An Opinion Dynamics Model with Increasing Self-Confidence

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Abstract

We propose an opinion dynamics model in which agents gradually increase their own self-confidence while interacting with each other. The relations between the newly proposed model and existing works of social learning, inertial opinion dynamics, Bayesian inference, and stochastic multi-armed bandits are demonstrated. We prove the convergence of the system with the existence of a truth under fixed and periodically changing social networks, and obtain tight convergence bounds related to the spectral gap of the graph Laplacian and the maximum total degree centrality, respectively. In the case of randomly generated social networks, an almost-sure convergence result is obtained. The dynamics of the model with multiple truths or zero truth is also discussed.

1 Introduction

In this paper, we propose an opinion dynamics model in which agents move via convex combinations of the positions of neighbors specified by a sequence of graphs. The key feature of our model is that each agent gradually increases its self-confidence represented by its weight in the convex combination. The intuition is that by constantly incorporating signals into its own belief, the agent becomes more and more confident about its own opinion. The motivation and validity of such a scheme are further testified by its relations to related works including social learning [1–4], Bayesian inference [5, 6], multi-armed bandits [7, 8], as well as opinion dynamics with stubborn or static agents [9–11]. We first give a brief introduction of related works, followed by formally presenting the model and its relations to the above areas. Our main results regarding consensus and convergence rate of the model under fixed and changing social networks are presented in Section 2 and Section 3, respectively. Section 4 concludes the paper.

Opinion dynamics model and network-based dynamical systems have received a surge of attention lately [12–16]. In these systems, typically, a group of agents will interact by communicating through a sequence of graphs. The original goal of opinion dynamics is to model the formation and propagation of opinions and knowledge in a crowd of interacting
individuals \[17,18\]. Later the model grows its popularity due to its widespread use in
economics and social sciences \[11,12,14\]. Opinion dynamics models are usually diffusive,
in the sense that the new opinion comes from a convex combination of the old ones \[19\].
Thus, in its matrix form, the dynamics of the opinions is equivalent to repeated stochastic
matrix multiplications to the opinions. We refer interested readers to \[20–23\] for more
detailed discussions on product of stochastic matrices and averaging processes.

The major question to ask regarding opinion dynamics systems is whether the agents
will achieve consensus, a state that all the agents share the same opinion. In fact, we will
show that, with the existence of a static agent (truth), the system achieves consensus
under mild assumptions about the network structure. For fixed graph, convergence is
achieved of polynomial order \(O(t^{-\nu})\), where \(\nu\) is the spectral gap of the graph Laplacian.
For periodically changing graphs, the system converges to consensus asymptotically of
order \(O(t^{-1/d_{\text{max}}})\), where \(d_{\text{max}}\) is the maximum total outdegree in a period. Both the
convergence bounds are tight. If the graphs are generated randomly, we obtain an
almost-sure convergence result. We note that related opinion dynamics systems usually
feature exponential convergence \[1,13,23\]. The slower convergence in our model origi-
nates from the increasing self-confidence: larger self-weight after repeated interactions
makes the agents reluctant to move.

1.1 The Model

The system consists of \(n\) agents represented by scalars or vectors \(x_1(t), x_2(t), \ldots, x_n(t)\),
where \(t = 0, 1, 2, \ldots\) is the time. The interactions between agents are captured by a
sequence of graphs \(G(0), G(1), G(2) \ldots\), where \(G(t)\) can be any directed graph over \(N\),
and self-loop is allowed. It should be noted that we do not assert constraint of how
\(G(t)\) is formed, and thus \(G(t)\) can be fixed, arbitrarily specified, randomly generated, or
even coupled with the opinions. A truth (static agent) refers to the agent which never
interacts with other agents, and thus it is stuck at its initial position eternally. Though
being interesting, it is not necessary for a truth to exist in the system; if it does, the
other mobile agents are usually referred to as the learners. We use \(N_i(t)\) to denote the
neighbor set of \(i\) at time \(t\), and \(j \in N_i(j)\) if and only if edge \((i, j) \in G(t)\). In this case,
\(i\) gets information from \(j\), or equivalently, \(j\) influences \(i\). The dynamics of the model is
written as

