A new characterization of symmetric $H^+$-tensors and $M$-tensors

Xin Shi$^a$, Luis F. Zuluaga$^a$

$^a$Department of Industrial and Systems Engineering, P.C. Rossin College of Engineering & Applied Science, Lehigh University, PA, 18015,

Abstract

In this work, we present a new characterization of symmetric $H^+$-tensors. It is known that a symmetric tensor is an $H^+$-tensor if and only if it is a generalized diagonally dominant tensor with nonnegative diagonal elements. By exploring the diagonal dominance property, we derive new necessary and sufficient conditions for a symmetric tensor to be an $H^+$-tensor. Based on these conditions, we propose a novel method that allows to check if a tensor is a symmetric $H^+$-tensor in polynomial time. Moreover, these results can be applied to the closely related and important class of $M$-tensors. In particular, this allows to efficiently compute the minimum $H$-eigenvalue of symmetric $M$-tensors. Furthermore, we show how this latter result can be used to provide tighter lower bounds for the minimum $H$-eigenvalue of the Fan product of two symmetric $M$-tensors.

Keywords: $H^+$-tensors, Generalized diagonally dominant tensors, Power cone optimization, Minimum $H$-eigenvalues

2010 MSC: 15A69

1. Introduction

Tensors can be regarded as a high-order generalization of matrices. For $m, n \in \mathbb{N}$, an $m$-order $n$-dimensional real tensor is a multidimensional array with the form

$$\mathcal{A} = (a_{i_1i_2...i_m}), \quad a_{i_1i_2...i_m} \in \mathbb{R}, \quad 1 \leq i_1, i_2, \ldots, i_m \leq n.$$ 

Matrices are tensors with order $m = 2$. Denote $\mathbb{T}_{m,n}$ as the space of all real tensors with order $m$ and dimension $n$. Then

$$\mathbb{T}_{m,n} = \mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n,$$

where $\otimes$ is the outer product. Denote $[n] = \{1, 2, \ldots, n\}$. The tensor $\mathcal{A} = (a_{i_1i_2...i_m}) \in \mathbb{T}_{m,n}$ is called symmetric if its entries $a_{i_1i_2...i_m}$ are invariant under any permutation of $(i_1, \ldots, i_m)$ for $i_j \in [n], j \in [m]$. Denote $\mathbb{S}_{m,n}$ as the set of symmetric tensors in $\mathbb{T}_{m,n}$. The entries $a_{i_1i_2...i_m}$ for any $i \in [n]$ are called diagonal elements (or entries) of $\mathcal{A}$.

Following [1-3], for $\mathcal{A} \in \mathbb{T}_{m,n}$, $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{A}$, if there exists an eigenvector $x \in \mathbb{C}^n \setminus \{0\}$ such that $\mathcal{A}x^{m-1} = \lambda x^{m-1}$, where $\mathcal{A}x^{m-1} \in \mathbb{C}^n$ is defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{1 \leq j_1 < \cdots < j_m \leq n} a_{i_1j_1...j_m}x_{j_1} \cdots x_{j_m},$$

and $x^{m-1} \in \mathbb{C}^n \setminus \{0\}$ is defined by $(x^{m-1})_i = x_i^{m-1}$ for all $i \in [n]$. In particular, if $x$ is real, then $\lambda$ is also real. In this case, we say that $\lambda$ is an $H$-eigenvalue of $\mathcal{A}$.

$^*$This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

$^*$Corresponding author

Email address: shi321xin@gmail.edu (Xin Shi), luis.zuluaga@lehigh.edu (Luis F. Zuluaga)
The comparison tensor of \( A \in \mathbb{T}_{m,n} \), denoted as \( M(A) \), is defined in [4, 5] as follows:

\[
M(A)_{i_1 \cdots i_m} = \begin{cases} |a_{i_1 \cdots i_m}| & \text{if } i_1 = \cdots = i_m, \\ -|a_{i_1 \cdots i_m}| & \text{otherwise}. \end{cases}
\]  

(1)

Following [4, 5], we introduce the following classes of tensors. A tensor is called a nonnegative tensor if all its entries are nonnegative and a tensor is called a diagonal tensor if all its off-diagonal elements are zero. A tensor \( A \in \mathbb{T}_{m,n} \) is said to be a Z-tensor if there exists a nonnegative tensor \( D \in \mathbb{T}_{m,n} \) and a nonnegative scalar \( s \) such that \( A = sI - D \), where \( I \in \mathbb{T}_{m,n} \) is a diagonal tensor with all diagonal elements equal to one. For tensor \( A \), denote \( \rho(A) \) as the largest modulus of its eigenvalues. A Z-tensor \( A = sI - D \) is said to be an M-tensor if \( s \geq \rho(D) \). If \( s > \rho(D) \), then \( A \) is called a strong M-tensor. A tensor is called an H-tensor if its comparison tensor is an M-tensor. A tensor is called a strong H-tensor if its comparison tensor is a strong M-tensor. An H-tensor with nonnegative diagonal elements is called an \( H^+ \)-tensor.

The authors in [4, Theorem 4.9] show that a symmetric tensor is an H-tensor if and only if it is a generalized diagonally dominant tensor (see Definition 1). The matrix version (i.e., when \( m = 2 \)) of this result is given in [6, Theorem 8] and [7]. Furthermore, the authors in [6] prove that a symmetric matrix is an \( H^+ \)-matrix if and only if it can be written as the sum of a number of positive semidefinite matrices which have a special sparse structure. Based on this result, the authors in [8] show that membership to the set of symmetric \( H^+ \)-matrices can be decided in polynomial time by solving a second-order cone optimization problem [see, e.g., 9].

In this work we generalize these results to symmetric \( H^+ \)-tensors. Namely, we prove that a symmetric tensor is an \( H^+ \)-tensor if and only if it can be written as the sum of a number of tensors which have a special sparse structure (see Theorem 11). Based on this result, we obtain (see Theorem 15) a novel characterization of \( H^+ \)-tensors that is amenable to the use of conic optimization techniques [see, e.g., 10]. In particular, we show (see Corollary 18 and 27) that membership to the set of symmetric \( H^+ \)-tensors can be decided in polynomial time by solving a power cone optimization problem [see, e.g., 11, 12].

A lot of effort has been made to characterize \( H^+ \)-tensors [see, e.g., 13–19]. However, these articles typically focus on studying sufficient conditions for a tensor to be an H-tensor. A notable exception is recent work based on the use of spectral theory of nonnegative tensors. Namely, the authors in [20] present a necessary and sufficient condition for strong H-tensors and propose an iterative algorithm for identifying strong H-tensors. In contrast from their methodology, here we study sufficient and necessary condition for a symmetric tensor to be an \( H^+ \)-tensor by exploring the diagonal dominance property. This type of characterization allows, unlike the recent results in [20], to directly optimize over the set of \( H^+ \)-tensors.

In particular, in Section 4, we consider the problem of computing the minimum \( H \)-eigenvalue of symmetric \( M \)-tensors, which play an important role in a wide range of interesting applications [see, 21, and the references therein]. In contrast with the problem of obtaining bounds on the minimum \( H \)-eigenvalue of \( M \)-tensors that has received significant attention in the literature [21–24], here, we use our characterization of \( H^+ \)-tensors to compute \( H \)-eigenvalues of symmetric \( M \)-tensors by solving a power cone optimization problem (see Corollary 21). A comparison of the \( H \)-eigenvalues obtained in this way with bounds proposed in the literature is provided in Table 1.

Further, in Section 5, we show that our characterization can be applied to obtain tighter lower bounds for the minimum \( H \)-eigenvalue of the Fan product of two symmetric \( M \)-tensors. Fan product is introduced by Ky Fan in 1964 [25]. One of the main characteristics of Fan product is that the Fan product of \( M \)-tensor is also an \( M \)-tensor [26]. Some bounds for the minimum \( H \)-eigenvalue of Fan product of \( Z \)-matrices (\( Z \)-tensors) are proposed in [26–28]. We contribute to this vein with the help of the proposed new characteristics of symmetric \( M \)-tensors. Specifically, we show both theoretically and empirically that our proposed lower bounds are tighter than any of the bounds provided in [26]. The corresponding numerical comparisons are provided in Table 3.

For ease of exposition, in what follows, we use small letters \( a, b, \ldots \) for scalars and vectors; capital letters \( A, B, \ldots \) for matrices; calligraphic letters \( \mathcal{A}, \mathcal{B}, \ldots \) for tensors and \( \mathcal{A}, \mathcal{B}, \ldots \) for index sets; and blackboard bold letters \( \mathbb{T}, \mathbb{D}, \ldots \) for other kinds of sets or spaces in this work.

The remaining of the article is structured as follows: Section 3 introduces additional notation, definitions and some basic results. In Section 4, the characterizations of symmetric \( H^+ \)-tensors are presented. With these characterizations, we provide a way to check if a tensor is a symmetric \( H^+ \)-tensor in polynomial time. In Section 5, we show how to obtain the minimum \( H \)-eigenvalue of a symmetric \( M \)-tensor by applying the methodology proposed in this work.
In Section 5, we further apply our results to obtain tighter lower bounds for the minimum $H$-eigenvalue of the Fan product of two symmetric $M$-tensors, than the ones proposed in the related literature. Section 6 concludes this work with some final remarks.

2. Preliminaries

First we introduce additional notation and some fundamental properties of tensors. Let $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ be the set of polynomials in $n$ variables with real coefficients. A polynomial $p \in \mathbb{R}[x]$ is called a sum of squares (SOS) if it can be written as $p = \sum q_i^2$ for a finite number of polynomials $q_i \in \mathbb{R}[x]$. Tensor $\mathcal{A} = (a_{i_1, \ldots, i_n}) \in \mathbb{S}_{m,n}$ is said to have an SOS-tensor decomposition if its corresponding polynomial

$$\mathcal{A}x^m = \sum_{i_1, i_2, \ldots, i_n = 1}^n a_{i_1 i_2 \ldots i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

is an SOS [see, e.g., 29]. A tensor is called a PSD-tensor if its corresponding polynomial is positive semidefinite. The authors in [30] show that every even order symmetric $H^t$-tensor has an SOS-tensor decomposition.

