Quantum correction of gravitational constant

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ABSTRACT: We suggest a scheme for considering the quantum correction of the gravitational constant. In the model, the gravitational constant originates from a coupling of the gravitational field with a scalar field. In this paper, we show that if the scalar field, as it should be in the real physical world, is a quantum field, then the gravitational constant will have a spacetime-dependent quantum correction, so that the quantum corrected physical constant is no longer a constant. The quantum correction of the gravitational constant is different in different spacetime. We calculate the quantum correction in the Schwarzschild spacetime, the $H_3$ (Euclidean $AdS_3$) spacetime, the $H_3/Z$ spacetime, the universe model, the de Sitter spacetime, and the Rindler spacetime.

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1 Introduction

The gravitational constant is determined phenomenologically. There is no generally accepted theory of the origin of the gravitational constant. In the frame of the induced gravity model, the gravitational constant originates from a coupling of the gravitational field and a scalar field. In the classical picture of the induced gravity, the scalar field is a classical field and the gravitational constant is a constant equaling to the gravitational constant in the Einstein gravity. However, in the real world, a scalar field, as it should be, must be a quantum field rather than a classical field. We show that if the scalar field is a quantum field, there will be a quantum correction on the gravitational constant. The quantum corrected gravitational constant is no longer a constant at the quantum level, but varies with time and space.

The quantum correction of the gravitational constant is different in different spacetimes. In this paper, as examples, we calculate the quantum correction of the gravitational constant in the Schwarzschild spacetime, the $H_3$ (Euclidean $AdS_3$) spacetime, the $H_3/Z$ spacetime, the universe models, the de Sitter spacetime, and the Rindler spacetime.

In this paper, we show that the gravitational constant may have a quantum correction. This correction is small, only at a quantum level. Nevertheless, there are studies show that even a small change in the gravitational constant will lead a remarkable influence on the
evolution of the universe, for big bang nucleosynthesis is very sensitive to the value of the gravitational constant [1]. This inspires us to study the influence of such a quantum level change of the gravitational constant in cosmology.

In this paper, the gravitational constant has a time- and space-dependent quantum correction. The result of the present paper provides a physical mechanism of nonconstant gravitational constants. The variation of gravitational constants with time has been discussed for a long time [2, 3]. The effect of a nonconstant gravitational constant in cosmology is also considered theoretically and experimentally [1, 4, 5]. In cosmology induced gravity theory has been applied, such as Bianchi-I cosmological models [6], the reconstruction technique of the potential for scalar fields in induced-gravity based cosmological models [7], induced gravity dark energy models [8], the quantum backreaction of the scalar and the tensor perturbations in induced gravity [9], and the exorcism of ghosts in induced gravity [10]. Black holes [11] and black hole entropies [12] in induced gravity are considered. Induced gravity with scalar fields and non-Abelian gauge interactions are studied [13, 14]. The Schwinger-DeWitt technique for the covariant curvature expansion of effective actions for brane induced gravity are developed [15, 16]. The criterion defining single-field models leading to Starobinsky-like inflation [17], the induced gravity on the Dvali-Gabadadze-Porrati brane [18–20], and the Wheeler-DeWitt equation for induced gravity models [21] are studied.

In section 2, we consider the quantum correction of the gravitational constant. As examples, we calculate the quantum correction of the gravitational constant in various spacetimes, including the Schwarzschild spacetime, the $H_3$ (Euclidean AdS$_3$) spacetime, the $H_3/Z$ spacetime, the universe model, the de Sitter spacetime, and the Rindler spacetime. Section 9 is devoted to conclusions and outlook.

2 Quantum correction of gravitational constant

The action of gravity is

$$I = \int d^4x \sqrt{-g} \frac{1}{16\pi G} R,$$

where $G$ is the gravitational constant and $R$ is the Ricci scalar. First construct an induced gravity model. Introduce a scalar field $\Phi$ and couple the scalar field $\Phi$ to the gravitational field:

$$I = \int d^4x \sqrt{-g} \left[ \chi(\Phi) R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \right].$$

Zee’s induced gravity [22] can be recovered by taking $\chi(\Phi) = \frac{1}{2} \epsilon \Phi^2$. Comparison with Eq. (2.1) shows that $\chi(\Phi)$ plays the role of the reciprocal of the gravitational constant $G$.

The equation of the scalar field $\Phi$, by the action (2.2), reads

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) = - \frac{dV_{\text{eff}}(\Phi)}{d\Phi},$$

where

$$V_{\text{eff}}(\Phi) = -\chi(\Phi) R + V(\Phi).$$
In the model, the gravitational constant $G$ is the reciprocal of $\chi (\Phi)$. In the classical picture, the gravitational constant $G$ is a constant, so $\chi (\Phi)$ must be a constant. This requires that the solution of the field $\Phi$ must be a stable classical ground-state solution: $\Phi = \Phi_0$, so that $\chi (\Phi) = \chi (\Phi_0)$ is a constant. The classical ground state $\Phi = \Phi_0$ is the position of the minimum value of $V_{\text{eff}} (\Phi)$ given by $V_{\text{eff}}' (\Phi_0) = 0$. In the classical picture, the gravitational constant is a constant and is independent of the gravitational field itself. This requires that though in Eq. (2.4) $V_{\text{eff}} (\Phi)$ depends on the Ricci scalar $R$, the position of the minimum value of $V_{\text{eff}} (\Phi)$ should not depend on $R$. For this purpose, we choose the position of the minimum value of $\chi (\Phi)$ and the position of the minimum value of $V (\Phi)$ are the same, so that the position of the minimum value of $V_{\text{eff}} (\Phi)$ is the same as the minimum value of $\chi (\Phi)$ and the minimum value of $V (\Phi)$. Consequently, the classical ground state $\Phi = \Phi_0$ which is the position of the minimum value of $V_{\text{eff}} (\Phi)$ is also the position of the minimum values of $\chi (\Phi)$ and $V (\Phi)$, i.e., $V_{\text{eff}}' (\Phi_0) = V' (\Phi_0) = \chi' (\Phi_0) = 0$.

