Identifiability of electrical and heat transfer parameters using coupled boundary measurements

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Abstract
In this paper, we show that a hybrid method using coupled boundary measurements can determine anisotropic electrical conductivity, anisotropic thermal conductivity, and the product of heat capacity and heat density within a bounded domain on the plane uniquely up to a boundary-fixing diffeomorphism.

Keywords: hybrid methods, uniqueness, coupled system

1. Introduction

It is a classical problem to try to determine the internal structure of an object by collecting external information. There are many examples of successful approaches to this type of question: like computed tomography (CT) in medical imaging, seismic imaging in geophysical prospection, and nondestructive testing in industry. In recent years, particularly in medical imaging, there has been interest in developing hybrid methods, i.e. combinations of different imaging modalities. This idea has been proven to be useful, since a given method can provide extra interior information for another which can be used to get a better resolution: see [4, 18, 20, 26, 24, 10].

In [17], the authors proved uniqueness for a new hybrid method they proposed to determine both the electrical and heat transfer parameters within some bounded domain in \( \mathbb{R}^n \) for \( n \geq 3 \), by using coupled boundary measurements. In this paper, we prove a similar result for the two dimensional case.

Given a bounded domain \( \Omega \subset \mathbb{R}^2 \) with smooth boundary, we consider two physical processes. First, we put a time-dependent voltage distribution \( f(x, t) \in C(\mathbb{R}^+, H^2(\partial \Omega)) \) at the boundary of \( \Omega \), and assuming there are no sinks or sources inside \( \Omega \). Then the resulting voltage distribution \( u(x, t) \) throughout the whole domain is governed by the conductivity equation
\[ \nabla \cdot (\gamma(x)\nabla u(x, t)) = 0, \quad x \in \Omega \quad \text{and} \quad u|_{\partial \Omega} = f, \]

where \( \nabla \) is the standard gradient operator, \( \nabla \cdot \) is the standard divergence operator and \( \gamma(x) \) is the electrical conductivity of \( \Omega \), the first unknown we want to determine. Since \( \gamma(x) \) may be anisotropic, it is represented by a positive definite matrix at each point, and we assume it is bounded below, i.e. \( \exists c > 0 \text{ s.t. } \| \gamma(x) \|_2 > c \) where \( \| \cdot \|_2 \) is the induced matrix 2-norm defined by \( \| A \|_2 \triangleq \sup_{\| x \|_2 = 1} |Ax|_2 \) and \( \| \cdot \|_2 \) on the right hand side is the usual vector 2-norm. Then \( u(x, t) \in C(\mathbb{R}_+, \mathcal{H}(\Omega)) \) since for any fixed time \( t \), equation (1) is a second order elliptic equation and the standard existence theory assures us that there exists a unique solution in \( \mathcal{H}(\Omega) \).

The second process is heat transfer. It is known that current generates heat which will diffuse throughout the domain. We model this by considering the temperature distribution \( \psi(x, t) \), which is governed by the heat equation

\[ \kappa^{-1}(x)\partial_t \psi(x, t) = \nabla \cdot (A(x)\nabla \psi(x, t)) + S(x, t), \quad x \in \Omega \quad \text{and} \quad t \geq 0, \]

\[ \psi(x, 0) = 0, \quad \forall x \in \Omega \quad \psi(x, t) = 0, \quad \forall x \in \partial \Omega \quad \text{and} \quad t \geq 0, \]

where \( \kappa(x) = c(x)^{-1}\rho(x)^{-1} \) is the reciprocal of the product of the heat capacity \( c(x) \) and density \( \rho(x) \), \( A(x) \) is the thermal conductivity and is represented by a positive definite matrix at each point. We assume that \( \kappa \) and \( \| A \|_2 \) are also bounded throughout the domain. The term \( S(x, t) \), which is called the energy density of the electrical field, is defined by \( S(x, t) = \nabla u(x, t) \cdot \gamma(x)\nabla u(x, t) \), where \( u(x, t) \) is the solution to (1) and \( S(x, t) \) will act as a source term in the heat equation. The initial and boundary condition (3) means that the temperature (after some shift) is zero at first and the boundary temperature is kept at 0 for all \( t \geq 0 \). We will assume that the heat transfer is sufficiently slow so that the quasistatic model (1) for the voltage \( u(x) \) is still realistic, as in [17].

Define coupled boundary measurements, namely the voltage-to-heat flow map \( \Sigma_{\gamma, \kappa, A} \), as follows:

\[ \Sigma_{\gamma, \kappa, A} : f(x, t) \Rightarrow \nu \cdot A \nabla \psi(x, t), \quad x \in \partial \Omega. \]

That is, we set a time-dependent voltage \( f(x, t) \) at the boundary, and measure the out-coming heat flow \( \nu \cdot A \nabla \psi(x, t) \). We study with what information about the internal parameters \( \gamma, \kappa \) and \( A \) we can recover from the boundary measurements \( \Sigma_{\gamma, \kappa, A} \).

The question to determine the inside conductivity by using voltage-to-current measurements at the boundary is known as Calderón’s inverse problem. It was first proposed by Alberto Calderón who came across it while working as an engineer in the 1940s and published his result [7] in 1980. Since then, Calderón’s inverse problem has both been applied in industry and become of further theoretical interest. In geophysical prospecting, the Schlumberger-Doll company was founded to find oil by using electromagnetic methods. In medical imaging, Calderón’s inverse problem is known as Electrical Impedance Tomography, which has been used for detecting pulmonary emboli (see [11]). Mathematically, there have been many results on uniqueness [7, 16, 31, 32, 19, 13, 12, 9, 23, 2], stability [1], reconstruction and the corresponding numerical methods [22, 21, 25, 5]. See [34, 35] for general reviews. For the question of determining the heat parameters through boundary measurements, there have also been some results [8, 14, 15].

Unlike [17], where the authors consider isotropic conductivity and show that \( \Sigma_{\gamma, \kappa, A} \) determines the parameters \( \gamma, \kappa \) and \( A \) uniquely, we allow the conductivity to be anisotropic and may encounter some nonuniqueness as a consequence. As we know for the anisotropic Calderón’s inverse problem, given any boundary-fixing diffeomorphism \( F \), we can define the
pushforward of $\gamma$ as $F_\gamma(x) = \frac{\partial F \circ F^{-1}}{\partial x} \circ F^{-1}(x)$, where $DF$ is the Jacobian of $F$, and it satisfies $\Lambda_{F_\gamma} = \Lambda_\gamma$ since the conductivity equation is independent of the choice of coordinates, where $\Lambda_\gamma$ is the standard Dirichlet-to-Neumann map that maps static boundary voltage (the Dirichlet boundary data) to the outcoming electrical current (the Neumann boundary data, i.e. the outer normal derivative of the solution). For the two dimensional case, it has been proved that this will be the only obstacle to uniqueness, see [33, 28, 3] for more details. Similarly we can apply the same change of coordinates to the heat equation, so for the two dimensional case, the best uniqueness result for this hybrid method one can hope for is that $\Sigma_{\gamma,A}$ determines the parameters uniquely up to a boundary-fixing diffeomorphism. The main purpose of this paper is to prove this is indeed the case.

**Theorem 1.** Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $\kappa_i(x) \in C^\infty(\overline{\Omega})$, $\gamma_i(x), \Lambda_i(x) \in C^\infty(\overline{\Omega}, S^2_{+2})$, where $S^2_{+2}$ stands for $2 \times 2$ positive definite matrices and $\kappa_i \|\gamma_i(x)\|_2 \leq g$ for some $g$. If $\Sigma_{\gamma_i,\Lambda_i} = \Sigma_{\gamma_1,\Lambda_1}$ then there exists a diffeomorphism $F : \Omega \rightarrow \overline{\Omega}$, which fixes the boundary, i.e. $F|_{\partial \Omega} = \text{id}$, such that $\kappa_2(x) = \kappa_1|DF| \circ F^{-1}(x)$, $\gamma_2(x) = \frac{\partial F \circ F^{-1}}{\partial x} \circ F^{-1}(x)$ and $\Lambda_2(x) = \frac{\partial F \circ F^{-1}}{\partial x} \circ F^{-1}(x)$, where $(DF)_i = \frac{\partial F_i}{\partial x_j}$ is the Jacobian matrix of $F$.

