BOTANY OF IRREDUCIBLE AUTOMORPHISMS OF FREE GROUPS

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Abstract. We give a classification of iwip outer automorphisms of the free group, by discussing the properties of their attracting and repelling trees.

1. Introduction

An outer automorphism $\Phi$ of the free group $F_N$ is fully irreducible (abbreviated as iwip) if no positive power $\Phi^n$ fixes a proper free factor of $F_N$. Being an iwip is one (in fact the most important) of the analogs for free groups of being pseudo-Anosov for mapping classes of hyperbolic surfaces. Another analog of pseudo-Anosov is the notion of an atoroidal automorphism: an element $\Phi \in \text{Out}(F_N)$ is atoroidal or hyperbolic if no positive power $\Phi^n$ fixes a nontrivial conjugacy class. Bestvina and Feighn [BF92] and Brinkmann [Bri00] proved that $\Phi$ is atoroidal if and only if the mapping torus $F_N \rtimes_{\Phi} \mathbb{Z}$ is Gromov-hyperbolic.

Pseudo-Anosov mapping classes are known to be “generic” elements of the mapping class group (in various senses). Rivin [Riv08] and Sisto [Sis11] recently proved that, in the sense of random walks, generic elements of $\text{Out}(F_N)$ are atoroidal iwip automorphisms.

Bestvina and Handel [BH92] proved that iwip automorphisms have the key property of being represented by (absolute) train-track maps.

A pseudo-Anosov element $f$ fixes two projective classes of measured foliations $[(\mathcal{F}^+, \mu^+)]$ and $[(\mathcal{F}^-, \mu^-)]$:

$$(\mathcal{F}^+, \mu^+) \cdot f = (\mathcal{F}^+, \lambda \mu^+) \quad \text{and} \quad (\mathcal{F}^-, \mu^-) \cdot f = (\mathcal{F}^-, \lambda^{-1} \mu^-)$$

where $\lambda > 1$ is the expansion factor of $f$. Alternatively, considering the dual $\mathbb{R}$-trees $T^+$ and $T^-$, we get:

$$T^+ \cdot f = \lambda T^+ \quad \text{and} \quad T^- \cdot f = \lambda^{-1} T^-.$$ 

We now discuss the analogous situation for iwip automorphisms. The group of outer automorphisms $\text{Out}(F_N)$ acts on the outer space $\mathcal{CV}_N$ and its boundary $\partial \mathcal{CV}_N$. Recall that the compactified outer space $\overline{\mathcal{CV}_N} = \mathcal{CV}_N \cup \partial \mathcal{CV}_N$ is made up of (projective classes of) $\mathbb{R}$-trees with an action of $F_N$ by isometries which is minimal and very small. See [Vog02] for a survey on outer space. An iwip outer automorphism $\Phi$ has North-South dynamics on $\overline{\mathcal{CV}_N}$: it has a unique attracting fixed tree $[T_\Phi]$ and a unique repelling fixed tree $[T_{\Phi^{-1}}]$ in the boundary of outer space (see [LL03]):

$$T_\Phi \cdot \Phi = \lambda_\Phi T_\Phi \quad \text{and} \quad T_{\Phi^{-1}} \cdot \Phi = \frac{1}{\lambda_{\Phi^{-1}}} T_{\Phi^{-1}},$$

where $\lambda_\Phi > 1$ is the expansion factor of $\Phi$ (i.e. the exponential growth rate of non-periodic conjugacy classes).
Contrary to the pseudo-Anosov setting, the expansion factor $\lambda_\Phi$ of $\Phi$ is typically different from the expansion factor $\lambda_{\Phi^{-1}}$ of $\Phi^{-1}$. More generally, qualitative properties of the fixed trees $T_\Phi$ and $T_{\Phi^{-1}}$ can be fairly different. This is the purpose of this paper to discuss and compare the properties of $\Phi$, $T_\Phi$ and $T_{\Phi^{-1}}$.

First, the free group, $F_N$, may be realized as the fundamental group of a surface $S$ with boundary. It is part of folklore that, if $\Phi$ comes from a pseudo-Anosov mapping class on $S$, then its limit trees $T_\Phi$ and $T_{\Phi^{-1}}$ live in the Thurston boundary of Teichmüller space: they are dual to a measured foliation on the surface. Such trees $T_\Phi$ and $T_{\Phi^{-1}}$ are called surface trees and such an iwip outer automorphism $\Phi$ is called geometric (in this case $S$ has exactly one boundary component).

The notion of surface trees has been generalized (see for instance [Bes02]). An $\mathbb{R}$-tree which is transverse to measured foliations on a finite CW-complex is called geometric. It may fail to be a surface tree if the complex fails to be a surface.

If $\Phi$ does not come from a pseudo-Anosov mapping class and if $T_\Phi$ is geometric then $\Phi$ is called parageometric. For a parageometric iwip $\Phi$, Guirardel [Gui05] and Handel and Mosher [HM07] proved that the repelling tree $T_{\Phi^{-1}}$ is not geometric. So we have that, $\Phi$ comes from a pseudo-Anosov mapping class on a surface with boundary if and only if both trees $T_\Phi$ and $T_{\Phi^{-1}}$ are geometric. Moreover in this case both trees are indeed surface trees.

In our paper [CH10] we introduced a second dichotomy for trees in the boundary of Outer space with dense orbits. For a tree $T$, we consider its limit set $\Omega \subseteq \overline{T}$ (where $\overline{T}$ is the metric completion of $T$). The limit set $\Omega$ consists of points of $\overline{T}$ with at least two pre-images by the map $Q : \partial F_N \to \overline{T} = \overline{T} \cup \partial T$ introduced by Levitt and Lustig [LL03], see Section 4A. We are interested in the two extremal cases: A tree $T$ in the boundary of Outer space with dense orbits is of surface type if $T \subseteq \Omega$ and $T$ is of Levitt type if $\Omega$ is totally disconnected. As the terminology suggests, a surface tree is of surface type. Trees of Levitt type where discovered by Levitt [Lev93].

