Algebraic Structures in Extended Geometry

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Abstract—Extended geometry is a unifying framework including exceptional field theory (XFT) and double field theory (DFT). It gives a geometric underpinning of the duality symmetries of M-theory. In this talk I give an overview of the surprisingly rich algebraic structures which naturally appear in the context of extended geometry. This includes Borcherds superalgebras, Cartan type superalgebras (tensor hierarchy algebras) and $L_\infty$ algebras. This is the written version of a talk based mainly on [1–6], presented at SQS 2017, Dubna, Aug. 2017.

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String theory/M-theory exhibits duality symmetries that mix gravitational and non-gravitational fields. Manifestation of such symmetries calls for a generalisation of the concept of geometry. It has been proposed that the compactifying space (torus) is enlarged to accommodate momenta (representing momenta and brane windings) in modules of a duality group. This leads to double geometry [7–31] in the context of T-duality, and exceptional geometry [32–52] in the context of U-duality. These classes of models are special cases of extended geometries, and can be treated in a unified manner [4, 6]. The duality group is in a certain sense present already in the uncompactified theory. It becomes “geometrised”.

In the present talk, I will
—Describe the basics of extended geometry, with focus on the gauge transformations;
—Describe the appearance of Borcherds superalgebras and Cartan-type superalgebras (tensor hierarchy superalgebras);
—Indicate why $L_\infty$ algebras provide a good framework for describing the gauge symmetries.
—Point out some questions and directions.

The focus will thus be on algebraic aspects, and less on geometric ones.

Consider compactification from 11 to 11 – $n$ dimensions on $T^n$. As is well known, fields and charges fall into modules of $E_n$.

To be explicit, take $n = 7$ as an example. The gauge parameters $\xi^M$ in 56 of $E_7$ decompose as:

$$\xi^m, \lambda_{mn}, \tilde{\lambda}_{mn}, \tilde{\lambda}_{mnpqr} \leftrightarrow \xi^M = 56.$$ 

We recognise the parameters for diffeomorphisms, gauge transformations of the 3-form and dual 6-form and a parameter for “dual diffeomorphisms”. The scalar fields are in the coset $E_7(7)/K(E_7(7)) = E_7(7)/(SU(8)/\mathbb{Z}_2)$. The dimension of coset is: $133 - 63 = 70$, and it is parametrised by

$$g_{mn}, C_{mn}, \tilde{C}_{mn}, \tilde{C}_{mnpqr} \leftarrow G_{MN} = 28 + 35 + 7 = 70.$$ 

From the point of view of $N=8$ supergravity in $D=4$, this is the scalar field coset. Now it becomes a generalised metric. There are also mixed fields (generalised graviphotons): 1-forms in $R_1$, etc.

The situation for T-duality is simpler. Compactification from 10 to 10 – $d$ dimensions gives the (continuous) T-duality group $O(d, d)$. The momenta are complemented with string windings to form the $2d$-dimensional module.

Note that the continuous duality group is not to be seen as a global symmetry. Discrete duality transformations in $O(d, d; \mathbb{Z})$ or $E_{n(0)}(\mathbb{Z})$ arise as symmetries in certain backgrounds, roughly as the mapping class

| $n$ | $E_{n(0)}$ | $R_1$ |
|-----|-------------|-------|
| 3   | $SL(3) \times SL(2)$ | (3, 2) |
| 4   | $SL(5)$     | 10    |
| 5   | $Spin(5, 5)$ | 16    |
| 6   | $E_6$       | 27    |
| 7   | $E_7(7)$    | 56    |
| 8   | $E_{8(8)}$  | 248   |
| 9   | $E_{9(9)}$  | Fund  |

Table 1. A list of U-duality groups

1 The article is published in the original.
group $SL(n, \mathbb{Z})$ arises as discrete isometries of a torus. The role of the continuous versions of the duality groups is analogous to that of $GL(n)$ in ordinary geometry (gravity).

One has to decide how tensors transform. The generic recipe is to mimic the Lie derivative for ordinary diffeomorphisms:

$$L_u V^m = U^a \partial_a V^m - \frac{\partial_a U^m}{g^a} V^n.$$

The first term is a transport term, and the second one a transformation.

In the case of $U$-duality, the role of $GL(n)$ is assumed by $E_{n(n)} \times \mathbb{R}^+$, and

$$\mathcal{L}_u V^M = L_u V^M + Y^{MN} \partial_N U^P V^Q = U^N \partial_N V^M + Z^{MN} \partial_N U^P V^Q,$$

where $Z^{MN} = -\alpha_n P^{M \alpha} Q, \quad N^P \beta_{\alpha} \delta^M \delta^N = Y^{MN} \delta^M - \delta^P \delta^Q$ projects on the adjoint of $E_{n(n)} \times \mathbb{R}^+$, so that the transformation term contains a parameter for an $e_n \otimes \mathbb{R}$ transformation.

