Relating the independence number and the dissociation number

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Abstract
The independence number $\alpha(G)$ and the dissociation number $\text{diss}(G)$ of a graph $G$ are the largest orders of induced subgraphs of $G$ of maximum degree at most 0 and at most 1, respectively. We consider possible improvements of the obvious inequality $2\alpha(G) \geq \text{diss}(G)$. For connected cubic graphs $G$ distinct from $K_4$, we show $5\alpha(G) \geq 3\text{diss}(G)$, and describe the rich and interesting structure of the extremal graphs in detail. For bipartite graphs, and, more generally, triangle-free graphs, we also obtain improvements. For subcubic graphs though, the inequality cannot be improved in general, and we characterize all extremal subcubic graphs.

KEYWORDS
cubic graph, dissociation set, independent set, triangle-free graph

1 | INTRODUCTION

We consider finite, simple, and undirected graphs, and use standard terminology. A set $D$ of vertices of a graph $G$ is a dissociation set in $G$ if the subgraph $G[D]$ of $G$ induced by $D$ has maximum degree at most 1, and the dissociation number $\text{diss}(G)$ of $G$ is the maximum order of a dissociation set in $G$. Clearly, dissociation sets and the dissociation number bear a close resemblance to the well known independent sets and the independence number $\alpha(G)$ of a graph $G$. Indeed, if $D$ is a dissociation set in a graph $G$, then every component of $G[D]$ has one or two vertices. Hence, selecting one vertex in each component yields an independent set of order at least $|D|/2$, which implies
\[ \alpha(G) \geq \frac{1}{2} \text{diss}(G) \text{ for every graph } G. \] (1)

While (1) is trivial, it is NP-hard [1] to recognize the extremal graphs for (1). The dissociation number is algorithmically hard even when restricted, for instance, to subcubic bipartite graphs [3, 14, 18]. Bounds [2, 4, 5, 9], fast exact algorithms [13], (randomized) approximation algorithms [12, 13], fixed parameter tractability [16], and the maximum number of maximum dissociation sets [17] have been studied for this parameter or its dual, the 3-path (vertex) cover number.

The starting point for the present paper were possible improvements of (1). As it turns out, (sub)cubic graphs play a special role in this context, and it is not difficult to construct arbitrarily large connected \( r \)-regular extremal graphs for (1) as soon as \( r \geq 4 \), cf. Section 2. For cubic graphs though, our first main result is the following best possible improvement of (1). Note that the connected cubic graph \( K_4 \) satisfies (1) with equality.

**Theorem 1.** If \( G \) is a connected cubic graph that is not isomorphic to \( K_4 \), then
\[ \alpha(G) \geq \frac{3}{5} \text{diss}(G). \]

The extremal graphs for Theorem 1 have a very interesting structure, which we elucidate in Theorem 7 at the end of Section 2. Their order is necessarily divisible by 18, and Figure 1 shows a smallest extremal graph.

For subcubic graphs, the inequality (1) cannot be improved. Nevertheless, the subcubic extremal graphs have a simple structure, and, as our second main result, we provide their constructive characterization: Let the set \( \mathcal{G} \) of connected graphs contain \( K_4 \) as well as all connected subcubic graphs that arise from the union of disjoint copies of the following three graphs, in which we mark certain vertices:

- \( K_2 \) with both vertices marked,
- \( K_3 \) with two of the three vertices marked, and
- the graph \( K_4^* \) that arises from \( K_4 \) with vertices \( a, b, c, \) and \( d \) by subdividing the edge \( ab \) twice, and marking the two vertices created by these two subdivisions as well as the vertices \( c \) and \( d \),

![Figure 1](image-url)

**FIGURE 1** A cubic graph \( G \) with \( \alpha(G) = 6 \) and \( \text{diss}(G) = 10 \); the encircled vertices indicate a maximum dissociation set. The orientation of some of the edges of \( G \) is explained in the context of Theorem 7 below.
by adding additional edges, each incident with at most one marked vertex. For a graph $G$ in $\mathcal{G}\setminus\{K_4\}$, let $D(G)$ be the set of all marked vertices. See Figure 2 for an illustration.

**Theorem 2.** A connected subcubic graph $G$ satisfies $2\alpha(G) = \text{diss}(G)$ if and only if it belongs to $\mathcal{G}$.

Having considered the influence of degree conditions on (1), we now add further structural conditions. For bipartite graphs, a simple argument yields the following.

**Proposition 3.** If $G$ is a connected bipartite graph of maximum degree at most $\Delta$, then

$$\alpha(G) \geq \frac{1}{2}\left(1 + \frac{1}{2(\Delta - 1)}\right)\text{diss}(G) - \frac{1}{2(\Delta - 1)}.$$ 

Proposition 3 is best possible, and its proof implies that all extremal graphs are trees. Generalizing the subcubic extremal tree shown in Figure 3 easily allows to construct arbitrarily large extremal trees of maximum degree $\Delta$ for all values of $\Delta \geq 2$.

For regular triangle-free graphs, we obtain the following result.

**Theorem 4.** If $G$ is a triangle-free $\Delta$-regular graph for some $\Delta \geq 3$, then

$$\alpha(G) \geq \frac{1}{2}\left(1 + \frac{(\Delta - 1)(\Delta + 1)}{2^\Delta \Delta^2 + (\Delta - 1)(\Delta + 1)}\right)\text{diss}(G) = \frac{1}{2}\left(1 + \Omega\left(\frac{1}{2^\Delta}\right)\right)\text{diss}(G).$$

All proofs are given in the following section, and in a concluding section, we mention some open problems.

## 2 PROOFS

We begin with the construction of $r$-regular extremal graphs for (1) for $r \geq 4$: Let $k$ and $\ell$ be positive integers at least 2. Let the graph $G(k, \ell)$ arise from the disjoint union of $\ell$ copies $K(1), \ldots, K(\ell)$ of $K_{2k}$, where we partition the vertex set of each $K(i)$ into two sets $L(i)$ and $R(i)$ of order $k$ for every $i$, by adding a matching $M(i)$ of size $k$ between $R(i)$ and $L(i + 1)$ for every $i$, where we consider $i$ modulo $\ell$. Since $G(k, \ell)$ is covered by $\ell$ cliques, we have $\alpha(G(k, \ell)) \leq \ell$ and $\text{diss}(G(k, \ell)) \leq 2\ell$. Since a set containing two vertices from $L(i)$ for every $i$ is a dissociation.
set, we have \(2\alpha(G(k, \ell)) \geq \text{diss}(G(k, \ell)) \geq 2\ell\). Altogether, it follows that \(2\alpha(G(k, \ell)) = \text{diss}(G(k, \ell))\). By construction, the graph \(G(k, \ell)\) is \(2k\)-regular. Let the graph \(G'(k, \ell)\) arise from \(G(k, \ell)\) by adding a matching of size \(k\) that is disjoint from \(M_i\) between \(R_i\) and \(L(i + 1)\) for every \(i\), where we consider \(i\) modulo \(\ell\). Arguing as above, it follows that \(G'(k, \ell)\) is \((2k + 1)\)-regular and satisfies \(2\alpha(G'(k, \ell)) = \text{diss}(G'(k, \ell))\). For the sake of completeness, let us point out that \(\alpha(G) = \text{diss}(G)\) if \(G\) is 0-regular, \(2\alpha(G) = \text{diss}(G)\) if \(G\) is 1-regular, \(\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil\), and \(\text{diss}(C_n) = \left\lfloor \frac{2n}{3} \right\rfloor\).

Our next goal is the proof of Theorem 1, which uses the following simple lemma.

**Lemma 5.** If \(G\) is a subcubic graph and \(D\) is a maximum dissociation set of \(G\) that maximizes the number \(p\) of isolated vertices in \(G[D]\), then \(G[V(G) \setminus D]\) has maximum degree at most 1.