In the classical picture, the gravitational constant is

$$G = \frac{1}{16\pi \chi (\Phi_0)}. \tag{2.5}$$

The scalar field $\Phi$ is in the classical ground state: $\Phi = \Phi_0$. The classical ground state $\Phi = \Phi_0$ is the position of the minimum values of $\chi (\Phi)$ and $V (\Phi)$ and the minimum values of $\chi (\Phi)$ and $V (\Phi)$ are the same.

Nevertheless, in the real physical world a scalar field must be a quantum field. We will show that a quantum scalar field leads to a spacetime-dependent quantum corrected gravitational constant.

Next we construct the equation of quantum correction.

The action of a scalar field in the spacetime with the metric $g_{\mu \nu}$ is

$$S [\Phi] = \int \sqrt{-g} d^4 x L (\Phi). \tag{2.6}$$

The classical field $\Phi_{\text{cl}}$ is given by the extreme value of the action functional:

$$\left. \frac{\delta S [\Phi]}{\delta \Phi} \right|_{\Phi = \Phi_{\text{cl}}} = 0, \tag{2.7}$$

i.e., $\Phi_{\text{cl}}$ is the solution of the classical field equation (2.7); note that the classical field equation with the action (2.2) is Eq. (2.3). Expanding the action $S [\phi]$ around the classical field $\phi_{\text{cl}}$, keeping the expansion to the second order, and applying the classical field equation (2.7) give

$$S [\Phi] = S [\Phi_{\text{cl}}] + \frac{1}{2} \int \sqrt{-g} (x_1) d^4 x_1 \sqrt{-g} (x_2) d^4 x_2 \frac{\delta^2 S [\Phi]}{\delta \Phi (x_1) \delta \Phi (x_2)} \bigg|_{\phi = \phi_{\text{cl}}} [\Phi (x_1) - \Phi_{\text{cl}} (x_1)] [\Phi (x_2) - \Phi_{\text{cl}} (x_2)]. \tag{2.8}$$

The generating functional $Z [J] = \int D\Phi \exp \left( \frac{i}{\hbar} (S [\Phi] + \int \sqrt{-g} d^4 x J \Phi) \right)$ with the action (2.8), by using the Gaussian integral formula $\int D\phi \exp \left( - \frac{1}{4 \hbar} (\Phi, A \Phi) + (b, \Phi) + c \right) =$
\[(\det A)^{-1/2} \exp \left[ \frac{1}{2} \left( b, A^{-1}b \right) - c \right], \text{ reads}\]

\[
Z [J] = e^{\frac{i}{\hbar} S[\Phi_{cl}] + \frac{i}{2} \int \sqrt{-g(x)} d^4x J(x) \Phi_{cl}(x)} \left[ \det \left( -\frac{i}{\hbar} \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \right) \right]^{-1/2} 
\times \exp \left( -\frac{i}{2\hbar} \int \sqrt{-g(x)} d^4x_1 \sqrt{-g_2(x_2)} d^4x_2 J(x_1) \left( \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \right)^{-1} J(x_2) \right) .
\]

(2.9)

The generating functional \( W[J] \), defined by \( Z [J] = e^{i W[J]} \), by Eq. (2.9), reads

\[
W [J] = -i\hbar \ln Z [J]
= S[\Phi_{cl}] + \int d^4x \sqrt{-g(x)} J(x) \Phi_{cl}(x) + \frac{i}{2} \hbar \text{ tr} \left( \ln \left( -\frac{i}{\hbar} \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \right) \right)
- \frac{1}{2} \int \sqrt{-g(x)} d^4x_1 \sqrt{-g_2(x_2)} d^4x_2 J(x_1) \left( \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \right)^{-1} J(x_2) .
\]

(2.10)

The effective action can be then obtained by the Legendre transform \( \Gamma [\Phi] = W [J] - \int \sqrt{-g(x)} d^4x \Phi(x) J(x) \):

\[
\Gamma [\Phi] = S[\Phi_{cl}] + \frac{i}{2} \hbar \text{ tr} \left[ \ln \left( -\frac{i}{\hbar} \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \right) \right]
- \int \sqrt{-g(x)} d^4x_1 \sqrt{-g_2(x_2)} d^4x_2 \frac{3}{2} \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \eta(x_1) \eta(x_2) .
\]

(2.11)

The extreme value of the effective action functional, by \( \frac{\delta \Gamma[\Phi]}{\delta \Phi} = 0 \), gives the equation of the quantum fluctuation \( \eta \):

\[
\int \sqrt{-g(x)} d^4x_1 \left( \frac{\delta^2 S[\Phi]}{\delta \Phi(1) \delta \Phi(2)} \bigg|_{\Phi = \Phi_{cl}} \right) \eta(x_1) = 0 . \]

(2.12)

This method is equivalent to writing

\[
\Phi = \Phi_{cl} + \eta , \]

(2.13)

and directly substituting Eq. (2.13) into the classical equation (2.3), as that in Ref. [23].

The equation of the quantum correction, with the action (2.2), up to the first-order contribution reads:

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \eta \right) + \left[ V''(\Phi_{cl}) - \chi''(\Phi_{cl}) R \right] \eta = 0 . \]

(2.14)

Expanding \( \chi(\Phi) \) around \( \eta = 0 \), we have

\[
\chi(\Phi) = \chi(\Phi_{cl} + \eta) = \chi(\Phi_{cl}) + \frac{1}{2} \chi''(\Phi_{cl}) \eta^2 + \cdots . \]

(2.15)
The reason why the leading contribution is the $\eta^2$ term is that the $\eta$ term vanishes for $\chi'(\Phi_{cl}) = 0$. The quantum corrected gravitational constant by Eqs. (2.5) and (2.15) then reads

$$G_Q = \frac{1}{16\pi \chi (\Phi)} \simeq \frac{1}{16\pi \chi (\Phi_{cl})} \left[ 1 - \frac{1}{2} \frac{\chi''(\Phi_{cl})}{\chi(\Phi_{cl})} \eta^2 \right].$$

(2.16)

The quantum correction $\eta$ is given by Eq. (2.14). In our case, the classical solution of the scalar field is the classical ground-state solution, i.e., $\Phi_{cl} = \Phi_0$, so

$$G_Q = G \left[ 1 - \frac{1}{2} \frac{\chi''(\Phi_0)}{\chi(\Phi_0)} \eta^2 \right]$$

(2.17)

with $G$ the Newtonian gravitational constant. The quantum corrected gravitational constant depends on the spacetime metric $g^\mu\nu$. In the classical picture, the gravitational constant in different spacetime is the same, but the quantum corrected gravitational constant in different spacetimes are different.

