ON THE CONTINUOUS DUAL HAHN PROCESS

WLODEK BRYC

Abstract. We extend the continuous dual Hahn process \((T_t)\) of Corwin and Knizel from a finite time interval to the entire real line by taking a limit of a closely related Markov process \((T_t)\). We also characterize processes \((T_t)\) by conditional means and variances under bidirectional conditioning, and we prove that continuous dual Hahn polynomials are orthogonal martingale polynomials for both processes.

This is an expanded version of the paper with additional material.

1. Introduction

In this paper we are interested in a family of Markov transition probabilities on the real line which are constructed from the orthogonality measures of the continuous dual Hahn polynomials. Together with the appropriate marginal laws that arise from a point mass as the initial law, these transition probabilities define a class of continuous time Markov processes \((T_t)\) which appeared in the construction of quadratic harnesses in [2, Section 3]. Together with an appropriate family of \(\sigma\)-finite positive measures as the entrance laws, see (4.4), these transition probabilities appeared in the description of the multipoint Laplace transform for stationary measures of the open KPZ equation in [8, Theorem 1.4(5)]. Following [8], we shall use the name the continuous dual Hahn process, and we will use their suggestive notation \((T_t)\). Our goal is to extend the time domain of the process \((T_t)\) from a finite interval described in [8, (1.10)], to the real line. The need for an extension of the time domain arose in [3, Theorem 1.3], although for the purposes of that paper the extension to \(t \in [0, \infty)\) would suffice. We accomplish our goal by analyzing the Markov process \((T_t)\) as one of its parameters diverges to \(\infty\).

The actual process \((T_t)\) constructed here differs slightly from the continuous dual Hahn process in [8]: the process that appears in Refs [3, 8] corresponds to \((4T_{s/2})\). On the other hand, \((T_t)\) as constructed in this note, is a direct extension of the family of Markov processes from [2] to a half-line as the time domain. We will obtain the entrance laws for the process \((T_t)\) by taking a limit of the appropriately scaled marginal laws for the process \((T_t)\).

Our approach to the construction, which relies on verification of the Chapman-Kolmogorov equations, is somewhat different than in Refs. [2, 8], which used explicit formulas for the orthogonality measures of the continuous dual Hahn polynomials. In the presence of atoms, such explicit formulas lead to proliferation of cases, which we avoid by relying on properties of the orthogonal martingale polynomials for \((T_t)\). In particular, as in [5, Section 3.2], we deduce the Chapman-Kolmogorov equations from the algebraic relations between two families of orthogonal polynomials.

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The paper is organized as follows. In Section 2 we recall the definition of the continuous dual Hahn polynomials and discuss the probability measures which make them orthogonal. In Section 3 we use these measures to construct the family of transition probabilities and marginal laws for Markov process \((T_t)\). Our main result, Theorem 3.5, establishes the Chapman-Kolmogorov equations. In Section 4 we introduce the \(\sigma\)-finite entrance laws that define process \((T_t)\) for all \(t \in (-\infty, \infty)\). In Section 5 we characterize the Markov process \((T_t)\) by the formulas for the conditional mean and the conditional variance.

2. Continuous dual Hahn polynomials

2.1. Favard’s Theorem. We first recall a version of Favard’s theorem in the form that encompasses in one statement orthogonality with respect to both finitely supported and infinitely supported measures. This form of Favard’s theorem is "well known" to the experts and it is implicit in many proofs, in particular in the argument presented in [10, Section 2.5]. The explicit reference (with a proof) is [5, Theorem A.1].

**Theorem 2.1** (Favard’s Theorem). Let \(\alpha_n, \beta_n\) be real, \(n \geq 0\). Consider monic polynomials \(\{p_n\}\) defined by the recurrence

\[
 xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \quad n \geq 0,
\]

with the initial conditions \(p_0(x) = 1, p_{-1}(x) = 0\). Then the following two conditions are equivalent:

(i) For all \(n \geq 1\),

\[
 \prod_{j=1}^{n} \beta_j \geq 0.
\]

(ii) There exists a (not necessarily unique) probability measure \(\nu\) with all moments such that for all \(m, n \geq 0\),

\[
 \int p_n(x)p_m(x)\nu(dx) = \delta_{m,n} \prod_{j=1}^{n} \beta_j.
\]

Furthermore, suppose that (2.2) holds. Then either \(\beta_n > 0\) for all \(n \geq 1\), and then measure \(\nu\) has infinite support, or there is a positive integer \(n \geq 1\) such that \(\beta_n = 0\). In the latter case, denote by \(N\) the first positive integer such that \(\beta_N = 0\). Then condition (2.2) contains no further restrictions on the values \(\beta_n\) for \(n > N\) and the orthogonality measure \(\nu(dx)\) is a (unique) discrete probability measure supported on the finite set of \(N \geq 1\) real and distinct zeros of the polynomial \(p_N(x)\).

2.2. The three step recurrence for the continuous dual Hahn polynomials. The continuous dual Hahn polynomials are monic polynomials which depend on three parameters. These parameters are traditionally denoted by \(a, b, c\), but to avoid confusion with the parameters \(a, b, c\) for the Markov process \((T_t)\), we will denote them by \(\alpha, \beta, \gamma\). We always assume that parameter \(\alpha\) is real, and that parameters \(\beta, \gamma\) are either both real or form a complex conjugate pair. Then sequences

\[
 A_n = (n + \alpha + \beta)(n + \alpha + \gamma), \quad C_n = n(n - 1 + \beta + \gamma), \quad n = 0, 1, \ldots
\]

are real. The continuous dual Hahn polynomials, see [11, (1.3.5)], are monic polynomials \(\{p_n(x|\alpha, \beta, \gamma)\}\) in real variable \(x\), defined by the three step recurrence relation

\[
 xp_n(x|\alpha, \beta, \gamma) = p_{n+1}(x|\alpha, \beta, \gamma) + (A_n + C_n - \alpha^2)p_n(x|\alpha, \beta, \gamma) + A_nC_np_{n-1}(x|\alpha, \beta, \gamma),
\]
\( n = 0, 1, \ldots, \) with the usual initialization \( p_{-1}(x|\alpha, \beta, \gamma) = 0,\ p_0(x|\alpha, \beta, \gamma) = 1; \) then (2.5) gives

\[
(2.6) \quad p_1(x|\alpha, \beta, \gamma) = x - \alpha \beta - \alpha \gamma - \beta \gamma.
\]

By comparing recursion (2.5) with [11, (1.3.4)], we get

\[
(2.7) \quad p_n(x|\alpha, \beta, \gamma) = (-1)^n(\alpha + \beta, \alpha + \gamma)_n \, \, _3F_2(-n, \alpha - \sqrt{-x}, \alpha + \sqrt{-x}; \alpha + \beta, \alpha + \gamma; 1),
\]

where

\[
(2.8) \quad _3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, a_3)_k}{(b_1, b_2)_k} \frac{z^k}{k!}
\]
denotes the generalized hypergeometric function. Here and throughout the paper we use the following notation for the products of Gamma functions and the Pochhammer symbols:

\[
\Gamma(a, b, \ldots, c) = \Gamma(a)\Gamma(b)\ldots\Gamma(c), \quad (a)_n = a(a+1)\ldots(a+n-1), \quad (a, b, \ldots, c)_n = (a)_n(b)_n\ldots(c)_n.
\]

With the above restrictions on the parameters, polynomials \( \{p_n(x|\alpha, \beta, \gamma)\} \) are well defined and real valued, but they do not have to be orthogonal. Favard’s theorem allows us to recognize for which choices of the parameters polynomials \( \{p_n(x|\alpha, \beta, \gamma)\} \) are orthogonal. If parameters \( \alpha, \beta, \gamma \) are such that (2.2) holds with \( \beta_n = A_{n-1}C_n \), then polynomials \( p_n(x|\alpha, \beta, \gamma) \) are orthogonal in the following sense: there exists a probability measure \( \nu(dx|\alpha, \beta, \gamma) \) such that

\[
(2.9) \quad \int_{\mathbb{R}} p_n(x|\alpha, \beta, \gamma) p_m(x|\alpha, \beta, \gamma) \nu(dx|\alpha, \beta, \gamma) = 0
\]

for \( m \neq n \). From (2.3) it is clear that in the case of a measure with \( N \) atoms, the integral (2.9) is zero also for \( m = n \geq N \).

Measures \( \nu(dx|\alpha, \beta, \gamma) \) play a prominent role in our construction, as we will define the transition probabilities and the marginal distributions for the Markov process \( (T_t) \) by specifying the parameters \( \alpha, \beta, \gamma \). We will refer to \( \nu(dx|\alpha, \beta, \gamma) \) as the orthogonality measure for the polynomials \( \{p_n(x|\alpha, \beta, \gamma)\} \).

For the constructions, we need to know that the orthogonality measure \( \nu(dx|\alpha, \beta, \gamma) \) is unique, i.e., that it is determined by moments. This fact should be known, but we did not find a published reference. So for completeness we adapt an argument from an unpublished preprint [6, Proposition 3.1], who considered a larger family of polynomials in a different parametrization.

