Relative Entropy via Distribution of Observables

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Abstract

We obtain formulas for Petz-Rényi and Umegaki relative entropy from the idea of distribution of a positive selfadjoint operator. Classical results on Rényi and Kullback-Leibler divergences are applied to obtain new results and new proofs for some known results about Petz-Rényi and Umegaki relative entropy. Most important among these, is a necessary and sufficient condition for the finiteness of the Petz-Rényi $\alpha$-relative entropy. All of the results presented here are valid in both finite and infinite dimensions. In particular, these results are valid for states in Fock spaces and thus are applicable to continuous variable quantum information theory.

**Keywords:** Distribution of a quantum observable, Petz-Rényi relative entropy, Rényi divergence, Nussbaum-Szkoła distributions, Umegaki relative entropy

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1 Introduction

Relative entropic quantities like Petz-Rényi relative entropy and Umegaki relative entropy are used in quantum information theory to distinguish between states. The distribution of an observable is a fundamental notion in quantum probability. In this article, we prove that Petz-Rényi and Umegaki relative entropy of quantum states can be obtained using the idea of distribution of observables.

In a seminal article [1], Nussbaum and Szkoła associated two probability distributions to any pair of finite dimensional quantum states and used it to study the Petz-Rényi relative entropy. These distributions are now known as Nussbaum-Szkoła distributions. In the article [2], these distributions have been generalized to infinite dimensions and were used to prove several results about quantum $f$-divergence. In the present article, we focus on two special cases of $f$-divergence: Petz-Rényi relative entropy and Umegaki relative entropy. In Section 2, we specialize the general formula for $f$-divergence given in [2] to the case of Petz-Rényi relative entropy and Umegaki relative entropy. These formulas are provided in Theorem 2.6. We use these formulas in Sections 3 and 4 to prove that the Petz-Rényi relative entropy and Umegaki relative entropy can be obtained from the distribution of observables. The results of this article hold in both finite and infinite dimensions. Hence our results are applicable to continuous variable quantum information theory [3, 4, 5]. In particular, Theorem 2.6 of this article is used to find the precise range of $\alpha$ where the Petz-Rényi $\alpha$-relative entropy of certain class of gaussian states is finite [6].

Another application of our formulas for Petz-Rényi relative entropy and Umegaki relative entropy is the fact that these quantum entropic quantities (in both finite and infinite dimensions) coincide with the classical Rényi and Kullback-Leibler divergence, respectively. This is a generalization of Nussbaum and Szkola’s result to infinite dimensions. This provides a general framework to obtain quantum results from the existing literature on classical divergences. We illustrate this in Section 5 by proving several results about Petz-Rényi and Umegaki relative entropy using existing results about corresponding classical divergences.

Now we describe some preliminaries and fix some notation for the rest of the article. Let $\mathcal{K}$ be a complex Hilbert space with $\dim \mathcal{K} = |\mathcal{I}|$, where $\mathcal{I} = \{1, 2, \ldots, n\}$ for some
Let $\rho$ and $\sigma$ be states on $\mathcal{K}$ with spectral decomposition

$$
\rho = \sum_{i \in \mathcal{I}} r_i |u_i\rangle\langle u_i|, \quad r_i \geq 0, \quad \sum_{i \in \mathcal{I}} r_i = 1, \quad \{u_i\}_{i \in \mathcal{I}} \text{ is an orthonormal basis of } \mathcal{K};
$$

$$
\sigma = \sum_{j \in \mathcal{I}} s_j |v_j\rangle\langle v_j|, \quad s_j \geq 0, \quad \sum_{j \in \mathcal{I}} s_j = 1, \quad \{v_j\}_{j \in \mathcal{I}} \text{ is an orthonormal basis of } \mathcal{K}.
$$

The next definition of Nussbaum-Szkoła distribution is a direct generalization of the original definition in [1].

**Definition 1.1.** (Nussbaum-Szkoła distributions.) Define the Nussbaum-Szkoła distribution $P$ and $Q$ associated with $\rho$ and $\sigma$ on $\mathcal{I} \times \mathcal{I}$ by,

$$
P(i, j) = r_i |\langle u_i|v_j\rangle|^2, \quad Q(i, j) = s_j |\langle u_i|v_j\rangle|^2, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}.
$$

Now we define the $f$-divergence of two states $\rho$ and $\sigma$ as in [7] Definition 2.1. A motivation for this definition can be seen in [3] Equations 3.9 and 3.12. It may be noted from the above references that the most general definition of $f$-divergence uses Araki’s relative modular operator $\Delta_{\rho,\sigma}$ [8], but our setting only involves density operators in $\mathcal{B}(\mathcal{K})$.

Hence, the following explicit spectral decomposition of $\Delta_{\rho,\sigma}$ may be used,

$$
\Delta_{\rho,\sigma} = \sum_{\{i, j: r_i \neq 0, s_j \neq 0\}} r_i s_j^{-1} |X_{ij}\rangle\langle X_{ij}|, \quad (1.3)
$$

where

$$
X_{ij} = |u_i\rangle\langle v_j| \in \mathcal{B}_2(\mathcal{K}), \quad \forall i, j \in \mathcal{I}.
$$

A proof of (1.3) can be seen in [2] Proposition B1. Before defining the $f$-divergence, we need to fix a few notations and conventions. Let $\tau$ be any state on a Hilbert space $\mathcal{B}(\mathcal{K})$, then $\Pi_\tau$ denote the orthogonal projection onto the support of $\sigma$. For a convex (or concave) function $f : (0, \infty) \to \mathbb{R}$, let

$$
f(0) := \lim_{t \downarrow 0} f(t), \quad f'(\infty) := \lim_{t \to \infty} \frac{f(t)}{t}.
$$

**Definition 1.2.** Let $\rho$ and $\sigma$ be states on a Hilbert space $\mathcal{K}$. If $f : (0, \infty) \to \mathbb{R}$ is a convex (or concave) function then the $f$-divergence $D_f(\rho||\sigma)$ of $\rho$ from $\sigma$ is defined as

$$
D_f(\rho||\sigma) = \int_0^{\infty} f(\lambda) \langle \sqrt{\sigma} | \xi_{\Delta_{\rho,\sigma}}(d\lambda) | \sqrt{\sigma} \rangle^2 + f(0) \text{ Tr } (\sigma \Pi_\rho^+) + f'(\infty) \text{ Tr } (\rho \Pi_\sigma^+), \quad (1.5)
$$

where $\xi_{\Delta_{\rho,\sigma}}$ denote the spectral measure associated with Araki’s relative modular operator $\Delta_{\rho,\sigma}$ as discussed in (1.3).

Now we state the main result in [2], which is crucial for the present article.

**Theorem 1.3.** [2] Let $\rho, \sigma$ be as in (1.4) and $P, Q$ denote the corresponding Nussbaum-Szkoła distributions. Let $f : (0, \infty) \to \mathbb{R}$ be a convex (or concave) function and $D_f(\rho||\sigma)$, $D_f(P||Q)$ respectively denote the quantum $f$-divergence of $\rho$ from $\sigma$ and the classical $f$-divergence of $P$ from $Q$. Then

$$
D_f(\rho||\sigma) = D_f(P||Q).
$$

(1.6)
Lemma 1.4. \cite{2} The $f$-divergence of the Nussbaum-Szkoła distributions can be computed as

$$D_f(P||Q) = \sum_{\{i,j: r_i s_j \neq 0\}} f\left(r_i s_j^{-1}\right) s_j |\langle u_i|v_j \rangle|^2 + f(0)Q(P = 0) + f'(\infty)P(Q = 0). \quad (1.7)$$

In several occasions below, we will use the following rearrangement trick for a sum of the form $\sum_k f(x_k)y_k$ with $y_k > 0$ for all $k$. Notice that if the sum of the negative terms in the series above is strictly bigger than $-\infty$ (or the sum of positive terms in the series is strictly less than $\infty$), then any rearrangement of the series produces the same sum. In particular, for $N \in \mathbb{N} \cup \{\infty\}$, if the sum of the negative terms in the series $\sum_{k=1}^N f(x_k)y_k$ is strictly bigger than $-\infty$, (or the sum of its positive terms is strictly less than $\infty$), we have

$$\sum_{k=1}^N f(x_k)y_k = \sum_{\lambda \in \{x_k:k=1,\ldots,N\}} f(\lambda) \sum_{\{\ell: x_\ell = \lambda\}} y_\ell. \quad (1.8)$$

Note that in the first sum on the right side of (1.8), every element $x$ in the sequence $(x_k)_{k=1}^N$ appears exactly once even if the terms $x_k$ are not distinct.

2 Petz-Rényi and Umegaki Relative Entropy of States

Having defined the $f$-divergences, the most economic way to define and study other entropic quantities is through $f$-divergences. In this section, we define the Petz-Rényi and Umegaki relative entropy using the $f$-divergences as in \cite{7}. Nevertheless, this definition coincides with other definitions seen in the literature, for example Araki in \cite{10} and Berta, Scholz and Tomamichel in \cite{11}.

