Factorization of the R-matrix and Baxter’s Q-operator

S.E. Derkachov

St.Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences,
Fontanka 27, 191023 St.Petersburg, Russia.
E-mail: derkach@euclid.pdmi.ras.ru

Abstract. The general rational solution of the Yang-Baxter equation with the symmetry algebra $sl(2)$ can be represented as the product of the simpler building blocks denoted as $R$-operators. The $R$-operators are constructed explicitly and have simple structure. Using the $R$-operators we construct the two-parametric Baxter’s Q-operator for the generic inhomogeneous XXX - spin chain. In the case of homogeneous XXX-spin chain it is possible to reduce the general Q-operator to the much simpler one-parametric Q-operator.
1 Introduction

The Yang-Baxter equation and its solutions play a key role in the theory of the completely integrable quantum models [1, 2, 3, 4]. The general $sl(2)$-invariant solution of the Yang-Baxter equation (R-matrix) is the operator $R(u)$ acting in a tensor product of two $sl(2)$ lowest weights modules $V_{\ell_1} \otimes V_{\ell_2}$. The Yang-Baxter equation is reduced to the simpler defining equation for the R-matrix [5]

$$R_{12}(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u - v)$$

where $L(u)$ is the Lax operator. We suggest the natural factorized expression for the general R-matrix. It can be represented as the product of the simple building blocks – $R$-operators [4]. The main idea is very simple. The Lax operator depends on two parameters: the spin of representation $\ell$ and the spectral parameter $u$. It is useful to change to other parameters $u_+ = u + \ell$ and $u_- = u - \ell$ and extract the operator of permutation $P_{12}$ from the R-matrix $R_{12} = P_{12}R_{12}$. The defining equation for the operator $R_{12}$ has the form

$$\tilde{R}_{12} \cdot L_1(u_+, u_-)L_2(v_+, v_-) = L_1(v_+, v_-)L_2(u_+, u_-) \cdot \tilde{R}_{12}.$$ 

The operator $\tilde{R}_{12}$ interchanges simultaneously $u_+$ with $v_+$ and $u_-$ with $v_-$ in the product of two Lax-operators. Let us perform this operation in two steps. In the first step we interchange the parameters $u_-$ with $v_-$ only. The parameters $u_+$ and $v_+$ remain the same. In this way one obtains the natural defining equation for the $R^-$-operator

$$R_{12}^- \cdot L_1(u_+, u_-)L_2(v_+, v_-) = L_1(v_+, v_-)L_2(u_+, u_-) \cdot R_{12}^-; \quad R_{12}^- = R_{12}^-(u_+, u_-|v_-).$$

In the case when $u_- = v_- = v$ there is no interchange of parameters so that it is naturally to expect that the operator $R_{12}^-(u_+, u_-|v_-)$ is reduced to the unit operator $R_{12}^-(u_+, v|v) = 1$. In the second step we interchange $u_+$ with $v_+$ but the parameters $u_-$ and $v_-$ remain the same. The defining equation for the $R^+$-operator is

$$R_{12}^+ \cdot L_1(u_+, u_-)L_2(v_+, v_-) = L_1(v_+, u_-)L_2(u_+, v_-) \cdot R_{12}^+; \quad R_{12}^+ = R_{12}^+(u_+, u_-|v_+).$$

In the case when $u_+ = v_+ = u$ there should be the similar degeneracy $R_{12}^-(u|u, v_-) = 1$. These equations appear much simpler then the initial defining equation for the R-operator and their solution can be obtained in a closed form. Finally, we construct the composite object - the R-matrix from the simplest building blocks - the $R$-operators

$$R_{12}(u_+, u_-|v_+, v_-) = P_{12}R_{12}^+(u_+, u_-|v_+, v_-)R_{12}^-(u_+, u_-|v_-).$$
There are two points of degeneracy \( u_- = v_- = v \) and \( u_+ = v_+ = u \) where the operator \( \mathbb{R}_{12} \) is reduced to a single \( \mathcal{R} \)-operator

\[
\mathbb{R}_{12}(u_+, v| v_+, v) = \mathbb{P}_{12} \mathcal{R}^+(u_+, v_+, v) ; \quad \mathbb{R}_{12}(u, u_-| u, v_-) = \mathbb{P}_{12} \mathcal{R}^-(u, u_-| v_-)
\]

The detailed discussion of the \( \mathbb{R} \)-matrix and its factorization is given in the Section 2.

The next natural step is to use the general operator \( \mathbb{R}_{12}(u_+, u_-| v_+, v_-) \) as building block in construction of Baxter’s \( Q \)-operator. In the case of the generic inhomogeneous periodic XXX spin chain the transfer matrix \( t(u) \) is constructed as follows

\[
t(u) = \text{tr} L_1(u + \delta) \cdot L_1(u + \delta_1) \cdot \cdots L_N(u + \delta_N)
\]

The most general transfer matrix \( Q(u, \ell_0) \) is constructed in a similar manner from the operators \( \mathbb{R}_{k0} \)

\[
Q(u, \ell_0) = \text{tr}_{V_0} \mathbb{P}_{10}(u + \delta_1) \mathbb{P}_{20}(u + \delta_2) \cdot \cdots \mathbb{P}_{N0}(u + \delta_N)
\]

The operator \( Q(u, \ell_0) \) depends on two parameters: the spectral parameter \( u \) and the spin in the auxiliary space \( V_0 = V_{\ell_0} \). It is useful to change to other parameters \( u_1 = 1 + u - \ell_0 \) and \( u_2 = u + \ell_0 \) such that \( Q(u, \ell_0) = Q(u_1|u_2) \). The operator \( Q(u_1|u_2) \) has the following properties:

- the operator \( Q(u_1|u_2) \) is \( s\ell(2) \)-invariant

- commutativity

\[
Q(u_1|u_2) \cdot Q(v_1|v_2) = Q(v_1|v_2) \cdot Q(u_1|u_2) ; \quad Q(u_1|u_2) \cdot t(v) = t(v) \cdot Q(u_1|u_2)
\]

- the operator \( Q(u_1|u_2) \) obeys the Baxter’s equation with respect to \( u_2 \)

\[
Q(u_1|u_2) \cdot t(u) = \Delta_+(u)Q(u_1|u_1 + 1) + \Delta_-(u)Q(u_1|u_1 - 1) ; \quad \Delta_{\pm}(u) = (u + \delta_1 \pm \ell_1) \cdots (u + \delta_N \pm \ell_N)
\]

- the operator \( Q(u_1|u_2) \) obeys the Baxter’s equation with respect to \( u_1 \)

\[
t(u) \cdot Q(u_2|u_2) = \frac{\Delta_+(u - 1) \Delta_-(u)}{\Delta_-(u - 1)} Q(u - 1|u_2) + \Delta_-(u)Q(u + 1|u_2)
\]

These properties allow to consider the operator \( Q(u_1|u_2) \) as two-parametric Baxter’s \( Q \)-operator [6, 8]. The proof of all these properties of the operator \( Q(u_1|u_2) \) is given in the Section 3.

In the case of homogeneous spin chain: \( \delta_k = 0 \) and \( \ell_k = \ell \) the points of degeneracy for all operators \( \mathbb{R}_{k0} \) coincide so that it is possible to remove half of the \( \mathcal{R} \)-operators. We obtain the following reductions of the two-parametric \( Q \)-operator: at the first point of degeneracy \( u_1 = 1 - \ell \)

\[
Q_-(u) = Q(1 - \ell|u) = \text{tr}_{V_0} \mathbb{P}_{10} \mathcal{R}_{10}^-(u_+, u_-|0) \cdot \mathbb{P}_{20} \mathcal{R}_{20}^-(u_+, u_-|0) \cdot \cdots \mathbb{P}_{N0} \mathcal{R}_{N0}^-(u_+, u_-|0)
\]

and at the second point of degeneracy \( u_2 = \ell \)

\[
Q_+(u) = Q(u|\ell) = \text{tr}_{V_0} \mathbb{P}_{10} \mathcal{R}_{10}^+(u_+, u_-|1) \cdot \mathbb{P}_{20} \mathcal{R}_{20}^+(u_+, u_-|1) \cdot \cdots \mathbb{P}_{N0} \mathcal{R}_{N0}^+(u_+, u_-|1)
\]

As the direct consequence of the equations for the general two-parametric operator \( Q(u_1|u_2) \) we immediately derive the following properties of the operators \( Q_+(u) \) and \( Q_-(u) \)

- operators \( Q_{\pm}(u) \) are \( s\ell(2) \)-invariant
• commutativity
\[ Q_\pm(u) \cdot Q_\pm(v) = Q_\pm(v) \cdot Q_\pm(u) ; \quad Q_+(u) \cdot Q_-(v) = Q_-(v) \cdot Q_+(u) ; \quad Q_\pm(u) \cdot t(v) = t(v) \cdot Q_\pm(u) \]

• Baxter equation for the \( Q_-(u) \)
\[ Q_-(u) \cdot t(u) = \Delta_+(u)Q_-(u+1) + \Delta_-(u)Q_-(u-1) ; \quad \Delta_\pm(u) = (u \pm \ell)^N \]

• Baxter equation for the \( Q_+(u) \)
\[ t(u) \cdot Q_+(u) = \frac{\Delta_+(u-1)\Delta_-(u)}{\Delta_-(u-1)}Q_+(u-1) + \Delta_-(u)Q_+(u+1) . \]

There exists the natural generalization of the operators \( Q_\pm(u) \) to the case of generic inhomogeneous periodic XXX spin chain: we use the local operators \( R_\pm^{k_0} \) as building blocks for the operator \( Q_\pm(u) \). These operators \( Q_\pm(u) \) obey the Baxter equations but they are not \( sl(2) \)-invariant and the commutation relations between \( Q_\pm(u) \) and between \( Q_\pm(u) \) and the transfer matrix \( t(u) \) are more complicated. The two-parametric operator \( Q(u_1|u_2) \) can be factorized on the product of these operators
\[ Q(u_1|u_2) = Q_+(u_1) \cdot P \cdot Q_-(u_2) \]
where \( P \) is the operator of cyclic shift.