**Proof.** Suppose, for a contradiction, that some vertex \(u\) in \(R = V(G) \setminus D\) has two neighbors in \(R\). The choice of \(D\) implies that \(u\) has a neighbor \(v\) in \(D\). Since \(G\) is subcubic, the vertex \(v\) is the only neighbor of \(u\) in \(D\), and the choice of \(D\) further implies that \(v\) has a unique neighbor \(w\) in \(D\). Now, replacing \(v\) by \(u\) within \(D\) yields a maximum dissociation set \(\tilde{D}\) with more than \(p\) isolated vertices in \(G[\tilde{D}]\), contradicting the choice of \(D\), and completing the proof. □

The following proof of Theorem 1 is set up in a way that allows to extract several structural features of the extremal graphs.

**Proof of Theorem 1.** Let \(G\) be a connected cubic graph of order \(n\) at least 6. Let \(D\) be a maximum dissociation set of \(G\) that maximizes the number \(p\) of isolated vertices in \(G[D]\). Let \(G[D]\) contain \(q\) components of order 2. By Lemma 5, the set \(\tilde{D} = V(G) \setminus D\) is also a dissociation set. Let \(G[\tilde{D}]\) contain \(r\) isolated vertices and \(s\) components of order 2, in particular, we have \(n = p + 2q + r + 2s\). Counting the edges between \(D\) and \(\tilde{D}\), we obtain \(3p + 4q = 3r + 4s\). By the choice of \(G\) and the structure of \(G[\tilde{D}]\), we obtain \(\text{diss}(G) = p + 2q\) and \(\alpha(G) \geq r + s\). Now,

\[
n = p + 2q + r + 2s = 3(p + 2q) + p - 2(r + s) \geq 3\text{diss}(G) + p - 2\alpha(G).
\]

(2)

Now, suppose that \(\frac{\alpha(G)}{\text{diss}(G)} \leq \frac{3}{5}\). Since \(G\) is connected and not isomorphic to \(K_4\), Brooks' theorem [6] implies \(\alpha(G) \geq n/3\), and (2) implies...
3\(\alpha(G) \geq n \geq 3 \text{diss}(G) + p - 2\alpha(G) \geq 3\alpha(G),\)

where the last inequality uses \(\frac{\alpha(G)}{\text{diss}(G)} \leq \frac{3}{5}\). It follows that equality holds throughout this inequality chain. This implies \(\alpha(G) = n/3, p = 0, r + s = \alpha(G),\) and \(2q = \text{diss}(G) = \frac{5}{3}\alpha(G) = \frac{5}{9}n\). Furthermore, using \(n = p + 2q + r + s + s\), we obtain \(s = \frac{n}{9}\) and \(r = \frac{2n}{9}\).

Theorem 7 concerning the extremal graphs for Theorem 1 requires quite some technical preparation, which is why we postpone it to the end of this section. We proceed to the proof of the characterization of the extremal subcubic graphs for (1).

**Proof of Theorem 2.** Let \(G\) belong to \(\mathcal{G}\). If \(G = K_4\), then \(2\alpha(G) = \text{diss}(G) = 2\). Now, let \(G \neq K_4\). Let \(G\) arise from \(n_2\) copies of \(K_2\), \(n_3\) copies of \(K_3\), and \(n_4\) copies of \(K_4^*\) by adding additional edges, each incident with at most one marked vertex. Since \(D(G)\) is a dissociation set by construction, we obtain \(2\alpha(G) \geq \text{diss}(G) \geq |D(G)| = 2n_2 + 2n_3 + 4n_4\). Since every independent set in \(G\) contains at most one vertex from each \(K_2\) and \(K_3\) as well as at most 2 vertices from each \(K_4^*\), we obtain \(\alpha(G) \leq n_2 + n_3 + 2n_4\). It follows that \(2\alpha(G) = \text{diss}(G)\); in particular, the set \(D(G)\) is a maximum dissociation set.

For the converse, let \(G\) be a connected subcubic graph with \(2\alpha(G) = \text{diss}(G)\) that is distinct from \(K_4\). Let \(D\) be a maximum dissociation set in \(G\). Since \(2\alpha(G) = \text{diss}(G)\), the graph \(G[D]\) consists of \(\alpha(G)\) copies of \(K_2\). If some vertex \(u\) in \(R = V(G) \setminus D\) has at most one neighbor in each \(K_2\) in \(G[D]\), then \(G\) has an independent set containing \(u\) as well as one vertex from each \(K_2\) in \(G[D]\), which implies the contradiction \(\alpha(G) > \text{diss}(G)/2\).

Hence, for every vertex \(u\) in \(R\), there is a \(K_2\) component in \(G[D]\), say \(K(\alpha\{}u\\alpha\}\), such that \(u\) is adjacent to both vertices of \(K(u)\). Since \(G\) is subcubic, the \(K_2\) component \(K(u)\) is uniquely determined for every \(u\) in \(R\), and, for every \(K_2\) component \(K\) in \(G[D]\), there are at most two vertices \(u\) in \(R\) with \(K = K(u)\).

Suppose that \(K(u) = K(v)\) for two distinct vertices \(u\) and \(v\) in \(R\). Since \(G\) is not \(K_4\), the vertices \(u\) and \(v\) are not adjacent. If every \(K_2\) component of \(G[D]\) that is distinct from \(K(u)\) contains a vertex that is adjacent neither to \(u\) nor to \(v\), then \(G\) has an independent set containing \(u\) and \(v\) as well as one vertex from each \(K_2\) in \(G[D]\) that is distinct from \(K(u)\), which implies the contradiction \(\alpha(G) > \text{diss}(G)/2\). Since \(G\) is subcubic, it follows that \(\alpha(G) \geq \text{diss}(G)/2\).

If there is some vertex \(w\) in \(R\) with \(K(w) = K(\{u, v\})\), then \(G\) has an independent set containing \(u, v, w\) as well as one vertex from each \(K_2\) in \(G[D]\) that is distinct from \(K(u)\) and \(K(w)\), which implies the contradiction \(\alpha(G) > \text{diss}(G)/2\). Hence, there is no vertex \(w\) in \(R\) with \(K(w) = K(\{u, v\})\). If there are two distinct vertices \(u'\) and \(v'\) in \(R\) with \(K(\{u, v\}) = K(\{u', v'\})\), then \(G\) has an independent set containing \(u, v, u', v'\) as well as one vertex from each \(K_2\) in \(G[D]\) that is distinct from \(K(u)\), \(K(u')\), and \(K(\{u, v\})\), which implies the contradiction \(\alpha(G) > \text{diss}(G)/2\). Hence, there are no two distinct vertices \(u'\) and \(v'\) in \(R\) with \(K(\{u, v\}) = K(\{u', v'\})\). Note that \(\{u, v\} \cup K(u) \cup K(\{u, v\})\) induces a \(K_4^*\) in \(G\).

We are now in a position to describe a decomposition of \(G\) into disjoint copies of \(K_2, K_3, \) and \(K_4^*\) as in the definition of \(\mathcal{G}\):
• For every $K_2$ component $K$ in $G[D]$ with $K = K(u) = K(v)$ for two distinct vertices $u$ and $v$ in $R$, the set $\{u, v\} \cup V(K) \cup K(\{u, v\})$ induces a $K^*_4$ in $G$, which will be part of the decomposition.

• For every $K_2$ component $K$ in $G[D]$ with $K = K(u)$ for exactly one vertex $u$ in $R$, the set $\{u\} \cup V(K)$ induces a $K_3$ in $G$, which will be part of the decomposition.

• For every $K_2$ component $K$ in $G[D]$ such that there is no vertex $u$ in $R$ with $K = K(u)$ and there are no two distinct vertices $u$ and $v$ in $R$ with $K = K(\{u, v\})$, the set $V(K)$ induces a $K_2$ in $G$, which will be part of the decomposition.

The above observations imply that all described copies of $K_2, K_3,$ and $K^*_4$ are disjoint. Furthermore, since $D$ is a dissociation set, all additional edges of $G$ are incident with at most one vertex in $D$. Since the vertices in $D$ correspond to the marked vertices in the definition of $\mathcal{G}$, we obtain that $G$ belongs to $\mathcal{G}$, which completes the proof. □

Our best possible result for bipartite graphs has quite a simple proof.

