Direct-exchange duality of the Coulomb interaction and collective excitations in graphene in a magnetic field

K. SHIZUYA

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan
shizuya@yukawa.kyoto-u.ac.jp

Received 31 January 2017
Accepted 27 March 2017
Published 8 June 2017

In a magnetic field two-dimensional (2d) electron systems host, with quenched kinetic energy, a variety of many-body correlation phenomena, such as interaction-driven new states and associated collective excitations over them. In a magnetic field the two-body operators pertinent to the 2d Coulomb interaction obey a crossing relation, with which the Coulomb interaction is also cast into the form of manifest exchange interaction. It is shown that active use of this direct/exchange duality of the interaction allows one to develop, within the framework of the single-mode approximation, a new efficient algorithm for handling a wide class of collective excitations. The utility of our algorithm is demonstrated by studying some examples of inter- and intra-Landau-level collective excitations in graphene and in conventional electron systems.

Keywords: collective excitations; quantum Hall effect; graphene.
PACS numbers: 73.43.Lp,72.80.Vp,71.10.Pm

1. Introduction

Two-dimensional (2d) electron systems such as GaAs heterostructures\(^1\) and graphene\(^2-4\) attract great attention in both applications and fundamental physics for their novel and promising features that reflect the dynamics specific to two dimensions and enriched with many-body correlations. In a magnetic field, in particular, the kinetic energy of electrons is quantized to form a tower of flat Landau levels and, along with such a large kinetic degeneracy, the Coulomb interaction between carriers essentially governs the physics of many-body correlations, such as the fractional quantum Hall (FQH) effect\(^5,6\) and some exotic states arising from the interplay of interaction and internal degrees of freedom (spin, valley, layer, etc.). Also of interest are collective excitations (such as spin waves and pseudospin waves) which such states support.

Inter-Landau-level excitations are also amenable to many-body effects. For (conventional) 2d electrons with quadratic dispersion, cyclotron resonance takes place...
only between adjacent levels and is scarcely affected by the Coulomb interaction, as implied by Kohn’s theorem. The situation is quite different for graphene which develops a quasi relativistic pattern of Landau levels, with a variety of cyclotron resonance within the conduction or valence band and across the two bands. Many-body corrections to such intra- and inter-band resonance, e.g., energy shifts and renormalization effects, reveal the nature of the underlying Dirac-like electrons, and have indeed been observed in experiment. Bilayer (and few-layer) graphene is even richer in “quasi-relativistic” effects, such as orbital degeneracy in the lowest Landau level and its lifting by many-body effects.

Among theoretical frameworks to handle such many-body effects are mean-field theory, Hartree-Fock (HF) approximation, the single-mode approximation (SMA), etc. In particular, the SMA, reformulated and adapted for quantum Hall systems by Girvin, MacDonald and Platzman, is a general and powerful means of studying many-body effects in a magnetic field. The purpose of this paper is to elaborate on the SMA and develop a new algorithm to facilitate actual calculations. We first note that in a magnetic field the two-body operators pertinent to the Coulomb interaction obey a crossing relation, with which the normal direct form of interaction is also cast into the form of manifest exchange interaction. Active use of this direct/exchange duality of the interaction allows one to effectively replace the calculation of two-body correlation functions (the static structure factors) crucial to the SMA by a far simpler calculation of the expectation values of some one-body charges. We study some examples of inter- and intra-Landau-level collective excitations to demonstrate the utility of the new algorithm and to supply some relevant techniques.

In Sec. 2 we refer to the case of graphene and set up notation for handling general 2d electrons in a magnetic field. In Secs. 3 and 4 we elaborate on the framework of the SMA, note the direct/exchange duality of the Coulomb interaction, and formulate our algorithm for general inter-Landau-level excitations. In Sec. 5 we examine many-body corrections to cyclotron resonance in graphene. In Sec. 6 we extend our algorithm to intra-Landau-level collective excitations in flavor (spin, valley, etc) space. In Sec. 7 we study how to handle genuine density fluctuations in our approach, such as those over the FQH states. Section 8 is devoted to a summary and discussion.

2. Electrons in a magnetic field

The electrons in graphene are described by two-component spinors on two inequivalent lattice sites \((A, B)\). They acquire a linear spectrum (with velocity \(v \sim 10^6 \text{m/s}\)) near the two inequivalent Fermi points \((K, K')\) in momentum space, and are described by an effective Hamiltonian of the form,

\[
H = \int dxdy \left\{ \psi^\dagger \mathcal{H}_+ \psi + \chi^\dagger \mathcal{H}_- \chi \right\},
\]

\[
\mathcal{H}_\pm = v (\Pi_1 \sigma^1 + \Pi_2 \sigma^2 \pm \delta m \sigma^3) - eA_0,
\]
where $\Pi_i = p_i + e A_i$ [with $i = (1, 2)$ or $(x, y)$] involve coupling to external potentials \((A_i, A_0)\) and $\sigma^i$ denote Pauli matrices. The Hamiltonians $\mathcal{H}_\pm$ describe electrons at two different valleys $a \in (K, K')$, and $\delta m$ stands for a possible tiny sublattice asymmetry; we take $\delta m > 0$, without loss of generality.

Let us place graphene in a uniform magnetic field $B_z = B > 0$ by setting $A_i = (-By, 0)$. The electron spectrum then forms an infinite tower of Landau levels of energy

$$\epsilon_n = s_n \omega_c \sqrt{|n| + \mu^2}$$

(2)

at each valley (with $s_n \equiv \text{sgn}[n] = \pm 1$), labeled by integers $n \in (0, \pm 1, \pm 2, \ldots)$ and $p_x$, of which only the $n = 0$ (zero-mode) levels split in valley (hence to be denoted as $n = 0_\pm$),

$$\epsilon_{0_\pm} = \mp v \delta m = \mp \omega_c \mu \text{ for } K/K'.$$

(3)

Here we have set, along with magnetic length $\ell \equiv 1/\sqrt{e B}$,

$$\omega_c \equiv \sqrt{2} v / \ell \approx 36.3 \times v [10^6 \text{m/s}] \sqrt{B [T]} \text{ meV}, \quad \mu \equiv \ell \delta m / \sqrt{2}.$$

(4)

The eigenmodes at each valley $a$ are written as

$$\psi_n^a = (|n| - 1) b_n^a, |n| e_n^a)^t$$

(5)

[here only the orbital eigenmodes are shown using the harmonic-oscillator basis \(|n\rangle\)] with \((b_n, e_n)\) given by

$$b_n^K = \frac{1}{\sqrt{2}} \left( \alpha_n^+, -s_n \alpha_n^- \right) \to \frac{1}{\sqrt{2}} (1, -s_n),$$

$$b_n^0 = (0, c_n),$$

(6)

where $\alpha_n^\pm = \sqrt{1 \pm s_n} \delta_n$ and $\delta_n = \mu / \sqrt{\mu^2 + |n|^2}$; $\delta_n < 1$ for $n \neq 0$ while $\delta_0 = 1$.

One can pass to another valley $K'$ by noting the relation $\sigma^3 \mathcal{H}_- \sigma^3 = -\mathcal{H}_+$. This means that the two valleys are related as

$$\epsilon_n^{K'} = -\epsilon_n^K, \quad (b_n^{K'}, e_n^{K'}) = (b_{-n}^K, -e_n^K).$$

(7)

Thus the Landau-level spectra as a whole are electron-hole symmetric. For $\delta m \to 0$ the two valleys differ only by the $n = 0_\pm$ modes, with $(b_{0_\pm}, c_{0_\pm}) = (0, \mp 1), (b_n, c_n) \to (1, -s_n)$ and $e_n \to s_n \omega_c \sqrt{|n|}$.

The Landau-level structure is made explicit by passing to the $|n, y_0\rangle$ basis (with $y_0 \equiv e^2 p_x$) via the expansion \((\psi, \chi) \equiv \sum_{n, y_0} \langle x|n, y_0\rangle \{\psi_n^{\alpha, a}(y_0)\}\), where $n$ refers to the Landau level, $a \in (K, K')$ to the valley and $\alpha \in (\downarrow, \uparrow)$ to the spin. The Lagrangian thereby reads

$$L = \int dy_0 \sum_n \sum_{a, \alpha} \langle \psi_n^{\alpha, a}| (i \partial_t - e_n^a) \psi_n^{\alpha, a}$$

(8)
and the charge density \( \rho_{-p} = \int d^2x e^{ip\cdot x} \rho \) with \( \rho = \psi^\dagger \psi + \chi^\dagger \chi \) is written as\(^{12} \)

\[
\rho_{-p} = \gamma_p \sum_{m,n=-\infty}^{\infty} g_{mn;\alpha}^p R_{\alpha\beta;\gamma}^{mn;\alpha\beta} \equiv \int d\gamma_0 \psi_{\alpha}^m(\gamma_0) e^{ip\cdot r} \psi_{\beta}^n(\gamma_0),
\]

(9)

with \( \gamma_p = e^{-i\ell^2 p^2/4} \). Here \( \mathbf{r} = (i\ell^2 \partial/\partial \gamma_0, \gamma_0) \) stands for the center coordinate with uncertainty \([r_x, r_y] = i\ell^2\). This leads to the composition law \( e^{ip\cdot r} e^{ik\cdot r} = e^{-i4\ell^2 p\times k e^{i(p+k)\cdot r}} \), or equivalently, the \( W_\infty \) algebra\(^{30} \) of the charge operators,

\[
[R_k^j, R_m^n] = \delta_{km} \eta_{k,p} R_{kj+p}^m - \delta_{mj} \eta_{p,k} R_{k+j}^m,
\]

(10)

with \( \eta_{k,p} \equiv e^{-i\ell^2 k^2 p^2} \) and \( k \times p \equiv k_x p_y - k_y p_x \). Here, for notational simplicity, we have suppressed spin and valley labels. Actually it is convenient to treat them collectively with the level label \( n \rightarrow (n, a, \alpha) \). One may regard, e.g., \( R_{k}^{j;k;ab} \equiv R_{k,\alpha;\beta}^{j;ab} \), and \( \delta_{jk} \) as \( \delta^{(j,a,a),(k,b,\beta)} = \delta_{jk} \delta_{ab} \delta_{\alpha\beta} \). The valley and spin labels are thereby properly recovered in Eq. (10). Accordingly we shall often suppress them in what follows.

