ALGORITHMS TO TEST OPEN SET CONDITION FOR SELF-SIMILAR SET RELATED TO P.V. NUMBERS

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Abstract. Fix a P.V. number \( \lambda^{-1} > 1 \). Given \( p = (p_1, \cdots, p_m) \in \mathbb{N}^m, b = (b_1, \cdots, b_m) \in \mathbb{Q}^m \), for the self-similar set \( E_{p,b} = \bigcup_{i=1}^m (\lambda^i E_{p,b} + b_i) \) we find an efficient algorithm to test whether \( E_{p,b} \) satisfies the open set condition (strong separation condition) or not.

1. Introduction

Suppose \( K = \bigcup_{i=1}^m f_i(K) \subset \mathbb{R}^d \) is a self-similar set where \( \{f_i\}_{i=1}^m \) are contracting similitudes, i.e., there are \( r_i \in (0,1) \) such that \( |f_i(x) - f_i(y)| = r_i|x-y| \) for all \( x, y \in \mathbb{R}^d \). If \( \sum_{i=1}^m (r_i)^s = 1 \), we call \( s \) the similarity dimension of \( K \) and denote it by \( \dim S \). We say that the open set condition (OSC) is fulfilled for \( \{f_i\}_{i=1}^m \), if there is a non-empty open set \( U \) such that \( \bigcup_{i=1}^m f_i(U) \subset U \) and \( f_i(U) \cap f_j(U) = \emptyset \) for all \( i \neq j \). The strong separation condition (SSC) is satisfied, if \( f_i(K) \cap f_j(K) = \emptyset \) for all \( i \neq j \).

Let \( \Sigma^* = \bigcup_{k=1}^\infty \{1, \cdots, m\}^k, f_i = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k} \) with \( i = i_1 i_2 \cdots i_k \in \{1, \cdots, m\}^k \).

Remark 1. The open set condition was introduced by Moran [28] and studied by Hutchinson [17]. Schief [36], Bandt and Graf [1] showed the relation between the open set condition and the positive Hausdorff measure.

Self-similar sets with overlaps have very complicated structures. Falconer [9] proved some “generic” result on Hausdorff dimension of self-similar sets without the assumption about the open set condition. One useful notion “transversality” to study self-similar sets (or measures) with overlaps can be found e.g. in Keane, Smorodinsky and Solomyak [13], Pollicott and Simon [31], Simon and Solomyak [37] and Solomyak [39]. For self-similar \( \lambda \)-Cantor set

\[ E_\lambda = E_\lambda/3 \cup (E_\lambda/3 + \lambda/3) \cup (E_\lambda/3 + 2/3), \]

a conjecture of Furstenberg says that \( \dim H E_\lambda = 1 \) for any \( \lambda \) irrational. Recently, Hochman [16] solved the Furstenberg conjecture, also see [40].

Kenyon [19] obtained that the OSC is fulfilled for \( E_\lambda \) if and only if \( \lambda = p/q \in \mathbb{Q} \) with \( p \equiv q \not\equiv 0 \pmod{3} \). Rao and Wen [34] also discussed the structure of \( E_\lambda \) with \( \lambda \in \mathbb{Q} \) using the key idea “graph-directed struture” introduced by Mauldin and Williams [27]. In particular, Rao and Wen [34] studied the self-similar sets with “complete overlaps”.

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In general, when considering the \( \{ f_i(x) = x/n + b_i \}_{i=1}^N \) with \( b_i \in \mathbb{Q}, n \in \mathbb{N} \) and \( n \geq 2 \), as mentioned in [50], we don’t have a general classification of those parameters \((n, b_1, \cdots, b_N)\) for which the OSC is satisfied. An interesting observation is that the weak separation property (WSP) fulfills for \( \{ f_i(x) \}_{i=1}^N \).

**Definition 1.** We say that the WSP fulfills iff the identity \( \text{id} \) is not an accumulation point of \( \{ f_i^{-1} f_j : i \neq j \in \Sigma^* \} \).

Lau and Ngai [21] introduced the WSP and Zerner [50] developed their theory. Feng and Lau [12] studied the multifractal formalism for self-similar measures with the WSP. It is clear that \( \text{SSC} \Rightarrow \text{OSC} \Rightarrow \text{WSP} \).

Ngai and Wang [30] introduced the finite type condition (FTC) and described a scheme for computing the exact Hausdorff dimension of self-similar set in the absence of the OSC. Nguyen [29] showed that the FTC implies the WSP. Lau and Ngai [22] introduced a generalized finite type condition (GFTC) which extended a more restrictive condition in [30] and proved that the generalized finite type condition implies the WSP. Lau, Ngai and Wang [23] also extended both WSP and FTC to include finite iterated function systems (IFSs) of injective \( C^1 \) conformal contractions on compact subsets of \( \mathbb{R}^d \). Then we have

\[ \text{FTC} \Rightarrow \text{GFTC} \Rightarrow \text{WSP}. \]

Das and Edgar [8] proved that the GFTC implies geometric WSP for graphs.

Lau and Ngai [21] revealed the connection between the WSP and P.V. number.

**Definition 2.** A Pisot-Vijayaraghavan number, also called simply a P.V. number, is a real algebraic integer greater than 1 such that all its Galois conjugates are less than 1 in absolute value.

The P.V. numbers were discovered by Thue in 1912 and rediscovered by Hardy in 1919 within the context of Diophantine approximation. They became widely known after the publication of Charles Pisot’s dissertation in 1938. Some elementary properties of P.V. number will offer help to us:

- Every positive integer except 1 is a P.V. number;
- For example, \( \sqrt{2} + 1 \) and \( \sqrt{2} + 1 \) are P.V. numbers;
- Garsia’s theorem [14]: Suppose \( \eta \) is a P.V. number and \( \eta_1, \cdots, \eta_l \) are algebraic conjugates of \( \eta \) with \( |\eta_i| < 1 \) for all \( i \). If \( L(x) = a_0 + a_1 x + \cdots + a_k x^k \) with integer coefficients, then

\[
L(\eta) = 0 \text{ or } L(\eta) \geq \left[ \max_i |a_i| \right]^{-l} \prod_{i=1}^l (1 - |\eta_i|); \tag{1.1}
\]

- Given a P.V. number \( \lambda^{-1} > 1 \), an interesting fact is that the IFS \( \{ \lambda^i x + b_i \}_{i=1}^n \) with \( p_i \in \mathbb{N} \) and \( b_i \in \mathbb{Q} \) for all \( i \), by Theorem 2.9 in [30], is of finite type, and then WSP is fulfilled [21].

### 1.1 Decidability on fractals.

It is well known (e.g. see [38]) that the halting problem for Turing machines, Wang’s tiling problem, Hilbert’s tenth problem and group isomorphism problem have no algorithms, i.e., these problems are undecidable. Here a decidable problem is a question such that there is an algorithm or computer program to answer “yes” or “no” to the question for every possible input.

There are some related results as follows.
(1) Based on the undecidability of Post correspondence problem (PCP), Dube \[6, 7\] discussed the undecidability of the problem on the invariant fractals of iterated function systems. For example, Theorem 5 of \[6\] actually shows that given a self-affine set in the plane, it is undecidable to test if it satisfies the SSC.

(2) With the WSP of similarities \(\{S_i\}_{i=1}^n\), Lau, Ngai and Rao \[24\] proved that the following problem is decidable to test if the self-similar measure \(\mu = \sum_{i=1}^m \rho_i \circ S_i^{-1}\) is absolutely continuous or not. In fact, they obtained the criterion based on the transition matrix.

An algorithm is said to be of polynomial time if its running time is upper bounded by a polynomial expression in the size of the input for the algorithm, i.e., \(T(n) = O(n^k)\) for some constant \(k\).

