The Number of Nodal Components of Arithmetic Random Waves

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Abstract

We study the number of nodal components (connected components of the set of zeroes) of functions in the ensemble of arithmetic random waves, that is, random eigenfunctions of the Laplacian on the flat $d$-dimensional torus $\mathbb{T}^d$ ($d \geq 2$). Let $f_L$ be a random solution to $\Delta f + 4\pi^2 L^2 f = 0$ on $\mathbb{T}^d$, where $L^2$ is a sum of $d$ squares of integers, and let $N_L$ be the random number of nodal components of $f_L$. By recent results of Nazarov and Sodin, $E\{N_L / L^d\}$ tends to a limit $\nu > 0$, depending only on $d$, as $L \to \infty$ subject to a number-theoretic condition - the equidistribution on the unit sphere of the normalized lattice points on the sphere of radius $L$. This condition is guaranteed when $d \geq 5$, but imposes restrictions on the sequence of $L$ values when $2 \leq d \leq 4$. We prove the exponential concentration of the random variables $N_L / L^d$ around their medians and means (unconditionally) and around their limiting mean $\nu$ (under the condition that it exists).

1 Introduction and presentation of the results

1.1 Toral eigenfunctions and arithmetic random waves

Let $\mathcal{H}_L \subset L^2(\mathbb{T}^d)$ be the real Hilbert space of Laplacian eigenfunctions on the torus, i.e. functions $f: \mathbb{T}^d \to \mathbb{R}$ satisfying the partial differential equation:

$$\Delta f + 4\pi^2 L^2 f = 0.$$ 

We consider $d \geq 2$ to be a fixed dimension (all “constants” mentioned below may depend on $d$); $L$ may vary. It is known that the spectrum of eigenvalues is discrete; eigenfunctions exist whenever $L^2$ can be expressed as a sum of $d$ squares of integers, and then, $\mathcal{H}_L = \text{Span} \{ \cos(2\pi \lambda \cdot x), \sin(2\pi \lambda \cdot x) : \lambda \in \Lambda_L \}$, where $\Lambda_L = \{ \lambda \in \mathbb{Z}_d : |\lambda| = L \}$. Each $\lambda$ generates the same functions as $-\lambda$, so $\dim \mathcal{H}_L = \#\Lambda_L$.

For any $f: \mathbb{T}^d \to \mathbb{R}$, we denote by $Z(f)$ its nodal set (the subset of $\mathbb{T}^d$ where $f$ vanishes), and by $N(f)$ the number of its nodal components (the connected components of the nodal set). In this paper, we address the question: What is the typical behavior of $N(f)$ for $f \in \mathcal{H}_L$, with fixed $d$ and large $L$?

Typically (when $f$ and $\nabla f$ do not vanish simultaneously), the number of nodal components almost equals the number of nodal domains (the connected components of $\mathbb{T}^d \setminus Z(f)$) - they cannot differ by more than $d - 1$. Thus, Courant’s nodal domain theorem gives a general upper bound $N(f) \lesssim L^d$, with an explicit constant. Unfortunately, a general, non-trivial lower bound cannot be obtained, as there are classical counterexamples with arbitrarily large $L$ and only two nodal domains, originally shown in [Ste25] (see also [BH15], [BF12]).

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It is expected, however, that such eigenfunctions with high eigenvalue but few nodal components are outliers, and \( N(f) \) is in the order of magnitude of \( L^d \) for “most” \( f \in \mathcal{H}_L \). To study the typical case, we refer to a probabilistic model that was introduced and investigated in [ORW08, RW08]. Consider the random function \( f_L : \mathbb{T}^d \to \mathbb{R} \):

\[
f_L(x) := \sqrt{\frac{2}{\dim \mathcal{H}_L}} \sum_{\lambda \in \Lambda^+_L} (a_\lambda \cos (2\pi \lambda \cdot x) + b_\lambda \sin (2\pi \lambda \cdot x)),
\]

where the set \( \Lambda^+_L = \Lambda_L / \pm \) is half of the set \( \Lambda_L \) (representatives of the equivalence \( \lambda \sim \pm \lambda \)), and \( a_\lambda, b_\lambda \) are random variables, i.i.d. \( \mathcal{N}(0, 1) \). The sequence of functions \( \{f_L\} \) is called the "ensemble of arithmetic random waves. The random function \( f_L \) may be viewed as a random element of the finite-dimensional space \( \mathcal{H}_L \), or as a centered, stationary Gaussian process, normalized such that \( \mathbb{E} \{ |f_L(x)|^2 \} = 1 \), with covariance kernel:

\[
K_L(x, y) := \mathbb{E} \{ f_L(x) f_L(y) \} = \frac{1}{\dim \mathcal{H}_L} \sum_{\lambda \in \Lambda_L} \cos (2\pi \lambda \cdot (x - y)).
\]

Note that due to rotation invariance, the definition of \( f_L \) does not depend on the choice of basis for \( \mathcal{H}_L \).

Under this probabilistic model, the number of nodal components \( N_L := N(f_L) \) becomes a random variable (we discuss its measurability in detail in Section 2) and the question of its behavior may be formulated in terms of expected value and concentration as \( L \to \infty \).

1.2 Asymptotic law for \( \mathbb{E} \{ N_L \} \)

Nazarov and Sodin [NS16] (see also lecture notes [Sod16]) proved, in a much more general setting of ensembles of Gaussian functions on Riemannian manifolds, an asymptotic law for the expected value of \( N(f) \). Our first theorem, Theorem 1.2, is simply a formulation of the Nazarov-Sodin theorem, applied to our case. There is one obstacle: The theorem requires the existence of a limiting spectral measure satisfying certain properties. In our case, this limiting spectral measure does not necessarily exist, and it depends on the following number-theoretic equidistribution condition:

**Definition 1.1.** A sequence of values of \( L \) that tends to infinity, with \( L^2 \) always a sum of \( d \) squares, is called an admissible sequence of \( L \) values if the integer points on the sphere of radius \( L \), when projected onto the unit sphere, become equidistributed as \( L \to \infty \). In other words,

\[
\frac{1}{\dim \mathcal{H}_L} \sum_{\lambda \in \Lambda_L} \delta_{\lambda/L} \Rightarrow \sigma_{d-1},
\]

where "\( \Rightarrow \)" indicates weak-* convergence of measures, and \( \sigma_{d-1} \) is the uniform measure on the unit sphere, with the normalization \( \sigma_{d-1} \left( S^{d-1} \right) = 1 \).

This equidistribution condition depends on the dimension \( d \). When \( d \geq 5 \), any sequence of \( L \) values is admissible, whereas in the low dimensions \( 2 \leq d \leq 4 \), some conditions must be satisfied. For more on the subject, see Appendix A.

**Theorem 1.2.** There is a constant \( \nu > 0 \) such that:

\[
\mathbb{E} \{ N_L \} \sim \nu L^d \quad \text{as} \quad L \to \infty \quad \text{through any admissible sequence}.
\]
1.3 Main result: Exponential concentration of $N_L$

Our main result is that $N_L$ concentrates around its median, mean, and limiting mean, exponentially in $\dim H_L$, i.e. the number of independent random variables. This is similar to a previous result by Nazarov and Sodin in the case of random spherical harmonics [NS09].

**Theorem 1.3.** Let $\varepsilon > 0$. There exist constants $C(\varepsilon), c(\varepsilon) > 0$ such that:

1. For any $L$:
   
   $$\mathbb{P}\left\{ \left| \frac{N_L}{L^d} - \text{Median}\left\{ \frac{N_L}{L^d} \right\} \right| > \varepsilon \right\} \leq C(\varepsilon) e^{-c(\varepsilon) \dim H_L}.$$ 

2. For any $L$:
   
   $$\mathbb{P}\left\{ \left| \frac{N_L}{L^d} - \mathbb{E}\left\{ \frac{N_L}{L^d} \right\} \right| > \varepsilon \right\} \leq C(\varepsilon) e^{-c(\varepsilon) \dim H_L}.$$ 

3. If $\mathbb{E}\left\{ N_L L^{-d} \right\}$ tends to a limit $\nu$ through some sequence of $L$ values, then for any large enough $L$ in this sequence:
   
   $$\mathbb{P}\left\{ \left| \frac{N_L}{L^d} - \nu \right| > \varepsilon \right\} \leq C(\varepsilon) e^{-c(\varepsilon) \dim H_L}.$$ 

In all cases, our proof yields $c(\varepsilon) \gtrsim \varepsilon (d+2)^2 - 1$.

**Remark 1.4.** In the low dimensions $2 \leq d \leq 4$, it is possible to have sequences of $L$ values for which $\dim H_L$ stays bounded as $L$ grows. However, if $\dim H_L$ is bounded, then all parts of Theorem 1.3 are completely trivial, and say nothing. This is unlike the case of random spherical harmonics discussed in [NS09], in which the dimension is a simple ascending function of the eigenvalue.

The second and third parts of Theorem 1.3 are straightforward consequences of the first part. This is proven in Subsection 5.2. When $d \geq 3$, the third part of the theorem only makes sense when $\nu$ is the same $\nu$ from Theorem 1.2. This is because under the assumption that $\dim H_L$ is not bounded from below (without which, the theorem says nothing anyway), the limit (1.3) holds and the sequence is admissible. However, when $d = 2$, we could have a limiting measure other than $\sigma_{d-1}$ in (1.3), so value of $\nu$ in the third part of Theorem 1.3 truly depends on the chosen sequence of $L$ values. See also Appendix A and [KW15].

1.4 Outline of the paper

In Section 2, we prove the Borel measurability of the random variable $N_L$ - the number of nodal components. We deduce it from a more general result - the measurability of the number of nodal components of more general random functions (Proposition 2.2), which may be of independent interest.

In Section 3, we show that Theorem 1.2 follows directly from the Nazarov-Sodin theorem.

In Section 4, we treat trigonometric polynomials in general, and give algebraic proofs to bounds on the sum of diameters of their connected components.

In Section 5, we present a proof of Theorem 1.3.

In Appendix A, we provide background and quote the known results on the problem of equidistribution of lattice points on spheres.

In Appendix B, we provide proofs for some of the claims used in the paper.

1.5 Notation

We reserve the letters $C$ and $c$ for positive constants (usually upper and lower bounds, respectively) which may vary from line to line; all constants may depend on the dimension $d$. When $A, B$ are
positive quantities, we denote by $A \lesssim B$, $A \gtrsim B$ and $A \simeq B$ that $A \leq CB$, $A \geq cB$ and $cB \leq A \leq CB$, respectively.

The notation $K_{\pm \delta}$ indicates the set of all points of distance at most $\delta$ from the compact set $K$, which may be a set in $T^d$ or $\mathbb{R}^d$ with Euclidean distance or $L^2\left(T^d\right)$ with the norm-induced distance.

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2 Measurability of $N_L$

The random process $f_L$ is, formally, a function $f_L : T^d \times \Omega \to \mathbb{R}$, where $\Omega$ is the Gaussian probability space. In this point of view, $N_L$ is the function on $\Omega$ given by $\omega \mapsto N(f_L(\cdot, \omega))$.

Proposition 2.1. $N_L$ is a random variable. In other words, the mapping $N_L : \Omega \to \mathbb{N} \cup \{0, \infty\}$ is measurable.

Naturally, the countable set $\mathbb{N} \cup \{0, \infty\}$ is equipped with the discrete (power set) $\sigma$-algebra.

There is an “analytic” way to prove that $N_L$ is at least Lebesgue measurable. Skipping some details, this proof is as follows: By Bulinskaya’s lemma, the event that $f_L$ does not have a “stable nodal set” is a subset of $\Omega$ of Lebesgue measure zero (see Definition 4.1 and Proposition 5.5 ahead). The complement of this event is open because, since $\dim H_L$ is finite, the norm $\|g\|$ of any small perturbation $g \in H_L$ bounds both $\max |g|$ and $\max |\nabla g|$. $N_L$ is locally constant in this open set (see Proposition 4.4 ahead). Thus, any event of the form $\{N_L = n\}$ is a union of an open set and a subset of an event of Lebesgue measure zero.