**Proof of Proposition 3.** Let $n$ be the order of $G$. Let $D$ be a maximum dissociation set in $G$. Let $G[D]$ contain $p$ isolated vertices and $q$ components of order 2, that is, $\text{diss}(G) = p + 2q$. The number $m$ of edges incident with the vertices in $R = V(G) \setminus D$ satisfies $m \leq \Delta |R| = \Delta(n - p - 2q)$. Removing these $m$ edges results in a graph with at most $1 + m$ components: $p + (n - p - 2q)$ isolated vertices and $q$ components of order 2. We obtain

$$1 + \Delta(n - p - 2q) \geq 1 + m \geq p + (n - p - 2q) + q = n - q,$$

which implies

$$\alpha(G) \geq \frac{n}{2} \geq \frac{\Delta}{2(\Delta - 1)}p + \frac{2\Delta - 1}{4(\Delta - 1)}2q - \frac{1}{2(\Delta - 1)} \geq \frac{2\Delta - 1}{4(\Delta - 1)} \text{diss}(G) - \frac{1}{2(\Delta - 1)}.$$

□

In the preceding proof, the hypothesis that $G$ is bipartite is only used to ensure that $\alpha(G) \geq \frac{n}{2}$.

The following proof of our main result about triangle-free graphs is based on a probabilistic argument, and recycles ideas from the probabilistic folklore proof of the Caro–Wei bound on the independence number.

**Proof of Theorem 4.** Let $D$ be a maximum dissociation set in $G$. Let $D$ contain $p$ isolated vertices and $q$ components of order 2. For $i \in \{0, 1\}$, let $D_i$ be the set of vertices of degree $i$ in $G[D]$. For $i \in \{0, 1, \ldots, \Delta\}$, let $R_i$ be the set of vertices in $V(G) \setminus D$ that have exactly $i$ neighbors in $D_1$, and let $r_i = |R_i|$. Since $G$ is $\Delta$-regular, we have

$$\sum_{i=1}^{\Delta} i r_i = 2(\Delta - 1)q.$$

(4)
Adding to $D_0$ one vertex from each of the $q$ components of $G[D_1]$ yields an independent set in $G$, which implies

$$\alpha(G) \geq p + q = \text{diss}(G) - q. \quad (5)$$

We now construct another independent set by the following random procedure: For every $K_2$ component in $G[D_1]$, we select independently one of its two vertices with probability $1/2$. Let $I_i$ be the set of selected vertices. Let $\pi$ be a linear ordering of the vertices in $V(G) \setminus D_1$ chosen uniformly at random. Let $I_2$ be the set of all vertices $u$ in $V(G) \setminus D_1$ such that

- if $u \in D_0$, then $u$ appears within $\pi$ before all its neighbors, and
- if $u \in V(G) \setminus D$, then $u$ appears within $\pi$ before all its neighbors within $V(G) \setminus D_1$, and $u$ has no neighbor in $I_1$.

Note that

$$P[u \in I_2] = \begin{cases} \frac{1}{\Delta + 1}, & \text{if } u \in D_0, \text{ and} \\ \frac{1}{2^i (\Delta - i + 1)}, & \text{if } u \in R_i \text{ for some } i \in \{0, 1, ..., \Delta\}. \end{cases}$$

By linearity of expectation,

$$E[I_2] = \frac{p \Delta + r_0}{\Delta + 1} + \sum_{i=1}^{\Delta} \frac{r_i}{2^i (\Delta - i + 1)} = \frac{p \Delta + r_0}{\Delta + 1} + \sum_{i=1}^{\Delta} \frac{in_i}{2^i (\Delta - i + 1)}.$$

Since

$$\max\{2^i(\Delta - i + 1) : i \in \{1, ..., \Delta\}\} = 2^{\Delta},$$

we obtain, using (4), that

$$E[I_2] \geq \frac{p \Delta + 2(\Delta - 1)q}{\Delta + 1}.$$

Altogether, considering the independent set $I_1 \cup I_2$, and using the first moment method, we obtain

$$\alpha(G) \geq q + \frac{p \Delta + 2(\Delta - 1)q}{\Delta + 1} \text{diss}(G) + \left(1 + \frac{2(\Delta - 1)}{2^{\Delta}} - \frac{2}{\Delta + 1}\right)q. \quad (6)$$

Now, multiplying (5) by \(1 + \frac{2(\Delta - 1)}{2^{\Delta}} - \frac{2}{\Delta + 1}\) and adding (6), the terms depending on $q$ cancel, which yields
\[\left(1 + \frac{2(\Delta - 1)}{2^\Delta} - \frac{2}{\Delta + 1}\right)\alpha(G) + \alpha(G) \geq \left(1 + \frac{2(\Delta - 1)}{2^\Delta} - \frac{2}{\Delta + 1}\right)\text{diss}(G) + \frac{1}{\Delta + 1}\text{diss}(G).\]

This is equivalent to

\[\alpha(G) \geq \frac{1}{2}\left(1 + \frac{(\Delta - 1)(\Delta + 1)}{2^\Delta^2 + (\Delta - 1)(\Delta + 1)}\right)\text{diss}(G),\]

which completes the proof. \(\square\)

We proceed to Theorem 7, which describes the structure of the extremal graphs for Theorem 1.

Let \(k\) be a positive integer. Let \(\mathcal{H}_k\) be the set of all connected subcubic multigraphs \(H\) with the following properties:

- \(H\) may contain parallel edges but no loops.
- \(3n_1 + 2n_2 + n_3 = 6k\), where \(n_i\) is the number of vertices of \(H\) of degree \(i\), and the degree of a vertex in \(H\) is the number of incident edges.
- \(H\) has an induced matching \(M\) of size \(k\) that covers all vertices of degree 1.
- \(H - M\) has an orientation \(\vec{H - M}\) such that every vertex \(u\) of \(H\) that is not incident with an edge in \(M\) has exactly two outgoing edges.

\(\mathcal{H}_k\) is quite a rich family of multigraphs; starting with an integral solution of \(3n_1 + 2n_2 + n_3 = 6k\), the construction of elements of \(\mathcal{H}_k\) is simple.

Let \(\mathcal{G}_k\) be the set of all graphs \(G\) that arise from a graph \(H\) in \(\mathcal{H}_k\) by

- replacing every vertex of \(H\) of degree 3 with a triangle as illustrated in Figure 4,
- replacing every vertex of \(H\) of degree 2 with a \(K_4^*\) as illustrated in Figure 5, and

**FIGURE 4** Replacing a vertex of degree 3.

**FIGURE 5** Replacing a vertex of degree 2.
• replacing every vertex of $H$ of degree 1 with one of the two graphs of order 9 as illustrated in Figure 6.

The graph $G$ in Figure 1 belongs to $\mathcal{G}_1$; the corresponding multigraph $H$ is obtained by contracting the six triangles, the matching $M$ consists of the edge between the two central triangles, and the orientation $\overrightarrow{H-M}$ is shown on the corresponding eight edges of $G$. Another example is shown in Figure 7.

**Lemma 6.** For every positive integer $k$, every graph $G$ in $\mathcal{G}_k$ is connected, cubic, and satisfies $\alpha(G) = 6k$ and $\text{diss}(G) = 10k$.

**Proof.** The construction of the graphs in $\mathcal{G}_k$ immediately implies that $G$ is connected and cubic. Since $\alpha(K_3) = 1, \alpha(K_6^*) = 2$, and the independence number of the two graphs replacing vertices of degree 1 shown in Figure 6 is 3, we obtain $\alpha(G) \leq 3n_1 + 2n_2 + n_3 = 6k$. Now, we describe a dissociation set $D$ in $G$:

(i) The $k$ edges in $M$ correspond in an obvious way to $k$ edges in $G$, and $D$ contains all $2k$ vertices incident with these edges; cf. the two encircled vertices in the two central triangles in Figure 1.
(ii) The edges in \( E(H) \setminus M \) correspond in an obvious way to edges in \( G \). We orient these edges exactly as in \( H - M \), and let \( D \) contain the vertices of \( G \) that have an outgoing oriented edge. This yields \( 2(n_1 + n_2 + n_3 - 2k) \) further vertices in \( D \), cf. the eight oriented edges in Figure 1.

