The obstacle problem in masonry structures and cable nets

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Abstract
We consider the problem of finding a net that supports prescribed forces applied at prescribed points, yet avoids certain obstacles, with all the elements of the net under compression (strut net) or under tension (cable web). In the case of masonry structures, for instance, this consists in finding a strut net that supports the forces, is contained within the physical structure, and avoids regions that may be not accessible due, for instance, to the presence of holes. We solve such a problem in the two-dimensional case, where the prescribed forces are applied at the vertices of a convex polygon, and we treat the cases of both single and multiple obstacles. By approximating the obstacles by polygonal regions, the task reduces to identifying the feasible domain in a linear programming problem. For a single obstacle we show how the region $\Gamma$ available to the obstacle can be enlarged as much as possible in the sense that there is no other strut net, having a region $\Gamma'$ available to the obstacle with $\Gamma \subset \Gamma'$. The case where some of the forces are reactive, unprescribed but reacting to the other prescribed forces, is also treated. It again reduces to identifying the feasible domain in a linear programming problem. Finally, one may allow a subset of the reactive forces to each act not at a prescribed point, but rather at any point on a prescribed line segment. Then the task reduces to identifying the feasible domain in a quadratic programming problem.

Keywords. Force networks, Unilateral response, Limit analysis

1 Introduction
The master ‘safe’ theorem of a masonry arch, formulated in a famous work by Jacques Heyman \cite{1} states that the collapse of such a structure does not occur if a line of thrust of the given external loads can be fit within the boundaries of the arch (see also, \cite{2,3,4}). The line of thrust coincides with the Hooke’s inverted chain of the given forces, as shown, e.g., by the Italian engineer Poleni in a study dated back to 1784, which was aimed at assessing the stability of St. Peter’s dome in the Vatican \cite{5}. Such a construction has been the foundation of the beautiful work of the Catalan architect Antoni Gaudí on funicular shapes of masonry structures \cite{6,7}. The use of truss networks within discrete element approaches to no-compression or no-tension bodies is frequent in the literature (refer to \cite{8,9,10,11,12} and references therein). Truss models are convenient, e.g., to tackle the equilibrium problem of masonry structures, described as no-tension bodies \cite{13,14}, and to assess the compatibility of external loads through graphical constructions, numerical methods, and hanging models \cite{10,11,12,15,16,17}. Polyhedral Airy stress functions have been employed in
two-dimensional problems, since such functions permit one to conveniently describe force networks through scalar potentials [18, 19]. The no-tension model of masonry by Heyman does not account for buckling analysis, since the material is supposed to behave rigidly up to collapse, with infinitely large compression strength [1]. Upon accounting for a geometrically nonlinear elastic response and a finite strength in compression of the masonry, it has been shown that second-order effects on the collapse load of circular arches tend to be negligible when the slenderness ratio of the arch radius over its thickness is sufficiently small [20].

A recent study has investigated the existence problem of systems of tensile forces that support given sets of nodal forces, coming to the conclusion that if such ‘cable webs’ exist then the given forces are supported by the complete web connecting pairwise their points of application (see [8], Theorem 1.1). In two-dimensions, and with forces at the vertices of a convex polygon, the use of polyhedral Airy stress functions shows alternatively that if such ‘cable webs’ exist then the given forces are supported by an open web with no internal loops (see [9], Theorem 1). The present work generalizes the above problem to the case of compression-only force networks (or ‘strut nets’) in two-dimensions, which support given sets of nodal forces, and are able to avoid ‘obstacles’ representing regions not accessible to the force network (e.g., holes or inclusions). We begin by deriving a result showing that a strut net supporting $q$ forces located strictly inside the convex hull of the points where forces are applied and avoiding $p$ obstacles can be replaced by a strut net with at most $p + q$ elementary loops. (Section 2). From then onwards we restrict our attention to sets of forces at the vertices of a convex polygon ($q = 0$). In Section 3 we provide a condition for the non-existence of strut nets that avoid a given obstacle, when such an obstacle intersects the open strut net associated to the given forces, and extends outside the convex hull of their points of application. We also provide an algorithm to establish if there exists a strut net avoiding multiple obstacles, and if so to construct one. Additionally, for a single obstacle, we show how the region available to the obstacle can be enlarged as much as possible (in a sense to be made more precise). Section 4 treats the case of multiple obstacles with the inclusion of reactive forces. Numerical applications are then presented in Section 5, by first dealing with a test problem characterized by a single active force, and then by treating a series of examples that analyze strut nets associated with the statics of masonry arches. We end in Section 6 by drawing concluding remarks and directions for future work.

2 Reducing the complexity of a strut net avoiding a given set of obstacles

To begin, we consider a set of points $x_1, x_2, \ldots, x_n$ at the vertices of a convex polygon $\Omega$, with external forces $t_1, t_2, \ldots, t_n$ acting on them. We assume that the points are numbered anti clockwise and we adopt the following convention: for any $i > n$, $x_i := x_{i-n}$ and $x_0 := x_n$. Letting

$$R_\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denote the matrix for a 90° rotation, we assume that, for all $j$ with $1 \leq j \leq n$ and for all $i$ with $j \leq i \leq j + n - 1$,

$$\sum_{k=j}^{i} (x_k - x_j) \cdot [R_\perp t_k] \geq 0. \tag{1}$$

This assumption ensures that there exists a truss structure supporting these forces with all the struts in the truss structure under compression [9]. The constraint (1) has a physical interpretation: the net anticlockwise torque around the point $x_j$ of the forces $t_k$ summed over any number of consecutive
points clockwise past the point $x_j$ is non-positive. If (1) is satisfied, the forces are supported by an open strut net with no internal loops, in other words, there are no polygons whose edges are the struts of the net.

A more general condition, applicable to any set of forces, either in two-dimensions or three-dimensions, is that if a strut net exists supporting the forces, then they will also be supported by a strut net connecting pairwise the points at which the forces are applied [8]. This leads to a linear programming problem for determining if a set of forces can be supported.

Condition (1) is derived using Airy stress functions. Indeed, for two-dimensional elasticity it is well known that, in the absence of body forces in a simply connected region $\Omega$, the divergence-free stress field $\sigma(x)$ can be represented in terms of the Airy stress function $\phi(x)$:

$$\sigma(x) = \begin{pmatrix} \frac{\partial^2 \phi(x)}{\partial x_1^2} & -\frac{\partial^2 \phi(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \phi(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 \phi(x)}{\partial x_2^2} \end{pmatrix} = R^T_\perp \nabla \nabla \phi(x) R_\perp,$$

in which $R^T_\perp = -R_\perp$ is the transpose of $R_\perp$. Since $\sigma(x)$ is negative semidefinite for all $x$, we see that $\nabla \nabla \phi(x)$ is negative semidefinite for all $x$. This implies that $\phi(x)$ is a concave function in any simply-connected two-dimensional region under compression, that may have subregions with zero stress [13]. Note that, when $\sigma(x)$ is zero in a region, as it is between the struts in a net, then $\nabla \nabla \phi(x) = 0$ which by (2) implies the Airy stress function is a linear function of $x$ in this region.

