MARTINGALE WEAK SOLUTION AND MARKOV SELECTION TO
2D AND 3D STOCHASTIC CAHN-HILLIARD-NAVIER-STOKES
EQUATIONS WITH THE DYNAMIC BOUNDARY

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Abstract. The existence of global martingale weak solution for the 2D and 3D stochastic Cahn-Hilliard-Navier-Stokes equations driven by multiplicative noise in a smooth bounded domain is established. In particular, the system is supplied with the dynamic boundary condition which accounts for the interaction between the fluid components and the rigid walls. The proof is completed by a three-level approximate scheme combining a fixed point argument and the stochastic compactness argument. Then, we prove the existence of an almost surely Markov selection to the associated martingale problem following the abstract framework established by F. Flandoli and M. Romito [16].

1. Introduction

Cahn-Hilliard-Navier-Stokes (CH-NS) equations describe the motion of two immiscible mixture fluid, which consist of the Navier-Stokes equations governing the fluid velocity and a convective Cahn-Hilliard equation for the relative density of atoms of one of fluids, for further details see [4, 24, 28, 29] and references therein. The motion of fluid could be easily affected by external random factor which accounts for numerical and physical uncertainties in applications ranging from climatology to turbulence theory. Therefore, the uncertainty and randomness must be taken into account for better understanding the dynamical phenomenon. Here we consider the following stochastic CH-NS system with multiplicative noise in a smooth bounded domain $\mathcal{D}$:

$$
\begin{aligned}
\frac{du}{dt} - (\text{div}(2\nu_0 D(u)) - (u \cdot \nabla) u - \nabla p + \mu \nabla \phi) dt &= h(u, \nabla \phi) dW, \\
\frac{d\phi}{dt} + (u \cdot \nabla) \phi dt - \Delta \mu dt &= 0, \\
\mu &= -\nu_1 \Delta \phi + \alpha f(\phi), \\
\nabla \cdot u &= 0,
\end{aligned}
$$

(1.1)

where $u, p$ and $\phi$ denote the velocity, pressure and the phase parameter, respectively. $D(u) = \frac{\nabla u + \nabla u^T}{2}$ is the viscous stress tensor, and $\nu_0$ is the viscosity of the fluid. Here we assume that $\nu_0 \equiv 1$. $W$ is a cylindrical Wiener process introduced later. $\mu$ stands for the chemical potential of binary mixtures which is derived from the variational derivative of the quantity (free energy functional):

$$
E^f(\phi) = \int_{\mathcal{D}} \frac{\nu_1}{2} \left| \nabla \phi \right|^2 + \alpha F(\phi) dx,
$$

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where $F(\xi) = \int_0^\xi f(r)dr$, constant $\nu_1$ is related to the thickness of the interface between the two fluids and parameter $\alpha = \frac{1}{\nu_1}$ describes the interaction between two phases, we also assume that $\alpha = \nu_1 \equiv 1$. In physical background, a representative example of $F$ is logarithmic type, that is, $F(s) = c_0[(1 + s) \ln(1 + s) + (1 - s) \ln(1 - s)] - c_1 s^2$, $c_1 > c_0 > 0$, where $s \in (-1, 1)$. Usually, we use a polynomial approximation of type $F(r) = c_2 r^4 - c_3 r^2$ taking place the type of logarithmic, where $c_2, c_3$ are two positive constants.

For the case of the Neumann boundary condition of $\phi$, abundant results of system (1.1) were published in both deterministic and stochastic areas, see [9, 10, 11, 13, 17, 19, 23, 24, 27, 32, 33, 35, 44] and references therein. Nevertheless, the Neumann boundary condition is incompatible with the moving contact line which describes the intersection of the incompressible two-phase flow interface with the solid wall, see [12]. Therefore, a new boundary condition was proposed by [37] (see also [38]) accounting for the physical phenomenon, that is, the velocity equation is equipped with the following generalized Navier boundary conditions which was introduced by Maxwell [34] in the kinetic theory of gases:

\begin{align}
\mathbf{u} \cdot \mathbf{n} &= 0, \\
2(D(u) \cdot \mathbf{n})_\tau + u_\tau &= K(\phi) \nabla_\tau \phi,
\end{align}

where $\mathbf{n}$ is the unit outward normal vector to the boundary $\partial D$, while the Cahn-Hilliard equation is equipped with a dynamic boundary and the Neumann boundary conditions:

\begin{align}
\partial_\mathbf{n} \mu &= 0, \\
\partial_t \phi + u_\tau \cdot \nabla_\tau \phi &= -K(\phi),
\end{align}

on $\partial D \times (0, \infty)$ and

\[ K(\phi) = -\triangle_\tau \phi + \partial_\mathbf{n} \phi + \phi + g(\phi). \]

Such a boundary condition (1.5) could be considered as a parabolic equation on boundary $\partial D$. The notation $\nabla_\tau$ represents the tangential gradient operator along tangential direction $\tau = (\tau_1, \ldots, \tau_{N-1})$ on the boundary $\partial D$ and we denote by $\triangle_\tau$ the Laplace-Beltrami operator on $\partial D$. $u_\tau$ is the tangential component of $u$ on the boundary $\partial D$. $g$ corresponds to the interfacial energy at the wall interface. For more descriptions of this kind of boundary, we refer to [12, 37, 38].

Note that, from the physical point of view it is significant to find the boundary conditions (1.2) and (1.4) guarantee the conservation form of $\phi$, that is

\[ \langle \phi(t) \rangle = \langle \phi(0) \rangle, \quad \text{for all } t \geq 0, \]

where $\langle \cdot \rangle$ denotes the mean value over $D$, that is, $\langle \phi(t) \rangle = |D|^{-1} \int_D \phi(t, x)dx$.

We supply the following initial conditions:

\[ u(0, x) = u_0, \quad \phi(0, x) = \phi_0, \]

where the random element $\omega$ is omitted.

When $h \equiv 0$, system (1.1)-(1.6) reduces to the deterministic one, Gal, Grasselli and Miranville [20] proved the existence of global energy weak solution, and also established the convergence of such solution to a single equilibrium. Moreover, Cherfils and Petcu [5] obtained the existence, uniqueness, and regularity of the solutions, in which the Navier boundary condition of $u$ was replaced by non-slip condition $u|_{\partial D} = 0$. Recently, Cherfils
et al. [6] considered the compressible case, and established the existence of global-in-time weak solutions for any finite energy initial data. For more researches, see [21, 22, 25, 26] and references therein.

According to what we know, the relative stochastic issue to system (1.1)-(1.6) has not been considered yet. We are the first to consider the stochastic version with dynamic boundary, and devoted to proving the global existence of martingale weak solution to 2D and 3D system (1.1)-(1.6) and an almost surely Markov selection to associated martingale problem. Note that even in 2D case, we can not establish the existence of global pathwise solution which is not a drawback of our approach but an intrinsic difficulty, mainly for the following reason: firstly, the arising of boundary term \( K(\phi)\nabla \tau \phi \) and the boundary advection term involve strong coupling of the system which fails the proof of uniqueness. Consequently, we can not use the Gyöngy-Krylov theorem to recover the convergence in original fixed probability space; Secondly, it is not allowed to apply the Galerkin scheme to establish the existence of approximate solutions due to the fact that in process of obtaining the a priori estimates, the test functions are not compatible any more, see Remark 3.2.

As a result, it is impossible to show the Galerkin-type approximations converge in mean square for passing the limit.

In view of these difficulties, we will adopt a mixed proof which combines a fixed point theorem and the stochastic compactness argument to establish the existence of global martingale weak solution. The specific strategy is as follows.

- **By the linearization and mollification technique, we transform the original system into a linear approximate modified system driven by additive noise with dynamic boundary.** In this step, we add cut-off function in nonlinear terms to control strong nonlinearity construction.

- **With linear approximate modified system established, we first show that the global well-posedness for the auxiliary system holds.**

- **Then, we could define a mapping \( M \) such that the well-posedness of approximate solutions of original system is equivalent to looking for a fixed point of this mapping.**

- **Applying the truncated skill and stopping time technique to show the contraction property of mapping \( M \).** In this procedure, we will design a two-level stopping time technique and an extra layer approximate scheme to surmount difficulties caused by random element \( \omega \) and cut-off function.

- **Finally, we establish the uniform a priori estimates of the regularized solution by the elliptic regularity theory, then the stochastic compactness, and identify the limit.**

How to establish the approximate solutions is more complex and challenging compared with the deterministic case [20]. In order to close the contraction estimate, here we have to consider an additional layer approximation and develop the cut-off skill. However, the cut-off operators bring further difficulty in establishing the contraction estimate, we will introduce stopping time technique to overcome it. Moreover, we have to choose the same cut-off operators in front of all nonlinear terms, making sure that the a priori estimates could be achieved.

A remarkable feature of dynamical system with uniqueness is the memoryless property, thus, the Markov property. As mentioned, even for 2D case, uniqueness for the system is unknown, therefore it is natural to consider if we could choose one law satisfying the Markov property. Existence of Markov selection is originally introduced by Krylov [31] for the stochastic differential equations in finite-dimensional setting. Then, Flandoli and
Romito [10] generalized the result to infinite-dimensional setting, establishing the Markov selection for the 3D incompressible Navier–Stokes equations with additive noise. Furthermore, Breit, Feireisl and Hofmanová [3] proved the existence of the Markov selection for the 3D stochastic compressible Navier–Stokes equations with a general multiplicative stochastic forcing. Dong and Zhai [11] considered the 3D stochastic incompressible Navier–Stokes equations with jump noise, obtaining the existence of martingale solution and an almost surely Markov selection.

A special point of the coupled system is that we only have regularity $\mu \in L^2_t H^1, \mathcal{K} \in L^2_t L^2(\Gamma)$ a.s. due to the involved structure. Therefore, quantities $\mu, \mathcal{K}$ are not processes in the classical sense and we have to understand them as $\mathcal{F}_t$-adapted random distribution. Following the idea of [3], we overcome the difficulty by introducing new variables $\mathcal{V}_1, \mathcal{V}_2$ satisfying equations $d\mathcal{V}_1 = \mu dt$ and $d\mathcal{V}_2 = \mathcal{K} dt$. Note that $\mathcal{V}_1$ and $\mathcal{V}_2$ are two continuous stochastic processes with trajectories in $W^{1,2}_t H^1$ and $W^{1,2}_t L^2(\Gamma)$, a.s. respectively and $\mu, \mathcal{K}$ are measurable functions of $\mathcal{V}_1, \mathcal{V}_2$. As a result, we could establish the existence of an almost surely Markov selection for $(u, \phi, \psi, \mathcal{V}_1, \mathcal{V}_2)$.

The rest of this paper is structured as follows. In section 2, we will give some deterministic and stochastic preliminaries needed throughout the paper. Our focus in section 3 will be to prove the existence of global martingale weak solution by four parts. We prove the existence of Markov selection to associated martingale problem in section 4. An appendix is included afterwards to state the results that are frequently used in the paper. Finally, two auxiliary results are given in section 6.

2. Preliminaries

Throughout the paper, we use the short notations $L^p_t X := L^p(0, T; X)$, $L^p_w X := L^p(\Omega; X)$ and $W^\alpha,p_t X := W^\alpha,p(0, T; X)$, $C^\alpha_t X := C^\alpha([0, T]; X)$ for $\alpha \in (0, 1], p \geq 1$, where $X$ is a Banach space and $\Omega$ is a sample space. Here, we use the fractional order Sobolev space with respect to $t$, since the evolution equations forced by noise only have $\alpha$-order Hölder continuous of $t$ for $\alpha \in (0, \frac{1}{2})$, see [8, 15] for more details of these spaces. Denote

$$(L^p_t X)_w := \text{the space } L^p(0, T; X) \text{ with weak topology}.$$

To simplify the notation, let $\Gamma := \partial D$, denote $H^s(D)$ and $H^s(\Gamma)$ norm by $\| \cdot \|_{H^s}$ and $\| \cdot \|_{H^s(\Gamma)}$ for any $s > 0$; when $s = 0$, the notation $L^2_\Gamma := L^2(\Gamma)$. Denote the dual of Banach space $X$ by $X'$ and the mathematical expectation by $\mathbb{E}$. Denote the trace operator from $H^s(D)$ into $H^{s-\frac{d}{2}}(\Gamma)$ by $\text{Tr}_D$.

Recall that the Laplace-Beltrami operator $A_\tau := -\triangle_\tau$ is a nonnegative self-adjoint operator in $L^2(\Gamma)$. Therefore, we could define

$$H^s(\Gamma) := \left\{ \psi \in L^2(\Gamma) : \psi \in D(A^\frac{s}{2}_\Gamma) \right\},$$

with the norm

$$\| \psi \|_{H^s(\Gamma)} = \| A^\frac{s}{2}_\Gamma \psi \|_{L^2(\Gamma)} + \| \psi \|_{L^2(\Gamma)}.$$

Introduce the Banach space $V^s$ for $s \geq 0$ corresponding to the parabolic equation on boundary $\Gamma$

$$V^s := \left\{ (\phi, \psi) \in H^s \times H^{s+\frac{d}{2}}(\Gamma) : \psi = \text{Tr}_D(\phi) \in H^s(\Gamma) \right\},$$
with the norm
\[ \|(\phi, \psi)\|_{V^s}^2 = \|\phi\|_{H^s}^2 + \|\psi\|_{H^s(\Gamma)}^2, \]
when \( s = 0 \), we write
\[ \|(\phi, \psi)\|_{V^0}^2 = \|\phi\|_{L^2}^2 + \|\psi\|_{L^2(\Gamma)}^2. \]
Also introduce the operator \( A_1 \phi = -\triangle \phi \) for any
\[ \phi \in D(A_1) = \{ \phi \in H^2(D) : \partial_n \phi|_{\Gamma} = 0 \}. \]
The Poincaré inequality implies \( H^2 \)-norm is equivalent to norm \( \|A_1(\cdot)\|_{L^2} + \|\cdot\| \). Then, introduce the operator
\[ A_2 \phi = -\triangle \phi, \text{ for } \phi \in D(A_2) = D(A_1) \cap L^2_0, \]
where \( L^2_0 \) is the space of \( L^2 \)-functions with 0-mean valued. Note that \( A_2^{-1} \) is a compact operator in \( L^2_0 \), and if \( \phi \in D(A_2) \), the operator \( A_2 \) is equivalent to \( A_1 \). For any \( \phi \in D(A_1) \), we have \( \phi - \langle \phi \rangle \in D(A_2) \), and then \( H^s \)-norm is equivalent to \( \|A_2^s(\phi - \langle \phi \rangle)\|_{L^2} + \|\langle \phi \rangle\|_{L^2} \). The Poincaré inequality again implies \( H^s \)-norm is equivalent to the norm \( \|A_2^s(\cdot)\|_{L^2} \) for all \( \phi \in D(A_2) \).

If \( L^2_{x}, H^{s,p} \) represent the solution spaces of the fluid velocity, we define them by the closure of
\[ \mathcal{V} = \{ u \in C^\infty_c(\mathcal{D}) : \text{div} u = 0 \text{ in } \mathcal{D}, u \cdot n = 0, \text{ on } \Gamma \}, \]
with respect to \( L^p(\mathcal{D}), H^{s,p}(\mathcal{D}) \)-norms. Considering the boundary condition, we introduce the bilinear form
\[ a_0(u, v) := 2(D(u), D(v)) + (u, v)_\Gamma, \text{ for } u, v \in H^1. \]
Note that, the bilinear form \( a_0 \) is coercive, continuous, symmetric and \( \sqrt{a_0(u, u)} \) is equivalent to \( H^1 \)-norm. Corresponding to \( a_0 \), we could define the Stokes operator by \( A_0 u = -P \text{div}(2D(u)) \) such that \( (A_0 u, v) = a_0(u, v) \) for all
\[ u \in D(A_0) := \{ u \in H^2 \cap H^1 : 2(D(u) \cdot n)_\tau + u_\tau = 0 \text{ on } \Gamma \}, \]
where \( P \) is the Helmholtz-Leray projection. By the definition, we could have that the operator \( A_0 \) is positive and self-adjoint on \( L^2_x \), and \( A_0^{-1} \) is compact.

To simplify the notation, we next define \( (b_0, b_1, b_2) : (L^p_{x} \times H^1, q)^3 \to (L^r_x)^3 \) with \( 1 - \frac{1}{p} = \frac{1}{r} + \frac{1}{q} = \frac{1}{r'} \) as the bilinear operators such that
\[ \langle b_0(u, v), \eta \rangle = \int_D (u \cdot \nabla v) \cdot \eta dx = B_0(u, v, \eta), \]
\[ \langle b_1(\mu, \phi), \eta \rangle = \int_D \mu (\nabla \phi \cdot \eta) dx = B_1(\mu, \phi, \eta), \]
\[ \langle b_2(u, \phi), \rho \rangle = \int_D (u \cdot \nabla \phi) \cdot \rho dx = B_2(u, \phi, \rho), \]
and also define the bilinear operator by \( b_{\Gamma} : L^p(\Gamma) \times H^{1,q}(\Gamma) \to L^{r'}(\Gamma) \) such that
\[ \langle b_{\Gamma}(u, \psi), \eta \rangle = \int_{\Gamma} (u_\tau \cdot \nabla_\tau \psi) \cdot \eta dS = B_{\Gamma}(u, \psi, \eta). \]
Recall that \( B_0(u, v, v) = 0 \) for \( u, v \in H^1 \).
Let $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \mathcal{W})$ be a fixed stochastic basis and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. $\mathcal{W}$ is a cylindrical Wiener process defined on the Hilbert space $\mathcal{H}$, which is is adapted to the complete, right continuous filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Namely, $\mathcal{W} = \sum_{k \geq 1} e_k \beta_k$ with $\{e_k\}_{k \geq 1}$ being the complete orthonormal basis of $\mathcal{H}$ and $\{\beta_k\}_{k \geq 1}$ being a sequence of independent standard one-dimensional Brownian motions. In addition, $L_2(\mathcal{H}, X)$ denotes the collection of Hilbert-Schmidt operators, the set of all linear operators $G$ from $\mathcal{H}$ to $X$, with norm $\|G\|^2_{L_2(\mathcal{H}, X)} = \sum_{k \geq 1} \|Ge_k\|^2_X$. Considering an auxiliary space $\mathcal{H}_0 \supset \mathcal{H}$, we define $\mathcal{H}_0$ by

$$\mathcal{H}_0 = \left\{ h = \sum_{k \geq 1} \alpha_k e_k : \sum_{k \geq 1} \alpha_k^2 k^{-2} < \infty \right\},$$

with the norm $\|h\|^2_{\mathcal{H}_0} = \sum_{k \geq 1} \alpha_k^2 k^{-2}$. Observe that the mapping $\Phi : \mathcal{H} \rightarrow \mathcal{H}_0$ is Hilbert-Schmidt. We also have that $\mathcal{W} \in \mathcal{C}([0, \infty), \mathcal{H}_0)$ for almost all $\omega$, see [7].

For an $X$-valued predictable process $\varpi \in L^2(\Omega; L^2_{loc}([0, \infty), L_2(\mathcal{H}, X)))$ by taking $\varpi_k = \varpi e_k$, the following Burkholder-Davis-Gundy inequality holds:

$$E \left( \sup_{t \in [0,T]} \left\| \int_0^t \varpi dW \right\|^p_X \right) \leq c_p E \left( \int_0^T \left\| \varpi \right\|^2_{L_2(\mathcal{H}, X)} dt \right)^{\frac{p}{2}} = c_p E \left( \int_0^T \sum_{k \geq 1} \|\varpi_k\|^2_X dt \right)^{\frac{p}{2}},$$

for any $p \geq 1$.

In the next sections, since 2D case is easier, we only focus on 3D case for the details of proof.

