A simple sketching algorithm for entropy estimation

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Abstract

We consider the problem of approximating the empirical Shannon entropy of a high-frequency data stream when space limitations make exact computation infeasible. It is known that $\alpha$-dependent quantities such as the Rényi and Tsallis entropies can be estimated efficiently and unbiasedly from low-dimensional $\alpha$-stable data sketches. An approximation to the Shannon entropy can be obtained from either of these quantities by taking $\alpha$ sufficiently close to 1. However, practical guidelines for the choice of $\alpha$ are lacking. We avoid this problem by going directly to the limit. We show that the projection variables used in estimating the Rényi entropy can be transformed to have a proper distributional limit as $\alpha$ approaches 1. The Shannon entropy can then be estimated directly from a data sketch based on this limiting distribution. We derive properties of the distribution, showing that it has a surprisingly simple characteristic function $\left(i\theta\right)^{\alpha}$ and that the $k$th moment of the exponential of such a variable is $k^k$ for all non-negative real values of $k$. These properties enable the Shannon entropy to be estimated directly from the associated data sketch as the logarithm of a simple average. We obtain the Fisher information for the statistical problem of recovering the entropy from the data sketch and hence a lower bound on the standard error of the estimated entropy. We show that our proposed estimator has theoretical statistical efficiency of 96.8% and confirm this with an empirical study. Finally we demonstrate that in order for the estimator to have $1 + \epsilon$ coverage with high probability the sketch must have size $O(1/\epsilon^2)$, in agreement with theoretical bounds.

1 Introduction

Streaming data is ubiquitous in a wide range of areas from engineering, and information technology, finance, and commerce, to atmospheric physics, and earth sciences. For background, see [19, 27]. The Shannon entropy provides an important characterisation of a data stream and many algorithms have been developed for its estimation [4, 5, 22]. Areas of application extend far beyond that of network traffic monitoring, e.g., entropy estimation of neural spike trains, or of images in video streams for the purpose of visual tracking. Our own particular interest is in developing convergence diagnostics when monitoring extensive Markov chain Monte Carlo simulations of posterior distributions in Bayesian inference [12].

This paper focuses on the estimation of Shannon entropy by $\alpha$-stable data sketching, i.e., transforming distinct stream elements online to distinct realizations of a strictly stable variable of index $\alpha$ (called $\alpha$-stable hereafter), and storing weighted linear combinations of these realizations, independently replicated $k$ times [18]. There is a long history of the use of the $\alpha$-stable distribution in the literature, e.g., in cardinality estimation [14, 3, 10, 13, 16, 8], norm and distance estimation [2, 11, 15, 20, 18, 23, 25, 9], and, more recently, entropy estimation [22, 12]. However none of these projection methods estimates the Shannon entropy directly. In contrast, they estimate alternative

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\( \alpha \)-dependent quantities such as the entropies of Rényi and Tsallis, and rely on these measures providing adequate approximations to the Shannon entropy when \( \alpha \) is close to 1.

Our contribution is to provide an easily implemented direct solution by forming a data sketch with one of the simplest stable distributions, the maximally skewed distribution with \( \alpha = 1 \). We show that this distribution arises in the limit when considering the \( \alpha \)-stable transformations involved in data sketches for the Rényi entropy. We derive this result in Section 2 and obtain a compact representation of the characteristic function of the limiting distribution along with moments of the exponential of a random variable with this distribution. The simple form of these moments is then exploited in Section 3.1 to derive a family of log-mean estimators of the Shannon entropy, indexed by \( \zeta > 0 \). We show that the optimal estimator is given by \( \zeta = 1 \) and has asymptotic efficiency of 97.8% in the sense that its variance is within 2.2% of the theoretic Cramér-Rao lower bound for the variance of any estimator based on the same sketch. The simple estimator with \( \zeta = 1 \) has asymptotic efficiency of 96.8%. We provide a table of small-sample bias corrections for both estimators in Section 3.1, and confirm the efficiency results with an empirical study in Section 3.2. Finally in Section 3.3 we establish that the probability of the estimator having error greater than \( \epsilon \) decreases exponentially with \( k \), where \( k \) is the length of the data sketch, when \( \zeta \leq 1 \). For small \( \epsilon \) it decreases exponentially with \( k\epsilon^2 \) which leads to a sample complexity bound, in the style of [25], on the length \( k \) of the data sketch, and hence on the storage requirements of the algorithm implementing this estimation procedure.

