Classification of Commutator Algebras
Leading to the New Type of
Closed Baker-Campbell-Hausdorff Formulas

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Abstract
We show that there are 13 Types of commutator algebras leading to the new closed forms of the Baker-Campbell-Hausdorff (BCH) formula
\[ e^X e^Z = e^{aX+bZ+c[X,Z]+dI}, \]
derived in [arXiv:1502.06589] and holding also in cases when \([X, Z]\) includes elements different from \(X\) and \(Z\). This follows by a rescaled version of the decomposition
\[ e^X e^Y e^Z = e^X e^{\alpha Y} e^{(1-\alpha)Y} e^Z, \]
with \(\alpha\) fixed in such a way that it reduces to \(e^X e^Y\), with \(\tilde{X}\) and \(\tilde{Y}\) satisfying the Van-Brunt and Visser condition \([\tilde{X}, \tilde{Y}] = \tilde{u}\tilde{X} + \tilde{v}\tilde{Y} + \tilde{c}I\).

It turns out that \(x := e^\alpha\) satisfies, in the generic case, an algebraic equation whose exponents depend on the parameters defining the commutator algebra. In nine Types of commutator algebras, such an equation leads to rational solutions for \(\alpha\).

We find all the equations that characterize the solution of the above decomposition problem by combining it with the Jacobi Identity.
1 Introduction

In [1] it has been shown that a simple algorithm extends, to a wider class of important cases, the remarkable simplification of the Baker-Campbell-Hausdorff (BCH) formula recently observed by Van-Brunt and Visser [2] (see also [3] for related work). In particular, in [1] it has been shown the following result.

If $X$, $Y$ and $Z$ are elements of an associative algebra satisfying the commutation relations

$$[X, Y] = uX + vY + cI,$$
$$[Y, Z] = wY + zZ + dI,$$  \hspace{1cm} (1.1)

with $c$, $d$, $u$, $v$ and $z$ complex numbers, then

$$e^X e^Y e^Z = e^{AX + BY + CZ + DI},$$  \hspace{1cm} (1.2)

where $A$, $B$, $C$ and $D$ are determined parameters depending on $c$, $d$, $u$, $v$, $w$ and $z$.

As a particular case, it has been shown that if the vector space over $\mathbb{C}$, spanned by $X$, $Y$, $Z$ and $I$, is closed under the commutation operation, then

$$e^X e^Z = e^{A'X + B'Y + C'Z + D'I},$$  \hspace{1cm} (1.3)

where $A'$, $B'$, $C'$ and $D'$ are determined parameters depending on $c$, $d$, $u$, $v$, $w$ and $z$.

The starting point of the algorithm in [1] has been to consider the decomposition

$$e^X e^Y e^Z = e^X e^\alpha Y e^{(1-\alpha)Y} e^Z,$$  \hspace{1cm} (1.4)

and then fix $\alpha$ in such a way that

$$e^X e^Y e^Z = e^X e^{\alpha Y} e^{(1-\alpha)Y} e^Z = e^{\tilde{X}} e^{\tilde{Y}},$$  \hspace{1cm} (1.5)

with $\tilde{X}$ and $\tilde{Y}$ satisfying the Van-Brunt and Visser condition

$$[\tilde{X}, \tilde{Y}] = \tilde{u}\tilde{X} + \tilde{v}\tilde{Y} + \tilde{c}I.$$  \hspace{1cm} (1.6)

This provides the solution of the BCH problem, since now

$$\exp(X) \exp(Y) \exp(Z) = \exp \left( \tilde{u}\tilde{X} + \tilde{v}\tilde{Y} + f(\tilde{u}, \tilde{v})[\tilde{X}, \tilde{Y}] \right),$$  \hspace{1cm} (1.7)

where $\alpha$, $\tilde{u}$ and $\tilde{v}$ are determined functions of the parameters defining the commutators between $X$, $Y$ and $Z$ [1] and

$$f(u, v) = \frac{(u - v)e^{u + v} - (ue^{u} - ve^{v})}{uv(e^{u} - e^{v})},$$  \hspace{1cm} (1.8)

is the Van-Brunt and Visser function [2].

The implementation of the Jacobi Identity is just a consequence of the splitting

$$e^Y = e^{\alpha Y} e^{(1-\alpha)Y}.$$  \hspace{1cm} (1.9)
The reason is that instead of considering the commutator between either \([X, Y], Z\) or \([X, [Y, Z]]\), (1.4) naturally leads to consider both \([X, Y], Z\) and \([X, [Y, Z]]\). It follows that the splitting (1.4) is the key to include in the game the Jacobi Identity

\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .
\] (1.10)

In this paper we study some of the consequences of the algorithm in [1]. In particular, solving the constraints coming from the Jacobi Identity is a key step to classify all commutator algebras leading to (1.7). More precisely, given the commutators (1.1), we consider the constraints on the parameters \(m, n, p\) and \(e\) in

\[
[X, Z] = mX + nY + pZ + eI ,
\] (1.11)

constrained by the linear system

\[
uw + mz = 0 ,
\]

\[
vm - wp + n(z - u) = 0 ,
\]

\[
pu + zv = 0 ,
\]

\[
c(w + m) + e(z - u) - d(p + v) = 0 ,
\] (1.12)

which follows by the Jacobi Identity.