The above proof has two weaknesses: The first is that it makes assumptions on the function space. To use Bulinskaya’s lemma, we require the smoothness of the functions, the finite dimension of $T^d$, and a condition on the probability density of the random process, and to continue the proof, we need the finite dimension of $H_L$. The second weakness is that the proof only yields Lebesgue measurability.

In this section, we present a strong generalization of Proposition 2.1 and prove it using only the most basic definitions in topology and measure theory:

Proposition 2.2. Let $X$ be a compact metric space, let $\Omega$ be a (not necessarily complete) probability space, and let $f : X \times \Omega \to \mathbb{R}$ be a random real-valued function on $X$ that is a.s. continuous. Then the number of nodal components of $f$ is a random variable, i.e. a measurable mapping $\Omega \to \mathbb{N} \cup \{0, \infty\}$.

This immediately implies Proposition 2.1 (where $X = T^d$) with Borel measurability. To prove Proposition 2.2, observe that the number of nodal components of $f$ is the composition of three maps:

$$
\Omega \xrightarrow{f} C(X) \xrightarrow{Z} F(X) \xrightarrow{\text{Count}} \mathbb{N} \cup \{0, \infty\}
$$

- The first map $\omega \mapsto f(\cdot, \omega)$ sends almost every $\omega \in \Omega$ to a corresponding function in $C(X)$.
- The second map $f \mapsto Z(f)$ sends a continuous real function to its (closed) zero set.
The sets on the right hand side are generating sets in the $\sigma$-preimage under $f$. Let $X$ be a compact metric space. The map $Z \in Y$ and open) subsets of $\sigma$-open sets, which is more easily expressed by generators of the $\sigma$-algebra of $X$.

We will show that for any compact measurable. $\Omega$ this is equivalent to the map $\sigma$-algebras on $\sigma$-algebras on $C$ and $F$.

The standard $\sigma$-algebra on $C$ is generated by the point-evaluation maps $\{ f \mapsto f(x) \}_{x \in X}$; that is, it is generated by the family of sets $\{ f \in C(X) : f(x) \in B \}$, where $x$ varies over all points in $X$ and $B$ varies over all Borel subsets of $\mathbb{R}$.

The standard $\sigma$-algebra on $F(X)$ is given by the following equivalent definitions (see also chapters 2.4 and 3.3 of [Sri98]):

1. The $\sigma$-algebra $\mathcal{F}_1$ generated by the family of sets $\{ F \in F(X) : F \cap U \neq \emptyset \}$, where $U$ varies over all open subsets of $X$.
2. The $\sigma$-algebra $\mathcal{F}_2$ generated by the family of sets $\{ F \in F(X) : F \subset U \}$, where $U$ varies over all open subsets of $X$.
3. The $\sigma$-algebra $\mathcal{F}_3$ generated by the family of sets $\{ F \in F(X) : F \cap K \neq \emptyset \}$, where $K$ varies over all compact subsets of $X$.

Proof that $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$. Any subset of $X$ is compact if and only if it is closed, so the sets $\{ F \in F(X) : F \subset U \}$ and $\{ F \in F(X) : F \cap K \neq \emptyset \}$ are complementary when taking $K = X \setminus U$, and we get $\mathcal{F}_2 = \mathcal{F}_3$.

Any open subset of $X$ is a countable union of compact subsets, so any set $\{ F \in F(X) : F \cap U \neq \emptyset \}$ is a countable union of sets $\{ F \in F(X) : F \cap K \neq \emptyset \}$, and we have $\mathcal{F}_1 \subset \mathcal{F}_3$. Similarly, any compact subset is a countable intersection of open subsets, so any set $\{ F \in F(X) : F \cap K \neq \emptyset \}$ is a countable intersection of sets $\{ F \in F(X) : F \cap U \neq \emptyset \}$, and we have $\mathcal{F}_3 \subset \mathcal{F}_1$.

It is evident that the first map $\Omega \rightarrow C(X)$ is measurable: By the definition of the $\sigma$-algebra on $C(X)$, this is equivalent to the map $\Omega \rightarrow \mathbb{R}$ given by $\omega \mapsto f(x, \omega)$ being measurable for any $x \in X$, which is precisely the definition of $f$ being a random function.

The second map is measurable by the following:

**Proposition 2.3.** Let $X$ be a compact metric space. The map $Z : C(X) \rightarrow F(X)$ given by $f \mapsto f^{-1}(\{0\})$ is measurable.

**Proof.** We will show that for any compact $K \subset X$, the set $\{ F \in F(X) : F \cap K \neq \emptyset \}$ has a measurable preimage under $Z$ - that is, that the set $\{ f \in C(X) : \exists x \in K : f(x) = 0 \}$ is measurable in $C(X)$. Let $A \subset K$ be countable and dense in $K$. By a standard continuity argument, we have:

$$\{ f \in C(X) : \exists x \in K : f(x) = 0 \} = \bigcap_{\varepsilon > 0} \bigcup_{x \in A} \{ f \in C(X) : |f(x)| < \varepsilon \}. \tag{2.1}$$

The sets on the right hand side are generating sets in the $\sigma$-algebra of $C(X)$, so the set on the left hand side is measurable, proving the proposition. \hfill \square

The measurability of the third map - the component counting function - is a little trickier. We first show a couple of lemmas that will help translate the number of components to a property of covers by open sets, which is more easily expressed by generators of the $\sigma$-algebra on $F(X)$.

For any topological space $Y$, we denote by Clopen($Y$) the Boolean algebra of clopen (that is, closed and open) subsets of $Y$. Note that $Y$ is connected if and only if Clopen($Y$) = $\{ \emptyset, Y \}$, and that if
A ∈ Clopen (Y) then Clopen (A) ⊂ Clopen (Y). Recall that any clopen set is a union of connected components, and that connected components are always closed.

**Lemma 2.4.** Let Y be a topological space and let N be a positive integer. The following are equivalent:

1. Y has strictly fewer than N connected components.
2. For any $Y_1, \ldots, Y_N \in$ Clopen (Y), if they are pairwise disjoint then one of them is empty.

**Proof.** The second condition follows from the first simply by the pigeonhole principle, since clopen sets are unions of connected components. Conversely, suppose the second condition holds. Let $N$ be a positive integer. The sets $\{F \cup \bigcup_{i=1}^{N} U_i : \text{there is a proper subcover (i.e. one of the sets } U_i \text{ does not intersect } F)\}$.

For the following, recall that any compact metric space $X$ admits a countable collection $\mathcal{U}$ of open sets which separates closed sets: For any closed, pairwise disjoint $F_1, \ldots, F_n \subset X$, there are pairwise disjoint $U_1, \ldots, U_n \in \mathcal{U}$ with $F_i \subset U_i$. We skip the proof of this, which follows easily from the fact that any compact metric space is second-countable and normal.

**Lemma 2.5.** Let $X$ be a compact metric space, let $\mathcal{U}$ be a collection of open sets in $X$ which separates closed sets, let $F \subset X$ be closed and let $N$ be a positive integer. The following are equivalent:

1. $F$ has strictly fewer than $N$ connected components.
2. For any pairwise disjoint $U_1, \ldots, U_N \in \mathcal{U}$ such that $F \subset \bigcup_{i=1}^{N} U_i$, there is a proper subcover (i.e. one of the sets $U_i$ does not intersect $F$).

**Proof.** Suppose the first condition holds. Let $U_1, \ldots, U_N \in \mathcal{U}$ be pairwise disjoint sets that cover $F$. Each $F \cap U_i$ is relatively open in $F$, and its $F$-complement $F \cap \left( \bigcup_{i \neq j} U_i \right)$ is also relatively open in $F$.

For the following, recall that any compact metric space X admits a countable collection U of open sets which separates closed sets: For any closed, pairwise disjoint $F_1, \ldots, F_n \subset X$, there are pairwise disjoint $U_1, \ldots, U_n \in \mathcal{U}$ with $F_i \subset U_i$. We skip the proof of this, which follows easily from the fact that any compact metric space is second-countable and normal.

**Proposition 2.6.** Let $X$ be a compact metric space. The map $\text{Count}: F(X) \to \mathbb{N} \cup \{0, \infty\}$, that counts the number of connected components in the given closed set, is measurable.

**Proof.** The sets $\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots$ generate the $\sigma$-algebra on $\mathbb{N} \cup \{0, \infty\}$, so it is enough to show that given a positive integer $N$, the following set is measurable:

$$F_{<N} := \{ F \in F(X) : F \text{ has strictly fewer than } N \text{ connected components} \}.$$

Let $\mathcal{U}$ be a countable collection of open sets separating closed sets. By Lemma 2.5, $F_{<N}$ may be written as the set of all $F \in F(X)$ such that for any pairwise disjoint $U_1, \ldots, U_N \in \mathcal{U}$, if $F \subset \bigcup_{i=1}^{N} U_i$
then one of the $U_i$ does not intersect $F$:

$$
F_{<N} = \bigcap_{U_1, \ldots, U_N \in \mathcal{U} \text{ pairwise disjoint}} \left\{ F \in \mathcal{F}(X) : \left( F \subset \bigcup_{i=1}^N U_i \right) \implies \exists i : F \cap U_i = \emptyset \right\}
$$

$$
= \bigcap_{U_1, \ldots, U_N \in \mathcal{U} \text{ pairwise disjoint}} \left\{ F \in \mathcal{F}(X) : \left( F \not\subset \bigcup_{i=1}^N U_i \right) \text{ or } \exists i : F \cap U_i = \emptyset \right\}
$$

$$
= \bigcap_{U_1, \ldots, U_N \in \mathcal{U} \text{ pairwise disjoint}} \left( \left\{ F \in \mathcal{F}(X) : F \not\subset \bigcup_{i=1}^N U_i \right\} \cup \bigcup_{i=1}^N \left\{ F \in \mathcal{F}(X) : F \cap U_i = \emptyset \right\} \right). \quad (2.2)
$$

Subsets of $\mathcal{F}(X)$ of the form $\{ F \in \mathcal{F}(X) : F \not\subset U \}$ and $\{ F \in \mathcal{F}(X) : F \cap U = \emptyset \}$, where $U \subset X$ is open, are basic measurable sets in the $\sigma$-algebra on $\mathcal{F}(X)$. Thus, the expression under the (countable) intersection in (2.2) evaluates to a measurable set, and we get that $F_{<N}$ is measurable. \qed

## 3 Proof of Theorem 1.2

The Nazarov-Sodin theorem [NS16] (see also lecture notes [Sod16]) gives an asymptotic law for the expected number of connected components of Gaussian functions under very general conditions. In this section, we show that Theorem 1.2 is a specialization of the Nazarov-Sodin theorem for our case.

We begin by computing the objects $K_{x,L}(u,v)$ and $C_{x,L}(u)$, as they are defined in [Sod16, Sections 2.2-2.3]. In our case, $K_L$ is given by (1.2). First, the scaled covariance kernel $K_{x,L}(u,v)$:

$$
K_{x,L}(u,v) = K_L \left( x + \frac{u}{L}, x + \frac{v}{L} \right) = K_L \left( \frac{u-v}{L} \right) = \frac{1}{\dim \mathcal{H}_L} \sum_{\lambda \in \Lambda_L} \cos \left( \frac{2\pi \lambda}{L} \cdot (u-v) \right). \quad (3.1)
$$

Note that this expression for $K_{x,L}(u,v)$ satisfies the definition for $C^{3,3}$-smoothness of the ensemble (see [Sod16, Definition 2], where it is called “separate $C^{3,3}$-smoothness”). To see this, it is enough to show that the partial derivative $\partial_u \partial_v K_{x,L}(u,v)$ with $0 \leq i, j \leq 3$ remains uniformly bounded. Note that this partial derivative is given by an expression similar to (3.1), where the function in the sum is either $\pm \cos$ or $\pm \sin$ (depending on $i+j$), and the addend corresponding to $\lambda$ is multiplied by $\pm 2\pi \lambda m L^{-1} \in [-2\pi, 2\pi]$ every time a derivative is taken with respect to $u_m$ or $v_m$; therefore, $|\partial_u \partial_v K_{x,L}(u,v)| \leq (2\pi)^6$.