Let \(G = (\mathcal{V}, \mathcal{E})\) be a directed graph with vertex set \(\mathcal{V}\) and directed-edge set \(\mathcal{E}\). Recall some polynomial time algorithms as follows:

(a) Dijkstra’s algorithm \[5\] finds the lowest cost path for a graph with nonnegative edge path costs, Dijkstra’s original algorithm does not use a min-priority queue and runs in \(O(|\mathcal{V}|^2)\);

(b) Kosaraju’s algorithm \[3\] is an algorithm with its running time \(O(|\mathcal{V}|^2)\) to find the strongly connected components of a directed graph;

(c) Kruskal’s algorithm \[20\] finds a minimum spanning tree for a connected weighted graph, this algorithm requires \(O(|\mathcal{E}| \log |\mathcal{E}|)\) time.

1.2. Main result.

In this paper, fix a P.V. number \(\lambda^{-1} > 1\), we consider the IFS

\[
S_i(x) = \lambda^p_i x + b_i \quad \text{for} \quad i = 1, \ldots, m,
\]

where \(p = (p_1, \ldots, p_m) \in \mathbb{N}^m\), \(b = (b_1, \ldots, b_m) \in \mathbb{Q}^m\). Let

\[
E_{p,b} = \bigcup_{i=1}^m (\lambda^{p_i} E_{p,b} + b_i) \subset \mathbb{R}^1
\]

be the self-similar set w.r.t. the IFS \(\{\lambda^{p_i} x + b_i\}_{i=1}^m\).

Although there is no general classification of parameters \((p,b)\) for which the OSC is satisfied, in this paper, we obtain an algorithm to test whether OSC is satisfied or not.

**Theorem 1.** Fix a P.V. number \(\lambda^{-1} > 1\). Given \(p = (p_1, \ldots, p_m) \in \mathbb{N}^m\), \(b = (b_1, \ldots, b_m) \in \mathbb{Q}^m\), the following problems on the self-similar set

\[
E_{p,b} = \bigcup_{i=1}^m (\lambda^{p_i} E_{p,b} + b_i)
\]

are decidable:

(1) whether \(E_{p,b}\) satisfies the open set condition;

(2) whether \(E_{p,b}\) satisfies the strong separation condition.

**Remark 2.** \[11\] deals with the case that \(\lambda^{-1}\) is an integer and \(p_1 = \cdots = p_m = 1\).

**Remark 3.** Without loss of generality, we assume that

\[
(b_1, \ldots, b_m) \in \mathbb{Z}^m.
\]

In fact, we can select \(a \in \mathbb{Z}\) such that \(ab_i \in \mathbb{Z}\) for all \(i\), suppose \(T_i(x) = \lambda^{p_i} x + ab_i\). Then the OSC (or SSC) holds for \(\{T_i\}\) if and only if the OSC (or SSC) holds for \(\{S_i\}\).

**Remark 4.** Since \(n \geq 2\) is a P.V. number, in an algorithmic point of view we answer the problem on \((n, b_1, \ldots, b_m)\) from \[50\].
Corollary 1. For $E_{p,b}$ as above, then $\dim_H E_{p,b} = \dim_S E_{p,b}$ if and only if OSC is fulfilled. Then the problem is decidable whether $\dim_H E_{p,b} = \dim_S E_{p,b}$.

Proof. Since $E_{p,b}$ is of finite type, we have $0 < H^{\dim_H E_{p,b}}(E_{p,b}) < \infty$ by Theorem 1.2 in [30]. Schief’s theorem [36] implies $\text{OSC} \iff 0 < H^{\dim_S E_{p,b}}(E_{p,b}) < \infty$. Then

$$\text{OSC} \iff \dim_H E_{p,b} = \dim_S E_{p,b}$$

and the decidability follows from Theorem [1].

We consider $E_{p,b}$ with $\dim_S E_{p,b} = 1$. Then, by Schief’s theorem, we have

$$\text{OSC} \iff 0 < H^{1}(E_{p,b}) < \infty \iff \text{int}(E_{p,b}) \neq \emptyset,$$

where $\text{int}(\cdot)$ denotes interior of set.

Corollary 2. For $E_{p,b}$ with $\dim_S E_{p,b} = 1$, then $\text{int}(E_{p,b}) \neq \emptyset$ if and only if OSC is fulfilled. Then the problem is decidable whether $\text{int}(E_{p,b}) \neq \emptyset$.

Remark 5. For $E = \bigcup_{i=1}^{m}(\frac{1}{m}E + b_i)$ with $b_i \in \mathbb{Q}$ we have $\dim_S(E) = 1$. See Example 3 for different types.

Lipschitz equivalence of fractals is also an interesting topic ([2], [11], [4], [32], [25], [35], [33]-[39]). Let’s consider the Lipschitz equivalence of the self-similar set

$$F = \bigcup_{i=1}^{m}(rF + b_i) \quad \text{with} \quad r < 1/m,$$

where $r^{-1} > 1$ is a P.V. number and $b_i \in \mathbb{Q}$.

The symbolic spaces are often regarded as an important tool for researching Lipschitz equivalence between sets. We use $\Sigma^r_m$ to denote the symbolic space \{1, \ldots, m\} with the metric

$$d(x, y) = \min\{|x_k - y_k|: k \neq k_0, \ldots, k_n\}.$$

It is pointed out in [49] that if $F$ fulfills OSC and totally disconnectedness, then $F$ is Lipschitz equivalent to $\Sigma^r_m$, denoted by $F \sim \Sigma^r_m$. Please refer to [46] and [51] for weaker versions of the above result. It is clear that

$$s := \dim_H(\Sigma^r_m) = \dim_S(F) = \frac{\log m}{\log r} < 1.$$ (1.5)

Corollary 3. For $F$ defined as (1.4), then $F$ is Lipschitz equivalent to $\Sigma^r_m$ if and only if OSC is fulfilled. Then the problem is decidable whether $F$ is Lipschitz equivalent to $\Sigma^r_m$.

Proof. Note that $\dim_H F \leq \dim_S F < 1$, which implies that $F$ is totally disconnected. Then by [49], it holds that $\text{OSC} \implies F \sim \Sigma^r_m$.

By a usual discussion, we have $0 < H^{r}(\Sigma^r_m) < \infty$, where $s$ is defined in (1.5). If suppose that $F \sim \Sigma_m$, then $0 < H^{r}(F) < \infty$. Hence, $F$ satisfies OSC by Schief’s theorem.

Therefore, the corollary is obtained from the item (1) in Theorem [1].

For example, let $\lambda = 1/5$, $p = (1, 1, 1)$ and $b = (0, 7/10, 4/5)$. Then Example 2 of [41] shows that

1. $E_{p,b}$ satisfies OSC;
2. $E_{p,b}$ does not satisfy SSC.

By Corollary 3, $E_{p,b} \sim \Sigma^{1/5}_3$. 


1.3. Graph and algorithm.

In fact, we can solve the above problems by constructing a directed graph $G$ and establishing the following criteria:

**Theorem 2.** The OSC fails for $\{\lambda^p x + b_i\}_{i=1}^m$ if and only if there is a finite path in $G$ starting from a point of $\Lambda$ and ending at $(0,0)$.

**Theorem 3.** The SSC fails for $\{\lambda^p x + b_i\}_{i=1}^m$ if and only if there is an infinite path in $G$ starting from a point of $\Lambda$.

**Remark 6.** Using Dijkstra’s algorithm, we can solve the existence of paths mentioned in Theorems 2–3, see Lemma 3.

Now, we will describe the graph $G$ and $\Lambda$ which is a subset of vertex set of $G$. Let $B = \{0\} \cup \{b_1, \ldots, b_m\}$. Denote $Z - Z = \{P : P$ is a polynomials with coefficients in $B - B\}$. Set $T = \max_i p_i$.