In order to solve Eq.(1.12), one has to consider thirteen different conditions, reported in the table, where, as explained later, \(cw = dv = 0\) corresponds to five different conditions.

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 \\
\hline
\(u = z = 0\) & \(u = 0, z \neq 0\) & \(u \neq 0, z = 0\) & \(u = z \neq 0\) & \(u \neq z, uz \neq 0\) \\
\hline
\(cw \neq dv\) & \(w = 0\) & \(v = 0\) & & \\
\hline
\(cw = dv \neq 0\) & \(w \neq 0\) & \(v \neq 0\) & & \\
\hline
\(cw = dv = 0\) & & & & \\
\hline
\end{tabular}
\end{center}

The thirteen cases of the Jacobi Identity

In section 2 we review the algorithm introduced in [1]. In section 3 we provide the explicit expressions of \(\tilde{u}, \tilde{v}, \tilde{c}\) and of the equation satisfied by \(\alpha\). This leads to the explicit expression of (1.7). It turns out that, in the generic case, the equation for \(\alpha\) is an algebraic one for \(x^u\) and \(x^v\), where \(x := e^\alpha\). In nine Types of commutator algebras, such an equation leads to rational solutions for \(\alpha\). The last section is devoted to a detailed classification of the types of commutator algebras leading to the closed BCH formula (1.7).
2 Review of the algorithm for BCH

Here we shortly review the algorithm leading, in relevant cases, to a new class of closed forms of the Baker-Campbell-Hausdorff (BCH) formula [1]. Let us start with the recent finding by Van-Brunt and Visser [2] (see also [3] for related results). They found a remarkable relation that simplifies considerably the BCH formula. Their result is the following. If

\[ [X, Y] = uX + vY + cI , \]  

then [2]

\[ e^X e^Y = e^{X+Y+f(u,v)[X,Y]} , \]  

where \( f(u, v) \) is the symmetric function

\[ f(u, v) = \frac{(u - v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)} . \]

We note that in deriving such a result there is a convergence condition for the Taylor series in Eq.(22) of [2], that is

\[ |1 - e^{v-tu}| < 1 , \]  

with \( t \in [0,1] \). Such a condition can be always satisfied by a suitable rescaling of \( X \) and \( Y \). In the following, when necessary, we tacitly assume such a rescaling.

In [1] it has been first considered the decomposition

\[ e^X e^Y e^Z = e^X e^{aY} e^{bZ} , \]  

where \( \alpha + \beta = 1 \). Note that replacing \( Y \) by \( \lambda_0 Y \)

\[ e^X e^Z = \lim_{\lambda_0 \to 0} e^X e^{\lambda-} Y e^{\lambda+} e^Z , \]  

where \( \lambda_\pm = \lambda_0 \alpha_\pm \), leads to a closed form for \( \ln(e^X e^Z) \) in some of the cases when \([X, Z] \) contains elements different from \( X \) and \( Z \). For all \( a \in \mathbb{C} \), set

\[ s(a) := \frac{\sinh(a/2)}{a/2} , \quad s_\alpha(a) := \frac{\sinh(\alpha a/2)}{a/2} . \]

Observe that if

\[ [X, Y] = uX + vY + cI , \quad [Y, Z] = wY + zZ + dI , \]  

then, by (2.2) and (2.4),

\[ e^X e^Y e^Z = e^{\tilde{X}} e^{\tilde{Y}} , \]

where

\[ \tilde{X} := \tilde{g}_\alpha(u, v)X + \tilde{h}_\alpha(u, v)Y + \tilde{l}_\alpha(u, v)cI , \]

\[ \tilde{Y} := \tilde{h}_\beta(z, w)Y + \tilde{g}_\beta(z, w)Z + \tilde{l}_\beta(z, w)dI . \]
with
\[ g_\alpha(u, v) := 1 + \alpha u f(\alpha u, v) = e^{\frac{\alpha u}{v}} \frac{s(v)}{s(v - \alpha u)}, \]
\[ h_\alpha(u, v) := \alpha (1 + v f(\alpha u, v)) = e^{\frac{v}{\alpha}} \frac{s_\alpha(u)}{s(v - \alpha u)}, \]
\[ l_\alpha(u, v) := \alpha f(\alpha u, v) = \frac{1}{u} \left( e^{\frac{\alpha u}{v}} \frac{s(v)}{s(v - \alpha u)} - 1 \right). \] (2.8)

Note that
\[ h_\alpha(u, v) = vl_\alpha(u, v) + \alpha, \quad g_\alpha(u, v) = ul_\alpha(u, v) + 1, \] (2.9)
and
\[ h_\alpha(u, v) = -\alpha g_\alpha(\frac{v}{\alpha}, \alpha u), \quad g_\alpha(u, v) = -\frac{1}{\alpha} h_\alpha(\alpha v, \frac{u}{\alpha}). \] (2.10)

