Quantization of empty space.

M.A. Zubkov\textsuperscript{a}

\textsuperscript{a} ITEP, B. Cheremushkinskaya 25, Moscow, 117259, Russia

Abstract

We suggest to use "minimal" choice of quantum gravity theory, that is the quantum field theory, in which space-time is seen as Riemannian space and metric (or vierbein field) is the dynamical variable. We then suggest to use the simplest acceptable action, that is the squared curvature action. The correspondent model is renormalizable, has the correct classical limit without matter and can be explored using Euclidian path integral formalism. In order to get non-perturbative results one has to put this model on the lattice. While doing so serious problems with measure over dynamical variables are encountered, which were not solved until present. We suggest to solve them using the representation of Riemannian space as a limiting case of Riemann-Cartan space, where the Poincare group connection plays the role of dynamical variable. We construct manifestly gauge invariant discretization of Riemann-Cartan space. Lattice realization of Poincare gauge transformation naturally acts on the dynamical variables of the constructed discretization. There exists local measure invariant under this gauge transformation, which could be used as a basic element of lattice path integral methods. The correspondent lattice model appears to be useful for numerical simulations.

March 25, 2022
1 Introduction

Quantum theory of gravity should describe behavior of the Early Universe. Information about its structure comes mostly from astrophysical observations. Unfortunately present data does not allow to make a definite choice between different approaches to construction of quantum gravity theory. Therefore we suggest to make a choice, which we would call ”minimal”. Namely, first we fix a functional integral method of quantization and do not consider Hamiltonian methods. Next, instead of using complicated mathematical structures we consider Riemannian manifold, which is used in general relativity. Therefore our dynamical variable is the field of metric. Then, there are two ingredients of the theory to be chosen: the action and the measure. In [1] we motivated that the squared curvature action is the most appropriate choice. In the next section we briefly repeat our arguments. It seems that this model should describe pure gravity without matter very naturally. And even if the correct theory is different, the constructed one may serve as a low energy approximation. Unfortunately, inclusion of matter is not straightforward. And in spite of we guess, that it could be done, the present paper is called ”quantization of empty space”.

When trying to use the chosen theory in practice, we have to put it on the lattice, which is the only way to explore it nonperturbatively. It is also the only way to define measure over dynamical variables. Here we encounter serious problems, which are related to the wideness of the gauge group (which is the group of general coordinate transformations). Namely, in the naive discretizations using rectangular lattices gauge invariance is lost. It was shown, that in ordinary gauge theories the discretization, which is not gauge invariant, is not appropriate (say, in nonabelian models there is no confinement of fundamental charges [2]). On the other hand, in Regge calculus, which is gauge invariant discretization of Riemannian manifold, the only measure whose appearance is motivated by reasonable (although not perfect) arguments is Lund - Regge measure, which corresponds to a certain metric\(^1\). This measure is nonlocal and so complicated, that practical numerical simulations seem to be impossible [3]. In [4] the way to solve this problem in two dimensions was suggested. In this approach partially gauge fixed version of the theory is used. So, we obtain local and simple\(^2\).

\(^1\)This metric is induced by the simplest possible continuum metric on the space of Riemannian geometries.

\(^2\)This metric is induced by the simplest possible continuum metric on the space of Riemannian geometries.
measure and instead lose a small part of gauge invariance (its conformal part). However, even this semi-solution can not be expanded to physically interesting case of four dimensions.

Alternative approach to discretization of Riemannian space was suggested in [1], where Riemannian manifold of simple topology is seen as Weitzenbock space. (Riemannian manifold and Weitzenbock space are just different ways to look at the same entirety.) In section 3 we remind this construction. In this discretization the inverse vierbein, i.e. a translational connection plays the role of fundamental variable. Measure over this variable is local. Negative feature of this discretization is that the action depends on Riemannian curvature, which should be expressed through the inverse vierbein in a non-trivial way. In [1] there were suggested to use the approximate expression, which tends to the correct one in the naive continuum limit. Thus, gauge invariance there is not exact. But even with this expression the model is not useful for numerical simulation as any discretization of the model with second derivatives in the action.

Therefore in this paper we suggest to apply another approach. We consider Riemann-Cartan space, in which both $O(4)$ connection\(^2\) and the inverse vierbein are the dynamical variables. We do not require vanishing of torsion (which would lead to appearance of Riemannian manifold) or vanishing of curvature (which would lead to appearance of Weitzenbock space). Instead we construct manifestly gauge invariant discretization of Riemann-Cartan space and suppose, that vanishing of torsion should be achieved dynamically. Elements of Poincare group that is the gauge group of the theory naturally act on the dynamical variables of the discretized model. The construction is described in section 4. In section 5 we consider the action of the model. It is worth mentioning, that discretization of Poincare gravity, which is based on the Riemann-Cartan space, has already been considered in literature (see, for example, [5]). However, the correspondent constructions were not manifestly gauge invariant as Regge discretization of Riemannian space or as the discretization suggested in this paper. Moreover, in [5] and related publications Poincare group was not the real gauge group of the lattice models.