**Lemma 2.2.** Orthogonality measures for the polynomials defined by (2.5) are determined uniquely by moments.

**Proof.** Since finitely supported measures are determined uniquely by moments, we only need to consider the case when \( A_k C_{k+1} > 0 \) for all \( k \geq 0 \). In particular, we assume that \( A_0 c_1 = (\alpha + \beta)(\alpha + \gamma)(\beta + \gamma) > 0 \). We will use a criterion that involves the numerator polynomials \( q_n(x) \), which solve recursion (2.5) with the initial conditions \( q_0(x) = 0, q_1(x) = 1 \) and \( n \geq 1 \) (see e.g. [10, Section 2.3] or [1, Section 2.1]). Let

\[
(2.10) \quad \tilde{p}_n(x) = \frac{1}{\sqrt{A_0 A_1 \ldots A_{n-1} C_1 C_2 \ldots C_n}} p_n(x|\alpha, \beta, \gamma)
\]

and

\[
\tilde{q}_n(x) = \frac{1}{\sqrt{A_0 A_1 \ldots A_{n-1} C_1 C_2 \ldots C_n}} q_n(x)
\]
be the corresponding normalized polynomials. By a theorem of Hamburger, [1, page 84], the moment problem is determined uniquely, if and only if at some point \( x_0 \in \mathbb{R} \) we have

\[
\sum_n |p_n(x_0)|^2 + \sum_n |\tilde{q}_n(x_0)|^2 = \infty.
\]

We shall verify that this condition holds with \( x_0 = -\alpha^2 \).

By (2.7),

\[
p_n(-\alpha^2|\alpha, \beta, \gamma) = (-1)^n(a + \beta, a + \gamma)_n.
\]

Therefore, noting that

\[
(2.12) \quad A_0A_1 \ldots A_{n-1}C_1C_2 \ldots C_n = n!(a + \beta, a + \gamma, \beta + \gamma)_n,
\]

we have

\[
|p_n(-\alpha^2)|^2 = \frac{(\alpha + \beta, \alpha + \gamma)_n}{n!(\beta + \gamma)_n} \sim \frac{\Gamma(\alpha + \beta + n, \alpha + \gamma + n)}{n!\Gamma(\beta + \gamma + n)} \sim \frac{1}{n^{1-2\alpha}},
\]

where we write \( a_n \sim b_n \) if \( \lim_{n \to \infty} a_n/b_n \in (0, \infty) \). Thus (2.11) holds if \( \alpha \geq 0 \).

To verify that (2.11) holds for \( \alpha \leq 0 \), we analyze the numerator polynomials. With \( x = -\alpha^2 \), recursion (2.5) simplifies to

\[
q_{n+1}(-\alpha^2) + A_nq_n(-\alpha^2) = -C_n \left( q_n(-\alpha^2) + A_{n-1}q_{n-1}(-\alpha^2) \right), \quad n \geq 1.
\]

So with \( q_0 = 0, q_1 = 1 \) we have

\[
q_{n+1}(-\alpha^2) + A_nq_n(-\alpha^2) = (-1)^n \prod_{k=1}^n C_k, \quad n \geq 0.
\]

It is easy to check that the solution of this recursion (with \( q_0 = 0, q_1 = 1 \)) is

\[
q_n(-\alpha^2) = (-1)^{n-1} \left( \prod_{k=1}^{n-1} A_k \right) \sum_{m=0}^n \prod_{k=1}^m \frac{C_k}{A_k}, \quad n \geq 0.
\]

After normalization, we get

\[
(2.13) \quad |\tilde{q}_n(-\alpha^2)|^2 = \frac{1}{A_0^2} \left( \prod_{k=0}^{n-1} \frac{A_k}{C_{k+1}} \right) \left( \sum_{m=0}^n \prod_{k=1}^m \frac{C_k}{A_k} \right)^2.
\]

To verify (2.11) we now use the fact that

\[
(2.14) \quad \frac{C_n}{A_n} = 1 - \frac{1 + 2\alpha}{n} + O(1/n^2), \quad \frac{A_n}{C_{n+1}} = 1 - \frac{1 - 2\alpha}{n} + O(1/n^2).
\]

Thus

\[
(2.15) \quad |\tilde{q}_n(-\alpha^2)|^2 \sim \prod_{k=0}^{n-1} \left( 1 - \frac{1 - 2\alpha}{k} \right) \left( \sum_{m=0}^n \prod_{k=1}^m \left( 1 - \frac{1 + 2\alpha}{k} \right) \right)^2 \sim \exp \left( -\sum_{k=0}^{n-1} \frac{1 - 2\alpha}{k} \right) \left( \sum_{m=0}^n \exp(-\sum_{k=1}^m \frac{1 + 2\alpha}{k}) \right)^2 \sim \frac{1}{n^{1-2\alpha}} \left( \sum_{m=0}^{n-1} \frac{1}{m^{1+2\alpha}} \right)^2 \sim \frac{1}{n^{1-2\alpha} n^{-4\alpha}} = \frac{1}{n^{1+2\alpha}}.
\]
Thus (2.11) holds also if \( \alpha \leq 0 \), completing the proof. \( \square \)

3. Transition probabilities and the marginal laws

Our first goal is to define a family of Markov kernels \( \{ p_{s,t}(x,dy) : s < t \} \) which will serve as the transition probabilities. These kernels will be chosen from the orthogonality measures of the continuous dual Hahn polynomials and depend on a real parameter \( c \). A subtle point in the construction is that due to restriction (2.2) these orthogonality measures cannot be defined for all real \( x \), and the excluded set of \( x \)'s depends on the value of \( s \). This leads to a rarely considered case of Markov processes with a "time-dependent" state space, as in [9, Sections 9, 10].

For \( s \in \mathbb{R} \), we introduce a family of sets

\[
E_s = \begin{cases} 
- (c-s)^2, \infty & s \leq c, \\
- (c-s+N)^2 : N = 0, 1, \ldots, N < s - c \cup [0, \infty) & s > c.
\end{cases}
\]

The orthogonality of the continuous dual Hahn polynomials allows us to define measures \( p_{s,t}(x,dy) \) at spatial locations \( x \in E_s \). We will extend artificially our definition to \( x \notin E_s \), but as in [9], the resulting Markov process will really be defined on the product of the sets \( E_s \). Thus sets \( E_s \) play an important role in the construction of the Markov family, and will appear in several statements below.

For \(-\infty < s < t < \infty\), we define the family of Markov kernels by inserting times \( s, t \) and location \( x \) into the parameters of the orthogonality measure introduced in formula (2.9). Let

\[
p_{s,t}(x,dy) := \begin{cases} 
\nu(dy|c-t,t-s-i\sqrt{x},t-s+i\sqrt{x}) & x \in E_s, x \geq 0, \\
\nu(dy|c-t,t-s-\sqrt{-x},t-s+\sqrt{-x}) & x \in E_s, x < 0, \\
\delta_{-(c-t)^2}(dy) & x \notin E_s.
\end{cases}
\]

We need to verify that these probability measures are well defined and that their supports are contained in the corresponding sets \( E_t \).

**Proposition 3.1.** Probability measures (3.2) are well defined for all \(-\infty < s < t < \infty\). Furthermore,

\[
p_{s,t}(x,E_t) = 1.
\]

**Proof.** It is clear that the conclusion holds if \( x \notin E_s \), so we only need to consider \( x \in E_s \). To verify that the probability measure is well defined, we analyze the factors in product (2.2) with \( \beta_n = A_{n-1}C_n \). With parameters as specified in (3.2), from (2.4) we get

\[
A_nC_{n+1} = (n+1)(n+2(t-s)) \left((c+n-s)^2 + x\right).
\]

First, consider the boundary case \( x = -(c-s)^2 \). Then \( A_0C_1 = 0 \), so by Theorem 2.1 measure \( p_{s,t}(x,dy) \) is concentrated at the root \( \alpha\beta + \alpha\gamma + \beta\gamma = -(c-t)^2 \) of polynomial (2.6). We get \( p_{s,t}(x,dy) = \delta_{-(c-t)^2}(dy) \). In particular, we have \( p_{s,t}(x,E_t) = 1 \).

Next, consider the non-boundary cases \( x \in E_s \) with \( x \geq 0 \). Then products (3.4) are strictly positive, so measure \( p_{s,t}(x,dy) \) is well defined and has infinite support (except for the already considered boundary case of \( x = 0, c = s \)). In (3.2), we have measure \( \nu(dy|c-t,t-s+i\sqrt{x},t-s-i\sqrt{x}) \) with the complex-conjugate pair of parameters with positive real part \( t-s \). From [11, Section 1.3] we see that \( p_{s,t}(x,dy) \) has absolutely continuous component with a density supported on \((0, \infty)\), and if \( t > c \) then in addition to the absolutely continuous component, measure \( p_{s,t}(x,dy) \) has also
a discrete component with atoms at points \( y_k = -(c - t + k)^2 \) for \( k = 0, 1, \ldots \) such that \( t > c + k \). Thus \( p_{s,t}(x, E_t) = 1 \).

Finally, we consider the non-boundary cases with \( x \in E_s \) such that \( -(c - s)^2 < x < 0 \).

(A) If \( s < c \), then products (3.4) are positive, as \( A_n C_{n+1} \geq A_0 C_1 > 0 \). So measure \( p_{s,t}(x, dy) \) is well defined and has infinite support. It remains to verify that \( p_{s,t}(x, E_t) = 1 \).

Since \( x < 0 \), we have \( x = -u^2 \) for some \( 0 < u < c - s \) and in (3.2), we have measure \( \nu(dy|x, c - t, t - s + v, t - s - v) \) with three real parameters. Since \( t - s + v > 0 \) and \( t - s - v > t - c \), at least two of the parameters of measure \( \nu(dy|x, c - t, t - s + v, t - s - v) \) are positive: the second positive parameter is either \( c - t > 0 \) or \( t - s - v > 0 \). From [11, Section 1.3] we see that \( p_{s,t}(x, dv) \) has absolutely continuous component with a density supported on \( (0, \infty) \) and with atoms at points \( y_k = -(c - t + k)^2 \), if \( c - t < 0 \), or at points \( \tilde{y}_k = -(t - s - v + k)^2 \) if \( t - s - v < 0 \), and then \( c - t > 0 \). It is clear that points \( y_k \) are in \( E_t \). On the other hand, if \( t - s - v < 0 \), then \( \tilde{y}_k \geq 0 \) and \( \tilde{y}_k \in E_t \), as we have \( c > t \) in this case.

