Abstract—A new graph dual formalism is presented for the analysis of line outages in electricity networks. The dual formalism is based on a consideration of the flows around closed cycles in the network. A new formula for the computation of Line Outage Distribution Factors (LODFs) is derived, which is not only computationally faster than existing methods, but also generalizes easily for multiple line outages. In addition, the dual formalism provides new physical insight for how the effects of line outages propagate through the network. For example, a single line outage can be shown to induce monotonically decreasing flow changes in the grid, which are mathematically equivalent to an electrostatic dipole field. A geometric heuristic for calculating LODFs is also derived that performs well compared to direct calculation.

Index Terms—Line Outage Distribution Factor, DC power flow, dual network, graph theory

I. INTRODUCTION

The robustness of the power system relies on its ability to withstand disturbances, such as line and generator outages. The grid is usually operated with ‘n−1 security’, which means that it should withstand the failure of any single component, such as a transmission circuit or a transformer. The analysis of such contingencies has gained in importance with the increasing use of generation from variable renewables, which has led to larger power imbalances in the grid and more situations in which transmission lines are loaded close to their thermal limits [1]–[6].

A crucial tool for contingency analysis is the use of Line Outage Distribution Factors (LODFs), which measure the sensitivity of active power flows in the network to outages of specific lines [7]. LODFs are not only used to calculate power flows after an outage, but are also employed in security-constrained linear optimal power flow (SCLOPF), where power plant dispatch is optimized such that the network is always n−1 secure [7].

LODF matrices can be calculated from Power Transfer Distribution Factors (PTDFs) [8], [9], which describe how power flows change when power injection is shifted from one node to another. In [10], a dual method for calculating PTDFs was presented. The dual method is based on an analysis of the flows around closed cycles in the network graph; for a plane graph, a basis of these closed cycles corresponds to the nodes of the weak dual graph [11]. In this paper the dual formalism is applied to derive a new direct formula for the LODF matrices, which can be easily extended to take account of multiple line outages. Moreover, the dual formalism is not just a calculational tool: it provides new insight into the physics of how the effects of outages propagate in the network. This leads to several useful results, including a heuristic method to estimate the LODF based on geometric considerations.

II. THE PRIMAL FORMULATION OF LINEARIZED NETWORK FLOWS

In this section the linear power flow formulation for AC networks is reviewed and a compact matrix notation is introduced.

For AC networks in which the following conditions hold:

• branch series reactances $x_ℓ$ for any branch $ℓ$ are much larger than their series resistances $r_ℓ$, so that the resistances may be neglected;
• branch shunt admittances may be assumed to vanish;
• voltages are maintained at nominal voltage at all nodes;
• voltage angle differences across the branches are small enough such that $\sin(\theta_m - \theta_n) \approx \theta_m - \theta_n$;

the flow of active power can be approximated as a linear function of nodal active power injections. These conditions usually hold in transmission networks with overhead lines. The usefulness of the linear approximation, often called the ‘DC approximation,’ is discussed in [12], [13].

Under these assumptions, the directed active power flow $F_ℓ$ on a line $ℓ$ from node $m$ to node $n$ can be expressed in terms of the line series reactance $x_ℓ$ and the voltage angles $θ_m, θ_n$ at the nodes

$$F_ℓ = \frac{1}{x_ℓ}(θ_m - θ_n) = b_ℓ(θ_m - θ_n), \tag{1}$$

where $b_ℓ = 1/x_ℓ$ is the susceptance of the line. In the following we do not distinguish between transmission lines and transformers, which are treated the same.

If the power flows of all lines are written in vector form, $F = (F_1, \ldots, F_L) \in \mathbb{R}^L$, and similarly for the nodal voltage angles $\theta = (θ_1, \ldots, θ_N) \in \mathbb{R}^N$, then equation (1) can be written compactly in matrix form

$$F = B_d K^t \theta, \tag{2}$$

where $B_d$ is the diagonal branch susceptance matrix $B_d = \text{diag}(b_1, \ldots, b_L) \in \mathbb{R}^{L \times L}$, $K$ is the incidence matrix $K$ and the superscript $t$ denotes the transpose of a matrix.

The incidence matrix $K \in \mathbb{R}^{N \times L}$ encodes how the nodes of the directed network graph are connected by the lines [14]. It has components

$$K_{n,ℓ} = \begin{cases} 1 & \text{if line } ℓ \text{ starts at node } n, \\ -1 & \text{if line } ℓ \text{ ends at node } n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$
In homology theory $K$ is the boundary operator from the vector space of lines $\cong \mathbb{R}^L$ to the vector space of nodes $\cong \mathbb{R}^N$.

