A CALCULATION OF L-SERIES IN TERMS OF JACOBI SUMS

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Abstract. Let us consider a cyclic extension of a function field defined over a finite field. For a character (non-trivial) of this extension, we calculate, as a linear combinations of products of Jacobi sums, the coefficients of the polynomial given by its Dirichlet L-series.

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1. Introduction

In [W] it is proved that the Hasse-Weil L-function for Fermat curves has an analytic continuation to the whole complex plane. It is proved bearing in mind that these curves are abelian coverings of the projective line, ramified at three points, and hence it is deduced that the p-local term of the Hasse-Weil L-function is the L-series associated with a Hecke character which came from a Jacobi sum.

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If we have cyclic ramified coverings of a smooth curve, defined over finite fields, then the $L$-series associated with the characters (non trivial) for these coverings are polynomials. In [D], it is proved that the constant term of these polynomials is given by Jacobi sums.

In this article we calculate, in terms of Jacobi sums, the coefficients of these polynomials. We shall show that these coefficients are explicitly given by linear combinations of products of Jacobi sums.

2. Previous notation and preliminaries

Let $\mathbb{F}_q$ be a finite field of $q = p^h$ elements, where $p$ is a prime integer. Let $m$ be a positive integer with $m | (q-1)$. Let us consider the primitive $(q-1)$-root of the unity $\epsilon_{q-1}$ and a prime ideal, $\mathfrak{p} \subset \mathbb{Z}[\epsilon_{q-1}]$, over the prime $p$. Let $Y, X$ be smooth, proper and geometrically irreducible curves over $\mathbb{F}_q$ and let $\Sigma_Y, \Sigma_X$ be the function fields of $Y$ and $X$, respectively. Now, let us consider $Y \to X$, a ramified Galois covering with Galois group $G := \mathbb{Z}/m\mathbb{Z}$, ramified at $(d + 1)$-different rational points of $X$, $T := \{x_0, \ldots, x_d\}$. We have that $\Sigma_Y = \Sigma_X(\sqrt[d]{T})$. Let $\text{div}(f)$ be the principal divisor associated with $f \in \Sigma_X$. We can choose $f$ such that $\text{div}(f) = a_0 \cdot x_0 + \cdots + a_d \cdot x_d + m \cdot D$, with $0 < a_i < m$, and where $D$ is a divisor on $X$ (note that $a_0 + \cdots + a_d \equiv 0 \mod m$).

We denote by $k(x_i)$ the residual field of $x_i$.

We consider the effective divisors $E := x_0 + x_1 + \cdots + x_d, E' := x_1 + \cdots + x_d$ and by $\mathfrak{m}, \mathfrak{m}'$ the ideals in $\mathcal{O}_X$ associated with $E$ and $E'$, respectively. An $\mathfrak{m}$-level structure over $X$ is a pair $(L, \iota_\mathfrak{m})$, where $L$ is a line bundle over $X$ and $\iota_\mathfrak{m} : L \to \mathcal{O}_X/\mathfrak{m}$ is an surjective morphism of $\mathcal{O}_X$-modules. We say that two level structures $(L, \iota_\mathfrak{m})$ and $(L', \iota'_\mathfrak{m})$ are isomorphic when there exists an isomorphism of $\mathcal{O}_X$-modules $\tau : L \to L'$ such that $\iota'_\mathfrak{m} \cdot \tau = \iota_\mathfrak{m}$.
It is not hard to see that the tensor product defines a group law for the level structures. We denote by $\text{Pic}^0_{X,m}(\mathbb{F}_q)$ the group of level structures $(L, \iota_m)$, where $\deg(L) = 0$. Moreover, by considering the morphism of forgetting $\iota_m$, $\theta(L, \iota_m) = L$, we have that $\text{Ker}(\theta) = (\mathcal{O}_X/m)^\times/\mathbb{F}_q^\times$ and hence, we obtain an exact sequence of groups

$$1 \to k(x_0)^\times \times \cdots \times k(x_d)^\times/\mathbb{F}_q^\times \xrightarrow{n} \text{Pic}^0_{X,m}(\mathbb{F}_q) \xrightarrow{m} \text{Pic}^0_X(\mathbb{F}_q) \to 1.$$  

