Near-Stability of a Near-Minimal Surface Noted Through a Tested Curvature Algorithm

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The differential geometry problem of finding the minimal surface for a known boundary, considering the traditional definition of minimal surfaces, is equivalent to solving the PDE obtained by setting the mean curvature equal to zero. We solve the problem through a variational algorithm based on the ansatz containing the mean curvature function numerator. The algorithm is first tested through the number of iterations it takes to return the minimal surface for a boundary for which a minimal surface (a solution of the PDE) is known. We consider two such instances: a hemiellipsoid surface with boundary as an ellipse and a hump-like surface spanned by four coplanar straight lines. Our algorithm achieved as much as 65 percent of the total possible decrease for the surface bounded by an ellipse. Having seen that our algorithm can expose the instability of a surface that is away from being a minimal, we apply it to a bilinear interpolation bounded by four non coplanar straight lines and find that in this case the area decrease achieved through our algorithm in computationally manageable two iterations is only 0.116179 percent of the original surface. (We implemented eight iterations but for a simpler one-dimensional case.) This suggests that the bilinear interpolation is already a near-minimal surface. A comparison of root mean square of mean curvature and Gaussian curvature in the three cases is also provided for further analysis of certain properties of these surfaces.

I. INTRODUCTION

By choosing a basis the abstract problem of finding a vector is converted to the concrete problem of finding the coefficients of the basis vectors. In comparison, in a variational problem, we write the form (termed ansatz) of a quantity to be minimized in such a way that what remains to be found is normally values of the variational parameters introduced in the ansatz. An important application of the variational approach is in finding characteristics of a surface locally minimizing its area for a known boundary, called a minimal surface. Minimal surfaces initially arose as the surfaces of minimal surface area subject to some boundary conditions/constraints, a problem characterized in variational calculus as Plateau problem \cite{1} \cite{2}, named after the Belgian physicist Joseph Plateau, that tries to find a surface of minimal area, spanned by a closed bounding curve. A trivial example of a minimal surface is a plane. Initial non-trivial examples of minimal surfaces, namely catenoid and helicoid, were found by Meusnier in 1776. Later in 1849, Joseph Plateau showed that minimal surfaces can be produced by dipping wire frame with certain closed boundaries into a liquid detergent. The problem of finding minimal surface attracted mathematicians like Schwarz \cite{3} (who investigated triply periodic surfaces with emphasis on the surfaces namely D (diamond), P (primitive), H (hexagonal), T (tetragonal) and CLP (crossed layers of parallels), Riemann \cite{1}, Weierstrass \cite{1}, R. Garnier \cite{1}. Later, significant results were obtained by L. Tonelli \cite{5}, R. Courant \cite{6} \cite{7}, C. B. Morrey \cite{8} \cite{9}, E. M. McShane \cite{10}, M. Shiffman \cite{11}, M. Morse \cite{12}, T. Tompkins \cite{12}, Osserman \cite{13}, Gulliver \cite{14} and Karcher \cite{15} and others.

The above mentioned characteristic of a minimal surface resulting from a vanishing variation in its area is that its mean curvature should be zero throughout; see, for example, section 3.5 of ref. \cite{16} for the partial differential equation (PDE) \( H = 0 \) this demand implies. This results in a prescription that for finding a minimal surface we should look for a surface for which the mean curvature is zero at each point of the surface. On the face of it, the level of abstractness resulting from this demand is still significantly more than that in a routine variational problem because we have to find a zero of the mean curvature function for each set of values of the parameters of the surface. In symbolic (analytic) form, it is not always possible to solve the problem for a given boundary. Numerically, we have to find a zero for each set of values, which means we have to solve a large number of problems on a large grid. We can convert the problem into a routine quantitative and even variational form by calculating integrals on the surface and minimizing them. Without help of the above mentioned result, such an integral would be the area functional given by eq. (6). This is highly non-linear and accordingly it is difficult to minimize it. Using the zero mean curvature

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prescription, we suggest another integral which is the mean square functional of the mean curvature numerator i.e. $\mu_n^2$ (given below eq. (23)). This integral is zero when the mean curvature function is zero for the whole surface. The functional of eq. (23) is non-negative by construction and thus has the same extremal (for its least value that is zero) as the above mentioned area integral i.e. a minimal surface. The technique of minimizing a different functional of the surface than the area integral is well known because of the Douglas’ suggestion of minimizing the Dirichlet integral that has the same extremal as the area functional. (A list of other possibilities of such functionals can be found in refs. [17] [18].) As with Dirichlet integral, the alternative functional $\mu_n^2$ that we use has no square root unlike the area integral.

In converting (see for example section 3.5 of ref. [16]) the abstract problem of finding (for a given boundary) a minimal surface to finding a zero of $\mu_n^2$, the concreteness achieved is that for $\mu_n^2$ we have a well written value (that is, zero) to be achieved; for the area integral we have to minimize without knowing what is the minimum value we have to obtain. In the remaining task, an efficient use of the variational technique would convert the problem to finding numerical values of at most a few variational parameters. For the whole scheme to work, we should have a good trial function (ansatz) of the variational parameters along with a good initial surface. In an earlier work [19], for a boundary composed of four straight lines, we took the initial surface to be bilinear interpolation and the ansatz for change in the surface was taken to be proportional to the numerator of the mean curvature function for the surface; other factors in the change were the variational parameter, a function of the surface parameters (not to be confused with the variational parameters) whose form vanishing at the boundary and a vector assuring that we add a 3-vector to the original surface in 3-dimensions. In this paper, we make an iterative use of the same ansatz that appears in eq. (18) below. This equation also introduces the variational parameter $t$ we calculate afresh for each iteration. When our $\mu_n^2$ eq. (23) is calculated for the resulting surface, this becomes for each iteration a polynomial in the variational parameter $t$.

In this paper we use these iterations to test our curvature algorithm by first considering cases where we already know a minimal surface for a boundary. In such a cases, if we take our initial surface to be a non-minimal one, an ideal test would be to use our algorithm to see in how many iterations it decreases its area to that of the target minimal surface for our fixed boundary. If the target surface is flat, achieving it is sufficient but not necessary. This is because, at least according to the traditional definition of a minimal surface as a surface with zero mean curvature, for a fixed boundary we may have more than one minimal surface because vanishing mean curvature is solution of an equation obtained by setting the derivative of the surface with respect to a parameter introduced in its modified expression. This means that a surface satisfying the resulting condition would be necessarily only a locally minimal in the set of all the surfaces generated by this prescription and hence is not proved to be unique. Thus another question about our algorithm is if it gets stuck in some other (locally) ”minimal” surface before it reaches the flat surface.

That much is about checking of the algorithm. As for the initial surface, if we know a sufficiently efficient algorithm and we can apply it for a boundary for which no minimal surface is known, we can inquire if the initial surface is near-stable against the otherwise area decreases that our algorithm can generate. A near-stability can be an indicator that our initial surface is a near-minimal one. If the first iteration does not appreciably decrease area, it is premature to conclude that the surface is a near-stable unless we have implemented further iterations and have demonstrated that even in a sufficient number of iterations there is still no appreciable decrease in the area.

Our previously used [19] algorithm for change can be simplified by replacing the numerator of the mean curvature function $H_0(u, v)$ there by one (see below eq. (33)). But then it would not re-iterate it and thus for most of the work in the present paper as well, we keep our previous algorithm and re-iterate it resulting in eq. (18) below. Others [20] [21] have converted Dirichlet integrals to a system of linear equations. As mentioned above, we convert the $ms$-mean curvature integral $\mu_n^2$ eq. (23) to a polynomial function of the variational parameter $t$ of the ansatz eq. (18). In each iteration, we keep integrating out the two usual parameters of a surface and thus minimize an expression ($ms$-mean curvature $\mu_n^2$ of eq. (23) below) of the variational parameter only. In the above, the number of iterations required to achieve a target minimal surface has been described as an important characteristic of the algorithm. In practically implementing this, computational problems may stop us from actually to get all the iterations. In a simpler situation we can somewhat avoid this impasse, namely using a curve rather than a surface. This means the target flat surface is replaced by a straight line, the mean curvature is replaced by a simple second derivative and a function of one curve parameter is sufficient to make variations at the end points (replacing boundary lines) zero. This gave us in actual implementation significantly more iterations than in the surface case.

We tested our ansatz for two cases with known minimal surfaces; of course the initial surfaces we chose for each case are not minimal. The first instance of a non-minimal initial surface considered is hemiellipsoid spanned by the an elliptic curve with the parametrization

$$x_0(u, v) = (\sin u \cos v, b \sin u \sin v, c \cos u), \quad (1)$$

with $b$ and $c$ being constants and $0 \leq u \leq \pi/2$ and $0 \leq v \leq 2\pi$ as shown in Fig. [1]. Secondly, for the initial surface we
FIG. 1: Hemiellipsoid spanned by an elliptic curve for $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$.

took a hump-like surface spanned by four arbitrary straight lines with parametrization

$$x_0(u, v) = (u, v, 16uv(1 - u)(1 - v))$$

(2)

where $0 \leq u, v \leq 1$ as shown in Fig. 2. The target minimal surface in this case is also a flat, but the boundary is a square rather than an ellipse for the hemiellipsoid case.

FIG. 2: A hump-like surface spanned by four straight lines with the parametrization $x_0(u, v) = (u, v, 16uv(1 - u)(1 - v))$ for $0 \leq u, v \leq 1$.

Having tested our algorithm, we applied it to an important case in surface theory, namely the bilinear interpolation where the corresponding minimal surface is not written explicitly. Minimal surfaces are the most important of the spanning surfaces with linear or bilinear interpolations. For the bilinear interpolations when they are near or related to the minimal surfaces, a thorough literature survey and related detail can be seen in the introductory section of our work [19] on variational Minimization of String rearrangement surfaces and [22] that extends these rearrangement surfaces to the string-breaking models that model string breaking probabilities for the physical problem of the ground states of the initial and final strings. These models find their nascent use in string-theories-related works of physics (two theories are worth mentioning in this context, theory of quantum chromodynamics (QCD) strings [23] and theory (or theories) in which string vibrations are supposed to generate different elementary particles of the present high energy physics [24]). The usefulness of the work extends to the emerging discipline of the computer aided geometric design (CAGD) [25], [26] along with physics and the differential geometry.

