Skein theory for $SU(n)$-quantum invariants

Adam S. Sikora

Abstract  For any $n \geq 2$ we define an isotopy invariant, $(\Gamma)_n$, for a certain set of $n$-valent ribbon graphs $\Gamma$ in $\mathbb{R}^3$, including all framed oriented links. We show that our bracket coincides with the Kauffman bracket for $n = 2$ and with the Kuperberg’s bracket for $n = 3$. Furthermore, we prove that for any $n$, our bracket of a link $L$ is equal, up to normalization, to the $SU_n$-quantum invariant of $L$. We show a number of properties of our bracket extending those of the Kauffman’s and Kuperberg’s brackets, and we relate it to the bracket of Murakami-Ohtsuki-Yamada. Finally, on the basis of the skein relations satisfied by $(\cdot)_n$, we define the $SU_n$-skein module of any 3-manifold $M$ and we prove that it determines the $SL_n$-character variety of $\pi_1(M)$.

AMS Classification  57M27; 17B37

Keywords  Kauffman bracket, Kuperberg bracket, Murakami-Ohtsuki-Yamada bracket, quantum invariant, skein module

1 Introduction

The $SU_2$-quantum invariant of links, known as the Jones polynomial, can be conveniently defined in terms the Kauffman bracket invariant, [Ka]. This approach has several advantages, for example, leading to definitions of skein modules and Khovanov homology$^1$ – two notions in the center of current active research – see for example [Bu FGL Ga Gd GI PS S2] and [APS BN Ga HK Jl K1 K2 KR Le Ra Yi]. In [Ku], Kuperberg constructs a bracket isotopy invariant of links and 3-valent graphs in $\mathbb{R}^3$, with properties analogous to those of the Kauffman bracket, and shows that it coincides with the $SU_3$-quantum invariant. We extend his work, by defining a bracket isotopy invariant $(\cdot)_n$ for any $n \geq 2$ and by showing that it determines the $SU_n$-quantum invariant. More specifically, for any $n \geq 2$ we consider the set $\mathcal{W}_n(\mathbb{R}^3)$ of $n$-webs which are ribbon graphs $\Gamma$ in $\mathbb{R}^3$ whose coupons are either $n$-valent sources or

---

1The Kauffman bracket skein relations allow a particularly simple definition of Khovanov’s $SU_2$-homology groups, [Vi].
$n$-valent sinks. In particular, $W_n(\mathbb{R}^3)$ contains all oriented framed links in $\mathbb{R}^3$ for any $n$. We define a bracket isotopy invariant of $n$-webs, $\langle \Gamma \rangle_n$, and show that it coincides with the Kauffman bracket for $n = 2$, and with the Kuperberg’s bracket for $n = 3$.

For reader’s convenience, we state three different definitions of $\langle \cdot \rangle_n$: by skein relations, (Theorem 1), by a state sum formula, (Proposition 2), and as a contraction of tensors, (Section 2). Furthermore, we show that for any $n$, $\langle \Gamma \rangle_n$ defines the $SU_n$-quantum invariant of $\Gamma$ with edges of $\Gamma$ labeled by the defining $SU_n$-representation and the sinks and the sources of $\Gamma$ labeled by the $q$-antisymmetrizer and its dual. The proofs are based on [RT].

We prove a number of properties of our bracket which extend those of the Kauffman’s and Kuperberg’s brackets. In particular, $\langle \cdot \rangle_n$ satisfies a skein relation which relates a crossing to its two “smoothings”, cf. Proposition 2. Furthermore, there is a state sum formula for $\langle \cdot \rangle_n$, Theorem 9, which has the “positivity” property analogous to that used in the construction of Khovanov and Khovanov-Rozansky homology groups, [K1, K2, KR], cf. Proposition 10. Our bracket can be used for an alternative definition of Khovanov-Rozansky homology groups; cf. Section 1.5.

There exists an alternative generalization of the Kauffman bracket due to Murakami, Ohtsuki, and Yamada. Their bracket is defined for certain 3-valent colored graphs with a flow, [MOY, Mn]. It is expressed in terms of our bracket in Sections 1.5 and 1.7. We believe that our bracket can be expressed in terms of Murakami-Ohtsuki-Yamada bracket as well. Nonetheless, both approaches have their advantages. Perhaps, an advantage of our approach is that $\langle \cdot \rangle_n$ is related directly to the representation theory of $U_q(sl_n)$, and that for $q = 1$ our skein relations are equivalent to the relations between characters of $SL(n)$-representations. Furthermore, our relations seem to be the most appropriate for the definition of $SU_n$-skein modules of 3-manifolds, cf. Section 3.1. (Our definition agrees with those of Ohtsuki and Yamada, [OY], and Frohman and Zhong, [FZ], for $n = 3$.) Several important properties of the Kauffman bracket skein modules have their generalizations to the $SU_n$-skein modules for any $n$.

In this paper, we show that $SU_n$-skein module of a manifold $M$ for $t = 1$ is a commutative ring isomorphic to the coordinate ring of the $SL_n$-character variety of $\pi_1(M)$. We postpone further study of the $SU_n$-skein modules to a forthcoming paper.

**Acknowledgments** The author was partially sponsored by NSF grants DMS-0307078 and DMS-0111298.
1.1 Webs

An \(n\)-web is a ribbon graph in \(\mathbb{R}^3\), cf. [RT], whose every coupon is either an \(n\)-valent sink or an \(n\)-valent source. We denote the coupons of the ribbon graphs by discs rather than rectangles and we use a marking point to represent the side of the coupon with no bands attached, \(\square = \bigcirc\).

![Figure 1: An example of a 3-web](image)

For reader’s convenience, we restate the definition of a web without invoking the notion of a ribbon graph. The role of “edges” of webs is played by bands which are embeddings of squares \([0,1] \times [0,1]\) into \(\mathbb{R}^3\). The segments \([0,1] \times \{0\}\) and \([0,1] \times \{1\}\) are the source and the target of the band, respectively. Their complement, \([0,1] \times (0,1)\), is the interior of the band.

An \(n\)-web is an oriented surface embedded in \(\mathbb{R}^3\) composed of a finite number of annuli, bands, and discs satisfying the following conditions:

(i) The annuli, disks, and the interiors of the bands are disjoint from each other.

(ii) The sources and the targets of bands are disjoint from each other and all of them lie in the boundaries of discs.

(iii) The boundary of every disk contains either precisely \(n\) sources and no targets of bands, in which case the disc is called a source, or it contains precisely \(n\) targets and no sources of bands. In the that case, the disk is called a sink of the web.

(iv) The marked boundary points of disks lie outside the sources and targets of bands.

Since each \(n\)-web retracts to its spine, which is an oriented graph, often the bands and discs of webs will be called its edges and vertices, respectively. In this terminology, each vertex \(v\) of a web is \(n\)-valent and all edges adjacent to \(v\) are either directed outwards, if \(v\) is a source, or inwards, if \(v\) is a sink. Notice that each web has an equal number of sources and sinks.

Our definition of \(n\)-webs is modeled on the notion of \(n\)-valent graphs considered in [S1], cf. Section 3.2. The \(n\)-webs extend the notion of webs for the geometric \(A_1\)-spider introduced in [Ku], cf. Section 1.4.
By analogy with the notion of a link diagram, an \textit{n-web diagram} is a projection \( \pi : \Gamma \to \mathbb{R}^2 \) of an \( n \)-web \( \Gamma \) into \( \mathbb{R}^2 \) which is an embedding of \( \Gamma \) except for a finite set of transverse (double) intersections of bands of \( \Gamma \) called crossings. We require that \( \pi \) preserves the orientation of \( \Gamma \) (considered as an oriented surface) and that it embeds the sinks and the sources into \( \mathbb{R}^2 \) away from the intersections. In particular, unlike in [RT], a web diagram is not allowed to have twists, \( \overset{\circ}{\text{w}} \), in their bands. Each web \( \Gamma \) is represented by a web diagram; for example:

\[
\begin{array}{c}
\text{Example web diagram}
\end{array}
\]

\subsection{1.2 The bracket isotopy invariant of \( n \)-webs}

For any permutation \( \sigma \in S_n \), define the \textit{length of} \( \sigma \), \( l(\sigma) \), to be the minimal number of factors in the decomposition of \( \sigma \) into elementary transpositions \((i, i + 1)\), \( i = 1, \ldots, n-1 \),

\[
l(\sigma) = \#\{(i, j) : 1 \leq i < j \leq n, \, \sigma(i) > \sigma(j)\}.
\]  

(1)

For \( \sigma \in S_n \), let \( \begin{overline{\sigma} \end{overline{\sigma}} } \) denote the positive braid with \( l(\sigma) \) crossings representing \( \sigma \). Such braid is unique. Let \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}\) and let \([n]! = [1] \cdot \ldots \cdot [n] \).

\textbf{Theorem 1} \hspace{1em} \textit{There exists a unique isotopy invariant of \( n \)-webs,} \( \langle \Gamma \rangle_n \in \mathbb{Z}[q^{\pm \frac{1}{n}}] \), \textit{satisfying the following conditions:}

\begin{enumerate}
  \item \[ q^n \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n - q^{-\frac{1}{n}} \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n = (q - q^{-1}) \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n. \]
  \item \[ \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n = q^{n-n-1} \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n, \quad \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n = q^{n-1-n} \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n. \]
  \item \[ \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n = q^{n(n-1)} \cdot \sum_{\sigma \in S_n} (-q^{1-n})^{l(\sigma)} \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n. \]
  \item \( \langle \Gamma \cup \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n = [n] \langle \Gamma \rangle_n \). Here \( \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \) denotes the trivial framed knot unlinked with \( \Gamma \).
  \item \( \langle \emptyset \rangle_n = 1 \) and, consequently, \( \langle \begin{array}{c}
  \begin{overline{\sigma} \end{overline{\sigma}} }
  \end{array} \rangle_n = [n] \).
\end{enumerate}
Proof The hard part of the statement – the existence of the bracket – follows from Theorem 17 stated in Section 2. The uniqueness of the bracket follows from the fact that each web $\Gamma$ has an equal number of sinks and sources, and, therefore, condition (iii) makes possible to represent $\langle \Gamma \rangle_n$ by a linear combination of brackets of framed links. On the other hand, the bracket for framed links is determined by conditions (i), (ii), (iv) and (v).