By class field theory, [S] VI, since $\Sigma_X(\sqrt[q]{T})/\Sigma_X$ is a cyclic extension of group $G$, ramified at $T$, there exists a surjective morphism of groups $\rho : \text{Pic}^0_{X,m}(\mathbb{F}_q) \to G$. Therefore, if $g \in \text{Pic}^0_{X,m}(\mathbb{F}_q)$ then $\rho(g)(\sqrt[q]{T}) = \lambda \cdot (\sqrt[q]{T})$ with $\lambda \in \mathbb{F}_q^\times$. We define the character $\chi_f$ of $\text{Pic}^0_{X,m}(\mathbb{F}_q)$ by $\chi_f(g) =: \chi_p(\lambda)$, $\chi_p$ being the Teichmüller character: If $\lambda \in \mathbb{F}_q$ then we set the $(q-1)$-root of the unity, $\chi_p(\lambda) \in \mathbb{Z}[\zeta_{q-1}]$, by the condition $\chi_p(\lambda) \equiv \lambda \mod p$. If we consider the injective morphism

$$\eta : k(x_0)^\times \times \cdots \times k(x_d)^\times/\mathbb{F}_q^\times \to \text{Pic}^0_{X,m}(\mathbb{F}_q)$$

we have that $\chi_f(\eta(z_0, \cdots, z_d)) = \chi_p^{-(\frac{a_0}{m})a_0}(z_0) \cdots \chi_p^{-(\frac{a_d}{m})a_d}(z_d)$.

Now we shall build a section for the morphism $\theta : \text{Pic}^0_{X,m}(\mathbb{F}_q) \to \text{Pic}^0_X(\mathbb{F}_q)$: Let $\pi_m : \mathcal{O}_X \to \mathcal{O}_X/m$ be the natural epimorphism. Let $D$ be a divisor on $X$. We shall fix an $m$-level structure for the line bundle $\mathcal{O}_X(D)$. If $D$ is an effective divisor on $X$ with support outside $T$ then by considering the natural morphisms $\mathcal{O}_X(-D) \to \mathcal{O}_X$, $\mathcal{O}_X \to \mathcal{O}_X(D)$ and the level structure $(\mathcal{O}_X, \pi_m)$ we obtain level structures $(\mathcal{O}_X(\pm D), \pi_m^{\pm D})$. Similarly, if $D' = D_1' - D_2'$ is a divisor with support away of $T$ and $D_1, D_2$ are effective divisors, then $(\mathcal{O}_X(D_1'), \pi_{D_1'}) \otimes (\mathcal{O}_X(-D_2'), \pi_{-D_2'})$ gives a level structure for $\mathcal{O}_X(D)$.

We now consider $D = r \cdot x_i + K$ with $x_i \in T$, $\text{supp}(K) \cap T = \emptyset$ and we set $t_{x_i}$ a local parameter for $x_i$, with $\text{supp}((\text{div}(t_{x_i}))) \cap T = \emptyset$. As above, we obtain a level structure for $t_{x_i}^{-r} \cdot \mathcal{O}_X(D) \simeq \mathcal{O}_X(K')$ since that $\text{supp}(K') \cap T = \emptyset$. Thus, from the isomorphisms of $\mathcal{O}_X$-modules
\( \mathcal{O}_X(D) \cong t_x^{-r} \cdot \mathcal{O}_X(D) \) and from the \( \mathfrak{m} \)-level structure for \( \mathcal{O}_X(K') \), we obtain an epimorphism \( \pi_{m}^{D} : \mathcal{O}_X(D) \to \mathcal{O}_X/\mathfrak{m} \). For each divisor \( F \) on \( X \) we denote \( t_{m}^{F} := (\mathcal{O}_X(F), \pi_{m}^{F}) \).