In section 6 we discuss problems with measure encountered in Regge discretization, where measure is constructed in order to correspond to the metric on space of Riemannian geometries. We demonstrate, using the finite

\(^2\)Here and further in the text we imply, that Wick rotation to Euclidean signature is performed. So, we deal with Euclidean path integral formalism.
dimensional analogy, that Lund - Regge measure may not be appropriate at all. Therefore, we construct measure over dynamical variables in the suggested lattice model using another approach. We use symmetry properties of the continuum model (that is a lattice gauge theory, in which Poincaren group plays the role of the gauge group). Gauge invariance of the measure is explored in order to determine it. This construction is considered in section 7. In section 8 we end with our conclusions.

2 "Minimal" quantum gravity

The simplest choice of the action would be Einstein-Hilbert action. However, the resulting theory appears to be nonrenormalizable. Moreover, after Wick rotation to Euclidian signature the action becomes unbounded below. This indicates that the vacuum state of the model should be highly curved with fractal dimension far from its physical value 4.

It is well-known, however, that ultraviolet divergences in quantum gravity with the action that contains additional squared curvature term can be absorbed by an appropriate renormalization of the coupling constants [6]. The model considered in [6] and related publications has the following action:

$$S = \int \{ \alpha (R_{AB}R_{AB} - \frac{1}{3}R^2) + \beta R^2 - \gamma m_p^2R + \lambda m_p^4 \}|E|d^4x, \quad \text{(1)}$$

where $|E| = \det E^A_{\mu}, E^A_{\mu}$ is the inverse vierbein, the tetrad components of Ricci tensor are denoted by $R_{AB}$, and $R$ is the scalar curvature. Coupling constants $\alpha, \beta, \gamma$ and $\lambda$ are dimensionless while $m_p$ is a dimensional parameter. Linearized theory (around flat background) contains graviton together with additional tensor and scalar excitations. The whole propagator behaves like $\frac{1}{q^4}$ in ultraviolet while $\alpha, \beta \neq 0$. Tensor excitation is a ghost, which leads to loss of unitarity.

The requirement that the action (1) is bounded below leads to appearance of a tachyon. This indicates that flat space is not real vacuum of the model. The tachyon would disappear if we construct perturbation expansion around the background that minimizes (1) [7]. In addition to the ultraviolet divergences the perturbation expansion may also contain infrared divergences. In order to separate their consideration from the consideration of the ultraviolet ones we have to use an additional regularization. This can be done,
for example, if the invariant volume $V$ of the space-time manyfold is kept constant. Then after usual regularization (say, the dimensional or lattice regularization) is removed and all the ultraviolet divergences are absorbed by redefinition of the coupling constants, each term of the perturbation expansion is finite. In the theory with fixed invariant volume cosmological constant does not influence the dynamics and the action is bounded below if $(\alpha \geq 0, \beta > 0, \gamma \neq 0)$ or $(\alpha \geq 0, \beta \geq 0, \gamma = 0)$. The renormalization group analysis shows [7] that at $\alpha, \beta > 0$ there exists a region of couplings such that the theory is asymptotic free in $\alpha$ and $\beta$ while $\gamma$ can be made constant (up to one-loop approximation). Divergences may appear also in the limit $V \to \infty$. However, similar divergences appear in QED but they are compensated by ejection of soft photons. Probably, the same mechanism may work here as well.

Unfortunately classical Newtonian limit cannot be obtained directly from the action (1) unless it is not bounded from below. However, if we start from pure gravity model with the action (1) (with $\lambda = 0$) and rotate it back to Minkowski signature, solutions of Einstein equations would satisfy the appeared classical equations of motion. They are not the only solutions of the equations of motion. However, at $\gamma = \lambda = 0$ Einstein spaces minimize the (Euclidean) action\(^3\). Therefore it could be interesting to consider the theory with the action (1) such that at some scale the renormalized couplings $\gamma$ and $\lambda$ vanish. Then massive point-like objects could be treated, as space-time singularities [8] and the Newtonian limit appears as an asymptotic of black hole solutions. Suppose that the line-like singularity is embedded into the space-time. Then Einstein equations in empty space would lead to Einstein equations in the presence of a particle moving along the mentioned singularity. Its mass is not fixed by the field equations but it is proved to be constant along the world trajectory [8]. This indicates that matter can be introduced into quantum gravity theory with the action (1) in such a way that it reproduces general relativity at $\alpha > 0, \beta > 0, \lambda = 0$.

\section{Discretization of Weitzenbock space.}

This approach is based upon teleparallel formulation of general relativity [9]. The correspondent geometrical construction is the so-called Weitzenbock space that appears as a limiting case of a more general concept - Riemann -

\footnotesize
\begin{flushleft}
\[3\text{And tachyon disappears.}\]
\end{flushleft}
Cartan space. The latter is a tangent bundle equipped with the connection from Poincare algebra. Poincare group consists of $O(4)$ rotations and translations\(^4\). Translational part of connection can be identified with the inverse vierbein and defines space-time metric. The correspondent part of the curvature becomes torsion. Riemannian geometry appears when torsion is set to zero. Weitzenbock geometry is an opposite limit: $O(4)$ part of Poincare curvature is set to zero while torsion remains arbitrary. Teleparallel gravity is the theory of Weitzenbock geometry, i.e. a translational gauge theory.

If the space-time manifold is parallelizable zero curvature $O(4)$ connection can be chosen equal to zero. Therefore the only dynamical variable is the inverse vierbein, treated as a translational connection. Usually action in teleparallel gravity is expressed through the translational curvature (torsion); space-time and internal indices can be contracted by the vierbein. The equivalence between continuum theories of Riemannian and Weitzenbock geometries can be set up if everything in Riemannian geometry is expressed through the inverse vierbein and the latter is identified with the translational connection.