The incidence matrix also relates the nodal power injections at each node $P = (P_1, \ldots, P_N) \in \mathbb{R}^N$ to the flows incident at the node

$$P = KF.$$  (4)

This is Kirchhoff’s Current Law expressed in terms of the active power: the net power flowing out of each node must equal the power injected at that node. (Active power is proportional to current, since all voltages are equal to the nominal voltage and reactive power is neglected.)

Combining (2) and (4), we obtain an equation for the power injections in terms of the voltage angles,

$$P = K B_d K^t \theta.$$  (5)

Through equations (2) and (5), there is now a linear relation between the line flows $F$ and the nodal power injections $P$.

It is useful to define the nodal susceptance matrix $B \in \mathbb{R}^{N \times N}$ by

$$B = K B_d K^t$$  (6)

such that the nodal power in can be expressed in terms of the voltage angles even more compactly,

$$P = B \theta.$$  (7)

The matrix $B \in \mathbb{R}^{N \times N}$ is a weighted Laplacian matrix

$$B_{m,n} = \begin{cases} \sum_{\ell \in \Lambda_m} b_{\ell} & \text{if } m = n; \\ -b_{\ell} & \text{if } m \text{ is connected to } n \text{ by } \ell, \end{cases}$$  (8)

where $\Lambda_m$ is the set of lines which are incident on $m$. Equation (7) is then a discrete Poisson equation.

For a connected network, $B$ has a single zero eigenvalue and therefore cannot be inverted directly. Instead, the Moore-Penrose pseudo-inverse $B^*$ can be used to solve (7) for $\theta$ and obtain the line flows directly as a linear function of the nodal power injections

$$F = B_d K^t B^* P.$$  (9)

This matrix combination is taken as the definition of the nodal Power Transfer Distribution Factor (PTDF) $\text{PTDF} \in \mathbb{R}^{L \times N}$

$$\text{PTDF} = B_d K^t B^*.$$  (10)

Next, the effect of a line outage is considered. Suppose the flows before the outage are given by $F_k$ and the line which fails is labeled $\ell$. The line flows after the outage of $\ell$, $F_k^{(\ell)}$ are linearly related to the original flows by the matrix of Line Outage Distribution Factors (LODFs) [7], [15]

$$F_k^{(\ell)} = F_k + \text{LODF}_{k\ell} F_\ell,$$  (11)

where on the right hand side there is no implied summation over $\ell$. It can be shown [3], [2] that the LODF matrix elements can be expressed directly in terms of the PTDF matrix elements as

$$\text{LODF}_{k\ell} = \frac{[\text{PTDF} \cdot K]_{k\ell}}{1 - [\text{PTDF} \cdot K]_{\ell\ell}}.$$  (12)

For the special case of $k = \ell$, one defines $\text{LODF}_{kk} = -1$. The matrix $[\text{PTDF} \cdot K]_{k\ell}$ can be interpreted as the sensitivity of the flow on $k$ to the injection of one unit of power at the from-node of $\ell$ and the withdrawal of one unit of power at the to-node of $\ell$.

III. CYCLES AND THE DUAL GRAPH

The power grid defines a graph $G = (V, E)$ with vertex set $V$ formed by the nodes or buses and edge set $E$ formed by all transmission lines and transformers. The orientation of the edges is arbitrary but has to be fixed because calculations involve directed quantities such as the real power flow. The directed cycles of a graph $G$ are combinations of directed edges of the graph which form a closed loop. A connected graph with $N$ nodes and $L$ edges has $L - N + 1$ independent cycles. A basis for the cycles can be defined using the cycle-edge incidence matrix $C \in \mathbb{R}^{L \times (L-N+1)}$

$$C_{\ell,c} = \begin{cases} 1 & \text{if edge } \ell \text{ is element of cycle } c, \\ -1 & \text{if reversed edge } \ell \text{ is element of cycle } c, \\ 0 & \text{otherwise.} \end{cases}$$  (13)

It is a result of graph theory, which can also be checked by explicit calculation, that the $L - N + 1$ cycles are a basis for the kernel of the incidence matrix $K$ [11].

$$KC = 0.$$  (14)

Using the formalism of cycles, the Kirchhoff Voltage Law (KVL) can be expressed in a concise way. KVL states that the sum of all angle differences along any closed cycle equals zero,

$$\sum_{(i,j) \in \text{cycle } c} (\theta_i - \theta_j) = 0.$$  (15)

Since the cycles form a vector space it is sufficient to check this condition for the $L - N + 1$ basis cycles. In matrix form this reads

$$C^t K^t \theta = 0,$$  (16)

which is satisfied automatically by virtue of equation (14).

