Combinatorial proof of the inversion formula on the Kazhdan–Lusztig $R$-polynomials

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Abstract In this paper, we present a combinatorial proof of the inversion formula on the Kazhdan–Lusztig $R$-polynomials. This problem was raised by Brenti. As a consequence, we obtain a combinatorial interpretation of the equidistribution property due to Verma stating that any nontrivial interval of a Coxeter group in the Bruhat order has as many elements of even length as elements of odd length. The same argument leads to a combinatorial proof of an extension of Verma’s equidistribution to the parabolic quotients of a Coxeter group obtained by Deodhar. As another application, we derive a refinement of the inversion formula for the symmetric group by restricting the summation to permutations ending with a given element.

Keywords Kazhdan–Lusztig $R$-polynomial · Inversion formula · Bruhat order

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1 Introduction

Let \((W, S)\) be a Coxeter system. For \(u, v \in W\), let \(R_{u,v}(q)\) be the Kazhdan–Lusztig \(R\)-polynomial indexed by \(u\) and \(v\). The following inversion formula was obtained by Kazhdan and Lusztig [8]:

\[
\sum_{u \leq w \leq v} (-1)^{\ell(w) - \ell(u)} R_{u,w}(q) R_{w,v}(q) = \delta_{u,v}, \tag{1.1}
\]

where \(\leq\) is the Bruhat order and \(\ell\) is the length function, see also Humphreys [6]. The aim of this paper is to present a combinatorial interpretation of this formula. This problem was raised by Brenti [3].

To give a combinatorial proof of (1.1), we start with Dyer’s combinatorial description of the \(R\)-polynomials in terms of increasing Bruhat paths [5]. Then we reformulate the inversion formula in terms of \(V\)-paths. For \(u \leq w \leq v\), a \(V\)-path from \(u\) to \(v\) with bottom \(w\) we mean a pair \((\Delta_1, \Delta_2)\) of Bruhat paths such that \(\Delta_1\) is a decreasing path from \(u\) to \(w\) and \(\Delta_2\) is an increasing path from \(w\) to \(v\). We construct an involution on \(V\)-paths. This leads to a combinatorial proof of (1.1).

We give two applications of the involution. First, we restrict the involution to \(V\)-paths from \(u\) to \(v\) with maximal length. This induces an involution on the interval \([u, v]\) with \(u < v\), which leads to a combinatorial proof of the equidistribution property that any nontrivial interval \([u, v]\) has as many elements of even length as elements of odd length. This property was proved inductively by Verma [12], which was used to deduce the Möbius function of the Bruhat order. Other proofs of the Möbius function formula for the Bruhat order can be found in [2, 4, 9, 11]. Recently, Jones [7] found a combinatorial proof of the equidistribution property by constructing an involution on the intervals of a Coxeter group \(W\). When \(W\) is finite, Jones [7] showed that this involution agrees with the construction of Rietsch and Williams [10] in their study of discrete Morse theory and totally nonnegative flag varieties.

The idea that we have used to prove Verma’s equidistribution can also be applied to Deodhar’s [4] extension to parabolic quotients. For \(J \subseteq S\), let \(W_J\) be the parabolic subgroup of \(W\) generated by \(J\), and let \(W^J\) be the quotient of \(W\) consisting of minimal representatives of the left cosets of \(W_J\) in \(W\), that is,

\[
W_J = \{ w \in W \mid \ell(ws) > \ell(w) \quad \text{for any} \quad s \in J \}. 
\]

The quotient \(W^J\) forms a subposet of \(W\) in the Bruhat order. For \(u \leq v \in W^J\), let

\[
[u, v]^J = [u, v] \cap W^J
\]

and let

\[
K_J(u, v) = \{ w \in [u, v]^J \mid [w, v]^J = [w, v] \}.
\]

When \(u < v\), Deodhar [4] showed that \(K_J(u, v)\) contains as many elements of even length as elements of odd length, from which the Möbius function of the Bruhat order on \(W^J\) can be easily deduced. When \(J = \emptyset\), Deodhar’s assertion reduces to Verma’s equidistribution. The Möbius function on \(W^J\) was rederived by Björner and Wachs [2] with the aid of topological techniques, and by Stembridge [11] by an algebraic approach. We construct an involution on \(K_J(u, v)\) that leads to a simple combinatorial interpretation of Deodhar’s equidistribution.