(iii) For every vertex \( u \) of degree 2 in \( H \), the set \( D \) contains the two vertices of the corresponding copy of \( K_4^* \) encircled in Figure 5. This yields \( 2n_2 \) further vertices in \( D \).

(iv) For every vertex \( u \) of degree 1 in \( H \), the set \( D \) contains the four vertices of the corresponding subgraph encircled in Figure 6. This yields \( 4n_1 \) further vertices in \( D \).

The construction implies that \( D \) is a dissociation set in \( G \), and

\[
|D| = 2k + 2(n_1 + n_2 + n_3 - 2k) + 2n_2 + 4n_1 = 6n_1 + 4n_2 + 2n_3 - 2k = 10k.
\]

By Theorem 1, we have \( 6k \geq \alpha(G) \geq \frac{3}{5} \text{diss}(G) \geq \frac{3}{5} |D| = 6k \), which implies \( \alpha(G) = 6k \) and \( \text{diss}(G) = 10k \). \( \square \)

We proceed to our final main result, the characterization of the extremal graphs for Theorem 1.

**Theorem 7.** A connected cubic graph \( G \) satisfies \( \alpha(G) = \frac{3}{5} \text{diss}(G) \) if and only if \( G \in \bigcup_{k \in \mathbb{N}} G_k \).

**Proof.** Let \( G \) be a connected cubic graph of order \( n \) at least 6 with \( \alpha(G) = \frac{3}{5} \text{diss}(G) \). In view of Lemma 6, it suffices to show that \( G \) belongs to \( \bigcup_{k \in \mathbb{N}} G_k \). Let \( D, D, p, q, r, \) and \( s \) be exactly as in the proof of Theorem 1. Recall that \( D \) was chosen to maximize \( p \), and that, using that \( G \) is extremal, we obtained that \( p = 0 \). This implies that every maximum dissociation set of \( G \) is a suitable choice for \( D \), and, hence, every maximum dissociation set shares the following properties with \( D \), where \( k = \frac{n}{18} \):

\[
p = 0, q = 5k, r = 4k, s = 2k, \text{ and } \alpha(G) = r + s.
\]

Let the set \( D_0 \) contain the \( r = 4k \) isolated vertices of \( G[D] \), and let the set \( D_1 \) contain the remaining \( 2s = 4k \) vertices from \( D \). A path or cycle in \( G \) is *alternating* if it alternates between the vertices from \( D \) and \( D \).

**Claim 1:** No induced alternating path has both its endpoints in \( D_1 \).

**Proof of Claim 1.** Suppose, for a contradiction, that \( P : \bar{u}_1 u_1 \bar{u}_2 u_2 \ldots \bar{u}_\ell u_\ell \bar{u}_{\ell + 1} \) is an induced alternating path with \( \bar{u}_1, \bar{u}_{\ell + 1} \in D_1 \). Since \( P \) is induced, the vertices \( \bar{u}_1 \) and \( \bar{u}_{\ell + 1} \) are not adjacent. Let \( v_1 \) and \( v_{\ell + 1} \) be the neighbors of \( \bar{u}_1 \) and \( \bar{u}_{\ell + 1} \) in \( D_1 \), respectively. Since \( P \) is induced, and the two neighbors of each \( u_i \) in \( D \) are \( u_i \) and \( u_{i+1} \), the set \( I \) containing \( v_1, u_1, \ldots, u_\ell, v_{\ell + 1} \) as well as one vertex from every component of \( G[D] \) that does not intersect \( P \) is independent. Since \( P \) intersects at most \( \ell + 1 \) of the \( r + s \) components of
Claim 2: No induced alternating cycle contains a vertex from $D_1$.

Proof of Claim 2. Suppose, for a contradiction, that $C : u_1 u_2 u_3 \ldots u_{\ell} u_1$ is an induced alternating cycle with $u_1 \in D_1$. Let $v_1$ be the neighbor of $u_1$ in $D_1$. Since $C$ is induced, the vertex $v_1$ does not lie on $C$, and the set $I$ containing $v_1, u_1, \ldots, u_\ell$ as well as one vertex from every component of $G[D]$ that does not intersect $C$ is independent. Since $C$ intersects at most $\ell$ of the $r + s$ components of $G[D]$, we obtain that $|I| = 1 + \ell + (r + s) - \ell = r + s + 1 > (7)$, which is a contradiction. □

For a set $X$ of vertices of $G$, let $d(X) = |X \cap D|$ and $\bar{d}(X) = |X \cap \overline{D}|$. For a subgraph $H$ of $G$, let $d(H) = d(V(H))$ and $\bar{d}(H) = \bar{d}(V(H))$. A $D_0$-alternating cycle is a (not necessarily induced) alternating cycle in $G$ that does not contain a vertex from $D_1$. For a $D_0$-alternating cycle $C$, the set $N_G[V(C) \cap \overline{D}]$ is the corresponding alternating cycle set. If $X = N_G[V(C) \cap \overline{D}]$ is an alternating cycle set, then every vertex in $V(C) \cap \overline{D}$ has at least two of its three neighbors in $V(C)$ and at most one neighbor outside of $V(C)$, which implies

$$d(X) \leq 2\bar{d}(X).$$  (8)

We now apply a recursive reduction to $G$: As long as the current graph contains a $D_0$-alternating cycle, iteratively remove the corresponding alternating cycle set. Let $G'$ be the resulting induced subgraph of $G$, that is, the graph $G'$ contains no $D_0$-alternating cycle. By construction, all vertices from $D_1$ belong to $G'$, and, for every vertex from $D$ in $G'$, its two neighbors in $\overline{D}$ belong to $G'$.

Let $uv$ be one of the $s = 2k$ edges in $G[D]$. A special $uv$-cycle is an induced cycle $C$ in $G'$ that contains $uv$ such that $C - uv$ is an alternating path. Let the bubble $B_{uv}$ of $uv$ be the union of $uv$ and all special $uv$-cycles. Claim 1 implies $(V(B_{uv}) \cap \overline{D}) \setminus \{u, v\} \subseteq D_0$. Our goal is to show that the bubbles of the edges in $G[D]$ are exactly the four different graphs of orders 3, 6, and 9 shown in Figures 4, 5, and 6 replacing certain vertices, in particular, the bubbles are disjoint.

By definition,

• every vertex in $V(B_{uv}) \cap D$ sends exactly two edges to $V(B_{uv}) \cap \overline{D}$,

• each of the two vertices $u$ and $v$ sends at most two edges to $V(B_{uv}) \cap D$, and

• every vertex in $(V(B_{uv}) \cap \overline{D}) \setminus \{u, v\}$ sends at most three edges to $V(B_{uv}) \cap D$.

Double-counting the edges between $V(B_{uv}) \cap D$ and $V(B_{uv}) \cap \overline{D}$, these observations imply

$$2d(B_{uv}) \leq 3\bar{d}(B_{uv}) - 2.$$  (9)

Claim 3: $d(B_{uv}) \leq 2\bar{d}(B_{uv}) - 3$ for every edge $uv$ in $G[D]$.

Proof of Claim 3. By (9), we have $\bar{d}(B_{uv}) \geq \left\lceil \frac{2}{3}(d(B_{uv}) + 1) \right\rceil$. For $d(B_{uv}) \geq 3$, this implies $\bar{d}(B_{uv}) \geq \frac{1}{2}(d(B_{uv}) + 3)$, and, hence, the claim. Since $\bar{d}(B_{uv}) \geq |\{u, v\}| = 2$, the claim
holds for $d(B_{uv}) \leq 1$. Now, let $d(B_{uv}) = 2$. Clearly, we may assume $\bar{d}(B_{uv}) = 2$. Let $V(B_{uv}) \cap D = \{u', v'\}$. By the definition of $B_{uv}$, the special $uv$-cycles are two triangles, and, hence, there are all possible four edges between $\{u, v\}$ and $\{u', v'\}$. Since $G$ is not a $K_4$, the vertices $u'$ and $v'$ are not adjacent. Now, the set $D' = (D \setminus \{u', v'\}) \cup \{u, v\}$ is a maximum dissociation set of $G$ such that $G[D']$ has two isolated vertices, which contradicts (7).

