On the dimension of angles and their units

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Abstract

We examine implications of angles having their own dimension, in the same sense as do lengths, masses, etc. The conventional practice in scientific applications involving trigonometric or exponential functions of angles is to assume that the argument is the numerical part of the angle when expressed in units of radians. It is also assumed that the functions are the corresponding radian-based versions. These (usually unstated) assumptions generally allow one to treat angles as if they had no dimension and no units, an approach that sometimes leads to serious difficulties. Here we consider arbitrary units for angles and the corresponding generalizations of the trigonometric and exponential functions. Such generalizations make the functions complete, that is, independent of any particular choice of unit for angles. They also provide a consistent framework for including angle units in computer algebra programs.
I. INTRODUCTION

Angle is a familiar concept that needs no formal definition. When two lines cross in a plane, two pairs of vertical angles are formed. Unless the lines are perpendicular, one pair of angles is acute and the other pair is obtuse. In trigonometry, the geometric properties of interior angles of right triangles may be used to derive general relations between various trigonometric functions. For example, the formula for the sine of the sum of two angles in terms of sine and cosine functions of each of the angles, does not depend on a quantitative specification of the angles. That is, when we write $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ we do not specify whether $\alpha$ and $\beta$ are measured in degrees or radians or grads, nor do we specify that the appropriate functions be used, according to the units of the arguments. This and other features of angles lead to the often-repeated (and often misunderstood) assertion that angles do not have units and therefore do not have a dimension (or that their dimension is unity).

By contrast, when angles appear in physical measurements, it is essential to specify the unit of measure for the angle. For example, it would be of no use to specify the angle subtended by the distance between two stars, or the angle between two laser beams, or the rotation angle of a wheel without giving units like arcseconds or milliradians or full rotations.

All of these comments about plane angles apply equally to phase angles. For example, the amplitude of an electromagnetic plane wave of fixed frequency and propagation vector (wave vector) is proportional to a sine function of an argument, a phase angle, whose value changes linearly with time and distance. If we want to report the difference in phase angle between two spacetime points, for example, we must specify the units—degrees or radians or cycles (full rotations).

Expressions for the frequency of periodic phenomena is a particularly important and troublesome example of the difficulties associated with units for angles. Frequency is the rate of change of phase angle with time. As such, it is just as important that the units of frequency reflect the units of angles as that they express the units of time. Frequencies are usually given in units of radians per second or hertz, meaning cycles (full rotations or revolutions of the phase angle) per second. Less often, frequencies might be given in degrees per second or revolutions per minute. Unfortunately, sometimes frequencies are simply expressed in inverse seconds, leading to confusion about what the unit really is.
If we agree that angles have units, we must also agree that they have a dimension, the dimension of angle, which is independent of other dimensions like length, time, or mass. Sometimes it is argued that angles do not have an independent dimension, or that the dimension is unity, because the dimension of angle is length divided by length. The justification given for this is that one may measure angles as arc length divided by radius vector length. However, this ratio gives only the numerical factor of the quantity expression for the angle. The complete quantity must include a unit, radian in this case, otherwise, the numerical factor could be the number of degrees of the angle, or anything else. Moreover, that is only one way in which angle might be measured. In fact, the most common way of measuring angles is with a protractor, whereby unknown angles are compared to known angles. This is exactly analogous to measuring an unknown length by comparing it to known lengths. Angles and lengths are manifestly different kinds of physical quantities, and each has its associated dimension.

While we argue that angles are physical quantities distinct from other physical quantities or combinations of such quantities, and that they therefore have their own dimension and unit, we recognize, as have many others, that this represents a choice, and that that choice is made in part for the sake of convenience. The number of base units in the metric system (now the SI, the International System of Units) has changed over time, because of changes in what was considered to be most useful and convenient. The scientific community could have decided to have a unit system in which temperature is measured in joules, but we find it to be more convenient to measure temperature in kelvins. In some sense we have the same kind of choice here with angles, and we argue that the advantage achieved by giving angles a separate dimension and unit outweigh any difficulties.

Our examination of the question of how to deal with units for angles is of interest not only for the important goal of a consistent, unambiguous use of units in physics, but also for systematically including units for angles in algebraic computer software applications. The current practice of sometimes ignoring the units and resolving any resulting ambiguity by human judgement is not suitable for the digital age.

There is a long history of discussions of the role of units for angles. The underlying ideas in this paper have, in essence, been considered in a multitude of earlier works. Here, we have synthesized many of those ideas, while paying particular attention to making the proposed reforms compatible with modern computer processing.
While it would be impractical for every argument in a numerical computation of trigonometric functions to explicitly carry an angle, having radians, degrees, \ldots, in general (compound) units that describe physical quantities, such as angular velocity or differential cross sections, are crucial for automated symbolic computations where results from one computation have to be dimensionally consistent with the next computation step. For instance, Mathematica\textsuperscript{57} considers angles to have an angle dimension. But in the absence of a well-defined standard, dimensional inconsistencies resulting from conversions such as “1 Hz = 1/s” are unavoidable and currently require hard-coded heuristics rather than a fully deterministic algorithmic treatment. The issue of the consistent presence of an angle dimension becomes even more amplified at the level of equations that involve physical quantities (see Secs. V D and V E below).

Although there is a diversity of opinions on the question, the overwhelming majority of papers on the subject acknowledge that angles should be regarded as having an independent dimension and associated units. The difficulty of consistently implementing this proposal is also often cited as a reason for maintaining the acknowledged unsatisfactory status quo.