Our discretized space is composed of flat pieces connected together. We consider two cases: when each such piece has the form of a simplex or when it has the hypercubic form. So, we talk about either simplicial or hypercubic lattice. For the further convenience we refer to simplices (hypercubes) as to elements of the lattice. Form of the lattice elements is fixed by the set of vectors $e_\mu$ connecting the center of the element with its vertices. The expression of $e_\mu$ through elements of the orthonormal frame $f_A (A = 1, 2, 3, 4)$ (common for all lattice elements) is the basic variable of the construction. So we have

$$e_\mu = \sum_A E_\mu^A f_A$$  \hspace{1cm} (2)

(Everywhere space-time indices are denoted by Greek letters contrary to the tetrad ones.) Contrary to \(\Pi\) we imply here that all of the vectors $e_\mu$ for the simplicial case and vectors $e_\mu, \mu = 0, 1, 2, 3, 4$ in the hypercubic case are independent. Also we imply that all vertices of lattice elements are ordered in some way. The other vectors in hypercubic case are defined in such a way,

\(^4\)Here and below we always remember, that Wick rotation to Euclidean signature is performed. Nevertheless for simplicity we shall call the group, which consists of $O(4)$ rotations and translations as Poincare group keeping in mind that back rotation to Lorentzian signature should be done after calculations using Euclidean path integral methods are performed.
that opposite sides of the lattice element are parallel to each other. The hypercubic lattice is periodic and the position of the starting point of each lattice element is always denoted by $e_0$.

Variables $E^A_\mu$ represent translations from the fixed point (which will be called later by the center of the lattice element) to vertices of the lattice element. Metric (or vierbein) is implied to be constant inside each lattice element. Its derivative is singular and is concentrated on the sides of the lattice elements. The translation along a path, which consists of the pieces that belong to different lattice elements is defined as the sum of the correspondent translations inside those lattice elements.

Here shift of the center of lattice element by a vector $v^A$ causes change in basic variables: $E^A_\mu \to E^A_\mu + v^A$, which is treated as gauge transformation with respect to the translational gauge group. It represents the translation of the given lattice element within the correspondent local map.

Our construction is the special case of Weitzenbock space with singular torsion, which is of $\delta$-functional type and is concentrated on the sides of the lattice elements. We do not discuss here how torsion is expressed through variables $E^A_\mu$ as it follows from the correspondent expressions of the next section in the limit of vanishing $O(4)$ connection.

4 Discretization of Riemann-Cartan space.

We construct the discretization of Riemann-Cartan space as the generalization of the construction considered in the previous section. Namely, again our space is composed of flat pieces (simplices or hypercubes). Now in addition to the translational connection, which is defined by the set of variables $E^A_\mu$, each shift from one lattice element to another is accompanied by the rotation in the four-dimensional tangent space. In other words, there is the $O(4)$ connection, which is singular and is concentrated on the sides of lattice elements. We denote by $U_{IJ}$ the $O(4)$ matrix, which is attached to the side that is common for the lattice elements $I$ and $J$.

The constructed Riemann-Cartan space has singular connection. In this case definitions of curvature and torsion become ambiguous. Therefore we

---

5 That are 3-dimensional subsimplices in simplicial case and cubes in hypercubic case.

6 In $\mathbb{H}$ we considered approximate expression for the torsion, which tends to the correct one in the naive continuum limit. Therefore in $\mathbb{H}$ lattice torsion was attached to the bones, that are 2-dimensional subsimplices in simplicial case and plaquettes in hypercubic case.
must fix one of the definitions in order to calculate them.

Connection is singular on the sides of lattice elements. $O(4)$ curvature is concentrated on the bones\(^7\). We choose the following integral equation as a definition of $O(4)$ curvature.

$$\exp(\int_{y \in \Sigma} \Omega(z, y) R_{\mu\nu}(y) \Psi(z, y) dy^\mu \wedge dy^\nu) = P\exp(\int_{z \in \partial \Sigma} \omega_\mu dx^\mu)$$

at $|\Sigma| \to 0$ (3)

Here $\omega_\mu$ is $O(4)$ connection\(^8\). $\Sigma$ is a small surface, that crosses the given bone, and $|\Sigma|$ is its area. $\partial \Sigma$ is the boundary of $\Sigma$. Its orientation corresponds to orientation of $\Sigma$. $\Omega(z, y) = P\exp(\int^y_0 \omega_\mu dx^\mu)$ is the parallel transport along the path that connects a fixed point on $\partial \Sigma$ with the point $y$. We choose this path in such a way, that it is winding around the given bone in the same direction as $\partial \Sigma$. The integral in the right hand side is over the path $\partial \Sigma$, which begins and ends at the point $z$.

It is worth mentioning, that the given definition does not contradict with the conventional one in case of smooth connection. And it gives us the possibility to calculate curvature in the case of the constructed singular piecewise linear manyfold.

