A general version of Price’s theorem

A tool for bounding the expectation of nonlinear functions of Gaussian random vectors

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Assume that $X_\Sigma \in \mathbb{R}^n$ is a random vector following a multivariate normal distribution with zero mean and positive definite covariance matrix $\Sigma$. Let $g : \mathbb{R}^n \to \mathbb{C}$ be measurable and of moderate growth, e.g. $|g(x)| \lesssim (1 + |x|)^N$. We show that the map $\Sigma \mapsto \mathbb{E}[g(X_\Sigma)]$ is smooth, and we derive convenient expressions for its partial derivatives, in terms of certain expectations $\mathbb{E}[(\partial^\alpha g)(X_\Sigma)]$ of partial (distributional) derivatives of $g$. As we discuss, this result can be used to derive bounds for the expectation $\mathbb{E}[g(X_\Sigma)]$ of a nonlinear function $g(X_\Sigma)$ of a Gaussian random vector $X_\Sigma$ with possibly correlated entries.

For the case when $g(x) = g_1(x_1) \cdots g_n(x_n)$ has tensor-product structure, the above result is known in the engineering literature as Price’s theorem, originally published in 1958. For dimension $n = 2$, it was generalized in 1964 by McMahon to the general case $g : \mathbb{R}^2 \to \mathbb{C}$. Our contribution is to unify these results, and to give a mathematically fully rigorous proof. Precisely, we consider a normally distributed random vector $X_\Sigma \in \mathbb{R}^n$ of arbitrary dimension $n \in \mathbb{N}$, and we allow the nonlinearity $g$ to be a general tempered distribution. To this end, we replace the expectation $\mathbb{E}[g(X_\Sigma)]$ by the dual pairing $\langle g, \phi_\Sigma \rangle_{S’,S}$, where $\phi_\Sigma$ denotes the probability density function of $X_\Sigma$.

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1 Introduction

In this section, we first present a precise formulation of Price’s theorem, the proof of which we defer to Section 4. We then briefly discuss the relevance of this theorem: In a nutshell, it is a useful tool for estimating the expectation of a nonlinear function $g(X_\Sigma)$ of a Gaussian random vector $X_\Sigma \in \mathbb{R}^n$ with possibly correlated entries. In Section 3 we consider a specific example application which illustrates this. The relation of our result to the classical versions [6, 5] of Price’s theorem is discussed in Section 2.

1.1 Our version of Price’s theorem

Let us denote by $\text{Sym}_n := \{ A \in \mathbb{R}^{n \times n} : A^T = A \}$ the set of symmetric matrices, and by $\text{Sym}_n^+ := \{ A \in \text{Sym}_n : \forall x \in \mathbb{R}^n \setminus \{0\} : \langle x, Ax \rangle > 0 \}$ the set of (symmetric) positive definite matrices, where we write $\langle x, y \rangle := x^T y$ for the standard scalar product of $x, y \in \mathbb{R}^n$. For $\Sigma \in \text{Sym}_n^+$, let

$$\phi_\Sigma : \mathbb{R}^n \to (0, \infty), \; x \mapsto [(2\pi)^n \cdot \det \Sigma]^{-\frac{1}{2}} \cdot e^{-\frac{1}{2} \langle x, \Sigma^{-1} x \rangle}, \tag{1}$$

and note that $\phi_\Sigma$ is the density function of a random vector $X_\Sigma \in \mathbb{R}^n$ which follows a joint normal distribution with covariance matrix $\Sigma$, i.e., $X_\Sigma \sim N(0, \Sigma)$, see e.g. [4, Chapter 5, Theorem 5.1].
It is not hard to see that \( \varphi_\Sigma \) belongs to the Schwartz class

\[
\mathcal{S}(\mathbb{R}^n) = \left\{ g \in C^\infty(\mathbb{R}^n; \mathbb{C}) : \forall \alpha \in \mathbb{N}_0^n \forall \beta \in \mathbb{N} \exists C > 0 \forall x \in \mathbb{R}^n : |\partial^\alpha g(x)| \leq C \cdot (1+|x|)^{-N} \right\}
\]

of smooth, rapidly decreasing functions; see e.g. [2] Chapter 8 for more details on this space. In fact, \( \varphi_\Sigma (x) = \mathbb{C}_\Sigma \cdot e^{-\frac{1}{2} x^T \Sigma^{-1/2} x} = \mathbb{C}_\Sigma \cdot \Phi(\Sigma^{-1/2} x) \), where \( \Phi \) is the usual Gaussian function \( \Phi(x) = e^{-\frac{1}{2}|x|^2} \), which is well-known to belong to \( \mathcal{S}(\mathbb{R}^n) \).

Thus, for an arbitrary tempered distribution \( g \in \mathcal{S}'(\mathbb{R}^n) \) (i.e., \( g : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} \) is a continuous linear functional, with respect to a certain topology on \( \mathcal{S}(\mathbb{R}^n) \), see [2] Chapter 8), the function

\[
\Phi_g : \text{Sym}_n^+ \to \mathbb{C}, \Sigma \mapsto \langle g, \varphi_\Sigma \rangle_{\mathcal{S}', \mathcal{S}}
\]

is well-defined, where \( \langle \cdot, \cdot \rangle_{\mathcal{S}', \mathcal{S}} \) denotes the (bilinear) dual pairing between \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{S}(\mathbb{R}^n) \).