II. UNIT NOTATION

Following Maxwell\textsuperscript{4} and the general practice of international metrology, we specify a physical quantity $Q$ by a coefficient and a unit as

$$Q = \{Q\} [Q],$$

where $\{Q\}$ is a real or complex number as in Maxwell’s definition (or more generally an operator, matrix, \textit{etc.}), and $[Q]$ is the unit. Or to be more precise, we may write

$$Q = \{Q\} [Q] [Q],$$

because the value of $\{Q\}$ depends on the unit $[Q]$.

For example, a length of 3 meters, $L = 3 \text{ m}$, corresponds to $\{L\} = 3$ and $[L] = \text{ m}$. The choice of units in Eq. [1] is not unique, so we may have

$$Q = \{Q\} [Q]_1 [Q]_1 = \{Q\} [Q]_2 [Q]_2.$$  \hspace{1cm} (3)

For the previous example, the length $L$ could (ill-advisedly) also be expressed in inches as

$$L = 3 \text{ m} = 118.11 \ldots \text{ in},$$

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where \([L_1] = m, \{L\}_m = 3, [L_2] = \text{in},\) and \(\{L\}_\text{in} = 118.11 \ldots\). The quantity \(Q\) is the same physical quantity, regardless of the unit in which it is expressed.

### III. Units for Angles

When we consider angles as measurable physical quantities, as with other such quantities, an angle \(\theta\) may be expressed as

\[
\theta = \{\theta\}_{\theta} \{\theta\},
\]

(5)

where \(\{\theta\}\) is one of any number of possible units. The list includes degree, minute, second, radian, revolution or cycle, grad, among other possible choices, although those mentioned are commonly used. We focus on degree, radian, and revolution \((i.e.,\) cycle or period\) as examples, with the understanding that generalization to other units is always possible.

An example of a particular angle, expressed in different units, is

\[
\theta = 45 \text{ deg} \quad \text{where} \ [\theta] = \text{deg} \quad \text{and} \quad \{\theta\}_\text{deg} = 45,
\]

(6)

\[
\theta = \frac{\pi}{4} \text{ rad} \quad \text{where} \ [\theta] = \text{rad} \quad \text{and} \quad \{\theta\}_\text{rad} = \frac{\pi}{4},
\]

(7)

\[
\theta = \frac{1}{8} \text{ rev} \quad \text{where} \ [\theta] = \text{rev} \quad \text{and} \quad \{\theta\}_\text{rev} = \frac{1}{8},
\]

(8)

where the notation \(45 \text{ deg} \equiv 45^\circ\) is used, and \(\text{rev}\) is the abbreviation for revolution. In these examples, the unit provides necessary information about the angle. If an angle were simply given as a number, say \(\theta = 45\), with no unit, it could taken to be 45 rad, which is a perfectly respectable angle, although it may not be the one the author had in mind.

The relationship between plane right triangles and the trigonometric functions is clear. For a given interior angle, the sine of this angle is the ratio of the length of the side opposite the angle to the length of the hypotenuse. This is a real number between 0 and 1. (Here we limit the discussion to real angles, although the extension to complex angles is considered below.) Thus the properties of trigonometric functions are based on geometrical angles, whereas the functions themselves are usually defined to have numbers as the argument. This apparent inconsistency will be addressed in Sec. [IV]

If we are working in an environment where only numbers are being used, such as a scientific calculator or traditional FORTRAN, then the unit ambiguity is avoided if it is recognized that the sine function itself depends on what units are being used. As is well
known, looking up the sine of a number representing an angle depends on its unit, and
vice versa. There are different trigonometric tables for angles in degrees and for angles in
radians. Similarly, when using a calculator to find the sine or inverse sine of a number, it
is necessary to specify which unit is being assumed for the input or expected as output by,
for example, touching the Deg/Rad key first. This dependence on the type of sine function
being used can be denoted by writing

\[
\sin_{\text{deg}} (45) = \frac{1}{\sqrt{2}}, \quad (9)
\]

\[
\sin_{\text{rad}} \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}, \quad (10)
\]

\[
\sin_{\text{rev}} \left( \frac{1}{8} \right) = \frac{1}{\sqrt{2}}, \quad (11)
\]

where the subscript on the name of the function indicates the unit being assumed for the
angle. Similarly, for the inverse functions, we have

\[
\arcsin_{\text{deg}} \left( \frac{1}{\sqrt{2}} \right) = 45, \quad (12)
\]

\[
\arcsin_{\text{rad}} \left( \frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}, \quad (13)
\]

\[
\arcsin_{\text{rev}} \left( \frac{1}{\sqrt{2}} \right) = \frac{1}{8}, \quad (14)
\]

where the particular arcsine function being used determines the value for the inverse. How-
ever, even if the type of arcsine function is specified, the result on the right-hand side of
Eqs. (12)-(14) has lost that information. As a result, the numbers 45, \( \pi/4 \), or \( 1/8 \) could be
degrees, radians, revolutions, or anything else. As humans, we can look back at our notes
and figure it out, but it is useful to retain that information along with the number, either
to provide complete information about the result or so that it can unambiguously be used
for the next step in a series of calculations. This is the rationale for using units in the first
place, and it is important for information to be processed by computers.

