Finite Energy Sum Rules in Potential Scattering

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We study scattering theory identities previously obtained as consistency conditions in the context of one-loop quantum field theory calculations. We prove the identities using Jost function techniques and study applications.

INTRODUCTION

In a recent study of the quantum energies of interfaces in field theory, we discovered a set of consistency conditions on scattering data that take the form of finite energy sum rules in potential scattering,

\[ \int_0^\infty \frac{dk}{\pi} k^{2n} \frac{d}{dk} \left[ \delta_\ell(k) - \sum_{\nu=1}^m \delta_\ell^{(\nu)}(k) \right] + \sum_j (-\kappa_{2j}^2)^n = 0, \quad m \geq n. \quad (1) \]

Here \( \delta_\ell(k) \) denotes the scattering phase shift in the channel with angular momentum \( \ell \), and \( \delta_\ell^{(\nu)}(k) \) is the \( \nu \)th Born approximation. The sum on \( j \) ranges over the bound states with angular momentum \( \ell \) and \( \kappa_{2j}^2 = -k_{2j}^2 \) is the binding energy. Note that for \( n = m = 0 \), eq. (1) is simply Levinson’s theorem. In fact, these identities are the natural generalizations of Levinson’s theorem. \( m = n \) is the minimal number of Born subtractions necessary to render the integral in eq. (1) finite. Since we may generally subtract further Born approximations, eq. (1) also implies “oversubtraction” rules such as

\[ \int_0^\infty \frac{dk}{\pi} k^{n} \frac{d}{dk} \delta_\ell^{(2)}(k) = 0. \quad (2) \]

In Ref. [1] these sum rules appeared as consistency conditions in quantum field theory leading to finite expressions for the Casimir energies of interfaces. Here we derive them within scattering theory, and consider applications and consequences in ordinary quantum mechanics. Our proof will employ Jost function techniques commonly used to prove Levinson’s theorem [2]. For sufficiently singular potentials, however, our sum rules fail, even though Levinson’s theorem continues to hold.

In Section II we derive the sum rules for the antisymmetric channel in one dimension, where the analysis is simplest. This derivation applies to the s-wave in three dimensions as well. The extension to higher partial waves in three dimensions is straightforward and is presented in the Appendix. In Section III we discuss generalizations. The generalization to fermion scattering (via the Dirac equation) is also straightforward and is left to the reader. We also mention the generalization to multichannel problems with internal symmetries. The symmetric channel in one dimension requires special consideration (as it does for Levinson’s theorem [2]) and is treated in detail in Section IV. In Section V we describe some singular potentials for which the sum rules do not hold. Finally, in Section VI we study the semiclassical limit, where the sum rules take a particularly compact form and have a simple physical interpretation.

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These sum rules could have been derived many years ago in the heyday of potential scattering theory. However, we have been unable to find them in the literature. They bear some resemblance to results based on the Gel’fand-Diki˘ı equation, obtained in the Russian literature [4], although the physical foundations and the resulting sum rules themselves are quite different. In our conclusion we mention this earlier work and compare it with our own. Also, for the special case of a separable potential a related sum rule was obtained in Ref. [5].

THE ANTISYMMETRIC CHANNEL IN ONE DIMENSION

Derivation of the Sum Rules

We consider the scattering of a spinless particle in a symmetric potential $V(x) = V(-x)$ in one dimension, described by the Schrödinger equation,

$$-\psi'' + V(x)\psi = k^2 \psi. \quad (3)$$

This is a two channel problem. The antisymmetric channel is specified by the boundary condition $\psi_-(0) = 0$. The symmetric channel corresponds to $\psi'_+(0) = 0$. Here we consider the antisymmetric channel. Let $\delta_-(k)$ denote the scattering phase shift in this channel, defined by the asymptotic form of the wave function $\psi_-(x)$ at large $x$,

$$\psi_-(x) \to e^{-ikx} - e^{2i\delta_-}(k)e^{ikx}. \quad (4)$$

Our goal is to derive the sum rules

$$\int_0^\infty \frac{dk}{\pi} k^{2n} \frac{d}{dk} \left[ \delta_-(k) - \sum_{\nu=1}^m \delta^{(\nu)}(k) \right] = - \sum_j (-\kappa_{-j}^2)^n, \quad m \geq n \quad (5)$$

where the sum on $j$ ranges over the antisymmetric bound states of $V(x)$ with binding energies $\kappa_{-j}^2 = -k_{-j}^2$. For the remainder of this section we suppress the subscript labeling the antisymmetric channel. For real $k$, the phase shift $\delta(k)$ is given in terms of the S-matrix $S(k)$, which in turn is related to the Jost function $F(k)$ by

$$\delta(k) = \frac{1}{2i} \ln S(k) = \frac{1}{2i} [\ln F(-k) - \ln F(k)]. \quad (6)$$

The Born approximation is an expansion of the phase shift $\delta$ (not the Jost function $F$ itself) in powers of the interaction $V(x)$. The Jost function is obtained from the Jost solution $f(k, x)$ to eq. (3), which is asymptotic to an outgoing wave at infinity,

$$\lim_{x \to \infty} e^{-ikx} f(k, x) = 1, \quad (7)$$

and $F(k) = f(k, 0)$. As is well known, the integral equation for $f(k, x)$ has a unique solution in the upper half $k$-plane, where it is holomorphic and continuous as $\text{Im } k \to 0$, provided that the potential $V(x)$ is locally integrable and from the so-called “Faddeev class”

$$\int_{-\infty}^{\infty} dx (1 + |x|)|V(x)| < \infty. \quad (8)$$

In addition, $F(k)$ has zeros at the bound states, $k = i\kappa_j$, on the positive imaginary axis. To quantitatively estimate the behavior of the Born approximation at large momenta $|k|$, we furthermore have to assume that the interaction $V(x)$ is bounded and sufficiently smooth to allow for integration by parts. Unless stated otherwise, we will restrict our analysis to non-singular potentials $V(x)$ with these properties.

To proceed, we take $m \geq n$ and introduce an auxiliary function, $F_m(k)$, with the following properties:

(a) $F_m(k)$ is analytic and has no zeros in the upper half $k$-plane including $k = 0$.