Some special values are
\[ f(0, v) = \frac{1}{1 - e^{-v}} - \frac{1}{v}, \]
\[ g_\alpha(u, 0) = \frac{\alpha u}{1 - e^{-\alpha u}}, \]
\[ h_\alpha(0, v) = \frac{\alpha v}{1 - e^{-v}}, \]
\[ l_\alpha(0, v) = \frac{\alpha}{1 - e^{-v}} - \frac{\alpha}{v}, \]
\[ l_\alpha(u, 0) = \frac{\alpha}{1 - e^{-\alpha u}} - \frac{1}{u}, \]
and
\[ f(0, 0) = \frac{1}{2}, \]
\[ g_\alpha(0, v) = g_\alpha(0, 0) = g_0(u, v) = 1, \]
\[ h_\alpha(u, 0) = h_\alpha(0, 0) = \alpha, \]
\[ l_\alpha(0, 0) = \frac{\alpha}{2}. \]
Next, note that if
\[ [\tilde{X}, \tilde{Y}] = \tilde{u}\tilde{X} + \tilde{v}\tilde{Y} + \tilde{c}I , \tag{2.11} \]
then, by (2.1) and (2.2),
\[ e^x e^y e^z = e^{\tilde{X} + \tilde{Y} + f(\tilde{u}, \tilde{v})[\tilde{X}, \tilde{Y}]} . \tag{2.12} \]

To determine the full solution, we need to fix \( \alpha, \tilde{u}, \tilde{v}, \tilde{c} \) and derive the constraints coming from the Jacobi Identity. Let us first consider such an identity
\[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 . \]

It implies
\[ [X, Z] = mX + nY + pZ + eI , \tag{2.13} \]
where \( m, n, p \) and \( e \) satisfy the linear system
\[ uw + mz = 0 , \]
\[ vm - wp + n(z - u) = 0 , \]
\[ pu + vz = 0 , \]
\[ c(w + m) + e(z - u) - d(p + v) = 0 . \tag{2.14} \]

Next, in order to determine \( \alpha, \tilde{u}, \tilde{v} \) and \( \tilde{c} \), we replace \( \tilde{X} \) and \( \tilde{Y} \) on the right hand side of (2.11) by their expressions (2.7), and compare the result with the direct computation, by (2.6) and (2.13), of \([\tilde{X}, \tilde{Y}]\). This leads to a linear system of equations whose compatibility condition fixes the equation for \( \alpha \). The latter is the basic equation of the algorithm
\[ h_\alpha(u, v)[h_\beta(z, w)u + z) + g_\beta(z, w)m - w)] + g_\alpha(u, v)[h_\beta(z, w)p - v) - g_\beta(z, w)n] = 0 . \tag{2.15} \]

The expressions for \( \tilde{u}, \tilde{v} \) and \( \tilde{c} \) follow by the other relations
\[ \tilde{u} = h_\beta(z, w)u + g_\beta(z, w)m , \]
\[ \tilde{v} = g_\alpha(u, v)p + h_\alpha(u, v)z , \]
\[ \tilde{c} = (h_\beta(z, w) - g_\beta(z, w)n)h_\alpha(u, v)m + (h_\alpha(u, v) - g_\alpha(u, v)d + g_\alpha(u, v)g_\beta(z, w)e . \tag{2.16} \]

In the next section, we will show that Eq.(2.15) is an algebraic one for \( x^u \) and \( x^z \), where \( x := e^\alpha \). It turns out that there are nine Types of commutator algebras with \( \alpha \) corresponding to a rational function of the parameters of the commutator algebra.
3 The explicit form of the new closed BCH formulas

The results in the previous section can be summarized by the following theorem.

**Theorem.** If $X$, $Y$ and $Z$ belong to an associative algebra satisfying the commutation relations

$$[X, Y] = uX + vY + cI, \quad [Y, Z] = wY + zZ + dI,$$

$c, d, u, v, w, z \in \mathbb{C}$, then

$$\exp(X) \exp(Y) \exp(Z) =$$

$$\exp \left\{ \frac{1}{s(\tilde{v} - \tilde{u})} \left( e^{\frac{\tilde{v} + \alpha u}{2}s(\tilde{v})} s(v) \right) X + \right.$$  

$$\left. + \left( \frac{e^{\frac{\tilde{v} + \alpha u}{2}s(\tilde{v})} s_\alpha(u)}{s(v - \alpha u)} + \frac{e^{\frac{\tilde{v} + \alpha u}{2}s(\tilde{v})} s_\beta(z)}{s(w - \beta z)} \right) Y + \right.$$  

$$\left. + \frac{e^{\frac{\tilde{v} + \beta z}{2}s(\tilde{v})} s(w)}{s(w - \beta z)} Z + \right.$$  

$$+ \left\{ \frac{e^{\frac{\tilde{v}}{u}} s(\tilde{v})}{u} \left( \frac{e^{\frac{\tilde{v}}{u}} s(v)}{s(v - \alpha u)} - 1 \right) c + \frac{e^{\frac{\tilde{v}}{z}} s(\tilde{u})}{z} \left( \frac{e^{\beta z} s(w)}{s(w - \beta z)} - 1 \right) d + \right.$$  

$$\left. + \frac{1}{\tilde{u}} \left( e^{\frac{\tilde{v}}{s(\tilde{v})}} - s(\tilde{v} - \tilde{u}) \right) \tilde{c} \right\} I \right\} \right\}$$