Relation (iii) appeared in an implicit form in [Bl, Yo] already. The skein relations of Theorem 1, appear in the most natural, but not necessarily, the simplest form. If $w(\Gamma)$ denotes the writhe (i.e. the sum of signs of crossings) of a web diagram $\Gamma$, and $v(\Gamma)$ is the number of sinks of $\Gamma$ then

$$P_n(\Gamma) = q^{(n^{-1} - n)w(\Gamma) - n(n-1)v(\Gamma)} \langle \Gamma \rangle_n$$

is invariant under all Reidemeister moves. Furthermore, it satisfies the standard skein relations of the $SU(n)$-quantum invariants, cf. [Tu, Thm 4.2.1]:

- $q^n P_n(\begin{tikzpicture}[baseline=-.5ex]
    \draw (-.1,0) -- (.2,0);
    \draw (.1,-.1) -- (-.1,-.4);
    \draw (.1,.1) -- (-.1,.4);
  \end{tikzpicture}) - q^{-n} P_n(\begin{tikzpicture}[baseline=-.5ex]
    \draw (0,0) circle (.1);
  \end{tikzpicture}) = (q - q^{-1}) P_n(\begin{tikzpicture}[baseline=-.5ex]
    \draw (0,0) circle (.1);
    \draw (.1,0) -- (-.2,0);
  \end{tikzpicture}),$
- $P_n(L \cup \circ) = [n] P_n(L).$

and the additional relation:

- $P_n(\begin{tikzpicture}[baseline=-.5ex]
    \draw (-.05,0) -- (.2,0);
    \draw (.05,-.1) -- (-.2,-.4);
    \draw (.05,.1) -- (-.2,.4);
    \draw (-.3,0) -- (-.7,-.4);
    \draw (-.3,0) -- (-.7,.4);
  \end{tikzpicture}) = \sum_{\sigma \in S_n} (-q^{n-1})^{l(\sigma)} P_n(\begin{tikzpicture}[baseline=-.5ex]
    \draw (-.05,0) -- (.2,0);
    \draw (.05,-.1) -- (-.2,-.4);
    \draw (.05,.1) -- (-.2,.4);
    \draw (-.3,0) -- (-.7,-.4);
    \draw (-.3,0) -- (-.7,.4);
    \draw (-.6,0) -- (-.6,.4);
    \draw (-.6,0) -- (-.6,-.4);
  \end{tikzpicture}).$

Proposition 2 (Proof in Section 6)

$$\langle \begin{tikzpicture}[baseline=-.5ex]
    \draw (-.05,0) -- (.2,0);
    \draw (.05,-.1) -- (-.2,-.4);
    \draw (.05,.1) -- (-.2,.4);
    \draw (-.3,0) -- (-.7,-.4);
    \draw (-.3,0) -- (-.7,.4);
  \end{tikzpicture} \rangle_n = q^{\frac{n-1}{n}} \langle \begin{tikzpicture}[baseline=-.5ex]
    \draw (0,0) circle (.1);
  \end{tikzpicture} \rangle_n - q^{\frac{n(n-1)}{2}} \frac{1}{[n-2]!} \langle \begin{tikzpicture}[baseline=-.5ex]
    \draw (-.05,0) -- (0,.3);
    \draw (0,-.3) -- (-.05,0);
  \end{tikzpicture} \rangle_n,$$

where the band labeled by $n-2$ represents $n-2$ parallel bands.

The above relation generalizes the Kauffman bracket skein formula and it makes possible to represent any link (or web) as a linear combination of webs with no crossings. A state-sum formula for the bracket of webs with no crossings is provided in Section 1.6. Note that various renormalizations of $\langle \cdot \rangle_n$ are possible, leading to a skein formula of Proposition 2 without fractional coefficients. Nonetheless, our definition seems to be the most natural one, cf. Section 2, and leading to the simplest state sum formula.

The following result shows that the bracket $\langle \Gamma \rangle_n$ for $n$ odd does not depend on the choice of marked points on the vertices of $\Gamma$. 

Algebraic & Geometric Topology, Volume 5 (2005)
Proposition 3  (Proof in Section 7) If $\Gamma, \Gamma'$ are $n$-webs which differ by the choice of marked points on the boundaries of their discs (vertices) only, then

(i) $\langle \Gamma \rangle_n = \langle \Gamma' \rangle_n$ if $n$ is odd, and

(ii) $\langle \Gamma \rangle_n = \langle \Gamma' \rangle_n \mod 2$ if $n$ is even.

1.3 The Kauffman bracket and $\langle \cdot \rangle_2$

The Kauffman bracket $[L] \in \mathbb{Z}[A^{\pm 1}]$ is an invariant of unoriented framed links $L \subset S^3$ given by the following skein conditions:

$[\text{ } ] = A [\text{ } ] + A^{-1} [\text{ \ } ], \quad [L \cup \bigcirc] = (-A^2 - A^{-2})[L], \quad [\emptyset] = 1.$

Theorem 4  For any $2$-web diagram $D$,

$$\langle D \rangle_2 = (-1)^{w(D) + c(D)}[D],$$

where $A = q^{\frac{1}{2}}$, $w(D)$ denotes the sum of signs of crossings of $D$ and $c(D)$ is the number of components of the link represented by $D$. (On the right side $D$ is considered as an unoriented framed link diagram).

Proof  The bracket $\langle \cdot \rangle_2$ for links is uniquely determined by conditions (i),(ii),(iv) and (v) of Theorem 11. Since $(-1)^{w(D) + c(D)}[D]$ satisfies these relations, the statement follows.

Note that the bracket of any $2$-web $\Gamma$ can be expressed by the bracket of a framed link by the following operations:

$$\langle \text{ } \rangle_2 = q \langle \text{ } \rangle_2, \quad \langle \text{ } \rangle_2 = -q^{\frac{1}{2}} \langle \text{ } \rangle_2.$$

These equations follow from Theorem 11. For example,

$$\left. \begin{array}{l}
\langle \text{ } \rangle_2 = q^{\frac{1}{2}} \left( q - q^{-\frac{1}{2}} \right)
\langle \text{ } \rangle_2 = q^2 \left( q^{-\frac{1}{2}} - q^{-\frac{1}{2}} q^{\frac{1}{2}} \right)
\langle \text{ } \rangle_2 = q^2 \left( q^{-\frac{1}{2}} q^{-\frac{1}{2}} - q^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\frac{1}{2}} \right)
\end{array} \right\} = \left. \begin{array}{l}
q^2 \left( q^{-\frac{1}{2}} q^{-\frac{1}{2}} - q^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\frac{1}{2}} \right)
\langle \text{ } \rangle_2 = -q \langle \text{ } \rangle_2.
\end{array} \right\} \cdot$$

Algebraic & Geometric Topology, Volume 5 (2005)
1.4 Kuperberg’s bracket and $\langle \cdot \rangle_3$

Kuperberg defined an invariant of framed graphs which are defined as our 3-webs but without marked points on their vertices, [Ku]. His bracket is defined by the following relations, in which we substituted his $q$ by $q^{-2}$:

\[(i) \quad \begin{array}{c}
\includegraphics[width=0.5cm]{fig1.png}
\end{array} = q^{-\frac{1}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig2.png}
\end{array} + q^{\frac{2}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig3.png}
\end{array}
\]

\[(ii) \quad \begin{array}{c}
\includegraphics[width=0.5cm]{fig4.png}
\end{array} = q^{\frac{1}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig5.png}
\end{array} + q^{-\frac{2}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig6.png}
\end{array}
\]

\[(iii) \quad \begin{array}{c}
\includegraphics[width=0.5cm]{fig1.png}
\end{array} = [3]
\]

\[(iv) \quad \begin{array}{c}
\includegraphics[width=0.5cm]{fig7.png}
\end{array} = -[2]
\]

\[(v) \quad \begin{array}{c}
\includegraphics[width=0.5cm]{fig8.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.5cm]{fig9.png}
\end{array} + \begin{array}{c}
\includegraphics[width=0.5cm]{fig10.png}
\end{array}
\]

Additionally, (i) and (ii) imply

\[(vi) \quad q^{\frac{1}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig1.png}
\end{array} - q^{-\frac{1}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig2.png}
\end{array} = (q - q^{-1}) \begin{array}{c}
\includegraphics[width=0.5cm]{fig3.png}
\end{array}
\]

and (i) and (iv) imply

\[(vii) \quad \begin{array}{c}
\includegraphics[width=0.5cm]{fig11.png}
\end{array} = q^{\frac{2}{3}} \begin{array}{c}
\includegraphics[width=0.5cm]{fig12.png}
\end{array}
\]

**Theorem 5** Kuperberg’s bracket of any web $\Gamma$ is equal to $(-q)^{-\frac{1}{2}v(\Gamma)} \langle \Gamma \rangle_3$, where $v(\Gamma)$ is the number of 3-valent vertices of $\Gamma$. (By Theorem 3(i), $\langle \Gamma \rangle_3$ is well defined.)

**Proof** It is straightforward to check that $q^{-v(\Gamma)} \langle \Gamma \rangle_3$ satisfies relations (i),(iii), (vi), and (vii). These equations uniquely determine Kuperberg’s bracket: (i) makes possible to express Kuperberg’s bracket of every Kuperberg’s web as a linear combination Kuperberg’s brackets of framed links. These are uniquely determined by (i),(iii),(vi), and (vii).
1.5 Bracket isotopy invariant of framed singular links

A singular framed link is a ribbon graph whose each vertex has two sinks and two sources. In particular, every oriented framed link is singular.

There is a map
\[ \Psi_n : \{ \text{singular framed links in } \mathbb{R}^3 \} \rightarrow \{ n \text{-webs in } \mathbb{R}^3 \}, \]

replacing each vertex in a singular framed link by two \( n \)-valent vertices connected by \( n-2 \) parallel edges:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{framed_sing_link}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{n_valent}
\end{array}
\end{array}
\]

For any singular link diagram \( D \), let
\[
(D)_n = \frac{\langle \Psi(D) \rangle_n}{\left[ n-2 \right] ! q^{n(n-1)/2} v(D) \cdot q^{w(D)/n}},
\]

where \( v(D) \) is the number of singularities of \( D \) (ie. 4-valent vertices) and \( w(D) \) is the number of positive crossings minus negative crossings.

Theorem 6 \( (\Gamma)_n \in \mathbb{Z}[q^{\pm n}] \), and
(i) \( \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{4_valent} \\
\end{array} \right)_n = q \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{4_valent} \\
\end{array} \right)_n - \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{4_valent} \\
\end{array} \right)_n, \)
(ii) \( \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{4_valent} \\
\end{array} \right)_n = q^{-1} \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{4_valent} \\
\end{array} \right)_n - \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{4_valent} \\
\end{array} \right)_n, \)
(iii) \( \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{n_valent} \\
\end{array} \right)_n = q^n \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{n_valent} \\
\end{array} \right)_n, \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{n_valent} \\
\end{array} \right)_n = q^{-n} \left( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{n_valent} \\
\end{array} \right)_n, \)
(iv) \( \left( L \cup \bigcirc \right)_n = [n](L)_n. \)
(v) \( (\emptyset)_n = 1 \) and, consequently, \( (\bigcirc)_n = [n]. \)

Proof (i) follows from Proposition 2 (ii) follows from (i) and Theorem 1(i). Parts (iii)–(v) follow from Theorem 1(ii),(iv) and (v).
Skein theory for $SU(n)$-quantum invariants

$(\cdot)$ is a version of the Kauffman-Vogel bracket, [KV]. Furthermore, it is related to the Murakami-Ohtsuki-Yamada bracket, [MOY §3] (see also [Mu]) in the following manner: Given a singular framed link $L$, label all its edges by 1 and replace each of its vertices by a pair of 3-valent vertices,

$$
\begin{array}{c}
\text{edge} \\
\text{vertex}
\end{array}
\rightarrow
\begin{array}{c}
\text{vertex} \\
\text{vertex}
\end{array}
.$$ Denote the colored ribbon graph obtained in this way by $\Phi(L)$.

**Proposition 7** $(L)_n$ is equal to the Murakami-Ohtsuki-Yamada bracket of $\Phi(L)$ when our $q$ is identified with $q^\frac{1}{2}$ in [MOY].

**Proof** It follows from [MOY] that the MOY bracket of $\Phi(L)$ satisfies conditions (i)-(v) of Theorem 6. These conditions determine $(\cdot)$ uniquely. 