(b) $|\ln F(k) - \ln F_m(k)|$ falls like $|k|^{-2m-1}$ as $|k| \to \infty$ in the upper half plane.
After completing the derivation of our sum rules we will construct \( F_m(k) \) and relate it to the Born approximation to \( \delta(k) \). For real \( k \) we introduce

\[
\delta_m(k) \equiv \frac{1}{2i} [\ln F_m(-k) - \ln F_m(k)] ,
\]

and consider

\[
I_{n,m} = \int_0^\infty \frac{dk}{\pi} k^{2n} \frac{d}{dk} (\delta(k) - \delta_m(k))
= -\frac{1}{2\pi i} \int_0^\infty dk k^{2n} \frac{d}{dk} (\ln F(k) - \ln F(-k) - \ln F_m(k) + \ln F_m(-k)) .
\]

Since the integrand is manifestly even in \( k \), we can extend the integration range to \(-\infty\). Applying the substitution \( k \to -k \) we obtain,

\[
I_{n,m} = -\frac{1}{2\pi i} \int_{-\infty}^\infty dk k^{2n} \frac{d}{dk} (\ln F(k) - \ln F_m(k))
= -\frac{1}{2\pi i} \oint_C dk k^{2n} \frac{d}{dk} (\ln F(k) - \ln F_m(k)) .
\]

where the contour \( C \) is the real axis plus the semicircle of infinite radius in the upper half plane. The semicircle gives no contribution to the integral because of property (b).

The contour integral can now be performed using Cauchy’s theorem by recognizing that \( d\ln F/dk \) has poles of unit residue at each bound state. By property (a), \( d\ln F_m/dk \) has no poles inside \( C \). The result is the sum rule, eq. (5).

**Construction of the Auxiliary Function**

In this section we construct an auxiliary function with the two properties required in the previous subsection. It is convenient to parameterize the Jost solution \( f(k, x) \) in terms of an exponent \( \beta(k, x) \),

\[
f(k, x) \equiv e^{ikx + i\beta(k, x)} .
\]

Substituting into the Schrödinger equation we find that the complex function \( \beta(k, x) \) satisfies

\[
- i\beta''(k, x) + 2k\beta'(k, x) + \beta'^2(k, x) + V(x) = 0,
\]

subject to the boundary condition

\[
\beta(k, \infty) = \beta'(k, \infty) = 0 ,
\]

where \( \beta'(k, x) = d\beta(k, x)/dx \). Combining eqs. (11) and (15) with the boundary condition \( \psi(0) = 0 \), it is easy to see that

\[
\delta(k) = -\text{Re} \beta(k, 0) .
\]

Eqs. (12) and (13) can be converted into a non-linear integro-differential equation,

\[
\beta(k, x) = \frac{1}{2k} \int_x^\infty dy \left(1 - e^{2ik(y-x)}\right) \Gamma(k, y) ,
\]

where

\[
\Gamma(k, x) = \beta'^2(k, x) + V(x).
\]

Note that by differentiation \( \beta'(k, x) \) obeys a similar equation,

\[
\beta'(k, x) = i \int_x^\infty dy e^{2ik(y-x)} \Gamma(k, y) .
\]
Denote the term in $\beta(k, x)$ that is $\nu$th order in the potential by $\beta^{(\nu)}(k, x)$ and define $\beta^{(\nu)}(k) \equiv \beta^{(\nu)}(k, 0)$. An equation for $\beta^{(\nu)}(k, x)$ can be obtained from eq. (13) by iteration

$$\beta^{(\nu)}(k, x) = \frac{1}{2k} \int_x^\infty dy \left( 1 - e^{2ik(y-x)} \right) \Gamma^{(\nu)}(k, y).$$  \hspace{1cm} (18)

Here $\Gamma^{(\nu)}$ is the term in the expansion of $\Gamma$ which is of $\nu$th order in the potential. For $\nu > 1$, $\Gamma^{(\nu)}$ involves only $\beta^{(\mu)}$ with $\mu < \nu$. Thus we are led to equations for $\beta^{(\nu)}(k)$, the first few of which are

$$\beta^{(1)}(k) = \frac{1}{2k} \int_0^\infty dy \left( 1 - e^{2iky} \right) V(y),$$  

$$\beta^{(2)}(k) = \frac{1}{2k} \int_0^\infty dy \left( 1 - e^{2iky} \right) [\beta^{(1)}(k, y)]^2,$$

$$\beta^{(3)}(k) = \frac{1}{2k} \int_0^\infty dy \left( 1 - e^{2iky} \right) 2\beta^{(1)}(k, y)\beta^{(2)}(k, y),$$  \hspace{1cm} (19)

and so forth. Similarly for the $\beta^{(\nu)}(k, x)$, which appear as sources in eqs. (14),

$$\beta^{(1)}(k, x) = i \int_x^\infty dy e^{2ik(y-x)} V(y),$$  

$$\beta^{(2)}(k, x) = i \int_x^\infty dy e^{2ik(y-x)} [\beta^{(1)}(k, y)]^2,$$

$$\beta^{(3)}(k, x) = i \int_x^\infty dy e^{2ik(y-x)} 2\beta^{(1)}(k, y)\beta^{(2)}(k, y).$$  \hspace{1cm} (20)

The exponential factors in eq. (18) guarantee that $\beta^{(\nu)}(k)$ is analytic in the upper half $k$-plane provided that $\Gamma^{(\nu)}$ is, and likewise for $\beta^{(\nu)}(k, x)$. Starting with $\Gamma^{(1)} = V(x)$ we derive the required analytic properties of $\beta^{(\nu)}$ and $\beta^{(\nu)}$ inductively. To obtain the large $|k|$ behavior of $\beta^{(\nu)}$ and $\beta^{(\nu)}$ from their respective integral representations, we integrate by parts once and estimate the remainder by sequentially applying the Riemann-Lebesgue lemma.$^1$ The result is that

- For the class, eq. (8) of potentials, $\beta^{(\nu)}(k)$ is holomorphic in the upper half plane including at $k = 0$;
- $|\beta^{(\nu)}(k)| \rightarrow \text{const} \cdot |k|^{-2\nu+1}$ as $|k| \rightarrow \infty$ in the upper half plane $\text{Im} k > 0$.

To complete the derivation, we define

$$F_m(k) = \exp \left[ i \sum_{\nu=1}^m \beta^{(\nu)}(k) \right].$$  \hspace{1cm} (21)

The required properties of $F_m(k)$ follow directly from those of $\beta^{(\nu)}(k)$ the convergence of the Born series $\beta(k) = \sum_{\nu=1}^\infty \beta^{(\nu)}(k)$ for sufficiently large $|k|$ in the upper half plane $\Re \text{Im} k > 0$ (see also the Appendix).

The quantity that enters the sum rule is $\delta_m(k)$, given by eq. (4). From eq. (12), it follows that $\beta^{(\nu)}(-k) = -\beta^{(\nu)}(k)$ for real $k$. As a result, we have

$$\delta_m(k) = -\text{Re} \sum_{\nu=1}^m \beta^{(\nu)}(k),$$  \hspace{1cm} (22)

so that $\delta_m(k)$ is the sum of the first $m$ terms in the Born expansion of $\delta(k)$. This completes the derivation of the sum rules in the antisymmetric channel in one dimension.

