ON LATENT POSITION INFERENCE FROM DOUBLY
STOCHASTIC MESSAGING ACTIVITIES

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Abstract. We model messaging activities as a hierarchical doubly stochastic point process with three main levels, and develop an iterative algorithm for inferring actors’ relative latent positions from a stream of messaging activity data. Each of the message-exchanging actors is modeled as a process in a latent space. The actors’ latent positions are assumed to be influenced by the distribution of a much larger population over the latent space. Each actor’s movement in the latent space is modeled as being governed by two parameters that we call confidence and visibility, in addition to dependence on the population distribution. The messaging frequency between a pair of actors is assumed to be inversely proportional to the distance between their latent positions. Our inference algorithm is based on a projection approach to an online filtering problem. The algorithm associates each actor with a probability density-valued process, and each probability density is assumed to be a mixture of basis functions. For efficient numerical experiments, we further develop our algorithm for the case where the basis functions are obtained by translating and scaling a standard Gaussian density.

Key words. Social network; Multiple doubly stochastic processes; Classification; Clustering

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1. Introduction. Communication networks are presenting ever-increasing challenges in a wide range of applications, and there is great interest in inferential methods for exploiting the information they contain. A common source of such data is a corpus of time-stamped messages such as e-mails or SMS (short message service). Such messaging data is often useful for inferring a social structure of the community that generates the data. In particular, messaging data is an asset to anyone who would like to cluster actors according to their similarity. A practitioner is often privy to messaging data in a streaming fashion, where the word streaming describes a practical limitation, as the practitioner might be privy only to the incoming data in a fixed summarized form without any possibility to retrieve past information. It is in the practitioner’s interest to transform the summarized data so that the transformed data is appropriate for detecting emerging social trends in the source community.

We mathematically model such streaming data as a collection of tuples of the form \(D = \{(t_\ell, i_\ell, j_\ell)\}\) of time and actors, where \(i_\ell\) and \(j_\ell\) are the \(\ell\)-th message and \(t_\ell\) \(\in \mathbb{R}_+\) represents the occurrence time of the \(\ell\)-th message. There are many models suitable for dealing with such data. The most notable are the Cox hazard model, the doubly stochastic process (also known as the Cox process), and the self-exciting process (although self-exciting processes are sometimes considered as special cases of the Cox hazard model). For references on these topics, see Andersen et al. [1995], Snyder [1975] and Bremaud [1981]. All three models are related to each other; however, the distinctions are crucial to statistical inference as they stem from different assumptions on information available for (online) inference. To transform \(D\) data to a data representation more suitable for clustering actors, we model \(D\) as a (multivariate) doubly stochastic process, and develop a
method for embedding $\mathcal{D}$ as a stochastic process taking values in $\mathbb{R}^d$ for some suitably chosen $d \in \mathbb{N}$.

2. Related works. For statistical inference when there is information available beyond $\mathcal{D}$, the Cox-proportional hazard model is a natural choice. In Heard et al. [2010] and Perry and Wolfe [2013], for instance, instantaneous intensity of messaging activities between each pair of actors is assumed to be a function of, in the language of generalized linear model theory, known covariates with unknown regression parameters. More specifically, in Heard et al. [2010], the authors consider a model where $\lambda_{ij}(t) = A_{ij}(t)(B_{ij}(t) + 1)$ with $A_{ij}(t)$ and $B_{ij}(t)$ representing independent counting processes, e.g., $A_{ij}(t)$ are Bernoulli random variables and $B_{ij}(t)$ are random variables from the exponential family. On the other hand, in Perry and Wolfe [2013], a Cox multiplicative model was considered where $\lambda_{ij}(t) = \xi_i(t) \exp\{\beta_0 x_{ij}(t)\}$. The model in Perry and Wolfe [2013] posits that actor $i$ interacts with actor $j$ at a baseline rate $\xi_i$ modulated by the pair’s covariate $x_{ij}$ whose value at time $t$ is known and $\beta_0$ is a common parameter for all pairs. In Perry and Wolfe [2013], it is shown under some mild conditions that one can estimate the global parameter $\beta_0$ consistently. In Stomakhin et al. [2011], the intensity is modeled for adversarial interaction between macro level groups, and a problem of nominating unknown participants in an event as a missing data problem is entertained using a self-exciting point process model. In particular, while no explicit intensity between a pair of actors (gang members) is modeled, the event intensity between a pair of groups (gangs) is modeled, and the spatio-temporal model’s chosen intensity process is self-exciting in the sense that each event can affect the intensity process.

When data $\mathcal{D}$ is the only information at hand, a common approach is to construct a time series of (multi-)graphs to model association among actors. For such an approach, a simple method to obtain a time series of graphs from $\mathcal{D}$ is to “pairwise threshold” along a sequence of non-overlapping time intervals. That is, given an interval, for each pair of actors $i$ and $j$, an edge between vertex $i$ and vertex $j$ is formed if the number of messaging events between them during the interval exceeds a certain threshold. This is the approach taken in Cortes et al. [2003], Eckmann et al. [2004], Adamic and Adar [2005], Lee and Maggioni [2011] and Ranola et al. [2010], to mention just a few examples. The resulting graph representation is often thought to capture the structure of some underlying social dynamics. However, recent empirical research, e.g., Choudhury et al. [2010], has begun to challenge this approach by noting that changing the thresholding parameter can produce dramatically different graphs.

Another useful approach when $\mathcal{D}$ is the only information available is to use a doubly stochastic process model in which count processes are driven by latent factor processes. This is the approach taken explicitly in Lee and Priebe [2011] and Tang et al. [2013], and this is also done implicitly in Chi and Kolda [2012]. In Lee and Priebe [2011] and Tang et al. [2013] interactions between actors are specified by proximity in their latent positions; the closer two actors are to each other in their latent configuration, the more likely they exchange messages. Using our model, we consider a problem of clustering actors “online” by studying their messaging activities. This allows us a more geometric approach afforded by embedding $O(n^2)$ data to an $O(n \times d)$ representation for some fixed dimension $d$.

In this paper, we propose a useful mathematical formulation of the problem as a filtering problem based on both a multivariate point process observation and a population latent position distribution.
3. Notation. As a convention, we assume that a vector is a column vector if its dimension needs to be disambiguated. We denote by $\mathcal{F}_i$ the filtration up to time $t$ that models the information generated by undirected communication activities between actors in the community, where “undirected” here means we do not know which actor is the sender and which is the receiver. We denote by $\mathcal{M}_1(\mathbb{R}^d)$ the space of probability measures on $\mathbb{R}^d$. For a probability density function defined on $\mathbb{R}^d$, $\phi(x; c, s)$ denotes the probability density function that is proportional to $\phi(s^{-1}(x - c))$ where the normalizing constant does not depend on $x$. The set of all $r \times c$ matrices over the reals is denoted by $\mathbb{M}_{r,c}$. For each $k_1 \times k_2$ matrix $M \in \mathbb{M}_{k_1,k_2}$, we write $\|M\|_F = (\sum_{r=1}^{k_1} \sum_{c=1}^{k_2} M_{r,c}^2)^{1/2}$. Given a vector $v \in \mathbb{R}^d$, we write $\|v\|$ for its Euclidean norm. Let $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{M}_{k_1,k_2}^+ := \{M \in \mathbb{M}_{k_1,k_2} : M_{r,c} \geq 0\}$. For each $M_1$ and $M_2 \in \mathbb{M}_{k_1,k_2}$, we write $M_1 \ast M_2$ for the Hadamard product of $M_1$ and $M_2$, i.e., $\ast$ denotes component-wise multiplication. Given vectors $v_1, \ldots, v_n$ in $\mathbb{R}^d$, the Gram matrix of the ordered collection $v = (v_1, \ldots, v_n)$ is the $d \times d$ matrix $G$ such that its $(r,c)$-entry $G_{r,c}$ is the inner product $\langle v_r, v_c \rangle = v_r^\top v_c$ of $v_r$ and $v_c$. Given a matrix $M \in \mathbb{M}_{d,d}$, $\text{diag}(M)$ is the column vector whose $k$-th entry is the $k$-th diagonal element of $M$. With a slight abuse of notation, given a vector $v \in \mathbb{R}^d$, we will also denote by $\text{diag}(v)$ the $d \times d$ diagonal matrix such that its $k$-th diagonal entry is $v_k$. We always use $n$ for the number of actors under observation and $d$ for the dimension of the latent space. We denote by $X$ the $n$-fold product $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ of $\mathbb{R}^d$. An element of $X$ will be written in bold face letters, e.g. $\mathbf{x} \in X$. Similarly, bold faced letters will typically be used to denote objects associated with the $n$ actors collectively. An exception to this convention is the identity matrix which is denoted by $I_d = I$, where the dimension is specified only if needed for clarification. Also, we write $\mathbf{1}$ as the column matrix of ones. With a bit of abuse of notation, we also write $\mathbf{1}$ for an indicator function, and when confusion is possible, we will make our meaning clear.

4. Hierarchical Modeling. Our actors under observation are assumed to be a subpopulation of a bigger population. That is, we observe actors $\{1, \ldots, n\}$ that are sampled from a population for a longitudinal study. We are not privy to the actors’ latent features that determine the frequency of pairwise messaging activities, but we do observe messaging activities $D_t = \{(t_\ell, i_\ell, j_\ell) : t_\ell \leq t\}$. A notional illustration of our approach thus far is summarized in Figure 4.1, Figure 4.2, and Figure 4.3. In both Figure 4.2 and Figure 4.3, $\tau_1$ represents the (same) initial time when there was no cluster structure, and $\tau_2$ and $\tau_3$ represent the emerging and fully developed latent position clusters which represent the object of our inference task.

Population density process level. The message-generating actors are assumed to be members of a community, which we call the population. The aspect of the population that we model in this paper is its members’ distribution over a latent space in which the proximity between a pair of actors determines the likelihood of the pair exchanging messages. The population distribution is to be time-varying and a mixture of component distributions.

The latent space is assumed to be $\mathbb{R}^d$ for some $d \in \mathbb{N}$, and the population distribution at time $t$ is assumed to have a continuous density $\mu_t$. To be more precise, we assume that the (sample) path $t \to \mu_t$ is such that for each $y \in \mathbb{R}^d$,

$$\mu_t(y) = \sum_\ell q_{t,\ell} \phi(y; \alpha_{t,\ell}, \gamma_{t,\ell})$$

where
Fig. 4.1: Hierarchical structure of the model. The top, middle and bottom layers correspond to the population level, the actor level and the messaging level, respectively. The top two levels, i.e., the population and the actor levels, are hidden and the bottom level, i.e., the messaging level, is observed. See also Figure 5.1 for a more detailed diagram.

- $q_{t,\ell}$ is a smooth sample path of a stationary (potentially degenerate) diffusion process taking values in $(0, 1)$,
- $\phi$ is a probability density function on $\mathbb{R}^d$ with convex support with its mean vector being the zero vector and its covariance matrix being a positive definite (symmetric) matrix,
- $c_{t,\ell}$ is a smooth sample path of an $\mathbb{R}^d$-valued (potentially non-stationary or degenerate) diffusion process,
- $\alpha_{t,\ell}$ is a smooth sample path of a stationary (potentially degenerate) diffusion process taking values in $(0, \infty)$.

Note that it is implicitly assumed that $\sum_{\ell} q_{t,\ell} = 1$, and additionally, we also assume that for each $k = 1, \ldots, d$ and $m \in \mathbb{N}$, the $m$-th moment $\langle \chi_k^m, \mu_t \rangle$ of the $k$-th coordinate of $\mu_t$ is finite, i.e., $\langle \chi_k^m, \mu_t \rangle := \int x_k^m \mu_t(x)dx < \infty$.

In this paper, we take $q_{t,\ell}, c_{t,\ell}$ and $\alpha_{t,\ell}$ as exogenous modeling elements. However, for an example of a model with yet further hierarchical structure, one could take a cue from a continuous time version of the classic “co-integration” theory, e.g., see Comte [1999]. The idea is that the location $c_{t,\ell}$ of the $\ell$-th center is non-stationary, but the inter-point distance between a combination of the centers is stationary. More specifically, one could further assume that there exist $d \times (d - d_0)$ matrix $\alpha_\perp$, $d \times d_0$ matrix $\alpha$ and $d_0 \times d$ matrix $\beta$ such that

- $(\alpha_\perp)^T \alpha$ is the $(d - d_0) \times d_0$ dimensional zero matrix,
- $(\alpha_\perp)^T c_{t,m}$ is a $(d - d_0)$ dimensional Brownian motion,
• $\beta^T c_{t,m}$ is a stationary diffusion process.

Thus, the position of centers are unpredictable, but the relative distance between each pair of centers are as predictable as that of a stationary process.

**Algorithm 1** Simulating a sample path of a population density process

**Require:** $((q(t), c(t), \alpha(t)) : t \in [0, T])$ and $\Delta t$

1: procedure PopulationProcess
2: \hspace{1em} $t \leftarrow 0$
3: \hspace{1em} while $t < T$ do
4: \hspace{2em} $\mu_t(x) \leftarrow \sum_m q_{t,m} \phi(x; c_{t,m}, \alpha_{t,m})$
5: \hspace{2em} $t \leftarrow t + \Delta t$
6: \hspace{1em} end while
7: end procedure

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**Fig. 4.2:** A notional depiction of the evolution of the full population and subpopulation latent position distributions. At each time $\tau_\ell$, the lightly-colored outer histogram represents the latent position distribution $\mu_t$ for the full population, and the darkly-colored inner histogram represents the distribution of the latent positions of actors under consideration. The illustrated temporal order is $\tau_1 < \tau_2 < \tau_3$.

