Entropic-gravity derivation of MOND

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A heuristic entropic-gravity derivation has previously been given of the gravitational two-body force of modified Newtonian dynamics (MOND). Here, it is shown that another characteristic of MOND can also be recovered, namely, the external field effect (implying a violation of the Strong Equivalence Principle). In fact, the derivation gives precisely the modified Poisson equation which Bekenstein and Milgrom proposed as a consistent nonrelativistic theory of MOND.

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1. Introduction

One possible explanation of the missing-mass problem on galactic scales is given by modified Newtonian dynamics (MOND). An extensive and up-to-date review of the observational data has been presented in Ref. [4]. Perhaps the experimentum crucis is provided by the nearby dwarf galaxy NGC 1560, where the fine-structure in the rotation curve is reproduced by MOND but not by a ‘virialized’ dark-matter halo. We thus have reason to investigate the theoretical basis of MOND.

MOND is, in fact, characterized by a new constant \( a_0 \) with the dimensions of acceleration: considering the simplest case, Newtonian dynamics is valid for accelerations \( |a| \gg a_0 \) and modified dynamics for accelerations \( |a| \ll a_0 \). The question is what the fundamental origin of \( a_0 \) is.

A recent proposal starts from Verlinde’s point of view that the standard Newtonian gravitational force is really an entropic force. Assuming the microscopic degrees of freedom on the holographic screen to have a nonzero minimum temperature \( T_{\text{min}} \), the MOND two-particle force between static test particles can be derived together with the result \( a_0 \sim (c/\hbar) k_B T_{\text{min}} \) and an explicit form of the interpolation function \( \hat{\mu}(x) \); see below for details. (A related paper on entropic gravity and MOND is Ref. [8]; additional references can be found in Ref. [7]. From a different point of view, several scaling relations and numerical coincidences involving MOND’s acceleration scale \( a_0 \) have been discussed in Ref. [9].)

A further characteristic of MOND is the so-called external field effect (EFE). Loosely speaking, the effect is that if a small weakly-bound gravitational system
(with internal acceleration $|a_{\text{int}}| \ll a_0$) is subjected to a large over-all acceleration (with $|a_{\text{ext}}| \gg a_0$), the internal gravitational dynamics becomes standard Newtonian even though $|a_{\text{int}}|/a_0$ is small. This EFE corresponds to a violation of the Strong Equivalence Principle, which suggests that the final relativistic theory is fundamentally different from general relativity.\(^{10}\)

In this Letter, we show that the entropic-gravity picture with $T_{\text{min}} > 0$ also explains the EFE. Moreover, we derive the Bekenstein–Milgrom modified Poisson equation\(^2\) which indeed provides a concrete realization of the EFE and has other applications in the context of missing-mass dynamics.\(^4\)

2. Derivation

It is possible to closely follow Verlinde’s argument\(^6\) here. The only new input is that, for a general mass distribution, the nonspherical closed holographic screen $\Sigma$ has a constant minimum temperature $T_{\text{min}} > 0$ for the fundamental degrees of freedom, in addition to the nonconstant temperature $T$ with $T \geq T_{\text{min}}$. In fact, it is the difference $T - T_{\text{min}}$ which corresponds to the effective localized mass that gravitates. The derivation proceeds in five steps.

First, along with Verlinde (Sec. 3.4 and Fig. 4 of Ref.\(^{6}\)), interpret the holographic screen as an equipotential surface, the quantity $-2\Phi/c^2$ being a measure of the entropy per degree of freedom on the screen, having set $\Phi = 0$ at spatial infinity (where, loosely speaking, there is no emerged space). Then, identify the acceleration of a test particle just outside the screen as the opposite of the potential gradient,

$$a = - \nabla \Phi.$$  

The basic idea is that inertia follows from the absence of an entropy gradient, acceleration from the presence of an entropy gradient. Specifically, the vector $\nabla \Phi$, if nonzero, is parallel to the outward normal unit-vector $n$ of the corresponding surface element $dA$ on the holographic screen, $\nabla \Phi \times n = 0$ and $\nabla \Phi \cdot n > 0$.

Second, recall the Davies–Unruh association between acceleration and temperature (references can be found in, e.g., Ref.\(^{6}\)). According to Eq. (6) of Ref.\(^{7}\), this gives\(^4\)

$$\hat{\mu} \left( \frac{|a|}{a_0} \right) |a| = 2\pi (c/\hbar) \left[ k_B T - k_B T_{\text{min}} \right],$$  

\(^{2a}\)The mathematical structure of (2a) is the same as the one from the Unruh-type temperature\(^{11,12}\) of an idealized detector with constant linear acceleration (for us, acceleration orthogonal to the screen) in a de-Sitter universe with Hubble constant $H_{\text{dS}}$ (for us, $H_{\text{dS}} \equiv 2\pi k_B T_{\text{min}}/\hbar$). As mentioned in the last two paragraphs of Sec. 3 in Ref.\(^{7}\) and referring to the discussion in Refs.\(^{11,12}\) the choice of a de-Sitter spacetime as the effective description of the minimum temperature of the fundamental degrees of freedom is consistent with the demands of local Lorentz invariance (special relativity), which, surprisingly, will also be seen to play a role in the third step of our derivation.
\[ \hat{\mu}(x) \equiv \sqrt{1 + (2x)^{-2} - (2x)^{-4}}, \]  
\[ a_0 \equiv 4\pi \left( c / \hbar \right) k_B T_{\text{min}}. \]  

