Second Variation of $F$-Einstein-Hilbert Functional

Ahmed Mohammed Cherif*

Abstract

This article describes a formula for second variation of generalized Einstein-Hilbert functional on Riemannian manifolds. This work extends the definition of stable Einstein manifolds, and we present some properties.

Keywords: Einstein manifolds, Einstein-Hilbert functional.
Subjclass: 53C25, 83C05.

1 Introduction

The Einstein-Hilbert functional $\mathcal{E}$ associates to each Riemannian metric $g$ the integral of its scalar curvature $S$, that is

$$\mathcal{E} : \mathcal{M} \longrightarrow \mathbb{R}, \quad g \longmapsto \mathcal{E}(g) = \int_{M} Sv^{g},$$

where $\mathcal{M}$ is the set of smooth Riemannian metrics on $M$, and $v^{g}$ the volume form with respect to $g$. It is the action functional that defines the dynamics of gravity in general relativity [3, 4, 5, 6, 8, 16].

One of the simplest modifications to general relativity is the $F(S)$ gravity in which the Lagrangian density $F$ is an arbitrary smooth function of the scalar curvature $S$ of a Riemannian manifold $(M, g)$. When $F(s) = s$, gives the classical Einstein-Hilbert functional, therefore the Einstein gravity, corresponds to $F(S) = S$. The Euler-Lagrange equation of the generalized Einstein-Hilbert functional (it is known by Einstein-Hilbert functional in $f(R)$ gravity, or briefly $F$-Einstein-Hilbert functional) with respect to $g$ is proved by A. D. Felice, S. Tsujikawa in [7], and T. P. Sotiriou, V. Faraoni in [17].

*University Mustapha Stamboul Mascara, Faculty of Exact Sciences, Mascara 29000, Algeria. Email: a.mohammedcherif@univ-mascara.dz
The second variation of Einstein-Hilbert functional at Einstein metrics was considered in [11]. In [9], K. Kröncke study the second variation of the Einstein-Hilbert functional on Einstein metrics, he find some conditions for stability of Einstein manifolds with respect to the Einstein-Hilbert functional, i.e., that the second variation of the Einstein-Hilbert functional at the metric is nonpositive in the direction of transverse-traceless tensors. Stability properties of compact Riemannian Einstein manifold play a role in mathematical general relativity [1], and in geometric analysis to understand rigidity of Riemannian structures, for example the dynamical behaviour of the Ricci flow.

In this paper, we extend the definition of the Einstein tensor, where we calculate the first variation of the $F$-Einstein-Hilbert functional, and we conclude the generalized Einstein tensor. We prove that the generalized Einstein tensor is divergence-free. We study the second variation of the $F$-Einstein-Hilbert functional on the Riemannian manifold. The second variation formula gives a tool/is a prerequisite for the study the stability of any generalized Einstein manifold, and to see if the $F$-Einstein-Hilbert functional has extremality properties at some critical points. The smooth function $F$ can be chosen for the existence and the stability of such Riemannian metrics which provide additional information on Riemannian manifolds.

2 $F$-Einstein-Hilbert functional

First, we give some definitions. Let $(M, g)$ be an $n$-dimensional Riemannian manifold, and let $X, X_1, \ldots, X_{q-1}, Y, Z \in \Gamma(TM)$. By $R$, $\text{Ric}$ and $S$ we denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of $(M, g)$. Thus $R$, $\text{Ric}$ and $S$ are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$\text{Ric}(X, Y) = g(R(X, e_i)e_i, Y), \quad S = \text{Ric}(e_i, e_i),$$

where $\nabla$ is the Levi-Civita connection with respect to $g$, $\{e_1, \ldots, e_n\}$ is an orthonormal frame.

Given a smooth function $f$ on $M$, the gradient of $f$ is defined by

$$g(\operatorname{grad} f, X) = X(f),$$

the Hessian of $f$ is defined by

$$(\operatorname{Hess} f)(X, Y) = g(\nabla_X \operatorname{grad} f, Y),$$
the Laplacian of $f$ is defined by
\[ \Delta f = -\text{Tr} (\text{Hess } f). \]  
(2.5)

The divergence of $(0,q)$-tensor $\alpha$ on $M$ is defined by
\[ (\delta \alpha)(X_1, ..., X_{q-1}) = -(\nabla_{e_i} \alpha)(e_i, X_1, ..., X_{q-1}). \]  
(2.6)

The formal adjoint of the divergence $\delta : \Gamma(\otimes^2 T^*M) \to \Gamma(T^*M)$ is the map $\delta^* : \Gamma(T^*M) \to \Gamma(\otimes^2 T^*M)$ defined by
\[ (\delta^* \alpha)(X,Y) = \frac{1}{2}(\nabla_X \alpha)Y + (\nabla_Y \alpha)X. \]  
(2.7)

The formal adjoint of the Levi-civita connection $\nabla$ is given by
\[ (\nabla^* \alpha)(X_1, ..., X_{q-1}) = -(\nabla_{e_i} \alpha)(e_i, X_1, ..., X_{q-1}), \]  
(2.8)

where $\alpha \in \Gamma(T^*M \otimes T^{(p,q)}M)$, and $\{e_1, ..., e_n\}$ is an orthonormal frame.

The composition of $T,Q \in \Gamma(\otimes^2 T^*M)$ is defined by
\[ (T \circ Q)(X,Y) = T(X,e_i)Q(Y,e_i), \]  
(2.9)

where $\{e_1, ..., e_n\}$ is an orthonormal frame on $M$.

For $T \in \Gamma(\otimes^2 T^*M)$, we define $T^\sigma \in \Gamma(\otimes^2 T^*M)$ by
\[ T^\sigma(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X)). \]  
(2.10)

We define an endomorphism $\hat{R} : \Gamma(\otimes^2 T^*M) \to \Gamma(\otimes^2 T^*M)$ by
\[ (\hat{R}T)(X,Y) = T(R(e_i,X)Y,e_i). \]  
(2.11)

For $T \in \Gamma(\otimes^2 T^*M)$, we define the Lichnerowicz Laplacian by
\[ \Delta_L T = \nabla^* \nabla T + \text{Ric} \circ T + T \circ \text{Ric} - 2\hat{R}T. \]  
(2.12)

(For more details, see for example [4, 15]).

