Mixing Time of Markov chain of the Knapsack Problem

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To find the number of assignments of zeros and ones satisfying a specific Knapsack Problem is \(\#P\) hard, so only approximations are envisageable. A Markov chain allowing uniform sampling of all possible solutions is given by Luby, Randall and Sinclair. In 2005, Morris and Sinclair, by using a flow argument, have shown that the mixing time of this Markov chain is \(O(n^{9/2+\epsilon})\), for any \(\epsilon > 0\). By using a canonical path argument on the distributive lattice structure of the set of solutions, we obtain an improved bound, the mixing time is given as

\[
\tau_{\epsilon}(n) \leq n^3 \ln(16\epsilon^{-1}).
\]

Keywords: Knapsack problem, Markov chain, mixing time

1 Introduction

Let \(a = (a_1, a_2, \ldots, a_n)\) be a vector in \(\mathbb{R}^n\) and let \(b\) be any real number. The Knapsack Problem, denoted by \(K(n, b)\), consists of finding a vector \(x = (x_1, x_2, \ldots, x_n)\), with \(x_i \in \{0, 1\}\), such that

\[
< a, x > = a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b. \quad (1)
\]

Such a vector \(x\) is called a feasible solution. In practical terms, the problem can be phrased as follows. Assume that a traveler disposes of a knapsack which can accommodate at most \(b\) kilograms. Let there be \(n\) items such that item \(i\) weighs \(a_i\) kg. How can we pack up so that the total weight does not exceed the weight accommodable in the knapsack? This example indicates the relevance of the problem in various walks of life. For instance, in small to medium-sized enterprises, the total weight allowed may be the total investment available for a list of possible projects. For everyday travelers, the problem consists as how we can pack up our luggage so not to exceed the total weight allowed by airlines regulations. There are many applications in business, two of these are explored in [8] (marketing) and [3] (e-commerce).

In mathematical terms, the problem consists of finding the number of solutions of the Inequality 1. The vector \((0, 0, \ldots, 0)\) is a trivial solution. Now, it is possible to check exhaustively all the \(2^n\) possibilities

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and see which of these satisfy Inequality 1. This scheme may work for small \( n \). In case \( n \) is large, say \( n > 100 \), to check exhaustively all the possible solutions will take a long time, and by the time we are finished, the problem is obsolete (the flight is gone or the investment opportunities wasted). Hence, it is advisable to sample a solution uniformly at random.

Let \( \Omega(n, b) \) be the set of all solutions of the problem \( K(n, b) \). One way to sample a solution at random is to construct a random walk which starts at a given element of \( \Omega(n, b) \), then moves to another element according to some simple rule which changes one element to another. One such simple change consists of changing the entry \( x_i \) to \( x_i + 1 \mod 2 \). We call such a change a \textit{flip} on the position \( i \). A flip at a point \( i \) is \textit{positive} if \( x_i \) goes from 0 to 1 and a flip at a point \( i \) is \textit{negative} if \( x_i \) goes from 1 to 0. With this terminology, we let a \textit{neutral flip} to be the fact of leaving an entry unchanged. After running the random walk for \( t \) steps, we output the element reached by the random walk and consider this element, the \( t^{th} \) state reached, as a random sample. Such a random walk is given in [6, 11] as follows.

Take the solution \( x = (0, 0, \ldots, 0) \) as the starting point. Stay on \( x \) with probability \( \frac{1}{2} \), and with probability \( \frac{1}{2} \) choose a position \( i \) in the vector \( x \) and perform a flip on the position \( i \). Call the new vector obtained by a single flip as \( y \). Now, if \( y \) satisfies Equation 1, move from \( x \) to \( y \), else stay at \( x \). It is routine to check that this random walk defines a Markov chain that can reach all the solutions and is aperiodic. Thus, it is an ergodic Markov chain that tends asymptotically towards a stationary state \( \pi \). So it may be used to sample uniformly a solution of \( K(n, b) \) problem. The key issue now is to know the time taken to reach the steady state.

