SOME PROBLEMS IN ADDITIVE NUMBER THEORY

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Abstract. In this article, we consider some problems in additive number theory. Elementary solutions for the Goldbach-Euler conjecture and the twin primes conjecture are presented. The method employed also makes it possible to obtain some interesting results related to the densities of sequences and gaps between primes. A stronger version of the Goldbach-Euler conjecture, in which the strengthening is established by reducing the set of all primes to the set of twin prime pairs, is also presented. The proof is based on the direct construction of the double sieve and does not use heuristic methods.

All results of the profoundest mathematical investigation must ultimately be expressible in the simple form of properties of integers.

Leopold Kronecker

Regarding the relative powers of elementary sieve methods and the analytical methods one usually considers that the latter should be more powerful... But history has shown that such views are not totally correct.

H.E. Richert, Lectures on Sieve Methods, Tata Institute of Fundamental Research, 1976

1. Background and conventions

The proof is based on some property of $\mathbb{Z}/6\mathbb{Z}$: the residue classes $\bar{1}_6$ and $\bar{5}_6$ contain all odd primes except 3; each of the even residue classes modulo 6 may be represented as a sum or a difference of $\bar{1}_6$ and/or $\bar{5}_6$; the sequences $\bar{1}_6$ and $\bar{5}_6$ are well-structured by all prime numbers. It allows us to construct the double sieve.

Let $P$ denote the set of all primes. We assume that $p \in P$, where $P = P \setminus \{2, 3\}$.

We use notation $\#S_m$ for the number of positive elements of the segment of sequence $S_m$.

We apologize to the reader for using a specific font to denote objects of a particular type. We have to do it in order to save a certain logical structure.

2. Preliminaries

2.1. Well-structured sequences.

Definition 1. We say that a sequence $S$ well-structured by number $q$ if the indices of terms of sequence $S$ that are divisible by $q$ form an arithmetic progression with the common difference $q$.

Let $A = \{ a_i : a_i = 6i - 1, i \in \mathbb{N} \}$, and let $B = \{ b_i : b_i = 6i + 1, i \in \mathbb{N} \}$:

$A = \{ 5, 11, 17, 23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95, 101, \ldots \}$,

$B = \{ 7, 13, 19, 25, 31, 37, 43, 49, 55, 61, 67, 73, 79, 85, 91, 97, 103, \ldots \}$.
A terms of sequences $A$ and $B$ either are primes $p \in \mathcal{P}$ or products of primes $p \in \mathcal{P}$. Obviously $A \cup B \supset \mathcal{P}$.

**Theorem 2.** The sequences $A$ and $B$ are well-structured by all primes $p \in \mathcal{P}$.

**Proof.** If $a_i$ are composite, then there exists $j \neq 0$, $k \neq 0$ such that

$$a_i = 6i - 1 = a_j b_k = (6j - 1)(6k + 1) = 36jk - 6k + 6j - 1.$$ 

In this case, we have two expressions for $i$:

$$i = k(6j - 1) + j = ka_j + j,$$

$$i = j(6k + 1) - k = jb_k - k,$$

that determine two families of arithmetic progressions

\begin{align*}
(1) & \quad a_j \mid a_i \Leftrightarrow i = +j + ka_j, \\
(2) & \quad b_k \mid a_i \Leftrightarrow i = -k + jb_k.
\end{align*}

If $b_i$ are composite, then there exists $j \neq 0$, $j' \neq 0$, or $k \neq 0, k' \neq 0$ such that at least one of the following equalities will hold:

\begin{align*}
b_i = 6i + 1 & = a_j a_{j'} = (6j - 1)(6j' - 1) = 6(6jj' - j - j') + 1, \\
b_i = 6i + 1 & = b_k b_{k'} = (6k + 1)(6k' + 1) = 6(6kk' + k + k') + 1.
\end{align*}

In that case, we also have two expressions for $i$:

$$i = j'(6j - 1) - j = j'a_j - j,$$

$$i = k'(6k + 1) + k = k'b_k + k,$$

that determine two families of arithmetic progressions

\begin{align*}
(3) & \quad a_j \mid b_i \Leftrightarrow i = -j + j'a_j, \\
(4) & \quad b_k \mid b_i \Leftrightarrow i = +k + k'b_k.
\end{align*}

Expressions (1), (2), (3), and (4) implies that the sequences $A$ and $B$ are well-structured by all numbers $a_j \in A$ and $b_k \in B$. The proof follows from the $A \cup B \supset \mathcal{P}$. \(\square\)

2.2. **Sieving.** That is, for any $p \in \mathcal{P}$ in each of the sequences $A$ and $B$ there is exactly one subsequence of terms are divisible by $p$ and form an arithmetic progression with the common difference equal to $6p$. The indices of these terms form an arithmetic progression with the common difference equal to $p$.

We define the sieving of a sequence $S$ by a number $p$ as replacing composite terms of sequence $S$ which are a multiple of $p$ by 0. A sequence $S$ sifted by $p$ we denote as $S\setminus \lambda p$. A sequence $S$ sifted by all $p \in \mathcal{P}$ we denote as $S\setminus \lambda \mathcal{P}$. We also write $S_m \setminus \lambda p$ and $S_m \setminus \lambda \mathcal{P}$ for sieving of a segment $S_m$.

