Quantum Scalar Fields on Anti-de Sitter Spacetime

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Abstract

We investigate the propagation of arbitrarily coupled scalar fields on the $N$-dimensional hyperbolic space $\mathbb{H}^N$. Using the ζ-function regularization we compute exactly the one loop effective action. The vacuum expectation value of quadratic field fluctuations and the one loop renormalized stress tensor are then computed using the recently proposed direct ζ-function technique. Our computation tests the validity of this approach in presence of a continuous spectrum. Our results apply as well to the $N$-dimensional anti-de Sitter spacetime, whose appropriate euclidean section is the hyperbolic space $\mathbb{H}^N$.

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I. INTRODUCTION

There has always been interest in anti-de Sitter (AdS) spacetime and quantum fields propagating on it. Being a maximally symmetric spacetime, it has been an excellent model to investigate questions of principle related to the quantization of fields propagating on curved background, the interaction with the gravitational field and the issues related to its lack of global hyperbolicity [1–4].

The importance of this theoretical work increased when it was realized that AdS spacetime emerges as a stable ground state solution of gauge extended supergravity [3] and Kaluza-Klein theories, in various dimensions. Stability was also established for gravity fluctuations about the AdS background [6]. Recently, there has been a revival of the interest in AdS spacetimes, due to the AdS/Conformal field theory-correspondence conjecture [7] and its relevance in the study of the large-$N$ limit of nonabelian gauge theories.

Moreover, due to the negative cosmological constant, black holes with nonspherical topology can be constructed on AdS background [8–16]. Some of them have a constant curvature and can be obtained as quotients of AdS by a discrete subgroup of its isometry group, $SO(N-1,2)$; the most popular are the Bañados-Teitelboim-Zanelli solutions in three dimensions [17], but higher dimensional generalizations exist [8,18]. These black holes are locally isometric to AdS, and quantum corrections due to the propagation of scalar fields on the background of these black holes have been considered by various authors, for the BTZ black hole [19–22], for the singular background of toroidal black holes in four dimensions [23], while the propagation of photons in topological black hole spacetimes has been investigated in [24].

In this paper we shall study the propagation of a scalar quantum field, with arbitrary coupling, on AdS spacetime, in the framework of euclidean field theory. The appropriate euclidean section of AdS spacetime is the hyperbolic space $H^N$ [25,26]. We shall compute the exact expressions at one loop of the effective action, the vacuum expectation value of the field fluctuations and the renormalized stress tensor in arbitrary dimension. We shall use the powerful formalism of $\zeta$-function renormalization [27–29]. In particular, the field fluctuations and the stress tensor will be computed with the recently proposed direct $\zeta$-function approach [30,31]. The equivalence of this approach with the more standard (euclidean) point-splitting procedure has been shown for compact spaces [32,33], and holds only formally in the noncompact case. We shall deal with operators in $H^N$ with continuous spectrum, and our results are an important test of the generalization of this equivalence when the hypothesis of compactness misses.

This paper is organized as follows. In Section I we study some spectral properties of Laplace-like operators in the hyperbolic spaces $H^N$ and compute the related $\zeta$-function. In Section II we compute the one loop effective action. The vacuum expectation value of the field fluctuations and of the renormalized stress tensor are computed using the direct $\zeta$-function approach, in Sections IV and V respectively. In Section VI we apply our formulae in various dimensions. We end drawing some conclusions in Section VII and with an Appendix, where an integral is analysed.
II. SPECTRAL ANALYSIS ON $\mathbb{H}^N$

The $N$-dimensional anti-de Sitter spacetime (AdS$_N$) with radius $a$ is the hyperboloid

$$x_1^2 + \cdots + x_{N-1}^2 - u^2 - v^2 = -a^2$$

(1)

embedded if flat $N + 1$ dimensional spacetime $\mathbb{R}^{N-1}$. It is an homogeneous, constant curvature manifold. This spacetime solves Einstein’s equations with negative cosmological constant $\Lambda$ and curvature scalar $R$ given respectively by

$$\Lambda = -\frac{(N - 1)(N - 2)}{2a^2}, \quad R = -\frac{N(N - 1)}{a^2}.$$  (2)

The definition of a quantum field theory on this manifold requires some care, and has been extensively studied. The main problem comes from the fact that it is not globally hyperbolic, and boundary conditions must be supplemented. The choice of the boundary conditions is not unique, and different inequivalent Fock representations exist [4]. Furthermore, the correct euclidean formulation of a field theory is not immediate, as the choice of the euclidean section on which to work is ambiguous. It is now an established fact that the appropriate euclidean section for AdS$_N$ spacetime is the $N$-dimensional hyperbolic space $\mathbb{H}^N$. The analytic continuation to the euclidean section automatically selects a particular representation for the quantum fields, corresponding to Dirichlet boundary conditions at infinity [23,24]. In the following we shall restrict ourselves to the euclidean theory, where the $\zeta$-function renormalization technique is available.

We shall study the propagation of a scalar field $\phi(x)$ on $\mathbb{H}^N$. Its action is given by

$$I[\phi] = -\frac{1}{2} \int \left( \nabla^\mu \phi \nabla_\mu \phi + m^2 \phi^2 + \xi R \phi^2 \right) \sqrt{g(x)} \, d^N x,$$

(3)

where $m$ is the mass of $\phi$ and $\xi$ determines the coupling with the scalar curvature $R$. The associated motion operator is $-\Delta_{\mathbb{H}^N} + m^2 + \xi R$. It is an elliptic Laplace-like operator, hence the first step is to study the spectral properties of the laplacian.