As a second application, we find a refinement of the inversion formula when \(W\) is the symmetric group \(S_n\). For a permutation \(w \in S_n\), we write \(w = w(1)w(2) \cdots w(n)\), where \(w(i)\) denotes the element in the \(i\)-th position. Let \(u\) and \(v\) be two permutations in \(S_n\) such
that \( u < v \) in the Bruhat order. For \( 1 \leq k \leq n \), let \([u, v]_k\) denote the set of permutations in the interval \([u, v]\) that end with \( k \), that is,
\[
[u, v]_k = \{ w \in [u, v] \mid w(n) = k \}.
\]
By using a variation of the involution, we show that the summation
\[
\sum_{w \in [u, v]_k} (-1)^{\ell(w) - \ell(u)} R_{u, w}(q) R_{w, v}(q)
\]
equals zero or a power of \( q \) up to a sign.

## 2 An involution on \( V \)-paths

Our combinatorial proof of the inversion formula is based on an equivalent formulation of (1.1) in terms of the \( \tilde{R} \)-polynomials. Let \((W, S)\) be a Coxeter system. For \( u, v \in W \) with \( u \leq v \), the \( \tilde{R} \)-polynomials \( \tilde{R}_{u, v}(q) \) were introduced by Dyer [5], which are connected to the \( R \)-polynomials via the following relation
\[
R_{u, v}(q) = q^{\frac{-\ell(v) + \ell(u)}{2}} \tilde{R}_{u, v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}),
\]
see also Björner and Brenti [1]. Thus the inversion formula (1.1) can be restated as
\[
\sum_{u \leq w \leq v} (-1)^{\ell(w) - \ell(u)} \tilde{R}_{u, w}(q) \tilde{R}_{w, v}(q) = \delta_{u, v}. \tag{2.1}
\]

To give a bijective proof of (2.1), we need a combinatorial interpretation of the \( \tilde{R} \)-polynomials due to Dyer [5] in terms of increasing Bruhat paths of a Coxeter group. For a Coxeter system \((W, S)\), let
\[
T = \{ ws w^{-1} \mid s \in S, w \in W \}
\]
be the set of reflections. The Bruhat graph \( BG(W) \) of \( W \) is a directed graph whose nodes are the elements of \( W \) such that there is an arc from \( u \) to \( v \) if \( v = ut \) for some \( t \in T \) and \( \ell(u) < \ell(v) \). We use \( u \rightarrow v \) to denote the arc from \( u \) to \( v \), where \( t \) is the reflection such that \( v = ut \). An increasing path in the Bruhat graph is defined based on the reflection ordering on the positive roots of \( W \). Let \( \Phi \) be the root system of \( W \), and \( \Phi^+ \) be the positive root system. A total ordering \( < \) on \( \Phi^+ \) is called a reflection ordering if for any \( \alpha < \beta \in \Phi^+ \) and two nonnegative real numbers \( \lambda, \mu \) such that \( \lambda \alpha + \mu \beta \in \Phi^+ \), then we have \( \alpha < \lambda \alpha + \mu \beta < \beta \). Since positive roots in \( \Phi^+ \) are in one-to-one correspondence with reflections, a reflection ordering induces a total ordering on the reflection set \( T \).

