Weak and quasi-polynomial tractability of approximation of infinitely differentiable functions

Jan Vybíral∗†

December 12, 2013

Dedicated to J. F. Traub and G. W. Wasilkowski
on the occasion of their 80th and 60th birthdays

Abstract
We comment on recent results in the field of information based complexity, which state (in a number of different settings), that approximation of infinitely differentiable functions is intractable and suffers from the curse of dimensionality. We show that renorming the space of infinitely differentiable functions in a suitable way allows weakly tractable uniform approximation by using only function values. Moreover, the approximating algorithm is based on a simple application of Taylor’s expansion about the center of the unit cube. We discuss also the approximation on the Euclidean ball and the approximation in the $L_1$-norm.

Key words: weak tractability, uniform approximation, infinitely differentiable functions, curse of dimensionality

1 Introduction
We consider different classes $F_d$ of infinitely-differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ and discuss algorithms using only function values of $f$ in order to approximate $f$ uniformly, or in the $L_1$-norm. We are especially interested in the case of large $d \gg 1$.

In the classical setting of approximation theory, the dimension of the Euclidean space $d$ is fixed. Furthermore, the decay of the minimal error $e(n)$ of approximation of smooth functions in a Lebesgue space norm is very well studied for both algorithms using $n$ arbitrary linear functionals and for algorithms using only $n$ function evaluations. We refer to [5, 6, 7, 12, 14, 16, 17] and references therein. The decay is usually polynomial, speeds up with increasing smoothness and slows down with increasing dimension. Furthermore, this terminology typically hides the dependence of the constants on the dimension $d$, which might be even exponential. This motivates the question, what happens if both the dimension $d$ and the smoothness parameter $s$ tend to infinity.

If $n(\varepsilon, d)$ denotes the minimal number of function values needed to approximate all functions from $F_d$ up to the error $\varepsilon > 0$, we say, that the problem suffers from curse of dimensionality, if $n(\varepsilon, d)$ grows exponentially in $d$. This means, that there are positive numbers $c, \varepsilon_0$ and $\gamma$, such that

$$n(\varepsilon, d) \geq c(1 + \gamma)^d$$

for all $0 < \varepsilon \leq \varepsilon_0$ and infinitely many $d \in \mathbb{N}$.

∗Technical University Berlin, Department of Mathematics, Secretary office MA 4-1, Street of 17. June 136, 10623 Berlin, Germany, e-mail: vybiral@math.tu-berlin.de
†Research supported by the DFG Research Center MATHEON “Mathematics for key technologies” in Berlin.
On the other hand, we say that the problem is weakly tractable if
\[
\lim_{\varepsilon \to 0^+} \varepsilon \ln n(\varepsilon, d) = 0.
\]
Furthermore, the problem is quasi-polynomially tractable in the sense of [1], if there exist two constants \( C, t > 0 \), such that
\[
n(\varepsilon, d) \leq C \exp \{ t(1 + \ln(1/\varepsilon))(1 + \ln(d)) \}
\]
for all \( 0 < \varepsilon < 1 \) and all \( d \in \mathbb{N} \). For the sake of completeness, we add that a problem is polynomially tractable if there exist non-negative numbers \( C, p \) and \( q \) such that
\[
n(\varepsilon, d) \leq C \varepsilon^{-p} d^q
\]
for all \( 0 < \varepsilon < 1 \) and \( d \in \mathbb{N} \).

If \( q = 0 \) above then a problem is strongly polynomially tractable. We refer to the monographs [15, 8] and [11] for a detailed discussion of these and other kinds of (in)tractability and the closely related field of information based complexity.