\[
x_i(t + 1) = \frac{w_i(t)x_i(t) + \sum_{j \in N_i(t)}x_j(t)}{w_i(t) + |N_i(t)|},
\]

where the weight \(w_i(t)\) is a scalar associated with agent \(i\) at time \(t\). \(w_i(t)\) is regarded as
\(i\)’s self-confidence, representing the degree of how much agent \(i\) believes in its current
opinion. In spite of various seemingly plausible ways of modeling the dynamics of the
self-confidence \(w_i(t)\), we assume in each step \(t\), \(w_i(t)\) increases by the number of \(i\)’s
neighbors \(|N_i(t)|\):

\[
w_i(t + 1) = w_i(t) + |N_i(t)|.
\]

The intuition is that after communicating with \(|N_i(t)|\) agents and obtaining \(|N_i(t)|\) sig-
nals, the amount of self-confidence should also increase in that amount. The modeling
of the dynamics of self-confidence [2] is further justified by the relations between the proposed model and existing works on social learning, Bayesian inference, inertial opinion dynamics, and multi-armed bandits in Section 1.2, 1.3, and 1.4. In the mean time, our results regarding the proposed model in Section 2 and 3 directly apply to the above fields.

1.2 Relation to Bayesian without Recall Social Learning

In the framework of social learning, a group of learners (mobile agents) tries to learn the state of the world denoted by a truth (static agent) via a social network. Recently, Rahimian and Jadbabaie proposed the so-called Bayesian without recall (BWR) model [24], in which the agents are assumed to be rational and memoryless. In the BWR model, each agent adopts an initial belief, updates the belief via Bayes’s rule based on signals transferred from the truth or other agents, while ignoring the mechanism behind the data generating process [25–27].

Assume the belief of agent \(i\) is Gaussian distributed \(N(\mu_i(t), \sigma^2_i(t))\), and the signal being transferred \(d_j(t)\) is noisy measurement of \(j’s belief \(d_j(t) \sim N(\mu_j(t), \sigma^2_j(t)) + \epsilon_j(t)\), where \(\epsilon_j(t) \sim N(0, \sigma^2)\) is independent Gaussian noise. In [24], the explicit update rule is demonstrated as

\[
\mu_i(t) = \frac{\tau_i(t)\mu_i(t) + \tau \sum_{j \in N_i(t)} \mu_j(t)}{\tau_i(t) + \tau|N_i(t)|},
\]

(3)

where \(\tau_i(t) := \sigma_i^{-2}(t)\) and \(\tau = \sigma^{-2}\) are inverse variances following update rule

\[
\tau_i(t + 1) = \tau_i(t) + \tau|N_i(t)|,
\]

(4)

By taking expectations on both sides of (3), it is clear that \(E\mu_i(t)\) together with the weight \(\tau_i(t)\) follows the proposed increasing self-confidence model.

1.3 Relation to Inertial Hegselmann-Krause System

In the famous Hegselmann-Krause (HK) system, each agent moves to the mass center of all the agents within a fixed distance \(R\) [17]. Stubborn agent in an HK system moves toward the mass center of its neighbors by any fraction of length:

\[
x_i(t + 1) = (1 - \lambda_i(t))x_i(t) + \frac{\lambda_i(t)}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t).
\]

(5)

Setting this fraction to zero makes the agent static. HK systems with stubborn or static agents attract much attention recently, for the model is more realistic and addresses issues like symmetry breaking and non-shrinking convex hull [11, 28, 29]. The factor \((1 - \lambda_i(t))\) in (5) can be regarded as the normalized self-confidence in the proposed model. In the extreme case of static agents, the corresponding self-confidence is infinity. We note that in [11], the inertial \(\lambda_i(t)\) is endowed with more degrees of freedom and may follow different dynamics other than (2).
The self-confidence $w_i(t)$ should be distinguished from the confidence bound. The former refers to the self-weight during the convex combination, while the latter is the cut-off threshold of the neighbor set.

