The control problem for the heat equation in the case of a composite material

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Abstract. In modern technology there are devices whose details are subject to thermal effects. As a result, the properties of materials may change in parts. In turn, this leads to the failure of the device. You can control surface temperature to protect components from excessive heat. The temperature measuring device cannot be placed at the point where the heating takes place, since the heating temperature is too high. On the other hand, the temperature of the outer side of the protective layer itself is of little interest: you need to know the temperature on the surface of the material. Therefore, one can either, knowing the temperature at the material-protective layer media section, solve the inverse problem, or solve the heating control problem. In this paper, we propose to consider the problem of controlling heating under the condition of controlling the temperature of the interface. We consider the problem on a half-line, and the following sections are identified in the problem: the heating region, a small thickness of the protective layer, the region occupied by the material, and the semi-infinite region in which the material interacts with the environment. At the same time, we consider such a coefficient of thermal conductivity that it is constant in each of the regions, but it experiences a first-order discontinuity at the interfaces. This makes it impossible to build a classical solution to the problem in the entire solution area, and we build solutions that are classic, each in its own subregion, and that meets the matching conditions at the interfaces. The control problem is as follows: the heating function, on the one hand, should be maximum all the time, and on the other, the temperature at the interface should not exceed a certain critical value (for example, the melting temperature of the material). The constructed solution of the problem is simple, but requires rather cumbersome proofs.

1. Introduction
In modern technology there are devices whose details are subject to thermal effects. As a result, the properties of materials may change in parts. In turn, this leads to the failure of the device [2].

You can control surface temperature to protect components from excessive heat. It is advisable to be able to solve the problem of controlling the heat so that the material would not be damaged. The temperature may be too high for direct measurement [3], and therefore it is necessary to solve the control problem.

A protective coating can also be applied. The coating protects the device for a while, but makes it difficult to solve the control problem for a composite material.

The difficulties of setting and solving the mathematical problem listed above can be overcome, for example, as follows: there is a classical solution for each of the areas. It is impossible to construct a classical solution for the entire area due to the discontinuities of the thermal...
diffusivity. Therefore, solutions that are classic within each of the areas can be combined into a common one by setting the matching conditions of the solution and its derivatives on media sections.

We restrict ourselves to considering the linear control problem in this paper, although the properties of materials can change in a more complex way.

2. Control problem statement

Consider the area of solution to the control problem. Let \( x \in [0; +\infty) \) be spatial coordinate, let \( t \in [0; +\infty) \) be time coordinate.

Let the solution domain be divided into four parts. The medium is heated in the first region, \( x \in [0; x_0] \). The time and space finite-function heating function \( F(x, t) \) is given in this region. The thermal diffusivity coefficient \( k_1(x, t) = a_1^2 \) in the first region is constant. The protective layer is heated from the wall in the second region, \( x \in [x_0; x_1] \). The thermal diffusivity \( k_2(x, t) = a_2^2 \) in the second region is also constant, but differs from the thermal conductivity in the first region. The material itself is heated in the third region \( x \in [x_1; x_2] \) from the common border with the protective layer. The thermal diffusivity \( k_3(x, t) = a_3^2 \) is constant. The material interacts with the environment in an unlimited fourth area. The thermal diffusivity \( k_4(x, t) = a_4^2 \) is also constant in the fourth region.

Here we used the notation: \( a_1, a_2, a_3, a_4 \) – some real values.

It is convenient to introduce a function \( u(x, t) \), such that

\[
  u(x, t) = \begin{cases} 
    u_1(x, t), & x \in [0; x_0], \ t \in [0; +\infty), \\
    u_2(x, t), & x \in [x_0; x_1], \ t \in [0; +\infty), \\
    u_3(x, t), & x \in [x_1; x_2], \ t \in [0; +\infty), \\
    u_4(x, t), & x \in [x_2; +\infty), \ t \in [0; +\infty). 
  \end{cases}
\]

We introduce the heat conductivity operator \( \Delta \), such that

\[
  \Delta u(x, t) = \begin{cases} 
    a_1^2 \frac{\partial^2 u_1(x, t)}{\partial x^2}, & x \in (0; x_0), \ t \in (0; +\infty), \\
    a_2^2 \frac{\partial^2 u_2(x, t)}{\partial x^2}, & x \in (x_0; x_1), \ t \in (0; +\infty), \\
    a_3^2 \frac{\partial^2 u_3(x, t)}{\partial x^2}, & x \in (x_1; x_2), \ t \in (0; +\infty), \\
    a_4^2 \frac{\partial^2 u_4(x, t)}{\partial x^2}, & x \in (x_2; +\infty), \ t \in (0; +\infty). 
  \end{cases}
\]

