Notional portfolios and normalized linear returns

Vic Norton
Department of Mathematics and Statistics
Bowling Green State University
mailto:vic@norton.name
http://vic.norton.name

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ABSTRACT

The vector of periodic, compound returns of a typical investment portfolio is almost never a convex combination of the return vectors of the securities in the portfolio. As a result the ex post version of Harry Markowitz’s ‘standard mean-variance portfolio selection model’ does not apply to compound return data. We propose using notional portfolios and normalized linear returns to remedy this problem.
1 The ex post standard model

Let us paraphrase the description of the “standard mean-variance portfolio selection model” ([Markowitz(1987), pp. 3–5]) for ex post return data.

Given an \( m \times n \) matrix \( R = [r_1, \ldots, r_n] \) of successive periodic returns (\( m \) returns for each of \( n \) securities), an investor is to choose the proportions \( p = [p_1, \ldots, p_n]^T \) invested in each security, the proportions being subject to the constraints \( p_j \geq 0 \) (\( j = 1, \ldots, n \)), \( \sum_{j=1}^n p_j = 1 \). We assume that the periodic returns, \( r_P \in \mathbb{R}^m \), of the corresponding investment portfolio satisfy the linear hypothesis

\[
   r_P = \sum_{j=1}^n r_j p_j = Rp. \tag{1}
\]

We also assume that mean or expected return is a linear function of periodic return: that the expected return of security \( k \) is given by \( e_j = \omega^T r_j \) for \( j = 1, \ldots, n \). Here the weight vector \( \omega \in \mathbb{R}^m \) should satisfy \( \omega_i > 0 \) (\( i = 1, \ldots, m \)) and \( \sum_{i=1}^m \omega_i = 1 \). Typically \( \omega_i = 1/m \) for \( i = 1, \ldots, m \): every periodic return of a given security contributes equally to its expected return.

Under these assumptions, the expected return of the investment portfolio is

\[
   e_P = \sum_{j=1}^n e_j p_j = E p, \tag{2}
\]

with \( E = [e_1, \ldots, e_n] = \omega^T R \), and the variance of portfolio return is

\[
   v_P = \sum_{j=1}^n \sum_{k=1}^n v_{jk} p_j p_k = p^T V p, \tag{3}
\]

where the \( n \times n \) covariance matrix \( V = [v_{jk}] \) is given by

\[
   v_{jk} = \sum_{i=1}^m \omega_i z_{ij} z_{ik} \quad (j, k = 1, \ldots, n), \tag{4}
\]

the deviation vectors \( z_j \in \mathbb{R}^m \) (\( j = 1, \ldots, n \)) being defined by

\[
   z_j = r_j - 1_m e_j, \tag{5}
\]

with \( 1_m \in \mathbb{R}^m \) representing the constant return vector of all 1’s.

2 An investment portfolio

We will illustrate the ideas in this paper with computations based on data for five iShares exchange traded funds (ETFs) over the year 2010. The funds are
1. IEF – iShares Barclays 7-10 Year Treasury Bond Fund
2. IWB – iShares Russell 1000 Index Fund
3. IWM – iShares Russell 2000 Index Fund
4. EFA – iShares MSCI EAFE Index Fund
5. EEM – iShares MSCI Emerging Markets Index Fund

Table 1 shows the 2010 performance of a hypothetical investment portfolio in the five funds. The portfolio, PORTF, is static in the sense that no sales or additional purchases were made during the one year period. The increases in the shares of its component funds are entirely due to (automatic) dividend reinvestment.

Table 1: A (static) investment portfolio

| fund | shares | price  | value  | proportion |
|------|--------|--------|--------|------------|
|      |        |        |        |            |
| At the close of Thursday, 2009-12-31 |
| IEF  | 395.03 | 88.60  | 35,000 | 35.00%     |
| IWB  | 652.42 | 61.31  | 40,000 | 40.00%     |
| IWM  | 0.00   | 62.44  | 0      | 0.00%      |
| EFA  | 452.24 | 55.28  | 25,000 | 25.00%     |
| EEM  | 0.00   | 41.50  | 0      | 0.00%      |
| PORTF|        |        | 100,000| 100.00%    |

At the close of Friday, 2010-12-31

| fund | shares | price  | value  | proportion |
|------|--------|--------|--------|------------|
|      |        |        |        |            |
| IEF  | 407.97 | 93.82  | 38,276 | 34.25%     |
| IWB  | 664.51 | 69.86  | 46,422 | 41.54%     |
| IWM  | 0.00   | 78.24  | 0      | 0.00%      |
| EFA  | 464.48 | 58.22  | 27,042 | 24.20%     |
| EEM  | 0.00   | 47.64  | 0      | 0.00%      |
| PORTF|        |        | 111,741| 100.00%    |

Remark. All results in this paper are based on data from the spreadsheet file ‘adj-close5_2010.csv’. Computations were done in double precision arithmetic with results rounded for presentation. Consequently certain numbers (e.g., the sums of the 2010-12-31 values and proportions in Table 1) may be seem to be off by 1 in the last digit.

In the standard model of portfolio selection an investor is to “choose proportions invested in each security.” But which proportions are appropriate for the portfolio PORTF of Table 1—the 2009-12-31-closing proportions, the 2010-12-31-closing proportions, or some set of market-day-closing proportions in between?

To examine this question we need a definition of what we mean by an “investment portfolio.” Say you invest a certain amount of money in n funds, and you let it ride, with all dividends reinvested automatically. At the close of market day one your investment is worth so much. At the close of the market day two it has another, probably different, value. And so it goes, market day after market day, value after value. The growth of your investment portfolio is described by this whole sequence of market-day-closing values indexed by the successive market days of your investment. These market-day-closing values are adjusted closing prices for your portfolio.
3 Adjusted closing prices and notional shares

We refer to any vector of positive numbers \( \mathbf{x} = [x_i] \), indexed by successive market days \( i \), as (a vector of) adjusted closing prices for a given security if the ratio \( x_i / x_{i-1} \) measures the growth in value of an investment in the security from the close of market day \( i-1 \) to the close of day \( i \), assuming dividends are automatically reinvested. Ordinarily \( x_i / x_{i-1} = c_i / c_{i-1} \), where \( c = [c_i] \) is the vector of closing prices for the security. However, on ex-dividend-days \( i \),

\[
x_i / x_{i-1} = \begin{cases} 
  c_i / (c_{i-1} - d) & \text{if the dividend is } d \text{ dollars per share;} \\
  (1 + s) c_i / c_{i-1} & \text{if the dividend is } s \text{ shares per share;} \\
  \tau c_i / c_{i-1} & \text{if shares are split } \tau : 1.
\end{cases} \quad (6)
\]

Clearly, if \( \mathbf{x} \) is a vector of adjusted closing prices for a security, then \( \mathbf{y} = \mathbf{x} \lambda \) is a vector of adjusted closing prices for the same security for any \( \lambda > 0 \). Moreover, all adjusted closing price vectors for the same security and covering the same time period can be expressed in the form \( \mathbf{y} = \mathbf{x} \lambda, \lambda > 0 \).

The spreadsheet file ‘adjclose5_2010.csv’. contains adjusted closing prices for our five sample ETFs over the 253 market days from Thursday, 2009-12-31, through Friday, 2010-12-31. The adjusted prices of each security are normalized at 100 on 2009-12-31. See [Norton(2010)] for a more extensive discussion of adjusted closing prices.

Let \( X = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \) be a matrix of adjusted closing prices for \( n \) securities over a certain time period. Assume the columns of \( X \) are linearly independent (\( \text{rank}(X) = n \)). Then adjusted closing prices \( \mathbf{x}_P \) can be defined for any (static) investment portfolio \( P \) in the \( n \) securities by

\[
\mathbf{x}_P = \sum_{j=1}^{n} \mathbf{x}_j s_j = X \mathbf{s} \quad \text{with} \quad s_j \geq 0 \quad (j = 1, \ldots, n) \quad \text{and} \quad \sum_{j=1}^{n} s_j > 0. \quad (7)
\]

Moreover, due to the independence of the \( \mathbf{x}_j \), the notional shares \( \mathbf{s} = [s_j] \) are uniquely determined by the adjusted closing prices \( \mathbf{x}_P \).

