A PENROSE TYPE INEQUALITY FOR GRAPHS OVER REISSNER-NORDSTRÖM-ANTI-DESITTER MANIFOLD

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ABSTRACT. In this paper, we use the inverse mean curvature flow to establish an optimal Minkowski type inequality, weighted Alexandrov-Fenchel inequality for the mean convex star shaped hypersurfaces in Reissner-Nordström-anti-deSitter manifold and Penrose type inequality for asymptotically locally hyperbolic manifolds in which can be realized as graphs over Reissner-Nordström-anti-deSitter manifold.

1. Introduction

The famous positive mass conjecture in general relativity states: any asymptotically flat Riemannian manifold with a suitable decay order and with nonnegative scalar curvature has the nonnegative ADM mass. Moreover, equality holds if and only if the manifold is isometric to the Euclidean space with the standard metric. The positive mass theorem first proved by Schoen and Yau [36] in 1979 using minimal surface techniques and then by Witten [42] in 1981 using spinors. Recently, Schoen and Yau [37] proved the positive mass theorem in all dimension.

The Penrose inequality in general relativity as refinement of the positive mass theorem states that the total mass of a spacetime is no less than the mass of its black holes. In the asymptotically flat case, which corresponds to a vanishing cosmological constant, the Riemannian Penrose inequality reads that

\[ m_{\text{ADM}} \geq \frac{1}{2} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \]

where \( m_{\text{ADM}} \) is the ADM mass of the asymptotically flat Riemannian manifold with horizon and \( |\Sigma| \) is the area of \( \Sigma \). The Riemannian Penrose inequality (1.1) has been established by Huisken–Ilmanen [23] by using inverse mean curvature flow for a connected horizon and Bray [3] by using conformal flow for an arbitrary horizon in dimension 3. Later, Bray’s approach was generalized to any dimension \( n \leq 7 \), as proven by Bray and Lee in [5]. For related results, see the excellent surveys [4] and [32]. Lam [27] proved (1.1) in all dimensions for an asymptotically flat manifold which is a graph over \( \mathbb{R}^n \). Mirandola and Vitório [33] generalized Lam’s result to arbitrary codimension graph with flat normal bundle. Recently, Ge et al. [16] introduced a new mass, which they called Gauss-Bonnet-Chern mass, and they proved Penrose type inequalities in this case. Wei, Xiong and the second author [29] obtained

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Penrose-type inequality of the second Gauss–Bonnet–Chern mass for the graphic manifold with flat normal bundle.

In recent years, extending the previous results to a spacetime with a negative cosmological constant attracts many authors’ interest. In the time symmetric case, \( (M^n, g) \) is now an asymptotically hyperbolic manifold with an outermost minimal boundary \( \Sigma \). The notion of mass for this class of manifolds was first defined mathematically by Wang [39] and Chruściel and Herzlich [8]. For the asymptotically hyperbolic manifolds, Chruściel and Herzlich in [8] introduced a mass-like invariant, which generalizes the ADM mass. See also [9, 21, 22, 34, 43]. For the mass in asymptotically hyperbolic manifolds, the corresponding Penrose conjecture states

\[
\left( \frac{\Sigma \Omega}{\omega_{n-1}} \right)^{\frac{n-1}{n}} + \left( \frac{\Sigma}{\Omega g} \right)^{\frac{n-1}{n}} \geq \frac{1}{2} \left[ \left( \frac{\Sigma}{\omega_{n-1}} \right)^{\frac{n-1}{n}} + \left( \frac{\Sigma}{\Omega g} \right)^{\frac{n-1}{n}} \right],
\]

if \( R_g + n(n-1) \geq 0 \), where \( R_g \) is the scalar curvature of \( g \). A recent result by Neves [35] shows that it is not possible to prove (1.2) only adapting Huisken and Ilmanen’s inverse mean curvature flow method in general. In dimension 3, Lee and Neves [31] were able to use the inverse mean curvature flow to establish a Penrose type inequality for asymptotically locally hyperbolic manifolds in the so-called negative mass range. Dahl-Gicquaud-Sakovich [10] and de Lima-Girão [13] proved (1.2) for asymptotically hyperbolic graphs over the hyperbolic space \( \mathbb{H}^n \) with the help of of the weighted hyperbolic Alexandrov-Fenchel inequality

\[
\int_{\Sigma} \cosh r H d\mu \geq (n-1)\omega_{n-1} \left[ \left( \frac{\Sigma}{\omega_{n-1}} \right)^{\frac{n-1}{n}} + \left( \frac{\Sigma}{\Omega g} \right)^{\frac{n-1}{n}} \right],
\]

provided \( \Sigma \) is star-shaped and strictly mean-convex (i.e. \( H > 0 \)). It was proved by de Lima and Girão in [13]. The sharp Alexandrov–Fenchel-type inequality has an independent interest. Recently, there have been considerable progress in establishing Alexandrov–Fenchel type inequalities, see for instance [7, 17, 18, 29, 40] and the references therein.

In this paper, we will consider the Penrose inequality for asymptotically locally hyperbolic graphic manifold over Reissner-Nordström-anti-deSitter (called "Reissner-Nordström-AdS" for short) manifold. Firstly, we briefly recall the definition of Reissner-Nordström-AdS manifold. We fix three positive numbers \( m, q \) and \( \kappa \), where \( q < m, \kappa << \infty \) such that the equation \( \epsilon + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n} = 0 \) has positive solutions, and let \( s_0 \) be the larger one. Let \( (N^{n-1}_{s_0}, \hat{g}) \) be a closed space form of constant sectional curvature \( \epsilon = 0, \pm 1 \) and \( f = \sqrt{\epsilon + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n}} \). The Reissner-Nordström-anti-deSitter manifold is a warped product manifold \( P = (s_0, \infty) \times N_{s_0} \) with the metric \( \hat{g} = \frac{1}{f^2} ds^2 + s^2 \hat{g} \), where \( \hat{g} \) is the standard metric on \( N_{s_0} \). The hypersurface \( \partial P = \{ s_0 \} \times N_{s_0} \) is referred to as the horizon. Here we must remark that the Reissner-Nordström-AdS manifold is referred to the manifold \( P = (s_0, \infty) \times N_{s_0} \) with \( \epsilon = 1 \) in [38]. In this case, Z.H. Wang [38] obtained a Minkowski type inequality by using the inverse mean curvature flow. Motivated by [38], we obtain

**Theorem 1.1.** Let \( \Sigma \) be a compact mean convex, star-shaped and embedded hypersurface in Reissner-Nordström-AdS manifold \( P \) and let \( \Omega \) denote the enclosed region by \( \Sigma \) and the horizon \( \partial P = \{ s_0 \} \times N_{s_0} \),
then we have
\[
\int_{\Sigma} fHd\mu - n(n-1)\kappa^2 \int_{\Omega} f dv \geq (n-1)\epsilon \partial_{n-1} \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^{\frac{1}{n-1}} - \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{1}{n-1}}
\]
(1.4)
\[
+ (n-1)q^2 \partial_{n-1} \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^{\frac{1}{n-1}} - \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{1}{n-1}}
\]
where \( \partial_{n-1} = |N_e| \) and \( \partial P \) is the area of horizon \( \{s_0\} \times N_e \). Equality in (1.4) holds if and only if \( \Sigma \) is a slice \( \{s\} \times N_e \) for \( s \in [s_0, \infty) \).

**Remark 1.2.** For \( \epsilon = 1 \) in Theorem 1.1, the inequality (1.4) is reduced to Theorem 1 in [38]. When \( q = 0 \) and \( \epsilon = 1 \), Theorem 1.1 reduces to the Minkowski inequality proved in [7]; when \( q = 0 \) and \( \epsilon = -1 \), Theorem 1.1 was proved in [18].

**Theorem 1.3.** Let \( \Sigma \) be a compact embedded hypersurface which is star-shaped with positive mean curvature in \( P \), then we have
\[
\int_{\Sigma} fHd\mu \geq (n-1)\kappa^2 \partial_{n-1} \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^{\frac{1}{n-1}} - \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{1}{n-1}}
\]
(1.5)
\[
+ (n-1)\epsilon \partial_{n-1} \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^{\frac{1}{n-1}} - \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{1}{n-1}}
\]
\[
+ (n-1)q^2 \partial_{n-1} \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^{\frac{1}{n-1}} - \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{1}{n-1}}
\]
where \( \partial P = \{s_0\} \times N \). Equality holds if and only if \( \Sigma \) is a geodesic slice.

**Remark 1.4.** If the parameter \( q \) vanishes, the Reissner-Nordström-AdS manifold becomes the Kottler space. Theorem 1.3 is reduced to Theorem 1.5 in [18].

Now we state that the inequality for mass in asymptotically hyperbolic graph manifolds over Reissner-Nordström-AdS manifold \( P \).

**Theorem 1.5.** Suppose \( M^n \subset Q \) is an ALH graph over \( P \) with inner boundary \( \Sigma \), associated to a smooth function \( u : P \setminus \Omega \rightarrow \mathbb{R} \). Assume that \( \Sigma \) is in a level set of \( u \) and \( |\nabla u| \rightarrow \infty \) as \( x \rightarrow \Sigma \). Then
\[
m(M, g) \geq m + c_n \left( \int_M (R_{\tilde{g}} - R_{g}) \left( \frac{\partial}{\partial t}, \xi \right) d\mu_M + \int_{\Sigma} fHd\mu \right)
\]
(1.6)
where \( H \) is the mean curvature of \( \Sigma \) in \((P, \tilde{g})\), \( \xi \) is the unit outer normal of \((M, g)\) in \((Q, \tilde{g})\), the constant \( c_n \) is defined by
\[
c_n = \frac{1}{2(n-1)\partial_{n-1}}
\]
and
\[
R_{\tilde{g}} = -n(n-1)\kappa^2 + (n-1)(n-2)q^2 s^{2-2n}
\]
is the scalar curvature of the Reissner-Nordström-AdS manifold $P$. Equality in (1.6) holds if and only if $M$ is rotationally symmetric. Moreover, if $R_g \geq \bar{R}$, we have

\begin{equation}
(1.7) \\
m(M, g) \geq m + c_n \int_{\Sigma} f \, H d\mu.
\end{equation}

From Theorem 1.3 and (1.7), we immediately deduce the Penrose inequality for ALH graphs.

**Theorem 1.6.** If $M \subset Q$ is a ALH graph as in Theorem 1.5, such that $\Sigma \subset (P, \bar{g})$ is star-shaped and mean-convex, moreover, if in addition $R_g \geq \bar{R}$, we have

\begin{equation}
(1.8) \\
m(M, g) \geq \frac{1}{2} \left( \kappa^2 \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^\frac{n}{n+1} + \epsilon \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^\frac{n-1}{n-2} + q^2 \left( \frac{|\Sigma|}{\partial_{n-1}} \right)^\frac{n-2}{n-1} \right).
\end{equation}

Equality in (1.8) is achieved by the Reissner-Nordström-AdS manifold $P$.

**Remark 1.7.** It’s easy to show that the Reissner-Nordström-AdS manifolds $(P, \bar{g})$ with constant $\epsilon, \kappa, m, q$ and $-\kappa^2 > c_\epsilon$ (see Remark 2.2) can be represented as an ALH graph over another Reissner-Nordström-AdS manifold with constant $\epsilon, \kappa, m’, q$. From (6.1) and (6.24) we konw that equality is achieved by the Reissner-Nordström-AdS manifolds. From the argument of Huang-Wu [25], we believe that the rigidity in Theorem 1.6 should hold, i.e., the equality holds in the Penrose type inequality (1.8) if and only if $M$ is exactly the Reissner-Nordström-AdS manifold.

