Abstract

The estimation of the condition numbers of Vandermonde matrices, motivated by applications to interpolation and quadrature, can be traced back to 1970s. Empirical study showed consistently that Vandermonde matrices are badly ill conditioned, with a narrow class of notable exceptions, such as the matrices of the discrete Fourier transform (DFT). So far analytic support for this empirical observation, however, has been limited to the matrices defined by the real set of knots. We achieve substantial advance by proving that any Vandermonde matrix of a large size is badly ill conditioned unless its knots are more or less equally spaced on or about the unit circle centered at the origin. This criterion is satisfied for the matrices of DFT, but is violated already for their rather natural extension into the Vandermonde matrices, whose knots form the so called quasi-cyclic sequence, as well as for some leading blocks of the DFT matrices. Consequently, we prove readily that these matrices and blocks are badly ill conditioned. The study of the latter blocks was motivated by an application to preconditioning with random circulant multipliers. We achieve our progress by means of the Vandermonde-to-Cauchy transformation of matrix structures (which demonstrates the power of that method of 1989 once again) and low-rank approximation of some reasonably large submatrices of Cauchy matrices.

Keywords: Vandermonde matrices; Condition number; Transformation to Cauchy matrices

1 Introduction

Our subject, motivated by applications to interpolation and quadrature, is the estimation of the condition number,

$$\kappa(V_s) = ||V_s|| ||V_s^{-1}||,$$

(1.1)

of an \( n \times n \) Vandermonde matrix

$$V_s = (s_j)^{n-1}_{i,j=0},$$

(1.2)

where \( s = (s_i)^{n-1}_{i=0} \) denotes the vector of \( n \) distinct knots and \( ||M|| = ||M||_2 \) denotes the spectral norm of a matrix \( M \). As a rule, the user would like to know first of all whether this number is reasonably bounded or large, that is, whether the matrix is well or ill conditioned, respectively.

It has been proven that the condition number \( \kappa(V_s) \) is exponential in \( n \) if all knots \( s_0, s_1, \ldots, s_{n-1} \) are real (see [GIS88, G90, T94]). This means that such a matrix is badly ill conditioned already for moderately large integers \( n \), but empirically almost all Vandermonde matrices with nonreal knots are badly ill conditioned as well, with a narrow class of exceptions.

Formal support for this observation turned out to be elusive, however. As Walter Gautschi writes in [G90 Chapter IV], the known lower and upper bounds on the condition number of a Vandermonde matrix,
results. In Section 3 we easily prove that the matrix based on the reduction of the computations to the case of matrices with Cauchy matrix structure and on matrix structures have been developed in [GKO95], [G98], [MRT05], [R06], [P15], [Pa], [XXG12], [XXCB14] and this is most relevant to our current study, some powerful technique has been developed in the papers up to scaling by 1 \sqrt{m}

Section IV], not supported analytically so far. One may note that of knots makes the matrices ill conditioned, however. Then again this is an empirical observation of [G90, equation (2.7)] to the hard task of the optimization of a nonsmooth function of n variables.

We propose alternative techniques and achieve substantial progress, supporting the cited empirical observations for a very general class of Vandermonde matrices. Namely we prove that the condition number of a Vandermonde matrix is exponential in n unless its knots are more or less equally spaced on or about the unit circle, C(0,1) = \{x : |x| = 1\}, which has its center at the origin.

Cyclic sequence of knots, equally spaced on this circle, define the Vandermonde matrices that are unitary (up to scaling by 1/\sqrt{n}) and thus perfectly conditioned, that is, their condition numbers reach the minimum value 1. In particular, this is the case for the matrix of the discrete Fourier transform,

\[ \Omega = (\omega^{ij})_{i,j=0}^{n-1}, \text{ for } \omega = \omega_n = \exp(2\pi \sqrt{-1}/n) \] (1.4)

denoting a primitive nth root of 1 (hereafter we use the acronym DFT). For the DFT matrix, the polynomial L(x) of (1.3) turns into x^n - 1. The slight straightforward modification into the so called quasi cyclic sequence of knots makes the matrices ill conditioned, however. Then again this is an empirical observation of [G90, Section IV], not supported analytically so far. One may note that L(x) of (1.3) turns into cyclophymic polynomials, which are much harder to handle than the polynomial x^n - 1. Our alternative technique, however, enables us to prove that the condition numbers \( \kappa(V) \) grows exponentially in this case.

Another application of our study is the proof that \( \kappa(V) \) is exponential in n where V is the leading block of the hn x hn matrix of DFT for h \approx 4, say. (Then again, in this case L(x) of (1.3) is a cyclophymic polynomial.) In [PQY15] we readily extend this result to prove that, with a probability close to 1, Gaussian elimination with no pivoting (GENP) is numerically unstable when it is applied to the matrix of DFT multiplied by a random circulant matrix. This is interesting because in our extensive tests random circulant multipliers have consistently stabilized GENP for a wide variety of inputs. We refer the reader to [PQY15] for these tests and to our Section 3 for the list of benefits of using GENP whenever it is numerically stable.

