ON SUBGRAPHS OF TRIPARTITE GRAPHS

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Abstract. In 1975 Bollobás, Erdős, and Szemerédi [1] investigated a tripartite generalization of the Zarankiewicz problem: what minimum degree forces a tripartite graph with $n$ vertices in each part to contain an octahedral graph $K_3(2)$? They proved that $n + 2^{-1/2}n^{3/4}$ suffices and suggested it could be weakened to $n + cn^{1/2}$ for some constant $c > 0$. In this note we show that their method only gives $n + (1 + o(1))n^{11/12}$ and provide many constructions that show if true, $n + cn^{1/2}$ is better possible.

1. Introduction

Let $K_t$ denote the complete graph on $t$ vertices. As a foundation stone of extremal graph theory, Turán’s theorem in 1941 [10] determines the maximum number of edges in graphs of a given order not containing $K_t$ as a subgraph (the $t = 3$ case was proven by Mantel in 1907 [6]). In 1975 Bollobás, Erdős, and Szemerédi [1] investigated a tripartite generalization of the Turán-type problem for multipartite graphs.

Problem 1. Given integers $n$ and $3 \leq t \leq r$, what is the largest minimum degree $\delta(G)$ among all $r$-partite graphs $G$ with parts of size $n$ and which do not contain a copy of $K_t$?

The $r = t$ case of Problem 1 had been a central topic in Combinatorics until it was finally settled by Haxell and Szabó [4], and Szabó and Tardos [8]. Recently Lo, Treglown, and Zhao [9] solved many $r > t$ cases of the problem, including when $r \equiv -1 \pmod{t - 1}$ and $r = \Omega(t^2)$.

For simplicity, let $G_r(n)$ denote an (arbitrary) $r$-partite graph with parts of size $n$. Let $K_r(s)$ denote the complete $r$-partite graph with parts of size $s$. In particular, $K_3(2)$ is known as the Octahedral graph. In the same paper Bollobás, Erdős, and Szemerédi [1] also asked the following question.

Problem 2. Given a tripartite graph $G = G_3(n)$, what $\delta(G)$ guarantees a copy of $K_3(2)$?

Problem 2 is a natural generalization of the well-known Zarankiewicz problem [11], whose symmetric version asks for the largest number of edges in a bipartite graph $G_2(n)$ that contains no $K_3(s)$ as a subgraph (in other words, $K_3(s)$-free).

In [1] Corollary 2.7 the authors stated that $\delta(G) \geq n + 2^{-1/2}n^{3/4}$ guarantees a copy of $K_3(2)$. This follows from [1] Theorem 2.6, which handles the general case of $K_3(s)$ for arbitrary $s$. Unfortunately, there is a miscalculation in the proof of [1] Theorem 2.6 and thus the bound $\delta(G) \geq n + 2^{-1/2}n^{3/4}$ is unverified. We follow the approach of [1] Theorem 2.6 and obtain the following result.

Theorem 3. Given an integer $s \geq 2$ and $\varepsilon > 0$, let $n$ be sufficiently large. If $G = G_3(n)$ satisfies $\delta(G) \geq n + (1 + \varepsilon)(s - 1)1/((3s^2))n^{1-1/((3s^2))}$, then $G$ contains a copy of $K_3(s)$.

In particular, Theorem 3 implies that every $G = G_3(n)$ with $\delta(G) \geq n + (1 + o(1))n^{11/12}$ contains a copy of $K_3(2)$. Using a result of Erdős on hypergraphs [3], we give a different proof of Theorem 3 under a slightly stronger condition $\delta(G) \geq n + (3n)^{1-1/((3s^2))}$. Thus $cn^{11/12}$ is a natural additive term for Problem 2 under typical approaches for extremal problems.

On the other hand, the authors of [1] conjectured that $\delta(G) \geq n + cn^{1/2}$ suffices for Problem 2. Although not explained in [1], they probably thought of Construction 10 a natural construction based on the one for the Zarankiewicz problem. We indeed find many non-isomorphic constructions, Construction 11 with the same minimum degree.

