Hypergraph Learning with Line Expansion

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Abstract

Previous hypergraph expansions are solely carried out on either vertex level or hyperedge level, thereby missing the symmetric nature of data co-occurrence, and resulting in information loss. To address the problem, this paper treats vertices and hyperedges equally and proposes a new hypergraph formulation named the line expansion (LE) for hypergraphs learning. The new expansion bijectively induces a homogeneous structure from the hypergraph by treating vertex-hyperedge pairs as “line nodes”. By reducing the hypergraph to a simple graph, the proposed line expansion makes existing graph learning algorithms compatible with the higher-order structure and has been proven as a unifying framework for various hypergraph expansions. We evaluate the proposed line expansion on five hypergraph datasets, the results show that our method beats SOTA baselines by a significant margin.

1 Introduction

This paper proposes a new hypergraph formulation, line expansion (LE), for the problem of hypergraph learning. The proposed LE is a topological mapping, transforming the hypergraph into a homogeneous structure, while preserving all the higher-order relations. LE allows all the existing graph learning algorithms to work elegantly on hypergraphs.

The problem of hypergraph learning is important. Graph-structured data are ubiquitous in practical machine/deep learning applications, such as social networks [1], protein networks [2], and co-author networks [3]. Intuitive pairwise connections among nodes are usually insufficient for capturing real-world higher-order relations. For example, in social networks, many relations (such as trust, friendship, or interest) are not transitive. Thus, it is difficult to infer trust or user interest groups from pairwise associations. For another example, in biology, proteins are bound by polypeptide chains, thus their relations are naturally higher-order. Hypergraphs allow modeling such multi-way relations, where edges could be incident to more than two nodes.

However, the research on spectral theory for hypergraphs is far less been developed [1]. Hypergraph learning was first introduced in [3] as a propagation process on hypergraph structure, however, [4] indicated that their Laplacian matrix is equivalent to pairwise operation. Since then, researchers explored non-pairwise relationships by developing nonlinear Laplacian operators [5, 6], utilizing random walks [1, 7] and learning the optimal weights [6, 8] of hyperedges. Essentially, all of these algorithms focus on vertices, viewing hyperedges as connectors, and they explicitly break the bipartite property of hypergraphs (shown in Fig. 1).

The investigation of deep learning on hypergraphs is also in a nascent stage. [9] developed Chebyshev formula for hypergraph Laplacians and proposed HGNN. Using a similar hypergraph Laplacian, [10] proposed HyperGCN while [11] generalized [2, 13] and defined two neural hypergraph operators. However, they in fact both constructed a simple weighted graph and applied mature graph learning algorithms by introducing only vertex functions, which is not sufficient for higher-order learning.

Our motivation stems from the lack of powerful tools for representing the hyper-structure. We are also motivated by the richness of literature on graphs as well as recent success of graph representation

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learning (GRL) with powerful neural operators (convolution, attention, spectral, etc). The point is that how can we develop a mapping from the hypergraph to a simple graph, without losing information, so that these powerful representation algorithms on graphs would be applicable and flexible for hypergraphs.

For hypergraphs, we propose line expansion (LE), which induces a new structure, where the “node” is a vertex-hyperedge pair, and “edges” between two “nodes” are constructed by either the common vertex or hyperedge. It is interesting that the new structure is (i) homogeneous, (i.e., a graph where nodes have the same semantics) and (ii) symmetrical to the original vertex and hyperedge. We further prove that LE is also (iii) bijective. To conduct hypergraph learning, we first transform the hypergraph to LE, where the actual learning happens. Features from vertices/hyperedges will be projected to nodes on the induced graph, and final representations of those nodes will be aggregated and back-projected to the original vertices/hyperedges.

The proposed line expansion of hypergraphs is novel and informative, compared to traditional formulations, where the hyperedges are usually transformed into cliques of edges (e.g., clique/star expansions) or hypergraph cuts, or the learning solely depends on edge connectivity (e.g., hyperedge expansions). Differently, LE treats vertices and hyperedges equally, thus preserving the nature of hypergraphs. Note that, LE is also significantly different from those theoretical hypergraph transformations, such as tensor based hyper-matrix representation, line graphs of hypergraphs, intersection graphs of hypergraphs, or middle graphs of hypergraphs. These formulations either require strong constraints (e.g., uniform hypergraphs) or result in heterogeneous topologies as well as other structures that complicate practical usage. For example, such formulations may restrict applicability of graph-based algorithms due to their special structures.

Further, this paper revisits the formulation of the standard star/clique expansion and simple graph learning algorithms. We conclude that they can be unified as special cases of LE. Empirically, this paper demonstrates the effectiveness of LE on five real-world hypergraphs. We apply the popular graph convolutional networks (GCNs) on LE as our method, and its performance is shown to consistently outperform other hypergraph learning baselines.

The paper is organized as follows. In Section 2 we introduce the general notations of hypergraphs and formulate our problem. In Section 3 we propose line expansion of hypergraphs and show some interesting observations. In Section 4 we generalize graph convolutional networks (GCNs) to hypergraphs by line expansion. We evaluate line expansion on three-experiments in Section 5 and conclude our work in Section 6.

2 Preliminaries

2.1 Hypergraphs

Research on graph-structured deep learning stems mostly from Laplacian matrix and vertex functions of simple graphs. Only recently, it became possible to learn on hypergraphs.

Hypergraphs. Let \( G_H = (V, E) \) denote a hypergraph, with vertex set \( V \) and edge set \( E \subset 2^V \). A hyperedge \( e \in E \) (we sometimes also call it “edge” interchangeably in this paper) is a subset of \( V \).

Given an arbitrary set \( S \), let \( |S| \) denote the cardinality of \( S \). A regular graph is thus a special case of a hypergraph, with \(|e|=2\) uniformly, which is also called a 2-regular hypergraph. A hyperedge \( e \) is said to be incident to a vertex \( v \) when \( v \in e \). One can represent a hypergraph by a \(|V| \times |E|\) incidence matrix \( H \) with its entry \( h(v, e) = 1 \) if \( v \in e \) and 0 otherwise. For each vertex \( v \in V \) and hyperedge \( e \in E \), \( d(v) = \sum_{e \in E} h(v, e) \) and \( \delta(e) = \sum_{v \in V} h(v, e) \) denote their degree functions, respectively. The vertex-degree matrix \( D_v \) of a hypergraph \( G_H \) is a \(|V| \times |V|\) matrix with each diagonal entry
corresponding to the node degree, and the edge-degree matrix $D_e$ is $|E| \times |E|$ and also diagonal, which is defined on the hyperedge degree.

