NEW GRADED METHODS IN THE HOMOLOGICAL ALGEBRA OF SEMISIMPLE ALGEBRAIC GROUPS

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ABSTRACT. Let $G$ be a semisimple algebraic group over an algebraically closed field $k$ of positive characteristic $p$. Under some restrictions on the size of $p$ (which in some cases require validity of the Lusztig character formula), the present paper establishes new results on the $G$-module structure of $\text{Ext}^\bullet_{G_1}(V, W)$ when $V, W$ belong to several important classes of rational $G$-modules, and $G_1$ denotes the first Frobenius kernel of $G$. For example, it is proved that, if $L, L'$ are ($p$-regular) irreducible $G_1$-modules, then $\text{Ext}^n_{G_1}(L, L')[-1]$ has a good filtration with computable multiplicities. This and many other results depend on the entirely new technique of using methods of what we call forced gradings in the representation theory of $G$, as developed by the authors in [28], [27] and [29], and extended here.

In addition to providing proofs, these methods lead effectively to a new conceptual framework for the study of rational $G$-modules, and, in this context, to the introduction of a new class of graded finite dimensional algebras, which we call $Q$-Koszul algebras. These algebras are similar to Koszul algebras, but are quasi-hereditary, rather than semisimple, in grade 0.

1. INTRODUCTION

Let $G$ be a semisimple, simply connected algebraic group over an algebraically closed field $k$ of positive characteristic $p$. The irreducible rational $G$-modules $L(\lambda)$ are indexed by the set $X(T)_+$ of dominant weights. When $\lambda \in X(T)_+$, $L(\lambda)$ occurs as the head (resp., socle) of the Weyl module $\Delta(\lambda)$ (resp., dual Weyl module $\nabla(\lambda)$). The structure and cohomology of the modules $\Delta(\lambda)$ and $\nabla(\lambda)$, for all $\lambda \in X(T)_+$, occupy a central place in the modular representation theory of semisimple groups. To give a recent example, write $\lambda = \lambda_0 + p\lambda_1$, where $\lambda_0$ is a restricted dominant weight and $\lambda_1$ is dominant, and define $\Delta^p(\lambda) := L(\lambda_0) \otimes \Delta(\lambda_1)^{[1]}$, where $\Delta(\lambda_1)^{[1]}$ denotes the Frobenius “twist” of $\Delta(\lambda_1)$. In 1980, Jantzen [17] asked if any Weyl module $\Delta(\lambda)$ has a $\Delta^p$-filtration, i.e., a filtration by $G$-submodules with sections $\Delta^p(\gamma)$, for various $\gamma \in X(T)_+$. In [29], the authors answered positively Jantzen’s question under the hypothesis that the Lusztig character formula (LCF) holds, and $p \geq 2h - 2$ is odd, where $h$ is the Coxeter number of $G$. The LCF is known to hold for very large $p$ depending on the root system of $G$ (see [3] and [14]).

[1] Williamson [37] has recently posted results stating that the original Lusztig conjecture with its proposed bound $p \geq h$ can fail for primes $p$ of this size—i.e., without stronger conditions on $p$. Williamson has stated that $p \geq f(h)$ is insufficient when $f(h)$ is linear in $h$, and he even proposes that a sufficient $f(h)$ must be
it holds, the modules $\Delta^p(\lambda)$ (resp., $\nabla_p(\lambda)$), $\lambda \in X(T)_+$, identify with certain modules $\Delta^{\text{red}}(\lambda)$ (resp., $\nabla^{\text{red}}(\lambda)$) arising from “reduction mod $p$” of the quantum enveloping algebra at a $p$th root of unity associated to $G$; see §2.4. This connection with quantum enveloping algebras plays an essential role in [29], fitting in well with the new forced-graded methods developed by the authors there and in [28] and [27].

The present paper builds on these methods, extending their scope from the module structure theory to the study of homological resolutions. Many new results for the homological algebra of rational $G$-modules emerge, as well as some promising forced-graded structures. Before elaborating on the latter, we briefly mention three specific new results.

First, let $G_1$ be the first Frobenius kernel of $G$. The representation theory of $G_1$ coincides with the representation theory of the restricted enveloping algebra $u = u(g)$ of $G$. Given rational $G$-modules $V, W$, the spaces $\text{Ext}^n_{G_1}(V, W)$, $n \geq 0$, carry the natural structure of twisted $G$-modules, that is, the natural action of $G_1$ through its containment $G_1 \subset G$ is trivial. Except in special cases, e.g., $n = 1$, little is known about the structure of the untwisted $G$-modules $\text{Ext}^n_{G_1}(V, W)^{-1}$, even when $V$ and $W$ are taken to be irreducible, Weyl or dual Weyl modules. Now assume that $p \geq 2h - 2$ is odd and that the LCF holds for $G$. Let $V = L$ and $W = L'$ be irreducible $G_1$-modules. Then $L \cong (L\lambda)_{G_1}$ and $L' \cong (L\mu)_{G_1}$ for restricted dominant weights $\lambda, \mu$ which we assume are $p$-regular. Theorem 5.3 establishes that $\text{Ext}^n_{G_1}(L(\lambda), L(\mu))^{-1}$ has a “good” or $\nabla$-filtration—that is, a filtration by $G$-submodules with sections of the form $\nabla(\gamma)$, for various $\gamma \in X(T)_+$. In addition, the multiplicity of any $\nabla(\gamma)$ as a section in this filtration can be combinatorially determined in terms of coefficients of Kazhdan-Lusztig polynomials for the affine Coxeter group of $G$; see Theorem 7.1.

To our knowledge, Theorem 5.3 and Theorem 7.1 give the first general results in the literature on the $G$-module structure of $\text{Ext}_{G_1}$-groups between irreducible modules.

Second, Theorem 6.2 proves, under the same assumptions about $p$, that, given any $p$-regular weight $\lambda \in X(T)_+$, restricted dominant weight $\mu$, and integer $n \geq 0$, the rational $G$-modules $\text{Ext}^n_{G_1}(\Delta(\lambda), L(\mu))^{-1}$ and $\text{Ext}^n_{G_1}(L(\mu), \nabla(\lambda))^{-1}$ both have $\nabla$-filtrations.

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1Exponential in $h$. To put this in perspective, however, the Weyl group order of $SL_n$ is $n! = h!$ which is exponential in $h$, but not “huge” in the sense of the sufficient bounds on $p$ given by Fiebig [14].

2The conclusion of Theorem 5.3 may fail if $p$ is small. For example, if $G$ has type $F_4$ and $p = 2$, then according to [32] 4.11,

$$\text{Ext}^1_{G_1}(L(0), L(\varpi_2))^{-1} \cong L(0) \oplus L(\varpi_1),$$

do not have a $\nabla$-filtration, since $L(\varpi_1) \neq \nabla(\varpi_1)$. We thank Peter Sin for pointing out his paper to us. See also David Stewart [34], which largely extends Sin’s $F_4$ calculations to twisted $F_4$. 
Again, the multiplicities of any $\nabla(\gamma)$ can be determined in terms of Kazhdan-Lusztig polynomial coefficients; see Theorem 7.2.

Third, let $a$ be the sum of the $p$-regular blocks in the restricted enveloping algebra $u$ of $G$. When $p > h$ and the LCF holds, an important result proved in [3] establishes that $a$ is a Koszul algebra, and so, in particular, it has a natural positive grading. The positive grading exists, inherited from the quantum analogue of $a$, without the LCF assumption; see [27]. Theorem 6.3 proves that, given any $\nu \in X_{\text{reg}}(T)_+$, the Weyl module $\Delta(\nu)$ has the structure of a graded $a$-module, provided $p \geq 2h - 2$ is odd. If, in addition, the LCF holds, this graded structure is linear. In other words, if $P_\bullet \twoheadrightarrow \Delta(\lambda)$ is a minimal graded $a$-projective resolution, then $\ker(P_{i+1} \to P_i)$ is generated by its terms of grade $i + 2$. This fact plays an important role in other results in this paper on the structure of $G$-module categories (see Corollary 3.8). In part, it grows out of a related result [28, Thm. 10.9], establishing $a$-gradings (but not linearity) for dual Weyl modules in some cases. But, surprisingly, it is the quantum version of [28, Thm. 8.7] of these results which we apply to study the $G$-module case here.

Our results on $G$-modules are modeled on (and require in a strong way) similar results for quantum enveloping algebras at roots of unity established by the authors in [28]. Underlying these results are gradings forced upon the algebras controlling the representation theory of the modules we study. Our philosophy has been that it is likely to be extremely difficult, if not impossible, to impose actual positive gradings on all of these algebras, although (as noted above) they do exist, under various assumptions, in the restricted enveloping algebra case. So, using filtrations related to the restricted enveloping algebra gradings, we pass to the associated graded algebras in all cases, thus forcing a grading. Once this is done, we do not immediately know, if any of the nice properties, e. g., quasi-heredity, carry over to the new graded algebras, or even if it is possible to work with them as a substitute for the original algebras in any meaningful way. However, from the start, a recent goal, continued in this paper, is to show that this is possible, thereby giving a genuinely viable alternative to finding from the start a positive grading. Indeed, because forced-graded structures come with built-in compatibility properties among the different algebras used, there are advantages to using them over actual gradings on the original algebras. There are, of course, some disadvantages. In particular, except in the case of the restricted enveloping algebra, there is no general “forget the grading” functor that allows passage back to the original algebras and modules. However, such a forgetful functor does exist for some graded modules, including all those which are completely reducible for the

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3Theorem 6.2 is suggested by the work [20] of Kumar, Lauritzen, and Thomsen (improving earlier work [2] of Andersen and Jantzen), showing that, if $p > h$, then $H^n(G_1, \nabla(\lambda))^{i-1} = \text{Ext}^n_{G_1}(k, \nabla(\tau))$ always has a $\nabla$-filtration. Our result, although it presently requires much larger values of $p$, considerably extends this result and rests on entirely different methods. The general question asking, given a rational $G$-module $V$, whether $H^n(G_1, V)^{i-1}$ has a $\nabla$-filtration goes back at least to Donkin’s paper [12] who conjectured a positive answer if $V$ has a $\nabla$-filtration. Counterexamples were later given by van der Kallen [?].

4By a slight abuse of terminology, a positively graded algebra has, by definition, nonzero grades only in grades $n \geq 0$. 
restricted enveloping algebra. We are able to use this functor to communicate from the forced-graded setting back to the original module categories. The results discussed above, as well as our $p$-Weyl filtration result [29], demonstrate the success of this approach, providing genuinely new advances in the structure and homological algebra of algebraic groups through proofs relying on forced-graded constructions.

The three results discussed above were chosen because the statements involve only the classical language of algebraic groups. But once the forced-graded framework is in place, many further results may be stated. Immediately, we observe from [27] that the new graded classical language of algebraic groups. But once the forced-graded framework is in place, through proofs relying on forced-graded constructions.

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$\Rightarrow$A weaker result was established in [28, Thm. 10.6], and the present result was promised there in Remark 10.7(a).
likely Q-Koszul\footnote{For example, this appears to be the case for $p = 2$ and with $A$ a Schur algebra $S(n, r), r \leq 5$. In another direction, the Humphreys-Verma conjecture on projective indecomposable $G_1$-modules becomes a theorem, valid for all $p$, in a forced-graded setting \cite{30}. At present, this conjecture is only known if $p \geq 2h - 2$. This is the main reason it is assumed that $p \geq 2h - 2$ in this paper.}. The authors intend to return to this topic in a later paper. Also, another sequel \cite{31} obtains some of the Q-Koszul results of this paper under weaker hypotheses, but still assuming that a version the LCF holds on a given poset of weights.

Many of the main results of this paper assume the validity of the Lusztig character formula (which is presently only known to hold for very large $p$, see footnote 1). However, even when the LCF is assumed to hold, many results are established for dominant weights outside the Jantzen region—giving homological and structural results not covered by the original conjecture or its immediate consequences. In addition, some results do not assume the LCF. For example, we mention again the deep Theorem 6.3(a) which shows that standard modules $\Delta(\lambda), \lambda$ $p$-regular, have a natural graded structure for $a$. Here we use the (positive) grading on a proved by the authors in \cite{30}, arising naturally, but non-trivially, from quantum group considerations when $p \geq 2h - 2$ is odd. Other examples include the quite satisfying identifications of Theorem 5.3 and Theorem 6.5(b), described above and proved under the same hypothesis.

1.1 Some Elementary Notation.

(1) $(K, \mathcal{O}, k)$: $p$-modular system. Thus, $\mathcal{O}$ is a DVR with maximal ideal $m = (\pi)$, fraction field $K$, and residue field $k$. An $\mathcal{O}$-lattice $\tilde{M}$ is, by definition, an $\mathcal{O}$-module which is free and of finite rank. A particular $p$-modular system will be required. Let $p > 0$ be a fixed odd prime. $\mathcal{O}$ will be a DVR with maximal ideal $m = (\pi)$, fraction field $K$ of characteristic 0, and residue field $\mathcal{O}/m \cong k = \mathbb{F}_p$. We can (and will) assume that $\mathcal{O}$ is complete and contains a $p$th root $\zeta \neq 1$ of unity. Let $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, the localization of the ring of integral Laurent polynomials in a indeterminate $v$ at the maximal ideal $n := (v - 1, p)$. Regard $\mathcal{A}$ as a subring of the function field $\mathbb{Q}(v)$. There is a natural ring homomorphism $\mathcal{A} \to \mathcal{O}$, $v \mapsto \zeta$.

(2) An $\mathcal{O}$-order is an $\mathcal{O}$-algebra $\tilde{A}$ which is also a $\mathcal{O}$-lattice. If $\tilde{A}$ is an $\mathcal{O}$-order, then an $\tilde{A}$-lattice is, by definition, an $\tilde{A}$-module $\tilde{M}$ which is also a $\mathcal{O}$-lattice. Let $\tilde{A}_K := K \otimes_{\mathcal{O}} \tilde{A}$ and $A := k \otimes_{\mathcal{O}} \tilde{A}$. More generally, if $\tilde{M}$ is an $\tilde{A}$-module, define $\tilde{M}_K := K \otimes_{\mathcal{O}} \tilde{M}$ and $M := \tilde{M}_K := k \otimes_{\mathcal{O}} \tilde{M}$.

(3) For an $\tilde{A}$-lattice $\tilde{M}$, define $\text{rad}^n \tilde{M} := \tilde{M} \cap \text{rad}^n \tilde{M}_K$, where $\text{rad}^n \tilde{M}_K$ denotes the $n$th-radical of the $\tilde{A}_K$-module $\tilde{M}_K$. Of course, $\text{rad}^n \tilde{M}_K = (\text{rad}^n \tilde{A}_K)\tilde{M}_K$.

Dually, let $\text{soc}^{-n} \tilde{M} := \text{soc}^{-n} \tilde{M}_K \cap \tilde{M}, n = 0, 1, \ldots$, where $\{\text{soc}^{-n} \tilde{M}_K\}_{n \geq 0}$ is the socle series of $\tilde{M}_K$. 
(4) If \( \tilde{M} \) is an \( \tilde{A} \)-lattice, then \( \text{gr} \tilde{M} := \bigoplus_{n \geq 0} \tilde{\text{rad}}^n \tilde{M}/\tilde{\text{rad}}^{n+1} \tilde{M} \) is a positively graded lattice for the \( \mathcal{O} \)-order

\[
\text{gr} \tilde{A} := \bigoplus_{n \geq 0} \tilde{\text{rad}}^n \tilde{A}/\tilde{\text{rad}}^{n+1} \tilde{A}.
\]

(1.1.1)

(5) A \( \tilde{A} \)-lattice \( \tilde{M} \) is called \( \tilde{A} \)-tight (or just tight, if \( \tilde{A} \) is clear from context) if

\[
(\tilde{\text{rad}}^n \tilde{A}) \tilde{M} = \tilde{\text{rad}}^n \tilde{M}, \quad \forall n \geq 0.
\]

(1.1.2)

Clearly, if \( \tilde{M} \) is \( \tilde{A} \)-projective, then it is tight. (Many other \( \tilde{A} \)-lattices are tight.)

(6) Now let \( \tilde{a} \) be an \( \mathcal{O} \)-subalgebra of \( \tilde{A} \). (More generally, we can assume that \( \tilde{a} \) is an order and \( \tilde{a} \to \tilde{A} \) is a homomorphism.) Then items (2)–(5) all make perfectly good sense using \( \tilde{a} \) in place of \( \tilde{A} \). If \( M \) is an \( \tilde{A} \)-lattice, then it is an \( \tilde{a} \)-lattice. In latter contexts (see, e. g., §2.3), it will usually be the case that \( (\text{rad}^n \tilde{a}_K) \tilde{A}_K = \text{rad}^n \tilde{A}_K \), for all \( n \geq 0 \). In that case, if \( \tilde{M} \) is an \( \tilde{A} \)-lattice, then \( \text{rad}^n \tilde{M} \) can be constructed viewing \( \tilde{M} \) as an \( \tilde{A} \)-lattice or as an \( \tilde{a} \)-lattice. Both constructions lead to identical \( \mathcal{O} \)-modules. Ambiguities of a formal nature may still arise as to whether it is more appropriate to use \( \tilde{a} \) or \( \tilde{A} \), but are generally resolved by context. Similar remarks apply for \( \text{gr} \tilde{M} \). Often the \( \tilde{A} \)-tightness of \( \tilde{M} \) is the same as its \( \tilde{a} \)-tightness; see [29, Cor. 3.8] and its elaboration at the end of §2.5 below.

(7) Finally, suppose that \( \tilde{a} \to \tilde{A} \) is a homomorphism of \( \mathcal{O} \)-orders. Assume that the image of \( \tilde{a} \) is normal in \( \tilde{A} \). Let \( A = \tilde{A}_k, a := \tilde{a}_k \), and consider an \( A \)-module \( M \). Define

\[
\begin{cases}
(1) \text{gr} M := \bigoplus_{n \geq 0} (\text{rad}^n A)M/(\text{rad}^{n+1} A)M; \\
(2) \text{gr}_a M := \bigoplus_{n \geq 0} (\text{rad}^n a)M/(\text{rad}^{n+1} a)M; \\
(3) \text{gr} \tilde{M} := \bigoplus_{n \geq 0} (\text{rad}^n \tilde{a})M/(\text{rad}^{n+1} \tilde{a})M.
\end{cases}
\]

(1.1.3)

Each of these is graded modules for \( \text{gr} A \) and \( \text{gr} \tilde{A} \). Though it will not often be used, (3) makes sense when \( M \) is replaced any \( \tilde{A} \)-lattice \( \tilde{M} \), i. e., we put \( \text{gr} \tilde{M} := \bigoplus_{n \geq 0} (\text{rad}^n \tilde{a})M/(\text{rad}^{n+1} \tilde{a})\tilde{M} \). It will often be the case that \( \text{gr} \tilde{M} \cong \text{gr} M \), which implies also \( \text{gr} M \cong (\text{gr} \tilde{M})_k \) if \( M = \tilde{M}_k \). A necessary and sufficient condition for either of these natural isomorphisms in the context of §2.3 is the \( \tilde{a} \)-tightness of \( \tilde{M} \); see [29, Lem. 3.5].

For a finite dimensional algebra \( A \) (over some field), let \( A\text{-mod} \) be the category of all finite dimensional \( A \)-modules. In the rest of this paragraph assume that \( A = \bigoplus_{n \geq 0} A_n \) is a positively graded algebra. Let \( A\text{-grmod} \) be the category of \( \mathbb{Z} \)-graded (finite dimensional) \( A \)-modules. Given graded \( A \)-modules \( M, N \) and \( n \in \mathbb{N} \), \( \text{ext}^n_A(M, N) \) denotes the space of \( n \)-fold extensions computed in the category \( A\text{-grmod} \). See Remark 8.4 for some elementary comments on the existence of projective covers in \( A\text{-grmod} \). When \( n = 0 \), the space of homomorphisms \( M \to N \) preserving grades is denoted \( \text{hom}_A(M, N) = \text{ext}^0_A(M, N) \).
For $M, N \in A\text{-mod}$ (not necessarily graded modules) and $n \in \mathbb{N}$, the space of $n$-fold extensions is denoted $\text{Ext}_A^n(M, N)$. The bifunctors $\text{ext}^\bullet$ and $\text{Ext}^\bullet$ are related as follows. If $M, N \in A\text{-grmod}$, then is a natural isomorphism

\begin{equation}
\text{Ext}_A^n(M, N) \cong \bigoplus_{r \in \mathbb{Z}} \text{ext}_A^n(M, N \langle r \rangle), \quad \forall n \in \mathbb{N}.
\end{equation}

In this expression, $N \langle r \rangle \in A\text{-grmod}$ is the $r$th shift of $N$, i.e., $N \langle r \rangle_i := N_{i-r}$.

2. **Varia**

This section collects together some useful material on several topics treated in this paper.

### 2.1 Algebraic groups.

Let $G$ be a simple, simply connected algebraic group defined and split over $\mathbb{F}_p$, where $p$ is a prime integer.\(^8\) Let $R$ be the root system of $G$ relative to a fixed maximal split torus $T$. Fix a Borel subgroup $B \supset T$ with opposite Borel subgroup $B^+$ determining a set $R^+$ of positive roots. Given $\lambda \in X(T)_+$ (the set of dominant weights), $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) will denote the Weyl module (resp., dual Weyl module) of highest weight $\lambda$. We generally follow the standard notation for $G$ and its representation theory as listed in [18 pp. 569–572] (except that $\Delta(\lambda)$ is denoted $W(\lambda)$ and $\nabla(\lambda)$ is denoted $H^0(\lambda)$ there).\(^9\) If $\lambda \in X(T)_+$ and $\lambda^* := -w_0 \lambda$ (where $w_0$ is the longest word in the Weyl group $W$ of $G$), then $\Delta(\lambda)$ has linear dual $\Delta(\lambda)^* \cong \nabla(\lambda^*)$.

For any affine algebraic group scheme $H$, let $H\text{-mod}$ be the category of finite dimensional rational (left) $H$-modules. The category $H\text{-mod}$ fully embeds into the category of finite dimensional modules for the distribution algebra $\text{Dist}(H)$ of $H$. See [18 Chps. 7,8]. In addition, if $H = G$, this embedding is an equivalence of categories; see [18 p. 171]. In this case, the classical Kostant $\mathbb{Z}$-form (an “order” of infinite rank) $\text{Dist}_\mathbb{Z}(G) := H_{\mathbb{Z}}(\mathfrak{g}_C)$ [15 Ch. 7] provides an integral form for $\text{Dist}(G)$, i.e., $\text{Dist}(G) \cong k \otimes_{\mathbb{Z}} \text{Dist}_\mathbb{Z}(G)$ (as Hopf algebras). For any commutative algebra $\mathcal{O}$, write $\text{Dist}_\mathcal{O}(G) := \mathcal{O} \otimes_{\mathbb{Z}} \text{Dist}_\mathbb{Z}(G)$. In particular, if $\mathcal{O} = K$ is a field of characteristic 0, then $\text{Dist}_K(G)$ is the universal enveloping algebra of the split semisimple Lie algebra $\mathfrak{g}_K$ over $K$, having the same root system as $G$.

For a positive integer $r$ and a rational $G$-module $V$, $V^{[r]}$ denotes the pull-back of $V$ through the $r$th power $F^r$ of the Frobenius morphism $F : G \to G$. Let $G_r$ be the scheme-theoretic kernel of $F^r$, and let $G_rT$ be the pull-back of $T$ through $F^r$. For $\lambda \in X(T)$, $\tilde{Q}_r(\lambda)$ denotes the injective envelope of the irreducible $G_rT$-module $\tilde{L}_r(\lambda)$ of highest weight $\lambda$.

Throughout this paper, we usually make the assumption that $p \geq 2h-2$. This means that, if $\lambda_0 \in X_r(T)$ (the set of $r$-restricted dominant weights), then the $G_rT$-module structure on $\tilde{Q}_r(\lambda_0)$ extends uniquely to a rational $G$-module structure. In the special case in which $r = 1$, this rational $G$-module will be denoted by $Q^\rho(\lambda_0)$; it the projective cover of $L(\lambda_0)$ in the subcategory of $G$-mod generated by $L(\gamma)$, with $\gamma \leq \lambda_0' := 2(p-1)\rho + w_0 \lambda_0$; see

\(^8\)The case when $G$ is semisimple, or even reductive, is easily reduced to the case when $G$ is simple.

\(^9\)Sometimes, in the context of quasi-hereditary algebras, $\Delta(\lambda)$ and $\nabla(\lambda)$ are called the “standard” and “costandard” modules, respectively, of highest weight $\lambda.$
We also generally assume that \( p \) is odd, so that previous results can be easily quoted. When \( p = 2 \geq 2h - 2 \), then \( G = SL_2 \), which is usually easy to treat directly.

Given \( \lambda \in X(T)_+ \), write \( \lambda = \lambda_0 + p\lambda_1 \in X(T)_+ \), where \( \lambda_0 \in X_1(T) \) and \( \lambda_1 \in X(T)_+ \). The indecomposable rational \( G \)-modules

\[
\begin{align*}
Q^\sharp(\lambda) &:= Q^\sharp(\lambda_0) \otimes \nabla(\lambda_1)_{[1]} \\
P^\sharp(\lambda) &:= Q^\sharp(\lambda_0) \otimes \Delta(\lambda_1)_{[1]}
\end{align*}
\]

will play an important role. Of course, the restrictions \( Q^\sharp(\lambda)|_{G_1T} \) and \( P^\sharp(\lambda)|_{G_1T} \) are injective and projective (but not indecomposable, unless \( \lambda_1 = 0 \)). By [29] Prop. 2.3, \( Q^\sharp(\lambda) \) (resp., \( P^\sharp(\lambda) \)) has a \( \nabla \)-filtration (resp., \( \Delta \)-filtration), namely, a filtration with sections of the form \( \nabla(\gamma) \) (resp., \( \Delta(\gamma) \)), for \( \gamma \in X(T)_+ \).

