Letter

Taming supersymmetric defects in 3d–3d correspondence

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Abstract
We study knots in 3d Chern–Simons theory with complex gauge group \(SL(N, \mathbb{C})\), in the context of its relation with 3d \(\mathcal{N} = 2\) theory (the so-called 3d–3d correspondence). The defect has either co-dimension 2 or co-dimension 4 inside the 6d \((2, 0)\) theory, which is compactified on a 3-manifold \(\hat{M}\). We identify such defects in various corners of the 3d–3d correspondence, namely in 3d \(SL(N, \mathbb{C})\) CS theory, in 3d \(\mathcal{N} = 2\) theory, in 5d \(\mathcal{N} = 2\) super Yang–Mills theory, and in the M-theory holographic dual. We can make quantitative checks of the 3d–3d correspondence by computing partition functions at each of these theories. This Letter is a companion to a longer paper [1], which contains more details and more results.

Keywords: 3d \(\mathcal{N} = 2\) theory, 3-manifold, complex Chern–Simons theory, knot

(Some figures may appear in colour only in the online journal)

Introduction

One lesson from history is that physics and mathematics often develop hand in hand; the development on one side facilitates development on the other, creating a virtuous cycle of feedback. The recently-discovered 3d–3d correspondence [2–7] is a perfect example for this interplay. The correspondence states that there exists a surprising connection between 3d \(SL(N, \mathbb{C})\) Chern–Simons (CS) theory defined on a 3-manifold \(M\) on the one hand, and a 3d \(\mathcal{N} = 2\) supersymmetric gauge theory (which we call \(T_N[M]\)) on the other.
This correspondence makes it possible to relate the strongly-coupled problems in 3d $\mathcal{N} = 2$ theories with rather different-looking problems in complex CS theory, which is a topological quantum field theory and contains no dynamical degrees of freedom. We can therefore expect to learn about the former from the latter, for example.

The physical origin of the 3d–3d correspondence is the compactification of the $N$ M5-branes on a closed 3-manifold $\hat{M}$, along which the theory is partially topologically twisted:

$$N \text{ M5s on } \mathbb{R}^{1,2} \times \hat{M}. \quad (1)$$

The 3d $\mathcal{N} = 2$ theory $T_N[\hat{M}]$ lives on $\mathbb{R}^{1,2}$, which is transverse to $\hat{M}$.

**Loop-like defects**

In this Letter we study the inclusion of supersymmetry-preserving defects to the 3d–3d correspondence. The defects fill in knots (or links) inside the closed 3-manifold $\hat{M}$, see figure 1.

There are two types of defects. They are easier to understand in terms of the M5-brane configuration, where the defect has either 2 or 4 co-dimensions:

$$\begin{array}{c|c|c|c|}
\text{Co-dim.} & \mathbb{R}^{1,2} & \hat{M} \\
\hline
2 \text{ M5} & 0 & 1 & 2 & 3 & 4 & 5 \\
4 \text{ M2} & 0 & 1 & 2 & 3 & 6 \\
4 \text{ M5} & 0 & 3 & 7 & 8 & 9 & \ddagger \\
\end{array} \quad (2)$$

The co-dimension 4 defect is described either by the M2-brane or its blow-up, the M5-brane (more comment on this later). We will include a co-dimension 2 defect along a knot $K$, and and co-dimension 4 along another knot $\bar{K}$.

There are several motivations for studying such defects in the 3d–3d correspondence. First, co-dimension 2 defects can be thought of as cutting out a tubular neighborhood of $K$ inside $\hat{M}$ and prescribe a particular holonomy along a cycle of the boundary torus, as have been studied in the mathematical literature on knot theory and hyperbolic geometry (e.g. [8]), and in fact most of the discussions on the 3d–3d correspondence in the literature already contain such a co-dimension 2 defect $K$. The 3-manifold $M$ is then taken to be a complement (exterior) of a knot inside a closed 3-manifold $\hat{M}$.
Second, 3d–3d correspondence motivates the study of knot invariants more general than those which have been studied so far in the mathematical literature. For example, when we consider the case of $N > 2$ M5-branes, a co-dimension 2 defect along $K$ could be of a ‘non-maximal’ type (as we will discuss later), and almost nothing is known about these cases. The resulting partition function, with the defects included, will be a quantity of both mathematical and physical interest, and gives a new invariant of a knot, in particular.