Let us fix the given lattice element. Inside it lattice curvature is equal to

$$R^{A}_{\mu\nu B}(y) = \frac{1}{D!} \sum_{b \in \text{bones}} \int_{x \in b} \epsilon_{\mu\rho\sigma} dx^\rho \wedge dx^\sigma \delta^{(4)}(y - x) [\log \prod_{i} U_{I_i, I_{i+1}}^b]^{A}_{B}, \quad (4)$$

Here the sum is over the bones that belong to the given lattice element. The integral is over the surface of the bone. The product of the rotation matrices, which are encountered, while going around the given bone $b$ is denoted as $\Pi_{I_i} U_{I_i, I_{i+1}}^b$. Here we imply, that this closed path begins within the given lattice element and has the minimal lattice length.

Now let us calculate torsion, which is concentrated on the sides of lattice elements. The torsion field $T^{A}_{\mu\nu}$ is defined by the integral equation

$$\int_{y \in \Sigma} \Omega^{A}_{B}(z, y) T^{B}_{\mu\nu}(y) dy^\mu \wedge dy^\nu = \int_{\partial \Sigma} \Omega^{A}_{B}(z, y) b^{B}_{\mu}(y) dy^\mu$$

(5)

\(^7\)That are 2 - dimensional subsimplices in simplicial case and plaquettes in hypercubic case.

\(^8\) $U_{IJ} = P\exp(\int \omega_\mu dx^\mu)$, where integral is over the path of minimal length, which connects centers of lattice elements $I$ and $J$. 

8
Here $b_A^\mu(x)$ is the field of inverse vierbein, which is expressed through our variables $E^A_\mu$ inside each lattice element if the given parametrization of the lattice element is chosen.

This equation is satisfied with the following expression (which is valid within the lattice element $I$):

$$T^A_{\mu\nu}(y) = \sum_{s \in \text{ sides}} \left[ \int_{x \in s} \frac{[U_{I}s]_B^A [U_{J}s]_B^A (x) \epsilon_{[\nu\tau\rho\sigma]} dx^\tau \wedge dx^\rho \wedge dx^\sigma \delta^{(4)}(y-x)}{D!} \right]$$

Here the first integral in the sum is over the given side $s$ (that is the image of $h_0$) is denoted as $E_{i}^{sA} = E_{ui}^{A} - E_{u0}^{A}$. The vector of unity length orthogonal to $E_{i}^{sA}$ (that is the image of $h_0$) is denoted as $E^{sA}_0 = \epsilon_{[ABC]D} E^{sB}_1 E^{sC}_2 E^{sD}_3 |_{x \in s}$. Then nonzero elements of the term (in the sum of (6)) correspondent to the given side $s$ enter the expressions for $T^A_{i0}(y)$:

$$\frac{1}{D} \int_{x \in s} ([U_{I}s]_B^A E^{sB}_i (J^s) - E^{sA}_i(I)) dx^i \delta^{(4)}(y-x),$$

where $E(I)$ is calculated inside lattice element $I$ while $E(J^s)$ is calculated inside lattice element $J^s$.

Next, in tetrad components we have

$$T^A_{CD}(y) = -T^A_{DC}(y) = \frac{1}{D} \sum_{s \in \text{ sides}} E^A_{[C}(I) E^A_{D]}(I) \int_{x \in s} ([U_{I}s]_B^A E^B(J^s) - E^{sA}_i(I)) dx^i \delta^{(4)}(y-x)).$$

where $E^A_i(I)$ are the elements of the inverse matrix $[E(I)]^{-1}$. 

\textbf{4.1 Simplicial case}

In order to simplify the last expression let us concentrate our attention on the given side $s$ (that is 3-dimensional simplex). Let us denote its vertices as $u_i$ ($i = 0, 1, 2, 3$). We fix the basis, which is composed of vectors $h_i = (u_0 u_i)$, $i = 1, 2, 3$ and the vector $h_0$ of unity length orthogonal to $h_i$, $i = 1, 2, 3$. Then we denote $E^{sA}_i = E^{A}_{u_i} - E^{A}_{u0}$. The vector of unity length orthogonal to $E^{sA}_i$ (that is the image of $h_0$) is denoted as $E^{sA}_0 = \epsilon_{[ABC]D} E^{sB}_1 E^{sC}_2 E^{sD}_3 |_{x \in s}$. Then nonzero elements of the term (in the sum of (6)) correspondent to the given side $s$ enter the expressions for $T^A_{i0}(y)$:
In order to calculate in tetrad components the expression for the curvature we fix the following basis within the lattice element $I$, which corresponds to the bone $b$. Let \( \{v^0_b, v^1_b, v^2_b, v^3_b\} \) be the vertices of the given lattice element, and \( \{v^0_b, v^1_b, v^2_b\} \) be the vertices of the given bone $b$. Then our basis consists of vectors \( h^i = (v^0_b v^i_b) \), $i = 1, 2, 3, 4$. Also we denote \( F_{bi}^A(I) = E_{v^0}^A - E_{v^i}^A \). Then

\[
R_{CFB}^A(y) = \sum_{b \in \text{bones}} \mathcal{F}_{C}^{b3}(I) \mathcal{F}_{F}^{b4}(I) \frac{2}{D!} \int_{x \in b} d^2x \delta^{(4)}(y - x) \Omega_B^{bA}(I),
\]

where \( \mathcal{F}_{bi}^A(I) \) are the elements of the inverse matrix \([\mathcal{F}^b(I)]^{-1}\) and the product of \(O(4)\) matrices around the given bone is denoted as \(\Omega_B^{bA}(I) = [\log\Pi_i U_{I,i+1}^b A B]^A\).