Using equation (2), the KVL in terms of the flows reads

$$C^t X_d F = 0,$$  (17)

where $X_d$ is the branch reactance matrix, defined by $X_d = \text{diag}(x_1, \ldots, x_L) = \text{diag}(1/b_1, \ldots, 1/b_L) \in \mathbb{R}^{L \times L}$.

In this paper, we are especially interested in planar graphs, i.e., graphs which can be drawn, or embedded in the plane $\mathbb{R}^2$ without edge crossings. Once such an embedding is fixed, the graph is called a plane graph. Power grids are naturally embedded in $\mathbb{R}^2$, and while transmission line crossings are possible, they are rare. In case of a plane graph we can construct the cycle basis from the faces of the graph and according to Mac Lane’s planarity criterion [11], all edges participate in at most two cycles. The simple topological properties of plane graphs are essential to derive the rigorous results obtained in section VI.

For a plane graph, the weak dual graph $DG$ of $G$ is formed by putting dual nodes in the middle of the cycle faces of $G$ as described above, and then connecting the dual points with dual edges across those edges where faces of $G$ meet [11], [14]. $DG$ has $L - N + 1$ nodes. The matrix $C$ is then the transpose of the incidence matrix of $DG$. 

IV. DUAL THEORY OF NETWORK FLOWS

In this section the linear power flow is defined in terms of dual cycle variables following [10], rather than the nodal voltage angles.

To do this, we define the linear power flow equations directly in terms of the network flows. The power conservation equation [4]

\[ K F = P, \]

provides \( N \) equations, of which one is linearly dependent, for the \( L \) components of \( F \). The solution space is thus given by an affine subspace of dimension \( L - N + 1 \).

In section [III] we discussed that the kernel of \( K \) is spanned by the cycle flows. Thus, we can write every solution of equation (18) as a particular solution of the inhomogeneous equation plus a linear combination of cycle flows:

\[ F = F^{(\text{part})} + C f, \quad f \in \mathbb{R}^{L-N+1}. \]  (19)

The components \( f_c \) of the vector \( f \) give the strength of the cycle flows for all basis cycles \( c = 1, 2, \cdots, L - N + 1 \). A particular solution \( F^{(\text{part})} \) can be found by taking the uniquely-determined flows on a spanning tree of the network graph [10].

To obtain the correct physical flows we need a further condition to fix the \( L - N + 1 \) degrees of freedom of \( F_c \). This condition is provided by the KVL in (17), which provides exactly \( L - N + 1 \) linear constraints on \( f \):

\[ C^t X_d C f = -C^t X_d F^{(\text{part})}. \]  (20)

Together with equation (18), this condition uniquely determines the power flows in the grid.

Equation (20) is the dual equation of (5). If the cycle reactance matrix \( A \in \mathbb{R}^{L-N+1 \times L-N+1} \) is defined by

\[ A = C^t X_d C, \]  (21)

then \( A \) also has the form,

\[ A_{cc'} = \begin{cases} \sum_{\ell \in \kappa_c} x_\ell & \text{if } c = c' ; \\
\pm x_\ell & \text{if } c \neq c, \end{cases} \]  (22)

where \( \kappa_c \) is the set of edges around cycle \( c \) and the sign ambiguity depends on the orientation of the cycles. The construction of \( A \) is very similar to the weighted Laplacian in equation [9]; for plane graphs where the cycles correspond to the faces of the graph, this analog can be made exact (see Section [V]). Unlike \( B \), the matrix \( A \) is invertible, due to the fact that the outer boundary cycle of the network is not included in the cycle basis. This is analogous to removing the row and column corresponding to a slack node from \( B \), but it is a natural feature of the theory, and not manually imposed.

V. DUAL COMPUTATION OF LINE OUTAGE DISTRIBUTION FACTORS

A. Single line outages

The dual theory of network flows derived in the previous section can be used to derive an alternative formula for the LODFs. For the sake of generality we consider an arbitrary change of the reactance of a transmission line \( \ell \),

\[ x_\ell \rightarrow x_\ell + \xi_\ell. \]  (23)

The generalization to multiple line outages is presented in the following section. The change of the network structure is described in terms of the branch reactance matrix

\[ \hat{X}_d = X_d + \Delta X_d = X_d + \xi_\ell u_\ell u_\ell^t, \]  (24)

where \( u_\ell \in \mathbb{R}^L \) is a unit vector which is 1 at position \( \ell \) and zero otherwise. In this section we use the hat to distinguish the line parameters and flows in the modified grid after the outage from the original grid before the outage.

