COLLAPSING AND ESSENTIAL COVERINGS

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Abstract. In the present paper, we consider the family of all compact Alexandrov spaces with curvature bound below having a definite upper diameter bound of a fixed dimension. We introduce the notion of essential coverings by contractible metric balls, and provide a uniform bound on the numbers of contractible metric balls forming essential coverings of the spaces in the family. In particular, this gives another view for Gromov’s Betti number theorem.

1. Introduction

It is an important problem to find the relation between curvature and topology of Riemannian manifolds. In the study of the finiteness of compact Riemannian manifolds with uniformly bounded curvature, originated by Cheeger [3] and Weinstein [16], it was a crucial idea to cover a manifold in a certain class by convex metric balls whose number is uniformly bounded depending only on the class. The minimal number of such metric balls covering a manifold represents the complexity of the manifold. For a certain class of compact Riemannian manifolds with a lower curvature bound, Grove, Petersen and Wu [9] used contractible balls in place of convex balls to get a topological finiteness of the manifolds in the class. Those are results in the non-collapsing cases.

In the present paper, we consider the collapsing case for compact Alexandrov spaces $M$ with curvature bounded below. The Perelman stability theorem [14] shows that a small metric ball around a given point of $M$ is homeomorphic to the tangent cone, and hence contractible.

If $M$ is collapsed, the sizes of contractible metric balls must be very small, and therefore the minimal number of contractible metric balls covering $M$ becomes large. In other words, it is not efficient to cover the whole $M$ by contractible metric balls. In the present paper, to overcome this difficulty, we introduce the notion of an essential covering of $M$ in place of a usual covering.

To illustrate the notion of essential covering, let us take a flat torus $T^2_0 = S^1(1) \times S^1(\epsilon)$ for a small $\epsilon > 0$, where $S^1(\epsilon)$ is the circle of length...
The torus $T^2_\epsilon$ can be covered by two thin metric balls $B_\alpha$, $\alpha \in \{1, 2\}$. Each ball $B_\alpha$ is isotopic to a much smaller concentric metric ball $\hat{B}_\alpha$ of radius, say $2\epsilon$. If one tries to cover $B_\alpha$ by contractible metric balls, we need too many, about $[1/\epsilon]$-pieces of such balls. In stead, we take a covering of $\hat{B}_\alpha$. It is possible to cover $\hat{B}_\alpha$ by two contractible metric balls $\{\hat{B}_{\alpha\beta}\}_{\beta=1}^2$. Thus we have a collection of four contractible metric balls $\{\hat{B}_{\alpha\beta}\}$, which is an essential covering of $T^2_\epsilon$. Although it is not a usual covering of $T^2_\epsilon$, deforming and enlarging $\hat{B}_{\alpha\beta}$ by isotopies, we obtain a covering $\{\hat{B}_{\alpha\beta}\}$ of $T^2_\epsilon$ by contractible open subsets $\hat{B}_{\alpha\beta}$. In that sense, the essential covering seems to contain an essential property of $T^2_\epsilon$.

In the general collapsing case with a lower curvature bound, it is believed that a collapsed space has a certain fiber structure in a generalized sense such that the fibers shrink to points (see [15], [17]). Although it is not established yet, a fiber which is not visible yet may shrink to a point with different scales in different directions, in general. This suggests that we have to repeat the above process of taking a smaller concentric metric ball and of covering it by much smaller metric balls at most $n$-times, $n = \dim M$, to finally reach contractible metric balls (see Examples 4.2 and 4.3). In this way, we come to the notion of an essential covering of $M$ with depth $\leq n$.

As illustrated above, an essential covering is not a usual covering of $M$, but it contains an essential feature on the complexity of the space $M$. Actually by deforming and enlarging the balls in the essential covering by isotopies of $M$ in a systematic way, we obtain a real open covering of $M$.

We define the geometric invariant $\tau_n(M)$ as the minimal number of contractible metric balls forming essential coverings of $M$ with depth $\leq n$. See Section 4 for a more refined formulation of $\tau_n(M)$. In particular, if $M$ is a Riemannian manifold, we can replace contractible metric balls by metric balls homeomorphic to an $n$-disk in the definition of $\tau_n(M)$.

For a positive integer $n$ and $D > 0$, we denote by $A(n, D)$ the isometry classes of $n$-dimensional compact Alexandrov spaces with curvature $\geq -1$ and diameter $\leq D$. In this paper, we shall prove

**Theorem 1.1.** For given $n$ and $D$, there is a positive integer $N(n, D)$ such that $\tau_n(M) \leq N(n, D)$ for all $M$ in $A(n, D)$.

This gives a new geometric restriction on the spaces in $A(n, D)$ even in the case of Riemannian manifolds. A more detailed information on the essential covering of $M$ is given in Theorem 4.4.

The minimal number $\text{cell}(M)$ of open cells needed to cover a compact manifold $M$ is an interesting topological invariant. It is known however that $\text{cell}(M) \leq \dim M + 1$ (see [4]). This suggests that to have better understanding of the complexity of a compact Riemannian manifold or a compact Alexandrov space concerning the minimal number of some
basic subsets needed to cover it, we have to consider a metric invariant rather than a topological invariant. This is the reason why we mainly consider coverings by metric balls.

Working with concentric coverings, Theorem 1.1 yields the following uniform bound on the total Betti number:

**Corollary 1.2** ([5], [13]). For given $n$ and $D$, there is a positive integer $C(n, D)$ such that if $M$ is in $A(n, D)$, then

$$\sum_{i=0}^{n} b_i(M; F) \leq C(n, D),$$

where $F$ is any field.

In the original work [5], Gromov developed the critical point theory for distance functions to obtain an explicit bound on the total Betti numbers for Riemannian manifolds. The argument in [13] is a natural extension of that in [5] to Alexandrov spaces. Unfortunately our bound is not explicit. However our approach provides a conceptually clear view of what the essence of Corollary 1.2 is like.

