Algebraic Structures on Valuations, Their Properties and Applications

Semyon Alesker

Abstract

We describe various structures of algebraic nature on the space of continuous valuations on convex sets, their properties (like versions of Poincaré duality and hard Lefschetz theorem), and their relations and applications to integral geometry.

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0. Introduction

The theory of continuous valuations on convex sets generalizes, in a sense, both the measure theory and the theory of the Euler characteristic. Roughly speaking one should think of a continuous valuation $\phi$ on a real linear space $V$ as a finite additive measure on a class of compact nice subsets of $V$ (say piecewise smooth submanifolds with corners) which satisfy the following additional property (instead of the usual sigma-additivity): the restriction of $\phi$ to the subclass of convex compact domains with smooth boundary extends by continuity to the class $\mathcal{K}(V)$ of all convex compact subsets of $V$. Here the continuity is understood in the sense of the Hausdorff metric on $\mathcal{K}(V)$. Remind that the Hausdorff metric $d_H$ on $\mathcal{K}(V)$ depends on the choice of the Euclidean metric on $V$ and it is defined as follows: $d_H(A, B) := \inf\{\varepsilon > 0 | A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\}$, where $(U)_\varepsilon$ denotes the $\varepsilon$-neighborhood of a set $U$. This condition of continuity turns out to be very strong restriction and has a lot of consequences on purely algebraic level. These properties will be discussed in this paper. The simplest examples of such valuations are any smooth measure on $V$ and the Euler characteristic. Also it turns out that one of the main tools used recently in investigations of valuations is the representation theory of real reductive groups and the Beilinson-Bernstein theory of $D$-modules.

Now let us give the formal definition of valuation.

*Department of Mathematics, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel. E-mail: semyon@post.tau.ac.il
0.1.1 Definition. a) A function \( \phi : \mathcal{K}(V) \to \mathbb{C} \) is called a valuation if for any \( K_1, K_2 \in \mathcal{K}(V) \) such that their union is also convex one has

\[
\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).
\]

b) A valuation \( \phi \) is called continuous if it is continuous with respect the Hausdorff metric on \( \mathcal{K}(V) \).

The linear space of all continuous valuations on \( V \) will be denoted by \( CVal(V) \). It is a Fréchet space with the topology of uniform convergence on compact subsets of \( \mathcal{K}(V) \). In Section 1 we discuss its dense subspace of polynomial smooth valuations (\( PVal(V) \))\( ^{sm} \) (it has the topology of inductive limit of Fréchet spaces). It turns out that this space has a natural structure of associative commutative unital algebra (when the unity is the Euler characteristic). In Section 2 we discuss the space \( Val(V) \) of translation invariant continuous valuations. Its dense subspace (\( Val(V) \))\( ^{sm} \) of so called smooth valuations is a subalgebra of (\( PVal(V) \))\( ^{sm} \). It has a natural grading and satisfies a version of Poincaré duality. This property follows from the Irreducibility Theorem 2.1.3 which is by itself key result in the investigation of valuations (see Subsection 2.1). Moreover even smooth translation invariant continuous valuations form a graded subalgebra of (\( Val(V) \))\( ^{sm} \) and satisfy a version of the hard Lefschetz theorem (Subsection 2.2). This property turns out to be closely related to the cosine transform problem in the (Gelfand style) integral geometry solved recently in [6]. These properties of valuations turn out to be useful to obtain new explicit classification results on valuations with additional invariance properties. The classical Hadwiger theorem describes explicitly \( SO(n) \)- and \( O(n) \)-invariant translation invariant continuous valuations on the Euclidean space \( \mathbb{R}^n \). The new result is the classification of unitarily invariant translation invariant continuous valuations on the Hermitian space \( \mathbb{C}^n \) (Subsection 2.3). The main application of the classification results on valuations is integral geometric formulas. Using our classification we obtain new results in (Chern style) integral geometry of real submanifolds of complex spaces (Section 3).