Let \( \Delta = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} u_r \) be a path from \( u \) to \( v \), where \( u_0 = u \) and \( u_r = v \). We say that \( \Delta \) is increasing if \( t_1 < t_2 < \cdots < t_r \), and \( \Delta \) is decreasing if \( t_1 > t_2 > \cdots > t_r \). Let \( \ell(\Delta) \) denote the length of \( \Delta \), that is, the number of arcs in \( \Delta \). Dyer [5] showed that for any fixed reflection ordering \( < \) on \( T \), we have
\[
\tilde{R}_{u, v}(q) = \sum_{\Delta} q^{\ell(\Delta)}, \tag{2.2}
\]
where the sum ranges over increasing Bruhat paths from \( u \) to \( v \) with respect to \( < \), see also Björner and Brenti [1]. By definition, the reverse of a reflection ordering is also a reflection ordering. So (2.2) can be restated as
where the sum ranges over decreasing Bruhat paths from \( u \) to \( v \) with respect to \( < \).

By a \( V \)-path from \( u \) to \( v \) with bottom \( w \), we mean a pair \((\Delta_1, \Delta_2)\) of Bruhat paths such that \( \Delta_1 \) is a decreasing path from \( u \) to \( w \) and \( \Delta_2 \) is an increasing path from \( w \) to \( v \). The sign of a \( V \)-path \((\Delta_1, \Delta_2)\) is defined as

\[
\text{sgn}(\Delta_1, \Delta_2) = (-1)^{\ell(\Delta_1)}.
\]

The length of a Bruhat path from \( u \) to \( w \) has the same parity as \( \ell(w) - \ell(u) \), see, e.g., Björner and Brenti [1]. It follows that

\[
\text{sgn}(\Delta_1, \Delta_2) = (-1)^{\ell(w) - \ell(u)},
\]

and so (2.1) can be rewritten as

\[
\sum_{(\Delta_1, \Delta_2)} \text{sgn}(\Delta_1, \Delta_2) q^{\ell(\Delta_1)+\ell(\Delta_2)} = \delta_{u,v},
\]

where the sum ranges over \( V \)-paths from \( u \) to \( v \).

We now define an involution \( \Phi \) on \( V \)-paths, which preserves the length, but reverses the sign of a \( V \)-path. This leads to a combinatorial proof of (2.3).

**An Involution \( \Phi \) on \( V \)-Paths:** For \( u < v \), let \((\Delta_1, \Delta_2)\) be a \( V \)-path from \( u \) to \( v \) with bottom \( w \). Write

\[
\Delta_1 = u_0 \overset{t_1}{\rightarrow} u_1 \overset{t_2}{\rightarrow} \cdots \overset{t_i}{\rightarrow} u_i \quad \text{and} \quad \Delta_2 = v_0 \overset{t'_i}{\rightarrow} v_1 \overset{t'_j}{\rightarrow} \cdots \overset{t'_j}{\rightarrow} v_j,
\]

where \( u_0 = u, u_i = v_0 = w \) and \( v_j = v \). The \( V \)-path \( \Phi(\Delta_1, \Delta_2) = (\Delta'_1, \Delta'_2) \) is constructed according to the following two cases.

**Case 1:** \( u = w \) or \( t_i > t'_i \). Set

\[
\Delta'_1 = u_0 \overset{t_1}{\rightarrow} u_1 \overset{t_2}{\rightarrow} \cdots \overset{t_i}{\rightarrow} u_i \rightarrow v_1 \quad \text{and} \quad \Delta'_2 = v_1 \overset{t'_i}{\rightarrow} \cdots \overset{t'_j}{\rightarrow} v_j.
\]

**Case 2:** \( v = w \) or \( t_i < t'_i \). Set

\[
\Delta'_1 = u_0 \overset{t_i}{\rightarrow} u_1 \overset{t_2}{\rightarrow} \cdots \overset{t_{i-1}}{\rightarrow} u_{i-1} \quad \text{and} \quad \Delta'_2 = u_{i-1} \overset{t_i}{\rightarrow} v_0 \overset{t'_i}{\rightarrow} v_1 \overset{t'_j}{\rightarrow} \cdots \overset{t'_j}{\rightarrow} v_j.
\]

It turns out that the involution \( \Phi \) yields a simple combinatorial interpretation of the following parity property of Verma [12].

