On finite groups factorized by $\sigma$-nilpotent subgroups*

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Abstract
Let $G$ be a finite group and $\sigma = \left\{ \sigma_i | i \in I \right\}$ be a partition of the set of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. A chief factor $H/K$ of $G$ is said to be $\sigma$-central in $G$, if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is a $\sigma_i$-group for some $i \in I$. The group $G$ is said to be $\sigma$-nilpotent if either $G = 1$ or every chief factor of $G$ is $\sigma$-central. In this paper, we study the properties of a finite group $G = AB$, factorized by two $\sigma$-nilpotent subgroups $A$ and $B$, and also generalize some known results.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a group. Moreover, $n$ is an integer, $\mathbb{P}$ is the set of all primes. The symbol $\pi(n)$ denotes the set of all primes dividing $n$ and $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$. And $\pi$ always denotes a set of primes. Following [6], we say that a finite group $G$ possesses the following properties: $E_\pi$ if $G$ contains a Hall $\pi$-subgroup; $C_\pi$ if $G$ enjoys $E_\pi$, and every two Hall $\pi$-subgroups of $G$

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are conjugate; \( D_\pi \) if \( G \) enjoys \( C_\pi \), and every one of its \( \pi \)-subgroups is contained in some Hall \( \pi \)-subgroup of \( G \).

In what follows, \( \sigma = \{ \sigma_i | i \in I \} \) is some partition of \( \mathbb{P} \), that is, \( \mathbb{P} = \bigcup_{i \in I} \sigma_i \) and \( \sigma_i \cap \sigma_j = \emptyset \) for all \( i \neq j \). \( \Pi \) is always supposed to be a non-empty subset of the set \( \sigma \) and \( \Pi' = \sigma \setminus \Pi \). We write \( \sigma(n) = \{ \sigma_i | \sigma_i \cap \pi(n) \neq \emptyset \} \) and \( \sigma(G) = \sigma(|G|) \).

Following [16] and [17], \( G \) is said to be \( \sigma \)-primary if \( G = 1 \) or \( |\sigma(G)| = 1 \). A chief factor \( H/K \) of \( G \) is said to be \( \sigma \)-central in \( G \) if the semidirect product \( H/K \rtimes G/C_G(H/K) \) is \( \sigma \)-primary. A set \( \mathcal{H} \) of subgroups of \( G \) is said to be a complete Hall \( \sigma \)-set of \( G \) if every non-identity member of \( \mathcal{H} \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( i \) and \( \mathcal{H} \) contains exactly one Hall \( \sigma_i \)-subgroup of \( G \) for every \( \sigma_i \in \sigma(G) \).

Recall that \( G \) is called [16]: (i) \( \sigma \)-nilpotent if either \( G = 1 \) or every chief factor of \( G \) is \( \sigma \)-central in \( G \), (ii) \( \sigma \)-soluble if either \( G = 1 \) or every chief factor of \( G \) is \( \sigma \)-primary. We use \( \mathfrak{N}_\sigma \) and \( \mathfrak{S}_\sigma \) to denote the classes of all \( \sigma \)-nilpotent and \( \sigma \)-soluble groups, respectively. Following [16] we call the product of all normal \( \sigma \)-nilpotent subgroups of \( G \) the \( \sigma \)-Fitting subgroup of \( G \) and denote it by \( F_\sigma(G) \). Clearly, \( F_\sigma(G) \) is also \( \sigma \)-nilpotent (see the below Lemma 2.8).

It has been established earlier by the work of Wielandt [20] and Kegel [9] that the product of two finite nilpotent groups is soluble (see for example [1, Theorem 2.4.3]). Moreover, the structure of finite groups factorized by two nilpotent subgroups has been investigated by several authors (see [1, Chapter 2]), and properties of such groups have also been discovered by many authors (see for example [2, 7, 8, 11, 12, 18]). On the other hand, very little is known about the properties of a product of two \( \sigma \)-nilpotent subgroups. So the aim of this paper is to extend the knowledge of properties of such products by some \( \sigma \)-nilpotent subgroups.