2.2 Problem Setup

In this paper, we are interested in the transductive problems on hypergraphs, specifically node classification, similar to [9, 11]. It aims to induce a labeling $f : V \rightarrow \{1, 2, \ldots, C\}$ from the labeled data as well as the geometric structure of the graph and then assigns a class label to unlabeled vertices by transductive inference.

Specifically, given a hypergraph $G_H = (V, E)$ with the labeled vertex subset $T \subset V$ and the label $L = \{1, 2, \ldots, C\}^T$, we propose minimizing the empirical risk,

$$f^* = \arg \min_{f(\theta)} \frac{1}{|T|} \sum_{v_t \in T} \mathcal{L}(f(v_t | \theta), L(v_t)),$$

where cross-entropy error [12] is commonly applied in $\mathcal{L}(\cdot)$. Intuitively, node similarity indicates similar labels on graphs. Given the bipartite symmetry in hypergraphs, we posit that vertex similarity and edge similarity are equally important. This work focuses on node classification problems on hypergraphs, but it also sheds some light on the applications of hyper-edge representation (e.g., relation mining) by exploiting symmetry.

3 Hypergraph Line Expansion

Most well-known graph-based algorithms [21, 22] are defined for graphs but not hypergraphs. Therefore, in real applications, hypergraphs are often transformed into simple graphs [3, 4] that are easier to handle.

3.1 Traditional Hypergraph Expansions

Two main ways of approximating hypergraphs by graphs are the clique expansion [23] and the star expansion [24]. The clique expansion algorithm (left side of Fig. 2) constructs a graph $G_c = (V, E_c)$ from the original hypergraph by replacing each hyperedge with a clique in the resulting graph (i.e., $E_c = \{(u, v) \mid u, v \in e, e \in E\}$), while the star expansion algorithm (right side of Fig. 2) constructs a new graph $G_s = (V_s, E_s)$ by augmenting the vertex set with hyperedges $V_s = V \cup E$, where vertices and hyperedges are connected by their incident relations (i.e., $E_s = \{(v, e) \mid v \in e, v \in V, e \in E\}$). Note that, the star expansion induces a heterogeneous graph structure.

Unfortunately, these two approximations cannot retain or well represent the higher-order structure of hypergraphs. Let us consider the co-authorship network, as $G_H$ in Fig. 2, where we view authors as nodes (e.g., $v_1, v_2$) and papers as hyperedges (e.g., $e_1$). Then we immediately know that author $v_1$ and $v_2$ have jointly written one paper $e_1$, and together with author $v_3$, they have another co-authored paper $e_2$. This hierarchical and multi-way connection is an example of higher-order relation. Assume we follow the clique expansion, then we obviously miss the information of author activity rate and whether the same persons jointly writing two or more articles. Though researchers have remedially used weighted edges [6, 11], the hyper-dependency still collapses or fuses into linearity. Star expansion express the whole incidence information, but the remaining heterogeneous structure (i) has no explicit vertex-vertex link and (ii) is too complicated for those well-studied graph algorithms, which are mostly designed for simple graphs. One can summarize [25] that these two expansion are not good enough for many applications.

3.2 Our Line Expansion

Since the commonly used expansions cannot give a satisfactory representation, we seek a new expansion that preserves all the original higher-order relations, while presenting an easy-to-learn graph structure. Motivated by the special symmetric structure of hypergraphs that vertices are connected to multiple edges and edges are conversely connected to multiple vertices, we treat vertices and edges equally and propose hypergraph Line Expansion (LE).

The Line Expansion of the hypergraph $G_H$ is constructed as follows (shown in Fig. 2, bottom): (i) each incident vertex-hyperedge pair is considered as a “line node”; (ii) “line nodes” are connected
when either the vertex or the hyperedge is the same. Essentially, the induced structure is a graph, where each node and each hyperedge (from the original hypergraph) induces a clique. We now formally define the line expansion $G_l$ of hypergraph $G_H$.

**Line Expansion.** Let $G_l = (V_l, E_l)$ denotes the graph induced by the line expansion of hypergraph $G_H = (V, E)$. The node set $V_l$ of $G_l$ is defined by vertex-hyperedge pair $\{(v, e) \mid v \in V, e \in E\}$ from the original hypergraph. The edge set $E_l$ and adjacency $A_l \in \{0, 1\}^{|V_l| \times |V_l|}$ is defined by pairwise relation with $A_l((u_l, v_l)) = 1$ if either $v = v'$ or $e = e'$ for $u_l = (v, e), v_l = (v', e') \in V_l$.

The construction of the line expansion follows the neighborhood feature sharing mechanism. For graph node representation learning, [15, 12] first encode local structure by aggregating information from a node’s immediate neighborhood. In line expansion, we view the incidence of vertex-hyperedge as a whole and generalize the “neighborhood” concept by defining that two line nodes are neighbors when they contain the same vertex (vertex similarity) or the same hyperedge (edge similarity). We argue that the line expansion consequently preserves higher-order associations.

### 3.3 Entity Projection

In this section, we define the projection matrices for hypergraph entities (i.e., vertices and hyperedges) for the topological map from $G_H = (V, E)$ to $G_l = (V_l, E_l)$.

In $G_l$, each line node $(v, e)$ could be viewed as a vertex with hyperedge context or a hyperedge with vertex context, which means that it encodes part of the vertex (related to that hyperedge) or part of the hyperedge (related to that particular vertex). In a word, the line expansion creates information linkage in the higher-order space.

To scatter the information, a vertex $v$ from $G_H$ is mapped to a set of line nodes $(v, \cdot) \in V_l$ in $G_l$. We introduce the vertex projection matrix $P_v \in \{0, 1\}^{|V| \times |V_l|}$,

$$P_v(v_l, v) = \begin{cases} 1 & v_l = (v, e), \exists e \in E, \\ 0 & \text{otherwise}, \end{cases} \quad (2)$$

where each entry encodes whether the line node contains the vertex. Similarly, we also define an edge projection matrix $P_e \in \{0, 1\}^{|V| \times |E|}$ that encodes the projection of hyperedges to sets of line nodes.

**Theorem 1.** Under the construction, for a hypergraph $G_H$ and its line expansion $G_l$, the mapping $\phi$ from hypergraph to line expansion (i.e., $\phi : G_H \to G_l$) is bijective.

The inverse mapping from $G_l$ to $G_H$ is guaranteed by Theorem 1 (proofs are in Appendix [B]), where the complete information of vertex $v \in V$ is re-obtained by aggregating all the the small parts $(v, \cdot) \in V_l$ from $G_l$. Naturally, the overall information from $(v, e)$ is shared (divided by edge degree $\delta(e)$) by vertices under hyperedge $e \in E$. 

