Intersections of adelic groups on a surface

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Abstract. We solve a technical problem related to adeles on an algebraic surface. Given a finite set of natural numbers, one can associate with it an adelic group. We show that this operation commutes with taking intersections if the surface is defined over an uncountable field, and we provide a counterexample otherwise.

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§ 1. Introduction

Adeles for surfaces were introduced by Parshin [1] as a generalization of classical adeles for global fields, in particular, fields of rational functions on curves (over finite fields). In this generalization, points on curves are replaced by flags, that is, chains of embedded irreducible subvarieties. One can define several adelic groups in accordance with the codimensions of members of flags. More precisely, given a surface $X$ and a subset $I \subset \{0, 1, 2\}$, one associates an adelic group $A_X(I, \mathcal{O}_X)$ (this is a particular case of an adelic group $A_X(I, \mathcal{F})$ defined for an arbitrary quasicoherent sheaf $\mathcal{F}$ on $X$). If $I \subset J$, then there is a canonical map $\varphi_{IJ} : A_X(I, \mathcal{O}_X) \rightarrow A_X(J, \mathcal{O}_X)$, which is injective if $X$ is regular (Proposition 1). Thus all adelic groups are subgroups of the biggest one, $A_X(\{0, 1, 2\}, \mathcal{O}_X)$, and a natural question is that of whether $A_X(I \cap J, \mathcal{O}_X)$ is equal to $A_X(I, \mathcal{O}_X) \cap A_X(J, \mathcal{O}_X)$. Note that the analogous question for rational (that is, incomplete) adeles is trivial [1], §2.

When $X$ is projective, a positive answer to this question was obtained in [2] by means of global methods. This was used later in [3] in order to prove the Riemann-Roch theorem for surfaces using adeles. In this paper we give a positive answer to the above question when the ground field is uncountable (Theorem 1) and provide a counterexample in the affine case when the ground field is countable (Theorem 2). Also, we give a positive answer to the above question in the projective case for an arbitrary locally free sheaf of finite rank. The proof is essentially the...
same as the proof in [2], Proposition 4.3, for the structure sheaf; we include it here with the kind permission of A. N. Parshin.

Generalization to a higher-dimensional case remains open. In fact, it is not even known whether the map $\varphi_{IJ}$ is injective (even for regular varieties). Notice that analogues of all statements of the paper for restricted adeles on projective Cohen-Macaulay varieties of arbitrary dimension hold true by [4], Theorems 4 and 5.

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§ 2. Statement of the main result

First we recall some general facts about adeles on Noetherian schemes. However, as our main result concerns surfaces only, the reader may easily restrict himself to these from the very beginning. Note that the definition of adeles on a surface has a much more explicit version [1], [5], [6].

Let $X$ be a Noetherian scheme and $\mathcal{F}$ be a quasicoherent sheaf on $X$. By $S(X)_p$, $p \geq 0$, denote the set of all nondegenerate length $p$ flags on $X$, that is, sequences of schematic points $(\eta_0, \ldots, \eta_p)$ such that $\eta_{i+1} \in \overline{\eta_i}$ and $\eta_{i+1} \neq \eta_i$, where $\overline{\eta}$ denotes the closure of a point $\eta$ in $X$. Given a subset $S \subset S(X)_p$, denote by $A_X(S, \mathcal{F})$ the corresponding group of adeles [7], [8]. The functor $\mathcal{F} \mapsto A_X(S, \mathcal{F})$ is exact and commutes with filtered colimits. There is a canonical embedding (see [8], Proposition 2.1.4)

$$A_X(S, \mathcal{F}) \hookrightarrow \prod_{\Delta \in S} A_X(\{\Delta\}, \mathcal{F}).$$

Thus an adele $a \in A_X(S, \mathcal{F})$ is uniquely determined by its local components $a_\Delta \in A_X(\{\Delta\}, \mathcal{F})$. By definition, put $A_X(\emptyset, \mathcal{F}) := \mathcal{F}(X) = H^0(X, \mathcal{F})$. We will use the following facts.