### 3. The existence of martingale weak solution

Our goal in this section is to prove the existence of global martingale weak solution to system (1.1)-(1.6). First, we formulate the definition.

**Definition 3.1.** Let $\mathcal{P}_0$ be a probability measure on space $L^2_x \times V^1$. The $(\mathcal{S}, u, \phi, \psi)$ is a global martingale weak solution to system (1.1)-(1.6) if the followings are satisfied:

1. $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \mathcal{W})$ is a stochastic basis, where $\mathcal{W}$ is a Wiener process;

2. $(u, \phi, \psi)$ are $\{\mathcal{F}_t\}_{t \geq 0}$ progressive measurable processes such that

$$u \in L^p_w(L^\infty_t L^2_x \cap L^2_t V^1), \quad (\phi, \psi) \in L^p_w(L^\infty_t V^1 \cap L^2_t V^2),$$

where $\psi = \text{Tr}_D(\phi)$, $\mathbb{P}$-a.s.,

3. $\mathcal{P}_0 = \mathbb{P} \circ \{(u_0, \phi_0, \psi_0)\}^{-1}$;

4. for any $v_0 \in H^1, v_1 \in H^1, v_2 \in L^2(\Gamma), t \in [0, T], (u, \phi, \psi)$ satisfy $\mathbb{P}$-a.s.

$$\begin{cases}
(u, v_0) + \int_0^t a_0(u, v_0) + B_0(u, u, v_0) - B_1(\mu, \phi, v_0) ds = \int_0^t (K(\psi) \nabla \psi, v_0, \tau) d\tau \\
+ \int_0^t h(u, \nabla \phi, v_0) d\mathcal{W}, \\
(\phi, v_1) + \int_0^t (\nabla \mu, \nabla v_1) + B_2(u, \phi, v_1) ds = 0, \\
(\psi, v_2) + \int_0^t (K(\psi), v_2) + B_3(u, \psi, v_2) ds = 0,
\end{cases}$$

with

$$\begin{cases}
\mu = -\Delta \phi + f(\phi), \quad \mathbb{P}$-a.s. in $\mathcal{D} \times (0, T), \\
K(\psi) = A \psi + \partial_n \phi + \psi + g(\psi), \quad \mathbb{P}$-a.s. on $\Gamma \times (0, T),
\end{cases}$$

\[V^1 = \left\{ v \in L^2(\mathcal{H}, X) \mid \left\| \int_0^T \left\| v_s \right\|_X^2 ds \right\|_X < \infty \right\}, \quad \mathbb{P}$-a.s. on $\mathcal{D} \times (0, T)\]
and $\mu, \mathcal{K}(\psi)$ are random distribution (for more details of definition see [3 Section 3]) adapted to $\mathcal{F}_t$ with regularity

$$\mu \in L^p_w(L^2_t^2 \mathcal{H}^1), \; \mathcal{K}(\psi) \in L^p_w(L^2_t^2 \mathcal{L}^2(\Gamma)),$$

5. the energy inequality

$$\frac{1}{2} \left[ E \left[ 1_A E^2 \right] \right]_s^t + \mathbb{E} \left[ 1_A \int_s^t E \left[ \| \nabla u, \nabla \mu \|_{L^2_x}^2 + \| (\phi, \psi) \|_{L^2_x}^2 + \| \mathcal{K}(\psi) \|_{L^2(\Gamma)}^2 \right] \, dr \right]$$

$$\leq \mathbb{E} \left[ 1_A \int_s^t E \left[ \| h(u, \nabla \phi) \|_{L^2(\mathcal{H};L^2_x)}^2 \right] \, dr \right] + \mathbb{E} \left[ 1_A \int_s^t \| h(u, \nabla \phi), u \|^2 \, dr \right]$$

holds for any $t \geq 0$ and a.e. $s, 0 \leq s \leq t$ and $A \in \mathcal{F}_s$, where

$$E := \frac{1}{2} \left( \| u \|_{L^2_x}^2 + \| \nabla \phi \|_{L^2_x}^2 + \| \psi \|_{L^2_x(\Gamma)}^2 + \| \nabla \phi \|_{L^2_x(\Gamma)}^2 \right) + \int_D F(\phi) dx + \int_{\Gamma} G(\psi) dS.$$

We next introduce the assumptions imposed on the noise intensity $h$: there exists constant $C$ such that

- $A_1.$ $\| h(u, \nabla \phi) \|_{L^2(\mathcal{H};L^2_x)} \leq C (1 + \| (u, \nabla \phi) \|_{L^2_x}^2)$, for $u \in L^2_x, \phi \in H^1$,
- $A_2.$ $\| h(u_1, \nabla \phi_1) - h(u_2, \nabla \phi_2) \|_{L^2(\mathcal{H};L^2_x)} \leq C \| (u_1 - u_2, \nabla (\phi_1 - \phi_2)) \|_{L^2_x}$, for $u \in L^2_x, \phi \in H^1$.

And the assumptions on $f, g$ are as follows: there exist positive constants $C_1, C_2$ such that

- $A_3.$ $|f(r)| \leq C_1 (1 + |r|^2)$ for $f \in C^1(\mathbb{R})$, $f(0) = 0$ and $F''(s) \geq -C_2, F(s) \geq -C_2$,
- $A_4.$ $|g'(r)| \leq C_1 (1 + |r|^m)$ for $g \in C^1(\mathbb{R})$, any $m \geq 1$, $g(0) = 0$ and $G''(s) \geq -C_2, G(s) \geq -C_2$, where $G(s) = \int_0^s g(r) dr$.

Here the norm $\| (u, v) \|_{L^2_x} := \| u \|_{L^2_x} + \| v \|_{L^2_x}$, for $u \in L^2_x, v \in L^2_x$.

Remark 3.1. The assumptions imposed on $f$ include the meaningful physical case $f(r) = r^3 - r$.

Now, we state our existence result as follows.

**Theorem 3.1.** Suppose that initial data $(u_0, \psi_0, \psi_0) \in L^p_w(L^2_t \times V^1)$ are $\mathcal{F}_0$-measurable random variables for any $p > 2$, and functions $f, g \in C^1(\mathbb{R})$ satisfy assumptions $A_3$ and $A_4$, operator $h$ satisfies assumptions $A_1$ and $A_2$. Then, there exists a global martingale weak solution to system (1.1)-(1.6) in the sense of Definition 3.1.

The proof of Theorem 3.1 is divided into four parts. The existence and uniqueness of approximate solutions are established by exploiting a fixed point argument in first part. Part 2 is devoted to obtaining the a priori estimates by selecting suitable test functions. We pass the limit with respect to $\varepsilon, \delta$ in the regularized system by the stochastic compactness argument in Parts 3, 4.

**Part 1.** The existence and uniqueness of approximate solutions to regularized system. Define by the space $X = X_1 \times X_2$, where

$$X_1 = \left\{ u : u \in L^p_w(L^\infty_t L^2_x \cap L^2_t \mathcal{H}^1), u|_{t=0} = u_0 \right\},$$

$$X_2 = \left\{ (\phi, \psi) : (\phi, \psi) \in L^p_w(L^\infty_t V^1 \cap L^2_t V^2 \cap W^{1,2}_t V^0), (\phi, \psi)|_{t=0} = (\phi_0, \psi_0) \right\},$$

for any $p > 2$. 
Let \( J_\varepsilon, J_c \) be the mollifiers on the corresponding spaces \( H^s(\mathcal{D}), H^s(\Gamma) \) for \( s \geq 0 \), respectively, for more properties and the construction of such operators see Lemma 5.5 and Remark 5.1. We consider the mollification system:

\[
\begin{cases}
(\partial_t \phi_{\varepsilon, \delta}, \xi) + (\mu_{\varepsilon, \delta} \nabla(J_\varepsilon \phi_{\varepsilon, \delta}), \xi)dt = (h(\varepsilon, \delta), \phi_{\varepsilon, \delta})dW, \\
(\partial_t \psi_{\varepsilon, \delta}, \xi_2) = 0,
\end{cases}
\]

for \( \xi \in H^1, \xi_1 \in H^1, \xi_2 \in L^2(\Gamma), \) where

\[
\mu_{\varepsilon, \delta} = -\Delta \phi_{\varepsilon, \delta} + \delta \partial_t \phi_{\varepsilon, \delta} + f(\phi_{\varepsilon, \delta}),
\]

\[
K(\psi_{\varepsilon, \delta}) = A_f \psi_{\varepsilon, \delta} + \partial_n \phi_{\varepsilon, \delta} + \psi_{\varepsilon, \delta} + g(\psi_{\varepsilon, \delta}).
\]

In order to surmount the difficulty caused by the random element \( \omega \), we introduce \( C^\infty \)-smooth cut-off function \( \Psi_R : [0, \infty) \rightarrow [0, 1] \) and

\[
\Psi_R(x) = \begin{cases} 
1, & \text{if } 0 < x < R, \\
0, & \text{if } x > 2R.
\end{cases}
\]

Then, define \( \Psi_R = \Psi_R(||u||_{L^2}) \cdot \Psi_R(||(\phi, \psi)||_{L^1}) \), add the cut-off function in front of non-linear terms of system (3.2), to obtain

\[
\begin{cases}
(\partial_t \psi_{\varepsilon, \delta}, \xi_2) = 0, \\
(\partial_t \psi_{\varepsilon, \delta}, \xi_2) = 0.
\end{cases}
\]

Considering its linearization

\[
\begin{cases}
(\partial_t \psi_{\varepsilon, \delta}, \xi_2) = 0, \\
(\partial_t \psi_{\varepsilon, \delta}, \xi_2) = 0.
\end{cases}
\]

where processes \( (v_{\varepsilon, \delta}, \varphi_{\varepsilon, \delta}, \eta_{\varepsilon, \delta}) \in X \) and

\[
\begin{align*}
\tilde{\mu}_{\varepsilon, \delta} &= -\Delta \varphi_{\varepsilon, \delta} + \delta \partial_t \varphi_{\varepsilon, \delta} + f(\varphi_{\varepsilon, \delta}), \\
\tilde{K}(\psi_{\varepsilon, \delta}) &= A_f \psi_{\varepsilon, \delta} + \partial_n \phi_{\varepsilon, \delta} + \psi_{\varepsilon, \delta} + g(\psi_{\varepsilon, \delta}), \\
\tilde{\mu}_{\varepsilon, \delta} &= -\Delta \phi_{\varepsilon, \delta} + \delta \partial_t \phi_{\varepsilon, \delta} + f(\phi_{\varepsilon, \delta}).
\end{align*}
\]

Assumptions \( A_3 \) and \( A_4 \) are insufficient for closing the estimate, therefore we use the functions \( f_\varepsilon, g_\varepsilon \in C^2 \) substituting for \( f, g \) in (3.2)–(3.10), having the properties:

(i) there exist two constants \( c_{f, \varepsilon}, c_{g, \varepsilon} \) such that

\[
|f_\varepsilon'(s)| \leq c_{f, \varepsilon}, \quad |g_\varepsilon'(s)| \leq c_{g, \varepsilon},
\]

for all \( s \in \mathbb{R} \),

(ii) on any compact set of \( \mathbb{R} \), the convergences \( f_\varepsilon \rightarrow f, g_\varepsilon \rightarrow g \) are uniform, see [20] for the construction of \( f_\varepsilon, g_\varepsilon \).

Define the mapping \( \mathcal{M} : X \mapsto X \), \( (v_{\varepsilon, \delta}, \varphi_{\varepsilon, \delta}, \eta_{\varepsilon, \delta}) \mapsto (u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, \psi_{\varepsilon, \delta}) \).
We show that the mapping is well-defined. Observe that, taking $\xi_1 \equiv 1$, we have $\langle \partial_t \phi_{\varepsilon, \delta} \rangle = 0$, a.e. $(0, T) \times \Omega$ which allow us to apply the Poincare inequality to get the desired estimates. By Property (i) and the mean value theorem, we have
\[
\|f_\varepsilon(\varphi_{\varepsilon, \delta})\|_{L^\infty L^2_2} \leq \|f_\varepsilon'(\theta \varphi_{\varepsilon, \delta})\varphi_{\varepsilon, \delta}\|_{L^\infty L^2_2} \leq c_{f, \varepsilon}\|\varphi_{\varepsilon, \delta}\|_{L^\infty L^2_2}, \tag{3.11}
\]
where $\theta \in (0, 1)$ is a constant. Similarly, we have
\[
\|g_\varepsilon(\eta_{\varepsilon, \delta})\|_{L^\infty L^2_2(\Gamma)} \leq c_{g, \varepsilon}\|\eta_{\varepsilon, \delta}\|_{L^\infty L^2(\Gamma)}. \tag{3.12}
\]
We use (3.7), estimate (3.11) and the fact $(v_{\varepsilon, \delta}, \varphi_{\varepsilon, \delta}, \eta_{\varepsilon, \delta}) \in X$ to find
\[
\tilde{\mu}_{\varepsilon, \delta} \in L^p_\omega L^2_1 L^2_2.
\]
And from (3.8), estimate (3.12) and the embedding $H^{m,p}(\mathcal{D}) \hookrightarrow H^{m-\frac{1}{p},p}(\Gamma)$, we get
\[
K(\eta_{\varepsilon, \delta}) \in L^p_\omega L^2_1 L^2_2(\Gamma).
\]
These regularity estimates along with the fact that
\[
J_\varepsilon v_{\varepsilon, \delta} \in L^p_\omega L^\infty_1 H^2, \quad J_\varepsilon \varphi_{\varepsilon, \delta} \in L^p_\omega L^\infty_1 H^3, \quad J_\varepsilon \eta_{\varepsilon, \delta} \in L^p_\omega L^\infty_1 H^3(\Gamma),
\]
and assumption $A_1$, Hölder’s inequality lead to
\[
h(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta}) \in L^p_\omega L^\infty_1 L^2_x,
\]
\[
f_1 := K(\eta_{\varepsilon, \delta}) \nabla \tau(J_\varepsilon \eta_{\varepsilon, \delta}) \in L^p_\omega L^2_1 L^2(\Gamma),
\]
\[
f_2 := -b_0(v_{\varepsilon, \delta}, J_\varepsilon \varphi_{\varepsilon, \delta}) + \tilde{\mu}_{\varepsilon, \delta} \cdot \nabla J_\varepsilon \varphi_{\varepsilon, \delta} \in L^p_\omega L^2_1 H^2,
\]
\[
g_1 := -b_1(v_{\varepsilon, \delta}, J_\varepsilon \varphi_{\varepsilon, \delta}) \in L^p_\omega L^2_1 L^2_x,
\]
\[
g_2 := -b_2(v_{\varepsilon, \delta}, J_\varepsilon(\eta_{\varepsilon, \delta})) \in L^p_\omega L^2_1 L^2(\Gamma).
\]
Thus, we verified the required conditions of the auxiliary Propositions 6.1 and 6.2 which are invoked to obtain the existence and uniqueness of approximate solutions to mollification system (3.6).

The fixed point theorem tells us once we show mapping $\mathcal{M}$ is contraction from $X$ into itself, we could obtain the existence and uniqueness of approximate solutions to mollification system (3.6). To this end, we first show:

**Step 1.** Proving that $\mathcal{M}$ is a mapping from $B_{K_1, K_2}$ into itself, where
\[
B_{K_1, K_2} := \{(v, \varphi, \eta) \in X : \|v\|_{X_1} \leq K_1, \|\theta(\varphi, \eta)\|_{X_2} \leq K_2\},
\]
for two positive constants $K_1, K_2$ choosing later.

