Kleene, Rogers and Rice Theorems
Revisited in C and in Bash

Salvatore Caporaso
(caporaso@di.uniba.it)
Nicola Corriero
(nicolacorriero@gmail.com)
Dipartimento d’Informatica dell’Università di Bari

Abstract

The recursion theorem in the weak form \(\{e\}(z) = x(e, z)\) (universal function not needed) and in Rogers form \(\phi_{\phi_{\phi_e}}(z) = \phi_z\), and Rice theorem are proved a first time using programs in C, and a second time with scripts in Bash.

1 Introduction

One of the cornerstones of recursion theory is the result known as \(S - m - n\) theorem (in honour of the original notation by Kleene, who called it Iteration Theorem) or as Parameter Theorem (after Schoenfield). Its proofs in Literature however are not fully satisfactory for a computer scientist. Some authors merely appeal to Church thesis (Rogers [6], Cutland [1], Enderton [2]). Some others arithmetize the metaprocessing, and this disguise the computation under a misleading plenty of numerical technicalities (Kleene [5], Smorynski[7]). The proof by Kechris and Moschovakis [3] use a universal function, which is not available when classes of total functions are discussed, like in the case, for example, of security or complexity classes. Among the consequences of the \(S - m - n\) we have the Kleene weak form of the Recursion Theorem (existence of a fixed-point value), the Rogers form (functional fixed-point), Rice theorem, the analysis by Thompson [8] in his Turing lecture of relationship between malware and Quine’s indirect self-referential paradox. We feel that simple idea need simple programs, and that, therefore, understanding these phenomena needs their revisitation in terms of real programming. In this paper we show that this is rather easy in C; and even easier in a language allowing quick writing of rough programs like Bash. To this purpose, we code, in both these languages, the procedures needed to prove the results mentioned above.

A by-result of this work is that one can show significant results in a couple of lectures in the context of a beginners programming course: there is no need of the cumbersome paraphernalia of abstract models of computation like TMs, recursive functions or functional programming.

Familiarity with C and/or Bash is not needed to follow the broad outlines of our discussion. To check the details one needs the small amount of information which is contained in Kernighan & Ritchie [4] Ch. 1 A Tutorial Introduction pp.5-30. All additional details about C, and the essential parts for Bash are explained by means of examples.
2 In C

1 Notation (1) \( \Sigma \) is the set of all strings in the alphabet of all characters that may occur in a C source file. \( \phi^{(n)} \) is a (partial) \( n \)-ary function such that \( \phi^{(n)} : \Sigma^n \rightarrow \Sigma \) (\( n = 1, 2 \) often omitted). \( \phi(y) = u \) is short for \((y,z) \in \phi \) or \( \phi(y) \uparrow \). (2) \( a, b \) and \( c \) are autonomous names for the three fixed identifiers consisting of resp. the 1st, 2d and 3d low-case letter of the Latin alphabet. They will play a crucial role throughout this paper. (3) \( r, \ldots, z \), possibly followed by decimal digits, are generic or constant strings. (4) We are going to discuss the behaviour of certain (C functions defined by) strings \( x \) of the form

\[
x_\_()\ldots
\]

where \( x_\_ \) denotes the identifier used in calls to \( x \). To this purpose, we write \( a=y \) to mean that the string variable \( a \) is assigned with \( y \). And we write \( x:y,z=u \) if after a call \( x_\_() \); with \( a=y \) and with \( b=z \) we get \( c=u \). Calls are tacitly assumed to be syntactically correct, and to include all needed directives and declarations.

2 Definition String \( x \) of the form \( A \) standard computes (s-computes) function \( \phi \) if we have

\[
x : y, z = u \quad \text{iff} \quad u = \phi(y,z)
\]

(\( z \) absent, and \( b \) immaterial for \( n = 1 \)).

Notation \( \phi_x \) is the function s-computed by \( x \).

