USING ALMOST-EVERYWHERE THEOREMS FROM ANALYSIS TO STUDY RANDOMNESS

KENSHI MIYABE, ANDRÉ NIES, AND JING ZHANG

Abstract. We study algorithmic randomness notions via effective versions of almost-everywhere theorems from analysis and ergodic theory. The effectivization is in terms of objects described by a computably enumerable set, such as lower semicomputable functions. The corresponding randomness notions are slightly stronger than Martin-Löf (ML) randomness. We establish several equivalences. Given a ML-random real \( z \), the additional randomness strengths needed for the following are equivalent.

1. all effectively closed classes containing \( z \) have density 1 at \( z \).
2. all nondecreasing functions with uniformly left-c.e. increments are differentiable at \( z \).
3. \( z \) is a Lebesgue point of each lower semicomputable integrable function.

We also consider convergence of left-c.e. martingales, and convergence in the sense of Birkhoff’s pointwise ergodic theorem. Lastly we study randomness notions for density of \( \Pi^0_n \) and \( \Sigma^1_1 \) classes.

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1. Introduction

Several theorems in analysis and ergodic theory state that all functions in a certain class are well-behaved at almost every point. For instance, Lebesgue proved the following in 1909.

**Theorem 1.1** (Lebesgue [25]). Let \( f : [0, 1] \to \mathbb{R} \) be a nondecreasing function. Then \( f \) is differentiable almost everywhere.

Another example is Birkhoff’s ergodic theorem; see e.g. [23, Thm. 2.3].

**Theorem 1.2** (Birkhoff [4]). Let \( T \) be a measure preserving operator on a probability space \( X \). Let \( f \) be an integrable function on \( X \). Then for almost every point \( z \in X \), at iterations of \( T \) that start from \( z \), the average of the first \( n \) values of \( f \) converges as \( n \to \infty \). If the operator is ergodic then this limit is the integral of \( f \).

The theorems state that there is a null set of exceptions which usually depends on given objects, such as \( T \) and \( f \) above. By an effective version of such a theorem, we mean the following. If the given objects are algorithmic in some sense, then the resulting null set is also algorithmic. (A slightly stronger effective version of such a theorem would also ask that the null set be obtained uniformly from a presentation of the given objects, without the assumption that it is algorithmic; this is usually the case for the examples we consider.)

Brattka, Miller and Nies [6], in Thm. 4.1 combined with Remark 4.7, show the following effective version of Lebesgue’s theorem. The given object is a computable function.

**Theorem 1.3** ([6]). Suppose a nondecreasing function \( f : [0, 1] \to \mathbb{R} \) is computable. Then there exists a computable martingale that succeeds on the binary presentation of each real \( z \) such that \( f'(z) \) fails to exist.

We clarify the terms used in this theorem.

(a) The computability of a function is taken in the usual sense of computable analysis [39]. As shown in the last section of the longer arXiv version [6], the weaker hypothesis is sufficient that \( f(q) \) be a computable real uniformly in a rational \( q \).

(b) In randomness theory, a martingale is a function \( M : 2^{<\omega} \to \mathbb{R}^+ \) such that \( 2M(\sigma) = M(\sigma 0) + M(\sigma 1) \). A martingale \( M \) succeeds on a bit sequence \( Z \) if the value of \( M \) on initial segments of \( Z \) is unbounded. The success set is a null set which is effective in case \( M \) is computable.

A real on which no computable martingale succeeds is called computably random, a notion introduced by Schnorr [37]; for a recent reference see e.g. [32, Ch. 7] or [11]. The theorem above shows that \( f'(z) \) exists for each computably random real \( z \) and nondecreasing computable function \( f \). Brattka et al. also show that conversely, if a real \( z \) is not computably random,
then some computable monotonic function \( f \) fails to be differentiable at \( z \).
In this way, this effective form of Lebesgue’s Theorem 1.1 is matched to computable randomness. This is an instance of a more general principle: effective versions of “almost everywhere” theorems often correspond to well-studied algorithmic randomness notions.

Pathak, Rojas and Simpson [34, Theorem 3.15] matched a particular effective form of the Lebesgue differentiation theorem to Schnorr randomness (the direction from functions to tests was independently proven in [18, Thm. 5.1]). We will discuss this in more detail in Subsection 5.1.

V’yugin [38], Gács et al. [19], Bienvenu et al. [3], Franklin et al. [17], and Franklin and Towsner [16] all studied effective versions of Birkhoff’s theorem. For instance, in the notation above, if an ergodic operator \( T \) is computable, and the integrable function \( f \) is lower semicomputable as defined below, then the corresponding notion is Martin-Löf randomness by [3, 17].

Matching such theorems to algorithmic randomness notions has been useful in two ways:

(a) to determine the strength of the theorem, and
(b) to understand the randomness notion.

For an example of (a), Demuth [10] (see [6] for a proof in modern language) showed that Jordan’s extension of Lebesgue’s result to functions of bounded variation corresponds to Martin-Löf randomness. This notion is stronger than computable randomness; so in a sense this extension is harder to obtain.

For an example of (b), Brattka et al. [6] used their results to show that computable randomness of a real does not depend on the choice of base in its digit expansion, even though martingales (which can also be defined with respect to bases other than 2) bet on such an expansion.

The main purpose of this paper is to examine effective versions of almost everywhere theorems that do not correspond to known randomness notions. This apparently occurred for the first time when Bienvenu et al. showed in [2, Cor. 5.10] that the randomness notion corresponding to the Denjoy-Young-Saks theorem implies computable randomness, but is incomparable with Martin-Löf randomness.

We base our study on Lebesgue’s theorems mentioned earlier, and on the following two results. The first, Lebesgue’s density theorem [25], asserts that for almost every point \( z \) in a measurable class \( C \subseteq [0, 1] \), the class is “thick” around \( z \) in the sense that the relative measure of \( C \) converges to 1 as one “zooms in” on \( z \). The second, Doob’s martingale convergence theorem [13], says that a martingale converges on almost every point.

The main given object will only be effective in the weak sense of computable enumerations: we consider the Lebesgue density theorem for effectively closed sets of reals (the complement is an open set that can be computable enumerated as a union of rational open intervals), and Doob’s convergence theorem for martingales that uniformly assign left-c.e. reals to strings.

A group of researchers working at the University of Wisconsin at Madison, consisting of Andrews, Cai, Diamondstone, Lempp, and Miller, showed in 2012 that for a real \( z \) the following two conditions are equivalent, thereby connecting these two theorems.
(1) $z$ is Martin-Löf random and every effectively closed class containing $z$ has density 1 at $z$.

(2) every left-c.e. martingale converges along the binary expansion of $z$.

In this paper we provide two further conditions on a real $z$ that are equivalent to the ones above. They are also linked to well-known classical results of the “almost everywhere” type where the main given object is in some sense computably enumerable. The conditions are:

(3) every interval-c.e. function $f$ is differentiable at $z$

(4) $z$ is Martin-Löf random and a Lebesgue point of each integrable lower semicomputable function $g: [0, 1] \to \mathbb{R} \cup \{\infty\}$.

By default, functions have domain $[0, 1]$. In (3), the relevant classical result is Lebesgue’s theorem on monotonic functions discussed above. To say that a monotonic $f$ is interval c.e. means that $f(0) = 0$ and $f(q) - f(p)$ is left-c.e. uniformly in rationals $p < q$. In (4), the classical result is Lebesgue’s differentiation theorem, which extends the density theorem. A function $g$ is lower semicomputable if \( \{x : g(x) > q\} \) is $\Sigma^0_1$ uniformly in a rational $q$. The new randomness notion identifying the strength of each of the conditions (1)–(4) will be called density randomness.

The analytic notion of density has already been very useful for resolving open problems on the complexity of sets of numbers, asked for instance in [29]. It was used in [8] to show that $K$-triviality coincides with ML-noncuppability. It was further used to solve the so-called covering problem that every $K$-trivial is Turing below an incomplete ML-random, and in fact below a single $\Delta^0_2$ one. See the survey [?] for more detail and references.

The first five sections of the paper are based on the almost-everywhere theorems that serve as an analytic background for our algorithmic investigations: Lebesgue density, martingale convergence, differentiability of monotonic functions [25], Lebesgue differentiation theorem [27], and Birkhoff’s theorem [4]. We will also study density of classes of complexity higher than $\Pi^0_1$.

2. LEBESGUE DENSITY THEOREM

This section presents background material and some initial results. We discuss the theorem that leads to the definition of two central notions for this paper, density-one points and density randomness. We also look at these notions in the setting of Cantor space. M. Khan and Joseph S. Miller [22] have shown that among the ML-random reals, this change of setting does not make a difference. We show that lowness for density randomness is the same as lowness for ML-randomness, or equivalently, $K$-triviality.

2.1. Density in the setting of reals. The definitions below follow [2]. Let $\lambda$ denote Lebesgue measure.

**Definition 2.1.** We define the lower Lebesgue density of a set $C \subseteq \mathbb{R}$ at a point $z$ to be the quantity

$$
\varrho(C|z) := \liminf_{\gamma, \delta \to 0^+} \frac{\lambda([z - \gamma, z + \delta] \cap C)}{\gamma + \delta}.
$$
Note that $0 \leq \rho(C|z) \leq 1$.

