Open inflation and the singular boundary

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Abstract

The singularity in Hawking and Turok’s model of open inflation has some appealing properties. We suggest that this singularity should be regularized with matter. The singular instanton can then be obtained as the limit of a family of “no-boundary” solutions where both the geometry and the scalar field are regular. Using this procedure, the contribution of the singularity to the Euclidean action is just 1/3 of the Gibbons-Hawking boundary term. Unrelated to this question, we also point out that gravitational backreaction improves the behaviour of scalar perturbations near the singularity. As a result, the problem of quantizing scalar perturbations and gravity waves seems to be very well posed.

Recently, Hawking and Turok [1,2] have suggested that an open universe can be created from nothing. This is an attractive possibility because it would allow to construct open models of inflation with very simple inflationary potentials (see also [3–5]).

The new ingredient that makes their construction possible is that they allow their instanton solution to be singular. There is some justification for this, since the Euclidean action

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is integrable near the singularity. Moreover, if we think of the singularity as the boundary of spacetime, the Gibbons-Hawking boundary term is non-vanishing and finite. This is rather coincidental, since it requires the extrinsic curvature of the boundary to increase just at the same rate as the inverse of its volume as the singularity is approached.

In this paper, we suggest that the singularity should be regularized with matter, so that the instanton can be obtained as the limit of a family of nonsingular geometries where the scalar field is also well behaved. The simplest way to do this is to introduce a membrane coupled to the scalar field. The Euclidean action is given by

$$S_E = \int d^4x \sqrt{g} \left[ \frac{1}{2} (\partial \phi)^2 + V(\phi) - \frac{R}{16\pi G} \right] + \int d^3\xi \sqrt{h} \mu(\phi).$$

Where

$$\mu(\phi) = \mu_0 - \alpha e^{\kappa \phi},$$

and $h$ is the determinant of the metric on the worldsheet of the membrane. The parameter $\mu_0 > 0$ is a positive tension which stabilizes the vacuum at $\phi = 0$, and $\alpha$ is a small coupling. These parameters will not play a role once the “singular” limit is taken, but for the time being there is no harm in thinking of them as physical. The parameter $\kappa$ will be specified below. We have not written a boundary term, since our geometries will not have a boundary.

Following we take an O(4)-symmetric ansatz for the metric and the scalar field:

$$ds^2 = d\sigma^2 + b^2(\sigma)(d\psi^2 + \sin^2 \psi d\Omega_2^2).$$

In the absence of a membrane, the field equations for $b(\sigma)$ and $\phi(\sigma)$ are

$$\phi'' + \frac{3b'}{b} \phi' = V_{,\phi},$$

$$\left( \frac{b'}{b} \right)^2 = \frac{8\pi G}{3} \left[ \frac{1}{2} \phi'^2 - V \right] + \frac{1}{b^2},$$

where primes stand for derivatives with respect to $\sigma$. 

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The instanton is regular at $\sigma = 0$, where $b \approx \sigma$ and $\phi' = 0$. As $\sigma$ is increased, $b$ grows to a maximum value and then decreases again, reaching a second zero at some $\sigma = \sigma_f$. However, this second zero is singular. The scale factor there behaves as

$$b^3 \approx C(12\pi G)^{1/2}(\sigma_f - \sigma)$$

and the scalar field as

$$\phi \approx -(12\pi G)^{-1/2}\ln(\sigma_f - \sigma) + \text{const.}$$

In spite of the singular behaviour of the scalar field and the geometry, the Euclidean action is integrable.

Here, we shall take the approach of modifying the solution so that it will be everywhere regular. The idea is to surround the singularity with a spherical membrane which will act as a source for the scalar field. The interior of the membrane is replaced with a ball of (nearly) flat space. At the center of the ball, $\sigma = \sigma_c$, we take $\phi' = 0$, $b' = -1$, and $\phi(\sigma_c)$ is chosen so that it matches the value of $\phi$ at the membrane. The membrane will also provide the energy momentum source necessary to match both geometries.

Substituting the O(3) symmetric ansatz into the Euclidean action and varying with respect to $\phi$, one easily finds the matching conditions for the scalar field at the membrane. The discontinuity in the first derivative is given by

$$[\phi'(\sigma_m)] = -\alpha \kappa e^{\kappa \phi(\sigma_m)},$$

where the square brackets indicate the difference between the values inside and outside, and $\sigma_m$ is the location of the membrane. Given that $\phi' \approx 0$ inside the membrane and using the asymptotic form of $\phi'$ near the external face we have

$$\frac{C}{(12\pi G)^{1/2}} \approx \alpha b^3(\sigma_m)e^{\kappa \phi(\sigma_m)}.$$  

The left hand side of this equation is constant. In order to obtain a nontrivial limit as $\sigma_m \to \sigma_f$ while keeping $\alpha$ finite we take

$$\kappa \equiv (12\pi G)^{1/2}.$$  

Let us now consider the backreaction of this membrane on the geometry. Einstein’s equations imply the matching condition

$$\left[\frac{b'}{b}\right] = -4\pi G\mu(\phi) = -4\pi G(\mu_0 - \alpha e^{\kappa\phi(\sigma_m)}).$$

(9)

Inside the membrane, the geometry is basically flat, and we have \((b'/b) \approx b^{-1}\). Outside the membrane, we have

$$\frac{b'}{b} \approx -\kappa C \frac{b^3}{3b^3}.$$  

(10)

Using (7) we find that the leading \(O(b^{-3})\) terms in (3) cancel out. The subleading terms are unimportant; they will not contribute once the size of the membrane is shrunk to zero.