### 2.2 Quantum enveloping algebras

Let \( \widehat{U}_\zeta' \) be the (Lusztig) \( \mathcal{A} \)-form of the quantum enveloping algebra \( U_v \) associated to the Cartan matrix of the root system \( R \) over the function field \( \mathbb{Q}(v) \). Put

\[
\widehat{U}_\zeta = \mathcal{O} \otimes_{\mathcal{A}} U_v'/\langle K_1^p - 1, \cdots, K_n^p - 1 \rangle.
\]

Finally, set \( U_\zeta = K \otimes_{\mathcal{O}} \widehat{U}_\zeta \), so that \( \widehat{U}_\zeta \) is an integral \( \mathcal{O} \)-form of the quantum enveloping algebra \( U_\zeta \) at a \( p \)th root of unity. Put \( \overline{U}_\zeta = \widehat{U}_\zeta/\pi \widehat{U}_\zeta \), and let \( I \) be the ideal in \( \overline{U}_\zeta \) generated by the images of the elements \( K_i - 1, 1 \leq i \leq n \). By [23, (8.15)],

\[
\overline{U}_\zeta/I \cong \text{Dist}(G).
\]

A rational \( G \)-module \( M \) is said to lift if there is a \( U_\zeta \)- lattice \( \widetilde{M} \) such that \( M \cong \widetilde{M}/\pi \widetilde{M} \) as rational \( G \)-modules.

The category \( U_\zeta \)-mod of finite dimensional and integrable type 1 modules is a highest weight category (in the sense of [5]) with irreducible (resp. standard, costandard) modules \( L_\zeta(\lambda) \) (resp., \( \Delta_\zeta(\lambda) \), \( \nabla_\zeta(\lambda) \)), \( \lambda \in X(T)_+ \). For \( \mu \in X(T)_+ \), \( \text{ch} \Delta_\zeta(\mu) = \text{ch} \nabla_\zeta(\mu) = \chi(\mu) \) (Weyl’s character formula).

There is a surjective (Hopf) algebra homomorphism

\[
\widetilde{F} : \overline{U}_\zeta \twoheadrightarrow \text{Dist}_{\mathcal{O}}(G),
\]

which, after base change to \( K \), defines the Frobenius morphism

\[
F : U_\zeta \rightarrow \text{Dist}_K(G).
\]

If \( M \) is a module for \( \text{Dist}_K(G) \), let \( M^{[1]} \) be the \( U_\zeta \)-module obtained by making \( U_\zeta \) act through \( F \). Similarly, if \( \widetilde{M} \) is a \( \text{Dist}_{\mathcal{O}}(G) \)-module, let \( \widetilde{M}^{[1]} \) be the \( \widehat{U}_\zeta \)-module obtained by making \( \widehat{U}_\zeta \) act through \( \widetilde{F} \). In particular, if \( \lambda \in X(T)_+ \), \( \overline{\Delta}(\lambda)_{[1]} \) (resp., \( \overline{\nabla}(\lambda)_{[1]} \)) is the \( \mathcal{O}_\zeta \)-module obtained from the Weyl (resp., dual Weyl) lattice \( \Delta(\lambda) \) (resp., \( \nabla(\lambda) \)) of the irreducible \( g_\zeta \)-module \( L_\zeta(\lambda) \) of highest weight \( \lambda_{[1]}^{[1]} \).

\[\text{[1]}\text{Thus, for example, } \overline{\Delta}(\lambda) := \text{Dist}_{\mathcal{O}}(G) \cdot v^+, \text{if } v^+ \in L_K(\lambda) \text{ is a highest weight vector.}\]
The rational $G$-modules $P^\otimes(\gamma)$ and $Q^\otimes(\gamma)$ defined in (2.1.1) lift to $\widetilde{U}_\zeta$-lattices, denoted $\widetilde{P}^\otimes(\gamma)$ and $\widetilde{Q}^\otimes(\gamma)$, respectively. If $\gamma = \gamma_0$ is restricted, these modules may be defined as the unique (up to isomorphism) $\widetilde{U}_\zeta$-lattices lifting $P^\otimes(\gamma_0)$ and $Q^\otimes(\gamma_0)$. We refer ahead to the discussion following display (4.0.4) for more details. In general, for $\gamma = \gamma_0 + p\gamma_1$, with $\gamma_0 \in X_1(T)_+$ and $\gamma_1 \in X(T)_+$, we have $\widetilde{P}^\otimes(\gamma) = \widetilde{P}^\otimes(\gamma_0) \otimes \Delta(\gamma_1)^{[1]}$ and $\widetilde{Q}^\otimes(\gamma) = \widetilde{Q}^\otimes(\gamma_0) \otimes \nabla(\gamma_1)^{[1]}$.

Let $u_\zeta$ be the small quantum enveloping algebra. It is a Hopf subalgebra of $U_\zeta$ and admits an integral form $\widetilde{u}_\zeta$, which is a subalgebra of $\widetilde{U}_\zeta$. As such $\widetilde{u}_\zeta$ is a lattice of rank $p^{\dim \vartheta}$. Let $u'_\zeta$ be the product of the $p$-regular blocks of $u_\zeta$ and define $\widetilde{u}'_\zeta := \widetilde{u}_\zeta \cap u'_\zeta$. Then $\widetilde{u}'_\zeta$ is a direct factor of $\widetilde{u}_\zeta$. In addition, $u' := k \otimes \widetilde{u}'_\zeta$ is the direct product of the regular blocks in the restricted enveloping algebra $u$ of $G$.

2.3 Finite dimensional algebras. A dominant weight $\lambda$ is $p$-regular if $(\lambda + \rho, \alpha^\vee) \neq 0$ mod $p$, for all roots $\alpha \in R$. The set $X_{\text{reg}}(T)_+$ of $p$-regular dominant weights is a poset, setting $\lambda \leq \mu \iff \mu - \lambda \in NR^+$. (There is a similar partial order on entire set $X(T)_+$ of dominant weights, though this paper focuses on the $p$-regular weights.) A subset $\Gamma$ of a poset $\Lambda$ is called an ideal if $\Gamma \neq \emptyset$ and, given $\lambda \in \Lambda$ and $\gamma \in \Gamma$, if $\lambda \leq \gamma$, then $\lambda \in \Gamma$. Write $\Gamma \trianglelefteq \Lambda$ in this case.

To a finite ideal $\Gamma$ in $X_{\text{reg}}(T)_+$, there is attached two finite dimensional algebras; the first, denoted $A_\Gamma$, is over $k$, and the second, denoted $A_{\zeta,\Gamma}$, is over $K$. These algebras capture some of the representation theory of $G$ and $U_\zeta$, respectively. Furthermore, $A_\Gamma$ and $A_{\zeta,\Gamma}$ are related by an $\zeta'$-order $\widetilde{A}_\Gamma$ with the properties that $\widetilde{A}_{\Gamma,\Delta} = \widetilde{A}_\Gamma/\pi\widetilde{A}_\Gamma \cong A_\Gamma$ and $(\widetilde{A}_\Gamma)_K \cong A_{\zeta,\Gamma}$. The “deformation theory” relating the representation theory of these algebras (and their graded versions) provides a major theme in earlier work, see [27] and [29], and it is continued in this paper. In the remainder of this subsection, we will sketch a few details.

Given $\Gamma \subseteq X(T)_+$, let $(G-\text{mod})[\Gamma]$ be the full subcategory of $G-\text{mod}$ generated by the irreducible modules $L(\gamma)$ having highest weight $\gamma \in \Gamma$. In particular, if $\Gamma$ is a finite ideal in $X_{\text{reg}}(T)_+$ (or, more generally, of $X(T)_+$), $(G-\text{mod})[\Gamma]$ is a highest weight category (in the sense of [3]) with weight poset $\Gamma$. The category $(G-\text{mod})[\Gamma]$ identifies with the category $A_\Gamma$-mod of finite dimensional modules for a certain finite dimensional algebra $A_\Gamma$. Specifically, let $I_\Gamma \leq \text{Dist}(G)$ be the annihilator ideal of all the modules $V \in (G-\text{mod})[\Gamma]$. Then $(G-\text{mod})[\Gamma] \cong A_\Gamma$-mod, the category of finite dimensional $A_\Gamma$-modules, putting $A_\Gamma := \text{Dist}(G)/I_\Gamma$.

There is a similarly constructed algebra $A_{\zeta,\Gamma}$ over $K$. It has the property that $A_{\zeta,\Gamma}$-mod is isomorphic to the full subcategory of $U_{\zeta}$-mod generated by the irreducible modules $L_{\zeta}(\gamma)$, $\gamma \in \Gamma$. The algebras $A_\Gamma$ and $A_{\zeta,\Gamma}$ are related by an $\zeta'$-order $\widetilde{A}_\Gamma$ which is defined to be the image of $\widetilde{U}_\zeta$ in $A_{\zeta,\Gamma}$. Necessarily, $\widetilde{A}_\Gamma/\pi\widetilde{A}_\Gamma \cong A_\Gamma$ and $(\widetilde{A}_\Gamma)_K \cong A_{\zeta,\Gamma}$.

The algebras $A_\Gamma$, $A_{\zeta,\Gamma}$, and $\widetilde{A}_\Gamma$ are all quasi-hereditary algebras (over $k$, $K$ and $\zeta'$, respectively) with poset $\Gamma$. For more details and properties of these algebras (as well as of
Then the PIMs for $\widetilde{\alpha}$, see [28], [27] and [29], as well as the earlier papers [5] and [6]. If $\Gamma$ is an ideal in finite ideal $\Lambda$ in the poset $X_{\text{reg}}(T)_+$, then there are surjective homomorphisms $A_\Lambda \rightarrow A_\Gamma$, $A_{\zeta,\Lambda} \rightarrow A_{\zeta,\Gamma}$, and $\widetilde{A}_\Lambda \rightarrow \widetilde{A}_\Gamma$. This induce full embeddings $i_* : A_\Gamma\text{-mod} \rightarrow A_\Lambda\text{-mod}$, etc. which preserve $\text{Ext}^*\text{-groups}$ (i.e., they induce full embeddings at the level of the bounded derived categories).

Let $\Lambda$ be any finite ideal of $p$-regular weights which contains all restricted $p$-regular weights. Assume also that if $\gamma$ is $p$-regular and restricted, then $2(p-1)\rho + w_0\gamma \in \Lambda$. Then the PIMs for $u'_\zeta$ are all $A_{\zeta,\Lambda}$-modules, so that the natural map $u'_\zeta \rightarrow A_{\zeta,\Lambda}$ is injective. Similarly, $u'$ maps isomorphically onto its image in $A_\Lambda$. It follows that the (isomorphic) image $\tilde{\alpha}$ of $u'_\zeta$ in $\widetilde{A}_\Lambda$ is pure in $\widetilde{A}_\Lambda$.

Of course, any poset ideal $\Gamma \subseteq X_{\text{reg}}(T)_+$ is contained in a poset $\Lambda$ as above, and this gives a natural map $\tilde{\alpha} \rightarrow \tilde{A}_\Gamma$. By [29 Cor. 3.9], $\widetilde{A}_\Gamma$ is a $\tilde{\alpha}$-tight in the sense of (1.1)(5), so that $\text{gr}\widetilde{A}_\Gamma = \text{gr}\tilde{A}_\Gamma$; see [29 Lem. 3.5]. By [27 Thm. 6.3], the algebra $\text{gr}\tilde{A}_\Gamma$ is quasi-hereditary over $\partial$. It has weight poset $\overline{\Gamma}$ and standard objects $\text{gr}\overline{\Gamma}(\gamma) = \text{gr}\tilde{\Delta}(\gamma)$. Thus, $\text{gr}\overline{\Gamma} = (\text{gr}\tilde{A}_\Gamma)_k = (\text{gr}\tilde{A}_\Gamma)_k$ is also quasi-hereditary. It is important to observe that $\text{gr}\overline{\Gamma}$ need not be the graded algebra $\text{gr}\overline{\Gamma}$ defined in (1.1)(3)(1). However, see Lemma 2.5 below.

If $\Gamma \subseteq \Lambda$ are any finite ideals in $X_{\text{reg}}(T)_+$, the surjective homomorphism $A_\Lambda \rightarrow \widetilde{A}_\Gamma$ above induces a surjective homomorphism $\text{gr}\overline{\Gamma} \rightarrow \text{gr}\overline{\Gamma}$. In addition, the corresponding map $\text{gr}\overline{\Gamma}\text{-mod} \rightarrow \text{gr}\overline{\Gamma}\text{-mod}$ induces a full embedding on the corresponding derived category (and the resulting equality of $\text{Ext}^*\text{-groups}$, just as described above in the ungraded cases. See [28 Cor. 3.16] for more discussion.

Another (more elementary) variant on the deformation theory described above also will be useful, replacing the triple $(A_{\zeta,\Gamma}, \tilde{A}_\Gamma, A_\Gamma)$ by a triple $(A_{\zeta,\Gamma}^\circ, \tilde{A}_\Gamma^\circ, A_\Gamma)$. In fact, define $A_{\Gamma,\Gamma}^\circ := \text{Dist}_K(G)/I_\Gamma^\circ$, where $I_\Gamma^\circ$ is the annihilator in $\text{Dist}_K(G) = U(g_K)$ of the irreducible modules for $g_K$ having highest weights in $\Gamma$. Thus, $A_{\Gamma,\Gamma}^\circ$ is a semisimple algebra over $K$ (in contrast to the fact that $A_{\zeta,\Gamma}$ is usually not semisimple). The image $\text{Dist}_\partial(G)$ in $A_{\Gamma,\Gamma}^\circ$ is denoted $\widetilde{A}_\Gamma^\circ$. It is an order over $\partial$ having the property that $(\widetilde{A}_\Gamma^\circ)/\pi\widetilde{A}_\Gamma^\circ \cong A_\Gamma$.

The terminology of §2.2 also applies in case of $\widetilde{A}_\Gamma^\circ$ and $A_{\Gamma,\Gamma}^\circ$. For example, if $\widetilde{M}$ is an $\widetilde{A}_\Gamma^\circ$-module, it is also a module for $\text{Dist}_\partial(G)$-module, and then, using (2.2.2), as a module for $\widetilde{U}_\zeta$, which is denoted $\widetilde{M}[1]$.

### 2.4 The Lusztig conjecture.

For $\lambda \in X(T)_+$, the irreducible $U_\zeta$-module $L_\zeta(\lambda)$ has two important “reductions mod $p$” from admissible $U_\zeta$-lattices $\widetilde{\Delta}^{\text{red}}(\lambda)$ and $\nabla^{\text{red}}(\lambda)$. Thus, $\Delta^{\text{red}}(\lambda) := \Delta^{\text{red}}(\lambda)/\pi\Delta^{\text{red}}(\lambda)$ and $\nabla^{\text{red}}(\lambda) := \nabla^{\text{red}}(\lambda)/\pi\nabla^{\text{red}}(\lambda)$. Both $\Delta^{\text{red}}(\lambda)$ and $\nabla^{\text{red}}(\lambda)$ are finite dimensional rational $G$-modules. Rather than defining these modules explicitly, see [21], [10], [27], and [29] for an extensive treatment. (Of course, there are other possible admissible lattices, leading to other rational $G$-modules, but $\Delta^{\text{red}}(\lambda)$ and $\nabla^{\text{red}}(\lambda)$ will only
be used in this paper.) If $\lambda = \lambda_0 + p\lambda_1$, $\lambda_0 \in X_1(T)$ and $\lambda_1 \in X(T)_+$, then

$$\Delta_{\text{red}}(\lambda) \cong \Delta_{\text{red}}(\lambda_0) \otimes \Delta(\lambda_1)^{[1]}, \quad \nabla_{\text{red}}(\lambda) \cong \nabla_{\text{red}}(\lambda_0) \otimes \nabla(\lambda_1)^{[1]}.$$  

See [21, Thm. 2.7] or [10, Prop. 1.7].

In addition, consider the rational $G$-modules $\Delta^p(\lambda) := L(\lambda_0) \otimes \Delta(\lambda_1)^{[1]}$ and $\nabla^p_p(\lambda) := L(\lambda_0) \otimes \nabla(\lambda_1)^{[1]}$. There are natural surjective (resp., injective) module homomorphisms $\Delta_{\text{red}}(\lambda) \to \Delta^p(\lambda)$ (resp., $\nabla^p_p(\lambda) \to \nabla_p(\lambda)$).

The following result indicates the importance of these modules to the representation theory of $G$, and, in particular, to the validity of the Lusztig modular character formula—a specific formula conjectured to hold for dominant weights in the Jantzen region. We do not repeat this formula here, but instead refer to [22] and [35]. Recall the Jantzen region is defined

$$(2.4.2) \quad \Gamma_{\text{Jan}} := \{\lambda \in X(T)_+ | (\lambda + \rho, \alpha^\vee_0) \leq p(p - h + 2)\} \subseteq X(T)_+.$$  

**Proposition 2.1.** If $p \geq 2h - 3$, then the validity of the Lusztig modular character formula of $G$ for $p$-regular weights $\lambda \in \Gamma_{\text{Jan}}$ is equivalent to requiring that

$$(2.4.3) \quad \Delta_{\text{red}}(\lambda) \cong \Delta^p(\lambda), \quad \forall \lambda \in X_{\text{reg}}(T)_+.$$  

See [29, Cor. 2.5] for the proof. It should be remarked that (2.4.3) holds for all $p$-regular weights if and only if it holds for $p$-regular weights in $\Gamma_{\text{Jan}}$. (In addition, if (2.4.3) holds then it also holds, for all $\lambda \in X(T)_+$, not just at the $p$-regular weights by an elementary translation functor argument.)

The lemma below will be important. It is a consequence of some basic Kazhdan-Lusztig theory [6] and homological properties of the modules $\Delta_{\text{red}}(\gamma)$ and $\nabla_{\text{red}}(\gamma)$, $\gamma \in X_{\text{reg}}(T)_+$. Write $\gamma = w \cdot \gamma'$ where $w$ belongs to the affine Weyl group $W_p = W \ltimes p\mathbb{Z}R$ of $G$, and $\gamma'$ belongs to the anti-dominant alcove $C^-_p$ containing $-2p$. Then put $l(\gamma) := l(w)$ (Coxeter length). It will be convenient to work inside the bounded derived category $\mathcal{D} := D^b(G\text{-mod})$ of $G$-mod. Let $[1]$ be shifting functor on $\mathcal{D}$. If $m > 0$, $[m] := [1] \circ \cdots \circ [1]$ (with the standard convention if $m < 0$). The category contains $G$-mod as a fully embedded subcategory. For $M, N \in G$-mod, $\text{Ext}^m_G(M, N) = \text{Hom}_{\mathcal{D}}^m(M, N) = \text{Hom}_{\mathcal{D}}(M, N[n])$.

We also need the full subcategories $\mathcal{E}^R$ and $\mathcal{E}^L$ of $\mathcal{D}$. For example, let $\mathcal{E}^R_0$ be the full subcategory of $\mathcal{E}$ consisting of objects which are isomorphic to direct sums $\nabla(\gamma)[r]$, with $r \equiv l(\gamma) \mod 2$. Having defined $\mathcal{E}^R_0$, define $\mathcal{E}^R_1$ to be the full, strict subcategory of $\mathcal{D}$ consisting of objects $X$ for which there is a distinguished triangle $Y \to X \to Z \to$, with $Y, Z \in \mathcal{E}^R_0$. Let $\mathcal{E}^R := \bigcup_{i \geq 0} \mathcal{E}^R_i$. The dual subcategory $\mathcal{E}^L$ is defined analogously, replacing the $\nabla(\gamma)$ by $\Delta(\gamma)$.

**Lemma 2.2.** Assume that $p \geq 2h - 3$ and that condition (2.4.3) holds. Let $M, N \in G$-mod. Assume that $M$ or $M[1]$ belongs to $\mathcal{E}^R$, and that $N$ or $N[1]$ belongs to $\mathcal{E}^L$. (Thus, the composition factors of $M$ and $N$ all have $p$-regular highest weights.) For any
\( \lambda \in X_{\text{reg}}(T)_+ \), the natural maps

\[(2.4.4) \begin{cases} 
(1) \quad \text{Ext}^n_G(\Delta^{\text{red}}(\lambda), M) \rightarrow \text{Ext}^n_G(\Delta(\lambda), M) \\
(2) \quad \text{Ext}^n_G(N, \nabla_{\text{red}}(\lambda)) \rightarrow \text{Ext}^n_G(N, \nabla(\lambda))
\end{cases}
\]

are surjective, for all \( n \geq 0 \).

**Proof.** First, consider statement \((2.4.4)(1)\). It is more convenient to prove \((2.4.4)(1)\) allowing \( M \) to be an arbitrary object in \( \mathcal{E}^R \) or \( \mathcal{E}^R[1] \) (rather than just a rational \( G \)-module). The condition \( p \geq 2h - 3 \) means that the restricted dominant weights are contained in the Jantzen region \( \Gamma_{\text{red}} \). Thus, since \((2.4.3)\) holds, \([10\text{ Thm. } 6.8(a)]\) implies that \( \Delta^{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^R \) and \( \nabla_{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^R \), for all \( \lambda \in X_{\text{reg}}(T)_+ \). Also, \((2.4.4)(1)\) holds trivially (using \([10\text{ Lem. } 2.2]\)) in case \( M = \nabla(\xi)[r] \), for some integer \( r \). Thus, \((2.4.4)(1)\) is valid for \( M \) or \( M[1] \) in \( \mathcal{E}^R \). Now assume that \( M \) or \( M[1] \) belongs to \( \mathcal{E}^R_{i+1} \) and the surjectivity of \((2.4.4)(1)\) holds with \( M \) replaced by objects in \( \mathcal{E}^R_i \) or \( \mathcal{E}^R[1], i \geq 1 \). But there is a distinguished triangle \( X \rightarrow M \rightarrow Y \rightarrow \) in which \( X \) or \( X[1] \) (resp., \( Y \) or \( Y[1] \)) belongs to \( \mathcal{E}^R_i \), so that surjectivity holds with \( M \) replaced by \( X \) or \( Y \). Now a standard long exact sequence argument (see the proof of \([6\text{ Thm. } 4.3]\)) completes the argument for \((2.4.4)(1)\).

The argument for the dual statement \((2.4.4)(2)\) is similar and is left to the reader. \( \Box \)

### 2.5 Graded structures.

Suppose \( B = \bigoplus_{n \geq 0} B_n \) is a positively graded finite dimensional algebra over a field. Let \( M \) be in the category \( B\text{-grmod} \) of \( \mathbb{Z} \)-graded \( B \)-modules. A resolution\(^{11}\)

\[ \cdots \rightarrow R_2 \rightarrow R_1 \rightarrow R_0 \rightarrow M \rightarrow 0 \]

in \( B\text{-grmod} \) is called \( B\text{-linear} \) (or just linear, if \( B \) is understood) if, for each nonnegative integer \( n \), the graded \( B \)-module \( R_n \) generated by its term \( R_{n,n} \) in grade \( n \). (In particular, \( M \) is generated by its grade 0-component \( M_0 \).) Call the graded \( B \)-module \( M \) resolution linear, or just linear, if it has a linear projective resolution\(^{12}\). (For the structure of projective objects in \( B\text{-grmod} \), see Remark \(8.4\) in §8 (Appendix I).) We remark that every such linear projective resolution is automatically linear and thus uniquely determined. The algebra \( B \) is a (finite dimensional) Koszul algebra provided every irreducible \( B \)-module (regarded as a graded module concentrated in grade 0) is resolution linear. In this case, the subalgebra \( B_0 \) is necessarily semisimple\(^{13}\).

\(^{11}\)In this resolution, the graded \( B \)-module \( R_i \) has cohomological degree \(-i\). For an integer \( j \), \( R_{i,j} \) denotes the \( j \)th grade of \( R_i \); thus, \( R_i = \bigoplus R_{i,j} \).

\(^{12}\)It is possible to define other useful notions of linearity, e. g., using graded \( \text{Ext} \) groups. While such \( \text{Ext} \) considerations play a role in this paper, there is no need here for a special terminology for them. In the Koszul case, these notions all coincide. See the next footnote.

\(^{13}\)When \( B \) is a Koszul algebra, a graded \( B \)-module \( M \) is resolution linear if and only if \( \text{Ext}^n_B(M, L(r)) \neq 0 \) \( \Rightarrow \) \( n = r \), for all irreducible \( B \)-modules \( L \) (concentrated in grade 0) and all \( n \in \mathbb{N}, r \in \mathbb{Z} \).
Finally, we mention that the definitions above of linear resolutions and modules easily carry over to graded lattices over a graded order (such as $\tilde{a}$ defined below). We leave further details to the reader.

If $p > h$, the sum $\tilde{a}_K = \bigoplus_{i \geq 0} \tilde{a}_{K,i}$ of the regular blocks in the small quantum group $u_{\zeta}$ is known to be Koszul [3]. Let $\tilde{a}_K = \bigoplus_{i \geq 0} \tilde{a}_{K,i}$ be the associated Koszul grading. By [27, §8], the $\mathcal{O}$-algebra $\tilde{a}$ has a positive grading $\tilde{a} = \bigoplus_{i \geq 0} \tilde{a}_i$ such that, for any $i \geq 0$, $K\tilde{a}_i = \tilde{a}_{K,i}$. Notice this implies that $\tilde{a} \cong \tilde{gr} \tilde{a}$. Putting $a_i = k \otimes \tilde{a}_i$,

$$\tag{2.5.1} a = \bigoplus_{r \geq 0} a_i$$

provides a positive grading of the $p$-regular part $u'$ of the restricted enveloping algebra of $G$, for all $p > h$. Also, $a \cong \tilde{gr} a$. In case (2.4.3) holds for $G$ with $p > h$, then, by [3], the algebras $u'$ and $u_{\zeta}$ are Koszul.