The coefficient matrix \( g_{mn;\alpha}^p \) at valley \( a \) is given by\( \text{E} \)

\[
g_{mn;\alpha}^p = b_m^a b_n^a f_{p}^{\left|m±1\right|,\left|n±1\right|} + c_m^a c_n^a f_{p}^{\left|m\right|,\left|n\right|},
\]

(11)

where

\[
f_{p}^{mn} = \sqrt{n! m!} (i\ell p/\sqrt{2})^{m-n} F_n^{(m-n)(\frac{1}{2}\ell^2 p^2)}
\]

(12)

for \( m \geq n \geq 0 \), and \( f_{p}^{mn} = (f_{-p}^{mn})^\dagger \); \( p = p_x + i p_y \); it is understood that \( f_{p}^{mn} = 0 \) for \( m < 0 \) or \( n < 0 \). In view of Eq. (11), \( g_{mn;\alpha}^p \) at the two valleys are related as

\[
g_{mn;K'}^p = g_{-m, -n;K}^p.
\]

(13)

Some explicit forms of \( g_{mn;\alpha}^p \) are

\[
g_{p}^{00} = 1, \quad g_{p}^{10} = 1 - (c_1)^2 \frac{1}{2}\ell^2 p^2, \quad g_{p}^{10} = i c_1 \ell p/\sqrt{2}, \quad g_{p}^{01} = i c_1 \ell p^1/\sqrt{2},
\]

(14)

with \( c_1^a \approx -(1+\mu/2)/\sqrt{2} \) for \( a = K/K' \).

From now on we frequently suppress summations over levels \( n \), valleys \( a \) and spins \( \alpha \), with the convention that the sum is taken over repeated indices. The one-body Hamiltonian \( H \) is thereby written as

\[
H = c_n^a R_{\beta;\alpha;0}^{\alpha;\beta;0} - \mu_Z (\frac{1}{2} \sigma^3)_{\alpha\beta} R_{\alpha\beta;0}^{mn;\alpha\beta}.
\]

(15)

Here, for generality, the Zeeman term \( \mu_Z = g^* \mu_B B \) is introduced.

\* We remark that an alternative choice of the \( U(1) \) phase of the harmonic-oscillator basis, \(|n\rangle \rightarrow (e^{i\alpha})^m|n\rangle \), allows one to replace \( c_a \rightarrow e^{i\alpha} c_a \) in Eq. (3) and \( p = p_y + i p_x \rightarrow e^{i\alpha} p \) in \( f_{p}^{mn} \), which is essentially a rotation in \( xy \) plane.
The Coulomb interaction \( V = \frac{1}{2} \sum_p v_p : \rho_p \rho_p : \) is written as

\[
V = \frac{1}{2} \sum_p v_p \sum_{k,a} g_{-p}^{mn;k,a} : R_{\alpha\alpha;-p}^{mn;k,a} : R_{\beta\beta;p}^{mn;bb} : ,
\]

with the potential \( v_p = 2\pi\alpha/(e_b|p|) \), \( \alpha \equiv e^2/(4\pi\epsilon_0) \) and the substrate dielectric constant \( \epsilon_b \); \( \sum_p \equiv \int d^2p/(2\pi)^2 \) and we set \( \delta_{p,0} \equiv (2\pi)^2\delta^2(p) \). As usual, normal ordering is defined as \( : R^{jk}R^{mn} : \propto (\psi^m)\dagger(\psi^j)\dagger\psi^k\psi^n \), with an obvious identity \( R^{jk}_{-p} R^{mn}_{p} = : R^{mn}_{p} R^{jk}_{-p} : \).

So far we have set up our notation for monolayer graphene but the total Hamiltonian of the form \( H^{\text{tot}} = H + V \) with Eqs. (15) and (16) applies to general electron systems in a magnetic field as well, so does our analysis below. For conventional 2d electrons, e.g., one may simply set \( \epsilon_n \to \omega_c(n + \frac{1}{2}) \) with \( \omega_c = eB/m^* \) and restrict orbital labels to \( n \in \{0, 1, 2, \cdots \} \) and \( g^{mn}_{p} \to f^{mn}_{p} \).

### 3. Collective excitations

Suppose now that a uniform ground state \( | \text{Gr} \rangle \) is realized at some filling factor in a magnetic field. Our task is to study collective excitations over this ground state using the Hamiltonian \( H + V \). For definiteness, let us consider interlevel excitations from \( \{j; a, \alpha\} \) to \( \{n; b, \beta\} \), using the SMA. The SMA is a variational method\(^{30-32}\) that adopts \( R_{\alpha\alpha \beta \beta}^{n,n} | \text{Gr} \rangle \) as the trial state for such an excitation. It is neatly systematized in the framework of effective Lagrangian.

Let \( \Xi_{\alpha \beta ; p}^{n \beta} \) be an interpolating field associated with the charge \( R_{\alpha \beta ; p}^{n \beta} \) and denote

\[
\Xi R = \sum_p \Xi_{p}^{n \beta} R_{-p}^{n \beta}
\]

for short; \( (\Xi_{\alpha \beta ; p}^{n \beta})^{\dagger} = \Xi_{\alpha \beta ; -p}^{n \beta} \) so that \( \Xi R \) is hermitian; here we consider general \( n \leftrightarrow j \) channels all together and the sum over orbital (and suppressed valley and spin) labels is understood. One then regards interlevel excitation as a \( W_{\infty} \)-rotation \( e^{-i\Xi R} | \text{Gr} \rangle \) of \( | \text{Gr} \rangle \) in the orbital space, and evaluates the associated energy. Note first that, via \( U = e^{i\Xi R} \), the field \( \psi^m \) turns into

\[
U \psi^m(y_0) U^{-1} = [U^{-1}]^{mn} \psi^n(y_0) \equiv \psi^m(y_0),
\]

where \( U = e^{i\Xi R} \) and \( (\Xi R)^{mn} = \sum_p \Xi_p^{mn} e^{ip \cdot r} \). Replacing \( \psi \) by \( \psi' = U^{-1} \psi \) in the Lagrangian \( 3 \) and taking the expectation value \( \langle \text{Gr} | \cdots | \text{Gr} \rangle \) then reveals the associated energy change in the form of Lagrangian\(^{34}\) for \( \Xi \),

\[
L_{\Xi} = \langle \text{Gr} | U(i\partial_t - H^{\text{tot}}) U^{-1} | \text{Gr} \rangle,
\]

with \( H^{\text{tot}} = H + V \); \( \partial_t \) acts on \( \Xi_p \) in \( U^{-1} = e^{-i\Xi R} \). [Note in this connection the relations \( \psi^m U H U^{-1} \psi = U (\psi^m H \psi) U^{-1} \) and \( \psi^m U i(\partial_t U^{-1}) \psi = U i\partial_t U^{-1} \).

Our task in this paper is to develop a general and efficient way to calculate the effective Lagrangian \( 19 \). To this end one first needs a set of ground-state expectation values, which we denote as

\[
\langle \text{Gr} | R_{\alpha \beta ; p}^{mn;ab} | \text{Gr} \rangle = \rho_n \delta_{\alpha \beta} \delta^{mn} \delta^{ab} \delta_{p,0}
\]
for good quantum numbers \( \{n, a, \alpha\} \), where \( \nu_n^{a\alpha} \) denotes the filling fraction of the \( \{n, a, \alpha\} \) level and \( \bar{\rho} \equiv 1/(2\pi L^2) \). If, for example, there arises mixing in valley, one has to first rotate the field \( \psi \) in valley space and define filling fractions only for a set of good (i.e., diagonal) valley labels. In what follows we regard one-body labels \( \{n, a, \alpha\} \) of \( H \) as good quantum numbers and study many-body effects to \( O(V) \).

The one-body part of \( L_\Xi \) in Eq. (19) is solely governed by the expectation values of rotated charges \((R_{mn}^{(a)})^d = U R_{mn}^{(a)} U^{-1} = e^{i\Xi R} R_{mn}^{(a)} e^{-i\Xi R} \). It is useful to write \( C_p^a(R_{-p})^{d} = C_p^{mn}(R_{mn}^{(a)})^d \), with an arbitrary function \( C_p^{mn} \). The rotated charges then read, to \( O(\Xi^2) \),

\[
\begin{align*}
C_p^a(R_{-p})^{d} &= C_p R_{-p} + \sum_{k} C_p^{(1)}_{k,p} R_{-k-p} + \sum_{q,k} C_p^{(2)}_{q,k,p} R_{-q-k+p} + \cdots, \\
C_p^{(1)}_{k,p} &= i\{\eta_{k,p} (\Xi_k C_p) - \eta_{p,k} (C_p \Xi_k)\}, \\
C_p^{(2)}_{q,k,p} &= \eta_{k,q,p} (\Xi_k C_p \Xi_q) - \frac{i}{2} \eta_{p,k,q} (\Xi_k \Xi_q C_p) - \frac{1}{2} \eta_{q,p,k} (C_p \Xi_k \Xi_q),
\end{align*}
\]

(21)

with \( \eta_{k,p} = e^{-i\xi_k^p x} \) and \( \eta_{q,p,k} = e^{-i\xi_q^p x} \). \( \langle \Xi_k C_p \rangle \) stands for the matrix product \( \Xi_k C_p \Xi_n = \Xi_k^b C^{b n} \), etc. In particular, the ground-state expectation values are neatly written as

\[
\langle (R_{-p}^{j n; a})^d \rangle = \bar{\rho} \left[ \nu_j^a \delta^{j n; b a} \delta_{p,0} + i(\nu_n^b - \nu_j^a) \Xi_{-p}^{j n; b a} + \sum_q e^{i\xi_k q x} \Gamma_{n j; b a}^{p q} + \cdots \right],
\]

(23)

where for generality we have restored valley indices. From now on expectation values \( \langle \langle Gr \cdots \rangle \rangle \) will be simply denoted by \( \langle \cdots \rangle \).