**Example 1.** If we consider \( X = \mathbb{P}_1 \), with \( \Sigma_{\mathbb{P}_1} = \mathbb{F}_q(x) \), \( \text{div}(x) = 0 - \infty \) and the local parameter \( x^{-1} \) for \( \infty \), then the \( \infty \)-level structure \( l_{\infty}^{\infty} := (\mathcal{O}_{\mathbb{P}_1}(\infty), \pi_{\infty}^{\infty}) \) is given by \( \pi_{\infty}^{\infty} : \mathcal{O}_{\mathbb{P}_1}(\infty) \to \mathcal{O}_{\mathbb{P}_1}/\mathfrak{m}_{\infty} \), with \( \pi_{\infty}^{\infty}(x) = 1 \). Recall that \( H^{0}(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(\infty)) = \mathbb{F}_q \cdot 1 \oplus \mathbb{F}_q \cdot x \oplus \cdots \oplus \mathbb{F}_q \cdot x^r \).

Let \( L \) be a line bundle on \( X \). By taking account the identification \( \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L) = H^{0}(X, L) \), we can consider the global sections of \( L \) as \( \mathcal{O}_X \)-module morphisms \( \mathcal{O}_X \to L \). We define the space of \( \mathfrak{m} \)-sections of a level structure, \( (L, \iota_{m}) \) as the subset \( H^{0}_{m}(X, (L, \iota_{m})) \subset H^{0}(X, L) \), of global sections of \( L \), \( s : \mathcal{O}_X \to L \) such that \( \iota_{m} \cdot s = \pi_{m} \).

We have that if \( s, s' \in H^{0}_{m}(X, (L, \iota_{m})) \) then \( s - s' \in H^{0}(X, L(-E)) \). Thus, \( H^{0}_{m}(X, (L, \iota_{m})) = s + H^{0}(X, L(-E)) \). We denote \( h^{0}_{m}(L, \iota_{m}) := \# H^{0}_{m}(X, (L, \iota_{m})) \). Note that either \( h^{0}_{m}(L, \iota_{m}) = 0 \) or \( h^{0}_{m}(L, \iota_{m}) = h^{0}(L(-E)) \).

### 3. L-series

Let \( F_y \) be the Frobenius element for \( y \in X \setminus T \). We consider the \( L \)-series associated with the character \( \chi_f \),

\[
L(t, \chi_f) = \prod_{y \in |X| \setminus T} (1 - \chi_f(F_{y}) \cdot t^{-deg(y)})^{-1}.
\]

We consider a divisor of degree 1, \( D_1 \) on \( X \) and we denote \( L(i) := L \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_1) \). In [Al, 3], by using [A] 4.1.1, this \( L \)-series is calculated in terms of the \( \mathfrak{m} \)-level structures

\[
(*) L(t, \chi_f) = \sum_{i=0}^{2g+d-1} \left[ \sum_{(L, \iota_{m})} h^{0}_{m}(L(i), \iota_{m}) \cdot \chi_f(\rho(L, \iota_{m})) \right] \cdot t^i.
\]

The second sum is over all level structures classes \( (L, \iota_{m}) \in \text{Pic}_{X, \mathfrak{m}}^{0}(\mathbb{F}_q) \).
If \( \lambda \in \mathbb{F}_q^* \) and \((L, \tau_m)\) is a \(m\)-level structure then \((L, \tau_m)\) and \((L, \lambda \cdot \tau_m)\) are isomorphic level structures. Moreover, as \(O_X/m \simeq k(x_0) \times O_X/m'\) we have that \(\tau_m = \tau_{x_0} \times \tau_{m'}\) with \(\tau_{x_0} : O_X \to k(x_0)\) and \(\tau_{m'} : O_X \to O_X/m'\) epimorphisms. Thus, in the isomorphism class of a level structure \((L, \tau_m)\) we can choose the element \((L, \pi_{x_0}^D \times \tau_{m'})\), with \(L \simeq O_X(D)\), and therefore we can fix the \(x_0\)-level structure in the classes of \(m\)-level structures. Let \(F\) be either \(E\) or \(E'\). We denote \(O_F := H^0(X, \frac{O_X}{\Omega_{X,-F}})\). Since we fix the \(x_0\)-level structure, the subgroup \(1 \times O_{E'}^\times \subset O_{E'}^\times\) acts transitively on the classes of \(m\)-level structures of the fiber \(\theta^{-1}(L)\).