Given a finite ideal $\Gamma$ in $X_{reg}(T) +$, any projective $\tilde{A} = \tilde{A}_{\Gamma}$-module is $\tilde{a}$-tight in the sense of (1.1)(5). In particular, $\tilde{A}$ is itself $\tilde{a}$-tight, as is any projective $\tilde{A}$-lattice. See [29, Cor. 3.9]. If $\tilde{X}$ is a $\tilde{A}$-lattice, it is $\tilde{a}$-tight if and only if it is $A$-tight, by [29, Cor. 3.8]. (The quoted result, as stated, applies to $A_{\Lambda}$ for a poset $\Lambda$, which may be assumed to contain $\Gamma$. In particular, $\tilde{A}$ is $\tilde{A}_{\Lambda}$-tight, and now the definitions show that $\tilde{A}$-tightness of $\tilde{X}$ is equivalent to $\tilde{A}_{\Lambda}$-tightness, and thus to $\tilde{a}$-tightness.) Thus, in this case, $\operatorname{gr} \tilde{X} = \tilde{gr} \tilde{X}$. In particular, $\operatorname{gr} \tilde{A} = \tilde{gr} \tilde{A}$. A similar argument, varying the poset, gives the tightness of $\tilde{\Delta}(\gamma)$, $\gamma \in \Gamma$, and $\tilde{gr} \tilde{\Delta}(\lambda) = \operatorname{gr} \tilde{\Delta}(\lambda)$.

2.6 The Jantzen region. The following result concerns the quasi-hereditary algebras $\operatorname{gr} A_{\Gamma}$.

**Lemma 2.3.** Assume that $p \geq 2h - 2$ is odd, and that (2.4.3) holds. Let $\Gamma \leq \Gamma_{\text{Jan}}$ consist of $p$-regular weights. Then (in the notation of (1.1.3))

$$\operatorname{gr} A_{\Gamma} = \operatorname{gr}_a A_{\Gamma} = \tilde{gr} A_{\Gamma}$$

and

$$\operatorname{gr} \Delta(\gamma) = \operatorname{gr}_a \Delta(\gamma) = \tilde{gr} \Delta(\gamma), \quad \forall \gamma \in \Gamma.$$

**Proof.** We first claim that

$$\tag{2.6.1} (\operatorname{rad} a)A_{\Gamma} = \operatorname{rad} A_{\Gamma}.$$

Observe $A_{\Gamma}$-modules are the same as finite dimensional rational $G$-modules which have composition factors $L(\gamma)$, for $\gamma \in \Gamma$. Thus, to prove (2.6.1), it’s enough to show that, given $M$ in $G\text{-mod}$, $M$ is completely reducible for $G$ if and only if it is completely reducible for $u'$. Because irreducible $G$-modules are completely reducible for the restricted enveloping algebra $u$ (or equivalently, for $G_1$), the “$\Rightarrow$” direction is obvious. Conversely, assume that $M$ is completely reducible for $G_1$. Let $L := \bigoplus L(\lambda_i)$ be the direct sum of the distinct
irreducible $G$-modules having restricted highest weights which, as $G_1$-modules, appear with nonzero multiplicity in $M|_{G_1}$. Then

$$\text{Hom}_{G_1}(L, M) \otimes L \xrightarrow{\sim} M, \quad f \otimes x \mapsto f(x)$$

is an isomorphism of rational $G$-modules. Also, $\text{Hom}_{G_1}(L, M) \cong N^0$, for a rational $G$-module $N$. (See [18, 3.16(1)].) Thus, if $L(\tau)$ is a $G$-composition factor of $N$, then $L(\lambda_i \otimes p\tau)$ is a composition factor of $M$. Thus, by hypothesis, $\lambda_i \otimes p\tau \in \Gamma_{\text{Jan}}$. A easy calculation shows that $(r + p, \alpha_i^\vee) \leq p$, i.e., $\tau$ belongs to the closure of the bottom $p$-alcove $C_p$ of $G$. Thus, $L(\tau) \cong \Delta(\tau) \cong \nabla(\tau)$, so that $N$ is a completely reducible $G$-module because $\text{Ext}_G^1(\Delta(\tau), \nabla(\sigma)) = 0$ for any $\tau, \sigma \in X(T)_+$. This proves our claim.

By (2.6.1), $(\text{rad}^n a)A_\Gamma = \text{rad}^n A_\Gamma$, for all nonnegative integers $n$. This implies that $\text{gr}A_\Gamma = \text{gr}_a A_\Gamma$. On the other hand, $\text{gr}_a A_\Gamma = \text{gr}_s A_\Gamma$ by [29, Cor. 5.6]. This proves the first assertion of the lemma. For the second assertion, $\text{rad}^n \Delta(\gamma) := (\text{rad}^n A_\Gamma)\Delta(\gamma) = (\text{rad}^n a)\Delta(\gamma)$, so that $\text{gr}\Delta(\gamma) = \text{gr}_a \Delta(\gamma) = \text{gr}_s \Delta(\gamma)$, as before. □

3. Q-KOSZULITY.

Q-Koszul algebras are introduced in Definition 3.3 of this section. Let $\Lambda$ be an arbitrary finite ideal of $p$-regular dominant weights, and let $B = \text{gr}_s A_\Lambda$ be the algebra defined in §2.3. Then, under favorable circumstances—which, for the present, means that $p \geq 2h - 2$ is odd and the LCF condition (2.4.3) holds—Theorem 3.5 states that $B$ is Q-Koszul. Its proof is postponed to §5. Next, Definition 3.6 formulates the notion of a "standard" Q-Koszul algebra, while Theorem 3.7 proves that the algebras $B$ are also standard Q-Koszul algebras. When $\Lambda$ is contained in the Jantzen region, Corollary 3.8 states that $B$-mod has a graded Kazhdan-Lusztig theory (in the sense of [7, §3]). The proofs of these last two results are presented at the end of §6. Thus, when $p \geq 2h - 2$ is odd, and when (2.4.3) holds, the following picture emerges: the graded algebras $B$ which "model" the representation theory of $G$ (on $p$-regular weights) are (standard) Koszul inside the Jantzen region $\Gamma_{\text{Jan}}$, but then become (standard) Q-Koszul as the weight poset $\Lambda$ expands outside $\Gamma_{\text{Jan}}$. Ultimately, we expect something similar to hold for small primes, and also for $p$-singular weights.

Suppose that $B = \bigoplus_{n \geq 0} B_n$ is a positively graded quasi-hereditary algebra with poset $\Lambda$. Since $B$ is quasi-hereditary, there is an increasing ("defining") sequence $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = B$ of idempotent ideals of $B$ with the following property: for $1 \leq i \leq n$, $J_i/J_{i-1}$ is a heredity ideal in the algebra $B/J_{i-1}$ [14]. Because $B$ is graded, [6, Prop. 4.2] says that the idempotent ideals $J_i$ are homogeneous; in fact, $J_i = Be_iB$ for some idempotent $e_i \in B_0$.

Each standard module $\Delta(\lambda)$, $\lambda \in \Lambda$, has a natural positive grading, described as follows. $\Delta(\lambda)$ is a projective (ungraded) module for an appropriate quotient algebra $B/J_i$—it

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14See [5, 6] and [11, §C.1] for further details. Recall that an idempotent ideal $J$ in a finite dimensional algebra $A$ (over the field $k$) is heredity provided that, writing $J = AeA$, for an idempotent $e$, the centralizer algebra $eAe$ is semisimple and multiplication $Ae \otimes_{eAe} eA \rightarrow AeA = J$ is an isomorphism (of vector spaces).
identifies with the projective cover of $L(\lambda)$ in $B/J_{-1}$-mod. By the previous paragraph, $B/J_i$ is also a graded quasi-hereditary algebra. Therefore, $\Delta(\lambda)$ is the projective cover in the $B/J_{-1}$-grmod of the irreducible module $L(\lambda)$ (viewed as a graded $B/J_{-1}$-module having pure grade 0). See Remark 8.4 in §8 (Appendix I) for more discussion of PIMs in $B$-grmod.

We have the following elementary result. See also [29, Cor. 3.2].

**Proposition 3.1.** (a) Suppose $B = \bigoplus_{n \geq 0} B_n$ is a positively graded quasi-hereditary algebra with poset $\Lambda$. Then the subalgebra $B_0$ is quasi-hereditary with poset $\Lambda$.

(b) In the special case that $\Lambda$ is a finite ideal of $p$-regular dominant weights, put $B := \widetilde{\text{gr}} A_\Lambda$. Then the modules $\Delta^\text{red}(\lambda)$ (resp., $\nabla^\text{red}(\lambda)$), $\lambda \in \Lambda$, are the standard (resp., costandard) modules for the quasi-hereditary algebra $B_0$. In particular, the rational $G$-modules $\Delta^\text{red}(\lambda)$ and $\nabla^\text{red}(\lambda)$ are naturally modules for all three algebra $B$, $B_0$ and $A_\Lambda$, all acting through the common quotient algebra $B_0$.

**Proof.** Let $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = B$ be a defining sequence of idempotent ideals in $B$ as described above. Each $J_i = Be_i B$, for an idempotent $e_i \in B_0$. Necessarily (by the axioms for a quasi-hereditary algebra), $e_i Be_1$ is a semisimple algebra, so that necessarily $e_1 B_0 e_1 = e_1 Be_1$ is semisimple. In addition, multiplication $Be \otimes e_1 Be \rightarrow Be B = J_1$ is an isomorphism of $k$-vector spaces. Taking the gradings into account, it follows that multiplication $B_0 e_1 \otimes e_1 B_0 e_1 \rightarrow e_1 B_0 e_1$ is an isomorphism. Therefore, $J_{0,1} = B_0 e_1 B_0$ is a heredity ideal in $B_0$. Continuing, we find that $0 \subseteq J_{1,0} \subseteq J_{2,0} \subseteq \cdots \subseteq J_{n,0}$ is a defining sequence of ideals in $B_0$. It follows that $B_0$ is quasi-hereditary with poset $\Lambda$, as required for (a).

Finally, to see (b), apply [29, Cor. 3.2], with standard (resp., costandard) modules the $\Delta^\text{red}(\lambda)$ (resp., $\nabla^\text{red}(\lambda)$), $\lambda \in \Lambda$. Notice that, for any $n > 0$, $(\text{rad}^n \tilde{a}) \Delta^\text{red}(\lambda) = (\text{rad}^n \tilde{a}) \nabla^\text{red} = 0$ because $\Delta^\text{red}(\lambda)$ and $\nabla^\text{red}(\lambda)$ are obtained by reductions mod $p$ of lattices in an irreducible $U_\zeta$-module. Hence, $\Delta^\text{red}(\lambda)$ and $\nabla^\text{red}(\lambda)$ are indeed $B_0 = \widetilde{\text{gr}} A/\text{rad} \tilde{A}$-modules. □

**Remark 3.2.** The above discussion extends to the $\mathcal{O}$-algebras $\tilde{A}_\Lambda$. In fact, since $\mathcal{O}$ is assumed to be complete, $\tilde{A} := \tilde{A}_\Lambda$ is a semi-perfect algebra (see [6]). In view of [6, Prop. 4.2], the idempotent ideals $\tilde{J}_i$ making up a defining sequence of $A_\Lambda$ are all homogeneous and have the form $\tilde{J}_i = \tilde{A} e_i \tilde{A}$, for some idempotent $e_i$. The argument is then completed as before. In particular, we note that $\tilde{\Delta}^\text{red}(\lambda)$ and $\tilde{\nabla}^\text{red}(\lambda)$ are modules for $\tilde{A}_\Lambda$, $\text{gr} \tilde{A}_\Lambda$, ($\text{gr} \tilde{A}_\Lambda)_0$, with the first two algebras acting through their common quotient algebra ($\text{gr} \tilde{A}_\Lambda)_0$.

We propose the following generalization of a Koszul algebra.

**Definition 3.3.** A finite dimensional, positively graded algebra $B = \bigoplus_{n \geq 0} B_n$ is called a Q-Koszul algebra provided the following conditions hold:

(i) the subalgebra $B_0$ is quasi-hereditary, with poset $\Lambda$ and standard (resp., costandard) modules denoted $\Delta^0(\lambda)$ (resp., $\nabla^0(\lambda)$), $\lambda \in \Lambda$; and
(ii) if $\Delta^0(\lambda)$ and $\nabla_0(\lambda)$ are given pure grade 0 as graded $B$-modules (through the homomorphism $B \to B/B_{\geq 1} \cong B_0$), then

$$\text{ext}_B^n(\Delta^0(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r, \quad \forall \lambda, \mu \in \Lambda, n \in \mathbb{N}, r \in \mathbb{Z}.$$ 

In the above definition, the algebra $B$ can be taken over any field, not necessarily our algebraically closed field $k$ of positive characteristic $p$.

**Remarks 3.4.** (a) A similar generalization—in the abstract—of Koszul algebras, using “tilting modules” has been proposed by Madsen [24].

(b) Koszul algebras and quasi-hereditary algebras provide rather trivial examples of Q-Koszul algebras. In the case in which $B$ is Koszul, the subalgebra $B_0 = B_{\geq 1}$ is semisimple and hence it is quasi-hereditary. In this situation, $\Delta^0(\lambda) \cong \nabla_0(\lambda), \lambda \in \Lambda,$ are irreducible. View them as graded $B$-modules having pure grade 0, condition (ii) is automatic from the definition of a Koszul algebra. Thus, $B$ is Q-Koszul. On the other hand, suppose that $B$ is an (ungraded) quasi-hereditary algebra. View $B$ as positively graded by setting $B_0 := B$. Then $B$ is Q-Koszul using the well-known fact that $\dim \text{Ext}_B^n(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu} \delta_{n,0}$ [7, Lem. 2.2].

Now return to the group $G$. The next result shows that there are more interesting examples of Q-Koszul algebras than those considered in Remark 3.4(b). The proof will be given in §5, immediately after the proof of Theorem 5.6.

**Theorem 3.5.** Assume that $p \geq 2h - 2$ is odd, and that condition (2.4.3) holds. Let $\Lambda$ be a finite ideal of $p$-regular dominant weights and form the graded algebra $B := \bigoplus \Lambda^0 A_{\lambda}$. Then $B$ is a Q-Koszul algebra with poset $\Lambda$, setting $\Delta^0(\lambda) = \Delta^\text{red}(\lambda)$ and $\nabla_0(\lambda) = \nabla^\text{red}(\lambda), \lambda \in \Lambda$.

Finally, there is the following notion of a standard Q-Koszul algebra. It is modeled on the notion of a standard Koszul algebras as used by Mazorchuk [25].

**Definition 3.6.** A positively graded algebra $B = \bigoplus_{n \geq 0} B_n$ is called a standard Q-Koszul algebra provided it is Q-Koszul[14] the following conditions are satisfied:

(i) $B$ graded quasi-hereditary algebra with weight poset $\Lambda$, and with standard (resp., costandard, irreducible) modules $\Delta^B(\lambda)$ (resp., $\nabla_B(\lambda)$, $L_B(\lambda)$), for $\lambda \in \Lambda$; and

(ii) given $\lambda, \mu \in \Lambda$, and positive integers $r, n$,

$$\begin{cases} 
\text{ext}_B^n(\Delta^B(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r; \\
\text{ext}_B^n(\Delta^B(\mu), \nabla_B(\lambda)\langle r \rangle) \neq 0 \implies n = r.
\end{cases}$$

15Mazorchuk quotes a paper [1] for the name standard Koszul, though the notion is not quite the same. In any case, the notion (but not the name) goes back to earlier work of Irving [16].

16It seems likely that the requirement that $B$ be Q-Koszul is already implied by conditions (i) and (ii) and thus is redundant. We intend to discuss this issue further elsewhere.
In (ii), \( \Delta^0(\mu) \) (resp., \( \nabla_0(\mu) \)), \( \lambda, \mu \in \Lambda \), are the standard (resp., costandard) modules for the quasi-hereditary algebra \( B_0 \). They are viewed as graded \( B \)-modules (concentrated in grade 0) through the homomorphism \( B \rightarrow B/B_{\geq 1} \cong B_0 \).

The complete proof of the theorem below is postponed to §6. The theorem requires that there is, by [28, 8.4], a natural duality \( \mathfrak{d} \) on the module categories \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \)-mod and \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \)-grmod. It arises from an anti-automorphism of the order \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \) and so induces an anti-automorphism on \( A_\Lambda \) and a graded anti-automorphism on \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \). Thus, it induces a duality on \( A_\Lambda \)-mod, \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \)-mod, and \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \)-grmod. This duality fixes irreducible modules and interchanges standard and costandard modules.

**Theorem 3.7.** Assume that \( p \geq 2h - 2 \) is odd, and that (2.4.3) holds. Let \( \Lambda \) be a finite ideal of \( p \)-regular dominant weights and form the graded algebra \( B := \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \). Then \( B \) is a standard \( Q \)-Koszul algebra (in the sense of Definition 3.6) with poset \( \Lambda \), setting

\[
\left\{
\begin{align*}
\Delta^B(\lambda) &= \tilde{\mathfrak{g}} \mathfrak{r} \Delta(\lambda), \\
\Delta^0(\lambda) &= \Delta^\text{red}(\lambda), \\
\nabla_B(\lambda) &= \mathfrak{d} \Delta^B(\lambda), \\
\nabla_0(\lambda) &= \nabla(\lambda),
\end{align*}
\right.
\]

for \( \lambda \in \Lambda \).

A graded quasi-hereditary algebra \( B \) has, by definition, a graded Kazhdan-Lusztig theory provided there is a length function \( l : \Lambda \rightarrow \mathbb{Z} \) such that, for \( \lambda, \mu \in \Lambda \), \( r, n \in \mathbb{Z} \), the non-vanishing of either \( \text{ext}^r_B(\Delta^B(\lambda), L_B(\mu)(r)) \) or of \( \text{ext}^r_B(L_B(\mu)(r), \nabla_B(\lambda)) \) implies that \( n = r \equiv l(\lambda) - l(\mu) \mod 2 \). See [6, §3] and [9, §2.1].

The usual length function \( l \) on the (affine) Coxeter group \( W_p \) of \( G \) leads to a length function \( l : X_{\text{reg}}(T)_+ \rightarrow \mathbb{N} \) as follows. For a \( p \)-regular dominant weight \( \lambda \), write \( \lambda = w \cdot \lambda^- \), where \( \lambda^- \in C_p^- \) (the unique alcove containing \( -2\rho \)) and \( w \in W_p \). Then put \( l(\lambda) := l(w) \).

The following corollary was promised in [28, Rem. 10.7(a)]. The proof is postponed to §6.

**Corollary 3.8.** Assume that \( p \geq 2h - 2 \) is odd, and that (2.4.3) holds (for \( p \) and \( G \)). Let \( \Lambda \) be a finite \( p \)-regular dominant weights contained in \( \Gamma_{\text{Jan}} \). Then \( \tilde{\mathfrak{g}} \mathfrak{r} \Delta(\lambda) \) is a linear module over \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \). Also, the graded quasi-hereditary algebra \( \tilde{\mathfrak{g}} \mathfrak{r} A_\Lambda \)-mod has a graded Kazhdan-Lusztig theory. In particular, \( \tilde{\mathfrak{g}} \mathfrak{r} A \) is Koszul.

4. \((\Gamma, a)\)-RESOLUTIONS.

This section begins the study of resolutions necessary for most of the main results of this paper. The detailed information obtained on filtrations of the syzygies in these resolutions are important in their own right.

We continue the notation of §§1,2. We will not quote any results from §3. Let \( \Gamma \) denote a finite ideal in \( X_{\text{reg}}(T)_+ \) and let \( A = A_\Gamma \). The reader should keep in mind that \( A \)-mod consists of finite dimensional rational \( G \)-modules whose composition factors have the form
$L(\gamma)$ for $\gamma \in \Gamma$. Let $M$ be a graded $\tilde{g}r \ A$-module. The main result of this section, given in Theorem 4.2, constructs a key specific resolution condition holds, that
\[(iii)\text{ above.}\]

$\tilde{M}$ filtration with sections $u$ of $\tilde{M}$ particular, apply in the integral case (over $M$ is a resolution of $\xi$).

In this statement, the graded vector spaces and graded maps with the following properties:
\[(i)\text{ there is an increasing chain } \Gamma = \Gamma_1 \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \text{ of finite ideals in } X_{reg}(T),\]
such that, for $i \geq 0$, $\Xi_i \in \tilde{g}r \ A_{\Gamma_i}$-mod;
\[(ii)\text{ the maps } \Xi_i \rightarrow \Xi_{i-1} \text{ are morphisms in the category } \tilde{g}r \ A_{\Gamma_i}$-grmod. (Set $\Xi_{-1} := M$.)

In this statement, the graded $\tilde{g}r \ A_{\Gamma_i}$-module $\Xi_{i-1}$ is regarded as a graded $\tilde{g}r \ A_{\Gamma_i}$-
module through the algebra surjection $\tilde{g}r \ A_{\Gamma_i} \rightarrow \tilde{g}r \ A_{\Gamma_i-1}$. See [27, Rem. 3.8].
\[(iii)\text{ for } i \geq 0, \text{ the } \tilde{g}r \ A_{\Gamma_i}$-module $\Xi_i$ has a graded filtration with sections of the form
$\tilde{g}r \ P^\ast(\gamma) \langle j \rangle, \gamma \in \Gamma_{i-1}, j \in \mathbb{N}. \text{(The module } P^\ast(\gamma)\text{ is defined in (2.1.1).)}$

Similarly, at level of orders and lattices over $\tilde{\vartheta}$, there is an analogous notion of a $(\Gamma, \tilde{\alpha})$-
projective resolution $\tilde{\Xi}_* \rightarrow \tilde{M}$ of a $\tilde{g}r \ A_{\Gamma}$-lattice $\tilde{M}$. Setting $\Xi_i := k \otimes_{\vartheta} \tilde{\Xi}_i$ and $M := k \otimes_{\vartheta} \tilde{M}$, it follows that $\Xi_i \rightarrow M$ is a $(\Gamma, \alpha)$-projective resolution of $M$. (Use $\tilde{g}r \tilde{P}^\ast(\gamma) = \tilde{g}r \ P^\ast(\gamma)$ in place of $\tilde{g}r \ P^\ast(\gamma)$.)

Continue in the context of Defn. 4.1. Suppose that $j > 0$, and let $\Omega_j := \ker(\Xi_{j-1} \rightarrow \Xi_{j-2})$. Recall that $\Xi_{-1} := M$. Define the $j$-truncated complex
\[(4.0.3) \quad \Xi^\dagger = \Xi^{j-1} : 0 \rightarrow \Omega_j \rightarrow \Xi_{j-1} \rightarrow \cdots \rightarrow \Xi_0 \rightarrow 0\]
in the category $\tilde{g}r \ A_{\Gamma_j}$-grmod. Observe that $(\Xi^\dagger)_j = \Omega_j$ and $\Xi^\dagger_{j-1} = 0$. By definition, $\Xi^\dagger \rightarrow M$ is a resolution of $M$. The syzygies $\Omega_j$ will play a role below. Similar considerations apply in the integral case (over $\tilde{\vartheta}$).

Now assume that $p \geq 2h - 2$ is odd, and that the LCF condition (2.4.3) holds. In particular, $\alpha = u'$ (the direct sum of the regular blocks of the universal enveloping algebra $u$ of $G$) is a Koszul algebra. A $(\Gamma, \alpha)$-projective resolution of $M$ gives a resolution of $M|_{\alpha}$ by graded and projective $\alpha$-modules $\Xi_i|_{\alpha}$. In fact, $\Xi_i|_{\alpha}$ has, by definition, a $\tilde{g}r \ A_{\Gamma_i}$-
filtration with sections $\tilde{g}r \ P^\ast(\gamma) \langle j \rangle$ and each $\tilde{g}r \ P^\ast(\gamma) \langle j \rangle$ is a projective graded $\alpha$-module. This resolution is $\alpha$-linear (in the sense of §2.5) if and only if $j = i$, for all the $\tilde{g}r \ A_{\Gamma_i}$-
modules $P^\ast(\gamma) \langle j \rangle$ which appear as sections (and hence as $\alpha$-summands) of $\Xi_i$ in condition (iii) above.
We will see in §§5, 6 that these resolutions can be used to compute, among other things, the spaces $\text{ext}^m_{\tilde{\frak{g}}} A(M, X)$ and $\text{Ext}^m_{\tilde{\frak{g}}} A(M, X)$ with $X = \nabla_{\text{red}}(\gamma)$, with $M$ as above. Theorem 4.2 below constructs these resolutions for suitable $M$. Integral versions are also obtained. In addition, the theorem shows that, in the presence of (2.4.3), the syzygy modules in suitable resolutions of the modules $\Delta_{\text{red}}(\lambda)$ (for a $p$-regular dominant weight $\lambda$) have $\Delta_{\text{red}}$-filtrations. Once Theorem 6.3 is established\(^{17}\), a similar result by be deduced from Theorem 4.2 for resolutions of $\tilde{\frak{g}} \Delta(\lambda)$, expanding a main theme of [29], which provided a $\Delta_{\text{red}}$-filtration of Weyl modules.

**Theorem 4.2.** Assume that $p \geq 2h - 2$ is odd, and that the LCF condition (2.4.3) holds. Let $\Gamma$ be any finite ideal in the set $X_{\text{reg}}(T)_+$ of $p$-regular dominant weights. Let $A = A_{\Gamma}$ and $\tilde{A} = \tilde{A}_{\Gamma}$.