### 1.4 Relation to Bayesian Inference and Multi-armed Bandits

To evaluate a quantity $\theta$ by repeated taking noisy measurement $d(t) = \theta + \epsilon(t)$, one can adopt a Gaussian estimator $N(\mu(t), \sigma^2(t))$ and update it sequentially according to Bayes’ rule by formula

$$\mu(t + 1) = \frac{\tau(t)\mu(t) + \tau d(t)}{\tau(t) + \tau},$$

where the inverse variance $\tau(t) = \sigma^{-2}(t)$ and $\tau = \sigma^{-2}$. Note that $E\mu(t)$ forms an increasing self-confidence model of a single truth and a single learner.

The Bayesian multi-armed bandit problem considers $K$ arms with unknown values $\theta_1, \ldots, \theta_K$. A learner sequentially picks arms and obtains noisy feedbacks. Based on (6), all the $E\mu_k(t)$ together with the arms form an increasing self-confidence model with $K$ leaners and $K$ truths. In multi-armed bandits, the goal is either to identify the best arm $\arg \max_k \mu_k$, or to maximize the expected cumulative rewards $\max \sum_{t \leq H} \theta_k(t)$, which lies on whether and how fast $E\mu_i(t)$ or $\mu_i(t)$ converges to $\theta_k$. In the language of opinion dynamics, the problem is whether and how fast consensus can be achieved for each pair $\theta_k$ and $E\mu_k$.

### 1.5 Dynamics in Matrix Form

Let $A(t)$ denote the associated adjacency matrix of $G(t)$, thus $a_{ij}(t) = 1$ if $(i, j) \in G(t)$, and otherwise $a_{ij}(t) = 0$. We use $D(t)$ to denote the outdegree matrix of $G(t)$: $D(t)$ is a diagonal matrix with its $i$-th diagonal element being the outdegree of $i$ in $G(t)$. The weight matrix $W(t) := \text{diag}(w_1(t), w_2(t), \ldots, w_n(t))$, can alternatively be defined as $W(t + 1) = W(t) + D(t)$ for $t \geq 0$, where $W(0)$ is the initial self-confidence.

We use the letter without the subscript $i$, namely $x(t)$, to denote the column vector $(x_1(t), x_2(t), \ldots, x_n(t))^T$. In this symbol system, the dynamics [1] is written as

$$x(t + 1) = (W(t) + D(t))^{-1} (W(t) + A(t)) x(t).$$

Notice that $D(t)$ is the outdegree matrix of $A(t)$, then $(W(t) + D(t))^{-1} (W(t) + A(t))$ is a row-stochastic matrix. For simplicity, whenever a row $i$ of matrix $W(t) + D(t)$ is zero, the dynamics [7] should be understood in the sense $x_i(t + 1) = x_i(t)$, since a zero row of matrix $W(t) + D(t)$ means the corresponding agent never interacts with anyone. Without loss of generality, we assume the initial self-confidence is 0 for each agent throughout this paper. For any matrix $M$ used in the paper, we use the small letter with double subscripts $m_{ij}$ to denote the $(i, j)$-th element of $M$. 
2 Truth Seeking in Fixed Social Network

In this section, we consider the increasing self-confidence model with a truth and fixed graph \( G(t) = G \). In this case, \( A(t) = A, \ D(t) = D, \ W(t) = tD \), and the matrix-form dynamics (7) becomes

\[
x(t + 1) = ((t + 1)D)^{-1}(tD + A)x(t).
\]

Notice that, unlike the DeGroot model [18], a sequence of the same graph does not lead to repeated multiplication of the same stochastic matrix. This is because the self-confidence of an agent will increase after it receives signals, and thus the update rule (1) does not remain the same for different \( t \).

Without loss of generality, we assume that agent 1 is the truth that stays at the origin forever. Thus, \( a_{1,j} = 0 \) for all \( j > 1 \). Notice that the value of \( a_{11} \) will not affect the dynamics, hence we assume \( a_{11} = 1 \) for convenience. By recursively adopting (8), we have

\[
(t + 1)Dx(t + 1) = Ax(t) + tDx(t)
\]

\[
= Ax(t) + Ax(t - 1) + (t - 1)Dx(t - 1)
\]

\[
= Ax(t) + \cdots + Ax(0).
\]

Let \( S(t) := x(t) + x(t - 1) + \cdots + x(0) \), then from (9), the dynamics of \( S(t) \) is written as

\[
S(t + 1) = \left( I + \frac{D^{-1}A}{t + 1} \right) S(t).
\]