Robinson and Stonehewer [14, Theorem 2] have shown that if the group \( G = AB \) is the product of two abelian subgroups \( A \) and \( B \), then every chief factor of \( G \) either is centralized by \( A \) or \( B \). Moreover, Stonehewer [18, Theorem 1] have proved that if \( G = AB \) is the product of two nilpotent subgroups \( A \) and \( B \), then for every minimal normal subgroup \( N \) of \( G \) one of the subgroups \( AN \) and \( BN \) is nilpotent. Our first main theorem generalizes these results to a finite group factorized by two \( \sigma \)-nilpotent subgroups.

**Theorem 1.1.** Let \( G \) be the product of two \( \sigma \)-nilpotent subgroups \( A \) and \( B \) and let \( N \) be a minimal normal subgroup of \( G \). Assume also that \( G \) is \( \sigma \)-soluble. Then \( AN \) or \( BN \) is \( \sigma \)-nilpotent.

Note that, the product of two finite nilpotent groups is always soluble by the well-known theorem of Wielandt [20] and Kegel [9]. However, the product of two finite \( \sigma \)-nilpotent groups is not necessarily \( \sigma \)-soluble. For example, let \( G = A_5 \) be the alternating group with degree 5, and let \( \sigma_1 = \{2, 3\}, \sigma_2 = \{5\} \) and \( \sigma_3 = \{2, 3, 5\}' \). Then it is clear that \( A_4 \) is \( \sigma \)-nilpotent.
and the Sylow 5-subgroup $P$ of $G$ is also $\sigma$-nilpotent. Then $G = A_4P$ is the product of two $\sigma$-nilpotent groups, but clear $G$ is not $\sigma$-soluble.

It was also proved by Robinson and Stonehewer [14, Theorem 1] that if a group $G$ has three abelian subgroups $A, B$ and $C$ such that $G = AB = BC = CA$, then every chief factor of $G$ is central in $G$. This result has been extended to nilpotent subgroups, that is, Kegel [10] have shown that if the finite group $G = AB = BC = CA$ is the product of three nilpotent subgroups $A, B$ and $C$, then $G$ is nilpotent. So we have the following theorem.

**Theorem 1.2.** Let $G$ be a finite group with $\sigma$-nilpotent subgroups $A, B$ and $C$ such that $G = AB = BC = CA$. Assume also that $G$ satisfies $D_{\sigma_i}$ for some $\sigma_i \in \sigma(G)$. Then $G$ is $\sigma$-nilpotent.

Later Cossey and Stonehewer studied the structure of the product of two nilpotent subgroups in which its Fitting subgroup is a $p$-group for some prime $p$, that is, Cossey and Stonehewer [2, Theorem 1] have proved that: Let $G$ be a soluble group for which $F(G)$ is a $p$-group (for some prime $p$). Then $G$ is the product of two nilpotent subgroups if and only if it has a nilpotent Hall $p'$-subgroup. Base on this fact, we can study the case of $\sigma$-nilpotent subgroups. So we have the following theorem.

**Theorem 1.3.** Let $G$ be a $\sigma$-soluble group with $F_{\sigma_i}(G)$ is a $\sigma_i$-group for some $\sigma_i \in \sigma(G)$. Then $G = AB$, where $A, B$ are $\sigma$-nilpotent subgroups of $G$ if and only if $G$ has a $\sigma$-nilpotent Hall $\sigma_i'$-subgroup.

All unexplained terminologies and notations are standard, as in [3] and [16].

## 2 Preliminaries

**Lemma 2.1.** (See [15, Lemma 2.1]) The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the $\sigma$-soluble group by a $\sigma$-soluble group is a $\sigma$-soluble group as well.

Recall that the product of all normal $\sigma$-soluble subgroups of $G$ called the $\sigma$-radical [15] of $G$ and denote it by $R_\sigma(G)$.

**Lemma 2.2.** (See [15, Theorem B ] ) Let $R = R_\sigma(G)$ be the $\sigma$-radical of $G$. Then $G$ is $\sigma$-soluble if and only if for any $\Pi$ the following hold: $G$ has a Hall $\Pi$-subgroup $E$, every $\Pi$-subgroup of $G$ is contained in some conjugate of $E$ and $E R$-permutes with every Sylow subgroup of $G$. 

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All unexplained terminologies and notations are standard, as in [3] and [16].
Lemma 2.3. (See [12, Lemma ]) Let $G = AB$ be a finite group satisfying $D_\pi$. If the subgroups $A$ and $B$ of $G$ possess normal Hall $\pi$-subgroups $A_\pi$ and $B_\pi$, respectively, then $A_\pi B_\pi = B_\pi A_\pi$ is a Hall $\pi$-subgroup of $G$ and $[A_\pi, B_\pi] \leq O_\pi(G)$. In particular, if $O_\pi(G) = 1$, then $[A_\pi^G, B_\pi^G] = 1$.