For example, choose a market day \( i_0 \), let \( x_{i_0,j} \) be the closing price of security \( j \) on that day, and let \( s_j \) be the number of shares of security \( j \) held in the portfolio on that day \( (j = 1, \ldots, n) \). Then \( x_{i_0,P} = \sum_{j=1}^{n} x_{i_0,j} s_j \) is the value of the portfolio at the close of day \( i_0 \). Now using (6), with \( d = 0 \) on non-ex-dividend-days \( i \), the \( x_{i_0,j} \) can be extended to adjusted closing prices for all \( n \) securities over the whole time period ([Norton(2010)]). The resulting \( x_{i_P} = \sum_{j=1}^{n} x_{ij} s_j \) are then the market-day-closing values of the investment portfolio \( P \).

This argument shows that a vector of closing values, \( \mathbf{x}_P \), of any investment portfolio \( P \) can be realized by equation (7)—with the specific matrix of adjusted closing prices, \( X \), and notional shares, \( \mathbf{s} \), described in the argument.
Now suppose that
\[ y_P = x_P \lambda_P, \quad Y = X \, \text{diag}(\lambda), \quad \lambda = [\lambda_1, \ldots, \lambda_n]^T, \]
(8)
with \( \lambda_P > 0 \) and \( \lambda_j > 0 \) for \( j = 1, \ldots, n \). Then \( y_P \) is a vector of adjusted closing prices for the portfolio \( P \), \( Y \) is a matrix of adjusted closing prices for the \( n \) securities in \( P \), and all matrices of adjusted closing prices for \( P \) and its \( n \) component securities can be represented this way. Moreover
\[ y_P = \sum_{j=1}^{n} y_j t_j = Y t \quad \text{with} \quad t = \text{diag}(\lambda)^{-1}s \lambda_P \]
(9)
as a consequence of
\[ y_P = x_P \lambda_P = X s \lambda_P = X \, \text{diag}(\lambda) \text{diag}(\lambda)^{-1}s \lambda_P = Y t. \]
Finally, the columns of \( Y \) are linearly independent if and only if the columns of \( X \) are linearly independent (\( \text{rank}(Y) = \text{rank}(X) \)). Thus equation (9) is effectively equivalent to equation (7), and equation (7) is valid in the general circumstances described.

Remark. One can also think of (7–9) as change of coordinate equations, with a fixed investment portfolio \( P \) being represented by different vectors of notional shares in different adjusted-closing-price systems.

4 Market-day-closing portfolios

Suppose an investment portfolio \( P \) is described by equation (7). Multiplying both sides of the equation by a positive constant if necessary, we may assume that \( x_P \) is the vector of market-day-closing values of the portfolio. Since the value of the portfolio at the close of day \( i \) is the sum of the values of its component securities,
\[ x_{iP} = \sum_{j=1}^{n} x_{ij} s_j = x_{iP} \sum_{j=1}^{n} \left( \frac{x_{ij}}{x_{iP}} \right) s_j = x_{iP} \sum_{j=1}^{n} p_{ij}^c \]
(10)
with
\[ p_{ij}^c = \left( \frac{x_{ij}}{x_{iP}} \right) s_j. \]
(11)
Apparently \( p_{ij}^c \) is the proportion of security \( j \) in the portfolio \( P \) at the close of day \( i \). In particular \( p_{ij}^c \geq 0 \) (\( j = 1, \ldots, n \)) and \( \sum_{j=1}^{n} p_{ij}^c = 1 \).

If \( y_P, Y, \) and \( t \) also represent \( P \), as in (8) and (9), then
\[ \left( \frac{y_{ij}}{y_{iP}} \right) t_j = \left( \frac{x_{ij} \lambda_j}{x_{iP} \lambda_P} \right) s_j \left( \frac{\lambda_P}{\lambda_j} \right) = \left( \frac{x_{ij}}{x_{iP}} \right) s_j = p_{ij}^c. \]
Thus the proportions \( p_{ij}^c \) of (10) and (11) are independent of the adjusted closing prices and notional shares used to represent \( P \). We refer to the matrix \( P^c = [p_{ij}^c] \), indexed by market days \( i \) and securities \( j \), as the matrix of *market-day-closing portfolios* for the investment portfolio \( P \).

Table A.1 shows end-of-week market-day-closing portfolios for the portfolio \( \text{PORTF} \) of Table 1 over the last forty weeks of 2010. The weeks are indexed by Fridays, even though Good Friday, 2010-04-02, and observed Christmas Friday, 2010-12-24, were market holidays. All forty of these closing portfolios are different. In fact, the market-day-closing portfolios of \( \text{PORTF} \) over the 253 market days from 2009-12-31 though 2010-12-31 are all distinct. This again begs the question—what proportions \( \mathbf{p} = [p_1, \ldots, p_n]^T \) represent the investment portfolio \( \text{PORTF} \) in the standard portfolio selection model?

5 Normalized adjusted closing prices and notional portfolios

Table B.1 shows weekly adjusted closing prices for the securities and the investment portfolio of Table 1 for the last 40 weeks of 2010. The prices of the securities and the portfolio have been normalized at $100 per notional share on 2009-12-31. As a consequence, the notional shares of \( \text{PORTF} \) with respect to these adjusted prices are the same as its [2009-12-31]-closing proportions:

\[
\text{PORTF} = 35.00\% \times \text{IEF} + 40.00\% \times \text{IWB} + 0\% \times \text{IWM} + 25.00\% \times \text{EFA} + 0\% \times \text{EEM}. \tag{12}
\]

Figure 1 shows the graphs of these adjusted closing prices over the whole of 2010. The [2009-12-31]-normalization is apparent: all graphs start at 100. We have not tried to distinguish \( \text{IWM} \) and \( \text{EEM} \) in this figure since these securities are not components of the investment portfolio \( \text{PORTF} \).

Table B.2 shows weekly adjusted closing prices for the same securities and the same investment portfolio, but the adjusted prices in Table B.2 have been normalized in an entirely different way: the average week-ending adjusted closing price over the last 13 weeks of 2010 is 100—for each security and the portfolio. The notional shares of \( \text{PORTF} \) with respect to this system of adjusted closing prices must also sum to one. In Table B.2

\[
\text{PORTF} = 35.65\% \times \text{IEF} + 40.35\% \times \text{IWB} + 0\% \times \text{IWM} + 24.01\% \times \text{EFA} + 0\% \times \text{EEM}. \tag{13}
\]

Figure 2 shows the graphs of these last-13-week-normalized adjusted closing prices over 2010, but with the graphs of \( \text{IWM} \) and \( \text{EEM} \) omitted. Each graph in Figure 2 has exactly the same shape as the corresponding graph in Figure 1 in the sense that the two adjusted-closing-price functions are positive multiples of one another.
We refer to the adjusted closing prices in Table B.2 and Figure 2 as being “α-normalized,” with \( \alpha \) the market-day-averaging-vector defined by

\[
\alpha_i = \begin{cases} 
\frac{1}{13} & \text{if } i \text{ corresponds to the last market day of one} \\
0 & \text{of the last 13 weeks of 2010;} \\
0 & \text{otherwise.}
\end{cases}
\]  

(14)
The adjusted closing prices $[X, x_P]$ of Table B.2 and Figure 2 are $\alpha$-normalized at 100 in the sense that $\alpha^T[X, x_P] = [100, \ldots, 100]$.

