NON-CHARACTERIZING SLOPES FOR HYPERBOLIC KNOTS

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ABSTRACT. A non-trivial slope \( r \) on a knot \( K \) in \( S^3 \) is called a characterizing slope if whenever the result of \( r \)-surgery on a knot \( K' \) is orientation preservingly homeomorphic to the result of \( r \)-surgery on \( K \), then \( K' \) is isotopic to \( K \). Ni and Zhang ask: for any hyperbolic knot \( K \), is a slope \( r = p/q \) with \( |p| + |q| \) sufficiently large a characterizing slope? In this article we answer this question in the negative by demonstrating that there is a hyperbolic knot \( K \) in \( S^3 \) which has infinitely many non-characterizing slopes. As the simplest known example, the hyperbolic knot \( 8_6 \) has no integral characterizing slopes.

1. Introduction

For a given knot \( K \) in the three sphere \( S^3 \), we call \( p/q \in \mathbb{Q} \) a characterizing slope for \( K \) if whenever the result of \( p/q \)-surgery on a knot \( K' \) in \( S^3 \) is orientation preservingly homeomorphic to the result of \( p/q \)-surgery on \( K \), then \( K' \) is isotopic to \( K \). For the trivial knot, Gordon [6] conjectured that every non-trivial slope \( p/q \in \mathbb{Q} \) is a characterizing slope. Kronheimer, Mrowka, Ozsváth and Szabó [12] proved this conjecture in the positive using Seiberg-Witten monopoles. See [19] and [22] for alternative proofs using Heegaard Floer homology. Furthermore, Ozsváth and Szabó [21] showed that for the trefoil knot and the figure-eight knot, every non-trivial slope is a characterizing slope.

On the other hand, it is known that many knots have non-characterizing slopes. The first such example was given by Lickorish [14]. Some torus knots have non-characterizing slopes. For instance, 21–surgeries on \( T_{5,4} \) and \( T_{11,2} \) produce the same oriented 3–manifold, and hence 21 is a non-characterizing slope for both \( T_{5,4} \) and \( T_{11,2} \) [17]. However, Ni and Zhang [17] prove that for a torus knot \( T_{r,s} \) with \( r > s > 1 \), a slope \( p/q \) is a characterizing slope if \( p/q > 30(r^2 - 1)(s^2 - 1)/67 \). This suggests that for a given knot \( K \), sufficiently large slopes are characterizing ones. For hyperbolic knots, Ni and Zhang ask the following:

**Question 1.1** (Ni and Zhang [17]). Let \( K \) be a hyperbolic knot. Is a slope \( r = p/q \) with \( |p| + |q| \) sufficiently large a characterizing slope of \( K \)? Equivalently, are there only finitely many non-characterizing slopes of \( K \)?

The purpose in this article is to answer Question 1.1 in the negative.

**Theorem 1.2.** There exists a hyperbolic knot which has infinitely many non-characterizing slopes.
To prove Theorem 1.2 we first give a general principle to produce knots with infinitely many non-characterizing slopes; see Theorem 2.1. (In fact, Corollary 2.3 shows this principle produces infinitely many such knots.) Then we apply this to present explicit examples. Recall that every non-trivial slope is a characterizing slope for a trefoil knot and the figure-eight knot \[21\], which are genus one, fibered knots. If we drop one of these conditions, we have:

**Example 1.3.**

1. Every integer except possibly 2 is not a characterizing slope for the knot 9\_42 in Rolfsen’s table. The knot 9\_42 is a fibered knot, but it has genus two.
2. Every integer except possibly 0 is not a characterizing slope for the pretzel knot \(P(-3,3,5)\). The pretzel knot \(P(-3,3,5)\) is a genus one knot, but it is not fibered.

**Question 1.4.** Is 2 a characterizing slope for 9\_42? Is 0 a characterizing slope for \(P(-3,3,5)\)?

A modification of the above example leads us to demonstrate:

**Theorem 1.5.** There exists a hyperbolic knot for which every integral slope is a non-characterizing slope. In particular, every integral slope is not a characterizing slope for the hyperbolic 8–crossing knot 8\_6 in Rolfsen’s table.

Further modifications produce the same result for prime satellite knots and composite knots.

**Theorem 1.6.**

1. Given a non-trivial knot \(k\), there exists a prime satellite knot with \(k\) a companion knot for which every integral slope is a non-characterizing slope.
2. Given a non-trivial knot \(k\), there exists a composite knot with \(k\) a connected summand for which every integral slope is a non-characterizing slope.

Among known examples, the knot 8\_6 is the simplest knot (with respect to crossing numbers) which has infinitely many non-characterizing slopes. So we would like to ask:

**Question 1.7.** Are there any knots of crossing number less than 8 that have infinitely many non-characterizing slopes?