This perturbation of the network topology will induce a change of the power flows

\[ \hat{F} = F + \Delta F. \]  (25)

We consider a change of the topology while the power injections remain constant. The flow change \( \Delta F \) thus does not have any source such that it can be decomposed into cycle flows

\[ \Delta F = C \Delta f. \]  (26)

The uniqueness condition (17) for the perturbed network reads

\[ C^t (X_d + \Delta X_d)(F + \Delta F) = 0. \]  (27)

Using condition (17) for the original network and the cycle flow decomposition equation (26) for the flow changes yields

\[ C^t \hat{X}_d C \Delta f = -C^t \Delta X_d F \]
\[ \Rightarrow \Delta f = - (C^t \hat{X}_d C)^{-1} C^t u_\ell \xi_\ell u_\ell^t F \]  (28)

such that the flow changes are given by

\[ \Delta F = C \Delta f = -C (C^t \hat{X}_d C)^{-1} C^t u_\ell \xi_\ell u_\ell^t F. \]  (29)

This expression suggests that we need to calculate the inverse separately for every possible contingency case, which would require a huge computational effort. However, we can reduce it to the inverse of \( A = C^t X_d C \) describing the unperturbed grid using the Woodbury matrix identity [16].

\[ (C^t \hat{X}_d C)^{-1} = (A + C^t u_\ell \xi_\ell u_\ell^t C)^{-1} = A^{-1} - A^{-1} C^t u_\ell \left( \xi_\ell^t + u_\ell^t C A^{-1} C^t u_\ell \right)^{-1} u_\ell^t C A^{-1}. \]

Thus we obtain

\[ (C^t \hat{X}_d C)^{-1} C^t u_\ell = A^{-1} C^t u_\ell \left( 1 + \xi_\ell u_\ell^t C A^{-1} C^t u_\ell \right)^{-1}. \]

We then obtain the induced cycle flows and flow change by inserting this expression into equation (29). We summarize our results in the following proposition.

Proposition 1. If the reactance of a single transmission line \( \ell \) is changed by an amount \( \xi_\ell \), the real power flows change as

\[ \Delta F = \frac{-\xi_\ell F_\ell}{1 + \xi_\ell u_\ell^t C A^{-1} C^t u_\ell} C A^{-1} C^t u_\ell. \]  (30)
If the line $\ell$ fails, we have $\xi_{\ell} \to \infty$. The line outage distribution factor for a transmission line $k$ is thus given by

$$LODF_{k,\ell} = \frac{\Delta F_k}{F_{\ell}} = -\frac{u_k^TCA^{-1}C^Tu_\ell}{u_{k\ell}^TCA^{-1}C^Tu_{\ell}}.$$  \hfill (31)

Finally, an example of failure induced cycle flows and the corresponding flow changes is shown in Figure 1. In the example, the node-edge incidence matrix is given by

$$K = \begin{pmatrix}
+1 & -1 & +1 & 0 & 0 & 0 \\
-1 & 0 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & -1 & +1 & 0 \\
0 & +1 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & +1
\end{pmatrix}.$$  \hfill (32)

The grid contains 2 independent cycles, which are chosen as cycle 1: line 2, line 6, line 3. cycle 2: line 1, line 4, line 5, reverse line 2. The cycle-edge incidence matrix thus reads

$$C' = \begin{pmatrix}
0 & +1 & +1 & 0 & 0 & +1 \\
+1 & -1 & 0 & +1 & +1 & 0
\end{pmatrix}. \hfill (33)$$

Thus, the flow changes can be written according to equation (26) with

$$\Delta f = (122.4 \text{ MW}, 64.4 \text{ MW})^t.$$  \hfill (34)

(c.f. also \cite{18} for a discussion of cycle flows in power grids).

The dual approach to the LODFs can be computationally advantageous for sparse networks as discussed in section \ref{sec:computational_aspects}. Furthermore, we will use it to prove some rigorous results on flow redistribution after transmission line failures in section \ref{sec:proofs}.

