A Smooth Distributed Feedback for Global Rendezvous of Unicycles

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Abstract—This paper presents a solution to the rendezvous control problem for a network of kinematic unicycles in the plane, each equipped with an onboard camera measuring its relative displacement with respect to its neighbors in body frame coordinates. A smooth, time-independent control law is presented that drives the unicycles to a common position from arbitrary initial conditions, under the assumption that the sensing digraph contains a reverse-directed spanning tree. The proposed feedback is very simple, and relies only on the onboard measurements. No global positioning system is required, nor any information about the unicycles’ orientations.

I. INTRODUCTION

This paper investigates rendezvous control of kinematic unicycles. The objective is to design smooth feedbacks for each robot so as to drive the group to a common position from arbitrary initial conditions. An important requisite is that the feedback be local and distributed. In other words, it is required that the feedback depend only on the relative displacement of each robot to its neighbors measured in the robot’s own body frame, so that the feedback can be computed using onboard sensing devices such as cameras or laser systems.

The solution to the rendezvous control problem proposed in this paper is time-independent and it does not require any information about the orientation of the unicycles, not even their relative orientation. To the best of our knowledge, this is the first solution having the property of being local and distributed, continuously differentiable, and time-independent. As we argue below, previous solutions require either time-varying or discontinuous feedback. For simplicity of exposition, the proposed solution relies on the assumption that the sensing digraph of the unicycles is time-invariant. However, it is only required to contain a reverse directed spanning tree, which is the minimal connectivity requirement.

The main difficulty in solving the rendezvous control problem comes from the fact that the unicycles are nonholonomic, in that their velocity is restricted to be parallel to the vehicle’s heading direction. To overcome this difficulty, the solution we present relies on a control structure made of two nested loops. An outer loop treats the vehicles as fully-actuated single integrators with a linear consensus controller providing a reference velocity. Here we leverage existing consensus algorithms for single integrators [1], [2], [3]. The desired velocity computed by the outer loop becomes a reference signal for the inner loop, which assigns local and distributed feedbacks that solve the rendezvous control problem. This methodology is inspired by our previous work in [4], [5] for rendezvous of rigid bodies in three dimensions.

As we illustrate through simulations, the proposed time-independent, continuously differentiable feedback has practical advantages over the time-varying feedback in [6] and the discontinuous feedback in [7] in that it induces a more natural behaviour in the ensemble of unicycles. The feedback in [6] makes the unicycle “wobble” indefinitely, a behaviour which would be unacceptable in practice. The feedback in [7] induces instantaneous changes in direction that are impossible to achieve with realistic implementations.

The paper is organized as follows. In Section II we present the notation and review basic graph theory and stability definitions. In Section III we formulate the rendezvous control problem. The solution of the rendezvous control problem is presented in Section IV together with an intuitive description of its operation. The proof of the main theorem is presented in Section V. Finally, in Section VI we make concluding remarks. Lemmas and claims related to the proof are in the appendix.

II. PRELIMINARIES

A. Notation

We use interchangeably the notation $v = [v_1 \cdots v_n]^\top$ or $(v_1, \ldots, v_n)$ for a column vector in $\mathbb{R}^n$. We denote by $1 \in \mathbb{R}^n$ the vector $(1, \ldots, 1)$. If $v, w$ are vectors in $\mathbb{R}^2$, we denote by $v \cdot w := v^\top w$ their Euclidean inner product, and by $\|v\| := \sqrt{v^\top v}$.

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Let \( \{e_1, e_2, \ldots \} \) denote the natural basis of \( \mathbb{R}^2 \). \( \text{SO}(2) := \{ M \in \mathbb{R}^{2 \times 2} : M^{-1} = M^T, \det(M) = 1 \} \) and let \( \mathbb{S}^1 \) denote the unit circle. If \( \Gamma \) is a closed subset of a geodesically complete Riemannian manifold \( \mathcal{X} \), and \( d : \mathcal{X} \times \mathcal{X} \to [0, \infty) \) is a distance metric on \( \mathcal{X} \), we denote by \( \|x_i\|_\Gamma := \inf_{\gamma \in \Gamma} d(x_i, \gamma) \) the point-to-set distance of \( x_i \in \mathcal{X} \) to \( \Gamma \). If \( \varepsilon > 0 \), we let \( B_\varepsilon(\Gamma) := \{ x_i \in \mathcal{X} : \|x_i\|_\Gamma < \varepsilon \} \) and \( N_\varepsilon(\Gamma) \) denote an open subset of \( \mathcal{X} \) containing \( \Gamma \). If \( A, B \subset \mathcal{X} \) are two sets, denote by \( A \setminus B \) the set-theoretic difference of \( A \) and \( B \). If \( I = \{i_1, \ldots, i_n\} \) is an index set, the ordered list of elements \( (x_{i_1}, \ldots, x_{i_n}) \) is denoted by \( (x_j)_{j \in I} \).

B. Graph Theory

We refer the reader to [9] for more details on the notions reviewed in this section. We denote a digraph by \( G = (V, E) \), where \( V \) is a set of nodes labelled as \( \{1, \ldots, n\} \) and \( E \) is the set of edges. The set of neighbors of node \( i \) is \( N_i := \{ j \in V : (i, j) \in E \} \).

Given positive numbers \( a_{ij} > 0 \), \( i, j \in \{1, \ldots, n\} \), the associated weighted Laplacian matrix of \( G \) is the matrix \( L := D - A \), where \( D \) is a diagonal matrix whose \( i \)-th diagonal entry is the sum \( \sum_{j \in N_i} a_{ij} \), and \( A \) is the matrix whose element \( (A)_{ij} \) is \( a_{ij} \) if \( j \in N_i \), and 0 otherwise.

A directed spanning tree is a graph consisting of \( n - 1 \) edges such that there exists a unique directed path from a node, called the root, to every other node. A reverse directed spanning tree is a graph which becomes a directed spanning tree by reversing the directions of all its edges. We identify the root of a reverse spanning tree with the root of its associated spanning tree. A graph \( G \) contains a reverse directed spanning tree if it has a subgraph which is a reverse directed spanning tree.

Proposition 1 ([1], [6]): The following conditions are equivalent for a digraph \( G \):

(i) \( G \) contains a reverse directed spanning tree.

(ii) For any set of positive gains \( a_{ij} > 0 \), \( i, j \in \{1, \ldots, n\} \)
the associated weighted Laplacian matrix \( L \) of \( G \) has rank \( n - 1 \), and \( \ker L = \text{span} \{1\} \).

A graph \( G = (V, E) \) is strongly connected if for any two nodes \( i, j \in V \) there exists a path from \( i \) to \( j \). A set of nodes \( S \subset V \) is an isolated component if it has no outgoing edges, i.e., for any edge \( (i, j) \in E \), if \( i \in S \) then \( j \notin S \). A graph \( G' = (V', E') \) is a subgraph of \( G \) if \( V' \subset V \) and \( E' \subset E \). A subgraph \( G' \) is an induced subgraph of \( G \) if for any two vertices \( i, j \in V' \), \( (i, j) \in E' \) if and only if \( (i, j) \in E \). A strongly connected component \( G' \) of \( G \) is a maximal strongly connected induced subgraph of \( G \). In other words, there does not exist any other strongly connected induced subgraph of \( G \) containing \( G' \). Letting \( G_0 = (V_0, E_0), \ldots, G_r = (V_r, E_r) \) be the strongly connected components of \( G \), the condensation digraph of \( G \), denoted \( C(G) = (V_C(G), E_C(G)) \), is defined as follows. The vertex set \( V_C(G) \) is the set of nodes \( \{v_i\}_{i \in \{0, \ldots, r\}} \) where the node \( v_i \) is a contraction of the vertex set \( V_i \) of the \( i \)-th strongly connected component \( G_i \). The edge set \( E_C(G) \) contains an edge \( (v_i, v_j) \) if there exist vertices \( i' \in V_i \) and \( j' \in V_j \) such that \( (i', j') \in E \). The following properties of the condensation digraph are found in [10].