It is convenient to work in the Poincaré half-space model of $\mathbb{H}^N$; the coordinates are $(y, x)$, where $y > 0$ is a “radial” coordinate and $x \in \mathbb{R}^{N-1}$ parametrizes the flat $(N - 1)$-dimensional transverse manifold. In these coordinates the metric is

$$ds^2 = \frac{a^2}{y^2} \left( dy^2 + dx_1^2 + \cdots + dx_{N-1}^2 \right)$$

(4)

and the Laplace operator on $\mathbb{H}^N$ reads

$$\Delta_{\mathbb{H}^N} = \frac{1}{a^2} \left[ y^2 \left( \partial_y^2 + \Delta_{N-1} \right) - (N - 2) y \partial_y \right],$$

(5)

where $\Delta_{N-1}$ is the Laplacian on the flat $(N - 1)$-dimensional transverse manifold. Let us define $\rho_N = (N - 1)/2$ and the operator $L = -\Delta_{\mathbb{H}^N} - \rho_N^2$. The eigenvalue equation for $L$, $a^2 L \psi = \lambda^2 \psi$, can be solved by separation of variables, with the ansatz
\[ \psi_{\lambda,k}(x) = \phi_{\lambda}(y) f_k(x). \] (6)

On the transverse manifold the eigenvalue equation becomes
\[ -\Delta_{N-1} f_k(x) = k^2 f_k(x), \quad f_k(x) = (2\pi)^{-\rho_N} e^{ik \cdot x}, \] (7)
and one gets for \( \phi_{\lambda}(y) \),
\[ \phi''_{\lambda} - \frac{N - 2}{y} \phi'_{\lambda} + \left( \frac{\lambda^2 + \rho_N^2}{y^2} - k^2 \right) \phi_{\lambda} = 0. \] (8)

This is a Bessel equation, and requiring the solutions to be well-behaved at infinity, one gets \( \phi_{\lambda}(y) = y^{\rho_N} K_{i\lambda}(ky) \), where \( K_{i\lambda}(x) \) is a Mc Donald function and \( k = |k| \). As a result, the spectrum is continuous and the generalized eigenfunctions are
\[ \psi_{\lambda,k}(x) = y^{\rho_N} K_{i\lambda}(ky) f_k(x). \] (9)

The associated spectral measure, defined by \( \langle \phi_{\lambda}, \phi_{\lambda'} \rangle = \delta(\lambda - \lambda')/\mu(\lambda) \), is
\[ \mu(\lambda) = \frac{2\lambda}{\pi^2} \sinh \pi \lambda. \] (10)

Hence the spectral theorem yields
\[ \langle x | F(L) | x' \rangle = \int_0^\infty d\lambda \mu(\lambda) F(\lambda^2/a^2) \int \frac{d^{N-1}k}{(2\pi)^{n-1}2^{\rho_N}} (yy')^{\rho_N} e^{ik \cdot u} K_{i\lambda}(ky) K_{i\lambda}(ky'), \] (11)
where \( u = x - x' \). The spectral measure \( \mu(\lambda) \) is related to the \( N \)-dimensional Plancherel measure \( p_N(\lambda) \), that arises from the spectral analysis on the hyperboloid model of \( \mathbb{H}^N \), by
\[ \Gamma(\rho_N + i\lambda) \Gamma(\rho_N - i\lambda) \mu(\lambda) = \frac{2^{2N-3}}{\pi^2} p_N(\lambda). \] (12)

The Plancherel measure is given by
\[ p_N(\lambda) = \beta_N \prod_{j=0}^{\rho_N-1} (\lambda^2 + j^2), \] (13)
for \( N \geq 3 \) odd, and by
\[ p_N(\lambda) = \beta_N \lambda \tanh(\pi \lambda) \prod_{j=\frac{1}{2}}^{\rho_N-1} (\lambda^2 + j^2), \] (14)
for \( N \) even (for \( N = 2 \) the product is omitted). The coefficients \( \beta_N \) are defined by
\[ \beta_N = \frac{\pi}{2^{2(N-2)} \left[ \Gamma \left( \frac{N}{2} \right) \right]^2}. \] (15)
It is useful to decompose the Plancherel measure in a sum of monomials, with coefficients $c_n^N$, according to

$$p_N(\lambda) = \beta_N \sum_{n=1}^{\rho N} c_{2n} \lambda^{2n}, \quad p_N(\lambda) = \beta_N \tanh(\pi \lambda) \sum_{n=0}^{N-1} c_{2n+1} \lambda^{2n+1},$$

(16)

for $N$ odd and $N$ even respectively.

The relevant motion operator for the scalar field theory (3) is $L_b = L + b$, where we have defined the coupling parameter $a^2 b = \rho^2_N + a^2 m^2 + \xi a^2 R$. The operator $L_b$ is positive definite as long as $a^2 b > 0$, that is

$$\xi < \frac{N - 1}{4N} + \frac{a^2 m^2}{N(N - 1)} \equiv \xi_{crit};$$

(17)

for $\xi > \xi_{crit}$ the ground state becomes unstable and the theory is not defined, while for $\xi = \xi_{crit}$ there is a continuous spectrum starting from zero and our procedure cannot be applied (not straightforwardly at least).

