CONTROLLABILITY OF NETWORK OPINION IN ERDŐS-RÉNYI GRAPHS USING SPARSE CONTROL INPUTS

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Abstract. This paper considers a social network modeled as an Erdős-Rényi random graph. Each individual in the network updates her opinion using the weighted average of the opinions of her neighbors. We explore how an external manipulative agent can drive the opinions of these individuals to a desired state with a limited additive influence on their innate opinions. We show that the manipulative agent can steer the network opinion to any arbitrary value in finite time (i.e., the system is controllable) almost surely when there is no restriction on her influence. However, when the control input is sparsity constrained, the network opinion is controllable with some probability. We lower bound this probability using the concentration properties of random vectors based on the Lévy concentration function and small ball probabilities. Further, through numerical simulations, we compare the probability of controllability in Erdős-Rényi graphs with that of power-law graphs to illustrate the key differences between the two models in terms of controllability. Our theoretical and numerical results shed light on how the controllability of the network opinion depends on the parameters such as the size and the connectivity of the network, and the sparsity constraints faced by the manipulative agent.

Key words. Social network opinion, linear propagation, sparse-controllability, Erdős-Rényi graph, concentration inequalities

1. Introduction. Consider the following problem regarding controllability of network opinion: the opinions propagated over an Erdős-Rényi random graph consisting of N people are influenced by an external agent (referred to as the manipulative agent henceforth) [14]. The goal of the manipulative agent is to influence the opinions of the people so that the network opinion is driven to a desired state. However, the manipulative agent is subject to sparsity constraints: it can only influence a few people in the network at each time.

Mathematically, our formulation is as follows: the network opinion at time k, denoted by \( x_k \in \mathbb{R}^N \) follows a DeGroot type linear propagation model [10]:

\[
x_k = \bar{A}x_{k-1} + u_k,
\]

where, \( \bar{A} \in \mathbb{R}^{N \times N} \) is the row-normalized weighted adjacency matrix of the network. In the most general setting, the randomness of the system (1.1) is induced by a hierarchical probability measure over the space of graphs with N nodes where, the locations of non-zero entries of \( \bar{A} \) are modeled using an Erdős-Rényi graph and the non-zero entries of each row of \( \bar{A} \) are drawn from a continuous distribution on the unit simplex. The control signal from the manipulative agent, \( u_k \in \mathbb{R}^N \) represents her influence at time k. We assume that the manipulative agent can influence only one of the predefined (overlapping) groups of people in the network at any time instant, and the size of each such group is small compared to the network size N. In other words, \( u_k \) is s–sparse with \( s \ll N \) (i.e., budget-constrained), and its support belongs to a set of admissible support sets \( \mathcal{A} \) (i.e., pattern-constrained).

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A fundamental question that arises in this context is: *Can the network opinion be driven to a desired value using the budget and pattern constrained inputs within a finite time duration?* The answer is that it is possible with some probability (that arises due to the randomness in Erdős-Rényi model). We use concentration inequalities to derive a lower bound for this probability which is a function of the number of nodes $N$, the sparsity $s$, the edge probability $p$ and the set of admissible support sets $A$ for two versions of the Erdős-Rényi graph model: a directed graph and an undirected graph. The main results of the paper are informally stated below:

**Theorem 1.1 (Informal statement of main results).** Consider the linear opinion formation model in (1.1) where $A$ is the row-normalized adjacency matrix of an Erdős-Rényi graph. Assume that the control signal $u_k$ is $s$-sparse with $0 < s \ll N$ and its support (defined in (2.6)) satisfies $\text{Supp} \{u_k\} \in A$ for some set $A$ which depends on the specific sparsity pattern (detailed in Subsection 2.3). Further, assume that the nonzero entries of each row $A$ is drawn from a continuous distribution on the unit simplex. Then, the following hold:

(a) Undirected graph: If the edge probability $p$ satisfies

\[(N - s)^{-1} \leq p \leq 1 - (N - s)^{-1},\]

then the probability of controllability of the network opinion in (1.1) is at least

\[o \left( \bar{Q}(A)(1 - p)^N \left[ 1 - e^{-c(p(N - s))} \right] \right).\]

(b) Directed graph: If the edge probability $p$ satisfies

\[C \frac{\log(N - s)}{N - s} < p \leq 1 - C \frac{\log(N - s)}{N - s},\]

then the probability of controllability of the network opinion in (1.1) is at least

\[o \left( \bar{Q}(A)(1 - p)^N \left[ 1 - e^{-c(p(N - s))} \right] \right),\]

where $\bar{Q}(A) \geq 1$ is an increasing function of $s$ and $N$ which depends on the set $A$. Here $C,c > 0$ are universal constants.

The key insights gained from Theorem 1.1 are as follows:

- The probability of controllability increases with the network size $N$ and sparsity $s$ when all other parameters are kept constant. In particular, the network opinion can be controlled almost surely, if the network size is sufficiently large ($N \to \infty$), irrespective of the sparsity patterns.
- The probability bound on controllability of network opinion is small when the network is either loosely connected (small values of $p$ which are close to 0) or densely connected (large values of $p$ which are close to 1) i.e., the probability of the system being controllable is larger for moderate values of the edge probability $p$.
- The dependence of the controllability on the sparsity pattern is captured by the function $\bar{Q}$. This relation allows us to compare the probability of controllability under various popular patterns like unconstrained sparsity, piecewise sparsity and block sparsity.\(^1\)

The proof technique we use relies on rank-related properties of binary random matrices (that model the adjacency matrix of Erdős-Rényi graph). The key mathematical tools used to derive these properties are the Lévy concentration function and small ball probability. The analysis characterizes the smallest singular value of the (unweighted)\(^2\)

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\(^1\)We discuss this point in Subsection 4.2.
binary adjacency matrix of an Erdős-Rényi graph, which can be of independent interest.

1.1. Practical context of the model and examples. Controllability of the network opinion has applications in marketing [25], targeted fake-news campaigns [33], and political advertising [9]. For such problems, the randomness of the Erdős-Rényi model captures the unknown structure of the underlying social network. The directed graphs represent social networks such as Twitter whereas the undirected graphs represent social networks such as Facebook.

The DeGroot type opinion propagation model (given in (1.1)) that we consider is an average based process that has been used widely in literature ([3,15,18,35]) to model consensus and learning in multi-agent systems. Here, each node $i \in V$ averages the opinions of her neighbors according to the non-uniform weights prescribed by the $i^{th}$ row of $A$.

The sparsity pattern models the limited influence of the manipulative agent in the following examples.

1. Consider a company that sends one salesperson each to $m$ different parts of a country for marketing their products by offering free samples. At each time instant, the company can afford a total of $s$ free samples (and hence, constraining the sparsity of the control signal to be $s$), and each salesperson can visit at most $s/m$ potential customers (and thus, constraining the sparsity pattern). Hence, the control input $u_k$ at each time instant $k$ is formed by concatenating $m$ sparse vectors. This is an example of the sparsity pattern called piece-wise sparsity (Definition 2.4).

2. Consider an election candidate visiting different parts of a country as part of a political campaign. At each time instant, she can only visit a particular area and influence the group of people located there. Therefore, the input $u_k$ at each time instant $k$ has nonzero entries occurring in clusters (corresponding to the individuals located in a particular area). This is an example of the sparsity pattern called block sparsity or group sparsity (Definition 2.5).

1.2. Related work. The linear opinion propagation model in (1.1) is similar to the widely used DeGroot model [10]. For such linear models, several works have explored the problem of controlling opinions in a social network by drawing tools from control theory [11, 12, 22, 30, 36–38]. However, the existing literature so far has not addressed the problem of the controllability of network opinion using sparse control inputs. A key reason for this literature gap is that the Kalman-rank based test for sparse-controllability (i.e., controllability of a linear dynamical system under sparsity constraints on inputs) is combinatorial which makes the analysis cumbersome. Recently, a simpler rank-based test which is similar to the classical PBH test [16] and equivalent to the Kalman type rank test has been presented in [19]. Therefore, we build upon the controllability conditions provided in [19] to derive sufficient conditions for sparse-controllability of network opinion.

1.3. Organization. Section 2 presents the network opinion dynamics model which has three parts: the opinion evolution model, the network connectedness, and the sparsity constraint on the input. Section 3 introduces the notion of generalized sparse-controllability and derives its necessary and sufficient conditions. This result is of independent interest, as it applies to a general linear dynamical system.

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2The model in [3,15] is limited to the case where the averaging is based on a uniform distribution whereas we consider the more general case of averaging based on social reputation.
Section 4 and Section 5 present our main results of bounds on the probability of sparse-controllability for undirected and directed graphs (and the practical insights that they yield), respectively. Finally, Section 6 presents numerical illustrations that complement and verify the main results of the paper.

**Notation:** Boldface lowercase letters denote vectors, boldface uppercase letters denote matrices, and calligraphic letters denote sets. Table 1 summarizes the parameters and variables used frequently throughout the paper.

### Table 1
**Summary of Notation**

| General notation |
|------------------|
| \( \mathbb{R} \) : Set of real numbers |
| \( \mathbb{C} \) : Set of complex numbers |
| \( S^N \) : Unit Euclidean sphere in \( \mathbb{R}^N \) |
| \( I \) : Identity matrix |
| \( 0 \) : All zero matrix (or vector) |
| \( A_i \) : \( i \)th column of \( A \) |
| \( A_{ij} \) : the \((i,j)\)th entry of \( A \) |
| \( A_S \) : Submatrix of \( A \) formed by the columns indexed by the set \( S \) |
| \( A_{S,:} \) : Submatrix of \( A \) formed by the rows indexed by the set \( S \) |
| \( \| \cdot \| \) : \( \ell_2 \) norm |
| \( \| \cdot \|_0 \) : \( \ell_0 \) norm |
| \( \text{Supp}\{\cdot\} \) : Support of a vector, \( \text{Supp}\{z\} = \{i : z_i \neq 0 \} \subseteq \{1, 2, \ldots, N\} \) |
| \( \odot \) : Hadamard product |
| \(|\cdot|\) : Cardinality of a set |
| \([a]\) : Set of integers \( \{1, 2, \ldots, a\} \) |
| \( \mathcal{P}(\cdot) \) : Power set |
| \( \text{dist}(a,S) \) : \( \ell_2 \) distance between the point \( a \) from the set \( S \) |

| Network Parameters |
|-------------------|
| \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) : Graph with set of nodes \( \mathcal{V} \) and set of edges \( \mathcal{E} \) |
| \( N \) : Number of nodes i.e., \( N = |\mathcal{V}| \) |
| \( A \in \{0, 1\}^{N \times N} \) : Adjacency matrix of \( \mathcal{G} \) where \( A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases} \) |
| \( W \in \mathbb{R}^{N \times N} \) : Weight matrix of \( \mathcal{G} \) |
| \( A \in \mathbb{R}^{N \times N} \) : Row-normalized weight matrix of \( \mathcal{G} \) |
| \( p \) : Edge probability |
| \( A \) : Set of admissible supports of the sparse control inputs |
| \( s \) : Sparsity of control inputs i.e., if \( S \in A, |S| \leq s \) |