Proposition 2 ([10]): Consider a graph \( G \) containing a reverse directed spanning tree. The condensation \( C(G) \) satisfies the following properties:

(i) \( C(G) \) is acyclic, i.e., there is no path in \( C(G) \) beginning and ending at the same node.

(ii) \( C(G) \) contains a reverse directed spanning tree \( T \) with a unique root \( v_0 \in V_C(G) \).

(iii) There exists at least one vertex \( v_i \in V_C(G) \) such that \( v_0 \) is the only neighbor of \( v_i \).

An example of a digraph \( G \) containing a reverse directed spanning tree is shown in Figure 1. The strongly connected components are boxed. The resulting acyclic condensation graph \( C(G) \) is shown in Figure 2. The vertex \( v_0 \) in the figure is the unique root of the reverse directed spanning tree in \( C(G) \).

![Fig. 1: Directed graph G containing a reverse directed spanning tree. The strongly connected components G0, ..., G3 are boxed.](image1)

![Fig. 2: Condensation digraph C(G) associated with the graph G in Figure 1(left) and reverse directed spanning tree contained in C(G) (right).](image2)
\( \bigcup_{k=0}^{k} \mathcal{L}_i \), by construction, the neighbors of any vertex in \( \mathcal{L}_i \) are contained in \( \mathcal{L}_{k-1} \). Therefore each node set \( \mathcal{L}_k \) is isolated.

For the example in Figure 2 we have \( \mathcal{L}_0 = \{1, 2, 3, 4, 5, 6\}, \mathcal{L}_1 = \{10\} \cup \{11, 12\} \) and \( \mathcal{L}_2 = \{7, 8, 9\} \).

C. Stability Definitions

The following stability definitions are taken from [11]. Let \( \Sigma : \dot{\chi} = f(\chi) \) be a smooth dynamical system with state space a geodesically complete Riemannian manifold \( \mathcal{X} \) with Riemannian distance \( L \).

For the example in Figure 2, we have \( \mathcal{L}_i \) is contained in \( \mathcal{L}_{k-1} \). Therefore each node set \( \mathcal{L}_k \) is isolated.

**Definition 1:** Consider a closed set \( \Gamma \subset \mathcal{X} \) that is positively invariant for \( \Sigma \), i.e., for all \( \chi_0 \in \Gamma \), \( \phi(t, \chi_0) \in \Gamma \) for all \( t > 0 \) for which \( \phi(t, \chi_0) \) is defined.

- \( \Gamma \) is **stable** for \( \Sigma \) if for any \( \varepsilon > 0 \), there exists a neighborhood \( N(\varepsilon) \subset \mathcal{X} \) such that, for all \( \chi_0 \in N(\varepsilon) \), \( \phi(t, \chi_0) \in B_{\varepsilon}(\Gamma) \), for all \( t > 0 \) for which \( \phi(t, \chi_0) \) is defined.
- \( \Gamma \) is **attractive** for \( \Sigma \) if there exists a neighborhood \( N(\varepsilon) \subset \mathcal{X} \) such that for all \( \chi_0 \in N(\varepsilon) \), \( \lim_{t \to \infty} \| \phi(t, \chi_0) \| = 0 \).
- The domain of attraction of \( \Gamma \) is the set \( \{ \chi_0 \in \mathcal{X} : \lim_{t \to \infty} \| \phi(t, \chi_0) \| = 0 \} \). \( \Gamma \) is **globally attractive** for \( \Sigma \) if it is attractive with respect to every point in \( \mathcal{X} \).
- \( \Gamma \) is **locally asymptotically stable (LAS)** for \( \Sigma \) if it is stable and attractive. The set \( \Gamma \) is globally asymptotically stable (GAS) for \( \Sigma \) if it is stable and globally attractive.

**Definition 2:** Let \( \Gamma_1 \subset \Gamma_2 \subset \mathcal{X} \) be two subsets of \( \mathcal{X} \) that are positively invariant for \( \Sigma \). Assume that \( \Gamma_1 \) is compact and \( \Gamma_2 \) is closed.

- \( \Gamma_1 \) is globally asymptotically stable relative to \( \Gamma_2 \) if it is GAS when initial conditions are restricted to lie in \( \Gamma_2 \).
- \( \Gamma_2 \) is locally stable near \( \Gamma_1 \) if for all \( c > 0 \) and all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x_0 \in B_{\delta}(\Gamma_1) \) and all \( t^* > 0 \), if \( \phi([0, t^*], x_0) \subset B_{\varepsilon}(\Gamma_1) \) then \( \phi([0, t], x_0) \subset B_{\varepsilon}(\Gamma_2) \).
- \( \Gamma_2 \) is locally attractive near \( \Gamma_1 \) if there exists a neighborhood \( N(\varepsilon) \subset \mathcal{X} \) such that, for all \( x_0 \in N(\varepsilon) \), \( \| \phi(t, x_0) \| \to 0 \) as \( t \to \infty \).

We present a reduction theorem used to derive our main result.

**Theorem 1 (Reduction Theorem [11], [12]):** Let \( \Gamma_1 \) and \( \Gamma_2 \), \( \Gamma_1 \subset \Gamma_2 \subset \mathcal{X} \), be two closed sets that are positively invariant for \( \Sigma \), and suppose \( \Gamma_1 \) is compact. Consider the following conditions: (i) \( \Gamma_1 \) is LAS relative to \( \Gamma_2 \); (ii) \( \Gamma_1 \) is GAS relative to \( \Gamma_2 \); (iii) \( \Gamma_2 \) is locally stable near \( \Gamma_1 \); (iii) \( \Gamma_2 \) is globally attractive; (iv) all trajectories of \( \Sigma \) are bounded.

Then, the following implications hold: (i) \( \wedge (ii) \implies \Gamma_1 \) is stable; (ii) \( \wedge (ii) \wedge (iii) \iff \Gamma_1 \) is LAS; (i) \( \wedge (ii) \wedge (iii) \wedge (iv) \iff \Gamma_1 \) is GAS.

III. RENDEZVOUS CONTROL PROBLEM

Consider a group of \( n \) kinematic unicycles. Let \( \mathcal{I} = \{i_x, i_y\} \) be an inertial frame in three-dimensional space and consider the \( i \)-the unicycle in Figure 4. Fix a body frame \( B_i = \{b_{ix}, b_{iy}\} \) to the unicycle, where \( b_{ix} \) is the heading axis, and denote by \( x_i \in \mathbb{R}^2 \) the position of the unicycle in the coordinates of frame \( \mathcal{I} \). The unicycle’s attitude is represented by a rotation matrix \( R_i \) whose columns are the coordinate representations of \( b_{ix} \) and \( b_{iy} \) in frame \( \mathcal{I} \). Letting \( \theta_i \in \mathbb{S}^1 \) be the angle between vectors \( b_{ix} \) and \( b_{ix} \), we have

\[
R_i = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i \\
\sin \theta_i & \cos \theta_i
\end{bmatrix}.
\]

The angular speed of robot \( i \) is denoted by \( \omega_i \). The unicycle dynamics are given by

\[
\dot{x}_i = u_i R_i e_1  \\
\dot{R}_i = R_i (\omega_i)^x, \quad i = 1, \ldots, n.
\]

In what follows, we refer to system (1)-(2) as \( \Sigma_i \). Its control inputs are the linear speed \( u_i \) and angular speed \( \omega_i \). The relative displacement of robot \( j \) with respect to robot \( i \) is \( x_{ij} := x_j - x_i \). If \( v \in \mathbb{R}^2 \) is the coordinate representation of a vector in frame \( \mathcal{I} \), then we denote by \( v^i := R_i^{-1} v \) the coordinate representation of \( v \) in body frame \( B_i \).

