Dynamics of interacting particle systems: stochastic process and field theory

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Abstract. We present an approach to the dynamics of interacting particle systems, which allows us to derive path integral formulae from purely stochastic considerations. We show that the resulting field theory is a dual version of the standard theory of Doi and Peliti. This clarifies both the origin of the Cole–Hopf map between the two approaches and the occurrence of imaginary noises in effective Langevin equations for reaction–diffusion systems. The advantage of our approach is that it focuses directly on the density field. We show some applications, in particular on the zero range process, hydrodynamic limits and large deviation functional.

Keywords: driven diffusive systems (theory), stochastic particle dynamics (theory), large deviations in non-equilibrium systems
1. Introduction

1.1. Generalities
Many problems of current interest in statistical physics of out-of-equilibrium systems involve strongly interacting particles exhibiting non-trivial collective phenomena. One example is that of supercooled liquids, where the dynamics slows down dramatically as the

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glass transition is approached, due to the increasingly collective nature of the dynamics, see, for example, [1]. Another example is given by systems of diffusing particles that branch and/or annihilate; depending on the relative strength of these effects a variety of non-equilibrium transitions and anomalous scaling behaviour appear [2]. A third example is provided by systems driven out of equilibrium by external sources. A celebrated example is the one-dimensional asymmetric exclusion process for which one finds phase transitions between different non-equilibrium steady states.

Developing theoretical techniques for such difficult problems is of great importance, in view of the diversity of situations in which they appear.

A natural framework to study these collective phenomena is field theory which has been crucial in understanding equilibrium phase transitions. In the context of non-equilibrium systems, it has been already applied successfully to reaction–diffusion systems. It has also been crucial to get an handle on strong or intermediate coupling problems where no perturbative technique is at one’s disposal. Two examples are the application of the exact renormalization group to the pair contact process [3,4] and the mode coupling theory of the glass transition [5].

A field theoretical formulation of interacting particle systems which has become standard is based on the Doi–Peliti formalism (DP) [6,7]. Starting from a second quantization representation of the Master equation, one obtains, after a rather elaborate coherent state representation, a field theory representation in terms of two fields $\phi$ and $\hat{\phi}$ (see below). This has furnished the starting point of a very large number of studies [3], including exact renormalization group calculations. However, besides its intrinsic difficulty, the formalism is not transparently related to stochastic equations for the particle evolutions. Actually, the action of the field theory corresponds to a reasonable looking Langevin equation for the density of particles, except that the noise is often complex or even pure imaginary! This suggests that the field $\phi$, despite its superficial resemblance with the density, in fact lacks a direct physical interpretation [8]. Other difficulties arise when one wants to treat systems of hard core particles or of particles with non-trivial diffusion constants. The relationship with stochastic equations on the particle trajectories is particularly important to study in the hydrodynamic limit and make a connection with the large deviation functional techniques developed in [9]–[11].

The aim of this paper is to discuss in full detail how stochastic questions on particle trajectories are related to field theory. We will unveil a dual version of the DP field theory that is naturally related to stochastic equations. This will shed new light on the underlying structure of DP field theory and allow us to re-obtain some recent results from a different perspective, e.g. the large deviation functional of Bertini et al [11]. We will also present some new applications, e.g. we will derive the stochastic equations characterizing the dynamics of the zero range process.

1.2. Issues and questions through a simple example

Let us illustrate what are the main questions and issues we want to address focusing on a simple example: particles A diffusing on a lattice with diffusion constant $\gamma$ and coalescing ($A + A \rightarrow A$) when they meet on the same site with rate $\lambda$ per unit of time. Following the DP formalism, which will be detailed below, the average density of particles in the systems...
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is given by the average of a field \( \phi \) in a path integral calculation with action

\[
S = \int_{t_i}^{t_f} dt \int d^D x \left[ \dot{\phi} \left( -\partial_t \phi + \gamma \Delta \phi \right) + \lambda \phi^2 \left( 1 - \dot{\phi}^2 \right) \right] + \text{boundary terms},
\]

after the extra field \( \dot{\phi} \) has been integrated out. The quadratic part comes from diffusion, while the rest comes from the pair coalescence. After a shift \( \phi \rightarrow \phi + 1 \), the above action becomes identical to the one obtained through the Martin–Siggia–Rose–DeDominicis–Janssen technique from the Langevin equation:

\[
\partial_t \phi = \gamma \Delta \phi - 2\lambda \phi^2 + \eta \langle \eta(x,t)\eta(x',t') \rangle = -2\lambda \phi^2.
\]

The first two terms of the RHS are exactly what one would expect naively: a diffusion term plus an annihilation term. The problem is that the white noise has a negative variance implying that \( \eta \) is purely imaginary! This problem, first observed in [12], demonstrates that the field \( \phi \) is not equal to the density field, as can be seen from a direct computation. Furthermore, it suggests that either the physical sources of fluctuations in the system are not Gaussian or \( \phi \) is not a direct probe of fluctuations, or both. Still, stochastic equations for the density field certainly exist. Actually, in many cases one starts from a phenomenological stochastic equation to get the field theory and not the other way round. But then, understanding the relationship between stochastic equations for the particle trajectories and the DP field theory is crucial. This is the main aim of our paper which is based on our joint recent work [5] with Andreanov and Bouchaud.

The main questions we want to address are:

- What are the stochastic equations governing the evolution of particles?
- How are they related to field theory? In particular, starting from these stochastic equations and using the MSRDJ technique what type of field theory is obtained? How is this field theory related to the standard DP one?
- What is this field theory useful for?

In the following we will answer the first two questions in detail. As for the third, we will show some applications, discuss some other promising ones and hope that the readers will find new ones.

1.3. Examples of systems of interacting particles

In the following we will introduce different classes of systems to which our theoretical approach applies and on which we shall focus in the next sections.

1.3.1. Diffusion on a lattice and the zero range process. The simplest example is that of non-(or very weakly) interacting particles diffusing on the hypercubic lattice \( \Lambda^D \). At each infinitesimal time step \((t, t + dt)\) a particle jumps to a nearest neighbor with probability \( \gamma dt/z \), where \( z \) is the site connectivity and \( \gamma \) the diffusion coefficient which may depend on the position.

The situation gets considerably more complex when \( \gamma \) is made explicitly dependent on the number of particles at each site or in the neighbourhood. Such is the case in the zero range process (ZRP), where a particle at site \( i \) jumps to any neighbour with probability
\[ \gamma u(n_i), \] where \( n_i \) is the occupation number of site \( i \) and \( u(n) \) is a function which vanishes for \( n = 0 \). This simple dynamic rule, although leading to factorized steady states, leads to interesting phenomena such as Bose–Einstein condensation [13]. Many variants exist, with several species [14], or diffusivity depending only on the target site [15].

1.3.2. Interacting particles. Another class of interesting models is that of point particles interacting via some potential, which for simplicity we shall consider to be pairwise. This includes lattice systems where sites cannot be occupied simultaneously by several particles, provided this constraint is respected by the initial condition. It also includes systems like gases, simple liquids or crystals, when molecules can be approximately considered as pointlike. The Hamiltonian of the system is

\[ \mathcal{H} = \sum_{i<j} v(x_i - x_j). \] (3)

There are several possible ways of modelling equilibrium dynamics for such systems, which are equivalent in the continuum limit. We will focus in particular on the following two:

(i) Particles jumping on lattice from sites to sites, using the Metropolis rule for jump acceptance,

(ii) Particles in free space obeying the Langevin equation:

\[ \frac{dx_i}{dt} = -\sum_{j \neq i} v(x_i - x_j) + \eta_i, \] (4)

where \( \eta_i \) is a Gaussian white noise with variance \( \langle \eta_i(t)\eta_j(t') \rangle = \delta_{i,j}\delta(t - t') \).