Lemma 2.4. (See [9, Satz 3]) Let $A$ and $B$ be the subgroups of $G$ such that $G \neq AB$ and $AB^x = B^xA$ for all $x \in G$. Then $G$ has a proper normal subgroup $N$ such that either $A \leq N$ or $B \leq N$.

Lemma 2.5. (See [4, Chapter 2, Lemma 4.4]) Let $G = AB$ be the product of the subgroups $A$ and $B$. If $L$ is a normal subgroup of $A$ and $L \leq B$, then $L \leq O_\pi(G)$.

Lemma 2.6. (See [13, Lemma 2.1]) Let $G$ be a finite group and $A$ its normal subgroup. If $G$ satisfies $D_\pi$, then $G/A$ satisfies $D_\pi$.

Lemma 2.7. (See [5, Proposition 3.4]) Any two of the following conditions are equivalent:

1. $G$ is $\sigma$-nilpotent.
2. $G = H_1 \times \cdots \times H_t$, where $\{H_1, \ldots, H_t\}$ is a complete Hall $\sigma$-set of $G$.
3. Every subgroup of $G$ is $\sigma$-subnormal in $G$.

Lemma 2.8. (See [16, Corollary 2.4]) If $A$ and $B$ are normal $\sigma$-nilpotent subgroups of $G$, then $AB$ is $\sigma$-nilpotent.

Lemma 2.9. (See [4, Chapter 1, Lemma 4.14]) Let $A$ and $B$ be proper subgroups of $G$ such that $G = AB$. Then:

1. $G_p = A_p B_p$ for some Sylow $p$-subgroups $G_p, A_p$ and $B_p$ of $G, A, B$, respectively;
2. $G = A^xB$ and $G \neq AA^x$ for all $x \in G$.

3 Proofs of Theorem 1.1, 1.2 and 1.3

Proof of Theorem 1.1. Assume that the result is false and let $G$ be a counterexample of minimal order. Since $G$ is $\sigma$-soluble and $N$ is a minimal normal subgroup of $G$, we have that $N < G$ and $N$ is a $\sigma_i$-group for some $i$. Now without loss of generality we may assume that $N$ is a $\sigma_1$-group, and let $\Pi = \sigma_1'$. Then we proceed the proof by the following steps.

1. $N$ is the unique minimal normal subgroup of $G$.

If not, then let $L$ be a minimal normal subgroup of $G$ different from $N$. Since $G/L = (AL/L)(BL/L)$ and $AL/L \cong A/(A \cap L), BL/L \cong B/(B \cap L)$ are all $\sigma$-nilpotent, we have that $G/L$ satisfies the hypothesis. Hence by the choice of $G$, we see that $(AL/L)(NL/L)$ or $(BL/L)(NL/L)$ is $\sigma$-nilpotent. If $(AL/L)(NL/L)$ is $\sigma$-nilpotent, then

$$AN/(AN \cap L) \cong ANL/L = (AL/L)(NL/L)$$
is $\sigma$-nilpotent. Clearly, $AN/N \cong A/(A \cap N)$ is $\sigma$-nilpotent. So

$$AN = (AN)/(AN \cap L \cap N) \subseteq (AN/N) \times (AN/AN \cap L)$$

is $\sigma$-nilpotent, a contradiction. Similarly, if $(BL/L)(NL/L)$ is $\sigma$-nilpotent, then we obtain that $BN$ is also $\sigma$-nilpotent, a contradiction. Hence (1) holds.

(2) $O_H(G) = 1$.

This is clear from (1).

Since $A$ and $B$ are $\sigma$-nilpotent, we get that $A$ and $B$ have normal Hall $\Pi$-subgroup $A_\Pi$ and $B_\Pi$, respectively.

(3) $A_\Pi \neq 1$ and $B_\Pi \neq 1$.

If $A_\Pi = 1$, then $A$ is a $\sigma_1$-group. This implies that $AN$ is also a $\sigma_1$-group, and so is $\sigma$-nilpotent, a contradiction. Hence $A_\Pi \neq 1$. Similarly, $B_\Pi \neq 1$. Hence (3) holds.