Thus, any strut net under compression that supports forces at the vertices of a convex polygon will have an associated Airy stress function whose graph is a concave polyhedron. The magnitudes of the slope discontinuities can be connected to the compression in the associated struts [19, 21].

If the strut net associated with a concave polyhedron has internal loops, we can replace the latter by a simpler concave polyhedron (supporting the same forces), by using the tangent planes to the Airy stress function at the boundary of the polygon $\Omega$ [9]. The strut net associated with this latter concave polyhedron will be an open strut net with no internal loops. To prove this, let us start by noticing that, since the strut net supports under compression the forces $t_i$ at the points $x_i$, it can support, for some $\epsilon > 0$, the same forces at points

$$\tilde{x}_i = x_i - \epsilon t_i,$$

the convex hull of which we call $\tilde{\Omega}$. To do this, one extends the strut net by attaching $n$ short struts of length $\epsilon > 0$ between $x_i$ and $\tilde{x}_i$, $i = 1, 2, \ldots, n$. The first step is to determine the Airy stress function $\phi(x)$ in the polygonal ring bounded on one side by the polygon joining the points $x_i$, and on the other side by the polygon joining the points $\tilde{x}_i$. When $\epsilon$ is sufficiently small there are no struts inside the quadrilateral $\Omega_i$ with vertices $\tilde{x}_{i-1}$, $x_{i-1}$, $x_i$, $\tilde{x}_i$ and since the stress vanishes there, the Airy stress function $\phi(x)$ inside that quadrilateral must be a linear function $L_i(x)$. Due to the stress in the wires (see, for example, [9]), these satisfy

$$R_\perp \left( \nabla L_{i+1} - \nabla L_i \right) = -t_i, \quad i = 1, 2, \ldots, n,$$

and, for all $i, j$,

$$L_i \geq L_j \quad \text{on } \Omega_j.$$

Equation (4) and the constraints

$$L_i(x_i) = L_{i+1}(x_i)$$

determine the sequence $(L_i)_{i=1}^n$ up to the addition of a global linear function. Uniqueness can be ensured by assuming, for instance, that $L_1(x) = 0$. The condition (5) is equivalent to (1) [9].
The stress measure $\sigma(x)$ associated to the Airy function $\phi(x)$, that is defined in the sense of
distributions by (2), is non-positive, concentrated along a finite union of segments, divergence-free
on $\Omega$ and, owing to the construction of $(L_i)_{i=1}^n$, the divergence in the sense of distributions, $\text{div}(\sigma|\Omega)$,
coincides with the discrete measure $\sum_{i=1}^n t_i \delta_{x_i}$ on $\partial\Omega$. Let us introduce the following definition:

**Definition 1.** A “strut net function” is a concave piecewise linear function $\phi(x)$ on $\tilde{\Omega}$ coinciding
with $L_i(x)$ on each $\Omega_i$.

We see that, in particular, the envelope of the tangent planes,

$$\phi_0(x) \equiv \min_i L_i(x),$$

(7)
is a strut net function: we call it the “open strut net function” as it is associated with a strut net
having no loops. Note also that concavity implies that any strut net function $\phi(x)$ satisfies, for all
$i$, $\phi(x) \leq L_i(x)$, and hence that $\phi(x) \leq \phi_0(x)$.

### 2.1 Forces at points inside the convex hull of all application points and multiple obstacles

Consider a strut net containing a convex elementary polygonal loop within which there are no
obstacles. By elementary we mean a loop with no interior struts. Then, the net forces (due
to adjoining struts or imposed forces) act on the vertices of the loop as in Figure 1(a), and the
polygonal loop can be replaced by an open strut net, as in Figure 1(b). In this way, if there are
many internal loops in the strut net, we decrease the number of elementary loops by one. This
reduction can be continued until any remaining elementary loop encloses one or more obstacles or
has an imposed force acting outwards from a vertex of the polygonal loop, so that the loop is not
convex.

Then, using the fact that any obstacle intersecting the boundary of the convex hull of all points
where forces are applied can never have a loop of struts under compression surrounding it, we
establish:

**Theorem 1.** A strut net that has $q$ imposed forces at points that lie strictly inside the convex hull
of all points where forces are applied and that avoids $p$ obstacles, $p_0$ of which intersect the boundary
of this convex hull, can be replaced by a strut net, supporting the same forces, that has at most
$q + p - p_0$ elementary loops.

This generalizes the result in [8] that used a similar argument to treat the case where $p = p_0 = 0
(q \neq 0)$.

If a loop contains an obstacle it may be possible to replace that loop by the associated open
strut net provided that open strut net does not collide with the obstacle. In the $q = 0$ case it
follows that, if a net exists that avoids a single obstacle, then there is a net with none or one loop
that also avoids the obstacle. Specifically, the zero loop case corresponds to the open strut net,
and if there is one loop that cannot be reduced, then that loop contains the obstacle. Note that, as
the struts in the loop are under compression, the loop cannot extend outside the convex hull of the
points where the forces are applied. Thus, if an obstacle intersects the open strut net and extends
outside the convex hull of the points where the forces are applied, then there is no strut net that
avoids the obstacle: see Figure 2.
Figure 1: Here (a) is a convex polygonal loop, possibly lying within a strut net that may contain many loops. The arrows denote the net forces acting on each vertex. Provided there are no obstacles inside the loop, it can be replaced by the open strut net as in (b) thus reducing the number of loops in the whole strut net by one.

Figure 2: An obstacle may lie partly outside the convex hull of where the forces are applied, but outside the open strut net, as in (a). However if the obstacle lies partly outside the convex hull of where the forces are applied and intersects the open strut net, as in (b), then there is no strut net that avoids the obstacle.
3 Solution for the obstacle problem with forces at the boundary of a convex polygon

Our objective is to find an algorithm for determining if a strut net exists that avoids a given set of simply connected obstacles and which supports a given set of forces at the boundary of a convex polygon. We will first assume that there is just one obstacle and we will enlarge the region it can occupy. Here by enlargement we use the following definitions:

Definition 2. Given a strut net avoiding an obstacle $O$, the region available to $O$ is the interior of the strut loop, having no internal struts, that contains $O$.

