MODULAR SUBVARIETIES AND BIRATIONAL GEOMETRY OF 
SU_C(r).

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Abstract. Let C be an algebraic smooth complex genus g > 1 curve. The object of this paper is the study of the birational structure of the coarse moduli space UC(r,0) of semi-stable rank r vector bundles on C with degree 0 determinant and of its moduli subspace SU_C(r) given by the vector bundles with trivial determinant. Notably we prove that UC(r,0) (resp. SU_C(r)) is birational to a fibration over the symmetric product C^{(rg)} (resp. over \( \mathbb{P}^{(r-1)g} \)) whose fibres are GIT quotients \( (\mathbb{P}^{r-1})^g // PGL(r) \). In the cases of low rank and genus our construction produces families of classical modular varieties contained in the Coble hypersurfaces.

1. Introduction

Let C be a genus g > 1 smooth complex algebraic curve, if g \( \neq \) 2 we will also assume that C is non-hyperelliptic. Let UC(r,0) be the moduli space of rank r semi-stable vector bundles on C with degree zero determinant and let us denote as usual SU_C(r) the moduli subspace given by vector bundles with trivial determinant. These moduli spaces appeared first in the second half of the last century thanks to the foundational works of Narashiman-Ramanan [NR69a] and Mumford-Newstead [MN68] and very often their study has gone along the study of the famous theta-map

\[
\theta : SU_C(r) \rightarrow |r\Theta|; \\
E \mapsto \Theta_E := \{L \in \text{Pic}^{g-1} | h^0(C, E \otimes L) \neq 0 \}.
\]

While we know quite a good deal about \( \theta \) for low genera and ranks, as the rank or the genus grow our knowledge decreases dramatically, see Sect. 2 for a complete picture of known results. The question of rationality is even more haunting. When rank and degree are coprime the situation is quite settled [KS99] but when the degree is zero (or degree and rank are not coprime) the open problems are still quite numerous. It is known that all the spaces SU_C(r) are unirational but the rationality is clear only for \( r = 2, g = 2 \), when in fact the moduli space is isomorphic to \( \mathbb{P}^3 \), [NR69a]. Some first good ideas about the birational structure for \( g = 2 \) were developed in [Ang04]. Then the \( r = 2 \) case was analyzed in any genus by the first named author and A.Alzati in [AB09] with the help of polynomial maps classifying extensions in the spirit of [Ber92]. In this paper we give a description for the higher rank cases.

**Theorem 1.1.** Let C be a smooth complex curve of genus g > 1, non-hyperelliptic if g > 2, then UC(r,0) (resp. SU_C(r)) is birational to a fibration over C^{(rg)} (resp. \( \mathbb{P}^{(r-1)g} // PGL(r) \)) whose fibers are GIT quotients \( (\mathbb{P}^{r-1})^g // PGL(r) \).

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In the case of $SU_C(r)$, Proposition 1.1 allows us to give a more precise explicit description of the fibration of $SU_C(r)$ in the case $r = 3, g = 2$. In fact $SU_C(3)$ is a double covering of $\mathbb{P}^8$ branched along a hypersurface of degree six $C_6$ called the Coble-Dolgachev sextic [Las96]. Our result is the following.

**Theorem 1.2.** The Coble-Dolgachev sextic $C_6$ is birational to a fibration over $\mathbb{P}^4$ whose fibers are Igusa quartics.

We recall that an Igusa quartic is a modular quartic hypersurface in $\mathbb{P}^4$ that is related to some classical GIT quotients (see e.g. [DO88]) and moduli spaces. Its dual variety is a cubic 3-fold called Segre cubic, that is isomorphic to the GIT quotient $(\mathbb{P}^1)^6//PGL(2)$.

If $r = 2$ and $g = 3$, then $SU_C(2)$ is embedded by $\theta$ in $\mathbb{P}^7$ as a remarkable quartic hypersurface $C_4$ called the Coble quartic, [NR69a]. Our methods also allow us to give a quick proof of the following fact, already shown in [AB09] by means of polynomial maps.

**Proposition 1.3.** The Coble quartic $C_4$ is birational to a fibration over $\mathbb{P}^3$ whose fibers are Segre cubics.

We underline that the cases of $C_4$ and $C_6$ are particularly interesting because one can interpret the beautiful projective geometry of the Igusa quartic and the Segre cubic in terms of vector bundles on $C$ (see Sect. 6). We hope that these results could help to shed some new light on the question of rationality of $SU_C(r)$ and on the properties of the theta map.

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**Description of contents.**

In section 2 we collect a few results on the theta map. In section 3 we outline the relation between theta maps and theta-linear systems by introducing the theta divisor of a vector bundle with integral slope. In section 4 we describe some properties of the evaluation and the determinant map for a vector bundle of any rank. In Section 5 we prove Theorem 1.1 and discuss briefly the relation between slope stability and GIT stability. Finally, in Section 6 we apply the results to the cases $g = 2, r = 3$ and $g = 3, r = 2$ and give an explicit description of the fibration.

2. The Theta map.

Let $C$ be a smooth complex algebraic curve of genus $g \geq 2$, we assume that it is non-hyperelliptic if $g > 2$. Let $\text{Pic}^d(C)$ be the Picard variety parametrizing line bundles of degree $d$ on $C$, $\text{Pic}^0(C)$ will be often denoted as $J(C)$. Let $\Theta \subset \text{Pic}^{g-1}(C)$ be the canonical theta divisor

$\Theta := \{ L \in \text{Pic}^{g-1}(C) | h^0(C, L) \neq 0 \}$.

For $r \geq 2$, let $SU_C(r)$ denote the coarse moduli space of semi-stable vector bundles of rank $r$ and trivial determinant on $C$. It is a normal, projective variety
of dimension \((r^2 - 1)(g - 1)\). It is well known that \(\mathcal{SU}_C(r)\) is locally factorial and that \(\text{Pic}(\mathcal{SU}_C(r)) = \mathbb{Z}[\mathcal{DN89}],\) generated by a line bundle \(L\) called the determinant bundle. On the other hand, for \(E \in \mathcal{SU}_C(r)\) we define

\[
\Theta_E := \{L \in \text{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\}.
\]

This is either a divisor in the linear system \(|r\Theta|\) or the whole \(\text{Pic}^{g-1}(C)\). For \(E\) a general bundle \(\Theta_E\) is a divisor, the theta divisor of \(E\). This means that one can define the rational theta map of \(\mathcal{SU}_C(r)\):

\[
(2.1) \quad \theta : \mathcal{SU}_C(r) \rightarrow |r\Theta|
\]

sending \(E\) to its theta divisor \(\Theta_E\). The relation between the theta map and the determinant bundle is given by the following fundamental result:

**Theorem 2.1.** [BNR89] There is a canonical isomorphism \(|r\Theta| \sim |L|^*\) which identifies \(\theta\) with the rational map \(\varphi_{\mathcal{L}} : \mathcal{SU}_C(r) \rightarrow |L|^*\) associated to the determinant line bundle.

The cases when \(\theta\) is a morphism or finite are of course very appealing. Notably, \(\theta\) is an embedding for \(r = 2\) [NR69a], [BV96], [vGI01] and it is a morphism when \(r = 3\) for \(g \leq 3\) and for a general curve of genus \(g > 3\), [Bea06], [Ray82]. Finally, \(\theta\) is generically finite for \(g = 2\) [Bea06], [BV07] and we know its degree for \(r \leq 4\), [Las96], [Pau08]. There are also good descriptions of the image of \(\theta\) for \(r = 2, g = 2,3\) [NR69b], [Pau02], \(r = 3, g = 2\) [Ort05], [Ngu07], \(r = 2, g = 4\) [OP99]. Moreover, it has recently been shown in [BV09] that if \(C\) is general and \(g >> r\) then \(\theta\) is generically injective.

