Ideal ballooning modes, shear flow and the stable continuum

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Abstract
There is a well-established theory of ballooning modes in a toroidal plasma. The cornerstone of this is a local eigenvalue \( \lambda \) on each magnetic surface—which also depends on the ballooning phase angle \( k \). In stationary plasmas, \( \lambda(k) \) is required only near its maximum, but in rotating plasmas its average over \( k \) is required. Unfortunately in many cases \( \lambda(k) \) does not exist for some range of \( k \), because the spectrum there contains only a stable continuum. This limits the application of the theory, and raises the important question of whether this ‘stable interval’ gives rise to significant damping. This question is re-examined using a new, simplified, model—which leads to the conclusion that there is no appreciable damping at small shear flow. In particular, therefore, a small shear flow should not affect ballooning mode stability boundaries.

(Some figures may appear in colour only in the online journal)

1. Introduction
Magnetohydrodynamics (MHD) ballooning modes, that is perturbations with short wavelength perpendicular to the local magnetic field, but long wavelength parallel to it, are a particularly insidious form of plasma instability [1]. In an axisymmetric toroidal plasma they are normal modes with large toroidal mode number \( n \). If the plasma is stationary (no toroidal flow) then after introducing an extended poloidal coordinate \( \eta \) (which removes explicit periodicity constraints [2]) these modes can be represented by an eikonal that describes the short wave behavior,

\[
\xi = \xi(x, \eta) \exp i n q'(x \eta + S(x)),
\]

where \( x \) is a minor radius coordinate, constant on each magnetic surface, and \( q' \) is the average magnetic shear.

In the limit \( n \to \infty \), each magnetic surface then decouples from its neighbors and has its own local growth rate \( \lambda(x, k) \), given as the eigenvalue of a differential equation in \( \eta \), with the surface label \( x \) and the phase \( k = dS/dx \) (related to the poloidal location of the mode) as parameters [2],

\[
\frac{d}{d\eta} P(\eta; x, k) \frac{d}{d\eta} \xi(\eta) + Q(\eta; x, k) \xi(\eta) = \lambda^2(x, k) R(\eta; x, k) \xi(\eta).
\]

The coefficients \( P, Q, R \) are defined by the plasma equilibrium and are periodic in the phase angle \( k \), as is the eigenvalue \( \lambda(x, k) \). The boundary condition is that \( \xi \) be bounded as \( |\eta| \to \infty \).

This local growth rate \( \lambda(x, k) \) contains the information necessary to construct global ballooning modes [2]. For example, near marginal stability, the ballooning mode growth rate is

\[
\Lambda = \lambda_{\text{max}} - \frac{1}{2nq'} \left[ \frac{\partial^2 \lambda}{\partial k^2} + \frac{\partial^2 \lambda}{\partial x^2} \right]^{1/2},
\]

where \( \lambda_{\text{max}} \) is the maximum of \( \lambda(x, k) \).

If there is a sheared toroidal rotation \( \Omega(x) \) in the equilibrium plasma, the eikonal (1) is no longer valid. Instead the appropriate form is [3]

\[
\xi(x, \eta) \exp[i n q'(x \eta + S(x) + \Omega(x)t)].
\]

In the limit \( n \to \infty \), each magnetic surface again decouples from its neighbors but its time dependence is no longer purely exponential. Instead it is of Floquet form, given [3] on each surface by

\[
\frac{\partial}{\partial t} R(\eta; x, \hat{k}) \frac{\partial}{\partial \eta} \xi(\eta, t) = \frac{\partial}{\partial \eta} P(\eta; x, \hat{k}) \frac{\partial}{\partial \eta} \xi(\eta, t) + Q(\eta; x, \hat{k}) \xi(\eta, t)
\]

where \( \hat{k} = k + s v_t \) and \( s_v = \Omega'/q' \) is the velocity shear.