2. **Opinion Dynamics Model.** This section details the network opinion dynamics model which has three key main components: propagation model, random graph model, and the sparsity model.
2.1. Propagation model. Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with binary adjacency matrix $A \in \{0, 1\}^{N \times N}$ where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E}$ are the set of nodes and the set of edges, respectively. The set $\mathcal{V}$ represents the individuals in the network, and the edges represent their connections with neighbors. The opinion $x_k[i]$ of individual $i \in \mathcal{V}$ at time $k$ is the weighted average of the opinions of her neighbors (defined by the set of edges) at time $k-1$ and the control input $u_k[i]$ which represents the influence of the manipulative agent. Thus, the opinions evolve according to (1.1) where the matrix $\bar{A}$ can be written as

$$\bar{A} = \Lambda (A \odot W),$$

with $\odot$ denoting the Hadamard product. Here, $A \in \{0, 1\}^{N \times N}$ is the adjacency matrix of the graph $\mathcal{G}$, and $W \in \mathbb{R}^{N \times N}$ is the weight matrix whose element $W_{ij} > 0$ is the weight of edge $(i, j) \in \mathcal{E}$ of the graph. The invertible diagonal matrix $\Lambda \in \mathbb{R}^{N \times N}$ is defined as

$$\Lambda_{ii} = \begin{cases} \sum_{j=1}^{N} A_{ij} W_{ij} & \text{if } \sum_{j=1}^{N} A_{ij} W_{ij} \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

The above model ensures that the zero entries of $\bar{A}$ and $A$ coincide, i.e., the Erdős-Rényi graph models the connectedness of the graph. Also, (2.2) implies that $\bar{A}$ is the row-normalized weighted adjacency matrix. We assume that $W = \mathbf{w} \mathbf{w}^T$ is a symmetric rank-one matrix where $\mathbf{w} \in \mathbb{R}^{N}$ is drawn from an arbitrary continuous distribution on nonnegative numbers. Since $W$ is rank-one matrix, the order of weights (entries of $\mathbf{w}$) in every row of $W$ is the same. As mentioned in Subsection 1.1, these weights model the reputation of each person in the network, i.e., a highly reputed person in the network always gets higher weightage while the people who are connected to her average the opinions of their neighbors. Also, since $\Lambda$ normalizes each row of $\bar{A}$, the entries of $W$ only determines the ratio of weights (i.e., $W = \mathbf{w} \mathbf{w}^T$ and $W = \mathbf{1} \mathbf{w}^T$ model the same system).

2.2. Random graph model. We assume that the network is modeled using the Erdős-Rényi model [13] where, each edge exists with probability is $0 < p < 1$ independently of all other edges. The effect of parameters $p$ and $N$ (number of nodes) on the connectivity and the emergence of a giant connected component is well studied [18]. We consider two different models for the connectedness: undirected and directed graphs as defined below:

**Definition 2.1 (Undirected Erdős-Rényi graph).** A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called an undirected Erdős-Rényi graph if for every $i \neq j$, an undirected edge from $i$ to $j$ is present with probability $p$, independently of everything else, and the graph contains no self loop. Therefore, the entries of the adjacency matrix $A \in \{0, 1\}^{N \times N}$ follows the model given below:

$$\mathbb{P} \{A_{ij} = 1\} = \begin{cases} p, & \text{if } i \neq j, i > j \\ 0, & \text{if } i = j \end{cases}$$

$$A_{ij} = A_{ji} \quad \text{if } i < j.$$

**Definition 2.2 (Directed Erdős-Rényi graph).** A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a directed Erdős-Rényi graph if for every $i \neq j$, a directed edge from $i$ to $j$ is present with probability $p$, independently of everything else, and the graph contains no self loop.
loop. Therefore, the entries of the adjacency matrix $A \in \{0,1\}^{N \times N}$ follows the model given below:

$$\mathbb{P}\{A_{ij} = 1\} = \begin{cases} p, & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

We note that the adjacency matrix $A \in \{0,1\}^{N \times N}$ is a (sparse) Bernoulli random matrix. Also, the adjacency matrix $A$ is symmetric for an undirected graph while it is not necessarily symmetric in the case of a directed graph. Thus, the undirected Erdős-Rényi graph (Definition 2.1) is a special case of the directed Erdős-Rényi graph (Definition 2.2).

2.3. Sparsity model. The sparsity model represents the constraints faced by the manipulative agent at each time instant, which restricts the set of nodes that are influenced (i.e. the support set of the control input $u_k$). We use a generalized sparsity model which we refer to as the pattern-and-budget-constraint model. The model restricts the support of the control input where the support of a vector $z \in \mathbb{R}^N$ is defined as:

$$\text{Supp}\{z\} = \{i : z_i \neq 0\} \subseteq \{1,2,\ldots,N\}.$$  

Let the set of all admissible supports be $\mathcal{A} = \{S_i\}_{i=1}^{A_1} \subseteq \mathcal{P}(\{1,2,\ldots,N\})$, i.e., $\text{Supp}\{u_k\} \subseteq S_i$ for some $S_i \in \mathcal{A}$ and for all values of $k$. We do not impose any special structure on $\mathcal{A}$ except that $|S| = s$, for all $S \in \mathcal{A}$ and $\bigcup_{S \in \mathcal{A}} = [N]$. Below, we define three specific sparsity patterns that serve as examples of this sparsity model:

**Definition 2.3 (Unconstrained sparsity).** Unconstrained sparsity refers to the restriction that the control signal can have at most $s$ nonzero entries. Here, we have $A = A_1$ where

$$A_1 \triangleq \{S \subset \{1,2,\ldots,N\} : |S| \leq s\}.$$  

Further, $|A_1| = \binom{N}{s}$.

**Definition 2.4 (Piece-wise sparsity).** Piece-wise sparsity refers to the sparsity model where each admissible control vector can be written as a concatenation of $m$ sparse vectors each with sparsity of at most $s/m$ (we assume that $m$ divides both $N$ and $s$). Here, we have $A = A_2$ where

$$A_2 \triangleq \left\{ \bigcup_{i=1}^{m} S_i : S_i \subseteq ((i-1)N/m + 1,\ldots,iN/m) \right\}.$$  

and $|S_i| \leq s/m$, $\forall i \in \{1,2,\ldots,m\}$.

Also, $|A_2| = \binom{N/m}{s/m}^m \leq \binom{N}{s}$.

**Definition 2.5 (Block sparsity).** Block sparsity refers to the sparsity model where the non-zero entries of the control signal form clusters of equal size $m \leq N$ (we assume that $m$ divides both $N$ and $s$). Here, we have $A = A_3$ where

$$A_3 \triangleq \left\{ \bigcup_{i=1}^{s/m} S_i : S_i \subseteq ((i-1)m + 1, (i-1)m + 2,\ldots,im) \right\}.$$  

3A sparse vector with unequal blocks of nonzero entries is also called a block spare vector. In this paper, we use equal block sizes.
In this case, $|A_3| = \binom{N}{s/m} \leq \binom{N}{s}$.

We make a few observations from the above models:

- Unconstrained sparsity model is the least restricted sparsity model, and thus, $A_2, A_3 \subset A_1$.
- We can rearrange the entries of a block sparse vector $z$ as follows:

  $\begin{bmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_m \\
  z_{m+1} \\
  \vdots \\
  z_{2m} \\
  \vdots \\
  z_{N-m+1} \\
  \vdots \\
  z_N
\end{bmatrix} \rightarrow \begin{bmatrix}
  z_1 \\
  z_{m+1} \\
  \vdots \\
  z_2 \\
  z_{N-m+1} \\
  \vdots \\
  z_{2m} \\
  \vdots \\
  z_{N-m+2} \\
  \vdots \\
  z_N
\end{bmatrix}$

  Block 1

  Block 2

  Block $\frac{N}{m}

The rearranged vector on the right-hand side is a piecewise sparse vector with $m$ blocks, each with at most sparsity $s/m$ and the same support. Since shuffling the entries of the control input does not change the properties of the system in (1.1), block sparsity can be seen as a special case of piecewise sparsity.

- As the number of blocks $m$ increases, piecewise constraint becomes more stringent and hence, $P$ decreases. We get $|A_2| = \binom{N}{s}$ when $m = 1$.
- As the block size $m$ increases (i.e., the number of blocks increases), the block sparse constraint becomes more stringent and hence, $P$ decreases. We get $|A_3| = \binom{N}{s}$ when $m = 1$.

In this paper, we explore the conditions under which the network opinion is controllable under the sparsity constraints on the support of the control inputs imposed by the different sparsity models (defined above) and with the two random graph models defined in Subsection 2.2. In the next section, we present a generalized sparse-controllability test that establishes the key ideas required to derive our main results.

3. Generalized Sparse-Controllability Results. The main result of this section is Theorem 3.1 which provides the necessary and sufficient conditions for sparse-controllability for a generalized sparsity-pattern (defined in Subsection 2.3). This result is an extension of the sparse-controllability result [19, Theorem 3.1.] (see Theorem D.2) to the case when the support set of the control input is constrained to a given set $A$ which is a collection of subsets of $\{1, 2, \ldots, N\}$. The original result [19, Theorem 3.1.] is applicable only to sparse signals whereas the result presented in this section is applicable under any other restriction on the support like block sparsity or piece-wise sparsity. We consider the following linear dynamical system given by

$$x_k = Ax_{k-1} + Bu_k, \quad k = 1, 2, \ldots,$$

where the state vector $x_k \in \mathbb{R}^N$, the transfer matrix $A \in \mathbb{R}^{N \times N}$, and the input matrix $B \in \mathbb{R}^{N \times L}$. For this system, the generalized sparse-controllability result is presented below:
Theorem 3.1. For an arbitrary linear system given by (3.1), let $\|u_k\|_0 \leq s$ and the supports $\text{Supp}\{u_k\} \in \mathcal{A} \subseteq \mathcal{P}\{1,2,\ldots,L\}$ with $|S| \leq s, \forall S \in \mathcal{A}$. For any initial state $x_0$, any final state $x_K$ and any set $\mathcal{A}$, there exists a sparse input sequence $u_k$, which steers the system from the state $x_0$ to $x_K$ for some finite $K$, if and only if the following two conditions hold:

(a) For all $\lambda \in \mathbb{C}$, the rank of the matrix $[\lambda I - A \ B_{\mathcal{M}}] \in \mathbb{R}^{N \times (N+|\mathcal{M}|)}$ is $N$, where the set $\mathcal{M} = \bigcup_{S \in \mathcal{A}} S \subseteq \{1,2,\ldots,L\}$.