**Theorem 2.2** (Lebesgue [25]). Let $C \subseteq \mathbb{R}$ be a measurable set. Then $\rho(C|z) = 1$ for almost every $z \in C$.

When $C$ is open, then the lower Lebesgue density is clearly 1. Thus, the simplest non-trivial case is when $C$ is closed. We use this case to motivate our central definition.

**Definition 2.3.** Consider $z \in [0, 1]$. We say that $z$ is a density-one point if $\rho(C|z) = 1$ for every effectively closed class $C$ containing $z$. We say that $z$ is density random if $z$ is a density-one point and Martin-Löf random.

As noted e.g. in [2], being a density-one point is in itself not a reasonable randomness notion: for instance, a 1-generic real is also a density-one point, but fails the law of large numbers.

By the Lebesgue density theorem and the fact that there are only countably many effectively closed classes, almost every real $z$ is density random. Recall that a weakly-2-random set is one that does not lie in any $\Pi^0_2$ null class. In fact, any such set is density random: for any effectively closed $C$ and rational $q < 1$, the null class $\{Z \in C : \rho_2(C | Z) \leq q\}$ is $\Pi^0_2$.

We say that $z$ is a positive density point if $\rho(C|z) > 0$ for every effectively closed class $C$ containing $z$. The distinction between positive and full density is typical for our algorithmic setting. In classical analysis null sets are usually negligible, so everything is settled by Lebesgue’s theorem. In effective analysis, for a ML-random real $z$, it is a stronger randomness condition to be a full density-one point, than to be a positive density point, by Day and Miller [9].

Bienvenu et al. [2] have shown that a ML-random real $z$ is a positive density point if and only if $z$ is Turing incomplete. In contrast, for density-one points, no characterisation in terms of computational complexity among the ML-random reals is currently known.

### 2.2. Density in the setting of Cantor space.

We let $2^\mathbb{N}$ denote the usual product probability space of infinite bit sequences. For $Z \in 2^\mathbb{N}$ we let $Z \upharpoonright_n$ (or $Z \upharpoonright_n$ in subscripts) denote the first $n$ bits of $Z$. Variables $\sigma, \tau, \eta$ range over strings in $2^{<\omega}$. For each $\sigma$ we let $[\sigma]$ denote the clopen set of extensions of $\sigma$. For $C \subseteq 2^\mathbb{N}$ we let $\lambda_{\sigma}(C) = 2^{\left|\sigma\right|} \lambda(C \cap [\sigma])$ denote the local measure of $C$ inside $[\sigma]$.

Consider a measurable set $C \subseteq 2^\mathbb{N}$ and $Z \in 2^\mathbb{N}$. The lower density of $Z \in 2^\mathbb{N}$ in $C$ is defined to be

$$\rho_\sigma(C|Z) = \liminf_{n \to \infty} \lambda_{Z \upharpoonright_n}(C)$$

We say that a real $z \in [0, 1]$ is a dyadic density-one point if its dyadic expansion is a density one point in Cantor space. We will use the following result.

**Theorem 2.4** (Khan and Miller [22]). Let $z$ be a ML-random dyadic density-one point. Then $z$ is a full density-one point.

Thus, by the usual identification of irrational real numbers in $[0, 1]$ with elements in Cantor space we can define density randomness for a real as in
Definition 2.3, or equivalently for the corresponding bit sequence in Cantor space using lower dyadic density.

2.3. Lowness for Density randomness. We say that a Turing oracle $A$ is low for density randomness if whenever $Z \in 2^\mathbb{N}$ is density random, $Z$ is already density random relative to $A$. Here, $z$ is density random relative to $A$ if $\rho(C|z)$ = 1 for every $A$-effectively closed class $C$ containing $z$ and $z$ is $A$-ML-random. We will show that this is equivalent to lowness for ML-randomness.

Recall that we denote by $W_{2R}$ the class of all weakly-2-random sets, i.e. sets that do not lie in any $\Pi^0_2$-null class of sets. Low($W_{2R}$, MLR) denotes the class of oracles $A$ such that $W_{2R} \subseteq MLR^A$. Downey, Nies, Weber and Yu [12] have shown that Low($W_{2R}$, MLR) = Low($MLR$).

Lemma 2.5 (Day and Miller [8]). Suppose $Z$ is Martin-Löf random, $A$ is low for ML-randomness, and $P$ is a $\Pi^0_1$ class containing $Z$. Then there exists $\Pi^0_1$ class $Q \subseteq P$ such that $A \in Q$.

Theorem 2.6. $A \in 2^\mathbb{N}$ is low for ML-randomness $\iff A$ is low for density randomness.

Proof. $\Leftarrow$: Let DenseR denote the class of density random sets. Since $W_{2R} \subseteq$ DenseR $\subseteq$ MLR and by the result in [12], we have $\text{Low}(\text{DenseR}) \subseteq \text{Low}(W_{2R}, \text{MLR}) = \text{Low}(\text{MLR})$.

$\Rightarrow$: Assume for a contradiction that $A$ is not low for density randomness, i.e., there exists a set $Z$ that is density random but not density random relative to $A$. If $Z$ is not even Martin-Löf random relative to $A$, then $A$ is not low for ML-randomness, which is a contradiction. Otherwise, $Z$ is Martin-Löf random relative to $A$ but not density random relative to $A$. Hence there exists a $\Pi^0_1$ class $P$ containing $Z$ such that $\rho(P|Z) < 1$. By Lemma 2.5, there is a $\Pi^0_1$ class $Q \subseteq P$ such that $Z \in Q$. Then $\rho(Q|Z) \leq \rho(P|Z) < 1$, contradicting with the fact the $Z$ is density random.

2.4. Upper density. The upper density of $C \subseteq 2^\mathbb{N}$ at $Z$ is:

$$\overline{\nu}_2(C|Z) = \limsup_{n \to \infty} \lambda_{Z|n}(C)$$

Bienvenu et al. [1, Prop. 5.4] have shown that for any effectively closed set $P$ and ML-random $Z \in P$, we have $\overline{\nu}_2(P \mid Z) = 1$. Actually ML-randomness of $Z$ was too strong an assumption. The right level seems to be the weaker notion of partial computable randomness, defined in terms of partial computable martingales. See [32, Ch. 7] for background on this notion.

Proposition 2.7. Let $P \subseteq 2^\mathbb{N}$ be effectively closed. Let $Z \in P$.

(i) If $Z$ is partial computably random, then $\overline{\nu}_2(P \mid Z) = 1$.

(ii) Suppose that, in addition, $\lambda P$ is computable. If $Z$ is Kurtz random, then $\overline{\nu}_2(P \mid Z) = 1$. 

Proof. Suppose that there is a rational \( q < 1 \) and an \( n^* \in \mathbb{N} \) such that \( \lambda_\eta(\mathcal{P}) < q \) for each \( \eta < Z \) with \( |\eta| \geq n^* \).

(i). We define a partial computable martingale \( M \) that succeeds on \( Z \). Let \( M(\eta) = 1 \) for all strings \( \eta \) with \( |\eta| \leq n^* \). Now suppose that \( M(\eta) \) has been defined, but \( M \) is as yet undefined on any extensions of \( \eta \). Search for \( t = t_\eta > |\eta| \) such that, where

\[
F = \{ \tau > \eta \colon |\tau| = t \land \tau \cap \mathcal{P}_t \neq \emptyset \},
\]

we have \( p := |F|2^{-(t-|\eta|)} \leq q \). If \( t_\eta \) and \( F \) are found, bet all the capital existing at \( \eta \) along the strings in \( F \). That is, for \( \tau \geq \eta, |\tau| \leq t \), let

\[
M(\sigma) = M(\eta) \cdot |\{ \sigma \in F \colon \sigma \geq \tau \}|/p.
\]

Then \( M(\sigma) = M(\eta)/p \geq M(\eta)/q \) for each \( \sigma \in F \). Now continue the procedure with all such strings \( \sigma \geq \eta \) of length \( t \).

For each \( \eta \prec Z \) of length at least \( n^* \), we have \( \lambda_\eta(\mathcal{P}) < q \), so a \( t \) will be found. Since \( Z \in \mathcal{P} \), \( M \) never decreases along \( Z \). Then, since \( q < 1 \), \( M \) succeeds on \( Z \).

(ii). Under the extra hypothesis on \( \mathcal{P} \), we can make \( M \) total and give a lower bound for its success rate at an infinite computable set of positions along \( Z \), which shows that \( Z \) is not Kurtz random (see Downey and Hirschfeldt [11, Theorem 7.2.13]).

Note that \( \lambda_\eta(\mathcal{P}) \) is a computable real uniformly in \( \eta \). Pick rationals \( q' < q < 1 \) and an \( n^* \in \mathbb{N} \) such that \( \lambda_\eta(\mathcal{P}) < q' \) for each \( \eta \prec Z \) with \( |\eta| \geq n^* \). In the same situation as above, search for \( t_\eta > |\eta| \) such that we see \( \lambda_\eta(\mathcal{P}) > q' \) at stage \( t_\eta \), or \( F \) is found. One of the cases must occur. If the former case is seen first, we let \( M(\tau) = M(\eta) \) for all \( \tau \geq \eta, \tau \leq t_\eta \). Otherwise, we proceed as above.