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**Figure 2: Hypergraph Expansions**
Afterwards, we fuse the higher-order information by defining the vertex back-projection matrix \( P_v' \in \mathbb{R}^{|V| \times |V|} \),
\[
    P_v'(v, v_l) = \begin{cases} \frac{1}{\delta(v)} & v_l = (v, e), \exists e \in E, \\ 0 & \text{otherwise.} \end{cases}
\]
(3)

Similarly, we could also get an edge back-projection matrix \( P_e' \in \mathbb{R}^{|E| \times |V|} \) to integrate all partial information of hyperedges into one piece.

### 3.4 Additional Properties and Discussion

In this section, we first present an interesting observation between characteristic matrices from \( G_H \) and \( G_l \). Then, we connect our line expansion with the “line graph” in graph theory, based on which, some sound properties could be derived.

**Observation 1.** Let \( H \) be the incidence matrix of a hypergraph \( G_H \). \( D_v \) and \( D_e \) are the vertex and hyperedge degree matrices. Let \( P_v \) and \( P_e \) be the vertex and edge projection matrices, respectively. \( A_l \) is the adjacency matrix of line expansion \( G_l \). Let \( H_r = [P_v', P_e'] \in \{0, 1\}^{|V| \times (|V| + |E|)} \), it satisfies the following equations,
\[
    H_v^\top H_r = \begin{bmatrix} D_v & H \\ H^\top & D_e \end{bmatrix},
\]
(4)
\[
    H_r H_r^\top = 2I + A_l.
\]
(5)

From Observation 1 (see proof in Appendix C), the left hand of both Eqn. 4 and Eqn. 5 are the projection matrices, and the right hand of these two equations are information respectively from the hypergraph and the line expansion. Essentially, they quantify the transition from \( G_H \) to \( G_l \). For Eqn. (5), we are interested in the product of \( H_r H_r^\top \) with two orders of self-loop, which would be useful in the analytical aspects of line expansion (shown in Appendix D).

**Theorem 2.** For a hypergraph, its line expansion \( G_l \) is equivalent to the line graph of its star expansion \( L(G_s) \), where \( L(\cdot) \) is a line graph notation from graph theory.

Theorem 2 is the foundation of Theorem 1. It provides a theoretical interpretation and enriches our expansion with sound graph theoretical properties (readers could refer to line graph theory [25]). That is why we name our formulation “line expansion”. Note that the line expansion is significantly different from the “line graph of hypergraphs” discussed in [18]. Instead, it is the line graph of the star expansion. Detailed proofs of Theorem 2 could be found in Appendix A.

Based on Theorem 2, we know that \( G_l \) is homogeneous and has the same connectivity with \( G_H \). The number of new edges in \( G_l \) could be calculated as \( |E_l| = \frac{\sum_v d(v)(d(v) - 1)}{2} + \frac{\sum_e d(e)(d(e) - 1)}{2} \) and new nodes as \( |V_l| = \frac{\sum_v d(v) + \sum_e \delta(e)}{2} \). In the worse case, for a fully-connected k-order hypergraph \( k \ll |V| \), \( |V_l| = \Theta(k |E|) \) and \( |E_l| = \Theta((\frac{k^2}{2} |E|^2) \). However, many real hypergraphs are indeed sparse (e.g., degrees of vertices and hyperedges follow long-tailed distribution, most of them have degree one, \(|V| \ll |E| \) or \(|E| \ll |V| \)), so that the cardinality could usually reduce to \(|V_l| = \Theta(|V| + |E|) \) and \(|E_l| = O(|V||E|) \).

### 4 Hypergraph Representation Learning

Transductive learning on graphs is successful due to the fast localization and neighbor aggregation [15][12][13]. It is easy to define the info-propagation pattern upon simple structures. For real-world cases, relationships among objects are usually more complex than pairwise. Therefore, to apply these algorithms, we need a succinct but informative representation of the higher order relations.

Shown in Section 3, the bijective map from \( G_H \) to \( G_l \) equipped with four entity projectors \( \{P_v, P_v', P_e, P_e'\} \) fills the conceptual gap between hypergraphs and graphs. With this powerful tool, it is possible to transfer the hypergraph learning problems into graph structures and address them by using well-studied graph representation algorithms. Note that, this work focuses on the generic hypergraphs without edge weights.
4.1 Hypergraph Learning with Line Expansion

In this section, we generalize graph convolution networks (GCNs) \cite{kipf2017semi} to hypergraphs and introduce a new learning algorithm defined on line expansion for hypergraph representation. Note that, on our proposed structure, other graph representation algorithms could be extended similarly \cite{Zhu2021Struc2vecHR, You2019HyperRepHL, Tang2018HydropathHG}.

Overall Pipeline. To address the transductive node classification problems on hypergraphs (in Section 2.2), we design the pipeline of our proposed model as the following three steps. First, vertices of the hypergraph will be mapped into relevant line nodes. Specifically, we use the proposed vertex projection matrix $P_v$ to conduct feature mapping. Second, we apply deep graph learning algorithms (e.g., GCNs) to learn the representation for each line node in higher-order space. Finally, the learned representation is fused by $P_l^r$, the vertex back-projection matrix, for each vertex in an inverse edge degree manner. The labelling of vertices is predicted on the fused representation.

4.2 Convolution on Line Expansion

Feature Projection. For $G_H$, given the initial state vector, $x \in \mathbb{R}^{|V| \times d_i}$ ($d_i$ is input dimension), we project it as the initial feature vector in $G_l$ by matrix $P_v \in \{0, 1\}^{|V| \times |V|}$,

$$h^{(0)} = P_v x \in \mathbb{R}^{|V| \times d_i},$$

which essentially scatters features from vertex of $G_H$ to feature vectors of line nodes in $G_l$. In line expansion, a line node could be adjacent to another line nodes that contain the same vertex (vertex similarity) or the same hyperedge (edge similarity). Let us denote $h_{(v,e)}^{(k)}$ as the representation of line node $(v,e)$ in the $k$-th layer.

Convolution Layer. By incorporating information from both vertex-similar neighbors and hyperedge-similar neighbors, the convolution is defined as,

$$h_{(v,e)}^{(k+1)} = \sigma \left( \sum_{e'} w_{v,e} h_{(v',e')}^{(k)} \Theta^{(k)} + \sum_{e'} w_{v,e} h_{(v',e')}^{(k)} \tilde{\Theta}^{(k)} \right),$$

where $\sigma(\cdot)$ is a non-linear activation function like ReLU \cite{kipf2017semi} or LeakyReLU \cite{Maas2013RectifierNL}. $\Theta^{(k)}$ is the filter parameters for layer $k$. Two hyper-parameters $w_v, w_e$ are what we used to parametrize vertex similarity and edge similarity. Specifically, in Eqn. (7), the first term (i.e., $\sum_{e'} w_{v,e} h_{(v',e')}^{(k)}$) convolves information from neighbors who share similar edges, whereas the second term (i.e., $\sum_{e'} w_{v,e} h_{(v',e')}^{(k)}$) convolves information from neighbors who share similar vertices.