Definition 2.1. 1. (Petz-Rényi $\alpha$-relative entropy.) For $\alpha \in (0,1) \cup (1,\infty)$, the Petz-Rényi $\alpha$-relative entropy of two states $\rho$ given $\sigma$ is

$$D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log D_{f_\alpha}(\rho||\sigma), \quad (2.1)$$

where

$$f_\alpha(\lambda) = \lambda^\alpha, \quad \lambda \in (0,\infty).$$

2. (Umegaki relative entropy.) The Umegaki relative entropy of $\rho$ given $\sigma$ is defined as

$$D(\rho||\sigma) = D_f(\rho||\sigma), \quad (2.2)$$

where

$$f(\lambda) = \lambda \log \lambda, \quad \lambda \in (0,\infty).$$

Now we state a lemma and two propositions which describe some relationships between the pairs $(\rho, \sigma)$ and $(P, Q)$ described in (1.1) and (1.2), respectively. Reader may refer to \cite{2} for proofs of these results.

Lemma 2.2. Let $\rho$ and $\sigma$ be as in (1.1). Then $\text{Supp} \rho \subseteq \text{Supp} \sigma$ if and only if $s_j = 0$ for some $j$ implies that for every $i$ at least one of the two quantities $\{\langle u_i|v_j \rangle, r_i\}$ is equal to zero.
Proposition 2.3. Let $\rho$ and $\sigma$ be as in (1.1) and let $P$ and $Q$ be as in (1.2) then

$$P = Q \Leftrightarrow \rho = \sigma.$$ 

Proposition 2.4. Let $\rho$ and $\sigma$ be as in (1.1) and let $P$ and $Q$ be as in (1.2), then

$$\text{Supp} \rho \subseteq \text{Supp} \sigma \Leftrightarrow P \ll Q.$$ 

Now we proceed to prove an important result in this article. It states that the Petz-Rényi $\alpha$-relative entropy and the Umegaki relative entropy of two states $\rho$ and $\sigma$ are the same as the corresponding classical divergences of the Nussbaum-Szkoła distributions. Also it provides a formula to compute these quantities. Previously, the result was known for finite dimensions by the work of Nussbaum and Szkoła in [1]. In the infinite dimensional setting, the result about Petz-Rényi relative entropy was only known for the special case of gauge invariant and translation invariant gaussian states and of orders of $\alpha$ in $(0,1)$ by the work of Mosonyi in [12]. We prove it in general for all possible orders of $\alpha$ and all states (not only gaussian states) which are given via trace duality using a density operator.

Towards our main goal, we will compute the the Rényi divergence and the Kullback-Leibler divergence of $P$ from $Q$ (Definition A.1) in the next lemma.

Lemma 2.5. Let $\rho$ and $\sigma$ be as in (1.1). Let $P$ and $Q$ denote the corresponding Nussbaum-Szkoła distributions as in (1.2). Then

1. For $\alpha \in (0,1) \cup (1,\infty)$ the Rényi divergence $D_\alpha(P||Q)$ is given by

$$D_\alpha(P||Q) = \frac{1}{\alpha - 1} \log \sum_{i,j} r_i^\alpha s_j^{1-\alpha} |\langle u_i|v_j\rangle|^2,$$

where for $\alpha > 1$, we adopt the conventions $0^{1-\alpha} = \infty$ and $0 \cdot \infty = 0$. In particular,

$$D_\alpha(P||Q) = \infty, \text{ if } P \not\ll Q. \hspace{1cm} (2.4)$$

2. The Kullback-Leibler divergence $D(P||Q)$ is given by

$$D(P||Q) = \sum_{i,j} r_i |\langle u_i|v_j\rangle|^2 \log \left( \frac{r_i}{s_j} \right),$$

where we use the conventions that $0 \log(0/q) = 0$, for $q \geq 0$ and $p \log(p/0) = \infty$ if $p > 0$. In particular,

$$D(P||Q) = \infty, \text{ if } P \not\ll Q. \hspace{1cm} (2.6)$$

Proof. [1] By Definition A.1 for $\alpha \in (0,1) \cup (1,\infty)$, we take $f_\alpha(\lambda) = \lambda^\alpha$, for $\lambda \in (0,\infty)$ to compute the Rényi divergence. It is enough to prove that the classical $f$-divergence satisfies

$$D_{f_\alpha}(P||Q) = \sum_{i,j} r_i^\alpha s_j^{1-\alpha} |\langle u_i|v_j\rangle|^2,$$

where for $\alpha > 1$, we adopt the conventions $0^{1-\alpha} = \infty$ and $0 \cdot \infty = 0$.
Thus the first part of the theorem is proved. In (2.9), we have 
\( \alpha (2.11) \) may not be valid in this case. Thus when 
\( \in \alpha \cdot \infty \)
and the third terms in (1.7). We have
\( f(0)Q(P = 0) = 0 \), therefore, the second term
\[ f(0)Q(P = 0) = 0 \tag{2.8} \]
in (1.7). To compute the term \( f'(\alpha)P(Q = 0) \) in (1.7) we consider three cases. Case (i) \( \alpha \in (0, 1) \), Case (ii) \( \alpha \in (1, \infty) \) and \( P \ll Q \); Case (iii) \( \alpha \in (1, \infty) \) and \( P \not\ll Q \). In Case (i),
\[ f'(\alpha) := \lim_{\lambda \to \infty} \frac{f_\alpha(\lambda)}{\lambda} = \lim_{\lambda \to \infty} \frac{1}{\lambda^{1-\alpha}} = 0. \]
Hence \( f'(\alpha)P(Q = 0) = 0 \) in this case. In Cases (ii) and (iii) we have \( f'(\alpha) = \infty \). In Case (ii) \( P(Q = 0) = 0 \) hence \( f'(\alpha)P(Q = 0) = 0 \) in Case (ii) as well. In Case (iii), there exists \( (i, j) \) such that \( Q(i, j) = 0 \) but \( P(i, j) \neq 0 \) therefore we have \( f'(\alpha)P(Q = 0) = \infty \) in this case. Finally,
\[ f'(\alpha)P(Q = 0) = \begin{cases} \ 0 & \text{if } \alpha \in (0, 1) \\ 0 & \text{if } \alpha \in (1, \infty) \text{ and } P \ll Q \\ \infty & \text{if } \alpha \in (1, \infty) \text{ and } P \not\ll Q. \end{cases} \tag{2.9} \]
Now we compute the first term in (1.7),
\[ \sum_{\{i, j: r_is_j \neq 0\}} f_\alpha(r_is_j^{-1})s_j|\langle u_i|v_j \rangle|^2 = \sum_{\{i, j: r_is_j \neq 0\}} r_i^\alpha s_j^{1-\alpha}|\langle u_i|v_j \rangle|^2. \tag{2.10} \]
Note that if \( \alpha \in (0, 1) \),
\[ \sum_{\{i, j: r_is_j \neq 0\}} r_i^\alpha s_j^{1-\alpha}|\langle u_i|v_j \rangle|^2 = \sum_{i, j} r_i^\alpha s_j^{1-\alpha}|\langle u_i|v_j \rangle|^2. \tag{2.11} \]
Also, by Proposition 2.4 and Lemma 2.2, the equation above is satisfied whenever \( P \ll Q \). Hence by (2.8) and (2.9), we have proved (2.7) for the cases when either \( \alpha \in (0, 1) \), or \( \alpha \in (1, \infty) \) and \( P \ll Q \). If \( P \not\ll Q \) then there exists \( (i, j) \) such that \( s_j = 0 \) but \( r_i|\langle u_i|v_j \rangle|^2 \neq 0 \), hence when \( \alpha \in (1, \infty) \) under our conventions that \( 0^{1-\alpha} = \infty \) and \( 0 \cdot \infty = 0 \), note that the sum on the right side of (2.11) above is equal to \( \infty \), even though (2.11) may not be valid in this case. Thus when \( \alpha \in (1, \infty) \) and \( P \not\ll Q \), by (2.8) and (2.9), we have
\[ D_{f_\alpha}(P||Q) = \sum_{\{i, j: r_is_j \neq 0\}} r_i^\alpha s_j^{1-\alpha}|\langle u_i|v_j \rangle|^2 + f(0)Q(P = 0) + f'(\alpha)P(Q = 0) \]
\[ = \sum_{\{i, j: r_is_j \neq 0\}} r_i^\alpha s_j^{1-\alpha}|\langle u_i|v_j \rangle|^2 + 0 + \infty \]
\[ = \infty \]
\[ = \sum_{i, j} r_i^\alpha s_j^{1-\alpha}|\langle u_i|v_j \rangle|^2. \]
Thus the first part of the theorem is proved.
By Definition 1.1, we take \( f(\lambda) = \lambda \log \lambda \), for \( \lambda \in (0, \infty) \) to compute the Kullback-Leibler divergence. Once again we will use equation (1.7) to compute \( D_f(P||Q) \). Since \( \lim_{\lambda \to 0} f(\lambda) = 0 \), we see that

\[
    f(0)Q(P = 0) = 0. \tag{2.12}
\]

Furthermore, (2.9) with \( f_\alpha \) replaced by \( f \) is satisfied in this case as well. To compute the first term in (1.7), note that

\[
    \sum_{\{i,j: r_is_j \neq 0\}} f \left( r_is_j^{-1} \right) s_j |\langle u_i|v_j \rangle|^2 = \sum_{\{i,j: r_is_j \neq 0\}} r_is_j^{-1} (\log r_is_j^{-1}) s_j |\langle u_i|v_j \rangle|^2
\]

\[
    = \sum_{\{i,j: r_is_j \neq 0\}} r_i (\log r_is_j^{-1}) |\langle u_i|v_j \rangle|^2. \tag{2.13}
\]

Now a similar argument as in the case of Rényi divergence completes the proof in this case as well.

