$S_3$ Symmetry and Tri-bimaximal Mixing

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Abstract

The near maximal value for the atmospheric neutrino mixing angle together with the fact that the solar mixing angle satisfies the relation $\sin^2 \theta_\odot \simeq \frac{1}{3}$ is the basis for the so called tri-bimaximal mixing when $\theta_{13} = 0$. In this note, we explore the possibility that tri-bimaximal mixing is an indication of a softly broken higher leptonic symmetry $S_3$, the permutation of three lepton families that embeds the $\mu - \tau$ exchange symmetry of leptons.
I. INTRODUCTION

Observation of nonzero neutrino masses and determination of two of their three mixing parameters by experiments have raised the hope that neutrinos may hold the key to unraveling the flavor puzzle for quarks and leptons [1]. In order to make progress towards realizing this goal, one must first decipher the underlying reason for the observed leptonic mixing pattern and then search for a unified description of quarks and leptons to understand the quark flavor puzzle.

An interesting possibility to explore is that lepton mixings are indications of underlying flavor symmetries. Two tantalizing hints in favor of this are the near maximal atmospheric mixing angle and vanishing mixing angle $\theta_{13}$. It has been speculated that they are consequences of an approximate discrete $\mu - \tau$ symmetry [2, 3] for leptons. Its presence can be tested under certain circumstances. Even though there is no such apparent "$\mu - \tau$" symmetry among quarks and charged leptons, it has been shown that unified description of quarks and leptons with this symmetry is possible [4]. A question raised by this is whether there are higher underlying symmetries of leptons.

A hint for a higher symmetry may be coming from the observation that the solar angle in the PMNS mixing matrix satisfies the relation $\sin^2 \theta_\odot \simeq \frac{1}{3}$. The resulting PMNS matrix has the simple form [5]:

$$U_{PMNS} = \begin{pmatrix}
\frac{\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}$$

and is called a tri-bimaximal mixing. The true nature of the symmetry responsible for this pattern is not clear, although there are many interesting suggestions [6, 7, 8].

In this brief note, we explore the possibility that the relevant symmetry may be the permutation symmetry $S_3$ of three lepton generations. Such connections have been considered in literature from phenomenological points of view [8] and it will be interesting to see to what extent a full gauge model for tri-bimaximal pattern based on $S_3$ group can be developed. In this note we take some steps in this direction. We show that a softly broken $S_3$ symmetry for leptons can lead to tri-bimaximal mixing pattern if we use a combination of type I and type II seesaw mechanism to understand the smallness of neutrino mass. This approach appears to be different from previous attempts at building models for tri-bimaximal mixing [6, 7, 8].
We proceed in two steps: we first show how in a basis where charged leptons are diagonal, one can derive the mixing pattern in Eq. (1) using softly broken $S_3$ symmetry under certain assumptions. We then show how this the $S_3$ symmetry combined with $Z_{2e} \times Z_{2\mu} \times Z_{2\tau}$ symmetry can lead to a diagonal charged lepton mass matrix. We then extrapolate the neutrino mass matrix from the seesaw scale to the weak scale and obtain constraints on the mass ratios $m_1/m_3$ and $m_2/m_3$ so that the mixing angles match the observations. We obtain a prediction for $\theta_{13}$, which turns out to be extremely small ($\sim 0.004$). We further show that if the neutrino masses are quasi-degenerate and have the same CP property (i.e. are all positive), then the radiative corrections in the extrapolation to the weak scale are so large that the solar mixing angle is in disagreement with observations. This implies that in supersymmetric theories with large $\tan\beta$, seesaw scale tri-bimaximal mixing and degenerate neutrinos are mutually exclusive.

II. AN $S_3$ MODEL

We start with the Majorana neutrino mass matrix whose diagonalization at the seesaw scale leads to the tri-bimaximal mixing matrix:

$$
\mathcal{M}_\nu = \begin{pmatrix}
a & b & b \\
b & a-c & b+c \\
b & b+c & a-c
\end{pmatrix}
$$

Diagonalizing this matrix leads to the $U_{PMNS}$ of Eq. (1) and the neutrino masses: $m_1 = a-b, m_2 = a+2b$ and $m_3 = a-b-2c$. Clearly if $|a| \simeq |b| \ll |c|$, we get a normal hierarchy for masses.