**Theorem 1 (30, Theorem 3.7).** Let $m, n \in \mathbb{N}$ and $\mathcal{A} \in \mathbb{S}_{m,n}$ be an $H^t$-tensor. If $m$ is even, then $\mathcal{A}$ has an SOS-tensor decomposition.

From Theorem 1 it follows that a symmetric $H^t$-tensor is also a PSD-tensor. On the other hand, symmetric $H^t$-tensors can be characterized using the notion of diagonally dominant tensors (see Definition 1). Most of the work related to $H^t$-tensors makes use of the diagonal dominance property [see, e.g., 13–15, 17, 19]. We will also make use of this property in this work. The definitions of diagonally dominant tensors and generalized diagonally dominant tensors are given below.

**Definition 1 (31, Definition 6.5).** Let $m, n \in \mathbb{N}$ and $\mathcal{A} = (a_{i_1, \ldots, i_n}) \in \mathbb{T}_{m,n}$.

(i) $\mathcal{A}$ is called a diagonally dominant (DD) tensor if

$$|a_{i_1, \ldots, i_n}| \geq \sum_{(i_2, \ldots, i_n) \neq (i_1, \ldots, i_1)} |a_{i_2, \ldots, i_n}|, \forall i \in [n]. \quad (2)$$

(ii) $\mathcal{A}$ is called a generalized diagonally dominant (GDD) tensor if there exists a positive diagonal matrix $D$ such that the tensor $\mathcal{AD}^{1-m}D \cdots D$ defined as

$$(\mathcal{AD}^{1-m}D \cdots D)_{i_1, \ldots, i_n} = a_{i_1, \ldots, i_n} d_{i_1}^{1-m} d_{i_2} \cdots d_{i_n}, \quad \forall i_1, \ldots, i_m \in [n], \quad (3)$$

is diagonally dominant, where $d_i = D_{ii}$ is the $i$th diagonal element of $D$.

From the definition of DD tensors and GDD tensors, one can derive an equivalent definition of GDD tensors that will be useful throughout the article.

**Proposition 2.** Let $m, n \in \mathbb{N}$, then $\mathcal{A} \in \mathbb{T}_{m,n}$ is a GDD tensor if and only if there exists a positive diagonal matrix $D$ such that the tensor $\mathcal{AD} \cdots D$ defined as

$$(\mathcal{AD} \cdots D)_{i_1, \ldots, i_n} = a_{i_1, \ldots, i_n} d_{i_1} d_{i_2} \cdots d_{i_m}, \quad \forall i_1, \ldots, i_m \in [n], \quad (4)$$

is diagonally dominant, where $d_i = D_{ii}$ is the $i$th diagonal element of $D$. If $\mathcal{A} \in \mathbb{S}_{m,n}$, then $\mathcal{AD} \cdots D \in \mathbb{S}_{m,n}$.

**Proof.** From Definition 1(ii) if $\mathcal{A} = (a_{i_1, \ldots, i_n}) \in \mathbb{T}_{m,n}$ is a GDD tensor, then there exists a positive diagonal matrix $D$ such that $\mathcal{AD}^{1-m}D \cdots D$ is a DD tensor. That is for all $i \in [n]$,

$$|(\mathcal{AD}^{1-m}D \cdots D)_{i_1, \ldots, i_n}| \geq \sum_{(i_2, \ldots, i_n) \neq (i_1, \ldots, i_1)} |(\mathcal{AD}^{1-m}D \cdots D)_{i_2, \ldots, i_n}|. \quad (5)$$
Note that (5) is equivalent to
\[ |a_{i_1}| \geq \sum_{(i_2, \ldots, i_m) \in (\ldots)} |a_{i_2}d_{i_2}^{i_2} \cdots d_{i_m}|. \] (6)

Considering that \( d_i > 0 \) for all \( i \in [n] \), and multiplying by \( d_i^m \) on both sides of (5), we have that
\[ |a_{i_1}|d_i^m \geq \sum_{(i_2, \ldots, i_m) \in (\ldots)} |a_{i_2}d_{i_2} \cdots d_{i_m}| \] (7)
for all \( i \in [n] \). Thus, the tensor ADD … D defined by (4) is a DD tensor.

For the other direction, if the tensor ADD … D defined by (4) is a DD tensor for a positive diagonal matrix \( D \), then inequality (7) holds for all \( i \in [n] \). Dividing both sides of (7) by \( d_i^m > 0 \), we have inequality (6) which is equivalent to (5) for all \( i \in [n] \) and indicates that \( \mathcal{A} \) is a GDD tensor.

For the remainder of this work, denote by \( DD_{m,n} \) and \( GDD_{m,n} \) the set of DD tensors and the set of GDD tensors in \( S_{m,n} \), respectively. DD and GDD tensors with nonnegative diagonal elements will be referred as DD\(^*\) and GDD\(^*\) tensors, respectively. Also, denote by \( DD^+_{m,n} \) and \( GDD^+_{m,n} \) the set of DD\(^*\) tensors and the set of GDD\(^*\) tensors in \( S_{m,n} \), respectively. The set of PSD-tensor in \( S_{m,n} \) is denoted as \( PSD_{m,n} \).

For \( n \in \mathbb{N} \), a set \( \mathcal{W} \subseteq \mathbb{R}^n \) is called a cone if \( 0 \in \mathcal{W} \) and \( x \in \mathcal{W} \) implies \( \lambda x \in \mathcal{W} \) for any \( \lambda \geq 0 \). A set \( \mathcal{W} \) is called a convex cone if it contains \( \lambda x + \mu y \) for any \( x, y \in \mathcal{W} \) and any \( \lambda, \mu \geq 0 \). Given a set \( \mathcal{W} \), let \( \text{cone}(\mathcal{W}) = \{ \lambda x \mid x \in \mathcal{W}, \lambda \geq 0 \} \) be the cone of \( \mathcal{W} \); and \( \text{convex}(\mathcal{W}) = \{ \lambda x + \mu y \mid x, y \in \mathcal{W}, \lambda, \mu \geq 0, \lambda + \mu = 1 \} \) be the convex hull of \( \mathcal{W} \).

Clearly, for \( m, n \in \mathbb{N} \), \( DD_{m,n} \) is a cone and \( DD^+_{m,n} \) is a convex cone. We will show that \( GDD^+_{m,n} \) is also a convex cone later (see Proposition 10). Next we present a characterization of symmetric H-tensors using symmetric GDD tensors.

**Theorem 3** (5) Theorem 4.9. Let \( m, n \in \mathbb{N} \) and \( \mathcal{A} \in S_{m,n} \). Then \( \mathcal{A} \) is an H-tensor if and only if \( \mathcal{A} \in GDD_{m,n} \).

**Corollary 4.** Let \( m, n \in \mathbb{N} \) and \( \mathcal{A} \in S_{m,n} \). Then \( \mathcal{A} \) is an H\(^*\) tensor if and only if \( \mathcal{A} \in GDD^+_{m,n} \).

From Theorem 3 and Corollary 4 if \( m \) is even, we have the following inclusion relationships:

\[ DD^+_{m,n} \subseteq GDD^+_{m,n} \subseteq PSD_{m,n}. \]

In light of Corollary 4 in what follows we will take the liberty to use both H\(^*\) and GDD\(^*\) interchangeably to refer to H\(^*\)-tensors.

Denote \( \text{card}(A) \) as the cardinality of the set \( A \). For \( m, n \in \mathbb{N} \), define the index sets
\[ \mathcal{P}^m_n = \{(i_1, \ldots, i_m) \mid 1 \leq i_1 \leq \cdots \leq i_m \leq n\} \cap \{(i_1, \ldots, i_m) : \text{card}(i_1, \ldots, i_m) > 1\}, \]
and
\[ \mathcal{F}^m_n = \{(i_1, \ldots, i_m) \mid i \in [n] \}. \]

For any index \( (i_1, \ldots, i_m) \in \mathcal{P}^m_n \cup \mathcal{F}^m_n \), denote \( \mathcal{P}_{i_1, \ldots, i_m} \) as the set of all permutations of \( i_1, \ldots, i_m \) and denote
\[ \mathcal{Q}_{i_1, \ldots, i_m} = \{(p, p, \ldots, p) \mid p \in \{i_1, \ldots, i_m\}\}. \]

Also, for \( (i_1, \ldots, i_m) \in \mathcal{P}^m_n \cup \mathcal{F}^m_n \), let \( \mathcal{D}^{i_1, \ldots, i_m}_{m,n} \in S_{m,n} \) be the set of sparse tensors defined as follows:
\[ \mathcal{D}^{i_1, \ldots, i_m}_{m,n} = \{(a_{j_1, \ldots, j_m}) \in S_{m,n} \mid a_{j_1, \ldots, j_m} = 0 \text{ if } (j_1, \ldots, j_m) \notin \mathcal{P}_{i_1, \ldots, i_m} \cup \mathcal{Q}_{i_1, \ldots, i_m}\}. \] (8)

Further, let
\[ \mathcal{D}_{m,n} = \bigcup_{(i_1, \ldots, i_m) \in \mathcal{P}^m_n} \mathcal{D}^{i_1, \ldots, i_m}_{m,n}. \]

To assist the proofs in this work, we introduce the following class of tensors.
Definition 2. For $m, n \in \mathbb{N}$ and any $(i_1, \ldots, i_m) \in \mathcal{D}^m_n$, $c \in \{0, 1\}$, denote $\mathcal{V}^c_{i_1 \ldots i_m} = (\mathcal{V}^c_{i_1 \ldots i_m}) \in \mathbb{D}^m_{i_1 \ldots i_m}$, as the tensor defined by:

(i) $\mathcal{V}^c_{j_1 \ldots j_m} = (-1)^c$ if $(j_1, \ldots, j_m) \in \mathcal{P}_{i_1 \ldots i_m}$.

