Close the Gaps: A Learning-while-Doing Algorithm for a Class of Single-Product Revenue Management Problems

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Abstract

In this work, we consider a retailer selling a single product with limited on-hand inventory over a finite selling season. Customer demand arrives according to a Poisson process, the rate of which is influenced by a single action taken by the retailer (such as price adjustment, sales commission, advertisement intensity, etc.). The relation between the action and the demand rate is not known in advance. The retailer will learn the optimal action policy “on the fly” as she maximizes her total expected revenue based on observed demand reactions.

Using the pricing problem as an example, we propose a dynamic “learning-while-doing” algorithm to achieve a near optimal performance. Furthermore, we prove that the convergence rate of our algorithm is almost the fastest among all possible algorithms in terms of asymptotic “regret” (the relative loss comparing to the full information optimal solution). Our result closes the performance gaps between parametric and non-parametric learning and between a post-price mechanism and a customer-bidding mechanism. Important managerial insights from this research are that the value of information on 1) the parametric form of demand function and 2) each customer’s exact reservation price are rather marginal. It also suggests the firms would be better off to perform concurrent dynamic learning and doing, instead of learning-first and doing-second in practice.

1 Introduction

Revenue management is one of the central problems for many industries such as airlines, hotels, and retailers who sell fashion goods. In revenue management problems, the availability of products is often limited in quantity and/or time, and the customer demand behavior is either unknown or uncertain. However, demands can be influenced by actions such as price adjustment, advertisement intensity, sales person compensation, etc. Thus, retailers are interested in finding an optimal action policy to maximize their revenue in such an environment.

Most existing research in revenue management assumes that the functional relationship between demand distribution (or the instantaneous demand rate) and retailers’ actions is known to the decision makers. This relationship is then exploited to derive optimal policies. However, in reality, decision makers seldom possess such information. This is especially true when a new product/service is provided at a new location or the market environment has changed. In light of this, some recent research has proposed learning methods that allow decision makers to learn the demand functions while optimizing their policies based on up-to-date information.

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There are two types of learning models: parametric and non-parametric. In parametric approaches, people assume that prior information has been obtained about which parametric family the demand function belongs to. Decision makers then take actions “on the fly” while updating their beliefs about the underlying parameters. On the other hand, in non-parametric approaches, people do not assume any structural properties of the demand function except some basic regularity conditions. And it is the decision maker’s task to learn the demand curve with very limited information. Intuitively, the non-parametric approaches are harder than the parametric counterparts since the non-parametric function space is much larger than the parametric one. However, the exact difference between these two models is not clear and several questions are to be studied: First, what are the “optimal” learning strategies for each setting? Second, what are the minimal revenue losses that have to be incurred over all possible strategies? Third, how valuable is the information that the demand function belongs to a parametric family? Besides, it seems quite advantageous for the retailer to be able to obtain each customer’s exact valuation rather than only observing a “yes-or-no” purchase decision. But how much value is added?

In this paper, we attempt to provide a complete answer to these questions using a pricing model as example where the retailer’s action is to control the price. The reason we choose the pricing problem is two-fold: First, the pricing problem is well-studied in the literature so that our results can be directly positioned and compared; Second, in the pricing problems, there are two mechanisms, the customer-bidding mechanism where the valuation of each customer is fully revealed, and the post-price mechanism where only a binary information of customer’s purchase decision is observed. Intuitively, the customer-bidding mechanism would be more efficient (given other conditions the same) since it extracts more information from each customer. However, one of the implications of our results indicates that the two mechanisms have the same level of efficiency.

In the pricing problem, a retailer is facing a given initial inventory and a finite selling season. The demand is formulated as a Poisson process whose intensity at each time is controlled by the prevailing price posted by the decision maker. We are interested in the case where the demand function is not known to the decision maker and the information regarding the demand function can only be obtained through observing realized demand. Specifically, we focus ourselves on the non-parametric setting where only some regularity conditions are assumed on the demand functions. We propose a dynamic price learning algorithm for this case and show that our algorithm is near “optimal” in the sense that no pricing policy can achieve a much better performance (which we will precisely define later) than the one generated by our algorithm. Consequently, the optimal learning strategy for the parametric and non-parametric cases are the same and the minimal revenue loss is of the same level.

As discussed in much literature, the key of a good pricing algorithm under demand uncertainty lies in its ability to balance the tension between demand learning (exploration) and near-optimal pricing (exploitation). The more time one spends in price exploration, the less time remains to exploit the knowledge to obtain the optimal revenue. On the other hand, if not enough price exploration is performed, then one may not be able to find a price good enough to achieve satisfactory revenue. This is especially true in the non-parametric setting where it is harder to infer structural information from the observed demand. Previously, researchers proposed price learning algorithms with separated learning and doing phases, where a grid of prices are tested and then the optimal one is used for pricing. Theoretic results are established to show that those algorithms achieve asymptotic optimality at a decent rate, see Besbes and Zeevi [8].

One of our main contributions in this paper is to propose a dynamic price learning algorithm that iteratively performs price experimentation within a shrinking series of intervals that always contain the optimal price (with high probability). We show that this dynamic price learning algorithm achieves better asymptotic revenue (in terms of the regret, which is the relative loss to
the case when the demand function is known exactly, and as problem size grows large) than the grid learning strategy. By showing a worst-case example for all possible policies, we prove that our algorithm provides the near best asymptotic performance over all possible pricing policies. To our best knowledge, this is the first time such an algorithm is proposed and analyzed. This result suggests that we should not separate price experimentation and exploitation, instead, we should combine “learning” and “doing” in a concurrent procedure, which might be of interest to revenue management practitioners.

In more detail, we summarize our contribution of this work in the following:

1. Under some mild regularity conditions on the demand function, we show that our pricing policy achieves a regret arbitrarily close to $O(n^{-1/2})$ (in terms of the order of $n$), uniformly across all possible demand functions. This result improves the best-known bound (of the asymptotic regret) by Besbes and Zeevi [8] for both the non-parametric learning (the best known bound was $O(n^{-1/3})$) and the parametric learning (the best known bound was $O(n^{-1/3})$ except for the case with only one parameter) in this context. Thus, it closes the efficiency gap between parametric and non-parametric learning in this setting. It implies that there is no additional cost associated with performing non-parametric learning, in terms of asymptotic regret. Therefore, our study suggests that firms could save time and effort on checking which class of parametric functions the demand belongs to and collecting data for curve fitting, which is widely used in practice.

2. Our result also closes the gap between two revenue management mechanisms: the customer-bidding mechanism and the post-price mechanism. In Agrawal et al [2], the authors obtained a dynamic learning algorithm with $O(n^{-1/2})$ regret under the former mechanism (in slightly different setting). However, under the post-price model, the previous best algorithm by Besbes and Zeevi [8] achieves a regret of $O(n^{-1/4})$. Our result asserts that although post-price mechanism extracts much less information from each individual customer’s valuation of the product, it can achieve the same order of asymptotic behavior as that in the customer-bidding mechanism. Therefore, our result reassures the usage of the post-price mechanism, which is more widespread in practice.

3. On the methodology side, our algorithm provides a new form of a dynamic learning method. In particular, we do not separate the “learning” and “doing” phases; instead, we integrate “learning” and “doing” together by considering a shrinking series of price intervals. This concurrent dynamic is actually the key to achieve a perfect balance between price exploration and exploitation, and thus achieve the near maximum efficiency in pricing. We believe that this method may be applied to problems with even more complex structure.

The rest of our paper is organized as follows: In Section 2, we review related literature in this field. In Section 3, we introduce our model and state our main assumptions. In Section 4, we present our dynamic price learning algorithm and the main theorems. We provide a sketch proof for our algorithm in Section 5. In Section 6, we show a lower bound of regret for all possible pricing policies. We show some numerical results in Section 7 and some extensions of this model in Section 8. We conclude this paper in Section 9. An Appendix is then followed for the detailed proofs for our technical results.

## 2 Literature Review

Pricing mechanisms have been an important research area in revenue management and there is abundant literature on this topic. For a comprehensive review on this subject, we refer the readers to Bitran and Caldentey [10], Elmaghraby and Keskinocak [14] and Talluri and van Ryzin [24]. Previously, research has mostly focused on the cases where the functional relationship between the price and demand (also known as the demand function) is given to the
decision maker. Gallego and van Ryzin [16] present a foundational work in such setting where the structure of the demand function is exploited and the optimal pricing policies are analyzed.

Although knowing the exact demand function is convenient for analysis, the decision makers in practice do not usually have such information. Therefore, much recent literature addresses the dynamic pricing problems under demand uncertainty. The majority of these work take the parametric approach, e.g., Lobo and Boyd [21], Bertsimas and Parekis [7], Araman and Caldentey [3], Avin and Pazgal [5], Carvalho and Puterman [13], Farias and Van Roy [15], Broder and Rusmevichientong [12] and Harrison et al [17]. Typically in these works, a prior knowledge of the parameters is assumed and a dynamic program with Bayesian updating of the parameters is formulated. Although such approach simplifies the problem to some degree, the restriction to a certain demand function family may incur model misspecification risk. As shown in Besbes and Zeevi [8], misspecifying the demand family may lead to revenue far away from the optimal. In such case, a non-parametric approach would be preferred since it does not commit to any family of demand function.

The main difficulty facing the non-parametric approach is its tractability and efficiency. And most research revolves around this idea. Several studies consider a model that the customers are chosen adversarially, e.g. Ball and Queyranne [6] and Perakis and Roels [22]. However, their models take a relatively conservative approach and no learning is involved. Rusmevichientong et al [23] consider static learning using historic data with no dynamic decision being made. In another paper by Lim and Shanthikumar [20], they consider dynamic pricing strategies that are robust to an adversarial at every point in the stochastic process. Again, this approach is quite conservative and the main theme is about robustness rather than demand learning.

The work that is closest to this paper is that of Besbes and Zeevi [8] where the authors considered demand learning in both parametric and non-parametric case. They proposed learning algorithms for both cases and showed that there is a gap in performance between them. They also provided a lower bound for the revenue loss in both cases. In this paper, we continue their work by improving the bound for both cases and closing the gap between them. In particular, they considered algorithms with separated learning and doing phases where price experimentation is performed exclusively during the learning phase (except the parametric case with single parameter). In our paper, the learning and doing is dynamically integrated: we keep shrinking a price interval that contains the optimal price and keep learning until we guarantee that the revenue achieved by applying the current price is near-optimal. Although our setting resembles theirs, our algorithm is quite different and the results are stronger.

Another paper that is related to ours is Kleinberg and Leighton [18]. In [18], they considered the online post-price auction with unlimited supply. They showed lower bounds for the revenue loss for three cases: 1) each customer has a same (deterministic) but unknown valuation, 2) each customer has an i.i.d. unknown valuation, and 3) the valuations of customers are chosen adversarially to the algorithm. They also provided algorithms that match these three lower bounds for each case. Specifically, for the case where each customer has an i.i.d. valuation, they presented an algorithm with the same level of regrets as ours. However, their model is different from ours in several ways. First, they considered an unconstrained revenue maximization problem (without inventory constraint) while we consider a constrained problem (with an inventory constraint). Second, they considered a discrete-time arrival model while we consider a continuous-time Poisson arrival model. As we will see in our algorithm, these differences are nontrivial and our analysis is fundamentally different from theirs.

Other related literatures that focus on the exploration-exploitation trade-off in sequential optimization under uncertainty are from the study of the multi-armed bandit problem: see Lai and Robbins [19], Agrawal [1] and Auer et al [4] and references therein. Although our study shares similarity in ideas with the multi-armed bandit problem, we consider a problem with continuous learning horizon (the time is continuous), continuous learning space (the possible demand function is continuous) and continuous action space (the price set is continuous). These
features in addition to the presence of inventory constraint make our algorithm and analysis quite different from theirs.

3 Problem Formulation

3.1 Basic Model and Assumptions

In this paper, we consider the problem of a monopolist selling a single product in a finite selling season \( T \). The seller has a fixed inventory \( x \) at the beginning and no recourse actions on the inventory can be made during the selling season. During the selling season, demand of this product arrives according to a Poisson process with intensity at time \( t \) being \( \lambda_t \) where \( \lambda_t \) is the instantaneous demand rate at time \( t \). In our model, we assume that \( \lambda_t \) is solely determined by the price offered at time \( t \), that is, we can write \( \lambda_t = \lambda(p(t)) \) as a function of \( p(t) \). At time \( T \), the sales will be terminated and there is no salvage cost for the remaining items (As shown in [16], the zero salvage cost assumption is without loss of generality)

We assume the feasible set of prices is an interval \([p, \bar{p}]\) with an addition cut-off price \( p_\infty \) such that \( \lambda(p_\infty) = 0 \). Regarding the demand rate function \( \lambda(p) \), we assume it is decreasing in \( p \), has an inverse function \( p = \gamma(\lambda) \), and the revenue rate function \( r(\lambda) = \lambda \gamma(\lambda) \) is concave in \( \lambda \). These assumptions are quite standard and such demand functions bear the name of “regular” demand function as defined in [16].

Besides being regular, we also make the following assumptions on the demand rate function \( \lambda(p) \) and the revenue rate function \( r(\lambda) \):

**Assumption A.** For some positive constants \( M, K, m_L \) and \( m_U \),

1. Boundedness: \( |\lambda(p)| \leq M \) for all \( p \in [p, \bar{p}] \);
2. Lipschitz continuity: \( \lambda(p) \) and \( r(\lambda(p)) \) are Lipschitz continuous with respect to \( p \) with factor \( K \). Also, the inverse demand function \( p = \gamma(\lambda) \) is Lipschitz continuous in \( \lambda \) with factor \( K \);
3. Strict concavity and differentiability: \( r''(\lambda) \) exists and \( -m_L \leq r''(\lambda) \leq -m_U < 0 \) for all \( \lambda \) in the range of \( \lambda(p) \) for \( p \in [p, \bar{p}] \).

In the following, let \( \Gamma = \Gamma(M, K, m_L, m_U) \) denote the set of demand functions satisfying the above assumptions with the corresponding coefficients. We briefly illustrate these assumptions as follows: The first assumption is just an upper bound on the demand rate. The second assumption says that when we change the price by a small amount, the demand and revenue rate will not change by too much, also the demand function does not have a “flat” period. These two assumptions are quite standard as they appear in most literature in revenue management with demand learning, e.g., in [16], [8] and [12]. The last assumption contains two parts, one being the smoothness of the revenue function, the other being strict concavity. The assumption on the existence of second derivatives is also made by [12] and [18], and the strict concavity assumption is made by [18]. And as shown in Appendix 10.1 our assumptions hold for several classes of commonly-used demand functions (e.g., linear, exponential, and logit demand functions).

In our model, we assume that the seller does not know the true demand function \( \lambda \), the only knowledge she has about \( \lambda \) is that it belongs to \( \Gamma \). Note that \( \Gamma \) doesn’t need to have any parametric representation. Therefore, our model is robust in terms of the choice of the demand function family.

3.2 Performance Analysis

To evaluate the performance of any pricing algorithm, we adopt the minimal regret objective formalized in [8]. Consider a pricing policy \( \pi \). At each time \( t \), \( \pi \) maps all the history price
and realized demand information into a current price $p(t)$. By our assumption that the demand follows a Poisson process, the cumulative demand up to time $t$ can be written as follows:

$$N^\pi(t) = N\left(\int_0^t \lambda(p(s))ds\right)$$  \hspace{1cm} (1)

where $N(\cdot)$ is a unit-rate Poisson process. In order to satisfy the inventory constraint, any admissible policy $\pi$ must satisfy:

$$\int_0^T dN^\pi(s) \leq x$$  \hspace{1cm} (2)

We denote the set of policy satisfying (2) by $\mathcal{P}$. Note the seller can always set the price to $p_\infty$, thus constraint (2) can always be met. The expected revenue generated by a policy $\pi$ is as follows:

$$J^\pi(x,T; \lambda) = E\left[\int_0^T p(s)dN^\pi(s)\right].$$  \hspace{1cm} (3)

Here, the presence of $\lambda$ in (3) means that the expectation is taken under the demand function $\lambda$. Given a demand function $\lambda$, we wish to find the optimal policy $\pi^*$ that maximizes the expected revenue (3) while subjected to the inventory constraint (2). In our model, since we don’t have perfect information on $\lambda$, we seek a pricing policy $\pi$ that performs as close to $\pi^*$ as possible.

