Non-perturbative corrections in $N = 2$ strings †

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Abstract

We investigate the non-perturbative equivalence of some heterotic/type IIA dual pairs with $N = 2$ supersymmetry. We compute $R^2$-like corrections, both on the type IIA and on the heterotic side. The coincidence of their perturbative part provides a test of duality. The type IIA result is then used to predict the full, non-perturbative correction to the heterotic effective action. We determine the instanton numbers and the Olive–Montonen duality groups.
All known different string constructions are conjectured to be part of a unique theory. This means in particular that compactifications with different field-content and supersymmetry correspond to different regions of a unique moduli space. The duality conjecture has received much attention in the past years, and has passed many tests, performed on different compactifications, for various dimensions and number of supersymmetries. However, in spite of this, very few tests have been performed by using string theory: most of them rely on classical arguments and on the properties of the supergravity effective theories, or on the geometry of the classical moduli space. The aim of our work is to provide a test of the duality for some specific string dual pairs, constructed at a point in the moduli space in which it is possible to explicitly solve the two-dimensional conformal theory and make one-loop string computations. The models we consider are type IIA/heterotic string pairs with \( N = 2 \) supersymmetry in four space-time dimensions. The starting point is the equivalence of all the \( N = 4 \) supergravity theories, and the (well-supported but still only conjectured) equivalence of the overlying string theories. The Hull–Townsend duality states the equivalence of the type IIA string compactified on \( T^2 \times K3 \) and the heterotic string compactified on the torus \( T^8 \). In both theories, the space of the moduli-field vacuum expectation values is spanned by 134 physical scalars, which are coordinates of the coset space:

\[
\left( \frac{SL(2, R)}{U(1)} \right)_S \times \left( \frac{SO(6, 6 + r)}{SO(6) \times SO(6 + r)} \right)_T, \quad r = 16. \tag{0.1}
\]

On the heterotic side the dilaton \( S_{\text{Het}} = S \) is in the gravitational multiplet, while on the type IIA side it is one of the moduli of the vector multiplet: \( S_{\text{II}} = T^1 \), \( T^1 \) being the volume form of the two-torus. The duality therefore is non-perturbative, involving an interchange of fields of the \( S \) and \( T \) manifolds. More \( N = 4 \) dual pairs, which involve the same field exchange, can be constructed by going to the \( T^4 / Z_2 \) orbifold limit of K3 and projecting out some states of the \( r = 16 \) theory with particular “freely-acting” projections, which remove some of the orbifold fixed points, without breaking supersymmetry further. In this way, the \( r = 8 \) and \( r = 4 \) dual pairs can be constructed. Dual pairs with \( N = 2 \) supersymmetry are then obtained by breaking the \( N = 4 \) supersymmetry with a further \( Z_2^2 \) projection, which still has the property of acting freely. The action is then a rotation on some coordinates accompanied by a translation in others. This implies that the supersymmetries are broken “spontaneously”, and it is possible to restore them by going to a specific corner in the moduli space of these orbifolds. The theories with \( N = 4 \) and \( N = 2 \) are then continuously related, and the duality of the \( N = 4 \) phase is naturally inherited by the \( N = 2 \) phase. The perturbative heterotic scalar manifolds of the models we consider are

\[
\frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2 + N_V)}{SO(2) \times SO(2 + N_V)} \quad \text{and} \quad \frac{SO(4, 4 + N_H)}{SO(4) \times SO(4 + N_H)}, \tag{0.2}
\]

with \( N_V = N_H = r = 8, 4, 2 \), for moduli respectively in the vector multiplets and hypermultiplets.

