The Pólya and Sisyphus lattice random walks with resetting – a first passage under restart approach

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We revisit the simple lattice random walk (Pólya walk) and the Sisyphus random walk in \( \mathbb{Z} \), in the presence of random restarts. We use a relatively direct approach namely First passage under restart for discrete space and time which was recently developed by us in PRE 103, 052129 (2021) and rederive the first passage properties of these walks under the memoryless geometric restart mechanism. Subsequently, we show how our method could be generalized to arbitrary first passage process subject to more complex restart mechanisms such as sharp, Poisson and Zeta distribution where the latter is heavy tailed. We emphasize that the method is very useful to treat such problems both analytically and numerically.

I. MOTIVATION

First passage under restart recently emerged as a useful framework to study arbitrary stochastic processes under arbitrary restart mechanisms [1]. Although the problems with continuous space and time set-ups have been studied in full glory, their discrete counterparts have been largely overlooked. To bridge this void, in a recent work [2], we developed a First Passage Under Restart formalism for discrete space and time processes. This comprehensive framework can be used to derive exact and useful formulae such as the generating functions for the survival and first passage, the mean first passage time (and higher order moments) etc. in the presence of arbitrary restart distributions. The derivation does not assume any specific form for the underlying first passage process or the restart process, hence can be ubiquitously used in a generic set-up.

In this note, power of this approach is illustrated using the following examples: Pólya’s lattice random walk on the integer points of the real line, and the one sided and double sided Sisyphus random walk in the presence of geometric restart – a memoryless distribution. These models were studied earlier and we will refer to these works in proper context. Moving forward, we show our method could be used to accommodate any first passage process that maybe subject to more complex restart distributions (and not necessarily memoryless) namely sharp, Poisson and Zeta. In the conclusions, we sketch out a three-step blue-print on how to use our recipe in a plug-n-play manner either analytically or numerically.

II. NOTATIONS

Before we proceed, it will be useful to introduce the notations used throughout the paper. We will use \( P_X(x) \), \((X)\), \( \sigma_X^2 \), and \( G_X(z) \equiv (z^X) \) to denote, respectively, the probability mass/density function (PMF/PDF), mean/expectation, variance, and the probability generating function (PGF) of a discrete random variable \( X \) taking values in the non-negative integers. In addition, the survival function and its corresponding generating function for the underlying process are denoted by \( Q_N(n) \) and \( G_{QN}(z) \) respectively. In the presence of restart, generating function for the survival function will be denoted by \( G_{RN}(z) \).

III. DISCRETE FIRST PASSAGE UNDER DISCRETE RESTART

Consider a generic discrete step first passage process that starts at the origin and, if allowed to take place without interruptions, ends after a random number of steps \( N \). The process is, however, restarted after some random number of steps \( R \). Thus, \( P_N(n) \) and \( P_R(n) \) are the probability density functions for the first passage and restart process respectively. If the process is completed only prior to the restart, we mark a completion of the event. Otherwise, the process will start from scratch and begin completely anew. This procedure repeats itself until the process reaches completion. Denoting the random completion number of steps of the restarted process by \( N_R \), it can be seen that

\[
N_R = \begin{cases} 
N & N < R \\
R + N'_R & N \geq R,
\end{cases}
\]

where \( N'_R \) is an independent and identically distributed copy of \( N_R \). Eq. (1) is the central renewal equation for first passage step under restart and assumes that after each restart, the memory is erased from the previous trial. To obtain the mean number of steps for the restarted process, we note that Eq. (1) can be written as \( N_R = \min(N, R) + I \{ N \geq R \} N'_R \), where \( I \{ N \geq R \} \) is an indicator random variable that takes the value 1 when \( N \geq R \) and zero otherwise. This approach is slightly different than the one used in [2] where we assumed that process can complete even when the steps for first passage and restart coincide i.e.,

\[ N < N_R \]
$N_R = N$ when $N \leq R$ unlike here [see Eq. (1) in above].