**Remark 1.8.** A similar argument appeared in [24], [20] and [28] implies that our main results in Theorem 1.1, Theorem 1.3 and Theorem 1.6 also hold for star-shaped and weakly mean convex hypersurface $\Sigma$ in Reissner-Nordström-AdS manifold $(P, \bar{g})$.

**Remark 1.9.** By use of the weak solution of inverse mean curvature flow (see [23]) and the techniques of [41], it is an interesting problem to prove the Minkowski-type inequality for outward minimizing hypersurfaces in Reissner-Nordström-AdS manifold $(P, \bar{g})$.

The paper is organized as follows. In Section 2, we collect some facts about the Reissner-Nordström-AdS, star-shaped hypersurfaces, inverse mean curvature flow and the asymptotically locally hyperbolic manifold; In Section 3, we establish the long-time existence and convergence result of the inverse mean curvature flow for star-shaped and strictly mean convex hypersurface in Reissner-Nordström-AdS manifold. In Section 4 and Section 5, we will prove Theorem 1.1 and Theorem 1.3, repectively. In the last section, we will give the proof of Theorem 1.5 and Theorem 1.6.

## 2. Preliminaries

In this section, we collect some facts about the Reissner-Nordström-AdS manifold, star-shaped hypersurfaces, inverse mean curvature flow and asymptotic locally hyperoblic (ALH) mass of graphs.
2.1. **Reissner-Nordström-AdS manifold.** We fix three positive numbers \( m, q \) and \( \kappa \) such that the equation \( \epsilon + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n} = 0 \) has positive solutions, and let \( s_0 \) be the largest one. Let \( (N_e^{-1}, \bar{g}) \) be a closed space form of constant sectional curvature \( \epsilon = 0, \pm 1 \). As in [38, 26], the Reissner-Nordström-AdS manifold is defined by \( P = (s_0, \infty) \times N_e \) with the metric

\[
\bar{g} = \frac{1}{\epsilon + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n}} ds^2 + s^2 \bar{g}.
\]

By a change of variable, the metric of Reissner-Nordström-AdS manifold can be rewritten as

\[
\tilde{g} = dr^2 + \lambda(r)^2 \bar{g},
\]

where \( \lambda(r) \) satisfies the ODE

\[
\lambda'(r) = \sqrt{\epsilon + \kappa^2 \lambda^2 - 2m\lambda^{2-n} + q^2\lambda^{4-2n}}.
\]

Defining \( f(\lambda) = \lambda' \), one can check that

\[
f'(\lambda) = \lambda'' = \kappa^2 \lambda + m(n-2)\lambda^{1-n} - (n-2)q^2\lambda^{3-2n}.
\]

Then the function \( f \) satisfies (see (1.2) in [38])

\[
(\tilde{\Delta}f)\bar{g} - \bar{\nabla}^2 f + f Ric_{\bar{g}} = (n-1)(n-2)q^2 \lambda(r)^{4-2n} f \bar{g},
\]

where \( Ric_{\bar{g}} \) is the Ricci curvature tensor, \( \tilde{\Delta} \) and \( \bar{\nabla}^2 \) are the Laplacian and Hessian operators with respect to the metric \( \tilde{g} \) of Reissner-Nordström-AdS manifold. In general, a Riemannian metric is called sub-static if \( (\tilde{\Delta}h)\bar{g} - \bar{\nabla}^2 h + h Ric_{\bar{g}} \geq 0 \) for some positive function \( h \).

**Remark 2.1.** When \( q \to 0 \), the Reissner-Nordström-AdS manifold reduces to Kottler manifold (see [18, 28]).

**Remark 2.2.** When \( q > 0 \), to ensure \( \psi(s) := \epsilon + \kappa^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n} = 0 \) to have positive root we need the following conditions (see [26]). One can check \( \psi(+\infty) = +\infty \), \( \psi(0^+) = +\infty \) and \( \psi'(s) = 0 \) always have a single positive solution \( s = a > 0 \). Then positive root for \( \psi(s) = 0 \) exists if and only if \( \psi(a) \leq 0 \). From \( \psi'(a) = 0 \), we get

\[
-k^2 = \frac{n-2}{a^2} \left( \frac{m}{a^{n-2}} - \frac{q^2}{a^{2n-4}} \right) \quad (=: g(a)).
\]

**Combining it with \( \psi(a) \leq 0 \) we have**

\[
0 \geq \epsilon - \frac{nm}{a^{n-2}} + \frac{(n-1)q^2}{a^{2n-4}}.
\]

If \( D := n^2m^2 - 4\epsilon(n-1)q^2 \geq 0 \), then

\[
\frac{1}{a^{n-2}} \leq \frac{1}{a_c^{n-2}} := \frac{nm + \sqrt{D}}{2(n-1)q^2}
\]

ensures \( \psi(a) \leq 0 \). Equivalently,

\[
-k^2 = g(a) \geq g(a_c) = \begin{cases}
  c_0 := -\frac{(n-1)^2}{(n-1)^2} \left( \frac{n}{n-1} \right) \frac{m^2}{\epsilon} m^{-\frac{n}{n-1}}, & \epsilon = 0, \\
  c_{-1} := -2\frac{n^2-2}{n-1} \left( \sqrt{D} - (n-2)m \right) \left( \sqrt{D} - nm \right)^{-\frac{n}{n-1}}, & \epsilon = -1, \\
  c_1 := -2\frac{n^2-2}{n-1} \left( \sqrt{D} - (n-2)m \right) \left( nm - \sqrt{D} \right)^{-\frac{n}{n-1}}, & \epsilon = 1.
\end{cases}
\]
When $\epsilon = 0, -1, D \geq 0$ automatically. When $\epsilon = 1, -k^2 \geq c_1$ implies $q^2 < m^2$. Therefore, $\psi(s) = 0$ has positive root when $-k^2 \geq c_\epsilon (c_\epsilon = c(\epsilon, n, m, q)$ as above) and $q^2 < m^2$ if $\epsilon = 1$. When $\epsilon = 1$, (2.5) can be written as $q < m, k << \infty$, which was observed by Z.H. Wang in [38].

**Remark 2.3.** One can check that when $\lambda \geq s_0$, where $s_0$ is the largest positive root of the equation $\epsilon + k^2 s^2 - 2ms^{2-n} + q^2 s^{4-2n} = 0$, then $\lambda'' \geq 0$.

### 2.2. Asymptotic behaviors of the Reissner-Nordström-AdS manifold.

Defining

$$r(s) = \int_{s_0}^{s} \frac{dt}{\sqrt{k^2 t^2 + \epsilon t}} - \int_{s_0}^{\infty} \left( \frac{1}{\sqrt{\epsilon + k^2 t^2 - 2mt^{2-n} + q^2 t^{4-2n}}} - \frac{1}{\sqrt{k^2 t^2 + \epsilon}} \right) dt,$$

we have $r(s)$ is the inverse function of $\lambda(r)$ up to a constant from (2.2). For $\epsilon = 1$, in [38], Z.H. Wang obtained

$$r(s) = k^{-1} \sinh(\epsilon r) - \frac{m}{n} k^{-3} s^{1-n} + k^{-1} O(s^{-n-2}).$$

By Taylor expansion,

$$\sinh(\epsilon r(s)) = \frac{m}{n} k^{-1} s^{1-n} + k^{-2} O(s^{-n-1})$$

$$= \frac{m}{n} k^{-1} (\sinh(\epsilon r))^1 + O((\sinh(\epsilon r))^{-1-n}).$$

Then

$$\lambda(r) = k^{-1} \sinh(\epsilon r) + \frac{m}{n} k^{-3} \sinh^{-1}(\epsilon r) + O(\sinh^{-1}(\epsilon r))$$

$$= O(\epsilon^{n}).$$

For $\epsilon = -1$ and $\epsilon = 0$, by similar calculations, we have

$$\lambda(r) = k^{-1} \cosh(\epsilon r) + \frac{m}{n} k^{-3} \cosh^{-1}(\epsilon r) + O(\cosh^{-1}(\epsilon r)) = O(\epsilon^{n}), \quad \text{for } \epsilon = -1,$$

$$\lambda(r) = \epsilon^{r} + \frac{m}{n} k^{-2} \epsilon^{-n+1} r + O(\epsilon^{-n+1} r) = O(\epsilon^{r}), \quad \text{for } \epsilon = 0.$$

Therefore, the function $\lambda(r)$ has the following asymptotic expansion

$$\lambda(r) = O(\epsilon^{r}).$$

Let $\tilde{\nabla}$ be the covariant derivative of $P$. The Riemann curvature tensor of $P$ is given by

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_X \tilde{\nabla}_Y Z + \tilde{\nabla}_{[X,Y]} Z.$$

Let $\{e_\alpha\}_{\alpha=0}^{n-1}$ be an orthonormal frame and let $\tilde{R}_{\alpha\beta\gamma\mu} = \tilde{g}(\tilde{R}(e_\alpha, e_\beta)e_\gamma, e_\mu)$ be the Riemannian curvature tensor and $\tilde{\nabla}$ be the covariant derivative of the Reissner-Nordström-AdS manifold $(P, \tilde{g})$, respectively. The asymptotic expansion of Riemannian curvature $\tilde{R}_{\alpha\beta\gamma\mu}$ and Ricci curvature $\text{Ric}_{\tilde{g}}$ are given by (see
Lemma 10 in [38])
\begin{align}
R_{\alpha\beta y} &= -\kappa^2 (\delta_{\alpha\gamma}\delta_{\beta\mu} - \delta_{\alpha\mu}\delta_{\beta\gamma}) + O(e^{-nkr}), \\
\nabla_{\mu} R_{\alpha\beta y} &= O(e^{-nkr}), \\
Ric_{\alpha\beta}(\partial_r, \partial_r) &= -(n-1)\kappa^2 + O(e^{-nkr}), \\
\lambda^{-2}Ric_{\alpha\beta}(\partial_i, \partial_j) &= -(n-1)\kappa^2 \hat{g}_{ij} + O(e^{-nkr}).
\end{align}

2.3. Star-shaped hypersurface in the Reissner-Nordström-AdS manifold. In this subsection, we recall some basic properties of star-shaped hypersurfaces in \((P, \tilde{g})\) (see in [14, 15, 28]). Let \(\theta = \{\theta_i\}_{i=1,\ldots,n-1}\) be a local coordinate systems on \(N_e\) and \(\partial_r\) be the radial vector fields. If \(\Sigma\) is a smooth closed hypersurface in the Reissner-Nordström-AdS manifold \((P, \tilde{g})\), the hypersurface \(\Sigma\) is called star-shaped if the support function \(\langle \lambda \partial_r, \nu \rangle > 0\) on \(\Sigma\), which implies that \(\Sigma\) could be parameterized by a graph
\[
\Sigma = \{(r(\theta), \theta) : \theta \in N_e\}.
\]
As in [7, 11, 14, 28], we define a function \(\varphi\) on \(N_e\) by \(\varphi(\theta) = \Phi(r(\theta))\), where \(\Phi(r)\) is a positive function satisfying \(\Phi'(r) = 1/\lambda(r)\). The tangential vector field along \(\Sigma\) can be expressed in the form
\[
X_i = \partial_i + r_i \partial_r = \partial_i + \lambda \varphi_i \partial_r
\]
and then the induced metric on \(\Sigma\) is given by
\[
g_{ij} = \lambda^2 (\hat{g}_{ij} + \varphi_i \varphi_j).
\]
The unit outward normal vector field \(\nu\) of the hypersurface \(\Sigma\) could be written as
\[
\nu = \frac{1}{v} \left( \partial_r - \frac{r^i}{\lambda^2} \partial_i \right), \quad v = \sqrt{1 + |\nabla \varphi|_{\tilde{g}}^2},
\]
where \(r = r^i \hat{g}_{ij}, (\hat{g}_{ij}) = (\hat{g}_{ij})^{-1}\) and \(\nabla\) denotes the Levi-Civita connection on \(N_e\).