Second, the scaled covariance matrix $C_{x,L}(u)$. By the following computation, we have that $C_{x,L}(u)$ is simply a constant multiple of the identity matrix:

$$
(C_{x,L}(u))_{ij} = \left. \partial_u \partial_v K_{x,L}(u,v) \right|_{v=u} = \frac{1}{\dim \mathcal{H}_L} \sum_{\lambda \in \Lambda_L} (2\pi)^2 \frac{\lambda_i \lambda_j}{L^2} \cos \left( 2\pi \cdot \frac{u-v}{L} \right) \bigg|_{v=u}
$$

$$
= \frac{1}{\dim \mathcal{H}_L} \frac{(2\pi)^2}{L^2} \sum_{\lambda \in \Lambda_L} \lambda_i \lambda_j = \begin{cases} 
0 & \text{if } i \neq j \\
\frac{4\pi^2}{d} & \text{if } i = j
\end{cases}
$$

In the last step we have used an orthogonality relation that can be seen easily by observing symmetries within the set $\Lambda_L$. Thus, it clearly satisfies the definition for non-degeneracy of the ensemble (see [Sod16, Definition 3]).

Finally, we introduce our target limiting spectral measure for the process: $\sigma_{d-1}$, the normalized Lebesgue measure on the sphere $S^{d-1}$, with $\sigma_{d-1}(S^{d-1}) = 1$. The target translation-invariant local
limiting covariance kernel $k$ is thus the Fourier (cosine) transform of $\sigma_{d-1}$:

$$k(x) := \int_{S^{d-1}} \cos(2\pi x \cdot \zeta) \, d\sigma_{d-1}(\zeta).$$

Under the assumption that $L \to \infty$ through an admissible sequence of $L$ values (Definition 1.1), we have $K_{x,L}(u,v) \to k(x)$ pointwise in $x \in \mathbb{R}^d$. Pointwise convergence implies compact convergence in $\mathbb{R}^d$ by a standard application of the Arzelà-Ascoli theorem, and we get translation-invariant local limits as in [Sod16, Definition 1].

Thus, Theorem 1.2 follows from [Sod16, Theorem 4], where the positivity of the constant $\nu$ for the spectral measure $\sigma_{d-1}$ follows from condition $(\rho 4)$ in [Sod16, Theorem 1].

4 Trigonometric polynomials and their nodal sets

4.1 Stability of nodal sets under small perturbations

We begin by introducing some notation and definitions for the discussion of the topological stability of a function’s nodal set under small perturbations. Let $f : \mathbb{T}^d \to \mathbb{R}$ be any continuous function. We define $Z(f), \hat{Z}(f)$ and $N(f)$ by:

$$Z(f) := \{ x \in \mathbb{T}^d : f(x) = 0 \} = f^{-1}(\{0\})$$

$$\hat{Z}(f) := \{ \text{connected components of } Z(f) \}$$

$$N(f) := \# \text{ connected components of } Z(f) = \# \hat{Z}(f)$$

$Z(f)$ is called the nodal set of $f$, and its connected components (which comprise $Z(f)$) are called the nodal components of $f$. The connected components of $\mathbb{T}^d \setminus Z(f)$ are called the nodal domains of $f$.

Definition 4.1. We say that a $C^1$-smooth function $f : \mathbb{T}^d \to \mathbb{R}$ has a stable nodal set if $\nabla f(x) \neq 0$ for all $x \in Z(f)$.

Remark 4.2. By compactness, the condition that $f$ has a stable nodal set is equivalent to the existence of $\alpha, \beta > 0$ such that for any $x \in \mathbb{T}^d$, $|f(x)| > \alpha$ or $|\nabla f(x)| > \beta$, and $\alpha, \beta$ may be chosen under a constraint $\alpha/\beta < \delta$ for any arbitrary $\delta > 0$. Note that if $f$ has a stable nodal set then $Z(f)$ is a $(d-1)$-dimensional smooth compact submanifold of $\mathbb{T}^d$ having finitely many connected components.

By the following two propositions, stable nodal sets are indeed stable under small perturbations. Proposition 4.3 discusses “local” stability in an open subset of the torus, and Proposition 4.4 is a “global” version (cf. [NS09, Corollary 4.3]). These propositions may be proven in a standard way, by studying the flow of the vector field $|\nabla f|^{-2} \nabla f$, and for completeness, we provide their proofs in Appendix B.

Proposition 4.3. Let $\alpha, \beta > 0$ and let $U$ be an open subset of $\mathbb{T}^d$. Let $f : U \to \mathbb{R}$ be a smooth function such that $|f(x)| > \alpha$ or $|\nabla f(x)| > \beta$ for any $x \in U$.

Let $g : U \to \mathbb{R}$ be a continuous function such that $|g(x)| < \alpha$ for any $x \in U$ (this is the “small perturbation”).

Then for each connected component $\Gamma$ of $\{ x \in U : f(x) = 0 \}$ that satisfies $\Gamma_{+\alpha/\beta} \subset U$, there is a connected component $\hat{\Gamma} \subset \Gamma_{+\alpha/\beta}$ of $\{ x \in U : f(x) + g(x) = 0 \}$. Furthermore, the mapping $\Gamma \mapsto \hat{\Gamma}$ is injective (that is, different components $\Gamma$ generate different components $\hat{\Gamma}$).

Proposition 4.4. Let $\alpha, \beta > 0$ and let $f : \mathbb{T}^d \to \mathbb{R}$ be a smooth function such that $|f(x)| > \alpha$ or $|\nabla f(x)| > \beta$ for any $x \in \mathbb{T}^d$.

Let $g : \mathbb{T}^d \to \mathbb{R}$ be a smooth function such that $|g(x)| < \alpha/2$ and $|\nabla g(x)| < \beta/2$ for any $x \in \mathbb{T}^d$. 
Then there is a bijection \( \mathbb{Z}(f) \to \mathbb{Z}(f + g) \) mapping each \( \Gamma \in \mathbb{Z}(f) \) to a corresponding \( \tilde{\Gamma} \in \mathbb{Z}(f + g) \), which satisfies:

\[
\text{diam} \, \Gamma \leq \frac{2\alpha}{\beta} + \text{diam} \, \tilde{\Gamma}.
\]

The following proposition allows us to change our focus from the number of nodal components to the number of nodal domains and vice versa, by showing that their difference is very small. It is proven in a standard way using singular homology theory. A proof is presented in Appendix B.

**Proposition 4.5.** Let \( f : \mathbb{T}^d \to \mathbb{R} \) be smooth with a stable nodal set.

1. If \( f \) has \( k \) nodal components and \( r \) nodal domains, then \( r - 1 \leq k \leq r + d - 1 \).

2. If \( f \) has \( k' \) nodal components and \( r' \) nodal domains lying completely inside some open ball of radius less than \( \frac{1}{2} \), then \( k' \leq r' \).

### 4.2 The number and sum of diameters of nodal components of trigonometric polynomials

We denote by \( \mathcal{P}_D \) the linear space of trigonometric polynomials on \( \mathbb{T}^d \) of degree at most \( D \):

\[
\mathcal{P}_D := \text{Span} \left\{ \cos (2\pi \lambda \cdot x), \sin (2\pi \lambda \cdot x) : \lambda \in \mathbb{Z}^d, \|\lambda\|_1 \leq D \right\}.
\]

**Proposition 4.6.** If \( T \in \mathcal{P}_D \) has a stable nodal set then \( N(T) \lesssim D^d \) and \( \sum_{\Gamma \in \mathbb{Z}(T)} \text{diam} \, \Gamma \lesssim D^{d-1} \).

The first result, the bound on the number of nodal components of \( T \), is a trigonometric version of a classical bound on the sum of the Betti numbers of the nodal hypersurface of a polynomial due to Oleinik and Petrovsky, Milnor, and Thom. We obtain this by result by algebraizing (that is, writing the trigonometric polynomials as algebraic ones) and applying elimination theory and Bézout’s theorem to count the number of critical points. This result is obtained along the way of proving the second result, the bound on the sum of diameters, which is shown using simple integral-geometric tools: The diameter of \( \Gamma \in \mathbb{Z}(T) \) is comparable to its average width, which may be computed by measuring the set of hypersurfaces that intersect it (a Crofton-type formula).

In [NS09], a different approach is taken to bound the sum of diameters of nodal components in dimension \( d = 2 \) - it is bounded by a well-known estimate on the total length of the nodal set. However, although this estimate may be generalized to higher dimensions (where length is replaced by hypersurface volume), when \( d > 2 \) it fails to bound the sum of diameters; a nodal component might be a long, thin “noodle” having large diameter and small hypersurface volume.

It is likely that the stability condition in Proposition 4.6 may be lifted, but we assume it as it makes the proof a little simpler, and for \( f_L \), this condition is almost surely satisfied (see Proposition 5.5 in the next section).

**Algebraic background.** Given polynomials \( P_1, \ldots, P_n : \mathbb{C}^k \to \mathbb{C} \), denote by \( \mathbb{Z}(P_1, \ldots, P_n) \) their common zero set. Subsets of \( \mathbb{C}^k \) of this form are called *algebraic*, and their complements *coalgebraic*. The family of algebraic sets (thus also coalgebraic sets) is closed under finite unions and finite intersections, and any coalgebraic set is either empty or dense in \( \mathbb{C}^k \) (since a nonzero polynomial cannot vanish in an open set).

Any homogeneous polynomial of positive degree \( P : \mathbb{C}^{n+1} \to \mathbb{C} \) has a trivial zero at the origin; other zeroes are called *nontrivial zeroes*. Note that if \( P(z_0, \ldots, z_n) = 0 \), then \( P(\lambda z_0, \ldots, \lambda z_n) = 0 \) for any \( \lambda \in \mathbb{C} \), so nontrivial zeroes extend at least to their complex spans, called *solution rays*. 
In what follows, we consider polynomials in which some (possibly all) of the coefficients are indeterminate; that is, they are parameters which may be assigned complex values, and on which we may impose conditions. When considering polynomials with indeterminate coefficients, they have a formal degree, that is, the degree of the polynomial with all the coefficients written explicitly; the actual degree may become lower if some coefficients become zero after assigning values.

We recall two classical theorems. See, for instance, [vdW50, Sections 80 and 83].

- **The fundamental theorem of elimination theory**: Given a system of homogeneous polynomials of positive formal degree, the existence of a nontrivial common zero is an algebraic condition on the coefficients.

- **Bézout’s theorem**: Given \( n \) homogeneous polynomials \( P_1, \ldots, P_n : \mathbb{C}^n \rightarrow \mathbb{C} \), if they have finitely many common solution rays, then the number of common solution rays is at most \( \prod_{i=1}^{n} \deg P_i \).

It is easy see that Bézout’s theorem continues to hold under the weakened hypothesis that the polynomials have at most countably many common solution rays; we will later name this result “Bézout’s theorem” as well.

**Coalgebraic condition for finiteness.** We begin by presenting a coalgebraic condition for the finiteness of the set of solutions to a system of polynomials which we will use later in a few settings.

First, we define the homogenization of a polynomial \( P : \mathbb{C}^n \rightarrow \mathbb{C} \) of formal degree \( D \) as the homogeneous polynomial \( \tilde{P}(z_0, \ldots, z_n) := z_0^D P \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right) \). Note that plugging in \( z_0 = 1 \) yields the original polynomial, so any zero \( (\zeta_1, \ldots, \zeta_n) \) of \( P \) yields a solution ray of \( \tilde{P} : \text{Span}_\mathbb{C} \{ (1, \zeta_1, \ldots, \zeta_n) \} \). However, \( \tilde{P} \) may have more solution rays not obtained this way - solution rays where \( z_0 = 0 \).