Now we have the theorem that was promised before the previous lemma.

**Theorem 2.6.** Let \( \rho \) and \( \sigma \) be as in (1.1). Let \( D_\alpha(P||Q) \) and \( D(P||Q) \) respectively denote the Rényi divergence of order \( \alpha \) and the Kullback-Leibler divergence of the Nussbaum-Szkoła distributions \( P \) and \( Q \) associated with \( \rho \) and \( \sigma \). Then,

1. the Petz-Rényi \( \alpha \)-relative entropy of \( \rho \) given \( \sigma \) is equal to the Rényi divergence of order \( \alpha \) of \( P \) from \( Q \), for every \( \alpha \in (0, 1) \cup (1, \infty) \), i.e.,

\[
    D_\alpha(\rho||\sigma) = D_\alpha(P||Q), \quad \forall \alpha \in (0, 1) \cup (1, \infty); \tag{2.14}
\]

2. the Umegaki relative entropy of \( \rho \) given \( \sigma \) is equal to the Kullback-Leibler divergence of \( P \) from \( Q \), i.e.,

\[
    D(\rho||\sigma) = D(P||Q). \tag{2.15}
\]

Moreover, we have the formulae

\[
    D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \sum_{i,j} r_i^\alpha s_j^{1-\alpha} |\langle u_i|v_j \rangle|^2, \quad \forall \alpha \in (0, 1) \cup (1, \infty), \tag{2.16}
\]

where for \( \alpha > 1 \), we adopt the conventions \( 0^{1-\alpha} = \infty \) and \( 0 \cdot \infty = 0 \), and

\[
    D(\rho||\sigma) = \sum_{i,j} r_i |\langle u_i|v_j \rangle|^2 \log \left( \frac{r_i}{s_j} \right), \tag{2.17}
\]

with the conventions that \( 0 \log(0/q) = 0 \), for \( q \geq 0 \) and \( p \log(p/0) = \infty \) if \( p > 0 \).

**Proof.** Let \( f_\alpha \) and \( f \) be as in the proof of Lemma 2.5. By Theorem 1.3 and Lemma 2.5 we have

\[
    D_{f_\alpha}(\rho||\sigma) = D_{f_\alpha}(P||Q) = \sum_{i,j} r_i^\alpha s_j^{1-\alpha} |\langle u_i|v_j \rangle|^2,
\]

\[
    D_f(\rho||\sigma) = D(P||Q) = \sum_{i,j} r_i |\langle u_i|v_j \rangle|^2 \log \left( \frac{r_i}{s_j} \right), \tag{2.18}
\]

which complete the proof.
Remark 2.7. In the setting of Theorem 2.6 and its proof, for \( i,j \in \mathcal{I} \) define \( A_{ij} = \langle u_i | v_j \rangle | u_i \rangle \langle v_j | \). Then \( \sum_{i,j} A_{ij}^\dagger A_{ij} = I = \sum_{i,j} A_{ij}^\dagger A_{ij} \). Hence both \( \{ A_{ij} A_{ij}^\dagger \} \) and \( \{ A_{ij}^\dagger A_{ij} \} \) are POVM’s. Furthermore, it may be noted that \( P(i,j) = \text{Tr} \rho A_{ij} A_{ij}^\dagger \) and \( Q(i,j) = \text{Tr} \sigma A_{ij}^\dagger A_{ij} \). Thus the probability measures \( P \) and \( Q \) in the previous theorem are precisely those measures that are obtained by measuring \( \rho \) and \( \sigma \), respectively in \( \{ A_{ij} A_{ij}^\dagger \} \) and \( \{ A_{ij}^\dagger A_{ij} \} \).

3 Petz-Rényi Relative Entropy using Pushforward Measure

The idea of distribution of a quantum random variable (observable or self adjoint operator) with respect to a state is as old as quantum mechanics itself. In this section we exploit a slight modification of this idea, i.e., the pushforward of a positive compact operator with respect to a selfadjoint operator to describe relative entropies. This terminology is motivated by the fact that the distribution of a classical random variable is simply the push forward of the probability measure with respect to the random variable. This approach has similarities with the approach of Haagerup in defining weights on noncommutative \( L^p \)-spaces [13 Example 1.2 and Proposition 1.11]. More details on relative entropies in terms Haagerup’s \( L^p \)-spaces is provided in Appendix A of [7].

Definition 3.1. Let \( X \) be a (possibly unbounded) selfadjoint operator defined on \( D(X) \subseteq \mathcal{K} \) with spectral measure \( \xi^X \) and let \( \tau \) be a bounded positive operator on \( \mathcal{K} \). Define a positive measure \( \mu^{\tau,X} \) on the Borel \( \sigma \)-algebra \( \mathcal{B}_\mathbb{R} \) of \( \mathbb{R} \), by

\[
\mu^{\tau,X}(E) := \text{Tr} \left\{ \tau^{1/2} \xi^X(E) \tau^{1/2} \right\}, \quad \forall E \in \mathcal{B}_\mathbb{R}.
\]

Then \( \mu^{\tau,X} \) is called the pushforward of \( \tau \) with respect to \( X \). If \( \tau \) is a state, then \( \mu^{\tau,X} \) is a probability distribution and it is called distribution of \( X \) with respect to \( \tau \).

Remark 3.2. Let \( \tau \) be a positive compact operator on \( \mathcal{K} \), with spectral decomposition

\[
\tau = \sum_i p_i \langle u_i | u_i \rangle, \quad (3.1)
\]

where \( p_i \geq 0 \) and \( \{ u_i \} \) an orthonormal basis in \( \mathcal{K} \). In this case,

\[
\mu^{\tau,X}(E) = \sum_i p_i \langle u_i | \xi^X(E) | u_i \rangle, \quad (3.2)
\]

for any Borel set \( E \subseteq \mathbb{R} \). Thus \( \mu^{\tau,X} \) is supported inside the spectrum of \( X \). Furthermore, if \( X \) has a spectral decomposition of the form

\[
X = \sum_j x_j | v_j \rangle \langle v_j |,
\]

where \( x_j \geq 0 \) and \( \{ v_j \} \) is an orthonormal basis, then \( \mu^{\tau,X} \) is supported on the eigenvalues \( \{ x_j \} \) and

\[
\mu^{\tau,X} \{ x_j \} = \sum_i p_i \left| \left| u_i \langle v_j | \sum_{k : x_j = x_k} | v_k \rangle \langle v_k | \right| u_i \right|^2 = \sum_i p_i \sum_{k : x_j = x_k} | \langle u_i | v_k \rangle |^2, \quad (3.3)
\]
When $\alpha > 1$, we note that $\sigma^{(1-\alpha)}$ is defined as the pseudo-inverse (also known as Moore-Penrose inverse) of $\sigma$ raised to the power $(\alpha - 1)$. If the spectral decomposition of $\sigma$ is $\sum_j s_j |v_j\rangle \langle v_j|$ with $s_j \geq 0$, then

$$
\sigma^{(1-\alpha)} := \sum_{\{j: s_j \neq 0\}} s_j^{1-\alpha} |s_j\rangle \langle s_j|.
$$

(3.4)

It may be noted from the spectral theorem [14, Theorem 12.4] that the pseudo-inverse as defined above is a selfadjoint operator (not necessarily bounded) because its spectral measure is supported on the real line. Furthermore, by (3.4), we also have $\sigma^{(1-\alpha)}$ is a positive operator.