In the following, we will not use the concrete function form of $\chi(\Phi)$ and $V(\Phi)$. The result applies to any $\chi(\Phi)$ and $V(\Phi)$ so long as they are lower bounded. In different theories $\chi(\Phi)$ and $V(\Phi)$ are different, e.g., in Zee’s induced gravity $\chi(\Phi) = \frac{1}{2} \epsilon \Phi^2$ [22] and in $\Phi^4$-theory $V(\Phi) = \lambda \Phi^4$.

The equation of the quantum fluctuation $\eta$, Eq. (2.14), is a linear homogeneous equation, so $\eta$ and $C \eta$ are both the solutions of Eq. (2.14). Therefore the quantum fluctuation $\eta$ can only be determined up to a constant factor. In other words, there is an undetermined constant in the quantum fluctuation $\eta$.

In physical experiments, at a spacetime point $x_0 = (t_0, x_0)$, one can measure the experimental value of the gravitational constant $G_{\text{experiment}}(x_0)$. For example, in the universe model one can choose $t_0$ as the current time and in Schwarzschild spacetime one can choose $x_0$ as the spacetime point where the observer stands. At $x_0$, the quantum corrected gravitational constant by Eqs. (2.5) and (2.16) is

$$G_Q(x_0) = \frac{1}{16\pi \chi (\Phi_{cl}(x_0))} \left[ 1 - \frac{1}{2} \frac{\chi''(\Phi_{cl}(x_0))}{\chi(\Phi_{cl}(x_0))} \eta^2(x_0) \right].$$

(2.18)

The classical solution $\Phi_{cl}(x)$ is given by solving the usual classical scalar field equation. Then we have

$$G_Q(x_0) = G_{\text{experiment}}(x_0).$$

(2.19)

The quantum fluctuation at $x_0$ then by Eq. (2.19) reads

$$\eta(x_0) = \sqrt{\frac{2 \chi(\Phi_{cl}(x_0))}{\chi''(\Phi_{cl}(x_0))} \left[ 1 - 16\pi \chi(\Phi_{cl}(x_0)) G_{\text{experiment}}(x_0) \right]}. $$

(2.20)

The undetermined constant in the quantum fluctuation $\eta$ can be solved from Eq. (2.20).

In our model, as discussed above, the world is in its ground state, so the classical solution is the classical ground-state solution: $\Phi_{cl}(x_0) = \Phi_0$. Then Eq. (2.20) becomes

$$\eta(x_0) = \sqrt{\frac{2 \chi(\Phi_0)}{\chi''(\Phi_0)} \left[ 1 - 16\pi \chi(\Phi_0) G_{\text{experiment}}(x_0) \right]}. $$

(2.20)
3 Schwarzschild spacetime

Consider the quantum correction of the gravitational constant in the Schwarzschild spacetime.

In the Schwarzschild spacetime,

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \]  

(3.1)

the radial equation of the field, by Eq. (2.3), after variable separation \( \Phi(t, r, \theta, \varphi) = \cos(\omega t)\phi(r)Y(\theta, \varphi) \), reads

\[ (1 - \frac{2M}{r})\phi''(r) + \frac{2(r - M)}{r^2}\phi'(r) - \frac{l(l + 1)}{r^2}\phi(r) + V'(\phi(r)) + \frac{\omega^2}{1 - \frac{2M}{r}}\phi(r) = 0, \]  

(3.2)

where we assume that the potential depends only on the radial part of the field, \( V(\Phi) = V(\phi(r)) \).

In the classical picture, the fact that the gravitational constant given by Eq. (2.5) is a constant requires that the classical solution is a stable classical ground-state solution. Therefore, we only concern ourselves with the classical ground state which corresponds to \( \omega = 0 \) and \( l = 0 \):

\[ (1 - \frac{2M}{r})\phi''(r) + \frac{2(r - M)}{r^2}\phi'(r) + V'(\phi(r)) = 0. \]  

(3.3)

The equation of \( \eta \) is given by substituting \( \phi = \phi_{cl} + \eta \) into Eq. (3.3) and keeping only the leading contribution:

\[ \frac{r - 2M}{r}\eta''(r) + \frac{2(r - M)}{r^2}\eta'(r) + V''(\phi_{cl})\eta(r) = 0. \]  

(3.4)

To solve Eq. (3.4), we use the variable substitution \( z = \frac{r}{M} - 1 \) to convert Eq. (3.4) to a confluent Heun equation [24]:

\[ \frac{d}{dz}\left[(z^2 - 1)\frac{dy(z)}{dz}\right] + \left[-p^2(z^2 - 1) + 2p\beta z - \lambda - \frac{m^2 + s^2 + 2msz}{z^2 - 1}\right]y(z) = 0, \]  

(3.5)

where \( p = M\sqrt{-V''(\phi_{cl})} \), \( m = s = 0 \), \( \beta = -M\sqrt{-V''(\phi_{cl})} \), and \( \lambda = -2V''(\phi_{cl})M^2 \). The solution of the confluent Heun equation (3.5) is the confluent Heun function \( y(z) = H^{(\alpha)}_c(p, \alpha, \gamma, \delta, \sigma; z) \) with \( \gamma = m + s + 1 \), \( \delta = m - s + 1 \), \( \alpha = -\beta + m + 1 \), and \( \sigma = \lambda - 2p(\beta - m - s - 1) - m(m + 1) \) [25]. Then the quantum fluctuation reads

\[ \eta = C H^{(\alpha)}_c\left(M\sqrt{-V''(\phi_{cl})}, \left[M\sqrt{-V''(\phi_{cl})} + 1\right], 1, 1, \left[-4M^2V''(\phi_{cl}) + 2M\sqrt{-V''(\phi_{cl})}\right] + \frac{r}{M} - 1\right). \]  

(3.6)

The quantum corrected gravitational constant by Eq. (2.16) is

\[ G_Q = \frac{1}{16\pi\chi(\phi_{cl})}\left\{1 - \frac{C^2}{2}\chi''(\phi_{cl})\left[H^{(\alpha)}_c\left(M\sqrt{-V''(\phi_{cl})}, \left[M\sqrt{-V''(\phi_{cl})} + 1\right], 1, 1, \left[-4M^2V''(\phi_{cl}) + 2M\sqrt{-V''(\phi_{cl})}\right] + \frac{r}{M} - 1\right]\right]\right\}. \]  

(3.7)
The gravitational constant in the Schwarzschild spacetime depends on the radial coordinate \( r \).