We also introduce the heat flux operator, \( D \). This operator is needed as a matching operator:

\[
  D u(x, t) = \begin{cases} 
    a_1 \frac{\partial u_1(x, t)}{\partial x}, & x \in (0; x_0), \ t \in (0; +\infty), \\
    a_2 \frac{\partial u_2(x, t)}{\partial x}, & x \in (x_0; x_1), \ t \in (0; +\infty), \\
    a_3 \frac{\partial u_3(x, t)}{\partial x}, & x \in (x_1; x_2), \ t \in (0; +\infty), \\
    a_4 \frac{\partial u_4(x, t)}{\partial x}, & x \in (x_2; +\infty), \ t \in (0; +\infty). 
  \end{cases}
\]  

(1)

The thermal conductivity problem has the form, taking into account the proposed notation:

\[
  \begin{cases} 
    \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) + F(x, t), x \in (0; +\infty), t \in (0; +\infty), \\
    u_1(0, t) = 0, t \in (0; +\infty), u_4(\infty, t) = 0, t \in (0; +\infty), \\
    u(x, 0) = 0, x \in [0; +\infty). 
  \end{cases}
\]  

(2)

Definition A solution to the problem (2) is a function \( u(x, t) \), such that:
(i) $u(x,t) \in C(x \in [0;+\infty), t \in [0;+\infty))$, 
(ii) $Du(x,t) \in C(x \in (0;+\infty), t \in [0;+\infty))$, 
(iii) $u(x,t) \in C^{(2,1)}(x \in (0;x_0) \cup (x_0;x_1) \cup (x_1;x_2) \cup (x_2;+\infty), t > 0)$, 
(iv) Function $u(x,t)$ satisfies all the equations (2),

Note that the function $F(x,t)$ on the right side of the equation (2) is not known to us. This function simulates the heating process from the point of view of the physical engineering setting.

Therefore, we choose it as follows:

$$F(x,t) = \begin{cases} F_0(t), & t \geq 0, \ x \in [0;x_0), \\ 0, & t \geq 0, \ x \in [x_0;+\infty), \\ F_0(t) \in C(t > 0), \exists t_0 > 0, \ \forall t > t_0: F_0(t) = 0, \\ \exists A_1 > 0, \ \forall t > 0: |F_0(t)| \leq A_1. \end{cases}$$

(3)

We also introduce the definition of the optimal solution to the problem (2-3):

**Definition** The function $u(x,t)$ is called the optimal solution to the problem (2) if:

(i) function $u(x,t)$ is a solution to (2),
(ii) $u(x_1,t) \leq T_{max} \ \forall t > 0, T_{max} > 0$,
(iii) $\int_0^t F_0(\tau)d\tau \rightarrow max \ \forall t$.

We construct the solution this way: first, we simply construct the solution to (2), then find out what the function $F_0(t)$ so that the solution to (2) is optimal.

3. Constructing a formal solution to (2-3)

The following lemma is true:

**Lemma 3.1.** Let the conditions (3) be satisfied. Then the solution to (2) has the form:

$$u_1(x,t) = \int_0^t \frac{F_0(\tau)d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{x_0}^{x_0} e^{-\frac{(k_1x-x_0)^2}{4a_3^2(t-\tau)}} - e^{-\frac{\tau}{4a_3^2(t-\tau)}} ds, x \in [0;x_0], t > 0,$$

$$u_2(x,t) = \int_0^t \frac{F_0(\tau)d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{x_0}^{x_0} e^{-\frac{(k_2x-x_0-x_1)^2}{4a_3^2(t-\tau)}} - e^{-\frac{\tau}{4a_3^2(t-\tau)}} ds, x \in [x_0;x_1], t > 0,$$

$$u_3(x,t) = \int_0^t \frac{F_0(\tau)d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{x_0}^{x_0} e^{-\frac{(k_3x-x_0-x_3)^2}{4a_3^2(t-\tau)}} - e^{-\frac{\tau}{4a_3^2(t-\tau)}} ds, x \in [x_1;x_2], t > 0,$$

$$u_4(x,t) = \int_0^t \frac{F_0(\tau)d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{x_0}^{x_0} e^{-\frac{(k_4x-x_0-x_3-x_4)^2}{4a_3^2(t-\tau)}} - e^{-\frac{\tau}{4a_3^2(t-\tau)}} ds, x \in [x_2;+\infty), t > 0,$$

where:

$k_1 = a_3/a_1$, $k_2 = a_3/a_2$, $k_3 = 1$, $k_4 = a_3/a_1$,

$x_{23} = (k_1 - k_2)x_0$, $x_{33} = x_{23} + (k_2 - k_3)x_1$, $x_{43} = x_{33} + (k_3 - k_4)x_2$.