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$^1$ This procedure is allowed by our assumptions on the potential $V(x)$. In particular, the Riemann-Lebesgue lemma requires the existence of $\langle |V|^{\nu} \rangle = \int_0^\infty |V(y)|^{\nu} dy$ for all $\nu$, as well as similar averages involving the derivative, e.g. $\langle |V V'| \rangle < \infty$. 
GENERALIZATIONS OF THE BASIC RESULT

The antisymmetric channel in one dimension is actually generic. The sum rules can easily be extended to scattering from a central potential in any number of space dimensions \( D > 1 \). The computation proceeds for each partial wave in analogy to the antisymmetric case in \( D = 1 \). Of course, the appropriate generalized Hankel functions must replace the simple exponentials that appear in one dimension. We summarize the derivation for \( D = 3 \) in the Appendix.

The sum rules also extend to the case of fermion scattering in a straightforward way. For a scalar potential, the Dirac equation decomposes into partial waves labeled by total spin \( j \) and parity \( \Pi \), and the sum rules again hold in each partial wave individually.

When there are internal symmetries, so that there are several channels \( \{ s \} \) in each partial wave, we expect that the sum rules will continue to hold with the phase shifts replaced by the sum of the eigenphases, which is given by the trace of the logarithm of \( S \):

\[
\delta(k) \rightarrow \sum_s \delta_s(k) = \frac{1}{2i} \text{Tr} \ln S(k).
\]

Similarly, the sum over bound states will include all bound states in the channel. For example, consider an isodoublet of fermions scattering in a background generated by an isodoublet scalar Higgs field in three dimensions. If the Higgs background is of the “hedgehog” form \( \phi(x) = \phi_0 \exp(i\vec{\tau} \cdot \vec{f}(r)) \), then the fermion spectrum will decompose into channels labeled by parity \( \Pi \) and grand spin \( G \). In each channel, \( S \) is a 2-by-2 matrix, which cannot be simultaneously diagonalized for all \( k \). (Each degree of freedom also appears with the usual \( 2G + 1 \) degeneracy.) If we introduce a chiral SU(2) gauge field that maintains grand spin conservation, then states with different parity but the same \( G \) will mix, leaving a 4-by-4 \( S \)-matrix labeled only by \( G \).

The symmetric channel in one dimension introduces additional subtleties, which are treated in the following section. The result is that the sum rules may be modified by an anomalous piece if too many subtractions are attempted. Specifically, the sum rules in the symmetric channel read

\[
\int_0^\infty \frac{dk}{k^{2n}} \left[ \frac{\delta_+(k) - \sum_{\nu=1}^m \delta_\nu(k)}{d} \right] = -\sum_j (\kappa_{+,j}^2)^n + f_{n,m}^{\text{anom}},
\]

for \( m \geq n \). The anomalous term vanishes if \( 2n > m \). As a result, the “minimally subtracted” form of the sum rules, where \( m = n \), hold without modification except for the case \( m = n = 0 \), which is Levinson’s theorem. In that case \( f_{0,0}^{\text{anom}} = \frac{1}{2} \) and we recover the extra term that appears in Levinson’s theorem in the symmetric channel \([3]\).

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We have checked these results numerically in a variety of simple, generic potentials. In one dimension, we have also checked them for the exactly solvable reflectionless scalar potentials of the form \( V(x) = -\ell(\ell + 1)\text{sech}^2 x \), with \( \ell \) integer \([4]\), and the corresponding potentials in the fermionic case \([3]\).

THE SYMMETRIC CHANNEL IN ONE DIMENSION

Derivation of the Sum Rules

As in the case of Levinson’s theorem, the symmetric channel requires special attention. The regular solution to the Schrödinger equation obeying the boundary conditions, \( \psi'(0) = 0 \) and \( \psi(0) = 1 \), can be written in terms of the Jost solution \( f(k, x) \),

\[
\psi(k, x) = \frac{1}{2ki} [G(k)f(-k, x) - G(-k)f(k, x)] ,
\]

where \( G(k) = df(k, x)/dx|_{x=0} \). Comparing to the \( S \)-matrix parameterization as \( x \rightarrow \infty \),

\[
\psi(k, x) \rightarrow e^{-ikx} + e^{2i\delta_+(k)}e^{ikx},
\]

we see that

\[
\delta_+(k) = \frac{1}{2i} \ln S(k) = \frac{1}{2i} [\ln(-G(-k)) - \ln G(k)] .
\]

The derivation proceeds exactly in analogy to the antisymmetric channel except, as we shall see, it is not possible to find an auxiliary function that is regular at \( k = 0 \). Instead we introduce an auxiliary function \( G_m(k) \) with the following properties:
(a) \( G_m(k) \) is analytic and has no zeros in the upper half \( k \)-plane excluding \( k = 0 \).

(b) \( |\ln G(k) - \ln G_m(k)| \) falls like \(|k|^{-2m-1} \) as \(|k| \to \infty \) in the upper half plane.

(c) At \( k = 0 \), \( k^{2n} d\ln G_m(k)/dk \) has a pole with residue \( 2I_{n,m}^{\text{anom}} \).

(d) \( I_{n,m}^{\text{anom}} = 0 \) for \( 2n > m \).

Then, using the properties of \( G \) and \( G_m \) we find,

\[
I_{n,m} = \int_0^\infty \frac{dk}{\pi} k^{2n} \frac{d}{dk} (\delta_+(k) - \delta_{+,m}(k))
\]

\[
= -\int_0^\infty \frac{dk}{2\pi i} k^{2n} \frac{d}{dk} \left[ \ln G(k) - \ln(-G(-k)) - \ln G_m(k) + \ln(-G_m(-k)) \right]
\]

\[
= -\frac{1}{2\pi i} \oint_{C} dk k^{2n} \frac{d}{dk} \left( \ln G(k) - \ln G_m(k) \right)
\]

\[
= \left\{ \begin{array}{ll}
-\sum_{j=1}^{-\infty} (-\kappa_j^2)^n & 2n > m \geq n \\
-\sum_{j=1}^{-\infty} (-\kappa_j^2)^n + I_{n,m}^{\text{anom}} & m \geq 2n
\end{array} \right.
\]

where

\[
\delta_{+,m}(k) = \frac{1}{2i} \left[ \ln(-G_m(-k)) - \ln G_m(k) \right].
\]

The factor of 2 difference between the residue of the pole at \( k = 0 \) and the anomalous term in the sum rule arises because the integration contour \( C \) passes through \( k = 0 \) and therefore captures only half the residue. Given the restrictions on \( m \) and \( n \) in eq. (28), it is clearly possible to derive a non-anomalous sum rule in the symmetric channel by making the minimal subtraction, \( m = n \). The only exception is Levinson’s theorem, \( n = m = 0 \), which we discuss in detail below. Otherwise anomalies arise if one attempts to “oversubtract” for a given \( n \). We consider specific examples after constructing the auxiliary function \( G_m(k) \).