(a) Assume that $M$ is a graded $\tilde{\frak{g}} A$-module such that each grade $M_a$ has a $\Delta_{\text{red}}$-filtration. Assume that $M|\tilde{\alpha}$ is a linear module. There exists a resolution (4.0.2) of $M$ which is both $\tilde{\alpha}$-linear and $(\Gamma, \tilde{\alpha})$-projective such that, for $i \geq 0$, $\Xi_i$ and $\Omega_{i+1} := \ker(\Xi_i \to \Xi_{i-1})$ have a $\Delta_{\text{red}}$-filtration, grade by grade.

(b) Assume that $M$ is a graded $\tilde{\frak{g}} \tilde{A}$-module such that each grade $\tilde{M}_s$ has a $\tilde{\Delta}_{\text{red}}$-filtration. Assume that $M|\tilde{\alpha}$ is a linear module. There exists an $\tilde{\alpha}$-linear $(\Gamma, \tilde{\alpha})$-projective resolution of $\tilde{M}$, analogous to (4.0.2), such that, for $i \geq 0$, $\Xi_i$ and $\Omega_{i+1} := \ker(\Xi_i \to \Xi_{i-1})$ have a $\tilde{\Delta}_{\text{red}}$-filtration, grade by grade.

Before proving the theorem, some further notation and a preliminary lemma are required.

For a finite ideal $\Gamma$ in $X_{\text{reg}}(T)_+$, let $r := r(\Gamma)$ be the minimal positive integer such that $\Gamma \subseteq X_r(T)$. For a positive integer $r$, put

$$L_r := \{ \lambda \in X_{\text{reg}}(T)_+ \mid (\lambda, \alpha_\gamma^\vee) < 2p^r(h - 1) \}.$$  

Thus, $L_r$ in an ideal in the poset of $p$-regular weights. If $r \geq r(\Gamma)$, then $\Gamma$ is an ideal in $L_r$. In addition, if $\gamma \in \Gamma$, the $G_rT$-projective cover $\tilde{Q}_r(\gamma)$ of the irreducible $G_rT$-module $\tilde{L}_r(\gamma)$ of highest weight $\gamma$ has a unique $G$-module structure with $G$-composition factors $L(\tau)$, $\tau \in L_r$. In [18], this $G$-module is denoted by the same symbol $\tilde{Q}_r(\gamma)$, but we write it as $P_r(\gamma)$. Given $\gamma \in X_1(T)$, $P_1(\gamma) = P^2(\gamma)$ in the notation of (2.1.1).

Let $A := A_{\Lambda_r}$. By [18, p. 333], $P_r(\gamma)$ is the projective cover of $L(\gamma)$ in the “$p^r$-bounded category" $A$-mod of rational $G$-modules having composition factors of highest weights in $L_r$.

Now pass to orders, and let $\tilde{A} := \tilde{A}_{\Lambda_r}$, where $r \geq r(\Gamma)$ as before. Given $\gamma \in \Gamma$, by [13, Thm. 3.2, Prop. 2.3 & p. 159], we can lift the projective $A$-module $P_r(\gamma)$ to an $\tilde{A}$-lattice $\tilde{P}_r(\gamma)$. Moreover, any such lifting is projective and unique.

Write $\gamma = \gamma_0 + p\gamma_1 \in \Gamma$, where $\gamma_0 \in X_1(T)$ and $\gamma_1 \in X(T)_+$. Then $P_1(\gamma_0) \in A_{\Lambda_1}$-mod lifts to a projective module for $\tilde{A}_{\Lambda_1}$ and, thus, to a $\tilde{U}_\zeta$-lattice $\tilde{P}_1(\gamma_0)$. The projective module

---

\(^{17}\)Part (a) of Theorem 6.3 does not assume the LCF and depends only on results from [27], while part (b) is derived from Theorem 4.2 applied to $\Delta_{\text{red}}(\lambda)$. 
Proof. \( \tilde{P}_r \) lifts to a Dist\(_F\)(G)-lattice \( \tilde{P}_r(\gamma) \). Therefore, pulling back through the Frobenius \( \tilde{F} \) in (2.2.2), we obtain the \( \tilde{U}_\zeta \)-lattice \( (\tilde{P}_r(\gamma))^{[1]} \), denoted \( \tilde{P}_r(\lambda)^{[1]} \) or simply \( \tilde{P}_r(\lambda)^{[1]} \) if it is convenient. There is a tensor product decomposition

(4.0.5) \[
\tilde{P}_r(\gamma) \cong \tilde{P}_1(\gamma_0) \otimes \tilde{P}_r(\gamma_1)^{[1]}.
\]

(The reductions mod \( \pi \) are isomorphic as rational \( G \)-modules, so they are integrally isomorphic.) The Hopf algebra structure on \( \tilde{U}_\zeta \) is required to view (4.0.5) as a \( \tilde{U}_\zeta \)-module.

The proof of the following lemma uses the fact that if \( \tilde{X} \) is a lattice for an integral quasi-hereditary algebra \( \tilde{B} \) with the property that \( \tilde{X}_k \) has a \( \Delta \)-filtration for the quasi-hereditary algebra \( \tilde{B} = \tilde{B}_k \), then \( \tilde{X} \) has a \( \Delta \)-filtration. This follows immediately from [29 Prop. 6.1] and a standard Nakayama’s lemma argument. The integral quasi-hereditary algebra will be \((\gr \tilde{A})_0\).

Lemma 4.3. Assume that \( p \geq 2h - 2 \) is odd, and that (2.4.3) holds.

Let \( \Gamma \subset X(T)_+ \) be a finite ideal in the poset of \( p \)-regular weights. Let \( \tilde{B} := \tilde{A}_\Gamma \). Suppose that \( \tilde{\Omega} := \bigoplus \tilde{\Omega}_s \) is a graded \( \gr \tilde{B} \)-module generated in grade \( m \), for some integer \( m \). View \( \tilde{\Omega}_m \) as a graded \( \gr \tilde{B} \)-module concentrated in grade \( m \), and assume that \( \tilde{\Omega}_m(-m) \) has a \( \tilde{\Delta}^{\text{red}} \)-filtration. Any \( \tilde{\Delta}^{\text{red}}(\mu), \mu \in \Gamma \), may viewed as a graded \( \gr \tilde{B} \)-module concentrated in grade \( 0 \); see Remark 3.2.

Let \( \Lambda = \Lambda_r \), for \( r \geq r(\Gamma) \) and set \( \tilde{A} := \tilde{A}_\Lambda \). The following statements hold.

(a) If

(4.0.6) \[
\tilde{\Omega}' := \ker \left( \gr \tilde{A} \otimes_{(\gr \tilde{A})_0} \tilde{\Omega}_m \to \tilde{\Omega} \right),
\]

then \( \tilde{\Omega}' \) is a graded \( \gr \tilde{A} \)-lattice vanishing in grades \( \leq m \). All composition factors of \( \tilde{\Omega}, \tilde{\Omega}' \) and \( \gr \tilde{A} \otimes_{(\gr \tilde{A})_0} \tilde{\Omega}_m \) have highest weights in \( \Lambda \).

(b) Moreover, \( \gr \tilde{A} \otimes_{(\gr \tilde{A})_0} \tilde{\Omega}_m \) has a graded filtration with sections of the form \( \gr \tilde{P}_s^{(\lambda)}(\langle m \rangle) \), \( \lambda \in \Gamma \). Any such filtration of \( \gr \tilde{A} \otimes_{(\gr \tilde{A})_0} \tilde{\Omega}_m \) induces a filtration of \( \tilde{\Omega}_m \) by modules \( \tilde{\Delta}^{\text{red}}(\lambda)(\langle m \rangle) \). All filtrations of \( \tilde{\Omega}_m \) with sections \( \tilde{\Delta}^{\text{red}}(\lambda)(\langle m \rangle) \), \( \lambda \in \Gamma \), arise this way.

(c) Suppose, for all \( s \in \mathbb{Z} \), that \( \tilde{\Omega}_s \) has a \( \tilde{\Delta}^{\text{red}} \)-filtration. Then \( \tilde{\Omega}'_s \) also has a \( \tilde{\Delta}^{\text{red}} \)-filtration.

Proof. We begin by proving (b). Let \( \gamma = \gamma_0 + p\gamma_1 \in \Gamma \), where \( \gamma_0 \in X_1(T) \) and \( \gamma_1 \in X_{r-1}(T) \). Form the exact sequences

\[
\begin{align*}
(1) \quad & 0 \to \jmath^{[1]} \to \tilde{P}_r(\gamma_1)^{[1]} \to \tilde{\Delta}(\gamma_1)^{[1]} \to 0, \\
(2) \quad & 0 \to \tilde{P}_1(\gamma_0) \otimes \jmath^{[1]} \to \tilde{P}_r(\gamma) \to \tilde{P}_r(\gamma) \to 0
\end{align*}
\]

of \( \tilde{U}_\zeta \)-modules. In (1), \( \jmath^{[1]} \) is defined as the kernel of the natural map \( \tilde{P}_r(\gamma_1)^{[1]} \to \tilde{\Delta}(\gamma_1)^{[1]} \). Then (2) is a sequence of \( \tilde{A} \)-modules, obtained (using the Hopf algebra \( \tilde{U}_\zeta \) by

applying $\tilde{\Delta}(\gamma_0) \otimes_{\mathcal{O}} -$ to (1). Also, (2) is $\tilde{\alpha}$-split, since $\tilde{P}_s^z(\gamma)$ is $\tilde{\alpha}$-projective. Hence, (2) remains an exact sequence in the category $\tilde{\mathcal{A}}$-grmod after $\tilde{\mathcal{A}}$ is applied. Observe from (the dual version of) [29, Lem. 4.1(c)], which uses (2.4.3), that $\tilde{\mathcal{A}}$-red $(\gamma)$ as a $(\tilde{\mathcal{A}})_0$-module. For convenience, put $\tilde{N} := \tilde{P}_1(\gamma_0) \otimes_{\mathcal{O}} \tilde{J}[1]$, and form the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \longrightarrow \tilde{\mathcal{A}} \otimes (\tilde{P}_r(\gamma)) \longrightarrow \tilde{\mathcal{A}} \otimes \tilde{\Delta}_{\text{red}}(\gamma) \longrightarrow 0 \\
0 \longrightarrow \tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \longrightarrow \tilde{\mathcal{A}} \otimes (\tilde{P}_r(\gamma)) \longrightarrow \tilde{\mathcal{A}} \otimes \tilde{\Delta}_{\text{red}}(\gamma) \longrightarrow 0
\end{array}
\]

where $\otimes = \otimes_{(\tilde{\mathcal{A}})_0}$ in the first row. As noted above, the second row is exact, and we also claim that the first row is also exact. This will follow provided that

\[
\text{Tor}_1^{(\tilde{\mathcal{A}})_0}(\tilde{\mathcal{A}}, \tilde{\Delta}_{\text{red}}(\gamma)) = 0.
\]

First, by [27, Thm. 6.3], $(\tilde{\mathcal{A}})$ is a quasi-hereditary algebra over $\mathcal{O}$ with poset $\Lambda$ and with standard right modules denoted $\tilde{\mathcal{A}} \Delta(\gamma)$, $\gamma \in \Gamma$. Now we work with the quasi-hereditary algebra $(\tilde{\mathcal{A}})_0$ which has right standard (resp., costandard) modules $\tilde{\Delta}_{\text{red}}(\tau)$ (resp., $\tilde{\nabla}_{\text{red}}(\tau)$), $\tau \in \Gamma$. By [29, Thm. 5.1], $\tilde{\mathcal{A}} \Delta(\gamma)$ has a $(\tilde{\Delta}_{\text{red}})$-filtration. Therefore,

\[
\text{Ext}_1^{(\tilde{\mathcal{A}})_0}(\tilde{\mathcal{A}} \Delta(\gamma), \tilde{\nabla}_{\text{red}}(\mu)) \cong \text{Ext}_1^{(\tilde{\mathcal{A}})_0}(\tilde{\mathcal{A}} \Delta(\gamma), \tilde{\nabla}_{\text{red}}(\mu)) = 0, \quad \forall \mu \in \Gamma.
\]

A standard Nakayama’s lemma argument gives that $\text{Ext}_1^{(\tilde{\mathcal{A}})_0}(\tilde{\mathcal{A}} \Delta(\gamma), \tilde{\nabla}_{\text{red}}(\mu)) = 0$, which means that $\tilde{\mathcal{A}}$, viewed as a right $(\tilde{\mathcal{A}})_0$, has a $(\tilde{\Delta}_{\text{red}})$-filtration by [29, Prop. 6.1]. Now (4.0.8) follows from Proposition 9.1 below (applied to the quasi-hereditary algebra $(\tilde{\mathcal{A}})_0 = (\tilde{\mathcal{A}})_0$).

The middle and left vertical maps in (4.0.7) are both surjective maps, since $\tilde{P}_1(\gamma_0)$ and $\tilde{P}_r(\gamma)$ are projective $\tilde{\alpha}$-modules. Thus, the right hand vertical map is surjective.

Next, the middle vertical map in (4.0.7) is, in fact, an isomorphism, since $\tilde{P}_r(\gamma)$ is $\tilde{\mathcal{A}}$-projective. The snake lemma now implies that the two remaining vertical maps are injective, hence they are also isomorphisms. (In particular, we record the isomorphism

\[
\tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \cong \tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}})\tilde{\Delta}_{\text{red}}(\gamma) \cong \tilde{\mathcal{A}} \otimes \tilde{\Delta}_{\text{red}}(\gamma)
\]

which will be used later.)

Consider the surjection $\tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \to \tilde{\Omega}$ of graded modules. A $\tilde{\Delta}_{\text{red}}$-filtration of $\tilde{\Omega}_m(-m)$ gives a filtration of $\tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \tilde{\Omega}_m$ with sections $\tilde{\mathcal{A}} \otimes \tilde{\Delta}_{\text{red}}(\gamma)(m)$, using Proposition [9.1] again and the right hand vertical isomorphism above. This filtration is a graded filtration of a graded module. Conversely, any graded filtration of $\tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \tilde{\Omega}_m$ results in a graded filtration of $\tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \tilde{\Omega}_m \cong (\tilde{\mathcal{A}} \otimes (\tilde{\mathcal{A}}) \tilde{\Omega}_m)_m$ by modules $\tilde{\Delta}_{\text{red}}(\gamma) \cong \tilde{\mathcal{A}} \otimes \tilde{\Delta}_{\text{red}}(\gamma)(m)_m$, proving (b).
Since each surjection 
$$\tilde{\text{gr}} P^0(\gamma) \to \tilde{\Delta}^{\text{red}}(\gamma)$$
has kernel with non-zero grades only in grades 1 or higher, it follows that \(\tilde{\Omega}'\) in \((4.0.6)\) vanishes in grades \(\leq m\). This establishes the first assertion of (a). The last assertion of (a) is clear, and so (a) is proved.

Finally, consider statement (c). For any \(s \in \mathbb{Z}\), \((4.0.6)\) gives a short exact sequence

$$0 \to \tilde{\Omega}'_s \to \tilde{X} := (\text{gr} \tilde{A} \otimes (\text{gr} \tilde{A})_0)_{s} \to \tilde{\Omega}_s \to 0,$$

of \((\text{gr} \tilde{A})_0\)-modules in which \(\tilde{\Omega}_s\) has a \(\tilde{\Delta}^{\text{red}}\)-filtration by hypothesis. By (b), \(\tilde{X}\) has a filtration with sections \((\text{gr} \tilde{P}^0(\lambda)(m))_s\). Also, \([29]\) Thm. 3.1 implies that \(\text{gr} \tilde{P}^0(\lambda)\) has a (graded) \(\text{gr} \tilde{\Delta}\)-filtration. Thus, \(\tilde{\text{gr}} P^0(\lambda) = \text{gr} \tilde{P}^0(\lambda)\) has a (graded) \(\text{gr} \tilde{\Delta}\)-filtration, and therefore, by \([29]\) Thm 5.1, each section \((\text{gr} \tilde{P}^0(\lambda)(m))_s\) has a (graded) \(\tilde{\Delta}^{\text{red}}\)-filtration.

Thus, the (graded) \((\text{gr} \tilde{A})_0\)-module \((\text{gr} \tilde{P}^0(\lambda))(m)_s\) (concentrated in grade \(s - m\)) has a \(\tilde{\Delta}^{\text{red}}\)-filtration. Thus, \(\tilde{X}\) has a \(\tilde{\Delta}^{\text{red}}\)-filtration, concentrated in grade \(s\). Now the long exact sequence of cohomology and \([29]\) Prop. 6.1] gives that \(\tilde{\Omega}'_s\) has a \(\tilde{\Delta}^{\text{red}}\)-filtration, completing the proof. 

---

**Proof of Theorem 4.2.** It suffices to prove part (b) of the theorem. Then part (a) is obtained by base change to the field \(k\). Define \(\Gamma_0 = \Lambda_{r(\Gamma_1)}\). Having defined \(\Gamma_i\), put \(\Gamma_{i+1} = \Lambda_{r(\Gamma_i)}\). The \((\Gamma, \tilde{a})\)-projective resolution \(\tilde{\Xi}_i \to \tilde{M}\) is constructed recursively. Let \(\tilde{\Xi}_0 = (\text{gr} \tilde{A}) \otimes (\text{gr} \tilde{A})_0 \tilde{M}_0\). Let \(\tilde{\Omega}_1\) to be the kernel of the natural map \(\tilde{\Xi}_0 \to \tilde{M}\). Both \(\tilde{\Xi}_0\) and \(\tilde{\Omega}_1\) are graded \(\text{gr} \tilde{A}\)-modules. Since \(\tilde{a}\) is a Koszul algebra and \(\Xi_0 = \tilde{\Xi}_k\) is a graded projective \(a\)-module generated by its grade 0-component, \(\tilde{\Omega}_1 := (\tilde{\Omega}_1)_k\) is generated by its grade 1-component. Therefore, by Nakayama’s lemma, the graded \(\tilde{a}\)-module \(\tilde{\Omega}_1\) is generated by its grade 1-component \(\tilde{\Omega}_{1,1}\). In any given grade \(s\), \(\tilde{\Omega}_{1,s}\) has a \(\tilde{\Delta}^{\text{red}}\)-filtration.

Now assume, for a given \(i > 0\), that graded \(\text{gr} \tilde{A}_{\Gamma_{j-1}}\)-modules \(\tilde{\Omega}_j\) and \(\tilde{\Xi}_{j-1}\) have been constructed, for \(0 < j \leq i\), such that

(i) there is an exact sequence \(0 \to \tilde{\Omega}_j \to \tilde{\Xi}_{j-1} \to \tilde{\Omega}_{j-1} \to 0\) (with \(\tilde{\Omega}_{j-1} = \tilde{M}\));
(ii) \(\tilde{\Omega}_{j-1}\) and \(\tilde{\Xi}_{j-1}\) are generated in grades \(j\) and \(j - 1\), respectively;
(iii) \(\tilde{\Xi}_{j-1}\) is a graded \(\text{gr} \tilde{A}_{j-1}\)-module, filtered by graded lattices with sections \(\text{gr} \tilde{P}^0(\gamma)\), \(\gamma \in \Gamma_{j-1} = \Lambda_{r(\Gamma_{j-2})}\);
(iv) \(\tilde{\Omega}_{j,s}\) has a \(\tilde{\Delta}^{\text{red}}\)-filtration, for each \(s \in \mathbb{Z}\). (Here \(\tilde{\Omega}_{j,s}\) is the grade \(s\)-component of \(\tilde{\Omega}_j\)).

Define \(\tilde{\Xi}_i = (\text{gr} \tilde{A}_{\Gamma_i}) \otimes (\text{gr} \tilde{A}_{\Gamma_i})_0 \tilde{\Omega}_{i,i}\), and set

$$\tilde{\Omega}_{i+1} = \ker ((\text{gr} \tilde{A}_{\Gamma_i}) \otimes (\text{gr} \tilde{A}_{\Gamma_i})_0) (\tilde{\Omega}_{i,i} \to \tilde{\Omega}_i).$$
Condition (i), with \( i + 1 \) replacing \( i \), clear from construction. Condition (ii) follows from the Koszulity of \( a \), together with (ii) for \( j \leq i \) and Nakayama’s lemma. Parts (iii) and (iv) follow from Lemma \[4.3\]. This completes the recursive construction and the proof of the theorem. \( \square \)

As a corollary of the proof of Lemma \[4.3\] we record the following result.

**Corollary 4.4.** Let \( \gamma \) be a \( p \)-regular weight which is \( r \)-restricted, for some positive integer \( r \). Let \( \tilde{A} = \tilde{A}_\Lambda \) (see (4.0.4)). Then, in the derived categories \( D^-(\tilde{\gr} \tilde{A}) \) and \( D^-(\gr A) \), we have

\[
\begin{align*}
\tilde{\gr} P^d(\gamma) &\cong \tilde{\gr} \tilde{A} \otimes L \tilde{\Delta}^{\text{red}}(\gamma); \\
\gr P^d(\gamma) &\cong \gr A \otimes L \Delta^{\text{red}}(\gamma).
\end{align*}
\]

Here \( \otimes = \otimes (\gr A)_b \).

**Proof.** This follows from (4.0.9) and the proof of (4.0.8), for \( n \geq 1 \) (replacing \( \text{Tor}_1 \) by \( \text{Tor}_n \), and again using Proposition \[9.1\] below). \( \square \)

### 5. Filtrations

The main result, Theorem \[5.3\], shows, under the hypotheses of Theorem \[4.2\] that, if \( \lambda, \mu \) are \( p \)-regular weights, the rational \( G \)-module

\[
\text{Ext}^n_G(\Delta^{\text{red}}(\lambda), \nabla^{\text{red}}(\mu))^{[-1]}
\]

has a \( \nabla \)-filtration.

Before beginning the proof of this theorem, we prove the following proposition which has independent interest and plays a key role in the proof of Theorem \[3.7\] in \$6\). The result is also based on Theorem \[4.2\] which guarantees the existence of the resolutions \( \Xi \) and \( \tilde{\Xi} \) in (a) below.

**Proposition 5.1.** Assume that \( p \geq 2h - 2 \) is odd and that (2.4.3) holds. Let \( \Gamma \) be finite ideal of \( p \)-regular weights. Let \( M \) be a graded \( \gr A_\Gamma \)-module which is \( \alpha \)-linear. Assume that each grade \( M_s \) has a \( \Delta^{\text{red}} \)-filtration and let \( \Xi \to M \) be as in display (4.0.2) as guaranteed by Theorem \[4.2\] (a). Similarly, let \( \tilde{M} \) be a graded \( \gr \tilde{A}_\Gamma \)-module which \( \tilde{\alpha} \)-linear, and such that each graded \( \tilde{M}_s \) has \( \tilde{\Delta}^{\text{red}} \)-filtration. Let \( \tilde{\Xi} \to \tilde{M} \) be an integral version of (4.0.2) as guaranteed by Theorem \[4.2\] (b).

(a) Then, setting \( A_i = A_\Gamma \) and \( \tilde{A}_i = \tilde{A}_\Gamma \),

\[
\begin{align*}
\text{Ext}^n_{\gr A_i}(\Xi_i, \nabla^{\text{red}}(\gamma)) &= 0; \\
\text{Ext}^n_{\gr \tilde{A}_i}(\tilde{\Xi}_i, \tilde{\nabla}^{\text{red}}(\gamma)) &= 0,
\end{align*}
\]

for all \( i \geq 0 \), all positive integers \( n \), and all \( \gamma \in \Gamma \).

(b) In particular, let \( j \geq n \) be nonnegative integers with \( j > 0 \), and let \( \Lambda \) be a finite poset of \( p \)-regular weights containing \( \Gamma_j \). Put \( A = A_\Lambda \) and \( \tilde{A} = \tilde{A}_\Lambda \). Then

\[
\text{Ext}^n_{\gr A_\Gamma}(M, \nabla^{\text{red}}(\lambda)) \cong H^n(\text{Hom}_{\gr \tilde{A}_\Gamma}(\Xi_i, \nabla^{\text{red}}(\lambda))
\]
where $\Xi^j_\bullet = \Xi^{\ast j}_\bullet$ is the $j$-truncated resolution \cite{A}. (c) In addition,
\[
\text{ext}^n_{\text{gr} A_i} (M, \nabla_{\text{red}} (\lambda) \langle r \rangle) \cong \text{H}^n (\text{hom}_{\text{gr} A} (\Xi^j_\bullet, \nabla_{\text{red}} (\lambda) \langle r \rangle)) \cong \begin{cases} 
\text{Hom}_{(\text{gr} A)_0} (\Omega_n / \text{rad} \Omega_n, \nabla_{\text{red}} (\lambda)), & n = r, \\
0 & \text{otherwise}. 
\end{cases}
\]

**Proof.** We first prove (a). Consider the integral case of $\text{gr} \tilde{A}_i$. The $\text{gr} \tilde{A}_i$-module $\tilde{\Xi}_i$ has a filtration by the modules $\text{gr} \tilde{P}^\xi (\lambda)$, $\lambda \in \Gamma$. Thus, it suffices to show that
\[
\text{Ext}^n_{\text{gr} \tilde{A}_i} (\text{gr} \tilde{P}^\xi (\lambda), \nabla_{\text{red}} (\gamma)) = 0, \ \forall n > 0,
\]
Using Cor. 4.4
\[
\text{Ext}^n_{\text{gr} \tilde{A}_i} (\text{gr} \tilde{P}^\xi (\lambda), \nabla_{\text{red}} (\gamma)) \cong \text{Hom}^n_{\tilde{D}^{-1} (\text{gr} \tilde{A}_i)} (\text{gr} \tilde{P}^\xi (\lambda), \nabla_{\text{red}} (\gamma)) \cong \text{Hom}^n_{\tilde{D}^{-1} (\text{gr} \tilde{A}_i)} (\tilde{\Delta}_{\text{red}} (\lambda), \nabla_{\text{red}} (\gamma)) \cong \text{Ext}^n_{\tilde{D}^{-1} (\text{gr} \tilde{A}_i)} (\tilde{\Delta}_{\text{red}} (\lambda), \nabla_{\text{red}} (\gamma)) = 0.
\]
This proves (a) for $\text{gr} \tilde{A}_i$. A similar argument works for $\text{gr} A_i$.