(3.1)

where $\alpha = 1 - \beta$ is solution of the equation

$$e^{\frac{\tilde{u}}{u}} s_\alpha(u) \left[ e^{\frac{\tilde{u}}{u}} s_\beta(z)(u + z) + e^{\frac{\tilde{u}}{u}} s(w)(m - w) \right] + e^{\frac{\tilde{u}}{u}} s(v) \left[ e^{\frac{\tilde{u}}{u}} s_\beta(z)(p - v) - e^{\frac{\tilde{u}}{u}} s(w)n \right] = 0$$

(3.2)

corresponding to Eq. (2.15), and
\[
\tilde{u} = \frac{e^{\frac{u}{2}}s_\beta(z)u + e^\frac{u}{2}s(w)m}{s(w - \beta z)}, \\
\tilde{v} = \frac{e^{\frac{v}{2}}s(v)p + e^\frac{v}{2}s_\alpha(u)z}{s(v - \alpha u)}, \\
\tilde{c} = \left(e - \frac{cm}{u} - \frac{dp}{z}\right) \frac{e^{\frac{u}{2}}s(v)}{s(v - \alpha u)} \frac{e^\frac{v}{2}s(w)}{s(w - \beta z)} + \\
+ \left[\left(\frac{w}{z} + \frac{m}{u}\right) \frac{e^{\frac{u}{2}}s(w)}{s(w - \beta z)} + \beta - \frac{w}{z}\right] c + \left[\left(\frac{v}{u} + \frac{p}{z}\right) \frac{e^{\frac{v}{2}}s(v)}{s(v - \alpha u)} + \alpha - \frac{v}{u}\right] d. \quad (3.3)
\]

The parameters \(e, m, n, p\) fix the commutator

\[ [X, Z] = mX + nY + pZ + eI, \]

and are constrained by the linear system (2.14) coming from the Jacobi Identity. \(\square\)

Set

\[ x := e^\alpha. \]

Note that if

\[ v - \alpha u \neq 2k\pi i \quad \text{and} \quad w - \beta z \neq 2k\pi i, \quad (3.4) \]

\(k \in \mathbb{Z}\ \backslash \{0\}\), then Eq. (2.15) is equivalent to

\[
x^{u+z} e^{\frac{u+z}{2}} \left( \frac{u + z}{uz} e^\frac{u}{2} + \frac{p - v}{z} s(v) \right) + \\
+ x^u \left( ns(v)s(w) - \frac{u + z}{uz} e^{\frac{u+z}{2}} - \frac{m - w}{u} s(w)e^\frac{v}{2} - \frac{p - v}{z} s(v)e^\frac{v}{2} \right) + \\
- x^z \frac{u + z}{uz} e^{\frac{v+z}{2}} + \\
+ \frac{u + z}{uz} e^{\frac{v+z}{2}} + \frac{m - w}{u} s(w)e^\frac{v}{2} = 0. \quad (3.5)
\]
4 Jacobi Identity and types of commutator algebras

In the following we provide the classification of the solutions of the BCH formula associated to the commutator algebras

\[ [X, Y] = uX + vY + cI \]
\[ [Y, Z] = zY + wZ + dI \]
\[ [X, Z] = mX + nY + pZ + eI \]

and derive the corresponding explicit expressions of Eq.(3.1). This is done by solving the Jacobi Identity, that is the linear system (2.14). The resulting constrained parameters fix the form of the equation for \( \alpha \), Eq.(3.2), and determine, by (3.3), the values of \( \tilde{u} \), \( \tilde{v} \) and \( \tilde{c} \). Finally, Eq.(3.1) provides the corresponding explicit expression of the closed form of the BCH formula. In the following, we will also assume the condition (3.4).

There are thirteen types of commutator algebras.

