_1, f_2, s, s^{-1}, c_2, \ldots, f_n, s, s^{-1}, c_n$.

The proof decomposes in three steps.

6.3 Step 1. The map $\mathcal{D}_1$ is a homomorphism of differential graded modules. Moreover, i) $\mathcal{D}_1$ is an isomorphism whenever $C = C^{\geq 0}$ is a free graded $\mathfrak{k}$-module of finite type.

Let $A = C^\vee$ and consider the following commutative diagram of graded $\mathfrak{k}$-modules.

\[
\begin{array}{cccc}
\text{Hom}_{C^\vee}(N, \Omega(C; C; C)) & \xrightarrow{t} & \text{Hom}_{A^e}(\Omega(C; C; C)^\vee, N^\vee) & \xrightarrow{\text{Hom}(\varnothing_{C,N^\vee})} \text{Hom}_{A^e}(\mathbb{B}(A; A; A), N^\vee) \\
\downarrow_{\Psi_{N,C}^{-1}} & & \downarrow & \downarrow_{\Phi_{A,N^\vee}} \\
(\text{Hom}(N, T(s^{-1}C)), D_0 + D_1) & \xrightarrow{\mathfrak{t}} & \text{Hom}((T(s^{-1}C))^\vee, N^\vee) & \xrightarrow{\text{Hom}(\varnothing_{T^e,N^\vee})} (\text{Hom}(TC(sA), N^\vee), D_0 + D_1) \\
\downarrow & & \downarrow & \downarrow \\
C^*(N; C) & & C^*(A; N^\vee)
\end{array}
\]

where $t$ and $\mathfrak{t}$ are the transposition maps $f \mapsto f^\vee$ and $\Theta_C : A \otimes T(sA) \otimes A \rightarrow (C \otimes T(s^{-1}C) \otimes C)^\vee$ is the unique homomorphism of $A$-bimodules extending $\Theta$. A tedious computation proves that the maps $\Theta : \mathbb{B}A \rightarrow (\Omega C)^\vee$ and $\Theta_C : \mathbb{B}(A; A; A) \rightarrow (\Omega(C; C; C))^\vee$ commute with the differentials. Since the maps on the upper line of the diagram are homomorphism of differential graded modules so is the composite $\mathcal{D}_1$ of the maps appearing in the lower line.

i) Suppose now that $C$ is free of finite type. Since $C = C^{\geq 0}$, $s^{-1}C = (s^{-1}C)^{\geq 1}$, $T(s^{-1}C)$ is also free of finite type. Therefore $t$ is an isomorphism. Since $sA \cong (s^{-1}C)^\vee$, $\Theta$ is also an isomorphism.

ii) Let $A$ be an augmented differential graded algebra such that $\overline{A}$ is $\mathfrak{k}$-free. Define the complex

\[
\overline{C}^*(A; A) := (\text{Hom}(TC(s\overline{A}, A)), D_0 + D_1)
\]

such that the canonical isomorphism $\overline{\Phi}_A : \text{Hom}_{A^e}(\overline{\mathbb{B}}(A; A; A), A) \rightarrow \overline{C}^*(A; A)$ is a morphism of differential graded modules. Let $p_A : \mathbb{B}(A; A; A) \rightarrow \overline{\mathbb{B}}(A; A; A)$ and $\overline{p} : \overline{\mathbb{B}}A \rightarrow \overline{\mathbb{B}}A$ be the canonical projections. Since $p_A$ is a quasi-isomorphism of $A^e$-semifree modules and since the following diagram of differential graded modules commutes,

\[
\begin{array}{cccc}
\text{Hom}_{A^e}(\overline{\mathbb{B}}(A; A; A), A) & \xrightarrow{\text{Hom}_{A^e}(p_A,A)} & \text{Hom}_{A^e}(\mathbb{B}(A; A; A), A) \\
\downarrow \overline{\Phi}_A & & \downarrow \Phi_A \\
\overline{C}^*(A; A) & \xrightarrow{\text{Hom}(p_A)} & C^*(A; A)
\end{array}
\]

then $\text{Hom}(p, A)$ is a quasi-isomorphism.

