Besov-type spaces with variable smoothness and integrability II

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Abstract

The aim of this paper is to study properties of Besov-type spaces with variable smoothness and integrability and it is a continuation of [12]. We show that these spaces are characterized by the \( \varphi \)-transforms in appropriate sequence spaces and we obtain atomic decompositions for these spaces.

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1 Introduction

The most known general scales of function spaces are the scales of Besov spaces and Triebel-Lizorkin spaces and it is known that they cover many well-known classical function spaces such as Hölder-Zygmund spaces and Sobolev spaces. These spaces play an important role in Harmonic Analysis.

The theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the study of partial differential equations. In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue, Sobolev spaces, Besov spaces, Triebel-Lizorkin spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [21], image restoration [1] and PDE with non-standard growth conditions.

Variable Besov-type spaces \( B_{\alpha(\cdot),\tau(\cdot)}^{p(\cdot),q(\cdot)} \) have been introduced in [12], where their basic properties are given, such as the Sobolev type embeddings and that under some conditions these spaces are just the Besov spaces \( B_{\infty,\infty}^{\alpha(\cdot)+n(1/\tau(\cdot)-1/p(\cdot))} \). For constant exponents, these spaces unify and generalize many classical function spaces including Besov spaces, Besov-Morrey spaces (see, for example, [27], Corollary 3.3).

The main aim of this paper is to present another essential property of the Besov-type spaces with variable smoothness and integrability such as the \( \varphi \)-transforms characterization and the atomic decomposition.

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The paper is organized as follows. First we give some preliminaries where we fix some notation and recall some basics facts on function spaces with variable integrability and we give some key technical lemmas needed in the proofs of the main statements. For making the presentation clearer, we give their proofs later in Section 5. We then define the Besov-type spaces $B^{\alpha(-),\tau(-)}_{p(-),q(-)}$ and $\tilde{B}^{\alpha(-),\tau(-)}_{p(-),q(-)}$ and give several basic properties such as the $\varphi$-transform characterization. The main statements are formulated in Section 4, where we give the atomic decomposition of these function spaces. It is shown that the element $f \in S'(\mathbb{R}^n)$ in the space $B^{\alpha(-),\tau(-)}_{p(-),q(-)}$ or $\tilde{B}^{\alpha(-),\tau(-)}_{p(-),q(-)}$ can be represented as

$$
 f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} q_{v,m}, \quad \text{converging in } S'(\mathbb{R}^n),
$$

where $q_{v,m}$’s are the so-called atoms and the sequence complex numbers $(\lambda_{v,m})$ belongs to an appropriate sequence space. Moreover, based on these sequence spaces equivalent quasi-norms for corresponding function spaces are derived. In this section we also give some key technical lemmas needed in the proofs of the main statements.

## 2 Preliminaries

As usual, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The Euclidean scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is given by $x \cdot y = x_1 y_1 + \ldots + x_n y_n$. The expression $f \lesssim g$ means that $f \leq c g$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to $x$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center $x$ and radius $r$. By $\text{supp } f$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function.

The symbol $S'(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions $\phi$ on $\mathbb{R}^n$, i.e., $\phi$ is infinitely differentiable and

$$
 \|\phi\|_{S_M} = \sup_{\gamma \in \mathbb{N}_0^n, |\gamma| \leq M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma \phi(x)| (1 + |x|)^{n+M+|\gamma|} < \infty
$$

for all $M \in \mathbb{N}$. We denote by $S'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in S(\mathbb{R}^n)$ by $\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx$. Its inverse is denoted by $\mathcal{F}^{-1} f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $S'(\mathbb{R}^n)$ in the usual way.

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on $L^1_{\text{loc}}$ by

$$
 \mathcal{M} f(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy.
$$

For $v \in \mathbb{Z}$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{v,m} = \{(x_1, \ldots, x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, \ldots, n\}$. For the collection of all such cubes $
we use \( Q = \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\} \). For each cube \( Q \), we denote by \( x_{Q_{v,m}} \) the lower left-corner \( 2^{-v}m \) of \( Q = Q_{v,m} \), its side length by \( l(Q) \) and for \( r > 0 \), we denote by \( rQ \) the cube concentric with \( Q \) having the side length \( rl(Q) \). Furthermore, we put \( v_Q = -\log_2 l(Q) \) and \( v_Q^+ = \max(v_Q, 0) \).

For \( v \in \mathbb{Z}, \varphi \in \mathcal{S}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), we set \( \varphi_v(x) = \varphi(-x) \), \( \varphi_v(x) = 2^m \varphi(2^vx) \), and

\[
\varphi_{v,m}(x) \equiv 2^m/2 \varphi(2^vx - m) = |Q_{v,m}|^{1/2} \varphi_v(x - x_{Q_{v,m}}) \quad \text{if} \quad Q = Q_{v,m}.
\]

By \( c \) we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. \( c(p) \) means that \( c \) depends on \( p \), etc.). Further notation will be properly introduced whenever needed.

The variable exponents that we consider are always measurable functions \( p \) on \( \mathbb{R}^n \) with range in \([c, \infty)\) for some \( c > 0 \). We denote the set of such functions by \( \mathcal{P}_0 \). The subset of variable exponents with range \([1, \infty[\) is denoted by \( \mathcal{P} \). We use the standard notation \( p^- = \text{ess-inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ = \text{ess-sup}_{x \in \mathbb{R}^n} p(x) \).

The variable exponent modular is defined by \( \varrho_{p(v)}(f) = \int_{\mathbb{R}^n} \varrho_{p(x)}(|f(x)|)dx \), where \( \varrho_{p}(t) = tp \). The variable exponent Lebesgue space \( L^{p(v)} \) consists of measurable functions \( f \) on \( \mathbb{R}^n \) such that \( \varrho_{p(x)}(\lambda f) < \infty \) for some \( \lambda > 0 \). We define the Luxemburg (quasi)-norm on this space by the formula \( \|f\|_{p(v)} = \inf \{\lambda > 0 : \varrho_{p(x)}\left(\frac{f}{\lambda}\right) \leq 1\} \). A useful property is that \( \|f\|_{p(v)} \leq 1 \) if and only if \( \varrho_{p(v)}(f) \leq 1 \), see [2], Lemma 3.2.4.

Let \( p, q \in \mathcal{P}_0 \). The mixed Lebesgue-sequence space \( \ell^{q(v)}(L^{p(v)}) \) is defined on sequences of \( L^{p(v)} \)-functions by the modular

\[
\varrho_{\ell^{q(v)}(L^{p(v)})}( (f_v) ) = \sum_v \inf \big\{ \lambda_v > 0 : \varrho_{p(v)}\left( \frac{f_v}{\lambda_v^{1/q(v)}} \right) \leq 1 \big\}.
\]

The (quasi)-norm is defined from this as usual:

\[
\|(f_v)\|_{\ell^{q(v)}(L^{p(v)})} = \inf \big\{ \mu > 0 : \varrho_{\ell^{q(v)}(L^{p(v)})}\left( \frac{1}{\mu} (f_v) \right) \leq 1 \big\}. \tag{1}
\]

If \( q^+ < \infty \), then we can replace (1) by the simpler expression \( \varrho_{\ell^{q(v)}(L^{p(v)})}( (f_v) ) = \sum_v \|f_v|^{q(v)}\|_{p(v)}^{q(v)} \). Furthermore, if \( p \) and \( q \) are constants, then \( \ell^{q(v)}(L^{p(v)}) = \ell^q(L^p) \).

The case \( p \equiv \infty \) can be included by replacing the last modular by \( \varrho_{\ell^{q(v)}(L^{\infty})}( (f_v) ) = \sum_v \|f_v|^{q(v)}\|_{\infty} \).

It is known, cf. [1] and [17], that \( \ell^q(L^p) \) is a norm if \( q(\cdot) \geq 1 \) is constant almost everywhere (a.e.) on \( \mathbb{R}^n \) and \( p(\cdot) \geq 1 \), or if \( \frac{1}{p(x)} + \frac{1}{q(x)} \leq 1 \) a.e. on \( \mathbb{R}^n \), or if \( 1 \leq q(x) \leq p(x) < \infty \) a.e. on \( \mathbb{R}^n \).

We say that \( g : \mathbb{R}^n \to \mathbb{R} \) is locally log-Hölder continuous, abbreviated \( g \in C^\log_{\text{loc}} \), if there exists \( c_{\log}(g) > 0 \) such that

\[
|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x - y|)} \tag{2}
\]
for all \( x, y \in \mathbb{R}^n \). We say that \( g \) satisfies the log-Hölder decay condition, if there exists \( g_\infty \in \mathbb{R} \) and a constant \( c_{\log} > 0 \) such that

\[
|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}
\]

for all \( x \in \mathbb{R}^n \). We say that \( g \) is globally-log-Hölder continuous, abbreviated \( g \in C_{\log}^\infty \), if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants \( c_{\log}(g) \) and \( c_{\log} \) are called the locally log-Hölder constant and the log-Hölder decay constant, respectively. We note that all functions \( g \in C_{\log}^\infty \) always belong to \( L^\infty \).

We define the following class of variable exponents

\[
P_{\log} = \left\{ p \in P : \frac{1}{p} \text{ is globally-log-Hölder continuous} \right\}.
\]

We define \( 1/p_\infty := \lim_{|x| \to \infty} 1/p(x) \) and we use the convention \( \frac{1}{p} = 0 \). Note that although \( \frac{1}{p} \) is bounded, the variable exponent \( p \) itself can be unbounded. It was shown in [7], Theorem 4.3.8 that \( M : L^p(\cdot) \to L^p(\cdot) \) is bounded if \( p \in P_{\log}^0 \) and \( p^\ast > 1 \). Also if \( p \in P_{\log} \), then the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) : \| \varphi * f \|_{p(\cdot)} \leq c \| \varphi \|_1 \| f \|_{p(\cdot)} \). We also refer to the papers [3] and [5], where various results on maximal function in variable Lebesgue spaces were obtained.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical \( L^p \) spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for \( p \in P_{\log} \) we have

\[
\| \chi_B \|_{p(\cdot)} \| \chi_B \|_{p^\prime(\cdot)} \approx |B|.
\]

(3)

Also,

\[
\| \chi_B \|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B
\]

(4)

for small balls \( B \subset \mathbb{R}^n (|B| \leq 2^n) \), and

\[
\| \chi_B \|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}}
\]

(5)

for large balls \( (|B| \geq 1) \), with constants only depending on the log-Hölder constant of \( p \) (see, for example, [7, Section 4.5]). Here \( p^\prime \) denotes the conjugate exponent of \( p \) given by \( \frac{1}{p(\cdot)} + \frac{1}{p^\prime(\cdot)} = 1 \). These properties are hold if \( p \in P_{\log}^0 \), since \( \| \chi_B \|_{p(\cdot)} = \| \chi_B \|_{p(\cdot)/a}^{1/a} \) and \( \frac{1}{a} \in P_{\log}^0 \) if \( p^\ast \geq a \).

Recall that \( \eta_{v,m}(x) = 2^{nv}(1 + 2^v |x|)^{-m} \), for any \( x \in \mathbb{R}^n \), \( v \in \mathbb{N}_0 \) and \( m > 0 \). Note that \( \eta_{v,m} \in L^1 \) when \( m > n \) and that \( \| \eta_{v,m} \|_1 = c_m \) is independent of \( v \).