3. Vector bundles and theta linear systems

The notion of theta divisor can be extended to vector bundles with integral slope. Let \(\mathcal{U}_C(r, rg)\) be the moduli space of semi-stable vector bundles on \(C\) with rank \(r\) and degree \(rg\). The tensor product defines a natural map:

\[
t : \mathcal{SU}_C(r) \times \text{Pic}^g(C) \rightarrow \mathcal{U}_C(r, rg);
\]

\[
(E, \mathcal{O}_C(D)) \mapsto E \otimes \mathcal{O}_C(D),
\]

which is étale, Galois, with Galois group \(J(C)[r]\), the group of \(r\)-torsion points of the Jacobian of \(C\).

Moreover, if one restricts \(t\) to \(\mathcal{SU}_C(r) \times \mathcal{O}_C(D)\) this yields an isomorphism \(t_D : \mathcal{SU}_C(r) \rightarrow \mathcal{SU}_C(r, \mathcal{O}_C(rD))\), where the latter is the moduli space of rank \(r\) semi-stable vector bundles with determinant \(\mathcal{O}_C(rD)\).

**Definition 3.1.** Let \(F \in \mathcal{U}_C(r, rg)\), then we define the theta divisor of \(F\) as follows:

\[
\Theta_F := \{L \in \text{Pic}^4(C) \mid h^0(C, F \otimes L^{-1}) \neq 0\}.
\]

Let \(E \in \mathcal{SU}_C(r)\) and \(\mathcal{O}_C(D) \in \text{Pic}^g(C)\). If \(F = E \otimes \mathcal{O}_C(D)\), then we have that \(\Theta_F = \mathcal{O}_C(D) - \Theta_E\); thus \(\Theta_F\) is a divisor if and only \(\Theta_E\) is a divisor. We define

\[
(3.1) \quad \Theta_D : = \{L \in \text{Pic}^1(C) \mid h^0(\mathcal{O}_C(D) \otimes L^{-1}) \geq 1\}.
\]
Then, for any \( r \geq 1 \), we have a natural isomorphism \( \sigma_D : |r\Theta| \to |r\Theta_D| \) given by the translation \( M \mapsto \mathcal{O}_C(D) - M \). Moreover, if \( \mathcal{O}_C(rD_1) \simeq \mathcal{O}_C(rD_2) \), then \( |r\Theta_{D_1}| = |r\Theta_{D_2}| \) so we conclude that if \( F \in \mathcal{U}_C(r, rg) \) admits theta divisor, then \( \Theta_F \in |r\Theta_D| \), for any line bundle \( \mathcal{O}_C(D) \in \text{Pic}^g(C) \) which is a \( r \)-root of \( \det F \). In this way we obtain a family of theta linear systems over the Picard variety \( \text{Pic}^r g(C) \), as the following shows.

**Lemma 3.2.** There exists a projective bundle \( T \) over \( \text{Pic}^r g(C) \):

\[
p : T \to \text{Pic}^r g(C),
\]
whose fiber over \( \mathcal{O}_C(M) \in \text{Pic}^r g(C) \) is the linear system \( |r\Theta_D| \), where \( \mathcal{O}_C(D) \in \text{Pic}^g(C) \) is any \( r \)-root of \( \mathcal{O}_C(M) \).

**Proof.** Remark first that the linear system \( |r\Theta_D| \) is well-defined since it does not depend on which \( r \)-root \( \mathcal{O}_C(D) \) of \( \mathcal{O}_C(M) \) we choose. Let us now consider the tensor product map:

\[
\delta : \text{Pic}^g(C) \times \text{Pic}^1(C) \to \text{Pic}^{g-1}(C);
\]

\[
(\mathcal{O}_C(D), L) \mapsto \mathcal{O}_C(D) \otimes L^{-1}.
\]

For any \( \mathcal{O}_C(D) \in \text{Pic}^g(C) \), we have: \( \delta^* r\Theta |_{\mathcal{O}_C(D) \times \text{Pic}^1(C)} \simeq \mathcal{O}_{\text{Pic}^1(C)}(r\Theta_D) \). Let \( p_1 : \text{Pic}^g(C) \times \text{Pic}^1(C) \to \text{Pic}^g(C) \) be the projection onto the first factor. Consider the sheaf \( \mathcal{F} := p_1_! \mathcal{O}_{\text{Pic}^g(C) \times \text{Pic}^1(C)}(\delta^*(r\Theta)) \). It is locally free and its fiber at \( \mathcal{O}_C(D) \in \text{Pic}^g(C) \) is canonically identified with the vector space \( H^0(\text{Pic}^1(C), \mathcal{O}_{\text{Pic}^1(C)}(r\Theta_D)) \).

We call \( \tilde{T} \) the projective bundle \( \mathbb{P}(\mathcal{F}) \) on \( \text{Pic}^g(C) \).

Moreover the vector bundle \( \mathcal{F} \) is \( J[r](C) \)-equivariant, hence by easy descent theory (see [Vis05] Thm. 4.46) it passes to the quotient by \( J[r](C) \), i.e. the image of the cover \( \rho : \text{Pic}^g(C) \to \text{Pic}^r g(C) \) given by taking the \( r \)-th power of each \( L \in \text{Pic}^g(C) \). The projectivized of the obtained bundle is the projective bundle \( T \) we are looking for. We denote by \( p : T \to \text{Pic}^r g(C) \) the natural projection on the base of the projective bundle.

The previous arguments allow us to define the rational *theta map* of \( \mathcal{U}_C(r, rg) \).

\[
\theta_{rg} : \mathcal{U}_C(r, rg) \to T;
\]

\[
F \mapsto \Theta_F.
\]

Let us denote by \( \theta_D \) the restriction of \( \theta_{rg} \) to the subspace \( \mathcal{S}\mathcal{U}_C(r, \mathcal{O}_C(rD)) \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{S}\mathcal{U}_C(r) & \xrightarrow{t_D} & \mathcal{S}\mathcal{U}_C(r, \mathcal{O}_C(rD)) \\
\theta \downarrow & & \theta_D \downarrow \\
|r\Theta| & \xrightarrow{\sigma_D} & |r\Theta_D|
\end{array}
\]

since \( t_D \) and \( \sigma_D \) are isomorphism, we can identify the two theta maps. Finally remark that the composed map \( p \circ \theta_{rg} \) is precisely the natural map \( \det : \mathcal{U}_C(r, rg) \to \text{Pic}^r g(C) \), which associates to each vector bundle \( F \) its determinant line bundle \( \det(F) \).
4. **The Fundamental Divisor of a Vector Bundle**

In the sequel, where we don’t state differently, we will be concentrating on semi-stable vector bundles on $C$ with rank $r$ and degree $rg$. Let $F \in \mathcal{U}_C(r, rg)$, note that $\chi(F) = r$ hence $h^0(F) \geq r$. We can associate to $F$ two natural maps. The first one is the *evaluation map* of $F$:

$$
ev_F: H^0(F) \otimes \mathcal{O}_C \rightarrow F; \\
(s, x) \mapsto s(x).$$

The second one is the *determinant map* of $F$:

$$d_F: \wedge^r H^0(F) \rightarrow H^0(\det F);
\quad s_1 \wedge s_2 \ldots \wedge s_r \mapsto (x \mapsto s_1(x) \wedge s_2(x) \wedge \ldots \wedge s_r(x)).$$

The image of the determinant map defines a linear subsystem $|\text{Im}(d_F)| \subset |\det(F)|$. These two maps are somehow dual to each other and some properties of the evaluation map can be translated in the language of the determinant map and vice versa, as the following Lemma shows.

