Efficient photon sorter in a high-dimensional Hilbert space

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An increase in the dimension of Hilbert space for quantum key distribution (QKD) can decrease its fidelity requirements while also increasing its bandwidth. A significant obstacle for QKD with qudits ($d \geq 3$) has been an efficient and practical quantum state sorter for photons with complex wavefronts. We propose such a sorter based on a multiplexed thick hologram constructed from photo-thermal refractive glass. We validate this approach using coupled-mode theory to simulate a holographic sorter for states spanned by three planewaves. The utility of such a sorter for broader quantum information processing applications can be substantial.

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We concern ourselves here with the secure distribution of a one-time only key from a sender (Alice) to a receiver (Bob). The three elements of any quantum key distribution (QKD) system are: (1) Alice must be able to prepare at will a single photon state chosen from a set of mutually-unbiased bases (MUB) $1, 2$, (2) each of these quantum amplitudes must be propagated from Alice to Bob with reasonable fidelity, and finally (3) Bob must have the ability to choose between one or another of the MUBs and, if he is lucky, be able to efficiently sort and detect each of these photon states. This QKD scenario has been exhaustively studied in the literature and is replete with security proofs for numerous protocols $3, 4, 5$. These security proofs have been extended in many cases to higher-dimensional Hilbert spaces $6$, and all of the protocols have been or are currently being implemented successfully $7$.

Conventional realizations of QKD today involve transmitting heavily-attenuated laser pulses from Alice to Bob and encoding qubit information in each packet by utilizing the spin of the photon. This allows Alice and Bob, who are suitably authenticated, the possibility to establish and share an arbitrarily-secure one-time only key between them. Here they have access to a two-dimensional Hilbert space and can therefore form three MUBs each with two orthogonal polarization states. Such a six-state QKD scheme $8$ has limited bandwidth and optical fidelity constraints. These constraints can be ameliorated by extending the QKD to higher-dimensional Hilbert space $6$.

The potential of extending photon-based QKD to higher dimensions was made possible in 1992 when Allen et. al. $8$ showed that Laguerre-Gaussian light beams possessed a quantized orbital angular momentum (OAM) of $\hbar l$ per photon. This opened up an arbitrarily high dimensional quantum space to a single photon$10$. Following this discovery Mair et. al. $11, 12$ unequivocally demonstrated the quantum nature of photon OAM by showing that pairs of OAM photons can be entangled using parametric down conversion. Shortly thereafter, Molina-Terriza et. al. $13$ introduced a scheme to prepare photons in multidimensional vector states of OAM commencing OAM QKD. Recently a practical method has been demonstrated to produce arbitrary OAM MUB states using computer-generated holography with a single spatial light modulator (SLM) $14$.

While the advantage of OAM QKD lies in its ability to increase bandwidth while simultaneously tolerating a higher bit error rate (BER) $6$, two potential problems confront this approach. First, such transverse photon wave functions are more fragile in propagation than the photon’s spin $15, 16$, and the divergence of the states ($\propto \sqrt{l}$) may require larger apertures. Despite this, multi-conjugate adaptive optical communication channels may be able to ameliorate these problems $17, 18$. A second obstacle involves the efficient sorting of OAM MUB-state photons with small Fock-state quantum numbers, and the paper will address this particular problem. Currently, the only solution to this problem is the use if a cascaded Mach-Zehnder interferometric system $13, 20, 21$. Proposed systems of this kind have been demonstrated only for 4-dimensions and are simply not practical. Other approaches that use crossed thin diffraction gratings are not efficient enough to establish a secure key.

We focus here on the efficient sorting of single photons with arbitrary complex wavefronts. Ideally, what is needed is an OAM version of a polarizer, i.e. a single optical element, one per MUB basis, that can efficiently sort each of the qudit states in that basis while equally distributing every other qudit state. Thick holographic gratings fortunately produce high diffraction efficiency in the first order $22, 23, 24$. If several predominant diffracted orders are required, as is the case for sorting, several independent fringe structures can exist in the emulsion. Such multiplexed holograms have been used for multiple-beam splitters and recombiners $24$ and more recently for wide-angle beam steering $25$. In this paper we propose such a MUB-state sorter based on a multiplexed thick holographic element constructed from commercially available photo thermal refractive (PTR) glass $26$. Due to the
unique properties of PTR glass the grating’s thickness can approach several mm and be highly Bragg selective. There is evidence that such sorters can be highly efficient, > 95% \[25\]. Our simulations presented here and empirical data on thick Bragg gratings indicate that they may provide an adequate solution to this critical and long standing problem.