For the proof of Theorem 1.1 we use the convergence and collapsing methods. If a space $M$ in $A(n, D)$ does not collapse, the stability theorem immediately yields the consequence. If $M$ collapses to a lower dimensional space, we use the rescaling method, which was used in [15] and [17] in some special cases. We first generalize those results to the general case. Using this rescaling method, we can grasp the proper size of a collapsed “fiber” although it is not visible. This enables us to have a covering $\{B_{\alpha_1}\}$ of $M$ such that each ball $B_{\alpha_1}$ is, under some rescaling of metric with the fiber size, close to a complete noncompact Alexandrov space $Y_1$ of nonnegative curvature with $\dim Y_1 > \dim X$ for the pointed Gromov-Hausdorff topology. If the “fiber” uniformly shrinks to a point, the new convergence $B_{\alpha_1} \rightarrow Y_1$ does not collapse. In the other collapsing case of $\dim Y_1 < n$, we again grasp the size of a new “fiber” in the collapsing $B_{\alpha_1} \rightarrow Y_1$ with the help of the rescaling method. From this, we see that a much smaller concentric subball $\hat{B}_{\alpha_1}$ of $B_{\alpha_1}$, which is isotopic to $B_{\alpha_1}$, can be covered by small metric balls $\{B_{\alpha_1\alpha_2}\}_{\alpha_2}$ whose number is uniformly bounded such that each $B_{\alpha_1\alpha_2}$ is, under some rescaling with the size of a new “fiber”, close to a complete noncompact Alexandrov space $Y_2$ of nonnegative curvature with $\dim Y_2 > \dim Y_1$ for the pointed Gromov-Hausdorff topology. Repeating this process at most $n - \dim X$ times and using the stability theorem, we finally get an essential covering of $M$ by contractible metric balls, as required.

Corollary 1.2 follows almost directly from Theorem 1.1 and the topological lemma of [5]. Actually we formulate and prove a more general result for every subset of an $n$-dimensional complete Alexandrov space in terms of $\delta$-content (see Theorem 5.2).
Acknowledgment. I would like to thank Vitali Kapovitch for bringing the paper [12] to my attention.

2. Preliminaries

We refer to Burago, Gromov and Perelman [2] for the basic materials on Alexandrov spaces with curvature bounded below.

Let $M$ be an Alexandrov space with curvature bounded below, say $\geq \kappa$. For two points $x$ and $y$ in $M$, a minimal geodesic joining $x$ to $y$ is denoted by $xy$ for simplicity. For any geodesic triangle $\Delta xyz$ in $M$ with vertices $x, y$ and $z$, we denote by $\tilde{\Delta xyz}$ a comparison triangle in the $\kappa$-plane $M^2_\kappa$, the simply connected complete surface with constant curvature $\kappa$. The angle between the geodesics $xy$ and $yz$ in $M$ is denoted by $\angle xyz$, and the corresponding angle of $\tilde{\Delta xyz}$ by $\tilde{\angle xyz}$. It holds that $\angle xyz \geq \tilde{\angle xyz}$.

Let $\Sigma_p = \Sigma_p(M)$ denote the space of directions at $p \in M$. Let $K_p = K_p(M)$ be the tangent cone at $p$ with vertex $o_p$, the Euclidean cone over $\Sigma_p$. For a closed set $A \subset M$ and $p \in M - A$, we denote by $A' = A'_p$ the subset of $\Sigma_p$ consisting of all the directions of minimal geodesics from $p$ to $A$.

From now on, we assume that $M$ is finite-dimensional. It is known that $\Sigma_p$ (resp. $K_p$) is a $(n-1)$-dimensional compact (resp. $n$-dimensional complete noncompact) Alexandrov space with curvature $\geq 1$ (resp. curvature $\geq 0$), where $n = \dim M$.

It is well-known that as $r \to 0$, $(\frac{1}{r}M, p)$ converges to $(K_p, o_p)$ with respect to the pointed Gromov-Hausdorff topology, where $\frac{1}{r}M$ denotes the rescaling of the original distance of $M$ multiplied by $\frac{1}{r}$.

We denote by $A_p(n)$ the isometry classes of $n$-dimensional complete pointed Alexandrov spaces $(M, p)$ with curvature $\geq -1$.

The following results play crucial roles in this paper.

Theorem 2.1 ([7], [8]). $A(n, D)$ (resp. $A_p(n)$) is relatively compact with respect to the Gromov-Hausdorff distance (resp. the pointed Gromov-Hausdorff topology).

Consider the distance function $d_p(x) = d(p, x)$ from a point $p \in M$. A point $q \neq p$ is a critical point of $d_p$ if $\hat{\angle}pqx \leq \pi/2$ for all $x \in M$.

For $0 < r < R$, $A(p; r, R)$ denotes the closed annulus $B(p, R) - B(p, r)$, where $B(p, r)$ is the open metric ball around $p$ of radius $r$.

Lemma 2.2 ([10], [5], [14]). If $d_p$ has no critical points on $A(p; r, R)$, then $A(p; r, R)$ is homeomorphic to $\partial B(p, r) \times [0, 1]$.

Theorem 2.3 ([14], cf. [11]). Let an infinite sequence $(M_i, p_i)$ in $A_p(n)$ converge to a space $(M, p)$ in $A_p(n)$ with respect to the pointed Gromov-Hausdorff topology. Take an $r > 0$ such that there are no critical points
of $d_p$ on $B(p, r) - \{ p \}$. Then $B(p_1, r)$ is homeomorphic to both $B(p, r)$ and $K_p$ for large $i$.

3. Rescaling metrics

Let a sequence $(M_i, p_i)$ in $A_p(n)$ converge to a pointed Alexandrov space $(X, p)$ with curvature $\geq -1$ with respect to the pointed Gromov-Hausdorff topology. It is a fundamental problem to find topological relation between $B(p_i, r)$ and $B(p, r)$ for a small but fixed positive number $r$ and large $i$.

In the case when $\dim X = n$, take $r > 0$ so that the distance function $d_p$ has no critical points on $B(p, r) - \{ p \}$. Then Theorem 2.3 shows that $B(p_i, r)$ is homeomorphic to $B(p, r)$ for large $i$.