**Lemma 4.1.** Let $F \in \mathcal{U}_C(r, rg)$, then we have:

1. $\text{rk}(\text{ev}_F)|_x \leq r - 1$ for a point $x \in C$ if and only if $x$ is a base point of $|\text{Im}(d_F)|$;
2. $\text{rk}(\text{ev}_F) \leq r - 1$ if and only if $d_F$ is the zero map;
3. if $h^0(F) = r$, then either $d_F$ is the zero map or $|\text{Im}(d_F)| = \{\sigma\}$, for a non zero section $\sigma \in |\det(F)|$. In this case $\text{ev}_F$ is generically surjective and its degeneracy locus is the zero locus of $\sigma$.

**Remark 4.2.** Let $g = 2$ and $F \in \mathcal{U}_C(r, 2r)$. The condition $h^0(F) > r$ is equivalent to $\text{Hom}(F, \omega_C) \neq 0$. Since $\mu(F) = \mu(\omega_C)$ this implies that $F$ is not stable and $F = \omega_C \oplus G$, for some $G \in \mathcal{U}_C(r - 1, 2(r - 1))$.

Let us now define the following subset of $\mathcal{U}_C(r, rg)$:

$$\mathcal{U} = \{F \in \mathcal{U}_C(r, rg) \mid h^0(F) = r \quad d_F \neq 0 \}.$$

First we show that $\mathcal{U} \subset \mathcal{U}_C(r, rg)$ is not empty.

**Lemma 4.3.** $\mathcal{U}$ is a non-empty open subset of $\mathcal{U}_C(r, rg)$.

**Proof.** Since the conditions that define $\mathcal{U}$ are open, it is enough to produce a semi-stable vector bundle $F \in \mathcal{U}$. For example, let $F = L_1 \oplus \ldots \oplus L_r$ be the sum of $r$ line bundles of degree $g$ with $h^0(L_i) = 1$ for any $i = 1, \ldots, r$. Then we have $H^0(F) = \bigoplus_{i=1}^r H^0(L_i)$ and $h^0(F) = r$. Moreover $d_F$ is just the natural multiplication map of global sections

$$\nu: H^0(L_1) \otimes H^0(L_2) \otimes \ldots H^0(L_r) \rightarrow H^0(L_1 \otimes \ldots \otimes L_r); \\
(s_1, \ldots, s_r) \mapsto s_1 \cdots s_r;$$

that is non-zero. †
Now suppose that we have a vector bundle $F \in \mathcal{U}$. Let $\{s_1, s_2, \ldots, s_r\}$ be a base of $H^0(F)$ and let $\sigma = d_F(s_1 \wedge \ldots \wedge s_r)$. Then Lemma 4.1 implies that the divisor $D_F := \text{Zeros}(\sigma) \in |\det F|$ is well defined.

**Definition 4.4.** We call $D_F$ the fundamental divisor of $F \in \mathcal{U}$.

Remark that if $F \in \mathcal{U}_C(r, rg)$ than it has integral slope and we can consider the theta divisor $\Theta_F = \{L \in \text{Pic}^1(C) \mid h^0(C, F \otimes L^{-1}) = 1\}$ of Def. 3.1. Now the natural Abel-Jacobi map $a: C \to \text{Pic}^1(C)$ embeds $C$ in $\text{Pic}^1(C)$, and with a little argument one can show the following.

**Proposition 4.5.** Let $F \in \mathcal{U}_C(r, rg)$. Then $F \in \mathcal{U}$ if and only if $F$ admits theta divisor $\Theta_F$ and $a(C) \not\subset \Theta_F$. In this case we have:

$$D_F = a^*(\Theta_F).$$

5. The fundamental map and its fibers

Let $C^{(rg)}$ be the $rg$-symmetric product of $C$. We recall [ACGH85] that for every $r \geq 2$, $C^{(rg)}$ has a natural structure of projective bundle over $\text{Pic}^{rg}(C)$, given by the natural Abel map $a_{rg}: C^{(rg)} \to \text{Pic}^{rg}(C)$. The fiber over $O_C(M) \in \text{Pic}^{rg}(C)$ is the complete linear system $|O_C(M)|$.

**Definition 5.1.** We call

$$\Phi: \mathcal{U} \longrightarrow C^{(rg)},$$

$$F \longrightarrow D_F$$

the fundamental map of $\mathcal{U}_C(r, rg)$.

The aim of this section is the description of the fibers of $\Phi$. We start by showing some basic properties of the map $\Phi$ itself. First of all, note that since $D_F \in |\det F|$, then we have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{U}_C(r, rg) & \xrightarrow{\Phi} & C^{(rg)} \\
\downarrow \text{det} & & \downarrow a_{rg} \\
\text{Pic}^{rg}(C) & \xrightarrow{\text{id}} & \text{Pic}^{rg}(C).
\end{array}$$

Let $D \in \text{Pic}^g(C)$, then the restriction of $\Phi$ to the moduli space $SU_C(r, O_C(rD))$ induces a map $\Phi_D: SU_C(r, O_C(rD)) \longrightarrow |O_C(rD)|$.

**Theorem 5.2.** For any $r \geq 2$ and $g \geq 2$, $\Phi$ is dominant. For any $O_C(D) \in \text{Pic}^g(C)$, $\Phi_D$ is defined and dominant.

**Proof.** Remark that if $G \subseteq C^{(rg)}$ can be written as the sum of $r$ non special effective divisor $G_i$ of degree $g$, $i = 1, \ldots, r$, then $G = \Phi(F)$, where $F = \bigoplus_{i=1}^r O_C(G_i)$. So the assertion follows once we prove that for any $r \geq 1$ a general divisor of $C^{(rg)}$ satisfies the above property. We prove, by induction on $r$, that the set $S_{rg} \subseteq C^{(rg)}$ of divisors which does not satisfy this property is a proper closed subvariety.

Let $r = 1$, then $S_g \subseteq C^{(g)}$ is the closed set of special divisor of degree $g$ and $\dim S_g = g - 2$ [ACGH85]. For any $r \geq 2$, let us consider the sum map:

$$m: C^{(g)} \times C^{((r-1)g)} \to C^{(rg)}, \quad (D_1, D_2) \to D_1 + D_2.$$
This is a surjective finite map and we have:

\[ S_{rg} = m(C^g \times S_{(r-1)g}) \cup m(S_g \times C^{((r-1)g)}). \]

By induction hypothesis \( S_{(r-1)g} \) is a proper closed subvariety of \( C^{((r-1)g)} \) and \( S_g \) is of codimension 2, hence \( S_{rg} \) is a proper closed subvariety of \( C^g \). Finally, we prove that for \( r \geq 2 \) and for any \( M \in \text{Pic}^r(C) \), the closed variety \( S_{rg} \) does not contain all the elements of the linear system \( |M| \). Let us consider

\[ m^*(|M|) = \{(D_1, D_2) \in C^g \times C^{((r-1)g)} | D_1 + D_2 \in |M|\}. \]

By Riemann-Roch the natural projection \( p_2 : m^*(|M|) \rightarrow C^{((r-1)g)} \) is surjective and \( \dim m^*(|M|) = \dim C^{((r-1)g)} \), thus \( p_2 \) is generically finite. Since the fiber of \( p_2 \) at \( D_2 \in C^{((r-1)g)} \) is the linear system \( |M(-D_2)| \), then it is connected. This in turn implies that for a general \( D_2, p_2^{-1}(D_2) = \{D_1\} \), hence \( D_1 \) is not special. This implies the claim.