If one chooses \( v(x) = 0 \) for \( x > 2a \) and \( v(x) = \infty \) for \( x \leq 2a \) and choose initial conditions such that all particles are at least at distances \( 2a \), the system is equivalent to hard spheres of radius \( a \). Furthermore, if the particles evolve on a hypercubic lattice with spacing \( 2a \), this is equivalent to the condition of single site occupation. In the so-called exclusion process, symmetric (SEP) or asymmetric (ASEP), particles diffuse freely on a lattice with the restriction that they cannot overlap and (in higher than one dimension) must avoid each other in order to cross. The system exchanges particles with some infinite reservoirs at its boundaries, with some rates, or may be closed. As shown, for instance, in [9] for the one-dimensional case, the role of boundary conditions is crucial.

In these models, as in those described in the previous paragraph, the number of particles is conserved. In a continuous description, this is expressed by a continuity equation \( \partial_t \rho = -\nabla \cdot J \), where \( J \) is the local particle current.

1.3.3. Reaction–diffusion systems. In many situations, where particles may appear, disappear, or be changed into something else, the number of particles is not conserved and no continuity equation can be written, making hydrodynamic descriptions in principle more complicated. Paradigmatic models for such phenomena are given by particle assemblies with reaction and diffusion. For instance, molecules deposed on a substrate, in a gas, or in a porous material may diffuse and react chemically. The problem is in general to study the concentration of molecules in the non-equilibrium steady state. Similar systems are those involving natural species, with birth and death processes, as well as predation, or epidemic spreading, where contamination occurs at contact. In general,
such systems can be easily mapped onto systems with reaction–diffusion. If we denote by $A$ and $B$ two different species, several reaction processes are possible:

(i) birth: $\emptyset \rightarrow A$,
(ii) death: $A \rightarrow \emptyset$,
(iii) coalescence: $2A \rightarrow A$, or more generally $mA \rightarrow nA$, with $m > n$,
(iv) contamination: $A + B \rightarrow 2A$, or $pA + qB \rightarrow p'A + q'B$, with $p + q = p' + q'$ and $q' < q$.
(v) transmutation: $A \rightarrow B$,
(vi) death at contact: $A + B \rightarrow A$,

this list being not exhaustive. Generically, when they contribute, these processes occur at some rate $\lambda$ per particle per unit of time.

A common simplification is to allow multiple occupation of sites and to consider only local reactions occurring at the same site. One can in principle consider particles with hard core repulsion or interactions occurring when the particles occupy neighbouring sites or compact clusters.

2. Stochastic process and field theory for interacting particle systems

In the following we show how to derive the field theory directly from the very definition of the stochastic process. This will bypass completely all the technical machinery based on coherent states used for DP field theory and it will make clear the relation between the stochastic process and field theory.

2.1. Basic processes

2.1.1. Particle disintegration. Let us start with the simple situation of a single site, occupied at initial time $t_0 = 0$ by $N_0$ particles. The time is cut in intervals $[t_i, t_{i+1}]$ ($i = 0, \ldots, N - 1$) of length $dt$, during which each particle disintegrates with probability $\lambda dt$. Our goal is to characterize the dynamic distribution of the number of particles $n(t)$ during the whole time interval $[0, T]$. To this aim, we introduce an auxiliary ‘jump process’ $J(t_i) = n(t_{i+1}) - n(t_i)$ and a conjugated field $\hat{n}(t_i)$, which will probe the generating function $Z[\{\hat{n}(t_i)\}] = \langle \exp(\lambda dt \sum_i \hat{n}(t_i) J(t_i)) \rangle$. For a reason which will be clear below, we temporarily forget that $J$ is related to the variations of the number of particles. Its statistics is very simple: $J(t_i) = -1$ with probability $n(t_i)\lambda dt$ and $J(t_i) = 0$ with probability $1 - n(t_i)\lambda dt$. This gives

$$Z[\{\hat{n}(t_i)\}] = \prod_i (1 - n(t_i)\lambda dt + n(t_i)\lambda dt e^{-\hat{n}(t_i)}) \approx \exp(\lambda dt \sum_i n(t_i) [e^{-\hat{n}(t_i)} - 1]),$$

when $N \rightarrow \infty$ and $T = N dt$ is kept fixed. Following the standard approach developed by Martin, Siggia, Rose, Janssen and de Dominicis (MSRJD) [16], the average of any observable $\langle \mathcal{O}\rangle = \frac{1}{Z} \left\langle \int \{dn(t_i)\} \mathcal{O}[\{n(t_i)\}] \prod_i \delta (n(t_{i+1}) - n(t_i) - J(t_i)) \right\rangle_f$.

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where $Z$ is such that $\langle 1 \rangle = 1$. Here, the number of particles is ensured to be an integer, as well as its variations, thanks to the delta functions and to the initial conditions. We remark that no infinitesimals like $dt$ appear explicitly yet, as they are hidden in the distribution of the $J$’s.

Using imaginary Fourier representations of the Dirac deltas, the average in (6) becomes

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \left\langle \int \{dn(t_i)d\hat{n}(t_i)\} \mathcal{O}\{\{n(t_i)\}\} \exp\left(\sum_i \hat{n}(t_i)(n(t_{i+1}) - n(t_i) - J(t_i))\right) \right\rangle
\]

\[
\times \frac{1}{Z} \left\langle \int \{dn(t_i)d\hat{n}(t_i)\} \mathcal{O}\{\{n(t_i)\}\} \exp\left(\sum_i \hat{n}(t_i)\left(n(t_{i+1}) - n(t_i) + \lambda dt \sum_i n(t_i) \left[e^{-\hat{n}(t)} - 1\right]\right)\right) \right\rangle.
\]

Taking the continuous time limit (i.e. $dt \to 0$), one gets that the probability of observing a given path $\{n(t')\}_{t' \in [0,T]}$ is

\[
\frac{1}{Z} \int \{d\hat{n}(t_i)\} e^{-S[\{n,\hat{n}\}]},
\]

with

\[
S[\{n,\hat{n}\}] = - \int_0^T dt \left(-\hat{n}(t)\partial_t n(t) + \lambda n(t) \left[e^{-\hat{n}(t)} - 1\right]\right).
\]

The action (9) is made of two parts. The first one, with time derivative, is a ‘kinetic’ term, which role is to fix the now infinitesimal increments of the field $n$. The second one comes from the jump process $J$ and encodes the dynamic fluctuations. This structure in two parts is generic, and thus the steps made above, which consist in obtaining the probability distribution of the paths from the generating function $Z[\{\hat{n}(t_i)\}]$ will be skipped in the other examples. We remark here that the introduction of the field $\hat{n}$ allows us to use real values for $n$ instead of integers only.

The generating function generates cumulants of the distributions of the variation of particle numbers. In particular, the formula

\[
\langle \partial_t n(t) \rangle = \left. \frac{\delta Z[\{\hat{n}\}]}{\delta \hat{n}(t)} \right|_{\hat{n}(t)=0}
\]

will be useful for the derivation of hydrodynamic equations.

2.1.2. Particle creation. Another simple process has been described briefly in the introduction. The calculation follows the lines of the previous paragraph and gives

\[
S[\{n,\hat{n}\}] = \int_0^T dt \left(\hat{n}(t)\partial_t n(t) + \lambda \left[e^{\hat{n}(t)} - 1\right]\right).
\]
2.1.3. Diffusion. The diffusion process can be either seen as the motion of a particle performing a sequence of jumps on connected sites, or as an exchange of particles between nearest neighbours. As we are interested in the density fluctuations, we adopt the second point of view, which amounts to considering only two neighbouring sites 1 and 2, with initial occupations \( n_1(0) \) and \( n_2(0) \). Particles may hop back and forth with rate \( \gamma \, dt \). The (integer) numbers of particles on the two sites at time \( t \) are \( n_1(t) \) and \( n_2(t) \). The variation of \( n_k \) between \( t \) and \( t+dt \) will again be denoted \( dJ_k(t) \), while particles hop from 1 to 2 with rate \( W_{12} \) and from 2 to 1 with rate \( W_{21} \). Of course, \( dJ_1(t) \) and \( dJ_2(t) \) are strongly correlated since a particle leaving site 1 lands on site 2 and vice versa. More precisely, \( dJ_1(t) = -dJ_2(t) = +1 \) with probability \( n_2(t)W_{21} \, dt \), \( dJ_1(t) = -dJ_2(t) = -1 \) with probability \( n_1(t)W_{12} \, dt \), and \( dJ_1(t) = dJ_2(t) = 0 \) otherwise. As before, the dynamical action is obtained through use of the generating function, leading to

\[
S\{\{n, \hat{n}\} \} = -\int dt \left\{ -\hat{n}_i \partial_t n_i - \hat{n}_j \partial_t n_j + n_1 W_{12} (e^{\hat{n}_2 - \hat{n}_1} - 1) + n_2 W_{21} (e^{\hat{n}_1 - \hat{n}_2} - 1) \right\}. \tag{12}
\]

On a lattice, the total MSRJD action is

\[
S\{\{n, \hat{n}\} \} = -\int dt \left\{ -\sum_i \hat{n}_i \partial_t n_i + \sum_{\langle ij \rangle} n_i W_{ij} (e^{\hat{n}_j - \hat{n}_i} - 1) \right\}, \tag{13}
\]

where the brackets restrict the summation to pairs of nearest neighbours. Later, we shall use this action to get continuum limits.

