Numerical Stochastic Perturbation Theory. Convergence and features of the stochastic process. Computations at fixed (Landau) Gauge.

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Concerning Numerical Stochastic Perturbation Theory, we discuss the convergence of the stochastic process (idea of the proof, features of the limit distribution, rate of convergence to equilibrium). Then we also discuss the expected fluctuations in the observables and give some idea to reduce them. In the end we show that also computation of quantities at fixed (Landau) Gauge is now possible.

1. Introduction

Numerical Stochastic Perturbation Theory (NSPT) was introduced in [1] and had successful applications [2] [3]. Here we want to describe the main features of the underlying stochastic process. In the last part we show that a deeper understanding of the process allows also computations at fixed (Landau) gauge.

NSPT was developed in the context of Stochastic Quantization [4]. In this approach perturbation theory is performed through a formal substitution of the expansion

\[ U_\eta(x,t) \rightarrow \sum_k g_k U_k(x,t), \]

in the Langevin equation:

\[ \partial_t U_\eta = \left[-i\nabla S[U_\eta] - i\eta\right] U_\eta \]

(\(\nabla\) is a suitably defined group derivative). The Langevin equation in the algebra variables reads:

\[ \frac{\partial}{\partial t} A_\mu^a(\eta, x, t) = D_\nu^b L_{\nu\mu}^b(\eta, x, t) + \eta_\mu^a(x, t) \]

which gives a system of equations that can be solved numerically.

2. Convergence and features of the stochastic process.

We now consider \( P_n[A^{(0)}, \ldots, A^{(n)}, t] \) : the distribution function of the first \( n \) perturbative components of the fields at Langevin time \( t \). \( P_n \) certainly exists for each \( n \) and \( t \) fixed. But we need to know: whether the limit distributions exist for \( t \to \infty \), to what kind of distribution they converge and at which rate.

At this stage the answer to the first question is certainly not. This is due to two reasons which become clear if we look at the formal solution of the (perturbatively developed) Langevin equation in the Fourier transformed space (with zero initial conditions):

\[ A^{(n)}_\mu(k, t) = T_{\mu\nu} \int_0^t ds e^{-k^2(t-s)} f^{(n)}_\nu(k, s) + L_{\mu\nu} \int_0^t ds f^{(n)}_\nu(k, s) \]

Colour indexes are left out, \( T_{\mu\nu} \) and \( L_{\mu\nu} \) are the transverse and longitudinal abelian projectors, and

\[ f^{(n)}_\nu(k, t) = g I^{(3)}(n-1)(k, t) + g^2 I^{(4)}(n-2)(k, t); \]

\[ f^{(0)}_\nu(k, t) = \eta_\nu(k, t). \]

With \( I^{(\cdot)} \) we mean the \( 3 \) (4) gluons interaction terms, which only depend on the fields till the \( n-1 \) (\( n-2 \)) perturbative order.

This process must behave like a random walk in correspondence of both the gauge degrees of freedom and the mode \( k = 0 \), since such degrees of freedom have no attractive force and consequently no damping factor. Thus they produce diverging fluctuations, even if their mean value shall be zero since the equivalence of Stochastic and Canonical quantization is true also when expanded in perturbation theory.

The idea is to introduce an attractive force which keep the norms of the fields under control, without affecting the observables [3]. To this end we interlace each step of Langevin dy-
namic with a step of gauge transformation defined by: \( U^w(x) = e^{w(x)U(x)e^{-w(x+\mu)}} \) (where \( w(x) = -\lambda \sum_\mu \partial_\mu A_\mu \)). One can prove that, by doing this, the system gains a force that drives it towards the Landau gauge (provided it is within the first Gribov horizon) [6].

We will come back to this point later when considering quantities in a fixed gauge. For the moment suffice it to say that all gauge degree of freedom now have an attractive force.

The problem of zero modes appears also in usual lattice perturbation theory: the propagator is singular at \( k = 0 \), and the common prescription is to exclude the zero-momentum degree of freedom of the fields.

In our context the problem is a bit more subtle since we work in configuration space. Moreover if we set the \( k = 0 \) mode to zero in the gaussian noise \( \eta_\mu(x,t) \), the free fields \( A^{(0)}(x,t) \) will have no \( k = 0 \) mode too, but this won’t be true for the higher perturbative components. In fact the interaction introduces a zero mode contribution in the fields \( A^{(p)}(x,t) \) even if it was not present in the lower perturbative order. For instance the three gluons term gives:

\[
\begin{align*}
A^a_\mu(0,t)|_{3\text{glu}} &= \frac{igf^{abc}}{2(2\pi)^n} \int dpdq \delta(p+q) \\
&\times A^b_\nu(-p,t)A^c_\sigma(-q,t)\eta_\mu^{(3)}(0,p,q)
\end{align*}
\]

\( \eta_\mu^{(3)}(k,p,q) = \delta_{\mu\nu}(k-p)\sigma + \text{perm.} \). We must subtract these 0 modes by hand at each step.

In principle one should subtract the zero mode component after the updating of each single perturbative component of each single link, and before evaluating the next perturbative order. But this would be extremely expensive. It is convenient, instead, to subtract the zero mode after each sweep of the whole lattice. In this way we just introduce another error of order \( \tau \), which is then extrapolated to zero.

Once implemented the two corrections described above, it is possible to prove the convergence of the process. By that we mean that any correlation function of any perturbative component of the fields \( \langle \prod_j A^{(p)}_\mu_j(x_j,t) \rangle \) has a finite limit for large Langevin time.