### 2.1 Some technical lemmas

In this subsection we present some results which are useful for us. The following lemma is from [13, Lemma 19], see also [6, Lemma 6.1].
Lemma 1 Let $\alpha \in C_{\text{loc}}^{\log}$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from (2) for $\alpha$. Then 
\[ 2^{\nu_\alpha(x)} \eta_{v,m+R}(x - y) \leq c \ 2^{\nu_\alpha(y)} \eta_{v,m}(x - y) \]
with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

\[ 2^{\nu_\alpha(x)} \eta_{v,m+R} * f(x) \leq c \eta_{v,m} * (2^{\nu_\alpha} f)(x). \]

Lemma 2 Let $r, R, N > 0$, $m > n$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have

\[ |\theta_R \ast \omega_N \ast g(x)| \leq c (\eta_{R,m} \ast |\omega_N \ast g|')(x)^{1/r}, \quad x \in \mathbb{R}^n, \]

where $\theta_R(\cdot) = R^r \theta(R \cdot)$, $\omega_N(\cdot) = N^m \omega(N \cdot)$ and $\eta_{R,m}(x) = R^n(1 + R|x|)^{-m}$.

This lemma is a slight variant of [22, Chapter V, Theorem 5], see also [6, Lemma A.7]. The following lemma is from [12, Lemma 2.11].

Lemma 3 Let $\tau \in \mathcal{P}_0^{\log}$ and $k \in \mathbb{Z}^n$.

(i) For any cubes $P$ and $Q$, we have

\[ \left\| \chi_{P+k l(Q)} \right\|_{\tau(\cdot)} \leq c \left( 1 + \frac{l(Q)}{l(P)} \right)^{c_{\log}(\frac{1}{r})} \]

with $c > 0$ independent of $l(Q)$, $l(P)$ and $k$.

(ii) For any cubes $P$ and $Q$, such that $P \subset Q$, we have

\[ C \left( \frac{|Q|}{|P|} \right)^{1/r^+} \leq \left\| \chi_Q \right\|_{\tau(\cdot)} \leq c \left( \frac{|Q|}{|P|} \right)^{1/r^-} \]

with $c, C > 0$ are independent of $|Q|$ and $|P|$.

Let $L^{p(\cdot)}_{\tau(\cdot)}$ be the collection of functions $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ such that

\[ \left\| f \mid L^{p(\cdot)}_{\tau(\cdot)} \right\| = \sup_{p(\cdot)} \left\| \frac{f \chi_P}{\| \chi_P \|_{\tau(\cdot)}} \right\|_{p(\cdot)} < \infty, \quad p, \tau \in \mathcal{P}_0, \]

where the supremum is taken over all dyadic cubes $P$ with $|P| \geq 1$. Also, the spaces $L^{p(\cdot)}$ is defined to be the set of all function $f$ such that

\[ \left\| f \mid L^{p(\cdot)} \right\| = \sup_{p \in \mathcal{P}_0} \| f \chi_P \|_{p(\cdot)} < \infty, \quad p \in \mathcal{P}_0, \]

where the supremum is taken over all dyadic cubes $P$ with $|P| = 1$. Notice that

\[ \left\| f \mid L^{p(\cdot)}_{\tau(\cdot)} \right\| \leq 1 \Leftrightarrow \sup_{P \in \mathcal{Q}, |P| \geq 1} \left\| \frac{f \chi_P}{\| \chi_P \|_{\tau(\cdot)}} \right\|^{q(\cdot)}_{p(\cdot)/q(\cdot)} \leq 1. \quad (6) \]

Let $\theta_R$ be as in Lemma 2.
Lemma 4 Let \( R > 0, \tau, p \in \mathcal{P}^\log_0, 0 < r < p^- \) and \( \theta, \omega \in \mathcal{S}(\mathbb{R}^n) \).

(i) For any \( f \in \mathcal{S}'(\mathbb{R}^n) \), any \( m > n + c_{\log} \left( \frac{1}{\tau} \right) r \) and any dyadic cube \( P \) with \( |P| \geq 1 \), we have
\[
\left\| \frac{\theta_R \ast \omega \ast f}{\|x_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leq c \max(1, (Rl(P))^{(n-m)/r}) \left\| \omega \ast f \mid L^{p(\cdot)}_{\tau(\cdot)} \right\|
\]
such that the right-hand side is finite, where \( c > 0 \) is independent of \( R \) and \( l(P) \).

(ii) For any \( f \in \mathcal{S}'(\mathbb{R}^n) \), any \( m > n \) and any dyadic cube \( P \) with \( |P| = 1 \), we have
\[
\left\| (\theta_R \ast \omega \ast f) \chi_P \right\|_{p(\cdot)} \leq c \max(1, R^{(n-m)/r}) \left\| \omega \ast f \mid \tilde{L}^{p(\cdot)} \right\|
\]
such that the right-hand side is finite, where \( c > 0 \) is independent of \( R \).

Proof. By similarity, we only consider \( L^{p(\cdot)}_{\tau(\cdot)} \). We use Lemma [2] in the form
\[
|\theta_R \ast \omega \ast f(x)| \leq c (\eta_{R,m} \ast |\omega \ast f|^r(x))^{1/r}.
\]
where \( 0 < r < p^- \), \( m > n + c_{\log} \left( \frac{1}{\tau} \right) r \) and \( x \in P \). We have
\[
\eta_{R,m} \ast |\omega \ast f|^r(x) = \int_{\mathbb{R}^n} \omega \ast f^r(z) \frac{1}{(1 + R \left| x - z \right|)^m} dz + \sum_{k=(k_1, \ldots, k_n) \in \mathbb{Z}^n, \max_{i=1, \ldots, n} |k_i| \geq 2} \int_{P^+ + kl(P)} \cdots dz
\]
\[
= J_R^1(\omega \ast f)(x) + \sum_{k=(k_1, \ldots, k_n) \in \mathbb{Z}^n, \max_{i=1, \ldots, n} |k_i| \geq 2} J_{R,k}^2(\omega \ast f)(x).
\]
Thus we obtain
\[
\left\| \frac{\theta_R \ast \omega \ast f}{\|x_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)}^r \leq \left\| \frac{J_R^1(\omega \ast f)}{\|x_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)/r} + \sum_{k=(k_1, \ldots, k_n) \in \mathbb{Z}^n, \max_{i=1, \ldots, n} |k_i| \geq 2} \left\| \frac{J_{R,k}^2(\omega \ast f)}{\|x_P\|_{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)/r}.
\]
(7)

Let us prove that the first norm on the right-hand side is bounded by
\[
c \left\| \frac{\omega \ast f}{\|x_P\|_{\tau(\cdot)}} \chi_{3P} \right\|_{p(\cdot)}^r.
\]
(8)

We have
\[
|J_R^1(\omega \ast f)(x)| \leq C \int_{\mathbb{R}^n} \frac{|\omega \ast f(z)|^r \chi_{3P}(z)}{(1 + R \left| x - z \right|)^m} dz.
\]

Now the function \( z \mapsto \frac{1}{(1 + |z|)^m} \) is in \( L^1 \) (since \( m > n \)), then using the majorant property for the Hardy-Littlewood maximal operator \( \mathcal{M} \), see E. M. Stein and G. Weiss [23, Chapter 2, (3.9)],
\[
\left( |g| \ast \frac{1}{(1 + |\cdot|)^m} \right)(x) \leq C \left\| \frac{1}{(1 + |\cdot|)^m} \right\|_1 \mathcal{M}(g)(x),
\]
where \( \mathcal{M}(g)(x) \) is the Hardy-Littlewood maximal function of \( g \).
it follows that for any \( x \in P \), \( |J_R^k(\omega*f)(x)| \leq C \cdot \mathcal{M}(\frac{\|\omega*f\|_r}{\|\chi_{3P}\|_{\tau(r)}})(x) \) where the constant \( C > 0 \) is independent of \( x \) and \( R \). Hence the first term of \((7)\) is bounded by

\[
c \left\| \mathcal{M} \left( \frac{|\omega*f|^r}{\|\chi_P\|_{\tau(r)}} \chi_{3P} \right) \right\|_{p(r)/r} \leq \left\| \frac{|\omega*f|^r}{\|\chi_P\|_{\tau(r)}} \chi_{3P} \right\|_{p(r)/r} = \left\| \frac{\omega*f}{\|\chi_P\|_{\tau(r)}} \chi_{3P} \right\|_r,
\]

after using the fact that \( \mathcal{M} : L^{p(r)/r} \rightarrow L^{p(r)/r} \) is bounded. Notice that \( 3P = \cup_{h=1}^{3^n} P_h \), where \( \{P_h\}_{h=1}^{3^n} \) are disjoint dyadic cubes with side length \( l(P_h) = l(P) \). Therefore \( \chi_{3P} = \sum_{h=1}^{3^n} \chi_{P_h} \) and the expression in \((8)\) can be estimated by

\[
c \sum_{h=1}^{3^n} \left\| \frac{\omega*f}{\|\chi_{P_h}\|_{\tau(r)}} \chi_{P_h} \right\|_r \leq c \left\| \frac{\omega*f}{\|\chi_P\|_{\tau(r)}} \right\|_r L^{p(r)},
\]

where we have used the fact that \( \frac{\|\chi_{P_h}\|_{\tau(r)}}{\|\chi_P\|_{\tau(r)}} \leq c \), see Lemma \(3\)(ii) and the proof of the first part is finished. The summation in \((7)\) can be rewritten as

\[
\sum_{k \in \mathbb{Z}} \sum_{|k| = 4\sqrt{n}} \cdots + \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} \cdots.
\]

The estimate of the first sum follows in the same manner as in the estimate of \( J_R^k(\omega*f) \), so we need only to estimate the second sum. Let us prove that

\[
\left\| \frac{|k|^{m-n-c \log(\frac{1}{r})}}{\|\chi_P\|_{\tau(r)}} J_{R,k}^r(\omega*f) \chi_P \right\|_{p(r)/r} \lesssim (Rl(P))^{n-m} \left\| \frac{\omega*f}{\|\chi_{P+kl(P)}\|_{\tau(r)}} \chi_{P+kl(P)} \right\|_{p(r)/r}.
\]

Let \( x \in P, z \in P + kl(P) \) with \( k \in \mathbb{Z}^n \) and \( |k| > 4\sqrt{n} \). Then \( |x-z| \geq \frac{3}{4} |k| l(P) \) and the term \( |J_{R,k}^r(\omega*f)(x)| \) is bounded by

\[
C |k|^{-m} R^{n-m} (l(P))^{-m} \int_{P+kl(P)} |\omega*f(z)|^r dz
\leq C |k|^{-m} R^{n-m} (l(P))^{-m} \int_{|z-x| \leq 2\sqrt{n} |k| l(P)} |\omega*f(z)|^r \chi_{P+kl(P)}(z) dz
\leq C |k|^{n-m} (Rl(P))^{n-m} \mathcal{M} \left( \frac{|\omega*f|^r}{\|\chi_{P+kl(P)}\|_{\tau(r)}} \right)(x).
\]

Hence the left-hand side of \((10)\) is bounded by

\[
(Rl(P))^{n-m} \left\| \frac{|k|^{m-n-c \log(\frac{1}{r})}}{\|\chi_P\|_{\tau(r)}} \frac{\frac{|\omega*f|^r}{\|\chi_{P+kl(P)}\|_{\tau(r)}}}{\chi_{P+kl(P)}} \right\|_{p(r)/r}
\lesssim (Rl(P))^{n-m} |k|^{-c \log(\frac{1}{r})} \left\| \frac{\omega*f}{\|\chi_{P+kl(P)}\|_{\tau(r)}} \right\|_{p(r)/r},
\]

after using the fact that \( \mathcal{M} : L^{p(r)/r} \rightarrow L^{p(r)/r} \) is bounded. By Lemma \(3\)(i), \( \frac{\|\chi_{P+kl(P)}\|_{\tau(r)}}{\|\chi_P\|_{\tau(r)}} \leq c|k|^{-c \log(\frac{1}{r})} \), with \( c > 0 \) independent of \( R, h \) and \( k \). Hence the last expression is bounded
by

\[ c(Rl(P))^{n-m} \left\| \frac{\omega \ast f}{\|\chi_{P+kl(P)}\|_{\tau(c)}} \right\|_{p(c)/r}. \]