The \( L_\infty \)-approximation of infinitely differentiable functions was studied in [4], where the authors showed that the problem is not strongly polynomially tractable. It was also discussed in [8], cf. Open Problem 2 therein. An essential breakthrough was achieved in [10] (which in turn is based on [9] and answers an open problem posed there), where uniform approximation of the functions from the class
\[
\mathcal{F}_d = \{ f : [-1/2, 1/2]^d \to \mathbb{R} : \sup_{\alpha \in \mathbb{N}_0^d} \| D^\alpha f \|_\infty \leq 1 \}
\]
was shown to satisfy \( n(\varepsilon, d) \geq 2^{d/2} \) for all \( 0 < \varepsilon < 1 \) and all \( d \in \mathbb{N} \) and this result is also true if arbitrary linear functionals are allowed as the information map about \( f \). Hence, the problem is intractable and suffers from the curse of dimensionality. In the context of weighted spaces of infinitely differentiable functions, the problem was also discussed in [18].

Multivariate integration of infinitely differentiable functions from the class \( \mathcal{F}_d \) was conjectured not to be polynomially tractable in [20] and was shown not to be strongly polynomially tractable in [19]. Furthermore, it is known (cf. [13] and [2]) that multivariate integration of functions from
\[
\mathcal{C}_d^k = \{ f : [0, 1]^d \to \mathbb{R} : \sup_{\alpha, |\alpha| \leq k} \| D^\alpha f \|_\infty \leq 1 \}
\]
suffers from the curse of dimensionality for all \( k \in \mathbb{N} \). Although multivariate integration of infinitely differentiable functions is also discussed in [2] and [3], it seems to be still an open problem if the curse of dimensionality holds also for multivariate integration and the class \( \mathcal{F}_d \).

The main result of this paper is the following.

**Theorem 1.** (i) Uniform approximation on the cube \([-1/2, 1/2]^d\) of functions from the class
\[
\mathcal{F}_d^1 = \{ f \in C^\infty([-1/2, 1/2]^d) : \sup_{k \in \mathbb{N}_0} \sum_{|\beta| = k} \| D^\beta f \|_\infty \leq 1 \}
\]
is quasi-polynomially tractable.

(ii) Uniform approximation on the balls \( B(0, r_d) = \{ x \in \mathbb{R}^d : \| x \|_2 \leq r_d \} \), where \( r_d \) are chosen in such a way, that the volume of \( B(0, r_d) \) is equal to one, and the functions are from the class
\[
\mathcal{F}_d^2 = \{ f \in C^\infty(B(0, r_d)) : \sup_{k \in \mathbb{N}_0} \| \partial_x^k f \|_\infty \leq 1 \}
\]
is weakly tractable. Here, \( (\partial_x^k f)(x) \) denotes the \( k \)-th derivative of \( f \) at \( x \neq 0 \) in the "normal" direction \( x/\| x \|_2 \).
(iii) Approximation in the $L_1$-norm on $B(0, r_d)$, where $r_d$ are as above and the functions are from the class

$$F_d^3 = \left\{ f \in C^\infty(B(0, r_d)) : \sup_{k \in \mathbb{N}_0} \int_0^{r_d} S(\partial^k_x f, r) dr \leq 1 \right\}$$

is weakly tractable. Here $S(\partial^k_x f, r)$ are the averages of $|\partial^k_x f|$ on the sphere $rS^{d-1}$.

Our method is rather simple and involves only the Taylor’s expansion of a smooth function about the center of the domain under consideration. The next two sections of this paper are devoted to the proof of this theorem. Finally, the last section presents some possible extensions of this method and mentions several closely related open problems.

2 Uniform approximation

We study first the uniform approximation of infinitely differentiable functions $f$ using only its function values. Let us recall, that the paper of Novak and Woźniakowski [10] shows intractability of this problem on the unit cube even for arbitrary linear functionals as the information map of $f$. We show that modifying the norm in a suitable way leads immediately to weak (and even quasi-polynomial) tractability, even when allowing only function values of $f$ as the admissible information and using only Taylor’s expansion about the center of the cube or ball, respectively. Of course, any tractability result for approximation using function values implies the same result also for algorithms using arbitrary linear information. On the other hand, we leave it as an open problem if algorithms using general linear information could achieve even better art of tractability.

