Amplitude Analysis

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Abstract

Even if a ‘complete set’ of experimental observables were measured for the elastic scattering or photo/electroproduction of pseudoscalar mesons, ambiguities would remain in the extracted partial-wave and isospin decomposed amplitudes. As these problems are not widely understood, the present work outlines the way model-dependence enters into analyses of data from both hadronic and electromagnetic facilities.

I. INTRODUCTION

In order to understand the significance of amplitude analysis, one must first recognize that there is generally a gap between the quantities ‘predicted’ through models and the observables which are actually ‘measured’ in experiments. The experimental quantities are generally cross sections, often with polarizations fixed or measured for the beam, target and recoiling particles. In contrast, the ‘predicted’ quantities (for example, the coupling constants, sigma terms, and resonance properties) typically require not only the underlying amplitudes, but also extrapolations of these into unphysical regions and/or separations into background and resonant contributions.

It is often assumed that at least the first step (data→amplitudes) in comparisons between theory and experiment could be carried out in a ‘model-independent’ way, if sufficient observables were available. This leads to a working definition of amplitude analysis:

“Amplitude analysis is a model-independent determination of the (helicity/transversity) amplitudes using only experimental data and the relations between amplitudes and observables, resulting in a set of amplitudes at discrete energies and angles.”

The usual goal in this type of analysis is the smallest set of observables removing all ambiguities in the extracted amplitudes, apart from an overall phase. Unfortunately, one rarely has the required set of observables. As a result, further theoretical input is generally required. In the following we will first review some of the most important (formal) results of these studies. We will then consider how relevant these results are for the N* program. Finally, since most analyses depart from the ideal defined above, we will consider how model-dependence enters into and effects a range of more standard analyses.

II. SOME FORMAL RESULTS

Most of the formal work on amplitude (and partial-wave) analysis has been confined to spinless and spin-0−spin-1/2 cases. A good source of references is the Introduction of
Höhler’s ‘bible’ on the subject [1]. Another very readable source is the review of Bowcock and Burkhardt [2]. More mathematical discussions are contained, for example, in the papers of Sabba Stefanescu [3] and the book of Chadan and Sabatier [4]. The basic problem is most easily demonstrated in spinless scattering. Here we have only the cross section

\[ \frac{d\sigma}{d\Omega} = |f|^2 \]  

which gives the scattering amplitude,

\[ f = |f|e^{i\phi(E,\theta)} \]  

up to an overall phase. The difficulty comes in determining \( \phi(E,\theta) \) with the addition of isospin, analyticity and unitarity constraints. It appears [4] that this is possible in spinless and spin-0–spin-1/2 scattering. Clearly the task is easier in the forward direction (optical theorem) and for elastic scattering.

When the scattered particles have spin, there are more amplitudes to determine and more observables to choose from. If we once again ignore the overall phase, the object is to pick a set of observables which fixes all the relative phases between the amplitudes. The allowable sets [4] have been determined by Dean and Lee [4] (for \( \pi N \) scattering), Chiang and Tabakin [8] (for pseudoscalar meson photoproduction), and Dmitrasinovic, Donnelly and Gross [9] (for scalar and pseudoscalar electroproduction).

**III. ARE WE ASKING THE RIGHT QUESTION?**

Suppose now that we have been given a complete set of observables for a particular reaction, and have constructed the helicity amplitudes up to an overall phase. Is this sufficient to determine all of the contributing \( N^\ast \) states? For two important reasons, the answer is negative. What we really want are the partial-wave amplitudes for each isospin state. Once the (model-dependent) separation of resonant and background contributions has been carried out, we can compare with model predictions. Unfortunately, the unknown overall phase is an important element in both the partial-wave and isospin decompositions.

To illustrate the first point, we can return to the simple spinless case, where the full amplitude

\[ f = \sum_l (2l + 1)f_l(E)P_l(x) \]  

may be written as an infinite sum of partial-wave amplitudes. Here \( x \) denotes \( \cos(\theta) \). Clearly, if \( f \) has a phase which is an unknown function of \( \theta \), this relation cannot be inverted to give unique values of \( f_l(x) \).

The effect of this overall phase on isospin decompositions is also serious, but has been discussed less often. This may be due to the fact that, in \( \pi N \) scattering, the potential ambiguities can be easily removed. Suppose, in the \( \pi N \) case, that we have constructed amplitudes for elastic \( \pi^\pm p \) scattering and charge-exchange \( (\pi^- p \rightarrow \pi^0 n) \) scattering. In principle, the amplitudes for each reaction have a different overall phase. Therefore we
cannot simply combine these quantities, using the usual relations, in order to obtain isospin amplitudes. However, we know one more piece of information. The amplitudes for the charge states satisfy a triangle relation

\[ A^- + \sqrt{2}A^0 = A^+ \]  

which, since we know the lengths of all 3 sides, fixes the relative angles.