**Definition 1 ([7, 17]).** We let $\mathcal{M}$ denote the space of Riemannian metrics on a closed orientable manifold $M$. The generalized Einstein-Hilbert functional (or $F$-Einstein-Hilbert functional) is defined by
\[ \mathcal{E}_F : \mathcal{M} \to \mathbb{R}, \quad g \mapsto \mathcal{E}_F(g) = \int_M F(S) v^g, \]  
(2.13)

where $S$ is the scalar curvature of $(M,g)$, and $F : \mathbb{R} \to \mathbb{R}$ is a non-constant smooth function.
The Definition 1, is a natural generalization of Einstein-Hilbert functional or the total scalar curvature, when $F$ is the identity map, then $E_F$ reduces to the usual Einstein-Hilbert functional whose second order infinitesimal behaviour is well understood (see [3, 4, 5, 6, 10, 11, 12, 13, 16]). Let $(M, g)$ be a closed orientable Riemannian manifold. Consider a smooth one-parameter variation of the metric $g$, i.e., a smooth family of metrics $(g_t)$ with $-\epsilon < t < \epsilon$, such that $g_0 = g$. Take local coordinates $(x^i)$ on $M$, and write the metric on $M$ in the usual way as $g_{ij}(t, x) dx^i \otimes dx^j$. Write $h = (\partial g_t / \partial t)_{t=0}$, then $h \in \Gamma(\otimes^2 T^*M)$ is a symmetric 2-covariant tensor field on $M$, we get the following.

**Theorem 2** ([7], [17]). The first variation of the $F$-Einstein-Hilbert functional in the direction of $h$ is given by the formula

$$\frac{d}{dt} E_F(g_t) \bigg|_{t=0} = - \int_M \langle E_F(g), h \rangle v^g, \quad (2.14)$$

where $\langle , \rangle$ is the induced Riemannian metric on $\otimes^2 T^*M$, 

$$E_F(g) = F'(S) \text{Ric} - \text{Hess} F'(S) - (\Delta F'(S) + \frac{1}{2} F(S)) g, \quad (2.15)$$

and $F'$ is the derivative of the function $F$.

**Definition 3.** $E_F(g)$ is called the generalized Einstein tensor (or $F$-Einstein tensor).

For the proof of Theorem 2, we need the following lemma.

**Lemma 4** ([14], [18]). Let $(M, g)$ be a Riemannian manifold. Then, the differential at $g$, in the direction of $h$, of the volume element and the scalar curvature are given by the following formulas

$$\frac{\partial v^g_t}{\partial t} \bigg|_{t=0} = \frac{1}{2} (\text{Tr } h) v^g = \frac{1}{2} \langle g, h \rangle v^g, \quad (2.16)$$

$$\frac{\partial S_t}{\partial t} \bigg|_{t=0} = \Delta (\text{Tr } h) + \delta (\delta h) - \langle \text{Ric}, h \rangle. \quad (2.17)$$

**Proof of Theorem 2**. First note that

$$\frac{d}{dt} E_F(g_t) \bigg|_{t=0} = \int_M \left[ \frac{\partial F(S_t)}{\partial t} v^g_t + F(S_t) \frac{\partial v^g_t}{\partial t} \right]_{t=0}, \quad (2.18)$$

for all $t \in (-\epsilon, \epsilon)$, we have

$$\frac{\partial F(S_t)}{\partial t} = F'(S_t) \frac{\partial S_t}{\partial t},$$
by the Lemma [4] we obtain

$$
\frac{\partial F(S_t)}{\partial t} \bigg|_{t=0} = F'(S) \Delta (\text{Tr} h) + F'(S) \delta(\delta h) - F'(S) \langle \text{Ric}, h \rangle.
$$

(2.19)

Calculating in a normal frame at \( x \in M \) we have

$$
F'(S) \Delta (\text{Tr} h) = -F'(S) e_i (e_i (\text{Tr} h))
= -e_i (F'(S) e_i (\text{Tr} h)) + e_i (F'(S)) e_i (\text{Tr} h)
= -e_i (F'(S)) e_i (\text{Tr} h) + e_i (e_i (F'(S)) \text{Tr} h)
= -e_i (e_i (F'(S))) \text{Tr} h,
$$

(2.20)

so, the first term in the right-hand side of (2.19), is given by

$$
F'(S) \Delta (\text{Tr} h) = \delta(F'(S)d(\text{Tr} h)) - \delta((\text{Tr} h)dF'(S))
+ \Delta(F'(S))\langle g, h \rangle.
$$

(2.21)

If \( f \in C^\infty(M) \) and \( \alpha \in \Gamma(T^*M) \), then (see [18], [15])

$$
\delta(f \alpha) = -\langle df, \alpha \rangle + f \delta \alpha,
$$

(2.22)

with \( \langle df, \alpha \rangle = \alpha(\text{grad} f) \). Applying this formula, gives

$$
F'(S) \delta(\delta h) = \delta(F'(S) \delta h) + \langle dF'(S), \delta h \rangle,
$$

(2.23)

by using the following formula (see [18])

$$
(\delta T)(Z) = \delta(T(\cdot, Z)) + \frac{1}{2} \langle T, \mathcal{L}_Z g \rangle,
$$

(2.24)

where \( \mathcal{L}_Z g \) is the Lie-derivative of \( g \) along \( Z \in \Gamma(TM) \) (see [15]), and \( T \in \Gamma(\otimes^2 T^*M) \), we get

$$
\langle dF'(S), \delta h \rangle = (\delta h)(\text{grad} F'(S))
= \delta(h(\cdot, \text{grad} F'(S))) + \frac{1}{2} \langle h, \mathcal{L}_{\text{grad} F'(S)} g \rangle
= \delta(h(\cdot, \text{grad} F'(S))) + \langle h, \text{Hess} F'(S) \rangle,
$$

(2.25)

by equations (2.23) and (2.25), the second term on the left-hand side of (2.19) is

$$
F'(S) \delta(\delta h) = \delta(F'(S) \delta h) + \delta(h(\cdot, \text{grad} F'(S)))
+ \langle h, \text{Hess} F'(S) \rangle.
$$