We will say that a vector $\alpha = [\alpha_i]$, indexed by successive market days $i$, is a market-day-averaging-vector if $\alpha_i \geq 0$, for all $i$, and $\sum \alpha_i = 1$. Given a market-day-averaging-vector $\alpha$, any vector of adjusted closing prices $x$ has a unique "$\alpha$-normalized-at-100" counterpart $x^\alpha$—a vector of adjusted closing prices for the same security that satisfies $\alpha^Tx^\alpha = 100$. Clearly

$$x^\alpha = x \cdot \frac{100}{\alpha^T x}. \tag{15}$$

Just as the adjusted closing prices of Table B.2 and Figure 2 are $\alpha$-normalized (at 100) for the $\alpha$ described by (14), the adjusted closing prices of Table B.1 and Figure 1 are $\alpha$-normalized (at 100) for the $\alpha$ that is 1 if $i \sim 2009-12-31$ and 0 at every other market day $i$.

We have “normalized” adjusted closing prices at 100 in this paper because 100 is convenient. For instance, if the adjusted closing price of a security or portfolio starts at 100 (as in Figure 1), then the change in adjusted price at some later time is exactly the percentage gain or loss from the start. But the value 100 is not really essential to our arguments. Any other positive value would work just as well, and we will delete the normalizing qualifier “at 100” in what follows.

Given $\alpha$-normalized adjusted closing prices $X^\alpha = [x_1^\alpha, \ldots, x_n^\alpha]$ and $x_P^\alpha$, for $n$-securities and an investment portfolio $P$ in these securities, and assuming that $\text{rank}(X^\alpha) = n$, the notional shares of $x_P^\alpha$ with respect to $X^\alpha$ are uniquely determined by $\alpha$ and sum to one. We denote the vector of these proportions by $p^\alpha = [p_1^\alpha, \ldots, p_n^\alpha]^T$ and refer to $p^\alpha$ as the $\alpha$-notional portfolio of $P$. Thus

$$x_P^\alpha = \sum_{j=1}^n x_j^\alpha p_j^\alpha = X^\alpha p^\alpha \quad \text{with} \quad p_j^\alpha \geq 0 \ (j = 1, \ldots, n) \quad \text{and} \quad \sum_{j=1}^n p_j^\alpha = 1. \tag{16}$$

This notional portfolio may (12) or may not (13) be the same as any of the market-day-closing portfolios in $P^c$.

If $\beta$ is a second market-day-averaging-vector, then

$$p_j^\beta = \left(\frac{\beta^T x_j^\alpha}{\beta^T x_P^\alpha}\right) p_j^\alpha \quad (j = 1, \ldots, n, P) \tag{17}$$

as a consequence of $x_j^\beta = x_j^\alpha \cdot [100/(\beta^T x_j^\alpha)]$ and (9).

6 Compound returns

Given a vector $x$ of periodic (e.g., weekly) adjusted closing prices for a particular security or portfolio, the percentage change in value, $r_i$, of the security or portfolio, over the period
from $i - 1$ to $i$, is given by

\[ r_i = \frac{x_i}{x_{i-1}} - 1. \]  \hspace{1cm} (18)

On the other hand, if one thinks of the adjusted closing prices as growing according to the exponential model $x_i = x_{i-1} \exp(r_i)$ over the period from $i - 1$ to $i$, then the continuous, periodic rate of growth, $r_i$, is

\[ r_i = \log \left( \frac{x_i}{x_{i-1}} \right). \]  \hspace{1cm} (19)

Note that $r_i(18)$ is just linear part of the series for $r_i(19)$:

\[ r_i(19) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r_i(18)^k \quad \text{for} \quad -1 < r_i(18) < 1. \]

Tables C.1 and C.2 show the weekly and continuously compounded weekly returns corresponding to the adjusted closing prices in Table B.1 or Table B.2. Either adjusted closing price table produces the same weekly return table since ratios of adjusted closing prices, $x_i/x_{i-1}$, depends only on the security and not on the particular adjusted closing prices used.

The returns in Tables C.1 and C.2 are quite comparable except that the weekly compounded returns (C.1) are consistently greater than the continuously compounded returns (C.2). In fact a weekly compounded return can only equal a continuously compounded return when $x_i = x_{i-1}$; then both returns are zero.

We can now settle the question of which vector of security proportions in the standard model corresponds to the investment portfolio PORTF. We simply need to solve

\[ \mathbf{r}_P = \sum_{j=1}^{n} r_j p_j = R \mathbf{p}. \]  \hspace{1cm} (1)

for $\mathbf{p} = [p_1, \ldots, p_n]^T$, using the periodic returns, $R$ and $\mathbf{r}_P$, of Table C.1 or Table C.2. Table 2 shows the results—after the requirement $\sum_{j=1}^{n} p_j = 1$ has been enforced.

**Table 2**: Solution security proportions in PORTF (compound returns)

| Table | IEF  | IWB  | IWM  | EFA  | EEM  | error |
|-------|------|------|------|------|------|-------|
| C.1   | 37.05% | 38.15% | 0.71% | 24.08% | 0.02% | 3.3% |
| C.2   | 37.22% | 38.60% | 0.32% | 23.60% | 0.25% | 2.6% |

The proportions in Table 2 are the least-squares solutions of (1) under the $\sum_{j=1}^{n} p_j = 1$ constraint. No matter that we started by investing exactly 35% of $100,000 in IEF, 40% in IWB, and 25% in EFA, with nothing in IWM or EEM, at the close of Thursday, 2009-12-31, there are no proportions $\mathbf{p}$ that can make the linear hypothesis (1) true with the weekly returns...
of either Table C.1 or Table C.2. With the compound returns of Table C.1, the theoretical portfolio returns, \( R_p \), can just get within 3.3\% of the investment portfolio returns, \( r_P \). With the continuous returns of Table C.2, \( R_p \) can get within 2.6\%. In either case, the least-squares-solution proportions of Table 2 are ridiculous. There is no way you can start out by investing nothing in IWM and EEM and end up with positive positions in both of those securities.

Our conclusion is simple. Forget about the standard portfolio selection model if you plan to use compound returns in the ex post version of the model. The linear hypothesis (1) will almost certainly not hold—no matter how you choose the portfolio proportions \( p \).

### 7 Linear returns

The problem with the periodically compounded periodic returns of the last section,

\[
    r_i = \frac{x_i}{x_{i-1}} - 1 = \frac{x_i - x_{i-1}}{x_{i-1}},
\]

is that the denominator, \( x_{i-1} \), varies from one period to the next. This non-linearity in the definition of the \( r_i \) is incompatible with the linear hypothesis of the standard model.

To remedy this problem one can choose a market-day-averaging-vector \( \alpha \), as defined on page 7, and define periodic returns \( r_i^\alpha \) by

\[
    \alpha\text{-denominated linear return:} \quad r_i^\alpha = \frac{x_i - x_{i-1}}{\alpha^T x}. \quad (20)
\]

Then \( r_i^\alpha \) is the percentage change in value of the security over the \( i-1 \) to \( i \) period relative to its \( \alpha \)-average value. Such \( \alpha \)-denominated or \( \alpha \)-normalized linear returns are compatible with the linear hypothesis of the standard model as we shall see.

Remark. In this section and in the previous one we have used \( i \) to index successive periods, e.g. weeks. However the index in the dot product \( \alpha^T x \) of (20) should not be restricted to periodic values but should be allowed to range over all market days.

Tables D.1 and D.2 show [2009-12-31]-denominated and \( \alpha \)-denominated linear returns corresponding to the adjusted closing prices in Tables B.1 and B.2. In fact the returns in Tables D.1 and D.2 are just the differences of the normalized adjusted closing prices in Tables B.1 and B.2, respectively. Here again \( \alpha \) is the last-13-week market-day-averaging-vector of (14).

