SPECIALIZATION OF POLYNOMIAL
COVERS OF PRIME DEGREE

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Abstract. Let $K$ be a complete field of unequal characteristics $(0,p)$. The aim of this paper is to describe the semi-stable models for covers $\mathbb{P}^1_K \rightarrow \mathbb{P}^1_K$ of degree $p$, unramified outside $r \leq p$ points and totally ramified above one of them, under the assumption that the ramification locus has a particular reduction type (which always occurs if $r \leq 4$). We are principally concerned with the minimal semi-stable models which separate the ramified fibers.

§0 Introduction

The study of the stable models for covers between algebraic curves over a complete discrete valuation ring is a subject which has been intensively developed during these last years. Roughly speaking, the problem can be summarized as follows: starting from a ramified cover $\beta : C \rightarrow D$ between non-singular projective curves defined over a complete discrete valued field $(K,v)$ of unequal characteristics, one would like to construct a semi-stable model $C \rightarrow D$ of this cover over its valuation ring $R$. The general theory, and in particular the semi-stable reduction theorem, asserts that such a model always exists, up to a finite extension of the base field (cf. [BLR] and [R1]). Its uniqueness is not ensured, but if we require that the model separates the ramified locus (that is, the ramification points specialize to pairwise distinct points), then there exists a minimal semi-stable model for the cover, which is unique, up to isomorphism. Moreover, this model behaves well under (finite) base change, so that it can be considered as an important birational invariant attached to the cover. In particular, there is a well defined notion of reduction type: the number of irreducible components of the special fiber, their intersection graph and the thicknesses of the singular points (cf. section 1).

Historically, the investigation started with the study of tame covers, i.e. covers such that the residual characteristic $p$ of $R$ does not divide the order of their monodromy group $G$. This first approach is somehow more simple, and one can use the results of [G], for example. The next case is where $p$ divides $|G|$. There are no complete results on this direction. Raynaud [R2] treats the case where $G$ has a $p$-Sylow subgroup of order $p$ and the cover is unramified outside three points. These results are sharpened by Wewers in [W]. Finally, Lehr [L] and Saïdi [S], study the stable model for a $p$-cyclic cover in full generality. In order to have a more complete reference on the subject, see also the results of Green-Matignon and Henrio.

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In this paper, we concentrate on the covers $\mathbb{P}^1 \to \mathbb{P}^1$ of prime degree $p$. The only restrictions we make on the cover are: there is at least one totally ramified point (wild ramification) and the branch locus has a particular reduction type (cf. section 5). Note that these covers need not to be Galois. It turns out that the stable models only depend on the ramification data and on the $v$-adic distance between the ramified points. There are four cases: if the branch locus has good reduction, we essentially recover Wewers’ results. The real interesting case is where the branch points have bad reduction. We choose two branch points $P$ and $Q$ and consider the semi-stable model $C \to D$ of the cover which separates these points. There are three cases: $P$ and $Q$ are $v$-adically far from each other, $P$ and $Q$ have medium distance from each other (the critical case) and $P$ and $Q$ are $v$-adically close to each other. In the first case, $P$ and $Q$ will specialize into two different irreducible components $C_1$ and $C_2$, and there will be only one irreducible component lying above each $C_i$. In the second case, $P$ and $Q$ will specialize in the same irreducible component $C$ and there will be only one component above $C$. In the last case, $P$ and $Q$ will again specialize into the same component $C$, but many irreducible components will lie over it.

The strategy is to start with a natural model for the cover $\mathbb{P}^1 \to \mathbb{P}^1$ over $R$ and then to blow it up several times until the ramification points are separated. This construction can be carried out explicitly because the covering curve is the projective line. This techniques can be carried over to the case where the degree of the cover is composite and even to the case where the curve $C$ has (potentially) good reduction, but we do not do it in this paper.

§1 Some notation and basic facts

Throughout this paper, $(K,v)$ will denote a complete valued field of unequal characteristics $(0,p)$. Its valuation ring $R$ is a complete discrete valuation ring with maximal ideal $p$ and residue field $k = \mathcal{O}_K/p$. The valuation $v : K^* \to \mathbb{Q}$ will be normalized by the condition $v(p) = 1$, so that it will extend the usual $p$-adic valuation. In particular $v(K^*) = e\mathbb{Z}$, where the positive integer $e$ is the absolute ramification index of $K$, that is $pR = p^e$.

Let $C$ be a projective, irreducible, non-singular curve of genus $g$ defined over $K$ and consider a finite subset $S$ of $C(K)$ of cardinality $n \geq 0$. The pair $(C,S)$ is called a pointed curve of type $(g,n)$. We will say that $(C,S)$ is hyperbolic if the inequality $2g - 2 + n > 0$ holds. A semi-stable model for $(C,S)$ over $R$ is a proper and flat scheme $\mathcal{C}/R$ satisfying the following conditions:

1. The generic fiber $\mathcal{C}_K = \mathcal{C} \otimes_R K$ is isomorphic to $C$.
2. The special fiber $\mathcal{C}_k = \mathcal{C} \otimes_R k$ is reduced and has only ordinary double points as singularities.
3. The elements of $S$ specialize to pairwise distinct non-singular points of $\mathcal{C}_k$.

If $S = \emptyset$, then this last condition must be ignored. We will say that $C$ is stable if, for any irreducible component $X$ of $\mathcal{C}_k$, we have the relation $2g(X) - 2 + n(X) + s(X) > 0$, where $g(X)$ denotes the genus of $X$, $n(X)$ is the cardinality of the subset of $S$ specializing in $X$ and $s(X)$ is the number of singular points of $X$. Semi-stable models behave well under base change, i.e. if $L/K$ is a finite extension, and $\mathcal{C}/R$ is a semi-stable model of $(C,S)$ over $R$, then $\mathcal{C} \otimes_R R'$ is a semi-stable model for $(C,S)$ over $R'$, where $R'$ is the ring of integers of $L$.

The semi-stable reduction theorem asserts that any pointed curve $(C,S)$ over
$K$ admits a semi-stable model over the valuation ring $R'$ of a finite extension $L$ of $K$. Moreover, if $(C, S)$ is hyperbolic, then there exists a stable model $C/R'$, which is unique, up to $R'$-isomorphism. The same kind of result holds for covers: more precisely, given a finite cover $\beta : C \to D$ of projective, non-singular curves over $K$ and a finite subset $S$ of $D(K)$, there exist a finite extension $L$ of $K$, two proper and flat schemes $C, D/R'$ and a finite morphism $C \to D$ such that the following conditions hold:

1. $C$ (resp. $D$) is a semi-stable model for the pointed curve $(C, \beta^{-1}(S))$ (resp. for the pointed curve $(D, S)$) over $R'$.
2. The generic morphism $C_L \to D_L$ is isomorphic to $\beta : C \to D$.
3. The specialized morphism $C_l \to D_l$ maps non-singular points to non-singular points (here, $l$ denotes the residue field of $L$).

In particular there exists a minimal semi-stable model satisfying the above properties, which is unique, up to $R'$-isomorphism. Since the behaviour of its special fiber is stable under finite base change, it can be considered as an important birational invariant of the cover. In this paper, we are not concerned with rationality questions, that’s why we will always suppose that the semi-stable model is already defined over $R$.

