Fermionic Casimir effect in Graphene

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Abstract

We investigate the Fermionic Casimir effect at finite temperature for two parallel chain of adatoms in a Graphene sheet, and the corresponding Casimir force is interpreted as an interaction between the adatom chains. We apply useful techniques to find asymptotically explicit expressions for the Casimir energy, for small as well as large temperatures (with respect to the effective temperature of the Graphene). We obtain a value in the order of \(10^{-2}N/m\) for the Casimir force between (per unit length of) the adatom chains being one nanometer apart, which is considerably noticeable e.g. in comparison to the experimental values for built-in tension of a suspended Graphene sheet.

Keywords
Fermionic Casimir effect, Graphene sheet, Adatoms, Built-in tension.

1 Introduction

Casimir effect, as an important macroscopic manifestations of the zero-point quantum oscillations \(^1\), nowadays has been investigated extensively for various configurations \(^2\)-\(^7\), and is expected to play a significant role in submicron dimensions \(^8\)-\(^10\). Graphene, a single atomic layer of Carbon atoms covalently bounded in a hexagonal lattice, as the thinnest known material with remarkable and exceptional properties \(^11\), \(^12\), is considered nowadays as an ideal candidates for constructing submicroscale systems, see \(^13\)-\(^17\) as reviews for various aspect and applications of Graphene. Hence investigating the Casimir effect for the Graphene, can have considerable importance e.g. for these ultra-miniature systems.

One of the most interesting features of the Graphene is the low-energy behavior of its electronic excitations: the electron-hole excitations of Graphene at low energy, behave as massless Dirac fermions in a 2-dimensional space, with the light-velocity replaced by the Fermi-velocity \(^18\). In this paper we investigate the fermionic Casimir effect originating from the zero-point energy of these Dirac-like fermions of the Graphene. Such
a Casimir-like interactions has been generally predicted in Ref. \cite{23}, by calculating the
interaction energy of the electron scattering between (single) adatoms in Graphene, using
the known tight-binding Hamiltonian, in which, the presence of adatoms has been
described by an additional potential term. Here we use the massless Dirac Hamiltonian
to calculate the Casimir energy between adatoms. In fact, as we know, the mentioned
electron-hole excitations are induced by hopping $\pi$-electrons of Graphene, hence con-
suming these $\pi$-electrons by e.g. some adatoms on Graphene, can break the isotropy
of the $\pi$-electron distribution, giving rise to a net interaction between adatoms. So,
in other words, taking these adsorbates as some external restricting boundaries on the
Dirac-like field in the Graphene, the resulting Casimir forces can be interpreted as inter-
actions between the adatoms. Note that this is different from the usual Casimir effect
as an interaction between two Graphene sheets (or e.g. between a Graphene sheet and a
substrate) which is arisen actually from the zero-point oscillations of the electromagnetic
field between the Graphene sheets, see \cite{19–22}. Here, the fermionic Casimir energy of
the Graphene comes from the zero-point oscillation energy of a Dirac-like spinor field
having the Graphene electron-hole as its quantum particle states (similar to the Photons
for the quantized electromagnetic field). We first find the zero-point energy of Dirac
spinors at finite temperature, by applying the conventional imaginary-time formalism.
Then, as a result, we would be able to calculate the Casimir energy between two parallel
chain of adatoms, in a (monolayer) Graphene sheet. The boundary conditions on the
Dirac-like fields in the Graphene sheet would be given by the familiar bag boundary
conditions on the adatom-chain lines. We use the known Riemann and Hurwitz zeta
functions to regularize the divergent zero-point sums, and apply some new useful tech-
niques to obtain asymptotically explicit expressions for the Casimir force for small as
well as large effective-temperatures, with respect to the room temperature. We compare
the values of the Casimir forces with the experimental values, given in Refs. \cite{28–30}, for
the pre-tensions of the Graphene at room temperature.