Let \(J\) be a divisor on \(X\). We consider the level structure \(l_m^J := (O_X(J), \pi_m^J)\), defined in the above section. We denote

\[
H_m^J := \pi_m^J \left( \frac{H^0(X, O_X(J))}{H^0(X, O_X(J - E))} \right) \subseteq O_E.
\]

**Proposition 3.1.** We have

\[
L(t, \chi_f) = \sum_{i=0}^{2g+d-1} \left[ \sum_{[D]} h^0(O_X(D + D_0 - E)) \cdot \chi_f(\rho(l_m^D)) \sum_{u} \chi_f(\eta(1 \times u)) \right] t^i.
\]

The second sum is over all the classes (with the algebraic equivalence) of divisors of degree 0, \(D\) on \(X \setminus T\) and the third sum is over all \(u \in O_{E'}^\times\) with \(1 \times u^{-1} \in H_m^{D+i.D_1} \subseteq O_E\).

**Proof.** By considering the level structure \(l_m^{D+i.D_1}\) and bearing in mind that if \((O_X(D)(i), \pi_{x_0}^{D+i.D_1} \times \tau_{m'})\) is another level structure for \(O_X(D)(i)\), then there exists \(u \in O_{E'}^\times\) with \(\pi_{x_0}^{D+i.D_1} \times \tau_{m'} = (1 \times u) \cdot \pi_m^{D+i.D_1}\). We have that the \(L\)-series (*) is equal to

\[
\sum_{i=0}^{2g+d-1} \left[ \sum_{[D]} h^0(O_X(D + D_0 - E)) \cdot \chi_f(\rho(l_m^D)) \sum_{u} \chi_f(\eta(1 \times u)) \right] t^i,
\]
where the third sum is over all \( u \in O_E \) with \( h^0_m((1 \times u) \cdot l^D_{m+D_1}) \neq 0 \), and therefore, \( h^0_m((1 \times u) \cdot l^D_{m+D_1}) = h^0(O_X(D + i \cdot D_1 - E)) \). Note that we have fixed the \( x_0 \)-level structure in the classes of \( m \)-level structures.

To conclude, it suffices to prove that \( h^0_m((1 \times u) \cdot l^D_{m+D_1}) \neq 0 \) if and only if \( 1 \times u^{-1} \in H^0_{m+D_1} \). If \( h^0_m((1 \times u) \cdot l^D_{m+D_1}) \neq 0 \) then there exists \( s \in H^0(X, O_X(D)(i)) \) such that the \( O_X \)-module morphism \( s : O_X \to O_X(D)(i) \) satisfies \( (1 \times u) \cdot \pi^D_{m+D_1} \cdot s = \pi_m \). Therefore, \( (1 \times u) \cdot \pi^D_{m+D_1}(s) = 1 \). Conversely, if \( 1 \times u^{-1} \in H^0_{m+D_1} \) there exists \( s \in H^0(X, O_X(D)(i)) \) with \( 1 \times u^{-1} = \pi^D_{m+D_1}(s) \). Thus, \( s \in H^0_m(X, (1 \times u) \cdot l^D_{m+D_1}) \) and we conclude.

\[ \square \]