We define the sensor digraph \( G = (V, \mathcal{E}) \), where each node represents a robot, and an edge from node \( i \) to node \( j \) indicates that robot \( i \) can sense robot \( j \). We assume that \( G \) has no self-loops and is time-invariant. Given a node \( i \), its set of neighbors \( N_i \) represents the set of vehicles that robot \( i \) can sense. If \( j \in N_i \), then we say that robot \( j \) is a neighbour of robot \( i \). If this is the case, then robot \( i \) can sense the relative displacement of robot \( j \) in its own body frame, i.e., the quantity \( x_{ij}^i \). Define the vector \( y_i := (x_{ij}^i)_{j \in N_i} \). The relative displacements available to robot \( i \) are contained in the vector \( y_i^i := (x_{ij}^i)_{j \in N_i} \). A local and distributed feedback \( (u_i, \omega_i) \) for robot \( i \) is a locally Lipschitz function of \( y_i^i \). We define the rendezvous manifold \( \Gamma := \{(x_i, R_i)_{i \in \{1, \ldots, n\}} \in \mathbb{R}^{2n} \times \text{SO}(2)^n : x_{ij} = 0, \forall i, j \} \).

We are now ready to state the rendezvous control problem.

**Rendezvous Control Problem:** For system (1)-(2) with sensor digraph \( G \), find local and distributed feedbacks \( (u_i, \omega_i)_{i \in \{1, \ldots, n\}} \) that globally asymptotically stabilize the rendezvous manifold \( \Gamma \).

IV. SOLUTION OF THE RENDEZVOUS CONTROL PROBLEM

In this section we present the solution of the rendezvous control problem. Consider the function

\[
f_i(y_i) := \sum_{j \in N_i} a_{ij} x_{ij},
\]
simulation is presented in Figure 6(a). The proposed feedback
Figure 5. For the feedback in (5), we pick
A. Simulation Results
The result below states that for sufficiently large $k_1$, the
feedbacks in (5) solve the rendezvous control problem if the
network of unicycles has a sensor digraph containing a reverse
directed spanning tree.

Theorem 2: The rendezvous control problem is solvable for
system (1) if, and only if, the sensor graph $G$ contains a
reverse directed spanning tree, in which case a solution is
as follows. There exists $k^*_1 > 0$ such that for any $k_1 > k^*_1$
feedback (5) with $f_i(y_i)$ in (4) solves the rendezvous control
problem.

The necessity portion of Theorem 2 was proved in [6]. The
sufficiency part, namely the fact that the feedback (5) solves
the rendezvous control problem, is proved in Section V.

The proposed control architecture is illustrated in the block
diagram of Figure 3. There are two nested loops. The outer
loop treats each robot as a single-integrator driven by the linear
consensus controller,

$$\dot{x}_i = f_i(y_i), \ i = 1, \ldots, n. \quad (6)$$

The set \(\{(x_i)_{i\in\{1,\ldots,n\}} \in \mathbb{R}^{2n} : x_{ij} = 0, \ \forall i,j\}\) is globally
asymptotically stable for (6) if the sensing graph has a reverse
directed spanning tree [2]. The signal $f_i(y_i(t))$ is computed in
the body frame $B_i$, and used as a reference signal for the
inner-loop thrust and direction controllers that assign the unicycle
control inputs in (5). The intuition behind these controllers
is shown in Figure 4. The speed input $u_i$ is the dot product
$u_i = \|f_i(y_i')\|f_i(y_i') \cdot e_1$. This is the projection of the reference
$\|f_i(y_i')\|f_i(y_i')$ onto the heading axis $b_{ix}$ of robot $i$. The angular
speed, on the other hand, is proportional to the dot product
between the reference $f_i(y_i)$ and the second body axis $b_{iy}$.
In Figure 3 one can see that $\omega_i = -k_1 \|f_i\| \sin(\phi_i)$ acts to
reduce the angle $\phi_i$ between $b_{ix}$ and $f_i(y_i)$ with a rate proportional
to the magnitude of $f_i$. Together, these control inputs drive
the robot velocity $u_i b_{ix}$ approximately to the reference
$\|f_i(y_i')\|f_i(y_i')$. The convergence is approximate because the
control inputs do not depend on the time derivative of $f_i$. It is
the difference in angle between $u_i b_{ix}$ and $\|f_i(y_i')\|f_i(y_i')$ as opposed
to the difference in magnitude that is important for
obtaining rendezvous. Since $\|f_i(y_i')\|f_i(y_i')$ is homogeneous
of degree two, as the robots approach consensus, $\omega_i$ converges
to zero slower than $u_i$. This allows $\omega_i$ to exert sufficient control
authority even as the robots converge to consensus, closing the
gap between the vectors $u_i b_{ix}$ and $\|f_i(y_i')\|f_i(y_i')$.

A. Simulation Results
We consider a group of five robots with sensor digraph in
Figure 5. For the feedback in (5), we pick $a_{ij} = 0.05$ for all $j \in \mathcal{N}_i$. The control gain $k_1$ is chosen to be $k_1 = 1$.
The initial conditions of the robots are shown in Table I. The
simulation is presented in Figure 5(a). The proposed feedback
has practical advantages over the time-varying feedback in (6)
and the discontinuous feedback in (7) whose simulation results
are shown in Figure 5(b) and Figure 5(c) respectively with the
same initial conditions in Table I and sensing graph in
Figure 5. The proposed feedback induces a more natural
behaviour in the ensemble of unicycles. The feedback in (6)
makes the unicycle “wiggle” indefinitely, a behaviour which
would be unacceptable in practice. The feedback in (7) induces
instantaneous changes in direction that are impossible to
achieve with realistic implementations.

V. PROOF OF THEOREM 2
This section presents the sufficiency proof of Theorem 2.
The necessity was proved in [6]. The key tool in our proof is the
condensation graph and the isolated node sets $\bar{\mathcal{L}}_k$ defined in
Section II-B. The same tool was employed in [10] for pose
synchronization (synchronization of positions and attitudes) of
fully actuated vehicles.

The dynamics of unicycles associated with an isolated node
set $\bar{\mathcal{L}}_k$ are independent of the nodes outside of this set because,
for any robot $i \in \bar{\mathcal{L}}_k$, the feedbacks $u_i$ and $\omega_i$ in (4), (5)
depend only on states of robots within $\bar{\mathcal{L}}_k$. Therefore, the
dynamics of the collection of unicycles in $\bar{\mathcal{L}}_k$,

$$\dot{x}_i = u_i R_i e_1 \quad (7)$$

$$\dot{R}_i = R_i (\omega_i)^\times, \ i \in \bar{\mathcal{L}}_k \quad (8)$$

define an autonomous dynamical system. Henceforth, the

dynamics in (7), (8) are denoted by $\Sigma_{\bar{\mathcal{L}}_k}$ and we define the
reduced rendezvous manifold $\Gamma_{\bar{\mathcal{L}}_k} := \{(x_i, R_i)_{i\in\bar{\mathcal{L}}_k} : x_{ij} = 0, \ \forall i,j \in \bar{\mathcal{L}}_k\}$. Recall from
Section II-B that the set $\bar{\mathcal{L}}_{-1}$ is empty, which implies that
the set $\Gamma_{\bar{\mathcal{L}}_{-1}}$ is also empty. We adopt the
convention that $\Gamma_{\mathcal{L}_{-1}}$ is GAS for $\Sigma_{\bar{\mathcal{L}}_{-1}}$.

The proof of Theorem 2 relies on an induction argument on
the node sets $\bar{\mathcal{L}}_k$. Key in the induction argument is the next

\begin{table}[H]
\centering
\caption{Simulation Initial Conditions}
\begin{tabular}{|c|c|c|}
\hline
Vehicle & $x_i(0)$ (m) & $\theta_i(0)$ (rad) \\
\hline
1 & (0, 10) & 0 \\
2 & (-10, -10) & 2\pi/5 \\
3 & (-50, 10) & 4\pi/5 \\
4 & (-10, 0) & 6\pi/5 \\
5 & (10, 0) & 8\pi/5 \\
\hline
\end{tabular}
\end{table}

Fig. 5: Sensor digraph used in the simulation results.