The one-body sum \( \sum_{n,j} \epsilon_j^a \langle (R_{pp=0}^{j n})^{d} \rangle \) term, in particular, leads to the \( O(\Xi^2) \) energy term

\[
\bar{\rho} \sum_{n>j} \sum_{a,b} \left( \nu_n^b - \epsilon_j^a \right) (\nu_j^a - \nu_n^b) \Xi_{-k}^{j n; a b} \Xi_{-k}^{j n; b a},
\]

(24)

where, in passing, we have cast all \( \sum_{n,j} \epsilon_j^a \langle (R_{pp=0}^{n; a})^{d} \rangle \) into the above form by noting the symmetry of \( \Xi_{-k}^{j n; a b} \Xi_{-k}^{j n; b a} \) under \( (b \leftrightarrow a, j \leftrightarrow n, k \leftrightarrow -k) \). Similarly, the \( U i\partial_t U^{-1} \) term \( \sim i\bar{\rho} \Xi_{-k}^{j n; a b} \Xi_{-k}^{j n; a b} \) yields, up to a total derivative,

\[
i\bar{\rho} \sum_k \sum_{n,j} \nu_j \Xi_{-k}^{j n; a b} \approx i\bar{\rho} \sum_k \sum_{n>j} \nu_j \Xi_{-k}^{j n; a b},
\]

(25)

where \( \hat{\Xi} \equiv \partial_t \Xi \). They combine to constitute the one-body part of the effective Lagrangian:

\[
L_{\Xi}^{1b} = \bar{\rho} \sum_{a,b} \sum_{n>j} \sum_k \left( \xi_{k}^{j n; a b} \right)^\dagger \left\{ i\partial_t - (\epsilon_j^a - \epsilon_n^b) \right\} \xi_{k}^{j n; a b},
\]

(26)

where we have rescaled \( \Xi_{k}^{j n; a b} = N_{j n}^{a b} \xi_{k}^{j n; a b} \) with \( N_{j n}^{a b} \equiv \sqrt{\nu_j^a - \nu_n^b} \). Clearly the field \( \xi_{k}^{j n; a b} \) describes an interlevel \( n \leftrightarrow j \) excitation of energy.
Fig. 1. Crossing relation for $R_{-p}^{jk} R_{p}^{mn}$. 

$\epsilon^n_b - \epsilon^a_j$. Note that $0 < N_{n,j}^{ba} \leq 1$ for each open $(n, b) \leftarrow (j, a)$ channel of excitation; $N_{n,j}^{ba} = 0$ for inactive channels so that only open channels appear in Eq. (26).

To handle the Coulomb interaction one needs the knowledge of the static structure functions $\langle \hat{R}_{ij}^{p} \hat{R}_{k\ell}^{p} \rangle$, to be examined in the next section.

4. Direct/exchange duality of the Coulomb interaction

In the $y_0$ basis the plane wave $e^{-i\mathbf{p} \cdot \mathbf{r}}$ is a unitary matrix, $(e^{-i\mathbf{p} \cdot \mathbf{r}}) = e^{i\mathbf{p} \cdot \mathbf{r}}$, with elements

$$\langle y_0 | e^{-i\mathbf{p} \cdot \mathbf{r}} | y'_0 \rangle = \delta(y_0 - y'_0 + \ell^2 p_z) e^{-i\frac{\ell^2 p_z}{2}(y_0+y'_0)}, \quad (27)$$

They obey the completeness relation

$$\sum_p \langle y'_0 | e^{i\mathbf{p} \cdot \mathbf{r}} | y_0 \rangle \langle z_0 | e^{-i\mathbf{p} \cdot \mathbf{r}} | z'_0 \rangle = \bar{\rho} \delta_{y_0,z_0} \delta_{y'_0,z'_0}, \quad (28)$$

as verified directly, where $\delta_{y_0,z_0} = \delta(y_0 - z_0)$ and $\bar{\rho} = 1/(2\pi \ell^2)$. This relation allows one to invert the charge operators $R_{-p}^{mn} = \sum_{y_0,y'_0} \psi^{m\dagger}(y_0) \langle y_0 | e^{i\mathbf{p} \cdot \mathbf{r}} | y'_0 \rangle \psi^n(y'_0)$ for the field products $\psi^{m\dagger} \psi^n$,

$$\bar{\rho} \psi^{m\dagger}(y_0) \psi^n(y'_0) = \sum_p \langle y'_0 | e^{i\mathbf{p} \cdot \mathbf{r}} | y_0 \rangle R_{-p}^{mn}. \quad (29)$$

This inversion formula has long been known.\textsuperscript{26,27}

Consider now the normal-ordered product $:R_{-p}^{jk} R_{p}^{mn}: \sim \psi^{m\dagger} \psi^{j\dagger} \psi^k \psi^n$, and let $\psi^{j\dagger}$ be paired with $\psi^n$ and $\psi^{m\dagger}$ with $\psi^k$, using Eq. (29). A little algebra then yields the crossing formula

$$:R_{-p}^{jk} R_{p}^{mn} := -\frac{1}{\bar{\rho}} \sum_k e^{i\ell^2 p \times k} :R_{-k}^{mk} R_{k}^{jn}: \quad (30)$$

See Fig. 1. For the regular product $R_{-p} R_{p}$ the crossing relation is somewhat complicated:

$$R_{-p}^{jk} R_{p}^{mn} = -\frac{1}{\bar{\rho}} \sum_k e^{i\ell^2 p \times k} R_{-k}^{mk} R_{k}^{jn} + \delta^{km} R_{0}^{jn} + \bar{\rho} \delta_{p,0} \delta^{kj} R_{0}^{mn}. \quad (31)$$

\textsuperscript{b} Note that $\int dy_0 \langle y_0 | e^{i\mathbf{p} \cdot \mathbf{r}} | y_0 \rangle = \bar{\rho} \delta_{p,0}$ and $\sum_k e^{i\ell^2 p \times k} = (\bar{\rho})^2 \delta_{p,0}$.
The Coulomb interaction $V$ in Eq. (16) has direct interaction and also exchange interaction at the quantum level. The crossing relation (30) allows one to rewrite $V$ in the form of exchange interaction,

$$V = -\frac{1}{2\bar{\rho}} \sum_{k} W_{jk}^{mn;ab} : R_{\beta\alpha,k}^{mk;ba} R_{\alpha\beta,k}^{jm;ab} :,$$

$$W_{jk}^{nm;ab} = \sum_{p} v_{p}^{2} \gamma_{p}^{jk,a} \gamma_{p}^{mn;b} e^{i\ell^{2}p \times k}.$$  \hspace{1cm} (32)

This makes manifest, on the operator level, the direct/exchange duality of the 2d interaction at the quantum level. The crossing relation (30) allows one to rewrite $V$ in the form of exchange interaction.

In view of Eq. (30) the normal-ordered static structure factors obey the relation

$$\langle : R_{-p}^{jk} R_{p}^{mn} : \rangle = -\frac{1}{\bar{\rho}} \sum_{k} e^{i\ell^{2}p \times k} \langle : R_{-k}^{mk} R_{k}^{jn} : \rangle.$$  \hspace{1cm} (33)

For a general uniform many-body state $\langle \text{Gr} \rangle$, the structure factors $\langle : R_{-p}^{jk} R_{p}^{mn} : \rangle$ are nonzero only for combinations $\delta^{jk} \delta^{mn}$ or $\delta^{mk} \delta^{jn}$ of labels. There are thus three cases to consider, (i) $\delta^{jk} \delta^{mn}$ and $j \neq m$, (ii) $\delta^{mk} \delta^{jn}$ and $j \neq m$, and (iii) $j = k = m = n$. For case (i), $\langle : R_{-p}^{jk} R_{p}^{mn} : \rangle$ equals $\langle \hat{R}_{-p}^{jk} \hat{R}_{p}^{mn} \rangle = \langle \hat{R}_{-p}^{jk} \hat{R}_{p}^{mn} \rangle$, i.e., proportional to $\delta_{p,0}$, where $\delta_{p,0} = \int d^{2}x$. For case (ii), Eq. (33) implies that $\langle : R_{-p}^{jm} R_{p}^{mn} : \rangle = -\bar{\rho} \nu_{m} \nu_{n} \delta_{p,0}$, i.e., a constant independent of $p$.