In this section, from now on, we consider the collapsing case when $1 \leq \dim X \leq n - 1$. Since we are concerned with the topology of a neighborhood of $p_i$, we may assume

Assumption 3.1. $B(\tilde{p}_i, r)$ is not homeomorphic to an $n$-disk for any $\tilde{p}_i$ with $d(p_i, \tilde{p}_i) \to 0$ and for any sufficiently large $i$.

The following is a generalization of the Key lemma 3.6 in [15] and Theorem 4.1 in [17].

Theorem 3.2. Under Assumption 3.1, there exist $\hat{p}_i \in B(p_i, r)$ and a sequences $\delta_i \to 0$ such that

1. $d(\hat{p}_i, p_i) \to 0$;
2. $d_{\hat{p}_i}$ has no critical points on $A(\hat{p}_i; R\delta_i, r)$ for every $R \geq 1$ and large $i$ compared to $R$. In particular, $B(\hat{p}_i, r)$ is homeomorphic to $B(\hat{p}_i, R\delta_i)$;
3. for any limit $(Y, y_0)$ of $\left( \frac{1}{\delta_i} M_i, \hat{p}_i \right)$, we have $\dim Y \geq \dim X + 1$.

The essential idea of the proof of Theorem 3.2 is the same as in [15]. In [15] however, we had to suppose that the function $f : K_p \to \mathbb{R}$ constructed there takes a strict local maximum at the vertex $o_p$ of $K_p$.

Since this does not hold in general, we must modify the construction. Some simplification of the proof is also made here.

For positive numbers $\theta$ and $\epsilon$ with $\epsilon \ll \theta \leq \pi/100$, take a positive number $r = r(p, \theta, \epsilon)$ such that

1. $\angle xpy - \tilde{\angle} xpy < \epsilon$ for every $x, y \in \partial B(p, 2r)$;
2. $\{ y' \}_{y \in \partial B(p, 2r)}$ is $\epsilon$-dense in $\Sigma_p$.

Note that the above (2) implies that there are no critical points of $d_p$ on $B(p, r) - \{ p \}$. Let $\{ x_\alpha \}_\alpha$ be a $\theta r$-discrete maximal system in $\partial B(p, r)$. For a small positive number $\epsilon$, take an $\epsilon r$-discrete maximal system $\{ x_{\alpha \beta} \}_\beta$, $1 \leq \beta \leq N_\alpha$, in $B(x_\alpha, \theta r) \cap \partial B(p, r)$. Let $\xi_{\alpha \beta} \in \Sigma_p$ be the direction of geodesic $px_{\alpha \beta}$. Note that $\{ \xi_{\alpha \beta} \}_\beta$ is $\epsilon/2$-discrete. A
standard covering argument implies that
\begin{equation}
N_α ≥ \text{const} \left( \frac{θ}{ε} \right)^{\text{dim} X - 1}.
\end{equation}

We consider the following functions \( f_α \) and \( f \) on \( M \):
\[
f_α(x) = \frac{1}{N_α} \sum_{β=1}^{N_α} d(x_{αβ}, x), \quad f(x) = \min_α f_α(x).
\]
A similar construction was made in [12] to define a strictly concave function on a neighborhood of a given point of an Alexandrov space. The effectiveness of the use of those functions was suggested to the author by Vitali Kapovitch.

**Lemma 3.3.** For every \( x ∈ B(p, r/2) \), we have \( f(x) ≤ r - d(p, x)/2 \). In particular, the restriction of \( f \) to \( B(p, r/2) \) has a strict maximum at \( p \).

**Proof.** Take \( y ∈ \partial B(p, r) \) and \( x_α \) with \( \angle xpy < ε \) and \( d(y, x_α) < θr \). It follows that \( \angle xpy < 5θ \). Let \( γ : [0, d] → X \) be a minimal geodesic joining \( p \) to \( x \). By the curvature assumption with trigonometry, we see that \( \angle xγ(t)x_{αβ} < π/4 \). The first variation formula then implies that \( d(x_{αβ}, x) ≤ r - d(p, x)/2 \), and therefore \( f(x) ≤ f_α(x) ≤ r - d(p, x)/2 \). \( \square \)

**Proof of Theorem 3.2.** Take a \( μ_i \)-approximation \( \phi_i : B(p, 1/μ_i) → B(p, 1/μ_i) \), with \( φ_i(p) = p_i \), where \( μ_i → 0 \) as \( i → ∞ \). Let \( x_i^α := φ_i(x_{αβ}) \), and define the functions \( f^i_α \) and \( f^i \) on \( M_i \) by
\[
f^i_α(x) = \frac{1}{N_α} \sum_{β=1}^{N_α} d(x_{αβ}^i, x), \quad f^i(x) = \min_α f^i_α(x).
\]
Note that \( f^i_α \circ φ_i → f_α \) and \( f^i \circ φ_i → f \). By Lemma 3.3, there is a point \( p_i ∈ B(p_i, r/2) \) such that

1. \( (p_i, p_i) → 0 \);
2. the restriction of \( f^i \) to \( B(p_i, r/3) \) takes a maximum at \( p_i \).

Consider the distance function \( d_{p_i} \). By Assumption 3.1, there is a critical point of \( d_{p_i} \) in \( B(p_i, r) \). Let \( δ_i \) be the maximum distance between \( p_i \) and the critical point set of \( d_{p_i} \) within \( B(p_i, r) \). Note that \( δ_i → 0 \). Let \( q_i \) be a critical point of \( d_{p_i} \) within \( B(p_i, r) \) realizing \( δ_i \). We may assume that \( \left( \frac{1}{δ_i} M_i, p_i \right) \) converges to a complete noncompact pointed Alexandrov space \( (Y, y_0) \) with nonnegative curvature. Let \( z_0 ∈ Y \) be the limit of \( q_i \) under this convergence. We denote by \( d = \frac{1}{δ_i} d \) the distance of \( \frac{1}{δ_i} M_i \). Consider the function
\[
h_{αβ}^i(x) := d(x_{αβ}^i, x) - d(x_{αβ}, p_i).
\]
which is 1-Lipschitz, and bounded on every bounded set. Therefore passing to a subsequence, we may assume that \( h_{\alpha\beta} \) converges to a 1-Lipschitz function \( h_{\alpha\beta} \) on \( Y \).