(B) If \( s > c \) and \( x \in E_s \), then \( x = -(c - s + N)^2 \) for some \( N \) such that \( s > c + N \). We see that (3.4) factors as

\[
A_n C_{n+1} = (n + 1)(n + 2(t - s))(2c + n + N - 2s)(n - N).
\]

Since \( 2c + n + N - 2s \leq 2(c + N - s) < 0 \) for \( n \leq N \), the last two factors are both negative for \( n = 0, 1, \ldots, N - 1 \), and \( A_N C_{N+1} = 0 \). So the products (2.2) are positive and then 0. By Theorem 2.1, measure \( p_{s,t}(x, dy) \) is atomic with \( N + 1 \) atoms at the roots of its \((N + 1)\)-th orthogonal polynomial \( p_{N+1}(y; \alpha, \beta, \gamma) \) as written in (3.12), which in view of (3.15) factors as

\[
Q_N(y; x, t, s) = (c - t - \sqrt{y}, c - t + \sqrt{y})_{N+1} = \prod_{k=0}^{N}((c - t + k)^2 + y).
\]

The roots of this polynomial are \(-(c - t)^2, -(c - t + 1)^2, \ldots, -(c - t + N)^2\) and they lie in \( E_t \).

\[\Box\]

Remark 3.2. From the proof of Proposition 3.1 we note that if \( x = -(c - s)^2 \), then measure \( p_{s,t}(x, dy) = \delta_{-(c - s)^2}(dy) \) is degenerate. As a consequence, a Markov process with transition probabilities \( p_{s,t}(x, dy) \) which is at location \( x \notin E_s \) at time \( s \), will follow the parabola \( t \mapsto -(c - t)^2 \) at the boundary of sets \( E_t \) for \( t > s \).

Remark 3.3. Wojciech Matysiak pointed out to us that for \( t > s > c \) the orthogonality measure \( \nu(dy|c - t, t - s - \sqrt{x}, t - s + \sqrt{x}) \) is well defined also for all \( x \in (\min_{k=0,1,\ldots}(c - s + k)^2, 0) \) which are not in \( E_s \). The form of the orthogonality measure for this case seems to be unknown, and we expect that in addition to the expressions listed in [11, Section 1.3] there is an additional component that allows for convergence to \( \delta_x \) as \( t \nearrow s \). However, under Assumption 3.1, such \( x \)'s are not within the support of the marginal laws, so we restrict our construction to \( x \in E_s \) and set \( p_{s,t}(x, dy) = \delta_{-(c - t)}(dy) \) for \( x \notin E_s \).

Marginal laws. The marginal laws for the Markov process are also defined using the orthogonality measures of the continuous dual Hahn polynomials. We use two additional parameters \( a, b \) which satisfy the following.

Assumption 3.1. We assume one of the following:

(a) \( a, b, c \) are real parameters such that \( a + c > 0 \), with \( b \geq a \), or
(b) \( c \) is real, and \( a, b \) are complex conjugates with \( \text{Im}(a) \neq 0 \).
(Since the expressions below are symmetric in parameters $a, b$, condition $b \geq a$ in Assumption 3.1(a) is just a convenient labeling convention.)

We use the orthogonality measure from formula (2.9) to define a family of marginal probability laws as follows:

$$(3.5) \quad p_t(dx \mid a, b, c) := \nu(dx|c-t, a+t, b+t).$$

We need to verify that the definition is correct and to state explicit formulas that will be needed in Section 4.

Proposition 3.4. Under Assumption 3.1, probability measures (3.5) are well defined for all $t \geq -(a+b)/2$. Furthermore, $p_t(E_t \mid a, b, c) = 1$, and the explicit formulas for the measures are as follows.

If $t = -(a+b)/2$, then $p_t(dx \mid a, b, c)$ is a degenerate measure $\delta_{-(a-b)^2/4}(dx)$.

If $t > -(a+b)/2$, then $p_t(dx \mid a, b, c) = p_t^{(e)}(dx \mid a, b, c) + p_t^{(d)}(dx \mid a, b, c)$ is the sum of the continuous and discrete components. The continuous component is supported on $(0, \infty)$ and is given by

$$(3.6) \quad p_t^{(e)}(dx \mid a, b, c) = \frac{1}{4\pi\Gamma(a+c, b+c, a+b+2t)} \cdot \frac{\Gamma(a+t+i\sqrt{x}, b+t+i\sqrt{x}, c-t+i\sqrt{x})^2}{\sqrt{x}|\Gamma(2\sqrt{x})|^2} 1_{x>0}dx.$$

The discrete component is either zero, or it has a finite number of atoms in $(-\infty, 0)$. The discrete component is non-zero in the following two cases.

(a) If $a$ is real and $t+a < 0$ then

$$(3.7) \quad p_t^{(d)}(dx \mid a, b, c) = \sum_{\{k \geq 0: a+t+k < 0\}} m_k(t)\delta_{a+t+k}^2(dx)$$

with

$$m_k(t) = \frac{\Gamma(-a+c-2t)}{\Gamma(-2(a+t))} \cdot \frac{(a+k)(a+c)_k(2(a+t))_k}{k!(a+t)(a-c+2t+1)_k} \cdot \frac{\Gamma(b-a)(a+b+2t)_k}{\Gamma(b+c)(a-b+1)_k}(-1)^k.$$

(b) If $t > c$ then

$$(3.8) \quad p_t^{(d)}(dx \mid a, b, c) = \sum_{\{k \geq 0: c+t+k < 0\}} M_k(t)\delta_{c+t+k}^2(dx)$$

with

$$M_k(t) = \frac{(c+k-t)}{k!(c-t)} \frac{\Gamma(a-c+2t)}{\Gamma(2(t-c))} \cdot \frac{(a+c, 2(c-t))_k}{(-a+c-2t+1)_k} \cdot \frac{(b-c+2t)}{\Gamma(a+b+2t)} \cdot \frac{(b+c)_k}{(-b+c-2t+1)_k}(-1)^k.$$

(To facilitate taking the limit as $b \to \infty$ in a later argument, factors with $b$ are separated at the end of the formulas.)

Note that if $a$ is real then Assumption 3.1 implies that $-a+c-2t = (a+c) - 2(a+t) > 0$ in case (a) and $b-c+2t \geq a-c+2t = (a+c) + 2(t-c) > 0$ in case (b), so the Gamma functions that appear in the formulas are well defined.
Proof. If \( t = -(a+b)/2 \), then in the product (2.2) we get \( A_0C_1 = 0 \), giving a degenerate measure at the root \( x_0 = \alpha \beta + \alpha \gamma + \beta \gamma \) of polynomial (2.6). We get \( p_t(dx \mid a, b, c) = \delta_{x_0} \) with \( x_0 = -(a-b)^2/4 \).

We now check that \( x_0 \in E_t \). If \( a = b \), then \( x_0 \geq 0 \), so \( x_0 \in E_t \). If the parameters are real, then \( b \geq a > -c \), so \( 0 \leq (b-a)/2 = -a - t < c - t \). Therefore, \( E_t = [-(c-t)^2, \infty) \) and

\[
x_0 = -(b-a)^2/2 > -(c-t)^2
\]
is in \( E_t \).

For \( t > -(a+b)/2 \), we have a measure \( \nu(dx \mid c - t, a + t, b + t) \) that either has to two complex-conjugate parameters \( a + t, b + t \) with positive real part (as \( t > -(a+b)/2 = -\text{Re}(a) = -\text{Re}(b) \)), or with three real parameters of which at least two are positive. Indeed, we have \( b+t \geq (a+b)/2+t > 0 \) and if \( a + t \leq 0 \) then \( c - t > c + a > 0 \). So the distribution can be read out from [11, Section 1.3].

(a) If either \(-a \leq t \leq c \) or \( \text{Im}(a) \neq 0 \) and \( t \leq c \) then the distribution has density (3.6) supported on \((0, \infty)\) and has no atoms.

(b) If \( a + t < 0 \), then in addition to the absolutely continuous component (3.6), there are atoms (3.7), which are in \( E_t \), as from \( 0 < -a - t < c - t \) we get \( E_t = [-(c-t)^2, \infty) \), see (3.1).

(c) If \( t > c \) then in addition to the absolutely continuous component (3.6), there are atoms (3.8) that are in \( E_t \).

So in all three cases, \( p_t(E_t \mid a, b, c) = 1 \). \( \square \)

We now verify that the family of probability measures \( p_t(dx \mid a, b, c) \) together with the family of Markov kernels \( p_{s,t}(x, dy) \) satisfy the Chapman-Kolmogorov equations.

Theorem 3.5. Suppose that parameters \( a, b, c \) satisfy Assumption 3.1. Let \( U \) be a Borel subset of \( \mathbb{R} \).

(i) For \(-\infty < s < t < u < \infty \) and \( x \in \mathbb{R} \), we have

\[
\int_{\mathbb{R}} p_{s,t}(x, dy)p_{t,u}(y, U) = p_{s,u}(x, U).
\] (3.9)

(ii) For \(-(a+b)/2 \leq s < t < \infty \),

\[
\int_{\mathbb{R}} p_s(dx \mid a, b, c)p_{s,t}(x, U) = p_t(U \mid a, b, c).
\] (3.10)

The following definition summarizes the above.