Definition 3. Given a region $\Gamma$ available to $O$ for some strut net, then $\Gamma'$ is an enlargement of it if $\Gamma'$ is available to $O$ for some strut net and $\Gamma' \supseteq \Gamma$.

Thus, if the obstacle is in a region $\Gamma$, then it is surely in the enlarged region $\Gamma'$: $O \subseteq \Gamma$ implies $O \subseteq \Gamma'$. While the procedure will be illustrated by figures in which there are only 4 applied forces, the analysis holds when there are any number of forces applied at the vertices of a convex polygon.

Given a strut net, and associated strut net function $\psi(x)$, we can modify the strut net to try to accommodate, or better accommodate, an obstacle $O$ by replacing $\psi(x)$ with

$$\phi(x) = \min\{\psi, L\}, \text{ with } L(x_k) \geq L_k(x_k) \text{ for all } k,$$

where $L(x)$ is a linear function to be chosen so that $\phi(x) = L(x)$ on $O$. Thus, $\psi(x) \geq L(x)$ on $O$. We call $L(x)$ the cleaving plane. Now, if for any $x$,

$$L(x) \leq \psi(x),$$

then surely because $\psi(x) \leq \phi_0(x)$, one has that

$$L(x) \leq \phi_0(x).$$

Thus, the region $\Gamma$ available to the obstacle is enlarged if the cleaving plane cleaves the open strut net function rather than any other strut net function supporting the forces: see Figure 3(a)-(d).

Hence, the existence of a strut net avoiding $O$ is equivalent to the existence of a linear function $L(x)$ satisfying

$$L(x_i) \geq L_i(x_i) \text{ and } L(x) \leq L_i(x) \text{ on } O \text{ for all } i.$$  \hfill (4)

3.1 One or more obstacles which are convex polygons, or which can be approximated by convex polygons

Note that, for any cleaving plane $L(x)$, the set $\{x : \phi_0(x) \geq L(x)\}$ is a convex set because of the concavity of the open strut net function $\phi_0(x)$. Hence, there is no loss of generality by assuming that $O$ is convex. Therefore, consider the case when $O$ is a convex polygon or is approximated by a convex polygon containing $O$. Let $y_p, p = 1, 2, \ldots, m$ be the vertices of this polygon. Then, (4) will hold if $L(x)$ satisfies

$$L(x_i) \geq a_i \equiv \phi_0(x_i) \text{ and } L(y_p) \leq b_p \equiv \phi_0(y_p) \text{ for all } i, p.$$  \hfill (5)
Figure 3: Here (a) shows the cleaving (dot-dash line) of a strut net function (solid and dashed lines), resulting in the strut net (b). If instead as in (c) one cleaves, with the same cleaving plane, the open strut net function, one enlarges the region that the obstacle can occupy, and the corresponding strut net (d) has a single elementary internal loop. Finally, by moving the cleaving plane downwards keeping the same orientation until it touches one of points \( L_k(x_k) \), as in (e), one further enlarges the region that the obstacle can occupy, as in (f).

Now, \( L(x) \) can be written as \( L(x) = v \cdot x + c \) and we are looking for \( v \) and \( c \) so that the inequalities are satisfied, that is, for all \( i \) and \( p \),

\[
v \cdot x_i + c \geq a_i, \quad v \cdot y_p + c \leq b_p.
\]  

The question becomes whether there is a feasible domain of \((v, c)\) pairs satisfying this system of linear inequalities: this is a standard problem in linear programming theory.

Note that this is easily generalized to the case of multiple inclusions \( O_q, q = 1, 2, \ldots, s \), with associated cleaving planes \( L^{(q)}(x) \). Then, (4) is replaced with

\[
L^{(q)} \geq L_i(x_i) \quad \text{and} \quad L^{(q)} \leq L_i, \quad L^{(q)} \leq L^{(r)} \quad \text{on} \ O_q, \ \text{for all} \ i, q, r. 
\]  

Treating each \( O_q \) as convex or approximating it by a convex polygon containing \( O_q \), we let \( y_p^{(q)}, p = 1, 2, \ldots, m(q) \) be its vertices. Then (7) will hold if the cleaving planes \( L^{(q)}(x) \) satisfy for all \( i, p, r \)

\[
L^{(q)}(x_i) \geq a_i \equiv \phi_0(x_i) = L_i(x_i),
\]  

and

\[
L^{(q)}(y_p^{(q)}) \leq b_p^{(q)} \equiv \phi_0(y_p^{(q)}), \quad L^{(q)}(y_p^{(q)}) \leq L^{(r)}(y_p^{(q)}).
\]  

Writing \( L^{(q)}(x) \) as \( L^{(q)}(x) = v^{(q)} \cdot x + c^{(q)} \) we arrive at the linear system of inequalities

\[
v^{(q)} \cdot x_i + c^{(q)} \geq a_i, \quad v^{(q)} \cdot y_p^{(q)} + c^{(q)} \leq b_p^{(q)}, \quad v^{(r)} \cdot y_p^{(q)} + c^{(r)} - v^{(q)} \cdot y_p^{(q)} + c^{(q)} \geq 0,
\]  

which must hold for all \( i, q \) and \( r \). This is again a standard problem of determining if there is a non-empty feasible domain in the \( 3s \) dimensional space consisting of \( s \) pairs \((v^{(q)}, c^{(q)}) q = 1, 2, \ldots, s \). If
it is non-empty, then associated with a \((v^{(q)}, c^{(q)})\) \(q = 1, 2, \ldots, s\) in the feasible domain are cleaving planes \(L^{(q)}(x)\), \(q = 1, 2, \ldots, s\) and when \(\phi_0(x)\) is cleaved by them we obtain the strut net avoiding the obstacles.

3.2 Enlarging as much as possible the region available to the obstacle

Let us return back to the case where there is a single obstacle and assume that the feasible domain associated with \(\overline{O}\) is non-empty. While the feasible domain allows us to identify all strut nets with one strut loop that contains \(\overline{O}\), it does not identify those strut nets with as much room as possible around \(\overline{O}\). By this we mean that the region \(\Gamma\) available to \(\overline{O}\) is maximal, in the sense that if \(\Gamma'\) is the region available to \(\overline{O}\) in another strut net and \(\Gamma \subseteq \Gamma'\), then \(\Gamma' = \Gamma\). Note that \(\Gamma\) is not necessarily unique, and a different \(\Gamma\) may have less room around the obstacle on one side and more on another side. Here we will show that if \(\overline{O}\) lies in a maximal \(\Gamma\) then

\[
L(x_k) = L_k(x_k) \quad \text{and} \quad L(x_\ell) = L_\ell(x_\ell),
\]

(11)

for some \(k\) and \(\ell\).

