The vector form of Kundu-Eckhaus equation and its simplest solutions

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Abstract

In our work a hierarchy of integrable vector nonlinear differential equations depending on the functional parameter $r$ is constructed using a monodromy matrix. The first equation of this hierarchy for $r = \alpha (p'q)$ is vector analogue of the Kundu-Eckhaus equation. When $\alpha = 0$, the equations of this hierarchy turn into equations of the Manakov system hierarchy. New elliptic solutions to vector analogue of the Kundu-Eckhaus and Manakov system are presented. In conclusion, it is shown that there exist linear transformations of solutions to vector integrable nonlinear equations into other solutions to the same equations.

Keywords: Monodromy matrix, spectral curve, derivative nonlinear Schrodinger equation, vector integrable nonlinear equation

Introduction

It is well known that the derived nonlinear Schrodinger equations \cite{8,15,23,31,33} have numerous applications in various fields of physics and mathematics. In this regard, studies of various types of solutions to these equations are constantly being carried out (see, for example, \cite{2,5,21,30,38,40}). At the same time, it should be noted that along with the Kaup-Newell \cite{23} equation

\begin{equation}
 ip_z + p_u + i(|p|^2 p)_t = 0,
\end{equation}

Chen-Lee-Liu equation \cite{8}

\begin{equation}
 ip_z + p_u + i |p|^2 p_t = 0,
\end{equation}

and Gerdjikov-Ivanov equation \cite{16,17}

\begin{equation}
 ip_z + p_u - ip^2 p_t^* + \frac{1}{2} |p|^4 p = 0,
\end{equation}
there exists the Kundu-Eckhaus equation \([6, 24, 25, 39, 41]\)

\[
i p_z + p_{tt} - 2\sigma |p|^2 p + \alpha^2 |p|^4 p + 2i\alpha \partial_t \left(|p|^2\right) p = 0, \quad \sigma = \pm 1. \tag{4}
\]

Equation (4), as well as equations (1)-(3), contains first derivatives and also has numerous applications.

However, there is a significant difference between equations (1)-(3) and (4). The first three equations are consequences of compatibility conditions of Lax pairs with quadratic in spectral parameter Lax operators. Equation (4), in contrast to equations (1)-(3), is a result of the gauge transformation

\[
p = \hat{p} e^{-i\theta}, \quad \theta = \alpha \int |p|^2 \, dt
\]

of a solution \(\hat{p}\) to the nonlinear Schrödinger equation

\[
i \hat{p}_z + \hat{p}_{tt} - 2\sigma |\hat{p}|^2 \hat{p} = 0.
\]

The ever-growing traffic in networks requires finding ways to increase bandwidth of optical fibers. Therefore, researchers are actively working on vector models of nonlinear optical waves propagation \([14, 15, 19]\). Many of these models have been known for a long time. One such model is the Manakov system \([12, 13, 18, 27, 32, 34]\)

\[
\partial_z p_1 = i \partial_t^2 p_1 - 2i\sigma(|p_1|^2 + |p_2|^2)p_1,
\]

\[
\partial_z p_2 = i \partial_t^2 p_2 - 2i\sigma(|p_1|^2 + |p_2|^2)p_2. \tag{5}
\]

Let us note that vector nonlinear Schrödinger equations also have derivative forms, one of which is the equations \([7, 20, 26, 28, 37]\)

\[
i \partial_z p_1 = -\partial_t^2 p_1 - \frac{2i}{3} \partial_t \left(|p_1|^2 + |p_2|^2\right) p_1,
\]

\[
i \partial_z p_2 = -\partial_t^2 p_2 - \frac{2i}{3} \partial_t \left(|p_1|^2 + |p_2|^2\right) p_2. \tag{6}
\]

In contrast to the above works, we decided to investigate a vector analogue of the Kundu-Eckhaus equation. In this paper, we use a monodromy matrix to derive equations from vector analogue of the Kundu-Eckhaus equation hierarchy and construct the simplest nontrivial solutions to first equation. We hope that the vector equation we have obtained, as well as the scalar one, will have numerous applications in physics and mathematics.

The work consists of an introduction, four sections, and concluding remarks. In the first section, we define the Lax operator

\[
i \Psi_t + U \Psi = 0,
\]

which depends on a functional parameter \(r \in \mathbb{R}\), and investigate properties of corresponding monodromy matrix. Since the spectral curve equation is a characteristic
equation of the monodromy matrix \[10\], it is not difficult to obtain properties of the spectral curves equations from properties of the monodromy matrix. As in the case of the Manakov system \[32\], the spectral curves equations are quite cumbersome, so we will not give them in this paper. Let us note only that, as is shown in \[32\], linear dependence of the functions $p_j$ leads to factorization of the spectral curve equation into separate components. Therefore, from our point of view, solutions with linearly independent $p_j$ are more interesting to study and use in applications.

In section 2 we derive stationary equations for multiphase solutions. These equations are analogs of the Novikov equations for the Korteweg-de Vries hierarchy. Also in this section, we define a hierarchy of the second operators of the Lax pair

$$i \Psi_{z_k} + W_k \Psi = 0,$$

which depend on the functional parameter $r_k$: $\partial_t r_k = \partial_{z_k} r$. The Lax pair compatibility conditions give a hierarchy of vector derivative nonlinear Schrödinger equations with an additional functional parameter $\phi$: $r = \partial_t \phi$ and $r_k = \partial_{z_k} \phi$. When $r = \alpha(p'q)$ these equations are vector analogue of the Kundu-Eckhaus equation and its higher forms. Another choice of the functional parameter leads to other vector nonlinear equations. For $\phi \equiv 0$ these equations will turn into equations from the Manakov hierarchy \[32\]. Let us note that an existence of a Lax pair makes it possible to use the Darboux transformation to construct new solutions to vector analogue of the Kundu-Eckhaus equation.