(2.26)
Substituting (2.21) and (2.26) in (2.19), we obtain
\[
\frac{\partial F(S_t)}{\partial t} \bigg|_{t=0} = \delta(F'(S)d(\text{Tr } h)) - \delta((\text{Tr } h)dF'(S)) + \Delta(F'(S))\langle g, h \rangle + \delta(F'(S)\delta h) + \delta(h(\cdot, \text{grad } F'(S))) + \langle h, \text{Hess } F'(S) \rangle - F'(S)\langle \text{Ric}, h \rangle.
\] (2.27)

From equation (2.27) and the Lemma 4, we have
\[
\left[ \frac{\partial F(S_t)}{\partial t} v^g + F(S_t) \frac{\partial v^g}{\partial t} \right]_{t=0} = \left\{ \delta(F'(S)d(\text{Tr } h)) - \delta((\text{Tr } h)dF'(S)) + \Delta(F'(S))\langle g, h \rangle + \delta(F'(S)\delta h) + \delta(h(\cdot, \text{grad } F'(S))) + \langle h, \text{Hess } F'(S) \rangle - F'(S)\langle \text{Ric}, h \rangle \right\} v^g + \frac{F(S)}{2} \langle g, h \rangle v^g.
\] (2.28)

Substituting the formula (2.28) in (2.18), and consider the divergence theorem (see [2]), the Theorem 2 follows.

Remark 5. Let $X, Y \in \Gamma(TM)$, we have
\[
\text{Hess } F'(S)(X,Y) = X(Y(F'(S))) - (\nabla_X Y)(F'(S)) = X(F''(S)Y(S)) - F''(S)(\nabla_X Y)(S) = X(F'''(S)Y(S) + F''(S)Y(Y(S)) - F''(S)(\nabla_X Y)(S)) = F'''(S)X(Y(S) + F''(S)(\text{Hess } S)(X,Y).
\]

According to this formula, the $F$-Einstein tensor is given by
\[
E_F(g) = F'(S)\text{Ric} - F''(S)\text{Hess } S - F'''(S)dS \otimes dS - (F''(S)\Delta S + F'''(S)|\text{grad } S|^2 + \frac{1}{2}F(S))g.
\] (2.29)

Remark 6. Let $(M,g)$ be a Riemannian manifold, we get the following

- If $F(s) = s$, for all $s \in \mathbb{R}$, the $F$-Einstein tensor is given by the formula (see [3], [14])
\[
E_F(g) = E(g) = \text{Ric} - \frac{S}{2}g.
\] (2.30)

is the Einstein tensor.
• If \( F(s) = s^2 \), for all \( s \in \mathbb{R} \), the \( F \)-Einstein tensor is given by (see [3], [5], [6])

\[
E_F(g) = 2S \text{Ric} - 2 \text{Hess} S - \left( 2\Delta S + \frac{S^2}{2} \right) g.
\]  

(2.31)

From Theorem 2

Theorem 7 ([7], [17]). A Riemannian metric \( g \) is a critical point of the \( F \)-Einstein-Hilbert functional if and only if

\[
F'(S) \text{Ric} - \text{Hess} F'(S) - \left( \Delta F'(S) + \frac{1}{2} F(S) \right) g = 0,
\]  

(2.32)

where \( F: \mathbb{R} \rightarrow \mathbb{R} \) is a non-constant smooth function.

By taking traces in (2.32), we obtain

\[
SF'(S) + (1 - n) \Delta F'(S) - \frac{n}{2} F(S) = 0.
\]  

(2.33)

Theorem 8. Let \((M, g)\) be a Riemannian manifold. Then, the divergence of the generalized Einstein tensor is zero (that is, \( \delta E_F(g) = 0 \)).

Proof. Let \( F: \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function, calculating in a normal frame \( \{e_i\} \) at \( x \in M \), with \( X = e_j \), we have

\[
\delta E_F(g)(X) = -(\nabla_{e_i} E_F(g))(e_i, X) = -e_i(E_F(g)(e_i, X)),
\]  

(2.34)

by the definitions of generalized Einstein tensor, and the Hessian tensor, we get

\[
E_F(g)(e_i, X) = F'(S) \text{Ric}(e_i, X) - g(\nabla_{e_i} \text{grad} F'(S), X)

- (\Delta F'(S) + \frac{1}{2} F(S)) g(e_i, X),
\]  

(2.35)

substituting (2.35) in (2.34), and consider the definition of gradient operator, we obtain

\[
\delta E_F(g)(X) = - \text{Ric}(\text{grad} F'(S), X) - F'(S) e_i (\text{Ric}(e_i, X))

+ g(\nabla_{e_i} \nabla_X \text{grad} F'(S), e_i) + X(\Delta F'(S)) + \frac{1}{2} X(F(S)),
\]

by the definitions of the divergence, and the curvature tensor, with \([e_i, X] = 0\), we conclude that

\[
\delta E_F(g)(X) = - \text{Ric}(\text{grad} F'(S), X) + F'(S)(\delta \text{Ric})(X)

+ g(R(e_i, X) \text{grad} F'(S), e_i) + g(\nabla_X \nabla_{e_i} \text{grad} F'(S), e_i)

+ X(\Delta F'(S)) + \frac{1}{2} X(F(S)),
\]  

(2.36)
note that
\[ \text{Ric}(\text{grad} F'(S), X) = g(R(e_i, X) \text{grad} F'(S), e_i), \]  
(2.37)
\[ F'(S)(\delta \text{Ric})(X) = -\frac{1}{2}X(F(S)) = -\frac{1}{2}F'(S)X(S), \]  
(2.38)
\[ g(\nabla_X \nabla_{e_i} \text{grad} F'(S), e_i) = -X(\Delta F'(S)). \]  
(2.39)
Substituting the formulas (2.37), (2.38) and (2.39) in (2.36), the Theorem 8 follows. \( \Box \)

Remark 9.

• If \( E_F(g) = fg \) for some function \( f \) on \( M \), then \( f \) is constant function on \( M \) (because \( \delta E_F(g) = 0 \)).