Definition 3.1. With parameters \( a, b, c \) which satisfy Assumption 3.1 and \( \tau = -(a+b)/2 \), we denote by \( (T_s)_{s \geq \tau} \) a Markov process marginal laws (3.5) and with transition probabilities (3.2). The process starts at time \( \tau = -(a+b)/2 \) at the deterministic location \( x_\tau = -(a-b)^2/4 \).

Remark 3.6. The continuous dual Hahn polynomials are a limiting case of the four-parameter family of Wilson polynomials [13]. It would be interesting to see how the four-parameter orthogonality measures of Wilson polynomials could be used to construct Markov processes. Two special cases are known: Ref. [2, Section 2] considered the absolutely continuous case, and Ref. [4] considered a purely atomic case.

The rest of this section is devoted to the proof of Theorem 3.5. In the proof we use the following two families of polynomials. Denote

\[
p_n(x; s) = p_n(x \mid c - s, a + s, b + s),
\] (3.11)
\[ Q_n(y; x, t, s) = p_n(y \mid c - t, t - s - \sqrt{-x}, t - s + \sqrt{-x}). \]

(With the convention that \( \pm \sqrt{-x} \) is \( \pm i\sqrt{x} \) when \( x > 0 \), compare (3.2).)

Polynomials \( p_n(x; s) \) are of course the monic orthogonal polynomials for the univariate laws \( p_s(dx \mid a, b, c) \). If \( x \in E_s \) and \( s < t \), then polynomials \( Q_n(y; x, t, s) \) are the orthogonal polynomials for the measures \( p_s(x, dy) \). However, recursion (2.5) defines polynomials \( Q_n(y; x, t, s) \) for all \( x, s, t \in \mathbb{R} \), and we will need this more general setting for some of the arguments.

The key step in the proof is the following algebraic fact about the connection coefficients between these two families of polynomials.

**Lemma 3.7.** If \( a, b, c \) satisfy Assumption 3.1, then there exist functions \( \{b_{n,k}(x, s) : 1 \leq k \leq n\} \) (in fact, polynomials in \( s, x \)) which do not depend on \( t \) such that \( b_{n,n}(x, s) = 1 \), and for all \( x, y \in \mathbb{R} \) we have

\[ Q_n(y; x, t, s) = \sum_{k=0}^{n} b_{n,k}(x, s)p_k(y; t), \quad n = 1, 2, \ldots \]

We remark that by linear independence, any set of monic polynomials can be expressed as a unique linear combination of the polynomials \( p_k(y; t) \), \( k = 0, 1, \ldots \). The main point of Lemma 3.7 is that the coefficients of the linear combination (3.13) do not depend on variable \( t \).

**Proof.** From (2.7), we see that each polynomial

\[ p_n(y; t) = (-1)^n(a + c, b + c)_n F_2 \left( -n, c - t - \sqrt{-y}, c - t + \sqrt{-y}; a + c, b + c; 1 \right) \]

is a linear combination of the linearly independent monic polynomials

\[ (c - t - \sqrt{-y}, c - t + \sqrt{-y})_k = \prod_{j=0}^{k-1}((c - t + j)^2 + y), \quad k = 0, 1, \ldots, n \]

in variable \( y \). From (2.8) we see that the coefficients of the linear combination,

\[ (-1)^n(a + c, b + c)_n(-n)_k, \quad k = 0, 1, \ldots, n, \]

do not depend on \( t \). By Assumption 3.1, either \( b + c \geq a + c > 0 \) or \( \text{Im}(a) \neq 0 \), so \( (a + c, b + c)_n \neq 0 \) and hence the coefficients are non-zero. This means that polynomials (3.14) can be written as linear combinations of polynomials \( p_0(y; t), p_1(y; t), \ldots, p_n(y; t) \) with the coefficients that do not depend on \( t \). Since

\[ Q_n(y; x, t, s) = F_2 \left( -n, c - t - \sqrt{-y}, c - t + \sqrt{-y}; c - s - \sqrt{-x}, c - s + \sqrt{-x}; 1 \right) \cdot (-1)^n(c - s - \sqrt{-x}, c - s + \sqrt{-x})_n, \]

from (2.8) we see that \( Q_n(y; x, t, s) \) is a linear combination of the monic polynomials (3.14), with the coefficients

\[ (-1)^n(-n)_k(c - s - \sqrt{-x}, c - s + \sqrt{-x})_n, \quad k = 0, 1, \ldots, n \]

that depend only on \( x, s \), but not on \( t \). Combining these two observations together, we get (3.13). The fact that \( b_{n,n}(x, s) = 1 \) is just a consequence of the fact that both sets of polynomials are monic.
We note that \( Q_n(x; x, s, s) = 0 \) for \( n \geq 1 \). To see this, as in [5], we use the three step recurrence relation (2.5) to verify that \( Q_1(x; x, s, s) = 0, Q_2(x; x, s, s) = 0 \), and then automatically (2.5) implies that \( Q_n(x; x, s, s) = 0 \) for all \( n \geq 3 \).

From (3.13), applied to each term in \( Q_n(y; x, t, s) = Q_n(y; x, t, s) - Q_n(x; x, s, s) \), after canceling the first term with \( p_0(y; t) = 1 \), we get

\[
Q_n(y; x, t, s) = \sum_{k=1}^{n} b_{n,k}(x, s)(p_k(y; t) - p_k(x; s)), \quad n = 1, 2, \ldots
\]

Formula (3.16) immediately implies that \( \{p_n(x; s)\} \) are in fact orthogonal martingale polynomials for the Markov process \( (T_s) \). This implication is known, see [5, Proposition 3.6] but we include proof for completeness.

**Proposition 3.8.** If \( x \in E_s \), then

\[
\int_{\mathbb{R}} p_n(y; t)p_{s,t}(x, dy) = p_n(x; s).
\]

**Proof.** The proof is by induction on \( n \). Trivially, (3.17) holds for \( n = 0 \), as \( p_0(x; s) = 1 \). For the induction step, suppose that the martingale property (3.17) holds for polynomials \( p_k(x; t) \) with \( k \leq n - 1 \), where \( n \geq 1 \). Since for \( x \in E_s \) polynomials \( Q_n(y; x, t, s) \) and \( Q_0(y; x, t, s) = 1 \) are orthogonal, from (3.16) we get

\[
0 = \int_{\mathbb{R}} Q_n(y; x, t, s)p_{s,t}(x, dy)
\]

\[
= \sum_{k=1}^{n-1} b_{n,k}(x, s) \int_{\mathbb{R}} (p_k(y; t) - p_k(x; s))p_{s,t}(x, dy) + b_{n,n}(x, s) \int_{\mathbb{R}} (p_n(y; t) - p_n(x; s))p_{s,t}(x, dy)
\]

\[
= 0 + \int_{\mathbb{R}} (p_n(y; t) - p_n(x; s))p_{s,t}(x, dy).
\]

where we used the induction assumption for all the terms with \( k \leq n - 1 \) and we used \( b_{n,n}(x, s) = 1 \) in the last term. Thus \( \int_{\mathbb{R}} (p_n(y; t) - p_n(x; s))p_{s,t}(x, dy) = 0 \), which ends the proof by induction. \( \square \)

We remark that for \( x \not\in E_s \), polynomials \( Q_n(y; x, t, s) \) do not have to be orthogonal, so (3.17) may fail if \( x \not\in E_s \).

**Proof of Theorem 3.5.** We note that (3.9) holds by default if \( x \not\in E_s \), as then the process is deterministic, see Remark 3.2.

To prove that (3.9) holds for \( x \in E_s \), following [5, page 1242] we introduce an auxiliary probability measure

\[
\mu(U) = \int_{\mathbb{R}} p_{s,t}(x, dy)p_{t,u}(y, U) = \int_{E_t} p_{s,t}(x, dy)p_{t,u}(y, U),
\]

as \( p_{s,t}(x, E_t) = 1 \). Then for \( n = 1, \ldots \), we have

\[
\int_{\mathbb{R}} Q_n(z; x, u, s)\mu(dz) = 0.
\]
Indeed,
\[
\int_{\mathbb{R}} Q_n(z; x, u, s) \mu(dz) = \int_{\mathbb{R}} Q_n(z; x, u, s) \int_{\mathbb{R}} p_{s,t}(x, dy) p_{t,u}(y, dz) \\
= \int_{\mathbb{R}} p_{s,t}(x, dy) \int_{\mathbb{R}} Q_n(z; x, u, s) p_{t,u}(y, dz) \\
= \sum_{k=1}^{n} b_{n,k}(x, s) \int_{\mathbb{R}} p_{s,t}(x, dy) \int_{\mathbb{R}} (p_k(z; u) - p_k(x; s)) p_{t,u}(y, dz) \\
\overset{(3.16)}{=} \sum_{k=1}^{n} b_{n,k}(x, s) \int_{\mathbb{R}} p_{s,t}(x, dy) \int_{\mathbb{R}} (p_k(z; u) - p_k(x; s)) p_{t,u}(y, dz) \\
\overset{(3.17)}{=} \sum_{k=1}^{n} b_{n,k}(x, s) \int_{\mathbb{R}} p_{s,t}(x, dy) (p_k(y; t) - p_k(x; s)) \overset{(3.17)}{=} 0,
\]
where we used martingale property first at \( y \in E_t \), and then again at \( x \in E_s \).