However, even if the demand function $\lambda$ is known, computing the expected value of the optimal policy is hard. It involves solving a Bellman equation resulting from a dynamic program. Fortunately, as shown in many previous literatures $[8], [16]$, we can obtain an upper bound for the expected value for any policy via considering a full-information deterministic optimization problem. Define:

$$J^D(x,T; \lambda) = \sup_{\pi \in \mathcal{P}} \int_0^T r(\lambda(p(s)))ds$$  \hspace{1cm} s.t. \hspace{1cm} \int_0^T \lambda(p(s))ds \leq x$$  \hspace{1cm} (4)

In (4) all the stochastic processes are substituted by their means. In $[8]$, the authors showed that $J^D(x,T; \lambda)$ provides an upper bound on the expected revenue generated by any admissible pricing policy $\pi$, that is, $J^\pi(x,T; \lambda) \leq J^D(x,T; \lambda)$, for all $\lambda \in \Gamma$ and $\pi \in \mathcal{P}$. With this useful relaxation, we can define the regret $R^\pi(x,T; \lambda)$ for any given demand function $\lambda \in \Gamma$ and policy $\pi \in \mathcal{P}$ to be

$$R^\pi(x,T; \lambda) = 1 - \frac{J^\pi(x,T; \lambda)}{J^D(x,T; \lambda)}.$$  \hspace{1cm} (5)

As we mentioned above, the deterministic optimal solution $J^D(x,T; \lambda)$ provides an upper bound of the expected value of any policy $\pi$, therefore $R^\pi(x,T; \lambda)$ is always greater than 0. And by definition, the smaller the regret, the closer $\pi$ is to the optimal policy. However, since the decision maker does not know the true demand function, it is attractive to obtain a pricing policy $\pi$ that achieves small regrets across all the underlying demand function $\lambda \in \Gamma$. In particular, we want to consider the “worst-case” regret, where the decision maker chooses a pricing policy $\pi$, and the nature picks the worst possible demand function for that policy:

$$\sup_{\lambda \in \Gamma} R^\pi(x,T; \lambda).$$  \hspace{1cm} (6)

Obviously, the seller wants to minimize the worst-case regret, i.e., we are interested in solving:

$$\inf_{\pi \in \mathcal{P}} \sup_{\lambda \in \Gamma} R^\pi(x,T; \lambda).$$  \hspace{1cm} (7)
Now our objective is clear. However, it is hard to evaluate (7) for any single problem. Therefore, in this work, we adopt the widely-used asymptotic performance analysis (the regime of high volume of sales). We consider a regime in which both the size of the initial inventory, as well as the potential demand, grow proportionally large. In particular, for a market of size \( n \), where \( n \) is a positive integer, the initial inventory and the demand function are given by

\[
x_n = nx\quad \text{and}\quad \lambda_n(\cdot) = n\lambda(\cdot).
\]

Denote \( J^D_n(x,T;\lambda) \) to be the deterministic optimal solution for the problem with size \( n \); it is easy to see that \( J^D_n = nJ^D_1 \). We also define \( J^\pi_n(x,T;\lambda) \) to be the expected value of a pricing algorithm \( \pi \) when it is applied to a problem with size \( n \). Therefore, we can define the regret for the size-\( n \) problem \( R^\pi_n(x,T;\lambda) \)

\[
R^\pi_n(x,T;\lambda) = 1 - \frac{J^\pi_n(x,T;\lambda)}{J^D_n(x,T;\lambda)},
\]

and our objective is to study the asymptotic behavior of \( R^\pi_n(x,T;\lambda) \) as \( n \) grows large and design an algorithm with small asymptotic regret.

4 Main Results: A Dynamic Pricing Algorithm

In this section, we introduce our main results: an optimal dynamic pricing algorithm. Before we state our theorems, it is useful to introduce some basic structural intuition of this problem.

4.1 Structural Insights

Consider the full-information deterministic problem \([4]\). As shown in \([8]\), the optimal solution to \([4]\) is given by

\[
p(t) = p^D = \max\{p^u, p^c\}
\]

where

\[
p^u = \arg\max_{p \in [p_u, p_c]} \{r(\lambda(p))\},
\]

\[
p^c = \arg\min_{p \in [p_u, p_c]} |\lambda(p) - \frac{x}{T}|.
\]

Here, the superscript \( u \) stands for “unconstrained” and superscript \( c \) stands for “constrained”. As shown in \([10]\), the optimal price is either the revenue maximizing price, or the inventory depleting price, whichever is larger. It is shown in \([16]\) that if one knows \( p^D \), then the revenue collected by using a fixed price \( p^D \) will be close to the deterministic optimal solution. Therefore, the goal of our algorithm will be to learn an estimate \( p^D \) close enough to the true one, using empirical observations at hand. We make one technical assumption about the value of \( p^D \) as follows.

**Assumption B.** There exists \( \epsilon > 0 \), such that \( p^D \in [p^D - \epsilon, p^D + \epsilon] \) for all \( \lambda \in \Gamma \).

Assumption B says that we require the optimal deterministic price to be in the interior of the price interval. This assumption is mainly for the purpose of analysis and is quite general, since one can always choose a large interval of \([p, \bar{p}]\) to start with.

4.2 An Optimal Dynamic Pricing Algorithm

We first state our main result as follows:
Theorem 1. Let Assumptions A and B hold for $\Gamma = \Gamma(M,K,m,L,m,U)$ and a fixed $\epsilon$. Then for any $\delta < 1/2$, there exists a policy $\pi_\delta \in \mathcal{P}$ generated by Algorithm DPA, such that for all $n \geq 1$,

$$\sup_{\lambda \in \Gamma} R_n^{\pi_\delta}(x,T;\lambda) \leq \frac{C \sqrt{\log n}}{n^{3.5}}$$

for some constant $C$.

Here $C$ only depends on $M,K,m,L,m,U,\epsilon$, the initial inventory $x$ and the length of time horizon $T$. However, the exact dependence is quite complex and thus omitted. A corollary of Theorem 1 follows from the relationship between the non-parametric model and the parametric one:

Corollary 1. Assume $\Gamma$ is a parameterized demand function family satisfying Assumption A and B for some coefficients. Then for any $\delta < 1/2$, there exists a policy $\pi_\delta \in \mathcal{P}$ generated by Algorithm DPA, such that for all $n \geq 1$,

$$\sup_{\lambda \in \Gamma} R_n^{\pi_\delta}(x,T;\lambda) \leq \frac{C \sqrt{\log n}}{n^{3.5}}$$

for some constant $C$.

We also establish a lower bound of asymptotic regret for any possible pricing policies:

Theorem 2. There exists a set of demand functions $\Gamma$ parameterized by a single parameter satisfying Assumption A and B with certain coefficients, such that for any admissible pricing policy $\pi$, for all $n \geq 1$

$$\sup_{\lambda \in \Gamma} R_n^{\pi}(x,T;\lambda) \geq \frac{C}{\sqrt{n}}$$

for some constant $C$ that only depends on the coefficients in $\Gamma, x$ and $T$.

Theorems 1 and 2 together provide a clear answer to the magnitude of regret the best pricing policy can achieve, under both parametric and non-parametric setting.

Now we describe our algorithm. As we mentioned in Subsection 4.1, we aim to learn $p^D$ through an iterative price experimentation. Specifically, our algorithm will be able to distinguish whether “$p^u$” or “$p^c$” is optimal. Meanwhile we keep shrinking an interval containing the optimal price until a certain accuracy is achieved.

Now we present our dynamic pricing algorithm. Explanations and illustration of the algorithm will follow right after.

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Algorithm DPA (Dynamic Pricing Algorithm):

Step 1. Initialization:

(a) Consider a sequence of $\tau^u_i, \kappa^u_i, i = 1, 2, ..., N^u$ and $\tau^c_i, \kappa^c_i, i = 1, 2, ..., N^c$ ($\tau$ and $\kappa$ represent the time of each learning period and the number of different prices considered in each learning period, respectively. Their values along with the value of $N^u$ and $N^c$ will be defined later). Define $p^u_0 = p^c_0 = p$ and $p^u_1 = p^c_1 = p$. Define $t^u_i = \sum_{j=1}^{i} \tau^u_j$, for $i = 0$ to $N^u$ and $t^c_i = \sum_{j=1}^{i} \tau^c_j$, for $i = 0$ to $N^c$;

Step 2. Learn $p^u$ or determine $p^c > p^u$: For $i = 1$ to $N^u$ do
(a) Divide \([p_u^i, \overline{p}_u^i]\) into \(\kappa^i_u\) equally spaced intervals and let \(\{p_{u,i,j}^i, j = 1, 2, ..., \kappa^i_u\}\) be the left endpoints of these intervals.

(b) Divide the time interval \([t_{u,i-1}^i, t_{u,i}^i]\) into \(\kappa^i_u\) equal parts and define

\[
\Delta^i_u = \tau^i_u \frac{t_{u,i}^i}{\kappa^i_u}, \quad t^i_{u,j} = t_{u,i-1}^i + j\Delta^i_u, \quad j = 0, 1, ..., \kappa^i_u
\]

(c) Apply \(p_{u,i,j}^i\) from time \(t_{u,i-1}^i\) to \(t^i_{u,j}\), as long as the inventory is still positive. If no more units are in stock, apply \(p_\infty\) until time \(T\) and STOP.

(d) Compute

\[
\hat{d}(p_{u,i,j}^i) = \text{total demand over } [t_{u,i-1}^i, t_{u,i,j}^i), \quad j = 1, ..., \kappa^i_u
\]

(e) Compute

\[
\hat{p}_i^u = \arg \max_{1 \leq j \leq \kappa^i_u} \{p_{u,i,j}^i \hat{d}(p_{u,i,j}^i)\}
\]

and

\[
\hat{p}_i^c = \arg \min_{1 \leq j \leq \kappa^i_u} |\hat{d}(p_{u,i,j}^i) - x/T|
\]

(f) If

\[
\hat{p}_i^c > \hat{p}_i^u + 2\sqrt{\log n} \cdot \frac{\overline{p}_i^u - \underline{p}_i^u}{\kappa^i_u} \quad (16)
\]

then Break from Step 2, Enter Step 3 and denote this \(i\) to be \(i_0\); Else, set \(\hat{p}_i = \max\{\hat{p}_i^u, \hat{p}_i^c\}\). Define

\[
\overline{p}_{i+1}^u = \hat{p}_i - \frac{\log n}{3} \cdot \frac{\overline{p}_i^u - \underline{p}_i^u}{\kappa^i_u} \quad (17)
\]

and

\[
\overline{p}_{i+1}^c = \hat{p}_i + \frac{2\log n}{3} \cdot \frac{\overline{p}_i^u - \underline{p}_i^u}{\kappa^i_u} \quad (18)
\]

And define the price range for the next iteration

\[
I_{i+1}^u = [\underline{p}_{i+1}^u, \overline{p}_{i+1}^u]
\]

Here we truncate the interval if it doesn’t lie inside the feasible set \([\underline{p}, \overline{p}]\).

(g) If \(i = N_u\), then Enter Step 4(a);

**Step 3. Learn** \(p^c\) **when** \(p^c > p^u\):

For \(i = 1\) to \(N^c\) do

(a) Divide \([\underline{p}_c^i, \overline{p}_c^i]\) into \(\kappa^c_i\) equally spaced intervals and let \(\{p_{c,i,j}^c, j = 1, 2, ..., \kappa^c_i\}\) be the left endpoints of these intervals.

(b) Define

\[
\Delta^c_i = \tau^c_i \frac{t_{c,i}^c}{\kappa^c_i}, \quad t_{c,i,j}^c = t_{c,i-1}^c + j\Delta^c_i + t_{i_0}, \quad j = 0, 1, ..., \kappa^c_i
\]

(c) Apply \(p_{c,i,j}^c\) from time \(t_{c,i-1}^c\) to \(t_{c,i,j}^c\), as long as the inventory is still positive. If no more units are in stock, apply \(p_\infty\) until time \(T\) and STOP.
(d) Compute  
\[ \hat{d}(p^c_{i,j}) = \frac{\text{total demand over } [t^c_{i,j-1}, t^c_{i,j})}{\Delta^c_i}, \quad j = 1, ..., \kappa^c_i \]

(e) Compute  
\[ \hat{q}_i = \arg \min_{1 \leq j \leq \kappa^c_i} |\hat{d}(p^c_{i,j}) - x/T| \]

Define  
\[ p^c_{i+1} = \hat{q}_i - \frac{\log n}{2} \cdot \frac{\hat{p}^c_i - p^c_i}{\kappa^c_i} \quad (19) \]

and  
\[ p^c_{i+1} = \hat{q}_i + \frac{\log n}{2} \cdot \frac{\hat{p}^c_i - p^c_i}{\kappa^c_i} \quad (20) \]

And define the price range for the next iteration  
\[ I^c_{i+1} = [p^c_{i+1}, \bar{p}^c_{i+1}] \]

Here we truncate the interval if it doesn’t lie inside the feasible set of \([\bar{p}, \bar{p}]\)

(f) If \(i = N^c\); then enter Step 4(b);

**Step 4. Apply the learned price:**

(a) Define \( \tilde{p} = \bar{p}_{N^c} + 2\sqrt{\log n} \cdot \hat{p}^c_{N^c} - \bar{p}^c_{N^c} \). Use \( \tilde{p} \) for the rest of the selling season until the stock runs out;

(b) Define \( \tilde{q} = \hat{q}_{N^c} \). Use \( \tilde{q} \) for the rest of the selling season until the stock runs out.

Now we explain this algorithm before we proceed to proofs. The idea of this algorithm is to divide the time interval into pieces, and in each piece, we test a grid of prices on a price interval. We find the empirical optimal price, then shrink the price interval to a smaller subinterval that still contains the optimal price (with high probability), and enter the next time interval with the smaller price range. We repeat the shrinking procedure until the price interval is small enough so that the desired accuracy is achieved.

Recall that the optimal deterministic price \( p^D \) is equal to the maximum of \( p^u \) and \( p^c \), where \( p^u \) and \( p^c \) are solved from \((11)\) and \((12)\) respectively. It turns out that \((11)\) and \((12)\) have quite different local behaviors around its optimal solution under our assumptions: \((11)\) resembles a quadratic function while \((12)\) resembles a linear function. This difference requires us to have different shrinking strategies for the case when \( p^u > p^c \) and \( p^c > p^u \). This is why we have two learning steps (Step 2 and 3) in our algorithm. Specifically, in Step 2, the algorithm works by shrinking the price interval until either a transition condition \((16)\) is triggered or the learning phase is terminated. As will be shown later, when the transition condition \((16)\) is triggered, with high probability, we are certain that the optimal solution to the deterministic problem is \( p^c \). Otherwise, if we terminate learning before the condition is triggered, we know that \( p^u \) is either the optimal solution to the deterministic problem or it is close enough so that using \( p^u \) will also give us a near-optimal revenue. When the transition condition \((16)\) happens, we switch to Step 3, where we use a new set of shrinking and price testing parameters. Note that in Step 3, we start from the initial price interval rather than the current interval obtained. This is not necessary but for the ease of analysis. Both Step 2 and Step 3 (if it is invoked) must terminate in a finite number of iterations (we prove this in Lemma \([1]\)).

After this learning-while-doing period ends, a fixed price is used for the remaining selling season (Step 4) until the inventory runs out. To help illustration, a high-level description of the algorithm is shown below. One thing to note is that the “next” intervals defined in \((17)\)
and (18) are not symmetric. Similarly in Step 4(a), we use an adjusted price for the remaining selling season. This adjustment is a trick to make sure that the inventory consumption can be adequately upper bounded. Meanwhile the adjustment is small enough so that the revenue is maintained. It is based on the different local behaviors of the revenue rate function and the demand rate function. The detailed reasoning of this adjustment will be clearly illustrated in Lemma 11.

High-level description of the Dynamic Price Learning Algorithm:

Step 1. Initialization:

(a) Initialize the time length and price granularity for each learning period. Set the maximum number of iterations $N^u$ and $N^c$ in Step 2 and 3;

Step 2. Learn $p^u$ or determine $p^c > p^u$:

(a) Set the initial price interval to be $[p, \bar{p}]$

(b) Test a grid of prices on the current price interval for a predetermined length of time, observe the demand for each price

(c) Compute the empirical optimal $p^c$ and $p^u$ using the observed demand

(d) If $p^c$ is “significantly” greater than $p^u$, then enter Step 3; otherwise, shrink the current interval to a subinterval containing the empirical optimal $p^D$

(e) Repeat (b)-(d) for $N^u$ times and then enter Step 4(a);

Step 3. Learn $p^c$ when $p^c > p^u$:

(a) Set the initial price interval to be $[p, \bar{p}]$

(b) Test a grid of prices on the current price interval for a predetermined length of time, observe the demand for each price

(c) Compute the empirical optimal $p^c$ using the observed demand

(d) Shrink the current interval to a subinterval containing the empirical optimal $p^c$

(e) Repeat (b)-(d) for $N^c$ times and then enter Step 4(b);

Step 4. Apply the learned price:

(a) Apply the last price in Step 2 until the stock runs out or the selling season finishes

(b) Apply the last price in Step 3 until the stock runs out or the selling season finishes.