2.2. Interacting particles and Dean equations

Let us consider particle diffusion on a lattice with fixed hopping rate, but feeling a potential \( U_i(t) \) at site \( i \). This potential can be external and time-dependent or can also be due to the interaction of particles at site \( i \) with all others, in particular \( U_i(t) = a^D \sum_{j \neq i} v(a |i - j|) n_i(t) n_j(t) \). A natural stochastic dynamics is to accept hops to neighbouring sites with the heat bath rule, that is by choosing

\[
W_{ij} = \frac{\gamma}{1 + e^{-\beta(U_i - U_j)}}, \tag{14}
\]

where \( \beta = \gamma^{-1} \) is the inverse temperature. In the continuum limit \( W_{ij} \) can be expanded in powers of \( a(i - j) \) and Langevin dynamics is recovered.

Before doing so, it is important to define the continuum limit properly. We start with off-lattice particles in a box of very large size \( L \) with given boundary conditions. We then divide this box into many tiny boxes of size \( a \) at position \( x = ai \). We use the number \( n_i \) of particles in this box to define the density \( \rho_x = n_i / a^D \) and its conjugate \( \dot{\rho}_x = \dot{n}_i \). We choose \( a \) very small in order to expand in powers of \( a \). At the lowest useful orders:

\[
e^{\hat{n}_j - \hat{n}_i} - 1 = \exp \left( ae_{ij} \cdot \nabla \hat{n}_i + \frac{a^2}{2} (e_{ij} \cdot \nabla)^2 \hat{n}_i \right) - 1 + O(a^2) \tag{15}
\]

\[
= ae_{ij} \cdot \nabla \hat{n}_i + \frac{a^2}{2} \left[ (e_{ij} \cdot \nabla)^2 \hat{n}_i + (e_{ij} \cdot \nabla \hat{n}_i)^2 \right] + O(a^2), \tag{16}
\]

where \( e_{ij} \) is the unit vector pointing from \( i \) to \( j \). One can also expand the expression in

\[\text{doi:10.1088/1742-5468/2007/07/P07024}\]
\[ W_{ij} = \gamma - \frac{1}{2} \alpha e_{ij} \cdot \nabla U_i + o(a). \]  

In addition, using \( x = ia \), one can replace \( a^D \sum_i \) by \( \int d^D x \). Terms of order \( a \) vanish by symmetry, and thus keeping terms of order \( a^2 \) and rescaling \( t \) by \( a^2 \), one gets the following action:

\[ S[\{n, \hat{n}\}] = -\int dt \int d^D x \left\{ \dot{\rho}_x \left( -\partial_t \rho_x + \gamma \Delta \rho_x + \nabla \cdot [\rho_x \nabla U_x] \right) + \gamma \rho_x (\nabla \dot{\rho}_x)^2 \right\}. \]  

This action can also be obtained from a MSRJD treatment of the following Langevin equation:

\[ \partial_t \rho_x = -\nabla \cdot J_x, \]  

with a fluctuating current

\[ J_x = -\gamma \nabla \rho_x - \rho_x \nabla U_x + \sqrt{\rho_x} \xi_x, \]  

where \( \xi_x(t) \) is a Gaussian field with variance

\[ \langle \xi_x(t) \xi_x'(t') \rangle = 2\gamma \delta(x - x') \delta(t - t'). \]

The deterministic part of the current can be expressed in the form \( \rho_x \nabla (\delta F / \delta \rho_x) \), with a free-energy functional

\[ F[\{\rho\}] = \int d^D x \left( T \rho_x \ln(a^D \rho_x) + \rho_x \nabla U_x \right). \]  

This stochastic differential equation (SDE) for the density field was first derived by Dean [17] directly from the Langevin equation for interacting particles by using the Itô formula. The free energy \( F \) is the same as that from mean-field theory. Our derivation gives further evidence that no coarse-graining is needed in order to get such free energy. In fact, the interaction \( U \) used here is often obtained as an effective interaction (e.g. for colloidal systems) and thus the free energy functional is actually different from the mean-field one, which contains the microscopic potential instead.

### 2.3. Newtonian dynamics

Now let us consider systems of particles with Newtonian dynamics and interacting via a pairwise potential \( v \). The equations of motion are

\[ \partial_t r_i = \frac{p_i}{m} \]  

\[ \partial_t p_i = -\sum_{j \neq i} \nabla v(r_i - r_j). \]  

We introduce the density field in phase space:

\[ \rho(r, p, t) = \sum_i \delta(r_i(t) - r) \delta(p_i(t) - p). \]  

It is easy to show that it obeys the deterministic equation

\[ \partial_t \rho(x, p, t) = -\frac{p}{m} \partial_x \rho(x, p, t) + \int d^D x' d^D p' \rho(x', p', t) \nabla V(x - x') \cdot \partial_p \rho(x, p, t), \]  

all randomness being in the initial conditions. One can write path integral formulae as before, introducing \( \hat{\rho}(x, p, t) \) fields. If one applies the reverse map of (60) to these fields, one gets the formulae originally obtained by Doi [6].

\[ \text{doi:10.1088/1742-5468/2007/07/P07024} \]
2.4. Exclusion processes

Dealing with hard cores is in general difficult. A first attempt was done in [18] where particles with hard core repulsion may be modelled in two ways. The first one is to add a pairwise interaction which is infinite if the distance between the particles is less than their radius and vanishes elsewhere. The second one is to include a constraint in the hopping rate of the particles on the lattice. From the stochastic approach which has been developed here, the latter is easier to deal with. We thus start with particles on a lattice with initial occupation number of site $i$ $n_i(0) \in \{0; 1\}$. The exclusion constraint is propagated by the dynamics, if the hopping rate from $i$ to $j$ is of the form

$$ n_i W_{ij} = A_{ij} n_i (1 - n_j), \quad (26) $$

which vanishes if site $i$ is empty or site $j$ is occupied. For the sake of clarity, we choose $A_{ij} = \gamma$, which corresponds to symmetric exclusion processes. Inserting (26) into (13), expanding like in (15) and using $n_j = n_i + a \epsilon_{ij} \cdot \nabla n_i + o(a)$, the action describing the dynamics becomes in the limit $a \to 0$:

$$ S[\{n, \hat{n}\}] = -a^2 D \int dt \int d^Dx \left\{ -\hat{n}_x \partial_t n_x + \gamma \hat{n}_x \Delta n_x + \gamma n_x (1 - n_x) (\nabla \hat{n}_i)^2 \right\}. \quad (27) $$

This looks like the hydrodynamic equation obtained rigorously by Bertini et al [11] for symmetric exclusion processes. However, our approach suggests that this equation is valid beyond the hydrodynamic limit. In addition, although we started from $n_x \in \{0; 1\}$, the measure gives also non-zero weights to paths such that $n_x (1 - n_x) \neq 0$. As we shall see, this is related to the fact that the hydrodynamic limit corresponds to a saddle point of the dynamic action which is located on continuous paths, and that the dynamics on slightly smaller scales is given by fluctuations around this saddle point.