**Construction of the Auxiliary Function**

As in the antisymmetric channel, the auxiliary function is obtained from the expansion of the Jost function in powers of the potential. The difference is that the relevant Jost function is \( G(k) \) defined by \( G(k) = df(k, x)/dx|_{x=0} \). Using the exponential parameterization of eq. (11), we find

\[
G(k) = i(k + \beta'(k))e^{i\beta(k)}.
\]

where, \( \beta'(k) = d\beta(k, x)/dx|_{x=0} \). The prefactor \( k + \beta'(k) \) gives the difference from the antisymmetric channel. Comparing with eq. (22) we are led to the ansatz

\[
\ln G_m(k) = [\ln(k + \beta'(k))]_m + \ln F_m(k)
\]

where the notation \([X]_m\) is an instruction to make the formal expansion of \( X \) in powers of the potential and keep all terms up to \( m^{th} \) order. For example,

\[
[\ln(k + \beta'(k))]_0 = \ln k
\]

\[
[\ln(k + \beta'(k))]_1 = \ln k + \beta^{(1)}(k)/k
\]

\[
[\ln(k + \beta'(k))]_2 = \ln k + \beta^{(1)}(k)/k + [\beta^{(1)}(k)]^2/k^2 + \beta^{(2)}(k)/k.
\]

This process is necessary to reproduce the asymptotic behavior of \( G(k) \) at large \(|k|\) as required by condition (b) above. The cost is the introduction of poles up to \( m^{th} \) order at \( k = 0 \).

It is straightforward to verify that \( G_m(k) \) defined in eq. (31) satisfies requirements (a) and (b) above. The argument is essentially the same as for the antisymmetric channel. It is clear from the definition of \( G_m(k) \) that its contribution to the integral along the real axis, proportional to \( \delta_{+,m}(k) \), is just the sum of the first \( m \) terms in the Born approximation to the phase shift in the symmetric channel.
It remains to characterize the singularity in $G_m(k)$ at $k = 0$. The term in the contour integral in eq. (28) which is potentially singular at $k = 0$ is proportional to

$$k^{2n} \frac{d}{dk} \ln(k + \beta'(k))_m = k^{2n} \left[ \frac{1 + \frac{d\beta'(k)}{dk}}{k + \beta'(k)} \right]_m$$

$$= k^{2n-1} \left[ \left( 1 + \frac{d\beta'(k)}{dk} \right) \sum_{p=0}^{\infty} \left( -\frac{\beta'(k)}{k} \right)^p \right]_m$$

(33)

An anomalous contribution to the sum rule will result if the $1/k$ singularities in the expansion of $(1 + \beta'(k))^{-1}$ to $m^\text{th}$ order in the potential overcome the prefactor of $k^{2n-1}$. Since the functions $\beta'(k)$ are all analytic in the vicinity of $k = 0$, the most singular term in eq. (33) comes from the term $k^{2n-1}(-\beta'(1)(k)/k)^m$, which is singular if $m \geq 2n$. If $m = 2n$ there is a simple pole at $k = 0$ from this term. If $m > 2n$ there are poles of higher order as well. It is straightforward (but increasingly tedious) to pull out the residue of the simple pole, which determines the anomalous contribution to the sum rule. Once having identified the residue, the expression for the anomalous contribution to $I_{n,m}$ is

$$I_{n,m} = \frac{1}{2} \text{Res} k^{2n-1} \left[ \left( 1 + \frac{d\beta'(k)}{dk} \right) \sum_{p=0}^{\infty} \left( -\frac{\beta'(k)}{k} \right)^p \right]_m$$

(34)

We illustrate this result with some important special cases:

- $n = m = 0$: Levinson’s Theorem
  For $n = m = 0$ we need the coefficient of $1/k$ in the term zeroth order in the potential in $d \ln(k + \beta'(k))/dk$, which is unity. Thus $I_{0,0} = 1/2$, and we obtain Levinson’s theorem in the symmetric channel:

$$\int_0^\infty \frac{dk}{\pi} \frac{d}{dk} \delta_+(k) = \frac{1}{\pi} (\delta_+(\infty) - \delta_+(0)) = \frac{1}{2} - \sum_j 1$$

(35)

- $n = m > 0$: Minimal subtraction
  For $n \neq 0$, the minimum Born subtraction we can make in order to render $I_{n,m}$ convergent is to take $m = n$. The most singular term in the expansion of eq. (33) through $m^\text{th}$ order is proportional to $(-\beta'_1(0))^{m}/k^{m+1}$. Thus the integrand goes like $k^{2n-m-1}$ near $k = 0$. This has no pole when $n = m > 0$. So $I_{n,n} = 0$ for $n > 0$, and the minimally subtracted form of the sum rules is not altered in the symmetric channel.

- $2n > m$: Oversubtraction without an anomaly:
  There is no singularity at $k = 0$ as long as $2n > m$. Therefore it is possible to subtract further Born approximations from the phase shift without introducing anomalies into the sum rules. For $n = 1$ the first Born approximation must be subtracted for convergence and no further subtraction is possible without anomaly. For $n = 2$ the first and second Born approximations must be subtracted and the third may be subtracted without anomaly, and so forth.

- $m = 2n$: Computation of the anomaly.
  For fixed $n$, as further subtractions are attempted, one finally reaches $m = 2n$, where an anomaly appears. The anomaly comes entirely from the term proportional to $(-\beta'^{(1)}(0))^{2n}$ in the expansion of the integrand. Referring back to the definition of $\beta'^{(1)}(k)$, we find

$$\beta'^{(1)}(0) = i \int_0^\infty dy V(y)$$

(36)

So

$$I_{n,2n} = \frac{(-)^n}{2} \left[ \int_0^\infty dy V(y) \right]^{2n}$$

(37)
In particular, the first non-trivial case is the $n = 1$ sum rule from where both the first and second terms in the Born approximation have been subtracted:

$$I_{1,2} = \int_0^\infty \frac{dk}{\pi} k^2 \frac{d}{dk} \left[ \delta_+(k) - \delta_+^{(1)}(k) - \delta_+^{(2)}(k) \right] = \sum_j \kappa_j^2 - \frac{1}{2} \left[ \int_0^\infty dy V(y) \right]^2$$

(38)

This result was first discovered in conjunction with the work of [1] by direct evaluation of the Feynman graph corresponding to the second Born approximation. Here we see that it follows from a careful analysis of the analytic properties of the Born approximation near $k = 0$ and has essentially the same origin as the extra factor of $1/2$ that appears in Levinson’s theorem for the symmetric channel.

**SINGULAR POTENTIALS**

Among the cases that we have checked numerically is the square well in one dimension. Even though its sharp edges seem to invalidate our use of complex analysis in the proof above, the sum rules still hold. As is the case with Levinson’s theorem, the difference between a sharp edge and a smooth, very steep edge can be made arbitrarily small.