**Actor position process level.** Figure 4.2 sketches the connection between actors and populations. We first define a process for a single actor. To begin, for each $t$, given $\mu_t$ and $(\omega_t, \sigma_t) \in (0, 1) \times (0, \infty)$, we write

$$A_t f(x) = \sum_{k=1}^d b_t^k(x) \frac{\partial}{\partial x_k} f(x) + \sum_{k_1, k_2=1}^d a_t^{k_1, k_2}(x) \frac{\partial^2}{\partial x_{k_1} \partial x_{k_2}} f(x),$$

where

$$\psi(z) = \phi(z)/\phi(0),$$

$$b_t^k(x) = 2(1 - \omega_t) \int \psi \left( \frac{y - x}{\sigma_t} \right) (y_k - x_k) \mu_t(y) dy,$$

$$a_t^{k_1, k_2}(x) = (1 - \omega_t)^2 \int \psi \left( \frac{y - x}{\sigma_t} \right) (y_{k_1} - x_{k_1})(y_{k_2} - x_{k_2}) \mu_t(y) dy.$$
The formulation here for the \( b^k_t \) and \( a^k_t \) is based on a quadratic Taylor-series approximation of a so-called “bounded confidence” model studied in Gomez-Serrano et al. [2012]. Here, the value of \( \omega_{t,i} \) represents the confidence level of actor \( i \) on its current position and \( \sigma_{t,i} \) represents the visibility of other actors' position by actor \( i \). Roughly speaking, an actor with a small value of \( \omega_{t,i} \) and a large value of \( \sigma_{t,i} \) will be influenced greatly by actors that are positioned both near and far in the latent space whereas an actor with a large value of \( \omega_{t,i} \) and a small value of \( \sigma_{t,i} \) will be influenced only a small amount by actors that are nearby in the latent space. For further discussion on our motivation for the form of \( \mathbb{A}_t \), see Appendix A.

For each actor \( i \), the deterministic path \( t \to (\omega_{t,i}, \sigma_{t,i}) \) is assumed to be continuous, taking values in a compact subset of \((0,1) \times (0, \infty)\). It is assumed that given \( t \to \mu_t \), each actor’s latent position process \( X_t = (X_{t,i} : t \in [0,T]) \) is a diffusion process whose generator is \( \mathbb{A}_t \), and moreover we assume that \( X_1, \ldots, X_n \) are mutually independent. For each \( t \), let

\[
X_t \equiv (X_{t,1}, \ldots, X_{t,n})^T,
\]

where each \( X_{t,i} \) is assumed to be a column vector, i.e., a \( d \times 1 \) matrix. In other words, the \( i \)-th row of \( X_t \) is the transpose of \( X_{t,i} \).

**Messaging process level.** Denote by \( N_{t,i\to j} \) the number of messages sent from actor \( i \) to actor \( j \). Also, denote by \( N_{t,ij} \) the number of messages exchanged between actor \( i \) and actor \( j \). Note that \( N_{t,ij} = N_{t,i\to j} + N_{t,j\to i}(t) \). For each actor \( i \), we assume that the path \( t \to \lambda_{t,i} \) is deterministic, continuous and takes values in \((0, \infty)\). For each \( t \), we assume that

\[
P[N_{t+dt,i\to j} = N_{t,i\to j} + 1 | \mathcal{F}_t, X_s, s \leq t] = (\lambda_{t,i} \lambda_{t,j} / 2) p_{t,i\to j}(X_t) dt + o(dt).
\]

For our algorithm development and Experiment 1 in Section 6, we take

\[
p_{t,i\to j}(x) := p_{t,i\to j}(x_i) := P[X_{t,j} = x_i | \mathcal{F}_t],
\]

but for Experiment 2 in Section 6, we take \( p_{t,i\to j}(x) = \exp(-\|x_i - x_j\|^2) \). Next, by way of assumption, for each pair, say, actor \( i \) and actor \( j \), we eliminate the possibility that both actor \( i \) and actor \( j \) send messages concurrently to each other. More specifically, we assume that

\[
P[N_{ij}(t + dt) = N_{ij}(t) + 1 | \mathcal{F}_t, X_{s,i}, X_{s,j}, s \leq t]
\]

\[
= (\lambda_i(t) \lambda_j(t) / 2)(p_{t,i\to j}(X_{t,i}) + p_{t,j\to i}(X_{t,j})) dt + o(dt).
\]
For future reference, we let
\[ \lambda_{t,i\to j}(x) := (\lambda_{t,i}\lambda_{t,j}/2)p_{t,i\to j}(x), \]  
(4.4) \[ \lambda_{t,ij} = \lambda_{t,i}\lambda_{t,j}\langle p_{t,i},p_{t,j}\rangle. \] (4.5)

**Algorithm 3** Simulating messaging activities during a near-infinitesimally-small time interval

**Require:** \( t \in \mathbb{R}_+, \Delta t \in \mathbb{R}_+ \) and \( \{(T_{ij}(t), \lambda_{ij}(t)) \in \mathbb{R}_+^2 : 1 \leq i < j \leq n\} \)

1: procedure **MessagingActivities** 
2: \( \ell \leftarrow 1 \) 
3: for \( i \leftarrow 1, \ldots, n-1 \) do 
4:   for \( j \leftarrow (i+1), \ldots, n \) do 
5:     \( T_{ij}(t+\Delta t) \leftarrow T_{ij}(t) - \lambda_{t,ij} \Delta t \) 
6:     if \( T_{ij}(t+\Delta t) \leq 0 \) then 
7:       **MESSAGES**[\( \ell \)] \( \leftarrow (t, i, j) \) 
8:       \( T_{ij}(t+\Delta t) \leftarrow \text{UNITEXPONENTIALVARIABLE} \) 
9: \( \ell \leftarrow \ell + 1 \) 
10:   end if 
11: end for 
12: end for 
13: \( t \leftarrow t + \Delta t \) 
14: end procedure

Fig. 4.3: One simulation’s Kullback-Leibler divergence of posteriors at times \( \tau_1 < \tau_2 < \tau_3 \). The horizontal (\( x \)) and vertical (\( y \)) values, ranging in \( 1, 2, \ldots, 30 \), label actors. The more red the cell at \( (x,y) \) is, the more similar vertex \( y \) is to vertex \( x \). The dissimilarity measure (KL) clearly indicates the emergence of vertex clustering.

5. **Algorithm for computing posterior processes.** We denote by \( \rho_t \) the conditional distribution of \( X(t) \) given \( \mathcal{F}_t \), i.e., for each \( B \in \mathcal{B}(\mathbb{X}) \),
\[ \rho_t(B) = \mathbb{P}[X_t \in B | \mathcal{F}_t]. \] (5.1)
For the rest of this paper, we shall assume that the (random) measure \( \rho_t(\text{d}x) \) is absolutely continuous with respect to Lebesgue measure with its density denoted by
filtering algorithm. Theorem 5.2, we obtain below in Theorem 5.3 the basic formula for our approximate
via the Galerkin method, cf. Gunther et al. [1997]. Following this idea, starting from
family, as in Brigo [2011], the projection filter is equivalent to an approximate filtering
statistics, cf. Bain and Crisan [2009]. When the space on which we project is a mixture
in a systematic way, the method being based on the differential geometric approach to
which provides an approximation to the conditional distribution of the latent process
A
\textit{\textasteriskcentered}

\textbf{5.1. Theoretical background.} In Theorem 5.1, the \textit{exact} formula for updating
the posterior is presented, and in Theorem 5.2, our \textit{working} formula used in our numerical experiments is given. We develop our theory for the case where \( \omega_t \) and \( \sigma_t \) are the same for all actors for simplicity, as generalization to the case of each actor having different values for \( \omega_t \) and \( \sigma_t \) is straightforward but requires some additional notational complexity.

\textsc{Theorem 5.1.} For each \( f \in C_b(\mathbb{X}) \) and \( t \in (0, \infty) \),

\[
\frac{d\rho_t}{dt} = \rho_t(\mathcal{A}_t f) dt + 1^\top \left( \int_{\mathbb{X}} \rho_t(dx) f(x) \left( \tilde{\lambda}_t(x) - 11^\top \right) \right) * dM_t, 1,
\]

where \( \tilde{\lambda}_t(x) = (\tilde{\lambda}_{t,ij}(x))_{i,j=1}^n \) is an \( n \times n \) matrix such that for each \( i \neq j \), \( \tilde{\lambda}_{t,ij}(x) = p_{t,ij}(x)/\{p_{t,i}, p_{t,j}\} \) and for each \( i = j \), \( \tilde{\lambda}_{t,ij}(x) = 1 \), and \( dM_t = (dM_{t,ij})_{i,j=1}^n \) is an \( n \times n \) matrix such that for each pair \( i \neq j \), \( dM_{t,ij} = dN_{t,ij} - \lambda_{t,ij} dt \) and for each pair \( i = j \), \( dM_{t,ij} = 0 \).

Hereafter, for developing algorithms further for efficient computations, we make the assumption that for each \( t \),

\[
\rho_{t,ijk} = \rho_{t,i} \rho_{t,j} \rho_{t,k}, \tag{5.2}
\]

where \( \rho_{t,ijk} \) denotes the joint density for actors \( i, j \) and \( k \).

\textsc{Theorem 5.2.} For each function \( f \in C_b(\mathbb{R}^d) \), we have

\[
\frac{d\rho_{t,i}}{dt} = \rho_{t,i}(\mathcal{A}_t f) dt + \sum_{j \neq i} \left( \frac{\int_{\mathbb{R}^d} \rho_{t,i} p_{t,j}}{\{p_{t,i}, p_{t,j}\} - p_{t,i}(f) \right) \left( dN_{t,ij} - \lambda_{t,ij} dt \right).
\]

Replacing \( f \) with a Dirac delta generalized function, Theorem 5.2 states that for each \( x \in \mathbb{R}^d \),

\[
\frac{d\rho_{t,i}}{dt} = \mathcal{A}_t^* \rho_{t,i}(x) dt + \rho_{t,i}(x) \sum_{j \neq i} \left( \frac{p_{t,j}(x)}{\{p_{t,i}, p_{t,j}\}} - 1 \right) \left( dN_{t,ij} - \lambda_{t,ij} dt \right),
\]

where \( \mathcal{A}_t^* \) denotes the formal adjoint operator of \( \mathcal{A} \). For use only within Algorithm 4,

\[
PDE\text{Term}_{t,i} = \mathcal{A}_t^* \rho_{t,i}(x) dt, \text{ and} \tag{5.3}
\]

\[
\text{Jump}\text{Term}_{t,i} = \rho_{t,i}(x) \sum_{j \neq i} \left( \frac{p_{t,j}(x)}{\{p_{t,i}, p_{t,j}\}} - 1 \right) \left( dN_{t,ij} - \lambda_{t,ij} dt \right). \tag{5.4}
\]

\textbf{5.2. A mixture projection approach.} The projection filter is an algorithm which provides an approximation to the conditional distribution of the latent process in a systematic way, the method being based on the differential geometric approach to statistics, cf. Bain and Crisan [2009]. When the space on which we project is a mixture family, as in Brigo [2011], the projection filter is equivalent to an approximate filtering via the Galerkin method, cf. Gunther et al. [1997]. Following this idea, starting from Theorem 5.2, we obtain below in Theorem 5.3 the basic formula for our approximate filtering algorithm.
Algorithm 4 Updating the posterior distribution of actors' latent position over a near-infinitesimally-small time interval

Require: $t \in \mathbb{R}_+$, $\Delta t \in \mathbb{R}_+$, $\mu$ and $\{(t, u, v) : t, u, v \in (t, t + \Delta t], 1 \leq u < v \leq n\}$

1: procedure EstimateActorPosterior
2:   for $i \leftarrow 1, \ldots, n$ do
3:       Compute $p_t + \Delta t, i$ from $p_t, i$ using PDETerm$_{t,i}$
4:   end for
5: $\ell \leftarrow \arg\min_m t_m$
6: $t_{\ell-1} \leftarrow t$
7: while $t_{\ell} \in (t, t + \Delta t]$ do
8:   $dt \leftarrow t_{\ell} - t_{\ell-1}$
9:   for $i \leftarrow 1, \ldots, n - 1$ do
10:      for $j \leftarrow (i + 1), \ldots, n$ do
11:         if $\{i, j\} = \{u, v\}$ then
12:            $dN_{t,ij} = 1$
13:         else
14:            $dN_{t,ij} = 0$
15:         end if
16:         Update $p_t + \Delta t, i$ using $(p_{t,m})^n_{m=1}$ and PDETerm$_{t,i}$ in (5.3)
17:         Update $p_t + \Delta t, j$ using $(p_{t,m})^n_{m=1}$ and JUMP Term$_{t,j}$ in (5.4)
18:      $\ell \leftarrow \ell + 1$
19:   end for
20: end while
21: end procedure

To be more specific, consider a set of probability density functions $\mathcal{S} \equiv \{\phi_k\}$. Then, let $\mathcal{F} \leq M_1(\mathbb{R}^d)$ be the space of all probability density functions that can be written as a probability-weighted sum of $\phi_1, \phi_2, \cdots$. That is, $f \in \mathcal{F}$ if and only if $f(x) = \sum_k w_k \phi_k(x)$ for some probability vector $(w_1, w_2, \cdots)^T$ on indices $1, 2, \cdots$. In particular, for deriving our algorithms, we will assume that for some systematic choice of $\mathcal{S}$, each probability density under consideration is a member of $\mathcal{F}$.

Among many possible choices for $\{\phi_k\}$ in Theorem 5.3 are a multivariate Haar wavelet model and a multivariate Daubechies basis. On the other hand, a Gaussian mixture model is pervasively used throughout statistical inference tasks such as clustering and classification in algorithms such as $k$-means clustering. As such, we develop our algorithms with an eye towards use with other Gaussian mixture model-based algorithms. In Appendix B, we further develop our algorithm under the assumption that $p_{t,i}(x) = \sum_k W_{t,i,k} \phi(x; \theta_k, s)$, where $\phi$ is the standard Gaussian density function defined on $\mathbb{R}^d$, $s \in \mathbb{R}_+$, and the finite sequence $\{\theta_k\} \subset \mathbb{R}^d$ is to be chosen judiciously prior to implementing the algorithm.

