Propagation of vortex Hermite-cosh-Gaussian beams in a gradient-index medium

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Abstract

Based on the Huygens–Fresnel diffraction integral, the analytical expression for a vortex Hermite-cosh-Gaussian beam (vHChGB) propagating in a gradient-index medium (GIM) is derived. From the obtained expression, the evolution of the intensity and the phase distributions of the vHChGB through GIM are analyzed numerically as a function of the gradient-index parameter $\beta$ and under the change of the incident beam parameters. The results show that the output beam evolves periodically versus the propagation distance, and the period of the self-repetition slows down as the parameter $\beta$ is increased. It is found that the self-repetition of the intensity and phase distributions of the field in GIM is altered by the incident beam parameters. The results obtained may be beneficial for applications in fiber communications and beam shaping.

Keywords Vortex Hermite-cosh-Gaussian beams · Gradient-index-medium · Self-repetition · Propagation properties

1 Introduction

The gradient index medium (GIM) is an optical material with continuous distribution and quadratic dependence of the refraction index. Such a medium allows the self-focusing of propagating light beams. Historically, the first model for a nonhomogeneous graded index optical medium is known as Maxwell fisheye lens, which was proposed by Maxwell in 1854 (Moore 1980). Much time later, the gradient index glass rods have been introduced in imaging devices (Uchida et al. 1970). Since then, the GIM has been the subject of extensive research for practical applications in optics, e.g., in imaging, optical communication, optical sensing and optical fibre manufacturing (Song et al. 1998; Leger and Kunkel 2016; Gomez-Reino et al. 2002). In the last few years, the propagation properties of laser beams in GIM have been extensively investigated,
including those of cosine-Gaussian beams (Song et al. 2011), Lorentz-Gaussian beams (Zhou 2013), Airy-Gaussian vortex beams (Zhao et al. 2016), hollow sinus hyperbolic-Gaussian beams (Zou et al. 2017), chirped Airy beams (Feng et al. 2017), Gaussian vortex beams (Yang et al. 2020) and higher-order cosine-hyperbolic-Gaussian beams (Saad and Belafhal 2021).

On the other hand, a beam model named as vortex Hermite-cosine hyperbolic Gaussian beam (vHChGB) has recently been introduced and its free space propagation characteristics have been investigated by Hricha et al. (2021a). This beam can be regarded as the extended form of the vortex cosh-Gaussian beam (vChGB) (Hricha et al. 2020). In the source plane, a vHChGB can be characterized by three key parameters, namely the beam index order \( n \) which is associated with the order of the Hermite polynomial, the decentered parameter \( b \) which is associated with the cosh part, and the vortex charge index \( M \) associated with the topological charge. The vHChGB cross section is square shaped, and its intensity profile depends strongly on the value of the parameter \( b \): For a small value of \( b \), the beam is hollow dark multi-petal-like with a dark central region surrounded by \( 4n \) lobes, while for a large value of \( b \) the beam profile is four petal-like. Under special parameters conditions, a vHChGB can reduce to well-known laser beams, such as the hollow vortex-Gaussian beam (Zhou et al. 2013), vortex Hermite-Gaussian beam (Kotlyar et al. 2015), vortex-cosh-Gaussian beam (Hricha et al. 2020, 2021b, 2021c, 2021d, 2021e), and so on. The high number of control parameters and the presence of topologic vortex charge in vChGBs allow potential applications in beam shaping and in micromanipulations (Simpson et al. 1997). Up to now, the propagation properties of vHChGBs in various optical media have been reported (Hricha et al. 2021f, 2021g), however, one can note that the propagation of these beams through the GIM has not been treated yet, to the best of our knowledge. Therefore, the present work is aimed at investigating the evolution properties of a vHChGB passing through a GIM. The remainder of the paper is structured as follows: the analytical expression for a vHChGB in GIM is derived in Sect. 2 based on the Huygens–Fresnel diffraction integral. In Sect. 3, the evolutions of the beam intensity and phase distributions are analysed numerically as functions of medium parameter \( \beta \) and under the change of the incident beam parameters \( b, n, M, \) and \( \omega_0 \). Finally, the main results obtained in this paper are summarized in the conclusion part.