1. General continuous valuations

In order to study general continuous valuations let us remind the definition of polynomial valuation introduced by Khovanskii and Pukhlikov [14], [15].

1.1.1 Definition. A valuation \( \phi \) is called polynomial of degree \( d \) if for every \( K \in \mathcal{K}(V) \) the function \( x \mapsto \phi(K + x) \) is a polynomial on \( V \) of degree at most \( d \).

Note that valuations polynomial of degree 0 are called translation invariant valuations. Polynomial valuations have many nice combinatorial-algebraic properties ([14], [15]). Also in [11] the author have classified explicitly rotation invariant polynomial continuous valuations on a Euclidean space.

Let us denote the space of polynomial continuous valuations on \( V \) by \( PVal(V) \). One has
1.1.2 Proposition ([5]). The space $PVal(V)$ of polynomial continuous valuations is dense in the space of all continuous valuations $CVal(V)$.

The proof of this proposition is rather simple; it is a tricky use of a form of the Peter-Weyl theorem (for the orthogonal group $O(n)$), and in particular the convexity is not used in any essential way.

Let us remind the basic definition of a smooth vector for a representation of a Lie group. Let $\rho$ be a continuous representation of a Lie group $G$ in a Fréchet space $F$. A vector $\xi \in F$ is called $G$-smooth if the map $g \mapsto \rho(g)\xi$ is infinitely differentiable map from $G$ to $F$. It is well known the the subset $F^{sm}$ of smooth vectors is a $G$-invariant linear subspace dense in $F$. Moreover it has a natural topology of a Fréchet space (which is stronger than that induced from $F$), and the representation of $G$ is $F^{sm}$ is continuous.

We will especially be interested in polynomial valuations which are $GL(V)$-smooth. This space will be denoted by $(PVal(V))^{sm}$.

**Example.** Let $\mu$ be a measure on $V$ with a polynomial density with respect to the Lebesgue measure. Let $A \in K(V)$ be a strictly convex compact subset with smooth boundary. Then $\phi(K) := \mu(K + A)$ is a continuous polynomial smooth valuation (here $K + A := \{k+a|k \in K, a \in A\}$).

Let us denote by $G(V)$ the linear space of valuations on $V$ which are finite linear combinations of valuations from the previous example. It can be shown (using Irreducibility Theorem 2.1.3) that $G(V)$ is dense in $(PVal(V))^{sm}$. Let $W$ be another linear real vector space. Let us define the exterior product $\phi \boxtimes \psi \in G(V \times W)$ of two valuations $\phi, \psi \in G(V)$. Let $\phi(K) = \sum_i \mu_i(K + A_i), \psi(L) = \sum_j \nu_j(L + A_j)$. Define

$$(\phi \boxtimes \psi)(M) := \sum_{i,j} (\mu_i \boxtimes \nu_j)(M + (A_i \times \{0\}) + (\{0\} \times B_j)),$$

where $\mu_i \boxtimes \nu_j$ denotes the usual product measure.

1.1.3 Proposition ([5]). For $\phi \in G(V), \psi \in G(W)$ their exterior product $\phi \boxtimes \psi \in G(V \times W)$ is well defined; it is bilinear with respect to each argument. Moreover

$$(\phi \boxtimes \psi) \boxtimes \eta = \phi \boxtimes (\psi \boxtimes \eta).$$

Now let us define a product on $G(V)$. Let $\Delta : V \leftrightarrow V \times V$ denote the diagonal imbedding. For $\phi, \psi \in G(V)$ let

$$\phi \cdot \psi := \Delta^\ast(\phi \boxtimes \psi),$$

where $\Delta^\ast$ denotes the restriction of a valuation on $V \times V$ to the diagonal.

1.1.4 Proposition ([5]). The above defined multiplication uniquely extends by continuity to $(PVal(V))^{sm}$. Then $(PVal(V))^{sm}$ becomes an associative commutative unital algebra where the unit is the Euler characteristic $\chi$. 
2. Translation invariant continuous valuations

For a linear finite dimensional real vector space \( V \) let us denote by \( \text{Val}(V) \) the space of translation invariant continuous valuations on \( V \). This is a Fréchet space with respect to the topology of uniform convergence on compact subsets of \( \mathcal{K}(V) \). In this section we will discuss properties of this space.