2.3. **Sequences $A$, $B$, $L$, and $R$.** Let $A = A \setminus \lambda \mathcal{P}$ and let $B = B \setminus \lambda \mathcal{P}$. We now set up a correspondence between sequences $\mathcal{L}$ and $\mathcal{R}$ and the sequences $A$ and $B$. The rules of correspondence is $\mathcal{L} = \{l_i \mid l_i = i \text{ if } a_i \in \mathcal{P}; \quad l_i = 0 \text{ if } a_i \notin \mathcal{P}\}$ and $\mathcal{R} = \{r_i \mid r_i = i \text{ if } b_i \in \mathcal{P}; r_i = 0 \text{ if } b_i \notin \mathcal{P}\}$, i.e.

$$A = \{5, 11, 17, 23, 29, 0, 41, 47, 53, 59, 0, 71, 0, 83, 89, 0, 101, \ldots\},$$

$$\mathcal{L} = \{1, 2, 3, 4, 5, 0, 7, 8, 9, 10, 0, 12, 0, 14, 15, 0, 17, \ldots\},$$

$$B = \{7, 13, 19, 0, 31, 37, 43, 0, 0, 61, 67, 73, 79, 0, 0, 97, 103, \ldots\},$$

$$\mathcal{R} = \{1, 2, 3, 0, 5, 6, 7, 0, 0, 10, 11, 12, 13, 0, 0, 16, 17, \ldots\}.$$
All nonzero terms of the sequences \( A \) and \( B \) are prime numbers. All nonzero terms of the sequences \( L \) and \( R \) are indices of proper nonzero terms of the sequences \( A \) and \( B \), respectively. Obviously, the sequences \( L \) and \( R \) inherit the structures of the sequences \( A \) and \( B \) in the sense of distribution of zero terms. The indices of zero terms of the sequences \( A \) and \( L \) determine by the right-hand sides of (1) and (2); the indices of zero terms of the sequences \( B \) and \( R \) determine by the right-hand sides of (3) and (4).

Now we define sequences \( A^{m'} = \{ a_i^{m'} : a_i^{m'} = a_{i+m'} \} \), \( B^{m'} = \{ b_i^{m'} : b_i^{m'} = b_{i+m'} \} \), as the remainder of sequences \( A \) and \( B \) after the \( m' \)-th term, as follows:

\[
A^5 = \{ 0, 41, 47, 53, 59, 0, 71, 0, 83, 89, 0, 101, 107, 113, 0, 0, 131, \ldots \}, \\
B^5 = \{ 43, 0, 0, 61, 67, 73, 79, 0, 0, 97, 103, 109, 0, 0, 127, 0, 139, \ldots \}.
\]

Just the same for sequences \( L^{m'} \) and \( R^{m'} \) we get

\[
L^5 = \{ 0, 7, 8, 9, 10, 0, 12, 0, 14, 15, 0, 17, 18, 19, 0, 0, 22, \ldots \}, \\
R^5 = \{ 7, 0, 0, 10, 11, 12, 13, 0, 0, 16, 17, 18, 0, 0, 21, 0, 23, \ldots \}.
\]

By \( \{g\} \) we denote a sequence with the constant term equal to even number \( g \).

2.4. Segments of sequences. Let \( S_m = \{ s_i \}^m_{i=1} \) denote the initial segment of length \( m \) of sequence \( S \), for example,

\[
A_{14} = (5, 11, 17, 23, 29, 0, 41, 47, 53, 59, 0, 71, 0, 83), \\
B_{14} = (7, 13, 19, 0, 31, 37, 43, 0, 0, 61, 67, 73, 79, 0).
\]

We call these segments the direct segments, and we call the segments

\[
A'_{14} = (83, 0, 71, 0, 59, 53, 47, 41, 0, 29, 23, 17, 11, 5), \\
B'_{14} = (0, 79, 73, 67, 61, 0, 0, 43, 37, 31, 0, 19, 13, 7),
\]

the inverse segments. Just the same for sequences \( L_m, R_m, L'_m \) and \( R'_m \):

\[
L_{14} = (1, 2, 3, 4, 5, 0, 7, 8, 9, 10, 0, 12, 0, 14), \\
R_{14} = (1, 2, 3, 0, 5, 6, 7, 0, 0, 10, 11, 12, 13, 0), \\
L'_{14} = (14, 0, 12, 0, 10, 9, 8, 7, 0, 5, 4, 3, 2, 1), \\
R'_{14} = (0, 13, 12, 11, 10, 0, 0, 7, 6, 5, 0, 3, 2, 1).
\]

2.5. Nonzero terms counting function. Let \( \pi(a, n) \) denote the number of primes not exceeding \( n \) that are of the form \( 6i - 1 \), and let \( \pi(b, n) \) denote the number of primes not exceeding \( n \) that are of the form \( 6i + 1 \). For the following discussion, we take \( n = 6m \). Then,

\[
\pi(a, n) = \#A_m = \#L_m = \sum_{l_i \leq m, l_i \neq 0} 1, \\
\pi(b, n + 1) = \#B_m = \#R_m = \sum_{r_i \leq m, r_i \neq 0} 1.
\]

2.6. Even numbers. Let \( G = \{ g \} \) be the set of all positive even numbers. We partition \( G \setminus \{2\} \) into three disjoint sequences (residue classes modulo 6) and assume that \( G^1 = \{ g_m^1 : g_m^1 = 6m - 2 \} \), \( G^2 = \{ g_m^2 : g_m^2 = 6m \} \), \( G^3 = \{ g_m^3 : g_m^3 = 6m + 2 \} \). Each integer \( m \) determines three consecutive even numbers, one from each of these classes.