### 2. Non-characterizing slopes and twist families of surgeries

#### 2.1. General construction.
In this subsection we establish the following general principle.

**Theorem 2.1.** Let \(k \cup c\) be a two-component link in \(S^3\) such that \(c\) is unknotted. Suppose that \((0,0)\)–surgery on \(k \cup c\) results in \(S^3\). Let \(K\) be the knot in \(S^3\) which is surgery dual to \(c\), the image of \(c\), in the surgered \(S^3\), and let \(k_n\) be the knot obtained from \(k\) by twisting \(n\) times along \(c\). Then \(K(n) \cong k_n(n)\) for all integers \(n\).

Moreover, if \(c\) is not a meridian of \(k\), then \(K \neq k_n\) for all but finitely many integers \(n\).
Proof. Since \((0, 0)\)–surgery on \(k \cup c\) is \(S^3\), a homology calculation shows that \(|\ell k(k, c)| = 1\). Performing \((-1/n)\)–surgery on \(c\) takes the knot \(k\) with the surgery slope 0 to a knot \(k_n\) with a surgery slope \(n = 0 + n(\ell k(k, c))^2\), i.e. \(-n\)–twist along \(c\) converts a knot-slope pair \((k, 0)\) into another knot-slope pair \((k_n, n)\); thus we obtain a twist family of knot-slope pairs \(\{(k_n, n)\}\). Let \(V\) be the solid torus \(S^3 - \mathcal{N}(c)\) which contains \(k\) in its interior. Observe that \(V(k; 0) \cong V(k_n; n)\) for all \(n\).

Let \(\langle \mu_c, \lambda_c \rangle\) be a preferred meridian-longitude pair of \(c \subset S^3\), oriented with the right-handed orientation (so that if \(c\) is oriented in the same direction as \(\lambda_c\) in \(\mathcal{N}(c)\), then \(\ell k(\mu_c, c) = 1\)). Note that \(\lambda_c\) represents the 0–slope on \(\mathcal{N}(c)\) and \(\lambda_c\) bounds a meridian disk of the solid torus \(V\). Let \(c_n\) be the surgery dual to the \((-1/n)\)–surgery on \(c\) (i.e. a core of the filled solid torus) with meridian \(\mu_n\), the \((-1/n)\)–surgery slope of \(c\) in \(\partial V\). These curves \(\mu_n\) are each longitudes of \(V\) and satisfy \([\mu_n] = -[\mu_c] + n[\lambda_c] \in H_1(\partial V); [\mu_0] = -[\mu_c]\).

Since \(k\) wraps algebraically once in \(V\), a preferred longitude of \(k \subset V \subset S^3\) is homologous to \(\mu_c\) in \(V - \mathcal{N}(k)\). Hence \(\mu_c\) is null-homologous in \(V(k; 0)\).

Let \(K\) be the surgery dual to \(c\) with respect to \(\lambda_c\)–surgery. (Adapting the above notation \(K\) may be regarded as \(c_{\infty}\).) Since \((0, 0)\)–surgery on \(k \cup c\) results in \(S^3\), \(K\) is a knot in this surgered \(S^3\) with exterior \(S^3 - \mathcal{N}(K) = V(k; 0)\) and meridian \(\lambda_c\). Because \(\mu_c\) is null-homologous in \(V(k; 0)\), \(\mu_c\) is the boundary of a Seifert surface for \(K\).

With right-handed orientation, a preferred meridian-longitude pair for \(K\) in \(S^3\) is given by \((\lambda_c, -\mu_c)\). Thus \([\mu_n] = -[\mu_c] + n[\lambda_c] = n[\lambda_c] + (-[\mu_c])\), which corresponds to a slope \(n\) with respect to the preferred meridian-longitude pair \((\lambda_c, -\mu_c)\). Therefore \(k_n(n) = K(n)\) for all integers \(n\).

If \(c\) is not a meridian of \(k\), since \(\ell k(k, c) \neq 0\), any disk bounded by \(c\) intersects \(k\) more than once. Then it follows from [11] that there are only finitely many \(n\) such that \(k_n\) is isotopic to \(K\). \(\square\)

Remark 2.2. Gompf-Miyazaki had previously utilized the mirror of the knot \(K\) associated to \(k\) as described in Theorem 2.1 for a satellite construction of ribbon knots that generalizes the connected sum of a knot and its mirror [4].

Let \(k \cup c\) be a link as in Theorem 2.1 where \(c\) is an unknot such that the result of \((0, 0)\)–surgery on \(k \cup c\) is \(S^3\) with surgery dual link \(C \cup K\) where \(K\) is dual to \(c\) and \(C\) is dual to \(k\). After 0–surgery on \(c\), \(k\) becomes some knot in \(c(0) = S^1 \times S^2\). Since a non-trivial surgery (corresponding to the 0–surgery) on \(k \subset S^1 \times S^2\) yields \(S^3\), due to Gabai [11 Corollary 8.3], it turns out that\(k(\subset S^1 \times S^2)\) is an \(S^1\) fiber in some product structure of \(S^1 \times S^2\). Since the product structure of \(S^1 \times S^2\) is unique up to isotopy, \(k\) is ambient isotopic to an \(S^1\) fiber in the original product structure of \(c(0) = S^1 \times S^2\). Thus the surgery dual \(C\) to \(k\) in \((k \cup c)(0, 0) = S^3\) is an unknot while the surgery dual \(K\) to \(c\) is not necessarily unknotted in this \(S^3\).