(ii) The value of $j$-th diagonal element is equal to the sum of the absolute values of the off-diagonal entries on the $j$-th slice (the diagonal elements are excluded in the sum); that is

$$
\mathcal{V}^c_{j_1 \ldots i_j \ldots j_m} = \sum_{(j_1', \ldots, j_j', \ldots, j_m)} |\mathcal{V}^c_{j_1' \ldots j_j' \ldots j_m}|, \quad \forall j \in [n].
$$

Further, for all $i \in [n]$, denote $\mathcal{V}^0_{i \ldots i}$ as the tensor where the only nonzero entry is $\mathcal{V}^0_{i \ldots i} = 1$; and $\mathcal{V}^1_{i \ldots i}$ as the tensor with all entries set to 0. Also, denote $\mathcal{E}_{m,n} = \{\mathcal{V}^c_{i_1 \ldots i_m} | c \in \{0, 1\}, (i_1, \ldots, i_m) \in \mathcal{D}^m_n \cup \mathcal{P}^m_n\}$.

Clearly, from Definition 2 it follows that for all $(i_1, \ldots, i_m) \in \mathcal{D}^m_n \cup \mathcal{P}^m_n$ and $c \in \{0, 1\}$, $\mathcal{V}^c_{i_1 \ldots i_m} \in DD^*_{m,n}$. For example, when $m = 2$ and $n = 4$, we have

$$
\mathcal{V}^{0,12} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{V}^{1,13} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

For ease of exposition, we also introduce an auxiliary notation for indices. For $m, n \in \mathbb{N}$, index $\vec{i} := (i_1, i_2, \ldots, i_m) \in \mathcal{D}^m_n$ and some $\vec{l} \in [m]$, we call

$$
((\vec{j}_1, \vec{j}_2, \ldots, \vec{j}_p), (\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_p)) \in [n]^{\vec{l}} \times [m]^p
$$

as the tight pair of $\vec{i}$ if $(\vec{j}_1, \vec{j}_2, \ldots, \vec{j}_p)$ and $(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_p)$ satisfy

$$
x_{i_1} x_{i_2} \ldots x_{i_m} = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \ldots x_{j_p}^{\alpha_p},
$$

where $1 \leq j_1 < j_2 < \cdots < j_p \leq n$. We will refer to $(\vec{j}_1, \vec{j}_2, \ldots, \vec{j}_p)$ as the tight index and to $(\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_p)$ as the tight power. However, we will routinely drop the upper $\vec{l}$ in the notation when the $\vec{l}$ we are referring to is clear from (or fixed in) the context. Also, denote $\epsilon_j$ as the unitary vector in the $j$th direction of appropriate dimensions.

3. New characterization of symmetric $H^*$-tensors

Next, we present a new characterization of symmetric $H^*$-tensors (or equivalently GDD$^+$ tensors (cf., Corollary 4)) based on the power cone (11, 12). First, we characterize DD$^*$ tensors with the following result.

Proposition 5. For $m, n \in \mathbb{N}$, $DD^*_{m,n} = \text{convex}(\text{cone}(\mathcal{E}_{m,n}))$ and each tensor in $\mathcal{E}_{m,n}$ generates an extreme ray of $DD^*_{m,n}$.

Proof. First, from Definition 2 it follows that $\mathcal{E}_{m,n} \subseteq DD^*_{m,n}$. This, together with the fact that $DD^*_{m,n}$ is a convex cone, implies that $\text{convex}(\text{cone}(\mathcal{E}_{m,n})) \subseteq DD^*_{m,n}$.

Second, for $\mathcal{A} = (a_{i_1 \ldots i_m}) \in DD^*_{m,n}$, denote $\mathcal{P}_+ = \{(i_1, \ldots, i_m) \in \mathcal{D}^m_n | a_{i_1 \ldots i_m} \geq 0\}$ and $\mathcal{P}_- = \{(i_1, \ldots, i_m) \in \mathcal{D}^m_n | a_{i_1 \ldots i_m} < 0\}$. Then

$$
\mathcal{A} = \sum_{l=1}^n \left( a_{ii \ldots i} - \sum_{(i_1, \ldots, i_n) \notin (i_1 \ldots i)} |a_{i_1 \ldots i_n}| \right) \mathcal{V}^0_{i \ldots i} + \sum_{(i_1, \ldots, i_n) \in \mathcal{P}_+} a_{i_1 \ldots i_n} \mathcal{V}^1_{i \ldots i} + \sum_{(i_1, \ldots, i_n) \in \mathcal{P}_-} (-a_{i_1 \ldots i_n}) \mathcal{V}^1_{i \ldots i}.
$$

Since $\mathcal{A} \in DD^*_{m,n}$, $a_{ii \ldots i} \geq \sum_{(i_1, \ldots, i_n) \notin (i_1 \ldots i)} |a_{i_1 \ldots i_n}|$ for all $i \in [n]$. Thus, $\mathcal{A}$ is in the convex hull of the conic hull of $\mathcal{E}_{m,n}$, after noticing that all the coefficients in the right hand side of (10) are nonnegative. That is $DD^*_{m,n} \subseteq \text{convex}(\text{cone}(\mathcal{E}_{m,n}))$. \qed
To give a similar characterization for GDD$^+$ tensors, we need Theorems 6 and 7 and Propositions 8 and 9.

**Theorem 6** ([32], Theorem 1 (a)). For $m, n \in \mathbb{N}$, if $D \in S_{m,n}$ is a nonnegative tensor, then $\rho(D)$ is an $H$-eigenvalue of $D$.

Denote the largest $H$-eigenvalue of tensor $A \in S_{m,n}$ as $\lambda_{max}(A)$.

**Theorem 7** ([32], Theorem 2). For $m, n \in \mathbb{N}$, if $A \in S_{m,n}$ is a nonnegative tensor, then

$$\lambda_{max}(A) = \max \left\{ A x^m : x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1 \right\}.$$

**Proposition 8.** For $m, n \in \mathbb{N}$, if both $A \in S_{m,n}$ and $B \in S_{m,n}$ are nonnegative tensors, then $\rho(A + B) \leq \rho(A) + \rho(B)$.

**Proof.** Let $D \in T_{m,n}$. From the definition of $\rho(D)$ and $\lambda_{max}(D)$, it clearly follows that $\rho(D) \geq \lambda_{max}(D)$. If $D$ is a symmetric nonnegative tensor, it then follows from Theorem 6 that

$$\rho(D) = \lambda_{max}(D). \quad (11)$$

Let $A \in S_{m,n}$, and $B \in S_{m,n}$ be nonnegative tensors. Then we have from equation (11) that $\rho(A) = \lambda_{max}(A)$ and $\rho(B) = \lambda_{max}(B)$. Furthermore, it follows from Theorem 7 that

$$\lambda_{max}(A + B) = \max \left\{ (A + B) x^m : x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1 \right\} = \max \left\{ A x^m + B y^m : x, y \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1, \sum_{i=1}^n y_i^m = 1, x = y \right\} \leq \lambda_{max}(A) + \lambda_{max}(B).$$

To finish, notice that $A + B$ is a symmetric nonnegative tensor. Thus after using equation (11) for the tensor $A + B$, we conclude that $\rho(A + B) = \lambda_{max}(A + B) \leq \lambda_{max}(A) + \lambda_{max}(B) = \rho(A) + \rho(B)$. \hfill \qed

**Proposition 9** ([3], Proposition 2.7). For $m, n \in \mathbb{N}$, let $A \in S_{m,n}$ be a Z-tensor such that $A \leq B$ where $A$ is an $M$-tensor. Then $B$ is also an $M$-tensor.

**Proposition 10.** For $m, n \in \mathbb{N}$, GDD$^*_{m,n}$ is a convex cone.

**Proof.** Let $A = (a_{i_1, \ldots, i_n}) \in GDD^*_{m,n}$ and $B = (b_{i_1, \ldots, i_n}) \in GDD^*_{m,n}$. From Corollary 2, both $A$ and $B$ are symmetric $H^*$-tensors. Thus $M(A)$ and $M(B)$ are symmetric $M$-tensors. That is, there exist nonnegative scalars $s_1, s_2$ and nonnegative tensors $D_1$ and $D_2$ such that $M(A) = s_1 I - D_1$, $M(B) = s_2 I - D_2$ and $s_1 \geq \rho(D_1)$, $s_2 \geq \rho(D_2)$. Then $M(A) + M(B) = (s_1 + s_2) I - (D_1 + D_2)$. Since $s_1 + s_2 \geq 0$ and $D_1 + D_2$ is a nonnegative tensor, $M(A) + M(B)$ is a symmetric Z-tensor. Also, from Proposition 8 if follows that $\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2) \leq s_1 + s_2$. Thus, $M(A) + M(B)$ is also a symmetric $M$-tensor.

Next, we prove that $M(A + B)$ is a Z-tensor. Recall that $M(A + B)$ is the comparison matrix of $A + B$. Thus, all its diagonal elements are nonnegative and all off-diagonal elements are nonpositive. Denote $s = \max |[a_{i_1, i_2}] + [b_{i_1, i_2}], i \in [n]|$. Then $M(A + B) = s I - (s I - M(A + B))$ where $s I - M(A + B)$ is a nonnegative tensor. Thus, $M(A + B)$ is a Z-tensor.