Combining together the two sets of properties, we introduced in [CH10] the following definitions. A tree $T$ in $\partial CV_N$ with dense orbits is

- a surface tree if it is both geometric and of surface type;
- Levitt if it is geometric and of Levitt type;
- pseudo-surface if it is not geometric and of surface type;
- pseudo-Levitt if it is not geometric and of Levitt type.

The following Theorem is the main result of this paper.

**Theorem 5.2.** Let $\Phi$ be an iwip outer automorphism of $F_N$. Let $T_\Phi$ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Then exactly one of the following occurs

1. The trees $T_\Phi$ and $T_{\Phi^{-1}}$ are surface trees. Equivalently $\Phi$ is geometric.
2. The tree $T_\Phi$ is Levitt (i.e. geometric and of Levitt type), and the tree $T_{\Phi^{-1}}$ is pseudo-surface (i.e. non-geometric and of surface type). Equivalently $\Phi$ is parageometric.
3. The tree $T_{\Phi^{-1}}$ is Levitt (i.e. geometric and of Levitt type), and the tree $T_\Phi$ is pseudo-surface (i.e. non-geometric and of surface type). Equivalently $\Phi^{-1}$ is parageometric.
4. The trees $T_\Phi$ and $T_{\Phi^{-1}}$ are pseudo-Levitt (non-geometric and of Levitt type).

Case (1) corresponds to toroidal iwips whereas cases (2), (3) and (4) corresponds to atoroidal iwips. In case (4) the automorphism $\Phi$ is called pseudo-Levitt.
Gaboriau, Jaeger, Levitt and Lustig [GJLL98] introduced the notion of an index $\text{ind}(\Phi)$, computed from the rank of the fixed subgroup and from the number of attracting fixed points of the automorphisms $\varphi$ in the outer class $\Phi$. Another index for a tree $T$ in $\overline{CV}_N$ has been defined and studied by Gaboriau and Levitt [GL95], we call it the geometric index $\text{ind}_{\text{geo}}(T)$. Finally in our paper [CH10] we introduced and studied the $Q$-index $\text{ind}_Q(T)$ of an $\mathbb{R}$-tree $T$ in the boundary of outer space with dense orbits. The two indices $\text{ind}_{\text{geo}}(T)$ and $\text{ind}_Q(T)$ describe qualitative properties of the tree $T$ [CH10]. We define these indices and recall our botanical classification of trees in Section 4A.

The key to prove Theorem 5.2 is:

**Propositions 4.2 and 4.4.** Let $\Phi$ be an iwip outer automorphism of $F_N$. Let $T_\Phi$ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Replacing $\Phi$ by a suitable power, we have

$$2 \text{ind}(\Phi) = \text{ind}_{\text{geo}}(T_\Phi) = \text{ind}_Q(T_{\Phi^{-1}}).$$

We prove this Proposition in Sections 4B and 4C.

To study limit trees of iwip automorphisms, we need to state that they have the strongest mixing dynamical property, which is called **indecomposability**.

**Theorem 2.1.** Let $\Phi \in \text{Out}(F_N)$ be an iwip outer automorphism. The attracting tree $T_\Phi$ of $\Phi$ is indecomposable.

The proof of this Theorem is quite independent of the rest of the paper and is the purpose of Section 2. The proof relies on a key property of iwip automorphisms: they can be represented by (absolute) train-track maps.

## 2. INDECOMPOSABILITY OF THE ATTRACTING TREE OF AN IWIP AUTOMORPHISM

Following Guirardel [Gui08], a (projective class of) $\mathbb{R}$-tree $T \in \overline{CV}_N$ is **indecomposable** if for all non degenerate arcs $I$ and $J$ in $T$, there exists finitely many elements $u_1, \ldots, u_n$ in $F_N$ such that

$$J \subseteq \bigcup_{i=1}^{n} u_i I$$

and

$$\forall i = 1, \ldots, n - 1, \quad u_i I \cap u_{i+1} I \text{ is a non degenerate arc.}$$

The main purpose of this Section is to prove

**Theorem 2.1.** Let $\Phi \in \text{Out}(F_N)$ be an iwip outer automorphism. The attracting tree $T_\Phi$ of $\Phi$ is indecomposable.

Before proving this Theorem in Section 2C, we collect the results we need from [BH92] and [GJLL98].

### 2A. Train-track representative of $\Phi$.

The rose $R_N$ is the graph with one vertex $*$ and $N$ edges. Its fundamental group $\pi_1(R_N, *)$ is naturally identified with the free group $F_N$. A **marked graph** is a finite graph $\Gamma$ with a homotopy equivalence $\tau : R_N \to \Gamma$. The marking $\tau$ induces an isomorphism $\tau_* : F_N = \pi_1(R_N, *) \xrightarrow{\cong} \pi_1(\Gamma, v_0)$, where $v_0 = \tau(*)$.

A homotopy equivalence $f : \Gamma \to \Gamma$ defines an outer automorphism of $F_N$. Indeed, if a path $m$ from $v_0$ to $f(v_0)$ is given, $a \mapsto mf(a)m^{-1}$ induces an automorphism $\varphi$ of $\pi_1(\Gamma, v_0)$,
and thus of $F_N$ through the marking. Another path $m'$ from $v_0$ to $f(v_0)$ gives rise to another automorphism $\varphi'$ of $F_N$ in the same outer class $\Phi$.