• The condition \( E_F(g) = \lambda g \) is equivalent to
\[ F'(S) \text{Ric} - \text{Hess} F'(S) = \mu g, \]  
(2.40)
for some function \( \mu \) on \( M \), it is also equivalent to
\[ F'(S) \text{Ric} - F''(S) \text{Hess} S - F'''(S)dS \otimes dS = \mu g, \]  
(2.41)
(see equation (2.24)).

• If \( F(s) = s \), for all \( s \in \mathbb{R} \), then \( E_F(g) = \lambda g \) if \( (M, g) \) is Einstein manifold, that is \( \text{Ric} = \mu g \) for some constant \( \mu \) (see [4]).

Example 10. Let \( M = (0, \infty) \times \mathbb{R}^3 \) equipped with the Riemannian metric \( g = dt^2 + t^2(dx^2 + dy^2 + dz^2) \). Let \( F(s) = s^\alpha \) for some constant \( \alpha \). Then, \( E_F(g) = 0 \) if and only if \( \alpha = \frac{1+\sqrt{3}}{2} \).

Example 11. Let \( M = S^n \subset \mathbb{R}^{n+1} \) and \( F : \mathbb{R} \rightarrow \mathbb{R} \) a non-constant smooth function. Then, the induced Riemannian metric \( g^{S^n} \) is a critical point of the \( F \)-Einstein-Hilbert functional if and only if \( F(s_0) = 0 \) and \( F'(s_0) = 0 \) where \( s_0 = n(n-1) \) is the scalar curvature of \( (S^n, g^{S^n}) \).

Remark 12. The previous examples prove the following results: There is no equivalence between \( E_F(g) = 0 \) and \( E(g) = 0 \) where \( F \) is a non-constant smooth function. There exist Riemannian Einstein metrics which are critical points of the \( F \)-Einstein-Hilbert functional where \( F(s) \neq s \).
3 The second variation of $\mathcal{E}_F$

Let $M$ be a closed orientable manifold. We denote by

$$\mathcal{M}_c = \{ g \in \mathcal{M} \mid \text{Vol}(M, g) = \int_M v^g = c \};$$

for some constant $c > 0$. This is a submanifold of $\mathcal{M}$ of codimension 1, and its tangent space at $g \in \mathcal{M}_c$ is given by

$$T_g \mathcal{M}_c = \{ T \in \Gamma(\otimes^2 T^* M) \mid \int_M \langle g, T \rangle v^g = 0 \}.$$

A Riemannian metric $g$ is a critical point of $\mathcal{E}_F|_{\mathcal{M}_c}$ if and only if $\mathcal{E}_F(g)$ is orthogonal to $T_g \mathcal{M}_c$, that is $\mathcal{E}_F(g) = \lambda g$ for some constant $\lambda$. In the following Theorem, we calculate the second derivative of $\mathcal{E}_F(g_t)$ at $t = 0$ where $(g_t) (-\epsilon < t < \epsilon)$ is a smooth one-parameter variation of such Riemannian metric $g$ which enables us to know the extremality properties of $\mathcal{E}_F$. Write

$$h = \left. \frac{\partial g_t}{\partial t} \right|_{t=0}, \quad k = \left. \frac{\partial^2 g_t}{\partial t^2} \right|_{t=0}, \quad (3.1)$$

then $h, k \in \Gamma(\otimes^2 T^* M)$. Under the notation above we have the following.

**Theorem 13.** Let $(M, g)$ be a closed orientable Riemannian manifold with volume $c$. Suppose that $\mathcal{E}_F(g) = \lambda g$, for some constant $\lambda$, then the second variation of $\mathcal{E}_F|_{\mathcal{M}_c}$ at $g$ in the direction of $h$ is given by

$$\left. \frac{d^2}{dt^2} \mathcal{E}_F(g_t) \right|_{t=0} = \int_M \langle T_0(h) + T_1(h), h \rangle v^g,$$

where $T_0(h), T_1(h)$ are defined by

$$T_0(h) = \left. -\frac{F'(S)}{2} \nabla^* \nabla h + F'(S) R h + F'(S) \delta^* (\delta h) + \frac{1}{2} F'(S) \text{Hess}(\text{Tr} h) \right| \left. + \frac{F'(S)}{2} \left[ \Delta(\text{Tr} h) + \delta (\delta h) \right] g - \frac{1}{2} \left( \lambda + \frac{1}{2} F(S) \right)(\text{Tr} h) g, \right.$$  

$$T_1(h) = \left. -f \text{Ric} + \text{Hess} f + \langle \Delta f \rangle g - h(\nabla \cdot \text{grad} F'(S), \cdot) - (\nabla \cdot \text{grad} F'(S)) \right| \left. + \frac{1}{2} \nabla \text{grad} F'(S) h - \langle \delta h + \frac{1}{2} d(\text{Tr} h), dF'(S) \rangle g - \frac{1}{2} (\Delta F'(S))(\text{Tr} h) g \right| \left. + \frac{1}{2} \langle \text{Hess} F'(S), h \rangle g, \right.$$  

and $f = F''(S) \left[ \Delta(\text{Tr} h) + \delta (\delta h) - (\text{Ric}, h) \right]$.  

9
For the proof of Theorem 13 we need the following lemmas.

**Lemma 14.** Let $T_t, Q_t \in \Gamma(\otimes^2 T^* M)$ all dependent of time $t \in (-\epsilon, \epsilon)$ with $T_0 = T$ and $Q_0 = Q$. Then

$$\frac{\partial}{\partial t} \bigg|_{t=0} \langle T_t, Q_t \rangle_t = \left\langle \frac{\partial T_t}{\partial t} \bigg|_{t=0}, Q \right\rangle + \left\langle T, \frac{\partial Q_t}{\partial t} \bigg|_{t=0} \right\rangle - 2\left\langle T, h \circ Q \right\rangle,$$

where $\langle \cdot, \cdot \rangle_t$ is the induced Riemannian metric (with respect to $g_t$) on $\otimes^2 T^* M$.