The linear interpolation in the one-dimensional case (a straight line) is already minimal. But the mean curvature function for the bilinear interpolation is not zero and checking how close this is to being a minimal is a non-trivial
problem. In equation form, the bilinear interpolation is a bilinear mapping from \((u, v)\) to \((x, y, z)\) given by

\[
x_0(u, v) = (r(u + v - 2uv), v, u)
\]

(3)

with the following corners:

\[
x(0, 0) = r_1, \quad x(1, 1) = r_2, \quad x(1, 0) = r_3, \quad x(0, 1) = r_4,
\]

(4)

for real scalars \(r\) and \(d\), we choose following configuration of the four corners

\[
r_1 = (0, 0, 0), \quad r_2 = (r, d, 0), \quad r_3 = (0, d, d), \quad r_4 = (r, 0, d).
\]

(5)

as shown in Fig. 3.

![Fig. 3: A surface spanned by four straight lines with the parametrization \(x_0(u, v) = (r(u + v - 2uv), v, u)\) for \(r = 1\) for \(0 \leq u, v \leq 1\).](image)

The paper is organized as follows: In section II we present basic definitions and a variational algorithm based on the ansatz containing the numerator of the mean curvature function to reduce the area of a surface spanned by a fixed boundary (including the boundary composed of finite number of curves). In the section III, the technique is applied to a hemiellipsoid surface (with boundary as an ellipse and a hump-like surface (spanned by four coplanar straight lines) to test the algorithm through the number of iterations it takes to return the minimal surface for a boundary for which a minimal surface is known. The technique is then applied to above mentioned bilinear interpolation bounded by four non coplanar straight lines, for which minimal surface is not known. The one-dimensional analogue of the ansatz eq. (18) is also provided and applied to a curve of given length, before we apply this technique to the above mentioned surfaces. Based on this comparison, the results and remarks are presented in the final section IV.

**II. MINIMAL SURFACES AND AN ANSATZ FOR VARIATIONAL IMPROVEMENT TO NON-MINIMAL SURFACES**

As mentioned above, for a closed boundary composed of finite number of curves, methods [27–33] of generating a spanning surface and finding different differential geometry-related properties for these surfaces are known. But that would not tell us anything about its characteristics in the differential geometry. For example, there is no guarantee that such a surface would be a minimal surface spanning its boundary. We can calculate the differential geometry related functions of the two parameters \(u\) and \(v\) of the surface and then do integrations with respect to these parameters to get numbers characterizing the surface. If the \(\text{rms}\) mean curvature (see below) of the generated surface is non-zero, the surface is not a minimal surface. For such a surface the solution of the PDE \(H = 0\), with \(H\) of eq. (11) below, is not achieved. The challenge is to modify it so that it either becomes a minimal surface solving \(H = 0\) or, if that is not possible, gets closer to being a minimal surface. We do this modification through the ansatz of eq. (18). As mentioned above, we calculate the \(\mu^2_n\) of eq. (23) and minimize it with respect to our variational parameter \(t\). We have checked that as we decrease our alternative functional \(\mu^2_n\), the area of the surface spanning our fixed boundary
also decreases. The \( \text{rms} \) mean curvature of both our starting surface and the one achieved after one variational area reduction remains zero and, as mentioned above, we re-use our variational algorithm for the resulting surface a number of times. Before discussing our iterative scheme eq. (18) for our successive surfaces \( \mathbf{x}_n(u, v) \) \( (n = 0, 1, 2, \ldots) \), we give the following definitions related to minimal surfaces fruitful in the sequel.

In the optimization problem we aim for here, we eventually try to find a surface of a known boundary that has a least value of area. Area is evaluated by the area functional:

\[
A(\mathbf{x}) = \int \int_D |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| \, du \, dv,  \tag{6}
\]

where \( D \subseteq \mathbb{R}^2 \) is a domain over which the surface \( \mathbf{x}(u, v) \) is defined as a map, with the boundary condition \( \mathbf{x}(\partial D) = \Gamma \) for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \), \( \mathbf{x}_n(u, v) \) and \( \mathbf{x}_v(u, v) \) being partial derivatives of \( \mathbf{x}(u, v) \) with respect to \( u \) and \( v \). It is known [16] that the first variation of \( A(\mathbf{x}) \) vanishes everywhere if and only if the mean curvature \( H \) of \( \mathbf{x}(u, v) \) is zero everywhere in it. The \( \text{rms} \) of Gaussian curvature allows us to visualize the surface in its specific range of curvature. For a locally parameterized surface \( \mathbf{x} = \mathbf{x}(u, v) \), let us denote

\[
\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}, \quad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}, \quad \mathbf{x}_{uv} = \frac{\partial^2 \mathbf{x}}{\partial u \partial v}, \quad \text{and} \quad \mathbf{x}_{vv} = \frac{\partial^2 \mathbf{x}}{\partial v^2}.  \tag{7}
\]

The coefficients of first and second fundamental form are

\[
E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle,  \tag{8}
\]

and

\[
e = \langle \mathbf{N}, \mathbf{x}_{uu} \rangle, \quad f = \langle \mathbf{N}, \mathbf{x}_{uv} \rangle, \quad g = \langle \mathbf{N}, \mathbf{x}_v \rangle,  \tag{9}
\]

with

\[
\mathbf{N}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|},  \tag{10}
\]

being the unit normal to the surface \( \mathbf{x}(u, v) \). The mean curvature \( H \) may be given by

\[
H = \frac{Ge - 2Ff + Eg}{2(EG - F^2)},  \tag{11}
\]

and the Gaussian curvature \( K \) is given by

\[
K = \frac{eg - f^2}{EG - F^2}.  \tag{12}
\]

Setting \( H = 0 \), gives us the PDE whose solution is defined to be the minimal surface. For example, for a locally parameterized surface \( \mathbf{x} : U \to \mathbb{R}^3 \) with \( U \) an open set in \( \mathbb{R}^2 \), the mean curvature of the Monge’s form of surface \( \mathbf{x} = (u, v, z(u, v)) \) may be found by substituting the coefficients (eqs. (8) and (9) above) of first and second fundamental forms in the eq. (11). The mean curvature in this case is given by

\[
H = \frac{(1 + z_v^2) z_{uu} - 2z_u z_v z_{uv} + (1 + z_u^2) z_{vv}}{2(1 + z_u^2 + z_v^2)}.  \tag{13}
\]

For the Monge’s form of surface, vanishing condition of the mean curvature \( H \) for a minimal surface is the solution of the following PDE

\[
(1 + z_v^2) z_{uu} - 2z_u z_v z_{uv} + (1 + z_u^2) z_{vv} = 0.  \tag{14}
\]

The mean square (\( \text{rms} \)) of the quantities \( H \) and \( K \), for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \) denoted by \( \mu \) and \( \nu \) are given by the following expressions,

\[
\mu^2 = \int_0^1 \int_0^1 H^2 \, du \, dv.  \tag{15}
\]
\[ \nu^2 = \int_0^1 \int_0^1 K^2 \, dudv. \]  
\[ \nu = \frac{\left( \int_0^1 \int_0^1 K^2 \, dudv \right)^{1/2}}{\int_0^1 \int_0^1 H^2 \, dudv}. \]

and a dimensionless quantity from (15) and (16) is,

\[ \frac{\nu}{\mu^2} = \frac{\left( \int_0^1 \int_0^1 K^2 \, dudv \right)^{1/2}}{\int_0^1 \int_0^1 H^2 \, dudv}. \]

For a minimal surface, the mean curvature is identically zero. For minimization we use only the numerator part of mean curvature \( H \) given by (11), as done in ref. 35 following ref. 11 who writes that “for a locally parameterized surface, the mean curvature vanishes when the numerator part of the mean curvature is equal to zero”. We use this numerator part \( H_0 \) to compute the rms curvature of the initial surface \( x_0(u, v) \) to be used in the ansatz eq. (28) to get first order variationally improved surface \( x_1(u, v) \) of lesser area. This process could be continued as an iterative process until a minimal surface is achieved. Having introduced the symbols we use, we can write in an equation form the ansatz we mentioned above. This is an iterative scheme for the successive non-minimal surfaces \( x_n(u, v) \) \( (n = 0, 1, 2, \ldots) \)

\[ x_{n+1}(u, v, t) = x_n(u, v) + t m_n(u, v) N_n, \]

where \( t \) is our variational parameter and

\[ m_n(u, v) = b(u, v) H_n. \]

Here \( b(u, v) \) is chosen so that the variation at the boundary curves is zero. \( H_n \) \( (n = 0, 1, 2, \ldots) \) denotes the numerator part of the mean curvature function eq. (11) of the non-minimal surface \( x(u, v) \) and is given by

\[ H_n = \epsilon_n G_n - 2 f_n f_n + g_n E_n. \]

For \( x_n(u, v) \), we denote \( E_n, F_n, G_n, \epsilon_n, f_n \) and \( g_n \) as the fundamental magnitudes given by eqs. (8) and (9) and \( N_n(u, v) \), the numerator part of unit normal given by eq. (10) to the surface \( x(u, v) \). The subscript \( n \) is used not only to denote numerator part of the quantities but also to denote the \( nth \) iteration. For \( n = 0 \), for the initial surface \( x_0(u, v) \), thus \( E_0(u, v), F_0(u, v), G_0(u, v), \epsilon_0(u, v), f_0(u, v) \) and \( g_0(u, v) \) denote the fundamental magnitudes, \( H_0(u, v) \) the numerator part of the mean curvature, \( m_0(u, v) = uv(1-u)(1-v) H_0 \), a function of \( u, v \) with no variation at its boundary curves and \( N_0(u, v) \) the unit normal. For \( n \neq 0 \), \( E_n(u, v, t), F_n(u, v, t), G_n(u, v, t), \epsilon_n(u, v, t), f_n(u, v, t) \) and \( g_n(u, v, t) \) depend not only the surface parameters but on the variational parameter as well. For \( x_n(u, v) \) eq. (18), for \( n \neq 0 \), the numerator \( H_n \) eq. (20) of mean curvature of \( x(u, v) \), is a polynomial in \( t \). It would have the following familiar expression

\[ H_n(u, v, t) = E_n g_n - 2 F_n f_n + G_n e_n + \sum_{i=0}^{6} (p_i(u, v)) t^i. \]