As an important special case, note that if \( g : \mathbb{R}^n \to \mathbb{C} \) is measurable and of moderate growth, in the sense that \( x \mapsto (1+|x|)^{-N} \cdot g(x) \in L^1(\mathbb{R}^n) \) for some \( N \in \mathbb{N} \), then

\[
\Phi_g (\Sigma) = \mathbb{E}[g(X_\Sigma)]
\]

is just the expectation of \( g(X_\Sigma) \), where \( X_\Sigma \sim N(0, \Sigma) \).

The main goal of this short note is to show that the function \( \Phi_g \) is smooth, and to derive an explicit formula for its partial derivatives. Thus, at least in the case of Equation (3), our goal is to calculate the partial derivatives of the expectation of a nonlinear function \( g \) of a Gaussian random vector \( X_\Sigma \sim N(0, \Sigma) \), as a function of the covariance matrix \( \Sigma \) of the vector space \( X_\Sigma \).

In order to achieve a convenient statement of this result, we first introduce a bit more notation: Write \( \underline{n} := \{1, \ldots, n\} \), and let

\[
I := \{(i, j) \in \underline{n} \times \underline{n} : i \leq j\}, \quad I_\ell := \{(i, i) \in \underline{n} \times \underline{n} \}, \quad I_{\leq} := \{(i, j) \in \underline{n} \times \underline{n} : i < j\},
\]

so that \( I = I_I \cup I_{\leq} \). Since for \( n > 1 \), the sets \( \text{Sym}_n \) and \( \text{Sym}_n^+ \) have empty interior in \( \mathbb{R}^{n\times n} \) (because they only consist of symmetric matrices), it does not make sense to talk about partial derivatives of a function \( \Phi : \text{Sym}_n^+ \to \mathbb{C} \), unless one interprets \( \text{Sym}_n^+ \) as an open subset of the vector space \( \text{Sym}_n \), rather than of \( \mathbb{R}^{n\times n} \). As a means of fixing a coordinate system on \( \text{Sym}_n \), we therefore consider the following isomorphism between \( \mathbb{R}^I \) and \( \text{Sym}_n \):

\[
\Omega : \mathbb{R}^I \to \text{Sym}_n, \quad (A_{i,j})_{1 \leq i \leq j \leq n} \mapsto \sum_{i \leq j} A_{i,j} E_{i,j} + \sum_{i > j} A_{j,i} E_{i,j}.
\]

Here, we denote by \( (E_{i,j})_{i,j \in \underline{n}} \) the standard basis of \( \mathbb{R}^{n\times n} \), i.e., \( (E_{i,j})_{k,\ell} = \delta_{i,k} \delta_{j,\ell} \) with the usual Dirac delta \( \delta_{i,k} \). Below, instead of calculating the partial derivatives of \( \Phi_g \), we will consider the function \( \Phi_g \circ \Omega|_U \), where \( U := \Omega^{-1}(\text{Sym}_n^+) \subset \mathbb{R}^I \) is open.

Finally, we introduce some notations concerning multiindices. For \( \alpha \in \mathbb{N}_0^n \), we use the usual notations \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) for \( z \in \mathbb{C}^n \), and \( \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} \). For multiindices \( \beta = (\beta(i,j))_{(i,j) \in I} \in \mathbb{N}_0^I \), we introduce a few nonstandard notations: We define the flattened version of \( \beta \) as

\[
\beta : \sum_{(i,j) \in I} \beta(i,j) (e_i + e_j) \in \mathbb{N}_0^n \quad \text{with the standard basis (}e_1, \ldots, e_n\text{) of} \mathbb{R}^n,
\]

and in addition to \( |\beta| = \sum_{(i,j) \in I} \beta(i,j) \), we will also use

\[
|\beta|_{\underline{n}} := \sum_{(i,j) \in I_{\leq}} \beta(i,j) = \sum_{i \in \underline{n}} \beta(i,i).
\]

Using these notations, our main result reads as follows:

**Theorem 1** (Generalized Price’s Theorem). Let \( g \in \mathcal{S}'(\mathbb{R}^n) \) be arbitrary. Then the function \( \Phi_g \circ \Omega|_U \) is smooth and its partial derivatives are given by

\[
\partial^\beta (\Phi_g \circ \Omega)(A) = (1/2)|\beta|_{\underline{n}} \cdot \left( \partial^\beta g, \phi_\Omega(A) \right)_{\mathcal{S}', \mathcal{S}} \quad \forall A \in U = \Omega^{-1}(\text{Sym}_n^+) \forall \beta \in \mathbb{N}_0^I.
\]

Here \( \partial^\beta g \) denotes the usual distributional derivative of \( g \). \( \blacktriangle \)
Remark. Note that even if one is in the setting of Equation (3) where \( g : \mathbb{R}^n \to \mathbb{C} \) is of moderate growth, so that \( \Phi_g (\Sigma) = \mathbb{E} [g (X_\Sigma)] \) is a “classical” expectation, it need not be the case that the derivative \( \partial^\beta g \) is given by a function, let alone one of moderate growth. Therefore, it really is useful to consider the formalism of (tempered) distributions.

1.2 Relevance of Price’s theorem

An important application of Price’s theorem is as follows: For certain values of the covariance matrix \( \Sigma \), it is usually easy to precisely calculate the expectation \( \mathbb{E} [g (X_\Sigma)] \), for example by using the independence of the entries of \( X_\Sigma \) if the respective covariances vanish, or conversely by using the linear dependence between the entries of \( X_\Sigma \) if the covariances are maximal. In addition, Price’s theorem can be used to obtain (bounds for) the partial derivatives of the map \( \Sigma \mapsto \mathbb{E} [g (X_\Sigma)] \). In combination with standard results from multivariable calculus, one can then obtain bounds for \( \mathbb{E} [g (X_\Sigma)] \) for general covariance matrices \( \Sigma \). Thus, Price’s theorem is a tool for estimating the expectation of a nonlinear function \( g (X_\Sigma) \) of a Gaussian random vector \( X_\Sigma \), even if the entries of \( X_\Sigma \) are correlated.