When the shear velocity is small the coefficients \( P, Q, R \) evolve slowly in time compared with the perturbation growth
rate and we can find an ‘adiabatic’ solution of equation (5) with exponential growth $\sim \exp(\int \check{\lambda}(x, t) \, dt)$, where
\[
\frac{\partial}{\partial \eta} P(\eta; x, \check{k}) \frac{\partial}{\partial \eta} \xi(\eta, t) + Q(\eta; x, \check{k}) \xi(\eta, t) = \check{\lambda}^2(x, t) R(\eta; x, \check{k}) \xi(\eta, t).
\] (6)

Thus $\check{\lambda}(x, t)$ is just the local growth rate of the static plasma (given by equation (2)), at a time dependent phase angle $(k + s, t)$, and the growth rate of ballooning modes in a rotating plasma therefore depends on the average of $\lambda$ over $k$.

So we see that, even with shear flow, ballooning instabilities can be described through the local growth rate $\lambda(x, k)$ of a static plasma. However, there is a crucial difference in the role of $\lambda$ when shear flow is significant. In the absence of shear flow, $\lambda(x, k)$ is required only in the vicinity of its maximum over $k$, say at $k_m$, whereas with shear flow it is required over the full range $0 < k < 2\pi$. This seemingly small difference introduces a fundamental difficulty: $\lambda(x, k)$ may not be defined for all $k$.

This problem arises when, as is often the case, equation (2) has an unstable eigenvalue only over a restricted range of phase angle $k$, around $k_m$; outside this range it has no discrete eigenvalues—only a stable continuum $\lambda = \pm \xi_0$. Within this ‘stable interval’ of $k$, the solution cannot be represented by an adiabatic form with exponential time dependence. (This depends on the highest eigenvalue being well separated from all others—a condition that is clearly violated when there is only the continuum.) It is then not clear how $\lambda$ should be interpreted (or replaced).

Formally therefore, the applicability of ballooning theory to plasmas with shear flow is severely limited. This limitation is particularly apparent in determining ballooning mode stability boundaries. By definition these are where a mode with some particular phase $k$ first becomes unstable; however, modes with any other phase angle will remain stable—so that near a stability boundary we expect there to be a significant stable interval in $k$. An important issue is whether, with sheared rotation, this stable interval leads to significant damping. If not, then although even a small rotation will reduce the growth rate of ballooning modes, it alone cannot stabilize them—consequently the formal stability boundaries for ballooning modes will be unchanged by small rotation.

The aim of this paper is to investigate the effect of a stable interval on ballooning modes in a plasma with small velocity shear. The next section describes a model of a toroidal plasma which incorporates a stable interval and introduces previous (contradictory) attempts to deal with it. Section 3 describes a new approach to the problem. This is discussed in detail in section 4. A summary and some conclusions are given in section 5.

2. The stable interval

There are conflicting views, leading to contradictory conclusions, on how to deal with the the stable interval. To illustrate the problem we invoke the well-known $(s, \alpha)$ model [4] of a large aspect ratio tokamak. ($\alpha$ is the plasma pressure gradient and $s = r q'/q$ is the magnetic shear.) In this model, equation (5) takes the specific form
\[
\frac{\partial}{\partial \eta} (1 + P^2) \frac{\partial}{\partial \eta} \xi(\eta, t) + Q \xi(\eta, t) = \frac{\partial}{\partial t} (1 + P^2) \frac{\partial}{\partial t} \xi(\eta, t)
\] (7)

with
\[
P = s \eta - \alpha \sin(\eta + s, t)
\]
\[
Q = \alpha [\cos(\eta + s, t) + P \sin(\eta + s, t)].
\] (8)

When the shear velocity $s$ is small, equation (7) can be reduced to the simpler ‘wave equation’ form
\[
\frac{\partial^2 \psi}{\partial \eta^2} + V(\eta, s, t) \psi = \frac{\partial^2 \psi}{\partial t^2}
\] (9)

where $\psi = \xi/\sqrt{1 + P^2}$ and the ‘potential’ $V$ is
\[
V(\eta, s, t) = \frac{(s - \alpha \cos(\eta + s, t))^2}{(1 + P^2)^2} + \frac{\alpha \cos(\eta + s, t)}{1 + P^2}.
\] (10)