**Proof.** We have

$$\langle T_t, Q_t \rangle_t = T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{jb},$$

so that

$$\frac{\partial}{\partial t} \bigg|_{t=0} \langle T_t, Q_t \rangle_t = \left\langle \frac{\partial T_t^{ij}}{\partial t} \bigg|_{t=0} Q_t^{ab} g_t^{ia} g_t^{jb} + T_t^{ij} \frac{\partial Q_t^{ab}}{\partial t} \bigg|_{t=0} g_t^{ia} g_t^{jb} \right\rangle + T_t^{ij} Q_t^{ab} \frac{\partial g_t^{ia}}{\partial t} \bigg|_{t=0} g_t^{jb} + T_t^{ij} Q_t^{ab} g_t^{ia} \frac{\partial g_t^{jb}}{\partial t} \bigg|_{t=0},$$

since $\frac{\partial g_t^{ia}}{\partial t} \bigg|_{t=0} = -g_t^{i u} g_t^{av} h_{uv}$ and $\frac{\partial g_t^{jb}}{\partial t} \bigg|_{t=0} = -g_t^{ju} g_t^{bv} h_{uv}$ (see [14]), we get

$$\frac{\partial}{\partial t} \bigg|_{t=0} \langle T_t, Q_t \rangle_t = \left\langle \frac{\partial T_t^{ij}}{\partial t} \bigg|_{t=0}, Q \right\rangle + \left\langle T, \frac{\partial Q_t}{\partial t} \bigg|_{t=0} \right\rangle - T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{jb} g_t^{av} h_{uv} g_t^{jb} - T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{ju} g_t^{bv} h_{uv},$$

note that

$$- T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{av} h_{uv} g_t^{jb} - T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{ju} g_t^{bv} h_{uv} = -2 T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{av} h_{uv} g_t^{jb},$$

on the other hand

$$-2 \langle T, h \circ Q \rangle = -2 T_t^{ij} (h \circ Q)^{ab} g_t^{ia} g_t^{jb} = -2 T_t^{ij} g_t^{av} h_{uv} Q_t^{ab} g_t^{ia} g_t^{jb}.$$

\hfill \Box

**Lemma 15.** Let $(f_t) (-\epsilon < t < \epsilon)$ be a time dependent family of smooth functions on $M$ with $f_0 = f$. Then, the first variation of the Hessian and the Laplacian are given by

$$\left. \frac{\partial \text{Hess}_t f_t}{\partial t} \right|_{t=0} = \text{Hess} \left. \left( \frac{\partial f_t}{\partial t} \right|_{t=0} \right) - \langle \nabla h \rangle (\cdot, \text{grad } f) + \frac{1}{2} \nabla_{\text{grad } f} h,$$

$$\left. \frac{\partial \Delta_t f_t}{\partial t} \right|_{t=0} = \Delta \left. \left( \frac{\partial f_t}{\partial t} \right|_{t=0} \right) - \langle \delta h + \frac{1}{2} d(\text{Tr } h), df \rangle + \langle \text{Hess}_t f, h \rangle,$$

where $\text{Hess}_t f_t$ (resp. $\Delta_t f_t$) is the Hessian (resp. Laplacian) of $f_t$ with respect to the metric $g_t$. 

10
Proof. By the definition of Hessian (2.4), we obtain
\[
\frac{\partial \text{Hess}_t f_t}{\partial t}(X, Y) = \frac{\partial}{\partial t} \left[ X(Y(f_t)) - (\nabla^t_X Y)(f_t) \right] \\
= \frac{\partial}{\partial t} \left[ X(Y(f_t)) - g(\nabla^t_X Y, \operatorname{grad} f_t) \right] \\
= X(Y(\frac{\partial f_t}{\partial t})) - g(\frac{\partial}{\partial t} \nabla^t_X Y, \operatorname{grad} f_t) - g(\nabla^t_X Y, \frac{\partial f_t}{\partial t}),
\]
(3.2)
where \( \nabla^t \) is the Levi-Civita connection with respect to \( g_t \). The first variation of the Levi-Civita connection in the direction of \( h \) is given by the formula
\[
g(\frac{\partial}{\partial t} \nabla^t_X Y \bigg|_{t=0}, Z) = \frac{1}{2} \left[ (\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) - (\nabla_Z h)(X, Y) \right],
\]
(3.3)
(see [4]). Here \( X, Y, Z \in \Gamma(TM) \) (all independent of time \( t \)). We conclude that
\[
\left. \frac{\partial \text{Hess}_t f_t}{\partial t} \right|_{t=0} = \text{Hess} \left. \left( \frac{\partial f_t}{\partial t} \right|_{t=0} \right) - (\nabla h)(\cdot, \operatorname{grad} f)^\sigma + \frac{1}{2} \nabla_{\operatorname{grad} f} h,
\]
(3.4)
By the Lemma [14], the first variation of \( \Delta_t f_t \) is given by
\[
\left. \frac{\partial \Delta_t f_t}{\partial t} \right|_{t=0} = -\frac{\partial}{\partial t} \left|_{t=0} \langle \text{Hess}_t f_t, g_t \rangle_t \right.
\]
\[= -\langle \frac{\partial}{\partial t} \text{Hess}_t f_t \bigg|_{t=0}, g \rangle - \langle \text{Hess} f, h \rangle + 2 \langle \text{Hess} f, h \circ g \rangle,
\]
(3.5)
by equations (3.4), (3.5), with \( h \circ g = h \), we have
\[
\left. \frac{\partial \Delta_t f_t}{\partial t} \right|_{t=0} = \Delta \left. \left( \frac{\partial f_t}{\partial t} \right|_{t=0} \right) + \operatorname{Tr}(\nabla h)(\cdot, \operatorname{grad} f)^\sigma
\]
\[-\frac{1}{2} \operatorname{Tr} \nabla_{\operatorname{grad} f} h + \langle \text{Hess} f, h \rangle,
\]
(3.6)
and note that
\[
\operatorname{Tr}(\nabla h)(\cdot, \operatorname{grad} f)^\sigma = -\langle \delta h, df \rangle,
\]
(3.7)
\[-\frac{1}{2} \operatorname{Tr} \nabla_{\operatorname{grad} f} h = -\frac{1}{2} \langle d(\operatorname{Tr} h), df \rangle.
\]
(3.8)
The proof is completed. \( \square \)
Proof of Theorem 1.3. First note that

\[
\frac{d^2}{dt^2}E_F(g_t)\bigg|_{t=0} = -\frac{d}{dt}\bigg|_{t=0} \int_M \langle E_F(g_t), \frac{\partial g_t}{\partial t} \rangle_{t=t^0} v^{g_t}, \quad (3.9)
\]

by the variational formulas in Lemma 1.4, we have

\[
\frac{d^2}{dt^2}E_F(g_t)\bigg|_{t=0} = -\int_M \langle \frac{\partial}{\partial t} E_F(g_t)\bigg|_{t=0}, h \rangle v^g
- \int_M \langle E_F(g), k \rangle v^g
+ 2 \int_M \langle E_F(g), h \circ h \rangle v^g
- \frac{1}{2} \int_M \langle E_F(g), (\text{Tr} h) v^g \rangle. \quad (3.10)
\]