Further, if \(c\) is a meridian of \(k\), then after we straighten \(k\) in \(c(0) = S^1 \times S^2\), the image \(K\) of \(c\) in \((k \cup c)(0, 0) = S^1 \times S^2\) intersects \(\{x\} \times S^2\) once for some \(x \in S^1\). This implies that the dual \(C\) to \(k\) is a meridian of \(K\) in \(S^3\); see [3] p.119. Conversely, if \(C\) is a meridian of \(K\), then \(C\) is a meridian of \(k\). Thus if \(c\) is not a meridian of \(k\), then \(C\) is not a meridian of \(K\) neither.
In the proof of Theorem 2.1, we observe that \((k \cup c)(0, -\frac{1}{n}) \cong (C \cup K)(\frac{1}{n}, n)\), \((k \cup c)(0, -\frac{1}{n}) \cong k_n(n)\) and \((C \cup K)(\frac{1}{n}, n) \cong K(n)\). Starting with \(m\)-surgery instead of 0–surgery on \(k\), the argument in the proof of Theorem 2.1 leads us the following generalization. In what follows, \(K_m\) denotes the knot obtained from \(K\) by twisting \(m\) times along \(C\).

**Corollary 2.3.** Let \(k \cup c\) be a link as in Theorem 2.1 with surgery dual link \(C \cup K\) where \(K\) is dual to \(c\) and \(C\) is dual to \(k\). Then \(K_m(n + m) \cong k_n(m + n)\) for any integers \(m, n\).

Moreover, if \(c\) is not a meridian of \(k\), then each family \(\{K_m\}\) and \(\{k_n\}\) contains infinitely many distinct knots, each of which has only finitely many integral characterizing slopes.

**Proof.** Observe that \(S^3 - N(k \cup c) = S^3 - N(C \cup K)\) and the meridian-longitude pairs \((\mu_k, \lambda_k)\) for \(k\) and \((\mu_c, \lambda_c)\) for \(c\) become meridian-longitude pairs \((\lambda_k, -\mu_k)\) for \(C\) and \((\lambda_c, -\mu_c)\) for \(K\). The latter correspondence was shown in the proof of Theorem 2.1. For the former correspondence, by definition, \(\lambda_k\) becomes a meridian of \(C\), the surgery dual to \(k\). Observe also that \(\mu_k\) is homologous to \(\lambda_c\), which bounds a disk of the filled solid torus after 0–surgery on \(c\). Thus \(\mu_k\) is a preferred longitude of \(C\). Now the orientation convention gives the desired result.

Then we have the following surgery relation

\[ K_m(n + m) \cong (C \cup K)(-\frac{1}{m}, n) \cong (k \cup c)(m, -\frac{1}{n}) \cong k_n(m + n) \]

as claimed.

If \(c\) is not a meridian of \(k\), then \(C\) is not a meridian of \(K\). Since \(\ell k(k, c) \neq 0\) and \(\ell k(K, C) \neq 0\), the wrapping numbers of \(k\) about \(c\) and \(K\) about \(C\) are at least 2. Then [11, Theorem 3.2] implies that each twist family of knots \(\{k_n\}\) and \(\{K_m\}\) partitions into infinitely many distinct knot types containing finitely many members. Therefore, since \(K_m(n + m) \cong k_n(m + n)\), each knot in these two families has only finitely many characterizing slopes.

**Remark 2.4.** Let \(k \cup c\) be a link in \(S^3\) such that \(c\) is unknotted and \((0, 0)\)–surgery on \(k \cup c\) yields \(S^3\) with surgery dual link \(C \cup K\). Then as observed above, \(C\) is also unknotted and \((0, 0)\)–surgery on \(C \cup K\) yields \(S^3\) with surgery dual \(k \cup c\). In particular, |\(\ell k(K, C)\)| = 1.

### 2.2. Multivariable Alexander polynomials

We take \(\Delta_{A \cup B}(x, y)\) to be the symmetrized multivariable Alexander polynomial of the oriented two-component link \(A \cup B\) where \(x\) corresponds to the oriented meridian \(\mu_A\) of \(A\) and \(y\) corresponds to the oriented meridian \(\mu_B\) of \(B\). Due to the symmetrization,

\[ \Delta_{A \cup B}(x, y) = \Delta_{A \cup B}(x^{-1}, y^{-1}) = \Delta_{-A \cup -B}(x, y). \]

However, in general, \(\Delta_{A \cup B}(x, y) \neq \Delta_{A \cup -B}(x, y)\).