**Theorem 2.1** (Verma [12]) Let \((W, S)\) be a Coxeter system and \( u < v \in W \). Then the interval \([u, v]\) has the same number of elements of odd length as elements of even length.

Indeed, for \( u < v \in W \), there exists a unique maximal increasing (or, decreasing) Bruhat path from \( u \) to \( v \) [5]. Thus, for any \( w \in [u, v] \) there is a unique maximal \( V \)-path from \( u \) to \( v \) with bottom \( w \). So the maximal \( V \)-paths from \( u \) to \( v \) are in one-to-one correspondence with elements in the interval \([u, v]\). Restricting the involution \( \Phi \) to the maximal \( V \)-paths from \( u \) to \( v \) induces an involution on the interval \([u, v]\), which reverses the parity of the length of each element in \([u, v]\). This proves Theorem 2.1.
The above argument also serves as a combinatorial interpretation of the following equi-distribution due to Deodhar [4]. Let us recall the common notation as mentioned in Introduction. For \( J \subseteq S \), let

\[
W^J = \{ w \in W \mid \ell(ws) > \ell(w) \text{ for any } s \in J \}.
\]

For \( u \leq v \in W^J \), let

\[
[u, v]^J = [u, v] \cap W^J
\]

and let

\[
K_f(u, v) = \{ w \in [u, v]^J \mid [w, v]^J = [w, v] \}.
\]

**Theorem 2.2** (Deodhar [4]) Let \((W, S)\) be a Coxeter system, and \( J \subseteq S \). Then, for \( u < v \in W \), the set \( K_f(u, v) \) has the same number of elements of odd length as elements of even length.

To construct an involution on \( K_f(u, v) \), we use a labeling on the edges of the poset \([u, v]^J\) introduced by Björner and Wachs [2], see also Björner and Brenti [1]. Let \( v = s_1 s_2 \cdots s_q \) be a given reduced expression of \( v \). We read a maximal chain in \([u, v]^J\) from top to bottom. Let \( v = w_0 \to w_1 \to \cdots \to w_r = u \) be a maximal chain in \([u, v]^J\), where \( r = \ell(v) - \ell(u) \). Then there is a unique sequence \((i_1, i_2, \ldots, i_r)\) of distinct integers such that for \( 1 \leq k \leq r \), \( w_k \) has a reduced expression obtained from \( s_1 s_2 \cdots s_q \) by deleting simple reflections indexed by \( i_1, i_2, \ldots, i_k \). Label the edge from \( w_k \) to \( w_{k+1} \) by \( i_k \). We denote the maximal chain with such a labeling by \( v = w_0 \overset{i_1}{\to} w_1 \overset{i_2}{\to} \cdots \overset{i_r}{\to} w_r = u \), and say that the chain \( v = w_0 \overset{i_1}{\to} w_1 \overset{i_2}{\to} \cdots \overset{i_r}{\to} w_r = u \) is increasing if \( i_1 < i_2 < \cdots < i_r \), and it is decreasing if \( i_1 > i_2 > \cdots > i_r \). The following theorem is due to Björner and Wachs [2].

**Theorem 2.3** (Björner and Wachs [2]) Let \( u < v \in W^J \), and let \( v = s_1 s_2 \cdots s_q \) be a given reduced expression of \( v \). Then there is a unique increasing maximal chain from \( v \) to \( u \) in \([u, v]^J\).

We remark that when \( J = \emptyset \), the proof of Theorem 2.3 can be employed to show that for any given reduced expression of \( v \), there is a unique decreasing maximal chain from \( v \) to \( u \) in \([u, v]^J\).

We are now ready to present an involution \( \Psi \) on \( K_f(u, v) \), which reverses the parity of the length. This leads to a combinatorial proof of Theorem 2.2.