Preparing for our next result in Theorem 5.3, we let $P$ be the symmetric matrix such that $P_{k,1} = \langle \phi_{k,1} \rangle$, and for each $k$, let $S_k$ be the symmetric matrix such that its $(r,c)$-entry $S_{k,rc}$ is $\langle \phi_{k,r} \phi_{c} \rangle$. Collectively, we denote by $S$ the three-way
tensor whose \((k, r, c)\) entry is \(S_{k, r, c}\). Let \(R_{t, i, \ell}\) be the matrix such that its \((r, c)\)-entry \(R_{t, i, \ell, r, c}\) is \(\langle A_{t, i, \ell} \phi_r, \phi_c \rangle\), where \(A_{t, i, \ell}\) is the differential operator such that

\[
A_{t, i, \ell} f(x) = \sum_{k=1}^d b_{t, i, \ell, k}(x) \frac{\partial}{\partial x_k} f(x) + \sum_{k_1, k_2=1}^d a_{t, i, \ell, k_1, k_2}(x) \frac{\partial^2}{\partial x_{k_1} \partial x_{k_2}} f(x),
\]

with

\[
b_{t, i, \ell, k}(x) = 2(1 - \omega_{t, i}) \int \psi \left( \sigma_{t, i}^{-1}(y - x) \right) (y - x)_k \phi(y; c_\ell, \alpha_{t, \ell}) dy,
\]

\[
a_{t, i, \ell, k_1, k_2}(x) = (1 - \omega_{t, i})^2 \int \psi \left( \sigma_{t, i}^{-1}(y - x) \right) (y - x)_{k_1} (y - x)_{k_2} \phi(y; c_\ell, \alpha_{t, \ell}) dy.
\]

**Theorem 5.3.** Suppose that for each \(t\), \(i\) and \(x\), \(p_{t, i}(x) = \sum_{k=1}^K W_{t, i, k} \phi_k(x)\). Let \(W_{t, i}\) denote the column vector whose \(k\)-th entry is \(W_{t, i, k}\). Then,

\[
P_{d} W_{t, i} = \sum_{\ell} q_{t, \ell} R_{t, i, \ell} W_{t, i} dt
+ \sum_{j \neq i} \left( \frac{(W_{t, i}^\top S_{t, i, j}) W_{t, i, j}^\top}{W_{t, i}^\top P W_{t, i}} - P W_{t, i} \right) \left( d N_{t, i j} - \lambda_{t, i j} \lambda_{t, j} W_{t, i}^\top P W_{t, j} dt \right).
\]

### 5.3. Algorithm for continuous embeddings.

#### 5.3.1. Classical multidimensional scaling.

In our application, our final analysis is completed by clustering the posterior distributions. Instead of working directly with posteriors, an infinite-dimensional object, we propose to work with objects in an Euclidean space each of which represents a particular actor. However, given \(p_{t, i}\) and \(p_{t, j}\), using their mean vectors or their KL distance for clustering can be uninformative. For example, if \(p_{t, i} = p_{t, j}\), then their mean vectors would be the same and their KL distance would be zero. However, if \(p_{t, i} = p_{t, j}\) is the density of, say, a normal random vector such that its mean is zero and its covariance matrix is \(v I\) for a large value of \(v\), then concluding that actor \(i\) and actor \(j\) are similar could be misleading.

To alleviate such situations in a clustering step of our numerical experiments, we propose using a multivariate statistical technique called classical multidimensional scaling (CMDS) to obtain a lower dimensional representation of \(p_t = (p_{t, i})^{n}_{i=1}\). More specifically, we achieve this by representing each actor as a point in \(\mathbb{R}^d\), where the configuration is obtained by solving the optimization problem

\[
\min_{x_1, \ldots, x_n \in \mathbb{R}^d} \sum_{i < j} \|x_i - x_j\|^2 - g(\langle p_{t, i}, p_{t, j} \rangle),
\]

where \(g\) denotes a strictly decreasing function defined on \([0, \infty)\) taking values in \(\mathbb{R}_+\). For example, one can take \(g(u) = \cos^{-1}(\omega u)\) where \(\omega \in \mathbb{R}_+\) is chosen so that \(\omega u \in [0, \pi/2]\) for all possible values \(u\) of \(\langle p_{t, i}, p_{t, j} \rangle\). Another possibility among many others is to choose \(g(u) = -\log(\omega u)\) if \(\omega\) is chosen so that \(\omega u \in (0, 1]\) for all possible values \(u\) of \(\langle p_{t, i}, p_{t, j} \rangle\).

We denote by \(\xi(p)\) the set of solutions to the optimization problem (5.9). Given a vector \(p\) of \(n\) probability densities on \(\mathbb{R}^d\), it can be shown that the solution set \(\xi(p)\) is not empty and is closed under orthogonal transformations.
In the classical embedding literature, ensuring continuous embeddings is neglected as it is not relevant to their applications. However, for our work, this is crucial as we study their evolution through time, i.e., ideally, we would like to see that a small change in time corresponds to a small change in latent location. In this section, we propose an extension to the CMDS algorithm to remedy the aforementioned non-uniqueness issue, and show that the resulting algorithm ensures continuity of embeddings.

In our numerical experiments, for each $p$, we choose a particular element $\xi^\star(p)$ of the solution set $\xi(p)$ so that $\xi^\star(p)$ depends on $p$ in a consistent manner.

**Algorithm 5** Compute a unique CMDS embedding of $M$ by minimizing the distance from a reference configuration $Z$ with full column rank

**Require:** a matrix $Z \in \mathbb{M}_{n \times d}$ with full column rank and a symmetric matrix $M \in \mathbb{M}_{n \times n}^+$ such that $\text{diag}(M) = 0$

1: $B \leftarrow$ any $n \times d$ classical MDS solution of $M$
2: Compute a singular value decomposition $UDV^\top$ of $Z^\top B$
3: Return $BVU^\top$

5.3.2. Continuous selection. By a dissimilarity matrix, we shall mean a real symmetric non-negative matrix whose diagonal entries are all zeros. First, fix $d$ such that $1 \leq d \leq n$. Then, for each $n \times n$ dissimilarity matrix $M$, define

$$\varrho(M) = -\frac{1}{2}(I - 11^\top/n)M^{(2)}(I - 11^\top/n),$$

$$\xi^\dagger_d(M) = \arg\min_{X \in \mathbb{R}^{n \times d}} \|\varrho(M) - XX^\top\|_F^2,$$

where $M^{(2)} = (M_{22}^2)$. The elements of $\xi^\dagger_d(M)$ are known as classical multidimensional scalings, and as discussed in Borg and Groenen [2005], it is well known that $\xi^\dagger_d(M)$ is not empty provided that the rank of $\varrho(M)$ is at least $d$. Our discussion in this section concerns making a selection $\xi^\star_d(M)$ from $\xi^\dagger_d(M)$ so that the map $M \rightarrow \xi^\star_d(M)$ is continuous over the set of dissimilarity matrices such that $\varrho(M)$ is of rank at least $d$.

Let $M$ be a dissimilarity matrix such that the rank of $\varrho(M)$ is at least $d$. We begin by choosing an element of $\xi^\dagger_d(M)$, say $\xi_d(M)$, through classical dimensional scaling. Let $UU^\top$ be the eigenvalue decomposition of $\xi_d(M)$, where $UU^\top = I$ and $\Sigma$ is the diagonal matrix whose entries are the eigenvalues in non-increasing order. By the rank condition, we have $\Sigma_{11} \geq \ldots \geq \Sigma_{dd} > 0$. First, we formally write

$$X_+ = U_+ \sqrt{\Sigma_+}, \quad (5.10)$$

where 
(i) $U_+$ is the $n \times d$ matrix with its $ij$ entry $U_{ij}$,
(ii) $\Sigma_+$ is the $d \times d$ diagonal matrix whose $i$-th diagonal entry is $\Sigma_{ii}$.

Dependence of $X_+$ on $M$ will be suppressed in our notation unless needed for clarity. Now, if the diagonals of $\Sigma_+$ are distinct, then $X_+$ is well defined. However, in general, due to potential geometric multiplicity of an eigenvalue, our definition of $X_+$ can be ambiguous. This is the main challenge in making a continuous selection and we resolve this issue in our following discussion.
For our remaining discussion, without loss of generality, we may assume that for each dissimilarity matrix $M$, $X_+$ is well-defined by making an arbitrary choice if there is more than one CMDS solution. Note that the mapping $M \mapsto X_+(M)$ may not be a continuous selection. We now remedy this. First, fix an $n \times d$ matrix $Z$ and let

$$
\xi_d(M) = \{X_+Q : QQ^T = I \} \subset \xi_d^\dagger(M),
$$

where $Q$ runs over all $d \times d$ real orthogonal matrices. Then, define

$$
\xi^*_d(M) \equiv \arg \min_{X \in \xi_d(M)} \|X - Z\|^2_F. \tag{5.11}
$$

Algorithm 5 yields the solution $\xi^*_d(M)$ and the proof of the following theorem, Theorem 5.4, can be found in Appendix E.

**Theorem 5.4.** Suppose that the $n \times d$ matrix $Z$ is of full column rank. Then, the mapping $M \mapsto \xi^*_d(M)$ yields a well-defined continuous function on the space of dissimilarity matrices such that $\varrho(M)$ is of rank at least $d$.

Fig. 5.1: A graphical representation of the dependence structure in the simulation experiment. The nodes that originate the dashed lines correspond to either one of constant model parameters ($\sigma_0, \omega_0$, and $\lambda_0$) or exogenous modeling element ($c_t$). The arrows are associated with influence relationships, e.g., $\mu_t \rightarrow \mu_{t+\Delta t}$ reads $\mu_t$ influences $\mu_{t+\Delta t}$.

5.4. **Technical observations.** Here we discuss some insightful facts related to our model given the assumption stated in the last section. First, we have the following:

**Theorem 5.5.** Fix $t \geq 0$ and suppose that $\mu_t(x) > 0$ for a.e. $x \in \mathbb{R}^d$. The operator $\mathcal{A}_t$ is elliptic, i.e., for each $x \in \mathbb{R}^d$, the matrix $a_t(x) = (a_t^{k_1,k_2}(x))$ is positive definite.
Proof. Note that for each $z \in \mathbb{R}^d$, 
\[
  z^T a_t(x) z = (1 - \omega_t)^2 \int \psi(\sigma_t^{-1}(y - x)) (z^T (y - x)(y - x)^T z) \mu_t(y) dy \\
  = (1 - \omega_t)^2 \int \psi(\sigma_t^{-1}(y - x)) (|y - x|^2 z)^2 \mu_t(y) dy.
\]
Note that $\psi(\sigma_t^{-1}(y - x)) > 0$ and $\mu_t(y) > 0$ for each $y \in \mathbb{R}^d$, and that $|y - x|^2 > 0$ away from a subspace of $\mathbb{R}^d$ whose Lebesgue measure is zero. It follows that $z^T a_t(x) z > 0$ for each $x \in \mathbb{R}^d$, whence $a_t(x)$ is positive definite. \qed

Now, we further examine Algorithm 1, Algorithm 2, Algorithm 3, Algorithm 4, Algorithm 5, and discuss some technical points behinds these algorithms.

In Algorithm 2, existence and uniqueness of $\sqrt{a_t(x)}$ follows from Theorem 5.5. The continuity of $x \to \sqrt{a_t(x)}$ follows from Theorem 5.5 and Strrook [2008]. In Algorithm 3, for simulating a sample path of $t \to N_{t,ij}$, we use the so-called time-change property; that is, we use the fact that the process given by $t \to N_{t,ij}^*$ is a unit-rate simple Poisson process, where $\Lambda_{t,ij} = \int_0^t \lambda_{t,ij} du$ and $\Lambda_{t,ij}^{-1} := \inf\{u \geq 0 : \Lambda_{u,ij} \geq t\}$ and $N_{t,ij}^* := N_{u,ij}|_{u=\Lambda_{t,ij}^{-1}}$. For simulation, we use its dual result, i.e., $t \to N_{t,ij}$ is a point process whose intensity process is $\lambda_{t,ij}$, where $u \to N_{u,ij}^*$ is a path of a unit-rate simple Poisson process. Also, we note that for computation of $\lambda_{t,ij}$, online inference is a necessary part of our simulation in Algorithm 3; that is, we need to compute $p_{t,i \to j}(x) = \mathbb{P}[X_{i,j} = x | \mathcal{F}_t]$. In Algorithm 3 and Algorithm 4, near-infinitesimally-small means $\Delta t$ so small that the likelihood of having more than one event during a time interval of length $\Delta t$ is practically negligible. Also, by \textsc{StandardNormalVector} in Algorithm 2 and \textsc{UnitExponentialVariable} in Algorithm 3, we mean generating, respectively, a single normal random vector with its mean vector being zero and its covariance matrix being the identity matrix, and a single exponential random variable whose mean is one.

6. Simulation experiments. In our experiments, we hope to detect clusters with accuracy and speed similar to that possible if the latent positions $X(t)$ were actually observed even though we use only information in $p_t = (p_{t,i})_{i=1}^m$ estimated from information contained in $\mathcal{F}_t$. We denote the end-time for our simulation as $T$. There are two simulation experiments presented in this section, and the computing environment used in each experiment is reported at the end of this section.

Experiment 1. We take $d = 1$ and we assume that for each $t \in [0,T]$ and actor $i = 1, \ldots, 8$, $\lambda_{t,i} = 5, \sigma_{t,i}^2 = 1/3$ and $\omega_{t,i} = 0.1$. For the population process we take for each $t \in [0,T]$
\[
  \mu_{t,i}(x) = \phi(x; \alpha_t, \omega_t),
\]
where
\[
  \alpha_t = 1/3, \quad \text{and} \quad c_{t,i} \equiv \begin{cases} 
  1 & \text{if } t \in [0,100\Delta t), \\
  0.5 & \text{if } t \in [100\Delta t, 250\Delta t), \\
  0 & \text{if } t \in [250\Delta t, 500\Delta t].
\end{cases}
\]
Then, we also consider $\mu_{t,II}$, where
\[
\alpha_t = 1/3, \quad \text{and}
\]
\[
c_{t,II} \equiv \begin{cases} 
1 & \text{if } t \in [0, 100\Delta t), \\
0 & \text{if } t \in [100\Delta t, 250\Delta t), \\
1 & \text{if } t \in [250\Delta t, 500\Delta t].
\end{cases}
\]

There is only one population density; in other words, $q_{t,i} = 1$. Note that even with one population center, we can have more than one empirical mode for the subpopulation. One of these modes is near zero, and another mode is near one. The reason for this is that because of the value of $\alpha_t^2 = 1/3$ and $\sigma_t^2 = 1/3$, when an actor is too far away from the mode $c_i$ of the population process, the population process affects the actors on its tail only by negligibly small amount. In Figure 6.4 and Figure 6.3, a sample path of the true latent position of each of eight actors is illustrated in black lines. It is apparent that in the $c_{t,II}$ case, all eight actors are equally informed of the population mode shift, but in the $c_{t,II}$ case, only the last three were able to adapt to the change, and the first five actors are surprised by the abrupt change at time 100$\Delta t$.