In the following, without loss of generality, we assume $T = 1$ and $\bar{p} - p = 1$. Now we define $\tau_i^u, \kappa_i^u, N^u, \tau_i^c, \kappa_i^c$ and $N^c$. We first show a set of equations we want $(\tau_i^u, \kappa_i^u)$ and $(\tau_i^c, \kappa_i^c)$ to satisfy. Then we explain the meaning of each equation and solve those equations. Finally we will prove our main theorem on the asymptotic performance of our algorithm.
Now we state the set of equations we want $\tau^u_i$ and $\kappa^u_i$ to satisfy

\[ \left( \frac{p^u_i - p^u_{i+1}}{\kappa^u_i} \right)^2 \sim \sqrt{\frac{\kappa^u_i}{n\tau^u_i}}, \forall i = 1, ..., N^u \] (21)

\[ p^u_i - p^u_{i+1} \sim \log n \cdot \frac{p^u_i - p^u_{i+1}}{\kappa^u_i}, \forall i = 1, ..., N^u - 1 \] (22)

\[ \tau^u_{i+1} \cdot \left( \frac{p^u_i - p^u_{i+1}}{\kappa^u_i} \right)^2 \cdot \sqrt{\log n} \sim \tau^u_i, \forall i = 1, ..., N^u - 1 \] (23)

Also we define

\[ N^u = \min_l \{ l | \left( \frac{p^u_l - p^u_1}{\kappa^u_l} \right)^2 \sqrt{\log n} < \tau^u_1 \} \] (24)

We then state the set of equations we want $\tau^c_i$ and $\kappa^c_i$ to satisfy:

\[ \frac{p^c_i - p^c_{i+1}}{\kappa^c_i} \sim \sqrt{\frac{\kappa^c_i}{n\tau^c_i}}, \forall i = 1, ..., N^c \] (25)

\[ p^c_i - p^c_{i+1} \sim \log n \cdot \frac{p^c_i - p^c_{i+1}}{\kappa^c_i}, \forall i = 1, ..., N^c - 1 \] (26)

\[ \tau^c_{i+1} \cdot \left( \frac{p^c_i - p^c_{i+1}}{\kappa^c_i} \right)^2 \cdot \sqrt{\log n} \sim \tau^c_i, \forall i = 1, ..., N^c - 1 \] (27)

Also we define

\[ N^c = \min_l \{ l | \left( \frac{p^c_l - p^c_1}{\kappa^c_l} \right)^2 \sqrt{\log n} < \tau^c_1 \} \] (28)

Before we explain these desired relationships, let us first examine where the revenue loss comes from in this algorithm. First, in each period, there is a so-called exploration bias, that is, the prices tested in each period may deviate from the optimal price, resulting in suboptimal revenue rate or suboptimal demand rate. These deviations multiplied by the time length of each period will be the “loss” for that period. Second, since we only explore a grid of prices, there is also a deterministic error associated with it. Thirdly, since the demand is essentially a stochastic process, the observed demand rate may deviate from the true demand rate, resulting in a stochastic error. Note that these three errors also exist in the learning algorithm proposed in [8]. However, in our dynamic learning case, each error does not simply appear once. For example, the deterministic error and stochastic error in one period may have impact on all the future periods. Thus, the design of our algorithm will revolve around the idea of balancing these errors in each step to achieve the maximum efficiency of learning. With this in mind, we explain the meaning of each equation above in the following:

- The first equation (21) (25), resp.) balances the deterministic error induced by only considering the grid points (in which the grid granularity is $\frac{p^u_i - p^u_{i+1}}{\kappa^u_i}$ (resp.)) and the stochastic error induced in the learning period which is $\frac{\kappa^u_i}{n\tau^u_i}$ (resp.). These two terms determine the price deviation in the next period and thus the exploration error of next period. We will show that under our assumptions, the loss due to price granularity is quadratic in Step 2, and linear in Step 3. We balance these two errors to achieve the maximum efficiency in learning.

\[ \sim \] Here $f \sim g$ means that $f$ and $g$ are of the same order of $n$
• The second equation \( \text{(22)} \) (\( \text{(26)} \), resp.) is used to make sure that with high probability, our learning interval \( I^u_i \) (\( I^c_i \), resp.) contains the optimal price \( p^D \). We have to guarantee that \( I^u_i \) (\( I^c_i \), resp.) contains \( p^D \), otherwise when we miss the optimal price, we will incur a constant exploration error in all periods afterwards. This relationship is actually given in the algorithm (see \( \text{(17)} \), \( \text{(18)} \), \( \text{(19)} \) and \( \text{(20)} \)). However, we include them here for the sake of completeness.

• The third equation \( \text{(23)} \) (\( \text{(27)} \), resp.) is used to bound the exploration error for each learning period. This is done by considering the multiplication of the revenue rate deviation (also demand rate deviation) and the length of the learning period, which in our case is \( \tau_{i+1} \sqrt{\log n} \cdot \left( \frac{p^u_i - p^u_i}{\sqrt{\kappa_i}} \right)^2 \) (\( \tau_{i+1} \sqrt{\log n} \cdot \left( \frac{p^c_i - p^c_i}{\kappa_i} \right)^2 \), resp.). We want this loss to be on the same order for each learning period (thus all equal to the loss in the first learning period, which is \( \tau_1 \)) to achieve the maximum efficiency of learning.

• The fourth equation \( \text{(24)} \) (\( \text{(28)} \), resp.) determines if the price we obtain is guaranteed to be close enough to optimal such that we can apply this price in the remaining selling season. We show that \( \sqrt{\log n} \cdot \left( \frac{p^u_i - p^u_i}{\sqrt{\kappa_i}} \right)^2 \) (\( \sqrt{\log n} \cdot \left( \frac{p^c_i - p^c_i}{\kappa_i} \right)^2 \), resp.) is the revenue rate and demand rate deviation of price \( \hat{p}_i \). When this is less than \( \tau_1 \), we can simply apply \( \hat{p}_i \) and the loss will not exceed the loss of the first learning period.

Now we solve for \( \kappa_i^u \) and \( \tau_i^u \) from the relations defined above. Define \( \tau_1^u = n^{-\delta} \cdot (\log n)^{3.5} \), one can solve \( \text{(21)} \), \( \text{(22)} \) and \( \text{(23)} \) and get:

\[
\kappa_i^u = n^{\frac{1}{2}(\frac{3}{2})^{i-1}(1-\delta)} \cdot \log n, \quad \forall i = 1, 2, ..., N^u
\]

\[
\tau_i^u = n^{1-2\delta-(\frac{5}{2})^{i-1}} \cdot (\log n)^5, \quad \forall i = 1, 2, ..., N^u
\]

And as a by-product, we have

\[
\hat{p}_i^u - p_i^u = n^{-\frac{1}{2}(1-(\frac{3}{2})^{i-1})}, \quad \forall i = 1, 2, ..., N^u
\]

Next we do a similar computation for \( \kappa_i^c \) and \( \tau_i^c \). Define \( \tau_1^c = n^{-\delta} \cdot (\log n)^{2.5} \). We have the following results:

\[
\kappa_i^c = n^{\frac{1}{2}(\frac{3}{2})^{i-1}(1-\delta)} \cdot \log n, \quad \forall i = 1, 2, ..., N^c
\]

\[
\tau_i^c = n^{1-2\delta-(\frac{3}{2})^{i-1}} \cdot (\log n)^3, \quad \forall i = 1, 2, ..., N^c
\]

and

\[
\hat{p}_i^c - p_i^c = n^{-\delta(1-(\frac{3}{2})^{i-1}), \quad \forall i = 1, ..., N^c
\]

5 Proof Outlines for Theorem 1

In this section, we give an outline of the proof of Theorem 1. We put most of the detailed proof in the Appendix, only the main steps and ideas are presented in this section.

As the first step of our proof, we show that given \( \delta < 1/2 \), our algorithm will stop within a finite number of iterations. Also the number of iterations is uniformly bounded. We have the following lemma:

**Lemma 1.** Fix \( \delta < 1/2 \). \( N^u \) and \( N^c \) defined in \( \text{(24)} \) and \( \text{(28)} \) exist. Moreover, there exists an \( N_\delta \) independent of \( n \) such that \( N^u \) and \( N^c \) are both bounded by \( N_\delta \).


Proof. See Appendix 10.3 □

Although Lemma 3 is simple, it is important since it allows us to treat the number of iterations of our algorithm as constant. In much of our analysis, we frequently need to take a union bound over the number of iterations, and Lemma 3 justifies such analysis. In our algorithm, it is important to make sure that the deterministic optimal price $p^D$ is always contained in our price interval. This is because when we miss the deterministic optimal price, we will incur a constant loss for all periods afterwards, and thus the algorithm can not achieve asymptotic optimality. The next lemma will show exactly such behavior of our algorithm.

Lemma 2. Assume $p^D$ is the optimal price for the deterministic problem and Assumption A and B hold for $\Gamma = \Gamma(M, K, m_L, m_U)$ and $\epsilon > 0$. Then with probability $1 - O\left(\frac{1}{n}\right)$,

- If we never enter Step 3, then $p^D \in I^u_i$ for all $i = 1, 2, \ldots, N^u$.
- If Step 2 stops at $i_0$ and the algorithm enters Step 3, then $p^D \in I^u_i$ for all $i = 1, 2, \ldots, i_0$ and $p^D \in I^u_j$ for all $j = 1, 2, \ldots, N^c$

Proof. Here we give a sketch of proof for the first part of this lemma. The detailed proof is given in Appendix 10.4.

We prove by induction on $i$. Assume $p^D \in I^u_i$. We consider the ($i+1$)th iteration. Define $u^i_n = \log n \cdot \max\left\{\left(\frac{p_i^u - p_i^c}{n^c}\right)^2, \sqrt{\frac{n^c}{ni}}\right\}$

((35))

Denote the unconstrained and constrained optimal solutions on the current interval to be $p_i^u$ and $p_i^c$. We can show (the details are in Appendix 10.4) that with probability $1 - O\left(\frac{1}{n}\right)$, $|p_i^u - p_i^c| < C\sqrt{u^i_n}$ and $|p_i^c - p_i^D| < C\sqrt{u^i_n}$ (in our analysis, for simplicity, we use $C$ to denote a generic constant, the relations between them may not be specified). Therefore, with probability $1 - O\left(\frac{1}{n}\right)$, $|\hat{p}_i - p^D| < C\sqrt{u^i_n}$. On the other hand, the length of the next price interval (the center is near $\hat{p}_i$) is of order $\sqrt{\log n}$ greater than $\sqrt{u^i_n}$. Therefore, with probability $1 - O\left(\frac{1}{n}\right)$, $p^D \in I^u_i$. Then we take a union bound over all $i$ and the first part of the lemma holds. □

Next we show that if condition (16) is triggered, then with probability $1 - O\left(\frac{1}{n}\right)$, $p^c > p^u$. An equivalent expression is that if $p^c \geq p^u$, then with probability $1 - O\left(\frac{1}{n}\right)$, condition (16) will not be triggered. We will use this fact many times in the future, so we formalize this into the following lemma.

Lemma 3. If $p^c \geq p^u$, then with probability $1 - O\left(\frac{1}{n}\right)$, our algorithm will not enter Step 3 before stopping;

Proof. The proof of this lemma follows from the proof of Part 2 in Lemma 2. See Appendix 10.4 □

Remark. Lemma 3 says that if $p^c \geq p^u$, then our algorithm will not enter Step 3. When $p^c > p^u$ however, it is also possible that our algorithm will not enter Step 3, but as we will show later, in that case, $p^u$ must be very close to $p^c$ so that the revenue collected is still near-optimal.

Now we have proved that with high probability, $p^D$ will always be in our price interval. Next we analyze the revenue collected for this algorithm and therefore prove our main theorem.

We first prove the case when $p^u \geq p^c$. We prove:

Proposition 1. When $p^u \geq p^c$, $\sup_{\lambda \in \Gamma} R^u_n(x, T; \lambda) \leq C(\log n)^{\frac{3}{2}} \frac{1}{n^\epsilon}$.

Consider a problem with size $n$. Recall that $J_n^u(x, T; \lambda)$ is the expected revenue collected by our algorithm given that the underlying demand function is $\lambda$. Define $Y^u_{ij}$ to be Poisson random variables with parameters $\lambda(p_{ij}^u)n\Delta^u_i$ ($Y^u_{ij} = N(\lambda(p_{ij}^u)n\Delta^u_i)$) Also define $\hat{Y}^u$ to be a Poisson random variable with parameter $\lambda(\hat{p})n(1 - t_{\lambda}^\epsilon)$ ($\hat{Y}^u = N(\lambda(\hat{p})n(1 - t_{\lambda}^\epsilon))$).

We remove the dependence on $n$ in the notation. If not otherwise stated, it is assumed we are talking a problem with size $n$. 

14
We define the following events ($I(\cdot)$ denotes the indicator function of a certain event):

$$A_1^n = \{ \omega : \sum_{i,j} Y_{ij}^u < nx \},$$

$$A_2^n = \{ \omega : \text{The algorithm never enters Step 3 and } p^D \in I_i^u, \forall i = 1, 2, ..., N^u \}.$$ We have

$$J_n(x, T; \lambda) \geq E\left[\sum_{i=1}^{N^u} \sum_{j=1}^{\kappa_j^u} p_{i,j}^u I(A_1^u) I(A_2^u)\right] + E[\hat{p} \min(\hat{Y}^u, (nx - \sum_{i,j} Y_{ij}^u)^+) I(A_1^u) I(A_2^u)]. \tag{36}$$

In the following, we will consider each term in (36). We will show that the revenue collected in both parts is “close” to the revenue generated by the optimal deterministic price $p^D$ on that part (and the consumed inventory is also near-optimal). We first have:

**Lemma 4.**

$$E\left[\sum_{i=1}^{N^u} \sum_{j=1}^{\kappa_j^u} p_{i,j}^u I(A_1^u) I(A_2^u)\right] \geq \sum_{i=1}^{N^u} p^D \lambda(p^D) n\tau_i^u - C n\tau_1^u. \tag{37}$$

**Proof.** This proof analyzes the exploration error in each iteration. Under $A_2^n$, the exploration error will be quadratic in the length of the price interval of each period. Summing up those errors will lead to this result. The detail of the proof is given in Appendix 10.5. □

Now we look at the other term in (36), we have

$$E[\hat{p} \min(\hat{Y}^u, (nx - \sum_{i,j} Y_{ij}^u)^+) I(A_1^u) I(A_2^u)]$$

$$= E[\hat{p}(\hat{Y}^u - \max(\hat{Y}^u - (nx - \sum_{i,j} Y_{ij}^u)^+, 0)) I(A_1^u) I(A_2^u)]$$

$$\geq E[\hat{p}\hat{Y}^u I(A_1^u) I(A_2^u)] - E[\hat{p}(\hat{Y}^u + \sum_{i,j} Y_{ij}^u - nx)^+] \tag{38}$$

For the first term, we have

$$E[\hat{p}\hat{Y}^u I(A_1^u) I(A_2^u)]$$

$$= E[n(1 - t_{N^u})\hat{p}\lambda(\hat{p}) I(A_1^u) I(A_2^u)]$$

$$\geq (1 - O\left(\frac{1}{n}\right))n(1 - t_{N^u})E[\hat{p}\lambda(\hat{p}) I(A_1^u) I(A_2^u)] \tag{39}$$

However, by our assumption on the bound of the second derivative of $r(\lambda)$, and 24 and 23 we know that

$$E[\hat{p}\lambda(\hat{p}) I(A_1^u) I(A_2^u)] \geq p^D \lambda(p^D) - C(p_{N^u+1}^u - E_{N^u+1}^u)^2 \geq p^D \lambda(p^D) - C n\tau_1^u. \tag{40}$$

Therefore,

$$E[\hat{p}\hat{Y}^u I(A_1^u) I(A_2^u)] \geq p^D \lambda(p^D) \cdot n(1 - t_{N^u}) - C n\tau_1^u. \tag{41}$$

Now we consider

$$E[\hat{p}(\hat{Y}^u + \sum_{i,j} Y_{ij}^u - nx)^+] \tag{42}$$
First we relax this to
\[ pE(\hat{Y}^u + \sum_{i,j} Y_{ij}^u - nx)^+. \] (43)

We have the following lemmas:

**Lemma 5.**
\[ E(\hat{Y}^u + \sum_{i,j} Y_{ij}^u - E\hat{Y}^u - \sum_{i,j} EY_{ij}^u)^+ \leq Cn\tau_1^u, \] (44)

where \( C \) is a properly chosen constant.