Using Itô’s formula to the function $\frac{1}{2}\|u_{\varepsilon, \delta}\|_{L^2_2}^2$, we have
\[
\frac{1}{2}\|u_{\varepsilon, \delta}\|_{L^2_2}^2 + a_0(u_{\varepsilon, \delta}, u_{\varepsilon, \delta})dt
= \left(-\Psi_R B_0(v_{\varepsilon, \delta}, J_\varepsilon \varphi_{\varepsilon, \delta}, u_{\varepsilon, \delta}) + \Psi_R B_1(\tilde{\mu}_{\varepsilon, \delta}, J_\varepsilon \varphi_{\varepsilon, \delta}, u_{\varepsilon, \delta}) + \Psi_R(K(\eta_{\varepsilon, \delta}) \nabla \tau(J_\varepsilon \eta_{\varepsilon, \delta}), u_{\varepsilon, \delta})(\Gamma)\right)dt
+ \frac{1}{2}\|P h(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta})\|_{L^2_2(\mathcal{D}; L^2_2)}^2 dt + (h(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta}), u_{\varepsilon, \delta})dW
= : \sum_{i=1}^4 I_i dt + (h(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta}), u_{\varepsilon, \delta})dW. \tag{3.13}
\]
By Hölder’s inequality, Sobolev’s embedding $H^2 \hookrightarrow L^\infty$, Lemmas $5.2$ and $5.3$ we obtain
\begin{equation}
|I_1| \leq C \Psi R \|u_{\varepsilon, \delta}\|_{L^2} \|v_{\varepsilon, \delta}\|_{L^2} \|J_\varepsilon v_{\varepsilon, \delta}\|_{H^1} \leq \frac{3}{16} \|u_{\varepsilon, \delta}\|^2_{L^2} + \frac{C}{\varepsilon^6} \Psi^2 R \|v_{\varepsilon, \delta}\|^4_{L^2},
\end{equation}
\begin{equation}
|I_2| \leq C \Psi R \|u_{\varepsilon, \delta}\|_{L^2} \|\bar{\mu}_{\varepsilon, \delta}\|_{L^2} \|J_\varepsilon F_{\varepsilon, \delta}\|_{H^1} \leq \frac{3}{16} \|u_{\varepsilon, \delta}\|^2_{L^2} + \frac{C}{\varepsilon^4} \Psi^2 R \|\bar{\mu}_{\varepsilon, \delta}\|^2_{L^2} \|\varphi_{\varepsilon, \delta}\|^2_{H^1},
\end{equation}
\begin{equation}
|I_3| \leq C \Psi R \|u_{\varepsilon, \delta, \tau}\|_{L^2(\Gamma)} \|\tilde{K}(\xi_{\varepsilon, \delta})\|_{L^2(\Gamma)} \|\nabla T(J_\varepsilon \eta_{\varepsilon, \delta})\|_{H^1(\Gamma)}
\leq \frac{1}{2} \|D(u_{\varepsilon, \delta})\|^2_{L^2} + 2\alpha \|u_{\varepsilon, \delta}\|^2_{L^2} + \frac{C}{\varepsilon^4} \Psi^2 R \|\tilde{K}(\xi_{\varepsilon, \delta})\|^2_{L^2(\Gamma)} \|\eta_{\varepsilon, \delta}\|^2_{H^1(\Gamma)}.
\end{equation}
Furthermore, by the Burkholder-Davis-Gundy inequality, for any $p \geq 2$, we get
\begin{equation}
\mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t (h(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta}, u_{\varepsilon, \delta}) dW_t) \right|^p \right) 
\leq C \mathbb{E} \left( \int_0^T \|u_{\varepsilon, \delta}\|^2_{L^2} (1 + \|(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta})\|^2_{L^2}) dt \right)^\frac{p}{2}
\leq C \mathbb{E} \left( \int_0^T \|u_{\varepsilon, \delta}\|^4_{L^2} + 1 + \|(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta})\|^4_{L^2} dt \right)^\frac{p}{2}
\leq CT^{\frac{p}{4}} \mathbb{E} \left( \int_0^T \|u_{\varepsilon, \delta}\|^{2p}_{L^2} dt + C T^{\frac{p}{4}} \left( 1 + \| (v_{\varepsilon, \delta}, \varphi_{\varepsilon, \delta}) \|^2_{X} \right) \right).
\end{equation}
Integrating of $t$, and then taking supremum, power $p$, expectation, we deduce from (3.13)-(3.17) that
\begin{equation}
\mathbb{E} \left( \sup_{t \in [0, T]} \|u_{\varepsilon, \delta}\|^{2p}_{L^2} \right) + \mathbb{E} \left( \int_0^T \|\nabla u_{\varepsilon, \delta}\|^2_{L^2} dt \right)^p
\leq \mathbb{E} \|u_0\|^{2p}_{L^2} + \left( \frac{C R^4 T}{\varepsilon^6} + \frac{C R^2 K_2}{\varepsilon^4} + C(K_1, K_2) T^{\frac{1}{2}} \right)^p
+ C(p) \left( \sqrt{2} T^{1-\frac{3}{2}} + T^{\frac{1}{2}-\frac{1}{p}} \right)^p \mathbb{E} \int_0^T \|u_{\varepsilon, \delta}\|^{2p}_{L^2} dt.
\end{equation}
In above, we also have used the Korn inequality $\|\nabla u\|_{L^2} \leq \sqrt{2} \|D(u)\|_{L^2} \leq \sqrt{2} \|\nabla u\|_{L^2}$. It follows from (3.18) and Gronwall’s inequality that
\begin{equation}
\mathbb{E} \left( \sup_{t \in [0, T]} \|u_{\varepsilon, \delta}\|^{2p}_{L^2} \right) + \mathbb{E} \left( \int_0^T \|\nabla u_{\varepsilon, \delta}\|^2_{L^2} dt \right)^p
\leq \left( C \mathbb{E} \|u_0\|^2_{L^2} + \frac{C R^4 T}{\varepsilon^6} + \frac{C R^2 K_2}{\varepsilon^4} + C(K_1, K_2) T^{\frac{1}{2}} \right)^p e^{C T \left( \sqrt{2} T^{1-\frac{3}{2}} + T^{\frac{1}{2}-\frac{1}{p}} \right)}.
\end{equation}
For fixed $R$, we first choose $K_2, K_1$ such that
\begin{equation}
C \mathbb{E} \|u_0\|^{2p}_{L^2} \leq \frac{K_1}{3}, \left( \frac{C R^2 K_2}{\varepsilon^4} \right)^p e^{C T \left( \sqrt{2} T^{1-\frac{3}{2}} + T^{\frac{1}{2}-\frac{1}{p}} \right)} \leq \frac{K_1}{3},
\end{equation}
and then $T$ small enough such that
\[
\left( \frac{CR^4 T}{\varepsilon^6} + C(K_1, K_2) T^{\frac{2}{3}} \right)^p e^{CT \left( \sqrt{T^2 T^{1-\frac{2}{3}}} + T^{\frac{2}{3}} \right)^p} \leq \frac{K_1}{3}.
\]
As (6.6) in appendix, taking power $p$, and then expectation, we obtain
\[
\mathbb{E} \left( \parallel \phi_{e, \delta}, \psi_{e, \delta} \parallel^2_{L^p_t(V^1)} + \parallel \phi_{e, \delta}, \psi_{e, \delta} \parallel^2_{L^2_t(H^1) \times L^2(\Gamma)} \right) + \delta \parallel \phi_{e, \delta}, \psi_{e, \delta} \parallel^2_{L^2_t(V^2)} \leq C_p \mathbb{E} \left( \parallel (\phi_0, \psi_0) \parallel^2_{V^1} + \parallel J \parallel^2_{L^2_t(V^2)} \right).
\]
Choosing (3.11) and (3.12), we have
\[
\parallel J \parallel^2_{L^2_t(V^2)} \leq C_p \mathbb{E} \left( \parallel (\phi_0, \psi_0) \parallel^2_{V^1} + \sum_{i=1}^{N} J_i \right)^p.
\]
We estimate terms $J_1$, $J_2$ using Hölder's inequality, Lemmas 5.2 and 5.3 to find
\[
|J_1| \leq \int_0^T \Psi_R^2 \parallel v_{e, \delta} \parallel^2_{L^2_t(V^2)} \parallel J \parallel^2_{H^1_t(V^2)} dt \leq \frac{C}{C^4} \int_0^T \Psi_R^2 \parallel v_{e, \delta} \parallel^2_{L^2_t(V^2)} \parallel J \parallel^2_{H^1_t(V^2)} dt \leq \frac{CR^4 T}{\varepsilon^6},
\]
\[
|J_2| \leq \int_0^T \Psi_R^2 \parallel v_{e, \delta} \parallel^2_{L^2_t(V^2)} \parallel J \parallel^2_{H^1_t(V^2)} dt \leq \frac{C}{C^4} \int_0^T \Psi_R^2 \parallel v_{e, \delta} \parallel^2_{L^2_t(V^2)} \parallel J \parallel^2_{H^1_t(V^2)} dt \leq \frac{C(K_1) R^2 T}{\varepsilon^4} + \frac{C(K_1) R^2 \varepsilon}{\varepsilon^4}.
\]
By (3.11) and (3.12), we have
\[
|J_3 + J_4| \leq C(K_2) T.
\]
Combining estimates (3.19), (3.22), we find
\[
\mathbb{E} \left( \sup_{t \in [0, T]} \parallel (\phi_{e, \delta}, \psi_{e, \delta}) \parallel^2_{V^1} + \parallel \phi_{e, \delta}, \psi_{e, \delta} \parallel^2_{L^2_t(H^1) \times L^2(\Gamma)} \right) + \delta \parallel \phi_{e, \delta}, \psi_{e, \delta} \parallel^2_{L^2_t(V^2)} \leq C_p \mathbb{E} \left( \parallel (\phi_0, \psi_0) \parallel^2_{V^1} + C(K_1, K_2, p) \left( \frac{R^4 T}{\varepsilon^4} + \frac{R^2 T}{\varepsilon^4 \varepsilon^2} + \frac{R^2 \varepsilon}{\varepsilon^4} + T \right)^p \right)
\]
Choosing
\[
3C_p \mathbb{E} \parallel (\phi_0, \psi_0) \parallel^2_{V^1} \leq K_2, \ \varepsilon < \frac{\varepsilon^4 K_2^\frac{1}{2}}{3T C(K_1, K_2, p) R^2},
\]
and then a small time $T$ such that
\[
C(K_1, K_2, p) \left( \frac{R^4 T}{\varepsilon^4} + \frac{R^2 T}{\varepsilon^4 \varepsilon^2} + T \right)^p \leq \frac{K_2}{3}.
\]
We conclude that $M$ is a mapping from $B_{K_1, K_2}$ into itself for $t \in [0, T]$ from (3.18), (3.23).
**Step 2.** Mapping $\mathcal{M}$ is contraction.

Let $(v^i_{\epsilon,\delta}, \varphi^i_{\epsilon,\delta}, \eta^i_{\epsilon,\delta}) \in B_{K_1, K_2}$ for $i = 1, 2$ and $(u^i_{\epsilon,\delta}, \phi^i_{\epsilon,\delta}, \psi^i_{\epsilon,\delta}) = \mathcal{M}(v^i_{\epsilon,\delta}, \varphi^i_{\epsilon,\delta}, \eta^i_{\epsilon,\delta})$. Denote by $(\bar{u}_{\epsilon,\delta}, \bar{\phi}_{\epsilon,\delta}, \bar{\psi}_{\epsilon,\delta})$, $(\bar{v}_{\epsilon,\delta}, \bar{\varphi}_{\epsilon,\delta}, \bar{\eta}_{\epsilon,\delta})$ the differences of $(u^i_{\epsilon,\delta}, \phi^i_{\epsilon,\delta}, \psi^i_{\epsilon,\delta})$, $(v^i_{\epsilon,\delta}, \varphi^i_{\epsilon,\delta}, \eta^i_{\epsilon,\delta})$ satisfying

\[
\begin{aligned}
\{ 
& d\bar{u}_{\epsilon,\delta} - \text{div}(2D(\bar{u}_{\epsilon,\delta}))dt + \Psi^1_R(b_0(v^1_{\epsilon,\delta}, J_{\bar{u}_{\epsilon,\delta}}) + b_0(\bar{v}_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}}))dt \\
& + \tilde{\Phi}_R b_0(v^2_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}})dt - (\Psi^1_R b_{\epsilon,\delta}) \nabla J_{\phi^1_{\epsilon,\delta}} - \Psi^2_R \bar{\mu}_{\epsilon,\delta} \nabla J_{\varphi^2_{\epsilon,\delta}})dt \\
& = (h(v^1_{\epsilon,\delta}, \nabla \varphi^1_{\epsilon,\delta}), h(v^2_{\epsilon,\delta}, \nabla \varphi^2_{\epsilon,\delta}))d\mathcal{W}, \\
& (D(\bar{u}_{\epsilon,\delta}) \cdot n)_{\tau} + \bar{u}_{\epsilon,\delta,\tau} \\
& = \Psi^1_R(\mathfrak{K}(\bar{\eta}_{\epsilon,\delta}) \nabla (J_{\bar{u}_{\epsilon,\delta}}) + K(\eta^1_{\epsilon,\delta}) \nabla (J_{\bar{u}_{\epsilon,\delta}})) + \tilde{\Phi}_R \mathfrak{K}(\eta^2_{\epsilon,\delta}) \nabla (J_{\bar{u}_{\epsilon,\delta}}), \\
& \partial_t \bar{\phi}_{\epsilon,\delta} + A_t \bar{\phi}_{\epsilon,\delta} + \Psi^2_R (b_1(v_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}}) + b_2(v^2_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}})) + \tilde{\Phi}_R b_2(v^2_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}}) = 0, \\
& \partial_t \bar{v}_{\epsilon,\delta} + \Psi^1_R (b_1(v_{\epsilon,\delta}, J_{\bar{\eta}_{\epsilon,\delta}}) + b_2(v^2_{\epsilon,\delta}, J_{\bar{\eta}_{\epsilon,\delta}})) + \tilde{\Phi}_R b_1(v^2_{\epsilon,\delta}, J_{\bar{\eta}_{\epsilon,\delta}}) + \mathfrak{K}(\bar{v}_{\epsilon,\delta}) \\
& = (-g(\eta^1_{\epsilon,\delta}) - g(\eta^2_{\epsilon,\delta}),) \\
& (\bar{u}_{\epsilon,\delta}(0), \bar{\phi}_{\epsilon,\delta}(0), \bar{v}_{\epsilon,\delta}(0)) = (0, 0, 0),
\end{aligned}
\]

with

\[
\begin{aligned}
\bar{\mu}_{\epsilon,\delta} &= -\triangle \bar{\phi}_{\epsilon,\delta} + \delta A_t \bar{\phi}_{\epsilon,\delta} + f_\epsilon(\varphi^1_{\epsilon,\delta}) - f_\epsilon(\varphi^2_{\epsilon,\delta}), \\
\mathfrak{K}(\bar{\eta}_{\epsilon,\delta}) &= A_t \bar{\eta}_{\epsilon,\delta} + \partial_t \bar{\phi}_{\epsilon,\delta} + \bar{\eta}_{\epsilon,\delta} + g_\epsilon(\eta^1_{\epsilon,\delta}) - g_\epsilon(\eta^2_{\epsilon,\delta}),
\end{aligned}
\]

and

\[
\begin{aligned}
\tilde{\Phi}_R &= \Psi^1_R - \Psi^2_R, \\
\Psi^1_R &= \Psi^1_R(\|v^1_{\epsilon,\delta}\|_{L^2}^2) \cdot \|v^1_{\epsilon,\delta}\|_{L^2}, \\
\Psi^2_R &= \Psi^2_R(\|v^2_{\epsilon,\delta}\|_{L^2}^2) \cdot \|v^2_{\epsilon,\delta}\|_{L^2}.
\end{aligned}
\]

Using Itô's formula to the function \(\frac{1}{2}\|\bar{u}_{\epsilon,\delta}\|_{L^2}^2\), we have

\[
\begin{aligned}
& \frac{1}{2} d\|\bar{u}_{\epsilon,\delta}\|_{L^2}^2 + a_0(\bar{u}_{\epsilon,\delta}, \bar{u}_{\epsilon,\delta})dt \\
& = (-\Psi^1_R b_0(v^1_{\epsilon,\delta}, J_{\bar{u}_{\epsilon,\delta}}) - \Psi^1_R b_0(\bar{v}_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}}) - \tilde{\Phi}_R b_0(v^2_{\epsilon,\delta}, J_{\bar{v}_{\epsilon,\delta}}))d\mathcal{W}, \\
& + \Psi^1_R(\Delta \varphi^1_{\epsilon,\delta} \nabla (J_{\phi^1_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta}) + \Psi^1_R(\Delta \varphi^2_{\epsilon,\delta} \nabla (J_{\phi^2_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta}) - \Psi^2_R \nabla (J_{\phi^1_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta})d\mathcal{W}, \\
& - \Psi^1_R(\partial_t \bar{\phi}_{\epsilon,\delta} \nabla (J_{\phi^2_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta}) - \Psi^1_R((f_\epsilon(\varphi^1_{\epsilon,\delta}) - f_\epsilon(\varphi^2_{\epsilon,\delta}))) \nabla (J_{\phi^2_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta})d\mathcal{W}, \\
& - \Psi^2_R(\bar{v}_{\epsilon,\delta} \nabla (J_{\bar{\eta}_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta}) - \tilde{\Phi}_R (v^2_{\epsilon,\delta} \nabla (J_{\bar{\eta}_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta})d\mathcal{W}, \\
& + \Psi^2_R(\mathfrak{K}(\eta^1_{\epsilon,\delta}) \nabla (J_{\bar{\eta}_{\epsilon,\delta}}), \bar{u}_{\epsilon,\delta})d\mathcal{W}, \\
& + \frac{1}{2} \|h(v^1_{\epsilon,\delta}, \nabla \varphi^1_{\epsilon,\delta}) - h(v^2_{\epsilon,\delta}, \nabla \varphi^2_{\epsilon,\delta})\|_{L^2(\mathcal{H}, L^2)}^2 dt \\
& + (h(v^1_{\epsilon,\delta}, \nabla \varphi^1_{\epsilon,\delta}) - h(v^2_{\epsilon,\delta}, \nabla \varphi^2_{\epsilon,\delta})), \bar{u}_{\epsilon,\delta})d\mathcal{W}.
\end{aligned}
\]

We proceed to estimate all terms $H_k, k = 1, \cdots, 14$. By Hölder’s inequality, the embedding $H^2 \hookrightarrow L^\infty$, and Lemma 5.5, we deduce

\[
|H_1| \leq C\Psi^1_R\|\bar{u}_{\epsilon,\delta}\|_{H^1} \|v^1_{\epsilon,\delta}\|_{L^2} \|\nabla J_{\bar{u}_{\epsilon,\delta}}\|_{H^2}.
\]
where $\varepsilon$ is a constant selected later. Hölder’s inequality, Property (i), the mean value theorem, and Lemma 5.5 yield

$$\begin{align*}
|H_6| + |H_7| & \leq \frac{C}{\varepsilon^2} \Psi_{\varepsilon}^1 \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}^2 \|\partial_t \tilde{\varphi}_{\varepsilon,\delta}\|_{L_2^1}^2 \|\varphi_{\varepsilon,\delta}\|_{H^1} + \frac{C}{\varepsilon^2} \Psi_{\varepsilon}^1 \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2} \|\partial_t \varphi_{\varepsilon,\delta}\|_{L_2^1} \|\varphi_{\varepsilon,\delta}\|_{H^1} \\
& \leq \frac{C}{\varepsilon^2} \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}^2 + \varepsilon_1 (\delta \Psi_{\varepsilon}^1)^2 \|\partial_t \tilde{\varphi}_{\varepsilon,\delta}\|_{L_2^1}^2 \|\varphi_{\varepsilon,\delta}\|_{H^1} + \varepsilon_1 (\delta \Psi_{\varepsilon}^1)^2 \|\partial_t \varphi_{\varepsilon,\delta}\|_{L_2^1}^2 \|\varphi_{\varepsilon,\delta}\|_{H^1},
\end{align*}$$

(3.30)

where $\varepsilon_1$ is a constant selected later. Hölder’s inequality, Property (i), the mean value theorem, and Lemma 5.5 yield

$$\begin{align*}
|H_8| & \leq C \Psi_{\varepsilon}^1 \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2} \|f_\varepsilon(\varphi_{\varepsilon,\delta}^1) - f_\varepsilon(\varphi_{\varepsilon,\delta}^2)\|_{L_2^2} \|\nabla \mathcal{J}_\varepsilon \varphi_{\varepsilon,\delta}^2\|_{H^2} \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2} \|\varphi_{\varepsilon,\delta}\|_{L_2^2} \|\nabla \mathcal{J}_\varepsilon \varphi_{\varepsilon,\delta}^2\|_{H^2} \\
& \leq \frac{C}{\varepsilon^2} \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2} \|\varphi_{\varepsilon,\delta}\|_{L_2^2} \|\nabla \mathcal{J}_\varepsilon \varphi_{\varepsilon,\delta}^2\|_{H^2} \\
& \leq \frac{1}{4} \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}^2 + \frac{C}{\varepsilon^6} \|\varphi_{\varepsilon,\delta}\|_{L_2^2}^2 \|\nabla \mathcal{J}_\varepsilon \varphi_{\varepsilon,\delta}^2\|_{H^2}.
\end{align*}$$

(3.31)

The term $H_9$ could be bounded easily. For term $H_{12}$, by Hölder’s inequality, Lemma 5.2 and Lemma 5.5, we deduce

$$\begin{align*}
|H_{12}| & \leq C \Psi_{\varepsilon}^1 \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} \|\nabla \tau(\mathcal{J}_\varepsilon \eta_{\varepsilon,\delta})\|_{H^2(\Gamma)} \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\nabla \tau(\mathcal{J}_\varepsilon \eta_{\varepsilon,\delta})\|_{H^2(\Gamma)} \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\nabla \tau(\mathcal{J}_\varepsilon \eta_{\varepsilon,\delta})\|_{H^2(\Gamma)} \left(\tilde{\varepsilon}\|D(\tilde{u}_{\varepsilon,\delta})\|_{L_2^2} + \tilde{\varepsilon}^{-\nu}\|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}\right) \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \left(\tilde{\varepsilon}\|D(\tilde{u}_{\varepsilon,\delta})\|_{L_2^2} + \tilde{\varepsilon}^{-\nu}\|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}\right) \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} + C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\eta_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \left(\tilde{\varepsilon}\|D(\tilde{u}_{\varepsilon,\delta})\|_{L_2^2} + \tilde{\varepsilon}^{-\nu}\|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}\right) \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} + C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \left(\tilde{\varepsilon}\|D(\tilde{u}_{\varepsilon,\delta})\|_{L_2^2} + \tilde{\varepsilon}^{-\nu}\|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2}\right) \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} + C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)}
\end{align*}$$

