Moduli and BPS configurations of the BLG theory

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Abstract

We study the moduli space of scalars in the BLG theory with and without a constant background four-form field. The classical vacuum moduli space is sixteen-dimensional in the absence of the four-form field. In its presence, however, the moduli space of BPS configurations may be reduced in dimension. We exemplify this with a BPS configuration having $SO(1,2)$ world-volume symmetry and $SO(4) \times SO(4)$ R-symmetry in the presence of a four-form field, by constructing an explicit solution.

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1 Introduction

The BLG theory, named after its inventors [1, 2, 3, 4, 5], is the maximally supersymmetric three-dimensional gauge theory with matter based on a ternary algebra. The BLG theory contains a Chern-Simons gauge field, eight scalars and a Majorana-Weyl spinor. The action has sixteen supersymmetries [6]. The global R-symmetry of the action is $SO(8)$; the eight scalars are thereby interpreted as the eight transverse directions to the three dimensions of the gauge theory. The gauge field as well as the matter fields are valued in a ternary algebra, satisfying the so-called fundamental identity. We shall concern ourselves with a realization of the theory furnished by a completely antisymmetric ternary product and a Euclidean metric. The BLG theory is deemed to describe the world-volume theory of M2-branes of the eleven-dimensional M-theory. Among the reasons for this expectation are its having sixteen supersymmetries, likely to be superconformal [6], and the existence of eight real scalars which may be interpreted as the eight directions transverse to an M2-brane. A third argument in favor of such an interpretation comes from the existence of a limit in which the BLG theory yields the super-Yang-Mills theory of D2-branes in the leading order in gauge-coupling, the latter being the vacuum expectation value of one of the eight scalars, thereby retaining only seven transverse directions to the D2-branes, as required [7]. Various aspects of the BLG theory as well as its variations have been studied [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. More recently, the BLG theory has been generalized to describe M2-branes on non-flat backgrounds containing a constant four-form field which contributes to the action of the theory with a mass term for the scalar fields and the fermion, as well as a flux term [19]. In another line of development, BPS configurations of the BLG theory have been studied leading to their classification based on R-symmetry breaking [20, 21], following a similar classification in Yang-Mills theories [22, 23].

Here we first study the classical vacuum moduli space of the BLG theory without the four-form field. The moduli space is written algebraically in terms of gauge-invariant quantities, furnishing a global description. Upon considering the scalar degree of freedom arising from the three-dimensional gauge field, the dimension of the moduli space is sixteen. We then analyze a BPS configuration of the modified theory with world-volume Lorentz symmetry $SO(1, 2)$ and an $SO(4) \times SO(4)$ R-symmetry in terms of the gauge-invariant variables to obtain an explicit solution for the scalars.

In the next section we write down the action and supersymmetry transformations to set up notation. In Section 3 we study the classical vacuum moduli space using gauge-invariant variables. In Section 4 we use the gauge-invariant variables to show that the moduli space reduces to a point in the presence of the four-form field. We then find an explicit configuration, before concluding in Section 5.
2 BLG theory

Let us begin with a discussion of some features of the BLG theory and its deformation by a background four-form field that will be used later. The BLG theory is an $\mathcal{N} = 8$ supersymmetric theory in three dimensions, given by the Lagrangian

$$
\mathcal{L} = \text{Tr} \left( -\frac{1}{2} (D_\mu X^I)(D^\mu X^I) + \frac{i}{2} \bar{\Psi} \gamma^\mu D_\mu \Psi + \frac{i}{3} \bar{\Psi} \Gamma_{IJK} \langle X^I, X^J, X^K \rangle - \frac{1}{12} \langle X^I, X^J, X^K \rangle^2 \right) + \frac{1}{2} \epsilon^{\mu\nu\lambda} \left( f_{abcd} A_{\mu ab} \partial_\nu A_{\lambda cd} + \frac{2}{3} f^a_{cd} f^{efgh} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \right). 
$$

(2.1)