Our simulation is discretized. Our unit time is $\Delta t = 0.05$, and in Figure 6.4, each tick in the horizontal axis corresponds to an integral multiple of $\Delta t$. The jump term in our update formula is quite sensitive to the number of actors being considered. As such, for updating the jump term, we further discretized $\Delta t$ into $n^2$ subintervals for numerical stability of our update iterations. For $n = 8$, each unit interval is associated with 64 sub-iterations, and the total number of the (main) iteration is 400, and we use $(N_{t+\Delta t,ij} - N_{t,ij})/n^2$ instead of $dN_{t+\Delta t}/n^2,ij - dN_{t+\Delta t-1}/n^2,ij$ in each $\ell$-th subiteration of each main iteration staring at time $t$.

To implement our mixture projection algorithm, we take $s^2 = 1/2^{12} = 1/4096$. The initial position of the $n = 8$ actors is sampled from the initial population distribution $\mu_0$. We take $p_{0,i}(x) = \phi(x; X_{0,i}, s)$. The discretized version $R_t$ of $A_t$ is illustrated in Figure 6.1. For inference during our experiment, we have dropped the second order term and used only the first order term to keep the cost of running our experiment low. On the other hand, for simulating the actors’ latent positions, we have used both the first and second order term of $A_t$. The value of $P^{-1}R_tW_{t,i}dt$ gives the first part of the change in $dW_{t,i}$. Note that in both Figure 6.1a and Figure 6.1b, the entries that are sufficiently far off from the diagonals are near zero.

For $c_t = c_{t,II}$, the time plot of the number of messages produced during interval $[\Delta t\ell, \Delta t(\ell+1)]$ is given in Figure 6.2, and shows transient behaviors of varying degrees of messaging intensity over the interval. Our set up for $c_t = c_{t,II}$ produced a simulation sample output of observing 2.5 messages amongst the $n = 8$ actors in unit time once the population center changed abruptly from $c_t = 1$ to $c_t = 0$ at the start of the 100-th unit time interval, i.e., $t = 5$. In other words, after $t \geq 5$, a single unit time is roughly associated with the amount of time during which the whole subpopulation of eight actors exchanges around 45 messages, or equivalently, during which each pair of actors exchange around 3 messages. On the other hand, in both $c_{t,II}$ and $c_{t,II}$ for the interval $[0, \Delta t] = [0, 5]$, the subpopulation messaging rate is relatively constant at the rate of 100 messages over each unit interval, and this is expected as all eight actors are tightly situated around 1.

Our experiments for $c_{t,II}$ and $c_{t,II}$ both show that the filtered positions for all eight actors are close to the exact positions.
On latent position inference from doubly stochastic messaging activities

(a) The population mean is at 1  
(b) The population mean is at 0

Fig. 6.1: The level plots of $P^{-1}R_t$ for the discretized version $R_t$ of $A_t$, used in the simulation experiment for two particular cases, where the horizontal axis is associated with the rows of $P^{-1}R_t$ and the vertical axis is associated with the columns of $P^{-1}R_t$.

Experiment 2. In this experiment, for each $t$, we have used the empirical distribution of

$$X_{t,n+1}, \ldots, X_{t,n+L}$$

to obtain an estimate $\hat{\mu}_t$ of $\mu_t$ by partitioning the latent space into sufficiently small intervals, where we place a uniform kernel of height equal to the proportion of $\{X_{n+i} : i = 1, \ldots, L\}$ that lies in that interval. Our inference is on $X_{t,1}, \ldots, X_{t,n}$. Recall that $n$ denotes the size of the subpopulation. The number $n + L$ is the size of the full population. This set-up is closer to the motivation for our work, the bounded confidence model, Gomez-Serrano et al. [2012], and the connection with our model in this paper is made in Appendix A. In theory, the general setup in Experiment 1 is comparable to the setup in Experiment 2 when $L$ in Experiment 2 is taken to be $\infty$.

We set $L = 70$, $n = 30$, $\Delta = .25$, and $\omega = .2$. We take the clustering based on $X_T$ as the ground truth. Note that $\Delta$ here is comparable to $\sigma_{t,i}$ in Experiment 1, or more generally, in our model. We set up the simulation to observe roughly 3000 messages amongst the $n$ actors in unit time. This translates to 10 per actor per unit time. Note that this is a rough estimate as the messaging intensity is time-dependent and stochastic. In Figure 6.7, we have snapshots of $X_t = (X_{t,1}, \ldots, X_{t,n})$ and those of $\hat{X}_t = (\hat{X}_{t,1}, \ldots, \hat{X}_{t,n})$ for a single simulation run. Denote as the latency

$$\Delta \xi \equiv \hat{\xi} - \xi,$$

where the dependency on our choice for a clustering algorithm is suppressed in our notation and for some $\varepsilon \in (0, 1)$,

$$\xi \equiv \inf\{t \in [0, T] : \text{MARI}(\kappa(X_s), \kappa(X_T)) \geq 1 - \varepsilon, \text{ for a.e. } s \in [t, T]\},$$

$$\hat{\xi} \equiv \inf\{t \in [0, T] : \text{MARI}(\psi^*(p_s), \kappa(X_T)) \geq 1 - \varepsilon, \text{ for a.e. } s \in [t, T]\},$$

where MARI denotes a moving average of the Adjusted Rand Index (c.f. Rand [1971] and Hubert and Arabie [1985]) and we fix $\kappa$ to be a $k$-means clustering algorithm for
concreteness. We use the latency as a performance measure for a clustering algorithm \( \kappa \) under our framework. For our projection, we use a Haar basis, i.e., a set of simple step functions, where the width of the intervals used in the experiment is \( \frac{1}{42} \). Also, unlike in Experiment 1, we take

\[
p_{t,i \rightarrow j}(X_{t,i}) = \exp(-\|X_{t,i} - X_{t,j}\|^2).
\]

These changes require us to modify our algorithm slightly. However, the necessary modifications are straightforward, and we leave the details to the reader.

It is important to note that we do not assume knowledge of the latent position of any individual, \( X_i(t) \); instead, we use only our knowledge of the overall population. As the number \( L \) gets larger, as shown in Gomez-Serrano et al. [2012], the dependence among

\[
X_{t,1}, \ldots, X_{t,n+L}
\]

diminishes, agreeing more closely with the model we specified in our framework. We investigate the behavior of our algorithm for small, medium and large values of \( L \), showing robustness of our framework in the face of limited information. Recall that Figure 6.7 shows results for \( L = 70 \). Figure 6.5 compares the latency for \( L = 30 \) and \( L = 70 \). The clarity and accuracy of the clustering suffers with significant reductions in information used to estimate the priors \( \mu_t \).

In Figure 6.7, we present snapshots of \( X_t \) and \( \hat{X}_t = \psi^*(p_t) \) for a single simulation run. Note that \( \hat{X}_t \) is a CMDS embedding of a dissimilarity matrix based on

---

**Fig. 6.2:** The \( c_t = c_{t,1} \) case. The number of messages per \( \Delta t \) across the time interval \([100\Delta t, 400\Delta t]\) for the subpopulation of actors 1, \ldots, 8.
Fig. 6.3: The $c_t = c_{t,1}$ case. The sample path of the true and estimated latent position of each of eight actors used for Experiment 1 in a black solid line and in a dashed red line, respectively.

the posteriors $p_t$. The colors denote the final cluster membership as determined from $k$-means clustering with $X_T$. It is clear that the emerging cluster structure of the $\hat{X}_t$ lags slightly behind that of $X_t$ in both accuracy and clarity; comparing the middle two figures, we can see that there are a few data points misclassified at time $\tau_2$. Indeed, Figure 6.6 shows that the clustering based on the embedded positions mirrors that possible with the true but unobserved latent positions with a small latency.

Computing Environment. For Experiment 1, we used R 2.14.1 (64 bit) under Ubuntu 12.0.4 on an Intel Core i7 CPU 870 @ 2.93 GHz × 8 machine with 16 GB RAM. For a single run for 8, 16 and 32 actors, our experiment took 190, 788 and 5384 seconds respectively. For Experiment 2, we used a Red Hat Linux cluster with 24 nodes with 24 × 2.5 MHz CPUs and 132 GB memory each. Each Monte Carlo replicate took a single slot. A single replicate took approximately 3000 seconds.

7. Conclusion and Future Work. We have described a strategy for clustering actors based on messaging activities. Our analysis is completed by clustering a CMDS embedding of posteriors. We have presented ways to simplify posterior analysis on two levels. The first level allows us to obtain an estimate of the posteriors in an online manner. The second level allows us to reduce our analysis to studying diffusion processes, which is often a starting point for addressing the optimal stopping problem.

We have illustrated in our numerical experiments that the assumptions used to derive our two simplified approaches are mild enough to be useful for our inference
task at hand, i.e., clustering.

We believe that our framework has potential for tackling the problems faced by the social network practitioner regarding emergence of structure. We intend to develop a measure of confidence for our inferred latent positions. This will be crucial to many applications, as it will provide the decision-maker with information about whether to act or to wait for more data to increase the confidence in the inferred positions. A measure of confidence would therefore be a way to establish a stopping rule. Noting that we took the parameters of our model to be exogenous, we will need to explore robustness of our inference to incorrect parameter choices and then make explicit an algorithm for parameter estimation. Making our algorithm more scalable is also an area of our interest. These areas of future work will be key to applying our framework on substantial problems.

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Appendix A. Motivation for the form of the differential operator $A_t$.

A.1. Bounded confidence model: an adaptation. Our work in this paper is in part influenced by a so-called bounded confidence model in Gomez-Serrano et al. [2012] which focuses on establishing a propagation of chaos property of the in-
Fig. 6.5: Latency ($\Delta \zeta$) distribution for 200 Monte Carlo experiments. The translucent grey histogram is based on $L = 30$, and the cross-hatch shaded histogram is based on $L = 70$. The latency is defined as the difference between the time $\hat{\zeta}$ at which the moving average of the predictive ARI maintained a level of $1 - \varepsilon$ for all $t \geq \hat{\zeta}$ and the time $\zeta$ at which true locations’ moving average of the ARI maintained a level of $1 - \varepsilon$ for all $t \geq \zeta$. The latency can be negative, but is generally small and positive.

Interacting particles model studied there. When denoting the actors’ latent positions $X_1(t), X_2(t), \ldots \in [0, 1]$, in the bounded confidence model, the opportunities for (latent) position changes that each actor experiences is modeled as a simple Poisson process. When there is a change at time $t$, the change is assumed to involve precisely two actors, say, actor $i$ and actor $j$, such that their position $X_i(t-)$ and $X_j(t-)$ differs by at most $\Delta$. This yields an inhomogeneity in the rate at which actors change their locations. Then, the exact amount of change is specified by the following formula:

$$X_i(t) = \omega X_i(t-) + (1 - \omega) X_j(t-),$$

$$X_j(t) = \omega X_j(t-) + (1 - \omega) X_i(t-),$$

where $w \in (0, 1)$ is a fixed constant. Roughly speaking, upon interaction, actor $i$ keeps $w \times 100$ percent of its original position, and is allowed to be influenced by $(1 - \omega) \times 100$ percent of the original position of actor $j$, and vice versa.

Fix constants $\Delta \in (0, 1)$ and $w \in (0, 1)$. Then, define $\mathcal{A}$ by letting for each $\mu$ and $f$,

$$\mathcal{A}(\mu)f(x) = 2 \int_{|x-y| \leq \Delta} f(\omega x + (1 - \omega)y) - f(x))\mu(dy).$$
Fig. 6.6: Moving average of the $k$-means clusterings of the embedded $\tilde{X}_t$ and $X(t)$ against the $k$-means clustering of $X_T$. Note that $\tau_2$ (cf. Figure 6.7) and $\eta^*$ are nearly identical.

Studied in Gomez-Serrano et al. [2012] particularly is the interaction between $\mu_t$ and $X = (X_1, \ldots, X_n)$ where $\mu_t$ is the empirical distribution of $X$. As shown in Gomez-Serrano et al. [2012], the bounded confidence model has an appealing feature that the parameter space for the underlying parameters $w$ and $\Delta$ can be partitioned according to the type of consensus that the population eventually reaches, namely, a total consensus and a partial consensus. In a total consensus regime, for sufficiently large $t$, everyone is expected to gather tightly around some fixed common point $x_0 \in [0,1]$. On the other hand, in a partial consensus regime, (depending on $w$ and $\Delta$), there is a finite collection of distinct values in $[0,1]$ separated by at least $\Delta$, to exactly one of which each actor’s position is attracted. In particular, the (asymptotic) position of actors yields a partition of the actor set when the exact locations of $X_1, X_2, \ldots$ are known. Generally, $(\mu_t : t \in [0,T])$ is contracting toward for some closed convex non-empty disjoint subsets $B_1$ and $B_2$ of $[0,1]$ in the sense that for some $t \in [0,T]$, $\mu_s(B_1 \cup B_2) \geq \mu_t(B_1 \cup B_2)$ for each $s \geq t$ and $\mu_T([0,1]) = 1$.