**Lemma 6.** If \( p^u \geq p^c \), then
\[ \sum_{i=1}^{N^u} \kappa_{ij}^u EY_{ij}^u + E\hat{Y}^u - nx \leq Cn\tau_1^u, \] (45)

where \( C \) is a properly chosen constant.

**Proof of Lemma 5 and 6.** The proof of Lemma 5 repeatedly uses Lemma 14 in Appendix 10.2, which bounds the tail values of Poisson random variables. The proof of Lemma 6 bounds the inventory consumed in the learning period. We show that by the way we define each learning interval in (17) and (18), with high probability, \( p^c \) is always to the left of the center of the price interval. Therefore, the excess inventory we consumed in each iteration (compared to the consumption by \( p^c \)) is at most a second order quantity of the length of the price range (the first order error has been canceled out, or a negative value is remaining). This important fact will give us the desired bound for the consumption of the inventory. The detailed proofs of these two lemmas are given in Appendix 10.6.

Now we combine Lemma 4, 5, and 6, we have
\[ R_n^u \leq 1 - \frac{np^D\lambda(p^D) - Cn\tau_1^u}{np^D\lambda(p^D)} \leq C\tau_1^u \] (46)

. Therefore, Proposition 1 follows. Next we consider the case when \( p^c > p^u \). We claim

**Proposition 2.** When \( p^c > p^u \), \( \sup_{x,T} R_n^u(x,T) \leq \frac{C(\log n)^3}{n^4} \).

When \( p^c > p^u \), we condition our analysis on the time the algorithm enters Step 3. We define \( Y_{ij}^u \) and \( \hat{Y}^u \) as before, and \( Y_{ij}^c \) to be Poisson random variables with parameters \( \lambda(p_{ij}^c)n\Delta_i^c \) (\( \Delta_i^c = N(\lambda(p_{ij}^c)n\Delta_i^c) \)). Also define \( \hat{Y}_{ij}^c \) to be a Poisson random variable with parameter \( \lambda(q)n(1 - t_{ij}^c - t_i^c) \) (\( \hat{Y}_{ij}^c = N(\lambda(q)n(1 - t_{ij}^c - t_i^c)) \)).

We define the following events:

\[ B_1 = \{ i_0 = 1 \} \]
\[ B_2 = \{ i_0 = 2 \} \]
\[ \vdots \]
\[ B_{N^u} = \{ i_0 = N^u \} \]
\[ B_{N^u+1} = \{ \text{The algorithm doesn’t enter Step 3} \}. \]

Then we have the following bound on \( J_n^u \) in this case:
\[ J_n^u(x,T,\lambda) \geq E[\sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_{ij}^c} p_{ij}^c Y_{ij}^c I(\cup_{i=1}^{N^u+1} B_i)] + E[\tilde{p}\min(\hat{Y}^u, (nx - \sum_{i,j} Y_{ij}^u)^+)I(B_{N^u+1})] \]
\[ + E[\sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_{ij}^c} p_{ij}^c Y_{ij}^c I(\cup_{i=1}^{N^u} B_i)] + \sum_{l=1}^{N^u} E[\tilde{q}\min(\hat{Y}_{ij}^c, (nx - \sum_{i,j} Y_{ij}^c - \sum_{i,j} Y_{ij}^u)^+)I(B^l)]. \] (48)
We will get a bound on each term. We prove the following lemmas:

**Lemma 7.**

$$E\left[\sum_{i=1}^{N^u} \sum_{j=1}^{\kappa^u_i} p_{ij}^u Y_{ij}^u I(\cup_{i=1}^{N^u+1} B_i)\right] \geq \sum_{i=1}^{N^u} n \tau_i p^D \lambda(p^D) P(\cup_{i=1}^{N^u+1} B_i) - C n \tau_i^u. \quad (49)$$

**Lemma 8.**

$$E[\hat{p} \min(\hat{Y}^u, (nx - \sum_{i,j} Y_{ij}^u)^+) I(B_{N^u+1})]\geq p^D \lambda(p^D) \cdot n (1 - t^u_{N^u}) P(B_{N^u+1}) - C n \tau_i^u. \quad (50)$$

**Lemma 9.**

$$\sum_{i=1}^{N^u} \sum_{j=1}^{\kappa^u_i} E[p_{ij}^u Y_{ij}^u I(\cup_{i=1}^{N^u+1} B_i)] \geq \sum_{i=1}^{N^u} n \tau_i^c p^D \lambda(p^D) P(\cup_{i=1}^{N^u+1} B_i) - C n \tau_i^c. \quad (51)$$

**Lemma 10.** For each $l = 1, \ldots, N^u$,

$$E[\hat{q} \min(\hat{Y}^c_l, (nx - \sum_{i,j} Y_{ij}^c l - \sum_{i,j} Y_{ij}^c l)^+) I(B_l)] \geq n p^D \lambda(p^D)(1 - t^u_l - t^c_l) P(B_l) - C n \tau_l^u. \quad (52)$$

**Proof.** The proofs of the above lemmas resemble the proofs for the case when $p^u \geq p^c$. They are given in Appendix [10.7].

We then combine Lemma 7, 8, 9 and 10, adding the right hand side together, and Proposition 2 follows. Theorem 1 thus follows from Proposition 1 and 2.

6 Lower Bound Example

In this section, we prove Theorem 2. We show that there exists a class of demand functions satisfying our assumptions, however no pricing policy can achieve an asymptotic regret less than $\Omega(\sqrt{n})$.

The proof involves statistical bounds on hypothesis testing, and it resembles the example discussed in [23] and [9]. However, since our model is different from that in [23] and [9], the proof is different in many ways. We will discuss this in the end of this section.

**Proposition 3.** Define a set of demand functions as follows. Let $\lambda(p; z) = 1/2 + z - z p$ where $z$ is a parameter taking value in $Z = [1/3, 2/3]$ (we denote this demand function set by $\Lambda$).

Assume that $\underline{p} = 1/2$ and $\overline{p} = 3/2$. Also assume that $x = 2$ and $T = 1$. Then we have

- This class of demand function satisfies Assumption A. Furthermore, for any $z \in [1/3, 2/3]$, the optimal price $p^D$ always equals to $p^u$ and $p^D \in [7/8, 5/4]$. Therefore, it also satisfies Assumption B with $\epsilon = 1/4$.
- For any admissible pricing policy $\pi$,

$$\sup_{z \in Z} R_n(\pi; x, T; z) \geq \frac{1}{3(48)^2 \sqrt{n}}, \quad \forall n \quad (53)$$

First, let me explain some intuition behind this example. Note that all the demand functions in $\Lambda$ cross at one common point, that is, when $p = 1$, $\lambda(p; z) = 1/2$. Such a price is called an “uninformative” price in [23]. When there exists an “uninformative” price, experimenting at that price will not gain information about the demand function. Therefore, in order to “learn”
the demand function (i.e., the parameter \(z\)) and determine the optimal price, one must at least perform some price experiments at prices away from the uninformative price; on the other hand, when the optimal price is indeed the uninformative price, doing price experimentations at a price away from the optimal price will incur some revenue losses. This tension between the loss during exploration and exploitation is the key reason for such a lower bound for the loss. Before we proceed, we list some general properties of the demand function set we defined in Proposition 3.

Lemma 11. For the demand function defined in Proposition 3, denote the optimal price \(p^D\) under parameter \(z\) to be \(p^D(z)\). We have:

1. \(p^D(z) = (1 + 2z)/(4z)\)
2. \(p^D(z_0) = 1\) for \(z_0 = 1/2\)
3. \(\lambda(p^D(z_0); z) = 1/2\) for all \(z\)
4. \(-4/3 \leq r''(p; z) \leq -2/3\) for all \(p, z\)
5. \(|p^D(z) - p^D(z_0)| \geq |z - z_0|/4\)
6. \(p^D(z) = p^n(z)\) for all \(z\).

Now in order to quantify the tension mentioned above, we need a notion of “uncertainty” about the unknown demand parameter \(z\). For this, we use the K-L divergence over two probability measures for a stochastic process.

For any policy \(\pi\), and parameter \(z\), let \(P_z^\pi\) denote the probability measure associated with the observations (the process observed when using policy \(\pi\)) when the true demand function is \(\lambda(p; z)\). We also denote the corresponding expectation operator by \(E_z^\pi\).

Given \(z\) and \(z_0\), the Kullback-Leibler (K-L) divergence between the two measures \(P_z^{\pi_0}\) and \(P_z^{\pi_1}\) over time 0 to \(T\) is given by the following (we refer to [1] for this definition):

\[
K(P_z^{\pi_0}, P_z^{\pi_1}) = E_z^{\pi_0} \left[ \int_0^T n \lambda(p(s); z) \log \frac{\lambda(p(s); z_0)}{\lambda(p(s); z)} + 1 - \lambda(p(s); z_0) \right] ds
\]

Note that the K-L divergence is a measure of distinguishability between probability measures: if two probability measures are close, then they have a small K-L divergence and vice versa. In terms of pricing policies, a pricing policy \(\pi\) is more likely to distinguish between the case when the parameter is \(z\) and the case when the parameter is \(z_0\) if the quantity \(K(P_z^{\pi_0}, P_z^{\pi_1})\) is large.

Now we show the following lemma, which gives a lower bound of the regret induced by any policy in terms of the K-L divergence; this means a pricing policy that is better able to distinguish different parameters will also be more costly.

Lemma 12. For any \(z \in Z\), and any policy \(\pi\) setting price in \(\mathcal{P}\),

\[
K(P_z^{\pi_0}, P_z^{\pi_1}) \leq 24n(z_0 - z)^2 R_n^\pi(x, T; z_0),
\]

where \(z_0 = 1/2\) and \(R_n^\pi(x, T; z_0)\) is the regret function defined in (5) with \(\lambda\) being \(\lambda(p; z_0)\).

Proof. The proof attempts to bound the final term in (54) and is given in Appendix 10.8.

Now we have shown that in order to have a policy that is able to distinguish between two different parameters, one has to give up some portion of the revenue. In the following lemma, we show that on the other hand, if a policy is not able to distinguish between two close parameters, then it will also incur a loss.
Lemma 13. Let $\pi$ be any pricing policy that sets prices in $[p, \overline{p}]$ and $p_\infty$. Define $z_0 = 1/2$ and $z_n^i = z_0 + \frac{1}{4n^2 \pi^2} \cdot (note \ z_n^i \in [1/3, 2/3] \ for \ all \ n \geq 2)$. We have for any $n \geq 2$

$$R_n^x(x, T; z_0) + R_n^x(x, T; z_n^i) \geq \frac{1}{3(48)^2 \sqrt{n}} e^{-K(P_{z_0}^\pi, P_{z_n^i}^\pi)}.$$ \hspace{2cm} (56)

Proof. The proof uses similar ideas as discussed in [9] and [23]. Here we give some sketches of the proof. We define two non-intersecting intervals around $p_{D}(z_0)$ and $p_{D}(z_n^i)$. We show that when the true parameter is $z_0$, pricing using $p$ in the second interval will incur a certain loss and the same order of loss will be incurred if we use $p$ in the first interval when the true parameter is $z_n^i$. At each time, we treat our policy $\pi$ as a hypothesis test engine, that maps the historic data into two actions:

- Choose a price in the first interval
- Choose a price outside the first interval

Then we can represent the revenue loss during the selling season by the “accumulated probability” of committing errors in those hypothesis tests. However, by the theory of the hypothesis test, one can lower bound the probability of the errors for any decision rule. Thus we can obtain a lower bound of revenue loss for any pricing policy. The complete proof is referred to Appendix 10.9.

Now we combine Lemma 12 and 13. By picking $z$ in Lemma 12 to be $z_n^i$ and add (55) and (56) together, we have:

$$2\{R_n^x(x, T; z_0) + R_n^x(x, T; z_n^i)\} \geq \frac{3}{32 \sqrt{n}} K(P_{z_0}^\pi, P_{z_n^i}^\pi) + \frac{1}{3(48)^2 \sqrt{n}} e^{-K(P_{z_0}^\pi, P_{z_n^i}^\pi)}$$

$$\geq \frac{1}{3(48)^2 \sqrt{n}} (K(P_{z_0}^\pi, P_{z_n^i}^\pi) + e^{-K(P_{z_0}^\pi, P_{z_n^i}^\pi)})$$

$$\geq \frac{1}{3(48)^2 \sqrt{n}}$$

The last inequality is because for any number $w > 0$, $w + e^{-w} \geq 1$. Therefore, we have shown that for any $n$, no matter what policy is used, we always have

$$\sup_{\lambda \in \Lambda} R_n^x(x, T; \lambda) \geq \frac{1}{3(48)^2 \sqrt{n}}$$

and Proposition 3 is proved.

Remark. Our proof is similar to the proof of the corresponding worst case examples in [8] and [23], but different in several ways. First, in [8], they considered only finite possible prices (though their proof is for a high-dimensional case, for the sake of comparison, here we compare our proposition with theirs in the one dimensional case). In our case, a continuous interval of prices is allowed. Therefore, the admissible policy in our case is much larger. And the K-L divergence function is thus slightly more sophisticated than the one used in their proof. In fact, the structure of our proof more closely resembles the one in [23] where they consider a worst-case example for a general parametric choice model. However, in their model, the time is discrete. Therefore, a discrete version of the K-L divergence is used and the analysis is based on the sum of the errors of different steps. In some sense, our analysis can be viewed as a continuous-case extension of the proof in [23].

7 Numerical Results

In this section, we perform numerical tests of the dynamic pricing algorithm discussed in previous sections. Specifically, we compare the results by using our dynamic price learning algorithm to
the algorithm proposed in [8].

In our numerical tests, we consider two underlying demand functions. One is linear with $\lambda_1(p) = 30 - 3p$ and the other is exponential with $\lambda_2(p) = 80e^{0.5p}$. These two demand functions are in accordance with the demand function chosen in the numerical tests in [8] where they considered $\lambda_1 = 30 - 3p$ and $\lambda_2 = 10e^{1 - 0.5p}$. The reason that we change the constant in $\lambda_2(p)$ is that we want to examine two different cases for our algorithm, one with $p^c > p^*\bar{p}$ and the other with $p^c < p^*\bar{p}$. Note that with underlying demand function $\lambda_1$, we have $p^D = p^a = 5 > p^c = 3$, and with underlying demand function $\lambda_2$, we have $p^D = p^f = 2\ln 4 > p^a = 2$. In both cases, we assume the initial inventory level is $x = 20$, the selling horizon $T = 1$, and the initial price interval is $[\bar{p}, \overline{\bar{p}}] = [0.1, 10]$. For each case, we run $10^3$ independent simulations, comparing the average of them to the deterministic optimal solution (the standard error for the cases of $n = 10$ and $n = 100$ is less than 0.4% of its mean and the standard errors for the remaining cases are less than 0.01% of its mean). Note that the above settings are exactly the same as those in [8].

We also make the following modifications to our algorithm in implementations:

- We remove the log $n$ factor in $\tau^a_i$ and $\tau^f_i$ in our numerical study. Otherwise the factors $(\log n)^5$ and $(\log n)^3$ in (30) and (33) are too big for the cases we study. Since the log $n$ factors are mainly used for analysis purposes, this modification is quite reasonable. In fact, this modification leads to better performance in revenue in the cases we study.

- Whenever our algorithm enters Step 3, instead of using $[\bar{p}, \overline{\bar{p}}]$ as the initial interval as we stated in our algorithm, we use $[p^a, p^f\overline{\bar{p}}]$, which is the last computed price interval. As we showed in Lemma 2 with high probability, this interval contains the deterministic optimal price, therefore intuitively this will also guarantee the asymptotic behavior we have (although we restart the process in our stated algorithm for ease of analysis). This would also make some improvement to the performance of our algorithm.

Before we show our comparison results, we first use an example to show how our algorithm actual works, that is, what is the time length for each step and how the price interval evolves. We take the linear demand function case with $\lambda_1(p) = 30 - 3p$ and $n = 10^5$ as an example. A sketch of a single run of our algorithm in this case is shown in Figure 1.

In Figure 1, we can see that our algorithm runs 4 iterations of price learning before entering the last step. The time spent in each iteration is increasing, which is in accordance with our definition of $\tau^a_i$ and is also intuitively true: since we are using a more accurate price for experimentation in each iteration, we can afford to spend more time without incurring extra losses. Besides, the number of prices tested in each interval is decreasing, along with the length of the price interval and the price granularity. Therefore, with time evolving, we are testing fewer prices on each interval, on finer grids, and test longer for each price. And finally, when $\tau^a_i > 1$, we apply the last learned price in the rest of the time horizon.

