Holographic Fisher Information Metric in Schrödinger Spacetime

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Abstract

In this paper we study the Fisher information metric on the space of the coupling constants on both sides of the duality between non-relativistic dipole field theories and string theory in Schrödinger spacetime. We consider the following setup. In the gauge theory side one can deform a given conformal field theory by a proper scalar operator and compute the quantum information metric via the two-point correlation function between two such operators. On the string side the deformation corresponds to a scalar field probing the background. In the large $N$ limit of the theory the probing can be done without backreaction on the original spacetime, thus one can construct a perturbative scheme for the calculation of the dual holographic Fisher information metric as shown by [1]. Considering the asymptotic behaviour of the holographic Fisher information metric close to the boundary of the Schrödinger spacetime we show that its divergence structure exactly matches its dual quantum counterpart up to the leading order, thus extending the holographic setup up to the non-relativistic case. One should note that the existence of other terms are not seen from the boundary theory to this level of approximation. Their behaviour near the boundary however, is pointing what kind of information from the boundary theory is missing to be able to reconstruct the bulk. Obviously more work is needed to refine and elucidate their meaning and interrelations in holographic setup.
1 Introduction

The advancement of string theory and the discovery of the AdS/CFT correspondence [2] have brought us a great deal of understanding about the nature of different high-energy physics models and their intricate interrelations. One particularly useful property of this correspondence is that it can relate perturbatively computable characteristics of a higher dimensional string theory to the degrees of freedom of lower dimensional strongly correlated quantum field theory. This is also valid in the other direction, when the quantum system is weakly coupled and its dual gravitational counterpart is at strong coupling. In this context, the string/gauge duality opens a window to study non-perturbative phenomena with well-known analytical techniques.

Recent attempts to generalize the AdS/CFT correspondence to strongly coupled non-relativistic field theories [3, 4] have led to the construction of various classes of background solutions. Particularly interesting examples include the non-relativistic Schrödinger spacetimes, where the isometry group of the solutions on the string side is the Schrödinger group. It consists of time and space translations, space rotations, Galilean boosts, dilatations and special conformal transformations. The quantum duals to such models fall in the class of the so called dipole gauge theories, which are characterized with non-locality (see for instance [5–7]). For a detailed group-theoretical perspective on non-relativistic holography see [8].

An important understanding of non-relativistic holography has been revealed in [9], where strong arguments for integrability and quantitative matching between string and gauge theory predictions have been presented. These studies have lead to a number of interesting applications of non-relativistic holography in condensed matter physics and string theory such as the description of ordinary $\mathcal{N} = 1$ SQCD-like gauge theories considered in the context of D-brane constructions [10–16], the Sachdev-Ye-Kitaev (SYK) model [17], Fermi unitary gas [18], and models with trapped supercooled atoms [3, 19], which in most cases are strongly correlated. For these reasons, and the fact that currently very little is known for dipole theories, we are
motivated to investigate the properties of such holographic models on both sides of the correspondence. Further studies of non-relativistic holography in Schrödinger spacetime can be found in [20–27].

Recently, an interesting subject in holography gained popularity after the conjecture of Ryu-Takayanagi [28, 29], suggesting a holographic relation between quantum entanglement and codimension two extremal surfaces in the dual bulk gravitational theory. Consequent studies of complexity and related concepts [30,31] brought further interest to the information-theoretic analysis of holographic systems. One of the key constructions in these investigations turns out to be the quantum Fisher information metric (QFIM), which plays an important role not only in quantum information theory, but also in high-energy physics. As advocated in [32] and subsequent works, the CFT QFIM is approximately dual to the volume of a codimension one time slice in AdS space (see however [33,34] for a recent critique of this proposal), which on the other hand was also conjectured in [30,31] to give a measure of complexity of the system under consideration.

Further fruitful applications of QFIM and its holographic dual metric include phenomena and models such as quantum information scrambling [35–37], quantum metrology [38], canonical energy-momentum tensor [39], quantum phase transitions [40, 41], entanglement entropy [40, 42–44], bulk reconstruction [45, 46], instantons [47] and many others (see for instance [48–50]). Additionally, whenever it is possible to extrapolate QFIM to certain thermodynamic limits, one can use it to describe the distance between classically measurable statistical macrostates with various applications in condensed matter physics [51] and black hole physics (see for example [52–56]). This further motivates us to extend the information-theoretic analysis of the AdS/CFT correspondence to the non-relativistic holography in Schrödinger spacetimes.

In general, the concept of “information metric” is based on the purely geometric idea that one can construct a well defined Riemannian distance (metric) [57,58] between various micro- or macrostates of the system. Due to the fact that geometry studies the mutual relations between elements, such as distance and curvature, one can naturally uncover essential features and gain valuable insights of the system under consideration. This is strongly evident in holography, where classical space-time geometry has the capacity to encode important properties of the dual quantum system.

In this paper, we show that within the context of non-relativistic holography in Schrödinger spacetime, the Fisher information metric (FIM) on the space of coupling constants of the model can be explicitly calculated on both sides of the correspondence. This will allow us to make quantitative and qualitative checks of the duality. Our investigation is based on similar studies, conducted in [1] and [59]. A perturbative scheme for computing the holographic Fisher information metric (HFIM) on the string side has been presented for AdS space. We also show that in certain limits HFIM in Schrödinger background fully reproduces the QFIM on the lower dimensional boundary of the spacetime, thus confirming the duality in this case.

The structure of this paper is as follows. In Section 2 we present the computational setup for the QFIM and its dual bulk counterpart, HFIM. In Section 3 we compute QFIM on the dipole gauge theory side via the 2-point correlation function between primary operators deforming the original conformal field theory. In Section 4 we use a perturbative method to explicitly calculate the dual HFIM in the bulk of the Schrödinger spacetime. In Section 5 we show how the leading divergence structure of HFIM on the gravity side reduces to QFIM on the lower dimensional boundary of the Schrödinger spacetime. In Section 6 we analyse the divergence structure of HFIM along the holographic direction. In Section 7 we give a short review of our results. At the end of the paper we present some Appendices with detailed calculations of the QFIM and HFIM integrals, where novel results for the incomplete gamma functions have been obtained.
2 Computational setup

In this section we briefly discuss the computational techniques used to calculate the Fisher information metric on both sides of the correspondence. We mainly follow the presentation of [1] and [60], where a general CFT on an Euclidean $\mathbb{R}^D = \mathbb{R}^d \times \mathbb{R}^1$ space is considered.

2.1 CFT construction

Let us start by considering an initially undeformed conformal field theory (CFT), living on an Euclidean $\mathbb{R}^D = \mathbb{R}^d \times \mathbb{R}^1$ space and described by an Euclidean Lagrangian $\mathcal{L}_0$, defined for Euclidean time $\tau \in (-\infty, 0)$. Consequently, we perturb $\mathcal{L}_0$ at $\tau = 0$ by some quantum operators $\mathcal{O}_a(\tau), a = 1, \ldots, n$, to a new theory with $\mathcal{L}_1$ for $\tau > 0$, i.e. $\mathcal{L}_1 = \mathcal{L}_0 + \delta \lambda^a \mathcal{O}_a$, where $\delta \lambda^a$ are given real coupling constants. By definition, the quantum information metric $\mathcal{G}_{ab}$, between the ground state $|\psi_0\rangle$ of the undeformed CFT and the ground state $|\psi_1\rangle$ of the deformed theory, can be obtained by expanding the quantum fidelity at temporal infinities in powers of $\delta \lambda$

$$\mathcal{F}(\lambda^a, \lambda^b + \delta \lambda^b) \equiv |\langle \psi_1(\tau \rightarrow \infty, \lambda^a + \delta \lambda^b)|\psi_0(\tau \rightarrow -\infty, \lambda^a)|^2 = 1 - \sum_{a,b=1}^n G_{ab} \delta \lambda^a \delta \lambda^b + \mathcal{O}(\delta \lambda^3),$$

where one can show that [1, 60]

$$G_{ab} = \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^\infty \eta_0 \int_0^\infty \int_0^\infty \langle \mathcal{O}_a(\tau_1, x_1) \mathcal{O}_b(\tau_2, x_2) \rangle - \langle \mathcal{O}_a(\tau_1, x_1) \rangle \langle \mathcal{O}_b(\tau_2, x_2) \rangle. \quad (2.2)$$

In order to arrive at this result one has to refer to the path-integral formalism, where the overlap $\langle \psi_1|\psi_0 \rangle$ can be written in the following way

$$\langle \psi_1|\psi_0 \rangle = \int_{\varphi(\tau=0)=\hat{\varphi}} \mathcal{D}\varphi \langle \psi_1 | \varphi \rangle \langle \varphi | \psi_0 \rangle = \frac{1}{\sqrt{Z_0 Z_1}} \int \mathcal{D}\varphi e^{-\int \mathcal{L}_0 + \frac{1}{2} \int \mathcal{L}_0 + \delta \lambda^a \mathcal{O}_a(\tau, x)} \int \mathcal{D}\varphi e^{-\int \mathcal{L}_0 + \delta \lambda^a \mathcal{O}_a(\tau, x)} \right]. \quad (2.3)$$

Here $|\varphi\rangle$ is a generic state, while $|\hat{\varphi}\rangle = |\varphi(\tau=0)\rangle$ is inserted at $\tau = 0$. The function $Z_0$ is the partition functions of the original theory,

$$Z_0 = \int \mathcal{D}\varphi e^{-\int \mathcal{L}_0}, \quad (2.4)$$

1We will refer to $t$ as the real time and $\tau = -it$ as the Wick-rotated time.

2In general, we consider the situation where the original Lagrangian $\mathcal{L}_0$ has an explicit dependence on a number of physical parameters $\lambda^a, a = 1, \ldots, n$, thus one can deform the theory by multiple operators with corresponding real coupling constants $\lambda^a \rightarrow \lambda^a + \delta \lambda^a$. However, it will be qualitatively informative to consider only one such deformation.

3In general, the complex quantity $G_{ab}$ is called the quantum geometric tensor. Its real part is the quantum information metric and its imaginary part is known as the Berry curvature. Usually, the quantum Fisher information metric is defined to be 4 times the quantum information metric, $F_{ab} = 4\mathcal{G}(G_{ab})$. However, with a slight abuse of notation, we will follow the terminology established by [1] and call $G_{ab}$ the quantum information metric or QIM.

4Fidelity is a quantum measure, which specifies the degree of change occurring in the system by turning on the deformations. Let us note that there is also another definition, $\mathcal{F} = |\langle \psi|\xi\rangle| = \sqrt{\mathcal{F}}$, which is sometimes referred to as quantum fidelity as well. However, $\mathcal{F}$ is more common, while $\mathcal{F}$ could be referred to as the square root fidelity.
while $Z_1$ represents the partition function of the deformed theory for $\tau \in (-\infty, \infty)$,

$$Z_1 = \int \mathcal{D}\varphi e^{\int_{-\infty}^{\infty} d\tau \int d^d x (L_0 + \delta \lambda^a O_a)}.$$  

(2.5)

Now one can take the squared overlap $|\langle \psi_1 | \psi_0 \rangle|^2$ and expand it in powers of the couplings $\delta \lambda^a$ to obtain Eq. (2.2), as shown in details in Appendix B.

When considering quantum field theories in general, the overlap (2.3) could be ill defined. Since the Lagrangian, governing the Euclidean propagation, changes discontinuously at $\tau = 0$, an UV divergence appears. As suggested in [1], one should renormalize the theory replacing for example $|\psi_1 \rangle$ by $|\psi_1(\epsilon) \rangle = e^{-\epsilon H_0} |\psi_1 \rangle / (\langle \psi_1 | e^{-2\epsilon H_0} |\psi_1 \rangle)^{1/2}$.

(2.6)

Here $H_0$ is the Euclidean Hamiltonian of the unperturbed theory and Eq. (2.3) now becomes

$$\langle \psi_1(\epsilon) | \psi_0 \rangle = \left( \frac{\exp \left( - \int_{-\epsilon}^{\epsilon} d\tau \int d^d x \delta \lambda^a O_a \right)}{\exp \left( - \left( \int_{-\epsilon}^{\epsilon} d\tau \int d^d x \delta \lambda^a O_a \right) \right)} \right)^{1/2}.$$  

(2.7)

The expectation values are taken with respect to the unperturbed state $|\psi_0 \rangle$. Consequently, equation (2.2) changes upon expansion of (2.7) in powers of $\delta \lambda$ by the cutoff $\epsilon$ replacing the infinities along $^5 \tau$. In order to be consistent with the original derivation of Eq. (2.2) in the context of gauge/gravity correspondence, the author of [1] considered CFT operators of non-zero dimension. In this case, their vacuum expectation value is $\langle O_a \rangle = 0$. Therefore, the formula for the quantum information metric (2.2) reduces to

$$G_{ab} = \frac{1}{2} \int_{V_{d+1}} d^d x_1 \int_{V_{d+1}} d^d x_2 \int_{-\epsilon}^{\epsilon} d\tau_1 \int_{-\epsilon}^{\epsilon} d\tau_2 \langle O_a(\tau_1, x_1) O_b(\tau_2, x_2) \rangle.$$  

(2.8)

Finally, one can define the quantum Fisher information metric by the real part of QIM, $F_{ab} = 4 \text{Re}(G_{ab})$, while its imaginary part corresponds to the Berry curvature $B_{ab} = 3(G_{ab})$.

