Non-Markovian coherent feedback control of quantum dot systems

Shibei Xue\textsuperscript{1,2,3}, Rebing Wu\textsuperscript{1,2}, Michael R Hush\textsuperscript{3} and Tzyh-Jong Tarn\textsuperscript{1,2,4}

\textsuperscript{1} Department of Automation, Tsinghua University, Beijing 100084, People’s Republic of China
\textsuperscript{2} Center of Quantum Information Science and Technology, TNList, Beijing 100084, People’s Republic of China
\textsuperscript{3} School of Engineering and Information Technology, University of New South Wales Canberra at the Australian Defence Force Academy, Canberra, ACT 2600, Australia
\textsuperscript{4} Department of Electrical and Systems Engineering, Washington University, St Louis, MO 63130, United States of America

E-mail: xueshibei@gmail.com

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Abstract

In this paper we present a non-Markovian coherent feedback scheme for decoherence suppression in single quantum dot systems. The feedback loop is closed via a quantum tunnelling junction between the natural source and drain baths of the quantum dot. The exact feedback-controlled non-Markovian Langevin equation is derived for describing the dynamics of the quantum dot. To deal with the nonlinear memory function in the Langevin equation, we analyse the Green’s function-based root locus, from which we show that the decoherence of the quantum dot can be suppressed via increasing the feedback coupling strength. The effectiveness of decoherence suppression induced by non-Markovian coherent feedback is demonstrated by a single quantum dot example bathed with Lorentzian noises.

1. Introduction

As a potential solid-state information carrier for quantum computation, quantum dot systems have attracted much attention in recent years [1–4]. As well as other quantum registers, the coherent manipulation of the quantum dot is vital for processing quantum information [5, 6], which is always deteriorated by the decoherence induced by interaction with the environments [7–10]. In quantum dot systems, the interaction occurs between the quantum dot and source and drain electrodes, the hyperfine interaction between electron spins of quantum dots and spins of nuclei, and the noise generated by the defects on the substrate materials [11–13].

When the memory time of the environment is ignorable, the Markovian approximation can be taken to simplify the analysis and design of open quantum control systems such as stabilising the current through nanostructures and purifying the state of quantum dot qubit via feedback control [15–17]. However, for general cases, the violation of Markovian approximation in solid-state systems may degrade the feedback control performance. Consequently, coloured noise disturbs the system of interest, whose spectrum is defined by a multiplication of the state density and the square norm of the coupling strength between the system and the environment [14]. This resulting non-Markovian effect can be harnessed by using a class of direct coherent feedback approach [18] where the structure of the environments is altered by the couplings between the modes of the environment, and the characteristics of correlated environments can modify the non-Markovianity of a quantum system [19, 20]. This coherent feedback scheme involving no measurement is expected to apply to a class of fermion quantum dots systems.

This paper studies the coherent feedback control of non-Markovian dynamics of single quantum dot systems, with application to the suppression of decoherence. The quantum dot system is a fermion system and its noise baths, i.e., the source and the drain, are two ensembles of fermions, i.e, electrons, which is quite different from the bosonic system considered in [18]. To modulate the quantum transport process of the electrons between the source and drain in this specific quantum system, a tunnelling junction is coupled with them to close the feedback loop. Thus, the characteristics of the noise baths are adjusted via modifying the
structure of the junction. With this loop topology, a direct coherent feedback scheme [18, 21] can be designed for suppression of non-Markovian decoherence and the mechanism behind it is expected to explore.

However, the circumstance for the quantum dots system is quite different due to the bias voltage applied on the source and drain baths. The resulting detuning between the central frequencies of the source and the drain leads to a memory kernel function that is nonlinearly dependent on the feedback coupling strength. This is beyond the scope of [18] where this relation is linear. Hence, in this paper, we develop a novel Green’s function-based root locus method [22, 23] to analyse the decoherence effect of the closed-loop system. This method can deal with the nonlinearity induced by the bias voltage. Furthermore, it reveals how the adjustable feedback coupling strength changes the modes of the system which determines the dynamics of the system in the time domain. We also compare our coherent feedback scheme with an open-loop control approach in the suppression of non-Markovian decoherence for the single quantum dot system. We find that the coherent feedback scheme is more effective at suppressing the non-Markovian decoherence, demonstrating the advantage of coherent feedback.

This paper is organised as follows. In section 2, the Hamiltonian of the coherent feedback loop is introduced. Starting from this Hamiltonian, we obtain an exact non-Markovian Langevin equation to describe the dynamics of the quantum dot in section 3. In section 4, via Green’s function-based root locus approach, the analysis for the dynamics of the controlled system is done in the frequency domain. An example of the quantum dot system is given in section 5. Finally, conclusions are drawn in section 6.

2. Coherent feedback loop Hamiltonian

Consider a single quantum dot [24] between two leads which are called the source (left) and the drain (right), respectively, and a bias voltage is applied on the two leads. The coupling strengths between the quantum dot and each mode of two electrodes are different which results in the non-Markovian decoherence dynamics of the quantum dot [12]. To effectively reject non-Markovian noises, non-Markovian coherent feedback scheme is introduced. To build a feedback loop, the source and the drain are joined together with a tunnelling junction where the tunnelling strength is tunable. This scheme is sketched in figure 1. This closed-loop design involves coherent feedback with two interconnected systems that always affect each other. Thus, the energy and information flow in this closed loop is in both clockwise and anticlockwise directions.