Here $\mu = 0, 1, 2$ designates the world volume directions, $I = 1, \ldots, 8$ indexes the flavors and $a = 1, 2, 3, 4$ the gauge algebra. $X^I$, $\Psi_a$ and $A_{\mu ab}$ are the scalars, the Majorana-Weyl spinor and the gauge field, respectively. $\gamma$ and $\Gamma$ are, respectively, the three- and eight-dimensional gamma matrices. The structure constants of the ternary algebra are denoted by $f_{abcd}$, while the ternary bracket is written as $\langle , , \rangle$. The repeated indices are summed in the above expression and in the following unless stated otherwise. Denoting the generators of the ternary algebra as $\tau_a$, the metric tensor raising and lowering gauge indices is written as

$$
h_{ab} = \text{Tr} \tau_a \tau_b. 
$$

(2.2)

We use the generators to write the fields valued in the ternary algebra as

$$
X^I = h^{ab} X^I_a \tau_b, 
$$

(2.3)

$$
\Psi = h^{ab} \Psi_a \tau_b. 
$$

(2.4)

The action obtained from the Lagrangian (2.1) is invariant under the supersymmetry transformations [3]

$$
\delta X^I = i \bar{\theta} \Gamma^I \Psi, 
$$

(2.5)

$$
\delta \Psi = D_\mu X^I \gamma^\mu \Gamma^I \theta - \frac{1}{6} \Gamma^{IJK} \langle X^I, X^J, X^K \rangle \theta, 
$$

(2.6)

$$
\delta A_\mu (\phi) = i \bar{\theta} \gamma_\mu \Gamma^I \langle \Psi, X^I, \phi \rangle, 
$$

(2.7)

where $\phi$ represents either a $X^I$ or $\Psi$ and $\theta$ denotes the parameter of supersymmetry variation. The supersymmetry transformations close on-shell up to translation and gauge transformation. A realization of the BLG theory, indeed, the only known finite-dimensional representation, is furnished by an $SO(4)$ gauge theory, in which the $X^I$ and the fermion $\Psi$ transform as vectors under the gauge group [1, 3, 24, 25]. In this instance the structure constant is taken to be the rank-four antisymmetric tensor

$$
f^{abcd} = \epsilon^{abcd}, 
$$

(2.8)

where we have set the level of the Chern Simons action to be unity, while the metric is taken to be Euclidean,

$$
h_{ab} = \delta_{ab}. 
$$

(2.9)
The ternary bracket then reads
\[ \langle X^I, X^J, X^K \rangle = \epsilon^{abcd} X^a_I X^b_J X^c_K \tau_d. \] (2.10)
Henceforth we shall only consider this realization and refer to it as “the BLG theory”.

The BLG theory has been generalized to incorporate a constant self-dual four-form field \( G \). The four-form field satisfies a self-duality condition
\[ \tilde{G}_{IJKL} = G_{IJKL}, \] (2.11)
where the dual of the four-form field \( G \) is defined as
\[ \tilde{G}_{IJKL} = \frac{1}{4!} \epsilon_{IJKLPQRS} G^{PQRS}. \] (2.12)

Incorporation of the four-form to the BLG theory is effected by adding a mass deformation term and a flux term to the Lagrangian. The modified Lagrangian reads
\[ L' = L + L_{\text{mass}} + L_{\text{flux}}, \] (2.13)
where the extra terms are
\[ L_{\text{mass}} = -\frac{1}{2} m^2 \delta^{IJ} \text{Tr}(X^I X^J) + \text{Tr}(\bar{\Psi} \Gamma^{IJKL} \Psi) \tilde{G}_{IJKL}, \] (2.14)
\[ L_{\text{flux}} = -c \tilde{G}_{IJKL} \text{Tr}(X^I \langle X^J, X^K, X^L \rangle), \] (2.15)
where the mass \( m \) is determined by the four-form field as \( m^2 = \frac{c^2}{768} G^2 \), with \( G^2 = G^{IJKL} G_{IJKL} \), while \( c \) is an arbitrary parameter.