Since \( m \) can be taken large enough such that \( m > n + c\log(\frac{1}{\tau})r \), then the second sum in (2) is bounded by

\[
(Rl(P))^{n-m} \sum_{k \in \mathbb{Z}^n, |k| > 4N} |k|^{n + c\log(\frac{1}{\tau})r - m} \left\| \frac{\omega \ast f}{\|\chi_{P+kl(P)}\|_{\tau(c)}} \right\|_{p(c)/r} \leq c(Rl(P))^{n-m} \sum_{k \in \mathbb{Z}^n, |k| > 4N} |k|^{n + c\log(\frac{1}{\tau})r - m} \left\| \omega \ast f \right\|_{L^{p(c)}(\tau(c))}^r \leq (Rl(P))^{n-m} \left\| \omega \ast f \right\|_{L^{p(c)}(\tau(c))}^r.
\]

Hence the proof is complete. □

We introduce the abbreviations

\[
\| (f_v)_v \|_{\ell^{q(c)}(\mathcal{L}^{p(c)}_{p(c)})} = \sup_{\{P \in \mathcal{Q}, |P| \leq 1\}} \left\| \left( \frac{f_v}{|P|^{1/p(c)} \chi_{P}} \right)_{v \geq v_p} \right\|_{\ell^{q(c)}(\mathcal{L}^{p(c)})}.
\]

\[
\| (f_v)_v \|_{\ell^{r(c),q(c)}(\mathcal{L}^{p(c)})} = \sup_{P \in \mathcal{Q}} \left\| \left( \frac{f_v}{\|\chi_{P}\|_{\tau(c)}} \chi_{P} \right)_{v \geq v_P^+} \right\|_{\ell^{r(c),q(c)}(\mathcal{L}^{p(c)})}.
\]

The following lemma is the \( \ell^{q(c)}(\mathcal{L}^{p(c)}_{p(c)})(\ell^{r(c),q(c)}(\mathcal{L}^{p(c)})) \)-version of Lemma 4.7 from A. Almeida and P. Hästö \[1\] (we use it, since the maximal operator is in general not bounded on \( \ell^{q(c)}(\mathcal{L}^{p(c)}) \), see \[1\] Example 4.1).

**Lemma 5** Let \( p \in \mathcal{P}^{\log} \) and \( q, \tau \in \mathcal{P}_{0}^{\log} \) with \( 0 < q^- \leq q^+ < \infty \).

(i) For \( m > n + c\log(1/\tau) + c\log(1/q) \), there exists \( c > 0 \) such that

\[
\left\| (\eta_{v,m} \ast f_v)_v \right\|_{\ell^{r(c),q(c)}(\mathcal{L}^{p(c)})} \leq c \left\| (f_v)_v \right\|_{\ell^{r(c),q(c)}(\mathcal{L}^{p(c)})}.
\]

(ii) For \( m > n + c\log(1/p) + c\log(1/q) \), there exists \( c > 0 \) such that

\[
\left\| (\eta_{v,m} \ast f_v)_v \right\|_{\ell^{q(c)}(\mathcal{L}^{p(c)}_{p(c)})} \leq c \left\| (f_v)_v \right\|_{\ell^{q(c)}(\mathcal{L}^{p(c)}_{p(c)})}.
\]

The proof (i) is given in \[12\] Lemma 2.12, their arguments are true to prove (ii) in view of the fact that \( \|\chi_{P}\|_{p(c)} \approx |P|^{1/p(c)} \), since the supremum taken with respect to dyadic cubes with side length \( \leq 1 \).

The next three lemmas are from \[6\] where the first tells us that in most circumstances two convolutions are as good as one.

**Lemma 6** For \( v_0, v_1 \in \mathbb{N}_0 \) and \( m > n \), we have

\[
\eta_{v_0,m} \ast \eta_{v_1,m} \approx \eta_{\min(v_0,v_1),m}
\]

with the constant depending only on \( m \) and \( n \).
The proof of Lemma 10 is postponed to the Appendix.

Lemma 8 Let $v, j \in \mathbb{N}_0$, $r \in (0, 1]$ and $m > \frac{n}{r}$. Then for any $Q \in \mathcal{Q}$ with $l(Q) = 2^{-v}$, we have

$$(\eta_{j,m} \ast \eta_{v,m} \ast \chi_Q)^r \approx 2^{(v-j)^+ n(1-r)} \eta_{j,m} \ast \eta_{v,m} \ast \chi_Q,$$

where the constant depends only on $m$, $n$ and $r$.

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 9 Let $0 < a < 1$, $J \in \mathbb{Z}$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}$ be a sequence of positive real numbers and denote $\delta_k = \sum_{j=J^+}^{\infty} a^{k-j} \varepsilon_j$, $k \geq J^+$. Then there exists constant $c > 0$ depending only on $a$ and $q$ such that

$$\left( \sum_{k=J^+}^{\infty} \delta_k^q \right)^{1/q} \leq c \left( \sum_{k=J^+}^{\infty} \varepsilon_k^q \right)^{1/q}.$$  

Lemma 10 Let $\alpha \in C_0^{\text{log}}$ and $p, q, \tau \in \mathcal{P}_{0}^{\text{log}}$ with $0 < q^- \leq q^+ < \infty$. Let $\{f_k\}_{k \in \mathbb{N}_0}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $v \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, let $g_v(x) = \sum_{k=0}^{\infty} 2^{-|k-v|\delta} f_k(x)$. Then there exists a positive constant $c$, independent of $\{f_k\}_{k \in \mathbb{N}_0}$ such that

$$\|g_v\|_{L^{p^+}(\Omega_{\alpha,p}(\Omega))} \leq c \|f_v\|_{L^{p^+}(\Omega_{\alpha,p}(\Omega))}, \quad \delta > \frac{n}{\tau}.$$  

and

$$\|g_v\|_{L^{p^+}(\Omega_{\alpha,p}(\Omega))} \leq c \|f_v\|_{L^{p^+}(\Omega_{\alpha,p}(\Omega))}, \quad \delta > \frac{n}{p^-}.$$  

The proof of Lemma 10 is postponed to the Appendix.

3 The spaces $\tilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ and $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$

In this section we present the Fourier analytical definition of Besov-type spaces of variable smoothness and integrability and we prove the basic properties in analogy to the Besov-type spaces with fixed exponents. Select a pair of Schwartz functions $\Phi$ and $\varphi$ satisfy

$$\supp \mathcal{F} \Phi \subset \overline{B(0, 2)} \text{ and } |\mathcal{F} \Phi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3} \quad (11)$$

and

$$\supp \mathcal{F} \varphi \subset \overline{B(0, 2)} \setminus B(0, 1/2) \text{ and } |\mathcal{F} \varphi(\xi)| \geq c \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \quad (12)$$

where $c > 0$. It easy to see that $\int_{\mathbb{R}^n} x^\gamma \varphi(x) dx = 0$ for all multi-indices $\gamma \in \mathbb{N}_0^n$.

Now, we define the spaces under consideration.
Definition 1 Let $\alpha : \mathbb{R}^n \to \mathbb{R}$, $p,q,\tau \in \mathcal{P}_0$ and $\Phi$ and $\varphi$ satisfy (11) and (12), respectively and we put $\varphi_v = 2^{vn}\varphi(2^{v})$.

(i) The Besov-type space $\widetilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{\widetilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} = \sup_{P \in \mathcal{Q}} \left\| \left( \frac{2^{v\alpha(\cdot)}\varphi_v*f}{|P|^{1/p(\cdot)}}\chi_P \right)_{v \geq v_P^*} \right\|_{p(\cdot)(L^q(\cdot))} < \infty,
$$

where $\varphi_0$ is replaced by $\Phi$.

(ii) The Besov-type space $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} = \sup_{P \in \mathcal{Q}} \left\| \left( \frac{2^{v\alpha(\cdot)}\varphi_v*f}{\|\chi_P\|_{\tau(\cdot)}}\chi_P \right)_{v \geq v_P^*} \right\|_{p(\cdot)(L^q(\cdot))} < \infty,
$$

where $\varphi_0$ is replaced by $\Phi$.

Using the system $\{\varphi_v\}_{v \in \mathbb{N}_0}$ we can define the norm

$$
\|f\|_{B_{p,q}^{\alpha,\tau}} = \sup_{P \in \mathcal{Q}} \left( \sum_{v = v_P^*}^{\infty} 2^{v\alpha \|f\|_{L^p(\cdot)}} \right)^{1/q}
$$

for constants $\alpha$ and $p,q \in (0,\infty]$. The Besov-type space $B_{p,q}^{\alpha,\tau}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{B_{p,q}^{\alpha,\tau}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $\{\varphi_v\}_{v \in \mathbb{N}_0}$ (up to equivalence of quasi-norms). Further details on the classical theory of these spaces can be found in [8], [9] and [27]; see also [11] for recent developments.

One recognizes immediately that if $\alpha, \tau, p$ and $q$ are constants, then $\widetilde{B}_{p,q}^{\alpha,\tau} = B_{p,q}^{1/p,\tau}$ and $B_{p,q}^{\alpha,\tau} = B_{p,q}^{\alpha,\tau}$. Then, $q \equiv \infty$ the Besov-type space $\widetilde{B}_{p,q}^{\alpha,\infty}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\sup_{P \in \mathcal{Q},v \geq v_P^*} \left\| \frac{2^{v\alpha(\cdot)}\varphi_v*f}{|P|^{1/p(\cdot)}}\chi_P \right\|_{p(\cdot)} < \infty
$$

and the Besov-type space $B_{p,q}^{\alpha,\infty}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\sup_{P \in \mathcal{Q},v \geq v_P^*} \left\| \frac{2^{v\alpha(\cdot)}\varphi_v*f}{\|\chi_P\|_{\tau(\cdot)}}\chi_P \right\|_{p(\cdot)} < \infty.
$$

Let $B_J$ be any ball of $\mathbb{R}^n$ with radius $2^{-J}$, $J \in \mathbb{Z}$. In the definition of the spaces $B_{p,q}^{\alpha,\tau}$ and $\widetilde{B}_{p,q}^{\alpha,\tau}$ if we replace the dyadic cubes $P$ by the balls $B_J$, then we obtain equivalent quasi-norms. From these if we replace dyadic cubes $P$ in Definition 1 by arbitrary cubes $P$, we then obtain equivalent quasi-norms.

Sometimes it is of great service if one can restrict $\sup_{P \in \mathcal{Q}}$ in the definition to a supremum taken with respect to dyadic cubes with side length $\leq 1$. 


Lemma 11 Let \( \alpha \in C^\text{log}_{0} \) and \( p, q, \tau \in \mathcal{P}_0^\text{log} \) with \( \tau_\infty \in (0, p^-] \) and \( 0 < q^+ < \infty \).

(i) A tempered distribution \( f \) belongs to \( \tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)} \) if and only if,

\[
\| f \|_{\tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)}} = \sup_{\{ P \in \mathcal{P} \mid |P| \leq 1 \}} \left( \frac{2^{\alpha(\cdot)} \varphi_{v^*} \ast f}{|P|^{1/p(-)} X_P} \right)_{v \geq v_P} < \infty.
\]

Furthermore, the quasi-norms \( \| f \|_{\tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)}} \) and \( \| f \|^\#_{\tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)}} \) are equivalent.