Solving

\[
    r_P = \sum_{j=1}^{n} r_j p_j = R p. \quad (1)
\]

for \( p \), with the \( R \) and \( r_P \) of Tables D.2 and D.2, simply reproduces the notional portfolios (12) and (13):
Table 3: Solution security proportions in PORTF (linear returns)

| Table | IEF  | IWB  | IWM  | EFA  | EEM  | error       |
|-------|------|------|------|------|------|-------------|
| D.1   | 35.00% | 40.00% | 0.00% | 25.00% | 0.00% | $5.9 \times 10^{-15}$ |
| D.2   | 35.65% | 40.35% | 0.00% | 24.01% | 0.00% | $7.6 \times 10^{-15}$ |

In general, the adjusted closing price equation
\[ x^α_P = X^α p^α \] (16)

implies
\[ \Delta Θ x^α_P = \Delta Θ X^α p^α, \] (21)

whence (dividing by the normalizing value, 100 in our case)
\[ r^α_P = R^α p^α. \] (1α)

In equation (21), Θ represents the $(m+1) \times (M+1)$ submatrix $[δ_{ik,j}]$ $(k = 0, 1, \ldots, m; \ i = 0, 1, \ldots, M)$ of the $(M+1) \times (M+1)$ identity matrix that picks out $m+1$ successive, periodic market days (rows) from a total of $M+1$ successive market days, and $\Delta$ is the $m \times (m+1)$ difference matrix $[δ_{ik} - δ_{i,k+1}]$ $(i = 1, \ldots, m; \ k = 0, 1, \ldots, m)$. (Here $δ_{ij}$ is the Kronecker delta: $δ_{ij} = 1$ if $i = j$; $δ_{ij} = 0$ otherwise.) Thus we see that $α$-normalized linear returns and their corresponding notional portfolios always satisfy the linear hypothesis (1) of the standard mean-variance portfolio selection model.

8 Some annualized statistics

We will close this paper with a brief discussion of some annualized weekly return statistics for the last 39 weeks of 2010. These statistics are based on the return Tables C.1–D.2 and uniform weighting: $ω_i = 1/39$ $(i = 1, \ldots, 39)$. To annualize expected weekly return or variance of weekly return one simply multiplies by 52. To annualize standard deviation of weekly return multiply by $\sqrt{52}$.

Table 4 shows portfolio proportions $p$, mean returns $e$, standard deviations of return $σ$, and return-risk ratios $e/σ$ for the five exchange traded funds and the investment portfolio PORTF. Portfolio proportions make no sense for the compound returns; we have filled these slots with question marks.

As noted earlier, continuously-compounded returns are “always” less than periodically-compounded returns. The mean compound returns of Table 4 reflect this relationship. Expected-values and standard-deviations of linear returns depend on their normalizations. This is clear in Table 4. However the linear return-risk ratio $e/σ$ is independent of normalization: normalizing factors cancel out, top and bottom. Correlations of linear returns are independent of normalization for the same reason.
We could compute the correlation coefficients \( c_{jk} \) corresponding to the compound returns of Tables C.1 and C.2. Then the compound covariance coefficients would be defined by

\[
v_{jk} = c_{jk} \sigma_j \sigma_k \quad (j, k = 1, \ldots, n) .
\]

But what is the point of computing such or \( c_{jk} \) and \( v_{jk} \) when the \( p_j \) and \( p_k \) in

\[
v_P = \sum_{j=1}^{n} \sum_{k=1}^{n} v_{jk} p_j p_k ,
\]

don’t even exist?

We will close this section and the paper by showing that the \( e/\sigma \) ratios and the correlation coefficients of normalized linear returns are independent of the normalization.

For the \( e/\sigma \) case start with an adjusted-closing-price vector \( x \) and a market-day-averaging-vector \( \alpha \). Let \( e^\alpha \) and \( \sigma^\alpha \) denote the expected value and standard deviation of the periodic return vector \( r^\alpha \). Then

\[
e^\alpha = \omega^T r^\alpha = \omega^T (\Delta \Theta x) \left( \frac{1}{\alpha^T x} \right) ,
\]

and

\[
\sigma^\alpha = \| r^\alpha - 1_m e^\alpha \|_\omega = \| (I_m - 1_m \omega^T) r^\alpha \|_\omega = \| (I_m - 1_m \omega^T) (\Delta \Theta x) \|_\omega \left( \frac{1}{\alpha^T x} \right) .
\]
Table 5: Correlation of linear returns

| fund | IEF  | IWB  | IWM  | EFA  | EEM  |
|------|------|------|------|------|------|
| IEF  | 1.000| −0.470| −0.497| −0.334| −0.296|
| IWB  | −0.470| 1.000| 0.948| 0.911| 0.887|
| IWM  | −0.497| 0.948| 1.000| 0.817| 0.829|
| EFA  | −0.334| 0.911| 0.817| 1.000| 0.904|
| EEM  | −0.296| 0.887| 0.829| 0.904| 1.000|

where \( \|z\|_\omega = \sqrt{\sum_{i=1}^{m} \omega_i z_i^2} \) and \( \Delta \) and \( \Theta \) are the matrices in of (21). The factors \( 1/(\alpha^T x) \) cancel in the \( e/\sigma \) ratio so that

\[
\frac{e}{\sigma} = \frac{e^\alpha}{\sigma^\alpha} = \frac{\omega^T (\Delta \Theta x)}{\|(I_m - 1_m \omega^T)(\Delta \Theta x)\|_\omega}
\]

(23)
does not depend on \( \alpha \) at all. Multiplying \( x \) by a positive number has no effect on the right side of (23); so the \( e/\sigma \) ratio depends only on the security, not on the particular adjusted closing prices that describe its growth. Of course, the \( e/\sigma \) ratio also depends on the weight system \( \omega \).

The "\( \alpha \)" correlation coefficient \( c^\alpha_{jk} \) is defined by

\[
c^\alpha_{jk} = \frac{(z^\alpha_j)^T \text{diag}(\omega) z^\alpha_k}{\|z^\alpha_j\|_\omega \|z^\alpha_k\|_\omega} \quad \text{with} \quad z^\alpha_{\ast} = r^\alpha_{\ast} - 1_m e^\alpha_{\ast}.
\]

Following the pattern of the \( e/\sigma \) demonstration we compute

\[
c_{jk} = c^\alpha_{jk}
\]

\[
= \frac{(\Delta \Theta x_j)^T (I_m - 1_m \omega^T)^T \text{diag}(\omega)(I_m - 1_m \omega^T)(\Delta \Theta x_k)}{\|(I_m - 1_m \omega^T)(\Delta \Theta x_j)\|_\omega \cdot \|(I_m - 1_m \omega^T)(\Delta \Theta x_k)\|_\omega}
\]

\[
= \frac{(\Delta \Theta x_j)^T [\text{diag}(\omega) - \omega \omega^T] (\Delta \Theta x_k)}{\|(I_m - 1_m \omega^T)(\Delta \Theta x_j)\|_\omega \cdot \|(I_m - 1_m \omega^T)(\Delta \Theta x_k)\|_\omega},
\]

(24)

with the normalizing factors, \( \alpha^T x_j \) and \( \alpha^T x_k \), canceling out. Since the expression (24) is unchanged when \( x_j \) and \( x_k \) are multiplied by positive numbers, \( c_{jk} \) is independent of the particular adjusted closing prices representing the \( j \) and \( k \) securities. \( \square \)
A Closing portfolios