![Fig. 4: Illustration of the control inputs $u_i$ and $\omega_i$ in (5).]
result stating that if the vehicles in $\tilde{L}_{k-1}$ achieve rendezvous, then so do the vehicles in $\tilde{L}_k$.

**Proposition 3:** Consider system (1), (2) and assume that the sensor graph $G$ contains a reverse directed spanning tree. Let $u_i$ and $\omega_i$ be as in (5) with $f_i(y)$ as in (4). Suppose that, for some integer $k > 0$, the set $\Gamma_{\tilde{L}_k-1}$ is globally asymptotically stable for the dynamics $\Sigma_{\tilde{L}_k-1}$. Then there exists $k^*_1 > 0$ such that choosing $k_1 > k^*_1$ in (5), implies $\Gamma_{\tilde{L}_k}$ is globally asymptotically stable for the dynamics $\Sigma_{\tilde{L}_k}$.

In Section VII-A we use the above proposition to prove Theorem 2 and in Section VII-B we prove Proposition 5.

**Corollary 1:** Consider system (1), (2) and assume that the sensor graph $G$ is strongly connected. Let $u_i$ and $\omega_i$ be as in (5) with $f_i(y)$ as in (4). There exists $k^*_1 > 0$ such that choosing $k_1 > k^*_1$ solves the rendezvous control problem.

**A. Proof of Theorem 2**

To begin with, the feedback in (5) is local and distributed because it is a smooth function of $y_i$ only. Consider a graph $G = (\mathcal{V}, \mathcal{E})$ containing a reverse directed spanning tree and the node sets $\mathcal{L}_k$ and $\tilde{L}_k$ defined in Section VII-B. By construction, the node sets $\tilde{L}_k$ are isolated, the subgraph $(\mathcal{V}_0, \mathcal{E}_0)$ is strongly connected, and $\tilde{L}_0 = \mathcal{L}_0 = \emptyset$.

The proof is by induction. Since the subgraph $(\mathcal{L}_0, \mathcal{E}_0)$ is strongly connected, by Corollary 1 there exists $l_0$ such that choosing $k_1 > l_0$ makes the set $\Gamma_{\tilde{L}_0}$ globally asymptotically stable for system $\Sigma_{\tilde{L}_0}$.

Now consider $\tilde{L}_k$ and suppose the reduced rendezvous manifold $\tilde{L}_{k-1}$ is globally asymptotically stable for system $\Sigma_{\tilde{L}_{k-1}}$. It holds from Proposition 3 that there exists $k_1$ such that choosing $k_1 > k_0$ makes the isolated node set $\Gamma_{\tilde{L}_k}$ globally asymptotically stable for system $\Sigma_{\tilde{L}_k}$. By part (ii) of Proposition 2, $C(G)$ contains a reverse directed spanning tree, so there is a path from every node of $C(G)$ to the unique root of $C(G)$. By part (i) of the same proposition, $C(G)$ is acyclic, which implies that the paths connecting the nodes of $C(G)$ to the unique root of $C(G)$ have a maximum length, $k^*$.

Recall that, by definition, $\tilde{L}_{k^*} = \sum_{i=1}^{k^*} L_i$ is the union of those strongly connected components $V_i$ of $\mathcal{V}$ that are associated with nodes $v_i$ of the condensation digraph $\mathcal{C}(G)$ with the property that the maximum path length from $v_i$ to the root $v_0$ is $\leq k^*$. As we argued earlier, the set of such nodes $v_i$ equals the entire condensation digraph, implying that $\tilde{L}_{k^*} = \mathcal{V}$. Let $k^*_1 > \max\{l_0, \ldots, l_k\}$. By induction, it must hold that choosing $k_1 > k^*_1$ makes $\Gamma_{\tilde{L}_k} = \Gamma$ globally asymptotically stable for system $\Sigma_{\tilde{L}_k} = \Sigma_\mathcal{V} = \Sigma$. We conclude that $\Gamma$ is globally asymptotically stable.

**B. Proof of Proposition 3**

We denote $A := \tilde{L}_{k-1} \in B := \tilde{L}_k$ and therefore $\tilde{L}_k = A \cup B$. By assumption, $\Gamma_A$ is globally asymptotically stable for the dynamics $\Sigma_A$ and the graph associated to the nodes in $B$ is strongly connected. We need to show that $\Gamma_{A \cup B}$ is globally asymptotically stable for the dynamics $\Sigma_{A \cup B}$. The proof relies on the following coordinate transformation.

1) **Coordinate Transformation:** For notational convenience, we collect the position vectors $x_i$ and rotation matrices $R_i$ into variables $x := (x_1, \ldots, x_n)$ and $R := (R_1, \ldots, R_n)$. We define the spaces $X := \mathbb{R}^{2n}$, $\mathcal{R} := SO(2) \times \cdots \times SO(2) (n \text{ times})$, so that $x \in X$ and $R \in \mathcal{R}$. For each $i \in \{1, \ldots, n\}$, define

$$X_i := f_i(y_i)/A_i,$$

where $A_i := \sum_{j \notin N_i} a_{ij}$, and let $X := (X_1, \ldots, X_n)$. We may express $X$ as

$$X = \text{diag}(1/A_1, \ldots, 1/A_n)(L \otimes I_2)x.$$

In the above, $\text{diag}(\ldots)$ is the diagonal matrix with diagonal elements inside the parenthesis; $L$ is the weighted Laplacian matrix of the sensor digraph associated with the gains $a_{ij}$; finally, $\otimes$ denotes the Kronecker product of matrices. Since the sensor digraph contains a reverse directed spanning tree, by Proposition 1 the matrix $L \otimes I_2$ has rank $2(n-1)$, and $\ker(L \otimes I_2) = \text{span}\{1 \otimes e_1, 1 \otimes e_2\}$ with $1 \in \mathbb{R}^n$. Let $\bar{x} := [I_2 \cdots I_2]x = \sum_i x_i$, then the linear map $T : X \rightarrow \mathbb{R}^2$, $x \mapsto (X, \bar{x})$ is an isomorphism onto its image. Under the action of $T$, the subspace $\{x \in X : x_1 = \cdots = x_n\}$ is mapped isomorphically onto the subspace $\{(X, \bar{x}) \in \text{Im}T : X = 0\}$. Since the feedbacks in (4)-(5) are local and distributed, it can be seen that the dynamics of the closed-loop unicycles in $(X, \bar{x}, R)$ coordinates are independent of $\bar{x}$. Moreover, as we have seen, in these coordinates the control specification is the global stabilization of $\{(X, \bar{x}, R) \in X \times \mathbb{R}^2 \times R : X = 0\}$, a set whose description is independent of $\bar{x}$. In light of these considerations, for the stability analysis we may drop the variable $\bar{x}$, and show that the set $\tilde{\Gamma} := \{(X, R) \in X \times R : X = 0\}$ is GAS for the $(X, R)$ dynamics.
From here on we will use the hat notation to refer to quantities represented in \((X, R)\) coordinates. Denote \(g_i(y_i) := \|f_i(y_i)\|_2\), the functions \(f_i\) and \(g_i\), and their body frame representations are given in \((X, R)\) coordinates by

\[
\dot{\hat{f}_i}(X_i) = A_i X_i, \quad \hat{g}_i(X_i) = A_i^2 \|X_i\|_2 X_i,
\]

and we can use these expressions to rewrite the feedback \((5)\) in new coordinates as

\[
\Phi(0, R) = \Phi(0, X, R) = \alpha > 0, \quad \text{and we can use these expressions to rewrite the feedback (5) in new coordinates as}
\]

and use of the reduction theorem (Theorem 1). We will first show that all solutions of the closed-loop system are bounded. The rotation matrices live in a compact set, therefore we only need to show that the states \(X_{AUB} = (X_i)_{i \in AUB}\) are bounded. Since \(A\) is isolated, \(\Sigma_A\) is an autonomous subsystem and by assumption, \(\hat{\Gamma}_A = \{(X_A, R_A) \in X_A \times R_A : X_A = 0\}\) (compact), is globally asymptotically stable. Therefore, \(X_A\) is bounded. From the inequality \(W(X_B, R_B) \geq \alpha^* \sqrt{V(X_B)}\) in part (iii) of Lemma 1 to show boundedness of \(V(X_B)\), it suffices to show that \(W(X_B, R_B)\) is bounded. Boundedness of \(V(X_B)\), in turn, implies boundedness of \(X_B\). From the bound on the derivative of \(W\) in \((15)\), and by Lemma 1 we obtain

\[
\frac{d}{dt} W(X_B, R_B) \leq -\sigma W(X_B, R_B)^2 + \Phi(X_A, R), \quad \sigma > 0.
\]