In case (iii) one encounters the static structure factor projectected to the $n$th level, defined as $(R_{-p}^{jm} R_{p}^{mn}) = \langle \hat{R}_{-p}^{jm} \hat{R}_{p}^{mn} \rangle + \delta_{p,0} \bar{\rho} \nu_{m} \hat{s}_{n}(p)$ (for fixed $n$), or

$$\langle : R_{-p}^{mn} R_{p}^{mn} : \rangle = \delta_{p,0} \bar{\rho} \nu_{n} \delta_{p,0} + \hat{s}_{n}(p) - 1;$$  \hspace{1cm} (34)

where $\bar{\rho} \delta_{p,0} = \bar{\rho} \int d^{2}x$ stands for the total number of electrons per filled level. (Normally it is $\gamma_{p}^{2} \hat{s}_{n}(p)$ that is defined as the projected structure factor.) Possible $p$ dependence $\hat{s}_{n}(p)$ comes from nontrivial correlations within a partially filled level $n$, and $\hat{s}_{n}(p) \to 0$ as $\nu_{n} \to 1$, i.e., for a filled level. Let us isolate a constant piece, $\hat{s}_{n}(p) = \hat{s}_{n}(\infty) + \delta \hat{s}_{n}(p)$, so that $\delta \hat{s}_{n}(p)$ has a Fourier transform. Substituting Eq. (34) into Eq. (33) then implies that

$$\hat{s}_{n}(\infty) = 1 - \nu_{n}, \quad \delta \hat{s}_{n}(p) = -(1/\bar{\rho}) \sum_{k} \delta \hat{s}_{n}(k) e^{i\ell^{2}p \times k}.$$  \hspace{1cm} (35)

Actually, $\hat{s}_{n}(\infty) = 1 - \nu_{n}$ is consistent with the HF approximation, which leads to $\hat{s}_{n}(p) = 1 - \nu_{n}$. Thus $\delta \hat{s}_{n}(p)$ stands for a possible deviation from the HF treatment, and such a deviation is present in the exact result. Indeed, for Laughlin’s wave function$^{29}$ for the FQH states with $\nu = 1/3, 1/5, \cdots$, e.g., one knows quite generally$^{30}$ that $\hat{s}_{n=0}(p) \to 0$ as $p \to 0$ while $\hat{s}_{0}(p) \to 1 - \nu$ for large $|p|$. Thus $\delta \hat{s}_{n}(p)$ arises for small $|p|$. It will be enlightening here to understand the HF-approximation result in the following way: Suppose that the $n$th level simply consists of filled modes $\{|y_{n}\}\text{fill}$ of fraction $\nu_{n}$ and empty modes $\{|y_{0}\}\text{emp}$ of fraction $1 - \nu_{n}$ in $y_{0}$ space. For such a configuration the structure factor associated with intralevel transitions $\{|y_{0}\}\text{fill} \to \{|y_{0}\}\text{emp} \to \{|y_{0}\}\text{fill}$ is calculated in essentially the same way as in case
under integrals over momenta $\langle U \rangle_{\text{rotated charge}}$, $(\mathbf{R})$

Associated with these are the delta-function pieces $\propto \delta \delta_0$. The crossing relation (30) is also promoted to the dressed form, $\langle \mathbf{R}_{-p} \rangle_{\text{dressed form}}$.

Let us now consider the Coulombic corrections $\langle V^U \rangle$ to $L_\Xi$ in Eq. (19), $V^U \equiv U V U^{-1}$ is given by Eq. (11) or Eq. (32) with each charge $R_p^k$ replaced by the rotated charge $\langle R_{-p}^k \rangle_U \equiv e^{-\Xi^U} R_p^k e^{-\Xi^U}$; the normal-ordered nature of the products is thereby left intact. Thus, e.g.,

$$\langle V^U \rangle = \frac{1}{2} \sum_{p} \gamma_p^2 g_{-p} \langle R_{-p}^k \rangle_U \langle R_{-p}^k \rangle_U$$

The crossing relation (34) is also promoted to the dressed form,

$$\langle R_{-p}^k \rangle_U \langle R_{-p}^k \rangle_U = \frac{1}{2} \sum_{p} e^{i \xi \mathbf{p} \times \mathbf{k}} \langle R_{-p}^k \rangle_U \langle R_{-p}^k \rangle_U$$

The $O(\Xi^r)$ piece of $\langle R_{-p}^k \rangle_U$ contains an operator of the form $R_{-\mathbf{p} - \mathbf{k}_1 - \cdots - \mathbf{k}_r}$ under integrals over momenta $\mathbf{k}_i$ of $r$ powers of $\Xi_{\mathbf{k}_i}$. Accordingly, the $O(\Xi^{r+s})$ terms of the structure factors $\langle R_{-p}^k \rangle_U \langle R_{-p}^k \rangle_U$ are built, under integrals over $(\mathbf{k}_1, \cdots, \mathbf{k}_{r+s})$, on factors of the form

$$\langle \mathbf{R}_{-p - \mathbf{k}_1 - \cdots - \mathbf{k}_r + \mathbf{k}_{r+1} + \cdots + \mathbf{k}_{r+s}} \rangle$$

Associated with these are the delta-function pieces $\propto \delta_{\mathbf{p} + \mathbf{k}_1 + \cdots + \mathbf{k}_r, 0}$. The remaining pieces are not “constants” any more and now consist of products of exponentials (sines and cosines) in $\mathbf{p}$, such as $e^{i \xi \mathbf{p} \times \mathbf{k}_1}$ and $e^{i \xi \mathbf{k}_2 \times \mathbf{p + k}_1}$, as seen from Eq. (22); that is, they are periodic functions in $\mathbf{p}$ and their Fourier images ($\mathbf{p} \rightarrow \mathbf{k}$) are a variety of monochromatic spectra, i.e., delta functions in dual variable $\mathbf{k}$. Thus the (nonsingular) oscillating pieces are again expressed in terms of the (singular) delta-function pieces in the dual expression. The relevant singular pieces

\[c\] See, e.g., K. Asano and T. Ando, in Ref. [8]
\( \propto \delta_{p+k_1+k_2+\ldots+k_m,0} \) are uniquely summarized by the expectation value \( \langle (R_p)_{\ell}^j \rangle \). This means that \( \langle (R_p)_{\ell}^j \rangle (p_{mn}^\ell) \) is calculable via \( \langle (R_p)_{\ell}^j \rangle \), i.e., Eq. (37) is also promoted to rotated charges,

\[
\langle (R_p)_{\ell}^j \rangle (p_{mn}^\ell) = Z^j_{p} Z_{p}^{-1} e^{i\ell^2 p \cdot k} Z_{p}^{mn} k \]

(42)

with \( Z^{jk}_{p} \equiv \langle (R_p)_{\ell}^j \rangle (p_{mn}^\ell) \). One can also write

\[
\langle V^{\ell j} \rangle = \frac{1}{2} \sum_p \rho_p g_{p}^{jk} g_{-p}^{mn} \langle (R_p)_{\ell}^j \rangle (p_{mn}^\ell) \langle (R_p)_{\ell}^j \rangle (p_{mn}^\ell) - \frac{1}{2} \sum_k W^{jk;mn;ab}_{k} \langle (R_{-k})_{\ell}^j \rangle (p_{mn}^\ell) \langle (R_{-k})_{\ell}^j \rangle (p_{mn}^\ell).
\]

(43)

Equation (42) is one of the key results of the present paper, and allows one to evaluate the structure factors \( \langle V^{\ell j} \rangle \) by a far simpler calculation handling only expectation values \( \langle R^{\ell j} \rangle \). Normally great labor is needed to calculate such structure factors of rotated charges \( R^{\ell j} \). They, when expanded in powers of \( \Xi \), proliferate rapidly in number and variety of terms, but many of them turn out to vanish on substituting the zeroth-order factors \( \langle RR \rangle \). Note that in Eq. (42) integration over \( \{k\} \) of \( \{\Xi k\} \), normally made toward the end of calculation, is carried out first in evaluating the expectation values \( \langle R^{\ell j} \rangle \). Our formula (42) or (43) thus neatly rearranges steps of calculations and achieves a most efficient approach to the goal.

5. Coulombic corrections

The calculation of many-body corrections \( \langle V^{\ell j} \rangle \) is greatly simplified by use of Eq. (43). See Appendix A for details. The \( O(\Xi) \) term vanishes, and the \( O(\Xi^2) \) term is

\[
\langle V^{\ell j} \rangle = \bar{\rho} \sum_{n>j} \sum_{b,a} \sum_{k} V_{k}^{nj},
\]

\[
V_{k}^{nj} = D_{k}^{nj;ba} (\xi_{k}^{nj;bb}) \xi_{k}^{na;ab} + E_{k}^{nj;ba} (\xi_{k}^{nj;ba}) (\xi_{k}^{nj;bb}),
\]

(44)

with

\[
E_{k}^{nj;ba} = -\sum_{p} \rho_p g_{p}^{nr;bb} \left( (v_p^{\ell^2} |g_p^{rr;bb}| - v_p^{\ell^2} |g_p^{rr;ab}|^2) \right),
\]

\[
D_{k}^{nj;ba} = N_{n^{aj};n^{bj}}^a (\hat{v}_k^{\ell^2} |g_k^{aj;bb}| |g_k^{aj;ab}|) \propto O(|k|^{n-j}),
\]

(45)

where \( c(p,k) \equiv \cos(\ell^2 p \cdot k) \). The first term \( \propto D^{nj;ba}_{k} \) comes from the direct interaction and is short-ranged as it vanishes for \( k \to 0 \). The exchange correction \( \propto E^{nj;ba}_{k} \) is composed of the self-energy corrections due to the filled levels and an attraction between the excited electron and hole pair. The effective Lagrangian is now given by \( L_{\Xi} = L_{\Xi}^{k} - \langle V^{\ell j} \rangle \).