(4) Final contradiction.

Since $G$ is $\sigma$-soluble, we have that $G$ satisfies $D_\Pi$ by Lemma 2.2. Clearly, $A_\Pi \leq A$ and $B_\Pi \leq B$. Then by (2) and Lemma 2.3, we see that $[A_\Pi^G, B_\Pi^G] = 1$, and so $A_\Pi^G \leq C_G(B_\Pi^G)$ and $B_\Pi^G \leq C_G(A_\Pi^G)$. From (1) and (3) we get that $N \leq A_\Pi^G \cap B_\Pi^G$. Hence $A_\Pi^G \leq C_G(N)$ and $B_\Pi^G \leq C_G(N)$. It is obvious that $A_\Pi$ is also a Hall $\Pi$-subgroup of $AN$. Since $A$ is $\sigma$-nilpotent, and $AN \leq A_\Pi^G \leq C_G(N)$, we have that $A_\Pi$ is the normal Hall $\Pi$-subgroup of $AN$. Clearly, $A_{\sigma_1}N$ is a Hall $\sigma_1$-subgroup of $AN$ and $A_{\sigma_1}N$ is normal in $AN$. Then $AN = A_\Pi \times A_{\sigma_1}N$ and so $AN$ is $\sigma$-nilpotent, a contradiction. Similarly, $BN$ is also $\sigma$-nilpotent, a contradiction. The final contradiction completes the theorem.

**Proof of Theorem 1.2.** First of all, we show that $G$ is $\sigma$-soluble. Assume that the assertion is false and let $G$ be a counterexample of minimal order. Without loss of generality we may assume that $\sigma_i \in \sigma(B)$. Then by Lemma 2.7, we see that $B$ has a nonidentity normal Hall $\sigma_i$-subgroup, denote it by $B_i$. If $\sigma_i \notin \sigma(A)$, then $|G : C| = |A : A \cap C|$ is a $\sigma_i$-number for $G = AC$. Since $G = BC$, we see that $|B : B \cap C| = |BC : C| = |G : C|$ is a $\sigma_i$-number. Hence $B_i \leq B \cap C$ for $B$ is $\sigma$-nilpotent. Therefore by Lemma 2.5, we have that $1 \neq B_i \leq C_G$. Now let $R$ be a minimal normal subgroup of $G$ which is contained in $C_G$. Then $R$ is $\sigma$-soluble. Clearly, by Lemma 2.6 we see that $G/R$ satisfies $D_{\sigma_i}$. Since

$$G/R = (AR/R)(BR/R) = (BR/R)(CR/R) = (CR/R)(AR/R)$$

and

$$AR/R \cong A/A \cap R, \; BR/R \cong B/B \cap R \text{ and } CR/R \cong C/C \cap R$$

are $\sigma$-nilpotent, we have that $G/R$ satisfies the hypothesis. Hence by the choice of $G$, we obtain that $G/R$ is $\sigma$-soluble. Then by Lemma 2.1 shows that $G$ is $\sigma$-soluble for $R$ is $\sigma$-soluble, a
contradiction. This contradiction shows that \( \sigma_i \in \sigma(A) \). Since \( A \) is \( \sigma \)-nilpotent, we have that \( A \) has a nonidentity Hall \( \sigma_i \)-subgroup by Lemma 2.7, and denote it by \( A_i \). Then by Lemma 2.3, we see that \( A_iB_i = B_iA_i \) is a Hall \( \sigma_i \)-subgroup of \( G \). It is obvious that \( A_iB_i < G \) and for any element \( x \in G \), by Lemma 2.9 \( G = AB^x \). And clearly, \( B^x \) is also \( \sigma \)-nilpotent and \( B_i^x \) is the normal Hall \( \sigma_i \)-subgroup of \( B^x \). Hence by Lemma 2.3 again, we have that \( A_iB_i^x = B_i^xA_i \) is a Hall \( \sigma_i \)-subgroup of \( G \). So by Lemma 2.4, we obtain that there exists a proper normal subgroup \( N \) of \( G \) such that \( A_i \leq N \) or \( B_i \leq N \). This means that \( G \) is not a simple group. Let \( R \) be a minimal normal subgroup of \( G \). Then \( G/R \) satisfies the hypothesis by Lemma 2.6. Hence by the choice of \( G \) we have that \( G/R \) is \( \sigma \)-soluble. Therefore \( R \) is the unique minimal normal subgroup of \( G \) and \( O_{\sigma_1}(G) = 1 \). So by Lemma 2.3, we see that \( [A^G_{\sigma_i}, B^G_{\sigma_i}] = 1 \), that is, \( A^G_{\sigma_i} \leq C_G(B^G_{\sigma_i}) \). Since \( A_{\sigma_i} \neq 1 \) and \( B_{\sigma_i} \neq 1 \) and \( R \) is the unique minimal normal subgroup of \( G \), we obtain that \( R \leq A^G_{\sigma_i} \cap B^G_{\sigma_i} \). So \( R \leq A^G_{\sigma_i} \leq C_G(B^G_{\sigma_i}) \leq C_G(R) \), which shows that \( R \) is abelian. Hence we also get \( G \) is \( \sigma \)-soluble for \( G/R \) is \( \sigma \)-soluble, a contradiction. This contradiction shows that \( G \) is \( \sigma \)-soluble.