Before we describe our proposed thick holographic MUB sorter we will briefly focus our attention on twelve-state QKD and work in a 3-dimensional Hilbert space. This is one more dimension than that available to photon polarization states and serves to illustrate our approach. Nevertheless, our work is equally applicable to higher dimensional Hilbert spaces. Its limitations will require further investigation.

In 3-dimensions there are a maximum of four MUBs which we refer to here as \(MUB_1, MUB_2, MUB_3\) and \(MUB_4\). Each of these orthonormal bases contain three state vectors. If we identify \(|a\rangle, |b\rangle\) and \(|c\rangle\) as the orthonormal k-vectors of \(MUB_1\), then the other nine qudit states from the other three MUB bases are specific linear combinations of these \([26]\).

For our application we can freely choose as our \(MUB_1\) any three pure OAM states \(|a\rangle, |b\rangle\) and \(|c\rangle\) corresponding to an angular momentum, \(l_a = a\hbar\), \(l_b = b\hbar\), and \(l_c = c\hbar\) with integers \(a, b\) and \(c\) being the azimuthal quantum numbers. For the purpose of our calculations we can simplify our analysis and retain the physical content by quantizing in the space of linear momentum (k-QKD) rather than in angular momentum (OAM-QKD). As a result we can freely choose as \(MUB_1\) any three non co-linear planewaves. In this case, our three integer quantum numbers will be the number of waves of tilt of these planewaves with respect to the normal of the holographic emulsion of aperture \(D\). These waves correspond to a transverse linear momentum \(p_a = a\hbar k^2\), \(p_b = b\hbar k^2\) and \(p_c = c\hbar k^2\); respectively. Here, \(k^2 = k\lambda/D\) is the x-component of a plane wave of frequency with one wave of tilt \((\tau \sim \lambda/D)\). In the frame of the hologram and in units where the speed of light is unity, the components of the 4-momentum \((p = \{p^x, p^y, p^z\})\) of each of our three photons can be expressed in terms of their transverse momentum \(k^2\) and wavenumber \((k)\).

\[
\begin{align*}
 p_a &= \hbar k_a = a\hbar k^2 \{1, 1, 0, \sqrt{(k/ak^2)^2 - 1}\}, \\
 p_b &= \hbar k_b = b\hbar k^2 \{1, 1, 0, \sqrt{(k/bk^2)^2 - 1}\}, \\
 p_c &= \hbar k_c = c\hbar k^2 \{1, 1, 0, \sqrt{(k/ck^2)^2 - 1}\}.
\end{align*}
\]

In the remainder of this paper the transverse linear momentum wavenumbers represent our three quantum numbers for k-QKD. These three planewaves define our first MUB.

\[
MUB_1 = \{|a\rangle, |b\rangle, |c\rangle\}.
\]

Each of these states represents a transverse Fourier mode of a photon; they are orthogonal \((\langle ij | j' \rangle = \delta_{i,j'})\) and define our 3-dimensional Hilbert space. The other nine MUB states \([26]\) can be obtained from these by linear superposition and will represent wavefronts with both amplitude and phase variations.

For each MUB we consider a multiplexed thick holographic sorter, i.e. a triple-exposed grating structure formed by the incoherent superposition of three gratings within a single emulsion. Here, each grating is formed by the superposition of the respective MUB state and its own unique plane reference wave. In this paper we concentrate on the construction of the \(MUB_4\) sorter \([21]\), as the other three MUB sorters will be of similar design.

We construct the \(MUB_4\) sorting in three steps. First, we record the interference pattern of our first signal wave \(|a_1\rangle\) with a corresponding reference planewave, \(|r_1\rangle\) having \(r_1 \gg 1\) waves of tilt. After this initial recording is complete, we then record the interference pattern of our second signal state from \(MUB_4\), namely \(|b_4\rangle\) with a second reference planewave \(|r_2\rangle\). Finally, we record a third independent set of fringe patterns by interfering the qudit signal state \(|c_4\rangle\) with a third reference planewave \(|r_3\rangle\). This produces a triple-multiplexed hologram.

