Abstract

We combine earlier investigations of linear systems with Lévy fluctuations [Physica 113A, 203, (1982)] with recent discussions of Lévy flights in external force fields [Phys.Rev. E 59,2736, (1999)]. We give a complete construction of the Ornstein-Uhlenbeck-Cauchy process as a fully computable model of an anomalous transport and a paradigm example of Doob’s stable noise-supported Ornstein-Uhlenbeck process. Despite the nonexistence of all moments, we determine local characteristics (forward drift) of the process, generators of forward and backward dynamics, relevant (pseudodifferential) evolution equations. Finally we prove that this random dynamics is not only mixing (hence ergodic) but also exact. The induced nonstationary spatial process is proved to be Markovian and quite apart from its inherent discontinuity defines an associated velocity process in a probabilistic sense.
1 Motivation

The casual understanding of the central limit theorem (in reference to the Boltzmann-Gibbs thermostatics), combined with the need to have clearly specified the mean features (moments and local conservation laws) of randomly implemented transport, at or off thermal equilibrium, resulted in an obvious predominance of Gaussian laws of probability and diffusion processes in typical statistical (eventually probabilistic, cf. the omnipresence of the Brownian motion conceptual background) analysis of physical phenomena.

Presently, we observe a continually growing recognition of the profound rôle (ubiquity, [1]) played by non-Gaussian Lévy distributions (probability laws) in both a consistent probabilistic interpretation of various experimental data and in a stochastic modelling of physical phenomena, followed by numerical and realistic experimentation attempts to verify (or rather falsify) probabilistic hypotheses.

Generically, Lévy’s probability laws appear in the context of anomalous diffusions (mostly subdiffusions that are modelled in terms of continuous random walks, [2]). On the other hand, under the name of Lévy flights, [2, 3] we encounter stochastic jump-type processes which are explicitly associated with those distributions. That allows in turn to model quite a variety of transport processes, cf. [3] which are either regarded as (non)typical phenomena of nonequilibrium statistical physics or as manifestations of a complex nonlinear dynamics with signatures of chaos, yielding an enhanced diffusion in particular.

We focus our attention on Lévy flights which are considered as possible models of primordial noise, [4, 5]. (Wiener noise or process is normally interpreted to represent a statistical ”state of rest” of the random medium). Generically, the variance and higher cumulants of those processes are infinite (nonexistent). There is also physically more singular subclass of such processes for which even the first moment (mean value) is nonexistent. Thus we need to relax the limitations of the standard Gaussian paradigm: we face here a fundamental problem of establishing other means (than variances and mean values) to characterise statistical properties of Lévy processes (frac-
tional moments of Ref. [6] are insufficient tools in this respect).

Specifically, if a habitual statistical analysis is performed on any experimentally available set of frequency data, there is no obvious method to extract a reliable information about tendencies (local mean values) of the random dynamics. Nonexistence of mean values and higher moments may also be interpreted as the nonexistence of observable (e.g. mean, like drifts or local currents) regularities of the dynamics. Moreover, the jump-type processes usually admit arbitrarily small jumps (with no lower bound) and finite, but arbitrarily large jump sizes (with no upper bound). Any computer simulation utilizes both the lower (coarse-graining) and upper bound on the jump size, [3, 4, 11], and any experimental data collection involves such limitations as well. Mathematically, that puts us in the framework of standard jump processes for which the central limit theorem is known to hold true in its Gaussian version (even if we account for the abnormally slow convergence to a Gaussian, in view of long tails of the probability distribution, [8]). Therefore, there is no clear-cut procedures allowing to attribute an unambiguous statistical interpretation in terms of Lévy processes to given phenomenological data. In a drastic contrast to a traditional Gaussian modelling. Mere scaling arguments, reflecting the self-similiar patterns of sample paths, are insufficient as well.

Although no realistic formulation of a fluctuation-dissipation theorem is possible in that case (nonexistence of variances), we can give a meaning to a theory of Lévy flights in external force fields, [3], under a simplifying assumption that force fields define linear processes with Lévy fluctuations. The corresponding velocity processes were introduced in Ref. [3] ( see also [11]), but we shall give a complete construction of the related jump-type stochastic process, together with a detailed characterization of the dynamics of induced spatial displacements. Our strategy is thus substantially different from that typically followed in the current literature, [3]. For example, the configuration space Langevin equation,

\[ \frac{d\mathbf{x}(t)}{dt} = \frac{\mathbf{F}(\mathbf{x})}{m\gamma} + \eta(t) \]

where \( m \) is the mass of transported particle, \( \gamma \) stands for the friction con-
stant and $\eta$ represents any conceivable generalization of the white-noise that employs Lévy stable statistics, [3], corroborates a tacit assumption that some kind of the the standard Smoluchowski projection (the large friction limit, normally employed in the Brownian motion context, [12]) from the phase-space to spatial only dynamics is possible. This is certainly not realizable in the non-Gaussian Lévy case.

Another delicate question is to settle possible physical origins of the spatial noise. That issue seems to be conceptually easier to handle on the velocity/momentum space level. However, another delicate problem is immediate: in case of the Brownian motion (Ornstein-Uhlenbeck process) spatial trajectories were by construction differentiable to give meaning to the velocity concept (even though accelerations were nonexistent anyway, somewhat conflicting with the naive but widely spread usage of the white noise-supported Langevin equation as the second Newton law analogue). The consequent exploitation of Lévy processes with their intrinsic discontinuities, seems to set an unresolvable obstacle in this (velocity notion) respect.

We shall demonstrate that this is not literally the case. For example, we can prove that the spatial random variable $x(t)$ of the Ornstein-Uhlenbeck-Cauchy process cannot possess derivatives in the sense of standard mathematical analysis, nonetheless this process has derivatives in a weaker, probabilistic sense. Hence, it is legitimate to interpret $u(t)$ as a velocity analogue attributed to the instantaneous (random) location $x(t)$, though this notion is more distant from classical intuitions than the velocity variable of the standard Ornstein-Uhlenbeck process (not differentiable, hence not yielding any analogue of a Newtonian acceleration).

Our analysis departs from a generalization of the Ornstein-Uhlenbeck process due to Doob, [7], where a symmetric Lévy (stable) noise was assumed to take place of the standard Wiener noise. A complete description of a concrete, computable in full detail Ornstein-Uhlenbeck-Cauchy process (with a familiar lorentzian as a probability law for velocity displacements) is our principal goal in the present paper. In addition we shall pay attention to intrinsic complications of the random dynamics by investigating a standard chain of its possible features (ordered with respect to the complication level):
ergodicity, mixing and exactness.