(3.32)

we choose $\tilde{\varepsilon} = \frac{\varepsilon^2}{\sqrt{C}}$. For term $H_{11}$, using (3.12), Lemma 5.2 and Hölder’s inequality, we obtain

$$\begin{align*}
|H_{11}| & \leq C \Psi_{\varepsilon}^1 \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} + \|g_\varepsilon(\eta_{\varepsilon,\delta}) - g_\varepsilon(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} \|\nabla \tau(\mathcal{J}_\varepsilon \eta_{\varepsilon,\delta})\|_{H^2(\Gamma)} \\
& \leq C \|\tilde{u}_{\varepsilon,\delta}\|_{L_2^2(\Gamma)} \|\mathcal{K}(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} + \|g_\varepsilon(\eta_{\varepsilon,\delta}) - g_\varepsilon(\eta_{\varepsilon,\delta})\|_{L_2^2(\Gamma)} \|\nabla \tau(\mathcal{J}_\varepsilon \eta_{\varepsilon,\delta})\|_{H^2(\Gamma)}
\end{align*}$$
\[ + \frac{C}{\varepsilon} \|\eta_{\varepsilon, \delta}\|_{L^2(\Gamma)} \left( \frac{1}{2}\|D(\bar{u}_{\varepsilon, \delta})\|_{L^2} + C_1 \|\bar{u}_{\varepsilon, \delta}\|_{L^2} \right) \|\bar{\eta}_{\varepsilon, \delta}\|_{L^2(\Gamma)} \]
\[ \leq \frac{C}{\varepsilon} \left( \varepsilon^2 \|D(\bar{u}_{\varepsilon, \delta})\|_{L^2}^2 + \varepsilon^{-2}\|\bar{u}_{\varepsilon, \delta}\|_{L^2}^2 \right) + \varepsilon_1 \left( \|\bar{\varphi}_{\varepsilon, \delta}, \bar{\eta}_{\varepsilon, \delta}\|_{V^2}^2 + c_1 \|\eta_{\varepsilon, \delta}\|_{L^2(\Gamma)}^2 \right) \]
\[ + \frac{1}{4} \|\bar{u}_{\varepsilon, \delta}\|_{L^2}^2 + \frac{1}{2} a_0 (\bar{u}_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) + \frac{C}{\varepsilon} \|\eta_{\varepsilon, \delta}\|_{L^2(\Gamma)} \|\bar{\eta}_{\varepsilon, \delta}\|_{L^2(\Gamma)}^2. \]  \hspace{1cm} (3.33)

For term \( H_{14} \), by assumption \( A_2 \), we have
\[ |H_{14}| \leq C \|(v_{\varepsilon, \delta}, \nabla \varphi_{\varepsilon, \delta})\|_{L^2}^2. \]

To surmount the difficulty caused by cut-off function, we further define stopping time
\[ \tau^i_R := \inf \left\{ t \geq 0; \|(v^i_{\varepsilon, \delta}, \varphi^i_{\varepsilon, \delta}, \eta^i_{\varepsilon, \delta})\|_X > 2R \right\} \]
for \( i = 1, 2 \). If the right-hand side set is empty, taking \( \tau^i_R = T \). Without loss of generality, we could assume that \( \tau^1_R \leq \tau^2_R \). Consequently, if \( t \geq \tau^2_R \), we have
\[ \|\nabla_R (\|(v^1_{\varepsilon, \delta}, \varphi^1_{\varepsilon, \delta}, \eta^1_{\varepsilon, \delta})\|_X) = 0, \ i = 1, 2. \]

For term \( H_5 \), by Lemma [5.3] and the Schwartz inequality, we have
\[
E \left( \int_0^T |H_3| dt \right)^p \leq E \left( \int_0^T |\nabla_R (\|(v^1_{\varepsilon, \delta})\|_{L^2}^2) - \nabla_R (\|(v^2_{\varepsilon, \delta})\|_{L^2}^2)\nabla_R (\|(\varphi^1_{\varepsilon, \delta}, \eta^1_{\varepsilon, \delta})\|_{V^1}^1) B_0 (v^2_{\varepsilon, \delta}, \nabla v^2_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) + \|\nabla_R (\|(\varphi^1_{\varepsilon, \delta}, \eta^1_{\varepsilon, \delta})\|_{V^1}^1) B_0 (v^2_{\varepsilon, \delta}, \nabla v^2_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) dt \right)^p \]
\[ \leq C E \left( \int_0^T \|J \varepsilon \eta_{\varepsilon, \delta}\|_{H^3}^2 \|\bar{u}_{\varepsilon, \delta}\|_{L^2}^2 + \|\nabla_R (\|(\varphi^1_{\varepsilon, \delta}, \eta^1_{\varepsilon, \delta})\|_{V^1}^1) B_0 (v^2_{\varepsilon, \delta}, \nabla v^2_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) dt \right)^p \]
\[ \leq C E \left( \int_0^T \|\nabla \varphi_{\varepsilon, \delta}\|_{L^2}^2 + \|\nabla \eta_{\varepsilon, \delta}\|_{L^2}^2 + \|\nabla \eta_{\varepsilon, \delta}\|_{L^2}^2 \right)^p \]
\[ \leq C (\tau^2_R)^p E \left( \|\nabla v_{\varepsilon, \delta}\|_{L^\infty L^2}^{2p} + \|\nabla (\varphi_{\varepsilon, \delta}, \eta_{\varepsilon, \delta})\|_{L^\infty V^1}^{2p} \right) + \frac{C (\tau^2_R)^p R^p}{\varepsilon} E \|\nabla \bar{v}_{\varepsilon, \delta}\|_{L^\infty L^2}^{2p} \]
\[ \leq C T^p E \left( \|\nabla v_{\varepsilon, \delta}\|_{L^\infty L^2}^{2p} + \|\nabla (\varphi_{\varepsilon, \delta}, \eta_{\varepsilon, \delta})\|_{L^\infty V^1}^{2p} \right) + \frac{C T^p R^p}{\varepsilon} E \|\nabla \bar{v}_{\varepsilon, \delta}\|_{L^\infty L^2}^{2p}. \]  \hspace{1cm} (3.34)

Similarly, we have
\[
E \left( \int_0^T |H_{10}| dt \right)^p \leq E \left( \int_0^T |\nabla_R (\|(v^1_{\varepsilon, \delta})\|_{L^2}^2) - \nabla_R (\|(v^2_{\varepsilon, \delta})\|_{L^2}^2)\nabla_R (\|(\varphi^1_{\varepsilon, \delta}, \eta^1_{\varepsilon, \delta})\|_{V^1}^1) (\mu_{\varepsilon, \delta} \nabla J \varphi^2_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) + \|\nabla_R (\|(\varphi^1_{\varepsilon, \delta}, \eta^1_{\varepsilon, \delta})\|_{V^1}^1) (\mu_{\varepsilon, \delta} \nabla J \varphi^2_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) dt \right)^p \]
\[
\begin{align*}
&\leq C\mathbb{E}\left( \int_{0}^{\tau_{R}^{2}} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{V^{1}} \right) \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}} \|\mu_{\varepsilon,\delta}^{2}\|_{L^{2}_{\varepsilon}} \|J_{\varepsilon}^{2}\|_{H^{3}}dt \right)^{p} \\
&\leq \frac{C}{\varepsilon_{1}^{2}} (\tau_{R})^{p} \mathbb{E} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}}^{2p} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}V^{1}}^{2p} \right) + \frac{\varepsilon_{1}R^{4p}}{\varepsilon^{4p}} \mathbb{E} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}}^{2p} \\
&\leq \frac{C}{\varepsilon_{1}^{2}} \mathbb{E} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}}^{2p} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}V^{1}}^{2p} \right) + \frac{\varepsilon_{1}R^{4p}}{\varepsilon^{4p}} \mathbb{E} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}}^{2p},
\end{align*}
\]

and
\[
\mathbb{E} \left( \int_{0}^{T} |H_{13}| dt \right)^{p}
\leq \mathbb{E} \left( \int_{0}^{T} \left| \mathbb{V}_{R}(\|v^{1}_{\varepsilon,\delta}\|_{L^{2}\varepsilon}) - \mathbb{V}_{R}(\|v^{1}_{\varepsilon,\delta}\|_{L^{2}\varepsilon}) \right| \left| \mathbb{V}_{R}(\|\varphi_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\|_{V^{1}}) \right| \left| \mathbb{V}_{R}(\|\varphi_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\|_{V^{1}}) - \mathbb{V}_{R}(\|\varphi_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\|_{V^{1}}) \right| \right)^{p}
\times \left( \mathbb{V}_{R}(\|\varphi_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\|_{V^{1}}) \right)^{p}
\leq \mathbb{E} \left( \int_{0}^{\tau_{R}^{2}} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}(\Gamma)} \|\mathbb{K}(\eta_{\varepsilon,\delta})\|_{L^{2}(\Gamma)} \|\nabla_{\tau}(J_{\varepsilon}\eta_{\varepsilon,\delta})\|_{H^{2}(\Gamma)} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}\varepsilon} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{V^{1}} \right) dt \right)^{p}
\leq \mathbb{E} \left( \int_{0}^{\tau_{R}^{2}} \frac{C}{\varepsilon^{3}} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}\varepsilon} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{V^{1}} \right) \|\eta_{\varepsilon,\delta}\|_{L^{2}(\Gamma)} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}(\Gamma)} dt \right)^{p}
\leq \mathbb{E} \left( \int_{0}^{\tau_{R}^{2}} \frac{C}{\varepsilon^{3}} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}\varepsilon} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{V^{1}} \right) \|\eta_{\varepsilon,\delta}\|_{L^{2}(\Gamma)} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}(\Gamma)} dt \right)^{p}
\leq \frac{(\tau_{R}^{2})^{p}}{\varepsilon_{1}^{2p}} \mathbb{E} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}}^{2p} + \frac{\varepsilon_{2p}}{\varepsilon^{6p}} \mathbb{E} \left( \int_{0}^{\tau_{R}^{2}} a_{0}(\bar{u}_{\varepsilon,\delta}, \bar{u}_{\varepsilon,\delta}) dt \right)^{p}
+ \frac{C\varepsilon_{1}R^{4p}}{\varepsilon^{6p}} \mathbb{E} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}V^{1}}^{2p} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{V^{1}}^{2p} \right).
\]

For the last term, the Burkholder-Davis-Gundy inequality and assumption $A_{2}$ give
\[
\begin{align*}
&\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_{0}^{t} h(v_{\varepsilon,\delta}^{1}, \nabla \varphi_{\varepsilon,\delta}^{1}) - h(v_{\varepsilon,\delta}^{2}, \nabla \varphi_{\varepsilon,\delta}^{2}) dW \right| \right)^{p} \\
&\leq C\mathbb{E} \left( \int_{0}^{T} \left( \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}}^{2} + \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{L^{2}}^{2} \right) dt \right)^{\frac{p}{2}} \\
&\leq CT^{\frac{p}{2}-1} \mathbb{E} \int_{0}^{T} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}}^{2p} dt + CT^{\frac{p}{2}} \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{X}^{2p} \\
&\leq CT^{\frac{p}{2}} \|\bar{u}_{\varepsilon,\delta}\|_{L^{2}_{\varepsilon}V^{1}}^{2p} + CT^{\frac{p}{2}} \|\varphi_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_{X}^{2p}.
\end{align*}
\]
Integrating of $t$, then taking supremum over $[0, T]$, power $p$ and expectation, combining all estimates (3.26)-(3.37), we have

$$
\frac{1}{2p} \mathbb{E} \left( \sup_{t \in [0, T]} \| \bar{u}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta}}^{2p} \right) + \left( \frac{1}{2} - 4\varepsilon_1 \right)^p \mathbb{E} \left( \int_0^T a_0(\bar{u}_{\varepsilon, \delta}, \bar{u}_{\varepsilon, \delta}) dt \right)^p \\
\leq T^p \left[ CT^{-\frac{8}{3}} + \left( \frac{CR^2 + R^4 + C}{\varepsilon_1 \varepsilon^{\delta}} \right)^p \right] \mathbb{E} \left( \| \bar{u}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta}}^{2p} + \| (\bar{\varphi}_{\varepsilon, \delta}, \bar{\eta}_{\varepsilon, \delta}) \|_{L^p_{\varepsilon, \delta}}^{2p} \right) \\
+ C(\varepsilon_1 R^2 + \varepsilon_1)^p \mathbb{E} \left( \| (\bar{\varphi}_{\varepsilon, \delta}, \bar{\eta}_{\varepsilon, \delta}) \|_{L^2_{\varepsilon, \delta}}^{2p} + \| \partial_t \bar{\varphi}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta}}^{2p} \right) + C\varepsilon_1^2 \mathbb{E} \left( \| (\bar{\varphi}_{\varepsilon, \delta}, \bar{\eta}_{\varepsilon, \delta}) \|_{L^p_{\varepsilon, \delta}}^{2p} \right) \\
+ T^p \left( \frac{1}{2} + \frac{C}{\varepsilon_1 \varepsilon^{\delta}} + \frac{C}{\varepsilon_1 \varepsilon^{2\nu}} + \frac{C}{\varepsilon_1 \varepsilon^{6\nu}} \right)^p \mathbb{E} \left( \| \bar{u}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta}}^{2p} + \| CT_\varepsilon^{\frac{R}{2}} \| \| \bar{u}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta}}^{2p} \right) \\
+ \frac{\varepsilon_1 R^p}{\varepsilon^{4p}} \mathbb{E} \left( \| \bar{u}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta}}^{2p} \right).
$$

We first choose $\varepsilon_1$ small and then take $T$ small enough such that

$$
\left[ \frac{1}{2} + \frac{C}{\varepsilon_1 \varepsilon^{\delta}} + \frac{C}{\varepsilon_1 \varepsilon^{2\nu}} + \frac{C}{\varepsilon_1 \varepsilon^{6\nu}} \right]^p \cdot (2T)^p + C(4T)^{\frac{4}{3}} \leq \frac{1}{4}, \quad C2^p(\varepsilon_1 R^2 + \varepsilon_1)^p \leq \frac{1}{8}, \\
\left[ CT^{-\frac{8}{3}} + \left( \frac{CR^2 + C}{\varepsilon_1 \varepsilon^{\delta}} \right)^p \right] \cdot (2T)^p + \frac{\varepsilon_1 R^p}{\varepsilon^{4p}} \leq \frac{1}{8}.
$$

From (3.38), for the small time $T$, we conclude that

$$
\| u_{\varepsilon, \delta}^1 - u_{\varepsilon, \delta}^2 \|_{X^1} \leq \frac{1}{2} \| (\bar{u}_{\varepsilon, \delta}, \bar{\varphi}_{\varepsilon, \delta}, \bar{\eta}_{\varepsilon, \delta}) \|_{X^2}^{2p}.
$$

Next, taking $L^2_{\varepsilon, \delta}, L^2(\Gamma)$ inner product with $A_1^{-1}(\partial_t \bar{\varphi}_{\varepsilon, \delta})$ and $\partial_t \bar{\varphi}_{\varepsilon, \delta}$ in (3.21)_{3, 4}, and also taking product with $\bar{\mu}_{\varepsilon, \delta} - \langle \bar{\mu}_{\varepsilon, \delta} \rangle$ in (3.21)_{3, 4}, and then taking expectation, we obtain

$$
\frac{1}{2} \mathbb{E} \left( \| \bar{\varphi}_{\varepsilon, \delta}, \bar{\psi}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta} L^2(\Gamma)} \right)^{2p} + \mathbb{E} \left( \| \partial_t \bar{\varphi}_{\varepsilon, \delta} \|_{L^2_{\varepsilon, \delta} L^2(\Gamma)} \right)^{2p} + \mathbb{E} \left( \| \bar{\varphi}_{\varepsilon, \delta} \|_{H^2 \times L^2(\Gamma)} \right)^{2p} \\
\leq \mathbb{E} \left( \int_0^T -\Psi_1^R \left( b_2(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \bar{\varphi}_{\varepsilon, \delta}) + b_2(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \bar{\varphi}_{\varepsilon, \delta}) A_1^{-1}(\partial_t \bar{\varphi}_{\varepsilon, \delta}) \right) dt \right)^p \\
+ \mathbb{E} \left( \int_0^T -\Psi_1^R \left( b_2(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \bar{\varphi}_{\varepsilon, \delta}) + b_2(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \bar{\varphi}_{\varepsilon, \delta}), \bar{\mu}_{\varepsilon, \delta} - \langle \bar{\mu}_{\varepsilon, \delta} \rangle \right) dt \right)^p \\
+ \mathbb{E} \left( \int_0^T -\Psi_1^R \left( b_T(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \eta_{\varepsilon, \delta}) + b_T(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \eta_{\varepsilon, \delta}), \partial_t \bar{\varphi}_{\varepsilon, \delta} \right) dt \right)^p \\
+ \mathbb{E} \left( \int_0^T \left( f_\varepsilon(\bar{\varphi}_{\varepsilon, \delta}) - g_\varepsilon(\bar{\varphi}_{\varepsilon, \delta}), \partial_t \bar{\varphi}_{\varepsilon, \delta} \right) dt \right)^p \\
+ \mathbb{E} \left( \int_0^T \left( f_\varepsilon(\bar{\varphi}_{\varepsilon, \delta}) - g_\varepsilon(\bar{\varphi}_{\varepsilon, \delta}) + \partial_t \bar{\varphi}_{\varepsilon, \delta} \right) dt \right)^p \\
+ \mathbb{E} \left( \int_0^T -\Psi_1^R \left( b_2(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \bar{\varphi}_{\varepsilon, \delta}) A_1^{-1}(\partial_t \bar{\varphi}_{\varepsilon, \delta}) \right) dt \right)^p \\
+ \mathbb{E} \left( \int_0^T -\Psi_1^R \left( b_2(\bar{v}_{\varepsilon, \delta}, J_\varepsilon \bar{\varphi}_{\varepsilon, \delta}), \bar{\mu}_{\varepsilon, \delta} - \langle \bar{\mu}_{\varepsilon, \delta} \rangle \right) dt \right)^p 
$$

(3.39)
Since $\langle \partial_t \psi_{e, \delta}^i \rangle = 0$, $\mathbb{P}$-a.s., $i = 1, 2$, so $\langle \partial_t \tilde{\psi}_{e, \delta} \rangle = 0$, $\mathbb{P}$-a.s., simple calculation yields

$$
(\tilde{\mu}_{e, \delta}) = \langle f_{e}(\varphi_{e, \delta}^1) - f_{e}(\varphi_{e, \delta}^2) \rangle - \frac{|\Gamma|}{|D|} \langle \tilde{K}(\tilde{\psi}_{e, \delta}) \rangle_{\Gamma} + \frac{|\Gamma|}{|D|} \langle \tilde{\psi}_{e, \delta} \rangle_{\Gamma}, \ \mathbb{P}\text{-a.s.},
$$

where $\langle \Lambda \Gamma \rangle = |\Gamma|^{-1} \int_{\Gamma} \text{AdS}$. From (3.34), we get

$$
\langle \tilde{\mu}_{e, \delta} \rangle^2 \leq C \left( \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|f_{e}(\varphi_{e, \delta}^1) - f_{e}(\varphi_{e, \delta}^2)\|_{L^2(\Gamma)}^2 + \|g_{\epsilon}(\eta_{e, \delta}) - g_{\epsilon}(\eta_{e, \delta})\|_{L^2(\Gamma)}^2 
+ \|b_{\epsilon}(\tilde{\psi}_{e, \delta}, J_{e}(\eta_{e, \delta})) + b_{\epsilon}(v_{e, \delta}, J_{e}(\eta_{e, \delta}))\|_{L^2(\Gamma)}^2 + \|\psi_{e, \delta}\|_{L^2(\Gamma)}^2 \right), \ \mathbb{P}\text{-a.s.},
$$

(3.40)

where $C$ depends on $|\Gamma|, |D|$ but independent of $\delta$. Combining Lemma 5.4, 5.39, 5.40, and the Poincaré inequality, we arrive at

$$
\frac{1}{2} \mathbb{E} \left( \|\tilde{\psi}_{e, \delta}, \tilde{\psi}_{e, \delta} \|_{L^2(\Gamma)}^{2p} \right) + \mathbb{E} \left( \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|f_{e}(\varphi_{e, \delta}^1) - f_{e}(\varphi_{e, \delta}^2)\|_{L^2(\Gamma)}^2 + \|g_{\epsilon}(\eta_{e, \delta}) - g_{\epsilon}(\eta_{e, \delta})\|_{L^2(\Gamma)}^2 \right)
+ \mathbb{E} \left( \|b_{\epsilon}(\tilde{\psi}_{e, \delta}, J_{e}(\eta_{e, \delta})) + b_{\epsilon}(v_{e, \delta}, J_{e}(\eta_{e, \delta}))\|_{L^2(\Gamma)}^2 \right),
$$

