REALIZING SULEIMANOVA SPECTRA VIA PERMUTATIVE MATRICES

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Abstract. A permutative matrix is a square matrix such that every row is a permutation of the first row. A constructive version of a result attributed to Suleimanova is given via permutative matrices. In addition, we strengthen a well-known result by showing that all realizable spectra containing at most four elements can be realized by a permutative matrix or by a direct sum of permutative matrices. We conclude by posing a problem.

Key words. Suleimanova spectrum, permutative matrix, real nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18, 15A29, 15B99.

1. Introduction. Introduced by Suleimanova in [13], the longstanding real nonnegative inverse eigenvalue problem (RNIEP) is to determine necessary and sufficient conditions on a set \( \sigma = \{ \lambda_1, \ldots, \lambda_n \} \subset \mathbb{R} \) so that \( \sigma \) is the spectrum an \( n \)-by-\( n \) entrywise nonnegative matrix.

If \( A \) is an \( n \)-by-\( n \), nonnegative matrix with spectrum \( \sigma \), then \( \sigma \) said to be realizable and the matrix \( A \) is called a realizing matrix for \( \sigma \). It is well-known that if \( \sigma \) is realizable, then

\[
 s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k \geq 0, \quad \forall k \in \mathbb{N} \tag{1.1}
\]

\[
 \rho(\sigma) := \max_{1 \leq i \leq n} |\lambda_i| \in \sigma. \tag{1.2}
\]

For additional background and results, see, e.g., [2, 9] and references therein.

A set \( \sigma = \{ \lambda_1, \ldots, \lambda_n \} \subset \mathbb{R} \) is called a Suleimanova spectrum if \( s_1(\sigma) \geq 0 \) and \( \sigma \) contains exactly one positive element. Suleimanova [13] announced (and loosely proved) that every such spectrum is realizable. Fiedler [3] showed that every Suleimanova spectrum is symmetrically realizable (i.e., realizable by a symmetric nonnegative matrix), however, his proof is by induction and does not explicitly yield a realizing matrix for all orders. In [6], Johnson and Paparella provide a constructive version of Fiedler’s result for Hadamard orders.

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Friedland [4] and Perfect [10] proved Suleǐmanova’s result via companion matrices (for other proofs, see references in [4]). In particular, the coefficients $c_0, c_1, \ldots, c_{n-1}$ of the polynomial $p(t) := \prod_{k=1}^{n}(t - \lambda_k) = t^n + \sum_{k=0}^{n-1} c_k t^k$ are nonpositive so that the companion matrix of $p$ is nonnegative. As noted in [11, p. 1380], the construction of the companion matrix of $p$ requires evaluating the elementary symmetric functions at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a computation with $O(2^n)$ complexity.

The computation of a realizing matrix for a realizable spectrum is of obvious interest for numerical purposes, but for many known theoretical results, a realizing matrix is not readily available. Indeed, according to Chu:

> Very few of these theoretical results are ready for implementation to actually compute [the realizing] matrix. The most constructive result we have seen is the sufficient condition studied by Soules [12]. But the condition there is still limited because the construction depends on the specification of the Perron vector – in particular, the components of the Perron eigenvector need to satisfy certain inequalities in order for the construction to work. [1, p. 18].

In this work, we provide a constructive version of Suleǐmanova’s result via permutative matrices. The paper is organized as follows: Section 2 contains notation and definitions; Section 3 contains the main results; in Section 4 we show that if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ satisfies (1.1) and (1.2), then $\sigma$ is realizable by a permutative matrix or by a direct sum of permutative matrices; and we conclude by posing a problem in Section 5.

2. Notation. The set of $m$-by-$n$ matrices with entries from a field $F$ (in this paper, $F$ is either $\mathbb{C}$ or $\mathbb{R}$) is denoted by $M_{m,n}(F)$ (when $m = n$, $M_{n,n}(F)$ is abbreviated to $M_n(F)$). For $A = [a_{ij}] \in M_n(\mathbb{C})$, $\sigma(A)$ denotes the spectrum of $A$.

The set of $n$-by-$1$ column vectors is identified with the set of all $n$-tuples with entries in $F$ and thus denoted by $F^n$. Given $x \in F^n$, $x_i$ denotes the $i^{th}$ entry of $x$.

For the following, the size of each object will be clear from the context in which it appears:

- $I$ denotes the identity matrix;
- $e$ denotes the all-ones vector; and
- $J$ denotes the all-ones matrix, i.e., $J = ee^\top$.

**Definition 2.1.** For $x \in \mathbb{C}^n$ and permutation matrices $P_2, \ldots, P_n \in M_n(\mathbb{R})$, a
permutative matrix is any matrix of the form
\[
\begin{bmatrix}
    x^\top \\
    (P_2x)^\top \\
    \vdots \\
    (P_nx)^\top
\end{bmatrix} \in M_n(\mathbb{C}).
\]

According to Definition 2.1, all one-by-one matrices are considered permutative.