2 Langevin equation with a linear (harmonic) force and Cauchy noise

The starting point for Ornstein and Uhlenbeck,[13, 14], was the dissipative Langevin equation

\[ \frac{du}{dt} = -\lambda u(t) + A(t) \]  

(2)

where \( u(t) \) is a random variable describing the velocity of a particle, \( \lambda > 0 \) is a dissipation constant, and \( A(t) \) is another random variable whose probabilistic features are determined by the probability distribution of \( u(t) \), which is assumed to satisfy a concrete law when \( t \to \infty \). Because \( u(t) \) may have no time derivative, equation (2) was soon replaced by another one, namely

\[ du(t) = -\lambda u(t)dt + dB(t), \quad u(0) = u_0 \]  

(3)

which received a rigorous interpretation within the framework of stochastic analysis [4]. In the case when the probability distribution of \( u(t), t \to \infty \), is the Maxwell one, \( B(t) \) must be a Gaussian process, and the formula (3) leads to the classical Ornstein-Uhlenbeck process.

Here, we discuss properties of the process \( u(t) \), and the corresponding process of displacements \( x(t) \), in the case when \( B = (B(t))_{t \geq 0} \) is the Cauchy process, that is when \( B \) satisfies the following conditions:

a) \( B \) has independent increments, i.e. given \( t_1 < ... < t_n \), the differences \( B(t_2) - B(t_1), B(t_3) - B(t_2),..., B(t_n) - B(t_{n-1}) \) are mutually independent random variables,

b) \( B \) has stationary increments, i.e. the probability distribution of \( B(t + \tau) - B(\tau) \) is independent of \( \tau \),

c) \( B \) is continuous in probability, that is \( \lim_{t \to s} B(t) = B(s) \) in probability,

d) the characteristic function of \( B \) is given by

\[ E[e^{ipB(t)}] = e^{-t\psi(p)} \]

where \( \psi(p) = \sigma^2|p| \).
All the above requirements form a mathematically consistent definition of the Markovian jump-type process in question, e.g. Cauchy process. Notice that a suitable modification of the condition d) (set $\psi(p) = -\alpha^2 p^2$ in the exponent; we refer to the general form of the Lévy-Khintchine formula) would leave us with the familiar Wiener process.

From a physical point of view, solutions of induced partial (here, pseudo-) differential equations are most important, and those incorporate transition probability densities and densities of the process.

Notice that the process of displacements is determined by $u(t)$ in the standard way

$$x(t) = x(0) + \int_0^t u(\tau) d\tau, \quad x(0) = x_0$$

(4)

Hence, we should be able to derive relevant densities and transition densities not only for the velocity process but also for the induced spatial process. Additionally, if we wish to interpret $u(t)$ as a genuine velocity field for the process of spatial displacements $x(t)$ (the mere formal attribution of the velocity name to our random variable $u(t)$ is highly misleading, in view of an apparent discontinuity of sample paths), a careful analysis of differentiability properties (in what sense?) of $x(t)$ is here necessary.

By integrating equation (2) we obtain that for $t \geq s$

$$u(t) = e^{-\lambda(t-s)}u(s) + e^{-\lambda t} \int_s^t e^{\lambda \tau} dB(\tau)$$

(5)

The integration of (3) yields

$$x(t) = x(s) + \int_s^t [e^{-\lambda(\tau-s)}u(s) + e^{-\lambda \tau} \int_s^\tau e^{\lambda \beta} dB(\beta)]d\tau$$

$$= x(s) + \frac{1 - e^{-\lambda(t-s)}}{\lambda} u(s) - \int_s^t d\tau \left( \frac{e^{-\lambda \tau}}{\lambda} \right)' \int_s^\tau e^{\lambda \beta} dB(\beta)$$

Integrating the last summand by parts and using the double integration formula involving a derivative with respect to the interior integral, cf. (3.12)
in [4], we get
\[
\int_s^t \frac{e^{-\lambda \tau}}{\lambda} \, d\tau \int_s^\tau e^{\lambda\beta} dB(\beta) = \frac{e^{-\lambda t}}{\lambda} \int_s^t e^{\lambda \tau} dB(\tau) - \frac{1}{\lambda} \int_s^t dB(\tau)
\]
Hence
\[
x(t) = x(s) + \frac{1 - e^{-\lambda(t-s)}}{\lambda} u(s) + \int_s^t \frac{1 - e^{-\lambda(t-\tau)}}{\lambda} dB(\tau)
\]
which mimics (is identical with respect to the form) a fairly traditional expression for a spatial random variable of the standard Ornstein-Uhlenbeck process (with Wiener increments put instead of the Cauchy increments in the last summand).

3 Probability densities and transition probability densities for \(u(t)\) and \(x(t)\)

There is a number of (equivalent) procedures to deduce a probability density of the process \(u(t)\) from the Cauchy increments statistics, see [3, 4]. We shall follow a direct probabilistic route.

In order to simplify the notation we write \(P[X = x]\) for the density of the probability distribution of a random variable \(X\), that is \(P[X \in \Gamma] = \int_{\Gamma} P[X = x] \, dx\) for \(\Gamma \subset \mathbb{R}\).

Suppose that \(f\) is a continuously differentiable function such that \(f(\tau) \geq 0\), and let \(X = \int_s^t f(\tau) dB(\tau)\). Terms of this functional form are encountered in formulas (4) - (6).

The random variable \(X\) is the limit of the following sum
\[
\sum_{k=0}^{n-1} f(\tau_k)[B(\tau_{k+1}) - B(\tau_k)]
\]
where \(s = \tau_0 < \tau_1 < \ldots < \tau_n = t\) is the partition of the interval \([s, t]\).

Because the process \(B\) has independent increments, the probability density of (7) is the convolution of densities of its summands which, since the process has stationary increments, are equal to
\[
\frac{1}{\pi f(\tau_k)} \frac{\sigma^2(\tau_{k+1} - \tau_k)}{(f(\tau_k))^2 + \sigma^2(\tau_{k+1} - \tau_k)^2} = \frac{1}{\pi} \frac{\sigma^2 f(\tau_k) \Delta \tau_k}{x^2 + (\sigma^2 f(\tau_k) \Delta \tau_k)^2}
\]
Because the Fourier transform maps the convolution to multiplication and
\[
\left( \frac{1}{\pi} \frac{\sigma^2 f(\tau_k) \Delta \tau_k}{x^2 + (\sigma^2 f(\tau_k) \Delta \tau_k)^2} \right)^\wedge(p) = e^{-\sigma^2 |p| f(\tau_k) \Delta \tau_k}
\]
we get
\[
P\left[\left( \sum_{k=0}^{n-1} f(\tau_k) [B(\tau_{k+1}) - B(\tau_k)] \right) = x \right]
= \prod_{k=0}^{n-1} e^{-\sigma^2 |p| f(\tau_k) \Delta \tau_k} \wedge(x)
= (\exp(-\sigma^2 |p| \sum_{k=0}^{n-1} f(\tau_k) \Delta \tau_k)) \wedge(x)
\]
where \(f^\wedge\) and \(f^\vee\) denote the Fourier transform and its inverse respectively. By taking the limit \(n \to \infty\) we obtain that
\[
P[X = x] = (\exp(-\sigma^2 |p| \int_s^t f(\tau) d\tau)) \vee(x)
\]
and so the general formula
\[
P[X = x] = \frac{1}{\pi} \frac{\sigma^2 \int_s^t f(\tau) d\tau}{x^2 + (\sigma^2 \int_s^t f(\tau) d\tau)^2}
\]  
(8)
is valid. We shall exploit Eq. (8) repeatedly in below.