The derived formula (2.8) for QIM on the CFT side of the correspondence will be the relevant expression for our study of non-relativistic Schrödinger holography throughout this paper.

### 2.2 Bulk holographic construction

Early calculations of the Fisher information metric on the gravity side with exactly marginal operators of the deformation have been conducted in [32,61]. Bulk holographic study for AdS$_{d+1}$ space lead to the development of a specific perturbative method [1], which allows us to deal with any primary operator with conformal dimension $\Delta$, provided that

$$2\Delta > d + 1.$$  

(2.9)

The basic idea of the suggested technique is to reinterpret the right hand side of equation (2.3) as a combination of partition functions, namely

$$\langle \psi_1 | \psi_0 \rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}}.$$  

(2.10)

$^5$The presence of the cutoff $\epsilon$ around $\tau = 0$ is necessary to address any ultraviolet divergences in case there is a discontinuity, when passing from the original to the deformed Lagrangian.
were $Z_2$ is the partition function for the deformed theory only for $\tau > 0$. The latter has been explicitly defined in the numerator of (2.3). Now, one can consider the large $N$ limit on the gravity side, where $Z_k = e^{-I_k}$, $k = 0, 1, 2$, with $I_k$ being the on-shell action of the gravity solution dual to the corresponding field theory configuration. Therefore, one can write

$$\langle \psi_1|\psi_0 \rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}} = e^{\frac{1}{2}(I_0 + I_1) - I_2}. \quad (2.11)$$

The deformation of the CFT by a single primary $O$ induces an interaction term $\delta \lambda O$ with a coupling $\delta \lambda$ in the CFT Lagrangian, thus changing the initial dual bulk gravitational action $I_0$ by $I_k = I_0 + \delta I_k(\delta \lambda)$ with $\delta I_0 = 0$. Hence, one has to compute

$$\langle \psi_1|\psi_0 \rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}} = e^{\frac{1}{2}(I_0 + I_1) - I_2} = e^{\frac{1}{2}(I_0 + I_0 + I_2 + I_2) - I_2} = e^{\frac{1}{2}I_2 - I_2}. \quad (2.12)$$

As a result, the final computation reduces to finding the variations $\delta I_1$ and $\delta I_2$ of the on-shell gravitational action. Let us assume that the bulk spacetime dynamics is governed by the action

$$I = -\frac{1}{k^2} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left( \frac{1}{2} (R - 2\Lambda) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 + V(\phi) \right) + I_{\partial \mathcal{M}}, \quad (2.13)$$

where $\phi(x)$ is a massive scalar field probing the background geometry $g_{\mu\nu}(x)$. The field $\phi(x)$ attains different profiles for $k = 0, 1, 2$. In particular, for the computation of $Z_0$, we notice that the massive field is turned off ($\phi_0 = 0$), thus the dual solution is the initial background geometry with metric $g_{\mu\nu}^{(0)}$, i.e. $Z_0 = \exp \left( -I_0[\phi_0, g_{\mu\nu}^{(0)}] \right)$. On the other hand, the profiles $\phi_{1, 2}(x)$ of the scalar field for $I_1$ and $I_2$ will in general be spacetime dependent and can be calculated by the corresponding bulk-to-boundary propagator. This is shown explicitly in the case of Schrödinger spacetime in Section 4 and Appendix D. Since we are interested in perturbative solutions in lower powers of $\delta \lambda$, we employ the following transformations of the fields

$$\phi(x) = \phi_0(x) + \varphi(x) \delta \lambda, \quad g_{\mu\nu}(x) = g_{\mu\nu}^{(0)}(x) + h_{\mu\nu}(x) \delta \lambda^2. \quad (2.14)$$

Here $\varphi(x)$ and $h_{\mu\nu}(x)$ are the corrections to the scalar field and the bulk metric acquired after turning on the deformation in the dual CFT. Notice that the metric receives corrections at order $\delta \lambda^2$ since the scalar field enters quadratically in the Einstein field equations. Hence, the variation of the bulk action $\delta I$ now can be computed in powers of $\delta \lambda$,

$$\delta I = I[\phi, g_{\mu\nu}] - I_0[\phi_0, g_{\mu\nu}^{(0)}] = I[\phi_0(x) + \varphi(x) \delta \lambda, g_{\mu\nu}^{(0)}(x) + h_{\mu\nu}(x) \delta \lambda^2] - I_0[\phi_0, g_{\mu\nu}^{(0)}]$$

$$= \delta \lambda \int \frac{\delta I}{\delta \phi(x)} \bigg|_{\delta \lambda = 0} \varphi(x) + \delta \lambda^2 \int \frac{\delta^2 I}{\delta \varphi(x) \delta \varphi(y)} \bigg|_{\delta \lambda = 0} \varphi(x) \varphi(y)$$

$$+ \delta \lambda^2 \int \frac{\delta I}{\delta g_{\mu\nu}(x)} \bigg|_{\delta \lambda = 0} h_{\mu\nu}(x) + O(\delta \lambda^3) \approx \delta \lambda^2 \int \frac{\delta^2 I}{\delta \phi(x) \delta \phi(y)} \bigg|_{\delta \lambda = 0} \varphi(x) \varphi(y). \quad (2.15)$$

The first and the last term vanish due to the field equations of motion. Higher order contributions are not taken into account, due to the fact that we are working in the probe limit and backreaction on the background geometry is considered negligible. Hence, we are in a situation where the scalar field probes the unperturbed background. Notice also that the boundary term of equation (2.13) cancels by the boundary terms coming from the integration by parts in obtaining the first and the third terms of Eq. (2.15). Therefore, we can write

$$\delta I_k = \frac{1}{2k^2} \int d^{d+1} x \sqrt{|g|} \left( g_{\mu\nu}^{(0)} \partial_\mu \phi_k \partial_\nu \phi_k + m^2 \phi_k^2 \right) = \frac{1}{2k^2} \int d^{d} x \sqrt{|\gamma|} \ n_\mu g_{\mu\nu}^{(0)} \phi_k \partial_\nu \phi_k, \quad (2.16)$$
with \( n_\mu \) being the unit normal vector and \( \gamma \) being the determinant of the induced metric on the boundary. Furthermore, \( \phi_k(x) \) are the scalar field configurations dual to the operators of the corresponding deformed and undeformed CFTs, while probing the fixed background \( g^{(0)}_{\mu\nu} \). Clearly, one can obtain these profiles by using the boundary-to-bulk propagator, which we will show in the following sections. The next step is to write the overlap (2.12) as

\[
\langle \psi_1 | \psi_0 \rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}} = \exp \left( \frac{1}{2} \delta I_1 - \delta I_2 \right) .
\]  

(2.17)

Finally, after expanding the exponent in (2.17) up to first order in \( \delta I_{1,2} \) and comparing to (2.1), one finds an expression for the holographic information metric

\[
G_{\lambda\lambda} = - \frac{1}{\delta \lambda^2} \left( \frac{\delta I_1}{2} - \delta I_2 \right) .
\]  

(2.18)

The real part of this expression leads to the holographic Fisher information metric (HFIM)

\[
F_{\lambda\lambda} = 4 \Re(G_{\lambda\lambda}) .
\]  

(2.19)

In what follows, we are going to calculate the quantum Fisher information metric (QFIM) and its dual holographic counterpart HFIM for a holographic system in Schrödinger spacetime, according to the computational procedures presented in this section.

### 3 Dual CFT quantum Fisher information metric

As is evident from Eq. (2.8), in order to find the QFIM, we must compute the two-point correlation function of operators with dimensions \( \Delta \) from the dual conformal field theory. In many cases, this can be achieved within the framework of the gauge/gravity correspondence. One has to look at the dynamics of a massive scalar field (2.13) propagating in the bulk geometry from the gravity side.

In our case, we consider a strongly coupled \((d+1)\)-dimensional conformal field theory with non-relativistic invariance\(^6\). It is assumed to be dual to a \((d+3)\)-dimensional Schrödinger spacetime with Lorentzian line element given by [3,4]

\[
ds^2_{\text{Schr},d+3} = L^2 \left( -\frac{dt^2}{r^4} + \frac{2d\xi dt + d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} \right).
\]  

(3.1)

The boundary of the background (3.1) is at \( r = 0 \) and the generator associated with translations along the compact \( \xi \) direction can be identified with the mass operator \( M = i \partial_\xi \). The latter is not a geometric dimension in the usual sense. Each operator of the boundary theory can have a fixed momentum (‘particle number’) conjugate to \( \xi \), where the compactification of \( \xi \) is usually taken so that the spectrum of possible momenta is discrete. Hence, the effective dimension of the boundary CFT has to be \((d+1)\), which is coordinazied by \((t, \vec{x})\) or \((\tau, \vec{x})\) in the Euclidean case.

Now, the 2-point function between primary operators from the boundary gauge theory with conformal dimension \( \Delta \) can be computed by a standard holographic procedure implemented in [62,63]. The explicit form of the correlator is

\[
A_{12} = \langle O(\tau_2, \vec{x}_2) O(\tau_1, \vec{x}_1) \rangle = \Delta c_\Delta \theta(\tau_2 - \tau_1) \left( \frac{1}{\tau_2 - \tau_1} \right)^\Delta \exp \left( i \frac{M(1 + i\varepsilon)(\vec{x}_2 - \vec{x}_1)^2}{2(\tau_2 - \tau_1)} \right) ,
\]  

(3.2)

\(^6\)It is conjectured to be a specific non-local field theory, namely dipole field theory (see Appendix A for more information).
where $c_\Delta$ is a normalization constant,

$$c_\Delta = \frac{\pi^{d/2} \Gamma \left( \Delta - \frac{d}{2} - 1 \right)}{\pi^{d/2} \Gamma \left( \Delta - \frac{d}{2} - 1 \right)},$$  

(3.3)

and $\Delta$ the conformal dimension, given by

$$\Delta = 1 + \frac{d}{2} + \sqrt{\left( 1 + \frac{d}{2} \right)^2 + m^2 + M^2}.$$  

(3.4)

Here, $d$ is the dimension of $\vec{x}$ space, $\theta(\tau_2 - \tau_1)$ is the unit step function for an Euclidean time interval and $M$ is a quantized momentum along the compact direction $\xi$ with radius $1/M$. We will consider the case $\tau_1 \leq \tau_2$, thus QIM for a single marginal deformation should be given by

$$G^{(CFT)}_{\lambda\lambda} = \frac{1}{2} \int d^d x_1 \int d^d x_2 \int d\tau_1 \int d\tau_2 \langle O(\tau_1, x_1) O(\tau_2, x_2) \rangle,$$  

(3.5)

where we have also taken into account that $\langle O \rangle = 0$ for an operator of non-zero dimension. Explicitly, one has

$$G^{(CFT)}_{\lambda\lambda} = \frac{\Delta c_\Delta}{2} \int d^d x_1 \int d^d x_2 \int d\tau_1 \int d\tau_2 \frac{e^{\frac{\Delta (1+i\epsilon)}{2} \frac{(\vec{x}_2 - \vec{x}_1)^2}{\tau_2 - \tau_1}}}{(\tau_2 - \tau_1)^\Delta},$$  

(3.6)

where $\epsilon$ is a regulator near $\tau = 0$, which is different from the regulator $\varepsilon$ in the correlation function. The integral over $\vec{x}$ space is Gaussian and can be easily computed by Eq. (C.3)

$$I_x = \int d^d x_1 \int d^d x_2 e^{\frac{M(1+i\epsilon)}{2} \frac{(\vec{x}_2 - \vec{x}_1)^2}{\tau_2 - \tau_1}} = e^{\frac{i\pi}{2} \left( 1 - \frac{d}{2} \right)} (2\pi)^\frac{d}{2} M^{\frac{d}{2}} (1 + i\epsilon)^{-\frac{d}{2}} (\tau_2 - \tau_1)^\frac{d}{2} V_{\mathbb{R}^d},$$  