The constant \( C \) is determined by observation. Suppose the observer stands at \( r_0 \). The gravitational constant measured by an observer standing at \( r_0 \) is \( G_{\text{experiment}} (r_0) \). Then \( G_Q (r_0) = G_{\text{experiment}} (r_0) \) determines the constant \( C \):

\[
C = \sqrt{\frac{2 \chi (\phi_0)}{\chi'' (\phi_0)} [1 - 16 \pi \chi (\phi_0) G_{\text{experiment}} (r_0)]} \left( H^{(a)}_1 \left( M \sqrt{-V'' (\phi_0)}, \left[ M \sqrt{-V'' (\phi_0)} + 1 \right], 1, 1, \left[ -4 M^2 V'' (\phi_0) + 2 M \sqrt{-V'' (\phi_0)} \right] ; \frac{r_0}{M} - 1 \right) \right)^{-1},
\]

where \( \phi_0 = \phi_{cl} (r_0) \).

For \( r \to \infty \), the asymptotics of the quantum fluctuation is [24–26]

\[
\eta \sim \frac{M}{r - M} \cos \left( \sqrt{-V'' (\phi_{cl})} (r - M) + M \sqrt{-V'' (\phi_{cl})} \ln 2 \sqrt{-V'' (\phi_{cl})} (r - M) \right)
\]

and for \( r \to 2M \), the asymptotics of the quantum fluctuation is

\[
\eta \sim e^{\sqrt{-V'' (\phi_{cl})} r} \left[ 1 - \frac{1}{2} \left( 4 M V'' (\phi_{cl}) + 2 \sqrt{-V'' (\phi_{cl})} \right) (r - 2M) \right].
\]

### 4 \( H_3 \) (Euclidean \( AdS_3 \)) spacetime

The \( AdS_3 \) space and the Euclidean BTZ black hole is important in gravity [27–31]. We consider the quantum correction of the gravitational constant in the three-dimensional hyperbolic space \( H_3 \) (Euclidean anti-de Sitter space \( AdS_3 \)). \( H_3 \) is a three-dimensional subspace of the flat four-dimensional spacetime: \( ds^2 = dX_1^2 - dT_1^2 + dX_2^2 + dT_2^2 \) with the constraint \( X_1^2 - T_1^2 + X_2^2 + T_2^2 = -l^2 \) [32, 33]. The metric is

\[
ds^2 = l^2 \left( d\rho^2 + \cosh^2 \rho d\phi^2 + \sinh^2 \rho d\theta^2 \right).
\]

The equation of the quantum fluctuation (2.14) with the metric (4.1) reads

\[
- \frac{\partial^2 \eta}{\partial \rho^2} - \frac{\partial^2 \eta}{\cosh \rho \sinh \rho} \frac{\partial \eta}{\partial \rho} - \frac{1}{\cosh^2 \rho} \frac{\partial^2 \eta}{\partial \phi^2} - \frac{1}{\sinh^2 \rho} \frac{\partial^2 \eta}{\partial \theta^2} - \left[ l^2 V'' (\Phi_{cl}) + 6 \chi'' (\Phi_{cl}) \right] \eta = 0.
\]

The variable separation \( \eta = f (\rho) \cos (k_1 \phi) \cos (k_2 \theta) \) gives the radial equation of the quantum fluctuation,

\[
- \frac{d^2 f (\rho)}{d \rho^2} - \frac{\cosh^2 \rho + \sinh^2 \rho}{\cosh \rho \sinh \rho} \frac{df (\rho)}{d \rho} - \left[ \frac{k_1^2}{\cosh^2 \rho} + \frac{k_2^2}{\sinh^2 \rho} + l^2 V'' (\Phi_{cl}) + 6 \chi'' (\Phi_{cl}) \right] f (\rho) = 0.
\]

It is reasonable to assume that the quantum fluctuation is in the ground state, so we only consider the case of \( k_1 = k_2 = 0 \):

\[
- \frac{d^2 f (\rho)}{d \rho^2} + \frac{\cosh^2 \rho + \sinh^2 \rho}{\cosh \rho \sinh \rho} \frac{df (\rho)}{d \rho} + \left[ l^2 V'' (\Phi_{cl}) + 6 \chi'' (\Phi_{cl}) \right] f (\rho) = 0.
\]
The solution of Eq. (4.4) is

\[ f (\rho) = C_1 P\left[ \sqrt{-\rho^2 V''(\Phi_{cl}) - 6 \chi''(\Phi_{cl}) + 1} \right] / 2 \left( \cosh 2\rho \right) + C_2 Q\left[ \sqrt{-\rho^2 V''(\Phi_{cl}) - 6 \chi''(\Phi_{cl}) + 1} \right] / 2 \left( \cosh 2\rho \right), \]

(4.5)

where \( P_u (z) \) is the Legendre function and \( Q_u (z) \) the Legendre function of the second kind. Since \( Q\left[ \sqrt{-\rho^2 V''(\Phi_{cl}) - 6 \chi''(\Phi_{cl}) + 1} \right] / 2 \left( \cosh 2\rho \right) \) diverges at \( \rho \to 0 \), we choose \( C_2 = 0 \):

\[ f (\rho) = C P\left[ \sqrt{-\rho^2 V''(\Phi_{cl}) - 6 \chi''(\Phi_{cl}) + 1} \right] / 2 \left( \cosh 2\rho \right), \]

(4.6)

where \( C = C_1 \). Then the quantum fluctuation

\[ \eta = C P\left[ \sqrt{-\rho^2 V''(\Phi_{cl}) - 6 \chi''(\Phi_{cl}) + 1} \right] / 2 \left( \cosh 2\rho \right). \]

(4.7)

By observation the gravitational constant measured at \( \rho_0 \) is \( G_{\text{experiment}} (\rho_0) \). Then \( G_Q (\rho_0) = G_{\text{experiment}} (\rho_0) \), by substituting Eq. (4.7) into (2.16), determines the constant \( C \):

\[ C = \sqrt{\frac{2 \chi(\Phi_{cl})}{\chi''(\Phi_{cl})}} \left[ 1 - 16 \pi \chi (\Phi_0) G_{\text{experiment}} (\rho_0) \right] P\left[ \sqrt{-\rho^2 V''(\Phi_{cl}) - 6 \chi''(\Phi_{cl}) + 1} \right] / 2 \left( \cosh 2\rho_0 \right), \]

(4.8)

where \( \Phi_0 = \Phi_{cl} (\rho_0) \).