The above proposition relates the Murakami-Ohtsuki-Yamada bracket with our bracket for some graphs only. We will see in Section 1.7 that Murakami-Ohtsuki-Yamada bracket of every 3-valent graph with a flow is determined by our bracket of a corresponding $n$-web.

Khovanov and Rozansky use $(\cdot)$ to define a homology theory whose extended Euler characteristic is the $SU_n$-quantum invariant, [KR].

### 1.6 State sum formula for the brackets of planar webs

An important future of Kauffman bracket is that it is given by a simple state sum formula. We describe a generalization of this formula for our bracket of $n$-webs below. An $n$-web diagram $\Gamma$ is planar if it has no crossings. Since Proposition 2 makes possible to express the bracket of any $n$-web as a linear combination of brackets of planar $n$-webs, we formulate a state sum formula planar webs only.

A state $S$ of a planar $n$-web diagram $\Gamma$ is a labeling of its annuli and bands $e$ by numbers $S(e) \in \{1, ..., n\}$ such that the bands attached to every disc are labeled by different numbers. (There is no restriction on labeling of annuli.)

Note that every state of $\Gamma$ determines an ordering of edges adjacent to every vertex $v$ of $\Gamma$. However, there is also a natural ordering of edges adjacent to $v$, which does not depend on the choice of a state: If $v$ is a sink then we order the edges from 1 to $n$ by starting at the base point of the disc $v$ and then by
Adam S. Sikora

Figure 3: The canonical ordering of bands adjacent to a sink and a source for $n = 3$

moving clockwise around its boundary. If $v$ is a source then we start at the base point of the disc $v$ and move counter-clockwise around its boundary.

For any state $S$ and a vertex $v$ of $\Gamma$, let $P(S, v)(i)$ denote the label associated by the state $S$ with the $i$th band attached to $v$. Hence $P(S, v) \in S_n$.

For any state $S$, we define the rotation index of $S$ as follows.

$$\text{rot}_n(S) = \sum_e \text{ind}(a)(2S(e) - n - 1),$$

(2)

where the sum is over all annuli and bands of $\Gamma$. If $e$ is an annulus, then $\text{ind}(e)$ is either +1 or −1 depending on whether $e$ is oriented anti-clockwise or clockwise. The indices $\text{ind}(e)$ for edges $e$ of $\Gamma$ are defined as follows: For each band $e$ in an $n$-web $\Gamma$ choose a smooth embedded arc

$$\alpha_e : [0, 1] \to \text{the band } e \cup \text{the sink of } e \cup \text{the source of } e$$

connecting the marked points of the sink and the source.

Choose the arcs $\alpha_e$ such that for different bands $e, e'$ the arcs $\alpha_e, \alpha_{e'}$ are disjoint, except possibly meeting at one or two of their endpoints. The union $\bigcup_e \alpha_e$ taken over all bands $e$ of $\Gamma$ forms an oriented $n$-valent graph $\Gamma'$ in $\mathbb{R}^2$.

We say that $\Gamma'$ is a core of $\Gamma$ if for every vertex $v$ of $\Gamma'$ the tangent vectors at $v$ of arcs having one of their endpoints at $v$ are pointing in the same direction.

Figure 4: A core in a 3-web

Given a core $\Gamma'$ of $\Gamma$, for every band $b$ in $\Gamma$ we define its winding number,

$$\text{ind}(b) = \frac{1}{2\pi i} \int_0^1 \frac{\alpha''_b(t)}{\alpha'_b(t)} dt,$$

Algebraic & Geometric Topology, Volume 5 (2005)
where we identify $\mathbb{R}^2$ with $\mathbb{C}$ and assume that $\alpha_b : [0, 1] \to \mathbb{C}$. Note that $\text{ind}(b) = \frac{\beta}{2\pi} \mod \mathbb{Z}$, where $\beta$ is the angle between the tangent vectors to $\alpha_b$ at its endpoints.

**Lemma 8** (Proof in Section 8) For any state $S$ of an $n$-web $\Gamma$, $\text{rot}_n(S)$ is independent of the choice of a core of $\Gamma$. Furthermore, $\text{rot}_n(S)$ is an isotopy invariant of $\Gamma$ and $\text{rot}_n(S) \in \mathbb{Z}$.

**Theorem 9** (Proof in Section 8) For any planar $n$-web $\Gamma$,

$$\langle \Gamma \rangle_n = \sum_S q^{\text{rot}_n(S)} \prod_v (-q)^{l(P(S, v))},$$

where the sum is taken over all states of $\Gamma$ and the product is over all its vertices.

We leave the proof of the following proposition to the reader.

**Proposition 10** Let $\Gamma$ be a planar $n$-web obtained by resolving all crossings of a link by the skein relation of Proposition 2 (In other words, let $\Gamma$ be one of the leaves of the skein tree of $L$). Then $\sum_v l(P(S, v))$ is even for any state $S$. Consequently, all coefficients of $\langle \Gamma \rangle_n \in \mathbb{Z}[q^{\pm 1}]$ are non-negative.

### 1.7 Murakami-Ohtsuki-Yamada colored 3-valent graphs

Murakami, Ohtsuki, and Yamada defined an $SU(n)$-quantum invariant of links by using 3-valent graphs with a flow. Inspired by their work, we say that a 3-valent oriented, framed graph embedded into $\mathbb{R}^3$ is a Murakami-Ohtsuki-Yamada graph (MOY-graph, for short) if the edges of $\Gamma$ are labeled by positive integers forming a flow on $\Gamma$:

![MOY-graph example]

We allow annuli embedded into $\mathbb{R}^3$ colored by positive integers as components of MOY-graphs as well.

An MOY-graph $\Gamma$ is an MOY$_n$-graph if the labels of its edges and annuli do not exceed $n$. The purpose of this section is to show that our bracket of $n$-webs defines a bracket invariant of MOY$_n$-graphs, which coincides (up to a normalization) with the Murakami-Ohtsuki-Yamada bracket.
For any \( MOY_n \)-graph diagram \( \Gamma \) with no crossings, let \( W(\Gamma) \) be a ribbon graph obtained by replacing all vertices of \( \Gamma \) as follows:

\[
\begin{align*}
\begin{array}{c}
i \quad j \\
i+j \quad \rightarrow \quad i+j \\
i+j \quad \rightarrow \quad i+j 
\end{array}
\end{align*}
\]

As before, an edge of a web labeled by \( i \) denotes \( i \) parallel edges. Let \( [\Gamma]_n = \langle W(\Gamma) \rangle_n \).

**Corollary 11** \([\Gamma]_n \) is an isotopy invariant of \( MOY_n \)-graphs.

We are going to show that \([\Gamma]_n \) is a renormalization of the Murakami-Ohtsuki-Yamada bracket of \( \Gamma \).

For any MOY-graph \( \Gamma \) denote the labels of edges \( e \) of \( \Gamma \) by \( |e| \). For any vertex \( v \) of \( \Gamma \), denote the adjacent edge with the largest label by \( e_v^0 \), and the left and the right of the two other adjacent edges by \( e_v^1 \) and by \( e_v^2 \) respectively. Hence the adjacent edges to \( v \) in \( \Gamma \) are either

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
i_1 \\
i_2 \\
i_0
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
i_3 \\
i_4
\end{array}
\end{array}
\end{array}
\end{align*}
\]

By the definition of a flow, \( |e_v^0| = |e_v^1| + |e_v^2| \) for any vertex \( v \).

We say that a function \( s \) assigning an \( |e| \)-element subset of \( \{1, \ldots, n\} \) to every edge \( e \) of \( \Gamma \) is an \( n \)-state (or, simply, a state) of \( \Gamma \) if \( s(e_1^0) \cap s(e_2^0) = \emptyset \) and \( s(e_1^1) \cup s(e_2^1) = s(e_1^0) \) for every vertex \( v \).

Note that our definition coincides with the definition of [MOY] if the sets \( s(e) = \{i_1, \ldots, i_{|e|}\} \) and \( \{i_1 - \frac{n-1}{2}, i_2 - \frac{n-1}{2}, \ldots, i_{|e|} - \frac{n-1}{2}\} \) are identified.

Any \( n \)-state \( s \) splits \( \Gamma \) into several simple closed loops (which may intersect each other), each labeled by an integer between 1 and \( n \). Following [MOY], let the rotation number of an \( n \)-state \( s \) be

\[
rot(s) = \sum_C (s(C) - \frac{n+1}{2}) \text{rot}(C) \in \frac{1}{2} \mathbb{Z},
\]

where the sum is over all simple closed loops \( C \) of the splitting of \( \Gamma \) by \( s \), \( s(C) \) is the label of \( C \), and \( \text{rot}(C) \) is either +1 or −1 depending on whether \( C \) is oriented anti-clockwise or clockwise.

As in [MOY], for any two sets \( s_1, s_2 \subset \{1, \ldots, n\} \) we denote by \( \pi(s_1, s_2) \) the number of pairs \( (i_1, i_2) \subset s_1 \times s_2 \) such that \( i_1 > i_2 \).
Skein theory for SU($n$)-quantum invariants

Proposition 12 (Proof in Section 9) For any MOY$_n$-graph diagram $\Gamma$ with no crossings

$$[\Gamma]_n = N(\Gamma) \sum_{n\text{-states } s} q^{2\text{rot}(s)} \prod_{\text{vertices } v} (-q)^{\pi(s(e^v_1), s(e^v_2))}.$$ 

$N(\Gamma)$ is a normalization factor,

$$N(\Gamma) = \prod_e [|e|]! \cdot \prod_v q^{\frac{n(n-1)-|e|+1}{2}[n-|e_0|]}!$$

where the first product is taken over all edges of $\Gamma$ and the second product is over all vertices of $\Gamma$. (Annuli of $\Gamma$ are not considered as edges.)

In order to avoid confusion with our bracket, we denote the $n$-th Murakami-Ohtsuki-Yamada bracket of an MOY$_n$-graph $\Gamma$ by $\{\Gamma\}_n$.

Proposition 13 (Proof in Section 10) For any MOY$_n$-graph diagram $\Gamma$ with no crossings,

$$\{\Gamma\}_n = q^{-\frac{1}{2}\sum_v |e_1^v||e_2^v|} \sum_{n\text{-states } s} q^{\text{rot}(s)} \prod_{\text{vertices } v} q^{\pi(s(e^v_1), s(e^v_2))/2}.$$ 

The following lemma is needed to relate the brackets $\{\cdot\}_n$ and $\langle\cdot\rangle_n$:

Lemma + Definition 14 For any MOY-graph $\Gamma$,

$$\eta_n(\Gamma) = 2\text{rot}(s) \mod 2$$

does not depend on the $n$-state $s$ of $\Gamma$.

