LIMITING BEHAVIOR OF DYNAMICS FOR STOCHASTIC
REACTION-DIFFUSION EQUATIONS WITH ADDITIVE NOISE
ON THIN DOMAINS

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Abstract. In this paper, we study the limiting behavior of dynamics for stochastic reaction-diffusion equations driven by an additive noise and a deterministic non-autonomous forcing on an $(n + 1)$-dimensional thin region when it collapses into an $n$-dimensional region. We first established the existence of attractors and their properties for these equations on $(n + 1)$-dimensional thin domains. We then show that these attractors converge to the random attractor of the limit equation under the usual semi-distance as the thinness goes to zero.

1. Introduction. This work is a continuation of our previous work [24] on the limit behavior of long term dynamics of stochastic partial differential equation on a spatial $(n + 1)$-dimensional thin domain when it collapses to an $n$-dimensional domain.

Let $Q \subset \mathbb{R}^n$ be a bounded $C^2$-domain and $\Omega_\varepsilon \subset \mathbb{R}^{n+1}$ be the domain

$$\Omega_\varepsilon = \{ x = (x^*, x_{n+1}) \mid x^* = (x_1, \ldots, x_n) \in Q \text{ and } 0 < x_{n+1} < \varepsilon g(x^*) \},$$

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where \(0 < \varepsilon \leq 1\) and \(g \in C^2(\overline{\mathcal{Q}}, (0, +\infty))\) which implies that there exist positive constants \(\gamma_1\) and \(\gamma_2\) such that
\[
\gamma_1 \leq g(x^*) \leq \gamma_2, \quad \forall x^* \in \overline{\mathcal{Q}}.
\]

Throughout this paper, we write \(\mathcal{O} = \mathcal{Q} \times (0, 1)\) and \(\tilde{\mathcal{O}} = \mathcal{Q} \times (0, \gamma_2)\). We consider the following stochastic non-autonomous reaction-diffusion equation with additive noise defined on the thin domain \(\mathcal{O}_\varepsilon\):
\[
\begin{cases}
\frac{d\tilde{u}^\varepsilon}{dt} - \Delta \tilde{u}^\varepsilon dt = (F(t, x, \tilde{u}^\varepsilon) + G(t, x)) dt + h(x) dw, & x \in \mathcal{O}_\varepsilon, \ t > \tau, \\
\frac{\partial \tilde{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, & x \in \partial \mathcal{O}_\varepsilon,
\end{cases}
\]
with initial condition
\[
\tilde{u}^\varepsilon(\tau, x) = \tilde{u}_\tau^\varepsilon(x), \quad x \in \mathcal{O}_\varepsilon,
\]
where \(\tau\) is the initial time, \(\nu_\varepsilon\) is the unit outward normal vector on \(\partial \mathcal{O}_\varepsilon\), \(F\) is a nonlinear function defined on \(\mathbb{R} \times \tilde{\mathcal{O}} \times \mathbb{R}\), \(G\) is a function defined on \(\mathbb{R} \times \tilde{\mathcal{O}}\), \(h\) is a given function defined on \(\tilde{\mathcal{O}}\), and \(w\) is a two-sided real-valued Wiener process on a probability space.

We will study the limiting behavior of system (2)-(3) when the \((n+1)\)-dimensional thin domain \(\mathcal{O}_\varepsilon\) degenerates into an \(n\)-dimensional domain \(\mathcal{Q}\) as \(\varepsilon \to 0\). In particular, we will construct a stochastic equation defined on the lower dimensional spatial domain \(\mathcal{Q}\) which captures the essential dynamics of the original higher dimensional stochastic equations. To justify our limiting system, we will prove not only the convergence of solutions but also the upper semi-continuity of tempered pullback random attractors \(\mathcal{A}_\varepsilon(\tau, \omega)\) of (2)-(3) as \(\varepsilon \to 0\), namely, the attractor \(\mathcal{A}_\varepsilon(\tau, \omega)\) converges to the tempered pullback random attractor \(\mathcal{A}_0(\tau, \omega)\) of the limit equation under the usual semi-distance as \(\varepsilon \to 0\).

As we will show later, the limiting dynamics of system (2)-(3) is determined by the following stochastic equation defined on \(\mathcal{Q}\):
\[
\begin{cases}
\frac{du^0}{dt} - \frac{1}{g} \sum_{i=1}^n \left( gv_{0, y_i}^0 \right) y_i dt = (F(t, y^*, 0, u^0) + G(t, y^*, 0)) dt \\
\quad + h(y^*, 0) dw, & y^* = (y_1, \ldots, y_n) \in \mathcal{Q}, \ t > \tau, \\
\frac{\partial u^0}{\partial \nu_0} = 0, & y^* \in \partial \mathcal{Q},
\end{cases}
\]
with the initial condition
\[
u_0^0(\tau, y^*) = u_\tau^0(y^*), \quad y^* \in \mathcal{Q},
\]
where \(\nu_0\) is the unit outward normal on \(\partial \mathcal{Q}\) and \(u_{y_i}^0\) stands for \(\frac{\partial u^0}{\partial y_i}\).

For the deterministic version of (2), Hale and Raugel [20] initiated a program to study the limiting behavior of long term dynamics for deterministic dissipative systems on thin domains and obtained results on upper semi-continuity of attractors. Their results have been extended to various problems arising in applications such as solid mechanics (thin rods, plates, shells), fluid dynamics (lubrication, meteorology problems, ocean dynamics), see for instance, [1, 3, 4, 5, 6, 14, 17, 21, 22, 26, 27, 28, 29].

The need for studying random dynamical systems was pointed out by Ulam and von Neumann [33] in 1945. It has flourished since 1980s due to the discovery that stochastic ordinary differential equations generate random dynamical systems through the efforts of Arnold, Harris, Elworthy, Baxendale, Bismut, Ikeda, Kunita, Watanabe, and others. The study of global random attractors dates back to [30]. The concept of pullback attractor for autonomous stochastic systems was introduced in [16, 18, 31] as an extension of the global attractor for deterministic equations in
Due to the unbounded fluctuations in the systems caused by the white noise, the concept of pullback global random attractor was introduced to capture the essential dynamics with possibly extremely wide fluctuations. This is significantly different from the deterministic case. There is an extensive literature on this subject, see for instance, [8, 10, 11, 15, 16, 18, 23, 34, 35, 36]. It is worth mentioning that the ergodicity of stochastic 3D Navier-Stokes equations in a thin domain was recently investigated in [12, 13], and the synchronization of semilinear parabolic stochastic equations in thin bounded tubular domains was studied in [9].

The outline of this paper is as follows. In the next section, we define a continuous cocycle in $L^2(O)$ for the stochastic equation defined on the fixed domain $O$. We also discuss the continuous cocycle in $L^2(\bar{Q})$ generated by the stochastic equation (4). Furthermore, we present the functional setting and the abstract formulation of the problem and give the main results of this paper. Section 3 is devoted to uniform estimates of the solutions for both system (2)-(3) and (4)-(5). The existence and uniqueness of tempered attractors for the stochastic equations are presented in section 4, and the upper semi-continuity of random attractors is finally proved in section 5.

2. Main results. In this section, we reformulate systems (2) and (4) and present our main results. Given $\tau \in \mathbb{R}$, consider the following stochastic equation driven by white noise and non-autonomous deterministic terms:

$$
\begin{cases}
\dot{\hat{u}} - \Delta \hat{u} dt = (F(t,x,\hat{u}) + G(t,x)) dt + h(x) dw, & x = (x^*, x_{n+1}) \in O_\varepsilon, \ t > \tau, \\
\frac{\partial \hat{u}}{\partial \nu_\varepsilon} = 0, & x \in \partial O_\varepsilon,
\end{cases}
$$

with the initial condition

$$
\hat{u}(\tau,x) = \hat{u}_\tau(x), \ x \in O_\varepsilon,
$$

where $\nu_\varepsilon$ is the unit outward normal vector on $\partial O_\varepsilon$, $G : \mathbb{R} \times \bar{O} \to \mathbb{R}$ belongs to $L^2_{loc}(\mathbb{R}, L^\infty(\bar{O}))$, $h \in C^2(\bar{Q} \times [0, \gamma_2])$ satisfies the boundary conditions, $w$ is a two-sided real-valued Wiener process on a probability space, $F$ is a nonlinear function satisfying the following conditions: for all $x \in \bar{O}$ and $t,s \in \mathbb{R}$,

$$
F(t,x,s) s \leq -\lambda_1 |s|^p + \varphi_1(t,x),
$$

$$
|F(t,x,s)| \leq \lambda_2 |s|^{p-1} + \varphi_2(t,x),
$$

$$
\frac{\partial F(t,x,s)}{\partial s} \leq \gamma,
$$

$$
\left| \frac{\partial F(t,x,s)}{\partial x} \right| \leq \varphi_3(t,x),
$$

where $p \geq 2$, $\lambda_1$, $\lambda_2$ and $\gamma$ are positive constants, $\varphi_1 \in L^1_{loc}(\mathbb{R}, L^\infty(\bar{O}))$, and $\varphi_2, \varphi_3 \in L^2_{loc}(\mathbb{R}, L^\infty(\bar{O}))$.

We now reformulate problem (6)-(7) by following the process of [24]. Let $\lambda \in (0, \lambda_1)$ be a fixed number and denote by

$$
f(t,x,s) = F(t,x,s) + \lambda s
$$

for all $x \in \bar{O}$ and $t,s \in \mathbb{R}$. Then it follows from (8)-(10) that there exist positive numbers $\alpha_1, \alpha_2, \beta, c_1$ and $c_2$ such that
Laplace operator in $\mathbb{R}^d$ where we denote by $\hat{u}$ By [20, 25] the gradient operator and the Laplace operator in $x$ belong to $\mathcal{O}$ and $t \in \mathbb{R}$.