The local $\zeta$-function associated to the operator $L_b$ acting on $\mathbb{H}^N$ can be obtained from the spectral theorem, using formula (11) with $F(x) = (x + b)^{-s}$, yielding

$$\zeta(s, x|L_b) = \frac{2^{N-3}}{\pi^{2N + 1}} a^{2s-N} \int_0^\infty \frac{\rho N(\lambda) d\lambda}{(\lambda^2 + a^2 b)^s};$$

(18)

note that the integrated $\zeta$-function coincides with the local $\zeta$-function, as $\mathbb{H}^N$ is a homogeneous manifold. We are interested in the meromorphic structure of the $\zeta$-function, dictated by the integral

$$I_N(s) = \int_0^\infty \frac{\rho N(\lambda) d\lambda}{(\lambda^2 + a^2 b)^s};$$

(19)

It is convenient here to split the Plancherel measure in a sum of monomials in $\lambda$ with coefficients $c_n^N$, and distinguish two cases according to the parity of $N$. For odd $N$, the integral can be computed explicitly in term of Euler’s gamma function, yielding

$$I_N(s) = \frac{\sqrt{\pi} \beta N}{2 \Gamma(s)} \sum_{n=1}^{\rho N} \frac{(2n - 1)!!}{2^n} c_{2n}^N (a^2 b)^{n+\frac{1}{2}-s} \Gamma \left( s - n - \frac{1}{2} \right),$$

(20)

and the odd-dimensional $\zeta$-function reads

$$\zeta(s, x|L_b) = \frac{a^{2s-N}}{2^N \pi^{\rho N} \Gamma \left( \frac{N}{2} \right) \Gamma(s)} \sum_{n=1}^{\rho N} \frac{(2n - 1)!!}{2^n} c_{2n}^N (a^2 b)^{n+\frac{1}{2}-s} \Gamma \left( s - n - \frac{1}{2} \right).$$

(21)

This function can be analytically continued to a meromorphic function with simple poles in $s = \frac{N}{2} - k$, with $k \in \mathbb{N}$. If $N$ is odd, the $\tanh(\pi \lambda)$ factor in the Plancherel measure complicates a little bit the computation; to show the analytic structure of $I_N(s)$ we shall split the integral in two according to the relation
\[
\tanh(\pi \lambda) = 1 - \frac{2}{e^{2\pi \lambda} + 1}.
\]  
(22)

We obtain

\[
I_N(s) = \beta_N \sum_{n=0}^{N-1} c_{2n+1} \left( \frac{1}{2} n! (a^2 b)^{n+1-s} \frac{\Gamma(s-n-1)}{\Gamma(s)} - 2H_n(s; a\sqrt{b}) \right),
\]
(23)

where we have defined the function

\[
H_n(s; \mu) = \int_{0}^{\infty} \frac{\lambda^{2n+1} d\lambda}{(e^{2\pi \lambda} + 1)(\lambda^2 + \mu^2)^s}.
\]
(24)

The exponential at denominator makes \(H_n(s; \mu)\) an analytic function on the whole complex \(s\)-plane. Some properties of this function are examined in the Appendix; in particular \(H_n(0; \mu)\) can be exactly calculated in terms of Bernoulli numbers.

From (23) we obtain the \(\zeta\)-function in an even-dimensional hyperbolic space

\[
\zeta(s, x|L_b) = \frac{a^{2s-N}}{2^{N-1} \pi^N \Gamma \left( \frac{N}{2} \right)} \sum_{n=0}^{N-1} c_{2n+1} \left( \frac{1}{2} n! (a^2 b)^{n+1-s} \frac{\Gamma(s-n-1)}{\Gamma(s)} - 2H_n(s; a\sqrt{b}) \right).
\]
(25)

Hence the meromorphic structure of the \(\zeta\)-function, shared with \(I_N(s)\), is again completely dictated by the Euler’s gamma functions; it can be analytically continued to a meromorphic function with simple poles located in \(s = 1, 2, \ldots, N/2\).

An important observation is that, in both cases, the \(\zeta\)-function is well-defined in \(s = 0\), and it is hence possible to proceed with the \(\zeta\)-function regularization.

**III. EFFECTIVE ACTION FOR SCALAR FIELDS**

In a path integral approach, the effective action for a scalar field can be formally expressed as the functional determinant of the operator \(L_b\) as

\[
I_{eff} = -\frac{1}{2} \ln \det(L_b/\mu^2),
\]
(26)

where \(\mu\) is an arbitrary renormalization mass scale coming from the path-integral measure. This determinant is however a formally divergent quantity and needs to be regularized. We shall proceed here with the \(\zeta\)-function renormalization. In this framework, the regularized determinant reads

\[
\ln \det(L_b/\mu^2) = -\zeta'(0|L_b) - \zeta(0|L_b) \ln \mu^2.
\]
(27)

Let us handle first the odd dimensional case. First of all, we note that \(\zeta(0|L_b) = 0\) in odd dimensions, and the dependence from the renormalization scale drops out. To compute the derivative of the \(\zeta\)-function in \(s = 0\), we note that it consists in a sum of terms of the form \(f(s)/\Gamma(s)\), with \(f(s)\) a smooth function of \(s\), and that
\[
\lim_{s \to 0} \frac{d}{ds} \left( \frac{f(s)}{\Gamma(s)} \right) = f(0). \tag{28}
\]