The Hamiltonian of the open-loop system (i.e., without tunnelling junction) can be written as

$$H_O = H_S + H_E + H_{SE}$$

where $H_S/h = \omega_S \hat{a} \hat{a}^\dagger$ is the quantum dot Hamiltonian with a working frequency $\omega_S$ and a fermion annihilation operator $\hat{a}$. The environment Hamiltonian $H_E$ describes two clusters of the electron bath (source and drain), i.e.,
with the frequencies for each mode $\omega_{k}\mathcal{E}$, where the symbols $\hat{b}_k$ ($\hat{c}_k$) are the fermion annihilation operator of the source (drain). Their couplings to the system are determined by the interaction Hamiltonian,

$$H_{SE} = \sum_k (V_{k\beta}^b \hat{b}_k + V_{k\beta}^d \hat{d}_k) + \sum_k (V_{k\beta}^c \hat{\varphi}_k + V_{k\beta}^{c\dagger} \hat{\varphi}^\dagger_k).$$

The coupling strengths between the system and each mode of the source (drain) are denoted as $V_{k\beta}$, which is different for each mode resulting in the non-Markovian decoherence dynamics. For simplicity, the nonlinear system–bath interactions are not considered here.

Note that the annihilation and creation operators for both the quantum dot system ($\hat{d}$ and $\hat{d}^\dagger$) and its source and drain ($\hat{b}_k$, $\hat{c}_k$ and $\hat{b}_k^\dagger$, $\hat{c}_k^\dagger$) and all fermion operators satisfying the anti-commutation relations, e.g.,

$$\{\hat{c}_k, \hat{c}^\dagger_{k'}\} = \delta_{kk'}I, \quad \{\hat{c}_k, \hat{c}_{k'}\} = 0,$$

where $[A, B] \equiv AB + BA$ is the anti-commutator for two arbitrary operators $A$ and $B$.

For the above system, the interaction between the system and the bath disturbs the system dynamics. The noise structure induced by the interaction determines how serious the non-Markovian decoherence is. In this paper, a tunnelling junction is introduced between the source and drain to efficiently modify the noise structure, which induces a coupling Hamiltonian between the source and drain as

$$H_F = \sum_k \sum_{k'} (F_{kk'} \hat{c}^\dagger_k \hat{b}_k + F_{kk'} \hat{d}^\dagger_k \hat{b}_k^\dagger),$$

where $F_{kk'}$ describes the tunnelling strength between the $k$-th source mode and the $k'$-th drain mode [25]. Here, the source together with the drain constitutes a structured bath for the system (as shown in figure 1), whose internal properties are expected to be modified via the tunable coupling strength $F_{kk'}$.

The $F_{kk'}$ will depend on the physical properties of the junction. In what follow we describe how to calculate $F_{kk'}$, assuming the electrons act as if they are free and are in one dimension. Consider an electron starting at the source with wave vector $k_0$ and ending at the drain with wave vector $k$, an initial energy and a final energy can be expressed as $E_b = \hbar^2 k_0^2 / 2m + eU_b$ and $E_c = \hbar^2 k^2 / 2m + eU_c$, respectively, where $U_b$ and $U_c$ are the voltages on each side of the junction, $e$ is the charge of the electron and $m$ is its mass. By energy conservation we can relate the initial and initial wave vector of the electron after crossing the junction as: $\hbar^2 (k_0^2 - k^2) / 2m = e(U_b - U_c)$.

On the other hand, we assume that the initial and final kinetic energy of the electron is much greater than the potential difference across the junction. In this case the probability is very low for the electron to be reflected. Hence, the characteristic central frequency $k_0$ of both baths is much higher than the potential difference across the junction, i.e., $\hbar^2 k_0^2 / 2m \gg e(U_c - U_b)$. Furthermore, we only consider perturbation modes whose frequency components are near this central frequency, as the off-resonant components have minor effects on the coherence of the quantum dot. Thus, one can see that $k_C - k_b \approx me(U_b - U_c) / \hbar^2 k_0$. Hence, the difference between the wave vectors is approximately a constant which is related to the potential difference across the junction. And thus the tunnelling strength between the source and drain can be expressed as

$$F_{kk'} = \begin{cases} f_k, & k - k' = l \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $l = me(U_b - U_c) / \hbar^2 k_0$. The coupling strengths $f_k$ can be engineered by treating the junction as a waveguide and changing its geometry.

Thus, the total Hamiltonian of our coherent feedback control system reads as

$$H_F = H_0 + H_F.$$  

The mechanism of the noise rejection via coherent feedback will be shown in the next sections.

### 3. Exact non-Markovian quantum Langevin equation

#### 3.1. Exact Langevin equation

In this paper, a class of an exact non-Markovian Langevin equation approach [18, 26] is adopted to describe the quantum dot system. The evolution of the fermion annihilation operator $\hat{d}(t)$ of the quantum dot is described by the following integral-differential exact non-Markovian quantum Langevin equation (for the details of the derivation, see appendix A),

$$\dot{\hat{d}}(t) = -i\omega_0 \hat{d}(t) - \int_0^t d\tau M(t - \tau) \hat{d}(\tau) - \hat{e}_s(t),$$
where the memory kernel function \( M(t) \) embedded with the noise spectrum determines the dissipation process; and the noise term \( \epsilon_n(t) \) corresponds to the equivalent noise injected from the two leads. Owing to the linearity of the integral-differential equation (6), the solution of \( \dot{\epsilon}(t) \) is expressed as

\[
\dot{\epsilon}(t) = g(t)\dot{\epsilon}(0) + \int_0^t g(t - \tau)\dot{\epsilon}_n(\tau) d\tau,
\]

where the first term characterises the dissipative evolution from its initial state \( \dot{\epsilon}(0) \) and the second term describes the dynamics excited by the noise \( \dot{\epsilon}_n(t) \). The complex coefficient \( g(t) \) satisfies the following integral-differential equation:

\[
g(t) = -i\omega_g g(t) - \int_0^t d\tau M(t - \tau)g(\tau), \quad g(0) = 1,
\]

where the absolute value of the Green’s function \( g(t) \) is the scaled amplitude of the system [26]. Note that the coherence of a quantum system can be defined as the first-order correlation function of a system operator [7]

\[
g^{(1)}(t) = \langle \dot{\epsilon}(0)\dot{\epsilon}(t) \rangle \quad \text{and} \quad \langle \dot{\epsilon}(0)\dot{\epsilon}(t) \rangle = \langle \dot{\epsilon}(t)\dot{\epsilon}(0) \rangle^*,
\]

where \( \langle \rangle \) is the quantum expectation of an operator. Substituting (7) into (9), we have \( g^{(1)}(t) = g(t) \), where we have assumed that the system and the noise field are initially uncorrelated. Also the absolute value of \( g(t) \) is relevant to the scaled amplitude of the system [26]. Hence, \( |g(t)| \) can be used to evaluate the coherence of the system.

### 3.2. The coherent feedback case

When the feedback couplings (3) are introduced, i.e., the total system is described by (5), both the kernel \( M(t) \) and the noise \( \dot{\epsilon}_n(t) \) are altered. Assume that the source and drain can be effectively coupled as expressed in equation (4) and denote \( \hat{H}_0 \) in the continuous limit and the polar coordinate as \( f(\omega) = r(\omega)e^{i\theta(\omega)} \). The memory kernel function is split by the feedback as \( M(t) = M_F(t) + M_I(t) \) with

\[
M^{\pm}(t) = \int_{-\infty}^{+\infty} d\omega f^{\pm}(\omega) e^{-i(\omega - \hbar \omega) t + i\omega t},
\]

where the noise spectral functions

\[
\begin{align*}
f^{+}(\omega) &= \rho(\omega) \left| V_B(\omega)e^{-i\theta(\omega)}\cos \frac{\alpha(\omega)}{2} + V_C(\omega)\sin \frac{\alpha(\omega)}{2} \right|^2, \\
f^{-}(\omega) &= \rho(\omega) \left| V_B(\omega)e^{-i\theta(\omega)}\sin \frac{\alpha(\omega)}{2} - V_C(\omega)\cos \frac{\alpha(\omega)}{2} \right|^2
\end{align*}
\]

are modulated by feedback parameters \( r(\omega) \) and \( \theta(\omega) \) and \( \alpha(\omega) = \arctan \frac{\theta(\omega)}{r(\omega)} \). The split in the memory kernel function shows that the noises can be modified by the tunnelling strength. The equivalent noise \( \dot{\epsilon}_n(t) = \dot{\epsilon}_{eq}(t) \) in equation (6) is

\[
\dot{\epsilon}_{eq}(t) = i \int_{-\infty}^{+\infty} d\omega \rho(\omega) \nu(\omega) \Phi(\omega, t) \dot{\epsilon}(\omega, 0),
\]

where \( \rho(\omega) \) is the density state and the definitions of the coupling strength vector \( \nu(\omega) \) and the feedback-induced modulation matrix \( \Phi(\omega, t) \) are given in equations (A.17) and (A.18). Note that we have assumed that the source and drain share the same density state \( \rho(\omega) \) and the effect of the feedback Hamiltonian \( H_f \) has been embedded in both Green’s function \( g(t) \) and the equivalent input \( \dot{\epsilon}_n(t) \).

### 3.3. The open-loop case

For comparison, we also consider a class of open-loop methods for suppressing the non-Markovian decoherence in [28]. In this case, the total system Hamiltonian is described by \( H_0 \) in equation (1) and the exact non-Markovian Langevin equation (6) can also be used to describe the system dynamics but with a different memory kernel function and a different noise term. The memory kernel function \( M(t) = M_0(t) = M_B(t) + M_C(t) \) with

\[
M_0(t) = \int_{-\infty}^{+\infty} d\omega \rho(\omega_B)|V_B(\omega_B)|^2 e^{-i\omega_B t},
\]

\[
M_B(t) = \int_{-\infty}^{+\infty} d\omega \rho(\omega_B)|V_B(\omega_C)|^2 e^{-i\omega_C t},
\]

where \( \rho(\omega_B) \) and \( \rho(\omega_C) \) are the state density functions of the source and the drain respectively; and the noise...
\[ \dot{\epsilon}_n(t) = \dot{\epsilon}_{\text{m}}(t) + \dot{\epsilon}_{\text{d}}(t) \]

is a summation of the noise arising from the source and the drain.

\[
\dot{\epsilon}_{\text{m}}(t) = i \int_{-\infty}^{+\infty} d\omega_B \varrho(\omega_B) V_B^*(\omega_B) e^{-i\omega_B t} \dot{b}(\omega_B, 0) \tag{14}
\]

\[
\dot{\epsilon}_{\text{d}}(t) = i \int_{-\infty}^{+\infty} d\omega_C \varrho(\omega_C) V_C^*(\omega_C) e^{-i\omega_C t} \dot{c}(\omega_C, 0). \tag{15}
\]

In the above expressions, \( \dot{b}(\omega_B, 0) \) and \( \dot{c}(\omega_C, 0) \) are the value of \( \dot{b}(\omega_B, t) \), \( \dot{c}(\omega_C, t) \) at \( t = 0 \), respectively.