As \( H_n^2(u, v, t) \) is a polynomial in \( t \), with real coefficients \( q_j(u, v) \) of \( t^j \) for \( j = 0, 1, 2, \ldots, 10 \), we rewrite eq. (21) in the form

\[ H_n^2(u, v, t) = \sum_{i=0}^{n} (q_i(u, v)) t^i. \]

Here \( n \) turns out to be 10; there being no higher powers of \( t \) in the polynomials as it can be seen from the expression for \( E_n(u, v, t), F_n(u, v, t) \) and \( G_n(u, v, t) \) which are quadratic in \( t \) and \( \epsilon_n(u, v, t), f_n(u, v, t) \) and \( g_n(u, v, t) \) which are cubic in \( t \). Integrating (numerically if desired) these coefficients w.r.t. \( u \) and \( v \) in the range \( 0 \leq u, v \leq 1 \) we get the following integral for mean square \( ms \) of mean curvature

\[ \nu^2(t) = \int_0^1 \int_0^1 H_n^2(u, v, t) \, dudv = t^2 \int_0^1 \int_0^1 \sum_{i=0}^{n} (q_i(u, v)) \, dudv. \]

The expression in the parentheses on right hand side of above equation turns out to be a polynomial in \( t \) of degree \( n \), which can be minimized w.r.t. \( t \) to find \( t_{min} \). The resulting value of \( t \) completely specify new surface \( x_{n+1}(u, v) \). For this value of \( t = t_{min} \), new surface \( x_{n+1}(u, v) \) is expected to have lesser area than that of surface \( x_n(u, v) \). Thus the
mean square (ms) eq. \(15\) of mean curvature eq. \(11\) of this \(H_n\), for \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\) and \(t = t_{min}\) can be calculated. That is,
\[
\mu_n^2 = \int_0^1 \int_0^1 H_n^2(u, v, t = t_{min}) \ dudv.
\] (24)

We denote the mean square (ms) eq. \(16\) of Gaussian curvature eq. \(12\) for \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\), as \(\nu_n\) given by the following expression,
\[
\nu_n^2(t) = \int_0^1 \int_0^1 K_n^2(u, v, t = t_{min}) \ dudv,
\] (25)

where \(K_n\) is the numerator part of \(K\) of eq. \(12\). Using eqs. \(23\) and \(25\), ratio of the \(rms\) of Gaussian curvature eq. \(17\) to the mean square of mean curvature takes the following form,
\[
\frac{\nu_n}{\mu_n^2} = \left(\frac{\int_0^1 \int_0^1 K_n^2(u, v, t = t_{min}) \ dudv}{\int_0^1 \int_0^1 H_n^2(u, v, t = t_{min}) \ dudv}\right)^{1/2}.
\] (26)

Now for \(x_n(u, v)\), we compute \(E_n(u, v, t = t_{min}), F_n(u, v, t = t_{min})\) and \(G_n(u, v, t = t_{min})\) and denote the area integral eq. \(6\) as \(A_n\) and it is given by the following expression
\[
A_n = \int_0^1 \int_0^1 \sqrt{E_n G_n - F_n^2} \ dudv.
\] (27)

The ansatz eq. \(18\), for the first order reduction (eq. \(28\)) in the area of a non-minimal surface \(x_0(u, v)\) is
\[
x_1(u, v, t) = x_0(u, v) + t m_0(u, v) N_0,
\] (28)

where \(t\) is our variational parameter. Here \(m_0(u, v), H_0(u, v)\) are given by eqs. \(19\), \(21\) and \(N_0(u, v)\) is the unit normal eq. \(10\) to the non-minimal surface \(x_0(u, v)\) for \(n = 0\). For \(n = 0\), the ratio \(29\) of the \(rms\) of Gaussian curvature to the \(ms\) of mean curvature is given by
\[
\frac{\nu_0}{\mu_0^2} = \left(\frac{\int_0^1 \int_0^1 K_0^2 \ dudv}{\int_0^1 \int_0^1 H_0^2 \ dudv}\right)^{1/2},
\] (29)

where \(K_0\) is the numerator part of \(K\), eq. \(12\). With the above notation (eqs. \(11\) to \(19\)), eq. \(6\) becomes, for \(x_0(u, v)\),
\[
A_0 = \int_0^1 \int_0^1 \sqrt{E_0 G_0 - F_0^2} \ dudv.
\] (30)

For \(n = 1, 2, \ldots\), the in eq. \(18\) gives us the surfaces \(x_1(u, v), x_3(u, v), \ldots\) of reduced area and related quantities may be computed from eqs. \(19\) to \(27\). Complexity in computer algebra system calculations did not allow us to compute third order reduction in area except for hemiellipsoid for \(H_0 = 1\) for the initial surface and \(H_n, n = 1, 2, \ldots\) the numerator part of the mean curvature defined as above by the eq. \(20\). It may be seen from the table \(1\) that for \(H_0\), the numerator part of the mean curvature as defined by the eq. \(20\), percentage decrease in area of the hemiellipsoid is slightly more than for the same hemiellipsoid when \(H_0 = 1\). In order to see a geometrically meaningful (relative) change in area we calculate the dimension less area. Let \(A_0\) be the initial area and \(A_f\) the area of the minimal surface (the subscript \(f\) denotes here that a minimal surface is flat). We can define maximum possible change to be achieved as \(\Delta A_{max} = A_0 - A_f\). Let \(A_i\) and \(A_j\) for \(i < j\) where \(i, j = 0, 1, 2, \ldots\) be the areas of the surfaces \(x_i(u, v)\) and \(x_j(u, v)\) obtained through \(ith\) and \(jth\) iterations respectively. Then the difference of the areas is denoted by \(\Delta A_{ij}\). When we know the minimal surface, the percentage decrease \(p_{ij}\) in area can be computed in such cases by multiplying quotient of \(\Delta A_{ij}\) and \(\Delta A_{max}\) by 100. Thus we have
\[
p_{ij} = 100 \frac{\Delta A_{ij}}{\Delta A_{max}} = 100 \frac{A_i - A_j}{A_0 - A_f} \text{ for } i < j \text{ and } i, j = 0, 1, 2, \ldots
\] (31)

When we do not know the minimal surface, the percentage decrease \(q_{ij}\) in area can be computed by using
\[
q_{ij} = 100 \frac{A_i - A_j}{A_0} \text{ for } i < j \text{ and } i, j = 0, 1, 2, \ldots
\] (32)
III. THE EFFICIENCY ANALYSIS OF THE ANSATZ EQ.  [18]

In this section we apply the technique introduced in the above section[11] to reduce the area of non-minimal surfaces
1) hemiellipsoid eq. (1), Fig. 1 spanned by an elliptic curve of which the minimal surface is known, namely the elliptic
disc. 2) surface eq. (2), Fig. 2 spanning a boundary comprising lines $0 \leq u, v \leq 1$ for which the minimal surface is
known, the plane bounded by the lines $0 \leq u, v \leq 1$, the area in this case is that of square of unit length, 3) surface
eq. (3), Fig. 3 spanned by four arbitrary straight lines lying in different planes, for which the corresponding minimal
surface is not known. The reduction in area is much lesser in this case showing that it is already a minimal surface. In
particular, if we take $H = 0$ and otherwise $H_n, n = 0, 1, 2, \ldots$ as the numerator part of mean curvature defined as above by the eq. (20). This is shown below in the following section and a comparison can be seen in the table[11]. In another choice we have taken $H_n, n = 0, 1, 2, \ldots$ as the numerator part of the mean curvature given by the eq. (30). In particular, if we take $H_n = 1, \forall n = 0, 1, 2, \ldots$, the ansatz once reduces the area of the surface but not usable for higher iterations as it can be seen from the following simple argument. For a convenient choice of $N(u, v) = k = constant vector (say)$ and $H_n = H_0, \forall n$, the ansatz eq. (18) reduces to the following expression expression

$$x_{n+1}(u, v, t) = x_0(u, v) + \tilde{t} m_0(u, v)k,$$

where $\tilde{t}$ is now variational parameter. The above ansatz may be used to reduce the area of the surface for generating
first variational surface $x_1(u, v)$ and in this case, as it can be seen from above eq. (33) that $x_2(u, v), x_3(u, v)$ and so
on are nothing but the replica of the first variational surface $x_1(u, v)$. The above ansatz eq. (33) is one time usable
iterative scheme.

Before, we apply this technique to reduce the area of above mentioned surfaces, we give below 1-dimensional
analogue of the ansatz eq. (15). We apply it to a curve of given length to find its reduced length given that the
shortest length between two points is a straight line. This gave us $X_i$ for $i = 1, 2, \ldots 8$ the curves of reduced lengths.