Technically we begin with reducing the estimation of the condition numbers of Vandermonde matrices to the same task for a subclass of Cauchy matrices, which we call CV matrices. Vandermonde matrices \( V_s = (s_i^{m-1,n-1})_{i,j=0}^{m,n-1} \) and Cauchy matrices \( C_s,t = (\frac{1}{x_i - t_j})_{i,j=0}^{m-1,n-1} \) are much different, and it is not obvious that their studies can be linked to one another. Unlike a Vandermonde matrix V, a Cauchy matrix C_s,t has rational entries and its structure is invariant in column interchange and in scaling and shifting all entries by the same scalar, that is, aC_s,t = C_s,t for a \neq 0 and C_{s,a \cdot t + c e} = C_s,t for e = (1,\ldots,1)^T. Moreover, and this is most relevant to our current study, some powerful technique has been developed in the papers [MRT05], [R06], [P15], [Pa], [XXG12], [XXCB14] for the approximation of reasonably large submatrices of Cauchy matrices by lower-rank matrices.

Highly efficient algorithms for computations with matrices having Vandermonde, Toeplitz and Hankel matrix structures have been developed in [GKO95], [G98], [MRT05], [R06], [P15], [Pa], [XXG12], [XXCB14] based on the reduction of the computations to the case of matrices with Cauchy matrix structure and on nontrivial exploitation of some of the cited properties peculiar to Cauchy matrices. Our work once again demonstrates the power of that approach, proposed in [PS9/90] and nontrivially developed in [MRT05], [R06], [P15], [Pa], [XXG12], [XXCB14].

We organize our presentation as follows. In the next section we recall some basic definitions and auxiliary results. In Section 3 we easily prove that the matrix V is exponential in n if at least one knot lies outside the unit disc D(0,1) = \{x : |x| \leq 1\} or if order of n knots lie strictly inside it. In Section 4 we define CV matrices and link them to Vandermonde matrices. In Sections 5 and 6 we prove our lower bounds on the condition numbers of a CV matrix and a Vandermonde matrix, respectively. In Sections 7 and 8 we study the Vandermonde matrices defined by the quasi-cyclic sequence of knots and the leading blocks of the DFT matrices, respectively. Section 9 is left for conclusions.

We cover rectangular \( m \times n \) Vandermonde matrices, for the sake of generality, but for simplicity the reader may assume that \( m = n \).
2 Basic Definitions and Auxiliary Results: General and DFT Matrices

\( \sigma_j(M) \) denotes the \( j \)th largest singular value of an \( m \times n \) matrix \( M \). \( \|M\| = \|M\|_2 = \sigma_1(M) \) is its spectral norm. \( \kappa(M) = \sigma_1(M)/\sigma_p(M) \) is its condition number provided that \( \rho = \text{rank}(M) \).

**Theorem 2.1.** (The Ekkart–Young Theorem.) \( \sigma_j(M) \) is equal to the distance \( \|M - M_{j-1}\| \) between \( M \) and its closest approximation by a matrix \( M_{j-1} \) of rank at most \( j - 1 \).

**Corollary 2.1.** Suppose that a matrix \( B \) has been obtained by appending \( k \) new rows or \( k \) new columns to a matrix \( A \). Then \( \sigma_j(A) \geq \sigma_{j+k}(B) \) for all \( j \).

\( \frac{1}{\sqrt{n}} \Omega \) is a unitary matrix, and so \( \|\Omega\|/n = 1/\sqrt{n} \) for the DFT matrix \( \Omega \), defined by \((1,4)\).

3 The Condition of Vandermonde Matrices: Simple Bounds

Hereafter assume Vandermonde matrices \( V_s = (s_i^j)_{i,j=0}^{m-1,n-1} \) with \( m \) distinct knots \( s_0, \ldots, s_{m-1} \) and recall that they have full rank.

In this section we prove that \( \kappa(V_s) \geq \nu^{n-1} \) if \( |s_i| \geq \nu > 1 \) for at least one knot \( s_i \), and that \( \kappa(V_s) \geq \nu^{k-1} \) if \( 1/|s_i| \geq \nu > 1 \) for at least \( k \) knots \( s_i, i = i_1, \ldots, i_k \) (see parts (iii) and (iv) of Corollary 3.1).