**Theorem 3.3.** The Petz-Rényi relative entropy satisfies

$$
D_\alpha(\rho||\sigma) = \begin{cases} 
\frac{1}{\alpha - 1} \log \left( \int_0^\infty \lambda \mu^{\rho^{\alpha},\sigma^{(1-\alpha)}}(d\lambda) \right), & \alpha \in (0, 1) \text{ or } \alpha \in (1, \infty) \text{ and } \text{Supp } \rho \subseteq \text{Supp } \sigma; \\
\infty, & \text{otherwise},
\end{cases}
$$

(3.5)

where $\mu^{\rho^{\alpha},\sigma^{(1-\alpha)}}$ is the pushforward of $\rho^{\alpha}$ with respect to $\sigma^{(1-\alpha)}$ as in Definition 3.1 and when $\alpha > 1$, $\sigma^{(1-\alpha)}$ is taken as the pseudo-inverse of $\sigma^{(\alpha-1)}$ (see (3.4)). Consequently,

$$
D_\alpha(\rho||\sigma) < \infty \text{ if and only if } \int_0^\infty \lambda \mu^{\rho^{\alpha},\sigma^{(1-\alpha)}}(d\lambda) < \infty,
$$

(3.6)

$\alpha \in (0, 1) \text{ or } \alpha \in (1, \infty) \text{ and } \text{Supp } \rho \subseteq \text{Supp } \sigma$.

**Proof.** Let $\rho$ and $\sigma$ be as in (1.1). Then

$$
\rho^{\alpha} = \sum_{i \in \mathcal{I}} r_i^{\alpha} |u_i\rangle \langle u_i|, \quad r_i \geq 0, \quad \sum_i r_i = 1, \quad \{u_i\}_i \text{ is an orthonormal basis};
$$

$$
\sigma^{(1-\alpha)} = \sum_{\{j: s_j \neq 0\}} s_j^{1-\alpha} |v_j\rangle \langle v_j|, \quad s_j > 0, \quad \sum_j s_j = 1, \quad \{v_j\}_j \text{ is an orthonormal set},
$$

(3.7)

Now by putting $\tau = \rho^{\alpha}$, $X = \sigma^{(1-\alpha)}$, $x_j = s_j^{1-\alpha}$, $p_i = r_i^{\alpha}$ in (3.3), we get

$$
\mu^{\rho^{\alpha},\sigma^{(1-\alpha)}}\{s_j^{1-\alpha}\} = \sum_i r_i^{\alpha} \sum_{\{k: s_k = s_j\}} |\langle u_i| v_k \rangle|^2, \quad \forall j \text{ such that } s_j \neq 0,
$$

$$
\mu^{\rho^{\alpha},\sigma^{(1-\alpha)}}\{0\} = \sum_i r_i^{\alpha} \sum_{\{k: s_k = 0\}} |\langle u_i| v_k \rangle|^2.
$$

Now for $\alpha \in (0, 1) \cup (1, \infty)$, using the convention $0 \cdot \infty = 0$, we avoid integrating at 0 and
obtain
\[
\int_0^\infty \lambda^\alpha \mu^{\alpha,\sigma}(1-\alpha) \, (d\lambda) = \sum_{\{j : s_j \neq 0\}} r^\alpha_i \sum_{\{k : s_k = s_j\}} |\langle u_i | v_k \rangle|^2
\]

Now, if we assume \( \text{Supp} \rho \subseteq \text{Supp} \sigma \) in the case \( \alpha > 1 \), we get
\[
\int_0^\infty \lambda^\alpha \mu^{\alpha,\sigma}(1-\alpha) \, (d\lambda) = \sum_{i,j} r^\alpha_i s^1_j |\langle u_i | v_j \rangle|^2 \quad \text{(by (1.8))}
\]

where for \( \alpha > 1 \), we use Lemma 2.2 and adopt the conventions \( 0^{1-\alpha} = \infty \) and \( 0 \cdot \infty = 0 \). The last sum above is same as \( D_f(\rho||\sigma) \) by (2.18) in the case \( \alpha \in (0,1) \) or \( \alpha \in (1,\infty) \) with \( \text{Supp} \rho \subseteq \text{Supp} \sigma \), where \( f_\alpha(\lambda) = \lambda^\alpha \).

\[\square\]

Remark 3.4. In the previous theorem, we integrated the function \( \lambda \) on \([0, \infty)\) using the measure \( \mu^{\alpha,\sigma}(1-\alpha) \) which is supported on the set \( \{s^1_j : s_j \neq 0\} \cup \{0\} \). Nevertheless the result can be obtained by integrating the function \( \lambda^{1-\alpha} \) on the same interval using the measure \( \mu^{\alpha,\sigma} \).

\section*{4 Umegaki Relative Entropy from Distribution of Observables}

A theorem similar to the following one was obtained in [15, Theorem 20] in a more general setting using a different proof. In the setting of \( B(\mathcal{K}) \), as we shall see now, it follows easily from the idea of distribution. Recall that if the spectral decomposition of an operator \( X \) is \( \sum_j x_j |f_j \rangle \langle f_j | \) with \( x_j \geq 0 \), then
\[
- \log X := \sum_{\{j : x_j \neq 0\}} - \log x_j |f_j \rangle \langle f_j | ,
\]

where we follow Umegaki [16, Pages 65-66] for defining the functional calculus of the logarithm.

\textbf{Theorem 4.1.} The Umegaki entropy \( D(\rho||\sigma) \) satisfies
\[
D(\rho||\sigma) = \int_0^\infty \lambda^{\rho,-\log \sigma} (d\lambda) - \int_0^\infty \lambda^{\rho,-\log \rho} (d\lambda) ,
\]

when \( \text{Supp} \rho \subseteq \text{Supp} \sigma \) and at least one of the two quantities
\[
\int_0^\infty \lambda^{\rho,-\log \sigma} (d\lambda) \text{ and } \int_0^\infty \lambda^{\rho,-\log \rho} (d\lambda)
\]

is finite, with the conventions \( \frac{0}{0} = 0 \), \( \frac{\infty}{\infty} = \infty \) if \( x > 0 \), and \( 0 \cdot \infty = 0 \).
Proof. Let $\rho$ and $\sigma$ be as in (1.1). We have by (3.3),
\[
\mu^{\rho,-\log \sigma \{ - \log s_j \}} = \sum_i r_i \sum_{\{k : s_j = s_k\}} |\langle u_i | v_k \rangle|^2 \quad \forall j \text{ such that } s_j \neq 0,
\]
and $\mu^{\rho,-\log \sigma}$ is zero everywhere else except possibly at $\{0\}$. Note that when $\text{Supp} \rho \subseteq \text{Supp} \sigma$, by Lemma 2.2, equation (1.8) and the conventions we have,
\[
\int_0^\infty \lambda \mu^{\rho,-\log \sigma} (d\lambda) = \sum_{\{j : s_j \neq 0\}} (-\log s_j) \sum_i r_i \sum_{\{k : s_j = s_k\}} |\langle u_i | v_k \rangle|^2
\]
\[
= \sum_{\{i,j : s_j \neq 0\}} r_i (-\log s_j) |\langle u_i | v_j \rangle|^2
\]
\[
= \sum_{i,j} -r_i (\log s_j) |\langle u_i | v_j \rangle|^2.
\]
Similar to the above situation, by (3.3) we have
\[
\mu^{\rho,-\log \rho \{ - \log r_i \}} = \sum_j r_j \sum_{\{k : r_i = r_k\}} |\langle u_j | u_k \rangle|^2
\]
\[
= \sum_j r_j \sum_{\{k : r_i = r_k\}} \delta_{jk} \quad \text{(where } \delta_{jk} \text{ denotes the Kronecker-}\delta \text{ function)}
\]
\[
= \sum_{\{k : r_i = r_k\}} r_k, \quad \forall i \text{ such that } r_i \neq 0,
\]
and $\mu^{\rho,-\log \rho}$ is zero everywhere else except possibly at $\{0\}$. Since $\sum_j |\langle u_i | v_j \rangle|^2 = \|u_i\| = 1$ for all $i$, we have,
\[
\int_0^\infty \lambda \mu^{\rho,-\log \rho} (d\lambda) = \sum_{\{i : r_i \neq 0\}} (-\log r_i) \sum_{\{k : r_i = r_k\}} r_k
\]
\[
= \sum_{\{i : r_i \neq 0\}} -r_i \log r_i \quad \text{(by (1.8))}
\]
\[
= \sum_i -r_i \log r_i \quad \text{(since } 0 \cdot \infty = 0)
\]
\[
= \sum_{i,j} -r_i \log r_i |\langle u_i | v_j \rangle|^2.
\]
If at least one of the quantities $\int_0^\infty \lambda \mu^{\rho,-\log \sigma} (d\lambda)$ and $\int_0^\infty \lambda \mu^{\rho,-\log \rho} (d\lambda)$ is finite, we can
combine the two summations in (4.3) and (4.4), and write

\[
\begin{align*}
&\int_0^\infty \lambda \rho_{\alpha} - \log \sigma (d\lambda) - \int_0^\infty \lambda \rho_{\alpha} - \log \rho (d\lambda) \\
&= \sum_{i,j} -r_i (\log s_j)|\langle u_i|v_j\rangle|^2 - \sum_{i,j} -r_i (\log r_i)|\langle u_i|v_j\rangle|^2 \\
&= \sum_{i,j} r_i (\log r_i)|\langle u_i|v_j\rangle|^2 - \sum_{i,j} r_i (\log s_j)|\langle u_i|v_j\rangle|^2 \\
&= \sum_{i,j} (r_i (\log r_i)|\langle u_i|v_j\rangle|^2 - r_i (\log s_j)|\langle u_i|v_j\rangle|^2).
\end{align*}
\]