Denoting \(h_{ij}\) by the components of the second fundamental form of the hypersurce \(\Sigma\). Then we have
\[
h_{ij} = \frac{\lambda}{v} \left( \lambda' (\hat{g}_{ij} + \varphi_i \varphi_j) - \varphi_{ij} \right), \quad h_i^j = \frac{1}{v \lambda} \left( \lambda' \hat{\sigma}_i^j - \hat{\sigma}_j^k \varphi_{ki} \right),
\]
where \(\varphi_{ki} = \hat{\nabla}_k \hat{\nabla}_i \varphi, g^{ij} = \lambda^{-2} \hat{\sigma}^{ij}\) and \(\hat{\sigma}^{ij} = \hat{g}^{ij} - \frac{\varphi \varphi}{v^2}\) with \(\varphi^j = \hat{g}^{jk} \varphi_k\). The mean curvature is given by
\[
H = g^{ij} h_{ij} = \frac{1}{\lambda v} ((n-1)\lambda' - \hat{\sigma}^{ij} \varphi_{ij}).
\]

2.4. The inverse mean curvature flow in Reissner-Nordström-AdS manifold. Let \(\Sigma_0\) be a smooth, strictly mean-convex, star-shaped closed embedded hypersurface in \((P, \tilde{g})\). The inverse mean curvature flow in the Reissner-Nordström-AdS manifold is a family of embeddings \(X : \Sigma \times [0, T) \rightarrow (P, \tilde{g})\) satisfying
\[
\begin{cases}
\frac{\partial}{\partial t} X = \frac{1}{H} \nu, \\
X_0 = \Sigma_0,
\end{cases}
\]
where $\nu$ is the unit outer normal to hypersurface $\Sigma_t = \chi(t, \Sigma)$ and $H$ is the mean curvature of $\Sigma_t$. If the initial hypersurface is strictly mean convex, the short time existence result (see, e.g., [14]) of (2.16) implies the flow exists on a maximum time interval $[0, T)$. Thus it remains to study the long time behavior of the flow (2.16).

The equation (2.16) is called the parametric form of the inverse mean curvature flow. For a graphic hypersurface, the equation (2.16) can be expressed in another form. Let $\Sigma_0 = \{(r_0(\theta), \theta) : \theta \in N_\epsilon\}$ be a star shaped hypersurface in Reissner-Nordström-AdS manifold $(P, \bar{g})$ defined on $N_\epsilon$. Assume that $\Sigma_t$ is the solution hypersurface and star shaped for (2.16). Then it can be parametrized as

$$\Sigma_t = \{(r(\theta, t), \theta) : \theta \in N_\epsilon\}.$$

As long as the solution of (2.16) exists and remains star shaped, it is equivalent to the following non-parametric form of the flow (cf. [7, 11, 15, 18])

$$\frac{\partial}{\partial t} r(\theta, t) = \frac{\nu}{H}.$$  

We have the following evolution equations (see [7, 14, 38, 44]).

**Proposition 2.4 (Evolution equations).** Along the inverse mean curvature flow (2.16), we have the following evolution equations:

\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= \frac{2}{H} h_{ij}, \\
\frac{\partial}{\partial t} g^{ij} &= -\frac{2}{H} h^{ij}, \\
\frac{\partial}{\partial t} \nu &= \frac{1}{H^2} \nabla H, \\
\frac{\partial}{\partial t} d\mu &= d\mu, \\
\frac{\partial}{\partial t} \phi &= \frac{\nu}{\lambda H} = \frac{\nu^2}{(n-1)\lambda' - \lambda^i \phi_{ij}}, \\
\frac{\partial}{\partial t} h_{ij} &= \frac{1}{H^2} \Delta h_{ij} - \frac{2}{H^3} H H_j - \frac{2}{H} R_{ijv} + \frac{1}{H^2} \left( |A|^2 + R^k_{\nu k v} \right) h_{ij} \\
&\quad - \frac{1}{H^2} \left( 2h^{'}^i R_{ij'} + h^i R^k_{i k p} + h^j R^k_{j k p} \right) - \frac{1}{H^2} \left( \hat{\nabla}_i R_{v j i}^k + \hat{\nabla}_j R_{v i j}^k \right), \\
\frac{\partial}{\partial t} H &= \frac{1}{H^2} \Delta H - \frac{2}{H^3} |\nabla H|^2 - \frac{|A|^2}{H} - \frac{\text{Ric}_{\bar{g}}(\nu, \nu)}{H},
\end{align*}

where $\hat{\hat{\sigma}}^i_j = \hat{g}^i_j - \frac{\phi^i}{\nu^2}, \hat{R}_{i j p}^k$ is the Riemannian curvature tensor and $\text{Ric}_{\bar{g}}$ is the Ricci tensor of $(P, \bar{g})$. In the non-parametric form the function $u := \frac{1}{H(\hat{\nu}^i \hat{\nu}^j)} = \frac{1}{H^2}$ evolves under

$$\frac{\partial}{\partial t} u = \frac{\Delta u}{H^2} - \frac{2|\nabla u|^2}{uH^2} - \frac{2\langle \nabla u, \nabla H \rangle}{H^3} - (n-1) \frac{\lambda' u}{\lambda H^2}.$$

2.5. Asymptotically locally hyperbolic manifold. In order to define the mass of asymptotically locally hyperbolic manifold, we recall from ([8, 13, 18]) the following definition. Fix $\epsilon = 0, \pm 1, \kappa > 0$ and suppose $(N^{n-1}, \hat{g})$ is a closed space form of sectional curvature $\epsilon$. Consider the product manifold $P_\epsilon = I_\epsilon \times N_\epsilon$, where $I_\epsilon = (\frac{1}{\epsilon}, \infty)$ and $I_0 = I_1 = (0, \infty)$ endowed with the warped product metric

\begin{equation}
(2.26) \quad b_\epsilon = \frac{ds^2}{V^2_\epsilon(s)} + s^2 \hat{g}, \ s \in I_\epsilon, \text{ and } V_\epsilon(s) = \sqrt{\kappa^2 s^2 + \epsilon}.
\end{equation}

We recall the following definition of asymptotically locally hyperbolic manifold and its mass which is a geometric invariant (see in [8]).

**Definition 2.5.** A Riemannian manifold $(M^n, g)$ is called a asymptotically locally hyperbolic (ALH) if there exists a compact subset $K$ and a diffeomorphism at infinity $\Phi : M \setminus K \rightarrow N \times (s_0, \infty)$, such that

\begin{equation}
(2.27) \quad \| (\Phi^{-1})^* g - b_\epsilon \|_{b_\epsilon} + \| \nabla^b_\epsilon ((\Phi^{-1})^* g) \|_{b_\epsilon} = O(s^{-\tau}), \quad \tau > \frac{n}{2},
\end{equation}

and

\begin{equation}
(2.28) \quad \int_M V_\epsilon |R_\epsilon + \kappa^2 n(n-1)|d\mu_M < \infty.
\end{equation}

Then the ALH mass can be defined as

\begin{equation}
(2.29) \quad m(M, g) = \frac{1}{2(n-1)\partial^{n-1}} \lim_{i \rightarrow \infty} \int_{N_\epsilon} \left( V_\epsilon (div^{b_{\epsilon}}e - d \iota tr^{b_{\epsilon}}e) + (tr^{b_{\epsilon}}e)dV_{b_{\epsilon}} - e(\nabla^{b_{\epsilon}} V_{b_{\epsilon}}, \cdot) \right) d\mu
\end{equation}

where $e := (\Phi^{-1})^* g - b_\epsilon, N_\epsilon = \{s\} \times N_\epsilon, \nu$ is the outer normal of $N_\epsilon$ induced by $b_\epsilon$ and $d\mu$ is the area element with respect to the induced metric on $N_\epsilon$.

The corresponding Reissner-Nordström-AdS manifold spacetime in general relativity is

\[-f^2 dt^2 + \tilde{g} = -f^2 dt^2 + \frac{1}{f^2} ds^2 + s^2 \hat{g}\]

where $m$ is the mass and $q$ is the charge. Now we consider its Riemannian version, namely $Q = \mathbb{R} \times P$ with metric

\begin{equation}
(2.30) \quad \tilde{g} = f^2 dt^2 + \tilde{g} = f^2 dt^2 + \frac{1}{f^2} ds^2 + s^2 \hat{g}.
\end{equation}

We identify $P$ with the slice $\{0\} \times P \subset Q$ and consider a graph over $P$ or over $P \setminus \Omega$ in $Q$, where $\Omega$ is a compact smooth subset containing $\{0\} \times \partial P$. A graph associated to a smooth function $u : P \setminus \Omega \rightarrow \mathbb{R}$ is a manifold $M^n$ with induced metric from $(Q, \tilde{g})$, i.e.

\begin{equation}
(2.31) \quad g = f^2(s)\tilde{\nabla} u \otimes \tilde{\nabla} u + \tilde{g}
\end{equation}

where $\tilde{\nabla}$ is the covariant derivative with respect to metric $\tilde{g}$.

**Definition 2.6.** We call that $M^n \subset Q$ is an ALH graph over $P \setminus \Omega$ (associated to a smooth function $u : P \setminus \Omega \rightarrow \mathbb{R}$) if there exists a compact subset $K$ and a diffeomorphism at infinity $\Phi : M \setminus K \rightarrow N_\epsilon \times (s_0, +\infty) \subset P \setminus \Omega$, such that

\begin{equation}
(2.32) \quad \| (\Phi^{-1})^* g - \tilde{g} \|_{\tilde{g}} + \| \tilde{\nabla}(\Phi^{-1})^* g \|_{\tilde{g}} = O(s^{-\tau}), \quad \tau > \frac{n}{2}
\end{equation}
or equivalently,
\begin{equation}
|f\bar{\nabla}u|_{g} + |f\bar{\nabla}^2u + \bar{\nabla}f \otimes \bar{\nabla}u|_{g} = O(s^{-\tau}), \quad \tau > \frac{n}{2}
\end{equation}
and
\begin{equation}
\int_{M} f|R_{g} - R_{\bar{g}}|d\mu_{M} < \infty.
\end{equation}

For such a graph over $P \setminus \Omega$, we can check definition 2.5 is equivalent to definition 2.6, i.e. any ALH graph is an ALH manifold (see [8, 18]).

3. Inverse mean curvature flows in the Reissner-Nordström-AdS manifold

In this section we establish a long time existence and asymptotic behavior of the inverse mean curvature flow. Our proofs use the similar arguments in [7, 18, 28, 38].

3.1. $C^0$ and $C^1$-estimates. By the parabolic maximum principle, we can obtain $C^0$ and $C^1$-estimates of the inverse mean curvature flow (2.16) in Reissner-Nordström-AdS manifold.