The functional behavior of (2b) is such that \( \hat{\mu}(x) \to x \) for \( x \to 0^+ \) and \( \hat{\mu}(x) \to 1 \) for \( x \to \infty \). Note that this particular function \( \hat{\mu}(x) \) has also been suggested in Ref. [3] but without the link between \( \Delta T \equiv T - T_{\text{min}} \) and inertia or gravity [see the discussion under Eq. (9) in Ref. [3]]. Precisely this link will appear in the next step of our derivation [see also the discussion in the first paragraph of Sec. 5 in Ref. [7]].

Third, replacing \( T \) by \( T - T_{\text{min}} \) in Eq. (4.3) of Ref. [6] gives for the total active (locally gravitating) energy

\[ E \equiv M c^2 = \int dN \frac{1}{2} \left[ k_B T - k_B T_{\text{min}} \right] = \frac{1}{2 f l^2} \int dA \left[ k_B T - k_B T_{\text{min}} \right], \]  

where we have used, in the first step, special relativity and, in the last step, the following holographic relation between the surface element \( dA \) and a subset \( dN \) of the total number \( N \) of degrees of freedom on the screen:

\[ dA = f l^2 dN, \]  

with \( l^2 \) the fundamental unit of area and \( f \) a numerical factor keeping track of possible internal degrees of freedom of the ‘atom of area.’

Fourth, combining (2a) and (3a) and recalling (1), together with the discussion below that equation, gives

\[ M = \frac{1}{4\pi G_N} \int_{\Sigma} \hat{\mu} \left( \frac{\left| \nabla \Phi \right|}{a_0} \right) \nabla \Phi \cdot n \ dA, \]  

with the identification

\[ G_N \equiv f c^3 l^2 / \hbar, \]  

already encountered in our previous work (quoted in Ref. [7]).

Fifth, consider a small mass \( \tilde{M} = \rho \tilde{V} \) from a small volume \( \tilde{V} \) with surface \( \tilde{\Sigma} = \partial \tilde{V} \) and approximately constant mass density \( \rho \), and bring this mass \( \tilde{M} \) close to the screen \( \Sigma \). According to Verlinde’s argument given below Eq. (4.4) in Ref. [6] the following local consistency condition holds for pushing \( \tilde{M} \) into the screen:

\[ \nabla \cdot \left[ \hat{\mu} \left( \frac{\left| \nabla \Phi \right|}{a_0} \right) \nabla \Phi \right] = 4\pi G_N \rho. \]  

The underlying mathematics is trivial (but not the physics): with \( M \) replaced by \( \tilde{M} \) and \( \Sigma \) by \( \tilde{\Sigma} \) (assuming the interior space with volume \( \tilde{V} \) to exist) immediately gives [6] by use of Gauss’ divergence theorem and in the limit \( \tilde{V} \to 0 \).

This completes our entropic-gravity derivation of the modified Poisson law (5).

The derivation is, of course, entirely heuristic. But, even without knowledge of

\[ ^{\text{b}}\text{The same equation has been obtained in Ref. [8] but with a different physics motivation. Here, the starting point is the holographic screen with its real physical degrees of freedom, not an embedding spacetime with curvature.} \]
the underlying microscopic theory, the derivation may still be valid, as it relies on general principles: thermodynamics and, in a more subtle way, special relativity.

3. Discussion

Equation (5) corresponds, in fact, to the particular modified Poisson law suggested by Bekenstein and Milgrom, now with the explicit interpolation function \( \hat{\mu}(x) \) from (2b) and the MOND constant \( a_0 \) from (2c).

As shown by Bekenstein and Milgrom, the nonstandard gravitational field equation (5) can be derived from a Lagrangian, whose invariance under spacetime translations guarantees energy and momentum conservation. (For us, this Lagrangian is not a fundamental object but an auxiliary quantity.) Also noted by Bekenstein and Milgrom (Sec. V in Ref. 2) is the external field effect (EFE) mentioned in the Introduction. It is possible to illustrate the EFE from (5) by a simple one-dimensional example.

