INFINITE ORDER LINEAR DIFFERENCE EQUATION SATISFIED BY A REFINEMENT OF GOSS ZETA FUNCTION

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Abstract. At the international congress of mathematicians in 1900, Hilbert claimed that the Riemann zeta function \( \zeta(s) \) is not the solution of any algebraic ordinary differential equations on its region of analyticity. Let \( T \) be an infinite order linear differential operator introduced by Van Gorder in 2015. Recently, Prado and Klinger-Logan [9] showed that the Hurwitz zeta function \( \zeta(s, a) \) formally satisfies the following linear differential equation

\[
T \left[ \zeta(s, a) - \frac{1}{a^s} \right] = \frac{1}{(s - 1)a^s - 1}.
\]

Then in [6], by defining \( T_p \), a \( p \)-adic analogue of Van Gorder’s operator \( T \), we constructed the following convergent infinite order linear differential equation satisfied by the \( p \)-adic Hurwitz-type Euler zeta function \( \zeta_{p,E}(s, a) \)

\[
T_p \left[ \zeta_{p,E}(s, a) - \langle a \rangle^{1-s} \right] = \frac{1}{s - 1} \left( (a - 1)^{1-s} - \langle a \rangle^{1-s} \right).
\]

In this paper, we consider this problem in the positive characteristic case. That is, by introducing \( \zeta_\infty(s_0, s, a, n) \), a Hurwitz type refinement of Goss zeta function, and an infinite order linear difference operator \( L \), we establish the following difference equation

\[
L \left[ \zeta_\infty \left( \frac{1}{T}, s, a, 0 \right) \right] = \sum_{\gamma \in \mathbb{F}_q} \frac{1}{(a + \gamma)^s}.
\]

1. Introduction

Let

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1
\]

be the Riemann zeta function. At the international congress of mathematicians in 1900, Hilbert [5] claimed that \( \zeta(s) \) is not the solution of any algebraic ordinary differential equations on its region of analyticity. In 2015, Van Gorder [12] showed that \( \zeta(s) \) formally satisfies an infinite order linear differential equation

\[
T[\zeta(s) - 1] = \frac{1}{s - 1}.
\]

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For $0 < a \leq 1$, $\text{Re}(s) > 1$, let

$$
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},
$$

be the Hurwitz zeta function (see [4]). This function can be analytically continued to a meromorphic function in the complex plane with a simple pole at $s = 1$. In 2020, Prado and Klinger-Logan [9] showed that the Hurwitz zeta function $\zeta(s, a)$ also formally satisfies an infinite order linear differential equation under the same operator $T$

$$
T \left[ \zeta(s, a) - \frac{1}{a^s} \right] = \frac{1}{(s-1)a^{s-1}}
$$

for $s \in \mathbb{C}$ satisfying $s+n \neq 1$ for all $n \in \mathbb{Z}_{>0}$. But unfortunately, in the same paper they proved that the operator $T$ applied to Hurwitz zeta function, does not converge at any point in the complex plane $\mathbb{C}$ (see [9, Theorem 8]).

The Hurwitz-type Euler zeta function

$$
\zeta_E(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}.
$$

is an alternating form of the Hurwitz zeta function $\zeta(s, a)$. This function can be analytically continued to the complex plane without any pole. Let $\zeta_{p,E}(s, a)$ be its $p$-adic analogue, which is defined from the integral transform

$$
\zeta_{p,E}(s, a) = \int_{\mathbb{Z}_p} \langle a + t \rangle^{-s} d\mu_{-1}(t)
$$

for $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, where

$$
I_{-1}(f) = \int_{\mathbb{Z}_p} f(t) d\mu_{-1}(t) = \lim_{r \to \infty} \sum_{k=0}^{p^r-1} f(k)(-1)^k.
$$

is the fermionic $p$-adic integral for a continuous function $f$ on $\mathbb{Z}_p$. For details on the definition and the properties of $\zeta_{p,E}(s, a)$, we refer to [7].

In a recent paper [6], by introducing an operator $T^a_p$, a $p$-adic analogue of Van Gorder’s operator $T$, we showed that the $p$-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s, a)$ satisfies an infinite order linear differential equation

$$
T^a_p \left[ \zeta_{p,E}(s, a) - \langle a \rangle^{1-s} \right] = \frac{1}{s-1} \left( \langle a - 1 \rangle^{1-s} - \langle a \rangle^{1-s} \right)
$$

(see [6, Theorem 3.5]). In contrast with the complex case, we proved that the left hand side of the above equation is convergent everywhere for $s \in \mathbb{Z}_p$ with $s \neq 1$ and $a \in F$ with $|a|_p > 1$, where $F$ is any finite extension of $\mathbb{Q}_p$ with ramification index over $\mathbb{Q}_p$ less than $p - 1$ (see [6, Corollary 3.8]).