Let us recall the standard multivariate notation, which we shall use with connection to the Taylor’s theorem. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex, we denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_d,$$

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad x \in \mathbb{R}^d,$$

$$\alpha! = \alpha_1! \cdots \alpha_d!,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad x \in \mathbb{R}^d.$$

2.1 Unit cube

In this part, we study uniform approximation of infinitely differentiable functions on $[-1/2, 1/2]^d$. Our algorithm is based on Taylor’s formula

$$f(x) = (T_k f)(x) + \sum_{|\beta|=k+1} (R_{\beta} f)(x) x^\beta,$$

where

$$(T_k f)(x) = \sum_{|\alpha| \leq k} \frac{(D^\alpha f)(0)}{\alpha!} x^\alpha,$$

$$(R_{\beta} f)(x) = \frac{|\beta|}{\beta!} \int_0^1 (1-t)^{|\beta|-1} (D^\beta f)(tx) dt.$$
The approximation of $f$ by $T_k f$ at the point $x \in [-1/2, 1/2]^d$ results into an error

$$|f(x) - T_k f(x)| \leq \sum_{|\beta|=k+1} |R_\beta f(x)| \cdot |x^\beta|$$

$$\leq \sum_{|\beta|=k+1} \frac{(k + 1) \cdot |x^\beta|}{\beta!} \int_0^1 (1 - t)^k |(D^\beta f)(tx)| dt$$

$$\leq \left(\frac{1}{2}\right)^{k+1} \sum_{|\beta|=k+1} \frac{(k + 1) \cdot |x^\beta|}{\beta!} \int_0^1 (1 - t)^k dt \cdot \|D^\beta f\|_\infty$$

$$= \left(\frac{1}{2}\right)^{k+1} \sum_{|\beta|=k+1} \frac{\|D^\beta f\|_\infty}{\beta!}.$$

Hence,

$$\sup_{f \in F_1^d} \|f - T_k f\|_\infty \leq \left(\frac{1}{2}\right)^{k+1}$$

for every $k \in \mathbb{N}_0$, where $F_1^d$ was defined in (2). To discuss the tractability of the problem, we need to estimate also the number of points needed to recover $T_k f$. As we are allowed to take only samples of $f$, and not of its derivatives, we are actually not able to recover $T_k f$ exactly. But using finite order differences, we may approximate it to an arbitrary precision using a bounded number of points.

To estimate the number of sampling points needed to approximate all derivatives up to the order $k$, we use induction. To evaluate $f(0)$ we need to sample $f$ at the point $A_0 = \{0\}$.

To calculate the first order differences, we need the points from the set

$$A_1 = A_0 \cup \bigcup_{j=1}^d (A_0 + he_j),$$

which has $d + 1$ points. For the second order differences, we need the values at

$$A_2 := A_1 \cup \bigcup_{j=1}^d (A_1 + he_j) = \{h(\alpha_1, \ldots, \alpha_d) : \alpha \in \mathbb{N}_0^d, |\alpha| \leq 2\},$$

which has $(d+2)^2$ points. By induction we obtain, that to evaluate all finite order differences up to the order $k$, we need the values at

$$A_k = \{h(\alpha_1, \ldots, \alpha_d) : |\alpha| \leq k\}$$

and that this set has

$$\sum_{j=0}^k \left(\begin{array}{c} d + j - 1 \\ j \end{array}\right) = \left(\begin{array}{c} d + k \\ k \end{array}\right)$$

points.