A topological representative of $\Phi \in \text{Out}(F_N)$ is an homotopy equivalence $f : G \to G$ of a marked graph $G$, such that:

(i) $f$ maps vertices to vertices,
(ii) $f$ is locally injective on any edge,
(iii) $f$ induces $\Phi$ on $F_N \cong \pi_1(G, v_0)$. Let $e_1, \ldots, e_p$ be the edges of $G$ (an orientation is arbitrarily given on each edge, and $e^{-1}$ denotes the edge $e$ with the reverse orientation). The transition matrix of the map $f$ is the $p \times p$ non-negative matrix $M$ with $(i, j)$-entry equal to the number of times the edge $e_i$ occurs in $f(e_j)$ (we say that a path (or an edge) $w$ of a graph $G$ occurs in a path $u$ of $G$ if it is $w$ or its inverse $w^{-1}$ is a subpath of $u$).

A topological representative $f : G \to G$ of $\Phi$ is a train-track map if moreover:

(iv) for all $k \in \mathbb{N}$, the restriction of $f^k$ on any edge of $G$ is locally injective,
(v) any vertex of $G$ has valence at least 3.

According to [BH92 Theorem 1.7], an iwip outer automorphism $\Phi$ can be represented by a train-track map, with a primitive transition matrix $M$ (i.e. there exists some $k \in \mathbb{N}$ such all the entries of $M^k$ are strictly positive). Thus the Perron-Frobenius Theorem applies. In particular, $M$ has a real dominant eigenvalue $\lambda > 1$ associated to a strictly positive eigenvector $u = (u_1, \ldots, u_p)$. Indeed, $\lambda$ is the expansion factor of $\Phi$: $\lambda = \lambda_\Phi$. We turn the graph $G$ to a metric space by assigning the length $u_i$ to the edge $e_i$ (for $i = 1, \ldots, p$). Since, with respect to this metric, the length of $f(e_i)$ is $\lambda$ times the length of $e_i$, we can assume that, on each edge, $f$ is linear of ratio $\lambda$.

We define the set $\mathcal{L}_2(f)$ of paths $w$ of combinatorial length 2 (i.e. $w = ee'$, where $e, e'$ are edges of $G$, $e^{-1} \neq e'$) which occurs in some $f^k(e_i)$ for some $k \in \mathbb{N}$ and some edge $e_i$ of $G$:

$$\mathcal{L}_2(f) = \{ ee' : \exists e_i \text{ edge of } G, \exists k \in \mathbb{N} \text{ such that } ee' \text{ is a subpath of } f^k(e_i) \pm 1 \}.$$ 

Since the transition matrix $M$ is primitive, there exists $k \in \mathbb{N}$ such that for any edge $e$ of $G$, for any $w \in \mathcal{L}_2(f)$, $w$ occurs in $f^k(e)$.

Let $v$ be a vertex of $G$. The Whitehead graph $W_v$ of $v$ is the unoriented graph defined by:

- the vertices of $W_v$ are the edges of $G$ with $v$ as terminal vertex,
- there is an edge in $W_v$ between $e$ and $e'$ if $e'e^{-1} \in \mathcal{L}_2(f)$.

As remarked in [BFH97 Section 2], if $f : G \to G$ is a train-track representative of an iwip outer automorphism $\Phi$, any vertex of $G$ has a connected Whitehead graph. We summarize the previous discussion in:

**Proposition 2.2.** Let $\Phi \in \text{Out}(F_N)$ be an iwip outer automorphism. There exists a train-track representative $f : G \to G$ of $\Phi$, with primitive transition matrix $M$ and connected Whitehead graphs of vertices. The edge $e_i$ of $G$ is isometric to the segment $[0, u_i]$, where $u = (u_1, \ldots, u_p)$ is a Perron-Frobenius eigenvector of $M$. The map $f$ is linear of ratio $\lambda$ on each edge $e_i$ of $G$.

**Remark 2.3.** Let $f : G \to G$ be a train-track map, with primitive transition matrix $M$ and connected Whitehead graphs of vertices. Then for any path $w = ab$ in $G$ of combinatorial length 2, there exist $w_1 = a_1b_1, \ldots, w_q = a qb_q \in \mathcal{L}_2(f)$ ($a, b, a_i, b_i$ edges of $G$) such that:
• $a_{i+1} = b^{-1}_i$, $i \in \{1, \ldots, q-1\}$
• $a = a_1$ and $b = b_q$.

2B. Construction of $T_\Phi$. Let $\Phi \in \text{Out}(F_N)$ be an iwip automorphism, and let $T_\Phi$ be its attracting tree. Following [GJLL98], we recall a concrete construction of the tree $T_\Phi$.

We start with a train-track representative $f : G \to G$ of $\Phi$ as in Proposition 2.2. The universal cover $\tilde{G}$ of $G$ is a simplicial tree, equipped with a distance $d_0$ obtained by lifting the distance on $G$. The fundamental group $F_N$ acts by deck transformations, and thus by isometries, on $\tilde{G}$. Let $\tilde{f}$ be a lift of $f$ to $\tilde{G}$. This lift $\tilde{f}$ is associated to a unique automorphism $\varphi$ in the outer class $\Phi$, characterized by

\begin{equation}
\forall u \in F_N, \forall x \in \tilde{G}, \quad \varphi(u)\tilde{f}(x) = \tilde{f}(ux).
\end{equation}

For $x, y \in \tilde{G}$ and $k \in \mathbb{N}$, we define:

\[ d_k(x, y) = \frac{d_0(\tilde{f}^k(x), \tilde{f}^k(y))}{\lambda^k}. \]

The sequence of distances $d_k$ is decreasing and converges to a pseudo-distance $d_\infty$ on $\tilde{G}$. Identifying points $x, y$ in $\tilde{G}$ which have distance $d_\infty(x, y)$ equal to 0, we obtain the tree $T_\Phi$.