Since the successive states of a Markov chain are not independent, an unbiased sample can only be obtained if the chain reaches stationarity, the stage when the probability of sampling a particular configuration is fixed in time. The \textit{mixing time} of a Markov chain is the number of steps necessary to reach that stationary state. To know the mixing time of a particular Markov chain is crucial to avoid either getting a biased sample (if stationarity is not reached), or to avoid the computational cost of running the chain more than necessary. In [7], it is shown that the mixing time of Markov chain defined above is polynomial on the length of the solutions of the problem. In this paper, by using a different approach which makes use of the lattice structure of the solution set \( \Omega(n, b) \), we obtain a better bound.

2 Preliminaries

Let \( M \) be a Markov chain on a set of states \( \Omega \). Let \( P \) be the matrix of transitions from one state to another. We can visualize the Markov chain \( M \) as a weighted directed graph \( G \), where the states are vertices of \( G \) and there is an edge of weight \( P(x, y) \) if the transition from state \( x \) to state \( y \) has probability \( P(x, y) \). A Markov chain is \textit{irreducible} if, for all pairs of states \( x \) and \( y \), there is an integer \( t \), depending on the pair \((x, y)\), such that \( P^t(x, y) > 0 \). In terms of the graph \( G \), the Markov chain is irreducible if there is a path between every pair of vertices of \( G \). A Markov chain is \textit{aperiodic} if for all states \( x \), there is an integer \( t \) such that for all \( t' \geq t \), \( P^{t'}(x, x) > 0 \). That is, after sufficient number of iterations, the chain has a positive probability of staying on \( x \) at every subsequent step. This ensures that the return to state \( x \) is not periodic. In terms of the graph \( G \), this can be achieved by having a loop at every vertex. A Markov chain is \textit{ergodic} if it is irreducible and aperiodic. If \( P \) is the matrix of transitions of an ergodic Markov chain, there is an integer \( t \) such that for all the pairs of states \((x, y)\), \( P^t(x, y) > 0 \). (Notice that \( t \) does not depend on the pair). It can also be proved that for every ergodic Markov chain with transition matrix \( P \), the largest eigenvalue of \( P \) is equal to 1. Using this, it can be proved that there is a unique probability vector \( \pi \), such that \( \pi P = \pi \). The vector \( \pi \) is the \textit{stationary} distribution of \( M \). A chain is \textit{reversible} if
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$P(x, y)\pi(x) = P(y, x)\pi(y)$ for every pair of states $x$ and $y$. If a chain is reversible and the matrix $P$ is symmetric, then $\pi(x) = \pi(y)$ for every pair $x$ and $y$. The stationary state is then said to be uniform. In the obvious way, we say a matrix $P$ is irreducible, ergodic, aperiodic, etc. if the Markov chain ruled by $P$ is irreducible, ergodic, aperiodic, etc.

Let $\lambda_1 = 1$ and $\lambda_2$ be the largest and the second largest eigenvalue of the transition matrix $P$. The real number $\lambda_1 - \lambda_2$, that is, $1 - \lambda_2$, is the spectral gap of the matrix $P$. It can be shown that the bigger the gap, the faster the mixing time of the chain. The analysis of the mixing time of a Markov chain is based on the intuition that a random walk on the graph $G$ mixes fast (i.e., reaches all states quickly) if $G$ has no bottleneck. That is, there is no cut between any set of vertices $S$ and its complement which blocks the flow of the Markov chain and thus prevents the Markov chain from reaching easily to some states. See [2, 4, 6, 11] for a better exposition on the topic, and for definitions in graph theory we refer to [10].