### 4.2 Hypercubic case

Let us define inside each lattice element the following variables: \( \mathcal{E}^A_\mu = E^A_\mu - E^A_0 \), $\mu = 1, 2, 3, 4$. Also we denote by \( \mathcal{E}^A_\mu \) elements of the inverse matrix \( \mathcal{E}^{-1} \).

In tetrad components we have:

\[
R_{CFB}^A(y) = \mathcal{E}^\mu_C \mathcal{E}_F^\nu \frac{1}{D!} \sum_{b \in \text{bones}} \int_{x \in b} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma \delta^{(4)}(y - x) \log\Pi_i U_{I,i+1}^b A B, \tag{10}
\]

Torsion is expressed as

\[
T_{CF}^A(y) = \frac{\mathcal{E}^\mu_C \mathcal{E}_F^\nu}{D!} \sum_{s \in \text{sides}} \int_{x \in s} dx^7 \wedge dx^\rho \wedge dx^\sigma \delta^{(4)}(y - x) \left( [U_{I,s}]^A B \mathcal{E}^B_\mu [\epsilon_\nu \epsilon_\tau \rho \sigma] - \mathcal{E}^A_\mu \epsilon_\nu [\epsilon_\tau \rho \sigma] \right) \tag{11}
\]

Here new subscript index has appeared in \( \mathcal{E} \) in order to make difference between this variable calculated in the given lattice element $I$ and in its neighbor $J^s$.

### 4.3 Gauge transformations

Now gauge transformations are represented by translations and \(O(4)\) rotations of the lattice elements, that result in the following change in basic variables:

\[
E^A_\mu \rightarrow \Theta_B^A E^B_\mu + v^A, \tag{12}
\]

where \( \Theta \) is the rotation matrix and \( v \) is the vector that represents translation.
We must notice, that torsion and curvature defined above become undefined on the bones and links respectively, where several sides (bones) intersect each other. It could be possible, in principle, to define these variables on such objects. Say, if we define torsion on the bone as the weighted sum of the torsions on the sides, incident on this bone, then we come to the definition of \[ \Pi \] (after \( O(4) \) connection is set to zero). We do not discuss here such a procedure for the curvature, but imply, that it could be made if necessary.

\section{The action}

\subsection{Simplicial lattice}

At this stage we consider Riemann-Cartan space only as the way to put Riemannian space on the lattice. Therefore, our action should be constructed in order to make the model close to the model based on Riemannian space with the action \( \Pi \) for some values of coupling constants. That’s why we consider the action in the form

\[
S = \int \{ \alpha (R_{AB} R_{AB} - \frac{1}{3} R^2) + \beta R^2 - \gamma m_p^2 R + \lambda m_p^4 \} |E| d^4x \\
+ \delta m_p^2 \int T^A_{BC} T^A_{BC} |E| d^4x,
\]

where the second term is added in order to suppress torsion at \( \delta \to \infty \).

In order to obtain compact form of the expression for the action on the simplicial lattice we define lattice tetrad components of curvature and torsion inside the given lattice element \( I \):

\[
\begin{align*}
R^A_{CFB}(I) &= \sum_{b \in \text{bones}} \mathcal{F}^b_{[C}(I) \mathcal{F}^b_{F]}(I) \Omega^b_A(I) \\
R_{FB}(I) &= R^A_{AFB}(I) \\
R(I) &= R_{AA}(I) \\
T^A_{CD}(I) &= \sum_{s \in \text{sides}} \varepsilon^s_{[C}(I) \varepsilon^s_{D]}(I) [U^A_{IJ}]^A B \varepsilon^s_j(I)^{A} - \varepsilon^s_A(I)]
\end{align*}
\]

Then the action has the form:

\[
S = \sum_{I \in \text{simplices}} \{ \bar{\alpha} R_{FB}(I) R_{FB}(I) + (\bar{\beta} - \frac{1}{3} \bar{\alpha}) R(I)^2 + \bar{\lambda} m_p^4 \\
- \bar{\gamma} m_p^2 R(I)^2 + \bar{\delta} m_p^2 T^A_{BC}(I) T^A_{BC}(I) \} |E(I)|,
\]

where the second term is added in order to suppress torsion at \( \delta \to \infty \).
where $|\mathbf{E}(\mathbf{I})|$ is the volume of the lattice element $\mathbf{I}$.

Here we introduce lattice couplings $\bar{\alpha}, \bar{\beta}, \bar{\delta}, \bar{\lambda}, \bar{\gamma}$ that differ from the original ones by the factors, which are formally infinite and come from delta-functions in expressions for torsion and curvature. We assume here, that a certain regularization is made, which makes these factors finite. Our supposition is that after the renormalization each physical quantity may be expressed through physical couplings, which differ from the bare ones (both lattice and continuum), and the infinity encountered here is absorbed into the renormalization factors.

It is easy to understand, that (12) is the symmetry of the action. So, we have lattice model with direct manifestation of Poincare gauge invariance.