By using Lemma 2.2 and the conventions \(\frac{0}{0} = 0, \frac{x}{\infty} = \infty\) if \(x > 0, 0 \cdot \infty = 0\), we get

\[
\begin{align*}
&\int_0^\infty \lambda \rho_{\alpha} - \log \sigma (d\lambda) - \int_0^\infty \lambda \rho_{\alpha} - \log \rho (d\lambda) \\
&= \sum_{i,j} r_i (\log r_i)|\langle u_i|v_j\rangle|^2 \log \frac{r_i}{s_j} \\
&= \sum_{i,j} r_i (\log r_i)|\langle u_i|v_j\rangle|^2 \log \frac{r_i}{s_j} |\langle u_i|v_j\rangle|^2 \\
&= D(P||Q) \\
&= D(\rho||\sigma)
\end{align*}
\]

where \(D(P||Q)\) is the Kullback-Leibler divergence of the Nussbaum-Szkoła distributions \(P\) and \(Q\) as in (2.5) and we use Theorem 2.6.

Remark 4.2. Equation (4.2) is same as a modified version of Umegaki’s definition of the relative entropy

\[
D(\rho||\sigma) = \text{Tr} \rho^{1/2} (\log \rho) \rho^{1/2} - \text{Tr} \rho^{1/2} (\log \sigma) \rho^{1/2},
\]

whenever \(\rho^{1/2} (\log \rho) \rho^{1/2}\) is a densely defined operator.

5 More Properties of Petz-Rényi Relative Entropy

In this section, we use the properties of classical divergences to prove the properties of their quantum counterparts. Some results in this section are already known with different proofs. We provide appropriate references whenever we reprove a known result. Nevertheless, our idea is to show the usefulness of Theorem 2.6 by showing that a number of results about quantum entropies follow trivially from corresponding classical results.

5.1 Limiting Cases

The definition of Petz-Rényi \(\alpha\)-relative entropy excludes the values 0, 1 and \(\infty\) of \(\alpha\). Nevertheless, we can give meaning to the entropic quantities corresponding to these values of \(\alpha\) and they are important in applications too \([17, 18, 19]\). First we prove that the Petz-Rényi \(\alpha\)-relative entropy is nondecreasing in \(\alpha\), which will help us to extend the definition of \(D_\alpha\) to the values 0, 1 and \(\infty\). The next Theorem is available in \([7\text{ part 4 of Proposition 5.3}]\).
Theorem 5.1. For \( \alpha \in (0,1) \cup (1,\infty) \) the Petz-Rényi entropy, \( D_\alpha(\rho||\sigma) \) is nondecreasing in \( \alpha \).

Proof. This is an easy consequence of Theorem 2.6 and Theorem A.3 \( \square \)

Theorem 5.1 enables us to extend the definition of \( D_\alpha(\rho||\sigma) \) to the values \( \alpha = 0,1 \) and \( \infty \) as in the following definition.

Definition 5.2. The Petz-Rényi relative entropies of orders 0,1 and \( \infty \) are defined as

\[
D_0(\rho||\sigma) = \lim_{\alpha \downarrow 0} D_\alpha(\rho||\sigma), \\
D_1(\rho||\sigma) = \lim_{\alpha \uparrow 1} D_\alpha(\rho||\sigma), \\
D_\infty(\rho||\sigma) = \lim_{\alpha \uparrow \infty} D_\alpha(\rho||\sigma).
\]

With the definition above, we have the following corollary.

Corollary 5.3. For \( \alpha \in [0,\infty] \), the function \( \alpha \mapsto D_\alpha(\rho||\sigma) \) is nondecreasing and thus

\[
D_0(\rho||\sigma) \leq D_\alpha(\rho||\sigma) \leq D_\infty(\rho||\sigma), \quad \forall \alpha \geq 0.
\]

Remark 5.4. Following the notations used in [17], the quantity \( D_0(\rho||\sigma) \) may also be written as \( D_{\min}(\rho||\sigma) \).

Our next result in this section shows that the Umegaki relative entropy is the limit at 1 of Petz-Rényi relative entropy. It is stated or proved in the references [7,11,20,21]. We show that this result is an easy consequence of the corresponding classical fact and our Theorem 2.6.

Theorem 5.5. The Umegaki relative entropy is the limit of the Petz-Rényi relative entropy, i.e.,

\[
D(\rho||\sigma) = \lim_{\alpha \uparrow 1} D_\alpha(\rho||\sigma) = D_1(\rho||\sigma).
\]

Moreover, if \( D(\rho||\sigma) = \infty \) or there exists \( \beta > 1 \) such that \( D_\beta(\rho||\sigma) < \infty \), then also

\[
\lim_{\alpha \downarrow 1} D_\alpha(\rho||\sigma) = D(\rho||\sigma).
\]

Proof. Recall from Theorem A.5 that the Rényi divergence and Kullback-Leibler divergence satisfy the (5.1) and (5.2) with \( \rho \) and \( \sigma \) replaced with \( P \) and \( Q \) respectively, where \( P \) and \( Q \) are the associated Nussbaum-Szkola distributions. Now the present theorem is an easy consequence Theorem 2.6 \( \square \)

Remark 5.6. It is possible that \( D_\alpha(\rho||\sigma) = \infty \) for all \( \alpha > 1 \), but \( D(\rho||\sigma) < \infty \), and hence (5.2) does not hold (See Example 5.10 below).

Now we discuss the limits at 0 and \( \infty \). If \( \rho = \sum r_i |u_i\rangle \langle u_i| \) is a spectral decomposition of \( \rho \), where \( \{u_i\} \) is an orthonormal basis of \( K \), then

\[
\text{Supp } \rho = \text{span } \{u_i|r_i \neq 0\}.
\]

In [17], Datta observes in the finite dimensional setting that,

\[
D_0(\rho||\sigma) = -\log \text{Tr } \Pi_\rho \sigma,
\]
where $\Pi_\rho$ is the projection onto the support of $\rho$, i.e., by keeping the notations as in (1.1)

$$\Pi_\rho = \text{Projection onto the span}\{u_i | r_i \neq 0\}.$$ 

In the finite dimensions, this result follows from our formula (2.16) as well, because we have

$$D_0(\rho||\sigma) = \lim_{\alpha \to 0} \frac{1}{\alpha - 1} \log \sum_{i,j} r_i^\alpha s_j^{1-\alpha} |\langle u_i | v_j \rangle|^2$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha - 1} \log \sum_{\{i,j : r_i \neq 0\}} r_i^\alpha s_j^{1-\alpha} |\langle u_i | v_j \rangle|^2$$

$$= - \log \sum_{\{i,j : r_i \neq 0\}} s_j |\langle u_i | v_j \rangle|^2$$

$$= - \log \text{Tr} \sum_{\{i|r_i \neq 0\}} |u_i\rangle\langle u_i| \sum_j s_j |v_j\rangle\langle v_j|$$

$$= - \log \text{Tr} \Pi_\rho \sigma. \hspace{1cm} (5.5)$$

Now we prove (5.4) in the infinite dimensional situation. A priori, the computation in (5.5) cannot go through in infinite dimensions because we need a limit theorem to pass the limit through the infinite sum. Nevertheless, Theorem A.4 helps us to prove the desired result and the following proof works in both finite and infinite dimensional setting.

**Theorem 5.7.** The Petz-Rényi relative entropy satisfies,

$$D_0(\rho||\sigma) = - \log \text{Tr} \Pi_\rho \sigma,$$ \hspace{1cm} (5.6)

where $\Pi_\rho$ is the projection onto $\text{Supp} \rho$.

**Proof.** Keeping the notations as in (1.1), we have by Theorem 2.6 and Theorem A.4

$$D_0(\rho||\sigma) = - \log (Q(\{P(i,j) > 0\}))$$

$$= - \log \sum_{\{i,j : r_i > 0, \langle u_i | v_j \rangle \neq 0\}} s_j |\langle u_i | v_j \rangle|^2$$

$$= - \log \sum_{\{i,j : r_i \neq 0\}} s_j |\langle u_i | v_j \rangle|^2$$

$$= - \log \text{Tr} \sum_{\{i|r_i \neq 0\}} |u_i\rangle\langle u_i| \sum_j s_j |v_j\rangle\langle v_j|$$

$$= - \log \text{Tr} \Pi_\rho \sigma. \hspace{1cm} \Box$$

**Remark 5.8.** On a related note, it may be recalled that a sandwiched Rényi relative entropy, $\tilde{D}_\alpha$ was introduced independently by Müller-Lennert et al. in [18] and Wilde et al. in [22]. Furthermore, Datta and Leditzky in their [19] Theorem 1] proved that

$$\lim_{\alpha \to 0} \tilde{D}_\alpha = D_0(\rho||\sigma),$$

whenever $\text{Supp} \rho = \text{Supp} \sigma$. 