Let

\[
h_\alpha = \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} h_{\alpha\beta}, \quad h = \min_{\alpha} h_\alpha,
\]

\[
h^i_\alpha = \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} h^i_{\alpha\beta}, \quad h^i = \min_{\alpha} h^i_\alpha.
\]

Since \( h^i = [f^i - f^i(\hat{p}_i)]/\delta_i \), \( h \) takes a maximum at \( y_0 \). Let \( x_{\alpha\beta}(\infty) \) denote the element of the ideal boundary \( Y(\infty) \) of \( Y \) defined by the limit ray, say \( \gamma_{\alpha\beta} \), from \( y_0 \) of the geodesic \( \hat{p}_i x_{\alpha\beta} \) under the convergence \((\frac{1}{\delta_i} M_i, \hat{p}_i) \to (Y, y_0)\). Let \( v_{\alpha\beta} \in \Sigma_{y_0} \) and \( v \in \Sigma_{y_0} \) denote the direction of \( \gamma_{\alpha\beta} \) and \( yz \) respectively. Since \( z_0 \) is a critical point of \( d_{y_0} \), we have \( \angle z_0 y_0 x_{\alpha\beta}(\infty) \leq \pi/2 \). Since \( Y \) has nonnegative curvature, it follows that \( \angle(v, v_{\alpha\beta}) \geq \angle z_0 y_0 x_{\alpha\beta}(\infty) \geq \pi/2 \), for every \( \alpha \) and \( \beta \). Choosing \( \alpha \) with \( h'_0(v) = (h_\alpha)'_{y_0}(v) \), we obtain

\[
0 \geq h'_0(v) = \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} -\cos \angle(v, v_{\alpha\beta}),
\]

and therefore \( \angle(v, v_{\alpha\beta}) = \pi/2 \). Since

\[
\angle(v_{\alpha\beta}, v_{\alpha\beta'}) = \lim_{t \to 0} \angle \gamma_{\alpha\beta}(t) y_0 \gamma_{\alpha\beta'}(t) \geq \angle x_{\alpha\beta} p x_{\alpha\beta'} \geq \epsilon/2,
\]

for every \( 1 \leq \beta \neq \beta' \leq N_\alpha \), \( \{v_{\alpha\beta}\}_{\beta=1}^{N_\alpha} \) is \( \epsilon/2 \)-discrete in \( \partial B(v, \pi/2) \subset \Sigma_{y_0} \). Since \( \Sigma_{y_0} \) has curvature \( \geq 1 \), there is an expanding map from \( \partial B(v, \pi/2) \) to the unit sphere \( S^{\text{dim} Y - 2}(1) \). It follows that

\[
(3.2) \quad N_\alpha \leq \text{const} \epsilon^{-(\text{dim} Y - 2)}.
\]

Since this holds for any sufficiently small \( \epsilon \), from (3.1) and (3.2) we can conclude \( \text{dim} Y \geq \text{dim} X + 1 \). This completes the proof of Theorem 3.2. \( \square \)

4. Isotopy Covering Systems and Essential Coverings

Let \( M \) be a compact \( n \)-dimensional Alexandrov space with curvature bounded below. For an open metric ball \( B \) of \( M \), we denote by \( \lambda B \) the concentric ball of radius \( \lambda r \). We call a concentric ball \( \hat{B} \subset B \) an isotopic subball of \( B \) if there is a homeomorphism \( M \to M \) sending \( \hat{B} \) onto \( B \).
and leaving the outside of a neighborhood of $\hat{B}$ fixed. For instance, this is the case when $d_\rho$ has no critical points on $\hat{B} - \hat{B}$ (Lemma 2.2).

Consider the following system $\mathcal{B} = \{B_{\alpha_1 \cdots \alpha_k}\}$ consisting of open metric balls $B_{\alpha_1 \cdots \alpha_k}$ of $M$, where the indices $\alpha_1, \ldots, \alpha_k$ range over

\[
1 \leq \alpha_1 \leq N_1, \quad 1 \leq \alpha_2 \leq N_2(\alpha_1), \\
1 \leq \alpha_k \leq N_k(\alpha_1 \cdots \alpha_{k-1}),
\]

and $1 \leq k \leq \ell$ for some $\ell$ depending on the choice of the indices $\alpha_1, \alpha_2, \ldots$. Note that the range of $\alpha_k$ also depends on $\alpha_1 \cdots \alpha_{k-1}$. Let $A$ be the set of all multi-indices $\alpha = \alpha_1 \cdots \alpha_k$ such that $B_{\alpha_1 \cdots \alpha_k} \in \mathcal{B}$. For each $\alpha = \alpha_1 \cdots \alpha_k \in A$, put $|\alpha| := k$ and call it the length of $\alpha$.

Let $X$ be a subset of $M$. We call $\mathcal{B}$ an isometry covering system of $X$ if it satisfies the following:

1. \( \{B_{\alpha_1}\}_{\alpha_1=1}^{N_1} \) covers $X$;
2. $B_{\alpha_1 \cdots \alpha_{k-1}} \supset B_{\alpha_1 \cdots \alpha_k}$;
3. $\{B_{\alpha_1 \cdots \alpha_k}\}_{\alpha_1=1}^{N_k(\alpha_1 \cdots \alpha_{k-1})}$ is a covering of an isotopic subball $\hat{B}_{\alpha_1 \cdots \alpha_{k-1}}$ of $B_{\alpha_1 \cdots \alpha_{k-1}}$;
4. there is a uniform bound $d$ such that $|\alpha| \leq d$ for all $\alpha \in A$.

We call $N_1$ the first degree of the system $\mathcal{B}$, and $N_k(\alpha_1 \cdots \alpha_{k-1})$ the $k$-th degree of $\mathcal{B}$ with respect to $\alpha_1 \cdots \alpha_{k-1}$.