### B. Multiple line outages

The dual approach can be generalized to the case of multiple damaged or perturbed transmission lines in a straightforward way. Consider the simultaneous perturbation of the $M$ transmission lines $\ell_1, \ell_2, \ldots, \ell_M$ according to

$$x_{\ell_1} \to x_{\ell_1} + \xi_{\ell_1}, x_{\ell_2} \to x_{\ell_2} + \xi_{\ell_2}, \ldots, x_{\ell_M} \to x_{\ell_M} + \xi_{\ell_M}.$$  

The change of the branch reactance matrix is then given by

$$\Delta X_d = \Xi \Xi^t,$$  \hfill (35)

where we have defined the matrices

$$\Xi = \text{diag}(\xi_{\ell_1}, \xi_{\ell_2}, \ldots, \xi_{\ell_M}) \in \mathbb{R}^{M \times M},$$

$$\mathcal{U} = (u_{\ell_1}, u_{\ell_2}, \ldots, u_{\ell_M}) \in \mathbb{R}^{N \times M}.$$  

The formula (29) for the flow changes then reads

$$\Delta F = -C' \hat{X}_d C \Xi \Xi^t C' \mathcal{U} \Xi \Xi^t C' \mathcal{U} F.$$  \hfill (36)

To evaluate this expression we again make use of the Woodbury matrix identity \cite{16}, which yields

$$\left(C' \hat{X}_d C\right)^{-1} = A^{-1} - A^{-1} C' \mathcal{U} \left(\Xi^{-1} + \mathcal{U}' CA^{-1} C' \mathcal{U}\right)^{-1} \mathcal{U}' CA^{-1}.$$  \hfill (37)

We then obtain the flow change by inserting this expression into equation (36) with the result

$$\Delta F = -CA^{-1} C' \mathcal{U} \left(\| + \Xi \mathcal{U}' CA^{-1} C' \mathcal{U}\right)^{-1} \Xi \mathcal{U}' F.$$  \hfill (38)

In case of a multiple line outages of lines $\ell_1, \ldots, \ell_m$ we have to consider the limit

$$\xi_{\ell_1}, \ldots, \xi_{\ell_m} \to \infty.$$  \hfill (39)

In this limit equation (37) reduces to

$$\Delta F = -CA^{-1} C' \mathcal{U} \left(\mathcal{U}' CA^{-1} C' \mathcal{U}\right)^{-1} \mathcal{U}' F.$$  \hfill (39)

### C. Computational aspects

The dual formula (31) for the LODFs can be computationally advantageous to the conventional approach. To calculate the LODFs via equation (12) we have to invert the matrix $B \in \mathbb{R}^{N \times N}$ to obtain the PTDFs. Using the dual approach the most demanding step is the inversion of the matrix $A = C' X_d C \in \mathbb{R}^{(L-N+1) \times (L-N+1)}$, which can be much smaller than $B$ if the network is sparse. However, more matrix multiplications need to be carried out, which decreases the potential speed-up. We test the computational performance of the dual method by comparing it to the conventional approach, which is implemented in many popular software packages such as for instance in MATLAB \cite{17}.

Conventionally, one starts with the calculation of the nodal PTDF matrix defined in Eq. (10). In practice, one usually does not compute the full inverse but solves the linear system of equations $PTDF \cdot B = B_d K$ instead. Furthermore, one fixes the voltage phase angle at a slack node $s$, such that one can omit the $s$th row and column in the matrix $B$ and the $s$th column in matrix $B_f = B_d K^T$ while solving the linear
TABLE I

COMPARISON OF CPU TIME FOR THE CALCULATION OF THE PTDFs IN SPARSE NUMERICS.

| Test Grid name | source | Grid Size cycles | speedup | nodes | \( \frac{L−N+1}{N} \) | \( \frac{L−N+1}{N} \) \( \frac{\text{Time}}{\text{Min}} \) |
|----------------|--------|------------------|--------|------|-----------------|-----------------|
| case300        | 17     | 300              | 1.83   | 110  | 0.37            | 1.83            |
| case1354pegase | 17     | 1354             | 4.43   | 357  | 0.26            | 4.43            |
| GNetwork       | 20     | 2224             | 2.09   | 581  | 0.26            | 2.09            |
| case2383wp     | 17     | 2383             | 4.20   | 504  | 0.21            | 4.20            |
| case2736sp     | 17     | 2736             | 3.27   | 760  | 0.28            | 3.27            |
| case2746wp     | 17     | 2746             | 2.97   | 760  | 0.28            | 2.97            |
| case2869pegase | 17     | 2869             | 2.79   | 1100 | 0.38            | 2.79            |
| case3012wp     | 17     | 3012             | 3.93   | 555  | 0.18            | 3.93            |
| case3120wp     | 17     | 3120             | 3.96   | 565  | 0.18            | 3.96            |
| case9241pegase | 17     | 9241             | 1.31   | 4967 | 0.54            | 1.31            |

system. The result is multiplied by the matrix \( K \) from the right to obtain the PTDFs between the endpoints of all lines. One then divides each column \( \ell \) by the value \( 1 − \text{PTDF}_{\ell\ell} \) to obtain the LODFs via formula (12). An implementation of these steps in MATLAB is listed in the supplement.