An example for this type of reasoning will be given in Section 3. The result which we derive there will be an important ingredient for the upcoming paper.

2 Comparison with the classical results

The original form of Price’s theorem as stated in [6] only considers the case when the nonlinearity \( g (x) = g_1 (x_1) \cdots g_n (x_n) \) has a tensor-product structure. Apart from this restriction, and up to notational differences, the formula derived in [6] is identical to the one which one gets from Theorem 1 for the special case \( g (x) = g_1 (x_1) \cdots g_n (x_n) \).

The tensor-product structure assumption concerning \( g \) was removed by McMahon [5]. Note that McMahon only considers the case \( n = 2 \), where the covariance matrix \( \Sigma \) satisfies \( \Sigma = \Sigma_\alpha = (\alpha \alpha^T) \) with \( \alpha \in (-1, 1) \). Precisely, if \( X_\alpha \sim \mathcal{N} (0, \Sigma_\alpha) \), then [5] states for \( g : \mathbb{R}^2 \to \mathbb{C} \) that

\[
\Theta_g : (-1, 1) \to \mathbb{C}, \alpha \mapsto \mathbb{E} [g (X_\alpha)] \quad \text{is smooth with} \quad \Theta_g^{(n)} (\alpha) = \mathbb{E} \left[ \frac{\partial^n g}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} (X_\alpha) \right].
\]

Finally, we mention the recent paper [7] in which a quantum-mechanical version of Price’s theorem is established. In Section II of that paper, the author reviews the “classical” case of Price’s theorem, and essentially derives the same formulas as in Theorem 1. Note though that in [7], it is assumed for calculating the \( k \)-th order derivatives of \( \Sigma \mapsto \mathbb{E} [g (X_\Sigma)] \) that the nonlinearity \( g \) is \( C^{2k} \), with a certain decay condition on the derivatives. This required classical smoothness of \( g \) is not fulfilled in many applications, see e.g. Section 3.

Despite their great utility, these three versions of Price’s theorem have some shortcomings—at least from a mathematical perspective:

- In [6, 5], the assumptions regarding the functions \( g_1, \ldots, g_n \) or \( g \) are never made explicit. The same holds for the nature of the derivatives of these functions. This is reflected in the proofs where it is assumed without justification that \( g_1, \ldots, g_n \) or \( g \) can be represented as the sum of certain Laplace transforms.

- In contrast, [7] imposes explicit assumptions concerning the nonlinearity \( g \) which ensure that the derivatives of \( g \) are defined in a classical sense; but these assumptions are rather strict, and in fact not satisfied in many applications, see Section 3.

Differently from [6, 5, 7], our version of Price’s theorem imposes precise, rather mild assumptions concerning the nonlinearity \( g \) (namely \( g \in S^r (\mathbb{R}^n) \)) and precisely explains the nature of the derivative \( \partial^\beta g \) that appears in the theorem statement: this is just a distributional derivative.

\[ \text{A general version of Price’s theorem} \]

3
Furthermore, maybe as a consequence of the preceding points, it seems that Price’s theorem is not as well-known in the mathematical community as it deserves to be. It is my hope that the present paper may promote awareness of this result.

Before closing this section, we prove that—assuming \( g \) to be a tempered distribution—the result of [5] is indeed a special case of Theorem [1] For the “classical” form of Price’s theorem considered in [6, 7], this is clear.

**Corollary 2.** Let \( f \in S' (\mathbb{R}^2) \). For \( \alpha \in (-1, 1) \), let \( \Sigma_\alpha := (\frac{1}{\alpha} \frac{\alpha}{\Omega}) \). Let

\[
\Theta_f : (-1, 1) \to \mathbb{C}, \alpha \mapsto \langle f, \phi_\Sigma \rangle_{S' S},
\]

where

\[
\phi_\Sigma : \mathbb{R}^2 \to (0, \infty), x \mapsto \left[ (2\pi)^2 \cdot \det \Sigma_\alpha \right]^{-\frac{1}{2}} \cdot e^{-\frac{1}{2} \langle x, \Sigma_\alpha^{-1} x \rangle} \\
= \left( 2\pi \cdot \sqrt{1 - \alpha^2} \right)^{-1} \cdot e^{-\frac{1}{2(1 - \alpha^2)}(x_1^2 + x_2^2 - 2\alpha x_1 x_2)}
\]
denotes the probability density function of \( X_\alpha \sim N (0, \Sigma_\alpha) \).

Then \( \Theta_f \) is smooth with \( n \)-th derivative \( \Theta_f^{(n)} (\alpha) = \left\langle \frac{\partial^{2n} f}{\partial x_1^{n} \partial x_2^{n}}, \phi_\Sigma \right\rangle_{S' S} \) for \( \alpha \in (-1, 1) \).