Substituting (12) into (6) we get for $t > \tau,$

$$\frac{d\hat{u}^\varepsilon}{dt} - (\Delta \hat{u}^\varepsilon - \lambda \hat{u}^\varepsilon) dt = (f(t, x, \hat{u}^\varepsilon) + G(t, x)) dt + h(x) dw, \quad x \in \mathcal{O},$$

with the initial condition

$$\hat{u}^\varepsilon(\tau, x) = \hat{u}^\varepsilon(x), \quad x \in \mathcal{O}.$$

To transform the $\varepsilon$-dependent domain $\mathcal{O}$ into the fixed domain $\mathcal{O}$, we define $T_\varepsilon : \mathcal{O} \to \mathcal{O}$ by $T_\varepsilon(x^*, x_{n+1}) = \left( x^*, \frac{x_{n+1}}{\varepsilon g(x^*)} \right)$ for $x = (x^*, x_{n+1}) \in \mathcal{O}$. Let $y = (y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1}).$ We obtain

$$x^* = y^*, \quad x_{n+1} = \varepsilon g(y^*) y_{n+1}.$$

The Jacobian matrix of $T_\varepsilon$ is

$$J = \frac{\partial (y_1, \ldots, y_{n+1})}{\partial (x_1, \ldots, x_{n+1})} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & & & & \\
-\frac{y_{n+1}}{g} y_{1} & -\frac{y_{n+1}}{g} y_{2} & \cdots & -\frac{y_{n+1}}{g} y_{n} & 0 \\
\end{pmatrix}.$$

The determinant of $J$ is $|J| = \frac{1}{\varepsilon g(y^*)}.$ Let $J^*$ be the transport of $J$. Then we have

$$JJ^* = \begin{pmatrix}
1 & 0 & \cdots & 0 & -\frac{y_{n+1}}{g} y_{1} \\
0 & 1 & \cdots & 0 & -\frac{y_{n+1}}{g} y_{2} \\
\vdots & & & & \\
-\frac{y_{n+1}}{g} y_{1} & -\frac{y_{n+1}}{g} y_{2} & \cdots & -\frac{y_{n+1}}{g} y_{n} & \sum_{i=1}^{n} \left( \frac{y_{n+1}}{g} y_{i} \right)^2 + \frac{1}{\varepsilon^2 g^2(y^*)} \\
\end{pmatrix}.$$

By [20, 25] the gradient operator and the Laplace operator in $x \in \mathcal{O}$ and in $y \in \mathcal{O}$ are related by

$$\nabla_x \hat{u}(x) = J^* \nabla_y u(y) \quad \text{and} \quad \Delta_x \hat{u}(x) = |J| \text{div}_y \left( |J|^{-1} J^* \nabla_y u(y) \right) = \frac{1}{\varepsilon} \text{div}_y (P_\varepsilon u(y)),$$

where we denote by $\hat{u}(x) = u(y)$, $\nabla_x$ and $\Delta_x$ are the gradient operator and the Laplace operator in $x \in \mathcal{O}$ respectively, $\text{div}_y$ and $\nabla_y$ are the divergence operator and the gradient operator in $y \in \mathcal{O}$ respectively, and $P_\varepsilon$ is the operator given by

$$P_\varepsilon u(y) = \begin{pmatrix}
gu_{u_1} - gu_{y_1} y_{n+1} u_{y_{n+1}} \\
\vdots \\
gu_{u_n} - gu_{y_n} y_{n+1} u_{y_{n+1}} \\
-\sum_{i=1}^{n} y_{n+1} g_{y_i} u_{y_i} + \frac{1}{\varepsilon^2 g^2(1 + \sum_{i=1}^{n} (\varepsilon y_{n+1} g_{y_i})^2)} u_{y_{n+1}} \\
\end{pmatrix}.$$
Given $y = (y^*, y_{n+1}) \in \mathcal{O}$ and $t, s \in \mathbb{R}$, denote by

$$
F_\varepsilon (t, y^*, y_{n+1}, s) = F (t, y^*, \varepsilon g(y^*) y_{n+1}, s), \quad F_0 (t, y^*, s) = F (t, y^*, 0, s),
$$

$$
f_\varepsilon (t, y^*, y_{n+1}, s) = f (t, y^*, \varepsilon g(y^*) y_{n+1}, s), \quad f_0 (t, y^*, s) = f (t, y^*, 0, s),
$$

$$
G_\varepsilon (t, y^*, y_{n+1}) = G (t, y^*, \varepsilon g(y^*) y_{n+1}), \quad G_0 (t, y^*) = G (t, y^*, 0),
$$

and

$$
h_\varepsilon (y^*, y_{n+1}) = h (y^*, \varepsilon g(y^*) y_{n+1}), \quad h_0 (y^*) = h (y^*, 0).
$$

In terms of $y = (y^*, y_{n+1}) \in \mathcal{O}$, system (16)-(17) can be written as, for $t > \tau$,

$$
\begin{align*}
\left\{ \begin{array}{ll}
\frac{du^\varepsilon}{dt} - (\frac{1}{2} \text{div}_y (P_\varepsilon u) - \lambda u^\varepsilon) dt &= (f_\varepsilon (t, y, u^\varepsilon) + G_\varepsilon (t, y)) dt + h_\varepsilon (y) dw, \ y \in \mathcal{O}, \\
P_\varepsilon u^\varepsilon \cdot \nu &= 0, \ y \in \partial \mathcal{O},
\end{array} \right.
\end{align*}
$$

with the initial condition

$$
u^\varepsilon (\tau, y) = \hat{u}^\varepsilon (\tau, y) = \hat{u}^\varepsilon (T_\varepsilon^{-1} (y)), \ y \in \mathcal{O},
$$

where $\nu$ is the unit outward normal vector on $\partial \mathcal{O}$. Note that the boundary condition in (18) follows from the original boundary condition in (16). Indeed, by the transformation $T_\varepsilon$ we find that $\nu_\varepsilon = J^* \nu$. Then by $\nabla_x \hat{u}^\varepsilon (t, x) = J^* \nabla_y u^\varepsilon (t, y)$, we get for $x \in \partial \mathcal{O}_{\varepsilon}$,

$$
\frac{\partial \hat{u}^\varepsilon (t, x)}{\partial \nu_\varepsilon} = \nabla_x \hat{u}^\varepsilon (t, x) \cdot \nu_\varepsilon = (J^* \nabla_y u^\varepsilon (t, y)) \cdot (J^* \nu) = J J^* \nabla_y u^\varepsilon (t, y) \cdot \nu = \frac{1}{g} (P_\varepsilon \hat{u}^\varepsilon (t, y)) \cdot \nu.
$$

Thus the boundary condition $\frac{\partial \hat{u}^\varepsilon (t, x)}{\partial \nu_\varepsilon} = 0$ in (16) yields $P_\varepsilon u^\varepsilon (t, y) \cdot \nu = 0$ as in (18).

We now define an inner product on $L^2(\mathcal{O})$ by

$$
(u, v)_{H_2(\mathcal{O})} = \int_{\mathcal{O}} g uv dy, \quad \text{for all } u, v \in L^2(\mathcal{O})
$$

and use $H_2(\mathcal{O})$ to denote $L^2(\mathcal{O})$ equipped with this inner product. Note that $g$ is continuous on $\mathcal{Q}$ and satisfies (1). We find that $H_2(\mathcal{O})$ is a Hilbert space with norm equivalent to the standard norm of $L^2(\mathcal{O})$.

For $0 < \varepsilon \leq 1$, we define a bilinear form $a_\varepsilon (\cdot, \cdot): H^1 (\mathcal{O}) \times H^1 (\mathcal{O}) \to \mathbb{R}$:

$$
a_\varepsilon (u, v) = (J^* \nabla_y u, J^* \nabla_y v)_{H_2(\mathcal{O})},
$$

where

$$
J^* \nabla_y u = (u_y - \frac{g y_1}{g} y_{n+1} u_{y_{n+1}}, \ldots, u_y - \frac{g y_n}{g} y_{n+1} u_{y_{n+1}}, \frac{1}{\varepsilon^2} u_{y_{n+1}}).
$$

For every $0 < \varepsilon \leq 1$ and $u \in H^1 (\mathcal{O})$ we further define

$$
\|u\|_{H_2^1(\mathcal{O})} = \left( \int_{\mathcal{O}} (|\nabla_y u|^2 + |u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy \right)^{\frac{1}{2}}.
$$

Then we find that there exist positive constants $\varepsilon_0$, $\eta_1$ and $\eta_2$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and $u \in H^1 (\mathcal{O})$,

$$
\eta_1 \int_{\mathcal{O}} (|\nabla_y u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy \leq a_\varepsilon (u, u) \leq \eta_2 \int_{\mathcal{O}} (\nabla_y u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy
$$

and

$$
\eta_1 \|u\|_{H_2^1(\mathcal{O})}^2 \leq a_\varepsilon (u, u) + \|u\|_{L^2(\mathcal{O})}^2 \leq \eta_2 \|u\|_{H_2^1(\mathcal{O})}^2.
$$
Let $A_\epsilon$ be an unbounded operator on $H_0^2(\Omega)$ given by

$$A_\epsilon v = -\frac{1}{\epsilon} \text{div} P_\epsilon v, \quad v \in D(A_\epsilon),$$

where $D(A_\epsilon) = \{ v \in H^2(\Omega), P_\epsilon v \cdot \nu = 0 \text{ on } \partial \Omega \}$. Thus we have

$$a_\epsilon (u, v) = (A_\epsilon u, v)_{H_0^2(\Omega)}, \quad \forall u \in D(A_\epsilon), \quad \forall v \in H^1(\Omega). \quad (24)$$

In terms of $A_\epsilon$, we can reformulate system (18)-(19) as

$$\begin{cases}
\frac{du^\epsilon}{dt} + A_\epsilon u^\epsilon + \lambda u^\epsilon = f_\epsilon (t, y, u^\epsilon) + G_\epsilon (t, y) + h_\epsilon (y) \frac{dw}{dt}, \quad y \in \Omega, \quad t > \tau, \\
u^\epsilon (\tau) = u^\epsilon_\tau.
\end{cases} \quad (25)$$

Similarly, we define an inner product $(\cdot, \cdot)_{H_0^2(\Omega)}$ on $L^2(\Omega)$ by

$$(u, v)_{H_0^2(\Omega)} = \int_\Omega g uv dy^*, \quad \forall u, v \in L^2(\Omega),$$

and use $H_0^2(\Omega)$ to denote $L^2(\Omega)$ equipped with this inner product. Define $a_0 (\cdot, \cdot): H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a_0 (u, v) = \int_\Omega g \nabla u \cdot \nabla v dy^*. \quad (26)$$

Let $A_0$ be an unbounded operator on $H_0^2(\Omega)$ given by

$$A_0 v = -\frac{1}{g} \sum_{i=1}^n (g v_i)_y, \quad v \in D(A_0),$$

where $D(A_0) = \{ v \in H^2(\Omega), \frac{\partial}{\partial y_0} v = 0 \text{ on } \partial \Omega \}$. Note that

$$a_0 (u, v) = (A_0 u, v)_{H_0^2(\Omega)}, \quad \forall u \in D(A_0), \forall v \in H^1(\Omega).$$

In terms of $A_0$, we can write system (4)-(5) as

$$\begin{cases}
\frac{du^0}{dt} + A_0 u^0 + \lambda u^0 = f_0 (t, y^*, u^0) + G_0 (t, y^*) + h_0 (y^*) \frac{dw}{dt}, \quad y^* \in \Omega, \quad t > \tau, \\
u^0 (\tau) = u^0_\tau.
\end{cases} \quad (26)$$

In this paper, we will use the standard probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{ \omega \in C (\mathbb{R}, \mathbb{R}) : \omega (0) = 0 \}, \mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $P$ is the Wiener measure on $(\Omega, \mathcal{F})$. As usual, we use $\{ \theta_t \}_{t \in \mathbb{R}}$ to denote the measure-preserving transformations on $(\Omega, \mathcal{F}, P)$ given by

$$\theta_t \omega (\cdot) = \omega (\cdot + t) - \omega (t), \quad \omega \in \Omega, \quad t \in \mathbb{R}. \quad (27)$$