A. The 1-Dimensional Analogue of the Curvature Algorithm Applied to a Curve of a Given Length

It is to be noted that for a two dimensional variational problem, the ansatz eq. (18) is meant for determining and
investigating a surface of given boundary, to find its smallest possible area and differential geometry related quantities
motivated by Variational calculus. In order to foresee that how much this ansatz may be effective, we write it for a
one dimensional variational problem of a known minimal straight line. This might determine the smallest possible
length of an arc of a curve bounded by two given end points. The ansatz eq. (34) for the one dimensional variational
problem to find the smallest possible length of an arc of a given non-minimal curve $x_n(u), n = 0, 1, 2, \ldots$ joining the
same two ends of the straight line, may be written in the form:

$$\chi_{n+1}(u) = \chi_n + tm_n(u)N_n, \quad n = 0, 1, 2, \ldots,$$

where

$$m_n(u) = (1-u)H_n$$

is chosen so that it is zero at $u = 0$ and $u = 1$, here $N_n$ may be interpreted as the normal to the curve at $u$, that is
simply nothing but the second derivative w.r.t. the curve parameter $u$ serving as the curvature multiple of the unit
vector in the direction normal to the curve, thus $H_n = 1$ is the convenient choice in this case. When the technique is
applied to a curve

$$\chi_0(u) = (u, u - u^8),$$

yields following expressions for variationally improved curves

$$\chi_1(u) = (u, u - 7.42857u^7 + 6.42857u^8),$$

$$\chi_2(u) = (u, u - 22.131u^6 + 40.2381u^7 - 19.1071u^8),$$

$$\chi_3(u) = (u, u - 40.6973u^5 + 122.159u^6 - 128.943u^7 + 46.4814u^8),$$

$$\chi_4(u) = (u, u - 128.943u^5 + 46.4814u^6 - 40.6973u^7 + 22.131u^8),$$

$$\chi_5(u) = (u, u - 40.2381u^5 + 6.42857u^6 - 7.42857u^7 + u^8),$$

$$\chi_6(u) = (u, u),$$

$$\chi_7(u) = (u, u - u^2),$$

$$\chi_8(u) = (u, u - u^3).$$
\[ \chi_4(u) = \left( u, u - 39.6743u^4 + 177.61u^5 - 320.449u^6 + 261.908u^7 - 80.3952u^8 \right), \]  

(40)

\[ \chi_5(u) = \left( u, u - 21.394u^3 + 141.344u^4 - 414.012u^5 + 605.86u^6 - 434.714u^7 + 121.916u^8 \right), \]  

(41)

\[ \chi_6(u) = \left( u, u - 5.02685u^2 + 50.0547u^3 - 249.339u^4 + 622.028u^5 - 820.916u^6 + 547.646u^7 - 145.446u^8 \right), \]  

(42)

\[ \chi_7(u) = \left( u, 0.623914u + 6.58385u^2 - 73.1064u^3 + 327.961u^4 - 764.604u^5 + 960.763u^6 - 617.46u^7 + 159.24u^8 \right), \]  

(43)

\[ \chi_8(u) = \left( u, 1.07777u - 8.98864u^2 + 77.659u^3 - 334.761u^4 + 755.916u^5 - 926.532u^6 + 583.748u^7 - 148.119u^8 \right). \]  

(44)

Corresponding lengths of the curves are \( \ell_0 = 1.7329 \), \( \ell_1 = 1.46525 \), \( \ell_2 = 1.30988 \), \( \ell_3 = 1.24103 \), \( \ell_4 = 1.20133 \), \( \ell_5 = 1.16958 \), \( \ell_6 = 1.1459 \), \( \ell_7 = 1.12682 \) and \( \ell_8 = 1.11081 \). Percentage decrease in length may be obtained by using

\[ \ell_{ij} = 100 \frac{\ell_i - \ell_j}{\ell_0 - 1}, \text{ where } i < j \text{ and } i, j = 0, 1, 2, \ldots \]  

(45)

We have been able to manage eight iterations summarized in the following table that yields the following graph for the comparison with the reduction in the initial length of the given curve \( \gamma \), whose initial length is computed to be \( \ell_0 = 1.73291 \) as shown in the Fig. [4].

### TABLE I: Reduction in Length of a Curve of Given Length

| \( \chi_i \) | \( \ell_i \) | \( \ell_{ij} \) |
|------------|------------|------------|
| \( \chi_0 \) | 1.7329 |    |
| \( \chi_1 \) | 1.46525 | 36.5206 |
| \( \chi_2 \) | 1.30988 | 21.1996 |
| \( \chi_3 \) | 1.24103 | 9.3933 |
| \( \chi_4 \) | 1.20133 | 5.41714 |
| \( \chi_5 \) | 1.16958 | 4.33218 |
| \( \chi_6 \) | 1.1459 | 3.2307 |
| \( \chi_7 \) | 1.12682 | 2.60332 |
| \( \chi_8 \) | 1.11081 | 2.18471 |

In the remaining part of this section we compute the reduction in area of above mentioned non-minimal surfaces and their related quantities using eqs. [18] to [27].
FIG. 4: Graph of variational improvement to arc length of a curve of given length, comparing the result of eight iterations, where γ is the initial curve and the remaining curves are the curves of reduced lengths.

B. The Hemiellipsoid with known minimal surface namely the elliptic disc

We have been able to produce only one iteration as reported in the ref. [19], to reduce the area of the hemiellipsoid eq. (1). However, to see an improvement in the area reduction of the hemiellipsoid, we present here other modifications of the ansatz (18) than that implied in the refs. [19, 22] e.g. we may replace there the numerator of the mean curvature function $H_0$ one, in the first variational surface (or in all the variational surfaces that gives one time usable iteration as mentioned above by the ansatz eq. (33)). For $b$ and $c$ being constants and $0 \leq u \leq \pi/2$ and $0 \leq v \leq 2\pi$, fundamental magnitudes of the initial surface hemiellipsoid eq. (1) are

$$
E_0 = \cos^2 u \left( b^2 \sin^2 v + \cos^2 v \right) + c^2 \sin^2 u, \\
F_0 = (b^2 - 1) \sin u \cos u \sin v \cos v, \\
G_0 = \sin^2 u \left( b^2 \cos^2 v + \sin^2 v \right),
$$

and

$$
e_0 = -bc \sin u, \quad f_0 = 0, \quad g_0 = -bc \sin^3 u.
$$

Here

$$
E_0 G_0 - F_0^2 = \sin^2 u \left( c^2 \sin^2 u \left( b^2 \cos^2 v + \sin^2 v \right) + b^2 \cos^2 u \right),
$$

$$
A_0 = \int_0^1 \int_0^1 \sqrt{\sin^2 u(b^2 \cos^2 u + c^2 \sin^2 u(b^2 \cos^2 v + \sin^2 v))} \ du \ dv,
$$

$$
H_0(u, v) = \frac{1}{8} \left( -2bc \sin^3 u \left( (b^2 - 2c^2 + 1) \cos(2u) + 3b^2 + 2c^2 + 3 \right) - 4b \left( b^2 - 1 \right) \cos^5 u \cos(2v) \right).
$$

In particular for $b = 1, c = 1$, these quantities reduce to $E_0 = 1, F_0 = 0, G_0 = \sin^2 u, e_0 = -\sin u, f_0 = 0, g_0 = -\sin^3 u, H_0 = -2\sin^3 u, A_0 = 6.283186$. Below, we give the reduction in area of the hemiellipsoid defining

$$
m_0(u, v) = b(u, v) = \sin(1 - u)(1 - v).
$$

Variational Improvement in Hemiellipsoid for $H_0$ Replaced by 1

We choose $H_0 = 1$ (say), otherwise $H_n$ is the numerator part of (11) for $x_n$ for $n = 1, 2$. A convenient possible choice of the unit normal $N(u, v)$ in this case is $k = (0, 1, 0)$ and thus $m_0 = \sin(u) \sin(v), \cos(u)$ as mentioned above. This gives us following expression for the first variational surface

$$
x_1(u, v, t) = (\sin(u) \cos(v), t(\pi - v) + \sin(u) \sin(v), \cos(u)).
$$
Thus the coefficients of $H_1$ are given by
\begin{align*}
p_0 &= -2 \sin^3(u), \\
p_1 &= -2 \sin^2(u)(\sin(v) + 2(\pi - 2v) \cos(v)), \\
p_2 &= -\frac{1}{4}(\pi - 2v)^2 \sin(u)(\cos(2u - v)) + \cos(2u + v) - 2 \cos(2u) + 6 \cos(2v) + 6), \\
p_3 &= -(\pi - 2v)^3 \cos(v),
\end{align*}
and
\begin{equation}
\mu_1^2(t) = 1270.43t^6 + 1724.78t^5 + 1465.01t^4 + 813.722t^3 + 317.473t^2 + 85.333t + 12.337. 
\end{equation}

Minimizing above w.r.t. $t$ gives us $t_{min1} = -0.351571$, so that
\begin{equation}
x_1(u, v) = (\sin(u) \cos(v), \sin(u) \sin(v) - 0.351571(\pi - v) \cos(u)).
\end{equation}

It can be seen that $A_1 = 4.70625$ and thus the percentage decrease in area is given by $p_{01} = 50.1954$. We find
\begin{equation}
\mu_2^2 = 21975.9t^6 - 9141.25t^5 + 5060.49t^4 - 1145.39t^3 + 363.123t^2 - 40.3821t + 1.73308.
\end{equation}

In this case $t_{min2} = 0.0706353$ and hence
\begin{equation}
x_2(u, v) = (\sin(u) \cos(v), 0.0706353(\pi - v) - v(0.888171(1.5078 - 1.1 \sin^2(u) \cos^2(u) \sin^2(v) + (\sin^2(u) \sin^2(v) + (sin(u) \cos(v)) + (0.703142 \sin(v) + (0.703142 - 1.10449)(\cos(v) - \sin(u))) - 1. \sin^2(u) \sin(v) - 0.351571(\pi - v) \cos(u)).
\end{equation}

Here, $A_2 = 4.44025$ and $p_{12} = 8.4671$ or percentage decrease in area w.r.t the maximum possible change is $p_{02} = 58.6625$. Here
\begin{equation}
\mu_3^2 = 6317.67t^6 - 656.284t^5 + 1498.48t^4 - 104.093t^3 + 239.168t^2 - 10.7401t + 0.401386.
\end{equation}

In this case $t_{min3} = 0.0226436$ and this can give us $x_3(u, v)$ (a lengthy expression not produced here) of which area comes out to be $A_3 = 4.4025$, $p_{23} = 1.20025$ or $p_{03} = 59.8627$. These results are presented in the table II.