Proposition 4. For any $n = q^2 + q + 1$ where $q$ is a prime power, there are many tripartite graphs $G = G_3(n)$ such that $\delta(G) \geq n + n^{1/2}$ and $G$ contains no $K_3(2)$.

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Theorem 3 and Proposition 4 together show that the answer for Problem 2 lies between \( n + n^{1/2} \) and \( n + n^{11/12} \). The truth may be closer to the lower bound. If this is the case, then verifying it may be hard given the presence of many non-isomorphic constructions.

We know less about the minimum degree of \( G_3(n) \) that forces a copy of \( K_3(s) \). Theorem 3 shows that \( \delta(G_3(n)) \geq n + cn^{1-1/(3s^2)} \) suffices. As shown in Remark 12 if there is a \( K_2(s) \)-free bipartite graph \( B = G_2(n) \) with \( \delta(B) = \Omega(n^{1-1/s}) \), then our constructions for Proposition 4 provide a tripartite \( K_3(s) \)-free graph \( G = G_3(n) \) with \( \delta(G) = n + \Omega(n^{1-1/s}) \).

2. Proof of Theorem 3

In order to prove Theorem 3 we need the following results from [1].

**Lemma 5.** [1] Theorem 2.3] Suppose every vertex of \( G = G_3(n) \) has degree at least \( n + t \) for some integer \( t \leq n \). Then there are at least \( t^3 \) triangles in \( G \).

**Lemma 6.** [1] Lemma 2.4] Let \( X = \{ 1, \ldots, N \} \) and \( Y = \{ 1, \ldots, p \} \). Suppose \( A_1, \ldots, A_p \) are subsets of \( X \) such that \( \sum_{i=1}^{p} |A_i| \geq pwN \) and \( (1-\alpha)wp \geq q \), where \( 0 < \alpha < 1 \) and \( N, p \) and \( q \) are natural numbers.

Then there are \( q \) subsets \( A_{i_1}, \ldots, A_{i_q} \) such that \( \left| \bigcap_{j=1}^{q} A_{i_j} \right| \geq N(\alpha w)^q \).

Let \( z(n, s) \) denote the largest number of edges in a bipartite \( K_2(s) \)-free graph with \( n \) vertices in each part. Kővári, Sós, and Turán [5] gave the following upper bound for \( z(n, s) \).

**Lemma 7.** \( z(n, s) \leq (s - 1)^{1/(s-1)}(n - s + 1)n^{1-1/s} + (s - 1)n \).

We are ready to prove Theorem 3.

**Proof of Theorem 3** Let \( G \) be a tripartite graph with three parts \( V_1, V_2, V_3 \) of size \( n \) each. Suppose \( \delta(G) \geq n + t \), where \( t = (1 + \varepsilon)(s - 1)^{1/(s-1)}n^{1-1/s} + (s - 1)n^{1-1/s} \). By Lemma 5, \( G \) contains at least \( t^3 \) triangles.

We apply Lemma 6 in the following setting. Let \( Y = V_1 = \{ 1, \ldots, n \} \) and \( X = V_2 \times V_3 \) be the set of \( n^2 \) pairs \( (x, y) \), \( x \in V_2, y \in V_3 \). For \( 1 \leq i \leq n \), let \( A_i \) be the set of pairs \( (x, y) \in X \) for which \( \{ i, x, y \} \) spans a triangle of \( G \). Then \( \sum_{i=1}^{n} |A_i| \) is the number of triangles in \( G \) so \( \sum_{i=1}^{n} |A_i| \geq t^3 \). Let \( N = n^2 \), \( p = n \), \( q = s \), \( w = t^3/n^3 \), and \( \alpha = 1/(1 + \varepsilon) \). The assumptions of Lemma 6 hold because \( pwN = t^3 \) and

\[
(1 - \alpha)wp = \frac{\varepsilon}{1 + \varepsilon} \left( \frac{t}{n} \right)^3 n > \frac{\varepsilon}{1 + \varepsilon} n^{1-1/s} n > s
\]

as \( n \) is sufficiently large. By Lemma 6 there are \( i_1, \ldots, i_s \in V_1 \) such that

\[
\left| \bigcap_{j=1}^{s} A_{i_j} \right| \geq N(\alpha w)^q = n^2 \left( \frac{t^3}{(1 + \varepsilon)n^3} \right)^s
\]