The corresponding equation for the local growth rate is
\[
\frac{\partial^2 \psi}{\partial \eta^2} + V(\eta, s, t) \psi = \lambda^2(s, t) \psi.
\] (11)

A stability diagram for the $s, \alpha$ model is shown in figure 1. As expected, there is a large region within the conventional ‘unstable’ zone where most phase angles are still stable.

One way to deal with the stable interval [5] is to represent the perturbation as an integral over the continuum modes. There are difficulties in justifying this approach, but the underlying picture is that continuum modes are excited as the phase $k + s, t$ enters the stable interval. The ‘depth’ of this excitation is proportional to $s^{-1}$ but the duration of the subsequent stable interval is proportional to $1/s_0$—so that phase mixing during the stable interval reduces the amplitude by a fixed amount per transit through the interval. Averaged over the whole interval this is equivalent to a damping rate that is proportional to $\Omega^2$ and vanishes as $\Omega \to 0$.

Another approach [6] is based on the fact that as $\eta \to \infty$ the solution of equation (9) will consist of outgoing and incoming, reflected, waves. If the reflected waves can be neglected it becomes appropriate to impose an ‘outgoing wave only’ boundary condition. This has no effect on eigenvalues of equation (11) in the unstable interval—because if $\lambda > 0$ solutions that match to outgoing waves are also those which decay as $\eta \to \infty$. However, in a stable interval the change in
boundary condition has a dramatic effect—because if \( \lambda < 0 \) a solution that matches to an outgoing wave would otherwise diverge as \( \eta \to \infty \) and be rejected as an eigenfunction. In fact, with the ‘outgoing wave’ boundary condition, equation (11) no longer has a continuum of eigenvalues \( \lambda = \pm i \omega \) in the stable interval; instead it will usually have a discrete negative eigenvalue. In this case the stable interval will introduce a significant ‘wave’ damping that is independent of \( \Omega' \) and persists as \( \Omega' \to 0 \).

This is essentially the view taken by Waelbroek and Chen [6], although they also introduced a further approximation. In essence they assumed that in equation (11) the range of \( \eta \) can be separated into an inner region where the inertial term \( \lambda^2 \) is negligible, and an outer region where \( V \) is negligible. One then solves equation (11) for \( \psi \) in the inner region (with \( \lambda^2 = 0 \)) and at its boundary matches its logarithmic derivative \( \Delta' \) to an outgoing wave. Then \( \lambda = \Delta' \) and the problem of finding an eigenvalue of the full equation (9) with the outgoing wave boundary condition is avoided.

Whether or not one adopts this additional approximation (which may be valid only for small \( \Delta' \)), the crucial question is whether reflected waves can be neglected. The fact that the potential \( V \) is small at large \( \eta \) certainly makes the reflection coefficient small. But if the perturbation were decaying exponentially during the stable interval, as it does when reflection is ignored, then any reflected wave would have been created when the outgoing wave was exponentially larger than it is when the reflected wave returns. This large exponential factor could outweigh the small reflection coefficient when \( s_v \) is small—making the reflected wave important during the stable interval.

### 3. A ‘Toy’ model

To investigate the importance of reflection we introduce a simple ‘Toy’ model. Suppose first that the potential vanishes outside some small region around \( \eta = 0 \), that is \( V(\eta, t) = -D(s_v, t)\delta(\eta) \) (so that \( D \) corresponds to \( \Delta' \) in the account above). In this Toy model equation (9) has an exact solution

\[
\psi = A \exp \int_{s_v'}^{s_v} D(s_v', t') \, ds' \tag{12}
\]

valid whether \( D \) is positive (unstable interval) or negative (stable interval) and for all \( s_v \). This is precisely the result one would get from the Waelbroek and Chen approximation and appears to support their picture of the damping. But of course the model does not yet address the question of reflections from an extended potential. To do this we modify it by adding a weak potential ‘tail’ \( v(\eta) \) that \( \to 0 \) as \( \eta \to \infty \). Then \( V(\eta, t) = D(s_v, t)\delta(\eta) + v(\eta) \).