**Proposition 2.5.** Assume \(k \cup c\) is an oriented two-component link with \(\ell k(k, c) = 1\) such that \(c\) is an unnot. Further assume \((0, 0)\)–surgery on \(k \cup c\) results in \(S^3\) with surgery dual \(C \cup K\) where \(K\) is dual to \(c\) and \(C\) is dual to \(k\), oriented so that \(\ell k(K, C) = 1\). Then \(\Delta_{K \cup C}(x, y) = \Delta_{k \cup c}(x, y^{-1})\), equivalently \(\Delta_{k \cup c}(x, y) = \Delta_{K \cup C}(x, y^{-1})\).

**Proof.** Let us write \(\mu_J\) and \(\lambda_J\) for the meridian and preferred longitude of an oriented knot \(J\) in \(S^3\) which we view as oriented curves in \(\partial N(J)\) such that \(\ell k(J, \mu_J) = 1\) and \(\lambda_J\) is homologous to
J. Let \( X = S^3 - N(k \cup c) \) be the exterior of the link \( k \cup c \). Since the linking number of \( k \cup c \) is 1, in \( H_1(X; \mathbb{Z}) \) we have that \([\mu_k] = [\lambda_c] \) and \([\mu_c] = [\lambda_k] \). Furthermore these homologies are realized by oriented Seifert surfaces \( \Sigma_c \) and \( \Sigma_k \) that are each punctured once by \( k \) and \( c \) respectively. In particular, restricting to \( X \), \( \partial \Sigma_c = \lambda_c - \mu_k \) and \( \partial \Sigma_k = \lambda_k - \mu_c \).

Since \( K \) is the surgery dual to \( c \) with respect to 0–surgery on \( c \) and \( C \) is the surgery dual to \( k \) with respect to 0–surgery on \( k \), \( X = S^3 - N(K \cup C) \). Upon surgery, the punctured Seifert surfaces \( \Sigma_k \) and \( \Sigma_c \) cap off to oriented Seifert surfaces \( \Sigma_K \) and \( \Sigma_C \) respectively for \( K \) and \( C \). Using these surfaces to orient \( K \) and \( C \) and thus their meridians and longitudes, we obtain that \((\mu_K, \lambda_K) = (\lambda_c, -\mu_c)\) and \((\mu_C, \lambda_C) = (\lambda_k, -\mu_k)\). Therefore \([\mu_K] = [\mu_k] \) and \([\mu_C] = [\mu_c] \) in \( H_1(X; \mathbb{Z}) \). However, since \([\lambda_C] = -[\mu_k] = -[\mu_K] \), we find that \( \ell k(K, C) = -1 \). To orient \( K \) and \( C \) so that \( \ell k(K, C) = 1 \), we must flip the orientation on \( C \), say. Then for this correctly oriented \( C \), we have \([\mu_C] = -[\mu_c] \). Hence \( \Delta_{K \cup C}(x, y) = \Delta_{K \cup C}(x, y^{-1}) \).

We recall also the following twisting formula for Alexander polynomials from \[11\] Theorem 2.1.

**Proposition 2.6** \([11]\). Let \( k \cup c \) be an oriented two-component link such that \( c \) is an unknot and \( \omega = \ell k(k, c) > 0 \). Denote by \( k_n \) a knot obtained from \( k \) by \( n \)-twist along \( c \). Then \( \Delta_{k_n}(t) = \Delta_{k \cup c}(t, t^{n\omega}) \).

Propositions 2.5 and 2.6 lead us some symmetry among Alexander polynomials of \( k_n \) and \( K_n \).

**Corollary 2.7.** Let \( k \cup c \) be a link as in Theorem 2.7 with surgery dual link \( C \cup K \) where \( K \) is dual to \( c \) and \( C \) is dual to \( k \). Then for the twist families of knots \( \{k_n\} \) and \( \{K_n\} \), we have \( \Delta_{k_n}(t) = \Delta_{K_{-n}}(t) \). In particular, \( \Delta_k(t) = \Delta_K(t) \).

**Proof.** We may orient \( k \) and \( c \) so that \( \ell k(k, c) = 1 \); see the proof of Theorem 2.1. Then Propositions 2.5 and 2.6 show that \( \Delta_{k_n}(t) = \Delta_{K_{-n}}(t) = \Delta_{K \cup C}(t, t^{-n}) = \Delta_{K_{-n}}(t) \). In particular, putting \( n = 0 \), we have \( \Delta_k(t) = \Delta_K(t) \).

### 3. Examples

In this section we will provide examples which satisfy the condition in Theorem 2.1 and hence Corollary 2.8. Example 3.3 follows from Examples 3.1 and 3.2. A slight modification gives a non-hyperbolic example, Example 3.4 that demonstrates Theorem 1.6. We will make a further modification of the first example to present Example 3.5 which implies Theorem 1.6.