To accomplish this, specification of the unit is added to the “output” of Eqs. (12)-(14).
Thus for the angle $\theta = 45 \text{ deg} = \pi/4 \text{ rad} = 1/8 \text{ rev}$, we have

$$\theta = \arcsin_{\text{deg}} \left( \frac{1}{\sqrt{2}} \right) \text{ deg} = 45 \text{ deg}, \quad (15)$$

$$\theta = \arcsin_{\text{rad}} \left( \frac{1}{\sqrt{2}} \right) \text{ rad} = \frac{\pi}{4} \text{ rad}, \quad (16)$$

$$\theta = \arcsin_{\text{rev}} \left( \frac{1}{\sqrt{2}} \right) \text{ rev} = \frac{1}{8} \text{ rev}. \quad (17)$$

Note that Eqs. (9)-(14) are mapping of numbers to numbers without dimensions, but Eqs.(15)-(17) provide angles with units included. These equations may be generalized to an angle $\theta = \{\theta\}[\theta]$ expressed in any unit $[\theta]$, as in Eq. (5), by writing

$$w = \sin_{[\theta]} \left( \{\theta\}[\theta] \right) \quad (18)$$

and

$$\{\theta\}[\theta] = \arcsin_{[\theta]} (w) \quad (19)$$

In general, because the sine function is periodic, an infinite number of angles map into a particular value of the sine function, so we assume that the formula for the inverse gives angles in the range: $|\{\theta\}[\theta]| \leq 90$, $\pi/2$, $1/4$, as appropriate.

In this section, we have considered the purely numerical sine and arcsine functions. That is, functions that have real numbers for both their domains and ranges. The extension to a sine function of angles with dimensions and any units and to the corresponding arcsine function that has dimensional angles as its range is examined in the next section. Of course, this generalization applies to the cosine and exponential functions and their inverses as well.

**IV. UNIT-INDEPENDENT TRANSCENDENTAL FUNCTIONS**

Here, we consider the concept of “complete” functions. That is, they are independent of the units in which the arguments are expressed. This is a property of trigonometric functions based on plane geometry, which may be derived with no reference to units for the angles. On the other hand, for physical applications, it is useful to specify units for both the arguments of the functions and for the functions themselves. This has been spelled out in detail above for the three examples of units of degrees, radians, and revolutions or cycles. For this purpose, it is useful to make a clear distinction between trigonometric functions that have
angles with dimensions as arguments and trigonometric functions that have real numbers as arguments, as considered in the preceding sections. In the latter case, the real number is the numerical factor of the angle when expressed in a particular unit. The terminology “geometrical angle” and “analytical angle” for these differing representations for angles was applied by Romain[8], although only radian units for the analytic form were considered at the time. This terminology has been repeated in many works since then. Here we examine the relation between complete trigonometric functions and the explicit unit forms considered in the foregoing.

A. Trigonometric functions

To describe complete transcendental functions, it is necessary to consider generalizations of their derivatives. Angles may be represented as quantities with the dimension of angle, but with an arbitrary unit. Infinitesimal changes in an angle correspond to infinitesimal changes in the ratios of the associated sides of the triangle that includes the angle. This relationship provides a quantitative relation between changes of the angle and changes of the trigonometric functions. However, the conventional formula for the derivative,

\[ \frac{d}{d\theta} \sin(\theta) = \cos(\theta) \]  

(20)
is problematic, because the equation is not dimensionally consistent. Both sine and cosine are dimensionless, as they are ratios of lengths of the sides of a triangle, so the left-hand side of the equation has the dimension of the inverse of an angle, and the right-hand side is dimensionless. As mentioned previously, angles are not intrinsically dimensionless.

To address this, we calculate the derivative as the defining limit by writing

\[ \lim_{\Delta \theta \to 0} \frac{\sin(\theta + \Delta \theta) - \sin(\theta)}{\Delta \theta} \]

(21)

From the conventional angle sum identity, based on the geometric properties of triangles and the notation in Fig. 1, we have

\[ \sin(\theta + \Delta \theta) = \sin(\theta) \cos(\Delta \theta) + \cos(\theta) \sin(\Delta \theta) \]

\[ = \sin(\theta) \frac{d}{r} + \cos(\theta) \frac{\Delta h}{r} \]

(22)

Also from Fig. 1, it is evident that
FIG. 1. The triangle (in red) relevant to Eqs. (21)-(24) with angle $\Delta \theta$. The hypotenuse has length $r$, the adjacent side has length $d$ and the opposite side has length $\Delta h$. The arc length of the circle of radius $r$ subtended by $\Delta \theta$ is $\Delta s$. 

\[
\lim_{\Delta \theta \to 0} \frac{d}{\Delta \theta} \to 1, \quad (23)
\]
\[
\lim_{\Delta \theta \to 0} \frac{\Delta h}{r} \to \frac{\Delta s}{r}. \quad (24)
\]

Thus,

\[
\frac{d}{d\theta} \sin(\theta) = C \cos(\theta), \quad (25)
\]

where

\[
C = \lim_{\Delta \theta \to 0} \frac{\Delta s}{r} \frac{1}{\Delta \theta} = \frac{1}{r} \frac{ds}{d\theta}. \quad (26)
\]