Let $\hat{A}$ be the set of all maximal multi-indices $\alpha_1 \cdots \alpha_\ell$ in $A$ in the sense that there are no $\alpha_{\ell+1}$ with $B_{\alpha_1 \cdots \alpha_\ell \alpha_{\ell+1}} \in \mathcal{B}$. Then $\mathcal{U} := \{B_{\alpha}\}_{\alpha \in \hat{A}}$ is called an essential covering of $\mathcal{B}$. In other words, $\mathcal{U}$ is the collection of the metric balls lying on the bottom of the system $\mathcal{B}$.

We show that the essential covering $\mathcal{U} = \{B_{\alpha}\}_{\alpha \in \hat{A}}$ produces a covering $\hat{U} = \{\hat{B}_{\alpha}\}_{\alpha \in \hat{A}}$ of $X$ such that $\hat{B}_{\alpha}$ is homeomorphic (actually isotopic) to $B_{\alpha}$. Let $h_{\alpha_1 \cdots \alpha_{k-1}} : M \rightarrow M$ be a homeomorphism sending $\hat{B}_{\alpha_1 \cdots \alpha_{k-1}}$ onto $B_{\alpha_1 \cdots \alpha_{k-1}}$ fixed and leaving the outside of a neighborhood of $\hat{B}_{\alpha_1 \cdots \alpha_{k-1}}$.

For each $\alpha = \alpha_1 \cdots \alpha_k \in \hat{A}$, consider the open set

$$\hat{B}_{\alpha} := h_{\alpha_1} \circ h_{\alpha_1 \alpha_2} \circ \cdots \circ h_{\alpha_1 \cdots \alpha_{k-1}}(B_{\alpha}).$$

For each $1 \leq \alpha_1 \leq N_1$, let $A(\alpha_1)$ be the set of all multi-indices $\alpha \in A$ of the forms $\alpha = \alpha_1 \cdots \alpha_k$ whose leading term is equal to $\alpha_1$ and $k \geq 2$.

From construction, we have

$$B_{\alpha_1} \subset \bigcup_{\alpha \in A(\alpha_1)} \hat{B}_{\alpha}$$

and therefore $\hat{U} = \{\hat{B}_{\alpha}\}_{\alpha \in \hat{A}}$ provides a covering of $X$.

We call

$$d_0 := \max_{\alpha \in \hat{A}} |\alpha|$$

the depth of both $\mathcal{B}$ and $\mathcal{U}$. Note that if $d_0 = 1$, then $\mathcal{B} = \mathcal{U}$ is a usual covering of $X$. 

Let $C(n)$ be the set of all isometry classes of the Euclidean cone over $(n-1)$-dimensional compact Alexandrov spaces with curvature $\geq 1$. We say that $\mathcal{B}$ and $\mathcal{U}$ are modeled on $C(n)$ if each $B_{\alpha}$ in $\mathcal{U}$ is homeomorphic to a space in $C(n)$.

For any positive integer $d$, we denote by $\tau_d(X)$ the minimal number of metric balls forming an essential covering $\mathcal{U}$ of $X$ with depth $\leq d$ modeled on $C(n)$. Note that $\tau_{d_1}(B) \geq \tau_{d_2}(B)$ if $d_1 \leq d_2$.

For open metric ball $B$ of $M$ having a proper isotopic subball, we set

$$\tau^*_d(B) = \min_{\hat{B}} \tau_d(\hat{B}),$$

where $\hat{B}$ runs over all isotopic subballs of $B$. If $B$ itself is homeomorphic to a space in $C(n)$, we define

$$\tau_0(B) = \tau^*_0(B) = 1.$$

From definition, we immediately have

**Lemma 4.1.** Suppose that $X$ is covered by metric balls $\{B_{\alpha_1}\}_{\alpha_1=1}^{N_1}$ having proper isotopic subballs. Then we have

$$\tau_{d+1}(X) \leq \sum_{\alpha_1=1}^{N_1} \tau^*_d(B_{\alpha_1}).$$

**Example 4.2.** For a positive number $\epsilon$, let us consider the flat torus

$$T^n_\epsilon = S^1(1) \times S^1(\epsilon) \times S^1(\epsilon^2) \times \cdots \times S^1(\epsilon^{n-1}).$$

An obvious observation similar to that in the introduction shows $\tau_n(T^n_\epsilon) \leq 2^n$. Note that $\lim_{\epsilon \to 0} \tau_d(T^n_\epsilon) = \infty$ for every $1 \leq d \leq n - 1$.

**Example 4.3.** Let $N$ be an $n$-dimensional simply connected Lie group, and $n$ its Lie algebra. Take a triangular basis $x_1, \ldots, x_n$ of $n$ in the sense that $[x_i, x] \in n_{i-1}$ for every $x \in \ell$, where $n_{i-1}$ is spanned by $x_1, \ldots, x_{i-1}$. For $\epsilon > 0$, put $\epsilon_i := \epsilon^{n-i}$, and define the inner product on $n$ by

$$||x||^2_\epsilon = \epsilon^2_1 a_1^2 + \cdots + \epsilon^2_n a_n^2,$$

for $x = \sum a_i x_i$. We equip $N$ the corresponding left invariant metric $g_\epsilon$. For a given uniform discrete subgroup $\Gamma$ of $N$, consider the quotient $M_\epsilon := (\Gamma \backslash N, g_\epsilon)$. Note that the sectional curvature of $M_\epsilon$ is uniformly bounded and $\delta^1_\epsilon := \text{diam}(M_\epsilon) \to 0$ as $\epsilon \to 0$ (see [6]). Now under the rescaling of metric $\frac{1}{\delta^1_\epsilon} M_\epsilon$ collapses to a circle. We then have a fibration

$$\Gamma_1 \backslash N_1 \to M_\epsilon \to S^1,$$

with a nilmanifold $\Gamma_1 \backslash N_1$ as fiber. Thus $M_\epsilon$ can be covered by two thin metric balls $B_{\alpha_1}$, $\alpha_1 \in \{1, 2\}$, each of which is homeomorphic to $\Gamma_1 \backslash N_1 \times [0, 1]$. Let $\delta^2_\epsilon := \text{diam}(\Gamma_1 \backslash N_1)$. Under the rescaling of metric $\frac{1}{\delta^2_\epsilon} B_{\alpha_1}$ collapses to $S^1 \times \mathbb{R}$. Now an isotopic subball $\hat{B}_{\alpha_1}$ of $B_{\alpha_1}$ has a fibration

$$\Gamma_2 \backslash N_2 \to \hat{B}_{\alpha_1} \to S^1 \times [0, 1],$$
with a nilmanifold \( \Gamma_2 \setminus N_2 \) as fiber. Thus \( \hat{B}_{\alpha_1} \) can be covered by two metric balls \( B_{\alpha_1 \alpha_2}, \alpha_2 \in \{1, 2\} \), each of which is homeomorphic to \( \Gamma_2 \setminus N_2 \times [0, 1]^2 \). Repeating this, we finally have \( \tau_n(M_i) \leq 2^n \). Note that \( \lim_{r \to 0} \tau_d(M_i) = \infty \) for every \( 1 \leq d \leq n - 1 \).

