An SIR mathematical model for *Dipterid* disease

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**Abstract.** In this paper we discuss SIR mathematical of diphtheria transmission by simplify the assumptions and find parameters that give contribution to endemic and nonendemic condition. The model assumes that infected humans can heal by itself. We try to analyse what happen in this simplest mathematical model of diphtheria transmission even though in practise, it is not realistic and cannot eliminate the disease. We use Basic Reproduction Number to analyse the endemic equilibrium. We open discussion if this model can be expanded by adding more complex parameters and variables.

1. Introduction

Indonesia’s Ministry of Health has confirmed the recent outbreak of Diphtheria based on their data. There are 95 regions and cities from 20 provinces that have reported cases from January to November 2017, in total there were 622 cases, 32 of which were fatal cases. Diphtheria is a very contagious and infectious disease, complications caused by diphtheria infection may cause death due obstruction of the airways and infection of the heart muscles, caused by bacteria called *Corynebacterium diphtheriae* and spreads through droplet infection from a sick patient or a carrier. Diphtheria is an infection caused by the bacterium *Corynebacterium diphtheriae*. Diphtheria is a terrible disease in which has caused thousands of deaths, and it is still endemic in undeveloped regions of the world. The survivors of this disease suffer from paralysis of certain muscles and permanent damage to the heart and kidneys. Children aged one to ten years are very sensitive to this disease [1]. One of the epidemic mathematical models for analyzing the spread of diseases is SIR [2]. The SIR mathematical model states that the population is divided into three groups: susceptible groups, infected groups, and groups of individuals who have recovered [2]. The purpose of this study is to analyze and interpret the basic reproduction number of SIR mathematical model of diphtheria disease.

2. Methods

The method used in this study is using SIR Mathematical Model and Basic Reproduction Number. The assumptions of the problem are needed to simplify the complex operation. Hence, it will be easier to determine steps in solving the mathematical model [2,3]. By studying related source, we obtain a general description of diphtheria, mathematical models, SIR epidemic models, differential equation systems, equilibrium points, eigenvalues and eigenvectors. Finally, the basic reproduction number will be analyzed.
3. Results and Discussion
In the spread of diphtheria, the human population can be classified into three classes consisting of susceptible classes including individuals susceptible denoted $S$, infected classes including infected individuals notated with $I$, and recovered classes including individuals who have been recovered notated with $R$ [2,3]. In this mathematical model, the spread of diphtheria is given limitations or assumptions such as: (1) The population is assumed to be small, (2) The birth rate and death rate are assumed to be the same, so that the total population is assumed to be constant, (3) The population is assumed to be closed (there is no emigration and immigration process). (4) There is no immune to the disease. (5) Diphtheria is transmitted through direct contact with patients. (6) It is assumed that there is only one disease that spreads in the population. (7) Infected individuals can recover from illness naturally and can die from illness. (8) Every child was born susceptible from passive immunity or maternal antibodies because it does not work effectively due to a relatively short time [4-7]. List of variables is presented in Table 1.

| Table 1. Variable list |
|------------------------|
| Variables | Remarks | Terms |
| $N(t)$ | Total population on $t$ | $N(t) \geq 0$ |
| $S(t)$ | Sum individuals are susceptible from infection disease at time $t$ | $S(t) \geq 0$ |
| $I(t)$ | Sum of individuals are infected with the disease at time $t$ | $I(t) \geq 0$ |

From model, we have the list of parameters on Table 2.

| Table 2. List of parameters |
|----------------------------|
| Parameters | Remarks | Term |
| $M$ | Birth rate and death rate | $\mu > 0$ |
| $B$ | Level of transmission between susceptible individuals and infected individuals | $\beta > 0$ |
| $\Gamma$ | Recovery rate | $\gamma > 0$ |

The schematic diagram of the spread of diphtheria is presented in figure 1.
The SIR epidemic model (system 1) is given as follows:

\[
\frac{ds}{dt} = \mu N - \mu S - \beta \frac{S}{N} I_v \\
\frac{di}{dt} = \beta \frac{S}{N} I_v - \mu I_v - \gamma I_v \\
\frac{dr}{dt} = \gamma I_v - \mu R
\]  

(1) (2) (3)