The four-form field \( \tilde{G}_{IJKL} \) contributes to the supersymmetry transformation of the fermion with a term linear in \( X \). The modified transformation assumes the form
\[ \delta \Psi = \gamma^\mu \Gamma^I D_\mu X^I \theta - \frac{1}{6} \Gamma^{IJK} \langle X^I, X^J, X^K \rangle \theta + \frac{c}{8} \Gamma^{IJKL} \Gamma^M \tilde{G}_{IJKL} X^M \theta. \] (2.16)
Comparing with Equation (2.6) we note that the modification of the transformation is in the last term. The supersymmetry transformations for the scalars and the gauge field remain unaltered from Equations (2.5) and (2.7). The BPS equation of the modified theory reads
\[ \delta \Psi = 0, \] (2.17)
which can be written equivalently as
\[ [D_\mu X^I \gamma^\mu \Gamma^I - \frac{1}{6} \langle X^I, X^J, X^K \rangle \Gamma^{IJK} + \frac{c}{8} \Gamma^{IJKL} \Gamma^M \tilde{G}_{IJKL} X^M] \theta = 0. \] (2.18)
Let us note that only the anti-self-dual combination of the four-form field appears in the last term on the left hand side, linear in \( X \). The \( R \)-symmetry in this formulation is realized explicitly in terms of the four-form field as
\[ R^I_J = \theta_2 \Gamma^{IJKLM} \theta_1 \tilde{G}_{KLMJ}, \] (2.19)
where \( \theta_1 \) and \( \theta_2 \) are two parameters of supersymmetry variation.
3 Classical vacuum moduli space

Let us now proceed to discuss the classical vacuum moduli space of scalars in the BLG theory without the four-form field $G$. The action we consider, then, is the one ensuing from the Lagrangian (2.1). The moduli space has been discussed earlier in the literature [3, 26, 27]. However, using gauge-invariant variables to describe the moduli space as a quotient allows us to write it as an algebraic variety over the field of real numbers; the coordinate ring of the quotient is simply the ring of invariant elements. Similar ideas of using gauge-invariant quantities in studying moduli spaces of D-branes have been used earlier [28, 29]. The essential difference of this approach from the earlier treatments is that in considering the part of the moduli space arising from the flavor degrees of freedom we do not proceed through an intermediate step of fixing the gauge symmetry partially.

The vacuum moduli space of scalars is the set of values of the scalars on which the bosonic potential vanishes, modulo gauge equivalence. Vanishing of the bosonic potential is ensured by the vanishing of the ternary bracket,

\[
\langle X^I, X^J, X^K \rangle = 0.
\]  

(3.1)

For the BLG theory satisfying Equations (2.8) and (2.9), with gauge group $G = SO(4)$, this condition reads

\[
\epsilon^{abcd} X_a^I X_b^J X_c^K = 0,
\]

(3.2)

where $I = 1, 2, \ldots, 8$ and $a = 1, 2, 3, 4$. We shall find the gauge-invariant moduli space of the 32 real scalars $X^I_a$ satisfying (3.2). Let us consider the ring

\[
\mathcal{R} = \mathbb{R}[X^I_a : I = 1, 2, \ldots, 8, a = 1, 2, 3, 4],
\]

where $\mathbb{R}$ denotes the field of real numbers and let us write $X = \text{Spec} \mathcal{R} = \mathbb{R}^8 \otimes \mathbb{R}^4$. We can construct 36 gauge-invariant local variables furnishing the coordinates of the moduli space, namely,

\[
y^{IJ} = \sum_{a=1}^{4} X_a^I X_a^J.
\]

(3.3)

Let us note that due to the condition (3.2) all higher-order invariants vanish. Denoting the ideal generated by (3.2) as

\[
\mathcal{J} = \langle \epsilon^{abcd} X_a^I X_b^J X_c^K \rangle \subset \mathcal{R}
\]

(3.4)

and the subspace of $X$ defined by this ideal as $Z = \text{Spec}(\mathcal{R}/\mathcal{J}) \hookrightarrow X$, we have the following maps:

\[
\mathbb{R}[y^{IJ}] \xrightarrow{f} \mathcal{R} \xrightarrow{q} \mathcal{R}/\mathcal{J}.
\]