The first explicit construction of the \( Q \)-operator using the general transfer matrix built from universal R-matrices was given by A.Yu.Volkov [13] in context of some simple q-deformed model. The very idea that the transfer matrix built from universal R-matrices can serve as a Baxter’s \( Q \)-operator probably belongs to E.K.Sklyanin [7, 8]. (The last sentence in fact coincides with footnote in Volkov’s paper.) Using universal R-matrix for the \( U_q(\widehat{sl}_2) \) affine algebra V.Bazhanov, S.Lukyanov and A.Zamolodchikov [14] constructed \( Q \)-operator for quantum KdV model. The general algebraic scheme how to derive the algebraic relations between different objects of QISM was formulated in the paper of A.Antonov and B.Feigin [15]. The Baxter \( Q \)-operators was constructed for different models in the papers [12, 16, 17, 18, 19]. The factorization of the R-matrix and Baxter’s \( Q \)-operator used in the present paper is very similar to the ones obtained in the context of the chiral Potts model [20, 21, 22].

The presentation is organized as follows. In section 2 we collect the standard facts about the algebra \( sl(2) \) and its representations. Next we consider the defining relation for the general R-matrix, i.e. the solution of the Yang-Baxter equation acting on tensor products of two arbitrary representations. We introduce the natural defining equations for the \( R \)-operators and show that the general R-matrix can be represented as the product of these much simpler operators. In the section 3 we construct Baxter’s \( Q \)-operator for the generic inhomogeneous periodic XXX-spin chain. In the section 4 we consider the reduction to the case of the homogeneous spin chain. In the section 5 we discuss the generalization of the operators \( Q_\pm(u) \) to the case of generic inhomogeneous spin chain and the factorization of the two-parametric operator \( Q(u_1|u_2) \). Finally, in section 6 we summarize.

2 The general \( sl(2) \)-invariant R-matrix

The Lie algebra \( sl(2) \) has three generators \( S , S_\pm \)
\[ [S, S_\pm] = \pm S_\pm , \quad [S_+, S_-] = 2S \]
the central element (Casimir operator \( C_2 \) being
\[ C_2 = S^2 - S + S_+S_- . \]
The Verma module $V_\ell$ is the generic lowest weight $\mathfrak{sl}(2)$-module with the lowest weight $\ell \in \mathbb{C}$ and Casimir $C_2 = \ell (\ell - 1)$. As a linear space $V_\ell$ is spanned by the basis $\{v_k\}_{k=0}^{\infty}$

$$v_k = S^k v_0, \quad S v_k = (\ell + k)v_k, \quad S_- v_k = -k(2\ell + k - 1)v_k$$

where the vector $v_0$ is the lowest weight vector: $S v_0 = 0$, $S v_0 = \ell v_0$. The module $V_\ell$ is irreducible, except for $\ell = -\frac{n}{2}$ where $n \in \{0, 1, 2, 3, \ldots\}$, when there exists an $(n + 1)$-dimensional invariant subspace $V_n \subset V_\ell$ spanned by $\{v_k\}_{k=0}^{n}$. We shall rely extensively on the explicit representation $V_\ell$ of $\mathfrak{sl}(2)$ as the space $\mathbb{C}[z]$ of polynomials in $z$ spanned by monomials $\{z^k\}_{k=0}^{\infty}$.

Then the lowest weight vector is polynomial $v_0 = 1$ and the action of $\mathfrak{sl}(2)$ in $V_\ell$ is given by the first-order differential operators:

$$S = z\partial + \ell, \quad S_- = -\partial, \quad S_+ = z^2\partial + 2\ell z.$$  \hspace{1cm} (2.1)$$

The generating function for the basis vectors can be calculated in closed form

$$e^{\lambda S_+} \cdot 1 = (1 - \lambda z)^{-2\ell} = \sum_{k=0}^{\infty} \lambda^k k! \cdot (2\ell)_k z^k; \quad (2\ell)_k \equiv \frac{\Gamma(2\ell + k)}{\Gamma(2\ell)}$$

This expression clearly shows that for generic $\ell \neq -\frac{n}{2}$ the module $V_\ell$ is an irreducible $\mathfrak{sl}(2)$-module isomorphic to $V_\ell$, the isomorphism being given by $v_k \leftrightarrow (2\ell)_k z^k$. For $\ell = -\frac{n}{2}$ where $n \in \{0, 1, 2, 3, \ldots\}$, we have the finite sum instead of infinite series so that there exists an invariant subspace $V_n \subset V_\ell$ spanned by $\{z^k\}_{k=0}^{n}$ which is isomorphic to $V_n$. For $\ell = -\frac{1}{2}$ one obtains the two-dimensional invariant subspace $V_1 \sim \mathbb{C}^2$ and the matrices of operators $S_-, S_+$ in the basis $e_1 = S_+ \cdot 1 = -z$, $e_2 = 1$ have the standard form of generators $s, s_\pm$ in the fundamental representation

$$s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (2.2)$$

Let $V_{\ell_1}$, $V_{\ell_2}$ and $V_{\ell_3}$ be lowest weight $\mathfrak{sl}(2)$-modules and consider three operators $R_{\ell_1, \ell_2}(u)$ which are acting in $V_{\ell_1} \otimes V_{\ell_2}$. The Yang-Baxter equation is the following three term relation

$$R_{\ell_1, \ell_2}(u - v)R_{\ell_1, \ell_3}(u)R_{\ell_2, \ell_3}(v) = R_{\ell_2, \ell_3}(v)R_{\ell_1, \ell_3}(u)R_{\ell_1, \ell_2}(u - v)$$  \hspace{1cm} (2.3)$$

We look for the general $\mathfrak{sl}(2)$-invariant solution $R_{\ell_1, \ell_2}(u)$ of this equation. The restriction of the operator $R_{\ell_1, \ell_2}(u)$ to the space $V_\ell \otimes \mathbb{C}^2$ coincides up to normalization and a shift of the spectral parameter with the fundamental Lax-operator $[4, 5]$

$$L(u) : V_\ell \otimes \mathbb{C}^2 \to V_\ell \otimes \mathbb{C}^2$$

It is (up to an additive constant) the Casimir operator $C_2$ for the tensor product of representations $V_\ell \otimes \mathbb{C}^2$ $[5]$

$$L(u) \equiv u + 2 \cdot S \otimes s + S_- \otimes s_+ + S_+ \otimes s_- = \begin{pmatrix} u + S & S_- \\ S_+ & u - S \end{pmatrix} = \begin{pmatrix} u + \ell + z\partial & -\partial \\ z^2\partial + 2\ell z & u - \ell - z\partial \end{pmatrix}$$

where $s, s_\pm$ are the generators in the fundamental representation $[2, 2]$ and $S, S_\pm$ are the generators $[2.1]$ in the generic representation $V_\ell$. The Lax operator acts in the space $\mathbb{C}[z] \otimes \mathbb{C}^2$ and despite of the compact notation $L(u)$ depends actually on two parameters: the spin $\ell$ and the spectral parameter $u$. We shall use extensively the parametrization $u_+ \equiv u + \ell$, $u_- \equiv u - \ell$ and show all parameters explicitly. There exists a very useful factorized representation for the L-operator $[7]$

$$L(u_+, u_-) \equiv \begin{pmatrix} u_+ + z\partial & -\partial \\ z^2\partial + (u_+ - u_-)z & u_- - z\partial \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_+ - 1 & -\partial \\ 0 & u_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}.$$  \hspace{1cm} (2.4)
We put $\ell_3 = -\frac{1}{2}$ in (2.8) and consider the restriction on the invariant subspace $V_{\ell_1} \otimes V_{\ell_2} \otimes \mathbb{C}^2$. In this way one obtains the defining equation for the operator $R_{12}(u)$ [5, 4]

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v).$$

The operator $L_k$ acts nontrivially on the tensor product $V_{\ell_k} \otimes \mathbb{C}^2$ which is isomorphic to $\mathbb{C}[z_k] \otimes \mathbb{C}^2$ and the operator $R_{12}(u)$ acts nontrivially on the tensor product $V_{\ell_1} \otimes V_{\ell_2}$ which is isomorphic to $\mathbb{C}[z_1] \otimes \mathbb{C}[z_2] = \mathbb{C}[z_1, z_2]$. It is useful to extract the operator of permutation $P_{12}$

$$P_{12}(z_1, z_2) = \Psi(z_2, z_1) ; \Psi(z_1, z_2) \in \mathbb{C}[z_1, z_2]$$

from the $R$-operator $R_{12}(u) = P_{12}R_{12}(u)$ and solve the defining equation for the $R$-operator

$$R_{12}(u_+|v_+, v_-)L_1(u_+, u_-)L_2(v_+, v_-) = L_1(v_+, v_-)L_2(u_+, u_-)R_{12}(u_+, u_-|v_+, v_-)$$

where $u_+ = u + \ell_1$ , $u_- = u - \ell_1$ , $v_+ = v + \ell_2$ , $v_- = v - \ell_1$. The $R$-operator can be factorized into the product of the simpler elementary building blocks - $R$-operators [9].