If we take the limit where the width of the well goes to zero with the area held fixed, we obtain the delta-function potential

$$V(x) = -\lambda \delta_D(x)$$

(39)

where we have written the Dirac delta function as $\delta_D(x)$ to distinguish it from a phase shift. The phase shift in this potential vanishes in the antisymmetric channel because $V(x)$ is localized at the origin, where the antisymmetric wavefunction vanishes. The symmetric channel phase shift and its first Born approximation are easily calculated:

$$\delta_+(k) = \arctan \frac{\lambda}{2k}$$

$$\delta_+^{(1)}(k) = \frac{\lambda}{2k}$$

(40)

The symmetric channel has a bound state at $\kappa = \frac{\lambda}{2}$.

Like the square well, this potential obeys the one-dimensional version of Levinson’s theorem relating the phase shifts to the number of bound states in each channel

$$\delta_-(0) = \pi n_- = 0$$

$$\delta_+(0) = \pi (n_+ - \frac{1}{2}) = \frac{\pi}{2}.$$  

(41)

However, sum rule with $m = n = 1$, eq. (24), fails. One expects $I_{1,1} = \kappa^2 = \lambda^2/4$, but obtains instead $I_{1,1} = \lambda^2/8$. Examining a sequence of square well potentials approaching the delta function reveals that for any square well, the sum rule is satisfied, but the support of the integral moves out to larger and larger $k$ as the potential gets narrower and deeper. The $\delta_D$-function limit and the $k$-integration do not commute.

It is instructive to examine more closely what goes wrong in the $\delta_D$-function case. A straightforward calculation shows that the proper Jost function for the symmetric channel in one dimension is

$$G(k) = ik + \lambda/2$$

(42)

Note that it has a zero at $k = i\lambda/2$ as expected and is analytic in the upper half $k$-plane. According to the symmetric channel analysis,

$$\ln G_0(k) = \ln ik$$

$$\ln G_1(k) = \ln ik - i\lambda/2k$$

(43)

The derivation of Levinson’s theorem using $\ln G - \ln G_0$ proceeds without difficulty. To derive the sum rule for $I_{1,1}$ it is necessary to consider $d(\ln G(k) - \ln G_1(k))/dk$. This quantity vanishes like $1/k^3$ for large $|k|$, not $1/k^4$ as expected on the basis of property (b) listed in the previous section. As a result, the integral around the semicircle at infinity does not vanish. Specifically,

$$I_{1,1}^\infty = -\frac{1}{2\pi i} \int_{C_{\infty}} dk k^2 \frac{d}{dk} \left[ \ln(ik + \lambda/2) - (\ln ik - i\lambda/2k) \right] = \frac{\lambda^2}{8}$$

(44)
where \( C_\infty \) is the semicircle at infinity in the upper half \( k \)-plane. We combine this result with the integral along the real axis,
\[
I_{1,1} = -\frac{1}{2m_1} \oint_C dk k^2 \frac{d}{dk} [\ln(ik + \lambda/2) - (\ln ik - i\lambda/2k)] - \lambda^2/8
\]
Now \( I_{1,1} \) can be evaluated by contour integration, yielding the same anomalous result obtained by direct integration of \( \delta - \delta^{(1)} \) along the real axis: \( I_{1,1} = \frac{1}{4}\kappa^2 = \lambda^2/8 \).

It remains to explain why \( \frac{d}{dk}(\ln G(k) - \ln G_1(k)) \) falls only like \( 1/k^3 \). Consider
\[
\ln G_1(k) = \ln k + \beta^{(1)}(k)/k + \ln F_1(k). \tag{46}
\]
Since \( \ln F_1(k) \) is proportional to \( \beta^{(1)}(k), \beta^{(1)} \) and \( \beta^{(1)} \) determine the large \( k \) behavior of \( G_1(k) \). From their definitions, eqs. (5) and (7),
\[
\beta^{(1)}(k) = \frac{1}{2k} \int_0^\infty dy \left( 1 - e^{2iky} \right) V(y) = 0, \tag{47}
\]
\[
\beta^{(1)}(k) = i \int_0^\infty dy e^{2iky} V(y) = -i\lambda/2,
\]
for \( V(x) = -\lambda V_D(x) \). In particular, as \( k \to \infty, \beta^{(1)}(k) \to \text{const} \). In contrast, if \( V(x) \) is any bounded function of \( x \), including a square well, \( \beta^{(1)}(k) \sim V(0)/k \). This is the ultimate source of the breakdown of the sum rule in the case of the \( \delta_D \) function, for which \( V(0) \) is ill-defined.

**WKB APPLICATIONS**

Our sum rules become especially simple in the WKB approximation. They provide formulas for the sum of powers of the binding energies as integrals over powers of the potential. In this way, the WKB approximation yields some insight into the physical origin of the sum rules. We have checked the accuracy of these results in some simple potentials.

We work in one dimension with a potential \( V(x) \) that is everywhere negative. We therefore define \( U(x) = -V(x) \). We assume that \( \int dx |U(x)|^n \) exists for all \( n \geq \frac{1}{2} \). The reflection coefficient is exponentially small in the WKB approximation, so the even and odd parity phase shifts are equal and are given by
\[
\delta(k) = \int_0^\infty dx \left( \sqrt{k^2 + U(x)} - k \right). \tag{48}
\]
The \( \nu^\text{th} \) Born approximation to \( \delta(k) \) is merely the term of order \( U^\nu \) in the expansion of the integrand.

The WKB approximation should be valid when \( d\lambda(x)/dx \ll 1 \), where \( \lambda(x) = 1/\sqrt{k^2 + U(x)} \) is the local de Broglie wavelength. For a deep, smooth potential this criterion is satisfied for all \( x \). The first correction to the WKB approximation gives only a modulation of the magnitude of the wavefunction and does not change its phase. So we expect eq. (18) to be quite a good approximation.

To evaluate the sum rule with minimal subtraction \( (m = n) \) we must calculate
\[
I_{n,n} = \sum_j \kappa_j^{2n} = (-1)^{n+1} \frac{2n}{\pi} \int_0^\infty k^{2n-1} \left( \delta(k) - \sum_{\nu=1}^n \delta^{(\nu)}(k) \right) dk. \tag{49}
\]
A straightforward calculation yields the WKB estimate
\[
\sum_j \kappa_j^{2n} \approx \frac{2^{n+1}}{\pi} \frac{n!}{(2n + 1)!!} \int_0^\infty dy [U(y)]^{n+\frac{1}{2}} \equiv I_{n,n}^{\text{WKB}}. \tag{50}
\]
Note that in the WKB approximation, the \( \nu^\text{th} \) Born approximation is proportional to \( 1/k^{2\nu+1} \) so oversubtraction of the sum rules is not allowed in this case. In Fig. 8 we show the relative error that arises due to the WKB approximation for various sum rules. For sufficiently strong potentials this error is indeed small.
FIG. 1: The relative error, defined as the difference of the right and left sides of eq. (50) divided by the sum, for the potential \( V(y) = -l(l+1)^2 \text{sech}^2(y) \) as a function of the coupling \( l \) for various sum rules that are labeled by \( n \).