We will end this section by introducing an important invariant attached to a semi-stable model $C/R$: the completion of the local ring of $C$ at a singular point $P$ of its special fiber is isomorphic to the power series ring $R[X, Y]/(XY - Z)$, with $Z \in p$. The thickness of $P$ is the valuation of $Z$, which only depends on $P$. Moreover, our normalization for the valuation $v : K^* \to \mathbb{Q}$ implies that the thickness of a singular point does not change after a finite base change. Finally, if $\beta : C \to D$ is a semi-stable model for a cover and $P$ is a singular point of the special fiber $C_k$ of $C$ of thickness $\nu$, then $Q = \beta(P)$ will be a singular point of the special fiber $D_k$ of $D$, of thickness $e_P\nu$, where $e_P$ is the ramification index at $P$ of the specialized cover $C_k \to D_k$.

We refer to [BLR] for a complete introduction to the theory of semi-stable curves and their application to the study of algebraic covers of curves. Two fundamental papers on the subject are [R1] and [R2].

§2 Polynomial covers. Normalized models

The main purpose of this paper is to construct semi-stable models for polynomial covers of prime degree $p$ over $K$ (i.e. covers $\beta : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree $p$ and totally ramified above one point) under certain assumptions on the reduction of the branch locus. We are principally concerned with the minimal models $\mathcal{X} \to \mathcal{Y}$ which separates the ramification locus. From now on, we will say that two such covers $\beta_1$ and $\beta_2$ are isomorphic, or equivalent (over $K$) if there exist two elements $\sigma, \tau \in \text{PGL}_2(K)$ such that $\sigma \circ \beta_1 = \beta_2 \circ \tau$, i.e. the following diagram

$$
\begin{array}{ccc}
\mathbb{P}^1_K & \xrightarrow{\tau} & \mathbb{P}^1_K \\
\beta_1 \downarrow & & \beta_2 \downarrow \\
\mathbb{P}^1_K & \xrightarrow{\sigma} & \mathbb{P}^1_K 
\end{array}
$$

is commutative. In this case, we easily see that the special fibers of the stable models of the covers $\beta_1$ and $\beta_2$ have the same behaviour. We will start by constructing a
“good” model of the cover over $K$. Let’s first of all reduce to the affine case: since $\beta : \mathbb{P}_K^1 \to \mathbb{P}_K^1$ is totally ramified above one point, we obtain a finite morphism $\beta : \mathbb{A}_K^1 = \text{Spec}(K[X]) \to \mathbb{A}_K^1 = \text{Spec}(K[T])$.

In terms of affine algebras, it corresponds to an injection $\beta^* : K[T] \to K[X]$, which is uniquely determined by the image $\beta(X)$ of $T$, which is a polynomial (that’s why we are speaking about polynomial covers). From now on, by eventually enlarging $K$, we will assume that the ramification locus of $\beta$ is contained in $\mathbb{P}_K^1$. For any $\lambda \in K$, we have a unique factorization (in $K$)

$$\beta(X) - \lambda = c \prod_{x \in \beta^{-1}(\lambda)} (X - x)^{e_x}$$

where $c \in K$ does not depend on $\lambda$ and $e_x$ is the multiplicity of $X - x$ in the factorization of $\beta(X) - \lambda$. We clearly have

$$\sum_{x \in \beta^{-1}(\lambda)} e_x = p$$

Furthermore, the Riemann-Hurwitz formula gives

$$\sum_{\lambda \in S} n_\lambda = (r - 2)p + 1$$

where $r$ is the cardinality of the branch locus $B = S \cup \{\infty\}$ of the cover and $n_\lambda$ is the cardinality of the fiber $\beta^{-1}(\lambda)$. Up to equivalence, and after a finite extension of $K$, we can reduce to the following case:

1. The finite branch locus $S = \{\lambda_1, \ldots, \lambda_{r-1}\} = \beta(\{x \in K \mid \beta'(x) = 0\})$ is contained in $R$ and $\{0, 1\} \subset S$.
2. The polynomial $\beta(X)$ is monic (i.e. $c = 1$) and $\beta(0) = 0$.

A polynomial $\beta(X) \in K[X]$ satisfying the above conditions will be called a normalized model for the cover. One easily checks that there exist finitely many normalized models associated to the same polynomial cover. More generally, we will say that $\beta(X)$ is a semi-normalized model if it satisfies the condition (2) above and the following property (which is weaker than the condition (1)):

3. The finite branch locus $S$ is contained in $R$ and there exists $\lambda \in R^*$ such that $\{0, \lambda\} \subset S$.

Remark that there exists infinitely many semi-normalized models associated to the same cover.

§3 Construction of the fundamental model

We will now start the construction of the semi-stable model associated to a polynomial cover of degree $p$. For any polynomial $h(X) \in R[X]$, we will denote by $\overline{h}(X)$ its canonical image in $k[X]$. The importance of the (semi-)normalized model introduced in the previous paragraph is related to the following lemma:
3.1 Lemma. Let \( h(X) \in K[X] \) be a monic polynomial of degree \( p \) such that \( h(0) = 0 \). If the associated cover \( h : \mathbb{A}^1_K \to \mathbb{A}^1_K \) is unramified outside a finite set \( S \subset R \), then \( h(X) \in \overline{R[X]} \) and \( \overline{h}(X) = X^p \).

Proof. The case \( h(X) = X^p \) is immediate. If \( h(X) \neq X^p \) then its Newton polygon is not reduced to a vertical line. Suppose that the last segment has positive slope, i.e. that there exists a root \( x \) of \( h(X) \) with (minimal) negative valuation. The degree of \( h(X) \) is equal to \( p \) and \( h(0) = 0 \), so that its Newton polygon coincides with the Newton polygon associated to \( Xh'(X) \), except for the last segment (see the following picture).

In particular, we see that there is a root \( y \) of \( h'(X) \) (a ramified point) such that its valuation is strictly less than the valuation of \( x \) (and thus, of any root of \( h(X) \)). We obtain \( v(h(y)) = pv(y) < 0 \), which is absurd since \( h(y) \in S \subset R \). We then have \( h(X) \in \overline{R[X]} \) and since the (finite) branch points belong to \( R \), we see that all the roots of \( h'(X) \) belong to \( R \). The leading coefficient of \( h'(X) \) is equal to \( p \), so that \( h'(X) = pg(X) \), with \( g(X) \in \overline{R[X]} \). This gives \( \overline{h'}(X) = 0 \), which leads to \( \overline{h}(X) = X^p \), since \( h(X) \) is monic, of degree \( p \) and satisfies \( h(0) = 0 \). \( \square \)

If \( \beta(X) \in K[X] \) is a (semi-)normalized model associated to a polynomial cover of degree \( p \) (as defined above), then it satisfies the hypothesis of Lemma 3.1, so that it belongs to \( \overline{R[X]} \) and we have the identity \( \overline{\beta}(X) = X^p \). This will be our starting point for the construction of the semi-stable model associated to the cover.