(b) There exists an index set $S \in \mathcal{A}$ such that rank of $[A \ B_{S}] \in \mathbb{R}^{N \times (N+|S|)}$ is $N$.

Proof. See Appendix A.

Note that both conditions in Theorem 3.1 have to be satisfied simultaneously for the system to be controllable under the constraints on the support of the input. The first condition is the same as the classical PBH test (see Theorem D.1) for the controllability of the reduced linear system described by the matrix pair $(A, B_{\mathcal{M}})$. Here, $B_{\mathcal{M}}$ is the effective control matrix, as the entries of the control inputs corresponding to $M'$ are always zero. Since all sparse-controllable systems are controllable, the necessity of the first condition is straightforward. The second condition distinguishes a system which is controllable (i.e., satisfies the first condition) from a system which is controllable under the constraints on the support. Also, Theorem 3.1 reduces to the existing sparse-controllability result [19, Theorem 3.1.], when $\mathcal{A}$ is assumed to be the set of all $s$-sized subsets of $\{1,2,\ldots,N\}$.

Next, we apply Theorem 3.1 to our stochastic setting and derive probabilistic results on sparse-controllability of network opinion dynamics. Before we launch into those results, we provide a result on the controllability of the network opinion using unconstrained (non-sparse) inputs.

Proposition 3.2. Consider the network opinion evolution model defined in (1.1) where the manipulative agent can influence any number of people ($s = N$) on the network at any time instant. Then, the network opinion is almost surely controllable under the two versions of Erdös-Rényi graphs: directed (given in (2.3) and (2.4)) and undirected (given in (2.5)).

Proof. Using the classical PBH test (see Theorem D.1) for controllability, it is easy to see that the system described by (1.1) is controllable for any arbitrary matrix $A$. If we restrict the influence of the manipulative agent to $s$ number of people in the network, the opinion of any $s$ people in the network is always controllable. Therefore, the interesting problem in this context is the controllability analysis for the sparse regime when $s \ll N$, which is presented in the next two sections.

4. Sparse Controllability of Opinions in an Undirected Graph. This section provides a lower bound on the probability with which the system in (1.1) is controllable for the case of undirected Erdös-Rényi graphs and, discusses the insights that it yields. In order to state the result, we define the function $Q : \{0,1,\ldots,s\} \times \mathcal{P}\{1,2,\ldots,N\} \rightarrow \mathbb{N}$ as follows:

$$Q(t, \mathcal{A}) \triangleq |\{I \subseteq S : S \in \mathcal{A} \text{ and } |I| = t\}|,$$

where $\mathbb{N}$ is the set of natural numbers. We recall that $\mathcal{A}$ is the set of all admissible supports of the control input. Also, if $\mathcal{A}^{(1)} \subset \mathcal{A}^{(2)}$, we obtain that $Q(t, \mathcal{A}^{(1)}) < \mathbb{N}$.
\( Q(t, \mathcal{A}^{(2)}) \). Therefore, the quantity \( Q(t, \mathcal{A}) \) that counts the number of \( t \)-sized subsets of \( \mathcal{S} \in \mathcal{A} \) can be seen as a measure of the flexibility of the sparsity pattern. If the sparsity model is less pattern-constrained, the size of \( \mathcal{A} \) increases which in turn increases \( Q(t, \cdot) \). Intuitively, if the control signal is less constrained, then the system is more likely to be controllable. This dependence of \( \mathcal{A} \) on the probability of controllability is captured by the function \( Q \) - the precise relation is presented in the following result and discussed in further detail in Subsection 4.2.

**Theorem 4.1 (Sparse controllability of network opinion in undirected Erdős-Rényi graph).** Consider the network opinion evolution model in (1.1) where the manipulative agent can influence at most \( s \ll N \) people in the network at any time instant. Let the support of the control input from the manipulative agent at any time instant be such that \( \text{Supp}\{u_k\} \subseteq \mathcal{S} \) for some \( \mathcal{S} \) that belongs to a given set \( \mathcal{A} \subseteq \mathcal{P} \{1, 2, \ldots, N\} \). Also, let \( |\mathcal{S}| = s \), for all \( \mathcal{S} \in \mathcal{A} \) and \( \cup S \subseteq \mathcal{A} = [N] \). Further, suppose that the adjacency matrix \( \mathbf{A} \) follows the model given in (2.3) and (2.4), and \( \mathbf{W} = \mathbf{w}\mathbf{w}^T \) where \( \mathbf{w} \) has entries sampled from a continuous distribution on nonnegative real numbers. If the following relation holds,

\[
(N - s)^{-1} \leq p \leq 1 - (N - s)^{-1},
\]

then the network opinion of the system can be steered to any desired value from any initial network opinion in finite time, with probability at least \( q \) where,

\[
q = \sum_{i=0}^{s} Q(i, \mathcal{A})(1 - p)^{i(2N-i-1)/2} \left[ 1 - C \exp \left( -c(p(N - i))^{1/32} \right) \right],
\]

for some constants \( C, c > 0 \), and \( Q \) is as defined in (4.1).

**Proof.** See Appendix B.

We derive the following corollary using the proof technique of Theorem 4.1. The corollary considers the model in [3,15] where the opinion of each person in the network is the non-weighted average of her neighbors (uniform weights).

**Corollary 4.2.** Consider the network opinion evolution model defined in (1.1) where the manipulative agent can influence at most \( s \ll N \) people on the network at any time instant. Let the support of the control input from the manipulative agent at any time instant be such that \( \text{Supp}\{u_k\} \subseteq \mathcal{S} \) for some \( \mathcal{S} \) that belongs to a given set \( \mathcal{A} \subseteq \mathcal{P} \{1, 2, \ldots, N\} \). Also, let \( |\mathcal{S}| = s \), for all \( \mathcal{S} \in \mathcal{A} \) and \( \cup S \subseteq \mathcal{A} = [N] \). Further, suppose that the adjacency matrix \( \mathbf{A} \) follows the model given in (2.3) and (2.4), and \( \mathbf{W} = \mathbf{1}\mathbf{1}^T \). If (4.2) holds, the network opinion of the system can be steered to any desired value from any initial network opinion in finite time, with probability at least \( q \) where \( q \) is defined in (4.3).

Next, we discuss several implications of the Theorem 4.1 that highlight its importance in practical contexts.

### 4.1. Dependence on parameters.

- **Sparsity:** The dependence of \( q \) on \( s \) is only through the first summation term. Clearly, \( q \) increases with the sparsity \( s \) as more terms are added to the summation term in (4.3). This observation is intuitive because as \( s \) increases, the restriction on the manipulative agent is less stringent and hence, the probability of controllability \( q \) increases.
• **Edge connectivity probability:** The result holds for moderate values of $p$, depending on the size of the network and the sparsity. This suggests that when the network is highly connected and sparsely connected, it is difficult to change the opinion of the entire network by influencing a few number of people. This is intuitive as it is not possible to influence the opinion of the network if $p = 0$ as the people on the network do not influence each other.

• **Number of people:** As network becomes larger (i.e., $N$ increases), the bound on the probability of controllability $q$ increases. This is evident from the following:

$$q \geq Q(0, \mathcal{A}) \left[ 1 - C \exp \left( -c(p(N))^{1/32} \right) \right] = 1 - C \exp \left( -c(p(N))^{1/32} \right),$$

since from (4.1), $Q(0, \mathcal{A}) = 1$ for any set $\mathcal{A}$. Therefore, as $N \to \infty$, and $s = o(N^b)$, for some $b \in \mathbb{R}$, $q$ goes to unity. This implies that the network opinion is controllable, almost surely, for any set of admissible support sets $\mathcal{A}$.

Intuitively, as the network size increases, the people who can be influenced by the manipulative people, are connected to more people in the network. This is because the Erdős-Rényi model ensures that the expected number of neighbors of every person on the network is $p(N - 1)$.

• **Giant connected component:** If $p < 1/N$, the undirected Erdős-Rényi graph almost surely has no connected component of size $O(\log(n))$ whereas, if $p > 1/N$, the undirected Erdős-Rényi graph has a unique giant connected component containing a positive fraction of the nodes almost surely [7, 18]. We note that the lower bound in (4.2) is larger than this threshold value $(1/N)$ of $p$ required for the almost sure existence of a unique giant component in the graph, and thus, our results hold only when a unique giant connected component exists almost surely.

### 4.2. Sparsity patterns.

The term $Q$ in (4.1) admits closed form expressions for the three specific sparsity patterns defined in Subsection 2.3:

• **Unconstrained sparsity (Definition 2.3):** Since there is no restriction apart from sparsity, clearly, $Q(i, \mathcal{A}_1) = \binom{N}{i}$.

• **Piece-wise sparsity (Definition 2.4):** In this case, the set defined in (4.1), $\{I \subseteq S : S \in \mathcal{A}_2 \text{ and } |I| = i\}$ is the collection of subsets of size $i$, with at most $s/m$ indices from each of the $m$ blocks of the sparse control inputs. Therefore, we are allowed to choose $0 \leq j_k \leq s/m$ indices from the $k^{th}$ block of the control vector such that $\sum_{k=1}^{m} j_k = i$.

$$Q(i, \mathcal{A}_2) = \sum_{0 \leq j_1, \ldots, j_m \leq s/m \; l=1}^m \prod_{j_1+j_2+\ldots+j_m=i}^{m} \binom{N/m}{j_l}.$$  

• **Block sparsity (Definition 2.5):** In this case, to choose subsets of $S \in \mathcal{A}$ of size $i$, we need to select $\lfloor im/N \rfloor \leq l \leq i$ blocks of the sparse control input. At least one entry is chosen from each of these $l$ blocks, and remaining $i - l$ entries can be chosen from any of the remaining $l(N/m - 1)$ entries.

$$Q(i, \mathcal{A}_3) = \sum_{l=\lfloor im/N \rfloor}^{\min\{i,sm/N\}} \binom{m}{l} (N/m)^l \binom{l(N/m - 1)}{i - l}.$$  

10
Since $A_2, A_3 \subseteq A$, it is easy to see that from (4.1) that for all values of $i$,

$$Q(i, A_2) \leq Q(i, A_1) \quad Q(i, A_3) \leq Q(i, A_1),$$

as expected.\footnote{Using the generalized Vandermonde’s identity \cite{2}, we have

$$Q(i, A_2) \leq \sum_{j_1 + j_2 + \ldots + j_m = i} \prod_{i=1}^{m} \binom{N/m}{j_i} = \binom{N}{i} = Q(i, A_1).$$}

This is intuitive since piece-wise sparsity and block sparsity are special cases of unconstrained sparsity, and therefore, they are more restrictive.