Proof Since $2\text{rot}(s) = (n+1)\sum_C \text{rot}(C) \mod 2$, it is enough to show that for any state $s$, the induced splitting of $\Gamma$ into simple loops $\{C\}$ is such that $\sum_C \text{rot}(C)$ does not depend on $s$. To prove that, consider all cups and caps, $c$, of $\Gamma$. Each of them, being a part of an edge of $\Gamma$, has an associated flow $|c|$. Let $\text{rot}(c)$ be either $+1/2$ or $-1/2$ depending on whether $c$ is oriented anti-clockwise or clockwise,

$$\text{rot}(c) = -1/2 \quad \text{rot}(c) = 1/2 \quad \text{rot}(c) = 1/2 \quad \text{rot}(c) = -1/2$$

Note that

$$\sum_C \text{rot}(C) = \sum_{\text{cups and caps: } c} |c| \text{rot}(c),$$

and hence the left hand side does not depend on $s$. }

Algebraic & Geometric Topology, Volume 5 (2005)
By Proposition 13,
\[ \{\Gamma\}_n q^{\frac{1}{4}} \sum_v |e_1^v| |e_2^v| \in \mathbb{Z}[q^{\pm \frac{1}{4}}] \]
and, by Proposition 12, substitution \( q^{\frac{1}{2}} \to -q \) gives
\[
\left( \{\Gamma\}_n q^{\frac{1}{4}} \sum_v |e_1^v| |e_2^v| \right)_{q^{\frac{1}{2}} \to -q} = \sum_{\text{states } s} q^{2 \text{rot}(s)} \prod_{\text{vertices } v} (-q)^{\pi(s(e_1^v),s(e_2^v))} =
\]
\[ [\Gamma]_n \cdot (-1)^{n_0} \gamma_n / \gamma(\Gamma) \].

**Corollary 15** The value of the Murakami-Ohtsuki-Yamada bracket of any MOY\(_n\)-graph \( \Gamma \) is determined by \( \langle W(\Gamma) \rangle_n \).

### 2 Definition of the bracket using tensors

We will state now another, more explicit definition of the bracket of \( n \)-webs, which utilizes the construction of Reshetikhin and Turaev, [RT]. Given a ribbon Hopf algebra \( H \), they constructed an isotopy invariant for ribbon graphs whose edges are labeled by representations of \( H \) and whose vertices are labeled by \( H \)-invariant tensors. We are going to see that our bracket \( \langle \Gamma \rangle_n \) is the Reshetikhin-Turaev quantum \( sl(n) \) invariant for \( \Gamma \) considered as a ribbon graph whose edges are decorated by the defining representation \( V \) and whose sinks and sources and decorated by an element of the 1-dimensional representation \( \wedge^n V \subset V \otimes V \) and by its dual, respectively.

Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{C}(q) \) with a basis \( e_1, ..., e_n \). Given a web diagram \( \Gamma \) decompose it into pieces with the following tensors associated with them:

\[
\begin{align*}
\begin{array}{c}
\bigg\|
\end{array} & \begin{array}{c}
Id_V
\end{array} & \begin{array}{c}
Id_{V^*}
\end{array} & \begin{array}{c}
\hat{R}
\end{array} & \begin{array}{c}
\hat{R}^{-1}
\end{array} & \begin{array}{c}
\bigg\|
\end{array}
\end{align*}
\]

where \( \hat{R} : V \otimes V \to V \otimes V \) is given by

\[
\hat{R}(e_i \otimes e_j) = q^{-\frac{1}{n}} \begin{cases} 
  e_j \otimes e_i & \text{if } i > j, \\
  q e_i \otimes e_j & \text{if } i = j, \\
  e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{if } i < j.
\end{cases}
\]

\[
\begin{align*}
\sum_{i} e_i \otimes e_j & \to \delta_{ij} \quad e_i \otimes e^j \to q^{-2i-1} \delta_{ij} \\
V^* \otimes V & \to \mathbb{C}(q) & V \otimes V^* & \to \mathbb{C}(q) & \mathbb{C}(q) & \to V \otimes V^* & \mathbb{C}(q) & \to V^* \otimes V
\end{align*}
\]

\[ \sum_{i} q^{n+1-2i} e_i \otimes e_i \]

\[ \begin{equation}
(3)
\end{equation} \]
\[ T_- : V^* \to \mathbb{C}(q), \quad T_+ : \mathbb{C}(q) \to V^* \quad (6) \]

where
\[ T_+(1) = T_+ = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \ldots \otimes e_{\sigma(n)}, \quad (7) \]

and
\[ T_-(e_{i_1} \otimes \ldots \otimes e_{i_n}) = \begin{cases} (-q)^{l(\sigma)} & \text{if } (1, \ldots, n) \to (i_1, \ldots, i_n) \text{ is a permutation } \sigma \\ 0 & \text{otherwise.} \end{cases} \quad (8) \]

**Definition 16** For any \( n \)-web diagram \( \Gamma \) decomposed into pieces as above, let \( \langle \Gamma \rangle_n \in \mathbb{Q}(q^n) \) be the scalar obtained by the contraction of the corresponding tensors.

**Theorem 17** (Proof in Section 5) (i) The bracket defined above is an isotope invariant of \( n \)-webs.

(ii) It satisfies the properties of the bracket stated in Theorem 1.

Consequently, Definition 16 coincides with the definition of the bracket given in Theorem 1.

### 3 The \( SU(n) \)-skein modules of 3-manifolds

#### 3.1 The definition of the skein module

Let \( M \) be an orientable 3-manifold, possibly with non-empty boundary, and let \( n \geq 2 \). Let \( W_n(M) \) denote the set of all isotopy classes of \( n \)-webs embedded into \( M \), including the empty web, \( \emptyset \). Consider a ring \( R \) with a specified invertible element \( t \). The \( SU(n) \)-skein module of \( M \) with coefficients in \( R \) is the quotient of the free \( R \)-module \( RW_n(M) \) by relations

(i) \[ t^n - t^{-1} - (t^n - t^{-n}) \]

(ii) \[ t^{n^2} - t^{1-n^2}, \quad t^{n^2-1} \]

(iii) \[ t^{n^3-n^2} \cdot \sum_{\sigma \in S_n} (-t^{1-n})^{l(\sigma)} \]
(iv) $\Gamma \cup \bigcirc - [n]\Gamma$.

Note that the above relations correspond to equations (i)-(v) of Theorem 1 after the substitution $t = q^{\frac{1}{n}}$. We denote the quotient module by $S_n(M; R, t)$.

The $SU_3$-skein module was defined independently in [FZ], and earlier, for 3-dimensional thickenings of surfaces, in [OY]. The definitions of Frohman-Zhong and Ohtsuki-Yamada are equivalent to ours, since their skein relations are the skein relations for $q^{-v(\Gamma)} \langle \Gamma \rangle_n$, when $A = q^{\frac{1}{3}}t = -t$, cf. Theorem 5.

Theorem 4 implies the following:

**Corollary 18** $S_2(M; R, t)$ is isomorphic to the Kauffman bracket skein module of $M$ with coefficients in $R$ and $A = t$.

It follows directly from the definition, that if $f : R \to R'$ is a homomorphism of rings such that $f(t) = t'$ then

$$S_n(M; R', t') = S_n(M; R, t) \otimes_R R'.$$

Since for any ring $R$ with an invertible element $t$ there is a map $f : \mathbb{Z}[t^{\pm 1}] \to R$, Theorem 1 can be restated as follows:

**Corollary 19** For any ring $R$ with an invertible element $t$, $S_n(\mathbb{R}^3; R, t) = R$.

Below, we describe the relation between $S_n(M; R, t)$ and $SL_n(R)$-representations of $\pi_1(M)$, which generalizes the theorems of Bullock, [Bu], and ours with J. Przytycki, [PS], for the Kauffman bracket skein modules. Further analysis of $SU(n)$-skein modules is postponed to [S2]. The discussion below and the results of [S2] show that $SU(n)$-skein modules have many properties analogous to those of the Kauffman bracket skein module.

### 3.2 $SU(n)$-skein modules and character varieties

Since the skein relation (i) above reduces to $\bigcirc \sim \bigcirc$ for $t = 1$, the $n$-webs in $M$, which are freely homotopic to each other, are identified in $S_n(M; R, 1)$. Consequently, the operation of taking the disjoint union, $\Gamma_1, \Gamma_2 \to \Gamma_1 \cup \Gamma_2$, extends to a well defined product in $S_n(M; R, 1)$ making this module a commutative $R$-algebra.

Furthermore, note that as an $R$-algebra, $S_n(M; R, 1)$ is isomorphic to $\mathbb{A}_n(M)$ (over the ring of coefficients $R$) defined in [S1]. Hence, by [S1] Theorem 3.7 we have:
Corollary 20 If $\mathbb{K}$ is an algebraically closed field of characteristic 0 then $S_n(M; \mathbb{K}, 1) \simeq \mathcal{O}(X_n(\pi_1(M)))$, where $\mathcal{O}(X(M))$ denotes the ring of global sections (the coordinate ring) of the $SL_n(\mathbb{K})$-character variety of $\pi_1(M)$.

The $SL_n(\mathbb{K})$-character variety, $X_n(G)$, of a group $G$ is an affine algebraic scheme over $\mathbb{K}$ “describing” the $SL_n(\mathbb{K})$-representations of $G$ up to conjugation. More precisely, the closed points of $X_n(G)$ (ie. the maximal ideals in $\mathcal{O}(X(G))$) correspond to the semi-simple $SL_n(\mathbb{K})$-representations of $G$ up to conjugation. For a precise definition of $SL_n$-character varieties see [SI, LM].

Up to nilpotent elements, the ring $\mathcal{O}(X(G))$ can be described as follows: A characteristic function $f : SL_n(\mathbb{K}) \to \mathbb{K}$ is any polynomial in the entries of the matrices in $SL_n(\mathbb{K})$ which is invariant under the conjugation by $SL_n(\mathbb{K})$. Characteristic functions of $SL_n(\mathbb{K})$ form a $\mathbb{K}$-algebra generated by the functions $f_n(A) = \text{tr}(A^n)$. Let $X'(G)$ be the set of all generalized $SL_n$-characters of $G$, that is $\mathbb{K}$-valued functions on $G$ of the form $\psi = f \circ \rho$, where $\rho : G \to SL_n(\mathbb{K})$ is a representation and $f$ is a characteristic function on $SL_n(\mathbb{K})$. With any $g \in G$ there is the associated “evaluation at $g$” function $\tau_g : X'(G) \to \mathbb{K}$, $\tau_g(\psi) = \psi(g)$. The $\mathbb{K}$-algebra $\mathcal{O}(X(G))/\sqrt{0}$ is isomorphic to the $\mathbb{K}$-algebra generated by all $\tau_g$ for all $g$. If we think of the functions $\tau_g$ as regular functions on $X'_n(G)$, then for any finitely generated group $G$, $X'_n(G)$ becomes an affine algebraic set whose coordinate ring is isomorphic to $\mathcal{O}(X(G))/\sqrt{0}$.

4 Preliminaries for the proofs

4.1 The quantum $sl_n$ group and its defining representation

Let $U_h = U_h(sl_n)$ be defined as in [KS Sect. 6.1.3]. Note that Reshetikhin’s and Turaev’s definition of $U_h(sl_n)$, [RT Sect. 7.1], coincides with our definition of $U_h$ after taking into account the following changes:

- $X_i^+$ and $X_i^-$ in [RT] are $E_i$ and $F_i$ in [KS],
- $h$ used by Reshetikhin and Turaev is equal to $2h$ in [KS].

In this paper we will use Klimyk-Schm"udgen $h$ only. Let $V = \mathbb{C}[[h]]^n$ be the defining representation of $U_h$, presented explicitly in [KS Sect. 8.4.1]. Additionally, consider the quantum group $U_q = U_q(sl_n)$, [KS Sect. 6.1.2], and its defining representation, $V = C(q)^n$, as defined in [KS Sect. 8.4.1]. This double meaning of $V$ will not lead to confusion since the $U_q$ and $U_h$ actions on $V$ agree if $q$ and $K_i$ are identified with $e^h$ and $e^{hH_i}$ respectively. In both
cases, $e_1, ..., e_n$ will be the weight basis of $V$ with the heighest weight vector $e_1$, see [KS, Sect. 8.4.1].