**An Involution** \( \Psi \) on \( K_f(u, v) \): Let \( w \in K_f(u, v) \), and let \( v = s_1 s_2 \cdots s_q \) be a fixed reduced expression of \( v \). Since \([w, v]^J = [w, v]\), by the above remark, there exists a unique decreasing maximal chain \( v = v_0 \overset{i_1}{\to} v_1 \overset{i_2}{\to} \cdots \overset{i_m}{\to} v_m = w \) from \( v \) to \( w \) in \([u, v]^J\). Let \( w = s_{k_1} s_{k_2} \cdots s_{k_p} \) be the reduced expression of \( w \) obtained from \( s_1 s_2 \cdots s_q \) by deleting the generators indexed by \( i_1, i_2, \ldots, i_m \), that is, \( 1 \leq k_1 < k_2 < \cdots < k_p \leq q \) and \( \{k_1, k_2, \ldots, k_p\} = \{1, 2, \ldots, q\} \setminus \{i_1, i_2, \ldots, i_m\} \). Assume that \( w = w_0 \overset{k_1}{\to} w_1 \overset{k_2}{\to} \cdots \overset{k_j}{\to} w_j = u \) is the unique increasing maximal chain in \([u, w]^J\) with respect to the reduced expression \( w = s_{k_1} s_{k_2} \cdots s_{k_p} \). Note that \( 1 \leq j_1 < \cdots < j_l \leq p \). Then \( \Psi(w) \) is defined according to the following two cases:

Case 1: \( u = w \) or \( i_m < k_{j_1} \). Set \( \Psi(w) = v_{m-1} \);

Case 2: \( v = w \) or \( i_m > k_{j_1} \). Set \( \Psi(w) = w_1 \).
The following theorem shows that $\Psi$ is an involution on $K_J(u, v)$. The proof relies on the following properties of the Bruhat order, see, for example, Björner and Brenti [1].

The Subword Property: Let $u, v \in W$. Then $u \leq v$ in the Bruhat order if and only if every reduced expression of $v$ has a subword that is a reduced expression of $u$.

The Lifting Property: Suppose that $u < v \in W$, and $s \in S$ is a simple reflection. If $\ell(sv) < \ell(v)$ and $\ell(su) > \ell(u)$, then $u \leq sv$ and $su \leq v$. Similarly, if $\ell(us) < \ell(v)$ and $\ell(us) > \ell(u)$, then $u \leq us$ and $us \leq v$.

**Theorem 2.4** The map $\Psi$ is an involution on $K_J(u, v)$.

**Proof** By the construction of $\Psi$, it suffices to show that for $w \in K_J(u, v)$, $\Psi(w)$ also belongs to $K_J(u, v)$. This is trivial when $u = w$ or $i_m < k_{j_1}$. Now we consider the case when $v = w$ or $i_m > k_{j_1}$. Let $w' = \Psi(w)$. Assume that $w = s_1 \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_q$ and $w' = s_1 \cdots \hat{s}_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q$, where for a simple reflection $s \in S$, $\hat{s}$ means that $s$ is missing. We aim to prove that $w' \in K_J(u, v)$.

Suppose to the contrary that $w' \notin K_J(u, v)$. Then there exists an element $w'' \in [w', v]$ such that $w'' \notin [w', v]^J$. By definition, there exists $s \in J$ such that $\ell(w's) < \ell(w'')$. Since $\ell(w's) > \ell(w')$, the lifting property implies that $w's \leq w''$. Thus we have $w's \leq v$. Since $\ell(ws) > \ell(v)$, we see that $w's \neq v$. It follows that $w's < v$, that is,

$$s_1 \cdots \hat{s}_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s < s_1 s_2 \cdots s_q.$$

It is easily checked that $\hat{s}_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s < s_{k_{j_1}} \cdots s_q$. By the lifting property, we deduce that $s_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s \leq s_{k_{j_1}} \cdots s_q$. Thus we have

$$ws = s_1 \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s \leq s_1 \cdots s_q = v,$$

which implies that $ws \in [w, v]$. On the other hand, it is obvious that $ws \notin [w, v]^J$. So we conclude that $w \notin K_J(u, v)$, contradicting the assumption that $w \in K_J(u, v)$. This completes the proof. \hfill $\Box$

From the proof of Theorem 2.4, we see that for any $w \in [u, v]^J$, $w \in K_J(u, v)$ if and only if there does not exist any $s \in J$ such that $ws \in [u, v]$. Notice that this characterization has been observed by Deodhar [4, Lemma 3].