Table A.1: Week-closing portfolios for the last 40 weeks of 2010

| Friday       | IEF | IWB | IWM | EFA | EEM | PORTF |
|--------------|-----|-----|-----|-----|-----|-------|
| 2010-04-02*  | 34.18% | 41.01% | 0% | 24.81% | 0% | 100% |
| 2010-04-09   | 33.94% | 41.35% | 0% | 24.71% | 0% | 100% |
| 2010-04-16   | 34.23% | 41.28% | 0% | 24.50% | 0% | 100% |
| 2010-04-23   | 33.88% | 41.89% | 0% | 24.23% | 0% | 100% |
| 2010-04-30   | 34.81% | 41.49% | 0% | 23.69% | 0% | 100% |
| 2010-05-07   | 37.05% | 40.51% | 0% | 22.44% | 0% | 100% |
| 2010-05-14   | 36.54% | 40.99% | 0% | 22.47% | 0% | 100% |
| 2010-05-21   | 37.82% | 39.92% | 0% | 22.26% | 0% | 100% |
| 2010-05-28   | 37.70% | 40.17% | 0% | 22.13% | 0% | 100% |
| 2010-06-04   | 38.49% | 39.76% | 0% | 21.75% | 0% | 100% |
| 2010-06-11   | 37.72% | 40.00% | 0% | 22.28% | 0% | 100% |
| 2010-06-18   | 37.12% | 40.22% | 0% | 22.66% | 0% | 100% |
| 2010-06-25   | 38.03% | 39.39% | 0% | 22.58% | 0% | 100% |
| 2010-07-02   | 39.44% | 38.22% | 0% | 22.34% | 0% | 100% |
| 2010-07-09   | 37.95% | 39.07% | 0% | 22.98% | 0% | 100% |
| 2010-07-16   | 38.52% | 38.66% | 0% | 22.81% | 0% | 100% |
| 2010-07-23   | 37.55% | 39.22% | 0% | 23.23% | 0% | 100% |
| 2010-07-30   | 37.71% | 39.02% | 0% | 23.27% | 0% | 100% |
| 2010-08-06   | 37.34% | 39.00% | 0% | 23.66% | 0% | 100% |
| 2010-08-13   | 38.71% | 38.44% | 0% | 22.86% | 0% | 100% |
| 2010-08-20   | 38.91% | 38.35% | 0% | 22.74% | 0% | 100% |
| 2010-08-27   | 38.91% | 38.20% | 0% | 22.89% | 0% | 100% |
| 2010-09-03   | 37.89% | 38.83% | 0% | 23.28% | 0% | 100% |
| 2010-09-10   | 37.64% | 38.96% | 0% | 23.40% | 0% | 100% |
| 2010-09-17   | 37.41% | 39.10% | 0% | 23.49% | 0% | 100% |
| 2010-09-24   | 37.07% | 39.14% | 0% | 23.79% | 0% | 100% |
| 2010-10-01   | 37.24% | 39.02% | 0% | 23.75% | 0% | 100% |
| 2010-10-08   | 37.05% | 39.00% | 0% | 23.95% | 0% | 100% |
| 2010-10-15   | 36.52% | 39.31% | 0% | 24.17% | 0% | 100% |
| 2010-10-22   | 36.53% | 39.45% | 0% | 24.02% | 0% | 100% |
| 2010-10-29   | 36.49% | 39.57% | 0% | 23.93% | 0% | 100% |
| 2010-11-05   | 35.86% | 39.92% | 0% | 24.21% | 0% | 100% |
| 2010-11-12   | 36.09% | 39.94% | 0% | 23.97% | 0% | 100% |
| 2010-11-19   | 35.82% | 40.03% | 0% | 24.14% | 0% | 100% |
| 2010-11-26   | 36.34% | 40.21% | 0% | 23.45% | 0% | 100% |
| 2010-12-03   | 35.29% | 40.81% | 0% | 23.89% | 0% | 100% |
| 2010-12-10   | 34.51% | 41.43% | 0% | 24.06% | 0% | 100% |
| 2010-12-17   | 34.52% | 41.51% | 0% | 23.97% | 0% | 100% |
| 2010-12-24*  | 34.18% | 41.71% | 0% | 24.12% | 0% | 100% |
| 2010-12-31   | 34.25% | 41.54% | 0% | 24.20% | 0% | 100% |

*Friday market-holiday = Thursday closing portfolio
## B Adjusted closing prices

Table B.1: [2009-12-31]-normalized adjusted closing prices for the last 40 weeks of 2010

\[
\text{PORTF} = 35.00\% \times \text{EIF} + 40.00\% \times \text{IWB} + 0\% \times \text{IWM} + 25.00\% \times \text{EFA} + 0\% \times \text{EEM}
\]

| Friday       | IEF  | IWB  | IWM  | EFA  | EEM  | PORTF   |
|--------------|------|------|------|------|------|---------|
| 2010-04-02*  | 101.41| 106.448 | 109.866 | 103.057 | 104.145 | 103.838 |
| 2010-04-09   | 101.424 | 108.118 | 112.788 | 103.383 | 105.494 | 104.591 |
| 2010-04-16   | 102.403 | 108.053 | 114.731 | 102.605 | 102.313 | 104.714 |
| 2010-04-23   | 102.005 | 110.345 | 119.017 | 102.135 | 103.759 | 105.374 |
| 2010-04-30   | 103.303 | 107.725 | 115.036 | 98.426  | 101.325 | 103.853 |
| 2010-05-07   | 104.986 | 100.440 | 104.937 | 89.020  | 92.024  | 99.176  |
| 2010-05-14   | 105.008 | 103.076 | 111.680 | 90.431  | 95.157  | 100.591 |
| 2010-05-21   | 106.722 | 98.590 | 104.471 | 87.952  | 89.976  | 98.777  |
| 2010-05-28   | 106.333 | 99.146 | 106.366 | 87.410  | 91.807  | 98.727  |
| 2010-06-04   | 107.181 | 96.871 | 102.047 | 84.805  | 89.639  | 97.463  |
| 2010-06-11   | 106.974 | 99.277 | 104.263 | 88.477  | 93.398  | 99.271  |
| 2010-06-18   | 107.135 | 101.586 | 107.249 | 91.552  | 96.193  | 101.020 |
| 2010-06-25   | 108.177 | 98.033 | 103.765 | 89.914  | 95.640  | 99.554  |
| 2010-07-02   | 109.331 | 92.703 | 96.359  | 86.674  | 91.565  | 97.016  |
| 2010-07-09   | 108.700 | 97.920 | 101.385 | 92.141  | 96.950  | 100.248 |
| 2010-07-16   | 110.078 | 96.669 | 98.372  | 91.257  | 93.748  | 100.009 |
| 2010-07-23   | 109.641 | 100.208 | 104.671 | 94.957  | 99.812  | 102.197 |
| 2010-07-30   | 110.571 | 100.109 | 104.735 | 95.546  | 100.419 | 102.630 |
| 2010-08-06   | 111.542 | 101.936 | 104.928 | 98.933  | 102.068 | 104.547 |
| 2010-08-13   | 112.912 | 98.101 | 98.372  | 93.337  | 98.672  | 102.094 |
| 2010-08-20   | 113.177 | 97.608 | 98.501  | 92.601  | 99.497  | 101.805 |
| 2010-08-27   | 112.912 | 96.982 | 99.307  | 92.987  | 98.211  | 101.559 |
| 2010-09-03   | 112.340 | 100.734 | 103.624 | 96.613  | 101.947 | 103.766 |
| 2010-09-10   | 111.705 | 101.162 | 102.641 | 97.202  | 102.505 | 103.862 |
| 2010-09-17   | 112.271 | 102.693 | 105.041 | 98.712  | 104.348 | 105.050 |
| 2010-09-24   | 113.437 | 104.798 | 108.180 | 101.933 | 107.016 | 107.105 |
| 2010-10-01   | 114.341 | 104.831 | 109.585 | 102.098 | 110.194 | 107.476 |
| 2010-10-08   | 115.672 | 106.551 | 111.878 | 104.675 | 112.134 | 109.274 |
| 2010-10-15   | 114.295 | 107.658 | 113.509 | 105.890 | 113.323 | 109.539 |
| 2010-10-22   | 114.561 | 108.237 | 113.558 | 105.448 | 111.649 | 109.753 |
| 2010-10-29   | 114.272 | 108.435 | 113.525 | 104.933 | 111.867 | 109.602 |
| 2010-11-05   | 115.274 | 112.287 | 119.129 | 108.964 | 117.616 | 112.502 |
| 2010-11-12   | 113.686 | 110.072 | 116.319 | 105.724 | 112.571 | 110.250 |
| 2010-11-19   | 112.700 | 110.204 | 116.997 | 106.332 | 112.813 | 110.110 |
| 2010-11-26   | 112.804 | 109.196 | 118.241 | 101.914 | 108.666 | 108.638 |
| 2010-12-03   | 111.489 | 112.800 | 122.197 | 105.669 | 114.341 | 110.558 |
| 2010-12-10   | 108.687 | 114.189 | 125.556 | 106.074 | 113.007 | 110.235 |
| 2010-12-17   | 109.012 | 114.701 | 125.992 | 105.964 | 112.546 | 110.526 |
| 2010-12-24*  | 108.547 | 115.906 | 127.698 | 107.240 | 113.942 | 111.164 |
| 2010-12-31   | 109.360 | 116.056 | 126.919 | 108.169 | 116.523 | 111.741 |