We present the parameterized adjacency matrix $A_l$ of $G_l$ (this parametrized version is used to derive a generic model and is convenient for the analysis in Appendix D). In the experiment, we set $w_v = w_e = 1$, which reduces to the binary version in Sec. 3.2.

$$A_l(u_l, v_l) = \begin{cases} w_e & u_l = (v, e), v_l = (v', e'), \ v = v', \\ w_e & u_l = (v, e), v_l = (v', e'), \ e = e', \\ 0 & \text{otherwise}, \end{cases}$$

and adopt the renormalized trick \cite{kipf2017semi} with the adjustment (two-orders of self-loop, referring to Section 3.4): $2I + D_l^{-\frac{1}{2}} A_l D_l^{-\frac{1}{2}} \to \hat{D}_l^{-\frac{1}{2}} \hat{A}_l \hat{D}_l^{-\frac{1}{2}}$ with $\hat{A}_l = 2I + A_l$ and $\hat{D}_{lii} = \sum_j \hat{A}_{lij}$, to make Eqn. 7 compact,

$$h^{(k+1)} = \sigma \left( \hat{D}_l^{-\frac{1}{2}} \hat{A}_l \hat{D}_l^{-\frac{1}{2}} h^{(k)} \tilde{\Theta} \right).$$

In real practice, we do not bother to calculate the adjacency $A_l$ or further add self-loops. An efficient trick is to directly use Eqn. 5 and re-normalize it to reach $\hat{D}_l^{-\frac{1}{2}} \hat{A}_l \hat{D}_l^{-\frac{1}{2}}$. 


**Representation Projection.** After the convolution layer, \( h^{(k+1)} \) is the representation matrix for all line nodes, from which we could derive fused representation for both vertices and hyperedges in \( G_H \). The representation for the vertex \( v \in V \) can be obtained by aggregating representations based on the reciprocal of edge degree by using back-projector \( P'_v \), formally,

\[
\hat{x} = P'_v h^{(k+1)} \in \mathbb{R}^{|V| \times d_o},
\]

where \( d_o \) is the dimension of output representation. Note that, in this work, we are only interested in the node representation. However, due to the symmetry of hypergraphs, this work also sheds some light on the applications of learning hyper-edges (e.g., relation mining) by using \( P_e, P'_e \) with similar classification algorithms. We leave it to future work.

In sum, the complexity of 1-layer convolution is of \( O(|E|d_id_o) \), since the convolution operation could be efficiently implemented as the a product of a sparse matrix with a dense matrix.

### 4.3 Unifying Hypergraph Expansion

As discussed in Sec. 3.1, for hypergraphs, common practices often collapse the higher order structure into simple graph structures by attaching weights on edges, and then the vertex operators are solely applied onto the remaining topology. Therefore, the interchangeable and complementary nature between nodes and edges are generally missing [30].

**Theorem 3.** Line expansion is a generalization of clique expansion, star expansion, as well as simple graph convolution when \( w_e = 0 \) (no message passing from hyperedge-similar neighbors) with additional conditions satisfied.

In this work, instead of designing a local vertex-to-vertex operator [9, 3, 31], we treat the vertex-hyperedge relation as a whole. Therefore, the neighborhood convolution on line expansion is equivalent to exchanging information simultaneously across vertices and hyperedges of \( G_H \). Our proposed line expansion is powerful in that it unifies clique and star expansions, as well as simple graph cases, stated in Theorem [3]. We theoretically formulate different hypergraph expansions and provide detailed proof in Appendix [D].

### 5 Experiments

#### 5.1 Experiment Setup

We empirically evaluated the representation power of line expansion (LE). Our experiments are three-fold: one main experiment and two case studies.

The first case study (shown in Appendix [E]) is for simple graphs (Cora, Citeseer, Pubmed). This experiment is to verify part of the “generalization” conclusion in Theorem [3]. We show that learning on LE will achieve at least the same accuracy as learning one the original graphs. The second case study (shown in Appendix [F]) is for two hyperparameters \( w_e \) and \( w_v \). A brief conclusion (from five datasets) is that hypergraphs have different sensitivities and preferences for \( w_e \) and \( w_v \), and it is always better to pass information from both edges and vertices (i.e., \( w_e \neq 0 \) and \( w_v \neq 0 \)). More specifically, from Appendix [F], a practical guide is to set a small \( w_e \) for hypergraphs with fewer hyperedges and set a large \( w_e \) for those with sufficient hyperedges. This case study also shows that \( w_v = w_e = 1 \) is a trivial but effective choice.

We present the main experiment below. This experiment is demonstrated on five real-world hypergraphs with eight baselines. All the experiments are conducted 50 times and mainly finished in a Linux server with 64GB memory, 32 CPUs and a single GTX-2080 GPU.

| Dataset   | Vertices | Hyperedges | Features | Class | Label rate | Training / Validation / Test |
|-----------|----------|------------|----------|-------|------------|-----------------------------|
| 20News    | 16,242   | 100        | 100      | 4     | 0.025      | 400 / 7,921 / 7,921         |
| Mushroom  | 8,124    | 112        | 112      | 2     | 0.006      | 50 / 4,062 / 4,062          |
| Zoo       | 101      | 42         | 17       | 7     | 0.650      | 66 / - / 33                 |
| ModelNet40| 12,311   | 12,321     | 2048     | 40    | 0.800      | 9,849 / 1,231 / 1,231       |
| NTU2012   | 2,012    | 2,012      | 2048     | 67    | 0.800      | 1,608 / 202 / 202           |

Table 1: Statistics of Hypergraphs
In this section, we employ four traditional and four SOTA hypergraph learning methods on five real-world datasets to evaluate hypergraph learning with line expansion. We apply GCN model on line expansion, named LE+GCN, introduced in Sec. 4. In the experiment, we set \( w_v = w_e = 1 \) for simplicity. Readers could refer to Appendix F for more intuitions about how to choose \( w_v \) and \( w_e \).