Second, we recall that the Jacobian of \( n \) polynomials \( P_1, \ldots, P_n : \mathbb{C}^n \rightarrow \mathbb{C} \) is the polynomial given by \( \det \left( \frac{\partial P_i}{\partial z_j} \right)_{1 \leq i,j \leq n} \). In case that the Jacobian is nonzero wherever \( P_1, \ldots, P_n \) are zero, we get by the implicit function theorem that there are at most countably many common zeroes of \( P_1, \ldots, P_n \).

**Condition 4.7** (Condition for finiteness). Polynomials \( P_1, \ldots, P_n : \mathbb{C}^n \rightarrow \mathbb{C} \) are said to satisfy the condition for finiteness if both of the following hold:

1. The homogenizations \( \tilde{P}_1, \ldots, \tilde{P}_n \) have no common nontrivial zero with \( z_0 = 0 \).
2. At any common zero of \( P_1, \ldots, P_n \), their Jacobian is nonzero.

**Lemma 4.8.** Condition 4.7 is a coalgebraic condition on the coefficients of the polynomials \( P_1, \ldots, P_n \), and whenever it holds, the number of common zeroes of \( P_1, \ldots, P_n \) is finite and bounded by \( \prod_{i=1}^{n} \deg P_i \).

**Proof.** Let \( P_{n+1} \) be the Jacobian of \( P_1, \ldots, P_n \). Replacing the second part of the condition with “\( \tilde{P}_1, \ldots, \tilde{P}_n, \tilde{P}_{n+1} \) have no nontrivial common zero” yields exactly the same condition, since by the first part of the condition, there cannot exist such a solution ray with \( z_0 = 0 \), and solution rays with \( z_0 \neq 0 \) are in correspondence with zeroes of \( P_1, \ldots, P_n, P_{n+1} \). By the fundamental theorem of elimination theory, this is a coalgebraic condition on the coefficients of \( P_1, \ldots, P_n \) (and also \( P_{n+1} \), but the coefficients of \( P_{n+1} \) are themselves polynomials of the coefficients of \( P_1, \ldots, P_n \)). Bézout’s theorem then gives the required bound for the number of common solution rays of \( \tilde{P}_1, \ldots, \tilde{P}_n \) with \( z_0 \neq 0 \), which are in correspondence with the zeroes of \( P_1, \ldots, P_n \). \( \square \)

**Algebraization of trigonometric polynomials.** Laplacian eigenfunctions on the torus are trigonometric polynomials, so it would be fruitful to consider trigonometric polynomials in general in order to analyze them. Recall that trigonometric polynomials are linear combinations of trigonometric
monomials - cosines or sines of $2\pi\lambda \cdot x$, where $\lambda \in \mathbb{Z}^d$. The degree of a trigonometric monomial is $\|\lambda\|_1 = \sum_{i=1}^{d} |\lambda_i|$, and the degree of a trigonometric polynomial is the maximal degree among its monomials. The polynomial is said to be homogeneous when its monomials all have the same degree.

We would like to use algebraic tools, such as Bézout’s theorem, to analyze trigonometric polynomials. Therefore, we algebraize them - convert them to simple algebraic polynomials in a different space.

In the following definition, we denote by $x_1, \ldots, x_d$ the coordinates of $\mathbb{T}^d$ and by $c_1, s_1, \ldots, c_d, s_d$ the coordinates of $\mathbb{R}^{2d}$.

**Definition 4.9** (algebraization of trigonometric polynomials).

1. Given a trigonometric polynomial $T: \mathbb{T}^d \to \mathbb{R}$, its algebraization is the polynomial $P: \mathbb{R}^{2d} \to \mathbb{R}$ such that $\deg P = \deg T$ and

   $$T(x_1, \ldots, x_d) = P(\cos(2\pi x_1), \sin(2\pi x_1), \ldots, \cos(2\pi x_d), \sin(2\pi x_d)).$$

2. Let $T_1, \ldots, T_r$ be a system of $r$ trigonometric polynomials $\mathbb{T}^d \to \mathbb{R}$. The algebraization of this system is a system of $r + d$ polynomials $\mathbb{R}^{2d} \to \mathbb{R}$, the first $r$ being the algebraizations of $T_1, \ldots, T_r$ and the last $d$ being the polynomials $c_i^2 + s_i^2 - 1$ for $i = 1, \ldots, d$.

**Remark.** Given a trigonometric polynomial $T$, it is easy to see that its algebraization $P$ exists and is unique. Moreover, the coefficients of $P$ are linear combinations of the coefficients of $T$, and if $T$ is homogeneous then $P$ is homogeneous. Regarding systems of polynomials, it is easy to see that the assignments $c_i = \cos(2\pi x_i)$ and $s_i = \sin(2\pi x_i)$ for $i = 1, \ldots, d$ yield a bijective correspondence between the set of zeroes of a system of trigonometric polynomials and the set of (real) zeroes of the algebraized system.

**Coalgebraic conditions for regularity of trigonometric polynomials.** Let $T$ be a real trigonometric polynomial in $d$ variables $x_1, \ldots, x_d$. We present two regularity conditions, viewed as conditions on the coefficients of $T$.

**Condition 4.10** (Condition for nodal regularity). The homogenization of the algebraization of the system $T$, $\frac{\partial T}{\partial x_1}, \ldots, \frac{\partial T}{\partial x_d}$ has no common nontrivial complex zeroes.

Condition 4.10 is coalgebraic by the fundamental theorem of elimination theory, and it implies that $T$ has a stable nodal set (recall Definition 4.1).

**Condition 4.11** (Condition for critical set regularity). The algebraization of the system $\frac{\partial T}{\partial x_1}, \ldots, \frac{\partial T}{\partial x_d}$ satisfies the condition for finiteness (Condition 4.7).

By Lemma 4.8, Condition 4.11 is coalgebraic and it implies that $\{\nabla T = 0\}$ is a finite set.

**Condition 4.12** (Condition for full regularity). Suppose $d > 1$. $T$ is said to be fully regular if:

1. $T$ satisfies both the condition for nodal regularity and the condition for critical set regularity.
2. For all $j = 1, \ldots, d$, the restriction of $T$ to the hyperplane $\{x_j = 0\}$, viewed as a trigonometric polynomial in the remaining $d - 1$ variables, satisfies both the condition for nodal regularity and the condition for critical set regularity.

It is clear that Condition 4.12 is coalgebraic in the coefficients of $T$. We will later need our trigonometric polynomial $T$ to have regular restrictions to almost any hyperplane of the form $\{x_j = t\}$, not just $\{x_j = 0\}$. The following lemma shows that the full regularity condition is enough.
Lemma 4.13. Suppose $T$ is fully regular. Let $1 \leq j \leq d$. For all but finitely many values of $t \in T$, the restriction of $T$ to the hyperplane $\{ x_j = t \}$, viewed as a trigonometric polynomial in the remaining $d - 1$ variables, satisfies both the condition for nodal regularity and the condition for critical set regularity.

Proof. For any $t \in T$, let $T_t$ be the restriction of $T$ to the hyperplane $H := \{ x_j = t \}$.

Put $\kappa := \cos (2\pi t)$ and $\sigma := \sin (2\pi t)$. In what follows, $\kappa$ and $\sigma$ are treated as indeterminate parameters, like $t$. The gradient $\nabla T_t$ comprises $d - 1$ trigonometric polynomials on $T^{d-1}$ (since the coordinate $x_j$ is omitted in the restriction to $H$), so the algebraization of the system $T_t, \nabla T_t$ is a system of $2d - 1$ polynomials in $2d - 2$ variables $(c_1, s_1, \ldots, c_d, s_d$ without $c_j, s_j)$ whose coefficients are polynomial expressions in the coefficients of $T$ (which are not indeterminate) and the parameters $\kappa$ and $\sigma$.

Condition 4.10 and Condition 4.11 are coalgebraic, therefore, there exists a system of polynomials $P_1 (\kappa, \sigma), P_2 (\kappa, \sigma), \ldots$ that vanishes whenever $\kappa$ and $\sigma$ are assigned values for which the conditions are not satisfied. At least one of these polynomials (w.l.o.g. $P_1$) is not identically zero, because both conditions are satisfied for $t = 0$ (i.e. for $\kappa = 1$ and $\sigma = 0$).

Consider the function $Q(t) := P_1 (\cos (2\pi t), \sin (2\pi t))$. Whenever $Q(t) \neq 0$, $t$ is a value for which $T_t$ satisfies both conditions. We also know that $Q(0) \neq 0$, so $Q$ is an analytic function that isn’t identically zero. Therefore, $Q$ has at most finitely many zeroes in $T$, proving the lemma.

Abundance of fully regular trigonometric polynomials. Since the condition for full regularity is coalgebraic, it is enough to show it is nonempty to see that it is, in fact, dense in $P_D$. This is the content of the following:

Lemma 4.14. There exists a $T \in P_D$ that is fully regular.

Proof. We will show that

$$T(x_1, \ldots, x_d) := \sum_{j=1}^d \sin (2\pi D x_j) + A,$$

where $A > d$ is constant, satisfies the condition for nodal regularity and the condition for critical set regularity. Since the restriction of $T$ to a hyperplane of the form $\{ x_j = 0 \}$ takes exactly the same form as $T$ with dimension smaller by 1, this is enough to imply that $T$ is fully regular.

Denote by $C_D(c, s)$ and $S_D(c, s)$ the algebraizations of $\cos (2\pi Dx)$ and $\sin (2\pi Dx)$. These are homogeneous polynomials of degree $D$ for which it is a simple exercise to prove:

(a) $(C_D(c,s))^2 + (S_D(c,s))^2 = (c^2 + s^2)^D$.

(b) The only solution in $c, s \in \mathbb{C}$ of $C_D(c,s) = S_D(c,s) = 0$ is $c = s = 0$.

(c) $\det \begin{pmatrix} \frac{\partial C_D}{\partial c} (c,s) & \frac{\partial C_D}{\partial s} (c,s) \\ c & s \end{pmatrix} = D S_D(c,s)$.

The algebraization of $T, \nabla T$ is the following system of $2d + 1$ polynomials in $2d$ variables $c_1, s_1, \ldots, c_d, s_d$:

$$\sum_{j=1}^d S_D(c_j, s_j) + A, \quad \frac{2\pi D C_D(c_j, s_j)}{1 \leq j \leq d}, \quad \frac{c_j^2 + s_j^2 - 1}{1 \leq j \leq d} \quad (4.1)$$

The condition for nodal regularity (Condition 4.10) requires this system to have no common nontrivial zeroes after homogenization. We introduce a homogenizing variable $z_0$, and split to two cases: $z_0 = 0$ and $z_0 \neq 0$. Since $S_D$ and $C_D$ are already homogeneous of degree $D$, letting $z_0 = 0$, system (4.1) becomes:

$$\sum_{j=1}^d S_D(c_j, s_j) + A, \quad \frac{2\pi D C_D(c_j, s_j)}{1 \leq j \leq d}, \quad \frac{c_j^2 + s_j^2 - 1}{1 \leq j \leq d} \quad (4.2)$$

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If $c_1, s_1, \ldots, c_d, s_d$ is any zero of the system (4.2), then $C_D(c_j, s_j) = 0$ and $c_j^2 + s_j^2 = 0$ for $j = 1, \ldots, d$, so by property (a) above, $S_D(c_j, s_j) = 0$; by property (b), $c_j = s_j = 0$ for $j = 1, \ldots, d$, i.e. the zero must be trivial.

The other case is $z_0 \neq 0$, and it suffices to search for solutions with $z_0 = 1$, i.e. solutions to the original system (4.1). If $c_1, s_1, \ldots, c_d, s_d$ is any zero of the system (4.1), then $C_D(c_j, s_j) = 0$ and $c_j^2 + s_j^2 = 1$ for $j = 1, \ldots, d$. By property (a), $|S_D(c_j, s_j)| = 1$ for $j = 1, \ldots, d$. But then $\sum_{j=1}^d S_D(c_j, s_j) + A$ cannot be zero, since $A > d$. Therefore there are no solutions with $z_0 \neq 0$.