Suppose now that $C$ is supplemented and $\overline{C} = \overline{C}^{\geq 2}$ is $\mathfrak{k}$-free of finite type. Since $A := C^\vee$ is $\mathfrak{k}$-free, then $\text{Hom}(p, C^\vee)$ is a quasi-isomorphism. Let $i : \overline{\Omega}C \rightarrow \Omega C$ be the canonical inclusion. Consider the unique morphism of graded modules $\overline{\varnothing} : \overline{\mathbb{B}}C^\vee \rightarrow (\Omega C)^\vee$ such that $\overline{\varnothing} \circ p = i^\vee \circ \varnothing$ and the linear map

\[
\mathcal{D}_1 : \overline{C}^*(C; C) \rightarrow \overline{C}^*(C^\vee; C^\vee), \quad f \mapsto f^\vee \circ \overline{\varnothing}.
\]
We have the commutative diagram of graded modules

\[
\begin{array}{ccc}
\mathcal{C}^*(C;C) & \xrightarrow{\mathcal{D}_1} & \mathcal{C}^*(C';C') \\
\mathcal{D}_1 & \cong & \mathcal{D}_1 \\
\text{Hom}(C,i) & \cong & \text{Hom}(p,C') \\
\mathcal{C}^*(C;C) & \xrightarrow{\mathcal{D}_1} & \mathcal{C}^*(C';C')
\end{array}
\]

Since the vertical maps are injective, \(\mathcal{D}_1\) is a morphism of differential graded modules. As in case i), \(\mathcal{D}_1\) is an isomorphism and so using Proposition A, \(\mathcal{D}_1\) is a quasi-isomorphism. 

6.4 Step 2. The linear map \(\mathcal{D}_1 : \mathcal{C}^*(C;C) \to \mathcal{C}^*(C';C')\) is a homomorphism of graded algebras.

Observe that, without finite type hypothesis, \(\Theta : TC(sA) \to (T(s^{-1}C))^\vee\) is not a homomorphism of graded coalgebras. Nevertheless, the composite,

\[\tilde{\Theta} : T(s^{-1}C) \hookrightarrow (T(s^{-1}C))^\vee \xrightarrow{(\Theta)^\vee} (TC(sA))^\vee\]

is a homomorphism of graded algebras. From the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(C,T(s^{-1}C)) & \xrightarrow{\tau} & \text{Hom}((T(s^{-1}C))^\vee,C') \\
\text{Hom}(C',\tilde{\Theta}) & \downarrow & \downarrow \text{Hom}(\tilde{\Theta},C') \\
\text{Hom}(C,(TC(sA))^\vee) & \xrightarrow{\tau} & \text{Hom}(TC(sA),C'),
\end{array}
\]

where \(\tau(f)\) is the composite \(TC(sA) \hookrightarrow (TC(sA))^\vee \xrightarrow{f^\vee} C'\), we deduce that \(\mathcal{D}_1 := \text{Hom}(\tilde{\Theta},C') \circ \tau = \tau \circ \text{Hom}(C,\tilde{\Theta})\) is a morphism of graded algebras.

6.5 Step 3. The homomorphism \(s\mathcal{D}_1 : s\mathcal{C}^*(C;C) \to s\mathcal{C}^*(C';C')\) is an homomorphism of graded Lie algebras.

We want to show that

\[
\beta_A \circ \mathcal{D}_1 \circ \gamma_C^{-1} : \text{Der}(\tilde{\Omega}C) \xrightarrow{\gamma_C^{-1}} \mathcal{C}^*(C;C) \xrightarrow{\mathcal{D}_1} \mathcal{C}^*(A;A) \xrightarrow{\beta_A} \text{Coder}(\tilde{\Omega}A)
\]

satisfies

\[
(\beta_A \circ \mathcal{D}_1 \circ \gamma_C^{-1})([\gamma_C(f),\gamma_C(g)]) = [(\beta_A \circ \mathcal{D}_1)(f), (\beta_A \circ \mathcal{D}_1)(g)], \quad f, g \in \text{Hom}(C,\tilde{\Omega}C). \quad (6.6)
\]

We consider the twisting cochains \(\tau_{\tilde{\Omega}C} : C \to \tilde{\Omega}(C)\) and \(\tau_{\tilde{\Omega}A} : \tilde{\Omega}A \to A\). By definition, \(\gamma_C(f)\) is the unique derivation such that \(f = (-1)^{|f|}\gamma_C(f) \circ \tau_{\tilde{\Omega}C}\), and \(\beta_A(g)\) is the unique coderivation such that \(\tau_{\tilde{\Omega}A} \circ \beta_A(g) = g\). On the other hand, we have clearly \(\tau_{\tilde{\Omega}A} = (\tau_{\tilde{\Omega}C})^\vee \circ \Theta\). Momentarily assuming that the next diagram is commutative (See Lemma 6.7 below).