Finally, the defects help us to better understand the proposed 3d $\mathcal{N} = 2$ theory $T_N[M]$. Since the $T_N[M]$ is associated with $N$ M5-branes, we expect the theory $T_N[M]$ to have non-Abelian gauge groups. However, most of the discussion in the literature uses Abelian gauge groups only. The considerations of our Letter and the forthcoming paper \cite{1} makes it even clearer that we need a non-Abelian description (e.g. \cite{2}) for the proper understanding of co-dimension 4 defects. In the rest of this Letter, let us discuss our defects in various different theories appearing in the 3d–3d correspondence:

\begin{equation}
\begin{aligned}
5d \mathcal{N} = 2 \text{SYM} & \quad \text{holographic dual} \\
6d (2,0) \text{ theory} & \\
3d \mathcal{N} = 2 \text{ theory } T_N[M] & \quad 3d SL(N) \text{ CS}
\end{aligned}
\end{equation}

5d

Let us begin with 5d $\mathcal{N} = 2$ SU($N$) super-Yang–Mills (SYM), which is the dimensional reduction of the 6d (2, 0) theory on $S^4$. When the 6d geometry contains an $S^1$ we expect that the physics of the remaining five directions will be captured by 5d $\mathcal{N} = 2$ SYM.

The advantage of this theory, when compared with the 6d (2, 0) theory, is that the theory has an explicit Lagrangian. In fact, when we replace $\mathbb{R}^{1,2}$ in (1) by an $S^1$-bundle over $S^5$, we can reduce the 6d theory along the $S^1$, to obtain 5d $\mathcal{N} = 2$ SYM on $S^5 \times \tilde{M}$. We can then perform the supersymmetric localization computation and show directly that the theory on $\tilde{M}$ is the 3d $SL(N, \mathbb{C})$ CS theory \cite{9, 10} (see also \cite{11}).

We can discuss the supersymmetric defects in this setup. A co-dimension 2 defect is realized by coupling the 5d $\mathcal{N} = 2$ SYM with the so-called $T_{\rho}[SU(N)]$ theory \cite{12, 13} (see \cite{14, 15}). Here $\rho$ is an embedding $\rho : su(2) \to su(N)$, or equivalently a partition of $N$:

\begin{equation}
\rho = [n_1, n_2, \ldots, n_s], \quad n_i \geq n_{i+1}, \quad \sum_{i=1}^s n_i = N.
\end{equation}

The defect is called ‘maximal’ (‘simple’) when $\rho = [1]^N = [1, 1, \ldots, 1]$ ($\rho = [N - 1, 1]$). The $T_{\rho}[SU(N)]$ theory has flavor symmetry $SU(N) \times H_\rho$, where $H_\rho$ is the commutant of the image of $\rho(SU(2))$ inside $SU(N)$. The 5d theory couples to $T_{\rho}[SU(N)]$ by gauging the $SU(N)$ part of this flavor symmetry.

A co-dimension 4 defect is realized by a supersymmetric Wilson loop in 5d $\mathcal{N} = 2$ SYM:
Here $\phi_i$ are three of the adjoint scalar fields of the 5d $\mathcal{N} = 2$ SYM, which after the topological twist are turned into a 1-form on $M$. The point $p \in S^2$ should be either at the north or south pole of the sphere, in order to preserve some supersymmetry. The co-dimension 4 defects are hence labeled by

$$R : \text{ a unitary representation of } SU(N).$$

### 3d Chern–Simons

Let us next consider the theory on $M$, the 3d $SL(N, \mathbb{C})$ CS theory [16]. The Lagrangian of the complexified CS theory is given by

$$S_{CS}[A, \overline{A}; \hbar, \tilde{\hbar}] = \frac{i}{2\hbar} \text{CS}[A] + \frac{i}{2\tilde{\hbar}} \text{CS}[\overline{A}],$$

where the CS functional defined by

$$\text{CS}[A] := \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

and we defined in general complex ‘Planck constants’ by

$$\hbar := \frac{4\pi k + \sigma}{i}, \quad \tilde{\hbar} := \frac{4\pi i}{k - \sigma},$$

with $k \in \mathbb{Z}$ and $\sigma \in \mathbb{R}$ or $i\mathbb{R}$. Mathematically, this theory gives a quantization of the moduli space of $SL(N, \mathbb{C})$-flat connections on $M$.