The defining representation $\rho : U_q \to \text{End}(V)$ is given by the matrices

\[
\rho(K_i) = q^{-1}E_{ii} + qE_{i+1,i+1} + \sum_{j \neq i, i+1} E_{jj}, \quad i = 1, 2, ..., n - 1, \quad (9)
\]

\[
\rho(E_i) = E_{i+1,i}, \quad \rho(F_i) = E_{i,i+1}, \quad i = 1, 2, ..., n - 1, \quad (10)
\]

where $E_{ij}$ denotes the matrix whose all entries are 0 except for the $(i,j)$th entry which is 1, cf. [KS, Sect. 8.4.1].

4.2 The Hecke algebra associated with $U_q$

For the purpose of studying $U_q(sl_n)$-actions on $V \otimes V$ we define the Hecke algebra, $H_k$, as follows. Let $H_k$ be the non-commutative, associative algebra over $\mathbb{C}(t)$ generated by elements $g_i^{\pm 1}, ..., g_k^{\pm 1}$ subject to the following relations: $g_ig_j = g_jg_i$ for $|i - j| \geq 2$, $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$, for $i = 1, ..., k - 1$, and $(g_i - t^n)(g_i + t^{-n}) = 0$ for all $i$. The algebra $H_k$ is an $k!$-dimensional space with basis vectors $h_\sigma$, for $\sigma \in S_k$, which satisfy the following multiplication rules

(i) $(h_{(i,i+1)} - t^{n-1})(h_{(i,i+1)} + t^{-n-1}) = 0$ for $i = 1, ..., n - 1$,

(ii) $h_\sigma h_\tau = h_{\sigma \tau}$ if $l(\sigma \tau) = l(\sigma) + l(\tau)$.

We have $g_i = t \cdot h_{(i,i+1)}$. Although various definitions of Hecke algebra appear in the literature, see for example [KS, Sect. 8.6.4], [CP, Sect. 12.3], [Gy], they are all isomorphic to our $H_k$ after a proper extension of base field. In fact, substituting the quadratic equation (i) above by any other quadratic equation with distinct roots leads an isomorphic algebra, after a finite extension of the base field. For example, Gyoja’s $q$ in [Gy] is our $t^{2n}$ and Gyoja’s $h(\sigma)$ is ours $t^{(n+1)!}\sigma h_\sigma$. Our somewhat cumbersome notation is chosen so that $h_{(i,i+1)}$’s satisfy the same quadratic equation as $\hat{R}$ for $q = t^n$. Therefore, from now on, we will assume that $q = t^n$. We summarize the basic relations between $H_n$ and the defining representation $V$ of $U_q$:

**Proposition 21**

(i) $H_k$ acts on $V \otimes V$ in such a way that the actions of $h_\sigma$ and of $\sigma \in S_k$ on $V \otimes V$ coincide.

(ii) The $H_k$ and $U_q$ actions on $V \otimes V$ commute.
(iii) (Frobenius-Schur duality) The images of the maps $U_q \rightarrow \text{End}(V^\otimes k)$ and $H_n \rightarrow \text{End}(V^\otimes k)$ are centralizers of each other. In particular, any $U_q$-equivariant endomorphism of $V^\otimes k$ is of the form $w \mapsto x \cdot w$ for a certain $x \in H_k$.

In [Gy, page 843], Gyoja defines two elements $e_+, e_-$ which in our notation are:

$$e_+ = \sum_{\sigma \in S_k} q^{\frac{n+1}{n}l(\sigma)} h_\sigma, \quad \text{and} \quad e_- = \sum_{\sigma \in S_k} (-q^{\frac{1-n}{n}})^l(\sigma) h_\sigma,$$

and shows that

$$h_\sigma e_+ = e_+ h_\sigma = q^{\frac{n-1}{n}l(\sigma)} e_+ \quad \text{and} \quad h_\sigma e_- = e_- h_\sigma = (-q^{\frac{n+1}{n}})^l(\sigma) e_-,$$

for any $\sigma \in S_k$. Furthermore, for $P_{\pm} = \sum_{\sigma \in S_k} q^{\pm 2l(\sigma)}$, $e_+/P_+$ are primitive idempotents of $H_k$: $e_+/P_+$ is the symmetrizer and $e_-/P_-$ is the antisymmetrizer.

5 Proof of Theorem 17

The isotopy invariance of the $n$-bracket follows from [RT, Theorem 5.1] and from the following proposition.

Proposition 22

(i) The tensors (5.1.1)-(5.1.3) in [RT] for the defining $U_h$-representation are given by our formulas (3)-(5). (Recall that $q = e^{h_{RT}/2}$, where $h_{RT}$ is the Reshetikhin-Turaev $h$.)

(ii) The map $\mathcal{T}_+: \mathbb{C}(q) \rightarrow V^\otimes n$, is $U_q$-equivariant.

(iii) The map $\mathcal{T}_-: (V^*)^\otimes n \rightarrow \mathbb{C}(q)$, is $U_q$-equivariant.

Proof of Proposition 22 (i) By [KS], the $R$-matrix acts on $V \otimes V$ by the matrix

$$q^{-\frac{1}{n}} \left( q \sum_i (E_{ii} \otimes E_{ii}) + \sum_{i \neq j} (E_{ii} \otimes E_{jj}) + (q - q^{-1}) \sum_{i>j} (E_{ij} \otimes E_{ji}) \right).$$

(This matrix is denoted by $R_{1,1}$ in [KS], cf. the first paragraph of Section 8.4.2 and (60) in [KS].) Here, as before, $E_{ij}$ represents the map $\delta_{i}^{j} : V \rightarrow V$, $\delta_{i}^{j}(e_k) = \delta_{j,k} e_i$. By composing the map represented by (13) with the transposition $\tau : V \otimes V \rightarrow V \otimes V$, $\tau(v_1, v_2) = (v_2, v_1)$, we obtain the map $\hat{R}$ given by (14).
The “cap” maps in [RT] (5.1.1-2) are given by the contraction map \( V^* \otimes V \to \mathbb{C}[h] \), \( (x, y) = x(y) \) and the map \( V \otimes V^* \to \mathbb{C}[h] \), \( (y, x) \to x(v^{-1}uy) \). The “cup” maps are their duals. Let us recall the meaning of \( v^{-1}u \) in [RT]. Let \( \rho \) be the half-sum \( \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \) of primitive roots of \( sl_n \) and let \( \rho_i \in \mathbb{Z} \) be the coordinates of \( \rho \) in the basis of the Cartan subalgebra of \( sl_n \) given by simple roots \( \alpha_1, ..., \alpha_{n-1} \). By [RT (7.1.1)], \( v^{-1}u = e^{2h \sum_{i=1}^{n-1} \rho_i H_i} \). (Recall that Reshetikhin-Turaev \( h \) is twice the \( h \) we use.) Therefore the following lemma completes the proof of Proposition 22(i).

**Lemma 23** \( v^{-1}u \) acts on \( V \) by sending \( e_k \) to \( q^{2k-n-1}e_k \)

**Proof** Positive roots in the Cartan subalgebra of \( sl_n \) are of the form \( \alpha_{ij} = \alpha_i + \alpha_{i+1} + ... + \alpha_j \), for \( 1 \leq i \leq j \leq n - 1 \). Therefore, \( \alpha_k \) appears \( \frac{1}{2} k(n-k) \) times in \( \rho \) and \( \rho_k = \frac{1}{2} k(n-k) \).

By [KS] Section 8.4.1, \( H_i e_k \) is \( -e_k \) if \( i = k \), \( e_k \) if \( i = k - 1 \), and 0 otherwise. Therefore,

\[
\exp(hH_i)e_k = \begin{cases} 
q^{-1}e_k & \text{if } i=k \\
q e_k & \text{if } i=k-1 \\
e_k & \text{otherwise}.
\end{cases}
\]

Consequently, \( (v^{-1}u)e_k = q^{-2\rho_k+2\rho_{k-1}}e_k = q^{(k-1)(n-k+1)-k(n-k)}e_k = q^{2k-n-1}e_k \).

**Proof of Proposition 22(ii) - \( U_q \)-equivariance of \( T_+ \)**

\( U_q \) acts on the \( n \)-th power of the defining representation \( V \), via the map

\[
U_q \xrightarrow{\Delta^{-1}} U_q^\otimes n \xrightarrow{\rho^\otimes n} \text{End}(V^\otimes n),
\]

where \( \Delta^{-1} : U_q \to U_q^n \) is the \((n-1)\)st power of the comultiplication in \( U_q \). The following explicit formulas for \( \Delta^{-1} \) follow by induction on \( n \) from the definition of \( \Delta \). [KS] Prop 6.1.2.5):

\[
\Delta^{-1}(K_i) = K_i \otimes ... \otimes K_i, \text{for } i = 1, ..., n - 1,
\]

\[
\Delta^{-1}(E_i) = \sum_{j=1}^{n} 1 \otimes ... \otimes 1 \otimes \bar{E}_i \otimes K_i \otimes ... \otimes K_i, \text{for } i = 1, ..., n - 1,
\]

where the index \( j \) over \( E_i \) means that it takes the \( j \)-th position in the tensor product. For \( \Delta^{-1}(F_i) \) we have a similar expression:

\[
\Delta^{-1}(F_i) = \sum_{j=1}^{n} K_i^{-1} \otimes ... \otimes K_i^{-1} \otimes F_i \otimes 1 \otimes ... \otimes 1.
\]
Since $U_q$ acts on $\mathbb{C}(q)$ by counit map, $\epsilon : U_q \to \mathbb{C}(q)$, which sends $E_i, F_i$ to 0 and $K_i$ to 1, we need to show that $\Delta^{n-1}(K_i)T_+ = T_+$, and $\Delta^{n-1}(E_i)T_+ = \Delta^{n-1}(F_i)T_+ = 0$ for $i = 1, \ldots, n - 1$. In order to prove the first equality notice that by (9) and (14) $\Delta^{n-1}(K_i)$ multiplies the $e_i$ component in $e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)}$ by $q^{-1}$ and it multiplies the $e_{i+1}$ component by $q$. Since it leaves all other components unchanged, $\Delta^{n-1}(K_i) \cdot e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)} = e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)}$ and, consequently, $\Delta^{n-1}(K_i)T_+ = T_+$. We complete the proof by showing that $\Delta^{n-1}(E_i)T_+ = 0$. The proof of $\Delta^{n-1}(F_i)T_+ = 0$ is analogous and left to the reader.