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**Theorem 5.9.** Let \( \rho \) and \( \sigma \) be as in (1.1). Then

\[
D_\infty(\rho||\sigma) = \log \sup \left\{ \frac{r_i}{s_j} : \langle u_i|v_j \rangle \neq 0 \right\},
\]

with the conventions that 0/0 = 0 and x/0 = \( \infty \) if \( x > 0 \).

**Proof.** Let \( P \) and \( Q \) be as in (1.2). By the definition of \( D_\infty(\rho||\sigma) \) and Theorem 2.6, we have

\[
D_\infty(\rho||\sigma) = \lim_{\alpha \to \infty} D_\alpha(\rho||\sigma) = \lim_{\alpha \to \infty} D_\alpha(P||Q) = D_\infty(P||Q).
\]

By equation (A.8),

\[
D_\infty(P||Q) = \log \sup_{(i,j) \in I \times I} \frac{r_i}{s_j} \langle u_i|v_j \rangle^2
\]

\[
= \log \sup \left\{ \frac{r_i}{s_j} : \langle u_i|v_j \rangle \neq 0 \right\}
\]

with the conventions that 0/0 = 0 and x/0 = \( \infty \) if \( x > 0 \). \( \square \)

### 5.2 Continuity, Positivity, Symmetry and Convexity

We begin this section with three examples which illustrate the behaviour of \( D_\alpha(\rho||\sigma) \) when \( \alpha \geq 1 \). These examples will help us to understand continuity points of \( D_\alpha(\rho||\sigma) \).

**Example 5.10** (\( D_1(\rho||\sigma) < \infty \) but \( D_\alpha(\rho||\sigma) = \infty, \forall \alpha > 1 \)). Let \( \{u_i\}_{i=1}^\infty \) be any orthonormal basis on \( K \). Take

\[
\rho = \sum_{i=1}^\infty 2^{-i} |u_i\rangle \langle u_i|,
\]

\[
\sigma = s^{-1} \sum_{j=1}^\infty 2^{-j^2} |u_j\rangle \langle u_j|,
\]

where \( s = \left( \sum_{j=1}^\infty 2^{-j^2} \right)^{1/2} \). In this case, keeping the notations in (1.1), and (1.2), \( r_i = 2^{-i} \), \( s_j = s^{-1}2^{-j^2} \), \( \langle u_i|v_j \rangle = \delta_{i,j} \), we have

\[
D_1(\rho||\sigma) = D(P||Q) = \sum_i r_i \log \left( \frac{r_i}{s_i} \right) = \sum_i 2^{-i} \log \left( \frac{s2^{-i}}{2^{-i^2}} \right)
\]

\[
= \sum_i 2^{-i} \log (s2^{-i+i^2}) = \sum_i 2^{-i} \log s + \sum_i 2^{-i} \log (2^{-i+i^2})
\]

\[
= \sum_i 2^{-i} \log s + \sum_i 2^{-i} (-i + i^2) < \infty.
\]

On the other hand, for \( \alpha > 1 \),

\[
\sum_i r_i^\alpha s_i^{(1-\alpha)} = s^{-1(1-\alpha)} \sum_i 2^{-\alpha i} 2^{-(1-\alpha)i^2} = s^{-1(1-\alpha)} \sum_i 2^{(\alpha-1)i^2 - \alpha i} = \infty
\]

because \( (\alpha - 1) > 0 \). Therefore,

\[
D_\alpha(\rho||\sigma) = D_\alpha(P||Q) = \frac{1}{\alpha - 1} \log \sum_i r_i^\alpha s_i^{1-\alpha} = \infty.
\]
Example 5.11 \( (D_\alpha(\rho||\sigma) < \infty \) for \( 1 < \alpha < 2 \), but \( D_2(\rho||\sigma) = \infty \)). Let \( \{u_i\}_{i=1}^\infty \) be any orthonormal basis on \( \mathcal{K} \). Take
\[
\rho = \sum_{i=1}^\infty 2^{-i} |u_i\rangle \langle u_i|,
\]
\[
\sigma = s^{-1} \sum_{j=1}^\infty 2^{-2j} |u_j\rangle \langle u_j|,
\]
where \( s = \left( \sum_j 2^{-2j} \right)^{1/2} \). In this case, keeping the notations in (1.1), and (1.2), \( r_i = 2^{-i} \), \( s_j = s^{-1}2^{-2j} \), \( \langle u_i|v_j \rangle = \delta_{i,j} \). We have for \( \alpha > 1 \),
\[
\sum_i r_i^\alpha s_i^{(1-\alpha)} = s^{-1(1-\alpha)} \sum_i 2^{-ai} 2^{(1-\alpha)2i} = s^{-1(1-\alpha)} \sum_i 2^{(a-1)2i-\alpha i} = s^{-1(1-\alpha)} \sum_i 2^{(a-2)i}.
\]
The above series converges for \( 1 < \alpha < 2 \) and diverges for \( \alpha = 2 \). Therefore,
\[
D_\alpha(\rho||\sigma) = D_\alpha(P||Q) = \frac{1}{\alpha-1} \log \sum_i r_i^\alpha s_i^{1-\alpha}
\]
is finite for \( 1 < \alpha < 2 \) and diverges for \( \alpha = 2 \).

Example 5.12 \( (D_2(\rho||\sigma) < \infty \), but \( D_\alpha(\rho||\sigma) = \infty \) for \( \alpha > 2 \)). Let \( \{u_i\}_{i=1}^\infty \) be any orthonormal basis on \( \mathcal{K} \). Take
\[
\rho = \sum_{i=1}^\infty 2^{-i} |u_i\rangle \langle u_i|,
\]
\[
\sigma = s^{-1} \sum_{j=1}^\infty j^2 2^{-2j} |u_j\rangle \langle u_j|,
\]
where \( s = \left( \sum_j j^2 2^{-2j} \right)^{1/2} \). In this case, keeping the notations in (1.1), and (1.2), \( r_i = 2^{-i} \), \( s_j = s^{-1}j^2 2^{-2j} \), \( \langle u_i|v_j \rangle = \delta_{i,j} \). We have for \( \alpha \geq 2 \),
\[
\sum_i r_i^\alpha s_i^{(1-\alpha)} = s^{-1(1-\alpha)} \sum_i 2^{-ai} i^2 2^{(1-\alpha)2i} = s^{-1(1-\alpha)} \sum_i i^{2(a-1)2i-\alpha i} = s^{-1(1-\alpha)} \sum_i i^{2(a-2)i}.
\]
The above series converges for \( \alpha = 2 \) and diverges for \( \alpha > 2 \). Therefore,
\[
D_\alpha(\rho||\sigma) = D_\alpha(P||Q) = \frac{1}{\alpha-1} \log \sum_i r_i^\alpha s_i^{1-\alpha}
\]
is finite for \( \alpha = 2 \) and diverges for \( \alpha > 2 \).

The following careful characterization of the continuity points in \( \alpha \) of \( D_\alpha(\rho||\sigma) \) in the infinite dimensions does not seem to be available in the literature.
Theorem 5.13. The Petz-Rényi relative entropy $D_\alpha(\rho||\sigma)$ is continuous in $\alpha$ on the set $A = [0, 1] \cup \{\alpha > 1 | D_\alpha(\rho||\sigma) < \infty\}$.

Proof. By Theorem 2.6 we know $D_\alpha(\rho||\sigma) = D_\alpha(P||Q)$ where $\rho$ and $\sigma$ are as in (1.1) and $P$ and $Q$ are as in (1.2). The result follows because the same result is true for the classical Rényi relative divergence (refer Theorem A.5). □

The following theorem is available in [7, 10 of Proposition 5.3] but our proof is different.

Theorem 5.14. For any order $\alpha \in [0, \infty]$, $D_\alpha(\rho||\sigma) \geq 0$. For $\alpha > 0$, $D_\alpha(\rho||\sigma) = 0$ if and only if $\rho = \sigma$. For $\alpha = 0$, $D_\alpha(\rho||\sigma) = 0$ if and only if $\text{Supp} \sigma \subseteq \text{Supp} \rho$.

Proof. Follows easily from Theorem A.9 and Proposition 2.3 because of Theorem 2.6. □

The following Proposition is available in [7, 5 Proposition 5.3].