(3.41)

where $C$ depends on $|\Gamma|, |D|$, but independent of $\delta$. By Hölder’s inequality, Lemma 5.2 and 5.3, we get

$$
|L_1| \leq \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^1(H^1)} + b_2(\tilde{\psi}_{e, \delta}, J_{e}\varphi_{e, \delta}) + b_2(v_{e, \delta}, J_{e}\varphi_{e, \delta})
\leq \varepsilon_2 \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^1(H^1)} + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|J_{e}\varphi_{e, \delta}\|_{L^2(\Gamma)}^2 \right)
\leq \varepsilon_2 \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^1(H^1)} + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|\varphi_{e, \delta}\|_{H^1}^2 \right),
$$

(3.42)

and

$$
|L_2| \leq \varepsilon_2 \|\nabla \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|\varphi_{e, \delta}\|_{H^1}^2 + \|\tilde{\psi}_{e, \delta}, \varphi_{e, \delta}\|_{L^2(\Gamma)}^2 \right),
$$

(3.43)

and

$$
|L_3| \leq \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} + J_{e}(\eta_{e, \delta})\right) + \|v_{e, \delta}\|_{L^2(\Gamma)} \|J_{e}(\eta_{e, \delta})\|_{H^1(\Gamma)}
\leq \frac{C}{\varepsilon_3} \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} + \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} \right) + \frac{C}{\varepsilon_3} \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} \left( \|\eta_{e, \delta}\|_{L^2(\Gamma)} \right)
\leq \varepsilon_2 \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|\eta_{e, \delta}\|_{L^2(\Gamma)}^2 \right) + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|\eta_{e, \delta}\|_{L^2(\Gamma)}^2 \right)
\leq \varepsilon_2 \|\partial_t \tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)} + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|\eta_{e, \delta}\|_{L^2(\Gamma)}^2 \right) + \frac{C}{\varepsilon_2} \left( \|\tilde{\psi}_{e, \delta}\|_{L^2(\Gamma)}^2 + \|\eta_{e, \delta}\|_{L^2(\Gamma)}^2 \right),
$$

(3.44)
In addition, by Lemma 5.3, we obtain
\[
\|b_T(\overline{v}_{\varepsilon,\delta}, J_\varepsilon(\eta_{\varepsilon,\delta})) + b_T(v^2_{\varepsilon,\delta}, J_\varepsilon(\overline{\eta}_{\varepsilon,\delta}))\|_{L^2(\Gamma)}^2
\leq \frac{C}{\varepsilon^3} (\|\overline{v}_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 \|\eta_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 + \|v^2_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 \|\overline{\eta}_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2)
\leq \frac{C}{\varepsilon^3} (\varepsilon \|D(\overline{v}_{\varepsilon,\delta})\|_{L^2(\Gamma)}^2 + \varepsilon^{-\nu} \|\overline{v}_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 \|\eta_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2)
\leq \frac{C}{\varepsilon^3} (\varepsilon \|\overline{v}_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 + \varepsilon^{-\nu} \|v^2_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 \|\eta_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2)
\]

(3.45)

For term $L_4$, using (3.11), (3.12), we have
\[
|L_4| \leq \|\partial_t \overline{\phi}_{\varepsilon,\delta}\|_{L^2(\Gamma)} \|f_\varepsilon(\varphi^1_{\varepsilon,\delta}) - f_\varepsilon(\varphi^2_{\varepsilon,\delta})\|_{L^2(\Gamma)} + \|\partial_t \overline{v}_{\varepsilon,\delta}\|_{L^2(\Gamma)} \|g_\varepsilon(\eta_{\varepsilon,\delta}) - g_\varepsilon(\overline{\eta}_{\varepsilon,\delta})\|_{L^2(\Gamma)}
\leq \frac{\varepsilon^2}{2} \|\partial_t \overline{\phi}_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 + \frac{C}{\varepsilon^2} \|f_\varepsilon(\varphi^1_{\varepsilon,\delta}) - f_\varepsilon(\varphi^2_{\varepsilon,\delta})\|_{L^2(\Gamma)}^2
\leq \frac{\varepsilon^2}{2} \|\partial_t \overline{\phi}_{\varepsilon,\delta}\|_{L^2(\Gamma)}^2 + \frac{C}{\varepsilon^2} \|f_\varepsilon(\varphi^2_{\varepsilon,\delta})\|_{L^2(\Gamma)}^2
\]

(3.46)

For term $L_5$, by Hölder’s inequality, Lemma 5.2 and Lemma 5.3 we get
\[
\mathbb{E} \left( \int_0^T L_5 dt \right)^p
\leq C \mathbb{E} \left( \int_0^T \left| \psi_R(\|v^1_{\varepsilon,\delta}\|_{L^2}) - \psi_R(\|v^2_{\varepsilon,\delta}\|_{L^2}) \right| \psi_R(\|\varphi^1_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\|_{V^1})
\times (b_2(v^2_{\varepsilon,\delta}, J_\varepsilon \varphi^2_{\varepsilon,\delta}), A^{-1}_1(\partial_t \overline{\phi}_{\varepsilon,\delta}))
\leq C \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2} + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\nabla J_\varepsilon \varphi^2_{\varepsilon,\delta}\|_{H^2} dt \right)^p
\leq C \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2} + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\varphi^2_{\varepsilon,\delta}\|_{H^1}^p dt \right)^p
\leq C \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2}^p + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\varphi^2_{\varepsilon,\delta}\|_{H^1}^p dt \right)^p
\leq \frac{C}{\varepsilon^2} \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2}^p + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\varphi^2_{\varepsilon,\delta}\|_{H^1}^p dt \right)^p
\leq \frac{C}{\varepsilon^2} \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2}^p + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\varphi^2_{\varepsilon,\delta}\|_{H^1}^p dt \right)^p
\leq \frac{C}{\varepsilon^2} \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2}^p + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\varphi^2_{\varepsilon,\delta}\|_{H^1}^p dt \right)^p
\leq \frac{C}{\varepsilon^2} \mathbb{E} \left( \int_0^T \left( \|\overline{v}_{\varepsilon,\delta}\|_{L^2}^p + \|f_\varepsilon(\varphi_{\varepsilon,\delta}, \overline{\eta}_{\varepsilon,\delta})\|_{H^1}^p \|v^2_{\varepsilon,\delta}\|_{L^2} \|\varphi^2_{\varepsilon,\delta}\|_{H^1}^p dt \right)^p
\]

(3.47)
\[
\leq C \mathbb{E}\left( \int_0^{\tau^R_{\bar{\varepsilon}}} \left( \|ar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_V + \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V \right) dt \right)^p
\]

\[
\leq C \mathbb{E}\left( \int_0^{\tau^R_{\bar{\varepsilon}}} \left( \|ar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_V + \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V \right)^2 dt \right)^p
\]

\[
\leq \frac{C}{\varepsilon^2} \left( \tau^R_{\bar{\varepsilon}} \right)^{2p} \mathbb{E}\left( \|ar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_V + \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V \right)^{2p} C_{\varepsilon^4} \mathbb{E}\left( \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V^2 \right)^{2p}
\]

\[
\leq \frac{C}{\varepsilon^2} \left( \tau^R_{\bar{\varepsilon}} \right)^{2p} \mathbb{E}\left( \|ar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_V + \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V \right)^{2p} C_{\varepsilon^4} \mathbb{E}\left( \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V^2 \right)^{2p}
\]

and

\[
E\left( \int_0^T L_7 dt \right)^p
\]

\[
\leq C \mathbb{E}\left( \int_0^{\tau^R_{\bar{\varepsilon}}} \left( \|ar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_V + \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V \right) dt \right)^p
\]

\[
\leq E\left( \frac{C}{\varepsilon^2} \int_0^{\tau^R_{\bar{\varepsilon}}} \left( \|ar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta}\|_V + \|
abla \bar{\varphi}_{\varepsilon,\delta}, \nabla \bar{\eta}_{\varepsilon,\delta}\|_V \right) dt \right)^p
\]

Choosing first \(\varepsilon_1, \varepsilon_2\) small enough, such that

\[
\frac{C_{\varepsilon^4}}{\varepsilon^4} + \frac{C}{\varepsilon^2} \leq \frac{1}{4},
\]

then, for fixed \(\varepsilon_1, \varepsilon_2\), choosing \(\bar{\varepsilon}\) small enough, such that

\[
\frac{\epsilon^p}{\delta^p} + \frac{C_{\varepsilon^4}}{\varepsilon^4} \leq \frac{1}{4},
\]

finally, for the small \(\varepsilon_1, \varepsilon_2\) and \(\bar{\varepsilon}\), we choose time \(T\) small enough such that

\[
\left( C + \frac{C(1 + R^2)}{\delta} \right) + C R^2 + \frac{C_{\varepsilon^2}}{\varepsilon^2} + \frac{C_{\varepsilon^6}}{\varepsilon^6} + \frac{C_{\varepsilon^4}}{\varepsilon^4} + \frac{C_{\varepsilon^8}}{\varepsilon^8} + \frac{C_{\varepsilon^2}}{\varepsilon^2} \right)^p T^p \leq \frac{1}{4}.
\]
Collecting all estimates (3.42)-(3.52), we conclude that for this small time $T$, 
\[
\|(\bar{\phi}_{\varepsilon,\delta}, \bar{\psi}_{\varepsilon,\delta})\|_{X}^{2} \leq \frac{1}{2}\|(\bar{v}_{\varepsilon,\delta}, \bar{\varphi}_{\varepsilon,\delta}, \bar{\eta}_{\varepsilon,\delta})\|_{X}^{2}.
\]
We obtain mapping $\mathcal{M}$ is contraction in $[0, T]$.

Hence, we have mapping $\mathcal{M}$ is contraction from $X$ into itself for the fixed constant $R$ and small time $T$. Thus, for every $\varepsilon, \delta, R$, there exists unique approximate solutions sequence $(u_{\varepsilon,\delta}^{R}, \phi_{\varepsilon,\delta}^{R}, \psi_{\varepsilon,\delta}^{R})$ to system (3.5) in time interval $[0, T]$. The following uniform estimates with respect to $\varepsilon, \delta, R$ estimates with respect to $\varepsilon, \delta, R$ allow us to extend the existence time to $[0, T]$ for any $T > 0$. Thus, we obtain the global existence and uniqueness of solutions $(u_{\varepsilon,\delta}^{R}, \phi_{\varepsilon,\delta}^{R}, \psi_{\varepsilon,\delta}^{R})$ for any fixed $\varepsilon, \delta, R$.

**Part 2. The a priori estimates and pass the limit as $R \to \infty$.**

In this part, we establish the a priori estimates for the approximate solutions uniformly in $\varepsilon, \delta, R$, and obtain the global existence of unique approximate solutions $(u_{\varepsilon,\delta}^{R}, \phi_{\varepsilon,\delta}^{R}, \psi_{\varepsilon,\delta}^{R})$ of system (3.2) for any fixed $\varepsilon, \delta$ by passing the limit as $R \to \infty$.

We first give the uniform estimates of $(u_{\varepsilon,\delta}^{R}, \phi_{\varepsilon,\delta}^{R}, \psi_{\varepsilon,\delta}^{R})$ with respect to $\varepsilon, \delta, R$. Here, it is admissible to test (3.5) and (3.53) with $\xi_{2} = \mu_{\varepsilon,\delta}^{R} \in H^{1}$ and $\xi_{3} = \mathcal{K}(\psi_{\varepsilon,\delta}^{R}) \in L^{2}(\Gamma)$, and also test (3.3) and (3.4) with $\partial_{\tau} \phi_{\varepsilon,\delta}^{R} \in L^{2}(x)$ and $\partial_{\tau} \psi_{\varepsilon,\delta}^{R} \in L^{2}(\Gamma)$. Applying Itô’s formula to the function $\frac{1}{2}\|u_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2}$, by the cancellation properties
\[
B_{1}(\mu, \phi, u) = B_{2}(u, \phi, \mu)
\]
and
\[
B_{\Gamma}(u, \psi, \mathcal{K}(\psi)) = (\mathcal{K}(\psi) \cdot \nabla_{\tau} \psi, u_{\tau})_{\Gamma},
\]
we conclude
\[
\frac{1}{2} \int D \left(\|u_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2} + \|\nabla \phi_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2} + \|\psi_{\varepsilon,\delta}^{R}\|_{L_{L_{x}^{2}}^{2}}^{2} + \|\nabla \psi_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2}\right) + 2 \int D F(\phi_{\varepsilon,\delta}^{R})dx
\]
\[
\begin{align*}
&= \frac{1}{2} \int D \left(\|u_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2} + \|\nabla \phi_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2} + \|\psi_{\varepsilon,\delta}^{R}\|_{L_{L_{x}^{2}}^{2}}^{2} + \|\nabla \psi_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2}\right) \right. \\
&\quad + \frac{1}{2} \int \mathcal{K}(\psi_{\varepsilon,\delta}^{R})\|\partial_{\tau} \phi_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2} + \|\partial_{\tau} \psi_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2})dt,
\end{align*}
\]
(3.53)

**Remark 3.2.** Note that, to achieve the a priori estimates we have to choose the functions $\xi_{2} = \mu$ and $\xi_{3} = \mathcal{K}(\psi)$ to test equations (3.3) and (3.4). In process of Galerkin approximate, we can not implement such choice since this procedure requires $\text{Tr}_{D} \mu = \mathcal{K}(\psi)$ at the level of finite-dimension.

Define the stopping time by
\[
\rho_{N} = \inf \left\{ t; \sup_{t \geq 0} \|u_{\varepsilon,\delta}^{R}\|_{L_{x}^{2}}^{2} \geq N \right\},
\]
and choose $\rho_{N} = T$, if the set is empty. Note that $\rho_{N}$ is non-decreasing with $\lim_{N \to \infty} \rho_{N} = T$. Assumption $A_{1}$ and the Burkholder-Davis-Gundy inequality give
\[
\mathbb{E} \left( \sup_{s \in [0, t \wedge \rho_{N}]} \left| \int_{0}^{s} (h(u_{\varepsilon,\delta}^{R}, \nabla \phi_{\varepsilon,\delta}^{R}), u_{\varepsilon,\delta}^{R})dW \right|^{p} \right)
\]
\[ \leq C \mathbb{E} \left( \int_0^{T \wedge \rho_N} \left\| h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}^R) \right\|_{L_2(\mathcal{H}; L_2^2)}^2 \left\| u_{\varepsilon, \delta}^R \right\|_{L_2^2}^2 ds \right)^{\frac{p}{2}} \tag{3.54} \]
\[
\leq \frac{1}{4} \mathbb{E} \left( \sup_{s \in [0, T \wedge \rho_N]} \left\| u_{\varepsilon, \delta}^R \right\|_{L_2^2}^{2p} \right) + C \mathbb{E} \int_0^{T \wedge \rho_N} \left( 1 + \left\| (u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}^R) \right\|_{L_2^2}^{2p} ds. \right.
\]

Again, assumption \( A_1 \) gives
\[
\frac{1}{2} \left\| h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}^R) \right\|_{L_2(\mathcal{H}; L_2^2)}^2 \leq C \left( 1 + \left\| (u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}^R) \right\|_{L_2^2}^2 \right). \tag{3.55} \]

By assumptions \( A_3 \) and \( A_4 \), the initial data \((\phi_0, \psi_0) \in L_\omega^p V^1_1\), we have \( F(\phi_0) \in L_\omega^p L_\omega^1 \), \( G(\psi_0) \in L_\omega^p L_\omega^1(\Gamma) \). Integrating of \( t \) in \( [3.53] \), taking supremum over \([0, T \wedge \rho_N]\) and then power \( p \), expectation, estimates \((3.54), (3.55)\) and Gronwall's inequality yield
\[
\mathbb{E} \left( \left\| u_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \nabla \phi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \nabla \tau \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 \right)^p
\]
\[
+ \mathbb{E} \left( \sup_{t \in [0, T \wedge \rho_N]} \int_\Gamma G(\psi_{\varepsilon, \delta}) \, d\mathcal{S} + \sup_{t \in [0, T \wedge \rho_N]} \int_{\mathcal{D}} F(\phi_{\varepsilon, \delta}) \, dx \right)^p
\]
\[
+ \mathbb{E} \left( \left\| (\nabla u_{\varepsilon, \delta}, \nabla \mu_{\varepsilon, \delta}) \right\|_{L_\omega^2 L_\omega^1}^2 + \left\| \mathcal{K}(\psi_{\varepsilon, \delta}) \right\|_{L_\omega^2 L_\omega^1}^2 + \delta \left\| \partial_t \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^2 L_\omega^1}^2 \right)^p \leq C,
\]

where \( C = C(p, T, \mathcal{D}, \Gamma) \) is a constant independence of \( \varepsilon, \delta, R, N \). Finally, the monotone convergence theorem yields as \( N \to \infty \)
\[
\mathbb{E} \left( \left\| u_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \nabla \phi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \nabla \tau \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 \right)^p
\]
\[
+ \mathbb{E} \left( \sup_{t \in [0, T]} \int_\Gamma G(\psi_{\varepsilon, \delta}) \, d\mathcal{S} + \sup_{t \in [0, T]} \int_{\mathcal{D}} F(\phi_{\varepsilon, \delta}) \, dx \right)^p \tag{3.56}
\]
\[
+ \mathbb{E} \left( \left\| (\nabla u_{\varepsilon, \delta}, \nabla \mu_{\varepsilon, \delta}) \right\|_{L_\omega^2 L_\omega^1}^2 + \left\| \mathcal{K}(\psi_{\varepsilon, \delta}) \right\|_{L_\omega^2 L_\omega^1}^2 + \delta \left\| \partial_t \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^2 L_\omega^1}^2 \right)^p \leq C.
\]

Using assumptions \( A_3, A_4 \) and estimate \((3.56)\), we get
\[
\mathbb{E} \left( \left\| u_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \nabla \phi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 + \left\| \nabla \tau \psi_{\varepsilon, \delta}^R \right\|_{L_\omega^p L_\omega^1 L_\omega^2}^2 \right)^p
\]
\[
+ \mathbb{E} \left( \left\| (\nabla u_{\varepsilon, \delta}, \nabla \mu_{\varepsilon, \delta}) \right\|_{L_\omega^2 L_\omega^1}^2 + \left\| \mathcal{K}(\psi_{\varepsilon, \delta}) \right\|_{L_\omega^2 L_\omega^1}^2 + \delta \left\| \partial_t \phi_{\varepsilon, \delta}^R \right\|_{L_\omega^2 L_\omega^1}^2 \right)^p \leq C. \tag{3.57}
\]

Since \( \langle \partial_t \phi_{\varepsilon, \delta}^R \rangle = 0 \), a.e. on \([0, T] \times \Omega\), we also have
\[
\langle \mu_{\varepsilon, \delta}^R \rangle = -\langle \nabla \phi_{\varepsilon, \delta}^R \rangle + \langle f_\varepsilon(\phi_{\varepsilon, \delta}^R) \rangle
\]
\[
= -|\mathcal{D}|^{-1} |\Gamma| \left( \langle \mathcal{K}(\psi_{\varepsilon, \delta}) \rangle_{\Gamma} - \langle \psi_{\varepsilon, \delta}^R \rangle + \langle g_\varepsilon(\psi_{\varepsilon, \delta}) \rangle + \langle f_\varepsilon(\phi_{\varepsilon, \delta}) \rangle \right)
\]
as well as \((3.57)\), assumption \( A_3 \), and \((3.11), (3.12)\), we infer that \( \langle \mu_{\varepsilon, \delta}^R \rangle \in L_\omega^2, \mathbb{P}\)-a.s. By \((3.57)\) again, the Poincaré inequality yields
\[
\mu_{\varepsilon, \delta}^R \in L_\omega^p L_\omega^2 H^1, \text{ uniformly in } \varepsilon, \delta, R.
\]

Moreover, \( (\phi_{\varepsilon, \delta}^R, \psi_{\varepsilon, \delta}^R) \) is the solution of the following elliptic boundary value problem:
\[
-\Delta \phi_{\varepsilon, \delta}^R = \Psi_{\varepsilon, \delta}^R, \text{ } \mathbb{P}\text{-a.s. in } \mathcal{D} \times (0, T),
\]
\[
A_\tau \psi_{\varepsilon, \delta}^R + \partial_t \phi_{\varepsilon, \delta}^R + \psi_{\varepsilon, \delta}^R = \bar{\Psi}_{\varepsilon, \delta}^R, \text{ } \mathbb{P}\text{-a.s. on } \Gamma \times (0, T),
\]
where
\[
\Psi_{\epsilon,\delta}^R = \mu_{\epsilon,\delta} - \delta \partial_t \phi_{\epsilon,\delta}^R - f_{\epsilon}(\phi_{\epsilon,\delta}^R) \in L_0^p L_\omega^1 L_2^2, \\
\bar{\Psi}_{\epsilon,\delta}^R = K(\psi_{\epsilon,\delta}^R) - g_{\epsilon}(\psi_{\epsilon,\delta}^R) \in L_0^p L_\omega^1 L_2^2(\Gamma).
\]
By the elliptic regularity theory and Lemma 5.4, we have
\[
E\| (\phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R) \|_{L_t^2 V^2}^p \leq C,
\]
where constant $C$ is independent of $\epsilon, \delta, R$.