Finally, (b) and (c) follow from a standard argument, using the spectral sequences associated to the Cartan-Eilenberg double complex resolution of $\Xi^j_\bullet$; see \cite{A3} Summary 5.7.9].

We will need the following preliminary lemma.

**Lemma 5.2.** Assume that $p \geq 2h - 2$ is odd, and that \cite{A1} holds. Let $X$ be a finite-dimensional $G$-module whose composition factors $L(\gamma)$ satisfy $\gamma \in X_{\text{reg}} (T)^{+}$. Assume $X$ is completely reducible for $G_1$ and has a $\Delta_{\text{red}}$-filtration as a $G$-module. Then $\text{Hom}_{G_1} (X, \nabla_{\text{red}} (\gamma))^{[-1]}$ has a $\nabla$-filtration, for any $\gamma \in X (T)^{+}$.

**Proof.** The statement is clearly true if $X = \Delta_{\text{red}} (\gamma')$, $\gamma \in X_{\text{reg}} (T)^{+}$, since
\[
\text{Hom}_{G_1} (\Delta^p (\gamma'), \nabla_{\mu} (\gamma)) \cong \begin{cases} 
\text{Hom}_k (\Delta (\gamma'[1]), \nabla (\gamma[1]), & \gamma_0 = \gamma_0 \\
0, & \text{otherwise}. 
\end{cases}
\]
In general, consider a short exact sequence $0 \to X' \to X \to X'' \to 0$ of rational $G$-modules in which $X'$ and $X''$ are nonzero modules having $\Delta_{\text{red}}$-filtrations. Observe this sequence is $G_1$-split. By an evident induction argument, we can assume the conclusion of the lemma holds with $X$ replaced by $X'$ or $X''$. Form the exact sequence
\[
0 \to \text{Hom}_{G_1} (X'', \nabla_{\text{red}} (\gamma))^{[-1]} \to \text{Hom}_{G_1} (X, \nabla_{\text{red}} (\gamma))^{[-1]} \to \text{Hom}_{G_1} (X', \nabla_{\text{red}} (\gamma))^{[-1]} \to 0
\]
of rational $G$-modules. By assumption, the right and left hand sides of this sequence have $\nabla$-filtrations. Thus, the middle term has a $\nabla$-filtration, as required.
We now establish the main result of this paper, part (a) of Theorem 5.3. The first step in its proof identifies $\text{Ext}^n_{G_1}(\Delta^\text{red}(\lambda), \nabla_\text{red}(\mu))$ as a vector space with a rational $G$-module $\text{Hom}_{G_1}(\Omega_n/\text{rad}_a\Omega_n, \nabla_\text{red}(\mu))[-1]$ (in the notation of Theorem 4.2), which can be easily shown to have a $\nabla$-filtration. Thus, it is necessary to show that this identification is an isomorphism of $G$-modules. This final step, which is delicate, requires the abstract setting of §8 (Appendix I).

Later, in §6, part (a) of the theorem below will be extended to the case in which $\Delta^\text{red}(\lambda)$ (resp., $\nabla_\text{red}(\mu)$) is replaced by $\Delta(\lambda)$ (resp., $\nabla(\mu)$); see Theorem 6.2 below. Part (b) will be similarly extended in Theorem 6.5 using $\text{gr}_G\Delta(\lambda)$ and a dual construction. We note that part (a) of the theorem below assumes the LCF condition (2.4.3), while part (b) does not. Parallel remarks hold for their respective extensions in §6.

**Theorem 5.3.** Assume that $p \geq 2h - 2$ is odd. Let $\lambda, \mu \in X_\text{reg}(T)_+.$

(a) Suppose that condition (2.4.3) holds. Then, for any integer $n \geq 0$, the rational $G$-module $\text{Ext}^n_{G_1}(\Delta^\text{red}(\lambda), \nabla_\text{red}(\mu))[-1]$ has a $\nabla$-filtration.

(b) Let $A = A_\Gamma$, for any finite ideal $\Gamma$ of $p$-regular dominant weights containing $\lambda, \mu$. For any integer $n \geq 0$, there are natural vector space isomorphisms

\[
\begin{align*}
\text{Ext}^n_{\text{gr}A}(\Delta^\text{red}(\lambda), \nabla_\text{red}(\mu)) & \cong \text{Ext}^n_A(\Delta^\text{red}(\lambda), \nabla_\text{red}(\mu)) \\
& \cong \text{Ext}^n_{G_1}(\Delta^\text{red}(\lambda), \nabla_\text{red}(\mu)).
\end{align*}
\]

(5.0.10)

**Proof.** Let $\Gamma$ be a finite ideal of $p$-regular weights containing $\lambda, \mu$. There is an algebra homomorphism $a \to A := A_\Gamma$ (which is an inclusion if $\Gamma$ is sufficiently large).

Using Theorem 4.2 and noting that $\nabla_\text{red}(\mu)|_\alpha$ is completely reducible, there are natural vector space isomorphisms (labelled for further discussion)

\[
\begin{align*}
\text{Ext}^n_{G_1}(\Delta^\text{red}(\lambda), \nabla_\text{red}(\mu)) & \cong H^n(\text{Hom}_a(\Xi_\bullet, \nabla_\text{red}(\mu))) \\
& \cong \text{Hom}_a(\Omega_n, \nabla_\text{red}(\mu)) \\
& \cong \text{Hom}_{G_1}(\Omega_n/\text{rad}_a\Omega_n, \nabla_\text{red}(\mu)).
\end{align*}
\]

(5.0.11)

The first term in (5.0.11) is obviously a rational $G$-module. On the other hand, the last term $\text{Hom}_{G_1}(\Omega_n/\text{rad}_a\Omega_n, \nabla_\text{red}(\mu))$ on the right is also a rational $G$-module. To see this, first observe that

$$\text{rad}_a\Omega_n = a_{\geq 1}\Omega_n = (a_{\geq 1}\text{gr}_a A)\Omega_n = (\text{gr}_a A)_{\geq 1}\Omega_n.$$  

(The first equality follows since $a$ is a Koszul algebra.) Now use the isomorphism

$$\text{gr}_a A/(\text{gr}_a A)_{\geq 1} \cong A/A_{\geq 1}$$

to make $\Omega_n/\text{rad}_a\Omega_n$ an $A$-module, thus, a $\text{Dist}(G)$-module, and, finally, a rational $G$-module.

Next, $\Omega_n\text{rad}_a\Omega_n \cong \Omega_n\Omega_n$ (the grade $n$-component of $\Omega_n$) has, by Theorem 4.2(a), a $\Delta^\text{red}$-filtration. Thus, Lemma 5.2 implies that $\text{Hom}_{G_1}(\Omega_n/\text{rad}_a\Omega_n, \nabla_\text{red}(\mu))[-1]$ has a $\nabla$-filtration.
Therefore, (a) will follow provided the composite

\[ (5.0.12) \quad (3) \circ (2) \circ (1): \text{Ext}^n_{G}(\Delta^{\text{red}}(\lambda), \nabla^{\text{red}}(\mu)) \to \text{Hom}_{G}(\Omega_n / \text{rad}_a \Omega_n, \nabla^{\text{red}}(\mu)) \]

doctrine the vector space isomorphisms (1), (2), (3) in (5.0.11) is a morphism of rational \(G\)-modules. While most readers will expect this to be true, a rigorous proof requires the constructions of §8 (Appendix I) below.

First, general methods imply that the left hand side of (5.0.12) can be calculated, as a rational \(G\)-module, by any a truncated resolution

\[ (5.0.13) \quad 0 \to E \to R_{n-1} \to \cdots \to R_0 \to \Delta^{\text{red}}(\lambda) \to 0 \]

doctrine \(\Delta^{\text{red}}(\lambda)\) by \(A\)-modules such that \(R_i|_{a}\) is \(a\)-projective, \(i = 0, \ldots, n - 1\). Here we use the fact that the category \(A\)-mod is the same as the category of finite dimensional rational \(G\)-modules having composition factors \(L(\gamma), \gamma \in \Gamma\). (This statement holds for any poset ideal \(\Gamma\). As we see below, the current \(\Gamma\) may need to be enlarged to for any given \(n\), to make the resolution construction possible.) That is,

\[ (5.0.14) \quad \text{Ext}^n_{G}(\Delta^{\text{red}}(\lambda), \nabla^{\text{red}}(\mu)) \cong \text{coker}(\text{Hom}_{a}(R_{n-1}, \nabla^{\text{red}}(\mu)) \to \text{Hom}_{a}(E, \nabla^{\text{red}}(\mu))) \]

in \(G\)-mod.\footnote{The isomorphism as vector spaces is elementary. Usually, given rational \(G\)-modules \(M, N\), the \(G\)-module structure of the spaces \(\text{Ext}^n_{G}(M, N)\) is obtained by computing these Ext-groups using a \(G\)-injective resolution \(N \to I*\) of \(N\). Necessarily, each \(I_j\) is also \(a\)-injective, defining a rational \(G\)-module structure on \(\text{Ext}^n_{G}(M, N)\), regarded as the \(n\)-cohomology of the complex \(\text{Hom}_{a}(M, I*\)). To see this \(G\)-action agrees with that defined by the isomorphism (5.0.14), temporarily denote \(E\) by \(R_n\) and form the double complex \(\text{Hom}(R_n, I*\)). Its total complex provides an \(a\)-injective resolution of \(\text{Hom}_{a}(\Delta^{\text{red}}(\lambda), \nabla^{\text{red}}(\mu))\). Both spectral sequences of the double complex collapse, one defining the usual \(G\)-action and the other defining the action using (5.0.14). Thus, the two actions are the same.} The construction will be used in the reader to make the connection. See also Remark 8.14. The construction will be used in the proof of (b) below and, again, in Theorem 6.5. Before getting started, we note the following lemma. Its proof also results from Appendix I.

**Lemma 5.4.** Suppose \(\Xi^*_{\ast} = \Xi^*_{\ast}\) is a \(n\)-truncated complex as in (4.0.3) in \(\tilde{\text{gr}} A_{\Gamma} \)-mod, so that \(\Xi^*_{\ast} \to M\) is a graded resolution of a \(\tilde{\text{gr}} A_{\Gamma}\)-module \(M\), and so that each \(\Xi^*_{j}\) is projective. Assume that \(M\) is \(a\)-linear and that \(\Xi^*_{j}\) is part of a linear (graded) \(a\)-resolution.

Let \(\Xi^*_{\ast}\) be a second \(n\)-truncated resolution of \(M\) with the same properties as listed above for \(\Xi^*_{\ast}\). Then \(\Xi^*_{\ast}\) \cong \Xi^*_{\ast}\) as graded \(\tilde{\text{gr}} A_{\Gamma_{\ast}}\)-resolutions of \(M\).

**Proof.** For the proof, simply break each complex into short exact sequences, e. g., \(0 \to \Omega_j \to \Xi_{j-1} \to \Omega_{j-1} \to 0\) and apply Theorem 8.5(d) with \(B = \tilde{\text{gr}} A_{\Gamma_j}\), for various choices of \(j\). Observe that, by construction, \(\Xi_j\) has a projective cover in \(B\)-grmod which remains
projective upon restriction to $a$. (This observation follows easily from the discussion involving [18, p.333] after the statement of Theorem 4.2) \hfill \Box

For the first step of the construction, put $N = \Delta^{\text{red}}(\lambda)$ and replace $A$ by $A_{\Gamma_1}$, where $\Gamma_1 := \Lambda_1(\Gamma)$ (so that $a \subset A$). Proposition 8.12 gives a short exact sequence $0 \to E' \to R' \to N' \to 0$ in $A_{\text{mod}}$ and, upon restricting to $a$, in $a_{\text{grmod}}$. Here $R|_a$ is projective in $a_{\text{grmod}}$. Moreover, all the objects $X$ in this sequence have the property that $X_{>s}$, $s \in \mathbb{Z}$, (as defined by the $a$-grading on $X$) are all $A$-modules, so we may construct a graded $\widetilde{\mathfrak{g}} A$-module

$$\widetilde{\text{Gr}} X = \bigoplus_{s \in \mathbb{Z}} X_{>s}/X_{>s+1}.$$ 

This construction guarantees that $\widetilde{\text{Gr}} X|_a \cong X|_a$ in $a_{\text{grmod}}$. Also, according to Proposition 8.12 there is exact sequence $0 \to E' \to R' \to N' \to 0$ in $\widetilde{\mathfrak{g}} A_{\text{mod}}$ and in $a_{\text{grmod}}$, where $N'$ (at the moment) is just $N$, $E'$ is a certain quotient of $\text{Gr} E$ (in $A_{\text{mod}}$ or in $a_{\text{grmod}}$) which is $a$-linear of degree 1 (in fact, the maximal such linear quotient). The conditions in Proposition 8.12 guarantee the hypotheses of Lemma 5.4 hold, for $m = n = 1$. In particular, $\Omega_1 \cong E'$ in $\mathfrak{g} A$-grmod (and in $a$-grmod).

Now enlarge $A$ to $A_{\Gamma_2}$, where $\Gamma_2 = \Lambda_1(\Gamma_1)$. Repeat the argument with $N$ replaced by $E$. The new $N'$ will not be the same as $E$, but will be $E'$. Continuing in this way, we obtain the sequence (5.0.13) in $A_{\text{mod}}$, for $A = A_\Lambda$ (for some large $\Lambda$). It is also an exact sequence in $a_{\text{grmod}}$. The top row of the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{\text{Gr}} E & \longrightarrow & \widetilde{\text{Gr}} R_{n-1} & \longrightarrow & \cdots & \longrightarrow & \widetilde{\text{Gr}} R_0 & \longrightarrow & \Delta^{\text{red}}(\lambda) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega_n & \longrightarrow & \Xi_{n-1} & \longrightarrow & \cdots & \longrightarrow & \Xi_0 & \longrightarrow & \Delta^{\text{red}}(\lambda) & \longrightarrow & 0.
\end{array}
$$

is exact in $\mathfrak{g} A$-grmod. The bottom row is just $\Xi^1_\bullet$ (obtained by repeatedly applying Lemma 5.4). Notice $\Omega_n \cong E'$ in Theorem 8.13(b), by its recursive construction. By Theorem 8.13, there is a natural isomorphism

$$\text{coker}(\text{Hom}_a(R_{n-1}, \nabla^{\text{red}}(\mu))) \to \text{Hom}_a(E, \nabla^{\text{red}}(\mu))) \cong \text{Hom}_a(\Omega_n, \nabla^{\text{red}}(\mu)),$$

easily seen to preserve the $G$-action on both sides. The key point is that $\text{Gr} E$ and $E$, as well as $\widetilde{\text{Gr}} R_{n-1}$ and $R_{n-1}$ share a (large) common quotient $\text{hd}^0 E$ in $A/A_{\geq 1}$-mod. This gives the isomorphism (5.0.12). This proves (a).

The proof of (b) relies on a similar construction, and uses Proposition 8.12(a) and Theorem 8.13(a). As is well-known, the identification of Ext-groups as cokernels such as those appearing in Theorem 8.13(a) works equally using projective resolutions or resolutions acyclic for $\text{Ext}^*_A(-, \nabla^{\text{red}}(\mu))$ or $\text{Ext}^*_A(-, \nabla^{\text{red}}(\mu))$. Since we are dealing with quasi-hereditary algebras, it is enough that each $R_j$ be projective for $A_{\Gamma_j}$ for an ideal $\Gamma_j$ in $\Gamma$ with $\mu \in \Gamma_j$. \hfill \Box
Lemma 6.1. Let $\Delta(\sigma)$ be a hereditary module with standard (resp., costandard) modules $\Delta(\sigma)$ (resp., $\nabla(\sigma)$) for all $\sigma \in \Lambda$. Then, for any nonnegative integer $n$ and any integer $r$,
\[
\text{ext}^n_{\text{gr} A}(\Delta(\sigma), \nabla(\mu) \langle r \rangle) \neq 0 \implies r = n.
\]

Proof. As in the proof of the previous theorem, for any integer $r$,
\[
\text{ext}^n_{\text{gr} A}(\Delta(\sigma), \nabla(\mu) \langle r \rangle) \cong \text{hom}_{(\text{gr} A)_0}(\Omega_n / \text{rad} \Omega_n, \nabla(\mu) \langle r \rangle)).
\]
But $\Omega_n / \text{rad} \Omega_n$ is pure of grade $n$, so if $\text{ext}^n_{\text{gr} A}(\Delta(\sigma), \nabla(\mu) \langle r \rangle) \neq 0$, then $r = n$. □

Proof of Theorem 5.6. First, by Proposition 3.1 applied to $B = \text{gr} A$, $(\text{gr} A)_0$ is quasi-hereditary with standard (resp., costandard) modules $\Delta^0(\lambda) = \Delta(\lambda) (\nabla^0(\lambda) = \nabla(\lambda))$, $\lambda \in \Lambda$. Thus, condition (i) follows in Definition 3.3. Finally, condition (ii) is implied by Theorem 5.6. This completes the proof. □

When $n = r$ in the theorem, the value of $\dim \text{ext}^n_{\text{gr} A}(\Delta(\sigma), \nabla(\mu) \langle r \rangle)$ can thus be calculated in terms of coefficients of Kazhdan-Lusztig polynomials; see [10, Thm. 5.4], which gives the corresponding calculation of $\text{Ext}^n$.

Proof. This follows from Theorem 5.6 and the fact that $\Delta(\lambda)$ is irreducible, for $\lambda \in \Gamma_{\text{reg}}$. □

6. Further Filtrations

This section gives certain variations on the results of §5. Explicitly, Theorem 6.2 shows that if $\lambda, \mu$ are $p$-regular dominant weights, then the $G$-modules $\text{Ext}^m_{G_1}(\Delta(\lambda), \nabla(\mu))[-1]$ and $\text{Ext}^m_{G_1}(\Delta(\lambda), \nabla(\mu))[-1]$ have $\nabla$-filtrations, for all $m \geq 0$. We also present proofs of Theorem 3.7 and its Corollary 3.8. This result requires Theorem 6.3 which shows that each $\nabla(\nu)$ can be naturally viewed as a graded $\alpha$-module, and that, as such, it is $\alpha$-linear.

In the following lemma, $B$ is a quasi-hereditary algebra with weight poset $\Lambda$, standard (resp., costandard) modules $\Delta(\lambda) = \Delta_B(\lambda)$ (resp., $\nabla(\lambda) = \nabla_B(\lambda)$), $\lambda \in \Lambda$. This lemma will be applied to the representation theory of $G$ in Theorem 6.2.

Lemma 6.1. Let $M \to N$ be a homomorphism of $B$-modules. Assume that $M$ has a $\nabla$-filtration, and that
\[
\text{Hom}_B(\Delta(\sigma), M) \to \text{Hom}_B(\Delta(\sigma), N) \quad \text{is surjective } \forall \sigma \in \Lambda.
\]
Then $N$ has a $\nabla$-filtration, and the map $M \to N$ is surjective.
Proof. Let $\lambda \in \Lambda$ be maximal, and put $\Gamma := \Lambda \setminus \{\lambda\}$. Let $M^\Gamma, N^\Gamma$ be the largest submodules of $M, N$, respectively, with all composition factors in $L(\gamma), \gamma \in \Gamma$. By induction, we may assume the result is true for quasi-hereditary algebras having posets of smaller cardinality that of $\Lambda$. In particular, if $J$ is the annihilator in $B$ of all modules with composition factors $L(\gamma), \gamma \in \Gamma$, then the lemma holds for the quasi-hereditary algebra $B' := B/J$. Thus, $N^\Gamma \in B'$ has a $\nabla_{B'}$-filtration and the map $M^\Gamma \to N^\Gamma$ is surjective. However, standard and costandard modules in $B'$-mod inflate to standard and costandard modules in $B$-mod, so $N^\Gamma$ has a $\nabla$-filtration in $B$-mod, as well. Now form the commutative diagram:

$$\begin{align*}
\text{Hom}_B(\Delta(\lambda), M) & \longrightarrow \text{Hom}_B(\Delta(\lambda), N) \\
\downarrow b & \quad \downarrow c
\end{align*}$$

By hypothesis (6.0.15), the map $a$ is surjective. Since $\lambda$ is maximal, $\Delta(\lambda)$ is projective in $B$-module, so that maps $b$ and $c$ are both surjective. Next, for $\sigma, \tau \in \Lambda$, $\text{Ext}_B^1(\nabla(\sigma), \nabla(\tau)) \neq 0$ implies that $\sigma > \tau$. Thus, because $M$ has a $\nabla$-filtration, $M/M^\Gamma$ is a direct sum of copies of the injective module $L(\lambda)$ which has socle $L(\lambda)$. On the other hand, clearly $N/N^\Gamma$ has socle which is a direct sum of copies of $L(\lambda)$. It follows the socle of $M/M^\Gamma$ maps surjectively onto the socle of $N/N^\Gamma$. Thus, we can choose a direct summand $X$ of $M/M^\Gamma$ which maps isomorphically onto a submodule of $N/N^\Gamma$ containing the socle of $N/N^\Gamma$. Since $X$ is injective, $X \cong N/N^\Gamma$. It follows that $N/N^\Gamma$ is isomorphic to a direct sum of copies of $\nabla(\lambda)$. Since $N^\Gamma$ has a $\nabla$-filtration, it follows that $N$ has a $\nabla$-filtration.

Finally, since we have shown that $M^\Gamma \to N^\Gamma$ and $M/M^\Gamma \to N/N^\Gamma$ are surjective maps, it follows that $M \to N$ is surjective. This completes the proof. \hfill \Box

We are now ready to prove the following result.

**Theorem 6.2.** Assume that $p \geq 2h - 2$ is odd and that (2.4.3) holds. Let $\nu, \mu \in X_{\text{reg}}(T)_+$ and $m \geq 0$.

(a) The rational $G$-module $\text{Ext}^m_{G_1}(\Delta(\nu), \nabla_{\text{red}}(\mu))^{[-1]}$ has a $\nabla$-filtration and the natural map

$$(6.0.16) \quad \text{Ext}^m_{G_1}(\Delta_{\text{red}}(\nu), \nabla_{\text{red}}(\mu)) \to \text{Ext}^m_{G_1}(\Delta(\nu), \nabla_{\text{red}}(\mu))$$

induced by $\Delta(\nu) \mapsto \Delta_{\text{red}}(\nu)$ is surjective.

(b) Dually, the rational $G$-module $\text{Ext}^m_{G_1}(\Delta_{\text{red}}(\mu), \nabla(\nu))^{[-1]}$ has a $\nabla$-filtration and the natural map

$$(6.0.17) \quad \text{Ext}^m_{G_1}(\Delta_{\text{red}}(\mu), \nabla_{\text{red}}(\nu)) \to \text{Ext}^m_{G_1}(\Delta_{\text{red}}(\mu), \nabla(\nu))$$

induced by $\nabla_{\text{red}}(\nu) \mapsto \nabla(\nu)$ is surjective.

**Proof.** We will only prove part (a), leaving the dual assertion (b) to the reader. We proceed by induction on $m$. 


First, consider the $m = 0$ case. By [29, Prop. 2.3(b)], the $G_1$-head of $\Delta(\nu)$ is isomorphic to $\Delta^{\text{red}}(\nu) \cong \Delta^p(\nu)$. Because $\nabla_{\text{red}}(\mu)|_{G_1}$ is completely reducible, it follows the map $\text{Hom}_{G_1}(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu)) \to \text{Hom}_{G_1}(\Delta(\nu), \nabla_{\text{red}}(\mu))$ is trivially an isomorphism (and so, in particular, a surjection). Write $\nu = \nu_0 + p\nu_1$ and $\mu = \mu_0 + p\mu_1$ as usual. If $\nu_0 \neq \mu_0$, then $0 = \text{Hom}_{G_1}(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu))[-1] = \text{Hom}_{G_1}(\Delta(\nu), \nabla_{\text{red}}(\mu))[-1]$, which has a $\nabla$-filtration. Thus, suppose that $\nu_0 = \mu_0$, so that the rational $G$-module

$$M^0 := \text{Hom}_{G_1}(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu))[-1] \cong \text{Hom}_k(\Delta(\nu_1), \nabla(\mu_1)) \cong \nabla(\nu_1^* \otimes \nabla(\mu_1))$$

is isomorphic to a tensor product of two costandard modules; thus, it has a $\nabla$-filtration. Therefore, $N_0 := \text{Hom}_{G_1}(\Delta(\nu), \nabla_{\text{red}}(\mu))[-1] \cong M^0$ has a $\nabla$-filtration. This completes the proof in the $m = 0$ case.