For our purpose, we need to convert the stochastic equations with additive noise into pathwise deterministic equations with a random parameter. To that end, we consider the one-dimensional Ornstein-Uhlenbeck equation:

$$dz + \lambda z dt = dw (t). \quad (28)$$

One may easily check that a solution to (28) is given by

$$z (\theta_t \omega) = -\lambda \int_{-\infty}^0 e^{\lambda \tau} (\theta_{-\tau} \omega) (\tau) d\tau, \quad t \in \mathbb{R}.$$
tempered. Let \( F_1 \) and \( P_1 \) be the restrictions of \( F \) and \( P \) on \( \tilde{\Omega} \), respectively. We will define a continuous cocycle for problem (18)-(19) over \((\tilde{\Omega}, F_1, P_1, \{ \theta_t \}_{t \in \mathbb{R}})\). For convenience, from now on, we will abuse the notation slightly and write the space \((\tilde{\Omega}, F_1, P_1)\) as \((\Omega, F, P)\). Let \( v^\varepsilon(t) = u^\varepsilon(t) - h_\varepsilon(y)z(\theta_t\omega) \) where \( u^\varepsilon \) is a solution of problem (25). Then \( v^\varepsilon \) satisfies

\[
\begin{align*}
\frac{dv^\varepsilon}{dt} + A\varepsilon v^\varepsilon + \lambda v^\varepsilon &= f_\varepsilon(t, y, v^\varepsilon + h_\varepsilon(y)z(\theta_t\omega)) \\
&\quad + G_\varepsilon(t, y) - A\varepsilon h_\varepsilon(y)z(\theta_t\omega), \quad y \in \mathcal{O}, \ t > \tau,
\end{align*}
\]

(29)

where \( \tau \in \mathbb{R} \). For the pathwise deterministic equation (29), we can show that if \( F \) satisfies (8)-(11), then for every \( \omega \in \Omega, \ \tau \in \mathbb{R} \) and \( v^\varepsilon \in L^2(\mathcal{O}) \), the equation has a unique solution \( v^\varepsilon(t, \tau, \omega, v^\varepsilon) \in \mathcal{C}([\tau, \infty), L^2(\mathcal{O})) \cap L^2_{\text{loc}}((\tau, \infty), H^1(\mathcal{O})) \) with \( v^\varepsilon(t, \tau, \omega, u^\varepsilon_0) = v^\varepsilon_0 \). In addition, the solution \( v^\varepsilon(t, \tau, \omega, v^\varepsilon) \) is \((\mathcal{F}, \mathcal{B}(L^2(\mathcal{O})))\)-measurable in \( \omega \in \Omega \) and continuous in \( v^\varepsilon \) in \( L^2(\mathcal{O}) \). Therefore, the function \( u^\varepsilon(t, \tau, \omega, u^\varepsilon_0) = v^\varepsilon(t, \tau, \omega, u^\varepsilon_0 - h_\varepsilon(y)z(\theta_\tau\omega)) + h_\varepsilon(y)z(\theta_\tau\omega) \) is an \((\mathcal{F}, \mathcal{B}(L^2(\mathcal{O})))\)-measurable solution of problem (25) which is continuous in \( t \geq \tau \) and \( u^\varepsilon \in L^2(\mathcal{O}) \).

We now define a mapping \( \Phi^\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \to L^2(\mathcal{O}) \) by

\[
\Phi^\varepsilon(t, \tau, \omega, u^\varepsilon_0) = u^\varepsilon(t + \tau, \tau, \omega, \theta_\tau\omega, u^\varepsilon_0)
\]

(30)

for all \( (t, \tau, \omega, u^\varepsilon_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \). By the properties of \( u^\varepsilon \), then it is easy to check that \( \Phi^\varepsilon \) is a continuous cocycle on \( L^2(\mathcal{O}) \) over \((\Omega, F, P, \{ \theta_t \}_{t \in \mathbb{R}})\). Define \( R^\varepsilon : L^2(\mathcal{O}) \to L^2(\mathcal{O}) \) by

\[
(R^\varepsilon \hat{u})(y) = \hat{u}(T_{-\tau}^{-1}y), \quad \forall \hat{u} \in L^2(\mathcal{O}).
\]

Then given \( t \in \mathbb{R}^+, \ \tau \in \mathbb{R}, \ \omega \in \Omega \) and \( \hat{u}^\varepsilon \in L^2(\mathcal{O}) \), we can define a continuous cocycle \( \Phi^\varepsilon \) for problem (16)-(17) by the formula

\[
\Phi^\varepsilon(t, \tau, \omega, \hat{u}^\varepsilon) = R^\varepsilon \Phi^\varepsilon(t, \tau, \omega, \hat{u}^\varepsilon),
\]

where \( \Phi^\varepsilon \) is the continuous cocycle for problem (25) on \( L^2(\mathcal{O}) \).

Similarly, by introducing a new variable \( v^0(t) = u^0(t) - h_0(y^*)z(\theta_t\omega) \), we can transform equation (26) into the following random partial differential equation on \( \mathcal{Q} \):

\[
\begin{align*}
\frac{dv^0}{dt} + A_0 v^0 + \lambda v^0 &= f_0(t, y^*, v^0 + h_0(y^*)z(\theta_t\omega)) \\
&\quad + G_0(t, y^*) - A_0 h_0(y^*)z(\theta_t\omega), \quad y^* \in \mathcal{Q}, \ t > \tau,
\end{align*}
\]

(31)

As above, one can prove that system (4)-(5) generates a continuous cocycle in \( L^2(\mathcal{Q}) \) which is denoted by \( \Phi^0(t, \tau, \omega, u^0_0) \) in the sequel.

Hereafter, we set \( X_\varepsilon = L^2(\mathcal{O}_\varepsilon), X_0 = L^2(\mathcal{Q}) \) and \( X_1 = L^2(\mathcal{O}) \). For every \( i = \varepsilon, 0 \) or 1, a family \( B_i = \{ B_i(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) of nonempty subsets of \( X_i \) is called tempered if for every \( c > 0 \), we have:

\[
\lim_{t \to -\infty} e^{ct} \| B_i(\tau + t, \theta_t\omega) \|_{X_i} = 0,
\]

where \( \| B_i \|_{X_i} = \sup_{x \in B_i} \| x \|_{X_i} \). The collection of all families of tempered nonempty subsets of \( X_i \) is denoted by \( D_i \), i.e.,

\[
D_i = \{ B_i = \{ B_i(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : B_i \text{ is tempered in } X_i \}.
\]
In the rest of this paper, we first prove the existence of $D_\varepsilon$-pullback attractor $\hat{A}_\varepsilon$ and $D_0$-pullback attractor $A_0$ for $\Phi_\varepsilon$ and $\Phi_0$, respectively, and then establish the upper semi-continuity of $A_\varepsilon$ at $\varepsilon = 0$, i.e., for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \sup_{\tilde{u} \in \hat{A}_\varepsilon} \inf_{\hat{u} \in \hat{A}_0} \varepsilon^{-1} \int_{\mathcal{O}_\tau} |\hat{u}_t - \hat{u}|^2 \, dx = 0. \quad (32)$$

To that end, we must show the cocycle $\Phi_\varepsilon$ has a $D_1$-pullback attractor $A_\varepsilon$ in $L^2(\mathcal{O})$ such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \text{dist}_{L^2(\mathcal{O})}(A_\varepsilon(\tau, \omega), A_0(\tau, \omega)) = 0,$$

which is the main result of the last section.

For the deterministic forcing terms, we assume:

$$\int_{-\infty}^\tau e^{\lambda s} \left( \|G(s, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_1(s, \cdot)\|_{L^\infty(\mathcal{O})} \right) + \|\varphi_2(s, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_3(s, \cdot)\|_{L^\infty(\mathcal{O})}^2 \, ds < \infty, \quad \forall \tau \in \mathbb{R}. \quad (33)$$

For the existence of tempered absorbing sets, we further assume the following tempered condition on the deterministic forcing terms: for all $\sigma > 0$,

$$\lim_{r \to -\infty} e^{\sigma r} \int_{-\infty}^0 e^{\lambda s} \left( \|G(s + r, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_1(s + r, \cdot)\|_{L^\infty(\mathcal{O})} \right) + \|\varphi_2(s + r, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_3(s + r, \cdot)\|_{L^\infty(\mathcal{O})}^2 \, ds = 0. \quad (34)$$

We remark that (33) and (34) do not require that $G(s, \cdot)$ and $\varphi_i$ be bounded in $L^\infty(\mathcal{O})$ for each $i = 1, 2, 3$ when $s \to \pm \infty$. As mentioned before, $\psi_1 = \varphi_1 + c_1$ and $\psi_2 = \varphi_2 + c_2$ for some positive constants $c_1$ and $c_2$. Based on this fact, we find that (33) and (34) are equivalent to the following conditions, respectively:

$$\int_{-\infty}^\tau e^{\lambda s} \left( \|G(s, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\psi_1(s, \cdot)\|_{L^\infty(\mathcal{O})} \right) + \|\psi_2(s, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\psi_3(s, \cdot)\|_{L^\infty(\mathcal{O})}^2 \, ds < \infty, \quad \forall \tau \in \mathbb{R}, \quad (35)$$

and for any $\sigma > 0$

$$\lim_{r \to -\infty} e^{\sigma r} \int_{-\infty}^0 e^{\lambda s} \left( \|G(s + r, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\psi_1(s + r, \cdot)\|_{L^\infty(\mathcal{O})} \right) + \|\psi_2(s + r, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\psi_3(s + r, \cdot)\|_{L^\infty(\mathcal{O})}^2 \, ds = 0. \quad (36)$$

We are now in a position to present our main results of this paper. We start with the existence and uniqueness of random attractors for the continuous cocycle $\Phi_\varepsilon$ and $\Phi_0$.

**Theorem 2.1.** Assume that (8)-(11), (33) and (34) hold. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, $\Phi_\varepsilon$ and $\Phi_0$ have a unique $D_1$- and $D_0$-pullback attractor $A_\varepsilon$ and $A_0$ in $L^2(\mathcal{O})$ and $L^2(\mathcal{Q})$, respectively. Moreover, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists a constant $L(\tau, \omega)$ such that for $0 < \varepsilon < \varepsilon_0$,

$$\|u\|_{H^1_\varepsilon(\mathcal{O})} \leq L(\tau, \omega), \quad \forall u \in A_\varepsilon(\tau, \omega). \quad (37)$$

If, in addition, $F$, $G$, $\varphi_1$, $\varphi_2$ and $\varphi_3$ are T-periodic with respect to $t$, then the attractors $A_\varepsilon$ and $A_0$ are also T-periodic.
Remark 1. Since $L^2(Q)$ can be embedded naturally into $L^2(O)$ as the subspace of functions independent of $y_{n+1}$, we can consider the cocycle $\Phi_0$ as a mapping from $L^2(O)$ into $L^2(O)$.

Next, we compare the dynamics of $\Phi_\varepsilon$ and $\Phi_0$. For this, we further assume the functions $G$ and $F$ satisfy, for all $x \in O$ and $t, s \in \mathbb{R}$,

$$
\|G_\varepsilon(t, \cdot) - G_0(t, \cdot)\|_{L^2(O)} \leq \kappa_1(t)\varepsilon, \quad (38)
$$

$$
\|F_\varepsilon(t, \cdot, s) - F_0(t, \cdot, s)\|_{L^2(O)} \leq \kappa_2(t)\varepsilon, \quad (39)
$$

$$
\left| \frac{\partial F(t, x, s)}{\partial s} \right| \leq \lambda_3 |s|^{p-2} + \varphi_4(t, x), \quad (40)
$$

where $\kappa_1(t), \kappa_2(t) \in L^2_{loc}(\mathbb{R})$, $\lambda_3$ is a positive constant and $\varphi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(O))$.