The free group $F_N$ still acts by isometries on $T_\Phi$. The quotient map $p : \tilde{G} \to T_\Phi$ is $F_N$-equivariant and 1-Lipschitz. Moreover, for any edge $e$ of $G$, for any $k \in \mathbb{N}$, the restriction of $p$ to $f^k(e)$ is an isometry. Through $p$ the map $\tilde{f}$ factors to a homothety $H$ of $T_\Phi$, of ratio $\lambda_\Phi$:

\[ \forall x \in \tilde{G}, \quad H(p(x)) = p(\tilde{f}(x)). \]

Property (2.3) leads to

\begin{equation}
\forall u \in F_N, \forall x \in T_\Phi, \quad \varphi(u)H(x) = H(ux).
\end{equation}

2C. Indecomposability of $T_\Phi$. We say that a path (or an edge) $w$ of the graph $G$ occurs in a path $u$ of the universal cover $\tilde{G}$ of $G$ if $w$ has a lift $\tilde{w}$ which occurs in $u$.

**Lemma 2.4.** Let $I$ be a non degenerate arc in $T_\Phi$. There exists an arc $I'$ in $\tilde{G}$ and an integer $k$ such that

• $p(I') \subseteq I$
• any element of $\mathcal{L}_2(f)$ occurs in $H^k(I')$.

**Proof.** Let $I \subset T_\Phi$ be a non-degenerate arc. There exists an edge $e$ of $\tilde{G}$ such that $I_0 = p(e) \cap I$ is a non-degenerate arc: $I_0 = [x, y]$. We choose $k_1 \in \mathbb{N}$ such that $d_\infty(H^{k_1}(x), H^{k_1}(y)) > L$ where

\[ L = 2 \max\{u_i = |e_i| : e_i \text{ edge of } G\}. \]

Let $x', y'$ be the points in $e$ such that $p(x') = x$, $p(y') = y$, and let $I'$ be the arc $[x', y']$. Since $p$ maps $f^{k_1}(e)$ isometrically into $T_\Phi$, we obtain that $d_0(f^{k_1}(x'), f^{k_1}(y')) \geq L$. Hence there exists an edge $e'$ of $\tilde{G}$ contained in $[f^{k_1}(x'), f^{k_1}(y')]$. Moreover, for any $k_2 \in \mathbb{N}$, the path $f^{k_2}(e')$ isometrically injects in $[H^{k_1+k_2}(x), H^{k_1+k_2}(y)]$. We take $k_2$ big enough so that any path in $\mathcal{L}_2(f)$ occurs in $f^{k_2}(e')$. Then $k = k_1 + k_2$ is suitable. \qed
Proof of Theorem 2.1. Let $I, J$ be two non-trivial arcs in $T_\Phi$. We have to prove that $I$ and $J$ satisfy properties (2.1) and (2.2). Since $H$ is a homeomorphism, and because of (2.4), we can replace $I$ and $J$ by $H^k(I)$ and $H^k(J)$, accordingly, for some $k \in \mathbb{N}$.

We consider an arc $I'$ in $\tilde{G}$ and an integer $k \in \mathbb{N}$ as given by Lemma 2.4. Let $x, y$ be the endpoints of the arc $H^k(J)$: $H^k(J) = [x, y]$. Let $x', y'$ be points in $\tilde{G}$ such that $p(x') = x$, $p(y') = y$, and let $J'$ be the arc $[x', y']$. According to Remark 2.3, there exist $w_1, \ldots, w_n$ such that:

- $w_i$ is a lift of some path in $L_2(f)$,
- $J' \subseteq \bigcup_{i=1}^n w_i$,
- $w_i \cap w_{i+1}$ is an edge.

Since Lemma 2.4 ensures that any element of $L_2(f)$ occurs in $H^k(I')$, we deduce that $H^k(I)$ and $H^k(J)$ satisfy properties (2.1) and (2.2), concluding the proof of Theorem 2.1. □

3. INDEX OF AN OUTER AUTOMORPHISM

An automorphism $\varphi$ of the free group $F_N$ extends to a homeomorphism $\partial \varphi$ of the boundary at infinity $\partial F_N$. We denote by $\text{Fix}(\varphi)$ the fixed subgroup of $\varphi$. It is a finitely generated subgroup of $F_N$ and thus its boundary $\partial \text{Fix}(\varphi)$ naturally embeds in $\partial F_N$. Elements of $\partial \text{Fix}(\varphi)$ are fixed by $\partial \varphi$ and they are called singular. Non-singular fixed points of $\partial \varphi$ are called regular. A fixed point $X$ of $\partial \varphi$ is attracting (resp. repelling) if it is regular and if there exists an element $u$ in $F_N$ such that $\varphi^n(u)$ (resp. $\varphi^{-n}(u)$) converges to $X$. The set of fixed points of $\partial \varphi$ is denoted by $\text{Fix}(\partial \varphi)$.

Following Nielsen, fixed points of $\partial \varphi$ have been classified by Gaboriau, Jaeger, Levitt and, Lustig:

Proposition 3.1 ([GJLL98, Proposition 1.1]). Let $\varphi$ be an automorphism of the free group $F_N$. Let $X$ be a fixed point of $\partial \varphi$. Then exactly one of the following occurs:

- (1) $X$ is in the boundary of the fixed subgroup of $\varphi$;
- (2) $X$ is attracting;
- (3) $X$ is repelling. □

We denote by $\text{Att}(\varphi)$ the set of attracting fixed points of $\partial \varphi$. The fixed subgroup $\text{Fix}(\varphi)$ acts on the set $\text{Att}(\varphi)$ of attracting fixed points.