Since \(X_A\) is bounded and \(R \in R\) lies on a compact set, it holds that \(\Phi(X_A, R)\) is bounded and therefore \(W\) is bounded, which implies that \(X_B\) is bounded. Therefore \(X_{AUB}\) is bounded, as claimed. Now define the set \(\hat{\Lambda} := \{(X_{AUB}, R_{AUB}) \in X_{AUB} \times R_{AUB} : X_A = 0\}\). Since the set \(\hat{\Gamma}_A\) is globally asymptotically stable for system \(\Sigma_A\) and \(X_{AUB}\) is bounded, it holds that \(\hat{\Lambda}\) is globally asymptotically stable for \(\Sigma_{AUB}\).

To show that the set \(\hat{\Gamma}_{AUB}\), which is compact, is globally asymptotically stable for the system \(\Sigma_{AUB}\), it suffices to show that \(\hat{\Gamma}_{AUB}\) is globally asymptotically stable relative to \(\hat{\Lambda}\). On the set \(\hat{\Lambda}\), \(\Phi(X_A, R)\) is equal to zero and the derivative of \(W\) is therefore given by \(\frac{d}{dt} W(X_B, R_B) \leq -\sigma W(X_B, R_B)^2, \quad \sigma > 0\).

By Lemma 1 all level sets of \(W(X_B, R_B)\) are compact and \(W^{-1}(0) = \{(X_B, R_B) : X_B = 0\}\). This implies \(\hat{\Gamma}_{AUB}\) is globally asymptotically stable relative to the set \(\hat{\Lambda}\). By Theorem \(\Gamma_{AUB}\) is globally asymptotically stable for \(\Sigma_{AUB}\).

This completes the proof.

**VI. CONCLUSION**

We have presented the first solution to the rendezvous control problem for a group of kinematic unicycles on the plane using continuous, time-independent feedback that is local and distributed. The solution assumes a fixed sensing digraph that contains a reverse-directed spanning tree. The control methodology is based on a control structure made of two nested loops. An outer loop produces a standard feedback for consensus of single integrators which becomes reference to an inner loop assigning the unicycle control inputs that rely only on onboard measurements. Information of the unicycle's local and distributed. The solution assumes a fixed sensing constraints.

The proof of Lemma 2 is presented in the appendix.

We will now show that choosing \(k_0 > 0\) implies \(\hat{\Gamma}_{AUB}\) is globally asymptotically stable for \(\Sigma_{AUB}\). The proof will make use of the reduction theorem (Theorem 1). We will first show that all solutions of the closed-loop system are bounded. The rotation matrices live in a compact set, therefore we only need to show that the states \(X_{AUB} = (X_i)_{i \in AUB}\) are bounded. Since \(A\) is isolated, \(\Sigma_A\) is an autonomous subsystem and by assumption, \(\hat{\Gamma}_A = \{(X_A, R_A) \in X_A \times R_A : X_A = 0\}\) (compact), is globally asymptotically stable. Therefore, \(X_A\) is bounded. From the inequality \(W(X_B, R_B) \geq \alpha^* \sqrt{V(X_B)}\) in part (iii) of Lemma 1 to show boundedness of \(V(X_B)\), it suffices to show that \(W(X_B, R_B)\) is bounded. Boundedness of \(V(X_B)\), in turn, implies boundedness of \(X_B\). From the bound on the derivative of \(W\) in (15), and by Lemma 1 we obtain

\[
\frac{d}{dt} W(X_B, R_B) \leq -\sigma W(X_B, R_B)^2 + \Phi(X_A, R), \quad \sigma > 0.
\]

Since \(X_A\) is bounded and \(R \in R\) lies on a compact set, it holds that \(\Phi(X_A, R)\) is bounded and therefore \(W\) is bounded, which implies that \(X_B\) is bounded. Therefore \(X_{AUB}\) is bounded, as claimed. Now define the set \(\hat{\Lambda} := \{(X_{AUB}, R_{AUB}) \in X_{AUB} \times R_{AUB} : X_A = 0\}\). Since the set \(\hat{\Gamma}_A\) is globally asymptotically stable for system \(\Sigma_A\) and \(X_{AUB}\) is bounded, it holds that \(\hat{\Lambda}\) is globally asymptotically stable for \(\Sigma_{AUB}\).

To show that the set \(\hat{\Gamma}_{AUB}\), which is compact, is globally asymptotically stable for the system \(\Sigma_{AUB}\), it suffices to show that \(\hat{\Gamma}_{AUB}\) is globally asymptotically stable relative to \(\hat{\Lambda}\). On the set \(\hat{\Lambda}\), \(\Phi(X_A, R)\) is equal to zero and the derivative of \(W\) is therefore given by \(\frac{d}{dt} W(X_B, R_B) \leq -\sigma W(X_B, R_B)^2, \quad \sigma > 0\).

By Lemma 1 all level sets of \(W(X_B, R_B)\) are compact and \(W^{-1}(0) = \{(X_B, R_B) : X_B = 0\}\). This implies \(\hat{\Gamma}_{AUB}\) is globally asymptotically stable relative to the set \(\hat{\Lambda}\). By Theorem 1 \(\hat{\Gamma}_{AUB}\) is globally asymptotically stable for \(\Sigma_{AUB}\).

This completes the proof.

**VI. CONCLUSION**

We have presented the first solution to the rendezvous control problem for a group of kinematic unicycles on the plane using continuous, time-independent feedback that is local and distributed. The solution assumes a fixed sensing digraph that contains a reverse-directed spanning tree. The control methodology is based on a control structure made of two nested loops. An outer loop produces a standard feedback for consensus of single integrators which becomes reference to an inner loop assigning the unicycle control inputs that rely only on onboard measurements. Information of the unicycle’s local and distributed relations is not required.

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APPENDIX

Throughout this appendix we will make use of functions $\mu_i$ and $\mu$ defined as follows. Recall that $V(X_B)$ is positive definite. Define the functions $\mu : X_B \setminus 0 \to \mu(X_B \setminus 0)$, $\mu(X_B) := X_B / \sqrt{V(X_B)}$, and $\mu_i : X_B \setminus 0 \to \mu_i(X_B \setminus 0)$, $\mu_i(X_B) := X_i / \sqrt{V(X_B)}$, $i \in B$. Since the numerator and denominator are both homogeneous of degree one, these functions are both homogeneous of degree zero with respect to $X_B$. Therefore, the images satisfy $\mu(X_B \setminus 0) = \mu(S^k)$ and $\mu_i(X_B \setminus 0) = \mu_i(S^k)$, where $S^k$ is the unit sphere in $X_B$. Since $\mu$ and $\mu_i$ are continuous functions and $S^k$ is a compact set, the images $\mu(X_B \setminus 0)$ and $\mu_i(X_B \setminus 0)$ are compact sets.

