ON THE POLYNOMIAL AUTOMORPHISMS OF A GROUP

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ABSTRACT. Let \( A(G) \) denote the automorphism group of a group \( G \). A polynomial automorphism of \( G \) is an automorphism of the form \( x \mapsto (v_1^{-1} x^\epsilon_1 v_1) \cdots (v_m^{-1} x^\epsilon_m v_m) \). We prove that if \( G \) is nilpotent (resp. metabelian), then so is the subgroup of \( A(G) \) generated by all polynomial automorphisms.

1. Introduction and main results

Let \( G \) be a group. We shall write \( A(G) \) for the automorphism group of \( G \). According to Schweigert [10], we say that an element \( f \in A(G) \) is a polynomial automorphism of \( G \) if there exist integers \( \epsilon_1, \ldots, \epsilon_m \in \mathbb{Z} \) and elements \( u_0, \ldots, u_m \in G \) such that

\[
    f(x) = u_0 x^{\epsilon_1} u_1 \cdots u_{m-1} x^{\epsilon_m} u_m
\]

for all \( x \in G \). Since \( f(1) = 1 \), it is easy to see that \( f(x) \) can be expressed as a ‘product’ of inner automorphisms, that is

\[
    f(x) = (v_1^{-1} x^{\epsilon_1} v_1) \cdots (v_m^{-1} x^{\epsilon_m} v_m).
\]

We shall write \( P_0(G) \) for the set of polynomial automorphisms of \( G \). Actually, Schweigert defines a polynomial automorphism in the context of finite groups. In particular, in this context, the set \( P_0(G) \) is clearly a subgroup of \( A(G) \). On the other hand, this is not necessarily the case when \( G \) is infinite. For instance, in the additive group of rational numbers, the set of polynomial automorphisms forms a monoid with respect to the operation of functional composition, which is isomorphic to the multiplicative monoid \( \mathbb{Z} \setminus \{0\} \).

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In this paper, we shall consider the subgroup $P(G) = \langle P_0(G) \rangle$ of $A(G)$, generated by all polynomial automorphisms of $G$. Hence $P_0(G) = P(G)$ when $G$ is finite, but for example $P(G)$ is distinct from $P_0(G)$ when $G$ is the additive group of rational numbers (note that $P(G) = A(G)$ in this last case).

It is easy to verify that $P_0(G)$ is a normal subset of $A(G)$. Thus $P(G)$ is a normal subgroup of $A(G)$; in addition, we have

$$I(G) \trianglelefteq P(G) \trianglelefteq A(G),$$

where $I(G)$ is the group of inner automorphisms of $G$. Also $P(G)$ contains the group of invertible elements of the monoid $P_0(G)$. It is worth noting that there exist finite groups $G$ such that the quotient $P(G)/I(G)$ is not soluble [7].

If $G$ is abelian, each polynomial automorphism is of the form $x \mapsto x^e$, and so $P(G)$ is abelian. When $G$ is a finite nilpotent group of class $k \geq 2$, it is proved in [4] that $P(G)$ is nilpotent of class $k - 1$ (see also [10, Satz 3.5]). We show here that this result remains true when $G$ is infinite.

**Theorem 1.1.** Let $G$ be a nilpotent group of class $k \geq 2$. Then $P(G)$ is nilpotent of class $k - 1$.

Notice that conversely, if $P(G)$ is nilpotent, then so is $G$ since $P(G)$ contains the group of inner automorphisms.

When $G$ is metabelian, it seems that nothing is known about $P(G)$, even in the context of finite groups. In this paper, we shall prove the following.

**Theorem 1.2.** Let $G$ be a metabelian group. Then the group $P(G)$ is itself metabelian.

In Section 3, we shall interpret a result of C. K. Gupta as a very particular case of this theorem (see Corollary 3.1 below).