Example Let \( \text{id} \) and \( \text{s1} \) denote resp. the strings (see \( \S 3 \) for \( 	ext{strcpy} \))

\[
\begin{align*}
\text{id}() & \\
& \quad \{ \\
& \quad \quad \text{strcpy} \ (c,a); \\
& \quad \}
\end{align*}
\]

\[
\begin{align*}
\text{s1}() & \\
& \quad \{ \\
& \quad \quad \text{strcpy}(c,a); \\
& \quad \quad \text{strcpy}(b,a); \quad \text{// comment: just to use two variables} \\
& \quad \}
\end{align*}
\]

we have \( \text{id}:y=y \) and \( \text{s1}:y,z=y \) and, therefore,

\[
\phi_{\text{id}}(y) = y \quad \phi_{\text{s1}}(y,z) = y
\]

3 Summary of string functions from the standard library Recall that the following functions are defined in \( < \text{string.h} > \)

\[
\begin{align*}
\text{strcpy}(s,t) & \quad s = t \quad \text{(i.e. \( t \) is copied into \( s \))} \\
\text{strcat}(s,t) & \quad s = st \quad \text{(concatenation)} \\
\text{strchr}(s, 'c') & \quad \text{locates the 1st occurrence in \( s \) of character 'c'} \\
\text{strncpy}(s,t,i) & \quad s = \text{first i characters of \( t \) (doesn't add '"\0')}
\end{align*}
\]

In what follows we need the C function defined by \( \text{fn=} \)
fn_(){
    int i;
    i=strchr(s,'()'); // pointer to the leftmost parenthesis
    strncpy(t,s,i);
    t[i]='$\0$'; // because strncpy doesn't do it
}

fn takes the definition of a function into the function name, in the sense that we have

\[ fn:x_()\{... = x_ \]

4 Diagonal Substitution Lemma (A variant of the s-m-n theorem). There is a C function ds which s-computes the function \( \sigma^{(1)} \) such that for all \( \phi_x^{(2)}(y,z) \) we have

\[ \phi_{\sigma(x)}(y) = \phi_x(x,y); \] or, in other terms, \( ds : x = u \) implies \( \phi_u(y) = \phi_x(x,y) \)

**Proof.** Define \( ds = \)

ds_(){
    fn_(); // c=x_
    strcpy(b,c); // b=x_
    strcpy(c,\"s_({})\{strcpy(b,a);strcpy(a,\""); // c=s_(){strcpy(b,a);strcpy(a,"
    strcat(c,\"a\"); // c=s_(){strcpy(b,a);strcpy(a,"x
    strcat(c,\"\"}); // c=s_(){strcpy(b,a);strcpy(a,"x\"");
    strcat(c,b); // c=s_(){strcpy(b,a);strcpy(a,"x\");x_
    strcat(c,\"}\"}); // c=s_(){strcpy(b,a);strcpy(a,"x\")\};x_()
    strcat(c,a); // c=s_(){strcpy(b,a);strcpy(a,"x\")\};x_();x
}

The comments above (at the left of the //'s) show that we have \( ds : x = u \) with (up to unnecessary new lines and indentations added for the sake of readability)

\texttt{\{s=}

s_(){
    strcpy(b,a);
    strcpy(a,\"x\");
    x_();
}
x

So, for \( a=x, (1) \) we put \( b=y \) and \( a=x \); and (2) we call \( x \) with these new values. Hence \( x : x, y=w \) implies, as promised, \( u : y = w \).

5 Kleene Theorem (A weak form of the Second Kleene Theorem) For each \( \phi_x^{(2)}(y,z) \) there is a fixed point \( u \) such that

\[ \phi_u(z) = \phi_x(u,z). \]

**Proof.** Given \( x \) in the form \([A] \), define a new C function by the string \( x0= \)
We have
\[ \phi_{x_0}(y, z) = \phi_x(\sigma(y), z) \] (B)
because (1) by calling \texttt{ds} with \( a = y \) we get \( c = \sigma(y) \); (2) by calling \texttt{x} with \( a = \sigma(y) \) (via \texttt{strcpy(a,c)}) and with \( b = z \) we get \( c = \phi_x(\sigma(y), z) \). Now define
\[ u = \sigma(x_0) \] (C)
The result follows because we have
\[ \phi_u(z) = \phi_{\sigma(x_0)}(z) = \phi_{x_0}(x_0, z) = \phi_x(\sigma(x_0), z) = \phi_x(u, z) \]
where we owe the first equality to definition (C) and the second to Lemma 4, and where we get the last two from resp. (B), and (C) again.