At this point it is convenient to express equation (9) (using Green’s function or more simply by introducing \( (\eta \pm t) \) as coordinates) in an integral form:

\[
\psi(\eta, t) = \frac{1}{2} \int_{s_v'}^{s_v} ds' \int_{\eta - (t - s')}^{\eta + (t - s')} \psi(\eta', t') V(\eta', t') \, d\eta'. \tag{13}
\]

Then, with \( V(\eta, t) = D(s_v, t)\delta(\eta) + v(\eta) \), the central perturbation, \( \psi(0, t) = \Psi(t) \) satisfies

\[
\frac{d\Psi(t)}{dt} = D(s_v, t)\Psi(t) + \frac{1}{2} \int_0^\infty \psi(\eta', t - \eta') v(\eta', t - \eta') \, d\eta'. \tag{14}
\]

Reflected waves arise from the integral term in equation (14). To calculate the first order reflected wave, linear in \( v \), we can replace \( \psi(\eta', t - \eta') \) in this integral by its form when \( v \) is ignored—then \( \psi(\eta, t) \) is constant along lines of fixed \( (t - \eta) \) and \( \psi(\eta', t - \eta') = \Psi(t - 2\eta') \). Thus we obtain a closed equation for the central perturbation \( \Psi(t) \).

\[
\frac{d\Psi(t)}{dt} = D(s_v, t)\Psi(t) + \frac{1}{2} \int_{-\infty}^{t'} \Psi(\eta') v((t - \eta')/2, (t + \eta')/2) \, d\eta'. \tag{15}
\]

(For a geometrical interpretation of equations (13)–(15) see figures 2 and 3).

### 4. Calculation

The effect of a weak, but extended, potential tail can be calculated explicitly in the case that it decays exponentially \( \sim \exp(-p\eta) \) and is independent of time. Then

\[
\frac{d\Psi(t)}{dt} = D(s_v, t)\Psi(t) + v_0 \int_{-\infty}^{t'} \Psi(\eta') \exp(-p\eta') \, d\eta'. \tag{16}
\]
With \( \tau = s_1 t \), this is equivalent to the equations

\[
\frac{d\Psi}{dt} - D(\tau) \Psi(\tau) = \Phi(\tau) : \frac{d\Phi}{dt} + p \Phi(\tau) = v_0 \Psi(\tau).
\]  
(17)

When \( s_e \) is small, equations (17) have the WKB solutions

\[\psi^P(\tau) = q(\tau) \exp\left( \frac{1}{s_e} \int_0^\tau D(\tau') d\tau' \right),\]

\[\psi^S(\tau) = \frac{1}{q(\tau)} \int_0^\tau D(\tau') + 1 \exp\left( - \frac{p}{s_e} \tau \right) \]

where

\[q(\tau) = \exp\left( \frac{v_0}{s_e} \int_0^\tau \frac{1}{D(\tau') + 1} d\tau' \right)\]

which, since \( v_0 \) is assumed small, is slowly varying compared with \( \psi^P(\tau), \psi^S(\tau) \).

Thus we see that perturbation \( \Psi \) has two components. One is essentially determined by the strength of the central potential \( D \); the other by rate of decay (not the strength!), of the potential tail. For convenience we will refer to these as the primary (P) and secondary (S) components, respectively. The two components are independent of each other except at the transition points where \( D + p = 0 \), that is near the beginning and end of a stable interval.