### 2. Proofs

As usual, in a group $G$, the commutator of two elements $x, y$ is defined by $[x, y] = x^{-1}y^{-1}xy$. Instead of $[[x, y], z]$, we shall write $[x, y, z]$. We denote by $[G, G]$ the derived subgroup of $G$. 

Lemma 2.1. Let $f, g$ be two functions over a group $G$, respectively defined by the relations
\[
f(x) = (v_1^{-1} x^{\epsilon_1} v_1) \ldots (v_m^{-1} x^{\epsilon_m} v_m),
g(x) = (w_1^{-1} x^{\eta_1} w_1) \ldots (w_n^{-1} x^{\eta_n} w_n)
\]
(we do not suppose that $f$ and $g$ are automorphisms). Let $t$ be an element of $G$ such that any two conjugates of $t$ commute. Then we have the relation
\[
f(g(t)) = m \prod_{i=1}^{m} \prod_{j=1}^{n} t^{\epsilon_i \eta_j} [t^{\epsilon_i \eta_j}, v_i] [t^{\epsilon_i \eta_j}, w_j] [t^{\epsilon_i \eta_j}, w_j, v_i]
\]
(notice that in this product, the order of the factors is of no consequence).

Proof. Using the fact that any two conjugates of $t$ commute, we can write
\[
f(g(t)) = \prod_{i=1}^{m} v_i^{-1} \left( \prod_{j=1}^{n} w_j^{-1} t^{\eta_j} w_j \right)^{\epsilon_i} v_i
\]
\[
= \prod_{i=1}^{m} \prod_{j=1}^{n} v_i^{-1} w_j^{-1} t^{\epsilon_i \eta_j} w_j v_i
\]
\[
= \prod_{i=1}^{m} \prod_{j=1}^{n} t^{\epsilon_i \eta_j} [t^{\epsilon_i \eta_j}, w_j, v_i].
\]
We conclude thanks to the relation $[x, yz] = [x, z][x, y][x, y, z]$. □

In a nilpotent group $G$ of class $\leq 2$, two conjugates of any element $t \in G$ commute. Therefore, as an immediate consequence of Lemma 2.1, we observe that any two polynomial automorphisms of $G$ commute. Since these automorphisms generate $P(G)$, we obtain:

Corollary 2.1. If $G$ is a nilpotent group of class $\leq 2$, then $P(G)$ is abelian.

We are now ready to prove our first theorem.

Proof of Theorem 1.1. Since $P(G)$ contains $I(G)$ (which is nilpotent of class $k - 1$ exactly), it suffices to show that $P(G)$ is nilpotent of class at most $k - 1$. We argue by induction on the nilpotency class $k$ of
\( G \). The case \( k = 2 \) follows from Corollary 2.1. Now suppose that our theorem is proved for an integer \( k \geq 2 \) and consider a nilpotent group \( G \) of class \( k + 1 \). Denote by \( \zeta(G) \) the centre of \( G \). One can define a homomorphism \( \Theta : P(G) \to A(G/\zeta(G)) \), where for each \( f \in P(G) \), \( \Theta(f) \) is the automorphism induced by \( f \) in \( G/\zeta(G) \). Clearly, if \( f \) is a polynomial automorphism of \( G \), then \( \Theta(f) \) is a polynomial automorphism of \( G/\zeta(G) \). Hence \( \Theta(P(G)) \) is a subgroup of \( P(G/\zeta(G)) \), and so, by induction, is nilpotent of class at most \( k - 1 \). Since \( \Theta(P(G)) \) and \( P(G)/\ker \Theta \) are isomorphic, it suffices to show that \( \ker \Theta \) is included in the centre of \( P(G) \) and the theorem is proved. For that, consider an element \( g \in \ker \Theta \) and put \( w(x) = x^{-1}g(x) \) for any \( x \) in \( G \). Thus \( g(x) = wx(x) \) and \( w(x) \) belongs to \( \zeta(G) \) for all \( x \in G \). Notice that \( w \) defines a homomorphism of \( G \) into \( \zeta(G) \) since
\[
w(xy) = y^{-1}x^{-1}g(x)g(y) = y^{-1}w(x)g(y) = w(x)w(y).
\]
In order to show that \( g \) belongs to the centre of \( P(G) \), it suffices to verify that \( g \) commutes with any polynomial automorphism \( f \) of \( G \). Suppose that \( f \) is defined by the relation
\[
f(x) = (v_1^{-1}x^{\epsilon_1}v_1) \cdots (v_m^{-1}x^{\epsilon_m}v_m).
\]
We have easily
\[
f(g(x)) = f(xw(x)) = f(x)f(w(x)) = f(x)w(x)^{\epsilon},
\]
where \( \epsilon = \epsilon_1 + \cdots + \epsilon_m \). In the same way, by using the fact that \( w \) is a homomorphism, we can write
\[
g(f(x)) = f(x)w(f(x))
\]
\[
= f(x)(w(v_1)^{-1}w(x)^{\epsilon_1}w(v_1)) \cdots (w(v_m)^{-1}w(x)^{\epsilon_m}w(v_m)),
\]
whence \( g(f(x)) = f(x)w(x)^{\epsilon} \). Thus \( g \) and \( f \) commute, as required, and the result follows.