Possible contributions from nontrivial intralevel correlations \( \delta_{p}^{2;j;}(p) \) are readily extracted from \( \langle V^{\ell j} \rangle \) in Eq. (43). In particular, when the initial level \( (j,a) \) is partially
filled, one can simply retain terms involving \( \langle :R^{j;j} R^{j;j}: \rangle \propto \bar{\rho}_j \delta \hat{s}_j \). The result is an addition to \( E_{nj}^{ba} \) of the form

\[
\delta E_{nj} = \sum_p v_p \gamma_p^2 \left[ \sigma^{ba} \delta \hat{s}_j(p + k) \left| g_{nj;p}^{n;j:a} \right|^2 + \delta \hat{s}_j(p) \{ e(p, k) g_{nj;p}^{n;j:b} g_{nj;p}^{n;j:a} - \left| g_{nj;p}^{n;j:a} \right|^2 \} \right].
\]

Equations (44) - (46) agree with and partly generalize some earlier calculations.\(^\text{10−12,31}\)

So far we have retained valley labels for practical applications as well as to show the generality of our method of calculation. In general, when one considers a specific \( n \leftarrow j \) interlevel excitation one also has to take into account all such related excitation channels that are strongly mixed with it via the short-ranged direct interaction \( \propto D_{nj}^{ba;k} \) and eventually go through a matrix diagonalization, as done in the literature.\(^\text{10,11}\)

Instead of handling such a general case, we from now on focus on cyclotron resonance, i.e., optical interlevel excitations at zero momentum transfer \( k \to 0 \), where no mixing takes place, with the selection rule\(^\text{9}\) \( \Delta |n| = \pm 1 \), i.e., (i) \( n + 1 \leftarrow \pm n \) for \( n = 0, 1, 2, \cdots \). The \( n \leftarrow j \) resonance energy is then simply written as

\[
\epsilon_{n\leftarrow j}^{\text{exc}} = \epsilon_n - \epsilon_j + \Delta \epsilon_{n;j},
\]

\[
\Delta \epsilon_{n;j} = - \sum_p v_p \gamma_p^2 \left[ \sum_r \nu_r \left\{ \left| g_{np;r}^{n;j:a} \right|^2 - \left| g_{np;r}^{n;j:a} \right|^2 \right\} + (\nu_j - \nu_n) g_{nj;p}^{n;j:a} \right]
\]

for each \((\text{valley, spin})\) channel. In particular, for the \( 1 \leftarrow 0 \) resonance

\[
\Delta \epsilon_{1;0} = - \sum_p v_p \gamma_p^2 \left[ \sum_{r \leq -1} \nu_r \left\{ \left| g_{np;r}^{1;0} \right|^2 - \left| g_{np;r}^{1;0} \right|^2 \right\} + \left\{ \nu_0 - \delta \hat{s}_0(p) \right\} \left( \left| g_{np;0}^{0;0} \right|^2 - \left| g_{np;0}^{0;0} \right|^2 + g_{np;0}^{0;1} g_{np;0}^{0;1} \right) \right],
\]

where the contribution of \( \delta E_{1;0}^k \) is also included. For conventional 2d electrons, one only has the last term \( \propto \{ \nu_0 - \delta \hat{s}_0 \} \), though it actually vanishes in accordance with Kohn’s theorem.\(^\text{7}\) It happens to vanish also for (\( \Delta \epsilon_{1;0} \) of) graphene, since \( \left| g_{np;0}^{0;0} \right|^2 - \left| g_{np;0}^{0;0} \right|^2 + g_{np;0}^{0;1} g_{np;0}^{0;1} \to 0 \), as one can verify using Eq. (14). Thus \( \Delta \epsilon_{1;0} \) consists solely of the self-energy correction due to the filled valence band and is actually logarithmically divergent.

Cyclotron resonance in graphene and bilayer graphene was studied earlier.\(^\text{10−12}\)

Here we briefly review it and present some basic formulas that clarify the structure of the self-energy corrections. Let us first note the completeness relation\(^\text{24}\)

\[
\sum_{k=-\infty}^{\infty} \left| g_{np;0}^k \right|^2 = e^2 4p^2 = 1/\gamma_p^2
\]
that, in general, holds for the eigenmodes of the one-body Hamiltonian. The infinite
sum in the self-energy corrections to level \( n \) is thereby rewritten as

\[
\gamma_p^2 \sum_{k \leq -1} |g_{n,k}^p|^2 = \frac{1}{2} - \frac{1}{2} s_n F_{|n|}(z) - \frac{1}{2} \gamma_p^2 |g_{n,0}^p|^2, \tag{50}
\]

where \( z = \frac{1}{2} e^2 \mathbf{p}^2, s_n = \text{sign}[n] \to \pm 1 \) and

\[
F_n(z) \equiv \gamma_p^2 \sum_{k = 1}^{\infty} \{ |g_{n,k}^p|^2 - |g_{n,-k}^p|^2 \} \quad (n > 0),
\]

\[
\mu \to 0 \quad e^{-z} \sum_{k = 1}^{\infty} \frac{k}{e^k - 1} \frac{n!}{k!} e^{-z} (0)^k (1)^k L_{n-1}^k(z) L_{n-1}^k(z),
\]

\[
F_0(z) \to 0. \tag{52}
\]

The resonance energy corrections \( \Delta \epsilon^{n,j} \) then reveal the underlying electron-hole
\((eh)\) symmetry,

\[
\Delta \epsilon^{n,j} = \sum_p \nu_p \left[ \frac{1}{2} \{ s_n F_{|n|}(z) - s_j F_{|j|}(z) \} - \gamma_p^2 G_{n,j}^p \right],
\]

\[
G_{n,j}^p \equiv \sum_k \nu[k] (|g_{n,k}^p|^2 - |g_{n,k}^j|^2) + (\nu_j - \nu_n) g_{n,j}^{n,j} g_{n,j}^{n,j},
\]

\[
\nu[k] \equiv \nu_k \theta(k \geq 1) - (1 - \nu_k) \theta(k \geq -1) + (\nu_0 - \frac{1}{2}) \delta^{k0}, \tag{53}
\]

where \( \theta(k \geq 1) = 1 \) for \( k \geq 1 \) and \( \theta(k \geq -1) = 0 \) otherwise; analogously for \( \theta(k \leq -1) \). Here \( G_{n,j}^p \) summarize corrections due to a finite number of electron or hole levels
around the \( n = 0 \) level, as seen from the definition of the \( eh\)-symmetric filling
factor \( \nu[k] \). The filled valence band also gives rise to \( eh\)-symmetric corrections
\( \propto s_n F_{|n|}(z) \). Formulas \((49) - (53)\) serve to provide compact analytic expressions
for some numerically-handled portions of an earlier analysis.\(^{12}\) They are equally
generalized to the case of few-layer graphene.

For simplicity, let us set tiny valley breaking \( \mu \to 0 \) below; actually, \( \mu \neq 0 \)
requires separate renormalization.\(^{12}\) One then finds that, for \( n \geq 1, F_n(z) > 0, \)
\( F_n(0) = 1 \) and \( F_n(z) \approx \sqrt{n/2}/(\ell |p|) \) for \( p \to \infty \), which reveals that the divergence
in \( \Delta \epsilon^{n,j} \) is proportional to the large-\( p \) behavior of \( s_n F_{|n|}(z) - s_j F_{|j|}(z) \), i.e., of the
form \( \propto (s_n \sqrt{|n|} - s_j \sqrt{|j|}) \log(\ell / \Lambda) \), with momentum cutoff \( \Lambda \).

This implies that the divergences in all \( \Delta \epsilon^{n,j} \) are removed via renormalization\(^{35}\)
of the velocity

\[
v = Z_v v^{\text{ren}} = v^{\text{ren}} + \delta v, \tag{54}
\]

with \( \delta v = (Z_v - 1) v^{\text{ren}} \). Indeed, setting \( \epsilon_n = \epsilon_n^{\text{ren}} + \delta \epsilon_n \) with renormalized energy
\( \epsilon_n^{\text{ren}} = s_n \sqrt{2 |n|} v^{\text{ren}} / \ell \) and the counterterm \( \delta \epsilon_n \propto s_n \sqrt{|n|} \delta v \), one can rewrite the excitation energy as

\[
\epsilon_{\text{exc}}^{n,j} = \epsilon_n^{\text{ren}} - \epsilon_j^{\text{ren}} + (\Delta \epsilon^{n,j})^{\text{ren}}. \tag{55}
\]
Fig. 2. Momentum profiles of (rescaled) many-body corrections $\Delta \epsilon^{n,j}/(s_n \sqrt{|n|} - s_j \sqrt{|j|})$ for some typical channels.

All the corrections $(\Delta \epsilon^{n,j})^{\text{ren}} \equiv \delta_{ct} \epsilon_n - \delta_{ct} \epsilon_j + \Delta \epsilon^{n,j}$ are now clearly made finite by a single choice of $\delta v$.