**Example 2.** As in Example 1, we consider \( X = \mathbb{P}_1 \), the line bundle \( O_{\mathbb{P}_1}(r \cdot \infty) \), \( E := \infty + x_1 + \cdots + x_d \), with \( r \leq d - 1 \), and \( f \) the polynomial of degree \( d \), \( p(x) \), where \( x_1, \cdots, x_d \) are given by the roots of this polynomial. We consider the \( m \)-level structure \( l^\infty_m \); recall that over \( m' \) this structure is obtained from the natural injective morphism \( O_{\mathbb{P}_1} \to O_{\mathbb{P}_1}(r \cdot \infty) \) and over \( \infty \) it is given by \( \pi^\infty_r \) by considering \( \pi^\infty_r(x^r) = 1 \)(Example 1). In this way, the subspace

\[ H^r_m := \pi^\infty_r(H^0(\mathbb{P}_1, O_{\mathbb{P}_1}(r \cdot \infty)) / H^0(\mathbb{P}_1, O_{\mathbb{P}_1}(r \cdot \infty - E)) \subseteq O_E = \mathbb{F}_q \times \mathbb{F}_q[x] / p(x) \]

is \( \{((h(x))_{x=\infty}, h(x)) \mid \deg(h(x)) \leq r \} \). Therefore, \( 1 \times u(x) \in H^r_m \) if and only if \( \deg(u(x)) = r \) and \( u(x) \) is monic.

4. **Jacobi sums and L-series**

Let us consider, \( \alpha_1, \cdots, \alpha_d, \alpha \in \mathbb{F}_q \) and the linear form \( \omega_d(z) = \alpha \cdot z_1 + \cdots + \alpha_d \cdot z_d + \alpha \), where \( z := (z_1, \cdots, z_d) \in \mathbb{F}_q^d \) and \( d \geq 2 \). We define the Jacobi sum with \( (a) = (a_1, \cdots, a_d) \in \mathbb{Z}^d \) by

\[ J_{\omega_d}^{(a)} := \sum_{z, \omega(z) = 0} \chi_p^{a_1}(z_1) \cdots \chi_p^{a_d}(z_d) \in \mathbb{Z}[\xi_{q-1}] \].

When \( z_i = 0 \), we set \( \chi_p^{a_i}(z_i) = 0 \).
Lemma 4.1. Let us consider \((z_1, \cdots, z_d) \in \mathbb{F}_q^d\), with \(z_i \neq 0\). We have the equality

\[
\chi_p^{a_d}(-\frac{\alpha_1}{\alpha_d} \cdot z_1 - \cdots - \frac{\alpha_{d-1}}{\alpha_d} \cdot z_{d-1} - \frac{\alpha}{\alpha_d}) = \frac{1}{(q-1)^{d-1}} \sum_{1 \leq i_1, \cdots, i_{d-1} \leq q-1} J_{\omega_d}^{(-i_1, \cdots, -i_{d-1}, a_d)} \cdot \chi_p^{i_1}(z_1) \cdots \chi_p^{i_{d-1}}(z_{d-1}),
\]

where \((z_1, \cdots, z_{d-1}) \in (\mathbb{F}_q^x)^{d-1}\), and the equality

\[
J_{\omega_d}^{(-i_1, \cdots, -i_{d-1}, a_d)} = \sum_{(z_1, \cdots, z_{d-1}) \in \mathbb{F}_q^{d-1}} \chi_p^{-i_1}(z_1) \cdots \chi_p^{-i_{d-1}}(z_{d-1}) \chi_p^{a_d}(-\frac{\alpha_1}{\alpha_d} \cdot z_1 - \cdots - \frac{\alpha_{d-1}}{\alpha_d} \cdot z_{d-1} - \frac{\alpha}{\alpha_d}).
\]

Proof. It suffices to consider the system of \((q-1)^{d-1}\)-linear equations, with variables \(X_{i_1, \cdots, i_{d-1}}\)

\[
\chi_p^{a_d}(-\frac{\alpha_1}{\alpha_d} \cdot z_1 - \cdots - \frac{\alpha_{d-1}}{\alpha_d} \cdot z_{d-1} - \frac{\alpha}{\alpha_d}) = \sum_{1 \leq i_1, \cdots, i_{d-1} \leq q-1} X_{i_1, \cdots, i_{d-1}} \cdot \chi_p^{i_1}(z_1) \cdots \chi_p^{i_{d-1}}(z_{d-1}),
\]