A. Mean

We take expectations on both sides of Eq. (1) and use the fact that $N$ and $R$ are independent of each other. This results in the following expression for the mean completion time under restart

$$\langle N_R \rangle = \frac{\langle \min (N, R) \rangle}{\Pr (N < R)}.$$  (2)

The numerator can be computed by noting that the probability $\Pr (\min (N, R) > n) = \Pr (N > n) \Pr (R > n)$ and thus

$$\langle \min (N, R) \rangle = \sum_{n=0}^{\infty} \Pr (\min (N, R) > n)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=n+1}^{\infty} P_N (k) \right) \left( \sum_{m=n+1}^{\infty} P_R (m) \right)$$

$$= \sum_{n=0}^{\infty} Q_N (n) Q_R (n),$$  (3)

where note that $Q_N (n)$ and $Q_R (n)$ are the survival functions for the first passage and restart processes respectively i.e.,

$$Q_N (n) = \Pr (N > n),$$  (4)

$$Q_R (n) = \Pr (R > n).$$  (5)

The denominator in Eq. (2) can also be computed easily

$$\Pr (N < R) = \sum_{n=0}^{\infty} P_N (n) \sum_{m=n+1}^{\infty} P_R (m).$$  (6)

B. Generating function for the restarted process

We now turn our attention to derive the generating function for the restarted process. The probability generating function of the discrete random variable $N_R$ taking values in the non-negative integers $0, 1, \ldots$ is defined as

$$G_{N_R} (z) \equiv \langle z^{N_R} \rangle = \sum_{n=0}^{\infty} P_{N_R} (n) z^n,$$  (7)

where $P_{N_R} (n)$ is the probability mass function of $N_R$. It will now prove useful to introduce the following conditional random variables

$$N_{min} \equiv \{ N | N < R \},$$  (8)

$$R_{min} \equiv \{ R | N \geq R \},$$  (9)

with their respective densities

$$P_{N_{min}} (n) = P_N (n) \sum_{m=n+1}^{\infty} P_R (m),$$  (10)

$$P_{R_{min}} (n) = P_R (n) \sum_{m=n+1}^{\infty} P_N (m),$$  (11)

where $\Pr (N \geq R) = 1 - \Pr (N < R)$. Using the renewal Eq. (1) and the new random variables in Eqs. (8)-(9), we can write

$$G_{N_R} (z) = \Pr (N < R) \langle z^{N_{min}} \rangle + \Pr (N \geq R) \langle z^{R_{min} + N_R} \rangle.$$  (12)

Now using the fact that $N'_R$ is an independent and identically distributed copy of $N_R$ in above, we arrive at the following expression for the generating function of the restarted process

$$G_{N_R} (z) = \frac{(1 - \Pr (N \geq R)) G_{N_{min}} (z)}{1 - \Pr (N \geq R) G_{R_{min}} (z)},$$  (13)

where $G_{X_{min}} (z)$ is the generating function for the random variable $X_{min}$. The above formula (13), being a simple analogue to Eq. (11) in [2], is extremely useful since it allows one to compute all the moments

$$\langle N_R^k \rangle = \left( \frac{\partial}{\partial z} \right)^k G_{N_R} (z) \bigg|_{z=1^-},$$  (14)

and, importantly, also the probability density function of $N_R$

$$P_{N_R} (n) = \Pr (N_R = n) = \frac{G_{N_R} (n)}{n!},$$  (15)

where $\langle N_R^k \rangle$ is the $k$-th moment and $G_{N_R}^{(n)} (z)$ is the $n$-th derivative of $G_{N_R} (z)$ with respect to $z$, and $n = 0, 1, 2, \ldots$. Note that in discrete set-up, computation of the first passage time probability mass requires only derivatives of the generating function as seen in Eq. (15), and thus is more accessible compared to the continuous time set-up [1].

C. First passage and survival probability

Depending on the quantities of interest, survival functions are often computed to infer first passage statistics. In $z$-space, the first passage and survival functions are almost trivially connected. For the random variable $N$, we can write

$$G_{Q_N} (z) \equiv \sum_{n=0}^{\infty} z^n Q_N (n) = \frac{1 - G_N (z)}{1 - z},$$  (16)

where recall $G_N (z) = \sum_{n=0}^{\infty} z^n P_N (n)$ is the PGF for first passage time density (see [2] for derivation). This relation is very general and useful, and holds for any non-negative random variable $X$ e.g., with restarts, we have

$$G_{Q_{N_R}} (z) \equiv \sum_{n=0}^{\infty} z^n Q_{N_R} (n) = \frac{1 - G_{N_R} (z)}{1 - z},$$  (17)

where $G_{N_R} (z)$ is given by Eq. (7).
IV. GEOMETRICALLY DISTRIBUTED RESTART

In this section, we consider a specific form of restart time distribution, namely the Geometric distribution. Here, a resetting step number is taken from the following distribution with parameter $p$ ($0 < p < 1$),

$$P_R(n) = (1 - p)^{n-1}p, \quad n \geq 1.$$  \hspace{1cm} (18)

In other words, restart will occur exactly at the $n$-th step with probability $p$, after $n-1$ unsuccessful attempts. Notably, this distribution is the discrete analog of the exponential distribution, being a discrete distribution possessing the memory-less property.