By repeatedly adopting (10), we obtain a formula of \( x(t) \) as

\[
x(t) = S(t) - S(t - 1) = (tD)^{-1}AS(t - 1)
\]

\[
= \frac{D^{-1}A}{t} \prod_{s=1}^{t-1} \left( I + \frac{D^{-1}A}{s} \right) x(0).
\]

Intuitively, if we replace the matrix \( D^{-1}A \) with a real number \( 0 < \rho < 1 \), then the product of matrices in (11) becomes

\[
\frac{\rho}{t} \prod_{s=1}^{t-1} \left( 1 + \frac{\rho}{s} \right) \leq \frac{\rho}{t} \exp \left( \rho \sum_{s=1}^{t-1} \frac{1}{s} \right) = O(t^{\rho - 1}).
\]

Therefore the vanishing speed is of polynomial order \( O(t^{\rho - 1}) \). Back to the dynamics of (11), we claim that whether \( x_i(t) \) vanishes (converges to the truth) depends on the difference between 1 and the modulus of the second largest eigenvalue of \( D^{-1}A \), which in fact is the spectral gap of \( G \). Formally, we claim:

**Theorem 2.1.** If the graph of the system \( G_t = G \) is fixed, and each learner has a path to the truth in \( G \), then the system converges to the truth in polynomial order \( O(t^{-\nu}) \), where \( \nu > 0 \) is the spectral gap of \( G \).
Notice that under the assumption that each learner has a path to the truth, the outdegree of each agent is positive, and thus $D$ is invertible.

**Proof.** The proof proceeds in two stages. We will first show that $\nu > 0$, then we will prove the convergence and estimate the convergence rate. Since $D^{-1}A$ is a row-stochastic matrix, therefore 1 is its largest eigenvalue. To prove that $\nu > 0$, it is sufficient to prove that $D^{-1}A$ does not have other eigenvalues with modulus 1. By regarding the truth and the learners as two groups, it is clear that $D^{-1}A$ is a block lower-triangular matrix. We use $E$ and $B$ to denote the remaining matrices by removing the first row and the first column of $D$ and $A$, respectively. What is left to show is that $E^{-1}B$ does not have an eigenvalue $\lambda^*$ on the unit circle in the complex plane.

Suppose otherwise $E^{-1}Bu = \lambda^*u$ for a non-zero vector $u$, then $(\lambda^*E - B)u = 0$. Recall that $D$ is the outdegree matrix of $A$, we have

$$|\lambda^*e_{ii} - b_{ii}| \geq e_{ii} - b_{ii} \geq \sum_{j \neq i} b_{ij},$$

and the second inequality is strict if $i$ has an edge to the truth. Suppose $|u_k| = \max_i |u_i|$, then from $(\lambda^*E - B)u = 0$ we have

$$0 = |(\lambda^*e_{ii} - b_{ii})u_i - \sum_{j \neq i} b_{ij}u_j| \geq |\lambda^*e_{ii} - b_{ii}||u_i| - \sum_{j \neq i} |b_{ij}||u_j| \geq 0.$$  \(14\)

One conclusion from (14) is that equalities hold in (13), which means $i$ does not have an edge to the truth. Furthermore, (14) implies that for all $b_{ij} \neq 0$, $|u_j| = |u_i|$ also has the largest modulus. Therefore, by repeating the same argument of $i$ to $j$, it is clear that $j$, and hence any learner reachable from $i$ in $G$, can not have edge to the truth. This contradicts the assumption in Theorem 2.1 that each learner has a path to the truth. Therefore, $|\lambda^*| < 1$ and thus we have proved that $\nu > 0$.