Unfortunately, this method doesn’t work in all cases. As an example, consider pion photoproduction. Here there are 4 different amplitudes for the production of \( \pi^0 p, \pi^+ n, \pi^- p, \) and \( \pi^0 n \). These can be constructed from 3 isospin amplitudes. In this case, we also have a relation

\[ \sqrt{2}A^{\pi^0 n} + A^{\pi^- p} + A^{\pi^+ n} = \sqrt{2}A^{\pi^0 p} \]  

between the charge states. However, knowing the lengths of the 4 sides is now not sufficient to determine the relative angles. This problem remains even if the overall phases have been fixed for 2 of the reactions.

Some readers may at this point be wondering how an ‘unmeasureable’ phase could be so important. It should first be noted that this phase is not equivalent to the unmeasureable phase of a wavefunction in quantum mechanics. Instead, this is the relative phase between the scattered and unscattered waves in the familiar relation

\[ \psi \sim e^{i\vec{k} \cdot \vec{x}} + f(\theta)e^{ikr} \]  

as \( r \to \infty \). (6)

It is actually possible to see effects of such phases via multiple scattering [1, 2] and to (in principle) measure some of them using the Hanbury-Brown–Twiss method [10]. In practice, however, some theoretical input is necessary to fix this phase.

IV. WHAT IS GENERALLY DONE

Given the problems with ‘pure’ amplitude analysis, it is not surprising that alternate approaches have been employed in the study of \( N^* \) physics. These include restricted multipole analyses, the use of dispersion relations, direct model fits to data, and fits based on Breit-Wigner plus background contributions.

Before discussing these, however, we should mention an approach to the amplitude analysis problem which has been applied to elastic \( \pi d \) scattering. In this work [11] there were insufficient observables for a model-independent amplitude analysis. As a result, amplitudes were given with 3 different levels of model-dependence. First, it was shown that certain combinations of amplitudes could be extracted directly from the existing data. Models then provided the additional information necessary to complete a full amplitude analysis. Finally, a model was used to fix the overall phase, thus allowing a partial-wave analysis. An analysis of this type in the \( N^* \) arena would also be interesting, particularly as more polarization data become available.

One way of avoiding the overall phase problem is to fit multipoles directly. In the simplest variant of this method, the partial-wave series is cut off after a few terms. Examples include analyses near threshold and over the first resonance region. Clearly this method won’t work
if the neglected terms sum up to a sizable contribution [12]. Often the higher waves are assumed to be well approximated by the Born terms. A particularly interesting study of this kind is the Grushin [13] fit. This analysis was carried out over the first resonance without the help of pion nucleon phases (which give the multipole phases via Watson’s theorem). Here the overall $\pi^+n$ phase was determined through interference with the (real) Born terms. A relation between the $\pi^0p$ and $\pi^+n$ multipoles was then used to fix the overall $\pi^0p$ phase. Such an analysis, free from $\pi N$ input, can form the basis for a relatively unbiased determination of the $\Delta^+$ resonance position [14].

Dispersion relations, applied either to the invariant or partial-wave amplitudes give the least model-dependent way to supplement experimental data. Analyticity, particularly for fixed-$t$, has been an important element in studies of uniqueness for spinless and spin-0–spin-1/2 scattering. These constraints are particularly useful in elastic scattering as they give the real parts of forward amplitudes, once the imaginary parts have been determined via the optical theorem and total cross sections. To the author’s knowledge, no formal studies (analogous to those carried out by Sabba Stefanescu for $\pi N$ scattering), have determined the minimal theoretical input required to fix all the phases for pseudoscalar meson photo and electroproduction. Here, even the appropriate question is less obvious, as there are many more observables which could be measured.

If the analytic properties of the amplitudes are assumed in advance (for example, in Breit-Wigner plus background or model-based fits), we are able to obtain results from minimal sets of data. A complete set of experiments is not required, and the results are obviously model dependent. Here the ambiguity lies in the choice of model, which must necessarily give only an approximation to the true analytic structure. This approach can also yield useful information, particularly when it fails. Failure within a model (hopefully one which builds in the most important constraints from analyticity and unitarity) indicates a missing element.

V. IMPLICATIONS FOR CLAS DATA ANALYSIS

At present, our knowledge of the $N^*$ spectrum comes from analyses of data from decades of measurements at numerous laboratories, carried out using a variety of techniques (each having different inherent systematic errors). The promise of CLAS data has been a set of measurements with high precision and linked systematic errors. This would greatly reduce one of the most difficult problems in amplitude and partial-wave analyses. It is therefore interesting to consider how much information we can obtain from CLAS data alone.