Let now \( F \in SU_C(r, \mathcal{O}_C(rD)) \) be a general bundle admitting theta divisor \( \Theta_F \).

By Proposition 4.5, the pull back defines a linear projection map

\[ (5.1) \quad a^* : |r\Theta_D| \rightarrow |\mathcal{O}_C(rD)|, \]

and the map \( \Phi_D \) is the composition of \( a^* \) with the theta map on \( SU_C(r, \mathcal{O}_C(rD)) \).

The projection of Eq. 5.1 can be globalized in a rational map \( A^* : T \rightarrow C^{(rg)} \) between projective bundles over \( \text{Pic}^g(C) \) that restricts to the corresponding projection on each fiber. This implies the following result.

**Proposition 5.3.** The map \( \Phi : U_C(r, rg) \rightarrow C^{(rg)} \) is the composition of \( \theta_{rg} \) with \( A^* \). In particular, the map \( \Phi_D : SU_C(r, \mathcal{O}_C(rD)) \rightarrow |\mathcal{O}_C(rD)| \) is the composition of \( \theta_D \) with \( a^* \).

As usual, let \( (\mathbb{P}^{r-1})^{rg}//\text{PGL}(r) \) denote the GIT quotient of \( (\mathbb{P}^{r-1})^{rg} \) by the diagonal action of \( \text{PGL}(r) \). We recall, ([DO88], Thm. 1 pag. 23), that a point \( v = (v_1, \ldots, v_g) \in (\mathbb{P}^{r-1})^{rg} \) is GIT semi-stable (resp. stable) if and only if for any subset \( \{v_1, \ldots, v_k\} \) of \( v \) we have \( \dim(\text{Span}(v_1, \ldots, v_k)) \geq \frac{k^2}{g} \) (resp. >). Then we have the following result:

**Theorem 5.4.** The general fiber of \( \Phi \) is birational to \( (\mathbb{P}^{r-1})^{rg}//\text{PGL}(r) \).

Let us remark the fact that in general we have only a birationality result, whereas (see Sect. 6) if \( g, r \leq 3 \) and \( g \neq r = 3 \) the general fiber is isomorphic to the corresponding GIT quotient.

**Proof.** Let \( B \in C^{(rg)} \) be a general divisor in the image of \( \Phi \). For simplicity, we assume \( B \) out of the big diagonal \( \Delta \), that is \( B = \sum_{i=1}^{rg} x_i \), with \( x_i \neq x_j \), then:

\[ \Phi^{-1}(B) = \{ F \in U : \exists F = B \}. \]

Since \( \Phi \) is dominant, we have:

\[ \dim \Phi^{-1}(B) = \dim(U_C(r, rg)) - \dim(C^{(rg)}) = (r^2 - r)g - (r^2 - 1), \]

which is actually the dimension of the variety \( (\mathbb{P}^{r-1})^{rg}//\text{PGL}(r) \). Let \( F \in \Phi^{-1}(B) \), since \( B \) is the degeneracy locus of the evaluation map, by dualizing we find the following exact sequence.
This means that, up to the choice of a basis of $H^0(F)$, $F^*$ is the kernel of a surjective morphism in $\text{Hom}(\mathcal{O}_C, \mathcal{O}_B)$. Let us now consider a vector space $V$ of dimension $r$ and the projective space $\mathbb{P}(V^*)$. By mimicking sequence 5.2, we can construct a flat family of vector bundles on $C$ over $(\mathbb{P}(V^*))^g//\text{PGL}(r)$ in the following way. We take $v = (v_1, \ldots, v_{rg}) \in \mathbb{P}(V^*)^g$ and let $\phi_v$ be the surjective morphism of sheaves $V \otimes \mathcal{O}_C \rightarrow \mathcal{O}_B$ that is the zero map out of the support of $B$ and that is obtained by taking one lift of $v_i$ to $V^*$ and applying it on the fiber of $V \otimes \mathcal{O}_C$ over $x_i \in B$. The morphism $\phi_v$ depends on the choice of the lift but the kernel of the sequence

\begin{equation}
0 \rightarrow \ker(\phi_v) \rightarrow V \otimes \mathcal{O}_C \xrightarrow{\phi_v} \mathcal{O}_B \rightarrow 0,
\end{equation}

is well defined over $\mathbb{P}(V^*)^g$. This implies that $E_v := \ker(\phi_v)^*$, for $v \in \mathbb{P}(V^*)^g$, is a family of rank $r$ vector bundles on $C$ with determinant $\mathcal{O}_C(B)$. Moreover, it is invariant under the diagonal action of $\text{PGL}(r)$ on $\mathbb{P}(V^*)^g$. We will abuse slightly notation by calling $E_v$ both the family of bundles over $\mathbb{P}(V^*)^g$ and, later, over its GIT quotient.

Remark that by definition, any semi-stable vector bundle in the fiber $\Phi^{-1}(B)$ can be written as $E_v$, for some set $v \in \mathbb{P}(V^*)^g$. We denote $A_k \subset \mathbb{P}(V^*)^g$ the open subset given by $v \in \mathbb{P}(V^*)^g$ s.t. $E_v \in \Phi^{-1}(B)$. Moreover, recall that $(\mathbb{P}^{r-1})^g//\text{PGL}(r)$ is the quotient of the open semi-stable subset of $\mathbb{P}(V^*)^g$. We denote $A_s \subset \mathbb{P}(V^*)^g$ this open subset. Then, by dimensional reasons, $A_k \cap A_s \neq \emptyset$. By passing to the quotient (recall that the construction of $E_v$ is $\text{PGL}(r)$-invariant) this implies that there exists at least one semi-stable $E_v$, for $v \in (\mathbb{P}^{r-1})^g//\text{PGL}(r)$. In turn this implies, by the openness of semistability, that the generic element of the family $E_v$, $v \in (\mathbb{P}^{r-1})^g//\text{PGL}(r)$, is a semi-stable vector bundle with fundamental divisor $B$. By the universal property of the coarse moduli space, this induces a birational map

\begin{equation}
(\mathbb{P}^{r-1})^g//\text{PGL}(r) \longrightarrow \mathcal{U}(r, rg).
\end{equation}

which is regular and one to one on the quotient of the open set $A_k \cap A_s$.

On the other hand, let us associate to $E \in \mathcal{U}$ a point set $v_E \in (\mathbb{P}(V^*))^g//\text{PGL}(r)$ s.t., keeping the notation of the previous proof, $E^*$ is the kernel of $\phi_{v_E}$.

**Lemma 5.5.** If $E \in \Phi^{-1}(B)$ is stable (resp. semi-stable) then $v_E \in (\mathbb{P}^{r-1})^g$ is GIT stable (resp. semi-stable).

**Proof.** Suppose that $E$ is semi-stable and $D_E = B$. For any subset $v' := \{v_1, \ldots, v_k\}$ of $v_E$, let us denote $V_{v'} := \text{Span}(v_1, \ldots, v_k) \subset V^*$ and $x_i$ the point of $B$ that correspond to $v_i$. Then we get a commutative diagram:
for some vector bundle $G^\ast$ with $rk(G^\ast) = \text{dim}(V_k) = s$. Now, in order to show that $v_E$ is GIT semi-stable, it is enough to show that $\text{dim}(V_k) \geq \frac{k}{g}$. Since we have that $\text{deg}(G^\ast) = -k$ and $E^\ast$ is semi-stable, then $\mu(G^\ast) = \frac{k}{g} \geq -g$, from which we obtain the desired inequality. The stable case is described in the same way but with strict inequalities. ♠

A part of Lemma 5.5 and Theorem 5.4 is proved for $g = 2$ in [Ang04]. Finally, we describe the quotient of $A_3 \cap A_5$ in $(\mathbb{P}^{r-1})^g//\text{PGL}(r)$. This is the sublocus of $(\mathbb{P}^{r-1})^g//\text{PGL}(r)$ where $E_v \in \Phi^{-1}(B)$.