2.7. Sequence’ segment summation. We define the addition (subtraction) of two sequences \( S' = \{ s'_i \} \) and \( S'' = \{ s''_i \} \) as sequence \( S = \{ s_i : s_i = s'_i \pm s''_i \text{ if } s'_i s''_i \neq 0; s_i = 0 \text{ if } s'_i s''_i = 0 \} \). Also, we define the addition (subtraction) of two segment of sequences \( S'_m = \{ s_i \}^m_{i=1} \) and \( S''_m = \{ s''_i \}^m_{i=1} \) as segment of sequence \( S_m = \{ s_i : s_i = s'_i \pm s''_i \text{ if } s'_i s''_i \neq 0; s_i = 0 \text{ if } s'_i s''_i = 0 \} \).

We assume that a sequence \( S = S' + S'' \) sifting by \( p \) if both sequences \( S' \) and \( S'' \) sifting by \( p \). In this case we write \( S \setminus \lambda p = S' \setminus \lambda p + S'' \setminus \lambda p \).
2.8. Binary additive problems.

2.8.1. Pairs of primes with a fixed difference. Let \( \pi_g(n) \) be a number of primes \( p \) not exceeding \( n \) and such that \( p' = p + g \) are also primes. All even numbers of each class may be represented as a difference of two odd integers from \( A \) and/or \( B \) in one and only one way, that is, \( g_1 = a + b \), \( g_2 = (a' + b) - b \), and \( g_3 = b' + a' - a \). These identities allow us to find the solution of this problem from the constructions

\[
\pi_{g_1}(n + 1) = \# \left( A_{m'} - B_{m'} \right) = \# \left( L_{m'} - R_{m'} \right),
\]

\[
\pi_{g_2}(n + 1) = \# \left( A_{m'} - A_m \right) + \# \left( B_{m'} - B_m \right)
\]

\[
= \# \left( L_{m'} - L_m \right) + \# \left( R_{m'} - R_m \right),
\]

\[
\pi_{g_3}(n) = \# \left( B_{m'} - A_m \right) = \# \left( R_{m'} - L_m \right).
\]

We note that \( \pi_{g_1}(n + 1) - \pi_{g_2}(n) \leq 1 \) and \( \pi_{g_2}(n + 1) - \pi_{g_2}(n) \leq 1 \).

Remark 3. If \( g \) then \( \pi_g(n) \) is equal to the sum of the numbers of nonzero terms in two segments.

For example, we find the number of pairs \( p, p + 28 \) where \( p \leq 126 + 1 \). We have \( 28 = g_5^1 \in G^1 \), \( m' = 5 \) and \( m = 126/6 = 21 \). Now apply construction (5), we get

\[
A_{21}^5 = 0, 41, 47, 53, 59, 0, 71, 0, 83, 89, 0, 101, 107, 113, 0, 0, 131, 137, 0, 149, 0
\]

\[
B_{21} = 7, 13, 19, 0, 31, 37, 43, 0, 0, 61, 67, 73, 79, 0, 0, 97, 103, 109, 0, 0, 127
\]

\[
\{28\} \setminus \lambda \mathcal{P} = 0, 28, 0, 0, 28, 0, 0, 28, 0, 0, 28, 0, 0, 28, 0, 0, 28, 0, 0
\]

The symbol \( 28^- \) indicates that number 28 is presented here as the difference of two odd numbers. Each nonzero term in segments \( \{28^-\} \setminus \lambda \mathcal{P} \) to indicate one of representation of number \( g_5^1 = 28 \) as a difference of two primes. Thus, \( \pi_{28}(127) = \#(28^- \setminus \lambda \mathcal{P}) = 9 \).

Substitution \( m' = 5 \) and \( m = 126/6 = 21 \) in constructions (6) and (7) give us the number of pairs of primes with differs equal to 30 and 32 properly.

2.8.2. Twin primes. An important special case of pairs of primes with a fixed difference are twin primes. For the twin primes we have the identity \( b_i - a_i = 2 \), and the construction

\[
\pi_2(n) = \# (B_m - A_m) = \# (R_m - L_m)
\]

for the number of pairs of twin primes \( \pi_2(n) \) not exceeding \( n = 6m \). Let \( B - A = T \). Since \( A = A \setminus \lambda \mathcal{P} \), \( B = B \setminus \lambda \mathcal{P} \) it follows

\[
B = 7, 13, 19, 0, 31, 37, 43, 0, 0, 61, 67, 73, 79, 0, 0, 97, 103, 109, 0, 0, 127, \ldots
\]

\[
A = 5, 11, 17, 23, 29, 0, 41, 47, 53, 59, 0, 71, 0, 83, 89, 0, 101, 107, 113, 0, 0, \ldots
\]

\[
T \setminus \lambda \mathcal{P} = 2, 2, 0, 0, 2, 0, 2, 0, 2, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots
\]

where each term equal to 2 in sequence \( T \setminus \lambda \mathcal{P} \) to indicate one of a twin primes pair.