A. Mean

To derive the mean completion time, we first note that the survival function for the restart can be written as

$$Q_R(n) = \Pr(R > n) = \sum_{k=n+1}^{\infty} (1 - p)^{k-1}p$$

$$= \sum_{k=1}^{\infty} (1 - p)^{k+n-1}p$$

$$= (1 - p)^n \sum_{k=1}^{\infty} (1 - p)^{k-1}p$$

$$= (1 - p)^n.$$  \hspace{1cm} (19)

We now use Eq. (19) and Eq. (16) to calculate $\langle \min (N, R) \rangle$, given in Eq. (3)

$$\langle \min (N, R) \rangle = \sum_{n=0}^{\infty} Q_N(n)Q_R(n)$$

$$= Q_N(0) + \sum_{n=1}^{\infty} Q_N(n)(1 - p)^n$$

$$= \sum_{n=0}^{\infty} Q_N(n)(1 - p)^n$$

$$= G_Q(1 - p)$$

$$= \frac{1 - G_N(1 - p)}{p}.  \hspace{1cm} (20)$$

Next, following Eq. (6), we have

$$\Pr (N < R) = \sum_{n=0}^{\infty} P_N(n) \sum_{m=n+1}^{\infty} P_R(m)$$

$$= P_N(0) + \sum_{n=1}^{\infty} P_N(n)(1 - p)^n$$

$$= G_N(1 - p),  \hspace{1cm} (21)$$

Substituting Eqs. (20) and (21) into the formula for the mean of the restarted process given in Eq. (2) yields

$$\langle N_R \rangle = \frac{1 - G_N(1 - p)}{pG_N(1 - p)},  \hspace{1cm} (22)$$

which gives us a simple plug-n-play solution for the mean completion time with the knowledge of the underlying first passage time distribution. We can also rewrite the mean first passage time in term of the PGF of the survival function. This is done by substituting Eq. (16) into Eq. (22)

$$\langle N_R \rangle = \frac{G_{Q_S}(1 - p)}{1 - pG_{Q_S}(1 - p)},  \hspace{1cm} (23)$$

which is identical to Eq. (7), derived by Kusmierz et al in [3].

B. Generating function for the restarted process

We now turn to the derivation of the PGF of the restarted process under geometric restart. We first compute the PGFs for the conditional random variables $N_{min}$ and $R_{min}$. Following a similar computation as in [2], we find

$$G_{N_{min}}(z) = \sum_{n=0}^{\infty} P_N(n) \frac{\sum_{m=n+1}^{\infty} P_R(m)}{\Pr (N < R)} z^n$$

$$= \sum_{n=0}^{\infty} P_N(n) \frac{(1 - p)^n}{G_N(1 - p)} z^n$$

$$= \frac{1}{G_N(1 - p)} \sum_{n=0}^{\infty} P_N(n)(1 - p)^n z^n$$

$$= \frac{G_N(z(1 - p))}{G_N(1 - p)},  \hspace{1cm} (24)$$

and

$$G_{R_{min}}(z) = \sum_{n=0}^{\infty} P_R(n) \frac{\sum_{m=n+1}^{\infty} P_N(m)}{\Pr (N \geq R)} z^n$$

$$= \sum_{n=1}^{\infty} (1 - p)^{n-1}p \frac{Q_N(n - 1)}{1 - G_N(1 - p)} z^n$$

$$= \frac{pz}{1 - G_N(1 - p)} \sum_{n=0}^{\infty} Q_N(n)(1 - p)^n z^n$$

$$= \frac{pzG_{Q_S}(1 - p)}{1 - G_N(1 - p)^2}$$

$$= \frac{p2G_{Q_S}(1 - p)z}{1 - (1 - G_N((1 - p)z))},  \hspace{1cm} (25)$$

where in the last step we once again used Eq. (16). Substituting Eqs. (24)-(25) into Eq. (13) we find