Let $uv$ be an edge in $G[D]$. An induced subgraph $H$ of $G'$ is a bubble extension of $B_{uv}$ if

- there is a special $uv$-cycle $C$ and
- an induced alternating path $P$ between a vertex $x$ from $V(C) \cap D_0$ and a vertex $y$ from $D$ such that
  - $V(P) \not\subseteq V(B_{uv})$,
  - $x$ is the only common vertex of $C$ and $P$,
  - $y$ is adjacent to some vertex $z$ from $V(C) \cap D$,
  - $H = G[V(C) \cup V(P)]$, and
  - $E(H) = E(C) \cup E(P) \cup \{yz\}$.

Note that $x$ and $z$ are the only vertices of $H$ that are of degree 3 in $H$, that all remaining vertices of $H$ have degree 2 in $H$, and that $d(H) = \bar{d}(H)$.

Claim 4: No bubble has a bubble extension.

Proof of Claim 4. Suppose, for a contradiction, that $H$ is a bubble extension of the bubble $B_{uv}$. Let $C, P, x, y, z$ be as above. By symmetry, we may assume that $z$ lies on the unique path in $C$ between $x$ and $v$ that avoids $u$. This implies that, for every vertex $w$ in $(V(H) \cap \bar{D}) \setminus \{u, v\}$, there is an induced alternating path between $w$ and $u$, and Claim 1 implies $(V(H) \cap \bar{D}) \setminus \{u, v\} \subseteq D_0$.

Let $H'$ be an inclusionwise maximal induced subgraph of $G'$ such that the following conditions (i), (ii), and (iii) hold:

(i) \begin{itemize}
  \item $H \subseteq H'$,
  \item $u$ and $v$ have degree 2 in $H'$,
  \item $(V(H') \cap \bar{D}) \setminus \{u, v\} \subseteq D_0$,
  \item $yz$ is the only edge in $G'[V(H') \cap D]$, and
  \item $d(H') = \bar{d}(H')$.
\end{itemize}

(ii) For every vertex $w$ in $V(H') \setminus \{u, v\}$, the graph $H'$ contains an alternating path $P_u(w)$ between $w$ and $u$ as well as an alternating path $P_v(w)$ between $w$ and $v$ such that
\begin{itemize}
  \item $P_u(w)$ is induced and
  \item either $P_v(w)$ is induced or $E[G[V(P_u(w))] \setminus E(P_v(w)) = \{yz\}$, and $P$ is a subpath of $P_u(w)$ as well as of $P_v(w)$.
\end{itemize}

(iii) Exactly one vertex $y'$ from $V(H') \cap D$ has exactly one neighbor in $V(H') \cap \bar{D}$, and all remaining vertices from $V(H') \cap D$ have exactly two neighbors in $V(H') \cap \bar{D}$.
Since \((H, y)\) satisfies all properties required for 
\((H', y')\) in (i), (ii), and (iii), the graph \(H'\) is well defined. The proof is completed by showing that 
\(H'\) is a proper subgraph of an 
induced subgraph \(H''\) of \(G'\) satisfying (i), (ii), and (iii), contradicting the maximality of \(H'\).

Let \(D' = V(H') \cap D\) and \(\overline{D}' = V(H') \cap \overline{D}\).

By (iii), the vertex \(y'\) has exactly one neighbor \(x'\) in \(\overline{D}\setminus \overline{D}'\). Note that \(y'\) is the only 
neighbor of \(x'\) in \(V(H')\), and, hence, extending the two paths \(P_u(y')\) and \(P_v(y')\) guaranteed for \(y'\) in (ii) by the edge towards \(x'\), yields two paths \(P_u(x')\) and \(P_v(x')\) with the 
properties stated in (ii) also for \(x'\). By Claim 1, the induced alternating path \(P_u(x')\) 
together with \(x'\) sends exactly three 
edges between \([x'] \cup (\overline{D}\setminus \{u, v\})\) and \(D\setminus D'\).

If some vertex in \(D\setminus D'\) has two neighbors in \([x'] \cup (\overline{D}\setminus \{u, v\})\), say \(a'\) and \(a''\), then the 
two paths \(P_0(a')\) and \(P_0(a'')\) share at least the edge incident with \(u\), and, hence, their 
union contains a \(D_0\)-alternating cycle, contradicting the choice of \(G'\). Hence, every vertex 
in \(D\setminus D'\) has at most one neighbor in \([x'] \cup (\overline{D}\setminus \{u, v\})\).

Since \(y'\) is an edge in \(D'\), and \(|D\setminus D'| = d(H') - 2\), this implies the existence of 
some vertex \(y''\) in \(D\setminus D'\) that has exactly one neighbor in \([x'] \cup (\overline{D}\setminus \{u, v\})\), say \(x''\), and 
no neighbor in \(D'\).

If \(y''\) is not adjacent to \(u\) or \(v\), then \(H'' = G[V(H') \cup \{x', y''\}]\) satisfies (i), (ii), and (iii), 
contradicting the choice of \(H'\). Note that extending the two paths \(P_u(x'')\) and \(P_v(x'')\) 
guaranteed for \(x''\) in (ii) by the edge towards \(y''\), yields two paths \(P_u(y'')\) and \(P_v(y'')\) 
with the properties stated in (ii) also for \(y''\). Hence, we may assume that \(y''\) is adjacent 
to \(u\) or \(v\).

If \(y''\) is adjacent to \(u\), then \(P_u(x'')\) together with \(y''\) and the two edges \(uy''\) and \(x''y''\) 
yields an induced alternating cycle that contains a vertex from \(D_1\), contradicting Claim 2.

If \(y''\) is adjacent to \(v\), and \(P_v(x'')\) is induced, then \(P_v(x'')\) together with \(y''\) and the two 
edges \(vy''\) and \(x''y''\) yields an induced alternating cycle that contains a vertex from \(D_1\), 
contradicting Claim 2. Hence, we may assume that \(y''\) is adjacent to \(v\), and that \(P_v(x'')\) is 
not induced. By (ii), this implies that \(P_u(x'')\) contains \(P\) as a subpath. Now, the path \(P_u(x'')\) 
together with the vertices \(v\) and \(y''\) and the edges \(uv, vy''\), and \(x''y''\) yields a special \(uv\)- 
cycle. By the definition of the bubble \(B_{uv}\), this implies that all vertices of \(P\) belong to \(B_{uv}\), 
contradicting the condition \(V(P) \notin V(B_{uv})\) in the definition of the bubble extension \(H\).

This completes the proof. □

Claim 5: Every two bubbles are disjoint.

Proof of Claim 5. Suppose, for a contradiction, that the two bubbles \(B_{uv}\) and \(B_{u'v'}\) are not 
disjoint, where \(uv\) and \(u'v'\) are distinct edges in \(G[\overline{D}]\). By Claim 1, we obtain 
\(u', v' \notin V(B_{uv})\) and \(u, v \notin V(B_{u'v'})\). Let \(x \in V(B_{uv}) \cap V(B_{u'v'})\). Let \(C\) be a special \(uv\)-cycle 
and let \(C'\) be a special \(u'v'\)-cycle such that \(x\) lies on \(C\) and on \(C'\). Let \(P\) be a shortest path 
in \(C'\) between \([u', v']\) and \(V(C)\). By Claim 1, there is at least one edge between \(V(P) \cap D\)
and $V(C) \cap D$. This implies that $C \cup P$ contains a bubble extension of $B_{uv}$, contradicting Claim 4.