In this paper, we consider this problem in the positive characteristic case by following the strategy of [6]. An analogue of Riemann zeta function $\zeta(s)$ in the positive characteristic is defined by Goss in 1979 [1], now known as the Goss zeta function. By introducing $\zeta_{\infty}(s_0, s, a, n)$, a Hurwitz type
refinement of Goss zeta function (see (2.8)), and an infinite order linear difference operator $L$ (see (2.13)), we prove that

\begin{equation}
L \left[ \zeta_\infty \left( \frac{1}{T}, s, a, 0 \right) \right] = \sum_{\gamma \in \mathbb{F}_q} \frac{1}{(a + \gamma)^s}.
\end{equation}

(see Theorem 2.1). It may be viewed as an analogue of (1.4) and (1.7) in the positive characteristic setting.

Our paper will be organized as follows. In Section 2, we first recall the definition of the Goss zeta function and introduce a Hurwitz type refinement of it, then we define an infinite order linear difference operator $L$. After these, we shall state our main result (Theorem 2.1). In Section 3, we will investigate the analytic properties of the function $\zeta_\infty(s_0, s, a, n)$ (see Proposition 3.3). In Section 4, after proving a shifted identity for $\zeta_\infty(s_0, s, a, n)$ (Lemma 4.1), we will get the main result.

2. Goss zeta function and its refinement

In this section, we first recall the definition of the Goss zeta function, then introduce a Hurwitz type refinement of it.

First we state some notations. Let $q = p^k$ be a power of a prime number $p$. Let $R = \mathbb{F}_q[T]$ be the polynomial ring with one variable over the finite field $\mathbb{F}_q$ and $K = \mathbb{F}_q(T)$ be the rational function field. For a polynomial $a \in R$, $\deg(a)$ denotes its degree. Let $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the completion of $K$ at the infinite place $\infty = (\frac{1}{T})$ and $K_\infty^* = K_\infty \setminus \{0\}$ be the multiplicative group of the field $K_\infty$. Let $\mathbb{Z}_p$ be the ring of $p$-adic integers. For an element $a \in K_\infty$, it has the expansion

\begin{equation}
a = a_m T^m + a_{m-1} T^{m-1} + \cdots + a_0 + a_{-1} T^{-1} + \cdots
\end{equation}

with $a_j \in \mathbb{F}_q$ ($j \leq m$) and $a_m \neq 0$. Let

\begin{equation}
\text{sgn}_\infty(a) = a_m, \ v_\infty(a) = -m
\end{equation}

and the absolute value

$$ |a|_\infty = \left( \frac{1}{e} \right)^{v_\infty(a)}. $$

Define

\begin{equation}
\langle a \rangle = a_m^{-1} T^{v_\infty(a)} a = 1 + a_m^{-1} a_{m-1} T^{-1} + \cdots
\end{equation}

and

\begin{equation}
\omega_\infty(a) = \frac{a}{\langle a \rangle} = a_m T^{-v_\infty(a)}.
\end{equation}

$\omega_\infty$ is an analogue of the Teichmüller character in the $p$-adic analysis. Furthermore let

$$ \mathbb{S} = K_\infty^* \times \mathbb{Z}_p, $$

which is an analogue of the complex plane $\mathbb{C}$. The set $\mathbb{S}$ has endowed with a product topology from $K_\infty^*$ and $\mathbb{Z}_p$. (See [1, Definition 2.1]).
Let $R_1$ be the set of monic polynomials in $R$, which is an analogue of the set of positive integers. For $a \in R_1$, $w = (s_0, s) \in \mathbb{S} = K^*_\infty \times \mathbb{Z}_p$, define
\begin{equation}
\label{eq Barnett2.5}
aw = s_0^{-v_\infty(a)} \langle a \rangle^s,
\end{equation}
which is an analogue of $n^s$ in the classical case.

From this, in 1979 Goss \cite{Goss79} introduced an analogue of Riemann zeta function $\zeta(s)$ in the characteristic $p$ case. In fact, he considered the series
\begin{equation}
\label{eq Barnett2.6}
\zeta_\infty(s_0, s) = \sum_{l=0}^{\infty} \sum_{a \in R_1 \, \deg_a = l} a^{-w} = \sum_{l=0}^{\infty} s_0^{-l} \sum_{a \in R_1 \, \deg_a = l} \frac{1}{\langle a \rangle^s}
\end{equation}
for $w = (s_0, s) \in \mathbb{S} = K^*_\infty \times \mathbb{Z}_p$ and showed that it is an entire function on $\mathbb{S}$. We will recall the concept of entirety for a function on $\mathbb{S}$ in Section 3 (see Definition 3.1).

Inspired by the definition of Hurwitz zeta functions $\zeta(s, a)$, in the following we introduce a refinement of the Goss zeta function. Denote by
\begin{equation}
\label{eq Barnett2.7}
\mathbb{A} = \{a \in K^*_\infty \mid |a|_\infty > 1\}.
\end{equation}
Let $\mathbb{F}_q[\frac{1}{T}]$ be the polynomial ring in $\frac{1}{T}$. If $k \in \mathbb{F}_q[\frac{1}{T}]$, then $\deg_\infty(k)$ denotes its degree as a polynomial in $\frac{1}{T}$. Now for a fixed $(a, n) \in \mathbb{A} \times \mathbb{Z}$ with $v_\infty(a) = -m$, we define the zeta function
\begin{equation}
\label{eq Barnett2.8}
\zeta_\infty(s_0, s, a, n) = \sum_{l=0}^{\infty} s_0^{-(m+l+1)n} \sum_{k \in \mathbb{F}_q[\frac{1}{T}] \, \deg_\infty(k) \leq l} \frac{1}{\langle k + a \rangle^s}
\end{equation}
on the plane $(s_0, s) \in \mathbb{S} = K^*_\infty \times \mathbb{Z}_p$.