Next, assume that assertion (a) is valid for smaller values of some fixed integer $m > 0$. Let $\lambda \in X(T)_+$. Consider two Hochschild-Serre spectral sequences

$$E_2^{a,b} = \text{Ext}_{G/G_1}^a(\Delta(\lambda)[1], \text{Ext}_{G_1}^b(\Delta(\nu), \nabla_{\text{red}}(\mu))) \Rightarrow \text{Ext}_{G}^{a+b}(\Delta(\lambda)[1] \otimes \Delta(\nu), \nabla_{\text{red}}(\mu)),$$

$$E_2^{a,b} = \text{Ext}_{G/G_1}^a(\Delta(\lambda)[1], \text{Ext}_{G_1}^b(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu))) \Rightarrow \text{Ext}_{G}^{a+b}(\Delta(\lambda)[1] \otimes \Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu)).$$

For $0 < \lambda < m$, \nabla_{\text{red}}(\mu)} \cong \text{Ext}_{G/G_1}^0(\Delta(\lambda), \text{Ext}_{G_1}^0(\Delta(\nu), \nabla_{\text{red}}(\mu))[1] = 0,$$ since, by induction, $\text{Ext}_{G_1}^0(\Delta(\nu), \nabla_{\text{red}}(\mu))[1]$ has a $\nabla$-filtration. In other words, $E_2^{a,b} = 0$ for $a > 0$ and $0 \leq b < m$, so that the edge map

$$E_\infty^m = \text{Ext}_{G/G_1}^m(\Delta(\lambda)[1] \otimes \Delta(\nu), \nabla_{\text{red}}(\mu)) \xrightarrow{\sim} E_2^{m,0} = \text{Hom}_{G/G_1}(\Delta(\lambda)[1], \text{Ext}_{G_1}^m(\Delta(\nu), \nabla_{\text{red}}(\mu)))$$

is an isomorphism. For the same reason, but now using Theorem 5.3(a), the edge map

$$E_\infty^m = \text{Ext}_{G}(\Delta(\lambda)[1] \otimes \Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu)) \xrightarrow{\sim} E_2^{m,0} = \text{Hom}_{G/G_1}(\Delta(\lambda)[1], \text{Ext}_{G_1}^m(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu)))$$

is also an isomorphism.

The natural surjection $\Delta(\nu) \twoheadrightarrow \Delta^{\text{red}}(\nu)$ induces a map $'E_* \to E_*$ of spectral sequences. This gives a commutative diagram

$$\text{Ext}_{G}^{m}(\Delta(\nu), \nabla_{\text{red}}(\mu) \otimes \nabla(\lambda^*)[1]) \xrightarrow{\alpha} \text{Hom}_{G}(\Delta(\lambda), \text{Ext}_{G_1}^{m}(\Delta(\nu), \nabla_{\text{red}}(\mu))[1]$$

in which the maps $\alpha$ and $\epsilon$ are isomorphisms (and are induced from the above edge maps, after identifying $G/G_1$ with $G$ and untwisting the appropriate modules).

By Theorem 5.3(a), $M^m := \text{Ext}_{G/G_1}^{m}(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu))[1]$ has a $\nabla$-filtration. Let $\Lambda$ be a large poset ideal in $X(T)_+$ containing all the dominant weights $\gamma$ such that $L(\gamma)$ appears as a composition factor of the $G$-modules $M^m$ and $N^m := \text{Ext}_{G_1}^{m}(\Delta^{\text{red}}(\nu), \nabla_{\text{red}}(\mu))[1]$.
let $B := A_A$. Then Lemma 6.1 will imply that $N^m$ has a $\nabla$-filtration and that (6.0.16) is surjective, provided that $\delta$ is surjective.

Equivalently, it suffices to show that the map $\beta$ is surjective. First, observe that

$$\nabla_{\text{red}}(\mu) \otimes \nabla(\lambda^*)^{[1]} \cong L(\mu_0) \otimes (\nabla(\mu_1) \otimes \nabla(\lambda^*)^{[1]})^\dagger.$$ 

Also, $\nabla(\mu_1) \otimes \nabla(\lambda^*)$ has a $\nabla$-filtration in which the sections $\nabla(\tau)$ satisfy $\tau \leq \mu_1 + \lambda^*$. Therefore, $\nabla_{\text{red}}(\mu) \otimes \nabla(\lambda^*)^{[1]}$ has a $\nabla_{\text{red}}$-filtration with sections $\nabla_{\text{red}}(\xi)$ in which $\mu + p\lambda^* - \xi \in p\mathbb{Z}R$. In particular, $\xi$ is $p$-regular and has the same parity as $\mu + p\lambda^*$, i.e., $l(\xi) \equiv l(\mu + p\lambda^*) \mod 2$. Since $\nabla_{\text{red}}(\xi)[-l(\xi)] \in \mathcal{E}^R$ by \cite{10} Thm. 6.8], $\nabla_{\text{red}}(\mu) \otimes \nabla(\lambda^*)^{[1]}$ belongs to $\mathcal{E}^R$ or to $\mathcal{E}^R[1]$ (depending on whether this parity is even or odd). Now Lemma 2.2 implies that $\beta$ is a surjection. \hfill \Box

Next, recall from (2.5.1) that $\lambda$ has a positive grading induced from a grading on $\tilde{\lambda}$, as long as $p > h$. We show, in part (a) of the theorem below, that the standard modules $\Delta(\nu), \nu \in X_{\text{reg}}(T)_+$, have a natural $\lambda$-grading, and as graded modules they satisfy $\Delta(\nu) \cong \text{gr} \Delta(\nu)$. This part of the theorem does not use the assumption (2.4.3) that the Lusztig character formula holds. However, if (2.4.3) is assumed to hold, then we show also that $\Delta(\nu)$ is linear over the Koszul algebra $\lambda$.

**Theorem 6.3.** Assume that $p \geq 2h - 2$ is odd.

(a) For $\lambda \in X_{\text{reg}}(T)_+$, the standard module $\Delta(\lambda)$ has a graded $\lambda$-module structure, isomorphic to $\text{gr} \Delta(\lambda)$ over $\text{gr} \lambda \cong \lambda$.

(b) Assume that (2.4.3) holds. With the graded structure given in (a), $\Delta(\lambda)$ is linear over $\lambda$.

**Proof.** We first prove (a). In fact, we will prove a stronger statement, namely, that the grading on $\Delta(\lambda)$ comes (via base change) from an $\tilde{\lambda}$-grading on $\tilde{\Delta}(\lambda)$. (The proof makes heavy, though implicit, use of a main result in \cite{28} Thm. 6.4) which establishes that, at the quantum enveloping algebra level, $\Delta_K(\lambda)$ has a $\tilde{\lambda}_K$-grading.)

First, \cite{27} Thm. 6.3 verifies the hypotheses of \cite{27} Thm. 5.3(ii)] in our context (ignoring the case $p = h = 2$.)\footnote{We take the opportunity to correct here several typos/omissions in \cite{27}. p. 257, l. 17 down: replace this line by "$R_\lambda \bigoplus (\text{rad} \lambda^{\dagger+1} \tilde{N})$," p. 266, l. 1 up: $P(\lambda)^\dagger = \tilde{P}(\lambda_0) + \sum_{\alpha \geq 1} \tilde{a}_\alpha \tilde{P}(\lambda)^\dagger = \tilde{P}(\lambda_0) + \sum_{\alpha \geq 1} \tilde{a}_\alpha \tilde{P}(\lambda)^\dagger + \sum_{\alpha \geq 2} \tilde{a}_\alpha \tilde{P}(\lambda)^\dagger$. p. 271, l. 14 down: Insert the sentence: "Note that $\tilde{P}(\lambda)^\dagger$ inherits the structure of a $\tilde{\lambda}$-graded module from $\tilde{Q}(\lambda_0)." before the expression "In general" p. 271, l. 24 down: $R \tilde{P}(\lambda)^\dagger \subseteq \tilde{A}_{K,0}$. p. 271, l. 26 down: ... as an $\tilde{A}_{K,0}$-module ...} The module $P_K(\lambda)$ in \cite{27} Thm. 5.3 is $\tilde{P}(\lambda)_K$ in this paper (see \S 2.2). The verification in \cite{27} Thm. 6.3 produces a lattice $\tilde{P}(\lambda)^\dagger$ in $P(\lambda)_K$ with certain properties, including an $\tilde{\lambda}$-grading. (In fact, $\tilde{P}(\lambda)^\dagger = \tilde{P}(\lambda_0) \otimes \tilde{\Delta}(\lambda)^{[1]}$, where $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1(T)$ and $\lambda_1 \in X(T)_+$. The grading of $\tilde{P}(\lambda)^\dagger$ is inherited from that of $\tilde{P}(\lambda_0)$.) The surjective map $\phi : P_K(\lambda) \to \Delta_K(\lambda)$ appearing in the proof of \cite{27} Thm. 5.3] is shown to satisfy:

(i) $\phi(\tilde{P}(\lambda)^\dagger) \cong \tilde{\Delta}(\lambda)$—see the last line of the proof;
Thus, $\tilde{\Delta}(\lambda)$ is a $\tilde{a}$-graded module. On the other hand, $\tilde{\Delta}(\lambda)$ is shown in [27] Thms. 5.3 & 6.3 to be $\tilde{a}$-tight; see also [27] Cor. 3.9. Hence, $\Delta(\lambda) = \bigoplus_{i \geq 0} a_i \Delta(\lambda) \cong \tilde{g}r \Delta(\lambda)$ as graded $a$-modules.

Next, we prove (b). It suffices to prove that if $\text{ext}_a^n(\Delta(\lambda), \nabla_{\text{red}}(\mu) \langle r \rangle) \neq 0$, then $n = r$. However, the surjection (6.0.16) induces a surjection

$$\text{ext}_a^n(\Delta_{\text{red}}(\lambda), \nabla_{\text{red}}(\mu) \langle r \rangle) \twoheadrightarrow \text{ext}_a^n(\Delta(\lambda), \nabla_{\text{red}}(\mu) \langle r \rangle).$$

Thus, $\text{ext}_a^n(\Delta_{\text{red}}(\lambda), \nabla_{\text{red}}(\mu) \langle r \rangle) \neq 0$, so $r = n$. \qed

**Remark 6.4.** We emphasize again that Theorem 6.3(a) does not require that the Lusztig modular conjecture (equivalent to (2.4.3)) hold. Also, under the hypothesis of (a), it is proved in [29, Cor. 3.2] that $\tilde{g}r \Delta(\lambda)_0 \cong \Delta_{\text{red}}(\lambda)$ as a rational $G$-module. In [29, Thm. 5.1], it proved under the hypothesis of (b) that $\tilde{g}r \Delta(\lambda)$ has a $\Delta_{\text{red}}$-filtration, section by section.

Suppose that $\Gamma$ is a finite non-empty ideal of regular weights and let $A = A_{\Gamma}$. For $\lambda \in \Gamma$, $\Delta(\lambda) \cong \tilde{g}r \Delta(\lambda)$ as $a \cong \tilde{g}r a$-modules by Theorem 6.3(a). On the other hand, $\tilde{g}r \Delta(\lambda)$ is a graded $\tilde{g}r A$-module. It follows easily that, for each nonnegative integer $i$, the $a$-submodule $\Delta(\lambda)_{\geq i}$ is $A$-stable. In the sense of Definition 8.7 below and its discussion, $\Delta(\lambda)$, together with its $a$-grading, has the structure of an admissible hybrid $A$-module. Each admissible hybrid $A$-module $N$ has an associated graded $\tilde{g}r A$-module

$$\tilde{G}r\ N = \bigoplus_{j \in \mathbb{Z}} N_{\geq j}/N_{\geq j+1}$$

as defined below Definition 8.7. It is important for our discussion to note that $N$ and $\tilde{G}r\ N$ have obviously isomorphic restrictions to $a$-grmod, and that $A/a_{\geq 1}A = (\tilde{g}r A)_0 \cong (\tilde{g}r A)/a_{\geq 1}(\tilde{g}r A)$ acts isomorphically on $N/a_{\geq 1}N \cong (\tilde{G}r\ N)/a_{\geq 1}(\tilde{G}r\ N)$. This latter isomorphism is a natural transformation of functors on the admissible hybrid $A$-module category. Next observe Corollary 8.8 can be applied after enlarging the poset $\Gamma$, using $\Delta(\lambda)$ as the module $N$ there. Then $N$ can be replaced by the admissible hybrid module $E$ obtained in that result. Once again, the weight poset can be enlarged and the process repeated. This process results in a resolution $R_\bullet \twoheadrightarrow \Delta(\lambda)$ by modules which are all projective over (various) quasi-hereditary algebras $A_{\Lambda}$ with $\Gamma \subseteq \Lambda$ and $(A_{\Lambda})_{\Gamma} = A_{\Gamma}$. All the differentials are maps of admissible hybrid $A_{\Lambda}$-modules for one of these posets $\Lambda$. In addition, $\tilde{G}r\ R_\bullet$ is a graded resolution of $\tilde{G}r \Delta(\lambda)$ by modules projective over the (various) associated quasi-hereditary graded algebras $\tilde{g}r A_{\Lambda}$ with $(\tilde{g}r A_{\Lambda})_{\Gamma} = \tilde{g}r A_{\Gamma}$. Consequently, for any $\mu \in \Gamma$, the resolution $R_\bullet|_{A_{\mu}}$ is by objects which are acyclic for the functor $\text{Hom}_{A_{\mu}}(-, \nabla_{\text{red}}(\mu))$. Similarly, $(\tilde{G}r\ R_\bullet)|_{\tilde{g}r A_{\mu}}$ is a resolution by objects acyclic for the functor $\text{Hom}_{\tilde{g}r A_{\mu}}(-, \nabla(\mu))$. Finally, using the isomorphisms $R_\bullet/a_{\geq 1}R_\bullet \cong (\tilde{G}r\ R_\bullet)/a_{\geq 1}(\tilde{G}r\ R_\bullet)$, it follows that the respective application of each of the two Hom functors to these respective resolutions by acyclic objects results, after making natural identifications, in exactly the same complex!
This gives the first half of the following important result. The proof of the second half, dual to the first, is left to the reader. The conclusions, of course, should be compared with Theorem 5.3(b). Observe that the LCF assumption (2.4.3) is not required in the proof.

**Theorem 6.5.** Assume that \( p \geq 2h - 2 \) is odd. Let \( \lambda, \mu \in X_{\text{reg}}(T)_+ \). Let \( A = A_\Gamma \), for any finite ideal \( \Gamma \) of \( p \)-regular dominant weights containing \( \lambda, \mu \). For any integer \( n \geq 0 \), there are natural vector space isomorphisms
\[
\Ext^n_{\gr A}(\gr \Delta(\lambda), \nabla_{\text{red}}(\mu)) \cong \Ext^n_A(\Delta(\lambda), \nabla_{\text{red}}(\mu)) \\
\cong \Ext^n_G(\Delta(\lambda), \nabla_{\text{red}}(\mu))
\]
and
\[
\Ext^n_{\gr A}(\Delta_{\text{red}}(\lambda), \gr \nabla(\mu)) \cong \Ext^n_A(\Delta_{\text{red}}(\lambda), \nabla(\mu)) \\
\cong \Ext^n_G(\Delta_{\text{red}}(\lambda), \nabla(\mu)).
\]

Now we are ready to complete the proof of several results from §3.

**Proof of Theorem 3.7** We use the notation of Theorem 3.7. By Theorem 3.5, \( B = \gr A \) is a \( Q \)-Koszul algebra with weight \( \Lambda \). As discussed in §2.3, the graded algebra \( B \) is also quasi-hereditary with weight poset \( \Lambda \) and with standard (resp., costandard) modules as indicated (in the statement of Theorem 3.7). In particular, condition (i) in Definition 3.6 holds.

It therefore remains to check condition (ii) in Definition 3.6 which is really two conditions. We will prove the first of these; the second follows by duality. Given \( \lambda \in \Lambda \), \( \Delta^0(\lambda) = \Delta_{\text{red}}(\lambda) \), again by Theorem 3.5. Also, \( \Delta^B(\lambda) = \gr \Delta(\lambda) \), as noted above. In turn, Theorem 6.3 implies (using both parts (a) and (b)) that \( \Delta^B(\lambda)|_a \) is linear. Also, by the main result [29 Thm. 5.1], each section \( \Delta^B(\lambda)|_a = (\gr \Delta(\lambda))|_a \) has a \( \Delta_{\text{red}} \)-filtration. Thus, \( M := \Delta^B(\lambda) \) satisfies the hypothesis of Theorem 4.2(a), using \( \Gamma = \Lambda \). These hypotheses appear again in Proposition 5.1 (which applies the construction of Theorem 4.2(a)). The vanishing in the conclusion of Proposition 5.1(c) now gives the desired result. □

**Proof of Corollary 3.8** First, suppose that \( \Ext^n_{\gr A}(\gr \Delta(\lambda), L(\mu)(\langle r \rangle)) \neq 0 \). Since \( \Lambda \subset \Gamma_{\text{Jan}}, L(\mu)(\nabla_{\text{red}}(\mu)) \), for all \( \mu \in \Lambda \). Then by Theorem 3.7, \( n = r \). Also, Theorem 6.5 implies that \( \Ext^n_{\gr A}(\Delta(\lambda), L(\mu)) \neq 0 \). Therefore, using [6], we obtain that \( l(\lambda) \equiv l(\mu) \) mod 2. It follows that \( \gr A \)-mod has a graded Kazhdan-Lusztig theory (and so is Koszul). In particular, \( \gr \Delta(\lambda), \lambda \in \Lambda \), is \( \gr A \)-linear. □

**Remark 6.6.** Theorem 6.5 is really quite general, and would hold with \( \Delta(\lambda) \) replaced by any other admissible hybrid \( A \)-module. A dual statement holds for \( \nabla(\mu) \).

7. Calculations

In this section, assume that \( p \geq 2h - 2 \) is odd, and that the Lusztig character formula holds (or, equivalently, that the isomorphisms (2.4.3) hold). If \( V \) is a (finite dimensional) rational \( G \)-module having a \( \nabla \)-filtration \( \mathcal{F} \), then the number of times that a given module \( \nabla(\gamma) \) appears as a section in \( \mathcal{F} \) depends only on \( V \) (and not on \( \mathcal{F} \)); this multiplicity equals \( \dim \Hom_G(\Delta(\gamma), V) \). This well-known observation is immediate since the
functor $\text{Hom}_G(\Delta(\gamma), -)$ is exact on the category of modules with a $\nabla$-filtration and since $\dim \text{Hom}_G(\Delta(\gamma), \nabla(\mu)) = \delta_{\gamma,\mu}$. If $V$ has a $\nabla$-filtration, let $[V : \nabla(\gamma)]$ denote the multiplicity of $\nabla(\gamma)$ as a section of $V$ in a $\nabla$-filtration.

Recall that Theorem 5.3(a) established that, if $\lambda, \mu \in X_{\text{reg}}(T)_+$ and $n$ is a nonnegative integer, then the rational $G$-module $\text{Ext}^n_{G_1}(\Delta(\lambda), \nabla(\mu))[-1]$ has a $\nabla$-filtration. Also, Theorem 6.2 showed that both $\text{Ext}^n_{G_1}(\Delta(\lambda), \nabla(\mu))[-1]$ and $\text{Ext}^n_{G_1}(\Delta(\mu), \nabla(\lambda))[-1]$ have $\nabla$-filtrations.

This section describes how the multiplicity of a $\nabla(\tau)$, $\tau \in X(T)_+$, in a $\nabla$-filtration of $\text{Ext}^n_{G_1}(\Delta(\lambda), \nabla(\mu))[-1]$ can be combinatorially determined in terms of the coefficients of certain Kazhdan-Lusztig polynomials $P_{x,y}$ for the the affine Weyl group $W_p$ of $G$, plus a well-known multiplicity result of Steinberg. We regard $P_{x,y}$ as a polynomial in $t := \sqrt{t}$—in fact, it is a polynomial in $t^2$. Also, in the formulas below, $\overline{T}_{x,y}$ is obtained from $P_{x,y}$ by replacing $t$ by $t^{-1}$.

First, consider $\text{Ext}^n_{G_1}(\Delta(\lambda), \nabla(\mu))[-1]$. Write $\lambda = \lambda_0 + pl_1$ and $\mu = \mu_0 + pm_1$, where $\lambda_0, \mu_0 \in X_1(T)$, $\lambda_1, \mu_1 \in X(T)_+$. Hence, $\Delta(\lambda) \cong L(\lambda_0) \otimes \Delta(\lambda_1)[1]$ and $\nabla(\mu) \cong L(\mu_0) \otimes \nabla(\mu_1)[1]$. Thus,

$$\text{Ext}^n_{G_1}(\Delta(\lambda), \nabla(\mu))[-1] \cong \text{Hom}_G(\Delta(\lambda_1), \nabla(\mu_1)) \otimes \text{Ext}^n_{G_1}(L(\lambda_0), L(\mu_0))$$

$$\cong \nabla(\lambda_1) \otimes \nabla(\mu_1) \otimes \text{Ext}^n_{G_1}(L(\lambda_0), L(\mu_0)).$$

It is well-known (and has been already used several times in this paper) that the tensor product of modules of the form $\nabla(\tau)$, $\tau \in X(T)_+$, has a $\nabla$-filtration, the terms of which can be determined by character-theoretic calculations, using Steinberg’s theorem [15, 24.2]. Thus, it suffices to determine the multiplicities of $\nabla$-sections in $\text{Ext}^n_{G_1}(L(\lambda_0), L(\mu_0))[-1]$. Observe that $L(\lambda_0) \cong \Delta(\lambda_0)$ and $L(\mu_0) = \nabla(\mu_0)$.

Thus, we can assume from the start that $\lambda = \lambda_0$ and $\mu = \mu_0$ are restricted dominant weights. Then, if $\tau \in X(T)_+$, the multiplicity of $\nabla(\tau)$ as a section in a $\nabla$-filtration of $\text{Ext}^n_{G_1}(\Delta(\lambda_0), \nabla(\mu_0))[-1]$ is

$$[(\text{Ext}^n_{G_1}(\Delta(\lambda_0), \nabla(\mu_0))[-1] : \nabla(\tau)] = \dim \text{Hom}_G(\Delta(\tau)[1], \text{Ext}^n_{G_1}(\Delta(\lambda_0), \nabla(\mu_0)))$$

$$= \dim \text{Ext}^n_{G_1}(\Delta(\lambda_0) \otimes \Delta(\tau)[1], \nabla(\mu_0))^G$$

$$= \dim (\text{Ext}^n_{G_1}(\Delta(\lambda_0 + p\tau), \nabla(\mu_0))[-1])^G$$

$$= \dim \text{Ext}^n_{G_1}(\Delta(\lambda_0 + p\tau), \nabla(\mu_0)).$$

The last equality holds because the Hochschild-Serre spectral sequence (using $G_1$ as the normal subgroup scheme) for computing $\text{Ext}^n_{G_1}(\Delta(\lambda + p\tau), \nabla(\mu_0))$ has $E^a_{2}^b$-term $(a + b = n)$ given by

$$E^a_{2}^b = H^a(G, \text{Ext}^b_{G_1}(\Delta(\lambda_0 + p\tau), \nabla(\mu_0))[-1]).$$

However, $E^a_{2}^b = 0$ if $a > 0$, since $H^a(G, V) = 0$, for $a > 0$ and any rational $G$-module $V$ having a $\nabla$-filtration.
Write \( \lambda' := \lambda_0 + pr = x \cdot \lambda^- \) and \( \mu_0 = y \cdot \mu^- \), where \( \lambda^-, \mu^- \) belong to the \( p \)-alcove \( C_p^- \) containing \(-2\rho\), and \( x, y \) are (uniquely determined) elements of \( W_p \). We can assume that \( \lambda^- = \mu^- \), otherwise all the Ext groups are 0 by the linkage principle. Then since \( p \geq 2h - 2 \) is odd and since (2.4.3) is assumed to hold, [10, Thms. 5.4 & 6.7] implies that

\[
\dim \text{Ext}^n_G(\Delta^\text{red}(\lambda'), \nabla(\nu)) = \sum_{m=0}^{n} \sum_{\nu} \dim \text{Ext}^m_G(\Delta^\text{red}(\lambda'), \nabla(\nu)) \cdot \dim \text{Ext}^{n-m}_G(\Delta(\nu), \nabla(\mu)).
\]

The dimensions of the Ext-groups appearing in the sum are all coefficients of Kazhdan-Lusztig polynomials, as shown in [10, §5]. More precisely, for a given \( \nu \), above Ext groups are 0, unless \( \nu = z \cdot \lambda^- \), for some \( z \in W_p \). Then

\[
t^{l(x)-l(z)} T_{z,x} = \sum_{n \geq 0} \dim \text{Ext}^n_G(\Delta^\text{red}(\lambda'), \nabla(z \cdot \lambda^-)) t^n
\]

(7.0.20)

\[
= \sum_{n \geq 0} \dim \text{Ext}^n_G(\Delta(z \cdot \lambda^-), \nabla(\lambda')) t^n.
\]

Thus, the multiplicity of \( \nabla(\tau) \) can be combinatorially calculated in terms of Kazhdan-Lusztig polynomial coefficients. We give the formula explicitly below, up to Steinberg’s formula for multiplicities in tensor products mentioned above, which calculates the multiplicities \( [\nabla(\lambda^\ast) \mathbin{\otimes} \nabla(\mu_1) \mathbin{\otimes} \nabla(\tau) : \nabla(\omega)] \) in (7.0.22). Given \( u, v \in W_p \) and \( s \in \mathbb{Z}, \) \( c(u, v, s) \) denotes the coefficient of \( t^s \) in \( P_{u,v} \). Thus,

(7.0.21)

\[
P_{u,v} = \sum_{s \geq 0} c(u, v, s) t^s.
\]

For \( p \)-regular dominant weights \( \lambda, \mu \), write \( \lambda = x \cdot \lambda^- \) and \( \mu = y \cdot \mu^- \), for unique \( x, y \in W_p \), and unique \( \lambda^-, \mu^- \in C^- \). Using (7.0.21), put

\[
C(\lambda, \mu, n) := \begin{cases} 
0, & \text{when } \lambda^- \neq \mu^-; \\
\sum_z \sum_{m=0}^{n} c(z, x, l(x) - l(z) - m) \cdot c(z, y, l(y) - l(z) - n + m), & \text{when } \lambda^- = \mu^-,
\end{cases}
\]

where \( \sum_z \) is the sum over all \( z \in W_p \) satisfying \( z \cdot \lambda^- \in X(T)_+ \).