In our adaptation, for analytic tractability, we replace the indicator function $1_{|x-y| \leq \Delta}$ with $\psi(z) = \exp(-\frac{1}{2}z^T z)$, take $\mu_t$ to be an exogenous modeling element, and take $w_t$ to be potentially time dependent, yielding the operator

$$A(\mu)f(x) = 2 \int \psi(y - x) \left(f(\omega x + (1 - \omega)y) - f(x)\right) \mu(y)dy.$$ (A.1)

The second numerical experiment in Section 6 focuses on the case where the community starts with no apparent clustering but as time passes, each actor becomes
Fig. 6.7: $X_{t,i}$ versus $\hat{X}_{t,i}$ at times $\tau_1 < \tau_2 < \tau_3$. The size of the population used to estimate the prior was $L = 70$. The first row shows the CMDS embedded positions ($k = 2$), and the second row shows the latent positions. Due to our CMDS embedding procedure (with rotation), the 1-dimensional embedding is the first coordinate of a 2-dimensional embedding. We show a 2-dimensional embedding for illustration purposes.

A.2. A quadratic Taylor series approximation. In this work, we use a model that that captures the action in (A.1) up to the second order. To begin, note that

$$f(z) = f(x) + Df(x) \cdot (z - x) + \frac{1}{2}(z - x)\T D^2f(x)(z - x) + \text{H.O.T.},$$

where $Df(x) \in \mathbb{R}^d$ and $Df(x) \in \mathbb{M}_{d \times d}$ denote respectively the gradient and the Hessian of $f$ at $x$, and H.O.T. denotes the higher order terms. Suppose that $\mu_t$ is
given. Now, we have
\[ A_t f(x) := A(\mu_t) f(x) \]
\[ = 2 \int \psi(y - x) Df(x) \cdot (1 - \omega)(y - x) \mu_t(y) dy \]
\[ + 2 \int \psi(y - x) \left( \frac{1}{2}(1 - \omega)^2(y - x)^T D^2 f(x)(y - x) \right) \mu_t(y) dy + \text{H.O.T.} \]
\[ = \left( \sum_{k} b_t^k(x) \partial_k f(x) + \sum_{k_1} \sum_{k_2} a_t^{k_1,k_2}(x) \partial_{k_1,k_2} f(x) \right) + \text{H.O.T.}, \]

where \( b_t(x) \in \mathbb{R}^d \) and \( a_t(x) \in \mathbb{R}^{d \times d} \) are given by the following:
\[ b_t^k(x) = 2(1 - \omega_t) \int \psi(y - x)(y - x)_k \mu_t(y) dy, \]
\[ a_t^{k_1,k_2}(x) = (1 - \omega_t)^2 \int \psi(y - x)(y - x)_{k_1}(y - x)_{k_2} \mu_t(y) dy. \]

Dropping the term associated with H.O.T., we obtain the following:
\[ A_t f(x) = \left( \sum_{k} b_t^k(x) \partial_k f(x) + \sum_{k_1} \sum_{k_2} a_t^{k_1,k_2}(x) \partial_{k_1,k_2} f(x) \right). \]

Appendix B. The mixture projection filter formula.

B.1. Proof of Theorem 5.3. For each \( \phi_r \), we see that
\[ \langle \phi_r, dp_{t,i} \rangle = \sum_c \langle \phi_r, \phi_c \rangle dW_{t,i,c} = e_r^T PdW_{t,i}. \]

We first consider the second term of the right side of (5.8).
\[
\begin{align*}
\sum_{j \neq i} \left( \frac{\phi_r(x)(p_{t,i}(x)\lambda_{t,i-j}(x) + p_{t,j}(x)\lambda_{t,j-i}(x))}{\lambda_{t,i,j}} - p_{t,i}(x) \right) dM_{t,i,j} \\
= \sum_{j \neq i} \left( \frac{\phi_r(x)(p_{t,i}(x)\lambda_{t,i-j}(x) + p_{t,j}(x)\lambda_{t,j-i}(x))}{\lambda_{t,i,j}} - p_{t,i}(x) \right) (dN_{t,i,j} - \lambda_{t,i,j} dt). 
\end{align*}
\]

Now, we have that
\[
\begin{align*}
\int \phi_r(x)(p_{t,i}(x)\lambda_{t,i-j}(x) + p_{t,j}(x)\lambda_{t,j-i}(x)) dx \\
= (\lambda_{t,i}\lambda_{t,j}/2) \int \phi_r(x)(p_{t,i}(x)p_{t,j}(x) + p_{t,j}(x)p_{t,i}(x)) dx \\
= (\lambda_{t,i}\lambda_{t,j}/2) \langle \phi_r, p_{t,i}p_{t,j} \rangle \\
= \lambda_{t,i}\lambda_{t,j} \langle \phi_r, p_{t,i}p_{t,j} \rangle 
\end{align*}
\]
and that
\[
\lambda_{t,i,j} = \lambda_{t,i}\lambda_{t,j}\langle p_{t,i}, p_{t,j} \rangle = \lambda_{t,i}\lambda_{t,j} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} W_{t,i}^{k_1} W_{t,j}^{k_2} (\phi_{k_1}, \phi_{k_2}) = \lambda_{t,i}\lambda_{t,j} W_{t,i}^T PW_{t,j}. 
\]
Hence,
\[
\langle \phi_r, dH_{t,i} \rangle = \sum_{j \neq i} \left( \langle \phi_r, p_{t,j} \rangle \frac{W_{t,i}^\top W_{t,j}}{W_{t,i}^\top P W_{t,j}} - \langle \phi_r, p_{t,i} \rangle \right) (dN_{t,ij} - \lambda_{t,i} \lambda_{t,j} W_{t,i}^\top P W_{t,j} dt) \\
= \sum_{j \neq i} \left( W_{t,i}^\top S_{t,j} W_{t,j} - e_r^\top P W_{t,i} \right) (dN_{t,ij} - \lambda_{t,i} \lambda_{t,j} W_{t,i}^\top P W_{t,j} dt).
\]

Next, for the first term of the right side of (5.8), we have
\[
\langle \phi_r, A_{t,i}^* p_{t,i} \rangle = \sum_{\ell} q_{t,\ell} (A_{t,i,\ell} \phi_r, p_{t,i}) \\
= \sum_{\ell} q_{t,\ell} \sum_c (A_{t,i,\ell} \phi_r, \phi_c) W_{t,i,c} \\
= \sum_{\ell} q_{t,\ell} \sum_c (A_{t,i,\ell} \phi_r, \phi_c) W_{t,i,c} \\
= \sum_{\ell} q_{t,\ell} e_r^\top R_{t,i,\ell} W_{t,i}.
\]

In summary, for each \( r \), we have
\[
e_r^\top P dW_{t,i} = e_r^\top R_{t} W_{t,i} dt + \sum_{j \neq i} \left( W_{t,i}^\top S_{t,j} W_{t,j} - e_r^\top P W_{t,i} \right) (dN_{t,ij} - \lambda_{t,i} \lambda_{t,j} W_{t,i}^\top P W_{t,j} dt),
\]
and our claim follows from this.

**B.2. Preliminary lemmas.** This section contains two formulas to be used in the next section. Our result and proof in Lemma B.2 is stated in the same notation as in Lemma B.1. Recall that \( \phi(z; \vartheta, \gamma) \propto \phi(\gamma^{-1}(z - \vartheta)) \).

**Lemma B.1.** Let \( \{ \vartheta_\ell \} \subset \mathbb{R}^d \) and \( \{ \gamma_\ell \} \subset \mathbb{R}_+ \). Then,
\[
\sum_\ell \| \gamma_\ell^{-1} (x - \vartheta_\ell) \|^2 = \left( \sum_\ell \gamma_\ell^{-2} \right) \| x - \sum_\ell \left( \frac{\gamma_\ell^{-2}}{\sum_m \gamma_m^{-2}} \right) \vartheta_\ell \|_2^2 - \frac{1^\top (\Gamma * (\Theta - \text{diag}(\Theta) 1^\top)) 1}{\sum_n \gamma_n^{-2}},
\]
where \( \Theta \) is the Gram matrix for \( \{ \vartheta_\ell \} \) and \( \Gamma \) is the matrix whose \((r,c)\)-entry is \( \gamma_r^{-2} \gamma_c^{-2} \).

**Proof.** Let \( C = \sum_\ell \gamma_\ell^{-2} \) and for each \( \ell \), let \( \rho_\ell = \gamma_\ell^{-2} / C \). First, note that
\[
\sum_\ell \| \gamma_\ell^{-1} (x - \vartheta_\ell) \|^2 = \sum_\ell \gamma_\ell^{-2} \left( x^\top x - 2x^\top \vartheta_\ell + \vartheta_\ell^\top \vartheta_\ell \right) \\
= \left( \sum_\ell \gamma_\ell^{-2} \right) \| x \|^2 - 2x^\top \left( \sum_\ell \gamma_\ell^{-2} \vartheta_\ell \right) + \sum_\ell \vartheta_\ell^\top \gamma_\ell^{-2} \vartheta_\ell \\
= \left( \sum_\ell \gamma_\ell^{-2} \right) \left( \| x \|^2 - 2x^\top \left( \sum_\ell \rho_\ell \vartheta_\ell \right) + \| \sum_\ell \rho_\ell \vartheta_\ell \|^2 - \| \sum_\ell \rho_\ell \vartheta_\ell \|^2 \right) + \sum_\ell \gamma_\ell^{-2} \vartheta_\ell^\top \vartheta_\ell \\
= C \left( x - \sum_\ell \rho_\ell \vartheta_\ell \right)^2 - C \| \sum_\ell \rho_\ell \vartheta_\ell \|^2 + \sum_\ell \gamma_\ell^{-2} \vartheta_\ell^\top \vartheta_\ell.
\]
Now,

\[ C\| \sum_{\ell} \rho_{\ell} \vartheta_{\ell} \|^2 - \sum_{\ell} \gamma_{\ell}^{-2} \vartheta_{\ell}^{\top} \vartheta_{\ell} \]

\[ = C \sum_{\ell} \sum_{c} \frac{\gamma_{\ell}^{-2}}{C} \frac{\gamma_{c}^{-2}}{C} \vartheta_{\ell}^{\top} \vartheta_{c} - \sum_{\ell} \gamma_{\ell}^{-2} \vartheta_{\ell}^{\top} \vartheta_{\ell} \]

\[ = \frac{1}{C} \sum_{r} \sum_{c \neq r} \gamma_{r}^{-2} \gamma_{c}^{-2} \vartheta_{r}^{\top} \vartheta_{c} + \sum_{r} \gamma_{r}^{-2} \frac{(\gamma_{r}^{-2} - 1) \vartheta_{r}^{\top} \vartheta_{r}}{C} \]

\[ = \frac{1}{C} \left( \sum_{r} \sum_{c \neq r} \gamma_{r}^{-2} \gamma_{c}^{-2} \vartheta_{r}^{\top} \vartheta_{c} + \sum_{r} \gamma_{r}^{-2} \frac{(\gamma_{r}^{-2} - C) \vartheta_{r}^{\top} \vartheta_{r}}{C} \right) \]

\[ = \frac{1}{C} \left( \sum_{r} \sum_{c \neq r} \gamma_{r}^{-2} \gamma_{c}^{-2} \vartheta_{r}^{\top} \vartheta_{c} - \sum_{r} \gamma_{r}^{-2} \sum_{c \neq r} \gamma_{c}^{-2} \vartheta_{r}^{\top} \vartheta_{r} \right) \]

\[ = \frac{1}{C} \sum_{\ell} \gamma_{\ell}^{-2} \left( \sum_{r} \sum_{c} \gamma_{r}^{-2} \gamma_{c}^{-2} \left( \vartheta_{r}^{\top} \vartheta_{c} - \vartheta_{r}^{\top} \vartheta_{r} \right) \right). \]

Our claim follows from this. \( \square \)

**Lemma B.2.** Let \( \phi \) be the standard multivariate normal density defined on \( \mathbb{R}^d \).
Also, fix a sequence \( \{ \gamma_m \}_{m=1}^{M} \subset \mathbb{R}_+ \), and a sequence \( \{ \vartheta_{\ell} \}_{\ell=1}^{M} \subset \mathbb{R}^d \).

\[
\prod_{m} \phi(x; \vartheta_m, \gamma_m) = \left( \frac{2\pi / \prod_m (2\pi)}{\sum_m \gamma_m^2 \prod_m \gamma_m^2} \right)^{d/2} \cdot \exp \left( \frac{1 \cdot 1^\top (\Gamma * (\Theta - \text{diag}(\Theta) 1^\top)) 1}{\sum \gamma_n^{-2}} \right) \cdot \phi \left( x; \sum_{\ell} \left( \frac{\gamma_{\ell m}^{-2}}{\sum_m \gamma_{m}^2} \right) \vartheta_{\ell}; \left( \sum_m \gamma_{m}^{-2} \right)^{-1/2} \right).
\]
Proof. Using Lemma B.1, we see that

\[
\prod_m \phi(x; \vartheta_m, \gamma_m) = \prod_m \frac{1}{\sqrt{2\pi\gamma_m^2}} \exp \left( -\frac{1}{2} \frac{\|x - \vartheta_m\|_2^2}{\gamma_m^2} \right) \\
= \frac{1}{\sqrt{\prod_m 2\pi\gamma_m^2}} \exp \left( -\frac{1}{2} \sum_m \frac{\|x - \vartheta_m\|_2^2}{\gamma_m^2} \right) \\
= \frac{1}{\sqrt{\prod_m 2\pi\gamma_m^2}} \exp \left( -\frac{1}{2} \left[ x - \sum_k \left( \frac{\gamma_k^{-2}}{\sum_m \gamma_m^{-2}} \vartheta_k \right) \right] \cdot \left[ \left( \sum_k \gamma_k^{-2} \right)^{-1/2} \right] \right) \\
\cdot \exp \left( \frac{1}{2} \frac{1_1^T \left( \Gamma \ast (\Theta - \text{diag}(\Theta)1^T) \right) 1}{\sum_n \gamma_n^2} \right) \\
= \frac{\left( \frac{2\pi}{\prod_m 2\pi\gamma_m^2} \right)^{d/2}}{\left( \sum_k \gamma_k^{-2} \prod_k \gamma_k^{-2} \right)} \phi \left( x; \sum_k \left( \frac{\gamma_k^{-2}}{\sum_m \gamma_m^{-2}} \vartheta_k \right), \left( \sum_k \gamma_k^{-2} \right)^{-1/2} \right) \\
\cdot \exp \left( \frac{1}{2} \frac{1_1^T \left( \Gamma \ast (\Theta - \text{diag}(\Theta)1^T) \right) 1}{\sum_m \gamma_m^2} \right) .
\]

\[\blacklozenge\]

B.3. Formula for \( R_{t,i,\ell} \) in a multivariate normal density case. Here, we assume, as done in Theorem 5.3, that \( \mu_t(y) = \sum_k q_t,\phi_t(y) \) and \( p_{t,i}(x) = \sum_k W_{t,i,\ell} \phi_t(y) \), where for simplicity, we have written \( \phi_t(z) := \phi(z; \theta_t, s) \propto \phi(s^{-1}(z - \theta_t)) \). In this section, we fix \( \phi \) to be the standard multivariate normal density defined on \( \mathbb{R}^d \) and recall that \( \psi(z) = \phi(z)/\phi(0) \). Also, we fix \( s \in \mathbb{R}^+ \), and a sequence \( \{\theta_t\} \subset \mathbb{R}^d \).