To check the condition for critical set regularity (Condition 4.11), we first write the polynomials in the algebraization of $\nabla T$ (as in system (4.1), excluding the first polynomial) in the following order:

$$2\pi D C_D(c_1, s_1), \ c_1^2 + s_1^2 - 1, \ldots, 2\pi D C_D(c_d, s_d), \ c_d^2 + s_d^2 - 1$$

(4.3)

This system should satisfy the condition for finiteness (Condition 4.7). We have already seen that its homogenization has no zero with $z_0 = 0$, and it remains to check that it has no zero common with its Jacobian. This Jacobian may be computed using property (c), and it equals $\mathcal{O}$. This Jacobian may be computed using property (c), and it equals $2\pi D C_D(c_1, s_1), \ c_1^2 + s_1^2 - 1, \ldots, 2\pi D C_D(c_d, s_d), \ c_d^2 + s_d^2 - 1$; this is impossible by property (a).

\[\Box\]

Proof of the main proposition for fully regular $T$. We may now prove a weak version of Proposition 4.6, assuming the full regularity condition; this assumption will later be lifted.

Background for the proof: Denote by $T^d$ the “round” torus - the $d$-dimensional submanifold of $\mathbb{R}^{2d}$ defined by the equations $c_j^2 + s_j^2 = 1, j = 1, \ldots, d$. Let $\varphi: T^d \to T^d$ be the natural diffeomorphism $\varphi(x_1, \ldots, x_d) := (\cos(2\pi x_1), \sin(2\pi x_1), \ldots, \cos(2\pi x_d), \sin(2\pi x_d))$. This diffeomorphism is not an isometry, but it induces a strongly equivalent metric; that is, there exist constants $a, b > 0$ such that for any $x, y \in T^d$, $a \text{ dist}(x, y) \leq |\varphi(x) - \varphi(y)| \leq b \text{ dist}(x, y)$.

We will need a little background in integral geometry. For any compact, connected set $K \subset \mathbb{R}^k$, denote by $W_j(K) := \max_{x,y \in K} |x_j - y_j|$ the width of $K$ along the $j$th axis, and by $W(K) := \frac{1}{k} \sum_{j=1}^k W_j(K)$ the average width\(^1\). It is an easy exercise to show that diam $K \lesssim W(K)$, and also that $W_j(K) = \int_{-\infty}^\infty I(K, \{x_j = \gamma\}) \, d\gamma$, where $I(A, B)$ is the intersection indicator function (1 if $A \cap B \neq \emptyset$, 0 otherwise). This is all we need; for more on the subject of integral geometry, see [KR97].

**Lemma 4.15.** Suppose $T \in \mathcal{P}_D$ is fully regular. Then $N(T) \lesssim D^d$ and $\sum_{\Gamma \in \mathcal{Z}(T)} \text{diam} \, \Gamma \lesssim D^{d-1}$.

**Proof.** By Lemma 4.8, the number of critical points of $T$ in $T^d$ is at most $O\left(D^d\right)$. There is at least one critical point in each nodal domain, since each nodal domain has a (necessarily nonzero) minimum or a maximum point by compactness. Therefore there are $O\left(D^d\right)$ nodal domains, and by Proposition 4.5, $O\left(D^d\right)$ nodal components, proving the first part.

For the second part, let $\Gamma \in \mathcal{Z}(T)$. We have:

\[\text{diam} \, \Gamma \lesssim \text{diam} \, \varphi(\Gamma) \lesssim W(\varphi(\Gamma)) = \frac{1}{2D} \sum_{j=1}^{2d} \int_{-\infty}^\infty I(\varphi(\Gamma), \{x_j = \gamma\}) \, d\gamma.\]

Since intersections are preserved by the bijection $\varphi$, the integrand $I(\varphi(\Gamma), \{x_j = \gamma\})$ can be written

\(^1\)This is a simpler version of the main width, defined similarly as an integral average of support functions.
Proposition 5.2. Suppose \( f \) is a Dirichlet eigenvalue of any nodal domain \( \Omega \). Any nodal domain \( \Omega \).

Proposition 5.1. Tools used in the proof

5.1 Proof of Theorem 1.3

Proof of Proposition 4.6. Let \( \alpha, \beta > 0 \) be such that \( |T(x)| > \alpha \) or \( |
\n\nProof of the main proposition without the regularity condition.

Proof of Proposition 4.6. Let \( \alpha, \beta > 0 \) be such that \( |T(x)| > \alpha \) or \( |\n\n\n\n\n\n5. Proof of Theorem 1.3

5.1 Tools used in the proof

Proposition 5.1. Any nodal domain \( \Omega \) of any \( f \in \mathcal{H}_L \) satisfies \( \text{vol}(\Omega) \gtrsim L^{-d} \).

Proposition 5.1 follows immediately from the classical Faber-Krahn inequality (note that the first Dirichlet eigenvalue of any nodal domain \( \Omega \) is \( 4\pi^2L^2 \)). See, for instance, [Cha84, Chapter IV].

Proposition 5.2. Suppose \( f : \mathbb{R}^d \to \mathbb{R} \) is smooth and satisfies \( \Delta f + 4\pi^2L^2 f = 0 \). Let \( x_0 \in \mathbb{R}^d \) and \( r > 0 \).
Then there is a constant $C = C(r, d) > 0$ such that:

\begin{align}
|f(x_0)|^2 & \leq CL^d \int_{B(x_0, r/L)} |f(x)|^2 \, dx \\
|\nabla f(x_0)|^2 & \leq CL^{d+2} \int_{B(x_0, r/L)} |f(x)|^2 \, dx \\
|\nabla^2 f(x_0)|^2 & \leq CL^{d+4} \int_{B(x_0, r/L)} |f(x)|^2 \, dx
\end{align}

Proposition 5.2 is a local property of functions on $\mathbb{R}^d$, and as such, it may be applied directly to functions in $\mathcal{H}_L$, viewed as functions on $\mathbb{R}^d$ periodic extension. These local bounds are special cases of very general classical local bounds on solutions of PDEs (see, for instance, [GT01, Chapter 8]), but for the sake of completeness, we provide a simple and easily readable proof for our case in Appendix B.

**Proposition 5.3.** $P \{ ||f_L|| > 2 \} \leq e^{-c \dim \mathcal{H}_L}$ for some absolute constant $c > 0$.

Considering $f_L$ as a random vector in $\mathbb{R}^n$ (where $n = \dim \mathcal{H}_L$) that is distributed like $\frac{X}{\sqrt{n}}$, where $X$ has standard multivariate normal distribution in $\mathbb{R}^n$, Proposition 5.3 may be formulated equivalently as $P \{ |X| > 2\sqrt{n} \} \leq e^{-cn}$. This is a special case of Bernstein’s classical inequalities and it may be proven by applying Chebyshev’s inequality on $\exp \left( \frac{1}{4} |X|^2 \right)$. We omit the details.

Again considering $f_L$ as a random vector in $\mathbb{R}^n$, the next proposition is a form of the Gaussian isoperimetric inequality (see [SC74], [Bor75]).

**Proposition 5.4.** Let $F \subset \mathcal{H}_L$ and for any $\rho > 0$, denote $F_\rho := \{ f \in \mathcal{H}_L : \text{dist}(f, F) \leq \rho \}$.

Suppose that $P \{ F_\rho \} \leq \frac{1}{2}$. Then $P \{ F \} \leq Ce^{-c\rho^2 \dim \mathcal{H}_L}$ for some absolute constants $C, c > 0$.

Next, recall that a smooth function is said to have a stable nodal set if it doesn’t have zeroes in common with its gradient (Definition 4.1). The following proposition follows either from Bulinskaya’s lemma [AT07, Lemma 11.2.10] or from [ORW08, Lemma 2.3].

**Proposition 5.5.** Almost surely, $f_L$ has a stable nodal set.

### 5.2 Concentration around the median implies concentration around the mean and limiting mean

We begin by showing that in Theorem 1.3, the first part implies the second and third parts. Throughout the remainder of this section, we denote $m_L := \text{Median} \{ N_L / L^d \}$.

Let $X_L := N_L / L^d - m_L$. By the first part of the theorem, $P \{ |X_L| \geq \varepsilon \} \leq C(\varepsilon) e^{-c(\varepsilon) \dim \mathcal{H}_L}$. The random variables $X_L$ are a.s. uniformly bounded, since $N_L / L^d$ are a.s. uniformly bounded by Courant’s nodal domain theorem (or alternatively by Proposition 4.6). By the law of total expectation:

$$
|E \{ X_L \}| \leq |E \{ X_L | X_L \geq \varepsilon \}| \cdot P \{ X_L \geq \varepsilon \}
+ |E \{ X_L | X_L \leq \varepsilon \}| \cdot P \{ |X_L| \leq \varepsilon \}
+ |E \{ X_L | X_L \leq -\varepsilon \}| \cdot P \{ X_L \leq -\varepsilon \}
$$

The first and third terms are bounded by $C(\varepsilon) e^{-c(\varepsilon) \dim \mathcal{H}_L}$ multiplied by the a.s. uniform bound on $X_L$, while the second term is bounded by $\varepsilon$. We may assume that $\dim \mathcal{H}_L$ is large enough (see Remark 1.4) such that the sum of the first and third terms is smaller than $\varepsilon$, and get $|E \{ X_L \}| \leq 2\varepsilon$. Therefore, by the triangle inequality:

$$
P \left( \frac{N_L}{L^d} - E \left\{ \frac{N_L}{L^d} \right\} > 3\varepsilon \right) = P \{ |X_L - E \{ X_L \}| > 3\varepsilon \} \leq P \{ |X_L| > \varepsilon \} \leq C(\varepsilon) e^{-c(\varepsilon) \dim \mathcal{H}_L}.$$

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Then the first part of Theorem 1.3 holds with constant $c$. We now present the construction of the set of instability, which we will later construct. This is an exponentially small set $E \subset \mathcal{H}_L$ such that outside this set, the number of nodal components is stable under sufficiently small perturbations.

**Proposition 5.6.** Suppose that for every $\varepsilon > 0$, there exist $\rho = \rho(\varepsilon) > 0$ and $\tau = \tau(\varepsilon) > 0$ such that for every $L$, there exists an “exceptional set of instability” $E = E(\varepsilon, L) \subset \mathcal{H}_L$ satisfying two conditions:

1. ($f_L$ has exponentially small probability to be exceptional.) For some constants $C(\varepsilon), c > 0$,
\[
\mathbb{P}(E) \leq \min\left\{\frac{1}{4}, C(\varepsilon) e^{-c\varepsilon^2 \dim \mathcal{H}_L}\right\}. \tag{5.4}
\]

2. ($N$ is lower semi-continuous for non-exceptional functions.) For any $f \in \mathcal{H}_L \setminus E$ and $g \in \mathcal{H}_L$ such that $\|g\| \leq \rho$,
\[
N(f + g) \geq N(f) - \varepsilon L^d. \tag{5.5}
\]

Then the first part of Theorem 1.3 holds with constant $c(\varepsilon)$ proportional to $\min\{\rho^2, \tau^2\}$.

**Proof.** Notice that $\{f \in \mathcal{H}_L : N(f) / L^d - m_L > \varepsilon\} = F \cup G$, where:
\[
F = \left\{f \in \mathcal{H}_L : N(f) > (m_L + \varepsilon) L^d\right\},
\]
\[
G = \left\{f \in \mathcal{H}_L : N(f) < (m_L - \varepsilon) L^d\right\}.
\]

First, we bound $\mathbb{P}(F)$. Let $h \in (F \setminus E)_{+\rho}$, that is, $h = f + g$ where $f \in F \setminus E$ and $g \in \mathcal{H}_L$ satisfies $\|g\| \leq \rho$. Then:
\[
N(h) = N(f + g) \geq N(f) - \varepsilon L^d, \quad \text{if } f \in F \setminus E.
\]

Therefore $(F \setminus E)_{+\rho} \subset \{h \in \mathcal{H}_L : N(h) / L^d > m_L\}$, the latter set having probability at most $\frac{1}{2}$.