5 \( H_3/Z \) spacetime

The \( H_3/Z \) spacetime, a quotient space of \( H_3 \), is the geometry of the Euclidean BTZ black hole \([33-35]\). The metric of the \( H_3/Z \) (Euclidean \( AdS_3 \)) spacetime is given by \([35]\)

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{dy^2 + dz d\bar{z}}{y^2} \]

(5.1)

with \( y = \rho \sin \theta, \ z = \rho \cos \theta e^{i\phi} \) and \( 1 \leq \rho < e^{2\pi}, \ 0 \leq \theta < \frac{\pi}{2}, \ 0 \leq \phi < 2\pi \):

\[ ds^2 = \frac{1}{\rho^2 \sin^2 \theta} d\rho^2 + \frac{1}{\sin^2 \theta} d\theta^2 + \cot^2 \theta d\phi^2. \]

(5.2)

The equation of the quantum fluctuation is obtained by substituting the metric (5.2) into Eq. (2.14),

\[ -\sin^2 \theta \left( \rho^2 \frac{\partial^2 \eta}{\partial \rho^2} + \rho \frac{\partial \eta}{\partial \rho} \right) - \left( \sin^2 \theta \frac{\partial^2 \eta}{\partial \theta^2} - \tan \theta \frac{\partial \eta}{\partial \theta} \right) - \tan^2 \theta \frac{\partial^2 \eta}{\partial \phi^2} - \left[ V'' (\Phi_{cl}) + 6 \chi'' (\Phi_{cl}) \right] \eta = 0. \]

(5.3)

The variable separation \( \eta = f (\rho) g (\theta) \cos (k \phi) \) gives

\[ -\rho^2 \frac{d^2 f (\rho)}{d\rho^2} - \rho \frac{df (\rho)}{d\rho} = \kappa f (\rho), \]

\[ \frac{d^2 g (\theta)}{d\theta^2} - \frac{1}{\sin \theta \cos \theta} \frac{dg (\theta)}{d\theta} + \left\{ \frac{1}{\sin^2 \theta} \left[ V'' (\Phi_{cl}) + 6 \chi'' (\Phi_{cl}) \right] - k^2 \frac{1}{\cos^2 \theta} \right\} g (\theta) = \kappa g (\theta). \]

(5.4)
It is reasonable to believe that the world is in its ground state, so we only consider the ground state of the quantum fluctuation, i.e., $k = 0$ and $\kappa = 0$:

$$
- \rho^2 \frac{d^2 f (\rho)}{d \rho^2} - \rho \frac{df (\rho)}{d \rho} = 0, \quad (5.5)
$$

$$
\frac{d^2 g (\theta)}{d \theta^2} - \frac{1}{\sin \theta \cos \theta} \frac{dg (\theta)}{d \theta} + \frac{1}{\sin^2 \theta} \left[ V'' (\Phi_{cl}) + 6 \chi'' (\Phi_{cl}) \right] g (\theta) = 0. \quad (5.6)
$$

The solution of Eq. (5.5) is

$$
f (\rho) = C_1 \ln \rho \quad (5.7)
$$

and the solution of Eq. (5.6) is

$$
g (\theta) = C_2 (\sin \theta) - \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} \, F \left( \left[ \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2, \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2 ; \sin^2 \theta \right] \right)
$$

$$
+ C_3 (\sin \theta) \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} \, F \left( \left[ \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2, \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2 ; \sin^2 \theta \right] \right). \quad (5.8)
$$

where $F \left( \frac{a, b; c}{z} \right) = 2 F_1 (a; b; c, z)$ is the hypergeometric function [36]. The first term in Eq. (5.8) diverges when $\theta = 0$, so we choose $C_2 = 0$ and then the solution reads

$$
g (\theta) = C_3 (\sin \theta) \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} \, F \left( \left[ \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2, \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2 ; \sin^2 \theta \right] \right). \quad (5.9)
$$

Then the quantum fluctuation $\eta = f (\rho) g (\theta)$ is

$$
\eta = C \ln \rho \left( \sin \theta \right) \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} \, F \left( \left[ \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2, \sqrt{-V'' (\Phi_{cl}) - 6 \chi'' (\Phi_{cl}) + 1 + 1} / 2 ; \sin^2 \theta \right] \right). \quad (5.10)
$$

By observation the gravitational constant measured at $(\rho_0, \theta_0)$ is $G_{\text{experiment}} (\rho_0, \theta_0)$. Then $G_Q (\rho_0, \theta_0) = G_{\text{experiment}} (\rho_0, \theta_0)$, by Eq. (2.16), determines the constant $C$:

$$
C = \sqrt{\frac{2 \chi (\Phi_0)}{\chi'' (\Phi_0)} \left[ 1 - 16 \pi \chi (\Phi_0) G_{\text{experiment}} (\rho_0, \theta_0) \right] \ln \rho_0 \left( \sin \theta_0 \right) \sqrt{-V'' (\Phi_0) - 6 \chi'' (\Phi_0) + 1 + 1} \times F \left( \left[ \sqrt{-V'' (\Phi_0) - 6 \chi'' (\Phi_0) + 1 + 1} / 2, \sqrt{-V'' (\Phi_0) - 6 \chi'' (\Phi_0) + 1 + 1} / 2 ; \sin^2 \theta_0 \right] \right)^{-1}, \quad (5.11)
$$

where $\Phi_0 = \Phi_{cl} (\rho_0, \theta_0)$.