Let

$$G'' = G' - \bigcup_{uv \in E(G[D])} V(B_{uv}).$$

**Claim 6:** $\mathcal{D} \cap V(G'')$ is a maximum independent set in $G''$, and $d(G'') \leq 2\bar{d}(G'')$.

**Proof of Claim 6.** Since $\alpha(G'') \geq \frac{1}{2}d(G'')$, the first part of the statement implies the second. Therefore, suppose, for a contradiction, that the independent set $\mathcal{D} \cap V(G'')$ is not a maximum independent set in $G''$. This implies that a maximum independent set $I''$ in $G''$ intersects $D$. Let $I''$ be chosen such that $|I'' \cap D|$ is as small as possible. Let $F = G[I'' \cup (V(G'') \cap \mathcal{D})]$. Since $G''$ contains no $\mathcal{D}_0$-alternating cycle, the graph $F$ is a forest.

If $V(G'') \cap \mathcal{D}$ is not empty, then let $T$ be a component of $F$ that contains a vertex from $\mathcal{D}$. By the choice of $I''$, we have $|V(T) \cap D| > |V(T) \cap \mathcal{D}|$, which implies that $T$ contains two distinct leaves from $D$. In this case, let $P$ be a path in $T$ between two distinct leaves $x$ and $x'$ that belong to $D$. If $V(G'') \cap \mathcal{D}$ is empty, then let $P$ be a path of length 0 consisting of a vertex $x = x'$ from $I''$.

There are two distinct edges $xy$ and $x'y'$ with $y, y' \in \mathcal{D} \setminus V(G'')$. By the definition of $G'$, it follows that $y$ and $y'$ both lie in bubbles, say $y \in B_{uv}$ and $y' \in B_{u'v'}$, where $uv = u'v'$ is possible.

If $uv \neq u'v'$, then $B_{uv} \cup B_{u'v'} \cup P$ contains an induced alternating path with both endpoints in $\mathcal{D}_1$ or a bubble extension of $B_{uv}$ or $B_{u'v'}$, which implies a contradiction to Claim 1 or Claim 4. Hence, we obtain that $uv = u'v'$. By Claim 2 and the definition of $B_{uv}$, we may assume, by symmetry, that $y$ is distinct from $u$ and $v$. Let $C$ be a special $uv$-cycle containing $y$.

First, suppose that $y' = u$. By Claim 2, the alternating cycle contained in $C \cup P$ that contains $P$ and intersects $\mathcal{D}_1$ in $u$ is not induced. By Claim 4, this implies that $x'$ has its neighbor in $D$ on the path in $C$ between $y$ and $u$ that avoids $v$. Since the path in $C$ between $y$ and $v$ that avoids $u$ together with $P$ and the edges $uv, xy$, and $x'y'$ is not a special $uv$-cycle, the union of $C$ and $P$ contains a bubble extension, contradicting Claim 4. Hence, we may assume, by symmetry, that $y'$ is distinct from $u$ and $v$. Let $C'$ be a special $u'v'$-cycle containing $y'$. Since $G'$ contains no $\mathcal{D}_0$-alternating cycle, we obtain $V(C) \cap V(C') = \{u, v\}$, that is, in particular, $y' \notin V(C)$ and $y \notin V(C')$. By Claim 4, there is no edge between $(V(C) \cup V(C')) \cap D$ and $V(P) \cap D$. Let $P_u$ be the path in $C$ between $y$ and $u$ avoiding $v$, let $P_v$ be the path in $C$ between $y$ and $u$ avoiding $v$, and let $P_u'$ be the path in $C'$ between $y'$ and $u$ avoiding $v$, and let $P_v'$ be the path in $C'$ between $y'$ and $v$ avoiding $u$. By Claim 2, there is an edge between $V(P_u) \cap D$ and $V(P_v') \cap D$ as well as between $V(P_u') \cap D$ and $V(P_v) \cap D$. By the definition of $B_{uv}$ and since no vertex of $P$ belongs to $B_{uv}$, there is an edge between $V(P_u) \cap D$ and $V(P_v') \cap D$ as well as between $V(P_v') \cap D$ and $V(P_v) \cap D$. Now, there is a bubble extension of $B_{uv}$ using $C, P$ and a proper subpath of $P_u'$, contradicting Claim 4.
Let $\mathcal{X}$ be the collection of all alternating cycle sets that were recursively removed from $G$ to construct $G'$. By (7), (8), Claim 3, and Claim 6, we obtain

$$
10k = d(G)
= \sum_{X \in \mathcal{X}} d(X) + \sum_{e \in E(G[D_1])} d(B_e) + d(G'')
\leq \sum_{X \in \mathcal{X}} 2\overline{d}(X) + \sum_{e \in E(G[D_i])} (2\overline{d}(B_e) - 3) + 2\overline{d}(G'')
= 2\overline{d}(G) - 3s
= 2(r + 2s) - 3s
= 10k.
$$

Since equality holds throughout this inequality chain, each individual estimate must be satisfied with equality, that is,

- $d(X) = 2\overline{d}(X)$ for every alternating cycle set $X$ in $\mathcal{X}$,
- $d(B_e) = 2\overline{d}(B_e) - 3$ for every edge $e$ of $G[D_1]$, and
- $d(G'') = 2\overline{d}(G'').$

Since, by Claim 6, we have $d(G'') \leq \text{diss}(G'') \leq 2\alpha(G'') = 2\overline{d}(G'') = d(G'')$, the subcubic graph $G''$ satisfies $\text{diss}(G'') = 2\alpha(G'')$, and, therefore, each of its components belongs to $G$ as described in Theorem 2. If $G''$ contains a copy of $K^*_4$, then this subgraph contains no two adjacent vertices from $D_1$, which easily implies the existence of a $D_0$-alternating cycle in that subgraph, contradicting the construction of $G'$. Since $d(G'') = 2\overline{d}(G'')$, the graph $G''$ arises from the union of disjoint copies of $K_3$ by adding additional edges as described for Theorem 2. Since $G''$ does not contain $D_0$-alternating cycles, these additional edges are actually all bridges in $G''$, that is, the disjoint copies of $K_3$ are the only cycles in $G''$. Note that, while copies of $K_3$ in a graph as in $\mathcal{X}$ contribute twice as many vertices to $D$ than to $D$, copies of $K_2$ only contribute vertices to $D$, which implies that the construction of $G''$ does not use any copy of $K_2$.

**Claim 7:** All bubbles are isomorphic to $K_3$, $K^*_4$, or one of the two graphs in Figure 8.

**Proof of Claim 7.** Let $uv$ be an edge of $G[D_1]$. Since $d(B_{uv}) = 2\overline{d}(B_{uv}) - 3 \geq \frac{4}{3}(d(B_{uv}) + 1) - 3$, we obtain that $d(B_{uv}) \in \{1, 3, 5\}$.

If $d(B_{uv}) = 1$, then $\overline{d}(B_{uv}) = 2$, and $B_{uv}$ is a triangle.

Now, let $d(B_{uv}) = 3$, and, hence, $\overline{d}(B_{uv}) = 3$. If $V(B_{uv}) \cap D$ is independent, then, since the vertices in $V(B_{uv}) \cap D$ have all their neighbors in $D$ within the bubble $B_{uv}$, the graph $G$ has an independent set containing the three vertices from $V(B_{uv}) \cap D$ as well as

![Figure 8](image-url) Two bubbles of order 9.
$r + s - 2$ further vertices, one from each component of $G[D]$ that does not intersect $B_{uv}$, which yields a contradiction to (7). Hence, the set $V(B_{uv}) \cap D$ contains exactly one edge. Let $V(B_{uv}) \cap D = \{u_1, u_2, u_3\}$ such that $u_1 u_2$ is an edge, and let $V(B_{uv}) \cap D = \{u, v, w\}$. Since $w$ lies on a special $uv$-cycle, we obtain, by symmetry, that $w$ is adjacent to $u_2$ and $u_3$, that $u_2$ is adjacent to $u$, and that $u_3$ is adjacent to $v$. Since $u_1$ lies on a special $uv$-cycle, the vertex $u_1$ is adjacent to either $u$ and $v$ (see the right part of Figure 9), or $u$ and $w$ (see the left part of Figure 9), or $v$ and $w$. In the third case, the independent set $\{u, u_1, u_3\}$ can be extended to an independent set in $G$ with more than $r + s$ vertices, contradicting (7).