In [GJLL98] the following index of the automorphism $\varphi$ is defined:

$$\text{ind}(\varphi) = \frac{1}{2}\#(\text{Att}(\varphi)/\text{Fix}(\varphi)) + \text{rank}(\text{Fix}(\varphi)) - 1$$

If $\varphi$ has a trivial fixed subgroup, the above definition is simpler:

$$\text{ind}(\varphi) = \frac{1}{2}\#\text{Att}(\varphi) - 1.$$ 

Let $u$ be an element of $F_N$ and let $i_u$ be the corresponding inner automorphism of $F_N$:

$$\forall w \in F_N, i_u(w) = uwu^{-1}.$$ 

The inner automorphism $i_u$ extends to the boundary of $F_N$ as left multiplication by $u$:

$$\forall X \in \partial F_N, i_u(X) = uX.$$
The group $\text{Inn}(F_N)$ of inner automorphisms of $F_N$ acts by conjugacy on the automorphisms in an outer class $\Phi$. Following Nielsen, two automorphisms, $\varphi, \varphi' \in \Phi$ are isogredient if they are conjugated by some inner automorphism $i_u$:

$$\varphi' = i_u \circ \varphi \circ i_u^{-1} = i_{u\varphi(u)^{-1}} \circ \varphi.$$ 

In this case, the actions of $\partial \varphi$ and $\partial \varphi'$ on $\partial F_N$ are conjugate by the left multiplication by $u$. In particular, a fixed point $X'$ of $\partial \varphi'$ is a translate $X' = uX$ of a fixed point $X$ of $\partial \varphi$. Two isogredient automorphisms have the same index: this is the index of the isogrediency class. An isogrediency class $[\varphi]$ is essential if it has positive index: $\text{ind}([\varphi]) > 0$. We note that essential isogrediency classes are principal in the sense of [FH06], but the converse is not true.

The index of the outer automorphism $\Phi$ is the sum, over all essential isogrediency classes of automorphisms $\varphi$ in the outer class $\Phi$, of their indices, or alternatively:

$$\text{ind}(\Phi) = \sum_{[\varphi] \in \Phi/\text{Inn}(F_N)} \max(0; \text{ind}(\varphi)).$$

We adapt the notion of forward rotationless outer automorphism of Feighn and Handel [FH06] to our purpose. We denote by $\text{Per}(\varphi)$ the set of elements of $F_N$ fixed by some positive power of $\varphi$:

$$\text{Per}(\varphi) = \bigcup_{n \in \mathbb{N}^*} \text{Fix}(\varphi^n);$$

and by $\text{Per}(\partial \varphi)$ the set of elements of $\partial F_N$ fixed by some positive power of $\partial \varphi$:

$$\text{Per}(\partial \varphi) = \bigcup_{n \in \mathbb{N}^*} \text{Fix}(\partial \varphi^n).$$

**Definition 3.2.** An outer automorphism $\Phi \in \text{Out}(F_N)$ is FR if:

(1) for any automorphism $\varphi \in \Phi$, $\text{Per}(\varphi) = \text{Fix}(\varphi)$ and $\text{Per}(\partial \varphi) = \text{Fix}(\partial \varphi)$;

(2) if $\psi$ is an automorphism in the outer class $\Phi^n$ for some $n > 0$, with $\text{ind}(\psi)$ positive, then there exists an automorphism $\varphi$ in $\Phi$ such that $\psi = \varphi^n$.

**Proposition 3.3.** Let $\Phi \in \text{Out}(F_N)$. There exists $k \in \mathbb{N}^*$ such that $\Phi^k$ is FR.

**Proof.** By [LL00, Theorem 1] there exists a power $\Phi^k$ with (FR1). An automorphism $\varphi \in \text{Aut}(F_N)$ with positive index $\text{ind}(\varphi) > 0$ is principal in the sense of [FH06, Definition 3.1]. Thus our property (FR2) is a consequence of the forward rotationless property of [FH06, Definition 3.13]. By [FH06, Lemma 4.43] there exists a power $\Phi^{k\ell}$ which is forward rotationless and thus which satisfies (FR2). □

4. Indices

4A. Botany of trees. We recall in this Section the classification of trees in the boundary of outer space of our paper [CH10].

Gaboriau and Levitt [GL95] introduced an index for a tree $T$ in $\overline{CV}_N$, we call it the geometric index and denote it by $\text{ind}_{\text{geo}}(T)$. It is defined using the valence of the branch points, of the $\mathbb{R}$-tree $T$, with an action of the free group by isometries:

$$\text{ind}_{\text{geo}}(T) = \sum_{[P] \in \overline{T}/F_N} \text{ind}_{\text{geo}}(P).$$
where the local index of a point $P$ in $T$ is
\[ \text{ind}_{\text{geo}}(P) = \#(\pi_0(T \setminus \{P\})/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2. \]

Gaboriau and Levitt [GL95] proved that the geometric index of a geometric tree is equal to $2N - 2$ and that for any tree in the compactification of outer space $\overline{\text{CV}}_N$ the geometric index is bounded above by $2N - 2$. Moreover, they proved that the trees in $\overline{\text{CV}}_N$ with geometric index equal to $2N - 2$ are precisely the geometric trees.

If, moreover, $T$ has dense orbits, Levitt and Lustig [LL03, LL08] defined the map $Q : \partial F_N \to \hat{T}$ which is characterized by

**Proposition 4.1.** Let $T$ be an $\mathbb{R}$-tree in $\overline{\text{CV}}_N$ with dense orbits. There exists a unique map $Q : \partial F_N \to \hat{T}$ such that for any sequence $(u_n)_{n \in \mathbb{N}}$ of elements of $F_N$ which converges to $X \in \partial F_N$, and any point $P \in T$, if the sequence of points $(u_nP)_{n \in \mathbb{N}}$ converges to a point $Q \in \hat{T}$, then $Q(X) = Q$. Moreover, $Q$ is onto.

Let us consider the case of a tree $T$ dual to a measured foliation $(\mathcal{F}, \mu)$ on a hyperbolic surface $S$ with boundary ($T$ is a surface tree). Let $\hat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal cover $\hat{S}$ of $S$. The boundary at infinity of $\hat{S}$ is homeomorphic to $\partial F_N$. On the one hand, a leaf $\ell$ of $\hat{\mathcal{F}}$ defines a point in $T$. On the other hand, the ends of $\ell$ define points in $\partial F_N$. The map $Q$ precisely sends the ends of $\ell$ to the point in $T$. The Poincaré-Lefschetz index of the foliation $\mathcal{F}$ can be computed from the cardinal of the fibers of the map $Q$. This leads to the following definition of the $Q$-index of an $\mathbb{R}$-tree $T$ in a more general context.