By (12), (39) and (40) we have, for all $x \in O$ and $t, s \in \mathbb{R}$,

$$
\|f_\varepsilon(t, \cdot, s) - f_0(t, \cdot, s)\|_{L^2(O)} \leq \kappa_2(t)\varepsilon \quad (41)
$$

and

$$
\left| \frac{\partial f(t, x, s)}{\partial s} \right| \leq \lambda_3 |s|^{p-2} + \psi_4(t, x), \quad (42)
$$

where $\psi_4(t, x) = \varphi_4(t, x) + \lambda$.

Based on these conditions, we will prove the convergence of $\Phi_\varepsilon$ to $\Phi_0$ as $\varepsilon \to 0$, which is stated below.

Theorem 2.2. Suppose (8)-(11), (33) and (38)-(40) hold. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and a positive number $L(\tau, \omega)$, if $u_\varepsilon^\tau \in H^1_\tau(O)$ such that $\|u_\varepsilon^\tau\|_{H^1_\tau(O)} \leq L(\tau, \omega)$, then we have, for any $t \geq \tau$,

$$
\lim_{\varepsilon \to 0} \|\Phi_\varepsilon(t, \tau, \omega, u_\varepsilon^\tau) - \Phi_0(t, \tau, \omega, \mathcal{G}u_\varepsilon^\tau)\|_{L^2(O)} = 0,
$$

where, for $u \in L^2(O)$, $\mathcal{G}u$ is the average function of $u$ in $y_{n+1}$ defined by

$$
\mathcal{G}u = \int_0^1 u(y^*, y_{n+1})dy_{n+1}.
$$

Finally, by using Theorem 2.1 and Theorem 2.2, we will prove our main result of this paper as given below.

Theorem 2.3. Suppose (8)-(11), (33), (34) and (38)-(40) hold. Then the pullback attractors $A_\varepsilon$ are upper-semicontinuous at $\varepsilon = 0$, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\lim_{\varepsilon \to 0} \text{dist}_{L^2(O)}(A_\varepsilon(\tau, \omega), A_0(\tau, \omega)) = 0.
$$

3. Uniform estimates of solutions. This section is devoted to uniform estimates of solutions of (25) in $H_\varepsilon(O)$ and $H^1(O)$. We start with the estimates in $H_\varepsilon(O)$.

Lemma 3.1. Assume that (8)-(11) and (33) hold. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_1$, there exists $T = T(\tau, \omega, D_1) > 0$, independent of $\varepsilon$, such that for all $t \geq T$, the solution $v_\varepsilon^\tau$ of (29) with $\omega$ replaced by $\theta_{-\varepsilon} \omega$ satisfies

$$
\|v_\varepsilon^\tau(\tau - t, \theta_{-\varepsilon} \omega, v_{\varepsilon-\tau})\|_{H_\varepsilon(O)}^2 \leq M + M \int_{-\infty}^0 e^{\lambda s} |z(\theta_{s}\omega)|^p ds
$$

$$
+ Me^{-\lambda \tau} \int_{-\infty}^\tau e^{\lambda s} \left( \|G(s, \cdot)\|_{L^\infty(O)} + \|\psi_1(s, \cdot)\|_{L^\infty(O)} + \|\psi_2(s, \cdot)\|_{L^\infty(O)} \right) ds
$$
and
\[
\int_{\tau-t}^{\tau} e^{\lambda s} \|v^\tau(s, \tau - t, \theta_{\tau-s}\omega, v^\tau_{\tau-t})\|^2_{H^1(\Omega)} ds \\
+ \int_{\tau-t}^{\tau} e^{\lambda s} \|v^\tau(s, \tau - t, \theta_{\tau-s}\omega, v^\tau_{\tau-t}) + h_\varepsilon(y)z(\theta_{\tau-s}\omega)\|^p_{L^p(\Omega)} ds \\
\leq M e^{\lambda \tau} + M e^{\lambda \tau} \int_{-\infty}^{0} e^{\lambda s} |z(\theta_s\omega)|^p ds \\
+ M \int_{-\infty}^{\tau} e^{\lambda s} \|G(s, \cdot)\|^2_{L^\infty(\partial)} ds + \|\psi_1(s, \cdot)\|_{L^\infty(\partial)} + \|\psi_2(s, \cdot)\|^2_{L^\infty(\partial)} ds,
\]
where \(v^\tau_{\tau-t} \in D_1(\tau - t, \theta_{-\omega})\) and \(M\) is a positive constant depending on \(\lambda\), but independent of \(\tau, \omega, \varepsilon\) and \(D_1\).

**Proof.** Taking the inner product of (29) with \(v^\tau\) in \(H_y(\Omega)\), we find that
\[
\frac{1}{2} \frac{d}{dt} \|v^\tau\|^2_{H_y(\Omega)} + a_\varepsilon(v^\tau, v^\tau) + \lambda(v^\tau, v^\tau)_{H_y(\Omega)} = (f_\varepsilon(t, y, v^\tau + h_\varepsilon(y)z(\theta_t\omega)), v^\tau)_{H_y(\Omega)} \\
+ (G_\varepsilon(t, y), v^\tau)_{H_y(\Omega)} - a_\varepsilon(h_\varepsilon(y)z(\theta_t\omega), v^\tau). \tag{43}
\]
For the nonlinear term, by (13)-(14) we obtain
\[
(f_\varepsilon(t, y, v^\tau + h_\varepsilon(y)z(\theta_t\omega)), v^\tau)_{H_y(\Omega)} \\
= \int_{\Omega} gf_\varepsilon(t, y, v^\tau + h_\varepsilon(y)z(\theta_t\omega)) v^\tau dy - \int_{\Omega} gf_\varepsilon(t, y, v^\tau) h_\varepsilon(y)z(\theta_t\omega)dy \\
\leq -\alpha_1 \gamma_1 \int_{\Omega} |u^\tau|^p dy + \gamma_2 \int_{\Omega} |\psi_1(t, y, \varepsilon g(y^*) y_{n+1})| dy \\
+ \gamma_2 \int_{\Omega} \left( \alpha_2 |u^\tau|^{p-1} + |\psi_2(t, y, \varepsilon g(y^*) y_{n+1})| \right) |h_\varepsilon(y)z(\theta_t\omega)| dy \\
\leq -\frac{1}{2} \alpha_1 \gamma_1 \|u^\tau\|^p_{L^p(\Omega)} + c(1 + |z(\theta_t\omega)|^p) \\
+ c\|\psi_1(t, \cdot)\|_{L^\infty(\partial)} + \|\psi_2(t, \cdot)\|^2_{L^\infty(\partial)} \tag{44}
\]
For the third term on the right-hand side of (43), we have
\[
(G_\varepsilon(t, y), v^\tau)_{H_y(\Omega)} \leq \frac{1}{4} \lambda \|v^\tau\|^2_{H_y(\Omega)} + \frac{1}{\lambda} \|G_\varepsilon(t, y)\|^2_{H_y(\Omega)} \\
\leq \frac{1}{4} \lambda \|v^\tau\|^2_{H_y(\Omega)} + c \|G(t, \cdot)\|^2_{L^\infty(\partial)} \tag{45}
\]
On the other hand, from (22), the last term on the right-hand side of (43) is bounded by
\[
-a_\varepsilon(h_\varepsilon(y)z(\theta_t\omega), v^\tau) \leq \frac{1}{2} a_\varepsilon(v^\tau, v^\tau) + \frac{1}{2} a_\varepsilon(h_\varepsilon(y)z(\theta_t\omega), h_\varepsilon(y)z(\theta_t\omega)) \\
\leq \frac{1}{2} a_\varepsilon(v^\tau, v^\tau) + c z^2 (\theta_t\omega) \int_{\Omega} \left| \nabla_y h(y^*, \varepsilon g(y^*) y_{n+1}) \right|^2 \\
+ \frac{1}{2} \left| h_{y_{n+1}}(y^*, \varepsilon g(y^*) y_{n+1}) \right|^2 dy \\
\leq \frac{1}{2} a_\varepsilon(v^\tau, v^\tau) + c z^2 (\theta_t\omega). \tag{46}
\]
Then it follows from (43)-(46) that
\[
\frac{d}{dt} \|v^\varepsilon\|^2_{H^1(\Omega)} + 3 \lambda \|v^\varepsilon\|^2_{H^1(\Omega)} + a_{\varepsilon}(v^\varepsilon, v^\varepsilon) + \alpha_1 \gamma_1 \|u^\varepsilon\|^p_{L^p(\Omega)} \leq c (1 + |z(\theta_\omega)|^p) + c(\|G(t, \cdot)\|^2_{L^\infty(\tilde{\Omega})} + \|\psi_1(t, \cdot)\|^2_{L^\infty(\tilde{\Omega})} + \|\psi_2(t, \cdot)\|^2_{L^\infty(\tilde{\Omega})}).
\]

Multiplying (47) by $e^{\lambda t}$ and then integrating the resulting inequality on $(\tau - t, \tau)$ with $t \geq 0$ and replacing $\omega$ by $\theta_{-t}\omega$, we get that for every $\omega \in \Omega$,
\[
\|v^\varepsilon(\tau, \tau - t, \theta_{-t}\omega, v^\varepsilon_{\tau - t})\|^2_{H^1(\Omega)} + \int_{\tau - t}^{\tau} e^{\lambda(s - t)} a_{\varepsilon}(v^\varepsilon(s, \tau - t, \theta_{-t}\omega, v^\varepsilon_{\tau - t}), v^\varepsilon(s, \tau - t, \theta_{-t}\omega, v^\varepsilon_{\tau - t})) \, ds 
+ \frac{1}{2} \lambda \int_{\tau - t}^{\tau} e^{\lambda(s - t)} \|v^\varepsilon(s, \tau - t, \theta_{-t}\omega, v^\varepsilon_{\tau - t})\|^2_{H^1(\Omega)} \, ds 
+ \alpha_1 \gamma_1 \int_{\tau - t}^{\tau} e^{\lambda(s - t)} \|u^\varepsilon(s, \tau - t, \theta_{-t}\omega, u^\varepsilon_{\tau - t})\|^p_{L^p(\Omega)} \, ds 
\leq e^{-\lambda \tau} \|v^\varepsilon_{\tau - t}\|^2_{H^1(\Omega)} + c \int_0^{\infty} e^{\lambda s} |z(\theta_\omega)|^p \, ds + c e^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \|G(s, \cdot)\|^2_{L^\infty(\tilde{\Omega})} \, ds 
+ c e^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \left(\|\psi_1(s, \cdot)\|^2_{L^\infty(\tilde{\Omega})} + \|\psi_2(s, \cdot)\|^2_{L^\infty(\tilde{\Omega})}\right) \, ds + c.
\]

Note that $v^\varepsilon_{\tau - t} \in D_1(\tau - t, \theta_{-t}\omega)$ and $D_1$ is tempered. We find that
\[
\limsup_{t \to \infty} e^{-\lambda \tau} \|v^\varepsilon_{\tau - t}\|^2_{H^1(\Omega)} \leq c \limsup_{t \to \infty} e^{-\lambda \tau} \|D_1(\tau - t, \theta_{-t}\omega)\|^2_{L^2(\Omega)} = 0.
\]

This shows that there exists $T = T(\tau, \omega, D_1) > 0$ such that for all $t \geq T$,
\[
e^{-\lambda \tau} \|v^\varepsilon_{\tau - t}\|^2_{H^1(\Omega)} \leq 1.
\]

Then the lemma follows immediately from (48).