**Lemma 3.1.** Let $r(t) = \inf_{N} r_{N}(\theta, t)$ and $\bar{r}(t) = \sup_{N} r_{N}(\theta, t)$. The solution $r$ of (2.17) satisfies
\begin{equation}
\lambda(r(0))e^{\frac{1}{n-1}t} \leq \lambda(r(\theta, t)) \leq \lambda(\bar{r}(0))e^{\frac{1}{n-1}t},
\end{equation}
which imply $e^{e^{t}} = O(e^{\frac{1}{n-1}t})$.

Next, we will give the lower and upper bounds of the mean curvature.

**Lemma 3.2.** Along the flow (2.16), we have $H \leq (n-1)\kappa + O(e^{-\frac{2}{n-1}t})$, and $H \geq Ce^{\frac{1}{n-1}t}$ for some constant $C$ depending only on the initial hypersurface $\Sigma_{0}$.

**Proof.** From (2.24), (2.10) and (2.11), we have
\begin{equation}
\frac{\partial}{\partial t}H^2 = \frac{1}{H^2}\Delta H^2 - \frac{3}{2H^2}|\nabla H^2|^2 - 2|A|^2 - 2Ric_{g}(\nu, \nu) \\
\leq \frac{1}{H^2}\Delta H^2 - \frac{3}{2H^2}|\nabla H^2|^2 - \frac{2}{n-1}H^2 + 2\kappa^2(n-1) + O(e^{-nk\tau})
\end{equation}
Using maximum principle and the estimate of $\lambda$ in Lemma 3.1, we can deduce
\begin{equation}
\frac{d}{dt}H_{\max}^2 \leq -\frac{2}{n-1}H_{\max}^2 + 2(n-1)\kappa^2 + O(e^{-2t})
\end{equation}
which implies $H^2 \leq (n-1)^2\kappa^2 + O(e^{-\frac{2}{n-1}t})$.

Denoting $H^2 = (n-1)^2\kappa^2 + O(e^{-\frac{2}{n-1}t})$.

**Lemma 3.3.**
\begin{equation}
\frac{1}{G} = \frac{\nu}{\lambda H} = \frac{\nu^2}{(n-1)\lambda' - \delta^{ij}\varphi_{ij}},
\end{equation}
where
\begin{equation}
\nu^2 = 1 + \varphi_{i}\varphi^{i}, \quad \text{and} \quad \nu\lambda h_{ij}^l = \lambda' \delta_{ij}^{l} - \delta_{ik}^{l}\varphi_{ik}.
\end{equation}
Since $\lambda'$ can be seen as a function of $\varphi$, therefore $G$ can be considered as a function of $\varphi$, $\hat{\nabla}\varphi$ and $\hat{\nabla}^2\varphi$. Taking derivative (2.22) with $t$, we get

\[
(3.4) \quad \frac{\partial}{\partial t} \varphi_i = -\frac{1}{G^2} G^{ij} (\varphi, i)_j - \frac{1}{G^2} G^i (\varphi_i, h) - \frac{n-1}{G^2} \lambda'' \lambda \varphi_i,
\]

where $G^{ij} = \frac{\partial G}{\partial \varphi_j} = -\frac{1}{v} \delta^{ij} < 0$ and $G^i = \frac{\partial G}{\partial \varphi_i}$. One can check that $\lambda'' \geq 0$ and $\varphi_i > 0$ according to it’s expanding. By using maximum principle, we have

\[
(3.5) \quad \varphi_i \leq C.
\]

From (2.22), (3.5) and Lemma 3.1, we obtain

\[
H \geq C \frac{\nu}{\lambda} \geq Ce^{-\frac{\lambda'}{\lambda''}}.
\]

**Lemma 3.3.** We have $|\hat{\nabla}\varphi_k| = O\left(e^{-\frac{1}{n-1}}\right)$.

**Proof.** Defining $w = \frac{1}{2} |\hat{\nabla}\varphi_k|^2$, then

\[
\frac{\partial}{\partial t} w = \frac{\partial}{\partial t} (\varphi_k) \varphi^k = \hat{\nabla}_k \left( \frac{\nu}{\lambda} \right) \varphi^k \leq \frac{1}{\lambda^2 H^2} \left( \hat{\nabla}^i \varphi_k - \nu \hat{\nabla}^i w_k - 2(n-1) \lambda \lambda'' w - 2(2(n-2)) w - \hat{\nabla}^i \varphi_k \varphi^k \right)
\]

where $G$ is defined by (3.3).

Similar in [18], by using $-\epsilon \leq \kappa^2 \lambda^2 - 2m \alpha^2 - 2q \alpha^2 + q \alpha^2$, (3.6), Lemma 3.1 and Lemma 3.2, we have

\[
\frac{d}{dt} w_{max} \leq \frac{1}{H^2} \left( -2(n-1) \frac{\lambda''}{\lambda} w_{max} + 2(n-2)(\kappa^2 - 2m \alpha^2 + q \alpha^2) w_{max} \right)
\]

\[
= \frac{1}{H^2} \left( -2 \kappa^2 - 2(n-2) (n-1) \alpha'' + 2(n-2) q \alpha^2 \right) w_{max}
\]

\[
\leq \left( -\frac{2}{(n-1)^2} + Ce^{-\frac{2}{\lambda''}} + Ce^{-\frac{\lambda''}{\lambda''}} + Ce^{-\frac{\lambda''}{\lambda''}} \right) w_{max}
\]

\[
\leq \left( -\frac{2}{(n-1)^2} + Ce^{-\frac{2}{\lambda''}} \right) w_{max}
\]

which implies $w = O\left(e^{-\frac{1}{n-1}}\right)$.

**Remark 3.4.** Since $\nu = \sqrt{1 + |\hat{\nabla}\varphi|^2}$ remains positive and then $\langle \partial r, \nu \rangle = \frac{1}{\nu} > 0$, Lemma 3.3 implies the hypersurface solution of the inverse mean curvature flow (2.16) remains star-shaped.

**Lemma 3.5.** There exists a positive constant $C$ depending on the initial data, such that $H \geq C$.

**Proof.** From the evolution equation (2.25), Lemma 3.1 and Lemma 3.2, we have

\[
\frac{d}{dt} u_{max} \leq -\left( n-1 \right) \frac{\lambda'' u_{max}}{\lambda H^2}
\]

\[
\leq \left( -\frac{1}{n-1} + Ce^{-\frac{2}{\lambda''}} \right) u_{max},
\]
which implies that $\frac{u}{H} = u \leq Ce^{-\frac{1}{n+1}}$. The assertion follows from Lemma 3.1. □

**Remark 3.6.** The solution of the inverse mean curvature flow (2.16) remains strictly mean convex from Lemma 3.5.

With the help of Lemma 3.5, we can improve the $C^1$-estimate in Lemma 3.3.

**Lemma 3.7.** We have $|\hat{\nabla}\varphi|_{g} = O(e^{-\frac{1}{n+1}})$.

**Proof.** From (3.6), we have, for $w = \frac{1}{2}|\hat{\nabla}\varphi|_{g}^{2}$,

$$\frac{d}{dt} w = \frac{1}{\lambda^{2}H^{2}} \left( \hat{\sigma}^{ij} w_{ij} - \nu^{2} G^{i} w_{i} - 2(n-1) \lambda \nu w - 2 \epsilon(n-2) w - \hat{\sigma}^{ij} \varphi_{j} \varphi_{i} \right).$$

Since the term $(-\epsilon) \frac{2(n-2)}{\lambda^{2}H^{2}} \leq 0$ in (3.7) for $\epsilon = 1$ and 0, we can infer that

$$\frac{d}{dt} w_{\max} \leq \left( -\frac{2}{n-1} + C e^{-\frac{1}{n+1}} \right) w_{\max}.$$

For $\epsilon = -1$, we have

$$\frac{d}{dt} w_{\max} \leq \left( -\frac{2(n-1)\nu}{\lambda H^{2}} + \frac{2(n-2)}{\lambda^{2}H^{2}} \right) w_{\max} \leq \left( -\frac{2(n-1)(\kappa^{2} + m(n-2)\lambda - q^{2}(n-2)\lambda^{2} - 2n)}{H^{2}} + C e^{-\frac{1}{n+1}} \right) w_{\max} \leq \left( -\frac{2}{n-1} + C e^{-\frac{1}{n+1}} \right) w_{\max}.$$

where the second inequality we used Lemma (3.5). By (3.8) and (3.9), we have always the estimates $|\hat{\nabla}\varphi|_{g} = O(e^{-\frac{1}{n+1}})$ for $\epsilon = 0, \pm 1$. □

3.2. $C^2$-estimates. Now we can give the $C^2$-estimates for the flow (2.16).

**Lemma 3.8.** The second fundamental form $h^{j}_{i}$ is uniform bounded, i.e. there exists a positive constant $C$ depending on the initial condition of the flow (2.16), such that $|A| \leq C$. Consequently, $|\hat{\nabla}^{2}\varphi|_{g} \leq C$.

**Proof.** We define $\eta^{j}_{i} := H h^{j}_{i}$. By (2.23), (3.1), (10.10) and (2.11), we have

$$\frac{\partial}{\partial t} h^{j}_{i} = \frac{\Delta h^{j}_{i}}{H^{2}} - \frac{2 \nabla_{i} H \nabla_{j} H}{H^{3}} + \frac{|A|^{2}}{H^{2}} h^{j}_{i} - \frac{2 h^{k}_{j} h^{i}_{k}}{H} + (n-1) \kappa^{2} h^{i}_{j} + \frac{|A| + 1}{H} O \left( e^{-\frac{1}{n+1}} \right).$$

Combining (2.24) and (3.10) gives

$$\frac{\partial}{\partial t} \eta^{j}_{i} = \frac{\partial}{\partial t} H h^{j}_{i} + H \frac{\partial}{\partial t} h^{j}_{i}$$

$$= \frac{\Delta \eta^{j}_{i}}{H^{2}} + 2 \nabla_{i} H \nabla_{j} H \frac{H}{H^{2}} - \frac{2 \nabla_{i} H \nabla_{j} H}{H^{2}} - \frac{2 \eta^{k}_{j} \eta^{i}_{k}}{H^{2}} + (n-1) \kappa^{2} \frac{\eta^{i}_{j}}{H^{2}} + \frac{|\eta|}{H} O \left( e^{-\frac{1}{n+1}} \right).$$

Let $\mu_{\max}$ be the maximal eigenvalue of $\eta^{j}_{i}$. Since the trace of $(\eta^{j}_{i})$ is positive, we have $|\eta| \leq C \mu_{\max}$. Noticing that $\nabla_{i} H \nabla_{j} H$ is non-negative definite and the mean curvature $H$ has a uniformly lower bound, we obtain

$$\frac{d}{dt} \mu_{\max} \leq - C \mu_{\max}^{2} + C \mu_{\max} + C.$$
Therefore, we have $\mu_{\text{max}} \leq C$ for some uniform constant $C$. Since the mean curvature $H$ has a uniformly lower bound, we conclude that $|A|$ is bounded. \hfill \square

With the above estimates, by standard parabolic Krylov and Schauder theory we can get the higher order estimate, which allows us to obtain the long time existence for the inverse mean curvature flow (2.16).

**Lemma 3.9.** The inverse mean curvature flow (2.16) exists for all $t \in [0, \infty)$.

### 3.3. Asymptotic behaviors of the mean curvature and second fundamental form.

In this subsection, we will use the results above to improve estimates of the mean curvature and second fundamental form.

**Lemma 3.10.** We have $H = (n - 1)\kappa + O(te^{-\frac{2}{n-1}t})$ and $|h^i_j - \kappa\delta^i_j| \leq O(t^2 e^{-\frac{2}{n-1}t})$.