(3.7)

where $V_{\mathbb{R}^d}$ is the volume of $\mathbb{R}^d$ space. The integrals over $\tau_1$ and $\tau_2$ now take the form

$$I_t = \int d\tau_1 \int d\tau_2 (\tau_2 - \tau_1)^\frac{d}{2} = \frac{2^{\frac{d}{2} - \Delta + 4}}{(d - 2\Delta + 2)(d - 2\Delta + 4)} e^{\frac{d^2}{2} - \Delta},$$  

(3.8)

with convergence condition given by

$$2\Delta > d + 4,$$  

(3.9)

which falls within the range specified in Eq. (2.9). Therefore, the QIM in the dual to the Schrödinger spacetime dipole CFT is given by

$$G^{(CFT)}_{\lambda\lambda} = \frac{\Delta 2^{d - 2\Delta + 3} e^{-\frac{\pi d}{4} (d + 2\Delta) M^{\Delta - \frac{d}{2}} - 1 V_{\mathbb{R}^d}}}{(d - 2\Delta + 4) \Gamma \left( \Delta - \frac{d}{2} \right)} e^{\frac{d^2}{2} - \Delta}.$$  

(3.10)

We can now safely turn off the regulator $\varepsilon = 0$, which is equivalent to switching back to real time $t$ [62], hence

$$G^{(CFT)}_{\lambda\lambda} = C e^{1 - a},$$  

(3.11)

where $C$ is a constant normalization factor given by

$$C = \frac{\Delta 2^{d - 2\Delta + 3} e^{-\frac{\pi d}{4} (d + 2\Delta) M^{\Delta - \frac{d}{2}} - 1 V_{\mathbb{R}^d}}}{(d - 2\Delta + 4) \Gamma \left( \Delta - \frac{d}{2} \right)}.$$  

(3.12)
Here, we have also defined the parameter
\[ a = \Delta - \frac{d}{2} - 1, \tag{3.13} \]
to better outline the divergence structure of the QIM. Since the divergence parameter \( \epsilon \) is a real parameter, the only difference between QIM and QFIM resides only in their normalization constants,
\[ F_{\lambda\lambda} = 4\Re(G_{\lambda\lambda}) = 4\Re(C)\epsilon^{1-a} = F\epsilon^{1-a}. \tag{3.14} \]

We can now proceed with the computation of the holographic Fisher information metric in the dual gravitational theory.

## 4 Bulk holographic Fisher information metric

We focus on the dynamics of the massive scalar field \( \phi(x) \) on the gravity side. In our setup we have the following picture. The initial theory is a CFT \( \mathcal{L}_0 \) with primary operators dual to a probe scalar field \( \phi_0(x) \) in the bulk defined for \( \tau \in (-\infty, +\infty) \). On the other hand, for the same \( \tau \in (-\infty, +\infty) \), there is another CFT with Euclidean Lagrangian \( \mathcal{L}_1 = \mathcal{L}_0 + \delta \lambda \mathcal{O} \), which is a deformation of the original theory by \( \delta \lambda \mathcal{O} \). Its operator content is dual to a new probe scalar field \( \phi_1(x) \) in the bulk. Finally, we consider a third theory with \( \mathcal{L}_2 = \mathcal{L}_1 \), defined only for \( \tau > 0 \). It is produced by deforming the initial CFT at an Euclidean time \( \tau = 0 \) and its content is dual to a bulk scalar field \( \phi_2(x) \).

As a consequence of Eq. (2.18) the computation of the holographic Fisher information metric requires the profiles only for \( \phi_1(x) \) and \( \phi_2(x) \). These solutions can be easily found by the method of Green’s functions (bulk-to-boundary propagators). In this case, the bulk-to-boundary propagator \( K(r, \vec{x}, \tau; \vec{x}_1, \tau_1) \) in Schrödinger spacetime is given by [62]

\[ K(r, \vec{x}, \tau; \vec{x}_1, \tau_1) = c_\Delta \theta(\tau - \tau_1) \frac{r^{\Delta}}{(\tau - \tau_1)^{\Delta}} \exp \left( i M r^2 + (\vec{x} - \vec{x}_1)^2 \frac{\tau}{\tau_1} \right), \tag{4.1} \]

where the normalization constant \( c_\Delta \) has been defined in Eq. (3.3). We choose to perform the integration over \( x_1 \) and the Euclidean time \( \tau_1 \) for \( \tau_1 \leq \tau \). The detailed computation has been presented in Appendix D. Hence, one can write the solution for \( \phi_1(x) \) in the deformed theory as

\[ \phi_1(r, \vec{x}, t) = \delta \lambda \int d^d x_1 \int_{-\infty}^{\tau} dr_1 K(r, \vec{x}, \tau; \vec{x}_1, \tau_1) \hat{\phi}_0(\vec{x}_1, \tau_1) = i \delta \lambda e^{-\frac{\pi M}{2}} r^{\Delta+2}. \tag{4.2} \]

It is valid only if the following convergence conditions for the integrals hold

\[ a > 0, \quad M > 0, \tag{4.3} \]

where \( a \) is given by Eq. (3.13) and \( M \) is the non-relativistic momentum associated to the compact Killing direction \( \xi \). The convergence condition \( a > 0 \) is weaker than (3.9), which translates to \( a > 1 \). However matching HFIM to QFIM, as shown in Section 5, requires \( a > 1 \) in the

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7Note that this picture is a bit different from the one in Section (2.1). When considering the dual gravitational theory we have to consider the full range of the CFT theory \( \mathcal{L}_0 \) for \( \tau \in (-\infty, +\infty) \). The same is valid also for the deformed theory with \( \mathcal{L}_1 \), which now should also be considered for \( \tau \in (-\infty, +\infty) \). Finally, a new theory with \( \mathcal{L}_2 = \mathcal{L}_1 \) only for \( \tau \in (0, +\infty) \) is required for consistency of the perturbative bulk holographic method.

8The source function \( \hat{\phi}_0(\vec{x}_1, \tau_1) = 1 \), which can be seen from the asymptotic expansion of the Bessel functions, defining the propagator near the boundary at \( r = 0 \), see [4].
bulk as well. One also notes that $\phi_1(r, \vec{x}, \tau) \equiv \phi_1(r)$ now becomes a function only of $r$, which considerably simplifies the subsequent computation of the HFIM. Furthermore, the choice of $\tau \to +\infty$ as an upper limit of the integral does not change the final result in (4.2).

The field $\phi_2$, for the deformed theory at $\tau > 0$, is given by

$$\phi_2(r, \vec{x}, t) = \delta \lambda \int d^d x_1 \int_0^\tau d\tau_1 K(r, \vec{x}, \tau; \vec{x}_1, \tau_1) \hat{\phi}_0(\vec{x}_1, \tau_1) = \frac{i \delta \lambda e^{-\frac{\epsilon \gamma}{2}}}{\Gamma(a)} r^{d-\Delta+2} \Gamma \left( a, \frac{\mu}{t} \right),$$

(4.4)

where in the final step we have restored the real time $\tau \to -it$, so that subsequent calculations with the incomplete gamma functions will attain the correct properties. The convergence condition for $\phi_2$ is $a > 0$ and we have also introduced the real parameter

$$\mu = \frac{Mr^2}{2}.$$  

(4.5)

This parameter is a convenient choice for a regulator along the holographic coordinate $r$, which we will employ in the sections below.

With the profiles of the fields $\phi_{1,2}$ at hand we can proceed with the computation of the HFIM integrals according to (2.16), adopted for the Schrödinger spacetime (3.1),

$$\delta I_k = \frac{1}{2\kappa r^{d-\Delta+2}} \lim_{\epsilon \to 0} \int d^d x \left( \int_{-\tau}^{-\epsilon} dt \sqrt{\gamma} n_\mu g^{\mu\nu} \phi_k \partial_\nu \phi_k + \int_\epsilon^\tau dt \sqrt{\gamma} n_\mu g^{\mu\nu} \phi_k \partial_\nu \phi_k \right).$$

(4.6)

Here $\gamma$ is the metric on the boundary, $\xi > 0$ is the regulator in the holographic direction $r$ near the boundary $r \to 0$, $n_\mu$ is the normal outward vector to the boundary, $\epsilon > 0$ is a regulator around $t = 0$, and $T$ is a cut-off at temporal infinity. On the boundary only the component $n_r$ is non-zero, thus $\sqrt{\gamma} n_r g^{rr} = L^d r^{1-d}$ (see Appendix D.2). However, one has to account for the fact that $\phi_1$ and $\phi_2$ span different ranges along $t$, which leads to

$$\delta I_1 = \frac{L^d V_{\text{red}}}{2\kappa} \lim_{\epsilon \to 0} \int_{-\tau}^{-\epsilon} dt \phi_1(r) \partial_t \phi_1(r) + \int_\epsilon^\tau dt r^{-1-d} \phi_1(r) \partial_r \phi_1(r),$$

(4.7)

$$\delta I_2 = \frac{L^d V_{\text{red}}}{2\kappa} \lim_{\epsilon \to 0} \int_\epsilon^\tau dt r^{-1-d} \phi_2(t, r) \partial_r \phi_2(t, r).$$

(4.8)

Now, Eq. (2.18) for the holographic Fisher information metric leads to the expression

$$G^{\text{Bulk}}_{\lambda\lambda} = -\frac{1}{\delta \lambda^2} \left( \frac{\delta I_1}{2} - \delta I_2 \right) = \frac{L^d V_{\text{red}}}{2\kappa} \lim_{\epsilon \to 0} \left( c_0 J_0 r^{d-2\Delta+2} + c_1 J_1 + c_2 J_2 r^{d-2\Delta+2} \right),$$

(4.9)

with the following integrals

$$J_0 = \int_{-\tau}^{-\epsilon} dt + \int_\epsilon^\tau dt = 2(T - \epsilon),$$

(4.10)

$$J_1 = \int_\epsilon^\tau dt t^{-a} e^{-\frac{\epsilon}{t}} \Gamma \left( a, \frac{\mu}{t} \right) = \int_{1/\epsilon}^\tau dx x^{-a-2} e^{-\mu x} \Gamma(a, \mu x),$$

(4.11)

$$J_2 = \int_\epsilon^\tau dt \Gamma^2 \left( a, \frac{\mu}{t} \right),$$

(4.12)
and the corresponding coefficients
\[ c_0 = \frac{e^{-i\pi d}}{2}(d - \Delta + 2), \quad c_1 = \frac{M^a e^{-i\pi d}}{2a - 1} \Gamma^2(a), \quad c_2 = -\frac{2c_0}{\Gamma^2(a)}. \] (4.13)

The non-trivial solutions to these integrals are presented in Appendix D. The final expression for the HFIM yields\(^9\)
\[
G_{\lambda\lambda}^{Bulk} = \frac{a_0}{\mu^a} (T - \epsilon) + a_1 \frac{T^{2-a} e^{-\frac{\mu}{T}}}{\mu} \Gamma(a - 2, \frac{\mu}{T}) + \frac{a_2}{\mu^{a-1}} \Gamma^2(a - 2, \frac{\mu}{T}) + \frac{a_3}{\mu^{a-1}} \Gamma^2\left(a - 1, \frac{\mu}{T}\right) + \frac{a_4}{\mu^{a-1}} \Gamma^2\left(a - 2, \frac{\mu}{\epsilon}\right) + \frac{a_5}{\mu^2} \Gamma^2\left(a - 3, \frac{\mu}{\epsilon}\right) + \frac{a_6}{\mu^2} \Gamma^2\left(a - 2, \frac{\mu}{\epsilon}\right).
\] (4.14)

This result is equpped with three different divergences near the Schrödinger boundary at \( r = 0 \), namely \( \epsilon \to 0, T \to \infty \) and \( \mu \sim \epsilon^2 \to 0 \). In the bulk spacetime the parameter \( \mu \) is finite, except at \( \mu \to \infty \), as schematically depicted on Fig. 1. Furthermore, one notes the relation \( a_i = -b_i \) between the coefficients \( a_i \) and \( b_i, i = 1, \ldots, 7 \), where
\[
\begin{align*}
  a_0 &= \frac{L^d V_{Rd}}{2a + 1} M^a \left[ 2c_0 + c_2 \left( \Gamma^2(a) - 2 \Gamma(2a) B_{1/2}(a, a) \right) \right], \\
  a_1 &= -b_1 = \frac{L^d V_{Rd}}{2\kappa } c_1(a - 1)(a - 2), \\
  a_2 &= -b_2 = \frac{L^d V_{Rd}}{4\kappa } c_1(a - 1)(a - 2)^2, \\
  a_3 &= -b_3 = -\frac{L^d V_{Rd}}{22a - 3\kappa } c_1(a - 1)(a - 2), \\
  a_4 &= -b_4 = \frac{L^d V_{Rd}}{23a - 1\kappa } (2a c_1 - 2c_2 M^a), \\
  a_5 &= -b_5 = \frac{L^d V_{Rd}}{22a - 2\kappa } c_1(a - 1), \\
  a_6 &= -b_6 = \frac{L^d V_{Rd}}{2a + 1\kappa } M^a c_2, \\
  a_7 &= -b_7 = \frac{L^d V_{Rd}}{2a + 1\kappa } M^a c_2(a - 1),
\end{align*}
\] (4.15)

and \( B_{1/2}(a, a) \) is the incomplete beta function. We have explicitly kept the different names for the coefficients \( a_i \) and \( b_i \) to reflect the different asymptotic behaviour of the arguments of the incomplete gamma functions.