6 Universe model

Next we consider the quantum correction of the gravitational constant in the universe model.
In the spacetime described by the Robertson-Walker metric,

\[ g_{\mu\nu} = \text{diag} \left( -1, \frac{a^2(t)}{1 - k r^2}, a^2(t) r^2, a^2(t) r^2 \sin^2 \theta \right) \]

(6.1)

with \( a(t) \) the scale factor and \( k \) the curvature, the field equation of the scalar field \( \Phi \) by the action (2.2) reads

\[ \ddot{\Phi} + 3 \frac{\dot{a}}{a} \dot{\Phi} - V'(\Phi) + 6 \chi'(\Phi) \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = 0. \]

(6.2)

Taking only the leading contribution into account, we here use the classical solution of the scale factor \( a(t) \).

The equation of the quantum fluctuation by Eq. (2.14) reads

\[ \ddot{\eta} + 3 \frac{\dot{a}}{a} \dot{\eta} - \left[ V''(\Phi_{\text{cl}}) - 6 \chi''(\Phi_{\text{cl}}) \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \eta = 0. \]

(6.3)

The scale factor \( a \) is given by the universe model.

6.1 Friedmann universe

For the Friedmann universe with the curvature \( k = 0 \), when the energy density of the universe is dominated by the non-relativistic matter, the scale factor \( a(t) \) is [37]

\[ a(t) = \left( \frac{3GM}{\pi} \right)^{1/3} t^{2/3}, \]

(6.4)

where \( M = 2\pi^2 a^4 \rho \) is the mass of the universe with \( \rho \) the density. With the scale factor given by Eq. (6.4), the equation of the quantum fluctuation reads

\[ \ddot{\eta} + 2 \frac{\dot{a}}{a} \dot{\eta} - \left[ \frac{V''(\Phi_{\text{cl}})}{3t^2} \right] \eta = 0. \]

(6.5)

Solving Eq. (6.5) gives

\[ \eta(t) = C_1 \frac{1}{\sqrt{t}} I_{\nu} \left( \frac{\sqrt{V''(\Phi_{\text{cl}})}}{\frac{4}{3} \chi''(\Phi_{\text{cl}})} \right) t + C_2 \frac{1}{\sqrt{t}} K_{\nu} \left( \frac{4}{3} \chi''(\Phi_{\text{cl}}) t \right), \]

(6.6)

where \( I_{\nu}(z) \) is the Modified Bessel function of the first kind and \( K_{\nu}(z) \) is the Modified Bessel function of the second kind. The Bessel functions \( I_{\nu}(z) \) and \( K_{\nu}(z) \) have the asymptotics \( I_{\nu}(z) \sim (z/2)^{\nu} \) and \( K_{\nu}(z) \sim (z/2)^{-\nu} \). Then for \( t \to 0 \) we have the asymptotics \( \eta(t) \sim C_1 \left( \frac{\sqrt{V''(\Phi_{\text{cl}})}}{\frac{4}{3} \chi''(\Phi_{\text{cl}})} \right) t^{\frac{4}{3} \chi''(\Phi_{\text{cl}}) - \frac{4}{3}} + C_2 \left( \frac{\sqrt{V''(\Phi_{\text{cl}})}}{\frac{4}{3} \chi''(\Phi_{\text{cl}})} \right) t^{-\frac{4}{3} \chi''(\Phi_{\text{cl}}) - \frac{4}{3}} \).

Here \( \sqrt{\frac{4}{3} \chi''(\Phi_{\text{cl}})} > \frac{4}{3} \) for \( \chi''(\Phi_{\text{cl}}) < 0 \), so the \( K_{\nu}(z) \) term diverges at \( t \to 0 \). This requires \( C_2 = 0 \). Then quantum fluctuation reads

\[ \eta(t) = C_1 \frac{1}{\sqrt{t}} I_{\nu} \left( \frac{\sqrt{V''(\Phi_{\text{cl}})}}{\frac{4}{3} \chi''(\Phi_{\text{cl}})} \right) \left( \sqrt{V''(\Phi_{\text{cl}})} t \right). \]

(6.7)
By observation the gravitational constant measured at $t_0$ is $G_{\text{experiment}}(t_0)$. Then $G_Q(t_0) = G_{\text{experiment}}(t_0)$, by Eq. (2.16), determines the constant $C$:

$$C = \frac{\sqrt{\frac{2\chi(\Phi_0)}{\chi'(\Phi_0)}} \left[ 1 - 16\pi \chi(\Phi_0) G_{\text{experiment}}(t_0) \right]}{\sqrt{\frac{1}{4\pi} \int_{\Phi_0}^{\Phi} \left( \sqrt{V''(\Phi)} t_0 \right) \dd \Phi}},$$

(6.8)

where $\Phi_0 = \Phi_{cl}(t_0)$.

### 6.2 Inflationary universe

In the inflationary universe epoch the scale factor is [37]

$$a(t) = \exp \left( \sqrt{\frac{8\pi G\rho_v}{3}} t \right),$$

(6.9)

where $\rho_v$ is the vacuum energy density and the curvature $k = 0$. Then by Eq. (6.3) the equation of the quantum fluctuation in the inflationary epoch reads

$$\ddot{\eta} + 3 \sqrt{\frac{8\pi G\rho_v}{3}} \dot{\eta} - \left[ V''(\Phi_{cl}) - 32\pi G\rho_v \chi''(\Phi_{cl}) \right] \eta = 0.$$

(6.10)

In the inflationary epoch, we concern ourselves with the epoch $0 < t < t_0$:

$$\eta = C \exp \left( -\sqrt{\frac{6\pi G\rho_v}{3}} \left[ 1 + \sqrt{1 - \frac{1}{6\pi G\rho_v} V''(\Phi_{cl}) - \frac{16}{3} \chi''(\Phi_{cl})} \right] t \right).$$

(6.11)

By observation the gravitational constant measured at $t_0$ is $G_{\text{experiment}}(t_0)$. Then $G_Q(t_0) = G_{\text{experiment}}(t_0)$, by Eq. (2.16), determines the constant $C$:

$$C = \frac{\sqrt{\frac{2\chi(\Phi_0)}{\chi'(\Phi_0)}} \left[ 1 - 16\pi \chi(\Phi_0) G_{\text{experiment}}(t_0) \right]}{\exp \left( -\sqrt{\frac{6\pi G\rho_v}{3}} \left[ 1 + \sqrt{1 - \frac{1}{6\pi G\rho_v} V''(\Phi_0) - \frac{16}{3} \chi''(\Phi_0)} \right] t_0 \right)},$$

(6.12)

where $\Phi_0 = \Phi_{cl}(t_0)$.