(ii) A tempered distribution \( f \) belongs to \( B^{\alpha(\cdot), \tau(\cdot)}_{p(-), q(-)} \) if and only if,

\[
\| f \|^\#_{B^{\alpha(\cdot), \tau(\cdot)}_{p(-), q(-)}} = \sup_{\{ P \in \mathcal{P} \mid |P| \leq 1 \}} \left( \frac{2^{\alpha(\cdot)} \varphi_{v^*} \ast f}{\|X_P\|_{\tau(\cdot)}} \right)_{v \geq v_P} < \infty.
\]

Furthermore, the quasi-norms \( \| f \|_{B^{\alpha(\cdot), \tau(\cdot)}_{p(-), q(-)}} \) and \( \| f \|^\#_{B^{\alpha(\cdot), \tau(\cdot)}_{p(-), q(-)}} \) are equivalent.

The proof is similar to that of [12]. We omit the details.

Remark 1 (i) We like to point out that this result with fixed exponents is given in [27, Lemma 2.2] with \( 1/\tau \) in place of \( \tau \).

(ii) Let \( \alpha \in C^\text{log}_{0} \) and \( p, q, \tau \in \mathcal{P}_0^\text{log} \) with \( \tau_\infty \in (0, p^-] \) and \( 0 < q^+ < \infty \). If \( (1/\tau - 1/p^-) > 0 \) or \( (1/\tau - 1/p^-) \geq 0 \) and \( q \equiv \infty \). As in [12], we obtain \( B^{\alpha(-), \tau(\cdot)}_{p(-), q(-)} = B^{\alpha(-)+n(1/\tau(-)-1/p(-))}_{\infty, \infty} \) and \( \tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)} = B^{\alpha(-)}_{\infty, \infty} \), with equivalent norms. Also we have

\[
2^{n(\alpha(x)+n(1/\tau(x)-1/p(x)))} |\varphi_{v^*} \ast f(x)| \lesssim \| f \|^\#_{B^{\alpha(-), \tau(\cdot)}_{p(-), q(-)}}
\]

for any \( x \in \mathbb{R}^n \), \( \alpha \in C^\text{log}_{0} \) and \( p, q, \tau \in \mathcal{P}_0^\text{log} \).

(iii) It is clear that if \( \alpha \) and \( p \) are constants, then \( \tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)} = F^\alpha \), see [10].

Let \( \alpha \in C^\text{log}_{0}, p, q \in \mathcal{P}_0^\text{log} \) and \( \alpha_0 < \alpha^- \). We obtain

\[
\tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)} \hookrightarrow B^{\alpha_0, p(-)}_{p(-)} = B^{\alpha_0}_{\infty, \infty} \hookrightarrow S'(\mathbb{R}^n).
\]

Let \( \alpha^+ < \alpha_1 \). We obtain

\[
S(\mathbb{R}^n) \hookrightarrow B^{\alpha_1}_{\infty, \infty} = B^{\alpha_1, p(-)}_{p(-), q(-)} \hookrightarrow \tilde{B}^{\alpha_1, p(-)}_{p(-), q(-)}.
\]

We use \( A^{\alpha(-), \tau(\cdot)}_{p(-), q(-)} \) to denote either \( \tilde{B}^{\alpha(-), p(-)}_{p(-), q(-)} \) or \( B^{\alpha(-), \tau(\cdot)}_{p(-), q(-)} \).

Theorem 1 Let \( \alpha \in C^\text{log}_{0} \) and \( p, q, \tau \in \mathcal{P}_0^\text{log} \) with \( 0 < q^+ < \infty \). Then

\[
S(\mathbb{R}^n) \hookrightarrow A^{\alpha(-), \tau(\cdot)}_{p(-), q(-)} \hookrightarrow S'(\mathbb{R}^n).
\]

Similar arguments of [12] can be used to prove the following Sobolev-type embeddings.
Theorem 2 Let $\alpha_0, \alpha_1 \in C_{\text{loc}}^{\log}$ and $p_0, p_1, q \in P_0^{\log}$ with $0 < q^+ < \infty$. If $\alpha_0 > \alpha_1$ and $\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$ with $\left(\frac{p_0}{p_1}\right)^- < 1$, then

$$\tilde{B}^{\alpha_0(\cdot), p(\cdot)}_{p_0(\cdot), q(\cdot)} \hookrightarrow \tilde{B}^{\alpha_1(\cdot), p(\cdot)}_{p_1(\cdot), q(\cdot)}.$$ 

Notice that the case of $B^{\alpha(\cdot), \tau(\cdot)}$ spaces is given in [12].

Let $\Phi$ and $\varphi$ satisfy, respectively (11) and (12). By [16, pp. 130–131], there exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (11) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (12) such that for all $\xi \in \mathbb{R}^n$

$$\mathcal{F} \tilde{\Phi}(\xi) \mathcal{F} \Psi(\xi) + \sum_{j=1}^{\infty} \mathcal{F} \tilde{\varphi}(2^{-j} \xi) \mathcal{F} \psi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^n. \quad (14)$$

Furthermore, we have the following identity for all $f \in \mathcal{S}'(\mathbb{R}^n)$; see [16, (12.4)]

$$f = \Psi * \tilde{\Phi} * f + \sum_{v=1}^{\infty} \psi_v * \tilde{\varphi}_v * f = \sum_{m \in \mathbb{Z}^n} \tilde{\Phi} * f(m) \Psi(\cdot - m) + \sum_{v=1}^{\infty} 2^{-vn} \sum_{m \in \mathbb{Z}^n} \tilde{\varphi}_v * f(2^{-v}m) \psi_v(\cdot - 2^{-v}m).$$

Recall that the $\varphi$-transform $S_\varphi$ is defined by setting $(S_\varphi)_{0,m} = \langle f, \Psi_m \rangle$ where $\Psi_m(x) = \Psi(x - m)$ and $(S_\varphi)_{v,m} = \langle f, \varphi_{v,m} \rangle$ where $\varphi_{v,m}(x) = 2^{vn/2} \varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse $\varphi$-transform $T_\psi$ is defined by

$$T_\psi \lambda = \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \psi_{v,m},$$

where $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, see [16].

For any $\gamma \in \mathbb{Z}$, we put

$$\|f\|_{B^{\alpha}(\cdot), p(\cdot)}^{*} = \sup_{P \in \mathcal{Q}, |P| \leq 1} \left\| \frac{2^{n\alpha(\cdot)} \varphi_{v,P} * f}{|P|^{1/p(\cdot)}} \chi_P \right\|_{L^p(P^{\gamma}(\cdot), \mathcal{L}^{\gamma}(P^{\gamma}(\cdot)))} < \infty$$

and

$$\|f\|_{B^{\alpha}(\cdot), \tau(\cdot)}^{*} = \sup_{P \in \mathcal{Q}} \left\| \frac{2^{n\alpha(\cdot)} \varphi_{v,P} * f}{\|\chi_P\|_{\tau(\cdot)}} \right\|_{L^p(P^{\gamma}(\cdot), \mathcal{L}^{\gamma}(P^{\gamma}(\cdot)))} < \infty,$$

where $\varphi_{-\gamma}$ is replaced by $\Phi_{-\gamma}$.

**Lemma 12** Let $\alpha \in C_{\text{loc}}^{\log}$ and $p, q, \tau \in P_0^{\log}$ and $0 < q^+ < \infty$. The quasi-norms $\|f\|_{A^{\alpha}(\cdot), p(\cdot), q(\cdot)}^{*}$ and $\|f\|_{A^{\alpha}(\cdot), \tau(\cdot), q(\cdot)}^{*}$ are equivalent with equivalent constants depending on $\gamma$.

**Proof.** By similarity, we only consider $B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}$ and the case $\gamma > 0$. First let us prove that $\|f\|_{B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}}^{*} \leq c \|f\|_{B^{\alpha(\cdot), p(\cdot), q(\cdot)}}^{*}$. By the scaling argument, it suffices to consider the
case \( \|f\|_{B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1 \) and show that the modular of \( f \) on the left-hand side is bounded. In particular, we will show that
\[
\sum_{v = v_P - \gamma}^{\infty} \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{\|X_P\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \chi_P \leq c
\]
for any dyadic cube \( P \). As in [27, Lemma 2.6], it suffices to prove that for all dyadic cube \( P \) with \( l(P) \geq 1 \),
\[
I_P = \sum_{v = -\gamma}^{0} \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{\|X_P\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \chi_P \leq c
\]
and for all dyadic cube \( P \) with \( l(P) < 1 \),
\[
J_P = \sum_{v = v_P - \gamma}^{v_P - 1} \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{\|X_P\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \chi_P \leq c.
\]
The estimate of \( I_P \), clearly follows from the inequality
\[
\left\| \frac{\varphi_v * f}{\|X_P\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c
\]
for any \( v = -\gamma, \ldots, 0 \) and any dyadic cube \( P \) with \( l(P) \geq 1 \). This claim can be reformulated as showing that
\[
\left\| \frac{\varphi_v * f}{\|X_P\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c. \tag{15}
\]
By (11) and (12), there exist \( \omega_v \in S(\mathbb{R}^n), v = -\gamma, \ldots, -1 \) and \( \eta_1, \eta_2 \in S(\mathbb{R}^n) \) such that
\[
\varphi_v = \omega_v * \Phi, \quad v = -\gamma, \ldots, -1 \quad \text{and} \quad \varphi = \varphi_0 = \eta_1 * \Phi + \eta_2 * \varphi_1.
\]
Hence \( \varphi_v * f = \omega_v * \Phi * f \) for \( v = -\gamma, \ldots, -1 \) and \( \varphi_0 * f = \eta_1 * \Phi * f + \eta_2 * \varphi_1 * f \). Applying Lemma 4, (6) and the fact that \( f \in L^{p(\cdot)}_{\tau(\cdot)} \), we then have
\[
J_P \leq c \sum_{v = v_P(\gamma)}^{v_P - 1} \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{\|X_P(\gamma)\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \chi_P(\gamma) \leq c.
\]
If \( 1 \leq v_P \leq \gamma \), we write \( J_P = \sum_{v = v_P - \gamma}^{v_P - 1} \cdots + \sum_{v = 0}^{v_P - 1} \cdots = J_P^1 + J_P^2 \). Let \( P(2^{v_P}) \) the dyadic cube containing \( P \) with \( l(P(2^{v_P})) = 2^{v_P} l(P) = 1 \), by the fact that
\[
\left\| \frac{\chi_P(2^{v_P})}{\|X_P\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 2^{nv_P/\tau^-} \leq 2^{n\gamma/\tau^-},
\]
see Lemma 3 we have
\[
J_P^2 \leq c \sum_{v = v_P(2^{v_P})}^{v_P - 1} \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{\|X_P(2^{v_P})\|_{\tau(\cdot)}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \chi_P(2^{v_P}) \leq c.
\]
By a similar argument to the estimate for $I_P$, we see that $J_P^1 \leq c$. For the converse estimate, it suffices to show that
\[
\left\| \frac{\Phi * f}{\| \chi_P \|_{\tau(\cdot)}(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c
\]
for all $P \in \mathcal{Q}$ with $l(P) \geq 1$ and all $f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ with $\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1$. This claim can be reformulated as showing that $\left\| \frac{\Phi * f}{\| \chi_P \|_{\tau(\cdot)}(\cdot)} \right\|_{p(\cdot)} \leq c$. Using the fact that there exist $\rho_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, \cdots, 1$, such that $\Phi * f = \rho_{-\gamma} \ast \Phi_{-\gamma} * f + \sum_{\gamma=1}^1 \rho_v \ast \varphi_v * f$, see [10] p. 130. Applying Lemma 4 we obtain
\[
\left\| \rho_{-\gamma} \ast \Phi_{-\gamma} * f \right\|_{L^{p(\cdot)}_{\tau(\cdot)}} \leq c \left\| \rho_{-\gamma} \ast \Phi_{-\gamma} * f \right\|_{L^{p(\cdot)}_{\tau(\cdot)}} \leq c,
\]
and
\[
\left\| \varphi_v \ast f \right\|_{L^{p(\cdot)}_{\tau(\cdot)}} \leq c \left\| \varphi_v \ast f \right\|_{L^{p(\cdot)}_{\tau(\cdot)}} \leq c, \quad v = 1 - \gamma, \cdots, 1
\]
by using (6) and the fact that $\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1$. The proof is complete. ■