**Lemma 5.6.** Let $v \in (\mathbb{P}^{r-1})^g//\text{PGL}(r)$, $E_v \in \Phi^{-1}(B)$ if and only if $h^0(E_v) = r$.

**Proof.** Let $v \in (\mathbb{P}^{r-1})^g//\text{PGL}(r)$. The only if part is clear. If $h^0(E_v) = r$, then by taking the dual of sequence 5.3, we have that $H^0(E_v) \simeq V^\ast$, the evaluation map of $E_v$ is generically surjective and its degeneracy locus is actually $B$. So it is enough to prove that $E_v$ is semi-stable. Suppose that there exists a proper rank $r$ sub-bundle $F \subset E_v$ with $\mu(F) > \mu(E_v) = g$. This by Riemann-Roch implies $h^0(F) \geq s+1$. Since $H^0(F) \subset H^0(E_v)$ this contradicts the generically surjectiveness of the evaluation map of $E_v$. ♠

As a corollary we have the following result:

**Theorem 5.7.** The moduli space $U_C(r,0)$ (resp. $SU_C(r)$) is birational to a fibration over $C^{(rs)}$ (resp. $\mathbb{P}^{(r-1)g}$) whose fibers are $(\mathbb{P}^{r-1})^g//\text{PGL}(r)$.

6. **Application to the Coble hypersurfaces**

When the genus of $C$ is 2 or 3 and the rank is small enough, the moduli spaces $SU_C(r)$ have very nice explicit descriptions related to certain hypersurfaces, called the **Coble hypersurfaces**.

The main theorems of this section show how these hypersurfaces are in fact fibrations over certain projective spaces whose fibers are isomorphic to classical modular varieties related to the moduli of 6 points on a line and on a plane: namely the **Segre Cubic** and the **Igusa quartic**.

The basic example of these fibrations, namely the $g = 2$, $r = 2$ case, is quite instructive. See [Bol09] for details.

6.1. **The Coble Sextic.** In this subsection we assume that $C$ is a curve of genus 2 and we consider the moduli space $SU_C(3)$ of semi-stable vector bundles on $C$ with rank 3 and trivial determinant. The theta map

$$\theta : SU_C(3) \rightarrow |3\Theta| \simeq \mathbb{P}^8$$
is a finite morphism of degree 2 and the branch locus is a sextic hypersurface \( C_6 \), called Coble-Dolgachev sextic \([\text{Las96}]\). The Jacobian variety \( J(C) \) is embedded in \( |3\Theta|^{*} \) as a degree 18 surface, and there exists a unique cubic hypersurface \( C_3 \subset |3\Theta|^{*} \) whose singular locus coincides with \( J(C) \): the Coble cubic, \([\text{Cob82}]\). It was then conjectured by Dolgachev, and subsequently proved in \([\text{Ort05}]\) and independently in \([\text{Ngu07}]\), that \( C_6 \) is the dual variety of \( C_3 \).

On the other hand the hypersurface known as Igusa quartic is the Satake compactification of the moduli space \( \mathcal{A}_2(2) \) of principally polarized abelian surfaces with a level two structure \([Igu64]\), embedded in \( \mathbb{P}^4 \) by fourth powers of theta-constants. Anyway in our context appears rather because of its relation with the GIT geometry of sets of points in the projective plane. The GIT quotient of \( (\mathbb{P}^2)^6 \) with respect to the diagonal action of \( \text{PGL}(3) \) is a degree two covering of \( \mathbb{P}^3 \) branched along a \( \Sigma_6 \)-invariant quartic hypersurface \( \mathcal{I}_4 \subset \mathbb{P}^4 \), which is exactly the Igusa quartic. The involution that defines the covering is the Gale transform (also called association, for details see \([\text{DO88}], [\text{EP00}]\)), which is defined as follows.

**Definition 6.1.** (\([\text{EP00}], \text{Def. 1.1}\))

Let \( r, s \in \mathbb{Z} \). Set \( \gamma = r + s + 2 \), and let \( \Gamma \subset \mathbb{P}^r \), \( \Gamma' \subset \mathbb{P}^s \) be ordered nondegenerate sets of \( \gamma \) points represented by \( \gamma \times (r + 1) \) and \( \gamma \times (s + 1) \) matrices \( G \) and \( G' \), respectively. We say that \( \Gamma' \) is the Gale transform of \( \Gamma \) if there exists a nonsingular diagonal \( \gamma \times \gamma \) matrix \( D \) s.t. \( G' \cdot D \cdot G = 0 \), where \( G'^T \) is the transposed matrix of \( G \).

The Gale transform acts trivially on the sets of 6 points in \( \mathbb{P}^2 \) that lie on a smooth conic. The branch locus of the double covering is then, roughly speaking, the moduli space of 6 points on a conic and henceforth a birational model of the moduli space of 6 points on a line. One can say even more, in fact the GIT compactification of the moduli space of 6 points on a line is a cubic 3-fold in \( \mathbb{P}^4 \), called the Segre cubic, and its projectively dual variety is the Igusa quartic (see \([\text{Koi03}], [\text{Hun96}]\) for details). From the projective geometry point of view the singular locus of \( \mathcal{I}_4 \) is an abstract configuration of lines and points that make up a \( 15_3 \) configuration. This means the following: there are 15 distinguished lines and 15 distinguished points. Each line contains 3 of the points and by each point pass 3 lines (see \([\text{Dol04} \text{ Sect. 9 for more}]\)). Moreover \( \mathcal{I}_4 \) is the only hypersurface with such a singular locus in the pencil of \( \Sigma_6 \)-invariant quartics in \( \mathbb{P}^4 \) (\([\text{Hun00}], \text{Example 7}\)).

Let us twist as customary the vector bundles in \( \mathcal{SU}_C(3) \) by the degree 2 canonical bundle \( K_C \) and get thus an isomorphism \( \mathcal{SU}_C(3) \cong \mathcal{SU}(3,3K_C) \). Let \( \Phi_K : \mathcal{SU}(3,3K_C) \to |3K_C| \) be the restriction of the fundamental map, then we have the following.

**Proposition 6.2.** Let \( L_\epsilon \) be the subsystem of theta divisors corresponding to decomposable bundles of type \( E \oplus K_C \), with \( E \in \mathcal{SU}_C(2,2K_C) \). The map \( \Phi_{K_C} \) is the composition of the theta map \( \theta_{K_C} : \mathcal{SU}(3,3K_C) \to |3\Theta| \) with the linear projection \( \pi_\epsilon : |3\Theta| \to |3K_C| \) whose center is \( L_\epsilon \cong \mathbb{P}^3 \).