Let $G$ be a directed finite graph with vertex set

$$\Xi = \{n \in \mathbb{Z} : |n| \leq T - 1\} \times \mathcal{C},$$

with

$$\mathcal{C} = \{x : |x| \leq \frac{(1 + \lambda^{-T+1}) \max_i |b_i|}{1 - \lambda}, \lambda^{-T+2} x = R(\lambda^{-1}) \text{ with } R \in Z - Z\}. \quad (1.6)$$

For a finite word $\sigma = i_1 i_2 \cdots i_k$ with letters in $\{1, \ldots, m\}$, then $S_\sigma = S_{i_1} \circ \cdots \circ S_{i_k}$. We suppose

$$S_\sigma x = \lambda^{p_\sigma} x + b_\sigma \text{ with } p_\sigma = p_{i_1} + \cdots + p_{i_k}.$$  

Given $(p, b), (p', b') \in \Xi$, there is a directed edge $e$ from $(p, b)$ to $(p', b')$, if and only if there are words $\alpha, \beta$ with $1 \leq p_\alpha, p_\beta \leq 2T - 1$ such that

$$p' = p_\alpha + p - p_\beta \text{ and } b' = \lambda^{-p_\beta} b + \lambda^{-p_\beta} p_\alpha - \lambda^{-p_\beta} b_\beta.$$  

Let $\Lambda = \Xi \cap \{(p_j - p_i, \lambda^{-p_i} b_j - \lambda^{-p_\beta} b_i) : i \neq j\}$.

**Remark 7.** In fact, for self-similar set of finite type, we have a scheme to test the OSC based on computing the exact Hausdorff dimension. In [30], for the IFS $\{\phi_j(x) = \rho_j R_j x + b_j, 1 \leq j \leq q\}$, Ngai and Wang described a scheme for computing the Hausdorff dimension of its attractors $K$:

- Step 1. select a suitable invariant open set $\Omega$;
- Step 2. determine neighborhood type $\Omega(v)$;
- Step 3. compute all neighborhood types;
- Step 4. calculate the incidence matrix $S$ of neighborhood type;
- Step 5. obtain the spectral radius $\lambda$ of $S$.

Then $\dim_H K = -\log \lambda / \log \rho$, where $\rho = \min_j \rho_j$. By Theorem 1.2 in [30] and Schief’s theorem, we have

$$\text{OSC} \iff \dim_S K = -\log \lambda / \log \rho.$$  

However, from the view of algorithm, when implementing the scheme we have to face some practical difficulties: (1) an exhaustive search in the words set $\mathcal{V}_k$ (defined as [30]) is inevitable to look for all neighborhood type $\Omega(v)$ of $v \in \mathcal{V}_k$ for every $(k \geq 0)$; (2) a heavy computation is inevitable to find all distinct neighborhood types, due to the unknownness of the number of types.
For example, for a class of IFSs with the integral parameter \( n > 0 \),
\[
T_1 = \frac{1}{3}x, \quad T_2 = \frac{1}{3}x + \frac{2}{3^{n+1}}, \quad T_3 = \frac{1}{3}x + \frac{2}{3}.
\]

Rao and Wen [34] obtained \( 2^n \) neighborhood types of \( \{T_i\} \) which implies that an exponential calculated quantity is need to differentiate neighborhood types of \( \{T_i\} \), also see [13] for details. However, our algorithm in Section 4 will avoid effectively these difficulties. Example 1 in Section 6 tells us that a path with length \( n \) in \( G \) starting from a point of \( \Lambda \) and ending at \((0,0)\) can be found quickly in a polynomial time, which leads to the OSC’s failing for \( \{T_i\} \) in (1.7) in virtue of Theorem 2.

The paper is organized as follows. Section 2 includes some preliminaries, for example, Lemma 4 describes the decidability of testing the existence of paths in Theorems 2 and 3. In Section 3, the recursive structure is introduced to construct the graph. In Section 4, we construct the graph and design these algorithms. In Section 5, we obtain a refinement algorithm for the OSC, especially we replace the upper bound \(|x| \leq \frac{(1+\lambda^{-n+1})\max\{b_i\}}{1-\lambda}\) in (1.6) by \(|x| < \frac{2\max\{b_i\}}{(1-\lambda)^2}\). In the last section, we will give some examples to illustrate these algorithms.

2. Preliminaries

2.1. WSP and OSC.

We always fix \( \mathbf{p} = (p_1, \ldots, p_m) \) and \( \mathbf{b} = (b_1, \ldots, b_m) \) and write \( E = E_{\mathbf{p}, \mathbf{b}} \) for notational convenience. Then \( E = \cup_{i=1}^{m} S_i(E) \) where
\[
S_i(x) = \lambda^{p_i}x + b_i
\]
with \( p_i \in \mathbb{N} \) and \( b_i \in \mathbb{Z} \). Here \( E \subset \{x : |x| \leq \frac{\max\{b_i\}}{1-\lambda}\} \). Write
\[
T = \max_i p_i.
\]

Set \( \Sigma = \{1, \ldots, m\}^\mathbb{N} \) and \( \Sigma^* = \cup_{k=1}^{\infty} \{1, \ldots, m\}^k \). If \( w = uv \) for words \( u \in \Sigma^* \) and \( w, v \in \Sigma^* \) or \( \Sigma \), we denote \( u \prec w \) and write \( w \backslash u = v \). Let \([i]^k\) denote \( i \cdots i \).

For \( \sigma \in \Sigma^* \), we write
\[
S_{\sigma}(x) = \lambda^{p_{\sigma}}x + b_{\sigma}.
\]

Given \( (p, b) \in \mathbb{Z} \times \mathbb{R} \), we let
\[
h_{(p,b)}(x) = \lambda^px + b.
\]

Then
\[
h_{(p_1,b_1)} \circ h_{(p_2,b_2)} = h_{(p_1+p_2, \lambda^{p_1}b_2 + b_1)},
\]
\[
h_{(p,b)} \circ h_{(-p, -\lambda^{-p}b)} = h_{(-p, -\lambda^{-p}b)} \circ h_{(p,b)} = h_{(0,0)}.
\]

Hence we obtain a group structure on \( \mathbb{Z} \times \mathbb{R} \) satisfying
\[
(p_1, b_1) \cdot (p_2, b_2) = (p_1 + p_2, \lambda^{p_1}b_2 + b_1), \quad (p, b)^{-1} = (-p, -\lambda^{-p}b).
\]

We will omit \( \cdot \) in the multiplication.

Let \( \pi_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{R} \to \mathbb{Z} \) and \( \pi_{\mathbb{R}} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) be the projections defined by
\[
\pi_{\mathbb{Z}}(p, b) = p \quad \text{and} \quad \pi_{\mathbb{R}}(p, b) = b.
\]
For a letter $i \in \{1, \ldots, m\}$, there is an element $(p_i, b_i) \in \mathbb{N} \times \mathbb{R}$ such that the contracting similitude $S_i(x) = h(p_i, b_i)$. For a finite word $\sigma = i_1 \cdots i_k$, then

$$(p_\sigma, b_\sigma) = (p_{i_1}, b_{i_1}) \cdots (p_{i_k}, b_{i_k})$$

and $S_\sigma = h(p_\sigma, b_\sigma)$.

Let $\mathcal{Z} = \{P : P$ is a polynomials with coefficients in $B\}$.

**Lemma 1.** If $|p_\sigma - p_\tau| \leq T - 1$, then

$$\pi_\mathbb{R}[(p_\tau, b_\tau)^{-1}(p_\sigma, b_\sigma)] = \lambda^{T-2}R(\lambda^{-1}),$$

where $R(\lambda^{-1})$. Then

$S_\sigma(x) = \lambda^{p_\tau}x + Q(\lambda),$ 

where $Q(x)$ is a polynomial with coefficients in $B$. 

**Proof.** In fact, for $\sigma = i_1 \cdots i_k$, using induction, we can obtain that

$S_\sigma(x) = \lambda^{p_\sigma}x + \pi \lambda^{T-2}R(\lambda_{-1}),$ 

where $\pi R(\lambda^{-1})$ holds if and only if the $WSP$ holds for $\{S_i\}_{i=1}^m$. 

We recall the following result.

**Claim 1.** (21) The $WSP$ fulfills for $\{S_i\}_{i=1}^m$.

In fact, using Lemma 1,

$S_\tau^{-1}S_\sigma(x) = \lambda^{p_\tau - p_\sigma}x + \pi \lambda^{T-2}R(\lambda_{-1})$ if $|p_\sigma - p_\tau| \leq T - 1$. By Garsia’s result [11], $\{R(\lambda_{-1}) : R \in \mathcal{Z} - \mathcal{Z}\}$ is discrete and $0 \in \{R(\lambda_{-1}) : R \in \mathcal{Z} - \mathcal{Z}\}$. 