Note that this provides a dimensionally consistent result for the derivative. The coefficient $C$ can be evaluated by the integration

\[
C \int_{0}^{\Theta} d\theta = \frac{1}{r} \int_{0}^{2\pi r} ds, \quad (27)
\]

where $\Theta$ is the angle of a complete revolution or period of the sine function

\[
\sin(\theta + \Theta) = \sin(\theta), \quad (28)
\]

and $2\pi r$ is the circumference of a circle of radius $r$, which yields

\[
C = \frac{2\pi}{\Theta}. \quad (29)
\]
Similarly, for the cosine function, we have
\[
\frac{d}{d\theta} \cos(\theta) = \lim_{\Delta\theta \to 0} \frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta},
\]  
where
\[
\cos(\theta + \Delta\theta) = \cos(\theta) \cos(\Delta\theta) - \sin(\theta) \sin(\Delta\theta)
= \cos(\theta) \frac{d}{r} - \sin(\theta) \frac{\Delta h}{r},
\]  
which gives
\[
\frac{d}{d\theta} \cos(\theta) = -C \sin(\theta).
\]  

**B. Exponential function**

The complete exponential function of an angle can be defined in terms of the complete trigonometric functions. This generalization recognizes that the exponential function expressed as Euler’s number e raised to a power is not a complete function. In that case the argument is the numerical part of the angle expressed in radian units, as shown in the next section.

The complete-function definition is
\[
\exp(i\theta) = \cos(\theta) + i \sin(\theta),
\]  
where \(i^2 = -1\). That is, we do not specify the number raised to a power. The derivative follows from the derivatives of the cosine and sine functions as
\[
\frac{d}{d\theta} \exp(i\theta) = -C \sin \theta + iC \cos \theta
= iC \exp(i\theta).
= i \frac{2\pi}{\Theta} \exp(i\theta).
\]  
We also have \(\exp(0) = 1\). With this generalization of the derivative of the exponential function, one obtains the power series
\[
\exp(i\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{2\pi i \theta}{\Theta} \right)^n,
\]  
or
\[
\exp(i\theta) = e^{2\pi i \theta/\Theta}.
\]
The expressions on the right-hand side of Eqs. (35) and (36) are well-defined, because the ratio \( \theta/\Theta \) is just a number with no unit.

The power series for the complete cosine and sine functions, the real and imaginary parts of Eq. (35), are

\[
\cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \frac{2\pi \theta}{\Theta} \right)^{2n}, \tag{37}
\]

\[
\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{2\pi \theta}{\Theta} \right)^{2n+1}. \tag{38}
\]

If

\[
z = \exp(i\theta) \tag{39}
\]

then the complete logarithmic function is

\[
\log(z) = i\theta + ik\Theta, \tag{40}
\]

where \( \theta \) is the geometric angle with the dimension of angle and \( k \) is an integer.

To gain some perspective, it is useful to consider differentiation and integration of the complete logarithmic function to check the consistency of viewing it as having the dimension of angle. From Eq. (33), we have

\[
\frac{dz}{d\theta} = \frac{2\pi i}{\Theta} z, \tag{41}
\]

so that

\[
\frac{d}{dz} \log(z) = \left( \frac{d}{d\theta} \right) i\theta = \left( \frac{d}{d\theta} \right) \left( \frac{d\theta}{dz} \right) = \frac{\Theta}{2\pi} \frac{1}{z}. \tag{42}
\]

Thus the derivative of the logarithmic function also has the correct dimension of angle. (Recall that \( z \) is a dimensionless number.) To check the integral of \( \log(z) \), we write

\[
\frac{d}{dz} \left( z \log(z) \right) = \log(z) + \Theta = \log(z) + \frac{d}{dz} \frac{\Theta}{2\pi} z, \tag{43}
\]

or

\[
\frac{d}{dz} \left( z \log(z) - \frac{\Theta}{2\pi} z \right) = \log(z) \tag{44}
\]

so that

\[
\int dz \log(z) = z \log(z) - \frac{\Theta}{2\pi} z + \text{constant}, \tag{45}
\]
where the constant has the dimension of angle. Thus the integral of the logarithmic function also has the proper unit of angle. Of course, this expression reduces to the special case commonly used, where the logarithmic function is assumed to have the base e and the radian is replaced by 1. That is, \( \log \rightarrow \ln \) and \( \Theta \rightarrow 2\pi \). In this form the integral of \( \ln(z) \) is \( z \ln(z) - z \).

The exponential function in Eq. (35) may be analytically continued to real values of the argument to give

\[
\exp(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{2\pi \phi}{\Theta} \right)^n, \tag{46}
\]

which provides the hyperbolic functions \( \cosh \) and \( \sinh \) as

\[
\cosh(\phi) = \frac{\exp(\phi) + \exp(-\phi)}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( \frac{2\pi \phi}{\Theta} \right)^{2n}, \tag{47}
\]

\[
\sinh(\phi) = \frac{\exp(\phi) - \exp(-\phi)}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( \frac{2\pi \phi}{\Theta} \right)^{2n+1}. \tag{48}
\]

### C. Explicit unit expressions

1. **Arbitrary units, \([\theta] = A\)**

   To connect to the earlier sections in which particular units are considered, the expressions in the previous section can be given for a particular, but arbitrary, choice of unit \([\theta] = A\). In this case, we employ the relations \( \Theta = \{\Theta\}_A \) A and \( \theta = \{\theta\}_A \) A to write

\[
\frac{\theta}{\Theta} = \frac{\{\theta\}_A}{\{\Theta\}_A}, \tag{49}
\]

a ratio that is independent of units. Equation (36) may be written as

\[
\exp(i \theta) = b_A^{i\{\theta\}_A}, \tag{50}
\]

where

\[
b_A = e^{2\pi/\{\Theta\}_A} \tag{51}
\]

is the base of the exponential function for the unit A. The corresponding logarithmic function is

\[
\log_{b_A}(z) = i\{\theta\}_A, \tag{52}
\]

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where

\[ z = b_A^i\theta. \] (53)

This can be compared to the complete logarithmic function in Eq. (40), which has the dimension of angle as its value.