**Variational Improvement in Hemiellipsoid when $H_0$ is the numerator of the mean curvature**

In this case we have been able to produce only two iterations. Here $H_n$ for $n = 0, 1$ are the numerator part of the mean curvature of the respective surfaces. For $b = 1$, $c = 1$, $b(u, v) = v(\pi - v)$, the expressions for $m_0(u, v)$, the first variational surface $x_1(u, v, t)$ and mean square of the mean curvature are
\begin{equation}
m_0(u, v) = -2(\pi - v)v \sin^3 u, 
\end{equation}
\begin{equation}
x_1(u, v, t) = (\sin u \cos v, \sin u \sin v - 2t (\pi - v) v \sin^3 u, \cos u),
\end{equation}
and
\begin{equation}
\mu_1^2 = 39774.1 \ t^6 - 41607.9 \ t^5 + 22816.4 \ t^4 - 7479.4 \ t^3 + 1683.28 \ t^2 - 219.911 \ t + 12.337.
\end{equation}

In this case $t_{min1} = 0.148252$, for which first variational surface is given by
\begin{equation}
x_1(u, v) = (\sin u \cos v, \sin u(0.698621 - 0.222378v) \cos^2 u + (0.074126v - 0.232874) v \sin^2 u + 0.222378 \ v^2 - 0.698621 v + \sin v), \cos u).
\end{equation}

Percentage decrease in the $rms$ mean curvature of the numerator of the mean curvature is 74.1475. Area $A_1 = 4.32641$, percentage decrease in area $p_{01} = 62.2861$, as summarised below in the table II. Continuing the process as above we find the mean square of mean curvature as a function of variational parameter $t$ given by
\begin{equation}
\mu_2^2 = 7735.45t^6 + 4011.85t^5 + 9417.35t^4 - 174.627t^3 + 701.03t^2 - 32.3979t + 0.824546.
\end{equation}

Here $t_{min2} = 0.0229728$ and the related results summarised in the table II. It may be seen from the table II that this choice of $H_0(u, v)$ give better reduction in the area even through the first iteration.
C. A Hump Like Surface \(2\) spanned by four boundary straight lines

We apply the ansatz eq. (18) along with eq. (19) mentioned in the section II to the surface \(x(u, v)\) given by eq. (2) bounded by four straight lines \(0 \leq u, v \leq 1\). A function \(b(u, v)\) whose variation at the boundary curves is zero is given by

\[
b(u, v) = uv(1-u)(1-v). \tag{64}
\]

A convenient possible choice of above mentioned unit normal \(N(u, v)\), making a small angle with it in case of surface given by eq. (2) is:

\[
k = (0, 0, 1). \tag{65}
\]

The surface given by eq. (2) is a non-minimal surface and the \(xy - plane\) bounded by \(0 \leq u, v \leq 1\) is a minimal surface in this case.

\[
E_0 = 1 + (16(1-u)(1-v)v - 16u(1-v)v)^2, \tag{66}
\]

\[
F_0 = (16(1-u)u(1-v) - 16(1-u)uv)(16(1-u)(1-v)v - 16u(1-v)v), \tag{67}
\]

\[
G_0 = 1 + (16(1-u)u(1-v) - 16(1-u)uv)^2, \tag{68}
\]

\[
e_0 = -32v + 32v^2, \tag{69}
\]

\[
f_0 = 16 - 32u - 32v + 64uv, \tag{70}
\]

\[
g_0 = -32u + 32u^2. \tag{71}
\]

Thus, eq. (20) for \(n = 0\), along with eqs. (66) to (71) gives

\[
H_0 = 32((1 + 256(-1 + u)^2)u^2(1 - 2v)^2(-1 + v)v - 256(1 - 2u)^2(-1 + u)u(1 - 2v)^2
\]

\[
(-1 + v)v + (-1 + u)u(1 + 256(-1 + u)^2(-1 + v)v^2)), \tag{72}
\]

as shown in the Fig. 5. Substituting value of \(H_0\) from eq. (72) along with eq. (64) in eq. (19), so that for \(n = 0\),

\[
x_1(u, v, t) = (u, v, 16(1-u)u(1-v)v + t(1-u)u(1-v)v(-2(16 - 32u - 32v + 64uv)
\]

\[
(16(1-u)u(1-v) - 16(1-u)uv)(16(1-u)(1-v)v - 16u(1-v)v) +
\]

\[
(-32v + 32v^2)(1 + (16(1-u)u(1-v) - 16(1-u)uv)^2) + (-32u + 32u^2)
\]

\[
(1 + (16(1-u)(1-v)v - 16u(1-v)v^2))). \tag{73}
\]

FIG. 5: \(H_0(u, v)\), numerator of mean curvature of the initial surface \(x_0(u, v)\) for \(0 \leq u, v \leq 1\).
Fundamental Magnitudes for this variational surface are $E_1(u,v,t)$, $F_1(u,v,t)$, $G_1(u,v,t)$, $e_1(u,v,t)$, $f_1(u,v,t)$, $g_1(u,v,t)$ are as follow:

$$E_1(u,v,t) = 1 + 256(1 - 2u)^2(-1 + v)^2v^2(1 + 2t((-1 + v)v + 1536u^3(-1 + v)v(3 + 8(-1 + v)v)$$
$$- 768u^4(-1 + v)v(3 + 8(-1 + v)v) + 2u(-1 + 256(-1 + v)v(13(-1 + v)v))$$
$$+ 2u^2(1 - 128(-1 + v)v(11 + 30(-1 + v)v))))^2, \tag{74}$$

$$F_1(u,v,t) = 256(-1 + u)u(-1 + 2u)(-1 + v)v(-1 + 2v)(-1 + 2t(-2(-1 + u)v + u - 256(-1 + u)u(-2 + 3u)$$
$$(-1 + 3u)v + (-1 + 256(-1 + u)u(8 + 33(-1 + u)u)v^2 - 3072(1 - 2u)^2(-1 + u)uv^3$$
$$+ 1536(1 - 2u)^2(-1 + u)uv^4))(-1 + 2t(-2(-1 + v)v - 3072u^3(1 - 2u)^2(-1 + v)v$$
$$+ 1536u^4(1 - 2u)^2(-1 + v)v + 2u(-1 + 256(-1 + v)v(-2 + 3v)$$
$$(-1 + 3v)) + u^2(-1 + 256(-1 + v)v(8 + 33(-1 + v)v))). \tag{75}$$

$$G_1(u,v,t) = 1 + 256(-1 + u)^3u^2(1 - 2v)^2(1 + 2t((-1 + u)v - 2v + 512(-1 + u)u(1 + 3(-1 + u)u)v$$
$$+ 2(1 - 128(-1 + u)v(11 + 30(-1 + u)u)v^2 + 1536(-1 + u)u(3 + 8(-1 + u)uv^3 - 768$$
$$(1 + 1 + 3(-1 + u)uv^3)^2, \tag{76}$$

$$e_1(u,v,t) = -32(-1 + v)v(-1 + 2t(-1 - 6u - 6u^2 - 15(17 + 256(-1 + u)u(13(-1 + u)u)v$$
$$+ 3(341 + 256(-1 + u)u(19 + 55(-1 + u)u)v^2 - 1536(1 - 2u)^2(1 + 10(-1 + u)uv^3$$
$$+ 768(1 - 2u)(1 + 10(-1 + u)uv^3), \tag{77}$$

$$f_1(u,v,t) = -16(-1 + 2u)(-1 + v)v(-1 + 4t(-1 + u)u + v - 256(-1 + u)u(-2 + 3u)(-1 + 3u)v$$
$$+ (-1 + 256(-1 + u)u(11 + 45(-1 + u)u)v^2 - 4608(1 - 2u)^2(-1 + u)uv^3$$
$$+ 2304(1 - 2u)^2(-1 + u)uv^4)), \tag{78}$$

$$g_1(u,v,t) = -32(-1 + u)u(-1 + 2t(-1 - 6(-1 + v)v - 1536u^3(1 - 2v)^2(1 + 10(-1 + v)v$$
$$+ 768u^4(1 - 2v)^2(1 + 10(-1 + v)v - 15u(17 + 256(-1 + v)v(1 + 3(-1 + v)v)$$
$$+ 3u^2(341 + 256(-1 + v)v(19 + 55(-1 + v)v))). \tag{79}$$

Plugging in above values of fundamental magnitudes in eq. \([21]\) for $n = 1$, we find the expression for $H_1(u,v,t)$, mean curvature of $x_1(u,v,t)$

$$H_1(u,v,t) = -32(-1 + v)v(-1 + 2t(-1 - 6u - 6u^2 - 15(17 + 256(-1 + u)u(13(-1 + u)u)v$$
$$+ 3(341 + 256(-1 + u)u(19 + 55(-1 + u)u)v^2 - 1536(1 - 2u)^2(1 + 10(-1 + u)uv^3$$
$$(1 + 1 + 3(-1 + u)uv^3)^2, \tag{80}$$

$$+ 256(-1 + u)(1 + 3(-1 + u)uv^3)^2, \tag{81}$$

$$+ 256(-1 + u)u(8 + 33(-1 + u)u)v^2 - 3072(1 - 2u)^2(-1 + u)uv^3$$
$$+ 1536(1 - 2u)^2(-1 + u)uv^4))(-1 + 2t((-1 + u)v - 2v + 512(-1 + u)u(13(-1 + u)u)v$$
$$+ 2(1 - 128(-1 + u)v(11 + 30(-1 + u)u)v^2 + 1536(-1 + u)u(3 + 8(-1 + u)uv^3 - 768$$
$$(1 + 1 + 3(-1 + u)uv^3)^2, \tag{76}$$

$$e_1(u,v,t) = -32(-1 + v)v(-1 + 2t(-1 - 6(-1 + v)v - 1536u^3(1 - 2v)^2(1 + 10(-1 + v)v$$
$$+ 768u^4(1 - 2v)^2(1 + 10(-1 + v)v - 15u(17 + 256(-1 + v)v(1 + 3(-1 + v)v)$$
$$+ 3u^2(341 + 256(-1 + v)v(19 + 55(-1 + v)v))). \tag{79}$$