**Lemma B.3.** Fix \( \theta_t, x \in \mathbb{R}^d \) and \( s \in \mathbb{R}^+ \). For each \( k \),

\[
b_{t,i,\ell}^k(x) = -2(1 - \omega_{t,i})(x - \theta_t)^k \left( \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} \sqrt{(2\pi\sigma_{t,i}^2)^d \phi(x; \theta_t, (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)^{1/2})} \right), \quad (B.1)
\]

\[
a_{t,i,\ell}^{k_1,k_2}(x) = (1 - \omega_{t,i})^2 \left( \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} \sqrt{(2\pi\sigma_{t,i}^2)^d \phi(x; \theta_t, (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)^{1/2})} \right) \\
\cdot \left( \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} \right)^2 \left( x - \theta_t \right)^{k_1} \left( x - \theta_t \right)^{k_2} + 1 \{k_1 = k_2\} \left( \frac{\sigma_{t,i}^2 \alpha_{t,\ell}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} \right), \quad (B.2)
\]
Proof. Let
\[
v_{t,i,\ell} = \frac{1}{(\sigma_{t,i}^{-2} + \alpha_{t,\ell}^{-2})} = \frac{\sigma_{t,i}^2 \alpha_{t,\ell}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2},
\]
\[
c_{t,i,\ell} = \frac{\alpha_{t,\ell}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} x + \frac{\alpha_{t,\ell}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} \theta_{t,i}
\]
where to simplify the expression of \(c_{t,i,\ell}\), we have used the fact that
\[
\sigma_{t,i}^2 \alpha_{t,\ell}^2 \times (\sigma_{t,i}^{-2} + \alpha_{t,\ell}^{-2}) = (\sigma_{t,i}^2 + \alpha_{t,\ell}^2).
\]
Also, note that
\[
\left(\sigma_{t,i}^{-2} + \alpha_{t,\ell}^{-2}\right)^{-1} \mathbf{1}^T (\Gamma \ast (\Theta - \text{diag}(\Theta) \mathbf{1}^T)) \mathbf{1}
\]
\[
= \left(\sigma_{t,i}^{-2} + \alpha_{t,\ell}^{-2}\right)^{-1} \mathbf{1}^T \left( \begin{bmatrix} \sigma_{t,i}^{-4} & \sigma_{t,i}^{-2} \alpha_{t,\ell}^{-2} \\ \sigma_{t,i}^{-2} \alpha_{t,\ell}^{-2} & \alpha_{t,\ell}^{-4} \end{bmatrix} \right) \mathbf{1}^T \begin{bmatrix} \sigma_{t,i}^2 \alpha_{t,\ell}^2 \alpha_{t,\ell}^2 \alpha_{t,\ell}^2 \end{bmatrix} \mathbf{1}
\]
\[
= \left(\sigma_{t,i}^{-2} + \alpha_{t,\ell}^{-2}\right)^{-1} \mathbf{1}^T \begin{bmatrix} -x^\top x & x^\top \theta_{t,i} & \theta_{t,i}^\top \theta_{t,i} \\ -x^\top x & -x^\top \theta_{t,i} & -\theta_{t,i}^\top \theta_{t,i} \\ x_{t,i}^\top \theta_{t,i} & x_{t,i}^\top \theta_{t,i} & -\theta_{t,i}^\top \theta_{t,i} \end{bmatrix} \mathbf{1}
\]
\[
= \left(\sigma_{t,i}^{-2} + \alpha_{t,\ell}^{-2}\right)^{-1} \left(\sigma_{t,i}^{-2} \alpha_{t,\ell}^{-2}\right) (-\|x - \theta_{t,i}\|^2)
\]
\[
= -\left(\sigma_{t,i}^2 + \alpha_{t,\ell}^2\right)^{-1} \|x - \theta_{t,i}\|^2.
\]
Using Lemma B.2 with \(x = y\), \(\theta_1 = x\) and \(\theta_2 = \theta_{t,i}\), we see that
\[
\exp \left( -\frac{1}{2} \frac{1}{\sigma_{t,i}^2} \|y - x\|^2 \right) \exp \left( -\frac{1}{2} \frac{1}{\alpha_{t,\ell}^2} \|y - \theta_{t,i}\|^2 \right) \sqrt{\frac{(2\pi \alpha_{t,\ell}^2)^d}{(2\pi \sigma_{t,i}^2)^d}}
\]
\[
= (2\pi \sigma_{t,i}^2)^{d/2} \left( \frac{2\pi}{(2\pi)^2} \right)^{d/2} \exp \left( -\frac{1}{2} \frac{1}{\sigma_{t,i}^2} \|x - \theta_{t,i}\|^2 \right) \exp \left( -\frac{1}{2} \frac{1}{\alpha_{t,\ell}^2} \|x - \theta_{t,i}\|^2 \right) \frac{\exp \left( -\frac{\|y - \theta_{t,i,\ell}\|^2}{2v_{t,i,\ell}} \right)}{(\sqrt{2\pi \alpha_{t,\ell}^2})^d}
\]
\[
= (2\pi \sigma_{t,i}^2)^{d/2} \frac{\exp \left( -\frac{\|x - \theta_{t,i}\|^2}{2(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)} \right)}{(\sqrt{2\pi (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)})^d}.
\]
Then, for our claim in (B.1), it is enough to see that
\[
\int \frac{1}{(\sqrt{2\pi \alpha_{t,\ell}^2})^d} \exp \left( -\frac{\|y - \theta_{t,i,\ell}\|^2}{2v_{t,i,\ell}} \right) (y - x) dy
\]
\[
= c - x
\]
\[
= \left( \frac{\alpha_{t,\ell}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} x + \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} \theta_{t,i} \right) - x
\]
\[
= \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \alpha_{t,\ell}^2} (\theta_{t,i} - x).
\]
Next, we show our claim in (B.2). Hereafter, to ease our notation, we write $c$ for $c_{t,i,\ell}$.
First, for $k_1 \neq k_2$, we have
\[
\int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{\|y - c\|^2}{2v_{t,i,\ell}} \right) y_{k_1} y_{k_2} \, dy \\
= \int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{(y_{k_1} - c_{k_1})^2}{2v_{t,i,\ell}} \right) y_{k_1} \, dy_{k_1} \cdot \int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{(y_{k_2} - c_{k_2})^2}{2v_{t,i,\ell}} \right) y_{k_2} \, dy_{k_2}
\]
= $c_k c_{k_2}$, and hence,
\[
\int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{\|y - c\|^2}{2v_{t,i,\ell}} \right) (y - x)_{k_1} (y - x)_{k_2} \, dy \\
= c_k c_{k_2} - x_{k_1} c_{k_2} - c_{k_1} x_{k_2} + x_{k_1} x_{k_2} \\
= (c - x)_{k_1} (c - x)_{k_2} \\
= \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} (\theta - x)_{k_1} \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} (\theta - x)_{k_2} \\
= \left( \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} \right)^2 (x - \theta)_{k_1} (x - \theta)_{k_2}.
\]
On the other hand, for $k_1 = k_2$, we have
\[
\int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{\|y - c\|^2}{2v_{t,i,\ell}} \right) y_{k_1} y_{k_2} \, dy \\
= \int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{(y - c_{k_1})^2}{2v_{t,i,\ell}} \right) y^2 \, dy \\
= v_{t,i,\ell} + c_{k_1}^2 \\
= \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} x_{k} + \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} \theta \, \, dy \\
= \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} x_{k} + \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} \theta_{k,\ell}
\]
and so, we have
\[
\int \frac{1}{\sqrt{2\pi v_{t,i,\ell}}} \exp \left( -\frac{\|y - c\|^2}{2v_{t,i,\ell}} \right) (y - x)_{k_1} (y - x)_{k_2} \, dy \\
= v_{t,i,\ell} + c_{k_1} c_{k_2} - x_{k_1} c_{k_2} - c_{k_1} x_{k_2} + x_{k_1} x_{k_2} \\
= v_{t,i,\ell} + (c - x)_{k_1} (c - x)_{k_2} \\
= \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} (\theta - x)_{k_1} \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} (\theta - x)_{k_2} \\
= \left( \frac{\sigma_{t,i}^2}{\sigma_{t,i}^2 + \sigma_{t,\ell}^2} \right)^2 (x - \theta)_{k_1} (x - \theta)_{k_2}.
\]
Our claim in (B.2) follows. \( \square \)
For Lemma B.4 and Lemma B.5, by \( \Theta_{\ell, r, c} \), we denote the Gram matrix for \((\theta_t, \theta_r, \theta_c)\), and define \( \Gamma_{t,i} \) to be as in Lemma B.1 for \( \gamma_1^2 = \sigma_{t,i}^2 + \alpha_{t,i,\ell}^2 \), \( \gamma_2^2 = s^2 \), \( \gamma_3^2 = s^2 \). Let

\[
C_0 = (\theta_t (\sigma_{t,i}^2 + \alpha_{t,i,\ell}^2)^{-1} + \theta_r s^{-2} + \theta_c s^{-2})/((\sigma_{t,i}^2 + \alpha_{t,i,\ell}^2)^{-1} + s^{-2} + s^{-2}),
\]

\[
C_1 = \frac{1}{\sigma_{t,i}^2 + \alpha_{t,i,\ell}^2} \left( \frac{1}{s^2} \right) = \frac{s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,i,\ell}^2}{(\sigma_{t,i}^2 + \alpha_{t,i,\ell}^2)s^2},
\]

\[
C_2 = \left( \frac{1/(2\pi)^{d/2}}{s^4 + 2s^2(\sigma_{t,i}^2 + \alpha_{t,i,\ell}^2)} \right)^{d/2} \exp \left( \frac{1}{2} C_1^T (\Gamma_{t,i} * (\Theta_{\ell, r, c} - \text{diag}(\Theta_{\ell, r, c}))^T) \right) 1.
\]

To simplify our notation, we let

\[
\xi_k = 1^T \left[ \begin{array}{c} -2 & 1 & 1 \\ \sigma_{t,i}^2 + \alpha_{t,i,\ell}^2 & \theta_{t,k} & \theta_{r,k} \\ \sigma_{t,i}^2 + \alpha_{t,i,\ell}^2 & \theta_{r,k} & \theta_{c,k} \end{array} \right] \begin{pmatrix} \theta_{t,k} \\ \theta_{r,k} \\ \theta_{c,k} \end{pmatrix} 1,
\]

\[
\Xi = 1^T (\Gamma_{t,i} * (\Theta_{\ell, r, c} - \text{diag}(\Theta_{\ell, r, c}))^T) 1.
\]

Define and note

\[
\xi := \sum_{k=1}^{K} \xi_k = 1^T \left[ \begin{array}{c} -2 & 1 & 1 \\ \sigma_{t,i}^2 + \alpha_{t,i,\ell}^2 & \theta_{t,k} & \theta_{r,k} \\ \sigma_{t,i}^2 + \alpha_{t,i,\ell}^2 & \theta_{r,k} & \theta_{c,k} \end{array} \right] \begin{pmatrix} \Theta_{\ell, r, c} \end{pmatrix} 1.
\]

Also, denote by \( h_{\ell, r, c}(x) \) the multivariate normal density defined on \( \mathbb{R}^d \) such that its mean vector is \( C_0 \) and its covariance matrix is \( C_1^{-1} I \). For \( f \in C(\mathbb{R}) \) and \( k = 1, \ldots, d \), we write

\[
\langle f(x_k) \rangle_{\ell, r, c} = \int_{\mathbb{R}^d} f(x_k) h_{\ell, r, c}(x) dx,
\]

and note that in particular,

\[
\langle x_k \rangle_{\ell, r, c} = C_{0,k},
\]

\[
\langle x_k^2 \rangle_{\ell, r, c} = C_{0,k} + C_{1}^{-1},
\]

\[
\langle x_k^3 \rangle_{\ell, r, c} = C_{0,k}^3 + 3C_{0,k}C_{1}^{-1},
\]

\[
\langle x_k^4 \rangle_{\ell, r, c} = C_{0,k}^4 + 6C_{0,k}^2C_{1}^{-1} + 3C_{1}^{-2}.
\]

Starting from (5.5), it is easy to see that

\[
\langle A_{t,i,\ell,\phi_r, \phi_c} \rangle = \sum_{k=1}^{d} \langle \gamma_{t,i,\ell}^k \theta_{k}\phi_r, \phi_c \rangle + \sum_{k_1=1}^{d} \sum_{k_2=1}^{d} \langle a_{t,i,\ell}^{k_1, k_2} \theta_{k_1} \partial_{k_2} \phi_r, \phi_c \rangle,
\]

and as a matter of definition, we have

\[
R_{t,i,\ell} = (R_{t,i,\ell, r,c})_{r,c=1}^{K} = (\langle A_{t,i,\ell,\phi_r, \phi_c} \rangle)_{r,c=1}^{K} \in M_{K,K}.
\]