From the definition of comparison tensors and the fact that $A, B$ have nonnegative diagonal elements, $M(A + B) \geq M(A) + M(B)$ componentwise. From the fact that $M(A) + M(B)$ is an $M$-tensor and $M(A + B)$ is a Z-tensor, it follows from Proposition 8 that $M(A + B)$ is also an $M$-tensor. Thus $A + B$ is a symmetric $H^*$-tensor, and from Corollary 2, $A + B$ is a GDD$^+$ tensor. Thus, $A + B \in GDD^*_{m,n}$. This, together with the fact that $A \in GDD^*_{m,n}$ implies $\lambda A \in GDD^*_{m,n}$ for any nonnegative scalar $\lambda$, implies that $GDD^*_{m,n}$ is a convex cone. \hfill \qed
Theorem 11. For $m, n \in \mathbb{N}$, $\mathcal{A} \in GDD^+_{m,n}$ if and only if $\mathcal{A} = \sum_{i=1}^r B_i$, where $r \in \mathbb{N}$ and $B_i \in \mathbb{D}_{m,n} \cap GDD^+_{m,n}$.

Proof. For $m, n \in \mathbb{N}$, let $\mathcal{A} \in GDD^+_{m,n}$. Then, from Proposition 1, there exists a positive diagonal matrix $D$ such that $\mathcal{B} := ADD \cdots D \in DD^+_{m,n}$. From Proposition 2 it follows that there exist $r \in \mathbb{N}$, $\lambda_i \geq 0_i, C_i \in \mathbb{P}_{m,n} \subset \mathbb{D}_{m,n} \cap DD^+_{m,n}$ for $i \in [r]$ such that $\mathcal{B} = \sum_{i=1}^r C_i$. Then $\mathcal{A} = \sum_{i=1}^r C_i D_i \cdots D^i D^i$. Let $B_i = C_i D_i \cdots D^i D^i$ for all $i \in [r]$. Then the only if statement follows after noticing that for all $i \in [r], B_i \in GDD^+_{m,n}$ and $B_i \in \mathbb{D}_{m,n}$ (as multiplying with positive numbers will not affect the sparse structure of tensors $C_i \in \mathbb{P}_{m,n}, i \in [r]$). For the if statement, note that if $\mathcal{A} = \sum_{i=1}^r B_i$ with $B_i \in \mathbb{D}_{m,n} \cap GDD^+_{m,n}$, then, from Proposition 10, we have $\mathcal{A} \in GDD^+_{m,n}$.

The matrix version of Theorem 11 has been presented in [6, 8].

Lemma 12 ([8], Lemma 3.8). For $n \in \mathbb{N}$, if matrix $A \in S_{2,n}$, then $A$ is a GDD$^+$ matrix if and only if $A = \sum_{i=1}^m M^i$ where each $M^i \in S_{2,n}$ with zeros everywhere except for four entries $(M^i)_i$, $(M^i)_j$, $(M^i)_{ij}$, $(M^i)_{ji}$ which make $M^i$ symmetric and positive semidefinite.

It is easy to see that $M^i$ in Lemma 12 is positive semidefinite if and only if $M^i$ is a GDD$^+$ matrix. Thus, Lemma 12 can be regarded as a special case of Theorem 11. In Theorem 13, we provide sufficient and necessary conditions for a tensor to be in $\mathbb{D}_{m,n} \cap GDD^+_{m,n}$ (i.e., a sparse GDD$^+$ tensor).

Theorem 13. Let $m, n \in \mathbb{N}$, $(i_1, \ldots, i_m) \in \mathcal{P}^m_n \cup \mathcal{U}^m_n$, and a tensor $\mathcal{B} = (b_{p_1 \ldots p_m}) \in \mathbb{D}_{m,n}$ be given. Then,

(i) if $(i_1, \ldots, i_m) \in \mathcal{P}^m_n$, $\mathcal{B} \in GDD^+_{m,n}$ if and only if its entries satisfy

$$
\prod_{i=1}^m b_{i_1 \ldots i_m}^a \geq c|b_{i_1 \ldots i_m}|^m,
$$

(12)

where $c = \prod_{i=1}^m (a_i - 1)^{a_i}$, and $((j_1, \ldots, j_m), \alpha = (\alpha_1, \ldots, \alpha_m))$ is the tight pair associated with $(i_1, \ldots, i_m)$, and

$$
b_{pp \ldots p} \geq 0, \quad \forall (p, p, \ldots, p) \in \mathcal{S}_{i_1 \ldots i_m}.
$$

(13)

(ii) If $(i_1, \ldots, i_m) \in \mathcal{P}^m_n$, $\mathcal{B} \in GDD^+_{m,n}$ if and only if $\mathcal{B}$ is a diagonal tensor satisfying $b_{i_1 \ldots i_m} \geq 0$.

Proof. Let $(i_1, \ldots, i_m) \in \mathcal{P}^m_n$ be given. Denote $((j_1, \ldots, j_m), \alpha = (\alpha_1, \ldots, \alpha_m))$ as the tight pair associated with $(i_1, \ldots, i_m)$. Let $\mathcal{B} \in \mathbb{D}_{m,n}$. Then, all the off-diagonal elements of $\mathcal{B}$ are zero except for the elements $b_{p_1 \ldots p_m}$, where $(p_1, \ldots, p_m) \in \mathcal{S}_{i_1 \ldots i_m}$. Then, using Proposition 1 it follows that $\mathcal{B} \in GDD^+_{m,n}$ if and only if its entries satisfy (13) and

$$
b_{i_1 \ldots i_m} d_{i_1 \ldots i_m}^m \geq \left(\frac{m-1}{\alpha - a_i}\right) |b_{i_1 \ldots i_m}| d_{i_1 \ldots i_m},
$$

(14)

for $k \in [l]$ and some $d_{i_k} > 0$, for all $k \in [l]$, after using (1), the sparsity pattern and symmetry of $\mathcal{B}$, and the fact that the number of equal summands in the right-hand side of (1) in this case is $m-1$.

Now note that if (13) and (15) hold then (13) and

$$
b_{i_1 \ldots i_m} d_{i_1 \ldots i_m}^m \geq \left(\frac{m-1}{\alpha - a_i}\right) |b_{i_1 \ldots i_m}| d_{i_1 \ldots i_m},
$$

(15)

hold for all $k \in [l]$, and some $d_{i_k} > 0$, for all $k \in [l]$; since (15) is obtained by taking the $a_i$th power on both sides of (14), whose (multiplicative) terms are all nonnegative. Given that both the left-hand side and the right-hand side of (15) are nonnegative, it follows, after multiplying the left-hand sides and the right-hand sides of (15) for all $k \in [l]$, and using the fact that $|x|_{a} = m$, that (13) and (15) imply (13) and

$$
\prod_{i=1}^m |b_{i_1 \ldots i_m}|^m (d_{i_1 \ldots i_m})^m,
$$

(16)
for some $d_k > 0$, for all $k \in [l]$. In turn, (16) is equivalent to (12), with $c := \prod_{k=1}^{l} (m-1)^{m_k}$, after noticing that from the definition of tight pair (8), it follows that

$$\prod_{k=1}^{l} d_k^{m_k} = d_i d_j \ldots d_k .$$

(17)

Now, to complete the proof, we show that (13) and (14) imply (15) and (16) (i.e., that $B$ is a $GDD^*_{m,n}$ tensor). First note that if for any $k \in [l]$, $b_{j_k \ldots k} = 0$, then (14) implies that $b_{i \ldots i} = 0$. Thus, in this case, given (13) and the fact that $d_k > 0$ for all $k \in [l]$, it follows that (14) is satisfied for all $k \in [l]$. Moreover, in the case where $b_{i \ldots i} = 0$, condition (14) follows from (13), given the fact that $d_k > 0$ for all $k \in [l]$. Thus, it is enough to consider the case in which $b_{j_k \ldots k} > 0$ for all $k \in [l]$, and $b_{i \ldots i} \not= 0$. In this case, using the fact that $d_k > 0$, we can write that

$$d_k = z_k \sqrt{\frac{(m-1)^{m_k}}{b_{j_k \ldots k}}} ,$$

(18)

for some $z_k > 0$, for all $k \in [l]$. Thus, for any $k \in [l]$, it follows that

$$|d_k| b_{i \ldots i} \ldots d_k = z_k^m |b_{i \ldots i}| \frac{c}{\prod_{i \neq k} b_{i \ldots i}} \leq \frac{b_{j_k \ldots k} d_k^m}{(m-1)} ,$$

(19)

where the first equality follows by using (17), (18), and the definition of $c$; the inequality follows from (12), and the last equality follows by using (18) again. After noticing that (19) is equivalent to (14), it then follows that (13) and (12) imply (13) and (14); that is, that $B \in GDD^*_{m,n}$.

If $(i_1, \ldots, i_m) \in \mathcal{F}^m$ and tensor $B = (b_{p_i \ldots p_n}) \in \mathcal{D}_{m,n}$, it follows from the definition of $\mathcal{D}_{m,n}$ (i.e., (8)) that $B$ is a diagonal tensor in which the only nonzero entry is $b_{i_1 \ldots i_m}$. Thus, $B \in GDD^*_{m,n}$ tensor if and only if $B$ is a diagonal tensor satisfying $b_{i_1 \ldots i_m} \geq 0$.

Next, we apply Theorem 11 and Theorem 13 to obtain sufficient and necessary conditions for a tensor $A \in \mathcal{S}_{m,n}$ to be an $H^*$-tensor (or equivalently $GDD^*$ tensor).

**Corollary 14.** Let $m, n \in \mathbb{N}$. Then $A = (a_{p_i \ldots p_n}) \in \mathcal{S}_{m,n}$ is a $GDD^*$ tensor if and only if there exist $b^i_j \geq 0$ for all $i \equiv (i_1, \ldots, i_m) \in \mathcal{G}_n^m$, $j \in \tilde{i}$ satisfying

(i) For $\tilde{i} \in \mathcal{G}_n^m$,

$$\prod_{k=1}^{l} (b^i_j)^{\alpha^k} \geq c(\tilde{i}) \alpha^m$$

(20)

where $c(\tilde{i}) := \prod_{k=1}^{l} (m-1)^{m_k}$, and $(\alpha^1, \alpha^2, \ldots, \alpha^l), \alpha^j = (\alpha^1, \alpha^2, \ldots, \alpha^l)$ is the tight pair associated with $\tilde{i}$.