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$$+ 768u^4(1 - 2v)^2(1 + 10(-1 + v)v - 15u(17 + 256(-1 + v)v(1 + 3(-1 + v)v)$$
$$+ 3u^2(341 + 256(-1 + v)v(19 + 55(-1 + v)v))). \tag{80}$$

$$+ 1536(1 - 2u)^2(-1 + u)uv^4))(-1 + 2t((-1 + u)v - 2v + 512(-1 + u)u(13(-1 + u)u)v$$
$$+ 2(1 - 128(-1 + u)v(11 + 30(-1 + u)u)v^2 + 1536(-1 + u)u(3 + 8(-1 + u)uv^3 - 768$$
$$(1 + 1 + 3(-1 + u)uv^3)^2, \tag{76}$$

$$e_1(u,v,t) = -32(-1 + v)v(-1 + 2t(-1 - 6(-1 + v)v - 1536u^3(1 - 2v)^2(1 + 10(-1 + v)v$$
$$+ 768u^4(1 - 2v)^2(1 + 10(-1 + v)v - 15u(17 + 256(-1 + v)v(1 + 3(-1 + v)v)$$
$$+ 3u^2(341 + 256(-1 + v)v(19 + 55(-1 + v)v))). \tag{79}$$

Plugging in above values of fundamental magnitudes in eq. \([21]\) for $n = 1$, we find the expression for $H_1(u,v,t)$, mean curvature of $x_1(u,v,t)$

$$H_1(u,v,t) = -32(-1 + v)v(-1 + 2t(-1 - 6(-1 + v)v - 1536u^3(1 - 2v)^2(1 + 10(-1 + v)v$$
$$+ 768u^4(1 - 2v)^2(1 + 10(-1 + v)v - 15u(17 + 256(-1 + v)v(1 + 3(-1 + v)v)$$
$$+ 3u^2(341 + 256(-1 + v)v(19 + 55(-1 + v)v))). \tag{80}$$
The coefficients \( p_i(u, v) \) of \( t^i \) for \( i = 0, 1, 2, 3 \) in the expansion of \( H_1(u, v, t) \) are

\[
p_0(u, v) = -32((-1 + u)u + v - 256((-1 + u)u(1 + 3((-1 + u)u)v + (-1 + 256((-1 + u)u)(4 + 11(-1 + u)u)v^2 - 512((-1 + u)u(3 + 8((-1 + u)u)v^3 + 256((-1 + u)u(3 + 8((-1 + u)u)v^4),
\]

\[
p_1(u, v) = 64((-768432u^7((-1 + v)^2v^2(21 + 4((-1 + v)v(31 + 48((-1 + v)v) + 196608u^8((-1 + v)^2v^2(21 + 4((-1 + v)v(31 + 48((-1 + v)v)) - (-1 + v)v(-1 + 3((-1 + v)v(85 + 256((-1 + v)v)) + u(-1 + 4((-1 + v)v(-3 + 64((-1 + v)v(19 + 54((-1 + v)v)) - 768u^5(-3 + 2((-1 + v)v(-27 + 2((-1 + v)v(7835 + 256((-1 + v)v(184 + 285((-1 + v)v)) + 256u^6(-3 + 2((-1 + v)v(-27 + 2((-1 + v)v(26651 + 768((-1 + v)v(206 + 319((-1 + v)v)) + 2u^2(-127 + 2((-1 + v)v(12) + 16((-1 + v)v(11765 + (-1 + v)v(72347 + 112320((-1 + v)v)))(119 + 32((-1 + v)v(11 + 12((-1 + v)v)) + 2359296u^{10}(-1 + v)^2v^2(14 + (-1 + v)v(119 + 32((-1 + v)v(11 + 12((-1 + v)v)) + u(-1 + v)v(-9 + 2((-1 + v)v(2029 + 2((-1 + v)v(7093 + 12672((-1 + v)v)) - 6144u^7((-1 + v)v(-33 + 16((-1 + v)v(4453 + 6((-1 + v)v(6383 + 128((-1 + v)v(149 + 164((-1 + v)v)v(14533 + 6((-1 + v)v(20663 + 128((-1 + v)v(479 + 524((-1 + v)v))) - u^3(31 + 8((-1 + v)v(-4561 + 8((-1 + v)v(143089 + 32((-1 + v)v(43313 + 48((-1 + v)v(2981 + 3528((-1 + v)v)) + u^2(6 + (-1 + v)v(-4049 + 2((-1 + v)v(367651 + 2((-1 + v)v(1912651 + 384((-1 + v)v(17455 + 21504((-1 + v)v))) - 3u^5(19 + 4((-1 + v)v(-23989 + 256((-1 + v)v(53227 + 12((-1 + v)v(39553 + 4((-1 + v)v(30469 + 34384((-1 + v)v))) + u^6(19 + 4((-1 + v)v(-83125 + 256((-1 + v)v(326171 + 12((-1 + v)v(237065 + 12((-1 + v)v(59735 + 66416((-1 + v)v)))) + u^4(63 + 2((-1 + v)v(-69931 + 16((-1 + v)v(1582433 + 32((-1 + v)v(456095 + 24((-1 + v)v(60259 + 69408((-1 + v)v)))))))),
\]
\[ p_3(u, v) = 65536(1 + u)(1 + v)(1 + v)(3 + v)(3 + 10(1 + v)(v) - 25367150592u^{13}(1 - 2v)^2
\]
\([-1 + v]^3v^2(3 + 8(1 + v)(v)(2(1 + v)(v)(9 + 16(1 + v)(v))) + 3623878656u^{14}(1 - 2v)^2
\]
\([-1 + v]^3v^3(3 + 8(1 + v)(v)(2(1 + v)(v)(9 + 16(1 + v)(v))) - u(1 + v)^2v^2(6 + (1 + v)(v)(2973
\]
\[+ 8(1 + v)(v)(255 + 4704(1 + v)(v))) + 2359296u^{12}(1 + v)^2v(213161
\]
\[+ 64(1 + v)v(35519 + (1 + v)v(155053 + 768(1 + v)v(421 + 355(1 + v)v)v)))) - 14155776u^{11}
\]
\([-1 + v]v^2(21 + (1 + v)v(63385 + 64(1 + v)v(11131 + (1 + v)v(48765 + 128(1 + v)v
\]
\[797 + 674(1 + v)v))))) - u^2(1 + v)v(6 + (1 + v)v(-5090 + (1 + v)v(999971 + 8(1 + v)v
\]
\[1355491 + 96(1 + v)v(51427 + 64(1 + v)v(11131 + (1 + v)v(48765 + 128(1 + v)v
\]
\[768u^9(1 + v)v(245 + 4(1 + v)v
\]
\[(-386905 + 8(1 + v)v(38922251 + 256(1 + v)v(1740737 + 24(1 + v)v(322558 + (1 + v)v(683777
\]
\[+ 584928(1 + v)v)))))) + 768u^{10}(1 + v)v(49 + 4(1 + v)v(-254789 + 8(1 + v)v(45198319 + 256
\]
\[(-1 + v)v(1999045 + 24(1 + v)v(367093 + (1 + v)v(772045 + 655968(1 + v)v)))) + u^3(3 - (1 + v)v
\]
\[(2781 + 4(1 + v)v(-248143 + 2(1 + v)v(14160189 + 128(1 + v)v(1627565 + 6(1 + v)v(1477367
\]
\[+ 384(1 + v)v(9583 + 9216(1 + v)v))))) - 8u^2(5 - 4(1 + v)v(14285 + 128(1 + v)v(-179703 + 4
\]
\[(-1 + v)v(15643685 + 32(1 + v)v(5816093 + 48(1 + v)v(556153 + 36(1 + v)v(33637 + 29392
\]
\[(-1 + v)v)))))) + u^4(-19 + (1 + v)v(28573 + 4(1 + v)v(-346681 + (1 + v)v(508303293 + 128(1 + v)v
\]
\[53765501 + 18(1 + v)v(15240289 + 512(1 + v)v(70459 + 65280(1 + v)v)))))) + 2u^8(-5 + 4(1 + v)v
\]
\[(49565 + 64(1 + v)v(-2244879 + 64(1 + v)v(1785581 + 4(1 + v)v(52018853 + 48(-1 + v)v
\]
\[(4883029 + 72(-1 + v)v(145395 + 125528(-1 + v)v)))))))) + u^5(49 - (1 + v)v(126811 + 4(1 + v)v
\]
\[(-1(1 + v)v(327257 + 295936(-1 + v)v)))))) + u^6(-63 + (1 + v)v(312153 + 4(-1 + v)v
\]
\[(-78350147 + 256(-1 + v)v(76888369 + 8(-1 + v)v(117926386 + 3(-1 + v)v(185099599
\]
\[+ 288(-1 + v)v(1428851 + 1267712(-1 + v)v))))))))
\]

(84)

\[ H_1(u, v, t) \] is a polynomial in \( t \) and thus \( H_1^2(u, v, t) \) is polynomial in \( t \) as well. We find the non-zero coefficients \( q_i(u, v) \) of \( t^i \) for \( i = 0, 1, 2, 3, 4, 5, 6 \) and integrate these coefficients for \( 0 \leq u, v \leq 1 \) as mentioned in eq. \((23)\) to get expression for mean square of mean curvature \( \mu^2_1 \) for \( n = 1 \), as a polynomial in \( t \), given by

\[ \mu^2_1(t) = 1637.65 - 20425t + 195725t^2 - 898809t^3 + 2.98414 \times 10^6t^4 - 5.10679 \times 10^6t^5 + 4.1912 \times 10^6t^6, \]