Because \(N(t)\) is constant, the system 1 can be simplified by calculating the proportions for each class. The proportion of many individuals in each group can be stated as follows:

\[
s = \frac{S}{N}, i = \frac{I_v}{N}, r = \frac{R}{N}
\]  

(4)

Obtained

\[
s + i + r = \frac{S}{N} + \frac{I_v}{N} + \frac{R}{N} = \frac{N}{N} = 1
\]  

(5)

So that it can be written as a system (2)

\[
\frac{ds}{dt} = \mu - \mu s - \beta \frac{s}{N} i \\
\frac{di}{dt} = \beta \frac{s}{N} i - \mu i - \gamma i \\
\frac{dr}{dt} = \gamma i - \mu r
\]  

(6) (7) (8)

\[
s + i + r = 1
\]  

(9)

The free diterid Equilibrium point is \(E_0 = (s, i, r) = (1, 0, 0)\)
The Endemic Equilibrium point $E_1$ with assumptions $i \neq 0$ so if $E_1 = (s_1, i_1, r_1)$

$$i_1 = -\frac{\mu(y-\beta+\mu)}{\beta(y+\mu)}, r_1 = -\frac{\gamma(y-\beta+\mu)}{\beta(y+\mu)}, s_1 = \frac{\gamma+\mu}{\beta}$$

(10)

3.1. Eigen value

$$MJ = \begin{bmatrix}
-\mu & -\beta & 0 \\
0 & \beta - \gamma - \mu & 0 \\
0 & \gamma & -\mu
\end{bmatrix}$$

(11)

Substitute the critical point that has been obtained, thus will be obtained by two Jacobian matrices. By knowing the Jacobian matrix, eigenvalues can be searched by:

$$\text{det}(MJ - \lambda I) = 0$$

(12)

Thus obtained eigenvalues for the critical point (i) are:

$$\lambda_1 = -\mu, \lambda_2 = -\mu, \text{ and } \lambda_3 = \beta - \gamma - \mu$$

(13)

While the eigenvalues for the critical point (ii) are:

$$\lambda_1 = -\mu, \lambda_2 = -\frac{1}{2}i\beta + \frac{1}{2}\beta s - \frac{1}{2}\gamma - \mu + \frac{1}{2}\sqrt{\beta^2i^2 - 2\beta^2is - 2\beta\gamma i - 2\beta\gamma s + \gamma^2}, \text{ and}$$

$$\lambda_3 = -\frac{1}{2}i\beta + \frac{1}{2}\beta s - \frac{1}{2}\gamma - \mu - \frac{1}{2}\sqrt{\beta^2i^2 - 2\beta^2is + \beta^2s^2 - 2\beta\gamma i - 2\beta\gamma s + \gamma^2}.$$

3.2. Basic reproduction number ($R_0$)

Basic reproduction number ($R_0$) can be obtained from system 1. Suppose that A is a derivative of $\frac{di}{dt}$ to i, where $A = M - D$. Thus, it can be known $R_0 = MD^{-1}$. Thus

$$A = \beta s - \gamma - \mu$$

(14)

Substitute the critical point ($s = 1, i = 0, r = 0$) in the equation above, therefore:

$$A = \beta - \gamma - \mu \text{ hence } M = \beta \text{ and } D = \gamma + \mu \text{. We obtained}$$

$$R_0 = MD^{-1} \text{ or } R_0 = \frac{\beta}{\gamma + \mu}$$

(15)

$R_0 < 1$ occurs when $\beta < \gamma + \mu$ while $R_0 > 1$ occurs when $\beta > \gamma + \mu$. Based on the results of $R_0$ obtained, to make $R_0 < 1$, Death due to diphtheria factor $\mu$ does not increase and the rate of healing $\gamma$ can increase. Therefore, this model depends on the $\beta \& \gamma$.

4. Conclusion

The SIR model of mathematics for the spread of diphtheria without treatment was examined by analytic method using the basic reproduction number method. The situation will be endemic if $R_0 > 1$ otherwise if $R_0 < 1$ the disease will not spread. The parameters that can be controlled to maintain $R_0 < 1$ are $\gamma$ recovery rate and $\beta$ which is the rate of transmission between susceptible humans and infected humans. On this study, we assume diphtheria can cure naturally. For further research, a comparison can be made with the natural cure rate and cure rate using drugs or vaccines.
5. References

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