It is well known, see e.g. [10, Exercise 2.5] that condition (3.18) implies that the moments of \( \mu(\text{dz}) \) are the same as the moments of the orthogonality measure \( p_{s,u}(x, \text{dz}) \) for the polynomials \( Q_n(z; x, u, s), n \geq 0 \). By Lemma 2.2, we get \( \mu(U) = p_{s,u}(x, U) \) for all Borel sets \( U \), proving (3.9).

To prove that (3.10) holds, we follow a similar plan. Recycling the same letter, we introduce another auxiliary probability measure
\[
\mu(U) = \int_{\mathbb{R}} p_s(\text{dx} \mid a, b, c) p_{s,t}(x, U) = \int_{E_s} p_s(\text{dx} \mid a, b, c) p_{s,t}(x, U).
\]
For \( n = 1, 2, \ldots \), we use martingale property (3.17) on \( E_s \) to compute
\[
\int_{\mathbb{R}} p_n(y; t) \mu(\text{dy}) = \int_{\mathbb{R}} p_n(y; t) \int_{\mathbb{R}} p_s(\text{dx} \mid a, b, c) p_{s,t}(x, dy) \\
= \int_{E_s} p_s(\text{dx} \mid a, b, c) \int_{\mathbb{R}} p_n(y; t) p_{s,t}(x, dy) \overset{(3.17)}{=} \int_{\mathbb{R}} p_n(x; s) p_s(\text{dx} \mid a, b, c) = 0,
\]
where in the last step we used orthogonality of polynomials \( p_n(x; s) \) and \( p_0(x; s) \equiv 1 \) with respect to \( p_s(\text{dx} \mid a, b, c) \). Since polynomials \( \{p_n(y; s)\} \) are orthogonal with respect to probability measure \( p_t(\text{dy} \mid a, b, c) \), by the uniqueness of the moment problem, \( \mu(dy) = p_t(dy \mid a, b, c) \), proving (3.10).

(In both proofs, the interchange of the order of integrals is allowed, as the measures have all moments.)

\[\square\]

4. A FAMILY OF \( \sigma \)-FINITE ENTRANCE LAWS

In this section we consider real parameters \( a, b, c \), under Assumption 3.1(a). (Parameter \( b \) will appear only in the proof.)

For \( -\infty < t < \infty \), we introduce a family of \( \sigma \)-finite measures \( p_t(\text{dx}) \), which depend on parameters \( a, c \). These measures were motivated by Ref. [8, Definition 7.8].
Definition 4.1. For real $t, a, c$ with $a + c > 0$, consider a family of positive $\sigma$-finite measures

\begin{equation}
 p_t(dx) = p_t^{(c)}(dx) + p_t^{(d)}(dx) := \frac{1}{4\pi} \frac{\Gamma(t + a + i\sqrt{x}, c - t + i\sqrt{x})^2}{\sqrt{x}\Gamma(2i\sqrt{x})^2} 1_{x > 0} dx + \sum_{j: j + a + t < 0} m_j(t) \delta_{-a+j+t}(dx) + \sum_{k: c-t+k < 0} M_k(t) \delta_{-(c-t+k+t)}(dx)
\end{equation}

with discrete masses given by

\begin{equation}
 m_j(t) = \frac{(a + j + t)}{j!(a + t)} \frac{\Gamma(a + c, c - a - 2t)}{\Gamma(-2(a + t))} \frac{(a + c, 2(a + t))_j}{(a - c + 2t + 1)_j}, \quad j \in \mathbb{Z} \cap [0, -a - t),
\end{equation}

\begin{equation}
 M_k(t) = \frac{(c + k - t)}{k!(c - t)} \frac{\Gamma(a - c + 2t, a + c)}{\Gamma(2(t - c))} \frac{(a + c, 2(c - t))_k}{(-a + c - 2t + 1)_k}, \quad k \in \mathbb{Z} \cap [0, t - c).
\end{equation}

Our goal is to show that measures $p_t(dx)$ are the entrance laws ([9, Sect. 10]) for the family of transition probabilities $p_{s,t}(x, dy)$ defined by (3.2), i.e., that for all Borel sets $U$ we have

\begin{equation}
 \int_{\mathbb{R}} p_s(dx)p_{s,t}(x, U) = p_t(U).
\end{equation}

The following extends [8, Lemma 7.11] to a larger time domain.

Theorem 4.1. Suppose $a, c$ are real and $a + c > 0$. Then for $-\infty < s < t < \infty$, and all Borel sets $U$, the entrance law formula (4.4) holds.

Theorem 4.1 follows from the explicit formulas for $p_t(dx \mid a, b, c)$ in Proposition 3.4. The idea of proof is that for fixed $t > -(a + b)/2$, we have

\[
\frac{\Gamma(a + c)\Gamma(b + c)\Gamma(a + b + 2t)}{\Gamma(b + t)^2} p_t(dx \mid a, b, c) \rightarrow p_t(dx)
\]

as $b \rightarrow \infty$, and that this limit preserves equation (3.10). To make this heuristics precise, we need to analyze separately the continuous and the discrete components.

Proof of Theorem 4.1. We first consider the continuous component. From (3.6) we see that the density of

\[
\frac{\Gamma(a + c)\Gamma(b + c)\Gamma(a + b + 2t)}{\Gamma(b + t)^2} p_t^{(c)}(dx \mid a, b, c)
\]

is

\[
\frac{\Gamma(a + t + i\sqrt{x}, c - t + i\sqrt{x})^2}{4\pi \sqrt{x}\Gamma(2i\sqrt{x})^2} \cdot \frac{\Gamma(b + t + i\sqrt{x})^2}{\Gamma(b + t)^2} = \frac{\Gamma(a + t + i\sqrt{x}, c - t + i\sqrt{x})^2}{4\pi \sqrt{x}\Gamma(2i\sqrt{x})^2} \prod_{k=0}^{\infty} \frac{1}{1 + \frac{x}{(b+t+k)^2}}
\]

as $b \rightarrow \infty$. 

\[
\frac{\Gamma(a + t + i\sqrt{x}, c - t + i\sqrt{x})^2}{4\pi \sqrt{x}\Gamma(2i\sqrt{x})^2} \prod_{k=0}^{\infty} \frac{1}{1 + \frac{x}{(b+t+k)^2}}
\]

as $b \rightarrow \infty$.
(Here we used \(|\Gamma(x)/\Gamma(x+iy)|^2 = \prod_k (1+y^2/(x+k)^2)\), see [12, 5.8.3].) The monotone convergence holds, because for \(x > 0\) we have

\[
0 < \sum_{k=0}^{\infty} \log \left(1 + \frac{x}{(b+t+k)^2}\right) < \sum_{k=0}^{\infty} \frac{x}{(b+t+k)^2} < \frac{x}{(b+t)^2} + \sum_{k=1}^{\infty} \frac{x}{(b+t+k)(b+t+k-1)} = \frac{x}{(b+t)^2} + \frac{x}{b+t} \to 0 \text{ as } b \to \infty.
\]

Next we consider the discrete components. We note that the locations of atoms do not depend on parameter \(b\). We compute the limit of masses of the atoms. For atoms in (3.7), as \(b \to \infty\) we have

\[
\frac{\Gamma(a+c)\Gamma(b+c)\Gamma(a+b+2t)}{\Gamma(b+t)^2} m_k(t) = \frac{\Gamma(a+c)\Gamma(-a+c-2t)}{\Gamma(-2(a+t))} \frac{(a+k+t)(a+c+k)(2(a+t))_k}{k!(a+t)(a-c+2t+1)} (-1)^k \frac{\Gamma(b-a)\Gamma(a+b+2t)}{\Gamma(b+2t)^2} \frac{(a+b+2t)_k}{(a-b+1)_k} \to \frac{\Gamma(a+c)\Gamma(-a+c-2t)}{k!(a+t)(a-c+2t+1)} \frac{(a+k+t)(a+c)_k(2(a+t))_k}{\Gamma(-2(a+t))}.
\]

This gives (4.2). For atoms in (3.8), as \(b \to \infty\) we have

\[
\frac{\Gamma(a+c)\Gamma(b+c)\Gamma(a+b+2t)}{\Gamma(b+t)^2} M_k(t) = \frac{(c+k-t)}{k!(c-t)} \frac{\Gamma(a-c+2t,a+c)}{\Gamma(2(t-c))} \frac{(a+c,2(c-t))_k}{(-a+c-2t+1)_k} (-1)^k \frac{\Gamma(b-c+2t)\Gamma(b+c)}{\Gamma(b+2t)^2} \frac{(b+c)_k}{(-b+c-2t+1)_k} \to \frac{(c+k-t)}{k!(c-t)} \frac{\Gamma(a-c+2t,a+c)}{\Gamma(2(t-c))} \frac{(a+c,2(c-t))_k}{(-a+c-2t+1)_k}.
\]

This gives (4.3).