$$G_{N_R}(z) = \frac{1 - (1 - p)z}{1 - z + pzG_N((1 - p)z)},  \hspace{1cm} (26)$$
from which one can derive the probability mass function of the restarted process by taking the derivatives of the generating function using Eq. (15). Now substituting (using Eq. (16))

\[ G_N(z) = (z-1)G_QN(z) + 1 \]  

and

\[ G_{N_R}(z) = (z-1)G_{QNR}(z) + 1 \]

into Eq. (26) we obtain a relation between the survival functions with and without restart

\[ G_{QNR}(z) = \frac{G_QN((1-p)z)}{1-pzG_QN((1-p)z)}, \]

which is identical to Eq. (5), derived by Kusmierz et al in [3]. While Eq. (26) connects the first passage time distributions, Eq. (29) relates the survival functions.

C. Criterion for restart to be beneficial

To derive the criterion, we first observe a first passage time process and turn on an infinitesimal restart probability \( p \to 0^+ \). If restart has to lower the mean time, it is sufficient enough to check whether \( d\langle N_R \rangle/dp|_{p=0} < 0 \), where \( \langle N_R \rangle \) is given by Eq. (22). A small \( p \) expansion of \( \langle N_R \rangle \) gives

\[ \langle N_R \rangle \approx G'_N(1) + \frac{1}{2} p \left( 2G''_N(1) - G'_N(1) \right). \]

Now noting that \( G'_N(1) = \langle N \rangle, \ G''_N(1) + G'_N(1) - G'_N(1)^2 = Var(N) \) and substituting into Eq. (30), the criterion can be recast as

\[ CV^2 > 1 + \frac{1}{\langle N \rangle} \]

where \( CV^2 = \frac{Var(N)}{\langle N \rangle^2} \) is the squared coefficient of variation of the underlying first passage process. This essentially means that whether restart would favour a completion depends on the underlying first passage process. Moreover, this criterion is also not sensitive to the entire density, but only to the first two moments of the underlying process. Note that this criterion is slightly different than from the one used in [2] where we assumed that process can complete even when the steps for first passage and restart coincide i.e., \( N_R = N \) when \( N \leq R \) unlike here [see Eq. (1) in above]. Also see [13] where they obtained the same criterion. We refer to a similar criterion that was derived for the continuous stochastic resetting case in [1].

V. APPLICATIONS

In this section, we revisit a few lattice restarted random walk problems which were studied recently by different groups (and that will be mentioned in context).

A. Simple random walk

Consider a simple random walker (starting at the origin) in the presence of only one absorbing boundary located at \( x \). In this case, the generating function for the underlying process can be easily obtained following steps from [5] and this reads

\[ G_N(z) = \left( \frac{1 - \sqrt{1 - z^2}}{z} \right)^{|x|}, \quad x \neq 0. \]

In fact, it is known that for large \( n \), the first passage time density has a power law tail \( n^{-3/2} \) [4–6], which is similar to the Lévy-Smirnov distribution for the first passage time of a Brownian walker in one dimension. This power law trivially leads to a diverging mean first passage time for a RW. However, motion of the walker is restarted to the origin with a probability \( p \). This simple restart mechanism can lead to finiteness of the mean first passage time (like the simple diffusion problem in [7] and see the recent review [8]). The mean completion time of a random walker on this geometry reads

\[ \langle N_R \rangle = \frac{1}{p} \left[ \left( \frac{1 + \sqrt{2p - p^2}}{1 - p} \right)^{|x|} - 1 \right], \]

which can be obtained by substituting Eq. (32) into Eq. (22). This result was previously obtained by Riascos et al [see Eq. (26) in [9]].

B. One sided Sisyphus random walk

In [10], Montero and Villarroel studied one sided Sisyphus random walk in the presence of geometric restarts. In this dynamics, the walker starts from the origin and makes a deterministic step to the right at each time step. However, the motion is stopped with probability \( p \) and the walker is returned to the origin. The process ends as soon as the walker reaches the threshold \( a > 0 \). The first passage time is recorded accordingly. Since the process is deterministic, for the underlying first passage process, we should have

\[ N = a, \]

so that \( P_N(n) = \delta_{n,a} \), where \( \delta_{a,b} \) is the Kronecker delta. The PGF of this first passage process is given by

\[ G_N(z) = \sum_{n=0}^{\infty} P_N(n)z^n = z^a. \]

Substituting the PGF of the underlying process given in Eq. (35) into Eq. (26) yields

\[ G_{N_R}(z) = \frac{(1 - (1-p)z)(z(1-p))^a}{1 - z(1-p)(1-p)^a}. \]
Substituting $p \to 1 - q$ (keeping the notation as in [10]) yields
\[ G_{NR}(z) = (qz)^a \frac{1 - qz}{1 - z + (1 - q)z(qz)^a}, \]  
which is identical to Eq. (16) in [10]. Furthermore, substituting the PGF into Eq. (22) gives us
\[ \langle N_R \rangle = \frac{1}{1 - q} \left( \frac{1}{q^a} - 1 \right), \]  
(again by replacing $p \to 1 - q$) which is identical to Eq. (18) in [10].