## 3 A refinement of the inversion formula for $S_n$

In this section, we use a variation of the involution $\Phi$ to give a refinement of the inversion formula for the symmetric group $S_n$. We introduce the notion of an $S$-interval. Let $u, v$ be two permutations in $S_n$ with $u < v$

$$D(u, v) = \{1 \leq i \leq n \mid u(i) \neq v(i)\}.$$ 

Suppose that $D(u, v) = \{a_1, a_2, \ldots, a_j\}_-$, that is, $D(u, v) = \{a_1, a_2, \ldots, a_j\}$ and $a_1 < a_2 < \cdots < a_j$. Let $b_1 < b_2 < \cdots < b_j$ be the values of $u(a_1), u(a_2), \ldots, u(a_j)$ listed in increasing order. We say that $[u, v]$ is an $S$-interval if it satisfies the following conditions:

1. $a_j = n$ and $u(a_j) = b_j$;
2. The values in $\{b_1, b_2, \ldots, b_j\}$ that are greater than $u(a_1)$ appear in increasing order in $u$, whereas the values in $\{b_1, b_2, \ldots, b_j\}$ that are less than $u(a_1)$ appear in decreasing order in $u$;
In the cycle notation, \( v = (b_1, b_2, \ldots, b_j) \) \( u \), that is, \( v \) is obtained from \( u \) by rotating the elements \( b_1, b_2, \ldots, b_j \) in \( u \).

Recall that for \( u < v \in S_n \), \([u, v]_k \) denotes the set of permutations in \([u, v] \) that end with \( k \). The following theorem gives a refinement of the inversion formula for \( S_n \).

**Theorem 3.1** Assume that \( u < v \in S_n \). Let \( m \) be the smallest index such that \( u(m) \neq v(m) \). If \([u, v] \) is an \( S \)-interval, and \( k = u(m) \) or \( k = v(m) \), then we have

\[
\sum_{w \in [u, v]_k} (-1)^{\ell(w) - \ell(u)} \tilde{R}_{u,w}(q) \tilde{R}_{w,v}(q) = (-1)^r q^{s-1},
\]

where \( s = |D(u,v)| \) and \( r = |\{j \in D(u,v) | u(j) > k \}| \).

Otherwise, we have

\[
\sum_{w \in [u, v]_k} (-1)^{\ell(w) - \ell(u)} \tilde{R}_{u,w}(q) \tilde{R}_{w,v}(q) = 0.
\]

For \( 1 \leq k \leq n \), let \( P_k(u, v) \) denote the set of \( V \)-paths from \( u \) to \( v \) with bottoms contained in \([u, v]_k \). To prove Theorem 3.1, we shall construct an involution \( \Omega \) on \( P_k(u, v) \). The reflection set \( T \) of \( S_n \) consists of transpositions of \( S_n \), that is,

\[
T = \{(i, j) \mid 1 \leq i < j \leq n \}.
\]

For two permutations \( u, v \) in \( S_n \), it is known that there is an arc from \( u \) to \( v \) in the Bruhat graph of \( S_n \) if \( v = u(i, j) \) and \( u(i) < u(j) \), see Björner and Brenti [1].