Alternatively, we can use zeta function regularization to define a regularized integral

\[
I_{WKB}^{n,n}(s) = \left(-1\right)^{n+1} \frac{2n}{\pi} \int_0^\infty k^{2n-1} \int_0^\infty dk \left(k^2 + U(y)\right)^s \, dk
\]

and evaluate for \( s < -n - 1 \), where it converges to a beta function. We then analytically continue to \( s = \frac{1}{2} \), and obtain the same result. Curiously, this result implies that in the WKB approximation, the contribution of the Born terms vanish in zeta function regularization.

Note that \( I_{WKB}^{0,0} \) gives the WKB approximation to Levinson’s theorem:

\[
\sum_j \kappa_j^{WKB} \approx \frac{2}{\pi} \int_0^\infty dy \sqrt{U(y)},
\]

and \( I_{WKB}^{1,1} \) gives a particularly simple formula for the sum of the binding energy of all bound states in the WKB approximation:

\[
\sum_j \kappa_j^{WKB} \approx \frac{4}{3\pi} \int_0^\infty dy \left[U(y)\right]^{\frac{3}{2}},
\]

Not surprisingly, eq. (50) has a simple semiclassical interpretation.\(^2\) We replace the sum over bound states by an integral over the density of states, \( \rho(k) \),

\[
I_{n,n} = \int d\rho(k)(-k^2)^n
\]

where the integral extends only over bound states \((k^2 < 0)\). If we approximate \( d\rho \) semiclassically by \( d\rho = dydp/2\pi \), where \( k^2 = p^2 + V(y) \), then

\[
I_{n,n} = \frac{1}{2\pi} \int_{-\infty}^\infty dy \int dy \sqrt{\frac{U(y)}{-U(y)}} \left[U(y) - p^2\right]^n.
\]

Direct evaluation of this simple integral yields eq. (50).

**DISCUSSION AND CONCLUSIONS**

Our sum rules are related to the results of Buslaev and Faddeev based on the Gel’fand-Dikii equation.\(^3\) They studied solutions to the Schrödinger equation on the half-line subject to the boundary condition \( \psi(0) = 0 \). In our

\(^2\) We thank J. Goldstone for this observation.

\(^3\) We thank G. Dunne for bringing this work to our attention.
language, this system is equivalent to the antisymmetric channel in one dimension, though without the restriction that the potential be smooth at the origin (so they can have \( V'(0) \neq 0 \)). They obtained a sequence of sum rules, beginning with\(^4\)

\[
\frac{2}{\pi} \int_0^\infty k \left( \delta(k) + \frac{1}{2k} \int V(x) \, dx \right) \, dk + \sum_j \kappa_j^2 = \frac{1}{4} V(0)
\]

\[
\frac{4}{\pi} \int_0^\infty k^3 \left( \delta(k) + \frac{1}{2k} \int V(x) \, dx + \frac{1}{(2k)^3} \left( V'(0) + \int V(x)^2 \, dx \right) \right) \, dk - \sum_j \kappa_j^4 = \frac{1}{8} \left( 2V(0)^2 - V''(0) \right) .
\] (56)

Their identities have a similar structure to our sum rules (after integrating by parts). There are some significant differences, however. Instead of subtracting the Born approximation as we have done, they instead have subtracted the leading local asymptotic expansion of the phase shift in powers of \( 1/k \). Their expressions are simpler — just integrals of the potential over space divided by powers of \( k \) — but more singular at the origin. As a result, it is not possible to form the oversubtracted versions of their identities. This difference also accounts for the need for extra terms proportional to the potential and its derivatives at the origin. In field theory applications, the Born subtractions arise naturally in the process of renormalization in a definite scheme.\(^4\)

We know of no similar application of the Buslaev-Faddeev results.

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**APPENDIX**

In this section, we extend the proof of the sum rules to spherically symmetric potentials in three dimensions. Our analysis follows the case of the antisymmetric channel in one dimension closely.

**Derivation of the Sum Rules**

We consider the scattering of a spinless particle in a central potential \( V(r) \) in three dimensions described by the radial Schrödinger equation,

\[
-\psi''_\ell + \left[ V(r) + \frac{\ell(\ell + 1)}{r^2} \right] \psi_\ell = k^2 \psi_\ell .
\] (57)

Let \( \delta_\ell(k) \) denote the scattering phase shift in the channel with angular momentum \( \ell \), and \( \delta^{(\nu)}_\ell(k) \) the \( \nu \)th Born approximation. Our goal is to derive the sum rules

\[
\int_0^\infty \frac{dk}{\pi} k^{2m} \frac{d}{dk} \left[ \delta_\ell(k) - \sum_{\nu=1}^m \delta^{(\nu)}_\ell(k) \right] = -\sum_j (-\kappa_j^2)^n , \quad m \geq n \] (58)

where the sum on \( j \) ranges over the bound states with angular momentum \( \ell \) and \( \kappa_j^2 = -k_j^2 \) is the binding energy. For real \( k \), the phase shift \( \delta_\ell(k) \) is given in terms of the S-matrix \( S_\ell(k) \), which in turn is related to the Jost function \( F_\ell(k) \) by

\[
\delta_\ell(k) = \frac{1}{2i} \ln S_\ell(k) = \frac{1}{2i} \left[ \ln F_\ell(-k) - \ln F_\ell(k) \right] .
\] (59)

\(^4\) We have corrected some apparently typographical sign errors in their results.
The Born approximation is an expansion of the phase shift $\delta_\ell$ in powers of the interaction $V(r)$.

The Jost function is obtained from the Jost solution $f_\ell(k,r)$, which is the solution to eq. (57) asymptotic to an outgoing wave at infinity

$$\lim_{r \to \infty} e^{-ikr} f_\ell(k,r) = i^\ell$$

(60)

Uniqueness of $f_\ell(k,r)$ and analyticity in the upper $k$-plane can be verified for locally integrable potentials $V(r)$ with the property

$$\int_0^\infty dr \left(1 + r\right) |V(r)| < \infty.$$  

(61)

It is convenient to introduce the free Jost solution

$$w_\ell(kr) = ikr h^{(1)}_\ell(kr),$$

(62)

where $h^{(1)}_\ell(z)$ is the spherical Hankel function. $F_\ell(k)$ is determined by the $r \to 0$ limit of $f_\ell(k,r)$,

$$F_\ell(k) = \lim_{r \to 0} f_\ell(k,r)/w_\ell(kr).$$

(63)

As in previous cases, $F_\ell(k)$ is analytic in the upper half $k$-plane with zeros at the bound state momenta, $k = i\kappa_j$.