In CS theory, a co-dimension 2 defect corresponds to a monodromy defect, which is to specify the holonomy along the boundary torus of the complement of $K$. More precisely, the manifold (3) has a boundary torus

$$\partial(\mathcal{M} \setminus K) = T^2.$$

The torus has two non-contractible cycles, the meridian $m$ (contractile in the tubular neighborhood of $K$) and the longitude $l$. In quantum theory it suffices to specify the holonomy for one of them, and the boundary holonomy along the meridian $m$ is taken to be

$$\text{Hol}_m(A) \sim \begin{pmatrix} e^{\text{nn}_1} & 0 & 0 & 0 \\ 0 & e^{\text{nn}_2} & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{\text{nn}_n} \end{pmatrix},$$

where the size of the block is determined by the partition $\rho$ (5), and $\sim$ denotes equivalence under the adjoint action of the gauge group$^5$. The partition function is defined by the path-integral

$^5$ More precisely, the holonomy on the left hand side should be in the closure of the orbit of the matrix on the right hand side. Since we are taking the closure, the limit of a closure of an orbit in general contains several different orbits, which is the reason why the discussion of $\rho$ is non-trivial. See appendix A of [1] for more detailed explanation.
with the boundary condition along $K$ as in (12). The path-integral (13) can be defined for arbitrary coupling constants $\hbar$, $\bar{\hbar}$ by enlarging the domain of integration and deforming the integration contour [17], as we will comment more in [1]. The existing literature has focused on the case of maximal puncture, see in particular [18–22] for the case $N > 2$.

On the other hand, a co-dimension 4 defect in 3d CS theory is a Wilson line along the knot $K$ in representation $R$ (more precisely, in $SL(N, \mathbb{C})$ CS theory $R$ becomes the natural holomorphic lift of the $SU(N)$ representation):

$$Z^\text{CS}_R = \langle W_R(K) \rangle = \int [DA] e^{iS_{\text{CS}}[A, \bar{A}, h, \bar{h}]} \langle = \oint_K A \rangle$$

We show in [1, section 6] (by generalizing [10]) that in the supersymmetric localization on $S^2 \times M$ the 5d Wilson loop (6) reduces to the 3d Wilson loop of (14), thus proving the equivalence of two Wilson lines.

We can evaluate the partition functions (13) and (14) either by the state-integral model ([23] for $N = 2$, and [22] for $N > 2$), or the ‘cluster partition function’ of [24] (which is based on [2, 3, 25]). As discussed in [1, sections 3 and 4], in both cases the answer can be written as an overlap of two states inside a certain quantum mechanical system with a finite degrees of freedom:

$$Z^\text{cluster}_R = \langle \mathcal{M}_\alpha, C_I = 0 | \psi_{\text{CS}} \rangle.$$ (15)

Here $| \psi_{\text{CS}} \rangle \in \mathcal{H}$ is a state representing $L$ free $\mathcal{N} = 2$ chiral multiplets. The parameters $\mathcal{M}_\alpha$ are the same as in (12), and $C_I = 0$ are the gluing constraints representing the gluing of the octahedra (or $\mathcal{N} = 2$ chiral multiplets). The choice of $\rho$ is reflected in the octahedron structures as well as the choice of gluing equations $C_I = 0$. The Hilbert space $\mathcal{H}$ is obtained by quantizing the space of flat connections on $M$. In this framework, co-dimension 4 defects are obtained by inserting a line operator $W_R(K)$ in between:

$$Z^\text{cluster}_R = \langle \mathcal{M}_\alpha, C_I = 0 | W_R(K) | \psi_{\text{CS}} \rangle.$$ (16)

Classically, the loop $W_R(K)$ can be computed by the snake rules of [22, 26]. The general rule for the quantization of the loops is not known, however we will provide a well-defined quantization rules of a class of loop operators in [1], by extending the formalism of [24] to incorporate Wilson lines.