For simplicity, denote $e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)} \in V^{\otimes n}$ by $e_{\sigma}$. By (14),

$$l((i, i + 1)\sigma) = \begin{cases} l(\sigma) + 1 & \text{if } \sigma^{-1}(i) < \sigma^{-1}(i + 1) \\ l(\sigma) - 1 & \text{otherwise.} \end{cases}$$

Therefore,

$$T_+ = \sum_{\sigma \in S_n, \sigma^{-1}(i) < \sigma^{-1}(i + 1)} (-q)^l(\sigma)(e_{\sigma} - qe_{(i, i+1)\sigma}),$$

and our goal is to prove that

$$\Delta^{n-1}(E_i) \cdot (e_{\sigma} - qe_{(i, i+1)\sigma}) = 0, \quad (17)$$

for $\sigma$ such that $\sigma^{-1}(i) < \sigma^{-1}(i + 1)$. We have

$$1 \otimes \ldots \otimes 1 \otimes E_{i} \otimes K_{i+1} \otimes K_{i} \cdot e_{\sigma} = \begin{cases} 0 & \text{if } \sigma(j) \neq i \\ e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(j - 1)} \otimes e_{i + 1}^j \otimes e_{\sigma(j + 1)} \otimes \ldots \otimes e_{\sigma(n)} & \text{if } \sigma(j) = i \text{ and } \sigma^{-1}(i + 1) < j \\ qe_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(j - 1)} \otimes e_{i + 1}^j \otimes e_{\sigma(j + 1)} \otimes \ldots \otimes e_{\sigma(n)} & \text{if } \sigma(j) = i \text{ and } \sigma^{-1}(i + 1) > j. \end{cases}$$

Therefore, if $\sigma^{-1}(i) = j$ then $\Delta^{n-1}(E_i) \cdot e_{\sigma} =$$

$$e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(j - 1)} \otimes e_{i + 1}^j \otimes e_{\sigma(j + 1)} \otimes \ldots \otimes e_{\sigma(n)} \begin{cases} q & \text{if } \sigma^{-1}(i) < \sigma^{-1}(i + 1) \\ 1 & \text{otherwise.} \end{cases}$$

This implies (17) and, therefore, completes the proof of $U_q$-equivariance of $T_+$.

**Proof of Proposition 22(iii) – $U_q$-equivariance of $T_-$**

We need to prove that for any $x \in U_q$ and any $w = v_{j_1} \otimes \ldots \otimes v_{j_n} \in V^{\otimes n}$, $\epsilon(x)T_-(w) = T_-(\Delta^{n-1}(x) \cdot w)$. This equality reduces to the following three sets of equations for $i = 1, \ldots, n - 1$:

$$T_-(\Delta^{n-1}(K_i) \cdot w) = T_-(w) \quad (18)$$
\[ T_-(\Delta^{n-1}(E_i) \cdot w) = 0 \] (19)
\[ T_-(\Delta^{n-1}(F_i) \cdot w) = 0. \] (20)

Both sides of (18) vanish if the numbers \((i_1, \ldots, i_n)\) are not distinct. On the other hand, if these numbers are distinct then \(\Delta^{n-1}(K_i) \cdot w = w\) and (18) follows. We will complete the proof by showing (19) – the proof of (20) is analogous.

Observe that \(\Delta^{n-1}(E_i) \cdot v_{j_1} \otimes \cdots \otimes v_{j_n}\) is a linear combination of terms \(v_{k_1} \otimes \cdots \otimes v_{k_n}\) such that the \(n\)-tuple \((k_1, \ldots, k_n)\) is obtained from \((j_1, \ldots, j_n)\) by changing one of the indices from \(i\) to \(i+1\). Since \(T_-(v_{k_1} \otimes \cdots \otimes v_{k_n}) = 0\) if the numbers \(k_1, \ldots, k_n\) are not a permutation of \(1, \ldots, n\), the left side of (19) vanishes unless \(j_1, \ldots, j_{l-1}, j_l+1, j_{l+1}, \ldots, j_n\) are a permutation \(\sigma\) of \(1, \ldots, n\), for some \(l\) such that \(j_l = i\). In this case \(j_k = i\) for some \(k \neq l\) and we can assume that \(k < l\). Under above assumptions,

\[
\Delta^{n-1}(E_i) \cdot (v_{j_1} \otimes \cdots \otimes v_{j_n}) = v_{j_1} \otimes \cdots \otimes v_{i+1} \otimes \cdots q^{-1} v_i \otimes \cdots \otimes v_{j_n} + v_{j_1} \otimes \cdots \otimes v_i \otimes \cdots v_{i+1} \otimes \cdots v_{j_n}.
\]

Hence, \(T_-(\Delta^{n-1}(E_i) \cdot (v_{j_1} \otimes \cdots \otimes v_{j_n})) = (-q)^{l((i,i+1)\sigma)} \cdot q^{-1} + (-q)^{l(\sigma)}\). Since \(l((i,i+1)\sigma) = l(\sigma) + 1\), the left hand side of (19) vanishes and the proof of Proposition 22(iii) is completed.

**Proof of Theorem 17(ii)** In the previous section, we proved that \(\langle \cdot \rangle_n\) is an isotopy invariant of \(n\)-webs. Now we are going to show that it satisfies properties (i)-(v) formulated in Theorem 1.

(i) Since Klimyk’s and Schmündgen’s \(\hat{R}\) is our \(q^\frac{1}{n} \hat{R}\), our \(\hat{R}\) satisfies

\[
(q^\frac{1}{n} \hat{R} - q)(q^\frac{1}{n} \hat{R} + q^{-1}) = 0
\]

by [KS] Proposition 8.4.24 and, hence,

\[
q^\frac{1}{n} \hat{R} - q^{-\frac{1}{n}} \hat{R}^{-1} = (q - q^{-1})I.
\]

This implies the skein relation (i) of Theorem 1

(ii) Since the two relations (ii) of Theorem 1 are inverses of each other, we will show the first of them only.

The “kink,” \(\text{\textcircled{}}}\), defines a map on \(V\) which is \(U_q\)-equivariant. Since \(V\) is an irreducible module, this map is a multiple of \(Id_V\) and, therefore, for our purpose

*Algebraic & Geometric Topology, Volume 5 (2005)*
it is enough to show that the kink maps $e_i$ to $q^{n-i} \cdot e_i$ for some (and hence for arbitrary) $i$. Choose $i = n$. Since the arc $\bigcirc$ maps $V$ to itself by sending $e_i$ to $q^{2i-n-1} e_i$, the kink maps $e_n$ to $C e_n$, where $C = \sum_{k=1}^{n} \hat{r}_{nk} q^{2k-n-1}$ and $\hat{r}_{nk}$ are the coefficients of the $\hat{R}$-matrix,

$$\hat{R}(e_n \otimes e_k) = \sum_{i,j} \hat{r}_{nk}^{ij} e_i \otimes e_j.$$ 

Since $\hat{r}_{nk}^{ij} = \begin{cases} q^{-\frac{i+j}{2}} + 1 & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$, 

$C = q^{n-\frac{i}{2}}$. 

(iii) This property will be proved in the next section.

(iv) The bracket of the trivial knot diagram is given by the contraction of the cup and the cap tensors, where the cup and the cap are chosen with coinciding orientations. Therefore

$$\langle \bigcirc \rangle_n = \sum_{i=1}^{n} q^{2i-n-1} = [n].$$

By the construction of the bracket, $\langle \Gamma_1 \cup \Gamma_2 \rangle_n = \langle \Gamma_1 \rangle_n \cdot \langle \Gamma_2 \rangle_n$, for disjoint (and hence unlinked) web diagrams $\Gamma_1, \Gamma_2$.

(v) This is obvious.

6 Proof of Proposition 2 and of Theorem 1(iii)

We will often use the following equality

$$\sum_{\sigma \in S_n} (-q)^{2l(\sigma)} = q^{n(n-1)/2} \cdot [n]!$$

following from [Gy, (3.1)].

Lemma 24

$$\langle \bigcirc \rangle_n = q^{n(n-1)/2} \cdot [n]!$$

Proof The above bracket is given by the contraction of $\mathcal{T}_-$ with $\mathcal{T}_+$,

$$\mathcal{T}_-(\mathcal{T}_+) = \sum_{\sigma \in S_n} (-q)^{2l(\sigma)} = q^{n(n-1)/2} \cdot [n]!,$$

by (21).
Consider the skein
\[ \Lambda_k = \sum_{\sigma \in S_k} (-q^{\frac{1-n}{n}})^{l(\sigma)} \sigma, \quad (22) \]
where, as before, \( \sigma \) is the unique positive braid with \( l(\sigma) \) crossings representing \( \sigma \). Given an \((k,k)\)-tangle \( T \), denote by \( \pi_k(T) \) the \((k-1,k-1)\)-tangle obtained from \( T \) by closing up its last string, \( T \). The definition of \( \pi_k(T) \) obviously extends to all skeins \( T \) being linear combinations of \((k,k)\)-tangles.

**Lemma 25** \( \pi_{k+1}(\Lambda_{k+1}) = \Lambda_k q^{-k}[n-k] \).

**Proof** Each permutation \( \sigma \in S_{k+1} \) can be written in the form
\[
(i_1, i_1 - 1, ..., i_1 - j_1)(i_2, i_2 - 1, ..., i_2 - j_2)...(i_l, i_l - 1, ..., i_l - j_l),
\]
where \( i_1 < i_2 < ... < i_l \). Furthermore, such presentation is unique. (These statements can be proved by induction on \( k \).) By splitting the set of all permutations \( \sigma \in S_{k+1} \) into those with \( i_l \leq k \) and those with \( i_l = k+1 \), we get
\[
\pi_{k+1}(\Lambda_{k+1}) = \Lambda_k [n] + \sum_{i=1}^{k} (-q^{\frac{1-n}{n}})^{k+1-i} \Lambda_kq^{-i} = \Lambda_k [n] + \Lambda_k q^{n-n^{-1}} \sum_{i=1}^{k} q^{-2(k-i)} = \Lambda_k [n] + \Lambda_k q^{n-1} \frac{1 - q^{-2k}}{1 - q^{-2}}.
\]

Algebraic & Geometric Topology, Volume 5 (2005)
\[ \Lambda_k \left( [n] - \frac{q^n - q^{n-2k}}{q - q^{-1}} \right) = \Lambda_k \left( \frac{q^{n-2k} - q^{-n}}{q - q^{-1}} \right) = \Lambda_k q^{-k} [n - k]. \]

Recall that \( T_+ \in V^\otimes n \) was defined in (7) and \( h_{i, i+1} \) in Section 4.2.