The exact non-Markovian Langevin equation above affords the basis for analysing the system dynamics with or without coherent feedback control. The feedback control parameters \( r(\omega) \) and \( \theta(\omega) \) are embodied in the memory kernel function \( M(t) \) in equation (6). Next we will study the way for effectively manipulating the memory kernel function \( M(t) \) is considered in the next section.

### 4. Green’s function-based root locus analysis for decoherence suppression

In our previous work [18], it is shown that the spectral modulation induced by coherent feedback can be used to suppress decoherence. However, this method is not directly extendable to the system discussed here due to the non-linearity of the control amplitude \( r(\omega) \) as shown in equation (10) resulting from the bias-voltage-induced central frequency difference between the source and drain. Hence, whether or not decoherence can be suppressed is not as obvious as in [18]. In this section, we will analyse it through Green’s function-based root locus method.

#### 4.1. Green’s function-based root locus

Root locus is a graphical method for describing the dependence of the modes on a changeable parameter of the controlled system (e.g., the gain) and thus determining the regime of the parameter that ensures the system stability [29]. Here, we analyse the root locus for the Green’s function to understand the mechanism of decoherence suppression induced by coherent feedback.

Transforming the dynamical equation of the Green’s function for the non-Markovian quantum system (8) into the complex frequency domain, the Laplace transform \( G(s) \) of the Green’s function \( g(t) \) is

\[
G(s) = \frac{1}{s + i\omega_0 + M(s), \tag{16}}
\]

where \( M(s) \) is the Laplace transform of the memory kernel function \( M(t) \). The poles of the Green’s function \( G(s) \) are defined as points of \( s \) at which \( G(s) \) is singular. The trajectories of the poles versus with a varying parameter are defined as the root locus of the Green’s function \( G(s) \) [22].

As shown in equation (16), the poles of the Green’s function are dependent on the memory kernel function \( M(s) \). For the simplest case \( M(s) = 0 \), i.e., the system is closed, the pole lies in the imaginary axis of the complex plane, which implies the coherence of the system are not destroyed. When the system is a Markovian quantum system, i.e., \( M(s) = \frac{1}{t} \) with a constant damping rate \( \gamma \), the pole is shifted to the left half of the complex plane with a negative real part corresponding to the damping. For a non-Markovian quantum system involving complicated noise spectrum, the distribution of the poles of the Green’s function becomes complicated, where we assume that \( M(s) \) can be expressed in a rational form. In the following, we will investigate its influence of the memory kernel function on the Green’s function in the case with or without our coherent feedback scheme, respectively, so as to observe the coherence of the system.

#### 4.2. The coherent feedback case

To explore the root locus of the Green’s function induced by the coherent feedback, we assume that the single quantum dot is equally strongly coupled to the source and drain, i.e., \( V_B(\omega) = V_C(\omega + 2\delta) = V(\omega) \). Note that due to the bias voltage between the source and drain, the frequency difference between the two baths is set to be \( 2\delta \). Also, the Lorentzian spectral density [12, 30] is adopted for noise spectral of fermion baths, i.e., the source and drain, which can be expressed as

\[
f(\omega) = 2\pi \varrho(\omega) |V(\omega)|^2 = \frac{\eta h^2}{(\omega - \omega_c)^2 + h^2}, \tag{17}
\]

where the parameters \( \eta \) and \( h \) are the strength and width of the noise spectrum, respectively. These system settings can be found in a recent quantum transport system [27].

Assume that the feedback coupling strength is independent on the frequency and express it in the polar coordinate as \( f = r e^{i\alpha} \), and thus the corresponding parameter \( \alpha \) is also independent of frequency. When the feedback coupling is applied, the memory kernel function \( M(t) \) is split into two branches in equation (10) which can be expressed as \( M(s) = M^+(s) + M^-(s) \) in the frequency domain (see appendix B) with
\[ M(s) = \frac{\frac{1}{2} \gamma h (1 \pm \cos \theta \sin \alpha)}{s + z_0 \pm i r \gamma}, \]  
(18)

where \( z_0 = h + i(\omega_S - \delta) \) and \( \gamma = \sqrt{\delta^2 + r^2} \). Physically, this means our coherent feedback can modify the noise spectrum, i.e., the structure of the environment can be engineered by the coherent feedback.

To see how the memory kernel \( M(s) \) affects the Green’s function, we can substitute equation (18) into equation (16) and then obtain

\[ G(s) = \frac{s^2 + \alpha_1 s + \alpha_2}{s^3 + \beta_1 s^2 + \beta_2 s + \beta_3}, \]  
(19)

where \( \alpha_1 = 2z_0 \), \( \alpha_2 = z_0^2 + \gamma^2 \), \( \beta_1 = 2z_0 + i\omega_S \), \( \beta_2 = z_0^2 + \gamma^2 + \eta h + i2\omega_S z_0 \), and \( \beta_3 = \eta h z_0 + i\omega_S (z_0^2 + \gamma^2) - i\eta h \gamma \cos \theta \sin \alpha \).