2 Zero-point energy of Dirac fermions at finite temper-

ature

From the functional formalism of the quantum field theory, the zero-point energy of a
field can be written (up to a total-time factor which would be automatically canceled in
the forthcoming calculations) in the form \cite{4,32}

$$E_0 \sim i \ln Z_0$$  \hspace{1cm} (1)

with $Z_0$ as the generating functional in the absence of any source, and we have used the
Planck unites. For massless Dirac fermions we simply have (up to an irrelevant constant)

$$Z_0 \sim \det (\gamma^\mu \partial_\mu),$$  \hspace{1cm} (2)

in which $\gamma^\mu$’s are the Dirac gamma matrices, and so the zero-point energy \cite{4}
for massless Dirac fermions takes the form

$$E_0 \sim i \text{Tr} \ln (\gamma^\mu \partial_\mu)$$  \hspace{1cm} (3)
where the trace is taken on a Dirac 4-spinor. Here in order to simplify the forthcoming calculations, we rewrite the above equation as

\[ E_0 \sim \frac{i}{2} \text{Tr} \ln (\gamma^\mu \partial_\mu)^2 \]

\[ \sim 2i \text{Tr} \ln (\partial^2) \]  

(4)

where, the second line has been multiplied by 4, since the trace in the second line would be taken on only one component of the Dirac 4-spinor. Then, in the presence of appropriate external boundaries on the Dirac field, the mode solutions would be discretized, and so the zero-point energy (4) takes the form

\[ E_0 = 2i \int \frac{d\omega}{2\pi} \sum_J \ln \left( -\omega^2 + k_J^2 \right). \]  

(5)

where, \(\omega\) and \(k\) represent the energy and the momentum of the Dirac modes, and the collective index \(J\) labels the mode numbers, and here the total-time factor would have been automatically canceled. One can show that the above equation is equivalent to the familiar form,

\[ E_0 = -2 \sum_J k_J, \]  

(6)

for the zero-point energy of the Dirac spinors. To show this, first we rewrite Eq. (5) as

\[ E_0 = 2i \sum_J \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \lim_{s \to 0} \left( -\frac{\partial}{\partial s} \right) \left( -\omega^2 + k_J^2 \right)^{-s} \]

\[ = -2i \lim_{s \to 0} \frac{\partial}{\partial s} \sum_J k_J^{-s} \int_{-\infty}^{\infty} \frac{dz}{2\pi} (1 - z^2)^{-s}, \]  

(7)

where in the second line, we have introduced a new variable \(\omega = zk_J\). Then using

\[ \int_{-\infty}^{\infty} dz (1 - z^2)^{-s} = i\sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)}, \]

see e.g. [36], we find

\[ E_0 = \frac{1}{\sqrt{\pi}} \lim_{s \to 0} \frac{\partial}{\partial s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_J k_J^{-s}, \]

which equals just to the zero-point energy in Eq. (6).

Now, to find the zero-point energy at finite temperature, we use the known imaginary-time formalism, in which, the temperature is introduced by a periodical condition on the field in the imaginary-time, through which, the temperature would appear in the Matsubara frequencies, see e.g. [2]–[5]. Since, through the the above calculations, we have not used the imaginary version for the time variable, here equivalently we can use an imaginary version for the Matsubara frequencies

\[ -i\omega_l = 2\pi l T \quad ; \quad l = 0, \pm 1, \pm 2, \ldots \]  

(8)
with $T$ as the temperature. As a result, the zero-point energy of the massless Dirac field (5), at finite temperature takes the form

$$E_0(T) = -2T \sum J \sum_{l=-\infty}^{\infty} \ln \left[ (2\pi T)^2 l^2 + k_J^2 \right].$$

(9)

Obviously for massive fermions, a squared mass term should be included in the logarithm argument. Note that, except for a coefficient ($-4$), the finite-temperature zero-point energy (9) for the spinor field, is equivalent to the well known zero-point energy of scalar field at finite temperature (see e.g. [4]), where the minus sign of the coefficient is a characteristic feature of the fermionic field, and the factor 4, turns up on account of the four component of the Dirac spinor, as previously mentioned.