The transformations form an “algebra” for $n \leq 7$:

$$[\mathcal{L}_U, \mathcal{L}_V] W^M = \mathcal{L}_{[U, V]} W^M,$$

where the “Courant bracket” is $[U, V]^M = \frac{1}{2} (\mathcal{L}_U V^M - \mathcal{L}_V U^M)$, provided that the derivatives fulfill a “section constraint”.

| $n$ | $R_1$ | $R_2$ |
|-----|-------|-------|
| 3   | (3, 2) | (3, 1) |
| 4   | 10    | 5     |
| 5   | 16    | 10    |
| 6   | 27    | 27    |
| 7   | 56    | 133   |
| 8   | 248   | 1 \otimes 3875 |

Fig. 1. The module $R_1$.

Fig. 2. The module $R_2$.

The section constraint ensures that fields locally depend only on an $n$-dimensional sub-space of the coordinates, on which a $GL(n)$ subgroup acts. It reads $Y^{MN} \partial_M \ldots \partial_N = 0$, or

$$(\partial \otimes \partial)[R_1] = 0.$$

For $n \geq 8$ more local transformations, so called “ancillary transformations” [4] emerge, which are constrained local transformations in $\mathfrak{g}$.

The interpretation of the section condition is that the momenta locally are chosen so that they may span a linear subspace of cotangent space with maximal dimension, such that any pair of covectors $p, p'$ in the subspace fulfill $\langle p \otimes p' \rangle_{R_1} = 0$.

The corresponding statement for double geometry is $\eta^{MN} \partial_M \otimes \partial_N = 0$, where $\eta$ is the $O(d, d)$-invariant metric. The maximal linear subspace is a $d$-dimensional isotropic subspace, and it is determined by a pure spinor $\Lambda$. Once a $\Lambda$ is chosen, the section condition can be written $\Gamma^M \Lambda \partial_M = 0$. An analogous linear construction can be performed in the exceptional setting. The section condition in double geometry derives from the level matching condition in string theory. Locally, supergravity is recovered. Globally, non-geometric solutions are also obtained.

There is a universal form [1, 3, 4] of the generalised diffeomorphisms for any Kac–Moody algebra and choice of coordinate representation. Let the coordinate representation be $R(\lambda)$, for $\lambda$ a fundamental weight dual to a simple root $\alpha$ (the construction can be made more general). Then

$$\sigma Y = -\eta_{AB} T^A \otimes T^B + (\lambda, \lambda) + \sigma - 1,$$

where $\eta$ is the Killing metric and $\sigma$ the permutation operator, $\sigma(a \otimes b) = (b \otimes a)\sigma$.

This follows from the existence of a solution to the section constraint in the form of a linear space:

—Each momentum must be in the minimal orbit.

Equivalently, $p \otimes p \in R(2\lambda)$.

—Products of different momenta may contain $R(2\lambda)$ and $R(2\lambda - \alpha)$, where $R(2\lambda - \alpha)$ is the highest representation in the antisymmetric product. Expressing these conditions in terms of the quadratic Casimir gives the form of $Y$.

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I will skip the detailed description of the generalised
gravity. It effectively provides the local dynamics of gravity and 3-form, which are encoded by a vielbein $E_M^A$ in the coset $(E_{n(n)} \times \mathbb{R})/K(E_{n(n)})$.

The $T$-duality case is described by a generalised metric in the coset $O(d, d)/(O(d) \times O(d))$, parametrised by the ordinary metric and $B$-field.

With some differences from ordinary geometry, one can go through the construction of connection, torsion, metric compatibility etc., and arrive at generalised Einstein’s equations encoding the equations of motion for all fields. (This has been done for $n \leq 8$.)

For $n \geq 8$, the coset $E_{n(n)}/K(E_{n(n)})$ contains higher mixed tensors that do not carry independent physical degrees of freedom. They are removed by ancillary transformations that arise in the commutator between generalised diffeomorphisms [3, 4, 45, 48, 49].

One may introduce (local) supersymmetry. In the case of $T$-duality, the superspace is based on the fundamental representation of an orthosymplectic supergroup $OSp(d, d)$ of the exceptional cases are unexplored, but will be based on $\infty$-dimensional superalgebras [53].

The generalised diffeomorphisms do not satisfy a Jacobi identity. On general grounds, it can be shown that the “Jacobiator”

$$[[U, V], W] + \text{cyc} \neq 0,$$

but is proportional to $((U, V), W) + \text{cyc}$, where $(U, V) = \frac{1}{2}(\mathcal{L}_U V + \mathcal{L}_V U)$.

It is important to show that the Jacobiator in some sense is trivial. It turns out that $\mathcal{L}_{U, V} W = 0$ (for $n \leq 7$), and the interpretation is that it is a gauge transformation with a parameter representing reducibility (for $n \leq 6$). (The limits on $n$ in the statements here are due to non-covariance of the derivative arising at some point in the tensor hierarchy, see below. I will not go into details.)

In double geometry, this reducibility is just the scalar reducibility of a gauge transformation: $\delta B_2 = d\lambda_1$, with the reducibility $\delta \lambda_1 = d\lambda_0$.