For the convergence of the system, let $\tilde{x}(t) = (x_2(t), \ldots, x_n(t))^T$ denote the dynamics of the learners, then it follows the update rule:

$$\tilde{x}(t) = \frac{E^{-1}B}{t} \prod_{s=1}^{t-1} \left( I + \frac{E^{-1}B}{s} \right) \tilde{x}(0).$$  \(15\)

Now let $U$ be an invertible matrix such that the similarity transformation $U^{-1}E^{-1}BU$ is the Jordan normal form of $E^{-1}B$, then for the vanishing speed of the product of matrices in (15), it is equivalent to estimate the vanishing speed of

$$Q^{(\Lambda)}(t) := \frac{\Lambda}{t} \left( I_{k \times k} + \frac{\Lambda}{t-1} \right) \cdots (I_{k \times k} + \Lambda),$$

where $\Lambda$ is any Jordan block of matrix $U^{-1}E^{-1}BU$:

$$\Lambda = \lambda I_{k \times k} + J, \quad J = \begin{pmatrix} 0_{(k-1) \times 1} & I_{(k-1) \times (k-1)} \\ 0 & 0_{1 \times (k-1)} \end{pmatrix},$$  \(16\)
and λ is an eigenvalue of $E^{-1}B$. Since $J^k = 0_{k \times k}$, and it commutes with multiples of $I_{k \times k}$ in the definition of $Q(\Lambda)(t)$ in (16), then for any $\lambda \neq 0$,
\[
\max_{ij} |q_{ij}(t)| = O\left(\frac{|\lambda|^{t-1}}{t} \prod_{s=1}^{t} \left(1 + \frac{|\lambda|}{s}\right)\right) = O(t^{\lambda^{-1}}).
\]

Notice that $U$ only depends on the graph $G$. As a result,
\[
\max_i |x_i(t)| = \max_{\lambda} O(t^{\lambda^{-1}}) = O(t^{\max |\lambda|^{-1}}) = O(t^{-\nu}),
\]
which completes the proof. A prevailing assumption of the graph structure in social learning is the network being strongly connected [1, 4], which guarantees the truth to be reachable by each learner. We note that the reachability of truth is similar to the definition of rooted graph in [23], in which Cao et al. carefully analyzed the convergence of the system with fixed self-confidence.

In addition to the convergence result, we prove that the bound of convergence is tight by constructing the following system. The initial positions $\bar{x}_i(0) = 1$ for $i \geq 2$; the outdegree of each learner is the same number $d \geq 1$; each learner has one edge to the truth; and $b_{ij} = 1$ if and only if $j = i, i + 1, \ldots, i + d - 2$, where the subscript is understood modulo $(n - 1)$. In the extreme case $d = 1$, $B$ is a zero matrix. Under this construction, $E = d I_{(n-1) \times (n-1)}$, and $E^{-1}B$ is a circulant matrix whose eigenvalues are straightforward to get: $\lambda_k = (1 + \omega_j + \cdots + \omega_j^{d-2})/d$, where $\omega_j = \exp(2\pi i j/(n - 1))$ is the $(n - 1)$-th root of unity and $i$ stands for the imaginary unit. It is clear to check that $|\lambda_k| \leq 1 - 1/d$ and thus $\nu = 1/d$.

On the other hand, since each learner starts at the same position and talks to the same number of learners, we have $x_2(t) = \cdots = x_n(t)$. In view of (11),
\[
x_2(t) = \prod_{s=1}^{t} \left(1 - \frac{1}{sd}\right) = \exp \left(\sum_{s=1}^{t} \log \left(1 - \frac{1}{sd}\right)\right).
\]
From Taylor expansion, we have $\log(1 - c) \geq -c - c^2/2$ for any $0 \leq c < 1$. Therefore
\[
x_2(t) \geq \exp \left(-\frac{1}{d} \sum_{s=1}^{t} \frac{1}{s} - \frac{1}{2d^2} \sum_{s=1}^{t} \frac{1}{s^2}\right) = \Omega(t^{\frac{1}{d}}) = \Omega(t^{-\nu}).
\]
We have proved

**Proposition 2.2** For any $n$, there exists a graph $G$ with spectral gap $\nu$ such that $\|x(t)\|_\infty = \Omega(t^{-\nu})$.