1.1 Notation

We follow the notation and terminology of [18, 10]. Let \( S_T \) denote a data stream of length \( T \), with elements of the form \((i_t, d_t)\), where the item type \( i_t \) belongs to a large or possibly infinite set \( D = \{c_1, c_2, \ldots, c_N\} \) and the associated quantity is \( d_t \) (either positive or negative), \( t = 1, 2, \ldots, T \).

**Definition 1.1.** The accumulation vector of \( S_T \) at stage \( T \) is \( a_T = (a_1, a_2, \ldots) \), where

\[
a_j = \sum_{t=1}^{T} d_t \mathbb{I}(i_t = c_j), \quad j = 1, 2, \ldots, N,
\]

is the cumulative quantity of elements of type \( c_j \) at stage \( T \) (the indicator function \( \mathbb{I}(i_t = c_j) \) equals 1 if \( i_t = c_j \) and 0 otherwise).

Note that we are assuming that the data types can be ordered by some convention, e.g., by the order of their first appearance. We also assume that at stage \( T \), each of the terms \( a_j \) is non-negative, and the relaxed strict-turnstile model introduced by Li [11].

The empirical entropies of Shannon, Rényi and Tsallis are then given respectively by

\[
H(p) = -\sum_{j=1}^{N} p_j \log p_j; \quad H_\alpha(p) = \frac{1}{1 - \alpha} \log \left( \sum_{j=1}^{N} p_j^\alpha \right) \quad \text{and} \quad S_\alpha(p) = \frac{1}{1 - \alpha} \left( \sum_{j=1}^{N} p_j^\alpha - 1 \right),
\]

where \( p_j = a_j / \sum_{i=1}^{N} a_i \) and by convention \( p \log(p) \) is defined to be 0 when \( p = 0 \). When the number of positive terms in the accumulation vector is large, storing \( a_T \) or equivalently \((p_1, p_2, \ldots, p_N)\) may be infeasible. This has motivated the use of data sketches and other synopsis construction techniques that store and update online a low dimensional representation of the stream [1].
2 Estimating entropy via random projections

Random projection methods require that each element type $c_j$ in the data stream can be transformed into a distinct random variable $R(c_j)$. In practice, this is achieved “to adequate approximation” by (i) hashing $c_j$ to an integer (or vector of integers), (ii) using these integers to seed a pseudo-random number generator, and (iii) using the seeded generator to simulate the random variable $R(c_j)$. The projection is then accumulated online as $\sum_{t=1}^{T} R(i_t) d_t = \sum_{j=1}^{N} R(c_j) a_j$. This provides a single element of the data sketch. A further $k - 1$ elements are generated independently in parallel to form the $k$-dimensional sketch.

Special properties of the $\alpha$-stable distributions [28] motivate their use in data sketching. For example, the $\alpha$-frequency moment $\sum_{j=1}^{N} a_j^\alpha$ can be recovered approximately from an $\alpha$-stable data sketch since $\sum_{j=1}^{N} a_j R(c_j)$ has the same distribution as $R(\sum_{j=1}^{N} a_j^\alpha)^{1/\alpha}$, where $R$ and $R(c_j), j = 1, 2, \ldots, N$, independently, each have the same $\alpha$-stable distribution [18]. Dividing by $\sum_{j=1}^{N} a_j = \sum_{i=1}^{T} d_i$, the total cumulative quantity of all items in the data stream at stage $T$, we have a similar distributional identity involving $B_\alpha = (\sum_{j=1}^{N} p_j^\alpha)^{1/\alpha}$, and hence this quantity can be estimated from a data sketch in the same way that a scale parameter can be estimated in an observed sample of size $k$ from an $\alpha$-stable distribution. Estimates of the Rényi and Tsallis entropies are then obtained by substituting the estimated value of $B_\alpha$ in [22], and by choosing $\alpha$ close to 1; these quantities provide an approximation to the Shannon entropy [22] as shown by the following result.