**Proposition 5.4** gives $\mathbb{P}(F \setminus E) \leq Ce^{-c\varepsilon^2 \dim \mathcal{H}_L}$. Together with $\mathbb{P}(E) \leq C(\varepsilon) e^{-c\varepsilon^2 \dim \mathcal{H}_L}$, we have $\mathbb{P}(F) \leq C(\varepsilon) e^{-c(\varepsilon) \dim \mathcal{H}_L}$ with $c(\varepsilon)$ proportional to $\min\{\rho^2, \tau^2\}$.

Next, we bound $\mathbb{P}(G)$. Let $h \in G_{+\rho} \setminus E$; that is, $h = f + g$ where $f \in G, g \in \mathcal{H}_L$ satisfies $\|g\| \leq \rho$, and $h \not\in E$. Then:
\[
(m_L - \varepsilon) L^d \geq N(h - g) \geq N(h) - \varepsilon L^d.
\]

Therefore $G_{+\rho} \setminus E \subset \{h \in \mathcal{H}_L : N(h) / L^d < m_L\}$, the latter set having probability at most $\frac{1}{2}$. Thus $\mathbb{P}(G_{+\rho}) \leq \frac{1}{4}$, and applying Proposition 5.4 again gives $\mathbb{P}(G) \leq Ce^{-c\varepsilon^2 \dim \mathcal{H}_L}$.

**5.4 Construction of the exceptional set $E$**

We now present the construction of the set $E = E(\varepsilon, L) \subset \mathcal{H}_L$, so throughout this part of the paper, $\varepsilon$ and $L$ are fixed. We introduce new small parameters $0 < a, b, \delta < 1$ and one large parameter $R > 2$ that all depend only on $\varepsilon$ in a way that will be determined later.
We will now prove (5.4) under certain assumptions that will arise from the proof. We introduce two small parameters $0 < \gamma, \tau < 1$ that depend only on $\varepsilon$ in a way that will be determined later.

It suffices to prove (5.4) for $E$ instead of $\hat{E}$ where $\hat{E} := E \cap \{ f \in H_L : \| f \| \leq 2 \}$ as it has comparatively negligible probability.

Thus, let $f \in E$ with $\| f \| \leq 2$. In each ball $3B_j$ that is unstable with respect to $f$, fix a point $x_j \in 3B_j$ such that $\| f(x_j) \| \leq a$ and $\| \nabla f(x_j) \| \leq \beta L$.

Using the local bound from Proposition 5.2 with arbitrary $x_0 \in B(x_j, \gamma L^{-1})$ and $r = 1$, we have:

$$\sup_{x_0 \in B(x_j, \gamma L^{-1})} |\nabla f(x_0)|^2 \lesssim \sup_{x_0 \in B(x_j, L^{-1})} \int_{B(x_0, L^{-1})} |f(x)|^2 \, dx \lesssim L^{d+4} \int_{B(x_j, 2L^{-1})} |f(x)|^2 \, dx.$$  \hfill (5.6)

Since $R > 2$, we have $B(x_j, 2L^{-1}) \subset 4B_j$, and the amount of balls $B(x_j, 2L^{-1})$ that any point $x \in \mathbb{T}^d$ belongs to is $O(1)$. Therefore, summing over all balls:

$$\sum_j \int_{B(x_j, 2L^{-1})} |f(x)|^2 \, dx \lesssim \| f \|^2 \lesssim 1.$$ \hfill (5.7)

Plugging (5.7) into (5.6) yields $\sum_j \sup_{B(x_j, \gamma L^{-1})} |\nabla f|^2 \lesssim L^{d+4}$. There are at least $\delta L^d$ summands, so the average summand is $O(L^4 \delta^{-1})$. Multiplying the hidden constant by 4, at least a proportion $\frac{3}{4}$ of all indexes $j$ satisfy:

$$\sup_{B(x_j, \gamma L^{-1})} |\nabla f| \lesssim L^2 \delta^{-1/2}. \hfill (5.8)$$

Now, introduce a perturbation $g \in H_L$ with $\| g \| \leq \tau$. By the same argument as above, using Proposition 5.2 and summing over $j$, we have $\sum_j \sup_{B(x_j, \gamma L^{-1})} |g|^2 \lesssim L^d \tau^2$, and at least a proportion $\frac{3}{4}$ of all indexes $j$ satisfy:

$$\sup_{B(x_j, \gamma L^{-1})} |g| \lesssim \tau \delta^{-1/2}. \hfill (5.9)$$

Using this argument for the third time, now with $\nabla g$, we have $\sum_j \sup_{B(x_j, \gamma L^{-1})} |\nabla g|^2 \lesssim L^{d+2} \tau^2$, so at least a proportion $\frac{3}{4}$ of all indexes $j$ satisfy:

$$\sup_{B(x_j, \gamma L^{-1})} |\nabla g| \lesssim \tau L^d \delta^{-1/2}. \hfill (5.10)$$

Thus, at least a proportion $\frac{1}{4}$ of all indexes $j$, i.e. at least $\frac{1}{4} \delta L^d$ indexes, satisfy all three (5.8), (5.9).
and (5.10). For such indexes \( j \), applying Taylor’s formula on \( f \) and on \( \nabla f \), we get:

\[
\sup_{B(x_j, \gamma L^{-1})} |f| \leq a + \beta \gamma + C \gamma^2 \delta^{-1/2}
\]

(5.11)

\[
\sup_{B(x_j, \gamma L^{-1})} |\nabla f| \leq \left( \beta + C \gamma \delta^{-1/2} \right) L
\]

(5.12)

Here, \( C \) is the hidden constant from (5.8).

By summing (5.9) + (5.11) and (5.10) + (5.12), we get:

\[
\sup_{B(x_j, \gamma L^{-1})} |f + g| \leq a + \beta \gamma + C \delta^{-1/2} \left( \gamma^2 + \tau \right)
\]

(5.13)

\[
\sup_{B(x_j, \gamma L^{-1})} |\nabla f + \nabla g| \leq \left( \beta + C \delta^{-1/2} (\gamma + \tau) \right) L
\]

Therefore, \( f + g \in U \), where:

\[
U := \left\{ h \in \mathcal{H}_L : \vol \left\{ x \in \mathbb{T}^d : |h(x)| \leq A, |\nabla h(x)| \leq BL \right\} \geq c \delta \gamma^d \right\}.
\]

For every fixed \( x \in \mathbb{T}^d \), we have \( \mathbb{P} \{ |f_L(x)| \leq A, |\nabla f_L(x)| \leq BL \} \lesssim AB^d \) due to the independence of the random variable \( f_L(x) \) and the random vector \( \nabla f_L(x) \) and the fact that they have bounded densities in \( \mathbb{R} \) and \( \mathbb{R}^d \), respectively. By Fubini’s theorem, we also have:

\[
\mathbb{E} \left\{ \vol \left\{ x \in \mathbb{T}^d : |h(x)| \leq A, |\nabla h(x)| \leq BL \right\} \right\} \lesssim AB^d.
\]

Thus, if we assume \( AB^d \leq C \delta \gamma^d \) for an appropriate constant \( C \), we get by Chebyshev’s inequality that \( \mathbb{P} (U) \leq \frac{1}{2} \) (or any other constant smaller than \( \frac{1}{2} \); since \( \tilde{E} \subset U \), we may decrease this constant to ensure \( \mathbb{P} (E) \leq \frac{1}{4} \)). Since furthermore \( \mathbb{P} \left( \tilde{E} + \tau \right) \leq \frac{1}{4} \), by Proposition 5.4 we get \( \mathbb{P} (E) \leq Ce^{-cr^2 \dim \mathcal{H}_L} \).

To conclude, we have showed (5.4) under the following assumption:

**Assumption 5.7.** \( (a + \beta \gamma + C \delta^{-1/2} (\gamma^2 + \tau)) \left( \beta + C \delta^{-1/2} (\gamma + \tau) \right)^d \lesssim \delta \gamma^d \).

### 5.6 Proof that \( N \) is lower semi-continuous outside of \( E \)

Next, we prove (5.5) for \( f \in \mathcal{H}_L \setminus E \) and \( g \in \mathcal{H}_L \) with \( ||g|| \leq \rho \), where \( 0 < \rho < 1 \) is a new parameter depending only on \( \epsilon \) in a way that will be determined later. Again, this proof will require a few assumptions.

We may assume \( f \) has a stable nodal set (this is a.s. true). For each nodal component \( \Gamma \in Z (f) \), we pick one ball \( B_j \) that intersects it, and call it the intersecting ball of \( \Gamma \). Now, assume three conditions:

(a) \( \text{diam} \ Gamma \leq RL^{-1} \); (b) \( 3B_j \) is a stable ball for \( f \); and (c) \( \sup_{3B_j} |g| < \alpha \). A component \( \Gamma \in Z (f) \) that satisfies these conditions (with \( B_j \) being its intersecting ball) is said to be a controllable component.

Since the radius of \( B_j \) if \( RL^{-1} \), we have \( \Gamma \subset 2B_j \), thus \( \Gamma \) is \( RL^{-1} \)-separated from the boundary of \( 3B_j \); under an additional assumption that \( R \geq \alpha/\beta \), we have \( \Gamma + \alpha/(\beta L) \subset 3B_j \). Since \( 3B_j \) is a stable ball for \( f \), we have \( |f(x)| > \alpha \) or \( |\nabla f(x)| > \beta L \) for any \( x \in 3B_j \), and also \( |g(x)| < \alpha \). By Proposition 4.3, \( \Gamma \) generates a component \( \tilde{\Gamma} \in Z (f + g) \) that is also contained in \( 3B_j \), and the mapping \( \Gamma \rightarrow \tilde{\Gamma} \) is injective for all \( \Gamma \) satisfying condition (a) above with intersecting ball \( B_j \). Thus, all controllable components \( \Gamma \) generate different components \( \tilde{\Gamma} \) under perturbation by \( g \), and their number does not decrease.

Thus, to prove (5.5) it remains to show that the number of components that are not controllable can be bounded by \( \epsilon L^d \).
First, by Proposition 4.6, the number of components $\Gamma$ with $\text{diam} \Gamma > RL^{-1}$ is $a.s.$ $O \left( L^d R^{-1} \right)$, and we get the required bound with an additional assumption: $R^{-1} \lesssim \epsilon$.

Second, suppose $\text{diam} \Gamma \leq RL^{-1}$ (thus $\Gamma \subset 2B_j$), but $3B_j$ is an unstable ball for $f$. The number of components $\Gamma$ that can fit into $2B_j$ is at most $O \left( R^d \right)$: If the radius of $2B_j$ is greater than half, then this is trivial (as there are no more than $O \left( L^d \right)$ components in general); otherwise, by Proposition 4.5 it suffices to bound the number of nodal domains, which has the required bound by Proposition 5.1. Since $f \notin E$, there are at most $\delta L^d$ unstable balls, so there are $O \left( \delta L^d R^d \right)$ components of this type, which has the required bound with an additional assumption: $\delta R^d \lesssim \epsilon$.

Finally, suppose $k$ of the balls $3B_j$ satisfy $\sup_{3B_j} |g| \geq \alpha$. For each such ball, fix $x_j \in 3B_j$ for which $|g(x_j)| \geq \alpha$. By Proposition 5.2, we have:

$$a^2 \leq |g(x_j)|^2 \lesssim L^d \int_{B(x_j,2L^{-1})} |g(x)|^2 \, dx.$$ 

Since a constant proportion of the balls $B(x_j,2L^{-1})$ may be dropped leaving a disjoint collection of balls, we get by summing over the remaining balls:

$$k \lesssim L^d \alpha^2 \rho^2.$$

We have seen that at most $O \left( R^d \right)$ components have $B_j$ as their intersecting ball, so the number of components for which the intersecting ball satisfies $\sup_{3B_j} |g| \geq \alpha$ is at most $O \left( L^d R^d \alpha^{-2} \rho^2 \right)$, which has the required bound with an additional assumption: $R^d \alpha^{-2} \rho^2 \lesssim \epsilon$.