The dual approach yields the direct formula (31) for the LODF, but solve a linear system of equations instead. The full LODF matrix is then obtained by dividing every column \( \ell \) by the factor \( M_{\ell\ell} \).

We evaluate the runtime for various test grids from [17], [19], [20] using a MATLAB program listed in the supplement. All input matrices are sparse, such that the computation is faster when using sparse numerical methods (using the command \texttt{sparse} in MATLAB and converting back to \texttt{full} at the appropriate time). Then MATLAB employs the high-performance supernodal sparse Cholesky decomposition solver \texttt{CHOLMOD 1.7.0} to solve the linear system of equations. We observe a significant speed-up of the dual method by a factor between 1.31 and 4.43 depending on how meshed the grid is (see Table I).

VI. TOPOLOGY OF CYCLE FLOWS

In this section the propagation of the effects of line outages are analyzed using the theory of discrete calculus and differential operators on the dual network graph. There is a wide body of physics and mathematics literature on discrete field theory (see, e.g., [21]). In the previous section we showed how the cycle flow decomposition can be used as a computational tool. In this section we turn back to the cycle flows themselves and derive some rigorous results. These results help to understand the effects of a transmission line outage and the spreading of failures in power grids. The starting point is a re-formulation of Proposition 1.

Lemma 1. The outage of a single transmission line \( \ell \) induces cycle flows which are determined by the linear system of equations

\[
\Delta f = q \quad \text{(40)}
\]

with \( q = F_{\ell}(u_{\ell}^t C A^{-1} C^t u_{\ell})^{-1} C^t u_{\ell} \) and \( A = C^t X_d C \).

Note that \( \Delta f, q \in \mathbb{R}^{(L−N+1)} \) and \( A \in \mathbb{R}^{(L−N+1) \times (L−N+1)} \). It will now be shown that this equation can be interpreted as a discrete Poisson equation for \( \Delta f \) with Laplacian operator \( A \) and inhomogeneity \( q \). This formulation is convenient to derive some rigorous results on flow rerouting after a transmission line failure.

We first note from the explicit construction of \( A \) in equation (22) that two cycles in the dual network are only coupled via their common edges. The coupling is given by the sum of the reactances of the common edges. Generally, the reactance of a line is proportional to its length. The coupling of two cycles is then directly proportional to the total length of their common boundary, provided that the lines are all of the same type. Since the inhomogeneity \( q \) is proportional to \( C^t u_{\ell} \), it is non-zero only for the cycles which are adjacent to the failing edge \( \ell \):

\[
q_c \neq 0 \quad \text{only if } \ell \text{ is an element of cycle } c. \quad (41)
\]

The matrix \( A \) typically has a block structure such that a failure in one block cannot affect the remaining blocks. The dual approach to flow rerouting gives a very intuitive picture of this decoupling. To see this, consider the example shown in Figure 2. The cycle at the top of the network is connected to the rest of the network via one node. However, it is decoupled in the dual representation because it shares no common edge with any other cycle. Thus, a failure in the rest of the grid will not affect the power flows in this cycle—the mutual LODFs vanish. This result is summarized in the following proposition, and a formal proof is given in the supplement.

Proposition 2. The line outage distribution factor LODF\(_{k,\ell}\) between two edges \( k = (i, j) \) and \( \ell = (s, r) \) vanishes if there is only one independent path between the vertex sets \( \{r, s\} \) and \( \{i, j\} \).
Some important simplifications can be made in case of a plane network. We can then define the cycle basis in terms of the interior faces of the graph which allows for an intuitive geometric picture of induced cycle flows as in Figures 2 and 3. For the remainder of this section we thus restrict ourselves to such plane graphs and fix the cycle basis by the interior faces and fix the orientation of all basis cycles to be counterclockwise. Thus equation (40) is formulated on the weak dual of the original graph.