(3.5)

Then the classical gauge-invariant moduli space written as a quotient is

\[
Z/G = \text{Spec}(\mathbb{R}[y^{IJ}] / \ker(q \circ f)).
\]

(3.6)
Let us observe that \( \ker(q \circ f) = f^{-1}(J) \). Since the one-forms in \( \mathbb{R}^4 \) that generate the ideal \( J \) are not invariant under the gauge group \( \mathcal{G} \), the preimage of \( f \) consists of the smallest functions of the \( y^{IJ} \) constructible from the one-forms, namely the metric contractions

\[
h_{ik} \epsilon^{abcd} \epsilon^{efgh} X_i^a X_k^b X_k^c X_i^d X_f^e X_g^f,
\]

(3.7)

which using (2.9) evaluates to an ideal generated by 1176 relations, viz.

\[
\mathcal{I}_{IJKLMN} = \left\langle y^{IJ}(y^{JM}y^{KN} - y^{JN}y^{KM}) - y^{JM}(y^{JL}y^{KN} - y^{JN}y^{KL}) + y^{JN}(y^{JM}y^{KM} - y^{JM}y^{KL}) \right\rangle,
\]

(3.8)

where \( I, J, K, L, M, N = 1, 2, \ldots, 8 \). A direct computation of \( \ker(q \circ f) \) using Macaulay 2 \([30]\) reproduces this ideal. The dimension of the space

\[
\mathcal{M} = \text{Spec} \left( \mathbb{R}[y^{IJ}] / \mathcal{I}_{L_1, \ldots, L_6} \right), \quad L_i = 1, \ldots, 8 \text{ for } i = 1, \ldots, 6.
\]

thus obtained is found to be fifteen, using Macaulay 2. The variety \( \mathcal{M} \) is singular at the origin, which is a fixed point of the \( SO(4) \) action.

In order to relate to earlier computations \([26]\) let us note that the purport of (3.8) is that all the \( 3 \times 3 \) minors of the \( 8 \times 8 \) matrix \( y^{IJ} \) vanish. Hence, the rank of the symmetric matrix \( y^{IJ} \) is at most two. This also makes the map \( f \) in (3.5) injective. We can, therefore, write the matrix \( y^{IJ} \) as

\[
y^{IJ} = a^I a^J + b^I b^J,
\]

(3.9)

where \( \{a^I\} \) and \( \{b^I\} \) are two linearly independent 8-vectors, \( I = 1, 2, \ldots, 8 \), chosen in a suitable basis. This correspond to the choice of gauge employed previously \([3, 26]\), leading to sixteen scalars. However, the \( SO(2) \) transformation

\[
a^I \longrightarrow a^I \cos \theta + b^I \sin \theta,
\]

\[
b^I \longrightarrow -a^I \sin \theta + b^I \cos \theta,
\]

(3.10)

where \( \theta \) is real, keeps the matrix \( y^{IJ} \) unaltered. Quotienting by this \( SO(2) \sim U(1) \) results in the fifteen-dimensional moduli space obtained above.

At this point let us note that so far our computations have proceeded in a manifestly gauge-invariant fashion. Let us also point out that had we considered gauging an \( SO(2) \) instead of \( SO(4) \), Equation (3.2) would not have imposed any condition on the scalars \( X_a^I \). This is effectively the same as what has been considered earlier \([3, 26]\). In this instance too we can form the gauge-invariant elements \( y^{IJ} \) similarly as above, and the moduli space turns out to be given by exactly the same equation as that for the \( SO(4) \) case. In other words, the moduli space of the \( SO(2) \)-theory is also \( \mathcal{M} \), with

\[
y^{IJ} = \sum_{a=1}^2 X_a^I X_a^J.
\]

This has been verified using Macaulay 2. This has indeed been the reason for obtaining fifteen scalar moduli from the flavors upon partially fixing the gauge group \( SO(2) \).