In section 3, we construct one-phase solutions to vector analogue of the Kundu-Eckhaus equation. The first three solutions are expressed in terms of elliptic Jacobi functions, and for $\phi \equiv 0$ they will be new elliptic solutions to the Manakov system. Let us recall, that obtained in \[32\] elliptic solutions to the Manakov system were expressed in terms of Weierstrass functions. Then we construct solutions expressed in terms of hyperbolic functions. Despite the fact that in the last case the spectral curve has a genus equal to 2, the corresponding solution is a traveling wave.

In the section 4, we show that there exist linear transformations of solutions to vector integrable nonlinear equations into other solutions to the same equations. Original and transformed solutions are associated with the same spectral curve, but they correspond to different Baker-Akhiezer functions. One of these Baker-Akhiezer functions differs from the other by an orthogonal matrix factor. I.e., the Baker-Akhiezer function for considered vector nonlinear Schrödinger equation is determined up to orthogonal transformation.

1 The monodromy matrix and its properties

Let first equation of a Lax pair have the form

$$i \Psi_t + U \Psi = 0,$$

where

$$U = U_0 + r J, \quad U_0 = -\lambda J + Q,$$
\[ J = \frac{1}{3} \begin{pmatrix} 2 & 0^t \\ 0 & -I \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & p^t \\ -q & 0 \end{pmatrix}, \]

\( p^t = (p_1, p_2), \quad q^t = (q_1, q_2), \quad I \) is identity matrix, \( r \in \mathbb{R} \) is some function, and \( \lambda \) is a spectral parameter.

Following \([10, 32]\), assume that there exists a monodromy matrix \( M \) such that the matrix function \( \Psi = M \Psi \) is also an eigenfunction of Lax operator \([7]\). Then the matrix \( M \) satisfies the equation

\[ iM_t + UM - MU = 0. \]

(8)

In the case of a finite-gap matrix potential \( Q \), the monodromy matrix \( M \) is a polynomial in the spectral parameter \( \lambda \) \([10, 32]\)

\[ M = \sum_{j=0}^{n} m_j(t) \lambda^j. \]

(9)

Substituting (9) in (8) and simplifying, we get that the matrix \( M \) has the following structure

\[ M = V_n + \sum_{k=1}^{n-1} c_k V_{n-k} + c_n U_0 + J_n, \]

where \( V_1 = \lambda U_0 + V_1^0, \)

\[ V_{k+1} = \lambda V_k + V_k^0, \quad V_k^0 = \begin{pmatrix} -F_k & H_k^t \\ G_k & F_k \end{pmatrix}, \quad k \geq 1, \]

\[ J_n = \begin{pmatrix} -c_{n+1} - c_{n+2} & 0 & 0 \\ 0 & c_{n+1} & c_{n+3} \\ 0 & c_{n+4} & c_{n+2} \end{pmatrix}, \]

\( c_j \) are some constants, \( F_k = \text{Tr} F_k \).

From equation (8) it follows that elements of the matrices \( V_k^0 \) satisfy recurrence relations

\[ H_1 = i \partial_t p + r p, \]
\[ G_1 = i \partial_t q - r q, \]
\[ F_k = -i \partial_t^{-1} (G_k p^t + q H_k^t), \]
\[ H_{k+1} = i \partial_t H_k + r H_k + (F_k + F_k I) p, \]
\[ G_{k+1} = -i \partial_t G_k + r G_k - (F_k + F_k I) q. \]

(10)

In particular, \( F_1 = q p^t, \quad F_1 = p^t q = q^t p, \)

\[ H_2 = -\partial_t^2 p + 2i r \partial_t p + (2 p^t q + r^2 - i \partial_t r) p, \]
\[ G_2 = \partial_t^2 q + 2i r \partial_t q - (2 p^t q + r^2 - i \partial_t r) q, \]
\[ F_2 = 2 (q p^t) r + i (q \partial_t p^t - \partial_t q p^t) = q H_1^t - G_1 p^t, \]
\[ F_2 = 2 (p^t q) r + i (q^t \partial_t p - p^t \partial_t q) = H_1^t q - p^t G_1. \]
\[ H_3 = -i\partial_t^3 p - 3r\partial_t^2 p + 3i(p'tq + r^2 + i\partial_t r)\partial_t p \\
+ (3i\partial_3 p'tq + r^3 + 6rp'tq + 3ir\partial_t r - \partial_t^2 r) p, \]
\[ G_3 = -i\partial_t^3 q + 3r\partial_t^2 q + 3i(p'tq + r^2 - i\partial_t r)\partial_t q \\
+ (3i\partial_t^3 q + r^3 - 6rp'tq + 3ir\partial_t r + \partial_t^2 r) q, \]
\[ F_3 = 3(qp'tq)^2 + 3i(q\partial_t p't - \partial_t qp't) + \partial_t qp'h p' \\
- q\partial_t^2 p't + \partial_t^2 qp't + 3(p'tq)qp't \\
= qH_2^t - G_2 p't - G_1 H_1^t - (p'tq)qp't, \]
\[ F_3 = 3(p'tq)^2 + 3i(q\partial_t p't - p't\partial_t q) + 3(p'tq)^2 \\
+ \partial_t p't\partial_t q - p't\partial_t^2 q - q\partial_t^2 p. \]

For \( r = 0 \), all the above equations turn into corresponding equations for the Manakov system [32].

2 Stationary and evolutionary equations

From equation (8) the following stationary equations follow:

\[ H_{n+1} + \sum_{k=1}^{n} c_k H_{n+1-k} + C_n^k p = 0, \]
\[ G_{n+1} + \sum_{k=1}^{n} c_k G_{n+1-k} - C_n q = 0, \]

where
\[ C_n = \begin{pmatrix} 2c_{n+1} + c_{n+2} & c_{n+3} \\ c_{n+4} & c_{n+1} + 2c_{n+2} \end{pmatrix}. \]

All multiphase solutions to evolutionary integrable nonlinear equations are simultaneously solutions to some stationary equations.