**Proposition 1** There exists operator $R_{12}^{+}$ which is the solution of defining equations

$$R_{12}^{+}L_1(u_+, u_-)L_2(v_+, v_-) = L_1(v_+, v_-)L_2(u_+, u_-)R_{12}^{+} \quad (2.5)$$

$$R_{12}^{+} = R_{12}^{+}(u_+|v_+, v_-) ; R_{12}^{+}(u_+|v_+, v_-) = R_{12}^{+}(u_+ + \lambda|v_+ + \lambda, v_- + \lambda).$$

The system of equations (2.5) for the operator $R_{12}^{+}$ is equivalent to the simpler system

$$R_{12}^{+} \cdot [L_1(u_+, u_-) + L_2(v_+, v_-)] = [L_1(v_+, u_-) + L_2(u_+, v_-)] \cdot R_{12}^{+} ; R_{12}^{+} \cdot z_1 = z_1 \cdot R_{12}^{+}. \quad (2.6)$$

These requirements fix the operator $R_{12}^{+}$ up to an overall normalization constant. Fixing the normalization in a such way that $R_{12}^{+} : 1 \mapsto 1$ we obtain

$$R_{12}^{+}(u_+|v_+, v_-) = \frac{\Gamma(v_+ - v_-)}{\Gamma(u_+ - v_-)} \frac{\Gamma(z_2 \partial_2 + u_+ - v_-)}{\Gamma(z_2 \partial_2 + v_+ - v_-)} ; z_2 = z_2 - z_1. \quad (2.7)$$

**Proposition 2** There exists operator $R_{12}^{-}$ which is the solution of defining equations

$$R_{12}^{-}L_1(u_+, u_-)L_2(v_+, v_-) = L_1(u_+, v_-)L_2(v_+, u_-)R_{12}^{-} \quad (2.8)$$

$$R_{12}^{-} = R_{12}^{-}(u_+|u_-, v_-) ; R_{12}^{-}(u_+|u_-, v_-) = R_{12}^{-}(u_+ + \lambda|u_- + \lambda, v_- + \lambda).$$

The system of equations (2.8) for the operator $R_{12}^{-}$ is equivalent to the simpler system

$$R_{12}^{-} \cdot [L_1(u_+, u_-) + L_2(v_+, v_-)] = [L_1(u_+, v_-) + L_2(v_+, u_-)] \cdot R_{12}^{-} ; R_{12}^{-} \cdot z_2 = z_2 \cdot R_{12}^{-}. \quad (2.9)$$

These requirements fix the operator $R_{12}^{-}$ up to an overall normalization constant. Fixing the normalization in a such way that $R_{12}^{-} : 1 \mapsto 1$ we obtain

$$R_{12}^{-}(u_+|u_-|v_-) = \frac{\Gamma(u_+ - u_-)}{\Gamma(u_+ - v_-)} \frac{\Gamma(z_1 \partial_1 + u_+ - v_-)}{\Gamma(z_1 \partial_1 + u_+ - u_-)} ; z_1 = z_1 - z_2. \quad (2.10)$$

**Proposition 3** The operator $\hat{R}$ can be factorized in the following way

$$\hat{R}_{12}(u_+, u_-|v_+, v_-) = R_{12}^{+}(u_+|v_+, u_-)R_{12}^{-}(u_+, u_-|v_-). \quad (2.11)$$
Note that the relations (2.6), (2.9) are simply the rules of commutation of the \( R \)-operators with \( sl(2) \)-generators written in a compact form. The \( R \)-operators change the spins of \( sl(2) \)-representations

\[
R_{12}^+(u_+|v_+, v_-) : V_{\ell_1} \otimes V_{\ell_2} \to V_{\ell_1-\xi_+} \otimes V_{\ell_2+\xi_+} ; \quad \xi_+ = \frac{u_+ - v_+}{2}
\]

\[
R_{12}^-(u_+, u_-|v_-) : V_{\ell_1} \otimes V_{\ell_2} \to V_{\ell_1+\xi_-} \otimes V_{\ell_2-\xi_-} ; \quad \xi_- = \frac{u_- - v_-}{2}
\]

while the general R-matrix

\[
R_{12}(u - v) = P_{12} R_{12}^+(u_+|v_+, u_-) R_{12}^-(u_+, u_-|v_-) \tag{2.12}
\]

appears automatically \( sl(2) \)-invariant \( [R_{12}(u), \vec{S}_1 + \vec{S}_2] = 0 \) where \( \vec{S}_k \equiv (S_k^+, S_k, S_k^-) \).

3 Construction of the Q-operator for the generic inhomogeneous periodic XXX spin chain.

The transfer matrix \( t(u) \) for the generic inhomogeneous periodic XXX spin chain is constructed as follows

\[
t(u) = \operatorname{tr} L_1(u_1^+, u_1^-) \cdots L_N(u_N^+, u_N^-) ; \quad L_k(u_k^+, u_k^-) \equiv \begin{pmatrix}
  u_k^+ + z_k \partial_k & -\partial_k \\
  z_k^2 \partial_k + (u_k^+ - u_k^-) z_k^- & u_k^- - z_k \partial_k
\end{pmatrix}, \tag{3.1}
\]

where \( u_k^\pm = u + \delta_k \pm \ell_k \) and the trace is taken in the auxiliary space \( \mathbb{C}^2 \). The most general transfer matrix \( Q(u_1|u_2) \) is constructed in a similar manner from the operators \( R_{k0} \)

\[
Q(u_1|u_2) = \operatorname{tr}_{V_0} R_{10}(u + \delta_1) R_{20}(u + \delta_2) \cdots R_{N0}(u + \delta_N)
\]

where we use the parameters \( u_1 = 1 + u - \ell_0 \) and \( u_2 = u + \ell_0 \) instead of the spectral parameter \( u \) and the spin parameter \( \ell_0 \) in the auxiliary space \( V_0 = V_{\ell_0} \). The explicit expression for the operator \( R_{k0} \) is (2.12)

\[
R_{k0}(u + \delta_k) = \frac{\Gamma(\ell_k + \ell_0 - u - \delta_k)}{\Gamma(\ell_k + \ell_0 + u + \delta_k)} \cdot P_{k0} \cdot \frac{\Gamma(z_{0k} \partial_0 + 2\ell_k)}{\Gamma(z_{0k} \partial_0 + \ell_k + \ell_0 - u - \delta_k)} \cdot \frac{\Gamma(z_{k0} \partial_k + \ell_k + \ell_0 + u + \delta_k)}{\Gamma(z_{k0} \partial_k + 2\ell_k)}
\]

and it is natural to simplify the notations: we omit the local parameters \( \ell_k, \delta_k \) in the chain and show the global parameters \( u_1 = 1 + u - \ell_0 \) and \( u_2 = u + \ell_0 \) only

\[
R_{k0}(u_1|u_2) = \frac{\Gamma(\ell_k + 1 - u_1 - \delta_k)}{\Gamma(\ell_k + u_2 + \delta_k)} \cdot P_{k0} \cdot \frac{\Gamma(z_{0k} \partial_0 + 2\ell_k)}{\Gamma(z_{0k} \partial_0 + \ell_k + 1 - u_1 - \delta_k)} \cdot \frac{\Gamma(z_{k0} \partial_k + \ell_k + u_2 + \delta_k)}{\Gamma(z_{k0} \partial_k + 2\ell_k)} \tag{3.2}
\]

The basic properties of the operator \( Q(u_1|u_2) \) are enumerated in the Introduction.

The \( sl(2) \)-invariance: \( [Q(u_1|u_2), \vec{S}_1 + \cdots + \vec{S}_N] = 0 \) follows immediately from the \( sl(2) \)-invariance of operators \( R_{k0} \): \( [R_{k0}(u), \vec{S}_k + \vec{S}_0] = 0 \) and the cyclicity property of the trace.

The commutativity \( [Q(u_1|u_2), Q(u_1'|u_2')] = 0 \) follows from the Yang-Baxter equation for the general R-matrix

\[
R_{00'}(u - v) R_{k0}(u) R_{k0'}(v) = R_{k0'}(u) R_{k0}(u) R_{00'}(u - v)
\]

where \( V_0 = V_{\ell_0} \) and \( V_{0'} = V_{\ell_0'} \) are two auxiliary spaces and \( V_k = V_{\ell_k} \) is the k-th quantum space.
The commutativity \([Q(u_1|u_2), t(v)] = 0\) follows from the special case of the general Yang-Baxter relation

\[ R_{k0}(u-v)L_k(u)L_0(v) = L_0(v)L_k(u)R_{k0}(u-v). \]

All these formulae are standard and well known. The really nontrivial is the derivation of the Baxter relations.