The thirteen cases of the Jacobi Identity

|   | 1 | 2        | 3 | 4 | 5          |
|---|---|----------|---|---|------------|
|   |   |   |   |   |            |
| 1 | \( u = z = 0 \) | \( u = 0, z \neq 0 \) | \( u \neq 0, z = 0 \) | \( u = z \neq 0 \) | \( u \neq z, uz \neq 0 \) |
|   | \( cw \neq dv \) | \( w = 0 \) | \( v = 0 \) | \( w \neq 0 \) | \( v \neq 0 \) |
|   | \( cw = dv \neq 0 \) | \( w \neq 0 \) | \( v \neq 0 \) |   |   |
|   | \( cw = dv = 0 \) |   |   |   |   |

The case \( u = z = 0 \) is composed of three Types of commutator algebras. These are the Type 1a, corresponding to \( cw \neq dv \), the Type 1b, corresponding to \( cw = dv \neq 0 \), and the Type 1c, corresponding to \( cw = dv = 0 \). In turn, the Type 1c is composed of five Types, depending on which of the parameters in the pairs \( cw \) and \( dv \) are vanishing. To illustrate this, we write only the non-vanishing components of the vector \((c, d, u, v, w, z)\). For example, \((d)\) corresponds to \((0, d \neq 0, 0, 0, 0, 0)\) and \((c, v)\) to \((c \neq 0, 0, 0, v \neq 0, 0, 0)\).

There are nine possible solutions of \( cw = dv = 0 \)

\[ \begin{align*}
  i : & \ (v, w) , \\
  ii : & \ (c, d) , \\
  iii : & \ (d, w) \text{ or } (d) \text{ or } (w) , \\
  iv : & \ (c, v) \text{ or } (c) \text{ or } (v) , \\
  v : & \ (0, 0, 0, 0, 0) .
\end{align*} \]

The cases \((d)\) and \((w)\) are grouped with \((d, w)\) because the corresponding solution of (2.14) holds also when \( w = 0 \) or \( d = 0 \). Similarly, \((c)\) and \((v)\) are grouped with \((c, v)\).
Let us introduce the following symbol

\[
\left[ \text{case} \middle| \text{Jacobi constraints} \middle| \text{parameters of } [X, Z] \text{ unfixed} \right]_D
\]

where the first slot specifies under which conditions, e.g., \( u = z = 0, \ cw \neq dv \), the linear system (2.14) is solved. This classifies the Types of commutator algebras. The second slot reports the constraints on the commutator parameters fixed by the Jacobi Identity, while the last one reports which ones, between the parameters \( m, n, p \) and \( e \) in \([X, Z]\), are unfixed by the Jacobi Identity. In some cases (2.14) also fixes \( v \) in \([X, Y]\) or \( w \) in \([Y, Z]\). \( D \) is the number of parameters in the commutators between \( X, Y \) and \( Z \), which are unfixed by the Jacobi Identity.

In this classification, we report the solution of (2.14) in the above symbol and then give the corresponding explicit expression for \( \alpha \), solution of Eq.(3.2), and of \( \tilde{u}, \tilde{v} \) and \( \tilde{c} \), solutions of Eq.(3.3). In this respect, it is useful to keep in mind that

\[
\lim_{a \to 0} s(a) = 1, \quad \lim_{a \to 0} s_\alpha(a) = \alpha.
\]

The first case corresponds to \( u = z = 0 \), which in turn leads to three different types of commutator algebras. Setting \( u = z = 0 \) in Eqs.(3.2)-(3.3) yields

\[
\alpha = s(v) \left( \frac{ns(w) - e^w (p - v)}{e^w s(w)(m - w) - e^w s(v)(p - v)} \right),
\]

\[
\tilde{u} = m,
\]

\[
\tilde{v} = p,
\]

\[
\tilde{c} = \alpha \left( \frac{dv - cm}{1 - e - v} + \frac{cm}{v} \right) + \beta \left( \frac{cw - dp}{1 - e - w} + \frac{dp}{w} \right) + e. \tag{4.1}
\]

The Jacobi Identity (2.14) gives

\[
vm = wp,
\]

\[
c(w + m) = d(p + v).
\]

In this case \( e \) and \( n \) are always unfixed. The are three types of commutator algebras, Type 1a, Type 1b and Type 1c. The latter is composed of five subtypes.

**Type 1a.**

\[
\left[ u = z = 0, \ cw \neq dv \middle| m = -w, \ p = -v \middle| e, n \right]_6
\]

\[
[X, Y] = vY + cI \quad [Y, Z] = wY + dI \quad [X, Z] = -wX + nY - vZ + eI
\]

\[
\alpha = s(v) \left( \frac{2e^w v + ns(w)}{2e^w vs(v) - e^w ws(w)} \right),
\]

\[
\tilde{u} = -w
\]

\[
\tilde{v} = -v
\]

\[
\tilde{c} = \left( \frac{\alpha}{1 - e^{-v}} + \frac{\beta}{1 - e^{-w}} \right)(cw + dv) - \alpha \frac{w}{v} c - \beta \frac{v}{w} d + e
\]


Type 1b.

\[
\begin{align*}
\[u = z = 0, \ cw = dv \neq 0 \mid p = \frac{vm}{w} \mid e, m, n\]_6 \\
[X, Y] &= vY + cI \quad [Y, Z] = w(Y + \frac{c}{v}I) \quad [X, Z] = mX + nY + \frac{v}{w}mZ + eI \\
\alpha &= s(v) \frac{e^wv(w - m) + nws(w)}{(m - w)(e^wsws(w) - e^wvs(v))} \\
\tilde{u} &= m \\
\tilde{v} &= \frac{mv}{w} \\
\tilde{c} &= \left[\left(\frac{\alpha}{1 - e^{-v}} - \frac{\alpha}{v}\right)(w - m) + \left(\frac{\beta}{1 - e^{-w}} - \frac{\beta}{w}\right)(v - m) + \left[1 + \alpha\left(\frac{w}{v} - 1\right)\right]c + e
\right]c + e
\end{align*}
\]

The next case is \(u = z = 0,\) with \(cw = dv = 0.\) By Eq.\((2,14),\) this leads to several subcases.

