STRONGLY FAR PROXIMITY AND HYPERSPACE TOPOLOGY

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Dedicated to the Memory of Som Naimpally

Abstract. This article introduces strongly far proximity \(\delta\), which is associated with Lodato proximity \(\delta\). A main result in this paper is the introduction of a hit-and-miss topology on \(\text{CL}(X)\), the hyperspace of nonempty closed subsets of \(X\), based on the strongly far proximity.

1. Introduction

Usually, when we talk about proximities, we mean Efremović proximities. Nearness expressions are very useful and also represent a powerful tool because of the relation existing among Efremović proximities, Weil uniformities and \(T_2\) compactifications. But sometimes Efremović proximities are too strong. So we want to distinguish between a weaker and a stronger forms of proximity. For this reason, we consider at first Lodato proximity \(\delta\) and then, by this, we define a stronger proximity by using the Efremović property related to proximity.

2. Preliminaries

Recall how a Lodato proximity is defined \([11, 12, 13]\) (see, also, \([16, 14]\)).

Definition 2.1. Let \(X\) be a nonempty set. A Lodato proximity \(\delta\) is a relation on \(\mathcal{P}(X)\) which satisfies the following properties for all subsets \(A, B, C\) of \(X\):

- \(P_0\) \(A \delta B \implies B \delta A\)
- \(P_1\) \(A \delta B \implies A \neq \emptyset\) and \(B \neq \emptyset\)
- \(P_2\) \(A \cap B \neq \emptyset \implies A \delta B\)
- \(P_3\) \(A \delta (B \cup C) \iff A \delta B \lor A \delta C\)
- \(P_4\) \(A \delta B\) and \(\{b\} \delta C\) for each \(b \in B\) \(\implies A \delta C\)

Further \(\delta\) is separated, if

- \(P_5\) \(\{x\} \delta \{y\} \implies x = y\).

When we write \(A \delta B\), we read \(A\) is near to \(B\) and when we write \(A \not\delta B\) we read \(A\) is far from \(B\). A basic proximity is one that satisfies \((P_0) \sim P_3\). Lodato proximity or LO-proximity is one of the simplest proximities. We can associate a topology with the space \((X, \delta)\) by considering as closed sets the ones that coincide with their own closure, where for a subset \(A\) we have

\[\text{cl}A = \{x \in X : x \delta A\}.\]
This is possible because of the correspondence of Lodato axioms with the well-known Kuratowski closure axioms.

By considering the gap between two sets in a metric space \( d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} \) or \( \infty \) if \( A \) or \( B \) is empty, Efremović introduced a stronger proximity called Efremović proximity or EF-proximity.

**Definition 2.2.** An EF-proximity is a relation on \( \mathcal{P}(X) \) which satisfies P0) through P3) and in addition

\[
A \not\in B \Rightarrow \exists E \subset X \text{ such that } A \not\in E \text{ and } X \setminus E \not\in B \text{ EF-property.}
\]

A topological space has a compatible EF-proximity if and only if it is a Tychonoff space.

Any proximity \( \delta \) on \( X \) induces a binary relation over the powerset \( \exp X \), usually denoted as \( \ll \delta \) and named the natural strong inclusion associated with \( \delta \), by declaring that \( A \) is strongly included in \( B \), \( A \ll \delta B \), when \( A \) is far from the complement of \( B \), \( A \not\in X \setminus B \).

By strong inclusion the Efremović property for \( \delta \) can be written also as a betweenness property

(EF) \( A \ll \delta B \), then there exists some \( C \) such that \( A \ll \delta C \ll \delta B \).

A pivotal example of EF-proximity is the metric proximity in a metric space \((X, d)\) defined by

\[ A \delta B \iff d(A, B) = 0. \]

That is, \( A \) and \( B \) either intersect or are asymptotic: for each natural number \( n \) there is a point \( a_n \) in \( A \) and a point \( b_n \) in \( B \) such that \( d(a_n, b_n) < 1/n \).

### 2.1. Hit and far-miss topologies.

Let \( \text{CL}(X) \) be the hyperspace of all non-empty closed subsets of a space \( X \). Hit and miss and hit and far-miss topologies on \( \text{CL}(X) \) are obtained by the join of two halves. Well-known examples are Vietoris topology \([22, 23, 24, 25]\) (see, also, \([1, 2, 4, 5, 6, 3, 7, 10, 15, 8, 17]\)) and Fell topology \([9, 10, 1]\). In this article, we concentrate on an extension of Vietoris based on the strongly far proximity.

**Vietoris topology**

Let \( X \) be an Hausdorff space. The **Vietoris topology** on \( \text{CL}(X) \) has as subbase all sets of the form

- \( V^- = \{ E \in \text{CL}(X) : E \cap V \neq \emptyset \} \), where \( V \) is an open subset of \( X \),
- \( W^+ = \{ C \in \text{CL}(X) : C \subset W \} \), where \( W \) is an open subset of \( X \).