Lemma 1. Let $f : X \to Y$ be a finite morphism between Noetherian schemes. Then for any subset $T \subset S(Y)_p$, $p \geq 0$, there is a canonical isomorphism

$$A_Y(T, f_* \mathcal{F}) \sim A_X(f^{-1}T, \mathcal{F}),$$

where $f^{-1}T \subset S(X)_p$ is defined in a natural way.

The proof of Lemma 1 is completely analogous to the proof of Proposition 3.1.7 in [9].

Lemma 2. If $X$ is affine, then for any subset $S \subset S(X)_p$, $p \geq 0$, there is a canonical isomorphism

$$A_X(S, \mathcal{O}_X) \otimes_{\mathcal{O}_X(\overline{X})} \mathcal{F}(X) \sim A_X(S, \mathcal{F}).$$

The proof of Lemma 2 is similar to the proof of Proposition 1.5 in [4]. Namely, since the functor $\mathcal{F} \mapsto A_X(S, \mathcal{F})$ commutes with filtered colimits, one may assume that $\mathcal{F}$ is coherent. Further, using the exactness of this functor and finite presentation of coherent sheaves, one reduces the problem to the obvious case $\mathcal{F} = \mathcal{O}_X$. 

In what follows we assume that $X$ is irreducible. Let $d$ be the dimension of $X$. Let $I$ be a subset in $\{0,1,\ldots,d\}$, that is, $I = \{i_0, \ldots, i_p\}$ for a strictly increasing sequence of integers $0 \leq i_0 < \cdots < i_p \leq d$, $0 \leq p \leq d$. Put

$$A_X(I, \mathcal{F}) := A_X(S(I), \mathcal{F}),$$

where $S(I)$ is the set of all flags $(\eta_0, \ldots, \eta_p)$ with $\text{codim}_X(\eta_j) = i_j$, $0 \leq j \leq p$. In particular, $A_X(\{0\}, \mathcal{F})$ is the fibre of $\mathcal{F}$ at the generic point of $X$. By $A_X(I, \mathcal{F})$ denote a flabby sheaf on $X$ defined by the formula

$$A_X(I, \mathcal{F})(U) := A_U(I, \mathcal{F}|_U),$$

where $U$ is an open subset of $X$. By definition, put $A_X(\emptyset, \mathcal{F}) := \mathcal{F}$. Lemma 2 immediately implies the following fact.

**Corollary 1.** For any subset $I \subset \{0,\ldots,d\}$, there is a canonical isomorphism of sheaves

$$A_X(I, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} A_X(I, \mathcal{F}).$$

Given subsets $I \subset J \subset \{0,\ldots,d\}$, there is a canonical map induced by boundary maps on adelic groups ([8], §2.2),

$$\varphi_{IJ}: A_X(I, \mathcal{F}) \to A_X(J, \mathcal{F}).$$

**Proposition 1.** If $X$ is a regular irreducible surface over a field and $\mathcal{F}$ is flat, then for all subsets $I \subset J \subset \{0,1,2\}$, the map $\varphi_{IJ}$ is injective.

**Proof.** The explicit definition of adeles on a regular surface ([1], §2, [5], §3.3, [6], §8.5) implies the proposition for $\mathcal{F} = \mathcal{O}_X$. Furthermore, the morphism of sheaves

$$A_X(I, \mathcal{O}_X) \to A_X(J, \mathcal{O}_X),$$

induced by $\varphi_{IJ}$, is injective as well. Combining flatness of $\mathcal{F}$, Corollary 1, and left exactness of $H^0$, we obtain the required result.

**Remark 1.** It is likely that Proposition 1 is also true when $X$ is a normal excellent two-dimensional irreducible Noetherian scheme.

Thus, under the conditions of Proposition 1, all groups $A_X(I, \mathcal{F})$ are canonically embedded into the group $A_X(\{0,1,2\}, \mathcal{F})$. The main question that we address in the paper is that of whether $A_X(I \cap J, \mathcal{F})$ is equal to $A_X(I, \mathcal{F}) \cap A_X(J, \mathcal{F})$ for arbitrary subsets $I$ and $J$ in $\{0,1,2\}$.

Recall that a flat coherent sheaf is the same as a locally free sheaf of finite rank [10], Proposition (3.G). Our main result is as follows.