In the case of reduction \( q = \sigma p^* \) (\( \sigma = \pm 1 \)), the following identities

\[ G_k^* = -\sigma H_k, \quad H_k^* = -\sigma G_k, \quad F_k^* = F_k^t, \quad F_k \in \mathbb{R} \]

follow from the recurrence relations (10). Therefore, the constants \( c_j \) in stationary equations (11) must satisfy the following conditions: \( c_k \in \mathbb{R} \) (\( 1 \leq k \leq n+2 \)), \( c_{n+4} = c_{n+3}^* \).

Let a second operator of a Lax pair have the form

\[ i\Psi_{z_k} + W_k \Psi = 0 \]

where \( W_k = V_k + r_k J \). Then from compatibility condition of equations (7) and (12), or from equation

\[ i\partial_t W_k - i\partial_{z_k} U + UW_k - W_k U = 0 \]
the evolutionary integrable nonlinear equations
\[ i\partial_{z_k} p = H_{k+1} - r_k p, \quad i\partial_{z_k} q = G_{k+1} + r_k q, \] (13)
and the additional relation
\[ \partial_{z_k} r = \partial_t r_k \] (14)
follow. It follows from (14) that there exists a function \( \phi \) such that
\[ r = \partial_t \phi, \quad r_k = \partial_{z_k} \phi. \]

The first systems of integrable nonlinear equations from hierarchy (13) have the form
\[
\begin{align*}
  i\partial_{z_1} p &= -\partial_t^2 p + 2ir\partial_t p + (2p'q + r^2 + i\partial_t r - r_1) p, \\
  i\partial_{z_1} q &= \partial_t^2 q + 2ir\partial_t q - (2p'q + r^2 - i\partial_t r - r_1) q,
\end{align*}
\] (15)
and
\[
\begin{align*}
  \partial_{z_2} p &= -\partial_t^3 p + 3ir\partial_t^2 p + 3(p'q + r^2 + i\partial_t r) \partial_t p \\
    &\quad + (3\partial_t p'q - ir^3 - 6irp'^2q + 3r\partial_t r + i\partial_t^2 r + ir_2) p, \\
  \partial_{z_2} q &= -\partial_t^3 q - 3ir\partial_t^2 q + 3(p'q + r^2 - i\partial_t r) \partial_t q \\
    &\quad + (3p'\partial_t q + ir^3 + 6irp'^2q + 3r\partial_t r - i\partial_t^2 r - ir_2) q.
\end{align*}
\] (16)

Equation (15) is one of a vector derivative nonlinear Schrödinger equations, and (16) is a vector modified Korteweg-de Vries equation. Both equations are parametrized by an arbitrary real function \( \phi \).

From (10) and (13) the following equalities
\[ \partial_{z_k} F_1 = \partial_t F_{k+1} \quad \text{and} \quad \partial_{z_k} F_1 = \partial_t F_{k+1} \]
follow. Therefore, there exist functions \( \Phi \) and \( \Phi \) such that
\[ F_1 = \partial_t \Phi, \quad F_{k+1} = \partial_{z_k} \Phi \quad \text{and} \quad F_1 = \partial_t \Phi, \quad F_{k+1} = \partial_{z_k} \Phi. \]

Therefore, if we put \( \Phi = \alpha \tilde{\phi} \) or
\[ r = \alpha F_1, \quad \text{and} \quad r_k = \alpha F_{k+1}, \] (17)
then equations (13), (17) will determine evolutionary integrable nonlinear equations from the hierarchy of a vector analogue of the Kundu-Eckhaus equation.

In particular, for \( k = 1 \), or for
\[ r = \alpha F_1 = \alpha (p'q) \quad \text{and} \quad r_1 = \alpha F_2 = 2\alpha^2(p'q)^2 + i\alpha (q'\partial_t p - p'\partial_t q), \]
equation (15) will have the form
\[
\begin{align*}
  i\partial_{z_1} p &= -\partial_t^3 p + 2i\alpha(p'q)\partial_t p + (2p'q - \alpha^2(p'q)^2 + 2i\alpha p'\partial_t q) p, \\
  i\partial_{z_1} q &= \partial_t^3 q + 2i\alpha(p'q)\partial_t q - (2p'q - \alpha^2(p'q)^2 - 2i\alpha \partial_t p'q) q.
\end{align*}
\] (18)
When \( q = Sp^*, \) \( S = \text{diag}(\sigma_1, \sigma_2), \) \( \sigma_j = \pm 1, \) equations (15) transform into a vector analogue of the Kundu-Eckhaus equation. It is not difficult to see that for \( \alpha = 0 \) equations (18) transform into the Manakov system [32].
3 One-phase solutions

3.1 Solutions in elliptic Jacobi functions

For $n = 1$ stationary equations have the form

\[ \begin{align*}
H_2 + c_1 H_1 + C_1^2 p &= 0, \\
G_2 + c_1 G_1 - C_1 q &= 0
\end{align*} \] (19)

or (for $c_4 = c_5 = 0$ and $r = \alpha F_1$, $r_1 = \alpha F_2$)

\[ \begin{align*}
\partial_t^2 p_1 &= i(c_1 + 2\alpha(p'q))\partial_t p_1 \\
&+ (2c_2 + c_3 + (2 + c_1\alpha)p'q + \alpha^2(p'q)^2 + i\alpha\partial_t(p'q)) p_1, \\
\partial_t^2 p_2 &= i(c_1 + 2\alpha(p'q))\partial_t p_2 \\
&+ (c_2 + 2c_3 + (2 + c_1\alpha)p'q + \alpha^2(p'q)^2 + i\alpha\partial_t(p'q)) p_2, \\
\partial_t^2 q_1 &= -i(c_1 + 2\alpha(p'q))\partial_t q_1 \\
&+ (2c_2 + c_3 + (2 + c_1\alpha)p'q + \alpha^2(p'q)^2 - i\alpha\partial_t(p'q)) q_1, \\
\partial_t^2 q_2 &= -i(c_1 + 2\alpha(p'q))\partial_t q_2 \\
&+ (c_2 + 2c_3 + (2 + c_1\alpha)p'q + \alpha^2(p'q)^2 - i\alpha\partial_t(p'q)) q_2.
\end{align*} \] (20)