5.2 Hypercubic lattice

In order to write down the needed lattice formula, we first drop to the dual lattice. Then our rotation matrices are attached to links while the inverse vierbein is attached to sites. Let us denote by $U_{\mu}(x)$ the matrix correspondent to the link, which begins at the site $x$ and points to the direction $\mu$ ($\mu = \pm 4, \pm 3, \pm 2, \pm 1$). We denote by $\Omega_{\mu\nu}(x) = U_{\mu}(x)...$ the product of link matrices along the boundary of the plaquette, which is placed in the $(\mu\nu)$ plane. The inverse vierbein, which is attached to the site $x$, is denoted as $E^A_{\mu} = E^A_{\mu} - E^A_0, \mu = 1, 2, 3, 4$. For negative values of $\mu$ we define $E^A_{-\mu} = -E^A_{\mu}$. The inverse matrix for positive values of indices is denoted by $E^A_{\mu}(x) = \{E(x)^{-1}\}^\mu_A$. We also expand this definition to negative values of indices: $E^A_{\mu}(x) = \text{sign}(\mu)\{[E(x)^{-1}]^\mu\}_A$. We shall denote by $\Delta x_{\mu}$ the shift on the lattice by one step in the $\mu$-th direction ($\Delta x_{-\mu} = -\Delta x_{\mu}$). So, $x + \Delta x_{\mu}$ is the site which is obtained via the shift from the site $x$ by one lattice spacing in the direction $\mu$ while $x - \Delta x_{\mu}$ is obtained by the shift in the opposite direction. Thus, $U_{-\mu}(x) = U_{\mu}^{-1}(x - \Delta x_\mu)$. Next, we define $\Delta_{\mu}E_{\nu}(x) = U_{\mu}(x)E_{\nu}(x + \Delta x_{\mu}) - E_{\nu}(x)$. Everywhere we imply summation over the repeated indices. The summation over space - time indices $\mu, \nu, ...$ is implied over $\pm 4, \pm 3, \pm 2, \pm 1$.

Further we introduce lattice tetrad components of torsion and curvature:

$$R^{A}_{CFB}(x) = E^\mu_{C}[E^\nu_{F}[\Omega_{\mu\nu}]]^A_B$$
$$R_{FB}(x) = R^{A}_{AFB}(x)$$
$$R(x) = R_{AA}(x)$$
$$T^{A}_{CF}(x) = \Delta_{[\mu}E^\nu_{\nu]}E^\mu_{C}E^\nu_{F}$$

(16)
Then the action has the form
\[
S = \sum_{x \in \text{sites}} \{ \bar{\alpha} R_{FB}(x) R_{FB}(x) + (\bar{\beta} - \frac{1}{3}\bar{\alpha}) R(x)^2 + \bar{\lambda} m_p^4 \\
- \bar{\gamma} m_p^2 R(x) + \bar{\delta} m_p^2 T_{BC}^A(x) T_{BC}^A(x) \} |E(x)|,
\]
where $|E(x)| = \det E$ is the volume of the lattice element correspondent to the site $x$. Again, lattice coupling constants contain infinities discussed above.

6 Problems with measure over discretized geometries

It is already a traditional point of view, that measure over Riemannian geometries should be defined in such a way, that it corresponds to a certain metrics on the space of geometries. In [4] we briefly remind how such a measure is constructed in case of finite dimensional space. It is often implied that this procedure could be extended to the infinite - dimensional space of Riemannian geometries. If so, measure over discretized spaces should be constructed in order to reproduce the correct measure over continuum geometries in the limit of infinite number of lattice elements. While moving in this direction the so - called Lund - Regge measure over Regge skeletons was suggested, which corresponds to the metric, which is induced on the space of Regge skeletons by the given metric on space of continuum geometries. Unfortunately, it appears, that in dimensions greater, than two, it is almost impossible to use this measure in real computer simulations as it is essentially nonlocal. The possible solution was suggested in two dimensions in [4], but it could not be extended to higher dimensions.

Moreover, below we show, that it is not straightforward, that Lund - Regge measure corresponds to the given metric in the limit of large number of lattice elements. We consider finite dimensional analogue of the situation, where definitely there is no such a correspondence. Namely, let $\Omega$ be the set in the coordinate plane $(xy)$: $\Omega = \{(x, y) : 0.5 < x < 1, |y| < 1\}$. Our aim is to calculate integral $\int_{\Omega} dx dy f(x, y)$. Let us now consider the sequence of curves
\[
y(x) = \sin\left(\frac{\pi n}{x}\right)
\]
correspondent to integer numbers $n$. Each curve approximates $\Omega$. It is placed within it more and more dense when $n$ tends to infinity. Therefore, it is
natural to suppose, that it is possible to represent integral over $\Omega$ as the limit of integrals over our curves at $n \to \infty$.

$$\int_{\Omega} dx dy f(x, y) = \lim_{n \to \infty} \int_{0.5}^{1} dx f(x, \sin(\frac{\pi n}{x})) \lambda(x)$$  \hspace{1cm} (19)

Here $\lambda(x) dx$ is the measure on the curve to be defined. This situation is the simplification of the problem of interest, when the given space of continuum geometries is approximated by the space of Regge skeletons. $\Omega$ is the analogue of the space of continuum geometries while each curve (18) is the analogue of a piecewise-continuum geometries with varying link lengths.