6 Note By applying the theorem to the \texttt{s1} of §2 we get a quine in C, i.e. a function definition that prints itself by means of a so-called \textit{indirect self-reference} of the form
\[ \ldots \text{what quoted is}\ldots \]
This quine includes a comment, which could be replaced by different actions of another kind.

7 The Universal Function One can write a string \texttt{univ = univ()}... defining a C function which \textit{s}-computes a universal function, in the sense that we have for all \( \phi^{(1)}_x \) and \( y \)
\[ \phi_{\texttt{univ}}(x, y) = \phi_x(y). \] (D)
The proof of Theorem 5 needs a few linear-time operations, and, therefore, it holds for almost all total fragments of C. We regard next theorem as a stronger form of that theorem, because its proof, being based on the existence of a universal function, fails with any class of total functions.

8 Rogers Theorem (A strong form of the Second Kleene Theorem) For each \( \phi^{(1)}_x(y) \) there is a value \( v \) such that
\[ \phi_{\phi_x(v)} = \phi_v \]
Proof. Given \( x \) in the form (A) define a new C function by means of the string \texttt{w=}
\[ w\{ \]
\[ x\{ \]
\[ \texttt{strcpy(a,c); } \]
\[ \texttt{univ(); } \]
\[ } \]
\[ \texttt{univ } \]
\[ x \]
We have
\[ \phi_v(y,z) = \phi_{\text{univ}}(\phi_x(y),z) \]  \hspace{1cm} (E)
because (1) by calling \( x \) with \( a=y \) we get \( c = \phi_x(y) \); (2) by copying \( c \) into \( a \) and calling \( \text{univ} \) with this value for \( a \) and with \( b=z \) we obtain (by (F) with \( \phi_x(y) \) as \( x \))
\[ c = \phi_v(y,z) = \phi_{\phi_v}(y(z)) \]  \hspace{1cm} (F)
Our assertion follows by taking as \( v \) the fixed-point for \( w \) which is granted by Theorem §5. Indeed, we then have
\[ \phi_v(z) = \phi_u(v,z) = \phi_{\text{univ}}(\phi_x(v),z) = \phi_{\phi_u}(v(z)) \]
where the first equality follows because \( v \) is the fixed-point for \( \phi_u \); the second by (E), and the last one by (F).

9 Rice Theorem \hspace{0.5cm} All not-trivial classes of \( s \)-computable functions are undecidable.

Proof. Assume (ad abs.) that there is a string \( x=\text{x_0} \{ \ldots \) that \( s \)-computes the characteristic function of \( A \), in the sense that we have
\[ \phi_x(z) = \text{"0"} \iff \phi_z \in A; \quad \phi_x(z) = \text{"1"} \iff \phi_z \notin A. \]  \hspace{1cm} (G)
Since the class is not trivial there exist \( s, t \) such that
\[ \phi_s \in A; \quad \phi_t \notin A. \]  \hspace{1cm} (H)
Define \( y=\text{y_0} \{ \text{x_0} \{ \ldots \) we have
\[ \phi_y(z) = \text{t} \iff \phi_z \in A; \quad \phi_y(z) = \text{s} \iff \phi_z \notin A. \]  \hspace{1cm} (I)
Let \( u \) be the string granted by Rogers Theorem, such that we have
\[ \phi_u = \phi_{\phi_y(u)} \]  \hspace{1cm} (J)
We get the following contradiction
\[ \begin{align*}
\phi_u \in A & \Rightarrow \phi_y(u) = \text{t} & \text{equation (I)} \\
& \Rightarrow \phi_u = \phi_t & \text{equation (I)} \\
& \Rightarrow \phi_u \notin A & \text{equation (H)} \\
\phi_u \notin A & \Rightarrow \phi_y(u) = \text{s} & \text{equation (I)} \\
& \Rightarrow \phi_u = \phi_s & \text{equation (I)} \\
& \Rightarrow \phi_u \in A & \text{equation (H)}. 
\end{align*} \]
3 In Bash