We now show that the mass matrix in Eq.(2) can be obtained from a softly broken $S_3$ symmetry in the neutrino sector. For this purpose, we assign the three lepton doublets of the standard model ($L_e, L_\mu, L_\tau$) to transform into each other under permutation. The three right handed neutrinos ($\nu_{R,i=1,2,3}$) transform under three permutation and two cyclic operations of $S_3$ as:

- $e \leftrightarrow \mu : \nu_{R,1} \leftrightarrow -\nu_{R,1}; \nu_{R,2} \leftrightarrow -\nu_{R,3}$
- $\mu \leftrightarrow \tau : \nu_{R,2} \leftrightarrow -\nu_{R,2}; \nu_{R,1} \leftrightarrow -\nu_{R,3}$
- $\tau \leftrightarrow e : \nu_{R,3} \leftrightarrow -\nu_{R,3}; \nu_{R,1} \leftrightarrow -\nu_{R,2}$
$e \to \mu \to \tau : \nu_{R,1} \to \nu_{R,2}; \nu_{R,2} \to \nu_{R,3}; \nu_{R,3} \to \nu_{R,1}$

$e \to \tau \to \mu : \nu_{R,1} \to \nu_{R,3}; \nu_{R,2} \to \nu_{R,1}; \nu_{R,3} \to \nu_{R,2}$  

In order to obtain the neutrino mass matrix, we assume that there is a standard model triplet Higgs field $\Delta$ with $Y = 2$ which is $S_3$ singlet that couples to the two lepton doublets and an $S_3$ singlet Higgs doublet field $H$ that gives the Dirac mass for the neutrinos. The triplet vev can be made small and of the desired order if the mass of the triplet Higgs field is around $10^{14}$ GeV or so[1].

The first point to note is that the most general $S_3$ invariant coupling of the triplet i.e. $f_{ab} L_a L_b \Delta$ is given by the coupling matrix:

$$ f = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \quad (4) $$

For the Dirac neutrino coupling we choose to keep the following $S_3$ invariant term:

$$ \mathcal{L}_D = h_\nu [\bar{\nu}_{R,1} H (L_e - L_\mu) + \bar{\nu}_{R,2} H (L_\mu - L_\tau) + \bar{\nu}_{R,3} H (L_\tau - L_e)] + h.c. \quad (5) $$

One other $S_3$ invariant coupling is set to zero. This is natural in a supersymmetric theory due to the nonrenormalization theorem. We then get for the Dirac mass matrix for neutrinos

$$ M_D = \begin{bmatrix} d & -d & 0 \\ 0 & d & -d \\ -d & 0 & d \end{bmatrix}. \quad (6) $$

where $d = h_\nu < H >$. If we now assume the following hierarchy among the right handed neutrinos, i.e. $M_{\nu_{R,1,3}} \gg M_{\nu_{R,2}}$ so that a single right handed neutrino dominates the type I contribution to the seesaw formula[10], then in the strict decoupling limit, using the mixed type I+II seesaw formula:

$$ M_\nu = M_0 - M_D^T M_{\nu_R}^{-1} M_D, \quad (7) $$

we get the desired form for the neutrino Majorana mass matrix (Eq.(2)) which leads to tri-bimaximal mixing. Note that the right handed neutrino masses being dimension three operators break the $S_3$ softly.

In this discussion we have assumed that the charged lepton mass matrix is diagonal. A major challenge for any model for neutrino mixings is to have a consistent picture for
both the charged lepton and neutrino sectors simultaneously so that the combination $U^\dagger_\ell U_\nu$ equals the observed PMNS matrix. Since in our case, the neutrino sector by itself gives the tri-bimaximal form for the PMNS matrix, the charged lepton sector should be diagonal or nearly so. We will now show that we can obtain a diagonal charged lepton mass matrix in a simple way using the $S_3$ symmetry, provided we choose only one of two allowed $S_3$ invariant Yukawa coupling terms.

In order to achieve this, we assume that there are three standard model Higgs doublets $(H_e, H_\mu, H_\tau)$ transforming like the lepton doublets above under $S_3$. We also assume that the right handed charged leptons $(e_R, \mu_R, \tau_R)$ transform under $S_3$ same way. We then assume a product of discrete symmetries $Z_{2e} \times Z_{2\mu} \times Z_{2\tau}$ under which all fields except the following are even: $(e_R, H_e)$ odd under only $Z_{2e}$ and similarly $(\mu_R, H_\mu)$ are odd only under $Z_{2\mu}$ and $(\tau_R, H_\tau)$ odd under $Z_{2\tau}$. The Yukawa couplings invariant under this are:

$$\mathcal{L}'_Y = h_e(\bar{L}_e H_e e_R + \bar{L}_\mu H_\mu \mu_R + \bar{L}_\tau H_\tau \tau_R) + h'_e(\bar{L}_e H_\mu \mu_R + \bar{L}_\mu H_e e_R$$

$$+ \bar{L}_\mu H_\tau \tau_R + \bar{L}_\tau H_\mu \mu_R + \bar{L}_e H_\tau \tau_R + \bar{L}_\tau H_\mu \mu_R + \bar{L}_e H_\mu \mu_R) + h.c. \quad (8)$$