Let us take a two component link \( k \cup c \) with \( |\ell k(k, c)| = 1 \) as in Figure 1. Then as shown in Figure 1 \((0, 0)\)-surgery on \( k \cup c \) yields \( S^3 \) and its surgery dual \( C \cup K \subset S^3 \). Thus \( k \cup c \) satisfies the condition in Theorem 2.1 and \( K(n) \cong k_n(n) \) does hold for all integers \( n \).

Furthermore, orienting \( k \cup c \) so that \( \ell k(k, c) = 1 \), one may calculate\(^1\) the multivariable Alexander polynomial of \( k \cup c \) to be

\(^1\)For a computer assisted calculation, one may first use PLink within SnapPy [3] to obtain a DowkerThistlethwaite code (DT code) for the link. Then the Knot Theory package [2] for Mathematica [23] can produce the multivariable Alexander polynomial from the DT code.
\[ \Delta_{k,c}(x, y) = -(x^{-1} - 2 + x)y^{-1} + 1 - (x^{-1} - 2 + x)y. \]

Hence by Proposition 2.6 we have:

\[ (\ast) \quad \Delta_{k_n}(t) = \Delta_{k \cup c}(t, t^n) = -(t^{-1} - 2 + t)t^{-n} + 1 - (t^{-1} - 2 + t)t^n. \]

In particular, since the Alexander polynomial of \( k_n \) varies depending on \( n \), \( c \) is not a meridian of \( k \).

\[ \text{Figure 1. } (0,0)\text{-surgery on } k \cup c \text{ results in } S^3 \text{ with its surgery dual } C \cup K. \]

Let us generalize this following Corollary 2.3. Let \( K_m \) be a knot obtained from \( K \) by \( m \)-twist along \( C \). Then Corollary 2.3 asserts that \( K_m(n + m) \cong k_n(m + n) \) for any integers \( m, n \). Figure 2 demonstrates this fact pictorially.

\[ \text{Figure 2. } (m + n)\text{-surgery on the knot } k_n \text{ is equivalent to } (m + n)\text{-surgery on } K_m. \]

Let us choose an integer \( m \) arbitrarily. Observe that, in this example, we have \( K_m = k_m \); see Figure 2. Hence, if \( k_n = K_m \) for some integer \( n \), then \( k_n = k_m \). Thus \( \Delta(k_n) \cong \Delta(k_m) \), and (\ast) implies that \( n = \pm m \). Thus at most \( k_m \) and \( k_{-m} \) can be isotopic to \( K_m \). Since \( K_m(n + m) \cong k_n(m + n) \) for all integers \( m, n \), we have the following:

- For a given integer \( m \), every integral slope except possibly 0 and 2\( m \) fails to be a characterizing slope for \( K_m \).
- If furthermore \( K_{-m} \neq K_m \), then 0 will fail to be a characterizing slope as well.
Example 3.1 \((m = 0)\). Let us choose \(m = 0\) in the above. Then \(K_0(n) = k_n(n)\) for all integers \(n\) and, as mentioned above, every non-zero integral slope fails to be a characterizing slope for \(K_0\). In Figure 3 we identify \(K_0 = k_0\) as the pretzel knot \(P(-5, -3, 3)\), which is known to be hyperbolic by [18].

\[
\text{Figure 3. The knot } k = k_0 \text{ is isotoped into a presentation as the pretzel knot } P(-5, 3, -3). \text{ The twisting circle } c \text{ is carried along with the isotopy.}
\]

Remark 3.2. Notably, the (mirror of the) knot \(P(-5, 3, -3)\) was the basic example of the first two families non-strongly invertible knots with a small Seifert fibered space surgery [15]. Indeed, \((-1)\)-surgery on \(P(-5, 3, -3)\) is the Seifert fibered space \(S^2(-2/5, 3/4, -1/3)\).

Since \(P(-5, 3, -3)\) is the knot \(K_0\) and \(K_0(n) = k_n(n)\) for all integers \(n\), we have \(K_0(-1) = k_{-1}(-1) = K_{-1}(-1)\). Thus \((-1)\)-surgery on \(K_{-1}\) is the same Seifert fibered space. SnapPy recognizes the complement of \(K_{-1}\) as the mirror of the census manifold \(\partial \Sigma_{4401}\). Furthermore, SnapPy reports this manifold as asymmetric, implying that \(K_{-1}\) is neither strongly invertible nor cyclically periodic, and hence cannot be embedded in a genus 2 Heegaard surface.