Let $T$ be an $\mathbb{R}$-tree in $\overline{\text{CV}}_N$ with dense orbits. The $Q$-index of the tree $T$ is defined as follows:
\[ \text{ind}_Q(T) = \sum_{[P] \in \hat{T}/F_N} \max(0; \text{ind}_Q(P)). \]
where the local index of a point $P$ in $T$ is:
\[ \text{ind}_Q(P) = \#(Q^{-1}(P)/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2 \]
with $Q^{-1}(P) = Q^{-1}(P) \setminus \partial \text{Stab}(P)$ the regular fiber of $P$.

Levitt and Lustig [LL03] proved that points in $\partial T$ have exactly one pre-image by $Q$. Thus, only points in $\hat{T}$ contribute to the $Q$-index of $T$.

We proved [CH10] that the $Q$-index of an $\mathbb{R}$-tree in the boundary of outer space with dense orbits is bounded above by $2N - 2$. And it is equal to $2N - 2$ if and only if it is of surface type.

Our botanical classification [CH10] of a tree $T$ with a minimal very small indecomposable action of $F_N$ by isometries is as follows

| Surface type | geometric | not geometric |
|--------------|-----------|---------------|
|              | $\text{ind}_Q(T) = 2N - 2$ | $\text{ind}_Q(T) < 2N - 2$ |

The following remark is not necessary for the sequel of the paper, but may help the reader’s intuition.

**Remark.** In [CHL08a, CHL08], in collaboration with Lustig, we defined and studied the dual lamination of an $\mathbb{R}$-tree $T$ with dense orbits:
\[ L(T) = \{(X, Y) \in \partial^2 F_N \mid Q(X) = Q(Y)\}. \]
The $Q$-index of $T$ can be interpreted as the index of this dual lamination.

Using the dual lamination, with Lustig [CHL09], we defined the compact heart $K_A \subseteq T$ (for a basis $A$ of $F_N$). We proved that the tree $T$ is completely encoded by a system of partial isometries $S_A = (K_A, A)$. We also proved that the tree $T$ is geometric if and only if the compact heart $K_A$ is a finite tree (that is to say the convex hull of finitely many points).

In our previous work [CH10], we used the Rips machine on the system of isometries $S_A$ to get the bound on the $Q$-index of $T$. In particular, an indecomposable tree $T$ is of Levitt type if and only if the Rips machine never halts.

**4B. Geometric index.** As in Section 2B, an iwip outer automorphism $\Phi$ has an expansion factor $\lambda_\Phi > 1$, an attracting $\mathbb{R}$-tree $T_\Phi$ in $\partial CV_N$. For each automorphism $\varphi$ in the outer class $\Phi$ there is a homothety $H$ of the metric completion $\overline{T_\Phi}$, of ratio $\lambda_\Phi$, such that

$$\forall P \in T_\Phi, \forall u \in F_N, \ H(uP) = \varphi(u)H(P)$$

In addition, the action of $\Phi$ on the compactification of Culler and Vogtmann’s Outer space has a North-South dynamic and the projective class of $T_\Phi$ is the attracting fixed point [LL03]. Of course the attracting trees of $\Phi$ and $\Phi^n$ ($n > 0$) are equal.

For the attracting tree $T_\Phi$ of the iwip outer automorphism $\Phi$, the geometric index is well understood.

**Proposition 4.2 ([GJLL98, Section 4]).** Let $\Psi$ be an iwip outer automorphism. There exists a power $\Phi = \Psi^k$ ($k > 0$) of $\Psi$ such that:

$$2 \text{ind}(\Phi) = \text{ind}_{\text{geo}}(T_\Phi),$$

where $T_\Phi$ is the attracting tree of $\Phi$ (and of $\Psi$).

**4C. $Q$-index.** Let $\Phi$ be an iwip outer automorphism of $F_N$. Let $T_\Phi$ be its attracting tree. The action of $F_N$ on $T_\Phi$ has dense orbits.

Let $\varphi$ an automorphism in the outer class $\Phi$. The homothety $H$ associated to $\varphi$ extends continuously to an homeomorphism of the boundary at infinity of $T_\Phi$ which we still denote by $H$. We get from Proposition 4.1 and identity 4.1:

$$\forall X \in \partial F_N, Q(\partial \varphi(X)) = H(Q(X)).$$

We are going to prove that the $Q$-index of $T_\Phi$ is twice the index of $\Phi^{-1}$. As mentioned in the introduction for geometric automorphisms both these numbers are equal to $2N - 2$ and thus we restrict to the study of non-geometric automorphisms. For the rest of this section we assume that $\Phi$ is non-geometric. This will be used in two ways:

- the action of $F_N$ on $T_\Phi$ is free;
- for any $\varphi$ in the outer class $\Phi$, all the fixed points of $\varphi$ in $\partial F_N$ are regular.

Let $C_H$ be the center of the homothety $H$. The following Lemma is essentially contained in [GJLL98], although the map $Q$ is not used there.