Let us choose $\delta v$ so that $(\Delta \epsilon^{1,0})^{\text{ren}} = 0$, or

$$\delta v = -\frac{1}{\sqrt{2}} \Delta \epsilon^{1,0} = -\frac{1}{8} \left( \alpha / \epsilon_b \right) \left[ \log(\Lambda^2 \ell^2) + \text{const.} \right]; \quad (56)$$

this defines $v^{\text{ren}}$ via $\epsilon^{1-0}_{\text{exc}} = \sqrt{2} v^{\text{ren}} / \ell$ at each value of $B$. The renormalized corrections $(\Delta \epsilon^{n,j})^{\text{ren}}$ are thereby given by $\Delta \epsilon^{n,j}$ in Eq. (53) with $F_{|m|} (z)$ and $F_{|j|} (z)$ replaced by the renormalized counterparts

$$F^{\text{ren}}_{|m|} (z) = F_{|m|} (z) - \sqrt{|m|} \left\{ F_1 (z) - \gamma^2 p \sigma^1_{-p} g^0_{-p} g^0_{p} \right\}. \quad (57)$$

A key effect of renormalization is the fact that, as implied by $v = v^{\text{ren}}|B + \delta v|_B$, the renormalized velocity runs with the magnetic field,

$$v^{\text{ren}}|B = v^{\text{ren}}|_{B_0} - \frac{\alpha}{8 \epsilon_b} \log(B/B_0). \quad (58)$$

Figure 2 shows momentum profiles of some rescaled corrections $\Delta \epsilon^{n,j}/(s_n \sqrt{|n|} - s_j \sqrt{|j|})$ associated with levels $n = 0, \pm 1, \pm 2$. Changes in profile relative to the $1 \leftarrow 0$ profile represent the genuine corrections after renormalization. The general tendency is that such renormalized corrections are negative for intraband resonance and positive for interband resonance. Actually this is consistent with an experiment which observed an appreciable deviation of the ratio of $\epsilon^{1-0}_{\text{exc}}$ and $\epsilon^{2-1}_{\text{exc}}$ from the tree-level value $1 : (1 + \sqrt{2})$. Running of the renormalized velocity has also been observed in experiment.

6. Intralevel excitations

In this section we examine some examples of collective excitations within a Landau level. The first example is a monolayer prototype with ordinary electrons of spin up and down occupying the $n = 0$ Landau level at half filling $\nu = 1$. Let us form the spin doublet $\psi = (\psi_0^0, \psi_0^1)^t$ and denote the spin charge as $\rho^\mu = \psi^\dagger \left( \sigma^\mu - 1 \right) \psi$ with $\sigma^0 = 1$ and Pauli matrices $\{ \sigma^a \}$; we denote the Fourier image as $\rho^\mu_{-p} = \gamma_p R^\mu_{-p}$ and
\[ R_{-p}^\mu = \int dy_0 \psi_1^\dagger \sigma^\mu e^{ip \cdot r} \psi . \] The ground state \(|Gr\rangle\) of such a system is characterized by density \(\langle \rho^0 \rangle = \frac{1}{2} \bar{\rho}\) and spin \(\langle \rho^3 \rangle = \frac{1}{2} \bar{\rho} n^3\) with \(n \equiv (n^1, n^2, n^3)\) pointing in a definite direction in spin space; \(n \cdot n = 1\).

The Zeeman energy \(-\mu_z \bar{\rho}^3\) naturally favors \(n^3 = 1\) whereas, in the \(\mu_z \to 0\) limit, spin \(n\) may take any direction, giving rise to spontaneous spin coherence. \(^{32}\) Such a ground state supports collective spin excitations via Coulomb interactions. Let us now study their spectra in some detail.

As before, we describe such spin waves as a local spin rotation \(U^{-1} |Gr\rangle\) of \(|Gr\rangle\) with \(U = e^{i \bar{\rho} R}\), where \(\bar{\rho} R\) now stands for \(\sum_{p} \sum_{a=1}^{3} \Xi^a_p R^a_{-p}\). We thus consider \(L_\Xi\) in Eq. (19) and now substitute for \(H^\text{tot}\) the Coulomb interaction acting within the \(n = 0\) level

\[ V_0 = 2 \sum_{\ell} v_{\ell} \gamma^2_{\ell} : R^0_{-\ell} R^0_{\ell} : . \] (59)

For \(R^\alpha_{-p}\) the crossing relation \(^{33}\) takes the form

\[ : R^\alpha_{-p} R^{\beta}_{p} : = -\frac{1}{\bar{\rho}} \sum_{k} \epsilon_{k} \ell^{\alpha} \epsilon_{k}^{\beta} \cdot k : R_{-k}^\mu R_{k}^\nu : T^{\mu \nu | \alpha \beta}, \] (60)

where \(T^{\mu \nu | \alpha \beta} = \frac{1}{4} \text{tr}(\sigma^\alpha \sigma^\mu \sigma^\beta \sigma^\nu)\) and Greek letters run over \((0, 1, 2, 3)\). \(V_0\) is thereby cast into the dual form

\[ V_0 = - (1/\bar{\rho}) \sum_{k} v_{k}^{\text{dual}} : R_{-k}^\alpha R_{k}^\alpha + R_{-k}^0 R_{k}^0 : \] (61)

with a sum over \(a \in (1, 2, 3)\) and

\[ v_{k}^{\text{dual}} = \sum_{p} v_{p} \gamma^2_{p} \epsilon_{p}^{\ell} \epsilon_{k}^{\beta} \cdot k = V_c \sqrt{\frac{\ell}{\bar{\rho}}} \int_0^1 (z) e^{-z}, \]

\[ = V_c \sqrt{\frac{\ell}{\bar{\rho}}} \left(1 - \frac{1}{4} \ell^2 k^2 + \frac{1}{24} (\ell | k|^4 + \ldots)\right), \] (62)

where \(z = \frac{1}{4} \ell^2 k^2\) and \(V_c \equiv \alpha/(\ell | k|)\).

Note now that the structure factors \(\langle R^\alpha_{-p} R^\beta_{p} \rangle\) consist of a \(\delta_{p,0}\) piece and a constant piece. This is clear for \(n = (0, 0, 1)\), in which case only \(\langle R_0^0 R^0_{-0} \rangle\) and \(\langle R^3_{-0} R^3_{0} \rangle\) are nonzero. For general \(n\), the associated structure factors are constructed from this \(\langle R^3_{-0} R^3_{0} \rangle\) by a global rotation in spin space, and naturally share the structure \(\propto \delta_{p,0} + \text{constant}\). As a result, analogues of Eqs. (37) and (42) again hold for \(R^\alpha_{-p}\), and one can calculate \(\langle R^\alpha_{-p} (R^\beta_{-p})^\dagger \rangle\) via the expectation values \(\langle (R^\alpha_{p})^\dagger \rangle\) alone. The latter read, to \(O(\Xi^2)\),

\[ \langle (R^\alpha_{-p})^\dagger \rangle = \frac{1}{2} \bar{\rho} \left\{ n^a \delta_{p,0} + \Xi_{-p}^{ab} n^b + \frac{1}{2} \sum_k c_{k,p} \Xi_{-k}^{ac} \Xi_{-p-k}^{cb} n^b \right\}, \]

\[ \langle (R^0_{-p})^\dagger \rangle = \frac{1}{2} \bar{\rho} \left\{ n^0 \delta_{p,0} + \frac{1}{2} \sum_k s_{k,p} \Xi_{-k}^{ab} \Xi_{-p-k}^{bc} n^b \right\}, \] (63)

where \(c_{k,p} \equiv \cos(\frac{1}{2} \ell^2 k \times p)\) and \(s_{k,p} \equiv \sin(\frac{1}{2} \ell^2 k \times p)\); \(\Xi_{-p}^{ab} \equiv \epsilon_{abc} \Xi_{-p}^{bc}\) and \(\epsilon_{abc}\) is the totally-antisymmetric tensor with \(\epsilon^{123} = 1\). Note that for \(p \to 0\) the
Direct-exchange duality of the Coulomb interaction . . .

\[ \sum_{k} s_{k,p} \Xi_{k}^{a} - \sum_{k-p} n^{b} \] term in \( \langle (R_{p}^{0})^{\dagger} d \rangle \) is reduced to a surface integral equal to \( 4\pi \ell^{2} Q_{\text{top}} \), where \( Q_{\text{top}} = (8\pi)^{-1} \int d^{2}x \epsilon_{ij} e^{abc} \Xi^{a}(\partial_{i} \Xi^{b})(\partial_{j} \Xi^{c}) \) is the topological charge \( 32 \) carried by the spin wave \( \Xi^{a} \).

Noting Eqs. (59) and (61), one can calculate \( V_{\text{eff}} \equiv \langle \hat{V} \hat{V}^{-1} \rangle \) via

\[ V_{\text{eff}} = \frac{1}{\hat{\rho}} \sum_{p} \left[ (2\hat{\rho} v_{p} \gamma_{p}^{2} - v_{p}^{\text{dual}}) Z_{p}^{00} - v_{p}^{\text{dual}} Z_{p}^{aa} \right], \quad (64) \]

with \( Z_{p}^{\mu\nu} = \langle (R_{p}^{\mu})^{\dagger} (R_{p}^{\nu}) \rangle \). The result to \( O(\Xi^{2}) \) is

\[ V_{\text{eff}} = -\frac{1}{2} v_{p}^{\text{dual}} \left( \int d^{2}x \hat{\rho} + Q_{\text{top}} \right) + \frac{1}{4} \hat{\rho} \sum_{p} (v_{p}^{\text{dual}} - v_{p}^{\text{dual}}) \langle \tilde{\Xi}^{ac} n^{c} \rangle \langle \tilde{\Xi}^{ab} n^{b} \rangle. \quad (65) \]

On the other hand, the \( U i \partial_{t} \hat{U}^{-1} \) term leads to \( L_{t} = -\frac{1}{\hat{\rho}} \sum_{p} e^{abc} n^{a} \Xi_{-p}^{b} \Xi_{-p}^{c} \), and the Zeeman energy \( H_{Z} = -\mu_{Z} \langle (R_{p}^{0})^{\dagger} d \rangle = -\frac{1}{2} \hat{\rho} \mu_{Z} \int d^{2}x n^{3} + \cdots \) selects the spin direction \( n^{3} = 1 \). One can now write down the effective Lagrangian \( L_{\text{eff}} \sim L_{t} - V_{\text{eff}} - H_{Z} \) for \( \Xi^{a} \),

\[ L_{\text{eff}} = \frac{1}{4} \hat{\rho} \sum_{p} \phi_{p}^{\dagger} \left[ i \partial_{t} - (\epsilon_{p} + \mu_{Z}) \right] \phi_{p}, \quad (66) \]

where \( \phi_{p} = \Xi_{p}^{1} + i \Xi_{p}^{2} \) and \( \phi_{p}^{\dagger} = \Xi_{p}^{1} - i \Xi_{p}^{2} \). The excitation spectrum consists of the Zeeman gap \( \mu_{Z} \) and the Coulombic energy

\[ \epsilon_{p} = v_{p}^{\text{dual}} - v_{p}^{\text{dual}} = \sqrt{\Xi^{2}} V_{r} \left\{ \frac{1}{4} \ell^{2} p^{2} + \cdots \right\}, \quad (67) \]

in agreement with an earlier result\( ^{28} \); this \( \epsilon_{p} \) rapidly rises with increasing \( |p| \) and approaches \( v_{0}^{\text{dual}} = \sqrt{\pi/2} V_{c} \) for \( p \to \infty \). In the limit \( \mu_{Z} \to 0 \), the spin \( n^{c} \) can point in any direction and spin waves have the spectrum \( \epsilon_{p} \).