**Proof.** Relations (4) are a solution to the differential heat equation (2). This statement follows from [1] if $k_m = a_m/a_3$ is true. It is easy to see that in this case the heat equation turns a true equality.
The existence of the second coordinate with respect to the coordinate in the interval $x \in (s_m; s_{m+1})$ and the uniform continuity of the first derivative $\frac{\partial u_m(x,t)}{\partial x}$ on the interval $x \in (s_m; s_{m+1})$ are proved in a manner similar to [1]. Thus, the solution inside each of the intervals $x \in (s_m; s_{m+1})$ is classical.

Let’s show that the boundary conditions are satisfied.

The equality $u_1(0, t) = 0$ is true $\forall t > 0$. This statement also follows from [1].

The equality of the limit for $x \to +\infty$ $u(t)^{(j)}(+\infty, t) \to 0$ also follows from [1].

Check the validity of the matching conditions. We restrict ourselves to the matching condition at the point $x = x_0$, $u_1^{(j)}(x_0, t) = u_2^{(j)}(x_0, t) \forall t > 0$, since in other cases proof can be carried out in this way.

Let us compare the first terms of the formulas $u_1(x, t)$ and $u_2(x, t)$ in (4) for $x = x_0$:

$$\int_0^t \frac{F_0(\tau) d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_0^{x_0} \exp\left(-\frac{(k_1x_0 - s)^2}{4a_3^2(t-\tau)}\right) ds = \int_0^t \frac{F_0(\tau) d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_0^{x_0} \exp\left(-\frac{(k_2x_0 - s + x_{23})^2}{4a_3^2(t-\tau)}\right) ds.$$

If the equality $\frac{k_1x_0 - s}{a_3\sqrt{t-\tau}} = \frac{k_2x_0 - s + x_{23}}{a_3\sqrt{t-\tau}}$ is true $\forall s \in (s_3; s_4), t, \tau$, then the matching condition is satisfied $u_1(x_0, t) = u_2(x_0, t) \forall t > 0$. From this statement it follows that $x_{23} = (k_1 - k_2)x_0$.

Under the same conditions, the matching condition is also true for the second terms.

Similarly, we verify that the matching of the derivatives $a_1 \frac{\partial u_1^{(j)}(x_0, t)}{\partial x} = a_2 \frac{\partial u_2^{(j)}(x_0, t)}{\partial x}$ is true under the same conditions.

Let’s check this statement for the first terms

$$\int_0^t \frac{a_1k_1F_0(\tau) d\tau}{8a_3^3\sqrt{\pi(t-\tau)^3}} \int_0^{x_0} \exp\left(-\frac{(k_1x_0 - s)^2}{4a_3^2(t-\tau)}\right) (k_1x_0 - s) ds = \int_0^t \frac{a_2k_2F_0(\tau) d\tau}{8a_3^3\sqrt{\pi(t-\tau)^3}} \int_0^{x_0} \exp\left(-\frac{(k_2x_0 - s + x_{23})^2}{4a_3^2(t-\tau)}\right) (k_2x_0 - s + x_{23}) ds.$$

It follows that if true $\frac{k_1x_0 - s}{a_3\sqrt{t-\tau}} = \frac{k_2x_0 - s + x_{23}}{a_3\sqrt{t-\tau}} \forall s \in (s_3; s_4), t, \tau$, then $u_1(x_0, t) = u_2(x_0, t) \forall t > 0$ is true. Everything can be proved in a similar way for the second terms and for other matching points $x = x_1$ and $x = x_2$.

3.1. Uniform continuity (4)

Now we prove that the formal solution constructed satisfies all the conditions for determining the solution. To do this, we prove that each of the integrals (4) converges uniformly, and its first and second derivatives converge uniformly.