Also, as discussed in Subsection 4.2, block sparsity can be seen as a special case of piecewise sparsity with a common support for all blocks, and therefore, we have

$$(4.4) \quad Q(i, A_3) \leq Q(i, A_2) \leq Q(i, A_1).$$

Thus, out of the three sparsity models we considered, the unconstrained sparse signals offer the highest probability of controlling network opinion, followed by the piece-wise sparse vectors.

5. Sparse Controllability of Opinions in a Directed Graph. In this section, we present a result analogous to Theorem 4.1 for directed Erdős-Rényi graphs which is stated below.

**Theorem 5.1 (Sparse controllability of network opinion in directed Erdős-Rényi graph).** Consider the network opinion evolution model in (1.1) where the manipulative agent can influence at most $s$ people on the network at any time instant. Let the support of the control input from the manipulative agent at any time instant belong to a given set $A \subseteq P(\{1, 2, \ldots, N\})$ such that $|S| = s$, for all $S \in A$ and $\cup_{S \in A} = [N]$. Further, suppose that the adjacency matrix $A$ follows the model given in (2.5) and $W = ww^T$ where $w$ has entries sampled from a continuous distribution on nonnegative real numbers. If the following relation holds,

$$(5.1) \quad C \frac{\log(N-s)}{N-s} < p \leq 1 - C \frac{\log(N-s)}{N-s},$$

the network opinion of the system can be steered to any desired value from any initial network opinion in finite time, with probability at least $q$ where

$$(5.2) \quad q = \sum_{i=0}^{s} Q(i, A)(1 - p)^{i(N-1)} \left[1 - \exp\left(-c(p(N-i))\right)\right],$$

for some constants $C, c > 0$, and $Q$ is as defined in (4.1).

**Proof.** See Appendix C. □

Using the proof technique of Theorem 5.1 we obtain a corollary similar to Corollary 4.2:

**Corollary 5.2.** Consider the network opinion evolution model defined in (1.1) where the manipulative agent can influence at most $s$ people on the network at any time instant. Let the support of the control input from the manipulative agent at any time instant belong to a given set $A \subseteq P(\{1, 2, \ldots, N\})$ such that $|S| = s$, for all
\( S \in A \) and \( \cup_{S \in A} = [N] \). Further, suppose that the adjacency matrix \( A \) follows the model given in (2.5) and \( W = 11^T \). If (5.1) holds, the network opinion of the system can be steered to any desired value from any initial network opinion in finite time, with probability at least \( q \) where \( q \) is given by (5.2).

We make similar observations on the result as those discussed in Section 4 about Theorem 4.1. Comparing Theorem 5.1 with Theorem 4.1, we make the following observations:

- The range of edge connectedness \( p \) is shorter for directed graphs compared to undirected graphs. Here, we require the graph to be connected with higher probability which is \( \log(N - s) \) times more than the undirected graph. For the same edge probability, the undirected graph is likely to have more number of connections and hence, it is more controllable.
- The bound on the probability of controllability is of similar order for directed graphs and undirected graphs, when the other parameters are kept the same. Therefore, the direction of information flow (uni-directional in directed graphs vs bi-directional in undirected graphs) does not have a significant effect on the probability bound.

**Extensions:** We note that the results presented in this paper (Proposition 3.2, Theorem 4.1, and Theorem 5.1) hold for the following extensions of the model:

- The influence of manipulative agent \( u_k \) in (1.1) is replaced with \( \Phi u_k \), for any invertible matrix \( \Phi \in \mathbb{R}^{N \times N} \) (the control input is sparse under any basis).
- The matrix \( A \) in (1.1) is replaced with \( \Phi D A \Phi^{-1} \), for any arbitrary invertible matrix \( \Phi \in \mathbb{R}^{N \times N} \), and any invertible diagonal matrix \( D \in \mathbb{R}^{N \times N} \).

The proofs are straightforward extensions and omitted.

**6. Numerical Experiments.** To give additional insights, we compute the probability of controllability of network opinion using Theorem 3.1 via numerical experiments, and compare the results with the bounds in Theorems 4.1 and 5.1. Also, we numerically evaluate the probability of controllability for a different model of social networks called a power-law model \([4, 24, 28]\) in order to understand how probability of controllability varies for different models.\(^5\)

### 6.1. Probability of controllability of the Erdős-Rényi model.

To evaluate the probability of controllability of the Erdős-Rényi model, we simulated 1000 independent realizations of both undirected and directed Erdős-Rényi graphs each (for each value of the network size \( N \) and connectivity \( p \)), and checked if the two conditions of Theorem 3.1 are satisfied with \( A \) as the adjacency matrix of the graph and \( B = I \). The fraction of the realizations that satisfy the conditions is used as the estimate of the probability of controllability. As we mentioned in Subsection 1.1, the randomness of the Erdős-Rényi model captures the unknown structure of the underlying social network. Several real world social networks such as high-school romantic partner networks have been shown to be similar to Erdős-Rényi model \([6]\).

\(^5\)Power-law graphs refer to undirected graphs where the probability \( p(k) \) that a uniformly sampled node has \( k \) neighbors (i.e., degree distribution evaluated at \( k \)) is proportional to \( k^{-\alpha} \) for a fixed value of the power-law exponent \( \alpha > 0 \). It has been shown that power-law degree distributions arise naturally from simple and intuitive generative processes such as preferential attachment whose power-law exponent \( \alpha \) lies in the range from 2 to 3 \([4, 24, 28]\). Hence, they have been widely compared with Erdős-Rényi graphs in the social network literature \([21, 26]\). The key difference between the two graph models (Erdős-Rényi and power-law) lies in the degree distribution (Erdős-Rényi graphs have Poisson degree distributions as opposed to power-law degree distributions) which is a key structural property of networks with implications in epidemic spreading, stability, friendship paradox and perception bias etc. \([1, 8, 29]\).
Further, even though the Erdős-Rényi model might not capture all the characteristics of other online social networks, it provides the simplest and most analytically tractable approximations for such networks (for example, the emergence of giant connected components) [18]. In this context, our numerical results in this section help to better understand the effect of the parameters of the Erdős-Rényi model on another such
sociologically important phenomena, namely the controllability of opinions in social networks.

The obtained results are presented in Figures 1 and 2 where Figures 1a, 1c, 1e, and 2a correspond to the undirected Erdős-Rényi graphs (Definition 2.1), Figures 1b, 1d, 1f, and 2b correspond to the directed Erdős-Rényi graphs (Definition 2.2). The labels in the figures correspond to the following settings:

- **Unconstrained Sparsity**: These curves correspond to unconstrained sparsity defined in Definition 2.3, and the corresponding sparsity \( s \) is specified in the labels.
- **Block Sparsity \( s=3, m=2 \)**: These curves in Figure 1 correspond to block sparsity defined in Definition 2.5 with sparsity \( s = 3 \) and block size \( m = 2 \).
- **Piecewise Sparsity \( s=3, m=2 \)**: These curves in Figure 1 correspond to piece-wise sparsity defined in Definition 2.4 with sparsity \( s = 3 \) and number of blocks \( m = 2 \).

**Figure 2.** Variation of the probability of controllability of the network opinion with network size \( N \). The figures confirm that the probability of the network opinion being not controllable decreases exponentially with the network size \( N \), as given by Theorems 4.1 and 5.1.

**Figure 3.** Variation of the probability of controllability of the network opinion of the power-law graph model with the power-law exponent \( \alpha \) and the network size \( N \). The figures show that the probability of controllability in the power-law model is significantly different from that of the Erdős-Rényi model.
The key observations from the numerical results are as follows:

- **Sparsity s and sparsity patterns:** Figures 1 and 2 confirm that as sparsity $s$ increases, the probability of controllability grows. This trend is in agreement with the bounds in Theorems 4.1 and 5.1 which also capture the monotonically increasing nature of the probability of controllability with $s$. Also, for $s = 3$, the unconstrained sparse signals offer the highest probability of controlling network opinion, followed by the piece-wise sparse vectors, and the block sparse vectors. This order verifies the relation given by (4.4).

- **Edge probability $p$:** Figure 1 shows that the probability of controllability first increases with $p$, reaches its maximum value, and then decreases. Also, the probability of controllability is one when $p$ is close to 1. For comparison, we note that the bounds on the probability of controllability (given by Theorems 4.1 and 5.1) approximately scale as $(1 - p)^{Ns}(1 - \exp((NP)^\alpha))$, for $\alpha > 0$. This bound is zero when $p = 0$, then increases with $p$ to attain a maximum value and diminishes thereafter. Thus, both the bounds in Theorems 4.1 and 5.1 and the curves in Figure 1 show similar behaviors. However, as $p$ approaches 1, the bound decreases, whereas the probability of controllability estimated in Figure 1 suddenly increases when $p = 1$. This difference in behavior is because the values of $p$ close to 1 lie outside the regime of the edge probability for which Theorems 4.1 and 5.1 holds. Also, this change in probability of controllability is not surprising because when $p = 1$, the adjacency matrix becomes $A = 11^T - I$ which is a deterministic full rank matrix. Therefore, both the conditions of Theorem 3.1 are satisfied by the system for all values of $s$ and all sparsity patterns. Hence, the probability of controllability is 1.

- **Network size $N$:** Figure 2 indicates that as the network size $N$ grows, the probability of the system not being controllable decreases exponentially. This observation corroborates the dependence of $N$ on the probability of controllability given by Theorems 4.1 and 5.1. Also, Theorems 4.1 and 5.1 imply the opinion of an asymptotically large network is controllable, almost surely, and the asymptotic behavior is attained in the regime $N > 30$ when $p = 0.2$. This observation is confirmed from the plots in Figure 1 that reveal that as $N$ becomes larger, the network opinion is controllable with high probability for a wider range of edge probability $p$ values.

- **Undirected and directed graphs:** Figures 1 and 2 show that the probability of controllability is larger for directed graphs compared to undirected graphs, in all settings. This is an additional insight which is not evident from the bounds in Theorems 4.1 and 5.1.

### 6.2. Probability of controllability of the power-law model.

The aim of this subsection is to show that power law networks behave very differently from Erdős-Rényi networks. Recall that an Erdős-Rényi network has a Poisson degree distribution, whereas a power-law network has a degree distribution of the form $p(k) = Ck^{-\alpha}$ where $C$ is the normalizing constant and $\alpha > 0$ is the power-law exponent. The simulation results presented below for power-law networks show that our theoretical results do not hold for this case, and there is a strong motivation to extend the results of this paper in future work.

To evaluate the probability of controllability for a power-law model, we simulated 1000 independent realizations of undirected power-law graphs using the so called configuration model [27] (for each value of the network size $N$ and power-law exponent $\alpha$).
More specifically, the configuration model generates \( k \) half-edges for each of the \( N \) nodes in the graph where \( k \) is the number obtained by rounding the realizations sampled independently from the power-law distribution i.e., \( k \sim Ck^{-\alpha} \) where \( C \) is the normalizing constant and \( \alpha > 0 \) is the power-law exponent. Then, each half-edge is connected to another randomly selected half-edge avoiding parallel edges and self-loops, yielding a graph with a power-law degree distribution. Finally, the fraction of the realizations that satisfy the conditions of Theorem 3.1 is used as the estimate of the probability of controllability. The results are presented in Figure 3 where the definition of the labels are the same as those in Figures 1 and 2 (see Subsection 6.1).