Let us suppose, that the measure over $\Omega$ is defined in accordance with the norm, that is $(\delta x)^2 + (\delta y)^2$. The induced norm on the $n$-th curve is $[(\frac{\pi n \cos(\frac{\pi n}{x})}{x^2})^2 + 1](\delta x)^2$. Then logic, which has led to the definition of Lund-Regge measure over discretized Riemannian geometries (seen as Regge manifolds)\(^3\), would lead us to the expression $\lambda(x) = \sqrt{[\frac{\pi n \cos(\frac{\pi n}{x})}{x^2}]^2 + 1}$, which is obviously incorrect!

The problem is that we missed that different pieces of the curve (18) are distributed with variable density within $\Omega$. So, the correct answer would be

$$\lambda(x) = \sqrt{[\frac{\pi n \cos(\frac{\pi n}{x})}{x^2}]^2 + 1} \frac{1}{\rho(x)},$$  \hspace{1cm} (20)

where $\rho$ is the density of the curve inside $\Omega$, which can be estimated as

$$\rho(x) \sim l_\epsilon(x) \sim \frac{\pi n}{x^2},$$  \hspace{1cm} (21)

where $l_\epsilon(x)$ is the length of the curve inside the $\epsilon$-vicinity of $x$ (for $\frac{1}{n} << \epsilon << 1$). This can also be proved by the following sequence of expressions:

$$\int_{\Omega} dx dy f(x, y) \hspace{1cm} = \hspace{1cm} \pi \lim_{n \to \infty} \int_{0.5}^{1} dx f(x, \sin(\frac{\pi n}{x})) \sqrt{[\cos^2(\frac{\pi n}{x}) + \frac{x^4}{\pi^2 n^2}]}
$$

$$\hspace{1cm} = \hspace{1cm} \pi \lim_{n \to \infty} \int_{0.5}^{1} dx f(x, \sin(\frac{\pi n}{x})) |\cos(\frac{\pi n}{x})|
$$

$$\hspace{1cm} = \hspace{1cm} \lim_{n \to \infty} \sum_{k=0,...,n-1} \int_{-1}^{1} dy f(\frac{1}{2 - \frac{k}{n}}, y) \frac{1}{(2 - \frac{k}{n})^2 n}$$  \hspace{1cm} (22)

Here the first expression is the equation (19) with $\lambda$ substituted in the form (20), where $\rho$ given by (21). Constant of proportionality is chosen to be equal
to π. The last expression is simply the discretization of integral over \( x \) with points \( x_k = \frac{1}{2-\frac{k}{n}} \) \( (k = 0, ..., n - 1) \).

So, the lesson of this example is that if we try to calculate measure over discretized geometries using the induced norm, then we may come to the incorrect result. In particular, Lund-Regge measure used in Regge discretizations of Riemannian spaces could be incorrect and further investigation should be made in order to determine there the analogue of density \( \rho \) of (20).

7 Another way to construct measure over discretized geometries

From the previous section we know, that it may not be possible to calculate lattice measure using the expression for the continuum norm. Therefore, we should find another solution. And this solution is right in front of us. Namely, let us remember QCD on the lattice, which is known to work perfectly. In this model there are two kinds of fields.

1. There are quarks and leptons. The correspondent measure over Grassmann variables is well defined and unique.

2. There is the gauge field. The correspondent measure on the lattice is unique as it is completely defined via symmetry properties: this is the local measure invariant under gauge transformations.

So, our solution comes easily. We must find symmetry properties of the continuum measure, which make the measure unique when it is transferred to lattice.

In our case there are two fields: \( O(4) \) connection and the translational connection. So, it is natural to use measures, which are invariant under lattice realization of the gauge transformation. Our choice of measure is measure, which is simultaneously invariant under lattice gauge transformations and is local.

Each piecewise linear manifold described above is itself a Riemann-Cartan space. Let the given discretization (with varying \( E \) and \( U \) be denoted as \( \mathcal{M} \). Then, let \( G_\mathcal{M} \) be the set of correspondent independent variables \( \{ E^A_\mu (I); U_{IJ} \} \). Gauge transformation corresponds to the shift of each lattice element by the vector \( v^A(I) \) and its rotation \( \Theta_I \in O(4) \). This transformation acts as \( \{ E^A_\mu (I); U_{IJ} \} \rightarrow \{ \Theta_I E^A_\mu (I) + v(I)); \Theta_I U_{IJ} \Theta_J^T \} \).

Here by locality of lattice measure we understand the following. The
whole measure should be represented as a product over the sides of lattice elements and over the links, that connect centers of lattice elements with their vertices, of measures over the matrices $U$ and vectors $E$ correspondingly:

$$D_M(E; U) = \prod_I \prod_\mu DE^A_\mu(I) \prod_{IJ} DU_{IJ},$$  \hspace{1cm} (23)$$

We call the lattice measure local if inside each lattice element $DE^A_\mu$ for the given $\mu$ depends upon $E^A_\nu$ only, and $DU_{IJ}$ for the given $I, J$ depends upon $U_{IJ}$ only. It is easy to understand, that this requirement together with gauge invariance fixes the only choice of $DE^A_\mu$ and $DU_{\nu}$: $DE^A_\mu = \Pi_{\nu A} dE^A_\mu$, while $DU$ is the invariant measure on $O(4)$.

We must mention, that another locality principle may be formulated. Say, we may thought that the measure is local if $DE^A_\mu$ may depend upon $E^A_\nu$ with $\nu \neq \mu$ but may not depend upon $E^A_\mu$ from another lattice element. Then gauge invariance does not fix measure precisely. However, our choice is more strong requirement, mentioned above, which gives us opportunity to fix the only local and gauge invariant measure. Future investigation must show is this choice correct or not.