**Hypergraph Datasets.** The first dataset 20Newsgroups is a modified version \(^7\) It contains 16,242 articles with binary occurrence values of 100 words. Each word is regarded as a hypergraph. The next two datasets are from the UCI Categorical Machine Learning Repository \(^{24}\): Mushroom, Zoo. For these two, a hyperedge is created by all data points which have the same value of categorical features. We follow the same setting for 20Newsgroups, Mushroom, Zoo in \(^{25}\). Other two are computer vision/graphics datasets: Princeton CAD ModelNet40 \(^{35}\) and National Taiwan University (NTU) 3D dataset \(^{36}\). We follow the same setting from \(^{29}\): 80\% of the data is used to train our model and the remaining 20\% as test. The construction of hyperedges is by MVCNN features with computer vision/graphics datasets: Princeton CAD ModelNet40 \(^{35}\) and National Taiwan University (NTU) 3D dataset \(^{36}\). We follow the same setting for 20Newsgroups, Mushroom, Zoo in \(^{25}\). Other two are computer vision/graphics datasets: Princeton CAD ModelNet40 \(^{35}\) and National Taiwan University (NTU) 3D dataset \(^{36}\). We follow the same setting from \(^{29}\): 80\% of the data is used to train our model and the remaining 20\% as test. The construction of hyperedges is by MVCNN features with computer vision/graphics datasets: Princeton CAD ModelNet40 \(^{35}\) and National Taiwan University (NTU) 3D dataset \(^{36}\).

**Baselines.** In this experiment, we carefully select baselines to compare with our LE +GCN. Logistic Regression (LR), works as a standard baseline, which only uses independent feature information. We apply GCN on clique/star expansions to be another two baselines. H-NCut \(^3\), equivalent to iH-NCut \(^6\) with uniform hyperedge cost, is a generalized spectral method for hypergraph partition. Line Hypergraph Convolution Network (LHCN) \(^{33}\), Hyper-Conv \(^{11}\), Hypergraph Neural Network (HGNN) \(^9\) and HyperGCN \(^{10}\) are four state-of-the-art graph-based hypergraph learning methods.

**Analysis.** As shown in Table 3, overall our models beat SOTA methods on all datasets consistently. Basically, every model works better than LR, which means transductive feature sharing helps in the prediction. The performance of traditional clique/star-GCN is not as good as SOTA graph-based baselines. H-Neut method depends on linear matrix factorization and it also cannot beat graph convolution methods, which are more robust and effective with non-linearity. The remaining four are all graph based deep learning methods, and in essence, they only utilize vertex functions on the flattened graph. These algorithms operate more quickly than our LE+GCN, but are much less effective in terms of learning representation.

In sum, current hypergraph deep learning operators are defined on a flattened topology (identical to clique expansion) with the specially designed edge weights. In Table 3 we calculate the edges and density for that topology, denoted as Exp. edge and Exp. density, and find that the scale of line expansion is within 5 times of the flattened topology, except for Zoo (flattened topology is a complete graph).
It is also interesting that for most of the datasets, the density of the line expansion graphs is less than $\frac{1}{2}$ of the flattened graph, especially for ModelNet40 and NTU2012, where the factor is about $\frac{1}{5}$. For each dataset, the training time of our method is within 3 times of these deep graph-based learning algorithms, which we think is acceptable when considering our state-of-the-art performance.

6 Discussion and Conclusion

In this paper, we proposed a novel hypergraph representation, Line Expansion (LE), which is able to utilize higher-order relations for message passing, without loss of any structure information. With line expansion and four entity projectors, we customize graph convolution and develop a novel transductive learning method, LE+GCN, for hypergraphs. We conduct extensive experiments on five real-world hypergraphs and show that LE+GCN can beat SOTA by a significant margin.

A possible future direction is to exploit the symmetry and apply LE for edge learning in complex graphs using node classification algorithms. Another interesting extension is to extend line expansion to directed graphs, where the relation between two nodes are not mutual.

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A Proof of Theorem \(\textbf{[2]}\)

Statement of the Theorem. For a hypergraph, its line expansion \(G_t\) is equivalent to the line graph of its star expansion \(L(G_s)\), where \(L(\cdot)\) is a line graph notation from graph theory.

We first state the notation: hypergraph (according to Definition 3).

We state the definition: hypergraph (same vertex or hyperedge in \(G\)). Theorem 4. (Whitney Graph Isomorphism Theorem.) Two connected graphs are isomorphic if and only if their line graphs are isomorphic. To prove the bijectivity of mapping \(\phi\) to \(G\).

To sum up, the line expansion \(G\) of its star expansion \(L\) is equivalent to \(L(G)\) (leaves and hyperedges on the right with edges from \(G\)).

A proof of Theorem \(\textbf{[2]}\) will be based on the following three definitions.

Definition 1. (Line Graph.) Given a graph \(G\), its line graph \(L(G)\) is a graph such that each vertex of \(L(G)\) represents an edge of \(G\); and two vertices of \(L(G)\) are adjacent if and only if their corresponding edges share a common endpoint (“are incident”) in \(G\).

Definition 2. (Star Expansion of Hypergraph.) Given a hypergraph \(G_H\), its star expansion \(G_s\) is a graph such that the vertex set consist of both vertices and hyperedges in \(G_H\) and the edges are defined on their incident relations of \(G_H\).

Definition 3. (Line Expansion of Hypergraph.) Given a hypergraph \(G_H\), its line expansion \(G_t\) is a graph such that each vertex of \(G_t\) represents an incident vertex-hyperedge pair; and two vertices of \(G_t\) are adjacent if and only if they share a common vertex or hyperedge in \(G_H\).

Proof. First, it is easy to conclude that the star expansion \(G_s\) of the hypergraph \(G_H\) is equivalent to the bipartite representation \(G_b\). According to Definition \(\textbf{[2]}\) when we re-range the nodes in \(G_s\), listing vertices on the left and hyperedges on the right with edges from \(G_s\), then the remaining structure is identical to \(G_b\), shown in Fig. 3 (right).

Second, when we view the bipartite representation \(G_b\) as a graph and take its line graph \(L(G_b)\), then according to Definition \(\textbf{[2]}\) the nodes of the resulting structure will be an incident vertex-hyperedge pair, and the nodes are adjacent if and only if their corresponding pair share a common endpoint (same vertex or hyperedge in \(G_H\)). This essentially constructs the same graph as line expansion (according to Definition \(\textbf{[3]}\)).

To sum up, the line expansion \(G_t\) of \(G_H\) is equivalent to the line graph of bipartite representation \(L(G_b)\), which is also equivalent to line graph of its star expansion \(L(G_s)\). \(\square\)

B Proof of Theorem \(\textbf{[1]}\)

Statement of the Theorem. Under the construction, for a hypergraph \(G_H\) and its line expansion \(G_t\) the mapping \(f\) from hypergraph to line expansion (i.e., \(f : G_H \rightarrow G_t\)) is bijective.