For utilising inverse Laplace transform to obtain an explicit solution of \( g(t) \) in the time domain, we express equation (19) in the form of a partial fraction decomposition as

\[ G(s) = \frac{q_1}{s - p_1} + \frac{q_2}{s - p_2} + \frac{q_3}{s - p_3}, \]  
(20)

where three poles

\[ p_1 = -\frac{\beta_1}{3} + \frac{1}{3\sqrt{2}} e^{i\theta} - \frac{\sqrt{2}A}{3l}, \]  
(21)

\[ p_2 = -\frac{\beta_1}{3} - \frac{1}{3\sqrt{2}} e^{-i(\theta - \frac{\pi}{2})} + \frac{\sqrt{2}A}{3l}, \]  
(22)

\[ p_3 = -\frac{\beta_1}{3} - \frac{1}{3\sqrt{2}} e^{i(\theta + \frac{\pi}{2})} + \frac{\sqrt{2}A}{3l}, \]  
(23)

with \( A = 3\beta_2 - \beta_1^2 \), \( B = 9(\beta_2\beta_1 - 3\beta_3) - 2\beta_1^3 \), and \( le^{i\theta} \equiv \sqrt{B + 4A^2 + B^2} \) are what we concern about. Their distribution determines the root locus of the Green’s function and thus the non-Markovian dynamics of the system. The complex coefficients \( q_1, q_2, q_3 \) can be calculated as

\[ q_1 = \frac{\alpha_2 + \alpha_1 p_1 + p_1^2}{(p_1 - p_2)(p_1 - p_3)}, \]  
\[ q_2 = \frac{-\alpha_2 - \alpha_3 p_2 + p_2^2}{(p_1 - p_2)(p_2 - p_3)}, \]  
\[ q_3 = \frac{\alpha_2 + \alpha_3 p_3 + p_3^2}{(p_1 - p_3)(p_2 - p_3)}. \]

With the help of equation (20), the solution of equation (8) can be obtained as

\[ g(t) = q_1 e^{p_1 t} + q_2 e^{p_2 t} + q_3 e^{p_3 t}, \]  
(24)

which will be used to observe the dynamics of \( g(t) \) under coherent feedback in the example of next section.

The number of poles of \( G(s) \) is increased to be 3. Their distribution will directly affect the dynamics of \( g(t) \). To qualitatively observe the effect of our coherent feedback on the distribution of the poles of the Green’s function \( G(s) \), we consider a limit case that the feedback coupling strength \( r \) approaches to infinity. Since the three poles \( p_{1,2,3} \) are functions of the feedback coupling strength \( r \), the limit value of \( p_{1,2,3} \) as \( r \) going to the infinity are calculated as

\[ \lim_{r \to +\infty} p_1 = 0 + i(-\omega_S), \]  
(25)

\[ \lim_{r \to +\infty} p_2 = -h + i(-\infty), \]  
(26)

\[ \lim_{r \to +\infty} p_3 = -h + i(+\infty). \]  
(27)

It is shown that one of the poles \( p_1 \) is pushed to be close to \( -i\omega_S \) by choosing a sufficiently large \( r \) and the real value of the other two poles are driven to be \( -h \) and the imaginary parts of them go to \( -\infty \) and \( +\infty \), respectively. Compared with \( p_2 \) and \( p_3 \) whose real parts are negative leading to quick damping, the pole \( p_1 \) for sufficiently large \( r \) is very close to the imaginary axis, which will keep its mode oscillating for a long time. It means \( |g(t)| \) can be kept on a high value close to 1. This indicates our coherent feedback scheme can suppress the decoherence. In practice, the feedback coupling strength can not be arbitrarily strong, and hence the dissipation process can only be slowed down only by the feedback.
In numerical simulations, we choose parameters that can be engineered in practice. The system working the same form Langevin equation (6) with a different memory kernel $M_0(t)$ and a noise term $\dot{\xi}_s(t)$ as given in section 3.3.

Transformed to the frequency domain, (see the appendix B), $M_0(s)$ in equation (16) is expressed as

$$M_0(s) = \frac{1}{\gamma} h \left( \frac{1}{s + h} + \frac{1}{s + h + i\omega_S} \right).$$

(28)

Substituting equation (28) into equation (16), a partial fraction decomposition form of $G_0(s)$ can be obtained as

$$G_0(s) = \frac{q_{01}}{s - p_{01}} + \frac{q_{02}}{s - p_{02}} + \frac{q_{03}}{s - p_{03}},$$

(29)

where three poles $p_{01}, p_{02}, p_{03}$ of $G_0(s)$ are equal to the values of $p_1, p_2, p_3$ as $r$ and $\theta$ being zero; and $q_{01}, q_{02}, q_{03}$ can also obtained in the same way. Hence, the behaviour of the Green’s function $g_0(t)$ can be evaluated by

$$g_0(t) = q_{01}e^{p_{01}t} + q_{02}e^{p_{02}t} + q_{03}e^{p_{03}t},$$

(30)

which can be obtained from equation (29) via inverse Laplace transform.

In [28], a scheme is proposed for realising strong couplings between the system and its environment to suppress non-Markovian decoherence for bosonic systems. With respect to our system, it is equivalent to increasing the noise strength $\eta$ in equation (17) for decoherence suppression. In the next section, we will numerically compare the method in [28] with our coherent feedback scheme.

Note that we use the Lorentzian noise here only as an example. The Green’s function-based root locus method can be extended to any non-Markovian quantum systems where the Laplace transform of the environment’s noise correlation function is a rational function. Moreover, it should be pointed out that a similar model can be derived for bosonic system in [18] when the baths are not resonant with each other, and thereby the root locus method can be used as well.

5. Numerical simulation

In numerical simulations, we choose parameters that can be engineered in practice. The system working frequency $\hbar \omega_S = 10 \mu eV$, the frequency difference between the source and drain $\hbar \delta = 0.05 \mu eV$ and the noise width $\hbar h = 0.3 \mu eV$. The phase of the feedback coupling is assumed to be $\theta = \frac{\pi}{2}$. Other varying parameters will be given below. The coherence of the system is measured by absolute value of the Green’s function $|g_s(t)|$ which can be analytically calculated as equation (24) or equation (30). Figure 2 shows the variations of the absolute value of open-loop Green’s function $g_s(t)$ with increasing the noise strength to realise strong couplings between the system and the bath. When the noise strength $\eta$ is set to be 0.4, $|g_s(t)|$ is oscillatingly damping as plotted in the blue line, which indicates that the dynamics of the system is in the non-Markovian regime. When $\eta$ is further increased, e.g., $\eta = 0.8$ or $\eta = 1.2$, the oscillation of $|g_s(t)|$ is enhanced. However, the damping of $|g_s(t)|$ can not be stopped. Even when $\eta$ reaches 1.6, the damping of $|g_s(t)|$ is still not changed. As pointed out by [28], the damping process can be slowed down via increasing the coupling strength between the system and the baths (equivalent to increase the noise strength $\eta$) for boson systems. Here, we observe that their strategy does not work for fermion systems.