Since

\[
t > (1 + \varepsilon)^{\frac{3}{4}}(s - 1)\frac{1}{s} n^{1 - \frac{1}{s}} \quad \text{and} \quad \frac{t^3}{(1 + \varepsilon)n^3} > (1 + \varepsilon)(s - 1)^{1/(s-1)}n^{1-1/s} \]

we have

\[
(1) \left| \bigcap_{j=1}^{s} A_{i_j} \right| > (1 + \varepsilon)^{s}(s - 1)^{1/s}n^{2-1/s} \geq (s - 1)^{1/s}n^{2-1/s} + (s - 1)n.
\]

Let \( B \) denote the bipartite graph between \( V_2 \) and \( V_3 \) with \( E(B) = \bigcap_{j=1}^{s} A_{i_j} \). By (1) and Lemma 7, \( B \) contains a copy of \( K_2(s) \). Since every edge of \( B \) forms a triangle with each of \( i_1, \ldots, i_s \in V_1 \), this copy of \( K_2(s) \) together with \( i_1, \ldots, i_s \) span a desired copy of \( K_3(s) \) in \( G \).

We now give another proof of Theorem 3 with slightly larger \( \delta(G) \) by a classical result of Erdős on hypergraphs [3]. An \( r \)-uniform hypergraph or \( r \)-graph is a hypergraph such that all its edges contain exactly \( r \) vertices. Let \( K^r_s \) denote the complete \( r \)-partite \( r \)-graph with \( s \) vertices in each part, namely, its vertex set consists of disjoint parts \( V_1, \ldots, V_r \) of size \( s \), and edges set consists of all \( r \)-sets \( \{v_1, \ldots, v_r\} \) with \( v_i \in V_i \) for all \( i \).

**Lemma 8.** [3] Theorem 1] Given integers \( r, s \geq 2 \), let \( n \) be sufficiently large. Then every \( r \)-graph on \( n \) vertices with at least \( n^{r-s+1} \) edges contains a copy of \( K^r_s \).
Proposition 9. Let \( s \geq 2 \) and \( n \) be sufficiently large. Every tripartite graph \( G = G_3(n) \) with \( \delta(G) \geq n + (3n)^{1-1/(3s^2)} \) contains a copy of \( K_3(s) \).

Proof. Suppose \( G = G_3(n) \) satisfies \( \delta(G) \geq n + (3n)^{1-1/(3s^2)} \). By Lemma 8, \( G \) contains at least \((3n)^{3-1/s^2}\) triangles. Let \( H \) be the 3-graph on \( V(G) \), whose edges are triangles of \( G \). Then \( H \) has \( 3n \) vertices and at least \((3n)^{3-s^{-2}}\) edges. By Lemma 8 with \( r = 3 \) and \( s = 2 \), \( H \) contains a copy of \( K_3^3(s) \), which gives a copy of \( K_3(s) \) in \( G \). \( \square \)

3. Proof of Proposition 4

In this section we prove Proposition 4 by constructing many tripartite \( K_3(2) \)-free graphs \( G_3(n) \) with \( \delta(G_3(n)) \geq n + n^{1/2} \).

One main building block is a bipartite \( K_3(2) \)-free graph \( G_0 = G_2(n) \) with \( \delta(G_0) \geq \sqrt{n} \). First shown in [7], such a graph exists when \( n = q^2 + q + 1 \) and a projective plane of order \( q \) exists. Indeed, two parts of \( V(G) \) correspond to the points and lines of the projective plane and a point is adjacent to a line if and only if the point lies on the line. It is easy to see that such graph contains no \( K_2(2) \) and is regular with degree \( q + 1 > \sqrt{n} \).