### 2 Theoretical model

In the present study, we consider an optical medium having a radial refractive-index distribution of the form (Moore 1980)

\[
n(r) = n_0 \left( 1 - \frac{1}{2} \beta^2 r^2 \right),
\]

where \( r = \sqrt{x^2 + y^2} \) is the radial coordinate with respect to the z-axis (the propagation axis), and \((x, y)\) are the transverse Cartesian coordinates. \( n_0 \) is value of the refractive index along the \( z \) axis, and \( \beta \) is the parameter associated with the parabolic dependence of the refractive index.

Within matrix optics, for a light beam propagating in GIM along the \( z \)-direction, the ABCD matrix associated with the optical system is given as (McMullin 1986)
The electric field of a symmetrical vHChGB propagating along the z-axis can be expressed in the source plane \( z = 0 \) as (Hricha et al. 2021a)

\[
E_0(x_0, y_0, z = 0) = E_p(x_0, z = 0) E_p(y_0, z = 0) (x_0 + iy_0)^M, \tag{3}
\]

where

\[
E_p(u, z = 0) = H_p\left(\frac{\sqrt{2}u}{\omega_0}\right) \cosh\left(\frac{b}{\omega_0}u\right) \exp\left(-\frac{u^2}{\omega_0^2}\right), \tag{4}
\]

with \( u = x_0 \) or \( y_0 \), and \((x_0, y_0)\) are the transverse coordinates at the source plane. The subscript \( p \) denotes the mode index associated with the order of Hermite polynomial \( H_p(\cdot) \), \( \cosh() \) stands for the hyperbolic-cosine function, \( \omega_0 \) is the waist size of the Gaussian part, \( b \) is the decentered parameter associated with the \( \cosh \) part, and \( M \) is an integer which denotes the vortex charge index.

Within the framework of the paraxial approximation, the propagation of a vHChGB through a GIM can be described by the Huygens-Fresnel diffraction integral of the form (Collins 1970)

\[
E(x,y,z) = \frac{ik}{2\pi B} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_0(x_0, y_0, 0) \exp\left\{ -\frac{ik}{2B} [A(x_0^2 + y_0^2) - 2(xx_0 + yy_0) + D(x^2 + y^2)] \right\} dx_0 y_0, \tag{5}
\]

where \( E_0(x_0, y_0, 0) \) and \( E(x, y, z) \) are the beam fields at the source and output planes, respectively. \( z \) is the distance from the initial plane to the output plane. \( A, B \) and \( D \) are the matrix elements associated with GIM, and \( k = \frac{2\pi}{\lambda} \) is the wave number, with \( \lambda \) being the wavelength of light in vacuum.

Substituting Eq. (3) into Eq. (5), and using the expanding series of the Hermite polynomials and the binomial formula following (Gradshteyn and Ryzhik 1994),

\[
H_j(x) = \sum_{p=0}^{[j/2]} \frac{(-1)^p j!}{p!(j - 2p)!} (2x)^{j-2p}, \tag{6a}
\]

\[
(x + iy)^M = \sum_{l=0}^{M} C_l^M x^{M-l}(iy)^{M-l}, \quad \text{with} \quad C_l^M = \frac{M!}{l!(M-l)!}. \tag{6b}
\]

Then, recalling the integral formula (Belafhal et al. 2020)

\[
\int_{-\infty}^{+\infty} x^n e^{-px^2+2qx} dx = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{p}\right) \left(\frac{1}{2i\sqrt{p}}\right)^n H_n\left(\frac{iq}{\sqrt{p}}\right), \tag{7}
\]

and after doing lengthy but straightforward algebraic calculations, one obtains
\[ E(x, y, z) = \frac{ik}{2\pi^\frac{1}{\beta}} \sin(\beta z) \exp \left[ -\frac{ik \cos(\beta z)}{2B} \left( x^2 + y^2 \right) \right] \left( \frac{\pi}{4\eta} \right)^M \sum_{l=0}^{M} C^M_l \right) 
\times \exp \left( \frac{f_z^2(x)}{\eta} \right) H_{l+p-2l} \left( \frac{if_z(x)}{\sqrt{\eta}} \right) 
\times \exp \left( \frac{f_z^2(y)}{\eta} \right) H_{M-l+p-2l} \left( \frac{if_z(y)}{\sqrt{\eta}} \right) 
\times \left[ \left( \frac{2\sqrt{2}}{\alpha_0} \right)^{p-2s} \left( \frac{1}{2i\eta} \right)^2 \left( \frac{1}{2i\eta} \right) \frac{\exp \left( \frac{f_z^2(x)}{\eta} \right)}{\eta} H_{l+p-2l} \left( \frac{if_z(x)}{\sqrt{\eta}} \right) 
\exp \left( \frac{f_z^2(y)}{\eta} \right) H_{M-l+p-2l} \left( \frac{if_z(y)}{\sqrt{\eta}} \right) 
\right]. \]

where

\[ \eta = \frac{1}{\omega_0^2} + \frac{ik\beta}{2\tan(\beta z)}, \] (9a)

and

\[ f_z(u) = \frac{ik\beta u}{2 \sin(\beta z)} \pm \frac{b}{2\omega_0}. \] (9b)