**Theorem 3.1.** For a Vandermonde matrix \( V_s = (s_i^j)_{i,j=0}^{m-1,n-1} \), write

\[
    s_+ = \max_{i=0}^{m-1} |s_i|, \quad |V_s| = \max\{1, s_+^{n-1}\}. \tag{3.1}
\]

Then

(i) \( |V_s| \leq |V_s| \leq \sqrt{mn} |V_s| \),

(ii) \( \sigma_n(V_s) \leq 1 \) if \( m \geq n \),

(iii) \( \sigma_n(V_s) \leq \max\{1, s_+^{n-m}\} \) if \( m \leq n \),

(iv) \( \sigma_n(V_s) \leq 1 \) if \( |s_i| \leq 1 \) for some \( i, 0 \leq i < m \),

(v) \( \sigma_n(V_s) \leq \nu^{k-1} \) if \( 1/|s_i| \geq \nu > 1 \) for \( i = 0, 1, \ldots, k-1 \).

**Proof.** Part (i) follows from [GL13, equation (2.3.7)]. Parts (ii)–(iv) follow from Theorem 2.1 perturb the matrix \( V_s \) by setting to 0 the entries of its first column, the first \( n - m \) columns, and the \( i \)th row, respectively. To prove part (v), replace by zeros all the entries of the matrix \( V_s \) lying in its first \( k \) rows but not in its first \( k - 1 \) columns. This is a perturbation of the matrix \( V_s \) by a matrix \( E \) with the row norm \( ||E||_\infty \) at most \( \nu^{1-k} \sum_{i=k}^n \nu^{k-i} \leq \frac{\nu^{2-k}}{\nu-1} \), and hence with the spectral norm at most \( \frac{\nu^{2-k}}{\nu-1} \).

**Corollary 3.1.** Under the assumptions of Theorem 3.1, define \( |V_s| \) by \((3.7)\) and fix a constant \( \nu > 1 \).

(i) If \( m \geq n \) or \( |s_i| \leq 1 \) for some \( i, 0 \leq i < m \), then \( \kappa(V_s) \geq |V_s| \),

(ii) if \( m \leq n \), then \( \kappa(V_s) \geq |V_s|/\max\{1, s_+^{n-m}\} \),

(iii) if \( s_+ \geq \nu \), then \( \kappa(V_s) \geq s_+^{n-m} \) for \( l = \min\{m, n\} \),

(iv) if \( 1/|s_i| \geq \nu > 1 \) for \( i = 0, 1, \ldots, k-1 \), then \( \kappa(V_s) \geq (\nu - 1)|V_s|/\nu^{2-k} \).

**Proof.** Combine parts (i)–(v) of Theorem 3.1.

In the following we prove that only under some strong assumptions on the set of knots \( \{s_0, \ldots, s_{m-1}\} \) a Vandermonde matrix \( V_s \) can be well conditioned. Parts (iii) and (iv) of Corollary 3.1 show that we can ignore the cases where \( s_+ \geq \nu \) or where \( 1/|s_i| \geq \nu \) for a constant \( \nu > 1 \), \( i = 0, 1, \ldots, k-1 \) and reasonably large \( k \), but we cover general case, making no such assumptions.
4 CV Matrices and a Link to Vandermonde Matrices

Fix a scalar \( f \) such that \( |f| = 1 \) and the CV matrices, \( C_{s,f} = (\frac{1}{s_i - t_j})_{i,j=0}^{n-1} \), which form a subclass in the class of Cauchy matrices \( C_{s,t} = (\frac{1}{s_i - t_j})_{i,j=0}^{n-1} \). The following equation, [CGS07, equation (8)], links them to Vandermonde matrices,

\[
V_s = f^{1-n} \text{diag} \left( s_i^n - f^n \right)_{i=0}^{n-1} C_{s,f} \text{diag}(\omega^j)_{j=0}^{n-1} (\Omega/n) \text{diag}(f^j)_{j=0}^{n-1}.
\]

Combine this equation with Theorem 2.1, substitute \( \|\Omega\|/n = 1/\sqrt{n} \), and obtain the following bounds.

**Theorem 4.1.** Fix a vector \( s = (s_i)_{i=0}^{n-1} \) and a scalar \( f \) such that \( |f| = 1 \) and define the CV matrix \( C_{s,f} \), the Vandermonde matrix \( V_s \), and the scalar \( s_+ \) of equation (4.1). Then for all positive integers \( \rho \), we have

\[
\sigma_\rho(V_s) \leq \sigma_\rho(C_{s,f})(1 + s_+^\rho)/\sqrt{n}.
\]

Divide \( \|V_s\| \) by both sides of this bound for \( \rho = n \) and obtain

\[
\kappa(V_s) \geq \frac{\|V_s\|\sqrt{n}}{(1 + s_+^n)} \sigma_n(C_{s,f}).
\]