Consider an external mass distribution \( \rho_{\text{ext}} \) which gives rise to the one-dimensional acceleration \( g_{\text{ext}} \) and a local mass density \( \rho \) which gives rise to the one-dimensional acceleration \( g \). The corresponding Newtonian accelerations (from the standard Poisson law \( \nabla^2 \Phi = 4\pi G_N \rho \)) are denoted \( g_{\text{N,ext}} \) and \( g_{\text{N}} \), respectively. Using (1) and considering all vectors to be one-dimensional (hence, no need for bold-face symbols), it follows from (5) that

\[
\nabla \cdot \left[ (g + g_{\text{ext}}) \hat{\mu}(\|g + g_{\text{ext}}\|/a_0) \right] = -4\pi G_N (\rho + \rho_{\text{ext}}) = \nabla \cdot (g_{\text{N}} + g_{\text{N,ext}}), \quad (6a)
\]

which gives

\[
(g + g_{\text{ext}}) \hat{\mu}(\|g + g_{\text{ext}}\|/a_0) - g_{\text{ext}} \hat{\mu}(\|g_{\text{ext}}\|/a_0) = g_{\text{N}}, \quad (6b)
\]

where the integration constant has been set to zero. With the explicit function (2b), we obtain the following relation between the local acceleration \( g \) and the corresponding Newtonian value \( g_{\text{N}} \), for given external acceleration \( g_{\text{ext}} \):

\[
(g + g_{\text{ext}}) \left[ \sqrt{1 + (a_0/2)^2} \|g + g_{\text{ext}}\|^{-2} - (a_0/2) \|g + g_{\text{ext}}\|^{-1} \right] - (g_{\text{ext}}) \left[ \sqrt{1 + (a_0/2)^2} \|g_{\text{ext}}\|^{-2} - (a_0/2) \|g_{\text{ext}}\|^{-1} \right] = g_{\text{N}}, \quad (7)
\]

which reduces to \( g = g_{\text{N}} \) for \( a_0 = 0 \) or \( T_{\text{min}} = 0 \) at a fundamental level.

For \( |g_{\text{ext}}| \gg a_0 \) and \( |g| \ll |g_{\text{ext}}| \), relation (7) gives the standard Newtonian dynamics (up to a small renormalization of Newton’s constant \( G_N \)),

\[
g \sim \left[ 1 + \frac{1}{8} \left( \frac{a_0}{g_{\text{ext}}} \right)^2 \right] g_{\text{N}}, \quad (8a)
\]

\(^c\)To a certain extent, inspiration is provided by Bohr’s derivation of the quantized energy levels of the atom before the discovery of quantum theory proper.
having kept only the leading $(g_{\text{ext}})^{-2}$ term. Observe that (8a) holds, in particular, for $|g| \ll a_0$ and does not display a fundamental change from the Newtonian behavior, which is precisely the content of the EFE. For $|g_{\text{ext}}| \ll |g| \ll a_0$, on the other hand, relation (7) does give the MOND-type behavior,

$$g |g|/a_0 \sim g N,$$  

which, as mentioned in the Introduction, may provide a solution to the missing-mass problem on galactic scales.\cite{14}

In a nonrelativistic flat-spacetime context, our heuristic explanation of the EFE is that a very large external acceleration of a small self-gravitating system corresponds to a temperature of the fundamental degrees of freedom very much larger than their inherent minimum temperature, so that deviations from the standard Newtonian gravitational behavior become negligible. Only if the external acceleration essentially vanishes and if the internal acceleration is small enough, is the net temperature of the fundamental degrees of freedom small enough to reveal the existence of an inherent (‘hard-wired’) minimum temperature $T_{\text{min}}$, with the corresponding non-Newtonian gravitational effects characterized by the MOND acceleration scale $a_0 \propto k_B T_{\text{min}}$.

Note that the previous entropic-gravity derivation\cite{7} of the MOND constant $a_0 \propto k_B T_{\text{min}}$ held for the special case of linear relative motion between the two test particles. This has now been generalized to arbitrary (nonrelativistic) motions, and the result is given by (2c). The connection of the local-dynamics constant $a_0$ to a genuinely cosmological quantity can be only qualitative, as the derivation, up till now, has been nonrelativistic. Still, this connection\cite{7} appears suggestive: $a_0 \sim 2 \, c \, H_{dS}$, from \cite{24} and the identification of $T_{\text{min}}$ as the Gibbons–Hawking temperature $T_{\text{GH}} = h / 2 \pi k_B$ of a de Sitter universe with Hubble constant $H_{dS}$ (see also the remarks in Ftn.\cite{4}).

With the external field effect from (5) implying a violation of the Strong Equivalence Principle\cite{21} it can be expected that the entropic-gravity derivation of the final relativistic version of MOND, if successful, will result in a theory significantly different from general relativity\cite{4}.

\cite{4}Repeating the steps of Sec. 2, Verlinde’s relativistic derivation (Secs. 5.1–5.2 of Ref.\cite{6}) can again be closely followed. This leads to a new version of his Eq. (5.9): $M = (4 \pi G N)^{-1} \int_{\Sigma} \exp[\phi/c^2] \tilde{\mu}([\nabla \phi] / a_0) \, \nabla \phi \cdot dA$, in terms of an effective redshift-related potential $\phi \equiv (c^2/2) \log[1 - \xi \cdot \xi]$ from the timelike Killing four-vector $\xi^b$ of the static spacetime considered (with signature $-+++$). It remains to be seen if this type of surface integral can give rise to a volume integral which matches the volume integral corresponding to the mass $M$. 
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