If, for any \(x\), (3) holds then surely

\[
L'(x) \leq \phi_0(x) \quad \text{with} \quad L' = L - \min_i (L(x_i) - a_i), \quad a_i = L_i(x_i),
\]

(12)
because \(L'(x) \leq L(x)\). In other words, the cleaving plane can be lowered while keeping its orientation fixed, expanding the region \(\Gamma\) that can be occupied by the obstacle. This can be continued until this cleaving plane \(L'\) first touches a point \(A = L_k(x_k)\), where \(i = k\) attains the equality in (12); see Figure 2.

Now we tilt the cleaving plane about the line through \(A\) and another point \(B \in L \cap L_k\) with \(B \neq A\). This line is where \(L'(x)\) and \(L_k(x)\) intersect. This gives a new cleaving plane

\[
L'' = L' + \alpha(L' - L_k), \quad \text{with} \quad \alpha > 0.
\]

(13)

We can continue this tilting by increasing \(\alpha\) until

\[
\alpha = \min_j \frac{L'(x_j) - L_j(x_j)}{L_k(x_j) - L'(x_j)}.
\]

(14)

when \(L''(x)\) first touches another point \(C = L_\ell(x_\ell)\) in which \(j = \ell\) achieves the equality in (14). It also touches \(A\). Note that if for any \(x\) the first inequality in (12) holds then surely

\[
L''(x) \leq \phi_0(x),
\]

(15)
because that first inequality implies \(L'(x) \leq L_k(x)\) at that point. This implies that as the cleaving plane is tilted until it touches \(B\), the region \(\Gamma\) that can be occupied by the obstacle enlarges further: see Figure 4(a)-(d).

One can subsequently rotate the cleaving plane about the line joining \(A\) with \(L_\ell(x_\ell)\) until the cleaving plane meets \(L_h(x_h)\) for some \(h\). In our example, rotation one way gives Figure 4(e) with \(L_h(x_h) = D\) and the corresponding strut net Figure 4(f), while rotation the other way gives Figure 4(g) with \(L_h(x_h) = E\) and the corresponding strut net Figure 4(h). However, this rotation does not generally enlarge the region \(\Gamma\) that can be occupied by the obstacle. The rotation tips the cleaving plane up on one side of the line \(L_k(x_k)\) with \(L_\ell(x_\ell)\) and down on the other side. The region \(\Gamma\), with the strut loop as its boundary, shrinks on the up side and expands on the down side.
Figure 4: Various configurations, (a), (c), (e), (g), (i), and (k) of the cleaving plane, denoted by dashed lines, that cleave $\phi_0(x)$, denoted by solid lines. The associated strut nets are below each of these subfigures. The procedure of tilting and then rotating the cleaving plane follows that described in the text.

The exception, as illustrated in Figure 4(i) with the corresponding strut net Figure 4(j) is when \( \ell = k \pm 1 \), i.e. when \( L_\ell(x_\ell) \) is on the same facet of \( \phi_0(x) \) as \( L_k(x_k) \). Then the region \( \Gamma \) lies on one side of the line joining \( x_k \) with \( x_\ell \) and, as illustrated in Figure 4(k) we may rotate the cleaving plane about the line so the plane tilts down on the side where \( \Gamma \) is located and touches \( D = L_h(x_h) \), producing the strut net of Figure 4(l).

This analysis provides an algorithm to generate all maximal regions \( \Gamma \). One starts with each \( i = 1, 2, \ldots, n \) and takes the cleaving plane through \( L_i(x_i), L_{i+1}(x_{i+1}) \) and a third point \( L_j(x_j) \) (with no points \( L_h(x_h), h = 1, 2, \ldots, n \) lying above the plane). One next rolls the cleaving plane, rotating it about the line joining \( L_i(x_i) \) and \( L_j(x_j) \) until it first hits another point \( L_k(x_k) \). Then one rolls it further about the line joining \( L_i(x_i) \) and \( L_k(x_k) \) until it first hits another point \( L_\ell(x_\ell) \). This is continued until the rolling cleaving plane hits \( L_{i-1}(x_{i-1}) \). As the rolling progresses the region enclosed by the strut net loop created by the cleaving plane is maximal and the region available to an obstacle placed within it cannot be enlarged.

4 Including reactive forces with multiple obstacles

Let us suppose that a subset \( I \subset \{1, \ldots, n\} \) of the \( n \) forces are reactive, while the remaining ones (corresponding to the subset of indices \( K := \{1, \ldots, n\} \setminus I \) are given. That means that the forces are not fixed at the points \( x_i \) for \( i \in I \) but that any forces at these points are admissible as soon as they balance through a strut net the forces \( t_i \) applied at points \( x_i \) for \( i \in K \). The planes \( L_k \) are no longer fixed by equations (4). When \( i \in I \), these equations, determine the reactive force \( t_i \):

\[
t_i \equiv -R_\perp(\nabla L_{i+1} - \nabla L_i), \quad i \in I
\]
and provide no constraints on the unknown \( L_{i+1} \) and \( L_i \) while, for the remaining \( i \in \mathcal{K} \), they provide the constraints

\[
R_{\perp}(\nabla L_{i+1} - \nabla L_i) = -t_i, \quad i \in \mathcal{K}.
\] (2)

Since we can add any linear function to the Airy stress function, we are free to assume, as before, that

\[
L_1 = 0.
\] (3)

Continuity of the Airy function requires that, for all \( i \in \{1, 2, \ldots, n\} \),

\[
L_{i+1}(x_i) = L_i(x_i).
\] (4)

Its concavity requires that, for all \( i \neq j \) in \( \{1, 2, \ldots, n\} \),

\[
L_j(x_i) \geq L_i(x_i).
\] (5)

As in [10], the existence of a strut net avoiding the \( s \) obstacles \( O_1, O_2, \ldots, O_s \), each \( O_q \) being the convex hull of points \( (y_1^q, y_2^q, \ldots, y_n^q) \), is equivalent to the existence, for all \( q \in \{0, 1, \ldots, s\} \), of linear functions \( L^{(1)}, L^{(2)}, \ldots, L^{(s)} \) satisfying, for any \( i \) in \( \{0, 1, \ldots, n\} \), any \( q \neq r \) in \( \{0, 1, \ldots, s\} \) and any \( p \) in \( \{0, 1, \ldots, n(q)\} \),

\[
L^{(q)}(x_i) \geq L_i(x_i), \quad L_i(y_p^{(q)}) \geq L^{(q)}(y_p^{(q)}), \quad L^{(r)}(y_p^{(q)}) \geq L^{(q)}(y_p^{(q)}).
\] (6)