We also have

\[
\cos_A(\{\theta\}_A) = \frac{b_A^i\theta + b_A^{-i\theta}}{2}, \tag{54}
\]

\[
\sin_A(\{\theta\}_A) = \frac{b_A^i\theta - b_A^{-i\theta}}{2i}. \tag{55}
\]

2. **Radian unit, \( A = \text{rad} \)**

If \( A \) is the radian, we have \( 2\pi/\{\Theta\}_\text{rad} = 1 \), \( b_\text{rad} = e \), and

\[
\exp(i\theta) = e^{i\theta}_\text{rad}, \tag{56}
\]

which is the conventional result. If

\[
z = e^{i\theta}_\text{rad}, \tag{57}
\]

then

\[
\log_e(z) = \ln(z) = i\{\theta\}_\text{rad}. \tag{58}
\]

Also

\[
\cos_{\text{rad}}(\{\theta\}_\text{rad}) = \frac{e^{i\theta}_\text{rad} + e^{-i\theta}_\text{rad}}{2}, \tag{59}
\]

\[
\sin_{\text{rad}}(\{\theta\}_\text{rad}) = \frac{e^{i\theta}_\text{rad} - e^{-i\theta}_\text{rad}}{2i}. \tag{60}
\]

As is evident from Eqs (56)-(60), where \( C = 1 \), the conventional trigonometric and exponential functions are implicitly assumed to be based on the radian unit. For example, the relation that the cosine function is the derivative of the sine function with no numerical coefficient assumes that the argument is the numerical factor of the angle expressed in the radian unit. Similarly, when the exponential function is written as the base \( e \) raised to the power of the argument, it is assumed that the argument is the numerical factor of the angle expressed in the radian unit.
3. Revolution or cycle unit, $A = \text{rev}$

In this case, $2\pi/\{\Theta\}_{\text{rev}} = 2\pi$, $b_{\text{rev}} = e^{2\pi}$, and

$$\exp(i\theta) = e^{2\pi i\{\theta\}_{\text{rev}}}.$$  

(61)

It is important to be aware of the units being considered, because it is also often implicitly assumed that the revolution (i.e., cycle or period), rather than the radian, is the suppressed unit, particularly where the angle denotes phase. For example, wavelength $\lambda$ is often taken to be the distance over which a spatially periodic function undergoes a phase change corresponding to one revolution or cycle rather than one radian. In the latter case, it is often called the reduced wavelength $\lambda$. The risk of not taking this into account can and does lead to errors of $2\pi$. Similarly, frequencies, are commonly expressed in Hz or cycles/s rather than radians/s, which can also lead to the same error.

4. Dimensionless functions

Various forms of the exponential and logarithmic functions are described above in terms of units for the dimension of angle or phase. For purely numerical (dimensionless) applications, the radian-based forms are commonly used with no reference to angle units. In such applications, we have

$$y = e^x,$$  

(62)

and

$$x = \ln(y),$$  

(63)

where $x$ and $y$ are dimensionless, possibly complex, numbers.

Another commonly used dimensionless base is 10, where

$$y = 10^x,$$  

(64)

and

$$x = \log_{10}(y).$$  

(65)

In either of these applications, there is no reference to angle or phase.
V. APPLICATIONS

The use of complete transcendental functions can resolve some apparent mismatches of angle units that appear in commonly used equations in physics. To examine this, it is useful to distinguish between “unit analysis” and “dimensional analysis”. As considered here, dimensional analysis is more general in the sense that a geometrical angle or phase angle may be expressed in various units, such as degrees, radians, and revolutions or cycles, but in all cases, it has the dimension of angle, which we denote by $A$. This can be expressed by writing $<\theta> = A$, where $\theta$ is an angle and the brackets are used to denote dimension, in contrast to the notation for the unit $[\theta] = \text{rad}$, for example. Other dimensions that we will consider here are time, length, mass, and charge denoted by $T$, $L$, $M$, and $Q$, respectively. (While the ampere, with dimension “current” is one of the traditional base units of the SI, the definition of the ampere involves the defining charge of the electron, Hence, it is natural, when considering electric units, to focus on the unit coulomb and its associated dimension, electrical charge.) In all cases, the dimensions on either side of an equation must be the same.

In the following, examples are given where a dimensional analysis based on complete functions resolves an apparent angle unit ambiguity.

A. Centripetal acceleration

A familiar relation in physics is the equation for centripetal acceleration $a_c$ of a mass, in uniform circular motion, conventionally given by

$$a_c = \frac{v^2}{r} = r\omega^2,$$

(66)

where $v$ is the linear velocity of the mass, $r$ is the radius of the path, and $\omega$ is the rotational frequency of the mass. The SI units of the two terms on the right-hand end of that equation are m s$^{-2}$ and m rad$^2$ s$^{-2}$. There is an obvious mismatch between these two terms, which is also reflected in the dimensional analysis which gives $L T^{-2}$ and $L A^2 T^{-2}$ for the latter two terms, whereas $a_c$ has the dimension $L T^{-2}$. This difference can be resolved by deriving the complete expression for the acceleration from basic principles.