**Proof.** The upper bound for the mean curvature $H$ is given in Lemma 3.2. It is sufficient to bound $H$ from below, i.e.,

$$H \geq (n - 1)\kappa - O(te^{-\frac{2}{n-1}t}).$$

Define $\chi = \frac{\varphi}{H} = \lambda \varphi_t$. From Lemma 3.2, Lemma 3.3 and Lemma 3.5, $\chi$ is bounded from above and below. Using $\varphi_t = \frac{1}{\vartheta} = \frac{1}{\lambda}$ and (3.4), we obtain

$$\frac{\partial}{\partial t} \chi = \lambda \frac{\partial}{\partial t} \varphi_t + \lambda' \varphi_t^2$$

(3.13)

$$= \frac{\chi^2}{\vartheta^2 \lambda^2} \left( \hat{\sigma}^{ij} \chi_{ij} - \frac{2}{\lambda} \hat{\sigma}^{ij} \lambda \chi_j - \vartheta^2 G^k \chi_k \right)$$

$$+ \frac{\chi^2}{\lambda^2 \vartheta^2} \left( -\frac{2\chi}{\lambda} \hat{\sigma}^{ij} \lambda \chi_j - \frac{\chi}{\lambda} \hat{\sigma}^{ij} \lambda \chi_j + \frac{\vartheta^2 \chi}{2\lambda} G^k \lambda k \right) + \frac{\chi'}{\lambda} - \frac{n - 1}{\vartheta^2} \lambda'^{\prime} \chi^3$$

where $G^k = -\frac{2}{\vartheta^2} \left( G \varphi^k - \frac{1}{\vartheta^2} \varphi' \varphi^k + \frac{1}{\vartheta^2} \varphi' \varphi' \varphi' \varphi_{ij} \right)$.

By (3.1) and (3.3), one can check that

$$\hat{\sigma}^{ij} \lambda \chi_j \leq Ce^{\frac{2}{n-1}t},$$

$$-\hat{\sigma}^{ij} \lambda \chi_j \leq C(\vartheta^2 \lambda' (n - 1) \lambda') + Ce^{\frac{2}{n-1}t},$$

$$\frac{\vartheta^2}{2} G^k \lambda k \leq Ce^{\frac{n}{n-1}t}.$$

Putting these estimates together, and the boundness of $H$ and $\vartheta$, we obtain that

$$\frac{d}{dt} \chi_{\text{max}} \leq \frac{2\lambda'}{\lambda} \chi_{\text{max}}^2 - \frac{n - 1}{\vartheta^2} \lambda \lambda'' + \frac{\lambda^2}{\lambda} \chi_{\text{max}}^3 + Ce^{\frac{2}{n-1}t}$$

$$\leq 2\kappa \chi_{\text{max}}^2 - 2(n - 1)\kappa^2 \chi_{\text{max}}^3 + Ce^{\frac{2}{n-1}t}.$$
whenever \( \chi_{\text{max}} \geq \frac{1}{(n-1)\kappa} \). From this fact, we deduce that \( \frac{\mu}{\eta} = \chi \leq \frac{1}{(n-1)\kappa} + O(t e^{-\frac{n}{n+1}}} \). Therefore, we can conclude that \( H \geq (n-1)\kappa - O(te^{-\frac{n}{n+1}}} \).

Now we will give the proof of the second statement. As above, we denote by \( \mu_{\text{max}} \) the maximal eigenvalue of \( (\eta^j_i) = (HH^j_i) \). From \( H = (n-1)\kappa + O(te^{-\frac{n}{n+1}}} \) and (3.11), we have

\[
\frac{d}{dt} \mu_{\text{max}} \leq -\frac{2}{(n-1)^2} \kappa^2 \mu_{\text{max}}^2 + \frac{2}{n-1} \mu_{\text{max}} + O(te^{-\frac{n}{n+1}}} \)
\]

(3.14)

where the last inequality we used Cauchy-Schwarz inequality.

Hence

\( \mu_{\text{max}} \leq (n-1)\kappa^2 + O(t^2 e^{-\frac{n}{n+1}}} \).

Since \( H = (n-1)\kappa + O(te^{-\frac{n}{n+1}}} \) and \( \eta^j_i = HH^j_i \), we have the largest eigenvalue of \( h^j_i \) is less than \( \kappa + O(t^2 e^{-\frac{n}{n+1}}} \). Furthermore, we can infer the smallest eigenvalue is greater than \( \kappa - O(t^2 e^{-\frac{n}{n+1}}} \). □

From (2.14) and Lemma 3.10, we have the following Lemma.

**Lemma 3.11.** \( |\hat{\nabla}^2 \varphi| \leq O(t^2 e^{-\frac{n}{n+1}}} \).

### 4. Proof of Theorem 1.1

In this section, we will prove the Minkowski type inequality in Theorem 1.1. When \( \epsilon = 1 \), Theorem 1.1 was proved in [38]. When \( \epsilon = 0, -1 \), the proof follows from a similar argument. For the convenience of the readers, we include it here. Now we consider a family of star-shaped hypersurface \( \Sigma \) evolving by the inverse mean curvature flow (2.16). Following [38], we define \( \Omega(t) \) as follows

\[
\Omega(t) = |\Sigma|^{-\frac{n}{2(n-1)}} \left[ \int_{\Sigma_t} fHd\mu - n(n-1)\kappa^2 \int_{\Omega_t} fdv + (n-1)\vartheta_{n-1}\epsilon \left( s_0^{n-2} - \left( \frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{-\frac{n}{n+1}} \right) \right]
\]

(4.1)

**Lemma 4.1.** Under the inverse mean curvature flow (2.16), we have

\[
\lim_{t \to \infty} \Omega(t) \geq 0.
\]

**Proof.** From (2.15), (3.3) and Lemma 3.11, we obtain

\[
H - (n-1)\kappa = \frac{(n-1)\lambda'}{\lambda} - \frac{1}{\lambda} \hat{\nabla}^{ij} \varphi_{ij} - (n-1)\kappa
\]

(4.2)

\[
= \frac{(n-1)\epsilon}{2\kappa} \lambda^{-2} - \frac{n-1}{2} \kappa |\hat{\nabla} \varphi|^2 - \lambda^{-1} \hat{\Delta} \varphi + O(e^{-\frac{n}{n+1}})
\]

and

\[
d\mu = \lambda^{n-1} \left( 1 + \frac{1}{2} |\hat{\nabla} \varphi|^2 + O(e^{-\frac{n}{n+1}}) \right) d\mu_N,
\]
we have
\begin{equation}
(4.3) \quad \int_{\Sigma} f (H - (n-1)\kappa) \, d\mu = \int_{\Sigma} \left( \kappa\lambda + O(e^{-\frac{\epsilon}{n-1}}) \right) \left( \frac{(n-1)\epsilon}{2\kappa} \lambda^2 - \frac{n-1}{2} \kappa |\nabla \varphi|^2 - \lambda^{-1} \hat{\Delta} \varphi + O(e^{-\frac{\epsilon}{n-1}}) \right) \, d\mu_N
\end{equation}
\begin{align*}
&\times \lambda^{n-1} \left( 1 + \frac{1}{2} |\nabla \varphi|^2 + O(e^{-\frac{\epsilon}{n-1}}) \right) \, d\mu_N \\
&= \frac{(n-1)\epsilon}{2} \int_{\Sigma} \lambda^{n-2} \lambda^{-1} \, d\mu_N + \frac{n-1}{2} \int_{\Sigma} \lambda^{n-4} |\nabla \lambda|^2 \, d\mu_N + O(e^{-\frac{\epsilon}{n-1}}),
\end{align*}
where the integration by parts is used in the last equality.

On the other hand, from asymptotic behaviors of $f = \lambda'$ and $\lambda''$, we have
\begin{equation}
(4.4) \quad (n-1) \int_{\Sigma} \left( \kappa f - \langle \nabla f, \nu \rangle \right) \, d\mu \geq \int_{\Sigma} (\kappa \lambda - (n-1) |\nabla f|) \, d\mu
\end{equation}
\begin{align*}
&= \int_{\Sigma} (\kappa \lambda' - \lambda'') \, d\mu \\
&= \int_{\Sigma} \left( \kappa^2 \lambda + \frac{\epsilon}{2} \lambda^{-1} - \kappa^2 \lambda + O(\lambda^{-2}) \right) \, d\mu \\
&= \frac{(n-1)\epsilon}{2} \int_{\Sigma} \lambda^{n-2} \lambda^{-1} \, d\mu_N + O(e^{-\frac{\epsilon}{n-1}})
\end{align*}
and
\begin{align}
\int_{\Sigma} (n-1) \langle \nabla f, \nu \rangle \, d\mu - n(n-1)\kappa^2 \int_{\Omega} f \, d\nu = (n-1) \left( \int_{\Sigma} \lambda^{n-1} \lambda' \, d\mu_N - \kappa^2 \int_{\Sigma} \lambda^{-1} d\mu_N \right) \\
= (n-1) \int_{\Sigma} (\lambda^{n-1} \lambda' - \kappa^2 \lambda) \, d\mu_N + (n-1) \kappa^2 \theta_{n-1} s_0^{e_1} + O(e^{-\frac{\epsilon}{n-1}}) + C.
\end{align}
\begin{equation}
(4.5)
\end{equation}
Therefore, (4.3)+ (4.4)+ (4.5) implies
\begin{equation}
\begin{align*}
\int_{\Sigma} f H d\mu - n(n-1)\kappa^2 \int_{\Omega} f \, d\nu &\geq (n-1) \epsilon \int_{\Sigma} \lambda^{n-2} \lambda^{-1} \, d\mu_N + \frac{n-1}{2} \int_{\Sigma} \lambda^{n-4} |\nabla \lambda|^2 \, d\mu_N + O(e^{-\frac{\epsilon}{n-1}}) \\
&\geq (n-1) \epsilon \theta_{n-1}^{\frac{\kappa^2}{n-1}} \left( \int_{\Sigma} \lambda^{n-1} \, d\mu_N \right)^{\frac{n-2}{n-1}} + O(e^{-\frac{\epsilon}{n-1}}),
\end{align*}
\end{equation}
where the last inequality above is from a non-sharp version of Beckner-Sobolev type inequality in [2] when $\epsilon = 1$ (see Proposition 21 in [38]); the inequality is trivial when $\epsilon = 0$; similarly in [18], when $\epsilon = -1$, we use the Hölder inequality
\begin{equation}
\int_{\Sigma} \lambda^{n-2} \leq \theta_{n-1}^{\frac{\kappa^2}{n-1}} \left( \int_{\Sigma} \lambda^{n-1} \right)^{\frac{n-2}{n-1}}.
\end{equation}
Noting that
\begin{align*}
|\Sigma| = \int_{\Sigma} \lambda^{n-1} (1 + O(e^{-\frac{\epsilon}{n-1}})) \, d\mu_N = \int_{\Sigma} \lambda^{n-1} \, d\mu_N + O(e^{-\frac{\epsilon}{n-1}}),
\end{align*}
we get
\[
\liminf_{t \to \infty} \frac{\int_{\Sigma_t} fH d\mu - n(n-1)\kappa^2 \int_{\Omega_1} f d\mu}{|\Sigma_t|^{\frac{n-2}{n}}} \geq (n-1)\epsilon \theta_{n-1}^{\frac{1}{n-1}}
\]
and
\[
(n-1) \lim_{t \to \infty} \frac{|\Sigma_t|^{\frac{n-2}{n}}}{\theta_{n-1}^{\frac{1}{n-1}}} \left( \partial_{n-1} - \left( \frac{|\Sigma_t|}{\theta_{n-1}} \right)^{\frac{n-2}{n}} \right) + \kappa^2 \partial_{n-1} \left( s_0^{n-2} - \left( \frac{|\Sigma_t|}{\theta_{n-1}} \right)^{\frac{n-2}{n}} \right) = -(n-1)\epsilon \theta_{n-1}^{\frac{1}{n-1}},
\]
which completes the proof of Lemma 4.1. \(\square\)

Now we prove that \(Q(t)\) has the following monotonicity along the inverse mean curvature flow (2.16).