Lemma B.4 and Lemma B.5 are associated, respectively, with the first and the second terms appearing in the right side of (B.3).
LEMMA B.4. For each \( \ell, r, c \) and \( i, t \), we have

\[
\sum_{k=1}^{d} \langle b_{i,t}^{k} \partial_k \phi_r, \phi_c \rangle = (2\pi \sigma_{t,i}^2)^{d/2} \left( \frac{1}{(2\pi)^d} \int \frac{1}{(2\pi \sigma_{t,i}^2)^{d/2}} \exp \left( -\frac{1}{2} \frac{(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^2}{s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,\ell}^2} \right) \right) \cdot \frac{1}{2} \frac{(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^2}{s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,\ell}^2}.
\]

Proof. To ease our notation, we first let

\[
\tilde{b}_{i,t}^{k}(x) = -(2s^2/C_2) \cdot \phi(x; \theta, (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)^{1/2})(x - \theta_k)_k.
\]

It follows that

\[
\langle \tilde{b}_{i,t}^{k} \partial_k \phi_r, \phi_c \rangle = -(2s^2/C_2) \cdot \frac{1}{2(1 - \omega_{t,i}) \sqrt{2\pi \sigma_{t,i}^2 \sigma_{r,i}^2 / (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)}} \cdot \langle b_{i,t}^{k} \partial_k \phi_r, \phi_c \rangle \cdot \frac{1}{(1 - \omega_{t,i}) \sqrt{2\pi \sigma_{t,i}^2 \sigma_{r,i}^2 / (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)}} \cdot \frac{1}{(2\pi \sigma_{t,i}^2)^{d/2}} \int \frac{1}{(2\pi \sigma_{t,i}^2)^{d/2}} \exp \left( -\frac{1}{2} \frac{(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^2}{s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,\ell}^2} \right) \right) \cdot \frac{1}{2} \frac{(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^2}{s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,\ell}^2}.
\]

We compute \( \langle \tilde{b}_{i,t}^{k} \partial_k \phi_r, \phi_c \rangle \) instead of directly working with (5.6). First, we observe that

\[
\langle x_{k}^{2} \rangle_{\ell,r,c} - \langle x_{k}^{2} \rangle_{\ell,r,c} = \frac{1}{C_1^2} = \frac{(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^2}{s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,\ell}^2},
\]

and that

\[
\frac{1}{C_1^2 (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^4} = \left( \frac{\sigma_{t,i}^2 + \alpha_{t,\ell}^2}s^2 \right)^2 \frac{1}{(\sigma_{t,i}^2 + \alpha_{t,\ell}^2)s^4} = \frac{1}{(s^2 + 2\sigma_{t,i}^2 + 2\alpha_{t,\ell}^2)^2}.
\]

Using Lemma B.1 on the third equality, we see that

\[
\int b_{i,t}^{k}(x) \partial_k \phi_r(x) \phi_c(x) dx
\]

\[
= \int -2s^2/C_2 \phi(x; \theta, (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)^{1/2})(x - \theta_k)_k \left( -\frac{1}{s^2}(x - \theta_r)_k \phi_r(x) \right) \phi_c(x) dx
\]

\[
= 2/C_2 \int \phi(x) \phi_c(x) \phi(x; \theta, (\sigma_{t,i}^2 + \alpha_{t,\ell}^2)^{1/2})(x - \theta_k)_k (x - \theta_r)_k dx
\]

\[
= 2 \int h_{\ell,r,c}(x)(x_k - x_k(\theta_k + \theta_r)_k + \theta_{k,\ell,\theta,r,k}) dx
\]

\[
= 2 \left( \langle x_{k}^{2} \rangle_{\ell,r,c} - \langle x_{k} \rangle_{\ell,r,c}^2 + (\langle x_{k} \rangle_{\ell,r,c} - \theta_{k,\ell,\theta,r,k})(\langle x_{k} \rangle_{\ell,r,c} - \theta_{r,k}) \right).
\]
Continuing with the calculation,
\[
(b_{k,i,j}^k, \theta_k \phi_r, \phi_c)
\]
\[
= 2 \left( \frac{1}{C_1} + \frac{1}{C_2^2} \left( \frac{k}{s^2} + \frac{\theta_r - \theta_t}{s^2} + \frac{\theta_t - \theta_r}{s^2} \right) \right)
\]
\[
= 2 \left( \frac{1}{C_1} + \frac{1}{C_2^2} \left( -2 \theta_t + \theta_r + \theta_c \right) \right) \frac{s^2 \theta_r - \left( \sigma_{i,\ell}^2 + \alpha_{i,\ell}^2 \right) \theta_r + \left( \sigma_{i,\ell}^2 + \alpha_{i,\ell}^2 \right) \theta_c = 1 \right)
\]  
(B.6)
\[
= \frac{(2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2) s^2}{s^2 + 2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2}
\]  
(B.7)
\[
\cdot 1^T \left( \begin{pmatrix} - (\sigma_{i,\ell}^2 + \alpha_{i,\ell}^2 + s^2) \\ \sigma_{i,\ell}^2 + \alpha_{i,\ell}^2 \end{pmatrix} \begin{bmatrix} 2 \ 1 \ 1 \end{bmatrix} \right) \begin{bmatrix} \theta_{t,k} \\ \theta_{r,k} \theta_{c,k} \end{bmatrix} \right) \right) 1.
\]  
(B.8)

Putting together (B.6), (B.7), (B.8) and (B.5), and plugging in the full expression for \( C_1 \), we see that

\[
\langle b_{t,i,j}^k, \theta_k \phi_r, \phi_c \rangle
\]
\[
= (2 \pi \sigma_{i,\ell}^2)^d/2 \left( 1 - \omega_{\mu,i} \right) \sigma_{i,\ell}^2 \frac{C_2}{s^2 \left( \sigma_{i,\ell}^2 + \alpha_{i,\ell}^2 \right)} \left( \frac{s^2 (2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2)}{s^2 + 2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2} + \frac{2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2}{s^2 + 2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2} \right)
\]  

Our claim follows after summing over \( k \) and replacing \( C_2 \) with its full expression. □

**Lemma B.5.** For each \( \ell, r, c, t \) and \( i, k \),

\[
\sum_{k_1=1}^{d} \sum_{k_2=1}^{d} \left( a_{i,\ell}^{k_1,k_2} \theta_{k_1} \phi_{r,k_2} \phi_c \right)
\]
\[
= \frac{1}{s^4} (1 - \omega_{\mu,i})^2 (2 \pi \sigma_{i,\ell}^2)^d/2 \left( \frac{\sigma_{i,\ell}^2 \alpha_{i,\ell}^2}{\sigma_{i,\ell}^2 + \alpha_{i,\ell}^2} \right) \sum_k \left( \begin{pmatrix} \theta_{t,k} \\ \theta_{r,k} \\ \theta_{c,k} \\ \theta_{r,k} \right) \begin{bmatrix} \sigma_{i,\ell}^2 + \alpha_{i,\ell}^2 \end{bmatrix} \begin{bmatrix} 2 \ 1 \ 1 \end{bmatrix} \right) \right) \right) 1
\]  
(B.9)
\[
+ \left( \frac{\sigma_{i,\ell}^2}{\sigma_{i,\ell}^2 + s^2} \right)^2 1^T \left( \sum_k \left( \begin{pmatrix} \theta_{t,k} \\ \theta_{r,k} \\ \theta_{c,k} \end{pmatrix} \begin{bmatrix} 2 \ 1 \ 1 \ 1 \end{bmatrix} \right) \right) \right) \right) 1
\]  
(B.10)
\[
\sum_k \sum_{k_1 \neq k_2} \left( \begin{pmatrix} \theta_{t,k} \\ \theta_{r,k} \theta_{c,k} \end{pmatrix} \begin{bmatrix} 2 \ 1 \ 1 \ end{bmatrix} \right) \right) \right) 1
\]  
(B.11)
\[
\cdot \frac{1}{(2 \pi \sigma_{i,\ell}^2)^2} \exp \left( \frac{1}{2} \frac{\sigma^2_{i,\ell} \alpha_{i,\ell}^2 s^2}{s^2 + 2 \sigma_{i,\ell}^2 + 2 \alpha_{i,\ell}^2} \right)
\]  
(B.12)

**Proof.** Note that

\[
a_{i,\ell}^{k_1,k_2}(x) = (1 - \omega_{\mu,i})^2 (2 \pi \sigma_{i,\ell}^2)^d/2 \left( \frac{\sigma_{i,\ell}^2}{\sigma_{i,\ell}^2 + \alpha_{i,\ell}^2} \right) \left( \begin{pmatrix} \sigma_{i,\ell}^2 \alpha_{i,\ell}^2 \alpha_{i,\ell}^2 \alpha_{i,\ell}^2 \end{pmatrix} \begin{bmatrix} 2 \ 1 \ 1 \end{bmatrix} \right) \right) \right) 1
\]  
(B.13)
where
\[ A_{t,\ell}^{k_1, k_2}(x) = \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2})(x - \theta_t)_{k_1} (x - \theta_t)_{k_2}, \]
\[ B_{t, \ell}^{k_1, k_2}(x) = \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) \mathbb{1}_{\{k_1 = k_2\}}. \]

We first compute the diagonal terms, i.e., the $k_1 = k_2$ cases. Note that
\[
\sum_{k_1} \sum_{k_2} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) \mathbb{1}_{\{k_1 = k_2\}} \partial^2 \phi_r(x) \phi_c(x) dx
\]
\[
= \sum_{k_1} \sum_{k_2} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) \mathbb{1}_{\{k_1 = k_2\}} \partial^2 \phi_r(x) \phi_c(x) dx
\]
\[
= \sum_{k} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) \partial^2 \phi_r(x) \phi_c(x) dx
\]
\[
= \frac{1}{s^4} \sum_{k} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) (x_k^2 - 2\theta_{r, k} x_k + \theta_{r, k}^2 - s^2) \phi_r(x) \phi_c(x) dx
\]
\[
= C_2 \frac{1}{s^4} \sum_{k} \int h_{\ell, r, c}(x)(x_k^2 - 2\theta_{r, k} x_k + \theta_{r, k}^2 - s^2) dx.
\]

and also that
\[
\sum_{k_1} \sum_{k_2} \int 1_{\{k_1 = k_2\}} A_{t, \ell}^{k_1, k_2}(x) \partial^2 \phi_{r, t} \phi_c(x) dx
\]
\[
= \sum_{k_1} \sum_{k_2} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) 1_{\{k_1 = k_2\}} \partial^2 \phi_r(x) \phi_c(x) dx
\]
\[
= \sum_{k} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) \partial^2 \phi_r(x) \phi_c(x) dx
\]
\[
= \frac{1}{s^4} \sum_{k} \int \phi(x; \theta_t, (\sigma_{t, i}^2 + \alpha_{t, \ell}^2)^{1/2}) (x_k^2 - 2\theta_{r, k} x_k + \theta_{r, k}^2 - s^2) \phi_r(x) \phi_c(x) dx
\]
\[
= C_2 \frac{1}{s^4} \sum_{k} \int h_{\ell, r, c}(x)(x_k^2 - 2\theta_{r, k} x_k + \theta_{r, k}^2 - s^2) dx.
\]

Next, we compute the off-diagonal terms, i.e., the $k_1 \neq k_2$ cases. First, using our calculation just above, we see that we note that
\[
\sum_{k_1} \sum_{k_2 \neq k_1} \int A_{t, \ell}^{k_1, k_2}(x) \partial^2 \phi_{r, k_2}(x) \phi_c(x) dx
\]
\[
= C_2 \frac{1}{s^4} \sum_{k_1} \sum_{k_2 \neq k_1} \int h_{\ell, r, c}(x)(x - \theta_t)_{k_1} (x - \theta_t)_{k_2} (x - \theta_r)_{k_1} (x - \theta_r)_{k_2} dx.
\]

Our claim follows from this after combining them together, and simplifying the combined term into a matrix notation.

\section*{Appendix C. Proof for Theorem 5.1.}

Here, we will take the convention that $X_t = (X_{t, i, k})_{i=1, k=1}^{n,d}$ is organized as a matrix. By the $i$-th row of $X_t$, we mean $X_{t, i} = (X_{t, i, 1}, ..., X_{t, i, d})$. Let
\[
\varphi_t(v) = \mathbb{P} \left[ e^{(v, X_t)} \mid \mathcal{F}_t \right] = \int \rho_t(dx) e^{(v, x)},
\]
where for each \( v \) and \( x \),

\[
(v, x) \equiv \sum_{i=1}^{n} \sum_{k=1}^{d} v_{ik} x_{ik}.
\]

In other words, \( \varphi_t \) is the (random) conditional characteristic function of \( X_t \). Note

\[
\pi_t(y) = \frac{1}{2\pi} \int e^{-i\langle v, y \rangle} \varphi_t(v) dv.
\]

Also, let, for each \( v \) and \( x \in \mathbb{R}^{n \times d} \), \( a_t(v | x) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E\left[ e^{i\langle v, X_{t+\varepsilon} - X_t \rangle} - 1 \middle| X_t = x \right] \). (C.1)

For each \( f \in B(X) \), \( f_{-i} \) denotes the function obtained by fixing all other indices different from the \( i \)-th actor indices but letting the \( i \)-th actor indices to be free, and if \( f_{-i} \) is in the domain of the operator \( A(\mu_t) \), with some abuse of notation, we write:

\[
A(\mu_t)f(z) = \sum_{i=1}^{n} (A(\mu_t)f_{-i}(z))(z_i).
\]

Similarly, for each \( v \in \mathbb{R}^d \), let

\[
\varphi_{t,i}(v) = E\left[ e^{i\langle v, X_{t,i}(t) \rangle} \middle| \mathcal{F}_t \right].
\]

In other words, \( \varphi_{t,i} \) denotes the conditional characteristic function of the \( i \)-th row \( X_k(t) \) of \( X(t) \), and also, let, for \( v \in \mathbb{R}^d \), and \( x \in X \),

\[
a_t(v | x) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E\left[ e^{i\langle v, X_{t,i} - X_{t,i} \rangle} - 1 \middle| X_{t,i} = x \right]. \quad \text{(C.2)}
\]

Note that the definition of \( a_t(v | x) \) is actually independent of a particular choice of vertex \( i \) as they are all identically distributed.