Consider the uniform circular motion of a mass described by its position in cylindrical
coordinates \( r(r, \phi) \). For motion in the \( \hat{i}, \hat{j} \) plane, we have

\[
\mathbf{r}(r, \phi) = r \cos(\phi) \hat{i} + r \sin(\phi) \hat{j},
\]

where \( \phi = \omega t \). The centripetal acceleration is

\[
a_c = \frac{d^2}{dt^2} \mathbf{r}(r, \phi) = \omega^2 \frac{d^2}{d\phi^2} \mathbf{r}(r, \phi).
\]

From Eqs. (25), (29), and (32), we have

\[
\frac{d^2}{d\phi^2} \cos(\phi) = -\left(\frac{2\pi}{\Theta}\right)^2 \cos(\phi),
\]

\[
\frac{d^2}{d\phi^2} \sin(\phi) = -\left(\frac{2\pi}{\Theta}\right)^2 \sin(\phi),
\]

so that

\[
a_c = -\left(\frac{2\pi \omega}{\Theta}\right)^2 \mathbf{r}(r, \phi),
\]

or

\[
a_c = r \left(\frac{2\pi \omega}{\Theta}\right)^2
\]

This has proper dimensionality of \( LT^{-2} \), which resolves the disagreement in Eq. (66).

For the radian as the angle unit, we have \( \Theta = 2\pi \) rad and

\[
a_c = r \omega^2_{\text{rad}},
\]

where \( \omega_{\text{rad}} \equiv \omega/(1 \text{ rad}) \). This means that \( a_c = r \omega^2 \) is not a complete equation, although \( a_c = v^2/r \) and Eq. (72) are, which resolves the mismatch.

### B. Volume integration

If we do a transformation of a Cartesian volume element to a volume element in spherical coordinates, we have

\[
<dx
dy
dz> = L^3,
\]

\[
<r^2 dr
d\theta
d\phi> = L^3 A^2.
\]

How can this be made consistent?
The transformed coordinates are

\[ x = r \sin(\theta) \cos(\phi) , \]  
\[ y = r \sin(\theta) \sin(\phi) , \]  
\[ z = r \cos(\theta) . \]  

(76)  

(77)  

(78)

The transformation is given by the Jacobian determinant

\[
dx \, dy \, dz = \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{vmatrix} \, dr \, d\theta \, d\phi
\]

\[
= \left(\frac{2\pi}{\Theta}\right)^2 \begin{vmatrix}
\sin(\theta) \cos(\phi) & r \cos(\theta) \cos(\phi) & -r \sin(\theta) \sin(\phi) \\
\sin(\theta) \sin(\phi) & r \cos(\theta) \sin(\phi) & r \sin(\theta) \cos(\phi) \\
\cos(\theta) & -r \sin(\theta) & 0
\end{vmatrix} \, dr \, d\theta \, d\phi
\]

\[
= \left(\frac{2\pi}{\Theta}\right)^2 r^2 \, dr \, \sin(\theta) \, d\theta \, d\phi ,
\]

(79)

which has the dimension \( L^3 \), as required. Thus, for an integral, we have

\[
\int_{-\infty}^{\infty} dx \, \int_{-\infty}^{\infty} dy \, \int_{-\infty}^{\infty} dz \, f(x, y, z) = \left(\frac{2\pi}{\Theta}\right)^2 \int_0^{\infty} dr \, r^2 \int_{-\Theta/2}^{\Theta/2} d\theta \sin(\theta) \int_0^\Theta d\phi \\
\times f(r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta)) .
\]

(80)

If the unit for angles is the radian, then \( \Theta = 2\pi \) rad, and

\[
\sin(\theta) = \sin_{\text{rad}}(\{\theta\}_{\text{rad}}) \equiv \sin_{\text{rad}}(\theta_{\text{rad}}) \quad \text{etc.}
\]

(81)

\[
\frac{2\pi}{\Theta} \, d\theta = d\{\theta\}_{\text{rad}} \equiv d\theta_{\text{rad}} \quad \text{etc.}
\]

(82)

This gives

\[
\int_{-\infty}^{\infty} dx \, \int_{-\infty}^{\infty} dy \, \int_{-\infty}^{\infty} dz \, f(x, y, z) = \int_0^{\infty} dr \, r^2 \int_{-\pi}^{\pi} d\theta_{\text{rad}} \sin_{\text{rad}}(\theta_{\text{rad}}) \int_0^{2\pi} d\phi_{\text{rad}} \\
\times f(r \sin_{\text{rad}}(\theta_{\text{rad}}) \cos_{\text{rad}}(\phi_{\text{rad}}), r \sin_{\text{rad}}(\theta_{\text{rad}}) \sin_{\text{rad}}(\phi_{\text{rad}}), r \cos_{\text{rad}}(\theta_{\text{rad}})) ,
\]

(83)

which reduces to the conventional result if the radian label is dropped:

\[
\int_{-\infty}^{\infty} dx \, \int_{-\infty}^{\infty} dy \, \int_{-\infty}^{\infty} dz \, f(x, y, z) = \int_0^{\infty} dr \, r^2 \int_{-\pi}^{\pi} d\theta \sin(\theta) \int_0^{2\pi} d\phi \\
\times f(r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta)) .
\]

(84)
C. Water waves

The problem of the phase velocity of shallow water waves provides an interesting case about unit compatibility.

The explicit conventional expression for the phase velocity is

\[ c_p = \frac{\omega}{k} = \frac{\sqrt{gk \tanh(kh)}}{k} = \sqrt{gh \tanh(kh)} , \]  

(85)
corresponding to

\[ \omega^2 = gk \tanh(kh) , \]  

(86)
where \( \omega \) is the frequency of the wave, \( k = 1/\lambda \) is the wavenumber, \( g \) is the acceleration of gravity, and \( h \) is the depth of the water. The corresponding dimensions are \( \langle \omega^2 \rangle = A^2 T^{-2} \), \( \langle k \rangle = A L^{-1} \), \( \langle g \rangle = L T^{-2} \), and \( \langle h \rangle = L \).