3 Casimir effect for quasi-spinors in Graphene

It has been demonstrated that the low-energy behavior of electronic excitations of the Graphene can be described by a massless Dirac-like equation in a 2-dimensional space, with the light velocity $c$ replaced by the Fermi velocity $v_F \approx c/300$, see [18]. As a result, the zero-point energy (4), for the quasi-spinors of Graphene can be directly written as

$$E_0 = 2i\hbar \text{Tr} \ln \left( \frac{\partial^2}{v_F^2} - \partial_x^2 - \partial_y^2 \right)$$

(10)

in which $(t, x, y)$ represent a (2+1) space-time, and we have restored the Planck constant $\hbar$. Now to find the discretized momentums, we must apply appropriate boundary condition on the Graphene quasi-spinors. But, as we previously discussed, the adatoms on Graphene can play the role of some external restricting boundaries on the Graphene Dirac-like field. Here we consider the configuration of figure 1 i.e. two parallel chain of adatoms in a Graphene sheet. These parallel adatom-chains can be regarded as a 1-dimensional version of the parallel planes considered in Refs. [4, 7, 31]. The mode solutions of a 2-dimensional Dirac equation, for the configuration of these adatom-chains, can be written as

$$\psi_{k_x,k_y}(x,y,t) = e^{-i\omega t}e^{ik_yy} \left( ue^{ik_x x} + ve^{-ik_x x} \right),$$

(11)

in which $u$ and $v$ are constant spinors. But the appropriate boundary condition for Dirac spinors is given by the known bag boundary condition [7,31], which is equivalent to requiring the normal component of the fermion current to vanish at the boundaries [4], or in other word, supposing the fermion current could not penetrate through the boundaries, as physically expected. Now considering the configuration of figure 1 the bag boundary condition for the mode solutions (11), can be written as

$$[i\gamma \cdot \hat{n} + 1] \psi(x,y)|_{x=0,a} = 0$$

(12)

where $\hat{n}$ is a unite vector normal to the boundaries, having $\hat{n} = -\hat{x}, \hat{x}$ for $x = 0, a$ respectively, and $\gamma$ is the spacelike component of the Dirac gamma matrices. Now
applying the boundary conditions (12) onto the solution (11), and after some calculations similar to those of Ref. [7], one can find the discretized modes with

\[ k_{x,n} = \left( n + \frac{1}{2} \right) \frac{\pi}{a}; \quad n = 0, 1, 2, \ldots \]  

(13)

As a result, the zero-point energy (5) takes the form

\[ E_0 = 2i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{L}{2\pi} dk_y \sum_{n=0}^{\infty} \ln \left[ -\omega^2 \frac{2}{v_F^2} + k_y^2 + \left( n + \frac{1}{2} \right)^2 \frac{\pi^2}{a^2} \right]. \]  

(14)

In which, \( L \) is the length of the adatom-chains \((L \gg a)\), see Fig. 1. Then using the Matsubara frequencies (8), the zero-point energy for the configuration of two parallel adatom-chains, can be written as

\[ E_0(a, T) = -2k_B T \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L}{2\pi} dk_y \sum_{n=0}^{\infty} \lim_{s \to 0} \frac{\partial}{\partial s} \delta^{-2s} \left[ \left( \frac{2\pi k_B T}{\hbar v_F} \right)^2 l^2 + k_y^2 + \left( n + \frac{1}{2} \right)^2 \frac{\pi^2}{a^2} \right] - s \]  

(15)

where \( \delta \) is an arbitrary parameter with the dimension of the length, introduced for dimensional requirements. Taking a Gaussian integral for \( k_y \), the above equation turns to

\[ E_0(a, T) = 2L \sqrt{\pi} \frac{(k_B T)^2}{\hbar v_F} \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{\hbar v_F}{2\pi k_B T \delta} \right)^{2s} \] 

\[ \times \int_{0}^{\infty} \frac{dt}{t} \frac{t^s}{\Gamma(s)} \exp \left( -t \left[ \left( \frac{2\pi k_B T}{\hbar v_F} \right)^2 l^2 + k_y^2 + \left( n + \frac{1}{2} \right)^2 \frac{\pi^2}{a^2} \right] \right) \]  

(16)

(17)
in which we have introduced the dimensionless variable \( \lambda_a = \theta_a/T \) having the effective temperature
\[
\theta_a \equiv \frac{h v_F}{2ak_B}.
\] (18)

3.1 \( T \gg \theta_a \)