We just give the main ideas and results. We first remark that all correlations of free fields converge at least like \( O(e^{-qt}) \) (if \( q \) is the lower momentum). Then we proceed by induction: It is convenient to re-write the solution \( (1) \) in discretized Langevin time \( t = N\tau \) distinguishing the memory of the past from the new contribution of the random process:

\[
\begin{align*}
A^{(0)}_\mu(k,t) &= e^{-k^2\tau}A^{(0)}_\mu(k,t-\tau) + \sqrt{\tau}\eta_\mu(k,t) \\
A^{(j)}_\mu(k,t) &= e^{-k^2\tau}A^{(j)}_\mu(k,t-\tau) + \tau f^{(j)}_\mu(k,t)
\end{align*}
\]

The insertion of this formula into a correlation function reduces it into others of lower perturbative order. It is not difficult to evaluate the sum of the relevant part which survives in the limit \( \tau \to 0 \) at \( t = N\tau \) fixed, and then take the limit \( t \to \infty \).

The result is the following: if every degree of freedom has an attractive force, as described above, then a limit distribution exists for each \( P_n \), and all their moments are finite (i.e. any correlation function of any perturbative component of fields is finite). Moreover convergence to equilibrium is damped by a factor \( t^\beta e^{-k^2\tau} \) where \( k \) is the lower momentum contributing and \( p \) is the global perturbative order of the correlation function.

### 3. About fluctuations: status and ideas for improvement

The fact that the limit distributions of these objects produce correlation functions which are all finite is not sufficient. We need to have an idea of how much such correlations can grow for a high number of fields or perturbative order. In fact some applications of NSPT need a knowledge of the perturbative coefficient with an extremely high precision. Since this is a stochastic method the result is known with an error which is essentially given by the intrinsic fluctuations of the correlation function one needs to calculate.

To gain some insight in this problem it is useful to think of the process in terms of the underlying gaussian process \( \eta(x,t) \). Any perturbative component \( O^{(p)} \) of an observable \( O \) may be seen as a sum of correlation functions of \( \eta \)'s. In fact \( \langle O^{(p)} \rangle \sim \sum_\sigma \langle \eta_{\pi_1} \cdots \eta_{\pi_M} \rangle + \langle (O^{(p)})^2 \rangle \sim \sum_\sigma \sum_\pi \langle \eta_{\pi_1} \cdots \eta_{\pi_M} \eta_{\sigma_1} \cdots \eta_{\sigma_M} \rangle \). The \( \sigma \)'s are some
choices from all the possible $\eta$’s in the process and $M$ is a number. Both depend on the theory, on the observable and on the perturbative order $p$. Since the fluctuations of $O^{(p)}$ clearly depend on the number $M$ and choices $\sigma$, it would be important to be able to say something about them.

It is quite easy to determine the number $M$ for a particular Theory and observable. For instance in the $\lambda\phi^4$ theory for $O(\phi) = \phi^2$ the relation between the perturbative order $p$ and the maximum number of correlated $\eta$’s is $M = 2p + 2$, while for the plaquette in gauge theory the relation is simply $M = p$.

This information is widely insufficient to determine the size of the fluctuations. There are other elements that strongly influence the size of the fluctuations, but we can be only qualitative about them. Consider for instance the $\lambda\phi^4$: the field interacts with itself in the same point. $\phi^{(1)} \sim (\phi^{(0)})^3$, $\phi^{(2)} \sim (\phi^{(0)})^2\phi^{(1)} \sim (\phi^{(0)})^5$. We have a strong contribution of correlations of the kind $\langle \eta^M \rangle$. We expect strong fluctuations quite soon. The situation for $\sigma$-model is similar. Consider instead gauge theories: Interaction is given by product of fields of different colours and directions. $A^{(1)}_{\mu} \sim gf^{abc}A^{(0)}_{\mu}A^{c(0)}_{\nu}$ etc. This makes the previous phenomenon much less severe.

Remark: If we want to study a fixed observable at a fixed perturbative order we do not really need the process $\eta$ to be gaussian. We just need a fixed number of its correlations to be gaussian. Higer moments could be chosen to be lower than those of a gaussian distribution. This is achieved, for instance, if one exploits combinations of Dirac delta’s: $p(x) = \sum_j w_j \delta(x - x_j)$ (see [5] for the general solution).

4. Computation of quantities at a fixed (Landau) Gauge.

The convergence of each correlation function imply in particular that not only gauge invariant quantities are computable but also those which depend on the gauge.

Actually the gauge fixing procedure which is realized here is not that of Faddeev-Popov [6] (which is possible but more expensive to perform [9]), but that introduced by Zwanziger (plus corrections of the order of the lattice spacing and of $\tau$). In fact the interleaved gauge transformations described above are equivalent to add to the Langevin equation a force

$$\lambda D_{\mu}^{ab} \partial \phi^b A_{\nu}^k = -\lambda \delta S_{GF}[A] \delta A_{\mu}^k + \lambda g f^{abc} A_{\mu}^k \partial \nu A_{\nu}^c$$

(where $S_{GF}[A] = \frac{1}{2} \int dx (\partial \lambda A_{\nu}^c)^2$). Corrections of the order of the lattice spacing are present, since the formula above is valid only in the continuum. A correction of order $\tau$ is expected to come from the procedure of interleaving a Langevin step with a gauge transformation.

Although this kind of gauge fixing is not that of Faddeev-Popov, the Landau choice of gauge can be reproduced. There are at least two ways of doing this. The first one is natural but maybe not efficient: one can perform the calculation at different value of the ratio $\frac{1}{\alpha} = \lambda/\tau$ and then extrapolate for large $\frac{1}{\alpha}$. The second method consists in performing many gauge transformations on a thermalized configuration. This should drive the system towards a stationary point, where - in fact - the Landau gauge condition is satisfied.

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