The problem is that the left-hand side of Eq. (86) has the dimension \( A^2 T^{-2} \) and the right-hand side has dimension \( A T^{-2} \), so that one angle dimension appears to have been lost in the derivation. This contradiction can be fixed by repeating the derivation with angles properly taken into account. Eq. (86) follows from solving the equation for the velocity potential given by

\[ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0 , \]  

(87)
with the boundary condition

\[ \left. \frac{\partial \phi}{\partial z} \right|_{z=-h} = 0 . \]  

(88)
The solution is

\[ \phi = \frac{\cosh k(z + h)}{k \sinh kh} \omega a \sin(k \cdot x - \omega t) , \]  

(89)
where \( a \) is a normalization constant and \( k_z = 0 \). Differentiation gives

\[ \frac{\partial^2 \phi}{\partial t^2} = -\left( \frac{2\pi \omega}{\Theta} \right)^2 \phi , \]  

(90)
so that

\[ \omega^2 = \frac{g}{\phi} \left( \frac{\Theta}{2\pi} \right)^2 \left. \frac{\partial \phi}{\partial z} \right|_{z=0} = gk \frac{\Theta}{2\pi} \tanh kh . \]  

(91)
The right-hand side has the correct dimensions of \( A^2 T^{-2} \).
D. Units for the cyclotron resonance frequency

The relevant conventional equation is

$$\omega = \frac{q B}{m}, \quad (92)$$

where $\omega$ is the frequency, $q/m$ is the charge to mass ratio of the particle, and $B_c$ is the classical magnetic field. The dimensions are $<\omega> = A \, T^{-1}$, $<q/m> = Q \, M^{-1}$, and $<B_c> = M \, T^{-1} \, Q^{-1}$.

The left-hand side of Eq. (92) has dimension $A \, T^{-1}$ and the right-hand side works out to $T^{-1}$, so there is a mismatch of the angle dimension and the equation is not complete. [Moreover, because the right-hand side has dimension $T^{-1}$, the current SI prescribes that the unit is s$^{-1}$, or Hz, which is not correct [see Eq. (102)]. As a result, Mathematica, which provides calculations with units, gets regular bug reports about this. To resolve this problem, we work out the complete version of this equation.]

For motion of a particle of mass $m$ and charge $q$ in the $\hat{i}$, $\hat{j}$ plane along a circular path of radius $r$, centered at $x = 0$, we have

$$x = r \cos(\omega t) \, \hat{i} + r \sin(\omega t) \, \hat{j} \quad (93)$$

$$v = \frac{d}{dt} x = \frac{d}{dt} \left[ r \cos(\omega t) \, \hat{i} + r \sin(\omega t) \, \hat{j} \right] \quad (94)$$

$$= \frac{2\pi \omega}{\Theta} \left[ -r \sin(\omega t) \, \hat{i} + r \cos(\omega t) \, \hat{j} \right] \quad (95)$$

$$a = \frac{d}{dt} v = -\left( \frac{2\pi \omega}{\Theta} \right)^2 \left[ r \cos(\omega t) \, \hat{i} + r \sin(\omega t) \, \hat{j} \right] \quad (96)$$

$$= -\left( \frac{2\pi \omega}{\Theta} \right)^2 x \quad (97)$$

For a magnetic field given by $B_c = -B_c \hat{k}$, the classical force is

$$F_c = q v \times B_c = -q B_c \frac{2\pi \omega}{\Theta} \left[ r \sin(\omega t) \, \hat{j} + r \cos(\omega t) \, \hat{i} \right] \quad (98)$$

$$= -q B_c \frac{2\pi \omega}{\Theta} x \quad (99)$$

Thus from Newton’s law, $F_c = m a$, we have

$$q B_c = m \frac{2\pi \omega}{\Theta} \quad (100)$$
FIG. 2. Classical pendulum

or

\[ \omega = \frac{q_B c \Theta}{m \ 2\pi} \]  

(101)

This is the complete equation, and the dimensions, \( \text{AT}^{-1} \), match.

In SI units, the frequency is

\[ \omega = \left\{ \frac{q_B c}{m} \right\} \times \left\{ \frac{\text{rad s}^{-1}}{2\pi} = \frac{\text{Hz}}{2\pi} \right\}. \]  

(102)

E. Classical pendulum

The conventional expression for the small-oscillation frequency \( \omega \) of a classical pendulum is

\[ \omega = \sqrt{\frac{g}{L}}, \]  

(103)

where \( g \) is the acceleration of gravity and \( L \) is the length of the pendulum arm. The current SI unit for \( \sqrt{g/L} \) is \( \text{s}^{-1} \), which suggests the frequency is in Hz, but, as for the case of the
cyclovoltor frequency above, this is not correct, as shown by considering the complete equation calculation.