**Theorem 1.** Let $X$ be a regular irreducible surface over a field $k$, $\mathcal{F}$ be a flat quasicoherent sheaf on $X$, and $I$ and $J$ be two subsets in $\{0,1,2\}$. Suppose that one of the following conditions is satisfied:

(i) $I \cap J = I \setminus \{0\}$;

(ii) the field $k$ is uncountable;

(iii) $\mathcal{F}$ is locally free of finite rank and $X$ is projective.
Then the equality

$$A_X(I \cap J, \mathcal{F}) = A_X(I, \mathcal{F}) \cap A_X(J, \mathcal{F})$$

holds, where the intersection is taken in $A_X(\{0, 1, 2\}, \mathcal{F})$.

The proof of the theorem under condition (i) is simple and is given in §3. The proofs of the theorem under conditions (ii) and (iii) are based on several auxiliary statements and are given in §4 and §5, respectively. Note that the proof of the theorem under condition (ii) uses only elementary facts from commutative algebra, while the proof of the theorem under condition (iii) uses Serre duality. In particular, for a projective regular surface over an uncountable field one has two different proofs.

§ 3. The case of condition (i)

Let $X$ and $\mathcal{F}$ be as in Theorem 1.

Lemma 3. Let $X$ be a regular irreducible surface over a field and let $\mathcal{F}$ be a flat sheaf. Suppose that for any affine open subset $U \subset X$, we have

$$A_U(I \cap J, O_U) = A_U(I, O_U) \cap A_U(J, O_U).$$

Then

$$A_X(I \cap J, \mathcal{F}) = A_X(I, \mathcal{F}) \cap A_X(J, \mathcal{F}).$$

Proof. Combine flatness of $\mathcal{F}$, Corollary 1 and left exactness of $H^0$.

Proof of Theorem 1, (i). Suppose that condition (i) is satisfied. By Lemma 3, it is enough to consider the case $\mathcal{F} = O_X$. Also, assume that $0 \in I$ (otherwise, there is nothing to prove). It follows from the explicit definition of adelic groups on a regular surface ([1], §2, [5], §3.3, [5], §8.5) that an element $a \in A_X(I, O_X)$ belongs to $A_X(I \setminus \{0\}, O_X)$ if and only if for any flag $\Delta = (\eta_X, \eta_1, \ldots, \eta_p)$ in $M(I)$, we have that $a_\Delta \in A_X(\Delta \setminus \eta_X, O_X)$, where $\eta_X$ is the generic point of $X$ (if $I = \{0\}$, then the last condition should be replaced by $a \in O_{X,x}$ for any point $x \in X$).

Since $X$ is a regular surface, the ring $A := A_X(\Delta \setminus \eta_X, O_X)$ is also regular (this follows from the explicit description of local factors in adeles on a regular surface [5], §3.3). Therefore, $a_\Delta \in A$ for all $\Delta \in M(I)$ if and only if $v_D(a) \geq 0$ for any prime divisor $D$ in Spec($A$); [11], Corollary 11.4. By the definition of adeles, $v_D(a) \geq 0$ for all $D$ that are not analytic components of a prime divisor on $X$. This implies that $a_\Delta \in A$ for all $\Delta \in M(I)$ if and only if $a \in A_X(\{1, 2\}, O_X)$. In other words, we have shown the equality

$$A_X(I \setminus \{0\}, O_X) = A_X(I, O_X) \cap A_X(\{1, 2\}, O_X).$$

Since $I \setminus \{0\} = I \cap J = I \cap \{1, 2\}$, the proof is finished.
§ 4. The case of condition (ii)

The next two lemmas are also used in § 5. Let $X$ and $\mathcal{F}$ be as in Theorem 1.

