Possibility of low energy signatures of Quantum Gravity from GW detection through Quasinormal Modes

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Abstract

We have analyzed the quasinormal mode (QNM) frequencies of a class of static spherically symmetric spacetime having a smeared matter distribution. Here we mainly focused on the QNM spectrum for the odd parity perturbation of this background geometry. The results as found with diffused mass distribution, capture significant changes in the QNM spectrum, which could be relevant for future generation experiments, specifically to distinguish the signals of GW from a non-singular source in contrast to a singular geometry. We also provide explicit numerical estimate for the (Gaussian) spread of mass distribution of typical globular cluster like spherical galaxies where the correction to the GW signal due to smeared distribution is accessible to present day observational precision. On the other hand, the correction induced by quantum gravity motivated smearing, of the order of Planck length, is expected to be very small for astrophysical objects (which, however, can be significant for the hypothetical Planck mass black holes).

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1 Introduction

The recent detection of Gravitational waves (GWs), from binary black hole (BH) mergers and Neutron stars by LIGO and Virgo collaboration [1, 2, 3, 4], has provided us with rich opportunities to understand physical processes at extreme conditions, where the role of gravity is of colossal importance over the other known forms of interactions in nature. As we do not know the metric to describe the gravitational collapse of a binary merger, therefore through numerical relativity simulations, we gain a fairly reasonable understanding of such realistic phenomena. So far these events are perfectly consistent with the theoretical predictions of general relativity (GR) [5]. These observations now firmly suggest studying the possibility of having numerous sources that are capable of generating GWs with different characteristic frequencies. Although an ultra-compact system of binaries is rather exotic to produce strong GW signal to be detected at large distances, less extreme objects nevertheless also produce GWs with frequencies that could be highly relevant for the next generation space-based GW detectors.

The popularly known GWs as observed by LIGO/Virgo collaboration are actually the QNMs [6, 7, 8]. The waveform of these GW signals consists of three parts: 1) inspiral, 2) merger and 3) ring-down. The ring-down phase shows characteristic frequencies of oscillation corresponding to damped resonances of the remnant BH. These damped frequencies or QNMs encode information about the BH source. Applying the linear perturbation results, the ring-down portion of the signal may be used to discriminate between BHs and other possible sources. The damped modes in turn possess a complex frequency whose real part corresponds to the oscillation frequency and whose imaginary part gives the lifetime. The QNM spectrum of a BH is completely characterized by the BH parameters, and does not depend on the initial conditions of the perturbations. However

In this work our aim is to investigate the QNM frequencies for a spherically symmetric geometry having a smeared matter source. An interesting approach was pioneered by Nicolini, Smailagic and Spallucci [9]. These authors introduced a (spherically symmetric) smeared source in the matter sector and solved the Einstein equation thereby obtaining a generalized form of Schwarzschild (black hole) metric that successfully cured the black hole singularity problem. It was also proposed to tentatively identify the smearing scale with Planck length so that the metric can play a role in the context of quantum gravity. (For an exhaustive review see [10].) Various aspects of this generalized black hole has been studied its thermodynamics [11], effect of the smearing on AdS/CFT correspondence [12], among others. As mentioned above, the conventional Dirac delta source term for matter is replaced with a new type of matter source with the energy density as given by a Gaussian distribution function. The resulting geometry will be helpful to understand the dynamics of objects, which have approximately a Gaussian mass profile. From astrophysical point of view, such
a Gaussian profile can be applied to study the dynamics of GWs for elliptical galaxies (e.g. globular clusters having dense matter core in the center [13]). Besides that, this distribution is also relevant for the Dark matter profiles within galaxies (e.g. Press-Schechter mass distribution that is extensively used in the context of dark matter distribution profile [14]). However, the mathematical formulation of this study with smeared matter source will be the same irrespective of whether it is inspired by QG or by astrophysical motivations. The only discrimination lies in the relevant length scales for these motivations. In QG this length scale is typically $O(M_{pl})$, whereas, for astrophysical objects the length scale would be $O(Ly)$ ($1Ly \sim 3 \times 10^{-7}Mpc$). In this paper, we correspond this length scale by the parameter $\Theta$.

In Sec. 2 first we have briefly reviewed about the basic aspects QNMs and an elementary method to obtain them for static spherically symmetric Schwarschild spacetime. Then in Sec. 3 we have studied the gravitational perturbation of a spherically symmetric QG-inspired spacetime. Here the study is made in three segments. Then in Sec 3.1 we have computed the odd parity gravitational perturbation of this QG-inspired spacetime. In Sec. 3.2 we discuss the results for the new QNMs and compared them with the standard Schwarzschild QNMs. Finally in Sec. 4 we conclude.