(ii) For $j \in [n]$,

$$a_{j \ldots j} \geq \sum_{\tilde{i} \in \mathcal{G}_n^m} b^i_j$$

(21)

**Proof.** Let $m, n \in \mathbb{N}$. From Theorem 11, $A = (a_{p_i \ldots p_n}) \in \mathcal{S}_{m,n}$ is a $GDD^*$ tensor if and only if

$$A = \sum_{\tilde{i} \in \mathcal{G}_n^m} B^\tilde{i}$$

(22)

and for $\tilde{i} \in \mathcal{G}_n^m \cup \mathcal{F}_n^m$, $B^\tilde{i} = (b^i_j)_{p_i \ldots p_n} \in \mathcal{D}_{m,n}$ satisfies conditions (11) and (12) in Theorem 13. Note that from the sparse structure of the tensors $B^\tilde{i}$ used in (22), it follows that for any $j \in [n]$,

$$a_{j \ldots j} = \sum_{\tilde{i} \in \mathcal{G}_n^m, j \equiv (j_1, \ldots, j_m) \in \mathcal{G}_n^m} b^i_j + b^i_{j \ldots j}$$

(23)
and for any \( \vec{t} \in D_n^m \),
\[
a'_i = b_{i}^\vec{t}
\]  
(24)

From Theorem [13][1] and (23), it follows that 
\[
c(\vec{t})|a_j^m| = c(\vec{t})|b_{i}^\vec{t}| \leq \prod_{k=1}^{t} (b_{ij}^\vec{t})^{\vec{t}_i} m^n
\]  
where
\[
c(\vec{t}) = \prod_{k=1}^{t} (\vec{t}^{m^n} - c_i)
\]  
\((\vec{t}_j, \vec{t}_j, \ldots, \vec{t}_j)\), \(a_j = (a_j^1, a_j^2, \ldots, a_j^n)\) is the tight pair associated with \( \vec{t} \), and \( b_{ij}^\vec{t} \) is a DD tensor. Thus, from Definition [13][1], \( b_{ij}^\vec{t} \geq 0 \) for all \((p, p, \ldots, p) \in D_p\). The statement then follows from this and (23), after noticing that from Theorem [13][1] \( b_{ij}^\vec{t} \geq 0 \) for all \( j \in [n] \), and after simplifying notation to let \( b_j^\vec{t} \geq 0 \) for any \( \vec{t} \in D_n^m : (jj) \in D_p \); that is, for any \( \vec{t} \in D_n^m : j \in \vec{t} \).

Now we provide an example to illustrate the results in Theorem [11] and Corollary [14].

**Example 1.** Consider the following symmetric tensor
\[
A = (a_{i,j,k,l}) = [A(1, 1, 1, 1); A(1, 2, 1, 1); A(2, 1, 1, 1); A(2, 2, 1, 1)] \in S_{4,2},
\]
where
\[
A(1, 1, 1, 1) = \begin{pmatrix} 4 & -2 & -2 & -1 \\ -2 & -1 & -1 & 64/3 \\ -2 & -1 & -1 & 64/3 \\ -1 & 64/3 & 1000 \end{pmatrix},
\]
\[
A(2, 1, 1, 1) = \begin{pmatrix} -2 & -1 & -1 & 64/3 \\ -1 & 64/3 & 1000 \end{pmatrix}
\]

Denote \( D_1 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}, D_2 = \begin{pmatrix} 1/2 & 0 & 0 & 4 \\ 0 & 1/3 & 0 & 4 \end{pmatrix}, \) Then, one can obtain
\[
A = \frac{1037}{1296} V^{0,1111} + 168 V^{0,2222} + 2 B^{(1112)} + 2 B^{(1122)} + 2 B^{(2222)},
\]
where
\[
B^{(1112)} = (b^{(1112)}_{j, j, j, j}) = V^{1,1112} D_1 D_1 D_1 D_1,
\]
\[
B^{(1122)} = (b^{(1122)}_{j, j, j, j}) = V^{1,1122} D_2 D_2 D_2 D_2,
\]
\[
B^{(2222)} = (b^{(2222)}_{j, j, j, j}) = V^{0,2222} D_3 D_3 D_3 D_3.
\]

Let \( b_j^\vec{t} = b_{ijj}^\vec{t} \geq 0 \), \( j \in \vec{t} \) for \( \vec{t} \in D_n^3 \). Then it is easy to show that these \( b_j^\vec{t} \), \( j \in \vec{t}, \vec{t} \in D_n^3 \) satisfy (21) and (20). As a result, from Corollary [14], \( A \) is a symmetric \( H^* \)-tensor (GDD\(^* \) tensor).

On the other hand, denote \( D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1/2 \end{pmatrix} \). Then
\[
A = ADDDD = [\tilde{A}(1, 1, 1, 1); \tilde{A}(1, 2, 1, 1); \tilde{A}(2, 1, 1, 1); \tilde{A}(2, 2, 1, 1)],
\]
where
\[
\tilde{A}(1, 1, 1, 1) = \begin{pmatrix} 324 & -27 & -27 & -9/4 \\ -27 & -9/4 & -9/4 & 8 \\ -27 & -9/4 & -9/4 & 8 \\ -9/4 & 8 & 8 & 125/2 \end{pmatrix},
\]
is a DD\(^* \) tensor. Thus, from Definition [14][1], \( A \) is a symmetric \( H^* \)-tensor (GDD\(^* \) tensor).
3.1. Checking membership with power cone optimization

Corollary \[12\] readily implies that membership in the set of \( H^* \)-tensors can be tested using tractable conic optimization techniques, and more precisely, the power cone \([\text{see, e.g., 11, 12}].\) To illustrate this, let us first introduce the high-dimensional power cone.

**Definition 3 (High-dimensional power cone [11, Sec. 4.1.2]).** For any \( \alpha \in \mathbb{R}_+^m \) such that \( e^\top \alpha = 1 \), the high-dimensional power cone is defined by

\[
\mathbb{K}_\alpha^{(m)} = \{ (x, z) \in \mathbb{R}_+^m \times \mathbb{R} : x_1^{\alpha_1} \cdots x_m^{\alpha_m} \geq |z| \}
\]  

(25)

Now, for any tensor \( \mathcal{A} \in \mathbb{S}_{m,n} \), let

\[
\mathbb{F}(\mathcal{A}) = \left\{ d_j^i \in \mathbb{R}, i \in \mathcal{D}_n, j \in \mathcal{I} : a_{ij} - \sum_{i \in \mathcal{D}_n, j \in \mathcal{I}} d_j^i \geq 0 \quad \forall j \in [n] \right\}
\]  

(26)

The result follows from Corollary 15, equation (27), and the fact that \( \mathbb{F}(\mathcal{A}) \) can be checked in polynomial time using appropriate interior point methods \([\text{see, e.g., 33}].\) To show this, we make use of the power cone, which is a lower-dimensional version of the high-dimensional power cone introduced in Definition 3. Namely, for any \( \alpha \in [0, 1] \), the power cone \( \mathbb{K}_\alpha := \mathbb{K}_\alpha^{2} \) is defined by

\[
\mathbb{K}_\alpha^2 = \{ (x, z) \in \mathbb{R}_+^2 \times \mathbb{R} : x_1^{\alpha_1} x_2^{1-\alpha_1} \geq |z| \}
\]  

(25)

Furthermore, the condition \( \mathbb{F}(\mathcal{A}) \neq \emptyset \) in Corollary 15 can be checked in polynomial time using appropriate interior point methods \([\text{see, e.g., 33}].\) To show this, we make use of the power cone, which is a lower-dimensional version of the high-dimensional power cone introduced in Definition 3. Namely, for any \( \alpha \in [0, 1] \), the power cone \( \mathbb{K}_\alpha := \mathbb{K}_\alpha^{2} \) is defined by

\[
\mathbb{K}_\alpha^2 = \{ (x, z) \in \mathbb{R}_+^2 \times \mathbb{R} : x_1^{\alpha_1} x_2^{1-\alpha_1} \geq |z| \}
\]  

(25)

As shown in [11, eq. (4.3), Sec. 4.1.2], the higher-dimensional power cone \( \mathbb{K}_\alpha^{(m)} \) can be decomposed into \( m \) \( \text{low-dimensional} \) power cones. Using this fact, we can rewrite (26) as follows:

\[
\mathbb{F}(\mathcal{A}) = \left\{ d_j^i \in \mathbb{R}, i \in \mathcal{D}_n, j \in \mathcal{I} \right\}
\]  

(27)

The relevance of introducing the power cone in (27) is that [11, 35, 36] provide different self-concordant barriers for the power cone. In short, this means that for any \( \mathcal{A} \in \mathbb{S}_{m,n} \), the nonsymmetric conic feasibility system defined by (27) can be solved in polynomial time using a primal-dual predictor-corrector method [10]. The reference to nonsymmetric, stems from the fact that the power cone is not symmetric if \( \alpha \neq \frac{1}{2} \) [12, 57].

**Theorem 16.** For \( m, n \in \mathbb{N} \), to check if a tensor in \( \mathbb{S}_{m,n} \) is an \( H^* \)-tensor \((GDD^* \text{ tensor})\) is equivalent to solve a power cone optimization problem of size polynomial in \( n \) for a fixed \( m \).

*Proof.* The result follows from Corollary 15 equation (27), and the fact that \( |\mathbb{S}_n| = \binom{n+m-1}{m} - n \). \( \square \)

For a detailed discussion of the properties of, and optimization over the power cone, we direct the reader to [11, 38].