For the lower bound on the success rate, define a computable function by

\[
g(n) = \max\{ t_\eta \colon n^* \leq |\eta| \leq n \},
\]

for \( n \geq n^* \), and \( g(n) = 0 \) otherwise. Let \( r(k) = g^{(2k)}(n^*) \). Then \( M(Z \mid r(k)) \geq q^{-k} \) for each \( k \).

We ignore at present whether the partiality of \( M \) in (i) is necessary.

Question 2.8. Is there a \( \Pi^0_1 \) class \( \mathcal{P} \) and a computably random \( Z \in \mathcal{P} \) such that \( \nu_2(\mathcal{P} \mid Z) < 1 \)?

In Subsection 5.1 we continue to study \( \Pi^0_1 \) sets \( \mathcal{P} \) of computable measure. We will show that such a set has density one at every Schnorr random member.

3. Martingale convergence theorem

For background on martingales in probability theory, see for instance Durrett [13, Ch. 4]. The martingale convergence theorem goes back to work of Doob. Recall that for a random variable \( Y \) one defines \( Y^+ = \max(Y, 0) \).

Theorem 3.1. Let \( \langle X_n \rangle_{n \in \mathbb{N}} \) be a martingale with \( \sup_n E X^+_n < \infty \). Then \( X(w) := \lim_n X_n(w) \) exists almost surely, and \( E|X| < \infty \).
The standard proof (see e.g. [13, Ch. 4, (2.10)]) uses Doob’s upcrossing inequality. In randomness theory, only a very restricted form of the powerful notion of a martingale has been used so far. The probability space is Cantor space with the usual product measure. The filtration $\langle F_n \rangle_{n \in \mathbb{N}}$ is defined by letting $F_n$ be the set of events that only depend on the first $n$ bits. If $\langle X_n \rangle$ is adapted to $\langle F_n \rangle_{n \in \mathbb{N}}$, then $X_n$ has constant value on each $\sigma$ for $|\sigma| = n$.

Let $M(\sigma)$ be this value. The martingale condition $E(X_{n+1} | F_n) = X_n$ now turns into $\forall \sigma M(\sigma 0) + M(\sigma 1) = 2M(\sigma)$. One also requires that the values be non-negative (so that one can reasonably define that a martingale succeeds along a bit sequence).

Note that $E X_0 = M(\langle \rangle) < \infty$. Thus, Theorem 3.1 turns into the following.

**Theorem 3.2.** Let $M : 2^{<\omega} \to \mathbb{R}_0^+$ be a martingale in the restricted sense above. Then for almost every $Z \in 2^\mathbb{N}$, $X(Z) = \lim_n M(Z | n)$ exists. Furthermore, $E X < \infty$.

If $\lim_n M(Z | n)$ exists, we say that $M$ converges along $Z$.

We can now analyze the theorem in the effective setting, according to the main plan of the paper. Firstly we discuss the effective form of Theorem 3.2 in terms of computable martingales. It is not hard to show that a computable martingale converges along any computably random bit sequence $Z$ (see [11, Theorem 7.1.3]). In other words, boundedness of all computable martingales along a bit sequence $Z$ already implies their convergence. For the converse see the proof of [18, Thm. 4.2], where success of a computable martingale is turned into oscillation of another. Thus, this effective form of Theorem 3.2 is matched to computable randomness.

Next, we weaken the effectiveness to a notion based on computable enumerability. A martingale $L : 2^{<\omega} \to \mathbb{R}_+^+$ is called left-c.e. if $L(\sigma)$ is a left-c.e. real uniformly in $\sigma$. Note that $Z$ is Martin-Löf-random iff every such martingale is bounded along $Z$ (see e.g. [32, Prop. 7.2.6]). Unlike the case of computable martingales, convergence requires a stronger form of algorithmic randomness than boundedness. For instance, let $U = [0, \Omega)$ where $\Omega$ is a left-c.e. Martin-Löf-random real, and let $L(\sigma) = \lambda(U | [\sigma])$ (as a shorthand we use $\lambda_U(\mathcal{U})$ for this conditional measure); then the left-c.e. martingale $L$ is bounded by 1 but diverges on $\Omega$ because $\Omega$ is Borel normal.

The following theorem matches left-c.e. martingale convergence to density randomness. It is due to 2012 work of the “Madison Group” consisting of Andrews, Cai, Diamondstone, Lempp and Joseph S. Miller.

**Theorem 3.3 (Madison group).** The following are equivalent for a ML-random real $z \in [0, 1]$ with binary expansion $0.Z$.

(i) $z$ is a dyadic density-one point.

(ii) Every left-c.e. martingale converges along $Z$.

A proof written by Nies based mostly on conversations with Miller and his talks at the Buenos Aires 2012 CCR semester can be found in [14, Section 13].
4. Differentiability of non-decreasing functions

We consider an effective version, in the sense of computable enumerability, of Lebesgue’s theorem that non-decreasing functions are almost everywhere differentiable. Freer, Kjos-Hanssen, Nies and Stephan [18] studied a class of non-decreasing functions they called interval-c.e. They showed (with J. Rute) that the continuous interval-c.e. functions are precisely the variation functions of computable functions.

**Definition 4.1.** A non-decreasing function \( f : [0, 1] \to \mathbb{R} \) is interval-c.e. if \( f(0) = 0 \), and \( f(y) - f(x) \) is a left-c.e. real, uniformly in rationals \( x < y \).

Our theorem matches the effective version of Lebesgue’s Theorem 1.1 in terms of interval-c.e. functions to density randomness. The proof of this result was first sketched in the conference paper [33]. We give a full proof here.

**Theorem 4.2.** \( z \in [0, 1] \) is density random \( \iff \) \( f'(z) \) exists for each interval-c.e. function \( f : [0, 1] \to \mathbb{R} \).

\( \Leftarrow \): If \( z \) is not density random then by Theorem 3.3 a left-c.e. martingale \( M \) diverges along the binary expansion of \( z \). Let \( \mu_M \) be the measure on \([0,1]\) corresponding to \( M \), which is given by \( \mu[\sigma] = 2^{-|\sigma|}M(\sigma) \), and let \( \text{cdf}_M(x) = \mu_M(0, x) \). Then \( \text{cdf}_M \) is interval-c.e. and \( (\text{cdf}_M)'(z) \) fails to exist.

The rest of this section is devoted to proving the implication \( \Rightarrow \). This combines purely analytical arguments with effectiveness considerations.

4.1. Slopes and martingales. First we need notation and a few definitions, mostly taken from [7] or [2]. For a function \( f : [0, 1] \to \mathbb{R} \), the slope at a pair \( a, b \) of distinct reals in its domain is

\[
S_f(a, b) = \frac{f(a) - f(b)}{a - b}.
\]

For a nontrivial interval \( A \) with endpoints \( a, b \), we also write \( S_f(A) \) instead of \( S_f(a, b) \).

We let \( \sigma, \tau \) range over (binary) strings. For such a string \( \sigma \), by \([\sigma] \) we denote the closed basic dyadic interval \([0, \sigma], 0.\sigma + 2^{-|\sigma|}] \). The corresponding open basic dyadic interval is denoted \((\sigma)\).

**Derivatives.** If \( z \) is in an open neighborhood of the domain of \( f \), the upper and lower derivatives of \( f \) at \( z \) are

\[
\overline{D}f(z) = \limsup_{h \to 0} S_f(z, z + h) \quad \text{and} \quad \underline{D}f(z) = \liminf_{h \to 0} S_f(z, z + h),
\]

where as usual, \( h \) ranges over positive and negative values. The derivative \( f'(z) \) exists if and only if these values are equal and finite.

We will also consider the upper and lower pseudo-derivatives defined by:

\[
\overline{D}f(x) = \limsup_{h \to 0^+} \{ S_f(a, b) \mid a \leq x \leq b \land 0 < b - a \leq h \},
\]

\[
\overline{D}f(x) = \liminf_{h \to 0^+} \{ S_f(a, b) \mid a \leq x \leq b \land 0 < b - a \leq h \},
\]

where \( a, b \) range over rationals in \([0, 1]\). We use them because in our arguments it is often convenient to consider (rational) intervals containing \( x \), rather than intervals with \( x \) as an endpoint.
Remark 4.3. Brattka et al. [7, after Fact 2.4] verified that
\[ D_f(z) \leq D_f(z) \leq \tilde{D}_f(z) \leq D_f(z) \]
for any real \( z \in [0, 1] \). The lower and upper pseudo-derivatives of \( f \) coincide with the usual lower and upper derivatives if \( f \) is nondecreasing. To show \( D_f(z) \leq \tilde{D}_f(z) \), given any real \( z \) and rationals \( a \leq z \leq b \) with \( a < b \),
\[ S_f(a, b) = \frac{b - a}{b - a} S_f(b, z) + \frac{z - a}{z - a} S_f(z, a) \leq D_f(z). \]
The other inequalities can be shown similarly.