where \((z_1, \cdots, z_{d-1}) \in (\mathbb{F}_q^x)^{d-1}\), and the equality

\[
J_{\omega_d}^{(-i_1, \cdots, -i_{d-1}, a_d)} = \sum_{(z_1, \cdots, z_{d-1}) \in \mathbb{F}_q^{d-1}} \chi_p^{-i_1}(z_1) \cdots \chi_p^{-i_{d-1}}(z_{d-1}) \chi_p^{a_d}(-\frac{\alpha_1}{\alpha_d} \cdot z_1 - \cdots - \frac{\alpha_{d-1}}{\alpha_d} \cdot z_{d-1} - \frac{\alpha}{\alpha_d}).
\]

\[\square\]

Lemma 4.2. With the above notations, there exists a finite subset \(A \subset \mathbb{Z}^{d-1}\), an affine subvariety \(H_r^{d-1} \subset \mathbb{F}_q^{d-1}\) of dimension \(r\) and a linear form \(\omega_d(z)\), such that

\[
J_{H_r^d}^{(a)} = \frac{1}{(q-1)^{d-1}} \sum_{(b) \in A} J_{\omega_d}^{(b,a_d)} \cdot J_{H_r^{d-1}}^{(-b) + (a_1, \cdots, a_{d-1})}.
\]

Proof. Let \(H_{r+1}^d \subset \mathbb{F}_q^d\) and \(\omega_d\) be such that \(H_r^d = H_{r+1}^d \cap \{\omega_d(z) = 0\}\). Suppose that \((q-1) \nmid a_d\), note that there exists \(\omega_d(z) = \alpha_1 \cdot z_1 + \cdots + \alpha_d \cdot z_d + \alpha\) with \(\alpha_d \neq 0\), in another case \(J_{H_r^d}^{(a)} = 0\). Therefore,
When Corollary 4.3.

For the first assertion it suffices to consider the isomorphism $P : \mathbb{F}_q^{d-1} \to \mathbb{F}_q^d$ defined by:

$$P(z_1, \ldots, z_{d-1}) = (z_1, \ldots, z_{d-1}, -\frac{\alpha_1}{\alpha_d} \cdot z_1 - \cdots - \frac{\alpha_{d-1}}{\alpha_d} \cdot z_{d-1} - \frac{\alpha}{\alpha_d})$$

and we denote $H_r^{d-1} = P^{-1}(H_{r+1}^d)$. Thus, we have

$$J_{H_r^{d-1}}^{(a)} = \sum_{(z_1, \ldots, z_{d-1}) \in H_r^{d-1}} \chi_p^a(z_1) \cdot \chi_p^{a_{d-1}}(z_{d-1}) \cdot \chi_p^{a_d}(\frac{\alpha_1}{\alpha_d} \cdot z_1 - \cdots - \frac{\alpha_{d-1}}{\alpha_d} \cdot z_{d-1} - \frac{\alpha}{\alpha_d})$$

We conclude, bearing in mind Lemma 4.1, by considering $A = \{(i_1, \ldots, i_{d-1}) \in \mathbb{Z}^{d-1} : 1 \leq i_1, \ldots, i_{d-1} \leq q - 1\}$. \[\Box\]

We denote by $Q : \mathbb{F}_q \times O_{E'} \to O_{E'}$ the projection morphism. Note that $1 \times u \in H_{m+D_1}^{d+i-D_1} \subset \mathbb{F}_q \times O_{E'}$ if and only if $u \in H_{m+D_1}^{d+i-D_1} := \mathbb{Z}[1 \times O_{E'}] \cap H_{m+D_1}^{d+i-D_1} \subset O_{E'} \simeq \mathbb{F}_q^d$. In the next Theorem we follow the notation of Proposition 3.1.