3d $\mathcal{N} = 2$ theory

Our defects can also be discussed in the context of the 3d $\mathcal{N} = 2$ theory $\mathcal{T}_N[\hat{M}]$. In this theory, the two types of defects play different roles. A co-dimension 2 defect fills the entire 3d, hence changes the 3d theory itself. We denote the resulting theory by $\mathcal{T}_N[\hat{M} \backslash K, \rho]$. By contrast a co-dimension 4 defect is a loop operator inside the 3d theory $\mathcal{T}_N[\hat{M} \backslash K, \rho]$ if it co-exists with a co-dimension 2 defect.

Quantitatively, we can compute the $(S^3/Z_h)_b$ [27–33] or $(S^2 \times S^1)_y$ [34, 35] partition function of $\mathcal{T}_N[\hat{M} \backslash K, \rho]$:

$$Z^{3d\mathcal{N} = 2}_\rho = Z_{(S^3/Z_h)_b} \text{ or } (S^2 \times S^1)_y [\mathcal{T}_N[\hat{M} \backslash K, \rho]],$$ (17)
which is to be identified with the CS partition function (13):
\[ Z^\text{CS}_\rho = Z^\text{3d}_\rho N=2, \]
under the identification of the parameters (recall (10)):
\[ (S^3/\mathbb{Z}_k)_b : k \in \mathbb{Z}_{>0}, \quad \sigma = k \frac{1 - b^2}{1 + b^2} \in \mathbb{R} \text{ or } i\mathbb{R}. \]
\[ (S^2 \times S^1)_b : k = 0, \quad \sigma \in i\mathbb{R}. \]
\( \sigma \) being real or imaginary depends if \( b \in \mathbb{R} \) [30] or \( |b| = 1 \) [32]. In fact, given the expression (15), we can reverse-engineer an Abelian gauge theory \( T_N[\hat{M}\backslash K, \rho] \), as has been done in [22, 23, 36] for maximal punctures and in our paper [1] for simple punctures.

It turns out that we run into problems for a co-dimension 4 defect, however. As long as we use the Abelian gauge theory description, it is hard to understand why such a defect is labeled by a representation \( R \) (7). This problem is solved by the non-Abelian description of the \( T_N[\hat{M}\backslash K, \rho] \) theory given in [2], which works when \( \rho \) is simple type.

In [2], the 3d theory is obtained as a duality domain wall between two 4d \( \mathcal{N} = 2^* \) theories whose complexified gauge-coupling constants are related by an element of the S-duality group \( \varphi \in SL(2, \mathbb{Z}) \). Since the 4d \( \mathcal{N} = 2^* \) theory is the 6d theory on a torus with a simple puncture, we are restricted to the case where \( \rho \) is simple. The resulting 3d theory, which was denoted by \( T_N[SU(N); \varphi] \) in [2], corresponds to a 3-manifold known as the mapping torus:
\[ M = (\Sigma_{1,1} \times S^1)_\varphi := \{(x, t) \in \Sigma_{1,1} \times [0, 1] \}/\sim, \]
where \( \varphi \) is an element of \( PSL(2, \mathbb{Z}) \) and the equivalence relation \( \sim \) is given by \( (x, 0) \sim (\varphi(x), 1) \). In this non-abelian gauge theory description the co-dimension 4 defects are the Wilson lines of the \( \mathcal{N} = 2 \) theory, so it is obvious why they carry a representation label (7). This resolves the paradox for the co-dimension 4 defects. Unfortunately, for general cases such a non-Abelian description of \( T_N[\hat{M}\backslash K, \rho] \) theory is not known.

**Quantitative checks**

Now that we have identified co-dimension 2 and 4 defects in various corners of (4), we can move to the quantitative checks of the 3d–3d correspondence, such as (18) or its counterpart for co-dimension 4 defects. Our companion paper [1] contains many such computations. Let us here consider one example, where we have a co-dimension 2 defect of simple type along the figure-eight knot 4_1 in \( \hat{M} = S^3 \) (figure 2).