Computing the derivative term by term in Eq. (21) we easily obtain

\[
\ln \det \left( L_b/\mu^2 \right) = \frac{a^{-N}}{2^{N-\frac{N}{2}-1} \Gamma \left( \frac{N}{2} \right)} \frac{\rho}{n+\frac{1}{2}} \sum_{n=1}^{n+1} \left( -1 \right)^n c_{2n}^N (a^2 b)^{n+\frac{1}{2}} \tag{29}
\]

for \( N \) odd. Let us turn now to the case of even dimensionality. Now the \( \zeta \)-function does not vanish anymore, and we have to keep the renormalization scale. This time we have to deal with a sum of functions of the form

\[
F_n(s) = \frac{\Gamma(s - n - 1)}{\Gamma(s)} = \prod_{k=1}^{n+1} \frac{1}{s - k}, \tag{30}
\]

that assume in \( s = 0 \) the value

\[
F_n(0) = \frac{(-1)^{n+1}}{(n+1)!}, \tag{31}
\]

inserting it into Eq. (25) we obtain \( \zeta(0|L_b) \). The computation of \( \zeta'(0|L_b) \) can be done using the relation

\[
F'_n(0) = \frac{(-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} \frac{1}{k}, \tag{32}
\]

and, with a bit of algebra, one obtains the effective action on an even-dimensional hyperbolic space

\[
\ln \det \left( L_b/\mu^2 \right) = \frac{a^{-N}}{2^{N-1} \pi^{N/2} \Gamma \left( \frac{N}{2} \right)} \sum_{n=0}^{N-1} c_{2n+1}^N \left[ \left( -1 \right)^n (a^2 b)^{n+1} \frac{d_{n+1} - \ln(b/\mu^2)}{2(n+1)} \right] + 2H'_n(0; a\sqrt{b}) + 2H_n(0) \ln \left( a^2 \mu^2 \right), \tag{33}
\]

where we have defined for convenience

\[
d_0 = 0, \quad d_n = \sum_{k=1}^{n} \frac{1}{k} \quad (n \geq 1). \tag{34}
\]

As expected, the arbitrary mass scale \( \mu \) combines with the radius \( a \) and the coupling parameter \( b \) to leave a dimensionless argument for the logarithm.
IV. Vacuum Expectation Value of the Field Fluctuations

The vacuum expectation value of the field fluctuations can be computed within the \( \zeta \)-function regularization scheme by means of the formula \[30\]

\[
\langle \phi^2(x) \rangle = \frac{d}{ds} \left. \frac{s}{\mu^2} \zeta(s + 1, x|L_b/\mu^2) \right|_{s=0} = \lim_{s \to 0} \left[ (1 + s \ln \mu^2) \zeta(s + 1, x|L_b) + s \zeta'(s + 1|L_b) \right],
\]

(35)

where \( \zeta(s, x|L_b) \) is the local \( \zeta \)-function and \( \mu \) is again the renormalization mass scale. Recently it has been shown that this procedure leads to the same results as the point-splitting technique \[32\]. The odd-dimensional case is simpler, as the local \( \zeta \)-function and its derivative are finite in \( s = 1 \): the field theory is super-\( \zeta \)-regular and the field fluctuations are simply given by the value in \( s = 1 \) of the local \( \zeta \)-function. From \(31\) we easily obtain the expectation value of the field fluctuations of a scalar field in an odd-dimensional hyperbolic space

\[
\langle \phi^2(x) \rangle = \frac{a^{2-N}}{2^{N-1}\pi^{N/2}\Gamma(\frac{N}{2})} \sum_{n=1}^{\rho_N} (-1)^n c_{2n}^N (a^2b)^{n-\frac{1}{2}}.
\]

(36)

The even-dimensional case has to be handled more carefully, because the associated local \( \zeta \)-function has a pole in \( s = 1 \). However, the poles cancel exactly in Eq. \(33\) as we shall see. The appearance of the poles is due to the presence of the function \( F_n(s) \) in the local \( \zeta \)-function. Near \( s = 1 \), \( F_n(s) \) and its derivative behave as

\[
F_n(1 + s) = \frac{(-1)^n}{n!} \frac{1}{s} + \mathcal{O}(s), \quad F'_n(1 + s) = \frac{(-1)^n}{n!} \left( \frac{1}{s^2} - \frac{d_n}{s} \right) + \mathcal{O}(s).
\]

(37)

Using these expressions, the behaviour of the local \( \zeta \)-function and its derivative near \( s = 1 \) is a simple matter of algebra, and one obtains

\[
\zeta(s + 1, x|L_b) = \frac{a^{2-N}}{2^{N-1}\pi^{N/2}\Gamma(\frac{N}{2})} \sum_{n=0}^{N-1} c_{2n+1}^N \left[ \frac{1}{2} (a^2b)^n \frac{1}{s} - 2H_n(1; a\sqrt{b}) \right] + \mathcal{O}(s),
\]

(38)

\[
\zeta'(s + 1, x|L_b) = \frac{a^{2-N}}{2^{N-1}\pi^{N/2}\Gamma(\frac{N}{2})} \sum_{n=0}^{N-1} c_{2n+1}^N \left[ -\frac{1}{2} (a^2b)^n \left( \frac{1}{s^2} - (d_n - \ln b) \frac{1}{s} \right) - 2H_n(1; a\sqrt{b}) \right] + \mathcal{O}(s).
\]

(39)