In conclusion, we have
\[
\left\{ \begin{array}{l}
u_{\epsilon,\delta}^R \in L_0^p (L_\omega^1 L_2^2 \cap L_\omega^2 H^1), \\
\mu_{\epsilon,\delta}^R \in L_0^p L_\omega^1 L_2^2 H^1, \\
\phi_{\epsilon,\delta}^R \in L_0^p L_\omega^1 L_2^2 V^1 \cap L_\omega^2 V^2, \\
K(\psi_{\epsilon,\delta}^R) \in L_0^p L_\omega^1 L_2^2 (\Gamma), \\
\delta^2 \partial_t \phi_{\epsilon,\delta}^R \in L_0^p L_\omega^1 L_2^2,
\end{array} \right. \tag{3.58}
\]
uniformly in $\epsilon, \delta, R$.

With these estimates in hand, we could build the existence and uniqueness of approximate solutions $(u_{\epsilon,\delta}, \phi_{\epsilon,\delta}, \psi_{\epsilon,\delta})$ for any fixed $\epsilon, \delta$ by passing $R \to \infty$ with the spirit of [2]. To this end, we define
\[
\mathcal{T}_R = \left\{ t \in [0, T] : \| u_{\epsilon,\delta}^R \|_{L_t^2 L_\omega^1 L_2^2 H^1} + \| (\phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R) \|_{L_t^2 V^1 \cap L_\omega^2 V^2}^2 \geq R \right\},
\]
if the set is empty, we take $\mathcal{T}_R = T$. We infer from above argument that $(u_{\epsilon,\delta}^R, \phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R)$ is a unique solution on $[0, \mathcal{T}_R]$. Note that $\mathcal{T}_R$ is an increasing sequence, thus, $\mathcal{T}_R' \geq \mathcal{T}_R$ when $R' > R$. Then, the uniqueness implies $(u_{\epsilon,\delta}^R, \phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R) = (u_{\epsilon,\delta}^R, \phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R)$ on $[0, \mathcal{T}_R)$. As a result, we can define $(u_{\epsilon,\delta}, \phi_{\epsilon,\delta}, \psi_{\epsilon,\delta}) = (u_{\epsilon,\delta}^R, \phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R)$ in $[0, \mathcal{T}_R)$. Finally, we need to show that
\[
\mathbb{P} \left\{ \sup_{R \in \mathbb{N}} \mathcal{T}_R = T \right\} = 1,
\]
that is, the blow-up does not occur within any finite time.

Indeed, using the Chebyshev inequality and estimates (3.58)
\[
\mathbb{P} \left\{ \sup_{R \in \mathbb{N}} \mathcal{T}_R = T \right\} = 1 - \mathbb{P} \left\{ \left( \sup_{R \in \mathbb{N}} \mathcal{T}_R = T \right)^c \right\} = 1 - \mathbb{P} \left\{ \sup_{R \in \mathbb{N}} \mathcal{T}_R < T \right\} \geq 1 - \mathbb{P} \{ \mathcal{T}_R < T \} = 1 - \mathbb{P} \left\{ \| u_{\epsilon,\delta}^R \|_{L_t^2 L_\omega^1 L_2^2 H^1} + \| (\phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R) \|_{L_t^2 V^1 \cap L_\omega^2 V^2}^2 \geq R \right\} \geq 1 - \frac{C}{R} \mathbb{E} \left( \| u_{\epsilon,\delta}^R \|_{L_t^2 L_\omega^1 L_2^2 H^1} + \| (\phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}^R) \|_{L_t^2 V^1 \cap L_\omega^2 V^2}^2 \right).
\]
Passing $R \to \infty$ on both sides, the result follows, making sure that we could define $(u_{\epsilon,\delta}, \phi_{\epsilon,\delta}, \psi_{\epsilon,\delta})$ in the whole time interval $[0, T]$ for any $T > 0$.

In order to show the energy inequality 5 in Definition 3.11, let
\[
E_{\epsilon,\delta}(t) := \frac{1}{2} \left( \| u_{\epsilon,\delta}^R \|^2_{L_2^2} + \| \nabla \phi_{\epsilon,\delta}^R \|^2_{L_2^2} + \| \psi_{\epsilon,\delta}^R \|^2_{L_2^2(\Gamma)} + \| \nabla \tau \psi_{\epsilon,\delta}^R \|^2_{L_2^2(\Gamma)} + 2 \int_D F(\phi_{\epsilon,\delta}^R) dx \right) + \int_\Gamma G(\psi_{\epsilon,\delta}^R) dS. \tag{3.59}
\]
Then after an application of Itô’s formula of $E_{\epsilon,\delta}^2(t)$, from (3.53) we have
\[
\frac{1}{2} E_{\epsilon,\delta}^2(t) + \int_0^t E_{\epsilon,\delta}(\|\nabla u_{\epsilon,\delta}^R, \nabla \mu_{\epsilon,\delta}^R\|_{L^2}^2 + \|K(\psi_{\epsilon,\delta})\|_{L^2(\Gamma)}^2 + \delta \|\partial_t \phi_{\epsilon,\delta}^R\|_{L^2}^2) ds \\
= \frac{1}{2} E_{\epsilon,\delta}^2(0) + \int_0^t E_{\epsilon,\delta}(h(u_{\epsilon,\delta}^R, \nabla \phi_{\epsilon,\delta}^R), u_{\epsilon,\delta}^R) dW + \frac{1}{2} \int_0^t E_{\epsilon,\delta} \|h(u_{\epsilon,\delta}^R, \nabla \phi_{\epsilon,\delta}^R)\|_{L^2(\mathcal{H}; L^2)}^2 ds \\
+ \frac{1}{2} \int_0^t (h(u_{\epsilon,\delta}^R, \nabla \phi_{\epsilon,\delta}^R), u_{\epsilon,\delta}^R)^2 ds.
\]
(3.60)

By the elliptic regularity theory and Lemma 5.4 again, from (3.60) we have
\[
\frac{1}{2} E_{\epsilon,\delta}^2(t) + \int_0^t E_{\epsilon,\delta}(\|\nabla u_{\epsilon,\delta}^R, \nabla \mu_{\epsilon,\delta}^R\|_{L^2}^2 + \|K(\psi_{\epsilon,\delta})\|_{L^2(\Gamma)}^2 + \delta \|\partial_t \phi_{\epsilon,\delta}^R\|_{L^2}^2 + \|\phi_{\epsilon,\delta}^R, \psi_{\epsilon,\delta}\|_{V^2}^2) ds \\
\leq E_{\epsilon,\delta}^2(0) + 2 \int_0^t E_{\epsilon,\delta}(h(u_{\epsilon,\delta}^R, \nabla \phi_{\epsilon,\delta}^R), u_{\epsilon,\delta}^R) dW + \int_0^t E_{\epsilon,\delta} \|h(u_{\epsilon,\delta}^R, \nabla \phi_{\epsilon,\delta}^R)\|_{L^2(\mathcal{H}; L^2)}^2 ds \\
+ \int_0^t (h(u_{\epsilon,\delta}^R, \nabla \phi_{\epsilon,\delta}^R), u_{\epsilon,\delta}^R)^2 ds.
\]
(3.61)

The inequality (3.61) will be used later.

**Part 3. Conduct the compactness argument**

In the stochastic setting, the embedding $L^2(\Omega; X) \hookrightarrow L^2(\Omega; Y)$ might not be compact, even if the embedding $X$ into $Y$ is compact. As a result, the usual compactness criteria, such as the Aubin or Arzelà-Ascoli type theorems, can not be used directly. We turn to invoke the Skorokhod representative theorem to obtain the compactness, which require us to show the tightness of the probability set induced by the law of sequence $\{u_{\epsilon,\delta}, \phi_{\epsilon,\delta}, \psi_{\epsilon,\delta}\}$.

**Claim 1.** The probability set $\mathcal{P}_{\epsilon,\delta}$ is tight on the corresponding space $Y_i$.

**Proof.** We first show that there exists a constant $\beta > 0$ such that,
\[
\mathbb{E}\|u_{\epsilon,\delta}\|_{H^{1,\delta}}^p \leq C,
\]
(3.62)
where constant $C$ is independent of $\epsilon, \delta$.

For any $v \in H^{1,\delta}$, we have
\[
(u_{\epsilon,\delta}, v) + \int_0^t a_0(u_{\epsilon,\delta}, v) ds + \int_0^t B_0(u_{\epsilon,\delta}, J_0 u_{\epsilon,\delta}, v) ds - \int_0^t B_1(\mu_{\epsilon,\delta}, J_0 \phi_{\epsilon,\delta}, v) ds \\
- \int_0^t (K(\psi_{\epsilon,\delta}) \nabla \tau(J_0 \phi_{\epsilon,\delta}), v) \tau ds = (u_0, v) + \int_0^t (h(u_{\epsilon,\delta}, \nabla \phi_{\epsilon,\delta}), v) dW.
\]
For a.s. $\omega$, and for any $\epsilon > 0$, there exist $t_1, t_2 \in [0, T]$ such that
Combining (3.64)-(3.67), we have
\[ \sup_{t_0 \neq t_0'} \left| \int_{t_0}^{t_0'} (h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}), v) dW \right| \leq \sup_{t_0 \neq t_0'} \left| \int_{t_1}^{t_2} (h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}), v) dW \right| + \epsilon, \]

and (3.68) give
\[ \langle \nabla \rangle_{\{H^1\}^p} \leq \langle H \rangle_{\{L^2\}^p} + \langle \nabla J \rangle_{\{L^2\}^p} + \langle \nabla \phi_{\varepsilon, \delta} \rangle_{\{L^2\}^p} + \langle \phi_{\varepsilon, \delta} \rangle_{\{L^2\}^p} \]

together with assumption A_1 and the Burkholder-Davis-Gundy inequality, we see
\[
\mathbb{E} \left( \sup_{t_0 \neq t_0'} \left| \int_{t_0}^{t_0'} (h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}), v) dW \right| \right)^p \leq \mathbb{E} \left( \sup_{t_0 \neq t_0'} \left| \int_{t_1}^{t_2} (h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}), v) dW \right| \right)^p + \epsilon^p \leq C(t_2 - t_1)^\beta \mathbb{E} \left( \sup_{t_0 \neq t_0'} \left| \int_{t_0}^{t_0'} (h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}), v) dW \right| \right)^p + \epsilon^p \]

where $\beta < \frac{1}{2}$. Applying Lemma 5.3, Lemma 5.5 and Hölder’s inequality, we obtain
\[
\|B_0(u_{\varepsilon, \delta}, J_{\varepsilon} u_{\varepsilon, \delta}, v)\|_{L^1_t L^2} \leq \|u_{\varepsilon, \delta}\|_{H^1} \|\nabla J_{\varepsilon} u_{\varepsilon, \delta}\|_{L^1_t L^2} \|v\|_{H^{1.3}} \leq C \|u_{\varepsilon, \delta}\|_{L^2_t H^1} \|u_{\varepsilon, \delta}\|_{L^\infty_t L^2} \]

(3.64)
\[
\|B_1(\mu_{\varepsilon, \delta}, J_{\varepsilon} \phi_{\varepsilon, \delta}, v)\|_{L^1_t L^2} \leq \|\mu_{\varepsilon, \delta}\|_{L^2_t H^1} \|\nabla J_{\varepsilon} \phi_{\varepsilon, \delta}\|_{L^1_t L^2} \|v\|_{L^2_t} \leq C \|\mu_{\varepsilon, \delta}\|_{L^2_t H^1} \|\phi_{\varepsilon, \delta}\|_{L^\infty_t H^1} \]

(3.65)

Choosing $s = \frac{1}{2}$ in Lemma 5.1 and using estimates (5.58), we infer that
\[
\mathbb{E} \|\nabla (J_{\varepsilon} \psi_{\varepsilon, \delta})\|_{L^1_t L^4(\Gamma)} \leq C \]

(3.66)

where $C$ is independent of $\varepsilon, \delta$. By Hölder’s inequality, obtaining
\[
\|\langle K(\psi_{\varepsilon, \delta}) \nabla (J_{\varepsilon} \psi_{\varepsilon, \delta}), v_{\tau} \rangle_{\Gamma}\|_{L^3_t} \leq \|v_{\tau}\|_{L^4(\Gamma)} \|\langle K(\psi_{\varepsilon, \delta}) \rangle_{L^1(\Gamma)} \|_{L^\infty_t} \|\nabla (J_{\varepsilon} \psi_{\varepsilon, \delta})\|_{L^4(\Gamma)} \|_{L^\infty_t} \]

(3.67)

Combining (3.64)-(3.67), we have
\[
\mathbb{E} \left\| u_{\varepsilon, \delta} - \int_0^t h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}) dW \right\|_{W^{1, q}_{t, \frac{q}{2}}(H^{1.3})} \leq C \]

(3.68)

The Sobolev embedding
\[
W^{\alpha, p}_t X \hookrightarrow C^\beta_t X, \text{ for } \beta < \alpha - \frac{1}{p},
\]

and (3.68) give
\[
\mathbb{E} \left\| u_{\varepsilon, \delta} - \int_0^t h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}) dW \right\|_{C^\beta_t(H^{1.3})} \leq C, \text{ for } \beta < \frac{1}{6}. \]

(3.69)

Estimates (3.63) and (3.69) yield the desired result (3.62).

Define the set
\[
B_{1, K} = \left\{ u_{\varepsilon, \delta} \in L^2_t H^1 \cap C^\alpha_t(H^{1.3})' : \|u_{\varepsilon, \delta}\|_{L^2_t H^1} + \|u_{\varepsilon, \delta}\|_{C^\beta_t(H^{1.3})} \leq K \right\}. \]

(3.70)
The Aubin-Lions compact embedding (see Lemma 5.6)

\[ L_t^2 H^1 \cap C_t^\alpha(H^{1,3})' \hookrightarrow L_t^2 L_x^2, \]

implies that the set \( B_{1,K} \) is relatively compact in \( L_t^2 L_x^2 \). Considering (3.58), (3.62), using the Chebyshev inequality, we have

\[
P_{\varepsilon, \delta}(B_{1,K}^c) \leq \mathbb{P}\left(\|u_{\varepsilon, \delta}\|_{L_t^2 H^1} > \frac{K}{2}\right) + \mathbb{P}\left(\|u_{\varepsilon, \delta}\|_{C_t^\alpha(H^{1,3})'} > \frac{K}{2}\right) 
\leq \frac{2}{K}E\left(\|u_{\varepsilon, \delta}\|_{L_t^2 H^1} + \|u_{\varepsilon, \delta}\|_{C_t^\alpha(H^{1,3})'}\right) \leq \frac{C}{K}. \tag{3.71}
\]

Owing to the Banach-Alaoglu theorem, the sets

\[ B_{2,K} := \left\{ u_{\varepsilon, \delta} \in L_t^2 H^1 : \|u_{\varepsilon, \delta}\|_{L_t^2 H^1} \leq K \right\}, \]

\[ B_{3,K} := \left\{ u_{\varepsilon, \delta} \in L_t^\infty L_x^2 : \|u_{\varepsilon, \delta}\|_{L_t^\infty L_x^2} \leq K \right\}, \]

are relatively compact on path spaces \((L_t^2 H^1)_w, (L_t^\infty L_x^2)_w\), respectively. Bound (3.58) and the Chebyshev inequality yield that

\[
P_{\varepsilon, \delta}(B_{1,K} \cap B_{2,K}^c) \leq P_{\varepsilon, \delta}(B_{1,K}^c) + P_{\varepsilon, \delta}(B_{2,K}^c) \leq \frac{C}{K}. \]

Then, we obtain the tightness of set \( P_{\varepsilon, \delta} \).

\[ \Box \]

**Claim 2.** The probability set \( P_{\varepsilon, \delta}^2 \) is tight on \( Y_2 \).

**Proof.** To this end, we first give the time regularity estimates of \((\phi_{\varepsilon, \delta}, \psi_{\varepsilon, \delta})\). For any \( v \in H^1 \), Hölder’s inequality, Lemmas 5.3 and 5.5 imply that

\[
\|B_2(u_{\varepsilon, \delta}, J_{\varepsilon} \phi_{\varepsilon, \delta}, v)\|_{L_t^2} \leq \|v\|_{H^1} \|u_{\varepsilon, \delta}\|_{L_t^2 H^1} \|J_{\varepsilon} \phi_{\varepsilon, \delta}\|_{H^2} \|_{L_t^2}
\leq \|v\|_{H^1} \|u_{\varepsilon, \delta}\|_{L_t^\infty L_x^2} \|\phi_{\varepsilon, \delta}\|_{L_t^2 H^2}. \tag{3.72}
\]

For any \( v \in L^2(\Gamma) \), using the trace embedding \( H^1(\mathcal{D}) \hookrightarrow H^{3/2}(\Gamma) \) and the embedding \( H^{3/2}(\Gamma) \hookrightarrow L^4(\Gamma) \), Lemma 5.5, and the Gagliardo-Nirenberg inequality

\[
\|u\|_{L^4(\Gamma)} \leq \|u\|_{L^2(\Gamma)} \|
abla u\|_{L^2(\Gamma)},
\]

we have

\[
\|B_{\Gamma}(u_{\varepsilon, \delta}, J_{\varepsilon} \psi_{\varepsilon, \delta}, v)\|_{L_t^{3/2}} \leq C \|v\|_{L^2(\Gamma)} \|u_{\varepsilon, \delta}\|_{L_t^2 L^2(\Gamma)} \|
abla J_{\varepsilon} \psi_{\varepsilon, \delta}\|_{L_t^4 L^4(\Gamma)}
\leq C \|v\|_{L^2(\Gamma)} \|u_{\varepsilon, \delta}\|_{L_t^2 H^1} \|
abla J_{\varepsilon} \psi_{\varepsilon, \delta}\|_{L_t^{3/2} L^4(\Gamma)}
\leq C \|v\|_{L^2(\Gamma)} \|u_{\varepsilon, \delta}\|_{L_t^2 H^1} \|
abla J_{\varepsilon} \psi_{\varepsilon, \delta}\|_{L_t^2 L^2(\Gamma)}^{1/3} \|
abla J_{\varepsilon} \psi_{\varepsilon, \delta}\|_{L_t^{3/2} L^2(\Gamma)}^{2/3}
\leq C \|u_{\varepsilon, \delta}\|_{L_t^2 H^1} \|
abla J_{\varepsilon} \psi_{\varepsilon, \delta}\|_{L_t^2 L^2(\Gamma)}^{1/3} \|
abla J_{\varepsilon} \psi_{\varepsilon, \delta}\|_{L_t^{3/2} L^2(\Gamma)}^{2/3}
\leq C \|u_{\varepsilon, \delta}\|_{L_t^2 H^1} \|
abla \psi_{\varepsilon, \delta}\|_{L_t^\infty L^2(\Gamma)}^{1/2} \|
abla \psi_{\varepsilon, \delta}\|_{L_t^2 H^1}^{1/2}. \tag{3.73}
\]

By estimates (3.72) and (3.73), estimates (3.58), we infer from equations (3.2), (3.2) \(^3\) that

\[
\mathbb{E}\|\partial_t(\phi_{\varepsilon, \delta}, \psi_{\varepsilon, \delta})\|_{L_t^{3/2}(H^1)^\times L^2(\Gamma)} \leq C,
\]
where \( C \) is independent of \( \varepsilon, \delta \). Then, using the Aubin-Lions compact embedding,

\[
L_t^2 V^2 \cap W_t^{1, \frac{4}{3}}(V') \hookrightarrow L_t^2 V^{2-\sigma},
\]

and bound \((3.58)_3\), we can deduce that the probability set \( \mathcal{P}_{\varepsilon, \delta}^2 \) is tight by the same argument as that of in Claim 1.