Now let \( N \) be a minimal normal subgroup of \( G \). Then since \( G \) is \( \sigma \)-soluble, we have that \( N \) is a \( \sigma_j \)-group for some \( \sigma_j \in \sigma(G) \). If \( N = 1 \), then \( G \) is a simple \( \sigma \)-soluble, and so \( G \) is \( \sigma_j \)-group, that is, \( G \) is \( \sigma \)-nilpotent. So we can always assume that \( N \neq 1 \). Now without loss of generality we may assume that \( N \) is a \( \sigma_1 \)-group. Then we can proceed to prove that \( G \) is \( \sigma \)-nilpotent by using induction on \( |G| \). Since

\[
G/N = (AN/N)(BN/N) = (BN/N)(CN/N) = (CN/N)(AN/N)
\]

and

\[
AN/N \cong A/A \cap N, \quad BN/N \cong B/B \cap N \quad \text{and} \quad CN/N \cong C/C \cap N
\]

are \( \sigma \)-nilpotent, and by Lemma 2.6 \( G/N \) satisfies \( D_{\sigma_i} \), we have that \( G/N \) satisfies the hypothesis. Hence by induction, we have that \( G/N \) is \( \sigma \)-nilpotent. If \( G \) has another minimal normal subgroup \( R \) of \( G \) which is different from \( N \), then by the same discussion as above, we also get \( G/R \) is \( \sigma \)-nilpotent. Hence \( G = G/(N \cap R) \cong G/N \times G/R \) is \( \sigma \)-nilpotent.

Now we can only consider that \( G \) has a unique minimal normal subgroup \( N \) of \( G \). By the discuss as above, we see that \( G/N \) is \( \sigma \)-nilpotent. So if \( N \leq \Phi(G) \), then \( G \) is also \( \sigma \)-nilpotent. Hence we can assume that \( N \notin \Phi(G) \). Since by Theorem 1.1 and \( G \) is \( \sigma \)-soluble, we have that at least two of \( AN \), \( BN \) and \( CN \) are \( \sigma \)-nilpotent. So we can assume that \( AN \) and \( BN \) are \( \sigma \)-nilpotent. This implies that \( A_{\sigma'_i} \leq C_G(N) \) and \( B_{\sigma'_i} \leq C_G(N) \) by Lemma 2.7. If \( A_{\sigma'_i} \neq 1 \), then \( A_{\sigma'_i} \leq C_G(N) \), it follows that \( C_G(N) \neq 1 \). But \( N \) is the unique minimal normal subgroup of \( G \), we have that \( N \leq C_G(N) \), that is, \( N \) is an elementary abelian \( p \)-group. Since \( N \notin \Phi(G) \), there exists a maximal subgroup \( M \) of \( G \) such that \( G = NM = C_G(N)M \). Clearly, \( N \cap M \leq G \) and \( C_G(N) \cap M \leq G \). This shows that \( N \cap M = 1 \) and \( C_G(N) \cap M = 1 \) for \( N \) is the unique
minimal normal subgroup of $G$. So $N = C_G(N)$. But $A_{\sigma'} \leq C_G(N) = N$ and $N$ is a $\sigma_1$-group, we have that $A_{\sigma'} = 1$, a contradiction. This contradiction shows that $A_{\sigma'} = 1$. If we assume that $B_{\sigma'} \neq 1$, then we can get a contradiction like above. So this contradiction shows that $B_{\sigma'} = 1$. Hence $A_{\sigma'} = 1$ and $B_{\sigma'} = 1$. But $A$ and $B$ are $\sigma$-nilpotent, we have that $A$ and $B$ are all $\sigma_1$-group, which means that $G = AB$ is also a $\sigma_1$-group. So $G$ is $\sigma$-nilpotent. Thus we have proved that $G$ is $\sigma$-nilpotent.