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\(^9\)In this section we prefer to directly call \( G_{\lambda\lambda}^{Bulk} \) HFIM, which in the correct asymptotic limits will be shown to reduce to QIM and thus equivalently to QFIM.
Figure 1: A schematic depiction of the Schrödinger spacetime and its boundary along the holographic direction $r$ ($\mu \sim r^2$). We have three important sectors capturing different divergence structures. To the left is the $(d+1)$-dimensional boundary at $r = 0$, where the dual CFT lives. The middle region defines the $(d+3)$-dimensional bulk of the considered space, where $r \neq 0$ is finite, and to the right is the limit $r \to \infty$.

5 Reduction of HFIM to the dual QFIM on the boundary

In order to compare HFIM from Eq. (4.14) to its CFT dual from Eq. (3.11) we have to consider the divergence structure of HFIM near the boundary at $r = 0$. In this case, we have effectively two competing divergences, namely $\mu \to 0$ when approaching $r = 0$ along $r$, and $\epsilon \to 0$ along $t$ near $t = 0$. Therefore, we can look at different situations, e.g. one in which $\mu$ goes to zero asymptotically faster than $\epsilon$, and the other case, where $\epsilon$ goes faster to zero than $\mu$. In both cases we have two possibilities for the cut-off $T$, i.e. $T \to \infty$ or finite $T$. Let us rewrite HFIM from Eq. (4.14) in the form

$$G_{\lambda \lambda} = G_{\lambda \lambda}^{\text{Bulk}} \epsilon^{a-1} = a_0 \frac{T \epsilon^{a-1} \mu^a}{\mu^a} - a_0 \frac{\epsilon^a}{\mu^a}$$

$$+ a_1 \frac{T^{2-a} \epsilon^{-2} \mu^{a-1}}{\mu^a} \Gamma(a - 2, \frac{\mu}{T}) + a_2 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma^2(a - 2, \frac{\mu}{T})$$

$$+ a_3 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma(2a - 4, \frac{2\mu}{T}) + a_4 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma(2a - 2, \frac{2\mu}{T})$$

$$+ a_5 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma(2a - 3, \frac{2\mu}{T}) + a_6 \frac{T \epsilon^{a-1} \mu^a}{\mu^a} \Gamma^2(a, \frac{\mu}{T}) + a_7 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma^2(a - 1, \frac{\mu}{T})$$

$$+ b_1 \frac{e^{-\mu}}{\mu} \epsilon \Gamma(a - 2, \frac{\mu}{\epsilon}) + b_2 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma^2(a - 2, \frac{\mu}{\epsilon})$$

$$+ b_3 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma(2a - 4, \frac{2\mu}{\epsilon}) + b_4 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma(2a - 2, \frac{2\mu}{\epsilon})$$

$$+ b_5 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma(2a - 3, \frac{2\mu}{\epsilon}) + b_6 \frac{\epsilon^a \mu^a}{\mu^a} \Gamma^2(a, \frac{\mu}{\epsilon}) + b_7 \frac{\epsilon^{a-1} \mu^{a-1}}{\mu^a} \Gamma^2(a - 1, \frac{\mu}{\epsilon}) \frac{\epsilon}{\mu}.$$ (5.1)

Our goal is to compare its divergence structure to the corresponding divergences in the dual metric from Eq. (3.11), which we write here again

$$G_{\lambda \lambda}^{(\text{CFT})} = C \epsilon^{1-a}, \quad a > 1,$$ (5.2)

with $C$ given in (3.12). We now show that there exists a certain well defined limit at which HFIM asymptotically matches $G_{\lambda \lambda}^{(\text{CFT})}$ on the boundary of the spacetime.
5.1 Dominant $\epsilon$

Let us first consider $\epsilon$ approaching zero sufficiently faster than $\mu \sim \tilde{\epsilon}^2$. In this case, the dominant regulator is given by $\epsilon$, thus one can use the asymptotic expansion

$$
\Gamma(a, z) \sim z^{a-1} e^{-z} \sum_{k=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a - k)} z^{-k}, \quad |z| \to \infty,
$$

which defines the explicit divergence structure of the $b_i$ terms. On the other hand, the asymptotic expansion for the $a_i$ terms is governed by

$$
\Gamma(a, z) = \Gamma(a), \quad z \to 0, \quad a > 0.
$$

Therefore, Eq. (5.1) acquires the following asymptotic form near $r = 0$:

$$
\mathcal{G}_{\lambda\lambda} = (a_0 + a_6 \Gamma^2(a)) \frac{T \epsilon^{a-1} \mu^a}{\mu^a} - a_0 \left( \frac{\epsilon}{\mu} \right)^a + a_1 \Gamma(a-2) \frac{T^{2-a} \mu^{-\frac{\epsilon}{2}} \epsilon^{a-1}}{\mu}
$$

$$
+ b_1 e^{-\frac{2a}{7}} \sum_{k=0}^{\infty} \frac{\Gamma(a-2)}{\Gamma(a-2-k)} \left( \frac{\epsilon}{\mu} \right)^{4-a+k} + b_2 e^{-\frac{2a}{7}} \sum_{k=0}^{\infty} \frac{\Gamma(2a-2)}{\Gamma(2a-2-k)} \left( \frac{\epsilon}{\mu} \right)^{5-a+k+p}
$$

$$
+ b_3 e^{-\frac{2a}{7}} \sum_{k=0}^{\infty} \frac{\Gamma(2a-4)}{\Gamma(2a-4-k)} 2^{2a-5-k} \left( \frac{\epsilon}{\mu} \right)^{2-a+k} + b_4 e^{-\frac{2a}{7}} \sum_{k=0}^{\infty} \frac{\Gamma(2a-2)}{\Gamma(2a-2-k)} 2^{2a-3-k} \left( \frac{\epsilon}{\mu} \right)^{2-a+k}
$$

$$
+ b_5 e^{-\frac{2a}{7}} \sum_{k=0}^{\infty} \frac{\Gamma(2a-6)}{\Gamma(2a-6-k)} 2^{2a-4-k} \left( \frac{\epsilon}{\mu} \right)^{3-a+k} + b_6 e^{-\frac{2a}{7}} \sum_{k=0}^{\infty} \frac{\Gamma(2a)}{\Gamma(2a-k)} \frac{\Gamma(2a-1)}{\Gamma(2a-1-k)} \frac{1}{(2a-1-k)!} \left( \frac{\epsilon}{\mu} \right)^{3-a+k+p}
$$

Note that the equals sign in Eq. (5.5) has to be understood as an asymptotic expansion. Obviously the divergence structure of any term depends on the range spanned by the parameter $a$. For example, the first term has a divergence structure $T \epsilon^{a-1} \mu^{-a}$ for $T \to \infty, \mu \to 0, \epsilon \to 0$. Therefore, one can take $T \epsilon^{a-1} \mu^{-a} = k = const$ for $a > 1$, thus it can be considered regular. This leads to a vanishing term $T^{2-a} \mu^{-\frac{\epsilon}{2}} \epsilon^{a-1} \mu^{-1} \to 0$. The terms with $\epsilon^a \mu^{-a}$ and $\epsilon^{-1} \mu^{1-a}$ also vanish for $a > 1$, because $\epsilon \to 0$ is dominant. The other terms look complicated, but fortunately they all have suppressing weight factors of $e^{-\frac{2a}{7}} \epsilon \to 0$, thus they all vanish for $a > 1$. This analysis suggests that one can recover the divergence structure of the dual CFT quantum information metric (5.2) from (5.5) for $a > 1$, namely

$$
G_{\lambda\lambda}^{Bulk} = (a_0 + a_6 \Gamma^2(a)) k \epsilon^{1-a} = K \epsilon^{1-a}, \quad a > 1,
$$

where $K = const$. A similar result is valid also for finite $T$,

$$
G_{\lambda\lambda}^{Bulk} = (a_0 + a_6 \Gamma^2(a)) Tk_1 \epsilon^{1-a} = K_1 \epsilon^{1-a}, \quad a > 1,
$$

where $k_1 = \epsilon^{a-1} \mu^{-a} = const$. In order to complete the analysis one can further require the normalization constants $K$ from (5.6) and $C$ from (3.12) to coincide, which can be used to fix $k$, namely

$$
k = \kappa \frac{(2a + d + 2) \Gamma(a - 1) \mu^d (d-a-1)}{2^{2a-1} L^d (2 - 2a + d) \Gamma(a) \Gamma(a+1) - \Gamma(2a+1) B_{1/2}(a,a)}.\quad (5.8)
$$

Insisting on holographic duality on both sides of the correspondence forces us to equate the coefficients in front of the leading singularities of both theories. Interestingly, being in perturbative regime in the bulk and staying close to the boundary the parameters characterizing gravity side shows the behaviour of the theory in the limiting case.
5.2 Dominant $\tilde{\epsilon}$

When $\tilde{\epsilon}$ approaches zero sufficiently faster than $\epsilon$, the dominant regulator is given by $\mu$. In this case, the second argument of the incomplete gamma function goes to zero, thus we can use (5.4) for all terms. Hence, one finds

$$G_{\lambda\lambda} = (a_0 + a_6 \Gamma^2(a)) \frac{T^{a-1}}{\mu^{a}} - a_0 \left(\frac{\epsilon}{\mu}\right)^a + a_1 \Gamma(a - 2) \frac{T^{2-a} \epsilon^{a-1}}{\mu}$$

$$+ (a_2 \Gamma^2(a - 2) + a_3 \Gamma(2a - 4) + a_4 \Gamma(2a - 2) + a_5 \Gamma(2a - 3) + a_7 \Gamma^2(a - 1)) \left(\frac{\epsilon}{\mu}\right)^a$$

$$+ (b_2 \Gamma^2(a - 2) + b_3 \Gamma(2a - 4) + b_4 \Gamma(2a - 2) + b_5 \Gamma(2a - 3) + b_7 \Gamma^2(a - 1)) \left(\frac{\epsilon}{\mu}\right)^a$$

$$+ b_1 \Gamma(a - 2) \frac{\epsilon}{\mu}.$$ \hspace{1cm} (5.9)

Due to the fact that $a_i = -b_i$, the terms with $a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5$ and $a_7, b_7$ cancel each other out, leaving us only with

$$G_{\lambda\lambda} = (a_0 + a_6 \Gamma^2(a)) \frac{T^{a-1}}{\mu^{a}} - (a_0 + a_6 \Gamma^2(a)) \left(\frac{\epsilon}{\mu}\right)^a + a_1 \Gamma(a - 2) \frac{T^{2-a} \epsilon^{a-1}}{\mu} - a_1 \Gamma(a - 2) \frac{\epsilon}{\mu}. \hspace{1cm} (5.10)$$

If $T \to \infty$, all terms in the metric are divergent for $1 < a \leq 2$, while for $a > 2$ only the third term can be considered regular. For finite $T$ all terms remain divergent for $a > 1$. This suggests that when $\mu$ dominates the divergent structure makes it impossible to represent HFIM in the form given by Eq.(5.2), thus one cannot obtain the dual QFIM from the bulk.

5.3 The case $\epsilon \sim \tilde{\epsilon}$

In this special case we have effectively one divergence, namely $0 < \eta \ll 1$. Hence, the asymptotic behaviour is the same as in the previous case, but with $\mu \sim \eta^2$ and $\epsilon \sim \eta$ replaced, thus

$$G_{\lambda\lambda} = \frac{2^a}{M^a} (a_0 + a_6 \Gamma^2(a)) \frac{T - \eta}{\eta^{a+1}} + \frac{2a_1}{M} \Gamma(a - 2) \left(\frac{T^{2-a} \eta^{3-a} - 1}{\eta}ight). \hspace{1cm} (5.11)$$

Taking $T \to \infty$ and $\eta \to 0$, all terms are divergent for $1 < a \leq 2$. If $2 < a < 3$, only the term $T^{2-a} \eta^{3-a}$ can be fixed to a constant. If $a \geq 3$, the third term vanishes. When $1 < a < 3$ and $T$ is considered finite, all terms diverge. If $a \geq 3$, the third terms is regular or vanishes for finite $T$. Therefore, it is impossible to reproduce the divergent structure of the dual CFT metric from the bulk HFIM.