Now we undertake the proof of our second theorem. First we need the following result, which is well known and easy to prove (see for example [8, Lemma 34.51] or [9, Part 2, p. 64]).

**Lemma 2.2.** In a metabelian group \( G \), if \( t \) is an element of the derived subgroup \( [G, G] \), we have the relation \( [t, x, y] = [t, y, x] \) for all \( x, y \in G \).
We arrive to the key lemma in the proof of Theorem 1.2. This lemma shows that when $G$ is metabelian, any element $h \in [P(G), P(G)]$ operates trivially on $[G, G]$ and on $G/[G, G]$.

**Lemma 2.3.** Let $G$ be a metabelian group. Suppose that $h$ is an element of the derived subgroup $[P(G), P(G)]$. Then

(i) $h(t) = t$ for all $t \in [G, G]$;
(ii) $x^{-1}h(x)$ belongs to $[G, G]$ for all $x \in G$.

**Proof.** (i) Consider the homomorphism $\Phi : P(G) \rightarrow A([G, G])$ defined like this: for any $f \in P(G)$, $\Phi(f)$ is the restriction of $f$ to $[G, G]$. We must show that $\ker \Phi$ contains $[P(G), P(G)]$. For that, first notice that any two conjugates of $t \in [G, G]$ commute since $G$ is metabelian. Now we apply Lemma 2.1. If $f$ and $g$ are polynomial automorphisms of $G$ defined as in this lemma, we obtain the equalities

$$f(g(t)) = \prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{k=1}^{k} t^{i+j}[t^{i+j}, v_i][t^{i+j}, w_j][t^{i+j}, v_j, w_i],$$

$$g(f(t)) = \prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{k=1}^{k} t^{i+j}[t^{i+j}, v_i][t^{i+j}, w_j][t^{i+j}, v_j, w_i],$$

and so, by Lemma 2.2, $f(g(t)) = g(f(t))$ for all $t \in [G, G]$. It follows that $[f, g]$ belongs to $\ker \Phi$. In other words, the images of $f$ and $g$ in $P(G)/\ker \Phi$ commute. Since $P(G)/\ker \Phi$ is generated by the images of the polynomial automorphisms, this quotient is abelian. It follows that $\ker \Phi$ contains $[P(G), P(G)]$, as desired.