2.5. Initial conditions

Up to now, we have not specified the initial conditions and considered only the dynamical changes of the density. In many cases, initial conditions are forgotten after some transient time and a stationary state is obtained. However, in the case of exclusion processes, our stochastic approach to the dynamics makes it mandatory—at least for consistency—to specify initial conditions with the good occupation numbers, in order to ensure that the paths supporting the measure are consistent with the dynamic rules. We assume a factorized initial state with $q_i$ particles on site $i$, drawn with a distribution $P_i(q_i)$. It is useful to define the generating function $g_i(z) = \ln(\sum_q P_i(q) z^q)$. In order to specify the initial conditions, we add a term $\langle \prod_i \delta(q_i - n_i(0)) \rangle$. Using integral representations of the $\delta$'s, this gives an extra term to the dynamic action, after averaging:

$$ S_I[\{n(0), \hat{n}(0)\}] = \sum_i n_i(0) \hat{n}_i(0) - \sum_i q_i \left( e^{\hat{n}_i(0)} \right). \quad (28) $$

A quite generic case is that of Poissonian initial state, with $P_i(n) = (\rho_{0,i} n!/n!) e^{-\rho_{0,i}}$, which leads to

$$ S_I[\{n(0), \hat{n}(0)\}] = \sum_i n_i(0) \hat{n}_i(0) - \sum_i \rho_{0,i} \left( e^{\hat{n}_i(0)} - 1 \right). \quad (29) $$

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If hard cores prevent several particles to sit on the same site, one may choose independent Bernoulli distributions with average $\rho_{0,i}$, leading to

$$S_I[\{n(0), \hat{n}(0)\}] = \sum_i n_i(0)\hat{n}_i(0) - \sum_i \ln \left(1 - \rho_{0,i} - \rho_{0,i} e^{\hat{n}_i(0)}\right).$$  \hfill (30)

### 2.6. Boundary conditions

It may happen that changing boundary conditions changes drastically the dynamical behaviour. One-dimensional exclusion processes are popular examples of such phenomena. Periodic boundary conditions are taken into account by imposing space periodicity on the fields. Taking into account open boundary conditions is more model-specific, and we explain how to do it in the example of an exclusion process in one dimension. We assume that the system is coupled to two reservoirs at its ends. Particles are added at boundary points 0 and $N$ with rates $\alpha$ and $\gamma$, and removed with rates $\beta$ and $\delta$. As the number of particles is preserved inside the system, its fluctuations are governed by exchanges of particles with the reservoirs. This gives an extra term to the dynamical action:

$$S_B[\{n_0, n_N, \hat{n}_0, \hat{n}_N\}] = -\int dt \left\{ \alpha (1 - n_0(t)) \left(e^{\hat{n}_0(t)} - 1\right) + \gamma (1 - n_N(t)) \left(e^{\hat{n}_N(t)} - 1\right) + \beta n_0(t)\left(e^{-\hat{n}_0(t)} - 1\right) + \delta n_N(t)\left(e^{-\hat{n}_N(t)} - 1\right) \right\}. \hfill (31)$$

### 3. Mapping between Doi–Peliti and stochastic approaches

This section is devoted to show mapping between the above approach and the more standard one based on the DP method. For self-consistency, we start by recalling briefly the derivation of path integral formula using the latter method.

### 3.1. Doi–Peliti path integral formulation

Let us consider particles with annihilation $A + A \to \emptyset$ on a single site at rate $2\lambda$, which will provide a benchmark to show our method. The initial number of particles is $n_0$. The probability of having $n$ particles at time $t$ evolves according to the master equation

$$\partial_t P(n, t) = \lambda (n + 2)(n + 1)P(n + 2, t) - \lambda n(n - 1)P(n, t). \hfill (32)$$

We map this onto an imaginary time Schrödinger equation by introducing a Fock space generated by $n$-particle states $|n\rangle$ and introducing the state ket of the system at time $t$:

$$|\psi(t)\rangle = \sum_n P(n,t)|n\rangle. \hfill (33)$$

We also use the lowering/raising operators $a$ and $a^+$ according to the standard definition

$$a|0\rangle = 0$$

$$a^+|n\rangle = |n + 1\rangle$$

$$a|n\rangle = n|n - 1\rangle, \hfill (36)$$

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which verify the commutation relation \([a, a^+] = 1\). The master equation (32) now becomes
\[
\partial_t |\psi(t)\rangle = -\hat{H} |\psi(t)\rangle,
\]
with
\[
\hat{H} = \lambda (a^+ a - 1) a^2.
\]
We obtain the state ket at any time:
\[
|\psi(t)\rangle = e^{-\hat{H}t} |\psi(0)\rangle,
\]
and
\[
P(n, t) = \frac{\langle n |\psi(t)\rangle}{n!}.
\]
The \(n\)-particle states verify \(\langle n |m\rangle = \delta_{n,m} n!\). As a consequence one can define coherent states
\[
|\phi\rangle = \frac{1}{\sqrt{\pi}} e^{-\phi^*/2} e^{\phi a^+} |0\rangle
\]
and
\[
\langle \phi | = \frac{1}{\sqrt{\pi}} e^{-\phi^*/2} \langle 0 | e^{\phi^* a}.
\]
The coherent states thus define the (over)complete relation:
\[
1 = \int d\phi d\phi^* |\phi\rangle \langle \phi |.
\]
The average of an observable \(A(n)\) is given by
\[
\langle A \rangle_t = \sum_n A(n) P(n, t)
\]
\[
= \langle P | \hat{A} |\psi(t)\rangle
\]
\[
= \langle P | \hat{A} e^{-\hat{H}t} |\psi(0)\rangle,
\]
where
\[
\hat{A} = \sum_n |n\rangle \frac{A(n)}{n!} \langle n |,
\]
and the ‘projection state’ \(\langle P | = \langle 0 | e^a\) is a left eigenvector of \(a^+\) with eigenvalue 1. Using the Trotter formula
\[
e^{-\hat{H}t} = (1 - \hat{H} dt + o(dt))^M,
\]
with \(M = t/dt\), and inserting (43) in between all factors, one obtains a product of terms of the form
\[
\langle \phi_{k+1} | \phi_k \rangle \left( 1 - dt \frac{\langle \phi_{k+1} | \hat{H} | \phi_k \rangle}{\langle \phi_{k+1} | \phi_k \rangle} \right).
\]
It is easy to show that
\[
\langle \phi_{k+1} | \hat{H} | \phi_k \rangle = \langle \phi_k | \phi_k \rangle H(\phi_{k+1}, \phi_k),
\]
where \( H(\phi^*, \phi) \) is obtained by replacing respectively \( a^+ \) and \( a \) by \( \phi^* \) and \( \phi \) in the expression of \( \hat{H} \): in terms of \( a^+ \) and \( a \). The expression \( \hat{H} \) is obtained from \( \hat{H} \) by normal ordering, i.e. commuting all operators until all \( a \)'s are on the right. In this example \( (38) \) has already been normal ordered, and thus
\[
H(\phi^*, \phi) = \lambda (\phi^{*2} - 1) \phi^2.
\]
We thus get:
\[
\langle A \rangle_t = \int \prod_{k=0}^{M} \text{d} \phi_k \text{d} \phi_k^* \langle P | \hat{A} | \phi_M \rangle \langle \phi_0 | \psi(0) \rangle \left\{ \prod_{k=1}^{M} \langle \phi_k | \phi_{k-1} \rangle (1 - \text{d} t \ H(\phi_k^*, \phi_{k-1})) \right\}.
\]
The operator \( \hat{A} \) can always be written in the normal ordered form \( A(a^+, a) = \hat{A}(a^+, a) \). In addition, using the identity \([e^{a}, f(a^+)] = f(a^+ + 1) e^{a}\), we get
\[
\langle P | \hat{A} | \phi_M \rangle = A(1, \phi_M) e^{\phi_M}.
\]
In addition:
\[
\langle \phi_0 | \psi(0) \rangle = \sum_q P(q) \langle \phi_0 | (a^+)^q | 0 \rangle \quad \text{(54)}
\]
\[
= \sum_q P(q) \phi_0^q \quad \text{(55)}
\]
\[
= e^{\phi(\phi_0^*)} \quad \text{(56)}
\]
In the limit \( \text{d} t \to 0 \), we introduce the continuous time \( s = \text{k} \text{d} t \) and thus
\[
\langle A \rangle_t = \int \{ \text{d} \phi^* \} \{ \text{d} \phi \} \ A(1, \phi_t)
\times \exp \left( \phi(t) + g(\phi_0^*) + \int_0^t \text{d} s \ H(\phi^*(s), \phi(s)) - \int_0^t \text{d} s \ \phi(s) \partial_s \phi^*(s) \right).
\]
Diffusion processes can be treated in the same way. The resulting Hamiltonian is
\[
H_{\text{dif}}[\{ \phi^* , \phi \}] = \gamma \sum_{\langle i,j \rangle} (\phi_i^* - \phi_j^*) (\phi_i - \phi_j).
\]
Correlation functions at different times can be computed in the same way. For instance:
\[
\langle n(t_1) \cdots n(t_p) \rangle = \langle P | a^+ a^+ \prod_{k=2}^{p} e^{-(t_k-t_{k-1})H(a^+, a)} a^+ a \rangle \sum_{q} P(q) | q \rangle.
\]
The average of the \( \phi^* \) field is 1, and in general the shift \( \overline{\phi} = \phi^* + 1 \) is carried out, in order to deal with a field with vanishing average. This also cancels the isolated term \( \phi(t) \) in (57).
3.2. The Cole–Hopf transformation