Inserting the trace of Einstein’s equations in (1), we find

$$S_E = -\int d^4x\sqrt{g}V(\phi) - \frac{1}{2}\int d^3\xi\sqrt{h}\mu(\phi).$$

(11)

The limit of the second term as the size of the membrane is shrunk to zero can be interpreted as the contribution of the singularity to the action of the instanton. It is given by

$$S_{sing} = \frac{\pi^2 C}{\kappa}.$$  

(12)

Taking into account that the trace of the extrinsic curvature of the membrane is \(K = 3(b'/b)\), and using (10), we find that this contribution is actually one third of the Gibbons-Hawking term evaluated on the external face of the membrane

$$S_{sing} = \frac{1}{3}S_{GH} = \frac{-1}{24\pi G} \int d^3\xi\sqrt{h}K_{ext}.$$  

(13)

This conclusion is rather general. The junction condition \([K] = -12\pi G\mu\) relates the value of \(\mu(\phi)\) in Eq. (11) to the jump in the trace of the extrinsic curvature. However, the jump in \(K\) is dominated by the extrinsic curvature on the external face, from which (13) follows.

Note that the result in (12) does not depend on the parameters \(\mu_0\) or \(\alpha\) characterizing the membrane. The reason is that \(\alpha\) has been elliminated in favour of \(C\) through equation
(4), whereas $\mu_0$ does not contribute in the limit $b(\sigma_m) \to 0$. In fact, there is no strong reason for using a coupling of the form (2). It has been chosen so that the regulator $\alpha$ remains finite as the singularity is approached. If we think of our membrane as a physical object, then for each $C$ and for each value of the cut-off $\sigma_m$, the solution only exists for specific values of $\mu_0$ and $\alpha$ determined by the matching conditions. One can extend this interpretation by taking the coupling $\alpha$ very small and allowing for a superposition of any number of membranes with positive and negative charges. In this case, the parameters $\mu_0$ and $\alpha$ can be thought of as continuous variables, which can be adjusted to satisfy (7) for any value of $C$ and $\sigma_m$.

The instability of flat space pointed out by Vilenkin [4] can be seen as the spontaneous creation of a membrane which is a source for the scalar field. Because $\phi$ is large near the membrane, its effective energy per unit area $\mu(\phi)$ is negative. This negative energy compensates for the positive energy in the scalar field configuration, so that the total energy is zero and tunneling is allowed. In Ref. [4], a massless scalar field was considered, and there was no minimum gap to be surmounted in order for tunneling to occur (the constant $C$ could be chosen arbitrarily small). This may also be true for a generic potential, and in this case it seems that the same regularization that makes the Hawking-Turok instanton acceptable also makes flat space unstable. There may be models, however, where there is a minimum height of the tunneling barrier. These models would make flat space metastable at least.

The solution of Hawking and Turok is also special with regard to the unrelated question of cosmological perturbations. In the approximation when the gravitational backreaction of the scalar field perturbations is neglected, Hawking and Turok [1] have argued that the quantization of fluctuations is marginally well defined in spite of the singularity. Indeed, after the rescaling $\phi = \chi/b$, and introducing the conformal coordinate $X = \int_{\sigma}^{\sigma_f} d\sigma/b(\sigma)$ the field modes obey a Schrodinger equation with a potential that behaves as $-(2X)^{-2}$ near the

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1 We could replace $\mu_0$ by $\mu_0 + \beta e^{\alpha/3}\phi_0$, and then $\mu_0$ and $\beta$ would also remain finite in the limit $b(\sigma_m) \to 0$. 
singularity. This is again very coincidental, since with a stronger singularity the quantum mechanical problem would certainly be ill posed [1,4].

Therefore it is important to check what happens when gravitational backreaction is included. The quantization of cosmological perturbations in $O(3,1)$ symmetric geometries (the analytic continuation of our instanton has this symmetry) was recently studied in Ref. [10]. The analysis is not straightforward because the $\sigma = \text{const.}$ surfaces of homogeneity and isotropy cannot be used as cauchy surfaces on which commutation relations can be imposed. Instead, one has to resort to inhomogeneous $\psi = \text{const}$ surfaces, where the disentanglement of scalar and tensor modes is complicated. However, the final result is rather simple. After a suitable rescaling [10], the gauge invariant scalar potential obeys a Schrodinger equation with effective potential given by

$$4\pi G\phi'^2 + \phi' \left( \frac{1}{\phi'} \right)'' .$$

(14)

Here, a prime indicates derivative with respect to the conformal coordinate $X$ introduced above. It is straightforward to show that the first term dominates near the singularity, behaving as $k/X^2$, with $k = 3/4$. Hence, the effective potential goes to plus infinity rather than minus infinity near the singularity. Interestingly, the coefficient $k = 3/4$ is again a critical one [9]. As mentioned above, for $k < -1/4$ the problem is not well posed. For $-1/4 < k < 3/4$ the problem is marginally well posed, since both solutions of the Schrodinger equation are square integrable near the singularity, but only one has a square integrable kinetic energy. Finally, for $k \geq 3/4$, the basis of functions is uniquely determined by the requirement of square integrability [3], which selects one solutions for each value of the energy. Thus, the problem of quantizing the perturbations seems much better posed thanks to backreaction. In particular, this seems to preclude the possibility of matter “streaming out” from the singularity into the universe [5]. The same comment applies to gravity waves, for which the corresponding effective potential reduces to the first term in (14) [11].

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