Figure 3a shows that the probability of controllability decreases monotonically with power-law exponent \( \alpha \) for all considered sparsity models. This observation is intuitive because a smaller power-law exponent \( \alpha \) implies that the network has larger number of high-degree nodes, making it easier to control. This is different from the non-monotone relation observed in Figure 1 for Erdős-Rényi graphs. However, it should also be noted that the parameter \( p \) of the Erdős-Rényi model and the parameter \( \alpha \) of the power-law model convey different information: \( p \) is the probability of the presence of an edge whereas \( \alpha \) is directly related to the degree of nodes. Further, Figure 3b shows that the probability of controllability decreases with the number of nodes \( N \) in power-law model indicating an opposite behavior to the Erdős-Rényi model shown in Figure 2a. Also, unlike the Erdős-Rényi model, the variation of the probability with \( N \) is not smooth. However, the cause of non-smoothness of the curve is not obvious, and we defer it as future work. To sum up, these differences suggest that the probability of controllability is an inherent property of the model.

7. Conclusion. This paper analyzed the controllability of network opinions modeled using a linear propagation framework with the additive influence of a sparsity constrained manipulative agent. The linear propagation was modeled using an Erdős-Rényi graph for two cases: the undirected and directed graphs. At every time instant, the manipulative agent can influence only a small (compared to the network size) number of people obeying some predefined sparsity patterns. The main results were Theorem 4.1 and Theorem 5.1 for the undirected and the directed graphs, respectively, which provide lower bounds on the probability with which the manipulative agent is able to drive the network opinion to any desired state starting from an arbitrary network opinion. Our results showed that in both cases, the probability increases with the network size, and the opinion of an asymptotically large network is controllable, almost surely.

One limitation of our results (Theorems 4.1 and 5.1) is that they are useful only if \( s \ll N \). As \( s \) becomes closer to \( N \), the range of edge probability \( p \) shrinks. In particular, our results do not cover the regime where \( s = N \), and generalizing the results for all values \( s \) is deferred to future work. Further, our main results were limited to the case where the weight matrix is rank one matrix and thus, relaxing this assumption to more general classes of weight matrices is an important direction for future work. Also, analyzing the systems with multiple competing external agents trying to manipulate the network opinion also is an interesting avenue for future work. Finally the numerical results in Subsection 6.2 showed that the results developed do not account for the behavior of power law networks. Further, the variation of controllability with network size for power-law model is non-smooth compared to the Erdős-Rényi model as seen in numerical results for reasons which are not obvious from our analysis. Studying these aspects of the power-law model remains an interesting direction for future research.
Appendix A. Proof of Theorem 3.1.
At a high level, the proof has three main steps:
A We first prove that controllability under the constraints given in the statement of the theorem is equivalent to the following: For some finite $K$, there exists a matrix $\tilde{B} \in B(K)$ such that $\operatorname{Rank} \{\tilde{B}\} = N$, where $B(K)$ is the set of all possible matrices of the following form:
\[
\begin{bmatrix}
A^{K-1}B_{S_1} & A^{K-2}B_{S_2} & \ldots & B_{S_K}
\end{bmatrix} \in \mathbb{R}^{N \times \sum_{k=1}^{K} |S_k|},
\]
and $S_k \in \mathcal{A}$, for all values of $k$.
B Next, we show that when one of the conditions, either Condition (a) or Condition (b) of Theorem 3.1 is not true, the condition given in Step A is violated. This is equivalent to showing that Conditions (a) and (b) of Theorem 3.1 are necessary for our notion of controllability to hold.
C Finally, we show that when the condition given in Step A does not hold, Conditions (a) and (b) of Theorem 3.1 are not true simultaneously. Thus, we show the sufficiency part of the result.

We present the proof for the above steps in the following subsections:

A.1. An equivalent condition. To characterize the controllability of the system as defined in Theorem 3.1, we consider the following equivalent system of equations:
\[
x_K - A^K x_0 = \sum_{k=1}^{K} A^{K-k} Bu_k
\]
\[
= \sum_{k=1}^{K} A^{K-k} B_{S_k} u_k, S_k,
\]
where $S_k \in \mathcal{A}$ is the support of $u_k$ and $B_{S_k} \in \mathbb{R}^{N \times |S_k|}$ is the submatrix of $B$ with columns indexed by $S_k$. Therefore, the system is controllable as defined in Theorem 3.1 if and only if the set $\mathcal{W}(K) = \mathbb{R}^{N}$ for some finite $K$, where
\[
\mathcal{W}(K) \triangleq \cup \tilde{B} \in B(K) \mathcal{CS}\{\tilde{B}\},
\]
where $\mathcal{CS}\{\cdot\}$ denote the column space of a matrix. However, a vector space over an infinite field ($\mathbb{R}^{N}$ in this case) cannot be a finite union of its proper subspaces. Therefore, $\mathcal{CS}\{\tilde{B}\} = \mathbb{R}^{N}$, for some $\tilde{B} \in \mathcal{B}$, and the desired result is proved.

A.2. Necessity. Suppose that one of the conditions in Theorem 3.1 does not hold:
(i) Suppose that Condition (i) does not hold. Then, from the classical PBH test for controllability without any constraints (Condition (iii) of Theorem D.1), the linear dynamical system defined by the transfer matrix-input matrix pair $(A, B_M)$ is not controllable. Therefore, the corresponding controllability matrix $\tilde{B}_{(K)}$ as defined below does not have full row rank for any finite $K$:
\[
\tilde{B}_{(K)} = \begin{bmatrix}
A^{K-1} B_M & A^{K-2} B_M & \ldots & B_M
\end{bmatrix}.
\]
Further, all matrices in $B(K)$ are submatrices of $\tilde{B}_{(K)}$, and therefore, the condition given in Step A is violated.
(ii) Suppose Condition (ii) does not hold. Then, for every index set \( S \in \mathcal{A} \), there exists a nonzero vector \( z \) such that \( z^T B_S = 0 \) and \( z^T A = 0 \). This implies that for any finite \( K \), there exits a vector \( z \) such that \( z^T \tilde{B} = 0 \), for all \( \tilde{B} \in \mathcal{B}_K \). Therefore, the condition given in Step A is violated.

Hence, we proved the necessity of the conditions given by Theorem 3.1.

A.3. Sufficiency. Suppose that condition given in Step A is not true. Then, we consider the following matrix of size \( N \times N \sum_{S \in \mathcal{A}} |S| \):

\[
\tilde{B}^* = [A^{pN-1} B_{S_1} \ A^{pN-2} B_{S_1} \ldots \ A^{(p-1)N} B_{S_1} \ldots \ A^{(p-1)N} B_{S_2} \ldots \ A^{(p-2)N} B_{S_2} \ldots \ldots \ A^{N-1} B_{S_P} \ldots B_{S_P}],
\]

where \( P = |\mathcal{A}| \) and \( \{ S_i \}_{i=1}^P = \mathcal{A} \). Since the condition given in Step A does not hold for any finite \( K \), \( \tilde{B}^* \) does not have full row rank. Next, we can rearrange the columns of \( \tilde{B}^* \) to get the following matrix which has the same rank as that of \( \tilde{B}^* \):

\[
\begin{bmatrix}
A^{N-1} B^* & A^{N-2} B^* & \ldots & B^*
\end{bmatrix},
\]

where \( B^* \in \mathbb{R}^{N \times ps} \) is defined as follows:

\[
B^* \triangleq [A^{(p-1)N} B_{S_1} \ A^{(p-2)N} B_{S_2} \ldots \ldots \ A^{N-1} B_{S_P}].
\]

Thus, using classical Kalman rank test for controllability without any constraints (Condition (ii) of Theorem D.1), the linear dynamical system defined by the transfer matrix-input matrix pair \((A, B^*)\) is not controllable. Then, the classical PBH test for controllability without any constraints (Condition (iii) of Theorem D.1), implies that the matrix \([A - \lambda I \ B^*]\) has rank less than \( N \), for some \( \lambda \in \mathbb{C} \). Therefore, there exists a nonzero vector \( z \in \mathbb{R}^N \) such that \( z^T A = \lambda z^T \) and \( z^T B^* = 0 \).

However, we have

\[
0 = z^T B^* = z^T \begin{bmatrix}
\lambda^{(p-1)N} B_{S_1} & \lambda^{(p-2)N} B_{S_2} & \ldots & B_{S_P}
\end{bmatrix}.
\]

So either \( \lambda = 0 \) and \( z^T B_{S_P} = 0 \), or, if \( \lambda \) is nonzero, then \( z^T B_{M} = 0 \). Since the ordering of the index sets in \( \mathcal{A} \) does not matter, we conclude that either \( \lambda = 0 \) and \( z^T B_S = 0 \) for all \( S \in S \), or, \( z^T B_{M} = 0 \) for some \( \lambda \in \mathbb{C} \). Therefore, the two conditions of Theorem 3.1 do not hold simultaneously. Thus, the proof is complete.

Appendix B. Proof of Theorem 4.1. Theorem 3.1 provides the necessary and sufficient conditions under which a system is controllable using sparse inputs. Therefore, the key idea of the proof of Theorem 4.1 is to derive the probability with which the conditions of Theorem 3.1 hold when \( B \) in Theorem 3.1 is set to be the identity matrix \( I \). The main tools used in the proof are the rank properties of the Hadamard product and a random symmetric binary matrix as stated below:

**Lemma B.1** (Invertibility of Hadamard product). For any matrix \( A \in \mathbb{R}^{N \times N} \) and a vector \( w \in \mathbb{R}^N \), the Hadamard product \( A \odot (ww^T) \) is invertible if and only if \( A \) is invertible and all entries of \( w \) are nonzero.

**Proof.** See Appendix E.

**Theorem B.2.** Let \( A \in \{0,1\}^{N \times N} \) be the adjacency matrix of an undirected Erdős-Rényi graph with the edge connectivity \( p \). Then, there exists finite positive constants \( C \) and \( c \) such that, for

\[
N^{-1} \leq p \leq 1 - N^{-1},
\]

Theorem 3.1 holds.
the following holds:

\[ P \{ A \text{ is non-singular} \} \geq 1 - C \exp \left( -c(pN)^{1/32} \right). \]

Proof. See Appendix F.