To summarize, (5.5) is proven assuming the following assumptions:

**Assumption 5.8.** $R \geq \alpha / \beta$.

**Assumption 5.9.** $R^{-1} \lesssim \epsilon$.

**Assumption 5.10.** $\delta R^d \lesssim \epsilon$.

**Assumption 5.11.** $R^d \alpha^{-2} \rho^2 \lesssim \epsilon$.

### 5.7 Choice of parameters

It remains to choose values for the small parameters $a, \beta, \delta, \gamma, \tau, \rho$ and large parameter $R$ that were used in the previous three subsections; firstly, to satisfy the five assumptions that arose from the proofs, and secondly, to maximize the value $c(\epsilon) \simeq \min \{ \rho^2, \tau^2 \}$ that appears in Theorem 1.3.

The conditions to satisfy are asymptotic inequalities, so we express each parameter asymptotically as a power of $\epsilon$: $a \simeq \epsilon^d, \beta \simeq \epsilon^b, \delta \simeq \epsilon^2k, \gamma \simeq \epsilon^g, \tau \simeq \epsilon^t, \rho \simeq \epsilon^h$ and $R \asymp \epsilon^{-r}$. Each of $a, b, k, g, t, h, r$ must be a positive real number, and the five asymptotic inequalities above can be expressed as the following constraints on the exponents:

\[
2k + dg \leq \min \{ a, b + g, 2g - k, t - k \} + d \cdot \min \{ b, g - k, t - k \} \\
\] (5.13)

\[
b \leq a + r \] (5.14)

\[
r \geq 1 \] (5.15)

\[
2k \geq 1 + rd \] (5.16)

\[
2h \geq 1 + 2a + rd \] (5.17)

That is, we want to find positive values that satisfy the above, and minimize $\max \{ h, t \}$. 


First, note that \( h \) is minimized simply by changing (5.17), the only inequality it appears in, to an equation, and we get \( 2h = 1 + 2a + rd \).

Next, observe that in (5.13), any choice of two minimal values on the right hand side creates a simple inequality. There are 12 such inequalities, and the inequality (5.13) is equivalent to all 12 occurring simultaneously. Of those 12 inequalities, 6 are constraints on \( t \) (in the other 6, \( t \) does not appear). In the 6 constraints on \( t \), we find that increasing \( b \), decreasing \( k \) or decreasing \( g \) either weakens or doesn’t change the constraint on minimizing \( t \). Therefore, \( b \) must be maximized and \( k \) and \( g \) must be minimized. To maximize \( b \), we may change the inequality (5.14) into an equation \( b = a + r \), and to maximize \( k \), (5.16) gives \( 2k = 1 + rd \).

Since \( b + g = a + r + g > a \), we may drop the term \( b + g \) from the first minimum in the right hand side of (5.13). This leaves only one inequality that bounds \( g \) from below - the one obtained by choosing \( 2g - k \) as the first minimum and \( g - k \) as the second. From this, we get \( 2g = (3 + d)k \).

Finally, decreasing \( r \) decreases \( h \) and does not affect any constraint on \( t \), so \( r \) must be minimized. The only remaining constraint on \( r \) is (5.16), so we get \( r = 1 \).

Collecting and simplifying all of the above equations, we have:

\[
\begin{align*}
b &= a + 1 \\
2k &= d + 1 \\
4g &= (d + 1)(d + 3) \\
2h &= d + 1 + 2a \\
r &= 1
\end{align*}
\]

Our target is to minimize \( \max \{ h, t \} \). Decreasing \( a \) decreases \( h \), but tightens the constraints in (5.13) on minimizing \( t \). Therefore, we set \( t = h \) and find minimal value of \( a \) by the constraint (5.13), which may now be written as:

\[
(d + 1) \left( d^2 + 3d + 4 \right) \leq \min \{ 4a, 2(d + 1)(d + 2) \} + d \cdot \min \left\{ 4a, (d + 1)^2 \right\}.
\]

The minimal solution to this inequality is:

\[
2a = (d + 1)(d + 2),
\]

and we can see that all constraints are satisfied, with \( 2h = 2t = (d + 2)^2 - 1 \); so, \( c(\varepsilon) \approx \varepsilon^{(d+2)^2-1} \).

### A Equidistribution of lattice points on spheres

For completeness, we present the known results on equidistribution of lattice points on spheres (as in Definition 1.1) in various dimensions \( d \).

In dimension \( d \geq 5 \), it was shown in [Pom59] that we have equidistribution unconditionally, and any sequence \( L \to \infty \) (with \( L^2 \in \mathbb{Z} \)) is admissible.

When \( d = 4 \), any natural number is a sum of four squares, but there are arbitrarily large values of \( L \) with few representations, and \( \dim \mathcal{H}_L \) may remain bounded as \( L \to \infty \). Requiring \( \dim \mathcal{H}_L \to \infty \) (for instance, by bounding the multiplicity of the prime 2 in \( L^2 \)) yields equidistribution, again by [Pom59] (see also [Mal57]).

In dimension \( d = 3 \), a congruence condition \( L^2 \not\equiv 0, 4, 7 \pmod{8} \) ensures \( \dim \mathcal{H}_L \to \infty \) (thus, bounding the multiplicity of the prime 2 in \( L^2 \) also ensures this). The question of whether \( \dim \mathcal{H}_L \to \infty \)
implies equidistribution is very difficult, and was answered affirmatively in [GF87] and [Duk88] following a breakthrough by Iwaniec [Iwa87]. See also [Duk07].

The equidistribution question in dimension $d = 2$ is trickier than higher dimensions. Any condition that simply ensures $\dim \mathcal{H}_L \to \infty$ must strongly depend on the prime decomposition of $L^2$, as can be concluded from Gauss’s classical formula for the number of representations of integers as sums of two squares. Furthermore, it turns out that the condition $\dim \mathcal{H}_L \to \infty$ is not strong enough to ensure equidistribution, and the limit measure may even be a sum of 4 atoms, as shown in [Cil93]. On the positive side, equidistribution can be proven for a subsequence of relative density 1 in the sequence of sums of two squares, as shown in [KK77] and [EH99] (see also [FKW06]).

In case that $d = 2$ and the integer points on the circle accumulate according to a non-uniform limiting measure, if it has no atoms, then the result of Theorem 1.2 still holds. However, the value of $\nu$ depends on the limiting measure. This situation is further investigated in [KW15]; see also [BW15].

Regarding the number of lattice points, when $d = 4$ we have $\dim \mathcal{H}_L \gtrsim L^{1-2}$ (under the correct assumptions when $d = 4$) by the classical Hardy-Littlewood circle method - see, for instance, [Gro85, Chapter 12]. When $d = 3$, we have (under the correct assumptions) $\dim \mathcal{H}_L \gtrsim c(d) L^{1-\delta}$ for any fixed $0 < \delta < 1$, due to Siegel - see [Dav80, Chapter 21]. Finally, when $d = 2$, we may find equidistributed subsequences of relative density 1 that satisfy $\dim \mathcal{H}_L \gtrsim (\log L)^\gamma$ for any fixed $0 < \gamma < \frac{1}{2} \log \frac{2}{\pi} \approx 0.226$, by [EH99].

### B Additional proofs

#### B.1 Stability of nodal sets - the “shell lemma” and Propositions 4.3 and 4.4

Let $\alpha, \beta > 0$ and let $U$ be an open subset of $\mathbb{T}^d$. Let $f : U \to \mathbb{R}$ be a smooth function such that $|f(x)| > \alpha$ or $|\nabla f(x)| > \beta$ for any $x \in U$.

The “shell lemma”, given below (cf. [NS09, Claim 4.2]), shows that each connected component of $\{x \in U : f(x) = 0\}$ which is not too close to $\partial U$ is contained in a “shell”, which is a connected component of $\{x \in U : |f(x)| < \alpha\}$, and the shells satisfy certain properties. Proposition 4.3 follows immediately from this lemma, and the proof of Proposition 4.4, given below, also follows from it.

Before presenting the lemma and its proof, we construct a vector field whose integral curves are used in the proof. See, for instance, [Lee13, Chapter 9] for the necessary background in the theory of integral curves and flows on smooth manifolds.

Let $M := \{x \in U : \nabla f(x) > \beta\}$. On the open submanifold $M$, define the following vector field:

$$V := \frac{\nabla f}{|\nabla f|^2}.$$  

For any $p \in M$, let $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ be the integral curve starting at $p$ with respect to the vector field $V$ (where $\mathcal{D}^{(p)}$ is an open interval containing zero, the curve’s maximal domain). It is easy to see that this integral curve has the following three properties (see Figure 1):

1. $f(\theta^{(p)}(t)) = f(p) + t$ for any $t \in \mathcal{D}^{(p)}$ (because the left side has constant derivative 1).

2. $\text{dist}(p, \theta^{(p)}(t)) \leq t/\beta$ (because $|V| \leq \beta^{-1}$).

3. If $p$ is such that $f(p) = 0$ and $\bar{B}(p, \alpha/\beta) \subset U$, then $[-\alpha, \alpha] \subset \mathcal{D}^{(p)}$.

Following the above construction, we now formulate and prove the shell lemma. The shells described by the lemma are illustrated in Figure 2.
Lemma B.1 (the shell lemma).

(i) Each connected component $\Gamma$ of \{ $x \in U : f(x) = 0$ \} that satisfies $\Gamma_{+\alpha/\beta} \subset U$ is contained in an open, connected "shell" $S_{\Gamma} \subset \{ x \in U : |f(x)| < \alpha \}$ whose boundary consists of two components, with $f = \alpha$ on one and $f = -\alpha$ on the other.

(ii) $S_{\Gamma} \subset \Gamma_{+\alpha/\beta}$, and for any point $p \in \Gamma$, the ball $B \{ p, \alpha/\beta \}$ contains a path through $p$ from one boundary component of $S_{\Gamma}$ to the other.

(iii) Given two such components $\Gamma_1 \neq \Gamma_2$, the shells $S_{\Gamma_1}, S_{\Gamma_2}$ are disjoint.

(iv) $S_{\Gamma}$ may be decomposed as $\Gamma \cup S_{\Gamma}^+ \cup S_{\Gamma}^-$, where $S_{\Gamma}^+ = \{ x \in S_{\Gamma} : f(x) > 0 \}, S_{\Gamma}^- = \{ x \in S_{\Gamma} : f(x) < 0 \}$.

In this decomposition, $S_{\Gamma}^+$ and $S_{\Gamma}^-$ are connected open sets.

(v) In case $U = \mathbb{T}^d$, the shells $\{ S_{\Gamma} \}_{\Gamma \in \mathcal{Z}(f)}$ are precisely the connected components of $\{ x \in \mathbb{T}^d : |f(x)| < \alpha \}$.

Proof. Let $\Gamma$ be a connected component of $\{ x \in U : f(x) = 0 \}$ that satisfies $\Gamma_{+\alpha/\beta} \subset U$. Define:

$$S_{\Gamma} := \{ \theta(x_0) (t) : |t| < \alpha, x_0 \in \Gamma \}.$$ 

For any $p \in S_{\Gamma}$, if $p = \theta(x_0) (t)$, then $t = f (p)$ and $x_0 = \theta(p) (-t)$. Therefore, the smooth map $(-\alpha, \alpha) \times \Gamma \to S_{\Gamma}$ given by $(t, x_0) \mapsto \theta(x_0) (t)$ has a smooth inverse, and we have that $(-\alpha, \alpha) \times \Gamma$ is diffeomorphic to $S_{\Gamma}$. Therefore $S_{\Gamma}$ is open, connected, and has exactly two boundary components, both diffeomorphic to $\Gamma$, with $f = \alpha$ on one and $f = -\alpha$ on the other, proving (i). The path described in (ii) is given by $\theta(p) ([-\alpha, \alpha])$. For (iii), note that for each $p \in S_{\Gamma}$ we have $\theta(p) (-f (p)) \in \Gamma$, so $p$ cannot be simultaneously in $S_{\Gamma_1}$ and $S_{\Gamma_2}$. (iv) follows from the fact that $S_{\Gamma}^+$ and $S_{\Gamma}^-$ are continuous images of the connected sets $(0, \alpha) \times \Gamma$ and $(-\alpha, 0) \times \Gamma$, respectively.

