New determination of $|V_{cb}|$ using the three-loop QCD corrections for the $B \to D^*$ semi-leptonic decays

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We present a new determination of the Cabibbo-Kobayashi-Maskawa matrix element $|V_{cb}|$ by using the three-loop perturbative QCD corrections for the $B \to D^*$ semi-leptonic decay. The decay width of $B \to D^*$ semi-leptonic decay can be factorized as perturbatively calculable short-distance part and the non-perturbative but universal long-distance part. We adopt the principle of maximum conformity (PMC) single-scale setting approach to deal with the perturbative series so as to achieve a precise fixed-order prediction for the short-distance parameter $\eta_A$. By applying the PMC, an overall effective $\alpha_s$ value is achieved by recursively using the renormalization group equation, which inversely results in a precise scale-invariant pQCD series. Such scale-invariant series also provides a reliable basis for predicting the contributions from uncalculated perturbative terms. We then obtain $\eta_A = 0.9225^{+0.0117}_{-0.0168}$, where the error is the squared average of those from $\Delta\alpha_s(M_Z) = \pm 0.0010$ and the uncertainties caused by the uncalculated higher-order perturbative terms. By using the data of $B \to D^*\ell\nu$, we finally obtain $|V_{cb}|_{\text{PMC}} = (40.60^{+0.53}_{-0.77}) \times 10^{-3}$, which is consistent with the PDG value within errors.

The $|V_{cb}|$ is an important element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, whose precise value is helpful for precision test of the Standard Model. Due to the recent theoretical progresses and a plentiful collection of the data on the $B$-meson semileptonic decays at the $B$ factories and the large hadronic collider (LHC), we are facing the chance of achieving more precise CKM matrix elements. Among them, the $B$-meson decays to charmed $D^*$-meson, $B \to D^*\ell\nu$, with $\ell$ being the light leptons, is helpful for extracting $|V_{cb}|$ [1–15]. Using the heavy-quark symmetry, it’s differential decay width $d\Gamma/dw$ can be written as [16]

$$
\frac{d\Gamma(B \to D^*\ell\nu)}{dw} = C_F^2 \frac{G_F^2}{48\pi^3} (m_B - m_{D^*})^2 \sqrt{\omega^2 - (\omega + 1)^2} \omega (m_B^2 - m_{D^*} m_B - m_{D^*}^2)}{\omega + 1} \times m_{D^*}^2 |V_{cb}|^2 F^2(\omega),
$$

where $F(\omega)$ is the hadronic form factor, $G_F$ is the Fermi constant, $m_B$ and $m_{D^*}$ are $B$-meson and $D^*$-meson masses, respectively. The kinematic variable $w = v \cdot v' = (m_B^2 + m_{D^*}^2 - q^2)/2m_B m_{D^*}$ represents the four-velocity transfer between the $B$-meson velocity $v$ and the $D^*$-meson velocity $v'$, where $q^2 = (p_B - p_{D^*})^2$.

Using the experimental data on the differential decay width $d\Gamma/dw$, one can fix the value of the combined parameter $|V_{cb}|F(\omega)$ at various momentum transfers. Then, if one has known precise value of the hadronic form factor $F(\omega)$, one can finally achieve precise value of $|V_{cb}|$. At present, $F(\omega)$ has been calculated under various approaches, such as the Lattice QCD [11–13], the QCD Sum Rule [17], the covariant light-front quark model [18, 19] and the Perturbative QCD (pQCD) [20, 21], and etc. Using the pQCD factorization approach, $F(\omega)$ can be expressed as a product of the short-distance coefficient $\eta_A$ and the long-distance hadronic dynamics $\xi(\omega)$, e.g. $F(\omega) = \eta_A \xi(\omega)$. Experimentally, one usually extracts the magnitude of $|V_{cb}|$ by using the product $|V_{cb}|F(\omega)$ at the zero recoil point $w = 1$ (or $q^2 = 0$) [4–10]. At this point, we have $\xi(1) = 1 + O(1/m_Q^2)$ by using the heavy quark effective theory [22], where $Q$ stands for $m_c$ or $m_b$ and the power correction have been considered in refs [23–25], at the $O(1/m_Q^2)$-order level is about $-5.5 \pm 2.5\%$. Thus the key component of improving the accuracy of the differential decay width is to achieve a precise prediction on the short-distance parameter $\eta_A$ at the zero recoil point, whose perturbative expressions can be expressed as

$$
\eta_A = 1 + C_F \sum_{i=1}^{\infty} \eta_A^{(i)} a_s^{(i)}(\mu_f),
$$

where $a_s = \alpha_s/\pi$, $C_F = N_c^2 - 1/(2N_c)$ for SU($N_c$) color group, and $\mu_f$ is the renormalization scale. The perturbative coefficients $\eta_A^{(i)}$ has been calculated up to three-loop QCD corrections under the conventional MS-scheme [21].