**Lemma 4.3.** Let $\Phi \in \text{Out}(F_N)$ be a FR non-geometric iwip outer automorphism. Let $T_\Phi$ be the attracting tree of $\Phi$. Let $\varphi \in \Phi$ be an automorphism in the outer class $\Phi$, and let $H$ be the homothety of $T_\Phi$ associated to $\varphi$, with $C_H$ its center. The $Q$-fiber of $C_H$ is the set of repelling points of $\varphi$. 9
Proof. Let $X \in \partial F_N$ be a repelling point of $\partial \varphi$. By definition there exists an element $u \in F_N$ such that the sequence $(\varphi^{-n}(u))_n$ converges towards $X$. By Equation 4.1

$$\varphi^{-n}(u)C_H = \varphi^{-n}(u)H^{-n}(C_H) = H^{-n}(uC_H).$$

The homothety $H^{-1}$ is strictly contracting and thus the sequence of points $(\varphi^{-n}(u)C_H)_n$ converges towards $C_H$. By Proposition 4.1, we get that $Q(X) = C_H$.

Conversely let $X \in Q^{-1}(C_H)$ be a point in the $Q$-fiber of $C_H$. Using the identity $4.2$, $\partial \varphi(X)$ is also in the $Q$-fiber. The $Q$-fiber is finite by [CH10, Corollary 5.4], $X$ is a periodic point of $\partial \varphi$. Since $\Phi$ satisfies property (FR1), $X$ is a fixed point of $\partial \varphi$. From [GJLL98, Lemma 3.5], attracting fixed points of $\partial \varphi$ are mapped by $Q$ to points in the boundary at infinity $\partial T_\Phi$. Thus $X$ has to be a repelling fixed point of $\partial \varphi$. □

**Proposition 4.4.** Let $\Phi \in \text{Out}(F_N)$ be a FR non-geometric iwip outer automorphism. Let $T_\Phi$ be the attracting tree of $\Phi$. Then

$$2\text{ind}(\Phi^{-1}) = \text{ind}_Q(T_\Phi).$$

Proof. To each automorphism $\varphi$ in the outer class $\Phi$ is associated a homothety $H$ of $T_\Phi$ and the center $C_H$ of this homothety. As the action of $F_N$ on $T_\Phi$ is free, two automorphisms are isogredient if and only if the corresponding centers are in the same $F_N$-orbit.

The index of $\Phi^{-1}$ is the sum over all essential isogredient classes of automorphism $\varphi^{-1}$ in $\Phi^{-1}$ of the index of $\varphi^{-1}$. For each of these automorphisms the index $2\text{ind}(\varphi^{-1})$ is equal by Proposition 4.3 to the contribution $\#Q^{-1}(C_H)$ of the orbit of $C_H$ to the $Q$ index of $T_\Phi$.

Conversely, let now $P$ be a point in $T_\Phi$ with at least three elements in its $Q$-fiber. Let $\varphi$ be an automorphism in $\Phi$ and let $H$ be the homothety of $T_\Phi$ associated to $\varphi$. For any integer $n$, the $Q$-fiber $Q^{-1}(H^n(P)) = \partial \varphi^n(Q^{-1}(P))$ of $H^n(P)$ also has at least three elements. By [CH10, Theorem 5.3] there are finitely many orbits of such points in $T_\Phi$ and thus we can assume that $H^n(P) = wP$ for some $w \in F_N$ and some integer $n > 0$. Then $P$ is the center of the homothety $w^{-1}H^n$ associated to $i_{w^{-1}} \circ \varphi^n$. Since $\Phi$ satisfies property (FR2), $P$ is the center of a homothety $uH$ associated to $i_u \circ \varphi$ for some $u \in F_N$. This concludes the proof of the equality of the indices. □

This Proposition can alternatively be deduced from the techniques of Handel and Mosher [HM06].

5. **Botanical classification of irreducible automorphisms**

**Theorem 5.1.** Let $\Phi$ be an iwip outer automorphism of $F_N$. Let $T_\Phi$ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Then, the $Q$-index of the attracting tree is equal to the geometric index of the repelling tree:

$$\text{ind}_Q(T_\Phi) = \text{ind}_{geo}(T_{\Phi^{-1}}).$$

Proof. First, if $\Phi$ is geometric, then the trees $T_\Phi$ and $T_{\Phi^{-1}}$ have maximal geometric indices $2N - 2$. On the other hand the trees $T_\Phi$ and $T_{\Phi^{-1}}$ are surface trees and thus their $Q$-indices are also maximal:

$$\text{ind}_{geo}(T_\Phi) = \text{ind}_Q(T_\Phi) = \text{ind}_{geo}(T_{\Phi^{-1}}) = \text{ind}_Q(T_{\Phi^{-1}}) = 2N - 2$$

We now assume that $\Phi$ is not geometric and we can apply Propositions 4.2 and 4.4 to get the desired equality.
From Theorem 5.1 and from the characterization of geometric and surface-type trees by the maximality of the indices we get

**Theorem 5.2.** Let $\Phi$ be an iwip outer automorphism of $F_N$. Let $T_\Phi$ and $T_{\Phi^{-1}}$ be its attracting and repelling trees. Then exactly one of the following occurs

1. $T_\Phi$ and $T_{\Phi^{-1}}$ are surface trees;
2. $T_\Phi$ is Levitt and $T_{\Phi^{-1}}$ is pseudo-surface;
3. $T_{\Phi^{-1}}$ is Levitt and $T_\Phi$ is pseudo-surface;
4. $T_\Phi$ and $T_{\Phi^{-1}}$ are pseudo-Levitt.

**Proof.** The trees $T_\Phi$ and $T_{\Phi^{-1}}$ are indecomposable by Theorem 2.1 and thus they are either of surface type or of Levitt type by [CH10, Proposition 5.14]. Recall, from [GL95] (see also [CH10, Theorem 5.9] or [CHL09, Corollary 6.1]) that $T_\Phi$ is geometric if and only if its geometric index is maximal:

$$\text{ind}_{\text{geo}}(T_\Phi) = 2N - 2.$$ 

From [CH10, Theorem 5.10], $T_\Phi$ is of surface type if and only if its $Q$-index is maximal:

$$\text{ind}_{Q}(T_\Phi) = 2N - 2.$$ 

The Theorem now follows from Theorem 5.1.