**Lemma 4.2.** The quantity \(Q(t)\) is monotone nonincreasing along the flow (2.16).

**Proof.** Using the evolution equations (2.21) and (2.24), the identity \(\Delta f = \tilde{\Delta} f - \tilde{\nabla}^2 f(\nu, \nu) - H \langle \tilde{\nabla} f, \nu \rangle\)
and the sub-static equation (2.4), we obtain
\[
\frac{d}{dt} \int_{\Sigma_t} fH d\mu = -\int_{\Sigma_t} \frac{1}{H} \Delta f d\mu - \int_{\Sigma_t} \frac{f}{H} \left( |\nabla|^2 + \text{Ric}_g(\nu, \nu) \right) d\mu + \int_{\Sigma_t} \left( \langle \tilde{\nabla} f, \nu \rangle + fH \right) d\mu
\]
\[
= -\int_{\Sigma_t} \frac{1}{H} \left( \tilde{\Delta} f - \tilde{\nabla}^2 f(\nu, \nu) + f\text{Ric}_g(\nu, \nu) \right) d\mu - \int_{\Sigma_t} \frac{f}{H} |\nabla|^2 d\mu
\]
\[
+ \int_{\Sigma_t} \left( 2\langle \tilde{\nabla} f, \nu \rangle + fH \right) d\mu
\]
\[
\leq \int_{\Sigma_t} \left( 2\langle \tilde{\nabla} f, \nu \rangle + \frac{n-1}{n} fH \right) d\mu.
\]

The hypersurfaces \(\Sigma_t\) can be seen as the graph \(\Sigma_t = \{(\omega, s_t(\omega))|\omega \in N_\epsilon\}\) of the space form \(N_\epsilon\). By the diverging theorem, we obtain
\[
\int_{\Sigma_t} \langle \tilde{\nabla} f, \nu \rangle d\mu = \kappa^2 \int_{N_\epsilon} s_0^n d\mu_{N_\epsilon} + m(n-2)\partial_{n-1} - (n-2)\kappa^2 \int_{N_\epsilon} s_0^{2-n} d\mu_{N_\epsilon}.
\]
Since
\[
n \int_{\Omega_\epsilon} f d\nu = n \int_{N_\epsilon} \int_{s_0} s_{n-1}^{\omega(\omega)} f \cdot s_{n-1}^{\omega} d\mu_{N_\epsilon} = \int_{N_\epsilon} s_0^n d\mu_{N_\epsilon} - s_0^n \partial_{n-1},
\]
we have
\[
\int_{\Sigma_t} \langle \tilde{\nabla} f, \nu \rangle d\mu = nk^2 \int_{\Omega_\epsilon} f d\nu + \kappa^2 s_0^n \partial_{n-1} + m(n-2)\partial_{n-1} - (n-2)\kappa^2 \int_{N_\epsilon} s_0^{2-n} d\mu_{N_\epsilon}.
\]
Therefore,
\[
\frac{d}{dt} \int_{\Sigma_t} fH d\mu \leq 2nk^2 \int_{\Omega_\epsilon} f d\nu + 2\kappa^2 s_0^n \partial_{n-1} + 2m(n-2)\partial_{n-1} - 2(n-2)\kappa^2 \int_{N_\epsilon} s_0^{2-n} d\mu_{N_\epsilon}
\]
\[
+ \frac{n-2}{n-1} \int_{\Sigma_t} fH d\mu.
\]
From a Heintze-Karcher type inequality proved by Brendle [6], we have
\begin{equation}
\frac{d}{dt} \int_{\Omega_t} f dv = \int_{\Sigma_t} f \frac{d\mu}{H} \geq \frac{n}{n-1} \int_{\Omega_t} f dv + \frac{1}{n-1} s_0^n \vartheta_{n-1}.
\end{equation}

Hence,
\begin{align*}
\frac{d}{dt} \left( \int_{\Sigma_t} f HD\mu - n(n-1)\kappa^2 \int_{\Omega_t} f dv \right) \\
\leq \frac{n-2}{n-1} \left( \int_{\Sigma_t} f HD\mu - n(n-1)\kappa^2 \int_{\Omega_t} f dv \right) \\
+ 2m(n-2)\vartheta_{n-1} - 2(n-2)q^2 \int_{N_t} s_{t}^{2-n} d\mu_{N_t}
\end{align*}
\begin{equation}
\leq \frac{n-2}{n-1} \left( \int_{\Sigma_t} f HD\mu - n(n-1)\kappa^2 \int_{\Omega_t} f dv \right) \\
+ (n-1)\vartheta_{n-1} \left( \epsilon s_0^{n-2} + q^2 s_0^{2-n} - 2(n-1)q^2 \int_{N_t} s_{t}^{2-n} d\mu_{N_t} \right).
\end{equation}

By the H\'lder inequality, we have
\begin{equation}
\vartheta_{n-1}^{\frac{2m-3}{n-1}} \left( \int_{N_t} s_{t}^{n-1} d\mu_{N_t} \right)^{\frac{n-2}{n-1}} \leq \int_{N_t} s_{t}^{2-n} d\mu_{N_t}.
\end{equation}

From $|\Sigma_t| \geq \int_{N_t} s_{t}^{n-1} d\mu_{N_t}$, we get
\begin{equation}
\vartheta_{n-1} \left( \frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{-\frac{n-2}{n-1}} \leq \int_{N_t} s_{t}^{-(n-2)} d\mu_{N_t}.
\end{equation}

From (2.21), (4.12) and (4.13), we have
\begin{align*}
\frac{d}{dt} \left( \int_{\Sigma_t} f HD\mu - n(n-1)\kappa^2 \int_{\Omega_t} f dv + (n-1)\epsilon \vartheta_{n-1} \left( s_0^{n-2} - \left( \frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n-2}{n-1}} \right) \right) \\
\leq \frac{n-2}{n-1} \left( \int_{\Sigma_t} f HD\mu - n(n-1)\kappa^2 \int_{\Omega_t} f dv + (n-1)\epsilon \vartheta_{n-1} \left( s_0^{n-2} - \left( \frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n-2}{n-1}} \right) \right) \\
+ (n-1)q^2 \vartheta_{n-1} \left( s_0^{n-2} - \left( \frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n-2}{n-1}} \right),
\end{align*}
which completes the proof of Lemma 4.2.

\textbf{Proof of Theorem 1.1.} By use of Lemma 4.1 and Lemma 4.2, we have
\[ Q(0) \geq \lim_{t \to \infty} \inf Q(t) \geq 0, \]
which gives the proof of Theorem 1.1.
5. Proof of Theorem 1.3

In this section, following the arguments in [13, 18], we will use Theorem 1.1 to give the proof of Theorem 1.3.

We define

\begin{equation}
\mathcal{J}(\Sigma_t) = nk^2 \int_{\Omega_t} f dv,
\end{equation}

\begin{equation}
\mathcal{K}(\Sigma_t) = \kappa^2 \partial_{n-1} \left( \left( \frac{\| \Sigma_t \|}{\partial_{n-1}} \right)^{\frac{2-n}{2}} - \left( \frac{\| \partial_t \Sigma_t \|}{\partial_{n-1}} \right)^{\frac{2-n}{2}} \right),
\end{equation}

\begin{equation}
\mathcal{L}(\Sigma_t) = |\Sigma_t|^{\frac{2-n}{2}} \left( \int_{\Sigma_t} f H \mu - (n-1) \mathcal{K}(\Sigma_t) + (n-1) \epsilon \partial_{n-1} \Theta_{n-2}^{n-2} \right)
\end{equation}

By using Lemma 2.4 and a Heintze-Karcher inequality (4.11), we can check that

\begin{equation}
\frac{d}{dt} \left( \frac{\mathcal{J}(\Sigma_t) - \mathcal{K}(\Sigma_t)}{|\Sigma_t|^{\frac{2-n}{2}}} \right) \geq 0.
\end{equation}

By using Lemma 2.4, (4.7), (4.8), (4.14) and $\epsilon + \kappa^2 s_0^2 - 2ms_0^{2-n} + q^2 s_0^{4-2n} = 0$, we have

\begin{align*}
\frac{d}{dt} \left( \int_{\Omega_t} f H \mu - (n-1) \mathcal{K}(\Sigma_t) + (n-1) \partial_{n-1} \Theta_{n-2}^{n-2} \right) \\
+ (n-1)q^2 \partial_{n-1} \left( s_0^{2-n} - \left( \frac{\| \Sigma_t \|}{\partial_{n-1}} \right)^{\frac{2-n}{2}} \right) \\
\leq 2\mathcal{J}(\Sigma_t) - n\mathcal{K}(\Sigma_t) - (n-2)\kappa^2 s_0^{2-n} \partial_{n-1} + 2m(n-2) \partial_{n-1} - 2(n-2)q^2 \int_{\Omega_t} s_0^{2-n} d\mu_{\Sigma_t} \\
+ \frac{n-2}{n-1} \int_{\Sigma_t} f H \mu + (n-2)q^2 \partial_{n-1} \left( \frac{\| \Sigma_t \|}{\partial_{n-1}} \right)^{\frac{2-n}{2}} \\
\leq 2(\mathcal{J}(\Sigma_t) - \mathcal{K}(\Sigma_t)) + \frac{n-2}{n-1} \int_{\Sigma_t} f H \mu - (n-1) \mathcal{K}(\Sigma_t) + (n-1) \partial_{n-1} \Theta_{n-2}^{n-2} \\
+ (n-1)q^2 \partial_{n-1} \left( s_0^{2-n} - \left( \frac{\| \Sigma_t \|}{\partial_{n-1}} \right)^{\frac{2-n}{2}} \right) \\
- (n-2) \partial_{n-1} \Theta_{n-2}^{n-2} - (n-2)q^2 \partial_{n-1} s_0^{2-n} - (n-2)\kappa^2 s_0^{2-n} \partial_{n-1} + 2m(n-2) \partial_{n-1} \\
= 2(\mathcal{J}(\Sigma_t) - \mathcal{K}(\Sigma_t)) + \frac{n-2}{n-1} |\Sigma_t|^{\frac{2-n}{2}} \mathcal{L}(\Sigma_t),
\end{align*}

which means that

\begin{equation}
\frac{d}{dt} \mathcal{L}(\Sigma_t) \leq 2 \frac{\mathcal{J}(\Sigma_t) - \mathcal{K}(\Sigma_t)}{|\Sigma_t|^{\frac{2-n}{2}}}
\end{equation}
It is sufficient to prove Theorem 1.3 when the initial surface $\Sigma_0$ satisfies $\mathcal{J}(\Sigma_0) < \mathcal{K}(\Sigma_0)$, otherwise the assertion follows directly from Theorem 1.1. In order to prove Theorem 1.3, we divide the proof into two cases.