Inserting these expressions in \(33\), we see that the poles disappear and the limit \( s \to 0 \) is smooth, yielding the expectation value for the field fluctuations in an even-dimensional hyperbolic space

\[
\langle \phi^2(x) \rangle_{\mu^2} = \frac{a^{2-N}}{2^{N-1}\pi^{N/2}\Gamma(\frac{N}{2})} \sum_{n=0}^{N-1} c_{2n+1}^N \left[ \frac{1}{2} (a^2b)^n \left( d_n - \ln(b/\mu^2) \right) - 2H_n(1; a\sqrt{b}) \right].
\]

(40)

Again, the coupling parameter \( b \) and the renormalization scale \( \mu \) combine to leave a dimensionless argument for the logarithm.
V. ONE LOOP RENORMALIZED STRESS TENSOR

Finally, we turn to the computation of the renormalized stress tensor for a quantum scalar field propagating in $\mathbb{H}^N$. It is possible to perform the computation directly in the framework of the $\zeta$-function regularization \[31\]. In this approach, one defines the analytic continuation of the tensor

$$\zeta_{\mu\nu}(s|L_b)(x) = \sum_n \lambda_n^{-s} T_{\mu\nu}[\phi_n^*, \phi_n](x),$$

in which $\phi_n$ are the eigenfunctions of the Laplace-like operator $L_b$, and $T_{\mu\nu}[\phi_n^*, \phi_n](x)$ is the classical stress tensor evaluated on the modes, defined as

$$T_{\mu\nu}[\phi^*, \phi](x) = \frac{2}{\sqrt{g}} \frac{\delta I[\phi^*, \phi]}{\delta g_{\mu\nu}(x)},$$

$I[\phi^*, \phi]$ being the associated classical action. Then, the vacuum expectation value of the stress tensor reads

$$\langle T_{\mu\nu}(x) \rangle = \lim_{s \to 0} \left[ \zeta_{\mu\nu}(s + 1, x|L_b) + \frac{1}{2} g_{\mu\nu} \zeta(s, x|L_b) + s \left[ \zeta'_{\mu\nu}(s + 1, x|L_b) + \zeta_{\mu\nu}(s + 1, x|L_b) \ln \mu^2 \right] \right].$$

This limit is smooth; the computation is simplified observing that \[31\]

$$\zeta_{\mu\nu}(s, x|L_b) = \zeta_{\mu\nu}(s, x|L_b) + L_{\mu\nu}\zeta(s, x|L_b) - \frac{1}{2} g_{\mu\nu} \zeta(s - 1, x|L_b),$$

where we have defined the operator

$$L_{\mu\nu} = -\xi \nabla_\mu \nabla_\nu + \left( \xi - \frac{1}{4} \right) g_{\mu\nu} \Delta + \xi R_{\mu\nu},$$

and $\tilde{\zeta}_{\mu\nu}(s, x|L_b)$ is the analytical continuation of the series

$$\tilde{\zeta}_{\mu\nu}(s, x|L_b) = C_{\mu\nu}^{-s} \left( \nabla_\mu \phi_n^* \nabla_\nu \phi_n + \nabla_\nu \phi_n^* \nabla_\mu \phi_n \right).$$

The equivalence of the direct $\zeta$-function approach to the computation of the one loop renormalized stress tensor with the point-splitting approach has been shown in \[33\]. Note that the proof of this equivalence has been carried out for compact spaces, where the spectrum is discrete. Here we are dealing with a continuous spectrum, and we shall check that this equivalence holds on hyperbolic spaces.

The continuous spectrum generalization of Eq. (46) reads, making use of the eigenfunctions \[3\],

$$\tilde{\zeta}_{\mu\nu}(s, x|L_b) = \frac{d^{2s}}{2} \int_0^\infty d\lambda \mu(\lambda) \int_{\mathbb{R}^{N-1}} d^{N-1}k \left( \lambda^2 + a^2 b \right)^{-s} \left( \nabla_\mu \phi_n^* \nabla_\nu \phi_n + \nabla_\nu \phi_n^* \nabla_\mu \phi_n \right).$$
This integral can be carried out without big difficulties, yielding

\[
\bar{\zeta}_{\mu\nu}(s, x|L_b) = \frac{\Gamma(\rho_N + 1)}{2\pi^\rho_N N!} a^{2s-2} \int_0^\infty \Gamma(\rho_N + i\lambda) \Gamma(\rho_N - i\lambda) \frac{\lambda^2 + \rho_N^2}{(\lambda^2 + a^2 b)^s} \mu(\lambda) \, d\lambda
\]

\[
= \frac{2^{N-3} \rho_N^{2s-2} \Gamma(\frac{N}{2})}{\pi^{N+1} N} g_{\mu\nu}(x) \int_0^\infty \frac{\lambda^2 + \rho_N^2}{(\lambda^2 + a^2 b)^s} \mu(\lambda) \, d\lambda,
\]

(48)

where we have used the relation (12) between the Plancherel measure and the measure \(\mu(\lambda)\). We recognise in the last integral the function \(I_N(s)\). Now the tensor \(\zeta_{\mu\nu}(s, x|L_b)\) follows from Eq. (14),

\[
\zeta_{\mu\nu}(s, x|L_b) = \frac{2^{N-3} \Gamma(\frac{N}{2})}{\pi^{N+1} N} a^{2s-N} \left(1 - \frac{N}{2}\right) a^{-2} I_N(s-1) - m^2 I_N(s) \right] g_{\mu\nu}(x). \tag{49}
\]