**Lemma 3.2.** Let the conditions (3) be valid for the function $F(x,t)$. Then each of the integrals (4) converges uniformly with its first and second derivatives with respect to $x \forall t > 0$, $x \in (0; x_0) \cup (x_0; x_1) \cup (x_1; x_2) \cup (x_2; +\infty)$.

**Proof.** The uniform convergence of the second, third, and fourth integrals on these sets is obvious, since the exponent argument inside the integral in each of these cases is negative over the entire integration area. Differentiation with respect to $x$ an arbitrary number of times also does not lead to divergence of any of the three integrals mentioned. $\square$
Therefore, we consider only the first of the integrals (4):

\[
u_1(x, t) = \int_0^t \frac{F_0(\tau)d\tau}{2a_1\sqrt{\pi(t-\tau)}} \int_0^{x_0} \left\{ e^{-\frac{(x-s)^2}{4a_1^2(t-\tau)}} - e^{-\frac{(x+s)^2}{4a_1^2(t-\tau)}} \right\} ds, x \in [0; x_0], t > 0. \tag{5}\]

It is easy to see that this integral \(\forall x > 0, \forall t > 0\) can be uniformly estimated. The difference of exponential functions with a non-positive exponent does not exceed two in absolute value; therefore, the following estimate is true:

\[
|u_1(x, t)| \leq \int_0^t \frac{F_0(\tau)d\tau}{2a_1\sqrt{\pi(t-\tau)}} \int_0^{x_0} 2ds. \]

We obtain a further uniform estimate due to the integrability of the integrand, the boundedness and finiteness of the function \(F_0(t)\)

\[
|u_1(x, t)| \leq \int_0^t \frac{F_0(\tau)d\tau}{2a_1\sqrt{\pi(t-\tau)}} \int_0^{x_0} \frac{2A_1x_0\sqrt{t_0}}{a_1\sqrt{\pi}}(\sqrt{t} - \sqrt{t_0}) \leq \frac{2A_1x_0\sqrt{t_0}}{a_1\sqrt{\pi}},
\]

where \(\exists t_0 > 0\) that \(\forall t > t_0\) \(F_0(t) = 0\).

Thus, \(u_1(x, t)\) is uniformly bounded \(\forall t > 0, x \in (0; x_0)\), and the integral converges uniformly to a continuous function.

Let us prove the same for the first derivative \(\frac{\partial u_1(x, t)}{\partial x}\). We differentiate the integrand formally with respect to \(x\) and prove integral uniform convergence on the set \(x \in (0; x_0), t > 0:\)

\[
\frac{\partial u_1(x, t)}{\partial x} = \int_0^t \frac{F_0(\tau)d\tau}{4a_1^2\sqrt{\pi(t-\tau)^3}} \int_0^{x_0} \left\{ -(x-s)e^{-\frac{(x-s)^2}{4a_1^2(t-\tau)}} + (x+s)e^{-\frac{(x+s)^2}{4a_1^2(t-\tau)}} \right\} ds, x \in [0; x_0], t > 0. \tag{6}\]

We obtain by calculating the internal integral:

\[
\frac{\partial u_1(x, t)}{\partial x} = \int_0^t \frac{F_0(\tau)d\tau}{2a_1\sqrt{\pi(t-\tau)^2}} \left\{ e^{-\frac{(x-s)^2}{4a_1^2(t-\tau)}} + e^{-\frac{(x+s)^2}{4a_1^2(t-\tau)}} \right\} |_{0}^{x_0}, x \in [0; x_0], t > 0.
\]

It is easy to see that the internal expression does not exceed 4. We obtain a uniform estimate, repeating the reasoning of the previous paragraph:

\[
\left| \frac{\partial u_1(x, t)}{\partial x} \right| \leq 4 \int_0^t \frac{F_0(\tau)d\tau}{2a_1\sqrt{\pi(t-\tau)}} \leq \frac{4A_1\sqrt{t_0}}{a_1\sqrt{\pi}}, x \in [0; x_0], t > 0.
\]

Now we prove that the second derivative \(\frac{\partial^2 u_1(x, t)}{\partial x^2}\) also converges uniformly \(\forall x \in (0; x_0), \forall t > 0\).