Remark 1: An apparent generalization of the previous observation is possible. Assume that \(B(t)\) is a Lévy stable process with the characteristic function
\[
\psi(p, t) = \exp(-\sigma^2 t |p|^\alpha), \quad 0 < \alpha \leq 2
\]
Then, there holds
\[
P[\int_s^t f(\tau) dB(\tau) = x] = (\exp(-\sigma^2 (\int_s^t f^\alpha(\tau) d\tau) |p|^\alpha)) \vee(x)
\]
Presently we shall use the formula (8) to calculate transition probability densities of processes \(u(t)\) and \(x(t)\).
Let \( f(\tau) = e^{-\lambda(t-\tau)} \). Then, by equation (5),

\[
P[u(t) = u | u(s) = v] = \frac{1}{\pi} \frac{\sigma^2(t-s)}{(u - ve^{-\lambda(t-s)})^2 + \sigma^4(t-s)}
\]

where \( \sigma^2(t-s) = \frac{e^2}{\lambda}(1 - e^{-\lambda(t-s)}) \), see e.g. also [6].

Since \( u(0) = u_0 \), the probability density of \( u(t) \) is given by

\[
P[u(t) = u] = \frac{1}{\pi} \frac{\sigma^2(t)}{(u - u_0e^{-\lambda t})^2 + \sigma^4(t)}
\]

We now turn to the process \( x(t) \). Since \( u(s) \) is independent of \( B(t) \) for all \( t \geq s \) [7], it is also independent of the integral \( \int_s^t f(\tau)dB(\tau) \). Therefore, the probability distribution of the sum

\[
\frac{1 - e^{-\lambda(t-s)}}{\lambda}u(s) + \int_s^t \frac{1 - e^{-\lambda(t-\tau)}}{\lambda}dB(\tau)
\]

is the convolution of its ingredients.

Let \( f(\tau) = \frac{1-e^{-\lambda(t-s)}}{\lambda} \). Because of

\[
\int_s^t f(\tau)d\tau = \frac{1}{\lambda^2}(e^{-\lambda(t-s)} - 1 + \lambda(t-s))
\]

by formula (8) there holds

\[
P\left[ \frac{1 - e^{-\lambda(t-s)}}{\lambda}dB(\tau) = x \right] = \frac{1}{\pi} \frac{(\frac{\sigma}{\lambda})^2(e^{-\lambda(t-s)} - 1 + \lambda(t-s))}{x^2 + (\frac{\sigma}{\lambda})^4(e^{-\lambda(t-s)} - 1 + \lambda(t-s))^2}
\]

On the other hand, by (10), we have

\[
P\left[ \frac{1 - e^{-\lambda(t-s)}}{\lambda}u(s) = u \right] = \frac{1}{\pi} \frac{\sigma^2(s)a(t-s)}{(u - u_0e^{-\lambda s}a(t-s))^2 + \sigma^4(s)a^2(t-s)}
\]

where \( a(t-s) = \frac{1-e^{-\lambda(t-s)}}{\lambda} \).

The Fourier transform of (11) and (12) are equal to

\[
\exp[-(\frac{\sigma}{\lambda})^2(e^{-\lambda(t-s)} - 1 + \lambda(t-s))]|p|
\]

and

\[
\exp[-\sigma^2(s)a(t-s)|p] \exp[-iu_0e^{-\lambda s}a(t-s)p]
\]
respectively.

Because of
\[
(\frac{\sigma}{\lambda})^2(e^{-\lambda(t-s)} - 1 + \lambda(t-s)) + \sigma^2(s)a(t-s) = (\frac{\sigma}{\lambda})^2(e^{-\lambda t} - e^{-\lambda s} + \lambda(t-s))
\]
the multiplication of transforms (13), and (14) followed by taking the inverse Fourier transform of the result, gives us a transition probability density of the spatial process

\[
P[\mathbf{x}(t) = y|\mathbf{x}(s) = x] = p(y, t|x, s) = \frac{1}{\pi} \frac{g(t, s)}{(y - x - u_0 h(t, s))^2 + g^2(t, s)}
\]

(15)

where
\[
g(t, s) = (\frac{\sigma}{\lambda})^2(e^{-\lambda t} - e^{-\lambda s} + \lambda(t-s))
\]

and
\[
h(t, s) = \frac{e^{-\lambda s} - e^{-\lambda t}}{\lambda}
\]

Finally, because \(\mathbf{x}(0) = x_0\), the probability density of the process \(\mathbf{x}(t)\) is given by

\[
P[\mathbf{x}(t) = x] = \frac{1}{\pi} \frac{(\frac{\sigma}{\lambda})^2(e^{-\lambda t} - 1 + \lambda t)}{(x - x_0 - u_0 \frac{1-e^{-\lambda t}}{\lambda})^2 + (\frac{\sigma}{\lambda})^4(e^{-\lambda t} - 1 + \lambda t)^2}
\]

(16)

Compare e.g. the corresponding formula for the displacements of the standard (Wiener noise-supported) Ornstein-Uhlenbeck process, \[12\].

4 Properties of the process \(u(t)\)

Considerations of the previous sections may leave us with an impression that a construction of the Ornstein-Uhlenbeck process supported by Cauchy noise is in fact complete. We have in hands not only Itô type stochastic differential equations that are amenable to direct computer simulations, \[10\], \[9\], but also explicit expressions for probability densities and transition probability densities for both processes: \(u(t)\) and \(x(t)\). In case of Markov processes such data are known to specify the process uniquely, \[15\].