10 Notation (1) foo() is the string stored in file foo. When foo() is a script x we display it along an indented column (with semicolon omitted according to Bash syntax). For example,

```bash
echo()=
echo echo hi! > hi
chmod 755 hi
hi
```
says that file echo contains a script that: redirects the output echo hi! of echo echo hi! from stdout (the monitor) to file hi (l1); grants the execution permissions to file hi()=echo hi! (l2); and calls it (l3).

(2) We use the sign -> to show the Bash prompt. A line like

```bash
-> comm arg1 ... argk
```
says that at the prompt command comm with arguments arg1 ... argk (k ≥ 0) enters from stdin (the console). Below such a line we list the h ≥ 0 lines that the command sends to stdout and the k ≥ 0 created files. The convention of part (1) allows the distinction between the former and the latter ones. For example, to say that echo creates file hi and prints hi!, we write

```bash
-> echo
hi()=
echo hi!
hi!
```

To summarize these notations:
(i) a not-indented column like

```bash
foo()
x
```
means that file foo stores x;
(ii) the same column, below -->...says that foo has been created by the command line ...;
(iii) foo() alone stands for x;
(iv) a not-indented string below a prompt is an output.
(4) comm1 args1 = comm2 args2 says that comm1 args1 and comm2 args2 print the same string — differences in their other effects (i.e. in the created files) do not matter.

11 Summary of useful Bash commands (1) Recall that Bash assigns its internal variables $1, $2,...,$n with the first, second,...,n-th argument of the script being currently executed. So, we have

```bash
id()=
echo $1
-> id foo
foo
```

Since the command cat foo bar sends to stdout the concatenation of foo() and bar(), we have
cat2()=
    cat $1 $2
-> cat2 id cat2
echo $1
cat $1 $2
(2) Assume
-> comm args
u
Bash interprets an expression like $(comm args) as a command substitution of that same expression with u. For example
-> echo $(cat id)
echo $1
(3) Command set arg1 ...argk assigns arg1,...,argk to $1,...,$k. So we have

cat2idcat2()=
    set id cat2
cat2
-> cat2idcat2
echo $1
cat $1 $2

12 Note In a script builder which produces another script built, we include in builder the line chmod 755 built. In all other cases, we tacitly assume that the execution permissions have been granted to the current script, when it has been edited.

13 Scripts arity The arity of script x is, by definition, $n \geq 0$ if the variables $\$1,...,$n occur in x. So, the arity of the previously introduced scripts cat2, id and hi is resp. 2, 1 and 0.

14 Notation $\phi_x^{(n)} (n = 1, 2)$ is the function $\phi : \Sigma^n \mapsto (\Sigma)$ such that we have
-> x y z
u
iff $u = \phi(y,z)$ ($z$ absent for $n = 1$).
Note that we don't need any standard of computation now.

15 Kleene Theorem (A uniform and weak version of the Second Kleene Theorem) There is a script uk such that for all binary scripts x we have
-> uk x
kx()

with kx such that, for all z we have (see §10(4) for this equality)
kx z = x kx z
In other terms, $u_k$ produces uniformly a script $k_x$ such that
\[ \varphi_{k_x}(z) = \varphi_x(k_x, z). \]
So, we can now get the fixed-point uniformly in $x$.

**Proof.** We have

\[
\begin{align*}
\text{uk}() &= \\
&= \text{echo "set k$1 \$1;$(cat $1)">k$1} \\
&\quad \text{chmod 755 k$1} \\
&\to \text{uk x} \\
\text{kx}() &= \\
&= \text{set kx $1} \\
&\quad \text{x()}
\end{align*}
\]

Indeed, when $x$ is assigned to $\$1$ the line \text{echo...} redirects (via a command substitution similar to the one under part (2) of §11) the string
\[ \text{set kx \$1};(x) \]
to file $k_x$. Since the second line of $u_k$ makes script $k_x$ executable, we may conclude, by the semantics of \text{set}, that $k_x z$ behaves like $x k_x z$.