We show that the hologram described above faithfully represents a quantum projection operator for \(MUB_4\).

\[
\mathcal{P}_4 = |r_1\rangle\langle a_4| + |r_2\rangle\langle b_4| + |r_3\rangle\langle c_4|
\] (1)

Its operation on any one of the 12 MUB states should produce the desired result. In other words, if the hologram is illuminated by MUB state \(|a_4\rangle, |b_4\rangle\) or \(|c_4\rangle\) it should produce a planewave in state \(|r_1\rangle, |r_2\rangle\) or \(|r_3\rangle\), respectively. If it is illuminated by any of the other nine MUB states it should then produce an equally weighted
response into all three reference states, e.g.

\[ \langle r_i | P_4 | b_3 \rangle = \frac{1}{\sqrt{3}}, \quad \forall i \in \{1, 2, 3\} \]

The unique property of PTR glass with its bulk index of refraction, \( n_0 = 1.4865 \), and depth of modulation, \( \Delta n/n_0 = 336 \text{ ppm} \), place it squarely in the realms of scalar diffraction theory and coupled-mode (CM) theory. Furthermore, a PTR hologram can be thick \( L \sim D \sim 1 \text{ cm} \) with Bragg-plane periods, \( \Lambda \sim \lambda \). Consequently, their Bragg selectivity \( \sim \Lambda/L \) can approach the diffraction limit of one wave of tilt across its aperture \( [25] \). For the gratings mentioned above, the wavenumber \( \kappa \) is assumed to be independent of \( x, z \) and to the corresponding amplitude and phase factors of one of the twelve corresponding signal states shown in Table I. For example, signal state \( |c_4 \rangle \) one would set \( \{ S_i \} = 1/\sqrt{3} \{1, z^2, 1\} \). The wavenumber \( k(x, z) \) of Eq. 2 represents the three incoherently recorded gratings mentioned above,

\[ k = n(x, z)k_0 = \sum_{\beta} \left(1 + \frac{\Delta n}{n_0} (I_{R1} + I_{R2} + I_{R3}) \right), \]

where \( I_{R1} \) is the intensity modulation of the \( i \)th grating, e.g.

\[ I_{R3} = 2 - |e^{ik_3 \cdot r} + \frac{1}{\sqrt{3}} (e^{ik_\omega \cdot r} + ze^{ik_3 \cdot r} + e^{i k_\omega \cdot r})|^2. \]

TABLE I: The four MUBs in our 3-dimensional Hilbert space. Note that phase \( z = \exp(i 2\pi/3) \) is a cube root of unity, and we have suppressed the normalizing factor of \( 1/\sqrt{3} \) in each of the nine MUB states in the last three columns.

Assuming that the interaction between the diffracted orders is slow, we can neglect second order terms and arrive at the CM equations for the mode amplitudes,

\[
\begin{pmatrix}
R_1' \\
R_2' \\
R_3' \\
S'_a \\
S'_b \\
S'_c \\
\end{pmatrix} = i\kappa^2 \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\Sigma_a} & \frac{1}{\Sigma_b} & \frac{1}{\Sigma_c} & 0 & 0 & 0 \\
\frac{1}{\Sigma_a} & \frac{1}{\Sigma_b} & \frac{1}{\Sigma_c} & 0 & 0 & 0 \\
\frac{1}{\Sigma_a} & \frac{1}{\Sigma_b} & \frac{1}{\Sigma_c} & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
R_1 \\
R_2 \\
R_3 \\
S_a \\
S_b \\
S_c \\
\end{pmatrix},
\]

where \( \kappa^2 \equiv \left( \frac{\beta^2}{6(3 + \psi)} \right) \left( \frac{\Delta n}{n} \right) \), and \( \rho_i = k_i^2 \) and \( \sigma_j = k_j^2 \) are the \( z \)-components of the wave vectors for the reference and signal states, respectively. The solution of this equation for each of the twelve initial signal states (MUB states) are shown in Fig. 2 and faithfully reproduce the desired projection operator of Eq. 1. We independently examined the far-field pattern for such gratings using a finite difference time domain solution of Maxwell’s equations and observed that the CW assumptions were valid, i.e. only the primary modes were dominant, and there were no relevant polarization changes in the field.

While the analysis presented here suggests that a high efficiency single optical element sorter is feasible with commercially available materials and holographic recording techniques, further work is needed. First, we need to understand the quantum nature of such an element. Second, we need to examine its sensitivity to alignment (linear and rotational) and examine the wavelength scaling issues involved in recording OAM photons. Our first step will be to produce a 3-state k-QKD MUB1 sorter. Our goal is to test its performance using states generated by a single phase SLM [14]. If the MUB-state sorters described here can be produced, they should have far more utility in quantum information processing than just QKD, e.g. as an essential element in linear quantum computing [27].

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