**Lemma 2.1.** As $\alpha \to 1$,

$$\frac{B_\alpha - 1}{1 - \alpha} - S_\alpha(p) \to 0, \quad \frac{\log B_\alpha}{1 - \alpha} - H_\alpha(p) \to 0.$$

**Proof.**

$$\lim_{\alpha \to 1} \left[ \frac{B_\alpha - 1}{1 - \alpha} - S_\alpha(p) \right] = \lim_{\alpha \to 1} (1 - \alpha)^{-1} \left[ (\sum_{j=1}^{N} p_j^\alpha)^{1/\alpha} - \sum_{j=1}^{N} p_j^\alpha \right]$$

$$= -\lim_{\alpha \to 1} \left\{ (\sum_{j=1}^{N} p_j^\alpha)^{1/\alpha} \left[ -\frac{1}{\alpha} \log \sum_{j=1}^{N} p_j^\alpha + \frac{1}{\alpha} \left( \sum_{j=1}^{N} p_j^\alpha \right)^{-1} \sum_{j=1}^{N} p_j^\alpha \log p_j \right] \right\}$$

$$- \sum_{j=1}^{N} p_j^\alpha \log p_j \right\}$$

$$= 0.$$

The second limit is found similarly. ∎

We now show that the limiting process can be carried out within the family of $\alpha$-stable variables leading to a projection variable that enables the Shannon entropy to be estimated directly. Suppose the $Z_\alpha$ is a positive, strictly stable random variable with index $0 < \alpha < 1$, having Laplace transform $e^{-\lambda^\alpha}$ for $\lambda \geq 0$. Let $(Z^{(1)}_\alpha, \ldots, Z^{(N)}_\alpha)$ be a vector of independent copies of $Z_\alpha$ and let $p = (p_1, \ldots, p_N)$ be a vector of frequencies with $\sum_{i=1}^{N} p_i = 1$. It follows that

$$\sum_{i=1}^{N} Z^{(i)}_\alpha p_i \sim Z_\alpha \left( \sum_{i=1}^{N} p_i^\alpha \right)^{1/\alpha} = Z_\alpha B_\alpha, \quad (3)$$

$$3$$
where the symbol \( \sim \) denotes equality in distribution, and \( Z_\alpha \to 1 \) as \( \alpha \to 1 \) (shown in proof of Lemma 2.2).

Starting from the Rényi entropy \( H_\alpha(p) = \alpha/(1-\alpha) \log B_\alpha \), we obtain that \( (1-B_\alpha)/(1-\alpha) = (1-\alpha)^{-1} [1-\exp\{(1-\alpha)H_\alpha(p)/\alpha]\} \to \delta \) as \( \alpha \to 1 \) by Lemma 2.1 where \( -\delta = -\sum_{i=1}^{N} p_i \log p_i \) is the Shannon entropy of \( p \).

Next, we define \( Y_{\alpha}^{(i)} = (1 - Z_{\alpha}^{(i)})/(1-\alpha) + \log(1-\alpha) \), and, using (3), we obtain

\[
\sum_{i=1}^{N} Y_{\alpha}^{(i)} p_i = \sum_{i=1}^{N} \left[ \frac{1 - Z_{\alpha}^{(i)}}{1-\alpha} + \log(1-\alpha) \right] p_i \sim \left[ \frac{1 - Z_{\alpha}}{1-\alpha} + \log(1-\alpha) \right] + Z_{\alpha} \frac{(1-B_{\alpha})}{1-\alpha}, \tag{4}
\]

so that taking limits \( \sum_{i=1}^{N} Y_{\alpha}^{(i)} p_i \to Y_1 + \delta \), provided \( Y_\alpha \) has a proper limit as \( \alpha \to 1 \). This is established in the following lemma.

**Lemma 2.2.** The random variable \( Y_\alpha \) has a proper limit \( Y_1 \) as \( \alpha \to 1 \). The characteristic function \( \phi(\theta) \) of \( Y_1 \) is \( (i\theta)^\theta \), and the kth moment of the random variable \( \exp(Y_1) \) is \( k^k \) for all \( k > 0 \).

See Appendix for proof, where we show that \( Y_1 \) has a maximally skewed stable distribution with \( \alpha = 1 \), and obtain the characteristic function in the equivalent form \( \phi(\theta) = \exp(-\frac{1}{2}\pi |\theta| + i\theta \log |\theta|) \). Denoting the distribution function of \( Y_1 + \delta \) by \( G(x; \delta) \) we have the following.

**Lemma 2.3.** Let \( X_1, \ldots, X_N \sim G(x; 0) \) be independent random variables, and let \( p_1, \ldots, p_N \) be positive constants satisfying \( \sum_{j=1}^{N} p_j = 1 \). Then,

\[
\sum_{j=1}^{N} p_j X_j \sim G\left(x; \sum_{j=1}^{N} p_j \log p_j \right).
\]

**Proof.** Straightforward using Lemma 2.2 and the fact that \( \sum_{j=1}^{N} p_j = 1 \). \( \square \)

Thus by projecting to maximally skewed stable random variables with distribution function \( G(x; 0) \), i.e., \( R(c_j) \sim G(x; 0) \), and storing projection elements of the form \( \sum_{i=1}^{T} R(i_c) d_t \), we reduce the problem of recovering the Shannon entropy to that of estimating a location parameter.