However, some alarm bells need to switched on at this point. The standard (stationary) Ornstein-Uhlenbeck velocity process is Markovian (in the
Gaussian case the Ornstein-Uhlenbeck process is the only continuous in probability stationary Markov process, but the induced (integrated) spatial process is not Markovian. Using an explicit expression for the transition probability density it is easy to verify that the Chapman-Kolmogorov identity does not hold true. Therefore, Markov property is normally attributed to a two-component, phase-space version of the Ornstein-Uhlenbeck process. In case of the Ornstein-Uhlenbeck-Cauchy process the situation is somewhat different.

4.1 Markovianess and stochastic continuity

First of all let us notice that \( u(t) \) is a time-homogeneous (but not stationary) Markov process. Markov property is clear from the construction since Chapman-Kolmogorov identity can be verified by inspection and it is a classic observation that nonnegative and normalized functions which obey the Chapman-Kolmogorov equation are necessarily Markovian transition probabilities.

Since the probability density (10) of the process depends explicitly on time, our process \( u(t) \) is not stationary.

**Remark 2:** That needs to be contrasted with the standard (Gaussian and stationary) Ornstein-Uhlenbeck process features where the transition probability density is time-homogeneous, while the density of the process does not depend on time at all. Indeed (we consider one spatial dimension and utilize dimensional units) the transition density

\[
p(y, t|x, s) = \left( \frac{\gamma}{2\pi D} \right)^{1/2} \cdot \exp\left( -\frac{\gamma\{x - y\exp[-\gamma(t - s)]\}^2}{2D\{1 - \exp[-2\gamma(t - s)]\}} \right)
\]

with \( s < t \), has an invariant density \( \rho(x) = \left( \frac{\gamma}{2\pi D} \right)^{-1/2} \cdot \exp(-\gamma x^2/2D) \).

The drift of the process reads \( b(x) = -\gamma x \) and \( p \) solves the Fokker-Planck (second Kolmogorov) equation \( \partial_t p = D\Delta_x p - \nabla_x(bp) \).

Now, we shall demonstrate an important property (mentioned before in connection with the Ornstein-Uhlenbeck process) of the so-called stochastic continuity which is a necessary condition to give a stochastic process an
unambiguous status, \[15, 17, 18\]. Namely, we need to show that for any \(\epsilon > 0\) the following equation is satisfied

\[
\lim_{t \to s} P[|u(t) - u(s)| \geq \epsilon] = 0
\]  

(17)

This equation is equivalent to

\[
\lim_{t \to 0} \int_{|u-v| \geq \epsilon} p_t(u|v)du = 0
\]  

(18)

Because of

\[
\int_{|u-v| \geq \epsilon} p_t(u|v)du = 1 - \frac{1}{\pi}[\arctan \frac{\epsilon + v(1 - e^{-\lambda t})}{\sigma^2(t)} + \arctan \frac{-\epsilon - v(1 - e^{-\lambda t})}{\sigma^2(t)}]
\]

and remembering that \(\sigma^2(t) = \frac{\sigma^2}{\lambda}(1 - e^{-\lambda t})\), the stochastic continuity property does follow.

It is perhaps not useless to emphasize that in typical Gaussian process investigations, stochastic continuity of the process is a necessary (but still insufficient) condition for the process to have continuous sample paths. Hence it is always explicitly mentioned in the context of diffusion processes, \[15\]. The Cauchy noise-supported process is surely not diffusive and its trajectories are discontinuous (jump-type) paths, \[4, 5\].

### 4.2 Local moments in the Cauchy case: forward drift – the existence issue

The nonexistence of moments of the probability measure in case of the Cauchy process is another source of difficulties, since the standard local characteristics of the diffusion-type process like the drift and the diffusion function (or coefficient) seem to be excluded in the present case.

However, for the considered Ornstein-Uhlenbeck-Cauchy process, the notion of the forward drift of the process proves to make sense (!). We shall first discuss the drift issue for the process \(u(t)\).

Let us start with the following definition. Suppose \(p(y, t|x, s)\), \(t \geq s\), is a Markov transition function and let \(X_t\) be the associated Markov process.
Guided by the analogy with diffusion processes we say that the process $X_t$ has a drift (in fact, forward drift) if the following limit
\[
\lim_{t \to s} \frac{1}{t-s} \int_{|y-x| \leq \epsilon} (y - x)p(y, t|x, s)dy
\] (19)
does not depend on the choice of $\epsilon > 0$. If so, then its value depending only on $(x, s)$ we denote by $b(x, s)$ and call it the drift coefficient.

Clearly, if $p$ is homogeneous in time, then the drift coefficient depends only on the variable $x$. Let us emphasize that in the above definition we do not require the process $X_t$ to have finite moments.

We claim that the jump-type Markov process $u(t)$ has a (forward) drift which reads $b(v) = -\lambda v$.

At first we calculate the indefinite integral
\[
I = \frac{1}{\pi} \int (u - v) \frac{\sigma^2(t)du}{(u - ve^{-\lambda t})^2 + \sigma^4(t)}
\]
Substituting $z = u - ve^{-\lambda t}$, we rewrite that integral as
\[
\frac{\sigma^2(t)}{\pi} \int \frac{zdz}{z^2 + \sigma^4(t)} + \frac{v}{\pi}(e^{-\lambda t} - 1) \int \frac{\sigma^2(t)dz}{z^2 + \sigma^4(t)}
\]
\[
= \frac{\sigma^2(t)}{2\pi} \log(z^2 + \sigma^4(t)) + \frac{v}{\pi}(e^{-\lambda t} - 1) \arctan\left(\frac{z}{\sigma^2(t)}\right)
\]
Hence
\[
I = \frac{\sigma^2(t)}{2\pi} \log[(u - ve^{-\lambda t})^2 + \sigma^4(t)] + \frac{v}{\pi}(e^{-\lambda t} - 1) \arctan\left(\frac{u - ve^{-\lambda t}}{\sigma^2(t)}\right)
\]
and consequently the limit
\[
\lim_{t \to 0} \frac{1}{t} I|_{u=v+\epsilon} = \lim_{t \to 0} \frac{1}{t} \frac{\sigma^2(t)}{2\pi} \log[(v + \epsilon - ve^{-\lambda t})^2 + \sigma^4(t)] - \log[(v - \epsilon - ve^{-\lambda t})^2 + \sigma^4(t)]
\]
\[
+ \lim_{t \to 0} \frac{1}{t} \frac{v}{\pi}(e^{-\lambda t} - 1) \arctan\left(\frac{v + \epsilon - ve^{-\lambda t}}{\sigma^2(t)}\right) - \arctan\left(\frac{v - \epsilon - ve^{-\lambda t}}{\sigma^2(t)}\right)
\]
exists and is $\epsilon$-independent. This is the forward drift of the process $u(t)$ which proves a consistency of the derived transition probability density with the stochastic differential equation (5).