Denoting the probability of $x$ at stationarity by $\pi(x)$ and the probability of moving from $x$ to $y$ by $P(x, y)$, the capacity of the arc $e = (x, y)$, denoted by $c(e)$, is given as $c(e) = \pi(x)P(x, y)$. Let $P_{x,y}$ denote the set of all simple paths $p$ from $x$ to $y$ (paths that contain every vertex at most once). A flow in $G$ is a function $\phi$, from the set of simple paths to the reals, such that $\sum_{p \in P_{x,y}} \phi(p) = \pi(x)\pi(y)$ for all vertices $x, y$ of $G$ with $x \neq y$. An arc-flow along an arc $e$, denoted by $\phi'(e)$, is then defined as $\phi'(e) = \sum_{p \ni e} \phi(p)$. For a flow $\phi$, a measure of existence of an overload along an arc is given by the quantity $\rho(e)$, where $\rho(e) = \frac{\phi'(e)}{c(e)}$, and the cost of the flow $\phi$, denoted by $\rho(\phi)$, is given by $\rho(\phi) = \max_e \rho(e)$.

If a network $G$ representing a Markov chain can support a flow of low cost, then it cannot have any bottlenecks, and hence its mixing time should be small. This intuition is confirmed by the following theorem [11].

**Theorem 2.1** [11] Let $\mathcal{M}$ be an ergodic reversible Markov chain with holding probabilities $P(x, x) \geq \frac{1}{2}$ at all states $x$. The mixing time of $\mathcal{M}$ satisfies

$$\tau_x(e) \leq \rho(\phi)|p| \left(\ln\frac{1}{\pi(x)} + \ln\frac{1}{e}\right),$$

where $|p|$ is the length of a longest path carrying non-zero flow in $\phi$.

Thus one way to prove that a Markov chain mixes fast is to produce a flow along some paths where the paths and the maximal overload on edges are polynomials on the size of the problem.

### 3 Canonical path and Mixing time

Let $G$ denote the graph whose vertices are all the solutions of the $K(n, b)$, and there is an edge connecting a vertex $x$ to a vertex $y$ if $x$ can be turned into $y$ by changing a single entry of the solution vector $x$. So $G$ is just an Hasse diagram, denoted as $L(n, b)$, which is a top-truncated lattice. For illustration, let $a_1 = 5$, $a_2 = 3$, $a_3 = 2$, $a_4 = 1$, and $b = 9$. The set of solutions of the inequality $5x_1 + 3x_2 + 2x_3 + x_4 \leq 9$ is given in Figure 1.

Now, we define formally a random walk on $G$ as follows. Start at any vertex $z$. If the walk is at $x$, move from $x$ to $y$ with probability $P_{x,y} = \frac{1}{2n}$ if there is an edge connecting $x$ to $y$ in $G$ and $x, y > \leq b$, and stay at $x$ otherwise. Obviously, the probability of staying at $x$ is at least $\frac{1}{2}$. Now, since $G$ is just the lattice of flats of the free matroid $U_{n,n}$ minus the upper part which contains the points $y$ such that $y, a > \leq b$.
so \(G\) is connected. Therefore the matrix \(P\) is irreducible. Moreover, since \(P(x, x) \geq \frac{1}{2}\), there is a loop on every point \(x\), the matrix \(P\) is aperiodic. So the random walk is an ergodic Markov chain. Finally, \(P\) is a symmetric matrix. Thus, the Markov chain has a stationary uniform distribution \(\pi\). We denote this Markov chain as \(M_{\text{knap}}\).

The following theorem is our main result.

**Theorem 3.1** Let \(\mathcal{L}(n, b)\) be the Hasse diagram of the solutions of the inequality \(\langle a, x \rangle \leq b\), for any real vector \(a\) of length \(n\). The Markov chain \(M_{\text{knap}}\) on \(\mathcal{L}(n, b)\) mixes fast and the mixing time \(\tau\), is given by

\[
\tau(\epsilon) \leq n^3 \ln(16\epsilon^{-1}).
\]

The proof is similar to that of papers [4, 5], and requires the following background. A path from \(v\) to \(w\) is a sequence of binary vectors \((x^{(1)}, x^{(2)}, \ldots, x^{(r)})\), such that \(x^{(1)} = v\) and \(x^{(r)} = w\), and \(x^{(i+1)}\) is obtained from \(x^{(i)}\) by a single flip. We prove this result by showing that, between any two vertices of \(\mathcal{L}(n, b)\), there is a path, the canonical path (defined below), that is not 'too long' and where no edge is overloaded by the 'probability fluid'. To put this in a more formal setting, we need the following definitions and lemmas.