Type 1c-i.

\[
\begin{align*}
\[(v, w) \mid p = \frac{mv}{w} \mid e, m, n\]_5 \\
[X, Y] &= vY \quad [Y, Z] = wY \quad [X, Z] = mX + nY + \frac{mv}{w}Z + eI \\
\alpha &= \frac{ns(v)ws(w)}{(m - w)(w - v)s(w - v)} \\
\tilde{u} &= m \\
\tilde{v} &= \frac{mv}{w} \\
\tilde{c} &= c + e
\end{align*}
\]

Type 1c-ii.

\[
\begin{align*}
\[(c, d) \mid p = \frac{cm}{d} \mid e, m, n\]_5 \\
[X, Y] &= cI \quad [Y, Z] = dI \quad [X, Z] = mX + nY + \frac{cm}{d}Z + eI \\
\alpha &= \frac{dn - cm}{m(d - c)} \\
\tilde{u} &= m \\
\tilde{v} &= \frac{cm}{d} \\
\tilde{c} &= \frac{dn}{m} + e
\end{align*}
\]
Type 1c-iii.

\[ (d, w) \text{ or } (d) \text{ or } (w) \mid p = 0 \mid e, m, n \]_{4 or 5}

\[ [X, Y] = 0 \quad [Y, Z] = wY + dI \quad [X, Z] = mX + nY + eI \]

\[ \alpha = \frac{n}{m - w} \]
\[ \tilde{u} = m \]
\[ \tilde{v} = 0 \]
\[ \tilde{c} = \frac{dn}{m - w} + e \]

Type 1c-iv.

\[ (c, v) \text{ or } (c) \text{ or } (v) \mid m = 0 \mid e, n, p \]_{4 or 5}

\[ [X, Y] = vY + cI \quad [Y, Z] = 0 \quad [X, Z] = nY + pZ + eI \]

\[ \alpha = 1 - \frac{n}{p - v} \]
\[ \tilde{u} = 0 \]
\[ \tilde{v} = p \]
\[ \tilde{c} = \frac{nc}{p - v} + e \]

Type 1c-v.

\[ (0, 0, 0, 0, 0) \| e, m, n, p \]_{4}

\[ [X, Y] = 0 \quad [Y, Z] = 0 \quad [X, Z] = mX + nY + pZ + eI \]

\[ \alpha = \frac{n - p}{m - p} \]
\[ \tilde{u} = m \]
\[ \tilde{v} = p \]
\[ \tilde{c} = e \]

The next case, corresponding to \( u = 0, z \neq 0 \), splits in two cases, \( w = 0 \) and \( w \neq 0 \).
Type 2a.

\[ u = w = 0, \ z \neq 0 \mid m = n = v = 0, \ e = \frac{pd}{z} \]

\[ [X, Y] = cI \quad [Y, Z] = zZ + dI \quad [X, Z] = p\left(Z + \frac{d}{z}\right)I \]

\[ \begin{align*}
\alpha &= -\frac{p}{z} \\
\tilde{u} &= 0 \\
\tilde{v} &= 0 \\
\tilde{c} &= \left(1 + \frac{p}{z}\right)c
\end{align*} \]

Type 2b.

\[ u = 0, \ w \neq 0, \ z \neq 0 \mid m = v = 0, \ p = \frac{nz}{w}, \ e = \frac{dn}{w} - \frac{cw}{z} \mid n \]

\[ [X, Y] = cI \quad [Y, Z] = wY + zZ + dI \quad [X, Z] = nY + \frac{nz}{w}Z + \left(\frac{dn}{w} - \frac{cw}{z}\right)I \]

\[ \begin{align*}
\alpha &= -\frac{n}{w} \\
\tilde{u} &= 0 \\
\tilde{v} &= 0 \\
\tilde{c} &= \left(1 + \frac{n}{w} - \frac{w}{z}\right)c
\end{align*} \]

The third case, corresponding to \( u \neq 0, \ z = 0 \), splits in two cases: \( v = 0 \) and \( v \neq 0 \).