**Corollary 4.1.** Under the assumptions of Theorem 4.1, it holds that \( \kappa(V_s) \geq \frac{\sqrt{n}}{(1 + \max\{s_i^\rho\}) \sigma_n(C_{s,f})} \), that is, \( \kappa(V_s) \geq \frac{\sqrt{n}}{2\sigma_n(C_{s,f})} \) if \( s_+ < 1 \); \( \kappa(V_s) \geq \frac{\sqrt{n}}{(s_+ + 1) \sigma_n(C_{s,f})} \) if \( s_+ \geq 1 \), and in particular

\[
\kappa(V_s) \geq \frac{\sqrt{n}}{2\sigma_n(C_{s,f})} \text{ if } s_+ = 1.
\]

In order to prove the corollary, combine (4.1) with the following lemma.

**Lemma 4.1.** For a Vandermonde matrix \( V_s = (s_i^{j+n-1})_{i,j=0}^{n-1} \), it holds that

\[(i) \|V_s\|/(1 + s_+^n) \geq 1/2 \text{ if } s_+ \leq 1 \text{ and}
(ii) \|V_s\|/(1 + s_+^n) \geq 1/(s_+ + 1) \text{ if } s_+ \geq 1.\]

\[\text{Proof: If } s_+ \leq 1, \text{ then } \|V_s\| \geq 1 \geq (1 + s_+^n)/2, \text{ and we obtain part (i). If } s_+ \geq 1, \text{ then } \|V_s\| \geq s_+^{n-1}. \text{ Hence if } \|V_s\|/(1 + s_+^n) \geq s_+^{n-1}/(1 + s_+^n) = 1/(s_+ + 1/s_+^{n-1}) \geq 1/(s_+ + 1), \text{ and obtain part (ii)}. \]

5 Low-Rank Approximation of Cauchy and CV Matrices

Our next goal is an exponential low bound on the reciprocal of the smallest singular value of a CV matrix, \( 1/\sigma_n(C_{s,f}) \), under some mild assumptions on the knots \( s_0, \ldots, s_{m-1} \). At first we deduce such a bound for a submatrix and then extend it to the matrix itself by applying Corollary 2.3.

**Definition 5.1.** (See [CGS07, page 125].) Two complex points \( s \) and \( t \) are \((\eta, c)\)-separated, for \( \eta > 1 \) and a complex center \( c \), if \( |\frac{s - c}{s - t}| \leq 1/\eta. \)

Two complex sets \( S \) and \( T \) are \((\eta, c)\)-separated if every pair of points \( s \in S \) and \( t \in T \) is \((\eta, c)\)-separated.

Hereafter \( |B| \) denotes the cardinality of a set \( B \).

**Theorem 5.1.** (Cf. [CGS07, Section 2.2], [PT15, Theorem 29].) Assume that, for a real \( \eta > 1 \) and a complex \( c \), a pair of \((\eta, c)\)-separated knot sets \( S_m = \{s_0, \ldots, s_{m-1}\} \) and \( T_l = \{t_0, \ldots, t_{l-1}\} \) define a Cauchy matrix \( C = (\frac{1}{s_i - t_j})_{i,j=0}^{m-1} \). Write \( \delta = \delta_{c,S} = \min_{i=0}^{m-1} |s_i - c| \). Then

\[1/\sigma_\rho(C) \geq (1 - \eta)\eta^{\rho-1} \delta \text{ for all positive integers } \rho. \]

We will apply the theorem only to a specific choice of the knot sets \( S_m \) and \( T_l \).
Theorem 5.2. Assume two positive integers m and n, a complex value f such that |f| = 1, and any set of complex knots \( S_m = \{s_0, \ldots, s_{m-1}\} \). Write \( t_j = f^{\omega^{j-1}} \), \( j = 0, \ldots, n-1 \), for \( \omega = \exp(2\pi\sqrt{-1}/n) \) and fix the knot set \( T_l = \{t_0, \ldots, t_{n-1}\} \) and the Cauchy matrix \( S_{m,n} = (\frac{1}{s_i-t_j})_{i \in S_m, j \in T_l} \).

Fix two integers \( j' \) and \( j'' \) in the range \([0, n-1]\) such that \( 0 < l = j'' - j' + 1 \leq n/2 \) and define the subset \( T_{j', j''} = \{f^{\omega^{j''-j'}} \}_{j''=j'}^{j''=j''} \) of the set \( T_l \).

Let \( c = 0.5(t_j' + t_j'') \) denote the midpoint of the line interval with the endpoints \( t_j' \) and \( t_j'' \) and let \( r = |c - t_j'| = |c - t_j''| = \sin((j'' - j')\frac{\pi}{n}) \) denote the distance from this midpoint to the points \( t_j' \) and \( t_j'' \). (Note that \( r \approx \frac{(j'' - j')\pi}{n} \) for large integers \( n \).)