The rest of the paper is organized as follows. In section 2 we explain the outline of the proof. In section 3 we do a brief review of the nonuniqueness caused by the change of coordinates. In section 4, we show that the conductivity is determined uniquely up to a boundary-fixing diffeomorphism. An important density argument, which is the core of this paper, is proved in section 5 and is later used in section 6 to determine the heat parameters up to the same diffeomorphism arises in section 4.

**Remark 1.** The regularity assumption in the theorem may not be optimal and may be improved by a more refined analysis. From now on, we will just assume everything is smooth without further mentioning.

2. Outline of the proof

In this section, we give the outline of the proof for theorem 1.

- First, we recover the quadratic forms related to the conductivity equation (1) which will be defined later from the voltage-to-heat flow map $\Sigma_{\gamma_i,\Lambda_i}$. It is done by taking special input data (static ones). The physical interpretation is that the quadratic form maps any static boundary voltage to the amount of energy needed to maintain that due to the energy loss caused by the heat generated by the current. So this energy should be equal to the heat flow coming out due to the conservation of energy. Then based on the exiting result for the two dimensional Calderón’s inverse problem, we prove that the two conductivities are the same up to certain boundary-fixing diffeomorphism.
- The conductivity equation (1) and the heat equation (2) are related through the electrical energy density $S(x,t) = \nabla u(x,t) \cdot \gamma(x) \nabla u(x,t)$, which is the outcome of (1) and acts as an input of (2). Now we take spacial input data $f$ which is separated in $x$ and $t$, i.e. $f(x,t) = h(x)g(t)$, then the corresponding solutions to (1) will have the form $u(x,t) = u_0(x)g(t)$, where $u_0$ are static solutions to (1) with boundary value $h(x)$, and $S(x,t) = (\nabla u_0(x) \cdot \gamma(x) \nabla u_0(x))g(t)$. Next we show the spaces spanned by $\{\nabla u_0 \cdot \gamma \nabla u_0\}$ are actually dense in $L^2(\Omega)$. So we can conclude that the voltage-to-heat flow map $\Sigma_{\gamma_i,\Lambda_i}$ are actually density in $L^2(\Omega)$. So we can conclude that the voltage-to-heat flow map $\Sigma_{\gamma_i,\Lambda_i}$ are actually dense in $L^2(\Omega)$.
contain information for putting arbitrary variable-separated source $S(x,t) = w(x)g(t)$ into (2) by the continuous dependence of the solution to (2) on $S(x,t)$. The most important part in this step, which is also the core of this paper, is to prove the density argument and it is obtained by using special solutions to the Schrödinger equation related with (1), including the Complex Geometric Optics solutions (CGO solutions for short) and the solutions constructed by Bukhgeim in [6].

- Based on the fact in the second step and a change of coordinates (arising from the first step), the problem has become that if for any variable-separated source $S(x,t) = w(x)g(t)$, the out-coming heat flows are the same for two systems, then their heat parameters must coincide. This has already been proven in [17]. The basic idea is once again to take special input data (pulse sources which converge to $x_{\delta}$) and solve the heat equations using eigenfunctions of the operator $\kappa_i \nabla \cdot (A_i(x)\nabla)$ in some weighted $L^2$ space. And with the help of some boundary determination results, it can be proved that all the eigenvalues and eigenfunctions of the two operator $\kappa_i \nabla \cdot (A_i(x)\nabla)$ are the same, which leads to $A_1 = A_2$ then $\kappa_1 = \kappa_2$.

Remark 2. Our proof for the argument in step one is similar with the one in [17] but emphasises more on the physical interpretation. The arguments in step three have already been proven in [17] for dimension $n \geq 2$, but their strategy to prove the density argument stated in step two can not work for the two dimensional case.

3. Obstacle to uniqueness

In this section, we review some basic facts about the obstacle to uniqueness caused by a boundary-fixing diffeomorphism of $\Omega$, or sometimes called a gauge transformation, since physic laws does not depend on the choice of coordinates. The results in this section are valid for dimension $n \geq 2$. In this whole section, we assume that $F : \Omega \rightarrow \tilde{\Omega}$ is a diffeomorphism of $\tilde{\Omega}$, which is a at least $C^2$ and fixes the boundary, i.e. $F|_{\partial \Omega} = id$. We first start with the conductivity equation (1) and have the following result.

**Lemma 1.** If $u(x)$ is a solution to (1), then $\tilde{u}(x) \triangleq u \circ F^{-1}(x) = u(F^{-1}(x))$ solves (1) with conductivity $\gamma = F_*\gamma \triangleq \frac{DF\gamma DF^T}{|DF|} \circ F^{-1}$, which is sometimes called the pushforward of $\gamma$ and $(DF)_{ij} = \frac{\partial F_i}{\partial x_j}$ is the Jacobian matrix of $F$ as usual. As a result, we have $\Lambda_{\gamma} = \Lambda_{F_*\gamma}$.

This can be shown in various ways instead of calculating directly. For example when $n \geq 3$, we can relate (1) with the Laplace–Beltrami operator $\Delta_{g_\gamma}$ for some properly chosen metric $g_\gamma$ associated with $\gamma$

$$g_\gamma \triangleq |\gamma|^{\frac{1}{n-2}} \gamma^{-1},$$  

then $\nabla \cdot (\gamma(x)\nabla u(x)) = |g_\gamma|^{\frac{1}{2}} \Delta_{g_\gamma} u = 0$ means that $u$ is a harmonic function on the Riemannian manifold $(\Omega, g_\gamma)$. Notice an important fact that

$$g_\gamma = g_{F_*\gamma} = (F^{-1})^* g_{\gamma},$$  

i.e. the metric corresponding to the push forward conductivity is just the pull back of the metric $g_\gamma$ by $F^{-1}$. Then since the Laplace–Beltrami operator is defined intrinsically, we have

$$0 = \Delta_{g_\gamma} u = \Delta_{(F^{-1})^* g_{\gamma}} u \circ F^{-1} = \Delta_{g_{\gamma}} \tilde{u},$$  

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which implies that \( \tilde{u} \) solves (1) with conductivity \( \tilde{\gamma} \). But we may have a problem when \( n = 2 \), since the metric \( g_{\gamma} \) in (5) will not be well defined. We will fix this problem later in the proof of lemma 2. Another equivalent way to see this is using the fact that \( u(x) \) solves (1) if and only if \( u(x) \) is the unique global minimizer of the energy functional \( E_{\gamma}(u) = \int_\Omega \nabla u(x) \cdot \gamma(x) \nabla u(x) \, dx \) within the class of functions with boundary value \( f \). An easy calculation using change of coordinates shows that \( E_{\gamma}(u) = E_{F_{\gamma_{\gamma}}}(u \circ F^{-1})(x) \). So \( u(x) \) minimizes \( E_{\gamma}(\cdot) \) is equivalent to \( u = F^{-1} \circ \tilde{\gamma} \circ \tilde{u} \) solves (1) with conductivity \( F_{\gamma_{\gamma}} \). To show that \( \Lambda_{\gamma} = \Lambda_{F_{\gamma_{\gamma}}} \), either use the Riemannian geometry approach which will be explained below, or just calculate directly and recall the fact that \( F \) is identity when restricted to the boundary.

Then we move on to the heat equation (2) and show the following result.