4. Minimum \( H \)-eigenvalue of \( M \)-tensors

The problem of obtaining bounds on the minimum \( H \)-eigenvalue of \( M \)-matrices and \( M \)-tensors has received significant attention in the literature [21, 24]. This is due to the important role the \( M \)-tensors play in a wide range of interesting applications \([\text{sec. 21, and the references therein}].\) However, these bounds are loose \([\text{see, e.g., 21, Table 1}].\) and even expensive to compute \([\text{see, e.g., 21, Table 2}].\) Next, we show that the characterization in Corollary 15 can be applied to obtain the exact minimum \( H \)-eigenvalue of symmetric \( M \)-tensors in polynomial time. Besides, this result can also be used to obtain lower bounds for the minimum \( H \)-eigenvalue of general \((\text{i.e. not necessarily symmetric})\) \( M \)-tensors in polynomial time. For that purpose, we first introduce the following results.
Thus, one can simplify (29) and obtain (28) for an \(M\)-tensor, where \(a\) and \(b\) are two real numbers. Then \(\mu\) is an eigenvalue (H-eigenvalue) of \(B\) if and only if \(\mu = a(\lambda + b)\) and \(\lambda\) is an eigenvalue (H-eigenvalue) of \(A\).

Lemma 18. For \(m, n \in \mathbb{N}\), if \(A = sI - D \in \mathbb{S}_{m,n}\) where \(D\) is a nonnegative tensor and \(s\) is a scalar, then \(s - \rho(D)\) is the minimum H-eigenvalue of \(A\).

Proof. First, from Theorem 6, it follows that \(\rho(D)\) is an H-eigenvalue of \(D\). Then, from Lemma 17, \(s - \rho(D)\) is an H-eigenvalue of \(A\). Assume that \(\lambda\) is an H-eigenvalue of \(A\). Then, \(s - \lambda\) is an H-eigenvalue of \(D\). Thus, \(\rho(D) \geq |s - \lambda| \geq s - \lambda\). That is, \(\lambda \geq s - \rho(D)\). Thus, \(s - \rho(D)\) is the minimum H-eigenvalue of \(A\).

In what follows, for any \(A \in \mathbb{S}_{m,n}\), let \(\lambda_{\min}(A)\) denote the minimum H-eigenvalue of \(A\).

Proposition 19. For \(m, n \in \mathbb{N}\), if \(A \in \mathbb{S}_{m,n}\) is an M-tensor, then for any \(\lambda \leq \lambda_{\min}(A), A - \lambda I\) is also an M-tensor.

Proof. Since \(A \in \mathbb{S}_{m,n}\) is an M-tensor, then there exist a nonnegative tensor \(D\) and nonnegative scalar \(s \geq \rho(D)\) such that \(A = sI - D\). Then, for any \(\lambda \leq \lambda_{\min}(A), A - \lambda I = (s - \lambda)I - D\).

From Lemma 18, \(\lambda_{\min}(A) = s - \rho(D)\). Thus for any \(\lambda \leq \lambda_{\min}(A), s - \lambda - \rho(D) \geq s - \lambda_{\min}(A) - \rho(D) = 0\). Furthermore, \(s - \lambda - \rho(D) \geq 0\). As a result, \(A - \lambda I\) is an M-tensor. Now, for some \(\lambda > \lambda_{\min}(A), A - \lambda I\) is an M-tensor. Then there exist a nonnegative tensor \(D\) and nonnegative scalar \(s \geq \rho(D)\) such that \(A - \lambda I = sI - D\). Thus \(A = (\lambda + s)I - D\).

From Lemma 18, \(\lambda_{\min}(A) = (\lambda + s) - \rho(D) \geq \lambda\) which contradicts the condition \(\lambda > \lambda_{\min}(A)\). Thus, \(A - \lambda I\) is not an M-tensor.

Note that from Corollary 15 and the definition of \(H^t\)-tensors in terms of the comparison tensor (cf., (1)), one obtains the following characterization for symmetric M-tensors.

Corollary 20. Let \(m, n \in \mathbb{N}\). Then \(A = (a_{i_1i_2...i_m}) \in \mathbb{S}_{m,n}\) is an M-tensor if and only if \(a_{i_1i_2...i_m} \leq 0\) for all \((i_1, i_2, \ldots, i_m) \in \mathcal{D}^m_\ast\), and \(F(A) \neq \emptyset\).

Proposition 19 and the characterization of M-tensors in Corollary 20, readily provide a way to compute the H-eigenvalue of symmetric M-tensors in polynomial time.

Corollary 21. For \(m, n \in \mathbb{N}\), if \(A \in \mathbb{S}_{m,n}\) is an M-tensor, then

\[
\lambda_{\min}(A) = \max \{\lambda : F(A - \lambda I) \neq \emptyset\}.
\]

(28)

Proof. From Proposition 19 it follows that

\[
\lambda_{\min}(A) = \max \{\lambda : A - \lambda I\text{ is an M-tensor}\}.
\]

Then using Corollary 20 to characterize the membership in the set of symmetric M-tensors, one can have

\[
\lambda_{\min}(A) = \max \{\lambda : F(A - \lambda I) \neq \emptyset, (A - \lambda I)_{i_1i_2...i_m} \leq 0, \forall (i_1, i_2, \ldots, i_m) \in \mathcal{D}^m_\ast\}.
\]

(29)

If \(A\) is an M-tensor, then for any \(\lambda \in \mathbb{R}, (A - \lambda I)_{i_1i_2...i_m} \leq 0, \forall (i_1, i_2, \ldots, i_m) \in \mathcal{D}^m_\ast\).

Thus, one can simplify (29) and obtain (28) for an M-tensor \(A\).
follows, for $A \in \mathbb{R}^{m \times n \times n}$, the results in [21] are particularly tight in comparison with the actual minimum $H$-eigenvalues of its corresponding symmetrized tensor. We prove this fact in the discussion next.

**Lemma 22 ([39], Lemma 2.3).** For $m, n \in \mathbb{N}$ and $M$-tensor $A \in \mathcal{S}_{m,n}$,

$$
\lambda_{\min}(A) = \min \left\{ \mathcal{A} x^m : x \in \mathbb{R}^n, \sum_{i=1}^n x_i^m = 1 \right\}.
$$

(30)

Let $\tau(\mathcal{A}) = \min \{\text{Re}(\lambda) : \lambda \in \sigma(\mathcal{A})\}$ where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of $\mathcal{A}$.

**Lemma 23 ([21], Theorem 3.4(a)).** If $m, n \in \mathbb{N}$ and $M$-tensor $\mathcal{A} \in \mathcal{T}_{m,n}$, then $\tau(\mathcal{A})$ is an $H$-eigenvalue of $\mathcal{A}$. That is $\lambda_{\min}(\mathcal{A}) = \tau(\mathcal{A})$.

In light of Lemma 23 in what follows, we use notation $\lambda_{\min}(\mathcal{A})$ to refer to $\tau(\mathcal{A})$ for an $M$-tensor $\mathcal{A}$. Note $\tau(\mathcal{A})$ is used in [21].

**Lemma 24.** If $\mathcal{A} \in \mathcal{T}_{m,n}$, then

$$
\lambda_{\min}(\mathcal{A}) \geq \lambda_{\min}(\text{sym}(\mathcal{A})).
$$

(31)

**Proof.** From the definition of $H$-eigenvalue of a tensor (see Section 1), if real value $\lambda$ is an $H$-eigenvalue of $\mathcal{A}$, then there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$
\mathcal{A} x^{m-1} = \lambda x^{m-1}.
$$

Thus, $\lambda$ satisfies $\mathcal{A} x^m = \lambda \sum_{i=1}^n x_i^m$. When $m$ is even, then clearly $\sum_{i=1}^n x_i^m > 0$. When $m$ is odd and assume $\sum_{i=1}^n x_i^m < 0$, one can set $y = -x$. Then $y$ and $\lambda$ satisfy

$$
\sum_{i=1}^n x_i^m > 0, \quad \mathcal{A} y^m = \lambda \sum_{i=1}^n y_i^m.
$$

Thus, for each $H$-eigenvalue $\lambda$ of $\mathcal{A}$, there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that

$$
\sum_{i=1}^n x_i^m > 0, \quad \mathcal{A} x^m = \lambda \sum_{i=1}^n x_i^m.
$$

---

Table 1: $H$-eigenvalues of symmetric $M$-tensors.

| symmetrized $M$-tensor | $m$ | $n$ | best lower bound [21] | value [28] | best upper bound [21] |
|-------------------------|-----|-----|------------------------|------------|-----------------------|
| Example 3.1 in [21]    | 3   | 3   | 1.1196                 | 5.8046     | 6.9383                |
| Example 3.2 in [21]    | 3   | 3   | 2.6088                 | 7.7442     | 9.1984                |

---

1Tensor $\mathcal{A}$ is called the symmetrized version of tensor $\mathcal{B}$ if their corresponding polynomials are the same and $\mathcal{A}$ is a symmetric tensor. In what follows, for $\mathcal{A} \in \mathcal{T}_{m,n}$, denote $\text{sym}(\mathcal{A})$ as the symmetrized version of $\mathcal{A}$. 

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Following this result, one can have

\[ \lambda \geq \min \left\{ \mathcal{A}x^m : x \in \mathbb{R}^n, \sum_{i=1}^n x_i^m = 1 \right\}. \]

Furthermore, \( \lambda \geq \lambda_{\min}(\text{sym}(\mathcal{A})) \) from Lemma 24.

To show the performance of the proposed method in obtaining the lower bound of the minimum \( H \)-eigenvalue of general \( M \)-tensors, we apply it to compute the lower bounds of the minimum \( H \)-eigenvalues of the \( M \)-tensors considered in Example 3.1 and Example 3.2 in [21] (i.e. Different from the tensors in Table 1 here the tensors are directly from [21]). Namely, in Table 2, we list the best upper and lower bounds for the minimum \( H \)-eigenvalue of these \( M \)-tensors obtained using (28).