\[\text{We thank an anonymous referee for this discussion.}\]
To reconcile with the expectations from supersymmetry, however, the moduli space is to have one more dimension. The extra modulus arises from the zero mode of the gauge field, as we now discuss. Since we quotient by the group $G = SO(4)$ in (3.6), we may consider a fixed non-vanishing choice of the $X$'s. In concordance with the choice (3.9) and previous results [3, 26], we may fix arbitrary real numbers $x^I_1$ and $x^I_2$, such that

$$X^I = \begin{pmatrix} x^I_1 \\ x^I_2 \\ 0 \\ 0 \end{pmatrix},$$

(3.11)

which correspond to a point in $\mathcal{M}$. This choice of gauge and the equations of motion, in turn, fix all the components of the gauge field, except $\tilde{A}_\mu^1$, where $\tilde{A}_\mu^a = \epsilon^{cda} A_\mu^c d$. The equation of motion for the latter now takes the form

$$\tilde{F}_{\mu\nu}^1 = \partial_\mu \tilde{A}_\nu^1 - \partial_\nu \tilde{A}_\mu^1 = 0,$$

(3.12)

as is expected of a $U(1)$ theory. The equation of motion is solved by taking

$$\tilde{A}_\mu^1 = \partial_\mu \varphi,$$

(3.13)

where $\varphi$ is a real scalar field. This scalar furnishes the sixteenth dimension of the moduli space

$$\mathcal{M}_{\text{tot}} = \mathbb{R} \times \mathcal{M}.$$  

(3.14)

We thus obtain a global description of the total moduli space of the BLG theory.

Let us now consider the regime in which one obtains the super-Yang-Mills theory of D2-branes from the BLG theory. Since the considerations for the gauge fields are as above, we shall only discuss the scalar moduli arising from the flavors. By assigning a constant vacuum expectation value to one of the transverse scalars, say $X^8$, and arranging the gauge field appropriately, one obtains the Yang Mills theory with fundamental matter [7]. One can consider the vacuum moduli space of the theory of D2-branes in this picture as well. The leading order term in the potential, proportional to the inverse square of the Yang-Mills coupling, vanishes provided

$$\epsilon^{abc} X^I_b X^J_c = 0,$$

(3.15)

where now $I, J = 1, 2, \ldots, 7$. This condition is sufficient, though not necessary, for (3.2), and is therefore more restrictive. Defining gauge-invariant coordinates as above, we obtain $\mathcal{M}$ as the seven-dimensional variety

$$\mathcal{M} = \text{Spec} (\mathbb{R}[y^{IJ}] / \mathcal{I}_{D2}),$$

(3.16)

where the ideal $\mathcal{I}_{D2}$ is given by

$$\mathcal{I}_{D2} = \langle (y^{IJ})^2 - y^I y^J \rangle, \text{ for } I, J = 1, 2, \ldots, 7.$$

(3.17)

Again, an eighth dimension is furnished by a zero mode of the gauge field, in accordance with supersymmetry.
We shall now discuss a half-BPS configuration of the BLG theory in the presence of the four-form field with $SO(1, 2)$ world-volume symmetry and $SO(4) \times SO(4)$ R-symmetry. We shall find that rewriting the BPS equations using the gauge-invariant variables introduced in the previous section facilitates the analysis of such configurations. The $SO(4) \times SO(4)$ R-symmetry is obtained by means of a projector $\Omega$ acting on the fermions [20, 23]. The projector is obtained by seeking a realization of the irreducible spinor representation of Spin(8) in a Clifford algebra formed by monomials of $\Gamma$-matrices. For the case at hand the projector is given by
\begin{equation}
\Omega = \frac{1}{4}(1 + \Gamma_9 - \Gamma_{1234} - \Gamma_{5678}),
\end{equation}
where $\Gamma_{1234}$ and $\Gamma_{5678}$ denote the $\Gamma$-matrices completely antisymmetrized in the indicated indices and $\Gamma_9 = \Gamma_1 \cdots \Gamma_8$.