Remark that our assumptions on \( K \) imply that, for any \( \lambda \in S \), the polynomial \( \beta(X) - \lambda \in K[X] \) totally splits. By construction, we moreover have \( \beta^{-1}(S) \subset \overline{R} \). Setting \( C_0 = \text{Spec}(\overline{R[X]}) \cong \mathbb{A}^1_{\overline{R}} \) and \( D_0 = \text{Spec}(\overline{R[T]}) \cong \mathbb{A}^1_{\overline{R}} \), we then obtain a morphism

\[ C_0 \to D_0 \]

In terms of affine algebras, it corresponds to the injection \( \overline{R[T]} \to \overline{R[X]} \) which sends \( T \) to \( \beta(X) \). It is a finite morphism, since the polynomial \( \beta(X) \) belongs to \( \overline{R[X]} \) and is monic. Let \( \gamma(X) \in \overline{R[X]} \) be the polynomial defined by the relation \( \gamma(X) = X^p \beta(X^{-1}) \). From a practical point of view, we have

\[ \gamma(X) = \prod_{x \in \beta^{-1}(0)} (1 - xX)^{e_x} \]

where \( e_x = e_x(\beta) \) is the multiplicity of \( X - x \) in the factorization of \( \beta(X) \). The relation \( \beta(0) = 0 \) implies that the degree of \( \gamma(X) \) is strictly less than \( p \). Moreover,
we have \( \gamma(0) = 1 \) and \( \gamma(X) = 1 \). Set \( \mathfrak{A} = R[T_\infty] \) and \( \mathfrak{B} = \mathfrak{A}[X_\infty]/(X_\infty'^{-1} - \gamma(X_\infty)T_\infty) \). The ring \( \mathfrak{B} \) is a finite \( \mathfrak{A} \)-module and thus, we obtain a finite morphism

\[
\mathcal{C}_1 \to \mathcal{D}_1
\]

where \( \mathcal{C}_1 = \text{Spec}(\mathfrak{B}) \) and \( \mathcal{D}_1 = \text{Spec}(\mathfrak{A}) \cong \mathbb{A}_R^1 \). One can easily check that the ring \( \mathfrak{B} \) is canonically isomorphic to \( R[X_\infty, \gamma(X_\infty)^{-1}] \). In particular, the generic fiber of \( \mathcal{C}_1 \) is isomorphic to \( \mathbb{P}^1_K - \beta^{-1}(0) \) and its special fiber is isomorphic to \( \mathbb{A}_k^1 \).

Consider now the affine scheme

\[
\mathcal{X}_V 
\]

we can consider the affine scheme

\[
\mathcal{C}_\mathcal{X}_V
\]

This is a semi-stable model for the cover over \( R \) (since its generic fiber is isomorphic to \( \beta \) over \( K \)) and we will call it the fundamental model associated to \( \beta \). Lemma 3.1 implies that its specialization is purely inseparable. In particular, for a given \( \lambda \in \mathcal{D}(K) \), the elements of \( \beta^{-1}(\lambda) \) all have the same specialization in the special fiber \( \mathcal{C}_k \) of \( \mathcal{C} \), so that this morphism will never separate the ramified fibers. A model \( \mathcal{X} \to \mathcal{Y} \) for \( \beta \) which separates the ramified fibers will be obtained from \( \mathcal{C} \to \mathcal{D} \) after a finite number of blow-ups. In order to get a minimal one, we may need to blow-down the original irreducible component \( \mathcal{C}_k \) (and \( \mathcal{D}_k \) of the special fiber of \( \mathcal{Y} \)). This last possibility will never occur. Indeed, this would mean that all the finite branch points have the same specialization in \( \mathcal{D} \), which is impossible, since we are assuming \( \{0, 1\} \in S \).

§4 Some complements on semi-stable models of genus zero

Before continuing, let’s make some general remarks. First of all, suppose that \( \mathcal{X}/R \) is any semi-stable model of a genus zero (pointed) curve and denote by \( \mathcal{X}_k \) its special fiber. We classically define the intersection graph \( \Gamma(\mathcal{X}_k) \) as the abstract graph whose vertices (resp. edges) correspond to the irreducible components of \( \mathcal{X}_k \) (resp. the singular points of \( \mathcal{X}_k \)).

In this special case, \( \Gamma(\mathcal{X}_k) \) is a tree, endowed with a metric, obtained by associating to any edge the thickness of the corresponding singular point. If \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \)
are two irreducible components of $\mathcal{X}_k$, we can consider the segment $[C_1, C_2]$, which is the minimal (linear) subtree of $\Gamma(\mathcal{X}_k)$ containing the vertices corresponding to $C_1$ and $C_2$. In particular, the distance $d(C_1, C_2)$ between $C_1$ and $C_2$ is defined as the sum of the lengths of the edges belonging to $[C_1, C_2]$. From a practical point of view, consider the model $\mathcal{X}'$ of the projective line over $R$ obtained by blowing down all the irreducible components of $\mathcal{X}_k$ different from $C_1$ and $C_2$. Then, the special fiber of $\mathcal{X}'$ is the union of $C_1$ and $C_2$, meeting at a unique singular point, of thickness $d(C_1, C_2)$.

Suppose now that $\mathcal{X} \rightarrow \mathcal{Y}$ is any semi-stable model over $R$ for a polynomial cover, obtained from its fundamental model $C \rightarrow D$ (cf. §3) after a finite number of blows-up. Denote by $C$ (resp. by $D$) the irreducible component of $\mathcal{X}_k$ (resp. of $\mathcal{Y}_k$) corresponding to the special fiber of $C$ (resp. of $D$). Let $C_1 \neq C$ be a tail of $C_k$ and denote by $D_1 \neq D$ its image in $D_k$. If $C_\infty$ is the irreducible component of $\mathcal{X}_k$ containing the specialization of the totally ramified point $\infty$, then we will suppose that the segments $[C, C_1]$ and $[C, C_\infty]$ have no common edges. Consider a point $P \in \mathcal{X}(K) = \mathbb{P}^1(K)$ specializing in a non-singular point of $C_1$. Since $\mathcal{X}(K) = \mathcal{C}(K)$, the point $P$ defines a well defined element $x$ of $\mathcal{C}(K) = \mathcal{C}(R)$. Moreover, the above assumption on the relative positions of $C$, $C_0$ and $C_\infty$ imply that $x$ belongs to $C_0(R) = \mathbb{A}^1(R)$ (cf. the end of §3) and thus, it can be assimilated to an element of $R$. Similarly the image of $P$ in $\mathcal{Y}$ defines an element $\lambda$ of $D_0(R) = R$. Let $\pi$ and $\pi'$ be two elements of $p$ such that $v(\pi) = d(C, C_1)$ and $v(\pi') = d(D, D_1)$. We then have a unique factorization

\[(*) \quad \beta(\pi X + x) - \lambda = \pi' \beta_0(X) \gamma_0(X)\]

where $\beta(X) \in R[X]$ is the normalized model from which the cover $C \rightarrow D$ was constructed, $\beta_0(X) \in R[X]$ is monic and $\gamma_0(X) \in R[X]$ satisfies $\gamma(X) \in k^*$, i.e. $\gamma_0(X) \in R^* + pR[X]$. Moreover, the finite morphism $A_1^k \rightarrow A_1^k$ obtained from $C_1 \rightarrow D_1$ by removing the singular points is (isomorphic to the one) induced by the inclusion $k[T] \rightarrow k[X]$ which maps $T$ to $\gamma_0(X)$. In particular, $C_1$ will be the only irreducible component of $\mathcal{X}_k$ lying above $D_1$ if and only if $\gamma_0(X)$ has degree 0, i.e. if $\gamma_0(X) = 1$. In this case, we get the relation $d(D, D_1) = pd(C, C_1)$.
§5 Simple reduction of the branch locus

We will now put some conditions on the reduction type of the branch locus $B = S \cup \{\infty\}$ of a polynomial cover over $K$. The curve $D$ of the fundamental model associated to it (cf. §2) is a model of the projective line over $R$ such that no point of $S$ specialize to $\infty$ and at least two points have distinct specializations. These two conditions uniquely determine $D$, up to $R$-isomorphism. The stable model $B$ (over $R$) for the pointed curve $(\mathbb{P}^1_K, B)$ is obtained from $D$ after a finite number of blows-up.