**Lemma 2.** If \( \psi(x, t) \) solves the heat equation (2) with the initial and boundary value condition (3). Define

\[
\bar{\psi}(x, t) \triangleq \psi(F^{-1}(x), t), \quad \bar{\kappa}(x) \triangleq |D(F^{-1}(x))| \kappa(F^{-1}(x)),
\]

\[
\bar{S}(x, t) \triangleq |D(F^{-1}(x))|^{-1} S(F^{-1}(x), t), \quad \bar{\Lambda}(x) = FA(x) \triangleq \frac{D F A D^2 F}{|D|} \circ F^{-1}(x),
\]

then \( \bar{\psi}(x, t) \) solves (2) with \( \bar{\psi}(x), \bar{\Lambda}(x) \) and \( \bar{S}(x, t) \). Also the two systems give the same out-coming heat flow at the boundary, i.e. \( \nu \cdot A \nabla \psi(x, t) = \nu \cdot \bar{\Lambda} \nabla \bar{\psi}(x, t), \forall x \in \partial \Omega \).

**Proof of lemma 2.** Denote \( y = F(x) \), according to the definition, the only thing needed to be shown here is that

\[
|D(F(x))|^{-1} \nabla_x \cdot (A(x) \nabla_x u(x)) = \nabla_y \cdot (\bar{\Lambda}(y) \nabla_y \tilde{u}(y)) = \nabla_y \cdot (FA(y) \nabla_y u(F^{-1}(y)))
\]

We will use the metric argument mentioned above. First when \( n \geq 3 \), we take the metric \( g_A \) associated with \( A(x) \) as in (5),

\[
g_A(x) \triangleq |A(x)|^{-\frac{2}{n-2}} A^{-1}(x)
\]

As we have already mentioned above, here are some basic facts which can be verified easily.

- \( |g_A| = |A|^{2 \frac{n}{n-2}} \)
- \( \nabla \cdot (A(x) \nabla u(x)) = |g_A|^{-\frac{1}{2}} \Delta_{g_A} u = |A|^{\frac{2}{n-2}} \Delta_{g_A} u \)
- The metric associated with \( \bar{\Lambda} = FA \) is the pullback of \( g_A \) by \( F^{-1} \), i.e. \( g_{FA} = (F^{-1})^* g_A \). And we have the following \( |FA(y)| = |DF(x)|^{2-n} |A(x)|, |FA(y)| = |DF(x)|^{-2} |g_A(x)| \).

Then it is easy to see that

\[
\nabla_x \cdot (A(x) \nabla x u(x)) = |g_A|^{\frac{1}{2}} \Delta_{g_A} u = \left( \frac{|g_A|}{|DF^{-1}(y) g_A|^{2}} \right)^{\frac{1}{2}} (F^{-1})^* g_A^{\frac{1}{2}} \Delta_{g_A} u \circ F^{-1} = \left( \frac{|g_A|}{|g_A|^{2}} \right)^{\frac{1}{2}} \Delta_{g_A} u \circ F^{-1} = |DF(x)| \nabla_y \cdot (\bar{\Lambda}(y) \nabla y \tilde{u}(y))
\]
To show that \( \nu \cdot A \nabla \tilde{\psi}(x, t) = \nu \cdot \tilde{A} \nabla \tilde{\psi}(x, t) \), we can either calculate directly, or relate it with the interior product of \( \nabla \) and the volume form \( d_v \) since we have the relation

\[
(\nabla_{\tilde{\psi}} \cdot d_v)_{|\partial \Omega} = (\nu \cdot A \nabla \psi(x, t))d_S,
\]
where \( d_S \) is the Euclidean volume form restricted to \( \partial \Omega \). Then we get what we want since \( \nabla_{\tilde{\psi}} \cdot d_v \) is independent of the choice of coordinates plus the fact that \( F \) is identity at the boundary.

When \( n = 2 \), we can not use the arguments above since (7) makes no sense, or from another perspective, in the Laplace–Beltrami operator \( \Delta_g = |g|^{-\frac{1}{2}} \nabla \cdot (|g|^{\frac{1}{2}} g^{-1}) \nabla , \) determinant of the matrix \( |g|^{\frac{1}{2}} g^{-1} \) is always one (\( \det(|g|^{\frac{1}{2}} g^{-1}) = (|g|^{\frac{1}{2}})^n |g^{-1}| = |g||g^{-1} = 1 \) since \( n = 2 \)), which is not satisfied by a general \( A \). We fix this by first normalizing \( A \), define

\[
A(x) = \frac{1}{|A(x)|^2} A(x) \triangleq |A(x)|^{-\frac{1}{2}} A_0(x),
\]
so \( |A_0| = 1 \) and

\[
\nabla \cdot (A \nabla u) = \nabla \cdot (|A|^2 A_0 \nabla u) = |A|^2 \nabla \cdot (A_0 \nabla u) + \nabla |A|^2 \cdot (A_0 \nabla u).
\]

Then just define the metric associated with \( A \) to be \( g_A = A^{-1} \), we get

\[
\nabla \cdot (A(x) \nabla u(x)) = \Delta_{g_A} u + \nabla \cdot (\nabla \log |A|^2 \cdot \nabla u) > g_A,
\]
and similarly,

\[
\nabla \cdot (\tilde{A}(y) \nabla \tilde{u}(y)) = \Delta_{g_A} \tilde{u} + \nabla \cdot (\nabla \log |\tilde{A}|^2 \cdot \nabla \tilde{u}) > g_A.
\]

Notice that when \( n = 2 \), \( g_{A_{\tilde{\psi}}} \triangleq \tilde{A}^{-1} \) does not coincides with \( (F^{-1})^*(g_A) \), but satisfies that \( g_{A_{\tilde{\psi}}} = |DF(x)|^{-1} g_A(\tilde{y}) \). Plus the fact that \( |A(x)| = |\tilde{A}(y)| \), we have

\[
\Delta_{g_{A_{\tilde{\psi}}}} \tilde{u} = |DF(x)|^{-1} \Delta_{g_A} u(x),
\]

\[
(\nabla \log |\tilde{A}|^2 \cdot \nabla \tilde{u})_{g_{A_{\tilde{\psi}}}} = |DF|^{-1} (\nabla \log |A|^2 \cdot \nabla u)_{g_A}.
\]

which finishes the proof that \( \tilde{\psi} \) solves (2) with \( \tilde{\kappa}, \tilde{A} \) and \( \tilde{S} \).

To show \( \nu \cdot A \nabla \tilde{\psi}(x, t) = \nu \cdot \tilde{A} \nabla \tilde{\psi}(x, t) \), we can also either calculate directly or use the same argument above with some small modification. Notice that when \( n = 2 \), relation (8) becomes

\[
|g_A|^{-\frac{1}{2}} (\nabla_{\tilde{\psi}} \cdot d_v)_{|\partial \Omega} = (\nu \cdot A \nabla \psi(x, t))d_S.
\]

Then notice that

\[
g_{A_{\tilde{\psi}}}(y) = |DF(x)|^{-1} g_A(y), \quad |g_A(y)| = |g_A(x)|,
\]

\[
d_{(F^{-1})^*g_A} V = |DF|^{-1} d_{g_A} V, \quad \nabla_{(F^{-1})^*g_A} \tilde{\psi} = |DF| \nabla_{g_A} \tilde{\psi},
\]
we get
\[ \nabla_x \psi \cdot d_{x^A} V|_x = (\nabla_{E^{-1}_{x^A}} \psi) \cdot d_{E^{-1}_{x^A}} V|_y = \nabla_x \psi \cdot d_{x^A} V|_y. \]

Combining the fact that \( F \) is identity at the boundary, we finish the proof.

\[ \square \]

4. Determination of the conductivity

In this section, we recover the conductivity from \( \Sigma_{\gamma, \kappa, A} \), most of the arguments follow from [17]. We start with a property that will be used to solve the heat equation. Consider the operator \( A \) defined in the weighted \( L^2 \) space \( L^2_{\kappa, \gamma}(\Omega) \) with weight function \( \kappa^{-1}(x) \), i.e. the space equipped with the weighted inner product \( \langle u, v \rangle_{\kappa, \gamma} = \int_\Omega u \bar{v} \kappa^{-1} \, dx \), first set the domain of \( A \) to be \( C_0^0(\Omega) \cap H^1_0(\Omega) \) by Friedrichs extension. It is known that, given \( \kappa, A \) both positive with lower bounds strictly greater than zero, \( A \) is positive self-adjoint, furthermore the spectrum of \( A \) consists of real positive eigenvalues \( \{ \lambda_i \}_{i=1}^\infty \) of finite multiplicity which accumulate at \( +\infty \) (we may assume that \( \lambda_i \) is non-decreasing), and the corresponding eigenfunctions \( \{ \phi_i \}_{i=1}^\infty \) form an orthonormal basis of \( L^2_{\kappa, \gamma}(\Omega) \).