2 Brief review on QNM and determination of QNM spectrum

(a) Quasi-normal mode:

A black hole posses characteristic frequencies which arise from perturbations in it’s spacetime geometry. Such perturbations of the BH geometry can originate in many different ways. For example a certain mass falling along the geodesic of the Schwarzschild spacetime can be considered as a perturbation on the background Schwarzschild geometry. In any of these ways of distorting the BH equilibrium, the BH system undergoes damped oscillations with complex frequencies. These frequencies are called quasi-normal modes (QNMs). Here the term ‘quasi’ is referring the fact that the frequencies are complex, thus they show strong damping. While the conventional normal modes of compact classical linear oscillating systems demand no dissipation, for black hole QNMs [6], the dissipations cannot be neglected, as the event horizon imposes necessary loss of energy.

The real part of this QNM frequency corresponds to the oscillation frequency, whereas the imaginary part corresponds to the damping rate. From astrophysical point of view, QNMs dominate an exponentially decaying ringdown phase at intermediate times in the GW signal from a perturbed BH [8]. Moreover, they also govern the ringdown phase of gravitational systems produced by the merger of a pair of black holes [15, 16]. Since these QNMs are
independent of the initial perturbation, hence we can infer crucial information regarding the fingerprints (e.g. mass, angular momentum of a BH [17], etc) of its geometry.

(b) Ferrari-Mashoon method for computing the QNM spectrum:

There are various methods to determine the Quasinormal mode (QNM) spectrum of a black hole spacetime. Here we will employ the semi-analytic method developed by Mashoon [18] for finding out the QNMs of QG motivated spherically symmetric BH space-time. This is a comparatively simpler tool to explore the essential features of QNM. Although, other methods vary in complexity and precision, nonetheless our approach is supposed to capture the immediate consequences of the quantum gravitational effects upon the QNM spectrum.

Ferrari-Mashoon method relies on the connection between the effective potential and the bound states of the inverted black hole effective potentials [18]. Let us briefly illustrate the method for the case of Schwarzschild spacetime. The Schwarzschild spacetime corresponding to the unperturbed solution of Einstein equation, is represented by the metric around a fixed spherically symmetrical center-of-mass $M$

$$
\begin{align*}
    ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2
\end{align*}
$$

where, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Now we consider a small non-spherical perturbations $h_{\mu\nu}$ such that the new perturbed metric is,

$$
\begin{align*}
    g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} \quad \text{where,} \quad \frac{|h_{\mu\nu}|}{|\bar{g}_{\mu\nu}|} << 1
\end{align*}
$$

Here we denote the static generic background metric by $\bar{g}_{\mu\nu}$. The inverse metric is then

$$
\begin{align*}
    g^{\mu\nu} &= \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)
\end{align*}
$$

The perturbed Christoffel symbols are given by

$$
\begin{align*}
    \Gamma^\alpha_{\mu\nu} &= \frac{1}{2}(\bar{g}^{\alpha\sigma} - h^{\alpha\sigma})(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}) \\
    &= \bar{\Gamma}^\alpha_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\sigma}(h_{\sigma\nu,\mu} + h_{\sigma\mu,\nu} - h_{\mu\nu,\sigma} - 2h_{\sigma\kappa}\bar{\Gamma}_{\mu\nu}^{\kappa}) + \mathcal{O}(h^2) \\
    \simeq \bar{\Gamma}^\alpha_{\mu\nu} + \delta\Gamma^\alpha_{\mu\nu}
\end{align*}
$$

where the $\bar{\Gamma}$ is the Christoffel symbol for the unperturbed metric $\bar{g}_{\mu\nu}$ and $\delta\Gamma$’s with respect to the perturbed metric is

$$
\begin{align*}
    \delta\Gamma^\alpha_{\mu\nu} &= \bar{g}^{\alpha\beta}\left(\nabla_\nu h_{\mu\beta} + \nabla_\mu h_{\nu\beta} - \nabla_\beta h_{\mu\nu}\right)
\end{align*}
$$

$^1$for a details and review about the other methods see for example [6, 7]
Exploiting this expression of $\delta \Gamma$, the Ricci tensor $R_{\mu\nu}$ can be written as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}$$  \hspace{1cm} (6)$$

where $\bar{R}_{\mu\nu}$ is the background Ricci tensor derived from $\bar{g}_{\mu\nu}$ and $\delta R_{\mu\nu}$ is its variation up to first order in $h_{\mu\nu}$. Therefore, the linearized vacuum Einstein field equation becomes

$$\delta R_{\mu\nu} = \frac{1}{2} (\nabla_\mu \nabla_\nu h - \nabla_\sigma \nabla_\mu h^\sigma_\nu - \nabla_\sigma \nabla_\nu h^\sigma_\mu + \Box h_{\mu\nu}) = 0$$  \hspace{1cm} (7)$$

where in eqn. (6) we have used the vacuum Einstein’s equation $\bar{R}_{\mu\nu} = 0$.