5.1 Definition. Consider the stable (separating) model $B$ associated to a pointed curve $(\mathbb{P}^1_K, B)$, where $B = S \cup \{\infty\}$. Denote by $D$ the irreducible component of its special fiber containing the specialization of the point $\infty$. An element $\lambda \in S$ is ordinary if it specializes in $D$. If $B$ has bad reduction, an irreducible component of $B_k$ is simple if it only meets $D$ and if there are exactly two elements of $S$ specializing in it. We will say that $B$ has simple reduction if any tail of $B_k$ (if there are any) is simple (see the following picture).

[Diagram of a line with points labeled $\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_r$, with $\infty$ at one end and $\lambda_{m+1}$ at the other end, illustrating the definition of simple reduction.]

Remark that if the cardinality of $B$ is less than or equal to four, then the pointed curve $(\mathbb{P}^1_K, B)$ has automatically simple reduction.

§6 Semi-stable model separating the fiber above an ordinary branch point

We will now describe the stable model for a polynomial cover of degree $p$ dominating its fundamental model $C \to D$ and separating the ramified fiber above an ordinary branch point.

6.1 Theorem. Let $\beta : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a polynomial cover of degree $p$. Denote by $B = S \cup \{\infty\}$ its branch locus and suppose that $\lambda \in S$ is ordinary. Then, the minimal semi-stable model $X \to Y$ of the cover which dominates the fundamental model $C \to D$ and separates the fiber above $\lambda$ has the following description:

1. The special fiber of $X$ is the union of two projective lines $C$ and $C'$ meeting at a unique singular point of thickness $(n_\lambda - 1)^{-1}$ where $n_\lambda$ is the cardinality of the fiber $\beta^{-1}(\lambda)$.
2. The special fiber of $Y$ is the union of two projective lines $D$ and $D'$ meeting at a unique singular point of thickness $p(n_\lambda - 1)^{-1}$.
3. The morphism $C \to D$ is purely inseparable, while $C' \to D'$ is generically étale, unramified outside two points, wildly ramified above one of them and tamely ramified above the other.
where \( n \) directly implies the inequality (4). From a practical point of view, we must show that two distinct roots of the polynomial \( \nu \), we would obtain \( \beta \) of the cover \( \beta \) with \( \pi \). We then obtain \( \beta \) derived of \( \beta \). Let \( \beta \) be an element of \( p \) such that \( v(\pi) = \nu \). By construction, we have the expression

\[
\beta(\pi X) = \pi^p \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1} x)^{r_x} \in R[X]
\]

with \( \pi^{-1} x \in R \) for any \( x \in \beta^{-1}(0) \). Following the notation introduced at the end of \( \S 4 \), we obtain \( -\lambda \) \( \beta \) \( \beta(X) = \pi^p \beta(\pi X) \) and \( \gamma_0(X) = 1 \). In particular, we have a model \( C' \to D' \) of the cover \( \beta \), where \( D' \) is the curve obtained from \( D \) after a blow-up at the origin, of thickness \( \nu \). We just have to prove that it separates the fiber above 0. From a practical point of view, we must show that two distinct roots of the polynomial \( \beta_0(X) \) have distinct specializations. First of all, if \( \lambda \) \( \pi \) \( S = \{ 0 \} \) then \( \beta^{-1}(\lambda) \subset R^\ast \). Indeed, we already have \( \beta^{-1}(\lambda) \subset R \). If there were \( x \in \beta^{-1}(\lambda) \) with positive valuation, we would obtain \( \beta = \beta(x) = \beta^p = 0 \), which is impossible since we are assuming that \( 0 \in S \) is ordinary. We have the following expression for the derivative of \( \beta(X) \):

\[
\beta'(X) = p \prod_{x \in \beta^{-1}(S)} (X - x)^{r_x - 1}
\]

We then obtain \( \beta_0'(X) = \pi^{1-p} \beta'(\pi X) \), which leads to

\[
\beta_0'(X) = p\pi^{1-n_0} \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1} x)^{r_x - 1} \prod_{x \in \beta^{-1}(S - \{0\})} (\pi X - x)^{r_x - 1}
\]

where \( n_0 \) is the cardinality of the fiber \( \beta^{-1}(0) \). We have \( \beta_0'(X) \in R[X] \), which directly implies the inequality \( (n_0 - 1) \nu \leq 1 \). A strict inequality would give \( \beta_0'(X) = 0 \) and thus \( \beta_0'(X) = X^p \) (recall that \( \beta_0(X) \) is monic, of degree \( p \) and satisfies \( \beta_0(0) = 0 \)). This is impossible since, by construction, there exists a root of \( \beta_0(X) \) belonging to \( R^\ast \). We then have \( \nu = \frac{1}{n_0 - 1} \). Suppose now that the elements \( x_1, \ldots, x_s \in \beta^{-1}(0) \) specialize to the same point \( t \) of \( C' \) and denote by \( e_1, \ldots, e_s \) their ramification indices. We have \( s < n_0 \), since there are at least two distinct roots of \( \beta_0(X) \). The monomial \( X - t \) will appear with a multiplicity \( e = e_1 + \cdots + e_s < p \) in the factorization of \( \beta_0'(X) \). In particular, the element \( t \) will be a root of \( \beta_0'(X) \) of multiplicity \( e - 1 \). But the above expression of \( \beta_0'(X) \) implies that this multiplicity is equal to \( e - s \), and so we have \( s = 1 \), i.e. the model \( C' \to D' \) separates the fiber above 0. □
6.2 Corollary. The notation and hypothesis being as in theorem 1, there exists a smooth model of the cover (over \( \mathbb{R} \)) which separates the fibers above \( \infty \) and \( \lambda \). In other words, the pointed curve \((\mathbb{P}^1_k, \beta^{-1}(\{\infty, \lambda\}))\) has good reduction.