We will use the subscript \( \omega \) to indicate that all the limit operations are restricted to the case of basic dyadic intervals containing \( z \). Thus,
\[ \tilde{D}_f(z) = \limsup \{ S_f(A) \mid x \in A \land A \text{ is basic dyadic interval} \}, \]
\[ D_f(z) = \liminf \{ S_f(A) \mid x \in A \land A \text{ is basic dyadic interval} \}. \]

4.2. Porosity and upper derivatives. We say that a set \( C \subseteq \mathbb{R} \) is porous at \( z \) via the constant \( \varepsilon > 0 \) if there exists arbitrarily small \( \beta > 0 \) such that \( (z - \beta, z + \beta) \) contains an open interval of length \( \varepsilon \beta \) that is disjoint from \( C \). We say that \( C \) is porous at \( z \) if it is porous at \( z \) via some \( \varepsilon > 0 \). This notion originated in the work of Denjoy. See for instance [5, 5.8.124] (but note the typo in the definition there).

Definition 4.4 ([2]). We call \( z \) a porosity point if some effectively closed class to which it belongs is porous at \( z \). Otherwise, \( z \) is a non-porosity point.

Clearly, if \( C \) is porous at \( z \) then \( \rho(C|z) < 1 \), so \( z \) is not a density-one point. The converse fails: every Turing incomplete Martin-Löf random real is a non-porosity point by [2], but not necessarily a density-one point [9].

We show that discrepancy of dyadic and full upper/lower derivatives at \( z \) implies that some closed set is porous at \( z \). This extends the idea in the proof of Theorem 2.4 due to Khan and Miller. We begin with the easier case of the upper derivative. The other case will be supplied in Subsection 4.4.

Proposition 4.5. Let \( f: [0, 1] \rightarrow \mathbb{R} \) be interval-c.e. If \( z \) is a non-porosity point, then \( \tilde{D}_f(z) = D_f(z) \).

Proof. Suppose that \( \tilde{D}_f(z) < p < D_f(z) \) for a rational \( p \). Choose \( k \in \mathbb{N} \) is such that \( p(1 + 2^{-k+1}) < D_f(z) \).

We denote by \( \sigma \leq \tau \) that \( \sigma \) is an initial segment of \( \tau \); \( \sigma \prec \tau \) denotes that \( \sigma \) is a proper initial segment of \( \tau \); \( \sigma \prec Z \) that \( \sigma \) is an initial segment of the infinite bit sequence \( Z \).

Let \( \sigma^* \prec Z \) be any string such that \( \forall \sigma \left[ \sigma^* \preceq \sigma \prec Z \Rightarrow S_f([\sigma]) \leq p \right] \). It is sufficient to establish the following.

Claim 4.6. The closed set
\[ C = \{
\sigma^* \} - \bigcup \{(\sigma) \mid S_f([\sigma]) > p \}, \]
which contains \( z \), is porous at \( z \).

Then, since \( f \) is interval-c.e., the function \( \sigma \rightarrow S_f([\sigma]) \) is a left-c.e. martingale. In particular, this set \( C \) is effectively closed, and porous at \( z \).
The proof of the claim is purely analytical, and only uses that \( f \) is nondecreasing. We show that there exists arbitrarily large \( n \) such that some basic dyadic interval \([a, \tilde{a}]\) of length \( 2^{-n-k} \) is disjoint from \( C \), and contained in \([z - 2^{-n+2}, z + 2^{-n+2}]\). In particular, we can choose \( 2^{-k-2} \) as a porosity constant.

By choice of \( k \) there is an interval \( I \ni z \) of arbitrarily short positive length such that \( p(1 + 2^{k+1}) < S_f(I) \). Let \( n \) be such that \( 2^{-n+1} > |I| \geq 2^{-n} \). Let \( a_0 \) be greatest of the form \( \ell 2^{-n-k}, \ell \in \mathbb{Z} \), such that \( a_0 < \min I \). Let \( a_u = a_0 + v 2^{-n-k} \). Let \( r \) be least such that \( a_r \geq \max I \).

Since \( f \) is nondecreasing and \( a_r - a_0 \leq |I| + 2^{-n-k+1} \leq (1 + 2^{-k+1}) |I| \), we have

\[
S_f(I) \leq S_f(a_0, a_r)(1 + 2^{-k+1}),
\]

and therefore \( S_f(a_0, a_r) > p \). Since \( S_f(a_0, a_r) \) is the average of the slopes \( S_f(a_u, a_u+1) \) for \( u < r \), there is a \( u < r \) such that

\[
S_f(a_u, a_u+1) > p.
\]

Since \((a_u, a_u+1) = (\sigma)\) for some string \( \sigma \), this gives the required ‘hole’ in \( C \) which is near \( z \in I \) and large on the scale of \( I \): in the definition of porosity at the beginning of this subsection, let \( \beta = 2^{-n+2} \) and note that we have \([a_u, a_u+1] \subseteq [z - 2^{-n+2}, z + 2^{-n+2}] \) because \( z \in I \) and \(|I| < 2^{-n+1}\).

4.3. Basic dyadic intervals shifted by \( 1/3 \). We will use a basic ‘geometric’ fact for instance observed by Morayne and Solecki [31]. For \( m \in \mathbb{N} \) let \( \mathcal{D}_m \) be the collection of intervals of the form

\[
[k2^{-m}, (k+1)2^{-m}]
\]

where \( k \in \mathbb{Z} \). Let \( \hat{\mathcal{D}}_m \) be the set of intervals \( (1/3) + I \) where \( I \in \mathcal{D}_m \).

\textbf{Lemma 4.7.} Let \( m \geq 1 \). If \( I \in \mathcal{D}_m \) and \( J \in \hat{\mathcal{D}}_m \), then the distance between an endpoint of \( I \) and an endpoint of \( J \) is at least \( 1/(3 \cdot 2^m) \).

To see this, assume that \(|k2^{-m} - (p2^{-m} + 1/3)| < 1/(3 \cdot 2^m)|. This yields \(|3k - 3p - 2^m|/(3 \cdot 2^m) < 1/(3 \cdot 2^m)|, and hence \( 3 \cdot 2^m \), a contradiction.

In order to apply Lemma 4.7, we may need values of nondecreasing functions \( f : [0, 1] \to \mathbb{R} \) at endpoints of any such intervals, which may lie outside \([0, 1] \). So we think of \( f \) as extended to \([-1, 2] \) via \( f(x) = f(0) \) for \(-1 \leq x < 0 \) and \( f(y) = f(1) \) for \( 1 < y \leq 2 \). The effectiveness properties we consider here, or interval-c.e. is preserved by this because it suffices to determine values of the function at rationals.

4.4. Porosity and lower derivatives. We complete the proof of the implication “\( \Rightarrow \)” in Theorem 4.2. We may assume that \( z > 1/2 \). Note that \( z - 1/3 \) is a ML-random density-one point, hence a dyadic density-one point. So, by Theorem 3.3, the real \( z - 1/3 \) is also a c.e. martingale convergence point. In particular, both \( z \) and \( z - 1/3 \) are non-porosity points.

Let \( M = M_f \) be the martingale given by \( \sigma \to S_f([\sigma]) \). Note that \( M \) converges on \( z \) by hypothesis (recall that we write \( \sigma(z) \) for the limit). Thus \( D_2 f(z) = \hat{D}_2 f(z) = M(z) \).

Let \( \hat{f}(x) = f(x + 1/3) \), and let \( \hat{M} = M_f \). Then \( \hat{M} \) converges on \( z - 1/3 \).

\textbf{Claim 4.8.} \( M(z) = \hat{M}(z - 1/3) \).
If \( M(z) < \hat{M}(z-1/3) \) then \( \tilde{D}_2 f(z) < \tilde{D} f(z) \). However, \( z \) is a non-porosity point, so this contradicts Proposition 4.5. If \( \hat{M}(z - 1/3) < M(z) \) we argue similarly using that \( z - 1/3 \) is a non-porosity point. This establishes the claim.

We have already established that \( D_2 f(z) = \tilde{D}_2 f(z) = \tilde{D} f(z) \), so to complete the proof of “\( \Rightarrow \)” in Theorem 4.2, it remains to be shown that \( \tilde{D} f(z) = D_2 f(z) \). Then, since \( f \) is nondecreasing, \( f'(z) \) exists by Remark 4.3.

Assume for a contradiction that if \( \tilde{D} f(z) < D_2 f(z) \). We will show that one of \( z, z - 1/3 \) is a porosity point. First we define porosity in Cantor space.