In the above construction \eqref{eq Barnett2.5}, since $a \in \mathbb{A}$, we have $|a|_\infty > 1$, and for $k \in \mathbb{F}_q[\frac{1}{T}]$, we have $|k|_\infty \leq 1$, which correspond to $a \in \mathbb{C}_p$ with $|a|_p > 1$, and $t \in \mathbb{Z}_p$ thus $|t|_p \leq 1$ in the definition of the $p$-adic Hurwitz zeta functions in the characteristic zero case, respectively (see \eqref{eq Barnett1.5}). These lead the applications of the binomial theorem to derive the shifted identities \eqref{eq Barnett4.1} \cite{Barnett15} \eqref{eq Barnett3.1} possible. It may be interesting to compare the proof of Lemma 4.1 below and the proof of Lemma 3.1 in \cite{Barnett15}, especially \eqref{eq Barnett4.6} \cite{Barnett15} \eqref{eq Barnett3.4}.

In the following, we shall introduce an infinite order linear difference operator $L$ from the forward difference operator $\Delta$.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. The forward difference operator $\Delta$ is defined by
\[ \Delta a_n = a_{n+1} - a_n, \]
and the higher order differences may be defined recursively by
\[ \Delta^h a_n = \Delta^{h-1} a_{n+1} - \Delta^{h-1} a_n. \]
Thus
\[ \Delta^h a_n = \sum_{l=0}^{h} (-1)^l \binom{h}{l} a_{n+h-l} \]
for \( h = 1, 2, 3, \ldots \) and a formula for the \((n+j)\)th term is given by

\[
a_{n+j} = (1 + \Delta)^j a_n := \sum_{h=0}^{j} \binom{j}{h} \Delta^h a_n
\]

(see [10, p. 10]).

Now given a function

\[
f : \mathbb{Z}_p \times \mathbb{Z} \to K_\infty
\]

\[(s, n) \mapsto f(s, n),\]

define a forward difference operator \( \Delta_{(s,n)} \) by

\[
\Delta_{(s,n)} f(s, n) = f(s + 1, n + 1) - f(s, n).
\]

Then from (2.9) we have

\[
f(s + j, n + j) = (1 + \Delta_{(s,n)})^j f(s, n)
\]

for \( j \in \mathbb{N} \). With the above forward difference operator \( \Delta_{(s,n)} \), we introduce an infinite order linear difference operator \( L \) by

\[
L := \text{id} + \sum_{j=1}^{\infty} \frac{(-s)}{j} (1 + \Delta_{(s,n)})^j.
\]

Now we are at the position to state our main result.

**Theorem 2.1.** For \( a \in A \) and \( s \in \mathbb{Z}_p \), \( \zeta_\infty(s_0, s, a, n) \) satisfies the following infinite order linear difference equation

\[
L \left[ \zeta_\infty \left( \frac{1}{T}, s, a, 0 \right) \right] = \sum_{\gamma \in \mathcal{F}_a} \frac{1}{(a + \gamma)^s}.
\]

3. The entirety of \( \zeta_\infty(s_0, s, a, n) \)

In this section, to investigate the analytic properties of \( \zeta_\infty(s_0, s, a, n) \), we first recall the concept of entirety for a function on \( \mathbb{S} = K_\infty^* \times \mathbb{Z}_p \) according to Goss's book [2, p. 248–249].

**Definition 3.1** (see [2, p. 248, Definition 8.5.1]). We define an entire function \( f(w) = f(s_0, s) \) on \( \mathbb{S} \) to be a continuous family of \( K_\infty \)-valued entire power series in \( s_0^{-1} \) which is parametrized by \( \mathbb{Z}_p \). Moreover, this family is required to be uniformly convergent on bounded subsets of \( K_\infty \).

The above definition means that a function \( f(w) = f(s_0, s) : \mathbb{S} \to K_\infty \) is said to be entire if and only if we have the following power series expansion

\[
f(w) = f(s_0, s) = \sum_{l=0}^{\infty} f_l(s) s_0^{-l},
\]

where \( f_l(s) : \mathbb{Z}_p \to K_\infty \) is continuous, and the above series is uniformly convergent for \( s_0 \) in any bounded subset of \( K_\infty \).

In Proposition 3.3 below, we will consider the entirety of \( \zeta_\infty(s_0, s, a, n) \). First, we need the following lemma.
Lemma 3.2. For a fixed $\alpha \in K^*_\infty$, $f(s) = \langle \alpha \rangle^s$ is a continuous function from $\mathbb{Z}_p$ to $K_\infty$.