Equation (8) is the main formula for a vHChGB propagating in GIM, from which we can see that the output beam depends on the initial beam parameters, the gradient-index parameter \( \beta \), and the propagation distance \( z \).

One can deduce from Eq. (8) that the output field is periodic versus the propagation distance \( z \), and the period is \( L = \frac{2\pi}{\beta} \).

As is known, the intensity \( I(x, y, z) \) of the output beam is defined as

\[ I(x, y, z) = |E(x, y, z)|^2. \] (10)

Thus, by inserting Eq. (8) into Eq. (10), one can directly obtain the analytical expression of the intensity for a vHChGB propagating in GIM.

According to the symmetry of the field expression derived in Eq. (8), one may expect that the period of self-repetition of the beam intensity in GIM will be less than or equal to \( \frac{L}{2} \).

By using the formulae obtained above one can conveniently analyze the effects of the gradient-index parameter \( \beta \) and the incident beam parameters on the self-repetition of the intensity and phase properties for a vHChGB in the GIM.

### 3 Numerical calculations and discussion

In this section, in order to analyse the evolution properties of a vHChGB in GIM, we have illustrated numerically, based on Eqs. (8) and (10), the transverse intensity and phase distributions of the output beam versus the propagation distance \( z \) under different beam
parameters conditions. In the numerical calculations, the parameters are set as \( \omega_0 = 1 \text{ mm} \), \( \lambda = 632.8 \text{ nm} \) and \( \beta = 0.3 \text{ mm}^{-1} \), except it is indicated. As it is previously reported, the initial vHChGB can have two types of profile depending on the magnitude of the parameter \( b \). That is, for a small value of \( b \) (typically, saying \( b = 0.1 \)), the vHChGB is an array pattern-like while for a large value of \( b \) (saying \( b = 4 \)), the beam profile is four petal-like. Therefore, in the following we will present the numerical examples corresponding to both initial beam configurations.

Figures 1, 2 and 3 show the intensity and phase distributions of the output beam at different propagation distances for incident vHChGBs (in small \( b \) and large \( b \) configurations) with \( M = 1 \) and \( p = 1 \) or 2.

From Fig. 1 one can clearly see that although the GIM cannot support the incident vHChGB as stationary solitons, it allows its intensity distributions to evolve periodically with the propagation distance \( z \), and the period is equal \( L/2 \) (with \( L = \frac{2\pi}{\beta} \)). Within the first half of period, i.e., \( 0 < z \leq L/4 \), the energy of the beam converges towards the center. The beam lobes bonds progressively and a new dark hollow structure is formed. The self-focused output beam attains its minimum spot width at \( z = L/4 \). Within the second half of the period, i.e., \( L/4 < z \leq L/2 \), the beam is defocused and its evolution behavior is the reverse process of that in the first half of period. The beam finally retrieves its initial shape at the plane \( z = L/2 \) and repeats the same evolution behavior periodically.

One can also notice that the profile of the output beam at half of the period \( (z = L/4) \) depends strongly on the combined values of the beam parameters \( b \), \( p \), and \( M \). Further numerical illustrations of vChGBs propagating in GIM are shown in Fig. 2, where we have depicted the beam propagation trajectory (with contour maps of the intensity distribution) in the \((x = y, z)\) plane. The plots of Fig. 2 clearly confirmed the characteristics of periodic evolution of vChGB in GIM.