Proposition 4 The following triangularity relations hold for the operators \(R_{12}^- (u_+, u_-|0)\) and \(R_{12}^+ (u_+|1, u_-)\)

\[
M_1^{-1} \cdot R_{12}^- (u_+, u_-|0) \cdot L_1 (u_+, u_-) \cdot M_2 =
\]

\[
= \begin{pmatrix}
  u_+ \cdot R_{12}^- (u_+ + 1, u_- + 1|0) & -R_{12}^- (u_+, u_-|0) \partial_1 \\
  0 & u_- \cdot R_{12}^- (u_+ - 1, u_- - 1|0)
\end{pmatrix},
\]

\[
M_1^{-1} \cdot L_2 (u_+, u_-) \cdot R_{12}^+ (u_+|1, u_-) \cdot M_2 =
\]

\[
= \begin{pmatrix}
  u_+ (u_+-1) \cdot R_{12}^+ (u_+ - 1|1, u_- - 1) & -\partial_1 R_{12}^+ (u_+|1, u_-) \\
  0 & u_- \cdot R_{12}^+ (u_+ + 1|1, u_- + 1)
\end{pmatrix},
\]

where

\[
R_{12}^- (u_+, u_-|0) = \frac{\Gamma(u_+ - u_-)}{\Gamma(u_+)} \frac{\Gamma(z_{12} \partial_1 + u_+)}{\Gamma(z_{12} \partial_1 + u_+ - u_-)}; \quad R_{12}^+ (u_+|1, u_-) = \frac{\Gamma(1 - u_-)}{\Gamma(u_+ - u_-)} \frac{\Gamma(z_{21} \partial_2 + u_+ - u_-)}{\Gamma(z_{21} \partial_2 + 1 - u_-)},
\]

\[
M_k \equiv \begin{pmatrix}
  1 & 0 \\
  z_k & 1
\end{pmatrix}.
\]

We prove the triangularity relation for the operator \(R_{12}^-\) and the proof for the operator \(R_{12}^+\) is very similar. We start directly from the defining equation (2.8). Using factorization (2.10) of the Lax operator and commutativity of \(R_{12}^-\) and \(z_2\) the defining equation for the \(R_{12}^-\)-operator can be represented in the form

\[
R_{12}^- \left( \begin{array}{ccc}
  u_+ + z_1 \partial_1 \\
  z_1^2 \partial_1 + (u_+ - u_-) z_1
\end{array} \right) \left( \begin{array}{ccc}
  -\partial_1 & 1 & 0 \\
  u_- - z_1 \partial_1 & z_2 & 1
\end{array} \right) \left( \begin{array}{ccc}
  1 & 0 & -\partial_2 \\
  0 & v_- & u_-
\end{array} \right) =
\]

\[
= \left( \begin{array}{ccc}
  1 & 0 \\
  z_1 & 1
\end{array} \right) \left( \begin{array}{ccc}
  u_+ - 1 & -\partial_1 \\
  0 & v_- & u_-
\end{array} \right) \left( \begin{array}{ccc}
  1 & 0 & -\partial_2 \\
  -z_1 & 1 & 0 & u_-
\end{array} \right) \left( \begin{array}{ccc}
  1 & 0 \\
  z_2 & 1
\end{array} \right) \left( \begin{array}{ccc}
  1 & 0 & -\partial_2 \\
  0 & v_- & u_-
\end{array} \right) R_{12}^-.
\]

Next we transform all this in the following simple way

\[
= v_-^{-1} \cdot \left( \begin{array}{ccc}
  u_+ - 1 & -\partial_1 \\
  0 & v_-
\end{array} \right) \left( \begin{array}{ccc}
  1 & 0 \\
  z_1 & 1
\end{array} \right) \left( \begin{array}{ccc}
  1 & 0 & -\partial_2 \\
  0 & u_-
\end{array} \right) R_{12}^- \left( \begin{array}{ccc}
  v_- & \partial_2 \\
  0 & 1
\end{array} \right) =
\]

\[
= \left( \begin{array}{ccc}
  u_+ + z_{12} \partial_1 \\
  -v_- \cdot z_{12} \partial_1
\end{array} \right) \cdot R_{12}^- \left( \begin{array}{ccc}
  u_+ - v_- - (u_+ - z_{12} \partial_1) \\
  0 & v_-
\end{array} \right) \cdot \frac{-R_{12}^- \partial_1}{u_+ - v_- - (u_+ - z_{12} \partial_1)}. \]

In the last transformation we use the explicit representation for the \(R_{12}^-\)-operator (2.10). The obtained matrix becomes triangular at the point \(v_- = 0\) and using the explicit expression for the operator \(R_{12}^-\) this matrix can be transformed to the form (3.3). From the triangularity relations for the \(R\)-operators immediately follow two triangularity relations for the general operator \(R_{12}^-\).
Proposition 5 The following triangularity relations for the operator $R_{12}(u_+, u_-|v_+, v_-)$ hold
\[
M_2^{-1} \cdot R_{12}(u_+, u_-|v_+, 0) \cdot L_1(u_+, u_-) \cdot M_2 = \left( \begin{array}{ccc} u_+ \cdot R_{12}(u_+, u_-|v_+, 1, 0) & 0 \\ 0 & u_- \cdot R_{12}(u_+, u_-|v_+, 1, 0) & ** \end{array} \right) \\
= \left( \begin{array}{ccc} u_+ & 0 \\ 0 & u_- \cdot R_{12}(u_+, u_-|v_+, 1, 1, 0) & ** \end{array} \right)
\]
(3.5)

The relation (3.5) is obtained from the relation (3.3) simply by multiplying with the operator $u$ from the left and using the expression (2.11) for the operator $R$. The parameter $u$ plays a passive role in this first relation. The relation (3.6) is obtained from the relation (3.4) simply by multiplying with the operator $R^{+}$ from the left and using the cyclisity of the trace one obtains the equation
\[
\text{M}_0^{-1} \cdot R_{k0}(u_1 - v|u) \cdot L_k(u + \delta_k) \cdot M_0 = \left( \begin{array}{ccc} u_k^+ \cdot R_{k0}(u_1 - v|u + 1) & 0 \\ 0 & u_k^- \cdot R_{k0}(u_1 - v|u - 1) & ** \end{array} \right)
\]
(3.6)

The operator $R^{+}$ plays a passive role. The relation (3.6) is obtained from the relation (3.4) simply by multiplying with the operator $P_{12}$ from the left and with the operator $R_{12}(u_+, u_-|v_-)$ from the right and using the expression (2.11) for the operator $R_{12}$. Now the operator $R^{-}$ plays a passive role.

Let us go to the proof of the Baxter equation
\[
Q(u_1|u) \cdot t(u) = \Delta_+(u)Q(u_1|u + 1) + \Delta_-(u)Q(u_1|u - 1); \quad \Delta_+(u) = (u + \delta_1 \pm \ell_1) \cdots (u + \delta_N \pm \ell_N)
\]

It is the direct consequence of the triangularity relation (3.5) and cyclicity of the trace. Let us choose the first space in (3.5) as $k$-th quantum space and the second space as the auxiliary space. We have in useful notations
\[
M_0^{-1} \cdot R_{k0}(u_1 - v|u) \cdot L_k(u + \delta_k) \cdot M_0 = \left( \begin{array}{ccc} u_k^+ \cdot R_{k0}(u_1 - v|u + 1) & 0 \\ 0 & u_k^- \cdot R_{k0}(u_1 - v|u - 1) & ** \end{array} \right)
\]

Multiplying these equalities for $k = 1, 2, \cdots N$, taking the traces in auxiliary spaces $C^2$ and $V_0$ and using the cyclisity of the trace one obtains the equation
\[
Q(u_1 - v|u) \cdot t(u) = \Delta_+(u)Q(u_1 - v|u + 1) + \Delta_-(u)Q(u_1 - v|u - 1)
\]

The parameter $u_1$ is arbitrary so that we obtain the needed relation. The Baxter equation with respect to parameter $u_2$ follows from the triangularity relation (3.6) and the derivation is very similar.