### 3 Changing Social Networks

In this section, we consider the increasing self-confidence model in changing social networks. In order to achieve consensus, information from the truth should be well spread to the learners. Indeed, if an agent is not able to get signals from the truth constantly, its dynamics should eventually be free from the influence of the truth. This intuition leads to the definition and analysis of influence indicator.
3.1 The Influence Indicator

We use $P(t)$ to denote the remaining matrix by removing the first row and the first column of $D^{-1}(t)A(t)$. Recall that whenever $d_{ii}(t) = 0$, $p_{ij}(t)$ is set to 0 for all $j$. Note that $P(t)$ is a sub-stochastic matrix since its row-sums are no greater than 1. If at time $t$, agent $i$ has an edge pointing to the truth, then $d_{ii}(t) = 1 + a_{12}(t) + a_{13}(t) + \cdots + a_{1n}(t)$. Thus, the corresponding row-sum is strictly less than one, implying non-zero impact from the truth. Indeed, if the row-sum is exactly 1, then the dynamics of the corresponding agent is completely determined by only the learners. Therefore, the difference between 1 and each row-sum of $P(t)$ is an indicator of the influence from the truth. Formally, we define the influence indicator from time $s$ to time $t$, denoted by $\alpha(t : s)$, as:

$$\alpha(t : s) = \xi - P(t : s)\xi,$$

where $\xi$ is the all-one column vector. When $t = s + 1$, (17) is reduced to

$$\alpha(s) = \xi - P(s)\xi.$$  \hspace{1cm} (18)

Notice that $\alpha(t) = 0$ indicates $1 \notin Ni(t)$. By repeatedly left-multiplying $P(s + 1), P(s + 2), \ldots, P(t - 1)$ to both sides of (18), we obtain

$$\alpha(t : s) = \sum_{k=s}^{t-1} P(t : k)\alpha(k),$$ \hspace{1cm} (19)

which builds up the relation between the single-step indicator $\alpha(k)$ and the multi-step indicator $\alpha(t : s)$.

The dynamics of the learners $\tilde{x}(t)$ is determined by the product of $P(s)$ for $s \leq t$: $\tilde{x}(t) = P(t : 0)\tilde{x}(0)$, and thus

$$\|\tilde{x}(t)\|_{\infty} \leq \|P(t : 0)\|_{\infty}\|\tilde{x}(0)\|_{\infty}. \hspace{1cm} (20)$$

On the other hand, taking infinity norm on both sides of (17) yields

$$\|P(t : s)\|_{\infty} = 1 - \min_i \alpha_i(t : s). \hspace{1cm} (21)$$

Therefore, whether $\tilde{x}$ converges to the truth depends on the minimum value of the indicator.

To estimate the indicator, notice that matrix $P(t)$ is non-negative, thus $p_{ii}(t : k) \geq p_{ii}(t - 1)\ldots p_{ii}(k)$. In addition, for any learner $i$ and time $r$, $p_{ii}(r) = (w_i(r) + a_{i1}(r))/w_i(t + 1) \geq w_i(r)/w_i(t + 1)$. Therefore, for any $k \geq s$,

$$p_{ii}(t : k) \geq \prod_{r=k}^{t-1} \frac{w_{ii}(r)}{w_{ii}(r + 1)} = \frac{w_{ii}(k)}{w_{ii}(t)} \geq \frac{w_{ii}(s)}{w_{ii}(t)}. \hspace{1cm} (22)$$

Again, when $w_{ii}(t) = 0$, then $w_{ii}(s) = 0$, and (22) should be understood as the trivial inequality $p_{ii}(t : k) \geq 0$. In view of (19), (22), and notice that the matrices and vectors involved are all non-negative, we obtain a lower bound estimate for the indicator. Formally, we have
Lemma 3.1 For any $t > s \geq 0$, the following inequality of the influence indicator holds:

$$\alpha(t:s) \geq W^{-1}(t)W(s) \sum_{k=s}^{t-1} \alpha(k).$$

(23)

The inequality (23) is element-wise. We will also use matrix inequality in the same sense for the rest of this paper.

3.2 Periodic Graph Sequence

In this subsection, we consider the case when the graph sequence is periodic: $G_{t+T} = G_t$ for $t \geq 0$, where $T \geq 1$ is the period. Note that the special case $T = 1$ reduces to the fixed graph scenario. We define the total outdegree of agent $i$ in a period $d_i := d_{ii}(1) + \cdots + d_{ii}(T)$, and its maximum $d_{\text{max}} := \min_{2 \leq i \leq n} d_i$. Since the graph sequence is periodic, the self-confidence $w_{ii}(kT) = k d_i$ grows linearly. We first state our main result in this subsection:

Theorem 3.2 If the graph sequence $G_t$ is periodic, and each learner has at least one edge to the truth in a period, then the system will converge to the truth in the order $O(t^{-1/d_{\text{max}}})$, and the bound is tight.