It has been found that the pQCD series (2) involves two mass scales $m_b$ and $m_c$, one usually sets the renormalization scale $\mu_f$ as the typical scale $Q = \sqrt{m_b m_c}$ and vary it within a certain range such as $[\sqrt{m_b m_c}/2, 2\sqrt{m_b m_c}]$ to ascertain its uncertainty. However as shall be shown below, large scale uncertainties persist for such simple treatment even when more loop terms have been in-

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included. This is due to the divergent renormalon terms such as $n!3^{n}α^{n}$ (The $β_i$-functions satisfy the approximation $β_i ≈ β_i^{(0)} + 1$) [26–28], and the mismatching of the $α_s$ and its perturbative coefficients. A valid pQCD prediction for a physical observable should be independent to any choices of renormalization scheme and renormalization scale. However a truncated perturbation series does not automatically satisfy these requirements. Especially, by using the guessed scale, it will generally violate the renormalization group invariance of the pQCD approxi-
mation scale. However a truncated perturbation series does tend to reanalyze this process, showing how the perturbative series; and as a byproduct, the perturbative coefficients. A valid pQCD predic-
tion scale-setting in pQCD to all orders, and it provides a systematical way to eliminate conventional scale-setting

$\eta_A = 1 - 0.667a_s^{\overline{MS}}(μ_r) + \left[ - 2.501 + 0.130n_f 
+ 0.349n_f \log \left( \frac{μ_r^2}{Q^2} \right) - 5.760 \log \left( \frac{μ_r^2}{Q^2} \right) \right] a_s^{\overline{MS},2}(μ_r)$

$+ \left[ - 32.612 + 4.782n_f - 0.096n_f^2 - 85.157 \log \left( \frac{μ_r^2}{Q^2} \right) 
+ 6.031n_f \log^2 \left( \frac{μ_r^2}{Q^2} \right) + 10.068n_f \log \left( \frac{μ_r^2}{Q^2} \right) 
- 0.183n_f^2 \log \left( \frac{μ_r^2}{Q^2} \right) \right] a_s^{\overline{MS},3}(μ_r) + O(a_s^{\overline{MS},4})$,}

where $Q = \sqrt{m_r m_b}$. It should be emphasized that the PMC method determines the correct momentum flow by absorbing the non-conformal $\{ β_i \}$-terms of the pertur-
bation sequence. Therefore, in order to correctly distin-

$y_1 = \frac{31}{12} + \frac{5}{18}n_f$, $y_2 = -\frac{499}{288} - \frac{9}{4}π^2 + \frac{9}{8}π^4 - \frac{33}{8}ζ_3 - \frac{11}{432}n_f$

$+ \frac{13}{12}ζ_5n_f + \frac{25}{256}n_f^2$,}

where $ζ_3$ is the Riemann zeta function. We are then ready to transform the $\overline{MS}$-scheme perturbative series (3) into

$1$ Here $m_b$ and $m_c$ are pole quark masses. Since when using the PMC, it is helpful to use the pole mass other than the $\overline{MS}$-mass in the perturbative series so as to find the correct $\{ β_i \}$-terms for fixing the $α_s$ running behavior.