**Theorem 1.** We have $\sum_{u \in H_{m+D_1}^{d+i-D_1}} \chi_f(\eta(1 \times u)) = J_{H_{m+D_1}^{d+i-D_1}}^{(a)}$. If we denote $r := \dim H_{m+D_1}^{d+i-D_1}$ and $e := \sum_{j=1}^{d-r-1} (d-j)$, then there exists a finite set $B = \{(b_d, \ldots, b_{d-r})\} \subset \mathbb{Z}^d \times \cdots \times \mathbb{Z}^{d-r}$ with $\#B = (q-1)^e$ and linear forms $\omega_d, \ldots, \omega_{d-r}$ over $\mathbb{F}_q^d, \ldots, \mathbb{F}_q^{d-r}$, respectively, such that

$$J_{H_{m+D_1}^{d+i-D_1}}^{(a)} = \frac{1}{(q-1)^e} \sum_{(b_d, \ldots, b_{d-r}) \in B} J_{\omega_d}^{(b_d)} \cdots J_{\omega_{d-r}}^{(b_{d-r})}.$$ 

**Proof.** For the first assertion it suffices to consider the isomorphism $O_{E'}^d \simeq \mathbb{F}_q \times \cdots \times \mathbb{F}_q^d$, and we conclude because $\chi_f(\eta(1, u_1 \cdots, u_d)) = \chi_p^{a_1(\frac{u_1}{m})}((u_1^{-1}) \cdots \chi_p^{a_d(\frac{u_d}{m})}(u_d^{-1})$. The second assertion follows by applying induction over $d-r$ in Lemma 4.2. \[\Box\]

**Corollary 4.3.** When $X = \mathbb{P}_1$, by following the notation of Example 2 we have that $L(t, \chi_f) = \sum_{r=0}^{d-1} J_{H_{m}^{d}}^{(a)} \cdot t^{d-1-r}$.
where $H^r_m = \{v(x)\} \subset \mathbb{F}_q[x]/(p(x))$ and $\{v(x)\}$ is the set of monic polynomials of degree $r$. Moreover, $(q - 1)^e \cdot J^{(\alpha)}_{T_m^r} \in \mathbb{F}_q[x]/(p(x))$, with $e := \sum_{j=1}^{d-r-1} (d - j)$, is a sum of $(q - 1)^e$ products of $d - r$ Jacobi sums.

**Proof.** By taking $D_1 = \infty$, this follows from Proposition 3.1, Example 2 and Theorem □

**Example 3:** Let us consider the elliptic curve $y^2 = x(x-1)(x-\lambda)$, $X = \mathbb{P}_1$, $\Sigma_X = \mathbb{F}_q(x)$, $p(x) = x(x-1)(x-\lambda)$ and $\Sigma_Y = \mathbb{F}_q(x, \sqrt{p(x)})$.

We have that $L(t, \chi_f) = t^2 + a_q \cdot t + q^2$.

By Examples 1, 2 and Corollary 4.3, the affine subvariety of degree 1, $H^1_m \subseteq \mathbb{F}_q[x]/(x-1)(x-\lambda)$, is given by the two linear equations

$$(\lambda^2 + \lambda) \cdot z_1 + (\lambda - \lambda^2) \cdot z_2 + (\lambda^2 - \lambda) \cdot z_3 = 1, \quad \lambda \cdot z_1 + (1 - \lambda) \cdot z_2 + (\lambda^2 - \lambda) \cdot z_3 = 0.$$  

By taking $\omega_3(z_1, z_2, z_3) = (\lambda^2 + \lambda) \cdot z_1 + (\lambda - \lambda^2) \cdot z_2 + (\lambda^2 - \lambda) \cdot z_3 - 1$ and $\omega_2(z_1, z_2) = \lambda^2 \cdot z_1 + (1 - \lambda)^2 \cdot z_2 - 1$, by Lemma 4.2 and Theorem □ we have

$$a_q = J^{(\omega_3, \omega_2)}_{T_m^1} \cdot \frac{1}{(q - 1)^2} \sum_{1 \leq i,j \leq q-1} J^{(\omega_3, \omega_2)}_{\omega_3}(i, j, q-1) .$$

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