### 7 de Sitter spacetime

The de Sitter spacetime is described by the Robertson-Walker metric with

$$a(t) = a_0 e^{\frac{t}{\sqrt{3}}},$$

(7.1)

and the curvature $k = 0$, where $\Lambda$ is the cosmological constant [37]. The equation of the field $\Phi$ by the action

$$I = \int d^4x \sqrt{-g} \left[ \chi(\Phi) \left( R - 2\Lambda \right) + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \right]$$

(7.2)

reads

$$\ddot{\Phi} + \frac{\dot{a}}{a} \dot{\Phi} - V'(\Phi) + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\Lambda}{3} \right) \chi'(\Phi) = 0.$$  

(7.3)
The equation of the quantum fluctuation by substituting \( \Phi = \Phi_{\text{cl}} + \eta \) into Eq. (7.3), up to the leading order, is

\[
\ddot{\eta} + \frac{3a}{a} \dot{\eta} - \left[ V''(\Phi_{\text{cl}}) - 6\chi''(\Phi_{\text{cl}}) \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\Lambda}{3} \right) \right] \eta = 0. \tag{7.4}
\]

For \( \Lambda > 0 \), by Eq. (7.1), we have

\[
\ddot{\eta} + 3\sqrt{\frac{2\Lambda}{3}} \dot{\eta} + \left[ 2\Lambda \chi''(\Phi_{\text{cl}}) - V''(\Phi_{\text{cl}}) \right] \eta = 0. \tag{7.5}
\]

The solution is

\[
\eta = C \exp \left( -\frac{1}{2} \left[ \sqrt{3\Lambda} + \sqrt{3\Lambda - 8\Lambda \chi''(\Phi_{\text{cl}})} + 4V''(\Phi_{\text{cl}}) \right] t \right). \tag{7.6}
\]

By observation the gravitational constant measured at \( t_0 \) is \( G_{\text{experiment}}(t_0) \). Then \( G_Q(t_0) = G_{\text{experiment}}(t_0) \), by Eq. (2.16), determines the constant \( C \):

\[
C = \frac{\sqrt{\frac{2\chi(\Phi_0)}{\chi'(\Phi_0)} \left[ 1 - 16\pi \chi(\Phi_0) G_{\text{experiment}}(t_0) \right]}}{\exp \left( -\frac{1}{2} \left[ \sqrt{3\Lambda} + \sqrt{3\Lambda - 8\Lambda \chi''(\Phi_{\text{cl}})} + 4V''(\Phi_{\text{cl}}) \right] t_0 \right)}, \tag{7.7}
\]

where \( \Phi_0 = \Phi_{\text{cl}}(t_0) \).

8 Rindler spacetime

The Rindler spacetime is a spacetime region that a uniformly accelerated observer sees. For a uniformly accelerated observer with the acceleration \( a \) in the Minkowski spacetime [38, 39], \( ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2 \), the Rindler spacetime is described by the metric

\[
ds^2 = \pm e^{2a\xi} \left( -dT^2 + d\xi^2 \right) + dY^2 + dZ^2, \tag{8.1}
\]

where "+" corresponds to the R and L regions and "-" corresponds to the F and P regions in the Rindler spacetime. In the R region, \( T = \frac{1}{a} e^{a\xi} \sinh (a\tau) \) and \( X = \frac{1}{a} e^{a\xi} \cosh (a\tau) \), and in the L region, \( T = -\frac{1}{a} e^{a\xi} \sinh (a\tau) \) and \( X = -\frac{1}{a} e^{a\xi} \cosh (a\tau) \). In the R and L regions, \( \tau \) is the time and \( \xi \) is the spatial coordinate with \( |T| \leq |X| \). In the F region \( T = \frac{1}{a} e^{a\xi} \cosh (a\tau) \) and \( X = \frac{1}{a} e^{a\xi} \sinh (a\tau) \), and in the P region \( T = -\frac{1}{a} e^{a\xi} \cosh (a\tau) \) and \( X = -\frac{1}{a} e^{a\xi} \sinh (a\tau) \). In the F and P regions, \( \xi \) is the time and \( \tau \) is the spatial coordinate with \( |T| \geq |X| \). Here the Rindler coordinates \((\tau, \xi)\) are the comoving coordinates of the uniformly accelerated observer along the \( X \) axis:

\[
\tau = \frac{1}{a} \arctanh \frac{T}{X}, \quad \xi = \frac{1}{a} \ln \left( a \sqrt{X^2 - T^2} \right). \tag{8.2}
\]

The equation of the unobservable field \( \Phi \), by Eq. (2.2), reads

\[
\mp e^{2a\xi} \frac{\partial^2 \Phi}{\partial \tau^2} \pm e^{-2a\xi} \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} + V'(\Phi) = 0. \tag{8.3}
\]
In the classical picture, the gravitational constant given by Eq. (2.5) is a constant and is related to the classical ground-state solution which is a constant solution. If the unobservable field $\Phi$ is a quantum field, then there is a quantum correction to the gravitational constant. Writing the quantum unobservable field as $\Phi = \Phi_{cl} + \eta$ and substituting into Eq. (8.3), after variable separation $\eta = \cos(\omega t) \phi(\xi) P(Y, Z)$, we arrive at a stationary equation of $\phi(\xi)$,

$$\pm \frac{\partial^2 \phi(\xi)}{\partial \xi^2} + \left[ \pm e^{-2a_\xi \omega^2} + V''(\Phi_{cl}) - \kappa_1^2 - \kappa_2^2 \right] e^{2a_\xi \phi(\xi)} = 0 \quad (8.4)$$

and $P(Y, Z) = \cos(\kappa_1 Y) \cos(\kappa_2 Z)$, where $\kappa_1$ and $\kappa_2$ are the momenta in the $Y$ and $Z$ directions.