Remember that we evaluate our algorithm by the regret function $R^a_n(x, T; \lambda)$ defined in (3). In Theorem 1, we showed that asymptotically, the regret is of order $n^{-\delta}$ (with a logarithmic factor), when we choose any fixed $\delta < 1/2$. In other words, $\log(R^a_n(x, T; \lambda))$ should be approximately a linear function of $\log n$ with slope $-\delta$. In the following, we choose $\delta = 0.49$, and we conduct numerical experiments for problems with different sizes of $n$ and study how $R^a_n$ changes with $n$. Specifically, we use a linear regression to fit the relationship between $\log R^a_n(x, T; \lambda)$ and $\log n$. The results are shown in Figure 2(a) and 2(b).

In Figure 2(a) and 2(b), the slopes of the best linear fit of the log-regret and log-$n$ are approximately 0.444 and 0.465, respectively. Although it is somewhat less than the stated $\delta = 0.49$, it is significantly larger than the slope of 0.25 obtained in [8] for the non-parametric policy, and even the slope of 0.33 obtained for the parametric policy in [8]. The deviation from $\delta$ however, may be due to the ignorance of the log-factor which is not insignificant in the cases we study. Besides the asymptotic behavior, we also compare our regrets to the one obtained in [8], which is shown in Figure 3.
Figure 1: Time and price evolvement of our algorithm with $\lambda(p) = 30 - 3p$. and the deterministic optimal price $p^D = p^u = 5$
Figure 2: Numerical results for the dynamic pricing algorithm. Diamonds show the performance of our algorithm and the solid line passing through the points is the best linear fit to those points.
As we can see in Figure 3, the regret obtained by our algorithm is well below the one obtained by the non-parametric policy. This means that by using a dynamic learning strategy, we can indeed improve the performance quite significantly compared to the policy when we only learn the price in one period. It can be seen that when $n$ achieves $10^6$, the performance of our algorithm also surpasses the parametric policy (with one learning period). Also note that the regrets obtained by our algorithm have larger deviation from the linear regression model than the one shown in [8]. This may be because when we use multiple period learning, the different particularities of each individual problem (e.g., where the grid points are positioned) have a larger impact on the results, since it might affect the positions of the later price intervals.

8 Extension

8.1 Other Applications

As we mentioned in the beginning, our work can be applied to a general class of single-product revenue management problems. In this section, we shed some light on the potential applications. We start with a case where the advertisement intensity is the decision variable.

Consider a company selling a single product over a finite time horizon. Due to the market competition, the price of the product is fixed. However, the company can choose its advertisement strategy to affect the demand rate. Assume the firm can choose an advertisement intensity parameter $a$; for example, in the online selling case, $a$ may be the pay-per-click price the company paid to search engines. The demand rate under advertisement intensity $a$ is denoted by $\lambda(a)$. In this setting, the company controls $a_t$ and the revenue collected is:

$$\int_{s=0}^{T} (p - a_t) dN^t,$$  \hspace{1cm} (57)
where \( N^t = N(\int_{t=0}^t \lambda(a_t) dt) \) is a Poisson random variable. Consider the following transformation:

\[
w_t = p - a_t \quad \hat{\lambda}(w_t) = \lambda(a_t).
\]

Then this problem will have the same form of the optimal pricing problem we discussed throughout the paper. Therefore, when the demand function satisfies the same set of conditions, our theorem will apply.

Besides advertisement intensity, the control variable can be viewed as the sales person compensation or other incentives of selling a product, as long as a similar formulation can be established. We believe there are more examples in practice that fit into this model.

8.2 When the second derivative assumption is not satisfied

In Assumption A, we assumed that \( r''(\lambda) \) exists and is bounded away from zero. This assumption is necessary in our analysis (at least locally at \( p^u \)) since we utilize the local quadratic behavior of the revenue functions. However, there are a few cases in practice when the demand function does not satisfy this assumption, e.g., when the demand function is piecewise linear and the revenue maximizing price \( p^u \) is exactly at one of the “kink” points of the piecewise function. In that case, at \( \lambda(p^u) \), \( r(\lambda) \) behaves more like a linear function. A natural question is: can we still achieve the same asymptotic behavior for those cases? The following theorem gives an assertive answer to this question, although it requires us to use another algorithm (Algorithm DPA2, see Appendix 10.10) to achieve this.

**Theorem 3.** Let Assumption A hold except for the third requirement. Let Assumption B hold for a fixed \( \epsilon > 0 \). Also, assume that for any \( p \in [p_L, p_U] \), \( L|p - p^u| \leq |r(\lambda(p)) - r(\lambda(p^u))| \). Then for any \( \delta < 1/2 \), there exists a policy \( \pi_\delta \in \mathcal{P} \) generated by Algorithm DPA2, such that for all \( n \geq 1 \),

\[
\sup_{\lambda \in \Gamma} R_n^{\pi_\delta}(x; T; \lambda) \leq \frac{C(\log n)^2}{n^\delta},
\]

for some constant \( C \).

Theorem 3 complements our main theorem in some cases when the demand function is not differentiable. In fact, in this case, a simpler learning algorithm (see Algorithm DPA2) involving only one learning step would work. However, to apply this theorem, one requires the advance knowledge that the demand function has a “kink” at optimal. How to combine this case and the case in our main theorem is one of our future work.

9 Conclusion and Future Work

In this paper, we present a dynamic pricing algorithm for a set of single-product revenue management problems. Our algorithm achieves an asymptotic regret arbitrarily close to \( O(n^{-1/2}) \) even if we have no prior knowledge on the demand function except some regularity conditions. By complementing with a worst-case bound, we show that our algorithm is almost the best possible in this setting, and it closes the performance gaps between parametric and non-parametric learning and between a post-price mechanism and a customer-bidding mechanism.

In terms of the algorithm itself, the dynamic learning algorithm integrates learning and doing in a concurrent procedure and may be of independent interest to the revenue management practitioners.

There are several open questions to explore including how to extend this result to high-dimensional problems (with high-dimensional control or inventory products). In high-dimensional problems, the structure of demand functions may be even more complicated and
the extension is not straightforward. Other directions may include models with competition among retailers and/or strategic behaviors of the customers.

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10 Appendix

10.1 Examples of demand functions

1. For linear demand functions $\lambda(p) = a - bp$ with $0 < a \leq \bar{a}$ and $0 < b \leq \bar{b}$, it is easy to see that all our assumptions hold with $M = \bar{a} - \frac{\bar{b}}{p}$, $K = \max\{\bar{b}, \frac{\bar{b}}{p}, \frac{a}{2}, \frac{\bar{b}}{p}\}$, $m_L = \frac{\bar{b}}{p}$ and $m_U = \frac{\bar{b}}{p}$.

2. For exponential demand functions $\lambda(p) = ae^{-bp}$ with $0 < a \leq \bar{a}$ and $0 < b \leq \bar{b}$, we have
   - $|\lambda(p)| \leq ae^{-bp}$
   - $\lambda(p)$ is Lipschitz continuous with coefficient $\bar{a} \cdot \frac{\bar{b}}{p}$, $r(p)$ is Lipschitz continuous with coefficient $\bar{a} \cdot \frac{\bar{b}}{p}$, and $\gamma(\lambda)$ is Lipschitz continuous with coefficient $\frac{\bar{a} \cdot \bar{b}}{p}$.
   - $r(\lambda) = -\frac{1}{b} \log \lambda + \frac{1}{b} \log a$ is second-order differentiable and $-\frac{7}{\bar{a} \cdot \bar{b}} \leq r''(\lambda) \leq -\frac{4}{\bar{a} \cdot \bar{b}}$.

3. For logit demand functions $\lambda(p) = e^{-a - bp} + \frac{1}{1 + e^{-a - bp}}$, with $0 < a \leq \bar{a}$ and $0 < b \leq \bar{b}$, we have
   - $|\lambda(p)| \leq 1$
   - $\lambda(p)$ is Lipschitz continuous with coefficient $\bar{b}$, $r(p)$ is Lipschitz continuous with coefficient $1 + \bar{b} \cdot \frac{1}{p}$, and $\gamma(\lambda) = \frac{1}{p} \log \frac{1 - \lambda}{\lambda} - a$ is Lipschitz continuous with coefficient $\frac{4}{\bar{b}}$
   - $r(\lambda) = \frac{1}{p} \log \frac{1 - \lambda}{\lambda} - a$ is second-order differentiable and $-7 - e^{-\frac{a - b}{\bar{b}}} \leq r''(\lambda) \leq -\frac{4}{\bar{b}}$.

10.2 A Lemma on the Deviation of Poisson Random Variables

In our proof, we will frequently use the following lemma on the tail behavior of Poisson random variables:

**Lemma 14.** Suppose that $\mu \in [0, M]$ and $r_n \geq n^\beta$ with $\beta > 0$. If $\epsilon_n = 2n^{1/2}M^{1/2}(\log n)^{1/2}r_n^{-1/2}$, then for all $n \geq 1$,

$$P(N(\mu r_n) - \mu r_n > r_n \epsilon_n) \leq \frac{C}{n^\eta}$$

and

$$P(N(\mu r_n) - \mu r_n < -r_n \epsilon_n) \leq \frac{C}{n^\eta}$$

for some suitably chosen constant $C > 0$.

We refer Lemma 2 in the online companion of [8] for the proof of this lemma.
10.3 Proof of Lemma 1

**Proof:** By plugging (29) and (31) into (24), we have

\[
N^u = \min_i \{l : n^{2\delta - 1 + (1-\delta) \cdot (\frac{3}{2})^l} < (\log n)^3 \}. \tag{61}
\]

Therefore, for any fixed \( \delta < \frac{1}{2} \), we can take

\[
N^1 = \lfloor \frac{1 - 2\delta}{1 - \delta} \rfloor + 1, \tag{62}
\]

where \( \lfloor x \rfloor \) is the largest integer less than \( x \). Similarly, by plugging (32) and (34) into (28), we have

\[
N^c = \min_i \{l : n^{2\delta - 1 + (1-\delta) \cdot (\frac{3}{2})^l} < (\log n)^3 \}. \tag{63}
\]

Therefore, for any fixed \( \delta < \frac{1}{2} \), we can take

\[
N^2 = \lfloor \frac{1 - 2\delta}{1 - \delta} \rfloor + 1.
\]

Thus \( N_{\delta} = \max\{N^1, N^2\} \) will be an upper bound for the number of iterations of our algorithm. Furthermore, \( N_{\delta} \) doesn’t depend on \( n \). \( \square \)

10.4 Proof of Lemma 2

**Part 1:** We first prove the first part, that is, when the algorithm runs within Step 2 until \( i \) reaches \( N^u \).

We prove by induction on \( i \). By the induction assumption, we know that at iteration \( i \), with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p_D \in I_i^u \). Now consider the next iteration. Define

\[
u_n^i = \log n \cdot \max \left\{ \left( \frac{p_D^u - p_i^u}{\kappa_i^u} \right)^2, \sqrt{\frac{\kappa_i^u}{n \tau_i^u}} \right\}.
\]

First, we establish a bound on the difference in revenue \( r(\lambda(p_i^u)) - r(\lambda(p_i^u)) \), where \( p_i^u \) is the revenue maximizing price on \( I_i^u \) and \( \hat{p}_i^u \) is the empirical revenue maximizing price defined in our algorithm. We assume \( \hat{p}_{i,j}^u \) is the nearest grid point to \( p_i^u \) in this iteration. We consider three cases:

- When \( p_i^u < p^u \): This is impossible since we know that \( p_D \geq p_i^u > p^u \) and by the induction assumption \( p_D \in I_i^u \). Therefore we must have \( p^u \in I_i^u \), and by definition, \( p^u \) achieves a larger revenue rate than \( p_i^u \), which is contradictory to the definition of \( p_i^u \).
- When \( p_i^u = p^u \): In this case, by the granularity of the grid at iteration \( i \), we have \(|p_i^u - p^u| \leq \frac{r_{\kappa_i}^u}{\kappa_i^u} \) and thus by assumption that \( r(\lambda) \) is bounded, we know that \(|r(\lambda(p^u)) - r(\lambda(p_{i,j}^u))| \leq m_1 K^2 \cdot \left( \frac{r_{\kappa_i}^u}{\kappa_i^u} \right)^2 \), therefore we have:

\[
\begin{align*}
& r(\lambda(p_i^u)) - r(\lambda(p_{i,j}^u)) \\
& = r(\lambda(p_i^u)) - r(\lambda(p_{i,j}^u)) + p_i^u \lambda(p_i^u) - p_i^u \lambda(p_{i,j}^u) + p_i^u \lambda(p_i^u) - p_i^u \lambda(p_{i,j}^u) + p_i^u \lambda(p_i^u) - p_i^u \lambda(p_{i,j}^u) \\
& \leq m_1 K^2 \left( \frac{r_{\kappa_i}^u}{\kappa_i^u} \right)^2 + 2 \max_{1 \leq j \leq \kappa_i^u} |p_i^u \lambda(p_{i,j}^u) - p_i^u \lambda(p_{i,j}^u)|. \tag{63}
\end{align*}
\]

In (63), \( \hat{\lambda} \) is the observed demand rate and the last inequality is due to the definition of \( \hat{p}_i^u \) and that \( p_i^u \) is among one of the \( p_{i,j}^u \).
By Lemma 14 in Appendix 10.2, we have

\[ P(|\hat{\lambda}(p_{i,j}^u) - \lambda(p_{i,j}^u)| > C\sqrt{\log n \cdot \frac{\kappa_i^u}{n \tau}}) \leq \frac{1}{n^2}, \]  \tag{64} \]

with some suitable constant \( C \). Therefore, with probability \( 1 - O\left(\frac{1}{n}\right) \), \( r(\lambda(p^u)) - r(\hat{\lambda}(p^u)) \leq Cu_n^u \). However, by our assumption that \( r''(\lambda) \leq m_U \) and that \( \gamma(\lambda) \) is Lipschitz continuous, with probability \( 1 - O\left(\frac{1}{n}\right) \), \( |p^u - \hat{p}^u| \leq C\sqrt{u_n^u} \) (here "\( C \)" represents some generic constant, and the relations are not always specified).

Now we consider the distance between \( \hat{p}_i^u \) and \( p_i^u \) (this part of result can also be found in Lemma 4 in the online companion of [8]). Assume \( p_{i,j}^u \) is the nearest grid point to \( p_i^u \). Then, using that we assumed \( T = 1 \), we have:

\[
|\lambda(\hat{p}^u_i) - x| \leq |\hat{\lambda}(\hat{p}^u_i) - x| + |\hat{\lambda}(\hat{p}^u_i) - \lambda(p_i^u)| \\
\leq |\hat{\lambda}(\hat{p}^u_{i,j}) - x| + |\hat{\lambda}(\hat{p}^u_{i,j}) - \lambda(\hat{p}^u_i)| + |\hat{\lambda}(\hat{p}^u_i) - \lambda(p_i^u)| \\
\leq |\hat{\lambda}(\hat{p}^u_{i,j}) - x| + |\lambda(p_i^u) - \lambda(p_{i,j}^u)| + 2\max_{1 \leq j \leq \kappa_i^u} |\hat{\lambda}(\hat{p}^u_{i,j}) - \lambda(p_{i,j}^u)|. \tag{65}
\]

And by the definition of \( p_i^u \), \( \lambda(p_i^u) - x \) and \( \lambda(\hat{p}^u_i) - x \) must have the same sign, otherwise there exists a point in between that achieves a smaller value of \( |\lambda(p) - x| \). Therefore we have

\[
|\lambda(p_i^u) - \lambda(p_{i,j}^u)| \leq |\lambda(p_i^u) - \lambda(p_{i,j}^u,^*)| + 2\max_{1 \leq j \leq \kappa_i^u} |\hat{\lambda}(p_{i,j}^u) - \lambda(p_{i,j}^u)|. \tag{66}
\]

By the Lipshitz continuity of \( \lambda \), we have

\[
|\lambda(p_i^u) - \lambda(p_{i,j}^u)| \leq K \frac{p_{i,j}^u - p_i^u}{\kappa_i^u}. \tag{67}
\]

Also by Lemma 14 in Appendix 10.2, we have with probability \( 1 - O\left(\frac{1}{n}\right) \),

\[
\max_{1 \leq j \leq \kappa_i^u} |\lambda(p_{i,j}^u)| \leq C\sqrt{\log n \cdot \frac{\kappa_i^u}{n \tau n^u}}. \tag{68}
\]

Therefore, with probability \( 1 - O\left(\frac{1}{n}\right) \), we have

\[
|\lambda(p_i^u) - \lambda(\hat{p}_i^u)| \leq C\sqrt{u_n^u} \tag{69}
\]

and by the Lipshitz continuity of \( \nu(\lambda) \), this implies that with probability \( 1 - O\left(\frac{1}{n}\right) \),

\[
|\hat{p}_i^u - p_i^u| \leq C\sqrt{u_n^u}. \tag{70}
\]

Therefore, we have

\[
P\{|\hat{p}_i^u - p_i^u| > C\sqrt{u_n^u}\} \leq P\{|\hat{p}_i^u - p_i^u| > C\sqrt{u_n^u}\} + P\{|p_i^u - p_i^u| > C\sqrt{u_n^u}\} \leq O\left(\frac{1}{n}\right). \tag{71}
\]

Here we used the fact that:

\[
|\max\{a, c\} - \max\{b, d\}| > u \quad \Rightarrow \quad |a - b| > u \quad \text{or} \quad |c - d| > u.
\]
Note that (71) is equivalent to saying that
\[ P(\hat{p}_i - C\sqrt{u_i}, \hat{p}_i + C\sqrt{u_i}) > 1 - O\left(\frac{1}{n}\right). \]  
(72)

Now also note that the interval \( I_{i+1} \) in our algorithm is chosen to be
\[ [\hat{p}_i - \frac{\log n \bar{p}_i - p_i^u}{\kappa_i^u}, \hat{p}_i + \frac{2\log n \bar{p}_i - p_i^u}{\kappa_i^u}], \]
which is of order \( \sqrt{\log n} \) greater than \( \sqrt{u_i} \) (and according to the way we defined \( \kappa_i^u \) and \( \tau_i^u \), the two terms in \( u_i^i \) are of the same order). Therefore we know that with probability
\[ 1 - O\left(\frac{1}{n}\right), \ p^D \in I_{i+1} \]

- \( p^u < p^c \): In this case, \( p^D = p^c \). With the same argument, but only the \( p^c \) part, we know that with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p^D \in I_{i+1}^u \)

Also, as claimed in the previous lemma, the number of steps \( N^u \) doesn’t depend on \( n \) when \( \delta < \frac{1}{2} \) is fixed. Therefore, we can take a union bound over \( N^u \) steps, and claim that with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p^D \in I_{i+1}^u \) for all \( i = 1, \ldots, N^u \).