**C. Two sided Sisyphus random walk**

In a recent preprint [11], Villarroel et al extended their study of one sided Sisyphus random walk to a two sided Sisyphus walk resulting in the walker in a 1D confinement (see [14, 15] for such other examples in continuous set-ups). The underlying first passage process in the interval $[-b, a]$ can be understood in the following way

\[ N = \begin{cases} 
  b & \text{prob. } 1 - \rho \\
  a & \text{prob. } \rho 
\end{cases}. \]  

This means that the walker deterministically moves to the right with probability $\rho$ and to the left with the complementary probability. The process ends as soon as the walker reaches one of the boundaries. The PGF of $N$ is then given by
\[ G_{N}(z) = \sum_{n=0}^{\infty} P_{N}(n) z^n = z^b(1 - \rho) + z^a\rho. \]  
Substituting the PGF of the underlying process given in Eq. (40) into Eq. (26), and substituting $p \to 1 - q$ yields the generating function for the entire process
\[ G_{NR}(z) = \frac{(1 - qz)((qz)^a\rho + (qz)^b(1 - \rho))}{1 - z + (1 - q)z((qz)^a\rho + (qz)^b(1 - \rho))}. \]  
This is a useful result since it helps one to derive not only the mean but also the higher order moments. For a symmetric box, Eq. (41) reduces to
\[ G_{NR}(z) = (qz)^a \frac{1 - qz}{1 - z + (1 - q)z(qz)^a}, \]  
which is identical to Eq. (37), as it should [also see Eq. (51) in [11]]. The results are identical since a walk directed to the left boundary and a walk directed to the right boundary are identically distributed for the case of a symmetric box. Thus, walks in opposite directions can be considered as different realizations of the same process.

Substituting the PGF given in Eq. (40) into Eq. (22), and $p \to 1 - q$, yields the mean escape time for the walker from the interval in the presence of restart
\[ \langle N_R \rangle = \frac{1 - (\rho q^n + (1 - \rho)q^b)}{(1 - q)(\rho q^n + (1 - \rho)q^b)}. \]  
In the case of symmetric box, the mean first passage reduces to
\[ \langle N_R \rangle = \frac{1}{1 - q} \left( \frac{1}{q^a} - 1 \right), \]  
which was also derived in [11] [see Eq. (55)]. It is important to note that our results hold for asymmetric interval, and thus are more general.

**VI. EXAMPLES OF OTHER RESTART PROTOCOLS**

In this section, we present exact results for the mean completion time of a generic first passage process (discrete space and time) under discrete restart protocols which are not necessarily memoryless. In other words, unlike geometric distribution, they may not be often thought as a probabilistic way. In fact, the protocols we choose can also come from a heavy-tailed distribution. We show how our formalism is robust to all these changes and finally the results are given in terms of the statistical metrics of the underlying first passage process.

**A. Sharp restart**

Consider now a strategy when restart events always take place after a fixed number of steps. This is often known as sharp, periodic or deterministic restart protocol (see [1, 16] for more details of this strategy). Since the resetting takes place always after a fixed period, the density can be written as
\[ P_R(n) = \delta_{n,r} = \begin{cases} 
  0, & n \neq r \\
  1, & n = r 
\end{cases}, \]  
where $\delta_{n,r}$ is the Kronecker delta. So, we will refer to this as a distribution with restart step or period $r$. We have studied this set-up in the preceding work [2]. Here, we recall the results for brevity. For sharp restart, we have
\[ \Pr (N < R) = \Pr (N < r) = \sum_{n=0}^{r-1} P_{N}(n). \]  

Furthermore, we find

\[
\langle \min(N, R) \rangle = \sum_{n=0}^{\infty} Q_N(n)Q_R(n)
\]

\[
= \sum_{n=0}^{\infty} Q_N(n) \sum_{m=n+1}^{\infty} P_R(m)
\]

\[
= \sum_{n=0}^{\infty} Q_N(n) \sum_{m=n+1}^{\infty} \delta_{m,r}
\]

\[
= \sum_{n=0}^{r-1} Q_N(n).
\]

Substituting Eq. (47) and Eq. (48) into Eq. (2) we get the following expression for the mean completion time

\[
\langle N_R \rangle = \frac{\sum_{n=0}^{r-1} Q_N(n)}{\sum_{n=0}^{\infty} P_N(n)},
\]

which is given in terms of the survival and first passage of the underlying process.