**Proof.** Recall from section 5 that \( \Phi_{K_C} \) factors through \( \pi_\epsilon \). Since \( \dim |3K_C| = 4 \) this implies that the center of the projection is a 3 dimensional linear subspace. By Remark 4.2 we know that the decomposable bundles of type \( E \oplus K_C \), \( E \in \mathcal{SU}_C(2,2K_C) \), have \( h^0 > 3 \). This locus in \( \mathcal{SU}(3,3K_C) \) has dimension 3, and it is
contained in the indeterminacy locus of $\phi_{K_C}$. Now, the hyperelliptic involution $h$ on $C$ defines the involution

$$\sigma : SU(3, \mathcal{O}_C) \rightarrow SU(3, \mathcal{O}_C);$$

$$F \mapsto h^* F^*;$$

which is associated to the 2:1 covering given by $\theta$ [Ngu07]. In [Ngu07] (Sect. 3 and 4), it is shown that the locus given by vector bundles $F = E \oplus \mathcal{O}_C$, $E \in SU_C(2, \mathcal{O})$, is contained in the fixed locus of $\sigma$ and it is embedded in $SU(3, \mathcal{O}_C)$. The twist by $K_C$ of these vector bundles give those of type $E \oplus K_C$ with $E \in SU(2, 2K_C)$. The image via $\theta$ of this locus is $L_e \cong \mathbb{P}^3$ and it is actually the center of the projection.

Now let us recall some results from the literature about $SU(3, \mathcal{O}_C)$. Of course the twist by $K_C$ is an isomorphism and it is easy to understand which is the analogue result for $SU_C(3, 3K_C)$. Since $\mathbb{P}^8$ is smooth, the image of the singular locus of $SU(3, \mathcal{O}_C)$ and the singular locus of the branch locus coincide, i.e. $Sing(C_6) = \theta(Sing(SU(3)))$. On Pic$^1(C)$ we have the involution $\lambda : L \mapsto K_C \otimes L^{-1}$ that leaves $\Theta$ invariant. Hence $\lambda$ induces an action on all the powers of $\Theta$ and in particular, on $|3\Theta|$. The linear system $|3\Theta|$ decomposes in two eigenspaces, respectively 4 and 3 dimensional. We call $\mathbb{P}^4$ the 4-dimensional eigenspace. It turns out that it cuts out on $C_6$ a reducible variety given by a double $\mathbb{P}^3$ (which is indeed contained in $Sing(C_6)$) and a quartic hypersurface $I \subset \mathbb{P}^4$. After the twist by $K_C$, the first component is precisely $L_e$, whereas the quartic 3-fold is an Igusa quartic.

**Lemma 6.3.** The intersection of the closure of the general fiber of $\pi_e$ with $Sing(C_6)$ is a 15$^3$ configuration of lines and points.

**Proof.** Recall that $Sing(C_6)$ is the locus of theta divisors corresponding to decomposable bundles, we will prove the claim by constructing explicitly these bundles. Let us denote $\Delta_{3K_C}$ the closed subset of $|3K_C|$ given by the intersection with the big diagonal of $C^6(6)$. Let us take $G = q_1 + \cdots + q_6 \in |3K_C| - \Delta_{3K_C}$, and let us consider the fiber of $\Phi_{K_C}$ over $G$. In order to guarantee the semi-stability of the vector bundles, the only totally decomposable bundles in the fiber of $G$ are all the 15 obtained by permuting the $q_i$'s in $\mathcal{O}_C(q_1 + q_2) \oplus \mathcal{O}_C(q_3 + q_4) \oplus \mathcal{O}_C(q_5 + q_6)$. Let us now consider the bundles that decompose as the direct sum of a line bundle $L$ and a rank two indecomposable bundle. By the previous argument of semi-stability then $L$ must be of the type $\mathcal{O}_C(p_i + p_j)$ for some $i, j \in \{1, \ldots, 6\}$ and $E$ should have fundamental divisor $D_E = \sum_{k \neq i,j} p_k$. Call $F$ the line bundle $\mathcal{O}_C(\sum_{k \neq i,j} p_k) \equiv 3K_C - p_i - p_j$. It is easy to see that $SU_C(2, F) \cong SU_C(2, \mathcal{O}_C) \cong \mathbb{P}^3$, the isomorphism being given by the tensor product by a square root $F'$ of $F$. Now we recall from [Bol09] the following description of the fundamental map $\Phi_{F'} : SU_C(2, F) \rightarrow |F|$. The linear system $|F|$ is a $\mathbb{P}^2$ and the fibers of $\Phi_{F'}$ are just lines passing by $D \in |F|$ and the origin $|\mathcal{O}_C \oplus \mathcal{O}_C|$. Now the composition of the following embedding

$$\zeta : SU_C(2, F) \hookrightarrow SU_C(3, 3K),$$

$$E \mapsto \mathcal{O}_C(p_i + p_j) \oplus E,$$

with the theta map is linear. In fact $\zeta(SU_C(2, F))$ is contained in the branch locus and the associated $3\Theta$ divisors form a three dimensional linear subsystem.
isomorphic to $|2\Theta|$. Then the image of $\zeta$ intersects the closure $\overline{\Phi^{-1}_K(G)}$ of the fiber over $G$ exactly along the fiber of $\Phi_K$ over the divisor $\sum_{k \neq j} p_k \in |F|$. By [Bol09] we know that this is a line and it is not difficult to see that it contains $3$ of the $15$ totally decomposable bundles. On the other hand each totally decomposable bundle with fundamental divisor $G$ is contained in three lines of this kind.

**Remark 6.4.** When the divisor $G$ is taken in $\Delta_{3K_C}$ then the configuration $15_3$ degenerates because some of the points and of the lines coincide.

**Theorem 6.5.** The closure of the general fiber of $\Phi_{K_C}$ is the GIT quotient $(\mathbb{P}^2)^6/\text{PGL}(3)$.

**Proof.** We recall that $L_e$ is contained in $\text{Sing} (\mathcal{C}_6)$ and in particular scheme-theoretically it is contained twice in $\mathcal{C}_6$. Since $\Phi_{K_C}$ factors through the projection with center $L_e$, then $\overline{\Phi^{-1}_K(G)}$, for $G \in |3K_C| - \Delta_{3K_C}$, is a degree two cover of $\mathbb{P}^3_e := \overline{\pi_e^{-1}(G)}$ ramified along the intersection of $\mathcal{C}_6$ with $\mathbb{P}^4_G$ which is residual to $2L_e$. This intersection is then a quartic hypersurface in $\mathbb{P}^8_G$. Notably, since $L_e \subset \mathbb{P}^4$, there exist a point $T \in |3K_C|$ s.t. $\overline{\pi_e^{-1}(T)} \cap \mathcal{C}_6$ is an Igusa quartic (see Prop. 5.2 of [OP96] or [NR03] Sect. 4).

Let us now blow up $|3\Theta|$ along $L_e$ and call $\overline{\mathbb{P}^8}$ the obtained variety, which contains canonically the blown up Coble sextic, that we denote $\overline{\mathcal{C}_6}$. Then the rational map $\pi_e$ resolves in a proper, flat map $\overline{\pi_e}$ as in the following diagram.

$$
\begin{array}{ccc}
\overline{\mathbb{P}^8} & \xrightarrow{\overline{\pi_e}} & |3\Theta| \\
|3\Theta| & \xrightarrow{\pi_e} & |3K_C|
\end{array}
$$

This boils down to saying that the restriction of $\overline{\pi_e}$ to $\overline{\mathcal{C}_6}$ is a flat family of quartic $3$-folds over $|3K_C|$ and for any $B \in |3K_C|$ we have an isomorphism $\overline{\pi_e^{-1}(B)} \cap \mathcal{C}_6 \cong \overline{\pi_e^{-1}(B)} \cap \mathcal{C}_6$. Hence also one fiber of $\overline{\pi_e|\mathcal{C}_6}$ is an Igusa quartic. Since the ideal of the singular locus of $\mathcal{I}_4$ is generated by the four polar cubics ([Hun96], Lemma 3.3.13) then the Igusa quartic has no infinitesimal deformations, i.e. it is rigid. This implies that the generic member of the flat family of quartics over $|3K_C|$ is an Igusa quartic. This in turn implies that the closure of the generic fiber of $\Phi_{K_C}$ is $(\mathbb{P}^2)^6/\text{PGL}(3)$.