The reason can be understood in the root locus plot figure 3 which shows the dependence of three poles of the open-loop Green’s function $g_0(t)$ with the noise coupling strength $\eta$ from 0.4 to 1.6. It is clearly shown $p_{01}$.
and \( p_{02} \) are towards the axis \( \text{Re } s = 0.15 \) with opposite directions and \( p_{03} \) is away from the imaginary axis. The three modes of \( g(t) \) have negative real parts no matter how large \( \eta \) is. The strategy in [28] can not decrease the real part value of poles approaching to the imaginary axis, which indicates the damping process can not be suppressed via increasing the couplings between the system and the bath. Note that the arrows in figure 3 indicate the varying direction of the poles with respect to \( \eta \).

The above phenomena can be well analysed from the variation of poles of \( g(t) \) with continuously increasing the feedback coupling strength \( r \) from 0 to \( 0.3\omega_S \) (see root locus plot figure 5). The three poles initially lie in the left-part of complex plane with negative real parts (as shown the starting point of three lines) causing the damping of \( |g(t)| \). When the feedback coupling strength is enhanced, the pole \( p_1 \) is driven to be close to the imaginary axis and the other poles \( p_2, p_3 \) are pushed to approach to \( \text{Re } s = 0.3 \), which are indicated by the arrows in figure 5. We can see that the real part of the pole \( p_1 \) is nearly decreased to be zero when the feedback coupling strength is sufficiently strong, which indicates that such weak damping mode can help \( g(t) \) to resist the decoherence. Compared with the open-loop strategy above, our coherent feedback scheme can effectively suppress the decoherence in quantum dot systems.

6. Conclusion

This paper presents a non-Markovian coherent feedback scheme to stabilise a single quantum dot whose natural noise baths (source and drain) are connected to form the tunable quantum tunnelling process. The mechanism of the decoherence suppression is analysed in the frequency domain via the root locus of the Green’s function which is extended from classical control theory. Compared with the open-loop strong coupling strategy, our coherent feedback scheme can suppress the damping of the system dynamics more efficiently.

Figure 3. Poles variation of the open-loop Green’s function \( G_s(s) \) versus increasing noise strength \( \eta \) from 0 to 1.6. The variation of the poles \( p_{01}, p_{02}, p_{03} \) in the non-Markovian regime corresponding to the figure 2 are plotted in red, green, and blue lines, respectively. This figure shows the scheme increasing noise strength as done in [28] can not effectively drive the poles to be close to the imaginary axis so as to slow down the damping. The arrows indicate the varying direction of the poles with respect to \( \eta \).

Figure 4. The dynamics of the absolute value of the Green’s function \( |g(t)| \) versus the increasing feedback coupling strength \( r(\omega) = r, \eta = 0.4 \). With the increasing \( r \), the value of \( |g(t)| \) is kept on a high value for a long time.
For future works, it is worthwhile to explore the application of the direct coherent feedback scheme to more complicated quantum dots systems, e.g., double quantum dot system where the Coulomb interaction between two dots exists.

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Appendix A. Derivation of non-Markovian Langevin equation under feedback

To facilitate the following derivation, we assume

\[ F_{kk'} = \begin{cases} f_k & k - k' = 1 \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.1} \]

which implies only two modes with mode difference \( l \) in each bath can be effectively coupled.

According to the Heisenberg equation in quantum mechanics

\[ \dot{\hat{o}}(t) = -\frac{i}{\hbar} [\hat{o}(t), H(t)] \tag{A.2} \]

for arbitrary operator \( \hat{o}(t) \). The motion equations for the system and bath modes can be obtained as

\[ \dot{\hat{d}}(t) = -i\omega_{25} \hat{d}(t) - i \sum_k V_{\text{hk}}^* \hat{b}_k(t) - i \sum_{k, l} V_{\text{ck} \rightarrow \text{ck}' \rightarrow \text{kl}} \hat{c}_{k-1}(t), \tag{A.3} \]

\[ \dot{\hat{b}}_k(t) = -i\omega_{2k} \hat{b}_k(t) - i f_k^* \hat{c}_{k-1}(t) - i V_{\text{hk}} \hat{d}(t), \tag{A.4} \]

\[ \dot{\hat{c}}_{k-1}(t) = -i\omega_{\text{ck} \rightarrow \text{ck}' \rightarrow \text{kl}} \hat{c}_{k-1}(t) - i f_k^* \hat{b}_k(t) - i V_{\text{ck} \rightarrow \text{ck}' \rightarrow \text{kl}} \hat{d}(t). \tag{A.5} \]

Note that to derive the above motion equations, we have used Leibniz rule, i.e., \([A, BC] = [A, B] C + B [A, C]\) for arbitrary fermion operators \( A, B, C \), where \([\cdot, \cdot]\) and \(\{\cdot, \cdot\}\) represent the commutation and anti-commutation relations, respectively.