Let us briefly summarize the results for HFIM near the boundary $r = 0$. When approaching asymptotically the boundary, primarily with $\epsilon \to 0$, we have been able to fully match the structure of the dual CFT QFIM by the structure of the bulk HFIM. On the other hand, for predominant $\tilde{\epsilon} \to 0$, HFIM is intrinsically divergent on the boundary with a structure that does not coincide with the dual CFT QFIM. Also, for the particular case $\epsilon \sim \tilde{\epsilon} \to 0$, the dual QFIM can not be reproduced by the bulk HFIM.

6 Analysis of the holographic Fisher information metric

In this section we are going to study the full divergence structure of HFIM in the bulk along the holographic coordinate $r$. This requires considering HFIM from Eq. (4.14).
6.1 HFIM near the boundary $r \to 0$

In Section 5 we have shown that one can reproduce the divergence structure and even match the normalization constant of the boundary CFT QFIM from the bulk HFIM under certain conditions. This was done by explicitly taking out a factor of $\epsilon^{a-1}$ and considering the divergence structure for the rest of the metric. Below we will restore this factor and consider the full HFIM near the boundary $r = 0$. Once again, one has several cases.

**Dominant $\epsilon$.** In this case, the bulk Fisher metric (4.14) has the following asymptotic form

$$G^B_{\lambda\lambda} = (a_0 + a_6 \Gamma^2(a)) \frac{T}{\mu^a} - a_0 \frac{\epsilon}{\mu^a} + a_1 \Gamma(a - 2) \frac{T^{2-a} \epsilon^{-\frac{\mu}{\epsilon}}}{\mu}$$

$$+ \left( a_2 \Gamma^2(a - 2) + a_3 \Gamma(2a - 4) + a_4 \Gamma(2a - 2) + a_5 \Gamma(2a - 3) + a_7 \Gamma^2(a - 1) \right) \frac{1}{\mu^{a-1}}.$$

By taking out a factor of $\epsilon^{a-1}$ one returns to the HFIM from (5.5), which was used to match the divergence structure of the dual QFIM on the boundary. On the other hand, when $T \to \infty$ and $\epsilon \to 0$ being dominant, all the $b_i$ terms in (6.1) are suppressed due to the exponential weight factors. Moreover, the second term with $a_0$ also vanishes, thus HFIM reduces to

$$G^B_{\lambda\lambda} = \left( a_0 + a_6 \Gamma^2(a) \right) \frac{T}{\mu^a} + a_1 \Gamma(a - 2) \frac{T^{2-a}}{\mu}$$

$$+ \left( a_2 \Gamma^2(a - 2) + a_3 \Gamma(2a - 4) + a_4 \Gamma(2a - 2) + a_5 \Gamma(2a - 3) + a_7 \Gamma^2(a - 1) \right) \frac{1}{\mu^{a-1}}.$$  \hspace{1cm} (6.2)

One notes that the first and third terms are intrinsically divergent for $a > 1$, while the second term is divergent only for $1 < a \leq 2$. The latter can be regular for $a > 2$. When $T$ is finite – all terms are divergent.

**Dominant $\tilde{\epsilon}$.** When $\tilde{\epsilon}$ goes to zero sufficiently faster than $\epsilon$, one finds

$$G^B_{\lambda\lambda} = \left( a_0 + a_6 \Gamma^2(a) \right) \frac{T}{\mu^a} - \frac{\epsilon}{\mu^a} + a_1 \Gamma(a - 2) \frac{T^{2-a} - \epsilon^{2-a}}{\mu}.$$  \hspace{1cm} (6.3)

Taking $T \to \infty$, all terms diverge for $a > 1$, except for the term with $T^{2-a}$, which is divergent only for $1 < a \leq 2$ and can be regular for $a > 2$. When $T$ is considered finite, all terms are divergent for $a > 1$.

**The case $\epsilon \sim \tilde{\epsilon}$**. When neither of the regulators dominates one effectively has $\epsilon = \tilde{\epsilon} = \eta$, hence

$$G^B_{\lambda\lambda} = \frac{2a}{M^a} (a_0 + a_6 \Gamma^2(a)) \frac{T - \eta}{\eta^{2a}} + \frac{2a_1}{M} \Gamma(a - 2) \frac{T^{2-a} - \eta^{2-a}}{\eta^2}.$$  \hspace{1cm} (6.4)
In the limit $T \to \infty$, all terms are divergent, except for the term with $T^{2-a}$, which is divergent only for $1 < a \leq 2$, and regular for $a > 2$. For finite $T$ – all terms diverge for $a > 1$.

### 6.2 HFIM in the bulk

In the bulk of Schrödinger spacetime the holographic coordinate $r$ is finite and the divergence structure of the metric is managed by $\tilde{\varepsilon}$ and $T$. The following cases are relevant.

**The case** $\mu \to \text{const} \neq 0, T \to \infty$. When we consider HFIM in the bulk $\mu \sim r^2$ is finite. If one takes $T \to \infty$ with $\varepsilon \to 0$, Eq. (4.14) becomes

$$G_{\lambda\lambda}^{\text{Bulk}} = \left( a_0 + a_6 \Gamma^2(a) \right) \frac{T}{\mu^a} + a_1 \Gamma(a - 2) \frac{T^{2-a}}{\mu} + \left( a_2 \Gamma^2(a - 2) + a_3 \Gamma(2a - 4) + a_4 \Gamma(2a - 2) + a_5 \Gamma(2a - 3) + a_7 \Gamma^2(a - 1) \right) \frac{1}{\mu^{a-1}}. \quad (6.5)$$

One notes that the divergence structure of the information metric is governed fully by $T$. For $1 < a \leq 2$ both terms are divergent, while for $a > 2$ only the first term is divergent.

**The case** $\mu \to \text{const} \neq 0, T \to \text{const}$. With both $\mu$ and $T$ finite the holographic Fisher information metric is completely regular for $a > 1$, although it has the same form as in Eq. (6.2). In other words, the information space now is non-singular.

### 6.3 HFIM near $r \to \infty$

Finally, we are going to consider HFIM on the second boundary at $r \to \infty$.

**The case** $\mu \to \infty, T \to \text{const}$. Here, the bulk HFIM (4.14) vanishes trivially. One way to interpret this result is that one cannot discern between deformed theory with coupling $\delta\lambda$ and the undeformed theory at $r \to \infty$. In other words, the Fisher distance between both theories in the space of coupling constants is zero.

**The case** $\mu \to \infty$ (dominant), $T \to \infty$. When we are approaching asymptotically the boundary at $r \to \infty$ with predominant $\mu \to \infty$, the holographic Fisher information metric (4.14) also vanishes trivially.

**The case** $\mu \to \infty, T \to \infty$ (dominant). We have only two relevant terms

$$G_{\lambda\lambda}^{\text{Bulk}} = \left( a_0 + a_6 \Gamma^2(a) \right) \frac{T}{\mu^a} + a_1 \Gamma(a - 2) \frac{T^{2-a}}{\mu}, \quad (6.6)$$

which can be considered divergent for $1 < a < 2$. When $a > 2$ the second term vanishes and we are left only with

$$G_{\lambda\lambda}^{\text{Bulk}} = \left( a_0 + a_6 \Gamma^2(a) \right) \frac{T}{\mu^a}, \quad (6.7)$$

which is regular if one considers $T\mu^{-a} = \text{const}$.

In summary, we have studied the divergence structure of the HFIM in the bulk of the entire Schrödinger spacetime, together with its boundaries, along the holographic direction $r$. We have shown that depending on how we approach the given sectors, different divergence structures arise, managed by the relevant regulators $\tilde{\varepsilon}, T$ or $\varepsilon$. In many of the considered cases
HFIM is intrinsically divergent. However, we have found two cases, in which the bulk HFIM is regular and finite, thus the information space over the couplings of the theory is now a well-defined non-singular Riemannian manifold. Finally, we have encountered two cases at $r \to \infty$, where the HFIM is zero, which suggest that we cannot discern between a deformed theory with coupling $\delta \lambda$ and the undeformed one.

7 Conclusion

In this study we have investigated the properties of the Fisher information metric on both sides of the duality between non-relativistic dipole gauge theory and bulk string theory in Schrödinger spacetime\(^{10}\). Our work extends the scope of information theory from the original AdS/CFT correspondence to non-relativistic holography. To the best of our knowledge this is the first attempt of application of information theory to Schrödinger/Dipole holography. The setup consists of a marginally deformed CFT by inserting a primary operator at Euclidean time $\tau = 0$. Holographically, this corresponds to a massive scalar field probing the Schrödinger background geometry on the string side, where backreaction on the original spacetime has been considered negligible. This is reflected by the perturbative technique used to calculate the holographic Fisher information metric as shown in [1].

On the gauge side, the quantum Fisher information metric on the space of coupling constants has been computed via the two-point correlation function between relevant deformation operators at different times $\tau_1$ and $\tau_2$. The computation has been done in the ground states of the deformed and the undeformed theories. The final expression (3.11) for the QFIM is independent of any spacetime coordinates and represents the information content difference between the originally undeformed CFT and the one generated after turning on the deformation. In this case, the divergence structure of the QFIM consists only of one temporal regulator $\epsilon \to 0$, coming from the fact that the CFT Lagrangian changes discontinuously at $\tau = 0$, which introduces an UV divergence.

On the string side, we have used the perturbation method suggested in [1], which is valid in the large $N$ limit of the theory, in order to compute the dual HFIM given in Eq. (4.14). This metric exhibits a richer divergence structure by introducing not one, but three different regulators – two temporal $\epsilon \to 0$ and $T \to \infty$, and one spacial regulator $\tilde{\epsilon}$ along the holographic direction $r$. As shown on Fig. 1, we can divide the Schrödinger background along the coordinate $r$ in three sectors. One of them includes the boundary at $r = 0$, where the dual CFT lives. The second one includes the bulk of spacetime, and the third sector is the limit $r \to \infty$. This naturally introduces several important cases.

The most important case for us is to compare the asymptotic divergent structure of HFIM to QFIM near the boundary at $r = 0$. The latter turn out to be a nontrivial task due to the competing divergences in the HFIM. Our analysis showed that when approaching asymptotically the boundary at $r = 0$, primarily with $\epsilon \to 0$, we have uncovered a full match between the leading divergences (5.6) of HFIM and the dual CFT QFIM. This result means that one can reconstruct the entire QFIM from the bulk HFIM close to the boundary. Furthermore, for predominant $\tilde{\epsilon} \to 0$, HFIM is intrinsically divergent on the boundary with a structure that does not coincide with the dual QFIM from (3.11). This is also true for the particular case of $\epsilon \sim \tilde{\epsilon} \to 0$. In summary, we showed that there exist only one asymptotic limit of HFIM, which leads to a consistent Schrödinger/Dipole theory duality.

We have further studied the divergence structure of HFIM in the bulk of the entire Schrödinger spacetime together with its boundaries. Depending on how we approach different sectors, we

\(^{10}\) In this case, the bulk theory is $(d+3)$-dimensional Schrödinger spacetime and the dual gauge theory lives on the $(d+1)$-dimensional boundary.
have found specific divergence structures managed by the relevant regulators \( \bar{\varepsilon}, T \) or \( \epsilon \). In many of the analysed cases HFIM is intrinsically divergent. However, there appeared two cases, where the bulk HFIM is regular and finite, thus the information space over the coupling constants of the theory is a non-singular Riemannian manifold. Finally, we have encountered two cases at \( r \to \infty \), where HFIM vanishes, thus one cannot discern between a deformed theory with coupling \( \delta \lambda \) and the undeformed one.

Although our study covers the computation of the holographic Fisher information metric and its dual quantum counterpart for non-relativistic holographic models in Schrödinger spacetime, there are still many unexplored research directions. For example, it would be interesting to consider key information-theoretic aspects of non-relativistic holography in the MERA and cMERA (continuous multi-scale entanglement renormalization ansatz) approach to the gauge/gravity correspondence [64], where HFIM in the bulk plays an important role. Furthermore, the HFIM for systems at finite temperature, which traditionally involves the presence of black holes, is also interesting to consider. Another interesting direction of investigation is to apply other relevant or irrelevant deformations on the theory with multiple operators. Examples of such deformations include \( T \bar{T}, JT, J \bar{T} \), etc. It could prove fruitful to consider models in different non-relativistic holographic solutions such as Lifshitz spacetimes\(^{11}\) or other similar backgrounds. Finally, it could be interesting to investigate the relation between the Fisher information metric and quantum complexity for such models.