Proof. The desired smooth model is obtained from \( C' \rightarrow D' \) by blowing down the irreducible components \( C \) and \( D \). \( \square \)

6.3 Corollary. With the above assumptions, suppose that the branch locus \( B = \{\infty, \lambda_1, \ldots, \lambda_{r-1}\} \) of the cover \( \beta \) has good reduction and that \( r \geq 3 \). Then, the minimal stable model \( X \rightarrow Y \) separating the ramified fibers has the following description:

1. \( X_k \) (resp. \( Y_k \)) is the union of \( r \) projective lines \( C, C_1, \ldots, C_{r-1} \) (resp. \( D, D_1, \ldots, D_{r-1} \)).
2. For any \( i \in \{1, \ldots, r-1\} \), the point \( \lambda_i \) specializes in \( D_i \).
3. The irreducible component \( C_i \) (resp. \( D_i \)) only meets \( C \) (resp. \( D \)). The corresponding singular point has thickness \((n_i - 1)^{-1}\) (resp. \(p(n_i - 1)^{-1}\)), where \( n_i \) denotes the cardinality of the fiber above \( \lambda_i \).
4. The morphism \( C \rightarrow D \) is purely inseparable and, for any \( i \in \{1, \ldots, r-1\} \), the cover \( C_i \rightarrow D_i \) is generically étale, unramified outside two points, wildly ramified above one of them (the singular point) and tamely ramified above the other (the specialization of \( \lambda_i \)).

\[
\begin{array}{c}
C_1 \quad \cdots \quad C_{r-1} \\
\| \| \\
\infty \quad (n_1^{-1})^{-1} \quad (n_{r-1}^{-1})^{-1} \\
\end{array}
\rightarrow
\begin{array}{c}
\beta \\
D_1 \quad \cdots \quad D_{r-1} \\
\| \| \\
\infty \quad p(n_1^{-1})^{-1} \quad p(n_{r-1}^{-1})^{-1} \\
\end{array}
\]

Proof. It suffice to repeat the construction in the proof of Theorem 6.1 for all the elements of \( S \). \( \square \)

§7 Semi-stable model separating the fibers above a simple tail

We will now study the case of simple reduction of the branch locus \( B = S \cup \{\infty\} \) of the cover \( \beta \). Let \( r \) be the cardinality of \( B \). We can assume \( r > 3 \), since for \( r \leq 3 \) the pointed curve \((\mathbb{P}^1, B)\) will always have good reduction. If \( \lambda \in S \) is an ordinary branch point, then the construction in the proof of theorem 1 leads to a stable model which separates the fiber above \( \lambda \). Suppose that \( \lambda, \lambda' \in S \) belong to a simple tail of the minimal stable model \( B \) of \( B \). In other words, viewing \( S \) as a subset of \( R = C_0(R) \), we have \( v(\lambda - \lambda') > 0 \) and \( v(\lambda - \lambda'') = v(\lambda' - \lambda'') = 0 \) for any \( \lambda'' \in S - \{\lambda, \lambda'\} \). The positive rational number \( \epsilon = v(\lambda - \lambda') \) is the thickness of the singular point connecting the simple tail with the rest of \( B_0 \). Since the reduction of the model \( C \rightarrow D \) is purely inseparable, all the elements of \( \beta^{-1}(\{\lambda, \lambda'\}) \) will have the same specialization.

a) The first separating blow-up. We will start the construction of the stable model separating the fibers above \( \lambda \) and \( \lambda' \) by considering the minimal model \( C' \) dominating \( C \) such that all the elements of \( \beta^{-1}(\{\lambda, \lambda'\}) \) specialize in the (smooth
specialization of polynomial covers

locus of the) same irreducible component and at least two of them have distinct specializations. In order to obtain an more explicit description, we can first of all reduce to the case $\lambda' = 0$ and $\beta(0) = 0$. Consider the rational number $\nu$ defined by

$$\nu = \min \left\{ v(x - y) \mid x \in \beta^{-1}(\{0, \lambda\}) \right\}$$

We have $\nu > 0$, since $\lambda' = 0$ and $\beta(X) = X^p$. Moreover, $v(\lambda) \geq p\nu$. The curve $\mathcal{C}'$ is obtained from $\mathcal{C}$ after a blow-up at the origin, of thickness $\nu$. Let $\pi$ be an element of $p$ such that $v(\pi) = \nu$. The relation (§) at the end of §4 becomes $\beta(\pi X) = \pi^p \beta_0(X) \gamma_0(X)$, with $\gamma_0(X) = 1$ and

$$\beta_0(X) = \pi^{-p} \beta(\pi X) = \prod_{x \in \beta^{-1}(0)} (X - \pi^{-1} x)^{e_x} = \pi^{-p} \lambda + \prod_{x \in \beta^{-1}(\lambda)} (X - \pi^{-1} x)^{e_x}$$

We then get a model $\mathcal{C}' \to \mathcal{D}'$ for $\beta$, where $\mathcal{D}'$ is the curve obtained from $\mathcal{D}$ after a blow-up at the origin, of thickness $p\nu$ (see the following picture).

The derivative of $\beta_0(X)$ can be expressed as

$$\beta'_0(X) = p\pi^{p+1-n_0-n_{\lambda}} \prod_{x \in \beta^{-1}(\{0, \lambda\})} (X - \pi^{-1} x)^{e_x-1} \prod_{x \in \beta^{-1}(S - \{0, \lambda\})} (\pi X - x)^{e_x-1}$$

where, for any $\lambda \in S$, $n_\lambda$ is the cardinality of the fiber $\beta^{-1}(\lambda)$. In particular, since $\beta'_0(X) \in R[X]$ and $\pi^{-1} x \in R$ for any $x \in \beta^{-1}(\{0, \lambda\})$, we obtain the relation

$$\nu \leq \frac{1}{n_0 + n_{\lambda} - p - 1}$$

b) First case: “far” points. Let’s start by supposing that this last inequality is strict. This directly implies that $\beta'_0(X) = X^p$. Now, by construction, the polynomial $h(X) = \beta'_0(X)(\beta_0(X) - \pi^{-p} \lambda)$ has at least two roots having distinct specializations, so that we obtain