Replacing functions $p_j$ and $q_j$ by formulas

\[ p_j = \hat{p}_je^{i\theta}, \quad q_j = \hat{q}_je^{-i\theta}, \quad \partial_t \theta = \frac{1}{2} c_1 + \alpha p'q. \]

we obtain the following equalities

\[ \begin{align*}
\partial_t^2 \hat{p}_1 &= \left(2\hat{p}'\hat{q} + 2c_2 + c_3 - \frac{1}{4} c_1^2 \right) \hat{p}_1, \\
\partial_t^2 \hat{p}_2 &= \left(2\hat{p}'\hat{q} + c_2 + 2c_3 - \frac{1}{4} c_1^2 \right) \hat{p}_2, \\
\partial_t^2 \hat{q}_1 &= \left(2\hat{p}'\hat{q} + 2c_2 + c_3 - \frac{1}{4} c_1^2 \right) \hat{q}_1, \\
\partial_t^2 \hat{q}_2 &= \left(2\hat{p}'\hat{q} + c_2 + 2c_3 - \frac{1}{4} c_1^2 \right) \hat{q}_2.
\end{align*} \] (21)

It is not difficult to see that functions $\hat{p}_j$ and $\hat{q}_j$ are solutions of the same second-order linear differential equations. Hence, their products $u_j = \hat{p}_j\hat{q}_j$ satisfy the corresponding Appel’s equations (\cite{35}, Part II, Chapter 14, Example 10; \cite{34})

\[ \begin{align*}
\partial_t^2 u_1 - (8u_1 + 8u_2 + 8c_2 + 4c_3 - c_1^2)\partial_t u_1 - 4\partial_t(u_1 + u_2)u_1 &= 0, \\
\partial_t^2 u_2 - (8u_1 + 8u_2 + 4c_2 + 8c_3 - c_1^2)\partial_t u_2 - 4\partial_t(u_1 + u_2)u_2 &= 0.
\end{align*} \] (22)
Let us introduce the notations: \( u_1 + u_2 = u, \ u_1 - u_2 = v \). In these notations, equations (22) take the form

\[
\begin{align*}
\partial_t^3 u + (c_1^2 - 6c_2 - 6c_3 - 12u)\partial_t u &= 2(c_2 - c_3)\partial_t v, \\
\partial_t^3 v + (c_1^2 - 6c_2 - 6c_3 - 8u)\partial_t v &= 2(c_2 - c_3 + 2v)\partial_t u.
\end{align*}
\] (23)

The simplest solutions of equations (23) can be obtained when \( v = (c_3 - c_2)/2 \). In this case, the function \( u \) satisfies the equation

\[
\partial_t^3 u + (c_1^2 - 6c_2 - 6c_3 - 12u)\partial_t u = 0
\]
or

\[
\partial_t^2 u + (c_1^2 - 6c_2 - 6c_3)u - 6u^2 = \hat{c}_1,
\] (24)
where \( \hat{c}_1 \) is an integration constant. Simplifying relation (24), we obtain the equation

\[(\partial_t u)^2 = 4u^3 - (c_1^2 - 6c_2 - 6c_3)u^2 + 2\hat{c}_1 u + \hat{c}_2,\] (25)
where \( \hat{c}_2 \) is a second integration constant. It is well known that solutions to equation (24) are elliptic functions or their degenerations.

It is easy to verify that one of the non-degenerate solutions to the equation (24) has the form

\[
u = k^2 \text{sn}^2(t; k) + \frac{1}{12} c_1^2 - \frac{1}{2}(c_2 + c_3) - \frac{1}{3}(1 + k^2),\] (26)
where \( \text{sn}(t; k) \) is an elliptic Jacobi function \([13]\), that satisfies the equation

\[[\text{sn}'(t)]^2 = (1 - \text{sn}^2(t))(1 - k^2 \text{sn}^2(t)).\]

The integration constant for solution (26) is equal to

\[
\hat{c}_1 = \frac{1}{24} c_1^4 - \frac{1}{2}(c_2 + c_3)c_1^2 + \frac{3}{2}(c_2 + c_3)^2 - \frac{2}{3}(1 - k^2 + k^4),
\]
\[
\hat{c}_2 = \frac{1}{432} c_1^6 + \frac{1}{24}(c_2 + c_3)c_1^4 - \frac{1}{4}(c_2 + c_3)^2 c_1^2 + \frac{1}{9} (1 - k^2 + k^4)c_1^2
\]
\[
+ \frac{1}{2}(c_2 + c_3)^3 - \frac{2}{3}(c_2 + c_3)(1 - k^2 + k^4) - \frac{4}{27}(2 - 3k^2 - 3k^4 + 2k^6).
\]

Knowing functions \( u \) and \( v \), we obtain functions \( u_j \)

\[
\begin{align*}
u_1 &= \frac{1}{2}(u + v) = \frac{k^2}{2} \text{sn}^2(t; k) + \frac{c_1^2}{24} - \frac{c_2}{2} - \frac{1 + k^2}{6}, \\
u_2 &= \frac{1}{2}(u - v) = \frac{k^2}{2} \text{sn}^2(t; k) + \frac{c_1^2}{24} - \frac{c_3}{2} - \frac{1 + k^2}{6}.
\end{align*}
\] (27)