**Lemma 4.** Suppose that $0 \notin I \cup J$, $I, J \subset \{0, 1, 2\}$, and that for any flat quasi-coherent sheaf $\mathcal{G}$ on $X$ we have

$$A_X(I \cap J, \mathcal{G}) = A_X(I, \mathcal{G}) \cap A_X(J, \mathcal{G}).$$

Then

$$A_X(\{0\} \cup (I \cap J), \mathcal{F}) = A_X(\{0\} \cup I, \mathcal{F}) \cap A_X(\{0\} \cup J, \mathcal{F}).$$

**Proof.** Let $\mathcal{F}_\eta$ denote the constant sheaf associated to the fibre of $\mathcal{F}$ at the generic point of $X$. By the definition of adeles, the equality (for which one does not require $\mathcal{F}$ to be coherent)

$$A_X(\{0\} \cup I, \mathcal{F}_\eta) = A_X(I, \mathcal{F}_\eta)$$

holds. We have $\mathcal{F}_\eta = \lim_D \mathcal{F}(D)$, where the limit is taken over all effective (not necessarily reduced) divisors $D$ on $X$. Since the functor $A_X(I, \cdot)$ is exact and commutes with filtered colimits, we obtain the equality

$$A_X(\{0\} \cup I, \mathcal{F}) = \lim_D A_X(I, \mathcal{F}(D)).$$

This implies the required statement after we apply the condition of the lemma to the sheaves $\mathcal{G} = \mathcal{F}(D)$ for various $D$.

**Lemma 5.** Suppose that for any flat quasicoherent sheaf $\mathcal{G}$ on $X$, we have

$$H^0(X, \mathcal{G}) = A_X(\{1\}, \mathcal{G}) \cap A_X(\{2\}, \mathcal{G}).$$

Then for all $I, J \subset \{0, 1, 2\}$, we have

$$A_X(I \cap J, \mathcal{F}) = A_X(I, \mathcal{F}) \cap A_X(J, \mathcal{F}).$$

**Proof.** By Theorem 1, (i), we need consider only pairs $(I, J)$ such that $I \cap J \neq I \setminus \{0\}$. Explicitly, it is enough to consider the following pairs $(I, J)$ (up to a transposition of $I$ and $J$):

$$\{1\}, \{2\}, \{0, 1\}, \{2\}, \{1\}, \{0, 2\}, \{0, 1\}, \{0, 2\}.$$

The second and third cases are reduced to the first as follows: one has the embedding

$$A_X(\{0, 1\}, \mathcal{F}) \cap A_X(\{2\}, \mathcal{F}) \subset A_X(\{0, 1\}, \mathcal{F}) \cap A_X(\{1, 2\}, \mathcal{F}) = A_X(\{1\}, \mathcal{F}),$$

where the last equality follows from Theorem 1, (i). Therefore, we see that

$$A_X(\{0, 1\}, \mathcal{F}) \cap A_X(\{2\}, \mathcal{F}) = A_X(\{1\}, \mathcal{F}) \cap A_X(\{2\}, \mathcal{F}).$$

The same reasoning works for the pair $(\{1\}, \{0, 2\})$. The fourth case is reduced to the first by Lemma 4.

**Remark 2.** As can be seen from the proofs, Lemmas 4 and 5 remain true after one replaces flat quasicoherent sheaves by locally free sheaves of finite rank.
The next lemma is the only place where we use the fact that \( k \) is uncountable.

**Lemma 6.** Let \( a \in k[[u, v]] \) be a Taylor series. Suppose that for any polynomial \( f \in k[u, v] \) with \( f(0) = 0 \), there is a polynomial \( b_f \in k[u, v] \) such that \( a \equiv b_f \) \((\text{mod } f)\). Also, suppose that \( k \) is uncountable. Then the series \( a \) is a polynomial: \( a \in k[u, v] \).

*Proof.* Let \( a = \sum_{i,j \geq 0} a_{i,j} u^i v^j \). For each \( \lambda \in k \), consider a linear polynomial \( f = u - \lambda v \). Then \( k[[u, v]]/(f) \cong k[[v]] \) and \( a \) \((\text{mod } f)\) coincides with the following Taylor series:

\[
\sum_{n \geq 0} \left( \sum_{i=0}^{n} a_{i,n-i} \lambda^i \right) v^n.
\]

Assume that \( a \) is not a polynomial. Equivalently, the sequence of polynomials

\[
p_n(t) := \sum_{i=0}^{n} a_{i,n-i} t^i
\]

in a formal variable \( t \) contains infinitely many nonzero elements. By the assumption of the lemma, for any element \( \lambda \), there exists \( n_0 \) such that for any \( n \geq n_0 \), we have \( p_n(\lambda) = 0 \). This contradicts the fact that \( k \) is uncountable, because each nonzero polynomial has finitely many roots and we have a countable set of polynomials \( p_n \).