It is well-known that that this knot complement is hyperbolic, and can be triangulated by two ideal tetrahedra. However, no state-integral model has been known in the literature for the case of a simple co-dimension 2 defect, and we need an alternative approach\(^6\). What helps us is that figure-eight knot complement is also a mapping torus (20), with \( \varphi = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \in PSL(2, \mathbb{Z}). \)

We can now use two different methods. One is to use the cluster partition function of [1, 24], applied to the quiver of figure 3 for \( N = 3 \) (see [37, 38]). In [1] we have checked the

\( ^6 \) Interestingly, our result from the cluster partition function in [1] does give a new octahedron-like decomposition for the case with simple punctures. This suggests the possibility of extending the state-integral model construction to more general co-dimension 2 defects.
consistency of the quiver, and found for example that the mapping class group \(\text{PSL}_2\) is indeed realized by a sequence of quiver mutations and permutations of the quiver vertices.

Another method is to use the \(\text{TSU}_N\) theory described previously. As established in \cite{2}, the 3d \(\mathcal{N} = 2\) theory \(T_\varphi[S^3\setminus 4_r, \rho = [2, 1]]\) for the figure-eight knot complement can be then constructed from two copies of the \(T[SU(3)]\) theory. The flavor symmetry group of \(T[SU(3)]\) includes a factor \(SU(3)_{\text{het}} \times SU(3)_{\text{hop}}\). First we define

\[
T[SU(3), \varphi = \text{STS}^{-1}T^{-1}]
= T[SU(3), \text{ST}] \odot T[SU(3), S^{-3}T^{-1}].
\]  

Here, the theories \(T[SU(3), \text{ST}]\) and \(T[SU(3), S^{-4}T^{-1}]\) are equivalent to \(T[SU(3)]\) plus some additional CS terms and the operation \(\odot\) is defined by identifying \(SU(3)_{\text{hop}}\) of the first factor with \(SU(3)_{\text{hop}}\) of the second and then gauging. Finally, \(T[S^3\setminus 4_r, \rho = [2, 1]]\) is constructed as

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The figure-eight knot inside \(M = S^3\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The proposed quiver for a once-punctured torus, where we take \(N = 3\) and the puncture is of the simple type \((\rho = [2, 1])\). The fundamental region of torus is chosen as the region surrounded by solid black lines.}
\end{figure}
\[ T_{N=3}[S^3(4), [2, 1]] = \text{Tr}(T [SU(3), STS^{-1}T^{-1}]), \]

where the operation \( \text{Tr}(\cdot) \) is defined by identifying the flavor symmetries \( SU(3)_{\text{bot}} \) and \( SU(3)_{\text{top}} \) of \( T [SU(3), \varphi] \) and then gauging.

In both cases, we can straightforwardly compute their supersymmetric partition functions. For the \( (S^1 \times S^3)_b \) partition function, we have verified that the two methods mentioned above give the same answer, at least up to certain orders in the \( q \)-expansion. For example

\[
\begin{align*}
Z_{(S^1 \times S^3)_b}(m_\eta = 0, \eta) &= 1 + \left( 2\eta + \frac{2}{\eta} \right) q^2 + \left( 8 + 2\eta^2 + \frac{2}{\eta^2} \right) q^4 \\
&\quad + \left( 6\eta + \frac{6}{\eta} \right) q^6 + \left( 2 - 3\eta^2 - \frac{3}{\eta^2} \right) q^8 + \ldots,
\end{align*}
\]

\[
\begin{align*}
Z_{(S^1 \times S^3)_b}(m_\eta = 1, \eta) &= \left( \frac{1}{\eta^2} + \frac{1}{\eta} + \eta + \eta^2 \right) q + \left( 6 + 3\eta + \frac{3}{\eta} \right) q^2 \\
&\quad + \left( -6 - \frac{1}{\eta^2} - \frac{3}{\eta} - 5 + 5\eta - 3\eta^2 - \eta^3 \right) q^3 + \ldots,
\end{align*}
\]

where \( (\eta, m_\eta) \) are the fugacity and the magnetic flux for a \( U(1) \) flavor symmetry. This symmetry corresponds geometrically to the puncture holonomy of the once-punctured torus, or more physically to a part of the R-symmetry of the \( \mathcal{N} = 4 T [SU(N); \varphi] \) theory, deforming the theory to \( \mathcal{N} = 2 \). The computation (23) is a highly non-trivial check for our proposed quiver (figure 3), as well as for the consistency check between Abelian and non-Abelian descriptions of \( T_n [\hat{M} \backslash K, \rho] \) theory.