$2$ Practically, the determined PMC scale may be close to or less than the critical scale $Λ_{QCD}$ under certain scheme, and a low-

energy $α_s$ model has to be chosen to get a reasonable prediction.
the $V$-scheme one, i.e.,
\begin{equation}
\eta_A = 1 + \sum_{i=1}^{3} r_i^V a_s^{V,i}(\mu_r) + \mathcal{O}(a_s^{V,4}),
\end{equation}
where
\begin{align}
r_1^V &= -0.667, \\
r_2^V &= -0.779 - 0.056n_f + 0.349n_f \log \left( \frac{\mu_r^2}{Q^2} \right) - 5.760 \log \left( \frac{\mu_r^2}{Q^2} \right), \\
r_3^V &= -9.558 + 1.872n_f - 0.076n_f^2 + 5.064n_f \log \left( \frac{\mu_r^2}{Q^2} \right) + 6.031n_f \log^2 \left( \frac{\mu_r^2}{Q^2} \right) + 0.058n_f^2 \log \left( \frac{\mu_r^2}{Q^2} \right) - 0.183n_f^2 \log^2 \left( \frac{\mu_r^2}{Q^2} \right) - 55.399 \log \left( \frac{\mu_r^2}{Q^2} \right) - 49.759 \log^2 \left( \frac{\mu_r^2}{Q^2} \right).
\end{align}

By further using the general QCD degeneracy relations among different orders \cite{50}, each perturbative coefficient can be written as a \{\beta_i\}-series and we obtain
\begin{align}
\eta_A &= 1 + r_{1,0}^V a_s^V(\mu_r) + r_{2,0}^V a_s^{V,2}(\mu_r) + (r_{3,0}^V + \beta_0 r_{2,1}^V) a_s^{V,3}(\mu_r) + \mathcal{O}(a_s^{V,4}),
\end{align}
where $r_{i,0}^V$ are scale-invariant conformal coefficients and $r_{i,j}(\neq 0)$ are generally scale-dependent coefficients, all of which can be derived from the above equations.

Following the standard PMC single-scale setting procedures \cite{51}, all non-conformal \{\beta_i\}-terms shall be adopted for determining the correct $\alpha_s$-value of the process, and the parameter $\eta_A$ becomes the following perturbative series which is free of RGE-involved \{\beta_i\}-terms, i.e.
\begin{equation}
\eta_A|_{\text{PMC}} = 1 + \sum_{i=1}^{3} r_{i,0}^V a_s^{V,i}(Q_s) + \mathcal{O}(a_s^{V,4}),
\end{equation}
where $Q_s$ represents the correct momentum flow of the process determined by the RGE, and by using the present known three-loop series, it can be fixed up to next-to-leading log (NLL) accuracy,
\begin{equation}
\ln \left( \frac{Q_s^2}{Q^2} \right) = T_0 + T_1 a_s^V(Q) + \mathcal{O}(a_s^{V,2}),
\end{equation}
where
\begin{equation}
T_0 = -\frac{r_{1,0}^V}{r_{1,0}^V}
\end{equation}
and
\begin{equation}
T_1 = \frac{2(r_{2,0}^V r_{2,1}^V - r_{1,0}^V r_{3,1}^V)}{r_{1,0}^V r_{2,1}^V} + \frac{(r_{2,1}^V - r_{2,0}^V r_{3,2}^V)}{r_{1,0}^V r_{2,1}^V} \beta_0.
\end{equation}

It is found that $Q_s$ is free of the renormalization scale $\mu_r$. This indicates that one can take any $\mu_r$ to finish the renormalization procedures for the parameter $\eta_A$, and the resultant PMC series shall be independent to this choice, well satisfying the requirement of RGI.

To do the numerical calculation, we adopt $\alpha_s(M_Z) = 0.1179 \pm 0.0010$ \cite{52} as the reference point for fixing $\alpha_s$ running behavior, which leads to $\Lambda_{\overline{MS}}^{\text{QCD}}|_{n_f=5} = 207.2^{+11.8}_{-11.4}$ MeV and $\Lambda_{QCD}^{\text{QCD}}|_{n_f=5} = 283.0^{+16.1}_{-15.6}$ MeV, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.pdf}
\caption{The parameter $\eta_A$ up to N$^3$LO-level as a function of the renormalization scale $\mu_r$ under conventional and PMC scale-setting approaches. The dashed and the solid lines are for conventional and PMC ones, respectively.}
\end{figure}

Firstly, we present the N$^3$LO-$\eta_A$ as a function of the renormalization scale $\mu_r$ in Fig. 1, which are calculated under conventional and PMC scale-setting approaches, respectively. There are large cancellations among different orders for conventional series, leading to a small scale error for the N$^3$LO-level prediction; e.g. by taking $\mu_r \in [Q/2, Q]$, the net scale error of $\eta_A$ is
\begin{equation}
\Delta \eta_A|_{\text{Conv.}} = (0.0444, -0.0408).
\end{equation}
This is clearly shown in Table I, which indicates that the scale errors of each loop terms are rather large. While the PMC predictions are independent to any choice of $\mu_r$ for either each loop terms or the total series.