Since \( E_F(g) = \lambda g \), we obtain

\[
2 \int_M \langle E_F(g), h \circ h \rangle v^g = 2\lambda \int_M |h|^2 v^g, \quad (3.11)
\]

\[
\frac{1}{2} \int_M \langle E_F(g), h \rangle (\text{Tr} h) v^g = -\frac{\lambda}{2} \int_M (\text{Tr} h)^2 v^g. \quad (3.12)
\]

Since \( \text{Vol}(M, g_t) = c \), we have

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \text{Vol}(M, g_t) = \int_M \frac{\partial^2 v^{g_t}}{\partial t^2} \bigg|_{t=0} = 0, \quad (3.13)
\]

by equation (3.13), and Lemma 4, we get

\[
\frac{1}{2} \int_M \frac{\partial}{\partial t} \bigg|_{t=0} \left[ (\text{Tr} g_t \frac{\partial g_t}{\partial t}) v^{g_t} \right] = 0, \quad (3.14)
\]

where \( \text{Tr} \frac{\partial g_t}{\partial t} \) is the trace of \( \frac{\partial g_t}{\partial t} \) with respect to \( g_t \), from equation (3.14), and the Lemmas 4 and 14, we obtain

\[
0 = \int_M \left[ \frac{\partial}{\partial t} (\text{Tr} g_t \frac{\partial g_t}{\partial t}) \right]_{t=0} v^g + (\text{Tr} h) \frac{\partial v^{g_t}}{\partial t} \bigg|_{t=0}
= \int_M \left[ \frac{\partial}{\partial t} (g_t, \frac{\partial g_t}{\partial t}) \right]_{t=0} v^g + \frac{1}{2} (\text{Tr} h)^2 v^g
= \int_M \left[ |h|^2 + (\text{Tr} k) - 2\langle g, h \circ h \rangle + \frac{1}{2} (\text{Tr} h)^2 \right] v^g
= \int_M \left[ - |h|^2 + (\text{Tr} k) + \frac{1}{2} (\text{Tr} h)^2 \right] v^g, \quad (3.15)
\]
by equation (3.15), the second term on the left-hand side of (3.10) is

\[- \int_M \langle E_F(g), k \rangle v^g = -\lambda \int_M (\text{Tr} k) v^g\]

\[= \int_M \left[ -\lambda |h|^2 + \frac{\lambda}{2} (\text{Tr} h)^2 \right] v^g. \quad (3.16)\]

We compute

\[\frac{\partial}{\partial t} E_F(g_t) \bigg|_{t=0} = \frac{\partial F'(S_t)}{\partial t} \bigg|_{t=0} \text{Ric} + \frac{1}{2} \frac{\partial F'(S_t)}{\partial t} \text{Hess} F'(S_t)\]

\[-\frac{\partial F'(S_t)}{\partial t} \bigg|_{t=0} - \lambda F'(S_t) \bigg|_{t=0} g - \frac{1}{2} F(S) h, \quad (3.17)\]

note that, from the Lemma 4, we have

\[\frac{\partial F(S_t)}{\partial t} \bigg|_{t=0} = F'(S) \left[ \Delta (\text{Tr} h) + \delta(\delta h) - \langle \text{Ric}, h \rangle \right], \quad (3.18)\]

\[\frac{\partial F'(S_t)}{\partial t} \bigg|_{t=0} = F''(S) \left[ \Delta (\text{Tr} h) + \delta(\delta h) - \langle \text{Ric}, h \rangle \right], \quad (3.19)\]

by Lemma 14 and the definition of Lichnerowicz Laplacian, we get

\[\frac{\partial \text{Ric}_t}{\partial t} \bigg|_{t=0} = \frac{1}{2} \Delta h - \delta^* (\delta h) - \frac{1}{2} \text{Hess (Tr h)} - \frac{1}{2} \text{Ric} \circ h + \frac{1}{2} h \circ \text{Ric}\]

\[-\delta^* (\delta h) - \frac{1}{2} \text{Hess (Tr h)}, \quad (3.20)\]

from equations (3.17), (3.18), (3.19), (3.20), and the Lemma 15, we have

\[\frac{\partial}{\partial t} E_F(g_t) \bigg|_{t=0} = \left[ f \text{Ric} + F''(S) \left[ \frac{1}{2} \nabla^* \nabla h - \hat{\text{Ric}} h + \frac{1}{2} \text{Ric} \circ h + \frac{1}{2} h \circ \text{Ric} \right. \right. \]

\[-\delta^* (\delta h) - \frac{1}{2} \text{Hess (Tr h)} \left] - \text{Hess } f + (\nabla, \text{grad } F''(S)) \sigma \right. \]

\[-\frac{1}{2} \nabla \text{grad } F''(S) h - (\Delta f) g + \langle \delta h + \frac{1}{2} d(\text{Tr h}), dF''(S) \rangle g \]

\[+ \langle \text{Hess } F''(S), h \rangle g - (\Delta F''(S)) h - \frac{F''(S)}{2} \left[ \Delta (\text{Tr h}) + \delta(\delta h) - \langle \text{Ric}, h \rangle \right] g - \frac{1}{2} F(S) h, \quad (3.21)\]
where \( f = F''(S)\left[\Delta(T_r h) + \delta(\delta h) - \langle \text{Ric}, h \rangle\right] \).