Using the definition of covariant derivative for the perturbed Christoffel symbol given in eqn. (5), we can also write eqn. (7) into a more convenient form as

$$\nabla_\beta \delta \Gamma^\beta_\mu_\nu - \nabla_\nu \delta \Gamma^\beta_\mu_\beta = 0.$$  \hspace{1cm} (8)$$

Finally, putting the expression for $\delta \Gamma$ into eqn. (8) and employing gauge freedom, we get the second order differential equation for $h_{\mu\nu}$,

$$\Box h_{\mu\nu} - 2\bar{R}^\rho_\sigma h^\sigma_\mu h^\rho_\nu = 0$$  \hspace{1cm} (9)$$

in the TT (transverse traceless) gauge, where

$$\nabla^\mu h_{\mu\nu} = 0 \text{ and } h^\mu_\mu = \bar{g}^{\mu\nu} h_{\mu\nu} = h = 0.$$  \hspace{1cm} (10)$$

A generic perturbation, $h_{\mu\nu}$ of the spherically symmetric metric can be broken up into odd ($h_{\mu\nu}^{\text{odd}}$) and even ($h_{\mu\nu}^{\text{even}}$) parity components according to their transformation properties under parity [19, 20, 21, 22]. Specifically, odd components are those that transform as $(-1)^{l+1}$, under a parity transformation $(\theta, \phi) \rightarrow (\pi - \theta, \pi + \phi)$, while the even ones transform as $(-1)^l$. However, due to linearity we can treat them separately. Here, we will focus on the odd-parity metric perturbations that are also called axial perturbations.

Let us now proceed for a further simplification of the components of $h_{\mu\nu}$ by using the residual freedom to choose a proper gauge. If we consider an infinitesimal coordinate transformation $x^\alpha \rightarrow x^\alpha + \xi^\alpha$, there will be new metric perturbations which are determined by specifying suitable conditions on $\xi^\alpha$. For odd parity perturbations, the gauge vector that simplifies their form is $\xi^\alpha = \{0, 0, \Lambda(t, r) \partial_\theta Y^m_l, \Lambda(t, r) \partial_\phi Y^m_l\}$. It is then customary to choose a gauge where the arbitrary function $\Lambda(t, r) = 0$; thus eliminating all the highest derivatives in the angles $(\theta, \phi)$ and thereby yielding a simplification of the equations. This is known as the Regge-Wheeler gauge. [20]

In this gauge, the true gauge invariant axial perturbations are described by the functions $h_0(t, r)$ and $h_1(t, r)$. The final form for the axial perturbations is then given by

$$h_{\mu\nu}^{\text{odd}} = \begin{bmatrix} 0 & 0 & 0 & h_0(t, r) \\ 0 & 0 & 0 & h_1(t, r) \\ h_0(t, r) & h_1(t, r) & 0 & 0 \end{bmatrix} \times e^{im\phi} (\sin \theta \partial \theta / \partial \theta) P_l(\cos \theta)$$
where $P_l(\cos \theta)$ is the Legendre polynomial of degree $l$.

The gravitational odd parity perturbations for this spherically symmetric spacetime are now described by the Regge-Wheeler equation

$$\frac{\partial^2 Q(t, r)}{\partial t^2} - \frac{\partial^2 Q(t, r)}{\partial r^2} + V_{\text{axial}}(r)Q(t, r) = 0$$  \hspace{1cm} (11)

where $Q(t, r)$ is the gauge-invariant odd-parity variable, also known as Regge-Wheeler variable, and it is defined as

$$Q(t, r) = \left(1 - \frac{2M}{r}\right) \frac{h_1(t, r)}{r}$$  \hspace{1cm} (12)

with $h_1(t, r)$ being an unknown function from the perturbation. Now the time dependence in $Q(t, r)$ can be extract by writing $Q(t, r) \sim e^{i\omega t}Q(r)$. Therefore, eqn. (11) takes the form

$$\frac{\partial^2 Q(r)}{\partial r_*^2} + (\omega^2 - V_{\text{axial}}(r))Q(r) = 0$$  \hspace{1cm} (13)

The function $V_{\text{axial}}(r)$ is given by

$$V_{\text{axial}}(r) = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$$  \hspace{1cm} (14)

The solutions of the eqn. (13) defines the QNMs of the black hole while the associated frequencies ($\omega$) to those modes are the QNM frequencies.