Since the sequence \( \mathcal{W} \) is only one element, the set \( \{ \mathcal{P}_W \} \) is weakly compact.

Consequently, we have the following result owing to the Skorokhod representative theorem (see Theorem 5.2).

**Proposition 3.1.** There exist a new sequence \((\tilde{u}_{\varepsilon, \delta}, \tilde{\phi}_{\varepsilon, \delta}, \tilde{\psi}_{\varepsilon, \delta}, \tilde{W}_{\varepsilon, \delta})\), a new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) and \( Y \)-valued processes \((\tilde{u}, \tilde{\phi}, \tilde{\psi}, \tilde{W})\) such that

1. the joint law of \((\tilde{u}_{\varepsilon, \delta}, \tilde{\phi}_{\varepsilon, \delta}, \tilde{\psi}_{\varepsilon, \delta}, \tilde{W}_{\varepsilon, \delta})\) is the same with \( \mathcal{P}_{\varepsilon, \delta} \),
2. the sequence \((\tilde{u}_{\varepsilon, \delta}, \tilde{\phi}_{\varepsilon, \delta}, \tilde{\psi}_{\varepsilon, \delta}, \tilde{W}_{\varepsilon, \delta})\) converges to \((\tilde{u}, \tilde{\phi}, \tilde{\psi}, \tilde{W})\) in the topology of \( Y \), \( \tilde{P} \)-a.s.,
3. the measure set \( \mathcal{P}_{\varepsilon, \delta} \) converges to \( \mathcal{P} \) where \( \mathcal{P} \) is a Radon measure with \( \tilde{P}(\tilde{u}, \tilde{\phi}, \tilde{\psi} \in \cdot) = \mathcal{P}(\cdot) \).

**Proposition 3.2.** The sequence \((\tilde{u}_{\varepsilon, \delta}, \tilde{\phi}_{\varepsilon, \delta}, \tilde{\psi}_{\varepsilon, \delta}, \tilde{W}_{\varepsilon, \delta})\) still satisfies system \((3.2)\) relative to the stochastic basis \( \tilde{S}_{\varepsilon, \delta} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}, \{ \tilde{\mathcal{F}}_t \}_{t \geq 0}, \tilde{W}_{\varepsilon, \delta}) \), where \( \tilde{\mathcal{F}}_t \) is a canonical filtration defined by

\[
\sigma \left( (\tilde{u}_{\varepsilon, \delta}(s), \tilde{\phi}_{\varepsilon, \delta}(s), \tilde{\psi}_{\varepsilon, \delta}(s), \tilde{W}_{\varepsilon, \delta}(s)) : s \leq t \right) \cup \left\{ \Sigma \in \tilde{\mathcal{F}} : \tilde{\mathcal{P}}(\Sigma) = 0 \right\},
\]

and also the sequence \((\tilde{u}_{\varepsilon, \delta}, \tilde{\phi}_{\varepsilon, \delta}, \tilde{\psi}_{\varepsilon, \delta}, \tilde{W}_{\varepsilon, \delta})\) shares the uniform estimates \((3.58)\).

**Proof.** The proof of first conclusion is standard, and similar highly to that of in \((33)\), here we omit the details. The second one is a result of Proposition 3.1.

**Part 4. Identify the limit.**

To simplify the notation, we still use \((u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, \psi_{\varepsilon, \delta}, W_{\varepsilon, \delta})\) standing for the new sequence. Using bound \((3.58)_5\), it holds

\[
\| \delta \partial_t \phi_{\varepsilon, \delta} \|_{L_t^p L_x^2 L_x^2} = \delta^\frac{1}{2} \left( E \| \delta^\frac{1}{2} \partial_t \phi_{\varepsilon, \delta} \|_{L_t^p L_x^2}^p \right)^{\frac{1}{p}} \rightarrow 0,
\]

as \( \varepsilon, \delta \rightarrow 0 \). By thinning the sequence (still denoted by \( \varepsilon, \delta \)), we have \( \tilde{P} \)-a.s.

\[
\delta \partial_t \phi_{\varepsilon, \delta} \rightarrow 0, \text{ in } L_t^2 L_x^2.
\]

Using Proposition 3.1 Property (ii) and bound \((3.58)_3\) again, we get \( \tilde{P} \)-a.s.

\[
f_{\varepsilon}(\phi_{\varepsilon, \delta}) \rightarrow f(\phi), \text{ in } L_t^2 L_x^2.
\]

From the convergence \((3.74)\), bound \((3.58)_2\) and Proposition 3.1, we know that \( \tilde{P} \)-a.s.

\[
\mu_{\varepsilon, \delta} \rightharpoonup \mu, \text{ in } L_t^2 H^1.
\]

For any \( v_0 \in D(A_0) \), decompose

\[
B_0(u_{\varepsilon, \delta}, J_{\varepsilon} u_{\varepsilon, \delta}, v_0) - B_0(u, u, v_0) = B_0(u, J_{\varepsilon} u_{\varepsilon, \delta} - u, v_0) + B_0(u_{\varepsilon, \delta} - u, J_{\varepsilon} u_{\varepsilon, \delta}, v_0).
\]
By the convergence $u_{\epsilon,\delta} \to u$ in $L^2_tL^2_x$, \(\bar{\mathbb{P}}\)\-a.s., Lemma [5.5] and bound (3.58), we have
\[
\int_0^t |B_0(u_{\epsilon,\delta} - u, J_\epsilon u_{\epsilon,\delta}, v_0)| ds \leq \|v_0\|_{H^2}\|u_{\epsilon,\delta} - u\|_{L^2_tL^2_x} \|J_\epsilon u_{\epsilon,\delta}\|_{L^2_tH^1} \to 0.
\]
Furthermore, the convergence $J_\epsilon u_{\epsilon,\delta} \to u$ in $L^2_tH^1$, \(\bar{\mathbb{P}}\)\-a.s. and bound (3.58) imply
\[
\int_0^t B_0(u, J_\epsilon u_{\epsilon,\delta} - u, v_0) ds \to 0, \quad \bar{\mathbb{P}}\)\-a.s.
\]
And
\[
B_1(\mu_{\epsilon,\delta}, J_\epsilon \phi_{\epsilon,\delta}, v_0) - B_1(\mu, \phi, v_0) = B_1(\mu_{\epsilon,\delta} - \mu, \phi, v_0) + B_1(\mu_{\epsilon,\delta}, J_\epsilon \phi_{\epsilon,\delta} - \phi, v_0).
\]
The convergence $J_\epsilon \phi_{\epsilon,\delta} \to \phi$ in $L^2_tH^{2-\sigma}$, $\mu_{\epsilon,\delta} \to \mu$ in $L^2_tH^1$, \(\bar{\mathbb{P}}\)\-a.s. and estimates (3.58) yield \(\bar{\mathbb{P}}\)\-a.s.
\[
\int_0^t B_1(\mu_{\epsilon,\delta} - \mu, \phi, v_0) ds \to 0,
\]
\[
\int_0^t |B_1(\mu_{\epsilon,\delta}, J_\epsilon \phi_{\epsilon,\delta} - \phi, v_0)| ds \leq \|v_0\|_{H^1}\|\mu_{\epsilon,\delta}\|_{L^2_tH^1}\|\nabla(J_\epsilon \phi_{\epsilon,\delta} - \phi)\|_{L^2_tL^2_x} \to 0.
\]
Using a same argument, we also have \(\bar{\mathbb{P}}\)\-a.s.
\[
\int_0^t B_2(u_{\epsilon,\delta}, J_\epsilon \phi_{\epsilon,\delta}, v_1) - B_2(u, \phi, v_1) ds \to 0,
\]
for any $v_1 \in L^2$.

Next, we focus on the nonlinear boundary integral terms. For any $v_2 \in L^2(\Gamma)$, decompose
\[
B_\Gamma(u_{\epsilon,\delta}, J_\epsilon \psi_{\epsilon,\delta}, v_2) - B_\Gamma(u, \psi, v_2) = B_\Gamma(u_{\epsilon,\delta} - u, \psi, v_2) + B_\Gamma(u_{\epsilon,\delta}, J_\epsilon \psi_{\epsilon,\delta} - \psi, v_2).
\]
The fact that $u_{\epsilon,\delta} \to u$ in $L^2_tH^1(\Gamma)$, \(\bar{\mathbb{P}}\)\-a.s. together with the bound $\psi \in L^2_tH^2(\Gamma)$, \(\bar{\mathbb{P}}\)\-a.s. yields
\[
\int_0^t B_\Gamma(u_{\epsilon,\delta} - u, \psi, v_2) ds \to 0.
\]
Moreover, by Hölder’s inequality, the Sobolev embedding $H^\frac{1}{2}(\Gamma) \hookrightarrow L^4(\Gamma)$ and the convergence $J_\epsilon \psi_{\epsilon,\delta} \to \psi$ in $L^2_tH^{2-\sigma}$, \(\bar{\mathbb{P}}\)\-a.s., we find \(\bar{\mathbb{P}}\)\-a.s.
\[
\int_0^t |B_\Gamma(u_{\epsilon,\delta}, J_\epsilon \psi_{\epsilon,\delta} - \psi, v_2)| ds \
\leq \|v_2\|_{L^2(\Gamma)}\|u_{\epsilon,\delta}\|_{L^2_tL^4(\Gamma)}\|J_\epsilon \psi_{\epsilon,\delta} - \psi\|_{L^2_tH^{1,4}(\Gamma)} \\
\leq \|v_2\|_{L^2(\Gamma)}\|u_{\epsilon,\delta}\|_{L^2_tH^\frac{1}{2}(\Gamma)}\|J_\epsilon \psi_{\epsilon,\delta} - \psi\|_{L^2_tH^{2-\sigma}} \to 0.
\]
(3.75)
The specific form of boundary term $\mathcal{K}(\psi_{\epsilon,\delta})\nabla_\tau(J_\epsilon \psi_{\epsilon,\delta})$ is
\[
\mathcal{K}(\psi_{\epsilon,\delta})\nabla_\tau(J_\epsilon \psi_{\epsilon,\delta}) = (A_\tau \psi_{\epsilon,\delta} + \partial_0 \phi_{\epsilon,\delta} + \psi_{\epsilon,\delta} + \sigma_\epsilon(\psi_{\epsilon,\delta}))\nabla_\tau(J_\epsilon \psi_{\epsilon,\delta}).
\]
Using $\psi_{\varepsilon,\delta} \to \psi$ in $L_t^2 H^{2-\sigma,\rho}$, $\mathbb{P}$-a.s. and convergence $g_{\varepsilon} \to g$ in $L_t^2 L^2(\Gamma)$, $\mathbb{P}$-a.s., we easily obtain $\mathbb{P}$-a.s.

$$
\int_0^t \left( (\psi_{\varepsilon,\delta} + g_{\varepsilon}(\psi_{\varepsilon,\delta})) \nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}), v_{0,\tau} \right) ds - \int_0^t \left( (\psi + g(\psi)) \nabla_\tau (\psi), v_{0,\tau} \right) ds,
$$

for any $v_0 \in H^1$.

By the trace theorem and Proposition 3.1, we infer that $\partial_n \phi_{\varepsilon,\delta} \to \partial_n \phi$ in $L_t^2 H^{\frac{1}{2}-\sigma}(\Gamma)$ for any $\sigma \in (0, \frac{1}{2})$, $\mathbb{P}$-a.s. as well as the convergence $J_{\varepsilon} \psi_{\varepsilon,\delta} \to \psi$ in $L_t^2 H^{2-\sigma}(\Gamma)$, bound (3.58), and the embedding $H^1(\mathcal{D}) \hookrightarrow H^{\frac{3}{2}}(\Gamma) \hookrightarrow L^4(\Gamma)$ give $\mathbb{P}$-a.s.

$$
\int_0^t (\partial_n \phi_{\varepsilon,\delta} \nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}), v_{0,\tau}) ds - \int_0^t (\partial_n \phi \nabla_\tau \psi, v_{0,\tau}) ds.
$$

Indeed,

$$
\int_0^t (\partial_n \phi_{\varepsilon,\delta} \nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}), v_{0,\tau}) ds - \int_0^t (\partial_n \phi \nabla_\tau \psi, v_{0,\tau}) ds
$$

$$
= \int_0^t \left( (\partial_n \phi_{\varepsilon,\delta} - \partial_n \phi) \nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}), v_{0,\tau} \right) ds + (\partial_n \phi (\nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}) - \nabla_\tau \psi), v_{0,\tau}) ds
$$

$$
\leq \|\partial_n \phi_{\varepsilon,\delta} - \partial_n \phi\|_{L_t^2 L^2(\Gamma)} \|\nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta})\|_{L_t^2 L^4(\Gamma)} \|v_{0,\tau}\|_{L^4(\Gamma)}
$$

$$
+ (\partial_n \phi (\nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}) - \nabla_\tau \psi), v_{0,\tau})ds
$$

$$
\leq \|\partial_n \phi_{\varepsilon,\delta} - \partial_n \phi\|_{L_t^2 H^{\frac{1}{2}-\sigma}(\Gamma)} \|\nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta})\|_{L_t^2 H^{2-\sigma}(\Gamma)} \|v_{0,\tau}\|_{L^4(\Gamma)}
$$

$$
+ (\partial_n \phi (\nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}) - \nabla_\tau \psi), v_{0,\tau})ds \to 0.
$$

Furthermore, by bound (3.58) and the convergence $\psi_{\varepsilon,\delta} \to \psi$ in $(L_t^2 H^2(\Gamma)_w$, we have $\mathbb{P}$-a.s.

$$
\int_0^t ((A_{\varepsilon} \psi_{\varepsilon,\delta} - A_{\varepsilon} \psi) \nabla_\tau \psi, v_{0,\tau}) ds \to 0.
$$

Also, bound (3.58), the convergence $J_{\varepsilon} \psi_{\varepsilon,\delta} \to \psi$ in $L_t^2 H^{2-\sigma}(\Gamma)$ and Hölder's inequality give $\mathbb{P}$-a.s.

$$
\int_0^t (A_{\varepsilon} \psi_{\varepsilon,\delta} (\nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}) - \nabla_\tau \psi), v_{0,\tau}) ds
$$

$$
\leq \|A_{\varepsilon} \psi_{\varepsilon,\delta}\|_{L_t^2 L^2(\Gamma)} \|J_{\varepsilon} \psi_{\varepsilon,\delta} - \psi\|_{L_t^2 H^{2-\sigma}(\Gamma)} \|v_{0,\tau}\|_{L^4(\Gamma)} \to 0.
$$

Combining all the convergence results, we conclude that $\mathbb{P}$-a.s.

$$
\int_0^t (\mathcal{K}(\psi_{\varepsilon,\delta}) \nabla_\tau (J_{\varepsilon} \psi_{\varepsilon,\delta}), v_{0,\tau}) ds - (\mathcal{K}(\psi) \nabla_\tau \psi, v_{0,\tau}) ds \to 0,
$$

(3.76)

for any $v_0 \in H^1$.

Finally, we pass the limit in stochastic integral term. Using assumption $A_2$, we obtain

$$
\|h(u_{\varepsilon,\delta}, \nabla \phi_{\varepsilon,\delta}) - h(u, \nabla \phi)\|_{L_4(H;L_t^2)} \leq C\|(u_{\varepsilon,\delta} - u, \nabla (\phi_{\varepsilon,\delta} - \phi))\|_{L_t^2},
$$

together with Proposition 3.1 yields that $h(u_{\varepsilon,\delta}, \nabla \phi_{\varepsilon,\delta}) - h(u, \nabla \phi)\|_{L_2(H;L_t^2)} \to 0$, a.e. $(t, \omega) \in [0,T] \times \Omega$. 

Collecting this convergence, (3.58) and the Vitali convergence theorem (see Theorem 5.1), we have

\[ h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}) \to h(u, \nabla \phi), \quad \text{in} \ L^p_1 L^2_2(\mathcal{H}; L^2_x), \]

which implies that the convergence holds in probability in \( L^2_1 L^2_2(\mathcal{H}; L^2_x) \). Applying Lemma 5.4, we infer that

\[ \int_0^T h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}) dW_{\varepsilon, \delta} \to \int_0^T h(u, \nabla \phi) dW, \]

in probability in \( L^2_1 L^2_2(\mathcal{H}; L^2_x) \).

Using estimates (3.58) and Hölder's inequality, we could claim that the sets

\[ \left\{ \int_0^T -B_0(u_{\varepsilon, \delta}, J_\varepsilon u_{\varepsilon, \delta}, v_0) + B_1(\mu_{\varepsilon, \delta}, J_\varepsilon \phi_{\varepsilon, \delta}, v_0) + (\mathcal{K}(\psi_{\varepsilon, \delta}) \nabla_\tau (J_\varepsilon \psi_{\varepsilon, \delta}), v_{0, \tau})_\Gamma ds \right\}_{\varepsilon, \delta}, \tag{3.77} \]

and

\[ \left\{ \int_0^T -B_2(u_{\varepsilon, \delta}, J_\varepsilon \phi_{\varepsilon, \delta}, v_1) ds \right\}_{\varepsilon, \delta}, \tag{3.78} \]

are uniformly integrable in \( L^p(\tilde{\Omega}; L^2(0, T)) \).