4 Q-operator for the homogeneous periodic XXX spin chain

The operator $R_{k0}(u_1|u_2)$ has two points of degeneracy: $u_1 = 1 - \delta_k - \ell_k$ and $u_2 = \ell_k - \delta_k$. In the case of homogeneous spin chain: $\delta_k = 0$ and $\ell_k = \ell$, the degeneration points for all operators $R_{k0}$ coincide so that it is possible to remove half of the $R$-operators in the two-parametric operator
\[
Q(u_1|u_2) = \text{tr}_{V_0} R_{10}(u_1|u_2) R_{20}(u_1|u_2) \cdots R_{N0}(u_1|u_2)
\]

We obtain the following reductions of the two-parametric Q-operator: at the first point of degeneracy $u_1 = 1 - \ell$
\[
Q_-(u) = Q(1 - \ell|u) = \text{tr}_{V_0} P_{10} R_{10}^{-}(u_+, u_-|0) \cdot P_{20} R_{20}^{-}(u_+, u_-|0) \cdots P_{N0} R_{N0}^{-}(u_+, u_-|0) = \frac{\Gamma^N(2\ell)}{\Gamma^N(\ell + u)} \cdot \text{tr}_{V_0} P_{10} \Gamma^N \frac{\Gamma(z_{10}\partial_1 + u + \ell)}{\Gamma(z_{10}\partial_1 + 2\ell)} \cdots P_{N0} \Gamma^N \frac{\Gamma(z_{N0}\partial_N + u + \ell)}{\Gamma(z_{N0}\partial_N + 2\ell)}
\]
and at the second point of degeneracy \( u_2 = \ell \)

\[
Q_+(u) = Q(u|\ell) = \text{tr}_{V_0} \mathcal{P}_{10} \mathcal{R}_{10}^+(u_1|1,u_-) \cdot \mathcal{P}_{20} \mathcal{R}_{20}^+(u_1|1,u_-) \cdots \mathcal{P}_{N_0} \mathcal{R}_{N_0}^+(u_1|1,u_-) = \frac{\Gamma^N(1 + \ell - u)}{\Gamma^N(2\ell)} \cdot \text{tr}_{V_0} \mathcal{P}_{10} \frac{\Gamma(z_{01} \partial_0 + 2\ell)}{\Gamma(z_{01} \partial_0 + 1 + \ell - u)} \cdots \mathcal{P}_{N_0} \frac{\Gamma(z_{0N} \partial_0 + 2\ell)}{\Gamma(z_{0N} \partial_0 + 1 + \ell - u)}.
\]

As the direct consequence of the equations for the general two-parametric operator \( Q(u_1|u_2) \) we immediately obtain the corresponding properties of the operators \( Q_+(u) \) and \( Q_-(u) \) itemized in Introduction. We construct the \( Q_\pm \)-operator as the trace of the products of \( \mathcal{R}^\pm \) operators in auxiliary space \( V_0 \). The whole construction is pure algebraic. In this Section we shall derive the explicit formulae for the action of the operator \( Q_- \) in the space of polynomials. The explicit expression for the second operator \( Q_+ \) is more complicated and we shall not consider it here. We have the following expression for the operator \( Q_- \)

\[
Q_-(u) = \text{tr}_{V_0} \mathcal{P}_{10} \mathcal{R}(z_{10} \partial_1) \cdots \mathcal{P}_{N_0} \mathcal{R}(z_{N_0} \partial_{N_0}) ; \quad \mathcal{R}(x) \equiv \frac{\Gamma(x + u + \ell)}{\Gamma(x + 2\ell)}.
\]

**Proposition 6** The action of the operator \( Q_-(u) \) on a polynomial \( \Psi(z_1 \cdots z_N) \) can be represented in the following equivalent forms

\[
\begin{align*}
\mathcal{R}(z_{01} \partial_0) \mathcal{R}(z_{12} \partial_1) \mathcal{R}(z_{23} \partial_2) \cdots \mathcal{R}(z_{N-1,N} \partial_{N_0}) \Psi(z_0, z_1, \cdots z_{N-1})|_{z_0 = z_N} = & \\
\left[ Q_-(u) \Psi \right](z_1, \cdots z_N) = & \\
\mathcal{R}(t_1 \partial_1) \mathcal{R}(t_2 \partial_2) \cdots \mathcal{R}(t_N \partial_N) \cdot \Psi(t_1 z_{N+1} + z_1, t_2 z_{N+2} + z_2 \cdots t_N z_{N-1}, z_N)|_{t_k = 1} = & \\
\left[ Q_-(u) \Psi \right](z_1, \cdots z_N) = & \\
\frac{\Gamma^N(2\ell)}{\Gamma^N(\ell + u) \Gamma^N(\ell - u)} \cdot \int_0^1 d\alpha_1 \left( 1 - \alpha_1 \right)^{\ell-u-1} \alpha_1^{\ell+u-1} \cdots \int_0^1 d\alpha_N \left( 1 - \alpha_N \right)^{\ell-u-1} \alpha_N^{\ell+u-1} \Psi(\alpha_1 z_{N+1} + z_1, \alpha_2 z_{N+2} + z_2 \cdots \alpha_N z_{N-1}, z_N).
\end{align*}
\]

The operator \( Q_-(u) \) maps polynomials in variables \( z_1 \cdots z_N \) to polynomials in variables \( u, z_1 \cdots z_N \)

\[
Q_-(u) : \mathbb{C}[z_1 \cdots z_N] \mapsto \mathbb{C}[u, z_1 \cdots z_N].
\]

Let \( z_0 \) be the variable in the auxiliary space \( V_0 \) and let the operator \( \Lambda \) act in the tensor product \( V_0 \otimes V_1 \cdots \otimes V_N \) and \( \Psi(z_1 \cdots z_N) \in V_1 \cdots \otimes V_N \). The trace of the operator \( \Lambda \) in auxiliary space \( V_0 = \mathbb{C}[z_0] \) can be calculated as follows

\[
\left[ \text{tr}_{V_0} \Lambda \right] \Psi(z_1 \cdots z_N) = \sum_{m=0}^{+\infty} \frac{1}{m!} \partial_0^m \Lambda \cdot z_0^m \cdot \Psi(z_1 \cdots z_N) \bigg|_{z_0 = 0}.
\]

In order to prove \( (4.1) \) it is useful to move all permutations to the right

\[
\mathcal{P}_{10} \mathcal{R}(z_{10} \partial_1) \mathcal{P}_{20} \mathcal{R}(z_{20} \partial_2) \cdots \mathcal{P}_{N_0} \mathcal{R}(z_{N_0} \partial_{N_0}) = \mathcal{R}(z_{01} \partial_0) \mathcal{R}(z_{12} \partial_1) \mathcal{R}(z_{23} \partial_2) \cdots \mathcal{R}(z_{N-1,N} \partial_{N_0}) \cdot \mathcal{P}_{10} \mathcal{P}_{20} \cdots \mathcal{P}_{N_0}.
\]

10
Then we have
\[ R(z_{01} \partial_{0})R(z_{12} \partial_{1})R(z_{23} \partial_{2}) \cdots R(z_{N-1,N} \partial_{N-1}) \cdot \mathbb{P}_{10} \mathbb{P}_{20} \cdots \mathbb{P}_{N0} \cdot z_{m}^{m} \cdot \Psi(z_{1} \cdots z_{N}) = \]
\[ = R(z_{01} \partial_{0})R(z_{12} \partial_{1})R(z_{23} \partial_{2}) \cdots R(z_{N-1,N} \partial_{N-1}) \cdot z_{N}^{m} \cdot \Psi(z_{0}, z_{1} \cdots z_{N-1}) = \]
\[ = z_{N}^{m_{0}} R(z_{01} \partial_{0})R(z_{12} \partial_{1})R(z_{23} \partial_{2}) \cdots R(z_{N-1,N} \partial_{N-1}) \Psi(z_{0}, z_{1} \cdots z_{N-1}). \]

The result of the operation \( \sum_{m=0}^{\infty} \frac{1}{m!} \partial_{0}^{m} \) can be calculated in closed form
\[ [Q_{-}(u) \Psi](z_{1} \cdots z_{N}) = \sum_{m=0}^{\infty} \frac{1}{m!} \partial_{0}^{m} z_{N}^{m_{0}} R(z_{01} \partial_{0})R(z_{12} \partial_{1}) \cdots R(z_{N-1,N} \partial_{N-1}) \Psi(z_{0}, z_{1} \cdots z_{N-1})|_{z_{0}=0} = \]
\[ = e^{z_{N} \partial_{0}} R(z_{01} \partial_{0})R(z_{12} \partial_{1})R(z_{23} \partial_{2}) \cdots R(z_{N-1,N} \partial_{N-1}) \Psi(z_{0}, z_{1} \cdots z_{N-1})|_{z_{0}=0} = \]
\[ = R(z_{01} \partial_{0})R(z_{12} \partial_{1})R(z_{23} \partial_{2}) \cdots R(z_{N-1,N} \partial_{N-1}) \Psi(z_{0}, z_{1} \cdots z_{N-1})|_{z_{0}=z_{N}} \]
and we obtain the first representation \([4.1]\).

The second formula is the simple consequence of the first one. Let us represent the action of the operator \( R(z_{k-1,k} \partial_{k-1}) \) in the form
\[ R(z_{k-1,k} \partial_{k-1}) \Psi(z_{k-1}) = R(t_{k} \partial_{k}) e^{z_{k} \partial_{k-1}} e^{-z_{k} \partial_{k-1}} \Psi(z_{k-1}) \Big|_{t_{k}=1} = \]
\[ = R(t_{k} \partial_{k}) \Psi(t_{k} z_{k-1,k} + z_{k}) \Big|_{t_{k}=1} \]
We have
\[ [Q_{-}(u) \Psi](z_{1} \cdots z_{N}) = R(t_{01} \partial_{1})R(t_{12} \partial_{2}) \cdots R(t_{Nt_{N}} \partial_{N}) \cdot \Psi(t_{1} z_{01} + z_{1}, t_{2} z_{12} + z_{2} \cdots t_{N} z_{N-1,N} + z_{N})|_{t_{k}=1}, z_{0}=z_{N} \]
which is just the second formula \([4.2]\).