Proof. For $\alpha \in K^*_\infty$ we have

$$\langle \alpha \rangle = 1 + \lambda_\alpha$$

with $|\lambda_\alpha|_\infty < 1$. Then by [2] p. 237, Definition 8.1.2, for $s \in \mathbb{Z}_p$

$$\langle \alpha \rangle^s = \sum_{j=0}^{\infty} \binom{s}{j} \lambda_\alpha^j.$$  

Here

$$\binom{s}{j} = \frac{s(s-1) \cdots (s-j+1)}{j!}$$

gives a continuous function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$. By reducing it modulo $(p)$, we can consider it as a continuous function with values in $\mathbb{F}_p \subset \mathbb{F}_q \subset K_\infty$ (see [2] p. 245, Definition 8.4.2). So

$$\langle \alpha \rangle^s < 1$$

and

$$|\langle \alpha \rangle^s|_\infty \leq \sum_{j=0}^{\infty} |\lambda_\alpha|^j.$$  

Since for a fixed $\langle \alpha \rangle = 1 + \lambda_\alpha$ with $|\lambda_\alpha|_\infty < 1$, from the Weierstrass test, the power series (3.1) is uniformly convergent for $s \in \mathbb{Z}_p$. By [8] p. 182, Theorem 3.2], $f(s) = \langle \alpha \rangle^s$ is a continuous function from $\mathbb{Z}_p$ to $K_\infty$. $\square$

Proposition 3.3 (The entirety of $\zeta_\infty(s_0, s, a, n)$). Fixed $(a, n) \in A \times \mathbb{Z}$, the function $\zeta_\infty(s_0, s, a, n)$ is an entire function for $(s_0, s) \in \mathbb{S} = K^*_\infty \times \mathbb{Z}_p$.

Proof. For $k \in \mathbb{F}_q[1_T]$ with deg$_\infty(k) \leq l$, we have the expansion

$$k = b_{-l}T^{-l} + b_{-l+1}T^{-l+1} + \cdots + b_1T^{-1} + b_0$$

with $b_j \in \mathbb{F}_q$ ($-l \leq j \leq 0$). And for $a \in A$ so $v_\infty(a) = -m < 0$ (see [2.7]), we have the expansion

$$a = a_mT^m + a_{m-1}T^{m-1} + \cdots + a_0 + a_{-1}T^{-1} + \cdots$$

with $a_j \in \mathbb{F}_q$ ($j \leq m$) and $a_m \neq 0$. Thus by (2.3) we have

$$\langle k + a \rangle = a_m^{-1}b_{-m}T^{-m-l} + a_{m-1}^{-1}b_{-m+1}T^{-m-l+1} + \cdots + a_0^{-1}b_0T^{-m}$$

$$+ 1 + a_m^{-1}a_{m-1}^{-1}T^{-1} + \cdots + a_0^{-1}a_0^{-1}T^{-m} + a_m^{-1}a_{-1}T^{-m-1} + \cdots$$

$$= (1 + a_m^{-1}a_{m-1}^{-1}T^{-1} + \cdots + a_0^{-1}a_0^{-1}T^{-m} + a_m^{-1}a_{-1}T^{-m-1} + \cdots)$$

$$+ (a_m^{-1}b_{-m}T^{-m-l} + a_{m-1}^{-1}b_{-m+1}T^{-m-l+1} + \cdots + a_0^{-1}b_0T^{-m}).$$

Let

$$x = 1 + a_m^{-1}a_{m-1}^{-1}T^{-1} + \cdots + a_0^{-1}a_0^{-1}T^{-m} + a_m^{-1}a_{-1}T^{-m-1} + \cdots$$

and

$$w = a_m^{-1}b_{-m}T^{-m-l} + a_{m-1}^{-1}b_{-m+1}T^{-m-l+1} + \cdots + a_0^{-1}b_0T^{-m}.$$
So by (3.4) we have

\[(3.5) \quad \langle k + a \rangle = x + w.\]

Note that if \(k\) varies over the set

\[(3.6) \quad C = \left\{ k \in \mathbb{F}[\frac{1}{T}] \mid \deg_{\infty}(k) \leq l \right\},\]

then \(w\) will run through the space

\[(3.7) \quad W = \{ \beta_{l}T^{-m} + \beta_{l-1}T^{-m-1} + \cdots + \beta_{0}T^{-m} \mid \beta_{0}, \beta_{1}, \ldots, \beta_{l} \in \mathbb{F} \}.\]