**Corollary 6.6.** The Coble sextic $\mathcal{C}_6$ is birational to a fibration over $\mathbb{P}^4$ whose fibers are Igusa quartics.

**Corollary 6.7.** Along the generic fiber of $\Phi_{K_C}$, the involution of the degree two covering $\text{SU}(3,\Omega_C) \rightarrow |3\Theta|$ coincides with the involution given by the association isomorphism on $(\mathbb{P}^2)^6/\text{PGL}(3)$.

The fact that the intersection of $\text{Sing}(\mathcal{C}_6)$ with the fibers over the open set $|3K_C| - \Delta_{3K_C}$ is precisely a $15_3$ configuration makes us argue that $|3K_C| - \Delta_{3K_C}$ should be the open locus where by rigidity the family of quartic three-folds is isotrivially isomorphic to the Igusa quartic. As already seen in Remark 6.4, if $B$ is an effective divisor out of this locus then, $\overline{\pi_e^{-1}(B)} \cap \text{Sing} (\mathcal{C}_6)$ is a degenerate $15_3$ configuration, in the sense that some of the $15$ points and lines come to coincide.
We are not able to prove the following, but it is tempting to say that this is all the singular locus of the special quartic three-folds over $\Delta_{3K}$. These would give very interesting examples of degenerate Igusa quartics. It would be interesting to study projective properties of these fibers such as the relation with the Segre cubic, or with the Mumford-Knudsen compactification $\overline{M}_{0,6}$ of the moduli space of 6 points on a line. For instance, do they come from linear systems on $\overline{M}_{0,6}$? If it is so, what linear systems on $\overline{M}_{0,6}$ do they come from?

The rational dual map of the Coble sextic has been thoroughly studied and described in [Ort05] and [Ngu07]. Let us denote by $X_0, \ldots, X_8$ the coordinates on $\mathbb{P}^8 \cong [3\Theta]$ and by $F(X_0: \cdots : X_8)$ the degree six polynomial defining $C_6$. Then the dual map is defined as follows:

$$D_6 : C_6 \rightarrow C_3; \quad x \mapsto \begin{bmatrix} \frac{\partial F}{\partial X_0} & \cdots & \frac{\partial F}{\partial X_8} \end{bmatrix}. $$

The polar linear system is given by quintics that vanish along $Sing(C_6)$. Now fix a general divisor $B \in |3K|$ and call $I_B$ the Igusa quartic defined by $(C_6 \cap \pi_1^{-1}(B)) \sim 2L_e$. Let us consider the restriction of $D_6$ to $I_B \subset \pi^{-1}(B) =: \mathbb{P}_B^4$ and denote by $A$ the 153 configuration of points and lines in $\mathbb{P}_B^4$. Let now $H$ be the class of $L_e$ in $\operatorname{Pic}(\mathbb{P}_B^4)$ and consider the 4-dimensional linear system $|I_A(3) + 2H|$ on $I_B$. We can show the following.

**Proposition 6.8.** The restricted dual map $D_6|_{I_B}$ is given by a linear system $|D_{I_B}|$ that contains $|I_A(3) + 2H|$ as a linear subsystem.

Remark that this means that for the general fiber $I_B$, there exists a canonical way to project the image $D(I_B) \subset C_3$ to a $\mathbb{P}_B^4$ where the image of $I_B$ is a Segre cubic. This is resumed in the following.

**Corollary 6.9.** The Coble cubic is birational to a fibration in Segre cubics over $\mathbb{P}^4$.

**Remark 6.10.** The birationality in itself is trivial, since $C_3$ is birational to $C_6$ which is birational to a fibration in Igusa quartics (which are in turn all birational to the Segre cubic) over $\mathbb{P}^4$. The projections on the linear systems $|I_A(3) + 2H|$ give a constructive canonical way to realize it.

### 6.2. The Coble quartic.

In this subsection we assume that $C$ is a curve of genus 3 and we consider the moduli space $SU(2, \mathcal{O}_C)$. We recall that the Kummer variety $Kum(C) := J(C)/\pm Id$ of $C$ is contained naturally in the $2\Theta$-linear series, whereas the moduli space $SU(2, \mathcal{O}_C)$ is embedded by $\theta$ in $\mathbb{P}^7 \cong |2\Theta|$ as the unique quartic hypersurface $C_4$ singular along $Kum(C)$. This hypersurface is called the Coble quartic. It is also known [Pau02] that the Coble quartic is projectively self-dual.

Now we need to introduce a second important modular variety, i.e. the Segre cubic. This is a nodal (and hence rational) cubic three-fold $S_3$ in $\mathbb{P}^4$ whose singular locus is given by ten double points. There is a natural action of $\Sigma_6$ on this projective space and $S_3$ is invariant with respect to this action. $S_3$ is in fact the GIT quotient $(\mathbb{P}^1)^6//\text{PGL}(2)$ [DO88]. Moreover, $S_3$ realizes the so-called Varchenko bound, that is, it has the maximum number of double points (ten) that a cubic threefold with
isolated singularities may have and this property identifies the Segre cubic in a unique way, up to projective equivalence. As already stated it is the projective dual variety of the Igusa quartic.

Our construction allows us to give a simple proof of the following result from [AB09].

**Proposition 6.11.** The moduli space $SU(2, \mathcal{O}_C)$ is birational to a fibration over $\mathbb{P}^3$ whose fibers are Segre cubics.

**Proof.** After the customary twist by a degree 3 divisor $D$, the fundamental map is $\Phi_D : SU(2, \mathcal{O}_C(2D)) \to |\mathcal{O}_C(2D)| \cong \mathbb{P}^3$. Since $\theta_D$ is an embedding, we identify $SU(2, \mathcal{O}_C(2D))$ with its image $\mathcal{C}_4 \subset \mathbb{P}^7$ and $\Phi_D$ with the linear projection onto $|2D|$. The center of the projection is the linear span of the locus of vector bundles $E$ s.t. $h^0(C, E \otimes \mathcal{O}_C(D)) > 2$. If $E$ is stable then $E \cong E^*$ and by an easy Riemann-Roch computation we find that $h^0(C, E \otimes \mathcal{O}_C(D)) > 2$ if and only if $h^0(C, E \otimes \mathcal{O}_C(K - D)) > 0$. As it is shown in [AB09] Prop 3.1 this is equivalent to the fact the $E$ lies in the $\mathbb{P}^3 \cong |3K - 2D|^\ast \subset \mathcal{C}_4$ that parametrizes vector bundles $E$ that can be written as an extension of the following type

$$0 \to \mathcal{O}_C(D - K) \to E \to \mathcal{O}_C(K - D) \to 0.$$ 

Let us denote $\mathbb{P}_4$ this projective space. $\mathcal{C}_4$ contains $\mathbb{P}_4$ with multiplicity one. This implies that the closure of any fiber of the projection $\Phi_D : \mathcal{C}_4 \dashrightarrow |2D|$ is a cubic 3-fold contained in the $\mathbb{P}^4$ spanned by $\mathbb{P}^3$ and a point of $|2D|$. Let us denote as usual $\Delta_D$ the intersection of the large diagonal with the linear system $|2D| \subset C^{(6)}$. Then suppose we fix a $B \in |2D| - \Delta_D$. Let us consider the intersection of the fiber of $\Phi_D$ over $B$ with the strictly semi-stable locus. By semi-stability it is easy to see that these points correspond to the partitions of the 6 points of $B$ in complementary subsets of 3 elements each. We have ten of them. As stated her above, a cubic 3-fold can not have more than ten ordinary double points and the Segre cubic is uniquely defined by this singular locus up to projective equivalence. 