### Table 2: \( H \)-eigenvalues of asymmetric \( M \)-tensors.

| \( M \)-tensor       | \( m \) | \( n \) | best lower bound [21] | lower bound [28] | best upper bound [21] |
|----------------------|--------|--------|------------------------|------------------|-----------------------|
| Example 3.1 in [21]  | 3      | 3      | 3.0738                 | 5.8046           | 6.8390                |
| Example 3.2 in [21]  | 3      | 3      | 4.0768                 | 7.7442           | 9.0313                |

Table 2 shows that the lower bounds obtained using (28) are much tighter than the lower bounds obtained in [21]. This empirically indicates that the proposed lower bound is able to provide high quality bounds when comparing with the methods in [21].

All the tests in Table 1 and 2 were implemented in MATLAB using the Systems Polynomial Optimization Toolbox (SPOT) [40], and the solver MOSEK [38], using an Intel computer Core i7-4770HQ with 2.20 GHz frequency and 16 GB RAM memory.

### 5. Minimum \( H \)-eigenvalue of the Fan product of symmetric \( M \)-tensors

For two tensors \( \mathcal{A} = (a_{i_1i_2,...,i_m}) \in \mathbb{T}_{m,n} \) and \( \mathcal{B} = (b_{i_1i_2,...,i_m}) \in \mathbb{T}_{m,n} \), Fan product \( \mathcal{A} \star \mathcal{B} \) is a tensor defined by

\[ (\mathcal{A} \star \mathcal{B})_{i_1i_2,...,i_m} = (-1)^{i_1i_2,...,i_m} a_{i_1i_2,...,i_m} b_{i_1i_2,...,i_m}. \]  

One of the main characteristics of Fan product is that the Fan product of \( M \)-tensors is also an \( M \)-tensor [26]. The authors in [26–28] propose bounds for the minimum \( H \)-eigenvalue of \( \mathcal{A} \star \mathcal{B} \) where \( \mathcal{A} \) and \( \mathcal{B} \) are \( Z \)-matrices (\( Z \)-tensors). With the help of the proposed new characteristics of symmetric \( M \)-tensors, we provide tighter bounds for the minimum \( H \)-eigenvalue of Fan product of symmetric \( M \)-tensors. Specifically, we show both theoretically and empirically that our proposed lower bounds are tighter than any of the bounds provided in [26].

For a symmetric \( M \)-tensor \( \mathcal{A} \), it follows from Theorem 11 and 13 and Corollary 21 that one can write \( \mathcal{A} \) as

\[ \mathcal{A} = \lambda_{\min}(\mathcal{A}) I + \sum_{i \in \mathcal{D}^+} \mathcal{A}_i, \]  

where \( \mathcal{A}_i = (a^i_{j_1j_2,...,j_m}) \in \mathbb{GDD}_{m,n}^+ \cap \mathbb{D}^+_m \). Similarly, for another symmetric \( M \)-tensor \( \mathcal{B} \), we can also decompose it as

\[ \mathcal{B} = \lambda_{\min}(\mathcal{B}) I + \sum_{i \in \mathcal{D}^+} \mathcal{B}_i, \]  

where \( \mathcal{B}_i = (b^i_{j_1j_2,...,j_m}) \in \mathbb{GDD}_{m,n}^+ \cap \mathbb{D}^+_m \). From Theorem 16 the decomposition of \( \mathcal{A} \) and \( \mathcal{B} \) can be done in polynomial time.
With the help of this decomposition, we are able to have tighter lower bounds of the minimum $H$-eigenvalue for Fan product of two symmetric $M$-tensors. Before presenting the lower bounds, we need the following result for the minimum $H$-eigenvalue of $M$-tensors.

**Lemma 25** ([26], Lemma 2.2). If $\mathcal{A} = (a_{i_1 \ldots i_n}) \in \mathbb{T}_{m,n}$ is an $M$-tensor, then

$$\min_{1 \leq i \leq n} \left( \frac{\langle \mathcal{A} v_i \rangle^m}{v_i^m} \right) \leq \lambda_{\min}(\mathcal{A})$$

for any $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_{++}$.

Then for $M$-tensor $\mathcal{A} = (a_{i_1 \ldots i_n}) \in \mathbb{T}_{m,n}$ and $\mathcal{B} = (a_{i_1 \ldots i_n}) \in \mathbb{T}_{m,n}$ and decomposition (33) and (34), let

$$\Omega_1(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \sum_{(i_2, \ldots, i_m) \in G^m} \frac{1}{(m-1)!} \sum_{(j_1, \ldots, j_m) \in G^m} a_{i_2 \ldots i_m} b_{i_1 \ldots i_m} \right),$$

(35)

$$\Omega_2(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \sum_{(i_2, \ldots, i_m) \in G^m} |a_{i_2 \ldots i_m}| b_{i_1 \ldots i_m} \right),$$

(36)

$$\Omega_3(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \sum_{(i_2, \ldots, i_m) \in G^m} |b_{i_2 \ldots i_m}| a_{i_1 \ldots i_m} \right),$$

(37)

and

$$\Omega_4(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \sum_{(i_2, \ldots, i_m) \in G^m} |b_{i_2 \ldots i_m}| a_{i_1 \ldots i_m} \right).$$

(38)

Let $\alpha_i(\mathcal{A}) = \max_{(i_2, \ldots, i_m) \in G^m} |a_{i_2 \ldots i_m}|$ for $i \in [n]$.

$$\Gamma_1(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} \lambda_{\min}(\mathcal{B}) + b_{i_1 \ldots i_n} \lambda_{\min}(\mathcal{A}) \right) - \lambda_{\min}(\mathcal{A}) \lambda_{\min}(\mathcal{B}),$$

(39)

$$\Gamma_2(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \alpha_i(\mathcal{A})(b_{i_1 \ldots i_n} - \lambda_{\min}(\mathcal{B})) \right),$$

(40)

$$\Gamma_3(\mathcal{B}, \mathcal{A}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \alpha_i(\mathcal{B})(a_{i_1 \ldots i_n} - \lambda_{\min}(\mathcal{A})) \right),$$

(41)

and

$$\Gamma_4(\mathcal{A}, \mathcal{B}) = \min_{1 \leq i \leq n} \left( a_{i_1 \ldots i_n} b_{i_1 \ldots i_n} - \alpha_i(\mathcal{A})(a_{i_1 \ldots i_n} - \lambda_{\min}(\mathcal{A})) \right).$$

(42)

Expressions (39), (40), (41) and (42) are proposed in [26] as lower bounds for the minimum $H$-eigenvalue of Fan product of $M$-tensors. In Theorem [26], we prove the expressions (35), (36), (37) and (38) tighten these lower bounds. As it will be illustrated in Table 3, which of the expression (35), (36), (37) and (38) provides the best lower bound depends on the specific tensors being considered.

**Theorem 26.** For symmetric $M$-tensors $\mathcal{A}$ and $\mathcal{B}$,

(i) $\Gamma_1(\mathcal{A}, \mathcal{B}) \leq \Omega_1(\mathcal{A}, \mathcal{B}) \leq \lambda_{\min}(\mathcal{A} \ast \mathcal{B})$

(ii) $\Gamma_2(\mathcal{A}, \mathcal{B}) \leq \Omega_2(\mathcal{A}, \mathcal{B}) \leq \lambda_{\min}(\mathcal{A} \ast \mathcal{B})$, $\Gamma_3(\mathcal{B}, \mathcal{A}) \leq \Omega_2(\mathcal{B}, \mathcal{A}) \leq \lambda_{\min}(\mathcal{A} \ast \mathcal{B})$

(iii) $\Gamma_3(\mathcal{A}, \mathcal{B}) \leq \Omega_3(\mathcal{A}, \mathcal{B}) \leq \lambda_{\min}(\mathcal{A} \ast \mathcal{B})$
Proof. First, for $M$-tensors $\mathcal{A}$ and $\mathcal{B}$, one can obtain the decomposition in (33) and (34). For each $I = (i_1, i_2, \ldots, i_m) \in G_{m}$ with $\alpha'$ as the tight power, since $\mathcal{A}' = (a'_{j_1j_2\ldots j_n}) \in GDO^*_{m,n} \cap D_{m,n}$, there exist $u_i > 0$ for $i \in [n]$ such that
\[
a'_i = \sum a^2_u = \left( \frac{m - 1}{\alpha' - e_i} \right) \prod_{j \in I} u_j.
\]
Similarly, there exist $v_i > 0$ for $i \in [n]$ such that
\[
b'_i = \sum b^2_u = \left( \frac{m - 1}{\alpha' - e_i} \right) \prod_{j \in I} v_j.
\]
To show (i), let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n_{+}$, where $z_i = u_i v_i$ for all $i \in [n]$. Then for $i \in [n]$, we have
\[
\frac{(\mathcal{A} \star \mathcal{B})^{(m-1)}_{i}}{z_i} = \frac{1}{z_i} \left( \sum a'_{i_1i_2\ldots i_n} b'_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\geq \frac{1}{z_i} \left( \sum a'_{i_1i_2\ldots i_n} b'_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\geq \frac{1}{z_i} \left( \sum a'_{i_1i_2\ldots i_n} b'_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\geq \frac{1}{z_i} \left( m - 1 \right) (a_{i_1} - A_{i_1}) \left( b_{i_1} - A_{i_1} \right).
\]
The last equality follows from the decompositions (33) and (34). Thus, it follows that
\[
\Gamma_1(\mathcal{A}, \mathcal{B}) \leq \Omega_1(\mathcal{A}, \mathcal{B}) \leq \min_{1 \leq m \leq n} \left( \sum a_{i_1i_2\ldots i_n} b_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\leq \lambda_{\min}(\mathcal{A} \star \mathcal{B}).
\]
To show (ii) let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n_{+}$, where $z_i = u_i$ for all $i \in [n]$. Then for $i \in [n]$, we have
\[
\frac{(\mathcal{A} \star \mathcal{B})^{(m-1)}_{i}}{z_i} = \frac{1}{z_i} \left( \sum a'_{i_1i_2\ldots i_n} b'_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\geq \frac{1}{z_i} \left( \sum a'_{i_1i_2\ldots i_n} b'_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\geq \frac{1}{z_i} \left( \sum a'_{i_1i_2\ldots i_n} b'_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\geq \frac{1}{z_i} \left( m - 1 \right) (a_{i_1} - A_{i_1}) \left( b_{i_1} - A_{i_1} \right).
\]
Thus,
\[
\Gamma_2(\mathcal{B}, \mathcal{A}) \leq \Omega_2(\mathcal{B}, \mathcal{A}) \leq \min_{1 \leq m \leq n} \left( \sum a_{i_1i_2\ldots i_n} b_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\leq \lambda_{\min}(\mathcal{A} \star \mathcal{B}).
\]
Similarly, it then follows that
\[
\Gamma_2(\mathcal{A}, \mathcal{B}) \leq \Omega_2(\mathcal{A}, \mathcal{B}) \leq \min_{1 \leq m \leq n} \left( \sum a_{i_1i_2\ldots i_n} b_{i_1i_2\ldots i_n} \prod_{j \in I} u_j \prod_{j \not\in I} v_j \right)
\leq \lambda_{\min}(\mathcal{A} \star \mathcal{B}).
\]
by setting $z_i = v_i$ for all $i \in [n]$. 