Let \(x_i\) denote the \(i\)th entry of \(x\). Let \(x\) and \(y\) be two solutions of the inequality \(\langle a, z \rangle \leq b\). We say that \(x\) and \(y\) are matched at the \(i\)th entry if \(x_i = y_i\). If \(x\) and \(y\) are not matched at the entry \(k\), to match the point \(k\) means to do the operation \(y_k = x_k + 1 \pmod{2}\). That is, if \(x_k = 0\), change it into 1, and vice-versa.

The canonical path from \(v\) to \(w\) is the path that consists of matching the entries in increasing order of the indices, starting from \(i = 1\) up to \(i = n\). But, if flipping the position \(i\) positively (changing its entry from 0 to 1) leads to a vector that is not in \(\mathcal{L}(n, b)\), then we first flip negatively (change 1 to 0) the nearest positions \(k_1, k_2, \ldots, k_r\), such that \(k_1 < k_2 < \ldots < k_r\), \(k_i > i\), for \(1 \leq j \leq r\), \(v_{k_j} = 1\) and \(w_{k_j} = 0\), and \(\sum_j a_{k_j} \geq a_i\). That is, the canonical path first flips as many nearest positions greater than \(i\) as possible, so that flipping afterwards the position \(i\) does not lead outside \(\mathcal{L}(n, b)\). In simpler words, the Canonical path is given by Algorithm 1.

**Algorithm 1.** To get from one feasible solution \(v = (v_1, \ldots, v_n)\) to another \(w = (w_1, \ldots, w_n)\), one scans from left to right, flipping variables from 0 to 1 and vice-versa, as required. Thus an intermediate
state is a feasible solution \((w_1, \ldots, w_{n-1}, v_i, \ldots, v_s)\). However, changing \(v_i = 0\) to \(w_i = 1\) might violate the linear inequality. In this case, one jumps \(v_j\), and continues scanning right, changing 1s to 0s, as necessary, until there is enough slack to allow \(v_i\) to be flipped.

(Informally, one may visualize the canonical path as follows. Suppose that the items one wants to pack up in a bag are ranked from 1 to \(n\), according to their importance. Let there be two bags, \(v\) and \(w\), both satisfying the requirement of not being overloaded. Assume that one would like to make \(v\) carry the same items as \(w\). Then starting from the most valuable item, say item 1, one checks whether or not the item \(i\) is contained in the bag \(w\). If not, and \(v\) also does not contain it, then nothing to do. In case \(v\) contains it, then remove this from \(v\). Suppose that \(w\) contains item \(i\). If \(v\) also contains it, then leave it alone. If not, check whether inserting item \(i\) into \(v\) would overload the bag. If there is no overload, insert item \(i\) into \(v\). If there is an overloading, then first remove from \(v\) the items less valuable than the item \(i\) and that are not in \(w\). One has to remove as many items as is necessary to avoid the item \(i\) to overload the bag.)

With this definition, suppose that Algorithm 1 is moving from \(v\) to \(w\), and at some instant, the highest entry flipped is \(v_k\). Then the sequence of indices \((1, 2, \ldots, n)\) can be partitioned in three parts. Part one, called the Matched Zone, is the ordered sequence of indices \((1, 2, \ldots, k)\) such that \(v_i = w_i\), for all \(i\) from 1 to \(k\). Part two, called the Heap Zone, is the sequence of indices \((k+1, k+2, \ldots, l)\), where \(l\) is the index of the last entry flipped along the canonical path, and where some entries are matched (flipped from 1 to 0 to avoid running into a binary vector that does not satisfy the inequality), and where some other entries are not flipped (they are jumped by Algorithm 1, either because \(v_i = 0\) or \(v_i = w_i = 1\)). Finally, part three, the Untouched Zone, consists of the sequence \((l+1, l+2, \ldots, n)\), the sequence of indices of entries not yet touched by Algorithm 1.