Thus, functions \( \hat{p}_1 \) and \( \hat{p}_2 \) are solutions to the equations

\[
\begin{align*}
\partial_t^2 \hat{p}_1 &= \left(2k^2 \text{sn}^2(t; k) - \frac{2}{3}(1 + k^2) - \frac{1}{12} c_1^2 + c_2\right) \hat{p}_1, \\
\partial_t^2 \hat{p}_2 &= \left(2k^2 \text{sn}^2(t; k) - \frac{2}{3}(1 + k^2) - \frac{1}{12} c_1^2 + c_3\right) \hat{p}_2.
\end{align*}
\] (28)
Since functions $\hat{q}_j$ satisfy the same equations as $\hat{p}_j$, and the Wronskian of these solutions are constant:

$$W[\hat{p}_j, \hat{q}_j] = 2iW_j,$$

functions $\hat{p}_j$ and $\hat{q}_j$ are equal to

$$\hat{p}_j = \sqrt{u_j} \exp \left\{ -iW_j \int \frac{dt}{u_j} \right\}, \quad \hat{q}_j = \sqrt{u_j} \exp \left\{ iW_j \int \frac{dt}{u_j} \right\}, \quad (29)$$

where $u_j$ are defined by formulas (27). Substituting expressions (29) into equation (28) and simplifying, we get

$$W^2_j = \frac{1}{6912} (c_1^2 - 12c_{j+1} - 4 - 4k^2)(c_1^2 - 12c_{j+1} + 8 - 4k^2)(c_1^2 - 12c_{j+1} - 4 + 8k^2). \quad (30)$$

It follows from equation (30) that there are three cases when $\hat{p}_j = \hat{q}_j$ and $\hat{p}_1 \neq \hat{p}_2$. If

$$c_2 = \frac{1}{12} (c_1^2 + 8 - 4k^2), \quad c_3 = \frac{1}{12} (c_1^2 + 8k^2 - 4),$$

then

$$\hat{p}_1 = \hat{q}_1 = \frac{i}{\sqrt{2}} \text{dn}(t; k), \quad \hat{p}_2 = \hat{q}_2 = \frac{ik}{\sqrt{2}} \text{cn}(t; k).$$

In this case, the solution to equations (18) has the form

$$p_1 = ip_1(t - c_1 z_1) e^{i\theta}, \quad q_1 = -p_1^*, \quad p_2 = ip_2(t - c_1 z_1) e^{i\theta}, \quad q_2 = -p_2^*, \quad (31)$$

where

$$p_1(T) = \frac{1}{\sqrt{2}} \text{dn}(T; k), \quad p_2(T) = \frac{k}{\sqrt{2}} \text{cn}(T; k), \quad \theta = \frac{c_1}{4} + \left( 1 - \frac{c_1^2}{4} \right) z_1 + \alpha \int \left( k^2 \text{sn}^2(t - c_1 z_1; k) - \frac{1 + k^2}{2} \right) dt.$$
\[ p_1(T) = \frac{k}{\sqrt{2}} \operatorname{cn}(T; k), \quad p_2(T) = \frac{k}{\sqrt{2}} \operatorname{sn}(T; k), \]
\[ \theta = \frac{c_1}{2} t + \left( 1 - k^2 - \frac{c_1^2}{4} \right) z_1 + \alpha \int \left( k^2 \operatorname{sn}^2(t - c_1 z_1; k) - \frac{k^2}{2} \right) dt. \]

If
\[ c_2 = \frac{1}{12} (c_1^2 + 8k^2 - 4), \quad c_3 = \frac{1}{12} (c_1^2 - 4k^2 - 4), \]
we have
\[ \hat{p}_1 = \hat{q}_1 = \frac{ik}{\sqrt{2}} \operatorname{cn}(t; k), \quad \hat{p}_2 = \hat{q}_2 = \frac{k}{\sqrt{2}} \operatorname{sn}(t; k). \]

In this case, the solution to equations (18) has the form
\[ p_1 = ip_1(t - c_1 z_1)e^{i\theta}, \quad q_1 = -p_1^*, \]
\[ p_2 = p_2(t - c_1 z_1)e^{i\theta}, \quad q_2 = p_2^*, \]
(33)

where
\[ p_1(T) = \frac{k}{\sqrt{2}} \operatorname{cn}(T; k), \quad p_2(T) = \frac{k}{\sqrt{2}} \operatorname{sn}(T; k), \]
\[ \theta = \frac{c_1}{2} t + \left( k^2 - 1 - \frac{c_1^2}{4} \right) z_1 + \alpha \int \left( k^2 \operatorname{sn}^2(t - c_1 z_1; k) - \frac{k^2}{2} \right) dt. \]

Dependency of solutions (31)-(33) from the variable \( z_1 \) was found from equations (18).

3.2 Solutions in hyperbolic functions

The equation (25) is well studied. In particular, it has the following solution
\[ u = a^2 \tanh^2(at) + \frac{1}{12} \left( (c_1^2 - 6c_2 - 6c_3) - 8a^2 \right). \]
Integration constants for a given function $u$ are equal to
\[
\hat{c}_1 = \frac{1}{24} \left( (c_1^2 - 6c_2 - 6c_3)^2 - 16a^4 \right),
\]
\[
\hat{c}_2 = \frac{1}{432} \left( 4a^2 + (c_1^2 - 6c_2 - 6c_3)^2 \left( 8a^2 - (c_1^2 - 6c_2 - 6c_3) \right) \right).
\]
In this case, functions $\hat{p}_j$ and $\hat{q}_j$ satisfy the following equations:
\[
\partial_t^2 \hat{p}_j = \left( 2at^2 \tanh(at) - \frac{4a^2}{3} - \frac{c_1^2}{12} + c_{j+1} \right) \hat{p}_j \tag{34}
\]
and
\[
u_j = \hat{p}_j \hat{q}_j = \frac{a^2}{2} \tanh^2(at) + \frac{1}{24} \left( c_1^2 - 8a^2 - 12c_{j+1} \right) \tag{35}
\]
Let us recall that functions $\hat{q}_j$ also satisfy equations \(34\).
Solving equations \(34\) with conditions \(35\), we get
\[
\hat{p}_j = \frac{1}{\sqrt{2}} (k_j + ia \tanh(at)) e^{ik_j t}, \quad \hat{q}_j = \frac{1}{\sqrt{2}} (k_j - ia \tanh(at)) e^{-ik_j t}, \tag{36}
\]
where
\[
k_j^2 = \frac{1}{12} \left( c_1^2 - 8a^2 - 12c_{j+1} \right) \quad \text{or} \quad c_{j+1} = \frac{1}{12} \left( c_1^2 - 8a^2 - 12k_j^2 \right). \tag{37}
\]
The corresponding solution to equations \(38\) has the form
\[
p_j = \frac{1}{\sqrt{2}} (k_j + ia [a(t - c_1 z_1)]) e^{i\theta_j}, \quad q_j = p_j^*,
\]
\[
\theta_j = \left( \frac{c_1}{2} + k_j \right) t + m_j z_1 + \alpha \int \left( a \tanh^2(at - ac_1 z_1) + \frac{k_j^2 + k_2^2}{2a} \right) dt,
\]
where
\[
m_j = -2a^2 - \frac{1}{4} (c_1 + 2k_j)^2 - k_1^2 - k_2^2 - \alpha(k_1 + k_2)(a^2 + k_1^2 - k_1k_2 + k_2^2). \tag{38}
\]
Magnitudes of solutions \(37\) are shown in Fig. 2.