Now we denote very important sequence \( T = \{t_i: t_i = i \text{ if } l_{ri} \neq 0; t_i = 0 \text{ if } l_{ri} = 0\} \):

\[
R = 1, 2, 3, 0, 5, 6, 7, 0, 10, 11, 12, 13, 0, 0, 16, 17, 18, 0, 0, 21, 0, 23, 0, 25 \ldots
\]

\[
L = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, 12, 0, 14, 15, 0, 17, 18, 19, 0, 0, 22, 23, 0, 25 \ldots
\]

\[
T = 1, 2, 3, 0, 5, 0, 7, 0, 10, 0, 12, 0, 0, 0, 0, 17, 18, 0, 0, 0, 23, 0, 25 \ldots
\]

Here, if \( t_i \neq 0 \) then \( 6t_i \mp 1 \) are both primes, i.e. form a pair of twin primes. It is easy to see that the sequence \( T \) inherits the structure of the sequence \( T \setminus \lambda \mathcal{P} \) in the sense of distribution of zero terms.
2.8.3. Representation of an even number as the sum of two primes. Let \( \pi^+(g) \) be the number of all representations of an even number \( g \) as the sum of two primes. All even numbers of each class may also be represented as a sum of two odd integers from \( A \) and/or \( B \) in one and only one way, \( g_m^1 = a_i + a_{m-i+1,} \) \( g_m^2 = a_i + b_{m-i+1,} \) and \( g_m^3 = b_j + b_{m-j+1} \). These identities allow us to find the solution of this problem for all even numbers \( g \geq 10 \) from the constructions

\[
\begin{align*}
\pi^+(g^1_m) & > 0.5 \cdot \#(A_{m-1} + A'_{m-1}) = 0.5 \cdot \#(L_{m-1} + L'_{m-1}) , \\
\pi^+(g^2_m) & = \#(A_{m-1} + B'_{m-1}) = \#(L_{m-1} + R'_{m-1}) , \\
\pi^+(g^3_m) & > 0.5 \cdot \#(B_{m-1} + B'_{m-1}) = 0.5 \cdot \#(R_{m-1} + R'_{m-1}) .
\end{align*}
\]

Remark 4. We take the coefficient 0.5 in (8) and (10) because the sum of the direct and inverse segments of the same classes of primes is a symmetrical segment where the terms symmetrical with respect to the center differ only by the order of summation. The sum of the segments of different classes of primes gives us representations that are all distinct.

We find, for example, the number of representation of even number 94 as the sum of two primes. We have \( g_{16} = 94 \in G^1, m = 16 \). Apply construction (8) we obtain

\[
\begin{align*}
A_{15} & = 5, 11, 17, 23, 29, 0, 41, 47, 53, 59, 0, 71, 0, 83, 89 . \\
A'_{15} & = 89, 83, 0, 71, 0, 59, 53, 47, 41, 0, 29, 23, 17, 11, 5 .
\end{align*}
\]

\[
\{94^+\} \setminus \lambda P = 94, 94, 0, 94, 0, 0, 94, 94, 0, 0, 94, 0, 94, 94 ,
\]

Here 0.5 \( \#(\{94^+\} \setminus \lambda P) = 4.5 \) while \( \pi^+(94) = 5 \). This result can be obtained by using last expression of (8):

\[
\begin{align*}
L_{15} & = 1, 2, 3, 4, 5, 0, 7, 8, 9, 10, 0, 12, 0, 14, 15 . \\
L'_{15} & = 15, 14, 0, 12, 0, 10, 9, 8, 7, 0, 5, 4, 3, 2, 1 .
\end{align*}
\]

Constructions (9) and (10) with \( m = 16 \) will give the numbers of representations as the sum of two primes the even numbers 96 and 98 properly.

Remark 5. If each of three consecutive even number \( g^1_m, g^2_m, \) and \( g^3_m \) may be represented as the sum of two primes, it is clear that number \( m \) may be represented in three ways: \( l + l, l + r \), and \( r + r \).

2.9. Double sieve. In the section 2.12.2.3 we showed that each of sequences \( A \) and \( B \) are well-structured by all primes \( p \in \mathcal{P} \). Thus, for all \( p \in \mathcal{P} \), in each of the sequences \( A, B, \mathcal{L}, \) and \( \mathcal{R} \) there exists one and only one infinite subsequence of zero terms, indices of which to form an arithmetic progression with the common difference equal to \( p \). For every \( p \in \mathcal{P} \) we have

\[
\begin{align*}
\#(A_m \setminus \lambda p) = |\{a_i : i \leq m, (p, a_i) = 1\}| \sim m (1 - 1/p) , \\
\#(B_m \setminus \lambda p) = |\{b_i : i \leq m, (p, b_i) = 1\}| \sim m (1 - 1/p) .
\end{align*}
\]

Definition 6. We say that sequence \( S \) is double sifted by prime \( p \) if there exists two various disjoint subsequences of zero terms and the indices of terms each of these subsequences to form an arithmetic progression with the common difference equal to \( p \).