The topology \( \tau_{V^-} \) generated by the sets of the first form is called **hit part** because, in some sense, the closed sets in this family hit the open sets \( V \). Instead, the topology \( \tau_{V^+} \) generated by the sets of the second form is called **miss part**, because the closed sets here miss the closed sets of the form \( X \setminus W \).

The Vietoris topology is the join of the two part: \( \tau_V = \tau_{V^-} \lor \tau_{V^+} \). It represents the prototype of hit and miss topologies.

The Vietoris topology was modified by Fell. He left the hit part unchanged and in the miss part, \( \tau_{F^+} \) instead of taking all open sets \( W \), he took only open subsets with compact complement.
Fell topology: \[ \tau_F = \tau_V^- \vee \tau_F^+ \]

It is possible to consider several generalizations. For example, instead of taking open subsets with compact complement, for the miss part we can look at subsets running in a family of closed sets \( \mathcal{B} \). So we define the hit and miss topology on \( CL(X) \) associated with \( \mathcal{B} \) as the topology generated by the join of the hit sets \( A^- \), where \( A \) runs over all open subsets of \( X \), with the miss sets \( A^+ \), where \( A \) is once again an open subset of \( X \), but more, whose complement runs in \( \mathcal{B} \).

Another kind of generalization concerns the substitution of the inclusion present in the miss part with a strong inclusion associated to a proximity. Namely, when the space \( X \) carries a proximity \( \delta \), then a proximity variation of the miss part can be displayed by replacing the miss sets with far-miss sets \( A^{++} = \{ E \in CL(X) : E \ll \delta A \} \).

Also in this case we can consider \( A \) with the complement running in a family \( \mathcal{B} \) of closed subsets of \( X \). Then the hit and far-miss topology, \( \tau_{\delta, \mathcal{B}} \), associated with \( \mathcal{B} \) is generated by the join of the hit sets \( A^- \), where \( A \) is open, with far-miss sets \( A^{++} \), where the complement of \( A \) is in \( \mathcal{B} \).

Fell topology can be considered as well an example of hit and far-miss topology. In fact, in any EF-proximity, when a compact set is contained in an open set, it is also strongly contained.

3. Main Results

Results for the strongly far proximity [19] (see, also, [18, 21, 20]) are given in this section. Let \( X \) be a nonempty set and \( \delta \) be a Lodato proximity on \( P(X) \).

**Definition 3.1.** We say that \( A \) and \( B \) are \( \delta \)-strongly far and we write \( A \ll \delta B \) if and only if \( A \not\subseteq B \) and there exists a subset \( C \) of \( X \) such that \( A \not\subseteq X \setminus C \) and \( C \not\subseteq B \), that is the Efremović property holds on \( A \) and \( B \).

**Example 3.2.** In the Fig. 3.1, let \( X \) be a nonempty set endowed with the Euclidean metric proximity \( \delta_e \) (e for Euclidean), \( C, E \subset X, A \subset C, B \subset E \). Clearly, \( A \parallel B \) (\( A \) is strongly far from \( B \)), since \( A \parallel C \) and \( C \parallel B \). Also observe that the Efremović property holds on \( A \) and \( B \). □

Observe that \( A \parallel B \) does not imply \( A \parallel B \). In fact, this is the case when the proximity \( \delta \) is not an EF-proximity.

**Example 3.3.** Let \( (X, \tau) \) be a non-locally compact Tychonoff space. The Alexandroff proximity is defined as follows: \( A \delta_A B \iff clA \cap clB \neq \emptyset \) or both \( clA \) and \( clB \) are non-compact. This proximity is a compatible Lodato proximity that is not an EF-proximity. So \( A \parallel_A B \) does not imply \( A \parallel_B B \). □

**Theorem 3.4.** The relation \( \parallel \) is a basic proximity.

**Proof.** Immediate from the properties of \( \delta \). □
Theorem 3.8. The hypertopologies $A$ where $A$ is a compact subset and $B$ are compatible with the previous choices. But $B \delta$ appears to be stronger than $\hat{\delta}$, but it is possible to observe the following relations.

Theorem 3.5. The relation $\hat{\delta}$ is stronger than $\hat{\delta}$, that is $A \hat{\delta} B \Rightarrow A \hat{\delta} B$.

Proof. Suppose $A \hat{\delta} B$. This means that there exists a subset $C$ of $X$ such that $A \hat{\delta} X \setminus C$ and $C \hat{\delta} B$. By the Lodato property $P4$ (see [11]), we obtain that $\text{cl}A \cap (X \setminus C) = \varnothing$ and $\text{cl}C \cap \text{cl}B = \varnothing$. So $\text{cl}A \subset \text{int}(C)$, $\text{cl}B \subset \text{int}(X \setminus C)$ and $\text{int}(C) \cap \text{int}(X \setminus C) = \varnothing$, that gives $A \hat{\delta} B$. $\square$

We now want to consider hit and far-miss topologies related to $\delta$ and $\hat{\delta}$ on $\text{CL}(X)$, the hyperspace of nonempty closed subsets of $X$.