The sum rules are derived once again by introducing an auxiliary function, $F_{\ell,m}(k)$, with the properties

(a) $F_{\ell,m}$ is analytic and has no zeros in the upper half $k$-plane including $k = 0$.

(b) $|\ln F_\ell(k) - \ln F_{\ell,m}(k)|$ falls asymptotically like $|k|^{-2m-1}$ as $|k| \to \infty$ in the upper half plane.

For real $k$ we define

$$\delta_{\ell,m}(k) = \frac{1}{2i} \left[ \ln F_{\ell,m}(-k) - \ln F_{\ell,m}(k) \right]$$

(64)

and the remainder of the derivation follows exactly as in the antisymmetric channel in one dimension: The sum rule eq. (58) follows from Cauchy’s theorem.

**Construction of the Auxiliary Function**

As in one dimension, it is convenient to parameterize $f_\ell(kr)$ in terms of an exponential,

$$f_\ell(k,r) = e^{i\beta_\ell(k,r)} w_\ell(kr)$$

(65)

Substituting into the radial Schrödinger equation, we find that the complex function $\beta_\ell(k,r)$ satisfies

$$-i\beta''_\ell(k,r) + 2k\eta_\ell(kr)\beta'_\ell(k,r) + \beta_\ell(k,r)^2 + V(r) = 0,$$

(66)

subject to the boundary condition

$$\beta_\ell(k,\infty) = \beta''_\ell(k,\infty) = 0$$

(67)

where $\beta''_\ell(k,r) = d\beta_\ell(k,r)/dr$. Here we have introduced $\eta_\ell(z)$,

$$\eta_\ell(z) \equiv -i \frac{w'_\ell(z)}{w_\ell(z)} = -i \frac{d}{dz} \ln \left[ zh^{(1)}_\ell(z) \right],$$

(68)

which is a simple rational function of $z$.

It is convenient to convert eq. (68) into a non-linear integro-differential equation,

$$\beta_\ell(k,r) = i \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \left( \frac{w_\ell(kr_2)}{w_\ell(kr_1)} \right)^2 \Gamma_\ell(k,r_2)$$

(69)
where
\[ \Gamma_{\ell}(k, r) = [V(r) + \beta_{\ell}^2(k, r)] . \] (70)

By definition the value of \( i\beta_{\ell}(k, r) \) at \( r = 0 \) is the logarithm of the Jost function,
\[ F_{\ell}(k) = \lim_{r \to 0} \frac{f_{\ell}(k, r)}{w_{\ell}(kr)} = e^{i\beta_{\ell}(k, 0)} . \] (71)

Furthermore, the analytic properties of the Jost solution imply \( \beta(-k, r) = -\beta^*(k^*, r) \), yielding the phase shift
\[ \delta_{\ell}(k) = -\Re \beta_{\ell}(k, 0) = \frac{1}{2} [\beta_{\ell}(k, 0) - \beta_{\ell}(k, 0)] \] (72)

for real \( k \). The Born series for \( \delta_{\ell}(k) \) is constructed by iterating eq. (66) and keeping track of powers of \( V(r) \) in the source \( \Gamma_{\ell}(k, r) \) on the r.h.s. of (66). With these definitions, we are prepared to construct \( F_{\ell,m}(k) \). We begin by proving two important properties of the \( \{\beta^{(\nu)}\} \) using induction.

First, we need to show that the \( O(V^\nu) \) approximation \( \beta^{(\nu)}_{\ell}(k, 0) \) of eqs. (66) and (67) is a holomorphic function of \( k \) in the upper half plane \( \Im(k) \geq 0 \). To see this, we go back to the initial value problem for \( \beta_{\ell} \) and rewrite eq. (66) as
\[ -i\beta^{(\nu)}_{\ell}(k, r) + 2k \eta_{\ell}(kr) \beta^{(\nu)}_{\ell}(k, r) = -\Gamma^{(\nu)}_{\ell}(k, r) \] (73)

with the boundary condition \( \beta^{(\nu)}_{\ell}(k, \infty) = \beta^{(\nu)}_{\ell}(k, \infty) = 0 \) for all \( k \). \( \Gamma^{(\nu)}_{\ell} \) is the \( O(V^\nu) \) term in the iteration of \( \Gamma_{\ell} = V + \beta_{\ell}^2 \), including all combinations of \( V \) and \( \beta^{(\nu)}_{\ell} \) that give a total order of \( V^\nu \):
\[ \Gamma^{(1)}_{\ell}(k, r) = V(r) \] (74)

and
\[ \Gamma^{(\nu)}_{\ell}(k, r) = \sum_{\sigma + \tau = \nu} \beta^{(\sigma)}_{\ell}(k, r) \beta^{(\tau)}_{\ell}(k, r) \quad \text{for } \nu \geq 2 . \] (75)

We proceed by induction and therefore assume that the right hand side of eq. (72) is holomorphic in the upper half \( k \)-plane. This is certainly the case for \( \nu = 1 \). Since the boundary condition is independent of \( k \), Poincaré’s theorem ensures that the solution \( \beta^{(\nu)}_{\ell}(k, r) \) of eq. (73) is in fact a holomorphic function of \( k \) in every domain where the coefficients are. It thus remains to show that \( \eta_{\ell}(z) \) as defined in eq. (68) is holomorphic in \( z \) for \( \Im z \geq 0 \). This will be the case if the free Jost solution \( w_{\ell}(z) \) is non-vanishing in the upper half plane. For \( \ell = 0 \) it is trivial, since \( w_0(z) = e^{iz} \). For \( \ell > 0 \), it suffices to note that any zero of \( w_{\ell} \) in the upper half plane would correspond to a bound state of the free Schrödinger equation, which is forbidden by the repulsive centrifugal barrier.