Clearly, Condition (a) of Theorem 3.1 holds with probability 1. So we focus on Condition (b) of Theorem 3.1. Let event \( \mathcal{E} \) denote that event that Condition (b) of Theorem 3.1 holds, i.e., \( \mathcal{E} \) is given by

\[
\mathcal{E} = \{ \exists S \in \mathcal{A} : \text{Rank} \left\{ \begin{bmatrix} \bar{A} & I_S \end{bmatrix} \right\} = N \}. \tag{B.1}
\]

In the following, we derive a lower bound on the probability \( P \{ \mathcal{E} \} \) which is also a lower bound on the probability with which the network opinion is controllable under given constraints.

For a given index set \( S \in \mathcal{A} \), we can rearrange the columns of the matrix in (B.1) as follows:

\[
\text{Rank} \left\{ \begin{bmatrix} \bar{A} & I_S \end{bmatrix} \right\} = \text{Rank} \left\{ \begin{bmatrix} \bar{A}_{S_c} & I \\ \bar{A}_{S_c} \end{bmatrix} \right\},
\]

where \( S^c = [N] \setminus S \) and \( |S^c| = N - s \). Also, \( \bar{A}_{S_c} \in \mathbb{R}^{s \times N} \) and \( \bar{A}_{S_c} \in \mathbb{R}^{N-s \times N} \) are the submatrices of \( \bar{A} \) formed by rows indexed by \( S \) and \( S^c \), respectively. Consequently, (B.1) can be further simplified as follows:

\[
\mathcal{E} = \{ \exists S \in \mathcal{A} : \bar{A}_{S_c} \text{ is full row rank} \}.
\]

Note that \( \mathcal{E} \) depends on the rank of a non-square matrix \( \bar{A}_{S_c} \). Since Lemma B.1 and Theorem B.2 deals with the invertibility of square matrices, we first lower bound \( P \{ \mathcal{E} \} \) in terms of probabilities with which certain square matrices are invertible. For this, we notice that

\[
\mathcal{E} \supseteq \{ \exists S \in \mathcal{A}, I \subseteq S : \bar{A}_{I_c} \text{ is full row rank} \}. \tag{B.2}
\]

where \( I_c = [N] \setminus I \supseteq S^c \), and \( \bar{A}_{I_c} \in \mathbb{R}^{N-|I| \times N} \) is the submatrix of \( \bar{A} \) formed by rows indexed by \( I^c \). Here, (B.2) follows because if all rows of \( \bar{A}_{I_c} \) are linearly independent, then, all rows of the submatrix \( \bar{A}_{S_c} \) of \( \bar{A}_{I_c} \) are also linearly independent. Next, we further bound (B.2) as follows:

\[
\mathcal{E} \supseteq \{ \exists S \in \mathcal{A}, I \subseteq S : \bar{A}_{I_c} = 0 \text{ and } \bar{A}_{I_c} \text{ is non-singular} \}. \tag{B.3}
\]

where \( \bar{A}_{I_c} \in \mathbb{R}^{|I| \times N} \) is the submatrix of \( \bar{A} \) formed by rows indexed by \( I \), and \( \bar{A}_{I_c} \in \mathbb{R}^{N-|I| \times N} \) is the (symmetric) principal submatrix of \( \bar{A} \) formed by the rows indexed by \( I^c \) and the corresponding columns. Therefore, we have

\[
P \{ \mathcal{E} \} \geq P \{ \exists S \in \mathcal{A}, I \subseteq S : \bar{A}_{I_c} = 0 \text{ and } \bar{A}_{I_c} \text{ is non-singular} \}.
\]

Hence, \( P \{ \mathcal{E} \} \) now depends on the invertibility of the symmetric square matrix \( \bar{A}_{I_c} \).

Next, from (2.1) and using the fact that \( \Lambda \) is an invertible diagonal matrix, we deduce that

\[
P \{ \mathcal{E} \} \geq P \{ \exists S \in \mathcal{A}, I \subseteq S : A_{I_c} \odot W_{I_c} = 0 \text{ and } A_{I_c} \odot W_{I_c} \text{ is non-singular} \}.
\]
The entries of $W_{\mathcal{T}, \mathcal{I}}$ are sampled from a continuous distribution, they are nonzero with probability one. Thus, Lemma B.1 leads to the following:

\[(B.4) \quad \mathbb{P}\{\mathcal{E}\} \geq \mathbb{P}\left\{ \exists \mathcal{S} \in \mathcal{A}, \mathcal{I} \subseteq \mathcal{S} : A_{\mathcal{I}, :} = 0 \text{ and } A_{\mathcal{I}^c, \mathcal{I}^c} \text{ is non-singular} \right\}.\]

Further, (B.4) can be further simplified using Theorem B.2. For this, we rewrite the right-hand side of (B.4) as follows:

\[(B.5) \quad \mathbb{P}\{\mathcal{E}\} \geq \mathbb{P}\left\{ \bigcup_{i=0}^{s} \mathcal{E}_i \right\},\]

where we define $\mathcal{E}_i$ as follows:

\[(B.6) \quad \mathcal{E}_i \triangleq \left\{ \exists \mathcal{S} \in \mathcal{A}, \mathcal{I} \subseteq \mathcal{S} : |\mathcal{I}| = i, A_{\mathcal{I}, :} = 0 \text{ and } A_{\mathcal{I}^c, \mathcal{I}^c} \text{ is non-singular} \right\}.\]

However, when $A_{\mathcal{I}, :}$ is invertible, all rows of $A$ indexed by $\mathcal{I}$ are nonzero. Therefore, $\mathcal{E}_i$ denote the event that $A$ has exactly $i$ zero rows (indexed by $\mathcal{I}$), and the remaining rows are linearly independent. Consequently, the events $\left\{ \mathcal{E}_i \right\}_{i=0}^{s}$ are disjoint, and so the union bound holds with equality. Therefore, (B.5) leads to

\[(B.7) \quad \mathbb{P}\{\mathcal{E}\} \geq \mathbb{P}\left\{ \bigcup_{i=0}^{s} \mathcal{E}_i \right\} = \sum_{i=0}^{s} \mathbb{P}\{\mathcal{E}_i\} .\]

Now, we simplify $\mathbb{P}\{\mathcal{E}_i\}$ by summing over all possible values of $\mathcal{I}$ (corresponding to zero rows of $A$) as follows:

\[(B.8) \quad \mathbb{P}\{\mathcal{E}_i\} = \sum_{\mathcal{I} \subseteq \mathcal{S}, \mathcal{S} \in \mathcal{A}, |\mathcal{I}| = i} \mathbb{P}\{A_{\mathcal{I}, :} = 0\} \mathbb{P}\{A_{\mathcal{I}^c, \mathcal{I}^c} \text{ is non-singular} \} ,\]

which we obtain using the fact that the entries of $A_{\mathcal{I}, :}$ and $A_{\mathcal{I}^c, \mathcal{I}^c}$ are independent.

The condition $A_{\mathcal{I}, :} = 0$ holds when all the independent Bernoulli variables in $A_{\mathcal{I}, :}$ are zeros. The number of independent random variables is

\[(N - 1) + (N - 2) + \ldots + (N - i) = i(N - (i + 1)/2) .\]

Therefore, we have

\[(B.9) \quad \mathbb{P}\{A_{\mathcal{I}, :} = 0\} = (1 - p)^i(N - (i + 1)/2) .\]

Further, the entries of $A_{\mathcal{I}^c, \mathcal{I}^c} \in \mathbb{R}^{N - i \times N - i}$ have the same distribution as that of $A$. Thus, we apply Theorem B.2 to get

\[(B.10) \quad \mathbb{P}\{A_{\mathcal{I}^c, \mathcal{I}^c} \text{ is non-singular} \} \geq 1 - C \exp \left( -cp(N - i)^{1/32} \right) ,\]

where $c > 0$ is universal constant. Combining (B.8), (B.9), and (B.10), we get that

\[(B.11) \quad \mathbb{P}\{\mathcal{E}_i\} \geq Q(i, A)p^i(N - (i + 1)/2) \left[ 1 - C \exp \left( -cp(N - i)^{1/32} \right) \right] ,\]

where $Q$ is as defined in the statement of the theorem (see (4.1)).

Finally, we complete the proof by combining (B.7) and (B.11).

**Appendix C. Proof of Theorem 5.1.** The proof technique used here is similar to that of Theorem 4.1. However, since Theorem B.2 does not hold in this case, we use an an analogous theorem for directed graphs which is as follows:
Theorem C.1. Let $A \in \{0, 1\}^{N \times N}$ be the adjacency matrix of a directed Erdős-Rényi graph with the edge connectivity $p$. Let $D$ be a real valued diagonal matrix independent of $A$ with $\|D\| \leq R\sqrt{pN}$ where $R \geq 1$. Then, there exists finite positive constants $C$ and $c$ that depend on $R$ such that for

$$C \frac{\log N}{N} \leq p \leq 1 - C \frac{\log N}{N},$$

it holds that

$$\mathbb{P}\{A + D \text{ is non-singular}\} \geq 1 - \exp(-cpN).$$

Proof. The result is an immediate corollary of [5, Theorem 1.11].

Using the arguments similar to those in the proof of Theorem 4.1, we see that all steps of the proof in Appendix B until (B.8) holds in this case. Hence, continuing from there, the condition $\mathbb{A}_{I, I} = 0$ holds when all the independent Bernoulli variables in $\mathbb{A}_{I, I}$ are zeros. The number of independent random variables is $i(N - 1)$. Therefore, we have

(C.1) \[ \mathbb{P}\{\mathbb{A}_{I, I} = 0\} = (1 - p)^{i(N - 1)}. \]

Further, the entries of $\mathbb{A}_{I, I} \in \mathbb{R}^{N - i \times N - i}$ have the same distribution as that of $\mathbb{A}$. Thus, we apply Theorem C.1 to get

(C.2) \[ \mathbb{P}\{\mathbb{A}_{I, I} \text{ is non-singular}\} \geq 1 - \exp(-cp(N - i)), \]

where $c > 0$ is universal constant. Combining (B.8), (C.1), and (C.2), we get that

(C.3) \[ \mathbb{P}\{\mathcal{E}_i\} \geq Q(i, \mathcal{A})(1 - p)^{i(N - 1)}[1 - \exp(-cp(N - i))], \]

where $\mathcal{E}_i$ is defined in (B.6) and $Q$ is as defined in the statement of the theorem (see (4.1)).

Finally, we complete the proof by combining (C.3) and (B.7) (as we mentioned in the beginning of the proof, (B.7) holds in this case).