Since the function

\[(3.8) \quad \mathbb{Z}_{p} \to K_{\infty} \quad j \mapsto \langle k + a \rangle^{j}\]

is continuous and the natural number system \(\mathbb{N}\) is dense in \(\mathbb{Z}_{p}\), we only need to consider the case for \(j \in \mathbb{N}\). By (3.5), for \(j \in \mathbb{N}\) we have

\[(3.9) \quad \sum_{k \in \mathbb{F}[\frac{1}{T}], \deg_{\infty}(k) \leq l} \langle k + a \rangle^{j} = \sum_{w \in W} (x + w)^{j} = \sum_{\beta_{0}, \beta_{1}, \ldots, \beta_{l} \in \mathbb{F}} (x + \beta_{l}T^{-m} + \beta_{l-1}T^{-m-1} + \cdots + \beta_{0}T^{-m})^{j}\]

\[= \sum_{0 \leq j_{-1}, j_{0}, j_{1}, \ldots, j_{l} \leq j} \left( \sum_{\beta_{l} \in \mathbb{F}} \beta_{l}^{j_{l}} \right) \left( \sum_{\beta_{l-1} \in \mathbb{F}} \beta_{l-1}^{j_{l-1}} \right) \cdots \left( \sum_{\beta_{0} \in \mathbb{F}} \beta_{0}^{j_{0}} \right) \left( x^{j} \right)\]

\[= \sum_{0 \leq j_{-1}, j_{0}, j_{1}, \ldots, j_{l} \leq j} \left( \sum_{\beta_{l} \in \mathbb{F}} \beta_{l}^{j_{l}} \right) \left( \sum_{\beta_{l-1} \in \mathbb{F}} \beta_{l-1}^{j_{l-1}} \right) \cdots \left( \sum_{\beta_{0} \in \mathbb{F}} \beta_{0}^{j_{0}} \right) \left( x^{j} \right)\]

\[= \sum_{\beta_{l} \in \mathbb{F}} \beta_{l}^{j_{l}} T^{(-m-l)j_{l}} \left( \sum_{\beta_{l-1} \in \mathbb{F}} \beta_{l-1}^{j_{l-1}} T^{(-m-l+1)j_{l-1}} \cdots \left( \sum_{\beta_{0} \in \mathbb{F}} \beta_{0}^{j_{0}} T^{(-m)j_{0}} \right) \right)\]

where

\[\left( \sum_{\beta_{j} \in \mathbb{F}} \beta_{j}^{j_{j}} \right) T^{(-m-j)j_{j}} = 0\]

and \(0^{0} = 1\) by convention.

For \(j_{h} < (q - 1)\), we have

\[\left( \sum_{\beta_{j} \in \mathbb{F}} \beta_{j}^{j_{j}} \right) T^{(-m-j)j_{h}} = 0.\]
For $j_h \geq q - 1$, we have

$$v_\infty \left( \left( \sum_{j \in F_q} \beta^j_j \right) T(-m-j)j_h \right) \geq v_\infty (T(-m-j)j_h)$$

(3.10)

$$= (m + j)j_h \geq (q - 1)(m + j)$$

and

$$\left| \left( \sum_{j \in F_q} \beta^j_j \right) T(-m-j)j_h \right|_\infty \leq \left( \frac{1}{e} \right)^{(q-1)(m+j)}$$

(3.11)

Since the valuations

$$\left| j \right|_{\infty} \leq 1$$

and $|x^{j-1}|_\infty = 1$,

by (3.9) and (3.11) we have

$$\left| \sum_{k \in F_q} \frac{1}{(k + a)^j} \right|_{\infty} \leq \left( \frac{1}{e} \right)^{(q-1)(2m+l)(l+1)}$$

(3.12)

Thus for any $s \in \mathbb{Z}_p$

$$\left| \sum_{k \in F_q} \frac{1}{(k + a)^s} \right|_{\infty} \leq \left( \frac{1}{e} \right)^{(q-1)(2m+l)(l+1)}$$

(3.13)

Notice that

$$v_\infty (s_0^{-(m+l+1)n}) = -(m + l + 1)n v_\infty (s_0)$$

and

$$\left| s_0^{-(m+l+1)n} \right|_{\infty} = \left( \frac{1}{e} \right)^{-(m+l+1)n v_\infty (s_0)}$$

(3.14)

Combining (3.13) and (3.15) we have

$$\left| \sum_{k \in F_q} \frac{1}{(k + a)^s} \right|_{\infty} \leq \left( \frac{1}{e} \right)^{(q-1)(2m+l)(l+1)}$$

(3.15)

Thus for any $s \in \mathbb{Z}_p$

$$\left| \sum_{k \in F_q} \frac{1}{(k + a)^{-(m+l+1)n}} \right|_{\infty} \leq \left( \frac{1}{e} \right)^{(q-1)(2m+l)(l+1)-(m+l+1)n v_\infty (s_0)}$$

(3.16)
So

\[
\lim_{l \to \infty} \left| s_0^{-m(l+1)n} \sum_{\substack{k \in \mathbb{N}[\frac{1}{T}] \\ \deg_\infty(k) \leq l}} \frac{1}{(k+a)^s} \right|_\infty = 0
\]

uniformly for \( s_0 \) in any bounded subset of \( K_\infty \). Thus for any fixed \((a, n) \in A \times \mathbb{Z}\) with \( v_\infty(a) = -m \), the series

\[
(3.18) \quad \zeta_\infty(s_0, s, a, n) = \sum_{l=0}^{\infty} s_0^{-(m+l+1)n} \sum_{\substack{k \in \mathbb{N}[\frac{1}{T}] \\ \deg_\infty(k) \leq l}} \frac{1}{(k+a)^s}
\]

is uniformly convergent for \( s_0 \) in any bounded subset of \( K_\infty \). So by Definition 3.1 and Lemma 3.2, we see that \( \zeta_\infty(s_0, s, a, n) \) is an entire function for \((s_0, s) \in S = K_\infty^* \times \mathbb{Z}_p\). □

4. Proof of the main result

In this section, we shall prove Theorem 2.1, which is implied by the following shifted identity for \( \zeta_\infty(s_0, s, a, n) \).