Proposition 5.15. For any $0 < \alpha < 1$, the Petz-Rényi relative entropy shows the following skew-symmetry property

$$D_\alpha(\rho||\sigma) = \frac{\alpha}{1 - \alpha} D_{1 - \alpha}(\sigma||\rho).$$

Proof. Follows from Proposition A.10. □

Note that in particular, Petz-Rényi relative entropy is symmetric for $\alpha = 1/2$, and that skew-symmetry does not hold for $\alpha = 0$ and $\alpha = 1$.

Theorem 5.16. For any $0 < \alpha \leq \beta < 1$,

$$\frac{\alpha}{\beta} \frac{1 - \beta}{1 - \alpha} D_\beta(\rho||\sigma) \leq D_\alpha(\rho||\sigma) \leq D_\beta(\rho||\sigma).$$

Proof. Follows from Theorems A.11 and 2.6. □

Remark 5.17. In the light of the Theorem 5.16 we can discuss about a topology on the set of states arising from $D_\alpha$. For a fixed $\alpha \in (0, 1)$, one can define $\alpha$-left open ball with center $\rho$ and radius $r > 0$ to be the set $\{\sigma | D_\alpha(\rho||\sigma) < r\}$, and subsequently define $\alpha$-left open sets to be the union of $\alpha$-left open balls. Notice that Theorem 5.16 yields that for $\alpha, \beta \in (0, 1)$, the $\alpha$-left topology is equivalent to the $\beta$-left topology. Similarly, one can define $\alpha$-right open balls and $\alpha$-right topologies by reversing the order of $\rho$ and $\sigma$ in the definition of $\alpha$-left topology. Proposition 5.15 combined with the fact that the $\alpha$-left topologies are all equivalent for $0 < \alpha < 1$, gives that the $\alpha$-left topologies are equivalent with the $\beta$-right topologies for all $\alpha, \beta \in (0, 1)$.

Theorem 5.18. The following conditions are equivalent:

1. $\text{Supp} \sigma \subseteq \text{Supp} \rho$,
2. $\text{Tr} \Pi_\rho \sigma = 1$, where $\Pi_\rho$ is the orthogonal projection onto $\text{Supp} \rho$,
3. $D_0(\rho||\sigma) = 0$. 

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4. \( \lim_{\alpha \downarrow 0} D_{\alpha}(\rho\|\sigma) = 0. \)

**Proof.** \( 1 \iff 3 \) follows from Theorem 5.14.

\( 3 \iff 2 \) follows from Theorem 5.7.

\( 3 \iff 4 \) follows from Theorem 5.13.

The following theorem, states that the Petz-Rényi \( \alpha \)-relative entropy of two states is infinity for some \( \alpha \in [0,1) \) if and only if \( \text{Supp} \rho \perp \text{Supp} \sigma \). This seems to be a very interesting consequence of the classical results on relative entropic quantities.

**Theorem 5.19.** The following conditions are equivalent:

1. \( \text{Supp} \rho \perp \text{Supp} \sigma \),

2. \( \text{Tr} \Pi_\rho \sigma = 0 \), where \( \Pi_\rho \) is the orthogonal projection onto \( \text{Supp} \rho \),

3. \( D_{\alpha}(\rho\|\sigma) = \infty \) for some \( \alpha \in [0,1) \),

4. \( D_{\alpha}(\rho\|\sigma) = \infty \) for all \( \alpha \in [0,\infty) \).

**Proof.** \( 1 \iff \text{Supp} \sigma \subseteq \text{Ran}(I - \Pi_\rho) \iff \text{Tr}(I - \Pi_\rho)\sigma = \text{Tr} \sigma = 1 \iff \text{the statement 2} \) by Theorem 5.7.

Finally, \( 3 \iff 4 \) because of Theorem 5.16.

We have the following corollary by combining Theorem 5.19 and Theorem 3.3.

**Corollary 5.20.**

1. \( \text{Supp} \rho \not\perp \text{Supp} \sigma \) if and only if \( D_{\alpha}(\rho\|\sigma) < \infty \) for all \( \alpha \in [0,1) \).

2. If \( \text{Supp} \rho \not\subseteq \text{Supp} \sigma \) then \( D_{\alpha}(\rho\|\sigma) = \infty \) for all \( \alpha > 1 \).

3. If \( \text{Supp} \rho \subseteq \text{Supp} \sigma \) and \( \alpha > 1 \), then \( D_{\alpha}(\rho\|\sigma) < \infty \) if and only if \( \int_0^\infty \lambda \mu^{\alpha} \nu^{\alpha(1-\alpha)} (d\lambda) < \infty \).

**Proposition 5.21.** The function \( [0,\infty] \ni \alpha \mapsto (\alpha - 1)D_{\alpha}(\rho\|\sigma) \) is convex, with the conventions that it is 0 at \( \alpha = 1 \) even if \( D(\rho\|\sigma) = \infty \) and that it is 0 at \( \alpha = \infty \) if \( \rho = \sigma \).

**Proof.** Follows from Corollary A.14 and Theorem 2.6.

### A Classical Divergences

In this section, we recall a few facts about the \( f \)-divergences and the Rényi divergence in the setting of classical probability. We refer to [23] and the survey article [24] for the following definitions and results which we state in this section. The results from [24] which we use in this article are repeated here for the ease of the reader. If \( \mu \) and \( \nu \) are two positive measures on a measure space \( (X,\mathcal{F}) \), then \( \nu \) is said to be absolutely continuous with respect to \( \mu \) and we write \( \nu \ll \mu \), if for every \( E \in \mathcal{F} \) such that \( \mu(E) = 0 \), then \( \nu(E) = 0 \).
Definition A.1. 1. The Rényi divergence of order $\alpha \in (0, 1) \cup (1, \infty)$ is defined as

\[
D_\alpha(P||Q) = \frac{1}{\alpha - 1} \log D_{f_\alpha}(P||Q), \quad (A.1)
\]

where

\[
f_\alpha(\lambda) = \lambda^\alpha, \quad \lambda \in (0, \infty).
\]

It may be noted that the quantity

\[
D_{f_\alpha}(P||Q) = \int_X p^\alpha q^{1-\alpha} d\mu,
\]

where for $\alpha > 1$, we adopt the conventions $0^{1-\alpha} = \infty$ and $0 \cdot \infty = 0$.

2. The Kullback-Leibler divergence of $P$ from $Q$ is defined as

\[
D(P||Q) = D_{f}(P||Q) \int p \log \frac{p}{q} d\mu, \quad (A.2)
\]

where

\[
f(\lambda) = \lambda \log \lambda, \quad \lambda \in (0, \infty).
\]

It may be noted that

\[
D_{f}(P||Q) = \int p \log \frac{p}{q} d\mu,
\]

with the conventions that $0 \log(0/q) = 0$, for $q \geq 0$ and $p \log(p/0) = \infty$ if $p > 0$.

Consequently,

\[
D(P||Q) = \infty, \quad \text{if } P \nless Q. \quad (A.3)
\]

Definition A.2. The Rényi divergences of orders 0, 1 and $\infty$ are defined as

\[
D_0(P||Q) = \lim_{\alpha \downarrow 0} D_\alpha(P||Q)
\]

\[
D_1(P||Q) = \lim_{\alpha \uparrow 1} D_\alpha(P||Q)
\]

\[
D_\infty(P||Q) = \lim_{\alpha \uparrow \infty} D_\alpha(P||Q)
\]

The limits in Definition A.2 always exist because Rényi divergence is nondecreasing in order.

Theorem A.3. [24, Theorem 3]. For $\alpha \in [0, \infty]$ the Rényi divergence $D_\alpha(P||Q)$ is nondecreasing in $\alpha$.

Theorem A.4. [24, Theorem 4].

\[
D_0(P||Q) = -\log(Q(\{p > 0\})). \quad (A.4)
\]

Theorem A.5. [24, Theorem 5]. The Kullback-Leibler divergence is the limit of the Rényi divergence, i.e.,

\[
D(P||Q) = D_1(P||Q). \quad (A.5)
\]

Moreover, if $D(P||Q) = \infty$ or there exists $\beta > 1$ such that $D_\beta(P||Q) < \infty$, then also

\[
\lim_{\alpha \downarrow 1} D_\alpha(P||Q) = D(P||Q). \quad (A.6)
\]
Remark A.6. It is possible that $D_\alpha(P||Q) = \infty$ for all $\alpha > 1$, but $D(P||Q) < \infty$, and hence [A.6] does not hold [24].

For any random variable $Y$, the essential supremum of $Y$ with respect to $P$ is $\text{ess sup}_P Y = \sup\{c | P(Y > c) > 0\}$.

Theorem A.7. [24, Theorem 6].

$$D_\infty(P||Q) = \log \sup_{A \in F} \frac{P(A)}{Q(A)} = \log \left( \text{ess sup}_P \frac{p}{q} \right),$$ \hspace{1cm} (A.7)

with the convention that $0/0 = 0$ and $x/0 = \infty$ if $x > 0$.