**Lemma 3.2** For all pairs of binary vectors \(v\) and \(w\) belonging to \(\mathcal{L}(n, b)\), there is a canonical path \((x^{(0)}, x^{(1)}, \ldots, x^{(r)})\) such that \(x^{(0)} = v\) and \(x^{(r)} = w\), and for all \(0 \leq j \leq r\), \(x^{(j)} \in \mathcal{L}(n, b)\).

**Proof.** Let \(v\) and \(w\) belong to \(\mathcal{L}(n, b)\) but there is no such a canonical path from \(v\) to \(w\). That is, Algorithm 1 gets stalled at some point. This is possible only if there is an entry \(v_i\) whose flipping positively will lead to a vector \(v'\) such that \(v' \notin \mathcal{L}(n, b)\), but for all \(j\) such that \(j > i\), \(v_j = 1\), \(w_j = 0\), we have \(a_j > \sum a_j\).

That is, even after Algorithm 1 had flipped negatively all such entries \(v_j\), flipping \(v_i\) would still lead to a binary vector \(v'\) where \(b < \sum v', a_j\). (That is, \(v'\) is outside \(\mathcal{L}(n, b)\)).

Now, suppose that the canonical path is at its \(t\)th iteration when it has to flip the entry \(v_i\). Thus, there are \(n - t\) positions left to match before reaching \(w\). Let there be \(s\) positions \(j\) such that \(j > i\), \(v_j = 1\), \(w_j = 0\) (points that may be flipped to avoid exiting \(\mathcal{L}(n, b)\)). Obviously, \(n - t \geq s\), since there may still be positions \(k\) such that \(v_k = 1\), \(w_k = 0\), or \(v_k = 0\), \(w_k = 1\). Since flipping all the \(s\) positions \(j > i\) such that \(v_j = 1\), \(w_j = 0\) still leads to a binary vector outside \(\mathcal{L}(n, b)\), we have \(a_j > \sum a_j\). This is clearly a contradiction. \(\square\)

One of the most important properties of Algorithm 1 is that once matched, an entry can not be unmatched anymore. That is, the matched zone only increases along the canonical path. This entails the following lemma.

**Lemma 3.3** For all binary vectors \(v\) and \(w\) that belong to \(\mathcal{L}(n, b)\), the length of the canonical path from \(v\) to \(w\) is at most \(n\).
Proof. Since every entry is matched at most once, the length of the path is at most the number of entries in $v$. $\square$

In order for the canonical paths argument to work, we need the number of paths through any edge of the upper-truncated lattice $\mathcal{L}(n, b)$ to be bounded above by a small polynomial of $n$ times $N$, the number of valid knapsack solutions. And indeed, in what follows, for any arc $e = (z, y)$ of $\mathcal{L}(n, b)$, the number of different canonical paths passing through $e$ is at most $2N$. But, before proving this crucial result in Corollary 3.7, we need the following definitions and lemmas. Let $v$ and $z$ be elements of $\mathcal{L}(n, b)$. We say an entry $v_i$ is determined by another entry $z_i$ if it is possible to know the value of $v_i$ from the value of $z_i$. That is, $v_i$ and $z_i$ are not independent, since either $v_i = z_i$, or $v_i = z_i + 1 \pmod{2}$. We say that the index $i$ is free in a vector $v$ if $v_i = 0$ or $v_i = 1$ independently of any entry of $z$. We recall that the canonical path $(v, \ldots, w)$ passes through $e = (z, y)$ means that to change $v$ into $w$, Algorithm 1 must make a sequence of flips that change $v$ into $z$, then $z$ into $y$, then $y$ into $w$. See Figure 2 for an illustration.

![Fig. 2: Different canonical paths that pass through the arc $e$.](image)

**Lemma 3.4** Let $G$ be the upper-truncated lattice whose vertices are all the elements of $\mathcal{L}(n, b)$. Let $z$ and $y$ be two such vertices, such that $z_i = y_i$ for all the indices except $r$, and let $e = (z, y)$ be the edge that represents the flip matching the $r$th entry of $z$ (That is, the flip transforming $z$ into $y$). For any two vertices $v$ and $w$, the canonical path $(v, \ldots, w)$ passes through $e$ only if the set of indices that are free in $v$ is disjoint from the set of indices that are free in $w$.