Let \( A(n) \) denote the set of all isometry classes of \( n \)-dimensional complete Alexandrov spaces with curvature \( \geq -1 \). Theorem 4.3 is an immediate consequence of the following

**Theorem 4.4.** For given \( n \) and \( D > 0 \), there are constants \( C_n \) and \( C_n(D) \) such that for every metric ball \( B \) of radius \( \leq D \) in \( M \in A(n) \), there is an isotopy covering system \( \mathcal{B} = \{ B_{\alpha_1 \cdots \alpha_k} \} \) of \( B \) with depth \( \leq n \) modeled on \( C(n) \) such that

1. the first degree \( \leq C_n(D) \);
2. any other higher degree \( \leq C_n \).

In particular \( \tau_n(B) \leq C_n(D)(C_n)^{n-1} \).

We first prove the local version of Theorem 4.4.

**Lemma 4.5.** There is a positive number \( C_n \) satisfying the following: For a given infinite sequence \( (M_i, p_i) \) in \( A_p(n) \) with \( \inf \text{diam}(M_i) > 0 \), there is a subsequence \( (M_j, p_j) \) for which we have a positive number \( r > 0 \) and \( \hat{p}_j \in M_j \) with \( d(p_j, \hat{p}_j) \to 0 \) such that \( \tau^*_{n-1}(B(\hat{p}_j, r)) \leq C_n \).

**Proof:** We prove it by contradiction. If the conclusion does not hold, we would have an infinite sequence \( (M_i, p_i) \) in \( A_p(n) \) such that for every \( r > 0 \) and every \( \hat{p}_i \in M_i \) with \( d(p_i, \hat{p}_i) \to 0 \), we have \( \tau^*_{n-1}(B(\hat{p}_j, r)) \to \infty \) for any subsequence \( \{j\} \) of \( \{i\} \). By Theorem 2.1, we have a subsequence \( \{j\} \) such that \( (M_j, p_j) \) converges to a pointed space \( (X, p) \). Set \( k = \dim X \).

We claim that \( \tau^*_{n-k}(B(\hat{p}_j, r)) \leq C \) for some \( r > 0 \) and constant \( C \) independent of \( j \), where \( \hat{p}_j \) is a point of \( M_j \) with \( d(p_j, \hat{p}_j) \to 0 \). Since this is a contradiction, this will complete the proof.

We prove the claim by the reverse induction on \( k \). If \( k = n \), then Theorem 2.3 shows that there is an \( r > 0 \) such that \( B(p_j, r) \) is homeomorphic to \( K_p \), yielding \( \tau_0^*(B(p_j, r)) = 1 \). Therefore together with the diameter assumption, we only have to investigate the case \( 1 \leq \dim X \leq n - 1 \). Suppose the claim holds for \( \dim X = k + 1, \ldots, n \), and consider the case of \( \dim X = k \). Take \( r = r_\delta > 0 \), \( \hat{p}_j \) and \( \delta_j \to 0 \) as in Theorem 3.2. Namely passing to a subsequence, we may assume that \( \left( \frac{1}{\delta_j} M_j, \hat{p}_j \right) \) converges to a pointed complete noncompact nonnegatively curved space \( (Y, y_0) \) with \( \dim Y \geq \dim X + 1 \) such that \( B(\hat{p}_j, R\delta_j) \) is an isotopic subball of \( B(\hat{p}_j, r) \) for every \( R \geq 1 \) and large \( j \) compared to \( R \). Applying the induction hypothesis to the convergence \( \left( \frac{1}{\delta_j} M_j, \hat{p}_j \right) \to (Y, y_0) \), we have the following: For each \( z \in B(y_0, 2) \), there are \( z^j \in \left( \frac{1}{\delta_j} M_j, \hat{p}_j \right) \) and \( r_z > 0 \) such that \( \tau^*_{n-k-1}(B(z^j, r_z; \frac{1}{\delta_j} M_j)) \leq C \) for some constant.
Note that \( \bigcup z \) conclude \( C \) independent of \( j \). By compactness, there are finitely many points \( z_\alpha \in B(y_0, 2) \) and \( z_j^{\alpha} \in M_j \) converging to \( z_\alpha \) together with \( r_\alpha > 0 \) such that

\[
\bigcup B(z_\alpha, r_\alpha/2) \supset B(y_0, 2), \quad \tau^*_{n-k-1}(B(z_j^{\alpha}, r_\alpha; 1/\delta_j M_j)) \leq C_\alpha.
\]

Note that \( \bigcup B(z_j^{\alpha}, r_\alpha; 1/\delta_j M_j) \supset B(\hat{p}_j, 2; 1/\delta_j M_j) \) for large \( i \). Thus we can conclude

\[
\tau^*_{n-k}(B(\hat{p}_j, r)) \leq \tau_{n-k}(B(\hat{p}_j, 2\delta_j)) \leq \sum \tau^*_{n-k-1}(B(z_j^{\alpha}, r_\alpha)) \leq \sum C_\alpha < \infty.
\]

\( \square \)

