Analytical design of optimal regulators for non-classic quality functionals in a degenerate formulation

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Abstract. The paper considers the problem of analytical design of optimal controllers in the most general formulation for nonlinear plants with affinely (linearly and additively) input states and/or controls. On the basis of a new approach to approximately optimal control synthesis, a generalized theory of analytical design of optimal controllers is being developed. The main theorems are formulated; new concrete forms of non-classical quality functionals are obtained.

Introduction.
For many years, the students of A.A. Krasovsky school developed highly efficient algorithmic support for the synthesis of optimal controls (for example, [1,2]) for complex (nonlinear and multidimensional) processes, focused, among other things, on the creation of promising integrated control complexes with advanced functions of adaptation, self-adjustment and self-organization [3-5]. A wide range of experimental studies and specific projects has confirmed, in general, high performance of the algorithms. At the same time, there still exist applied problems in which the named algorithms either have unnecessarily cumbersome form, or are effective in very narrow ranges of application conditions.

In articles [6-10] for the first time we rigorously substantiated the functional of the generalized work of A.A. Krasovsky and showed his role and significance in the development of the modern control theory. To develop the method associated with minimizing the functional of the generalized work, it is proposed to transform the original problem into a degenerate problem and search for a solution based on the choice of a suitable strategy. Necessary and sufficient conditions of optimality for a degenerate problem of analytical design are obtained, which make it possible to distinguish a class of objects without singularities while minimizing the functional of the generalized work. Illustrative examples of oscillatory process control are given.

The principle of minimum generalized work of A.A. Krasovsky in a degenerate formulation. The question of the existence of a support functional with the properties of the Bellman-Krotov function remains open due to the non-uniqueness of the solution of the Cauchy problem defining it. However, in the 60s A.A. Krasovsky proposed to use the isoperimetric condition, which has an energetic meaning, due to which the support functional acquires the properties of the Lyapunov function in its classical definition. This integral equality guarantees the uniqueness of the solution to the Cauchy problem regardless of the dimension of the system state vector and means the generalized operation of controls in the optimal system. The criterion extended in this way is called the generalized work functional (GWF). GWF is generally known as a non-classical quality functional [2].
For the optimization problem, we present the most general formulation of the principle of minimum generalized work. Remaining within the framework of the developed theory of approximately optimal nonlinear synthesis, we set ourselves the goal of finding out under what assumptions, instead of sufficient Bellman - Krotov optimality conditions, it is possible to use simpler sufficient conditions in the form of the first-order nonlinear equation - the Lyapunov equation

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f_1 + Q = 0, \quad V(t, x_{on}(t)) = V_i(x_{on}(t_x))
\]  

(1)

Here \( f_1(t, x_{on}), Q(t, x_{on}) \) are some vector and scalar functions, \( V(t, x_{on}) \) is the Lyapunov scalar function \( x_{on} = x_0 \). The solution of the Cauchy problem (1), due to the fact that the gradient \( \frac{\partial V}{\partial x} \) enters the Lyapunov equation linearly, always uniquely determines the optimal control vector and does not depend on the dimension of the equations describing the differential system. Let \( \tilde{I} \) be a support functional with the properties of the Lyapunov function. Then, to solve the original problem of approximately optimal synthesis, it is required to extend it to a support functional \( I \) with the properties of the Bellman - Krotov function. This can be done in a non-unique way, following the schemes of weak and strong improvement and an analogue of the scheme for solving problems of variational calculus [11, 12]:

1) for a differential dynamic programming scheme

\[
I = \tilde{I} + J_4(u_0), \quad J_4(u_0) > 0
\]

(2)

2) for an approximation scheme in the policy space

\[
I = \tilde{I} + J_5(x_0), \quad J_5(x_0) > 0
\]

(3)

3) for an analogue of the variational scheme

\[
I = \tilde{I} + J_4(u_0) + J_5(x_0),
\]

(4)

where \( J_4(u_0), J_5(x_0) \) are some positive definite functionals that have the meaning of isoperimetric conditions: \( J_4(u_0) = C_1, J_5(x_0) = C_2, C_1, C_2 \) are constants corresponding to the displacement of the zero of the functional.