2.1. Irreducibility theorem and Poincaré duality

It was shown by P. McMullen [17] that the space \( \text{Val}(V) \) of translation invariant continuous valuations on \( V \) has a natural grading given by the degree of homogeneity of valuations. Let us formulate this more precisely.

2.1.1 Definition. A valuation \( \phi \) is called homogeneous of degree \( k \) if for every convex compact set \( K \) and for every scalar \( \lambda > 0 \)

\[
\phi(\lambda K) = \lambda^k \phi(K).
\]

Let us denote by \( \text{Val}_k(V) \) the space of translation invariant continuous valuations homogeneous of degree \( k \).

2.1.2 Theorem (McMullen [17]).

\[
\text{Val}(V) = \bigoplus_{k=0}^{n} \text{Val}_k(V),
\]

where \( n = \dim V \).

Note in particular that the degree of homogeneity is an integer between 0 and \( n = \dim V \). It is known that \( \text{Val}_0(V) \) is one-dimensional and is spanned by the Euler characteristic \( \chi \), and \( \text{Val}_n(V) \) is also one-dimensional and is spanned by a Lebesgue measure [10]. The space \( \text{Val}_n(V) \) is also denoted by \( \text{Dens}(V) \) (the space of densities on \( V \)). One has further decomposition with respect to parity:

\[
\text{Val}_k(V) = \text{Val}^{ev}_k(V) \oplus \text{Val}^{odd}_k(V),
\]

where \( \text{Val}^{ev}_k(V) \) is the subspace of even valuations (\( \phi \) is called even if \( \phi(-K) = \phi(K) \) for every \( K \in \mathcal{K}(V) \)), and \( \text{Val}^{odd}_k(V) \) is the subspace of odd valuations (\( \phi \) is called odd if \( \phi(-K) = -\phi(K) \) for every \( K \in \mathcal{K}(V) \)). The Irreducibility Theorem is as follows.

2.1.3 Theorem ([3], [2]). The natural representation of the group \( \text{GL}(V) \) on each space \( \text{Val}_k^{ev}(V) \) and \( \text{Val}_k^{odd}(V) \) is irreducible.

This theorem is the main tool in further investigations of valuations and classification of them (see Subsection 2.3). This immediately implies so called McMullen’s conjecture [18]. Its proof is heavily based on the use of the representation theory of real reductive groups and the Beilinson-Bernstein theory of D-modules. Another key tool in the proof of this result is the Klain-Schneider characterization of simple translation invariant continuous valuations [12], [20].
By the results of Section 1 \((Val(V))^{sm}\) is a subalgebra of \((PVal(V))^{sm}\). It is easy to see that the algebra structure is compatible with the grading, namely
\[
(Val_i(V))^{sm} \otimes (Val_j(V))^{sm} \rightarrow (Val_{i+j}(V))^{sm}.
\]
In particular we have
\[
(Val_i(V))^{sm} \otimes (Val_{n-i}(V))^{sm} \rightarrow Dense(V).
\]
A version of the Poincaré duality theorem says that this is a perfect pairing. More precisely

\begin{theorem} \cite{5} \end{theorem}

The induced map
\[
(Val_i(V))^{sm} \rightarrow (Val_{n-i}(V)^{*})^{sm} \otimes Dense(V)
\]
is an isomorphism.

2.2. Even translation invariant continuous valuations

Let us denote by \(Val^{ev}(V)\) the subspace of even translation invariant continuous valuations. Then clearly \((Val^{ev}(V))^{sm}\) is a subalgebra of \((Val(V))^{sm}\). It turns out that it satisfies a version of the hard Lefschetz theorem which we are going to describe.