Fix a constant \( \eta > 1 \) and partition the set \( S_m \) into the two subsets \( S_{m,-} \) and \( S_{m,+} \) of cardinalities \( m_- \) and \( m_+ \), respectively, such that \( m = m_- + m_+ \), the knots of \( S_{m,-} \) lie in the open disc \( D(c, \eta r) = \{z : |z - c| < \eta r \} \), and the knots of \( S_{m,+} \) lie in the exterior of that disc.

(i) Then the two sets \( S_{m,+} \) and \( T_{j', j''} \) are \((c, \eta)\)-separated and

(ii) \( 1/\sigma_{\rho + m_- + l}(C_{m,n}) \geq \eta^\rho (\eta - 1)r \) for all positive integers \( \rho \).

Proof. Note that distance(\( c, T_{j', j''} \)) = \( r \) and deduce part (i).

Define the two following submatrices of the matrix \( C_{m,n} \),

\[
C_{m,j', j''} = (\frac{1}{s_i - t_j})_{i \in S_m, j \in T_{j', j''}} \quad \text{and} \quad C_{m,j', j''} = (\frac{1}{s_i - t_j})_{i \in S_m, j \in T_{j', j''}}.
\]

Apply Theorem 5.2 to the matrix \( C = C_{m,j', j''} \) and \( \delta = \eta r \) and obtain that

\[
1/\sigma_{\rho}(C_{m,j', j''}) \geq \eta^\rho (\eta - 1)r \quad \text{for all positive integers} \ \rho.
\]

Combine this bound with Corollary 5.1 applied for \( k = m_- \) and obtain that

\[
1/\sigma_{\rho + m_-}(C_{m,j', j''}) \geq \eta^\rho (\eta - 1)r \quad \text{for all positive integers} \ \rho.
\]

Combine the latter bound with Corollary 5.1 applied for \( k = n - l \) and obtain part (ii) of the theorem. \( \square \)

Corollary 5.1. Under the assumptions of Theorem 5.2 let \( m = n \) and \( \rho = l - m_- > 0 \). Then \( C_{s,f} = C_{m,n} \) and

\[
1/\sigma_n(C_{s,f}) \geq \eta^\rho (\eta - 1)r.
\]

6 A Typical Vandermonde Matrix is Badly Ill Conditioned

Corollaries 4.1 and 5.1 combined enable us to estimate the condition number of a Vandermonde matrix.

Corollary 6.1. Under the assumptions of Corollary 5.1 it holds that

\[
\kappa(V_s) \geq \eta^\rho (\eta - 1)r \sqrt{n} / \alpha(s_+) \quad \text{for} \quad s_+ \quad \text{of equation (6.4)} \quad \text{and} \quad \alpha(s_+) = 1 + \max\{1, s_+\}, \quad \text{where} \quad \alpha(s_+) \leq 3 \quad \text{for} \quad s_+ \leq 2 \quad \text{and} \quad \alpha(1) = 2.
\]

The corollary shows that, under the assumptions of Corollary 5.1, the condition number \( \kappa(V_s) \) of an \( n \times n \) Vandermonde matrix \( V_s \) grows exponentially in \( n \) if \( \rho \) grows proportionally to \( n \).

To specify this implication, fix any real \( \eta > 1 \) and any pair of integers \( j' \) and \( j'' \) in the range \([0, n-1]\) such that \( 0 < j'' - j' \leq n/2 \). Then the discs \( D(c, r) \) contains precisely \( l = j'' - j' + 1 \) knots from the set \( T \), and \( m_- \) denotes the number of the knots of the set \( S \) in the disc \( D(c, \eta r) \).

Now we can see that Corollary 5.1 implies that \( \kappa(V_s) \) is exponential in \( n \) unless \( l - m_- = o(n) \), for all such triples of \( \eta \), \( j' \) and \( j'' \) with \( l = j'' - j' + 1 \) of order \( n \), that is, unless the number of knots \( s_i \) in the disc \( D(c, \eta r) \) exceeds, matches or nearly matches the number of knots \( t_j \) in the disc \( D(c, \eta r) \) for all such triples.

Loosely speaking, this property is only satisfied for all such triples of \( \eta \), \( j' \) and \( j'' \) if the associated knot set \( S \) is more or less evenly spaced on or about the unit circle \( C((0,1)) = \{z : |z| = 1\} \).