Hence we notice that if $S_\tau^{-1}S_\sigma$ is very closed to the identity $id$, then $p_\sigma = p_\tau$ and $\pi \lambda^{T-2}R(\lambda_{-1}) = 0$, i.e., $S_\tau = S_\sigma$. Therefore, the $WSP$ fulfills. Furthermore we have the following result.

**Lemma 2.** The $OSC$ holds for $\{S_i\}_{i=1}^m$.

Remark 8. Proposition 1 of [21] says that when $K = \cup_i f_i(K) \subset \mathbb{R}^n$ does not lie in a hyperplane, then the $OSC$ holds if and only if the $WSP$ fulfills and $f_i \neq f_j$ for all $i \neq j$. We can also verify Lemma 2 by using this result. In fact we only need to deal with the case that $E$ is a singleton. Suppose $E = \{x_0\}$ is a singleton and the $OSC$ fails. Then $S_i(x_0) = S_j(x_0) = x_0$ for different letters $i, j$. Let $\sigma = [i]^{p_\sigma}$ and $\tau = [j]^{p_\tau}$, then $S_\sigma(x - x_0) = x_0 + \lambda^{p_\tau}x - x_0 = S_\tau(x - x_0)$ for all $x$. Hence $S_\sigma = S_\tau$.

Set

$$C = \{x : |x| \leq \frac{(1 + \lambda^{-T+1}) \max |b_i|}{1 - \lambda} \text{ and } \lambda^{-T+2}x = R(\lambda_{-1}) \text{ with } R \in \mathcal{Z} - \mathcal{Z}\}.$$
Lemma 3. \( C \) is a finite set with cardinality
\[
\#C \leq \frac{2^l(1 + \lambda^{-T+1})(\max_i |b_i|)^{l+1}}{\lambda^{T-2}(1 - \lambda) \prod_{i=1}^l (1 - |\eta_i|)}.
\]
where \( \eta_1, \ldots, \eta_l \) are algebraic conjugates of \( \lambda^{-1} \) with \( |\eta_i| < 1 \) for all \( i \).

Proof. Notice that the height of \( R \) is not greater than \( 2 \max_i |b_i| \).
Using Garsia’s result (1.1), for any \( c, c' \in C \), we have
\[
c = c' \text{ or } |c - c'| \geq \lambda^{T-2}\delta
\]
where the constant
\[
\delta = [2 \max_i |b_i|]^{-l} \prod_{i=1}^l (1 - |\eta_i|).
\]

2.2. Graph algorithm.

Given a vertex \( i_0 \) in a nonnegative weighted and directed graph, Dijkstra’s algorithm [5] finds the lowest cost path starting from \( i_0 \). Dijkstra’s original algorithm does not use a min-priority queue and runs in \( O(|V|^2) \), where \( V \) is the vertex set and \( |V| \) is the number of vertices. Furthermore, we can calculate all the lowest costs \( \{c(i, j)\}_{(i, j) \in V \times V} \) with running time \( O(|V|^3) \).

In this paper, we will meet the following questions:

Question (A): Given a directed graph \( G \), for two different vertexes \( i_0 \) and \( j_0 \) in \( G \), is there a directed path in \( G \) starting from \( i_0 \) and ending at \( j_0 \)?

Question (B): Given a directed graph \( G \), for a vertex \( i_0 \) in \( G \), is there an infinity directed path in \( G \) starting from vertex \( i_0 \)?

The following lemma is easy, but we give its proof here just to make this paper self-contained.

Lemma 4. Questions (A) and (B) have polynomial time algorithms with running time \( O(|V|^2) \) and \( O(|V|^3) \) respectively.

Proof. Suppose \( G = (V, E) \) with vertex set \( V \) and directed edge set \( E \).

We will obtain construct a weighted graph \( G' \) with vertex set \( V \). For any ordered pair \( (i, j) \in V \times V \), we give the following nonnegative weight
\[
w(i, j) = \begin{cases} 1 & \text{if there exists an edge in } E \text{ from } i \text{ to } j, \\ |V| + 1 & \text{otherwise}. \end{cases}
\]

Using Dijkstra’s algorithm for \( G' \), we can calculate the lowest cost \( c(i_0, j_0) \) from \( i_0 \) to \( j_0 \). Then there exists a directed path in \( G \) starting from \( i_0 \) and ending at \( j_0 \) if and only if the lowest cost \( c(i_0, j_0) \leq |V| \). This algorithm requires \( O(|V|^2) \) time.

It is easy to find an equivalent question for Question (B): given a directed graph \( G \), for a vertex \( i_0 \) in \( G \), is there a directed path in \( G \) starting from vertex \( i_0 \) and ending at a vertex in a loop? To solve this question, we can use the above \( G' \) again to calculate all the lowest costs \( \{c(i, j)\}_{(i, j) \in V \times V} \) with running time \( O(|V|^3) \).

Considering all such points lying in loops, i.e.,
\[
\Delta = \{i : c(i, i) \leq |V|\},
\]
we find out that there is a path starting from vertex \( i_0 \) and ending at a vertex in \( \Delta \) if and only if \( \min_{i \in \Delta} c(i_0, i) \leq |V| \). \( \square \)
Let \[ \Xi = \{ n \in \mathbb{Z} : |n| \leq T - 1 \} \times \mathcal{C}. \]

### 3. Strong separation condition.

**Proposition 1.** Suppose \( i_1 i_2 \cdots, j_1 j_2 \cdots \in \Sigma. \) If \( S_{i_1 i_2 \cdots}(E) = S_{j_1 j_2 \cdots}(E), \) then there exist \( \{ \sigma_k \}_k, \{ \tau_k \}_k \subset \Sigma^* \) such that for all \( k \geq 1, \sigma_k \prec i_1 i_2 \cdots \) and \( \tau_k \prec j_1 j_2 \cdots \) satisfying \( p_{\sigma_k}, p_{\tau_k} \in ((k - 1)T, kT], \)

\[ (p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k}) \in \Xi. \]

**Proof.** For any \( k \geq 1, \) we pick up the shortest prefix \( \sigma_k \) of \( i_1 i_2 \cdots \) such that \( p_{\sigma_k} \in (kT, (k + 1)T) \) and pick up \( \tau_k \) for \( j_1 j_2 \cdots \) in the same way. Since \( \pi_{\Xi}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})] = p_{\sigma_k} - p_{\tau_k} \)
and \( p_{\sigma_k}, p_{\tau_k} \in (kT, (k + 1)T), \) we have \( \pi_{\Xi}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})] \leq T - 1. \)

It follows from \( S_{i_1 i_2 \cdots}(E) = S_{j_1 j_2 \cdots}(E) \) that \( S_{\sigma_k}(E) \cap S_{\tau_k}(E) \neq \emptyset, \)
i.e.,

\[ E \cap S_{\sigma_k}^{-1} S_{\tau_k}(E) \neq \emptyset, \]
where \( S_{\sigma_k}^{-1} S_{\tau_k}(x) = \lambda^{p_{\tau_k} - p_{\sigma_k}} x + \pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})]. \) Suppose that

\[ x_1 = \lambda^{p_{\tau_k} - p_{\sigma_k}} x_2 + \pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})] \tag{3.1} \]
with \( x_1, x_2 \in E. \) Using the fact \( E \subseteq \{ x : |x| \leq \frac{\max |b_i|}{1 - \lambda} \}, \) we have

\[ |\pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})]| \leq |x_1| + \lambda^{p_{\tau_k} - p_{\sigma_k}} |x_2| \leq (1 + \lambda^{p_{\tau_k} - p_{\sigma_k}}) \max_i |b_i| \tag{3.2} \]

By Lemma 1 we obtain \( \pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})] = \lambda^{T - 1} R(\lambda^{-1}) \) directly. \( \square \)