In exceptional geometry, the reducibility turns out to be more complicated, leading to an infinite (but well defined) reducibility, containing the modules of tensor hierarchies, and providing a natural generalisation of forms (having connection-free covariant derivatives).

The reducibility continues, and there are ghosts at all levels $> 0$. The representations are those of a “tensor hierarchy”, the sequence of representations $R_n$ of $n$-form gauge fields in the dimensionally reduced theory.

$$R_1 \leftarrow \partial \quad R_2 \leftarrow \partial \quad R_3 \leftarrow \partial \quad \ldots$$

Example, $n = 5$:

$$16 \leftarrow \partial \; 10 \leftarrow \partial \; \overline{16} \leftarrow \partial \; 45 \leftarrow \partial \; 144 \leftarrow \partial \ldots$$

$$16 - 10 + 16 - 45 + 144 - \ldots = 11,$$

(suitably regularised) which is the number of degrees of freedom of a pure spinor. The representations $\{R_n\}_{n=1}^{\infty}$ agree with

—The ghosts for a “pure spinor” constraint (a constraint implying an object lies in the minimal orbit);

—The positive levels of a Borcherds superalgebra $\mathcal{B}(E_n)$.

Indeed, the denominator appearing in the denominator formula for $\mathcal{B}(E_n)$ is identical to the partition function of a “pure spinor” [54].

$$\mathcal{B}(D_n) = 0 \delta \mathcal{P}(n, n)2,$$

$$\mathcal{B}(A_n) = \delta(1(n + 1)1),$$

The modules $R_1, \ldots, R_{8-n}$ behave like forms. The “exterior derivative” is connection-free (for a torsion-free connection), and there is a wedge product [43].

The modules show a symmetry: $R_{8-n} = R_n^B$. There is another extension to negative levels that respects this symmetry, and seems more connected to geometry: tensor hierarchy algebras [2, 5].

In the classification of finite-dimensional superalgebras by Kac, there is a special class, “Cartan-type superalgebras”. The Cartan-type superalgebra $W(n)$, which I prefer to call $W(A_{n-1})$, is asymmetric between positive and negative levels, and (therefore) not defined through generators corresponding to simple roots and Serre relations.

\begin{table}
\centering
\caption{A list of compact subgroups}
\begin{tabular}{|l|l|l|}
\hline
$n$ & $E_{n(n)}$ & $K(E_{n(n)})$ \\
\hline
3 & $SL(3) \times SL(2)$ & $SO(3) \times SO(2)$ \\
4 & $SL(5)$ & $SO(5)$ \\
5 & $Spin(5, 5)$ & $(Spin(5) \times Spin(5))/\mathbb{Z}_2$ \\
6 & $E_{6(6)}$ & $USp(8)/\mathbb{Z}_2$ \\
7 & $E_{7(7)}$ & $SU(8)/\mathbb{Z}_2$ \\
8 & $E_{8(8)}$ & $Spin(16)/\mathbb{Z}_2$ \\
9 & $E_{9(9)}$ & $K(E_{9(9)})$ \\
\hline
\end{tabular}
\end{table}
$W(A_{n-1})$ is the superalgebra of derivations on the superalgebra of (pointwise) forms in $n$ dimensions.

Any operation $\omega \rightarrow \Omega \wedge V \omega$ where $\Omega$ is a form and $V$ a vector, belongs to $W(A_{n-1})$. A basis is given by

$$\begin{align*}
\text{Level} = 1 & & \iota_a \\
0 & & e^a \\
-1 & & e^b e^b \iota_a \\
-2 & & e^b e^c e^b \iota_a \\
& & \ldots \\
\text{The level decomposition of } W(A_{n-1}).
\end{align*}$$

A subalgebra $S(A_{n-1})$ contains traceless tensors. The positive levels agree with $\mathfrak{B}(A_{n-1}) = \mathfrak{B}(n|1)$. Note that the representations of torsion and torsion Bianchi identity appear at levels $-1$ and $-2$.

In spite of the absence of a Cartan involution, there is a way to give a systematic Chevalley–Serre presentation of the superalgebra, based on the same Dynkin diagram as the Borcherds superalgebra [5].

The construction can be extended to $W(D_n)$, and, most interestingly, $W(E_n)$ (and the corresponding $S(\mathfrak{g})$). The statements about torsion and Bianchi identities remain true (but we still lack a good geometric argument).

Back to the Jacobi identity. Expressed in terms of a fermionic ghost in $R_1$,

$$[[c,c],c] \neq 0.$$

How is this remedied? The most general formalism for gauge symmetries is the Batalin–Vilkovisky formalism, where everything is encoded in the master equation $(S, S) = 0$.

If transformations are field-independent, one may consider the ghost action consistently. An $L_\infty$ algebra is a (super)algebraic structure which provides a perturbative solution to the master equation.

Let $C$ denote all ghosts. Then the master equation states the nilpotency of a transformation

$$\delta C = (S, C)$$

$$= \partial C + [C, C] + [C, C, C] + [C, C, C, C] + \ldots.$$
What can be learned about the full string theory/M-theory?

...Thank you for your attention.

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