To describe the coupling between P and S components at the transition points it is convenient to write \( \Psi(\tau) = \chi(\tau) \exp(-pt/s_e) \). Then

\[s_e \frac{d^2 \chi}{d\tau^2} - (D(\tau) + p) \frac{d\chi}{d\tau} - \left( D'(\tau) + \frac{v_0}{s_e} \right) \chi = 0.\]

At the first transition, near the start of a stable interval, \( D + p \sim -at \) with \( a > 0 \) and

\[\frac{d^2 \chi}{d\tau^2} + u \frac{d\chi}{d\tau} + (1 - b) \chi = 0,\]

where \( u = \sqrt{a/s_e} \tau, \ b = v_0/bs_e \) and the primary and secondary components are \( \chi^P(u) = u^b \exp(-u^2/2), \chi^S(u) = u^{b-1} \).

The general solution of equation (21) can be expressed in terms of parabolic cylinder functions as

\[\chi = [A D_{-b}(u) + B D_{b-1}(iu)] \exp(-u^2/4)\]

(22)

and the asymptotic expansions of the PCFs link the amplitudes of \( \chi^P(u) \) and \( \chi^S(u) \) before and after the transition. In particular, \( D_{-b}(u) \) links \( \chi^P \), which is sub-dominant before the transition, to the combination \( \chi^P + C_1 \chi^S \) after the transition. That is

\[\chi^P(u) \rightarrow D_{-b}(u) \exp(-u^2/4) \rightarrow \chi^P(u) + C_1 \chi^S(u)\]

(23)

with the coupling coefficient \( C_1 = \sqrt{2\pi / \Gamma(b)} \).

After the transition the primary component decays rapidly, so this transition effectively converts the exponentially growing perturbation in the unstable interval into a slowly varying perturbation during the stable interval. In terms of \( \tau; \)

\[\tau^{-b} \exp(-a\tau^2/2s_e) \rightarrow C_1 \cdot (a/s_e)^{(b-1)/2} \tau^{(b-1)}\]

(24)

At the second transition, near the end of the stable interval, \( D + p \sim +at \) and

\[\frac{d^2 \chi}{d\tau^2} - u \frac{d\chi}{d\tau} - (1 + b) \chi = 0.\]

(25)

The primary and secondary components at this transition are \( \chi^P(u) = u^b \exp(u^2/2), \chi^S = u^{b-1} \), and the general solution of (25) is

\[\chi = [A D_b(iu) + B D_{-(b+1)}(-u)] \exp(+u^2/4).\]

(26)

In this case the asymptotic expansions show that \( D_{-(b+1)}(u) \) links \( \chi^S \), sub-dominant before the transition, to the combination \( \chi^P + C_2 \chi^S \) after the transition.

\[\chi^S(u) \rightarrow D_{-(b+1)}(u) \exp(u^2/4) \rightarrow (\chi^P + C_2 \chi^S(u))\]

(27)

with coupling coefficient \( C_2 = \sqrt{2\pi / \Gamma(1+b)} \). After this transition the secondary component rapidly becomes negligible compared with the exponentially growing primary component. This transition therefore transforms the slowly varying perturbation during the stable interval back to exponential growth in the next unstable interval. In terms of \( \tau\)

\[\tau^{-(b+1)} \rightarrow C_2 \cdot (a/s_e)^{(b+1)/2} \tau^{b} \exp(at^2/2s_e).\]

(28)

To calculate the full effect of passing through a stable interval we also need to take into account the change in the secondary component across the interval, that is from \( \chi(\tau_1) = \int_{\tau_1}^{t_1} \chi(\tau) \exp(-pt/s_e) \) at a time \( \tau_1 \) immediately after the first transition to \( \chi(\tau_2) = \int_{\tau_2}^{\tau_2} \chi(\tau) \exp(-pt/s_e) \) at a time \( \tau_2 \) immediately before the second transition. From equations (18) and (19) this is given by

\[\frac{\chi(\tau_2)}{\chi(\tau_1)} = \mu^2 \left( \frac{\tau_2^{-(b+1)}}{\tau_1^{-(b+1)}} \right).\]

(29)

Here \( \mu \) is a numerical coefficient that depends on the precise form of \( D(\tau) \) in the stable interval. (When \( D(\tau) = -a \sin(\tau), \mu = 2 \).