Finally, we prove (v). Suppose $U = \mathbb{T}^d$, the shells $S_{\Gamma}$ are among the connected components of $\{ x \in \mathbb{T}^d : |f(x)| < \alpha \}$, and it remains to show that there is no other connected component $S$. We have $f = \pm \alpha$ on $\partial S$, but $f$ cannot be constant on $\partial S$ or there would have to be a point inside $S$ where $\nabla f = 0$, a contradiction. Therefore there must be some $\Gamma \in \mathcal{Z}(f)$ lying inside $S$, and we have $S = S_{\Gamma}$.  

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Proof of Proposition 4.4. By applying Proposition 4.3 directly, we get \( N(f) \leq N(f + g) \), and by applying Proposition 4.3 on the function \( f + g \) with perturbation \( -g \), we get \( N(f + g) \leq N(f) \). Therefore, the mapping \( \Gamma \mapsto \tilde{\Gamma} \) given in Proposition 4.3 is a bijection, and each \( \tilde{\Gamma} \) is the only component of \( Z(f + g) \) lying inside \( S_{\Gamma} \). Let \( p, q \in \Gamma \) be a pair of points realizing the diameter: \( \text{diam} \, \Gamma = \text{dist} \, (p, q) \). By property (ii) in Lemma B.1, the ball \( B(p, \alpha/\beta) \) contains a point \( \tilde{p} \in Z(f + g) \), which must then belong to \( \tilde{\Gamma} \). Similarly, the ball \( B(q, \alpha/\beta) \) contains a point \( \tilde{q} \in \tilde{\Gamma} \). By the triangle inequality,

\[
\text{diam} \, \Gamma = \text{dist} \, (p, q) \leq \text{dist} \, (p, \tilde{p}) + \text{dist} \, (\tilde{p}, \tilde{q}) + \text{dist} \, (\tilde{q}, q) \leq \frac{\alpha}{\beta} + \text{diam} \, \tilde{\Gamma} + \frac{\alpha}{\beta}.
\]

B.2 Counting nodal components vs. counting nodal domains - Proposition 4.5

We prove Proposition 4.5 using an elementary concept in singular homology theory - the Mayer-Vietoris sequence (see, for instance, [Hat02, Section 2.2]). We denote by \( H_n(X) \) the \( n \)th singular homology group of the topological space \( X \), and by \( \cong \) an isomorphism of groups.

Proof of Proposition 4.5. Let \( \alpha, \beta > 0 \) be such that \( |f(x)| > \alpha \) or \( |\nabla f(x)| > \beta \) for any \( x \in T^d \) (see Remark 4.2). Define two sets \( A, B \subset T^d \) by:

\[
A := \left\{ x \in T^d : f(x) \neq 0 \right\}, \quad B := \left\{ x \in T^d : |f(x)| < \alpha \right\}.
\]

\( A \) and \( B \) are both open, and \( A \cup B = T^d \). We count the connected components of \( A, B \) and \( A \cap B \):

- The connected components of \( A \) are precisely the nodal domains, so \( A \) has \( r \) components.
- The connected components of \( B \) are precisely the shells \( S_{\Gamma} \) defined in Lemma B.1, which are in correspondence with the nodal components, so \( B \) has \( k \) components.
- By Lemma B.1 (iv), excluding \( \Gamma \) from any shell \( S_{\Gamma} \) leaves it with exactly two components, so \( A \cap B \) has \( 2k \) components.
Consider the last four terms of the Mayer-Vietoris sequence, and name the nonzero maps \( \phi_1, \phi_2, \phi_3 \):

\[
\cdots \rightarrow H_1 \left( \mathbb{T}^d \right) \xrightarrow{\phi_1} H_0 \left( A \cap B \right) \xrightarrow{\phi_2} H_0 \left( A \right) \oplus H_0 \left( B \right) \xrightarrow{\phi_3} H_0 \left( \mathbb{T}^d \right) \rightarrow 0.
\]

Written explicitly:

\[
\cdots \rightarrow \mathbb{Z}^d \xrightarrow{\phi_1} \mathbb{Z}^{2k} \xrightarrow{\phi_2} \mathbb{Z}^{r+k} \xrightarrow{\phi_3} \mathbb{Z} \rightarrow 0.
\]

\( \phi_1, \phi_2, \phi_3 \) are group homomorphisms, and they may be extended naturally to linear maps between \( \mathbb{Q} \)-vector spaces, so the rank-nullity theorem applies and we get:

\[
d = \text{rank ker } \phi_1 + \text{rank im } \phi_1
\]

\[
2k = \text{rank ker } \phi_2 + \text{rank im } \phi_2 \quad \text{(B.1)}
\]

\[
r + k = \text{rank ker } \phi_3 + \text{rank im } \phi_3
\]

By the exactness of the Mayer-Vietoris sequence:

\[
\text{im } \phi_1 = \text{ker } \phi_2
\]

\[
\text{im } \phi_2 = \text{ker } \phi_3 \quad \text{(B.2)}
\]

\[
\text{im } \phi_3 = \mathbb{Z}
\]

Plugging equations (B.2) into equations (B.1), we get:

\[
\text{rank ker } \phi_1 = d - k + r - 1.
\]

Since \( \text{ker } \phi_1 \) is a subgroup of \( \mathbb{Z}^d \), we have \( 0 \leq \text{rank ker } \phi_1 \leq d \), leading to the required conclusion.

For the second part, let \( \Gamma_1, \ldots, \Gamma_{k'} \in \mathbb{Z}(f) \) be the nodal components of \( f \) that lie completely inside some open ball \( U \) with radius less than \( \frac{1}{2} \). Suppose the set \( A := U \setminus (\Gamma_1 \cup \cdots \cup \Gamma_{k'}) \) has \( s \) components. Exactly one touches the (connected) boundary of \( U \). The other \( s - 1 \) have \( f = 0 \) on their boundary, and each of them must contain at least one nodal domain of \( f \) that lies completely inside \( U \), so \( s - 1 \leq r' \).

We conclude by showing that \( s = k' + 1 \). Let \( a, \beta > 0 \) be such that \( |f(x)| > a \) or \( |\nabla f(x)| > \beta \) for any \( x \in \mathbb{T}^d \), and dist \((\Gamma_i, \partial U) \geq a/\beta \) for any \( 1 \leq i \leq k' \) (see Remark 4.2). By Lemma B.1, each \( \Gamma_i \) is contained in a shell \( S_{\Gamma_i} \subset U \). Define \( B := S_{\Gamma_1} \cup \cdots \cup S_{\Gamma_{k'}} \). As in the first part, \( A, B \) are open and \( A \cup B = U \), \( A \) has \( s \) components, \( B \) has \( k' \) components, and \( A \cap B \) has \( 2k' \) components. \( U \) is simply-connected (because its radius is less than \( \frac{1}{2} \)), so the last four terms of the Mayer-Vietoris sequence form the short exact sequence \( 0 \rightarrow \mathbb{Z}^{2k'} \rightarrow \mathbb{Z}^{s+k'} \rightarrow \mathbb{Z} \rightarrow 0 \), and we get \( s = k' + 1 \).

\[\square\]

### B.3 Local bounds on eigenfunctions - Proposition 5.2

**Proof of Proposition 5.2.** W.l.o.g. we assume that \( x_0 = 0 \), and also that \( L = 1 \) - for the latter, simply apply the result to \( \tilde{f}(x) := f(x/L) \). Thus, \( f \) satisfies \( \Delta f + 4\pi^2 f = 0 \).

We utilize a trick to view \( f \) as a harmonic function in one more dimension. Define \( u : \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) by:

\[
u(x_1, \ldots, x_d, x_{d+1}) := f(x_1, \ldots, x_d) \cosh(2\pi x_{d+1}).
\]

Clearly, \( u \) is harmonic and \( u(0) = f(0), \frac{\partial u}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(0) \) and \( \frac{\partial^2 u}{\partial x_i \partial x_j}(0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \) for \( 1 \leq i, j \leq d \).
Using the mean value property in the ball $B^{d+1} \left( 0, r/\sqrt{d} \right)$, we get:

$$f (0) = C \int \cdots \int f (x_1, \ldots, x_d) \cosh (2\pi x_{d+1}) \, dx_1 \cdots dx_{d+1},$$

where $C$ is a positive constant depending only on $r, d$. Taking the absolute value and using the inclusions

$$B^{d+1} \left( 0, \frac{r}{\sqrt{d}} \right) \subset \left[ -\frac{r}{\sqrt{d}}, \frac{r}{\sqrt{d}} \right]^{d+1} \quad \text{and} \quad \left[ -\frac{r}{\sqrt{d}}, \frac{r}{\sqrt{d}} \right]^d \subset B^d (0, r),$$

we get:

$$|f (0)| \lesssim \int \cdots \int |f (x_1, \ldots, x_d)| |\cosh (2\pi x_{d+1})| \, dx_1 \cdots dx_{d+1} \lesssim \int_{B^d (0, r)} |f (x)| \, dx,$$

where the implied constant in $\lesssim$ depends only on $r, d$. By the Cauchy-Schwarz inequality, we get (5.1).

Next, we turn to (5.2) and (5.3). Note that for any $|x| < \frac{r}{2}$ in $\mathbb{R}^{d+1}$, we have:

$$u (x) = \left( \frac{r}{2} \right)^{d-1} \int_{|\zeta|=1} u \left( \frac{1}{2} r \zeta \right) P \left( \frac{1}{2} r \frac{x}{\zeta} \right) d\sigma_d (\zeta),$$

where $P (x, y) := |x|^2 - |y|^2 \quad |x-y|^{d+1}$ is the $(d+1)$-dimensional Poisson kernel. Differentiating under the integral sign with respect to $x_j$, we get:

$$\frac{\partial f}{\partial x_j} (0) = \frac{\partial u}{\partial x_j} (0) = \left( \frac{r}{2} \right)^{d-1} \int_{|\zeta|=1} u \left( \frac{1}{2} r \zeta \right) \frac{\partial P}{\partial x_j} \left( 0, \frac{1}{2} r \zeta \right) d\sigma_d (\zeta).$$

The Poisson kernel is smooth when its two parameters are separated, so $\zeta \mapsto \frac{\partial P}{\partial x_j} \left( 0, \frac{1}{2} r \zeta \right)$ is a continuous function on $\{|\zeta| = 1\}$. Its integral depends only on $r, d$. Therefore:

$$\left| \frac{\partial f}{\partial x_j} (0) \right| \lesssim \sup_{|\zeta|=1} \left| u \left( \frac{1}{2} r \zeta \right) \right| \lesssim \sup_{|x| \leq r/2} \sup_{|t| \leq r/2} |f (x)| \sup_{|y| \leq r/2} |\cosh (2\pi t)| \lesssim \sup_{|x| \leq r/2} |f (x)|.$$

Using (5.1) applied on $f$ at point $x$, this gives:

$$|\nabla f (0)|^2 = \sum_{j=1}^d \left| \frac{\partial f}{\partial x_j} (0) \right|^2 \lesssim \sup_{|x| \leq r/2} |f (x)|^2 \lesssim \sup_{|x| \leq r/2} \int_{B^d (x, r/2)} |f (y)|^2 \, dy \leq \int_{B^d (0, r)} |f (y)|^2 \, dy.$$

This proves (5.2). Similarly, applying (5.2) on $\frac{\partial f}{\partial x_j}$ at point 0 and then applying (5.1) on $\frac{\partial f}{\partial x_j}$ at point $x$, we get:

$$\left| \nabla \frac{\partial f}{\partial x_j} (0) \right|^2 \lesssim \int_{B^d (0, r/2)} \left| \frac{\partial f}{\partial x_j} (x) \right|^2 \, dx \lesssim \sup_{|x| \leq r/2} \left| \frac{\partial f}{\partial x_j} (x) \right|^2 \lesssim \int_{B^d (0, r)} |f (y)|^2 \, dy.$$

Summing over $j$ gives (5.3). □
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