Lemma 4.1. For \( a \in A \) and \( s \in \mathbb{Z}_p \), \( \zeta_\infty(s_0, s, a, n) \) satisfies the following identity

\[
(4.1) \quad \sum_{\gamma \in \mathbb{F}_q} \frac{1}{(a + \gamma)^s} = \zeta_\infty \left( \frac{1}{T}, s, a, 0 \right) + \sum_{j=1}^{\infty} \binom{-s}{j} \zeta_\infty \left( \frac{1}{T}, s + j, a, j \right).
\]

Remark 4.2. This is a positive characteristic analogue of two known identities in the characteristic zero case. One is the complex identity for Hurwitz zeta function \( \zeta(s, a) \) (see [9, Lemma 1]), the other is the \( p \)-adic identity for \( p \)-adic Hurwitz-type Euler zeta function \( \zeta_{p,E}(s, a) \) (see [6, Lemma 3.1]).

Remark 4.3. It may be interesting to compare (4.1) with the following recurrence relation for the special values of the Goss zeta function

\[
\zeta_\infty(-N) = 1 - \sum_{j=0}^{N-1} \binom{N}{j} (-1)^{(N-j)} \zeta_\infty(-j)
\]

for \( N \in \mathbb{N} \) (see [11, Theorem 5.6]).

Proof of Lemma 4.1. For \( s \in \mathbb{Z}_p \) and \( N \in \mathbb{N} \) we have

\[
(4.2) \quad \sum_{\gamma \in \mathbb{F}_q} \frac{1}{(a + \gamma)^s} = \sum_{l=0}^{N} \sum_{\substack{k \in \mathbb{N}[\frac{1}{T}] \\ \deg_\infty(k) \leq l}} \frac{1}{(k+a)^s} - \sum_{l=1}^{N} \sum_{\substack{k \in \mathbb{N}[\frac{1}{T}] \\ \deg_\infty(k) \leq l}} \frac{1}{(k+a)^s}.
\]

Denote by

\[
(4.3) \quad \tilde{W} = \{ \beta_{l-1}T^{-m-l+1} + \cdots + \beta_0T^{-m} | \beta_0, \beta_1, \ldots, \beta_{l-1} \in \mathbb{F}_q \}.
\]
Obviously,
\[ \dim_{\mathbb{F}_q} \widetilde{W} = \dim_{\mathbb{F}_q} W - 1 = (l + 1) - 1 = l. \]

By Eqs. (3.5), (3.6) and (3.7) we have
\begin{equation}
\sum_{k \in \mathbb{F}_q \setminus \{1\}} \frac{1}{(k + a)^s} = \sum_{w \in W} \frac{1}{(x + w)^s} = \sum_{\beta \in \mathbb{F}_q} \sum_{\beta \in \mathbb{F}_q} \frac{1}{(x + \bar{w} + \beta T^{-m-l})^s}.
\end{equation}

So (4.2) implies that
\begin{equation}
\sum_{\gamma \in \mathbb{F}_q} \frac{1}{(a + \gamma)^s} = \sum_{l=0}^{N} \sum_{w \in W} \frac{1}{(x + w)^s} - \sum_{l=1}^{N} \sum_{w \in W} \frac{1}{(x + w)^s} = \sum_{l=0}^{N} \sum_{w \in W} \frac{1}{(x + w)^s} - \sum_{l=0}^{N-1} \sum_{\beta_{l+1} \in \mathbb{F}_q} \sum_{w \in W} \frac{1}{(x + w + \beta_{l+1} T^{-m-l-1})^s}.
\end{equation}

Since \( x + w \) is a 1-unit, and \( |a|_\infty > 1 \) so \( v_\infty(a) = -m < 0 \), we have
\[ \left| \frac{\beta_{l+1} T^{-m-l-1}}{x + w} \right|_\infty < 1 \]
for \( l \in \mathbb{N} \cup \{0\} \). So
\begin{equation}
\left( \frac{x + w}{x + w + \beta_{l+1} T^{-m-l-1}} \right)^s = \left( 1 + \frac{\beta_{l+1} T^{-m-l-1}}{x + w} \right)^{-s} = \sum_{j=0}^{\infty} \binom{-s}{j} \beta_{l+1}^j \left( \frac{1}{T} \right)^{(m+l+1)j} \frac{1}{(x + w)^j}.
\end{equation}
and

\[(4.7)\]

\[
\sum_{\beta_{l+1} \in \mathbb{F}_q} \left( \frac{x + w}{x + w + \beta_{l+1} T^{-m-l-1}} \right)^s = \sum_{\beta_{l+1} \in \mathbb{F}_q} \left( 1 + \frac{\beta_{l+1} T^{-m-l-1}}{x + w} \right)^{-s} = \sum_{j=0}^{\infty} \binom{-s}{j} \left( \sum_{\beta_{l+1} \in \mathbb{F}_q} \beta_{l+1}^j \right) \left( \frac{1}{T} \right)^{(m+l+1)j} \frac{1}{(x + w)^j}.
\]