**Definition 1.** A nonclassical quality functional is a functional of the form

\[
I = V(t_0, x(t_0), t_x, x(t_x)) + \int_{t_0}^{t_x} L(t, x(t), u(t), u_{on}(t)) dt, \quad I \in \mathbb{R}^l
\]

(5)

where \( Q \) is a given non-negative function, \( L \) is a given function of the indicated arguments such that

\[
f_0(t, x(t), u(t)) - [Q(t, x(t)) + L(t, x(t), u(t), u_{on}(t))] = \begin{cases} 0 & \text{if } \forall x = x_{on}(t), u = u_{on}(t) \\ > 0 & \text{if } \forall x \neq x_{on}(t), u \neq u_{on}(t) \end{cases}
\]

(6)

for an analogue of the general scheme for solving variational problems,

\[
f_0(t, x_{on}(t), u(t)) - [Q(t, x_{on}(t)) + L_1(t, u(t), u_{on}(t))] = \begin{cases} 0 & \text{if } \forall u = u_{on}(t) \\ > 0 & \text{if } \forall u \neq u_{on}(t) \end{cases}
\]

(7)

\[L = L_1(t, u(t), u_{on}(t))\]

for a differential dynamic programming circuit,

\[
f_0(t, x(t), u_{on}(t)) - [Q(t, x(t)) + L_2(t, x(t), x_{on}(t))] = \begin{cases} 0 & \text{if } \forall x = x_{on}(t) \\ > 0 & \text{if } \forall x \neq x_{on}(t) \end{cases}
\]

(8)

\[L = L_2(t, x(t), x_{on}(t))\]

for an approximation scheme in the policy space.
When conditions (6) - (8) are satisfied, it is always possible to select a support functional from the nonclassical functional (5), with the properties of the Lyapunov function
\[ I = \int_{t_0}^{t} Q(t, x(t)) dt, \quad \bar{I} \in \mathbb{R}^1, \]
(9)
satisfying the original formulation of the optimization problem for a specific form of representation of the right-hand sides of the differential relations. This functionality is called the main part of the GWF (5): \( I = \bar{I} \). Wherein \( V_1(t_0, x(t_0), t, x(t)) = V_1'(t_0, x(t_0)) = V_1'(x(t)). \)

According to the differential dynamic programming scheme, the main theorem of the principle of minimum generalized work corresponds to the following statement.

**Theorem 1.** Optimal control, minimizing GWF (5), in the local sense, with the control cost function \( L_1(t, u(t), u_{on}(t)) \) for a differential system
\[ \dot{x} = f_1(t, x) + \varphi_1(t, x)u \]
(10)
given that
\[ L_1(t, u(t), u_{on}(t)) - \pi_1^T(t, u_{on})(u) = \begin{cases} 0 & \text{if } u = u_{on}(t) \\ > 0 & \text{if } u \neq u_{on}(t) \end{cases} \]
(11)
defined by the expression
\[ \pi_1(t, u_{on}) = -\varphi_1^T(t, x_{on}) \frac{\partial V^*(t, x_{on})}{\partial x} \]
(12)
for any admissible vector \( u \), including for \( u = u_{on} \). Here the function \( V(t, x_{on}) \) is the solution to the Lyapunov equation (1).

Thus, we have the following construction of the nonclassical quality functional:
\[ I = V_1(x(t)) + \int_{t_0}^{t} [Q(t, x(t)) + \pi_1^T(t, u_{on}(t))u(t)] dt, \]
(13)
which, in the absence of constraints on the controls \( U = \mathbb{R}^m \) in the differential relation (10), corresponds to the degenerate formulation of the optimal control synthesis problem.

**Conclusion.**
Thus, the degeneracy of the statement of the synthesis problem according to the differential dynamic programming scheme is manifested in the fact that the control \( u \) enters the functional \( I \) and the differential relation linearly, and the stationarity condition is satisfied trivially (\( u_{on} \) does not depend on the vector function \( u \)), which makes it possible to choose a suitable control strategy when formulating a problem. We called the functional (13) the criterion of weighted generalized work [6]. Similar statements are formulated for the approximation scheme in the policy space and the analogue of the variational scheme.

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