This Lagrangian \( L_{\text{eff}} \) applies to some other cases as well. (i) Valley pseudospin waves in graphene. Consider, e.g., the \( \nu = 0 \) vacuum state in graphene with degenerate valley \( \langle \delta m \to 0 \rangle \) and frozen spin (via the Zeeman energy). The half-filled \( n = 0 \) level will then support a local valley excitation which is described by this \( L_{\text{eff}} \) with \( \mu_{Z} \to 0 \). (ii) Layer excitations in bilayer systems. Consider a bilayer system of conventional electrons with frozen spin, and replace, in the above analysis, the spin by the layer, with \( \langle \uparrow, \downarrow \rangle \) now reinterpreted as (upper layer, lower layer). Then \( L_{\text{eff}} \) (with \( \mu_{Z} \to 0 \)) describes local layer-pseudospin excitations in the half-filled \( n = 0 \) level (at \( \nu = 1 \)) of such a bilayer system\( ^{32} \) (in the limit of zero separation).

Let us here look into the latter case in more detail, especially to see how our present framework streamlines actual calculations. In bilayer systems the difference between intralayer and interlayer Coulomb potentials \( (v_{p} \text{ and } e^{-d|p|} v_{p} \text{ with interlayer separation } d) \) gives rise to layer \( SU(2) \) breaking, which drives spontaneous layer coherence,\( ^{32} \) with equal population of electrons in both layers. To verify this let us try to improve \( L_{\text{eff}} \) in Eq. (66).

For a bilayer with separation \( d \), the Coulomb interaction takes the form of \( V_{0} \) in Eq. (59) with \( v_{p} R_{-p}^{0} R_{p}^{0} \) replaced by \( v_{p}^{+} R_{-p}^{0} R_{p}^{0} + v_{p}^{-} R_{-p}^{3} R_{p}^{3} \), where \( v_{p}^{+} = \frac{1}{2}(1 \pm \).
$e^{-d|p|} \psi_p$. Thus the addition to $V_0$ is of the form

$$\Delta V = 2 \sum_p \psi_p^{-2} : (R_3^3 R_3^3 - R_3^0 R_3^0) : .$$

(68)

Since $\psi_p^{-2} > 0$, $\Delta V$ favors layer pseudospin $n^3 = 0$, i.e., equal population in both layers. Its dual form is

$$\Delta V = - \frac{2}{\rho} \sum_k \psi_k^{-2} \psi_k^{-dual} : (R_3^3 R_3^3 - R_0^0 R_0^0) : ,$$

(69)

with $\psi_k^{-dual} \equiv \sum_p \psi_p^{-2} e^{i\ell}\psi_k\psi_p^{2}$, or

$$\psi_k^{-dual} = \frac{1}{2} V_c \{ \hat{d} - \frac{1}{2} \sqrt{\frac{\pi}{2}} d^2 - (\frac{1}{2} \hat{d} - \frac{3}{8} \sqrt{\frac{\pi}{2}} d^2) q^2 + \cdots \},$$

(70)

where $\hat{d} = d/\ell$ and $q = \ell|p|$. It is now a simple task to calculate the layer breaking correction $\Delta V_{\text{eff}} = \langle (\Delta V)^{d/4} \rangle$ via the formula

$$\Delta V_{\text{eff}} = \frac{2}{\rho} \sum_p [ - \bar{\rho} \psi_p^{-2} \psi_p^{00} + \psi_p^{-dual} \psi_p^{aa} + (\bar{\rho} \psi_p^{-2} - \psi_p^{-dual}) \psi_p^{33}] .$$

(71)

The result is

$$\Delta V_{\text{eff}} = \frac{1}{4} \rho \sum_p \{ \beta_p^{(2)} \hat{z}^2_p - \hat{z}^2_p - \beta_p^{(3)} \hat{x}^3_p \hat{x}^3_p \},$$

$$\beta_p^{(2)} \equiv 2 (\rho_0 \psi_p^{-2} - \hat{z}^2_p) = V_c \{ \frac{1}{2} \hat{d} - \frac{1}{2} q^2 + \cdots \},$$

$$\beta_p^{(3)} \equiv 2 (\psi_p^{-dual} - \hat{z}^2_p) = - V_c \{ \frac{1}{4} \hat{d} q^2 + \cdots \},$$

$$\hat{z}^2_p = n^1 \hat{z}^1_p - n^2 \hat{z}^3_p,$$

(72)

where only $O(\hat{x}^2)$ terms of our concern are shown.

The full effective Lagrangian is again cast in the form of $L_{\text{eff}}$ in Eq. (66) with the spectrum and field,

$$\epsilon_p^{\text{exc}} = \sqrt{(\epsilon_p + \beta_p^{(2)})(\epsilon_p + \beta_p^{(3)})},$$

$$\phi_p = \alpha_p \hat{z}^2_p + i(1/\alpha_p) \hat{z}^3_p,$$

(73)

where $\alpha_p = [ (\epsilon_p + \beta_p^{(2)}) / (\epsilon_p + \beta_p^{(3)}) ]^{1/4}$. Note, in particular, that the presence of the $O(|p|^0)$ term in $\beta_p^{(2)}$ critically changes the long-wavelength property of the pseudospin wave, i.e., from $\epsilon_p \sim O(p^2)$ to

$$\epsilon_p^{\text{exc}} \approx |p| V_c \sqrt{\frac{\pi}{2}} d (1 - \sqrt{8/\pi} d/\ell)^{1/2} .$$

(74)

These results reproduce and partly generalize those of earlier studies.\textsuperscript{32}
7. Intralevel density excitations

So far we have handled collective excitations described as rotations in orbital space or in spin or valley space. In this section we consider how to treat intralevel density excitations such as those over the FQH states. For definiteness let us consider (single-spin) density fluctuation within the partially-filled nth level (with $0 < \nu_n < 1$), described by a trial state $R^n_{-\mathbf{p}}|\text{Gr}\rangle$. Such a density excitation is not described by a phase rotation alone and requires an amplitude modulation as well. Indeed, for a real phase $(\Xi^{nn}_p)^\dagger = \Xi^{nn}_p$, Eq. (19) fails to work as the resulting $\sum_{\mathbf{p}} \Xi^{nn}_p i \partial_t \Xi^{nn}_p$ term vanishes (up to a total derivative).

Let us recall that, for a trial state of the form $R_{\mathbf{p}}$, with $(\Xi^{nn}_p)^\dagger = \Xi^{nn}_p$, Eq. (19) fails to work as the resulting $\sum_{\mathbf{p}} \Xi^{nn}_p i \partial_t \Xi^{nn}_p$ term vanishes (up to a total derivative).

Let us recall that, for a trial state of the form $R_{\mathbf{p}}$, the original SMA excitation spectrum reads

$$\epsilon_n(\mathbf{p}) = \langle R^n_{\mathbf{p}} [H^{\text{tot}}, R^n_{-\mathbf{p}}]/\langle R^n_{\mathbf{p}} R^n_{-\mathbf{p}} \rangle, \quad (75)$$

where $\langle \cdots \rangle \equiv \langle \text{Gr}| \cdots |\text{Gr} \rangle$. One can thus suppose a Lagrangian of the form

$$L_\Xi^{\text{eff}} \sim \bar{\rho} \nu_n \int d\mathbf{y} \delta(\mathbf{p}) \Re \hat{\Xi} \frac{1}{2} \Xi \hat{R} \left[ H^{\text{tot}}, \Xi \hat{R} \right], \quad (76)$$

with $(\Xi^{nn}_p)^\dagger \neq \Xi^{nn}_p$, i.e., a complex field in real space. Actually this $L_\Xi^{\text{eff}}$ follows if one sets

$$L_\Xi^{\text{eff}} = \langle \hat{\Xi} \hat{R} \hat{\Xi} R - \frac{1}{2} \hat{\Xi} R [H^{\text{tot}}, \hat{\Xi} R] \rangle, \quad (77)$$

with $\Xi R = \sum_{\mathbf{p}} \Xi^{nn}_\mathbf{p} R^n_{\mathbf{p}}$ and $\hat{\Xi} R \equiv \langle \Xi R \rangle = \sum_{\mathbf{p}} \langle \Xi^{nn}_\mathbf{p} \rangle R^n_{\mathbf{p}}$, $\hat{\Xi} \equiv \partial_t \Xi$. We set $\Xi^{nn}_\mathbf{p} \rightarrow (\Xi^{nn}_\mathbf{p})^\dagger$ in $\Xi R$ to extract the $\Xi^\dagger \Xi$ term properly. In contrast, the $\frac{1}{2} \Xi[R, [H^{\text{tot}}, \Xi R]]$ term, which is essentially equivalent to $\Xi R [H^{\text{tot}}, \Xi R]$, is unambiguously cast into $\Xi \Xi^\dagger$ form by setting $\Xi^{nn}_\mathbf{p} \rightarrow (\Xi^{nn}_\mathbf{p})^\dagger$.