$$v(\lambda) = p\nu$$

Indeed, the relation $v(\lambda) > p\nu$ would give $\overline{h}(X) = X^{2p}$, which has only one root. In particular, this first blow-up separates the branch points 0 and $\lambda$ but does not separate the corresponding fibers.
The above inequality now reads as
\[ v(\lambda) < \frac{p}{n_0 + n_\lambda - p - 1} \]
In order to obtain a model separating the fiber above 0, let \( \nu_0 \) be the positive integer defined by
\[ \nu_0 = \min \{ v(x) \mid x \in \beta_0^{-1}(0) \} \]
We clearly have \( \nu + \nu_0 = \min \{ v(x) \mid x \in \beta^{-1}(0) \} \). Consider the curve \( C'' \) obtained from \( C' \) after a blow-up at the origin of the irreducible component \( C' \), of thickness \( \nu_0 \). If \( C'' \) denotes the new irreducible component of \( C''_k \), then we have \( d(C'', C) = \nu + \nu_0 \). Let \( \pi_0 \in p \) such that \( v(\pi_0) = \nu_0 \) and set \( \pi_1 = \pi\pi_0 \) (as in the previous paragraph, \( \pi \) has valuation \( \nu \)). We then obtain the expression \( \beta(\pi_1X) = \pi^p\beta_0(\pi_0X) = \pi^p\beta_1(X)\gamma_1(X) \), with \( \gamma_1(X) = 1 \) and
\[
\beta_1(X) = \pi_1^{-p}\beta_1(X) = \pi^{-p}\beta_0(\pi_0X) = \\
= \prod_{x \in \beta^{-1}(0)} (X - \pi_1^{-1}x)^{e_x} = \prod_{x \in \beta_0^{-1}(0)} (X - \pi_0^{-1}x)^{e_x}
\]
By construction, at least two roots of \( \beta_1(X) \) have distinct specializations. If \( D'' \) is the curve obtained from \( D' \) after a blow-up at the origin of \( D' \), of thickness \( pv_0 \), we then have a model \( C'' \to D'' \) of the cover \( \beta \). Setting \( u = n_0 + n_\lambda - p - 1 \), \( u_0 = n_0 - 1 \) and \( \delta = 1 - u\nu - u_0\nu_0 \), we have the expression
\[
\beta_1'(X) = p\pi^u\pi_0^{n_0} \prod_{x \in \beta^{-1}(0)} (X - \pi_1^{-1}x)^{e_x - 1} \prod_{x \in \beta^{-1}(\lambda)} (\pi_0X - \pi^{-1}x)^{e_x - 1} \times \\
\times \prod_{x \in \beta^{-1}(S \setminus (0, \lambda))} (\pi_1X - x)^{e_x - 1}
\]
which implies that \( \delta = 0 \) (otherwise, we would obtain \( \beta_1(X) = X^p \) and all the roots of \( \beta_1(X) \) would have the same specialization). Since \( v(\lambda) = pu \), we obtain
\[ \nu_0 = \frac{p - (n_0 + n_\lambda - p - 1)v(\lambda)}{p(n_0 - 1)} < \frac{1}{n_0 - 1} \]
Proceeding exactly as in the end of the proof of theorem 1, we finally check that this model separates the fiber above 0, i.e. that any two distinct roots of \( \beta_1(X) \) have distinct specializations. The same procedure leads to a stable model separating the fiber above \( \lambda \). Summarizing, we have just proved the following result:

**7.1 Theorem.** Suppose that the branch locus \( B = S \cup \{ \infty \} \) of the polynomial cover \( \beta \) has bad reduction and that \( D' \) is a simple tail of the special fiber of the stable model \( B \) associated to the pointed curve \( (\mathbb{P}^1, B) \). Denote by \( \lambda_1, \lambda_2 \in S \) the two branch points specializing in \( D' \) and let \( \epsilon = v(\lambda_1 - \lambda_2) \) be the thickness of the corresponding singular point of \( B_k \). For any \( \lambda \in S \), denote by \( n_\lambda \) the cardinality of the fiber \( \beta^{-1}(\lambda) \). If the inequality
\[ \epsilon < \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1} \]
holds, then, the minimal semi-stable model $X \rightarrow Y$ of $\beta$ dominating $C \rightarrow D$ and separating the fibers above $\lambda_1$ and $\lambda_2$ has the following description:

1. The curve $X_k$ (resp. $Y_k$) is the union of four projective lines $C, C', C_1$ and $C_2$ (resp. $D, D', D_1$ and $D_2$).
2. The specializations of the points $\infty, \lambda_1$ and $\lambda_2$ belong respectively to $D, D_1$ and $D_2$.
3. The irreducible component $C$ (resp. $D$) only meets $C'$ (resp. $D'$), at a singular point of thickness $\frac{p}{p}$ (resp. of thickness $\epsilon$).
4. For any $i \in \{1, 2\}$, the irreducible component $C_i$ (resp. $D_i$) only meets $C'$ (resp. $D'$), at a singular point of thickness $\nu_i = \frac{p - (n\lambda_1 + n\lambda_2 - p - 1)\epsilon}{p(n\lambda_i - 1)}$ (resp. of thickness $p\nu_i = \frac{p - (n\lambda_1 + n\lambda_2 - p - 1)\epsilon}{n\lambda_i - 1}$).
5. The morphisms $C \rightarrow D$ and $C' \rightarrow D'$ are purely inseparable, while the covers $C_1 \rightarrow D_1$ and $C_2 \rightarrow D_2$ are generically étale, unramified outside two points, wildly ramified above one of them and tamely ramified above the other.

7.2 Corollary. The notation and hypothesis being as above, if $\epsilon < \frac{1}{n\lambda_1 + n\lambda_2 - p - 1}$ then, for any $i \in \{1, 2\}$, the pointed curve $(P^1, \beta^{-1}(\{\infty, \lambda_i\}))$ has good reduction.

Proof. It suffices to take the smooth $R$-curve obtained from $X$ by blowing down the irreducible components $C, C'$ and $C_{2-i}$ of $C_k$. □

c) Second case: the critical distance. Keeping the above notation and hypothesis, suppose now that the inequality (**) at the end of §7a is in fact an equality, i.e. that $\nu = \frac{1}{n\lambda_1 + n\lambda_2 - p - 1}$, so that $\beta_0(X) \neq 0$. In particular, the morphism $C' \rightarrow D'$ is generically étale. We already know that $\nu(\lambda) \geq p\nu$. In this paragraph, we will assume that this last inequality is an equality. As before, the cover $C' \rightarrow D'$ will separate the branch points $\infty$ and $\lambda$. Moreover, proceeding as at the end of the proof of Theorem 6.1, we can easily show that this model also separates the fibers above these two points. In particular, we have the following result:

7.3 Theorem. The notation and hypothesis being as in Theorem 7.1, suppose that

$$\epsilon = \frac{p}{n\lambda_1 + n\lambda_2 - p - 1}$$

Then, the minimal semi-stable model $X \rightarrow Y$ of $\beta$ dominating $C \rightarrow D$ and separating the fibers above $\lambda_1$ and $\lambda_2$ has the following description:

1. The curve $X_k$ (resp. $Y_k$) is the union of two projective lines $C$ and $C'$ (resp. $D$ and $D'$) meeting at a singular point of thickness $\frac{p}{p}$ (resp. of thickness $\epsilon$).
(2) The specializations of the points $\lambda_1$ and $\lambda_2$ belong $D'$ and $\infty$ specializes in $D$.

(3) The morphism $C \rightarrow D$ is purely inseparable while the cover $C' \rightarrow D'$ is generically étale, unramified outside three points, wildly ramified above one of them and tamely ramified above the others.

\[ \xymatrix{ C' \ar[r] ^-\beta & C \ar@{.>}[r] ^-\infty \ar^-\epsilon \ar@{-}[r] & D \ar@{.>}[r] ^-\lambda_1 \ar^-\epsilon \ar@{-}[r] & D' \ar^-\lambda_2 } \]

**7.4 Corollary.** The notation and hypothesis being as above, if $\epsilon = \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}$ then the pointed curve $\left( \mathbb{P}^1, \beta^{-1}(\{ \infty, \lambda_1, \lambda_2 \}) \right)$ has good reduction.