Firstly, the motion coupling equations between two bath (A.4) and (A.5) can be jointly solved as

\[ \hat{\xi}_k(t) = \Phi_k(t) \hat{\xi}_k(0) - i \int_0^t \Phi_k(t - \tau) \nu_k \hat{d}(\tau) d\tau, \tag{A.6} \]

where

\[ \hat{\xi}_k(t) = \begin{bmatrix} \hat{b}_k(t) \\ \hat{c}_{k-1}(t) \end{bmatrix}, \quad \nu_k = \begin{bmatrix} V_{\text{hk}} \\ V_{\text{ck} \rightarrow \text{ck}' \rightarrow \text{kl}} \end{bmatrix} \]

Expressing the feedback coupling strength in the polar coordinate as \( f_k = \eta_k e^{i\phi_k} \), the transition matrix is calculated as

\[ \text{Figure 5. Root locus of the Green’s function } G(s) \text{ with respect to the feedback coupling strength } r(\omega) = r \text{ from 0 to 0.3, with the noise strength } \eta = 0.4. \text{ The pole } p_1 \text{ is pushed to be close to the imaginary axis so as to afford a very slow damping mode of } g(t), \text{ which indicates that the decoherence is effectively suppressed by our coherent feedback scheme.} \]
\[ \Phi(t) = \exp \left[ -it \left( \frac{\omega_B}{\omega} f_k^* \right) \right] = \left[ \chi e^{-i\lambda_1 t} - \chi e^{-i\lambda_2 t} \right. \]
\[ \left. \kappa(e^{-i\lambda_1 t} - e^{-i\lambda_2 t}) - \chi e^{-i\lambda_1 t} + \chi e^{-i\lambda_2 t} \right] \]  
(A.7)

where \( \chi = \frac{i}{2} (\cos \alpha \pm 1) \) and \( \kappa = \frac{i}{2} \sin \alpha e^{i\delta} \) with \( \alpha = \arctan \frac{\alpha}{\gamma} \) and frequency difference \( 2\delta = \omega_B - \omega_{\text{cl}} \). Eigenvalues \( \lambda_\pm \) of the matrix \( \left[ \begin{array}{cc} \omega_B & f_k^* \\ f_k & \omega_{\text{cl}} \end{array} \right] \) are expressed as

\[ \lambda_\pm = \frac{\omega_B + \omega_{\text{cl}} \pm \sqrt{(\omega_B - \omega_{\text{cl}})^2 + 4\gamma^2}}{2} \]  
(A.8)

Then, substituting (A.6) into (A.3), we can get the system Langevin equation as

\[ \dot{d}(t) = -i\omega B \hat{d}(t) - \int_0^t d\tau M(t - \tau) \hat{d}(\tau) - \hat{\epsilon}_n(t), \]  
(A.9)

where the memory kernel function and the equivalent noise are defined as

\[ M(t) = \sum_k \chi_k \hat{\Phi}_k(t) \nu_k, \quad \hat{\epsilon}_n(t) = i \sum_k \chi_k \hat{\Phi}_k(t) \hat{\epsilon}_k(0), \]  
(A.10)

respectively.

The memory kernel function \( M(t) \) can be further expressed as

\[ M(t) = \sum_k V_{\text{Bk}} e^{-i\delta} \cos \frac{\alpha_k}{2} + V_{\text{Ck} - 1} \sin \frac{\alpha_k}{2} \int e^{-i\lambda_1 t} \]
\[ + \sum_k V_{\text{Bk}} e^{-i\delta} \sin \frac{\alpha_k}{2} - V_{\text{Ck} - 1} \cos \frac{\alpha_k}{2} \int e^{-i\lambda_2 t} \]  
(A.11)

which are modulated by \( r_k \) and \( \theta_k \).

Under the continuous limit that the modes of the baths are too dense, the memory kernel function \( M(t) \) is expressed in a frequency continuous form and can be further decomposed as \( M(t) = M^+(t) + M^-(t) \) with

\[ M^\pm(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega J^\pm(\omega) e^{-i\lambda_{\pm}(\omega)t} \]  
(A.12)

where the noise spectral functions are

\[ J^+(\omega) = g^+(\omega) \left| V_{\text{B}}(\omega) e^{-i\delta} \cos \frac{\alpha_{\omega}}{2} + V_{\text{C}}(\omega) \sin \frac{\alpha_{\omega}}{2} \right|^2 \]  
(A.13)

\[ J^-(\omega) = g^-(\omega) \left| V_{\text{B}}(\omega) e^{-i\delta} \sin \frac{\alpha_{\omega}}{2} - V_{\text{C}}(\omega) \cos \frac{\alpha_{\omega}}{2} \right|^2 , \]  
(A.14)

and

\[ \lambda_{\pm}(\omega) = \omega - \delta \pm \sqrt{\delta^2 + \tau(\omega)^2} , \]  
(A.15)

with the state density \( \varrho(\omega) \). The noise term \( \hat{\epsilon}_n(t) \) in equation (6) is

\[ \hat{\epsilon}_n(t) = i \int_{-\infty}^{+\infty} d\omega g(\omega) v(\omega) \Phi(\omega, t) \hat{\epsilon}(\omega, 0), \]  
(A.16)

where

\[ \hat{\epsilon}(\omega, t) = \begin{bmatrix} \hat{b}(\omega, t) \\ \hat{\epsilon}(\omega - 2\delta, t) \end{bmatrix}, \quad v(\omega) = \begin{bmatrix} V_{\text{B}}(\omega, t) \\ V_{\text{C}}(\omega, t) \end{bmatrix}, \]  
(A.17)
and

\[
\Phi(\omega, t) = \left[ \chi_+ e^{-i\lambda_+(\omega)t} - \chi_- e^{-i\lambda_-(\omega)t} \right] + \frac{\kappa^2}{4} \left( e^{-i\lambda_+(\omega)t} - e^{-i\lambda_-(\omega)t} \right),
\]

where \(\chi_{\pm} = \frac{1}{2} (\cos \alpha(\omega) \pm 1)\) and \(\kappa = \frac{1}{2} \omega \sin \alpha(\omega)\) with \(\alpha(\omega) = \arctan \frac{\phi(\omega)}{\omega}\) and frequency difference \(2\delta = \omega_h(\omega) - \omega_c(\omega - 2\delta)\).