**B. Poisson restart**

Here we consider the restart steps to be drawn from the Poisson distribution namely

\[
P_R(n) = \frac{\lambda^{n-1}e^{-\lambda}}{(n-1)!}, \quad n = 1, 2, 3, \ldots
\]

where \(\lambda\) is a positive rate parameter. Here, we use a Poisson distribution with rate parameter \(\lambda\), shifted by 1, so the support of the restart distribution is strictly positive, i.e., avoiding restart at 0. Under this resetting protocol, we get

\[
\Pr(N < R) = \sum_{n=0}^{\infty} P_N(n) \sum_{m=n+1}^{\infty} P_R(m)
\]

\[
= P_N(0) + \sum_{n=1}^{\infty} P_N(n) \left( 1 - \sum_{m=1}^{n} P_R(m) \right)
\]

\[
= \sum_{n=0}^{\infty} P_N(n) - \sum_{n=1}^{\infty} P_N(n) \sum_{m=1}^{n} P_R(m)
\]

\[
= 1 - \sum_{n=1}^{\infty} P_N(n) \sum_{m=1}^{n} P_R(m)
\]

\[
= 1 - \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} P_N(n)P_R(m)
\]

\[
= 1 - \sum_{m=1}^{\infty} P_R(m) \sum_{n=m}^{\infty} P_N(n)
\]

\[
= 1 - \sum_{m=1}^{\infty} P_R(m)Q_N(m-1)
\]

\[
= 1 - \sum_{m=0}^{\infty} P_R(m+1)Q_N(m)
\]

\[
= 1 - e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} Q_N(m),
\]

and

\[
\langle \min(N, R) \rangle = \sum_{n=0}^{\infty} Q_N(n)Q_R(n)
\]

\[
= Q_N(0) + \sum_{n=1}^{\infty} Q_N(n)Q_R(n)
\]

\[
= Q_N(0) + \sum_{n=1}^{\infty} Q_N(n) \left( 1 - \sum_{m=1}^{n} P_R(m) \right)
\]

\[
= \sum_{n=0}^{\infty} Q_N(n) - \sum_{n=1}^{\infty} Q_N(n) \sum_{m=1}^{n} P_R(m)
\]

\[
= \langle N \rangle - \sum_{n=1}^{\infty} Q_N(n) \sum_{m=1}^{n} \frac{\lambda^{m-1}e^{-\lambda}}{(m-1)!}.
\]

\[
= \langle N \rangle - \sum_{n=1}^{\infty} Q_N(n) \sum_{m=0}^{n-1} \frac{\lambda^m}{m!} e^{-\lambda}
\]

\[
= \langle N \rangle - e^{-\lambda} \sum_{n=1}^{\infty} Q_N(n) \sum_{m=0}^{n-1} \frac{\lambda^m}{m!}.
\]

Substituting Eq. (51) and Eq. (52) into Eq. (2) we get the following formula for the mean first passage time under
Poissonian resetting

\[
\langle N_R \rangle = \frac{\langle N \rangle - e^{-\lambda} \sum_{n=1}^{\infty} Q_N(n) \sum_{m=0}^{n-1} \frac{\lambda^m}{m!}}{1 - e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} Q_N(m)},
\]

which is given in terms of the survival function of the underlying first passage time process.