To our knowledge, such consistency check has never been performed before in discussions of Lévy flights and anomalous diffusion processes.

On the other hand, it is well known that for Markovian diffusion processes all local characteristics of motion (conditional expectation values that yield drifts and variances) are derivable from transition probability densities, supplemented (if needed) by the density of the process, cf. \[15\]. We have demonstrated that, in the non-Gaussian context, the nonexistence of moments does not necessarily imply the nonexistence of local characteristics (drifts) of the process.

As a consequence, once a formal definition is adopted of a stochastic differential equation whose deterministic driving term (functionally unrestricted drift) is subject to perturbations by Lévy flights, the process may still possess local characteristics (forward drift) that are in turn derivable by means of its transition density. Our derivation in the Cauchy noise case is limited to linear functions of random variables (linear systems, \[6, 3\]). Possible generalizations to stochastic differential equations with driving terms represented by nonlinear and possibly time-dependent functions need to be carefully examined.

This is an uncomfortable situation, since a formal computer experimentation may not indicate any inconsistency of the formalism. Even worse, the uncommented visualization effectively may convey misleading or entirely wrong messages if uncritically accepted. (The rigorous existence theorems available in the mathematical literature pertain to linear systems as well, \[19, 20\] and extend to perturbations by general Lévy processes.)
4.3 Markov generators and Kolmogorov (Fokker-Planck type) equations

Once densities and transition probability densities have been obtained from the first principles, we can invert the problem (that is a commonly shared viewpoint in the physics-oriented research) and ask for differential (evolution) equations obeyed by them. The Fokker-Planck equation is an obvious example in case of Markovian diffusion processes, while various forms of the Master equation were adopted to extend the standard jump-processes (Poisson or more generally-point processes) framework to more singular step or jump-type ones.

In the case of unperturbed (free) Lévy processes basically all interesting (covering stable laws of probability) evolution equations were classified by means of Fourier transform techniques, [6, 21, 3, 22, 4]. A disregarded point was that in case of Markov processes a single (Fokker-Planck or Master equation - type) evolution equation does not characterise the process uniquely. Both forward and backward evolution equations need here to be involved, cf. [10, 18]. Except for Refs. [6, 3] no attempt was made to investigate such equations for deterministically driven Lévy systems.

To elucidate that issue, we shall next consider the generator of a Markov transition function \( p_t(y|x) \) for the Cauchy-perturbed process (cf. the previous Section).

Let us recall that it is defined by

\[
(Lf)(x) = \lim_{t \to 0} \frac{1}{t} \left[ \int_{-\infty}^{\infty} p_t(y|x) f(y) dy - f(x) \right] \tag{20}
\]

where the domain of definition consists of all functions \( f \in C_0(\mathbb{R}) \), whose limit on the right hand side in (19) exists uniformly with respect to the variable \( x \).

It is worth noting that when the transition function is stochastically continuous (see the previous section), then the corresponding semigroup \( T_t \) in \( C_0(\mathbb{R}) \) defined by

\[
(T_t f)(x) = \int_{-\infty}^{\infty} p_t(y|x) f(y) dy \tag{21}
\]
is strongly continuous, and so its generator $L$ is densely defined.

In such a case we can also define an adjoint semigroup $T_t^*$ acting on the space of (probability) densities $L^1(\mathbb{R}, dx)$,

$$(T_t^*\rho)(u) = \int_{-\infty}^{\infty} p_t(u|v)\rho(v)dv$$ (22)

Its generator we denote by $L^*$.

Arguments of the present section involve a little bit of a mathematical formalism to stay in conformity with the classic work of Feller and Dynkin on evolution equations for Markov processes, cf. [5, 18] for references.

Suppose $L$ is the generator of the semigroup associated with the process $u(t)$ and let $L^*$ be its adjoint.

We wish to demonstrate that

$$L = L_0 + b\nabla$$ (23)

and

$$L^* = L_0 - \nabla(b\cdot)$$ (24)

where $L_0$ is the generator of the Cauchy process $B$ (we have used an explicit notation $L_0 = |\nabla|$ in Refs. [4, 5], see also [3]) and $b(v) = -\lambda v$.

To this end, we first observe that for $p_t(u|v)$ given by the formula (9) the associated semigroup $T_t$ maps $C_0(\mathbb{R})$ to $C_0(\mathbb{R})$. Since $p_t(u|v)$ is stochastically continuous, $T_t$ is strongly continuous.

Next, we calculate the Fourier transform of equation (20).

$$(Lf)^\wedge(p) = \lim_{t \to 0} \frac{1}{t} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sigma^2(t)}{\pi (u - ve^{-\lambda t})^2 + \sigma^4(t)} f(u)e^{-ipv}dudv - \hat{f}(p) \right]$$

Substituting $z = ve^{-\lambda t} - u$, $dv = e^{\lambda t}dz$, we obtain that

$$(Lf)^\wedge(p) = \lim_{t \to 0} \frac{1}{t} \int_{-\infty}^{\infty} dz \frac{1}{\pi} \exp(-izp) \frac{\sigma^2(t)}{z^2 + \sigma^4(t)} \int_{-\infty}^{\infty} du \exp(-ius) f(u) - \hat{f}(p)$$
By taking the inverse Fourier transform and using the identity

\[(p \hat{f'})^\vee(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipv} p \hat{f'}(p) dp = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{ipv})' \hat{f}(p) dp\]

we arrive at

\[Lf(v) = L_0 f(v) - \lambda v f'(v)\]

where \(L_0 = -\sigma^2 |\nabla|\).

Hence \(L = L_0 + b \nabla\).

Because of

\[L^*(\rho)(u) = \lim_{t \to 0} \frac{1}{t} \int_{-\infty}^{\infty} p_t(u|v) \rho(v) dv - \rho(u)\]

so, by similar calculations as above, we obtain

\[L^*(\rho)(u) = L_0 \rho(u) - \nabla(b(u) \rho(u))\]

That ends the demonstration.

As a consequence, since \(T_t : C_0(\mathbb{R}) \to C_0(\mathbb{R})\) is strongly continuous, almost all paths of the process \(u(t)\) are cadlag, that is they are continuous from the right and have finite left-hand limits (see chap.II sec.4 in [17], vol.II).

There follows also from Eqs. (23) and (24) that the transition probability function of the process \(u(t)\) satisfies the backward equation

\[\frac{\partial p_t(u|v)}{\partial t} = L_0 p_t(u|v) + b(v) \nabla_v p_t(u|v)\]

and the forward equation (the Fokker-Planck equation analogue)

\[\frac{\partial p_t(u|v)}{\partial t} = L_0 p_t(v|u) - \nabla_u [b(u)p_t(u|v)]\]
for definitions see, for example, chap. 15 sec. 4 in [16].