We define a ratio $k_i$ to show the convergent behavior of the perturbative series, e.g.,
\begin{equation}
k_i = \Gamma^{N^3LO}/\Gamma^{LO}.
\end{equation}
Table II shows the ratios $k_i$ up to N$^3$LO-level under conventional and PMC scale-setting approaches, respectively. The PMC series is scale-invariant and convergent, and its ratios are $k_1 = -6.44\%, k_2 = -1.58\%$ and $k_3 = 0.27\%$ for any choice of $\mu_r$. The conventional series is scale-dependent, if setting $\mu_r = Q$ to remove the renormalon terms, the ratios are close to the PMC ones, i.e., $k_1 = -6.25\%, k_2 = -0.93\%$ and $k_3 = -0.17\%$; however when taking $\mu_r \in [Q/2, 2Q]$, those ratios change greatly, e.g., the uncertainties become $\Delta k_1 = (-3.68\%, +3.28\%)$, $\Delta k_2 = (10.14\%, -2.84\%)$ and $\Delta k_3 = (-2.42\%, -3.08\%)$, respectively.
 According to Eq. (13), we obtain $Q_\ast = 2.662$ GeV. Because the effective scale $Q_\ast$ is of perturbative nature, its unknown terms shall lead to the first kind of residual scale dependence [53]. At present, we don’t have too much reliable ways for predict unknown higher order terms. Based on the renormalization scale-independent PMC series, the Padé approximation approach (PAA) seems to be realistic. In this process, the PMC scale $Q_\ast$ is at NLL level as shown in Eq.(13), we need more known terms to do PAA. The PMC scale $Q_\ast$ is determined by absorbing all the RGE related non-conformal terms into the effective coupling $\alpha_s(Q_\ast)$. From Eq.(13), the PMC scale is of perturbative and it would suffers from both exponential-suppression and $\alpha_s(Q)$-suppression. Thus, it’s reliable to make the NLL-term as missing higher-order term, in other words, as a conservative estimation of first kind of residual scale dependence, we take the absolute value of the last known term as the magnitude of the unknown next-to-next-to-leading-log terms (N$^2$LL), i.e. $Q_\ast$ up to N$^2$LL-level takes the form

$$\ln \left( \frac{Q^2}{Q^2_\ast} \right) = 0.5 - 6.669 a_s^{V}(Q) \pm 6.669 a_s^{V}(Q),$$

which leads to a small scale shift $\Delta Q_\ast = (0.976 \pm 0.714)$ GeV, and hence a small error ($\sim 1\%$) to the ratio $\eta_A$, e.g.

$$\Delta \eta_A|_{\text{PMC}} = (0.011 \pm 0.016).$$

Secondly, it is helpful to have an estimation of the contribution from the uncalculated higher-order terms. In the literature, the Padé approximation approach [54-56] (PAA) provides an effective do such a prediction. For the PAA, a pQCD approximate $\rho_0$ is expressed as the following $[N/M]$-type form:

$$\rho_0^{[N/M]} = a_s^0 \times \frac{b_0 + b_1 a_s + \cdots + b_N a_s^N}{1 + c_1 + \cdots + c_M a_s^M},$$

$$= \sum_{i=1}^{N} C_i a_s^{p+i-1} + C_{n+1} a_s^{p+n} + \cdots,$$

where $p$ is the $\alpha_s$-order of the leading-order terms, which is equal to 0 for the present case of $\eta_A$. The input parameters $b_i \in [0,N]$ and $c_i \in [1,M]$ can be expressed by using the known perturbative coefficients $C_i\in [1,N]$; while the first unknown $(n + 1)$-$\alpha_s$-order coefficient $C_{n+1}$ can be expressed by $b_i \in [0,N]$ and $c_i \in [1,M]$; and hence by the known coefficients $\{C_1, \ldots, C_n\}$. For the present considered N$^3$LO-level $\eta_A$, since it already shows good convergence for both conventional and PMC series, we take the preferable $[0/n-1]$-type PAA to estimate the contribution of the unknown terms [57], which is consistent with the “Generalized Crewther Relations” [58] and the Gell-Mann-Low method for fixing the coupling constant of quantum electrodynamics [59]. Specifically, the N$^4$LO-term for either the conventional series or the PMC series is