We now show how the action obtained from purely stochastic considerations can be mapped onto the one derived in the previous paragraph. The particle number operator is $\hat{n} = a^+ a$. Thus, it is tempting to express the density field as $\rho = \phi^* \phi$. However, in order to go from operators $a, a^+$ to fields $\phi, \phi^*$, one must normal order. For instance $\hat{n}^2 = (a^+ a)^2 + a^+ a$, and thus in the field theoretic formalism, one should rather choose $\rho^2 = (\phi^* \phi)^2 + \phi^* \phi$ instead of $\rho^2 = (\phi^* \phi)^2$. Before dealing with this ambiguity, we will proceed to a naive mapping between fields in the different formulations, and apply it to situations where it is correct. Let us define

\begin{align}
\rho_i(s) &= \phi_i^* \phi_i(s), \\
\dot{\rho}_i(s) &= \ln \phi_i^*(s). 
\end{align}

The mapping (60) has unit determinant and thus does not change the measure. In addition, it is valid both on lattice and in the continuum. Plugging (60) into (58), one gets

$$
\gamma \sum_{(i,j)} \int_0^t ds \rho_i(s) \left(e^{\dot{\rho}_j(s)} - \dot{\rho}_i(s) - 1\right).
$$

This is precisely the term resulting from the calculation of the generating function in the stochastic approach. The extra kinetic term in (57) becomes under (60)

$$
- \sum_i \int_0^t ds \rho_i(s) \partial_s \dot{\rho}_i(s).
$$

This shows that the field defined as $\rho = \phi^* \phi$ is a ‘good’ density field, as it formally corresponds to the one of the stochastic method. The nature of density fluctuations is thus hidden in the DP formalism. Thanks to the Cole–Hopf mapping detailed above one obtains a dual version of the field theory which turns out to be the DP one as the reader can easily check for all processes considered in this paper. There is only one subtle point needed for this derivation that we shall discuss below. For concreteness let us come back to the reaction $A + A \rightarrow \emptyset$. The non-kinetic bulk part of the dynamic action is in the DP form

$$
S_{A+A,\text{DP}} = - \int_0^t ds \lambda \phi^2(s) \left(1 - \phi^*(s)^2\right).
$$

The equivalent term in the stochastic approach is

$$
S_{A+A,\text{MSRJD}} = - \int_0^t ds \lambda \rho(s) \left(\rho(s) - 1\right) \left(e^{-2\dot{\rho}(s)} - 1\right).
$$

However, if we use the mapping (60) into (64), one gets

$$
- \int_0^t ds \lambda \rho(s)^2 \left(e^{-2\dot{\rho}(s)} - 1\right),
$$

which differs from (65). In order to understand the origin of this difference, we recall that the fields $\phi$ and $\phi^*$ are quantum fields. As a consequence the products of fields at
the same time is a quantity that depends crucially on the underlying discretization. For example \( \phi^*(t^+)\phi(t^-) \neq \phi(t^+)\phi^*(t^-) \) as it can be readily checked. In order to understand which discretization allows one to map the field theory derived in the previous section to the DP one we carry a mapping equivalent to (60) at the operator level:

\[
a = e^{-\rho^+}\rho
\]

\[
a^+ = e^{\rho^+}.
\]

Operators \( \rho \) and \( \rho^+ \) have canonical commutation rules:

\[
[\rho, \rho^+] = a^+ [a, \ln a^+] = a^+ \partial \ln a^+/\partial a^+ = 1.
\]

(69)

It is thus natural to use them as lowering/raising operators. We define a new set of Fock space generating vectors as follows:

\[
|\tilde{0}\rangle = |0\rangle
\]

\[
|\tilde{n}\rangle = (\rho^+)^n|\tilde{0}\rangle
\]

(70)

(71)

and corresponding left vectors:

\[
\langle \tilde{0}| = \langle P|
\]

\[
\langle \tilde{n}| = \langle 0|\rho^m.
\]

(72)

(73)

We point out that \( \rho \) and \( \rho^+ \) are not Hermitian conjugates, and thus neither are \( \langle \tilde{n}| \) and \( |\tilde{n}\rangle \). We can redo the DP derivation of the path integral formulae, but using these operators instead of operators \( a \) and \( a^+ \), writing

\[
\hat{H}(a^+, a) = \hat{H}_{\text{stoc}}(\rho^+, \rho).
\]

(74)

Normal ordering with respect to \( \rho^+ \) and \( \rho \) and replacing respectively \( \rho^+ \) and \( \rho \) with \( \hat{\rho} \) and \( \rho \), one gets the equivalent of (57), where the term at time \( s \) is

\[
-\hat{\rho}(s)\partial_s \rho(s) + H_{\text{stoc}}(\hat{\rho}(s), \rho(s)).
\]

(75)

Initial conditions are also taken into account:

\[
\sum_q P(q)\langle \rho_q^m|\tilde{n}\rangle = \sum_q P(q)e^{\rho q^m} = e^{\rho^0}.
\]

(76)

Gathering terms from (75) and (76), we get exactly the dynamic action derived from the purely stochastic approach. From the point of view of the time discretization the above derivation clarifies what was the origin of the discrepancy between (65) and (66). The field theory derived directly from the stochastic equation gives, after field transformation, \( \rho(t)\rho(t) \rightarrow \lim_{\epsilon \to 0} \phi^*(t + 4\epsilon)\phi(t + 3\epsilon)\phi^*(t + 2\epsilon)\phi(t + \epsilon) \). Instead within the DP field theory the corresponding term is \( \phi^*(t)\phi^*(t)\phi(t)\phi(t) = \lim_{\epsilon \to 0} \phi^*(t + 4\epsilon)\phi^*(t + 3\epsilon)\phi(t + 2\epsilon)\phi(t + \epsilon) \).

As a consequence, in order to map one onto the other one has to use also the same time discretization.

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We remark that we have not tried to justify the existence of the operators $\rho$ and $\rho^+$. The change (67) may not be well defined at the operator level. However, it is a posteriori justified by the fact that it gives exactly the action obtained from another, stochastic, approach. Furthermore, the same result can be obtained alternatively just by discretizing the field theory.