Now for the Schwarzschild geometry, eqn. (13) involves the so called tortoise co-ordinate ($r_*$) and it’s defined as

$$dr_* = e^{\frac{1}{2}\lambda - \frac{1}{2}\nu} = \frac{1}{1 - \frac{2M}{r}}$$  \hspace{1cm} (15)

where, $\lambda$ and $\nu$ are given by the expression for the metric written in an alternate form as

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2.$$  \hspace{1cm} (16)

Thus, one gets

$$e^\nu = e^{-\lambda} = 1 - \frac{2M}{r}$$  \hspace{1cm} (17)

After integrating eqn. (15) one obtains for $r_*$

$$r_* = r + 2M \ln \left(\frac{r}{2M} - 1\right)$$  \hspace{1cm} (18)

\(^2\text{The other component } h_0(t, r) \text{ can be removed by using the } \delta R_{\theta\phi} \text{ components of eqn. (6) \cite{20}}\)
Since \( r_\ast \to \infty \) as \( r \to \infty \) and \( r_\ast \to -\infty \) as \( r \to 2M \), so tortoise co-ordinate will be help full in this context for it does not suffer from coordinate singularity near the event horizon at \( r = 2M \) (since \( r_\ast \) is pushed to \( -\infty \) at horizon).

The main purpose is getting the form of the quantity \( \omega \) and hence the possible values it can take. Now eqn.(13) can be considered as a one-dimensional Schrödinger equation in a scattering potential barrier \( V^{\text{axial}}(r) \). But this equation can not be solved analytically for getting the bound states of the system. Therefore, to proceed further and obtain some realistic results, one replaces this potential with another similar potential function whose bound states are exactly known [18]. The behavior of this new potential mimics the Schwarzschild potential fairly accurately but now becomes more convenient to handle at infinity.

It turns out that [18] for \( V^{\text{axial}}(r) \), the ground state and first few excited states can be approximated by the Pöschl-Teller potential

\[
V_{PT}(r_\ast) = \frac{V_0}{\cosh^2 \alpha(r_\ast - \tilde{r}_\ast)},
\]

where \( \tilde{r}_\ast \) is the point of extremum of the potential. Here \( V_0 \) and \( \alpha \) are related to the height and curvature of the potential respectively and they are,

\[
V_0 = V_{PT}(\tilde{r}_\ast) \quad \text{and} \quad \alpha^2 = \frac{1}{2V_0} \left[ \frac{d^2V_{PT}}{dr_\ast^2} \right]_{r_\ast=\tilde{r}_\ast}.
\]

The bound state frequencies \( \Omega(V_0, \alpha) \) corresponding to this potential are known eaxactly to be

\[
\Omega(V_0, \alpha) = \alpha \left[ - \left( n + \frac{1}{2} \right) + \left( \frac{1}{4} + \frac{V_0}{\alpha^2} \right)^{1/2} \right]
\]

The proper QNM frequencies of the original potential \( V^{\text{axial}}(r) \) in eqn. (14) are then obtained from eqn. (21) by the parameter replacement \( (V_0, \alpha) \to (V_0, -i\alpha) \) [18] and are given by the expression

\[
\omega = \pm \sqrt{\left( V_0 - \frac{\alpha^2}{4} \right) + i\alpha \left( n + \frac{1}{2} \right)}
\]

### 3 Spacetime for a smeared (Gaussian) matter distribution

In this section, we consider the metric of a spherically symmetric spacetime geometry with a Gaussian distributed matter source (this kind of matter distribution in general has two-fold motivation as mentioned in Sec.1). Our aim is to compute the form of the potential for odd parity perturbations of this background geometry and subsequently the associated QNM frequencies.
3.1 Perturbation of the QG-inspired spacetime geometry

Let us now start with the QG-inspired spherically symmetric spacetime. The metric of such spacetime is given by

\[ ds^2 = - \left(1 - \frac{4M}{r} \gamma(3/2, r^2/4\Theta)\right) dt^2 + \left(1 - \frac{4M}{r} \gamma(3/2, r^2/4\Theta)^{-1} dr^2 + r^2 d\Omega^2 \right) \]

(23)

where, \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2, \) \( \sqrt{\Theta} \) is some minimal length scale (may be identified with Planck length) which removes the singularity of the usual Schwarzschild spacetime, and

\[ \gamma(3/2, r^2/4\Theta) = \int_0^{r^2/4\Theta} \sqrt{t} e^t \; dt \]

(24)

is the lower incomplete Gamma function. We now expand the incomplete gamma function of eqn. (23) in the limit \( r^2 >> 4\Theta \) and obtain an approximate form of the metric

\[ ds^2 = - \left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right)^{-1} dr^2 + r^2 d\Omega \]