**Corollary 26**

\[ \sum_{\sigma \in S_n} (-q^{1-n})^{l(\sigma)} \left\langle \begin{array}{c} \sigma \\ \end{array} \right\rangle_n = q^{-\frac{n(n-1)}{2}} [n]! \]

**Lemma 27** \( h_{i,i+1}T_+ = (-q^{-\frac{n+1}{n}})T_+ \).

**Proof** Denote \( e_{\sigma(1)} \otimes \ldots e_{\sigma(n)} \in V^\otimes n \) by \( e^\sigma \) as before. By Proposition 21(1) and (1),

\[ h_{i,i+1}e^\sigma = q^{-\frac{1}{n}}e_{\sigma(i)\sigma(i+1)\sigma} + \begin{cases} q^{-\frac{1}{n}}(q - q^{-1})e^\sigma & \text{if } \sigma(i) < \sigma(i+1) \\ 0 & \text{otherwise} \end{cases} \]

Let \( A_1 \) be the set of these permutations \( \sigma \in S_n \) for which \( \sigma(i) < \sigma(i+1) \), and let \( A_2 = S_n \setminus A_1 \). Furthermore, let

\[ T_i = \sum_{\sigma \in A_i} (-q)^{l(\sigma)}e^\sigma, \]

for \( i = 1, 2 \). Then

\[ h_{i,i+1}T_2 = q^{-\frac{1}{n}} \sum_{\sigma \in A_2} (-q)^{l(\sigma)}e_{(\sigma(i),\sigma(i+1))\sigma} = q^{-\frac{1}{n}} \sum_{\sigma \in A_2} (-q)^{l((\sigma(i),\sigma(i+1))\sigma) + 1}e_{(\sigma(i),\sigma(i+1))\sigma} = q^{-\frac{1}{n}}(-q)T_1. \]

Similarly,

\[ h_{i,i+1}T_1 = q^{-\frac{1}{n}} \sum_{\sigma \in A_1} (-q)^{l(\sigma)}e_{(\sigma(i),\sigma(i+1))\sigma} + q^{-\frac{1}{n}}(q - q^{-1}) \sum_{\sigma \in A_1} (-q)^{l(\sigma)}e^\sigma. \]

Note that \( \sigma \in A_1 \Leftrightarrow (\sigma(i),\sigma(i+1))\sigma \in A_2 \) and \( l((\sigma(i),\sigma(i+1))\sigma) = l(\sigma) + 1 \), for \( \sigma \in A_1 \). Therefore, after substituting \( \tau = (\sigma(i),\sigma(i+1))\sigma \) in the first sum of (25) we get

\[ h_{i,i+1}T_1 = q^{-\frac{1}{n}} \sum_{\tau \in A_2} (-q)^{l(\tau)}e_\tau + q^{-\frac{1}{n}}(q - q^{-1})T_1 \]

\[ = q^{-\frac{1}{n}}(-q^{-1})T_2 + q^{-\frac{1}{n}}(q - q^{-1})T_1. \]

Hence, by (24), \( h_{i,i+1}(T_1 + T_2) = (-q^{-\frac{n+1}{n}})(T_1 + T_2) \).
Proof of Theorem 1(iii) We need to prove that the skeins $S = \Lambda^n$ and $q^{n(n-1)} \Lambda_n$ coincide as operators on $V^\otimes n$. Since $S$ is $U_q$-equivariant, by Proposition 21(iii), it is equal to the map $w \to x \cdot w : V^\otimes n \to V^\otimes n$, for certain $x \in H_n$. Since the image of $S$ is 1-dimensional, $x$ is either a multiple of $e_+$ or $e_-$. (This statement follows from the fact that $H_n$ is isomorphic to the group ring of $S_n$ over $\mathbb{C}(t)$.) Lemma 27 indicates that $S$ is a multiple of $e_-$, i.e. $S = e_-$ for certain $c \in \mathbb{C}(q^\frac{1}{n})$. On the other hand, $\Lambda_n$ coincides with $e_-$ as an operator on $V^\otimes n$. Therefore, we need to prove that $c = q^{n(n-1)}$, and for that it is enough to consider the closures of $S$ and $\Lambda_n$. Now, the statement follows from Lemma 24 and Corollary 26.

Proof of Proposition 2 By Theorem 1(iii) and Lemma 25,
\[
\frac{n-2}{2} = q^{n(n-1)} \cdot \pi_n(\ldots \pi_3(\Lambda_n)) = q^{n(n-1)} q^{-\frac{n(n-1)}{2} + 1} [n-2]! \Lambda_2.
\]
By substituting $\frac{n-2}{2}$ for $\Lambda_2$, we get the statement of Proposition 2.

7 Proof of Proposition 3

We prove the statement for sources only – the proof for sinks is analogous. We begin with two preliminary results.

Recall that for any two sets of integers, $S_1, S_2$,
\[
\pi(S_1, S_2) = \#\{(i_1, i_2) \in S_1 \times S_2 : i_1 > i_2\}.
\]

The proof of the following lemma is left to the reader:

Lemma 28 If $S = \{s_1, \ldots, s_k\} \subset N = \{1, \ldots, n\}$ and $S' = N \setminus S$, then $\pi(S, S') = \sum_{i=1}^{k} s_i - \frac{k(k+1)}{2}$.

Let
\[
\tau_{n,k} = \begin{pmatrix}
1 & \ldots & k & k+1 & \ldots & n \\
\ldots & n-k+1 & \ldots & n & 1 & \ldots & n-k
\end{pmatrix}.
\]

Lemma 29 For any $\sigma \in S_n$,
\[
\ell(\sigma) = \ell(\sigma \tau_{n,k}) + 2 \sum_{i=1}^{k} \sigma(i) - k(n+1).
\]

Algebraic & Geometric Topology, Volume 5 (2005)
Skein theory for SU(n)-quantum invariants

Proof Fix $k < n$. By (1), $l(\sigma) = A + B + C$, where

$$A = \# \{ i < j \leq n - k : \sigma(i) > \sigma(j) \}, \quad B = \{ n - k < i < j \leq n : \sigma(i) > \sigma(j) \},$$

$$C = \# \{ i \leq n - k < j : \sigma(i) > \sigma(j) \}.$$

Similarly,

$$l(\sigma\tau_{n,k}) = \# \{ i < j \leq k : \sigma\tau_{n,k}(i) > \sigma\tau_{n,k}(j) \} + \# \{ k < i < j \leq n : \sigma\tau_{n,k}(i) > \sigma\tau_{n,k}(j) \} + \# \{ i \leq k < j : \sigma\tau_{n,k}(i) > \sigma\tau_{n,k}(j) \}.$$

Note that the first, second, and the third summands above are equal $B$, $A$, and $k(n - k) - C$ respectively. By Lemma 28, $C = \sum_{i=1}^{k} \sigma(i) - \frac{k(k+1)}{2}$, and hence the statement follows.

Let $\mathcal{T}' : V^\otimes n \to \mathbb{Z}(q)$ be the tensor associated with the above graph:

The tensor $\mathcal{T}'$ is given by the contraction of cups and caps placed on strings $1, \ldots, k$ with the tensor $\mathcal{T}_\sigma$. Hence $\mathcal{T}' : V^\otimes n \to \mathbb{Z}(q)$ equals $\mathcal{T}_\sigma \Psi$, where

$$\Psi(e_\sigma) = e_{\sigma\tau_{n,k}} \cdot q^2 \sum_{i=1}^{k} \sigma(i) - k(n+1).$$

Since $\mathcal{T}_\sigma(e_\sigma) = (-q)^{l(\sigma)}$ and

$$\mathcal{T}_\sigma(\Psi(e_\sigma)) = (-q)^{l(\sigma\tau_{n,k})} q^2 \sum_{i=1}^{k} \sigma(i) - k(n+1).$$

Now the statement follows from Lemma 29.

8 Proofs of Lemma 8 and Theorem 9:

For any core $\Gamma'$ of an $n$-web $\Gamma$ let

$$\text{rot}_{n,\Gamma'}(S) = \sum_e \text{ind}(e)(2S(b) - n - 1) \in \mathbb{Z}. \quad (27)$$

Lemma 30 For any core $\Gamma'$ of $\Gamma$, $\text{rot}_{n,\Gamma'}(S) \in \mathbb{Z}$. 

Algebraic & Geometric Topology, Volume 5 (2005)
Proof Let $\Gamma'$ be a core of $\Gamma$ and let $v$ be the marked point of a disc of $\Gamma$. Suppose that $\Gamma$ is isotoped to $\tilde{\Gamma}$ and $\Gamma'$ is isotoped to a core $\tilde{\Gamma}'$ of $\tilde{\Gamma}$ such that the tangents at the endpoints of edges of $\tilde{\Gamma}'$ are unchanged, except for those at $v$. Then the indices, $\text{ind}(e)$, remain unchanged, except for those edges $e$ which are adjacent to $v$. For these edges $\text{ind}_{\tilde{\Gamma}'}(e) = \text{ind}_{\Gamma'}(e) + \beta$, for some $\beta$ (which is the same for all edges $e$ adjacent to $v$). Since

$$\sum_{e \text{ adjacent to } v} \beta(2S(b) - n - 1) = 0,$$

$$\text{rot}_{n,\tilde{\Gamma}'}(S) = \text{rot}_{n,\Gamma'}(S).$$

Fix a vector $\vec{w}$. By performing appropriate isotopies of $\Gamma$ and of $\Gamma'$ we may assume that $\Gamma'$ is such that the tangent vector to any endpoint $v$ of every edge of $\Gamma'$ is either $\vec{w}$ or $-\vec{w}$ depending if $v$ is the marked point of a source or a sink. In this situation, $\text{ind}(e) \in \mathbb{Z}$ for all edges $e$ of $\Gamma'$ and, consequently,

$$\text{rot}_{n,\Gamma'}(S) \in \mathbb{Z}.$$
An enhanced state $S$ of MOY-$n$-graph $\Gamma$ is a function which assigns to each edge $e$ a sequence $S(e) = (i_1, \ldots, i_{|e|})$ of $|e|$ distinct elements of the set $\bar{n}$, such that $\bar{S}(e_1^v) \cup \bar{S}(e_2^v) = \bar{S}(e_0^v)$ for any vertex $v$. Any enhanced state $S$ defines a state $\bar{S}$ of $\Gamma$ labeling every edge $e$ of $\Gamma$ by the set $\bar{S}(e)$.

For a sequence $a$ of numbers $(a_1, \ldots, a_k)$, which does not contain any repetitions, we denote by $l(a)$ the length of the permutation which puts the numbers of the sequence in the increasing order.

Denote the tensors associated with the graphs

\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{graph1} \\
\includegraphics[width=0.2\textwidth]{graph2}
\end{array}
\end{align*}

by $T_1 : V^k \otimes V^l \rightarrow V^k \otimes V^l$, $T_2 : V^k \otimes V^l \rightarrow V^{k+l}$. There are obtained by a partial contraction of four tensors: a cap, a cup, a sink, and a source, and are given by the following formulas

\begin{align*}
T_1(c) &= \sum_{a,b,d} t(a, b, c, d) a \otimes b, \\
T_2(a, b) &= \sum_{c,d} t(a, b, c, d) c,
\end{align*}

where $a \in \bar{n}^k$, $b \in \bar{n}^l$, $c \in \bar{n}^{k+l}$, $d \in \bar{n}^m$, $m = n - k - l$ and $t(a, b, c, d)$ is defined as follows:

(i) $\bar{a} \cup \bar{b} \cup \bar{d} = \bar{n}$

(ii) $\bar{c} = \bar{a} \cup \bar{b}$.