**Type IIA**

On the type IIA side, these models correspond to self-mirror Calabi–Yau threefolds with Hodge numbers \( h^{1,1} = N_V + 3 = h^{2,1} = N_H + 3 \), which are K3 fibrations. This last property is
a necessary condition for the existence of the heterotic duals \[15, 16\], and is directly related to the spontaneous breaking of the \(N = 4\) supersymmetry \[10, 17\]. The gauge group is Abelian:

\[
G = [U(1)^2] \times U(1)'
\]

(within square brackets, we indicate the gauge symmetry corresponding to the bosons of the untwisted sector of the orbifold). The \(r = 8\) model is obtained from the \(N = 4, r = 16\), with the only SUSY-breaking projection \(Z_2^f\). The \(r = 4\) and \(r = 2\) orbifolds are then obtained by applying respectively one and two “\(Z_2^D\)”-projections; these latter act on the \(Z_2\) orbifolds by modding out the states that are odd under the \(D\)-symmetries, which exchange twist fields \(\sigma_{\pm}\) and project the untwisted vacua \(V_{nm}\) \[18\]:

\[
D: (\sigma_+, \sigma_-, V_{nm}) \rightarrow (\sigma_-, \sigma_+, (-)^{nm}V_{nm})
\]  

\[0.4\]

\textbf{Heterotic}

The heterotic duals are constructed by following analogous steps, starting from the \(N = 4, "E_8 \times E_8"\) string compactified on \(T^6\). However, on the heterotic side, in order to obtain the same spectrum as on the type IIA side, we need to introduce also \(Z_2\)-projections (discrete Wilson lines), which break the initial gauge group to \([U(1)^2] \times U(1)^{16}\). The heterotic analogue of the \(Z_2^f\)-projection acts on the coordinates of the compact space \(T^6 = T^2 \times T^4\) as a rotation on \(T^4\) accompanied by a translation in \(T^2\). On the \(c = (0, 16)\) part of the currents, it acts instead as a lattice exchange \[[1]\]. The result of this operation is the spontaneous breaking of the \(N = 4\) supersymmetry to \(N = 2\) and a reduction of the rank of the gauge group from \(r = 16\) (we omit for simplicity the gauge group of the torus) to \(r = 8\). Out of the Abelian point, the rank \(r\) part of the gauge group is realized at the level 2. The models with \(r = 4\) and \(r = 2\) are then constructed by applying further projections, the heterotic analogue of the \(Z_2^D\)-projections, which act on the currents as a further lattice exchange, and as a translation on some specific directions of \(T^4\). Out of the \(U(1)^r\) point, the gauge group is realized at the level \(16/r\). The action of the translations associated to the above projections is not completely arbitrary, being constrained by modular invariance. There is, however, some freedom in choosing the specific embedding of those, and it turns out that there exists a particular choice that corresponds to the same region of the perturbative moduli space as that selected in the type IIA duals. This renders possible the identification of the map between the moduli of the heterotic and type IIA dual constructions (for details, see \[[10]\]).

\textbf{The gravitational corrections}

In the particular region of moduli space we are considering, it is possible to construct the partition function of these orbifolds and to make a direct computation of the string corrections to any term of the effective action. The particular term we consider here is a well-defined combination of the \(R^2 = \langle R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \rangle\) gravitational amplitude \[20, 22\] and of

\[1\]This construction is similar to that of \[19\]. For the details, see \[10\].
gauge amplitudes, namely the $F_{\mu\nu}F^{\mu\nu}$ amplitude corresponding to the gauge bosons of the $U(1) \times U(1)$ symmetry originating from the heterotic untwisted torus $T^2$ (or, equivalently, on the type IIA side, from the untwisted sector) and the amplitude corresponding to the gauge bosons of the rank $r$ (level $16/r$) part. The $F_{\mu\nu}F^{\mu\nu}$ terms, although vanishing at a generic point in the moduli space, turn out to be necessary for a comparison of the type IIA and heterotic theories. These amplitudes are computed by inserting specific operators in the partition function.

On the type IIA side, the correction to the $R^2$ term is computed at one loop and is obtained by inserting the operator $2Q_2\overline{Q}^2$, $Q$ and $\overline{Q}$ being respectively the left-handed and right-handed helicity operators. Owing to the world-sheet symmetry between left and right movers, the insertion of the helicity operators amounts eventually to a scalar derivative on modular forms, considered as functions of the modular parameter $\tau$ of the world-sheet torus. This amplitude depends on the moduli $T^1$, $T^2$, $T^3$, but owing to the absence of $\Delta N_V \neq 0$ and/or $\Delta N_H \neq 0$ singularities, it is regular. The gauge amplitudes, on the other hand, vanish trivially, being the gauge-bosons Ramond–Ramond states. After specifying the direction of the $Z_f^2$ and $Z_D^2$ translations, which we choose to act on the momenta, we obtain:

\[
\frac{16 \pi^2}{g^2_{\text{grav}} (\mu^{(II)})} = - \frac{3N_V}{4} \log \frac{\mu^{(II)} \text{Im} T^1}{M^{(II)}} \left| \eta (T^1) \right|^4 - \left( 2 - \frac{N_V}{4} \right) \log \frac{\mu^{(II)} \text{Im} T^1}{M^{(II)}} \left| \vartheta_4 (T^1) \right|^4 \\
- 2 \log \frac{\mu^{(II)} \text{Im} T^2}{M^{(II)}} \left| \vartheta_4 (T^2) \right|^4 - 2 \log \frac{\mu^{(II)} \text{Im} T^3}{M^{(II)}} \left| \vartheta_4 (T^3) \right|^4 + \text{const.} ,
\]

with $\mu^{(II)}$ and $M^{(II)}$ the infrared cut-off and the string scale of type IIA, respectively.

On the heterotic side, both genus-0 and genus-1 contribute to the $R^2$ term. However, the genus-1 contribution, which depends on the moduli $T$ and $U$, respectively the Kähler class and the complex structure modulus of the untwisted torus $T^2$, is not regular as a function of these fields. In fact, its beta-function jumps at the special values of these moduli for which new vector- or new hyper-multiplets appear in the spectrum (in particular, this happens when the $U(1)$ factors from the torus are enhanced to $SU(2)$). Moreover, on the heterotic side there is no world-sheet symmetry between left and right movers, and the insertion of the “gravitational” operator in the partition function does not amount, after saturation of the fermion zero-modes, to a scalar but to a covariant derivative on modular forms $[26]–[30]$:

\[
D_\tau = \partial_\tau - \frac{id}{2 \text{Im} \tau} ,
\]

where $d$ is the weight of the form. In order to obtain a regular amplitude, we correct the $R^2$ term by adding to it a term proportional to the $F^2$ amplitude of the torus, whose beta-function vanishes for generic values of the moduli $T$ and $U$, but jumps at the singular points, in a way opposite to the jumping of the $R^2$ beta-function. This amplitude is now regular, but it still corresponds to an operator that acts as a covariant derivative. In order to obtain a scalar operator, we add also a term proportional to the $F^2$ amplitude of the level $16/r$ part of

\[
\text{const.} .
\]

\footnote{For details, see [2] [13] [23] [24] [25] [26].}

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the gauge group. The latter has vanishing beta-function and is regular, for any value of the moduli \( T \) and \( U \). The proper combination, which has the required properties of regularity and "holomorphicity" (i.e. no \( \text{Im} \tau \)-covariantization), is therefore:

\[
Q_{\text{grav}}^2 = R^2 + \frac{1}{12} (F_{\mu
u}F^{\mu\nu})_T^2 + \frac{5}{36N_V} (F_{\mu
u}F^{\mu\nu})_{G(N_V)} .
\] (0.7)

Saturation of the fermion zero-modes implies that only the so-called \( N = 2 \) sector of the heterotic orbifolds contributes to the one-loop amplitude. This sector corresponds to the terms of the partition function that are projected and twisted by \( Z_2^f \). A remarkable fact is that this sector is universal: namely, it is the same for all three models, and it does not depend on the extension of the rank \( r \) part of the gauge group, \( G(N_V) : U(1)^r, (SU(2))_{16/r} \), etc. As a consequence, the one-loop amplitude of (0.7) is the same for all models [10].

Putting together the tree-level and the one-loop contributions, we get, for all the models:

\[
\frac{16 \pi^2}{g_{\text{grav}}^2 (\mu_{\text{Het}})} = 16 \pi^2 \text{Im}S - 2 \log \text{Im}T |\vartheta_4 (T)|^4 - 2 \log \text{Im}U |\vartheta_4 (U)|^4 + 4 \log \frac{M^{\text{(Het)}}}{\mu^{\text{(Het)}}} + \text{const.} ,
\] (0.8)

where \( S \) is the dilaton–axion field

\[
\text{Im}S = \frac{1}{g_{\text{Het}}^2} ,
\] (0.9)