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**A A short note on how to generate Schrödinger spacetimes and their dual CFT theories**

The symmetry of the free Schrödinger equation

\[
\frac{\partial^2}{\partial \vec{r}^2} \phi - 2im \frac{\partial}{\partial t} \phi = 0, \quad (A.1)
\]

is the so called Schrödinger group. In \( n \) dimensional spacetime the group consists of spatial translations indicated by \( \vec{A} \), rotations given by the matrix \( \Omega \), and Galilean boosts with velocity \( \vec{v} \),

\[
t \to t' = \frac{at + b}{ct + d}, \quad \vec{r} \to \vec{r}' = \frac{\Omega \vec{r} + \vec{v}t + \vec{A}}{ct + d}, \quad ad - bc = 1. \quad (A.2)
\]

In addition, one has dilatation, where time and space scale differently

\[
t \to \lambda^2 t, \quad \vec{r} \to \lambda \vec{r}, \quad (A.3)
\]

and one additional special conformal transformation

\[
t \to \frac{t}{1 + \lambda t}, \quad \vec{r} \to \frac{\vec{r}}{1 + \lambda t}. \quad (A.4)
\]

\(^{11}\)Such investigations for holographic entanglement entropy, mutual information and entanglement of purification over holographic Lifshitz theory have been already initiated in [65].
From group theory point of view, the Schrödinger group can be thought of as a non-relativistic analogue of the conformal group. In fact, the Schrödinger group can be embedded into the relativistic conformal group $SO(2, n + 2)$ in $n + 1$ dimensions [3, 4, 66], as well as a particular contraction of the conformal group.

For purposes of the holographic correspondence it is important to consider spaces with the Schrödinger group being the maximal group of isometries\(^{12}\). Such spaces are called Schrödinger spaces. There is a specific way to obtain the spacetime geometry equipped with this symmetry via TsT (T-duality-shift-T-duality) transformations.

Our starting point is the AdS metric, which is invariant under the whole conformal group and deform it to reduce the symmetry down to the Schrödinger group.

Being a particular case of the so called Drinfel’d-Reshetikhin twist, the TsT procedure has been used for generating many backgrounds keeping partial or full integrability of the system. Specific point in generating Schrödinger backgrounds via TsT transformations is to include one of the light-cone variables. This particular deformation is also known as the null-Melvin twist. This procedure can be implemented by the following key steps:

- Represent the theory in light-cone coordinates and identify a Killing direction, say $\psi$,
- Perform a T-duality along the chosen Killing direction $\psi$,
- Boost the geometry in the Killing direction by $\hat{\mu}$, i.e. $x^- \rightarrow x^- - \hat{\mu}\tilde{\psi}$, where $\tilde{\psi}$ is the T-dualized coordinate $\psi$,
- Finally, perform a T-duality back to IIA/IIB along $\tilde{\psi}$.

In order to accomplish the desired result consider general background in the form $AdS_n \times X^m$,

$$ds^2 = \ell^2 \frac{2dx^+ dx^- + dx^i dx_i + dz^2}{z^2} + ds^2_{X^m},$$  \hspace{1cm} (A.5)

Now we perform a null Melvin twist along a Killing vector $K$ on $X^m$. The result is

$$ds^2 = ds^2_{Schr} + ds^2_{X^m},$$ \hspace{1cm} (A.6)

where the Schrödinger metric yields

$$ds^2_{Schr} = - \frac{\Omega}{z^4} + \frac{1}{z^2} (2dx^+ dx^- + \hat{dx}^2 + dz^2), \quad \Omega = ||K||^2 = \star (K \wedge \star K).$$  \hspace{1cm} (A.7)

Here the Hodge star operator $\star$ is taken with respect to the metric on $X^m$. It is clear that $\Omega$ is non-negative being a square length of a Killing vector\(^{13}\). For particular examples see [20, 21, 67–69]\(^{14}\).

An important remark is that in order to make holographic sense of these solutions one has to impose some conditions. In particular for these to be holographic duals to non-relativistic field theories the light-cone coordinate $x^-$ should be periodic, $x^- \sim x^- + 2\pi r_\times$ [3, 4, 19]. The momentum along this compact direction is quantized in units of the inverse radius $r_\times^{-1}$.

The dual theory is conjectured to be a specific field theory, namely dipole field theory. The TsT transformation, which produces the bulk theory corresponds to a particular deformation

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\(^{12}\)For a detailed group-theoretical perspective on non-relativistic holography see [8].

\(^{13}\)Together with the metric the procedure also generates a non-zero $B$-field of the form

$$B_{(2)} = \frac{1}{z^2} K \wedge dx^+.$$ 

\(^{14}\)The original Schrödinger background (3.1), used in this paper, was first derived in [3, 4].
(Drinfel’d-Reshetikhin twist) on field theory side. It translates to the dual theory as a star product. When the directions involved in TsT are transverse to the stack of branes from which geometry descends, it produces a twist in the field theory having ordinary product. If however, one of the directions is along the branes, then the star product is non-trivial and the theory becomes dipole one

\[
(\Phi_1 \star \Phi_2)(x) = \Phi_1(x + L_1)\Phi_2(x - L_2),
\]

where \( L = L_1 + L_2 \) is the dipole length associate with R-charges. The big advantage of holographic model with Schrödinger symmetry is that they could be integrable. Proving that in the case of \( S\text{chr}_5 \times S^5 \) for example, the authors of [70] were able to map the composite operators of monomial form to a spin chain. To study field theory side one must just to replace the ordinary product with the star one

\[
\mathcal{O} = \text{tr}(\Phi_1 \star \Phi_2 \star \cdots).
\]

Here, one can choose to work either using Seiberg-Witten map or working directly with the star product. A nice analysis has been presented in [70].

**B Derivation of the CFT quantum geometric tensor**

Let \( |\varphi\rangle \) be a generic state, inserted at \( \tau = 0 \). The overlap between the original ground state \( |\psi_0\rangle \) at \( \tau \to -\infty \) and this new state at \( \tau \to 0 \) is given by

\[
\langle \tilde{\psi} | \psi_0 \rangle = \frac{1}{\sqrt{Z_0}} \int D\varphi e^{-\int_0^\infty d\tau \int d^d x L_0},
\]

where \( \tilde{\varphi} = \varphi(\tau = 0) \) and the partition function of the initial undeformed theory is defined by

\[
Z_0 = \int D\varphi e^{-\int_0^\infty d\tau \int d^d x L_0}.
\]

In a similar fashion, one can consider the evolution from \( \tau = 0 \), where \( |\tilde{\varphi}\rangle \) is inserted, to \( \tau \to \infty \), where we are placing the perturbed state \( |\psi_1\rangle \),

\[
\langle \psi_1 | \tilde{\varphi} \rangle = \frac{1}{\sqrt{Z_1}} \int D\varphi e^{-\int_0^\infty d\tau \int d^d x L_1} = \frac{1}{\sqrt{Z_1}} \int D\varphi e^{-\int_0^\infty d\tau \int d^d x (L_0 + \delta \lambda^a O_a)}.
\]

As usually,

\[
Z_1 = \int D\varphi e^{-\int_0^\infty d\tau \int d^d x (L_0 + \delta \lambda^a O_a)}
\]

is the partition function of the deformed theory. Now, the overlap between both states in the above notations can be formally written as

\[
\langle \psi_1 | \psi_0 \rangle = \int_{\varphi(\tau = 0) = \tilde{\varphi}} D\varphi \langle \psi_1 | \varphi \rangle \langle \varphi | \psi_0 \rangle = \frac{1}{\sqrt{Z_0 Z_1}} \int D\varphi e^{-\int d^d x \left[ \int_0^\infty d\tau L_0 + \int_0^\infty d\tau (L_0 + \delta \lambda^a O_a) \right]}.
\]

Let us write this overlap in a more convenient way [71]

\[
\langle \psi_1 | \psi_0 \rangle = \frac{\exp \left( -\int_0^\infty d\tau \int d^d x \delta \lambda^a O_a \right) }{\sqrt{\frac{Z_1}{Z_0}}} = \frac{\exp \left( -\int_0^\infty d\tau \int d^d x \delta \lambda^a O_a \right) }{\sqrt{\frac{Z_1}{Z_0}}}.
\]
We can now easily expand this expression in power series

\[
|\langle \psi_1 | \psi_0 \rangle|^2 = \frac{\exp \left( - \int_0^{\infty} d\tau \int \mathcal{V} \delta(\alpha) \mathcal{O}_a(\tau) \right) \exp \left( - \int_{-\infty}^{0} d\tau \int \mathcal{V} \delta(\alpha) \mathcal{O}_a(\tau) \right)}{\exp \left( - \int_{-\infty}^{\infty} d\tau \int \mathcal{V} \delta(\alpha) \mathcal{O}_a(\tau) \right)}.
\]

We can now easily expand this expression in power series

\[
|\langle \psi_1 | \psi_0 \rangle|^2 = 1 + \frac{1}{2} \int_{\mathcal{V}}^{d^d} \xi_1 \int_{\mathcal{V}}^{d^d} \xi_2 \left[ \int_{0}^{\infty} d\tau_1 \int_{0}^{\infty} d\tau_2 \langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(\tau_2) \rangle \right]
\]

\[
+ \int_{-\infty}^{0} d\tau_1 \int_{-\infty}^{0} d\tau_2 \langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(\tau_2) \rangle - \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(\tau_2) \rangle
\]

\[
+ 2 \int_{-\infty}^{0} d\tau_1 \int_{-\infty}^{0} d\tau_2 \langle \mathcal{O}_a(\tau_1) \rangle \langle \mathcal{O}_b(\tau_2) \rangle \delta \lambda^a \delta \lambda^b + \mathcal{O}(\delta \lambda^3).
\]

Since we have time reversal symmetry of the correlator,

\[
\langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(-\tau_2) \rangle = \langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(\tau_2) \rangle,
\]

and the fact that

\[
\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 = \int_{-\infty}^{0} d\tau_1 \int_{-\infty}^{0} d\tau_2 + \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 + \int_{-\infty}^{0} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 + \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{0} d\tau_2,
\]

one finds

\[
|\langle \psi_1 | \psi_0 \rangle|^2 = 1 - \frac{1}{2} \int_{\mathcal{V}}^{d^d} \xi_1 \int_{\mathcal{V}}^{d^d} \xi_2 \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \left( \langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(\tau_2) \rangle - \langle \mathcal{O}_a(\tau_1) \rangle \langle \mathcal{O}_b(\tau_2) \rangle \right) \delta \lambda^a \delta \lambda^b.
\]

Using the definition of the quantum fidelity, we extract the expression for QIM from Eq. (2.2).