To this purpose, call $\tau_\delta$ the topology having as subbase the sets of the form:

- $V^- = \{ E \in \text{CL}(X) : E \cap V \neq \varnothing \}$, where $V$ is an open subset of $X$,
- $A^{++} = \{ E \in \text{CL}(X) : E \n E \setminus A \}$, where $A$ is an open subset of $X$.

and $\tau_\hat{\delta}$ the topology having as subbase the sets of the form:

- $V^- = \{ E \in \text{CL}(X) : E \cap V \neq \varnothing \}$, where $V$ is an open subset of $X$,
- $A_\hat{\delta} = \{ E \in \text{CL}(X) : E \n E \setminus A \}$, where $A$ is an open subset of $X$.

It is straightforward to prove that these are admissible topologies on $\text{CL}(X)$. The following results concern comparisons between them. From this point forward, let $X$ be a $T_1$ topological space.

Lemma 3.6. Let $A, B, C \in \text{CL}(X)$. If $A \notin B \Rightarrow A \notin B$ for all $A \in \text{CL}(X)$, then $C \subseteq B$. That is $(X \setminus B)^{++} \subseteq (X \setminus C)_\hat{\delta} \Rightarrow C \subseteq B$.

Proof. By contradiction, suppose $C \notin B$. Then there exists $x \in C : x \notin B$. So $x \notin B$ but $x \in C$, which is absurd. $\square$

Lemma 3.7. Let $\delta = \delta_A$, the Alexandroff proximity on a non-locally compact Tychonoff space, and let $H$ and $E$ be open subsets of $X$. Then $H = E^{++} \Leftrightarrow H \subseteq E$.

Proof. $\Rightarrow$. By contradiction, suppose that $H \notin E$. Then we can choose $X \setminus H$ as compact subset and $X \setminus E$ non-compact. Take another closed subset $B$ noncompact and suppose $B \hat{\delta} X \setminus H$. So there exists $D : B \notin X \setminus D$ and $D \notin X \setminus H$, and this is compatible with the previous choices. But $B \delta_A X \setminus E$, being both non-compact sets.

$\Leftarrow$. For any $B \in \text{CL}(X)$, $B \hat{\delta} X \setminus H \Rightarrow B \hat{\delta} X \setminus E \Rightarrow B \notin X \setminus E$. $\square$

Now let $\tau_\delta^{++}$ be the hypertopology having as subbase the sets of the form $A^{++}$, where $A$ is an open subset of $X$, and let $\tau_\hat{\delta}^+$ the hypertopology having as subbase the sets of the form $A_\hat{\delta}^+$, again with $A$ an open subset of $X$.

Theorem 3.8. The hypertopologies $\tau_\delta^{++}$ and $\tau_\hat{\delta}^+$ are not comparable.
Proof. First we want to prove that, in general, $\tau_\omega^+ \not\subseteq \tau_\omega^{++}$. Consider the space of rational numbers $X = \mathbb{Q}$ and the Alexandroff proximity $\delta_A$ (see example 3.3). Let $H$ be an open subset of $X$ with $cl(X \setminus H)$ non-compact and suppose $E \in CL(X)$. We ask if there exists a $\tau_\omega^{++}$-open set, $K^{++}$, such that $E \in K^{++} \subseteq H_\omega$. We have two cases: $cl(X \setminus K)$ compact or not. First, suppose $cl(X \setminus K)$ compact and $A \in K^{++}$ with $clA$ non-compact. Then it must be $clA \cap cl(X \setminus K) = \emptyset$. But $A \delta A X \setminus H$, because for all $D$, $A \delta A X \setminus D$ or $D \delta A X \setminus H$. In fact if $clD$ is compact, then $cl(X \setminus D)$ is not compact. So either both $clA$ and $cl(X \setminus D)$ are non-compact, or both $clD$ and $cl(X \setminus H)$ are non-compact. Instead, suppose $cl(X \setminus K)$ non-compact. So, being $A \notin A X \setminus K$, we have $clA$ compact and $clA \cap cl(X \setminus K) = \emptyset$. To obtain $A \delta A X \setminus H$, by lemma 2.6 we should have $K \subseteq H$. So we need a set $K$ such that $clA \subseteq K \subseteq H$ and more with $clK$ compact and $clA \subseteq K \subseteq clK \subseteq H$. But we are in a non-locally compact space, so it could be not possible.

Conversely, we want to prove that $\tau_\omega^{++} \not\subseteq \tau_\omega^+$. Consider again the space of rational numbers $X = \mathbb{Q}$ and the Alexandroff proximity $\delta_A$. Take $E^{++} \in \tau_\omega^{++}$ and $A \in E^{++}$, with $E$ open subset of $X$. To identify a $\tau_\omega^+$-open set, $H_\omega$, such that $A \in H_\omega \subseteq E^{++}$, by lemma 2.7 we need $H \subseteq E$. But we can choose $A$ and $E \setminus H$ in such a way that EF-property does not hold. So EF-property does not hold either for $A$ and $X \setminus H$, for each $H \subseteq E$. Hence $A$ cannot belong to any $H_\omega$ included in $E^{++}$. □

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