And now we return to the topic of this section, determining the conductivity through \( \Sigma_{\gamma, \kappa, A} \).

The method used here has a strong physical interpretation. Recall that in the two dimensional Calderón’s inverse problem, the conductivity is determined by the Dirichlet-to-Neumann map \( \Lambda_\gamma \) up to a boundary-fixing diffeomorphism. And knowing the Dirichlet-to-Neumann map \( \Lambda_\gamma \) is actually equivalent to knowing the quadratic form \( Q_\gamma \), or the energy functional \( E_\gamma \),

\[ Q_\gamma(f) = E_\gamma(u) = \Lambda_\gamma f(f) = \int_{\partial \Omega} u(\gamma \nabla u \cdot \nu) \, dS = \int_\Omega \nabla u \cdot \gamma \nabla u \, dx, \]

where \( u \) is the solution to (1) with boundary value \( f \). The physical interpretation of \( Q_\gamma(f) = E_\gamma(u) \) is that, it is equal to the amount of energy needed to maintain the boundary voltage \( f \). We know that energy is conservative, so it must transfer into some other form, which in fact is heat. So it is quite nature to expect that in the static case, the energy used to maintain the boundary voltage should be equal to the heat energy coming out through the boundary, i.e.

\[ Q_\gamma(f) = E_\gamma(u) = -\int_{\partial \Omega} A \nabla \psi_0 \cdot \nu \, dS, \]

where \( u(x), \psi_0(x) \) solve the static conductivity and heat equation. This can be verified as follows,

\[ \int_{\partial \Omega} A \nabla \psi_0 \cdot \nu \, dS = \int_{\partial \Omega} \nabla \cdot (A \nabla \psi_0) \, dx = -\int_{\Omega} S(x) \, dx = -\int_{\Omega} \nabla u \cdot \gamma \nabla u \, dx = Q_\gamma(f). \]

So we can recover the quadratic form \( Q_\gamma \) from the static out-coming heat flow. This reminds us to relate the static out-coming heat flow with \( \Sigma_{\gamma, \kappa, A} \), but get into a problem that the static solution \( \psi_0(x) \) doesn’t satisfy the initial condition. We can fix this by the following result.

**Lemma 3.** If we take \( f(x, t) = f(x) \), then \( \lim_{t \to +\infty} \int_{\partial \Omega} (\Sigma_{\gamma, \kappa, A} f)(x, t) \, dS = -Q_\gamma(f). \)

**Proof of Lemma 3.** Assume \( u(x) \) solves (1) with the static boundary data \( f(x, t) = f(x) \), then the source term will also be static (independent of \( t \)), \( S(x, t) = S(x) = \nabla u(x) \cdot \gamma \nabla u(x) \) and \( Q_\gamma(f) = \int_{\Omega} S(x) \, dx \). After that we solve the heat equation (2) by decomposing \( \psi(x, t) = \psi_0(x) + \psi_1(x, t) \), where \( \psi_0(x) \) solves the static heat equation with the static source term,
\[ \nabla \cdot (A \nabla \psi_0(x)) + S(x) = 0, \ x \in \Omega, \ \psi_0(x) = 0, \ \forall x \in \partial \Omega, \]  
(10)\]

and \( \psi_t(x, t) \) fixes the initial condition,

\[ \kappa^{-1} \partial_t \psi_t = \nabla \cdot (A \nabla \psi_t), \ \psi_t(x, 0) = -\psi_0(x), \ \forall x \in \Omega, \ \psi_t(x, t) = 0, \ \forall x \in \partial \Omega. \]

(11)

From the standard existence theory, we know there is always a unique \( \psi_t(x) \in H^1_0(\Omega) \) solves (10). Also notice that

\[ \int_{\partial \Omega} A \nabla \psi_0 \cdot \nu \ dS = \int_{\Omega} \nabla \cdot (A \nabla \psi_0) \ dx = -\int_{\Omega} S(x) \ dx = -Q_{\gamma}(f). \]

And for \( \psi_t \), we solve it by using the eigenfunction expansion as follows. First we do the eigenfunction decomposition for the initial value \(-\psi_0\),

\[ -\psi_0(x) = \sum_{i=1}^{\infty} a_i \phi_i(x), \ \ a_i = \langle -\psi_0, \phi_i \rangle_{C^1} = -\int_{\Omega} \psi_0 \phi_i \kappa^{-1} \ dx. \]  
(12)

Substituting (12) into (11), we get

\[ \psi_t(x, t) = \sum_{i=1}^{\infty} a_i e^{-\lambda t} \phi_i(x). \]

Since \( \lambda_i \geq \lambda > 0 \), we know that \( \psi_t \) converges to 0 exponentially as \( t \to +\infty \), so is the out-coming heat flow corresponding to \( \psi_t \). Then we have

\[ \lim_{t \to +\infty} \int_{\partial \Omega} (\Sigma_{\gamma, \kappa} Af)(x, t) \ dS = \int_{\partial \Omega} A \nabla \psi_0 \cdot \nu \ dS = -Q_{\gamma}(f) \] 

□

This way we recover the quadratic form \( Q_{\gamma} \), then according to the theory for the two dimensional Calderón’s inverse problem (see [3] for reference), this determines the conductivity up to a boundary-fixing diffeomorphism, and we also have some recovering algorithms for it. Also since we are working with smooth conductivities, the diffeomorphism is at least \( C^3 \) (see [29] for more details) which proves the following result.

**Lemma 4.** If \( \gamma = \gamma \), then there exists a boundary-fixing diffeomorphism \( F \) of \( \bar{\Omega} \) which is at least \( C^3 \), s.t. \( \gamma_2 = F \gamma_1 \).

### 5. A density argument

In this section, we prove the density argument mentioned in section 2.

**Lemma 5.** Assume \( \gamma \) is smooth up to the boundary and bounded below as above, then the space spanned by \( \nabla u(x) \cdot \gamma(x) \nabla u(x) \), where \( u(x) \) is arbitrary static solution to (1), is dense in \( L^2(\Omega) \).

It is easy to see that

\[ \text{span} \{ \nabla u \cdot \gamma \nabla u \} = \text{span} \{ \nabla u \cdot \gamma \nabla v \}, \]  
where \( u, v \) are arbitrary static solutions to (1).

So \( \text{span} \{ \nabla u \cdot \gamma \nabla u \} \) is dense if and only if \( \text{span} \{ \nabla u \cdot \gamma \nabla v \} \) is dense, and we will prove the second statement in this section.

\( L^2(\Omega) \) is a Hilbert space equipped with the standard \( L^2 \) inner product, then any linear subspace is dense if and only if its orthogonal complement contains only the 0 element. So it is equivalent to show that \( \text{span} \{ \nabla u \cdot \gamma \nabla v \} = 0 \), i.e. \( \forall f(x) \in L^2(\Omega) \) satisfies that

\[ \int_{\Omega} f(x) \nabla u(x) \cdot \gamma(x) \nabla v(x) \ dx = 0, \]  
then \( f = 0 \) (strictly speaking we should use \( \tilde{f} \) in the inner product, but it doesn’t matter here since our final goal is to show \( f = 0 \)).
First we claim that since we are now working on the two dimensional case, we may assume the conductivity \( \gamma \) is isotropic, i.e. \( \gamma = \gamma(x) I \). Since it has been proven in [3] that \( \forall \gamma \in L^\infty(\Omega, S^2_{++}) \), there exits a quasiconformal homeomorphism \( F \in W^{1,\infty}(\mathbb{C}; \mathbb{C}) \) of the whole complex plane with asymptotic behaviour \( F(z) = z + O(1/z) \) as \( |z| \to +\infty \), such that \( F \gamma = \text{det}(\gamma \circ F^{-1}) I \). This means that for any anisotropic conductivity \( \gamma \), we can find a change of coordinates \( F \) such that the push forward of \( \gamma \) is isotropic. We also have \( F \) is at least \( C^3 \) when \( \gamma \) is smooth (see [29] for more details). It is easy to verify that 

\[
\int_\Omega f (\nabla u \cdot \gamma \nabla v) \, dx = \int_{\partial \Omega} f \circ F^{-1} (\nabla (u \circ F^{-1}) \cdot F \gamma \nabla (v \circ F^{-1})) \, dy,
\]

so the density of \( \text{span}\{\nabla u \cdot \gamma \nabla v\} \) in \( L^2(\Omega) \) is equivalent to the density of \( \text{span}\{\nabla (u \circ F^{-1}) \cdot F \gamma \nabla (v \circ F^{-1})\} \) in \( L^2(F(\Omega)) \), where \( F \gamma \) is an isotropic conductivity. So from now on, we will assume that \( \gamma \in C^\infty(\Omega) \) is isotropic.