From now on, we choose the reflection ordering \( \prec \) on \( T \) to be the lexicographic ordering:

\[
(1, 2) < (1, 3) < \cdots < (1, n) < (2, 3) < \cdots < (n-1, n).
\]

For a Bruhat path \( \Delta = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} u_r \), let

\[
L(\Delta) = (t_1, t_2, \ldots, t_r).
\]

**An Involution \( \Omega \) on \( P_k(u, v) \):** Let \( (\Delta_1, \Delta_2) \) be a \( V \)-path in \( P_k(u, v) \) with bottom \( w \). Write \( \Delta_1 = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_l} u_l \) and \( \Delta_2 = v_0 \xrightarrow{t'_1} v_1 \xrightarrow{t'_2} \cdots \xrightarrow{t'_j} v_j \), where \( u_0 = u, u_i = v_0 = w \) and \( v_j = v \). Let \( t = \min\{t_i, t'_j\} \). Then the \( V \)-path \( \Omega(\Delta_1, \Delta_2) = (\Delta'_1, \Delta'_2) \) is defined as follows. We consider three cases.

Case 1: \( t \) is an internal transposition, that is, \( t = (a, b) \) and \( 1 \leq a < b < n \). In this case, set \( (\Delta'_1, \Delta'_2) = \Phi(\Delta_1, \Delta_2) \).

Case 2: \( t \) is a boundary transposition, that is, \( t = (a, n) \) for some \( a < n \), and there is an internal transposition among the transpositions \( t_1, \ldots, t_i, t'_1, \ldots, t'_j \). Let \( \tilde{t} \) be the smallest internal transposition among \( t_1, \ldots, t_i, t'_1, \ldots, t'_j \). By the choice of the reflection ordering in (3.1), it is easy to check that \( \tilde{t} \) belongs to either \( \{t_1, \ldots, t_i\} \) or \( \{t'_1, \ldots, t'_j\} \), but not both. So we have the following two subcases.

Subcase 1: \( \tilde{t} \) belongs to \( \{t_1, \ldots, t_i\} \). Assume that \( t_{i_0} = \tilde{t} \), where \( 1 \leq i_0 \leq i \). Let \( \Delta'_1 \) be the path such that \( L(\Delta'_1) \) is the sequence obtained from \( L(\Delta_1) \) by deleting \( t_{i_0} \), and let \( \Delta'_2 \) be the

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path such that $L(\Delta'_2)$ is the sequence obtained from $L(\Delta_2)$ by inserting $t_{j_0}$ such that $L(\Delta'_2)$ remains increasing.

Subcase 2: $\tilde{r}$ belongs to $\{t'_1, \ldots, t'_j\}$. Assume that $t'_{j_0} = \tilde{r}$, where $1 \leq j_0 \leq j$. Let $\Delta'_2$ be the path such that $L(\Delta'_2)$ is the sequence obtained from $L(\Delta_2)$ by deleting $t'_{j_0}$, and let $\Delta'_1$ be the path such that $L(\Delta'_1)$ is the sequence obtained from $L(\Delta_1)$ by inserting $t'_{j_0}$ such that $L(\Delta'_1)$ remains decreasing.

Case 3: The transpositions $t_1, \ldots, t_i, t'_1, \ldots, t'_j$ are all boundary transpositions. In this case, set $(\Delta'_1, \Delta'_2) = (\Delta_1, \Delta_2)$.

It is easy to verify that $\Omega$ is a length preserving involution on $P_k(u, v)$, and it is clear that $\Omega$ reverses the sign of $(\Delta_1, \Delta_2)$ unless $(\Delta_1, \Delta_2)$ is a fixed point. To prove Theorem 3.1, we also need the following property.

**Proposition 3.2** Assume that $u < v \in S_n$ and $1 \leq k \leq n$. Then the involution $\Omega$ on $P_k(u, v)$ has at most one fixed point. Moreover, $\Omega$ has a fixed point if and only if $[u, v]$ is an $S$-interval and $k = u(m)$ or $k = v(m)$, where $m$ is the smallest integer such that $u(m) \neq v(m)$.