**Proof.** Suppose that the canonical path from $v$ to $w$ passes through the arc $e$ at the $r$th entry of $z$. Consider the Matched Zone, the Untouched Zone and the Heap Zone at the instant when the flip is performed at $r$. We aim at showing the following.

1. The Matched Zone (from index 1 to $k$) is free in $v$ and determined by $y$ in $w$.
2. The Untouched Zone (from index $l + 1$ to $n$) is free in $w$ and determined by $z$ in $v$.
3. In the Heap Zone, entries that are flipped carry 0 in $z$, and must carry 1 in $v$.
4. In the Heap Zone, entries that are not flipped and carry 0 in $z$ must carry 0 in $v$. In the Heap Zone, entries that are not flipped and carry 1 in $z$ must carry 1 in $v$. 

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Indeed, the canonical path from \( v \) to \( w \) passes through \( e = (z, y) \) only if the sequence of flips from the first one to the \( t \)th flip change \( v \) to \( z \). Thus, (1), all the entries of \( v \) in the Untouched Zone must be equal to the corresponding entries in \( z \), since there are untouched, but they are free in \( w \). Now, (2), after changing \( z \) to \( y \), the remaining flips must change \( y \) to \( w \). Since the canonical path does not un-match positions that are already matched, entries of \( w \) in the Matched Zone must be equal to the corresponding entries in \( y \), while they are free in \( v \). In the Heap zone, entries are either flipped or left untouched (jumped). If (3), they were flipped, thus they are 0 in \( z \) and they must be 1 in \( v \). Those are the entries that are flipped from 1 to 0 to get some slack. If (4), they are jumped, they may be 0 or 1 in \( z \). If they are 0 in \( z \), they must be 0 in \( v \), and if they are 1 in \( z \), they are also 1 in \( v \), since the Canonical Path from \( v \) to \( w \) reaches \( z \) at that instant means that the untouched entries are equal. Hence in the Heap zone, entries of \( v \) are determined by entries of \( z \).

**Lemma 3.5** Let \( \Omega(n, b) \) be the set of solutions of the inequality \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b \) and let \( |\Omega| = N \). Then

\[
N = N' + N'',
\]

where \( N' \) is the number of solutions of the inequality \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b \) and \( N'' \) is the number of solutions of the inequality \( a_2 x_2 + \cdots + a_n x_n \leq b - a_1 \).

**Proof.** (First, note that there is nothing special with choosing to remove \( x_1 \). Any other variable would do as well.) The set of solutions of the inequality \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b \) can be partitioned into the sets \( A \) and \( B \), where \( A \) is the set of solutions with \( x_1 = 0 \) and \( B \) is the set of solutions with \( x_1 = 1 \). The set \( A \) is in one-one correspondence with the set of solutions of the inequality \( a_2 x_2 + \cdots + a_n x_n \leq b \), and the set \( B \) is in one-one correspondence with the set of solutions of inequality \( a_2 x_2 + \cdots + a_n x_n \leq b - a_1 \).

**Corollary 3.6** Let \( z \) be a given feasible solution, let \( \alpha + \beta = n \), let \( |\Omega| = N \), and let \( V \) be the set of all the solutions where the first \( \beta \) entries are determined by \( z \). Then \( |V| \) is less or equal to \( (2N)^{\frac{\alpha}{\beta}} \).