Note that

\[
\sum_{\beta_{l+1} \in \mathbb{F}_q} \beta_{l+1}^j = 0 \text{ if } (q - 1) \not| j \text{ or } j = 0
\]

and

\[
\sum_{\beta_{l+1} \in \mathbb{F}_q} \beta_{l+1}^j = q - 1 = -1 \text{ if } (q - 1) | j \text{ and } j \geq 1,
\]

(4.7) implies

\[(4.8)\]

\[
\sum_{\beta_{l+1} \in \mathbb{F}_q} \left( \frac{x + w}{x + w + \beta_{l+1} T^{-m-l-1}} \right)^s = -\sum_{j=1}^{\infty} \binom{-s}{j} \left( \frac{1}{T} \right)^{(m+l+1)j} \frac{1}{(x + w)^j}.
\]

Substituting the above expansion into (4.5), we have

\[(4.9)\]

\[
\sum_{\gamma \in \mathbb{F}_q} \frac{1}{(a + \gamma)^s} = \sum_{l=0}^{N-1} \sum_{w \in W_{\dim_q W = l+1}} \frac{1}{(x + w)^s} \left( 1 + \sum_{j=1}^{\infty} \binom{-s}{j} \left( \frac{1}{T} \right)^{(m+l+1)j} \frac{1}{(x + w)^j} \right)
\]

\[
+ \sum_{k \in \mathbb{F}_q} \frac{1}{(k + a)^s} = \sum_{l=0}^{N-1} \sum_{w \in W_{\dim_q W = l+1}} \frac{1}{(x + w)^s} + \sum_{j=1}^{\infty} \binom{-s}{j} \sum_{l=0}^{N-1} \left( \frac{1}{T} \right)^{(m+l+1)j} \sum_{w \in W_{\dim_q W = l+1}} \frac{1}{(x + w)^{s+j}}
\]

\[
+ \sum_{k \in \mathbb{F}_q} \frac{1}{(k + a)^s} = \sum_{l=0}^{N-1} \sum_{k \in \mathbb{F}_q_{\deg_k(k) \leq l}} \frac{1}{(k + a)^s} + \sum_{j=1}^{\infty} \binom{-s}{j} \sum_{l=0}^{N-1} \left( \frac{1}{T} \right)^{(m+l+1)j} \sum_{k \in \mathbb{F}_q_{\deg_k(k) \leq l}} \frac{1}{(k + a)^{s+j}}
\]

\[
+ \sum_{k \in \mathbb{F}_q} \frac{1}{(k + a)^s}.
\]
Taking $N \to \infty$ in the above equality, by (3.13), we have

$$\lim_{N \to \infty} \left| \sum_{k \in \mathbb{F}_q \left[ \frac{1}{T} \right]} \frac{1}{\langle k + a \rangle^s} \right|_{\infty} = 0$$

and

$$\sum_{\gamma \in \mathbb{F}_q} \frac{1}{\langle a + \gamma \rangle^s} = \lim_{N \to \infty} \sum_{l=0}^{N-1} \sum_{k \in \mathbb{F}_q \left[ \frac{1}{T} \right]} \frac{1}{\langle k + a \rangle^s}$$

$$+ \sum_{j=1}^{\infty} \left(-\frac{s}{j}\right) \lim_{N \to \infty} \sum_{l=0}^{N-1} \left(\frac{1}{T}\right)^{(m+l+1)j} \sum_{k \in \mathbb{F}_q \left[ \frac{1}{T} \right]} \frac{1}{\langle k + a \rangle^{s+j}}$$

(see Proposition 4.4)

$$= \sum_{l=0}^{\infty} \sum_{k \in \mathbb{F}_q \left[ \frac{1}{T} \right]} \frac{1}{\langle k + a \rangle^s}$$

$$+ \sum_{j=1}^{\infty} \left(-\frac{s}{j}\right) \sum_{l=0}^{\infty} \left(\frac{1}{T}\right)^{(m+l+1)j} \sum_{k \in \mathbb{F}_q \left[ \frac{1}{T} \right]} \frac{1}{\langle k + a \rangle^{s+j}}.$$
is uniformly convergent for $N \in \mathbb{N}$ and

$$
\lim_{N \to \infty} \sum_{j=1}^{\infty} \frac{(-s)^j}{(q-1)j} \sum_{l=0}^{N-1} T^{-(m+l+1)j} \sum_{k \in \overline{F_q[\frac{1}{T}]}} \frac{1}{\langle k+a \rangle^{s+j}} \cdot
$$