Let us fix on \(V\) a scalar product. Let \(D\) denote the unit ball with respect to this product. Let us define an operator \(\Lambda : Val(V) \rightarrow Val(V)\). For a valuation \(\phi \in Val(V)\) set
\[
(\Lambda \phi)(K) := \lim_{\varepsilon \rightarrow 0} \phi(K + \varepsilon D).
\]
(Note that by a result of P. McMullen \cite{17} \(\phi(K + \varepsilon D)\) is a polynomial in \(\varepsilon > 0\) of degree at most \(n\).) It is easy to see that \(\Lambda\) preserves the parity of valuations and decreases the degree of homogeneity by 1. In particular
\[
\Lambda : Val^{ev}_k(V) \rightarrow Val^{ev}_{k-1}(V).
\]
The following result is a version of the hard Lefschetz theorem.

\begin{theorem} \cite{4} \end{theorem}

Let \(k > n/2\). Then
\[
\Lambda^{2k-n} : (Val^{ev}_k(V))^{sm} \rightarrow (Val^{ev}_{n-k}(V))^{sm}
\]
is an isomorphism. In particular for \(1 \leq i \leq 2k - n\) the map
\[
\Lambda^i : (Val^{ev}_k(V))^{sm} \rightarrow (Val^{ev}_{k-i}(V))^{sm}
\]
is injective.

Note that the proof of this result is based on the solution of the cosine transform problem due to J. Bernstein and the author \cite{6}, which is the problem from (Gelfand style) integral geometry motivated by stochastic geometry and going back to G. Matheron \cite{16}.
2.3. Valuations invariant under a group

Let $G$ be a subgroup of $GL(V)$. Let us denote by $Val^G(V)$ the space of $G$-invariant translations invariant continuous valuations. From the results of [2] and [4] follows the following result.

2.3.1 Theorem. Let $G$ be a compact subgroup of $GL(V)$ acting transitively on the unit sphere. Then $Val^G(V)$ is a finite dimensional graded subalgebra of $(Val(V))^m$. It satisfies the Poincaré duality, and if $-Id \in G$ it satisfies the hard Lefschetz theorem.

It turns out that $Val^G(V)$ can be described explicitly (as a vector space) for $G = SO(n), O(n)$, and $U(n)$. In the first two cases it is the classical theorem of Hadwiger [10], the last case is new (see [4]). In order to state these results we have to introduce first sufficiently many examples.

Let $\Omega$ be a compact domain in a Euclidean space $V$ with a smooth boundary $\partial \Omega$. Let $n = \dim V$. For any point $s \in \partial \Omega$ let $k_1(s), \ldots, k_{n-1}(s)$ denote the principal curvatures at $s$. For $0 \leq i \leq n-1$ define

$$V_i(\Omega) := \frac{1}{n} \left( \frac{n - 1}{n - 1 - i} \right)^{-1} \int_{\partial \Omega} \left\{ k_{j_1}, \ldots, k_{j_{n-1-i}} \right\} d\sigma,$$

where $\left\{ k_{j_1}, \ldots, k_{j_{n-1-i}} \right\}$ denotes the $(n-1-i)$-th elementary symmetric polynomial in the principal curvatures, $d\sigma$ is the measure induced on $\partial \Omega$ by the Euclidean structure. It is well known that $V_i$ (uniquely) extends by continuity in the Hausdorff metric to $K(V)$. Define also $V_0(\Omega) := \text{vol}(\Omega)$. Note that $V_0$ is proportional to the Euler characteristic $\chi$. It is well known that $V_0, V_1, \ldots, V_n$ belong to $Val^{O(n)}(V)$.

It is easy to see that $V_k$ is homogeneous of degree $k$. The famous result of Hadwiger says

2.3.2 Theorem (Hadwiger, [10]). Let $V$ be $n$-dimensional Euclidean space. The valuations $V_0, V_1, \ldots, V_n$ form a basis of $Val^{SO(n)}(V) (= Val^{O(n)}(V))$.