Now we can state

**Theorem 7.1.** Let \( \lambda, \mu \in X_{\text{reg}}(T) \) and let \( n \) be a nonnegative integer. For any \( \omega \in X(T)_+ \),

(7.0.22)

\[
[\text{Ext}^n_G(\Delta^\text{red}(\lambda), \nabla(\mu))^{[-1]} : \nabla(\omega)] = \sum_{\tau \in X(T)_+} C(\lambda_0 + pr, \mu_0, n) [\nabla(\lambda_1^\ast) \mathbin{\otimes} \nabla(\mu_1) \mathbin{\otimes} \nabla(\tau) : \nabla(\omega)].
\]
For the case of $\text{Ext}_{G_1}^n(\Delta(\lambda), \nabla_{\text{red}}(\mu))$, the calculations are easier (but use Theorem 6.2) and are left to the reader. Given $p$-regular weights $\lambda = x \cdot \lambda^-$ and $\mu = y \cdot \mu^-$ as above, define, for $n \in \mathbb{Z}$,

$$ c(\lambda, \mu, n) := \begin{cases} 0, & \text{when } \lambda^- \neq \mu^-; \\ c(x, y, l(x) - l(y) - n), & \text{when } \lambda^- = \mu^- . \end{cases} $$

**Theorem 7.2.** Let $\lambda, \mu \in X_{\text{reg}}(T)$ and let $n$ be a nonnegative integer. For any $\omega \in X(T)_+$,

$$ [\text{Ext}_{G_1}^n(\Delta(\lambda), \nabla_{\text{red}}(\mu))[-1] : \nabla(\omega)] $$

(7.0.23) $$ = \sum_{\tau \in X(T)_+} c(\lambda, \mu_0 + p\tau, n)[\nabla(\tau) \otimes \nabla(\mu_1) : \nabla(\omega)]. $$

and

$$ [\text{Ext}_{G_1}^n(\Delta(\mu), \nabla(\mu))[-1] : \nabla(\omega)] $$

(7.0.24) $$ = \sum_{\tau \in X(T)_+} c(\mu, \lambda_0 + p\tau, n)[\nabla(\lambda_1) \otimes \nabla(\tau) : \nabla(\omega)]. $$

**Remarks 7.3.** (a) Choose a total ordering $\lambda_0 < \lambda_1 < \cdots$ of $X_{\text{reg}}(T)_+$ with the property that $\lambda \leq \mu \implies \lambda < \mu$. Since $\text{Ext}_{G_1}^1(\nabla(\lambda), \nabla(\mu)) 
eq 0$ implies that $\mu < \lambda$, an explicit description (in some sense) of a $\nabla$-filtration of any of the above $\text{Ext}_{G_1}^n$-groups can be given once the $\nabla$-multiplicities are calculated.

(b) Observe that

$$ \begin{cases} \dim \text{Ext}_{G_1}^n(\Delta(\lambda), \nabla_{\text{red}}(\mu_0 + p\tau)) = \dim \text{Ext}_{G_1}^n(\Delta_\xi(\lambda), L_\xi(\mu_0 + p\tau)), \\ \dim \text{Ext}_{G_1}^n(\Delta(\mu), \nabla(\mu_0)) = \dim \text{Ext}_{G_1}^n(\Delta_\xi(\lambda), L_\xi(\mu_0)) \end{cases} $$

(c) In (7.0.24), if $\lambda = 0$, we find, using [10] Lem. 4.1(b)] that the total multiplicity of $\nabla(\tau)$ as a section in a $\nabla$-filtration of $H^*(G_1, \nabla(\mu))[\nabla(\omega)]$ equals the dimension $\dim \Delta(\tau)_\xi$ of the $\xi$-weight space in $\Delta(\tau)$. Here we write $\mu = w \cdot 0 + p\xi, \sigma \in C_p$.

8. **APPENDIX I: SYZYGIES**

This appendix coordinates the representation theory of a positively graded “subalgebra” $\mathfrak{a}$ with that of a larger algebra, which is allowed to be graded or ungraded. In fact, both cases arise, and we will use $B$ for an algebra which is graded, and $A$ for an algebra that may not have a grading. We will assume that $\mathfrak{a}$ is an actual subalgebra of $A$, but, for $B$ we require only that we have only a natural homomorphism $\mathfrak{a} \rightarrow B$ of graded algebras, which might well also be injective. In applications, $B$ will arise as the graded algebra $\bar{A}$ associated with a filtration of $A$, and the map $\mathfrak{a} \rightarrow B$ will occur naturally from this construction. We set this up in reasonable generality in §8.2, which is aimed at coordinating the representation theory of all three algebras. The first §8.1 deals with the graded algebras $\mathfrak{a}$ and $B$ only. We largely have in mind here the case where $\mathfrak{a}$ is a Koszul algebra, though the results
are formulated under only the assumption that \( a \) is positively graded. A central issue addressed is how to formulate the notion of a nice resolution in \( B\)-grmod of a module which, in \( a\)-grmod, has a linear resolution. This leads to the notation of a semilinear resolution, formulated below. Another concept in §8.1 is the notion of the “flat” radical of a (graded or ungraded) module over a graded algebra. When \( a \) and \( B \) are sufficiently closely related (see Definition 8.1), the flat radical \( \operatorname{rad}^b M \) of any \( B \)-module \( M \), whether taken with respect to \( a \) or \( B \), give the same subspace. In §8.2, the quotient module \( \operatorname{hd}^b M := M / \operatorname{rad}^b M \) is also a \( A \)-module. In this way, the representation theories of \( A \), \( a \) and \( B \) can be coordinated. The consequent results—here all cast in an abstract finite dimensional algebra setting—play an important role in the algebraic group results in §5. This is discussed more at the end of this section.

In this section, all algebras and modules for them will always be finite dimensional over the field \( k \).

### 8.1 Syzygies of graded modules.

Let \( a = \bigoplus_{n \geq 0} a_n \) be a positively graded algebra. Generalizing slightly the terminology of §2.5, a graded \( a \)-module \( M \) is said to be linear of degree \( m \in \mathbb{Z} \) if the following conditions hold:

1. \( M \) is generated by its grade \( m \)-component \( M_m \), and
2. if \( M \) has a graded projective resolution \( \cdots \to P_{m+1} \to P_m \to M \to 0 \) such that, for each \( i \geq m \), \( \Omega_{i+1} := \ker(P_i \to P_{i-1}) \) is generated by its grade \( i+1 \)-component \( \Omega_{i+1,i+1} \) (Here \( P_{m-1} := M \)).

Clearly, \( M \) is linear of degree \( m \) if and only if it satisfies condition (i), and condition (ii) holds for its minimal graded projective resolution. In this case, \( \Omega_{i+m} \) is called the \( i \)th syzygy module of \( M \).

Thus, the usual notion of a linear (or Koszul) module is the same as that of an \( a \)-module which is linear of degree 0. The \( m \)th syzygy of such a module is linear of degree \( m \).

It is useful to have a notion which applies to syzygies in more general resolutions. A graded \( a \)-module \( M \) will be called semilinear of degree \( m \) if \( M \) is a direct sum \( M = N \oplus P \), where \( N \) is linear of degree \( m \) and \( P \) is projective and is generated by its components in grades \(< m \), i.e., \( P = a(P_{<m}) \), where \( P_{<m} := \bigoplus_{i<m} P_i \). Many important resolutions that we encounter of linear modules have \( m \)th syzygies which are semilinear of degree \( m \). We are able to show this by proving that semilinearity is “inherited” in the short exact sequences building the resolutions we require, and it provides considerable structure for these resolutions. Before stating the main theorem in this direction, we introduce more notation and give some general preliminary results.

**Definition 8.1.** Let \( E \) be any graded or ungraded \( a \)-module. Define the “flat radical” of \( E \) to be

\[
\operatorname{rad}^b E := a_{\geq 1} E := \sum_{i \geq 1} a_i E.
\]

Also, the “flat head” of \( E \) is

\[
\operatorname{hd}^b E := E / \operatorname{rad}^b E.
\]
Observe that $\text{rad}^b E = (\text{rad}^b a)E \subseteq (\text{rad} a)E = \text{rad} E$, since $a_{\geq 1} = \text{rad}^b a$ is a nilpotent ideal of $a$.

Now suppose that $E$ is graded $a$-module. Both $\text{rad}^b E$ and $\text{hd}^b E$ are graded $a$-modules, and $\text{hd}^b E$ decomposes as an $a$- (or $a_0$-) module as $\text{hd}^b E = \bigoplus_{i \in \mathbb{Z}} (\text{hd}^b E)_i$. There is also a natural identification $(\text{hd}^b E)_i = E_i / \sum_{j > 0} a_j E_{i-j}$, for each $i \in \mathbb{Z}$.

For any graded $a$-module $E$, and $s \in \mathbb{Z}$, define graded $a$-submodules

$$
\begin{align*}
E^s &:= aE_s, \\
E^{s \leq} &:= \sum_{j \leq s} E^j, \\
E^{s<} &:= E^{s-1}, \\
E^{s#} &= E^{s\leq} / E^{s<}.
\end{align*}
$$

(8.1.1)

There is a natural filtration

$$
\cdots \subseteq E^{s \leq} \subseteq E^{s \leq +1} \subseteq \cdots
$$

with, of course, only finitely many distinct terms. There is a corresponding filtration of the graded quotient module $\text{hd}^b E$ of $E$, and we have, for each $s \in \mathbb{Z}$, natural isomorphisms

$$
\begin{align*}
\text{hd}^b E^{s \leq} &\cong (\text{hd}^b E)^{s \leq}, \\
\text{hd}^b E^{s#} &\cong (\text{hd}^b E)^{s#} \cong (\text{hd}^b E)_s.
\end{align*}
$$

(8.1.3)

Any homomorphism $E \to F$ of graded $a$-modules induces maps $E^{s \leq} \to F^{s \leq}$ and $E^{s#} \to F^{s#}$, both surjections whenever the original map is a surjection.

**Definition 8.2.** If $a \to B$ is a morphism of graded algebras, we say that $B$ is (left) tight over $a$ if $a B_0 = B$. (There is, of course, a corresponding right hand notion.

When $B$ is tight over $a$, and $E = E'|_a$, for a graded $B$-module $E'$, then all the graded $a$-modules listed in (8.1.1) inherit natural graded $B$-module structures from $E'$, for any $s \in \mathbb{Z}$. In fact, $E^s = E'^s|_a$, etc.

**Lemma 8.3.** Suppose that $M$ is a graded semilinear $a$-module of degree $m$.

(a) All the inclusions in the filtration (8.1.3) are split as graded $a$-modules, and there is a direct sum decomposition $M \cong \bigoplus_{s \in \mathbb{Z}} M^{s#}$ in which $M^{s#}$ is linear of degree $m$, $M^{s#}$ is projective (and generated in grade $s$) for $s \neq m$, and $M^{s#} = 0$ for $s > m$.

(b) Moreover, $M^{s#}$ naturally inherits a $B$-module structure $M'^{s#}$, whenever $B$ is a graded algebra which is tight over $a$ and $M = M'|_a$, for a graded $B$-module $M'$. Also, the natural surjection $M \twoheadrightarrow M^{s#}$ agrees by restriction with the natural surjection $M' \twoheadrightarrow M'^{s#}$.

---

20The word “tight” in this paper is an adjective applying in many not necessarily related contexts. In particular, $B = a$ is always tight over $a$, but $a$ is not necessarily a tightly graded algebra—which means that it is generated by $a_0$ and $a_1$. 

Proof. By definition, $M \cong N \oplus P$, where $N$ is linear of degree $m$ and $P$ is a graded projective $a$-module generated in grades $< m$. Of course, $M^{< m}$ is also generated in grades $< m$, hence projects to 0 in $N$, which has $N_s = 0$ for $s < m$. Therefore, $M^{< m} = P$, $N \cong M^m$, and $M \cong M^m \oplus M^{< m}$ with $M^{< m} = P$. The projective module $P$ qualifies as a graded semilinear $a$-module of degree $m - 1$, so the process can be repeated, obtaining $M^{< n} \cong M^m \oplus M^{< m}$ with $M^{< m}$ projective, etc. This proves (a).

Finally, (b) follows from the discussion preceding the statement of the lemma.  

Remarks 8.4. We have implicitly assumed that projective covers exist in the category of graded $B$-modules, for any positively graded algebra $B$. We will elaborate on this a little.

(a) First, consider the case of the category $B$-mod of ungraded $B$-modules. Observe that the exact restriction functor $B$-mod $\rightarrow B_0$-mod has a right exact left adjoint $B \otimes_{B_0} -$. Thus, if $P$ be any projective $B_0$-module, then $B \otimes_{B_0} P$ is a projective $B$-module. Every projective $B$-module has this form. In fact, the irreducible $B$-modules naturally identify with the irreducible $B_0$-modules. If $L$ is an irreducible $B_0$-module with projective cover $P$ in $B_0$-mod, then $B \otimes_{B_0} P$ is the projective cover of $L$ regarded as an $B$-module.

(b) Second, a similar construction works at the graded level. First, regard $B_0$ as a positively graded algebra concentrated in grade 0. The graded projective $B_0$-modules are just projective $B_0$-modules $P$ equipped with a direct sum decomposition $P = \bigoplus_{i \in \mathbb{Z}} P_i$ in $B_0$-mod, with $P_i$ viewed as a graded $B_0$-module concentrated in grade $i$. In this way, $P$ is a graded $B_0$-module. Again, the exact restriction functor $B$-$\mathrm{gr}$-mod $\rightarrow B_0$-$\mathrm{gr}$-mod has right exact left adjoint $B \otimes_{B_0} -$. In fact, if $X = \bigoplus_{i \in \mathbb{Z}} X_i \in B_0$-$\mathrm{gr}$-mod, then $(B \otimes_{B_0} X)_j := \bigoplus_{i \in \mathbb{Z}} B_i \otimes_{B_0} X_{j-i}$, for each $j \in \mathbb{Z}$, defines $B \otimes_{B_0} X$ as a graded $B$-module. If $X = P$ is projective in $B_0$-$\mathrm{gr}$-mod, then $R := B \otimes_{B_0} P$ is projective in $B$-$\mathrm{gr}$-mod. We have

$$R = \bigoplus_{s \in \mathbb{Z}} R^s$$

and $R^s \cong B \otimes_{B_0} P_s$, $(s \in \mathbb{Z})$.

If $R \rightarrow N$ is a homomorphism in $B$-$\mathrm{gr}$-mod, then the image of $R^s$ is contained in $N^{s \leq s}$. Moreover, $R \rightarrow N$ is surjective if and only if all the composite maps $R^s \rightarrow N^{s \leq s} \rightarrow N^s$ are surjective. If $P$ is the projective cover in $B_0$-$\mathrm{gr}$-mod of $\mathrm{hd}^p N = \bigoplus_{s \in \mathbb{Z}} \mathrm{hd}^p N^s$, then $R = B \otimes_{B_0} P$ is the projective cover of $N$ in $B$-$\mathrm{gr}$-mod. So each $R^s \rightarrow N^s$ is surjective in this case. While $R^s$ is a direct summand of $R$, the module $N^s$ is, in general, only a section of $N$. Finally, forgetting the gradings, $R$ is the projective cover of $\hat{N}$ in $B$-mod.

We now state the main theorem of this subsection.

Theorem 8.5. Suppose that $a \rightarrow B$ is morphism of positively graded algebras such that $B$ is tight over $a$. Let $N$ be a graded $B$-module such that $N|_a$ is semilinear of degree $m$. Suppose there is given a projective $B$-module $P$ such that $P|_a$ is also projective and such that there is a surjection $P \rightarrow N$ in $B$-mod. Then the following statements hold:

(a) Let $R \rightarrow N$ be the projective cover in the category $B$-$\mathrm{gr}$-mod. Then $R|_a$ projective in $a$-$\mathrm{gr}$-mod.
(b) In the short exact sequence $0 \to E \to R \to N$ in $B\text{-grmod}$, $E := \ker(R \to N)$ is semilinear of degree $m + 1$ in $a\text{-grmod}$. 
(c) The graded $B$-modules $E^{m+1}$ and $N^m$, when restricted to $a$, are linear of degrees $m + 1$ and $m$, respectively.
(d) There is, up to isomorphism, a unique graded $B$-module $P'$ for which there is a graded $B$-module homomorphism $P' \to N^m$ becoming a projective cover upon restriction to $a$. The kernel of this map is isomorphic to $E^{m+1}$, and the resulting short exact sequence $0 \to E^{m+1} \to P' \to N^m \to 0$ in $B\text{-grmod}$ is unique up to isomorphism (assuming $P'|_a$ is projective).
(e) The short exact sequences in (b) and (d) (in $B\text{-grmod}$) fit into a commutative diagram with exact rows and natural surjective vertical maps:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & E^{m+1} & \longrightarrow & P' & \longrightarrow & N^m & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & R^m & \longrightarrow & N^m & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \longrightarrow & R & \longrightarrow & N & \longrightarrow & 0.
\end{array}
\]

Proof. Consider (a). First, $R$ is the projective cover of $N$ in $B\text{-mod}$, so $R$ is a $B$-direct summand of $P$. Since $P'|_a$ is projective, we conclude that $R|_a$ is projective in $a\text{-mod}$. Hence, it is projective as a graded $a$-module.

For parts (b)—(d), by Remark 8.4, $R = \bigoplus_{s \in \mathbb{Z}} R^s$, and, in this case, $R^s = 0$ if $s > m$ (the semilinearity degree of $N$). In addition to (a), this is the main property of $R$ that will be needed.

Now we prove (c). Observe, by Remark 8.4, the surjection $R \to N$ induces surjections $R^s \to N^s$, for all $s$. Also, because $N$ is semilinear of degree $m$, $N|_a = (N|_a)^m \oplus (N|_a)^{m+1}$. Also, $(N|_a)^m = \bigoplus_{s < m} N^s$.

First, let $X := \ker(R^m \to N^m)$. The module $N^m|_a$ is linear by Lemma 8.3 and the map from $R^m$ is surjective. So $X|_a$ must be the direct sum of a linear module of degree $m + 1$ and a graded projective $a$-module, the latter a summand of $R^m$. (This is a standard argument using minimal projective covers in $a\text{-grmod}$, and it is left to the reader.) All summands of $R^m$ are generated in grade $m$, so that $X|_a \cong (X|_a)^{m+1} \oplus (X|_a)^m$, and $(X|_a)^{m+1}$ is projective in $a\text{-grmod}$.

Second, let $Y = \ker(R^m \to N^{<m})$. As noted above, $N|_a = (N|_a)^{m} \oplus (N|_a)^{<m}$. Clearly, $Y|_a$ is a direct sum of projective modules generated in grades $< m$.

However, the given surjection $R \to N$ in $B\text{-grmod}$ need not be the direct sum of above surjections $R^m \to N^m$ and $R^{<m} \to N^{<m}$, i.e., there is a (possibly) different graded $a$-module surjection $R|_a \to N|_a$. But Schanuel’s lemma and the Krull-Schmidt theorem in $a\text{-grmod}$, $X|_a \oplus Y|_a \cong E|_a$. Consequently, $E|_a$ is semilinear of degree $m + 1$. This proves (b). We also obtain that $X|_a$ is semilinear of degree $m + 1$. 

By the above decomposition of $E|_a$ and of $X|_a$, together with Lemma 8.3, $E^{#m+1}|_a \cong X^{#m+1}|_a$ is linear of degree $m + 1$, and that $N^{#m}|_a \cong (N|_a)^{#m}$ is linear of degree $m$. This proves (c).

To prove (d), observe that $X^m$ is a $B$-grmod submodule of $R^{#m}$, and, as noted above (with the proof left to the reader), an $a$-summand of $R^{#m}|_a$. In fact, the same analysis shows that the inclusion $X^m \subseteq R^{#m}$ is split upon restriction to $a$, with $P' := R^{#m}/X^m$ projective upon restriction to $a$. This gives the existence of an exact sequence $0 \to E^{#m+1} \to P' \to N^{#m} \to 0$ as required in the existence part of (d). Here we have used the identifications of graded $B$-modules

$$E^{#m+1} = E/E^{<m} \cong (E/E^{<m})/(E^{\leq m}/E^{<m})$$

$$\cong (E/E^{<m})/(E/E^{<m})^{\leq m}$$

$$\cong X/X^{\leq m} = X^{#m+1}.$$

Next, suppose that $P^\dagger \to N^{#m}$ is any surjection in $B$-grmod with $P^\dagger|_a \to N^{#m}|_a$ a projective cover. Then $P^\dagger|_a$ is generated in grade $m$, so there a commutative diagram (in $B$-grmod)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega & \longrightarrow & P^\dagger & \longrightarrow & N^{#m} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & R^{#m} & \longrightarrow & N^{#m} & \longrightarrow & 0
\end{array}
$$

with horizontal rows exact. The middle vertical map arises from the projectivity of $R \in B$-grmod, the fact that $P^\dagger$ is generated in grade $m$, and the description $R^{#m} = R^{\leq m}/R^{<m}$. This middle vertical map is surjective by Nakayama’s lemma. (Note that $P^\dagger|_a \to N^{#m}|_a$ is a projective cover as an ungraded map, whether given as a graded or ungraded cover, by Remark 8.4.) The module $\Omega|_a$, as a first syzygy, in a minimum graded projective resolution of $N^{#m}|_a$, is necessarily linear of degree $m + 1$. So the vertical map $X|_a \to \Omega$ must kill $X^m$. Thus, there is an induced commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega & \longrightarrow & P^\dagger & \longrightarrow & N^{#m} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & | & & | \\
0 & \longrightarrow & X^{#m+1} & \longrightarrow & P' & \longrightarrow & N^{#m} & \longrightarrow & 0,
\end{array}
$$

where $P' = R^{#m}/X^m$ is as constructed above. Since $P'|_a$ and $P^\dagger|_a$, as projective covers of $N^{#m}|_a$, both have the same dimension, the surjective middle vertical map is an isomorphism. We have already identified $X^{#m+1} \cong E^{#m+1}$, so $\Omega \cong E^{#m+1}$. If we are given any exact sequence $0 \to E^{#m+1} \to P'' \to N^{#m} \to 0$ with $P''|_a$ projective, then $P''$ has the same dimension as $P'$, so the above argument gives both an isomorphism $P'' \cong P'$, and a similar isomorphism of exact sequences with end terms $E^{#m+1}$ and $N^{#m}$. This proves (d).
Finally, (e) is easily obtained from the descriptions of \( X = \ker(R^m \to N^m) \), and \( P' = R^m/X^m \) in the discussion above. The map \( E \to X \) is surjective by a snake lemma argument. (Note that \( R^m \to N^m \) is surjective.)

8.2 Gradings induced by graded subalgebras. An important case occurs when the grading of the algebra \( B \) results from a filtration of another algebra \( A \), induced by a sufficiently “normal” graded subalgebra \( \mathfrak{a} \). More precisely, throughout this subsection, the following conditions are in force:

(i) \( \mathfrak{a} \) is positively graded and \( \mathfrak{a} \to A \) is a homomorphism of algebras.

(ii) For each \( j \geq 0 \), put \( \mathfrak{a}_{\geq j} := \bigoplus_{i \geq j} \mathfrak{a}_i \). Then \( \mathfrak{a}_{\geq j} A \) is required to be an ideal in \( A \). That is, \( A\mathfrak{a}_{\geq j} A = \mathfrak{a}_{\geq j} A \). (In applications, \( A\mathfrak{a}_{\geq 1} A = \mathfrak{a} \).)

(iii) Define \( B = \tilde{gr} A := \bigoplus_{j \geq 0} \mathfrak{a}_{\geq j} A/\mathfrak{a}_{\geq j+1} A \).

Condition (ii) implies that the algebra \( B \) defined above is positively graded. There is a graded morphism \( \mathfrak{a} \to B \) such that \( \mathfrak{a}_{\geq j} B_0 = B_j \), for each \( j \geq 0 \). That is, \( B \) is tight over \( \mathfrak{a} \), as per Definition 8.2. In most applications, the map \( \mathfrak{a} \to B \) will be an inclusion.

Every \( A \)-module \( M \) is naturally an \( \mathfrak{a} \)-module, so the \( \mathfrak{a} \)-modules \( \text{hd}^\mathfrak{a} M \) and \( \text{rad}^\mathfrak{a} M \) are defined, using Definition 8.1. By (ii), they are also modules for \( A/\mathfrak{a}_{\geq 1} A = B_0 \).

Definition 8.6. An \( A \)-module equipped with a fixed graded \( \mathfrak{a} \)-module structure will be called hybrid. Morphisms of hybrid \( A \)-modules are just morphisms of \( A \)-modules which preserve the given \( \mathfrak{a} \)-gradings.

The hybrid \( A \)-modules form an abelian category, exactly embedded in the category of \( A \)-modules. A hybrid \( A \)-module \( N \) is admissible if each subspace \( N_{\geq j} := \bigoplus_{i \geq j} N_i \) is an \( A \)-submodule. The admissible objects form a full abelian subcategory of the category of hybrid \( A \)-modules. Given an admissible hybrid \( A \)-module \( N \), one can form a graded \( B \)-module

\[ \tilde{\text{Gr}} N := \bigoplus_{j \in \mathbb{Z}} N_{\geq j}/N_{\geq j+1}. \]

Here the capitalizing \( \tilde{\text{Gr}} \) is used to help distinguish this module from

\[ \tilde{\text{gr}} N := \bigoplus_{j \geq 0} \mathfrak{a}_{\geq j} N/\mathfrak{a}_{\geq j+1} N \]

defined in (1.1.3).