**Definition 4.9.** For a closed set \( C \subseteq 2^\mathbb{N} \), we say that \( C \) is porous at \( Y \in C \) if there is \( r \in \mathbb{N} \) as follows: there exists arbitrarily large \( m \) such that
\[
C \cap [(Y | m)^\tau] = \emptyset \text{ for some } \tau \text{ of length } r.
\]
Clearly this implies that \( C \) viewed as a subclass of \([0, 1]\) is porous at \( 0.Y \) (in which case “holes” on both sides of \( 0.Y \) would be allowed). We will actually define \( \Pi_0^1 \) classes \( \mathcal{E} \) and \( \hat{\mathcal{E}} \) in Cantor space such that \( \mathcal{E} \) is porous at the binary expansion of \( z \), or \( \hat{\mathcal{E}} \) is porous at the binary expansion of \( z - 1/3 \).

We employ a method similar to the one in Subsection 4.2, but now take into account both dyadic intervals, and dyadic intervals shifted by \( 1/3 \) of the same length. Recall that \( D_2 f(z) = M(z) \).

We can choose rationals \( p, q \) such that
\[
\tilde{D} f(z) < p < q < M(z) = \hat{M}(z - 1/3).
\]
Let \( k \in \mathbb{N} \) be such that \( p < q(1 - 2^{-k+1}) \). Let \( u, v \) be rationals such that
\[
q < u < M(z) < v \text{ and } v - u \leq 2^{-k-3}(u - q).
\]
Recalling the notation in Subsection 4.3, let \( n^* \in \mathbb{N} \) be such that for each \( n \geq n^* \) and any interval \( A \in D_n \cup D_n \) containing \( z \), we have \( S_f(A) \geq u \). Let
\[
\mathcal{E} = \{X \in 2^\mathbb{N} : \forall n \geq n^* M(X | n) \leq v\}
\]
\[
\hat{\mathcal{E}} = \{W \in 2^\mathbb{N} : \forall n \geq n^* \hat{M}(W | n) \leq v\}
\]
Since \( f \) is interval-c.e., \( M \) and \( \hat{M} \) are left-c.e. martingales, so these classes are effectively closed.

Let \( Z \) be the bit sequence such that \( z = 0.Z \). By the choice of \( n^* \) we have \( Z \in \mathcal{E} \). Let \( Y \) be the bit sequence such that \( 0.Y = z - 1/3 \). We have \( Y \in \hat{\mathcal{E}} \).

Consider an interval \( I \ni z \) of positive length \( \leq 2^{-n^* - 3} \) such that \( S_f(I) \leq p \). Let \( n \) be such that \( 2^{-n^*+1} > |I| \geq 2^{-n} \). Let \( a_0 \) be least of the form \( w2^{-n-k} \) where \( w \in \mathbb{Z} \), such that \( a_0 \geq \min(I) \). Similarly, let \( b_0 \) be least of the form \( w2^{-n-k} + 1/3 \) such that \( b_0 \geq \min(I) \). Let
\[
a_i = a_0 + i2^{-n-k} \text{ and } b_j = b_0 + j2^{-n-k}.
\]
Let \( r, s \) be greatest such that \( a_r \leq \max(I) \) and \( b_s \leq \max(I) \).

Since \( f \) is nondecreasing and
\[
a_r - a_0 \geq |I| - 2^{-n-k+1} \geq (1 - 2^{-k+1})|I|,
\]
we have \( S_f(I) \geq S_f(a_0, a_r)(1 - 2^{-k+1}) \), and therefore \( S_f(a_0, a_r) < q \). Then there is an \( i < r \) such that \( S_f(a_i, a_{i+1}) < q \). Similarly, there is a \( j < s \) such that \( S_f(b_j, b_{j+1}) < q \).
Claim 4.10. One of the following is true.
(i) \( z, a_i, a_{i+1} \) are all contained in a single interval from \( D_{n-3} \).
(ii) \( z, b_j, b_{j+1} \) are all contained in a single interval from \( D_{n-3} \).

For suppose that (i) fails. Then there is an endpoint of an interval \( A \in D_{n-3} \) (that is, a number of the form \( w2^{-n+3} \) with \( w \in \mathbb{Z} \)) between \( \min(z, a_i) \) and \( \max(z, a_{i+1}) \). Note that \( \min(z, a_i) \) and \( \max(z, a_{i+1}) \) are in \( I \).

By Fact 4.7 and since \( |I| < 2^{-n+1} \), there can be no endpoint of an interval \( A \in D_{n-3} \) in \( I \). Then, since \( b_j, b_{j+1} \in I \), (ii) holds. This establishes the claim.

Suppose \( I \) is an interval as above and \( 2^{-n+1} > |I| \geq 2^{-n} \), where \( n \geq n^*+3 \).

Let \( \eta = Z|_{n-3} \) and \( \hat{\eta} = Y|_{n-3} \).

If (i) holds for this \( I \) then there is a string \( \alpha \) of length \( k+3 \) (where \( [\eta \alpha] = [a_i, a_{i+1}] \)) such that \( M(\eta \alpha) < q \). So by the choice of \( q < u < v \) and since \( M(\eta) \geq u \) there is \( \beta \) of length \( k+3 \) such that \( M(\eta \beta) > v \). (The decrease along \( \eta \alpha \) of the martingale \( M \) must be balanced by an increase along some \( \eta \beta \).) This yields a “hole” in \( E \), large and near \( Z \) on the scale of \( I \), as required for the porosity of \( E \) at \( Z \); in the notation of the Definition 4.9 above, \( E \) is porous at \( Z \) via \( m = |\eta| \) and \( r = k+3 \).

Similarly, if (ii) holds for this \( I \) then there is a string \( \alpha \) of length \( k+3 \) (where \( [\eta \alpha] = [b_j, b_{j+1}] \)) such that \( M(\eta \alpha) < q \). So by the choice of \( q < u < v \) and since \( M(\hat{\eta}) \geq u \), there is a string \( \beta \) of length \( k+3 \) such that \( M(\hat{\eta} \beta) > v \). This yields a hole, large and near \( Y \) on the scale of \( I \), as required for the porosity of \( \hat{E} \) at \( Y \).

Thus, if case (i) applies for arbitrarily short intervals \( I \), then \( E \) is porous at \( Z \), whence \( z \) is a porosity point. Otherwise (ii) applies for intervals below a certain length. Then \( \hat{E} \) is porous at \( Y \), whence \( z - 1/3 \) is a porosity point. Both cases are contradictory. This concludes the proof of Theorem 4.2.

5. Lebesgue differentiation theorem

This section is centred around an effective version in the c.e. setting of a result obtained by Lebesgue in 1904 (see [26]).

Definition 5.1. Given an integrable non-negative function \( g \) on \([0, 1]\), a point \( z \) in the domain of \( g \) is called a weak Lebesgue point if

\[
\lim_{Q \to z} \frac{1}{\lambda(Q)} \int_Q g
\]

exists, where \( Q \) ranges over open intervals containing \( z \) with length \( \lambda(Q) \) tending to 0; \( z \) is called a Lebesgue point if this value equals \( g(z) \).

We note that also a variant of this definition can be found in the literature, where \( Q \) is centred at \( z \). This is in fact equivalent to the definition given here; see for instance [36, Thm. 7.10]

Theorem 5.2 (Lebesgue [26]). Suppose \( g \) is an integrable function on \([0, 1]\). Then almost every \( z \in [0, 1] \) is a Lebesgue point of \( g \).

Equivalently, the function \( f(z) = \int_{[0,z]} g d\lambda \) is differentiable at almost every \( z \), and \( f'(z) = g(z) \).

Several years later, Lebesgue [27] extended this result to higher dimensions; the variable \( Q \) now ranges over open cubes containing \( z \).
5.1. Effective Lebesgue Differentiation Theorem via \(L_1\)-computability.

Pathak, Rojas and Simpson [34, Theorem 3.15] studied an effective version of Lebesgue’s theorem, where the given function is \(L_1\)-computable, as defined in [35] (or see [34, Def. 2.6]). They showed that

\[ z \text{ is Schnorr random} \iff z \text{ is a weak Lebesgue point of each } L_1\text{-computable function.} \]

The implication “\(\Rightarrow\)” was independently obtained in [18, Thm. 5.1]. Using this result, we observe that if a \(\Pi^0_1\) class has computable measure, it has density 1 at every Schnorr random member.

**Proposition 5.3.** Let \( P \subseteq [0, 1] \) be an effectively closed set such that \( \lambda P \) is computable. Let \( z \in P \) be a Schnorr random real. Then \( \hat{\rho}(P | z) = 1 \).

**Proof.** Let \( P = \bigcap_s P_s \) for a computable sequence \( \langle P_s \rangle \) of finite unions of closed intervals. There is a computable function \( g \) such that \( \lambda(P_g(n) - P) \leq 2^{-n} \). Hence the characteristic function \( 1_{\overline{P}} \) is \( L_1 \)-computable. Now by [34, Theorem 3.15] or [18, Thm. 5.1], the density of \( P \) at \( z \) exists, that is \( \overline{\rho}(P | z) = \rho(P | z) \).

The binary expansion \( Z \) of the real \( z \) is Kurtz random, so by Proposition 2.7(ii) we have \( \overline{\rho}_2(P | Z) = 1 \). Therefore \( \overline{\rho}(P | Z) = 1 \).