Remark: We note that bound in Theorem 5.1 is not as tight as the result in Theorem 4.1 because of the bound in (B.3) used in the proof of Theorems 4.1 and 5.1 (see Figures 1 and 2). To be specific, for both cases, we claim that $\mathcal{F}_1 \supseteq \mathcal{F}_2$ where

$$\mathcal{F}_1 \triangleq \{\exists S \in \mathcal{A}, I \subseteq S : \mathbb{A}_{I, I} \text{ is full row rank}\}$$

$$\mathcal{F}_2 \triangleq \{\exists S \in \mathcal{A}, I \subseteq S : \mathbb{A}_{I, I} = 0 \text{ and } \mathbb{A}_{I^c, I^c} \text{ is non-singular}\}.$$

We recall that $\mathbb{A}_{I^c, I^c} \in \mathbb{R}^{N - \{1, 2, \ldots, N\} \setminus I \times \{1, 2, \ldots, N\} \setminus I}$ are the submatrices of $\mathbb{A}$ formed by rows indexed by $I^c = \{1, 2, \ldots, N\} \setminus I$ and $I$, respectively. Also, $\mathbb{A}_{I^c, I^c} \in \mathbb{R}^{N - \{1, 2, \ldots, N\} \setminus I \times \{1, 2, \ldots, N\} \setminus I}$ is the principal submatrix of $\mathbb{A}$ formed by the rows indexed by $I^c$ and the corresponding columns. To understand the difference between the directed and the undirected graph cases, we define another event $\mathcal{F}_3$ as follows:

$$\mathcal{F}_3 \triangleq \{\exists S \in \mathcal{A}, I \subseteq S : \mathbb{A}_{I, I} = 0 \text{ and } \mathbb{A}_{I^c, I^c} \text{ is full row rank}\}.$$

Clearly, $\mathcal{F}_1 \supseteq \mathcal{F}_3 \supseteq \mathcal{F}_2$. However, for undirected graphs, $\mathbb{A}$ is a symmetric matrix, and so if $\mathbb{A}_{I, I} = 0$, we have

$$\begin{bmatrix} \mathbb{A}_{I, I} = 0 \\ \mathbb{A}_{I^c, I^c} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{A}_{I^c, I^c} \end{bmatrix}.$$
Hence, when \( A_{xc} = 0 \), we have \( A_{xc} = [0 \ A_{xc}] \), and therefore, \( A_{xc} \) has full row rank if and only if \( A_{xc} \) has full row rank. Further, since \( A_{xc} \) is a square matrix, this is equivalent to \( A_{xc} \) being non-singular. Hence, \( F_3 = F_2 \) for the undirected graph case. However, for directed graphs, \( F_3 \supset F_2 \), and thus, the bound is not as tight as the bound for the undirected case.

**Appendix D. Controllability-related results.** In this section, we review some results related to the controllability of a linear dynamical system. To state the results, we define a general linear dynamical system as given by (3.1). The first result gives conditions under which the system is controllable without any constraints on the input vector.

**Theorem D.1.** For an unconstrained system defined in (3.1), the following conditions are equivalent:

(i) The system is controllable, i.e., it can be driven to any final state starting from any initial state in finite time.

(ii) Kalman rank test [20]: Rank of the matrix \( \tilde{B}(K) \) is \( N \) for some finite \( K \) where

\[
\tilde{B}(K) = \begin{bmatrix} A^{K-1}B & A^{K-2}B & \ldots & B \end{bmatrix} \in \mathbb{R}^{N \times KL}
\]

(iii) Popov-Belevitch-Hautus (PBH) test [16]: For all \( \lambda \in \mathbb{C} \), rank of the matrix

\[
[\lambda I - A \ B] \in \mathbb{R}^{N \times (N+L)}
\]

is \( N \).

The next result is on the necessary and sufficient conditions for the controllability of a linear dynamical system with sparsity constraints on the input:

**Theorem D.2 ([19, Theorem 3.1]).** Consider a system as defined in (3.1). For any initial state \( x_0 \) and any final state \( x_K \), there exists an input sequence \( u_k \), \( k = 1, 2, \ldots, K \) such that \( \|u_k\|_0 \leq s \) and \( 0 < s \leq L \), which steers the system from the state \( x_0 \) to \( x_K \) for some finite \( K \), if and only if the following two conditions hold:

(i) For all \( \lambda \in \mathbb{C} \), rank of the matrix

\[
[\lambda I - A \ B] \in \mathbb{R}^{N \times (N+L)}
\]

is \( N \).

(ii) There exists an index set \( S \subseteq \{1, 2, \ldots, L\} \) with \( s \) entries such that rank of

\[
[A \ B_S] \in \mathbb{R}^{N \times (N+s)}
\]

is \( N \).

**Appendix E. Proof of Lemma B.1.**

Using Schur’s unitary triangularization [17, Theorem 2.3.1], we have

\[
A = ULU^H = \sum_{1 \leq i \leq j \leq N} L_{ij}U_i U_j^H,
\]

where \( U \in \mathbb{C}^{N \times N} \) is a unitary matrix, and \( L \in \mathbb{C}^{N \times N} \) is an upper triangular matrix with the diagonal entries of \( A \) along the diagonal. Also, \((\cdot)^H\) denotes the conjugate...
transpose of a matrix (or vector). Therefore,

\[
\text{det} \{ A \otimes W \} = \text{det} \left\{ \sum_{1 \leq i \leq j \leq N} L_{ij} \left( U_i U_j^H \right) \otimes \left( ww^T \right) \right\}
\]

\[
= \text{det} \left\{ \sum_{1 \leq i \leq j \leq N} L_{ij} \left( U_i \otimes w \right) \left( U_j^* \otimes w \right)^T \right\}
\]

(E.1)

\[
= \text{det} \{ VLV^T \} = \text{det} \{ L \} \text{det} \{ VV^H \}
\]

\[
= \text{det} \{ L \} \text{det} \left\{ \left( UU^H \right) \otimes \left( ww^T \right) \right\}
\]

(E.2)

\[
= \text{det} \{ L \} \text{det} \left\{ I \otimes \left( ww^T \right) \right\}
\]

(E.3)

where in (E.1), we define \( V \in \mathbb{R}^{N \times N} \) such that \( V_i = U_i \otimes w \). Also, we use the fact that \( U \) is unitary to obtain (E.2). Also, we obtain (E.3) using the fact that the eigenvalues of \( L \) and \( A \) are the same.

Thus, from (E.3), we conclude that \( \text{det} \{ A \otimes W \} \neq 0 \) if and only if \( \text{det} \{ A \} \neq 0 \), and all entries of \( w \) are nonzero. Hence, the proof is complete.

Appendix F. Proof of Theorem B.2. The probability with which a symmetric random matrix with iid above-diagonal entries\(^6\) is invertible is studied in [34]. However, [34] assumes that the sparse random matrix has entries with zero mean and unit variance. Also, the constants in the result depend on the fourth moments of the entries of the matrix, and the dependence is not explicitly defined. Our result is a generalization of [34, Theorem 1.5] which is modified to handle the adjacency matrix of an undirected Erdős-Rényi graph with edge probability \( p \). However, our proof is not based on the concentration of quadratic forms using small ball probabilities used in [34, Theorem 1.5]. Our approach is similar to that in [5] which uses the concentration of inner product using small ball probabilities.

Before we launch into the proof, we introduce some notation, definitions and list a few useful results from the literature.

F.1. Toolbox. In this section, we present a concept called small ball probability which describes the spread of a distribution in space. The results on small ball probabilities requires us to define two other quantities called Lévy concentration function and least common denominator (LCD). The definition of the Lévy concentration function is as follows:

**Definition F.1 (Lévy function).** The Lévy concentration of a random vector \( x \in \mathbb{R}^N \) for any \( \epsilon > 0 \) is defined as

\[
L(x, \epsilon) = \sup_{z \in \mathbb{R}^N} \mathbb{P}\{ \|x - z\| \leq \epsilon \}.
\]

One simple result on the Lévy concentration which we later use to define the LCD is as follows:

\(^6\)The above diagonal entries refers to the entries in the upper triangular portion of a matrix other than the diagonal entries.
LEMMA F.2. Let \( \xi \) be a random variable with unit variance and finite fourth moment, and \( \zeta \in \{0, 1\} \) be another random variable independent of \( \xi \) such that \( p = \mathbb{P}\{\zeta = 1\} \). Then, there exists constants \( 0 < \delta_0, \epsilon < 1 \) such that the Lévy function (in Definition F.1) satisfies

\[
\mathcal{L}(\zeta\xi, \epsilon) \leq 1 - \delta_0p.
\]

Proof. The proof follows from [32, Lemma 3.2].

We need some other definitions to introduce the concept of LCD. Let \( S^{N-1} \subset \mathbb{R}^N \) denote the unit Euclidean sphere. We define a subset of \( S^{N-1} \) parameterized by \( \rho \in (0, 1) \) based on sparsity as

\[ T_{\text{incomp}}(N, \rho) \triangleq \{ x \in S^{N-1} : \not\exists y \in \mathbb{R}^N \text{ such that } \| y \|_0 \leq \frac{N}{(pN)^{1/16}} \| x - y \| \leq \rho \}. \]

**Definition F.3 (Regularized LCD [23, Definition 6.3]).** Let \( \alpha \in (0, 1) \), \( x \in T_{\text{incomp}}(N, \rho) \) and \( \mathbb{Z} \) be the set of integers. We define the regularized LCD of \( (x, \alpha) \) as

\[
\hat{D}(x, \alpha) = \max_{I \subseteq \{1, \ldots, N\}} D(x_I/\|x_I\|)
\]

\[
D(x) = \inf\{ \theta > 0 : \text{dist}(\theta x, \mathbb{Z}^N) < \gamma \}
\]

\[
\gamma = (\delta_0p)^{-1/2} \sqrt{\log_+ \left( \sqrt{\delta_0 p} \theta \right)};
\]

where \( D(x) \) is called the LCD of \( x \) and \( \delta_0 \) is given by Lemma F.2.

With the above definitions, we are ready to state the result on small ball probability using the Lévy concentration function and LCD:

**Proposition F.4 ([23, Proposition 6.7]).** Let \( \mathbf{a} \in \mathbb{R}^N \) be a random vector with iid coordinates which are products of two independent random variables \( a_i = \zeta_i \xi_i \) where \( \mathbb{P}\{\zeta_i = 1\} = 1 - \mathbb{P}\{\zeta_i = 0\} = p \), and \( \xi \) is a random variable with unit variance and finite fourth moment. Then, for any \( x \in S^{N-1} \) and \( \epsilon > 0 \), we have

\[
\mathcal{L}(x^T \mathbf{a}, \sqrt{p}\epsilon) \leq C_1 \left( \epsilon + \frac{1}{\sqrt{p}D(x)} \right),
\]

where \( D \) is the LCD given by Definition F.3.

To state the other results used in the proof, we define a subset of \( T_{\text{incomp}}(N, \rho) \) in (F.1) based on the regularized LCD (see Definition F.3) as follows:

\[ T_{\text{large}}(N, \rho) \triangleq \{ x \in T_{\text{incomp}}(N, \rho) : \hat{D}(x, (pN)^{-1/16}) > \exp \left( (pN)^{1/32} \right) \}, \]

where \( \hat{D} \) is the regularized the least common denominator (see Definition F.3).