If the dimension \(N\) of the hyperbolic space is odd, the function \(I_N(s)\), given in (20), is finite in \(s = 0\) and \(s = 1\), where it assumes the values \(I_N(0) = 0\) and

\[
I_N(1) = \frac{1}{2\pi \beta_N} \sum_{n=1}^{\rho_N} (-1)^n c_{2n}^N (a^2 b)^{n-\frac{1}{2}};
\]

(50)

the theory is hence super-\(\zeta\)-regular and there are no divergent terms in (13), that leads to the vacuum expectation value of the stress tensor in odd dimensions

\[
\langle T_{\mu\nu}(x) \rangle_{\mu^2}^H = \frac{m^2 a^{2-N}}{2^{N-1} \pi^{N/2 - 1} N \Gamma(\frac{N}{2})} \left(\sum_{n=1}^{\rho_N} (-1)^n c_{2n}^N (a^2 b)^{n-\frac{1}{2}}\right) g_{\mu\nu}(x). \tag{51}
\]

In the even-dimensional case, the computation is more delicate as divergent terms appear in Eq. (13); however they cancel and the limit can be performed without excessive difficulty, leading to the following vacuum expectation value for the stress tensor in an even-dimensional hyperbolic space,

\[
\langle T_{\mu\nu}(x) \rangle_{\mu^2}^H = \frac{a^{-N}}{2^{N-1} \pi^{N/2 - 1} N \Gamma(\frac{N}{2})} g_{\mu\nu}(x) \sum_{n=0}^{N-1} c_{2n+1}^N \left[(-1)^{n+1} \frac{(a^2 b)^{n+1}}{2(n+1)} + \frac{(-1)^n}{2} a^2 m^2 (a^2 b)^n \left(d_n - \ln(b/\mu^2)\right) + 2a^2 m^2 H_n(1; a\sqrt{b}) - 2H_n(0)\right]. \tag{52}
\]

Correctly, the renormalization scale \(\mu\) combines with the coupling parameter \(b\) to give a dimensionless argument for the logarithm.

We always find a stress tensor proportional to the metric, as expected for an homogeneous manifold; this property implies that it is automatically conserved.

We shall now proceed with some checks of this result. First of all there is a general formula relating the vacuum expectation value of the field fluctuations with the trace of the renormalized stress tensor [30]

\[
\langle T^\mu_{\mu}(x) \rangle = \zeta(0, x|L_b) - \left[m^2 + \frac{\xi - \xi_N}{4\xi_N - 1} \Delta\right] \langle \phi^2(x) \rangle, \tag{53}
\]

10
where $\xi_N = (N-2)/4(N-1)$ is the conformal coupling parameter $^1$. Noting that $\Delta \langle \phi^2(x) \rangle = 0$, it is easy to check that this relation is verified, both in odd and even dimensions. $\mathbb{H}^N$ being a homogeneous manifold, the stress tensor is completely determined by its trace, and $\langle T_{\mu\nu} \rangle$ can be computed directly from the field fluctuations. We proceeded however to a direct $\zeta$-function computation to verify its validity in presence of a continuous spectrum. Another check can be done in the conformally coupled case. This is defined by $m = 0$ and $\xi = \xi_N$, that is $a^2b = 1/4$ for any dimension. In odd dimension the stress tensor (51) is proportional to $m^2$ and the conformal anomaly correctly vanishes, while in even dimension we get, taking the trace of (52),

$$\langle T_{\mu\mu}(x) \rangle = \frac{a^{-N}}{2^{N-1} \pi^{N/2} \Gamma\left(\frac{N}{2}\right)} \sum_{n=0}^{N-1} c_{2n+1}^N \left[ \frac{(-1)^{n+1}}{2(n+1)} 4^{-n-1} - 2H_n(0) \right].$$  (54)

This can be compared with the spectral coefficient $a_{N/2}(x|L_h)$, related to the conformal anomaly by $\langle T_{\mu\mu}(x) \rangle = a_{N/2}(x)(4\pi)^{N/2}$ $^3$. This coefficient can be computed making an heat kernel expansion in the local $\zeta$-function; it turns out that $a_{N/2}(x|A) = (4\pi)^{N/2}\text{Res}[\Gamma(s)\zeta(s,x|A)]_{s=0}$. This residue can be directly read in equation (25), and coincides with the trace of the renormalized stress tensor. We stress the fact that this last check, in the conformally coupled case, is completely independent from the $\zeta$-function regularization.

VI. APPLICATION TO VARIOUS DIMENSIONS

In this Section, we shall apply the previous results to compute the effective action, the field fluctuations and the stress tensor in various dimensions. In the even dimensional case we shall report only the results in 2 and 4 dimensions, that can be compared with analogous expressions obtained with other methods. Higher dimensional cases can be computed easily from our general formulae; however, as the resulting expressions increase considerably in complexity with the dimension, we shall omit them. The odd dimensional case is simpler, and we shall give the expressions of the effective action and the stress tensor up to $N = 11$.