**Part 2:** Using the same argument as in Part 1, we know that with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p^D \in I_{i+1}^u \) for \( i = 1, \ldots, i_0 \). Now at \( i_0 \), condition (16) is triggered. We first claim that whenever (16) is satisfied, with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p^D = p^c \).

By the argument in Part 1, we know that with probability \( 1 - O\left(\frac{1}{n}\right) \),
\[ |\hat{p}^c_{i_0} - p^c_{i_0}| \leq \sqrt{\log n} \cdot \frac{p^u_{i_0} - p^c_{i_0}}{\kappa_{i_0}^c} \]
\[ |\hat{p}^c_{i_0} - p^c_{i_0}| \leq \sqrt{\log n} \cdot \frac{p^u_{i_0} - p^c_{i_0}}{\kappa_{i_0}^c}. \]

Therefore, if
\[ \hat{p}^c_{i_0} > \hat{p}^c_{i_0} + 2\sqrt{\log n} \cdot \frac{p^u_{i_0} - p^c_{i_0}}{\kappa_{i_0}^c}, \]  
(73)
holds, then with probability \( 1 - O\left(\frac{1}{n}\right) \),
\[ p^c_{i_0} > p^u_{i_0}. \]

And when (73) holds, \( p^c_{i} \) is not the left end-point of \( I_{i_0}^u \) and \( p^u_{i_0} \) is not the right end-point of \( I_{i_0}^u \), which means
\[ p^u \leq p^u_{i_0} < p^c_{i_0} \leq p^c = p^D. \]

Now we consider the procedure in Step 3 of our algorithm and show that with probability
\[ 1 - O\left(\frac{1}{n}\right), \ p^D = p^c \in I_{i_0}^c \]  
for all \( i = 1, 2, \ldots, N^c \).

We again prove by induction. By the induction assumption, we can assume that with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p^D = p^c \in I_{i_0}^c \). Now we consider \( \hat{q}_i^c \) (which is the optimal empirical solution in Step 3 in our algorithm). Define in this case
\[ v_i^c = \sqrt{\log n} \cdot \max\left\{ \frac{p_c^c - p_c^u}{\kappa_i^c}, \sqrt{\frac{\kappa_i^c}{n\tau_i^c}} \right\}. \]

Using the same discussion as in Part 1, we have with probability \( 1 - O\left(\frac{1}{n}\right) \) that
\[ |\hat{q}_i^c - p_c^c| = |\hat{q}_i^c - p^c| \leq Cv_i^c. \]

29
However, remember that in our algorithm, the next interval is defined to be
\[ I_{t+1}^c = [\hat{q}_t - \frac{\log n}{2} \cdot \frac{1}{\kappa_t^c} \cdot \hat{q}_t + \frac{\log n}{2} \cdot \frac{1}{\kappa_t^c}], \]
which is of order $\sqrt{\log n}$ larger than $\kappa_n^c$. Therefore with probability $1 - O \left( \frac{1}{n} \right)$, $p^D = p^c \in I_{t+1}^c$. Again, taking a union bound over these $N_\delta$ steps results in this lemma. \(\square\)

### 10.5 Proof of Lemma 4

**Proof.** Define $A_{ij}^u = \{ \omega : Y_{ij}^u - EY_{ij}^u \leq EY_{ij}^u \}$. First we show that

\[ \bigcap_{i,j} A_{ij}^u \subset A_1^u. \quad (74) \]

To show this we only need to show that $2 \sum_{i,j} EY_{ij}^u \leq n \epsilon$. Recall that $|\lambda(p)| \leq M$, for all $p \in [p, \bar{p}]$. We have:

\[ \sum_{i,j} EY_{ij}^u \leq M \sum_{i,j} n \Delta_i = Mn \sum_{i=1}^{N_\delta} \tau_i^u \leq MnN_\delta \tau_N^u. \quad (75) \]

By our definition of $\tau_i^u$, we know that for every fixed $\delta$, $\tau_N^u$ is of order less than 1 (otherwise the algorithm is stopped at the previous iteration). Therefore, when $n$ is large enough, $2 \sum_{i,j} EY_{ij}^u \leq n \epsilon$, i.e., \((74)\) holds uniformly in $n$.

However, by Lemma 14 we know that $P(A_{ij}^u) \geq 1 - O \left( \frac{1}{n^2} \right)$ and thus

\[ P(A_1) \geq 1 - O \left( \frac{1}{n^2} \right), \]

since each $\kappa_n^u \leq n$ and $N_\delta$ is independent on $n$.

Now we can rewrite the left-hand side of \((37)\) as:

\[ E \left[ \sum_{i=1}^{N_\delta} \sum_{j=1}^{N_\delta} p_{ij}^u Y_{ij}^u I(A_1^u) I(A_2^u) \right] \]

\[ = \sum_{i=1}^{N_\delta} \sum_{j=1}^{N_\delta} E[p_{ij}^u Y_{ij}^u I(A_1^u) I(A_2^u)] \]

\[ = \sum_{i=1}^{N_\delta} \sum_{j=1}^{N_\delta} E[p_{ij}^u I(A_1^u) I(A_2^u) E[Y_{ij}^u I((A_1^u)^c \cup (A_2^u)\c)]]. \quad (76) \]

However, by Cauchy-Schwartz inequality and the property of Poisson distribution ($EN(\lambda) = \lambda, E[N(\lambda)^2] = \lambda^2 + \lambda$)

\[ E[Y_{ij}^u I((A_1^u)\c \cup (A_2^u)\c)] \leq \sqrt{E(Y_{ij}^u)^2} \cdot P((A_1^u)\c \cup (A_2^u)\c) \leq O \left( \frac{1}{\sqrt{n}} \right) E[Y_{ij}^u]. \quad (77) \]
Now we plug this back into (76) and obtain

\[
E\left[\sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} p_{i,j}^n Y_{ij}^n I(A_i^n) I(A_2^n)\right]
\]

\[
\geq \left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right) \sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} E[p_{i,j}^n E Y_{ij}^n I(A_i^n) I(A_2^n)]
\]

\[
= \left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right) \sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} E[p_{i,j}^n \lambda(p_{i,j}^n) n \Delta_i^n I(A_i^n) I(A_2^n)]
\]

\[
= \left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right) \sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} E[p_{i,j}^n \lambda(p_{i,j}^n) n \Delta_i^n |I(A_i^n) I(A_2^n)|] P(A_i^n A_2^n)
\]

\[
= \left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right) \sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} E[p_{i,j}^n \lambda(p_{i,j}^n) n \Delta_i^n |I(A_i^n) I(A_2^n)|].
\]  

(78)

Now we consider

\[
\sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} E[p_{i,j}^n \lambda(p_{i,j}^n) n \Delta_i^n |I(A_i^n) I(A_2^n)|].
\]  

(79)

By the bound on the second derivative of the function \( r(\lambda) \) and the assumption that \( p^D = p^u \):

\[ p_{i,j}^u \lambda(p_{i,j}^u) = r(\lambda(p_{i,j}^u)) \geq r(\lambda(p^D)) - m_L (\lambda(p_{i,j}^u) - \lambda(p^D))^2 \geq p^D \lambda(p^D) - m_L K^2 (\bar{p}_i^u - \underline{p}_i^u)^2 \]  

(80)

Therefore, we have

\[
(1 - O\left(\frac{1}{\sqrt{n}}\right)) \sum_{i=1}^{N^n} \sum_{j=1}^{\kappa_i^n} E[p_{i,j}^u \lambda(p_{i,j}^u) n \Delta_i^n |I(A_i^n) I(A_2^n)|]
\]

\[
\geq (1 - O\left(\frac{1}{\sqrt{n}}\right)) \sum_{i=1}^{N^n} E[p^D \lambda(p^D) n \tau_i^n - m_L K^2 (\bar{p}_i^u - \underline{p}_i^u)^2 n \tau_i^n]
\]

\[
\geq (1 - O\left(\frac{1}{\sqrt{n}}\right)) \sum_{i=1}^{N^n} E[p^D \lambda(p^D) n \tau_i^n - m_L K^2 N^u n \tau_i^n]
\]

\[
\geq \sum_{i=1}^{N^n} p^D \lambda(p^D) n \tau_i^n - C n \tau_i^n,
\]  

(81)

where the second to last step is due to (22) and (23), and the last step is because \( \frac{1}{\sqrt{n}} \tau_i^n = n^{1/2-25-(1-6)(3/5)^{i-1}} (\log n)^5 < \tau_i^n \), for \( i = 1, ..., N^u \). Therefore, the lemma holds. \( \square \)

### 10.6 Proof of Lemma 5 and 6

**Proof of Lemma 5.** First we show that with probability \( 1 - O\left(\frac{1}{n}\right) \),

\[
\sum_{i,j} Y_{ij}^u + \hat{Y}^u - \sum_{i,j} E Y_{ij}^u - E \hat{Y}^u \leq C n \tau_1^n.
\]  

(82)
Then by Cauchy-Schwartz inequality,
\[ E[(\hat{Y}_i^n + \sum_{i,j} Y_{ij}^n - E\hat{Y}_i^n - \sum_{i,j} EY_{ij}^n)^+ \cdot I(\hat{Y}_i^n + \sum_{i,j} Y_{ij}^n - E\hat{Y}_i^n - \sum_{i,j} EY_{ij}^n > Cn\tau_1^n)] \]
\[ \leq O \left( \frac{1}{\sqrt{n}} \right) (E\hat{Y}_i^n + \sum_{i,j} EY_{ij}^n) \leq Cn\tau_1^n \] (83)
where the second to last inequality is because that for a Poisson random variable $N(\mu)$, $\text{Var}(N(\mu)) = \mu^2$ and the last inequality is because the demand rate is bounded and $\tau_1^n > n^{-1/2}$. Therefore, we have
\[ E(\hat{Y}_i^n + \sum_{i,j} Y_{ij}^n - E\hat{Y}_i^n - \sum_{i,j} EY_{ij}^n)^+ < Cn\tau_1^n \] (84)
which implies our lemma.

To show (82), we apply Lemma 14. For each given $i, j$, by Lemma 14 we have
\[ P(Y_{ij}^u - EY_{ij}^u > 2M\sqrt{n\Delta_i^n \log n}) \leq \frac{C}{n^2}. \]
By taking a union bound over all $i, j$, we get
\[ P(\sum_{i,j} Y_{ij}^u - \sum_{i,j} EY_{ij}^u > 2M\sum_{i,j} \kappa_i^n \sqrt{n\Delta_i^n \log n}) \leq \sum_{i=1}^{N_u} \sum_{j=1}^{N_u} P(Y_{ij}^u - EY_{ij}^u > 2M\sqrt{n\Delta_i^n \log n}) \leq O\left( \frac{1}{n} \right), \] (85)
where the last step is because $\kappa_i^n < n$ and $N_i \geq N_u$ is a constant with respect to $n$.

On the other hand, by the definition of $\tau_i^n$ and $\kappa_i^n$, we have
\[ 2M \sum_i \kappa_i^n \sqrt{n\Delta_i^n \log n} = 2M \sqrt{n \log n} \sum_{i=1}^{N_u} \sqrt{\kappa_i^n \tau_i^n} \leq C N^\delta n \tau_1^n, \] (86)
where the last inequality follows from the definition of $\kappa_i^n$ and $\tau_i^n$ in (29) and (30). We then consider $\hat{Y}_u - E[\hat{Y}_u]$, again use inequality (59). We have
\[ P(\hat{Y}_u - E[\hat{Y}_u] > 2M\sqrt{n \log n}) < \frac{C}{n^2}, \]
since $\sqrt{n \log n} < n\tau_1^n$ when $\delta < \frac{1}{2}$, the lemma holds.  \(\square\)

**Proof of Lemma 6** By definition, we have
\[ EY_{ij}^u = n\lambda(p_{i,j}^u)\Delta_i^n, \]
where
\[ p_{i,j}^u = p_i^u + (j - 1) \frac{p_i^u - p_j^u}{\kappa_i^u} = \hat{p}_{i-1} - \frac{\log n}{3} \cdot \frac{p_{i-1}^u - p_i^u}{\kappa_{i-1}^u} + (j - 1) \frac{p_i^u - p_j^u}{\kappa_i^u}. \]
And by our above discussion, with probability $1 - O\left( \frac{1}{\sqrt{n}} \right)$, condition (16) doesn’t hold, i.e.,
\[ \hat{p}_{i-1} \geq \hat{p}_{i-1}^c - 2\sqrt{\log n} \cdot \frac{p_{i-1}^u - p_i^u}{\kappa_{i-1}^u}. \] And as we showed in (70), \[ \hat{p}_{i-1}^c \geq \hat{p}_{i-1}^c - \sqrt{\log n} \cdot \frac{p_{i-1}^u - p_i^u}{\kappa_{i-1}^u}. \]
Also since $p^u \geq p^c$, we must have $\hat{p}_{i-1}^c \geq \hat{p}_{i-1}^c$. Therefore we have,
\[ p_{i,j}^u \geq p^c + (j - 1) \frac{p_i^u - p_j^u}{\kappa_i^u} = \frac{\log n}{3} \cdot \frac{p_{i-1}^u - p_i^u}{\kappa_{i-1}^u} - 3\sqrt{\log n} \cdot \frac{p_{i-1}^u - p_i^u}{\kappa_{i-1}^u} \]
\[ \geq p^c + (j - 1) \frac{p_i^u - p_j^u}{\kappa_i^u} = \frac{\log n}{2} \cdot \frac{p_{i-1}^u - p_i^u}{\kappa_{i-1}^u} \] (87)
when $\sqrt{\log n} \geq 6$. Using the Taylor expansion for $\lambda(p)$, we have that

$$\lambda(p_{i,j}) \leq \lambda(p^c) + ((j - 1) \frac{p_{i,j}^u - p_{i,j}^\mu}{\kappa_{i,j}^u} - \frac{\log n}{2} \cdot \frac{p_{i,j}^u - p_{i,j}^{\mu - 1}}{\kappa_{i,j}^{\mu - 1}}) \lambda'(p^c) + \sup \{ \lambda''(p^c) \} (p_{i,j}^u - p_{i,j}^\mu)^2.$$ (88)

The last inequality uses the fact that $\lambda''(p)$ is bounded by a constant which is not hard to derive from Assumption A. Therefore

$$\sum_{i,j} EY_{ij}^u \leq n\lambda(p^c) t_{N^u}^u + Cn \sum_{i=1}^{N^u} \tau_i^u (p_{i,j}^u - p_{i,j}^c)^2 \leq n\lambda(p^c) t_{N^u}^u + Cn \tau_{i}^u,$$ (89)

where the last equation follows from (22) and (23).