Altogether, the bubble $B_{uv}$ is isomorphic to $K^*_4$.

Finally, let $d(B_{uv}) = 5$, and, hence, $\overline{d}(B_{uv}) = 4$. Similarly as above, we obtain that $V(B_{uv}) \cap D$ contains exactly two edges. In view of the definition of a bubble, it is again easy to verify that $B_{uv}$ is isomorphic to one of the two graphs in Figure 8. Counting the edges of $B_{uv}$ between $D$ and $\overline{D}$ actually implies that there is only one edge of $G$ leaving $V(B_{uv})$, and that both endpoints of this edge belong to $D$. \[\square\]

By the final remark of the previous proof, the unique vertices of degree two in the bubbles $B_e$ of order 9 shown in Figure 8 both lie in $D$, and their neighbors outside of $B_e$ also lie in $D$. For bubbles $B_e$ that are copies of $K^*_4$, these observations easily imply that $V(B_e) \cap D$ is as shown in Figure 9.

Claim 8: For $X \in \mathcal{X}$ with $|X| > 6$, the set $D \cap X$ is a maximum independent set in $G[X]$.

Proof of Claim 8. By definition and since $d(X) = 2\overline{d}(X)$, the induced subgraph $G[X]$ of $G$ contains a $D_0$-alternating cycle $C : u_1 a_1 u_2 a_2 ... a_\ell u_\ell u_1$, where $\{a_1, ..., a_\ell\} = \overline{D} \cap X$, and the set $W$ defined as $X \setminus V(C)$ consists of $\ell$ distinct vertices $w_1, ..., w_\ell$ from $D$, where $w_i$ is the third neighbor of $u_i$ in $D$ for $i \in [\ell]$. Since $|X| > 6$, we have $\ell \geq 3$.

First, we show that, for every $i \in [\ell]$, there is an induced alternating path $P_i$ between $w_i$ and a vertex from $\overline{D}$ in some bubble $B_i$, where $V(P_i) \cap X = \{w_i\}$ and $|V(P_i) \cap V(B_j)| = 1$. Let $i \in [\ell]$. Let $y_i$ be the neighbor of $w_i$ in $D$ that is distinct from $u_i$. Since $w_i \in X$, the vertex $y_i$ cannot belong to some alternating cycle set in $X$. If $y_i$ belongs to some bubble, then $P_i : w_i y_i$ is the desired path. Otherwise, the vertex $y_i$ belongs to $G''$, and, hence, lies in a triangle $y_i w'_i w''_i$ in $G''$ with $w'_i, w''_i \in D$. Let $y'_i$ be the neighbor of $w'_i$ in $\overline{D}$ that is distinct from $y_i$. Again, either $y'_i$ belongs to some bubble, in which case $P_i : w_i y_i w'_i y'_i$ is the desired path, or $y'_i$ lies in a second triangle in $G''$. Continuing this
reasoning, it follows that the desired path $P_i$ can be obtained using edges in distinct triangles in $G''$ as illustrated in Figure 10.

Note that, if $y_i$ does not belong to a bubble but to the triangle $y_i w_i' w_i''$ as in Figure 10, then we have a choice of using either $w_i'$ or $w_i''$ to construct the path $P_i$. This leads to an alternative choice $P_i'$ for the path $P_i$ that is disjoint from $P_i$ after $y_i$, and leads to some bubble $B_i'$. Note that $B_i$ can only coincide with $B_i'$ if $B_i$ is a triangle, and the two paths $P_i$ and $P_i'$ use the two disjoint edges into $B_i$ that are incident with the vertices of $B_i$ from $D$.

If the unique vertex in $V(B_i) \cap V(B_j)$ belongs to $D$, then let $Q_i = P_i$. Otherwise, the bubble $B_i$ and its intersection with $D$ is necessarily as in the right part of Figure 9. In this case, let the path $Q_i$ arise by extending $P_i$ by one vertex in $V(B_i) \cap D$ that has no neighbors outside of $V(B_i)$, and one vertex in $V(B_i) \cap D_1$.

In view of the claimed statement, we suppose, for a contradiction, that $G[X]$ has a maximum independent set $I$ that is strictly bigger than $\cap I V D_1$. Let $I$ be chosen such that $|I \cap D|$ is as small as possible. Let $F = G[I \cup (D \cap X)]$.

First, suppose that $F$ is not a forest. By construction, the only possible cycle in $F$ is $C$, that is, $F$ consists of $C$ and at least one vertex, say $w_1$, from $W$. Now, the union of $I' = (V(Q_i) \cup V(C)) \cap D$ with a maximal independent set in $G[D \setminus N_C(I')]$ is an independent set in $G$ that contains more than $r + s$ vertices, contradicting (7). Hence, the graph $F$ is a forest.

Let $T$ be a component of $F$ that contains a vertex, say $w_1$, from $W$. By construction, the order of $T$ is at least 2. If $w_1$ is the only endvertex of $T$ from $W$, then $I \Delta V(T) = (I \setminus V(T)) \cup (V(T) \setminus I)$ is a maximum independent set of $G[X]$ containing less vertices from $D$, contradicting the choice of $I$. Hence, the tree $T$ contains an alternating induced path $P$ between $w_1$ and another vertex, say $w_t$, from $W$. By symmetry, we may assume that $P : w_1 u_i u_2 \ldots u_t u_i w_t$.

If the two bubbles $B_i$ and $B_t$ are distinct, then $Q_i, P,$ and $Q_t$ yield an induced alternating path with both endpoints in $D_1$, contradicting Claim 1. Hence, we obtain that $B_i = B_t$. Since the path $P_i$ and $P_t$ are disjoint, the bubble $B_i$ must be a triangle $y y' z$, the path $P_i$ ends in $y$, the path $P_t$ ends in $y'$, and $z$ belongs to $D$. Since assuming that $B_i$ and $B_t$ differ leads to a contradiction, the comment concerning the different choices for $P_i$ after Figure 10 implies that $P_i$ and $P_t$ both have length one, that is, the paths $P_i$ and $P_t$ consist of the edges $w_1 y$ and $w_t y'$, respectively.

If $i \geq 3$, then $B_2$ is distinct from $B_1$. In this case, the paths $P_1$ or $P_t$, a suitable part of $P$, and $Q_2$ yield an induced alternating path with both endpoints in $D_1$, contradicting Claim 1. Hence, we obtain that $t = 2$, that is, the length of $P$ is four.

Figure 11 illustrates the setup for the following arguments.

If the set $I' = \{u_2, w_1, w_2, z\}$ is independent, then the union of $I'$ with a maximal independent set in $G[D \setminus N_C(I')]$ is an independent set in $G$ that contains more than

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**Figure 10** An induced alternating path $P_i$ between $w_i$ and some vertex in a bubble $B_i$. 
If the neighbor of $w_1$ in $D$ does not lie in $X \setminus \{w_2\}$, then, using $P_1$, and the fact that $B_1$ is distinct from $B_\ell, B_{\ell-1}, \ldots, B_3$, repeated applications of Claim 1 imply in turn that $w_2$ is adjacent to $u_1$, that $w_{\ell-1}$ is adjacent to $u_\ell$, that $w_{\ell-2}$ is adjacent to $u_{\ell-1}$, and so forth up to that $w_3$ is adjacent to $u_4$. Now, using the paths $yw_2u_2u_3u_3w_3$ and $Q_3$, Claim 1 implies that $w_2$ is adjacent to $u_3$. Yet now, the paths $yw_1u_1u_2u_2u_3u_3w_3$ and $Q_3$ yield a contradiction to Claim 1. Hence, by symmetry, the neighbors in $D$ of both vertices $w_1$ and $w_2$ lie in $X \setminus \{w_1, w_2\}$. This implies that the neighbor of $z$ in $D$ is $u_2$.