These results, which also generalize the results obtained in Section 3, can be summarized in the following theorem:

**Theorem 2.** The existence of a strut net balancing forces \( t_i \) applied at points \( (x_i)_{i \in \mathcal{K}} \) with the help of reacting points \( (x_i)_{i \in \mathcal{T}} \) and avoiding obstacles \( O_1, O_2, \ldots, O_s \), each \( O_q \) being the convex hull of points \( (y_1^q, y_2^q, \ldots, y_n^q) \), is equivalent to the existence of linear functions \( (L_i)_{i=1}^{n} \) and \((L^{(q)})_{q=1}^{s}\) satisfying, for any \( i \) in \( \{0, 1, \ldots, n\} \), any \( q \neq r \) in \( \{0, 1, \ldots, s\} \) and any \( p \) in \( \{0, 1, \ldots, n(q)\} \), the constraints (2), (3), (4), (5) and (6). Setting\n
\[
L_i(x) = w_i \cdot x + d_i \quad \text{and} \quad L^{(q)}(x) = v^{(q)} \cdot x + c^{(q)},
\] (7)

the existence of such a strut net is equivalent to the existence of a solution \( (w_i, d_i)_{i=1}^{n} \) and \((v^{(q)}, c^{(q)})_{q=1}^{s}\) of the following linear programming problem \( R_{\perp}(w_{i+1} - w_i) = -t_i, \) for \( i \in \mathcal{K} \),

\[
\begin{align*}
w_1 &= 0, \quad d_1 = 0, \\
w_{i+1} \cdot x_i + d_{i+1} &= w_i \cdot x_i + d_i, \quad \text{for } 1 \leq i \leq n, \\
w_j \cdot x_i + d_j &\geq w_i \cdot x_i + d_i, \quad \text{for } i \neq j \text{ in } \{1, \ldots, n\}, \\
v^{(q)} \cdot x_i + c^{(q)} &\geq w_i \cdot x_i + d_i, \quad \text{for } 1 \leq i \leq n, 1 \leq q \leq s, \\
w_i \cdot y_p^{(q)} + d_i &\geq v^{(q)} \cdot y_p^{(q)} + c^{(q)}, \quad \text{for } 1 \leq i \leq n, 1 \leq q \leq s, 1 \leq p \leq n(q), \\
v^{(r)} \cdot y_p^{(q)} + c^{(r)} &\geq v^{(q)} \cdot y_p^{(q)} + c^{(q)}, \quad \text{for } q \neq r \text{ in } \{1, \ldots, s\}, 1 \leq p \leq n(q). \quad \text{(8)}
\end{align*}
\]

Recall that, as soon as admissible quantities \( (w_i, d_i)_{i=1}^{n} \) have been chosen, the reacting forces are determined by (4).

We must emphasize the fact that, when a solution exists for the previous linear programming problem, generally an infinity of solutions exists. Indeed we have not fixed any objective function to minimize. We can fix in very different ways such a linear objective function. But even when
it is given uniqueness is not guaranteed. This fact is clearly illustrated by the following example. Assume that all bars in the strut net have a thickness proportional to the force they carry, in such a way that the stress remains constant in the whole structure. Then the total volume $\mathcal{V}$ of the structure is proportional to the integral over $\Omega$ of the Laplacian of the strut net function (a negative measure). This linear function of the unknowns, which from here onwards we will call the total weight of the structure, seems to be a good candidate for the objective function. However one can notice that, using the divergence theorem, $\mathcal{V}$ can be computed from the normal derivative of the strut net function on the boundary of the domain. Therefore it does not involve the $L^{(q)}$ functions: its minimization can help fixing the $L_i$ functions, that is the reactive forces, but it cannot entirely determine a unique strut net.

We can also treat the case when a subset $J \subseteq \mathcal{I}$ of the points at which the reactive forces act, rather than being fixed, are confined to some line segments. These segments, for example, could be sections of supporting walls or ground. Then, there are the constraints

$$x_j \cdot (R \perp g_j) = z_j, \quad g_j^- \leq x_j \cdot g_j \leq g_j^+, \quad \text{for } j \in J,$$

where the unit vectors $g_j$ and constants $z_j, g_j^-, g_j^+$ are given, but only defined for $j \in J$. These segments should be such that the $x_i, i = 1, 2, \ldots, n$ are still the vertices, going anticlockwise, of a convex polygon for all choices of points $x_j, j \in J$ anywhere on the line segments. Now we need to add the $x_j, j \in J$ to the unknowns. Then, those equations in (8) that involve $x_j, j \in J$ provide quadratic (and non-convex) constraints on the unknowns, and we are left with determining the feasible domain associated with a quadratic programming problem, involving (8) and (9), again a standard problem.

It could be the case that reactive forces act at, say, a pair of neighboring points $x_i$ and $x_{i+1}$ with $i \in J$ and $i + 1 \in J$ and share the same line segment. Then, $g_i = g_{i+1}, \quad z_i = z_{i+1}, \quad g_i^- = g_{i+1}^-, \quad g_i^+ = g_{i+1}^+$, and to maintain the anticlockwise order of points (9) needs to be supplemented by the additional constraint that

$$(x_i - x_{i+1}) \cdot g_i \geq 0, \quad (11)$$

by, if necessary, reversing the direction of $g_i$, the signs of $z_i, g_i^-$ and $g_i^+$, and reversing the inequalities in (9), while maintaining (10). The generalization to the case of more reactive forces acting at neighboring points sharing the same interval is straightforward.

There is an alternative to allowing the points $x_j, j \in J$ to range along these line segments: one can distribute along each line segment sufficiently many additional fixed points $x_i$ at which reactive forces act. This has the advantage of replacing the quadratic programming problem with a linear one, generally at the sacrifice of having more unknowns. This is what has been done in the numerical examples in Sect. 5.3 where 200 points with reactive forces have been distributed along two supporting segments.