**Proposition 2.** Suppose \( i_1 i_2 \cdots, j_1 j_2 \cdots \in \Sigma. \) If for all \( k \geq 1, \) there exist \( \sigma_k \prec i_1 i_2 \cdots \) and \( \tau_k \prec j_1 j_2 \cdots \) such that \( p_{\sigma_k}, p_{\tau_k} \in ((k - 1)T, kT] \) and

\[ |\pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})]| \leq (1 + \lambda^{-T + 1}) \max_i |b_i| \]
then \( S_{i_1 i_2 \cdots}(E) = S_{j_1 j_2 \cdots}(E). \)

**Proof.** Considering the distance between \( S_{\sigma_k}(0) \) and \( S_{\tau_k}(0), \) we have \( S_{\tau_k}^{-1} S_{\sigma_k}(x) = \lambda^{p_{\tau_k} - p_{\sigma_k}} x + \pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})], \)
i.e.,

\[ S_{\tau_k}^{-1} S_{\sigma_k}(0) = \pi_{\mathbb{R}}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})], \]
which implies

\[ |S_{\sigma_k}(0) - S_{\tau_k}(0)| = \lambda^{p_{\tau_k}} |S_{\tau_k}^{-1} S_{\sigma_k}(0) - 0| \leq \lambda^{(k-1)T} (1 + \lambda^{-T + 1}) \max_i |b_i| \rightarrow 0 \text{ as } k \rightarrow \infty. \]
Letting $k \to \infty$, we obtain that
\[
S_{i_1i_2\cdots}(0) = S_{j_1j_2\cdots}(0)
\]
and thus $S_{i_1i_2\cdots}(E) = S_{j_1j_2\cdots}(E)$. \hfill \square

3.2. **Open set condition.**

According to Lemma 2, we only need to check whether $S_{\sigma} = S_{\tau}$ for some distinct $\sigma, \tau \in \Sigma^*$ or not?

**Proposition 3.** If $S_{\sigma} = S_{\tau}$, then there exists a positive integer $M$ and
\[
\{\sigma_k \}_{k=1}^M, \{\tau_k \}_{k=1}^M \subset \Sigma^*
\]
such that
\[
\sigma_M = \sigma, \tau_M = \tau
\]
and for all $k \geq 1$, $\sigma_k \prec \sigma$ and $\tau_k \prec \tau$ satisfying $p_{\sigma_k}, p_{\tau_k} \in ((k-1)T, kT]$, $$(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k}) \in \Xi.$$ 

**Proof.** Suppose that $p_{\sigma} = p_{\tau} \in ((M-1)T, MT]$. For $k < M$, we can take shortest prefix $\sigma_k$ (or $\tau_k$) of $\sigma$ (or $\tau$) such that $p_{\sigma_k}, p_{\tau_k} \in ((k-1)T, kT]$. Take $\sigma_M = \sigma, \tau_M = \tau$. Let $\alpha_0 = \sigma_1$ and $\beta_0 = \tau_1$ with $|\alpha_0| = |\beta_0| = 1$. Set $\alpha_k = \sigma_{k+1}\backslash\sigma_k$ and $\beta_k = \tau_{k+1}\backslash\tau_k$ for $k \geq 1$. Then we also have
\[
1 \leq p_{\alpha_k}, p_{\beta_k} \leq 2T - 1
\]
for all $k \geq 0$.

Since $p_{\sigma_k}, p_{\tau_k} \in ((k-1)T, kT]$, we have $|p_{\sigma_k} - p_{\tau_k}| \leq T - 1$ for all $k$, i.e.,
\[
|\pi_x[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})]| \leq T - 1.
\]
As $S_{\sigma}(E) = S_{\tau}(E)$, then $S_{\sigma_k}(E) \cap S_{\tau_k}(E) \neq \emptyset$. As in the proof of Proposition 1, we have
\[
|\pi_x[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})]| \leq (1 + \lambda^{-T+1})\max_i |b_i|\frac{1}{1 - \lambda}.
\]
By Lemma 3, we also have $\pi_x[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})] \in \lambda^{T-2}R(\lambda^{-1})$. \hfill \square

3.3. **Recursive structure.**

**Definition 3.** Suppose $\{\alpha_k\}_{k=0}^t, \{\beta_k\}_{k=0}^t \subset \Sigma^*$ with $t \in \mathbb{N}$ or $t = \infty$. We say that
\[
\{(q_k, c_k)\}_{k=1}^{t+1}
\]
has the recursive structure w.r.t. $\{\alpha_k\}_{k=1}^t, \{\beta_k\}_{k=0}^t$, if
\[
(q_1, c_1) = (p_{\beta_0}, b_{\beta_0})^{-1}(p_{\alpha_0}, b_{\alpha_0}),
(q_{k+1}, c_{k+1}) = (p_{\beta_k}, b_{\beta_k})^{-1}(q_k, c_k)(p_{\alpha_k}, b_{\alpha_k}) \text{ for } 1 \leq k \leq t.
\]

By Propositions 1, 2, we have the following result on the SSC.

**Proposition 4.** The SSC fails for $\{\lambda^p + b_i\}_{i=1}^n$ if and only if there are
\[
\{\alpha_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty} \subset \Sigma^*
\]
with $1 \leq p_{\alpha_k}, p_{\beta_k} \leq 2T - 1$ for all $k$ and $|\alpha_0| = |\beta_0| = 1$, $\alpha_0 \neq \beta_0$, such that
\[
(q_k, c_k) \in \Xi \text{ for all } k \geq 1,
\]
where $\{(q_k, c_k)\}_k$ has the recursive structure w.r.t. $\{\alpha_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}$. 

Proof. If the strong separation condition fails, then there are \( i_1, i_2, \ldots, j_1, j_2, \ldots \in \Sigma \) with \( i_1 \neq j_1 \) such that
\[
S_{i_1 i_2 \cdots} = S_{j_1 j_2 \cdots}.
\]
Using the method in Proposition \( \text{[1]} \) we obtain that \( \{ \sigma_k \}_k, \{ \tau_k \}_k \) with \( \sigma_1 = i_1 = \alpha_0 \) and \( \tau_1 = j_1 = \beta_0 \). Let \((q_k, c_k) = (p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})\), and \( \alpha_k = \sigma_{k+1} \setminus \sigma_k \) and \( \beta_k = \tau_{k+1} \setminus \tau_k \) where
\[
1 \leq p_{\alpha_k}, p_{\beta_k} \leq 2T - 1.
\]
Since \( S_{\tau_{k+1}}^{-1} S_{\sigma_{k+1}} = S_{\beta_k}^{-1}(S_{\tau_k}^{-1} S_{\sigma_k}) S_{\alpha_k} \), we have
\[
(q_{k+1}, c_{k+1}) = (p_{\tau_{k+1}}, b_{\tau_{k+1}})^{-1}(p_{\sigma_{k+1}}, b_{\sigma_{k+1}})
= (p_{\beta_k}, b_{\beta_k})^{-1}[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})](p_{\alpha_k}, b_{\alpha_k})
= (p_{\beta_k}, b_{\beta_k})^{-1}(q_k, c_k)(p_{\alpha_k}, b_{\alpha_k}).
\]
Then by Proposition \( \text{[1]} \) for all \( k \geq 1 \), we have
\[
(q_k, c_k) \in \Xi.
\]

On the other hand, if there are such \( \{ \alpha_k \}_{k \geq 0}, \{ \beta_k \}_{k \geq 0} \subset \Sigma^* \), let \( \sigma_k = \alpha_0 * \alpha_1 * \cdots * \alpha_{k-1} \) and \( \tau_k = \beta_0 * \beta_1 * \cdots * \beta_{k-1} \). Letting \( k \to \infty \), we obtain \( i_1, i_2, \ldots, j_1, j_2, \ldots \in \Sigma \) such that \( \sigma_k \prec i_1 i_2 \cdots \) and \( \tau_k \prec j_1 j_2 \cdots \) for all \( k \), and
\[
|p_\beta[p_{\tau_k}, b_{\tau_k}]^{-1}(p_{\sigma_k}, b_{\sigma_k})| \leq (1 + \lambda^{-T+1}) \max_k |b_k| \frac{1}{1 - \lambda}.
\]
By Proposition \( \text{[2]} \) we find that the strong separation condition fails. \( \square \)

By Proposition \( \text{[3]} \) we have the following result on the OSC.