*Friday market-holiday = Thursday adjusted closing prices
Table B.2: \( \alpha \)-normalized adjusted closing prices for the last 40 weeks of 2010

\[
\text{PORTF} = 35.65\% \times \text{IEF} + 40.35\% \times \text{IWB} + 0\% \times \text{IWM} + 24.01\% \times \text{EFA} + 0\% \times \text{EEM}
\]

| Friday         | IEF  | IWB  | IWM  | EFA  | EEM  | PORTF |
|----------------|------|------|------|------|------|-------|
| 2010-04-02*    | 90.277 | 95.681 | 92.056 | 97.294 | 92.039 | 94.142 |
| 2010-04-09     | 90.287 | 97.182 | 94.504 | 97.602 | 93.231 | 94.825 |
| 2010-04-16     | 91.158 | 97.123 | 96.132 | 96.868 | 90.419 | 94.936 |
| 2010-04-23     | 90.804 | 99.184 | 99.723 | 96.424 | 91.697 | 95.534 |
| 2010-04-30     | 91.960 | 96.829 | 96.387 | 92.922 | 89.546 | 94.155 |
| 2010-05-07     | 93.458 | 90.281 | 87.926 | 84.042 | 81.327 | 89.915 |
| 2010-05-14     | 93.477 | 92.650 | 93.575 | 85.374 | 84.095 | 91.198 |
| 2010-05-21     | 95.003 | 88.618 | 87.535 | 83.034 | 79.517 | 89.553 |
| 2010-05-28     | 94.657 | 89.117 | 89.123 | 82.522 | 81.135 | 89.599 |
| 2010-06-04     | 95.412 | 87.073 | 85.504 | 80.063 | 79.219 | 88.362 |
| 2010-06-11     | 95.227 | 89.235 | 87.361 | 83.530 | 82.541 | 90.001 |
| 2010-06-18     | 95.371 | 91.311 | 89.863 | 86.433 | 85.011 | 91.587 |
| 2010-06-25     | 96.298 | 88.117 | 86.944 | 84.886 | 84.522 | 90.258 |
| 2010-07-02     | 97.326 | 83.326 | 80.738 | 81.828 | 80.921 | 87.957 |
| 2010-07-09     | 96.764 | 88.015 | 84.949 | 86.989 | 85.680 | 90.887 |
| 2010-07-16     | 97.991 | 86.891 | 82.425 | 86.154 | 82.850 | 90.671 |
| 2010-07-23     | 97.602 | 90.072 | 87.703 | 89.647 | 88.209 | 92.654 |
| 2010-07-30     | 98.429 | 89.983 | 87.756 | 90.203 | 88.746 | 93.047 |
| 2010-08-06     | 99.294 | 91.625 | 87.918 | 93.401 | 90.203 | 94.785 |
| 2010-08-13     | 100.513 | 88.178 | 82.425 | 88.118 | 87.202 | 92.561 |
| 2010-08-20     | 100.749 | 87.735 | 82.533 | 87.423 | 87.931 | 92.299 |
| 2010-08-27     | 100.513 | 87.172 | 83.208 | 87.788 | 86.794 | 92.076 |
| 2010-09-03     | 100.004 | 90.545 | 86.825 | 91.211 | 90.096 | 94.077 |
| 2010-09-10     | 99.439 | 90.929 | 86.002 | 91.767 | 90.589 | 94.164 |
| 2010-09-17     | 99.943 | 92.306 | 88.013 | 93.192 | 92.218 | 95.241 |
| 2010-09-24     | 100.981 | 94.198 | 90.643 | 96.233 | 94.576 | 97.104 |
| 2010-10-01     | 101.785 | 94.227 | 91.820 | 96.389 | 97.384 | 97.441 |
| 2010-10-08     | 102.970 | 95.773 | 93.741 | 98.822 | 99.099 | 99.071 |
| 2010-10-15     | 101.745 | 96.768 | 95.108 | 99.969 | 100.150 | 99.311 |
| 2010-10-22     | 101.981 | 97.289 | 95.149 | 99.552 | 98.670 | 99.505 |
| 2010-10-29     | 101.724 | 97.467 | 95.121 | 99.066 | 98.863 | 99.368 |
| 2010-11-05     | 102.616 | 100.929 | 99.817 | 102.871 | 103.944 | 101.997 |
| 2010-11-12     | 101.202 | 98.938 | 97.462 | 99.812 | 99.485 | 99.955 |
| 2010-11-19     | 100.325 | 99.057 | 98.031 | 100.386 | 99.699 | 99.828 |
| 2010-11-26     | 100.417 | 98.151 | 99.073 | 96.215 | 96.034 | 98.494 |
| 2010-12-03     | 99.247 | 101.390 | 102.388 | 99.760 | 101.049 | 100.235 |
| 2010-12-10     | 96.752 | 102.639 | 105.202 | 100.143 | 99.870 | 99.941 |
| 2010-12-17     | 97.042 | 103.099 | 105.567 | 100.039 | 99.463 | 100.205 |
| 2010-12-24*    | 96.628 | 104.182 | 106.997 | 101.244 | 100.697 | 100.784 |
| 2010-12-31     | 97.351 | 104.317 | 106.344 | 102.121 | 102.978 | 101.307 |

*Friday market-holiday = Thursday adjusted closing prices
## C Compound returns

Table C.1: Compound weekly returns (%) for the last 39 weeks of 2010.