### 3 The log-mean estimator

#### 3.1 Derivation

The following lemma introduces a family of log-mean estimators of \( \delta \) indexed by \( \zeta \), where \( -\delta = \zeta \) is the Shannon entropy. See Appendix for proof. We qualify the performance of these estimators in terms of asymptotic relative efficiency (ARE), defined as the ratio of the variance of the limiting distribution of the maximum likelihood estimator (MLE) to the variance of the limiting distribution of the estimator in question, as the sample size increases [21]. The former attains the theoretical lower bound called the Cramér-Rao lower bound, so the ARE compares the performance of the estimator to the best possible performance. For practical purposes the simplest of these estimators with \( \zeta = 1 \) is to be recommended, and we begin by describing it below.

Let \( y_1, \ldots, y_k \) be independent samples from the \( G(y; \delta) \) distribution. Simulating from the maximally skewed stable distribution \( G(y; 0) \) via the algorithm of Chambers et al. [6] shows that this distribution has very heavy negative tails, so we exponentiate to flatten them.
Consider the transformation \( w_j = \exp(y_j) = \exp(\delta + z_j) \), where \( z_j \sim G(z; 0) \) independently, \( j = 1, \ldots, k \), with characteristic function \( \phi(\theta) = \mathbb{E}\exp(i\theta z_j) = (i\theta)^d \), for \( \theta \in \mathbb{R} \). It follows that \( \mathbb{E}w_j = e^\delta \phi(-i) = e^\delta \), so \( k^{-1} \sum_{j=1}^k \exp(y_j) \) is unbiased for \( e^\delta \) and has variance \( 3e^{2\delta}/k \). The Cramér-Rao lower bound for estimating \( \delta \) is approximately \( 0.3445 \), so the Fisher information about \( e^\delta \) is \( 0 \). Hence, the ARE of the mean estimator to the MLE is \( 0.3445 \times 3 \approx 0.968 \).

By taking the logarithm, we obtain the log-mean estimator of \( \delta \): \( \log(k^{-1} \sum_{j=1}^k \exp(y_j)) \). Lemma 3.1 presents the family of log-mean estimators indexed by \( \zeta > 0 \), where \( w_j = \exp(\zeta y_j) \).

**Lemma 3.1.** Let \( y_1, \ldots, y_k \) be independent samples from the \( G(y; \delta) \) distribution, and \( \zeta > 0 \) constant. The bias-corrected, log-mean estimator of \( \delta \) is

\[
\hat{\delta}_{lm,bc} = \zeta^{-1} \log \left( \zeta^{-\zeta} k^{-1} \sum_{j=1}^k \exp(\zeta y_j) \right) - BC,
\]

where \( BC \) is the additive bias in small samples:

\[
BC = \zeta^{-1} \mathbb{E} \log \left( \zeta^{-\zeta} k^{-1} \sum_{j=1}^k \exp(\zeta z_j) \right),
\]

and \( z_j \sim G(z; 0) \) i.i.d. For \( \zeta = 1.15 \), the estimator is near-optimal with largest ARE of 0.978; for \( \zeta = 1.0 \), the estimator has ARE of 0.968.

The log-mean estimator has finite variance in small samples for all values of \( \zeta > 0 \), but it has exponentially decreasing tail bounds only for \( \zeta \leq 1 \). See Sections 3.2 and 3.3 for details. When \( \zeta \leq 1 \), the maximum ARE of 0.968 is attained at \( \zeta = 1 \). In other words, \( \hat{\delta}_{lm,be} \) with \( \zeta = 1 \) is 96.8% as efficient as the best possible estimator, the MLE.

Table II presents the small-sample bias for \( \zeta = 1 \) and \( \zeta = 1.15 \), and various sample sizes \( k \), approximated via simulations. The standard error, computed as the sample standard deviation divided by \( \sqrt{n} \), where \( n \) is the number of repetitions, appears in brackets.