### 3.3 Two-gap one-phase solutions

If $n = 1$, $v \neq \text{const}$, and $c_3 \neq c_2$, then from \(23\) we have
\[
v = \frac{1}{2(c_2 - c_3)} (u_{tt} - 6u^2 + (c_1^2 - 6c_2 - 6c_3)u) + C_1, \tag{38}
\]
and
\[
u_{ttt} + 2(c_1^2 - 6c_2 - 6c_3 - 10u)u_{ttt} - 10 (u_t)^2 + 40u^3 - 12(c_1^2 - 6c_2 - 6c_3)u^2
+ (c_1^4 - 12c_1^2(c_2 + c_3) + 8 (4c_2^2 + 10c_2c_3 + 4c_3^3 - C_1 c_2 + C_1 c_3)) u + C_2 = 0. \tag{39}
\]
where $C_1$ and $C_2$ are integration constants.

We can rewrite equation (39) in the form

$$I_2 + 2AI_1 + (A^2 - 4B^2 + 8C_1B)u + C_2 = 0,$$

(40)

where $A = c_1^2 - 6(c_3 + c_2)$, $B = c_3 - c_2$,

$$I_2 = u_{ttt} - 20u_{tt} - 10(u_t)^2 + 40u^3, \quad I_1 = u_{tt} - 6u^2.$$

(41)

It follows from equations (40) and (41) that the function $2u(t)$ is a two-gap potential of the Schrödinger operator [11, 29]

$$\psi_{tt} - 2u\psi = E\psi.$$  

(42)

It is well known that linear independent solutions to equation (42) with two-gap potential $2u(t)$ can be written as

$$\psi_{1,2} = \varepsilon_{1,2}\sqrt{\Psi} \exp \left\{ \pm \nu(E) \int dt \frac{\Psi}{\Psi} \right\},$$

(43)

where

$$\Psi = E^2 + (\gamma_1 - u)E + \gamma_2 - \gamma_1u - \frac{1}{4}I_1,$$

and

$$\gamma_1 = \frac{A}{2}, \quad \gamma_2 = \frac{1}{16} \left( A^2 - 4B^2 + 8C_1B \right).$$

Substituting (43) in (42) and simplifying, we obtain an equation for spectral curve of two-gap potential $u(t)$:

$$\nu^2 = E^3 + 2\gamma_1 E^4 + (\gamma_1^2 + 2\gamma_2)E^3 + \left( 2\gamma_1\gamma_2 - \frac{1}{8}C_2 \right) E^2 + C_3 E + C_4,$$

(44)
where
\[
C_3 = \frac{1}{8} u_t u_{tt} - \frac{1}{16} u_{tt}^2 + \frac{1}{4} (\gamma_1 - 5u) u_t^2 + \frac{5}{4} u_t^4 - \gamma_1 u^3 + \gamma_2 u^2 + \frac{1}{8} C_2 u + \gamma_2^2 - \frac{1}{8} \gamma_1 C_2,
\]
\[
C_4 = \frac{1}{64} u_{ttt}^2 + \frac{1}{8} (\gamma_1 - 3u) u_t u_{ttt} - \frac{1}{8} u_{tt}^2 + \frac{1}{32} (C_2 + 16\gamma_2 u - 24\gamma_1 u^2 + 40u^3 + 2u_t^2) u_{tt}
+ \frac{1}{8} (2\gamma_1^2 - 2\gamma_2 - 10\gamma_1 u + 15u^2) u_t^2 - 3u^5 + \frac{7}{2} \gamma_1 u^4 - (\gamma_1^2 + 2\gamma_2) u^3
+ \frac{1}{16} (16\gamma_1 \gamma_2 - 3C_2) u^2 + \frac{1}{8} \gamma_1 C_2 u - \frac{1}{8} \gamma_2 C_2.
\]

From (38) we have
\[
u_1 = \frac{1}{2} (u + v) = -\frac{1}{4B} I_1 + \left(\frac{1}{2} - \frac{A}{4B}\right) u + \frac{1}{2} C_1, \tag{45}
\]
\[
u_2 = \frac{1}{2} (u - v) = \frac{1}{4B} I_1 + \left(\frac{1}{2} + \frac{A}{4B}\right) u - \frac{1}{2} C_1.
\]

Therefore, functions \(\hat{\nu}_j\) and \(\hat{\pi}_j\) are solutions to equations
\[
\partial_t^2 \hat{\nu}_1 - 2u \hat{\nu}_1 = -\frac{1}{4} (A + 2B) \hat{\nu}_1,
\]
\[
\partial_t^2 \hat{\nu}_2 - 2u \hat{\nu}_2 = -\frac{1}{4} (A - 2B) \hat{\nu}_2. \tag{46}
\]

It is following from (42), (43), (45), and (46) that
\[
\hat{\nu}_1 = \epsilon_{11} \sqrt{\Psi} \exp \left\{ \nu(E) \int dt \frac{\Psi}{\Psi} \right\}_{E=E_1},
\]
\[
\hat{\nu}_2 = \epsilon_{21} \sqrt{\Psi} \exp \left\{ -\nu(E) \int dt \frac{\Psi}{\Psi} \right\}_{E=E_1},
\]
\[
\hat{\nu}_2 = \epsilon_{21} \sqrt{\Psi} \exp \left\{ \nu(E) \int dt \frac{\Psi}{\Psi} \right\}_{E=E_2},
\]
\[
\hat{\nu}_2 = \epsilon_{22} \sqrt{\Psi} \exp \left\{ -\nu(E) \int dt \frac{\Psi}{\Psi} \right\}_{E=E_2},
\]

where
\[
\epsilon_{11} \epsilon_{12} = \frac{1}{B}, \quad \epsilon_{21} \epsilon_{22} = -\frac{1}{B}, \quad E_1 = -(A + 2B)/4, \quad E_2 = -(A - 2B)/4.
\]

It is not difficult to see that functions \(\hat{\nu}_j\) and \(\hat{\pi}_j\) will be bounded when \(E_j\) satisfies to conditions \(\text{Re}(\nu(E_j)) = 0\).