Definition 7. We say that sequence \( S \) is a realization double sieve if sequence \( S \) is double sifted by all \( p \in \mathcal{P} \); We say that sequence \( S \) is a realization double sieve if sequence \( S \) is double sifted by all but finitely many \( p \in \mathcal{P} \).

Theorem 8. Sequences \( \mathcal{T \setminus \lambda P} \) and \( \mathcal{T} \) are double sieve realizations.
Proof. Sequences $T \setminus \lambda P$ and $T$ have the same structure in the sense of distribution of zero terms. We consider structure of the sequence $T \setminus \lambda P$:

$$
\begin{align*}
B &= 7, 13, 19, 0, 31, 37, 43, 0, 0, 61, 67, 73, 79, 0, 0, 97, 103, 109, 0, 0, 127, \ldots \\
A &= 5, 11, 17, 23, 29, 0, 41, 47, 53, 59, 0, 71, 0, 83, 89, 0, 101, 107, 113, 0, 0, 127, \ldots \\
T \setminus \lambda P &= 2, 2, 2, 0, 2, 0, 2, 0, 0, 2, 0, 2, 0, 0, 0, 0, 2, 2, 0, 0, 0, \ldots
\end{align*}
$$

Terms of both the sequences $A$ and $B$ which designate as 0 mapping on $T \setminus \lambda P$ as 0 too. Therefore the indices of zero terms of sequence $T \setminus \lambda P$ will be determined by all expressions (1), (2), (3), and (4). Combine (1) with (3) and (2) with (4), we get two families of arithmetic progressions, $\lambda \in \mathbb{Z}^+$,

$$
\begin{align*}
i &= \lambda a_j \pm j, \\
i &= \lambda b_k \pm k.
\end{align*}
$$

Equation (11) asserts that in sequence $T \setminus \lambda P$ for each $p \in A$ there are two disjoint subsequences of zero terms the indices of them forms arithmetic progressions with the common difference equal to $p$. Equation (12) asserts that in sequence $T \setminus \lambda P$ for each $p \in B$ there are two disjoint subsequences of zero terms the indices of them forms arithmetic progressions with the common difference equal to $p$. The proof follows from the statement $A \cup B \supset P$. □

Thus, the sum of the sequences $A$ and $B$ and the sum of the sequences $L$ and $R$ are double sieve realizations. For all $p \in P$ we have

$$
\#(T_m \setminus \lambda p) \approx m \left(1 - 2/p\right).
$$

2.9.1. General case of double sieve. In a general case, that is, when we summarize sequences derived from $A$ and $B$ or $L$ and $R$, we can get a sequence where two subsequences of zero terms which are arithmetic progressions with the same common difference may be congruent and degenerate to one subsequence.

All the binary constructions that were considered in 2.8.1, 2.8.2 and 2.8.3 have a general view $S'_m \pm S''_m = S_m = \{g\}_{i=1}^{m}$ and are generated by the identity $s'_i \pm s''_i = s_i = g$. In all cases, if $p \mid g$, then $p \mid s'_i$ if and only if $p \mid s''_i$. Let $p, q \in P$ and let $p \mid g$ and $q \nmid g$; then

$$
\begin{align*}
\#(S_m \setminus \lambda p) &\approx m \left(1 - 1/p\right), \\
\#(S_m \setminus \lambda q) &\approx m \left(1 - 2/q\right).
\end{align*}
$$

We give two examples to illustrate these assertions. Let us $\{28^-\} = A^5 - B$. The difference between the proper terms of these sequences equal to $g = 28$ have the only one prime factor $p = 7$.

If we have sieved this construction by a prime $p = 7$, $7 \mid 28$, we get

$$
\begin{align*}
A^5 \setminus \lambda 7 &= 0, 41, 47, 53, 59, 65, 71, 0, 83, 89, 95, 101, 107, 113, 0, 125, 131, 137, 143, \ldots \\
B \setminus \lambda 7 &= 7, 13, 19, 25, 31, 37, 43, 0, 55, 61, 67, 73, 79, 85, 0, 97, 103, 109, 115, \ldots \\
\{28^-\} \setminus \lambda 7 &= 0, 28, 28, 28, 28, 28, 0, 28, 28, 28, 28, 28, 0, 28, 28, 28, 28, 28, \ldots
\end{align*}
$$

where in compliance with (13) from sequence $\{28^-\}$ sieved out one from each of seven terms. If we have sieved this construction by a prime $p = 5$, $5 \nmid 28$, we get

$$
\begin{align*}
A^5 \setminus \lambda 5 &= 0, 41, 47, 53, 59, 0, 71, 77, 83, 89, 0, 101, 107, 113, 0, 131, 137, 143, \ldots \\
B \setminus \lambda 5 &= 7, 13, 19, 0, 31, 37, 43, 49, 0, 61, 67, 73, 79, 0, 91, 97, 103, 109, 0, \ldots \\
\{28^-\} \setminus \lambda 5 &= 0, 28, 28, 0, 28, 28, 0, 28, 28, 0, 28, 28, 0, 28, 28, 0, 28, 28, \ldots
\end{align*}
$$

where in compliance with (14) from sequence $\{28^-\}$ sieved out two from each of five terms. Such a result we get for any $p \nmid 28$. 