**Proof.** It suffices to take the smooth $R$-curve obtained from $X$ by blowing down the irreducible component $C$ of its special fiber. □

d) Third case: “near” points. We now have to study the most critical case, that is, when $\nu = \frac{1}{n_0 + n_\lambda - p}$ and $\nu(\lambda) > p\nu$. This situation occurs if and only if the model $C' \rightarrow D'$ introduced at the beginning of this section does not separate the elements 0 and $\lambda$ of the branch locus $B$ of the cover. As in the previous paragraph, the derivative of $\beta_0(X)$ will not identically vanish, so that the cover $C' \rightarrow D'$, which is (isomorphic to the one) induced by the polynomial $\beta_0(X) \in k[X]$, is generically étale, unramified outside two points, wildly ramified above one of them (the intersection with $D$) and tamely ramified above the other. There exist at least two roots of $\beta_0(X)$ having distinct specializations. Indeed, the contrary would give $\beta_0(X) = X^p$, so that the cover $C' \rightarrow D'$ would be purely inseparable, which is a contradiction. The points 0 and $\lambda$ are the only elements of $B$ specializing in $C'$ and their fibers with respect to $\beta$ can be assimilated to the fibers $\beta_0^{-1}(0)$ and $\beta_0^{-1}(\lambda_0)$, where $\lambda_0 = \pi^{-p}\lambda \in p$. Set

\[ \overline{\beta_0}(X) = \prod_{i=1}^{s} (X - w_i)^{d_i} \]

with $s > 1$, $d_1 + \cdots + d_s = p$ and $w_i \neq w_j$ for any $i \neq j$. We then obtain two partitions $\beta_0^{-1}(0) = S_{1,0} \cup \cdots \cup S_{s,0}$ and $\beta_0^{-1}(\lambda_0) = S_{1,\lambda_0} \cup \cdots \cup S_{s,\lambda_0}$, where we have set, for a fixed $t \in p$,

\[ S_{i,t} = \{ x \in \beta_0^{-1}(t) \mid x = w_i \} \]

For any $i \in \{1, \ldots, s\}$ we then have the identity

\[ d_i = \sum_{x \in S_{i,t}} e_x \]
where $e_x$ is the multiplicity of the root $x$ of the polynomial $\beta_0(X) - t$. Since the cover $C' \to D'$ is generically étale and unramified outside the set $\{0, \infty\}$, we have

$$\beta_0'(X) = c \prod_{i=1}^{s} (X - w_i)^{d_i - 1}$$

with $c \in k^*$. On the other hand, the expression of the derivative of $\beta_0(X)$ given previously leads to

$$\beta_0'(X) = c \prod_{i=1}^{s} (X - w_i)^{2d_i - n_i,0 - n_i,\lambda}$$

where $n_i,0$ (resp. $n_i,\lambda$) denotes the cardinality of $S_{i,0}$ (resp. of $S_{i,\lambda}$). Combining these two expressions we finally obtain the identity

$$n_i,0 + n_i,\lambda = d_i + 1$$

which holds for any $i \in \{1, \ldots, s\}$. In order to completely separate the ramified fibers, we need to blow-up some projective lines at the points $w_1, \ldots, w_s$ belonging to the irreducible component $C'$ of $C$. In other words, we have to separate the elements of $S_{i,0} \cup S_{i,\lambda}$, for any $i \in \{1, \ldots, s\}$. Let’s start with $i = 1$: since $0 \in S$ is a branch point and $\beta_0(0) = \beta(0) = 0$, we can assume, without any loss of generality, $w_1 = 0$. Set

$$\nu_1 = \text{Min} \{v(x - y) \mid x, y \in S_{1,0} \cup S_{1,\lambda}, x \neq y\} > 0$$

and consider the $R$-curve $C''$, obtained from $C'$ by blowing-up a projective line $C_1$ at $w_1$, of thickness $\nu_1$. Let $\pi_2 \in p$ such that $v(\pi_2) = \nu_1$ and set $\pi_3 = \pi \pi_2$. We then have the identity

$$\beta(\pi_3 X) = \pi^p \beta_0(\pi_2 X) = \pi^p \beta_1(X)\gamma_1(X)$$

where

$$\beta_1(X) = \prod_{x \in S_{1,0}} (X - \pi_2^{-1} x)^{e_x}$$

and

$$\gamma_1(X) = \prod_{x \in \beta_0^{-1}(0) \setminus S_{1,0}} (\pi_2 X - x)^{e_x}$$

Since we are assuming $w_1 = 0$, an element $x \in \beta_0^{-1}(0)$ belongs to $S_{1,0}$ if and only if $v(x) > 0$. In particular, $\gamma_1(X) = c \in k^*$. We then obtain a $R$-morphism $C'' \to D''$ (which is not finite), where $D''$ is the $R$-curve obtained from $D'$ by blowing-up a projective line $D_1$ at the origin of $D'$, of thickness $\epsilon_1 = p\nu_1$. For any $x \in S_{1,\lambda}$, we have $v(x) > \nu_1$, from which we easily deduce the inequality $v(\lambda) \geq \epsilon_1$, i.e. $v(\lambda) \geq \epsilon_0 + \epsilon_1$, with $\epsilon_0 = p\nu$. The specialized morphism $C_1 \to D_1$ has degree $d_1$ and is (isomorphic to the cover $P^1_k \to P^1_k$) induced by the polynomial $\beta_1(X)$. The integer $d_1$ being strictly less than $p$, the derivative $\beta_1'(X)$ does not identically vanish. More explicitly, using once again the expression of the derivative of $\beta_0(X)$, we obtain

$$\beta_1'(X) = u \prod_{x \in S_{1,0} \cup S_{1,\lambda}} (X - \pi_2^{-1} x)^{e_x - 1}$$
with \( u \in k^* \). The strict inequality \( v(\lambda) > \epsilon_0 + \epsilon_1 \) would imply that the cover \( P^1_k \to P^1_k \) induced by \( \beta_1(X) \) is tame, of degree \( d_1 \) and unramified outside 0 and \( \infty \), and thus \( \beta_1(X) = uX^{d_1} \). In particular, all the elements of \( S_{1,0} \) and \( S_{1,\lambda_0} \), would specialize to the same element of \( C_1 \). This is impossible, since \( C'' \) is the minimal model dominating \( C' \) and separating at least two elements of \( S_{1,0} \cup S_{1,\lambda_0} \). We then have the identity

\[
v(\lambda) = \epsilon_0 + \epsilon_1 = \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1} + d_1 \nu_1
\]

which implies that the model \( D'' \) separates the branch points 0 and \( \lambda \). Then, using the above expression for the derivative of \( \beta_1(X) \) and the same arguments at the end of the proof of Theorem 6.1, one easily shows that \( X \to Y \) stable model and thus \( D \). It follows that the model \( D \) specialize to the same element of \( C \). Iterating this construction for all the roots of \( \beta_0(X) \), we obtain a semi-stable model \( \lambda \to \lambda \) (this time the morphism is finite) which dominates \( C \to D \) and separates the fiber above 0 and \( \lambda \). Summarizing, we have proved the following result:

**7.5 Theorem.** The notation and hypothesis being as in theorem 2, suppose that

\[
\epsilon > \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1}
\]

Then, the special fiber of the minimal stable model \( \lambda \to \lambda \) of the cover \( \beta \) dominating \( C \to D \) and separating the fibers above \( \lambda_1 \) and \( \lambda_2 \) has the following description:

1. The curve \( \lambda \) is the union of three projective lines \( D, D' \) and \( D_1 \).
2. The curve \( D \) (resp. \( D_1 \)) meets \( D' \) at a singular point of thickness \( \epsilon_0 = \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1} \) (resp. of thickness \( \epsilon_1 = v(\lambda) - \frac{p}{n_{\lambda_1} + n_{\lambda_2} - p - 1} \)).
3. The specializations of the points \( \lambda_1 \) and \( \lambda_2 \) belong \( D_1 \) and \( \infty \) specializes in \( D \).
4. The curve \( \lambda \) has \( s + 2 \geq 4 \) irreducible components \( C, C', C_1, \ldots, C_s \).
5. The curve \( C \) meets \( C' \) at a singular point of thickness \( \frac{p}{p} \).
6. There exist two partitions

\[
\beta^{-1}(\lambda_1) = S_{1,1} \cup \cdots \cup S_{s,1} \quad \text{and} \quad \beta^{-1}(\lambda_2) = S_{1,2} \cup \cdots \cup S_{s,2}
\]

satisfying the identities

\[
\sum_{x \in S_{i,1}} e_x = \sum_{x \in S_{i,2}} e_x = d_i \quad \text{and} \quad n_{i,1} + n_{i,2} = d_i + 1
\]

where \( e_x \) denotes the ramification index of \( \beta \) at a point \( x \in \mathbf{P}^1(K) \) and, for any \( i \in \{1, \ldots, s\} \) and any \( j \in \{1, 2\} \), \( n_{i,j} \) is the cardinality of the set \( S_{i,j} \).
7. For any \( i \in \{1, \ldots, s\} \), the curve \( C_i \) meets \( C \) at a unique singular point, of thickness \( \frac{p}{d_i} \) and the elements of \( S_{i,1} \cup S_{i,2} \) specialize to pairwise distinct points of \( C_i \).
8. The morphism \( C \to D \) is purely inseparable and the cover \( C' \to D' \) is generically étale, unramified outside two points, wildly ramified above one of them (the intersection with \( D \)) and tamely ramified above the other (the intersection with \( D_1 \)). For any \( i \in \{1, \ldots, s\} \), the cover \( C_i \to D_1 \) is generically étale, of degree \( d_i \), unramified outside three points (the specializations of \( \lambda_1 \) and \( \lambda_2 \) and the intersection with \( C' \)) over which the ramification is tame.
The theorems 6.1, 7.1, 7.3 and 7.5 allow us to completely classify the reduction type of the minimal semi-stable model $X \to Y$ (separating the ramified fibers) for a polynomial cover $\beta$ over $K$ of degree $p$, under the assumption of simple reduction of its branch locus $B$. More precisely, the above results show that the behaviour of this model essentially depends only on the ramification data of the cover and on the thicknesses of the singular points of the special fiber of the stable (separating) model $B$ associated to the pointed curve $(\mathbb{P}^1, B)$.

§8 An example

We will end this paper with an explicit example. Let $K$ be a 5-adic field, i.e. a finite extension of the field $\mathbb{Q}_5$. We will describe the semi-stable model for a polynomial cover $\beta : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree 5 having the following ramification data:

1. The cover is unramified outside the set $\{\infty, 0, \lambda_1, \lambda_2\}$ with $\lambda_1, \lambda_2 \in K^*$ and $\lambda_1 \neq \lambda_2$.
2. There is only one point above $\infty$.
3. There are three points above 0, one of them, denoted by $P_0$ has ramification index 3, while the others are unramified.
4. There are four points above $\lambda_1$ (resp. above $\lambda_2$), one of them, denoted by $P_1$ (resp. $P_2$) has ramification index 2 while the other three are unramified.

Up to equivalence (cf. §2), and since the behaviour of the ramification above $\lambda_1$ and $\lambda_2$ is the same, we can reduce to the case $\lambda_1 = 1$ and $\lambda_2 = \lambda \in R - \{0, 1\}$. The results of this paper allow us to describe the semi-stable model for $\beta$ without any direct computation. More precisely, if the branch locus has good reduction, i.e. if $v(\lambda) = v(\lambda - 1) = 0$ then we can apply Theorem 6.1, which leads to the following model:

The branch locus of the cover $\beta$ has bad reduction if either $v(\lambda) > 0$ or $v(\lambda - 1) > 0$. In the first case, there are three possibilities, leading to four different semi-stable models: if $\epsilon = v(\lambda) < 5$, we can apply Theorem 7.1 and obtain the following model:
As it is shown in the following picture, and applying Theorem 7.3, the simplest case occurs for \( \epsilon = 5 \): 

Finally, for \( \epsilon > 5 \) we find two possibilities, depending on the partitions of the ramification indices \((1, 1, 3)\) and \((1, 1, 1, 2)\) of the fibers above 0 and \( \lambda \) (cf. Theorem 7.5). The first partition is given by \( S_0 = \{\{1, 3\}, \{1\}\} \) and \( S_1 = \{\{1, 1, 2\}, \{1\}\} \), which leads to the following semi-stable model:

If we consider the second partition \( S_0 = \{\{3\}, \{1, 1\}\} \) and \( S_1 = \{\{1, 1, 1\}, \{2\}\} \), we then obtain the following model:

In order to completely classify the possible semi-stable models for \( \beta \), we have to study the last case of bad reduction of the branch locus, i.e. when \( \epsilon = v(\lambda - 1) > 0 \). As before, there will be three different cases, depending on \( \epsilon \), leading to four different situations. First of all, for \( \epsilon < \frac{5}{2} \) we obtain the following semi-stable model (cf. Theorem 7.1):
The next picture describes the case $\epsilon = \frac{5}{2}$ (cf. Theorem 7.3):

For $\epsilon > \frac{5}{2}$, according to Theorem 7.5, there are two possible partitions of the ramification indices above 1 and $\lambda$. The first is given by $S_1 = S_\lambda = \{\{1, 2\}, \{1\}, \{1\}\}$ and leads to the following semi-stable model:

Finally, the next picture describes the semi-stable model associated to the second partition $S_1 = S_\lambda = \{\{2\}, \{1, 1\}, \{1\}\}$:

References

[BLR] Jean-Benoît Bost, François Loeser and Michel Raynaud. (eds.), *Courbes semi-stables et groupe fondamental en géométrie algébrique*, Proceedings of the Conference on Fundamental Group of Curves in Algebraic Geometry held in Luminy, November 30–December 4, 1998. (French); Progress in Mathematics, vol. 187, Birkhäuser Verlag, Basel, 2000.

[G] Grothendieck, Alexandre, *Revêtements étalés et groupe fondamental*. (French) Séminaire de Géométrie Algébrique du Bois Marie (SGA 1), vol. 224; Springer-Verlag, Berlin-New York, 1971.

[L] Lehr, Claus, *Reduction of $p$-cyclic covers of the projective line*., Manuscripta Math. 106 (2001), no. 2, 151–175.
[R1] Raynaud, Michel, *Revêtements de la droite affine en caractéristique p > 0 et conjecture d’Abhyankar*, Invent. Math. **116** (1994), no. 1-3, 425–462. (French)

[R2] Raynaud, Michel, *Spécialisation des revêtements en caractéristique p > 0*, Ann. Sci. cole Norm. Sup. (4) **32** (1999), no. 1, 87–126. (French)

[S] Saïdi, Mohamed, *Galois covers of degree p: semi-stable reduction and Galois action*, Preprint, math.AG/0106249.

[W] Wewers, Stefan, *Three point covers with bad reduction*, Preprint, math.AG/0205026.

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