If the sample space $X$ is countable, then with the notational conventions of Theorem A.7, the $P$-essential supremum of $\frac{p}{q}$ reduces to the ordinary supremum of $\frac{p}{q}$, which in turn is equal to the supremum of $\frac{p}{q}$, and we have

$$D_\infty(P||Q) = \log \sup_x \frac{P(x)}{Q(x)},$$ \hspace{1cm} (A.8)

with the convention that $0/0 = 0$ and $x/0 = \infty$ if $x > 0$.

Theorem A.8. [24, Theorem 7]. The Rényi divergence $D_\alpha(P||Q)$ is continuous in $\alpha$ on $A = \{\alpha \in [0, \infty] | 0 \leq \alpha \leq 1 \text{ or } D_\alpha(P||Q) < \infty\}$.

Theorem A.9. [24, Theorem 8]. For any order $\alpha \in [0, \infty]$

$$D_\alpha(P||Q) \geq 0.$$ For $\alpha > 0$, $D_\alpha(P||Q) = 0$ if and only if $P = Q$. For $\alpha = 0$, $D_\alpha(P||Q) = 0$ if and only if $Q \ll P$.

Proposition A.10. [24, Proposition 2]. For any $0 < \alpha < 1$, the Rényi divergence shows the following skew-symmetry property

$$D_\alpha(P||Q) = \frac{\alpha}{1 - \alpha} D_1(Q||P).$$

Note that in particular, Rényi divergence is symmetric for $\alpha = 1/2$, and that skew-symmetry does not hold for $\alpha = 0$ and $\alpha = 1$.

Theorem A.11. [24, Theorem 16]. For any $0 < \alpha \leq \beta < 1$, 

$$\frac{\alpha}{\beta} \frac{1 - \beta}{1 - \alpha} D_\beta(P||Q) \leq D_\alpha(P||Q) \leq D_\beta(P||Q).$$

Theorem A.12. [24, Theorem 23]. The following conditions are equivalent:

1. $Q \ll P$,
2. $Q(\{p > 0\}) = 1$,
3. $D_0(P||Q) = 0$, 

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4. \( \lim_{\alpha \to 0} D_\alpha(P||Q) = 0. \)

**Theorem A.13.** [24, Theorem 24]. The following conditions are equivalent:

1. \( P \perp Q, \)
2. \( Q(\{p > 0\}) = 0, \)
3. \( D_\alpha(P||Q) = \infty \) for some \( \alpha \in [0, 1), \)
4. \( D_\alpha(P||Q) = \infty \) for all \( \alpha \in [0, \infty]. \)

**Corollary A.14.** [24, Corollary 2]. The function \( [0, \infty] \ni \alpha \mapsto (1 - \alpha)D_\alpha(P||Q) \) is concave, with the conventions that it is 0 at \( \alpha = 1 \) even if \( D(P||Q) = \infty \) and that it is 0 at \( \alpha = \infty \) if \( P = Q. \)

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**References**

[1] M. Nussbaum and A. Szkoła, “The Chernoff lower bound for symmetric quantum hypothesis testing,” *Ann. Statist.*, vol. 37, no. 2, pp. 1040–1057, 2009. [Online]. Available: https://doi.org/10.1214/08-AOS593

[2] G. Androulakis and T. C. John, “Quantum \( f \)-divergences via Nussbaum-Szkoła Distributions and Applications to \( f \)-divergence Inequalities,” *Reviews in Mathematical Physics*, Jul. 2023. [Online]. Available: https://doi.org/10.1142/S0129055X23600024

[3] S. L. Braunstein and P. van Loock, “Quantum information with continuous variables,” *Rev. Mod. Phys.*, vol. 77, pp. 513–577, Jun 2005. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.77.513

[4] G. Adesso, S. Ragy, and A. R. Lee, “Continuous variable quantum information: Gaussian states and beyond,” *Open Syst. Inf. Dyn.*, vol. 21, no. 1-2, pp. 1440001, 47, 2014. [Online]. Available: https://doi.org/10.1142/S1230161214400010

[5] A. Serafini, *Quantum Continuous Variables: A Primer of Theoretical Methods*. Taylor & Francis Group, 2021.
[6] G. Androulakis and T. C. John, “Petz-Rényi relative entropy of thermal states and their displacements,” Submitted to journal, 2023. [Online]. Available: https://doi.org/10.48550/arXiv.2303.03380

[7] F. Hiai, “Quantum f-divergences in von Neumann algebras. I. Standard f-divergences,” J. Math. Phys., vol. 59, no. 10, pp. 102202, 27, 2018. [Online]. Available: https://doi.org/10.1063/1.5039973

[8] F. Hiai and M. Mosonyi, “Different quantum f-divergences and the reversibility of quantum operations,” Rev. Math. Phys., vol. 29, no. 7, pp. 1750023, 80, 2017. [Online]. Available: https://doi.org/10.1142/S0129055X17500234

[9] H. Araki, “Relative entropy of states of von Neumann algebras,” Publications of the Research Institute for Mathematical Sciences, vol. 11, no. 3, pp. 809–833, 1976.

[10] ———, “Relative entropy for states of von Neumann algebras ii,” Publications of the Research Institute for Mathematical Sciences, vol. 13, no. 1, pp. 173–192, 1977.

[11] M. Berta, V. B. Scholz, and M. Tomamichel, “Rényi divergences as weighted non-commutative vector-valued Lp-spaces,” Ann. Henri Poincaré, vol. 19, no. 6, pp. 1843–1867, 2018. [Online]. Available: https://doi.org/10.1007/s00023-018-0670-x

[12] M. Mosonyi, “Hypothesis testing for Gaussian states on bosonic lattices,” J. Math. Phys., vol. 50, no. 3, pp. 032105, 17, 2009. [Online]. Available: https://doi.org/10.1063/1.3085759

[13] U. Haagerup, “Operator-valued weights in von Neumann algebras. I,” J. Functional Analysis, vol. 32, no. 2, pp. 175–206, 1979. [Online]. Available: https://doi.org/10.1016/0022-1236(79)90053-3

[14] K. R. Parthasarathy, An introduction to quantum stochastic calculus, ser. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1992, [2012 reprint of the 1992 original] [MR1164866]. [Online]. Available: https://doi.org/10.1007/978-3-0348-0566-7

[15] A. Łuczak, H. Podseďkowska, and R. Wieczorek, “Relative modular operator in semifinite von Neumann algebras and its use,” arXiv preprint arXiv:1912.09633, 2019. [Online]. Available: https://doi.org/10.48550/arXiv.1912.09633

[16] H. Umegaki, “Conditional expectation in an operator algebra. IV. Entropy and information,” Kodai Math. Sem. Rep., vol. 14, pp. 59–85, 1962. [Online]. Available: http://projecteuclid.org/euclid.kmj/1138844604

[17] N. Datta, “Min- and max-relative entropies and a new entanglement monotone,” IEEE Trans. Inf. Theor., vol. 55, no. 6, p. 2816–2826, jun 2009. [Online]. Available: https://doi.org/10.1109/TIT.2009.2018325

[18] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, “On quantum Rényi entropies: a new generalization and some properties,” J. Math. Phys., vol. 54, no. 12, pp. 122203, 20, 2013. [Online]. Available: https://doi.org/10.1063/1.4838856
[19] N. Datta and F. Leditzky, “A limit of the quantum Rényi divergence,” *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 4, p. 045304, Jan 2014. [Online]. Available: https://doi.org/10.1088/1751-8113/47/4/045304

[20] V. Jakšić, Y. Ogata, C.-A. Pillet, and R. Seiringer, “Quantum hypothesis testing and non-equilibrium statistical mechanics,” *Rev. Math. Phys.*, vol. 24, no. 6, pp. 1230002, 67, 2012. [Online]. Available: https://doi.org/10.1142/S0129055X12300026

[21] M. Ohya and D. Petz, *Quantum entropy and its use*, ser. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1993. [Online]. Available: https://doi.org/10.1007/978-3-642-57997-4

[22] M. M. Wilde, A. Winter, and D. Yang, “Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy,” *Comm. Math. Phys.*, vol. 331, no. 2, pp. 593–622, 2014. [Online]. Available: https://doi.org/10.1007/s00220-014-2122-x

[23] F. Liese and I. Vajda, “On divergences and informations in statistics and information theory,” *IEEE Transactions on Information Theory*, vol. 52, no. 10, pp. 4394–4412, 2006. [Online]. Available: https://doi.org/10.1109/TIT.2006.881731

[24] T. van Erven and P. Harremos, “Rényi divergence and Kullback-Leibler divergence,” *IEEE Transactions on Information Theory*, vol. 60, no. 7, pp. 3797–3820, 2014. [Online]. Available: http://doi.org/10.1109/TIT.2014.2320500

[25] G. Androulakis and T. C. John, “Quantum f-divergences via Nussbaum-Szkoła distributions with applications to Petz-Rényi and von Neumann relative entropy,” 2022. [Online]. Available: https://arXiv:2203.01964