**Proof of Theorem 4.4** The proof is by contradiction. If the conclusion does not hold, we would have an infinite sequence of metric balls \( B_i \) of spaces \( M_i \in A(n) \) such that for every essential covering system \( B^i \) of \( B_i \) with depth \( \leq n \) modeled on \( C(n) \), either \( \liminf N^1_k = \infty \) or \( \liminf N^1_k > C_n \), where \( N^1_k \) are the degrees of \( B^i \), and \( C_n \) is the positive constant given in Lemma 4.5. Let \( p_i \) be the center of \( B_i \). By Theorem 2.1, we may assume that \((M_i, p_i)\) converges to a pointed complete Alexandrov space \((X, p)\) with curvature \( \geq -1 \) with respect to the pointed Gromov-Hausdorff topology. We may also assume that \( B_i \) converges to a metric ball \( B \) around \( p \) under this convergence. If \( X \) is a point, we rescale the metric of \( M_i \) so that the new diameter is equal to 1. Thus we may assume that \( 1 \leq \dim X \leq n \). Applying Lemma 4.5 to the convergence \( B_i \to B \), we obtain finitely many points \( \{x_\alpha\}_{\alpha=1}^N \) of \( B \) and positive numbers \( r_\alpha \) with \( B \subset \bigcup B(x_\alpha, r_\alpha/2) \) such that for a subsequence \( \{j\} \) of \( \{i\} \), we get \( p_\alpha^j \in M_j \) converging to \( x_\alpha \) with \( \tau^*_{n-1}(B(p_\alpha^j, r_\alpha)) \leq C_n \) for every \( 1 \leq \alpha \leq N \). Together with the covering \( \{B(p_\alpha^j, r_\alpha)\}_\alpha \) of \( M_j \), this enables us to obtain an essential covering system \( B^j \) of \( B_j \) with depth \( \leq n \) modeled on \( C(n) \) such that \( N^1_j \leq N \) and \( N^1_k \leq C_n \). This is a contradiction. \( \square \)

**Remark 4.6** Let \( M(n) \) denote the subfamily of \( A(n) \) consisting of Riemannian manifolds. By Theorem 4.4, each metric ball of radius \( \leq D \) in \( M \in M(n) \) has an essential covering with depth \( \leq n \) modeled on \( C(n) \) whose number is uniformly bounded. In this case, one can easily check from the proof that each metric ball in the essential covering is homeomorphic to an \( n \)-disk. Namely, for \( M(n) \), we can take the single \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) as the model family in stead of \( C(n) \).

**Remark 4.7** Let \( \delta > 0 \) be given. Under the situation of Theorem 4.4 if we restrict ourselves to metric balls of radii \( < \delta \), we can construct an
isotopy covering system $\mathcal{B} = \{B_{\alpha_1, \ldots, \alpha_k}\}$ of $B$ with depth $\leq n$ modeled on $C(n)$ such that

1. the radius of $B_{\alpha_1}$ is less than $\delta$ for every $1 \leq \alpha_1 \leq N_1$;
2. the first degree $N_1 \leq C_n(D, \delta)$ for some uniform constant $C_n(D, \delta)$;
3. any other higher degree $\leq C_n$.

In particular $\tau_n(B) \leq C_n(D, \delta)(C_n)^{n-1}$.

Perhaps Examples 4.2 and 4.3 will be ones of maximal case.

**Conjecture 4.8.** Let $M$ be an $n$-dimensional compact Alexandrov space with nonnegative curvature. Then $\tau_n(M) \leq 2^n$.

## 5. Betti numbers

In this section, we apply Theorem 4.4 to prove Corollary 1.2. We consider homology groups with any coefficient field $F$. Let $\beta(\ )$ denote the total Betti number for simplicity.

We make use of the following machinery in [5], whose proof is based on Leray’s spectral sequence.

**Lemma 5.1** (Topological lemma ([5])). Let $B_i, 0 \leq i \leq n+1$, be open subsets of an $n$-dimensional space $M$, with $B_0 \subset B_1 \subset \cdots \subset B_{n+1}$, and set $A^i := \bigcup_{i=1}^N B_i^i$. Let $I_+$ denote the set of all multi-indices $(\alpha_1, \ldots, \alpha_m)$ with $1 \leq \alpha_1 < \ldots < \alpha_m \leq N$ and with non-empty intersection $\bigcap_{i=1}^m B_{\alpha_i}^{n+1}$. For each $\mu = (\alpha_1, \ldots, \alpha_m) \in I_+$, let $f_\mu^i : H_*(B_{\alpha_1}^i \cap \cdots \cap B_{\alpha_m}^i) \to H_*(B_{\alpha_1}^{i+1} \cap \cdots \cap B_{\alpha_m}^{i+1})$ be the inclusion homomorphism. Then the rank of the inclusion homomorphism $H_*(A^0) \to H_*(A^{n+1})$ is bounded above by the sum

$$
\sum_{0 \leq i \leq n, \mu \in I_+} \text{rank } f_\mu^i
$$

For any subset $X \subset M$ and $\delta \geq 0$, let $U_\delta(X) := \{x \mid d(x, X) \leq \delta\}$. We define $\delta$-content, denoted by $\delta$-cont$(X)$ of $X$ as the rank of the inclusion homomorphism $H_*(X) \to H_*(U_\delta(X))$. Observe that $0$-cont$(X) = \beta(X)$ may be infinite. However we have

**Theorem 5.2.** For given $n, D > 0$ and $\delta > 0$, there is a positive integer $C(n, D, \delta)$ such that if $X$ is a subset of diameter $\leq D$ in an $n$-dimensional complete Alexandrov space $M$ with curvature $\geq -1$, then