The third formula is the simple consequence of the second one. We use the integral representation for the operator
\[ R(t_{k} \partial_{k}) \Phi(t_{k}) \Big|_{t_{k}=1} = \frac{\Gamma(2\ell)}{\Gamma(\ell + u) \Gamma(\ell - u)} \cdot \int_{0}^{1} d\alpha_{k}(1 - \alpha_{k})^{\ell - u - 1} \alpha_{k}^{\ell + u - 1} \Phi(t_{k}) \Big|_{t_{k}=1} = \]
\[ = \frac{\Gamma(2\ell)}{\Gamma(\ell + u) \Gamma(\ell - u)} \cdot \int_{0}^{1} d\alpha_{k}(1 - \alpha_{k})^{\ell - u - 1} \alpha_{k}^{\ell + u - 1} \Phi(t_{k}) \Big|_{t_{k}=1} = \]
This allows now to obtain \([4.3]\) from \([4.2]\).

The most useful for the proof of the last property is the formula \([4.2]\). Let us consider the action of \( Q_{-}(u) \) on the monomial \( z_{1}^{m_{1}} \cdots z_{N}^{m_{N}} \)
\[ Q_{-}(u) z_{1}^{m_{1}} \cdots z_{N}^{m_{N}} = R(t_{1} \partial_{1}) \cdots R(t_{N} \partial_{N}) \cdot (t_{1} z_{1N} + z_{1})^{m_{1}} (t_{2} z_{12} + z_{2})^{m_{2}} \cdots (t_{N} z_{N-1,N} + z_{N})^{m_{N}} \Big|_{t_{k}=1} \]
The left hand side is the sum of monomials \( t_{1}^{k_{1}} \cdots t_{N}^{k_{N}} \) with polynomials coefficients from \( \mathbb{C}[z_{1} \cdots z_{N}] \) so that it is sufficient to prove that
\[ R(t_{1} \partial_{1})R(t_{2} \partial_{2}) \cdots R(t_{N} \partial_{N}) \cdot t_{1}^{k_{1}} t_{2}^{k_{2}} \cdots t_{N}^{k_{N}} \Big|_{t_{k}=1} = R(k_{1})R(k_{2}) \cdots R(k_{N}) \]
is polynomial in \( u \). We have
\[ R(k) = \frac{\Gamma(2\ell)}{\Gamma(u + \ell) \Gamma(k + 2\ell)} = \frac{\Gamma(2\ell)}{\Gamma(k + 2\ell)} \cdot (u + \ell)(u + \ell + 1) \cdots (u + \ell + k - 1) \]
In the generic situation all is well defined and one obtains a polynomial in \( u \). Note that the operator \( Q_{-}(u) \) coincides with Q-operator constructed in \([10]\) by using Pasquier-Gaudin method \([12]\). There exists another equivalent representation for the operator \( Q_{-}(u) \) \([11]\).
5 Back to the inhomogeneous chain: factorization of the two-parametric operator \( Q(u_1 | u_2) \)

There exists the natural generalization of the operators \( Q_\pm(u) \) to the case of the generic inhomogeneous periodic XXX spin chain. The natural local building blocks for the operators \( Q_\pm(u) \) are the operators \( R_{k_0}^\pm \):

\[
R_{k_0}^-(u) = \text{tr}_{V_0} P_0 R_{10}^- (u + \delta_1) \cdots P_{N_0} R_{N_0}^- (u + \delta_N)
\]

\[
R_{k_0}^+(u) = \text{tr}_{V_0} P_0 R_{10}^+ (u + \delta_1) \cdots P_{N_0} R_{N_0}^+ (u + \delta_N)
\]

where we used the notations

\[
R_{k_0}^- (u) = \frac{\Gamma (2 \ell_k)}{\Gamma (u + \ell_k)} \cdot \frac{\Gamma (z_{k_0} \partial_k + u + \ell_k)}{\Gamma (z_{k_0} \partial_k + 2 \ell_k)}
\]

\[
R_{k_0}^+ (u) = \frac{\Gamma (1 + \ell_k - u)}{\Gamma (2 \ell_k)} \cdot \frac{\Gamma (z_{k_0} \partial_0 + 2 \ell_k)}{\Gamma (z_{k_0} \partial_0 + 1 + \ell_k - u)}
\]

Note that explicit expressions for the operator \( Q^- (u) \) can be obtained by repeating step by step all calculations from the previous Section.

**Proposition 7** The operator \( Q^- (u) \) obeys the Baxter’s equation

\[
Q^- (u) \cdot t(u) = \Delta_+ (u) Q^- (u + 1) + \Delta_- (u) Q^- (u - 1) ; \quad \Delta_\pm (u) = (u + \delta_1 \pm \ell_1) \cdots (u + \delta_N \pm \ell_N)
\]

and the operator \( Q^+ (u) \) obeys the Baxter’s equation

\[
t(u) \cdot Q^+ (u) = \frac{\Delta_+ (u-1) \Delta_- (u)}{\Delta_- (u-1)} Q^+ (u - 1) + \Delta_- (u) Q^+ (u + 1)
\]

We prove again the first relation only and the proof of the second ones is similar. It is the direct consequence of the triangularity relation \( [\text{XXX}] \) and cyclicity of the trace. Let us choose the first space in \( [\text{XXX}] \) as k-th quantum space and the second space as the auxiliary space. Then we have

\[
P_{k_0} R^-_{k_0} (u + \delta_k) \cdot L_k (u_k^+, u_k^-) = M_0 \cdot \left( \begin{array}{cc} u_k^+ & -P_{k_0} R_k^- (u + \delta_k) \partial_k \\ 0 & u_k^- \cdot P_{k_0} R_k^- (u - 1 + \delta_k) \end{array} \right) M_0^{-1}
\]

The triangularity relation for the \( R^- \)-operator allows to transform the following product to the triangular form

\[
P_{10} R^-_{10} (u + \delta_1) \cdots P_{N_0} R^-_{N_0} (u + \delta_N) \cdot L_1 (u_1^+, u_1^-) \cdots L_N (u_N^+, u_N^-) =
\]

\[
= P_{10} R^-_{10} (u + \delta_1) L_1 (u_1^+, u_1^-) \cdots P_{N_0} R^-_{N_0} (u + \delta_N) L_N (u_N^+, u_N^-) =
\]

\[
= M_0 \left( \begin{array}{cc} u_1^+ & \cdots P_{10} R^-_{10} (u + 1 + \delta_1) \\ 0 & u_1^- \cdot P_{10} R^-_{10} (u - 1 + \delta_1) \end{array} \right) \cdots
\]

\[
\cdots \left( \begin{array}{cc} u_N^+ & \cdots P_{N_0} R^-_{N_0} (u + 1 + \delta_N) \\ 0 & u_N^- \cdot P_{N_0} R^-_{N_0} (u - 1 + \delta_N) \end{array} \right) M_0^{-1}
\]

To derive the Baxter’s equation it remains to calculate the traces in the auxiliary spaces \( V_0 \) and \( \mathbb{C}^2 \) using the following simple statement. Let operators \( A_{ij} \) act in the tensor product \( V_0 \otimes V_1 \cdots \otimes V_n \). Then we have

\[
\text{tr}_{V_0} \text{tr} \left( \begin{array}{cc} 1 & 0 \\ z_0 & 1 \end{array} \right) \left( \begin{array}{cc} A_{11} & A_{12} \\ 0 & A_{22} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ -z_0 & 1 \end{array} \right) = \text{tr}_{V_0} A_{11} + \text{tr}_{V_0} A_{22}.
\]
The proof is straightforward and uses the cyclic property of the trace only
\[
\text{tr}_{V_0} A_{12} \cdot z_0 = \text{tr}_{V_0} z_0 \cdot A_{12}.
\]

Next we consider the commutation relations between the operators \(Q\pm(u)\) and the transfer matrix \(t(u)\).