\[
\frac{\partial^2 u_1(x, t)}{\partial x^2} = -\int_0^t \frac{F_0(\tau)d\tau}{4a_1^2\sqrt{\pi(t-\tau)^3}} \left\{ (x-s)e^{-\frac{(x-s)^2}{4a_1^2(t-\tau)}} + (x+s)e^{-\frac{(x+s)^2}{4a_1^2(t-\tau)}} \right\} |_{0}^{x_0}, x \in (0; x_0), t > 0.
\]
We choose the first term and carry out all the calculations for it. The calculations do not differ significantly for the remaining terms.

\[
\frac{\partial^2 J_1(x,t)}{\partial x^2} = -\int_0^t \frac{F_0(\tau)(x-x_0)d\tau}{4a_1^2 \sqrt{\pi}(t-\tau)^\frac{3}{2}} e^{-\frac{(x-x_0)^2}{4a_1^2(t-\tau)}}, x \in (0; x_0), t > 0,
\]

or

\[
\frac{\partial^2 J_1(x,t)}{\partial x^2} = -\int_0^t \frac{F_0(\tau)}{2a_1 \sqrt{\pi}} e^{-\frac{(x-x_0)^2}{4a_1^2(t-\tau)}} d \left( \frac{x-x_0}{2a_1 \sqrt{t-\tau}} \right), x \in (0; x_0), t > 0,
\]

We obtain an estimate due to the boundedness of \( F_0(t) \):

\[
\left| \frac{\partial^2 J_1(x,t)}{\partial x^2} \right| \leq \frac{A_1}{2a_1 \sqrt{\pi}} e^{-t^2} \int_0^t \left( \frac{x-x_0}{2a_1 \sqrt{t-\tau}} \right)^2 \, d\xi \leq \frac{A_1}{a_1}, x \in (0; x_0), t > 0.
\]

Thus, the estimate obtained is uniform and the second derivative with respect to the coordinate is a continuous function \( \forall x \in (0; x_0), t > 0 \). Similarly, we prove a uniform estimate for all three remaining integrals.

So, the second derivative exists and is a continuous function \( \forall x \in (0; x_0), t > 0 \).

**Theorem 3.3.** Let the conditions

(i) \( F_0(t) \in C[0; +\infty) \).

(ii) \( \exists A_1 > 0, \forall t \geq 0 \left| F_0(t) \right| \leq A_3 \).

(iii) \( \exists t_0, \forall t > t_0 F_0(t) = 0 \).

for the function \( F_0(t) \) be true. Then the solution to (2-3) exists, has the form of (4) and is unique.

**Proof.** In the lemma 3.1-3.2 we proved that the function (4) is a solution to the problem (2-3). Therefore, this solution exists.

We show that it is unique. Suppose this is not true, and there are two solutions to (2-3): \( w(x, t) \) and \( v(x, t) \). We construct the difference of these solutions \( z(x, t) = w(x, t) - v(x, t) \) and show that the difference \( z(x, t) \) is equal to zero.

For the function \( z(x, t) \) we get the following problem:

\[
\begin{aligned}
\frac{\partial z(x,t)}{\partial t} &= \Delta z(x,t), x \in (0; +\infty), t \in (0; +\infty), \\
z_1(0,t) &= 0, t \in (0; +\infty), \quad z_4(+\infty, t) = 0, t \in (0; +\infty), \\
z(x,0) &= 0, x \in [0; +\infty).
\end{aligned}
\]  

(7)

The solution to the problem (7) exists. It was shown above that this solution is limited. Therefore, there is the Laplace transform \( \hat{z}(x,p) = \int_0^{+\infty} e^{-pt} z(x,t) dt \) of the function \( z(x,t) \) and of its derivatives. Then the equation (7) is transformed into a system of ordinary differential equations:

\[
\begin{aligned}
p \hat{z}(x,p) &= \Delta \hat{z}(x,p), x \in (0; +\infty), p = \sigma + i\zeta, \sigma, \zeta \in (-\infty; +\infty), \\
\hat{z}_1(0,p) &= 0, \hat{z}_4(+\infty, p) = 0, p = \sigma + i\zeta, \sigma, \zeta \in (-\infty; +\infty).
\end{aligned}
\]

This is an ordinary linear differential equation with completely homogeneous initial conditions given on the half-line. The solution to this equation can only be zero \( \forall x > 0 \).

Therefore, the solution to (2-3) is unique.
4. Conclusion. Control problem

We now solve the control problem. Note that the solution to the control problem turned out to be simple in our case and does not require cumbersome calculations.