Actually Eq. (77) applies to full modes $\Xi^{nn}_\mathbf{p}$. For off-diagonal modes $\Xi^{mn}_\mathbf{p}$, $(\Xi' R) \hat{\Xi} R \rightarrow \frac{1}{2} \Xi R \hat{\Xi} R$, and this $L_\Xi^{\text{eff}}$ agrees with $L_\Xi$ in Eq. (19) [to $O(\Xi^2)$]. This in turn implies that the spectrum $\epsilon_n(\mathbf{p})$ is still calculable through $(\hat{V}^{\text{eff}})$ in Eq. (69) by setting $\Xi R \rightarrow \sum_{\mathbf{p}} \Xi^{nn}_\mathbf{p} R^n_{\mathbf{p}}$ there, as done for $\delta E^{nn}_k$ in Eq. (40). This step simplifies actual calculation. The result is

$$\epsilon_n(\mathbf{p}) = \sum_k \epsilon_k |g_k|^2 \{ 1 - \cos(2k \cdot \mathbf{p}) \} M_{\mathbf{p}, k}, \quad M_{\mathbf{p}, k} = \{ \mathbf{s}_n(\mathbf{p} + k) - \mathbf{s}_n(\mathbf{k}) \} / \mathbf{s}_n(\mathbf{p}). \quad (78)$$

This agrees with and somewhat generalizes an earlier result.

8. Summary and discussion

In this paper we have developed a new efficient algorithm for studying many-body effects on 2d electrons in a magnetic field, and verified its utility by examining inter-Landau-level excitations in graphene and some intra-Landau-level collective excitations in monolayer and bilayer systems. The key observation is the fact that in a magnetic field the Coulomb interaction obeys the crossing relation in Eq. (30),
which relates via the Fourier transform the small- and large-momentum behavior of the direct and exchange interactions. This direct/exchange duality of the Coulomb interaction, when adapted to the SMA-dressed interaction [in Eq. (40)], efficiently rearranges steps of calculations and leads to an SMA algorithm, based on formula (42), that greatly simplifies the calculation of many-body corrections at integer filling. For a partially-filled level further corrections arise via the portion \([\delta \hat{s}_n(p)]\) of the static structure factors that comes from nontrivial intralevel correlations, as we have seen, especially in Sec. 7.

The basic crossing relation in Eq. (30) or (40) takes a natural and simple form for normal-ordered products: rather than regular products \(R_p R_{-p}\). As noted in the course of our discussion, normal ordering is maintained within the algebra of charge operators, e.g., \(\{ V, R_p \} = \{ V, R_{-p} \}\). This suggests that normal-ordered products are the natural entity to handle in the formulation and practice of the SMA.

The usefulness of the present algorithm will be further appreciated when one handles some more complex systems of 2d electrons, such as few-layer graphene in which a host of new many-body effects come into play, and the case of collective excitations over non-uniform ground states. Research in this direction will be reported elsewhere.

Acknowledgements

This work was supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan (Grant No. 17540253).

Appendix A. Coulombic corrections \((V U)\)

In this appendix we outline the calculation of the Coulombic correction \(\langle V U \rangle\) using Eq. (43). Note first that, on integrating factors \(g_{jk}^p g_{mn}^p = g_{jk}^p (g_{pn}^m)^*\) over \(p\) symmetrically, only combinations with \(j - k = n - m\) survive. One then immediately finds that the \(O(\Xi)\) term in \(\langle V U \rangle\) vanishes.

As for the \(O(\Xi^2)\) terms let us begin with the direct interaction: Extracting the \(O(\Xi) \times O(\Xi)\) piece out of the first term in Eq. (43) and selecting the \(j \rightarrow n \rightarrow j\) process gives rise to \(E^{nj;ba}_k\) of Eq. (45). Off-diagonal processes \((j \rightarrow n, n' \rightarrow j')\) contribute to yet higher-order corrections in \(V\). The remaining \(O(\Xi^0) \times O(\Xi^2)\) term involves \(v_{p \rightarrow 0}\) and is removed when the neutralizing background is taken into account.

The exchange interaction has the main structure

\[
I = \sum_{p} g_{jk}^{p;ba} \sum_{mn} \langle \langle R_{jk}^{mn;ba};\ell \rangle \rangle e^{i p \cdot \ell} \left( n - m \right) \sum_{q} \Xi^{q;j;ca} \Xi^{q;j;ca}.
\]

(A.1)

This leads to the selfenergy term in \(E^{nj;ba}_k\) of Eq. (45).
On the other hand, selecting the diagonal process out of the $O(\Xi) \times O(\Xi)$ portion yields

$$I^{(1,1)} \approx \bar{\rho}^2 \sum_{n,j} g_{nn}^{ij;ba} g_{pp}^{nn;bb} (\nu_n - \nu_j)^2 \Xi_{-k}^{nj;ab} \Xi_{k}^{nj;ba} e^{i \ell p \cdot k},$$

(A.2)

which leads to the remaining attraction term in $E_{-k}^{nj;ab}$.

References

1. *The Quantum Hall effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, Berlin, 1987).

2. K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, *Nature* (London) **438**, 197 (2005).

3. Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, *Nature* (London) **438**, 201 (2005).

4. A. K. Geim and K. S. Novoselov, *Nat. Mater.* **6**, 183 (2007).

5. D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).

6. R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

7. W. Kohn, *Phys. Rev.* **123**, 1242 (1961).

8. For many-body corrections to cyclotron resonance of conventional 2d electrons, see, A. H. MacDonald and C. Kallin, *Phys. Rev. B* **40**, 5795 (1989); K. Asano and T. Ando, *Phys. Rev. B* **58**, 1485 (1998).

9. D. S. L. Abergel and V. I. Fal’ko, *Phys. Rev. B* **75**, 155430 (2007).

10. A. Iyengar, J. Wang, H. A. Fertig, and L. Brey, *Phys. Rev. B* **75**, 125430 (2007).

11. Yu. A. Bychkov and G. Martinez, *Phys. Rev. B* **77**, 125417 (2008).

12. K. Shizuya, *Phys. Rev. B* **81**, 075407 (2010); *ibid.* **B84**, 075409 (2011).

13. S. Viola Kusminskiy, D. K. Campbell, and A. H. Castro Neto, *Euro. Phys. Lett.* **85**, 58005 (2009).

14. Z. Jiang, E. A. Henriksen, L. C. Tung, Y.-J. Wang, M. E. Schwartz, M. Y. Han, P. Kim, and H. L. Stormer, *Phys. Rev. Lett.* **98**, 197403 (2007).

15. E. A. Henriksen, Z. Jiang, L.-C. Tung, M. E. Schwartz, M. Takita, Y.-J. Wang, P. Kim, and H. L. Stormer, *Phys. Rev. Lett.* **100**, 087403 (2008).

16. D. C. Elias, R. V. Gorbachev, A. S. Mayorov, S. V. Morozov, A. A. Zhukov, P. Blake, L. A. Ponomarenko, I. V. Grigorieva, K. S. Novoselov, F. Guinea, and A. K. Geim, *Nat. Phys.* **7**, 701 (2011).

17. T. Ohta, A. Bostwick, T. Seyller, K. Horn, and E. Rotenberg, *Science* **313**, 951 (2006).

18. E. McCann and V. I. Fal’ko, *Phys. Rev. Lett.* **96**, 086805 (2006).

19. M. Koshino and T. Ando, *Phys. Rev. B* **73**, 245403 (2006).

20. F. Guinea, A. H. Castro Neto, and N. M. R. Peres, *Phys. Rev. B* **73**, 245426 (2006).

21. M. Koshino and E. McCann, *Phys. Rev. B* **83**, 165443 (2011).

22. Y. Barlas, R. Côté, K. Nomura, and A. H. MacDonald, *Phys. Rev. Lett.* **101**, 097601 (2008).

23. D. S. L. Abergel and T. Chakraborty, *Phys. Rev. Lett.* **102**, 056807 (2009).

24. K. Shizuya, *Phys. Rev. B* **86**, 045431 (2012); *ibid.* **B87**, 085413 (2013).

25. J. Sári and C. Tóke, *Phys. Rev. B* **87**, 085432 (2013).

26. H. Fukuyama, P. M. Platzman, and P. W. Anderson, *Phys. Rev. B* **19**, 5211 (1979).

27. A. H. MacDonald, *Phys. Rev. B* **30**, 4392 (1984).

28. C. Kallin and B. I. Halperin, *Phys. Rev. B* **30**, 5655 (1984).

29. A. H. MacDonald, H.C.A. Oji, and S. M. Girvin, *Phys. Rev. Lett.* **55**, 2208 (1985).

30. S. M. Girvin, A. H. MacDonald, and P. M. Platzman, *Phys. Rev. B* **33**, 2481 (1986).
31. A. H. MacDonald and S.-C. Zhang, Phys. Rev. B49, 17208 (1994).
32. K. Moon, H. Mori, K. Yang, S.M. Girvin, A.H. MacDonald, L. Zheng, D. Yoshioka, and S.-C. Zhang, Phys. Rev. B51, 5138 (1995).
33. G. W. Semenoff, Phys. Rev. Lett. 53, 2449 (1984).
34. K. Shizuya, Int. J. Mod. Phys. B17, 5875 (2003).
35. J. Gonzalez, F. Guinea, and M. A. H. Vozmediano, Nucl. Phys. B424, 595 (1994).