Type 3a.

\[ v = z = 0, \ u \neq 0 \mid n = p = w = 0, \ e = \frac{cm}{u} \mid m \]

\[ [X, Y] = uX + cI \quad [Y, Z] = dI \quad [X, Z] = m\left(X + \frac{c}{u}\right)I \]

Eq. (3.2) yields

\[ s_\alpha(u)(\beta u + m) = 0 \]

Note that (3.4) excludes the solutions \( \alpha_k = 2k\pi i/u, \ k \in \mathbb{Z}\backslash\{0\} \). Therefore

\[ \begin{align*}
\alpha &= \frac{m + u}{u} \\
\tilde{u} &= 0 \\
\tilde{v} &= 0 \\
\tilde{c} &= \frac{m + u}{u}d
\end{align*} \]
Type 3b.

\[
\begin{align*}
[z = 0, \ u \neq 0, \ v \neq 0] \ p = w = 0, \ m = \frac{nu}{v}, \ e = \frac{cn}{v} - \frac{dv}{u} n
\end{align*}
\]

\[
[X, Y] = uX + vY + cI \quad [Y, Z] = dI \quad [X, Z] = \frac{nu}{v}X + nY + \left(\frac{cn}{v} - \frac{dv}{u}\right)I
\]

In this case (3.2) yields

\[
e^{\alpha u} = e^v
\]

that, taking into account (3.4), fixes \(\alpha = v/u\). Therefore

\[
\alpha = \frac{v}{u}
\]

\[
\tilde{u} = u - v + \frac{nu}{v}
\]

\[
\tilde{v} = 0
\]

\[
\tilde{c} = \left(1 - \frac{v}{u} + \frac{n}{v}\right)c
\]

Type 4.

\[
[u = z \neq 0] \ m = -w, \ p = -v \ e, n
\]

\[
[X, Y] = uX + vY + cI \quad [Y, Z] = wY + zZ + dI \quad [X, Z] = -wX + nY - vZ + eI
\]

In this case (3.5) leads to an equation of second degree in \(x^u\), where \(x := e^\alpha\),

\[
x^2 + x^u \left(\frac{nu}{2} s(v) s(w) e^{u+w-w} - e^u - e^v - e^{u+v} - e^{u+v-w}\right) + e^{u+v-w} = 0,
\]

that is

\[
x^u_{\pm} = \frac{-b \pm \sqrt{b^2 - 4e^{u+v-w}}}{2}, \quad (4.2)
\]

where

\[
b := \frac{nu}{2} s(v) s(w) e^{u+w-w} - e^u - e^v - e^{u+v} - e^{u+v-w}.
\]

By (3.3)

\[
\tilde{u} = \beta u - w
\]

\[
\tilde{v} = \alpha u - v
\]

\[
\tilde{c} = \left(e + \frac{cw + dv}{u}\right) \frac{e^w s(v) s(w)}{s(v - \alpha u) s(w - \beta u)} - \frac{cw + dv}{u} + \beta c + \alpha d \quad (4.3)
\]
**Type 5.**

\[
\begin{aligned}
&\left[u \neq z, uz \neq 0\right] m = -\frac{uuw}{z}, n = -vw \left(\frac{1}{u} + \frac{1}{z}\right), p = -\frac{vz}{u}, e = -\frac{cw}{z} - \frac{dv}{u} \\
&[X, Y] = uX + vY + cI \quad [Y, Z] = zY + wZ + dI \\
&[X, Z] = -\frac{uw}{z} X - vw \left(\frac{1}{u} + \frac{1}{z}\right) Y - \frac{vz}{u} Z - \left(\frac{cw}{z} + \frac{dv}{u}\right) I
\end{aligned}
\]

Eq. (3.5) yields

\[
(x^u - e^v)(x^z - e^{z-w}) = 0
\]

By (3.4), the possible solutions are

\[
\alpha = \frac{v}{u}
\]

and

\[
\alpha = 1 - \frac{w}{z}
\]

On the other hand, by (3.3)

\[
\tilde{u} = (\beta z - w)\frac{u}{z} \\
\tilde{v} = (\alpha u - v)\frac{z}{u} \\
\tilde{c} = \left(\beta z - w\right)\frac{c}{z} + \left(\alpha u - v\right)\frac{d}{u} = \tilde{u}\frac{c}{u} + \tilde{v}\frac{d}{z}
\]

Therefore, the two solutions are

\[
\alpha = \frac{v}{u} \\
\tilde{u} = u - v - \frac{uuw}{z} \\
\tilde{v} = 0 \\
\tilde{c} = \left(1 - \frac{v}{u} - \frac{w}{z}\right)d
\]

and

\[
\alpha = 1 - \frac{w}{z} \\
\tilde{u} = 0 \\
\tilde{v} = z - w - \frac{vz}{u} \\
\tilde{c} = \left(1 - \frac{v}{u} - \frac{w}{z}\right)d
\]
This concludes the classification of the commutator algebras leading to the closed Baker-Campbell-Haussdorf formula. Such commutator algebras appear in several contexts of mathematics and physics. Here we show how the solution of the BCH problem for the \( \mathfrak{sl}_2(\mathbb{R}) \) algebra, derived in [1], follows immediately applying the explicit general formula (3.1).