According to Mac Lane’s planarity criterion, every edge in a plane graph belongs to at most two cycles such that \( q \) has at most two non-zero elements: One non-zero element \( q_{c_1} \) if \( \ell \) is at the boundary and two non-zero elements \( q_{c_2} = -q_{c_2} \) if the line \( \ell \) is in the interior of the network. Furthermore, the matrix \( A \) is a Laplacian matrix in the interior of the network \( [14] \). That is, for all cycles \( c \) which are not at the boundary we have

\[
\sum_{d \neq c} A_{dc} = -A_{cc}. \tag{42}
\]

Up to boundary effects, equation (40) is thus equivalent to a discretized Poisson equation on a complex graph with a dipole source (monopole source if the perturbation occurs on the boundary).

For plane networks we now prove some rigorous results on the orientation of cycle flows (clockwise vs. counterclockwise) and on their decay with the distance from the failing edge. In graph theory, the (geodesic) distance of two vertices is defined as the number of edges in a shortest path connecting them \([11]\). Similarly, the distance of two edges is defined as the number of vertices on a shortest path between the edges.

**Proposition 3.** Consider the cycle flows \( \Delta f \) induced by the failure of a a single line \( \ell \) in a plane linear flow network described by equation (40). The weak dual graph can be decomposed into at most two connected subgraphs (‘domains’) \( D_+ \) and \( D_- \), with \( \Delta f_c \geq 0 \forall c \in D_+ \) and \( \Delta f_c \leq 0 \forall c \in D_- \).

The domain boundary, if it exists, includes the perturbed line \( \ell \), i.e. the two cycles adjacent to \( \ell \) belong to different domains.

A proof is given in the supplement. The crucial aspect of this proposition is that the two domains \( D_+ \) and \( D_- \) must be connected. The implications of this statement are illustrated in Figure 3 in panel (2), showing the induced cycle flows when the dashed edge is damaged. The induced cycle flows are oriented clockwise above the domain boundary and counterclockwise below the domain boundary. If the perturbed edge lies on the boundary of a finite plane network, then there is only one domain and all cycle flows are oriented in the same way.

With this result we can obtain a purely geometric view of how the flow of all edges in the network change after the outage. For this, we need some additional information about the magnitude of the cycle flows in addition to the orientation. We consider the upper and lower bound for the cycle flows \( \Delta f_c \) at a given distance to the cycle \( c_1 \) with \( q_{c_1} > 0 \) and the cycles \( c_2 \) with \( q_{c_2} < 0 \), respectively:

\[
u_d = \max_{c, \text{dist}(c,c) = d} \Delta f_c,
\ell_d = \min_{c, \text{dist}(c,c) = d} \Delta f_c. \tag{43}\]

Here, dist denotes the graph-theoretic distance between two cycles or faces, i.e. the length of the shortest path between the two faces in the dual graph. We then find the following result.

**Proposition 4.** The maximum (minimum) value of the cycle flows decreases (increases) monotonically with the distance \( d \) to the reference cycles \( c_1 \) and \( c_2 \), respectively:

\[
u_d \leq \nu_{d-1}, \quad 1 \leq d \leq d_{\text{max}},
\ell_d \geq \ell_{d-1}, \quad 1 \leq d \leq d_{\text{max}}. \tag{44}\]

A proof is given in the supplement. Strict monotonicity can be proven when some additional technical assumptions are satisfied, which are expected to hold in most cases. For a two-dimensional lattices with regular topology and constant weights the cycle flows are proportional to the inverse distance (see supplement for details). However, irregularity of the network topology and line parameters can lead to a stronger, even exponential, localization \([22]\). Hence, the response of the grid is strong only in the ‘vicinity’ of the damaged transmission line, but may be non-zero everywhere in the connected component.

However, it has to be noted that the distance is defined for the dual graph, not the original graph, and that the rigorous results hold only for plane graphs. The situation is much more involved in non-planar graphs, as a line can link regions which would be far apart otherwise. Examples for the failure induced cycles flows and the decay with the distance are shown in Figure 2.

Finally, we can use the analogy to electrostatics to prove a discrete version of Green’s theorem:

**Proposition 5.** Consider a plane network and let \( S \) be an arbitrary closed path in the plane and \( E_S \) the set of all lines
crossing the path $S$. Then

$$\sum_{\ell \in S} \Delta F_\ell = 0. \quad (45)$$

We remark that all lines of the network must be included, also the failing lines. That is, if the surface $S$ crosses a failing line $\ell$, then the term $\Delta F_\ell = -F_\ell$ must be included in the sum. The formal proof is given in the supplement.