Let $\Phi \in \text{Out}(F_N)$ be an iwip outer automorphism.

The outer automorphism $\Phi$ is geometric if both its attracting and repelling trees $T_\Phi$ and $T_{\Phi^{-1}}$ are geometric. This is equivalent to saying that $\Phi$ is induced by a pseudo-Anosov homeomorphism of a surface with boundary, see [Gui05] and [HM07]. This is case 1 of Theorem 5.2.

The outer automorphism $\Phi$ is parageometric if its attracting tree $T_\Phi$ is geometric but its repelling tree $T_{\Phi^{-1}}$ is not. This is case 2 of Theorem 5.2.

The outer automorphism $\Phi$ is pseudo-Levitt if both its attracting and repelling trees are not geometric. This is case 4 of Theorem 5.2.

We now bring expansion factors into play. An iwip outer automorphism $\Phi$ of $F_N$ has an expansion factor $\lambda_\Phi > 1$: it is the exponential growth rate of (non fixed) conjugacy classes under iteration of $\Phi$.

If $\Phi$ is geometric, the expansion factor of $\Phi$ is equal to the expansion factor of the associated pseudo-Anosov mapping class and thus $\lambda_\Phi = \lambda_{\Phi^{-1}}$.

Handel and Mosher [HM07] proved that if $\Phi$ is a parageometric outer automorphism of $F_N$ then $\lambda_\Phi > \lambda_{\Phi^{-1}}$ (see also [BBC08]). Examples are also given by Gautero [Gau07].

For pseudo-Levitt outer automorphisms of $F_N$ nothing can be said on the comparison of the expansion factors of the automorphism and its inverse. On one hand, Handel and Mosher give in the introduction of [HM07] an explicit example of a non geometric automorphism with $\lambda_\Phi = \lambda_{\Phi^{-1}}$; thus this automorphism is pseudo-Levitt. On the other hand, there are examples of pseudo-Levitt automorphisms with $\lambda_\Phi > \lambda_{\Phi^{-1}}$. Let $\varphi \in \text{Aut}(F_3)$ be the automorphism such that

$$\varphi: \begin{align*}
a & \mapsto b \\
      b & \mapsto ac \\
      c & \mapsto a
\end{align*}$$

and

$$\varphi^{-1}: \begin{align*}
a & \mapsto c \\
b & \mapsto a \\
c & \mapsto c^{-1}b
\end{align*}$$

Let $\Phi$ be its outer class. Then $\Phi^6$ is FR, has index $\text{ind}(\Phi^6) = 3/2 < 2$. The expansion factor is $\lambda_\Phi \approx 1.3247$. The outer automorphism $\Phi^{-3}$ is FR, has index $\text{ind}(\Phi^{-3}) = 1/2 < 2$. 

The expansion factor is $\lambda_{\Phi^{-1}} \simeq 1, 4655 > \lambda_{\Phi}$. The computation of these two indices can be achieved using the algorithm of \cite{Jul09}.

Now that we have classified outer automorphisms of $F_N$ into four categories, questions of genericity naturally arise. In particular, is a generic outer automorphism of $F_N$ iwip, pseudo-Levitt and with distinct expansion factors? This is suggested by Handel and Mosher \cite{HM07}, in particular for statistical genericity: given a set of generators of $\text{Out}(F_N)$ and considering the word-metric associated to it, is it the case that

$$\lim_{k \to \infty} \frac{\#(\text{pseudo-Levitt iwip with } \lambda_{\Phi} \neq \lambda_{\Phi^{-1}}) \cap B(k)}{\#B(k)} = 1$$

where $B(k)$ is the ball of radius $k$, centered at $1$, in $\text{Out}(F_N)$?

5A. **Botanical memo.** In this Section we give a glossary of our classification of automorphisms for the working mathematician.

For a FR iwip outer automorphism $\Phi$ of $F_N$, we used 6 indices which are related in the following way:

$$2 \text{ ind}(\Phi) = \text{ ind}_{\text{geo}}(T_{\Phi}) = \text{ ind}_{\text{Q}}(T_{\Phi^{-1}})$$

$$2 \text{ ind}(\Phi^{-1}) = \text{ ind}_{\text{geo}}(T_{\Phi^{-1}}) = \text{ ind}_{\text{Q}}(T_{\Phi})$$

All these indices are bounded above by $2N - 2$. We sum up our Theorem \ref{thm:5.2} in the following table.

| Automorphisms | Trees | Indices |
|---------------|-------|---------|
| $\Phi$ geometric | $T_{\Phi}$ and $T_{\Phi^{-1}}$, geometric | $\text{ ind}(\Phi) = \text{ ind}(\Phi^{-1}) = N - 1$ |
| $\Phi^{-1}$ geometric | $T_{\Phi}$ surface | |
| | $T_{\Phi^{-1}}$ surface | |
| $\Phi$ parageometric | $\begin{cases} T_{\Phi}$ geometric \\ and \\ $T_{\Phi^{-1}}$ non geometric \end{cases}$ | $\begin{cases} \text{ ind}(\Phi) = N - 1 \\ \text{ and} \\ \text{ ind}(\Phi^{-1}) < N - 1 \end{cases}$ |
| | $T_{\Phi}$ Levitt | $\uparrow$ |
| | $T_{\Phi^{-1}}$ pseudo-surface | $\uparrow$ |
| $\Phi$ pseudo-Levitt | $T_{\Phi}$ and $T_{\Phi^{-1}}$, non geometric | $\begin{cases} \text{ ind}(\Phi) < N - 1 \\ \text{ and} \\ \text{ ind}(\Phi^{-1}) < N - 1 \end{cases}$ |
| $\Phi^{-1}$ pseudo-Levitt | $T_{\Phi}$ pseudo-Levitt | $\uparrow$ |
| | $T_{\Phi^{-1}}$ pseudo-Levitt | $\uparrow$ |

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