(ii) Here, we consider the homomorphism $\Psi : P(G) \rightarrow A(G/[G, G])$, where for any $f \in P(G)$, $\Psi(f)$ is the automorphism induced in $G/[G, G]$ by $f$. Since a polynomial automorphism of $G$ induces in $G/[G, G]$ a polynomial automorphism of $G/[G, G]$, $\Psi(P(G))$ is a subgroup of $P(G/[G, G])$. But $P(G/[G, G])$ is abelian (see for instance Corollary 2.1 above) and $\Psi(P(G))$ is isomorphic to $P(G)/\ker \Psi$. Hence $P(G)/\ker \Psi$ is abelian. Consequently, $\ker \Psi$ contains $[P(G), P(G)]$ and the result follows. $\square$

**Proof of Theorem 1.2.** Let $f, g$ be two elements of $[P(G), P(G)]$. For any $x \in G$, put $v(x) = x^{-1}f(x)$ and $w(x) = x^{-1}g(x)$. By Lemma 2.3,
v(x) and w(x) belong to [G, G]. Applying again Lemma 2.3, we can write
\[ f(g(x) = f(xw(x)) = f(x)f(w(x)) = xv(x)w(x). \]
In the same way, we have \( g(f(x)) = xw(v(x)) = xv(w(x)). \) It follows that \( f \) and \( g \) commute. Thus \([P(G), P(G)]\) is abelian, and so \( P(G) \) is metabelian.

\[ \square \]

3. IA-Automorphisms of Two-Generator Metabelian Groups

By way of illustration, we apply Theorem 1.2 to IA-automorphisms of a two-generator metabelian group. We recall that an automorphism of a group \( G \) is said to be an IA-automorphism if it induces the identity automorphism on \( G/[G, G] \). In a free metabelian group of rank 2, each IA-automorphism is inner \([1]\), and so is a polynomial automorphism. It turns out that in any two-generator metabelian group, each IA-automorphism is polynomial. This result is implicit in \([2]\) with a different terminology. For convenience, we give a proof since this one is short and elementary.

**Proposition 3.1.** Each IA-automorphism of a two-generator metabelian group is polynomial.

To prove this proposition, we shall use the following result.

**Lemma 3.1.** In a metabelian group \( G \), each function \( \varphi \) of the form
\[ x \mapsto \varphi(x) = x[x, v_1]^{\eta_1} \cdots [x, v_n]^{\eta_n} \quad (v_i \in G, \eta_i \in \mathbb{Z}) \]
is an endomorphism.

**Proof.** Thanks to the relation \([xy, z] = y^{-1}[x, z]y[y, z]\), we get
\[ \varphi(xy) = xy \prod_{i=1}^{n} (y^{-1}[x, v_i]y[y, v_i])^{\eta_i}. \]
But since the derived subgroup of $G$ is abelian, we can write
\[
\varphi(xy) = xy \prod_{i=1}^{n} (y^{-1}[x, v_i]^{\eta_i}) \prod_{i=1}^{n} [y, v_i]^{\eta_i}
\]
\[
= xy \left( y^{-1} \left( \prod_{i=1}^{n} [x, v_i]^{\eta_i} \right) y \right) \prod_{i=1}^{n} [y, v_i]^{\eta_i}
\]
\[
= \varphi(x)\varphi(y),
\]
as required.

Proof of Proposition 3.1. Suppose that $G$ is a two-generator metabelian group generated by $a$ and $b$. If $f$ is an IA-automorphism of $G$, we have $f(a) = av$ and $f(b) = bw$, where $v$ and $w$ belong to the derived subgroup $[G, G]$. Now notice that $[G, G]$ is the normal closure of $[a, b]$. Therefore, $[G, G]$ is generated by $[a, b]$ and the elements of the form $[a, b, u]$, with $u \in G$. Hence $v$ and $w$ can be written in the form
\[
v = [a, b]^\alpha \prod_{i=1}^{n} [a, b, v_i]^{\lambda_i},
\]
\[
w = [a, b]^\beta \prod_{i=1}^{n} [a, b, w_i]^{\mu_i},
\]
where $\alpha, \beta, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ are integers (possibly equal to 0). By using the relation $[x, y, z] = [x, y]^{-1}[x, z]^{-1}[x, yz]$, we obtain
\[
v = [a, b]^\alpha \prod_{i=1}^{n} [a, v_i]^{-\lambda_i} [a, bv_i]^{\lambda_i},
\]
\[
w = [a, b]^\beta \prod_{i=1}^{n} [a, w_i]^{-\mu_i} [a, bw_i]^{\mu_i},
\]
where $\lambda = \lambda_1 + \cdots + \lambda_n$ and $\mu = \mu_1 + \cdots + \mu_n$. Now put
\[
\varphi(x) = x[x, b]^\alpha [x, a]^{-\lambda} \prod_{i=1}^{n} [x, v_i]^{-\lambda_i} [x, bv_i]^{\lambda_i} [x, w_i]^{\mu_i} [x, aw_i]^{-\mu_i}.
\]
By Lemma 3.1, $\varphi$ is an endomorphism of $G$. Moreover, we have
\[
\varphi(a) = a[a, b]^\alpha \prod_{i=1}^{n} [a, v_i]^{-\lambda_i} [a, bv_i]^{\lambda_i} = av = f(a)
\]
since \([a, w_i] = [a, aw_i]\).