*Proof of Theorem 1, (ii).* Suppose that condition (ii) is satisfied. By Lemma 5, it is enough to show the equality \( H^0(X, \mathcal{F}) = \mathbf{A}_X(\{1\}, \mathcal{F}) \cap \mathbf{A}_X(\{2\}, \mathcal{F}) \). By Lemma 3, we may assume that \( X \) is affine and that \( \mathcal{F} = \mathcal{O}_X \). By Noether normalization, there exists a finite morphism \( \pi : X \to \mathbb{A}^2 \). By Lemma 1, the statement of the lemma for \((X, \mathcal{O}_X)\) is equivalent to that for \((\mathbb{A}^2, \pi_* \mathcal{O}_X)\).

Since \( \mathbb{A}^2 \) is regular and \( X \) is equidimensional, \( k[X] = (\pi_* \mathcal{O}_X)(\mathbb{A}^2) \) is a projective module over \( k[\mathbb{A}^2] \): [11], Corollary 18.17. Therefore, by Lemma 2, we are reduced to the case of \((\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2})\).

For brevity, put \( A_i := \mathbf{A}_{\mathbb{A}^2}(\{i\}, \mathcal{O}_{\mathbb{A}^2}) \), \( i = 1, 2 \). Let \( a \in A_1 \cap A_2 \). For any irreducible curve \( C \subset \mathbb{A}^2 \), the restriction \( a|_C := (a|_{C,x})|_{x \in C} \) of \( a \in A_2^2(\{1, 2\}, \mathcal{O}_{\mathbb{A}^2}) \) is an element in \( \mathbf{A}_C(\{0, 1\}, \mathcal{O}_C) \). Since \( a \in A_1 \), we see that \( a|_C \) belongs to the field of rational functions \( k(C) \). On the other hand, we have \( a \in A_2 \), whence \( a|_C \) is regular on \( C \) (cf. the proof of Proposition 2 below).

Now let \( x \in \mathbb{A}^2 \) be a \( k \)-point. Then the component \( a_x \in \mathcal{O}_{\mathbb{A}^2,x} \) of \( a \in A_2 \) satisfies the conditions of Lemma 6, whence \( a_x \in \mathcal{O}_{\mathbb{A}^2,x} \). Since \( a \in A_1 \), the rational function \( a_x \) is the same for all points \( x \in X \), which completes the proof.

§ 5. The case of condition (iii)

First we recall the constructions of adelic complexes and of the inverse image map between them. Given a Noetherian scheme \( X \) and a quasicoherent sheaf \( \mathcal{F} \) on \( X \), the degree \( p \) term of the corresponding adelic complex is defined by the formula

\[
\mathbf{A}(X, \mathcal{F})^p := \mathbf{A}_X(S(X)_p, \mathcal{F}).
\]

There exist canonical isomorphisms ([18], Theorem 4.2.3 and Proposition 5.1.3) \( H^p(X, \mathcal{F}) \cong H^p(\mathbf{A}(X, \mathcal{F})^\bullet), p \geq 0 \). In particular, the group of cocycles in \( \mathbf{A}(X, \mathcal{F})^0 \) is equal to \( H^0(X, \mathcal{F}) \).
Given a morphism of finite type $f: Y \to X$ between Noetherian schemes and a quasicoherent sheaf $\mathcal{F}$ on $X$, one has well-defined inverse image maps ([12], p. 178)

$$f^*: \mathbb{A}_X(S(X)_p, \mathcal{F}) \to \mathbb{A}_Y(S(Y)_p, f^* \mathcal{F}), \quad p \geq 0. \quad (5.1)$$