As an implication, we prove in the next section that the condition numbers of the Vandermonde matrices defined by the so called quasi cyclic sequence of knots grows exponentially, even though the deviation of such knots from the above requirement may seems to be rather innocent.
7 The Case of Quasi-Cyclic Sequence of Knots

If all the knots \( s_0, \ldots, s_{n-1} \)

(a) lie on the unit circle \( C(0,1) \) and

(b) are equally spaced on it,

then we arrive at perfectly conditioned Vandermonde matrices, such as the DFT matrix \( \Omega \), satisfying \( \kappa(\Omega) = 1 \). Even some apparently minor deviations from assumption (b), however, can make the matrix ill conditioned, as \([G90]\) shows empirically and we prove formally next.

Example 7.1. The quasi-cyclic sequence of knots. In applications to interpolation and quadrature, one needs to have a sequence of \( n \times n \) well conditioned Vandermonde matrices for \( n = 2, 3, \ldots \); such that all the knots \( s_i \) are reused recursively as \( n \) increases (cf. [G90, Section IV]). For \( n = 2^k \), we choose the DFT matrices, defined by the equally spaced knots \( s_i = \exp(2\pi\sqrt{-1}/2^k) \) for \( i = 0, 1, \ldots, 2^k - 1 \). They contain the DFT knots also for \( n = 2^k \), \( h = 0, \ldots, k-1 \), and we can keep them for all other \( n \) as well, but how should we choose the remaining knots \( s_i \) for the other \( n \)? We can try the straightforward choice, called quasi-cyclic sequence in [G90, Section IV]. This sequence \( s_i = \exp(2\pi f_i\sqrt{-1}) \), \( i = 0, 1, \ldots, n-1 \), is defined by the following sequence of the fractions \( f_i \),

\[ 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{7}{8}, \frac{7}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{16}, \frac{1}{16}, \frac{11}{16}, \frac{11}{16}, \frac{15}{16}, \frac{15}{16}, \ldots \]

In other words one uses the DFT matrix for \( n = 2^k \), and then, as \( n \) increases, one extends the set \( s_0, \ldots, s_{2^k-1} \) of the \( 2^k \)th roots of 1 recursively by adding, one by one, the remaining \( 2^k+1 \)st roots of 1 in counter-clockwise order.

For \( n = 2^k \) these are the unitary matrices of DFT, but for \( n = 3 \cdot 2^k \) their condition number becomes large already for \( k \geq 4 \), exceeding 1,000,000 for \( n = 48 \) (cf. [G90, Section IV]). This has been just empirical observation since 1990, but next we apply our Corollary 6.7 and readily deduce that \( \log(\kappa(V_n)) \) grows proportionally to \( 2^k \), that is \( \kappa(V_n) \) grows exponentially in \( n \), although for \( n = 48 \) our bound is well below 1,000,000.

Note that, for \( n = 3 \cdot 2^k-1 \), the sequence has \( 2^k \) knots \( s_i = \omega_{2^k}^i, i = 0, 1, \ldots, 2^k-1 \), equally spaced on the unit circle \( C(0,1) = \{ z : |z| = 1 \} \), and \( 2^k-1 \) additional knots \( s_i = \omega_{2^k+1}^i, i = 0, 1, \ldots, 2^k-1 \), equally spaced on the upper half of that circle. Recall that \( \kappa(V_n) \geq \kappa_\eta = \sqrt{n} \eta^\rho(\eta-1)r/\alpha(s_+) \) Apply Corollary 6.7 for \( j' = 17n/32 \) and \( j'' = 31n/32 \) fix the set \( S \) of \( 2^k+1 \) knots \( s_i \) lying on the upper half of the unit circle (excluding the \( 2^k-1 \) knots that lie on its lower half), and fix the set \( T_{j',j''} \) made up of \( 15n/16+1 \) knots \( t_\nu = \omega_\nu^i \) for \( \nu = j', \ldots, j'' \). Calculate that \( r = \sin(15\pi/32+1) \approx 0.98, \eta = \sqrt{1-r^2}/r \approx 1.04, \) and \( \eta - 1 \approx 0.04 \). Substitute also \( s_+ = 1, \alpha(s_+) = 2, \) and \( \rho \approx (\frac{1}{16} - \frac{1}{3})n = 5n/48 \), and obtain the exponential lower bound

\[ \kappa(V_n) \geq \kappa_\eta = \eta^\rho(\eta-1)r\sqrt{n}/\alpha(s_+) \approx 0.02 \cdot 1.04^{5n/48} \sqrt{n}. \]

For comparison, also apply Corollary 6.1 for \( j' = 9n/16 \) and \( j'' = 15n/16 \), calculate that \( r = \sin(7\pi/16+1) \approx 0.924, \eta = \sqrt{1-r^2}/r \approx 1.17, \) and \( \eta - 1 \approx 0.17 \). Substitute also \( s_+ = 1, \alpha(s_+) = 2, \) and \( \rho \approx (\frac{1}{16} - \frac{1}{3})n = n/24 \), and obtain the exponential lower bound

\[ \kappa(V_n) \geq \kappa_\eta = \eta^\rho(\eta-1)r\sqrt{n}/\alpha(s_+) \approx 0.08 \cdot 1.17^{n/24} \sqrt{n}. \]