5 Numerical Results

The first part of this section deals with some numerical applications of Theorem 1 of Section 2, the second part treats the issue of enlarging the region where the obstacle can be placed, while the last part discusses numerical examples of the linear programming procedure described in Theorem 2 of Section 4. Specifically, in Section 5.1, we consider two examples in which the internal loops of a strut net that avoids an obstacle are simplified into open strut nets, up to a certain number
of irreducible elementary loops. Indeed, according to Theorem 1 in Section 2, the number of loops that cannot further be simplified depends on the number of points of application of the forces that lie inside the convex hull and on the position of these forces and the obstacle itself. In Section 5.2, we implement the algorithm, presented at the end of Section 3, that allows one to enlarge the area available to place an obstacle. Finally, in Section 5.3 strut nets are generated that support sets of active applied forces at given points and reactive forces at other given points that act in response to the applied forces. The first example gives a strut net with two triangular cells that supports a single active force and avoids four obstacles. Four subsequent examples examine strut net models for masonry arches modeled as rigid no-tension bodies [11, 12], which are subject to different loading conditions and are required to avoid various obstacles. In all numerical examples, we use abstract units for lengths and forces.

5.1 Loop reduction

5.1.1 Reduction to a funicular arch strut net

Let us consider strut net models of a semi-circular masonry arch with horizontal span of the extrados equal to 18, a rise at the intrados of 7.6 and a thickness of 1.4. The arch is loaded by a set of 9 active forces with unit magnitude directed downwards, uniformly distributed at equal angles of \( \pi/20 \) along the extrados, and 2 reactive forces acting at the ends of the arch. An initial strut net featuring \( \ell = 13 \) closed loops was obtained by employing the numerical procedure presented in [22] under the action of the given active forces (a unique material was assigned to all the struts, see Fig. 5(a)). Such a strut net was next reduced to an open strut net (\( \ell = 0 \)), employing the loop reduction procedure presented in [8] (see Figs. 5(b)-(e) and Movie S1). Since the active and reactive external forces of the current example are applied at the vertices of a convex polygon (\( q = 0 \)), it is easily observed that \( \ell \leq q + p - p_0 \), in agreement with Theorem 1 of Section 2. It is easy to show that the open strut net coincides with the funicular polygon of the active forces, which passes through the end points of the arch and has the initial and final segments parallel to the lines of the reactive forces [23]. Note that the open strut net function \( \phi_0(x) \) for this example, plotted in Figure 6(a) for \( L_1 = 0 \), provides the open strut net in Figure 6(b), which coincides with the one represented in Figure 6(a), obtained through loop reduction.

5.1.2 Loop reduction for a strut net with an internal force and that avoids an obstacle

Here we consider the example depicted in Figure 7(a), in which one of the forces, \( f_4 = (0.874, 0.574) \), is applied inside the polygon formed by the remaining forces. Indeed, the point \( x_4 = (5, 5) \), lies inside the polygon formed by \( x_1 = (10, 0) \), \( x_2 = (0, 0) \), \( x_3 = (-5, 10) \), and \( x_5 = (12, 7) \) where the forces \( f_1 = (2.751, -2.319) \), \( f_2 = (-3.415, -1.933) \), \( f_3 = (-3.149, 1.786) \), and \( f_5 = (2.938, 1.891) \) are applied. According to the theory presented in [8], a strut net supporting such forces with all the elements under compression exists. An example is provided by the strut net connecting all the points pairwise as showed in Figure 7(a). Now suppose that an obstacle is placed inside one of the loops of the strut net . As stated by Theorem 1 in Section 2, in which \( q = 1 \), \( p = 1 \), and \( p_0 = 0 \), we can replace the strut net connecting the points pairwise by one that has at most \( q + p - p_0 = 2 \) elementary loops, as showed in the supplementary Movie S2. The result of the loop reduction is portrayed in Figure 7(b), where the two remaining elementary loops cannot be reduced further.
Figure 5: An arched strut net that avoids a half disk (represented in gray) and supports 9 vertical forces of equal magnitude acting downwards along the arch, with the help of two supporting reactive forces (marked in red) acting at the end points of the arch. An open strut net is obtained from an initial net exhibiting $\ell = 13$ closed loops (panel (a)), through the loop reduction procedure presented in [8] (panels (b)-(f)). The masonry arch containing the sequence of strut nets is colored light blue. Note that the reactive forces at the end points of the arch have been scaled by a factor of 0.2. See also Movie S1 in the supplementary material.
Figure 6: (a) Here is the open strut net function $\phi_0(x)$ associated with the example considered in Figure 5. (b) represents the related open strut net, that coincides with Figure 5(f). Note that the reactive forces at the end points of the arch have been scaled by a factor of 0.5.

Figure 7: (a) A strut net that supports the applied forces and avoids the obstacle (here represented by the gray circle) is the one connecting the points pairwise. By applying loop reduction, as illustrated step by step in Movie S2 in the supplementary material, it is possible to reduce the number of elementary loops with at most $q + p - p_0 = 2$ remaining loops, as illustrated in (b).

5.2 Enlarging the region available to the obstacle

In this section, we provide a numerical example that illustrates the algorithm presented at the end of Section 3 to enlarge the region available to the obstacle. We start with 7 forces, $f_1 = (-1, -4)$,
\[ f_2 = (-3, -1), f_3 = (-2, 2), f_4 = (-3, 5), f_5 = (1, 1), f_6 = (6, 2), \text{and} f_7 = (2, -5), \]

applied respectively at the points \( x_1 = (0, 0), x_2 = (-16, 7), x_3 = (-13, 16), x_4 = (2, 20), x_5 = (12, 19), x_6 = (12, 13), \) and \( x_7 = (10, 0) \). The open strut net function for this example is shown in Figure 8(a), and the associated open strut net is represented in Figure 8(b). In order to find the maximal regions \( \Gamma \), we start with point \( x_1 \) and consider the cleaving plane passing through \( L_1(x_1), L_2(x_2), \) and \( L_3(x_3) \), see Figure 8(c), which provides the region \( \Gamma \) depicted in black in Figure 8(d). Then, we roll the cleaving plane about the line connecting \( L_1(x_1) \) and \( L_3(x_3) \), until the plane touches the point \( L_4(x_4) \), Figure 8(e)-(f), and this is continued until the plane touches \( L_7(x_7) \), as shown in Figure 8(g)-(h). Movie S3 in the supplementary material shows the rolling of the plane from the initial configuration Figure 8(c) to the final configuration Figure 8(h): as the rolling occurs, the region in which the obstacle can be placed is maximal.