If we take the Graphene temperature to be near the room temperature, \( T \approx 300K \), then the assumption \( \theta_a \ll T \approx 300 \) corresponds to a configuration of two parallel adatom-chains with a separation distance e.g. in the order of few micrometers, see Eq. (18) (with \( v_F \approx 10^6, h \approx 10^{-34} \) and \( k_B \approx 10^{-23} \)). An asymptotically suitable expression for the zero-point energy for high temperatures, would be provided generally by applying the familiar heat kernel expansion, see e.g. [4]. However here we use a new useful technique to obtain an exact expression, for the zero-point energy, being asymptotically suitable for sufficiently high temperatures (respecting the effective temperature). First separating \( l = 0 \) from the \( l \)-sum, we rewrite Eq. (17) as
\[
E_0(a,T) = 2L \sqrt{\pi} \left( \frac{k_B T}{h v_F} \right)^2 \lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{h v_F}{2\pi k_B T \delta} \right)^{2s} \times \int_0^\infty \frac{dt}{t} \frac{t^{s-1/2}}{\Gamma(s)} \left[ 1 + 2 \sum_{l=1}^\infty \exp \left( -tl^2 \right) \sum_{n=0}^\infty \exp \left( -t\lambda_a^2 \left( n + \frac{1}{2} \right)^2 \right) \right]
\]
\[
= 2L \sqrt{\pi} \left( \frac{k_B T}{h v_F} \right)^2 \lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{h v_F}{2\pi k_B T \delta} \right)^{2s} \left( \frac{\zeta_H(2s - 1, \frac{1}{2}) \Gamma(s - \frac{1}{2})}{\Gamma(s) \lambda_a^{2s-1}} \right)
\]
\[
+ 2 \int_0^\infty \frac{dt}{t} \frac{t^{s-1/2}}{\Gamma(s)} \sum_{l=1}^\infty \exp \left( -tl^2 \right) \sum_{n=0}^\infty \exp \left( -t\lambda_a^2 \left( n + \frac{1}{2} \right)^2 \right) \right) \] (19)

where in the third line, we have utilized the known Hurwitz zeta function
\[
\zeta_H(s, r) = \sum_{n=0}^\infty (n + r)^{-s}.
\] (20)

Then we use the following formula from Ref. [33]:
\[
\sum_{n=0}^\infty \exp \left[ -c \left( n + d \right)^2 \right] = \sum_{n=0}^\infty \frac{(-1)^n}{n!} c^n \zeta_H(-2n, d) + \frac{1}{2} \sqrt{\frac{\pi}{c}}
\]
\[
+ \sqrt{\frac{\pi}{c}} \cos(2\pi d) \sum_{n=1}^\infty \exp \left[ -\frac{n^2 \pi^2}{c^2} \right] ; \quad c > 0, \quad d \geq 0.
\] (21)

For \( d = 1/2 \), the above equation would be simplified to
\[
\sum_{n=0}^\infty \exp \left[ -c \left( n + \frac{1}{2} \right)^2 \right] = \frac{1}{2} \sqrt{\frac{\pi}{c}} - \sqrt{\frac{\pi}{c}} \sum_{n=1}^\infty \exp \left[ -\frac{n^2 \pi^2}{c^2} \right],
\] (22)
where we have used \( \zeta_H(-2n,1/2) = 0 \), \( n = 0, 1, \ldots \), see e.g. [34]. Now, by applying Eq. (22) to the \( n \)-sum in the fourth line of Eq. (19), and after some calculations we find

\[
E_0(a,T) = 2L\sqrt{\frac{\pi}{2\pi}} \left( \frac{k_B T}{\hbar v_F} \right)^2 \lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{\hbar v_F}{2\pi k_B T \delta} \right)^{2s-1} \times \left( \frac{\zeta_H(2s-1,1/2) \Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s) \lambda_a^{2s-1}} + \frac{\sqrt{\pi} \Gamma(s-1)}{\lambda_a} \zeta_R(2s-2) \right)
\]

\[
- \frac{4\sqrt{\pi}}{\lambda_a \Gamma(s)} \sum_{l,n=1}^{\infty} \left( \frac{l}{n \pi / \lambda_a} \right)^{-s+1} K_{-s+1} \left( \frac{2ln\pi}{\lambda_a} \right)
\]