VII. GEOMETRIC ESTIMATES FOR LODFs

The insights obtained in the previous section can be used to understand and predict the effects of a line outage on a purely geometrical basis. In particular, we can use the results from Proposition [4] to estimate the orientation of the induced cycle flows. Furthermore, we know that the strength of the cycle flows decays with the distance according to Proposition [4]. Combining these results we can predict the sign (i.e., the direction) and the value of all LODFs in a plane grid on a purely geometrical basis.

Before stating an explicit algorithm we note that the decay of the induced cycle flows depends on the distance in the dual network. For two transmission lines $k$ and $\ell$ we thus define the cycle induced distance $d_c$, as the minimum distance of adjacent cycles in the dual graph. If the two edges belong to exactly the same two cycles the cycle distance is defined as zero:

$$d_c(\ell,k) = \begin{cases} 0 & \text{if } |C_{c,\ell}| = |C_{c,k}| \forall c \\ \min_{c_0, c_1 \in L} \text{dist}(c_0, c_1) + 1 & \text{otherwise}. \end{cases}$$

We then propose the following algorithm to predict LODFs.

**Algorithm 1. (Prediction of flow changes)**

1. Assign a cycle flow to the cycles directly adjacent to the failed edge $\ell$, denoted by $c_1$ and $c_2$ in the following. These cycle flows are anti-parallel to $F_\ell$.
2. Predict the orientation of all other cycle flows from the distance to the cycles $c_1$ and $c_2$. If $d(c,c_1) < d(c,c_2)$ then cycle flow $\Delta f_c$ has the same orientation as $\Delta f_{c_1}$. If $d(c,c_1) > d(c,c_2)$ then cycle flow $\Delta f_c$ has the same orientation as $\Delta f_{c_2}$. If both distances are equal, then the orientation cannot be decided.
3. The direction of flow change of each edge is then given by the direction of the strongest cycle flow in the two adjacent cycles. The flow change over a bridge and the flow changes in disconnected cycles are zero.
4. Predict the strength of the flow changes in terms of the distance, that is for some line $k$, $|\Delta F_k| = g(d_{c,k}(\ell,\ell)) \times |F_\ell|$, where $g: \mathbb{N}_0 \to \mathbb{R}_+$ is a monotonically decreasing function of the distance with $g(0) = 1$, for example an exponential decay.

The steps of the algorithm are illustrated in Figure [3] and an example is given in Figure [4]. In particular, this figure shows that the cycle flow approach accurately predicts the direction of flow changes and it illustrates the correspondence of the magnitude of flow changes and the cycle-induced distance.

The algorithm is tested for the IEEE 30-bus test case included in [17] for all possible contingency cases and the results are shown in Figure [4]. We observe that the sign of the LODFs is predicted correctly with only very few exceptions at remote locations. The correlation coefficient between the predicted and the numerically exact values is $R = 0.938$.

We stress that the algorithm does not include any matrix computation. It is purely geometric and can be carried on a sheet of paper if necessary. Its predictions are not perfect but very accurate for such a simple approach. Therefore, the value of the algorithm does not lie in the performance of any actual calculations. Instead, it strengthens our understanding of network vulnerability and resilience and can be used as a basis for heuristic algorithms for security assessment and grid operation.

VIII. CONCLUSIONS

Line Outage Distribution Factors are important for assessing the stability of a power system, in particular with the recent rise of renewables. In this paper, we described a new dual formalism for calculating LODFs, that is based on using power flows through the closed topological cycles of the network instead of using nodal voltage angles.

The dual theory yields a compact formula for the LODFs that only depends on real power flows in the network. In particular, the formula lends itself to a straightforward generalization for the case of multiple line outages. Effectively, using cycle flows instead of voltage angles changes the dimensionality of the matrices appearing in the formulae from $N \times N$ to $(L - N + 1) \times (L - N + 1)$. In cases where the network is very sparse (i.e., it contains few cycles but many nodes), this can lead to a significant speedup in LODF computation time.
a critical improvement for quick assessment of real network contingencies.

The dual theory not only yields improvements for numerical computations, it also provides a novel viewpoint of the underlying physics of power grids, in particular if they are (almost) planar. Within the dual framework, it is easy to show that single line contingencies induce flow changes in the power grid which decay monotonically in the same way as an electrostatic dipole field. This enables the construction of a simple algorithm to estimate the LODFs given only the contingent line and the network topology.

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