| Friday       | IEF | IWB | IWM | EFA | EEM | PORTF |
|--------------|-----|-----|-----|-----|-----|-------|
| 2010-04-09   | 0.011 | 1.569 | 2.660 | 0.316 | 1.295 | 0.726 |
| 2010-04-16   | 0.965 | -0.060 | 1.723 | -0.753 | -3.015 | 0.117 |
| 2010-04-23   | -0.389 | 2.121 | 3.736 | -0.458 | 1.413 | 0.630 |
| 2010-04-30   | 1.272 | -2.374 | -3.345 | -3.631 | -2.346 | -1.443 |
| 2010-05-07   | 1.629 | -6.763 | -8.779 | -9.556 | -9.179 | -4.503 |
| 2010-05-14   | 0.021 | 2.624 | 6.426 | 1.585 | 3.405 | 1.427 |
| 2010-05-21   | 1.632 | -4.352 | -6.455 | -2.741 | -5.445 | -1.804 |
| 2010-05-28   | -0.364 | 0.564 | 1.814 | -0.616 | 2.035 | -0.050 |
| 2010-06-04   | 0.797 | -2.295 | -4.061 | -2.980 | -2.361 | -1.281 |
| 2010-06-11   | -0.193 | 2.484 | 2.172 | 4.330 | 4.193 | 1.855 |
| 2010-06-18   | 0.151 | 2.326 | 2.864 | 3.475 | 2.993 | 1.762 |
| 2010-06-25   | 0.973 | -3.498 | -3.249 | -1.789 | -0.575 | -1.451 |
| 2010-07-02   | 1.067 | -5.437 | -7.137 | -3.603 | -4.261 | -2.549 |
| 2010-07-09   | -0.577 | 5.628 | 5.216 | 6.308 | 5.881 | 3.332 |
| 2010-07-16   | 1.268 | -1.278 | -2.972 | -0.959 | -3.303 | -0.239 |
| 2010-07-23   | -0.397 | 3.661 | 6.403 | 4.054 | 6.468 | 2.187 |
| 2010-07-30   | 0.848 | -0.099 | 0.061 | 0.620 | 0.608 | 0.424 |
| 2010-08-06   | 0.878 | 1.825 | 0.184 | 3.545 | 1.642 | 1.868 |
| 2010-08-13   | 1.228 | -3.762 | -6.248 | -5.656 | -3.327 | -2.347 |
| 2010-08-20   | 0.235 | -0.503 | 0.131 | -0.789 | 0.836 | -0.283 |
| 2010-08-27   | -0.234 | -0.641 | 0.818 | 0.417 | -1.293 | -0.242 |
| 2010-09-03   | -0.507 | 3.869 | 4.347 | 3.899 | 3.804 | 2.173 |
| 2010-09-10   | -0.565 | 0.425 | -0.949 | 0.610 | 0.547 | 0.093 |
| 2010-09-17   | 0.507 | 1.513 | 2.338 | 1.553 | 1.798 | 1.144 |
| 2010-09-24   | 1.039 | 2.050 | 2.988 | 3.263 | 2.557 | 1.957 |
| 2010-10-01   | 0.797 | 0.031 | 1.299 | 0.162 | 2.970 | 0.346 |
| 2010-10-08   | 1.164 | 1.641 | 2.092 | 2.524 | 1.761 | 1.673 |
| 2010-10-15   | -1.190 | 1.039 | 1.458 | 1.161 | 1.060 | 0.242 |
| 2010-10-22   | 0.233 | 0.538 | 0.043 | -0.417 | -1.477 | 0.196 |
| 2010-10-29   | -0.252 | 0.183 | -0.029 | -0.488 | 0.195 | -0.137 |
| 2010-11-05   | 0.877 | 3.552 | 4.936 | 3.841 | 5.139 | 2.645 |
| 2010-11-12   | -1.378 | -1.973 | -2.359 | -2.973 | -4.280 | -2.002 |
| 2010-11-19   | -0.867 | 0.120 | 0.583 | 0.575 | 0.215 | -0.127 |
| 2010-11-26   | 0.092 | -0.915 | 1.063 | -4.155 | -3.676 | -1.336 |
| 2010-12-03   | -1.166 | 3.300 | 3.346 | 3.684 | 5.222 | 1.767 |
| 2010-12-10   | -2.513 | 1.231 | 2.749 | 0.383 | -1.167 | -0.293 |
| 2010-12-17   | 0.299 | 0.448 | 0.347 | -0.104 | -0.408 | 0.264 |
| 2010-12-24   | -0.427 | 1.051 | 1.354 | 1.204 | 1.240 | 0.577 |
| 2010-12-31   | 0.749 | 0.129 | -0.610 | 0.866 | 2.265 | 0.519 |
Table C.2: Continuous weekly returns (%) for the last 39 weeks of 2010.

| Friday     | IEF | IWB  | IWM  | EFA  | EEM  | PORTF |
|------------|-----|------|------|------|------|-------|
| 2010-04-09 | 0.011 | 1.557 | 2.625 | 0.316 | 1.287 | 0.723 |
| 2010-04-16 | 0.961 | −0.060 | 1.708 | −0.755 | −3.062 | 0.117 |
| 2010-04-23 | −0.389 | 2.099 | 3.668 | −0.459 | 1.403 | 0.628 |
| 2010-04-30 | 1.264 | −2.403 | −3.402 | −3.699 | −2.374 | −1.454 |
| 2010-05-07 | 1.616 | −7.002 | −9.188 | −10.044 | −9.628 | −4.608 |
| 2010-05-14 | 0.021 | 2.591 | 6.228 | 1.573 | 3.348 | 1.417 |
| 2010-05-21 | 1.619 | −4.450 | −6.673 | −2.780 | −5.599 | −1.820 |
| 2010-05-28 | −0.365 | 0.562 | 1.798 | −0.618 | 2.015 | −0.050 |
| 2010-06-04 | 0.794 | −2.321 | −4.145 | −3.026 | −2.390 | −1.289 |
| 2010-06-11 | −0.193 | 2.453 | 2.148 | 4.239 | 4.108 | 1.838 |
| 2010-06-18 | 0.150 | 2.299 | 2.824 | 3.416 | 2.949 | 1.746 |
| 2010-06-25 | 0.968 | −3.560 | −3.302 | −1.805 | −0.577 | −1.462 |
| 2010-07-02 | 1.061 | −5.590 | −7.405 | −3.670 | −4.354 | −2.583 |
| 2010-07-09 | −0.579 | 5.475 | 5.084 | 6.117 | 5.715 | 3.278 |
| 2010-07-16 | 1.260 | −1.286 | −3.017 | −0.964 | −3.359 | −0.239 |
| 2010-07-23 | −0.398 | 3.596 | 6.207 | 3.974 | 6.268 | 2.164 |
| 2010-07-30 | 0.845 | −0.099 | 0.061 | 0.618 | 0.606 | 0.423 |
| 2010-08-06 | 0.874 | 1.809 | 0.184 | 3.484 | 1.629 | 1.851 |
| 2010-08-13 | 1.221 | −3.835 | −6.452 | −5.823 | −3.384 | −2.375 |
| 2010-08-20 | 0.234 | −0.504 | 0.131 | −0.792 | 0.833 | −0.283 |
| 2010-08-27 | −0.234 | −0.643 | 0.815 | 0.416 | −1.301 | −0.243 |
| 2010-09-03 | −0.508 | 3.796 | 4.255 | 3.825 | 3.733 | 2.150 |
| 2010-09-10 | −0.567 | 0.424 | −0.953 | 0.608 | 0.546 | 0.093 |
| 2010-09-17 | 0.505 | 1.502 | 2.311 | 1.542 | 1.782 | 1.137 |
| 2010-09-24 | 1.033 | 2.029 | 2.945 | 3.211 | 2.525 | 1.938 |
| 2010-10-01 | 0.794 | 0.031 | 1.290 | 0.162 | 2.926 | 0.346 |
| 2010-10-08 | 1.157 | 1.627 | 2.071 | 2.493 | 1.745 | 1.659 |
| 2010-10-15 | −1.198 | 1.034 | 1.447 | 1.154 | 1.055 | 0.242 |
| 2010-10-22 | 0.232 | 0.536 | 0.043 | −0.418 | −1.488 | 0.195 |
| 2010-10-29 | −0.253 | 0.183 | −0.029 | −0.490 | 0.195 | −0.137 |
| 2010-11-05 | 0.873 | 3.491 | 4.818 | 3.770 | 5.011 | 2.611 |
| 2010-11-12 | −1.387 | −1.992 | −2.387 | −3.019 | −4.384 | −2.022 |
| 2010-11-19 | −0.871 | 0.120 | 0.581 | 0.573 | 0.215 | −0.127 |
| 2010-11-26 | 0.092 | −0.919 | 1.058 | −4.244 | −3.745 | −1.345 |
| 2010-12-03 | −1.173 | 3.247 | 3.291 | 3.618 | 5.091 | 1.752 |
| 2010-12-10 | −2.545 | 1.224 | 2.712 | 0.383 | −1.174 | −0.293 |
| 2010-12-17 | 0.299 | 0.447 | 0.347 | −0.104 | −0.409 | 0.264 |
| 2010-12-24 | −0.427 | 1.045 | 1.345 | 1.197 | 1.233 | 0.576 |
| 2010-12-31 | 0.746 | 0.129 | −0.612 | 0.863 | 2.240 | 0.518 |
## D Linear returns

Table D.1: [2009-12-31]-denominated weekly returns (%) for the last 39 weeks of 2010.