Suppose there exists an estimate
\[ \pi_g(n) \sim M \prod_{5 \leq p \leq n} \left( 1 - \frac{2}{p} \right). \]

Than, if \( d \mid g, d \in \mathcal{P} \), and \( 3 \nmid g \), that \( \pi_g(n)/\pi_2(n) \sim (d - 1)/(d - 2) \). In common case we get
\[ \eta_2(g) = \frac{\pi_g(n)}{\pi_2(n)} = \kappa \prod_{d \mid g, d \in \mathcal{P}} \left( \frac{d - 1}{d} : \frac{d - 2}{d} \right) = \kappa \prod_{d \mid g, d \in \mathcal{P}} \frac{d - 1}{d - 2}. \]

Here, in compliance with the remark 3 in section 2.8.1 (p.4), \( \kappa = 1 \) if \( 3 \nmid g \) and \( \kappa = 2 \) if \( 3 \mid g \). Also, there exists the same relation for the number of representations of an even number \( g \) as the sum of two primes, where in compliance with the remark in section 2.8.3 (p.5), \( \kappa = 1 \) if \( 3 \mid g \) and \( \kappa = 0 \) if \( 3 \nmid g \). We will show in section 3.2 that the number of representations of an even number \( g \) as the sum of two primes can be presented by \( \pi_2(g) \).

3. Additive problems II

3.1. Twin prime conjecture.

**Proposition 9.** There exists the function \( H_m \) such that \( \pi_2(6m) > m H_m \) for all sufficiently large \( m \) and \( m H_m \to \infty \) as \( m \to \infty \).

By the asymptotic Mertens’ formula (See e.g. Ingham [2], pp. 22-24.)
\[ \prod_{p \leq n} \left( 1 - \frac{1}{p} \right) \sim \frac{e^{-\gamma}}{\log n} \]
and the inequality \( n/\log n < \pi(n) \) that holds for all \( n \geq 17 \) (Rosser, Schoenfeld, [3]), we have for all sufficiently large \( m = n/6 \)
\[ 2me^\gamma \prod_{5 \leq p \leq 6m} \left( 1 - \frac{1}{p} \right) \approx \frac{6m}{\log 6m} < \pi(a, 6m) + \pi(b, 6m) \]
and then apply Dirichlet’s theorem on primes in arithmetic progressions to get
\[ me^\gamma \prod_{5 \leq p \leq 6m} \left( 1 - \frac{1}{p} \right) < \pi(a, 6m) \approx \pi(b, 6m). \]

Relation between single sieving \( \prod_{5 \leq p \leq 6m} (1 - 1/p) \) and double sieving \( \prod_{5 \leq p \leq 6m} (1 - 2/p) \) determine by asymptotic equality
\[ \prod_{5 \leq p \leq 6m} \left( 1 - \frac{2}{p} \right) \sim C_{1.2} \prod_{5 \leq p \leq 6m} \left( 1 - \frac{1}{p} \right)^2, \]
where

\[ C_{1;2} = \prod_{5 \leq p \leq 6m} \frac{p(p - 2)}{(p - 1)^2} = \frac{4}{3} \prod_{3 \leq p \leq 6m} \frac{p(p - 2)}{(p - 1)^2}. \]

Finally, using this asymptotic equality we obtain

\[ (16) \quad H_m = e^{2\gamma} \prod_{5 \leq p \leq 6m} \left( 1 - \frac{2}{p} \right) \sim C_{1;2} e^{2\gamma} \prod_{5 \leq p \leq 6m} \left( 1 - \frac{1}{p} \right)^2. \]

The inequality \( \pi_2(6m) > mH_m \) holds for all \( m > 5 \), although this approximation is not sufficient. For example, for \( n = 10^6 \), we have \( \pi_2(6m) - mH_m > 1251 \), while \( \pi_2(10^6) = 8168 \).

3.1.1. The first Hardy-Littlewood conjecture. We may get the approximation of the number of twin primes through the number of primes in the following way: assume that there are functions \( \varphi_{a,m} \) and \( \varphi_{b,m} \) such that

\[ m\varphi_{a,m} = \pi(a, 6m), \quad m\varphi_{b,m} = \pi(b, 6m). \]

Then, the right-hand side of (16) can be written as \( C_{1;2} \varphi_{a,m} \varphi_{b,m} \) and we get

\[ (18) \quad \pi'_2(6m) = C_{1;2} \frac{\pi(a, 6m)}{m} \cdot \frac{\pi(b, 6m)}{m}. \]

This is a better approximation of \( \pi'_2(6m) \) than \( mH_m \). Thus, \( \pi_2(10^6) - \pi'_2(10^6) = 32.5356\ldots \)

When we substitute \( m = n/6 \), \( \pi(a, 6m) \cdot \pi(b, 6m) \sim \pi^2(n)/4 \), and \( C_{1;2} = 4C_2/3 \) where \( C_2 \) is the twin prime constant in to (18) we obtain

\[ (19) \quad \pi'_2(6m) = 2nC_2 \left[ \frac{\pi(n)}{n} \right]^2 \sim 2C_2 \frac{n}{(\log n)^2} \sim 2C_2 \int_2^n \frac{dt}{(\log t)^2}, \]

and therefore, (18) is equivalent to the first Hardy-Littlewood conjecture (see [4] for example). The last term here expresses \( \pi_2(n) \) by the density of primes. Thus, for the number of pairs of prime numbers not exceeding \( n + g \) with difference is equal to \( g \), we have the asymptotic formula

\[ \frac{\pi_g(n)}{\eta_2(g)} \sim 2C_2 \int_2^n \frac{dt}{\log t \log(t + g)}. \]

The density of prime numbers in one of summand segments here decreases from \( 1/\log 2 \) to \( 1/\log n \) and the other with decrease from \( 1/\log(2 + g) \) to \( 1/\log(n + g) \).