$$r_4^V = \frac{2 r_1 r_2 r_3^V - r_2^V}{r_1^2},$$

and

$$r_{4,0}^V = \frac{2 r_1^V r_2^V r_3^V - r_2^V}{r_1^2},$$

respectively. Then the predicted N$^4$LO-terms for conventional and PMC series of $\eta_A$ are $\eta_A|_{\text{Conv}} = r_4^V a_s^{V,4}(\mu_r)$ and $\eta_A|_{\text{PMC}} = r_{4,0}^V a_s^{V,4}(Q_\ast)$, respectively.

| $r_4^V$ | $r_{4,0}^V$ |
|------------------|------------------|
| -0.0252 | -0.0228 |
| 0.0388 | 0.0444 |

TABLE III: The parameters $b_i \in [0,N]$ and $c_i \in [1,M]$ can be expressed by using the known perturbative coefficients $C_i\in [1,N]$; while the first unknown $(n + 1)$-$\alpha_s$-order coefficient $C_{n+1}$ can be expressed by $b_i \in [0,N]$ and $c_i \in [1,M]$; and hence by the known coefficients $\{C_1, \ldots, C_n\}$. For the present considered N$^3$LO-level $\eta_A$, since it already shows good convergence for both conventional and PMC series, we take the preferable $[0/n-1]$-type PAA to estimate the contribution of the unknown terms [57], which is consistent with the “Generalized Crewther Relations” [58] and the Gell-Mann-Low method for fixing the coupling constant of quantum electrodynamics [59]. Specifically, the N$^4$LO-term for either the conventional series or the PMC series is
\begin{align}
\Delta \eta_A^{\text{High order}} &= (+0.0570, -0.0499),
\Delta \alpha_s(M_Z) &= (+0.0115, -0.0166),
\end{align}

which shows that the PMC series has a much smaller scale uncertainty due to uncalculated terms.

Thirdly, except for the scale uncertainty due to uncalculated perturbative terms, there is also uncertainty caused by the error of \(\alpha_s\) fixed-point error \(\Delta \alpha_s(M_Z)\). By taking \(\Delta \alpha_s(M_Z) = \pm 0.0010\) \([52]\), we obtain

\begin{align}
\Delta \eta_A^{\text{Conv.}}(M_Z) &= (+0.0020, -0.0021),
\Delta \alpha_s(M_Z) &= (+0.0024, -0.0025). 
\end{align}

For conventional scale-setting approach, the error caused by \(\Delta \alpha_s(M_Z)\) is about twenty-five times smaller than the error (24) caused by the unknown higher-order terms. While for the PMC single-scale setting approach, the error caused by \(\Delta \alpha_s(M_Z)\) is at the same order of the one (25) caused by unknown higher-order terms. Since the PMC uses the RGE to fix the correct \(\alpha_s\)-running behavior of \(\eta_A\), and inversely, its prediction depends heavily on the precise value of \(\alpha_s(M_Z)\). This explains why \(\Delta \eta_A^{\text{PMC}}(M_Z)\) is slightly larger than \(\Delta \eta_A^{\text{Conv.}}\). Thus more precise measurements on \(\alpha_s(M_Z)\) is important for achieving more precise pQCD predictions.

In combination of all the above mentioned errors, we finally obtain

\begin{align}
\eta_A^{\text{Conv.}} &= 0.9265_{-0.0117}^{+0.0057},
\eta_A^{\text{PMC}} &= 0.9225_{-0.0167}^{+0.0024}. 
\end{align}

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
 & PMC & Conv. \\
\hline
OPAL partial reco \([4]\) & 42.72_{-1.34}^{+1.85} & 42.83_{-2.92}^{+7.18} \\
BELLE \([10]\) & 39.94_{-1.81}^{+1.18} & 40.04_{-2.41}^{+2.69} \\
BABAR global fit \([9]\) & 40.6_{-3.3}^{+3.3} & 40.77_{-3.83}^{+3.83} \\
OPAL excl \([4]\) & 41.92_{-2.72}^{+2.37} & 42.03_{-4.36}^{+3.36} \\
BABAR excl \([7]\) & 39.53_{-1.31}^{+1.31} & 39.63_{-3.49}^{+3.49} \\
BABAR D^{\ast+} \([8]\) & 40.89_{-3.19}^{+3.19} & 41.00_{-5.56}^{+5.56} \\
DELPHI partial reco \([5]\) & 40.44_{-2.92}^{+2.92} & 44.05_{-4.45}^{+4.45} \\
DELPHI excl \([6]\) & 41.33_{-4.16}^{+4.16} & 41.45_{-4.16}^{+4.16} \\
\hline
\end{tabular}
\caption{The values of \(|V_{cb}|\) \((\times 10^{-3})\) using the PMC and conventional (Conv.) pQCD predictions for \(\eta_A\), which are derived by using the data given by various experiments group and under CLN parameterization \([4-10]\).}
\end{table}