Example 3.3 \((m = 1)\). By choosing \(m = 1\) instead of 0, we obtain a knot \(K_1\) for which we have \(K_1(n + 1) = k_n(1 + n)\) for all integers \(n\). As we mentioned, every integral slope other than 0, 2 are non-characterizing slope for \(K_1\). In Figure 4 we recognize the knot \(K_1\) as the 9-crossing Montesinos knot \(M(1/3, -1/2, 2/5)\) which is known to be the knot 9_42 in Rolfsen’s table [24]. Following [18], \(K_1\) is a hyperbolic knot.

Now let us show that 0–slope is also a non-characterizing slope for \(K_1\). Since \(K_1(0) \cong k_{-1}(0)\), it is sufficient to see that \(K_1 \neq k_{-1}\). Recall that \(K_m = k_m\) for any \(m\). Alexander polynomials distinguish \(k_1\) from \(k_n\) for all \(n \neq \pm 1\); see (⋆). The Jones polynomials will however distinguish \(k_1 = K_1\) and \(k_{-1}\):

\[
V_{k_1}(q) = q^{-3} - q^{-2} + q^{-1} - 1 + q - q^2 + q^3
\]

\(\text{Kodama’s software KNOT [10] was used confirm the Jones polynomials of knots.}\)
while

\[ V_{k-1}(q) = q^{-1} + q^{-3} - q^{-6} - q^{-8} + q^{-9} - q^{-10} + q^{-11}. \]

(As noted in Remark 3.2, SnapPy also identifies the complement of \( K_{-1} = k_{-1} \) as distinct from the complement of \( K_1 = 9_{42} \), thereby distinguishing these knots.) Hence all integers except possibly 2 are non-characterizing slopes for the hyperbolic knot \( K_1 = 9_{42} \).

\[ \text{Figure 4. The knot } K_1 \text{ in Figure 2 is isotoped into a presentation as the 9 crossing Montesinos knot } M(1/3, -1/2, 2/5) \text{ which may be recognized as the knot } 9_{42} \text{ in Rolfsen’s table [24].} \]

Next we provide examples of non-hyperbolic knots with all integral slopes are non-characterizing slopes, from which Theorem 1.6 follows.

**Example 3.4 (Non-hyperbolic example).** Given any non-trivial knot \( k'' \), let us take a two component link \( k \cup c \) as in Figure 5, where \( k \) is a connected sum of a knot \( k' \) (which is \( k \) in Figure 1, the closure of the 1–string tangle \( \tau' \)) and the non-trivial knot \( k'' \) (the closure of the 1–string tangle \( \tau'' \)).

Then as in Figure 1 we see that \((0, 0)\)–surgery on \( k \cup c \) gives \( S^3 \) with the surgery dual \( C \cup K \). Actually, we follow the isotopy and “light bulb” moves as indicated in Figure 1 to obtain the sixth figure, in which \( k \) is almost an \( S^1 \) fiber, but it has the connected summand \( k'' \) (i.e. the knotted arc \( \tau'' \)). Then we apply further “light bulb” moves to \( k \) so that it becomes an \( S^1 \) fiber; \( K \) becomes a satellite knot with \( k'' \) as a companion knot. Then by Corollary 2.3, \( K_m(n + m) \cong k_n(m + n) \) for all integers \( m, n \).

It is easy to observe that \( k_n \) is a connected sum \( k'_n \sharp k'' \), where \( k'_n \) is a knot obtained from \( k' \) by \( n \)–twist along \( c \). For instance, \( k_0 = P(-5, 3, -3) \sharp k'' \) and \( k_1 = 9_{42} \sharp k'' \). Since \( k'_n \) is non-trivial for all integers \( n \) by (⋆), \( k_n \) is not prime for all integers \( n \).