Also in this case, if $B \in \Delta_D$ then the intersection $\overline{\Phi_D^{-1}(B)} \cap Kum(C)$ is set-theoretically a finite set of points of cardinality strictly smaller then 10: the singular locus seems to degenerate. It is tempting, like in the case of Igusa quartics, to say that some of these points have multiplicity bigger than one and we obtain degenerate Segre cubics over $\Delta_D$.

As we have already remarked, also in the case of $\mathcal{C}_4$ the polar map is well known and described. Let $Y_i$ be the coordinates on $\mathbb{P}^7 = |2\Theta|$ and $G(Y_0 : \cdots : Y_7)$ the quartic equation defining $\mathcal{C}_4$, then the (self) polar rational map of $\mathcal{C}_4$ is defined in the following way.

$$D_4 : \mathcal{C}_4 \dashrightarrow \mathcal{C}_4 : x \mapsto \left[ \frac{\partial G}{\partial Y_0} : \cdots : \frac{\partial G}{\partial Y_7} \right].$$

Let $B \in |2D| - \Delta_D$ and let $\mathbb{P}_B^4$ be the linear span of the point corresponding to $B$ and of $\mathbb{P}^3$. It turns out that the restriction of $D_4$ to $\mathbb{P}_B^4$ behaves in a way very similar to the case of the sextic (see Prop. 6.8). Let $S_{3B} \subset \mathbb{P}_B^4$ be the Segre cubic such that $\mathcal{C}_4 \cap \mathbb{P}_B^4 = S_{3B} \cup \mathbb{P}^3_c$. We denote $J$ the set of 10 nodes of $S_{3B}$. Then the linear series $|I_J(2)|$ on $S_{3B}$ is the polar system of the Segre cubic.

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Proposition 6.12. The restricted dual map $\mathcal{D}_{4|S_3B}$ is given by a linear system $|\mathcal{D}_{S_3}|$ that contains $|I_j(2) + H|$ as a linear subsystem.

As in the case of $C_6$ this implies that we have a canonical way to construct the birational map of the following corollary via the polar map $\mathcal{D}_4$.

Corollary 6.13. The Coble quartic is birational to a fibration in Igusa quartics over $\mathbb{P}^3$.

References

[AB09] Alberto Alzati and Michele Bolognesi, A structure theorem for SU(2) and the moduli of pointed genus zero curves, 1 – 17, (preprint, http://arxiv.org/abs/0903.5515).

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, Geometry of algebraic curves. Vol. I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985.

[Ang04] Cristian Anghel, Fibrés vectoriels semi-stables sur une courbe de genre deux et association des points dans l’espace projectif, Serdica Math. J. 30 (2004), no. 2-3, 103–110.

[Bea06] Arnaud Beauville, Vector bundles and theta functions on curves of genus 2 and 3, Amer. J. Math. 128 (2006), no. 3, 607-618.

[Ber92] A. Bertram, Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space, J. Differential Geom. 35 (1992), no. 2, 423-469.

[BNR89] Arnaud Beauville, M. S. Narasimhan, and S. Ramanan, Spectral curves and the generalised theta divisor, J. Reine Angew. Math. 398 (1989), 169–179.

[Bol09] Michele Bolognesi, A conic bundle degenerating on the Kummer surface, Math. Z. 261 (2009), no. 1, 149–168.

[BV96] Sonia Brivio and Alessandro Verra, The theta divisor of $\text{su}_c(2,2d)$ is very ample if $c$ is not hyperelliptic, Duke Math. J. 82 (1996), 503–552.

[BV07] Plucker forms and the theta map, 1–20, (preprint http://arxiv.org/abs/0910.5630).

[Cob82] Arthur B. Coble, Algebraic geometry and theta functions, American Mathematical Society Colloquium Publications, vol. 10, American Mathematical Society, Providence, R.I., 1982, Reprint of the 1929 edition.

[DN89] J.-M. Drezet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. 97 (1989), no. 1, 53–94.

[DO88] Igor Dolgachev and David Ortland, Point sets in projective spaces and theta functions, Astérisque, vol. 165, Société Mathématique de France, 1988.

[Dol04] Igor V. Dolgachev, Abstract configurations in algebraic geometry, The Fano Conference, Univ. Torino, Turin, 2004, pp. 423–462.

[EP00] David Eisenbud and Sorin Popescu, The projective geometry of the Gale transform, J. Algebra 230 (2000), no. 1, 127–173.

[Hun96] B. Hunt, The geometry of some special arithmetic quotients, Lecture Notes in Mathematics, vol. 1637, Springer, Berlin, 1996.

[Hun00] Bruce Hunt, Nice modular varieties, Experiment. Math. 9 (2000), no. 4, 613–622.

[Igu64] J.-I. Igusa, On the graded ring of theta-constants. I, Amer. J. Math. 86 (1964), 219–246.

[Koi03] Kenji Koike, Remarks on the Segre cubic, Arch. Math. (Basel) 81 (2003), no. 2, 155–160.

[KS99] Alastair King and Aidan Schofield, Rationality of moduli of vector bundles on curves, Indag. Math. (N.S.) 10 (1999), no. 4, 519–535.

[Las96] Yves Laszlo, Local structure of the moduli space of vector bundles over curves, Comment. Math. Helv. 71 (1996), no. 3, 373–401.

[MN68] D. Mumford and P. Newstead, Periods of a moduli space of bundles on curves, Amer. J. Math. 90 (1968), 1200–1208.
Quang Minh Nguyen, Vector bundles, dualities and classical geometry on a curve of genus two, Internat. J. Math. 18 (2007), no. 5, 535–558.

M. S. Narasimhan and S. Ramanan, Moduli of vector bundles on a compact Riemann surface, Ann. of Math. (2) 89 (1969), 14–51.

Quang Minh Nguyen and Slawomir Rams, On the geometry of the Coble-Dolgachev sextic, Matematiche (Catania) 58 (2003), no. 2, 257–275 (2005).

W. M. Oxbury and C. Pauly, SU(2)-Verlinde spaces as theta spaces on Pryms, Internat. J. Math. 7 (1996), no. 3, 393–410.

William Oxbury and Christian Pauly, Heisenberg invariant quartics and SUc(2) for a curve of genus four, Math. Proc. Cambridge Philos. Soc. 125 (1999), no. 2, 295–319.

Angela Ortega, On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic, J. Algebraic Geom. 14 (2005), no. 2, 327–356.

Christian Pauly, Self-duality of Coble’s quartic hypersurface and applications, Michigan Math. J. 50 (2002), no. 3, 551–574.

Rank four vector bundles without theta divisor over a curve of genus two, 1–8, (to appear on Adv. in Geom.; http://arxiv.org/abs/0804.3001).

Michel Raynaud, Sections des fibrés vectoriels sur une courbe, Bull. Soc. Math. France 110 (1982), no. 1, 103–125.

B. van Geemen and E. Izadi, The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian, J. Algebraic Geom. 10 (2001), no. 1, 133–177.

Grothendieck topologies, fibered categories and descent theory, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

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