**Proposition 5.** The OSC fails for \( \{ \lambda^n x + b_i \}_{i=1}^m \) if and only if there exists a positive integer \( M \) and
\[
\{ \alpha_k \}_{k=0}^M, \{ \beta_k \}_{k=0}^M \subset \Sigma^*
\]
with \( 1 \leq p_{\alpha_k}, p_{\beta_k} \leq 2T - 1 \) for all \( k \) and \( |\alpha_0| = |\beta_0| = 1, \alpha_0 \neq \beta_0 \), such that
\[
(q_{M+1}, c_{M+1}) = (0, 0) \quad \text{and} \quad (q_k, c_k) \in \Xi \quad \text{for all} \quad k \geq 1,
\]
where \( \{ (q_k, c_k) \}_{k=1}^{M+1} \) has the recursive structure w.r.t. \( \{ \alpha_k \}_{k=0}^M, \{ \beta_k \}_{k=0}^M \).

### 4. Directed Graph and Algorithm

Consider a graph \( G \) with vertex set
\[
\Xi = \{ n \in \mathbb{Z} : |n| \leq T - 1 \} \times \mathcal{C}.
\]
We will equip the graph with directed edge as follows. Given \( (p, b), (p', b') \in \Xi \), there is a directed edge \( e \) from \( (p, b) \) to \( (p', b') \), if and only if there are words \( \alpha, \beta \) with \( 1 \leq p_\alpha, p_\beta \leq 2T - 1 \) such that
\[
(p', b') = (p_\beta, b_\beta)^{-1}(p, b)(p_\alpha, b_\alpha).
\]
Notice that \((p_i, b_i)^{-1}(p_j, b_j) = (p_j - p_i, \lambda^{-p_i} b_j - \lambda^{-p_i} b_i)\). Let
\[
\Lambda = \{ (p_i, b_i)^{-1}(p_j, b_j) : i \neq j \text{ and } \lambda^{-p_i} |b_j - b_i| \leq (1 + \lambda^{-T+1}) \max_i |b_i| \frac{1}{1 - \lambda} \}.
\]
We have \( \Lambda \subset \Xi \) since \( \lambda^n(b_j - b_i) \in \mathbb{Z} \setminus \mathbb{Z} \).
Using Propositions 4-5, we complete the proofs of Theorems 2-3. Furthermore, Theorem 4 follows from Theorems 2-3 and Lemma 4.

4.1. Realization of algorithm.

Step 1: Set \( (b_1, \ldots, b_m) \in \mathbb{Z}^m \)
If \( (b_1, \ldots, b_m) \in \mathbb{Q}^m \setminus \mathbb{Z}^m \), then as in Remark 3, we take \( a \in N \) such that \( b_i = ab_i \in \mathbb{Z} \) for all \( i \), and thus we replace the IFS \( \{ \lambda^p x + b_i \}_{i=1}^m \) by \( \{ \lambda^p x + b'_i \}_{i=1}^m \).

Step 2: Calculate the set \( C \)
Let \( y = \lambda^{-T+2} x \), then \( C = \lambda^{T-2} D \) where
\[
D = \{ y : |y| \leq d \text{ and } x = R(\lambda^{-1}) \text{ with } R \in \mathbb{Z} - \mathbb{Z} \}.
\]
where
\[
d = \lambda^{-T+2} \frac{(1 + \lambda^{-T+1}) \max_i |b_i|}{1 - \lambda}.
\] (4.1)
Since \( d \geq \frac{2 \max_i |b_i|}{\lambda} \), we have
\[
|y| > d \implies |\lambda^{-1} y + b| > d \text{ for all } b \in B - B.
\] (4.2)
Let
\[
D_k = \{ y \in D : x = R(\lambda^{-1}) \text{ with } \deg(R) \leq k \}.
\] (4.3)
For any \( R_k \in \mathbb{Z} - \mathbb{Z} \) with \( \deg(R_k) = k \), there is a polynomial \( R_{k-1} \in \mathbb{Z} - \mathbb{Z} \) with \( \deg(R_{k-1}) = k - 1 \) and \( b \in B - B \) such that
\[
R_k(\lambda^{-1}) = \lambda^{-1} R_{k-1}(\lambda^{-1}) + b.
\]
It follows from (4.2) that \( R_{k-1}(\lambda^{-1}) \in D_{k-1} \) if \( R_k(\lambda^{-1}) \in D_k \). Therefore,
\[
D_k = \{ y : |y| \leq d \} \cap (\lambda^{-1} D_{k-1} + (B - B)).
\] (4.4)

We have the following recursive algorithm:
(1) Let \( D_0 = \{ y : |y| \leq d \} \cap (B - B) \);
(2) For \( k \geq 1 \), let
\[
D_k = \{ y : |y| \leq d \} \cap (\lambda^{-1} D_{k-1} + (B - B)),
\]
or
\[
D_k = \{ y : |y| \leq d \} \cap (\lambda^{-1} (D_{k-1} \setminus D_{k-2}) + (B - B));
\] (4.5)
(3) If we find a smallest \( k_0 \) such that \( D_{k_0+1} = D_{k_0} \), then let \( D = D_{k_0} \);
(4) Let \( C = \lambda^{T-2} D \).

Step 3: Draw all edges in graph
We obtain \( C \) in step 2, then we have the vertex set
\[
\Xi = ([T + 1, T - 1] \cap \mathbb{Z}) \times C.
\] (4.6)
We draw a directed edge \( e \) from \((p, b) \in \Xi \) to \((p', b') \in \Xi \), if there are words \( \alpha, \beta \) with \( 1 \leq p_\alpha, p_\beta \leq 2T - 1 \) such that
\[
(p', b') = (p_\alpha + p - p_\beta, \lambda^{-p_\alpha + p_\beta} b_\alpha + \lambda^{-p_\beta} b - \lambda^{-p_\beta} b_\beta).
\]
Let
\[
\Lambda = \Xi \cap \{ (p_j - p_i, \lambda^{-p_\beta} b_j - \lambda^{-p_\beta} b_i) : i \neq j \}.
\] (4.7)

Step 4: Test the existence of corresponding paths
Using the methods stated in Lemma 4, we can test the existence of paths.
Then we solve the problems, since the SSC fails if and only if we find an infinite path starting from a point of $\Lambda$, and the OSC fails if and only if there is a finite path starting at a point of $\Lambda$ and ending at $(0,0)$.

4.2. The case that $T = 1$.

Suppose $T = 1$, i.e., $p_1 = \cdots = p_m = 1$. Then
\[ F = \{ \sigma : 1 \leq p_\sigma \leq 2T - 1 \} = \{ 1, \cdots, m \}. \]

Since $\Lambda = \Xi \cap \{ (0, \lambda^{-1}(b_j - b_i) : i \neq j) \}$ and
\[ \pi_2((1, b_1)^{-1}(0, c)(1, b_j)) = 0, \]
we conclude that any vertex $(q, c)$ in a directed path starting from a point of $\Lambda$ must have
\[ q = 0. \tag{4.8} \]

Since $p_{\sigma_k} = p_{r_k} = k$ in (3.1), as in (3.2) we have
\[ |\pi_k[(p_{\tau_k}, b_{\tau_k})^{-1}(p_{\sigma_k}, b_{\sigma_k})]| = |x_1 - x_2| \leq \frac{\max_{b \in B - \bar{B}} |b|}{1 - \lambda}, \]
where $x_1, x_2 \in E$. Then we can restrict the vertices in
\[ \{0\} \times C', \tag{4.9} \]
where
\[ C' = \{ x : |x| \leq \frac{\max_{b \in B - \bar{B}} |b|}{1 - \lambda} \} \text{ and } x = \lambda^{-1}R(\lambda^{-1}) \text{ with } R \in \mathcal{Z} - \mathcal{Z}. \tag{4.10} \]