A. $N = 2$

In two dimensions $\rho_2 = 1/2$, the coupling parameter is $a^2b = 1/4 - 2\xi + a^2m^2$ and $c_1^2 = 0$. The effective action (26) reads

$$I_{eff}^{\mathbb{R}^2} = \frac{1}{2\pi a^2} \left[ \left( \frac{1}{8} - \xi + \frac{1}{2} a^2 m^2 \right) (1 - \ln(a^2b)) + 2H'_0(0; a\sqrt{b}) \right].$$

$^1$The coefficient $1/2\xi_D$ which appears in (13) of $^3$ is misprinted and has to be replaced by $1/(4\xi_D - 1)$. See also Theorem 2.4. of $^3$.
and gives the correct conformal anomaly dictated by the spectral coefficient \( a_2(x) \mid L_{\text{conf}} \).

\[
\langle \phi^2(x) \rangle_{H^4} = -\frac{1}{2\pi} \left[ \psi \left( a\sqrt{b} + \frac{1}{2} \right) - \ln(a\mu) \right],
\]

(56) the expectation value of the field fluctuations reads

\[
\langle T_{\mu\nu}(x) \rangle_{H^4} = \frac{1}{4\pi a^2} \left[ \xi - \frac{1}{2} - \frac{1}{2}a^2m^2 \left( 1 - 2\psi \left( a\sqrt{b} + \frac{1}{2} \right) - \ln(a^2\mu^2) \right) \right] g_{\mu\nu}^{H^4}(x).
\]

(57) Furthermore, for a conformally coupled field, \( m = 0, a^2b = 1/4 \), the stress tensor is

\[
\langle T_{\mu\nu}(x) \rangle_{H^4} = -\frac{1}{960\pi^2a^4} g_{\mu\nu}^{H^4}(x),
\]

(61) and gives the correct conformal anomaly dictated by the spectral coefficient \( a_2(x) \mid L_{\text{conf}} \).
C. Odd Dimensions

We report here the effective action and the stress tensor for odd dimensions, up to \( N = 11 \). We shall not report the expressions of the field fluctuations, as in odd dimensions Eq. (53) reduces to a simple proportionality relation,

\[
\langle T_{\mu}^{\mu}(x) \rangle = -m^2 \langle \phi^2(x) \rangle .
\]

The effective actions are in accord with those computed in \[37\].

1. \( N = 3 \)

In three dimensions we have \( \rho_3 = 1, a^2b = 1 - 6\xi + a^2m^2 \) and \( c_3^2 = 1 \); the effective action reads

\[
I_{\text{eff}}^{\mathbb{H}^3} = \frac{1}{12\pi a^3} (1 - 6\xi + a^2m^2)^{3/2} ,
\]

and the expectation value of the stress tensor (51) reads

\[
\langle T_{\mu\nu}(x) \rangle_{\mu^2}^{\mathbb{H}^3} = \frac{m^2}{12\pi a} \sqrt{1 - 6\xi + a^2m^2} g_{\mu\nu}^{\mathbb{H}^3}(x).
\]

2. \( N = 5 \)

In five dimensions we have \( \rho_5 = 2, a^2b = 4 - 20\xi + m^2a^2 \) and \( c_5^7 = c_4^5 = 1 \); the effective action reads

\[
I_{\text{eff}}^{\mathbb{H}^5} = -\frac{1}{360\pi^2 a^5} (7 - 60\xi + 3a^2m^2) (4 - 20\xi + a^2m^2)^{3/2} ,
\]

and the expectation value of the stress tensor (51) reads

\[
\langle T_{\mu\nu}(x) \rangle_{\mu^2}^{\mathbb{H}^5} = -\frac{m^2}{120\pi^2 a^5} (3 - 20\xi + a^2m^2) \sqrt{4 - 20\xi + a^2m^2} g_{\mu\nu}^{\mathbb{H}^5}(x).
\]

3. \( N = 7 \)

In seven dimensions \( \rho_7 = 3, a^2b = 9 - 42\xi + a^2m^2, c_7^7 = 4, c_4^7 = 5 \) and \( c_6^7 = 1 \); the effective action reads

\[
I_{\text{eff}}^{\mathbb{H}^7} = -\frac{1}{5040\pi^3 a^7} (82 + 33m^2a^2 + 3m^4a^4 - 1386\xi - 252m^2a^2\xi + 5292\xi^2) \\
\times (9 - 42\xi + a^2m^2)^{3/2} ,
\]

and the expectation value of the stress tensor (51) reads

\[
\langle T_{\mu\nu}(x) \rangle_{\mu^2}^{\mathbb{H}^7} = \frac{m^2}{1680\pi^3 a^5} (40 + 13a^2m^2 + a^4m^4 - 546\xi - 84a^2m^2\xi + 1764\xi^2) \\
\times \sqrt{9 - 42\xi + a^2m^2} g_{\mu\nu}^{\mathbb{H}^7}(x).
\]
4. $N = 9$

In nine dimensions $\rho_9 = 4$, $a^2b = 16 - 72\xi + a^2m^2$, $c_2^9 = 36$, $c_4^9 = 49$, $c_6^9 = 14$ and $c_8^9 = 1$; the effective action reads

$$I_{\text{eff}}^{9} = -\frac{1}{151200\pi^4a^9} (-540 + 441a^2b - 90a^4b^2 + 5a^6b^3) (a^2b)^{3/2},$$

and the expectation value of the stress tensor (69) reads

$$\langle T_{\mu\nu}(x) \rangle_{\mu^2}^{9} = \frac{m^2}{30240\pi^4a^7} (36 - 49a^2b + 14a^4b^2 - a^6b^3) a\sqrt{b} g^{9\mu\nu}(x).$$