### 3.2 Small sample performance

Figure 1 compares the performance of the log-mean estimator in terms of relative mean square error (MSE) in small samples, alongside the Cramér-Rao lower bound given by \( (k \times 0.3445)^{-1} \). The MSE is defined as the ratio of \( \mathbb{E}(\hat{\delta}_{lm,be} - \delta)^2 \) to \( \delta^2 \), where the expectation is approximated via simulations. The log-mean estimators with \( \zeta = 1 \) and \( \zeta = 1.15 \) have very good small sample performance for sample sizes \( k \geq 20 \). Furthermore, the difference in MSE between the two estimators is negligible.

Figure 2 compares the Rényi entropy estimator with \( \alpha = 0.98 \) using the optimal quantile estimator for the \( l_\alpha \) quasi-norm; this is the recommended estimator in [22]. Following [1], we obtain the Cramér-Rao lower bound, and plot it for comparison. As an approximate estimator of entropy, the Rényi estimator has good small sample performance for \( k \geq 20 \). However, it is an estimator of entropy only in the limit as \( \alpha \to 1 \), whereas the log-mean estimator approximates the entropy directly. For \( \alpha > 0.98 \), we encountered numerical instabilities in the R contributed package fBasics, used to simulate from the stable distribution, and to approximate the density, distribution, and quantile functions. These instabilities were also pointed out in [21].
Table 1: Approximate small-sample bias $BC$ for the log-mean estimator computed over $n = 5 \times 10^5$ replicates. The standard error of the approximations appears in brackets.

| $k$ | $\zeta = 1$ | $\zeta = 1.15$ | $k$ | $\zeta = 1$ | $\zeta = 1.15$ |
|-----|-------------|----------------|-----|-------------|----------------|
| 10  | -0.1617     | -0.1810        | 90  | -0.01662    | -0.01886       |
|     | $(8.8584 \times 10^{-4})$ | $(8.529 \times 10^{-4})$ |     | $(2.605 \times 10^{-4})$ | $(2.590 \times 10^{-4})$ |
| 20  | -0.07795    | -0.08792       | 100 | -0.01514    | -0.01719       |
|     | $(5.745 \times 10^{-4})$ | $(5.710 \times 10^{-4})$ |     | $(2.470 \times 10^{-4})$ | $(2.457 \times 10^{-4})$ |
| 30  | -0.05113    | -0.05786       | 110 | -0.01316    | -0.01504       |
|     | $(4.616 \times 10^{-4})$ | $(4.588 \times 10^{-4})$ |     | $(2.355 \times 10^{-4})$ | $(2.342 \times 10^{-4})$ |
| 40  | -0.03857    | -0.04365       | 120 | -0.01278    | -0.01451       |
|     | $(3.973 \times 10^{-4})$ | $(3.950 \times 10^{-4})$ |     | $(2.253 \times 10^{-4})$ | $(2.241 \times 10^{-4})$ |
| 50  | -0.03060    | -0.03469       | 130 | -0.01170    | -0.01326       |
|     | $(3.530 \times 10^{-4})$ | $(3.509 \times 10^{-4})$ |     | $(2.163 \times 10^{-4})$ | $(2.151 \times 10^{-4})$ |
| 60  | -0.02501    | -0.02841       | 140 | -0.01070    | -0.01216       |
|     | $(3.216 \times 10^{-4})$ | $(3.197 \times 10^{-4})$ |     | $(2.083 \times 10^{-4})$ | $(2.072 \times 10^{-4})$ |
| 70  | -0.02170    | -0.02459       | 150 | -0.009971   | -0.01133       |
|     | $(2.967 \times 10^{-4})$ | $(2.950 \times 10^{-4})$ |     | $(2.008 \times 10^{-4})$ | $(1.997 \times 10^{-4})$ |
| 80  | -0.01851    | -0.02109       |     |             |                |
|     | $(2.766 \times 10^{-4})$ | $(2.749 \times 10^{-4})$ |     |             |                |

Figure 1: Comparison in terms of MSE of the log-mean estimators $\hat{\delta}_{lm,bc}$ with $\zeta = 1$ and $\zeta = 1.15$ ($10^5$ replicates), alongside the Cramér-Rao lower bound.
3.3 Tail bounds

The length of the data sketch vector, \( k \), is determined by the behaviour of the tail bounds. Given arbitrary parameters \( \epsilon > 0 \), and \( 0 < \gamma < 1 \), we require that the estimation error be bounded as follows: \( \mathbb{P}(|\hat{\delta}_{lm} - \delta| \geq \epsilon) \leq \gamma \), where \( \hat{\delta}_{lm} \) is the estimator without the small-sample bias removed. Note that the absolute estimation error is independent of \( \delta \) since the bias of \( \hat{\delta}_{lm} \) is additive. Lemma 3.2 shows that for \( \zeta \leq 1 \), the log-mean estimator has exponentially decreasing tail bounds. See Appendix for proof.