Construction 10. Suppose \( G = G_3(n) \) has parts \( V_1, V_2 \) and \( V_3 \) each of size \( n \). Let the bipartite graphs between \( V_1 \) and \( V_2 \) and between \( V_1 \) and \( V_3 \) be complete, while the bipartite graph between \( V_2 \) and \( V_3 \) is \( G_0 \) defined above.

Since \( \deg_{G_0}(v) \geq \sqrt{n} \) for \( v \in V_2 \cup V_3 \), we have \( \delta(G) \geq n + \sqrt{n} \). Furthermore, \( G \) contains no \( K_3(2) \) because by the definition of \( G_0 \), there is no \( K_2(2) \) between \( V_2 \) and \( V_3 \).

We now provide a family of constructions with the same properties.

Construction 11. Let \( G = G_3(n) \) be a tripartite graph with parts \( V_1, V_2, \) and \( V_3 \) of size \( n \) each. Partition \( V_2 = X_2 \cup Y_2 \) arbitrarily such that \( \sqrt{n} \leq |X_2| \leq |Y_2| \). Partition \( V_3 = X_3 \cup Y_3 \) arbitrarily such that \( |X_3| = |Y_2| \) and \( |Y_3| = |X_2| \).

The bipartite graphs \( (V_1, X_2), (X_2, Y_3), (Y_3, Y_2), (Y_2, X_3) \), and \( (X_3, V_1) \) are complete, in other words, \( V_1, X_2, Y_3, Y_2, X_3 \) form a blowup of \( C_5 \). Let the bipartite graph between \( V_1 \) and \( Y_2 \cup Y_3 \) be isomorphic to \( G_0 \) (note that \( |X_2| + |Y_2| = |X_3| + |Y_3| = n \)).

For any vertex \( v \in X_2 \), \( \deg(v) = |V_1| + |Y_3| \geq n + \sqrt{n} \). The vertices \( v \in X_2 \) satisfy \( \deg(v) = |V_1| + |Y_3| \geq n + n/2 \). For any \( v \in Y_2 \), \( \deg(v) \geq |V_3| + \delta(G_0) \geq n + \sqrt{n} \). The same holds for the vertices of \( Y_3 \). At last, every vertex \( v \in V_1 \) satisfies \( \deg(v) \geq |X_2| + |X_3| + \delta(G_0) \geq n + \sqrt{n} \). These together show that \( \delta(G) \geq n + \sqrt{n} \).

Suppose \( G \) contains a copy of \( K_3(2) \) with vertex set \( S \). Then \( |S \cap V_i| = 2 \) for \( i = 1, 2, 3 \). Since there is no edge between \( X_2 \) and \( Y_3 \), either \( S \cap X_2 = \emptyset \) or \( S \cap X_3 = \emptyset \). Suppose, say, \( S \cap X_2 = \emptyset \), which forces \( |S \cap Y_2| = 2 \). Hence \( S \cap Y_2 \) and \( S \cap V_1 \) span a copy of \( K_2(2) \), contradicting the definition of \( G_0 \).

If letting \( X_2 = \emptyset = Y_3 \) in Construction 11 then we obtain Construction 10. Nevertheless, we prefer viewing Constructions 10 and 11 as different constructions because after removing \( o(n^2) \) edges, Construction 11 contains many 5-cycles while Construction 10 does not.

\[ \text{Figure 1. Graph from Construction 11} \]
Remark 12. If we replace $G_0$ by a $K_2(s)$-free bipartite graph with $n$ vertices in each part in Constructions 10 and 11, then we obtain a $K_3(s)$-free tripartite graph $G_3(n)$. It has been conjectured that there exist a $K_2(s)$-free bipartite graph with $n$ vertices in each part and $\Omega(n^{s+1/s})$ edges (this is known for $s = 2, 3$ [2, 7]). If such bipartite graph exists and is regular, then (revised) Constructions 10 and 11 provide a $K_3(s)$-free tripartite graph $G = G_3(n)$ with $\delta(G) = n + \Omega(n^{1-1/s})$.

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