To extract the value of \(|V_{cb}|\), we refer to the same parameterization method as the Heavy Flavor Averaging Group (HFLAV) for the parameterization of the form factor function, i.e. the Caprini, Lellouch, and Neubert (CLN) \([3]\) parameterization. Table IV gives the values of \(|V_{cb}|\) \((\times 10^{-3})\) using PMC and conventional (Conv.) pQCD predictions for \(\eta_A\), which are derived by using the data given by CLN experimental parameterization \([4-10]\). In Table IV, the errors are calculated by using the following formula,

\begin{equation}
|V_{cb}| = \frac{\exp{\mathcal{F}(1)}}{\mathcal{F}(1)} \sqrt{(\Delta_{\exp}\mathcal{F}(1))^2 + (\Delta_{\mathcal{F}(1)}\mathcal{F}(1))^2},
\end{equation}

where \(\Delta_{\mathcal{F}(1)}\) and \(\Delta_{\exp}\mathcal{F}(1)\) represent theoretical and experimental uncertainties, respectively. Table IV shows that the PMC predictions are more precise than the conventional ones due to much less scale uncertainties. This fact can be more clearly shown by Fig. 2, in which comparisons of the various predicted values of \(|V_{cb}|\) are given. The weighted average of those theoretical values can be calculated by using the method described in detail in Ref.\([52]\), i.e.

\begin{equation}
\overline{x} \pm \delta \overline{x} = \frac{\sum_i w_i x_i}{\sum_i w_i} \pm \left(\sum_i w_i\right)^{1/2},
\end{equation}

where \(\overline{x}\) is the central value of the concern parameter (e.g. \(|V_{cb}|\)) and \(\delta \overline{x}\) is its uncertainty. Here \(x_i\) stands for the \(i\)th given value, \(\delta_i\) is the uncertainty of the \(i\)th value, and \(w_i = 1/\delta_i^2\) is the weight factor. We then obtain the weighted average of \(|V_{cb}|\),

\begin{align}
|V_{cb}|^{\text{Conv.}} &= (40.90_{-1.06}^{+1.05}) \times 10^{-3},
|V_{cb}|^{\text{PMC}} &= (40.60_{-0.57}^{+0.52}) \times 10^{-3}.
\end{align}

Those two values are consistent with the PDG average value \(|V_{cb}|_{\text{PDG}} = (40.8 \pm 1.4) \times 10^{-3}\) within errors \([60]\).

As a comparison, the recent value given by the HFLAV is \(|V_{cb}|_{HFLAV} = (38.76 \pm 0.55) \times 10^{-3}\) \([61]\), which deviates from the PDG average value by 0.164σ, the present PMC prediction by 1.309σ and the present conventional prediction by 1.072σ, respectively.

As a summary, in this paper, we have given a detailed analysis on the perturbative nature of the parameter \(\eta_A\) for the \(B \to D^* l \nu\) process up to \(N^3\)LO-level.

By applying the PMC single-scale setting approach, the conventional scale ambiguity is removed. Thus the
PMCs scale-invariant series provides a better platform for achieving a more precise fixed-order pQCD prediction. To compare with a larger uncertainty caused by unknown terms, \( \Delta \eta_{\text{High order}}^{\text{Conv}} = (0.0570, 0.0499) \) for conventional series, the PMC series has a much smaller error, \( \Delta \eta_{\text{High order}}^{\text{PMC}} = (0.0117, 0.0166) \). Then a more accurate prediction on \( |V_{cb}| \) can be achieved, which is consistent with the PDG average within errors. Thus our present results emphasize the necessity of a proper renormalization scale-setting approach during the pQCD calculation.

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