As a consequence of Lemma 3.1, we have the following inequality which is useful for deriving the uniform estimates of solutions in $H^1(\Omega)$.

**Lemma 3.2.** Assume that (8)-(11) and (33) hold. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, $\omega \in \Omega$, and $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_1$, there exists $T = T(\tau, \omega, D_1) \geq 1$, independent of $\varepsilon$, such that for all $t \geq T$, the solution $v^\varepsilon$ of (29) with $\omega$ replaced by $\theta_{-t}\omega$ satisfies
\[
\int_{\tau - 1}^{\tau} \|v^\varepsilon(s, \tau - t, \theta_{-t}\omega, v^\varepsilon_{\tau - t})\|^2_{H^1(\Omega)} + \|u^\varepsilon(s, \tau - t, \theta_{-t}\omega, u^\varepsilon_{\tau - t})\|^p_{L^p(\Omega)} \, ds 
\leq M + M \int_0^{\infty} e^{\lambda s} |z(\theta_\omega)|^p \, ds + M e^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \|G(s, \cdot)\|^2_{L^\infty(\tilde{\Omega})} \, ds 
+ M e^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \left(\|\psi_1(s, \cdot)\|^2_{L^\infty(\tilde{\Omega})} + \|\psi_2(s, \cdot)\|^2_{L^\infty(\tilde{\Omega})}\right) \, ds,
\]
where $v^\varepsilon_{\tau - t} \in D_1(\tau - t, \theta_{-t}\omega)$ and $M$ is a positive constant depending on $\lambda$, but independent of $\tau$, $\omega$, $\varepsilon$ and $D_1$. 

We first estimate the nonlinear term in (50) for which, by (14) and Lemma 3.3, we find that for \( t \geq 1, \)
\[
e^{\lambda(t-1)} \int_{\tau-1}^{\tau} \left( \|v^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 + \|u^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, u^\varepsilon_{t-\tau})\|_{L^p(O)}^p \right) ds
\]
\[
\leq \int_{\tau-1}^{\tau} e^{\lambda s} \left( \|v^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 + \|u^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, u^\varepsilon_{t-\tau})\|_{L^p(O)}^p \right) ds
\]
\[
\leq \int_{\tau-1}^{\tau} e^{\lambda s} \left( \|v^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 + \|u^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, u^\varepsilon_{t-\tau})\|_{L^p(O)}^p \right) ds,
\]
from which and Lemma 3.1 the proof is completed.

Proof. Since \( e^{\lambda(t-1)} \leq e^{\lambda s} \leq e^{\lambda t} \) for all \( \tau - 1 \leq s \leq \tau, \) we find that for \( t \geq 1, \)
\[
e^{\lambda(t-1)} \int_{\tau-1}^{\tau} \left( \|v^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 + \|u^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, u^\varepsilon_{t-\tau})\|_{L^p(O)}^p \right) ds
\]
\[
\leq \int_{\tau-1}^{\tau} e^{\lambda s} \left( \|v^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 + \|u^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, u^\varepsilon_{t-\tau})\|_{L^p(O)}^p \right) ds
\]
\[
\leq \int_{\tau-1}^{\tau} e^{\lambda s} \left( \|v^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 + \|u^\varepsilon(s, \tau - t, \theta_{-\tau} \omega, u^\varepsilon_{t-\tau})\|_{L^p(O)}^p \right) ds,
\]
the uniform estimates of solutions in \( H^1_s(O). \)

Lemma 3.3. Assume that (10)-(11) hold. Then we have for all \( u \in D(A_\varepsilon), \)
\[
(f_\varepsilon(t, y, u), A_\varepsilon u)_{H^1_s(O)} \leq M \left( a_\varepsilon(u, u) + \|\varphi_3(t, \cdot)\|_{L^\infty(\tilde{\omega})}^2 \right),
\]
where \( M \) is a positive constant independent of \( \varepsilon. \)

Lemma 3.4. Assume that (8)-(11) and (33) hold. Then there exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0, \) \( t \in \mathbb{R}, \omega \in \Omega, \) and \( D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_1, \)
there exists \( T = T(\tau, \omega, D_1) \geq 1, \) independent of \( \varepsilon, \) such that for all \( t \geq T, \) the solution \( v^\varepsilon \) of (29) with \( \omega \) replaced by \( \theta_{-\tau} \omega \) satisfies
\[
\|v^\varepsilon(\tau, \tau - t, \theta_{-\tau} \omega, v^\varepsilon_{t-\tau})\|_{H^1_s(O)}^2 \leq M + M \int_{-\infty}^{\tau} e^{\lambda s} (|\varphi_3(s, \cdot)|^2_{L^\infty(\tilde{\omega})} + \|\psi_1(s, \cdot)\|_{L^\infty(\tilde{\omega})} + \|\psi_2(s, \cdot)\|_{L^\infty(\tilde{\omega})}) ds
\]
\[
+ Me^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \left( \|G_s(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 \right) ds,
\]
where \( v^\varepsilon_{t-\tau} \in D_1(\tau - t, \theta_{-\tau} \omega) \) and \( M \) is a positive constant depending on \( \lambda, \) but independent of \( \tau, \omega, \varepsilon \) and \( D_1. \)

Proof. Taking the inner product of (29) with \( A_\varepsilon v^\varepsilon \) in \( H^1_s(O), \) we find that
\[
\frac{1}{2} \frac{d}{dt} a_\varepsilon(v^\varepsilon, v^\varepsilon) + \|A_\varepsilon v^\varepsilon\|_{H^1_s(O)}^2 + \lambda a_\varepsilon(v^\varepsilon, v^\varepsilon)
\]
\[
= (f_\varepsilon(t, y, v^\varepsilon + h_\varepsilon(y) z(\theta_{-\tau} \omega)), A_\varepsilon v^\varepsilon)_{H^1_s(O)} + (G_\varepsilon(t, y), A_\varepsilon v^\varepsilon)_{H^1_s(O)}
\]
\[
- (A_\varepsilon h_\varepsilon(y) z(\theta_{-\tau} \omega), A_\varepsilon v^\varepsilon)_{H^1_s(O)}. \tag{50}
\]
We first estimate the nonlinear term in (50) for which, by (14) and Lemma 3.3, we have
\[
(f_\varepsilon(t, y, v^\varepsilon + h_\varepsilon(y) z(\theta_{-\tau} \omega)), A_\varepsilon v^\varepsilon)_{H^1_s(O)}
\]
\[
= (f_\varepsilon(t, y, u^\varepsilon), A_\varepsilon u^\varepsilon)_{H^1_s(O)} - (f_\varepsilon(t, y, u^\varepsilon), A_\varepsilon h_\varepsilon(y) z(\theta_{-\tau} \omega))_{H^1_s(O)}
\]
\[
\leq c a_\varepsilon(u^\varepsilon, u^\varepsilon) + c \|\varphi_3(t, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \alpha_2 \int_{\tilde{\omega}} g |u^\varepsilon|^{p-1} |A_\varepsilon h_\varepsilon(y) z(\theta_{-\tau} \omega)| dy
\]
\[
+ \int_{\tilde{\omega}} g |\psi_2(t, y^\varepsilon, z(g^\varepsilon(y) y_{\mu}, y_{\nu+1}) | |A_\varepsilon h_\varepsilon(y) z(\theta_{-\tau} \omega)| dy
\]
\[
\leq c \left( a_\varepsilon(v^\varepsilon, v^\varepsilon) + \|u^\varepsilon\|_{L^p(\tilde{\omega})}^p \right) + c \left( 1 + |z(\theta_{-\tau} \omega)|^p \right)
\]
\[
+ c\|\psi_2(t, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\varphi_3(t, \cdot)\|_{L^\infty(\tilde{\omega})}^2. \tag{51}
\]
On the other hand, the last two term on the right-hand side of (50) is bounded by
\[
\frac{1}{2} \|A_\varepsilon v^\varepsilon\|_{H^2_\mbox{s}(\mathcal{O})}^2 + \|G_\varepsilon(t,y)\|_{H^2_\mbox{s}(\mathcal{O})}^2 + \|A_\varepsilon h_\varepsilon(y)z(\theta_t\omega)\|_{H^2_\mbox{s}(\mathcal{O})}^2. \tag{52}
\]
By (50)-(52), we can get that
\[
\frac{d}{dt} a_\varepsilon(v^\varepsilon, v^\varepsilon) + \|A_\varepsilon v^\varepsilon\|_{H^2_\mbox{s}(\mathcal{O})}^2 + 2\lambda a_\varepsilon(v^\varepsilon, v^\varepsilon)
\leq c \left( a_\varepsilon(v^\varepsilon, v^\varepsilon) + \|v^\varepsilon\|_{L^p(\mathcal{O})}^p \right) + c \left( 1 + |z(\theta_t\omega)|^p \right)
+ c \left( \|\psi_2(t, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_3(t, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|G(t, \cdot)\|_{L^\infty(\mathcal{O})}^2 \right),
\]
which implied that
\[
\frac{d}{dt} a_\varepsilon(v^\varepsilon, v^\varepsilon) \leq c \left( a_\varepsilon(v^\varepsilon, v^\varepsilon) + \|v^\varepsilon\|_{L^p(\mathcal{O})}^p \right) + c \left( 1 + |z(\theta_t\omega)|^p \right)
+ c \left( \|\psi_2(t, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_3(t, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|G(t, \cdot)\|_{L^\infty(\mathcal{O})}^2 \right). \tag{54}
\]
Given \( t \geq 1, \tau \in \mathbb{R}, \omega \in \Omega \) and \( s \in (\tau - 1, \tau) \), integrating (54) on \((s, \tau)\) we get
\[
a_\varepsilon(v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau - t}), v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau - t}))
\leq a_\varepsilon(v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau - t}), v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau - t}))
\]
\[
+ c \int_\tau^\tau a_\varepsilon(v^\varepsilon(\xi, \tau - t, \omega, v^\varepsilon_{\tau - t}), v^\varepsilon(\xi, \tau - t, \omega, v^\varepsilon_{\tau - t}))) \, d\xi
\]
\[
+ c \int_\tau^\tau \|v^\varepsilon(\xi, \tau - t, \omega, v^\varepsilon_{\tau - t})\|_{L^p(\mathcal{O})}^p \, d\xi + c \int_\tau^\tau (1 + |z(\theta_t\omega)|^p) \, d\xi
\]
\[
+ c \int_\tau^\tau \left( \|\psi_2(\xi, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_3(\xi, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|G(\xi, \cdot)\|_{L^\infty(\mathcal{O})}^2 \right) \, d\xi.
\]
Now integrating the above with respect to \( s \) over \((\tau - 1, \tau)\) and replacing \( \omega \) by \( \theta_{-\tau}\omega \), we obtain that
\[
a_\varepsilon(v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t}), v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t}))
\leq \int_{\tau - 1}^\tau a_\varepsilon(v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t})), v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t}))) \, ds
\]
\[
+ c \int_{\tau - 1}^\tau a_\varepsilon(v^\varepsilon(\xi, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t}), v^\varepsilon(\xi, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t}))) \, d\xi
\]
\[
+ c \int_{\tau - 1}^\tau \|v^\varepsilon(\xi, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t})\|_{L^p(\mathcal{O})}^p \, d\xi + c \int_{\tau - 1}^\tau (1 + |z(\theta_t\omega)|^p) \, d\xi
\]
\[
+ c \int_{\tau - 1}^\tau \left( \|\psi_2(\xi, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|\varphi_3(\xi, \cdot)\|_{L^\infty(\mathcal{O})}^2 + \|G(\xi, \cdot)\|_{L^\infty(\mathcal{O})}^2 \right) \, d\xi.
\]
Let \( T = T(\tau, \omega, D_t) \geq 1 \) be the positive number found in Lemma 3.2. Then it follows from the above inequality and Lemma 3.2 that, for all \( t \geq T \) and \( \omega \in \Omega \),
\[
a_\varepsilon(v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t}), v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau - t})))
\leq c \int_{\tau - 1}^\tau |z(\theta_{-\tau}\omega)|^p \, d\xi + c \int_{-\infty}^0 e^{\lambda s} |z(\theta_s\omega)|^p \, ds + c.
\]
of the stochastic equation (25) by using those estimates for the solution \( v \).