As a consequence, although path integral formulae can be obtained both using the simple MSRJD stochastic method or the more abstract DP one, the previous discussion highlights that care must be taken in the interpretation of the field theory. Indeed, depending on the physical situation, the sets of fields $\phi, \phi^*$ and $\rho, \rho^*$ are more adapted than the other. However, different sets are adapted to different approaches and going from one to the other using (60) may be dangerous. In fact, this mapping has been often referred to [3, 12], [18]–[20], in a rather loose way, in particular without attention to the fact that the commutation rules of the fields are important.

3.3. Physical origin of fluctuations

Let us discuss in more detail the origin of the difference between the two field theories obtained in the previous sections. The noise generated by the stochastic dynamics is intrinsically Poissonian at the microscopic level. For example, consider particles created on a site (empty at $t = 0$) with rate $\lambda$ per unit of time. The generating function of $N(t)$, the number of particles at time $t$, is $f(s) = \langle e^{sN(t)} \rangle$. It is a straightforward exercise to get

$$f(s) = \exp [\lambda t (e^s - 1)]$$

(77)

in the limit $dt \to 0$ keeping $t$ fixed. Similar expressions, are obtained for diffusions and more complicated processes. The intrinsic Poissonian nature of the noise, i.e. the fact that with very small probability variables change by a finite amount, makes the logarithm of the generating function a complicated function of the argument. In particular it leads generically to exponential terms. This is the main reason why the action of the field theory in the $\rho, \rho^*$ variable is complicated even for simple processes. On the other hand, the field theory in the $\phi, \phi^*$ variables is simple for simple processes as creation, annihilation or diffusion. In particular for diffusion the action is quadratic. The drawback is that $\phi$ is not the density field. This is the reason why stochastic equations on $\phi$ do not make sense physically and lead to inconsistencies, such as the presence of imaginary noise.

The usefulness of each of the two field theories depends on the particular physical problem one is interested in. One focuses directly on the physical variables but it is complicated (non-Gaussian) already for simple processes. The DP one is instead simple in these cases; however, computing physical observables may be quite complicated because they correspond to high order multipoint correlation functions. Thus, for renormalization group treatment of simple processes the DP one seems, and has been proven to be, very useful. On the other hand, in cases where one is interested in two-point density correlation functions the other field theory seems to be more useful because it allows us to develop approximations directly for these functions avoiding dealing with high order multipoint (more than two points) correlation functions, which is in general a quite cumbersome task. Finally, we remark that, on sufficiently large length scales the noise becomes Gaussian. In this case the field theory on $\rho, \rho^*$ leads directly to hydrodynamic descriptions away from critical points. Hydrodynamic descriptions correspond to coarse-graining the dynamics on mesoscopic scales much bigger than the correlation length, with corresponding rescaling of
time. From coarse-graining results a law of large numbers in the form of a large deviation theory, providing the probability of occurrence of coarse-grained trajectories, or of density profiles in the steady state.

4. Applications

We now give some applications of the formalism which we have explained in detail. We start with the zero range process for which we derive continuum stochastic equations.

4.1. Zero range process

Zero range processes (ZRP) provide a simple model for the study of particles undergoing an out-of-equilibrium process. Indeed, the dynamic rules, although quite simple, may lead to a rather rich phenomenology. In addition, in the simplest cases, the steady state measure can be computed and is factorized. For a detailed review, see [13].

4.1.1. Single species ZRP. The model is defined as follows. Particles hop on a lattice with a rate $u(n_i(t))$ depending only on the number of particles at the considered site $i$ at the moment of the jump; $u$ is a function which vanishes at 0 (no motion from site $i$ if no particle). Here the continuum limit is obtained in a way slightly different from before. The lattice spacing is $a$, and is sent to zero, while the density is defined as before from the particle number. It is now a straightforward exercise to generalize what has been done previously to get the following action:

$$S[\{\rho, \hat{\rho}\}] = -\int dt \int d^Dx \left\{ \hat{\rho}_x (-\partial_t \rho_x + \Delta u_a(\rho_x)) + u_a(\rho_x) (\nabla \hat{\rho}_x)^2 \right\},$$  \hspace{1cm} (78)

with $u_a(\rho) = a^{-D} u(a^D \rho)$, after rescaling of the microscopic time $t \to t/a^2$. Integrating out the $\hat{\rho}$ field like previously, one gets that the dynamics of the density field follows the SDE

$$\partial_t \rho_x = -\nabla \cdot J_x$$ \hspace{1cm} (79)

$$J_x = -\nabla u_a(\rho_x) + \sqrt{u_a(\rho_x)} \xi_x,$$ \hspace{1cm} (80)

where $\xi$ has the same statistics as in (20). The steady state free energy can be obtained from the Fokker–Planck equation. We introduce the instantaneous probability distribution of paths:

$$P[\{\hat{\rho}\}, t] = \left\langle \prod_x \delta(\hat{\rho}_x - \rho_x(t)) \right\rangle,$$ \hspace{1cm} (81)

which obeys the following Fokker–Planck equation:

$$\partial_t P[\{\hat{\rho}\}, t] = -\left\langle \frac{\delta}{\delta \hat{\rho}_x} \Delta u_a(\hat{\rho}_x) P[\{\hat{\rho}\}, t] + \nabla \cdot (u_a(\hat{\rho}_x) \nabla P[\{\hat{\rho}\}, t]) \right\rangle,$$ \hspace{1cm} (82)

where we used the notation $\langle f, g \rangle = \int d^Dx f_x g_x$. Writing formally

$$\Delta u_a(\hat{\rho}_x) = \nabla \cdot \left( u_a(\hat{\rho}_x) \nabla \frac{\delta \mathcal{F}[\{\hat{\rho}\}]}{\delta \hat{\rho}_x} \right).$$ \hspace{1cm} (83)
Equation (82) is
\[
\partial_t \mathcal{P} = -\left\langle \frac{\delta}{\delta \rho_x} \cdot (u_a(\hat{\rho}_x) \nabla \frac{\delta \mathcal{F}[\{\hat{\rho}\}]}{\delta \rho_x}) \right\rangle \mathcal{P} + \nabla \cdot (u_a(\hat{\rho}_x) \nabla \mathcal{P}) \right\rangle. \tag{84}
\]

It is straightforward to check that the steady state distribution is obtained from the solution \(\mathcal{F}\) of (83), which is
\[
\mathcal{F}[\{\hat{\rho}\}] = \int d^D x \int \hat{\rho}_x \, dr \ln u_a(r), \tag{85}
\]
via
\[
P[\{\hat{\rho}\}] \propto \exp (-\mathcal{F}[\{\hat{\rho}\}]). \tag{86}
\]
This is the continuous analogue of
\[
\ln \left( \prod_{i} w!_{n_i} \right), \tag{87}
\]
where \(w!_{n_i} = \prod_{k=1}^{n_i} u_k\). The possibility to solve (83) by direct integration is related to the fact that the steady state measure is factorized. In general, this measure is not factorized, and it is more difficult to read \(\mathcal{F}\) in the Fokker–Planck equation. We will illustrate such difficulty in the context of the two-species ZRP.

We want to end this section with a remark on the continuum stochastic equations derived for the ZRP. They are valid on scales \(l\) such that \(a \ll l\). On the other hand, there is no assumption on the correlation length \(\xi\). Coarse-graining the previous stochastic equations on scales much larger than \(\xi\) would lead to hydrodynamic equations (see the next sections). The main interest of equation (79) is that they provide at the same time a continuum description and can describe critical and out-of-equilibrium properties of ZRP, i.e. they have the same status as the model A stochastic equations for the (non-conserved) dynamics of the Ising model.

We remark that we have taken spatial derivatives of the fields without justification of their smoothness. In order to make this more rigorous, one would have to consider the density fields as distributions acting on the space of smooth functions with compact support on the discrete lattice. This is in fact a rigorous but much less clear way of doing the coarse-graining which we refer to.