### C Computation of the dual CFT quantum Fisher metric

One has to compute the following integral expression

\[
G^{(CFT)}_{\lambda \lambda} = \frac{i \Delta M^{\Delta-1} e^{-\frac{i \Delta}{4}}} {\pi^{d/2} \Gamma(\Delta - \frac{d}{2} - 1)} \int_{\mathcal{V}}^{d^d} \xi_1 \int_{\mathcal{V}}^{d^d} \xi_2 \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \frac{1}{(\tau_2 - \tau_1)^{\Delta}} e^{i \frac{M^{1+i\varepsilon} (\tau_2 - \tau_1)^2}{2(\tau_2 - \tau_1)}},
\]

which is a Gaussian integral over $\bar{x}$ space, i.e.

\[
I_x = \int_{\mathcal{V}}^{d^d} \xi_1 \int_{\mathcal{V}}^{d^d} \xi_2 e^{i \frac{M^{1+i\varepsilon} (\tau_2 - \tau_1)}{2(\tau_2 - \tau_1)}} = e^{i \frac{\pi}{4} (1-d/2)} (2\pi)^{d/2} M^{-d} \left( 1 + i\varepsilon \right)^{-d} (\tau_2 - \tau_1)^{d} V_{\mathcal{V}}^{d}.\]
where $V_{\mathbb{R}^d}$ is the volume of $\mathbb{R}^d$ space and we have resorted to the standard Gaussian integral [72]:

$$
\int d^d x \, e^{i\beta \overline{x}^2 - 2i\xi \overline{x}} = e^{i\xi (1-\frac{d}{2}) \pi^{d/2} \beta^{d/2} e^{-i\delta/2}}.
$$

(C.3)

We can check this by setting $\xi = \frac{M(1+i\epsilon)}{2(t_2-t_1)}$, thus

$$
I_x = \int d^d x_1 \int d^d x_2 \, e^{i\xi (x_2-\overline{x}_1)^2} = \int d^d x_1 \int d^d x_2 \, e^{i\xi (x_2^2 - 2\overline{x}_1 \cdot x_2)}
$$

$$
= \int d^d x_1 \, e^{i\xi \overline{x}_1^2} \left( \int d^d x_2 \, e^{i\xi (x_2^2 - 2\overline{x}_1 \cdot x_2)} \right) = e^{i\xi (1-\frac{d}{2}) \pi^{d/2} \xi^{d/2}} \int d^d x_1 \, e^{i\xi \overline{x}_1^2} e^{-i\xi \overline{x}_1^2}
$$

$$
= e^{i\xi (1-\frac{d}{2}) 2^{d/2} \pi^{d/2} e^{-i\xi \overline{x}_1^2}} V_{\mathbb{R}^d}.
$$

The integrals over $\tau_1$ and $\tau_2$ now take the form

$$
I_t = \int_{-\infty}^{\epsilon} d\tau_1 \int_{\epsilon}^{\infty} d\tau_2 (\tau_2 - \tau_1)^{4-\Delta} = \frac{2^{2+\Delta}}{(d - 2\Delta + 2)(d - 2\Delta + 4)} \epsilon^{\Delta - \frac{d}{2}} e^{-\frac{d}{2} \Delta - \frac{d}{2}},
$$

(C.4)

where the integrals are convergent only if

$$
2 + \frac{d}{2} - \Delta < 0.
$$

(C.5)

Therefore, QIM in the dual CFT to Schrödinger spacetime yields

$$
G_{\lambda\lambda}^{(CFT)} = \frac{\Delta 2^{d-2\Delta+3} e^{-\frac{d}{2} (d+2\Delta) M^{d-\frac{d}{2}} - 1} V_{\mathbb{R}^d}}{(d - 2\Delta + 2) \Gamma \left( \Delta - \frac{d}{2} \right) (1 + i\epsilon)^{\frac{d}{2}} - \Delta} = C \epsilon^{\Delta - \frac{d}{2}},
$$

(C.6)

where at the final expression we have removed the regulator $\epsilon$ from the correlation function.

**D** Computation of the bulk holographic Fisher metric

**D.1** Computation of the fields $\phi_{1,2}$

We will be integrating the bulk-to-boundary propagator $K(r, \overline{x}, \tau; \overline{x}_1, \tau_1)$ over $x_1$ and $\tau_1$, when $\tau_1 < \tau$. Therefore, the field $\phi_1$ is given by

$$
\phi_1(r, \overline{x}, \tau) = \delta \lambda \int d^d x_1 \int_{-\infty}^{\tau} d\tau_1 K(r, \overline{x}, \tau; \overline{x}_1, \tau_1) \hat{\phi}_0(\overline{x}_1, \tau_1)
$$

$$
= \delta \lambda c_\Delta r^\Delta \int_{-\infty}^{\tau} d\tau_1 \frac{1}{(\tau - \tau_1)_\Delta} e^{\frac{M^2}{2(\tau - \tau_1)}} e^{\frac{M^2}{2(\tau - \tau_1)}} e^{\frac{M^2}{2(\tau - \tau_1)}} \int d^d x_1 \, e^{\frac{M^2}{2(\tau - \tau_1)} (\overline{x}_1^2 - 2\overline{x}_1 \cdot \overline{x})}
$$

$$
= \delta \lambda c_\Delta r^\Delta \int_{-\infty}^{\tau} d\tau_1 \frac{1}{(\tau - \tau_1)_\Delta} e^{\frac{M^2}{2(\tau - \tau_1)}} e^{\frac{M^2}{2(\tau - \tau_1)}} e^{\frac{M^2}{2(\tau - \tau_1)}} e^{i\xi (1-\frac{d}{2}) \pi^{d/2}} \left( \frac{M}{2(\tau - \tau_1)} \right)^{-d/2}
$$

15Here $\hat{\phi}_0(\overline{x}_1, \tau_1) = 1$. 

22
\[ = \delta \lambda c_\Delta r^\Delta 2^{d/2} M^{-d/2} e^{\frac{i\mu}{2} (1 - \frac{d}{2})} \pi^{d/2} \int_{-\infty}^{\tau} d\tau_1 e^{-\frac{i\mu}{2} (\tau - \tau_1)} (\tau - \tau_1)^{\frac{d}{2} - \Delta}, \]

where we have introduced the notation
\[ \mu = \frac{Mr^2}{2}. \quad (D.1) \]

Let us calculate the last integral. For this purpose, we change the variables to \( y = \tau - \tau_1 \) with boundaries
\[ y = \tau - \tau_1 = \begin{cases} 0, & \tau_1 \to \tau, \\ \infty, & \tau_1 \to -\infty, \end{cases} \quad (D.2) \]
hence,
\[ \int_{0}^{\infty} dy e^{\frac{i\mu}{2} y} y^{\frac{d}{2} - \Delta} = \Gamma \left( \Delta - \frac{d}{2} - 1 \right) (-i\mu)^{\frac{d}{2} - \Delta + 1}, \quad (D.3) \]
where one has the convergence condition
\[ \Delta > \frac{d}{2} + 1, \quad M > 0. \quad (D.4) \]
The field \( \phi_1 \) now becomes
\[ \phi_1(r, \vec{x}, \tau) = i\delta \lambda e^{-\frac{i\pi d}{2} r^{d-\Delta+2}}. \quad (D.5) \]

We can do a similar computation for the filed \( \phi_2 \), namely
\[ \phi_2(r, \vec{x}, t) = \delta \lambda c_\Delta r^\Delta 2^{d/2} M^{-d/2} e^{\frac{i\mu}{2} (1 - \frac{d}{2})} \pi^{d/2} \int_{0}^{\tau} d\tau_1 e^{-\frac{i\mu}{2} (\tau - \tau_1)} (\tau - \tau_1)^{\frac{d}{2} - \Delta} \]
\[ = \delta \lambda c_\Delta r^\Delta 2^{d/2} M^{-d/2} e^{\frac{i\mu}{2} (1 - \frac{d}{2})} \pi^{d/2} \int_{0}^{\tau} dy e^{-\frac{i\mu}{2} y} y^{\frac{d}{2} - \Delta} \]
\[ = \delta \lambda c_\Delta r^\Delta 2^{d/2} M^{-d/2} e^{\frac{i\mu}{2} (1 - \frac{d}{2})} \pi^{d/2} \mu^{\frac{d}{2} - \Delta + 1} \int_{\tilde{\mu}/r}^{\infty} dx e^{-x} x^{\mu - 1} = \frac{i\delta \lambda e^{-\frac{i\pi d}{2} r^{d-\Delta+2}}}{\Gamma(a)} \Gamma(a, \frac{\mu}{t}), \quad (D.6) \]
where at the end we have returned to real time \( \tau \to -it \), and we have defined the parameter
\[ \tilde{\mu} = -i\frac{Mr^2}{2}. \quad (D.7) \]

### D.2 Computation of the HFIM integrals

In order to compute Eqs. (4.7) and (4.8) we have to consider the metric \( \gamma_{ab}, a, b = 0, \ldots, d \), on the \((d + 1)\)-dimensional boundary of the Schrödinger background, namely
\[ ds^2_{(\partial \text{Schr})_{d+1}} = -L^2 \frac{dt^2}{t^4} + L^2 \frac{d\vec{x}^2}{r^2}, \quad (D.8) \]
where \( \vec{x} \) is a \( d \)-dimensional vector. Therefore, one finds
\[ \sqrt{|\gamma|} = \frac{L^{d+1}}{r^{d+2}}, \quad (D.9) \]
On the other hand, only the \( r \) component of the normal vector to the boundary contributes, which can be explicitly calculated from

\[
1 = g_{\mu\nu}n^{\mu}n^{\nu} = g_{rr}n^r n^r = \frac{L^2}{r^2} (n^r)^2. \tag{D.10}
\]

Hence \( n^r = r/L \) and consequently \( n_r = g_{rr}n^r = L/r \). Finally, one has

\[
\sqrt{\left| \gamma \right|} n_{\mu}g^{\mu\nu} = \sqrt{\left| \gamma \right|} n_r g_{rr} = L^2 r^{-1-d}. \tag{D.11}
\]

Now, we can proceed with the computation of the bulk holographic Fisher information metric, which can be written by

\[
G_{\text{Bulk}}^{\lambda\lambda} = -\frac{1}{\delta\lambda^2} \left( \frac{\delta I_1}{2} - \delta I_2 \right) = \frac{L^d V_{d-2}}{2\kappa} \lim_{r \to T} \left( c_0 (J_0 r^{d-2\Delta+2} + c_1 J_1 + c_2 J_2 r^{d-2\Delta+2}) \right), \tag{D.12}
\]

where the coefficients \( c_i \) are given by

\[
c_0 = \frac{e^{-\mu T}}{2} (d - \Delta + 2), \quad c_1 = \frac{M^a e^{-id\pi}}{2a - 1 \Gamma^2(a)}, \quad c_2 = -\frac{2c_0}{\Gamma^2(a)}. \tag{D.13}
\]

Consequently, one has to compute the following integrals

\[
J_0 = \int_{-T}^{-\epsilon} dt + \int_{\epsilon}^{T} dt = 2(T - \epsilon), \tag{D.14}
\]

\[
J_1 = \int_{\epsilon}^{T} dt \frac{1}{t-a} e^{-\frac{a}{t} \Gamma \left( a, \frac{\mu}{t} \right)} = \int_{1/T}^{1/\epsilon} dx x^{a-2} e^{-\mu x} \Gamma (a, \mu x), \tag{D.15}
\]

\[
J_2 = \int_{\epsilon}^{T} dt \frac{1}{\Gamma^2 \left( a, \frac{\mu}{t} \right)}. \tag{D.16}
\]

In order to solve \( J_1 \) we transform the incomplete gamma function in the following way

\[
\Gamma(a, \mu x) = (a - 1)(a - 2) \Gamma(a - 2, \mu x) + (a - 1) \mu^{a-2} x^{a-2} e^{-\mu x} + \mu^{a-1} x^{a-1} e^{-\mu x}, \tag{D.17}
\]

which follows directly from

\[
\Gamma(a, z) = \frac{\Gamma(a)}{\Gamma(a - n)} \Gamma(a - n, z) + z^{a-1} e^{-z} \sum_{k=0}^{n-1} \frac{\Gamma(a)}{\Gamma(a - k)} z^{-k} \tag{D.18}
\]

for \( n = 2 \) and we have also used \( \Gamma(a) = (a - 1)\Gamma(a - 1) \). Hence, the integral \( J_1 \) now becomes

\[
J_1 = (a - 1)(a - 2) \int_{1/T}^{1/\epsilon} e^{-\mu x} x^{a-2} \Gamma(a - 2, \mu x) \, dx
\]

\[
+ (a - 1) \mu^{a-2} \int_{1/T}^{1/\epsilon} x^{a-4} e^{-2\mu x} \, dx + \mu^{a-1} \int_{1/T}^{1/\epsilon} x^{a-3} e^{-2\mu x} \, dx
\]

\[
= \frac{1}{\mu} (a - 1)(a - 2) T^{2-a} e^{-\mu T} \Gamma \left( a - 2, \frac{\mu}{T} \right) + \frac{1}{2 \mu^{a-1}} (a - 1)(a - 2)^2 \Gamma^2 \left( a - 2, \frac{\mu}{T} \right)
\]

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The integral $J_2$ is more complicated, but also analytically solvable if we use the following power reduction formula [73]:

$$\Gamma^2(a, z) = \Gamma^2(a) - 2 \int_0^{1/2} d\omega \gamma \left(2a, \frac{z}{\omega} \right) \omega^{a-1}(1 - \omega)^{a-1}. \quad (D.20)$$