The BPS equations can now be obtained by operating $\Omega$ on (2.18):
\begin{equation}
\left(\frac{1}{6}\langle X^I, X^J, X^K \rangle \Gamma^{IJK} - \frac{3}{8}(\tilde{G}_{IJKL}\Gamma^{IJKL}M^M)\right) \Omega = 0.
\end{equation}
Expressing the projector explicitly in terms of the antisymmetrized products of the $32 \times 32$ gamma matrices of $SO(8)$, the elements of the BPS matrix yield the following set of equations.
\begin{align}
\langle X^1, X^2, X^3 \rangle &= -\eta X^4, \\
\langle X^1, X^3, X^4 \rangle &= -\eta X^2, \\
\langle X^1, X^2, X^4 \rangle &= \eta X^3, \\
\langle X^2, X^3, X^4 \rangle &= \eta X^1, \\
\langle X^5, X^6, X^7 \rangle &= -\eta X^8, \\
\langle X^5, X^7, X^8 \rangle &= -\eta X^6, \\
\langle X^5, X^6, X^8 \rangle &= \eta X^7, \\
\langle X^6, X^7, X^8 \rangle &= \eta X^5,
\end{align}
where $\eta = 3c(\tilde{G}_{1234} - \tilde{G}_{5678})$, in terms of the dual four-form field,
\begin{align}
\langle X^i, X^5, X^6 \rangle + \langle X^i, X^7, X^8 \rangle &= 0, \\
\langle X^i, X^5, X^7 \rangle - \langle X^i, X^6, X^8 \rangle &= 0, \\
\langle X^i, X^5, X^8 \rangle + \langle X^i, X^6, X^7 \rangle &= 0,
\end{align}
with $i = 1, 2, 3, 4$, and
\begin{align}
\langle X^r, X^1, X^2 \rangle + \langle X^r, X^3, X^4 \rangle &= 0, \\
\langle X^r, X^1, X^3 \rangle - \langle X^r, X^2, X^4 \rangle &= 0, \\
\langle X^r, X^1, X^4 \rangle + \langle X^r, X^3, X^2 \rangle &= 0,
\end{align}
with $r = 5, 6, 7, 8$. The equations have manifest $SO(4) \times SO(4)$ R-symmetry.

In studying the moduli space of this deformed theory, let us first note that due to the Equations (4.3) and (4.4), we have to consider the order four gauge-invariants, unlike the previous case of vanishing $\eta$. Let us denote two of the 70 order four invariants as

$$
\xi_1 = \epsilon^{abcd} X^a_1 X^2_b X^3_c X^4_d,
\xi_2 = \epsilon^{abcd} X^5_a X^6_b X^7_c X^8_d.
$$

(4.7)

Upon multiplying both sides of (4.3) and (4.4) with a suitable $X'_a$ and summing over the gauge indices, we derive:

$$
y^{11} = y^{22} = y^{33} = y^{44} = -\xi_1 / \eta
y^{55} = y^{66} = y^{77} = y^{88} = -\xi_2 / \eta
$$

(4.8)

In the same fashion we also derive from these equations that

$$
y^{ij} = 0, \text{ for } i \neq j \text{ and } i, j = 1, 2, 3, 4, \text{ and}
y^{rs} = 0, \text{ for } r \neq s \text{ and } r, s = 5, 6, 7, 8.
$$

(4.9)

Next, from the first equation in (4.3) we have

$$
- \eta y^{45} = \epsilon^{abcd} X^a_1 X^2_b X^3_c X^5_d.
$$

(4.10)

Multiplying $X^3$ on both sides of the first equation in (4.6) with $r = 5$ then yields, upon summing over the gauge indices,

$$
\epsilon^{abcd} X^1_a X^2_b X^3_c X^5_d = - \epsilon^{abcd} X^5_a X^3_b X^4_c X^3_d = 0.
$$

(4.11)

Hence we conclude that $y^{45} = 0$. Proceeding similarly with the other equations, we obtain

$$
y^{ri} = 0 \text{ for all } i = 1, 2, 3, 4 \text{ and } r = 5, 6, 7, 8.
$$

(4.12)