In the same way, we get

\[
\varphi(b) = b[a, b]^{\beta-\mu} \prod_{i=1}^{n} [b, w_i]^{\mu_i} [b, aw_i]^{-\mu_i}.
\]

By using the identity \([a, w_i]^{-1}[a, bw_i] = [b, aw_i]^{-1}[b, w_i]\) (valid in any group), we obtain

\[
\varphi(b) = b[a, b]^{\beta-\mu} \prod_{i=1}^{n} [a, w_i]^{-\mu_i} [a, bw_i]^{\mu_i} = bw = f(b).
\]

Thus \(f = \varphi\) and the proof is complete. \(\square\)

We remark that Proposition 3.1 cannot be extended to three-generator metabelian groups. For example, in the free metabelian group of rank 3 freely generated by \(a, b, c\), consider the IA-automorphism \(f\) defined by \(f(a) = a, f(b) = b\) and \(f(c) = c[a, b]\). Suppose that \(f\) is polynomial. Since \([a, b] = c^{-1} f(c)\), the commutator \([a, b]\) would be in the normal closure of \(c\), hence would be a product of conjugates of \(c^{\pm 1}\). Substituting 1 for \(c\) in this expression gives then \([a, b] = 1\), a contradiction. Therefore \(f\) is an IA-automorphism which is not polynomial.

As a consequence of Theorem 1.2 and Proposition 3.1, we obtain an alternative proof of a result due to C. K. Gupta [6] (see also [3]).

**Corollary 3.1** ([6]). In a two-generator metabelian group, the group of IA-automorphisms is metabelian.

Let \(M_d\) denote the free metabelian group of rank \(d\). By a result of Bachmuth [1], if \(d \geq 3\), the group of IA-automorphisms of \(M_d\) contains a subgroup which is (absolutely) free of rank \(d\). Thus Corollary 3.1 fails in a \(d\)-generator metabelian group when \(d \geq 3\). Also Bachmuth’s result shows once again that the group of IA-automorphisms of \(M_d\) is not included in \(P(M_d)\) (if \(d \geq 3\), since \(P(M_d)\) is metabelian.

In conclusion we mention that the metabelian groups constitute an important source of polynomial endomorphisms and automorphisms. Indeed, by Lemma 3.1, each function of the form

\[
x \mapsto x[x, v_1]^{\eta_1} \cdots [x, v_n]^{\eta_n} \ (v_i \in G, \ \eta_i \in \mathbb{Z})
\]
is an endomorphism in a metabelian group $G$. Besides, when $G$ is metabelian and nilpotent, such an endomorphism is an automorphism since in a nilpotent group, every function of the form

$$x \mapsto u_0x^{\epsilon_1}u_1 \cdots u_{m-1}x^{\epsilon_m}u_m \ (u_i \in G, \ \epsilon_i \in \mathbb{Z})$$

is a bijection if $\epsilon_1 + \cdots + \epsilon_m = \pm 1$ (see [5, Theorem 1]).

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