3. Main Results. We begin with the following lemmas.

Lemma 3.1. For \( x \in \mathbb{C}^n \), let

\[
P = P_x = \begin{bmatrix}
1 & 2 & \cdots & i & \cdots & n \\
1 & x_1 & x_2 & \cdots & x_i & \cdots & x_n \\
2 & x_2 & x_1 & \cdots & x_i & \cdots & x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
i & x_i & x_2 & \cdots & x_1 & \cdots & x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n & x_n & x_2 & \cdots & x_i & \cdots & x_1
\end{bmatrix}
\]

where \( P_{\alpha_i} \) is the permutation matrix corresponding to the permutation \( \alpha_i \) defined by \( \alpha_i(x) = (1i) \), \( i = 2, \ldots, n \). Then \( \sigma(P) = \{s, \delta_2, \ldots, \delta_n\} \), where \( s := \sum_{i=1}^{n} x_i \) and \( \delta_i := x_1 - x_i \), \( i = 2, \ldots, n \).

Proof. Since every row sum of \( P \) is \( s \), it follows that \( Pe = se \), i.e., \( s \in \sigma(P) \).

Since
\[
P - \delta_i I = \begin{bmatrix}
1 & 2 & \cdots & i & \cdots & n \\
1 & x_i & x_2 & \cdots & x_i & \cdots & x_n \\
2 & x_2 & x_i & \cdots & x_i & \cdots & x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
i & x_i & x_2 & \cdots & x_i & \cdots & x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n & x_n & x_2 & \cdots & x_i & \cdots & x_i
\end{bmatrix},
\]

it follows that the homogeneous linear system \( (P - \delta_i I)\hat{x} = 0 \) has a nontrivial solution (notice that the first and \( i \)th rows of \( P - \delta_i I \) are identical). Thus, \( \delta_i \in \sigma(P) \).

\footnote{Terminology due to Charles R. Johnson.}
Moreover, if
\[ v_i := \begin{bmatrix} 1 & x_i \\ \vdots & \vdots \\ i-1 & x_i \\ i & x_1 - s \\ i+1 & x_i \\ \vdots & \vdots \\ n & x_i \end{bmatrix}, \quad i = 2, \ldots, n \]
then
\[ P v_i = \begin{bmatrix} 1 & x_i(s - x_i) + x_i(x_1 - s) \\ \vdots & \vdots \\ i-1 & x_i(s - x_i) + x_i(x_1 - s) \\ i & x_i(s - x_1) + x_i(x_1 - s) \\ i+1 & x_i(s - x_i) + x_i(x_1 - s) \\ \vdots & \vdots \\ n & x_i(s - x_i) + x_i(x_1 - s) \end{bmatrix} \begin{bmatrix} x_i \\ \vdots \\ x_i \end{bmatrix} = \begin{bmatrix} x_1 - x_i \\ x_1 - s \end{bmatrix} = \delta_i v_i, \]
so that \((\delta_i, v_i)\) is a right-eigenpair for \(P\).

**Lemma 3.2.** If
\[ M = M_n := \begin{bmatrix} 1 & e^\top \\ e & -I \end{bmatrix} \in M_n(\mathbb{R}), \ n \geq 2, \]
then
\[ M^{-1} = M_n^{-1} = \frac{1}{n} \begin{bmatrix} 1 & e^\top \\ e & J - nI \end{bmatrix}. \]

**Proof.** Clearly,
\[ nM M^{-1} = \begin{bmatrix} 1 & e^\top \\ e & -I \end{bmatrix} \begin{bmatrix} 1 & e^\top \\ e & J - nI \end{bmatrix} = \begin{bmatrix} n & e^\top + e^\top (J - nI) \\ 0 & nI \end{bmatrix}, \]
but \(e^\top + e^\top (J - nI) = e^\top + (n - 1)e^\top - ne^\top = 0\); dividing through by \(n\) establishes the result.

**Theorem 3.3 (Suleimanova).** Every Suleimanova spectrum is realizable.
Proof. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a Suleimanova spectrum and assume, without loss of generality, that $\lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $\lambda := [\lambda_1 \lambda_2 \cdots \lambda_n]^{\top} \in \mathbb{R}^n$, then, following Lemma 3.2, the solution $x$ of the linear system
\[
\begin{aligned}
x_1 + x_2 + \cdots + x_n &= \lambda_1 \\
x_1 - x_2 &= \lambda_2 \\
&\quad \vdots \\
x_1 - x_n &= \lambda_n
\end{aligned}
\]
is given by
\[
x = M^{-1}\lambda = \frac{1}{n} \begin{bmatrix}
s_1(\sigma) \\
s_1(\sigma) - n\lambda_2 \\
& \vdots \\
s_1(\sigma) - n\lambda_n
\end{bmatrix},
\]
which is clearly nonnegative. Following Lemma 3.1, the nonnegative matrix $P$ realizes $\sigma$. \qed