Note that, from the definitions of the composition (2.9) and the \( F \)-Einstein tensor (2.15), and the condition \( E_F(g) = \lambda g \), we get

\[
\frac{F''(S)}{2} (\text{Ric} \circ h + h \circ \text{Ric}) = \left[ \lambda + \Delta F'(S) + \frac{1}{2} F(S) \right] h + h(\nabla \text{grad} F'(S), \cdot)^\sigma, \tag{3.22}
\]

\[
\frac{F'(S)}{2} \langle \text{Ric}, h \rangle g = \frac{1}{2} \left[ \lambda + \Delta F'(S) + \frac{1}{2} F(S) \right] (\text{Tr} h) g + \frac{1}{2} \langle \text{Hess} F'(S), h \rangle g. \tag{3.23}
\]

From equations (3.10), (3.11), (3.12), (3.16), (3.21), (3.22) and (3.23), we have

\[
\frac{d^2}{dt^2} E_F(g_t) \bigg|_{t=0} = \int_M \left\langle - f \text{Ric} - \frac{F'(S)}{2} \nabla^* \nabla h + F'(S) \circ h, \partial_h \circ h \right\rangle + h(\nabla \text{grad} F'(S), \cdot)^\sigma + F'(S) \delta^*(\delta h) + \frac{1}{2} F'(S) \text{Hess}(\text{Tr} h) + \text{Hess} f - (\nabla h)(\cdot, \text{grad} F'(S))^\sigma + \frac{1}{2} \nabla \text{grad} F'(S) h + (\Delta f) g - \langle \delta h + \frac{1}{2} d(\text{Tr} h), dF'(S) \rangle g + \frac{F'(S)}{2} [\Delta(\text{Tr} h) + \delta(\delta h)] g - \frac{1}{2} [\lambda + \Delta F'(S)] (\text{Tr} h) g + \frac{1}{2} \langle \text{Hess} F'(S), h \rangle g, h \rangle v^g, \tag{3.24}
\]

the Theorem follows from equation (3.24).

**Remark 16.** If \( F(s) = s \), for all \( s \in \mathbb{R} \). Note that, the condition \( E_F(g) = \lambda g \) is equivalent to \( \text{Ric} = [\lambda + \frac{S}{2}] g \). That is, \( g \) is Einstein Riemannian metric with constant \( \mu = \lambda + \frac{S}{2} \). In this case, we have

\[
T_0(h) = -\frac{1}{2} \nabla^* \nabla h + \circ h + \delta^*(\delta h) + \frac{1}{2} \text{Hess}(\text{Tr} h) + \frac{1}{2} [\Delta(\text{Tr} h) + \delta(\delta h)] g - \frac{\mu}{2} (\text{Tr} h) g,
\]

and \( T_1(h) = 0 \). From the formula

\[
\langle \text{Tr} h \rangle \delta(\delta h) = \delta(\langle \text{Tr} h \rangle \delta h) + \delta(h(\cdot, \text{grad}(\text{Tr} h))) + \langle \text{Hess}(\text{Tr} h), h \rangle,
\]

14
and the divergence theorem (see [2]), the second variation of $\mathcal{E}_F|_{\mathcal{M}}$ at $g$ in the direction of $h$ is given by (see [4], [9])

$$
\left. \frac{d^2}{dt^2} \mathcal{E}_F(g_t) \right|_{t=0} = \int_M \left\langle -\frac{1}{2} \nabla^* \nabla h + \circ Rh + \delta^*(\delta h) + \frac{1}{2} \Delta (\text{Tr} h) g + \delta(\delta h) g - \frac{\mu}{2} (\text{Tr} h) g, h \right\rangle v^g.
$$

**Definition 17.** A Riemannian manifold $(M, g)$ is said to be $F$-Einstein if $E_F(g) = \lambda g$ for some constant $\lambda$, where $F : \mathbb{R} \to \mathbb{R}$ is a non-constant smooth function. We call $\lambda$ the $F$-Einstein constant of $g$. We say that a closed orientable $F$-Einstein manifold is stable (resp. strictly stable) if for any $h \in TT = \text{Tr}^{-1}(0) \cap \delta^{-1}(0)$ (such tensors are called transverse traceless or $TT$-tensors)

$$
\mathcal{E}_F''(h) = \int_M \left\langle \hat{T}_0(h) + \hat{T}_1(h), h \right\rangle v^g \leq 0 \quad (\text{resp.} < 0),
$$

where $\hat{T}_0, \hat{T}_1$ are the restrictions of $T_0, T_1$ to $TT$ respectively, given by

$$
\hat{T}_0(h) = -\frac{F'(S)}{2} [\nabla^* \nabla h - 2 \circ Rh],
$$

$$
\hat{T}_1(h) = -\hat{f} \text{Ric} + \frac{1}{2} \nabla_{\text{grad} F'}(S) h,
$$

and $\hat{f} = f|_{TT} = -F''(S)(\text{Ric}, h)$.

**Remark 18.**

- In the Definition [7], $\text{Tr}^{-1}(0)$ (resp. $\delta^{-1}(0)$) denotes the space of symmetric $(0,2)$-tensor fields, whose trace (resp. divergence) vanishes on $(M, g)$.

- By using $\delta h = 0$ and symmetry of $h$, we obtain the following formulas

$$
\langle \text{Hess} \hat{f}, h \rangle = -\delta [h(\text{grad} \hat{f}, \cdot)];
$$

$$
-\langle h(\nabla \cdot \text{grad} F'(S), \cdot), h \rangle - \langle (\nabla h)(\cdot, \text{grad} F'(S))^\sigma, h \rangle
$$

$$
= \delta [(h \circ \text{grad} F'(S), \cdot)] .
$$

This explains the disappearance of these terms in $\langle \hat{T}_1(h), h \rangle$ after integration over $M$. 15
• **The Definition** is a natural generalization of stable Einstein manifold (see \[4, 9, 11, 12\]).

• We call the operator \( \Delta^F_E(h) = -2(\hat{T}_0(h) + \hat{T}_1(h)) \) the F-Einstein operator. Thus, an F-Einstein manifold \((M, g)\) is stable, if the F-Einstein operator is nonnegative on TT-tensors, and strictly stable if it is positive on TT-tensors. If \( F(s) = s \) for all \( s \in \mathbb{R} \), then the F-Einstein operator reduces to the usual Einstein operator \( \Delta_E(h) = \nabla^*\nabla h - 2\hat{R}h \).