The paper \([1]\) also contains quantitative results for the co-dimension 4 defects from the \( T [SU(N = 2)] \) theory and the cluster partition function.

### Large \( N \)

Finally, we can discuss the large \( N \) limit. The \( D = 11 \) supergravity background takes the form of a warped product \( AdS_4 \times \hat{M} \times \tilde{S}^4 \) \([39-42]\):

\[
ds_{11}^2 = l_p^2 (2\pi N)^{2/3} (1 + \sin^2 \theta)^{2/3} \\
\times [ds^2(AdS_4) + ds^2(H^3) + (1 + \sin^2 \theta)^2 ds^2(\tilde{S}^4)],
\]

where \( l_p \) is the \( D = 11 \) Planck constant. The warp factors depend on \( \theta \), which is one of the coordinates of the squashed 4-sphere \( \tilde{S}^4 \). In addition to that, \( \tilde{S}^4 \) contains a round \( S^2 \) which is fibered over \( H^3 \). Here the \( H^3 \) is the local representation of the closed 3-manifold \( \hat{M} = H^3 / \Gamma \), where \( \Gamma \) is a torsionless discrete subgroup of \( PSL(2, \mathbb{C}) \). \( \hat{M} \) thus obtained has a finite volume, and it is free of orbifold singularities.

The supergravity computation gives, for the \( S_6^3 \) partition function with maximal defect,

\[
\log (Z_p^{\text{SUGRA}}) = \frac{N^3}{12\pi} (b + b^{-1})^2 \text{vol}(\hat{M} \backslash K).
\]

The co-dimension 2 maximal defect has the \( N^3 \) scaling, and back-reacts to the geometry, replacing the geometry \( \hat{M} \) by \( \hat{M} \backslash K \) (see \([43]\)).
A co-dimension 4 defect is described by the M2/M5-branes (recall (2)). Our analysis in [1] shows the following: the Wilson line in the fundamental representation ($R = \square$) corresponds to a probe M2-brane filling the cycle $\mathcal{K}$ inside $M$. When we consider the Wilson in $k$th anti-symmetric representation ($R = A_k$) with $K$ of order $N$, then M2-brane blows up into the probe M5-brane (see [44–46]). In the latter case, the supergravity computation gives the leading large $N$ answer to be [1]

$$\log(Z^{\text{SUGRA}}_R) = \frac{1 + b^2}{2} N^2 \ell(\mathcal{K}) \left(\frac{K}{N} - \frac{K}{N}ight),$$

where $\ell(\mathcal{K})$ is the hyperbolic length of the knot $\mathcal{K}$. In [1] we reproduced the $b^0$ term of this result from the large $N$ analysis of the CS theory. We also computed the exact $S_0^{Q=1}$ partition function of the $\text{Tr}(T [SU(N), \varphi])$ theory (see [47–50]). More generally, these large $N$ results give fascinating predictions for the asymptotic behavior of the perturbative CS partition functions and the cluster partition functions. It would be interesting to explore the implications of this large $N$ analysis in more depth.