To prove the bijectivity of mapping \(\phi : G_H \rightarrow G_t\), we present a graph isomorphism theorem \(\textbf{[37]}\) below. Node that the bipartite representation of hypergraph from \(G_H \rightarrow G_b\) is an one-to-one mapping. \(G_b\) is a graph with heterogeneous nodes. In the following, we will use \(G_b\) to present \(G_H\).

Theorem 4. (Whitney Graph Isomorphism Theorem.) Two connected graphs are isomorphic if and only if their line graphs are isomorphic, with a single exception: \(K_3\), the complete graph on three vertices, and the complete bipartite graph \(K_{1,3}\), which are not isomorphic but both have \(K_3\) as their line graph.

Definition 4. (Maximum Independent set.) An independent set is a set of vertices in a graph, no two of which are adjacent. A maximum independent set is an independent set of largest possible size for a given graph \(G\).
Figure 4: The Exception of Whitney’s Theorem

Proof. For the bipartite representation of the hypergraph, it could be unconnected when parts of the vertices are only incident to parts of the hyperedges. In that case, we could consider it as a union of several disjoint connected components and prove them one by one. So we mainly discuss the case that $G_b$ is connected.

The proof consists of three parts. First, we show that for the class of bipartite graphs, Theorem 4 holds without exception. Second, we will show how to construct a line expansion $G_l$ from the bipartite representation $G_b$. Third, we show how to recover the bipartite graph $G_b$ from $G_l$.

First, for the exception in Whitney’s theorem, it is obvious that $K_3$ (in Fig. 4) cannot be the bipartite representation of any hypergraph. Therefore, for bipartite graphs, Theorem 4 holds without exception.

(Injectivity) Second, according to Definition 1, the line expansion $G_l$ of the hypergraph is equivalent to line graph of star expansion $G_s$, which is the line graph of bipartite representation $G_b$, i.e., $L(G_b)$. Also, Theorem 4 guarantees that the topology of $L(G_b)$ is unique. The actual construction is given by Definition 1 or Definition 3.

(Surjectivity) Third, given a line graph topology (of a bipartite graph), we know from Theorem 4 immediately that the original bipartite structure is unique. We now provide a construction from $G_l$ to $G_b$. Given a line graph structure, we first find a maximum independent set (in Definition 4) and color them in red (shown in Fig. 5 (a)). [38] proves that it could be found in polynomial time.

Since every vertex and hyperedge from $G_b$ spans a clique in $L(G_b)$. Let us think about the node in this topology, it is potentially a vertex-hyperedge pair in the original hypergraph. Therefore, each node $(v, e)$ must be connected to exactly two cliques: one spanned by vertex $v$ and one spanned by hyperedge $e$. Essentially, we try to project these cliques back to original vertex or hyperedges in $G_b$.

In fact, for each colored node (three in Fig. 5 (a)), we choose one of two cliques connected to it so as to make sure: i) the selected cliques have no intersections (there is only two choices. In this case, choose cliques with 1 on their edges or cliques with 0 on their edges) and ii) the set of cliques cover all nodes in the topology, shown in Fig. 5 (b).
on edges or the set of cliques with 0 on edges that satisfies i) and ii), guaranteed by Definition 1. Conceptually, due to the bipartite nature, one set will be the cliques spanned by original hyperedges and another set will be the cliques spanned by original vertices. Either will work for us. Note that the set of cliques with 0 on edges also includes two size-1 clique, denoted as \( v_4 \) and \( v_5 \) in Fig. 5(b). They seem to only connect to one clique with 1 on edges, i.e., \( e_3 \) clique, however, they are actually size-1 cliques spanned by original vertices which belongs to only one hyperedge in \( G_b \).

To construction of the bipartite representation \( G_b \) is as follows: essentially each clique in the given topology will be either a vertex or a hyperedge in \( G_b \). Suppose we have choose the set of cliques with 1 on edges, we transform each selected clique as a hyperedge of \( G_b \). The vertex set is created two-folded: i) a clique with 0 on its edges is a vertex in \( G_b \) (in this case, we have three size-2 cliques with 0 on their edges, i.e., \( v_1, v_2, v_3 \)), and the vertex will be connected to the according hyperedges. For example, in Fig. 5(c), \( v_2 \) will connected to \( e_1 \) and \( e_2 \) in \( G_b \), because \( v_2 \) represents a size-2 clique with 0 on its edges, \( e_1 \) and \( e_2 \) represent two selected cliques, and \( v_2 \) clique is connected to both \( e_1 \) clique and \( e_2 \) clique in this topology; ii) nodes only connected to one clique with 1 on their edges are also designated to be vertices in \( G_b \), and we connect them to the represented hyperedge. For example, two nodes denoted as \( v_4 \) and \( v_5 \) in Fig. 5(b). They are indeed two size-1 clique with 0 on their edges (it is not so obvious because size-1 clique has no edge). Indeed, these size-1 cliques are spanned by vertices in original \( G_b \) where they only connect to one hyperedge.

So far, we have reconstructed the bipartite representation \( G_b \) in Fig. 5(c) from a given line graph structure in Fig. 5(a). For those unconnected bipartite representation, we do the same reconstruction for each connected components. Thus, we conclude the bijecivity of line expansion. □

### C Proof of Observation 1

#### Statement of the Observation.
Let \( H \) be the incidence matrix of a hypergraph \( G_H \). \( D_v \) and \( D_e \) are the vertex and hyperedge degree matrices. Let \( P_v \) and \( P_e \) be the vertex and edge projection matrix, respectively. \( A_l \) is the adjacency matrix of line expansion \( G_l \). Let \( H_r = [P_v, P_e] \in \{0, 1\}^{V_l \times (|V|+|E|)} \), it satisfies the following equations,

\[
H_r^\top H_r = \begin{bmatrix} D_v & H \\ H^\top & D_e \end{bmatrix},
\]

(11)

\[
H_r H_r^\top = 2I + A_l.
\]

(12)

The detailed meaning of these matrices is shown in notation Table 4. Let us provide an example to give more sense. For the hypergraph shown in Fig. 5, we list the matrices below. It is easy to verify that they satisfy Eqn. (11) and Eqn. (12).

\[
H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[ D_v = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ D_e = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \]

\[ P_v^\top = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ P_e^\top = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \]

\[ H_r^\top = \begin{bmatrix} P_v^\top \\ P_e^\top \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ A_l = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \]

**Proof.** For Eqn. (11),

\[ H_r^\top H_r = \begin{bmatrix} P_v^\top \\ P_e^\top \end{bmatrix} [P_v P_e] = \begin{bmatrix} P_v^\top P_v \\ P_e^\top P_e \end{bmatrix} \begin{bmatrix} P_v P_e \end{bmatrix} = \begin{bmatrix} D_v & H \\ H^\top & D_e \end{bmatrix}, \]

where the last equation is easy to verify since i) \( P_v^\top P_v \) implies the vertex degree matrix, which is \( D_v \). ii) \( P_e^\top P_e \) implies the hyperedge degree matrix, which is \( D_e \); iii) \( P_v^\top P_e \) implies the vertex-hyperedge incidence, which is \( H \).