A. Proof of Lemma 2

Recall the definition of $W(X_B, R_B)$,

$$W = \alpha \sqrt{V(X_B)} + \sum_{i \in B} \hat{f}_i(X_i, R_i) \cdot e_1 = \sqrt{V(X_B)} \left( \alpha + \sum_{i \in B} \hat{f}_i(X_i, R_i) \cdot \frac{e_1}{\sqrt{V(X_B)}} \right).$$

Using the fact that $\hat{f}_i(X_i, R_i)$ is homogeneous with respect to its first argument, we have $W = \sqrt{V(X_B)} \left( \alpha + \sum_{i \in B} \hat{f}_i(\mu_i(X_B), R_i) \cdot e_1 \right)$. Since $\hat{f}_i$ is continuous, $\mu_i(X_B)$ is bounded, and $R_B \in R_B$, a compact set, it follows that the function $\sum_{i \in B} \hat{f}_i(\mu_i(X_B), R_i) \cdot e_3$ has a bounded supremum. Accordingly, let $\alpha^* = \sup_{(X_B, R_B) \in X_B \times R_B} \sum_{i \in B} \hat{f}_i(\mu_i(X_B), R_i) \cdot e_1$. For all $\alpha > 2\alpha^*$, we have $W(X_B, R_B) \geq W(X_B, R_B) := \alpha^* \sqrt{V(X_B)} \geq 0$. This inequality implies that $W \geq 0$ and $W^{-1}(0) \subset W^{-1}(0)$. But $W = 0$ if and only if $V(X_B) = 0$ (i.e., $X_B = 0$). Thus $W^{-1}(0) \subset \{(X_B, R_B) : X_B = 0\}$. Conversely, on the set $\{(X_B, R_B) : X_B = 0\}$, $X_B = 0$ and hence $W = 0$, and therefore $\{(X_B, R_B) : X_B = 0\} \subset W^{-1}(0)$. It follows that $W^{-1}(0) = \{(X_B, R_B) : X_B = 0\}$ proving part (i).

For part (ii), note that for all $c > 0$, $W_c \subset \{W(X, R) \leq c\}$. Since the sublevel sets of $W$ are compact and $R_B \in R_B$, a compact set, the set $W_c$ is bounded. Continuity of $W$ implies that $W_c$ is compact.

For part (iii), it has already been shown that $W(X_B, R_B) \geq \alpha^* \sqrt{V(X_B)}$. It also holds that $W = \sqrt{V(X_B)} \left( \alpha + \sum_{i \in B} \hat{f}_i(\mu_i(X_B), R_i) \cdot e_1 \right) \leq \sqrt{V(X_B)} \left( \alpha + \alpha^* \sqrt{V(X_B)} \right) \leq 2\alpha^* \sqrt{V(X_B)}$. □

B. Proof of Lemma 2

We first compute inequalitites for $W_{\text{tran}}$ and $W_{\text{cat}}$ for system (11) and (12). We then combine them to derive (15). Consider unicycle $i \in B$. The dynamics of $X_i$ in (11) are split into two terms, for neighboring robots $j \in N_i \cap A$ and $j \in N_i \cap B$ respectively,

$$X_i = \sum_{j \in N_i \cap A} a_{ij} (u_{ij} R_j e_1 - u_i R_i e_1) \frac{A_i}{A_i} + \sum_{j \in N_i \cap B} a_{ij} (u_{ij} R_j e_1 - u_i R_i e_1) \frac{A_i}{A_i}.$$  (16)

For simplicity of notation, we drop the arguments of $g_i(X_i)$ and $\tilde{g}_i(X_i, R_i)$. Adding and subtracting the term,

$$\sum_{j \in N_i \cap B} a_{ij} (g_i - \tilde{g}_i) - \sum_{j \in N_i \cap A} a_{ij} g_i \frac{A_i}{A_i}$$

to (16) yields,

$$X_i = \sum_{j \in N_i \cap B} a_{ij} (g_{ij} - \tilde{g}_i) - \sum_{j \in N_i \cap A} a_{ij} g_i \frac{A_i}{A_i} + \sum_{j \in N_i \cap B} a_{ij} (u_{ij} R_j e_1 - u_i R_i e_1) - \sum_{j \in N_i \cap B} a_{ij} (g_i - \tilde{g}_i) \frac{A_i}{A_i} + \sum_{j \in N_i \cap B} a_{ij} (u_{ij} R_j e_1 - u_i R_i e_1) - \sum_{j \in N_i \cap B} a_{ij} (u_{ij} R_j e_1 - \tilde{g}_i) \frac{A_i}{A_i} + \sum_{j \in N_i \cap A} a_{ij} (u_{ij} R_j e_1 - \tilde{g}_i) \frac{A_i}{A_i} - \sum_{j \in N_i \cap A} a_{ij} (\tilde{g}_i - u_i R_i e_1) \frac{A_i}{A_i} + \sum_{j \in N_i \cap A} a_{ij} (\tilde{g}_i - u_i R_i e_1) \frac{A_i}{A_i}.$$
The time derivative of $W_{\text{tran}} = \sqrt{V(X_B)}$ in (13) yields,

$$W_{\text{tran}} = \frac{1}{2 \sqrt{V(X_B)}} \left[ \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} (a_i(X_B) + b_i(X_B, R)) + c_i(X_B, R) \right] + \frac{1}{2 \sqrt{V(X_B)}} \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} \dot{d}_i(X_A, R).$$

(17)

The derivative of the first term is considered in Claim 1.

Claim 1: There exist gains $\gamma_i$ in (13) and a negative definite function $r(X_B)$, homogeneous of degree three, such that $\sum_{i \in B} \frac{\partial}{\partial X_i} a_i(X_B) \leq r(X_B)$.

The proof of Claim 1 is presented in Section C of this Appendix. Let the gains $\gamma_i$ be as in Claim 1. The derivative of the remaining terms in the square brackets of (17) satisfies,

$$\sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} (b_i(X_B, R) + c_i(X_B, R))$$

\leq \sum_{i \in B} \frac{1}{A_i} \frac{\partial V(X_B)}{\partial X_i} \left[ \sum_{j \in N_i \cap B} a_{ij} \left\| (\hat{g}_j \cdot e_1) e_1 - \hat{g}_j \right\| + \sum_{j \in N_i \cap A} a_{ij} \left\| (g_j \cdot e_1) e_1 - (g_j \cdot e_1) e_1 \right\| \right]

\leq \sum_{i \in B} \frac{1}{A_i} \frac{\partial V(X_B)}{\partial X_i} \left[ \sum_{j \in N_i \cap B} a_{ij} \left\| (\hat{g}_j \cdot e_1) e_1 - \hat{g}_j \right\| + \sum_{j \in N_i \cap A} a_{ij} \left\| (g_j \cdot e_1) e_1 - (g_j \cdot e_1) e_1 \right\| \right].

We claim that $\left\| (g_j \cdot e_1) e_1 \right\| = \left\| (g_j \cdot e_1) e_1 \right\| = \left\| (g_j \cdot e_1) e_1 \right\|$. Indeed, writing $g_j = (g_j \cdot e_1) e_1 + (g_j \cdot e_1) e_1$, we have $g_j = (g_j \cdot e_1) e_1 + (g_j \cdot e_1) e_1$. Since the vector $g_j = (g_j \cdot e_1) e_1$ is parallel to $e_2$, $\left\| (g_j \cdot e_1) e_1 \right\| \cdot e_2 = \left\| (g_j \cdot e_1) e_1 \right\|$, so that $\left\| (g_j \cdot e_1) e_1 \right\| = \left\| (g_j \cdot e_1) e_1 \right\|$. Then,

$$\sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} (b_i(X_B, R) + c_i(X_B, R))$$

\leq \sum_{i \in B} \frac{\overline{a}}{A_i} \left\| \frac{\partial V(X_B)}{\partial X_i} \right\| \left( \sum_{j \in B} \left\| (\hat{g}_j \cdot e_1) e_2 + n |g_j \cdot e_2| \right\| \right) + \Phi_{\text{tran}}(X_A, R).