Lokewise, by choosing \( j' = 13n/24 \) and \( j' = 23n/24 \), we obtain \( r = \sin(5\pi/12+1) \approx 0.966, \eta = \sqrt{1-r^2}/r \approx 1.07, \eta - 1 \approx 0.07, \) and \( \rho \approx (\frac{1}{16} - \frac{1}{3})n = n/12, \) and so

\[ \kappa(V_n) \geq \kappa_\eta = \eta^\rho(\eta-1)r\sqrt{n}/\alpha(s_+) \approx 0.035 \cdot 1.07^{n/12} \sqrt{n}. \]

Can one still define a desired sequence of well conditioned \( h \times h \) Vandermonde matrices \( V_h \) for \( h = 1, 2, 3, \ldots, n \), which would reuse the equally knots defined where \( h = 2^k \) is a power of 2 and would consists of only \( n \) knots overall? [G90, Section IV] proves that such a sequence of well conditioned \( h \times h \) Vandermonde matrices is obtained if we define the knots \( s_i = 2\pi d_i\sqrt{-1}, i = 0, 1, \ldots, n-1 \), by the following van der Corput sequence of fractions \( d_i \), which reorders the knots of quasi-cyclic sequence in a zigzag manner,

\[ 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{7}{8}, \frac{7}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{16}, \frac{1}{16}, \frac{11}{16}, \frac{11}{16}, \frac{15}{16}, \frac{15}{16}, \ldots \]
The knots \( s_1, \ldots, s_n \) are equally spaced on the unit circle \( C(0, 1) \) for \( n = 2^k \) being the powers of 2, are distributed on it quite evenly for all \( n \), and Gautschi in [390] Section 4] has proved the following estimate.

**Theorem 7.1.** Given the van der Corput sequence of \( n \times n \) Vandermonde matrices \( V_s \) for \( n = 1, 2, 3, \ldots \), it holds that \( \kappa(V_s) < \sqrt{2n} \) for all \( n \).

**8 Gaussian Elimination with No Pivoting (GENP) for the Matrices of Discrete Fourier Transform**

Gaussian elimination with no pivoting can fails or run into numerical problems, but otherwise is practically more efficient than the customary Gaussian Elimination with partial pivoting, that is, with appropriate row interchange (hereafter we use the acronyms GENP and GEPP). (The paper [PQY13] lists the following practical problems with pivoting: it interrupts the stream of arithmetic operations with foreign operations of comparison, involves book-keeping, compromises data locality, complicates parallelization of the computations, and increases communication overhead and data dependence.)

GENP fails or runs into numerical problems if and only if some leading blocks of the input matrix \( A \) are singular or ill conditioned (cf. [PQZ13] Theorem 5.1]). According to our previous sections, the matrix \( A = V_s \) is typically rank deficient or ill conditioned, but according to our next theorem, GENP fails even when it is applied to the perfectly conditioned matrix \( \Omega \) of DFT of a large size.

**Theorem 8.1.** GENP is numerically unsafe even for the perfectly conditioned \( n \times n \) matrix \( \Omega \) if \( n \) is large.

Of course, we do not need GENP to solve a linear system of \( n \) equations, \( \Omega x = b \), because \( x = \frac{1}{\sqrt{n}} \Omega^T b \), but the result has further impact on the study of randomized structured preconditioning in [PQY13]: it implies that Gaussian random circulant multipliers support numerically safe application of GENP to the matrix \( \Omega \) only with a probability near 0, even though empirically they have consistently supported numerical GENP in our extensive tests with matrices of various other classes (cf. [PQZ13], [PQY13]).

In order to prove Theorem 8.1 apply Corollary 6.1 to the \( k \times k \) leading block \( \Omega_n^{(k)} \) of the matrix \( \Omega_n \) for \( k = n/4 \) (assume for convenience that \( n \) is divisible by 8) and to the knot sets \( S = \{ s_0, \ldots, s_{k-1} \} \) and \( T = \{ t_0, \ldots, t_{k-1} \} \), where \( s_i = \omega^i, t_j = \omega^j, \) and \( \omega \) denotes exp\((2\pi \sqrt{-1}/q)\) for \( q = k \) and \( q = n \). Choose \( j' = 0 \), and \( j'' = n/4 - 1 \), and so \( t_{j'} = 1, t_{j''} = \sqrt{-1}, l = k = n/4, c = (1 + \sqrt{-1})/2 \) and \( r = 1/\sqrt{2} \). Choose the sets \( S_{m,-} \) and \( S_{m,+} \) separated by the line passing through the points \((-1 + \sqrt{-1})\sqrt{2} \) and \((1 - \sqrt{-1})\sqrt{2} \) (also through the origin \(0\)) and and estimate that \( m_- = m_+ = 2k/2 - 1 = n/8 + 1 \), and \( \eta = \sqrt{3}, \eta - 1 = \sqrt{3} - 1 \approx 0.732, (\eta - 1)\sqrt{7}/4 > 0.25 \).