5.3 Construction of strut nets avoiding obstacles

5.3.1 A strut net avoiding triangular and square inclusions

Let us search for a strut net that supports a single active vertical force (pointing downward) applied to the point with coordinates \((0, 4)\) in a given Cartesian plane. We examine two supporting segments delimited by the points \((-4, 0)\) and \((3, 4)\) (segment 1), and the points \((3, 0)\) and \((4, 0)\) (segment 2), which are placed on the opposite extremities of the strut net. Each of such segments is composed of 100 fixed reaction points uniformly spaced. The searched strut net needs to avoid four obstacles, which are formed by three triangles \((T_1, T_2\) and \(T_3)\) and one rectangle \((R)\). The coordinates of the vertices of such polygons are defined as follows

- \( T_1 \) \([1, 2]; (3, 2); (3, 3)\]; \( T_2 \) \([0, 1.3]; (0.2, 3.5); (-0.2, 3.5)\]; \( T_3 \) \([-2.5, 0.5]; (-1, 0.5); (-1, 2)\];
- \( R \) \([-2, 1.2]; (-2, 3); (-3, 3); (-3, 1.2)\].

Figure 9 illustrates four strut nets obtained through the linear programming algorithm of Section 4 using different objective functions. The result in Figure 9(a) was obtained by minimizing the total weight of the structure, i.e., the integral over the boundary of the domain of the normal derivative of the strut net function. In this case, the optimal strut net is composed of 7 struts and 2 elementary loops. The strut nets in Figure 9(b,c,d) were instead obtained by minimizing the heights of the cleaving planes of the obstacles \( T_1, T_2, \) and \( T_3 \) at their centers of mass, respectively. By minimizing one of such functions, the goal is to enlarge the room available to the corresponding obstacle. It does not always achieve this goal because the minimization can tilt the cleaving plane, or move the surrounding facets of the Airy stress function. The strut net in Figure 9(b) is again composed of 7 struts and 2 elementary loops, as that in (a). The strut net in (c) is instead formed by five struts and one loop, while that in (d) shows ten struts and two loops. Observe that in most directions more space is available to obstacle 3 in (c) rather than (d). Solutions like those in (c), where there are no struts separating two obstacles, can be enforced, if possible, by treating the two obstacles as a single obstacle. Also, it is worth noting that one could remove the last fork on the left of the strut net in Figure 9(d) and replace it by a straight line that touches the support midway. This is another way to enlarge the room around \( T_3 \) (not shown in the figure).

5.3.2 Arched strut nets avoiding elliptical inclusions

Let us now pass to examine strut net models of masonry arches loaded by vertical forces acting downwards on top of the arch. We consider two different sets of 23 vertical active forces, equally spaced along the horizontal line segment connecting the point \((-2.75, 4.20)\) with the point
Figure 8: (a) Here’s the open strut net function $\phi_0(x)$ associated with the open strut net depicted in (b), when $L_1 = 0$. In (c), a cleaving plane (represented in black) passing through the points $L_1(x_1)$, $L_2(x_2)$, and $L_3(x_3)$, intersects $\phi_0(x)$ to create a region displayed in black in (d) where an obstacle can be placed. We roll, then, the cleaving plane about the line connecting $L_1(x_1)$ and $L_3(x_3)$, until the plane touches the point $L_4(x_4)$, see panels (e) and (f). At this point, we let the plane roll about the line connecting $L_1(x_1)$ and $L_4(x_4)$, until it reaches the point $L_6(x_6)$. Lastly, we let the cleaving plane roll about the line connecting $L_1(x_1)$ and $L_6(x_6)$ until it touches $L_7(x_7)$, as showed in (g) and (h). The rolling of the plane is illustrated step by step in Movie S3 in the supplementary material. Note that, by applying this procedure to the remaining points $x_i$, $i = 2, ..., 7$, we can find the maximal feasible areas that an obstacle can occupy.

(2.75, 4.20) of a given Cartesian plane (‘loading segment’ running along the extrados of the arch). We also examine two supporting segments, which coincide with those analyzed in the previous example. The first obstacle to be avoided by the strut net is a half-disk of radius $R = 2.9$, centered at the origin of the Cartesian plane. It represents the central region that the arch needs to cross (‘arch obstacle’). The second and third obstacles are two elliptical regions, which are respectively centered at the point $(1.2, 3.5)$ and the point $(-1.2, 3.5)$. These obstacles exhibit vertical semi-major axis equal to 0.40 and horizontal semi-minor axis equal to 0.25. They are intended to reproduce holes that need to be drilled within the masonry arch (‘hole obstacles’). The masonry structure that contains the strut nets as ‘thrust lines’ (or internal resisting structures) are colored light brown in the figures that follow. It is composed of two triangular abutments with 1.10 width at
Figure 9: Strut nets resulting from the linear programming problem: a single active vertical force (marked in green) is supported by the shown strut net (blue in color) with the help of 200 reacting points (placed along the segments marked in red). The net must avoid 4 regions (marked in gray). The strut net in (a) minimizes the total weight of the structure, while those in (b,c,d) minimize the heights of the cleaving planes of the obstacles $T_1$, $T_2$ and $T_3$ at their centers of mass, respectively.

The strut nets lie within the masonry, which implies that the structure is stable under the examined loading conditions, according to the master safe theorem of masonry arches [1]. All the strut nets of the examples that follow have been obtained through the linear programming algorithm of Section 4 by minimizing the total weight of the structure. In the simulations the tension in the outermost struts is very small and these struts disappear if the objective function to be minimized is suitably perturbed.

We begin by considering the case when the 23 active vertical forces are of equal (unit) magnitude and there is only the arch obstacle. The strut net obtained for this set of forces is illustrated in Figure 10. It consists of an open strut net formed by 49 struts, which is symmetric with respect to the $y$ axis. As a second example, we again consider 23 vertical active forces of equal magnitude, as in the previous case, but this time we search for a strut net that avoids both the arch obstacle and the two ellipse obstacles defined above. Figure 11 shows the solution obtained for the current example, which is composed of 53 struts, again symmetrically distributed with respect to the $y$
axis. The elliptical obstacles are embraced by closed loops of the strut net.

Figure 10: An arch that avoids a half disk (represented in gray), supporting 23 vertical forces of equal magnitude acting downwards on the top (in green) with the help of two supporting segments (in red). A feasible strut net (in dark blue) is obtained by using our linear programming formulation where each supporting segment has been replaced by 100 fixed reactive points, and the half disk is approximated by a 101 sided polygon. The masonry arch containing the examined strut net has been colored light brown.