Let $\epsilon = T_1$ and $T = T_2$, hence

$$J_2 = \int_{T_1}^{T_2} dt \Gamma^2 \left(a, \frac{\mu}{t} \right) = \int_{T_1}^{T_2} dt \left[ \Gamma^2(a) - 2 \int_0^{1/2} d\omega \gamma \left(2a, \frac{\mu}{\omega t} \right) \omega^{a-1}(1 - \omega)^{a-1} \right]$$

$$= (T_2 - T_1) \Gamma^2(a) - 2 \int_0^{1/2} d\omega \omega^{a-1}(1 - \omega)^{a-1} \int_{T_1}^{T_2} dt \gamma \left(2a, \frac{\mu}{\omega t} \right)$$

$$= (T_2 - T_1) \Gamma^2(a) - 2 \int_0^{1/2} d\omega \omega^{a-1}(1 - \omega)^{a-1} \left( \int_{T_1}^{T_2} dt \Gamma(2a) - \int_{T_1}^{T_2} dy \Gamma \left(2a, \frac{\mu}{\omega y} \right) \right)$$

$$= (T_2 - T_1) \left[ \Gamma^2(a) - 2\Gamma(2a) \mathcal{B}_{1/2}(a, a) \right] + 2 \int_0^{1/2} d\omega \omega^{a-1}(1 - \omega)^{a-1} \mathcal{J}(\omega).$$

The inner integral is

$$\mathcal{J}(\omega) = \int_{T_1}^{T_2} dt \Gamma \left(2a, \frac{\mu}{\omega t} \right) = T_2 \Gamma \left(2a, \frac{\mu}{\omega T_2} \right) - T_1 \Gamma \left(2a, \frac{\mu}{\omega T_1} \right) + \frac{\mu}{\omega} \left[ \Gamma \left(2a - 1, \frac{\mu}{\omega T_1} \right) - \Gamma \left(2a - 1, \frac{\mu}{\omega T_2} \right) \right], \quad (D.21)$$

thus

$$J_2 = (T_2 - T_1) \left[ \Gamma^2(a) - 2\Gamma(2a) \mathcal{B}_{1/2}(a, a) \right] + I_1 + I_2 + I_3 + I_4, \quad (D.22)$$

where

$$I_1 = -2T_1 \int_0^{1/2} d\omega \omega^{a-1}(1 - \omega)^{a-1} \Gamma \left(2a, \frac{\mu}{\omega T_1} \right), \quad (D.23)$$

$$I_2 = 2T_2 \int_0^{1/2} d\omega \omega^{a-1}(1 - \omega)^{a-1} \Gamma \left(2a, \frac{\mu}{\omega T_2} \right), \quad (D.24)$$
\[ I_3 = 2 \mu \int_0^{1/2} d\omega \omega^{a-2} (1 - \omega)^{a-1} \Gamma \left( 2a - 1, \frac{\mu}{\omega T_1} \right), \quad (D.25) \]

\[ I_4 = -2 \mu \int_0^{1/2} d\omega \omega^{a-2} (1 - \omega)^{a-1} \Gamma \left( 2a - 1, \frac{\mu}{\omega T_2} \right). \quad (D.26) \]

The solution is the following. Let \( b_{1,2} = \mu/T_{1,2} \), then

\[ I_{1,2} = \mp 2T_{1,2} \int_0^{1/2} d\omega \omega^{a-1} (1 - \omega)^{a-1} \Gamma \left( 2a, \frac{b_{1,2}}{\omega} \right) \]

\[ = \mp 2T_{1,2} \int_0^{1/2} d\omega \omega^{a-1} (1 - \omega)^{a-1} \left( \frac{b_{1,2}}{\omega} \right)^{2a} e^{-\frac{b_{1,2}}{\omega}} \int_0^{\infty} \frac{e^{-\frac{b_{1,2}}{\omega}}}{(1 + z)^{1-2a}} dz \]

\[ = \mp 2T_{1,2} \int_0^{\infty} dz (1 + z)^{2a-1} b_{1,2}^{2a} \int_0^{1/2} d\omega \frac{1}{\omega^2} \left( \frac{1}{\omega} - 1 \right)^{a-1} e^{-\frac{b_{1,2}}{\omega} (1+z)} \]

\[ = \mp 2T_{1,2} \int_0^{\infty} dz (1 + z)^{2a-1} b_{1,2}^{2a} \int_0^{1/2} d \left( \frac{-1}{\omega} \right) \left( \frac{1}{\omega} - 1 \right)^{a-1} e^{-\frac{b_{1,2}}{\omega} (1+z)} \]

\[ = \left\{ x = \frac{1}{\omega} = \left\{ \begin{array}{ll} 2, & \omega = 1/2, \\
0, & \omega = 0 \end{array} \right. \right\} = \left\{ x = \frac{1}{\omega} = \left\{ \begin{array}{ll} 2, & \omega = 1/2, \\
0, & \omega = 0 \end{array} \right. \right\} \]

\[ = \mp 2T_{1,2} \int_0^{\infty} dz (1 + z)^{a-1} b_{1,2}^a e^{-b_{1,2}(1+z)} \Gamma(a, b_{1,2}(1 + z)) \]

\[ = \mp 2T_{1,2} \int_0^{\infty} dz \frac{d}{dz} \Gamma(a, b_{1,2}(1 + z)) \bigg|_{z \to \infty} = \mp T_{1,2} \Gamma^2(a, b_{1,2}) \bigg|_{z \to 0} = \mp T_{1,2} \Gamma^2(a, b_{1,2}). \]

Therefore, one finds

\[ I_{1,2} = \mp T_{1,2} \Gamma^2 \left( a, \frac{\mu}{T_{1,2}} \right). \quad (D.27) \]

In the previous computations we have used the following integral representation of the incomplete gamma function

\[ \Gamma(a, t) = t^a e^{-t} \int_0^{\infty} \frac{e^{-zt}}{(1 + z)^{1-a}} dz. \quad (D.28) \]

With similar calculations for \( I_{3,4} \), one finds

\[ I_{3,4} = \pm 2\mu \int_0^{1/2} d\omega \omega^{a-2} (1 - \omega)^{a-1} \Gamma \left( 2a - 1, \frac{b_{1,2}}{\omega} \right) \]

\[ = \pm 2\mu \int_0^{1/2} d\omega \omega^{a-2} (1 - \omega)^{a-1} \left( \frac{b_{1,2}}{\omega} \right)^{2a-1} e^{-\frac{b_{1,2}}{\omega}} \int_0^{\infty} \frac{e^{-\frac{b_{1,2}}{\omega}}}{(1 + z)^{2-2a}} dz \]
Therefore, one has

$$I = 2 \mu \int_0^\infty dz (1 + z)^{2a-2} b_{1,2}^{2a-1} \int_0^{1/2} d\omega \frac{1}{\omega^2} \left( \frac{1}{\omega} - 1 \right)^{a-1} e^{-\frac{b_{1,2}}{\omega}(1+z)}$$

$$= 2\mu \int_0^\infty dz (1 + z)^{2a-2} b_{1,2}^{2a-1} \int_0^{1/2} d\omega \left( \frac{-1}{\omega} \right) \left( \frac{1}{\omega} - 1 \right)^{a-1} e^{-\frac{b_{1,2}}{\omega}(1+z)}$$

$$= \left\{ x = \frac{1}{\omega} = \left\{ 2, \omega = 1/2, \infty, \omega = 0 \right\} \right\}$$

$$= 2\mu \int_0^\infty dz (1 + z)^{2a-2} b_{1,2}^{2a-1} \int_2^\infty dx (x - 1)^{a-1} e^{-b_{1,2}(1+z)x}$$

$$= 2\mu \int_0^\infty dz (1 + z)^{a-2} b_{1,2}^{a-1} e^{-b_{1,2}(1+z)} \Gamma(a, b_{1,2}(1 + z)) .$$

Now we use

$$\Gamma(a, b(z + 1)) = e^{-b(z+1)} b^{a-1}(z + 1)^{a-1} - \frac{1}{2}(a - 1)(z + 1)^{2a} - b b^{1-a} e^{b(z+1)} \frac{d}{dz} \Gamma^2(a - 1, b(z + 1)),$$

thus

$$I_{3,4} = \mp(a - 1) \mu \int_0^\infty dz \frac{d}{dz} \Gamma^2(a - 1, b_{1,2}(z + 1)) \pm 2\mu b_{1,2}^{2a-2} \int_0^\infty dz (z + 1)^{2a} e^{-2b_{1,2}(z+1)}$$

$$= \mp(a - 1) \mu \Gamma^2(a - 1, b_{1,2}(z + 1)) \bigg|_{z=\infty}^{z=0} \pm 2^{3-2a} \mu \Gamma(2a - 2, 2b_{1,2})$$

$$= \pm(a - 1) \mu \Gamma^2(a - 1, b_{1,2}) \pm 2^{3-2a} \mu \Gamma(2a - 2, 2b_{1,2}).$$

Therefore, one has

$$I_{3,4} = \pm \mu \left[ (a - 1) \Gamma^2(a - 1, \frac{\mu}{T_{1,2}}) + 2^{3-2a} \Gamma \left( 2a - 2, \frac{2\mu}{T_{1,2}} \right) \right] .$$

Putting everything together we get the complete expression for $J_2$, namely

$$J_2 = (T - \epsilon) \left( \Gamma^2(a) - 2\Gamma(2a) B_{1/2}(a, a) \right) + T \Gamma^2(a, \frac{\mu}{T}) - \epsilon \Gamma^2(a, \frac{\mu}{\epsilon})$$

$$+ \mu \left( (a - 1) \Gamma^2(a - 1, \frac{\mu}{\epsilon}) + 2^{3-2a} \Gamma \left( 2a - 2, \frac{2\mu}{\epsilon} \right) \right)$$

$$- \mu \left( (a - 1) \Gamma^2(a - 1, \frac{\mu}{T}) + 2^{3-2a} \Gamma \left( 2a - 2, \frac{2\mu}{T} \right) \right) .$$

The final expression for the holographic Fisher metric yields

$$G_{\lambda\lambda}^{Bulk} = \frac{a_0}{\mu^a} (T - \epsilon) + a_1 \frac{T^{2-a} e^{-\frac{\mu}{T}}}{\mu} \Gamma \left( a - 2, \frac{\mu}{T} \right) + \frac{a_2}{\mu^{a-1}} \Gamma^2 \left( a - 2, \frac{\mu}{T} \right)$$

$$+ \frac{a_3}{\mu^{a-1}} \Gamma \left( 2a - 4, \frac{2\mu}{T} \right) + \frac{a_4}{\mu^{a-1}} \Gamma \left( 2a - 2, \frac{2\mu}{T} \right) + \frac{a_5}{\mu^{a-1}} \Gamma \left( 2a - 3, \frac{2\mu}{T} \right)$$

$$+ a_6 \frac{T}{\mu^a} \Gamma^2 \left( a, \frac{\mu}{T} \right) + \frac{a_7}{\mu^{a-1}} \Gamma^2 \left( a - 1, \frac{\mu}{T} \right) .$$
\[ + b_1 e^{2-a} \frac{\mu}{e} \Gamma(a-2, \frac{\mu}{e}) + \frac{b_2}{\mu^{a-1}} \Gamma^2(a-2, \frac{\mu}{e}) \]
\[ + \frac{b_3}{\mu^{a-1}} \Gamma(2a-4, \frac{2\mu}{e}) + \frac{b_4}{\mu^{a-1}} \Gamma(2a-2, \frac{2\mu}{e}) + \frac{b_5}{\mu^{a-1}} \Gamma(2a-3, \frac{2\mu}{e}) \]
\[ + \frac{b_6}{\mu^a} \Gamma^2(a, \frac{\mu}{e}) + \frac{b_7}{\mu^{a-1}} \Gamma^2(a-1, \frac{\mu}{e}), \]
(D.32)

with coefficients \( a_i = -b_i, i = 1, \ldots, 7 \), where
\[ a_0 = \frac{L^d V_{Rd}}{2^{a+1} \kappa} M^a \left[ 2c_0 + c_2 \left( \Gamma^2(a) - 2\Gamma(2a) B_{1/2}(a,a) \right) \right], \]
\[ a_1 = -b_1 = \frac{L^d V_{Rd}}{2\kappa} c_1(a-1)(a-2), \]
\[ a_2 = -b_2 = \frac{L^d V_{Rd}}{4\kappa} c_1(a-1)(a-2)^2, \]
\[ a_3 = -b_3 = \frac{L^d V_{Rd}}{2^{2a-3} \kappa} c_1(a-1)(a-2), \]
\[ a_4 = -b_4 = \frac{L^d V_{Rd}}{2^{3a-1} \kappa} (2^a c_1 - 2c_2 M^a), \]
\[ a_5 = -b_5 = \frac{L^d V_{Rd}}{2^{2a-2} \kappa} c_1(a-1), \]
\[ a_6 = -b_6 = \frac{L^d V_{Rd}}{2^{a+1} \kappa} M^a c_2, \]
\[ a_7 = -b_7 = \frac{L^d V_{Rd}}{2^{a+1} \kappa} M^a c_2(a-1), \]
(D.33)

where \( B \) is the incomplete beta function.

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