Now Corollary 6.1 implies that

\[
\kappa(\Omega_n^{(k)}) \geq 2(\eta - 1)\eta \sqrt{n}/(s_+ > \kappa_-(\Omega_n^{(k)}) = 0.25 \sqrt{n} 3^{n/16}. \quad (8.1)
\]

Finally we state the following observation.

**Theorem 8.2.** Given the van der Corput sequence of \( n \times n \) Vandermonde matrices \( V_s \) for \( n = 1, 2, 3, \ldots \), the leading block of any of the matrices in the sequence is also a matrix in the same sequence.

Combine this observation with Theorem 7.1 and obtain the following result.

**Corollary 8.1.** All leading blocks of any Vandermonde matrix with van der Corput sequence of knots are nonsingular and well conditioned.

**9 Conclusions**

We estimated from below the condition numbers \( \kappa(V_s) \) of \( n \times n \) Vandermonde matrices and defined the classes for which \( \kappa(V_s) = (s_j^{n-1})_{j=0}^{n-1} \) is exponential in \( n \), implying that the matrices are badly ill conditioned already for moderately large integers \( n \). This turned out to be the usual case for Vandermonde matrices, with a narrow class of exceptions, which we specified.
We began with simple estimates of Corollary 3.1, which showed that the condition number \( \kappa(V_s) \) is exponential in \( n \) unless all knots \( s_0, \ldots, s_{n-1} \) lie in or near the unit disc \( D(0,1) = \{ x : |x| \leq 1 \} \) and mostly on or near the unit circle \( C(0,1) = \{ x : |x| = 1 \} \).

Independently of these results, Corollaries 4.1 and 4.5 combined imply that the condition number \( \kappa(V_s) \) is exponential in \( n \) if there are fewer than \( c_n \) knots in a neighborhood of a disc covering an arc of the unit circle \( C(0,1) \) of length \( 2\pi c_+/n \), for a pair of positive constants \( c_+ \) and \( c_- < c_+ \). This property holds (for all such discs) only if the knots knots \( s_0, \ldots, s_{n-1} \) are more or less evenly spaced on or about the unit circle \( C(0,1) \). The results formally support the well known empirical observation and is in good accordance with our tests, although empirically the condition numbers \( \kappa(V_s) \) tend to grow faster as \( n \) increases than their lower bounds. The discrepancy leads to the challenge of sharpening our estimates, although this may be not easy to do in view of the long history of slow progress in the area of estimating the condition numbers \( \kappa(V_s) \).

Actually our results and proofs also apply to \( m \times n \) Vandermonde matrices.

Our proofs exploit the transformation of a Vandermonde matrix into a CV matrix (they form a special subclass of the class of Cauchy matrices) and approximation of its submatrices by low-rank matrices. This is a new demonstration of the power of an approach traced back to [P90], which has served as the basis for the design of highly efficient practical algorithms for computations with structured matrices.

A natural research challenge is the extension of our results to polynomial Vandermonde matrices \( V_{P,s} = (p_j(x_i))_{i,j=0}^{n-1} \) where \( P = (p_j(x))_{i=0}^{n-1} \) is any basis in the space of polynomials of degree less than \( n \), for example, the basis made up of Chebyshev polynomials. Equation (4.1) is extended to this case as follows (cf. [P01 equation (3.6.8)]),

\[
C_{s,t} = \left( \frac{1}{s_i - t_j} \right)_{i,j=0}^{n-1,n-1} = \text{diag}(t(s_i)^{-1})_{i=0}^{n-1} V_{P,s} V_{P,t}^{-1} \text{diag}(t'(t_j))_{j=0}^{n-1}.
\]

Here \( t(x) = \prod_{j=0}^{n-1} (x-t_j) \) and \( t = (t_j)_{j=0}^{n-1} \) denotes the coefficient vector of the polynomial \( t(x) = x^n \). The next step could be the selection of a proper set of knots \( T \) or \( S \), which would enable us to reduce the task to the estimation of the condition numbers \( \kappa(C_{s,t}) \) of Cauchy matrices. [G90] Sections 5 and 6 can give us some initial guidance for this selection, which would extend the set of equally spaced knots on the unit circle, used in the case of Vandermonde matrices. It would remain to extend our estimates for CV matrices to the new subclass of Cauchy matrices.

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