We see therefore, that if we follow the perturbation from an unstable interval, through the following stable interval and into the next unstable interval, its amplitude is changed by a factor

\[\tilde{C} = (2\pi / \Gamma(1+b)) \mu^{2b} (a/s_e)^{2b} \]

(30)

For small shear this is equivalent to a time constant

\[\Lambda_{\text{stable}} = s_e \log \tilde{C} \sim \frac{2v_0}{a} \left( \log \left( \frac{2a^2}{v_0^2} \right) + 1 \right) - \frac{1}{6} \left( \frac{s_e^2}{v_0} \right) \]

(31)

throughout the stable interval.

5. Summary and conclusion

The theory of ballooning modes in a toroidal plasma [1, 2] is based on a local growth rate \( \lambda \)—an eigenvalue of equation (2)—that is periodic in the phase angle \( k \). If the plasma is at rest, \( \lambda \) is required only near a maximum. However, in a toroidally rotating plasma \( k \) increases continuously and \( \lambda \) is required over the full range \( 0 < k < 2\pi \). In many cases the plasma is stable for some range of \( k \) and an important question is whether this ‘stable interval’ leads to significant damping.
In rotating plasmas, ballooning modes are governed by the wave-equation-like equation (5), and a determining factor is whether, at large $\eta$, reflected waves can be neglected. If so, then an ‘outgoing wave’ boundary condition is appropriate and leads to the conclusion [6] that there is a negative eigenvalue $\lambda(k)$ in the stable interval. This would represent a significant damping—indeed independent of $/Omega_1'$ and persisting as $/Omega_1' \to 0$.

The crucial question is whether reflected waves can indeed be neglected. The analysis in this paper, based on a model that takes specific account of reflections, suggests otherwise. It appears that no matter how small they may be, reflections determine the behavior during the stable interval—and the negative eigenvalues of $\lambda$ are therefore not relevant. Instead, at the start of the stable interval there is a transition from exponential growth to a near constant amplitude throughout the interval, and a transition back to exponential growth at its end. These transitions, and the stable interval itself, change the perturbation by a factor $\hat{C}$ (equation (30)) each time the phase $k(t)$ passes through the stable interval. At small shear this change is negligible compared with the decay that would occur if reflections were ignored.

The behavior described above is clearly seen in numerical solutions of the model equations (17). Figures 4 and 5 show that the instantaneous growth rate closely follows $\exp \int (D(\tau)/s_x) \, d\tau$ during the unstable interval. It becomes briefly negative at the start of the stable interval (at this transition the coupling coefficient is less than unity), increases slowly during the stable interval and then returns smoothly to exponential growth at the start of the next unstable interval. Figures 6 and 7 show that as $s_x$ decreases the perturbation becomes effectively constant throughout the stable interval (so that effectively the growth rate is $\lambda(k)$ in an unstable region and zero in a stable interval).

These features are remarkably similar to what is seen in numerical computations of ballooning modes by Furukawa and colleagues [7–9]. These computations simulate the behavior of ballooning modes in realistic Tokamak configurations, using initial value codes. The example in figure 8 (adapted from [9]), shows the logarithmic growth rate of the kinetic energy of an $n \to \infty$ ballooning mode in a tokamak with toroidal rotation. (The ‘Mach’ numbers $M_i$ are proportional to the shear flow velocity.) This closely resembles the model solutions in figures 5 and 7.

In conclusion, it appears that, in contrast to some previous results, a stable interval in $k$ does not lead to significant damping of ballooning modes in a plasma with small toroidal...
shear flow. Interestingly, it therefore does not change formal ballooning mode stability boundaries.

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