(85)

shown in Fig. 6. Minimizing above polynomial eq. \((85)\) for \( t \), giving \( t_{\text{min} 1} = 0.088933 \), we find the variationally improved surface \( x_1(u, v) \) eq. \((28)\) for this minimum value of \( t \), that is,

\[ x_1(u, v) = (u, v, -(1 + u)u(1 + v)v(16 + v(-2.84585 + 2.84585v) + u^4v(2185.61 + v(-8013.91
\]
\[+ (1165.6 - 5828.3v)v)) + u^3v(-3318.72 + v(160207.8 + v(-23313.2 + 1165.6v)))
\]
\[+ u^2(2.84585 + v(2914.15 + v(-10928.1 + (160207.8 - 8013.91v)v))) + u(-2.84585
\]
\[+ v(-728.537 + v(2914.15 + v(-4371.22 + 2185.61v)))))). \]

(86)

FIG. 6: Mean square of mean curvature \( \mu^2_1(t) \) of the surface \( x_1(u, v) \) as a function of \( t \).
shown in Fig. 7. For this \( t_{\text{min1}} \) mean curvature of \( x_1(u, v) \) is shown in Fig. 8. Initial area of surface \( x_0(u, v) \) (using eq. (30)) is 2.494519 units and that of surface (86) comes out to be 2.11589 units as given by eq. (27) \((n = 1)\) for \( t_{\text{min1}} = 0.088933 \). Percentage decrease in original area in this case comes out to be \( p_01 = 15.1784 \). Substituting \( H_1(u, v) \) (shown in Figure 8) in eq. (19) along with eq. (64), so that for \( n = 1 \), eq. (18) results in expression for variational surface \( x_2(u, v, t) \). We find fundamental Magnitudes \( E_2(u, v, t) \), \( F_2(u, v, t) \), \( G_2(u, v, t) \), \( e_2(u, v, t) \), \( f_2(u, v, t) \), \( g_2(u, v, t) \) for this variational surface \( x_2(u, v, t) \) and insert these fundamental coefficients in eq. (21) to get the expression for \( H_2(u, v, t) \), mean curvature of \( x_2(u, v) \), and thus mean square of mean curvature \( H_2(u, v, t) \) (eq. (23) for \( n = 2 \)) gives the following expression,

\[
\mu_2^2(t) = 897.323 - 14022.3t + 207068t^2 - 1.07708 \times 10^6t^3 + 6.45464 \times 10^6t^4 - 9.9155 \times 10^6t^5 + 1.99266 \times 10^7t^6,
\]

(87)

and minimizing this expression results in \( t_{\text{min2}} = 0.044086 \). For this \( t_{\text{min2}} = 0.044086 \) we find variationally improved surface \( x_2(u, v) \) shown in Fig. 9 and mean curvature of \( x_2(u, v) \) is shown in Fig. 10 and that of quantity \( m_2(u, v) \) is shown in Fig. 11. The summary of related results is provided in the table II.

D. The Bilinear Interpolation Surface eq. (3) spanned by four boundary straight lines

In above part of section we have applied the algorithm eq. (18) along with eq. (19) (section II) to the surfaces where the corresponding minimal surface is known to see that how much it is efficient in these cases of known minimal surfaces. Below, we apply the above mentioned algorithm to an important class of surfaces, namely the bilinear interpolation where the corresponding minimal surface is not explicitly known. Here, we have taken the initial surface \( x_0(u, v) \) given by eq. (3) bounded by four straight lines \( 0 \leq u, v \leq 1 \). A convenient possible choice of the above mentioned unit normal \( \mathbf{N}(u, v) \) in the eq. (18), making a small angle with it in case of surface given by eq. (3) is:
FIG. 9: Surface \( x_2(u,v,t) \) for \( t_{min2} = 0.044086 \) for \( 0 \leq u,v \leq 1 \).

FIG. 10: Mean curvature of the surface \( x_2(u,v) \) for \( t_{min2} = 0.044086 \) for \( 0 \leq u,v \leq 1 \).

FIG. 11: \( m_2(u,v) \) for the surface \( x_3(u,v) \) for \( 0 \leq u,v \leq 1 \).

\[ k = (-1,0,0). \] (88)

The corresponding function \( b(u,v) \) such that its variation at the boundary curves \( u = 0, u = 1, v = 0, v = 1 \) is zero is given by

\[ b(u,v) = uv(1-u)(1-v). \] (89)
The surface given by (3) is a non-minimal surface spanned by a boundary composed of non-coplanar straight lines for $0 \leq u, v \leq 1$. The fundamental magnitudes of the initial surface $x_0(u, v)$ eq. (3) in this case are,

$$E_0 = 1 + (1 - 2v)^2, F_0 = (1 - 2u)(1 - 2v), G_0 = 1 + (1 - 2u)^2, e_0 = 0, f_0 = 2, g_0 = 0. \quad (90)$$

For $n = 0$, eq. (20) along with eqs. (90) gives

$$H_0 = -4(1 - 2u)(1 - 2v), \quad (91)$$
as shown in the Fig. 12.

![Fig. 12](image)

FIG. 12: $H_0(u, v)$, numerator of mean curvature of the initial surface $x_0(u, v)$ of bilinear interpolation for $0 \leq u, v \leq 1$.

The variationally improved surface eq. (18) along with its fundamental magnitudes as function of $u, v$ and $t$ are

$$x_1(u, v, t) = (u + v - 2uv + 4uv(1 - u)(1 - v)(1 - 2u)(1 - 2v)t, v, u), \quad (92)$$

$$E_1(u, v, t) = 1 + (1 - 2v)^2 \left( 1 + 4t \left( 1 - 6u + 6u^2 \right) (-1 + v)v \right)^2, \quad (93)$$

$$F_1(u, v, t) = (-1 + 2u)(-1 + 2v) \left( -1 + 4t \left( 1 - 6u + 6u^2 \right) (-1 + v)v \right) \left( -1 + 4t(-1 + u)u \left( 1 - 6v + 6v^2 \right) \right), \quad (94)$$

$$G_1(u, v, t) = (1 - 2u)^2 \left( 1 - 4t(u - 1)u \left( 6v^2 - 6v + 1 \right) \right)^2 + 1, \quad (95)$$

$$e_1(u, v, t) = -24t(-1 + 2u)v \left( 1 - 3v + 2v^2 \right), \quad (96)$$

$$f_1(u, v, t) = -2 (-1 + 2t \left( 1 - 6u + 6u^2 \right) \left( 1 - 6v + 6v^2 \right)), \quad (97)$$

$$g_1(u, v, t) = -24t(u \left( 1 - 3u + 2u^2 \right) (-1 + 2v). \quad (98)$$

Thus, eq. (21) along with eqs. (93) to (98) gives us

$$H_1(u, v, t) = 16(1 - 2u)^2(1 - 2v)^2, \quad (99)$$

The Coefficients $p_i(u, v)$ of $t^i$ for $i = 0, 1, 2, 3$ in the expansion of $H_1(u, v, t)$ as mentioned in the eq. (21) are

$$p_0(u, v) = 16(1 - 2u)^2(1 - 2v)^2, \quad (100)$$
\begin{align}
p_1(u, v) &= -64(1 - 2u)^2(1 - 2v)^2 \left(1 - 2v + 2v^2 + u(-2 + 36v - 36v^2) + u^2(2 - 36v + 36v^2)\right), \\
p_2(u, v) &= 64(1 - 2u)^2(1 - 2v)^2(1 - 8v + 36v^2 - 56v^3 + 28v^4 - 8u(1 - 17v + 65v^2 - 96v^3 + 48v^4) \\
&\quad - 8u^3(7 - 96v + 636v^2 - 1080v^3 + 540v^4) + 4u^4(7 - 96v + 636v^2 - 1080v^3 + 540v^4) + 4u^2 \\
&\quad (9 - 130v + 766v^2 - 1272v^3 + 636v^4)), \\
p_3(u, v) &= -512(1 - 2u)^2(1 - 2v)^2(v(-1 + 9v - 28v^2 + 44v^3 - 36v^4 + 12v^5) + u(-1 + 20v - 174v^2 \\
&\quad + 692v^3 - 1306v^4 + 1152v^5 - 384v^6) - 36u^5(1 - 32v + 278v^2 - 1212v^3 + 2406v^4 - 2160v^5 \\
&\quad + 720v^6) + 12u^6(1 - 32v + 278v^2 - 1212v^3 + 2406v^4 - 2160v^5 + 720v^6) + u^2(9 - 174v \\
&\quad + 1432v^2 - 5852v^3 + 11266v^4 - 10008v^5 + 3336v^6) - 4u^3(7 - 173v + 1463v^2 \\
&\quad - 6216v^3 + 12198v^4 - 10908v^5 + 3636v^6) + 2u^4(22 - 653v + 5633v^2 - 24396v^3 + \\
&\quad 48288v^4 - 43308v^5 + 14436v^6)).
\end{align}

\(H_1(u, v, t)\) is a polynomial in \(t\) and thus \(H_1^2(u, v, t)\) is polynomial in \(t\) as well. We find the non-zero coefficients \(q_i(u, v)\) of \(t^i\) for \(i = 0, 1, 2, 3, 4, 5, 6\) and integrate these coefficients for \(0 \leq u, v \leq 1\) as mentioned in eq. (23) to get expression for mean square of mean curvature as a polynomial in \(t\), given by

\begin{equation}
\mu_1^2(t) = 1.77778 - 6.826667t + 6.42612t^2 + 0.696599t^3 + 0.264823t^4 + 0.00760335t^5 + 0.00089932t^6.
\end{equation}

Minimizing above polynomial eq. (104) for \(t\), giving us \(t_{\text{min}} = 0.483636\). Mean square of the mean curvature of the surface \(x_1(u, v)\) as a function of \(t\) is shown in Fig. 15 and for this \(t_{\text{min}} = 0.483636\) mean curvature of \(x_1(u, v)\) is shown in Fig. 14.