In the classical picture the gravitational constant corresponds to the classical ground state, so we only need to consider the quantum correction of the ground state. For the ground state, $\omega = 0$, $\kappa_1 = \kappa_2 = 0$ and then $\eta = \phi(\xi)$. Eq. (8.4) becomes

$$\pm \frac{\partial^2 \phi(\xi)}{\partial \xi^2} + V''(\Phi_{cl}) e^{2a_\xi \phi(\xi)} = 0. \quad (8.5)$$

In the R and L regions, Eq. (8.5) becomes

$$\frac{\partial^2 \phi(\xi)}{\partial \xi^2} - V''(\Phi_{cl}) e^{2a_\xi \phi(\xi)} = 0. \quad (8.6)$$

The solution is

$$\phi(\xi) = C_1 K_0 \left( \sqrt{V''(\Phi_{cl})} \frac{e^{a_\xi}}{a} \right) + C_1' I_0 \left( \sqrt{V''(\Phi_{cl})} \frac{e^{a_\xi}}{a} \right), \quad (8.7)$$

where $I_\nu(z)$ is the modified Bessel function of the first kind and $K_\nu(z)$ is modified Bessel function of the second kind. Requiring that $\phi(\xi)$ is finite at $\xi \to \infty$, we have

$$\eta = \phi(\xi) = CK_0 \left( \sqrt{V''(\Phi_{cl})} \frac{e^{a_\xi}}{a} \right). \quad (8.8)$$

where $C = C_1$. Then the quantum correction to the gravitational constant, by Eqs. (2.15) and (2.16), is

$$G_Q = \frac{1}{16\pi \chi(\Phi_{cl})} \left[ 1 - \frac{C^2}{2} \frac{\chi''(\Phi_{cl})}{\chi(\Phi_{cl})} K_0 \left( \sqrt{V''(\Phi_{cl})} \frac{e^{a_\xi}}{a} \right) \right]. \quad (8.9)$$

By observation the gravitational constant measured at $\xi_0$ is $G_{\text{experiment}}(\xi_0)$. Then $G_Q(\xi_0) = G_{\text{experiment}}(\xi_0)$ determines the constant $C$:

$$C = \sqrt{\frac{2\chi'(\Phi_0)}{\chi(\Phi_0)}} \frac{\left[ 1 - 16\pi \chi(\Phi_0) G_{\text{experiment}}(\xi_0) \right]}{K_0 \left( \sqrt{V''(\Phi_0)} \frac{e^{a_\xi_0}}{a} \right)}, \quad (8.10)$$

where $\Phi_0 = \Phi_{cl}(\xi_0)$. 

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In the F and P regions, Eq. (8.5) becomes
\[
\frac{\partial^2 \phi (\xi)}{\partial \xi^2} + V'' (\Phi_{\text{cl}}) e^{2a\xi} \phi (\xi) = 0.
\] (8.11)

The solution is
\[
\phi (\xi) = C_2 J_0 \left( \sqrt{V'' (\Phi_{\text{cl}})} e^{a\xi} \right) + C'_2 Y_0 \left( \sqrt{V'' (\Phi_{\text{cl}})} e^{a\xi} \right).
\] (8.12)

where \( J_\nu (z) \) is the Bessel function of the first kind and \( Y_\nu (z) \) is the Bessel function of the second kind.

Dropping the \( Y_0 \) term which diverges at \( \xi \to 0 \) when \( a \) is large, we have
\[
\eta = \phi (\xi) = C J_0 \left( \sqrt{V'' (\Phi_{\text{cl}})} e^{a\xi} \right),
\] (8.13)

where \( C = C_2 \).

Then the quantum correction of the gravitational constant, by Eqs. (2.15) and (2.16), is
\[
G_Q = \frac{1}{16 \pi \chi (\Phi_{\text{cl}})} \left[ 1 - \frac{C^2 \chi'' (\Phi_{\text{cl}}) J_0^2 \left( \sqrt{V'' (\Phi_{\text{cl}})} e^{a\xi} \right)}{2 \chi (\Phi_{\text{cl}}) J_0 \left( \sqrt{V'' (\Phi_{\text{cl}})} e^{a\xi} \right)} \right].
\] (8.14)

By observation the gravitational constant measured at \( \xi_0 \) is \( G_{\text{experiment}} (\xi_0) \). Then \( G_Q (\xi_0) = G_{\text{experiment}} (\xi_0) \) determines the constant \( C \):
\[
C = \frac{\sqrt{2 \chi (\Phi_0) [1 - 16 \pi \chi (\Phi_0) G_{\text{experiment}} (\xi_0)]}}{J_0 \left( \sqrt{V'' (\Phi_0)} e^{a\xi_0} \right)},
\] (8.15)

where \( \Phi_0 = \Phi_{\text{cl}} (\xi_0) \).

9 Conclusions and outlook

In this paper, we suggest a scheme for considering the quantum correction of the gravitational constant. In the model, the gravitational constant is induced by the coupling of the gravitational field and a scalar field. When the scalar field is a quantum field, there will be a time- and space-dependent quantum correction of the gravitational constant. The order of magnitude of the quantum correction is of the magnitude of Planck’s constant.

The focus of this paper is to give a conceptual and qualitative discussion on the quantum correction of the gravitational constant. In order to illustrate the main concept clearly, we construct the toy models as simply as possible. In a more realistic and therefore more complex case, the choice of the scalar field will take into account more factors, such as symmetry breaking. But the qualitative conclusion remains the same. When needed, along this line of thought, such a consideration can be applied to various models.

Moreover, the quantum-corrected gravitational constant varies with time and space, which will influence the black hole and the Hawking radiation. In further work, we will consider the black hole solution under the quantum-corrected gravitational constant and its
influence on the Hawking radiation, especially the species problem of the black hole entropy [40]. The gravitational wave with a quantum corrected gravitational constant is also worth considering [41]. The gravitational constant in the model originates from a coupling between gravity and a scalar field. In principle we need to consider the solution of a scalar field in a certain spacetime. Previously, the solution of a scalar field in a spacetime is solved under the premise that the gravitational constant is a constant [42–45]. The quantum-corrected gravitational constant, however, is no longer a constant, so we need to reconsider the solution of the scalar field in curved spacetime under the time- and space-dependent quantum-corrected gravitational constant. The measurement of the gravitational constant is always a basic experimental problem in physics [46–48].

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