**Remark.** In particular, if both \( f \) and the (distributional) derivative \( \frac{\partial^{2n} f}{\partial x_1^{n} \partial x_2^{n}} \) are given by functions of moderate growth, then Equation (9) holds, i.e.,

\[
\frac{d^n}{d\alpha^n} \mathbb{E} [ f (X_\alpha) ] = \mathbb{E} \left[ \frac{\partial^{2n} f}{\partial x_1^{n} \partial x_2^{n}} (X_\alpha) \right].
\]

**Proof of Corollary 2.** In the notation of Theorem [1] we have

\[
\Theta_f (\alpha) = (\Phi_f \circ \Omega) (A^{(\alpha)}) \quad \text{with} \quad A^{(\alpha)}_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ \alpha, & \text{if } i \neq j \end{cases} \quad \text{for } (i, j) \in I = \{(1, 1), (1, 2), (2, 2)\}.
\]

Since \( \Omega (A^{(\alpha)}) = \Sigma_\alpha \) is easily seen to be positive definite, we have \( A^{(\alpha)} \in U \). Thus, setting \( \beta := n \cdot e_{(1, 2)} \in \mathbb{N}_0 \) (with the standard basis \( e_{(1, 1)}, e_{(1, 2)}, e_{(2, 2)} \) of \( \mathbb{R}^d \)), the chain-rule shows that \( \Theta_f \) is smooth, with

\[
\Theta_f^{(n)} (\alpha) = \frac{d^n}{d\alpha^n} [(\Phi_f \circ \Omega) (A^{(\alpha)})] = \left[ \frac{\partial^{\beta} (\Phi_f \circ \Omega)}{\partial x_1^{\alpha} \partial x_2^{\alpha}} \right] (A^{(\alpha)}) \\
= \left\langle \frac{\partial^{\beta} f}{\partial x_1^{\alpha} \partial x_2^{\alpha}}, \phi_{\Omega(A^{(\alpha)})} \right\rangle_{S' S}.
\]

**3 An example application of Price’s theorem**

In this section, we derive bounds for the expectation \( \mathbb{E} [ g (X_\alpha, Y_\alpha) ] \), where \( X_\alpha, Y_\alpha \) follow a joint normal distribution with covariance matrix \( (\frac{1}{\alpha} \frac{\alpha}{\Omega}) \), and where the nonlinearity \( g = g_\tau \) is just a componentwise truncation (or clipping) to the interval \([-\tau, \tau]\). We remark that this example has already been considered by Price [6] himself, but that his arguments are not completely mathematically rigorous, as explained in Section [2]. Precisely, we obtain the following result:

**Lemma 3.** Let \( \tau > 0 \) be arbitrary, and define

\[
f_\tau : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} \tau, & \text{if } x \geq \tau, \\ x, & \text{if } x \in [-\tau, \tau], \\ -\tau, & \text{if } x \leq -\tau. \end{cases}
\]

For \( \alpha \in [-1, 1] \), set \( \Sigma_\alpha := (\frac{1}{\alpha} \frac{\alpha}{\Omega}) \), and let \( (X_\alpha, Y_\alpha) \sim N (0, \Sigma_\alpha) \). Finally, define

\[
F_\tau : [-1, 1] \to \mathbb{R}^2, \alpha \mapsto \mathbb{E} \left[ f_\tau (X_\alpha) \cdot f_\tau (Y_\alpha) \right].
\]

Then \( F_\tau \) is continuous and \( F_\tau|_{[0, 1]} \) is convex with \( F_\tau (0) = 0 \). In particular, \( F_\tau (\alpha) \leq \alpha \cdot F_\tau (1) \) for all \( \alpha \in [0, 1] \).
Proof. It is easy to see that \( f_x \) is bounded and Lipschitz continuous, i.e., \( f_x \in W^{1, \infty} (\mathbb{R}) \) with weak derivative \( f'_x = \mathbf{1}_{(-\tau, \tau)} \). Therefore, also \( g_x := f_x \otimes f_x \in W^{1, \infty} (\mathbb{R}^2) \subset S' (\mathbb{R}^2) \), with weak derivative \( \frac{\partial^2 g_x}{\partial x_1 \partial x_2} = \mathbf{1}_{(-\tau, \tau)} \otimes \mathbf{1}_{(-\tau, \tau)} = \mathbf{1}_{(-\tau, \tau)^2} \). Directly from the definition of the weak derivative, in combination with Fubini’s theorem and the fundamental theorem of calculus, one can derive \( \frac{\partial^4}{\partial x_1 \partial x_2^2} g_x = \delta (\tau, \tau) - \delta (\tau, -\tau) - \delta (-\tau, \tau) + \delta (-\tau, -\tau) \). We leave the details to the interested reader.

Now, Corollary 2 shows that \( F_x \rvert_{(-1, 1)} = \Theta g_x \) is smooth with

\[
F''_x (\alpha) = \left( \frac{\partial^4}{\partial x_1 \partial x_2^2} g_x, \phi_{\Sigma_\alpha} \right)_{S', S} = \phi_{\Sigma_\alpha} (\tau, \tau) - \phi_{\Sigma_\alpha} (-\tau, \tau) - \phi_{\Sigma_\alpha} (\tau, -\tau) + \phi_{\Sigma_\alpha} (-\tau, -\tau)
\]

for \( \alpha \in (-1, 1) \). We want to show \( F''_x (\alpha) \geq 0 \) for \( \alpha \in [0, 1] \). Since \( \phi_{\Sigma_\alpha} \) is symmetric, it suffices to show \( \phi_{\Sigma_\alpha} (\tau, \tau) - \phi_{\Sigma_\alpha} (-\tau, \tau) \geq 0 \), which is easily seen to be equivalent to

\[
\exp \left( -\frac{1}{2} \frac{1}{(1 - \alpha^2)} (2\tau^2 - 2\alpha^2) \right) \geq \exp \left( -\frac{1}{2} \frac{1}{(1 - \alpha^2)} (2\tau^2 + 2\alpha^2) \right)
\]

\( \iff 2\tau^2 + 2\alpha^2 \geq 2\tau^2 - 2\alpha^2 \)

\( \iff 4\alpha^2 \geq 0, \)

which clearly holds for \( \alpha \in [0, 1] \).