![FIG. 13: The surface \(x_1(u, v)\) at \(t = t_{\text{min}} = 0.483636\) for \(0 \leq u, v \leq 1\).](image)

We find the variationally improved surface eq. (28) for this \(t_{\text{min}} = 0.483636\),

\begin{equation}
x_1(u, v) = \{u + v - 2uv + 1.93454(-1 + u)u(-1 + 2u)(-1 + v)v(-1 + 2v), v, u\},
\end{equation}

shown in Fig. 16. Initial area of the surface \(x_0(u, v)\) (eq. (3)) (using eq. (30)) comes out to be \(A_0 = 1.280789\) and that of surface (105) is \(A_1 = 1.27936\).
FIG. 14: Mean curvature of the surface $x_1(u,v)$ for $t = t_{\text{min}} = 0.483636$ for $0 \leq u, v \leq 1$

FIG. 15: Mean square of the mean curvature of the surface $x_1(u,v)$ as a function of $t$

FIG. 16: $x_1(u,v,t)$ for $t = t_{\text{min}} = 0.483636$ for $0 \leq u, v \leq 1$
In the similar way, we find the second order variation of the above surface $x_0(u,v)$ eq. \[3\] is given by

$$x_2(u,v,t) = (u + v - 2uv + 1.9345(-1 + u)u(-1 + 2u)(-1 + v)v(-1 + 2v) - t(1-u)u(1-v)v(-2(1+(-0.0655 - 11.6073u + 11.6073u^2)v + (-5.8036 + 34.8218u - 34.8218u^2)v^2 + (3.8691 - 23.2145u + 23.2145u^3)(1 + u^2(-5.8036 + 34.8218u - 34.8218u^2) + u(-0.0655 - 11.6073v + 11.6073v^2) + u^3(3.8691 - 23.2145v + 23.2145v^2)))(0.06546 + 11.6073v - 11.6073v^2 + u^2(-11.6073 + 69.6435v - 69.6435v^2) + u(11.6073 - 69.6435v + 69.6435v^2)) + u(11.6073 + u^2(23.2145 - 46.429v) - 23.2145 + u(-34.8218 + 69.6435v))))(1 + (1 + (-0.0656 - 11.6073u + 11.6073u^2)v + (-5.803636 + 34.8218u - 34.8218u^2)v^2 + (3.86908 - 23.2145v + 23.2145v^2) + v(11.6073 - 34.8218v + 23.2145v^2 + u(-34.8218 + 69.6435v - 46.429v^2)))(1 + (1 + u^2(-5.8036 + 34.8218v - 34.8218v^2) + u(-0.0655 - 11.6073v + 11.6073v^2) + u^3(3.8691 - 23.2145v + 23.2145v^2))v),u),\) \[106\]

and eq. \[23\] gives mean square $ms$ of mean curvature

$$\mu_s^2(t) = \frac{0.072754 - 1.29519t + 7.35239t^2 - 0.00580625t^3 + 0.0146511t^4 - 0.000111494t^5 + 0.0000269367t^6,\] \[107\]

as shown in the Fig. \[17\] as a function of $t$. Here it comes out $t_{min2} = 0.088086$ and for this $t_{min2} = 0.088086$, $x_2(u,v)$

\[108\]

**FIG. 17:** Mean square of the mean curvature $\mu_s^2(t)$ of the bilinear interpolation surface $x_2(u,v)$ as a function of $t$.

is given by the following expression

$$x_2(u,v) = (u(1-v) + (1-u)v + 1.9345(1-2u)(1-u)u(1-2v)(1-v)v - 0.088071(1-u)u(1-v)v(-2(0.0655 + 11.6073u - 11.6073u^2 + 11.6073v - 69.6435uv + 69.6435u^2v - 11.6073v^2 + 69.6435uv^2 - 69.6435u^2v^2)) + (1 - 2u - 0.4836(-4(1 - 2u)(1-u)u(1-2v)(1-v) + 4(1 - 2u)(1-u)u(1-2v)v + 8(1 - 2u)(1-u)

u(1-v)(1-v)v/(1-2v - 0.4836(-4(1 - 2u)(1-u)(1-2v)(1-v)^2 + 4(1 - 2u)(1-u)v + 8(1 - 2u)v + 8(1 - u)

u(1-2v)(1-v)v)) + (11.6073v - 23.2145uv - 34.8218v^2 + 69.6435uv^2 + 23.2145v^3 - 46.429uv^3)) + (1 + (1 - 2u - 0.4836(-4(1 - 2u)(1-u)(1-2v)(1-v)^2 + 4(1 - 2u)(1-u)v + 8(1 - 2u)v + 8(1 - u)

u(1-2v)(1-v)v)) + (11.6073u - 34.8218u^2 + 23.2145u^3 - 23.2145uv + 69.6435u^2v - 46.429au^3) + (1 + (1 - 2u - 0.4836(-4(1 - 2u)(1-u)(1-2v)(1-v)^2 + 4(1 - 2u)(1-u)v + 8(1 - u)(1-2v)

v(1-v)^2)))/(1-2u + 0.4836(-4(1 - 2u)(1-u)(1-2v)(1-v)^2 + 4(1 - 2u)(1-u)v + 8(1 - u)(1-2v)

v(1-v)^2)) + 108,\)] \[108\]

as shown in the Fig. \[18\] and its curvature for this $t_{min2} = 0.0880861$ is shown in the Fig. \[19\]. Initial area (eq. \[27\] for $n = 0$) of surface $x_0(u,v)$ eq. \[3\] is $1.280789$ and that (eq. \[27\] for $n = 1$) of $x_1(u,v)$ eq. \[105\] is $1.27936$ units for $t_{min} = 0.483636$. The area (eq. \[27\] for $n = 2$) of surface $x_2(u,v)$ eq. \[106\] comes out to be $1.279301$, percentage decrease in area w.r.t that of eq. \[105\] comes out to be 0.004382. We have not been able to find higher order variational surfaces for $n \geq 3$ in this case, however to foresee that further reduction is possible the variational quantity $m_2(u,v)$ for $x_3(u,v)$ is shown in Fig. \[20\]. The algorithm reduces area less significantly showing that it is already near-minimal surface. The summary of the results is presented in the table \[11\].
FIG. 18: $x_2(u, v, t)$ for $t_{min2} = 0.088086$ for $0 \leq u, v \leq 1$.

FIG. 19: Mean curvature of the surface $x_2(u, v)$ for $t_{min2} = 0.088086$.

FIG. 20: $m_2(u, v)$ for the surface $x_3(u, v)$ for $0 \leq u, v \leq 1$.

In the above table it may be seen that the area reduction for the cases of hemiellipsoid (in both the cases when $H_0 = 1$ and when $H_0$ is the mean curvature as defined by the eq. (20)) and hump-like surface are significant as compared to the area reduction in the bilinear interpolation surface. The ratio of $rms$ of Gaussian curvature to the $ms$ of the mean curvature shows decrease in the surface of hemiellipsoid and whereas this ratio increases for the hump-like surface and bilinear interpolation surface.
TABLE II: Summary of results for the surfaces namely Hemiellipsoid, Hump-Like and Bilinear Interpolation

| Surface                  | $\lambda_n$ | $p_{ij}$ | $q_{ij}$ | $p_n/m_n^2$ | $t_{min}$ |
|--------------------------|-------------|----------|----------|--------------|-----------|
| Hemiellipsoid ($H_0 = 1$) | 6.283186    | -        | -        | 0.389848     | -         |
|                          | 4.70625     | 50.1954  | -        | 0.013686     | -0.351571 |
|                          | 4.44025     | 8.46711  | -        | 0.013079     | 0.0706355 |
|                          | 4.4025      | 1.20025  | -        | 0.013073     | 0.0226436 |
| Hemiellipsoid            | 6.283186    | -        | -        | 0.389848     | -         |
|                          | 4.32641     | 62.2861  | -        | 0.196444     | 0.148252  |
|                          | 4.23731     | 2.83624  | -        | 0.0944857    | 0.0440856 |
| Hump Like Surface        | 2.49452     | -        | -        | 0.034564     | -         |
|                          | 2.11589     | 25.3341  | -        | 0.052073     | 0.0889327 |
|                          | 1.92788     | 12.58    | -        | 0.10095      | 0.0440856 |
| Bilinear Interpolation   | 1.280789    | -        | -        | 2.25         | -         |
|                          | 1.27936     | -        | 0.111572 | 58.9258      | 0.483636  |
|                          | 1.279301    | -        | 0.004607 | 272.152      | 0.088086  |

IV. CONCLUSIONS

We have discussed a technique to get variationally improved surfaces $x_i(u,v)$ for appropriate number of iterations from eq. (18), which give reduction in area of non-minimal surfaces given by eqs. (1), (2) and (3). The first two cases are meant to test our curvature algorithm and the third one is to apply it to a surface for which there is no corresponding known minimal surface. The percentage decrease in area of the hemiellipsoid eq. (1) is given for the two cases, namely for which $H_0$ is replaced by 1 and for which $H_0$ is the usual numerator part of the mean curvature defined by eq. (20). In the notation of eq. (31), the area reduction in the former case comes out to be $p_{01} = 50.1954$, $p_{02} = 58.6625$, $p_{03} = 59.86276$, and in the latter case it is $p_{01} = 62.2861$, $p_{02} = 65.1223$. For the hump-like surface eq. (2), the percentage decrease in area is $p_{01} = 25.3341$, $p_{02} = 37.9141$. This indicates that the algorithm attains at least a local minimum of the area in these two cases. The percentage decrease (resulting through the corresponding slightly different definition given by eq. (32)) in area of the surface eq. (3) is much lesser i.e. $q_{01} = 0.111572$ and $q_{02} = 0.116176$. This means that in the case of bilinear interpolation with the known minimal surface, the variational improvement does not result in a significant decrease in the area. This saturation indicates that the bilinear interpolation is at least a local minima. Thus we get an area stability for the bilinear interpolation, in contrast to the first two cases where the area is reduced appreciably through our algorithm. Thus bilinear interpolation is indicated as (or may well be) a critical point of the area.

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