(The first condition implies that $\bar{a}$, $\bar{b}$, and $\bar{d}$ are disjoint and the sequences $a, b, d$ have no repeating elements.) If these conditions are satisfied then

\begin{align*}
t(a, b, c, d) &= (-q)^{l(a)+l(b)+l(c)+\pi(\bar{a}, \bar{b})} \cdot (-q)^{l(c)+l(d)+\pi(\bar{a}, \bar{d})} \cdot q^{2 \sum_{i=1}^{m} d_i - (n+1)m}.
\end{align*}

The first two factors above come from the tensors associated with the vertices of graphs $\Gamma$. The third factor corresponds to caps and cups of graphs $\Gamma$. And since $\pi(\bar{a} \cup \bar{b}, \bar{d}) = \pi(\bar{c}, \bar{d})$, we get

\begin{align*}
t(a, b, c, d) &= (-q)^{l(a)+l(b)+l(c)+\pi(\bar{a}, \bar{b})} \cdot q^{2l(d)+2\pi(\bar{c}, \bar{d})+2 \sum_{i=1}^{m} d_i - (n+1)m}.
\end{align*}

(29)

If conditions (i) and (ii) above are satisfied then by Lemma $\Gamma$

\begin{align*}
2(\bar{c}, \bar{d}) + 2 \sum_{i=1}^{m} d_i - (n+1)m = 2 \sum_{i=1}^{k+l} c_i - (k+l)(k+l+1) + 2 \sum_{i=1}^{m} d_i - (n+1)m =
\end{align*}

Algebraic & Geometric Topology, Volume 5 (2005)
By (21), \(\sum_{\sigma \in S_m} q^{2\ell(\sigma)} = q^{\frac{m(m-1)}{2}}[m]!\). Therefore, if we denote \(\sum_t t(a, b, c, d)\) by \(t(a, b, c, d)\), then

\[
t(a, b, c, d) = (-q)^{l(a)+l(b)+l(c)+\pi(\bar{a}, \bar{b})} \cdot q^{\frac{m(m-1)+(n-k-l)(k+l)}{2}}[n - k - l]!
\]

By (21), \(\sum_{\sigma \in S_m} q^{2\ell(\sigma)} = q^{\frac{m(m-1)}{2}}[m]!\). Therefore, if we denote \(\sum_t t(a, b, c, d)\) by \(t(a, b, c, d)\), then

\[
t(a, b, c, d) = (-q)^{l(a)+l(b)+l(c)+\pi(\bar{a}, \bar{b})} \cdot q^{\frac{m(m-1)+(n-k-l)(k+l)}{2}}[n - k - l]!
\]

The complete contraction of tensors associated with vertices, cups, and caps of \(W(\Gamma)\) produces

\[
[\Gamma]_n = \langle W(\Gamma) \rangle_n = \sum_S \Psi(S),
\]

where

\[
\Psi(S) = q^{rot(S)} \cdot \prod_v t(S(e^v_1), S(e^v_2), S(e^v_0))
\]

(The first of the above factors is the contraction of the tensors associated with “caps” and “cups” in \(\Gamma\).) Hence

\[
\Psi(S) = q^{rot(\bar{S})} \cdot (-q)^{\sum_e 2\ell(S(e)) + \sum_e \pi(\bar{S}(e^v_1), \bar{S}(e^v_2))} \prod_v q^{\frac{(n-|e^v_0|)(n+|e^v_0| - 1)}{2}}[n - |e^v_0|]!
\]

where the sum \(\sum_e\) is taken over all edges of \(\Gamma\) which are not annuli. For any state \(s\) of \(\Gamma\), we denote sum \(\sum \Psi(S)\) over all enhanced states \(S\) such that \(\bar{S} = s\), by \(\Phi(s)\). Hence, by (21),

\[
\Phi(s) = \mathcal{N}(\Gamma) \cdot q^{rot(s)} \cdot (-q)^{\sum_e \pi(s(e^v_1), s(e^v_2))},
\]

where

\[
\mathcal{N}(\Gamma) = \prod_v q^{\frac{|v|(|v| - 1)}{2}}[|v|!] \cdot \prod_v q^{\frac{(n-|e^v_0|)(n+|e^v_0| - 1)}{2}}[n - |e^v_0|]!
\]

Since

\[
\frac{1}{2}(n - |e^v_0|)(n + |e^v_0| - 1) = \frac{n(n-1) - |e^v_0|^2}{2}
\]

\[
= \frac{1}{2}n(n - 1) + \frac{1}{4}(|e^v_0| + |e^v_1| + |e^v_2| - |e^v_0|^2 - |e^v_1|^2 - |e^v_2|^2) - \frac{1}{2}|e^v_1| \cdot |e^v_2|,
\]

and each edge appears as \(e^v_i\) for two different vertices \(v\),

\[
\mathcal{N}(\Gamma) = \prod_v q^{\frac{|v|^2 - |v|}{2}}[|v|!] \cdot \prod_v q^{\frac{|e^v_i - |e^v_0|}{2}} \prod_v q^{\frac{n(n-1) - |e^v_0|^2}{2}}[n - |e^v_0|]!
\]

*Algebraic & Geometric Topology, Volume 5 (2005)*
Hence,
\[ \mathcal{N}(\Gamma) = \prod_e |e|! \cdot \prod_v q^{\frac{n(n-1)-|e_v^1||e_v^2|}{2}} [n - |e_v^0|]! \]

\[\square\]

10 Proof of Proposition \[13\]

For any
\[ s : \{\text{edges of } \Gamma\} \rightarrow \text{subsets of } \{1, \ldots, n\} \]

let
\[ \hat{s} : \{\text{edges of } \Gamma\} \rightarrow \text{subsets of } \{-\frac{n+1}{2}, -\frac{n}{2} + 1, \ldots, \frac{n-1}{2}\}, \]

be such that \( \hat{s}(e) = \{i_1 - \frac{n+1}{2}, \ldots, i_k - \frac{n+1}{2}\} \) if \( s(e) = \{i_1, \ldots, i_k\} \). We observed in Section \[17\] already, that \( s \) is a state of \( \Gamma \) if and only if \( \hat{s} \) is a MOY-state of \( \Gamma \).

Let \( \Gamma \) be an \( MOY_n \)-graph diagram with no crossings. We can assume that \( \Gamma \) is composed of caps, cups, and vertices of the following form:

Since these pictures are obtained by rotating the pictures of [MOY, Fig. 1.3] by 180° and exchanging \( e_1 \) with \( e_2 \), the Murakami-Ohtsuki-Yamada weight associated to the vertices above is
\[ q^{\frac{|e_1^v||e_2^v|}{4} - \pi(s(e_2),s(e_1))/2}. \]

But since
\[ \pi(s(e_2),s(e_1)) + \pi(s(e_1),s(e_2)) = |e_1| \cdot |e_2|, \]

this weight equals to
\[ q^{-|e_1||e_2|/4 + \pi(s(e_1),s(e_2))/2}. \]

Since the Murakami-Ohtsuki-Yamada rotation index of \( \hat{s} \) is \( rot(s) \),
\[ \{\Gamma\}_n = \sum_{\text{states}} q^{rot(s)} \prod_v q^{-|e_v^1||e_v^2|/4 + \pi(s(e_v^1),s(e_v^2))/2}, \]

and the statement follows.
References

[APS] M M Asaeda, J H Przytycki, A S Sikora, Categorification of the Kauffman bracket skein module of I-bundles over surfaces, Algebr. Geom. Topol. 4 (2004) 1177–1210 MR2113902

[BN] D Bar-Natan, On Khovanov’s categorification of the Jones polynomial, Algebr. Geom. Topol. 2 (2002) 337–370 MR1917056

[Bl] C Blanchet, Hecke algebras, modular categories and 3-manifolds quantum invariants, Topology 39 (2000) 193–223 MR1710999

[Bu] D Bullock, Rings of SL_2(C)-characters and the Kauffman bracket skein module, Comment. Math. Helv. 72 (1997) 521–542 MR1600138

[BFK] D Bullock, C Frohman, Joanna Kania-Bartoszyńska, Understanding the Kauffman bracket skein module, J. Knot Theory Ramifications 8 (1999) 265–277 MR1691437

[CP] V Chari, A Pressley, A guide to quantum groups, Cambridge University Press, Cambridge (1994) MR1300632

[FZ] C Frohman, J Zhong, The Yang-Mills measure in the SU_3 skein module, preprint (2004)

[FGL] C Frohman, R Gelca, W Lofaro, The A-polynomial from the noncommutative viewpoint, Trans. Amer. Math. Soc. 354 (2002) 735–747 MR1862565

[Ga] S Garoufalidis, Difference and differential equations for the colored Jones function, arXiv:math.GT/0306229

[GL] S Garoufalidis, T T Q Le, The colored Jones function is q-holonomic, Geom. Topol. 9 (2005) 1253–1293

[Ge] R Gelca, On the relation between the A-polynomial and the Jones polynomial, Proc. Amer. Math. Soc. 130 (2002) 1235–1241 (electronic) MR1873802

[Go] B Gornik, Note on Khovanov link cohomology, arXiv:math.QA/0402266

[Gy] A Gyoja, A q-analogue of Young symmetrizer, Osaka J. Math. 23 (1986) 841–852 MR873212

[HK] R S Huerfano, M Khovanov, Categorification of some level two representations of sl(n), arXiv:math.QA/0204333

[Ja] M Jacobsson, An invariant of link cobordisms from Khovanov’s homology theory, Algebr. Geom. Topol. 4 (2004) 1211–1251 MR2113903

[Ka] L H Kauffman, State models and the Jones polynomial, Topology 26 (1987) 395–407 MR899057

[KV] L H Kauffman, P Vogel, Link polynomials and a graphical calculus, J. Knot Theory Ramifications 1 (1992) 59–104 MR1155094

[K1] M Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (2000) 359–426 MR1740682

Algebraic & Geometric Topology, Volume 5 (2005)
Skein theory for $SU(n)$-quantum invariants

[1] M Khovanov, sl(3) link homology, Algebr. Geom. Topol. 4 (2004) 1045–1081

[KR] M Khovanov, L Rozansky, Matrix factorizations and link homology, arxiv:math.QA/0401268

[KS] A Klimyk, K Schm"udgen, Quantum groups and their representations, Texts and Monographs in Physics, Springer-Verlag, Berlin (1997) MR1492989

[Ku] G Kuperberg, Spiders for rank 2 Lie algebras, Comm. Math. Phys. 180 (1996) 109–151 MR1403861

[Le] E S Lee, On Khovanov invariant for alternating links, arxiv:math.GT/0210213

[LM] A Lubotzky, A R Magid, Varieties of representations of finitely generated groups, Mem. Amer. Math. Soc. 58 (1985) xi+117 MR818915

[Mu] H Murakami, A quantum introduction to Knot Theory, preprint (2003)

[MOY] H Murakami, T Ohtsuki, S Yamada, Homfly polynomial via an invariant of colored plane graphs, Enseign. Math. (2) 44 (1998) 325–360 MR1659228

[OY] T Ohtsuki, S Yamada, Quantum SU(3) invariant of 3-manifolds via linear skein theory, J. Knot Theory Ramifications 6 (1997) 373–404 MR1457194

[Pr] J H Przytycki, Fundamentals of Kauffman bracket skein modules, Kobe J. Math. 16 (1999) 45–66 MR1723531

[PS] J H Przytycki, A S Sikora, On skein algebras and $SL_2(\mathbb{C})$-character varieties, Topology 39 (2000) 115–148 MR1710996

[Ra] J A Rasmussen, Khovanov homology and the slice genus, arxiv:math.GT/0402131

[RT] N Yu Reshetikhin, V G Turaev, Ribbon graphs and their invariants derived from quantum groups, Comm. Math. Phys. 127 (1990) 1–26 MR1036112

[S1] A S Sikora, SL_n-character varieties as spaces of graphs, Trans. Amer. Math. Soc. 353 (2001) 2773–2804 (electronic) MR1828473

[S2] A S Sikora, Skein modules and TQFT, from: “Knots in Hellas ’98 (Delphi)”, Ser. Knots Everything 24, World Sci. Publishing, River Edge, NJ (2000) 436–439 MR1865721

[Tu] V G Turaev, The Yang-Baxter equation and invariants of links, Invent. Math. 92 (1988) 527–553 MR939474

[Vi] O Viro, Remarks on definition of Khovanov homology, e-print (2002) arxiv:math.GT/0202195

[Yo] Y Yokota, Skeins and quantum SU(N) invariants of 3-manifolds, Math. Ann. 307 (1997) 109–138 MR1427678

Department of Mathematics, University at Buffalo
Buffalo, NY 14260-2900, USA

Email: asikora@buffalo.edu

Received: 23 July 2004

Algebraic & Geometric Topology, Volume 5 (2005)