**Lemma 3.2.** Exponentially decreasing tail bound exist for \( \zeta \leq 1 \) and arbitrary \( \epsilon > 0 \), with

\[
\mathbb{P}(\hat{\delta}_{lm} - \delta \geq \epsilon) < \exp\left(-k \frac{\epsilon^2}{G_R}\right), \quad \text{and} \quad \mathbb{P}(\hat{\delta}_{lm} - \delta \leq -\epsilon) < \exp\left(-k \frac{\epsilon^2}{G_L}\right),
\]

where \( G_R = \epsilon^2 / \sup_{t>0} Q_\zeta(t, \epsilon) \), \( G_L = \epsilon^2 / \sup_{t>0} Q_\zeta(-t, -\epsilon) \) and

\[
Q_\zeta(t, \epsilon) = -\log \left( \sum_{j=0}^{\infty} \frac{t^j \zeta^j}{j!} \right) + t \epsilon \zeta.
\]

Furthermore as \( \epsilon \rightarrow 0 \) both \( G_R \) and \( G_L \) tend to \( 2(4^\zeta - 1)/\zeta^2 \).

A sample complexity bound in the style of \[25\] follows, stating that it suffices to let \( k \) be of order \( O(1/\epsilon^2) \) for the absolute error to be at most \( \epsilon \) with probability exceeding \( 1 - \gamma \). This result agrees with theoretical lower bounds for similar estimation problems \[18, 20, 23, 25\].

Figure 3 plots approximations to the left and right tail bound constants for various \( \epsilon \), showing that these constants are small.
4 Conclusion

The Shannon entropy of a data stream is a useful summary statistic in analyzing the evolving nature of the data, and has found many areas of application. The quest for efficient and accurate estimation in a one-pass algorithm using small time and space requirements is motivating ongoing research. We propose the log-mean estimator of entropy based on the method of data sketching via hashing to maximally skewed $\alpha$-stable random variables. Our approach is similar to that in [22] based on compressed counting, and estimating the entropy in the limit as $\alpha \to 1$, but we estimate the entropy directly to the limit, avoiding the problem of how to choose $\alpha$ in practice.

The data is processed online, in a one-pass algorithm, storing and updating a sketch vector of length $k$. Each item type is hashed to a maximally skewed $\alpha$-stable random variable with $\alpha = 1$, and a linear combination, weighted by the associated quantity, is updated; this processing is performed $k$ times via independent hash functions. The running time is $O(k)$ per data element, assuming each hashing operation requires constant time. The storage requirement for the data sketch is $O(k)$.

We derive the characteristic function of the maximally skewed stable distribution with $\alpha = 1$, and the moments of the exponential of such a variable. Exploiting these properties, we derive a family of log-mean estimators of the Shannon entropy indexed by $\zeta$, and recommend the estimator with $\zeta = 1$. We explain that this estimator is near optimal, having asymptotic relative efficiency of 96.8%, and show empirically that it has good performance in small samples. Finally, we prove that for small $\epsilon$, the error probability of this estimator decreases exponentially with $k\epsilon^2$ which leads to a sample complexity bound on $k$, and on the storage requirements of the algorithm implementing this estimation procedure.
Appendix

Following [23], the stable distribution has four parameters: index \( \alpha \in (0, 2] \), skewness parameter \( \beta \in [-1, 1] \), location parameter \( \delta \in \mathbb{R} \), and scale parameter \( \gamma > 0 \), denoted by \( F(x; \alpha, \beta, \gamma, \delta) \). If \( X \) has distribution \( F(x; \alpha, \beta, \gamma, \delta) \), then its characteristic function \( \phi(\theta) \) (c.f.), \( \theta \in \mathbb{R} \), is given by

\[
\phi(\theta) = \mathbb{E}\exp(i\theta X) = \begin{cases} 
\exp \left( \gamma \frac{\theta}{|\theta|} + i\theta |\theta|^{-1} \right) & \text{if } \alpha \neq 1 \\
\exp \left( \gamma \frac{\theta}{|\theta|} - i\theta \beta (2/\pi) \log |\theta| + i\delta \theta \right) & \text{if } \alpha = 1,
\end{cases}
\]

where \( \mathbb{E} \) denotes expected value, and \( i = \sqrt{-1} \). If \( \beta = \pm 1 \), the distribution is called maximally skewed.