Second, we have to establish the convergence of the Born series. The iteration of the integral equation for \( f_{\ell}(k, r) \) yields an expansion in the interaction \( \sum_{\nu=1}^{\infty} f^{(\nu)}_{\ell}(k, r) \) that is uniformly and absolutely convergent in the upper half plane \( \Im k > 0 \). From the bound \[ |f_{\ell}(k, r) - w_{\ell}(kr)| < \left( \frac{|k|r}{1 + |k|r} \right)^\ell e^{-\Im kr} \int_r^\infty |V(r_1)| dr_1 \]
and the limit \( \lim_{|k| \to \infty} w_{\ell}(kr) = 1 \) in the upper half plane \( \Im k \geq 0 \), it is clear that for any given \( r \geq 0 \), we can always find a radius \( \rho_r \) such that \( |F_{\ell}(k, r) - 1| > \frac{1}{2} \) for \( |k| > \rho_r \). For such large \( |k| \), the argument of the logarithm in \( i\beta_{\ell}(k, r) = \ln F_{\ell}(k, r) \) is entirely contained in the circle around unity of radius 1/2, where the logarithm is holomorphic. The absolute convergence of \( \sum_{\nu=1}^{\infty} f^{(\nu)}_{\ell}(k, r) \) thus implies the convergence of the Born series for \( \beta_{\ell}(k, r) \) outside a semi-circle of sufficiently large radius \( |k| > \rho_r \). Thus the (Born) series \( \sum_{\nu=1}^{m} \beta^{(\nu)}_{\ell}(k, r) \equiv \beta_{\ell}(k, r) \) converges absolutely and uniformly at sufficiently large \( |k| \) in the upper half plane.

Finally, we need to show that the difference \( |\beta_{\ell}(k, 0) - \sum_{\nu=1}^{m} \beta^{(\nu)}_{\ell}(k, 0)| \) vanishes at least as \( O(1/|k|^{2m+1}) \) for large \( |k| \) in the upper half \( k \)-plane. We proceed inductively from eq. (70) using integration by parts. What we will actually show is that the approximation \( |\beta^{(\nu)}_{\ell}(k, 0)| \) decays as \( |k|^{-2\nu+1} \). From the convergence of the Born series it is then clear that the remainder \( |\ln F - \ln F_m| = |\beta_{\ell} - \sum_{\nu=1}^{m} \beta^{(\nu)}_{\ell}| = |\sum_{m+1}^{\infty} \beta^{(\nu)}_{\ell}| \) vanishes at least as the leading term
\[ \beta^{(m+1)}(k, r) \sim |k|^{-(2m+1)}. \]

To estimate the large-\( k \) behavior of \( \beta_{\ell}^{(\nu)}(k, r) \), we rewrite the integral equation, eq. (70) in the form

\[ \beta_{\ell}(k, r) = i \int_{r}^{\infty} dr_{1} K_{\ell}(k, r_{1}) \quad (76) \]

where

\[ K_{\ell}(k, r) \equiv \int_{r}^{\infty} dr_{1} \left( \frac{w_{\nu}(kr_{1})}{w_{\nu}(kr)} \right)^{2} \Gamma_{\ell}(k, r_{1}) = \int_{r}^{\infty} dr_{1} \exp \left( 2ik \int_{r}^{r_{1}} \eta_{\nu}(kr_{2}) dr_{2} \right) \Gamma_{\ell}(k, r_{1}). \quad (77) \]

Integrating by parts once and estimating the remainder by the Riemann-Lebesgue lemma, the leading asymptotic behavior of the kernel \( K_{\ell} \) is easily found to be

\[ K_{\ell}(k, r) = \Gamma_{\ell}(k, r) \left[ \frac{1}{2ik\eta_{\nu}(kr)} + O(k^{-2}) \right]. \quad (78) \]

From this estimate and the limit \( \lim_{|z| \to \infty} \eta_{\nu}(z) = 1 \) in the upper half plane, we infer

\[ \beta_{\ell}^{(\nu)}(k, r) = \frac{1}{2k} \int_{r}^{\infty} dr_{1} \Gamma_{\ell}^{(\nu)}(k, r_{1}) \left[ 1 + O(k^{-1}) \right]. \quad (79) \]

The starting iteration for the source is \( \Gamma_{\ell}^{(1)}(k, r) = V(r) \), which is independent of \( k \) (and \( \ell \) ), whence

\[ \beta_{\ell}^{(1)}(k, r) = \frac{1}{2k} \int_{r}^{\infty} dr_{1} V(r_{1}) \left[ 1 + O(k^{-1}) \right]. \quad (80) \]

As a check, the large \( k \) behavior of the first Born approximation in the \( s \)-channel is correctly predicted as

\[ \delta_{0}^{(1)}(k) = -\text{Re} \beta_{0}^{(1)}(k, 0) = -\frac{1}{2k} \int_{0}^{\infty} dr \ V(r) + O(k^{-2}). \quad (81) \]

For the next iteration, we note \( \Gamma_{\ell}^{(2)}(k, r) = \left[ \frac{d}{dr} \beta_{\ell}^{(1)}(k, r) \right]^{2} \sim k^{-2} \) (with a real constant factor), so that \( |\beta_{\ell}^{(2)}(k, r)| \sim |k|^{-3} \). The general iteration step follows from the assumption that \(|\beta_{\ell}^{(\nu)}(k, r)| \sim |k|^{-2\nu}\) for \( \mu < \nu \). To find the behavior of \( \beta_{\ell}^{(\nu)}(k, r) \), we need to consider the \( \nu \)th order term \( \Gamma_{\ell}^{(\nu)}(k, r) \) that is given in eq. (75). By assumption, each \( |\beta_{\ell}^{(\nu)}(k, r)| \) decays as \( |k|^{-2\nu} \) at large \( k \). Hence all the terms in eq. (75) are of order \( |k|^{-1-2\nu} \sim |k|^{-2} \). Thus the source \(|\Gamma_{\ell}^{(\nu)}(k, r)|\) vanishes as \(|k|^{2(1-\nu)}\). From eq. (79), we easily complete the induction, \( |\beta_{\ell}^{(\nu)}(k, r)| \sim |k|^{-2\nu} \), as claimed.

It should be noted that the \( O(k) \) estimates in eq. (78) require the potential \( V(r) \) to be bounded and sufficiently smooth to allow for integration by parts. Moreover, the Riemann-Lebesgue lemma imposes certain restrictions on \( V \) and its derivatives, as discussed in the main text. These restrictions are certainly satisfied for smooth bounded potentials from the Faddeev class (8), but more general cases such as a step function may also be handled.

Having established these two properties of the \( \{\beta_{\ell}^{(\nu)}(k)\} \), it is clear that the auxiliary function satisfying (a) and (b) in the previous subsection takes the same form as in the antisymmetric channel in one dimension,

\[ F_{\ell, m}(k) = \exp \left[ i \sum_{\nu=1}^{m} \beta_{\ell}^{(\nu)}(k) \right]. \quad (82) \]

The required properties of \( F_{m}(k) \) follow directly from those of \( \beta_{\ell}^{(\nu)}(k) \) and the convergence of the Born series for sufficiently large \( |k| \). The quantity that enters the sum rule is \( \delta_{\ell, m}(k) \), given by eq. (64). Clearly \( \delta_{\ell, m}(k) \) is the sum of the first \( m \) terms in the Born expansion of \( \delta_{\ell}(k) \). This completes the derivation of the sum rules in \( \ell \)th partial wave in three dimensions.

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