(25)

The perturbed Einstein eqn. (6) in vacuum is

\[ R_{\mu\nu} = \delta R_{\mu\nu} = 0 \quad \text{as,} \quad \bar{R}_{\mu\nu} = 0. \]

(26)

This equation has ten components. It turns out that only three of them (corresponding to the components \( \delta R_{r\phi}, \delta R_{t\phi} \) and \( \delta R_{\theta\phi} \)) survive and they are respectively given below in explicit form

\[ \left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right)^{-1} \left[ \partial_{\phi}^2 h_0 - \partial_{\phi} \partial_{\theta} h_1 + \frac{2}{r} \partial_{\theta} h_1 \right] - \frac{l(l+1)}{r^2} h_0 \]

\[ + \frac{2}{r} \left( \frac{2M}{r} - \frac{M \sqrt{\pi} \Theta e^{-r^2/4\Theta}}{r} \right) h_0 = 0 \]

(27)

\[ \left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right)^{-1} \left[ \partial_{\theta}^2 h_1 - \partial_{\theta} \partial_{\phi} h_0 + \frac{2}{r} \partial_{\phi} h_0 \right] + \frac{(l+2)(l-1)}{r^2} h_1 = 0 \]

(28)

\[ \partial_r \left[ \left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right) h_1 \right] + \frac{\partial_t h_0}{\left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right)} = 0 \]

(29)

Let us now introduce a \( \Theta \)-dependent generalized Regge-Wheeler variable \( Q_\Theta \) as,

\[ Q_\Theta(t, r) = \left(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi} \Theta} e^{-r^2/4\Theta}\right) \frac{h_1(t, r)}{r} \]

(30)
With the help of eqn. (29) we can eliminate $h_0(t,r)$ and thus the final equation for the axial perturbation assumes the following simple form,

$$\frac{\partial^2}{\partial t^2}Q_\Theta(t,r) - \frac{\partial^2}{\partial r^2}Q_\Theta(t,r) + V^{\text{axial}}(r)Q_\Theta(t,r) = 0,$$

(31)

where $Q_\Theta(t,r)$ is given in eqn. (30), the potential function is

$$V^{\text{axial}}(r) = \left( 1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi}\Theta} e^{-r^2/4\Theta} \right) \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} + \frac{M}{\Theta\sqrt{\pi}\Theta} e^{-r^2/4\Theta} \right. $$

$$\left. + \frac{4M}{\sqrt{\pi}\Theta r^2} e^{-r^2/4\Theta} \right]$$

(32)

and we define the new co-ordinate $r_\Theta$, in analogy with the definition of eqn. (15), as

$$\frac{dr_\Theta}{dr} = \frac{1}{(1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi}\Theta} e^{-r^2/4\Theta})}$$

(33)

Therefore, likewise eqn. (17), for the QG-inspired spherically symmetric geometry we would have

$$e^\nu = e^{-\lambda} = \left( 1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi}\Theta} e^{-r^2/4\Theta} \right).$$

(34)

Let us now extract the time dependence in $Q_\Theta(t,r)$ as $Q_\Theta(t,r) \sim e^{i\omega t}Q_\Theta(r)$. So, eqn. (31) can be written as,

$$\frac{\partial^2}{\partial r_\Theta^2}Q_\Theta(r) + \left[ \omega_\Theta^2 - V(r) \right]Q_\Theta(r) = 0$$

(35)

So we have obtained the form of the potential for the odd parity perturbation in QG-corrected BH spacetime. Now we are going to see whether we can implement the method described in Sec. 2 to get the QNMs spectrum for this QG-corrected potential.

### 3.2 QNM of QG-inspired spherically symmetric space-time

To obtain the QNMs for the potential in eqn. (32) in Ferrari-Mashoon method, first we have to see whether this potential can also be mapped with the so-called Pöschl-Teller potential of eqn. (19). For this, we write the modified potential for axial perturbation after incorporating the QG-correction as,

$$V^{\Theta}_{\text{axial}}(r) = V_{\text{axial}}(r) + V^{\text{extra}}_{\text{axial}}(r)$$

(36)

where $V_{\text{axial}}$ is given in eqn. (14) and

$$V^{\text{extra}}_{\text{axial}}(r) = \frac{2M}{\sqrt{\pi}\Theta} e^{-r^2/4\Theta} \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} + \left( \frac{1}{2\Theta} + \frac{2}{r^2} \right) \left( 1 - \frac{2M}{r} \right) \right]$$

(37)
Figure 1: Graph showing the variations of both $r_*$ and $r_\Theta$ as a function of the coordinate $r$.