It is trivial to check by inspection that the transition probability function of the (free) Cauchy process obeys both those equations with $b(u)$ set equal identically 0.

4.4 Asymptotics: ergodicity, mixing, exactness

We have in hands an explicit expression for the density of the process $u(t)$. One of efficient ways to investigate the complexity of the involved random dynamics is not necessarily via a direct recourse to sample paths, but rather via studying asymptotic properties of probability densities, cf. [25].

Let us consider the asymptotic properties of the process $u(t)$. By direct calculations we check that the density

$$\rho_0(u) = \frac{1}{\pi u^2 + \sigma^4} = \frac{\sigma^2/\lambda}{u^2 + \sigma^4/\lambda^2}$$

is stationary with respect to the dual semigroup $T_t^\ast$. Therefore, irrespectively of its initial probability distribution, $P[u(\infty) = u] = \rho_0(u)$.

**Remark 3**: Normally, if we get a convergence of a density in the asymptotic ($t \to \infty$) regime to a unique density, we say about an asymptotic stability. In case when for every initial density we get a set spanned by a finite number of densities, we say about an asymptotic periodicity. We may also have a situation that every initial density is dispersed under the action of a Markov operator. That is related to the concept of sweeping.

Our knowledge of the explicit formula of the transition probability density for the semigroup $T_t^\ast$ allows us to examine its ergodic properties in a more detailed way. For example, a point-wise convergence of the Cauchy process transition density to the stationary Cauchy probability density was investigated in [3].

Let us recall (see [2] for the definition and more details) that a Markov semigroup $T_t^\ast$ is mixing, if for any density $\rho$, $T_t^\ast \rho$ tends to a stationary density in a weak sense, and exact, if this limit holds in the $L^1$-norm.
Hence exactness is a stronger property and implies mixing, ergodicity of the dynamics being a straightforward consequence.

(It might be worth noting that strong mixing properties of the standard Ornstein-Uhlenbeck velocity fields were discussed and visualized by computer simulations in [23]. The point-wise convergence of the probability density of the process to its stationary limit was established in [24].)

The dynamics induced by Cauchy noise (and other stable noises) shows higher level of complications and is not only mixing, but also exact. We shall provide a demonstration of this property in the spirit of Ref. [25].

In fact, what we claim is that $T^*_t$ is exact (hence both mixing and ergodic).

Since $\rho_0$ is a stationary density we have to show that

$$\lim_{t \to \infty} \|\rho_t - \rho_0\|_1 = 0$$

where $\rho_t(u) = \int p_t(u|v)\rho(v)dv$ and $\rho(v)$ is an arbitrary initial density. To this end we need an auxiliary lemma which comprises the most technical segment of the paper.

Lemma

For $t \to \infty$, $\|p_t(\cdot|v) - \rho_0\|_1 \to 0$ uniformly in $v$ on compact sets.

Proof: We shall show that $\forall N \in \mathbb{N} \forall \epsilon > 0 \exists t_0 > 0$ such that

$$\forall t > t_0 \forall v \in [-N, N] \|p_t(\cdot|v) - \rho_0\|_1 < \epsilon$$

Let us begin from

$$\|p_t(\cdot|v) - \rho_0\|_1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sigma^2(t)}{(u - ve^{-\lambda})^2 + \sigma^4(t)} - \frac{\sigma^2(\infty)}{u^2 + \sigma^4(\infty)} \right| du$$

$$\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sigma^2(t) - \sigma^2(\infty)}{(u - ve^{-\lambda})^2 + \sigma^4(t)} \right| du +$$

$$\frac{\sigma^2(\infty)}{\pi} \int_{-\infty}^{\infty} \frac{1}{(u - ve^{-\lambda})^2 + \sigma^4(t)} - \frac{1}{u^2 + \sigma^4(t)} |du$$
\[+ \frac{\sigma^2(\infty)}{\pi} \int_{-\infty}^{\infty} \frac{1}{u^2 + \sigma^4(t)} - \frac{1}{u^2 + \sigma^4(\infty)} \, du = \frac{\sigma^2(\infty) - \sigma^2(t)}{\sigma^2(t)}

+ \frac{\sigma^2(\infty)}{\pi} \int_{-\infty}^{\infty} \left|u^2 - (u - ve^{-\lambda t})^2\right| \, du + \frac{\sigma^2(\infty)[\sigma^4(\infty) - \sigma^4(t)]}{\pi} \int_{-\infty}^{\infty} \frac{du}{u^2 + \sigma^4(t)[u^2 + \sigma^4(\infty)]}

= \frac{\sigma^2(\infty)[\sigma^4(\infty) - \sigma^4(t)]}{\pi} \int_{-\infty}^{\infty} \frac{\lambda t}{\sigma^6(\infty)} \cdot \frac{4\pi}{\sigma^6(\infty)} = 8e^{-\lambda t}

Therefore, for any \( t > t_3 = \max(\frac{4}{\lambda} \log 2, \frac{1}{\lambda} \log \frac{24}{e}) \) there holds \( I_3 \leq \epsilon/3 \).

Finally, we estimate the second summand, denoted by \( I_2 \). At first let us notice that

\[ I_2 = \frac{\sigma^2(\infty)}{\pi} \int_{-\infty}^{\infty} \left| \frac{|v|e^{-\lambda t}2u - ve^{-\lambda t}}{(u - ve^{-\lambda t})^2 + \sigma^4(t)[u^2 + \sigma^4(t)]} \right| \, du
\]

\[= 8|v|e^{-\lambda t}\sigma^2(\infty) \int_{-\infty}^{\infty} \frac{|x| \, dx}{(x - ve^{-\lambda t})^2 + 4\sigma^4(t)[(x + ve^{-\lambda t})^2 + 4\sigma^4(t)]}
\]

where \( x = 2u - ve^{-\lambda t} \).