The category of hybrid \( A \)-modules is equipped with natural grade shifting functors \( N \mapsto N(\mathfrak{r}) \), for every \( \mathfrak{r} \in \mathbb{Z} \). Recall from §1.1 that \( N(\mathfrak{r})_j := N_{j-\mathfrak{r}} \). If \( N \) is admissible, so is \( N(\mathfrak{r}) \), and

\[ \tilde{\text{Gr}} N(\mathfrak{r}) = (\tilde{\text{Gr}} N)(\mathfrak{r}), \quad \mathfrak{r} \in \mathbb{Z}. \]

Finally,

\[ (\tilde{\text{Gr}} N)|_\mathfrak{a} \cong N|_\mathfrak{a}, \quad \text{in } \mathfrak{a}-\text{grmod}. \]
We now construct some admissible hybrid modules. Suppose that \( R \) is an \( A \)-module equipped with a decreasing filtration by \( A \)-submodules \( \{i^1R\}, i \in \mathbb{Z} \); thus, \( \cdots \supset i^0R \supset i^1R \supset \cdots \). Assume that \( i^1R/i^0R \) is projective as an \( a \)-module, for each \( i \). Also, assume that \( i^1R = R \), for \( i \) sufficiently small, and \( i^1R = 0 \), for \( i \) sufficiently large. In particular, each \( i^1R \) is projective as an \( a \)-module and has a decomposition \( i^1R = i^1R/i^0R \oplus i^0R \) as a direct sum of projective modules. Choose any \( a_0 \)-stable complement \( h_i \) to \( i^1R + \operatorname{rad}^b(i^1R) \) in \( i^1R \). (Such an \( h_i \) exists because all projective \( a \)-modules \( X \) may be given a grading \( X \cong a \otimes a_0 \operatorname{hd}^bX \) corresponding to any \( a_0 \)-grading of \( \operatorname{hd}^bX \).) Then \( h := \sum_{i \in \mathbb{Z}} h_i \cong \bigoplus_{i \in \mathbb{Z}} h_i \) is an \( a_0 \)-submodule of \( R \) and a complement to \( \operatorname{rad}^bR \). As an \( a \)-module,

\[
R = ah \cong a \otimes a_0 h \cong \bigoplus_i a \otimes a_0 h_i.
\]

We now give \( R \) an \( a \)-grading by assigning each \( h_i \) grade \( i \). The resulting hybrid structure on \( R \) is admissible, since

\[
R_{\geq s} = sR + a_{\geq 1}(s^1R) + a_{\geq 2}(s^2R) + \cdots.
\]

The flexibility to choose the \( a_0 \)-submodules \( h_i \) generating the \( a \)-grading is quite useful.

**Proposition 8.7.** Let the \( A \)-module \( R \) have a decreasing filtration \( \{i^1R\}_{i \in \mathbb{Z}} \) as above. Suppose that \( N \) is an admissible hybrid \( A \)-module, and \( \phi : R \to N \) is a surjection of \( A \)-modules such that \( \phi(i^1R) = N_{\geq i} \), for each \( i \in \mathbb{Z} \). Then there is a choice of \( a_0 \)-submodules \( h_i \) so that, as above, \( h_i \) is an \( a_0 \)-stable complement to \( i^1R + \operatorname{rad}^b(i^1R) \) in \( i^1R \), and, additionally, \( \phi(h_i) \subseteq N_i \) (\( i \in \mathbb{Z} \)). The induced \( a \)-grading on \( R \) gives \( R \) an admissible hybrid \( A \)-module and \( \phi \) becomes a surjective homomorphism of admissible hybrid \( A \)-modules.

**Proof.** The proposition is trivial if \( R = 0 \), in which case just take all \( h_i = 0 \). We may, thus, proceed by induction on \( \dim R \). Let \( m \in \mathbb{Z} \) be maximal with \( m^1R = R \). Thus \( m^1R \nsubseteq R \). So, the proposition holds, by induction, when \( m^1R, N_{\geq m+1} \), and \( \phi|_{m^1R} \) play the roles of \( R \), \( N \), and \( \phi \), respectively. This gives \( h_{m+1}, h_{m+2}, \cdots \) contained in \( m^1R \), \( m^2R, \cdots \), respectively, such that each \( h_i \) is an \( a_0 \)-stable complement to \( i^1R + \operatorname{rad}^b(i^1R) \) in \( i^1R \) (\( i \geq m + 1 \)). We need to find an \( h_j \) for \( j = m \) with this property.

Put \( S = \phi^{-1}(N_m) \). Then \( \phi(S + m^1R) = N_{\geq m} = \phi(m^1R) = \phi(R) \). Since \( \ker \phi \subseteq S \subseteq S + m^1R \), we must have \( S + m^1R = R = m^1R \). The surjection

\[
S \to m^1R/m^1R \to \operatorname{hd}^b(m^1R/m^1R)
\]

is \( a_0 \)-split, since the projective \( a \)-module \( X := (m^1R/m^1R) \) has \( \operatorname{hd}^bX = X/a_{\geq 1}X \) as a natural projective \( a_0 \cong a/a_{\geq 1} \)-quotient module. Let \( h_m \) be the image in \( S \) of the splitting. By construction, the image

\[
(h_m + m^1R + \operatorname{rad}^b m^1R) / (m^1R + \operatorname{rad}^b m^1R)
\]

of \( h_m \) under the map

\[
S \subseteq m^1R \to m^1R/m^1R \to \operatorname{hd}^b(m^1R/m^1R) = m^1R/(m^1R + \operatorname{rad}^b m^1R)
\]
Corollary 8.8. Assume that each \( \phi \) is an \( a \)-graded projective \( a \)-module, where \( h = \sum_{j \in \mathbb{Z}} h_j = \bigoplus_{j \in \mathbb{Z}} h_j \). This proves the proposition, since \( \phi(h_m) \subseteq \phi(S) \subseteq N_m \). This proves the proposition, since \( \phi \) now becomes an \( a \)-graded projective \( a \)-module on the constructed \( a \)-grading of \( R \). (Recall that \( R = ah \cong a \otimes_{a_0} h \) as an \( a \)-module, where \( h = \sum_{j \in \mathbb{Z}} h_j = \bigoplus_{j \in \mathbb{Z}} h_j \).

\[ \square \]

**Corollary 8.8.** Let \( N \) be an admissible hybrid \( A \)-module, and suppose, for each \( s \in \mathbb{Z} \), there is a projective \( a \)-module \( \#^s R \) and a surjection

\[ \phi_s : \#^s R \twoheadrightarrow N_{\geq s}/N_{\geq s+1}. \]

Assume that each \( \#^s R|_a \) is projective and that \( \#^s R = 0 \), for \( |s| \gg 0 \). Lift each \( \phi_s \) in any way to a \( A \)-module homomorphism \( \phi_{\geq 0} : \#^s R \to N_{\geq s} \). Put \( R := \bigoplus_{s \in \mathbb{Z}} \#^s R \) and let \( \phi : R \to N \) denote the sum of the maps \( \phi_{\geq s} \).

Then \( R \) has the structure of an admissible hybrid \( A \)-module in such a way that \( \phi \) becomes a surjective homomorphism of admissible hybrid \( a \)-modules. In particular, if \( E = \ker \phi \), then \( E \) is admissible, and

\[ 0 \to E \longrightarrow R \longrightarrow N \to 0 \]

remains exact upon applying the functors \(-|_a \) and \( \widetilde{\text{Gr}} \), giving graded exact sequences in each case (in \( a \)-grmod and \( B \)-grmod, respectively). The modules \( R|_a \) and \( \widetilde{\text{Gr}} R \) are projective in \( a \)-grmod and \( B \)-grmod, respectively.

**Proof.** Put \( ^{i}R = \bigoplus_{s \geq j} \#^s R \subseteq R \), for each \( j \in \mathbb{Z} \). The hypotheses of Proposition 8.7 are then satisfied. So there are admissible hybrid \( A \)-module structures on the object \( R \) and the morphism \( \phi \), required in the first assertion. The second assertion is just a property of all exact sequences in the category of admissible hybrid \( A \)-modules, and has been essentially previously noted below Definition 8.7 (and is obvious, in any case). The final assertion, regarding graded projectivity, follows from the following isomorphisms of graded \( B \)-modules:

\[ \widetilde{\text{Gr}}(\#^s R)/\widetilde{\text{Gr}}(\#^{s+1} R) \cong (\#^s R)(s), \quad s \in \mathbb{Z}, \]

which is easily obtained by inspecting the construction. The right hand side is clearly projective both as a graded \( B \)-module and as a graded \( a \)-module. This completes the proof. \( \square \)

**Remark 8.9.** We can sometimes trim some of the terms \( \#^s R \) from \( R \). Let \( m \) be an integer such that \( N_{\geq m} = aN_m \). (If \( a \) is tightly graded—that is, if \( a \) is generated by \( a_0 \) and \( a_1 \)—and if \( N|_a \) is semilinear of degree \( m \), then this equality holds.) In this case, we do not need any \( \#^s R \) with \( s > m \), and we may redefine \( \#^s R = 0 \) and \( \phi_s = 0 \) in the definition of \( R \) and \( \phi \), ignoring the requirements in the hypothesis of Corollary 8.8 and Proposition 8.7, so that \( \phi(^i R) = N_{\geq i} \) is assumed only for \( i \leq m \) with \( ^i R = 0 \) assumed for \( i > m \). The modified analogue of Proposition 8.7 is proved essentially as above, but beginning the argument by observing, for \( S := \phi^{-1}(N_m) \cap ^m R \),

\[ \phi(S + a_{\geq 1}^m R) = N_m + a_{\geq 1}N_{\geq m} = N_{\geq m} = \phi(^m R). \]
The revised corollary then follows as before from the modified proposition. For the convenience of the reader, we state these two results as Proposition 8.10 and Corollary 8.11, without further details of their proofs.

**Proposition 8.10.** Let the \( A \)-module \( R \) have a decreasing filtration \( \{^i R\}_{i \in \mathbb{Z}} \) as above the statement of Proposition 8.7. Suppose that \( N \) is an admissible hybrid \( A \)-module, and \( m \) is an integer with \( aN_m = N_{\geq m} \). Suppose \( \phi : R \rightarrow N \) is a surjection of \( A \)-modules such that \( \phi(^i R) = N_{\geq i} \), for each \( i \in \mathbb{Z} \) with \( i \leq m \), and \( ^i R = 0 \), for \( i > m \). Then there is a choice of \( a_0 \)-submodules \( h_i \) so that \( h_i \) is an \( a_0 \)-stable complement to \( ^{i+1} R + \text{rad}^a_i R \) in \( ^i R \), and, additionally, \( \phi(h_i) \subseteq N_i \) (\( i \in \mathbb{Z} \)). The induced \( a \)-grading on \( R \) gives \( R \) an admissible hybrid structure and \( \phi \) becomes a surjective homomorphism of admissible hybrid \( A \)-modules.

**Corollary 8.11.** Let \( N \) be an admissible hybrid \( A \)-module, and let \( m \in \mathbb{Z} \) be such that \( aN_m = N_{\geq m} \). Suppose, for each \( s \leq m \), there is given a projective \( A \)-module \( #^s R \) and a surjection \( \phi_s : #^s R \rightarrow N_{\geq s} / N_{\geq s+1} \). Assume that \( #^s R|_a \) is projective and that \( #^s R = 0 \), for \( |s| \gg 0 \) (or \( s > m \)). Lift each \( \phi_s \) in any way to an \( A \)-module homomorphism \( \phi_{\geq s} : #^s R \rightarrow N_{\geq s} \). Put \( R := \bigoplus_{s \in \mathbb{Z}} #^s R \) and let \( \phi : R \rightarrow N \) denote the sum of the maps \( \phi_{\geq s} \).

Then \( R \) has the structure of an admissible hybrid \( A \)-module in such a way that \( \phi \) becomes a surjective homomorphism of admissible hybrid \( a \)-modules. In particular, if \( E = \ker \phi \), then \( E \) is admissible, and

\[
0 \rightarrow E \rightarrow R \rightarrow N \rightarrow 0
\]

remains exact upon applying the functors \(-|_a\) and \( \widetilde{\text{Gr}} \), giving graded exact sequences in each case (in \( a\text{-grmod} \) and \( B\text{-grmod} \), respectively). The modules \( R|_a \) and \( \widetilde{\text{Gr}} R \) are projective in \( a\text{-grmod} \) and \( B\text{-grmod} \), respectively.

We reformulate these latter conclusions in part (a) of the proposition below. The hypotheses of Corollary 8.11 above are assumed, as they are in Theorem 8.13.

**Proposition 8.12.** Let \( N \) be an admissible hybrid \( A \)-module. Let \( \phi : R \rightarrow N \) be the morphism of admissible hybrid \( A \)-modules constructed above, with \( R = \bigoplus #^s R \). (In particular, \( R \) is \( A \)-projective.) Let \( E = \ker(\phi) \) and form the exact sequence

\[
(8.2.1) \quad 0 \rightarrow E \rightarrow R \rightarrow N \rightarrow 0
\]

in the category of admissible hybrid \( A \)-modules, as in Corollary 8.11.

(a) The modules \( R, R|_a, \widetilde{\text{Gr}} R \) are projective in \( A\text{-mod}, a\text{-grmod}, \) and \( \widetilde{\text{gr}} A\text{-grmod}, \) respectively. Further, \((8.2.1)\) gives rise to three exact sequences

\[
\begin{align*}
0 \rightarrow E|_a \rightarrow R|_a \rightarrow N|_a & \rightarrow 0; \\
0 \rightarrow \widetilde{\text{Gr}} E \rightarrow \widetilde{\text{Gr}} R \rightarrow \widetilde{\text{Gr}} N & \rightarrow 0
\end{align*}
\]
in the categories \(A\text{-mod}\), \(\mathfrak{a}\text{-grmod}\), and \(B\text{-grmod}\), respectively.

(b) In addition, if \(N|_{\mathfrak{a}}\) is semilinear of degree \(m\) and if \(\mathfrak{a}\) is Koszul (or just tight—generated by \(a_0\) and \(a_1\)), then \(E|_{\mathfrak{a}}\) is also semilinear, of degree \(m + 1\) (as is \(\widetilde{\text{Gr}} E|_{\mathfrak{a}} \cong E|_{\mathfrak{a}}\)). Let \(E', N'\) denote the maximal linear quotients of \(\text{Gr} E\) and \(\text{Gr} N\) of degrees \(m + 1\) and \(m\), respectively. Then there is an induced exact sequence

\[
0 \rightarrow E' \rightarrow R' \rightarrow N' \rightarrow 0
\]

in \(B\text{-grmod}\). Here \(R' = \widetilde{\text{Gr}} R/X\), where \(X\) is the image in \(\widetilde{\text{Gr}} R\) of \(\ker(\widetilde{\text{Gr}} E \rightarrow E')\). Also, \(R'|_{\mathfrak{a}}\) is projective in \(\mathfrak{a}\text{-grmod}\).

Proof. Everything except the last assertion has been outlined in the Remark 8.9. For that last assertion, the argument in the proof of Theorem 8.5(d) may be used. \(\square\)

Before stating the second main theorem of this section, note that, for any admissible hybrid \(A\)-module \(X\), there is an obvious ungraded isomorphism

\[
\text{hd}^p(\widetilde{\text{Gr}} X) \cong \text{hd}^p X \quad \text{in } A/\mathfrak{a}_{\geq 1} A\text{-mod}.
\]

The algebra \(A/\mathfrak{a}_{\geq 1} A\) is \((\text{gr} A)_0\), by definition. There is even a natural isomorphism of graded \((\text{gr} A)_{0}\text{-modules}

\[
\text{hd}^p(\widetilde{\text{Gr}} X) \cong \widetilde{\text{Gr}} (\text{hd}^p X).
\]

**Theorem 8.13.** Let \(0 \rightarrow E \rightarrow R \rightarrow N \rightarrow 0\) be as in (8.2.1) and let \(V\) be any \(B_0 = A/\mathfrak{a}_{\geq 1} A\text{-module}. Then the following statements hold.

(a) There is a natural isomorphism

\[
(8.2.3) \quad \text{coker}(\text{Hom}_A(R, V) \rightarrow \text{Hom}_A(E, V)) \cong \text{coker}(\text{Hom}_B(\widetilde{\text{Gr}} R, V) \rightarrow \text{Hom}_B(\widetilde{\text{Gr}} E, V)).
\]

(b) If \(\mathfrak{a}\) is tightly graded, if \(N|_{\mathfrak{a}}\) is semilinear of degree \(m\), and if \(0 \rightarrow E' \rightarrow R' \rightarrow N' \rightarrow 0\) is as in (8.2.2), then there is a natural isomorphism

\[
\text{Hom}_\mathfrak{a}(E', V) \cong \text{coker}(\text{Hom}_\mathfrak{a}(R, V) \rightarrow \text{Hom}_\mathfrak{a}(E, V))
\]

of vector spaces induced by the quotient maps \(\widetilde{\text{Gr}} E \rightarrow E', \widetilde{\text{Gr}} R \rightarrow R'\), together with the analogue of (8.2.3) for \(\mathfrak{a}\).

Proof. Assertion (a) is a consequence of the natural isomorphisms

\[
\text{Hom}_A(X, V) \cong \text{Hom}_{A/\mathfrak{a}_{\geq 1} A}(\text{hd}^p X, V)
\]

\[
\cong (\text{Hom}_{(\text{gr} A)_0}(\text{hd}^p X, V)
\]

\[
\cong \text{Hom}_B(\text{Gr} X, V),
\]

for all admissible hybrid \(A\)-modules \(X\).

For (b), apply Theorem 8.5 (though not with the same notation). First, as above,

\[
\text{coker}(\text{Hom}_\mathfrak{a}(R, V) \rightarrow \text{Hom}_\mathfrak{a}(E, M)) \cong \text{coker}(\text{Hom}_\mathfrak{a}(\text{Gr} R, V) \rightarrow \text{Hom}_\mathfrak{a}(\text{Gr} E, M)).
\]
Next, we construct a projective cover $P \to \widetilde{\gr} N$ in $\widetilde{\gr} A\text{-}\text{grmod}$, as in Remark 8.4, with $P = P(\text{hd}^{\text{deg}} \widetilde{\gr} N)$. Note that $P = \bigoplus_{s \in \mathbb{Z}} P^{s}$ by construction, using the notation of Remark 8.4, except that $P^{s}$ is the projective cover of $\text{hd}^{s} \widetilde{\gr} N$. Recall that $R$, constructed as in Remark 8.9, is generated in grades $\leq m$ over $\mathfrak{a}$, as is $\widetilde{\gr} R$ (over $\mathfrak{a}$ or over $\widetilde{\gr} A$). The surjection $\gr R \to \widetilde{\gr} N$ lifts to a split surjection $\widetilde{\gr} R \to P$ in $\widetilde{\gr} A\text{-}\text{grmod}$. Let $F$ be its kernel. Standard diagram arguments show that $\widetilde{\gr} E \cong F \oplus \ker(P \to \widetilde{\gr} N)$ in $\widetilde{\gr} A\text{-}\text{grmod}$ (and in $\text{grmod}$, consequently). By Theorem 8.5, $\ker(P \to \widetilde{\gr} N)$ is semilinear of degree $m + 1$. Clearly, $F|_{\mathfrak{a}}$ is projective in $\mathfrak{a}\text{-}\text{grmod}$ and is generated in grades $\leq m$ (properties inherited from $\widetilde{\gr} R$). Therefore, $\ker(P \to \widetilde{\gr} N)|_{\mathfrak{a}}$ and $(\widetilde{\gr} E)|_{\mathfrak{a}} \cong E|_{\mathfrak{a}}$ share the same maximal quotient $E'$ which is linear of degree $m + 1$. Also, $E'$ carries the same $\widetilde{\gr} A$-module structure from $\gr R$ as from $\ker(P \to \widetilde{\gr} N)$. Theorem 8.5 guarantees that there is, up to isomorphism, a unique exact sequence

$$0 \to E' \to P' \to N' \to 0 \quad \text{in} \quad \widetilde{\gr} A\text{-}\text{grmod},$$

with $N'$ the degree $m$ maximal linear quotient of $\widetilde{\gr} N$, $P'$ an object in $\widetilde{\gr} A\text{-}\text{grmod}$ with $P'|_{\mathfrak{a}}$ projective. The commutative diagram in Theorem 8.5(a) may now be used to produce a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & E' & \longrightarrow & P' & \longrightarrow & N' & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widetilde{\gr} E & \longrightarrow & \widetilde{\gr} R & \longrightarrow & \widetilde{\gr} N & \longrightarrow & 0
\end{array}
$$

in $\widetilde{\gr} A\text{-}\text{mod}$, with exact rows and with all vertical maps surjective. Both $\widetilde{\gr} R|_{\mathfrak{a}}$ and $P'|_{\mathfrak{a}}$ are projective in $\mathfrak{a}\text{-}\text{grmod}$, and $P$ is (consequently) generated in grades $\leq m$. So $P'_{\geq m+1} \subseteq a_{\geq 1}P'$. Thus,

$$\text{coker}((\text{Hom}\mathfrak{a}(P', V) \to \text{Hom}\mathfrak{a}(E', V))) \cong \text{coker}(\text{Hom}\mathfrak{a}(\text{hd}^{\text{deg}}(P', V) \to \text{Hom}\mathfrak{a}(\text{hd}^{\text{deg}}E', V)))$$

$$\cong \text{Hom}\mathfrak{a}(\text{hd}^{\text{deg}}E', V).$$

Finally, it is clear that the complex consisting of the bottom row is the direct sum of a complex of projective modules in $\mathfrak{a}\text{-}\text{grmod}$. So there is a natural isomorphism

$$\text{coker}(\text{Hom}\mathfrak{a}(P', V) \to \text{Hom}\mathfrak{a}(E', V)) \cong \text{coker}(\text{Hom}\mathfrak{a}(\widetilde{\gr} R, V) \to \text{Hom}\mathfrak{a}(\widetilde{\gr} E, V)).$$

Together with the identifications previously noted. This proves the second assertion of the theorem. \hfill \Box

Remark 8.14. The theory resulting from Proposition 8.12 and Theorem 8.13 is a recursive one, for the purpose of building resolutions one step at a time. The recursive design is made necessary by the hypothesis, appearing in Corollary 8.11 (and earlier), that requires sufficient projective $A$-modules exist with projective restrictions to $\mathfrak{a}$ to be able to make the constructions. In the situations we must deal with in Section 5, this generality requires enlarging the algebra $A$. We refer the reader to the proofs of Theorem 5.3 and Theorem
for cases where this can be done successfully, the latter requiring results of this section only through Corollary 8.8. The algebras \( A \) are various \( A_f \)'s and \( B = \tilde{\gr} A \). The algebra \( a \) is introduced in §2.5. The hypothesis in Theorem 8.5 that \( B \) is tight over \( a \) follows from the definition of \( \tilde{\gr} A \) and the tightness of \( \tilde{A} \) over \( \tilde{a} \).

9. Appendix II: Vanishing of Tor

This appendix proves the following general fact about integral quasi-hereditary algebras. Let \( \tilde{A} \) be an integral quasi-hereditary over \( \mathcal{O} \) with poset \( \Lambda \). Recall that given \( \lambda \in \Lambda \), \( \tilde{\Delta} (\lambda) \) is the standard left module defined by \( \lambda \). Let \( \tilde{\Delta} (\lambda)^o \) be the standard right module defined by \( \lambda \).

**Proposition 9.1.** Let \( \tilde{A} \) be a split quasi-hereditary algebra over \( \mathcal{O} \) with weight poset \( \Lambda \). For \( \lambda, \mu \in \Lambda \), we have

\[
\text{Tor}^\tilde{A}_n (\tilde{\Delta}(\lambda)^o, \tilde{\Delta}(\mu)) \cong \begin{cases} \mathcal{O} & \text{when } n = 0 \text{ and } \lambda = \mu; \\ 0 & \text{otherwise}. \end{cases} \quad \forall n \in \mathbb{N}.
\]

A similar result holds when \( \mathcal{O} \) is replaced by a field.

**Proof.** One can reduce easily to the case of a quasi-hereditary algebra \( A \) over a field \( F \). (The argument is similar to that used in [10, p. 5243].) For convenience, we assume that the poset \( \Lambda \) is linear.

First, consider the case \( n = 0 \). We assume that \( \lambda \geq \mu \) (and leave the other case to the reader). Without loss, we can replace \( A \) by a quotient quasi-hereditary \( B \) with \( \lambda \) maximal in the poset of \( B \). Thus, \( \Delta(\lambda)^o \) is a projective indecomposable module \( eB \) with \( e \) a primitive idempotent. Then \( \Delta(\lambda)^o \otimes_B \Delta(\mu) = eB \otimes_B \Delta(\mu) \cong e\Delta(\mu) \). If \( \lambda \neq \mu \), then \( e\Delta(\mu) = 0 \). If \( \lambda = \mu \), then \( e\Delta(\mu) = eBe = F \), since \( B \) is split. This completes the proof when \( n = 0 \).

Now assume that \( n > 0 \). Again, we treat only the case \( \mu \geq \lambda \), leaving the other case to the reader. Then then the projective indecomposable \( A \)-module \( P(\mu) \) has a \( \Delta \)-filtration with top section \( \Delta(\lambda) \) and lower sections of the form \( \Delta(\tau) \), for \( \tau > \mu \). Now an evident induction on \( \mu \) completes the proof. (Observe the assertion is trivial if \( \mu \) is maximal.) \( \Box \)

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