Just by inspection of eqn. (37) we can see that at large distance from the horizon at $r = 2M$, the correction terms fall exponentially fast. Therefore, this effective potential is not expected to deviate much from $V_{PT}$. To see that, let us first find out the minimum of the $\Theta$-corrected effective potential in eqn. (36).

We assume that the extremum of the new potential is perturbatively shifted to the point,

$$r_0^\Theta \simeq r_0 + \frac{2M}{\sqrt{\pi \Theta}} e^{-r^2/4\Theta} r'$$

where $r'$ is so far unknown and $r_0$ is the minima of the effective potential of eqn. (36) and is given by

$$\frac{r_0}{M} = \sqrt{9(l^2 + l + 3)^2 - 96l(l + 1)} + \frac{3(l^2 + l + 3)}{2l(l + 1)}.$$  \hspace{1cm} (39)

Now taking the derivative of eqn. (36) and using eqn. (38), we solve for $r'$ to get the new extremum located at

$$r_0^\Theta \simeq r_0 + \frac{2M}{\sqrt{\pi \Theta}} e^{-r^2/4\Theta} \times \frac{1}{3\Delta} \left[ r_0^2 A + \frac{r_0^4}{4\Theta} B \right]$$

where,

\begin{align*}
A &= \left[ l(l + 1)r_0 - M(9 + 4r_0^2) \right], \\
B &= \left[ r_0(l^2 + l + 4r_0^2) - 2M(3 + 4r_0^2) \right], \\
\Delta &= l(l + 1)r_0^2 + 40M^2 - 4(l^2 + l + 3)Mr_0.
\end{align*}

Now following the same prescription of eqn. (20), the maximum of the potential $V_0^{\text{axial}}$ and the curvature parameter $\alpha_0^{\text{new}}$ are then given by,

$$V_0^\Theta = V_{\text{axial}}(r)|_{r = r_0^\Theta}$$

$$\alpha^\Theta = \frac{1}{2V_0^\Theta} \left[ \frac{d^2V_{\text{axial}}(r)}{dr_0^2} \right]_{r = r_0^\Theta}$$
where \( \frac{d^2}{dr_\Theta^2} \) can be found by using the new transformation rule of eqn. (33),

\[
\frac{d^2 V^{\Theta}_\text{axial}(r(r_\Theta))}{dr_\Theta^2} = \left( 1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi \Theta}} e^{-r^2/4\Theta} \right) \frac{d^2 V^{\Theta}_\text{axial}(r)}{dr^2} + \\
\frac{dV^{\Theta}_\text{axial}(r)}{dr} \left\{ \left( 1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi \Theta}} e^{-r^2/4\Theta} \right) \cdot \left( \frac{2M}{r^2} - \frac{4Mr}{\sqrt{\pi \Theta}} e^{-r^2/4\Theta} \right) \right\}
\]

3.2.1 Brief note on the behaviour of the co-ordinate \( r_\Theta \):

At this point let us illustrate one important issue. Without the \( \Theta \)-correction, the Pöschel-Teller potential was a function of the tortoise coordinate \( r_* \), while the potential, \( V_{\text{axial}} \) for the odd parity perturbation of Schwarzschild geometry is a function of \( r \). In that case, the behaviour of the potentials \( V_{PT} \) and \( V_{\text{axial}} \) are very similar. But, when we use the \( \Theta \)-corrected metric of eqn. (25), then the potential for the axial perturbation of this new geometry is changed to \( V^{\Theta}_{\text{axial}} \). Therefore, now the corresponding transformation between the co-ordinates is not given by eqn. (15) but instead through eqn. (34). Here the new co-ordinate, which we call \( r_\Theta \) is a function of the parameter \( \Theta \). The relation can now be integrated numerically.
to obtain the behaviour of \( r_\Theta \) as a function of \( r \). This is plotted in Fig. 1. We can see clearly that the behaviour of this new coordinate \( r_\Theta \) is not markedly different from \( r_* \), since \( \frac{dr_*}{dr} \sim \frac{dr_\Theta}{dr} \) (though it is shown for \( \Theta = 0.4 \), but for other values of \( \Theta \) which are used to plot Fig. 2, we have checked numerically that this conclusion is not altered). Therefore,