If $w_2$ is adjacent to $w_j$ for some $j \in \{3, \ldots, \ell\}$, then, similarly as above, repeated applications of Claim 1 imply that $u_3w_3, \ldots, u_{j-1}w_{j-1}$ as well as $u_1w_1, \ldots, u_{j+2}w_{j+1}$ are edges in $G$, and either $yw_1u_1u_2u_2u_3 \ldots u_j$ or $yw_1u_1u_1u_\ell u_\ell \ldots u_j$ together with $Q_j$ yields a contradiction to Claim 1. Hence, we may assume that $w_2$ is adjacent to $u_j$ for some $j \in \{\ell\} \setminus \{2\}$. Repeated applications of Claim 1 imply that $u_3w_3, \ldots, u_{j-1}w_{j-1}$ as well as $u_1w_1, \ldots, u_{j+1}w_j$ are edges in $G$. Now, the paths $yw_1u_1u_2u_2u_3 \ldots u_j$ and $Q_j$ yield a contradiction to Claim 1, which completes the proof.

Claim 8 implies that $\text{diss}(G[X]) = 2\alpha(G[X])$ for every $X$ in $\mathcal{X}$ with $|X| > 6$. Hence, each $G[X]$ belongs to $\mathcal{G}$, and, hence, arises from the disjoint union of copies of $K_3$ and $K_4^*$ by adding edges as specified for $\mathcal{G}$.

Claim 9: For $X \in \mathcal{X}$ with $|X| = 6$, the subgraph $G[X]$ of $G$ is either $K_4^*$, or arises from the disjoint union of two copies of $K_3$ by adding edges, or is completely contained in an induced subgraph of $G$ that is isomorphic to the graph shown in the left of Figure 8.

Proof of Claim 9. Using the notation from the proof of Claim 8, let $X = \{w_1, w_2, u_1, u_1, u_2, u_2\}$, where $C : u_1u_1u_2u_2u_1$. Similarly as in the proof of Claim 8, we obtain induced alternating paths $P_i$ and $Q_i$ between $w_i$ and some bubble $B_i$ for $i \in \{2\}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{$B_1, P_1, P_2,$ and $C$.}
\end{figure}
First, suppose that $u_1$ and $u_2$ are adjacent. If $w_1$ and $w_2$ are adjacent, then $G[X]$ is a copy of $K_4^*$. Hence, we may assume that $w_1$ and $w_2$ are not adjacent. Claim 1 implies that $B_1 = B_2$, which implies that $B_1$ is a triangle $yy'z$, $P_1$ ends in $y$, and $P_2$ ends in $y'$. The observation after the definition of $P_i$ in Claim 8 implies that the paths $P_1$ and $P_2$ consist of the edges $w_1y$ and $w_2y'$, respectively. If the set $I' = \{z, w_1, w_2, u_1\}$ is independent, then the union of $I'$ with a maximal independent set in $G[D \setminus N_G(I')]$ is an independent set in $G$ that contains more than $r + s$ vertices, contradicting (7). Hence, by symmetry, we may assume that $z$ is adjacent to $w_1$ and $G[X]$ is completely contained in the induced subgraph $G[X \cup V(B_i)]$ of $G$ that is isomorphic to the left graph shown in Figure 8. Hence, we may assume that $u_1$ and $u_2$ are not adjacent.

If $\cap I = \{u_1, u_2\} \cup (V(Q_1) \cap D)$ is independent, then the union of $I'$ with a maximal independent set in $G[D \setminus N_G(I')]$ is an independent set in $G$ that contains more than $r + s$ vertices, contradicting (7). Hence, by symmetry, we obtain that $w_1$ is adjacent to $u_1$, and that $w_2$ is adjacent to $u_2$, that is, the graph $G[X]$ arises from the disjoint union of two copies of $K_3$ by adding edges. This completes the proof of the claim.

At this point of the proof we know that $V(G)$ can be partitioned into sets $V_i$ such that each induced subgraph $G[V_i]$ is one of the graphs illustrated in Figure 12, where we also illustrate the possible intersections of $V_i$ with $D$ (up to symmetry).

Let $n_3$ be the number of indices $i$ such that $G[V_i]$ is a triangle, and let $k_3$ be the number of those $G[V_i]$ with $|V_i \cap D| = 1$. Let $n_2$ be the number of indices $i$ such that $G[V_i]$ is a $K_4^*$, and let $k_2$ be the number of those $G[V_i]$ with $|V_i \cap D| = 3$. Let $n_1$ be the number of indices $i$ such that $G[V_i]$ has order 9. Contracting each $G[V_i]$ to a single vertex yields a connected subcubic multigraph $H$. Since $18k = n = 3n_3 + 6n_2 + 9n_1$, we have $6k = n_3 + 2n_2 + 3n_1$. The edges of $H$ that correspond to edges of $G$ between vertices from $D$ that belong to different sets $V_i$ form a matching $M$ in $H$ with $2|M| = k_3 + k_2 + n_1$ that covers all vertices of degree 1 in $H$. Since $10k = |D| = 2n_3 - k_3 + 4n_2 - k_2 + 6n_1 - n_1$, we obtain $k_3 + k_2 + n_1 = 2(n_3 + 2n_2 + 3n_1) - 10k = 2k$, that is, the matching $M$ of $H$ has size exactly $k$. The multigraph $H$ has exactly $\frac{1}{2}(3n_3 + 2n_2 + n_1)$ edges. Furthermore, the multigraph $H$ has exactly $2(n_3 - k_3) + 2(n_2 - k_2)$ edges $e$ that correspond to an edge $uv$ of $G$ such that $u \in D$ and $v \in \overline{D}$, and $u$ and $v$ lie in different $V_i$. Since

![Figure 12](image-url)  
The induced subgraphs $G[V_i]$ and their possible intersection with $D$; the vertices that belong to $D$ are encircled.
$2(n_3 - k_3) + 2(n_2 - k_2) + |M| = \frac{1}{2}(3n_3 + 2n_2 + n_1)$, the multigraph $H$ actually has no edge $e$ that corresponds to an edge $uv$ of $G$ such that $u, v \in D$, and $u$ and $v$ lie in different $V_j$. This implies that the matching $M$ is induced. Furthermore, it implies that orienting the edges $e$ of $H$ that do not belong to $M$ away from the set $V_i$ containing the element of $D$ in $e$ yields an orientation $\overrightarrow{H - M}$ of $H - M$ such that every vertex of $H$ that is not incident with an edge in $M$ has exactly two outgoing edges. Altogether, it follows that $G$ belongs to $\mathcal{G}_k$, which completes the proof.

3 | CONCLUSION

While Theorem 7 provides deep structural insights concerning the extremal graphs for Theorem 1, it remains unclear whether it leads to a polynomial time recognition algorithm for these graphs.

There are several natural graph classes for which it seems interesting to study lower bounds on the independence number in terms of the dissociation number. Note that the inequality (2) from the proof of Theorem 1 implies $\alpha(G) \geq 3\text{diss}(G) - 2\alpha(G)$, and actually holds for every cubic graph $G$ of order $n$. Now, for a triangle-free cubic graph $G$ of order $n$, it is known that

$$\alpha(G) \geq \begin{cases} \frac{5n}{14}, & \text{if } G \text{ is connected} \quad [10, 15], \\ \frac{11n - 4}{30}, & \text{if } G \text{ is planar} \quad [11] \text{ or } G \text{ is 2-connected and } n \geq 23 \quad [7], \\ \frac{3n}{8}, & \text{if } G \text{ is planar} \quad [11] \text{ or } G \text{ is 2-connected and } n \geq 23 \quad [7]. \end{cases}$$

and combining these estimates with $n \geq 3\text{diss}(G) - 2\alpha(G)$ yields

$$\alpha(G) \geq \begin{cases} \frac{5 \text{diss}(G)}{8}, & \text{if } G \text{ is connected}, \\ \frac{33 \text{diss}(G) - 4}{52}, & \text{if } G \text{ is planar or } G \text{ is 2-connected and } n \geq 23. \end{cases}$$

It remains to determine what are the best possible estimates for these classes of triangle-free cubic graphs. For connected triangle-free subcubic graphs $G$, we conjecture $\alpha(G) \geq \frac{5}{8}\text{diss}(G) - \frac{1}{4}$.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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