For the first term on the right-hand side of (55), we have

\[
\int_{-\infty}^{\tau} \left( \| \psi_2 (\xi, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \varphi_3 (\xi, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| G (\xi, \cdot) \|_{L^\infty(\mathcal{O})}^2 \right) d\xi
\]

\[
+ c \int_{-\infty}^{\tau} e^{\lambda t} \left( \| G (s, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_1 (s, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \psi_2 (s, \cdot) \|_{L^\infty(\mathcal{O})}^2 \right) ds
\]

The second term on the right-hand side of (55) satisfies

\[
\int_{-\infty}^{\tau} e^{\lambda t} \left( \| \psi_2 (\xi, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| \varphi_3 (\xi, \cdot) \|_{L^\infty(\mathcal{O})}^2 + \| G (\xi, \cdot) \|_{L^\infty(\mathcal{O})}^2 \right) d\xi
\]

Thus, Lemma 3.4 follows from (55)-(57) and Lemma 3.1.

We are now in a position to establish the uniform estimates for the solution \( u^\varepsilon \) of the stochastic equation (25) by using those estimates for the solution \( v^\varepsilon \) of (29).

Notice that for each \( \tau \in \mathbb{R}, t \geq 0, \) and \( \omega \in \Omega, \)

\[
u^\varepsilon (\tau, t, \theta_{-t} \omega, u^\varepsilon_{t-}) = v^\varepsilon (\tau, t, \theta_{-t} \omega, v^\varepsilon_{t-}) + h_\varepsilon (y) z (\omega),
\]

where \( v^\varepsilon_{t-} = u^\varepsilon_{t-} - h_\varepsilon z (\theta_{-t} \omega). \) Suppose \( D = \{ D (\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is a family of nonempty subsets of \( L^2 (\mathcal{O}). \) Based on \( D, \) given \( c > 0, \) define a family \( D_c \) by

\[
D_c (\tau, \omega) = \left\{ u \in L^2 (\mathcal{O}) : \| u \|_{L^2 (\mathcal{O})} \leq 2 \| D (\tau, \omega) \|_{L^2 (\mathcal{O})} + c z^2 (\omega) \right\}.
\]

If \( D \) is tempered, then we can verify that \( D_c \) given by (59) is also tempered. In addition, if \( u^\varepsilon_{t-} \in D (\tau - t, \theta_{-t} \omega), \) then we have \( v^\varepsilon_{t-} = u^\varepsilon_{t-} - h_\varepsilon (y) z (\theta_{-t} \omega) \in D_c (\tau - t, \theta_{-t} \omega) \) for some \( c > 0. \) This fact allows us to get uniform estimates on \( u^\varepsilon \) immediately from (59) and those estimates on \( v^\varepsilon \) as established by Lemma 3.1 and Lemma 3.4.

**Lemma 3.5.** Assume that (8)-(11) and (33) hold. Then there exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0, \tau \in \mathbb{R}, \omega \in \Omega, \) and \( D_1 = \{ D_1 (\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_1, \) there exists \( T = T (\tau, \omega, D_1) \geq 1, \) independent of \( \varepsilon, \) such that for all \( t \geq T, \) the solution \( u^\varepsilon \) of (25) with \( \omega \) replaced by \( \theta_{-t} \omega \) satisfies

\[
\| u^\varepsilon (\tau, t, \theta_{-t} \omega, u^\varepsilon_{t-}) \|_{H^1 (\mathcal{O})} \leq M \left( 1 + \| z (\omega) \|^2 \right) + M \int_{-\infty}^{0} e^{\lambda s} \| z (\theta_{t} \omega) \|^2 ds
\]

\[
+ M e^{-\lambda t} \int_{-\infty}^{\tau} e^{\lambda s} \left( \| G (s, \cdot) \|_{L^\infty (\mathcal{O})}^2 + \| \psi_1 (s, \cdot) \|_{L^\infty (\mathcal{O})}^2 + \| \psi_2 (s, \cdot) \|_{L^\infty (\mathcal{O})}^2 \right) ds,
\]

where \( u^\varepsilon_{t-} \in D_1 (\tau - t, \theta_{-t} \omega) \) and \( M \) is a positive constant depending on \( \lambda, \) but independent of \( \tau, \omega, \varepsilon \) and \( D_1. \)
4. Existence and uniqueness of pullback attractors. In this subsection, we establish the existence of $D_1$-pullback attractor for $\Phi_z$. To that end, we must show that (25) has a tempered pullback absorbing set, which is given as follows.

**Lemma 4.1.** Suppose (8)-(11), (33) and (34) hold. Then the continuous cocycle $\Phi_z$ associated with problem (25) has a closed measurable $D_1$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subseteq D_1$.

**Proof.** We first notice that, by Lemma 3.5, $\Phi_z$ has a closed $D_1$-pullback absorbing set $K$ in $L^2(\Omega)$. More precisely, given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let

$$K(\tau, \omega) = \left\{ u \in L^2(\Omega) : \|u\|_{L^2(\Omega)}^2 \leq L(\tau, \omega) \right\},$$

(61)

where $L(\tau, \omega)$ is the constant given by the right-hand side of (60). It is evident that, for each $\tau \in \mathbb{R}$, $L(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. In addition, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 \subseteq D_1$, there exists $T = T(\tau, \omega, D_1) \geq 1$, independent of $\varepsilon$, such that for all $t \geq T$,

$$\Phi_z(t, \tau - t, \theta_{-t} \omega, D_1 (\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega).$$

Thus we find that $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable set and pullback absorbs all elements in $D_1$. We now verify that $K$ is tempered. Let $\sigma$ be an arbitrary positive number. Without loss of generality, we may assume $\sigma < \lambda$. Then, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have by (61)

$$e^{\sigma \tau} \|K(\tau + r, \theta_{r} \omega)\|_{L^2(\Omega)} \leq Me^{\sigma \tau} \left(1 + z^2(\theta_{-r} \omega)\right) + M \int_{-\infty}^{0} e^{\sigma(s+r)} \|z(\theta_{s+r} \omega)\|^p ds$$

$$+ Me^{-\sigma \tau} e^{\sigma(r+r)} \int_{-\infty}^{0} e^{\lambda s} \left(\|G(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})}^2 + \|\psi_1(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})} + \|\psi_2(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})}^2 \right) ds$$

$$+ M e^{-\sigma \tau} e^{\sigma(r+r)} \int_{-\infty}^{0} e^{\lambda s} \|\varphi_3(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})}^2 ds.$$  

(62)

Since $\|z(\theta_{r} \omega)\|$ is tempered, we get

$$\lim_{r \to -\infty} \left( Me^{\sigma \tau} \left(1 + z^2(\theta_{-r} \omega)\right) + M \int_{-\infty}^{0} e^{\sigma(s+r)} \|z(\theta_{s+r} \omega)\|^p ds \right)$$

$$= \lim_{r \to -\infty} \left( Me^{\sigma \tau} \left(1 + z^2(\theta_{-r} \omega)\right) + M \int_{-\infty}^{r} e^{\sigma s} \|z(\theta_{s} \omega)\|^p ds \right) = 0.$$  

(63)

In addition, by (34) we obtain that

$$\limsup_{r \to -\infty} Me^{-\sigma \tau} e^{\sigma(r+r)} \int_{-\infty}^{0} e^{\lambda s} \left(\|G(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})}^2 + \|\psi_1(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})} + \|\psi_2(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})}^2 \right) ds$$

$$+ \limsup_{r \to -\infty} Me^{-\sigma \tau} e^{\sigma(r+r)} \int_{-\infty}^{0} e^{\lambda s} \|\varphi_3(s + r + \tau, \cdot)\|_{L^\infty(\tilde{\Omega})}^2 ds = 0.$$  

(64)

Then it follows from (62)-(64) that, for every $\sigma < \lambda$,

$$\lim_{r \to -\infty} e^{\sigma \tau} \|K(\tau + r, \theta_{r} \omega)\|_{L^2(\Omega)} = 0,$$

and hence $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered. The proof is complete. 

Lemma 5.1. Assume that (8)-(11) hold. Then there exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( v_{\varepsilon}^\tau \in H_g(\mathcal{O}) \), the solution \( v^\tau \) of (29) satisfies, for all \( T > 0 \),

\[
\int_\tau^{\tau+T} \| v^\tau (s, \tau, \omega, v_{\varepsilon}^\tau) \|_{H^1_0(\mathcal{O})}^2 ds + \int_\tau^{\tau+T} \| u^\tau (s, \tau, \omega, u_{\varepsilon}^\tau) \|_{L^p(\mathcal{O})}^p ds \\
\leq M \| v_{\varepsilon}^\tau \|_{H^1_0(\mathcal{O})}^2 + M \int_\tau^{\tau+T} |z(\theta_s \omega)|^p ds + M \\
+ M \int_\tau^{\tau+T} (\| G(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2 + \| \psi_1(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2 + \| \psi_2(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2) ds,
\]

where \( M > 0 \) is a constant depending on \( \lambda \) and \( T \), but independent of \( \tau, \omega \) and \( \varepsilon \).