To finish the proof, we only need to show that \( F_x \) is continuous with \( F_x (0) = 0 \). To see this, let \( (X, Z) \sim N (0, I_2) \), with the the 2-dimensional identity matrix \( I_2 \). For \( Y_\alpha := \alpha X + \sqrt{1 - \alpha^2} Z \), it is not hard to see that \( (X, Y_\alpha) \sim N (0, \Sigma_\alpha) \), so that

\[
|F_x (\alpha) - F_x (\beta)| = \left| E \left[ g_x (X, Y_\alpha) \right] - E \left[ g_x (X, Y_\beta) \right] \right|
\]

\( \quad \text{(since } |f_x (X)| \leq \tau \text{)} \)

\( \quad \leq \tau \cdot E \left| f_x (Y_\alpha) - f_x (Y_\beta) \right| \)

\( \quad \leq \tau \cdot |\alpha - \beta| \cdot E |X| + \tau \cdot \sqrt{1 - \alpha^2 - \beta^2 - 2 \beta (1 - \alpha^2)} \cdot E |Z| \xrightarrow{\beta \to \alpha} 0, \)

which shows that \( F_x \) is indeed continuous. Furthermore, we see by independence of \( X, Z \) that

\[
F_x (0) = E \left[ f_x (X) \cdot f_x (Z) \right] = E \left[ f_x (X) \right] \cdot E \left[ f_x (Z) \right] = 0,
\]

since \( E \left[ f_x (X) \right] = -E \left[ f_x (-X) \right] = -E \left[ f_x (X) \right], \) because of \( X \sim -X \) and \( f_x (x) = -f_x (-x) \) for \( x \in \mathbb{R} \).

\( \square \)

4 The proof of Theorem 1

The main idea of the proof is to use Fourier analysis, since the Fourier transform \( \mathcal{F} \phi_S \) of the density function \( \phi_S \) will turn out to be much easier to handle than \( \phi_S \) itself. This is similar but slightly different from the approach in [11, 12], where the Laplace transform is used instead. For this, we will use the normalization

\[
\mathcal{F} \phi (\xi) := \hat{\phi} (\xi) := \int_{\mathbb{R}^n} \phi (x) \cdot e^{-i (x, \xi)} \, dx \quad \text{for } \xi \in \mathbb{R}^n \text{ and } \phi \in L^1 (\mathbb{R}^n).
\]

It is well-known that the restriction \( \mathcal{F} : S (\mathbb{R}^n) \to S (\mathbb{R}^n) \) of \( \mathcal{F} \) is a well-defined homeomorphism, with inverse \( \mathcal{F}^{-1} : S (\mathbb{R}^n) \to S (\mathbb{R}^n) \), where \( \mathcal{F}^{-1} \phi (x) = (2\pi)^{-n} \cdot \mathcal{F} \varphi (-x) \). By duality, the Fourier transform also extends to a bijection \( \mathcal{F} : S' (\mathbb{R}^n) \to S' (\mathbb{R}^n) \) defined by \( \langle \mathcal{F} g, \varphi \rangle_{S', S} := \langle g, \mathcal{F} \varphi \rangle_{S', S} \text{ for } g \in S' (\mathbb{R}^n) \) and \( \varphi \in S (\mathbb{R}^n) \). Further, it is well-known for the

\( ^1 \text{This definition is motivated by the identity } \int f (x) \cdot g (x) \, dx = \int \int f (\xi) e^{-i (x, \xi)} g (x) \, d\xi dx = \int f (\xi) \hat{g} (\xi) \, d\xi \)

which is valid for \( f, g \in L^1 (\mathbb{R}^n) \) thanks to Fubini’s theorem.
distributional derivatives $\partial^n g$ of $g \in S' (\mathbb{R}^n)$ defined by $(\partial^n g, \varphi)_{S', S} = (-1)^{|\alpha|} \cdot (g, \partial^n \varphi)_{S', S}$ that if we set \[ X^\alpha \cdot \varphi : \mathbb{R}^d \to \mathbb{C}, x \mapsto x^\alpha \cdot \varphi (x) \] and \[ (X^\alpha \cdot g, \varphi)_{S', S} = (g, X^\alpha \cdot \varphi)_{S', S} \]
for $g \in S' (\mathbb{R}^n)$ and $\varphi \in S (\mathbb{R}^n)$, then we have
\[ \mathcal{F} [\partial^n g] = i^{|\alpha|} \cdot X^\alpha \cdot \mathcal{F} g \quad \forall g \in S' (\mathbb{R}^n). \] (10)

These results can be found e.g. in [1, Chapter 14], or (with a slightly different normalization of the Fourier transform) in [2, Sections 8.3 and 9.2].

Finally, we will use the formula
\[ (2\pi)^n \cdot \mathcal{F}^{-1} \phi_\Sigma (\xi) = \int_{\mathbb{R}^n} e^{i (x, \xi)} : \phi_\Sigma (x) \, dx = \mathbb{E} \left[ e^{i \langle \xi, X_\Sigma \rangle} \right] = e^{\frac{1}{2} \langle \xi, \xi \rangle} =: \psi_\Sigma (\xi) \quad \text{for } \xi \in \mathbb{R}^n, \] (11)
which is proved in [1, Chapter 5, Theorem 4.1]; in probabilistic terms, this is a statement about the characteristic function of the random vector $X_\Sigma \sim N (0, \Sigma)$.