To show that the problem is quasi-polynomial tractable, we proceed in the following way. Given an $1 > \varepsilon > 0$, we choose the smallest $k \in \mathbb{N}_0$, such that

$$\varepsilon \geq \left(\frac{1}{2}\right)^{k+1},$$

i.e.

$$k + 1 := \left\lceil \frac{\ln(1/\varepsilon)}{\ln 2} \right\rceil,$$

4
where \( [a] \) denotes the smallest integer, which is larger than or equal to the real number \( a \in \mathbb{R} \). Together with the estimate
\[
n(\varepsilon, d) \leq \left( \frac{d + k}{k} \right) \leq \left( \frac{e(d + k)}{k} \right)^k,
\]
this gives that there is \( t > 0 \), such that
\[
\ln n(\varepsilon, d) \leq k(1 + \ln(d + k) - \ln k) \leq t(1 + \ln(1/\varepsilon))(1 + \ln d).
\]
This immediately implies (1), i.e. the quasi-polynomial tractability.

### 2.2 Euclidean ball

In this section, we discuss the uniform approximation of an infinitely differentiable function \( f \) on an Euclidean ball in \( \mathbb{R}^n \) with radius \( r_d > 0 \). As all the sets of infinitely differentiable functions under consideration include all constant functions with values between -1 and 1, the initial error of approximation is always 1 for every sequence \( (r_d)_{d \in \mathbb{N}} \).

We start again with Taylor’s expansion of an infinitely differentiable function \( f \) about zero. Let \( x \in B(0, r_d), x \neq 0 \) and let \( g_x(t) = f(tx), 0 \leq t \leq 1 \). Then
\[
f(x) = g_x(1) = \sum_{j=0}^{k} \frac{g_x^{(j)}(0)}{j!} + \frac{1}{k!} \int_{0}^{1} (1 - t)^k g_x^{(k+1)}(t)dt.
\]
Iterating the formula
\[
g_x^{(k+1)}(t) = ((\nabla f)(tx), x) = ((\nabla f)(tx), \frac{x}{\|x\|_2})\|x\|_2 = (\partial_x f)(tx) \cdot \|x\|_2
\]
we obtain
\[
g_x^{(k+1)}(t) = (\partial_x^{k+1} f)(tx) \cdot \|x\|_2^{k+1}, \quad 0 < t < 1,
\]
where \( (\partial_x f)(x) \) denotes the derivative of \( f \) at \( x \neq 0 \) in the direction \( x/\|x\|_2 \).

We denote
\[
\tilde{T}_k f(x) = \sum_{j=0}^{k} \frac{g_x^{(j)}(0)}{j!},
\]
which allows to estimate the error of approximation of \( f(x) \) by the Taylor’s polynomial about \( x \) by
\[
|f(x) - \tilde{T}_k f(x)| \leq \|x\|_2^{k+1} \frac{1}{k!} \int_{0}^{1} (1 - t)^k \|\partial_x^{k+1} f(tx)\|_2 dt
\leq \frac{r_d^{k+1} \|\partial_x^{k+1} f\|_\infty}{(k+1)!} \leq \frac{r_d^{k+1}}{(k+1)!},
\]
if \( f \in F_2^d \), where \( F_2^d \) is as in (4).

Finally, we observe that we need again \( \binom{d+k}{k} \) points to approximate the derivatives \( D^\alpha f(0) \) for every \( |\alpha| \leq k \), which again allows to approximate all the derivatives \( g_x^{(j)}(0) \) for all \( x \in B(0, r_d) \) and all \( 0 \leq j \leq k \).

Hence, if \( d \in \mathbb{N} \) is fixed and \( 1 > \varepsilon > 0 \) is given, we choose first the smallest \( k \), for which
\[
\varepsilon \geq \frac{r_d^{k+1}}{(k+1)!}.
\]
This is always possible, as the right hand side goes to zero for \( d \) fixed and \( k \to \infty \). On the other side, let us mention that if (6) holds for \( \varepsilon < 1 \), then \( k \) is at least of the order \( r_d \). Using the estimate

\[
\left( \frac{e^r_d}{k+1} \right)^{k+1} \geq \frac{r_d^{k+1}}{(k+1)!}
\]

we obtain that (6) is satisfied any time we have

\[
k \geq \max(e^2r_d, \ln(1/\varepsilon)).
\]