**Proof** To prove that $\Omega$ has at most one fixed point, assume that $(\Delta_1, \Delta_2) \in P_k(u, v)$ is a $V$-path that is fixed by $\Omega$. We proceed to show that $(\Delta_1, \Delta_2)$ is uniquely determined. Let $\Delta_1 = u_0 \overset{t_1}{\rightarrow} u_1 \overset{t_2}{\rightarrow} \cdots \overset{t_i}{\rightarrow} u_i$ and $\Delta_2 = v_0 \overset{t'_1}{\rightarrow} v_1 \overset{t'_2}{\rightarrow} \cdots \overset{t'_j}{\rightarrow} v_j$. By the construction of $\Omega$, we see that $t_1, \ldots, t_i$ and $t'_1, \ldots, t'_j$ are all boundary transpositions. Assume that $t_1 = (p_1, n), \ldots, t_i = (p_i, n)$ and $t'_1 = (p'_1, n), \ldots, t'_j = (p'_j, n)$. Since $\Delta_1$ and $\Delta_2$ are Bruhat paths, we see that

$$u(n) > u(p_1) > \cdots > u(p_i) = k = w(n) > w(p'_1) > \cdots > w(p'_j).$$

(3.2)

Noting that $t_1 > t_2 > \cdots > t_i$ and $t'_1 < t'_2 < \cdots < t'_j$, we find that $n > p_1 > \cdots > p_i$ and $p'_1 < \cdots < p'_j < n$.

By (3.2) together with the relation $w = u(p_1, n) \cdots (p_i, n)$, it is easily seen that

$$\{p_1, \ldots, p_i\} \cap \{p'_1, \ldots, p'_j\} = \emptyset.$$

This yields that $w(p'_1) = u(p'_1), \ldots, w(p'_j) = u(p'_j)$, and so (3.2) becomes

$$u(n) > u(p_1) > \cdots > u(p_i) = k = w(n) > u(p'_1) > \cdots > u(p'_j).$$

(3.3)

Observe that

$$\{p_1, \ldots, p_i\} \cup \{p'_1, \ldots, p'_j\} \cup \{n\} = D(u, v).$$

In view of (3.3), we deduce that given $u$, $v$ and $k$, the values of $i$, $j$ as well as the elements $p_1, \ldots, p_i, p'_1, \ldots, p'_j$ are uniquely determined. In other words, the $V$-path $(\Delta_1, \Delta_2)$ is uniquely determined.

It remains to prove that $\Omega$ has a fixed point if and only if $[u, v]$ is an $S$-interval and $k = u(m)$ or $k = v(m)$. By the above argument, we see that if $\Omega$ has a fixed point, then $[u, v]$ is an $S$-interval and $k = u(p_i) = v(p'_j)$. Since $m = \min\{p_i, p'_j\}$, we obtain that $k = u(m)$ if $p_i < p'_j$ and $k = v(m)$ if $p_i > p'_j$. Conversely, if $[u, v]$ is an $S$-interval, it is easy to construct a $V$-path in $P_k(u, v)$ fixed by $\Omega$, where $k = u(m)$ or $k = v(m)$. This completes the proof. □

We are now ready to complete the proof of Theorem 3.1.
Proof of Theorem 3.1  By Proposition 3.2, we only need to consider the case when \([u, v]\) is an S-interval and \(k = u(m)\) or \(k = v(m)\). In this case, we have

\[
\sum_{w \in [u,v]} (-1)^{\ell(w) - \ell(u)} \tilde{R}_{u,w}(q) \tilde{R}_{w,v}(q) = (-1)^{\ell(\Delta_1)} q^{\ell(\Delta_1) + \ell(\Delta_2)},
\]

where \((\Delta_1, \Delta_2)\) is the unique V-path in \(P_k(u, v)\) that is fixed by \(\Omega\). Evidently,

\[
\ell(\Delta_1) + \ell(\Delta_2) = |D(u, v)| - 1.
\]

It is also clear that

\[
\ell(\Delta_1) = |\{ j \in D(u, v) \mid u(j) > k \}|.
\]

Hence the proof is complete. \(\square\)

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