From Fig. 3, one can see that the beam phase distributions take initially on diverging rays and morph gradually into distorted concentric ringed and spiral patterns for \( p = 1 \) and \( p = 2 \), respectively. With the increase of propagation distance, the phase distributions undergo rotation and combination. One can note that the period of the phase evolution is

![Figure 1](image-url)  
**Fig. 1** The intensity distribution of vHChGBs in GIM at different propagation distances \( z \), for \( M = 1 \), with \( \omega_0 = 1 \text{ mm} \), \( \lambda = 632.8 \text{ nm} \) and \( \beta = 0.3 \text{ mm}^{-1} \). Rows a, b for \( p = 1 \) and c, d for \( p = 2 \).
two times that of the intensity evolution. One can see that the vHChGB has the same pattern at the positions of $z=0$, $z=L/2$ and for $z=L/8$, $z=5L/8$. In addition, it can be clearly seen that during propagation, the rotation of the phase spiral changes from a clockwise to counterclockwise sense.

The effects of the topological charge $M$ on the evolution of the intensity and phase distributions of a vHChGB in GIM are plotted in Figs. 4 and 5, respectively. It is seen from Fig. 4 that the intensity distributions at the output plane is quite similar for the three values of $M$ ($M=1, 2$ and $3$), except at the positions $z=L/4$ and $z=3L/4$ for which the profile of the beam is strongly dependent on the value of $M$. One can note that at these output plane

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**Fig. 2** Propagation trajectory of vHChGBs through GIM in the $(x = y, z)$ plane, for $M = 1$, with $\omega_0 = 1 \text{ mm}$, $\lambda = 632.8 \text{ nm}$ and $\beta = 0.3 \text{ mm}^{-1}$. Rows a, b for $p = 1$ and c, d for $p = 2$.

**Fig. 3** The phase distributions of vHChGBs in the GIM. The conditions are the same as those in Fig. 1.
positions (i.e., for $z=L/4$ and $z=3L/4$) the beam has a bright center for $M=2$, in contrast with the case $M=3$ where the beam has a dark center.

Figure 5 indicates that the phase evolution is dependent on the value of $M$, particularly we can see that when $M=3$ the period is $L/2$, whereas for $M=2$ the period equals to $L$.

Figure 6 displays the intensity distributions of a vHChGB for three values of $\beta$ ($\beta=0.1$, $0.2$ and $0.3$ mm$^{-1}$). One can see that period of evolution of the output beam slows down as $\beta$ is increased. The influence of the value of $\beta$ on the beam behaviour in GIM is clearly demonstrated in Fig. 7, where we can see in obvious manner the period of evolution decreased as $\beta$ is increased.

In Fig. 8, we have illustrated the effect of the beam waist $\omega_0$ on the intensity distributions in the x-direction for a vHChGB in the GIM at the self-focus plane (i.e., the output plane corresponding to the minimum beam width $z=0.3$ m). The plots show that the dark central region of the beam at the self-focus plane increases as $\omega_0$ is increased. Thus, from

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Fig. 4 The normalized intensity distribution of vHChGBs in the GIM at different propagation distances $z$, with $\omega_0 = 1$ mm, $\lambda = 632.8$ nm, $\rho = 1$ and $\beta = 0.3$ mm$^{-1}$. Lines a, b for $M=2$, and lines c, d for $M=3$

Fig. 5 Phase distributions of vHChGBs in GIM at different propagation distances $z$. The conditions are the same as those in Fig. 3
the above numerical simulations, it is demonstrated that the self-repetition properties and the intensity and the phase distributions of a vHChGB in GIM are altered by the combined effects of the incident beam parameters $b$, $p$, $M$ and $\omega_0$ in addition to the gradient-index parameter $\beta$. The control properties and the richness of the output beam can be beneficial for applications in beam shaping, micromanipulation of particles, fiber communications, and so on.

4 Conclusion

In summary, we have investigated the propagation characteristics of a vHChGB propagating in the GIM. The analytical expression for a vHChGB passing through a GIM has been derived in detail. From the obtained expression, the evolution of the intensity and the phase distributions at the output plane are illustrated numerically as functions of the propagation distance and under the change of the gradient-index parameter $\beta$ and the source beam parameters. The evolution properties of a vHChGB in GIM are shown to exhibit the self-repetition characteristics in the inhomogeneous media. It is demonstrated that the gradient-index parameter $\beta$ determines the period of the evolution of the intensity and the phase distributions. The period of evolution of the output beam
becomes shorter when $\beta$ is increased. The period of the phase distribution is affected by the topological charge $M$. The period is equal $L/2$ and $L$ for $M=2$ and $M=3$, respectively. The results obtained might have applications in beam shaping, micromanipulation of particles, fiber communications, and so on.
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