**16 Example** Let us apply the theorem with $\text{cat2}$ as $x$

\[
\begin{align*}
\to \text{uk cat2} & \quad \text{uk with cat2 as x creates executable script kcat2} \\
\to \text{kcat2 id} & \quad \text{kcat2 by input id behaves like cat2 cat2 id} \\
\text{set kcat2 \$1;\text{cat \$1 \$2} & \quad \text{prints (kcat2)} \\
\text{echo \$1} & \quad \text{(id)} \\
\to \text{cat kcat2} & \quad \text{to check this let’s use cat to print directly kcat2} \\
\text{set kcat2 \$1;\text{cat \$1 \$2} & \quad \text{indeed this equals the first output of kcat2}
\end{align*}
\]

**17 Quine** By replacing in the example above $\text{cat2}$ with

\[
\begin{align*}
\text{self}() &= \\
&= \text{cat \$1}
\end{align*}
\]

we get the rather compact \text{quine}

\[
\begin{align*}
\text{kself}() &= \\
&= \text{set kself \$1;\text{cat \$1} \\
\to \text{kself} \\
\text{set kself \$1;\text{cat \$1}}
\end{align*}
\]

But of course the quine can bring some extra luggage.
18 Definition  A *script-maker* is a unary script $x$ that for each string $y$ prints an executable script $u = \varphi_x(y)$ which, in turn, computes a function $\varphi_u(x)$. That is to say that for all $x, y$ there is $u$ such that we have

-> $x$ $y$

$u()$

and for all $z$ there is $w$ such that

-> $u$ $z$

$w$

or $\varphi_u$ is not defined at $z$.

19 Rogers Theorem  (A strong and uniform variant of the Second Kleene Theorem) There is a script $ur$ that for each script-maker $x$ yields a script $krx$ such that, for all $z$ we have

$$\varphi_{krx}(z) = \varphi_{\varphi_x(krx)}(z)$$

*Proof.* Define

$ur() =$

```bash
echo "$1 \$1 > ${1}_;chmod 755 ${1}_; ${1}_ \$2" > r$1
uk r$1
```

For $s1=x$ the two lines of this script produce the two scripts below, one by redirection to $r$1=rx, and the other by application to rx of the uniform procedure of §15

-> $ur$ $x$

$rx()$

$krx()$

the form of the former is

$rx() =$

```bash
x $1 > x_
chmod 755 x_
x_ $2
```
and we have
\[ \varphi_{rx}(y, z) = \varphi_{x}(z) = \varphi_{\varphi_{x}(y)}(z) \] (K)
because the script above, when called with arguments \( y, z \), sends \( u = \varphi_{x}(y) \) to \( x_\omega \) then computes \( \varphi_{u}(z) \) (line \( x_\omega \$2 \) with \$2 = z).
The result now follows by Kleene Theorem since we have
\[ \varphi_{krx}(z) = \varphi_{rx}(krx, z) \]

References
[1] Cutland, N.J. *Computability* Cambridge University Press (1980).
[2] Enderton, H.B.: *Elements of Recursion Theory*. In J. Barwise (ed.) *Handbook of Mathematical Logic*. North-Holland (1977).
[3] Kechris, A.S. and Moschovakis, Y.N.: *Recursion in Higher Types*. In J. Barwise (ed.) *Handbook of Mathematical Logic*. North-Holland (1977).
[4] Kernighan, B.W. and Ritchie D.M.: *The C Programming Language*. Prentice Hall, Second Edition (1988).
[5] Kleene, S.C.: *Introduction to Metamathematics*. North-Holland (1952).
[6] Rogers, H.: *Theory of Recursive Functions and Effective Computability*. McGraw-Hill (1967).
[7] Smorynski, C.: *Logical Number Theory I*. Springer (1980).
[8] Thompson, Ken.: *Reflections on Trusting Trust*. Communications. of the ACM. 27(1984).761-763.