By softly breaking the global $S_3$ symmetry in the Higgs potential for the $H_{e,\mu,\tau}$, we can get $<H_e> \ll <H_\mu> \ll <H_\tau>$ which allows us to obtain a realistic diagonal charged lepton mass matrix if we assume $h'_e = 0$. This model then gives us a tri-bimaximal neutrino mixing at the seesaw scale.

III. SOME IMPLICATIONS

In order to compare this model with observations, we need to extrapolate the seesaw scale neutrino mass matrix in Eq.(2) down to the weak scale and then calculate the masses and mixing angles. This extrapolation depends on the mass hierarchy of the neutrinos. So comparing with observations, we can put limits on the mass hierarchy at low scale. From the expressions for the neutrino masses derived after Eq.(2), one might think that degenerate masses are compatible with tri-bimaximal pattern since there are three parameters and three masses to be fitted. However, in supersymmetric models, mixing angles can receive substantial contributions from RGE effects (specially for large $\tan\beta$) and will in general lead to distortion of the mixing angles away from the tri-bimaximal values. For the specific case of $\tan\beta = 50$ we calculate the radiative corrections to the solar mixing angle $\theta_{12}$ in Fig. 1. We
plot $\sin^2\theta_{12}$ against $m_2/m_3$ with the input constraint being that $\Delta m^2_{\odot}/\Delta m^2_{\text{ATM}}$ is within $3\sigma$ of its present value i.e. $0.024 \leq \Delta m^2_{\odot}/\Delta m^2_{\text{ATM}} \leq 0.060^{[1]1}$. We see that for $m_2/m_3 > 0.3$ or so, the solar mixing angle goes outside the observed range and the agreement gets worse for larger values of this mass ratio which corresponds to quasi-degenerate neutrino spectrum. This leads us to conclude that tri-bimaximal mixing at the seesaw scale is incompatible with quasi-degenerate neutrinos for large values of $\tan \beta$.

For the same value of $\tan \beta$, we show in Fig.2 the allowed ranges for the neutrino mass ratios for the case of normal hierarchy and in Fig.3, the prediction for $\theta_{13}$ for this model. In these figures, we have used the above $3\sigma$ experimental bounds for $\Delta m^2_{\odot}/\Delta m^2_{\text{ATM}}$ and also $3\sigma$ bounds for $0.23 \leq \sin^2 \theta_{12} \leq 0.38$ and $0.34 \leq \sin^2 \theta_{23} \leq 0.68^{[1]1}$. We find that the prediction for $\theta_{13} \sim 0.004$ which is much too small to be observable in near future. This is because the low energy theory in the absence of radiative corrections is $\mu-\tau$ symmetric. Clearly observation of $\theta_{13}$ higher than this value will rule out this model and indeed any simple model for tri-bimaximal mixing at the seesaw scale for the case of normal mass hierarchy.

![FIG. 1: sin$^2\theta_{12}$ at the weak scale for the case of quasi- degenerate neutrinos. Note that the higher the ratio $m_2/m_3$, the more degenerate the neutrinos are and further off the prediction for sin$^2\theta_{12}$ is from the observed value.](image)

**IV. CONCLUSION**

In conclusion, in this brief note, we have presented a new way to obtain the tri-bimaximal mixing pattern for neutrinos by embedding $\mu-\tau$ symmetry of the neutrino mass matrix into a softly broken $S_3$ permutation symmetry for leptons and using a simple combination
of the type I and type II seesaw formulae along with the dominance of a single right handed neutrino \cite{10}. We also find that tri-bimaximal mixing at the seesaw scale is incompatible with degenerate neutrino spectrum due to large radiative correction effects for large $\tan \beta$.

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\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{Allowed ranges of mass ratios at the weak scale for normal hierarchy case.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig3}
\caption{Distribution of $\sin \theta_{13}$ value.}
\end{figure}

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