Figure 11: An arched strut net avoiding three obstacles. Like the previous example, it supports 23 vertical downward forces of equal magnitude. Additionally, the structure avoids the two elliptical regions, each approximated by a 20 sided polygon.
We now pass to analyze a system of active forces composed of 11 vertical forces with magnitude 0.3 uniformly spaced on the half of the loading segments placed along the negative $x$-axis; and 12 vertical forces with unit magnitude equally spaced along the complementary half of the loading segment. The strut net returned by the linear programming algorithm for the present case is shown in Figure 12. It is composed of 52 struts that are not symmetrically distributed with respect to the $y$ axis, and exhibits three closed loops.

![Figure 12: An arched strut net (in dark blue) avoiding three obstacles (in grey), which supports 11 vertical downward forces with magnitude 0.3, and 12 vertical downward forces with magnitude 1.0.](image)

6 Conclusions

Given a two-dimensional set of forces at the vertices of a convex polygon, some of which are prescribed forces and the remainder reactive forces, our main goal was to determine if a strut net exists avoiding a given set of obstacles, and if so, to construct one such strut net. By approximating each obstacle by a convex polygon, possibly with many sides, we successfully devised an algorithm for doing this. It reduced to a linear programming procedure and was based on finding a suitable concave polyhedral Airy stress function associated with the strut net when all struts are under compression. Additionally, under the assumption that there is just one obstacle, we devised an algorithm for generating regions that provide the maximum amount of space available to the obstacle, as shown in Figure 8 and Movie S3. By this we mean that the region available to the obstacle cannot be enlarged in the sense of Definitions 3 and 2. We also obtained partial results (Theorem 1) in the case where there are $q$ forces strictly inside the convex hull of all points where forces are applied. We established that a strut net supporting these forces, while avoiding $p$ obstacles, $p_0$ of which intersect this convex hull, can be replaced by a strut net, supporting the same forces and avoiding the same obstacles, that has at most $q + p - p_0$ elementary loops. In Figure 5 and Movie S1, $q = 0$, and $p = p_0 = 1$ and the number of loops can be reduced until one obtains a final structure that is an open strut net. In Figure 7 and Movie S2, instead, $q = p = 1$, 

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and $p_0 = 0$, and the final net presents two elementary loops.

Our work can be used to determine if a given masonry structure (modeled as an incompressible, no-tension body [1]) can support, under compression, one or more families of given forces (and convex combinations of them). For example, the masonry arch colored light brown in Figures 11 and 12 can support at least 2 families of forces (and convex combinations of them), namely those in the figures, while avoiding the two elliptical obstacles. Alternatively, we may use the strut nets that support the desired families of forces to efficiently design the masonry structures themselves, eliminating unnecessary regions. Constraints on the boundary of the masonry structure can be imposed by introducing appropriate obstacles such as, for instance, the semicircular grey obstacle beneath the arch in Figures 10, 11 and 12. Our analysis also applies to structures built from unreinforced concrete since it supports compression but not tension.

We address the following generalization of the present study in future work: given a force set $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and a vector set $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, determine the extreme values of $\lambda$ such that a strut net avoiding the obstacles supports both $\mathbf{t}$ and $\mathbf{t} + \lambda \mathbf{f}$. Here $\mathbf{f}$ could represent an additional forcing, say, e.g., due to an earthquake [4, 24].

There are many other avenues that have not been explored in the present paper. An extension of the results to three dimensions would obviously be important, but perhaps very difficult. The stress does have a matrix valued function, the Beltrami stress function, that is analogous to the two-dimensional Airy stress function. However, it is difficult to interpret the constraints on the Beltrami stress function imposed by a negative semidefinite stress field. Another extension is to find a way of generating strut nets supporting $q$ forces strictly inside the convex hull of all points where forces are applied, while avoiding $p$ obstacles, without assuming one starts with a strut net achieving these goals. Further goals are outlined in [25]. One, with or without obstacles, is to allow for some elasticity in the struts, and to determine the possible sets of displacements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ at the vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ when a force set $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ is applied. (Assuming the force set is such that at least one strut net supporting the forces, yet avoiding any obstacles, exists.) Another is to allow for finite deformations, with or without obstacles. That is, given moving points $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ and balanced forces $\mathbf{f}_1(t), \mathbf{f}_2(t), \ldots, \mathbf{f}_n(t)$ that are dependent on time $t$, when does there exist a single strut net under tension that supports these forces and avoids any obstacles for some given interval of $t$? Here, single strut net means a strut net wherein the angles between struts change with $t$ but the topology and strut lengths do not; all struts also remain under compression and do not collide. An even more challenging problem results if some of the struts collide as $t$ changes.

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Data availability

Three movies are provided as supplementary material. Movie S1 illustrates the step-by-step loop reduction procedure for the example depicted in Figure 5, whereas Movie S2 treats the loop reduction of the strut net represented in Figure 7. Finally, Movie S3 illustrates the rolling of the cleaving plane to create maximal regions where the obstacle can be placed, as illustrated in Figure 8.
Supplementary materials

Movie_S1.mp4

Caption for Movie S1

An arched strut net featuring $\ell = 13$ closed loops, which avoids an semi-circular arch obstacle ($p = 1$), is reduced to an open strut net ($\ell = 0$) by employing the loop reduction procedure presented in [S]. The active and reactive external forces are at the vertices of a convex polygon ($q = 0$) and the obstacle intersects this polygon ($p_0 = 1$). It is worth observing that the final arch satisfies $\ell \leq q + p - p_0$, in agreement with Theorem 1 of Section 2.

Movie_S1.mp4

Caption for Movie S2

We consider an example in which one of the 5 applied forces lies inside the convex hull formed by the points where the forces are applied. A strut net that supports such forces and avoids the obstacle represented by the gray circle is the one connecting the points pairwise. The movie shows how each elementary loop forming such a net can be replaced by an open strut net. In agreement with Theorem 1 of Section 2, the reduction can be done until at most 2 remaining elementary loops remain, as shown in the last frame of the movie.

Movie_S2.mp4

Caption for Movie S3

We consider 7 forces applied at the vertices of a convex polygon and we use the algorithm proposed at the end of Section 3 to determine the maximal regions in which an obstacle can be placed so that a strut net avoiding it exists. The movie starts with a picture of the open strut net function on the left-hand side, and the associated open strut net on the right-hand side. Secondly, the cleaving plane passing through the points $L_1(x_1)$, $L_2(x_2)$, and $L_3(x_3)$ is depicted on the left-hand panel, and the corresponding region $\Gamma$ is depicted in black in the right-hand panel. Pivoting around point $L_1(x_1)$, the cleaving plane rolls about the line connecting $L_1(x_1)$ and the last point touched by the plane (initially $L_3(x_3)$), spanning over the maximal elements $\Gamma$.

Movie_S3.mp4

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