Hence, for all \( t \geq t_4 = \frac{1}{\lambda} \log(1 + \frac{N}{2\sigma^2(\infty)}) \) there holds \( |v|^2e^{-2\lambda t} \leq 4\sigma^4(t) \) for all \( v \in [-N, N] \), and so

\[ I_2 \leq 8e^{-\lambda t}\sigma^2(\infty) \int_{-\infty}^{\infty} \frac{|x| \, dx}{x^4 + (4\sigma^4(t))^2} = \frac{1}{\sigma^2(\infty)} \frac{e^{-\lambda t}}{(1 - e^{-\lambda t})^2}
\]

Therefore, \( I_2 < \epsilon/3 \) for all \( t > t_2 = \max(t_4, t_5) \), where \( t_5 \) is determined by \( 3e^{-\lambda t_5} = \sigma^2(\infty)(1 - e^{-\lambda t_5})^2\epsilon \). Thus \( \|p_t(\cdot|v) - \rho_0\|_1 < \epsilon \) for all
Now, we are ready to address the exactness issue for $T_t^*$. We observe that

$$\|\rho_t - \rho_0\|_1 = \int \int_{-\infty}^{\infty} |p_t(u|v)\rho(v)dv - \rho_0(u)|du =$$

$$\int \int_{-\infty}^{\infty} (p_t(u|v) - \rho_0(u))\rho(v)dv|du \leq \int_{-\infty}^{\infty} \|p_t(\cdot|v) - \rho_0\|_1 \rho(v)dv$$

$$\leq \int_{-N}^{N} \|p_t(\cdot|v) - \rho_0\|_1 \rho(v)dv + 2 \int_{[-N,N]^c} \rho(v)dv$$

where $[-N, N]^c$ denotes the complement of the interval $[-N, N]$.

Let us choose $N$ such that the second integral in the above is less than $\epsilon/4$, and next, in conjunction with Lemma, we choose $t_0$ such that for all $t > t_0$, there holds $\|p_t(\cdot|v) - \rho_0\|_1 < \epsilon/2$ for all $v \in [-N, N]$.

Then

$$\int_{-N}^{N} \|p_t(\cdot|v) - \rho_0\|_1 \rho(v)dv \leq \frac{\epsilon}{2}$$

and so $\|\rho_t - \rho_0\|_1 < \epsilon$, which completes the exactness demonstration for $T_t^*$.

**Remark 4:** In our considerations of the exactness issue the stationary Ornstein-Uhlenbeck-Cauchy process has been employed. This process is a direct Lévy stable analogue of the standard Gaussian Ornstein-Uhlenbeck process. There is however an important difference, [20]. In the Gaussian case all stationary Markov processes are Ornstein-Uhlenbeck (which in turn is unique). In particular, the standard process coincides with its reverse (fully anticipating) version. It turns out that in the Cauchy case there exist at least two different stationary Markov processes, since the reverse one does not coincide with the forward (nonanticipating) one. This means that the so-called statistical inversion of the Markovian dynamics (cf. a discussion and references in the closing section of Ref. [26]), in case of stable Lévy processes, makes a distinction between the ”time arrow” direction. See e. g. also
time reversal and time adjointness problems encountered in Refs. [4, 5]. Presumably all that derives from the exactness property. In modern attempts to devise a theoretical framework for nonequilibrium statistical physics, based on exploitation of classically chaotic systems, most irregular and “most irreversible” dynamical phenomena are known to be generated by exact systems.

We could as well consider Eq. (1) with a nondissipative \( \lambda < 0 \) factor. However, then the asymptotic properties of the semigroup \( T_t^\ast \) change in an essential way. Indeed, since now the transition probability density can be written as

\[
p_t(u|v) = \frac{1}{\pi (u - ve^{\lambda t})^2 + (\frac{\sigma^2}{|\lambda|}(e^{\lambda t} - 1))^2}
\]

then for any \( N \in \mathbb{N} \) and any density \( \rho \) there holds

\[
\lim_{t \to \infty} \int_{-N}^{N} p_t(u|v)\rho(v)dv = 0
\]

In consequence \( T_t^\ast \) is sweeping, [27]. It means that \( T_t^\ast \) has no stationary density and, in consequence, there is no probability law at all for the \( t \to \infty \) limit of the process \( u(t) \).

5 Properties of the process \( x(t) \)

5.1 Markovianess and forward drift

In case of the classic Ornstein-Uhlenbeck process, it is well known that the spatial random variable does not represent a Markov process.

A little bit surprisingly, in the present (Cauchy noise) case, we can prove that \( x(t) \) is a Markov process which is (like the previous \( u(t) \)) stochastically continuous. Moreover, while being a discontinuous process, nonetheless it has a forward drift equal \( b(s) = u_0e^{-\lambda s} \).

To show Markov property it suffices to check the Chapman-Kolmogorov equation for the transition function given by equation (15). That imme-
ately follows due to the additivity properties of functions \( g(t, s) \) and \( h(t, s) \) which enter the formula for \( p(y, t|x, s) \)

\[
g(t, t') + g(t', s) = g(t, s)
\]

\[
h(t, t') + h(t', s) = h(t, s)
\]

The stochastic continuity can be shown by a direct verification of the formula (18) which, in the nonstationary case, reads

\[
\lim_{t \to s} \int_{|y-x| \geq \epsilon} p(y, t|x, s) dy = 0
\]

Also, by direct calculations we check that the limit

\[
\lim_{t \to s} \frac{1}{t - s} \left[ \int_{|y-x| \geq \epsilon} (y - x)p(y, t|x, s) dy \right]
\]

does not depend on the chosen \( \epsilon \) cutoff, and equals \( u_0 e^{-\lambda s} \).

**Remark 5**: It is worth pointing out that Markov property of the pure spatial process \( x(t) \) is a distinguishing feature of the Cauchy noise. It does not hold for other \( \alpha \)-stable Lévy processes, in particular for the Gaussian one (the standard Ornstein-Uhlenbeck process). The reason of this exception is rooted in a particularly simple form of the probability distribution of the process \( \int_s^t f(\tau) dB(\tau) \) when \( \alpha = 1 \), see e.g. our Remark 1.

In contrast to the velocity process our spatial process is no longer time-homogeneous. In the inhomogeneous case instead of a one-parameter semigroup we have a two-parameter family of operators \( T_{t,s} \) defined by

\[
(T_{t,s} f)(x) = \int_{-\infty}^{\infty} p(y, t|x, s) f(y) dy \tag{28}
\]

which satisfy the composition rule \( T_{t,u} T_{u,s} = T_{t,s} \).

Therefore, we can introduce a time dependent generator by the following formula

\[
(M(s)f)(x) = \lim_{t \to s} \frac{1}{t - s} \left[ \int_{-\infty}^{\infty} p(y, t|x, s) f(y) dy - f(x) \right] \tag{29}
\]
In analogy with our previous considerations, we can readily identify an explicit form of the generator $M(s)$. Namely, let us assume that $p(y, t|x, s)$ is the transition function of the process $x(t)$ and $T_{t,s}$ are operators associated with this function. Then

$$M(s) = -\sigma^2(s)|\nabla| + b(s)\nabla$$

(30)

where $\sigma^2(s)$ is as (8) and $b(s) = u_0 e^{-\lambda s}$.