For any flag $\Delta \in S(Y)_p$ and $a \in \mathbb{A}_X(S(X)_p, \mathcal{F})$, one has $(f^* a)_\Delta = f^*(a_{f(\Delta)})$, where we set $a_{f(\Delta)}$ to be zero if $f(\Delta)$ is not a nondegenerate flag, and we use the canonical map

$$f^*: \mathbb{A}_X(\{f(\Delta)\}, \mathcal{F}) \to \mathbb{A}_Y(\{\Delta\}, f^* \mathcal{F})$$

otherwise. The maps (5.1) define morphisms between the adelic complexes

$$f^*: \mathbb{A}(X, \mathcal{F})^* \to \mathbb{A}(Y, f^* \mathcal{F})^*.$$

**Proposition 2.** If $\mathcal{F}$ is a locally free sheaf of finite rank on a regular surface $X$, then there is a canonical isomorphism

$$\mathbb{A}_X(\{1\}, \mathcal{F}) \cap \mathbb{A}_X(\{2\}, \mathcal{F}) \cong \lim_{\leftarrow Z} H^0(Z, \mathcal{F}|_Z),$$

where the projective limit is taken over all closed subschemes $Z \subset X$ (not necessarily reduced or irreducible) such that $Z \neq X$.

**Proof.** Since $\mathcal{F}$ is coherent, we have the equalities

$$\mathbb{A}_X(\{1\}, \mathcal{F}) = \prod_{C \subset X} \mathcal{F}_C, \quad \mathbb{A}_X(\{2\}, \mathcal{F}) = \prod_{x \in X} \mathcal{F}_x,$$

where $C$ runs through all irreducible curves on $X$, $x$ runs through all closed points on $X$, and $\mathcal{F}_C$, $\mathcal{F}_x$ denote the completions of the stalks $\mathcal{F}_C$ and $\mathcal{F}_x$, respectively. Therefore, there is a well-defined map

$$\lim_{\leftarrow Z} H^0(Z, \mathcal{F}|_Z) \to \mathbb{A}_X(\{1\}, \mathcal{F}) \cap \mathbb{A}_X(\{2\}, \mathcal{F}),$$

induced by taking limits over subschemes in $X$ whose support is either a curve $C$ or a point $x$. Let us construct an inverse map.

Given $a \in \mathbb{A}_X(\{1\}, \mathcal{F}) \cap \mathbb{A}_X(\{2\}, \mathcal{F})$, consider the element

$$\tilde{a} := (0, a, a) \in \mathbb{A}_X(\{0\}, \mathcal{F}) \oplus \mathbb{A}_X(\{1\}, \mathcal{F}) \oplus \mathbb{A}_X(\{2\}, \mathcal{F}) = \mathbb{A}(X, \mathcal{F})^0.$$

We have that

$$d\tilde{a} \in \mathbb{A}_X(\{01\}, \mathcal{F}) \oplus \mathbb{A}_X(\{02\}, \mathcal{F}) \subset \mathbb{A}(X, \mathcal{F})^1.$$

Let $i: Z \to X$ be a closed subscheme such that $Z \neq X$. Since $Z$ has dimension at most 1, the inverse image map

$$i^*: \mathbb{A}_X(S(X)_1, \mathcal{F}) \to \mathbb{A}_Z(S(Z)_1, \mathcal{F}|_Z)$$

vanishes on the subspaces $\mathbb{A}_X(\{01\}, \mathcal{F})$ and $\mathbb{A}_X(\{02\}, \mathcal{F})$ in $\mathbb{A}_X(S(X)_1, \mathcal{F})$. Therefore, $i^*(d\tilde{a}) = 0$ and $i^*(\tilde{a})$ is a cocycle in $\mathbb{A}(Z, \mathcal{F}|_Z)^0$. The isomorphism
between $H^0(Z, F|_Z)$ and $H^0(A(Z, F|_Z)^\bullet)$ implies that $\iota^*(\tilde{a})$ corresponds to a unique element $s_z \in H^0(Z, F|_Z)$. One may check that the collection $\{s_z\}$ defines an element in the projective limit and that this gives the desired map

$$A_X(\{1\}, F) \cap A_X(\{2\}, F) \to \lim_{\leftarrow Z} H^0(Z, F|_Z).$$

This completes the proof.

**Proof of Theorem 1.(iii).** Suppose that condition (iii) is satisfied. By Lemma 5, it is enough to show the equality $H^0(X, F) = A_X(\{1\}, F) \cap A_X(\{2\}, F)$. By Proposition 2, we need to show that the natural embedding

$$H^0(X, F) \to \lim_{\leftarrow Z} H^0(Z, F|_Z)$$

is an isomorphism, where the limit is taken over all closed subschemes $Z \subset X$ (not necessarily reduced or irreducible) such that $Z \neq X$.