Proof of Theorem 1.3. On one hand, if $G$ has a $\sigma$-nilpotent Hall $\sigma'_i$-subgroup $K$. Then since $G$ is $\sigma$-soluble, we have that there exists a Hall $\sigma_i$-subgroup $H$ of $G$. Clearly, $G = HK$ and $H, K$ are all $\sigma$-nilpotent.

On the other hand, suppose that $G = AB$ with $A, B$ are $\sigma$-nilpotent subgroups of $G$ and $F_\sigma(G)$ is a $\sigma_i$-group. First, we claim that $O_{\sigma'}(G) = 1$. In fact, if $O_{\sigma'}(G) \neq 1$, then let $N$ be a minimal normal subgroup of $G$ contained in $O_{\sigma'}(G)$. So $N$ is a $\sigma_j$-group for some $\sigma_j \in \sigma(G) \setminus \sigma_i$, and so $N \leq F_\sigma(G)$. This contradicts to the fact that $F_\sigma(G)$ is a $\sigma_i$-group. Hence we have $O_{\sigma'}(G) = 1$. Since $A$ and $B$ are $\sigma$-nilpotent and by Lemma 2.7, we have that $A$ and $B$ have normal Hall $\sigma'_i$-subgroup $A_{\sigma'_i}$ and $B_{\sigma'_i}$, respectively. By Lemma 2.2, we see that $G$ satisfies $D_{\sigma'}$ for $G$ is $\sigma$-soluble. Hence by Lemma 2.3, we obtain that $G_{\sigma'} = A_{\sigma'}B_{\sigma'}$ is the Hall $\sigma'_i$-subgroup of $G$ and $[A_{\sigma'}^G, B_{\sigma'}^G] = 1$. If $A_{\sigma'} = 1$ or $B_{\sigma'} = 1$, then $G_{\sigma'} = B_{\sigma'} \leq B$ or $G_{\sigma'} = A_{\sigma'} \leq A$, which all shows that $G_{\sigma'}$ is $\sigma$-nilpotent for $A$ and $B$ are $\sigma$-nilpotent. If $A_{\sigma'} \neq 1$ and $B_{\sigma'} \neq 1$, then since $[A_{\sigma'}^G, B_{\sigma'}^G] = 1$, we have that $A_{\sigma'} \leq A_{\sigma'}^G \leq C_G(B_{\sigma'}^G) \leq C_G(B_{\sigma'})$. This implies that $A_{\sigma'} \leq G_{\sigma'}$ and $B_{\sigma'} \leq G_{\sigma'}$. Thus by Lemma 2.8, we have that $G_{\sigma'}$ is $\sigma$-nilpotent. Therefore in any case, we always get that $G$ has a $\sigma$-nilpotent Hall $\sigma'_i$-subgroup.

Remark 1. Note that the Theorem 1.1 does not hold if ”$G$ is $\sigma$-soluble” is replaced by ”$G$ satisfies $D_{\sigma_i}$, for every $\sigma_i \in \sigma(G)$”. We will give a counterexample to show this.

Example 3.1. Let $G = PSL_2(7)$ be the projective special linear group of degree 2 over $F_7$, a simple group with order $168 = 2^3 \cdot 3 \cdot 7$. Let $\pi = \{3, 7\}$. Then by [19, Condition III], we see that $G$ satisfies $D_{\pi}$. Now let $H$ be a Hall $\pi$-subgroup of $G$, and $P$ be a Sylow 2-subgroup of $G$, and $\sigma_1 = \{2\}$, $\sigma_2 = \{3, 7\}$ and $\sigma_3 = \{2, 3, 7\}$. So $P$ is a Hall $\sigma_2$-subgroup of $G$ and $G = HP$. Clearly, $G$ satisfies $D_{\sigma_1}$ and $D_{\sigma_2}$. The Theorem 1.1 does not hold for $G$. However $G$ is not $\sigma$-soluble.

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