To show \( f = 0 \), we use some special solutions to (1).

- First we use Liouville transform and the standard CGO solutions to the related Schrödinger equation, which was first introduced in [30], to show that \( f_{\chi_\Omega} \in H^{2+\epsilon}(\mathbb{R}^2) \) for \( 0 < \epsilon \leq 1 \), where \( \chi_\Omega \) is the standard characteristic function of \( \Omega \) so that \( f_{\chi_\Omega} \) is a well defined function on \( \mathbb{R}^2 \). By Sobolev embedding \( H^{2+\epsilon}(\mathbb{R}^2) \subset C^4(\mathbb{R}^2) \), we have \( f_{\chi_\Omega} \in C^4(\mathbb{R}^2) \), which implies that \( f, \partial_i f \) must vanish at \( \partial \Omega \) for \( i \leq 3 \).

- Based on the facts above, we can apply integration by parts to the assumption,

\[
0 = \int_\Omega f \nabla u \cdot \gamma \nabla v \, dx = \frac{1}{2} \int_\Omega \nabla \cdot (\gamma \nabla (uv)) \, dx = \frac{1}{2} \int_\Omega \nabla \cdot (\gamma \nabla f) uv \, dx.
\]

Then we use Liouville transform again and the solutions Bukhgeim constructed in [6], to show that \( \frac{1}{\gamma} \nabla \cdot (\gamma \nabla (f)) = 0 \) in \( \Omega \) by the same arguments used in [6].

- Finally based on \( \frac{1}{\gamma} \nabla \cdot (\gamma \nabla (f)) = 0 \), plus the fact that both \( f \) and \( \partial_i f \) vanish at the boundary, we conclude that \( f = 0 \) by the unique continuation results for elliptic equations.

Step two and three have already been proven, so the only missing part here is the following.

**Lemma 6.** Assume \( \gamma \) is smooth up to boundary as above, \( \forall f \in L^2(\Omega) \), s.t. \( f \perp \text{span}\{\nabla u \cdot \gamma \nabla v\} \), where \( u, v \) are arbitrary static solutions to (1), then \( f_{\chi_\Omega} \in H^{2+\epsilon}(\mathbb{R}^2) \) for any \( 0 < \epsilon \leq 1 \).

Recall the standard Liouville transform that transform (1) into a related Schrödinger equation with potential \( q = \frac{\Delta u}{\gamma^2} \), we have \( u \) is a solution to (1) if and only if \( u = \gamma^{-1} u_{S} \) where \( u_{S} \) solves the Schrödinger equation

\[
(-\Delta + q)u_{S} = 0, \quad u_{S}|_{\partial \Omega} = \gamma^{-1} f.
\]  

(13)

In [27], the authors showed that if \( g \perp \text{span}\{u_{S}v_{S}\} \), where \( u_{S} \) and \( v_{S} \) are arbitrary solutions to (13), then \( g_{\chi_\Omega} \in H^{4}(\mathbb{R}^2) \) for \( 0 < s \leq 1 \). They proved this by using pairs of CGO solutions whose product was approximately \( e^{ik \cdot x} \) to derive estimates for the Fourier transform of \( g_{\chi_\Omega} \) when \( |k| \) was large. We extend this result and show that when the potential \( q \) is smooth, \( g_{\chi_\Omega} \) belongs to \( H^{s}(\mathbb{R}^2) \) for \( 0 < s < 5 \), and we can use almost the same arguments to show that \( f_{\chi_\Omega} \) in Lemma 6 belongs to \( H^{s}(\mathbb{R}^2) \), for \( 0 < s < 5 \).

First we review and modify the construction of CGO solutions to the Schrödinger equation (13), which were first introduced in [30]. For \( \forall \eta \in C^1 \) s.t. \( \eta \cdot \eta = 0 \) and \( |\eta| \) large enough,
we construct CGO solution $u_6(x,k) = e^{\eta \cdot x}(1 + r(x,k))$, where the correction term $r(x,k)$ is small in certain norm (like the $W^p_{\infty}$ norm or the $H^{n/2+k+\epsilon}$ norm which will be discussed later).

We introduce some notation here before we move on. $\forall \eta \in \mathbb{C}^2$ satisfies $\eta \cdot \eta = 0$ if and only if $\eta = \frac{1}{2}(\pm k^\perp + ik)$ for some $k \in \mathbb{R}^2$ (the factor $\frac{1}{2}$ is just for notation simplicity), where $k^\perp$ stands for the vector obtained by rotating $k$ clockwise by $\frac{\pi}{2}$. We do the analysis for $\eta = \frac{1}{2}(k^\perp + ik)$ first, and all the results hold for $\eta = \frac{1}{2}(-k^\perp + ik)$ by almost the same arguments with minor adjustments. We use $k_{kk_{ij}}(\eta)$ to denote the vector and $k_{kk_{ij}}(\eta)$ to denote the complex number, similarly $x$ denotes the vector $x_{kk_{ij}}(\eta)$ and $z_{kk_{ij}}(\eta)$ denotes the complex number. It is easy to see that $\eta \cdot x = \frac{1}{2}kz$ and if we use the standard notation
\[
(\partial \partial + \partial \partial = \partial \partial + \partial \partial = (14)
\]

This was solved in [30] by constructing $r(x,k)$ explicitly with the following expansion,
\[
r(x,k) = \frac{a(x)}{ik} + \frac{b(x,k)}{(ik)^2},
\]
where we have uniform bounds (in $|k|$) for the norm of $b(x,k)$. And based on this, [27] showed that $\forall g \perp \text{span}\{u_0, \chi\}$, $g_{kk_{ij}}(\Omega) \in H^s(\mathbb{R}^2)$ for $0 < s < 1$. To extend their result to any $0 < s < 5$, we need to modify the expansion (15) with the following result.

**Lemma 7.** Assume $q$ is smooth up to the boundary as above, for $|k|$ large enough, we can construct CGO solutions by solving (14) with $r(x,k)$ in the following form,
\[
r(x,k) = \sum_{j=1}^5 a_j(x)(ik)^j + \frac{b_0(x,k)}{(ik)^6},
\]
where $a_j = (-2\partial + Pq)^{j-1}Pq$. $j < 6$ are smooth, do not depend on $k$ and $P$ stands for the Cauchy transform which will be introduced later.