The following result shows that, with high probability, the eigenvectors of \( A \) (defined in Theorem B.2) belong to \( T_{\text{large}}(N, \rho) \).

**Lemma F.5.** There exists constants \( C, \tilde{C}, c > 0 \) and \(-1/2 \leq \alpha < 1/2 \) such that if \( N^{-1} < p < 1 - N^{-1} \), for any \( \lambda \in \mathbb{R} \),

\[
\mathbb{P}\left\{ \exists x \in S^N \setminus T_{\text{large}}(N, \rho) : \|(A - \lambda I)x\| \leq \tilde{C} \rho(pN)^{\alpha} \right\} \leq \exp(-c pN),
\]

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where \( A \in \mathbb{R}^{N \times N} \) and \( p \) are defined in Theorem B.2. Also, \( \mathbb{S}^{N-1} \subset \mathbb{R}^N \) is the unit Euclidean sphere, \( T_{\text{large}}(N, \rho) \) is defined in (F.2), and

\[
\rho = C^{-\frac{\log \frac{1}{(8p)\sqrt{pN}}}{\log \sqrt{pN}}}.
\]

**Proof.** The proof is a straightforward adaptation of the proof of [23, Theorem 2.2] and [23, Corollary 5.5.] using the fact that the Lévy concentration function (see Definition F.1) is invariant to shifts, and hence, we omit the details.

The last result of this subsection bounds the infimum of \( \|Ax\| \) over incompressible vectors for a general random matrix \( A \).

**Lemma F.6.** Let \( A \in \mathbb{R}^{N \times N} \) be any random matrix with iid columns. Let \( H \subseteq \mathbb{R}^N \) denote the span of all columns of \( A \) except the first column. Then, for every \( \epsilon > 0 \), it holds that

\[
P\left\{ \inf_{x \in T_{\text{incomp}}(N, \rho)} \|Ax\| \leq \frac{\epsilon \rho \sqrt{N}}{\sqrt{pN}} \right\} \leq (pN)^{1/16} P\{ \text{dist}(A_1, H) \leq \epsilon \},
\]

where \( T_{\text{incomp}}(N, \rho) \) is defined in (F.1).

**Proof.** The result is obtained from [31, Lemma 3.5] by choosing the first parameter of the compressible set as \((pN)^{1/16}\) and the fact that columns of \( A \) are iid.

Having presented all the mathematical tools, in the next subsection, we formally prove Theorem B.2.

**F.2. Proof of Theorem B.2.** We obtain the probability with which \( A \) is invertible by computing the probability with which the smallest singular value of \( A \) is positive. Using the union bound and with \( \mathbb{S}^{N-1} \subset \mathbb{R}^N \) denoting the unit Euclidean sphere, we have

\[
P\{A \text{ is singular}\} = P\left\{ \inf_{x \in \mathbb{S}^{N-1}} \|Ax\| = 0 \right\}
\]

\[
\leq P\left\{ \inf_{x \in T_{\text{large}}(N, \rho)} \|Ax\| = 0 \right\} + P\left\{ \inf_{x \in \mathbb{S}^{N-1} \setminus T_{\text{large}}(N, \rho)} \|Ax\| = 0 \right\},
\]

where \( T_{\text{large}}(N, \rho) \) and \( \rho \) are given by (F.2) and (F.3), respectively. So in the following, we upper bound the two terms in (F.4).

Using Lemma F.5, there exists for constant \( c_1 > 0 \) such that

\[
P\left\{ \inf_{x \in \mathbb{S}^{N-1} \setminus T_{\text{large}}(N, \rho)} \|Ax\| = 0 \right\} \leq \exp(-c_1 pN).
\]

Next, to bound the first term in (F.4), we consider the following:

\[
P\left\{ \inf_{x \in T_{\text{large}}(N, \rho)} \|Ax\| = 0 \right\} = \lim_{\epsilon \to 0^+} P\left\{ \inf_{x \in T_{\text{large}}(N, \rho)} \|Ax\| \leq \epsilon \rho \sqrt{N} \right\}.
\]

Due to (F.2), we know that \( T_{\text{large}}(N, \rho) \subset T_{\text{incomp}}(N, \rho) \) which is defined in (F.1). Thus, we deduce that for every \( \epsilon > 0 \),

\[
P\left\{ \inf_{x \in T_{\text{large}}(N, \rho)} \|Ax\| \leq \epsilon \rho \sqrt{N} \right\} \leq P\left\{ \inf_{x \in T_{\text{incomp}}(N, \rho)} \|Ax\| \leq \epsilon \rho \sqrt{N} \right\}.
\]
We simplify the above expression using Lemma F.6, and for that, we write the symmetric matrix $A$ as

$$A = \begin{bmatrix} 0 & a^T \in \mathbb{R}^{1 \times N-1} \\ a \in \mathbb{R}^{N-1 \times 1} & A_{\text{sub}} \in \mathbb{R}^{N-1 \times N-1} \end{bmatrix},$$

Then, applying Lemma F.6 with the random matrix as $A$, we obtain that

$$P \left\{ \inf_{x \in \mathbb{T}_{\text{large}}(N, \rho)} \|Ax\| \leq \epsilon \rho \sqrt{\frac{p}{N}} \right\} \leq (pN)^{1/16} P \left\{ \left\| 0 \right\|_{A_{\text{sub}}} \right\} \leq \sqrt{p \epsilon},$$

where $CS\{\cdot\}$ denote the column space of a matrix. The distance term on the right hand side simplifies as follows:

$$\text{dist} \left( \begin{bmatrix} 0 \\ a \end{bmatrix}, CS \left\{ a^T A_{\text{sub}} \right\} \right) \geq \text{dist} \left( a, CS \{ A_{\text{sub}} \} \right) = \min_{v \in CS\{A_{\text{sub}}\}} \| a - v \| = \max_{v \in \mathbb{S}^{N-2}} v^T a.$$

Therefore, from (F.7), for any $\epsilon > 0$, we have

$$P \left\{ \inf_{x \in \mathbb{T}_{\text{large}}(N, \rho)} \|Ax\| \leq \epsilon \rho \sqrt{\frac{p}{N}} \right\} \leq (pN)^{1/16} \left\{ \max_{v \in \mathbb{S}^{N-2}} v^T a \leq \sqrt{p \epsilon} \right\} + (pN)^{1/16} \left\{ \exists v \in \mathbb{T}_{\text{large}}(N-1, \rho') : A_{\text{sub}}v = 0 \right\}.$$

where $\rho' \triangleq \sqrt{\frac{1}{\log \sqrt{p(N-1)}}}$ for the constant $C$ is same as the constant in (F.5). Next, we use Lemmas F.2 and F.5 to simplify the two probability terms in (F.8).

Since the entries of $A_{\text{sub}} \in \mathbb{R}^{N-1 \times N-1}$ have the same distribution as that of $A$, we again apply Lemma F.5 to get

$$P \left\{ \exists v \in \mathbb{S}^{N-2} \setminus \mathbb{T}_{\text{large}}(N-1, \rho') : A_{\text{sub}}v = 0 \right\} \leq \exp(-c_1 p(N-1)),$$

The second term in (F.8) can be simplified as follows:

$$P \left\{ \exists v \in \mathbb{T}_{\text{large}}(N-1, \rho') : v^T a \leq \sqrt{p \epsilon} \right\} \leq \sup_{v \in \mathbb{T}_{\text{large}}(N-1, \rho')} P \left\{ |v^T a| \leq \sqrt{p \epsilon} \right\} \leq \sup_{v \in \mathbb{T}_{\text{large}}(N-1, \rho')} \sup_{z \in \mathbb{R}} P \left\{ |v^T a - z| \leq \sqrt{p \epsilon} \right\} \leq \sup_{v \in \mathbb{T}_{\text{large}}(N-1, \rho')} \mathcal{L} \left( v^T a, \sqrt{p \epsilon} \right).$$

Further, we note that the entries of $a$ have the same distribution as $\zeta \xi$, where $\zeta, \xi \in \{0, 1\}$ are Bernoulli random variables with probabilities of being 1 as $1/2$ and $2p$. 

1

\[ 0 \quad a^T \in \mathbb{R}^{1 \times N-1} \]
\[ a \in \mathbb{R}^{N-1 \times 1} \]
\[ A_{\text{sub}} \in \mathbb{R}^{N-1 \times N-1} \]

\[ \text{dist} \left( \begin{bmatrix} 0 \\ a \end{bmatrix}, CS \left\{ a^T A_{\text{sub}} \right\} \right) \geq \text{dist} \left( a, CS \{ A_{\text{sub}} \} \right) = \min_{v \in CS\{A_{\text{sub}}\}} \| a - v \| = \max_{v \in \mathbb{S}^{N-2}} v^T a. \]
respectively. Thus, applying Proposition F.4 with $x$ as $v$ and the parameter as $\epsilon/\sqrt{2}$, we get that there exists a constant $C_1 > 0$ such that

$$
P \left\{ \exists v \in T_{\text{large}}(N - 1, \rho') : v^T a \leq \sqrt{p} \epsilon \right\} \leq \sup_{v \in T_{\text{large}}(N - 1, \rho')} C_1 \left[ \frac{\epsilon}{\sqrt{2}} + \frac{1}{\sqrt{2} \sqrt{D(v)}} \right],
$$

where the last step follows from the definition of $T_{\text{large}}(N - 1, \rho')$ and the fact that $\hat{D}(x, \alpha) \leq D(x)$, for any $0 < \alpha < 1$. Combining (F.8), (F.9) and (F.10), we get that

$$
P \left\{ \inf_{x \in T_{\text{large}}(N, \rho)} \|Ax\| \leq \epsilon \sqrt{p} \right\} \leq (pN)^{1/16} \exp(-c_1 p(N - 1))$$

$$+ C_1 \sqrt{2} (pN)^{1/16} \left[ \frac{\epsilon}{\sqrt{2}} + \frac{1}{\sqrt{p} \exp \left( ((N - 1)p)^{1/32} \right)} \right]$$

$$\leq C_2 \sqrt{2} (pN)^{1/16} + C_2 \exp \left( -c_2 (pN)^{1/32} \right),
$$

for some constants $C_2, c_2 > 0$.

Substituting (F.11) in (F.6) by setting $\epsilon$ to zero, we get

$$
P \left\{ \inf_{x \in T_{\text{large}}(N, \rho)} \|Ax\| = 0 \right\} \leq C_2 \exp \left( -c_2 (pN)^{1/32} \right),
$$

Finally, combining the above equation with (F.4) and (F.5), we conclude that

$$
P \left\{ \inf_{x \in S_{N-1}} \|Ax\| > 0 \right\} \geq 1 - C_3 \exp \left( -c_3 (pN)^{1/32} \right),$$

for some constants $C_3, c_3 > 0$. Thus, the proof is complete.

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