and we expressed the infrared running in terms of the heterotic string scale \( M^{\text{(Het)}} \equiv 1/\sqrt{\alpha^{\prime}_{\text{Het}}} \) and cut-off \( \mu^{\text{(Het)}} \). The coefficient 2 of the terms containing the contribution of the moduli \( T, U \) is actually the “gravitational beta-function”, \( b_{\text{grav}} = (24 - N_V + N_H)/12 \). A comparison of (0.5) and (0.8) in the heterotic perturbative limit, \( \text{Im}S \rightarrow \infty \), leads to the identification of \( T^2 / Z_2^\tau \) with \( 16 \tau_S^2 / N_V \) (\( \tau_S = 4 \pi S \)). This is consistent with the interpretation of the compact spaces in the type IIA models as orbifold limits of K3 fibrations, with base respectively \( T^2 / Z_2^\tau = \mathbf{P}^1, \mathbf{P}^1 / Z_2^\eta \) and \( \mathbf{P}^1 / (Z_2^\eta \times Z_2^{\eta'}) \) for \( N_V = 8, 4, 2 \). The heterotic “dilaton” \( \tau_S \) is then the volume form of the base of these fibrations. Moreover, the perturbative corrections, as a function of moduli \( T^2 \) and \( T^3 \) on the type IIA side and as a function of \( T \) and \( U \) on the heterotic side, coincide with the identification of \( T^2, T^3 \) with \( T, U \). These identifications provide a test of the type IIA/heterotic duality.

Once established the precise duality map, we use the type IIA result to predict the full, perturbative and non-perturbative correction to the heterotic effective action. This is therefore given by the expression (0.5), with the above-mentioned substitutions of the moduli of the type IIA compactification with their heterotic duals. The contribution of moduli \( T, U \) to the effective coupling constant, \( 16 \pi^2 / g_{\text{grav}}^2 \), is then universal and coincides with the one computed at one loop, while the dilaton dependence, for the various models, is:

\[
- \frac{3N_V}{4} \log \left| \eta \left( \frac{16 \tau_S}{N_V} \right) \right|^4 - (2 - \frac{N_V}{4}) \log \left| \vartheta_4 \left( \frac{16 \tau_S}{N_V} \right) \right|^4 - \left( \frac{B_4 - B_2}{3} - 2b_{\text{grav}} \right) \log \text{Im}\tau_S + \frac{B_4 - B_2}{3} \log \frac{M_{\text{Planck}}}{\mu} ,
\] (0.10)
where we have expressed the infrared running in terms of the Planck mass and the physical cut-off $\mu$, related to the string scales and cut-offs by:

$$\frac{M_{\text{II}}^{(\text{II})}}{\mu^{(\text{II})}} = \frac{M_{\text{Het}}^{(\text{Het})}}{\mu^{(\text{Het})}} = \frac{M_{\text{Planck}}}{\mu}. \tag{0.11}$$

The coefficients of the terms in the second line of (0.10) are given in terms of the massless contribution to the helicity supertraces $B_3 = \text{Str}(Q + \bar{Q})^2$, $B_4 = \text{Str}(Q + \bar{Q})^4$, and the gravitational beta-function. Since the amplitude of $Q^2_{\text{grav}}$ is regular in the $(T,U)$-plane, for these quantities the values have to be taken at a generic point in the moduli space.

The dilaton contribution (0.10) can be interpreted as a series that contains the perturbative (linear) contribution and the exponential, instantons contributions. The instantons are due to Euclidean five-branes also wrapped around the time direction. The instanton numbers $k$ are the powers in the expansion of this expression in the parameter $q_S = \exp 2\pi i \tau_S$. We obtain $k = 2n$, $n \in N$, for $N_V = 8, 4$ and $k = 4n$ for $N_V = 2$. As a consequence of the non-perturbative $Z^2_1$ and $Z^2_D$ translations in the plane of the dilaton, also the Montonen–Olive $SL(2, Z)_{S}$-duality group is broken. The unbroken subgroup depends on $N_V$: it is a $\Gamma(2)$ subgroup for $N_V = 8$, $\Gamma(8)$ for $N_V = 4$, and $\Gamma(16)$ for $N_V = 2$.

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3For notation and conventions see for instance [4, 11, 31, 32].
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