3.2. The Goldbach-Euler conjecture.

Proposition 10. There exists the function \( H'_m \) such that for all sufficiently large \( m \) numbers of representations as a sum of two primes apiece of the three successive even numbers, \( g^1_m, g^2_m, g^3_m \) are not less than \( mH'_m \), and \( mH'_m \to \infty \) as \( m \to \infty \).

The last term in (19) expresses \( \pi_g(n) \) by the density of prime numbers. The density of prime numbers in both summand segments here decreases from \( 1/\log 2 \) to \( 1/\log n \). The constructions for \( \pi^+(g) \) contain two segments of primes, one with decreasing density of primes from \( 1/\log 2 \) to \( 1/\log n \) and the other with increasing density of primes from \( 1/\log n \) to \( 1/\log 2 \). In this way, by analogy with (19), to estimate the number of representations of an even number \( g \) as the sum of two primes, we have

\[ \frac{\pi^+(g)}{\eta_2(g)} \sim 2C_2 \int_2^n \frac{dt}{\log t \log(n - t)}. \]
In general case, we have \( \Pi_m \) the quotient patterns of four primes of any sort.

For \( n \) sufficiently large \( g \) we may suppose that \( n = g \) and make a suitable choice of \( \kappa \) in \( \eta_2(g) \). The quotient

\[
\mu_2(n) = \frac{\sum_2^{n-2} \frac{1}{\log t \log(n-t)}}{\sum_2^{n-2} \frac{1}{(\log t)^2}} \approx \frac{\int_2^{n-2} \frac{dt}{\log t \log(n-t)}}{\int_2^{n-2} \frac{dt}{(\log t)^2}},
\]

has a minimum value 0.706... at \( n = 32 \) and attains value 0.972... at \( n = 10^5 \). Thus, function \( H'_m = \mu_2(6m)H_m \) satisfy the Goldbach-Euler conjecture. Finally, we have \( \pi^+(g) \approx \pi_2(g)\mu_2(g)\eta_2(g) \).

3.3. Patterns with two pairs of twin primes. Let \( \Pi_{m'}(n) \) be the number of twins \( p, p + 2 \) not exceeding \( n \) such that \( p' = p + 6m', p' + 2 \) are also twins. We have the construction for the solution of this problem,

\[
\Pi_{m'}(6m) = \# \{T_{m'} + T_m \}.
\]

When \( m' = 1 \), we obtain the number of prime quadruplets. We will then show Proposition 11.

**Proposition 11.** There exists the function \( Q_m \) such that the inequality \( \Pi_1(6m) > mQ_m \) holds for all sufficiently large \( m \), and \( mQ_m \rightarrow \infty \) as \( m \rightarrow \infty \).

As we mentioned in Section 2.9.1, we must sum \( k = 4 \) segments, each of which contains composite numbers, distributed in certain arithmetic progressions. If \( m' = 1 \), then \( a_i, b_i, a_i+m' \) and \( b_i+m' \) are all relatively prime numbers. In this case,

\[
|\{i: i \leq m, (p, a_i \cdot b_i \cdot a_i+1 : b_i+1) = 1\}| \approx m \left(1 - \frac{4}{p}\right)
\]

for all \( p \). Having used a method analogous to the twofold sieve on twin primes in section 3.1 and applied a fourfold sieve, we have

\[
mQ_m = m e^{4\gamma} \prod_{5 \leq p \leq 6m} \left(1 - \frac{4}{p}\right) < \Pi_1(6m).
\]

This inequality holds for all \( m > 1 \) and \( \Pi_1(6m) - mQ_m = 52.07... \) when \( 6m = 10^6 \). A better approximation can be obtained by using the technique in section 3.1. Then, we have

\[
\Pi_1(6m) \sim C_{1:4} m \left[ \frac{\pi(a, 6m)}{m}, \frac{\pi(b, 6m)}{m} \right]^2
\]

\[
\sim C_{2:4} m \left[ \frac{\pi_2(6m)}{m} \right]^2,
\]

where

\[
C_{1:4} = \prod_{5 \leq p \leq 6m} \left[ \frac{p - 4}{p} : \frac{(p - 1)^4}{p^4} \right] = \prod_{5 \leq p \leq 6m} \frac{(p - 4)p^2}{(p - 1)^4},
\]

\[
C_{2:4} = \prod_{5 \leq p \leq 6m} \left[ \frac{p - 4}{p} : \frac{(p - 2)^2}{p^2} \right] = \prod_{5 \leq p \leq 6m} \frac{(p - 4)p}{(p - 2)^2}.
\]

For \( n = 10^6 \), we have the error 8.3904... in formula (21) and the error 7.1272... in formula (22). In general case, we have \( \Pi_{m'}(6m) = \eta_4(m')\Pi_1(6m) \).