The equation of motion is

\[ m \frac{d^2s(t)}{dt^2} = -mg \sin(\theta(t)), \quad (104) \]

where \( \theta \) and \( s \) are the angle and arc length from the lowest position of the pendulum, as indicated in Fig. 2. The mass cancels out of the equation. We consider small displacements from the lowest point, so that

\[
\sin \left( \theta(t) \right) = \frac{2\pi \theta(t)}{\Theta} + \ldots, \quad (105)
\]

according to Eq. (38). From Eqs. (26) and (29), we have

\[
\frac{2\pi}{\Theta} \int d\theta = \frac{1}{L} \int ds, \quad (106)
\]

which provides the relation

\[
s(t) = \frac{2\pi L}{\Theta} \theta(t). \quad (107)
\]

We thus have

\[
\frac{d^2\theta(t)}{dt^2} + \frac{g}{L} \theta(t) = 0. \quad (108)
\]

The general solution is

\[
\theta(t) = a \sin(\omega t) + b \cos(\omega t), \quad (109)
\]

and so the complete second derivative

\[
\frac{d^2\theta(t)}{dt^2} = -\left( \frac{2\pi \omega}{\Theta} \right)^2 \theta(t) \quad (110)
\]

gives

\[
\omega = \sqrt{\frac{g}{L} \frac{\Theta}{2\pi}}, \quad (111)
\]

which has the dimension \( \text{AT}^{-1} \). Finally, the proper SI expression for \( \omega \) is

\[
\omega = \sqrt{\left\{ \frac{g}{L} \right\}} \text{ rad s}^{-1}. \quad (112)
\]
F. Jacobi elliptic functions

The Jacobi elliptic functions $sn(u, k)$ and $cn(u, k)$ may be viewed as generalizations of the sine and cosine functions because

\[ sn(u, 0) = \sin u, \tag{113} \]
\[ cn(u, 0) = \cos u. \tag{114} \]

It is natural to seek the generalization of the differentiation formula in Eq. (25).

The functions, together with $dn(u, k)$, are defined by

\[ u = \int_{0}^{\phi} \frac{d\phi'}{\sqrt{1 - k^2 \sin^2 \phi'}}, \tag{115} \]
\[ sn(u, k) = \sin \phi, \tag{116} \]
\[ cn(u, k) = \cos \phi, \tag{117} \]
\[ dn(u, k) = \sqrt{1 - k^2 \sin^2 \phi}. \tag{118} \]

From Eq. (115), we have

\[ \frac{du}{d\phi} = \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}}, \tag{119} \]

and so

\[ \frac{d}{du} sn(u, k) = \frac{d}{du} \sin \phi = \left( \frac{d}{d\phi} \sin \phi \right) \frac{d\phi}{du} \]
\[ = \frac{2\pi}{\Theta} \cos \phi \sqrt{1 - k^2 \sin^2 \phi} \]
\[ = \frac{2\pi}{\Theta} cn(u, k) \cdot dn(u, k), \tag{120} \]

which reduces to the conventional result for the derivative of $sn(u, k)$ if rad $\rightarrow 1$ and agrees with Eq. (25) if $k = 0$.

VI. CONCLUSION

Here, we review some key points relevant to the treatment of angles and their units in equations of physics. In this context, the term angle is used to mean both plane angle and phase angle.
A guiding principle is that the use and properties of units (e.g. their dimensions) is a matter of choice. Units are not a property of nature, but rather they provide a rational method for quantitatively describing natural phenomena with mathematical equations. Of course, the choice is not completely arbitrary, because the units must fit into a logically consistent framework.

We may choose to assign to angles an independent dimension with associated units, or not. The current practice is generally to treat angles simply as numbers and view them as dimensionless. However, as should be evident from the forgoing sections, this point of view makes unstated assumptions. In particular, it assumes that the number representing an angle is the numerical factor of the angle expressed in the radian unit. This view also assumes that the radian unit is not actually a unit but is just the number one. An advantage of this approach is that it represents the status quo and as such requires no further thought or analysis. However, this leads to ambiguities and errors as described in the forgoing examples. Because the radian is a coherent unit in the SI, there is no numerical factor when the radian is restored, as illustrated by Eqs. (102) and (112). On the other hand, Hz is not a coherent unit as seen in Eq. (102) where there is an additional factor of $\frac{1}{2\pi}$.

Such ambiguities and errors can be avoided by assigning an independent dimension to angles and treating this dimension in a consistent way, insuring the correct dimension for the equations of physics, as well as insuring the proper units among the various units that might be used for angles. The use of complete functions and complete equations is a powerful tool to insure a consistent treatment of dimension and units. An important advantage is that it eliminates errors of a factor of $2\pi$ as shown in the examples. The assignment of a dimension to angles also raises awareness that there are built-in hidden assumptions in the widely accepted way in which angles and trigonometric functions of angles are treated. Moreover, the analysis of the consequences of assigning a dimension to angles provides a generalization of the relevant equations, so that they become complete equations. As already noted, complete equations do not assume that the quantities appearing in them are expressed in particular units.

To elaborate on this point, we consider the example, often used by theoretical physicists, of setting the speed of light $c$ to 1. When this is done, $c$ disappears from equations that would otherwise show it. Then we have $E = m$ instead of $E = mc^2$, which introduces an ambiguity between energy and mass. Restoration of $c$ at the end of a calculation requires
judgement by the practitioner. This is completely analogous to the practice of setting \( C = \frac{2\pi}{\Theta} \), as defined in the preceding sections, equal to 1, which is the default for the “radian assumption”. If \( C \) is restored, it shows up as an additional parameter in some equations, just as \( c \) shows up when it is not 1.

The cost of this restoration of \( C = \frac{2\pi}{\Theta} \) is the task of determining where it belongs in equations in which it has been replaced by 1, by default. In this paper, the generalization has been worked out for transcendental functions and applied in various examples.

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