\( \square \)

5.2. Dyadic Lebesgue points and integral tests. Recall that an open basic dyadic interval in \([0, 1]\) has the form \((i2^{-n}, (i + 1)2^{-n})\) where \( i < 2^n \). If a string \( \sigma \) of length \( n \) is the binary expansion of \( i \), we also write \((\sigma)\) for this interval. We say that \( z \) is a (weak) dyadic Lebesgue point if the limit in Definition 5.1 exists when \( Q \) is restricted to open basic dyadic intervals.

As usual let \( \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \). For a function \( f : [0, 1] \to \overline{\mathbb{R}} \) and \( z \in [0, 1] \), let

\[ E(f, \sigma) = \frac{\int_{[\sigma]} f \, d\lambda}{2^{-n}}. \]

Then, \( z \) is a dyadic Lebesgue point iff \( \lim_n E(f, Z |_n) = f(z) \) where \( z = 0.Z \).

Recall from the introduction that a function \( g : [0, 1] \to \mathbb{R} \cup \{\infty\} \) is lower semi-computable if \( f^{-1}(\{z : z > q\}) \) is effectively open, uniformly in a rational \( q \). (This is an effective version of lower semicontinuity.) It is well known that such functions can be used to characterise Martin-Löf randomness; see for instance Li and Vitányi [28, Subsection 4.5.6].

**Definition 5.4.** An integral test is a non-negative lower semi-computable function \( g : [0, 1] \to \overline{\mathbb{R}} \) such that \( \int g \, d\lambda < \infty \).

**Theorem 5.5** (Levin). A real \( z \) is Martin-Löf random if and only if \( g(z) < \infty \) for each integral test \( g \).

Note that if \( f \) is an integral test, the function \( \sigma \mapsto E(f, \sigma) \) is a left-c.e. martingale. Since \( f \) is integrable, \( f^{-1}(\{\infty\}) \) is a null set.

In Definition 5.1 of [weak] Lebesgue points, we allow functions \( g \) that can take the value \( \infty \). For \( z \) to be a (weak) Lebesgue point, the limit as the intervals approach \( z \) is required to be finite. First we show that for an integral test \( g \), the dyadic versions of the weak and strong conditions in Def. 5.1 coincide at a ML-random real \( z \).
Lemma 5.6. Let $g$ be an integral test, and let $z$ be a Martin-Löf random real. If $z$ is a dyadic weak Lebesgue point of $g$, then $z$ is in fact a dyadic Lebesgue point of $g$.

Proof. Let $\langle g_n \rangle_{n \in \mathbb{N}}$ be an increasing computable sequence of step functions with dyadic points of discontinuity and rational values such that $\sup g_n(z) = g(z)$ for each dyadic irrational (see Miyabe [30, Lemmas 4.6, 4.8]). Then, there is a non-decreasing computable function $u : \mathbb{N} \to \mathbb{N}$ such that for each $\sigma$ with $|\sigma| \geq u(s)$

$$E(g_n, \sigma) = E(g_n, \sigma 0) = E(g_n, \sigma 1).$$

Unless $z$ is a dyadic rational, we have $g_t(z) = \lim_n E(g_t, Z | n)$, where, as usual, $0.Z$ is the binary expansion of $z$.

By hypothesis, $\lim_n E(g_t, Z | n) =: r$ exists. Clearly $g(z) \leq r$, because for each $t$

$$g_t(z) = \lim_n E(g_t, Z | n) \leq \lim_n E(g_t, Z | n). \tag{1}$$

Suppose for a contradiction that $g(z) < r$, and let $q$ be a rational number such that $g(z) < q < r$. We build an integral test $h$ such that $h(z) = \infty$, which contradicts our assumption that $z$ is ML-random. To do so, we define a uniformly c.e. sequence of sets $S_n \subseteq 2^{<\omega} \times \omega$. Let $S_0 = \{((\epsilon), 0)\}$. Suppose now that $n \geq 1$ and $S_{n-1}$ has been defined. Uniformly in $(\sigma, s) \in S_{n-1}$, let $B \subseteq 2^{<\omega}$ be a c.e. antichain of strings of length $\geq u(s)$ such that

$$[B]^\prec = \{[\tau \succ \sigma : |\tau| \geq u(s) \land \exists t E(g_t, \tau) > q]\}^\prec.$$

For each $\tau \in B$ let $t > s$ be the least corresponding stage and put $(\tau, t)$ into $S_n$.

Let $1_A$ denote the characteristic function of a set $A$. For each $(\tau, t) \in S_n$, let

$$h_{\tau} = (q - E(g_s, \tau))1_{[\tau]}$$

where $(\sigma, s) \in S_{n-1}$ and $\sigma < \tau$. We define $h$ by

$$h = \sum_{n} \sum_{(\tau, t) \in S_n} h_{\tau}.$$

We aim to show that $h$ is an integral test and $h(z) = \infty$. So $z$ is not ML-random contrary to our assumption.

To see that $h$ is an integral test, note that $h$ is lower semicomputable. So it suffices to show that, for every $N$,

$$\sum_{n=0}^{N} \sum_{(\tau, t) \in S_n} \int h_{\tau} d\lambda \leq \int g d\lambda < \infty.$$

If $(\tau, t) \in S_n$, for $n > 0$, let $(\sigma_{\tau, t}, s_{\tau, t}) \in S_{n-1}$ be the corresponding element for which $(\tau, t)$ is enumerated into $S_{n+1}$.

Notice that

$$\int h_{\tau} d\lambda \leq (E(g_t, \tau) - E(g_{s_{\tau, t}}, \tau))2^{-|\tau|} = \int_{|\tau|}^{\infty} (g_t - g_{s_{\tau, t}}) d\lambda,$$
whence
\[ \sum (\tau,t) \in S_n \int h_\tau d\lambda \leq \sum (\tau,t) \in S_n \int (g - g_{s_\tau,t}) d\lambda \leq \sum (\sigma,s) \in S_{n-1} \int [\sigma] (g - g_s) d\lambda. \]

Then, in case \( N \geq 2 \),
\[ \sum_{n=N-1}^{N} \sum (\tau,t) \in S_n \int h_\tau d\lambda \leq \sum (\tau,t) \in S_{N-1} \int [\tau] (g - g_\tau) d\lambda + \sum (\tau,t) \in S_{N-1} \int [\tau] (g_\tau - g_{s_\tau,t}) d\lambda \]
\[ \leq \sum (\tau,t) \in S_{N-1} \int [\sigma] (g - g_\tau) d\lambda \]
\[ \leq \sum (\tau,t) \in S_{N-2} \int [\sigma] (g - g_\tau) d\lambda. \]

By iterating this argument for sums starting at \( N - 2, N - 3, \ldots, 2 \), we have
\[ \sum_{n=0}^{N} \sum (\tau,t) \in S_n \int h_\tau d\lambda \leq \sum (\tau,t) \in S_0 \int [\tau] (g - g_s) d\lambda = \int g d\lambda < \infty. \]

Finally, since \( \lim_n E(g,Z|_n) = r \), for each \( n \) there exists \((\tau_n, t_n) \in S_n\) such that \( \tau_n < z \). Then
\[ h(z) = \sum_n (q - E(g_s, \tau_n)) \geq \sum_n (q - g(z)) = \infty. \]

\[ \square \]

Remark 5.7. The proofs of Lemma 5.6 and of Thm. 3.3 in [14, Section 13] are related. In the notation of Lemma 5.6, we have a left-c.e. martingale \( L(\sigma) = E(g,\sigma) \) and uniformly computable martingales \( L_s(\sigma) = E(g_s,\sigma) \) so that \( L(\sigma) = \sup_s L_s(\sigma) \). By definition of the \( g_s \) as dyadic step functions, we have a computable function \( u \) on \( \mathbb{N} \) such that \( L_s(\tau) = L_s(\tau|_{u(s)}) \) whenever \( |\tau| \geq u(s) \). Let us say that a left-c.e. martingale \( L \) of this kind is stationary in approximation. The obvious inequality
\[ \sup_s L(Z|_{u(s)}) \leq \liminf_n L(Z|_n) \]
corresponds to \( f(z) \leq r \) before (1). If \( Z \) is density random then \( \lim_n L(Z|_n) \) exists, and equals \( \sup_s L(Z|_{u(s)}) \) by an argument similar to the one in the proof of Lemma 5.6.

5.3. Effective Lebesgue Differentiation Theorem via lower semi-computability. We show that density randomness is the same as being a Lebesgue point of each integral test. We use as a basic fact: if \( g \) is a non-negative integrable function, then \( \sigma \to E(g,\sigma) \) is a martingale. By definition, \( z \) is a weak dyadic Lebesgue point of \( g \) iff this martingale converges along \( Z \).

Theorem 5.8. The following are equivalent for \( z \in [0,1] \):

(i) \( z \) is density random.
(ii) \( z \) is a dyadic Lebesgue point of each integral test.
(iii) \( z \) is a Lebesgue point of each integral test.
We could equivalently formulate (ii) and (iii) in terms of integrable lower semicomputable functions, rather than the seemingly more restricted integral tests. For, any lower semicontinuous function on a compact domain is bounded below. So any integrable lower semicomputable function on $[0, 1]$ becomes an integral test after adding a constant.