| \( n \) | \( \Theta \) | \( r_\Theta \) (normal case) | \( r_* \) (QG-inspired case) |
|-------|----------|----------------|------------------|
| 2     | 0.2      | 0.756547 + i 0.181062 | 0.769551 + i 0.183625 |
|       | 0.3      | 0.817874 + i 0.176333  |
|       | 0.4      | 0.911319 + i 0.0762563  |
| 1     | 0.2      | 0.756547 + i 0.543186  | 0.769551 + i 0.550875 |
|       | 0.3      | 0.817874 + i 0.529     |
|       | 0.4      | 0.911319 + i 0.228769  |
| 3     | 0.2      | 1.20484 + i 0.186741  | 1.2255 + i 0.189122  |
|       | 0.3      | 1.29923 + i 0.182999   |
|       | 0.4      | 1.42385 + i 0.12858    |
| 1     | 0.2      | 1.20484 + i 0.560224  | 1.2255 + i 0.567365  |
|       | 0.3      | 1.29923 + i 0.548996   |
|       | 0.4      | 1.42385 + i 0.38574    |
| 2     | 0.2      | 1.20484 + i 0.933707  | 1.2255 + i 0.945609  |
|       | 0.3      | 1.29923 + i 0.914993   |
|       | 0.4      | 1.42385 + i 0.642901   |

Table 1: Comparison between the QNMs for the gravitational perturbation of Schwarzschild spacetime (in unit of \((2M)^{-1}\)) and spherically symmetric spacetime (in unit of \((2\tilde{M})^{-1}\), where \(\tilde{M} \simeq M(1 + \frac{2M}{\sqrt{\pi}e^{r^2/\Theta}})\) with smeared matter source.

suffice would it be here to use the corresponding formula for the \( V_\Theta^{\alpha} \) and \( \alpha^{\Theta} \) as given by eqns. (41),(42).

In Fig. 2 we have plotted the variation of the Pöschel-Teller potential and the potential for the axial perturbation as a function of the co-ordinate \( r_\Theta \). We have taken three different values of \( \Theta \) to show that the form of the axial perturbation potential \( V^{\Theta}_{\text{axial}} \) does not differ drastically from the form of \( V_{\text{PT}} \) with respect to the new transformed co-ordinate \( r_\Theta \). Since the QNMs of \( V^{\Theta}_{\text{axial}} \) are related with the asymptotical behaviour of the potential, therefore we can legitimately use the previous mapping scenario to get the QNMs for this potential.
Therefore, the QNMs for the QG-corrected potential for axial perturbation will be given as,

\[ \omega^\Theta = \pm \sqrt{\left(V^\Theta_0 - \frac{(\alpha^\Theta)^2}{4}\right) + i\alpha^\Theta \left(n + \frac{1}{2}\right)} \]  \hspace{1cm} (43)

where \(V^\Theta_0\) and \(\alpha^\Theta\) are given by eqns. (41) and (42) respectively.

### 3.2.2 Results of QNM

The following table shows the values of QNM frequencies for the QG-inspired space-time calculated using eqn. (43).

From Tab. 1, we see that for a given \(l\) both \(\text{Re}[\omega^\Theta]\) and \(\text{Re}[\omega]\) are independent of \(n\). However, we found that for values of \(l = 2, 3, 4\) etc., the \(\text{Re}[\omega^\Theta]\) corresponding to the QNMs of QG-inspired geometry always has a value larger than the \(\text{Re}[\omega]\) of the QNMs for the standard Schwarzschild geometry. Though here we have shown the QNMs for vales of \(l\) upto 3, but this result is true in general for other higher values of \(l\) also. In Fig. 3 we plot the frequencies \(\text{Re}[\omega^\Theta]\) and \(\text{Re}[\omega]\) as a function of the parameter \(\Theta\).

Here the colored dashed plots correspond to the variation of \(\text{Re}[\omega^\Theta]\) as a function of \(\Theta\) for values of \(l\) ranging from 2 (bottom) to 5 (top) respectively. Also, the associated frequencies \(\text{Re}[\omega]\) in this graph are shown by colored contiguous plots. Since \(\text{Re}[\omega]\) being independent of \(\Theta\) for all values \(l\), so they appear to be parallel straight lines for the same set of \(l\) values. From Fig. 3, we also note that the difference \((\text{Re}[\omega^\Theta] - \text{Re}[\omega])\) is found to increase as we keep on moving towards higher \(l\) values.
A sharp contrast between $\omega^\Theta$ and $\omega$ is that for line element of eqn. (1), the QNM frequencies $\omega$ can be expressed in unit of $(2M)^{-1}$, since $\omega(M,n,l) = M.f(n,l)$ i.e. its overall dependence on $M$ can be factored out. However this is not the case with the QNM frequencies $\omega^\Theta$, where it depends on $M$ in a complicated way as as $\omega^\Theta(n,l,M,\Theta) \neq M.g(n,l,\Theta)$. This is because $\omega^\Theta$ now involves a dimensionful parameter $\Theta$ and $\dim[\sqrt{\Theta}] = \mathcal{O}(M_{pl})$. So, whenever $\sqrt{\Theta}$-parameter appears in $\omega^\Theta$, there must come up a factor of $M$ so that the dimension of $\omega^\Theta$ remains preserved.