For Eqn. (12), each row of \( H_r \) is a \( 0 - 1 \) vector of size \(|V| + |E|\) with each dimension indicating a vertex or a hyperedge. Therefore, the vector has exactly two 1s, which is due to that a line node contains exactly one vertex and one hyperedge.

For the \((i, j)\)-th entry of \( H_r H_r^\top \), it is calculated by the dot product of row \( i \) (line node \( i \)) and row \( j \) (line node \( j \)) of \( H_r \). If \( i = j \), then this entry will get 2 (dot product of the same \( 0 - 1 \) vector with two 1s). If \( i \neq j \), the result will be 0 if line node \( i \) and line node \( j \) has no common vertex or hyperedge and be 1 if they have either common vertex or hyperedge (the corresponding dimension gives 1 and 0 for other dimensions, summing to 1). This is defined by Definition 3. In sum, \( H_r H_r^\top \) is equal to the adjacency \( A_l \) with 2-order self-loops, quantitatively,

\[ H_r H_r^\top = 2I + A_l. \]

\[ \square \]

**D Proof of Theorem 3**

In this section, we show that our proposed line expansion is powerful in that it unifies clique and star expansions, as well as simple graph adjacency cases.
D.1 Clique Expansion and Star Expansion

Given a hypergraph $G_H = (V, E)$, consider the clique expansion $G_c = (V, E_c)$. For each pair $(u, v) \in E_c$, 

$$A_c(u, v) = \frac{w_c(u, v)}{\sqrt{d_c(u) d_c(v)}},$$

where in standard clique expansion, we have,

$$w_c(u, v) = \sum h(u, e) h(v, e),$$

$$d_c(u) = \sum h(u, e) \sum_{v \in E \setminus \{u\}} w_c(u, v),$$

$$= \sum h(u, e)(\delta(e) - 1).$$

For the same hypergraph $G_H = (V, E)$, star expansion gives $G_s = (V_s, E_s)$. We adopt adjacency formulation from [4], formally,

$$A_s(u, v) = \sum_{e \in E} \frac{h(u, e) h(v, e)}{\delta(e)^2 \sqrt{\sum_e h(u, e) \sqrt{\sum_e h(v, e)}}}. \quad (20)$$

D.2 Line Expansion

To analyze the message passing on line expansion, we begin by introducing some notations. Let us use $h^{(k)}_{v, e}$ (in short, $h^{(k)}_{v, e}$) to denote the representation of line node $(v, e) \in V_l$ at the $k$-th layer. The convolution operator on line expansion, in Eqn. (7), can be presented,

$$h^{k+1}_{v, e} = \frac{w_v \sum_{e'} h^{k}_{v, e'} + w_u \sum_{e'} h^{k}_{u, e'}}{w_v(d(v) - 1) + w_u(\delta(e) - 1)}. \quad (21)$$

We augment Eqn. (21) by applying 2-order self-loops (mentioned in Section 4.1), and it yields,

$$h^{k+1}_{v, e} = \frac{w_v \sum_{e'} h^{k}_{v, e'} + w_u \sum_{e'} h^{k}_{u, e'} + w_u d(v) + w_v (\delta(e))}{w_u d(v) + w_v(\delta(e))}. \quad (22)$$

It is hard and unfair to directly compare the proposed algorithm with clique/star expansions, since our graph operator is not defined on hypergraph vertices. Thus, we calculate the expected representation for vertex $u$ denoted as $x^{k}_{u}$, i.e., aggregating line node representations by back-projector $P^v$, 

$$x^{k+1}_{u} = \frac{\sum_{e} h(u, e) \frac{1}{\delta(e)} \frac{w_v \sum_{e'} x^{k}_{e'} + w_u \sum_{e'} x^{k}_{e'} + w_v \sum_{e'} x^{k}_{e'} + w_u d(u)}{w_v \delta(e)}}{\sum_{e} h(u, e) \frac{1}{\delta(e)}}. \quad (23)$$

After organizing the formula, we calculate that for each hypergraph vertex pair $(u, v) \in V \times V$, they are adjacent by,

$$A_l(u, v) = \frac{w_u h(u, e) h(v, e)}{\sum_{e} h(u, e) \frac{1}{\delta(e)}}, \quad (24)$$

or by the following form after symmetric re-normalization,

$$A_l(u, v) = \frac{w_u h(u, e) h(v, e)}{\sqrt{\sum_{e} h(u, e) \frac{1}{\delta(e)}} \sqrt{\sum_{e} h(v, e) \frac{1}{\delta(e)}}}. \quad (25)$$

D.3 Analysis of Unification

We already show that line expansion enables to exchange information beyond 0-chain and thus can utilize the higher order relation. In this subsection, we illustrate why line expansion is more powerful at the actual message passing.
Unifying Star and Clique Expansion. We start by considering the clique expansion graph with weighting function,

\[ w_c(u, v) = \sum_{e \in E} \frac{h(u, e)h(v, e)}{(\delta(e) - 1)^2}. \] (26)

Note that this is equivalent to vanish Eqn. (18) by a factor of \(\frac{1}{(\delta(e) - 1)^2}\). We plug the value into Eqn. (19), then adjacency of clique expansion transforms into,

\[ A_c(u, v) = \frac{\sum_e h(u, e)h(v, e)}{\sqrt{\sum_e h(u, e)\frac{1}{(\delta(e) - 1)^2} \sqrt{\sum_e h(v, e)\frac{1}{(\delta(e) - 1)^2}}}}. \] (27)

Note that when we set \(w_c = 0\) (no message passing from hyperedge-similar neighbors). The higher-order relation of line expansion, in Eqn. (25) degrades into,

\[ A_l(u, v) = \frac{\sum_e h(u, e)h(v, e)}{2\sqrt{d(u)d(v)}}. \] (28)

The Eqn. (28) is exactly the adjacency of star expansion in Eqn. (20), and Eqn. (27) (adjacency of clique expansion) is the 1-order self-loop form of the degraded line expansion.