**Proposition 8** The operators \(Q\pm(u)\) have the following commutation relations with the transfer matrix \([3,4]\)

\[
Q_- (\lambda) \cdot \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) = \\
= \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) \cdot Q_- (\lambda), \\
(5.7)
\]

\[
Q_+ (\lambda) \cdot \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) = \\
= Q_+ (\lambda) \cdot \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_1^-) \cdots L_n (u_n^+, u_{n-1}^-). \\
(5.8)
\]

Let us prove the equation \([5.7]\) for example. We start from the commutation relation

\[
\mathbb{P} R_{10}^+ (u + \delta_1 - \lambda_-) \cdots \mathbb{P} R_{N0}^+ (u + \delta_N - \lambda_-) \cdot L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) \cdot L_0 (\lambda_+, \lambda_-) = \\
= L_0 (u_1^+, \lambda_-) \cdot L_1 (u_2^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (\lambda_+, u_n^-) \cdot \mathbb{P} R_{10}^+ (u + \delta_1 - \lambda_-) \cdots \mathbb{P} R_{N0}^+ (u + \delta_N - \lambda_-)
\]

which is derived directly from the defining relation \([2.8]\) for the operator \(R^-\). Next we put \(\lambda_+ = u_1^+\) and multiply both sides of the equation by the \(L_0^{-1} (u_1^+: \lambda_-)\). It remains to calculate the traces in the auxiliary spaces \(V_0 \otimes V_1 \cdots \otimes V_n\).

\[
\text{tr}_{V_0} \text{tr} L_0 (u^+; u^-) \cdot \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} \cdot L_0^{-1} (u^+; u^-) = \text{tr}_{V_0} A_{11} + \text{tr}_{V_0} A_{22}. \\
(5.9)
\]

The proof is straightforward and uses the cyclic property of the trace. After all one obtains the commutation relation \([5.7]\) for \(\lambda = u - \lambda_-\) which is equivalent \([5.7]\) due to arbitrariness of \(\lambda_+\).

It is possible to build from the operators \(Q\pm\) some composite operator commuting with transfer matrix \(t(u)\). The operator \(Q_+ (\lambda) \cdot \mathbb{P} \cdot Q_- (\mu)\), where \(\mathbb{P}\) is the operator of cyclic shift

\[
\mathbb{P} \psi (z_1, z_2 \cdots z_n) = \psi (z_2, z_3 \cdots z_n, z_1)
\]

commutes with the generic transfer matrix \([3,1]\)

\[
Q_+ (\lambda) \cdot \mathbb{P} \cdot Q_- (\mu) \cdot \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) = \\
= \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) \cdot Q_+ (\lambda) \cdot \mathbb{P} \cdot Q_- (\mu)
\]

The commutativity follows from the commutation relations \([5.7]\) and \([5.8]\)

\[
Q_+ (\lambda) \cdot \mathbb{P} \cdot Q_- (\mu) \cdot \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) = \\
= Q_+ (\lambda) \cdot \mathbb{P} \cdot \text{tr} L_1 (u_2^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) \cdot Q_- (\mu) = \\
= Q_+ (\lambda) \cdot \text{tr} L_2 (u_2^+, u_1^-) L_3 (u_3^+, u_2^-) \cdots L_1 (u_1^+, u_1^-) \cdot \mathbb{P} \cdot Q_- (\mu) = \\
= Q_+ (\lambda) \cdot \text{tr} L_1 (u_1^+, u_0^-) L_2 (u_2^+, u_1^-) \cdots L_n (u_n^+, u_{n-1}^-) \cdot \mathbb{P} \cdot Q_- (\mu) = \\
= \text{tr} L_1 (u_1^+, u_1^-) L_2 (u_2^+, u_2^-) \cdots L_n (u_n^+, u_n^-) \cdot Q_+ (\lambda) \cdot \mathbb{P} \cdot Q_- (\mu)
\]
The operator $Q_+ (\lambda) \cdot P \cdot Q_- (\mu)$ obeys both Baxter equations with respect to each parameter. It is a simple consequence of Baxter’s equations for the operators $Q_+ (\lambda)$ and $Q_- (\mu)$.

After all it is evident that in the case of homogeneous spin chain: $\delta_k = 0$ and $\ell_k = \ell$ we have the points of degeneracy $\lambda = 1 - \ell$ and $\mu = \ell$ where operators $Q_\pm$ reduced to the operator $P$: $Q_+ (1 - \ell) = P$ and $Q_- (\ell) = P$. So that we have

$$Q_+ (1 - \ell) \cdot P \cdot Q_- (\mu) = Q_- (\mu) \cdot Q_+ (\lambda) \cdot P \cdot Q_- (\ell) = Q_+ (\lambda)$$

We see that the composite operator $Q_+ (u_1) \cdot P \cdot Q_- (u_2)$ has the same properties as the two-parametric operator $Q(u_1|u_2)$. Of course this is not accidental because these operators indeed coincide.

**Proposition 9**

$$Q(u_1|u_2) = Q_+ (u_1) \cdot P \cdot Q_- (u_2) \quad (5.10)$$

The direct proof which we have is rather technical and does not illuminate the origin of this factorization. For brevity we omit the proof and hope that this factorization looks sufficiently natural.

Finally we consider the commutation relations between operators $Q_+ (u_1)$ and $Q_- (u_2)$.

**Proposition 10** The operators $Q_+ (u_1)$ and $Q_- (u_1)$ have the following commutation relations

$$Q_+ (u_1) \cdot P \cdot Q_- (u_2) \cdot Q_+ (v_1) = Q_+ (v_1) \cdot P \cdot Q_- (u_2) \cdot Q_+ (u_1) \quad (5.11)$$

$$Q_- (u_2) \cdot Q_+ (v_1) \cdot P \cdot Q_- (v_2) = Q_- (v_2) \cdot Q_+ (v_1) \cdot P \cdot Q_- (u_2) \quad (5.12)$$

These commutation relations are the consequence of the commutation relations for the two-parametric operators

$$Q(u_1|u_2) \cdot Q(v_1|v_2) = Q(v_1|u_2) \cdot Q(u_1|v_2) \quad (5.13)$$

$$Q(u_1|u_2) \cdot Q(v_1|v_2) = Q(u_1|v_2) \cdot Q(v_1|u_2) \quad (5.14)$$

and the factorization of the two-parametric operator

$$Q(u_1|u_2) = Q_+ (u_1) \cdot P \cdot Q_- (u_2).$$

We shall prove the commutation relations (5.13) and (5.14). These commutation relations follow from some local three-term relations which are similar to the Yang-Baxter relation for the $R$-operators. To derive the needed relation we shall proceed in close analogy with the well known derivation of the Yang-Baxter relation from the defining equation

$$\hat{R}_{12}(u_+, u_-|v_+, v_-) L_1(u_+, u_-) L_2(v_+, v_-) = L_1(v_+, v_-) L_2(u_+, u_-) \hat{R}_{12}(u_+, u_-|v_+, v_-)$$

We recall that the commutativity of the diagram

$$\begin{align*}
L_1(v_+, v_-) L_2(u_+, u_-) L_3(w_+, w_-) &\xrightarrow{\hat{R}_{23}(u_+, u_-|w_+, w_-)} L_1(v_+, v_-) L_2(w_+, w_-) L_3(u_+, u_-) \\
L_1(u_+, u_-) L_2(v_+, v_-) L_3(w_+, w_-) &\xrightarrow{\hat{R}_{12}(u_+, u_-|v_+, v_-)} L_1(v_+, v_-) L_2(v_+, v_-) L_3(u_+, u_-) \\
L_1(u_+, u_-) L_2(v_+, v_-) L_3(w_+, w_-) &\xrightarrow{\hat{R}_{23}(v_+, v_-|w_+, w_-)} L_1(w_+, w_-) L_2(v_+, v_-) L_3(u_+, u_-) \\
L_1(u_+, u_-) L_2(w_+, w_-) L_3(v_+, v_-) &\xrightarrow{\hat{R}_{12}(u_+, u_-|w_+, w_-)} L_1(w_+, w_-) L_2(v_+, v_-) L_3(v_+, v_-) \\
\end{align*}$$
results in the Yang-Baxter equation for $\mathcal{R}$-operators

$$
\mathcal{R}_{12}(v_+, v_-|w_+, w_-)\mathcal{R}_{23}(u_+, u_-|v_+, v_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-) = \\
= \mathcal{R}_{23}(u_+, u_-|v_+, v_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-)\mathcal{R}_{23}(v_+, v_-|w_+, w_-).
$$

In the same way the commutativity of the diagram

\[
\begin{array}{ccc}
L_1(v_+, v_-) & \mathcal{R}_{23}(u_+, u_-|w_+, w_-) & L_1(v_+, v_-) \\
\mathcal{R}_{12}(u_+, u_-|v_+, v_-) & & \mathcal{R}_{12}(v_+, v_-|w_-) \\
L_1(v_+, u_-) & \mathcal{R}_{23}(v_+, v_-|w_-) & L_1(u_+, u_-) \\
\mathcal{R}_{23}(v_+, v_-|w_-) & \mathcal{R}_{23}(u_+, u_-|w_+, w_-) & L_1(v_+, v_-) \\
\end{array}
\]

results in three term relation for one $\mathcal{R}^-$ and two $\mathcal{R}$-operators

$$
\mathcal{R}_{12}^-(v_+, v_-|w_-)\mathcal{R}_{23}(u_+, u_-|w_+, w_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-) = (5.15) \\
= \mathcal{R}_{23}(u_+, u_-|w_+, v_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-)\mathcal{R}_{23}(v_+, v_-|w_-)
$$

In a similar way one obtains the relation for one $\mathcal{R}^+$ and two $\mathcal{R}$-operators

$$
\mathcal{R}_{12}^+(v_+|w_+, w_-)\mathcal{R}_{23}(u_+, u_-|w_+, w_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-) = (5.16) \\
= \mathcal{R}_{23}(u_+, u_-|v_+, w_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-)\mathcal{R}_{23}(v_+, w_+, w_-)
$$