Our main result is the change in QNM frequencies due to a change the spacetime geometry induced by an smeared distribution of mass. In particular we found that the real part of QNM frequency $\omega^\Theta$ for smeared mass density to be greater than the same $\omega$ for conventional point mass density. A possible reason for this behavior might be understood in a qualitative way from the smearing effect of the distribution profile. For a smeared gaussian matter source the mass density is diffused within the length $\sqrt{\Theta}$. The density decreases exponentially as we move out from the central bulge of the distribution. This means the scattering of the GW would continue to take place even if we go further away from the high source density region thus hinting at a possibility of an enhanced oscillation frequency.

3.2.3 Contact with measurements

So far, we have depicted the change in the QNM frequency spectrum when the source has smeared matter distribution. Here, we discuss the relevance of this result as regard with the observational aspects. As an example, let us consider the fundamental GW mode (for $l = 2, n = 0$) of Schwarzschild geometry. The associated real part of the frequency $\omega_{re} = 0.756547$ from Tab. 1 can be expressed in Hz unit as

$$f = \frac{\omega_{re}}{4\pi M} \times \frac{c^3}{G} = \left(\frac{\omega_{re}}{4\pi} \times \frac{c^3}{GM_{\odot}}\right) \times \frac{M_{\odot}}{M}$$

where $M_{\odot}$ is the solar mass. Using this formula with $M = 1 M_{\odot}$, the frequency $f$ for the fundamental mode turns out to be 12 kHz.

Now there exists compact spherical star clusters (e.g globular clusters) that approximately follow a Gaussian matter distribution. A typical order of magnitude estimate for the mass ($\tilde{M}$) of this cluster is $10^5 M_{\odot}$. With $M = \tilde{M}$, it can be shown from eqn. (44) that if the corresponding smeared distribution has a spread $\sqrt{\Theta} \sim 10^4$ km (which matches with a $\Theta \sim 0.29$ within the range $\Theta \sim 0.20 - 0.30$ of Tab. 1), then it yields a signal having frequency $f \sim 13$ kHz. As a first clue, this small change in frequency is significant to infer the nature of the source, that is to say, whether a GW detected with this frequency is associated to a class of point mass or diffused mass pattern. On the other hand, for $\sqrt{\Theta}$ close to Planck length (which comes from its interpretation as a scale of QG), the change
in frequency turns out to be too small for present day observational precision. However, there are theoretical predictions of the existence of micro-black holes with mass of the order of Planck mass $M \sim O(M_{pl})$ (see for example [23, 24]). Collapse of such micro-black holes can produce signals which might be detectable in future as the associated frequencies, $f \approx \left( \frac{\omega_{ce}}{\pi} \frac{c^3}{GM_{pl}} \right) \cdot \frac{M_{pl}}{M} \mid_{M=M_{pl}}$ will be nearly $10^6$ THz.

4 Conclusion

In this work, we have studied the QNM frequency spectrum for the static spherically symmetric spacetime having a smeared (Gaussian type) matter distribution. This type of matter distribution, involving a length scale, can be motivated either from a (phenomenological) quantum theory of gravity or from astrophysical perspectives (in the context of star clusters). Hence our result can be relevant for both the above scenarios depending on a proper choice of the length of smearing scale. Also, such a length scale is crucial to identify the character of the source density. As a demonstration with astrophysical objects, we found that the resulting frequency change due to smearing is within the current limits of the GW detectors. Besides that, if we consider the existence of quantum black holes then spacetime geometries might have a non trivial scale where stringy effects would be a salient feature. In this case, the signals would be in very high frequency regime, thus enabling the possibility to be detected in future.

In summary, our analysis here focuses on the gravitational perturbations of a background geometry, that are odd multipoles under parity transformation. The gravitational perturbations do posses an even parity component as well. For the case of conventional spherically symmetric Schwarzschild geometry with a delta-function source, there is a special property for the perturbation spectrum that ensures that the QNM spectra for odd and even parity perturbations are equal. In technical terms, one says that the QNM spectrum of odd parity perturbations is iso-spectral with the QNM spectrum of even parity perturbations [19]. However, it is not clear whether the same would hold true for the QG-induced spacetime studied here. With the Gaussian distributed matter density, the potential for even-parity perturbation would have a new scale dependence. This feature may restrict the validity of the iso-spectral character between perturbations of opposite parity. This requires an explicit computation of the QNM spectrum of even-parity perturbations that we left as a future work.
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