**Definition 2** Let $p, q, \tau \in \mathcal{P}_0$ and let $\alpha : \mathbb{R}^n \to \mathbb{R}$. Then for all complex valued sequences $\lambda = \{\lambda_{e,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ we define
\[
\overline{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} = \{ \lambda : \|\lambda\|_{\overline{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} < \infty \},
\]
where
\[
\|\lambda\|_{\overline{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} = \sup_{P \in \mathcal{Q}} \left\| \left( \sum_{m \in \mathbb{Z}^n} \frac{2^{v(\alpha) + n/2} \lambda_{e,m} \chi_{v,m}}{P_{1/p(\cdot)}} \chi_P \right) \right\|_{L^{p(\cdot)}_{\tau(\cdot)}}
\]
and
\[
b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} = \{ \lambda : \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} < \infty \}
\]
where
\[
\|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} = \sup_{P \in \mathcal{Q}} \left\| \left( \sum_{m \in \mathbb{Z}^n} \frac{2^{v(\alpha) + n/2} \lambda_{e,m} \chi_{v,m}}{P_{1/p(\cdot)}} \chi_P \right) \right\|_{L^{p(\cdot)}_{\tau(\cdot)}}.
\]
If we replace dyadic cubes $P$ by arbitrary balls $B_J$ of $\mathbb{R}^n$ with $J \in \mathbb{Z}$, we then obtain equivalent quasi-norms, where the supremum is taken over all $J \in \mathbb{Z}$ and all balls $B_J$ of $\mathbb{R}^n$. In the definition of $\overline{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$, the supremum can be taken over all dyadic cube $P$, with $|P| \leq 1$. Similarly, we use $a_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ to denote either $b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ or $\overline{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$. Let $\alpha_0, \alpha_1 \in C_{loc}^{2\log}$ and $p_0, p_1, q, \tau \in \mathcal{P}_0^{lo\geq0}$ with $0 < q^+ < \infty$. If $0 < \alpha_1$ and $\alpha_0(x) - \frac{\tau_n}{p_0(x)} = \alpha_1(x) - \frac{\tau_n}{p_1(x)}$, then we can prove the following Sobolev-type embeddings
\[
a_{p_0(\cdot),q(\cdot)}^{\alpha_0(\cdot),\tau(\cdot)} \hookrightarrow a_{p_1(\cdot),q(\cdot)}^{\alpha_1(\cdot),\tau(\cdot)}.
\]
Lemma 13  Let $\alpha \in C_{log}^{\log}$, $p,q,\tau \in P_{0}^{\log}$, $0 < q^{+} < \infty$, $v \in \mathbb{N}_{0}$, $m \in \mathbb{Z}^{n}$, $x \in Q_{v,m}$ and $\lambda \in a_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$. Then there exists $c > 0$ independent of $v$ and $m$ such that

$$|\lambda_{v,m}| \leq c \, 2^{-v(\alpha(x)+n/2)} \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \|X_{v,m}\|_{\tau(\cdot)} \|X_{v,m}\|_{p(\cdot)}^{-1}$$

and

$$|\lambda_{v,m}| \leq c \, 2^{-v(\alpha(x)+n/2)} \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}.$$ 

**Proof.** By similarity, we only consider $b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$. Let $\lambda \in b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$, $v \in \mathbb{N}_{0}$, $m \in \mathbb{Z}^{n}$ and $x \in Q_{v,m}$, with $Q_{v,m} \in \mathcal{Q}$. Then $|\lambda_{v,m}|^{p^{-}} = |Q_{v,m}|^{-1} \int_{Q_{v,m}} |\lambda_{v,m}|^{p^{-}} \chi_{v,m}(y)dy$. Using the fact that $2^{v(\alpha(x)-\alpha(y))} \leq c$ for any $x,y \in Q_{v,m}$ and $|Q_{v,m}|^{1/p(x)} \approx \|X_{v,m}\|_{p(\cdot)}$; see (11), we obtain

$$\frac{2^{v(\alpha(x)+n/2)p^{\tau}}}{|Q_{v,m}|^{p^{-}/p(x)}} |\lambda_{v,m}|^{p^{-}} \leq c |Q_{v,m}|^{-1} \int_{Q_{v,m}} \frac{2^{v(\alpha(y)+n/2)p^{\tau}}}{|\lambda_{v,m}|^{p^{-}} \chi_{v,m}(y)dy} \leq c |Q_{v,m}|^{-1} \int_{Q_{v,m}} \frac{2^{v(\alpha(y)+n/2)p^{\tau}}}{|\lambda_{v,m}|^{p^{-}} \chi_{v,m}(y)dy}.$$ 

Applying Hölder’s inequality to estimate this expression by

$$c |Q_{v,m}|^{-1} \left\| \frac{2^{v(\alpha(y)+n/2)p^{\tau}}}{|\lambda_{v,m}|^{p^{-}} \chi_{v,m}} \right\|_{p^{-}/p^{-}} \left\| \chi_{v,m} \right\|_{p^{-}/p^{-}} \leq c \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \|X_{v,m}\|_{p(\cdot)}^{-1},$$ 

where we have used (3). Therefore for any $x \in Q_{v,m}$

$$|\lambda_{v,m}| \leq c \, 2^{-v(\alpha(x)+n/2)} |Q_{v,m}|^{1/p(x)} \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \|X_{v,m}\|_{\tau(\cdot)} \|X_{v,m}\|_{p(\cdot)}^{-1} \leq c \, 2^{-v(\alpha(x)+n/2)} \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}},$$

again by (3), which completes the proof. 

Lemma 14  Let $\alpha \in C_{log}^{\log}$ and $p,q,\tau \in P_{0}^{\log}$ and $\Psi, \psi \in \mathcal{S}^{\prime}(\mathbb{R}^{n})$ satisfy, respectively, (11) and (12). Then for all $\lambda \in a_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$

$$T_{\psi} \lambda = \sum_{m \in \mathbb{Z}^{n}} \lambda_{0,m} \Psi_{m} + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^{n}} \lambda_{v,m} \psi_{v,m},$$

converges in $\mathcal{S}^{\prime}(\mathbb{R}^{n})$; moreover, $T_{\psi} : a_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \rightarrow \mathcal{S}^{\prime}(\mathbb{R}^{n})$ is continuous.

**Proof.** By similarity, we only consider $b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$. Let $\lambda \in b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ and $\phi \in \mathcal{S}(\mathbb{R}^{n})$. Observe that

$$|\lambda_{v,m}| \leq \frac{c \, 2^{-v(\alpha^{+}+n/2)} \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \|X_{v,m}\|_{\tau(\cdot)}}{\|X_{v,m}\|_{p(\cdot)}} \leq c \, 2^{-v(1/p^{\tau}+1/2-\alpha^{+}/n-1/\tau^{+})} \|\lambda\|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}$$

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for all dyadic cubes $Q_{v,m}$. Let $M > \max(n, n/p - \alpha^- - n/\tau^+ - n)$. We see that,
\[
\sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}||\langle \Psi_m, \phi \rangle| \leq c \|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \sum_{m \in \mathbb{Z}^n} \int |\Psi(x - m)||\phi(x)|dx \\
\leq c \|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \|\Psi\|_{S_{2M}} \|\phi\|_{S_{2M}} \sum_{m \in \mathbb{Z}^n} (1 + |m|)^{-n-M}.
\]
On the other hand, by [27, Lemma 2.4], we obtain
\[
\sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}||\langle \psi_{v,m}, \phi \rangle| \\
\leq c \|\psi\|_{S_{M+1}} \|\phi\|_{S_{M+1}} \|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \frac{2^{-vn(a^- + 1/\tau^+ - 1/p^- + 1/M)/2n}}{(1 + |2^{-v}m|)^{n+M}} \\
\leq c \|\psi\|_{S_{M+1}} \|\phi\|_{S_{M+1}} \|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}},
\]
which completes the proof.

For a sequence $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}, 0 < r \leq \infty$ and a fixed $d > 0$, set
\[
\lambda^*_{v,m,r,d} = \left(\sum_{h \in \mathbb{Z}^n} \frac{|\lambda_{v,h}|^r}{(1 + 2^v|2^{-v}h - 2^{-v}m|)^d}\right)^{1/r}
\]
and $\lambda^*_{r,d} = \{\lambda^*_{v,m,r,d} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

**Lemma 15** Let $\alpha \in C^\log_{loc}, p, q, \tau \in \mathcal{P}^\log_0, 0 < q^+ < \infty$ and $a = \max(2c_{\log}(q) + c_{\log}(\alpha), 2(\frac{1}{q} - \frac{1}{q^+}) + \alpha^+ - \alpha^-)$. Then
\[
\|\lambda^*_{r,d}\|_{a^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \approx \|\lambda\|_{a^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}}
\]
where
\[
d > \left\{ \begin{array}{ll}
\quad n + a_k + n/\tau^- & \text{if } a^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)} = b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}
\quad n + a_k + c_{\log}(\frac{1}{p}) & \text{if } a^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)} = b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}
\end{array} \right.
\]

**Proof.** By similarity, we only consider $b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}$. Obviously,
\[
\|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \leq \|\lambda^*_{r,d}\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}}
\]
Let us prove that $\|\lambda^*_{r,d}\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \leq c \|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}}$. By the scaling argument, it suffices to consider the case $\|\lambda\|_{b^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} = 1$ and show that the modular of a constant times the sequence on the left-hand side is bounded. In particular, we will show that
\[
\sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(a^- + 1/\tau^+ - 1/p^- + 1/M)/2n} \lambda^*_{v,m,r,d} \lambda_{v,m} \chi_{B_J} \|\psi\|_{r(\cdot)} \leq 1
\]
for any ball $B_J$ centered at $x_0 \in \mathbb{R}^n$ and of radius $2^{- J}$, $J \in \mathbb{Z}$. It suffices to prove that

$$
\| c \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)+n/2)} \lambda_{\nu,m,r,d} \chi_{B,J} \|_{q(\cdot)} \leq \delta, \quad \nu \geq 2^J,
$$

where, $\epsilon = (n - d + a_k + n/\tau^-)/2$. This claim can be reformulated as showing that

$$
\| c \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)+n/2)} \lambda_{\nu,m,r,d} \chi_{B,J} \|_{q(\cdot)} \leq \epsilon \quad \| \chi_{B,J} \|_{q(\cdot)} = 1,
$$

which is equivalent to

$$
\| \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)+n/2)} \lambda_{\nu,m,r,d} \chi_{B,J} \|_{q(\cdot)} \leq \epsilon.
$$

(16)