4.3. Running time of algorithms.

Using Lemma 3 and $(1 + \lambda^{-T+1}) \leq 2\lambda^{-T+1}$, we have
\[ \#C \leq \frac{2^l(1 + \lambda^{-T+1})(\max_i |b_i|)^{l+1}}{\lambda^{T-2}(1 - \lambda) \prod_{i=1}^l (1 - |\eta_i|)} \leq a_\lambda b_T c_B, \tag{4.11} \]
where
\[ a_\lambda = 2^{l+1}(1 - \lambda)^{-1} \prod_{i=1}^l (1 - |\eta_i|)^{-1}, \quad b_T = \lambda^{-2T+3} \text{ and } c_B = (\max_i |b_i|)^{l+1}. \]

Notice that
\[ \#\Xi = (2T - 1)\#C \leq a_\lambda(2T - 1)b_T c_B. \]

(1) Time for calculating $C$:
According to Step 3, we need running time
\[ t_1 \leq \sum_{i=0}^{k_0} \#(B - B) \cdot \#(D_k \setminus D_{k-1}) \leq \#(B - B) \cdot \#C \]
i.e.,
\[ t_1 \leq (m + 1)^2 \cdot \#C. \]

(2) Time for finding all edges:
According to Step 4, we need to calculate
\[ (p_\alpha + p - p_\beta, \lambda^{-p_\alpha + p_\beta}b_\alpha + \lambda^{-p_\beta}b_\beta) \]
for all $(p, b) \in \Xi$ and words $\alpha, \beta \in F = \{ \sigma : 1 \leq p_\sigma \leq 2T - 1 \}$ with $\#F \leq m + m^2 + \cdots + m^{2T-1} < m^{2T}$. Then running time
\[ t_2 \leq \#\Xi \cdot (\#F)^2 \leq m^{4T}(\#\Xi). \]
If \( (#F) \) is large, we need more time to calculate.

(3) Time for finding corresponding paths:

By Lemma 4 when the graph has been constructed, using Dijkstra’s algorithm we can test the non-existences of the corresponding paths w.r.t. the OSC and the SSC with running time

\[
t_3 \leq C(#\Xi)^2 \quad \text{and} \quad t_3' \leq C(#\Xi)^3\]

respectively, where \( C \) is the constant related to the original Dijkstra’s algorithm.

Therefore we have

\[
t_1 + t_2 + t_3 \leq (m + 1)^2 C + m^{(4)(#\Xi)} + C(#\Xi)^2,\]

\[
t_1 + t_2 + t_3' \leq (m + 1)^2 C + m^{(4)(#\Xi)} + C(#\Xi)^3,
\]

where \((#\Xi) = (2T - 1)C\) and \(#C \leq \alpha_b c_B\).

**Remark 9.** If \( T = 1, \#C \) in (4.11) can be replaced by \( \#C' \) in (4.9), and

\[
#C' \leq 2\alpha c_B' + 1
\]

where \( \alpha = (1 - \lambda)^{-1} \) and \( c_B' = \max_{b \in B-B} |b| \).

**Remark 10.** When \( \lambda^{-1} \) is an integer, by (4.11) and (4.9), the cardinality of \( C \) satisfies that

\[
#C = #D_k \leq 2d + 1 = 2bTc_B + 1
\]

where \( bT = (\lambda^{-2} + \lambda^{-3} - 2\lambda^{-1})(1 - \lambda)^{-1} \) and \( c_B = \max_i |b_i| \).

5. Refinement Algorithm on OSC

5.1. Recursive structure.

Suppose

\[
S_{\tau} = S_{\tau}
\]

with \( \sigma = i_1 \cdots \in \Sigma^* \) and \( \tau = j_1 \cdots \in \Sigma^* \). Without loss of generality, we assume that \( p_{i_1} \geq p_{j_1} \).

We will use \( \sigma, \alpha, \beta \) with different meaning as above, but they have the same structure. We say that two numbers \( a, b \) have different signs if \( ab \leq 0 \).

Let \( \sigma_1 = \alpha_0 = i_1 \) and \( \tau_1 = \beta_0 = j_1 \). Take \( \beta_1 \) the shortest prefix of \( \tau \setminus \beta_0 \) such that

\[
p_{\sigma_1} - p_{\tau_1}, p_{\sigma_1} - p_{\tau_1} \beta_1
\]

have different signs. Let \( \tau_2 = \tau_1 \beta_1 \). Take \( \alpha_1 \) the shortest prefix of \( \sigma \setminus \alpha_0 \) such that

\[
p_{\sigma_1} - p_{\tau_2}, p_{\sigma_1} \alpha_1 - p_{\tau_2}
\]

have different signs and let \( \sigma_2 = \sigma_1 \alpha_1 \). Inductively, we have

\[
p_{\sigma_1} - p_{\tau_1}, p_{\sigma_1} - p_{\tau_2}, p_{\sigma_2} - p_{\tau_2}, \cdots, p_{\sigma_k} - p_{\tau_k}, p_{\sigma_k} - p_{\tau_k}, \cdots
\]

have alternative signs, where

\[
p_{\sigma_k} - p_{\tau_k} \geq 0. \quad (5.1)
\]

There exists an integer \( M \) such that \( \sigma_M = \sigma \). We will distinguish two following cases \( \tau_M = \tau \) or \( \tau_M \neq \tau \). For the latter, we will replace \( \tau_M \) by \( \tau \). Since \( p_{\tau} = p_{\tau} \),

\[
p_{\sigma_1} - p_{\tau_1}, \cdots, p_{\sigma_{k-1}} - p_{\tau_{k-1}}, p_{\sigma_k} - p_{\tau_k}, \cdots, p_{\sigma_{M-1}} - p_{\tau_{M-1}}, p_{\sigma} - p_{\tau}
\]

have alternative signs. Notice that

\[
\sigma = \alpha_0 \cdots \alpha_M, \sigma_k = \alpha_0 \cdots \alpha_{k-1} \quad \text{and} \quad \tau = \beta_0 \cdots \beta_M, \tau_k = \beta_0 \cdots \beta_{k-1},
\]
and

\[ 1 \leq p_{\alpha_k}p_{\beta_k} \leq 2T - 1 \quad \text{and} \quad 0 \leq p_{\sigma_k} - p_{\tau_k} \leq T - 1. \quad (5.2) \]

Suppose \( \{(q_k, c_k)\}_k \) has recursive structure w.r.t. \( \{(\alpha k)_{k=0}^M, \{\beta k\}_{k=0}^M\} \). Then

\[ 0 \leq q_k \leq T - 1 \quad \text{and} \quad c_k = \lambda^{T-2} R(\lambda^{-1}) \quad \text{with} \quad R \in \mathbb{Z} - \mathbb{Z} \]
due to Lemma 11 and (5.2). Notice that \( (q_{k+1}, c_{k+1}) = (p_{\beta_k}, b_{\beta_k})^{-1}(q_k, c_k)(p_{\alpha_k}, b_{\alpha_k}) \)
where

\[ q_k, q_k - p_{\beta_k}, q_{k+1} = q_k - p_{\beta_k} + p_{\alpha_k} \]
have alternative signs. Let

\[ d^* = \frac{2}{1 - \lambda} \left( \max_i |b_i| \right). \]

**Lemma 5.** Suppose \( (q', c') = (p_{\beta}, b_{\beta})^{-1}(q, c)(p_{\alpha}, b_{\alpha}) \) with \( q \geq 0 \). If \( |c| \geq d^* \), then \( |c'| \geq |c| \).

**Proof.** Suppose \( a > 1 \) and \( |b| \leq t \). If \( |x| \geq \frac{t}{a-1} \), then

\[ |ax + b| \geq a|x| - t \geq |x|. \quad (5.3) \]

Note that

\[ c' = \lambda^{-p} a + \lambda^{-p} b \]

with \( |b_{\alpha}|, |b_{\beta}| \leq \max_i |b_i| \). Applying \( a = \lambda^{-p}\), \( b = \lambda^{-p} (\lambda^q b_{\alpha} - b_{\beta}) \) and \( t = 2\lambda^{-p} \max_i |b_i| \), we notice that

\[ \text{if} \quad |c| \geq \frac{2\lambda^{-p}}{\lambda^{-p} - 1} \left( \max_i |b_i| \right), \quad \text{then} \quad |c'| \geq |c|. \]

Since the function \( \frac{2\lambda^{-p}}{\lambda^{-p} - 1} \) is decreasing and \( p_{\beta} \geq 1 \), we take \( d^* \) for \( p_{\beta} = 1 \) and complete the proof. \( \square \)

Notice that \( \text{id} = h_{(0, 0)} \), we have the following corollary.