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References

[1] Gang D, Kim N, Romo M and Yamazaki M 2015 To appear arXiv:1510.05011 [hep-th]
[2] Terashima Y and Yamazaki M 2011 J. High Energy Phys. JHEP08(2011)135
[3] Terashima Y and Yamazaki M 2013 Phys. Rev. D 88 026011
[4] Dimofte T, Gukov S and Hollands L 2011 Lett. Math. Phys. 98 225
[5] Dimofte T and Gukov S 2013 J. High Energy Phys. JHEP05(2013)109
[6] Dimofte T, Gaioatto D and Gukov S 2014 Commun. Math. Phys. 325 367
[7] Cecotti S, Cordova C and Vafa C 2011 arXiv:1110.2115 [hep-th]
[8] Thurston W P 1978–79 The geometry and topology of three-manifolds (unpublished)
[9] Cordova C and Jafferis D L 2012 arXiv:1305.2891 [hep-th]
[10] Lee S and Yamazaki M 2013 J. High Energy Phys. JHEP12(2013)035
[11] Yagi J 2013 J. High Energy Phys. JHEP08(2013)017
[12] Gaiotto D and Witten E 2009 J. Stat. Phys. 135 789
[13] Gaiotto D and Witten E 2009 Adv. Theor. Math. Phys. 13 721
[14] Bullimore M and Kim H-C 2015 J. High Energy Phys. JHEP05(2015)048
[15] Yonekura K 2014 J. High Energy Phys. JHEP01(2014)142
[16] Witten E 1991 Commun. Math. Phys. 137 29
[17] Witten E 2011 Chern–Simons gauge theory: 20 years after Proc. Workshop (Bonn, Germany, 3–7 August, 2009) (AMS/IP Stud. Adv. Math. vol 50) p 347 arXiv:1001.2933 [hep-th]
[18] Bergeron N, Falbel E and Guilloux A 2011 arXiv:1101.2742 [math.GT]
[19] Garoufalidis S, Thurston D P and Zickert C K 2011 arXiv:1111.2828 [math.GT]
[20] Garoufalidis S, Goerner M and Zickert C K 2015 Algebr. Geom. Topol. 15 565
[21] Garoufalidis S and Zickert C K 2013 arXiv:1310.2497 [math.GT]
[22] Dimofte T, Gabella M and Goncharov A B 2013 arXiv:1301.0192 [hep-th]
[23] Dimofte T 2013 Adv. Theor. Math. Phys. 17 479
[24] Terashima Y and Yamazaki M 2014 *Progress Theor. Exp. Phys.* **023** B01
[25] Nagao K, Terashima Y and Yamazaki M 2011 arXiv:1112.3106 [math.GT]
[26] Fock V V and Goncharov A B 2003 arXiv:math/0311149
[27] Kapustin A, Willett B and Yaakov I 2010 *J. High Energy Phys.* JHEP03(2010)089
[28] Gang D 2009 arXiv:0912.4664 [hep-th]
[29] Jafferis D L 2012 *J. High Energy Phys.* JHEP05(2012)159
[30] Hama N, Hosomichi K and Lee S 2011a *J. High Energy Phys.* JHEP03(2011)127
[31] Hama N, Hosomichi K and Lee S 2011b *J. High Energy Phys.* JHEP05(2011)014
[32] Imamura Y and Yokoyama D 2012 *Phys. Rev.* D **85** 025015
[33] Benini F, Nishioka T and Yamazaki M 2012 *Phys. Rev. D* **86** 065015
[34] Kim S 2009 *Nucl. Phys.* B **821** 241
[35] Kim S 2012 *Nucl. Phys.* B **864** 884 (erratum)
[36] Dimofte T 2015 *Commun. Math. Phys.* **339** 619
[37] Xie D 2012 arXiv:1203.4573 [hep-th]
[38] Fomin S and Pylyavskyy P 2014 *Proc. Natl Acad. Sci. USA* **111** 9680
[39] Gauntlett J P, Kim N and Waldram D 2001 *Phys. Rev. D* **63** 126001
[40] Donos A, Gauntlett J P, Kim N and Varela O 2010 *J. High Energy Phys.* JHEP12(2010)003
[41] Gang D, Kim N and Lee S 2014 *Phys. Lett.* B **733** 316
[42] Gang D, Kim N and Lee S 2015 *J. High Energy Phys.* JHEP04(2015)091
[43] Bah I, Gabella M and Halmagyi N 2014 *J. High Energy Phys.* JHEP11(2014)112
[44] Yamaguchi S 2006 *J. High Energy Phys.* JHEP05(2006)037
[45] Gomis J and Passerini F 2006 *J. High Energy Phys.* JHEP08(2006)074
[46] Assel B, Estes J and Yamazaki M 2014 *Ann. Henri Poincaré* **15** 589
[47] Nishioka T, Tachikawa Y and Yamazaki M 2011 *J. High Energy Phys.* JHEP08(2011)003
[48] Benvenuti S and Pasquetti S 2012 *J. High Energy Phys.* JHEP05(2012)099
[49] Gulotta D R, Herzog C P and Pufu S S 2011 *J. High Energy Phys.* JHEP12(2011)077
[50] Assel B, Estes J and Yamazaki M 2012 *J. High Energy Phys.* JHEP09(2012)074