We call (16) CGO solutions expanded up to the 5th order, so (15) are CGO solutions expanded up to the 1st order. Equation (14) is solved by first formally setting $r = \sum_{j=1}^\infty r_j$, where $r_j$ satisfies $(4\partial \partial + 2ik\partial \partial)r_j = qr_j$ with $r_0 = 1$, then proving the summation converges and so solves the equation. Instead of solving (14) inside $\Omega$, we extend $q \in C^\infty(\Omega)$ to $\mathbb{R}^2$ and solve (14) on the whole plane. Notice that, there are various way to extend $q$ to the whole plane continuously, which can be done by the standard extension technique, and we require the extension $q_{\text{ext}}$ belongs to $C^\infty(\mathbb{R}^2)$ at least, the reason will be explained later. We will denote $q_{\text{ext}}$ simply by $q$ from now on. Then the key step is to solve the following equation on the whole plane,
\[
(4\partial \partial + 2ik\partial \partial)r = 2\partial(2\partial + ik\partial)r = f.
\]
We can first invert the $\partial$ equation by the standard Cauchy transform $P$,
\[
Pf(z) = \frac{1}{2\pi} \int \frac{f(w)}{z - w} \, dw.
\]
where $dw$ is the usual Lebesgue measure on the plane (we have modified the standard definition by a factor of $\frac{1}{2}$ for simplicity). The properties of $P$ have been studied thoroughly, like

$$2\partial(Pf) = f,$$

where $1 < p < \infty$, $-\frac{2}{p} < \delta < 1 - \frac{2}{p}$ (see [36] for more details), and $L^p_{\delta+1}$, $W^{1,p}_{\delta}$ are the completion of $C_c^\infty(\mathbb{R}^2)$ under the norm

$$\|f\|_{L^p_{\delta+1}} = \left(\int_{\mathbb{R}^2} |(f(x))(1 + |x|^2)^{\frac{\delta+1}{2}}|^p \, dx\right)^{\frac{1}{p}}, \quad \|f\|_{W^{1,p}_{\delta}} = \|f\|_{L^p_{\delta+1}} + \|\nabla f\|_{L^p_{\delta+1}}.$$  

Then we are trying to solve equations like

$$(2\partial + i\kappa)r = g,$$  

where $g = Pf$ for (17). Notice that $2\partial(e^{ik\cdot x}) = i\kappa e^{ik\cdot x}$, using integration factor, we have

$$(2\partial + i\kappa)r = e^{-ik\cdot x}2\partial e^{ik\cdot x}r,$$

which means $2\partial + i\kappa$ is actually $2\partial$ conjugated by $e^{ik\cdot x}$. Then we can solve (19) using $P$, which is defined as convolution with $\frac{1}{\kappa}$ and has similar properties as $P$ does,

$$r = e^{-ik\bar{P}(e^{ik}g)}.$$  

But if we solve (19) this way, we can not control the norm of $r$ by $|k|$ thus can not prove the convergence of $r = \sum_1^\infty n$. Instead, [30] solved (19) by noticing

$$r = \frac{g}{i\kappa} + \frac{r^{(1)}}{ik}, \quad \text{where} \quad (2\partial + i\kappa)r^{(1)} = (-2\partial)g,$$  

which can be verified by substitute (21) into (19) directly. Then we have the correct norm control

$$\|r\|_{W^{1,p}_{\delta}} \leq \frac{C}{|k|}\|g\|_{W^{1,p}_{\delta}},$$  

from which we can conclude that the summation $r = \sum_1^\infty n$ converges and we get CGO solutions expanded up to 1st order.

But we can not get CGO solutions expanded up to the 5th order if we solve (19) this way. Instead, when $g$ is regular enough, we can modify (21) a little bit and derive (16). If we define $r^{(n)}$ by the following,

$$r = \sum_{j=1}^n \frac{(-2\partial)^{j-1}g}{(ik)^j} + \frac{r^{(n)}}{(ik)^n},$$

then substitute (23) into (19), we get

$$(2\partial + i\kappa)r = g, \quad \Leftrightarrow \quad (2\partial + i\kappa)r^{(n)} = (-2\partial)^n g.$$  

And we call (23) expanding $r$ up to the $n$th order. Notice that (20), (21) and (23) are all the unique solution to (19) in $W^{1,p}_{\delta}$ (see [30] for the uniqueness proof), but written in different forms, in particular, we will get the same solution (23) no matter which order we expand up to, i.e. $\forall n, m \in \mathbb{N}_+$.

$$r = \frac{g}{ik} + \frac{-2\partial g}{(ik)^2} + ... + \frac{(-2\partial)^{n-1}g}{(ik)^n} + \frac{r^{(n)}}{(ik)^n} = \frac{g}{ik} + \frac{-2\partial g}{(ik)^2} + ... + \frac{(-2\partial)^{m-1}g}{(ik)^m} + \frac{r^{(m)}}{(ik)^m}.$$
as long as $(-2\partial)^n g$ and $(-2\partial)^n g$ exist, which is why (23) requires $g$ to be regular enough. And now we are ready to prove lemma 7.

**Proof of lemma 7.** As mentioned above, we first extend $q \in C^\infty(\overline{\Omega})$ regularly to the whole plane (at least in $C^\infty_c(\mathbb{R}^3)$), then solve (14) in the same way [30] did, i.e. set $r = \sum_{j=0}^\infty r_j$ with $r_j$ satisfies $4\partial\partial + 2ik\partial r_j = qr_j$, where $r_0 = 1$. Since we are constructing the same solutions as [30] and [4] did, all their convergence analysis and norm bounds apply to our solutions as well and we will not repeat the proof but mention some important facts here.

- For $q \in L^\infty$ compactly supported, $|k|$ large enough, (14) has a unique solution $r \in W^1_{k,p}(\mathbb{R}^3)$ satisfies the norm estimate $\|r\|_{W_{k+1}^1} < \frac{C}{|k|}\|q\|_{L^\infty}$, where the constant $C$ does not depend on $k$, see [30] for more details.
- Introduce the space $H^s_k$ as the completion of $C^\infty_c(\mathbb{R}^3)$ under the norm
  \[ \|f\|_{H^s_k} = \left( \int_{\mathbb{R}^3} \left( (I - \Delta)\frac{1}{|x|^2}\right)^s \|f\|^2 \right)^{1/2}, \]
  where $(I - \Delta)\frac{1}{|x|^2}\hat{f}(\xi) = (1 + |\xi|^2)\frac{1}{|\xi|^2}\hat{f}(\xi)$. Then for $q \in H^s_{k+1+\varepsilon}$ compactly supported, $|k|$ large enough, the unique solution $r$ belongs to $H^s_{k+1+\varepsilon}$, which further implies $r \in C^\infty(\overline{\Omega})$, and we have norm control $\|r\|_{H^s_{k+1+\varepsilon}} < \frac{C}{|k|}\|q\|_{H^s_{k+1+\varepsilon}}$, where the constant $C'$ does not depend on $k$, see [4] for more details.
- In both cases, the norm of $r_j$ decays like $\|r_j\| \sim O\left(\frac{1}{|k|}\right)$.

To derive CGO solutions expanded up to the 5th order, all we need to do is expanding $r_j$ up to higher orders for $j \leq 5$. Namely, we will need to expand $r_j$ up to the $(7 - j)$th order for $j \leq 5$ and leave the rest of $r_j$ unchanged. In fact, for $n \in \mathbb{N}_+$, we can derive CGO solutions expanded up to the $n$th order as long as $r_i$ can be expanded up to order $n + 2 - i$ for all $i \leq n$, which require more regularity in $q$ as $n$ increases. As an example, now we show how to derive CGO solutions expanded up to the 2nd order and what we need to do here is just expanding $r_1$ up to the 3rd order and $r_2$ up to the 2nd order.

\[
\begin{align*}
    r_1 &= \frac{Pq}{ik} + \frac{-2\partial Pq}{(ik)^2} + \frac{(-2\partial)^2 Pq}{(ik)^3} + \frac{r_1^{(3)}}{(ik)^4}, \\
    r_2 &= \frac{P(qr_1)}{ik} + \frac{-2\partial P(qr_1)}{(ik)^2} + \frac{r_2^{(2)}}{(ik)^3} \\
    &= \frac{1}{ik} \frac{P(qr_1)}{ik} + \frac{-2\partial P(qr_1)}{(ik)^2} + \frac{1}{(ik)^2} (-2\partial)^2 P(qr_1) + \frac{r_1^{(3)}}{(ik)^3} + \frac{r_2^{(2)}}{(ik)^4}, \\
\end{align*}
\]

where $r_1^{(0)}$ solves $(2\partial + ik)r_1^{(0)} = (-2\partial)^2 Pq$, $r_1^{(2)}$ solves $(2\partial + ik)r_1^{(2)} = (-2\partial)^2 P(qr_1)$. Then the norm of $r_2^{(2)}$ is of order $O\left(\frac{1}{|k|^2}\right)$ since the norm of $r_1$ is of order $O\left(\frac{1}{|k|^3}\right)$. This means the only term that is of order $O\left(\frac{1}{|k|^4}\right)$ in the last line of (25) is $\frac{P(qr_1)}{(ik)^2}$ and we have

\[
r_2 = \frac{Pq}{(ik)^2} + O\left(\frac{1}{|k|^4}\right),
\]