First of all after multiplication with the permutation operators the three term relation (5.15) can be rewritten as follows

$$
P_{23}\mathcal{R}_{23}^-(v_+, v_-|w_-)\mathcal{R}_{13}(u_+, u_-|w_+, w_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-) = \\
= R_{12}(u_+, u_-|w_+, v_-)R_{13}(u_+, u_-|v_+, v_-)P_{23}\mathcal{R}_{23}^-(v_+, v_-|w_-)
$$

Next we choose the first space $V_k$, the second space $V_0$ and the third space $V_{0'}$. We have

$$
P_{00'}R_{00}^-(v_+, v_-|w_-)R_{k0'}(u_+, u_-|w_+, w_-)R_{k0}(u_+, u_-|v_+, v_-) = \\
= R_{k0}(u_+, u_-|v_+, v_-)R_{k0'}(u_+, u_-|v_+, w_-)P_{00'}R_{00}^-(v_+, v_-|w_-)
$$

These local relations imply in the standard way the following commutation relation

$$Q(u_1|u_2)Q(v_1|v_2) = Q(v_1|u_2)Q(u_1|v_2)$$

In a similar way choosing the first space $V_k$, the second space $V_0$ and the third space $V_{0'}$ we rewrite the equation

$$
\mathcal{R}_{12}^+(v_+|w_+, w_-)\mathcal{R}_{23}(u_+, u_-|w_+, w_-)\mathcal{R}_{12}(u_+, u_-|v_+, v_-) = 
$$
\[ \mathbb{R}_{23}(u_+, u_-|v_+, w_-) \mathbb{R}_{12}(u_+, u_-|w_+, v_-) \mathbb{R}_{23}^+(v_+|w_+, w_-) \]

in the form

\[ \mathbb{P}_{00}^+ \mathbb{R}_{00}^+(v_+|w_+, w_-) \mathbb{R}_{k0}(u_+, u_-|w_+, v_-) \mathbb{R}_{k0}(u_+, u_-|v_+, v_-) = \mathbb{R}_{k0}(u_+, u_-|v_+, v_-) \mathbb{P}_{00}^+ \mathbb{R}_{00}^+(v_+|w_+, w_-). \]

These local relations imply in the standard way the following commutation relation

\[ Q(u_1|u_2)Q(v_1|v_2) = Q(u_1|v_2)Q(v_1|u_2). \]

6 Conclusions

Using the universal R-matrices \( \mathbb{R}_{k0}(u) \) as building blocks it is possible to construct the two-parametric Baxter’s Q-operator in the case of generic inhomogeneous periodic XXX spin chain

\[ Q(u_1|u_2) = \text{tr}_V \mathbb{R}_{10}(u + \delta_1) \mathbb{R}_{20}(u + \delta_2) \cdots \mathbb{R}_{N0}(u + \delta_N). \]

This operator is factorized on the product of simpler operators \( Q_+(u_1) \) and \( Q_-(u_2) \)

\[ Q(u_1|u_2) = Q_+(u_1) \cdot \mathbb{P} \cdot Q_-(u_2). \]

In the general case of inhomogeneous spin chain the operators \( Q_+(u_1) \) and \( Q_-(u_2) \) are not \( sl(2) \)-invariant. There are nontrivial commutation relations between \( Q_+ \) and \( Q_- \) and between \( Q_\pm \) and transfer matrix \( t(u) \). In the special case of homogeneous spin chain the operators \( Q_+(u), Q_-(u) \) and \( t(u) \) become commuting operators and the \( sl(2) \)-invariance of the operators \( Q_+(u) \) and \( Q_-(u) \) is restored. The operators \( Q_+(u) \) and \( Q_-(u) \) have all properties of Baxter’s Q-operator in the special case of homogeneous spin chain but in the case of inhomogeneous spin only the composite operator \( Q_+(u_1) \cdot \mathbb{P} \cdot Q_-(u_2) \) has the needed properties.

The factorization of the general R-matrix can be generalized to the more complicated situation when the symmetry algebra is \( sl(3) \) [9]. We hope that there exists the similar construction of the Baxter’s Q-operator in this case.

7 Acknowledgments

I would like to thank R.Kirschner, G.Korchemsky, P.Kulish, A.Manashov, E.Sklyanin and V.Tarasov for the stimulating discussions and critical remarks on the different stages of this work. I thank also the Center of theoretical science of Leipzig University for hospitality in the stage of writing up this paper. This work was supported by the grant 03-01-00837 of the Russian Foundation for Fundamental Research and partially by a travel grant of Deutsche Forschungsgemeinschaft.

References.

[1] P.P. Kulish and E.K. Sklyanin, ”On the solutions of the Yang-Baxter equation” Zap.Nauchn.Sem. LOMI 95 (1980) 129

[2] M. Jimbo, ”Introduction to the Yang-Baxter equation”, Int. J. Mod. Phys A 4, (1983) 3759

”Yang-Baxter equation in integrable systems”, M. Jimbo ed., Adv. Ser. Math. Phys., 10, World Scientific (Singapore) 1990
[3] V.G. Drinfeld, "Hopf algebras and Yang-Baxter equation," Soviet Math. Dokl. 32 (1985), 254
V.G. Drinfeld, "Quantum Groups" in "Proc. Int. Congress Math., Berkeley, 1986", AMS, Providence RI (1987), p 798

[4] P.P. Kulish and E.K. Sklyanin, "Quantum spectral transform method. Recent developments", Lect. Notes in Physics, v 151, (1982), 61,
L.D. Faddeev, "How Algebraic Bethe Anstz works for integrable model", Les-Houches lectures 1995, hep-th/9605187
E.K. Sklyanin, "Quantum Inverse Scattering Method. Selected Topics", in "Quantum Group and Quantum Integrable Systems" (Nankai Lectures in Mathematical Physics), ed. Mo-Lin Ge, Singapore: World Scientific, 1992, pp. 63-97; hep-th/9211111

[5] P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, "Yang-Baxter equation and representation theory", Lett. Math. Phys. 5 (1981) 393-403

[6] R.J. Baxter, Exactly solved models in statistical mechanics, London: Academic Press (1982), Ch 9-10

[7] E.K. Sklyanin, private communication

[8] E.K. Sklyanin, Backlund transformations and Baxter’s Q-operator, In: Integrable systems: from classical to quantum (Montreal, QC, 1999), 227-250, CRM Proc. Lecture Notes 26, Amer. Math. Soc., Providence, RI, 2000

[9] S.E. Derkachov, Factorization of the R-matrix. I Zapiski nauchnuch seminarov POMI vol 335 p 134, math.QA/0503396

[10] S.E. Derkachov, Baxter’s Q-operator for the homogeneous XXX spin chain, J. Phys. A: Math. Gen. 32 (1999) 5299-5316, solv-int/9902015

[11] S.E. Derkachov, G.P. Korchemsky, A.N. Manashov Separation of variables for the quantum SL(2,R) spin chain, JHEP 0307 (2003) 047, hep-th/0210216

[12] V. Pasquier and M. Gaudin, The periodic Toda chain and a matrix generalization of the Bessel function recursion relations, J. Phys. A: Math. Gen. 25 (1992), 5243-5252

[13] A. Yu. Volkov, "Quantum lattice KdV equation", Lett. Math. Phys. 39 (1997), 313 - 329, hep-th/9509024

[14] V. Bazhanov, S. Lukyanov, A. Zamolodchikov, "Integrable structure of Conformal Field Theory II. Q-operator and DDV equation", Commun. Math. Phys. 190 (1997), 247-278, hep-th/9604044

[15] A. Antonov, B. Feigin, "Quantum group representation and Baxter equation", Phys. Lett. B 392 (1997), 115-122, hep-th/9603105

[16] V. Kuznetsov, M. Salerno, E. K. Sklyanin, Quantum Backlund transformation for the integrable DST model, J. Phys. A 33 (2000), 171-189, solv-int/9908002

[17] G. P. Pronko, On the Baxter’s Q-operator for the XXX spin chain, Commun. Math. Phys. 212: 687-701, 2000, hep-th/9908179
A.E. Kovalsky and G. P. Pronko Baxter Q-operators for integrable DST chain, nlin.SI/0203030
A.E. Kovalsky and G. P. Pronko Baxters Q-operators for the simplest q-deformed model, nlin.SI/0307040
[18] M.Rossi, R.Weston, A Generalized Q-operator for $U_q(\hat{sl}_2)$ Vertex Models, J.Phys.A 35(2002) 10015-10032, [math-ph/0207004]

[19] A.Zabrodin Commuting difference operators with elliptic coefficients from Baxter’s vacuum vectors, J.Phys.A 33(2000) 3825, [math.QA/9912218]

[20] V.V.Bazhanov and Yu.G.Stroganov, Chiral Potts model as a descendant of the six vertex model, J.Stat.Phys.51(1990) 799-817

[21] V.O.Tarasov, Cyclic monodromy matrices for $sl(n)$ trigonometric R-matrices, Commun.Math.Phys.158 (1993)459-483

[22] A.A.Belavin, A.V.Odessky, R.A.Usmanov, New relations in the algebra of the Baxter Q-operators, [hep-th/0110126]