If finally \( r_d \approx \sqrt{d} \), this implies the weak tractability of the problem by

\[
\lim_{\varepsilon^{-1}+d \to \infty} \frac{\ln(n(\varepsilon, d))}{\varepsilon^{-1}+d} \leq \lim_{\varepsilon^{-1}+d \to \infty} \frac{k(1 + \ln(d + k) - \ln k)}{\varepsilon^{-1}+d} \leq \lim_{\varepsilon^{-1}+d \to \infty} \frac{k\ln(d+k)}{\varepsilon^{-1}+d} = 0.
\]

Unfortunately, the calculation above does not give quasi-polynomial tractability in this case.

### 3 Approximation in the \( L_1 \)-norm

In this section we prove the last part of Theorem 1. Of course, on a domain with volume one, the error of approximation in the \( L_1 \)-norm may be bounded from above by the error of uniform approximation. Therefore, the problem is weakly tractable for the \( F_d^1 \) class considered above. Using again (5) for all \( x \in B(0, r_d), x \neq 0 \), we obtain

\[
\int_{B(0, r_d)} |f(x) - \tilde{T}_k f(x)| dx \leq \int_{B(0, r_d)} \frac{1}{k!} \|x\|_2^{k+1} \int_0^1 (1-t)^k |(\partial_{\nu}^{k+1} f)(tx)| dt dx
\]

\[
= \frac{1}{k!} \int_{B(0, r_d)} \|y\|_2^{k+1} \cdot |(\partial_{\nu}^{k+1} f)(y)| \cdot \int_{\|y\|_2/r_d}^1 t^{-(k+1)} (1-t)^k t^{-d} dt dy
\]

\[
= \frac{1}{k!} \int_{B(0, r_d)} \|y\|_2^{k+1} \cdot |(\partial_{\nu}^{k+1} f)(y)| \cdot \int_{\|y\|_2/r_d}^1 t^{-(k+1)} (1-t)^k t^{-d} dt dy
\]

\[
\leq \frac{1}{k!} \int_{B(0, r_d)} \|y\|_2^{k+1} \cdot |(\partial_{\nu}^{k+1} f)(y)| \cdot \frac{1}{k+d} \left( \frac{\|y\|_2}{r_d} \right)^{-k-d} dy
\]

\[
= \frac{1}{k!} \cdot \frac{r_d^{k+d}}{k+d} \int_{B(0, r_d)} \|y\|_2^{1-d} \cdot |(\partial_{\nu}^{k+1} f)(y)| dy
\]

\[
= \frac{1}{k!} \cdot \frac{r_d^{k+d}}{k+d} \int_0^{r_d} \int_{S^{d-1}} |(\partial_{\nu}^{k+1} f)(y)| d\sigma(y) dr,
\]

where \( \sigma \) is the \( d-1 \) dimensional Hausdorff measure in \( \mathbb{R}^d \) and \( S^{d-1} = \{ x \in \mathbb{R}^d : \|x\|_2 = 1 \} \) is the unit sphere in \( \mathbb{R}^d \). If we denote by \( \omega_{d-1} \) the surface area of \( S^{d-1} \), i.e. \( \omega_{d-1} = \sigma(S^{d-1}) \), and by

\[
S(\partial_{\nu}^{k+1} f, r) = \frac{1}{\omega_{d-1} r^{d-1}} \int_{S^{d-1}} |\partial_{\nu}^{k+1} f(y)| d\sigma(y)
\]

the averages of \( |\partial_{\nu}^{k+1} f| \) on the sphere \( r S^{d-1} \), we obtain