For each $k \in \mathbb{N}_0$ we define $\Omega_k = \{ h \in \mathbb{Z}^n : 2^{k-1} < 2^v |2^{-v} h - 2^{-v} m| \leq 2^k \}$ and $\Omega_0 = \{ h \in \mathbb{Z}^n : 2^v |2^{-v} h - 2^{-v} m| \leq 1 \}$. Then for any $x \in Q_{v,m} \cap B_J$, $\sum_{h \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)+n/2)} \lambda_{\nu,m,r,d} \chi_{B,J}$ can be rewritten as

$$
\sum_{k=0}^{\infty} \sum_{h \in \Omega_k} \delta^{- \frac{v}{\tau^-}} 2^{v r \alpha(x)} \frac{\lambda_{\nu,h}}{(1 + 2^v |2^{-v} h - 2^{-v} m|)^d} \leq c \sum_{k=0}^{\infty} 2^{-dk} \sum_{h \in \Omega_k} \delta^{- \frac{v}{\tau^-}} 2^{v r \alpha(x)} \frac{\lambda_{\nu,h}}{d}.
$$

(17)

Let $x \in Q_{v,m} \cap B_J$ and $y \in \bigcup_{z \in \Omega_k} Q_{v,z}$, then $y \in Q_{v,z}$ for some $z \in \Omega_k$ and $2^{k-1} < 2^v |2^{-v} z - 2^{-v} m| \leq 2^k$. From this it follows that $y$ is located in some ball $B(x, 2^{k-v+h_n})$. In addition, from the fact that

$$
|y - x_0| \leq |y - x| + |x - x_0| \leq \sqrt{n} 2^{-J} + 2^{k-v+h_n} \leq 2^{k-J+h_n}, \quad h_n \in \mathbb{N},
$$

we have $y$ is located in some ball $B_{J-k-h_n}$. Since $1/q$ is log-Hölder continuous and $\delta \in [2^{-v}, 1 + 2^{-v}]$, we have

$$
\delta \frac{1}{q(x)} - \frac{1}{q(y)} \leq 2 \frac{1}{q(x)} - \frac{1}{q(y)} \leq 2^{c_{\log}(q)(2v+1)} \frac{\epsilon^{- \frac{v}{\tau^-}} 2^{v r \alpha(x)}}{d} \leq c 2^{c_{\log}(q)(2v+1)}
$$

where $c_{\log}(q)$ is the constant associated with the log-Hölder continuity of $1/q$. Therefore, we have

$$
\delta \frac{1}{q(x)} - \frac{1}{q(y)} \leq c 2^{c_{\log}(q)(2v+1)}
$$

and

$$
\delta \frac{1}{q(x)} - \frac{1}{q(y)} \leq c 2^{c_{\log}(q)(2v+1)}
$$

Thus, we have

$$
\delta \frac{1}{q(x)} - \frac{1}{q(y)} \leq c 2^{c_{\log}(q)(2v+1)}
$$
for any \( k < \max(0, v - h_n) \) and any \( y \in B(x, 2^{k-v+h_n}) \). If \( k \geq \max(0, v - h_n) \) then since again \( \delta \in [2^{-v}, 1 + 2^{-v}], \delta \frac{1}{\max(q,v)} - \frac{1}{\min(q,v)} \leq c 2\left\lfloor \frac{1}{\max(q,v)} - \frac{1}{\min(q,v)} \right\rfloor (2^{v+1}) \leq c 2^{2\left(\frac{1}{q} - \frac{1}{v}\right)} \). Also since \( \alpha \) is log-Hölder continuous we can prove that

\[
2^{v(\alpha(x)-\alpha(y))} \leq c \times \begin{cases} 2^{c\log(\alpha)}k & \text{if } k < \max(0, v - h_n), \\ 2^{(\alpha^+-\alpha^-)k} & \text{if } k \geq \max(0, v - h_n), \end{cases}
\]

where \( c > 0 \) not depending on \( v \) and \( k \). Therefore, (17) does not exceed

\[
\sum_{k=0}^{\infty} 2^{(n-d+a)k+(v-k)n} \int_{B(x,2^{k-v}+h_n)} \delta - \frac{1}{\min(q,v)} 2^{\alpha(y)r} \sum_{h \in \Omega_k} |\lambda_{v,h}|^{r} \chi_{v,h}(y) \chi_{B_{J-k-h,n}} dy
\]

\[
\leq c \sum_{k=0}^{\infty} 2^{(n-d+a)k} \mathcal{M} \left( \sum_{h \in \Omega_k} \delta - \frac{1}{\min(q,v)} 2^{\alpha(y)r} |\lambda_{v,h}|^{r} \chi_{v,h} \chi_{B_{J-k-h,n}} \right)(x).
\]

Hence the left-hand side of (15) is bounded by

\[
c \left\| \frac{1}{\|X_{B_{J}}\|_{\tau(\cdot)}} \sum_{k=0}^{\infty} 2^{(n-d+a)k} \mathcal{M} \left( \sum_{h \in \Omega_k} \delta - \frac{1}{\min(q,v)} 2^{\alpha(y)r} |\lambda_{v,h}|^{r} \chi_{v,h} \chi_{B_{J-k-h,n}} \right) \right\|_{L^{p(r)}(\tau)}^{1/r}
\]

\[
\leq c \left( \sum_{k=0}^{\infty} 2^{ek} \right)^{1/r} \left( \sum_{i=1}^{\infty} 2^{\epsilon i} \sum_{h \in \Omega_i} \delta - \frac{1}{\min(q,v)} 2^{\alpha(y)r} |\lambda_{v,h}|^{r} \chi_{v,h} \chi_{B_{J-k-h,n}} \right)^{1/r}
\]

\[
\leq c \left( \sum_{k=0}^{\infty} 2^{ek} \right)^{1/r} ,
\]

where on the first estimate we use Lemma 3 and the boundedness of the maximal function on \( L^{p/r} \) (we choose \( r < p^- \)), and for the last estimate we use the fact that

\[
\left\| \sum_{i=0}^{\infty} 2^{\epsilon i} \sum_{h \in \Omega_i} \delta - \frac{1}{\min(q,v)} 2^{\alpha(y)r} |\lambda_{v,h}|^{r} \chi_{v,h} \chi_{B_{J-k-h,n}} \right\|_{L^{p(r)}(\tau)} \approx 1
\]

and \( d \) sufficiently large such that \( d > n + a_k + n/\tau^- \). The proof of the lemma is thus complete.

Theorem 3. Let \( \alpha \in C^{\log}_{loc} \) and \( p, q, \tau \in \mathcal{P}^{\log}_{0}, 0 < q^+ < \infty \). Suppose that \( \Phi, \Psi \in \mathcal{S}(\mathbb{R}^n) \) satisfying (1) and \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \) satisfy (12) such that (14) holds. The operators \( S_{\varphi} : A^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to A^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \) and \( T_{\psi} : A^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to A^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \) are bounded. Furthermore, \( T_{\psi} \circ S_{\varphi} \) is the identity on \( A^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \).

Proof. By similarity, we only consider \( A^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \) and \( B^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \). For any \( f \in \mathcal{S}(\mathbb{R}^n) \) we put \( \sup(f) = \{ \sup_{v,m}(f) : v \in \mathbb{N}_0, m \in \mathbb{Z}^n \} \) where

\[
\sup_{v,m}(f) = 2^{-vn/2} \sup_{y \in Q_{v,m}} |\varphi_v * f(y)|
\]
if \( v \in \mathbb{N}, m \in \mathbb{Z}^n \) and
\[
\sup(f) = \sup_{y \in Q_{0,m}} |\Phi * f(y)|
\]
if \( m \in \mathbb{Z}^n \). For any \( \gamma > 0 \), we define the sequence \( \inf_{\alpha,\gamma}(f) = \{\inf_{v,m,\gamma}(f) : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \) by setting
\[
\inf_{v,m,\gamma}(f) = 2^{-\gamma m/2} \sup_{h \in \mathbb{Z}^n} \{ \inf_{y \in Q_{v+h,\gamma}} |\tilde{\varphi}_v * f(y)| : Q_{v+\gamma,h} \cap Q_{v,m} \neq \emptyset \}
\]
if \( v \in \mathbb{N}, m \in \mathbb{Z}^n \) and
\[
\inf_{0,m,\gamma}(f) = \inf_{h \in \mathbb{Z}^n} \{ \inf_{y \in Q_{\gamma,h}} |\Phi * f(y)| : Q_{\gamma,h} \cap Q_{0,m} \neq \emptyset \}
\]
if \( m \in \mathbb{Z}^n \). Here \( \tilde{\varphi}_j(x) = 2^{jn} \varphi(-2^j x) \) and \( \tilde{\Phi}(x) = \Phi(-x) \). As in Lemma A.5 of [16] we obtain
\[
\|\inf_{\gamma}(f)\|_{B^{\alpha}(\mathbb{R}^n)} \leq c \|f\|_{B^{\alpha}(\mathbb{R}^n)}
\]
for any \( \alpha \in C^\infty_{\text{loc}} \) and \( p, q, r \in \mathbb{P}_0 \), \( 0 < q^+ < \infty \) and \( \gamma > 0 \) sufficiently large. Indeed, we have
\[
\|\inf_{\gamma}(f)\|_{B^{\alpha}(\mathbb{R}^n)} = c \sup_{P \in \mathcal{Q}} \left\| \left( \sum_{m \in \mathbb{Z}^n} 2^{j(n/2+m/2)} \inf_{j=m,\gamma} f \chi_{j=m,\gamma} \chi_P \right) \right\|_{L^q(\mathbb{R}^n)}
\]
Define a sequence \( \{\lambda_{i,k}\}_{i \in \mathbb{N}_0, k \in \mathbb{Z}^n} \) by setting \( \lambda_{i,k} = 2^{-in/2} \inf_{y \in Q_{i,k}} |\tilde{\varphi}_i * f(y)| \) and \( \lambda_{0,k} = \inf_{y \in Q_{\gamma,k}} |\tilde{\Phi} * f(y)| \). We have
\[
\inf_{j=m,\gamma} f = 2^{\gamma n/2} \sup_{h \in \mathbb{Z}^n} \{ \lambda_{j,h} : Q_{j,h} \cap Q_{j,m} \neq \emptyset \}
\]
and
\[
\inf_{0,m,\gamma} f = \sup_{h \in \mathbb{Z}^n} \{ \lambda_{0,h} : Q_{\gamma,h} \cap Q_{0,m} \neq \emptyset \}.
\]
Let \( h \in \mathbb{Z}^n \) with \( Q_{j,h} \cap Q_{j,m} \neq \emptyset \). Then \( \lambda_{j,h} \leq c \ 2^{\gamma n/r} \lambda_{j,k,r,d} \) for any \( k \in \mathbb{Z}^n \) with \( Q_{j,k} \cap Q_{j,m} \neq \emptyset \). Hence
\[
\sum_{m \in \mathbb{Z}^n} \inf_{j=m,\gamma} f \chi_{j=m,\gamma} \leq c \sum_{k \in \mathbb{Z}^n} \lambda_{j,k,r,d} \chi_{j,k}
\]
and
\[
\|\inf_{\gamma}(f)\|_{B^{\alpha}(\mathbb{R}^n)} \leq c \sup_{P \in \mathcal{Q}} \left\| \left( \sum_{k \in \mathbb{Z}^n} 2^{j(n/2+m/2)} \lambda_{j,k,r,d} \chi_{j,k} \chi_P \right) \right\|_{L^q(\mathbb{R}^n)}
\]
Notice that \( P = \bigcup_{m=1}^{2^n} P_m \) where \( \{P_m\}_{m=1}^{2^n} \) are disjoint dyadic cubes with side length \( l(P_m) = 2^{-(v_p+\gamma)} \). Therefore, taking \( 0 < s < \frac{1}{2} \min(p^-, q^- 2) \) and applying Lemmas b
and \[15\]

\[
\|\inf_{\gamma}(f)\|_{B_{p,q}(\phi,\tau)}^s \leq c \sum_{m=1}^{2^n} \left( \sup_{P \in \mathbb{Q}} \left( \sum_{k \in \mathbb{Z}^n} 2^{j(r(m-n/2)+1/2)} \lambda_{j,k} \chi_P \right) \right)^{\gamma} \leq c \sup_{P \in \mathbb{Q}} \left( \sum_{k \in \mathbb{Z}^n} 2^{j(\alpha(m-n/2)+1/2)} \lambda_{j,k} \chi_P \right)^{\gamma}.
\]

By Lemma 12 we obtain

\[
\|\inf_{\gamma}(f)\|_{B_{p,q}(\phi,\tau)}^s \leq c \|f\|_{B_{p,q}(\phi,\tau)}^s \leq c \|f\|_{B_{p,q}(\phi,\tau)}^s.
\]

Applying Lemma A.4 of [16], see also Lemma 8.3 of [2], we obtain \(\inf_{\gamma}(f)_{r,d} \approx \sup(f)_{r,d}^{\ast}\). Hence for \(\gamma > 0\) sufficiently large we obtain by applying Lemma 15

\[
\|\inf_{\gamma}(f)\|_{B_{p,q}(\phi,\tau)}\approx \sup(f)_{r,d}^{\ast} \approx \inf_{\gamma}(f)_{r,d}^{\ast} \approx \|\sup(f)\|_{B_{p,q}(\phi,\tau)} \text{ for any } \alpha \in C_{\text{loc}}^{\log} \text{ and } p, q, \tau \in \mathcal{P}_0, 0 < q^+ < \infty.
\]

Therefore,

\[
\|\inf_{\gamma}(f)\|_{B_{p,q}(\phi,\tau)} \approx \|f\|_{B_{p,q}(\phi,\tau)} \approx \|\sup(f)\|_{B_{p,q}(\phi,\tau)}.
\]

Use these estimates and repeating the proof of Theorem 2.2 in [16] or Theorem 2.1 in [27], then complete the proof of Theorem 3.