C. Zeta restart

We now consider restart steps which are taken from Zeta distribution

\[
P_R(n) = \frac{n^{-s}}{\zeta(s)}, n = 1, 2, 3, \ldots
\]

where \( s \) is a positive integer, and \( \zeta(s) \) is the Zeta function, defined as

\[
\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \text{Re}(s) > 1.
\]

The Zeta distribution is a heavy-tailed distribution. Under this resetting protocol, following similar steps to the Poisson case, we get

\[
\Pr(N < R) = \sum_{n=0}^{\infty} P_N(n) \sum_{m=n+1}^{\infty} P_R(m)
\]

\[
= P_N(0) + \sum_{n=1}^{\infty} P_N(n) \sum_{m=n+1}^{\infty} P_R(m)
\]

\[
= P_N(0) + \sum_{n=1}^{\infty} P_N(n) \left(1 - \sum_{m=1}^{n} P_R(m)\right)
\]

\[
= \sum_{n=0}^{\infty} P_N(n) - \sum_{n=1}^{\infty} P_N(n) \sum_{m=1}^{n} P_R(m)
\]

\[
= 1 - \sum_{n=1}^{\infty} P_N(n) \sum_{m=1}^{n} P_R(m)
\]

\[
= 1 - \sum_{n=1}^{\infty} \sum_{m=1}^{n} P_N(n) P_R(m)
\]

\[
= 1 - \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} P_N(n) P_R(m)
\]

\[
= 1 - \sum_{m=1}^{\infty} P_R(m) \sum_{n=m}^{\infty} P_N(n)
\]

\[
= 1 - \sum_{m=1}^{\infty} P_R(m) Q_N(m-1)
\]

\[
= 1 - \frac{1}{\zeta(s)} \sum_{m=1}^{\infty} Q_N(m-1)m^{-s},
\]

and

\[
\langle \min(N, R) \rangle = \sum_{n=0}^{\infty} Q_N(n) Q_R(n)
\]

\[
= Q_N(0) + \sum_{n=1}^{\infty} Q_N(n) Q_R(n)
\]

\[
= Q_N(0) + \sum_{n=1}^{\infty} Q_N(n) \left(1 - \sum_{m=1}^{n} P_R(m)\right)
\]

\[
= \sum_{n=0}^{\infty} Q_N(n) - \sum_{n=1}^{\infty} Q_N(n) \sum_{m=1}^{n} P_R(m)
\]

\[
= \langle N \rangle - \sum_{n=1}^{\infty} Q_N(n) \sum_{m=1}^{n} \frac{m^{-s}}{\zeta(s)}
\]

\[
= \langle N \rangle - \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} Q_N(n) H_{n,s},
\]

where \( H_{n,s} \) is the generalized harmonic number of order \( s \) of \( n \). Substituting Eq. (56) and Eq. (57) into Eq. (2) we get the following formula for the mean first passage time under Zeta distributed resetting

\[
\langle N_R \rangle = \frac{\langle N \rangle - \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} Q_N(n) H_{n,s}}{1 - \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} Q_N(m-1) m^{-s}}
\]

which is given in terms of the survival function of the underlying process.

VII. CONCLUDING PERSPECTIVE

In this note, we presented a simple, and completely general recipe to study statistical properties of discrete first passage processes under arbitrary discrete restart steps. The stages of the computation are as follows:

- **Survival functions**: compute the survival functions \( Q_N(n) \) and \( Q_R(n) \) from the given first passage and restart time distributions namely \( P_N(n) \) and \( P_R(n) \) respectively [use Eqs. (4) and (5)].

- **Mean completion time under restart**: next, compute \( \Pr(N < R) \) and its complementary probability \( \Pr(N \geq R) \) using Eq. (6). The mean completion time can then be obtained from Eq. (2).

- **Generating function of the restarted first passage process**: compute the generating functions for the random variable \( N_{min} \) and \( R_{min} \) and plug in Eq. (13) to derive the generating function for the restarted process from which all the moments can be derived. In some cases, the first passage time density of the compound process can be obtained making use of Eq. (15). This was
demonstrated in [2].

The three-step algorithm prescribed above gives a systematic way to compute first passage quantities and investigate the effects of different restart mechanisms. Since the formalism holds for any first passage process (Markov or non-Markov) under any restart mechanism (Markov or non-Markov), we anticipate that the method would be useful to develop synergy between discrete stochastic process and resetting which are much less studied compared to the continuous space-time processes.

### Numerical treatment.

Finally, it is worth emphasizing that the machinery presented above also works to treat the problems numerically without having to solve the exact dynamics of the stochastic process of interest. What one needs is two datasets: one for the underlying first passage time (which is known for many problems numerically) and the other for resetting (which is an external input). The above-mentioned three-step recipe can then be implemented numerically, and relevant quantities may be computed. To this end, a supplemented **MATHEMATICA** file will be available on the Notebook Archive.

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