$$
\delta \text{-cont}(X) \leq C(n, D, \delta).
$$

Corollary 1.2 is a direct consequence of Theorem 5.2. Although it is not explicitly stated in [5] or [1], Theorem 5.2 also follows from the methods there. Below we give the proof of Theorem 5.2 based on Theorem 4.4.
For a subset $X$ of diameter less than $D$ in a space $M \in \mathcal{A}(n)$, let $B$ an open metric $D$-ball in $M$ containing $X$. For $\delta > 0$, take an isotopy covering system $\mathcal{B} = \{B_{\alpha_1 \cdots \alpha_k}\}$ of $B$ with depth $\leq n$ modeled on $\mathcal{C}(n)$ satisfying the conclusion of Theorem 4.4 and Remark 4.7 such that the radii of $B_{\alpha_1}$ are less than $10^{-(n+2)}\delta$ for all $1 \leq \alpha_1 \leq N_1$. To apply Lemma 5.1, we let $\lambda_i := 10^i$ for $0 \leq i \leq n + 1$, and put
\[ B_{\alpha_1 \cdots \alpha_k}^i := \lambda_i B_{\alpha_1 \cdots \alpha_k}. \]
In view of the conclusion (2) of Theorem 3.2, we may assume that
\begin{enumerate}
\item $B_{\alpha_1 \cdots \alpha_k}^{n+1} \subset B_{\alpha_1 \cdots \alpha_k-1};$
\item $B_{\alpha_1 \cdots \alpha_k}^i$ is an isotopic subball of $B_{\alpha_1 \cdots \alpha_k}^{i+1}$,
\end{enumerate}
for each $1 \leq \alpha_k \leq N_k(\alpha_1 \cdots \alpha_k-1)$ and $0 \leq i \leq n + 1$.
Let $\mathcal{U} = \{B_{\alpha}\}_{\alpha \in \hat{A}}$ be the essential covering of $B$ associated with $\mathcal{B}$.

**Lemma 5.3.** For every $\alpha = \alpha_1 \cdots \alpha_\ell \in \hat{A}$ and every $1 \leq k \leq \ell$ we have
\[ \beta(B_{\alpha_1 \cdots \alpha_k}) \leq C_n. \]

**Proof.** We prove it by the reverse induction on $k$. The case $k = \ell$ is clear since $B_{\alpha_1 \cdots \alpha_\ell}$ is contractible. Suppose the conclusion $\beta(B_{\alpha_1 \cdots \alpha_{k+1}}) \leq C_n$ for all $1 \leq \alpha_{k+1} \leq N_{k+1}$. Let $\hat{B}_{\alpha_1 \cdots \alpha_k}$ be the isotopic subball of $B_{\alpha_1 \cdots \alpha_k}$ such that
\[ \hat{B}_{\alpha_1 \cdots \alpha_k} \subset \bigcup_{\alpha_{k+1}=1}^{N_{k+1}} B_{\alpha_1 \cdots \alpha_{k+1}}. \]
Since $(\alpha_1, \ldots, \alpha_k)$ is fixed, we put
\[ \hat{B} := \hat{B}_{\alpha_1 \cdots \alpha_k}, \quad B := B_{\alpha_1 \cdots \alpha_k}, \quad B_{\alpha} := B_{\alpha_1 \cdots \alpha_\ell}, \quad B_{\alpha}^i := \lambda_i B_{\alpha}, \]
for each $1 \leq \alpha \leq N_{k+1}$. Let $A_i = \bigcup_{\alpha=1}^{N_i} B_{\alpha}$. From the inclusions $\hat{B} \subset A_0 \subset A_1^{n+1} \subset B$, we have $\beta(\hat{B}) \leq \operatorname{rank}(H_*([B_{\alpha_1 \cdots \alpha_k} \rightarrow H_*])$. Let $I_+$ denote the set of multi-indices of intersection for the covering $\{B_{\alpha_1}^{n+1}\}_\alpha$. For each $\mu = (\gamma_1, \ldots, \gamma_m) \in I_+$, let $B_{\gamma_\mu}$ have minimal radius among $\{B_{\gamma_j}^{n+1}\}$. Let $f_{\mu} : H_*([B_{\gamma_1} \cap \cdots \cap B_{\gamma_m}^{n+1}]) \rightarrow H_*([B_{\gamma_1} \cap \cdots \cap B_{\gamma_m}^{n+1}])$ be the inclusion homomorphism. From the inclusions
\[ B_{\gamma_1} \cap \cdots \cap B_{\gamma_m} \subset B_{\gamma_m}^i \subset \frac{1}{2} B_{\gamma_m}^{i+1} \subset B_{\gamma_1} \cap \cdots \cap B_{\gamma_m}^{i+1}, \]
we have
\begin{align*}
\operatorname{rank}(f_{\mu}^i) &\leq \operatorname{rank}(H_*([B_{\gamma_m}^{i+1} \rightarrow H_*])
= \beta(B_{\gamma_\mu}) \\
&\leq C_n.
\end{align*}
Lemma 5.1 then shows $\beta(B) \leq (n+1)2^{C_n}C_n$. \qed
Proof of Theorem 5.2. Without loss of generality, we may assume that \( \{B_{\alpha_1}^{N(X)}\}_{\alpha_1=1}^{N(X)} \) is a covering of \( X \) for some \( N(X) \leq N_1 \). By Lemma 5.3, we have in particular \( \beta(B_{\alpha_1}) = \beta(B_{\alpha_1}^i) \leq C_n \) for all \( 1 \leq \alpha_1 \leq N(X) \) and \( 0 \leq i \leq n+1 \). Therefore applying Lemma 5.1 to the concentric coverings \( \{B_{\alpha_1}^i\}_{\alpha_1=1}^{N(X)} \) of \( X \) together with \( X \subset N(X) \bigcup_{\alpha_1=1}^{N(X)} B_{\alpha_1}^i \subset U_\delta(X) \), we have

\[
\delta\text{-cont}(X) \leq (n + 1)2^{C_n(D,\delta)}C_n.
\]

This completes the proof of Theorem 5.2. \( \square \)

For a subset \( X \) of a metric space, we define the homological injectivity radius of \( X \), denoted by \( \text{hom inj}(X) \), as the supremum of \( \delta \geq 0 \) such that the inclusion homomorphism \( H_*(X) \to H_*(U_\delta(X)) \) is injective for any coefficient field.

The following is an immediate consequence of Theorem 5.2.

**Corollary 5.4.** For a space \( M \) in \( A(n) \), let \( X_i \) be a sequence of subsets of \( M \) with \( \lim \beta(X_i) = \infty \). Then one of the following must occur:

1. \( \lim \inf \text{hom inj}(X_i) = 0 \);
2. \( \lim \sup \text{diam}(X_i) = \infty \).

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