**Case 1:** There exists some $t_0 \in (0, \infty)$ such that $\mathcal{J}(\Sigma_{t_0}) = \mathcal{K}(\Sigma_{t_0})$ and $\mathcal{J}(\Sigma_t) - \mathcal{K}(\Sigma_t) \leq 0$ for $t \in [0, t_0]$.

From (5.4) and (1.4), we obtain

$$
\mathcal{L}(\Sigma_0) \geq \mathcal{L}(\Sigma_{t_0})
$$

$$= |\Sigma_{t_0}|^{-\frac{1}{n-1}} \left( \int_{\Sigma_{t_0}} f Hd\mu - (n-1)\mathcal{J}(\Sigma_{t_0}) + (n-1)\epsilon \theta_{n-1} \right)
$$

(5.5)

$$+(n-1)q^2 \theta_{n-1} \left( \frac{2}{n} - \left( \frac{|\Sigma_{t_0}|}{\theta_{n-1}} \right)^{-\frac{1}{n-1}} \right)
$$

$$\geq (n-1)\epsilon \theta_{n-1}^{\frac{1}{n-1}}.
$$

**Case 2:** For all $t \in [0, \infty)$, we have

$$\mathcal{J}(\Sigma_t) - \mathcal{K}(\Sigma_t) < 0.
$$

Since $\mathcal{L}(t)$ is monotone non-increasing in $t \in [0, \infty)$ from (5.4), we obtain

$$
\mathcal{L}(\Sigma_0) \geq \mathcal{L}(\Sigma_\infty) = \lim_{t \to \infty} \frac{\int_{\Sigma_t} f Hd\mu - (n-1)\mathcal{K}(\Sigma_t)}{|\Sigma_t|^{-\frac{1}{n-1}}}.
$$

By (4.3), we have

$$
\int_{\Sigma_t} f Hd\mu - (n-1)\mathcal{K}(\Sigma_t) = \int_{\Sigma_t} f(H - (n-1)\kappa)d\mu + (n-1) \left( \int_{\Sigma_t} \kappa f d\mu - \mathcal{K}(\Sigma_t) \right)
$$

(5.7)

$$= \frac{(n-1)\epsilon}{2} \int_{N_t} \lambda^{n-2} d\mu_{N_t} + \frac{n-1}{2} \int_{N_t} \lambda^{n-4} |\nabla \lambda|^2 d\mu_{N_t}
$$

$$+ (n-1) \left( \int_{\Sigma_t} \kappa f d\mu - \mathcal{K}(\Sigma_t) \right) + O\left( e^{\frac{n-1}{n+1}} \right).
$$

By the H"older inequality, we have

$$
\left( \frac{|\Sigma_t|}{\theta_{n-1}} \right)^{-\frac{1}{n-1}} \leq \frac{\int_{N_t} \lambda^n v^{\frac{n}{n+1}} d\mu_{N_t}}{\theta_{n-1}}.
$$

(5.8)

By (3.1), Lemma 3.7 and (5.8), we obtain

$$
(n-1) \left( \int_{N_t} \kappa f d\mu - \mathcal{K}(\Sigma_t) \right) = (n-1) \left( \int_{\Sigma_t} \kappa f \lambda^{n-1} v d\mu_{N_t} - \kappa^2 \theta_{n-1} \left( \frac{|\Sigma_t|}{\theta_{n-1}} \right)^{-\frac{1}{n-1}} \right)
$$

(5.9)

$$\geq (n-1) \int_{N_t} \left( \kappa f \lambda^{n-1} v - \kappa^2 \lambda v^{\frac{n}{n+1}} \right) d\mu_{N_t}
$$

$$= (n-1) \int_{N_t} \left( \frac{\epsilon}{2} \lambda^{n-2} - \frac{1}{2(n-1)} \lambda^{n-4} |\nabla \lambda|^2 \right) d\mu_{N_t} + O\left( e^{\frac{n-1}{n+1}} \right).
This implies
\[ \int_{\Sigma_t} f H d\mu - (n - 1) \mathcal{K}(\Sigma_t) \geq (n - 1) \varepsilon \int_N \lambda^{n-1} d\mu_N + \frac{n - 2}{2} \int_N \lambda^{n-1} |\nabla \lambda|^2 d\mu_N + O\left(\varepsilon^{\frac{n-3}{4}}\right) \]
(5.10)
\[ \geq (n - 1) \varepsilon \theta^{\frac{1}{n-1}} \left( \int_N \lambda^{n-1} d\mu_N \right)^{\frac{n-2}{2(n-1)}} + O(\varepsilon^{\frac{n-3}{4}}), \]
where the Beckner-Sobolev type inequality [2, 7] is used in second inequality when \( \varepsilon = 1 \); the inequality is trivial when \( \varepsilon = 0 \); it comes from the Hölder inequality when \( \varepsilon = -1 \).

Finally, by (5.6), (5.7), (5.9) and (5.10), we deduce that
\[ \mathcal{L}(\Sigma_0) \geq (n - 1) \varepsilon \theta^{\frac{1}{n-1}} \lim_{\varepsilon \to 0} \frac{\left( \int_N \lambda^{n-1} d\mu_N \right)^{\frac{n-2}{2(n-1)}}}{|\Sigma_t|^{\frac{n}{2(n-1)}}}, \]
which completes the proof of Theorem 1.3.

6. Proofs of Theorem 1.5 and Theorem 1.6

We will compute the mass of the ALH graph over Reissner-Nordström-AdS manifold. With the Alexandrov-Fenchel inequality in Theorem 1.3, we now can give a Penrose inequality of ALH mass.

Lemma 6.1. The Reissner-Nordström-AdS manifold \((P, \bar{g})\) is a ALH manifold with ALH mass

\[ m(P, \bar{g}) = m. \]

Proof. Let \( \{x^a\}_{a=1}^{n-1} \) be a local coordinate system on \( N_\varepsilon \) and \( x^0 = s \). According to (2.26), one can check that the Christoffel symbols with respect to the metric \( b_\varepsilon \) are

\[ \left\{ \begin{array}{l}
\Gamma^0_{00} = -\frac{s^3}{s^2 + \varepsilon}, \\
\Gamma^0_{ab} = -sV^2\bar{g}_{ab}, \quad a, b > 0, \\
\Gamma^b_{a0} = s^{-1}\bar{g}_{ab}, \quad a, b > 0, \\
\Gamma^0_{a0} = \Gamma^a_{00} = 0, \quad a > 0.
\end{array} \right. \]

(6.2)

Since \( d\mu = s^{n-1}d\mu_N \), \( e = \left( \frac{ds^2}{s^2} - \frac{dV^2}{V^2} \right) \), one can compute directly

\[ m(M, g) = \frac{1}{2(n - 1)\theta^{n-1}} \lim_{\varepsilon \to 0} \int_{N_\varepsilon} \left( V_\varepsilon (d div^{b_\varepsilon} e - d tr^{b_\varepsilon} e) + (tr^{b_\varepsilon} e) dV_\varepsilon - e(\nabla^{b_\varepsilon} V_\varepsilon \cdot \cdot) \right) d\mu_{b_\varepsilon} \]

(6.3)
\[ = \frac{1}{2(n - 1)\theta^{n-1}} \int_{N_\varepsilon} \left( 2m(n - 1) + \frac{2m}{k} - \frac{2m}{k} \right) d\mu_N, \]
and

\[ \| (\Phi^{-1}) \ast \bar{g} - b_\varepsilon \|_{b_\varepsilon} + \| \nabla^{b_\varepsilon} ((\Phi^{-1}) \ast \bar{g}) \|_{b_\varepsilon} = O(s^{-n}) \]

(6.4)

By using the static equation (2.4), we have

\[ R_{\bar{g}} = -n(n - 1)\kappa^2 + q^2(n - 2)(n - 1)s^{2-2n} \]
which implies
\[ \int_P V_e |R_g| + n(n-1)\kappa^2 d\mu_M = q^2(n-2)(n-1) \int_{\mathbb{S}_0} \int_{N_e} V_e s^{2-n} V_e s^{n-1} d\mu_{N_e} < \infty. \]

Hence \((P, \bar{g})\) is an ALH manifold. \( \square \)

**Lemma 6.2.** We have
\begin{equation}
(6.5) \quad m(M, g) = m(P, \bar{g}) + \frac{1}{2(n-1)\partial_{n-1}} \lim_{s \to \infty} \int_{N_s} \left( f(div^\bar{g} \hat{e} - d tr^\bar{g} \hat{e}) + (tr^\bar{g} \hat{e}) d\nu - \hat{e}(\bar{\nabla} f, \cdot) \right) \hat{v} d\mu
\end{equation}

where \(\hat{e} = (\Phi^{-1}) \ast g - \bar{g}, \hat{v}\) is the outward unit normal of \(N_s\) induced by \(\bar{g}\) and \(d\mu\) is the area element induced by \(\bar{g}\).

**Proof.** Since \(e = \hat{e} + g - \bar{g}\), we have
\[
m(M, g) = m(P, \bar{g}) + \frac{1}{2(n-1)\partial_{n-1}} \lim_{s \to \infty} \int_{N_s} \left( V_e (div^b \hat{e} - d tr^b \hat{e}) + (tr^b \hat{e}) d\nu - \hat{e}(\nabla^b V_e, \cdot) \right) v d\mu_{b_s}
\]
\[
= m + \frac{1}{2(n-1)\partial_{n-1}} \lim_{s \to \infty} \int_{N_s} \left( V_e (div^b \hat{e} - d tr^b \hat{e}) + (tr^b \hat{e}) d\nu - \hat{e}(\nabla^b V_e, \cdot) \right) v d\mu_{b_s}.
\]

Recall that the Reissner-Nordström-AdS manifold is ALH, we have (6.4). Then one can replace \(V_e, b_s, \nu, d\mu_{b_s}\) by \(f, \bar{g}, \hat{v}, d\mu\), respectively, will not change the value of the limit, i.e.
\[
m(M, g) = m + \frac{1}{2(n-1)\partial_{n-1}} \lim_{s \to \infty} \int_{N_s} \left( f(div^\bar{g} \hat{e} - d tr^\bar{g} \hat{e}) + (tr^\bar{g} \hat{e}) d\nu - \hat{e}(\bar{\nabla} f, \cdot) \right) \hat{v} d\mu.
\]
\( \square \)

**Proof of Theorem 1.5.** The proof of this theorem follows in the spirit of the one in [12, 18]. Denote the outward unit normal of \((M, g) \subset (Q, \bar{g})\) by \(\xi\) and let \(B = -\bar{\nabla} \xi\) be the second fundamental form of \(M\). The Newton tensor is inductively given by
\[ T_r = S_r I - BT_{r-1}, \quad T_0 = I, \]
where \(S_r\) is the \(r\)-th elementary symmetric polynomial of principle curvature of \(M\) with respect to the unit normal \(\xi\).