Next, by assumption of Theorem 4 we have $g \in S' (\mathbb{R}^n)$ and hence $\mathcal{F} g \in S' (\mathbb{R}^n)$. Thus, by the structure theorem for tempered distributions (see e.g. [1, Theorem 17.10]), there are $L \in \mathbb{N}$, certain $\alpha_1, \ldots, \alpha_L \in \mathbb{N}_0^n$ and certain polynomially bounded, continuous functions $f_1, \ldots, f_L$ satisfying $\mathcal{F} g = \sum_{\ell=1}^L \partial^{\alpha_\ell} f_\ell, \ i.e., \ g = \sum_{\ell=1}^L \mathcal{F}^{-1} (\partial^{\alpha_\ell} f_\ell)$. Since both sides of the target identity \[ \Phi_g (\Sigma) = (g, \phi_\Sigma)_{S', S} = (\mathcal{F} g, \mathcal{F}^{-1} \phi_\Sigma)_{S', S} = (f, \psi_\Sigma)_{S', S} \] (12)
are linear, we thus assume without loss of generality that $g = \mathcal{F}^{-1} (\partial^n f)$ for some $\alpha \in \mathbb{N}_0^n$ and some continuous $f : \mathbb{R}^n \to \mathbb{C}$ which is polynomially bounded, say $|f (x)| \leq C \cdot (1 + |x|)^N$ for all $x \in \mathbb{R}^n$ and certain $C > 0, N \in \mathbb{N}_0$.

Our first goal in the remainder of the proof is to show that one can justify “differentiation under the integral” with respect to $A_{i,j}$ with $\Sigma = \Omega (A)$ in the last integral in Equation (12).

It is easy to see that $A \mapsto \psi_{\Omega (A)} (\xi)$ is smooth with \[ \partial_{A_{i,j}} \psi_{\Omega (A)} (\xi) = e^{\frac{1}{2} \langle \xi, \Omega (A) \xi \rangle} \cdot \partial_{A_{i,j}} \left( -\frac{1}{2} \cdot \sum_{j, \ell=1}^n [\Omega (A)]_{j, \ell} \cdot \xi_j \xi_\ell \right) \]
\[ = \begin{cases} -\frac{1}{2} \cdot \xi_i \xi_j \cdot \psi_{\Omega (A)} (\xi), & \text{if } i = j, \\ -\xi_i \xi_j \cdot \psi_{\Omega (A)} (\xi), & \text{if } i < j \end{cases} \]
for all $\xi \in \mathbb{R}^n$ and arbitrary $(i, j) \in I$ and $A \in U$. Based on this identity, a straightforward induction shows (with the notations $|\beta|_n$ and $\beta_n$ from Equations (11) and (12)) that
\[ \partial^n \psi_{\Omega (A)} (\xi) = (-1)^{|\beta|} \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot \xi_\beta \cdot \psi_{\Omega (A)} (\xi) \quad \forall \beta \in \mathbb{N}_0^n, \xi \in \mathbb{R}^n, A \in U. \] (13)

Next, let us show for arbitrary $\gamma \in \mathbb{N}_0^n$ by induction on $|\alpha|$ that for each $\alpha \in \mathbb{N}_0^n$ there is a polynomial $p_{\alpha, \gamma} = p_{\alpha, \gamma} (\Xi, \Lambda, B)$ in the variables $\Xi, \Lambda \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$ that satisfies \[ \partial^n \psi_{\Omega (A)} (\xi) = p_{\alpha, \gamma} (\xi, \Sigma \xi, \Sigma) \cdot \psi_{\Omega (A)} (\xi) \quad \forall \Sigma \in \text{Sym}_n^+, \xi \in \mathbb{R}^n. \] (14)

For $\alpha = 0$, this is trivially true for $p_{\alpha, \gamma} (\Xi, \Lambda, B) := \Xi \gamma$. Now, assume that Equation (14) holds for some $\alpha \in \mathbb{N}_0^n$, and let $i \in \mathbb{N}$ be arbitrary. We want to show that Equation (14) also holds for $\alpha + e_i$ instead of $\alpha$. To this end, note for arbitrary $\ell \in \mathbb{N}$ that
\[ p_{\alpha, \gamma}^{(\ell, 1)} (\Xi, \Lambda, B) := \partial_{\xi_i} p_{\alpha, \gamma} (\Xi, \Lambda, B) \quad \text{and} \quad p_{\alpha, \gamma}^{(\ell, 2)} := \partial_{\Lambda_{i, \ell}} p_{\alpha, \gamma} (\Xi, \Lambda, B). \]
are polynomials, and that
\[
\partial_{\xi_i} \psi_{\Sigma} (\xi) = e^{- \frac{1}{2} (\xi; \Sigma \xi)} \cdot \partial_{\xi_i} \left[ - \frac{1}{2} \cdot \sum_{j, \ell = 1}^{n} \Sigma_{j, \ell} \cdot \xi_j \xi_{\ell} \right]
\]

(product rule) \[= - \frac{1}{2} \cdot \psi_{\Sigma} (\xi) \cdot \sum_{j, \ell = 1}^{n} [\Sigma_{j, \ell} \cdot (\delta_{i,j} \xi_{\ell} + \xi_j \delta_{i,\ell})]
\]

\[= - \frac{1}{2} \cdot \psi_{\Sigma} (\xi) \cdot \left( \sum_{i = 1}^{n} [\Sigma_{i,i} \xi_i] + \sum_{j = 1}^{n} [\Sigma_{j,j} \cdot \xi_j] \right)
\]

(symmetry of \(\Sigma\)) \[= - \frac{1}{2} \cdot \psi_{\Sigma} (\xi) \cdot \left( (\Sigma \xi)_i + (\Sigma \xi)_i \right) = - \psi_{\Sigma} (\xi) \cdot (\Sigma \xi)_i.
\]