Unifying Simple Graph. The convolution operator \([12]\) on a simple graph can be briefly present,

\[ A(u, v) = \frac{\sum_e h(u, e)h(v, e)}{\sqrt{d(u)d(v)}}. \] (29)

A graph could be regarded as a 2-order hypergraph, where hyperedge \(e\) has exactly two vertices, i.e., \(\delta(e) = 2\) and each pair of vertices \((u, v) \in V \times V\) has at most one common edge. Plugging the value into Eqn. (28), and it yields,

\[ A_l(u, v) = \frac{\sum_e h(u, e)h(v, e)}{2\sqrt{d(u)d(v)}}. \] (30)

Comparing Eqn. (29) and (30), the only difference is a scaling factor 2, which could be absorbed into filter \(\Theta\).

To sum up, we prove that clique and star expansions and simple graph adjacency could all be unified as a special class of line expansion, where there is no information sharing between hyperedge-similar neighbors.

E Simple Citation Network Classification

Since simple graphs are a special case of hypergraphs, we apply line expansion to three citation networks. Cora dataset has 2,708 vertices and 5.2% of them have class labels. Nodes contain sparse bag-of-words feature vectors and are connected by a list of citation links. Another two datasets, Citeseer and Pubmed, are constructed similarly \([39]\). Basic statistics are reported in Table 5.

| Dataset | Nodes | Edges | Features | Class | Label rate |
|---------|-------|-------|----------|-------|------------|
| Cora    | 2,708 | 5,429 | 1,433    | 7     | 0.052      |
| Citeseer| 3,327 | 4,732 | 4,732    | 6     | 0.036      |
| Pubmed  | 19,717| 44,338| 500      | 3     | 0.003      |

Table 5: Overview of Citation Network Statistics

We consider the popular deep end-to-end learning methods GCN \([12]\) and well-known graph representation methods SpectralClustering (SC) \([21]\), Node2Vec \([22]\), DeepWalk \([27]\) and LINE \([28]\). We follow the same experimental setting from \([40]\). Note that, GCNs could input both features and the network, whereas other methods only use structural information.

Cora Dataset has 2,708 vertices and 5.2% of them have class labels. Each node contains sparse bag-of-words feature vector, and they are connected by a list of citation links. Likewise, Citeseer has 3,327 vertices with 3.6% labeled data and Pubmed is a little bit larger with 19,717 vertices and even smaller label rate 0.3%.
| Model          | Cora   | Citeseer | Pubmed |
|----------------|--------|----------|--------|
| **Numbers from literature:** |        |          |        |
| Planetoid      | 75.7   | 64.7     | 77.2   |
| DeepWalk       | 67.2   | 43.2     | 65.3   |
| GCN            | 81.5   | 70.3     | 79.0   |
| **Our implementations:** |        |          |        |
| SC             | 53.3 ± 0.2 | 50.8 ± 0.7 | 55.2 ± 0.4 |
| Planetoid      | 75.0 ± 0.9 | 64.0 ± 1.3 | 76.7 ± 0.6 |
| ICA            | 74.5 ± 0.6 | 63.4 ± 0.6 | 72.9 ± 1.0 |
| Node2Vec       | 66.3 ± 0.3 | 46.2 ± 0.7 | 71.6 ± 0.5 |
| DeepWalk       | 62.8 ± 0.6 | 45.7 ± 1.2 | 63.4 ± 0.4 |
| LINE           | 27.7 ± 1.1 | 30.8 ± 0.2 | 53.5 ± 0.8 |
| GCN            | 81.5 ± 0.7 (3s) | 70.5 ± 0.3 (9s) | 78.2 ± 0.6 (12s) |
| **Our variants:** |        |          |        |
| LE+SC          | 56.9 ± 0.2 | 50.7 ± 0.2 | 71.9 ± 0.7 |
| LE+Planetoid   | 76.6 ± 0.4 | 66.0 ± 0.7 | 77.0 ± 0.2 |
| LE+ICA         | 72.7 ± 0.4 | 68.6 ± 0.5 | 73.3 ± 0.7 |
| LE+Node2Vec    | 74.3 ± 0.4 | 46.2 ± 0.1 | 74.3 ± 0.4 |
| LE+DeepWalk    | 68.3 ± 0.1 | 50.4 ± 0.4 | 68.0 ± 0.8 |
| LE+LINE        | 51.7 ± 0.2 | 34.9 ± 0.5 | 57.5 ± 0.3 |
| LE+GCN         | 82.6 ± 0.7 (3s) | 70.4 ± 0.3 (11s) | 78.7 ± 0.4 (31s) |

Table 6: Results for Citation Network Node Classification (%)

![Figure 6: Ablation Study on \(w_e\) and \(w_v\)](image)

**Analysis.** The results of transductive node classification for citation networks are shown in Table 6. The experiment clearly demonstrates that LE shows comparable results in graph node classification tasks. Specifically for those non-end-to-end methods, they consistently outperform the original algorithm on simple graphs. LE enriches the plain structure by providing a finer-grained structure and makes nodes edge-dependent, which might explain the improvement in structure-based non-end-to-end models.

End-to-end GCNs can reach a much higher accuracy compared to other baselines. We observe that LE+GCN tie with original GCN on the three datasets. However, the expansion of original network indeed provides lower variance consistently and contributes to more robust models.

F Ablation Experiments on \(w_e\) and \(w_v\)

We provide further ablation studies on \(w_v\) and \(w_e\). Since only the fraction \(\frac{w_v}{w_e}\) matters, we symmetrically choose \(\frac{w_v}{w_e} = 0, 0.1, 0.2, 0.5, 1, 2, 5, 10, \infty\) and conduct the experiments. The following figure could provide some intuitions about how to select proper \(w_v\) and \(w_e\) in real hypergraph tasks.
From Fig. 6, we can conclude that these hypergraphs have different sensitivities and preferences for \( w_e \) and \( w_v \). However, we do find all the curves follow a first-rise-and-then-down pattern, which means the maximum value is always in the mid, i.e., it is always better to convolve information from both edges and vertices. Specifically, we find that for those hypergraphs with fewer hyperedges, e.g., 20News and Mushroom, the peak appears before \( \frac{w_v}{w_e} = 1 \), and for those hypergraphs with sufficient hyperedges, e.g., ModelNet40 and NTU2012, the peak appears after \( \frac{w_v}{w_e} = 1 \). Therefore, one empirical guide for practical usage is to set smaller \( w_e \) when there are fewer hyperedges and set larger \( w_v \), vice versa. After all, using the binary version (i.e., \( w_e = w_v = 1 \)) seems to be a simple and effective choice.