Since $r(X_B)$ is homogeneous of degree three. We can write,

$$r(X_B) = \frac{\sqrt{V(X_B)} V(X_B)}{\sqrt{V(X_B) V(X_B)}} r(X_B)$$

$$= \frac{\sqrt{V(X_B)}}{\sqrt{V(X_B)}} \frac{\sqrt{V(X_B) V(X_B)}}{r \left( \sqrt{V(X_B)} \right)}$$

$$= \sqrt{V(X_B) V(X_B)} \Phi_{\text{tran}}(X_A, R).$$

Since $r$ is continuous and negative definite and $\mu(X_B)$ lies on a compact set $S_1$, it follows that $r(\mu(X_B))/2$ has bounded maximum $-M_2 < 0$. Similarly, the function $\frac{\overline{a}}{A_i} \left\| \frac{\partial V(\mu(X_B))}{\partial X_i} \right\| \left( \sum_{j \in B} \left\| (\hat{g}_j \cdot e_1) e_2 \right\| + n |g_j \cdot e_2| \right) + \Phi_{\text{tran}}(X_A, R)$.
has a maximum. Letting $M_1 := n \max_{i,B} \frac{\partial}{\partial x_i} \left| \frac{\partial V(\theta)}{\partial x_i} \right|$ yields,

$$W_{\text{tran}} \leq V(X_B) \left[ -M_2 + \frac{M_1}{2n} \sum_{i,B} \left( \sum_{j,B} \left| g_j^{(i)}(\mu_j(X_B), R_j) \right| \cdot e_2 \right) + n \left| g^{(i)}_i(\mu_i(X_B), R_i) \cdot e_2 \right| \right] + \Phi_{\text{tran}}(X_A, R)
\leq V(X_B) \left[ -M_2 + \frac{M_1}{2n} \sum_{i,B} \left( n \left| g^{(i)}_i(\mu_i(X_B), R_i) \cdot e_2 \right| \right) + n \left| g^{(i)}_i(\mu_i(X_B), R_i) \cdot e_2 \right| \right] + \Phi_{\text{tran}}(X_A, R)
\leq V(X_B) \left[ -M_2 + M_1 \sum_{i,B} \left| g^{(i)}_i(\mu_i(X_B), R_i) \cdot e_2 \right| \right] + \Phi_{\text{tran}}(X_A, R).
$$

This proves the first inequality. We now turn to the second. Recall the definition of $W_{\text{rot}}$, $W_{\text{rot}}(X_B, R_B) = \sum_{i,B} \bar{f}_i^T(X_i, R_i) \cdot e_1$. The time derivative of $W_{\text{rot}}$ along the vector field in (11)-(12) is $W_{\text{rot}} = \sum_{i,B} \left( \frac{d}{dt} \bar{f}_i^T \right) \cdot e_1$. To express $(d/dt)\bar{f}_i^T$, recall that $\bar{f}_i^T(X_i, R_i) = R_i^{-1} \hat{f}_i(X_i)$. Then, $\frac{d}{dt} \bar{f}_i^T = \left( \frac{d}{dt} R_i^{-1} \right) \bar{f}_i^T + R_i^{-1} \frac{d}{dt} R_i^{-1}$. We will denote the derivative of $\hat{f}_i(X_i) = A_i X_i$ by

$$h_i(X, R) := (d/dt) \hat{f}_i(X_i) = A_i (a_i(X_B) + b_i(X_B, R) + c_i(X_B, R) + d_i(X_A, R))$$
where the first three terms are homogeneous of degree two with respect to $X_B$ and the last term is homogeneous of degree two with respect to $X_A$. Consistently with our notational convention, we will let $h_i^T(X, R) := R_i^{-1} h_i(X, R)$.

Returning to the derivative of $\hat{f}_i$, we have

$$\frac{d}{dt} \bar{f}_i^T = -(\omega_i)^T R_i^{-1} \hat{f}_i(X_i) + R_i^{-1} h_i(X, R) = -\begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \bar{f}_i^T(X_i, R_i) + h_i^T(X, R).$$

We substitute the above identity in the expression for $W_{\text{rot}}$,

$$W_{\text{rot}} = \sum_{i,B} \left( -e_1^{T} \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \bar{f}_i^T(X_i, R_i) + h_i^T(X, R) \cdot e_1 \right) = \sum_{i,B} \left( \bar{f}_i^T(X_i, R_i) \cdot e_2 \right) \omega_i + h_i^T(X, R) \cdot e_1.$$

Substituting the feedback $\omega_i = -k_i (\bar{f}_i^T(X_i, R_i) \cdot e_2)$ and taking norms, we arrive at the inequality

$$W_{\text{rot}} \leq \sum_{i,B} \left[ -k_i \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + h_i^T(X, R) \cdot e_1 \right].$$

This gives,

$$W_{\text{rot}} \leq \left[ -k_i \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + \ell(X_B, R) \right] + \Phi_{\text{rot}}(X_A, R)$$

where

$$\ell(X_B, R) := \sum_{i,B} A_i R_i^T (a_i(X_B) + b_i(X_B, R) + c_i(X_B, R)) \cdot e_1$$
and $\Phi_{\text{rot}}(X_A, R) := \sum_{i,B} A_i R_i^T d_i(X_A, R) \cdot e_1$. Note that $\sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|$ and $\ell(X_B, R)$ are homogeneous of degree two with respect to $X_B$. The function $\Phi_{\text{rot}}(X_A, R)$ does not depend on $X_B$ and $\Phi_{\text{rot}}(0, R) = 0$. This yields,

$$W_{\text{rot}} \leq V(X_B) \left[ -k_i \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + \ell(X_B, R) \right] + \Phi_{\text{rot}}(X_A, R)
\leq V(X_B) \left[ -k_i \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + \ell(X_B, R) \right] + \Phi_{\text{rot}}(X_A, R).$$

$|\ell(\mu(X_B), R)|$ has a bounded supremum. Letting $M_3 = \sup_{(\theta, R) \in S_1 \times R} |\ell(\theta, R)|$, we conclude that,

$$W_{\text{rot}} \leq V(X_B) \left[ -k_i \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + M_3 \right] + \Phi_{\text{rot}}(X_A, R).$$

By using the inequalities (20) and (21) we now bound the derivative of $W$ to derive (15). Notice that

$$\dot{W} = \alpha W_{\text{tran}} + W_{\text{rot}} \leq V(X_B) \left[ -\alpha M_2 + \alpha M_1 \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right| \right] - k_i \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + M_3 + \Phi(X_A, R),$$

where $\Phi(X_A, R) := \alpha \Phi_{\text{tran}}(X_A, R) + \Phi_{\text{rot}}(X_A, R)$.

Choose $\alpha > 3M_3/M_2$. This implies,

$$\dot{W} \leq V(X_B) \left[ -2M_3 + \alpha M_1 \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right| \right] - k_i \sum_{i,B} \left| \bar{f}_i^T(X_i, R_i) \cdot e_2 \right|^2 + \Phi(X_A, R).$$

Since $\bar{f}_i^T(X_i, R_i)$ is homogeneous with respect to $X_i$, we have, $\bar{f}_i^T(X_i, R_i) = \frac{\sqrt{\|\bar{g}_i^T(\mu_i(X_B), R_i)\|}}{\|\bar{g}_i^T(\mu_i(X_B), R_i)\|} \bar{g}_i^T(\mu_i(X_B), R_i)$. Plugging the last expression into (22) yields

$$\dot{W} \leq V(X_B) \left[ -2M_3 + \alpha M_1 \sum_{i,B} \left| \bar{g}_i^T(\mu_i(X_B), R_i) \cdot e_2 \right| \right] - k_i \sum_{i,B} \left( \frac{\sqrt{\|\bar{g}_i^T(\mu_i(X_B), R_i)\|}}{\|\bar{g}_i^T(\mu_i(X_B), R_i)\|} \bar{g}_i^T(\mu_i(X_B), R_i) \cdot e_2 \right)^2 + \Phi(X_A, R).$$

$$\dot{W} \leq V(X_B) \left[ -2M_3 + \alpha M_1 \sum_{i,B} \left| \bar{g}_i^T(\mu_i(X_B), R_i) \cdot e_2 \right| \right] - k_i \sum_{i,B} \left( \frac{1}{\|\bar{g}_i^T(\mu_i(X_B), R_i)\|} \|\bar{g}_i^T(\mu_i(X_B), R_i) \cdot e_2 \right)^2 + \Phi(X_A, R).$$
Since $\hat{g}^{i}_i(\mu_i(X_B), R_i)$ is a continuous function of its arguments and $\mu_i(X_B)$ is compact, $|\hat{g}^{i}_i(\mu_i(X_B), R_i)|$ has a maximum $M_i$. This implies,

$$W \leq V(X_B) \left[ -2M_2 + \alpha M_1 \sum_{i \in B} |\hat{g}^{i}_i(\mu_i(X_B), R_i) \cdot e_2| - k_1 \sum_{i \in B} \frac{1}{M_i} |\hat{g}^{i}_i(\mu_i(X_B), R_i) \cdot e_2|^2 \right] + \Phi(X_A, R).$$