**Corollary 4.** Suppose \( S_{\sigma} = S_{r} \) and \( \{(q_k, c_k)\}_k \) has recursive structure w.r.t. \( \{(\alpha k)_{k=0}^M, \{\beta k\}_{k=0}^M\} \) as above. Then \( |c_k| < d^* \) for all \( k \).

### 5.2. Graph and criterion.

Let

\[ \Theta = \{n \in \mathbb{Z} : 0 \leq n \leq T - 1\} \times \mathcal{A} \]

where

\[ \mathcal{A} = \{x : |x| < \frac{2 \max_i |b_i|}{(1 - \lambda)^2}, \quad \lambda^{-T+2} x = R(\lambda^{-1}) \quad \text{with} \quad R \in \mathbb{Z} - \mathbb{Z}\}. \]

**Remark 11.** When \( T \geq 3 \), \( (1 + \lambda^{-T+1}) \max_i |b_i| \geq \frac{2 \max_i |b_i|}{(1 - \lambda)^2} \). When \( T \) is large enough,

\[ (1 + \lambda^{-T+1}) \max_i |b_i| \approx \frac{2 \max_i |b_i|}{(1 - \lambda)^2}. \]

Then \( \# \mathcal{A} \) seems to be much less than \( \# \mathcal{C} \).
Given two vertices \((q, c), (q', c')\) ∈ \(\Theta\), we draw a directed edge from \((q, c)\) and \((q', c')\), if and only if there are words \(\alpha, \beta\) with \(1 \leq p_{\alpha}, p_{\beta} \leq 2T - 1\) such that
\[ (q', c') = (p_{\beta}, b_{\beta})^{-1}(q, c)(p_{\alpha}, b_{\alpha}) \]
and \(q, q - p_{\beta}, q'\) have alternative signs.

Then we obtain a graph \(G^*\) with vertex set \(\Theta\) and edge set defined above. Let
\[ \Psi = \{(p_i, b_i)^{-1}(p_j, b_j) : p_j \geq p_i, \ i \neq j \ \text{and} \ \lambda^{-p_i}|b_j - b_i| < \frac{2\max_i|b_i|}{(1 - \lambda)^2}\} \]
Then \(\Psi \subset \Theta\). We have the following criterion.

**Theorem 4.** The OSC fails if and only if there is a directed path in \(G^*\) starting at a point of \(\Psi\) and ending at \((0, 0)\).

### 6. Examples

**Example 1.** Let \(\lambda = \frac{1}{3}\). For given \(n \in \mathbb{N}\), we consider the self-similar set \(E_{p, b}\) with
\[ p = (1, 1, 1), \ b = \left(0, \frac{2}{3n+1}, \frac{2}{3}\right) \]
Notice that \(\lambda^{-1} = 3\) is a P.V. number. By Remark 8, take \(a = 3^{n+1}\) and replace \((0, \frac{2}{3n+1}, \frac{2}{3})\) by \(b = (0, 2, 2 \cdot 3^n)\). Then
\[ B = \{0, 2, 2 \cdot 3^n\} \]
and
\[ B - B = \{-2 \cdot 3^n, 2(1 - 3^n), -2, 0, 2, 2(3^n - 1), 2 \cdot 3^n\} \]
Since \(T = 1\), by the discussion of Subsection 4.2 we have
\[ C' = \{m : m \ \text{is even and} \ m \in [-3^{n+1}, 3^{n+1}] \cap \mathbb{Z}\} \]
and \(C' \cap \{\lambda^{-1}(b_i - b_j)\}_{i \neq j} = \{-6, 6\}, \ i.e., \)
\[ \Lambda = \{(0, -6), (0, 6)\} \]
Considering the vertices \(\{0\} \times C'\) and using step 3 and step 4 of algorithm, we can obtain a path
\[ (0, 6) \rightarrow (0, 2 \cdot 3^2) \rightarrow (0, 2 \cdot 3^3) \rightarrow \cdots \rightarrow (0, 2 \cdot 3^n) \rightarrow (0, 0) \rightarrow (0, 0) \rightarrow (0, 0) \cdots \]
starting from \((0, 6)\). Then neither OSC nor SSC is fulfilled. And thus \(\dim_H E_{p, b} \neq \dim_S E_{p, b} = 1\) and \(\text{int}(E_{p, b}) = \emptyset\) due to Corollary 11 and 2.

**Example 2.** Let \(\lambda = \frac{1}{3}\). We consider the self-similar set \(E_{p, b}\) with
\[ p = (1, 2, 1), \ b = \left(0, \frac{11}{18}, \frac{2}{3}\right) \]
Take \(a = 18\) and replace \(\left(0, \frac{11}{18}, \frac{2}{3}\right)\) by \(b = (0, 11, 12)\). Then
\[ B = \{0, 11, 12\} \]
and
\[ B - B = \{-12, -11, -1, 0, 1, 11, 12\} \]
Notice that \(T = 2\). Then we have, by \([4.7]\),
\[ d = \lambda^{-T+2}(1 + \lambda^{-T+1})\max_{b \in B}|b| = 72, \]

Using step 2 of algorithm, we have
\[ C = [-72, 72] \cap \mathbb{Z}, \]
and
\[ \Xi = \{-1, 0, 1\} \times C. \]
By (4.7) we also have
\[ \Lambda = \{(-1, 9), (0, -36), (0, 36), (1, -3), (1, 33)\}. \]
Using step 3 and step 4 of algorithm, we find that there exists no a directed path starting from any element in \( \Lambda \) and ending at \((0, 0)\). Hence, \( E_p, b \) satisfies \( OSC \).

Example 3. Let \( \lambda = \sqrt{2} - 1 \). Then \( \lambda^{-1} = \sqrt{2} + 1 \) is a P.V. number called the silver ratio. We consider the self-similar set \( E_p, b \) with
\[ p = (1, 2, 1), \quad b = \left(0, \frac{2}{5}, \frac{1}{2}\right). \]
Take \( a = 10 \) and replace \( (0, \frac{2}{5}, \frac{1}{2}) \) by \( b = (0, 4, 5) \). Then
\[ B = \{0, 4, 5\} \]
and \( B - B = \{-5, -4, -1, 0, 1, 4, 5\} \).
Notice that \( T = 2 \). Then by (4.1) we have
\[ d = \lambda^{-T+2} \frac{(1 + \lambda^{-T+1}) \max_{b \in B} |b|}{1 - \lambda} = 15 + 10\sqrt{2}, \]
And we can obtain, using step 2 of algorithm, the set \( \Xi \) containing 1059 elements which will not be listed here. By (4.7) we also have
\[ \Lambda = \{(1, 4 + 4\sqrt{2}), (0, 5 + 5\sqrt{2}), (-1, -12 - 8\sqrt{2}), (-1, 3 + 2\sqrt{2}), (0, -5 - 5\sqrt{2}), (1, -1 - \sqrt{2})\}. \]
In virtue of Dijkstra’s method, using step 3 and step 4 of algorithm, we can obtain a path
\[ (-1, 3 + 2\sqrt{2}) \rightarrow (-1, -9 - 6\sqrt{2}) \rightarrow (-1, -6 - 5\sqrt{2}) \rightarrow (0, 0) \rightarrow (0, 0) \rightarrow (0, 0) \cdots, \]
starting from \( (-1, 3 + 2\sqrt{2}) \in \Lambda \), which implies that neither \( OSC \) nor \( SSC \) is fulfilled. Using \( \dim_S(E_p, b) = 1 \), we have \( \text{int}(E_p, b) = \emptyset \) due to Corollary 2.
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