Therefore, we get by induction
\[
\partial_{\xi}^{\alpha+\varepsilon_i} [\xi^T \cdot \psi_{\Sigma} (\xi)]
\]
\[= \partial_{\xi_i} [p_{\alpha,\gamma} (\xi, \Sigma \xi, \Sigma) \cdot \psi_{\Sigma} (\xi)]
\]
\[= \psi_{\Sigma} (\xi) \cdot \partial_{\xi_i} [p_{\alpha,\gamma} (\xi, \Sigma \xi, \Sigma)] + p_{\alpha,\gamma} (\xi, \Sigma \xi, \Sigma) \cdot \partial_{\xi_i} \psi_{\Sigma} (\xi)
\]
\[= \psi_{\Sigma} (\xi) \cdot \left[ p_{\alpha,\gamma}^{(i,1)} (\xi, \Sigma \xi, \Sigma) + \sum_{\ell = 1}^{n} p_{\alpha,\gamma}^{(\ell,2)} (\xi, \Sigma \xi, \Sigma) \cdot \partial_{\xi_i} (\Sigma \xi)_\ell \right] - p_{\alpha,\gamma} (\xi, \Sigma \xi, \Sigma) \cdot (\Sigma \xi)_i \cdot \psi_{\Sigma} (\xi)
\]
\[= \psi_{\Sigma} (\xi) \cdot \left( p_{\alpha,\gamma}^{(i,1)} (\xi, \Sigma \xi, \Sigma) - p_{\alpha,\gamma} (\xi, \Sigma \xi, \Sigma) \cdot (\Sigma \xi)_i + \sum_{\ell = 1}^{n} p_{\alpha,\gamma}^{(\ell,2)} (\xi, \Sigma \xi, \Sigma) \cdot \Sigma_{i,\ell} \right)
\]
with \(p_{\alpha+\varepsilon_i,\gamma} (\Xi, \Lambda, B) := p_{\alpha,\gamma}^{(i,1)} (\Xi, \Lambda, B) - p_{\alpha,\gamma} (\Xi, \Lambda, B) \cdot \Lambda_i + \sum_{\ell = 1}^{n} p_{\alpha,\gamma}^{(\ell,2)} (\Xi, \Lambda, B) \cdot B_{i,\ell}.

Now we are ready to justify differentiation under the integral (as in [2 Theorem 2.27]) for the last integral appearing in Equation (12), with \(\Sigma = \Omega (A)\), i.e., for
\[
A \nrightarrow \int_{\mathbb{R}^n} f (\xi) \cdot (\partial^\alpha \psi_{\Omega (A)} (\xi)) \, d\xi.
\]

Indeed, let \(A_0 \in U\) be arbitrary. Since \(U\) is open, there is some \(\varepsilon > 0\) satisfying \(\overline{B_\varepsilon (A_0)} \subset U\), for the closed ball \(B_\varepsilon (A_0) = \{ A \in \mathbb{R}^I : |A - A_0| \leq \varepsilon \}\), with the euclidean norm \(|\cdot|\) on \(\mathbb{R}^I\). The open ball \(B_\varepsilon (A_0)\) is defined similarly.

Now, with
\[
\sigma_{\min} (A) := \inf_{x \in \mathbb{R}^n} \langle x, Ax \rangle \quad \text{for } A \in \mathbb{R}^{n \times n}
\]
we have for \(A, B \in \mathbb{R}^{n \times n}\) and arbitrary \(x \in \mathbb{R}^n\) with \(|x| = 1\) that
\[
\sigma_{\min} (A) \leq \langle x, Ax \rangle = \langle x, Bx \rangle + \langle x, (A - B) x \rangle \leq \langle x, Bx \rangle + \| A - B \|
\]
Since this holds for all \(|x| = 1\), we get \(\sigma_{\min} (A) \leq \sigma_{\min} (B) + \| A - B \|\), and by symmetry \(|\sigma_{\min} (A) - \sigma_{\min} (B)| \leq \| A - B \|\). Therefore, the continuous function \(A \mapsto \sigma_{\min} (\Omega (A))\) has a (positive) minimum on the compact set \(\overline{B_\varepsilon (A_0)}\), so that \(\langle \xi, \Omega (A) \xi \rangle \geq c \cdot |\xi|^2\) for all \(\xi \in \mathbb{R}^n\) and \(A \in \overline{B_\varepsilon (A_0)}\), for a positive \(c > 0\). Furthermore, there is some \(K = K (A_0) > 0\) with \(\| \Omega (A) \| \leq K\) for all \(A \in \overline{B_\varepsilon (A_0)}\).

Now, since the map \(U \times \mathbb{R}^n \ni (A, \xi) \mapsto \psi_{\Omega (A)} (\xi) \in \mathbb{C}\) is smooth, we have (in view of Equations (13) and (14)) for arbitrary \(\beta \in \mathbb{N}_0^I\) and \(\xi \in \mathbb{R}^n\) that
\[
\partial_A^\beta [f (\xi) \cdot (\partial^\alpha \psi_{\Omega (A)} (\xi))] = f (\xi) \cdot \partial_{\xi}^\beta \left[ \partial_A^\beta \psi_{\Omega (A)} (\xi) \right]
\]
(Eq. 13) \[= f (\xi) \cdot (-1)^{|\beta|} \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot \partial_{\xi}^\beta [\xi^\beta \cdot \psi_{\Omega (A)} (\xi)]
\]
(Eq. 14) \[= f (\xi) \cdot (-1)^{|\beta|} \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot p_{\alpha,\beta} (\xi, \Omega (A) \xi, \Omega (A)) \cdot \psi_{\Omega (A)} (\xi).
\]
Using the polynomial growth restriction concerning \( f \), we thus see that there is a constant \( C_{\alpha,\beta} > 0 \) and some \( M_{\alpha,\beta} \in \mathbb{N} \) with

