GAPS OF POWERS OF CONSECUTIVE PRIMES AND SOME CONSEQUENCES

DOUGLAS AZEVEDO AND TIAGO REIS

Abstract. Let \( \{q_n\} \) be a sequence of positive numbers and \( x \in \mathbb{R} \). In this note we prove that the inequality

\[
q_n p_{n+1}^x - q_{n+1} p_n^x < p_n^x p_{n+1}^{x-1},
\]

holds for infinitely many values of \( n \). As it is shown, the key ingredient to obtain this behaviour is a consequence of an extension of the Kummer’s characterization of convergent series of positive terms.

1. Background and main result

The behaviour of the prime numbers is one of the most interesting issues in mathematics and many great mathematicians have been working on this subject, for instance, we indicate [3, 4, 7, 8, 10], and references therein. In particular, the investigation of gaps between consecutive prime numbers, that is, the behaviour of the sequence \( \{g_n\} \), defined as \( g_n = p_{n+1} - p_n \), for all positive integer \( n \), in which, \( p_n \) denotes the \( n \)th prime number, is among one of the most important unsolved problems in number theory.

In this note we present some information about the sequence \( \{p_{n+1}^x - p_n^x\} \), in which \( x \) is a positive real number. Our main result is as follows

Theorem 1.1. Let \( x \in \mathbb{R} \). For any sequence of positive terms \( \{q_n\} \), the inequality

\[
q_n p_{n+1}^x - q_{n+1} p_n^x < \frac{p_n^x}{p_{n+1}^{1-x}}
\]

holds for infinitely many values of \( n \).

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It is clear that Theorem 1.1 provides a general inequality involving powers of prime numbers and by exploring this inequality, for suitable choices of $x \in \mathbb{R}$ we are able to obtain interesting information about the sequence $\{p_{n+1}^x - p_n^x\}$. Note also that this result is related to the main result of [2], which deals with the case $x = 1$.

The method to prove Theorem 1.1 depends on an extension of the Kummer’s test for convergence of series of positive terms. This test is actually a theoretical characterization of convergent series of positive terms, that is, it provides necessary and sufficient conditions that ensures convergence of series of positive terms. For more information about the original Kummer’s test we refer to [12].

Another important result that plays a fundamental role in the proof of Theorem 1.1 is the well known divergence of the series of the reciprocal of the primes numbers. Roughly speaking, the idea behind the proof of Theorem 1.1 is to combine the divergence of this series with the contrapositive argument of an extension of the Kummer’s test.

Let us now present the results that will be needed to prove our main result and then extract some interesting information about gaps of powers of prime numbers.

We start by presenting an extension of the Kummer’s test which characterizes the convergence of series of positive terms.

**Lemma 1.2.** Let $\{a_n\}$ and $\{b_n\}$ are sequences of positive terms. The series $\sum a_n b_n$ converges if and only if there exist a sequence $\{q_n\}$ of positive terms and a positive integer $N$ such that

$$q_n \frac{a_n}{a_{n+1}} - q_{n+1} \geq b_{n+1},$$

for $n > N$.

**Proof.** Let us show that $\sum_{n=1}^{\infty} a_n b_n$ converges. For this, note that the condition

$$q_n \frac{a_n}{a_{n+1}} - p_{n+1} \geq b_{n+1}, \quad n \geq N$$
implies that

(1.1) \[ a_n q_n \geq a_{n+1}(q_{n+1} + b_{n+1}), \quad n \geq N. \]

Hence, for any fixed \( k \geq 0 \) we have that

\[ a_{N+k}q_{N+k} \geq a_{N+k+1}q_{N+k+1} + a_{N+k+1}b_{N+k+1}, \]

as so,

\[ a_{N}q_{N} + \sum_{i=1}^{k} a_{N+i}q_{N+i} \geq \sum_{i=1}^{k} a_{N+i}b_{N+i} + \sum_{i=1}^{k} a_{N+i}q_{N+i} + a_{N+k+1}q_{N+k+1}. \]

Therefore we obtain

\[ a_{N}q_{N} \geq \sum_{i=1}^{k} a_{N+i}b_{N+i} + a_{N+k+1}q_{N+k+1} \geq \sum_{i=1}^{k} a_{N+i}b_{N+i} > 0, \]

for all integer \( k \geq 0 \). This implies the convergence of \( \sum_{n=1}^{\infty} a_n b_n \).

Conversely, suppose that \( S := \sum_{n=1}^{\infty} a_n b_n \) and define

(1.2) \[ q_n = \frac{S - \sum_{i=1}^{n} a_i b_i}{a_n}, \quad n \geq N. \]

For this \( \{q_n\} \), clearly \( q_n > 0 \) for all \( n \geq 1 \) and it is easy to check that

\[ q_n \frac{a_n}{a_{n+1}} - q_{n+1} = b_{n+1}, \quad n \geq 1. \]

\( \square \)

As we have mentioned before, the key argument in the proof of Theorem 1.1 is the contrapositive of Lemma 1.2 which is presented below.

**Lemma 1.3.** Let \( \{a_n\} \) and \( \{b_n\} \) are sequences of positive terms. The series \( \sum a_n b_n \) diverges if and only if for each \( \{q_n\} \) of positive terms we have that: for all positive integer \( N \) there exist \( n > N \) such that

\[ q_n \frac{a_n}{a_{n+1}} - q_{n+1} < b_{n+1}. \]
2. **Proof Theorem 1.1 and some consequences**

In this section we prove Theorem 1.1 and present some interesting consequences.