Two-soliton potential of operator (42) has the form (\(b > a > 0\))
\[
u(t) = \frac{(a^2 - b^2)(b^2 - a^2 + a^2 \cosh(2bt) + b^2 \cosh(2at))}{2(b \cosh(bt) \cosh(at) - a \sinh(bt) \sinh(at))^2}. \tag{47}
\]
Substituting (47) into (40), we get
\[ A = -2(a^2 + b^2), \quad C_1 = -\frac{(a^2 - b^2)^2 - B^2}{2B}, \quad C_2 = 0. \]
The spectral curve (44) of potential (47) is determined by the equation
\[ \nu^2 = E(E - a^2)^2(E - b^2)^2. \]
Calculating \( \nu^2(E_j) \), we have
\[ \nu^2(E_1) = \frac{1}{32}(a^2 + b^2 - B) \left( (a^2 - b^2)^2 - B^2 \right), \]
\[ \nu^2(E_2) = \frac{1}{32}(a^2 + b^2 + B) \left( (a^2 - b^2)^2 - B^2 \right). \]
From conditions \( \nu^2(E_j) \leq 0 \) we have \( B = \pm(b^2 - a^2) \).

If \( B = b^2 - a^2 \) then \( E_1 = b^2, E_2 = a^2 \),
\[ c_2 = \frac{1}{12}(c_1^2 + 8b^2 - 4a^2), \quad c_3 = \frac{1}{12}(c_1^2 + 8a^2 - 4b^2), \]
and
\[ \hat{p}_1 = \hat{q}_1 = \frac{ib\sqrt{b^2 - a^2} \cosh(at)}{(b \cosh(bt) \cosh(at) - a \sinh(bt) \sinh(at))}, \quad \hat{q}_1 = -\hat{p}_1, \]
\[ \hat{p}_2 = \hat{q}_2 = \frac{ia\sqrt{b^2 - a^2} \sinh(bt)}{(b \cosh(bt) \cosh(at) - a \sinh(bt) \sinh(at))}, \quad \hat{q}_2 = -\hat{p}_2. \]
The corresponding solution to equations (38) has the form
\[ p_1(t, z_1) = ip_1(t - c_1z_1)e^{i\theta_1(t, z_1)}, \quad q_1(t, z_1) = -p_1^*(t, z_1), \]
\[ p_2(t, z_1) = ip_2(t - c_1z_1)e^{i\theta_2(t, z_1)}, \quad q_2(t, z_1) = -p_2^*(t, z_1), \]
(48)
where
\[ p_1(T) = \frac{b\sqrt{b^2 - a^2} \cosh(aT + t_a)}{(b \cosh(bt + t_b) \cosh(aT + t_a) - a \sinh(bt + t_b) \sinh(aT + t_a))}, \]
\[ p_2(T) = \frac{a\sqrt{b^2 - a^2} \sinh(bt + t_b)}{(b \cosh(bt + t_b) \cosh(aT + t_a) - a \sinh(bt + t_b) \sinh(aT + t_a))}, \]
and
\[ \theta_1(t, z_1) = \frac{c_1^2 t}{2} + \left( b^2 - \frac{1}{4} c_1^2 \right) z_1 - \alpha \int \left( p_1^2(T) + p_2^2(T) \right) dt, \]
\[ \theta_2(t, z_1) = \frac{c_1^2 t}{2} + \left( a^2 - \frac{1}{4} c_1^2 \right) z_1 - \alpha \int \left( p_1^2(T) + p_2^2(T) \right) dt. \]
Here \((t_a, t_b)\) is an initial two-dimensional phase. Magnitudes of solutions (31) are shown in Fig. 3.
4 Orthogonal transformation

Since the matrix $J$ has two equal diagonal elements, an orthogonal transformation of the vectors of solutions to equations (13) again gives solutions to these equations. To prove this statement, we consider the equation

$$i\tilde{\Psi}_t + \tilde{U}\tilde{\Psi} = 0,$$

where

$$\tilde{U} = \tilde{U}_0 + rJ, \quad \tilde{U}_0 = -\lambda J + \tilde{Q},$$

$$\tilde{\Psi} = \tilde{T}\Psi, \quad \tilde{T} = \begin{pmatrix} 1 & 0' \\ 0 & T \end{pmatrix}.$$

From equations (13) and (49) it follows that $\tilde{Q}\tilde{T} = \tilde{T}\tilde{Q}$. Therefore, the following equalities hold:

$$\tilde{q} = Tq, \quad \tilde{p} = (T^t)^{-1}p.$$  

Thus, if the matrix $T$ satisfies to the condition

$$TST^t = S,$$  

then vectors $p$ and $q$ ($\tilde{p}$ and $\tilde{q}$) are connected by the relation

$$q = Sp^*, \quad \tilde{q} = S\tilde{p}^*.$$

The following equalities

$$\tilde{G}_k = TG_k, \quad \tilde{H}_k = (T^t)^{-1}H_k, \quad \tilde{F}_k = TF_kT^{-1}, \quad \bar{F}_k = F_k.$$
follow from the recurrence relations (10). Therefore, if vectors $p$ and $q$ are solutions to evolutionary equations (13), then vectors $\tilde{p}$ and $\tilde{q}$ are also solutions to the same equations. Thus, using formula (50) with the matrix

$$T_1 = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$$

and solution (31), it is possible to construct new elliptic solutions to equation (18):

$$\tilde{p}_1 = \frac{i}{\sqrt{2}} (\cos \varphi \text{dn}(t - c_1 z_1; k) + k \sin \varphi \text{cn}(t - c_1 z_1; k)) e^{i\theta}, \quad \tilde{q}_1 = -\tilde{p}_1^*, \quad \tilde{p}_2 = -\frac{i}{\sqrt{2}} (\sin \varphi \text{dn}(t - c_1 z_1; k) - k \cos \varphi \text{cn}(t - c_1 z_1; k)) e^{i\theta}, \quad \tilde{q}_2 = -\tilde{p}_2^*. \quad (52)$$