The \( \mathfrak{sl}_2(\mathbb{R}) \) algebra is a particular case of the Type 4 commutator algebras. In the following we focus on
\[
X := \lambda_{-k}L_{-k}, \quad Y := \lambda_0L_0, \quad Z := \lambda_kL_k, \quad (4.4)
\]
where the \( L_k \)'s are the Virasoro generators
\[
[L_j, L_k] = (k - j)L_{j+k} + \frac{c}{12} (k^3 - k) \delta_{j+k} I, \quad (4.5)
\]
j, k \in \mathbb{Z}. The \( \mathfrak{sl}_2(\mathbb{R}) \) algebra corresponds to set \( k = 1 \) in (4.4). We have
\[
[X, Y] = k\lambda_0X, \quad [Y, Z] = k\lambda_0Z,
\]
\[
[X, Z] = \lambda_{-k}\lambda_k \left[ \frac{2k}{\lambda_0} Y + \frac{c}{12} (k^3 - k) \right],
\]
so that
\[
u = z = k\lambda_0, \quad n = \lambda_{-k}\lambda_k \frac{2k}{\lambda_0}, \quad c = \lambda_{-k}\lambda_k \frac{c}{12} (k^3 - k), \quad (4.6)
\]
where the central charge \( c \) in (4.6) should not be confused with the \( c = 0 \) in the commutator \([X, Y]\). By the Jacobi Identity,
\[
m = p = 0,
\]
and by (4.3)
\[
\tilde{u} = \beta k\lambda_0, \quad \tilde{v} = \alpha k\lambda_0, \quad \tilde{c} = e\alpha (k\lambda_0, 0)g_\beta (k\lambda_0, 0) = \lambda_{-k}\lambda_k \frac{\alpha k\lambda_0}{1 - e^{-\alpha k\lambda_0}} \frac{\beta k\lambda_0}{1 - e^{-\beta k\lambda_0}} \frac{c}{12} (k^3 - k). \quad (4.7)
\]
Setting \( \tilde{u} = \beta u \) and \( \tilde{v} = \alpha u \) in Eq.(3.1) yields
\[
\exp(X) \exp(Y) \exp(Z) = \frac{(\alpha - \beta)u}{e^{-\beta u} - e^{-\alpha u}} \left[ X + \left( e^{-\frac{\alpha u}{2}} s_\alpha(u) + e^{-\frac{\beta u}{2}} s_\beta(u) \right) Y + Z + \frac{1}{\beta u} \left( e^{-\frac{\alpha u}{2}} s(\alpha u) - e^{-\frac{\beta u}{2}} s((\alpha - \beta)u) \right) \tilde{c} I \right]. \quad (4.8)
\]
Set $\lambda = \lambda_0 \alpha$. By (4.8)

$$
\exp(\lambda_k \mathcal{L}_k) \exp(\lambda_0 \mathcal{L}_0) \exp(\lambda_k \mathcal{L}_k) =
\exp\left\{ \frac{\lambda_+ - \lambda_-}{e^{-k\lambda_+} - e^{-k\lambda_-}} \left[ k\lambda_- \mathcal{L}_0 + \left( 2 - e^{-k\lambda_+} - e^{-k\lambda_-} \right) \mathcal{L}_0 + k\lambda_k \mathcal{L}_k + c_k I \right] \right\},
$$

(4.9)

where by (4.2)

$$
e^{-k\lambda_\pm} = \frac{1 + e^{-k\lambda_0} - k^2 \lambda_- \lambda_+ \pm \sqrt{(1 + e^{-k\lambda_0} - k^2 \lambda_- \lambda_+)^2 - 4 e^{-k\lambda_0}}}{2},
$$

(4.10)

that, together with (4.7), gives

$$
c_k = \frac{\lambda_- \lambda_k}{\lambda_+ - \lambda_-} \left( \frac{\lambda_+}{1 - e^{-k\lambda_+}} - \frac{\lambda_-}{1 - e^{-k\lambda_-}} \right) \frac{c}{12} (k^4 - k^2).
$$

(4.11)

We note that $\lambda_\pm$ corresponds to $\lambda_{\mp}$ in [1], this is irrelevant since the Type 4 commutator algebras are invariant under the interchange $\lambda_\pm \leftrightarrow \lambda_{\mp}$.

Finally, we report the case $\lambda_0 = 0$

$$
\exp(\lambda_- \mathcal{L}_-k) \exp(\lambda_k \mathcal{L}_k) = \exp\left[ \frac{\lambda_+}{\sinh(k\lambda_+)} (k\lambda_- \mathcal{L}_{-k} + k^2 \lambda_- \lambda_k \mathcal{L}_0 + k\lambda_k \mathcal{L}_k + c_k I) \right],
$$

(4.12)

where now

$$
c_k = \lambda_- \lambda_k \frac{c}{24} (k^4 - k^2).
$$

(4.13)

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References

[1] M. Matone, arXiv:1502.06589

[2] A. Van-Brunt and M. Visser, arXiv:1501.02506

[3] A. Van-Brunt and M. Visser, arXiv:1501.05034