This, in turn, makes all the order four invariants apart from $\xi_1$ and $\xi_2$ vanish. Furthermore, from the first equation in (4.3) we get, upon squaring,

$$
\epsilon^{abcd} \epsilon^{pqr} X^1_a X^2_b X^3_c X^5_d X^1_p X^2_q X^3_r = \eta^2 X^4_d X^4_d,
$$

(4.13)

where repeated indices are summed. This leads to

$$
y^{11} y^{22} y^{33} = \eta^2 y^{44},
$$

(4.14)

which, using (4.8), yields

$$
\xi_1 (\xi_1^2 - \eta^4) = 0.
$$

(4.15)

Similarly, from (4.4) and (4.8) we derive

$$
\xi_2 (\xi_2^2 - \eta^4) = 0.
$$

(4.16)
Finally, by squaring $\epsilon^{abcd} X_a^1 X_b^2 x_c^3 X_d^5$ and using (4.11), we obtain

$$y^{11} y^{22} y^{33} y^{55} = 0,$$

that is,

$$\xi_1^3 \xi_2 = 0.$$  \hspace{1cm} (4.18)

This equation, along with (4.15) and (4.16), implies that one of the $\xi$’s have to vanish. If we choose $\xi_2 = 0$, then using (4.8) and recalling that $y^{II}$, for each $I$, is a sum of squares of real numbers, we conclude that

$$X^5 = X^6 = X^7 = X^8 = 0.$$  \hspace{1cm} (4.19)

From (4.15) we then have $\xi_1 = \pm \eta^2$ and by (4.8) we are thus left with

$$y^{11} = y^{22} = y^{33} = y^{44} = \pm \eta,$$  \hspace{1cm} (4.20)

with the sign chosen such that these invariants are positive-definite for a non-vanishing $\eta$. All other gauge invariants vanish. We conclude that the gauge-invariant moduli space is zero-dimensional. An explicit BPS configuration is then furnished by, for example, choosing the four scalars $X^i$, $i = 1, 2, 3, 4$ to be proportional to the four basis vectors in the Euclidean space $\mathbf{R}^4$ of the gauge indices.

5 Conclusion

We have examined the classical vacuum moduli space of scalars of the BLG theory. By considering gauge-invariant combinations of scalars we obtain a global description of the sixteen-dimensional moduli space as the product of a fifteen-dimensional real singular algebraic variety $\mathcal{M}$ and the real line, the latter arising from a zero-mode of the three-dimensional gauge field. We then use the gauge-invariant variables to study a half-BPS configuration of the BLG theory deformed by a four-form background field. The configuration has $SO(1, 2)$-world-volume symmetry and $SO(4) \times SO(4)$ R-symmetry. In this case, however, we have to consider invariants higher than order two in the scalars $X$. We obtain an explicit solution derived from the fact that $\mathcal{M}$ is zero-dimensional, parametrized by the four-form field. We find that the BPS equations allow only four of the eight scalars, corresponding to one of the $SO(4)$ R-symmetry factors, to be non-zero.

This method of forming gauge invariants is extremely useful in studying moduli spaces of gauge theories. They serve also as a useful tool for studying BPS configurations by constructing explicit solutions. It seems that configurations with various other R-symmetries can also be studied in this fashion. We hope to report on these issues in future.
Acknowledgments

TK and KR are partially supported by Grant 2008/R2 of the Royal Society. They would like to thank the PIs Elizabeth Gasparim and Pushan Majumdar for being generous with the grant under which they enjoy a fruitful exchange between the IACS, Calcutta and the University of Edinburgh, UK. SPC is supported by the Department of Science and Technology, Government of India. SC wishes to thank the Department of Physics, IACS, for warm hospitality during a major part of this work. We thank Anirban Basu, Bobby Ezhuthachan, José Figueroa-O’Farrill, Elizabeth Gasparim, Sachin Jain, Sudipta Mukherji for several helpful discussions and the anonymous referees for important suggestions.

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