**Theorem 19.** Let \( F \in C^\infty(\mathbb{R}) \). We assume that \( F'(s) \geq 0 \) (resp. \( F'(s) > 0 \)) for all \( s \in \mathbb{R} \). Then, any Einstein manifold of negative sectional curvature is stable (resp. strictly stable) F-Einstein manifold.

**Proof.** Let \((M, g)\) be an Einstein manifold with Einstein constant \( \mu \), i.e., \( \text{Ric} = \mu g \). Thus, \((M, g)\) is F-Einstein manifold with F-Einstein constant \( \lambda = \mu F'(S) - \frac{1}{2} F(S) \). Moreover, \( \hat{f} = 0 \) and the F-Einstein operator becomes

\[
\Delta^F_E(h) = F'(S)[\nabla^*\nabla h - 2\hat{R}h].
\]

Hence, if \( F'(S) \geq 0 \) (resp. \( F'(S) > 0 \)) and the sectional curvature of \((M, g)\) is negative, then \((M, g)\) is stable (resp. strictly stable) F-Einstein manifold (see \[9, 10\]).

**Remark 20.** Let \((M, g)\) be an F-Einstein manifold with F-Einstein constant \( \lambda \). We assume that \((M, g)\) has constant scalar curvature. If \( F'(S) > 0 \), according to \([2, 15]\), the Riemannian manifold \((M, g)\) is Einstein with Einstein constant \( \mu = F'(S)^{-1} \left( \lambda + \frac{1}{2} F(S) \right) \). Moreover, if the sectional curvature of \((M, g)\) is negative, then \((M, g)\) is strictly stable. Here, if the manifold \( M \) is even-dimensional, by using \([2, 33]\) with \( E_F(g) = \lambda g \), we can consider the smooth function \( F(s) = -2\lambda + c s^{n/2} \) for some \( c \in \mathbb{R} \).

**Theorem 21.** Let \( F \in C^\infty(\mathbb{R}) \) and \((M, g)\) be a closed orientable F-Einstein manifold of constant sectional curvature \( c > 0 \). We assume that \( F'(s) \geq 0 \) and \( F''(s) \leq 0 \) for all \( s \in \mathbb{R} \). Then, \((M, g)\) is stable. Moreover, if \( F'(s) > 0 \) for all \( s \in \mathbb{R} \), then \((M, g)\) is strictly stable.

**Proof.** A straightforward calculation shows that if \( h \in TT \),

\[
-\frac{F'(S)}{2} \langle \nabla^*\nabla h, h \rangle = -\frac{1}{2} \delta [F'(S)\langle \nabla.h, h \rangle] - \frac{1}{2} \langle \nabla_{\text{grad} F'(S)} h, h \rangle - \frac{F'(S)}{2} \text{Tr} \langle \nabla.h, \nabla.h \rangle.
\]  

(3.25)
By using $\text{Tr} \ h = 0$, we find that

$$F'(S) \langle \nabla h, h \rangle = -c F'(S)|h|^2.$$  \hspace{2em} (3.26)

From equations (3.25) and (3.27), we conclude that

$$\mathcal{E}_F''(h) = \int_M \left[ -\frac{1}{2} F'(S) \text{Tr} (\nabla h, \nabla h) - c F'(S)|h|^2 
+ F''(S) \langle \text{Ric}, h \rangle^2 \right] v^g.$$  \hspace{2em} (3.27)

Theorem 21 follows from equation (3.27), the assumptions $F' \geq 0$, $F'' \leq 0$, and $c > 0$.

**Corollary 22.** The $n$-dimensional unit sphere $S^n$ is a strictly stable $F$-Einstein manifold for all $F \in C^\infty(\mathbb{R})$ such that $F' > 0$ and $F'' \leq 0$.

**Conflict of interest statement**

The author declares no conflict of interest.

**References**

[1] L. Andersson, V. Moncrief, *Einstein spaces as attractors for the Einstein flow*, J. Differential Geom., 89 (2011), no. 1, 1-47.

[2] P. Baird, J.C. Wood, *Harmonic Morphisms between Riemannain Manifolds*, Clarendon Press, Oxford, 2003.

[3] M. Berger, *Quelques formules de variation pour une structure riemannienne*, Ann. Sci. Ecole Norm. Sup., 4 (1970), no. 3, 285-294.

[4] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.

[5] E. Calabi, *Extremal Kähler metrics*, Ann. Math. Stud., 102 (1982), 259-290.

[6] E. Calabi, *Extremal Kähler metrics*, II, in: Differential Geometry and Complex Analysis, Springer, Berlin (1985), 95-114.

[7] A. D. Felice, S. Tsujikawa, *$f(R)$ Theories*, arXiv:1002.4928v2 [gr-qc] (2010).
[8] D. Hilbert, *Die grundlagen der Physik*, Nachr. Ges. Wiss. Göttingen (1915), 395-407.

[9] K. Kröncke, *Stability of Einstein Manifolds*, Doctoral thesis, University of Potsdam, 2013.

[10] N. Koiso, *Nondeformability of Einstein metrics*, Osaka J. Math., 15 (1978), 419–433.

[11] N. Koiso, *On the second derivative of the total scalar curvature*, Osaka J. Math., 16 (1979), 413–421.

[12] N. Koiso, *Rigidity and stability of Einstein metrics-the case of compact symmetric spaces*, Osaka J. Math., 17 (1980), 51–73.

[13] N. Koiso, *Rigidity and infinitesimal deformability of Einstein metrics*, Osaka J. Math., 19 (1982), 643–668.

[14] R. Müller, *Differential Harnack Inequalities and the Ricci Flow*, European Mathematical Society, 2006.

[15] O’Neil, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[16] R. M. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in calculus of variations, Lect. 2nd Sess., Montecatini/Italy (1987), Lect. Notes Math. 1365 (1989), 120-154.

[17] T. P. Sotiriou, V. Faraoni, *f(R) theories of gravity*, arXiv:0805.1726v4 [gr-qc] (2010).

[18] P. Topping, *Lectures on the Ricci Flow*. Number 325 in London Mathematical Society Lecture Note Series. Cambridge University Press, October, 2006.