Proof. Multiplying (47) by \( e^{\lambda t} \) and then integrating the resulting inequality on \((\tau, \tau + T)\), we get that for every \( \omega \in \Omega \),

\[
\| v^\tau (\tau + T, \tau, \omega, v_{\varepsilon}^\tau) \|_{H^1_0(\mathcal{O})}^2 + \int_\tau^{\tau+T} e^{\lambda(s-(\tau+T))} a_{\varepsilon} (v_{\varepsilon}^\tau (s, \tau, \omega, v_{\varepsilon}^\tau), v^\tau (s, \tau, \omega, v_{\varepsilon}^\tau)) ds \\
+ \frac{1}{2} \lambda \int_\tau^{\tau+T} e^{\lambda(s-(\tau+T))} \| v^\tau (s, \tau, \omega, v_{\varepsilon}^\tau) \|_{H^1_0(\mathcal{O})}^2 ds \\
+ \alpha_1 \gamma_1 \int_\tau^{\tau+T} e^{\lambda(s-(\tau+T))} \| u^\tau (s, \tau, \omega, u_{\varepsilon}^\tau) \|_{L^p(\mathcal{O})}^p ds \\
\leq e^{-\lambda T} \| v_{\varepsilon}^\tau \|_{H^1_0(\mathcal{O})}^2 + c \int_\tau^{\tau+T} e^{\lambda(s-(\tau+T))} |z(\theta_s \omega)|^p ds + c \\
+ c \int_\tau^{\tau+T} e^{\lambda(s-(\tau+T))} (\| G(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2 + \| \psi_1(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2 + \| \psi_2(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2) ds \\
\leq \| v_{\varepsilon}^\tau \|_{H^1_0(\mathcal{O})}^2 + c \int_\tau^{\tau+T} |z(\theta_s \omega)|^p ds + c \\
+ c \int_\tau^{\tau+T} (\| G(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2 + \| \psi_1(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2 + \| \psi_2(s, \cdot) \|_{L^{\infty}(\overline{\mathcal{O}})}^2) ds \tag{65}
\]

which along with the same argument as Lemma 3.2 completes the proof. \( \square \)

Similarly, one can prove

Lemma 5.2. Assume that (8)-(11) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( v_{\varepsilon}^\tau \in H_g(\mathcal{O}) \), the solution \( v^0 \) of (31) satisfies, for all \( T > 0 \),
\[
\int_{\tau}^{\tau+T} \left\| v^0(s, \tau, \omega, v_0) \right\|_{H^1(Q)}^2 ds + \int_{\tau}^{\tau+T} \left\| v^0(s, \tau, \omega, u^0_\tau(\omega)) \right\|_{L^p(Q)}^p ds \\
\leq M \left\| v^0_\tau(\omega) \right\|_{H^1(Q)}^2 + M \int_{\tau}^{\tau+T} |\varepsilon(\theta_\tau\omega)|^p ds + M \\
+ M \int_{\tau}^{\tau+T} \left( \left\| G(s, \cdot) \right\|_{L^2(Q)}^2 + \left\| \psi_1(s, \cdot) \right\|_{L^\infty(Q)} + \left\| \psi_2(s, \cdot) \right\|_{L^\infty(Q)}^2 \right) ds,
\]
where \( M \) is a positive constant depending on \( \lambda \) and \( T \), but independent of \( \tau \) and \( \omega \).

The following result on the average function can be found in [20].

**Lemma 5.3.** If \( u \in H^1(O) \), then \( G u \in H^1(Q) \) and

\[
\| u - Gu \|_{H^1(Q)} \leq c \varepsilon \| u \|_{H^1(O)},
\]
where \( c \) is a constant, independent of \( \varepsilon \).

We now prove Theorem 2.2 regarding the convergence of \( \Phi_\varepsilon \) as \( \varepsilon \to 0 \).

**Proof of Theorem 2.2.** Taking the inner product of (31) with \( \phi_l \), where \( l \in H^1(Q) \), we find

\[
\int_{Q} g \frac{du^0}{dt} dy^* + \sum_{i=1}^{n} \int_{Q} g y^0_{i,1} y_i dy^* + \lambda \int_{Q} g v_0^0 l dy^*
\]

\[
= \int_{Q} g f_0(t, y^*, v^0 + h_0(y^*) \varepsilon(\theta_\tau\omega)) l dy^* + \int_{Q} g G_0(t, y^*) l dy^*
\]

\[
- \sum_{i=1}^{n} \int_{Q} g h_0 y_i(y^*) \varepsilon(\theta_\tau\omega) l y_i dy^*.
\]

As \( \int_{0}^{1} \zeta(y^*, y_{n+1}) dy_{n+1} \) belongs to \( H^1(Q) \) if \( \zeta \) is in \( H^1(O) \), the above equality becomes, for any \( \zeta \in H^1(O) \),

\[
\left( \frac{dv^0}{dt}, \zeta \right)_{H^1(Q)} + \sum_{i=1}^{n} \left( y^0_{i,1}, \zeta_{y_i} \right) \left( G_{i,0}, \zeta \right)_{H^1(Q)} + \left( G_{i,0}, \zeta \right)_{H^1(Q)}
\]

\[
= \left( f(t, y^*, 0, v^0 + h_0(y^*) \varepsilon(\theta_\tau\omega)) , \zeta \right)_{H^1(Q)} + \left( G(t, y^*, 0), \zeta \right)_{H^1(Q)}
\]

\[
- \sum_{i=1}^{n} \left( h_0(y^*) \varepsilon(\theta_\tau\omega), \zeta_{y_i} \right)_{H^1(Q)}.
\]

Since \( v^0 \) is independent of \( y_{n+1} \), the above equality gives, for any \( \zeta \in H^1(O) \) and \( 0 < \varepsilon \leq 1 \),

\[
\left( \frac{dv^0}{dt}, \zeta \right)_{H^1(Q)} + a_\varepsilon \left( v^0, \zeta \right)_{H^1(Q)} + \lambda \left( v^0, \zeta \right)_{H^1(Q)}
\]

\[
= \left( f(t, y^*, 0, v^0 + h_0(y^*) \varepsilon(\theta_\tau\omega)) , \zeta \right)_{H^1(Q)} + \left( G(t, y^*, 0), \zeta \right)_{H^1(Q)}
\]

\[
- a_\varepsilon \left( h_0(y^*) \varepsilon(\theta_\tau\omega), \zeta \right)_{H^1(Q)} - \sum_{i=1}^{n} \left( y^0_{i,1} y_i, y_{n+1} \zeta_{y_{n+1}} \right)_{H^1(Q)}
\]

\[
+ \sum_{i=1}^{n} \left( g y_i, h_0 y_i(y^*) \varepsilon(\theta_\tau\omega), y_{n+1} \zeta_{y_{n+1}} \right)_{H^1(Q)}.
\]

(66)
Due to (66) and (29), the function \( v^\varepsilon - v^0 \) satisfies the equation, for any \( \zeta \in H^1(\mathcal{O}) \),
\[
\frac{dv^\varepsilon}{dt} - \frac{dv^0}{dt}, \zeta)_{H^1(\mathcal{O})} + a_{\varepsilon} (v^\varepsilon - v^0, \zeta)_{H^1(\mathcal{O})} + \beta (v^\varepsilon - v^0, \zeta)_{H^1(\mathcal{O})}
\]
\[
= (f_{\varepsilon}(t, y^*, y_{n+1}, v^\varepsilon + h_{\varepsilon}(y)z(\theta t\omega)) - f(t, y^*, 0, v^0 + h_0(y^*)z(\theta t\omega)), \zeta)_{H^1(\mathcal{O})} + g(t, y^*, 0, \zeta)_{H^1(\mathcal{O})} - H_{\varepsilon}(t, y^*, 0, \zeta)_{H^1(\mathcal{O})} - \zeta)
\]
\[
+ \sum_{i=1}^{n} \left( \frac{g_{y_i}}{g} \right) y_{n+1} \zeta_{y_i}, y_{n+1} \right)_{H^1(\mathcal{O})} - \sum_{i=1}^{n} \left( \frac{g_{y_i}}{g} \right) h_{0y_i}(y^*)z(\theta t\omega)_{H^1(\mathcal{O})}.
\]
(67)

We replace \( \zeta \) by \( v^\varepsilon - v^0 \) and estimate all the terms on the right-hand side of this equality. We have from (15) and (39)-(40)
\[
\left( f_{\varepsilon}(t, y^*, y_{n+1}, v^\varepsilon + h_{\varepsilon}(y)z(\theta t\omega)) - f(t, y^*, 0, v^0 + h_0(y^*)z(\theta t\omega)), v^\varepsilon - v^0 \right)_{H^1(\mathcal{O})}
\]
\[
\leq \beta \| v^\varepsilon - v^0 \|^2_{H^1(\mathcal{O})} + c \varepsilon \| z(\theta t\omega) \|_{L^p} \left( \| u^\varepsilon \| + |u^0| \right)^{p-2} \| h_{\varepsilon}(y) - h_0(y^*) \| \| v^\varepsilon - v^0 \| dy
\]
\[
+ c \varepsilon \| z(\theta t\omega) \| \int_{\mathcal{O}} \psi_4 |h_{\varepsilon}(y) - h_0(y^*)| |v^\varepsilon - v^0| dy + c\kappa_2(t)\varepsilon \| v^\varepsilon - v^0 \|_{H^1(\mathcal{O})}.
\]
(68)

Since \( h \in C^2(\mathcal{Q} \times [0, \gamma_2]) \) we have
\[
|h_{\varepsilon}(y) - h_0(y^*)| = |h(y^*, \varepsilon g(y^*)y_{n+1}) - h(y^*, 0)| = |\varepsilon h_{x_{n+1}} g(y^*)y_{n+1}| \leq c\varepsilon.
\]
(69)

By (69) we get
\[
|z(\theta t\omega)\|_{L^p(\mathcal{O})} \left( \| u^\varepsilon \| + |u^0| \right)^{p-2} \| h_{\varepsilon}(y) - h_0(y^*) \| \| v^\varepsilon - v^0 \| dy
\]
\[
\leq |z(\theta t\omega)\|_{L^p(\mathcal{O})} \left( \| u^\varepsilon \| + |u^0| \right)^{p-1} \| h_{\varepsilon}(y) - h_0(y^*) \| dy
\]
\[
+ \int_{\mathcal{O}} (|u^\varepsilon| + |u^0|)^{p-2} (h_{\varepsilon}(y) - h_0(y^*)) z(\theta t\omega) \|_{L^p(\mathcal{O})} \| z(\theta t\omega) \|_{L^p(\mathcal{Q})}.
\]
(70)

Similarly, we have
\[
|z(\theta t\omega)| \int_{\mathcal{O}} \psi_4 |h_{\varepsilon}(y) - h_0(y^*)| |v^\varepsilon - v^0| dy \leq c\varepsilon(1 + |z(\theta t\omega)|^p + \| v^\varepsilon - v^0 \|^2_{H^1(\mathcal{O})}).
\]
(71)