Proof. (ii) $\Rightarrow$ (i). By definition $g(z)$ is finite for each integral test $g$, whence $z$ is ML-random.

Let $C$ be a $\Pi^0_1$ class containing $z$. Clearly the function $g = 1 - 1_C$ is an integral test. Since $z$ is a Lebesgue point for $g$, $C$ has dyadic density one at $z$. Then, by Theorem 2.4, $z$ is a density-one point.

(i) $\Rightarrow$ (ii). Let $g$ be an integral test. Then $\sigma \to E(g, \sigma)$ is a left-c.e. martingale. By Theorem 3.3, $\lim_n E(g, Z \upharpoonright n)$ exists, whence $z$ is a dyadic weak Lebesgue point for $g$. By Lemma 5.6, $z$ is a dyadic Lebesgue point for $g$.

(ii) $\Rightarrow$ (iii). Let $g$ be an integral test. Then the function $f(x) = \int_{[0,x]} g \, d\lambda$ is interval-c.e. by the aforementioned result of Miyabe [30, Lemmas 4.6, 4.8]. Thus, $f'(z)$ exists using Theorem 4.2, since $z$ is density random by (ii) $\Rightarrow$ (i). In particular, $\lim_{Q \to z} \lambda(Q)^{-1} \int_Q f \, d\lambda$ exists and is equal to $\lim_n 2^n \int_{[Z,n]} f \, d\lambda = f(z)$. Hence, $z$ is a Lebesgue point for $g$.

The implication (iii) $\Rightarrow$ (ii) holds by definition. □

6. Birkhoff’s theorem

We give an effective version, in the c.e. setting, of Birkhoff’s Theorem 1.2. Franklin and Towsner [16] considered the case of a not necessarily ergodic operator $T$ and a lower semicomputable function $f$. Working in the setting of Cantor space with the uniform measure, they showed that the ergodic limit of the averages at iterations exists for each weakly 2-random point $z$.

Under an additional, hypothetical assumption, in [16, Thm. 5.6] they were able to obtain convergence on the weaker assumption that $z$ is balanced random in the sense of [15].

We work in the setting of Cantor space $2^\mathbb{N}$ with a computable probability measure $\mu$. That is, $\mu[\sigma]$ is a left-c.e. real uniformly in a string $\sigma$. By Hoyrup and Rojas [20, Thm. 5.1.1], for any computable probability space $(X, \rho)$, there is a computable isomorphism with an appropriate computability space $(2^\mathbb{N}, \mu)$ for an appropriate computable measure $\mu$. So the restriction to Cantor space is inessential.

Bienvenu, Greenberg, Kucera, Nies, and Turetsky [1, Def. 2.5] introduced a randomness notion that implies density randomness. A left-c.e. bounded test is a nested sequence $\langle V_n \rangle$ of uniformly $\Sigma^0_1$ classes such that for some computable sequence of rationals $\langle \beta_n \rangle$ and $\beta = \sup_n \beta_n$ we have $\mu(V_n) \leq \beta - \beta_n$ for all $n$. $Z$ fails this test if $Z \in \bigcap_n V_n$. $Z$ is Oberwolfach (OW) random if it passes each left-c.e. bounded test.

It is known that balanced randomness in the sense of [15] implies OW randomness, which implies density randomness. The converse of the first implication holds as noted in [1]: some low ML-random is not balanced random [15]; on the other hand, any such set is OW random. It is unknown whether the converse of the second implication holds.
In the following let $\mu$ be a probability measure on $2^\mathbb{N}$ which is computable in the strong sense of [38] that $\mu[\sigma]$ is a computable real uniformly in a string $\sigma$. Note that this is equivalent to the weaker condition above that $\mu[\sigma]$ is uniformly left-c.e., in case the boundary of any open set is null. Recall that a (total) function $f: 2^\mathbb{N} \to \mathbb{R}$ is lower semicomputable if \{x: g(x) > q\} is $\Sigma_0^1$ uniformly in a rational $q$.

**Theorem 6.1.** Let $T$ be a computable measure preserving operator on $(2^\mathbb{N}, \mu)$. Let $f$ be a non-negative integrable lower semicomputable function on $X$. Let $A_n f(x)$ be the usual ergodic average

$$\frac{1}{n} \sum_{i<n} f \circ T^i(x).$$

For every Oberwolfach random point $z \in X$, $\lim_n A_n f(z)$ exists.

Note that we do not assume the operator $T$ is total. However, being computable measure preserving, its domain is conull. Then, since the domain is also $\Pi_0^0$, $T(x)$ is defined whenever $x$ is weakly (i.e. Kurtz) random.

**Proof.** By [30, Lemma 4.6], we have $f(z) = \sup_t f_t(z)$ for every weakly random $z \in X$, where $(f_t)$ is a computable non-decreasing sequence of simple functions (namely, the range is a finite subset of $\mathbb{Q}$, and for each $q \in \mathbb{Q}$, the preimage is $\Sigma_0^1$ uniformly in $q$).

By V’yugin [38], $\lim_n A_n f_t(x)$ exists for each $t$ and each ML-random point $x$. By the maximal ergodic inequality (see e.g. Krengel [23, Cor. 2.2]), for each non-negative integrable function $g$ and each $r > 0$, we have

$$\mu \{ x: \exists n A_n g(x) > r \} < \frac{1}{r} \int g d\mu.$$ 

Since $z$ is weakly random, for each $n$ the value $A_n f(z)$ exists. Thus, if $\lim_n A_n f(z)$ fails to exists, there are reals $a < b$ such that $A_n f(z) < a$ for infinitely many $n$, and $A_n f(z) > b$ for infinitely many $n$.

Let

$$\mathcal{V}_t = \{ x: \exists k A_k (f - f_t) > b - a \}.$$ 

Then $\mathcal{V}_t$ is uniformly $\Sigma_0^1$ nested sequence of open sets in $X$. By the maximal ergodic inequality we have $\mu \mathcal{V}_t \leq 1/(b-a) \int (f - f_t) d\mu$. Finally $\lim_n A_n f_t(z)$ exists for each $t$, and $\lim_n A_n f_t(z) \leq a$. Therefore $z \in \bigcap_t \mathcal{V}_t$. □

7. **Density-one points for $\Pi_0^n$ classes and $\Sigma_1^1$ classes**

In this section we work in the setting of Cantor space. So far we have looked at the density of $\Pi_0^n$ classes at points. Now we will consider classes of higher descriptional complexity. Firstly, we look at $\Pi_0^n$ classes. It turns out that if $Z$ is density random relative to $0^{(n-1)}$, then each $\Pi_0^n$ class has density 1 at $Z$.

Thereafter we consider the density of $\Sigma_1^1$ classes at $Z$. This complexity forms a natural bound for our investigation because $\Sigma_1^1$ classes are measurable (Lusin; see e.g. [32, Thm. 9.1.9]), which is no longer true within ZFC for more complex classes.
7.1. Density of $\Pi^0_n$ classes at a real. Recall that $Z$ is $n$-random if $Z$ is ML-random relative to $\emptyset^{(n-1)}$. By a $\Pi^0_n$ class we mean a $\Pi^0_1$ class relative to $X$. Every $\Pi^0_1,\emptyset^{(n-1)}$ class is $\Pi^0_n$. We show that for an $n$-random $Z$, it is sufficient to consider $\Pi^0_1,\emptyset^{(n-1)}$ classes in order to obtain that every $\Pi^0_n$ class has density one at $Z$. To do so, we rely on a lemma about the approximation in terms of measure of $\Pi^0_n$ classes by $\Pi^0_1,\emptyset^{(n-1)}$ subclasses. This can be seen as an effective form of regularity for Lebesgue measure. See [11, Thm. 6.8.3] for a recent write-up of the proof.

**Lemma 7.1** (Kurtz [24], Kautz [21]). From an index of a $\Pi^0_n$ class $P$ and $q \in \mathbb{Q}^+$, $\emptyset^{(n-1)}$ can compute an index of a $\Pi^0_1,\emptyset^{(n-1)}$ class $V \subseteq P$ such that $\lambda(P) - \lambda(V) < q$.

**Theorem 7.2.** Suppose $n \geq 1$ and $Z \in 2^{\mathbb{N}}$ is density random relative to $\emptyset^{(n-1)}$. Let $P$ be $\Pi^0_n$ class such that $Z \in P$. Then $g_n(P|Z) = 1$.

**Proof.** Let $P = \bigcap_s U_s$ where $\langle U_s : s \in \omega \rangle$ is a nested sequence of uniformly $\Sigma^0_{n-1}$ classes. It suffices to show that there exists a $\Pi^0_1,\emptyset^{(n-1)}$ class $Q \subseteq P$ such that $Z \in Q$.

We define a Solovay test relative to $\emptyset^{(n-1)}$. By Lemma 7.1 we $\emptyset^{(n-1)}$ effectively obtain an index of a $\Pi^0_1,\emptyset^{(n-1)}$ class $Q_s \subseteq U_s$ such that $\lambda(U_s) - \lambda(Q_s) < 2^{-n}$.