**Proof of Theorem 1.1.** It is clear that

\[
\sum_{n=1}^{\infty} \frac{1}{p_n} = \sum_{n=1}^{\infty} \frac{1}{p_n^{x-1}}
\]

holds for any real number \(x\). Since \(\sum \frac{1}{p_n}\) diverges, a direct application of Lemma 1.3 with \(a_n = \frac{1}{p_n}\) and \(b_n = p_n^{x-1}\) give us that, for all \(\{q_n\}\) of positive terms, we have that for all positive integer \(N\) there exists \(n > N\) such that

\[q_n \left(\frac{1}{p_n^{x}}\right) - q_{n+1} < p_n^{x-1}.
\]

That is, for every sequence \(\{q_n\}\) of positive numbers, the inequality

\[q_n p_n^x - q_{n+1} p_n^x < \frac{p_n^x}{p_n^{1-x}},
\]

holds for infinitely many values of \(n\). \(\square\)

Let us now extract some consequences of Theorem 1.1.

**Corollary 2.0.1.** Let \(x\) be a real number and \(\{q_n\}\) any sequence of positive numbers. The inequality

\[p_{n+1}^x - p_n^x < \left(p_n p_{n+1}\right)^x \frac{q_n}{q_n p_{n+1}} + p_n^x q_{n+1} - q_n
\]

holds for infinitely many values of \(n\).

**Proof.** Note that

\[q_n \left(p_{n+1}^x - p_n^x\right) = q_n p_{n+1}^x - q_n p_n^x + q_{n+1} p_n^x - q_n p_{n+1} p_n^x < \frac{p_n^x}{p_{n+1}^{1-x}} q_{n+1} - q_n p_n^x + q_{n+1} p_n^x.
\]

This implies that

\[p_{n+1}^x - p_n^x < \left(p_n p_{n+1}\right)^x \frac{q_n}{q_n p_{n+1}} + p_n^x q_{n+1} - q_n
\]
The next result presents a general bound for gaps of prime numbers which was obtained previously in [2].

**Corollary 2.0.2.** Let \( \{q_n\} \) be a sequence of positive numbers. The inequality

\[
p_{n+1} - p_n < p_n \frac{q_{n+1} - q_n + 1}{q_n},
\]

holds for infinitely many values of \( n \).

**Proof.** It is enough to take \( x = 1 \) in Theorem 1.1. \(\square\)

In particular, from the previous corollary we have the following behaviour.

**Corollary 2.0.3.** If \( x > 0 \) then

\[
\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{p_n^x} = 0.
\]

**Proof.** Let \( x > 0 \) be fixed and take as \( q_n = n \) in Corollary 2.0.2. From the Prime Number Theorem we have that \( \lim_{n \to \infty} \frac{p_n}{n \log(n)} = \lim \frac{\log(p_n)}{\log(n)} = 1. \)
Hence

\[
\frac{p_{n+1} - p_n}{p_n^x} < \frac{p_n}{n \log(n)} \frac{n \log(n)}{p_n^x} \frac{2}{n} = 2 \frac{p_n}{n \log(n)} \frac{\log(n)}{p_n^x}
\]
holds for infinitely many values of \( n \). As so

\[
\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{p_n^x} \leq 2 \liminf_{n \to \infty} \frac{p_n}{n \log(n)} \frac{\log(n)}{p_n^x} = 0.
\] \(\square\)

The next result is related to the celebrated estimate obtained in [3] which states that

\[
\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log(p_n)} = 0.
\]

Although we prove a weaker version, note that we use quite more elementary methods.
Corollary 2.0.4. If $x > 0$ then

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log(p_n)^{1+x}} = 0.$$  

Proof. Let $x > 0$ be fixed and again take as $q_n = n$ in Corollary 2.0.2. By the same arguments used in the previous proof we have that

$$\frac{p_{n+1} - p_n}{\log(p_n)^{1+x}} < \frac{p_n}{n \log(n) \log(n)^x} \cdot \frac{2}{n},$$

holds for infinitely many values of $n$. This implies that

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log(p_n)^{1+x}} \leq 2 \liminf_{n \to \infty} \frac{p_n}{n \log(n) \log(n)^x} = 0.$$

We close the paper with a result related to the Andrica’s conjecture.

Corollary 2.0.5. If $0 \leq x < 1$ then

$$(2.2) \quad \liminf_{n \to \infty} p_n^x - p_n = 0.$$  

Proof. In Corollary 2.0.4 let $q_n = p_n$, for all $n \geq 1$. Then the inequality

$$p_n^x - p_{n+1}^x < \frac{1}{(p_{n+1}p_n)^{1-x}} \cdot \frac{p_{n+1} - p_n}{p_n^{1-x}},$$

holds for infinitely many values of $n$. Since $0 \leq x < 1$ we have that

$$\lim_{n \to \infty} \frac{1}{(p_{n+1}p_n)^{1-x}} = 0.$$  

Also, from Corollary 2.0.3 note that

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{p_n^{1-x}} = 0,$$

thus

$$\liminf_{n \to \infty} p_n^x - p_{n+1}^x \leq \liminf_{n \to \infty} \frac{1}{(p_{n+1}p_n)^{1-x}} + \frac{p_{n+1} - p_n}{p_n^{1-x}} = 0.$$

The proof is concluded. □
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UTFPR-CP. AV. ALBERTO CARAZZAI 1640, CENTRO, CAIXA POSTAL 238, 86300000, CORNELIO PROCOPIO, PR, BRASIL.

E-mail address: dgs.nvn@gmail.com