(4.12)

$$
= \sum_{j=1}^{\infty} \frac{(-s)^j}{(q-1)j} \lim_{N \to \infty} \sum_{l=0}^{N-1} T^{-(m+l+1)j} \sum_{k \in \overline{F_q[\frac{1}{T}]}} \frac{1}{\langle k+a \rangle^{s+j}}.
$$

Proof. Let

$$
U = 1 + \frac{1}{T} \overline{F_q[\frac{1}{T}]}
$$

be the group of 1-units in $K_\infty$. By [3, p. 98, the second paragraph], for any $u = 1 + \omega \in U$ with $|\omega|_\infty < 1$ and $s = \sum_{j=j_0}^{\infty} c_j p^j \in \mathbb{Z}_p$, we have

$$
u^s := \prod_j (1 + \omega^p)^{c_j}.
$$

Thus the map

$$
f : U \times \mathbb{Z}_p \to K_*^\infty \quad \quad (u, s) \mapsto u^s
$$

(4.14)

is continuous. Since the sets $U$ and $\mathbb{Z}_p$ are compact, the function $f(u, s) = u^s$ is uniformly bounded for $(u, s) \in U \times \mathbb{Z}_p$, that is, there exists a constant $M > 0$ such that

$$
|u^s|_\infty \leq M
$$

(4.15)

for $u \in U$ and $s \in \mathbb{Z}_p$.

So for any $k \in \overline{F_q[\frac{1}{T}]}, s \in \mathbb{Z}_p$ and $j \in \mathbb{N}$

$$
\left| \frac{1}{\langle k+a \rangle^{s+j}} \right|_\infty \leq M
$$

(4.16)

and for any $l \in \mathbb{N} \cup \{0\}, j \in \mathbb{N}$ and $s \in \mathbb{Z}_p$ we have

$$
\left| T^{-(m+l+1)j} \sum_{k \in \overline{F_q[\frac{1}{T}]}} \frac{1}{\langle k+a \rangle^{s+j}} \right|_\infty \leq \left( \frac{1}{e} \right)^{(m+l+1)j} M.
$$

(4.17)
Then combining the estimations (3.2) and (4.17) we see that

\[
\left( -s \right) \left( \begin{array}{c}
    j \\
    \end{array} \right) N-1 \sum_{l=0}^{T-(m+l+1)j} \sum_{k \in \mathbb{F}_q[\frac{1}{T}]} \frac{1}{\langle k + a \rangle^{s+j}} \left\| \nabla \right\|_{\infty}^{j} \\
\right.
\]

\[
\left( -s \right) \left( \begin{array}{c}
    j \\
    \end{array} \right) N-1 \sum_{l=0}^{T-(m+l+1)j} \sum_{k \in \mathbb{F}_q[\frac{1}{T}]} \frac{1}{\langle k + a \rangle^{s+j}} \left\| \nabla \right\|_{\infty}^{j} \\
\right.
\]

\[
\leq M \left( \frac{1}{e} \right)^{(m+l+1)j}. \\
\]

The limit

\[
\lim_{j \to \infty} M \left( \frac{1}{e} \right)^{(m+l+1)j} = 0
\]

implies that the series

\[
M \sum_{j=1}^{\infty} \left( \frac{1}{e} \right)^{(m+l+1)j}
\]

is convergent. Finally by (4.18), (4.19) and the Weierstrass test (see \(8, p. 230, \text{Theorem 5.1}\)), we see that the series

\[
\sum_{j=1}^{\infty} \left( \frac{1}{e} \right)^{(m+l+1)j}
\]

is uniformly convergent for \(N \in \mathbb{N}\). Then applying \(8, p. 185, \text{Theorem 3.5}\) we conclude that the limit \(N \to \infty\) can be moved to the inside of the above series, which is the desired result.

**Proof of Theorem 2.1**

With the definition of the forward difference operator \(\Delta_{(s,n)}\) (see (2.11)), the shifted identity (4.1) can be rewritten as

\[
\sum_{j=1}^{\infty} \left( \frac{1}{e} \right)^{(m+l+1)j}
\]

is uniformly convergent for \(N \in \mathbb{N}\). Then applying \(8, p. 185, \text{Theorem 3.5}\) we conclude that the limit \(N \to \infty\) can be moved to the inside of the above series, which is the desired result.

**Proof of Theorem 2.1**

With the definition of the forward difference operator \(\Delta_{(s,n)}\) (see (2.11)), the shifted identity (4.1) can be rewritten as

\[
\sum_{\gamma \in \mathbb{F}_q} \frac{1}{\langle a + \gamma \rangle^s} = \zeta_{\infty} \left( \frac{1}{T}, s, a, 0 \right) + \sum_{j=1}^{\infty} \left( \frac{s}{j} \right) (1 + \Delta_{(s,n)})^j \zeta_{\infty} \left( \frac{1}{T}, s, a, 0 \right).
\]

Then from the definition of the infinite order linear difference operator \(L\) (see (2.13)), it is equivalent to

\[
L \left[ \zeta_{\infty} \left( \frac{1}{T}, s, a, 0 \right) \right] = \sum_{\gamma \in \mathbb{F}_q} \frac{1}{\langle a + \gamma \rangle^s},
\]

which is the desired result.
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