Example 3.4. If $\sigma = \{10, -1, -2, -3\}$, then $\sigma$ is realizable by
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 3 & 4 \\
3 & 2 & 1 & 4 \\
4 & 2 & 3 & 1
\end{bmatrix}.
\]

Corollary 3.5. If $\sigma = \{\lambda_1, -\lambda_2, \ldots, -\lambda_n\}$ is a Suleimanova spectrum such that $s_1(\sigma) = 0$ and $\lambda_1 > 0$, then the $n$-by-$n$ nonnegative matrix
\[
P := \begin{bmatrix}
0 & \lambda_2 & \cdots & \lambda_i & \cdots & \lambda_n \\
\lambda_2 & 0 & \cdots & \lambda_i & \cdots & \lambda_n \\
& \vdots & \ddots & \vdots & \ddots & \vdots \\
\lambda_i & \lambda_2 & \cdots & 0 & \cdots & \lambda_n \\
& \vdots & \ddots & \vdots & \ddots & \vdots \\
\lambda_n & \lambda_2 & \cdots & \lambda_i & \cdots & 0
\end{bmatrix}
\]
realizes $\sigma$.

Example 3.6. If $\sigma = \{6, -1, -2, -3\}$, then $\sigma$ is realizable by
\[
\begin{bmatrix}
0 & 1 & 2 & 3 \\
1 & 0 & 2 & 3 \\
2 & 1 & 0 & 3 \\
3 & 1 & 2 & 0
\end{bmatrix}.
\]
4. Connection to the RNIEP. It is well-known that for $1 \leq n \leq 4$, conditions (1.1) and (1.2) are also sufficient for realizability (see, e.g., [6, 7]). In this section, we strengthen this result by demonstrating that the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

**Theorem 4.1.** If $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ and $1 \leq n \leq 4$, then $\sigma$ is realizable if and only if $\sigma$ satisfies (1.1) and (1.2). Furthermore, the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

**Proof.** Without loss of generality, assume that $\rho(\sigma) = 1$.

The case when $n = 1$ is trivial, but it is worth mentioning that $\sigma = \{1\}$ is realized by the permutative matrix $[1]$.

If $\sigma = \{1, \lambda\}, -1 \leq \lambda \leq 1$, then the permutative matrix

$$
\begin{pmatrix}
\frac{1}{2} \left[ 1 + \lambda & 1 - \lambda \\
1 - \lambda & 1 + \lambda \\
1 - \lambda & 1 + \lambda \\
\lambda & 1 - \lambda \\
\end{pmatrix}
$$

realizes $\sigma$.

As established in [4], if $\sigma = \{1, \mu, \lambda\}$, where $-1 \leq \mu, \lambda \leq 1$, then the matrix

$$
\begin{pmatrix}
\frac{(1 + \lambda)}{2} & \frac{(1 - \lambda)}{2} & 0 \\
\frac{(1 - \lambda)}{2} & \frac{(1 + \lambda)}{2} & 0 \\
0 & 0 & \mu
\end{pmatrix}
$$

realizes $\sigma$ when $1 \geq \mu \geq \lambda \geq 0$ or $1 \geq \mu \geq 0 > \lambda$. Notice that this matrix is a direct sum of permutative matrices. If $0 > \mu \geq \lambda$, then, following Theorem 3.3, $\sigma$ is realizable by a permutative matrix.

When $n = 4$, all realizable spectra can be realized by matrices of the form

$$
\begin{pmatrix}
a + b & a - b & 0 & 0 \\
a - b & a + b & 0 & 0 \\
0 & 0 & c + d & c - d \\
0 & 0 & c - d & c + d
\end{pmatrix}
$$
or

$$
\begin{pmatrix}
a & b & c \\
b & a & d \\
c & d & a \\
d & c & b
\end{pmatrix}
$$

(for full details, see [5] pp. 10–11). □

5. Concluding Remarks. In [4], Fiedler posed the symmetric nonnegative inverse eigenvalue problem (SNIEP), which requires the realizing matrix to be symmetric. Obviously, if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is a solution to the SNIEP, then it is a solution to the RNIEP. In [5], Johnson, Laffey, and Loewy that showed that the RNIEP strictly contains the SNIEP when $n \geq 5$. It is in the spirit of this problem that we pose the
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following.

Problem 5.1. Can all realizable real spectra be realized by a permutative matrix or by a direct sum of permutative matrices?

At this point there is no evidence that suggests an affirmative answer to Problem 5.1 however, a negative answer could be just as difficult. One possibility to establish a negative answer, communicated to me by R. Loewy, is to find an extreme nonnegative matrix with a real spectrum that cannot be realized by a permutative matrix, or a direct sum of permutative matrices.

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