\[
\left| \partial^3_A \left[ f (\xi) \cdot (\partial^a \psi_{\Omega(A)} (\xi)) \right] \right| = \left| f (\xi) \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot p_{\alpha,\beta,} (\xi, \Omega (A) \xi, \Omega (A)) \cdot \psi_{\Omega(A)} (\xi) \right|
\leq C \cdot (1 + |\xi|)^N \cdot C_{\alpha,\beta} \cdot (1 + |\xi| + |\Omega (A) \xi| + \| \Omega (A) \|)^{M_{\alpha,\beta}} \cdot e^{-\frac{1}{2} |\xi| \Omega (A) \xi}
\leq C_{\alpha,\beta} C \cdot (1 + |\xi|)^N \cdot (1 + |\xi| + K \cdot |\xi| + K)^{M_{\alpha,\beta}} \cdot e^{-\frac{1}{2} |\xi|^2}
=: h_{\alpha,\beta,A_0,f} (\xi),
\]

for all \( \xi \in \mathbb{R}^n \) and all \( A \in B_\varepsilon (A_0) \). Since \( h_{\alpha,\beta,A_0,f} \) is independent of \( A \in B_\varepsilon (A_0) \) and we clearly have \( h_{\alpha,\beta,A_0,f} \in L^1 (\mathbb{R}^n) \), [2] Theorem 2.27 shows that the map

\[
B_\varepsilon (A_0) \to \mathbb{C}, A \mapsto \int_{\mathbb{R}^n} f (\xi) \cdot (\partial^a \psi_{\Omega(A)} (\xi)) \, d\xi
\]
is smooth, with

\[
\partial^3_A \left[ \int_{\mathbb{R}^n} f (\xi) \cdot (\partial^a \psi_{\Omega(A)} (\xi)) \, d\xi \right] = \int_{\mathbb{R}^n} f (\xi) \cdot \partial^3_A (\psi_{\Omega(A)} (\xi)) \, d\xi
\]

(derivatives commute for smooth fcts.) = \[
(\text{Eq. (10)}) = (-1)^{|\beta|} \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot \int_{\mathbb{R}^n} f (\xi) \cdot \partial^2_A \left( \xi^\beta \cdot \psi_{\Omega(A)} (\xi) \right) \, d\xi
= (-1)^{|\beta|} \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot \left( f, \partial^a \left[ X^{\beta_1} \psi_{\Omega(A)} \right] \right)_{S', \mathcal{S}}
\]

(\text{Eq. (11)}) = \[
(-1)^{|\beta| + |\alpha|} \cdot \left( \frac{1}{2} \right)^{|\beta|} \cdot (2\pi)^n \cdot \left[ X^{\beta_1} \cdot \partial^a f, F^{-1} \phi_{\Omega(A)} \right]_{S', \mathcal{S}}
\]

(\text{Eq. (12)}) = \[
\left( \frac{1}{2} \right)^{|\beta|} \cdot (2\pi)^n \cdot (-1)^{|\alpha|} \cdot \left[ F \left[ \partial^{\beta_1} g \right], F^{-1} \phi_{\Omega(A)} \right]_{S', \mathcal{S}}.
\]

Here, the step marked with (\text{*}) is valid, since we assumed above that \( g = F^{-1} (\partial^a f) \), so that Equation (10) shows

\[
F \left[ \partial^{\beta_1} g \right] = i^{\beta_1} \cdot X^{\beta_1} \cdot F f = (-1)^{|\beta_1|} \cdot X^{\beta_1} \cdot \partial^a f, \quad \text{since} \quad |\beta_1| = 2 |\beta|.
\]

Now, recall from Equation (12) that

\[
\Phi_g (\Omega (A)) = (-1)^{|\alpha|} \cdot (2\pi)^{-n} \cdot \int_{\mathbb{R}^n} f (\xi) \cdot (\partial^a \psi_{\Omega(A)} (\xi)) \, d\xi.
\]

In combination, this shows that \( \Phi_g \circ \Omega \) is smooth on \( B_\varepsilon (A_0) \), with partial derivatives given by

\[
\partial^\beta \left[ \Phi_g \circ \Omega \right] (A) = \left( \frac{1}{2} \right)^{|\beta|} \cdot \left( F \left[ \partial^{\beta_1} g \right], F^{-1} \phi_{\Omega(A)} \right)_{S', \mathcal{S}} = \left( \frac{1}{2} \right)^{|\beta|} \cdot \left( \partial^{\beta_1} g, \phi_{\Omega(A)} \right)_{S', \mathcal{S}},
\]
as claimed. Since \( A_0 \in U \) was arbitrary, the proof is complete. \( \square \)

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This paper originated from an inspiring discussion with Martin Genzel: He was interested in a proof for the result of Lemma 3 which will be crucially used in the upcoming paper [3]. We didn’t make any progress on this until our colleague Ali Hashemi pointed us to Price’s theorem. Since an application of this theorem for the given example really leads to distributional derivatives (as seen in the proof of Lemma 3), and since the assumptions in [3] are not explicitly formulated, it was not entirely clear whether the result really is applicable. This caused me to examine Price’s theorem more closely, and eventually led to the present paper.

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