**Proof.** (Obviously, \( |V| \) is just the number of solutions \( a_{\beta+1} x_{\beta+1} + \cdots + a_n x_n \leq b - a_1 - a_2 - \cdots - a_\beta \). The proof uses induction on \( \alpha \), the number of indices that are free. If \( \alpha = 0 \) (i.e., all the variables \( x_i \) are determined), the number of solutions is less or equal to \( N^0 = 1 \). (We may also check the formula by taking \( \alpha = n \). If all the variables are free, there are \( N^\alpha = N \) possible feasible solutions.) For induction, assume that the result holds for \( \alpha - 1 \). Let \( M \) denote the number of feasible solutions where \( \alpha \) variables are free, and suppose that \( x_n \) is free. Then, by Lemma 3.5, \( M = M' + M'' \), where \( M' \) is the number of solutions of the inequality \( a_1 x_1 + \cdots + a_n x_{n-1} \leq b \) and \( M'' \) is the number of solutions of the inequality \( a_1 x_1 + \cdots + a_n x_{n-1} \leq b - a_n \). We have

\[
M = M' + M''
\leq N'\frac{a-1}{a-2} + N''\frac{a-1}{a-2}
\text{by inductive hypothesis}
\leq (2N')^\frac{a-1}{a-2}
\text{since } N' \geq N''
\leq (2N)^\frac{\alpha}{\beta}.
\]

\( \square \)
Corollary 3.7 If \( e = (z, y) \) is an arc of \( L(n, b) \), the number of different canonical paths passing through \( e \) is at most \( 2N \), where \( N \) is the number of vertices of \( L(n, b) \).

Proof. Let the canonical path from \( v \) to \( w \) passes through the arc \( e \). By Lemma 3.4, the set of indices that are free in \( v \) is disjoint from the set of indices that are free in \( w \). Let \( \alpha \) and \( \beta \), with \( \alpha + \beta \leq n \), be the number of indices that are free in \( v \) and \( w \), respectively. By Corollary 3.6, for a fixed feasible solution \( w' \), there are at most \( N^{\alpha} \) vectors \( v \) such that the canonical path \( (v, \ldots, w') \) passes through \( e = (z, y) \), and for a fixed vertex \( v' \), there are at most \( N^{\beta} \) vectors \( w \) such that the canonical path \( (v', \ldots, w) \) passes through \( e = (z, y) \). Therefore, for a fixed vector \( z \) and fixed index \( k \) such that \( e \) is the edge consisting of flipping the \( k \)th entry of \( z \), the total number of canonical paths passing through \( e \) is at most \( (2N)^{\alpha} (2N)^{\beta} \leq 2N \). \( \square \)

Proof of Theorem 3.1. In order to prove Theorem 3.1 by using Theorem 2.1, we show that there is a flow \( \phi \) such that \( \rho(\phi) \) is a polynomial in \( n \), the size of a vector solution of \( K(n, b) \). Indeed, if \( x \) and \( y \) are two vertices of \( G \), let \( \hat{p}_{xy} \) denote the canonical path from \( x \) to \( y \) and let \( \phi \) be a flow consisting of injecting \( \pi(x)\pi(y) \) units of flow along \( \hat{p}_{xy} \). Then, for all arcs \( e \), we have

\[
\phi'(e) = \sum \pi(x)\pi(y),
\]

where the sum is over all the pairs \( \{x, y\} \) such that \( e \in \hat{p}_{xy} \). Since by Corollary 3.7, there are at most \( 2N \) canonical paths through \( e \), and \( \pi(x) = \pi(y) = \frac{1}{N} \), as the distribution \( \pi \) is uniform, we have

\[
\phi'(e) \leq 2N\pi(x)\pi(y) \leq \frac{2}{N}.
\]

Moreover, since \( P_{x,y} = \frac{1}{2n} \) if there is an edge from \( x \) to \( y \), we have

\[
c(e) = \pi(x)P_{x,y} \geq \frac{1}{2nN}.
\]

Thus,

\[
\rho(\phi) \leq \max_e \phi'(e) \leq 4n.
\]

Now, using Theorem 2.1, Lemma 3.3 and Equation 4, we have

\[
\tau_x(\epsilon) \leq (n)(4n)(\ln \frac{1}{\pi(x)} + \ln \frac{1}{\epsilon}).
\]

Finally, using the fact that \( \frac{1}{\pi(x)} = N \leq 2^n \), we have

\[
\tau_x(\epsilon) \leq 4n^2(n \ln 2 + \ln \frac{1}{\epsilon}) \leq n^3 \ln(16\epsilon^{-1}).
\]

\( \square \)

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