We can now prove (4.4). Fix a Borel set \(U\) and \(s < t\). We will use Theorem 3.5(ii), i.e., formula (3.10). Since the convergence of the densities is monotone, the locations of atoms do not depend on \(b\), and there are only finitely many atoms, by the monotone convergence theorem for the continuous component, and by convergence of the finite sums for the discrete component, we have

\[
\int_{\mathbb{R}} p_s(dx)p_{s,t}(x,U) = \int_{\mathbb{R}} p_s^{(c)}(dx)p_{s,t}(x,U) + \int_{\mathbb{R}} p_s^{(d)}(dx)p_{s,t}(x,U)
\]

\[
= \int_{\mathbb{R}} \lim_{b \to \infty} p_s^{(c)}(dx | a, b, c)p_{s,t}(x,U) + \int_{\mathbb{R}} \lim_{b \to \infty} p_s^{(d)}(dx | a, b, c)p_{s,t}(x,U)
\]

\[
= \lim_{b \to \infty} \left(\int_{\mathbb{R}} p_t^{(c)}(dx | a, b, c)p_{s,t}(x,U) + \int_{\mathbb{R}} p_t^{(d)}(dx | a, b, c)p_{s,t}(x,U)\right)
\]

\[
= \lim_{b \to \infty} \int_{\mathbb{R}} p_s(dx | a, b, c)p_{s,t}(x,U) \overset{(3.10)}{=} \lim_{b \to \infty} p_t(U | a, b, c) = p_t(U). \]

\[
\square
\]

Formulas (3.9) and (4.4) extend the formulas in [8, Lemma 7.11] to a larger time domain and hence extend the continuous dual Hahn process \((T_s)\) to \(s \in \mathbb{R}\).
**Definition 4.2.** A continuous dual Hahn process \((T_s)_{-\infty<s<\infty}\) with real parameters \(a, c\) such that 
\(a + c > 0\) is a family of Markov transition probabilities \((3.2)\) with the family of \(\sigma\)-finite entrance laws \((4.1)\).

Since \((T_s)_{-\infty<s<\infty}\) is not a "stochastic process" in the usual probabilistic sense, we remark that for any real \(\tau\) and a measurable function \(\varphi > 0\) such that
\[
\mathcal{E} := \int_{\mathbb{R}} \varphi(x) p_\tau(dx) < \infty,
\]
the family of entrance laws \((4.1)\) can be used to construct a Markov process \((X_t)\) on \((-\infty, \tau]\) with probability measures as the marginal laws. Versions of such constructions are well known; we follow [9, Section 10]. It is easy to see that for \(t_0 < t_1 < \cdots < t_n < t_{n+1} = \tau\), the consistent family of the joint probability distributions for \((X_{t_0}, X_{t_1}, \ldots, X_{t_n}, X_\tau)\) \(\in \mathbb{R}^{n+2}\) is
\[
\frac{1}{\mathcal{E}} p_{t_0}(dx_0)p_{t_0, t_1}(x_0, dx_1) \cdots p_{t_{n-1}, t_n}(x_{n-1}, dx_n)p_{t_n, \tau}(x_n, dx_{n+1})\varphi(x_{n+1}).
\]
Thus Kolmogorov’s extension theorem guarantees existence, and Markov property holds with the initial law
\[
P(X_{t_0} = dx) = \frac{1}{\mathcal{E}} h_{t_0}(x)p_{t_0}(dx)
\]
and with transition probabilities
\[
P(X_t = dy|X_s = x) = \frac{1}{h_s(x)} p_{s,t}(x, dy)h_t(y), \quad t_0 \leq s < t \leq \tau, x \in \mathbb{R},
\]
where for \(t < \tau\) we define
\[
h_t(x) := \int_{\mathbb{R}} \varphi(y) p_t(x, dy)
\]
with \(h_t(x) := \varphi(x)\). Note that \(\varphi > 0\) implies \(h_t(x) > 0\) for all \(x \in \mathbb{R}\); in particular, if \(x \notin E_s\), then \(h_t(x) = \varphi(-(c-t)^2)\).

5. Characterization by the conditional means and variances

For a process \((X_t)_{t \geq 0}\) and \(s < u\) consider the two-sided sigma fields \(\mathcal{F}_{s,u}\) generated by \(\{X_r: r \in (0, s] \cup [u, \infty)\}\). We will also use the past sigma-fields \(\mathcal{F}_s\) generated by \(\{X_r: r \in (0, s]\}\).

The following is a characteristic property of the Markov process introduced in Definition 3.1.

**Theorem 5.1.** Let \((X_t)_{t \geq 0}\) be a square-integrable process such that for all \(t, s \geq 0\),
\[
\mathbb{E}[X_t] = 0, \quad \mathbb{E}[X_sX_t] = \min\{t, s\},
\]
Assume that for \(t > 0\), random variable \(X_t\) has infinite support. Then the following conditions are equivalent:
(i) For all \(0 \leq s < t < u\),
\[
\mathbb{E}[X_t|\mathcal{F}_{s,u}] = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u,
\]
and there exist \(\eta > 0, \theta > -2\) such that
\[
\text{Var}[X_t|\mathcal{F}_{s,u}] = \frac{(u-t)(t-s)}{1 + u-s} \left(1 + \eta \frac{uX_u - sX_s}{u-s} + \theta \frac{X_u - X_s}{u-s} + \frac{(X_u - X_s)^2}{(u-s)^2}\right).
\]
(ii) The law of \((X_t)_{t \geq 0}\) is the same as the law of the process

\[
\frac{T_t/2-(a+b)/2}{\sqrt{(a+c)(b+c)}} + \frac{t^2 - 2(a + b + 2c)t + (a - b)^2}{4\sqrt{(a+c)(b+c)}},
\]

where \((T_t)\) is a Markov process from Definition 3.1 with parameters \(a, b, c\) such that \(c \in \mathbb{R}\) and

\[
a + c = \frac{\theta - \sqrt{\theta^2 - 4}}{2\eta}, \quad b + c = \frac{\theta + \sqrt{\theta^2 - 4}}{2\eta}.
\]

We note that the expression in (5.5) should be interpreted as:

\[
\frac{\theta \pm \sqrt{\theta^2 - 4}}{2\eta} = \begin{cases} 
\frac{\theta \pm \sqrt{\eta} \sqrt{2}}{2\eta}, & \theta \geq 2, \\
\frac{\theta \pm \sqrt{\eta} \sqrt{2}}{2\eta}, & -2 < \theta < 2.
\end{cases}
\]

Conditions \(\eta > 0\), \(\theta > -2\) ensure that Assumption 3.1 holds.

Formulas (5.5) are equivalent to

\[
\eta = \frac{1}{\sqrt{(a+c)(b+c)}}, \quad \theta = \frac{2c + a + b}{\sqrt{(a+c)(b+c)}},
\]

compare \([2, (3.5)\) and \((3.6)\)]. In particular, if \(a = \alpha - i\beta\) and \(b = \alpha + i\beta\) with \(\beta \neq 0\) as in Assumption 3.1(ii), then \(\theta = 2(c + a)/[c + a + i\beta]\), so \(|\theta| < 2\). On the other hand, under Assumption 3.1(i), \(2c + a + b \geq 2(a + c) > 0\) so (5.3.1) gives \(\theta > 0\).

In the terminology of [7], Theorem 5.1 says that the following conditions are equivalent:

(i) \((X_t)_{t \geq 0}\) is a quadratic harness in standard form with parameters \(\sigma = 0\), \(\tau = 1\), \(\eta > 0\), \(\theta > -2\) and infinite support for \(X_t\), \(t > 0\).

(ii) \((X_t)_{t \geq 0}\) is a deterministic transformation (5.4) of the Markov process \((T_t)\) with parameters (5.5).

Theorem 5.1 also relates the finite dimensional distributions of Markov processes \((T_t)\) with the same values of sums \(a + c\) and \(b + c\).

**Corollary 5.2.** If \((T_s)_{s \geq -(a+b)/2}\) and \((T_{s'})_{s' \geq -(a'+b')/2}\) are two Markov processes from Definition 3.1 with parameters \(a, b, c\) and \(a', b', c'\) respectively such that \(a + c = a' + c'\) and \(b + c = b' + c'\), then the laws of \((T_s)\) and \((T_{s'})\) differ only by a deterministic shift of time and a deterministic shift of space:

\[
\mathcal{L} \left( \frac{(a'-b')^2}{4} + T_{s}(a'+b')/2 \right)_{s \geq 0} = \mathcal{L} \left( \frac{(a-b)^2}{4} + T_{s-(a+b)/2} \right)_{s \geq 0}.
\]

**Remark 5.3.** By inspecting formulas for the \(\sigma\)-finite entrance law and transition probabilities, we see that the finite-dimensional \((\sigma\text{-finite})\) joint distributions for the process \((T_{s+(c-a)/2})_{s \in \mathbb{R}}\) depend only on \(a + c\).

5.1. **Proof of Theorem 5.1(ii)⇒(i).** For \(-a < s < t < u < c\), this implication was verified in [2, Section 3] by a direct computation with the conditional densities, and formula (5.4) comes from that paper. This is not a feasible approach for the time interval where the conditional laws may be of mixed type, so we will rely on properties of polynomials, see [7] and [5, Section 3]. (Even with this technique, the proof still involves several long calculations that we will omit.)
From explicit expressions for polynomials $p_1(x; t)$ and $p_2(x; t)$, see (3.11), we read out the first two moments:

\[(5.6) \quad E[T_t] = bc + ab + ac + 2ct - t^2, \quad \text{Var}[T_t] = (a + c)(b + c)(a + b + 2t)\]

Thus

\[(5.7) \quad E[T_{t-(a+b)/2}] = \frac{(a + b + 2c)t - t^2 - (a - b)^2}{4}, \quad \text{Var}[T_{t-(a+b)/2}] = 2t(a + c)(b + c).\]

This shows that the first two moments of $X_t$ and of random variable (5.4) are the same, as claimed.

Since $Q_1(y; x, t, s) = 2c(s - t) - s^2 + t^2 - x + y$ we see that $T_t - E(T_t)$ is a martingale. Thus $\text{Cov}[T_s, T_t] = \text{Var}[T_{\min(s,t)}]$, and the covariance of process (5.4) matches the covariance in (5.1).