Now let us describe unitarily invariant valuations on a Hermitian space. Let $W$ be a Hermitian space, i.e. a complex vector space equipped with a Hermitian scalar product. Let $m := \dim C W$ (thus $\dim R W = 2m$). For every non-negative integers $p$ and $k$ such that $2p \leq k \leq 2m$ let us introduce the following valuations:

$$U_{k,p}(K) = \int_{E \in AGr_{m-p}} V_{k-2p}(K \cap E) \cdot dE.$$

Then $U_{k,p} \in Val^{U(m)}_k(W)$.

2.3.3 Theorem ([4]). Let $W$ be a Hermitian vector space of complex dimension $m$. The valuations $U_{k,p}$ with $0 \leq p \leq \frac{\min\{k,2m-k\}}{2}$ form a basis of the space $Val^{U(m)}_k(W)$.

It turns out that the proof of this theorem is highly indirect, and it uses everything known about even translation invariant continuous valuations including
the solution of McMullen’s conjecture, cosine transform, hard Lefschetz theorem for valuations, and also results of Howe and Lee [11] on the structure of certain $GL_n(\mathbb{R})$-modules. Namely in order to describe explicitly the (finite dimensional) space of unitarily invariant valuations it is necessary to study the (infinite dimensional) $GL_{\mathbb{R}}(W)$-module $Val^\text{ev}(W)$.

Note that as algebra $Val^{SO(n)}(V)$ is isomorphic to $\mathbb{F}[x]/(x^{n+1})$. The algebra structure of $Val^{U(m)}(W)$ is not yet computed.

3. Applications to integral geometry

In this section we state new results from (Chern style) integral geometry of Hermitian spaces. They are obtained by the author in [4] using the classification of unitarily invariant valuations described in Subsection 2.3 of this paper. They can be considered as a generalization of the classical kinematic formulas due to Chern, Crofton, Santaló, and others (see e.g. [7], [8], [9], [13], [19]).

Let us remind first the principal kinematic formula following Chern [7]. Let $ISO(n)$ denote the group of affine isometries of the Euclidean space $\mathbb{R}^n$. Let $\Omega_1$, $\Omega_2$ be compact domains with smooth boundary in $\mathbb{R}^n$. Assume also that $\Omega_1 \cap U(\Omega_2)$ has finitely many components for all $U \in ISO(n)$.

3.1.1 Theorem ([7]).

$$\int_{U \in ISO(n)} \chi(\Omega_1 \cap U(\Omega_2))dU = \sum_{k=0}^{n} \kappa_k V_k(\Omega_1)V_{n-k}(\Omega_2),$$

where $\kappa_k$ are constants depending on $k$ and $n$ only which can be written down explicitly.

For the explicit form of the constants $\kappa_k$ we refer to [7] or [19], Ch.15 §4.

Let us return back to the Hermitian situation. Let $IU(m)$ denote the group of affine isometries of the Hermitian space $\mathbb{F}^m$ preserving the complex structure (then $IU(m)$ is isomorphic to $\mathbb{F}^m \rtimes U(m)$). Let $\Omega_1$, $\Omega_2$ be compact domains with smooth boundary in $\mathbb{F}^m$ such that $\Omega_1 \cap U(\Omega_2)$ has finitely many components for all $U \in IU(m)$. The new result is

3.1.2 Theorem ([4]).

$$\int_{U \in IU(m)} \chi(\Omega_1 \cap U(\Omega_2))dU = \sum_{k_1+k_2=2m} \sum_{p_1,p_2} \kappa(k_1,k_2,p_1,p_2)U_{k_1,p_1}(\Omega_1)U_{k_2,p_2}(\Omega_2),$$

where the inner sum runs over $0 \leq p_i \leq k_i/2$, $i = 1,2$, and $\kappa(k_1,k_2,p_1,p_2)$ are certain constants depending on $m,k_1,k_2,p_1,p_2$ only.

Remark. We could compute explicitly the constants $\kappa(k_1,k_2,p_1,p_2)$ only in $\mathbb{F}^{2^2}$.

For more integral geometric formulas of this and other type for real domains in $\mathbb{F}^m$ we refer to [4].
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