\[
\int_{B(0, r_d)} |f(x) - \tilde{T}_k f(x)| dx \leq \frac{1}{k!} \cdot \frac{r_d^{k+d}}{k+d} \cdot \omega_{d-1} \int_0^{r_d} S(\partial_{\nu}^{k+1} f, r) dr.
\]
Assuming finally, that the volume of the $B(0, r_d)$ is equal to $\omega_d r_d^d = 1$, we may further reduce this to
\[
\int_{B(0,r_d)} |f(x) - \tilde{T}_k f(x)| dx \leq \frac{r_d^k}{k!} \int_0^{r_d} S(\partial^k f, r) dr,
\]
which is smaller than $r_d^k/k!$ for every $f \in F_3^d$, cf. (1), or smaller than $r_d^{k+1}/k!$ for every $f \in F_3^d$, where
\[
F_3^d = \left\{ f \in C^\infty(B(0, r_d)) : \sup_{k \in \mathbb{N}_0} \sup_{0 < r \leq r_d} S(\partial^k f, r) \leq 1 \right\}.
\]

The proof of weak tractability follows in both cases from these estimates exactly as in Section 2.2.

4 Extensions and closing remarks

The aim of this paper was to discuss the use of Taylor’s theorem for tractability of approximation of infinitely differentiable functions. Therefore, we have restricted ourselves in Theorem 1 to the classes $F_1^d$, $F_2^d$, $F_3^d$ and simple domains as balls and cubes. Nevertheless, the method used here may be directly generalized also to other situations.

Remark (Possible extensions): (i) The point (i) of Theorem 1 which deals with uniform approximation on the cube for the class $F_1^d$, holds also for the classes
\[
F_1^d(c) = \left\{ f \in C^\infty([-1/2, 1/2]^d) : \sum_{|\beta| = k} \frac{\|D_\beta^c f\|_\infty}{\beta!} \leq c \right\}
\]
where $c < 2$ is a fixed number. The analysis done in Section 2.1 applies literally also to this setting.

(ii) The statements of points (ii) and (iii) of Theorem 1 hold true also when $F_2^d$ and $F_3^d$ are replaced by
\[
F_2^d(c) = \left\{ f \in C^\infty(B(0, r_d)) : \|\partial^c f\|_\infty \leq c \right\}
\]
or
\[
F_3^d(c) = \left\{ f \in C^\infty(B(0, r_d)) : \int_0^{r_d} S(\partial^k f, r) dr \leq c \right\}
\]
respectively. Here, $c < \infty$ is arbitrary. Actually, one observes that even $c = d^{1/2-\delta}$ for some fixed $\delta > 0$ is still admissible. Again, also in these cases, the analysis done in Section 2.2 or Section 3 applies.

(iii) The analysis of Section 2.2 was done for a sequence of balls $B(0, r_d) \subset \mathbb{R}^d$. Although we concentrated in the very end on the case, when the volume of $B(0, r_d)$ is equal to one (i.e. $r_d \approx \sqrt{d}$), the same calculation applies also to the case when $r_d \leq C d^{1-\delta}$ for two universal constants $C > 0$ and $\delta > 0$. Furthermore, the same is true for star-shaped subsets of such balls with zero in their center. Therefore, the statement applies also to unit cube $[-1/2, 1/2]^d$.

(iv) When trying to generalize the analysis of Section 3 to other domains, we encounter several problems. The crucial calculation (7) made a heavy use of spherical coordinates and they were also used in the definition of $S(\partial^k f, r)$. Although these obstacles can be overcome by measuring the size of $\partial^k f$ in a certain weighted space, we avoid the technicalities and do not give the details.

Remark (Open problems): (i) We have provided only the upper bounds on $n(\varepsilon, d)$, which in turn led to tractability results for the classes $F_1^d$, $F_2^d$, and $F_3^d$, respectively. It would
be interesting to know, if these results are optimal, i.e. to show that these results can not be improved. We leave this as an open problem.

(ii) We studied only algorithms using the function values of \( f \). Of course, the presented tractability results also apply to the larger class of algorithms using arbitrary linear functionals. Nevertheless, we leave as an open problem if such algorithms may achieve a better art of tractability than the one presented here.