The sequence of uniformly $\Sigma^0_1,\emptyset^{(n-1)}$ classes

$\langle U_s \setminus Q_s : s \in \mathbb{N} \rangle$

is a Solovay test relative to $\emptyset^{(n-1)}$ since $\lambda(U_s \setminus Q_s) \leq 2^{-s}$. Notice $Z \in P \subseteq U_s$ for each $s \in \mathbb{N}$. Since $Z$ is Martin-Löf random relative to $\emptyset^{(n-1)}$, there exists $k \in \mathbb{N}$ such that for all $j \geq k$, $Z \in Q_j$. Since $\langle Q_j : j \geq k \rangle$ is a uniform sequence of $\Pi^0_1,\emptyset^{(n-1)}$ classes, $V = \bigcap_{j \geq k} Q_j$ is itself a $\Pi^0_1,\emptyset^{(n-1)}$ class. Also $V \subseteq \bigcap_{i \in \mathbb{N}} U_i = P$ because $Q_j \subseteq U_j$. We have found a $\Pi^0_1,\emptyset^{(n-1)}$ class $V \subseteq P$ that contains $Z$. \hfill $\square$

Relativizing Theorem 3.3 to $\emptyset^{(n-1)}$ we obtain:

**Corollary 7.3.** An $n$-random set $Z$ is a density one point for $\Pi^0_n$ classes if and only if every left-$\emptyset^{(n-1)}$.c.e. martingale converges along $Z$.

7.2. Higher randomness. The adjective “higher” indicates that algorithmic tools are replaced by tools from effective descriptive theory. See [32, Ch. 9] for background. The work of the Madison group described in Section 3 can be adapted to this setting. For a higher version of density randomness, instead of $\Pi^0_1$ classes we now look at $\Sigma^1_r$ classes containing the real in question. Similar to the foregoing case of $\Pi^0_n$ classes, it does not matter whether the $\Sigma^1_1$ class is closed.

We use the following fact due to Greenberg (personal communication). It is a higher analog of the original weaker version of Prop. 2.7(i) proved in Bienvenu et al. [1, Prop. 5.4]. The hypothesis on $Z$ could be weakened to a higher notion of partial computable randomness as well.
Proposition 7.4 (N. Greenberg, 2013). Let $C \subseteq 2^\mathbb{N}$ be $\Sigma^1_1$. Let $Z \in C$ be $\Pi^1_1$-ML-random. Then $\mathcal{T}_2(C \mid Z) = 1$.

Proof. If $\mathcal{T}_2(C \mid Z) < 1$ then there is a positive rational $q < 1$ and $n^*$ such that for all $n \geq n^*$ we have $\lambda_{Z \upharpoonright n}(C) < q$. Choose a rational $r$ with $q < r < 1$. We define $\Pi^1_1$-antichains in $U_n \subseteq 2^{<\omega}$, uniformly in $n$. Let $U_0 = \{Z \upharpoonright n^*\}$. Suppose $U_n$ has been defined. For each $\sigma \in U_n$, at a stage $\alpha$ such that $\lambda_\sigma(C_\alpha) < q$, we obtain effectively a hyper-arithmetic antichain $V$ of extensions of $\sigma$ such that $\mathcal{C}_\alpha \cap [\sigma] \subseteq [V]^{-}$ and $\lambda_\sigma([V]^{-}) < r$. Put $V$ into $U_{n+1}$.

Clearly $\lambda(U_n)^{<} \leq r^n$ for each $n$. Also, $Z \in \bigcap_n[U_n]^<$, so $Z$ is not $\Pi^1_1$-ML-random.

A martingale $L : 2^{<\omega} \to \mathbb{R}$ is called left-$\Pi^1_1$ if $L(\sigma)$ is a left-$\Pi^1_1$ real uniformly in $\sigma$.

Theorem 7.5. Let $Z$ be $\Pi^1_1$-ML-random. The following are equivalent.

(i) $\mathcal{T}_2(C \mid Z) = 1$ for each $\Sigma^1_1$ class $C$ containing $Z$.

(ii) $\mathcal{T}_2(C \mid Z) = 1$ for each closed $\Sigma^1_1$ class $C$ containing $Z$.

(iii) Each left-$\Pi^1_1$ martingale converges along $Z$ to a finite value.

Proof. (i) $\to$ (i): The measure of a $\Sigma^1_1$ set is left-$\Sigma^1_1$ in a uniform way (see e.g. [32, Thm. 9.1.10]). Therefore $M(\sigma) = 1 - \lambda_\sigma(C)$ is a left-$\Pi^1_1$ martingale. Since $M$ converges along $Z$, and since by Prop. 7.4 $\liminf M(Z \upharpoonright n) = 0$, it converges along $Z$ to 0. This shows that $\mathcal{T}_2(C \mid Z) = 1$.

(ii) $\to$ (iii). We follow the proof of the Madison group’s Theorem 3.3 given in [14, Section 13]. All stages $s$ are now interpreted as computable ordinals. Computable functions are now functions $\omega_1^{CK} \to L_{\omega_1^{CK}}$ with $\Sigma_1$ graph. Constructions are now assignments of recursive ordinals to instructions.

Definition 7.6. A $\Pi^1_1$-Madison test is a $\Sigma_1$ over $L_{\omega_1^{CK}}$ function $\langle U_s \rangle_{s < \omega_1^{CK}}$ mapping ordinals to (hyperarithmetical) subsets of $2^{<\omega}$ such that $U_0 = \emptyset$, for each stage $s$ we have $\text{wt}(U_s) \leq c$ for some constant $c$, and for all strings $\sigma, \tau$,

- (a) $\tau \in U_s - U_{s+1} \to \exists \sigma < \tau [\sigma \in U_{s+1} - U_s]$,
- (b) $\text{wt}(\sigma^< \cap U_s) > 2^{-|\sigma|} \to \sigma \in U_s$.

Also $U_t(\sigma) = \lim_{s < t} U_s(\sigma)$ for each limit ordinal $t$.

The following well-known fact can be proved similar to [32, 1.9.19].

Lemma 7.7. Let $A \subseteq 2^\mathbb{N}$ be a hyperarithmetical open. Given a rational $q$ with $q > \lambda A$, we can effectively determine from $A, q$ a hyperarithmetical open $S \supseteq A$ with $\lambda S = q$.

Lemma 7.8. Let $Z$ be a $\Pi^1_1$ ML-random such that $\mathcal{T}_2(C \mid Z) = 1$ for each closed $\Sigma^1_1$ class $C$ containing $Z$. Then $Z$ passes each $\Pi^1_1$-Madison test.

The proof follows the proof of the analogous Lemma 13.3 in [14, Section 13]. The sets $A^k_{\sigma, s}$ are now hyperarithmetical open sets computed from $k, \sigma, s$. Suppose $\sigma \in U_{s+1} - U_s$. The set $A^k_{\sigma, s}$ is defined as before. To effectively obtain $A^k_{\sigma, s+1}$, we apply Lemma 7.7 to add mass from $[\sigma]$ to $A^k_{\sigma, s+1}$ in order to ensure that $\lambda(A^k_{\sigma, s+1}) = 2^{-|\sigma|-k}$. 

\[ \text{(ii)} \]
As before, let \( S^k_t = \bigcup_{\sigma \in U_t} A^k_{\sigma,t} \). Then \( S^k_t \subseteq S^k_{t+1} \) by condition (a) on \( \Pi^1_1 \)-Madison tests. Clearly \( \lambda S^k_t \leq 2^{-k} \text{wt}(U_t) \leq 2^{-k} \). So \( S^k = \bigcup_{t<\omega} S^k_t \) determines a \( \Pi^1_1 \) ML-test.

By construction \( g_\omega(2^N - S^k \mid Z) \leq 1 - 2^{-k} \). Since \( Z \) is ML-random we have \( Z \notin S^k \) for some \( k \). So \( g_\omega(C \mid Z) < 1 \) for the closed \( \Sigma^1_1 \) class \( C = 2^N - S_k \) containing \( Z \).

The analog of Lemma 13.4 in [14, Section 13] also holds.

**Lemma 7.9.** Suppose that \( Z \) passes each \( \Pi^1_1 \)-Madison test. Then every left-\( \Pi^1_1 \) martingale \( L \) converges along \( Z \).

The proof of Lemma 13.4 in [14, Section 13] was already set up so that this works. If \( L \colon 2^{<\omega} \to \mathbb{R} \) is a left-\( \Pi^1_1 \) martingale, then \( L(\sigma) = \sup_s L_s(\sigma) \) for a non-decreasing sequence \( \{L_s\} \) of hyperarithmetic martingales computed uniformly from \( s < \omega_1^{CK} \). The labelling functions \( \gamma_s \colon U_s \to \omega_1^{CK} \) are now uniformly hyperarithmetical.

We may assume that \( L_t(\sigma) = \lim_{s \downarrow t} L_s(\sigma) \) for each limit ordinal \( t \). This implies \( U_t(\sigma) = \lim_{s \downarrow t} U_s(\sigma) \) for each limit ordinal \( t \) as required in the definition of higher Madison tests. \( \square \)

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