To show (111), let $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n_+$, where $z_i = (u_i)^{\frac{1}{2}}(v_i)^{\frac{1}{2}}$ for all $i \in [n]$. Then for $i \in [n],$

$$
\frac{(\langle A \star B \rangle_{z^{m-1}})_i}{z_{i}} = a_{ii} - b_{ii} - \sum_{(i_2, \ldots, i_m) \in \mathbb{P}^m_i} \left( a_{i_2 \ldots i_m} a_{i_2 \ldots i_m}^{\frac{1}{2}} b_{i_2 \ldots i_m}^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( b_{i_2 \ldots i_m} b_{i_2 \ldots i_m}^{\frac{1}{2}} a_{i_2 \ldots i_m}^{\frac{1}{2}} \right)^{\frac{1}{2}} \geq a_{ii} - b_{ii} - \sum_{(i_2, \ldots, i_m) \in \mathbb{P}^m_i} |a_{i_2 \ldots i_m}|^2 (a_{i_2 \ldots i_m}^{\frac{1}{2}} b_{i_2 \ldots i_m}^{\frac{1}{2}})^{\frac{1}{2}} (b_{i_2 \ldots i_m} b_{i_2 \ldots i_m}^{\frac{1}{2}} a_{i_2 \ldots i_m}^{\frac{1}{2}})^{\frac{1}{2}} \\
= \Gamma_3(A, B).
$$

The second inequality follows from the Cauchy–Schwarz inequality. Thus,

$$
\Gamma_3(A, B) \leq \Omega_3(B, A) \leq \min_{1 \leq i \leq n} \frac{(\langle A \star B \rangle_{z^{m-1}})_i}{z_{i}} \leq \lambda_{\min}(A \star B).
$$

To illustrate how the new bounds introduced in Theorem 26 tighten the bounds introduced in [26], we compute bounds (i.e. (35), (36), (37) and (38)) of the minimum $H$-eigenvalue for the Fan product of symmetrized tensors in Example 3.9 in [26] and compare the results with the values obtained with methods (include (39), (40), (41) and (42)) proposed in [26].

**Example 2.** In this example, a tensor $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{R}^{[4,2]}$ is written as an unfolded form

$$
\mathcal{A} = \begin{bmatrix}
a_{1111} & a_{1112} & a_{1121} & a_{1122} & a_{1211} & a_{1221} & a_{1222} & a_{2111} & a_{2112} & a_{2121} & a_{2122} & a_{2211} & a_{2212} & a_{2221} & a_{2222}
\end{bmatrix}.
$$

Symmetric $M$-tensors $\mathcal{A}_i, \mathcal{B}_i \in \mathbb{R}^{[4,2]}$ for $i = 1, 2, 3$ are given as follows

$$
\mathcal{A}_1 = \begin{bmatrix}
3 & -0.5 & -0.5 & -0.5 & 0 & 0 & 0 & 0 & -2.5 & -0.25 & -0.25 & -0.25
\end{bmatrix},
\mathcal{B}_1 = \begin{bmatrix}
1.5 & -0.125 & 0 & -0.125 & 0 & 0 & 0 & 0 & -0.625 & -0.625 & -0.625 & 2.5
\end{bmatrix},
$$

$$
\mathcal{A}_2 = \begin{bmatrix}
3.8 & -0.5 & -0.5 & -0.5 & -13/30 & -13/30 & -0.5 & -0.5 & -0.5 & -0.5 & 3.9
\end{bmatrix},
\mathcal{B}_2 = \begin{bmatrix}
3.2 & -0.675 & -0.675 & -0.675 & -1/3 & -0.675 & -1/3 & -0.675 & -1/3 & -0.35 & -0.35 & -0.35 & 3.9
\end{bmatrix},
$$

$$
\mathcal{A}_3 = \begin{bmatrix}
3.8 & -0.575 & -0.575 & -0.575 & -11/30 & -11/30 & -0.4 & -11/30 & -0.4 & -0.4 & 3.7
\end{bmatrix}.
$$

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\[ B_3 = \begin{bmatrix} 3.5 & -0.35 & -0.35 & -23/60 & -0.35 & -23/60 & -0.525 & -23/60 & -0.525 & -0.525 \\ -0.35 & -23/60 & -23/60 & -0.525 & -23/60 & -0.525 & -0.525 & -23/60 & -0.525 & 3.1 \end{bmatrix}, \]

Similar to Table 1 in [26], Table 3 shows the bounds for the minimum $H$-eigenvalue of Fan product of $A_i$ and $B_i$, $i = 1, 2, 3$ obtained with the expressions (35), (36), (37), (38) and the lower bound expressions from [26]. For $A_i$ and $B_i$, $i = 1, 2, 3$, the lower bounds from (35) are 4.0717, 10.8346 and 10.3187 respectively which are larger than the values from all the methods in [26]. Besides, the lower bounds from (36), (37) and (38) are also tighter than the lower bounds from (3.4), (3.5) and (3.6) (i.e. (40), (41), and (42)) in [26], respectively. This empirically validates the fact that the proposed bounds (35), (36), (37), (38) are tighter lower bounds for the minimum $H$-eigenvalue of Fan product of two symmetric $M$-tensors. The proposed bounds (35), (36), (37), (38) contain more information comparing the bounds proposed in [26]. As a result, they are able to provide the tighter lower bounds. Note also that the expression among (35), (36), (37), (38) that provides the best lower bound depends on the specific tensors. For example in the first column ($A_1$ and $B_1$), the best lower bound is given by expression (36) while for the second column ($A_2$ and $B_2$) it is expression (35).

\[
\begin{array}{cccc}
\lambda_{\text{min}}(A) & 0.9723 & 0.54995 & 0.6970 \\
\lambda_{\text{min}}(B) & 0.5000 & 0.41253 & 0.3717 \\
\lambda_{\text{min}}(A \ast B) & 4.2762 & 11.3818 & 12.0646 \\
\end{array}
\]

Lower bounds on $\lambda_{\text{min}}(A \ast B)$ from [26]

| Lower bounds | $\lambda_{\text{min}}(A \ast B)$ |
|--------------|----------------------------------|
| (3.1) in [26] | 2.4722 3.1006 3.2768 |
| (3.3) in [26] | 4.0000 10.7663 9.9012 |
| (3.4) in [26] | 3.2327 9.9662 9.8934 |
| (3.5) in [26] | 4.0000 10.7663 9.9012 |
| (3.6) in [26] | 3.7040 10.4114 9.8973 |
| (3.7) in [26] | 2.5000 10.2250 2.6294 |

Proposed lower bounds on $\lambda_{\text{min}}(A \ast B)$

| Proposed lower bounds | $\lambda_{\text{min}}(A \ast B)$ |
|-----------------------|----------------------------------|
| (35) | 4.0717 10.8346 10.3187 |
| (36) | 4.1562 10.8177 10.3682 |
| (37) | 4.0717 10.5605 10.1657 |
| (38) | 4.1169 10.6959 10.2691 |

Table 3: Lower bounds for the minimum $H$-eigenvalues of the Fan product of symmetric $M$-tensors.

6. Conclusions

In this work, a new characterization of symmetric $H^+$-tensors is presented (see Corollary 14). As a result of this characterization, it follows that one can decide whether a tensor is a symmetric $H^+$-tensor in polynomial time (see Theorem 16). Comparing other characterizations which typically focus on sufficient conditions for a tensor to be an $H^+$-tensor, our characterization provides sufficient and necessary conditions. Besides, the set of symmetric $H^+$-tensors is described using tractable convex cones; in particular, the power cone.

We apply the new characterization of symmetric $H^+$-tensors in computing the minimum $H$-eigenvalue of symmetric $M$-tensors. In particular, we show how these $H$-eigenvalues; which can be computed in polynomial time, compare with the best bounds for the minimum $H$-eigenvalues of symmetric $M$-tensors proposed in the related literature. Furthermore, we illustrate how this new characterization of symmetric $H^+$-tensors can be used to obtain tighter lower bounds for the minimum $H$-eigenvalue of Fan product of two symmetric $M$-tensors. We show both theoretically and
empirically that the proposed bounds are tighter compared to the bounds proposed in [26]. Besides these two applications on tensor analysis, we believe more interesting results can be obtained with the proposed new characterization of $H^*$-tensors and $M$-tensors.

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