From Theorem 3 we obtain the next important property of spaces \(A_{p,q}(\phi,\tau)\).

**Corollary 1** The definition of the spaces \(A_{p,q}(\phi,\tau)\) is independent of the choices of \(\Phi\) and \(\varphi\).

### 4 Decomposition by atoms

In recent years, it turned out that atomic and sub-atomic, as well as wavelet decompositions of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. But this applies equally to questions of mapping properties of pseudo-differential operators and to trace problems, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. The idea of atomic decompositions leads back to M. Frazier and B. Jawerth in their series of papers [15], [16], see also [25].

The main goal of this section is to prove an atomic decomposition result for \(B_{p,q}(\phi,\tau)\) and \(\tilde{B}_{p,q}(\phi,\tau)\). Atoms are the building blocks for the atomic decomposition.

**Definition 3** Let \(K \in \mathbb{N}_0, L + 1 \in \mathbb{N}_0\) and let \(\gamma > 1\). A \(K\)-times continuous differentiable function \(a \in C^K(\mathbb{R}^n)\) is called \([K, L]\)-atom centered at \(Q_{v,m}\), \(v \in \mathbb{N}_0\) and \(m \in \mathbb{Z}^n\), if
If the atom \( a \) and if
\[
\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad \text{for } 0 \leq |\beta| \leq L \text{ and } v \geq 1.
\] (20)

If the atom \( a \) located at \( Q_{v,m} \), that means if it fulfills (18), then we will denote it by \( a_{v,m} \). For \( v = 0 \) or \( L = -1 \) there are no moment conditions (20) required.

For proving the decomposition by atoms we need the following lemma, see Frazier & Jawerth [15, Lemma 3.3].

**Lemma 16** Let \( \Phi \) and \( \varphi \) satisfy, respectively, (11) and (12) and let \( \rho_{v,m} \) be an \([K,L]\)-atom. Then
\[
|\varphi_j * \rho_{v,m}(x)| \leq c \ 2^{(v-j)K+en/2} \left(1 + 2^v |x - x_{Q_{v,m}}|\right)^{-M}
\]
if \( v \leq j \), and
\[
|\varphi_j * \rho_{v,m}(x)| \leq c \ 2^{(j-v)(L+n+1)+en/2} \left(1 + 2^j |x - x_{Q_{v,m}}|\right)^{-M}
\]
if \( v \geq j \), where \( M \) is sufficiently large, \( \varphi_j = 2^{in} \varphi(2^j \cdot) \) and \( \varphi_0 \) is replaced by \( \Phi \).

Now we come to the atomic decomposition theorem.

**Theorem 4** Let \( \alpha \in C_{bc}^{\log} \) and \( p,q,\tau \in P_0^{\log} \) with \( 0 < q^- \leq q^+ < \infty \). Let \( 0 < p^- \leq p^+ \leq \infty \) and let \( K, L + 1 \in \mathbb{N}_0 \) such that
\[
K \geq (\lceil n^+ / \tau^- \rceil + 1)^+,
\] (21)
respectively
\[
K \geq (\lceil n^+ / p^- \rceil + 1)^+
\]
and
\[
L \geq \max(-1,\lceil n(\frac{1}{\min(1,p^-)} - 1) - \alpha^-\rceil).
\] (22)

Then \( f \in S'(\mathbb{R}^n) \) belongs to \( B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \), respectively to \( \widetilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \), if and only if it can be represented as
\[
f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \rho_{v,m}, \quad \text{converging in } S'(\mathbb{R}^n),
\] (23)
where \( \rho_{v,m} \) are \([K,L]\)-atoms and \( \lambda = \{\lambda_{v,m} \in \mathbb{C} : \lambda_{v,m} \in \mathbb{N}_0, m \in \mathbb{Z}^n \} \) is \( B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \), respectively \( \widetilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \). Furthermore, \( \inf \|\lambda\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \), respectively \( \inf \|\lambda\|_{\widetilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \), where the infimum is taken over admissible representations (23), is an equivalent quasi-norm in \( B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \), respectively \( \widetilde{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \).
The convergence in $\mathcal{S}'(\mathbb{R}^n)$ can be obtained as a by-product of the proof using the same method as in [25, Corollary 13.9], so the convergence is postponed to the Appendix.

If $p, q, \tau$, and $\alpha$ are constants, then the restriction (24), and their counterparts, in the atomic decomposition theorem are $K \geq ([\alpha + n/\tau] + 1)^+$ and $L \geq \max(-1, n[1/\min(1,p) - 1] - \alpha])$, which are essentially the restrictions from the works of [11, Theorem 3.12], with $1/\tau$ in place of $\tau$.

**Proof.** By similarity, we only consider $B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}$. The proof follows the ideas in [15, Theorem 6].

**Step 1.** Assume that $f \in B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}$. Using the same of the arguments used in [10, Theorem 3] we obtain a sequence $\{\lambda_{v,m}\}$ and $\rho_{v,m}$ (atoms in the sense of Definition 3) such that $f = \sum_{v=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} e_{v,m}$ and $\|\lambda\|_{B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \leq c \|f\|_{B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}}$.

**Step 2.** Assume that $f$ can be represented by (23), with $K$ and $L$ satisfying (21) and (22), respectively. We will show that $f \in B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}$ and that for some $c > 0$,$$
\|f\|_{B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}} \leq c \|\lambda\|_{B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}}.
$$

We write $f = \sum_{v=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \rho_{v,m} = \sum_{v=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \rho_{v,m} = \sum_{v=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \rho_{v,m}$

Recalling the definition of $B^{\alpha(\cdot), \tau(\cdot)}_{p(\cdot), q(\cdot)}$ space, it suffices to estimate

$$
\left(\sum_{v=0}^j \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| \varphi_j \ast \rho_{v,m}(x)\right)
$$

and

$$
\left(\sum_{v=0}^\infty \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| \varphi_j \ast \rho_{v,m}(x)\right)_{j \geq 0}
$$

in $\ell^{\tau(\cdot), q(\cdot)}(L^{p(\cdot)})$-norm. From Lemma [16] we have for any $M$ sufficiently large and any $v \leq j$

$$
\sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| \varphi_j \ast \rho_{v,m}(x)
\leq 2^{(v-j)(K-\alpha^+)} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha^)+n/2} |\lambda_{v,m}| \left(1 + 2^v |x - xQ_{v,m}|\right)^{-M}
\leq 2^{(v-j)(K-\alpha^+)} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha^-)-n/2} |\lambda_{v,m}| \eta_{v,M}(x - xQ_{v,m})
\leq 2^{(v-j)(K-\alpha^+)} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha^-)+1/2} |\lambda_{v,m}| \eta_{v,M} * \chi_{v,m}(x),
$$

by Lemma [7] Lemma [1] gives $2^{\alpha(\cdot)} \eta_{v,M} * \chi_{v,m} \lesssim \eta_{v,T} * 2^{\alpha(\cdot)} \chi_{v,m}$, with $T = M - c_{\log}(\alpha)$ and since $K > \alpha^+ + n/\tau^-$ we apply Lemma [10] to obtain

$$
\left\|\left(\sum_{v=0}^j 2^{(v-j)(K-\alpha^+)} \eta_{v,T} * 2^{v(\alpha^)+n/2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}\right)\right\|_{\ell^{\tau(\cdot), q(\cdot)}(L^{p(\cdot)})}
\lesssim \left\|\left(\eta_{v,T} * 2^{v(\alpha^-)+n/2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}\right)\right\|_{\ell^{\tau(\cdot), q(\cdot)}(L^{p(\cdot)})}.
$$

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The right-hand side can be rewritten as

$$
\sup_{P \in Q} \left\| \left( \eta_{v,T} * 2^{v(\alpha) + n/2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right)^r \frac{\|\lambda_P\|_{\ell^r(\nu)}}{\|\chi_P\|_{\ell^r(\nu)}} \right\|^{1/r}_{L^p(\nu)}^{1/r(L^p(\nu)/r)}
$$

by Lemma 8 since $\eta_{v,T} \approx \eta_{v,T} * \eta_{v,T}$ and $0 < r < \min(1, p^-)$. The application of Lemma 5 and the fact that $\|g_v\|_{L^p(\nu)}^{1/r} \leq \|\|\lambda\|_{L^p(\nu)}^{1/r}\|_{(L^p(\nu)/r)}$ give that the last expression is bounded by $\|\lambda\|_{L^p(\nu)}^{1/r}$. Now from Lemma 16, we have for any $M$ sufficiently large and $v \geq j$

$$
\sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| |\varphi_j * \rho_{v,m}(x)| \leq 2^{j-v(L+1+n/2)} \sum_{m \in \mathbb{Z}^n} 2^{j(\alpha(x) + n/2)} |\lambda_{v,m}| \left(1 + 2^j |x - x_{Q_v,m}|\right)^{-M}
$$

$$
= c \cdot 2^{j-v(L+1+n/2)} \sum_{m \in \mathbb{Z}^n} 2^{j(\alpha(x) - n/2)} |\lambda_{v,m}| \eta_{j,M}(x - x_{Q_v,m})
$$

$$
\leq 2^{j-v(L+1+n/2)} \sum_{m \in \mathbb{Z}^n} 2^{j(\alpha(x) - n/2)} |\lambda_{v,m}| \eta_{j,M} * \eta_{v,M}(x - x_{Q_v,m}),
$$

where the last inequality follows by Lemma 3 since $\eta_{j,M} = \eta_{\min(v,j),M}$. Again by Lemma 7 we have

$$
\eta_{j,M} * \eta_{v,M}(x - x_{Q_v,m}) \leq 2^m \eta_{j,M} * \eta_{v,M} * \chi_{v,m}(x).
$$