One can prove the next result by directly following Snyder [1975], but one needs to adapt to the fact that the underlying process can now be a time-inhomogeneous non-linear Markov process. The proof details are left to the reader. For a survey of similar techniques, see also Kunita [1997] and Bain and Crisan [2009].

**Proposition C.1.** For each \( v \in \mathbb{R}^{n \times d} \) and \( t \in (0, \infty) \),

\[
d\varphi_t(v) = \langle \rho_t, e^{i\langle v, \cdot \rangle} a_t(v | \cdot) \rangle dt + \mathbf{1}^\top (\rho_t, e^{i\langle v, \cdot \rangle} (\widetilde{\Lambda} - \mathbf{1} \mathbf{1}^\top)') dM_t \mathbf{1}.
\]

Our proof of Theorem 5.1 is by brute force calculation, starting from Proposition C.1. In particular, our claim in Theorem 5.1 follows from Proposition C.1 by directly applying Lemma C.2, Lemma C.3 and Lemma C.4 which we list and prove now.

**Lemma C.2.** For each \( t \in [0, \infty) \),

\[
a_t(v | x) = A(\mu_t) e^{i\langle v, -x \rangle}(x).
\]
Proof. Fix $t, i, v$ and $x$. Then, for each $\varepsilon > 0$, we have:

$$
\mathbb{E}\left[e^{i\langle v, X_{t+i} - x \rangle} | X_{t,i} = x\right] = 1 + \int_0^\varepsilon \mathbb{E}\left[\mathcal{A}(\mu_{t+s}) e^{i\langle v, -x \rangle} (X_{t+s,i}) | X_{t,i} = x\right] ds.
$$

We have

$$
\sup_{y \in \mathbb{X}} |\mathcal{A}(\mu_{t+s}) e^{i\langle v, -x \rangle} (y)| \leq |e^{i\langle v, -x \rangle}| = 1,
$$

and hence,

$$
\left|\mathbb{E}\left[\mathcal{A}(\mu_{t+s}) e^{i\langle v, -x \rangle} (X_{t+s}) | X_{t,i} = x\right]\right| \leq 1.
$$

It follows that

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left[\mathcal{A}(\mu_{t+s}) e^{i\langle v, -x \rangle} (X_{t+s}) | X_{t,i} = x\right] ds
$$

$$
= \mathbb{E}\left[\mathcal{A}(\mu_{t}) e^{i\langle v, -x \rangle} (X_{t}) | X_{t,i} = x\right]
$$

$$
= \mathcal{A}(\mu_{t}) e^{i\langle v, -x \rangle} (x).
$$

\[\square\]

Lemma C.3. For each $f \in C_b(\mathbb{X})$, we have:

$$
\int_{\mathbb{X}} f(y) \left(\frac{1}{2\pi} \int e^{-i\langle v, y \rangle} \langle \rho_t, e^{i\langle v, \cdot \rangle} a_t(\cdot) \rangle dv\right) dy = \int_{\mathbb{X}} (\mathcal{A}(\mu_{t}) f)(z) \rho_t(dz).
$$

Proof. Fix $y \in \mathbb{X}$ and note:

$$
h(y) = \frac{1}{2\pi} \int e^{-i\langle v, y \rangle} \langle \rho_t, e^{i\langle v, \cdot \rangle} a_t(\cdot) \rangle dv
$$

$$
= \frac{1}{2\pi} \int e^{-i\langle v, y \rangle} \rho_t(dz) e^{i\langle v, z \rangle} a_t(\cdot) d\nu(z)
$$

$$
= \int \rho_t(dz) \left(\frac{1}{2\pi} \int e^{-i\langle v, y \rangle} e^{i\langle v, z \rangle} a_t(\cdot) d\nu(z)\right),
$$

and that

$$
\frac{1}{2\pi} \int e^{i\langle v, z - y \rangle} a_t(\cdot) d\nu(z) = \frac{1}{2\pi} \int e^{i\langle v, z - y \rangle} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[e^{i\langle v, X_{t+s} - z \rangle} - 1 | X_t = z\right] dv
$$

$$
= \frac{1}{2\pi} \int \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[e^{i\langle v, X_{t+i} - y \rangle} - e^{i\langle v, z - y \rangle} | X_t = z\right] dv.
$$
Treating $h(y)$ as a generalized function (i.e. a tempered distribution), we have:

$$\int h(y)f(y)dy = \int \rho_i(dz) \left( \int f(y) \left( \frac{1}{2\pi} \int \lim_{\varepsilon \to 0} E \left[ e^{i(y,v,X_{i+\varepsilon}-y)} | X_i = z \right] dv \right) dy \right)$$

$$= \int \rho_i(dz) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( E \left[ \int f(y) \left( \frac{1}{2\pi} \int e^{i(y,v,X_{i+\varepsilon}-y)} dv \right) dy | X_i = z \right] - \int f(y) \left( \frac{1}{2\pi} \int e^{i(y,v,z-y)} dv \right) dy \right)$$

Lemma C.4. For each $f \in C_b(\mathbb{X})$, we have:

$$\int_{\mathbb{X}} f(y) \left( \frac{1}{2\pi} \int e^{-i(y,v)} \langle \rho_t, e^{i(v,\cdot)} (\lambda(\cdot) - \mathbf{11}^T) \rangle dv \right) dy = \int_{\mathbb{X}} \rho_i(dz) f(z) (\lambda(z) - \mathbf{11}^T).$$

Proof. Note

$$\int_{\mathbb{X}} f(y) \left( \frac{1}{2\pi} \int e^{-i(y,v)} \langle \rho_t, e^{i(v,\cdot)} (\lambda(\cdot) - \mathbf{11}^T) \rangle dv \right) dy = \int \rho_i(dz) \left( \lambda(z) - \mathbf{11}^T \right) \left( \int f(y) \left( \frac{1}{2\pi} \int e^{i(v,z-y)} dv \right) dy \right) = \int \rho_i(dz) \left( \lambda(z) - \mathbf{11}^T \right) f(z).$$

Appendix D. Proof of Theorem 5.2. Recall that for each $f \in \mathcal{B}(\mathbb{X})$, $f_{-i}$ denotes the function obtained by fixing all other indices different from the $i$-th actor indices but letting the $i$-th actor indices to be free. Fix $u \in \{1, \ldots, n\}$. Let $f \in \mathcal{B}(\mathbb{X})$ be such that $f(z) = f_{-u}(z_u)$ for all $z \in \mathbb{X}$. For each $t \in (0, \infty)$,

$$d\langle \rho_t, f \rangle = \langle \rho_t, A(\mu_t) f_{-u} \rangle dt$$

$$+ \sum_{i < j, i \neq u, j \neq u} \int_{\mathbb{X}} \rho_{t,i,j}(dz_u, dz_i, dz_j) f_{-u}(z_u) \left( \frac{p_{t,j}(z_i)}{p_{t,i,j}} - 1 \right) dM_{t,ij}$$

$$+ \sum_{i \neq u} \int_{\mathbb{X}} \rho_{t,i,u}(dz_i, dz_u) f_{-u}(z_u) \left( \frac{p_{t,j}(z_i)}{p_{t,i,j}} - 1 \right) dM_{t,iu}.$$

Then, the claimed formula follows from our assumption in (5.2).

Appendix E. Proof of Theorem 5.4. Suppose that $M_\varepsilon \to M_0$ as $\varepsilon \to 0$ and that for each $\varepsilon \geq 0$, $M_\varepsilon$ satisfies the rank condition, i.e., $\varrho(M_\varepsilon)$ is of rank at least $d$. 
Note that each $\xi_d(M_\varepsilon)$ is a non-empty compact subset of $\mathbb{R}^{n \times d}$ since $\|X_{\varepsilon, +}Q\|^2_F = \|X_{\varepsilon, +}\|^2_F$ for any real orthogonal matrix $Q$. In particular, for sufficiently small $\varepsilon_0$, we may assume that $\sup_{\varepsilon \in [0, \varepsilon_0]} \|\xi_d(M_\varepsilon)\|^2_F < \infty$. It is enough to show that for each arbitrary convergent subsequence of $\{\xi_d(M_\varepsilon)\}$,

$$\lim_{\varepsilon \to 0} \xi_d(M_\varepsilon) = \xi_d(M_0).$$

(E.1)

Consider an arbitrary convergent subsequence of $\{\xi_d(M_\varepsilon)\}$. We begin by observing some linear algebraic facts. First, any sequence of real orthogonal matrices has a convergent subsequence whose limit is also real orthogonal. Next, since both $X_{\varepsilon, +}$ and $Z$ are of rank $d$, there exists a unique real orthogonal $d \times d$ matrix $Q_{\varepsilon, +}$ such that

$$\xi_d(M_\varepsilon) = X_{\varepsilon, +}Q_{\varepsilon, +},$$

and in fact, $Q_{\varepsilon, +} = U_{\varepsilon, +}V_{\varepsilon, +}^\top$ where $X_{\varepsilon, +}^\top Z = U_{\varepsilon, +}S_{\varepsilon, +}V_{\varepsilon, +}^\top$ is a singular value decomposition of $X_{\varepsilon, +}^\top Z$, and $U_{\varepsilon, +}V_{\varepsilon, +}^\top$ is the corresponding unique right factor in the polar decomposition of $X_{\varepsilon, +}^\top Z$. Note that this implies the well-definition part of our claim on $\xi_d^\ast$. Also, since $M_\varepsilon \to M_0$, we have that

$$\lim_{\varepsilon \to 0} \sum_{i=1}^d |\Sigma_{\varepsilon, ii} - \Sigma_{0, ii}|^2 \leq \lim_{\varepsilon \to 0} \|M_\varepsilon - M_0\|^2_F = 0.$$

For relevant linear algebra computation details for these facts, see Horn and Johnson [1985, pg. 69, pg. 370, pg. 412, and pg. 431].

Now, by taking a subsequence if necessary, we also have that for some $n \times d$ matrix $U_\ast$ such that $U_\ast^\top U_\ast = I$, $\lim_{n \to \infty} U_{\varepsilon, +} = U_\ast$. Then,

$$\lim_{\varepsilon \to 0} X_{\varepsilon, +} = \lim_{\varepsilon \to 0} U_{\varepsilon, +}\Sigma_{\varepsilon, +}^{1/2} = U_\ast \Sigma_{0, +}^{1/2} \equiv X_\ast.$$

Next, note that if $\Sigma_{0, +}$ has distinct diagonal elements, then we also have $U_\ast = U_{0, +}$ so that $X_\ast = X_{0, +}$. On the other hand, more generally, i.e., even when there are some repeated diagonal elements, we can find a $d \times d$ matrix $Q_\ast$ such that $X_\ast = X_{0, +}Q_\ast$. To see this, note that the $i$-th column of $U_\ast$ is also an eigenvector of $\varrho(M_0)$ for the eigenvalue $\Sigma_{0, +, ii}$, and $U_\ast^\top U_\ast = I$, and hence it follows that for some $d \times d$ real orthogonal matrix $Q_\ast^\top$, we have $U_\ast Q_\ast^\top = U_{0, +}$. Moreover, exploiting the block structure of $\Sigma_{0, +}$ owing to algebraic multiplicity of eigenvalues, we can in fact choose $Q_\ast^\top$ so that $Q_\ast \Sigma_{0, +}^{1/2} = \Sigma_{0, +}^{1/2} Q_\ast$.

Then,

$$X_\ast = U_\ast \Sigma_{0, +}^{1/2} = U_\ast Q_\ast^\top Q_\ast \Sigma_{0, +}^{1/2} = U_{0, +} \Sigma_{0, +}^{1/2} Q_\ast = X_{0, +} Q_\ast.$$

Now, we have

$$\xi_d^\ast(M_0) = X_{0, +}Q_{0, +} = X_\ast Q_\ast^\top Q_{0, +} = \lim_{\varepsilon \to 0} (X_{\varepsilon, +}Q_{\varepsilon, +}Q_{\varepsilon, +}^\top) Q_\ast Q_{0, +} = \lim_{\varepsilon \to 0} \xi_d^\ast(M_\varepsilon)(\lim_{\varepsilon \to 0} Q_{\varepsilon, +}^\top Q_{\varepsilon, +}) Q_{0, +} = \lim_{\varepsilon \to 0} \xi_d^\ast(M_\varepsilon)Q.$$
where $\bar{Q} \equiv (\lim_{\varepsilon \to 0} Q^{\top}_{\varepsilon,+})_{Q_{0,+}}$ is a $d \times d$ real orthogonal matrix and and implicitly the limit was taken along a further subsequence when necessary. Moreover,

$$
\| \lim_{\varepsilon \to 0} \xi^*_d(M_{\varepsilon}) - Z \|_F^2 \geq \| \xi^*_d(M_0) - Z \|_F^2 \\
= \| \lim_{\varepsilon \to 0} \xi^*_d(M_{\varepsilon}) \bar{Q} - Z \|_F^2 \\
= \lim_{\varepsilon \to 0} \| \xi^*_d(M_{\varepsilon}) \bar{Q} - Z \|_F^2 \\
\geq \lim_{\varepsilon \to 0} \| \xi^*_d(M_{\varepsilon}) - Z \|_F^2 \\
= \| \lim_{\varepsilon \to 0} \xi^*_d(M_{\varepsilon}) - Z \|_F^2.
$$

In summary, we have:

$$
\| \lim_{\varepsilon \to 0} \xi_d(M_{\varepsilon}) - Z \|_F^2 = \| \xi^*_d(M) - Z \|_F^2.
$$

By definition of $\xi^*_d(M_0)$, along with the facts that (i) all of the convergent subsequences share the common limit, (ii) each subsequence has a convergent subsequence, and (iii) $X_{0,+}$ and $Z$ have of full column rank $d$, we have (E.1).

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