**Appendix B. Expression of memory kernel function \(M(t)\) in the frequency domain**

Inserting the Lorentzian spectral density (17) into equation (12), we obtain

\[
M_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta h^2 \frac{e^{-i\omega t}}{h^2 + (\omega - \omega_S)^2} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\eta h^2}{h^2 + (\omega - \omega_S + 2\delta)^2} e^{-i\omega t} d\omega.
\]

The integration of equation (B.1) can be solved as

\[
M_0(t) = \frac{1}{2} \eta he^{-h|t-i\omega_S t|} + \frac{1}{2} \eta he^{-h|t-i(\omega_S-2\delta)|} t.
\]

Further, via Laplace transform, we can obtain \(M_0(s)\) as

\[
M_0(s) = \frac{1}{2} \eta h s + \frac{1}{2} \eta h s + \frac{1}{2} \eta h s + i(\omega_S - 2\delta).
\]

Following the same idea, we also assume \(\omega_S \geq \sqrt{\delta^2 + \gamma^2}\) are much larger than the noise width \(h\). Therefore, we directly write transformed \(\tilde{M}(t)\) as

\[
\tilde{M}(s) = \frac{1}{2} \eta h (1 \pm \cos \theta \sin \alpha) s + \frac{1}{2} \eta h (1 \pm \cos \theta \sin \alpha) s.
\]

**Appendix C. Definition of Laplace transform**

Laplace and its inverse transform for arbitrary operator \(\hat{\delta}(t)\) are defined as

\[
\hat{O}(s) = \int_0^{\infty} \delta(t) e^{-st} dt, \quad (C.1)
\]

\[
\delta(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{O}(s) e^{st} ds, \quad (C.2)
\]

respectively, where \(s = \sigma + i\omega\).

**References**

[1] Loss D and DiVincenzo D P 1998 Phys. Rev. A 57 120
[2] Burkard G, Loss D and DiVincenzo D P 1999 Phys. Rev. B 59 2070
[3] Elzerman J M, Hanson R, Van Beveren L H W, Witkamp B, Vandersypen L M K and Kouwenhoven L P 2004 Nature 430 431
[4] Koutvar M, Ducommun Y, Heiss D, Bichler M, Schuh D, Abstreiter G and Finley J J 2004 Nature 432 81
[5] Man Z X, Xia Y J and Franco R L 2015 Sci. Rep. 5 13843
[6] Franco R L 2015 New J. Phys. 17 081004
[7] Gardiner C W and Zoller P 2004 (Berlin: Springer Verlag)
[8] Lee M T and Zhang W M 2008 J. Chem. Phys. 129 224106
[9] Wang K, Zhao X and Yu T 2014 Phys. Rev. A 89 042320
[10] Cui W, Xi Z R and Pan Y 2008 Phys. Rev. A 77 032117
[11] Chirolli L and Burkard G 2008 Adv. in Phys. 57 225
[12] Tu M Y and Zhang W M 2008 Phys. Rev. B 78 235311
[13] Xue S, Zhang J, Wu R B, Li C W and Tarn T J 2011 J. Phys.: B: At. Mol. Phys. 44 154016
[14] Leggett A J, Chakravarty S, Dorsey A T, Fisher M P A, Garg A and Zwerger W 1987 Rev. Mod. Phys. 59 1
[15] Brandes T 2010 Phys. Rev. Lett. 105 060602
[16] Pöltl C, Emary C and Brandes T 2011 Phys. Rev. B 84 085302
[17] Bluhm H, Foletti S, Mahalu D, Umansky V and Yacoby A 2010 Phys. Rev. Lett. 105 216803
[18] Xue S, Wu R B, Zhang W M, Zhang J, Li C W and Tarn T J 2012 Phys. Rev. A 86 052304
[19] Man Z X, Xia Y J and Lo Franco R 2015 Phys. Rev. A 92 012315
[20] Zhu Q S, Ding C C, Wu S Y and Lai W 2015 Eur. Phys. J. D 69 231
[21] Lloyd S 2000 Phys. Rev. A 62 022108
[22] Xue S, Wu R B, Tarn T J and Petersen I R 2015 Quantum Inf. Process. 14 2657
[23] Xue S and Petersen I R 2016 Quantum Inf. Process. 15 1001
[24] Reimann S M and Manninen M 2002 Rev. Mod. Phys. 74 1283
[25] Mahan G D 2000 Many-Particle Physics 3rd edn (New York: Kluwer Academic/Plenum Publishers)
[26] Tan H T and Zhang W M 2011 Phys. Rev. A 83 032102
[27] Jin J S, Tu M W Y, Zhang W M and Yan Y J 2010 New J. Phys. 12 083013
[28] Lei C U and Zhang W M 2011 Phys. Rev. A 84 052116
[29] Ogata K 1996 Modern Control Engineering (Englewood Cliffs: PrenticeHall)
[30] Zhang W M, Lo P Y, Xiong H N, Tu M W Y and Nori F 2012 Phys. Rev. Lett. 109 170402