| Friday         | IEF  | IWB  | IWM  | EFA  | EEM  | PORTF |
|----------------|------|------|------|------|------|-------|
| 2010-04-09     | 0.011| 1.670| 2.922| 0.326| 1.349| 0.753 |
| 2010-04-16     | 0.979| −0.065| 1.943| −0.778| −3.181| 0.122 |
| 2010-04-23     | −0.398| 2.292| 4.286| −0.470| 1.446| 0.660 |
| 2010-04-30     | 1.298| −2.620| −3.981| −3.709| −2.434| −1.521|
| 2010-05-07     | 1.683| −7.285| −10.099| −9.406| −9.301| −4.676|
| 2010-05-14     | 0.022| 2.636| 6.743| 1.411| 3.133| 1.415 |
| 2010-05-21     | 1.714| −4.486| −7.209| −2.479| −5.181| −1.814|
| 2010-05-28     | −0.389| 0.556| 1.895| −0.542| 1.831| −0.049|
| 2010-06-04     | 0.848| −2.275| −4.319| −2.605| −2.168| −1.264|
| 2010-06-11     | −0.207| 2.406| 2.216| 3.672| 3.759| 1.808 |
| 2010-06-18     | 0.161| 2.309| 2.986| 3.075| 2.795| 1.749 |
| 2010-06-25     | 1.042| −3.553| −3.484| −1.638| −0.553| −1.466|
| 2010-07-02     | 1.154| −5.330| −7.406| −3.240| −4.075| −2.538|
| 2010-07-09     | −0.631| 5.217| 5.026| 5.467| 5.385| 3.233 |
| 2010-07-16     | 1.378| −1.251| −3.013| −0.884| −3.202| −0.239|
| 2010-07-23     | −0.437| 3.539| 6.299| 3.700| 6.064| 2.188 |
| 2010-07-30     | 0.930| −0.099| 0.064| 0.589| 0.607| 0.433 |
| 2010-08-06     | 0.971| 1.827| 0.193| 3.387| 1.649| 1.917 |
| 2010-08-13     | 1.370| −3.835| −6.556| −5.596| −3.396| −2.453|
| 2010-08-20     | 0.265| −0.493| 0.129| −0.736| 0.825| −0.288|
| 2010-08-27     | −0.265| −0.626| 0.806| 0.386| −1.286| −0.247|
| 2010-09-03     | −0.572| 3.752| 4.317| 3.626| 3.736| 2.207 |
| 2010-09-10     | −0.635| 0.428| −0.983| 0.589| 0.558| 0.096 |
| 2010-09-17     | 0.566| 1.531| 2.400| 1.510| 1.843| 1.188 |
| 2010-09-24     | 1.166| 2.105| 3.139| 3.221| 2.668| 2.055 |
| 2010-10-01     | 0.904| 0.033| 1.405| 0.165| 3.178| 0.371 |
| 2010-10-08     | 1.331| 1.720| 2.293| 2.577| 1.940| 1.798 |
| 2010-10-15     | −1.377| 1.107| 1.631| 1.215| 1.189| 0.265 |
| 2010-10-22     | 0.266| 0.579| 0.049| −0.442| −1.674| 0.214|
| 2010-10-29     | −0.289| 0.198| −0.033| −0.515| 0.218| −0.151|
| 2010-11-05     | 1.002| 3.852| 5.604| 4.031| 5.749| 2.899 |
| 2010-11-12     | −1.588| −2.215| −2.810| −3.240| −5.045| −2.252|
| 2010-11-19     | −0.986| 0.132| 0.678| 0.608| 0.242| −0.140|
| 2010-11-26     | 0.104| −1.008| 1.244| −4.418| −4.147| −1.471|
| 2010-12-03     | −1.315| 3.604| 3.956| 3.755| 5.675| 1.920 |
| 2010-12-10     | −2.802| 1.389| 3.359| 0.405| −1.334| −0.324|
| 2010-12-17     | 0.325| 0.512| 0.436| −0.110| −0.461| 0.291 |
| 2010-12-24     | −0.465| 1.205| 1.706| 1.276| 1.396| 0.638 |
| 2010-12-31     | 0.813| 0.150| −0.779| 0.929| 2.581| 0.577 |
Table D.2: $\alpha$-denominated weekly returns (%) for the last 39 weeks of 2010.

| Friday       | IEF  | IWB  | IWM  | EFA  | EEM  | PORTF |
|--------------|------|------|------|------|------|-------|
| 2010-04-09   | 0.010| 1.501| 2.448| 0.308| 1.192| 0.683 |
| 2010-04-16   | 0.871| −0.058| 1.628| −0.734| −2.811| 0.111 |
| 2010-04-23   | −0.354| 2.060| 3.591| −0.444| 1.278| 0.598 |
| 2010-04-30   | 1.155| −2.355| −3.336| −3.502| −2.151| −1.379 |
| 2010-05-07   | 1.498| −6.548| −8.462| −8.880| −8.220| −4.240 |
| 2010-05-14   | 0.020| 2.369| 5.650| 1.332| 2.769| 1.283 |
| 2010-05-21   | 1.526| −4.032| −6.040| −2.340| −4.579| −1.645 |
| 2010-05-28   | −0.346| 0.500| 1.588| −0.512| 1.618| −0.045 |
| 2010-06-04   | 0.755| −2.045| −3.619| −2.459| −1.916| −1.146 |
| 2010-06-11   | −0.184| 2.163| 1.857| 3.467| 3.322| 1.639 |
| 2010-06-18   | 0.143| 2.075| 2.502| 2.903| 2.470| 1.585 |
| 2010-06-25   | 0.928| −3.194| −2.919| −1.546| −0.489| −1.329 |
| 2010-07-02   | 1.027| −4.791| −6.205| −3.059| −3.601| −2.301 |
| 2010-07-09   | −0.562| 4.689| 4.211| 5.161| 4.759| 2.931 |
| 2010-07-16   | 1.227| −1.124| −2.525| −0.835| −2.830| −0.217 |
| 2010-07-23   | −0.389| 3.181| 5.278| 3.493| 5.359| 1.983 |
| 2010-07-30   | 0.828| −0.089| 0.054| 0.556| 0.536| 0.393 |
| 2010-08-06   | 0.864| 1.642| 0.162| 3.198| 1.457| 1.738 |
| 2010-08-13   | 1.220| −3.447| −5.493| −5.283| −3.001| −2.224 |
| 2010-08-20   | 0.236| −0.443| 0.108| −0.695| 0.729| −0.262 |
| 2010-08-27   | −0.236| 0.563| 0.675| 0.364| −1.137| −0.224 |
| 2010-09-03   | −0.509| 3.372| 3.617| 3.423| 3.302| 2.001 |
| 2010-09-10   | −0.565| 0.385| −0.824| 0.556| 0.493| 0.087 |
| 2010-09-17   | 0.504| 1.376| 2.011| 1.426| 1.629| 1.077 |
| 2010-09-24   | 1.038| 1.892| 2.630| 3.041| 2.358| 1.863 |
| 2010-10-01   | 0.805| 0.030| 1.177| 0.156| 2.809| 0.336 |
| 2010-10-08   | 1.185| 1.546| 1.921| 2.433| 1.714| 1.630 |
| 2010-10-15   | −1.226| 0.995| 1.367| 1.147| 1.051| 0.240 |
| 2010-10-22   | 0.237| 0.520| 0.041| −0.417| −1.479| 0.194 |
| 2010-10-29   | −0.257| 0.178| −0.028| −0.486| 0.193| −0.137 |
| 2010-11-05   | 0.892| 3.462| 4.696| 3.806| 5.081| 2.629 |
| 2010-11-12   | −1.414| −1.991| −2.354| −3.059| −4.459| −2.042 |
| 2010-11-19   | −0.878| 0.119| 0.568| 0.574| 0.214| −0.127 |
| 2010-11-26   | 0.093| −0.906| 1.042| −4.171| −3.665| −1.334 |
| 2010-12-03   | −1.171| 3.239| 3.315| 3.545| 5.015| 1.741 |
| 2010-12-10   | −2.494| 1.249| 2.814| 0.382| −1.179| −0.294 |
| 2010-12-17   | 0.289| 0.460| 0.365| −0.104| −0.407| 0.264 |
| 2010-12-24   | −0.414| 1.083| 1.429| 1.205| 1.234| 0.579 |
| 2010-12-31   | 0.724| 0.135| −0.653| 0.877| 2.281| 0.523 |
References

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