Also we have

$$EY^u = \lambda(\hat{p}) n(1 - t_{N^u}^u),$$

and with probability $1 - O \left( \frac{1}{n} \right)$,

$$\hat{p} = \hat{p}_{N^u} + 2\sqrt{\log n} \cdot \frac{p_{N^u}^u - p_{N^u}^c}{\kappa_{N^u}^u} \geq \max(\hat{p}_{N^u}, \hat{p}_{N^u}^c) + 2\sqrt{\log n} \cdot \frac{p_{N^u}^u - p_{N^u}^c}{\kappa_{N^u}^u} \geq p^D \geq p^c,$$ (90)

where the first equation is due to the definition of $\hat{p}$, the second one is due to (71), and the last one is by Lemma 2. Therefore,

$$EY^u \leq \lambda(p^c) n(1 - t_{N^u}^u)$$

and thus

$$\sum_{i,j} EY_{ij} + EY^u \leq n x + Cn \tau_{i}^u.$$

□

### 10.7 Proof of Lemma 7, 8, 9, and 10

**Proof of Lemma 7.** Define $A_1 = \{ \omega : p^D \in I_i^u \text{ for all } i \}$. By Lemma 2, we know that $P(A_1) \geq 1 - O \left( \frac{1}{n} \right)$, and we have

$$E[\sum_{i=1}^{N^u} \sum_{j=1}^{\kappa_i^u} p_{i,j}^u Y_{ij}^u I(\cup_{l=1}^{N^u+1} B_l)] \geq \sum_{i=1}^{N^u} \sum_{j=1}^{\kappa_i^u} E[p_{i,j}^u E[Y_{ij}^u I(\cup_{l=1}^{N^u+1} B_l)] \cdot I(A_i)]$$ (91)

However, note that $\cup_{l=1}^{N^u+1} B_l = (\cup_{i=1}^{i-1} B_l)^c$ only depends on the realization up to period $i - 1$. 

33
Therefore, we know that $Y_{ij}^u$ given $p_{ij}^u$ is independent of $\cup_{l=i}^{N^u+1} B_l$. Therefore, we have

$$E\left[\sum_{i=1}^{N^u} \sum_{j=1}^{k^u} p_{ij}^u Y_{ij}^u I(\cup_{l=i}^{N^u+1} B_l)\right] \geq \sum_{i=1}^{N^u} \sum_{j=1}^{k^u} E[n \Delta_i^u p_{ij}^u \lambda(p_{ij}^u) I(\cup_{l=i}^{N^u+1} B_l) I(A_1)]$$

$$\geq (1 - O\left(\frac{1}{n}\right)) \sum_{i=1}^{N^u} \sum_{j=1}^{k^u} E[n \Delta_i^u p_{ij}^u \lambda(p_{ij}^u) I(\cup_{l=i}^{N^u+1} B_l) I(A_1)]$$

$$\geq (1 - O\left(\frac{1}{n}\right)) \sum_{i=1}^{N^u} (n \tau^u_i p^D \lambda(p^D) P(\cup_{l=i}^{N^u+1} B_l) - n \tau^u_i m L K^2 \left(\frac{p^u_i - p^u_l}{\kappa^u_i}\right)^2)$$

$$\geq \sum_{i=1}^{N^u} n \tau_i p^D \lambda(p^D) P(\cup_{l=i}^{N^u+1} B_l) - Cn \tau_1,$$

(92)

where the second to last inequality is because of the bounded second derivative of the revenue function and that $A$ holds; and the last inequality is because of the relation of $[23]$ and that $N^u$ is bounded by a constant $N^u$ which is independent with $n$. □

**Proof of Lemma 8** Define

$$A_2 = \{ \omega : \text{For each } i, j, Y_{ij}^u \leq EY_{ij}^u + 2M \sqrt{n \Delta_i^u \log n} \text{ and } \hat{Y}_{ij}^u \leq E\hat{Y}_{ij}^u + 2M \sqrt{n \log n} \}. \quad (93)$$

By Lemma 14, $P(A_2) = 1 - O\left(\frac{1}{n}\right)$. Similar to (38), we have the following relation:

$$E[\hat{p} \min(\hat{Y}_{ij}^u, (nx - \sum_{i,j} Y_{ij}^u)^+) I(A_2) I(B_{N^u+1})]$$

$$\geq E[\hat{p} \hat{Y}_{ij}^u I(A_2) I(B_{N^u+1})] - E[\hat{p}(\hat{Y}_{ij}^u + \sum_{i,j} Y_{ij}^u - nx)^+ I(A_2) I(B_{N^u+1})].$$

(94)

And by the same argument as (41) we have

$$E[\hat{p} \hat{Y}_{ij}^u I(A_2) I(B_{N^u+1})] \geq (1 - O\left(\frac{1}{n}\right)) (p^D \lambda(p^D) \cdot n(1 - t_{N^u}) P(B_{N^u+1})) - Cn \tau^u_i$$

$$\geq p^D \lambda(p^D) \cdot n(1 - t_{N^u}) P(B_{N^u+1}) - Cn \tau^u_i.$$

(95)

Now we consider

$$E[\hat{p}(\hat{Y}_{ij}^u + \sum_{i,j} Y_{ij}^u - nx)^+ I(A_2) I(B_{N^u+1})].$$

(96)

We first relax it to

$$\hat{p} E[(\hat{Y}_{ij}^u + \sum_{i,j} Y_{ij}^u - nx)^+ I(A_2) I(B_{N^u+1})].$$

(97)

Conditional on $A$, $Y_{ij}^u \leq EY_{ij}^u + 2M \sqrt{n \Delta_i^u \log n}$ for all $i, j$ and $\hat{Y}_{ij}^u \leq E\hat{Y}_{ij}^u + 2M \sqrt{n \log n}$, and by the argument in Lemma 3, we have $\sum_{i,j} 2M \sqrt{n \Delta_i^u \log n} + 2M \sqrt{n \log n} \leq Cn \tau^u_1$. Also, by the same argument as in Lemma 4, we have $\sum_{i,j} EY_{ij}^u + E\hat{Y}_{ij}^u - nx \leq Cn \tau^u_1$. Therefore, the lemma holds. □

34
Proof of Lemma 9

Define $A_3 = \{ \omega : p^D \in I_i^c, \forall i \}$. By Lemma 2, we know that $P(A_3) = 1 - O \left( \frac{1}{n} \right)$. Also note that each $Y_{ij}^c$ given $p_{ij}^c$ is independent with $\cup_{i=1}^{N^c} B_i$. Therefore we have,

$$\sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_i^c} E[ p_{ij}^c Y_{ij}^c I(\cup_{i=1}^{N^c} B_i)]$$

$$\geq \sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_i^c} E[ p_{ij}^c E[Y_{ij}^c I(\cup_{i=1}^{N^c} B_i)] I(\cup_{i=1}^{N^c} B_i) I(A_3)]$$

$$= \sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_i^c} E[ n \Delta_i^c p_{ij}^c \lambda(p_{ij}^c) I(\cup_{i=1}^{N^c} B_i) I(A_3)]$$

$$\geq (1 - O \left( \frac{1}{n} \right) ) (\sum_{i=1}^{N^c} n \tau_i^c p^D \lambda(p^D) P(\cup_{i=1}^{N^c} B_i) - \sum_{i=1}^{N^c} M(p_i^c - p_i^c) \tau_i^c n)$$

$$\geq \sum_{i=1}^{N^c} n \tau_i^c p^D \lambda(p^D) P(\cup_{i=1}^{N^c} B_i) - C n \tau_1^c,$$

(98)

where the equality is because of the independence of $Y_{ij}^c$ and $B_i$ as we argued above, the second inequality is because $p^D \in I_i^c$ for all $i$ and the Lipschitz continuity of the revenue rate function. The last inequality is because of the relation (27) and that $N^c$ is bounded by a constant $N^\delta$ that is not dependent on $n$. □

Proof of Lemma 10

Define

$$A_4 = \{ \omega : Y_{ij}^u \leq EY_{ij}^u + 2M \sqrt{n \Delta_i^u} \log n \text{ and } Y_{ij}^c \leq EY_{ij}^c + 2M \sqrt{n \Delta_i^c} \log n, \forall i, j; \hat{Y}_i^c \leq E \hat{Y}_i^c + 2M \sqrt{n} \log n \text{ and } p^D \in I_i^c, \forall i \}.$$ (99)

By Lemma 14, we have that $P(A_4) = 1 - O \left( \frac{1}{n} \right)$.

By the same transformation as in (94), we have

$$E[ \hat{q} \min(\hat{Y}_i^c, (nx - \sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_i^c} Y_{ij}^u - \sum_{i=1}^{N^c} \sum_{j=1}^{\kappa_i^c} Y_{ij}^c)] I(B_i)]$$

$$\geq E[ \hat{q} \hat{Y}_i^c I(B_i) I(A_4)] - E[ \hat{q} \sum_{i,j} Y_{ij}^c + \sum_{i,j} \hat{Y}_{ij}^c - nx)^+ I(B_i) I(A_4)].$$

(100)

For the first term, we have

$$E[ \hat{q} \hat{Y}_i^c I(B_i) I(A_4)]$$

$$\geq (1 - O \left( \frac{1}{n} \right) ) E(\hat{q} E[\hat{Y}_i^c] I(B_i) I(A_4))$$

$$\geq (1 - O \left( \frac{1}{n} \right) ) E(n(1 - t_i^u - t_i^c) \hat{q} \lambda(\hat{q}) I(B_i))$$

$$\geq (1 - O \left( \frac{1}{n} \right) ) (n(1 - t_i^u - t_i^c) p^D \lambda(p^D) P(B_i) - n(1 - t_i^u - t_i^c)(\bar{p}_{N^c+1} - \bar{p}_{N^c+1}))$$

$$\geq n(1 - t_i^u - t_i^c) p^D \lambda(p^D) P(B_i) - C n \tau_1^c,$$

(101)

35
where the third inequality is because of the definition of $\tilde{q}$ and the Lipschitz continuity of the revenue rate function. And the last inequality is due to condition (28).

For the second term, we first relax it to

$$\bar{p}E[(\sum_{i,j} Y_{ij}^c + \sum_{i=1}^l \sum_{j=1}^{\kappa^u_i} Y_{ij}^u + \tilde{Y}_i^c - nx)^+ I(B_l)I(A_4)].$$

And by the definition of $A_4$, we have,

$$E[(\sum_{i,j} Y_{ij}^c + \sum_{i=1}^l \sum_{j=1}^{\kappa^u_i} Y_{ij}^u + \tilde{Y}_i^c - nx)^+] \leq \left(\sum_{i,j} EY_{ij}^c + \sum_{i=1}^l \sum_{j=1}^{\kappa^u_i} EY_{ij}^u + E\tilde{Y}_i^c - nx\right)^+ + Cn\tau^c_i \leq Cn\tau^c_i.$$

(102)

The first inequality is because as we argued in Lemma 5, \sum_{i,j} 2M \sqrt{n} \Delta u_i \log n + \sum_{i,j} 2M \sqrt{n} \Delta c_j \log n + M \sqrt{n} \log n \leq Cn\tau^u_i, and the second inequality is because

$$\sum_{i,j} EY_{ij}^c + \sum_{i=1}^l \sum_{j=1}^{\kappa^u_i} EY_{ij}^u + E\tilde{Y}_i^c \leq \sum_{i=1}^l n\tau^u_i \lambda(p_i^u) + \sum_{i=1}^l n\tau^c_i \lambda(p_i^c) + n(1 - t^u_i - t^c_i) \lambda(\tilde{q})$$

$$\leq \sum_{i=1}^l Cn\tau^u_i \lambda(p_i^D) + C_1 n\tau^c_i + n(1 - t^u_i - t^c_i) \lambda(p_i^D) + C_2 n\tau^c_i$$

$$\leq nx + Cn\tau^u_i,$$

(103)

where the first inequality is by definition, the first part of the second inequality is by Lemma 6, the second and third part of the second inequality is due to the continuity of the demand rate function and that $p_i^D \in [\tilde{q}_i, p_i^c]$; so that $p_i^c - p_i^D \leq M(p_i^c - p_i^u)$ and the relation in (27). And the last inequality is because when $p_i^c > p_i^u$, $\lambda(p_i^D)$ is $x$ (remember that we assumed $T = 1$).

Combining (101) and (103) together proves that the lemma holds. $\square$

### 10.8 Proof of Lemma 12

**Proof.** Consider the final term in (54). Note that we have the following simple inequality:

$$\log(x + 1) \geq x - \frac{x^2}{2(1 - |x|)}, \quad \forall x < 1.$$  

Therefore, we have

$$-\log x - 1 \leq -x + \frac{(x - 1)^2}{2(2 - x)}, \quad \forall 0 < x < 2.$$  

Apply this relationship to the final term in (54) and note that for any $z \in [1/3, 2/3]$ and $p(s) \in [1/2, 3/2]$,

$$\frac{2}{3} \leq \frac{1}{2} + z - zp(s) \leq 2,$$  

(104)
we have
\[ K(P_{z_0}^\pi, P_z^\pi) \leq n E_{z_0}^\pi \int_0^1 \frac{1}{2(2 - \frac{1}{2+z_0-z_0p(s)})} (z_0 - z)^2 (1 - p(s))^2 \, ds \leq n E_{z_0}^\pi \int_0^1 (z_0 - z)^2 (1 - p(s))^2 \, ds. \] (105)

Also, for \( z \in [1/3, 2/3] \) and \( p(s) \in [1/2, 3/2] \), we have
\[ \frac{1}{2} + z_0 - z_0 p(s) \geq \frac{1}{4}. \] (106)

Therefore, we have
\[ K(P_{z_0}^\pi, P_z^\pi) \leq 16 n (z_0 - z)^2 E_{z_0}^\pi \int_0^1 (1 - p(s))^2 \, ds. \]

However, under the case when the parameter is \( z_0 \), we have \( p^D = 1 \) and
\[ R_n^\pi(x, T; z_0) = 1 - \frac{J_0^\pi(x, T; z_0)}{J_t^\pi(x, T; z_0)} \geq \frac{E_{z_0}^\pi \int_0^1 (r(p^D) - r(p(s))) \, ds}{E_{z_0}^\pi \int_0^1 r(p^D) \, ds} \geq \frac{2}{3} E_{z_0}^\pi \int_0^1 (1 - p(s))^2 \, ds, \] (107)

where the first inequality follows from the definition of \( J^D \) and that we relaxed the inventory constraint, and the second inequality is because of the 4th condition in Lemma 11. Therefore,
\[ K(P_{z_0}^\pi, P_z^\pi) \leq 24 n (z_0 - z)^2 R_n^\pi(x, T; z_0), \] (108)

and Lemma 12 holds. \( \square \)

10.9 Proof of Lemma 13

Proof.
We define two intervals \( C_{z_0} \) and \( C_{z_1} \) as follows:
\[ C_{z_0} = [p^D(z_0) - \frac{1}{48n^{1/2}}, p^D(z_0) + \frac{1}{48n^{1/2}}] \text{ and } C_{z_1} = [p^D(z_1^\pi) - \frac{1}{48n^{1/2}}, p^D(z_1^\pi) + \frac{1}{48n^{1/2}}]. \]

Note that by the 5th property in Lemma 11 we know that \( C_{z_0} \) and \( C_{z_1} \) are disjoint.

By the 4th property in Lemma 11 we have for any \( z \),
\[ r(p^D(z); z) - r(p; z) \geq \frac{2}{3} (p - p^D(z))^2. \] (109)

Also by the definition of the regret function, we have
\[ R_n^\pi(x, T; z_0) \geq \frac{E_{z_0}^\pi \int_0^1 \{r(p^D(z_0); z_0) - r(p(s); z_0)\} \, ds}{E_{z_0}^\pi \int_0^1 r(p^D(z_0); z_0) \, ds} \geq \frac{4}{3(48)^2 \sqrt{n}} E_{z_0}^\pi \int_0^1 I(p(s) \in C_{z_1}) \, ds = \frac{4}{3(48)^2 \sqrt{n}} \int_0^1 P_{z_0}^\pi(s)(p(s) \in C_{z_1}) \, ds, \] (110)
where the first inequality is because we relaxed the inventory constraint when using $\pi$, and the second inequality is because of [109], the definition of $C_{z_1^n}$ and that the denominator is 1/2. In the last equality, $P_{z_0}^{(s)}$ is the probability measure under policy $\pi$ and up to time $s$ (with underlying demand function has parameter $z_0$). Similarly, we have

$$R_n^\pi(x, T; z_1^n) \geq \frac{E_{z_1^n}\int_0^1 \{ r(p^{D}(z_1^n); z_1^n) - r(p(s); z_1^n) \}ds}{E_{z_1^n}\int_0^1 r(p^{D}(z_1^n); z_1^n)ds}$$

$$\geq \frac{2}{3(48)^2\sqrt{n}} E_{z_1^n}\int_0^1 I(p(s) \notin C_{z_1^n})ds$$

$$= \frac{2}{3(48)^2\sqrt{n}} \int_0^1 P_{z_1^n}(p(s) \notin C_{z_1^n})ds,$$

(111)

where in the second inequality, we use the fact that the denominator is less than 1, and in the last equality, $P_{z_1^n}$ is defined similarly as the one in [110].