It follows from (68)-(71) that
\[
\left( f_{\varepsilon}(t, y^*, y_{n+1}, v^\varepsilon + h_{\varepsilon}(y)z(\theta t\omega)) - f(t, y^*, 0, v^0 + h_0(y^*)z(\theta t\omega)), v^\varepsilon - v^0 \right)_{H^1(\mathcal{O})}
\]
\[
\leq \beta \| v^\varepsilon - v^0 \|^2_{H^1(\mathcal{O})} + c\varepsilon \left(1 + \| u^\varepsilon \|_{L^p(\mathcal{O})}^p + \| u^0 \|_{L^p(\mathcal{Q})}^p + |z(\theta t\omega)|^p\right)
\]
\[
+ c\kappa_2(t)\varepsilon \| v^\varepsilon \|_{H^1(\mathcal{O})}^2 + \| v^0 \|_{H^1(\mathcal{Q})}^2.
\]
(72)

By (38) we obtain
\[
\left( G_{\varepsilon}(t, y^*, y_{n+1}) - G(t, y^*, 0), v^\varepsilon - v^0 \right)_{H^1(\mathcal{O})}
\]
\[
\leq \| G_{\varepsilon}(t, y^*, y_{n+1}) - G(t, y^*, 0) \|_{H^1(\mathcal{O})} \| v^\varepsilon - v^0 \|_{H^1(\mathcal{O})}
\]
\[
\leq c\kappa_1(t)\varepsilon + c\varepsilon \| v^\varepsilon \|^2_{H^1(\mathcal{O})} + \| v^0 \|^2_{H^1(\mathcal{Q})}.
\]
(73)
On the other hand, by (22) we obtain, for $0 < \varepsilon \leq \varepsilon_0$,

$$-\alpha_{\varepsilon}(h_\varepsilon(y)z(\theta_t \omega) - h_0(y^*)z(\theta_t \omega), v^\varepsilon - v^0)$$

$$\leq c\varepsilon^2 \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle dy + \frac{1}{2} \alpha_{\varepsilon} \langle (v^\varepsilon - v^0, v^\varepsilon - v^0) \rangle$$

$$\leq c\varepsilon^2 \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle dy + \frac{1}{2} \alpha_{\varepsilon} \langle (v^\varepsilon - v^0, v^\varepsilon - v^0) \rangle$$

$$\leq c\varepsilon^2 \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle dy + \frac{1}{2} \alpha_{\varepsilon} \langle (v^\varepsilon - v^0, v^\varepsilon - v^0) \rangle.$$  

(74)

Since $h$ satisfies the boundary condition in (6) by assumption, we find that

$$h_{x_n+1}(y^*, 0) = h_{x_n+1}(x^*, 0) = 0.$$  

(75)

By (75) and the fact $h \in C^2(\Omega \times [0, \gamma_2])$ we get

$$\int (h_{x_n+1}(y^*, \varepsilon g(y^*)y_{n+1}) - h(x, y^*, 0))^2 dy \leq c\varepsilon^2.$$  

(76)

Similarly, we also have, for each $i = 1, \ldots, n$,

$$\int (h_{x_i}(y^*, \varepsilon g(y^*)y_{n+1}) - h(x, y^*, 0))^2 dy \leq c\varepsilon^2.$$  

(77)

By (74) and (76)-(77) we obtain

$$-\alpha_{\varepsilon}(h_\varepsilon(y)z(\theta_t \omega) - h_0(y^*)z(\theta_t \omega), v^\varepsilon - v^0) \leq c\varepsilon^2 \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle dy + \frac{1}{2} \alpha_{\varepsilon} \langle (v^\varepsilon - v^0, v^\varepsilon - v^0) \rangle.$$  

(78)

By (21), we have

$$\sum_{i=1}^n \int \frac{\partial h_{y_i}}{\partial y_i} (y^*, \varepsilon g(y^*)y_{n+1}) \leq c\varepsilon \|u^0\|_{H^1(\Omega)} \|v^\varepsilon - v^0\|_{H^1(\Omega)}$$

$$\leq c\varepsilon \|v^\varepsilon\|^2_{H^1(\Omega)} + \|v^0\|^2_{H^1(\Omega)}.$$  

(79)

Similarly, we also have

$$\sum_{i=1}^n \int \frac{\partial h_{y_i}}{\partial y_i} (y^*, \varepsilon g(y^*)y_{n+1}) \leq c\varepsilon \|v^\varepsilon\|^2_{H^1(\Omega)} + \|v^0\|^2_{H^1(\Omega)} + \varepsilon^2 \langle (\varepsilon_{y_1} + \varepsilon_{y_2}) \rangle.$$  

(80)

By (72), (73), (78)-(80) as well as (67), we obtain, for $t \geq \tau$,

$$\frac{d}{dt} \|v^\varepsilon - v^0\|^2_{H^1(\Omega)} \leq 2\beta \|v^\varepsilon - v^0\|^2_{H^1(\Omega)} + c\varepsilon (1 + \|z(\theta_t \omega)\|^p) + c\varepsilon \|H^2(t) + \kappa^2(t)\)$$

$$+ c\varepsilon \left(\|u^\varepsilon\|^2_{L^p(\Omega)} + \|u^0\|^p_{L^p(\Omega)} + \|v^\varepsilon\|^2_{H^1(\Omega)} + \|v^0\|^2_{H^1(\Omega)} \right).$$  

(81)
Integrating (81) on \((\tau, t)\) we obtain
\[
\|\varphi(t) - \varphi(0)\|_{H^s(\Omega)}^2 \leq e^{2\beta(t-\tau)} \|\varphi(\tau) - \varphi(0)\|_{H^s(\Omega)}^2 + c\varepsilon \int_\tau^t e^{2\beta(t-s)} (1 + |z(\theta_\omega)|^p) ds + c\varepsilon \int_\tau^t e^{2\beta(t-s)} (\kappa_1^2(s) + \kappa_2^2(s)) ds
\]
\[
+ c\varepsilon e^{2\beta(t-\tau)} \left( \|\varphi\|_{L^p(\Omega)}^2 + \|\varphi(0)\|_{L^p(\Omega)}^2 + \int_\tau^t (\kappa_1^2(s) + \kappa_2^2(s)) ds \right)
\]
\[
+ \int_\tau^t \left( \|G(s, \cdot)\|_{L^\infty(\partial\Omega)}^2 + \|\psi_1(s, \cdot)\|_{L^\infty(\partial\Omega)}^2 + \|\psi_2(s, \cdot)\|_{L^\infty(\partial\Omega)}^2 \right) ds.
\]
Then we have for all \(t \in [\tau, \tau + T]\),
\[
\|\varphi(t) - \varphi(0)\|_{H^s(\Omega)}^2 \leq e^{2\beta T} \|\varphi(\tau) - \varphi(0)\|_{H^s(\Omega)}^2 + c\varepsilon e^{2\beta T} \int_\tau^{\tau + T} (1 + |z(\theta_\omega)|^p) ds
\]
\[
+ c\varepsilon e^{2\beta T} \left[ \|\varphi\|_{H^s(\Omega)}^2 + \|\varphi(0)\|_{H^s(\Omega)}^2 + \int_\tau^{\tau + T} (\kappa_1^2(s) + \kappa_2^2(s)) ds \right]
\]
\[
+ \int_\tau^{\tau + T} \left( \|G(s, \cdot)\|_{L^\infty(\partial\Omega)}^2 + \|\psi_1(s, \cdot)\|_{L^\infty(\partial\Omega)}^2 + \|\psi_2(s, \cdot)\|_{L^\infty(\partial\Omega)}^2 \right) ds.
\]
which together with Lemma 5.3 completes the proof.

Now we are in the position to prove our main result.

**Proof of Theorem 2.3.** The proof is based on Theorem 2.2 by following a standard argument, which is presented here just for the reader’s convenience. Given \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), by the invariance of \(A_\varepsilon\) and (37) we find that there exists \(\varepsilon_0 > 0\) such that
\[
\|u\|_{H^s(\Omega)}^2 \leq L(\tau, 0) \quad \text{for all} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad u \in A_\varepsilon(\tau, \omega),
\]
where \(L(\tau, \omega)\) is a constant independent of \(\varepsilon\). Let \(K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) be the \(D_1\)-pullback absorbing set of \(\Phi_\varepsilon\) obtained in Lemma 4.1 and denote by \(K_0 = \{K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) with \(K_0(\tau, \omega) = \{u : u \in K(\tau, \omega)\}\). Then \(K_0\) is tempered in \(L^2(\Omega)\) and hence \(K_0 \in D_0\). Since \(A_0\) is the \(D_0\)-pullback attractor of \(\Phi_0\) in \(L^2(\Omega)\), given \(\eta > 0\), we infer that there exists \(T = T(\eta, \tau, \omega) \geq 1\) such that
\[
\text{dist}_{L^2(\Omega)}(\Phi_0(T, \tau - T, \theta_{-T}\omega, K_0(\tau - T, \theta_{-T}\omega)), A_0(\tau, \omega)) < \frac{1}{2}\eta.
\]
By the invariance of $\mathcal{A}_\epsilon(\tau, \omega)$, we see that for any $x_\epsilon \in \mathcal{A}_\epsilon(\tau, \omega)$, there exists $y_\epsilon \in \mathcal{A}_\epsilon(\tau - T, \theta_T \omega)$ such that

$$x_\epsilon = \Phi_\epsilon(T, \tau - T, \theta_T \omega, y_\epsilon).$$

(87)

By (85) and Theorem 2.2 we get

$$\lim_{\epsilon \to 0} \|\Phi_\epsilon(T, \tau - T, \theta_T \omega, y_\epsilon) - \Phi_0(T, \tau - T, \theta_T \omega, \mathcal{G} y_\epsilon)\|_{L^2(\Omega)} = 0,$$

and hence there exists $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\epsilon < \epsilon_1$,

$$\|\Phi_\epsilon(T, \tau - T, \theta_T \omega, y_\epsilon) - \Phi_0(T, \tau - T, \theta_T \omega, \mathcal{G} y_\epsilon)\|_{L^2(\Omega)} < \frac{1}{2}\eta.$$  

(88)

Since $y_\epsilon \in \mathcal{A}_\epsilon(\tau - T, \theta_T \omega)$ and $\mathcal{A}_\epsilon(\tau - T, \theta_T \omega) \subseteq K(\tau - T, \theta_T \omega)$, we know $\mathcal{G} y_\epsilon \in K_0(\tau - T, \theta_T \omega)$, which along with (86) implies

$$\text{dist}_{L^2(\Omega)}(\Phi_0(T, \tau - T, \theta_T \omega, \mathcal{G} y_\epsilon), \mathcal{A}_0(\tau, \omega)) < \frac{1}{2}\eta.$$  

(89)

By (88) and (89) we have, for all $\epsilon < \epsilon_1$,

$$\text{dist}_{L^2(\Omega)}(\Phi_\epsilon(T, \tau - T, \theta_T \omega, y_\epsilon), \mathcal{A}_0(\tau, \omega)) < \eta.$$  

(90)

By (87) and (90) we obtain, for all $\epsilon < \epsilon_1$,

$$\text{dist}_{L^2(\Omega)}(x_\epsilon, \mathcal{A}_0(\tau, \omega)) < \eta$$

for all $x_\epsilon \in \mathcal{A}_\epsilon(\tau, \omega)$.

This indicates that for all $\epsilon < \epsilon_1$,

$$\text{dist}_{L^2(\Omega)}(\mathcal{A}_\epsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) \leq \eta.$$

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