Denote $\beta_i(\mu_i(X_B), R_i) := |\hat{g}^{i}_i(\mu_i(X_B), R_i) \cdot e_2|$, and $\beta := (\beta_i(\mu_i(X_B), R_i))_{i \in B}$. Then,

$$W \leq V(X_B) \left[ -2M_2 + \alpha M_1 \sum_{i \in B} \beta_i \right] + \Phi(X_A, R) = V(X_B) \left[ 1^T \beta^T \begin{bmatrix} -\frac{2M_2}{\alpha M_1 I} & \frac{M_i}{\alpha M_1 I} \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} \right] + \Phi(X_A, R).$$

There exists $k^*_1 > 0$ such that choosing $k_1 > k^*_1$, the matrix above is negative definite and therefore the first term satisfies,

$$V(X_B) \left[ 1^T \beta^T \right] \left[ \begin{bmatrix} -\frac{2M_2}{\alpha M_1 I} & \frac{M_i}{\alpha M_1 I} \end{bmatrix} \right] \begin{bmatrix} 1 \\ \beta \end{bmatrix} \leq -\sigma V(X_B),$$

where $\sigma > 0$. This concludes the proof of Lemma 2.$\square$

C. Proof of Claim 1

Recalling that $V(X_B) = \gamma_i X_i^T X_i$ with $X_i = \hat{f}_i / A_i$ and defining $b_{ij} := \frac{a_{ij}}{A_i^2}$, it holds that,

$$\sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} a_i(X_B) = 2 \sum_{i \in B} \gamma_i \frac{\hat{f}_i}{A_i} \cdot a_i(X_B) \leq 2 \sum_{i \in B} \gamma_i \left( \sum_{j \in N_i \cap B} b_{ij}(\|\hat{f}_j\|\|\hat{f}_i\| - \|\hat{f}_i\|\hat{f}_j) - \sum_{j \in N_i \cap A} b_{ij}(\|\hat{f}_j\|\hat{f}_i) \right) \leq 2 \sum_{i \in B} \gamma_i \left( \sum_{j \in N_i \cap B} b_{ij}(\|\hat{f}_j\| + \|\hat{f}_j\| \hat{f}_i) - \sum_{j \in N_i \cap A} b_{ij}(\|\hat{f}_j\|^3) \right) \leq \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap B} b_{ij} \left( \frac{4}{3}\|\hat{f}_j\|^3 + \frac{4}{3}\|\hat{f}_j\|^3 \right) + \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap B} b_{ij} \left( \left(\|\hat{f}_j\| + \|\hat{f}_j\| \hat{f}_i \right) - \frac{4}{3}\|\hat{f}_j\|^3 \right) \leq -2 \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap A} b_{ij}(\|\hat{f}_j\|)$$

The first term equals $\frac{4}{3} \gamma^T M \bar{h}$ with $\bar{h} := (\|\hat{f}_j\|)_{i \in B}$. $M$ is the $(r \times r)$-matrix whose $(i, j)$-th component is $\sum_{k \in N_i \cap B} b_{ik}$ for $i = j$, $b_{ij}$ for $j \in N_i \cap B$, and zero otherwise for $i, j \in \{1, \ldots, r\}$ where it is assumed without loss of generality that $B = \{1, \ldots, r\}$. Choose $\gamma = (\gamma_1, \ldots, \gamma_n)$ as the left eigenvector associated to the zero eigenvalue of $M$. Since $B$ corresponds to a collection of strongly connected components with no links from one to the other, the zero eigenvalue is unique and all components of $\gamma$ are positive (see Proposition D.5 in [10]). Therefore,

$$\sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} a_i(X_B) \leq \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap B} b_{ij} \left( -\frac{2}{3}\|\hat{f}_j\|^3 + 2\|\hat{f}_j\|\hat{f}_j \cdot \hat{f}_i - \frac{4}{3}\|\hat{f}_j\|^3 \right) - 2 \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap A} b_{ij}(\|\hat{f}_j\|^3) =: \mathbf{r}(X_B).$$

The term

$$\mathbf{r}(X_B) = \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap B} b_{ij} \left( -\frac{2}{3}\|\hat{f}_j\|^3 + 2\|\hat{f}_j\|\hat{f}_j \cdot \hat{f}_i - \frac{4}{3}\|\hat{f}_j\|^3 \right)$$

is less than or equal to zero with equality only when $\hat{f}_i = \hat{f}_j$ for all $i, j \in B$ and as such $\mathbf{r}(X_B)$ is less than or equal to zero with equality only when $\hat{f}_i = \hat{f}_j$ for all $i, j \in B$.

Now we prove that $\mathbf{r}(X_B) = 0$ only if $\hat{f}_i = \hat{f}_j$ for all robots $i \in B$. In the case that $A$ is not empty, the inequality $\mathbf{r}(X_B) \leq -2 \sum_{i \in B} \gamma_i \sum_{j \in N_i \cap B} b_{ij}(\|\hat{f}_j\|^3)$ implies $\mathbf{r}(X_B) = 0$ only if $\hat{f}_i = \hat{f}_j$ for all $i \in B$ with a neighbor in $A$. As such, by the previous arguments, $\mathbf{r}(X_B) = 0$ only if $\hat{f}_i = \hat{f}_j$ for all $i \in B$. On the other hand, if $A$ is empty, then $B$ is isolated and strongly connected. Therefore $\mathbf{r}(X_B) = \mathbf{r}_1(X_B)$ is equal to zero only if $\mathbf{r}_1(X_B) = 0$ which is the case only if $\hat{f}_i = \hat{f}_j$ for all $i, j \in B$. This implies that $(L \otimes I_2)x \in \text{span}\{1 \otimes e_1, 1 \otimes e_2\}$. Since $B$ is a strongly connected component there exists a unique vector $\bar{\gamma}$ (with positive entries) such that $\bar{\gamma}^T (L \otimes I_2) = 0$. Since $\bar{\gamma}^T (L \otimes I_2)x = \bar{\gamma}^T 1 \otimes (\alpha e_1 + \beta e_2)$ for some $\alpha, \beta \in \mathbb{R}$, it holds that $\bar{\gamma}^T 1 \otimes (\alpha e_1 + \beta e_2) = 0$. Since all entries of $\bar{\gamma}$ are positive, this implies $\alpha = \beta = 0$ and $(L \otimes I_2)x = 0$. Therefore $x \in \text{span}\{1 \otimes e_1, 1 \otimes e_2\}$ or, equivalently, that $\hat{f}_i = \hat{f}_j$ for all $i \in B$.

Therefore $\mathbf{r}(X_B) = 0$ only if $X_i = 0$ for all $i \in B$ and as such $\mathbf{r}(X_B)$ is negative definite. Note that $\mathbf{r}(X_B)$ is homogeneous of degree three with respect to $X_B$ because $\hat{f}_i$ is homogeneous of degree one with respect to $X_B$ for all $i \in B$. This completes the proof of the claim.