Now consider any decision rule that maps historical demand observations up to time $s$ into one of the following two sets:

$$H_1 : p(s) \in C_{z_1^n}$$

$$H_2 : p(s) \notin C_{z_1^n}.$$

By Theorem 2.2 in [25], we have the following bound on the probability error of any decision rule:

$$P_{z_0}^{(s)}(\{p(s) \in C_{z_1^n}\} + P_{z_1^n}^{(s)}(\{p(s) \notin C_{z_1^n}\}) \geq \frac{1}{2} e^{-K(P_{z_0}^{(s)}, P_{z_1^n}^{(s)})}.$$

(112)

However, by the definition of the K-L divergence [54], we know that

$$K(P_{z_0}^{(s)}, P_{z_1^n}^{(s)}) \geq E_{z_0}\int_0^1 \frac{1}{2(2 - \frac{1}{2+z_0-z_0p(s)})} (z_0 - z)^2 (1 - p(s))^2 ds \geq 0,$$

(113)

where the last inequality is because of [106]. Therefore, we have

$$P_{z_0}^{(s)}(\{p(s) \in C_{z_1^n}\} + P_{z_1^n}^{(s)}(\{p(s) \notin C_{z_1^n}\}) \geq \frac{1}{2} e^{-K(P_{z_0}^{(s)}, P_{z_1^n}^{(s)})}$$

(114)

Now we add (110) and (111) together. We have

$$R_n^\pi(x, T; z_0) + R_n^\pi(x, T; z_1^n) \geq \frac{2}{3(48)^2\sqrt{n}} \int_0^1 \{ P_{z_0}^{(s)}(p(s) \in C_{z_1^n}) + P_{z_1^n}^{(s)}(p(s) \notin C_{z_1^n}) \}ds$$

$$\geq \frac{1}{3(48)^2\sqrt{n}} e^{-K(P_{z_0}^{(s)}, P_{z_1^n}^{(s)})}.$$

Thus, Lemma 13 holds. □

10.10 Proof of Theorem 3

We first define Algorithm DPA2.

Algorithm DPA2:

Step 1. Initialization
(a) Consider a sequence of $\tau_i, \kappa_i, i = 1, 2, ..., N$ ($N$ will be defined later). Define $p^1 = p$ and $\overline{p}^1 = \overline{p}$. Define $t_i = \sum_{j=1}^{i} \tau_j$, for $i = 0$ to $N$;

**Step 2. Dynamic Learning**

For $i = 1$ to $N$ do

(a) Divide $[\underline{p}, \overline{p}]$ into $\kappa_i$ equally spaced intervals and let $\{p_{i,j}, j = 1, 2, ..., \kappa_i\}$ be the left endpoints of these intervals

(b) Divide the time interval $[t_{i-1}, t_i]$ into $\kappa_i$ parts and define

$$\Delta_i = \frac{\tau_i}{\kappa_i}, \quad t_{i,j} = t_{i-1} + j\Delta_i, \quad j = 0, 1, ..., \kappa_i$$

(c) Apply $p_{i,j}$ from time $t_{i,j-1}$ to $t_{i,j}$, as long as the inventory is still positive. If no more units are in stock, apply $p_{\infty}$ until time $T$ and STOP

(d) Compute

$$\hat{d}(p_{i,j}) = \frac{\text{total demand over } [t_{i,j-1}, t_{i,j}]}{\Delta_i}$$

(e) Compute

$$\hat{p}_i^u = \arg \max_{1 \leq j \leq \kappa_i} \{p_{i,j}\hat{d}(p_{i,j})\}$$

and

$$\hat{p}_i^c = \arg \min_{1 \leq j \leq \kappa_i} |\hat{d}(p_{i,j}) - x/T|$$

(f) Set $\hat{p}_i = \max\{\hat{p}_i^u, \hat{p}_i^c\}$. Define

$$p^{i+1} = p_i - \frac{\log n}{2} \cdot \frac{\overline{p}^i - p^i}{\kappa_i}$$

and

$$\overline{p}^{i+1} = p_i + \frac{\log n}{2} \cdot \frac{\overline{p}^i - p^i}{\kappa_i}$$

And the price range for the next iteration

$$I_{i+1} = [\underline{p}^{i+1}, \overline{p}^{i+1}]$$

Here we truncate the interval if it doesn’t lie inside the feasible set of $[\underline{p}, \overline{p}]$;

**Step 3. Applying the optimal price**

(a) Use $\hat{p}_N$ for the rest of time until the stock is run out.

Similarly as we study algorithm DPA, we first list the set of equations we want the parameters to satisfy:

$$\frac{\overline{p}_i - p_i}{\kappa_i} \sim \sqrt{\frac{\kappa_i}{n\tau_i}}, \quad \forall i = 1, ..., N$$

$$\overline{p}_{i+1} - p_{i+1} \sim \log n \cdot \frac{\overline{p}_i - p_i}{\kappa_i}, \quad \forall i = 1, ..., N - 1$$

$$\tau_{i+1} \cdot \frac{\overline{p}_i - p_i}{\kappa_i} \cdot \sqrt{\log n} \sim \tau_1, \quad \forall i = 1, ..., N - 1.$$
Also we define
\[ N = \min \{ I : \sqrt{\log n} \cdot \frac{\sqrt{n_i - p_i}}{\sqrt{\kappa_i}} < \tau_i \}. \quad (118) \]

The meaning of each of these equations is similar to the one in DPA (but since we have a different local behavior of the revenue function under this alternative assumption, we have different relationships between these parameters). We solve \( \kappa_i \) and \( \tau_i \) from the above relations. Define \( \tau_1 = n^{-\delta} \cdot (\log n)^3 \). We get
\[
\kappa_i = n^{2(1-\delta)}i(\frac{3}{4})^{i-1} \log n, \quad \forall i 
\]
\[
\tau_i = n^{1-2\delta-i(1-\delta)}i^{i-1} (\log n)^3, \quad \forall i 
\]
And as a by-product, we have
\[
\bar{p}_i - \hat{p}_i = n^{-1-\delta}i(1-\frac{3}{4})^{i-1}, \quad \forall i 
\]

Now we prove some lemmas similar to those we used to prove Theorem 1.

**Lemma 15.** Fix \( \delta < 1/2 \). \( N \) defined in (118) exists. Moreover, \( N \) is independent of \( n \).

**Proof.** We plug (119) and (121) into (118). We get \( N = \log \frac{\sqrt{n} \sqrt{\log n}}{\sqrt{\kappa_i}} \) which does not depend on \( n \).

Now we prove a lemma similar to Lemma 2 showing that the price range for each learning period contains the actual optimal price, with high probability.

**Lemma 16.** Denote the optimal deterministic price by \( p^D \). And the assumption that \( L|p^u - p| \leq |r(\lambda(p^u)) - r(\lambda(p))| \) hold. Then with probability \( 1 - O \left( \frac{1}{n^2} \right) \), \( p^D \in I_i \) for any \( i = 1, \ldots, N \).

**Proof.** Like the proof of Lemma 2 we prove by induction. Assume that with probability \( 1 - O \left( \frac{1}{n^2} \right) \), \( p^D \in I_i \). Now consider the \((i+1)\)th interval. Define
\[
u_i = \sqrt{\log n} \max \left\{ \frac{\bar{p}_i - \hat{p}_i}{\kappa_i}, \frac{\sqrt{\kappa_i}}{n\tau_i} \right\}. \]

(122)

We consider three cases:

- \( p^u_i < p^u \): This is impossible since we know that \( p^D \geq p^u > p^u_i \) and by the induction assumption \( p^D \in I_i \). Therefore, we must have \( p^u \in I_i \) and by definition, \( p^u \) achieves larger revenue rate than \( p^u_i \), which is contradictory to the definition of \( p^u_i \).

- \( p^u_i = p^u \): We assume \( p_{i,j}^* \) is the nearest grid point to \( p^u_i \) in this iteration. In this case, by the granularity of the grid at iteration \( i \), we have \( |p_{i,j} - p^u| \leq \frac{\bar{p}_i - \hat{p}_i}{\kappa_i} \) and thus by our assumption that \( r(p) \) is Lipschitz continuous, we know that \( |r(\lambda(p^u)) - r(\lambda(p_{i,j}^*)))| \leq C \cdot \frac{\bar{p}_i - \hat{p}_i}{\kappa_i} \), therefore we have:
\[
r(\lambda(p^u)) - r(\lambda(p_{i,j}^*))) = r(\lambda(p^u)) - r(\lambda(p_{i,j}^*))) + p^u_{i,j} \lambda(p_{i,j}^*) - p^u_{i,j} \hat{\lambda}(p_{i,j}^*) + (\hat{p}_i^u \lambda(p_i^u) - \hat{p}_i^u \hat{\lambda}(p_i^u)) + p_{i,j}^* \hat{\lambda}(p_{i,j}^*)) - \hat{p}_i^u \hat{\lambda}(p_i^u) 
\leq C \frac{\bar{p}_i - \hat{p}_i}{\kappa_i} + 2 \max_{1 \leq j \leq \kappa_i} |p_{i,j} \lambda(p_{i,j}) - p_{i,j} \hat{\lambda}(p_{i,j})|. \]

(123)

In (123), \( \hat{\lambda} \) is the observed demand rate, and the last inequality is due to the definition of \( \hat{p}_i^u \) and that \( \hat{p}_i^u \) is among one of the \( p_{i,j} \).

By Lemma 14 in Appendix 10.2 we have
\[
P(|\hat{\lambda}(p_{i,j}) - \lambda(p_{i,j})| > \sqrt{\log n} \cdot \sqrt{\frac{\kappa_i}{n\tau_i}} \leq \frac{1}{n^2}) \leq C \sqrt{\log n} \cdot \sqrt{\frac{\kappa_i}{n\tau_i}} \leq \frac{1}{n^2} \]

(124)

40
Therefore, we have

Now also note that the interval

which is of order

which is of order

Note that (131) is equivalent of saying that

Therefore, with probability 1

by Lemma 14 in Appendix 10.2, we have with probability 1

by the Lipshitz continuity of \( \lambda \), we have

Also by Lemma 14 in Appendix 10.2, we have with probability 1

Therefore, with probability 1

and by the Lipschitz continuity of \( \nu(\lambda) \), this implies that with probability 1

Therefore, we have

Here we used the fact that:

Note that (131) is equivalent of saying that

Now also note that the interval \( I_{i+1} \) in our algorithm is chosen to be

which is of order \( \sqrt{\log n} \) greater than \( \sqrt{u_n^i} \) (and according to the way we defined \( \kappa_i \) and \( \tau_i \), the two terms in \( u_n^i \) are of the same order). Therefore we know that with probability

1 – \( O \left( \frac{1}{n} \right), p^D \in I_{i+1} \).
• $p^u < p^w$: In this case, $p^D = p^c$. With the same argument, but only the $p^c$ part, we know that with probability $1 - O\left(\frac{1}{n}\right)$, $p^D \in I_i$ for all $i = 1, ..., N$. □

Also, as claimed in the previous lemma, the number of steps $N$ doesn’t depend on $n$ when $\delta < \frac{1}{2}$ is fixed. Therefore, we can take a union bound over $N$ steps, and claim that with probability $1 - O\left(\frac{1}{n}\right)$, $p^D \in I_i$ for all $i = 1, ..., N$.

Now we have proved that with high probability, $p^D$ will always be in our interval. Next we will analyze the revenue collected by this algorithm.

Define $Y_{ij}$ to be the Poisson random variable with parameter $\lambda(p_{ij})n\Delta_i$ ($Y_{ij} = N(\lambda(p_{ij})n\Delta_i)$). Also define $\hat{Y}$ to be a Poisson random variable with parameter $\lambda(\hat{p}_N)n(1-t_N)$ ($\hat{Y} = N(\lambda(\hat{p}_N)n(1-t_N))$). We define the following event:

$$A_1 = \{\omega : \sum_{i,j} Y_{ij} < nx\}.$$ 

We have

$$J_N^\alpha(x, T; \lambda) \geq E[\sum_{i,j=1}^{N} p_{ij} Y_{ij} \mathbb{1}(A_1)] + E[p \min(\hat{Y}, (nx - \sum_{i,j} Y_{ij})^+) \mathbb{1}(A_1)].$$

(133)

In the following, we will consider each term in (133). We will show that the revenue collected in both parts is “close” to the revenue generated by the optimal deterministic price $p^D$ in its corresponding part (and the consumed inventory is also near-optimal). We first have:

**Lemma 17.**

$$E[\sum_{i,j=1}^{N} p_{ij} Y_{ij} \mathbb{1}(A_1)] \geq \sum_{i,j=1}^{N} p^D \lambda(p^D)n\tau_i - Cn\tau_1.$$  

(134)

**Proof.** The proof of this lemma is almost the same of the proof of Lemma II in Appendix 10.5 except that in (80), $mLK(p_i - \bar{p}_i)^2$ is replaced by $L(p_i - \bar{p}_i)$. In (81), We use the relationship defined in (117), the result follows. □

Now we look at the other term in (133). We have

$$E[p \min(\hat{Y}, (nx - \sum_{i,j} Y_{ij})^+)] = E[p \min(\hat{Y} - \max(\hat{Y} - (nx - \sum_{i,j} Y_{ij})^+), 0))$$

$$\geq E[p \hat{Y} - E[p \hat{Y} + \sum_{i,j} Y_{ij} - nx^+]].$$

For $E[\hat{p} \hat{Y}] = E[\hat{p} \lambda(\hat{p})n(1-t_N)]$, we apply the same argument as we proved in Lemma 17 and we have:

$$E[r(\hat{p} \lambda(\hat{p}))] \geq r(\lambda(p^D)) - Cu_N,$$

(135)

where $u_N = \sqrt{\log n} \cdot \max\{\frac{p_n - p_N}{\kappa_N}, \sqrt{\frac{\kappa_N}{n\tau_N}}\}$. Therefore, $E[p \hat{Y}] \geq p^D \lambda(p^D) \cdot n(1-t_n) - Cnu_N \geq p^D \lambda(p^D) \cdot n(1-t_n) - Cn\tau_1$.

(136)

Now we consider

$$E[p \hat{Y} + \sum_{i,j} Y_{ij} - nx^+].$$

(137)

First we relax this to

$$\bar{p}E[\hat{Y} + \sum_{i,j} Y_{ij} - nx^+]$$

We claim that:

42
Lemma 18. 

\[ E(\hat{Y} + \sum_{i,j} Y_{ij} - E\hat{Y} - \sum_{i,j} EY_{ij})^+ \leq Cn\tau_1, \]  

(138)

where \( C \) is a properly chosen constant.

and

Lemma 19. 

\[ \sum_{i,j} EY_{ij} + E\hat{Y} - nx \leq Cn\tau_1, \]  

(139)

where \( C \) is a properly chosen constant.

The proof of Lemma 18 is exact the same as the proof of Lemma 5. For Lemma 19, we have:

\[ EY_{ij} = \lambda(p_{i,j})n\Delta, \]  

(140)

\[ E\hat{Y} = \lambda(\hat{p}_N)n(1 - t_n). \]  

(141)

By our assumption that \( p^D \) is in the interior and Lemma 16, we know that with probability \( 1 - O\left(\frac{1}{n}\right) \), \( p^c < \overline{p}_i \) for all our price ranges. Therefore, with probability \( 1 - O\left(\frac{1}{n}\right) \), we have

\[ p^c - p_{i,j} \leq \overline{p}_i - \underline{p}_i. \]  

(142)

By the Lipschitz condition on \( \lambda(p) \), this implies that \( \lambda(p_{i,j}) - \lambda(p^c) \leq C(\overline{p}_i - \underline{p}_i) \). Therefore, with probability \( 1 - O\left(\frac{1}{n}\right) \),

\[ \sum_{i,j} EY_{ij} - nx t_N = \sum_{i,j} (EY_{ij} - \lambda(p^c)n\Delta_i) \leq Cn\Delta_i \sum_{i,j} (\overline{p}_i - \underline{p}_i) \Delta_i \]

= \[ Cn\tau_1. \]  

(143)

Similarly, we have that with probability \( 1 - O\left(\frac{1}{n}\right) \),

\[ \lambda(\hat{p}_N) - \lambda(p^c) \leq \frac{\overline{p}_N - \underline{p}_N}{\kappa_N} \leq C\tau_1. \]

And therefore,

\[ E\hat{Y} - nx (1 - t_N) = E\hat{Y} - \lambda(p^c)n(1 - t_n) \leq Cn\tau_1. \]  

(144)

Thus the lemma holds. \( \square \)

We combine Lemma 17, 18 and 19 Theorem 3 holds.