Let \(\{x_0, \ldots, x_{n-1}\}\) be a local coordinate system of \(P\). The induced metric \(g_{ij}\) and the unit normal vector field \(\xi\) of \(M\), respectively, are given by
\begin{equation}
(6.6) \quad g_{ij} = f^2 u_i u_j + \bar{g}_{ij},
\end{equation}
\begin{equation}
(6.7) \quad \xi = \frac{1}{\sqrt{1 + f^2 |\bar{\nabla} u|^2}} \left( f^{-1} \frac{\partial}{\partial t} - f \bar{\nabla} u \right),
\end{equation}
where \(\bar{\nabla}\) is the covariant derivative with respect to \(\bar{g}\), \(u_k = \bar{\nabla}_k u, u^l = \bar{g}^{ik} u_k\) and \(|\bar{\nabla} u|^2 = u_i u_j \bar{g}^{ij}\). The components of second fundamental form \(B_{ij}\) is given by
\begin{equation}
(6.8) \quad B_{ij} = \frac{f}{\sqrt{1 + f^2 |\bar{\nabla} u|^2}} \left( \bar{\nabla}_i \bar{\nabla}_j u + \frac{u_i f_j + u_j f_i}{f} + f \left( \bar{\nabla} u, \bar{\nabla} f \right) \bar{g}_{ij} u_i u_j \right).
\end{equation}
From (6.7), we deduce that the tangential part of $\frac{\partial}{\partial t}$ is

$$
\left(\frac{\partial}{\partial t}\right)^T = \frac{f^2\nabla u}{1 + f^2|\nabla u|^2} + \frac{f^2|\nabla u|^2}{1 + f^2|\nabla u|^2} \partial_t
$$

(6.9)

From (6.6) and (6.8), we get

$$
\left[ f(div^\delta \hat{e} - d \nabla^\delta \hat{e}) + (tr^\delta \hat{e}) df - \hat{e}(\nabla f, \cdot) \right]_{\hat{\nabla}} + f^2\nabla u_i \nabla_k u + f^3 \Delta u \nabla_i u - f^2|\nabla u|^2 f_i
$$

(6.10)

Note that since $M$ is ALH, by Definition 2.6, we have for some $\tau > \frac{4}{7}$

$$
f^2|\nabla u|^2 = O(s^{-2\tau}), \quad g_{ij} = \tilde{g}_{ij} + O(s^{-2\tau}), \quad g^{ij} = \tilde{g}^{ij} + O(s^{-2\tau}).
$$

Since $d\mu = s^{n-1} d\mu_{\mathcal{N}_t}$, we have

$$
d\mu_{\mathcal{N}_t} = \sqrt{det(s^2 \tilde{g}_{ab} + f^2 u_a u_b)} dx^1 \cdots dx^{n-1} = d\mu(1 + O(s^{-2\tau})).
$$

Because the tangent space of graph $|_{\mathcal{N}_t} u$ is spanned by $e_a := u_a \partial_t + \partial_a, a = 1, \ldots, n-1$. The normal vector of graph $|_{\mathcal{N}_t} u$ can be set as $\tilde{e}_0 = e_0 - t_a e_a$, where $e_0 = u_a \partial_t + \partial_s$. Then from $g(e_a, e_0) = 0$, we have $t_a = O(s^{-2\tau})$. Note that $|e_a| = O(s), |e_0| = O(s^{-1}), f \partial_s = \hat{\nu}$, so we can see

$$
\nu_{\mathcal{N}_t} = \frac{e_0}{|e_0|} = \frac{O(s^{-2\tau}) \partial_t + \partial_s + O(s^{-2\tau}) \Sigma \partial_a}{\sqrt{O(s^{-2\tau}) + \frac{1}{2} + O(s^{-4\tau})}} = \hat{\nu} + O(s^{-2\tau}).
$$

Hence from (6.10) and above, when $s \to \infty$,

$$
\lim_{s \to \infty} \int_{\mathcal{N}_t} \left( f(div^\delta \hat{e} - d \nabla^\delta \hat{e}) + (tr^\delta \hat{e}) df - \hat{e}(\nabla f, \cdot) \right) \hat{\nu} d\mu
$$

(6.11)

$$
= \lim_{s \to \infty} \int_{\mathcal{N}_t} \left( B_m \delta^m_{ij} - B_i \right) \frac{f^2 \nabla i}{1 + f^2|\nabla u|^2} (\nu_{\mathcal{N}_t}) d\mu_g
$$

$$
= \lim_{s \to \infty} \int \left( T_1 \left( \frac{\partial}{\partial t} \right)^T, \nu_{\mathcal{N}_t} \right) d\mu_g.
$$

By integration by parts, we get (see [19])

$$
\lim_{s \to \infty} \int_{\mathcal{N}_t} \left( T_1 \left( \frac{\partial}{\partial t} \right)^T, \nu_{\mathcal{N}_t} \right) d\mu_g = \int_M \text{div}_g \left( T_1 \left( \frac{\partial}{\partial t} \right)^T \right) d\mu_M + \int_{\Sigma} \left( T_1 \left( \frac{\partial}{\partial t} \right)^T, \nu_{\Sigma} \right) d\mu_{\Sigma}
$$

where $\nu_{\Sigma}, d\mu_{\Sigma}$ are the unit outer normal and area element of graph $|_{\mathcal{N}_t} u$ induced by metric $g$, respectively.

Using the fact that $\frac{\partial}{\partial t}$ is a Killing vector field, one can check (refer [1] for the proof)

$$
\text{div}_g \left( T_1 \left( \frac{\partial}{\partial t} \right)^T \right) = \left( \text{div}_g T_1, \left( \frac{\partial}{\partial t} \right)^T \right)_g + 2 S_2 \left( \frac{\partial}{\partial t}, \xi \right).
$$

(6.13)
A direct computation (see [1]) gives

\[(6.14) \quad \text{div}_g(T_r) = -B\text{div}_g(T_{r-1}) - \sum_i \left( R_g(\xi, T_{r-1}(\partial_i))\partial_i \right)^T \]

where \{\partial_i\} is a tangent frame of \(M\), \(R_g(\cdot, \cdot)\) is the Riemannian curvature tensor of \((Q, \tilde{g})\). In particular, for \(r = 1\) we have

\[(6.15) \quad \left( \text{div}_g T_1, \left( \frac{\partial}{\partial t} \right)^T \right)_g = \text{Ric}_\tilde{g}(\xi, \left( \frac{\partial}{\partial t} \right)^T). \]

From the Gauss equation, we have

\[(6.16) \quad 2S_2 = R_g - R_{\tilde{g}} + 2\text{Ric}_\tilde{g}(\xi, \xi) \]

where \(R_g = -n(n+1)\kappa^2 - (n-2)(n-3)q^2\lambda^{2-2n}\) is the scalar curvature of the manifold \((Q, \tilde{g})\). After a change of variable as in (2.1), the metric \(\tilde{g}\) has the following form

\[ \tilde{g} = f^2 dt^2 + dr^2 + \lambda^2 \tilde{g}, \quad f = \lambda'. \]

Now we have

\[(6.17) \quad \text{Ric}_\tilde{g}(\xi, \xi) = -nk^2 + \frac{(n-2)q^2\lambda^{2-2n}}{1 + f^2|\tilde{\nabla}u|^2} \left( -(n-2)(1 + f^2 u_r^2) + f^2(|\tilde{\nabla}u|^2 - u_r^2) \right), \]

and

\[(6.18) \quad \text{Ric}_\tilde{g}(\xi, \left( \frac{\partial}{\partial t} \right)^T) = -\left( \frac{\partial}{\partial t}, \xi \right) \left( \text{Ric}_\tilde{g}(\xi, \xi) + nk^2 + (n-2)q^2\lambda^{2-2n} \right). \]

Combining (6.13), (6.15), (6.17) and (6.18), we obtain

\[(6.19) \quad \text{div}_g \left( T_1 \left( \frac{\partial}{\partial t} \right)^T \right) = 2S_2 \left( \frac{\partial}{\partial t}, \xi \right) + \text{Ric}_\tilde{g} \left( \xi, \left( \frac{\partial}{\partial t} \right)^T \right) \]

where \(R_g = -n(n-1)\kappa^2 + (n-1)(n-2)q^2\lambda^{2-2n}\) is the scalar curvature of the Reissner-Nordström-AdS manifold.

To calculate the integration on \(\Sigma\), we use the assumption that \(\Sigma\) is in a level set of \(u\) and \(|\tilde{\nabla}u| \to \infty\) as \(x \to \Sigma\). Let \(\{\phi^\alpha\}_{\alpha = 0}^{\nu-1}\) be a new coordinate system in a neighborhood of graph \(_{\Sigma}u\) in \(M\), such that \(\frac{\partial}{\partial \nu^\alpha}\) is the unit outer normal direction of \(\Sigma\) and \(\{\frac{\partial}{\partial \nu^0}, \cdots, \frac{\partial}{\partial \nu^{\nu-1}}\}\) span the tangent space of \(\Sigma\). Then on \(\Sigma\)

\[ \partial_a u = 0, \quad |\partial_0 u| = |\tilde{\nabla}u| = +\infty, \quad 1 \leq a \leq n - 1 \]

We set \(e_a = u_a \partial_t + \frac{\partial}{\partial \nu^a}, \quad e_0 = u_0 \partial_t + \frac{\partial}{\partial \nu^0}\) be the corresponding coordinate frame on \(M\). Then

\[ v_\Sigma = \frac{e_0}{|e_0|}, \quad |e_0| = \sqrt{f^2|u_0|^2 + 1}, \quad d\mu_\Sigma = d\mu, \quad g_{ab} = \tilde{g}_{ab} \]
where $d\mu$ is the area element of $\Sigma$ with respect to $\bar{g}$.

From (6.8), we get on $\Sigma$,

(6.20) \[ B_{ab} = \frac{fu_0}{\sqrt{1 + f^2|\nabla u|^2}} h^2_{ab}, \quad 0 < a, b < n. \]

where $\Gamma^0_{\alpha\beta}$ is the Christoffel symbols of $(P, \bar{g})$ with respect to the coordinate $\{y^\alpha\}_{\alpha=0}^{n-1}$ and $h^2_{ab}$ is the second fundamental form of $\Sigma$ in $(P, \bar{g})$. From (6.20) and $\lim_{x \to \Sigma} f / |\nabla u| \sqrt{1 + f^2} H = 1$, we infer

(6.21) \[
\left\langle T_1 \left( \frac{\partial}{\partial t} \right)^T, \nu_\Sigma \right\rangle_g = \left( tr B - B^0 \right) \left( \frac{f^2 u_0}{1 + f^2 |\nabla u|^2} e_0, e_0 \right) \\
= \frac{f u_0}{\sqrt{1 + f^2 |\nabla u|^2}} H \frac{f^2 u_0}{1 + f^2 |\nabla u|^2} |e_0| \\
= H f,
\]

which yields

(6.22) \[
\int_\Sigma \left\langle T_1 \left( \frac{\partial}{\partial t} \right)^T, \nu_\Sigma \right\rangle_g d\mu_\Sigma = \int_\Sigma H f d\mu.
\]

Finally, combining (6.12), (6.19) and (6.22) together, we get

(6.23) \[
m(M, g) \geq m + c_n \left( \int_M (R_g - R_{\bar{g}}) \left( \frac{\partial}{\partial t} \xi \right) d\mu_M + \int_\Sigma H f d\mu \right).
\]

From (6.19) we know equality holds if and only if $u_0^2 = |\nabla u|^2$, which means that $M$ is rotationally symmetric. \hfill \square

Finally, we can prove the Penrose type inequality for ALH graphs.

**Proof of Theorem 1.6.** It’s easy to check that on the horizon $\partial P$,

(6.24) \[
2m = \epsilon \left( \frac{|\partial P|}{\partial_{n-1}} \right)^{\frac{\pi n^2}{n-2}} + \kappa^2 \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{2n}{n+2}} + q^2 \left( \frac{\partial P}{\partial_{n-1}} \right)^{\frac{2n+2}{n+4}}.
\]

Combining (6.24), (1.7) with Theorem 1.3, we obtain the Penrose type inequality (1.8). \hfill \square

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