in which, \( \zeta_R \) denotes the well known Riemann zeta function;

\[
\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}.
\]

and we have used the integral relation

\[
\int_0^{\infty} t^r \exp \left[ -x^2 t - y^2 / t \right] dt = 2(x/y)^{r-1} K_{r-1}(2xy).
\]

with \( K \) as a Bessel function of the second kind. Then, after some calculations, the zero-point energy (23) takes the form

\[
E_0(a,T) = -2L\left( \frac{k_B T}{\hbar v_F} \right)^2 \left( \frac{2\pi \zeta'(-2)}{\lambda_a} + \frac{\pi \lambda_a}{12} + 4 \sum_{l,n=1}^{\infty} \frac{l}{n} K_1 \left( \frac{2ln\pi}{\lambda_a} \right) \right)
\]

in which, “prime” denotes the differentiating with respect to the argument. To obtain the Casimir energy, one must subtract the contribution of free (2-dimensional) space from Eq. (26). To obtain the zero-point energy contribution of the free (i.e. unbounded) space, one just need to turn the discrete modes to continuous ones (see e.g. [4]), which for the zero-point energy (15), we have

\[
E_0^{\text{free}}(a,T) = -2k_B T \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L}{2\pi} dk_y \int_{-\infty}^{\infty} \frac{a}{2\pi} dk_x \ln \left( \frac{2\pi k_B T}{\hbar v_F} \right)^2 l^2 + k_x^2 + k_y^2.
\]

Using similar calculations, one can show that the zero-point contribution (27) is exactly equal to the first term of Eq. (26).

\[
E_0^{\text{free}}(a,T) = -L \left( \frac{k_B T}{\hbar v_F} \right)^2 4\pi \zeta'(-2) \frac{\lambda_a}{2}
\]

Thus subtracting this term, we find the fermionic Casimir energy of the Graphene:

\[
E_C(a,T) = -L \left( \frac{\pi k_B T}{12a} + \frac{8k_B^2 T^2}{\hbar v_F} \sum_{n,l=1}^{\infty} \frac{l}{n} K_1 \left( 4ln\pi a \frac{k_B T}{\hbar v_F} \right) \right)
\]
see Eq. (18). So the fermionic Casimir force between adatom-chains, \( F_C \equiv -\frac{\partial}{\partial a} E_C \), would be obtained as

\[
F_C(a, T) = -L \left( \frac{\pi k_B T}{12a^2} + \frac{16\pi k_B^3 T^3}{\hbar^2 v_F^2} \sum_{n,l=1}^{\infty} l^2 \left[ K_0 \left( 4l \pi a \frac{k_B T}{\hbar v_F} \right) + K_2 \left( 4l \pi a \frac{k_B T}{\hbar v_F} \right) \right] \right) \tag{30}
\]

Note that, Eqs. (29) and (30) are exact expressions for the Casimir energy and the Casimir force, however, they provide asymptotically explicit expressions for temperatures sufficiently larger than the effective temperature. In fact for sufficiently large temperatures (in comparison to the effective temperature (18), the Casimir energy (29) and the Casimir force (30), can be approximated with their first term. The above useful approach can be generalized directly to find asymptotic expressions for the Casimir energy in spaces with arbitrary dimensions. Note also that the above equation represents an attractive Caimir interaction between two (parallel) chains of adatoms, which is in agreement to the result of Ref. [23] for the Casimir interaction between two single adatoms.

### 3.2 \( T \ll \theta_a \)

At room temperature, the assumption \( T \ll \theta_a \) corresponds to a configuration of two parallel adatom-chains with a separation distance e.g. in the order of few nanometers. In order to find an asymptotically explicit expression for the zero-point energy (15) for small temperatures, we apply Eq. (21) with \( d = 0 \),

\[
\sum_{n=0}^{\infty} \exp \left[ -cn^2 \right] = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{\pi}{c}} + \sqrt{\frac{\pi}{c}} \sum_{n=1}^{\infty} \exp \left[ -\frac{n^2 \pi}{c^2} \right], \tag{31}
\]