5. $N = 11$

In eleven dimensions $\rho_{11} = 5$, $a^2b = 25 - 110\xi + a^2m^2$, $c_2^{11} = 576$, $c_4^{11} = 820$, $c_6^{11} = 273$, $c_8^{11} = 30$ and $c_{10}^{11} = 1$; the effective action reads

$$I_{\text{eff}}^{11} = -\frac{1}{1995840\pi^5a^{11}} (-6336 + 5412a^2b - 1287a^4b^2 + 110a^6b^3 - 3a^8b^4) (a^2b)^{3/2},$$

and the expectation value of the stress tensor (71) reads

$$\langle T_{\mu\nu}(x) \rangle_{\mu^2}^{11} = \frac{m^2}{665280\pi^5a^9} (576 - 820a^2b + 273a^4b^2 - 30a^6b^3 + a^8b^4) a\sqrt{b} g^{11\mu\nu}(x).$$

VII. CONCLUSION

In this paper we have obtained exact expressions at one loop for the effective action and the vacuum expectation values of the field fluctuations and the stress tensor for a scalar field propagating on an $N$-dimensional hyperbolic space. Our expressions hold for massless as well as massive fields, with an arbitrary coupling with the scalar curvature. The computation of the stress tensor has been carried out with the recently developed direct $\zeta$-function approach, which is known to be equivalent to the point-splitting in compact spaces. Comparison of our results with the known expressions in three and four dimensions sustains the equivalence of the $\zeta$-function and point-splitting approaches also in presence of a continuous spectrum.

The computation presented here is the first step to the study of physically more interesting cases. Making use of Selberg-like trace formulae to extend this work to quotient spaces $\mathbb{H}^N/\Gamma$, it is possible to investigate finite temperature effects on AdS spacetime, and quantum corrections to the metric and entropy of the higher dimensional constant curvature black holes.

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APPENDIX A: THE FUNCTION $H_N(S; \mu)$

In this section we shall study the integral $H_n(s; \mu)$ defined in Eq. (24). It defines an analytic function on the whole complex $s$-plane. We are interested in the values it takes, with its derivative, in $s = 0$ and $s = 1$.

The integral can be exactly computed in $s = 0$ using Eq. (3.411.4) of [38]

$$H_n(0; \mu) = (1 - 2^{-2n-1}) \frac{|B_{2n+2}|}{4n+4}, \quad (A1)$$

where the $B_n$ are the Bernoulli numbers. In $s = 1$, we have

$$H_0(1; \mu) = \frac{1}{2} \psi \left( \mu + \frac{1}{2} \right) - \frac{1}{4} \ln(\mu^2); \quad (A2)$$

to evaluate $H_n(1; \mu)$ in $n = 1, 2, \ldots$, we start from the identity

$$\int_0^\infty \frac{x^{2n+1}}{e^{2\pi x} + 1} \ln(\alpha x^2 + \mu^2) \, dx = \ln \alpha \int_0^\infty \frac{x^{2n+1}}{e^{2\pi x} + 1} \, dx + \int_0^\infty \frac{x^{2n+1}}{e^{2\pi x} + 1} \ln(x^2 + \mu^2/\alpha) \, dx. \quad (A3)$$

Taking the derivative with respect to $\alpha$ and setting $\alpha = 1$, one obtains the recurrence relation

$$H_{n+1}(1; \mu) = X_{n+1} - \mu^2 H_n(1; \mu), \quad (A4)$$

where we have defined

$$X_n = (1 - 2^{1-2n}) \frac{|B_{2n}|}{4n}. \quad (A5)$$

From equation (A4) one finds finally

$$H_n(1; \mu) = \sum_{k=1}^{n} X_k (-\mu^2)^{n-k} + (-\mu^2)^n H_0(1; \mu). \quad (A6)$$

In particular we shall need the case $n = 1$, for which,

$$H_1(1; \mu) = \frac{1}{48} - \frac{1}{2} \mu^2 \psi \left( \mu + \frac{1}{2} \right) + \frac{1}{4} \mu^2 \ln(\mu^2). \quad (A7)$$

Let us turn now to the derivative of $H_n(s; \mu)$; let

$$H_n'(0; \mu) = - \int_0^\infty \frac{\lambda^{2n+1} \ln(\lambda^2 + \mu^2)}{e^{2\pi \lambda} + 1} \, d\lambda; \quad (A8)$$

integrating in the variable $\mu^2$ the relation

$$\frac{\partial H_n'(0; \mu)}{\partial (\mu^2)} = -H_n(1; \mu), \quad (A9)$$

using (A6), one obtains
\[
H_n'(0; \mu) = H_n'(0; 0) + \sum_{k=1}^{n} X_k \frac{(-\mu^2)^{n-k+1}}{n-k+1} + (-1)^n \mu^{2n+2} \left[ \frac{\ln \mu}{2n+2} - \frac{1}{4(n+1)^2} \right] + (-1)^{n+1} \int_0^{\mu} \mu^{2n+1} \psi \left( \mu + \frac{1}{2} \right) d\mu, \quad (A10)
\]

where \( H_n'(0; 0) \) is the constant
\[
H_n'(0; 0) = -2 \int_0^{\infty} \frac{\lambda^{2n+1} \ln \lambda}{e^{2\pi \lambda} + 1} d\lambda
= -2 \frac{(2n+1)!}{(2\pi)^{2n+2}} (1 - 2^{-2n-1}) \zeta'(2n + 2)
- \left[ (1 - 2^{-2n-1})(d_{2n+1} - \gamma \ln 2\pi) + 2^{-2n-1} \ln 2 \right] \frac{B_{2n+2}}{2n + 2}. \quad (A11)
\]

Note that the integral in Eq. (A10) can evaluated in terms of multi-gamma functions (see Appendix of [37]).
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