**Proof of Lemma 3.1.** The c.f. of \( Z_{\alpha} \) can be written as

\[
\phi(\theta) = \exp \left\{ -|\theta|^\alpha \cos(\pi \alpha/2) + i|\theta|^\alpha \text{sgn}(\theta) \sin(\pi \alpha/2) \right\},
\]

where \( \gamma^\alpha = \cos(\pi \alpha/2) \) in (5), and \( \text{sgn}(\theta) = \theta/|\theta| \) for \( \theta \neq 0 \), and 0 otherwise. As \( \alpha \to 1 \), \( \phi(\theta) \to \exp(i\theta) \), so \( \lim_{\alpha \to 1} Z_{\alpha} = 1 \). It follows that the limit \( Y_1 = \lim_{\alpha \to 1} Y_\alpha \) exists.

Next, we show that the moment generating function (m.g.f.) of \( Y_\alpha \), defined by \( \mathbb{E}\exp(\theta Y_\alpha) \), exists for \( \theta > 0 \), and use a result in [26] to conclude that the c.f. of \( Y_\alpha \) equals the m.g.f. evaluated at \( i\theta \). For \( \theta > 0 \),

\[
\mathbb{E}\exp(\theta Y_\alpha) = (1 - \alpha)^\theta e^{\theta/(1-\alpha)} \mathbb{E}\exp\{-\theta Z_{\alpha}/(1 - \alpha)\} = (1 - \alpha)^\theta e^{-[\theta/(1-\alpha)]^\alpha + \theta/(1-\alpha)},
\]

where the last equality is given by the Laplace transform of \( Z_{\alpha} \). So, the c.f. of \( Y_\alpha \) is then given by

\[
\exp \left\{ i\theta \left[ 1/(1 - \alpha) + \log(1 - \alpha) \right] - [i\theta/(1-\alpha)]^\alpha \right\}.
\]

Letting \( \alpha \to 1 \) in (7), we have the desired limit. The moments of \( \exp(Y_\alpha) \) follow by taking limits as \( \alpha \to 1 \) in (6).

**Proof of Lemma 3.1.** Consider the following transformation: \( w_j = e^{\zeta y_j} = e^{\zeta \delta + \zeta z_j} = e^{\zeta \delta} e^{\zeta z_j} \), where \( z_j \sim G(z; 0) \) i.i.d. have characteristic function \( \phi(\theta) = \mathbb{E}\exp(i\theta z_j) = (i\theta)^\theta \), for \( \theta \in \mathbb{R} \). Then, from Lemma 3.1 \( \mathbb{E}w_j = e^{\zeta \delta} \zeta^\delta \). Let \( \eta = e^{\zeta \delta} \). The estimator \( \hat{\eta} = \zeta^{-\delta} e^{\zeta \delta} k^{-1} \sum_{j=1}^k w_j \) is unbiased for \( \eta \) and has variance \( \eta^2 k^{-1}(4^\zeta - 1) \). Moreover, by the Central Limit Theorem, as \( k \to \infty \),

\[
\sqrt{k} \eta^{-1}(\hat{\eta} - \eta) \to \text{Normal}(0, 4^\zeta - 1).
\]

The log-mean estimator of \( \delta \) is

\[
\hat{\delta}_{lm} = \zeta^{-1} \log \hat{\eta} = \zeta^{-1} \log \left( \zeta^{-\delta} e^{\zeta \delta} k^{-1} \sum_{j=1}^k \exp(\zeta y_j) \right).
\]

By the Delta Method applied to (8), as \( k \to \infty \), \( \sqrt{k}(\hat{\delta}_{lm} - \delta) \to \text{Normal}(0, \zeta^{-2}(4^\zeta - 1)) \). The small-sample bias of \( \hat{\delta}_{lm} \) equals

\[
\mathbb{E}\hat{\delta}_{lm} - \delta = \zeta^{-1} \mathbb{E} \log \left( \zeta^{-\delta} e^{\zeta \delta} k^{-1} \sum_{j=1}^k \exp(\zeta z_j) \right).
\]
Finally, we want to find the optimal value of $\zeta$ that maximizes the ARE of $\hat{\delta}_{lm}$ relative to the MLE of $\delta$. The Cramér-Rao lower bound for estimating $\delta$ is approximately $(0.3445)^{-1}$, so the ARE is $\zeta^2/[0.3445(4^\zeta - 1)]$. This is a concave function that attains a maximum value of 0.978 when $\zeta \approx 1.15$. When $\zeta = 1.0$, the ARE evaluates to 0.968.