![Diagram of graded Lie algebras](image-url)
We have \([\gamma_C(f), \gamma_C(g)]^V = (-1)^{|h_C(f)||h_C(g)}[\gamma_C(g)^V, \gamma_C(f)^V] = -[\gamma_C(f)^V, \gamma_C(g)^V]\). Therefore the following diagram commutes

\[
\begin{array}{ccc}
\widetilde{IB}A & \xrightarrow{\Theta} & (\overline{\Omega}C)^V \\
\downarrow [\beta_A(f^V \circ \Theta), \beta_A(g^V \circ \Theta)] & & \downarrow [-\gamma_C(f), \gamma_C(g)]^V \\
\widetilde{IB}A & \xrightarrow{\Theta} & (\overline{\Omega}C)^V \\
\end{array}
\]

This implies that the coderivation \([\beta_A(f^V \circ \Theta), \beta_A(g^V \circ \Theta)]\) coincides with the coderivation \(\beta_A \left((\gamma_C^{-1}[\gamma_C(f), \gamma_C(g)])^V \circ \Theta\right)\). Therefore (6.6) is proved since \(D_1(h) = h^V \circ \Theta\).

\[\square\]

**Lemma 6.7.** Let \(f : C \to \overline{\Omega}C\) be a linear map and \(A = C^V\). With the previous notations, then the next diagram commutes

\[
\begin{array}{ccc}
\widetilde{IB}A & \xrightarrow{\Theta} & (\overline{\Omega}C)^V \\
\beta_A(f^V \circ \Theta) & \downarrow & -\gamma_C(f)^V \\
\widetilde{IB}A & \xrightarrow{\Theta} & (\overline{\Omega}C)^V \\
\end{array}
\]

**Proof.** Recall the extension maps \(\beta_A : \text{Hom}(\widetilde{IB}A, A) \to \text{Coder}(\widetilde{IB}A)\) (3.5), \(\gamma_C : \text{Hom}(C, \overline{\Omega}C) \to \text{Der}(\overline{\Omega}C)\) (4.2) and if \(A = TV\) the universal derivation \(S_V\) (4.3). Let \(S_V^V\) be the universal coderivation of \(TC(V)\)-bicomodules: \(S_V^V : TC(V) \otimes V \otimes TC(V) \to TC(V), \alpha \otimes v \otimes \beta \mapsto \alpha v \beta\). The morphisms \(S(V^V)\) and \((S_V)^V\) are related by the following diagrams where \(i_1\) and \(i_2\) denote standard maps.

\[
\begin{array}{ccc}
TC(V^V) \otimes V^V \otimes TC(V^V) & \xrightarrow{i_1} & (TV \otimes V \otimes TV)^V \\
S(V^V) \downarrow & & \downarrow (S_V)^V \\
TC(V^V) & \xrightarrow{i_2} & (TV)^V \\
\end{array}
\]

Then \(\gamma_C(f)\) (resp. \(\beta_A(g)\)) is defined as the composite

\[
\gamma_C(f) : \overline{\Omega}C \xrightarrow{S} \overline{\Omega}C \otimes s^{-1}C \otimes \overline{\Omega}C \xrightarrow{1 \otimes ((-1)^{1/f}(f^\circ s) \otimes 1)} (\overline{\Omega}C)^{\otimes 3} \xrightarrow{\mu^2} \overline{\Omega}C
\]

(resp. \(\beta_A(g) : \widetilde{IB}A \xrightarrow{\Delta^2} (\widetilde{IB}A)^{\otimes 3} \xrightarrow{1 \otimes (s \otimes g) \otimes 1} \widetilde{IB}A \otimes sA \otimes \widetilde{IB}A \xrightarrow{S^{sA} \circ (1 \otimes s \otimes 1)} \widetilde{IB}A\)).

The claim follows then from the commutativity of the following diagram

\[
\begin{array}{ccc}
\widetilde{IB}A & \xrightarrow{\Theta} & (\overline{\Omega}C)^V \\
\downarrow \Delta^2_{\widetilde{IB}A} & & \downarrow (\mu^2)^V \\
(\widetilde{IB}A)^{\otimes 3} & \xrightarrow{\Theta \otimes 3} & (\overline{\Omega}C)^{\otimes 3} \\
\downarrow 1 \otimes (f^V \circ \Theta) \otimes 1 & & \downarrow (1 \otimes f \circ 1)^V \\
\widetilde{IB}A \otimes A \otimes \widetilde{IB}A & \xrightarrow{\Theta \otimes 1 \otimes \Theta} & (\overline{\Omega}C \otimes C \otimes \overline{\Omega}C)^V \\
\downarrow S^{sA} \circ (1 \otimes s \otimes 1) & & \downarrow -(S_{s^{-1}C} \circ (1 \otimes s \otimes 1))^V \\
\widetilde{IB}A & \xrightarrow{\Theta} & (\overline{\Omega}C)^V
\end{array}
\]
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