(26)
The norm of the rest of $r_i$, $i \geq 3$ is no more than order $O(\frac{1}{|k|^2})$, so we have derived CGO solutions expanded up to the 2nd order, i.e.
\[
r(x, k) = \sum_{j=1}^{\infty} r_j = \frac{Pq}{ik} + \frac{(-2\partial + Pq)Pq}{(ik)^2} + \frac{b_3(x_k, k)}{(ik)^3}.
\] (27)

By exactly the same process, but with more terms involved, we can construct CGO solutions expanded up to the 5th order (even up to the nth order) as long as we have enough regularity in $q$. And for our case, $q \in C^{2,1}$ (the extended version) will be enough since we need $C^6$ for expanding $r_1$ up to the 6th order, then another $C^5$ for $r_2$ up to the 5th order, etc. The expression for $a_j$ can be derived by some easy observation when you write out more terms.

**Remark 3.** With minor adjustments, for each $\tilde{\eta} = \frac{1}{2}(-k^2 + ik)$, we can construct CGO solutions expanded up to the 5th order (or even nth order),
\[
\tilde{r}(x, k) = \sum_{j=1}^{5} \tilde{a}_j(x) + \tilde{b}_0(x_k, k)\frac{(-2\partial + Pq)^{j-1}Pq}{(ik)^j}, \quad \tilde{a}_j = (-2\partial + Pq)^{j-1}Pq,
\] (28)
as long as we have enough regularity in $q$.

For lemma 6, we first prove a generalization of the similar result in [27] for the Schrödinger equation. In [27], the authors showed that $\forall g \perp \text{span}\{u_5v_5\}$, $g\chi_\Omega$ was in $H^s(\mathbb{R}^2)$ for $0 < s < 1$. They used the standard CGO solutions,
\[
u_5 = e^{\frac{1}{2}(k^2 + ik)\cdot x}(1 + r(x, k)), \quad v_5 = e^{\frac{1}{2}(k^2 + ik)\cdot x}(1 + \tilde{r}(x, k)),
\] (29)
for $|k|$ sufficiently large. Substituting (29) back into the assumption $\int_{\Omega} g u_5 v_5 \, dx = 0$, they got
\[
\int_{\Omega} g e^{ik\cdot x}(1 + r + \tilde{r} + r\tilde{r}) \, dx = 0 \quad \Rightarrow \quad \nabla g\chi_\Omega(-k) = -\int_{\Omega} g e^{ik\cdot x}(r + \tilde{r} + r\tilde{r}) \, dx.
\] (30)

Then they used (16) and (28) up to the 1st order to show that $|(-k)^s \nabla g\chi_\Omega(-k)|$ was in $L^2(|k| > R)$ for $R$ sufficiently large, which implied $g\chi_\Omega \in H^s(\mathbb{R}^2)$ since only the large $k$ behaviour of $g\chi_\Omega$ mattered. They stopped here since it was enough for their purpose, while we can actually prove $g\chi_\Omega \in H^s$ for some larger $s$ by expanding (16), (28) up to higher order together with a bootstrap argument, given that $q$ is regular enough.

For instance, for $|k|$ large enough, if we use CGO solutions (16), (28) expanded up to the 2nd order, (30) becomes
\[
\begin{align*}
-\nabla g\chi_\Omega(-k) &= \frac{1}{ik} \tilde{a}_3 g\chi_\Omega(-k) + \frac{1}{ik} \tilde{a}_2 g\chi_\Omega(-k) \\
+ \frac{1}{(ik)^2} \tilde{a}_2 g\chi_\Omega(-k) + \frac{1}{(ik)^2} \tilde{a}_3 g\chi_\Omega(-k) + \frac{1}{(ik)^3} \tilde{a}_4 g\chi_\Omega(-k) \\
+ \frac{1}{(ik)^3} \int_{\Omega} g e^{ik\cdot x} b_3(1 + \tilde{r}) + \frac{1}{(ik)^4} \int_{\Omega} g e^{ik\cdot x} \tilde{b}_3(1 + r) - \frac{1}{(ik)^5} \int_{\Omega} g e^{ik\cdot x} \tilde{b}_3
\end{align*}
\] (31)

Then combining with the following facts, we can show that $\forall 0 < s < 2$, $|(-k)^s \nabla g\chi_\Omega(-k)| \in L^2(|k| > R)$, so $g\chi_\Omega \in H^s(\mathbb{R}^2)$. 


The terms in the first line of (31), like \( \frac{|-k^r|}{|a_1g\chi_\Omega|} \), belongs to \( L^2(|k| > R) \) is based on the fact that it has already been shown in [27] that \( g\chi_\Omega \in H^t(\mathbf{R}^2) \) for any \( 0 < t < 1 \), plus \( g\chi_\Omega \) is compactly supported and \( a_1 = Pq \) is a smooth function which does not depend on \( k \), then \( a_1g\chi_\Omega \in H^t(\mathbf{R}^2) \), which means \( |-k^r| a_1g\chi_\Omega \in L^2(|k| > R) \). Then since \( \frac{|-k^r|}{|a_1g\chi_\Omega|} \leq |k|^{-1} \), \( |a_1g\chi_\Omega| \) belongs to \( L^2(|k| > R) \).

The terms in the second and third line of (31), like \( \frac{|-k^r|}{|a_2g\chi_\Omega|} \), belongs to \( L^2(|k| > R) \) since \( \frac{|-k^r|}{|a_2g\chi_\Omega|} < 1 \) and the Fourier transform of \( a_2g\chi_\Omega = (-2\partial + Pq)Pq \cdot g\chi_\Omega \), which means \( \frac{|-k^r|}{|a_2g\chi_\Omega|} \) belongs to \( L^2(|k| > R) \). The leading order term \( \chi_{1,1,e} \), \( \chi_{1,1} \) \( = \frac{-\Delta \ln \gamma \cdot \chi}{\ln(1 + \hat{\gamma}(x,k))} \) \( \in |k| > \Omega \), which means \( \frac{-\Delta \ln \gamma \cdot \chi}{\ln(1 + \hat{\gamma}(x,k))} \) \( \in L^2(|k| > R) \). Then since \( \frac{-\Delta \ln \gamma \cdot \chi}{\ln(1 + \hat{\gamma}(x,k))} \) \( \in L^2(|k| > R) \), \( \chi_{0,1,1} \) \( \in |k| > \Omega \), \( \chi_{0,1} \) \( \in |k| > \Omega \). Then as long as \( g \) is regular enough to expand CGO solutions to higher order, we can keep running this bootstrap process to get more and more regularities in \( \chi_{0,1,1} \). Similar with lemma 6, if we want to show \( g\chi_\Omega \) belongs to \( H^t \) for \( s < 5 \), then we will need to use CGO solutions expanded up to the 5th order which requires \( q \) (the extended version) belongs to \( C^2_0(\mathbf{R}^2) \) at least as stated above. Now we come back to the conductivity equation and prove lemma 6.