Note. As a corollary of Proposition 11, we obtain the assertion that there are infinitely many patterns of four primes of any sort.
3.4. The strengthened Goldbach-Euler conjecture. Because the Goldbach–Euler conjecture and the twin prime conjecture hold true, we must answer the following question: how many even numbers are the sum of two primes, each of which has a twin? It is clear that if one of three consecutive even numbers $g_m^1, g_m^2, g_m^3$ has such a representation, then the other two also have similar representations. The sum of two pairs of twin primes gives us two representations for $g_m^2$ and one representation each for $g_m^1$ and $g_m^3$. We can think of this set of representations as a single representation. The construction for this statement in entry notation will be $T_m + T'_m = G_{m+1}^T$. Thus, $T_m = L_m + R_m$, and then formally,

$$\#G_{m+1}^T = \#(T_m + T'_m) = \#\left((R_m - L_m) + (R'_m - L'_m)\right).$$

We checked 30,000 triples of even numbers and found that only 12 of them lack such representations. We therefore claim Proposition 12.

**Proposition 12.** (The strengthened Goldbach-Euler conjecture). All even numbers except $g_m^1, g_m^2, g_m^3$ for $m = 1, 16, 67, 86, 131, 151, 186, 191, 211, 226, 541, 701$ are the sum of two primes, each of which has a twin. In addition, there exists the function $Q'_m$ such that the inequality $G_{m+1}^T > mQ'_m$ holds for all sufficiently large $m$, and $mQ'_m \to \infty$ as $m \to \infty$.

Now we proceed as in section 3.2. In analogy to (20), we may now expect that

$$\#G_{m+1}^T \sim m \cdot \eta_4(m) \cdot \mu_4(m) \cdot C_{1.4} \left[\frac{\pi(6m)}{m}\right]^4 \sim \eta_4(m) \cdot \mu_4(m) \cdot C_{1.4} \int_2^m \frac{dt}{(\log t)^2 (\log (n-t))^2}.$$  

A quotient

$$\mu_4(n) = \frac{\sum_{2}^{n-2} \frac{1}{(\log t)^2 (\log (n-t))^2}}{\sum_{2}^{n-2} \frac{1}{(\log t)^2}} \approx \frac{\int_2^{n-2} \frac{dt}{(\log t)^2 (\log (n-t))^2}}{\int_2^{n-2} \frac{dt}{(\log t)^2}}$$

has its minimum 0.136278... at $n = 227$ and attains the value 0.57533... at $n = 1.2 \cdot 10^5$. Thus, we obtain the relation $Q'_m = \mu_4(6m)Q_m$. The inequality $Q'_m > 1$ holds for all $m \geq 947 \geq 701$, and therefore, every even number greater than $4208 = 6 \times 701 + 2$ may be represented as the sum of two primes, each of which has a twin.

Proposition 12 can be extended to a set of prime pairs with any difference $d$ if $3 \nmid d$.

To complete the proof of all propositions, we need to demonstrate Theorem 13.

**Theorem 13.** If $m \to \infty$, then

$$m \prod_{5 \leq p \leq 6m} \left(1 - \frac{4}{p}\right) \to \infty.$$  

**Proof.** We can apply the asymptotic relation (Rosser, Schoenfeld, [3])

$$\prod_{\alpha \leq p \leq x} \left(1 - \frac{\alpha}{p}\right) \approx \frac{c(\alpha)}{(\log x)^\alpha}$$

to obtain

$$I_m = m \prod_{5 \leq p \leq 6m} \left(1 - \frac{4}{p}\right) \approx \frac{c(4) \cdot m}{(\log 6m)^4}.$$
The proof follows from the well-known relation \( \lim_{x \to +\infty} x^{-n} \log x = 0 \), which holds for every \( n > 0 \). (See e.g. Hardy \([I]\), p. 381.)

3.5. **Density of certain sequences.** A natural density (or asymptotic density) \( \delta(S) \) and the Schnirelman density \( d(S) \) of the sequence \( S_m \) are defined as

\[
\delta(S) = \lim_{m \to \infty} \frac{|S(m)|}{m}, \quad d(S) = \liminf_{m \to \infty} \frac{|S(m)|}{m}.
\]

Obviously, \( \delta(P) = d(P) = 0 \), \( \delta(L) = \delta(R) = \delta(T) = 0 \) and \( d(L) = d(R) = d(T) = 0 \). It is easy to see that if the Goldbach-Euler conjecture is true, then \( \delta(P + P) > 0.5 \). The method we present here for the proof of the Goldbach-Euler conjecture allows us to assert that

\[
d(L + L) = d(L + R) = d(R + R) = 1,
\]

that is, there exist sequences with density 0 that are bases of order 2. From the strengthened Goldbach-Euler conjecture, it follows that \( \delta(T + T) = 1 \), that is, sequence \( T \) is an asymptotic basis of order 2.

The sequence of primes with 1 is the basis of order 3. The set of twin prime pairs is the asymptotic basis of order 3.

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