Proof. Since each learner has at least one edge to the truth in a period, then $a_{i1}(kT) + a_{i1}(kT + 1) + \cdots + a_{i1}((k + 1)T - 1) \geq 1$ for any $2 \leq i \leq n$, and thus each element of the indicator $\alpha((k + 1)T : kT)$ is positive. More precisely,

$$\alpha_i((kT + l) = \frac{a_{i1}(kT + l)}{w_{ii}(kT + l)} \geq \frac{a_{i1}(kT + l)}{w_{ii}((k + 1)T)} = \frac{a_{i1}(kT + l)}{(k + 1)d_i},$$

and thus

$$\sum_{r=kT}^{(k+1)T-1} \alpha(r) \geq \frac{1}{(k + 1)d_i} \sum_{r=kT}^{(k+1)T-1} a_{i1}(r) \geq \frac{1}{(k + 1)d_i}.$$ 

Based on Lemma 3.1 for $k \geq 1$:

$$\min_{2 \leq i \leq n} \alpha_i((k + 1)T : kT) \geq \frac{k}{(k + 1)^2d_{\text{max}}} \geq \frac{1}{(k + 3)d_{\text{max}}}.$$ 

In view of the last inequality and (21),

$$\|P((k + 1)T : kT)\|_{\infty} \leq 1 - \frac{1}{(k + 3)d_{\text{max}}}. \quad (24)$$

Now let $t = rT$, by the sub-multiplicativity of the matrix infinity norm,

$$\|P(rT : 0)\|_{\infty} \leq \prod_{k=1}^{r-1} \|P((k + 1)T : kT)\|_{\infty}$$

$$\leq \prod_{k=4}^{r+2} \left(1 - \frac{1}{kd_{\text{max}}}\right) \leq \exp \left(-\frac{1}{d_{\text{max}}} \sum_{k=4}^{r+2} \frac{1}{k}\right)$$

$$\leq O(r^{-1/d_{\text{max}}}) = O(t^{-1/d_{\text{max}}}). \quad (25)$$

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This completes the proof of convergence.

For any positive integer \( d_{\max} \) and period \( T \), we build a system with graph sequence \( G(t) \) such that \( G(kT) \) is the graph in the proof of Proposition 2.2 and \( G(kT + r) \) is an empty graph for \( 1 \leq r \leq T - 1 \). Then we have

\[
\|x(t)\|_\infty = \Omega((t/T)^{-1/d_{\max}}) = \Omega(t^{-1/d_{\max}}),
\]

which completes the proof. Note that \( d_{\max} \) can be interpreted as the maximum total degree centrality of all the learners, which indicates a slower convergence rate when highly-important learner exists.

### 3.3 Random Graph Sequence

Now we consider the scenario when the graph \( G(t) \) is randomly sampled. There are various ways of generating a random graph, and we adopt the following scheme. In each step, every learner randomly picks \( d_i \) agents as its neighbors, where \( 1 \leq d_i \leq n \) is a fixed integer, and we define \( d_{\max} = \max_{2 \leq i \leq n} d_i \). We will show an almost-sure convergence result:

**Theorem 3.3** Assume in each step, each learner \( i \) independently picks \( d_i \) agents as its neighbors uniformly at random, then almost surely, the system converges to the truth, and the convergence rate is polynomial in \( t \).

**Proof.** Since in each step, the outdegree of agent \( i \) is the fixed number \( d_i \), then the self-confidence \( w_{ii}(t) = d_i t \). In view of (21),

\[
\|P(t)\|_\infty = 1 - \min_i \alpha_i(t) \leq 1 - \frac{1}{d_{\max} t} \min_i a_{i1}(t). \tag{26}
\]

Notice that \( a_{i1}(t) \) is a Bernoulli random variable with \( \mathbb{P}[a_{i1}(t) = 1] = d_i/n \). Define random process \( \beta(t) = \min_i a_{i1}(t) \). Since \( a_{i1}(t) \) and \( a_{j1}(t) \) are independent for \( i \neq j \), then \( \beta(t) \) is again a Bernoulli random variable, and

\[
q : \mathbb{P}[\beta(t) = 1] = d_2 d_3 \ldots d_n / n^{n-1}.
\]

Based on (26),

\[
\|B(t : 0)\|_\infty \leq \prod_{s=1}^t \left( 1 - \frac{\beta(s)}{sd_{\max}} \right) \leq \exp \left( -\frac{1}{d_{\max}} \sum_{s=1}^t \frac{\beta(s)}{s} \right).
\]

We only need to show that, with probability one, the sum of the series \( \beta(s)/s \) goes to infinity.