4.1.2. Two-species ZRP: This version of the ZRP involves two species \(A\) and \(B\), with occupation numbers \(n^A_i(t)\) and \(n^B_i(t)\). The difference with the single-species case is that the hopping rate from site \(i\) of particles of a given species is a function of the local number of particles of the other species, which we denote by \(u^A(n^A_i)\) and \(u^B(n^B_i)\). In the continuum limit, the density fields verify the SDE
\[
\partial_t \rho^A_x = \Delta u^A_a(\rho^A_x, \rho^B_x) + \nabla \cdot \left( \sqrt{u^A_a(\rho^A_x, \rho^B_x)} \xi^A_x \right), \tag{88}
\]
\[
\partial_t \rho^B_x = \Delta u^B_a(\rho^A_x, \rho^B_x) + \nabla \cdot \left( \sqrt{u^B_a(\rho^A_x, \rho^B_x)} \xi^B_x \right). \tag{89}
\]
If the steady state is factorized, the following conditions must be verified:

\[
\ln u_a^A(\tilde{\rho}_x^A, \tilde{\rho}_x^B) = \frac{\delta F}{\delta \tilde{\rho}_x^A}, \tag{90}
\]

\[
\ln u_a^B(\tilde{\rho}_x^A, \tilde{\rho}_x^B) = \frac{\delta F}{\delta \tilde{\rho}_x^B}. \tag{91}
\]

This gives a necessary condition for factorization of the steady state:

\[
u_a^B(\tilde{\rho}_x^A, \tilde{\rho}_x^B) \frac{\partial u_a^A(\tilde{\rho}_x^A, \tilde{\rho}_x^B)}{\partial \tilde{\rho}_x^B} = u_a^A(\tilde{\rho}_x^A, \tilde{\rho}_x^B) \frac{\partial u_a^B(\tilde{\rho}_x^A, \tilde{\rho}_x^B)}{\partial \tilde{\rho}_x^A}. \tag{92}
\]

This is the continuous analogue of the condition given by Evans and Hanney for the model on-lattice [14].

4.2. Hydrodynamic limits

We now show how hydrodynamic limits can be obtained within our field theory approach. Taking hydrodynamic limits consists in splitting the volume of the system into boxes of equal sizes, with volumes a fraction of the total volume of the system. This amounts to using a coarse-graining length \(l\) much smaller than the system size \(L\), but much larger than the lattice spacing \(a\). In one dimension, boxes are thus located between \(xL\) and \(xL + l\), with \(x \in \{0, l/L, 2l/L, \ldots, (L - l)/L\}\). In the hydrodynamic limit, where both \(l\) and \(L\) go to infinity, \(x\) becomes a real coordinate in \([0, 1]\) (this corresponds to \(dx = l/L\)).

In order to avoid artificial slowness, the time is rescaled: \(t \rightarrow t/L^2\). The hydrodynamic limit is in principle different from the much less controlled continuum limit taken before. There are in fact two possibilities. The first one is that the hydrodynamic limit exists, in which case the limits are connected; this happens for instance away from criticality, where correlation lengths are finite. In this case the large time and length scale behaviour of the continuum field theory leads to the hydrodynamic limit. The other possibility is that the hydrodynamic limit does not exist, while the continuum limit of the field theory still exists, although great care must be taken; this typically happens at criticality where the field theory in the continuum allows for renormalization group calculations.

There are now standard routes for rigorously deriving hydrodynamic limits for particle systems on lattices, such as ZRP or exclusion processes. We will not repeat them, as they can be found in textbooks and recent papers. Instead, we will show how our field theoretic approach naturally leads to hydrodynamic equations and large deviation functionals. Although the following results are not new, we think that the derivation we present gives an interesting new perspective on these problems.

The basic idea underlying hydrodynamic equations is the existence of a law of large numbers for density profiles. Let us imagine a system with a finite correlation length \(\xi\), coarse-grained using a coarse-graining length \(l \gg \xi\). Inside all boxes, the density field is the sum of a large number of independent identically distributed processes and is thus Gaussian. It can thus be characterized by first and second moments and a large deviation functional is easily obtained (see last section).
4.2.1. Zero range and exclusion processes. Let us start by a simple example, single-species ZRP in one dimension. In such models, the stationary measure is factorized, which makes it possible and simple to compute local averages from the marginals of the total joined distribution. Coarse-graining on the scale \( l \) is equivalent to averaging out all fields while fixing the value of the density inside each box at each time to its average value \( \overline{\rho}(x, t) \). This amounts to using constant conjugated field \( \hat{n} \) inside each box. However, instead of restricting the path integral to piecewise constant \( \hat{n} \) fields, it is simpler to restrict it to slowly varying \( \hat{n} \). More precisely, we restrict the measure to fields which significant variations are on scales larger than \( l \). The dynamical action when the lattice spacing \( a = 1/L \) goes to zero is

\[
- \sum_i \int dt \left\{ -\hat{n}_i \partial_t n_i + a^2 u(n_i) [(\nabla \hat{n}_i)^2 + \Delta \hat{n}_i] \right\}.
\]  

(93)

Now comes the fundamental hypothesis of local equilibrium. The variation of the total number of particle inside large boxes is only due to particle flows at its boundaries. If the size is large enough, one may assume that the timescale for any significant variation is also large and the system is locally at equilibrium. The local average density evolves slowly following the hydrodynamic equation which we want to derive, and we make the following approximation:

\[
\frac{1}{l} \sum_{i \in B_x} u(n_i(t)) \left[ (\nabla \hat{n}_i(t))^2 + \Delta \hat{n}_i(t) \right] \approx \frac{1}{l} \sum_{i \in B_x} \overline{u(n)}(x, t) \left[ (\nabla \hat{n}_i(t))^2 + \Delta \hat{n}_i(t) \right],
\]  

(94)

where

\[
\overline{u(n)}(x, t) = \frac{1}{l} \sum_{i \in B_x} u(n_i(t))
\]  

(95)

is the local mean value of \( u(n_i(t)) \). The assumption of local equilibrium also gives this mean value:

\[
\overline{u(n)}(x, t) = \langle u(n) \rangle_{\overline{\rho}(x, t)},
\]  

(96)

where \( \langle \cdot \rangle_{\overline{\rho}} \) stands for the average using the marginal with mean \( \overline{\rho} \). It can be obtained from the grand canonical partition function that

\[
\Theta(z) = \sum_n p_n z^n,
\]  

(97)

with \( p_n = 1!/n! \):

\[
\overline{\rho} = z \frac{\partial \Theta}{\partial z}.
\]  

(98)

Inverting (98) defines \( R(\overline{\rho}) = z \). Then we get:

\[
\overline{u(n)}_{\overline{\rho}} = z = R(\overline{\rho}).
\]  

(99)

In addition, \( \hat{n} \) being a very slow varying field, one has in one dimension:

\[
\sum_{i \in B_x} \left[ (\nabla \hat{n}_i)^2 + \Delta \hat{n}_i \right] \approx l[(\nabla x \hat{\rho}_x)^2 + \Delta \rho_x]
\]  

(100)
where \( \hat{n}_i \approx \hat{\rho}_x \) inside the box of size \( l \) around \( x \) and \( \nabla_x \) is now the gradient at the intermediate scale \( x \), i.e. the variation between two points at distance \( l \) divided by \( l \). Carrying the same analysis with the term \(- \sum_i \hat{n}_i \partial_i n_i\), and rescaling the time by \( a^2 = 1/L^2 \) the action finally is

\[
S_{\text{hydro}} = -L \int dt \int_0^1 dx \left\{ \hat{\rho}_x \partial_t \bar{\rho}_x + \hat{\rho}_x \Delta R(\bar{\rho}_x) + R(\bar{\rho}_x) (\nabla \bar{\rho}_x)^2 \right\}. \tag{101}
\]

The result (101) is a good illustration of the emergence of a law of large numbers at the hydrodynamic level, where the empirical current has mean \( \nabla \bar{R}(\bar{\rho}) \) and mean square \( R(\bar{\rho}) \). This can be seen either from a direct interpretation in terms of an effective Langevin equation or by using (10) and second derivatives of the generating function. In particular (10) gives the hydrodynamic equation:

\[
\partial_t \bar{\rho} = \Delta R(\bar{\rho}). \tag{102}
\]

The action (101) gives the large deviation functional for density fluctuations, by integration on the field \( \bar{\rho} \). If one formally defines \( \sigma(\bar{\rho}) = -2 \nabla \cdot (R(\bar{\rho}) \nabla) \), then the weight of any—coarse-grained—trajectory is given by

\[
P[\{\bar{\rho}\}] \propto \int \{d\hat{\rho}_x(t)\} e^{-S_{\text{hydro}}[\bar{\rho},\hat{\rho}]}, \tag{103}
\]

which gives

\[
P[\{\bar{\rho}\}] \propto e^{-\bar{S}_{\text{hydro}}[\bar{\rho}]} , \tag{104}
\]

with

\[
\bar{S}_{\text{hydro}}[\{\bar{\rho}\}] = L \int dt \int_0^1 dx \frac{[\partial_1 \bar{\rho}_x - \Delta R(\bar{\rho}_x)]^2}{2\sigma(\bar{\rho})}. \tag{105}
\]

Due to the \( L \) factor in front of the integral, the path integral is dominated by the hydrodynamic trajectory, which minimizes the large deviation functional.