Therefore, $\sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| |\varphi_j * \rho_{v,m}(x)|$ is bounded by

$$
c \cdot 2^{j-v(L+1+n/2)} \sum_{m \in \mathbb{Z}^n} 2^{j(\alpha(x) + n/2)} |\lambda_{v,m}| \eta_{j,M} * \eta_{v,M} * \chi_{v,m}(x)
$$

$$
\leq 2^{j-v(L+1+n/2)} \sum_{m \in \mathbb{Z}^n} 2^{j(\alpha(x) + n/2)} |\lambda_{v,m}| \chi_{v,m}(x),
$$

by Lemma 11 with $T = M - c_{\log}(\alpha)$. Let $0 < r < \min(1, p^-)$ be a real number such that $L > n/r - 1 - \alpha - n$. We have

$$
\left( \sum_{v=j}^{\infty} 2^{j-v(L+1+n/2)} \eta_{j,T} * \eta_{v,T} * 2^{v(\alpha(x) + n/2)} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right)^r
$$

$$
\leq \sum_{v=j}^{\infty} 2^{j-v(L+1+n/2)} \left( \eta_{j,T} * \eta_{v,T} * 2^{v(\alpha(x) + n/2)} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right)^r
$$

$$
\leq \sum_{v=j}^{\infty} 2^{j-v(L+n/r + 1 - \alpha - n)} \eta_{j,T} * \eta_{v,T} * 2^{v(\alpha(x) + n/2)} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}.\]
where the first estimate follows by the well-known inequality \( \left( \sum_{j=0}^{\infty} |a_j| \right)^\sigma \leq \sum_{j=0}^{\infty} |a_j|^\sigma \), with \( \{a_j\}_j \subset \mathbb{C} \), \( \sigma \in [0, 1] \) and the second inequality is by Lemma 8. The application of Lemma 5 gives

\[
\left\| \left( \sum_{v=j}^{\infty} 2^{(j-v)(L+1-\alpha^-)} \eta_{j,T} \right) \sum_{m \in \mathbb{Z}^n} |\lambda_v,m| X_{v,m} \right\|_{L^p(\mathbb{R}^d)} \leq \left\| \left( \sum_{v=j}^{\infty} 2^{(j-v)(L+1-\alpha^-)} \eta_{j,T} \right) \sum_{m \in \mathbb{Z}^n} |\lambda_v,m|^r X_{v,m} \right\|_{\ell^r(\mathbb{R}^d) / \ell^r(\mathbb{R}^d)}.
\]

where \( H = L - n/r + n + 1 - \alpha^- \). Observing that \( H > 0 \), an application of Lemma 10 (this is possible, see the proof of this lemma) yields that the last expression is bounded by

\[
\left\| \left( \sum_{v=j}^{\infty} 2^{(j-v)(L+1-\alpha^-)} \eta_{j,T} \right) \sum_{m \in \mathbb{Z}^n} |\lambda_v,m|^r X_{v,m} \right\|_{\ell^r(\mathbb{R}^d) / \ell^r(\mathbb{R}^d)} \leq \left\| \lambda \right\|_{\ell^q(\mathbb{R}^d)}.
\]

where we used again Lemma 5 and hence the proof is complete.

5 Appendix

First let us prove Lemma 10. By similarity, we only consider \( B^{\alpha-,\tau(-)}_{p(\cdot),q(\cdot)} \). Let \( P \in \mathcal{Q} \). In view of the proof of Lemma 5, the problem can be reduced to the case when \( \ell^q(\mathbb{R}^d) \) is a normed space. Then

\[
\left\| \left( \sum_{k=0}^{v_p^+} \frac{2^{-|k-v|\delta} f_k}{\|X_P\|_{\tau(\cdot)}} \right)_{v \geq v_p^+} \right\|_{\ell^q(\mathbb{R}^d)} \leq \left\| \left( \sum_{k=0}^{v_p^+} \frac{2^{-|k-v|\delta} f_k}{\|X_P\|_{\tau(\cdot)}} \right)_{v \geq v_p^+} \right\|_{\ell^q(\mathbb{R}^d)} + \left\| \left( \sum_{k=v_p^+}^{v} \cdots \right)_{v \geq v_p^+} \right\|_{\ell^q(\mathbb{R}^d)} + \left\| \left( \sum_{k=v}^{v_p^+} \cdots \right)_{v \geq v_p^+} \right\|_{\ell^q(\mathbb{R}^d)}.
\]

The first norm is bounded by

\[
\sum_{k=0}^{v_p^+} 2^{(v_p^+-v)\delta} \left\| \left( \sum_{k=v_p^+}^{v} \cdots \right)_{v \geq v_p^+} \right\|_{\ell^q(\mathbb{R}^d)}.
\]
Let $Q_{kh}$ be a dyadic cube such that $P \subset Q_{kh}$. Obviously $v_{Q_h}^+ = k$ and by Lemma 3 we have $\frac{\|x_{Q_h}\|_{\tau(\cdot)}}{\|x_P\|_{\tau(\cdot)}} \lesssim 2^{n(v_P^+ - k)/\tau^-}$. Therefore the last sum is bounded by

$$\sum_{k=0}^{v_P^+} 2^{(k-v_P^+)(\delta-n/\tau^-)} \left\| \left( \frac{f_j}{\|X_{Q_h}\|_{\tau(\cdot)}} X_{Q_h} \right) \right\|_{\ell^\infty(\ell^p(\cdot))} \lesssim \sum_{k=0}^{v_P^+} 2^{(k-v_P^+)(\delta-n/\tau^-)} \left\| (f_v)_v \right\|_{\ell^\infty(\ell^q(\cdot))(\ell^p(\cdot))} \lesssim \left\| (f_v)_v \right\|_{\ell^\infty(\ell^q(\cdot))(\ell^p(\cdot))},$$

since $\delta > n/\tau^-$. Let $\sigma > \max(q^+, \frac{q^+}{p^-})$ and $\| (f_v)_v \|_{\ell^\infty(\ell^q(\cdot))(\ell^p(\cdot))} = 1$. Then

$$\sum_{v=v_P^+}^{\infty} \left\| \sum_{k=v_P^+}^{v} 2^{(k-v)^\delta} f_v \right\|_{\ell^\infty(\ell^p(\cdot))} \left\| \left( \frac{f_v}{\|X_P\|_{\tau(\cdot)}} X_P \right) \right\|_{\ell^\infty(\ell^p(\cdot))}^{\sigma} \lesssim \sum_{v=v_P^+}^{\infty} \left\| \sum_{k=v_P^+}^{v} 2^{(k-v)^\delta} f_v \right\|_{\ell^\infty(\ell^p(\cdot))} \left\| \left( \frac{f_v}{\|X_P\|_{\tau(\cdot)}} X_P \right) \right\|_{\ell^\infty(\ell^p(\cdot))}^{\sigma} \lesssim c \sum_{v=v_P^+}^{\infty} \left\| \frac{f_v}{\|X_P\|_{\tau(\cdot)}} X_P \right\|_{\ell^\infty(\ell^p(\cdot))} \left\| \left( \frac{f_v}{\|X_P\|_{\tau(\cdot)}} X_P \right) \right\|_{\ell^\infty(\ell^p(\cdot))} \lesssim 1,$$

by Lemma 9. The desired estimate is completed by the scaling argument. Now the last norm in \eqref{24} is bounded by

$$\left\| \left( \sum_{i=0}^{\infty} 2^{-i\delta} \frac{f_{i+v}}{\|X_P\|_{\tau(\cdot)}} X_P \right) \right\|_{\ell^\infty(\ell^p(\cdot))} \lesssim \sum_{i=0}^{\infty} 2^{-i\delta} \left\| \left( \frac{f_k}{\|X_P\|_{\tau(\cdot)}} X_P \right) \right\|_{\ell^\infty(\ell^p(\cdot))} \lesssim \sum_{i=0}^{\infty} 2^{-i\delta} \left\| \left( \frac{f_k}{\|X_P\|_{\tau(\cdot)}} X_P \right) \right\|_{\ell^\infty(\ell^p(\cdot))} \lesssim \left\| (f_v)_v \right\|_{\ell^\infty(\ell^q(\cdot))(\ell^p(\cdot))}.$$
The last factor in the integral can be uniformly estimated from the above by
\[ c \ 2^{-v(L+1)} (1 + |y|^2)^{-M/2} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{M/2} \sum_{|\beta| \leq L+1} |\partial^\alpha \varphi(x)|, \]
where \( M > 0 \) is at our disposal. Let \( 0 < t < \min(1, p^-) = 1 + p^- - \frac{p^-}{\min(1, p^-)} \) and \( s(x) = \alpha(x) + \frac{n}{p(x)}(t-1) \) be such that \( n(1-\frac{1}{\min(1, p^-)}) + \alpha^- > s^- > -1 - L \). Since \( g_{v,m} \) are \([K, L]-\text{atoms}\), then for every \( S > 0 \), we have \( |g_{v,m}(y)| \leq c 2^{m/2} (1 + 2^n |y - x_{Q_v,m}|)^{-S} \). Therefore,
\[
\left| \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} g_{v,m}(y) \varphi(y) dy \right|
\leq c \ 2^{-v(L+1+\alpha^-)} \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha^- + n/2)} |\lambda_{v,m}| (1 + 2^n |y - x_{Q_v,m}|)^{-S} (1 + |y|^2)^{-M/2} dy
= 2^{-v(L+1+\alpha^-)} \sum_{i=0}^\infty \int_{C_i} \cdots dy,
\]
where \( C_0 = \{ y \in \mathbb{R}^n : |y| < 1 \} \) and \( C_i = \{ y \in \mathbb{R}^n : 2^{i-1} \leq |y| < 2^i \} \) for any \( i \in \mathbb{N} \). Applying Lemma 7 to obtain
\[
\sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| (1 + 2^n |y - x_{Q_v,m}|)^{-S}
= 2^{-vn} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \eta_{v,S} (y - x_{Q_v,m}) \leq \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \eta_{v,S} \chi_{v,m}(y).
\]
We split \( M \) into \( R + S \). Since we have in addition the factor \((1 + |y|^2)^{-S/2}\), Hölder’s inequality and fact that \( \|\chi_{B_{-\lambda}}\|_{\tau(\cdot)} \|\chi_{B_{-\lambda}}\|_{(p(\cdot)/t)'} \approx 2^{in(1-t/p_\infty + 1/\tau_\infty)} \) give that the term \( \left| \int_{\mathbb{R}^n} \cdots dy \right| \) is bounded by
\[
\begin{align*}
c \ 2^{-v(L+1+\alpha^-)} & \sum_{i=0}^\infty \left( \frac{2^{(\alpha^- + n/2)v} |\lambda_{v,m}| \chi_{v,m}}{\|\chi_{B_{-\lambda}}\|_{\tau(\cdot)} \|\chi_{B_{-\lambda}}\|_{(p(\cdot)/t)}} \right) \\
& \leq c \ 2^{-v(L+1+\alpha^-)} \sup_{B_{-\lambda}, j \geq j^+} \left( \frac{2^{(\alpha^- + n/2)j} \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}| \chi_{j,m}}{\|\chi_{B_{-\lambda}}\|_{\tau(\cdot)} \|\chi_{B_{-\lambda}}\|_{L^{p(\cdot)/t}}} \right) \\
& \leq c \ 2^{-v(L+1+\alpha^-)} \|\lambda\|_{p^*(\cdot)/t, \infty}^{\alpha^*(\cdot)/t, \infty},
\end{align*}
\]
where the first inequality follows by Lemma 5 and by taking \( R \) large enough. Since \( L + 1 + \alpha^- > 0 \), the convergence of \((23)\) is now clear by the embeddings \( \|\lambda\|_{p^*(\cdot)/q(\cdot)} \hookrightarrow \|\lambda\|_{p^*(\cdot)/q(\cdot)} \hookrightarrow \|\lambda\|_{p^*(\cdot)/t, \infty} \). The proof is completed.
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