It remains to confirm that the first two conditional moments match. We begin by removing the quadratic component from the mean, so we will be working with process

\[(5.8) \quad Y_t := T_t + t^2.\]

We have

\[(5.9) \quad E[Y_t] = bc + ab + ac + 2ct, \quad \text{and Var}[Y_t] = (a + c)(b + c)(a + b + 2t) > 0.\]

Note that formula (5.5) implies that $(a + c)(b + c) = 1/q^2 > 0$.

Our goal is to show that $(Y_t)$ is a (general) quadratic harness, i.e., that it has conditional mean

\[(5.10) \quad E[Y_t|Y_s, Y_u] = \frac{u - t}{u - s}Y_s + \frac{t - s}{u - s}Y_u, \quad s < t < u,\]

and conditional variance

\[(5.11) \quad \text{Var}[Y_t|Y_s, Y_u] = \frac{(u - t)(t - s)}{1 + 2(u - s)} \left( 4 \frac{uY_s - sY_u}{u - s} + \frac{(Y_u - Y_s)^2}{(u - s)^2} \right), \quad s < t < u.\]

Since $(Y_t)$ is a Markov process, the formulas for conditional moments (5.10) and (5.11) are of course the same if we condition with respect to the two-tail $\sigma$-field $F_{s,u}$ generated by $(Y_t)$. Once (5.10) and (5.11) are established, routine but cumbersome calculations based on moments (5.9) then verify that the deterministic transformation

\[(5.12) \quad X_t = \frac{Y_{t/2-\tau} - E(Y_{t/2-\tau})}{\sqrt{\text{Var}(Y_{t/2-\tau})}} = \frac{Y_{t/2-\tau} - (ab + ct)}{\sqrt{(a + c)(b + c)}}\]

with $\tau = (a + b)/2$, converts process $(Y_t)$ into a quadratic harness $(X_t)$ in standard form: the moments of $X_t$ become (5.1), formula (5.2) is a direct consequence of (5.10), and a longer calculation verifies that (5.3) follows from (5.11). For the latter, we apply (5.12) to the left hand side of (5.3). From expression

\[4 \frac{uY_s - sY_u}{u - s} + \frac{(Y_u - Y_s)^2}{(u - s)^2}\]

on the right hand side of (5.11), we get

\[4 \frac{(u/2 - \tau)Y_{s/2-\tau} - (s/2 - \tau)Y_{u/2-\tau}}{u/2 - s/2} + 4 \frac{(Y_{u/2-\tau} - Y_{s/2-\tau})^2}{(u - s)^2}.\]

Applying the inverse of transformation (5.12), after a tedious but elementary calculation, we arrive at (5.3), up to a multiplicative constant. (Transformation (5.12) is just a different way of writing (5.4). One can also apply formulas in [2, Proposition 1.1].) We omit the details of this calculation.
To verify (5.10) and (5.11), we use the orthogonal martingale polynomials for the process \((Y_t)\). The three step recursion for these polynomials is recalculated from the recursion for polynomials (3.11). Recall that if monic orthogonal polynomials \(\{p_n\}\) for (the law of) a random variable \(T\) satisfy recursion
\[
x p_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),
\]
then the monic orthogonal polynomials \(\{q_n\}\) for (the law of) \(Y = (T - \mu)/\sigma\) satisfy recursion
\[
x q_n(x) = q_{n+1}(x) + \frac{\beta_n - \mu}{\sigma} q_n(x) + \frac{\gamma_n}{\sigma^2} q_{n-1}(x).
\]

After a calculation, we verify that monic orthogonal martingale polynomials \(q_n(x; t)\) for the process \((Y_t)\) satisfy recursion
\[
x q_n(x; t) = q_{n+1}(x; t) + b_n(t) q_n(x; t) + c_n(t) q_{n-1}(x; t), \quad n \geq 0,
\]
with \(q_{-1}(x; t) = 0, q_0(x; t) = 1\), where the coefficients
\[
b_n(t) = \alpha_n + \beta_n t, \quad c_n(t) = \gamma_n + \delta_n t
\]
are given by
\[
\begin{align*}
\alpha_n &= a b + a c + b c + (2(a + b + c) - 1)n, \\
\beta_n &= 2(c + n), \\
\gamma_n &= n(a + b + n - 1)(a + c + n - 1)(b + c + n - 1), \\
\delta_n &= 2n(a + c + n - 1)(b + c + n - 1).
\end{align*}
\]

To determine the conditional moments we use the following criterion.

**Lemma 5.4.** Fix \(k = 1, 2, \ldots\). Suppose that \(\varphi(x, y)\) is a polynomial such that for all \(n = 0, 1, \ldots\) we have
\[
\mathbb{E}[Y_t^k q_n(Y_u; u)|Y_s] = \mathbb{E}[\varphi(Y_s, Y_u) q_n(Y_u; u)|Y_s].
\]
Then
\[
\mathbb{E}[Y_t^k|Y_s, Y_u] = \varphi(Y_s, Y_u).
\]

**Proof.** By Dynkin’s \(\pi - \lambda\) lemma, it is enough to show that for any pair of bounded measurable functions \(f, \varphi : \mathbb{R} \to [0, \infty)\) we have
\[
\mathbb{E}[f(Y_s) Y_t^k g(Y_u)] = \mathbb{E}[f(Y_s) g(Y_u) \varphi(Y_s, Y_u)].
\]

By our assumption, (5.20) holds if \(g(Y_u)\) is replaced by a polynomial \(p(Y_u)\), a linear combination of the polynomials \(\{q_n(Y_u; u) : n = 0, 1, \ldots\}\).

By Lemma 2.2, the law of \(Y_u\) is determined by moments. Recall that for a probability measure determined by moments, polynomials are dense in \(L_2\), [1, Corollary 2.3.3]. So if \(g\) is an arbitrary bounded measurable function, then for any \(\varepsilon > 0\), there exists a polynomial \(p\) such that \(\mathbb{E}[|g(Y_u) - p(Y_u)|^2] < \varepsilon^2\). Since (5.20) holds for the polynomial \(p\), the difference between the left hand side and the right hand side of (5.20) is arbitrarily small. Indeed,
\[
\begin{align*}
|\mathbb{E}[f(Y_s) Y_t^k g(Y_u)] &- \mathbb{E}[f(Y_s) g(Y_u) \varphi(Y_s, Y_u)]| \\
&\leq |\mathbb{E}[f(Y_s) Y_t^k (g(Y_u) - p(Y_u))]| + |\mathbb{E}[f(Y_s) (g(Y_u) - p(Y_u)) \varphi(Y_s, Y_u)]|. \\
\end{align*}
\]

By the Cauchy-Schwarz inequality, the first term can be bounded by
\[
|\mathbb{E}[f(Y_s) Y_t^k]|^{1/2} |\mathbb{E}[(g(Y_u) - p(Y_u))^2]|^{1/2} \leq (\mathbb{E}[f^2(Y_s) Y_t^{2k}])^{1/2} \varepsilon,
\]

and similarly the second term is at most
\[ \left( \mathbb{E}[f^2(Y_s)\varphi^2(Y_s, Y_u)] \right)^{1/2} \leq \left( \mathbb{E}[g(Y_u) - p(Y_u)]^2 \right)^{1/2} \leq \left( \mathbb{E}[f^2(Y_s)\varphi^2(Y_s, Y_u)] \right)^{1/2} \varepsilon. \]
Since \( \varepsilon > 0 \) is arbitrary, (5.20) follows.

For the subsequent calculations, we write recursion (5.13) in the vector form as
\begin{equation}
(5.21) \quad x\bar{q}(x; t) = \bar{q}(x; t)J(t),
\end{equation}
where \( \bar{q}(x; t) = [q_0(x; t), q_1(x; t), \ldots] \) and \( J(t) \) is the Jacobi matrix
\[
J(t) = \begin{bmatrix}
  b_0(t) & c_1(t) & 0 & 0 & \ldots & 0 & \ldots \\
  1 & b_1(t) & c_2(t) & 0 & \ldots & 0 & \ldots \\
  0 & 1 & b_2(t) & c_3(t) & \ddots & \vdots & \\
  0 & 0 & 1 & b_3(t) & \ddots & \ddots & \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & c_n(t) & \ddots \\
  0 & 0 & 0 & 1 & b_n(t) & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]
with the diagonal entries given by (5.14). From (5.14) we see that the Jacobi matrix depends linearly on \( t \),
\begin{equation}
(5.22) \quad J(t) = Y + tX.
\end{equation}
Ref. [7] indicates that linearity of regression (5.10) (the so called harness property) is related to the fact that \( J \) depends linearly on \( t \), and that quadratic conditional variance (5.11) is related to a quadratic relation between the matrices \( X, Y \). However, process \( (Y_t) \) here has a different covariance, and the emphasis in [7] was on the converse implication, so we shall work out the formulas that are pertinent to our case anew.