8 Discussion and conclusions.

In this paper we suggest to choose the simplest possible way to quantize pure gravity using Euclidean path integral formalism. The resulting model is well-known in literature. It deals with Riemannian space, and its only dynamical variable is the field of vierbein (or, metrics). After making this “minimal” choice we encounter the following:

1. The model is renormalizable. This means, that it is sensible, at least on the level of perturbation expansion.

2. There exists the region of coupling constants, where the model is asymptotic free. Fortunately, it is this region, where the Euclidean action is bounded below. So, the model can be naturally explored using Euclidean path integral formalism.

3. If we choose bare couplings in such a way, that at low enough energy scale renormalized $\lambda$ and $\gamma$ vanish, then at this scale classical Einstein equations give global minimum to Euclidean action. So, classical limit is achieved in the absence of matter.

4. If we add matter to the given model in a traditional way, then classical Newtonian limit cannot be achieved. Nevertheless, in principle, there exists
the way to overcome this difficulty. Namely, according to original ideas of
Einstein and Infeld, it is possible to consider point-like matter objects
as the singularities of space-time. Then classical solutions around such
singularities would give rise to classical Newtonian potential.

5. The model suffer from loss of unitarity. However, at $\lambda = \gamma = 0$
perturbative ghost disappears. And the only problem is the high derivative
action. There exists the possibility to represent the theory in such a way,
that only first derivatives enter the action. Namely, Riemannian space can
be considered as the limit of Riemann-Cartan space with vanishing torsion.
We consider Poincare connection as the basic variable and do not require
vanishing of torsion from the very beginning. Instead we add to the action
the term, that forces torsion to be close to zero at high enough value of the
new coupling constant. The resulting action contains only first derivatives of
Poincare connection and reproduces the original one, when torsion vanishes
dynamically.

6. In order to give sense to Euclidean path integral formalism one should
put the model onto the lattice and choose the way to determine measure
over dynamical variables. It is well-known, that if we try to construct
measure over Riemannian geometries in such a way, that it corresponds to a
certain simple choice of metric on the space of geometries, then we would get
nonlocal and rather complicated measure over dynamical variables in Regge
discretization of the model. This measure is in fact so complicated, that
real numerical simulations seem to be almost impossible. Moreover, it may
contain such a factor, that we have no idea how to calculate it at all.

7. We suggest to overcome difficulty with measure in discretized gravity
as follows. We again use Riemann-Cartan space instead of Riemannian one.
Our dynamical variable is Poincare connection.

8. Next, we construct manifestly gauge invariant discretization of Rie-
mann-Cartan space, which may use rectangular or simplicial lattices. Poincare
Gauge transformation naturally acts on the dynamical variables of the con-
structed lattice model. We use gauge invariance together with the lattice
locality principle in order to determine measure over the dynamical vari-
ables. Such a measure is shown to exist. Moreover, our definition of locality
fixes only one gauge invariant measure.

9. Finally, we have manifestly gauge invariant discretization, local mea-
sure, and the action, which does not contain second derivatives. The model,
therefore, is expected to be useful for numerical simulations.

The author is grateful to S.Kofman for numerous discussions of vari-
ous theoretical problems and to A.I.Veselov for kind support. This work was partly supported by RFBR grants 03-02-16941, 05-02-16306, and 04-02-16079, by Federal Program of the Russian Ministry of Industry, Science and Technology No 40.052.1.1.1112.

References

[1] M.A. Zubkov, Phys. Lett. B 582, 243 (2004);

[2] A. Patrascioiu, E. Seiler, and I.O. Stamatescu, Phys. Lett. B 107, 364 (1981);
E. Seiler, I.O. Stamatescu, and D. Zwanziger, Nucl. Phys. B 239, 177 (1984);
Y. Yotsuyanagi, Phys. Lett. B 135, 141 (1984);
K. Cahill, S. Prasad, R. Reeder, and B. Richter, Phys. Lett. B 181, 333 (1986);
K. Cahill, Phys. Lett. B 231, 294 (1989)

[3] J. Ambjorn, J. Nielsen, J. Rolf, G. Savvidy, Class.Quant.Grav. 14 (1997) 3225-3241

[4] M.A. Zubkov, Phys. Lett. B 616, 221 (2005);

[5] E.T.Tomboulis, Phys. Rev. Lett. 52, 1173 (1984)

[6] K.S.Stelle, Phys. Rev. D 16 , 953 (1977)

[7] I. G. Avramidi, Soviet Journal of Nuclear Physics, 44 (1986) 160-164, ”Heat Kernel and Quantum Gravity”, Lecture Notes in Physics, Series Monographs, LNP: m64 (Berlin: Springer-Verlag, 2000), [hep-th/9510140]
I. G. Avramidi and A. O. Barvinsky, Physics Letters B, 159 (1985) 269-274

[8] L.Infeld, Rev. Mod. Phys. 29, 398 (1957)

[9] K.Hayashi, T.Shirafuji, Phys. Rev. D19, 3524 (1979)
Yu. N. Obukhov, J. G. Pereira, Phys.Rev. D67 (2003) 044016
V. C. de Andrade, J. G. Pereira, Phys.Rev. D56 (1997) 4689-4695