Proof of Lemma 3.2. For $\epsilon > 0$ and $t > 0$,  
\[
P(\hat{\delta}_{lm} - \delta \geq \epsilon) = P\left(\epsilon^{-\zeta}k^{-1} \sum_{j=1}^{k} \exp(\epsilon z_j) \geq e^{\epsilon t}\right) \leq \exp\left\{-t \epsilon e^{\epsilon t}\right\},
\]
by the Chernoff bound [17], provided the right hand side converges. Define $T_j = t \epsilon^{-\zeta} e^{\epsilon z_j}$, $j = 1, \ldots, k$. Then,
\[
E \exp \left(\sum_{j=1}^{k} T_j\right) = \left(E \exp(T_1)\right)^k = \left(\sum_{j=0}^{\infty} \frac{t^j \epsilon^j}{j!}\right)^k.
\]
By the Ratio Test, the series is absolutely convergent for all $t > 0$ if $0 < \zeta < 1$, and for $0 < t < e^{-1}$ if $\zeta = 1$. If $\zeta > 1$, the series is divergent. Define $T \equiv \{t; t > 0\}$ for $0 < \zeta < 1$, and $T \equiv \{t; 0 < t < e^{-1}\}$ for $\zeta = 1$. It follows that, if $\zeta \leq 1$, then $\hat{\delta}_{lm}$ has an exponentially decreasing right tail bound that satisfies
\[
P(\hat{\delta}_{lm} - \delta \geq \epsilon) < \exp\left(-k \frac{\epsilon^2}{G_R}\right),
\]
where
\[
\frac{\epsilon^2}{G_R} = \sup_{t \in T} \left\{-\log \left(\sum_{j=0}^{\infty} \frac{t^j \epsilon^j}{j!}\right) + t \epsilon e^{\epsilon t}\right\}.
\]
(9)
It is straightforward to show that the function maximized in (9) is concave. The result follows similarly for the left tail bound. Furthermore by expanding the series in (9) for small values of $t$ we can show that as $\epsilon \to 0$ both $G_R$ and $G_L$ converge to $2(4^\zeta - 1)/\zeta^2$. The details are as follows.

Define
\[
M_\zeta(t) = \sum_{j=0}^{\infty} \frac{t^j \epsilon^j}{j!},
\]
and consider
\[
K_\zeta(s, \epsilon) = \left(M_\zeta(s \epsilon) \exp\left(-s \epsilon e^{\epsilon t}\right)\right)^{1/\epsilon^2}, \quad s > 0.
\]
$K_\zeta(s, \epsilon)$ is a convex function [17], so it follows that $\inf_{s > 0} K_\zeta(s, \epsilon) \to \inf_{s > 0} K_\zeta^*(s)$, where $K_\zeta^*(s)$ is the pointwise limit of $K_\zeta(s, \epsilon)$ as $\epsilon \to 0$, provided this limit exists.

Furthermore, since $1/G_R = -\log \left(\inf_{s > 0} K_\zeta(s, \epsilon)\right)$, it follows that
\[
\lim_{\epsilon \to 0} G_R = -\frac{1}{\log(\inf_{s > 0} K_\zeta^*(s))}.
\]
To establish the pointwise limit, first note that if $s \epsilon \in T$, then
\[
\sum_{j=3}^{\infty} \frac{(s \epsilon)^j \epsilon^j}{j!} \leq \epsilon^3 \sum_{j=3}^{\infty} \frac{s^j \epsilon^j}{j!} = o(\epsilon^2).
\]
So that expanding in powers of $\epsilon$, we have that

$$\log(K_\zeta(s, \epsilon)) = \frac{1}{\epsilon^2} \left[ \log\left(1 + s\epsilon + \frac{(s\epsilon)^2}{2} + o(\epsilon^3)\right) - s\epsilon(1 + \zeta \epsilon) + o(\epsilon^3) \right]$$

$$= \frac{s^2}{2} \{4\zeta - 1\} = K_\zeta^*(s)$$

Differentiating with respect to $s$, we obtain

$$\inf_{s>0} K_\zeta^*(s) = -\frac{\zeta^2}{2(4\zeta - 1)},$$

as required, where the convexity ensures a unique minimum.

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