**Proof of (5.10).** We use Lemma 5.4 with \( k = 1 \) and \( \varphi(x, y) = \frac{y-x}{u-x} x + \frac{u-y}{u-x} y \). To verify assumption (5.19), we write it in vector form as
\begin{equation}
(5.23) \quad \mathbb{E}[Y_t\bar{q}(Y_u; u)|Y_s] = \mathbb{E} \left( \frac{y-x}{u-x} Y_s + \frac{u-y}{u-x} Y_u \right) \bar{q}(Y_u; u)|Y_s.
\end{equation}
To verify (5.23), we use the vector form of the martingale property, component-wise. That is, we write martingale identity
\[
\mathbb{E}[Y_tq_\alpha(Y_u; u)|Y_s] = \mathbb{E} [Y_t \mathbb{E}[q_\alpha(Y_u; u)|Y_s] |Y_s] = \mathbb{E}[Y_tq_\alpha(Y_u; t)|Y_s]
\]
in the vector form, and combine it with the vector form (5.21) of the three step recursion. We get
\begin{equation}
(5.24) \quad \mathbb{E}[Y_t\bar{q}(Y_u; u)|Y_s] = \mathbb{E}[Y_t\bar{q}(Y_s; t)|Y_s] = \mathbb{E}[\bar{q}(Y_s; t)J(t)|Y_s] = \mathbb{E}[\bar{q}(Y_s; t)J(t)|Y_s]J(t) = \bar{q}(Y_s; s)J(t).
\end{equation}
Similarly, we have
\[
\mathbb{E}[Y_u\bar{q}(Y_u; u)|Y_s] = \bar{q}(Y_s; s)J(u).
\]
Since \( Y_u\bar{q}(Y_s; s) = \bar{q}(Y_s; s)J(s) \), we see that formula (5.23), is a consequence of a simple algebraic identity
\begin{equation}
(5.25) \quad J(t) = \frac{u-t}{u-s} J(s) + \frac{t-s}{u-s} J(u).
\end{equation}
left-multiplied by $\vec{q}(Y_s; s)$. Identity (5.25) holds as $J(t) = tX + Y$ is linear in variable $t$. Hence, by Lemma 5.4 formula (5.10) holds.

Proof of (5.11). This proof is based on a similar plan: we shall deduce (5.11) from Lemma 5.4 with $k = 2$ using an algebraic identity

$$XY - YX = \frac{1}{2}X^2 + 2Y$$

for the two components of the Jacobi matrix (5.22), compare [7, formula (1.1)]. The arguments rely on several cumbersome calculations which we shall only indicate. (We used a computer algebra system to complete several calculations, with (5.26) verified by representing infinite matrices $X, Y$ as (5.29) and (5.30)).

The first step is to rewrite formula (5.11) in expanded form. A calculation shows that (5.11) is equivalent to the following

$$E[Y_t^2|Y_s, Y_u] = \frac{(1 + 2u - 2t)(u-t)}{(1 + 2u - 2s)(u-s)} Y_s^2 + \frac{(1 + 2t - 2s)(t-s)}{(1 + 2u - 2s)(u-s)} Y_u^2 + \frac{4(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} Y_s Y_u + \frac{4u(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} \vec{q}(Y_s; s) - \frac{4s(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} \vec{q}(Y_u; u).$$

So assumption (5.19) in vector form is:

$$E[Y_t^2 q(Y_t; u)|Y_u] = \frac{(1 + 2u - 2t)(u-t)}{(1 + 2u - 2s)(u-s)} Y_s^2 q(Y_s; s) + \frac{(1 + 2u - 2s)(t-s)}{(1 + 2u - 2s)(u-s)} E[Y_s^2 q(Y_u; u)|Y_s] + \frac{4(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} Y_s q(Y_s; s) - \frac{4s(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} E[Y_s q(Y_u; u)|Y_s].$$

Next, we note that by the martingale property for the sequence of polynomials $q(x; t)$, we have

$$E[Y_t^2 q(Y_t; u)|Y_u] = E[Y_t^2 E[q(Y_t; u)|Y_t]|Y_u] = E[Y_t^2 q(Y_t; t)|Y_u] = q(Y_s; s) J^2(t).$$

Similar calculations apply to each of the terms on the right hand side of (5.28). So to deduce (5.27), and hence (5.11) from Lemma 5.4, it is enough to show that the Jacobi matrices satisfy identity

$$J^2(t) = \frac{(1 + 2u - 2t)(u-t)}{(1 + 2u - 2s)(u-s)} J^2(s) + \frac{(1 + 2t - 2s)(t-s)}{(1 + 2u - 2s)(u-s)} J^2(u) + \frac{4(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} J(u) + \frac{4u(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} J(u) - \frac{4s(t-s)(u-t)}{(1 + 2u - 2s)(u-s)} J(u)$$

for all $s < t < u$. After substituting (5.22) into this expression, a lengthy calculation shows that for $s < t < u$, the identity is equivalent to (5.26). (For a similar result of this type, see [7, Proposition 4.1].)

The final step is to prove that (5.26) holds. Here, we use the explicit form of the matrices $X, Y$.

Matrix $X$ is bi-diagonal, with the sequence $(\beta_0, \beta_1, \ldots)$ on the main diagonal, and $(\delta_1, \delta_2, \ldots)$ above the main diagonal. Matrix $Y$ is tri-diagonal with 1’s below the main diagonal, $(\alpha_0, \alpha_1, \ldots)$ on the main diagonal, and $(\gamma_1, \gamma_2, \ldots)$ above the main diagonal. Relation (5.26) becomes a system of recursions for these coefficients. Using the explicit formulas (5.15-5.18), another lengthy calculation
Remark 5.5. A somewhat more conceptual approach to (5.26) is described in [7, Section 4.4]. In this approach, one represents Jacobi matrix as a matrix of an operator on polynomials in variable $z$ in the basis of monomials. The three step recursion (5.13) is then encoded by $J_t = tX + Y$ with (5.29)

$$X = 2(c + z\partial_z) + 2(a + c + z\partial_z)(b + c + z\partial_z)\partial_z,$$

(5.30) $Y = z + (ab + ac + bc) + (2(a + b + c) - 1)z\partial_z + (a + b + z\partial_z)(a + c + z\partial_z)(b + c + z\partial_z)\partial_z,$

where $z^n$ represents the $n$-th orthogonal polynomial. In principle, verification of (5.26) in this form is just a long calculation which uses the product rule $\partial_z(fz) = f(z) + zf'(z)$ to swap the order of operators $\partial_z z = 1 + z\partial_z$ multiple times. The plan here is to rewrite all mixed products of operators on both sides of the identity (5.26) in normal (Wick) order, i.e., to express both sides as linear combinations of the operators $z^n\partial_z^k$, $m, k \geq 0$, and then compare the coefficients. Instead, we used a computer algebra system to verify (5.26) by this technique.

Mathematica code for Remark 5.5. A reader who has attempted a direct verification of (5.26) as described in our sketch of proof might appreciate an implementation in a symbolic computer algebra system. (Some longer outputs are suppressed.)

Define the operators:

$$XX[z] = 2(C[#] + zD[#], z]) + 2(A + C)(B + C)D[#], z] + 2(A + B + 2C)zD[#], [z, 2]] + 2zD[zD[#], [z, 2]], z] &$$

$$YY[z] = (z + AB + AC + BC)\# + 2(A + B + C) - 1)zD[#], z] + 2zD[zD[#], z], z] + (A + B)(A + C)(B + C)D[#], z] + (A + B)(A + C) + (A + B)(B + C) + (A + C)(B + C)zD[#], [z, 2]] + (A + B) + (A + C) + (B + C)zD[zD[#], [z, 2]], z] &$$

Confirm match with (5.22) and (5.13), where $a = A, b = B, c = C$:

Collect[YY[z]z^n + t * XX[z]z^n], {t, z}, FullSimplify]

$n(-1 + A + B + n)(-1 + A + C + n)(-1 + B + C + n)z^{-1 + n} + (-n + 2n(C + n) + B(C + 2n) + A(B + C + 2n))z^{n + 1} + (2n(-1 + A + C + n)(-1 + B + C + n)z^{-1 + n} + 2(C + n)z^n)

Define the left hand side LHS and the right hand side RHS of (5.26), acting on $f(z)$:

LHS = XX[z][YY[z][f[z]]] - YY[z][XX[z][f[z]]]/FullSimplify

RHS = 1/2XX[z][XX[z][f[z]]] + 2YY[z][f[z]]/FullSimplify

Prove (5.26):

LHS – RHS/FullSimplify

5.2. Proof of Theorem 5.1(i)⇒(ii). Process $(X_t)$ is defined on $[0, \infty)$ and satisfies (5.1-5.3), so by [7, Theorem 2.5] it has finite moments of all order. Then [7, Theorem 4.1] implies that process $(X_t)$ has orthogonal martingale polynomials $\{p_n(t; x)\}$, and their three step recursion is determined uniquely by the coefficients in (5.1), (5.2) and (5.3). However, by the first part of the theorem,
the same holds for the process $\tilde{X}$ obtained by transformation (5.4) of the continuous dual Hahn process. So both processes have the same orthogonal martingale polynomials. It remains to show that processes $X$ and $\tilde{X}$ have the same finite dimensional distributions.

For any $t > 0$, both processes have the same orthogonal polynomials and hence the same moments

$$E[X^n_t] = E[\tilde{X}^n_t].$$

In view of Lemma 2.2, this means that both processes have the same univariate laws. In addition, by martingale property of the orthogonal martingale polynomials, for every $n$, one can find a polynomial $\varphi_n$ such that

$$E[X^n_t | F_s] = \varphi_n(X_s) \quad \text{and} \quad E[\tilde{X}^n_t | \tilde{F}_s] = \varphi_n(\tilde{X}_s).$$

(Here $F_s$ and $\tilde{F}_s$ are the past $\sigma$-fields.) By Lemma 2.2, this proves that conditional laws are the same, at every point of the support of $X_s$. Since $\tilde{X}$ is a Markov process, process $X$ is also Markov, with the same transition probabilities. So the finite dimensional distributions for both processes are the same. \qed

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Włodzimierz Bryc, Department of Mathematical Sciences, University of Cincinnati, 2815 Commons Way, Cincinnati, OH, 45221-0025, USA.
Email address: wlodek.bryc@gmail.com