On the other hand, we show that \( K_m \) is prime for all integers \( m \). (We note that, by construction, \( K_m \) has \( k'' \) as a companion knot for every integer \( m \).) In the following we fix an integer \( m \) arbitrarily. First we observe that \( k_n(m + n) \) is obtained by gluing \( E(k'_n) \) and \( E(k'') \) along their boundary tori. Recall that the exterior \( E(k_n) \) may be expressed as the union of the 2-fold composing space \( X \) (i.e. [disk with 2–holes] × \( S^1 \)) and two knot spaces \( E(k'_n), E(k'') \). We
note that $\partial X$ consists of $\partial E(k_m), \partial E(k'_n)$ and $\partial E(k''_n)$ and a regular fiber in $\partial X \cap \partial E(k_m)$ is a meridian of $k_n$. Since the surgery slope $m+n$ is integral, the corresponding Dehn filling of $X$ results in $S^1 \times S^1 \times [0,1]$ and $k_n(m+n)$ can be viewed as the union of $E(k'_n)$ and $E(k''_n)$. Hence $K_m(n+m) \cong k_n(m+n) = E(k'_n) \cup E(k''_n)$ for all integers $n$. It should be noted here that $E(k''_n)$ is independent of $n$, but the topological type of $E(k'_n)$ depends on $n$. Now assume for a contradiction that $K_m$ is not prime and express $K_m = t_1 \# \cdots \# t_p$ where $t_i$ is a prime knot for $1 \leq i \leq p$. Then $E(K_m)$ is the union of the $p$–fold composing space $Y = \{ \text{disk with } p \text{-holes} \} \times S^1$ and $p$ knot spaces $E(t_1), \ldots, E(t_p)$, where a regular fiber in $\partial Y \cap \partial E(K_m)$ is a meridian of $K_m$. Since the surgery slope $n+m$ is integral, the corresponding Dehn filling of $Y$ results in $(p-1)$–fold composing space $Y' = \{ \text{disk with } (p-1) \text{-holes} \} \times S^1$. Hence $K_m(n+m)$ is expressed as the union $Y'' \cup E(t_1) \cup \cdots \cup E(t_p)$. If necessary, decomposing each $E(t_i)$ further by essential tori, we obtain a torus decomposition of $K_m(n+m)$ in the sense of Jaco-Shalen-Johannson [8, 9]. Note that identifications of $Y'$ and $E(t_i)$ ($1 \leq i \leq p$) depends on $n$, but the topological type of $E(t_i)$ ($1 \leq i \leq p$) does not depend of $n$. To make precise, let us focus on the case of $n = 0, 1$. Then $K_m(n+m) \cong k_n(m+n) = E(k'_n) \cup E(k''_n)$ and $E(k'_n)$ admits a hyperbolic structure in its interior: $E(k'_0)$ is the exterior of the hyperbolic knot $P(-5,3,-3)$ and $E(k'_1)$ is the exterior of the hyperbolic knot $9_{42}$. If $E(k''_n)$ is neither hyperbolic nor Seifert fibered, we decompose $E(k''_n)$ by essential tori to obtain a torus decomposition of $K_m(n+m) \cong k_n(m+n)$ in the sense of Jaco-Shalen-Johannson. Since $E(k'_0) \neq E(k'_1)$, uniqueness of the torus decomposition of $K_m(n+m)$ shows that some $E(t_i)$ changes according as $n = 0, 1$. This is a contradiction. It follows that $K_m$ is a prime knot. Since $K_m$ is prime, while $k_n$ is not prime for all integers $m,n$, we have $\{ K_m \} \cap \{ k_n \} = \emptyset$. Thus every integral slope fails to be a characterizing slope for a prime satellite knot $K_m$ (with a given knot $k''_n$ a companion knot) for any integer $m$, establishing Theorem 1.6(1). Similarly, every integral slope fails to be a characterizing slope for a composite knot $k_n$ (with a given knot $k''_n$ a connected summand) for any integer $n$. This establishes Theorem 1.6(2).
Example 3.5 (Proof of Theorem 1.5). Figure 6 shows a sequence of transformations relating $(m + n, \infty)$–surgery on a link $k_n \cup c$ to $(\infty, n + m)$–surgery on a link $C \cup K_m$. In particular, it gives two twist families of knots $\{k_n\}$ and $\{K_m\}$ such that $k_n(m + n) = K_m(n + m)$.

Replace $(m + n, \infty)$–surgery on $k_n \cup c$ by $(0, 0)$–surgery on $k_0 \cup c$, and follow isotopies and “light bulb” moves as indicated in Figure 6 to see that $(0, 0)$–surgery on $k_0 \cup c$ yields $S^3$ with surgery dual $C \cup K_0$ where $K_0$ is dual to $c$ and $C$ is dual to $k_0$.

As Figure 6 demonstrates, the knot $k_1$ is the hyperbolic 8–crossing Montesinos knot $M(3, 1/3, 1/2)$. It is the knot 8_6 in Rolfsen’s table, the two-bridge knot $\frac{23}{11}$. Following [18] (cf. [7, 16]) it is a hyperbolic knot.
Using $n = 0$, we may calculate that
\[(**) \quad \Delta_{k_0 \cup c}(x, y) = (x^{-1} - 2 + x)y^{-1} + (x^{-2} - 4x^{-1} + 5 - 4x + x^2) + (x^{-1} - 2 + x)y,\]
which is equal to $\Delta_{K_0 \cup C}(x, y^{-1})$ by Proposition 2.6. Note also that $\Delta_{k_0 \cup c}(x, y) = \Delta_{k_0 \cup c}(x, y^{-1})$; see (**) Hence $\Delta_{k_0 \cup c}(t, t^n) = \Delta_{k_0 \cup c}(t, t^{-n}) = \Delta_{K_0 \cup C}(t, t^n)$, and it follows from Proposition 2.6 that
\[
\Delta_{k_1}(t) = \Delta_{K_1}(t) = (t^{-1} - 2 + t)t^{-n} + (t^{-2} - 4t^{-1} + 5 - 4t + t^2) + (t^{-1} - 2 + t)t^n.
\]
Thus Alexander polynomials distinguish $k_1$ from $K_m$ for all integers $m \neq \pm 1$.