Because the major steps of the demonstration are essentially the same as in the case of the process $u(t)$, we skip them here.

Furthermore, let us notice that in view of $\lim_{s \to \infty} M(s) = \frac{\sigma^2}{\lambda}|\nabla|$ so, for large $t$ (i.e. asymptotically) the process $x(t)$ converges to the Cauchy process with the transition function given by

$$p_{t}(y|x) = \frac{1}{\pi} \frac{t\sigma^2/\lambda}{(y - x)^2 + t^2\sigma^4/\lambda^2}$$

Hence, the dissipation constant $\lambda > 0$ does the job "as usual", though with no recourse to the standard Maxwell-Boltzmann notion of thermal equilibrium and fluctuation-dissipation theorems.

5.2 Sample paths features

Let us turn to a brief discussion of the properties of sample paths of the process $x(t)$. (We remember that sample paths of the jump-type process $u(t)$ were cadlag.)

In his seminal paper, [7], Doob proved that the displacements of the standard Ornstein-Uhlenbeck process satisfy

$$\lim_{t \to \infty} \frac{x_{OU}(t) - x_{OU}(0)}{t} = 0$$

(31)

almost surely, i.e. the above limit holds for almost all sample paths (with probability 1). That is interpreted as an ergodic theorem applied to the
velocity process to give the strong law of large numbers, \[4\], and at the same
time as a statement about the (sample) path of a single particle.

In the final remark on p.369, he also concluded that Eq. (31) holds true
also in the case when the noise \( B \) is a stable process with the characteristic
\( \mu \geq 1 \). Hence, for the Cauchy process as well.

However, this conjecture appears to be wrong, in view of the estimates
we present in below.

Because of

\[
P[|x(t) - x(0)| > \epsilon t] = \int_{|x| > \epsilon} \frac{1}{\pi} \frac{g(t, 0)dx}{(x - u_0h(t, 0))^2 + g^2(t, 0)}
\]

\[
= 1 - \frac{1}{\pi} \left[ \text{arctan} \frac{\epsilon t + u_0h(t, 0)}{g(t, 0)} + \text{arctan} \frac{\epsilon t - u_0h(t, 0)}{g(t, 0)} \right]
\]

we have

\[
\lim_{t \to \infty} P\left[ \frac{|x(t) - x(0)|}{t} > \epsilon \right] = 1 - \frac{2}{\pi} \arctan \frac{\epsilon \lambda}{\sigma^2} > 0
\]

Therefore, \( \frac{x(t) - x(0)}{t} \) does not tend to zero even in probability.

That means that sample paths of the process \( x(t) \) diverge to infinity faster
than time \( t \) (up to dimensional constants).

**Remark 6:** (i) We can generalize slightly the discussion and allow the
parameter \( \lambda \) to depend on time

\[
du(t) = -\lambda(t)u(t)dt + dB(t)
\]

where \( \lambda(t) \) is a continuous function. Then, by integrating the above equation,
we obtain

\[
u(t) = b(t, s)u(s) + \int_s^t b(t, \tau)dB(\tau)
\]

where \( b(t, s) = \exp[-\int_s^t \lambda(\tau)d\tau] \). By invoking our previous arguments it is
easy to find the probability distribution and transition function for the new
process \( u(t) \) (and the new process of displacements).

We can also consider an \( n \)-dimensional situation, when \( \ddot{B}(t) \) is an \( \mathbb{R}^n \)-valued
Cauchy process, and the Langevin equation (1) is replaced by the following one

$$d\vec{u}(t) = -A\vec{u}(t)dt + d\vec{B}(t)$$  \hspace{1cm} (32)

where \( A \) denotes now \( n \times n \) matrix with real coefficients. The existence of the solution for (32) in a general setting of \( \mathcal{H} \)-valued processes, \( \mathcal{H} \) being a real and separable Hilbert space, was established in [19].

(ii) Sample paths of the process \( x(t) \) are also cadlag.

5.3 Interpreting \( u(t) \) as a velocity variable for \( x(t) \):

**Limitations**

Finally, we shall discuss the relation between the process of velocities \( u(t) \) and the process of displacements \( x(t) \). In the Ornstein-Uhlenbeck case the process \( u_{OU}(t) \) is continuous in the mean square, that is

$$\lim_{h \to 0} E[|u_{OU}(t + h) - u_{OU}(t)|^2] = 0$$

That follows from the continuity of its covariance function

$$(t_1, t_2) \to E[u_{OU}(t_1)u_{OU}(t_2)] = \sigma^2 e^{-\lambda|t_1 - t_2|}$$

Therefore, \( x_{OU}(t) \) exists as a limit in mean square of the corresponding Riemann sums. Moreover, since sample paths of \( u_{OU}(t) \) are continuous, the integral exists also almost surely and they both coincide. Hence \( x_{OU}(t) \) is not only mean square differentiable but is also differentiable in the sense of conventional mathematical analysis and its derivative is \( u_{OU}(t) \).

For the Cauchy process \( B(t) \) the situation is different. Because the moments of \( x(t) \) do not exist so \( x(t) \) is not even continuous in mean square. Moreover, since its sample paths are not continuous, they have no derivatives either.

However, \( x(t) \) has a velocity field \( u(t) \) in a probabilistic sense.

Indeed, because \( u(t) \) is stochastically continuous so for \( h \to 0 \) there holds

$$\frac{x(t + h) - x(t)}{h} = \frac{1}{h} \int_t^{t+h} u(\tau)d\tau \to u(t)$$
in probability, compare e.g. Eq. (17).

That demonstrates that our naming (a priori) of $u(t)$ the velocity random variable, can still be maintained to a certain extent, once we pass to the induced spatial variable and try to recover back a built-in information about the velocity process. This feature is slightly amusing and somewhat counterintuitive, since both the Cauchy noise-supported velocity process and the induced process of spatial displacements are discontinuous with probability 1. Anyway, the notion of velocity in the standard Ornstein-Uhlenbeck process has its own limitations as well: its nondifferentiability and thus the nonexistence of accelerations is not resolved but merely bypassed by invoking the white noise calculus.

One should perhaps recall at this point that our theoretical framework reduces to a stochastic modelling of physical phenomena. That constitutes a fine-tuned approximation, in terms of stochastic processes, of a generically robust Reality. Surely, we have not attempted her genuine reproduction. The obtained description is much too detailed and thus necessarily involves a number of artefacts (the nonexistence of accelerations in the standard Ornstein-Uhlenbeck process is one of them, an unbounded variation of the Wiener process sample paths just another).

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