(iii) One could also modify the classes under consideration in a way used recently in [3]. This approach uses a sequence \( L = (L_d)_{d \in \mathbb{N}} \) to define, for example,

\[
F_d^2(L) = \{ f \in C^\infty(B(0, r_d)) : \sup_{k \in \mathbb{N}_0} \| \partial^k f \|_\infty \leq L_d \}.
\]

Of course, the calculations given above could be to some extent transferred also to this setting, and we could really prove similar results for sequences which do not grow too quickly. Unfortunately, due to the lack of lower estimates, we could not hope for being able to characterize sequences \( L \), for which weak (or quasi-polynomial) tractability still holds.

Acknowledgment: I would like to thank to Aicke Hinrichs, Erich Novak, Mario Ullrich, Markus Weimar and Henryk Woźniakowski for useful comments, remarks and hints to an earlier version of the manuscript. Furthermore, I would like to thank to anonymous referees for their valuable comments, which helped to improve the presentation of the paper.

References

[1] M. Gnewuch and H. Woźniakowski, Quasi-polynomial tractability, J. Complexity 27, 312–330, 2011.

[2] A. Hinrichs, E. Novak, M. Ullrich, and H. Woźniakowski, The curse of dimensionality for numerical integration of smooth functions, preprint (2012), available at: http://arxiv.org/abs/1211.0871

[3] A. Hinrichs, E. Novak, M. Ullrich, and H. Woźniakowski, The curse of dimensionality for numerical integration of smooth functions II, preprint (2013).

[4] F. L. Huang and S. Zhang, Approximation of infinitely differentiable multivariate functions is not strongly tractable, J. Complexity 23, 73–81, 2007.

[5] S. N. Kudryavtsev, The best accuracy of reconstruction of finitely smooth functions from their values at a given number of points, Izv. Math. 62(1), 19–53, 1998.

[6] E. Novak, Deterministic and stochastic error bounds in numerical analysis, Lecture Notes in Mathematics, 1349, 1988.

[7] E. Novak and H. Triebel, Function spaces in Lipschitz domains and optimal rates of convergence for sampling, Constr. Approx. 23, 325–350, 2006.

[8] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems, Volume I: Linear Information, European Math. Soc. Publ. House, Zürich, 2008.

[9] E. Novak and H. Woźniakowski, Optimal order of convergence and (in)tractability of multivariate approximation of smooth functions, Constr. Appr. 30 (2009), 457–473.

[10] E. Novak and H. Woźniakowski, Approximation of infinitely differentiable multivariate functions is intractable, J. Compl. 25 (2009), 398–404.

[11] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems, Volume II: Standard Information for Functionals, European Math. Soc. Publ. House, Zürich, 2010.
[12] A. Pinkus, *n-widths in approximation theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3.7, Springer, Berlin, 1985.

[13] A. G. Sukharev, *Optimal numerical integration formulas for some classes of functions of several variables*, Soviet Math. Dokl. 20, 472–475, 1979.

[14] V. N. Temlyakov, *Approximation of periodic functions*, Nova Science, New York, 1993.

[15] J. F. Traub, G. W. Wasilkowski and H. Woźniakowski, *Information-Based Complexity*, Academic Press, 1988.

[16] J. Vybíral, *Sampling numbers and function spaces*, J. Compl. 23 (2007), 773–792.

[17] J. Vybíral, *Widths of embeddings in function spaces*, J. Compl. 24 (2008), 545–570.

[18] M. Weimar, *Tractability results for weighted Banach spaces of smooth functions*, J. Compl. 28 (2012), 59–75.

[19] O. Wojtaszczyk, *Multivariate integration in $C^\infty([0,1]^d)$ is not strongly tractable*, J. Complexity 19, 638–643, 2003.

[20] H. Woźniakowski, *Open problems for tractability of multivariate integration*, J. Complexity 19, 434–444, 2003.