If one is able to control the beam and target polarization, while measuring in both $\theta$ and $\phi$, it has been shown [13] that one can obtain the type-$S$ (cross section and single-polarization) and $BT$ (beam-target double-polarization) data, including the recoil polarization $P$. In pion photo- or electroproduction, for example, this leaves only one relative phase undetermined, apart from the overall phase. How one could obtain $P$ without recoil-polarization measurements is also evident in the relation

$$FG - EH = P - T\Sigma$$

between the type-$S$ and $BT$ observables [15].
The problem, from an ‘amplitude analysis’ point of view, is that this remaining relative phase requires further double-polarization measurement involving recoil polarization. As CLAS was not designed for such measurements, it might seem that non-CLAS data would be required to complete a ‘model-independent’ analysis. However, as we have argued above, in order to extract partial-wave amplitudes, the overall phase must be fixed. By fixing this overall phase, relative phases are also restricted. It seems reasonable to ‘conjecture’ that any physical input sufficient to fix this overall phase will simultaneously remove (at least) the one relative phase undetermined from type-$S$ and $BT$ experiments. The remaining question then has a more theoretical nature. What (minimal) theory input is required to fix the overall phase in photo- electroproduction reactions?

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REFERENCES

[1] G. H"ohler: *Pion-Nucleon Scattering*. Edited by H. Schopper, Landolt-B"ornstein, Vol. I/9b2 (Springer-Verlag, Berlin, 1983).

[2] J.E. Bowcock and H. Burkhardt: *Principles and problems of phase-shift analysis*, Rep. Prog. Phys. 38, 1099 (1975).

[3] I. Sabha Stefanescu: *On the determination of πN phase shifts from isospin constraints and fixed t analyticity*, J. Math. Phys. 23, 1190 (1982).

[4] K. Chadan and P.C. Sabatier: *Inverse problems in quantum scattering theory*. New York: Springer-Verlag 1989.

[5] While the work of Ref. [3] claims a unique solution *in principle*, whether current solutions are truly unique has recently been questioned by I.G. Alekseev et al., Phys. Rev. C 55, 2049 (1997).

[6] Note that forward and backward scattering are special cases for the amplitude reconstruction problem. For example, in πN scattering and photoproduction, the spin observables are either zero or ±1 at these angles. Therefore we need only a cross section measurement to determine these (essentially degenerate) transversity amplitudes up to an overall phase.

[7] N.W. Dean and P. Lee: *Ambiguities in scattering amplitudes resulting from insufficient experimental data*, Phys. Rev. D 5, 2741 (1972).

[8] W.-T. Chiang and F. Tabakin: *Completeness rules for spin observables in pseudoscalar meson photoproduction*, Phys. Rev. C 55, 2054 (1997).

[9] V. Dmitrasinovic, T.W. Donnelly and F. Gross: *Complete measurements of scalar and pseudoscalar electroproduction*, in *Research Program at CEBAF (III)*, RPAC III, edited by F. Gross (CEBAF, Newport News, 1988), p. 547.

[10] This is briefly discussed in Refs. [1] and [2]. A modern perspective is given by G. Baym: *The physics of Hanbury Brown—Twiss interferometry: from stars to nuclear collisions*, lectures at the XXXVII Zakopane School, June 1997; [nucl-th/9804023](https://arxiv.org/abs/nucl-th/9804023).

[11] H. Garcilazo, E.T. Boschitz, W. Gyles, W. List, C.R. Ottermann, R. Tacik, and M. Wessler: *Polarization observables in πd elastic scattering: Amplitude analysis*, Phys. Rev. C 39, 942 (1989).

[12] The neglect of high partial waves can alter the resulting low partial waves, as has been discussed by A. Donnachie: *Partial wave analysis and baryon resonances*, Rep. Prog. Phys. 36, 695 (1973).

[13] V.F. Grushin, A.A. Shikanyan, E.M. Leiken, and A. Ya. Rotvain: *Determination of the isotopic components of the multipole amplitudes for the γp → πN process in the photon-energy range 300-420 MeV from experimental data on photoproduction alone*, Yad. Fiz. 38, 1448 (1983) [Sov. J. Nucl. Phys. 38, 881 (1983)].

[14] R.L. Workman: *Δ+ mass reexamined*, Phys. Rev. C 56, 1645 (1997).

[15] I.S. Barker, A. Donnachie and J.K. Storrow: *Complete experiments in pseudoscalar photoproduction*, Nucl. Phys. B95, 347 (1975).