Let us note that the vectors $\tilde{p}$ and $\tilde{q}$ satisfy the stationary equation (11) with a non-diagonal matrix $\tilde{C}_n = T C_n T^{-1} \cdot$ 

For solution (52) the matrix $\tilde{C}_1$ has the form

$$\tilde{C}_1 = \frac{1}{2} \begin{pmatrix} 3(c_3 + c_2) - (c_3 - c_2) \cos(2\varphi) & (c_3 - c_2) \sin(2\varphi) \\ (c_3 - c_2) \sin(2\varphi) & 3(c_3 + c_2) + (c_3 - c_2) \cos(2\varphi) \end{pmatrix}.$$ 

Therefore, for transformed solutions the constants $\tilde{c}_j$ are equal:

$$\tilde{c}_2 = \frac{1}{2}(c_3 + c_2) - \frac{1}{2}(c_3 - c_2) \cos(2\varphi),$$

$$\tilde{c}_3 = \frac{1}{2}(c_3 + c_2) + \frac{1}{2}(c_3 - c_2) \cos(2\varphi),$$

$$\tilde{c}_4 = \tilde{c}_5 = \frac{1}{2}(c_3 - c_2) \sin(2\varphi).$$

At the same time, since the monodromy matrices of the functions $\tilde{\Psi}$ and $\Psi$ are similar

$$\tilde{M} = \tilde{T} M \tilde{T}^{-1},$$

original and transformed solutions are associated with the same spectral curve. Therefore, the Baker-Akhiezer function $\Psi$ can be constructed from a spectral curve only up to a linear transformation $\tilde{T}$.

**Discussions and conclusions**

In many works that previously investigated finite-gap solutions of the Manakov system (see, for example [9, 12, 13, 22, 34, 36]), following factors that we found recently were not taken into account. First, as we showed in [32], if functions $p_j$ are linearly dependent, then eigenfunctions of Lax operator (7) are determined on two separated spectral curves.
Secondly, as the authors know, other researchers have not previously taken into account preserving spectral curves orthogonal transformations of solutions.

And finally, as we have seen from the considered in the section \(3\) examples, the number of phases of the solution is less than the genus of the corresponding spectral curve. Indeed, it follows from equations (11) and (13) that the following relations hold:

\[
\partial z_n^p = -\sum_{k=1}^{n-1} c_k \partial z_{n-k}^p - c_n \partial t^p + i \left( r_n + \sum_{k=1}^{n-1} c_k r_{n-k} + c_n r_n \right) p + i C_n^p,
\]

\[
\partial z_n^q = -\sum_{k=1}^{n-1} c_k \partial z_{n-k}^q - c_n \partial t^q - i \left( r_n + \sum_{k=1}^{n-1} c_k r_{n-k} + c_n r_n \right) q - i C_n^q.
\]

Therefore, solutions \( p_j^p \) and \( q_j^q \) up to exponential multipliers are \(n\)-phase functions (functions with \(n\) arguments):

\[
p_j^p(t, z_1, \ldots, z_n) = p_j^p(t - c_n z_n, z_1 - c_n z_n, \ldots, z_{n-1} - c_n z_n) e^{i\theta_j(t, z_1, \ldots, z_n)},
\]

\[
q_j^q(t, z_1, \ldots, z_n) = q_j^q(t - c_n z_n, z_1 - c_n z_n, \ldots, z_{n-1} - c_n z_n) e^{-i\theta_j(t, z_1, \ldots, z_n)}.
\]

An equation for a spectral curve \( \Gamma = \{ (\mu, \lambda) \} \) has the form

\[
\mathcal{R}(\mu, \lambda) = \det(\mu I - M) = \mu^3 + A(\lambda)\mu + B(\lambda) = 0,
\]

where

\[
A(\lambda) = -\frac{1}{3} \lambda^{2n+2} - \frac{2c_1}{3} \lambda^{2n+1} + \sum_{j=2}^{2n+2} A_j \lambda^{2n+2-j},
\]

\[
B(\lambda) = \frac{2}{27} \lambda^{3n+3} + \frac{2c_1}{9} \lambda^{3n+2} + \sum_{j=2}^{3n+3} B_j \lambda^{3n+3-j}.
\]

If \(n \leq 3\), then a discriminant of (53), as a polynomial of \(\mu\), is equal to:

\[
\Delta(\lambda) = (c_{n+1} - c_{n+2})^2 \lambda^{4n+4} + \sum_{j=1}^{4n+4} D_j \lambda^{4n+4-j}.
\]

Perhaps equality (54) is also true for other values of \(n\). It follows from equation (54) that when condition \(c_{n+1} \neq c_{n+2}\) is fulfilled, the curve \(\Gamma\) has \((4n + 4)\) branching points. Using the Riemann-Hurwitz formula

\[
g = \frac{M}{2} - N + 1,
\]

where \(M\) is a number of branching points, \(N\) is a number of sheets of a covering, we get that the genus of the spectral curve \(\Gamma\) is equal

\[
g = \frac{4n + 4}{2} - 3 + 1 = 2n.
\]

Thus, to construct finite-gap solutions to the Manakov system or to the vector Kundu-Eckhaus equation, it is necessary to use trigonal curves, the genus of which is twice the number of phases of solutions.
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