Let \( \gamma(s) := \beta(s) - \mathbb{E}\beta(s) \), then \( \mathbb{E}\gamma(s) = 0 \) and \( \var\gamma(s) = \var\beta(s) = q(1-q) \). Define the random process

\[
V(s) := \sum_{k=1}^s \gamma(s)/s
\]
and $\mathcal{F}(s)$ the sigma algebra generated by $V(0), \ldots, V(s)$. Then

$$\mathbb{E}[V_{s+1}|\mathcal{F}(s)] = \frac{\mathbb{E}v_s}{s} = 0,$$

hence $V(s)$ is a martingale. In addition,

$$\mathbb{E}V_s^2 = \sum_{k=1}^{s} \frac{\mathbb{E}v^2(s)}{s^2} \leq \sum_{k=1}^{s} \frac{1}{s^2} < \infty.$$ 

Therefore by Martingale convergence theorem, the random variable $V_\infty =: \lim_{s \to \infty} V_s$ exists and has finite variance [30]. Now let $\beta := \mathbb{E}\beta(s)$, then almost surely

$$\|B(t:0)\|_\infty \leq \exp \left( - \frac{1}{d_{\text{max}}} \sum_{s=1}^{t} \frac{\beta + \gamma(s)}{s} \right) \leq O(t^{-\frac{\beta}{d_{\text{max}}}}),$$

which completes the proof.

We simulate the system with $n = 20, 50, 100$ agents with $m = 1, 5, 10$. The initial positions of the learners are uniformly sampled in the unit interval. For each pair $(n,m)$, we simulate 100 independent systems and calculate the averaged error $\|x(t)\|_\infty$. The log-log curve of each case is demonstrated in Figure 1. It is clear that the curves eventually become straight lines, indicating a convergence rate of polynomial order.

![Figure 1: Log-log curve of $\|x(t)\|_\infty$ and $t$ for $n = 20, 50, 100$ and $m = 1, 5, 10$. The curves eventually become straight lines with negative slopes.](image)

We further use linear regression to get the slope $f_{n,m}$ of each curve after it becomes steady, and obtain a rough relation: $f_{n,m} \approx -1/n$, which does not depend on $m$. For
learner $i$, when $m$ increases, it is more likely for the truth to be a neighbor of $i$, which contributes the convergence. But in the mean time, there are more learners in the neighbor set of $i$, which harms the convergence since the information from other learners is less perfect compared to the information from the truth.

### 3.4 Discussions: Multiple Truths and Zero Truth

The presence of multiple truths will immediately complicates the behavior of the system. For example, if a learner communicates with truth 1 enough times in order to be in the vicinity of truth 1, and then starts to communicate with truth 2 and does the same, by repeating this process, the learner will oscillate between the two truths forever. If the system contains no truth, then the previous example could still happen: we only need to replace each truth by two colliding agents. Nevertheless, from the proof of Theorem 2.1, it is clear that the components of each learner that is perpendicular to $S$ should vanish in polynomial order, where $S$ is the space spanned by all the truths. The dynamics in the perpendicular space $S$ is more involved in the graph sequence $G(t)$. We note that more powerful techniques are required for such general cases.

### 4 Conclusion

In this paper, we proposed an opinion dynamics model with increasing self-confidence. The growing confidence of an agent after it repeatedly communicates with others is reflected in its increasing self-weight. We proved that, with fixed or periodically changing social network and a single truth, the system achieves consensus asymptotically and a tight convergence rate of polynomial order is obtained. If each learner randomly selects a fixed number of neighbors, then the system is proved to converge to the truth almost surely. We also discussed the behavior of the model when zero truth or multiple truths are present, which requires delicate analysis in the future.

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