where we have used \( \zeta_H(0, 0) = \frac{1}{2}, \zeta_H(-2n, 0) = 0 : n = 1, 2, \ldots \). Note that the above equation is equivalent just to the familiar Poisson sum formula, see e.g. [34]. Now applying the formula (31) to the \( l \)-sum in the fourth line of Eq. (19), and using Eqs. (20) and (25), one finds

\[
E_0(a, T) = 2\pi L \left( \frac{k_B T}{\hbar v_F} \right)^2 \lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{\hbar v_F}{2\pi k_B T \delta} \right)^{2s} \left( \frac{\zeta \left( 2s - 2, \frac{1}{2} \right) \Gamma(s - 1)}{\Gamma(s) \lambda s^{2s - 2}} \right) +
\]

\[
+ \frac{4}{\Gamma(s)} \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \left( \frac{(2n + 1) \lambda a}{2l \pi} \right)^{-s+1} K_{-s+1} \left[ l \left( 2n + 1 \right) \pi a \right] \tag{32}
\]

Then after some similar calculations as before, the above equation takes the form

\[
E_0(a, T) = 2\pi L \left( \frac{k_B T}{\hbar v_F} \right)^2 \left( \frac{3\pi^2 \lambda^2 c^2 \zeta'(-2)}{2} + 2\lambda a \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{2n + 1}{l} K_1 \left[ l \left( 2n + 1 \right) \pi a \right] \right) \tag{33}
\]

Then subtracting the contribution of free space, Eq. (28), from the above equation, we obtain the Casimir energy

\[
E_C(a, T) = -L \left( \frac{3\zeta_R(3) \hbar v_F}{16\pi a^2} + \frac{4\zeta_R(3) (k_B T)^3 a}{\pi \left( \hbar v_F \right)^2} \right) + \frac{2k_B T}{a} \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{(2n + 1)}{l} K_1 \left( l(2n + 1) \pi \frac{\hbar v_F}{2ak_B T} \right) \tag{34}
\]
where we have used
\[ \zeta_R'(-2) = -\frac{\zeta_R(3)}{4\pi^2}; \quad \zeta_R(3) \approx 1.2 \]  
(35)
see e.g. [34]. To find asymptotic expression for the Casimir energy for sufficiently small temperatures (in comparison to the effective temperature [18]), the last term of [34] can be neglected.

### 3.3 Influence on the initial tension of a suspended Graphene

It has been experimentally demonstrated that a Graphene sheet suspended over a trench, possesses a rather large initial tension, which is arisen from e.g. the strong Van der Waals interactions between the Graphene and the sidewalls of the trench, and/or from the external forces during the fabrication of the Graphene sheet, see e.g. [24–30]. Such a large built-in tension plays an important role in the resonance frequencies of the Graphene resonators, see e.g. [24–27]. Here we want to show that the the fermionic Casimir forces can have considerable influence on the the mentioned built-in tension of a suspended Graphene. In fact considering a suspended Graphene sheet with an initial tension, and with two parallel adatom-chains as given in Fig. 1 then the fermionic Casimir force between the adatom chains, can be considered as a (quantum) contribution term to the initial tension of Graphene. Note however that the above consideration is meaningful, if the separation distance between adatom-chains is sufficiently smaller than sidelengths of the Graphene sheet. As a result, the change in the initial tension (per unit length) of Graphene sheet due to the fermionic Casimir forces between (per unit length) of adatom chains, at room temperature, using Eq. (34), can be written as
\[ \Delta \tau(a) \approx \frac{3\zeta_R(3)}{8\pi} \frac{\hbar v_F}{a^3}; \quad a < \text{few nanometer} \]  
(36)
in which \( \tau \) is the initial tension of the suspended Graphene sheet. The above equation obviously means that the attractive Casimir force between the adatom chains, increases the initial tension of the Graphene, as physically expected. For \( a = 1\text{nm} \) in Eq. (36) one finds a value of the order of \( 10^{-2}N/m \) for \( \Delta \tau \), which is considerably noticeable in comparison to the experimental values of the order of \( 10^{-1} - 10^{-4}N/m \), given in Refs. [28–30], for the initial tension of the Graphene sheet with sidelengths in the order of few micrometers.

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