It is easy to see that \( \int_0^T F_0(\tau) d\tau \to \text{max} \), if \( F_0(t) \) immediately reaches the maximum allowable value (in the physical sense, the heating function \( F_0(t) \) cannot be unlimited). Let us find out whether such a value exists and whether the condition \( u_2(x_1, t) \leq T_{\text{max}} \) is true.

Suppose that \( F_0(t) = 1 \), and construct a solution using the formula (6):

\[
u_2(x_1, t) = F_0 \int_0^t \frac{d\tau}{2a_2 \sqrt{\pi (t-\tau)}} \left\{ e^{-\frac{(x_1-x)^2}{4a_2^2(t-\tau)}} - e^{-\frac{(x_1+\tau)^2}{4a_2^2(t-\tau)}} \right\} ds =
\]

\[
= \frac{4F_0a_2^2t}{x_1^2} \text{erf} \left( \frac{x_1}{2a_2 \sqrt{t}} \right) - \frac{2F_0a_2^2t}{(x_1-x_0)^2t} \text{erf} \left( \frac{x_1-x_0}{2a_2 \sqrt{t}} \right) - \frac{2F_0a_2^2t}{(x_1+x_0)^2t} \text{erf} \left( \frac{x_1+x_0}{2a_2 \sqrt{t}} \right) +
\]

\[
+ \frac{2a_2 \sqrt{t}}{x_1 \sqrt{\pi}} e^{-\frac{x_1^2}{4a_2^2t}} - \frac{a_2 \sqrt{t}}{(x_1-x_0) \sqrt{\pi}} e^{-\frac{(x_1-x_0)^2}{4a_2^2t}} - \frac{a_2 \sqrt{t}}{(x_1+x_0) \sqrt{\pi}} e^{-\frac{(x_1+x_0)^2}{4a_2^2t}} +
\]

\[
- 2F_0 \text{erf} \left( \frac{x_1}{2a_2 \sqrt{t}} \right) + F_0 \text{erf} \left( \frac{x_1-x_0}{2a_2 \sqrt{t}} \right) + F_0 \text{erf} \left( \frac{x_1+x_0}{2a_2 \sqrt{t}} \right)
\]

If the value of the variable \( t \) is large enough, then the approximate equality is true:

\[
u_2(x_1, t) \approx 2F_0 \frac{1 + \sqrt{\pi} x_0^2}{\sqrt{\pi} x_1(x_1^2 - x_0^2) \sqrt{t}}.
\]

Thus, \( u_2(x_1, t) = A_2 \sqrt{t} \), where \( A_2 = 2F_0 \frac{1 + \sqrt{\pi} x_0^2}{\sqrt{\pi} x_1(x_1^2 - x_0^2)} \), and the time taken to reach the maximum allowable temperature at the boundary \( T_{\text{max}} \), is found from the equality: \( t_0 = \left( \frac{T_{\text{max}}}{A_2} \right)^2 \).

The temperature should drop after the highest permissible temperature value at the boundary has been reached. We will look for a solution to the control problem in the form: \( F_0(t) = B/\sqrt{t} \quad \forall t > t_0 \), where \( B \) is a number, \( B = A_2 \sqrt{t_0} \).

In this case, approximate calculations lead to an expression of the form:

\[
u_2(x_1, t) \approx \frac{B}{a_2} \left( \frac{\pi}{2} - \arcsin \left( \frac{\sqrt{t_0}}{\sqrt{t}} \right) \right), \quad t > t_0,
\]

\[
F_0(t) = \begin{cases} 
F_0, & t \leq \left( \frac{T_{\text{max}}}{A_2} \right)^2, \\
\frac{2A_2 \sqrt{t_0}}{\sqrt{\pi} \sqrt{t}}, & t \geq t_0.
\end{cases}
\]

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References

[1] Sveshnikov A G, Bogolyubov A N and Kravtsov V V 1993 Lektsii po matematicheskoi fizike: ucheb. posobie [Lectures on mathematical physics: textbook] (Moscow: Moscow state University Press) p 352
[2] Carslow H S, Jaeger J C 1959 Conduction of heat in solids (Oxford University Press) p 517.
[3] Alifanov O M, Artyukhin E A and Rymantsev S V 1988 Extremalnie metodi reshsenya nekorrektnikh zadach i ih prilozheniya k zadacham teploobmena [Extreme methods for solving ill-posed problems and their application to inverse heat transfer problems] (Moscow, Science) p 288