Since $X$ is projective, by Serre duality, there exists a very ample invertible sheaf $O_X(1)$ on $X$ such that

$$H^0(X, F(-n)) = H^1(X, F(-n)) = 0$$

for any natural number $n > 0$.

Let $D$ and $E$ be the zero schemes of nonzero sections of $O_X(m)$ and $O_X(n)$ for some natural numbers $m, n > 0$. Exact sequences of cohomology groups imply that the restriction $H^0(X, F) \to H^0(D, F|_D)$ is an isomorphism and the restriction $H^0(E, F|_E) \to H^0(D \cap E, F|_{D \cap E})$ is injective, where $D \cap E$ denotes the schematic intersection.

Let now $\{s_z\}$ be an element in $\lim_{\leftarrow Z} H^0(Z, F|_Z)$. Fix $D$ as above and let $s \in H^0(X, F)$ be the element that restricts to $s_D$ on $D$. Then for any $E$ as above, $s$ restricts to $s_E$ on $E$, as both $s$ and $s_E$ restrict to the same element in $H^0(D \cap E, F|_{D \cap E})$.

We claim that $s$ restricts to $s_{\tilde{x}}$ for any closed subscheme $\tilde{x}$ in $X$ whose support is a closed point $x \in X$. Indeed, it is sufficient to consider the case $\tilde{x} = \text{Spec}(O_X,N_x/m_x^N)$, $N \geq 1$. There exists $E$ as above, passing through $x$. Then $\tilde{x}$ is a closed subscheme in $N \cdot E$, which is the zero scheme of the corresponding section of $O_X(Nn)$. As shown above, $s$ restricts to $s_{N \cdot E}$ on $N \cdot E$, whence $s$ restricts to $s_{\tilde{x}}$ on $\tilde{x}$.

By Proposition 2, this implies that $\{s_z\}$ is equal to the image of $s$ under the map $H^0(X, F) \to \lim_{\leftarrow Z} H^0(Z, F|_Z)$.

### § 6. Counterexample

We provide a counterexample to Theorem 1 when $X$ is affine and $k$ is countable. With this aim we use the following construction.

Let $X$ be an affine surface over a field $k$. Suppose that a series $\sum_{n \geq 1} f_n$, $f_n \in k[X]$, converges in the complete local ring $\widehat{O}_{X,n}$ for any schematic point
\[ a \in A_X(\{1\}, \mathcal{O}_X) \cap A_X(\{2\}, \mathcal{O}_X). \]

**Theorem 2.** Let \( X \) be an affine regular surface over a countable field \( k \). Then

\[ k[X] \subsetneq A_X(\{1\}, \mathcal{O}_X) \cap A_X(\{2\}, \mathcal{O}_X). \]

**Proof.** By the construction before the theorem, it is enough to find a convergent series \( \sum_{n \geq 1} f_n \) as above, which does not converge to an element in \( k[X] \).

There are countably many prime ideals in \( k[X] \) (use Hilbert’s basis theorem and countability of \( k[X] \)). Let \( p_1, \ldots, p_n, \ldots \) be a sequence of all nonzero prime ideals in \( k[X] \) and take the sequence of ideals

\[ a_1 := p_1, \quad a_n := a_{n-1}^2 \cdot p_n, \quad n \geq 2. \]

Choose nonzero elements \( g_n \in a_n, \ n \geq 1 \). Choose a closed point \( x \in X \) and a nonzero function \( t \in k[X] \) which vanishes at \( x \). Let \( (t^a), \ a \geq 1 \), denote the ideal \( t^a \cdot \mathcal{O}_{X,x} \). Choose an increasing exhaustive filtration of \( k[X] \) by finite-dimensional \( k \)-subspaces:

\[ F_1 \subset F_2 \subset \cdots \subset F_i \subset F_{i+1} \subset \cdots \subset k[X]. \]