The case of ZRP is relatively simple as the stationary distribution is factorized, and thus \( u(n) \) is easily computed. In general, the whole product measure must be used.

Starting from (27), the fluctuating hydrodynamic equation is obtained for exclusion processes, where a drift velocity \( v \) can be added:

\[
\partial_t \bar{\rho} = -\nabla \cdot (v \bar{\rho}(1 - \bar{\rho})) + \Delta \bar{\rho} + \nabla \cdot \left( \sqrt{\bar{\rho}(1 - \bar{\rho})} \eta \right), \tag{106}
\]

where \( \eta \) is as usual a normal Gaussian noise field. This equation contains initial symmetries of the microscopic model, i.e. particle–hole symmetry.

### 4.3. Systems with reaction processes

Reaction processes can provide an example of hydrodynamic limits in which the time has not to be rescaled because there are no conserved quantities.

Again, we take for a canonical example the pair annihilation \( A + A \rightarrow {} \), but without diffusion. Using the notations used until here, one gets

\[
\lambda \sum_i \int ds \, n_i(s)(n_i(s) - 1)(e^{-2\hat{n}_i(s)} - 1) \approx L \lambda \int dt \int_0^1 d\bar{\rho}_x \langle n(n - 1) \rangle_{\bar{\rho}_x} (e^{-2\hat{\rho}_x(s)} - 1) \tag{107}
\]

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and thus

\[ S_{\text{dyn}} \approx -L \int ds \int_0^1 d^D x \left\{ \lambda \langle n(n-1) \rangle_{\rho_x(s)} \left( e^{-2\hat{\rho}_x(s)} - 1 \right) - \hat{\rho}_x(s) \partial_s \rho_x(s) \right\} . \] (108)

The difficulty is still to compute the static average. For the simple example given here, there is no steady state except the empty one, but occupation numbers remain Poissonian at all times, and thus

\[ \langle n(n-1) \rangle_{\bar{\rho}(s)} = \bar{\rho}(s)^2 . \] (109)

As \( L \) is large, the measure is dominated by saddle points of the action. Thus the hydrodynamic equation is

\[ \partial_t \bar{\rho}_x(t) = -2\lambda \bar{\rho}_x(t)^2 e^{-2\hat{\rho}_x(t)} , \] (110)

where \( \hat{\rho}_x(t) \) is a solution of

\[ \partial_t \hat{\rho}_x(t) = -2\bar{\rho}_x(t) \left( e^{-2\hat{\rho}_x(t)} - 1 \right) . \] (111)

This equation describes the behaviour of the system on large length scales but timescales of order of one. There is no need for rescaling the time because the density is not conserved so the coarse-grained density still evolves on timescales of order one.

4.4. Large deviation functional

Recently there have been a lot of interest and works on large deviation functional for out of equilibrium driven systems. In the following we would like to show how this large deviation functional appears naturally within our framework. Our approach is not rigorous compared to the previous ones [11, 9]. On the other hand it shows clearly in our opinion the key ingredients and it can be easily generalized to systems more complicated than the ones considered up to now. In the following, for the sake of simplicity, we will focus on one dimensional driven stochastic lattice gases. The driving can be due either to the boundary conditions or to a force not deriving from a potential.

We shall focus on the probability that the system follows a given path \( \{n(x, t)\} \) in configuration space. This can be obtained formally by integrating out the auxiliary field \( \hat{n} \) but this is in principle not feasible. However, if one is only interested in the probability of a given path after coarse-graining then the task simplifies a lot. Let us rescale time and length scales in the hydrodynamic way described before and focus on the probability of smooth paths. This can be obtained with a construction very similar to the one used to obtain Feynman path integral in quantum mechanics:

\[ P[\{\rho(x, t)\}] \simeq \prod_{\Delta x, \Delta t} P_{\Delta x}[\rho; t \rightarrow t + \Delta t] \] (112)

where \( P_{\Delta x}[\rho; t \rightarrow t + \Delta t] \) is the probability inside a small coarse-grained box of the evolution of the density on a small coarse-grained time (\( \rho \) is the notation for the coarse-grained density). If the hydrodynamic limit exists then each \( P_{\Delta x}[\rho; t \rightarrow t + \Delta t] \) can be replaced by its hydrodynamic expression because although the global fluctuation of the density can be large, on each small (coarse-grained) box they are small. This expression equals the corresponding current distribution \( P_{\Delta x}[J; t \rightarrow t + \Delta t] \), up to a constant.
Jacobian. Repeating the procedure explained previously for the hydrodynamic limit one obtains a field theory such that the auxiliary field can be integrated out leading to:

\[ P_{\Delta x}[J; t \to t + \Delta t] \approx \int D\rho \exp \left[ -L\Delta x \Delta t \frac{(J - J_{\text{av}}(\rho))^2}{\chi(\rho)} \right] \tag{113} \]

where \( J_{\text{av}}(\rho) = -D(\rho)\nabla \rho + \chi(\rho)E \) is the empirical current; \( D, \chi \) are the transport coefficients characterizing the hydrodynamic limit, i.e. diffusion coefficient and mobility (\( L \) is the size of the system) and \( E \) is the external driving field. Putting together all the probability on small boxes one obtains the large deviation functional:

\[ P[\{\rho(x, t)\}] \approx \int D\rho \exp \left[ -L \int dt \langle \partial_t \rho + \nabla \cdot J_{\text{av}}(\rho), \eta(\rho)(\partial_t \rho + \nabla \cdot J_{\text{av}}(\rho)) \rangle \right] , \tag{114} \]

where \( \eta(\rho) \) is the inverse of the operator \( -\nabla \cdot (\chi(\rho) \nabla) \). Note that this field theory corresponds to a stochastic equations for the density field:

\[ \partial_t \rho(x, t) = -\nabla \cdot J_{\text{av}}(\rho(x, t)) + \nabla \cdot [\sqrt{\chi(\rho(x, t))} \eta(x, t)] \tag{115} \]

where \( \eta \) is a white noise in space and time, \( \langle \eta(x, t)\eta(x', t') \rangle = 2\delta(t - t')\delta(x - x')/L \), and the multiplicative noise must be interpreted in the Ito sense.

This derivation although certainly not rigorous has the virtue of showing in a simple manner why only hydrodynamic transport coefficients matter for the derivation of the large deviation functional and why this is related to a stochastic equation which leads to a non-linear fluctuating hydrodynamics. It can be straightforwardly generalized to systems with a more complicated hydrodynamics, e.g. real fluids where density, energy and momentum are conserved.

5. Conclusion

The aim of this paper was to discuss and present a field theoretical approach to interacting particle systems. We wanted to show a dual version of the Doi–Peliti field theory that can be obtained from the stochastic process and that is directly related to stochastic equations. The advantage of this field theory is that it focuses on the physical density fields as we have shown in some applications, e.g. ZRP continuum stochastic equations, large deviation functional and hydrodynamic limits. We think that this approach will help to tackle difficult problems in interacting particle systems.

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