We further calculate the Jones polynomials of $k_1$, $K_1$, and $K_{-1}$ to be

\[
V_{k_1}(q) = \frac{1}{q^4} - \frac{2}{q^3} + \frac{3}{q^2} - \frac{4}{q} + \frac{4}{q^2} - \frac{4}{q^3} + \frac{3}{q^4} - 1 + q,
\]

\[
V_{K_1}(q) = -\frac{1}{q^{15}} + \frac{1}{q^{14}} + \frac{1}{q^{11}} - \frac{1}{q^9} + \frac{1}{q} - \frac{3}{q^2} + \frac{3}{q^3} - \frac{4}{q^4} + \frac{5}{q^5} - \frac{4}{q^6} + \frac{3}{q},
\]

and

\[
V_{K_{-1}}(q) = -\frac{1}{q^{21}} + \frac{1}{q^{20}} - \frac{1}{q^{17}} + \frac{1}{q^{14}} + \frac{2}{q^{13}} - \frac{1}{q^{11}} + \frac{1}{q^{10}} + \frac{1}{q^9} - \frac{1}{q^8} + \frac{1}{q^7} + \frac{1}{q^6} - \frac{2}{q^5} + \frac{3}{q^4} - \frac{4}{q^3} - \frac{3}{q^2} + \frac{2}{q}
\]
to conclude that $k_1 \neq K_{\pm 1}$. Thus $k_1$ is an 8–crossing hyperbolic knot for which every integral slope is not a characterizing slope.

4. Further discussions

Let $k \cup c$ be a two-component link such that $c$ is unknotted and $(0, 0)$–surgery on $k \cup c$ yields $S^3$ with surgery dual link $C \cup K$. Denote by $k_n$ a knot obtained from $k$ by $n$–twist along $c$, similarly denote by $K_m$ a knot obtained from $K$ by $m$–twist along $C$. Thus we obtain twist families of knots \{k_n\} and \{K_n\}, which enjoy $K_m(n + m) \cong k_n(m + n)$ for all integers $m, n$. See Corollary 2.3.

Since $(0, 0)$–surgery on $k \cup c$ results in $S^3$, the linking number between $k$ and $c$ must be $\pm 1$. Now assume that $k = k_0$ is an L-space knot. Then $k_0(m)$ is an L-space for infinitely many integers.
n [20, Proposition 2.1], and since \( k_0(m) = K_m(m) \) for all integers \( m \), the twist family \( \{(K_m, m)\} \) contains infinitely many L-space surgeries. Furthermore, it follows from [11, Proposition 1.10] that \( K_m \) have the same Alexander polynomial for all \( m \in \mathbb{Z} \).

Based on [11, Conjecture 1.9], we expect a negative answer to the following question.

**Question 4.1.** Does there exist a link \( k \cup c \) of an L-space knot \( k \) and unknot \( c \) such that \( c \) is not a meridian of \( k \) and \((0, 0)\)-surgery on \( k \cup c \) is \( S^3 \)?

Recall that the non-zero coefficients of the Alexander polynomial of an L-space knot are \( \pm 1 \) and alternate in sign [20, Corollary 1.3]. Hence it turns out that our knots with infinitely many non-characterizing slopes given in Section 3 are not L-space knots.

So we may expect a positive answer to the following:

**Question 4.2.** Does an L-space knot have only finitely many non-characterizing slopes?

If the answer to Question 4.2 is positive, then \( k_0 \) has only finitely many non-characterizing slopes. Since \( k_0(m) = K_m(m) \) for all integers \( m \), \( K_m \) must be isotopic to \( k_0 \) except for at most finitely many integers \( m \). Then it follows from [11] that \( c \) is a meridian of \( k = k_0 \). Thus the positive answer to Question 4.2 enables us to answer Question 4.1 in the negative.

More generally, we ask:

**Question 4.3.** For which knots \( k \) does there exist a link \( k \cup c \) of \( k \) and an unknot \( c \) such that \( c \) is not a meridian of \( k \) and \((0, 0)\)-surgery on \( k \cup c \) is \( S^3 \)?

Our technique cannot work for non-integral slopes. So we would like to propose a modified version of Ni-Zhang’s question:

**Question 4.4.** For a hyperbolic knot \( K \), is a non-integral slope \( r = p/q \) with \(|p| + |q| \) sufficiently large a characterizing slope?

It should be noted here that Lackenby [13] shows that for each atoroidal, homotopically trivial knot \( K \) in a 3-manifold \( Y \) with \( H_1(Y; \mathbb{Q}) \neq \{0\} \), there exists a number \( C(Y, K) \) such that \( p/q \) is a characterizing slope for \( K \) if \(|q| > C(Y, K) \).

It is also reasonable to ask:

**Question 4.5.** Does every knot \( K \) have a characterizing slope? More strongly, does every knot have infinitely many characterizing slopes?

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