**Proof of lemma 6.** As we mentioned above, we take solutions to the conductivity equation (1) obtained by using Liouville transform and the CGO solutions constructed to the corresponding Schrödinger equation (13), i.e.

\[
\begin{align*}
\gamma u &= \gamma^{-\frac{1}{2}} e^{\frac{1}{2}(1 + \hat{\gamma}(x,k))} (1 + r(x,k)), \\
\gamma v &= \gamma^{-\frac{1}{2}} e^{\frac{1}{2}(1 - k^r + ik\cdot\nabla)} (1 + \hat{r}(x,k)).
\end{align*}
\]

Substituting (32) into the assumption \( \int_{\Omega} f \nabla \cdot \gamma \nabla v \, dx = \frac{1}{2} \int_{\Omega} f \nabla (\gamma (\nabla (uv))) \, dx = 0 \), we get

\[
\begin{align*}
\int_{\Omega} f \nabla e^{ik\cdot\nabla} &\cdot \nabla x \, dx = - \int_{\Omega} f R e^{ik\cdot\nabla} x \, dx + \frac{2i}{|k|^2} k \cdot \int_{\Omega} f \nabla R e^{ik\cdot\nabla} x \, dx \\
&\quad - \int_{\Omega} f (1 + R) \nabla \log \gamma e^{ik\cdot\nabla} x \, dx \frac{1}{|k|^2} (\int_{\Omega} f (1 + R) \Delta \log \gamma e^{ik\cdot\nabla} x \, dx \\
&\quad + \int_{\Omega} f \nabla R \cdot \nabla \log \gamma e^{ik\cdot\nabla} x \, dx + \int_{\Omega} f \Delta R e^{ik\cdot\nabla} x \, dx),
\end{align*}
\]

where \( R = R(x,k) = r(x,k) + \hat{r}(x,k) + \hat{r}(x,k) \hat{r}(x,k) \hat{r}(x,k) \). The leading order term \( \int_{\Omega} f R e^{ik\cdot\nabla} x \, dx \) is exactly the same as what we get in (30), so we can run the same bootstrap arguments to prove
$f_{\Omega} \in H^s(\mathbb{R}^3)$ for $s < 5$. And as we just mentioned above, we need the potential $q = \gamma^{-\frac{1}{2}} \Delta \gamma^{\frac{1}{2}}$ belongs to $C^2(\bar{\Omega})$, which is always satisfied since we assume $\gamma$ to be smooth.

Finally we are ready to prove lemma 5.

**Proof of lemma 5.** The lemma is equivalent to that $\forall f \in L^2(\Omega)$, s.t. $f \perp \text{span}\{\nabla u \cdot \gamma \nabla v\}$, where $u, v$ are arbitrary static solutions to (1), then $f = 0$.

As mentioned above, we may assume the conductivity $\gamma$ is isotropic. By lemma 6, we have $f_{\Omega} \in H^{4+\varepsilon}(\mathbb{R}^3)$, for $0 < \varepsilon < 1$. By Sobolev embedding, $f_{\Omega} \in C^3(\mathbb{R}^3)$, which implies $f, \partial_i f$ ($i \leq 3$) all vanish at the boundary. Then apply integration by parts to the assumption $f \perp \text{span}\{\nabla u \cdot \gamma \nabla v\}$, we get

$$0 = \int_\Omega f \nabla u \cdot \gamma \nabla v \, dx = \frac{1}{2} \int_\Omega f \nabla \cdot (\gamma \nabla (uv)) \, dx = \frac{1}{2} \int_\Omega \nabla \cdot (\gamma \nabla (f)uv) \, dx,$$

for $u, v$ at least $C^2$. Use the standard Liouville transform, i.e. $u = \gamma^{-\frac{1}{2}} u_S$, where $u_S$ is the solution to the related Schrödinger equation (13). We have $\int_\Omega \frac{1}{\gamma} \nabla \cdot (\gamma \nabla (f)u_Sv_S) \, dx = 0$ for any $C^2$ solutions $u_S, v_S$. Since $f \in C^3$ and $\gamma$ is smooth, $\frac{1}{\gamma} \nabla \cdot (\gamma \nabla (f)) \in C^1(\bar{\Omega})$, then by taking $u_S, v_S$ to be the special solutions constructed in [6] (which are $C^2$ when the potential is smooth), we can prove $\frac{1}{\gamma} \nabla \cdot (\gamma \nabla (f)) = 0$ by the stationary phase arguments used in [6].

Finally by the unique continuation results for elliptic equations, $\frac{1}{\gamma} \nabla \cdot (\gamma \nabla (f)) = 0$ together with both $f$ and $\partial_i f$ vanish at the boundary, we conclude that $f = 0$. \qed

6. Determination of the heat parameters

With all the preparations above, we can now determine the heat parameters and finish theorem 1.

**Proof of theorem 1.** By lemma 4, we have already proved that there exists a boundary-fixing diffeomorphism of $\bar{\Omega}$, denote by $F$, such that $\gamma_i = F_i \gamma_i$. By lemma 1, we know that $u_i(F^{-1}(x), t)$ solves (1) with conductivity $F_i \gamma_i$, which is equal to $\gamma_i$ now. By the uniqueness for the elliptic Dirichlet boundary value problems and the fact that $u_i(F^{-1}(x), t)\big|_{\partial \Omega} = f$ since $F$ fixes the boundary, we have $u_2(x, t) = u_2(F^{-1}(x), t)$ $S_2(x, t) = \nabla u_2(x, t) \cdot \gamma_2(x) \nabla u_2(x, t) = \nabla u_1(F^{-1}(x), t) \cdot F_2 \gamma_2(x) \nabla u_1(F^{-1}(x), t)$

$$= \frac{1}{|DF(F^{-1}(x))|} S_1(F^{-1}(x), t).$$

(35)

On the other hand, if we consider a new system with parameters $\tilde{\gamma}_i, \tilde{\kappa}_i, \tilde{A}_i$ defined as in lemmas 1 and 2,
and compare ($\tilde{\gamma}_1, \tilde{\kappa}_1, \tilde{A}_1$) with ($\gamma_2, \kappa_2, A_2$). We have already pointed out that, $\tilde{\gamma}_1 = \gamma_2 \triangleq \gamma$, so for any input boundary data $f(x, t), \tilde{u}(x, t) = u(x, t) \triangleq u(x, t)$ which implies $S_t(x, t) = S_2(x, t) \triangleq S(x, t)$. And we have $\Sigma_{\gamma_1, \kappa_1, A_1} = \Sigma_{\gamma_2, \kappa_2, A_2}$, where the first equality is by lemmas 1 and 2 while the second one is our assumption, which means that $\forall S(x, t), S_2(x, t) \triangleq S(x, t)$, the out-coming heat flows are the same for the two systems. Now we take special boundary input data which is separated in $x$ and $t$, i.e. $f(x, t) = h(x)g(t)$, then the resulting source term $S(x, t) = (\nabla u(x) \cdot \gamma(x)\nabla u(x,t))g(t)$, where $u(x)$ is the static solution to (1) with boundary value equals $h$. By lemma 5 and continuous dependence of the solutions to (2) on $S(x, t)$, we know that the out-coming heat flows are the same for any source term in the form $S(x, t) = w(x)g(t)$, where $w(x) \in L^2(\Omega)$. From here, we can conclude that

$$\kappa_2 = \tilde{\kappa}_1, \quad A_2 = \tilde{A}_1,$$

which has already been proven in [17] for dimension $n \geq 2$. We are not going to repeat the proof here but the main idea is to take special source term close to $S(x, t) = w(x)\delta(t)$ and solve the equations using eigenfunction expansions. Then by the definition of $\tilde{\kappa}_i, \tilde{A}_i$, we finish the proof.

7. Summary

In summary, we prove the uniqueness result for a new hybrid method proposed in [17] for the two dimensional case. The main difficulty is to show a density argument, which is proved in section 5. Since the two dimensional anisotropic Calderón’s inverse problem is well understood, in this paper we allow the conductivity to be anisotropic and so we can only expect to have uniqueness up to a boundary-fixing diffeomorphism. The hybrid method actually doesn’t provide any interior information, so we should not expect any improvement in stability or higher-resolution, but on the other hand, we can recover three coefficients all together and the coupled measurement may be more efficient in application. Further work may include requiring less regularity of the parameters, numerical reconstruction, improve the model for time-varying boundary voltage, etc.

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