Since all \( F_i \) are finite-dimensional and \( \bigcap_{a \geq 0} (t^a) = 0 \), we see that for any \( l \), there is a natural number \( a(l) \) such that

\[ F_i \cap (t^{a(l)}) = 0, \quad (6.1) \]

where the intersection is taken in \( \mathcal{O}_{X,x} \). We can also assume that \( a(l) \) is strictly increasing in \( l \). Put \( f_1 := g_1 \). Now define recursively natural numbers \( l(n), \ n \geq 2 \), and \( f_n \in k[X] \) as follows:

\[ \sum_{i=1}^{n-1} f_i \in F_{l(n)}, \quad f_n := t^{a(l(n))} \cdot g_n, \quad n \geq 2. \]

In particular, the sequence \( l(n) \) is strictly increasing in \( n \) and \( f_n \in F_{l(n+1)} \). The series \( \sum_{n \geq 1} f_n \) converges in the complete local ring \( \mathcal{O}_{X,\eta} \) of any schematic point \( \eta \in X \) except for the generic point of \( X \). Also, this series is rather rarefied.

Denote by \( f \) the sum \( \sum_{n \geq 1} f_n \) in \( \mathcal{O}_{X,x} \). We show that \( f \) does not belong to \( k[X] \subset \mathcal{O}_{X,x} \). Assume the opposite. Since \( l(n) \) is strictly increasing, we have that \( f \in F_{l(n)} \) for some \( n \). Since \( \sum_{i=1}^{n-1} f_i \in F_{l(n)} \) and, by construction, \( \sum_{i \geq n} f_i \in (t^{a(l(n))}) \), we conclude by condition (6.1) that \( \sum_{i \geq n} f_i = 0 \). On the other hand, since \( f_i \in (t^{a(l(n+1))}) \) for \( i > n \), we see that

\[ \sum_{i \geq n} f_i \equiv f_n \pmod{t^{a(l(n+1))}}. \]

Since \( 0 \neq f_n \in F_{l(n+1)} \), we conclude by condition (6.1) that \( f_n \) does not vanish modulo \( t^{a(l(n+1))} \). Therefore, \( \sum_{i \geq n} f_i \neq 0 \) and we get a contradiction.
The following proposition describes the intersection $\mathbb{A}_X(\{1\}, \mathcal{O}_X) \cap \mathbb{A}_X(\{2\}, \mathcal{O}_X)$ and is interesting in itself.

**Proposition 3.** Let $X$ be an affine surface over an arbitrary field $k$.

(i) There is a canonical isomorphism

$$\mathbb{A}_X(\{1\}, \mathcal{O}_X) \cap \mathbb{A}_X(\{2\}, \mathcal{O}_X) \cong \varprojlim_a k[X]/a,$$

where the projective limit is taken over all nonzero ideals $a \subset k[X]$.

(ii) Any element in $\mathbb{A}_X(\{1\}, \mathcal{O}_X) \cap \mathbb{A}_X(\{2\}, \mathcal{O}_X)$ is obtained from a series

$$\sum_{n \geq 1} f_n, \, f_n \in k[X],$$

as in the construction before Theorem 2.

**Proof.** First, (i) is a particular case of Proposition 2.

Let us prove (ii). If $k$ is uncountable, then by Theorem 1, (ii), there is nothing to prove. Assume that $k$ is countable. Let $a_1, \ldots, a_n, \ldots$ be the sequence of ideals in $k[X]$ constructed in the proof of Theorem 2. By (i), an element $a \in \mathbb{A}_X(\{1\}, \mathcal{O}_X) \cap \mathbb{A}_X(\{2\}, \mathcal{O}_X)$ defines a compatible collection of elements $g_n \in k[X]/a_n$. Moreover, by the construction of $a_n$, the collection $\{g_n\}$ defines $a$ uniquely. Let $h_n \in k[X], \, n \geq 1$, be any lift of $g_n$ and put

$$f_1 := h_1, \quad f_n := h_n - h_{n-1}, \quad n \geq 2.$$

Then the series $\sum_{n \geq 1} f_n$ converges in the complete local ring $\widehat{\mathcal{O}}_{X, \eta}$ of any schematic point $\eta \in X$ except for the generic point and corresponds to $a$, because for any $m \geq 1$, we have that $f_{m+1} \in a_m$ and $\sum_{n=1}^{m} f_n \equiv g_m \pmod{a_m}$.

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