We have everything in hand to pass the limit. Define the functionals as

\[
\begin{align*}
F_1(u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, \psi_{\varepsilon, \delta}, W_{\varepsilon, \delta}, v_0) &= (u_0, v_0) - \int_0^t a(u_{\varepsilon, \delta}, v_0) ds - \int_0^t B_0(u_{\varepsilon, \delta}, J_\varepsilon u_{\varepsilon, \delta}, v_0) ds \\
&+ \int_0^t B_1(\mu_{\varepsilon, \delta}, J_\varepsilon \phi_{\varepsilon, \delta}, v_0) ds + \int_0^t (\mathcal{K}(\psi_{\varepsilon, \delta}) \nabla_\tau (J_\varepsilon \psi_{\varepsilon, \delta}), v_{0, \tau})_\Gamma ds \\
&+ \left( \int_0^t h(u_{\varepsilon, \delta}, \nabla \phi_{\varepsilon, \delta}) dW_{\varepsilon, \delta}, v_0 \right), \\
\end{align*}
\]

\[
\begin{align*}
F_2(u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, v_1) &= (\phi_0, v_1) - \int_0^t (\nabla \mu_{\varepsilon, \delta}, \nabla v_1) + B_2(u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, v_1) ds, \\
\end{align*}
\]

\[
\begin{align*}
F_3(u_{\varepsilon, \delta}, \psi_{\varepsilon, \delta}, v_2) &= (\psi_0, v_2) - \int_0^t B_1(u_{\varepsilon, \delta}, J_\varepsilon \psi_{\varepsilon, \delta}, v_2) + (\mathcal{K}(\psi_{\varepsilon, \delta}), v_2)_\Gamma ds, \\
\end{align*}
\]

and

\[
\begin{align*}
F_1(u, \phi, \psi, W, v_0) &= (u_0, v_0) - \int_0^t a(u, v_0) ds - \int_0^t B_0(u, v_0) ds + \int_0^t B_1(\mu, \phi, v_0) ds \\
&+ \int_0^t (\mathcal{K}(\psi) \nabla_\tau \psi, v_{0, \tau})_\Gamma ds + \left( \int_0^t h(u, \nabla \phi) dW, v_0 \right), \\
F_2(u, \phi, v_1) &= (\phi_0, v_1) - \int_0^t (\nabla \mu, \nabla v_1) + B_2(u, \phi, v_1) ds, \\
F_3(u, \psi, v_2) &= (\psi_0, v_2) - \int_0^t B_1(u, \psi, v_2) + (\mathcal{K}(\psi), v_2)_\Gamma ds.
\end{align*}
\]
Combining all the convergence results and the uniformly integrability of sets, we could summarize from the Vitali convergence theorem that

\[
\tilde{\mathbb{E}} \int_0^T 1_A(F_1(u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, \psi_{\varepsilon, \delta}, W_{\varepsilon, \delta}, v_0) - F_1(u, \phi, \psi, W, v_0))dt \to 0, \tag{3.79}
\]

\[
\tilde{\mathbb{E}} \int_0^T 1_A(F_2(u_{\varepsilon, \delta}, \phi_{\varepsilon, \delta}, v_1) - F_2(u, \phi, v_1))dt \to 0, \tag{3.80}
\]

\[
\tilde{\mathbb{E}} \int_0^T 1_A(F_3(u_{\varepsilon, \delta}, \psi_{\varepsilon, \delta}, v_2) - F_3(u, \psi, v_2))dt \to 0, \tag{3.81}
\]

for any set \( A \subset [0, T] \), \( v_0 \in D(A_0), v_1 \in H^1, v_2 \in L^2(\Gamma) \), where \( 1 \) is the indicator function.

Considering (3.79)–(3.81) and Proposition 3.2 we conclude that \((u, \phi, \psi)\) satisfy

\[
\begin{align*}
(u, v_0) + \int_0^t a_0(u, v_0) + B_0(u, u, v_0) - B_1(\mu, \phi, v_0)ds &= \int_0^t (K(\psi) \nabla_t \psi, v_0, \tau)dr + \int_0^t (h(u, \nabla \phi), v_0)dW, \\
(\phi, v_1) + \int_0^t (\nabla \mu, \nabla v_1) + B_2(\phi, \phi, v_1)ds &= 0, \\
(\psi, v_2) + \int_0^t (K(\psi), v_2) + B_T(u, \psi, v_2)ds &= 0,
\end{align*}
\]

\( \tilde{\mathbb{P}} \)-a.s. for any \( v_0 \in D(A_0), v_1 \in H^1, v_2 \in L^2(\Gamma), t \in [0, T] \). Furthermore, using a density argument, we deduce that system (3.1) holds for any \( v_0 \in H^1, v_1 \in H^1, v_2 \in L^2(\Gamma) \).

Finally, we show that the energy inequality holds. By proposition 3.2 the energy inequality (3.61) still holds for the new approximate sequence on the new probability space. In the limit procedures \( \varepsilon, \delta \to 0 \), using the lower semi-continuity on the left hand-side for any time and using the strong convergence of the approximate sequence for a.e. time, we could have for any \( t > 0 \), there exists a nullset \( T_t \) such that

\[
\mathbb{E} [1_A \mathcal{U}(u, \phi, \psi)(t)] \leq \mathbb{E} [1_A \mathcal{U}(u, \phi, \psi)(s)] \tag{3.82}
\]

for all \( s \in T_t \) and \( A \in \mathcal{F}_s \), where

\[
\mathcal{U}(u, \phi, \psi)(t) := \frac{1}{2} E^2(t) + \int_0^t E \left( \| \nabla u, \nabla \mu \|_{L^2}^2 + \| \phi, \psi \|_{V^2}^2 + \| K(\phi) \|_{L^2(\Gamma)}^2 \right) dr \\
- \int_0^t E \| h(\nabla \phi), u \|_{L^2(H^1, V^2)}^2 dr - \int_0^t (h(u, \nabla \phi), u)^2 dr.
\]

Set \( T = \bigcup_{t \in C} \), where the set \( C \) is countable and dense in \([0, \infty)\). Then for \( s \notin T \) and \( s \leq t \), there exists a sequence \( t_m \subset C \) with \( t_m \to t \), then letting \( m \to \infty \) in (3.82), the lower semi-continuity of mapping \( t \mapsto E(t) \) gives the energy inequality, for further details see [16, Lemma A.3] and [3]. We complete the proof.

4. Markov Selection

4.1. Preliminary. Denote by \( \Omega = C([0, \infty); \mathbb{X}) \) where \( \mathbb{X} \) is a Polish space and by \( \tilde{\mathcal{B}} \) the Borel \( \sigma \)-field of \( \Omega \) with Skorokhod topology and by \( \text{Pr}(\Omega) \) the set of all probability measures on \((\Omega, \tilde{\mathcal{B}}) \). For any \( t > 0 \), let \( \Omega_t = C([0, t]; \mathbb{X}) \) and \( \Omega^t = C([t, \infty); \mathbb{X}) \), then denote by \( \tilde{\mathcal{B}}_t, \tilde{\mathcal{B}}^t \)
the Borel σ-field of \( \Omega_t, \Omega^t \) with Skorokhod topology respectively. Define the canonical process \( q : \Omega \to \mathbb{R} \) as \( q_t(\omega) = \omega(t) \). Define by the shift map \( \theta_t : \Omega \to \Omega^t \) as

\[
\theta_t(\omega)(s) = \omega(s - t), \quad s \geq t,
\]

establishes a measurable isomorphism between \( (\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s \geq 0}) \) and \( (\Omega^t, \mathcal{F}^t, \{\mathcal{F}^t_s\}_{s \geq t}) \) where the filtration \( \mathcal{F}^t_s := \sigma\{\omega(r); t \leq r \leq s\} \) and \( \mathcal{F}^t := \vee_{s} \mathcal{F}^t_s \), we have the fact that \( \mathcal{F}^t = \mathcal{B}^t \), also establishes an isomorphism between \( \text{Pr}(\Omega) \) and \( \text{Pr}(\Omega^t) \), that is, if \( \mathbb{P} \in \text{Pr}(\Omega) \), then \( \mathbb{P} \circ \theta_t^{-1} \in \text{Pr}(\Omega^t) \), if \( \mathbb{P} \in \Omega^t \), then \( \mathbb{P} \circ \theta_t \in \text{Pr}(\Omega) \).

Given \( \mathbb{P} \in \text{Pr}(\Omega) \) and \( t > 0 \), define by \( \mathbb{P}(\cdot | \mathcal{F}_t) \) a regular conditional probability distribution of \( \mathbb{P} \) with respect to \( \mathcal{F}_t \). Also \( \mathbb{P}(\cdot | \mathcal{F}_t) \) is a probability measure on \( (\Omega, \mathcal{F}) \) and for any bounded \( \mathcal{F} \)-measurable function \( f \)

\[
\mathbb{E}^\mathbb{P}(f | \mathcal{F}_t) = \int_\Omega f(y) \mathbb{P}(dy | \mathcal{F}_t), \mathbb{P} \text{-a.s.}
\]

and there exists a \( \mathbb{P} \)-null set \( \mathcal{N} \in \mathcal{F} \) such that for any \( \omega \notin \mathcal{N} \)

\[
\mathbb{P}(\theta_t = \omega(t), s \in [0, t] | \mathcal{F}_t) = 1. \quad (4.1)
\]

Particularly, by (4.1), we could consider \( \mathbb{P}(\cdot | \mathcal{F}_t) \) as a measure on \( (\Omega^t, \mathcal{F}^t) \), that is,

\[
\mathbb{P}^w_{|\mathcal{B}^t_t} := \mathbb{P}(\cdot | \mathcal{F}_t) \in \text{Pr}(\Omega^t).
\]

And for any \( A \in \mathcal{B}^t_t \) and \( B \in \mathcal{B}^t_t \), it holds

\[
\mathbb{P}(A \cap B) = \int_A \mathbb{P}^w_{|\mathcal{B}^t_t}(B) \mathbb{P}(d\omega).
\]

We recall the following reconstruction result which is an inverse procedure of above disintegration.

**Proposition 4.1.** [41, Lemma 6.1.1] For any \( \mathbb{P} \in \text{Pr}(\Omega) \) and \( t > 0 \), \( Q \) is a map from \( \Omega \to \text{Pr}(\Omega^t) \) such that for any \( B \in \mathcal{F}^t \), \( \omega \mapsto Q(\omega)(B) \) is \( \mathcal{F}^t \)-measurable and

\[
Q(\omega)(\theta_t = \omega(t)) = 1, \text{ for all } \omega \in \Omega.
\]

Then there exists a unique probability measure \( \mathbb{P} \otimes_t Q \) on \( \Omega \) such that
1. \( \mathbb{P} \otimes_t Q \) and \( \mathbb{P} \) agree on \( \mathcal{F}_t \);
2. \( Q \) is a regular conditional probability measure distribution of \( \mathbb{P} \otimes_t Q \) on \( \mathcal{F}_t \).

The following definition is a generalization of supermartingale, see [16, Definition 3.2].

**Definition 4.1.** Let \( \mathcal{G} \) be \( \mathcal{B}^t_t \)-adapted real-valued stochastic process, we call \( \mathcal{G} \) is an almost surely \((\mathcal{B}^t_t, \mathbb{P})\)-supermartingale if

\[
\mathbb{E}^{\mathbb{P}}[1_A \mathcal{G}_t] \leq \mathbb{E}^{\mathbb{P}}[1_A \mathcal{G}_s], \quad (4.2)
\]

for a.e. \( s \geq 0 \) and \( s \leq t \), and all \( A \in \mathcal{B}^t_t \).

We call time point \( s \) by regular time of \( \mathcal{G} \) if (4.2) holds, and by exceptional time of \( \mathcal{G} \) if (4.2) fails.

**Definition 4.2.** Let \( \mathcal{H} \) be a Polish space, if there exists an \( B \in \mathcal{F} \) with \( \mathbb{P}(B) = 1 \) such that \( B \subset \{ \omega \in \mathcal{H}, \omega(t) \in \mathcal{H}, \forall t \geq 0 \} \), then we say \( \mathbb{P} \in \text{Pr}_{\mathcal{H}}(\Omega) \subset \text{Pr}(\Omega) \) is concentrated on the paths with values in \( \mathcal{H} \).
Definition 4.3. Let \( \{ \mathbb{P}_x \}_{x \in \mathbb{H}} \) be a family of probability measures in \( \text{Pr}_{\mathbb{H}}(\Omega) \), which is \( \mathcal{B}/\mathcal{B}[0,1] \)-measurable, and if for each \( x \in \mathcal{H} \), there exists a Lebesgue null measure set \( \mathcal{N}_x \) such that for all \( t \notin \mathcal{N}_x \)
\[
\mathbb{P}_x (\cdot | \mathcal{F}_t) = \mathbb{P}_{\omega(t)} \circ \theta_t^{-1}, \text{ for } \mathbb{P}_x \text{-a.s.}
\]
We say the family of \( \{ \mathbb{P}_x \}_{x \in \mathbb{H}} \) is an almost surely Markov family.

Denote by \( \text{Comp}(\text{Pr}(\Omega)) \) the space of all compact subsets of \( \text{Pr}(\Omega) \), which is a metric space after endowed with the Hausdorff metric.

Definition 4.4. Consider a measure map \( \mathcal{C} : \mathbb{H} \to \text{Comp}(\text{Pr}(\Omega)) \cap \text{Pr}_{\mathbb{H}}(\Omega) \), the family \( \{ \mathcal{C}(x) \}_{x \in \mathbb{H}(\Omega)} \) is almost surely pre-Markov if for each \( x \in \mathcal{H} \) and \( \mathbb{P} \in \mathcal{C}(x) \), there exists a Lebesgue null measure set \( \mathcal{N}_x \) such that for all \( t \notin \mathcal{N}_x \), it holds

1. (Disintegration) there exists \( \mathbb{P} \)-null set \( \bar{N} \) such that for all \( \omega \notin \bar{N} \)
\[
\omega(t) \in \bar{N}, \mathbb{P}(\theta_t(\cdot)|\mathcal{F}_t) \in \mathcal{C}(\omega(t));
\]
2. (Reconstruction) for each measurable map \( \omega \to Q_\omega : \Omega \to \text{Pr}(\Omega') \) such that there exists a \( \mathbb{P} \)-null set \( \bar{N} \subseteq \mathcal{F}_t \) such that all \( \omega \notin \bar{N} \)
\[
\omega(t) \in \bar{N}, Q_\omega \circ \theta_t \in \mathcal{C}(\omega(t)),
\]
then \( \mathbb{P} \otimes Q \in \mathcal{C}(x) \).

The following abstract Markov selection result was given in [16, Theorem 2.8].

Proposition 4.2. Let \( \{ \mathcal{C}(x) \}_{x \in \mathbb{H}} \) be an almost surely pre-Markov family with non-empty convex values. Then, there exists a measurable map \( x \mapsto \mathbb{P}_x \in \text{Pr}_{\mathbb{H}}(\Omega) \) such that \( \mathbb{P}_x \in \mathcal{C}(x) \) for all \( x \in \mathbb{H} \) and \( \{ \mathbb{P}_x \}_{x \in \mathbb{H}} \) has almost surely Markov property.

4.2. The solution to martingale problem of system (1.1)-(1.6). In view of the result established in section 3, choosing \( X_1 = H^1 \times V^2 \) and \( Y = Y_1 \times Y_2 \), where \( Y_1 = L^2 \times V^1, Y_2 = H^1 \times L^2(\Gamma) \). Introducing by \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) two new variables
\[
\mathcal{V}_1 = \mathcal{V}_1(0) + \int_0^t \mu ds, \quad \mathcal{V}_2 = \mathcal{V}_2(0) + \int_0^t K ds,
\]
for arbitrary initial data \( \mathcal{V}_1(0) \in H^1, \mathcal{V}_2(0) \in L^2(\Gamma) \). From the regularity of \( \mu, \mathcal{K} \), we easily see \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are two continuous stochastic processes with trajectories in \( W^{1,2}_t H^1 \) and \( W^{1,2}_t L^2(\Gamma), \mathbb{P}\text{-a.s.} \) respectively.

Denote by \( \bar{\Omega} = C([0, \infty); \mathcal{X}) \) with \( \mathcal{X} = (D(A_0) \times V^2 \times H^1 \times L^2(\Gamma))' \). Let
\[
\varrho = (\varrho^1, \varrho^2, \varrho^3, \varrho^4, \varrho^5)
\]
be the canonical process with
\[
\varrho_t(\omega) = (\varrho^1, \varrho^2, \varrho^3, \varrho^4, \varrho^5)(\omega) = \omega(t) \in \mathcal{X}.
\]
For Markov selection, we re-define the solution to martingale problem (1.1)-(1.6).

Definition 4.5. Given \( y \in Y \), a probability measure \( \mathcal{P} \) on \( (\bar{\Omega}, \bar{\mathcal{B}}) \) is a solution to martingale problem (1.1)-(1.6) if

The process is given by
\[ M_t^{v_0} = (\varphi^1_1 - \varphi^1_0, v_0) + \int_0^t a_0(\varphi^1_1, v_0) - B_0(\varphi^1_1, v_0, \varphi^2_1) - B_1(\partial_t \varphi^3_1, \varphi^2_1, v_0)ds \]
\[ - \int_0^t (\partial_t \varphi^5_1 \nabla_\tau \varphi^3_1, v_0, \tau) ds, \]

is a continuous square integrable martingale with respect to \( \mathcal{P} \), its quadratic variation process is given by
\[ \langle M_t^{v_0} \rangle := \frac{1}{2} \int_0^t (h(\varphi^1_1, \nabla \varphi^2_1), v_0)^2 ds; \]

for every \( t \geq 0 \), it holds \( \mathcal{P} \)-a.s.
\[ (\varphi^2_1, v_1) + \int_0^t (\nabla \partial_t \varphi^3_1, \nabla v_1) - B_2(\varphi^1_1, v_1, \varphi^2_1)ds = (\varphi^2_0, v_1); \]

for every \( v_2 \in H^1 \) and \( t \geq 0 \), it holds \( \mathcal{P} \)-a.s.
\[ (\varphi^2_1, v_2) + \int_0^t (\partial_t \varphi^5_1, \varphi^2_2) - B_1(\varphi^1_1, v_2, \varphi^3_1)ds = (\varphi^2_0, v_1); \]

the process
\[ \mathcal{G} : [\omega, t] \rightarrow \frac{1}{2} \mathcal{E}_t + \int_0^t \mathcal{E}_t \left( \| \nabla \varphi^1_1 \|^2_{L^2} + \| \nabla \varphi^2_1 \|^2_{L^2} + \| \varphi^2_1 \|^2_{L^2} + \| \partial_t \varphi^3_1 \|^2_{L^2} \right) ds - \frac{1}{2} \mathcal{E}_0^2 \]
\[ - \int_0^t \mathcal{E}_t \| h(\varphi^1_1, \nabla \varphi^2_1) \|^2_{L^2(\mathbb{H}; L^2_\tau)} ds \]
\[ - \int_0^t (h(\varphi^1_1, \nabla \varphi^2_1), \varphi^2_2)^2 ds, \]

is an almost surely \((\mathcal{F}_t, \mathcal{P})\)-supermartingale and 0 is a regular time, where
\[ \mathcal{E}_t := \frac{1}{2} \left( \| \varphi^1_1 \|^2_{L^2} + \| \nabla \varphi^2_1 \|^2_{L^2} + \| \varphi^2_1 \|^2_{L^2(\mathbb{H}; L^2_\tau)} + \| \nabla \varphi^3_1 \|^2_{L^2(\mathbb{H}; L^2_\tau)} \right) + \int_\mathcal{D} F(\varphi^3_1)dx + \int_\Gamma G(\varphi^3_1)ds; \]

it holds \( \mathcal{P} \)-a.s.
\[ \frac{\partial_t \varphi^4_1}{\partial_x \varphi^2_1} + \frac{\partial_t \varphi^5_1}{\partial_x \varphi^3_1} = A \varphi^3 + \partial_x \varphi^2 + \varphi^3 + g(\varphi^3). \]

**Proposition 4.3.** Definition 3.1 and Definition 4.5 are equivalent in the following sense:
(i) if \((S, u, \phi, \psi)\) is a global martingale weak solution to system (1.1)-(1.6), then for every \( H^1, L^2(\Gamma) \)-valued \( \mathcal{F}_0 \) measurable random variables \( V_1(0), V_2(0) \), we could have a probability measure \( \mathcal{P} = \mathcal{L}(u, \phi, \psi, V_1, V_2) \) is a solution to the martingale problem associated to (1.1)-(1.6);
(ii) if a probability measure \( \mathcal{P} \) is a solution to the martingale problem associated to (1.1)-(1.6), then there exists a global martingale weak solution \((S, u, \phi, \psi)\) to system (1.1)-(1.6) and \( H^1, L^2(\Gamma) \)-valued \( \mathcal{F}_0 \) measurable random variables \( V_1(0) \) and \( V_2(0) \) such that \( \mathcal{L}(u, \phi, \psi, V_1, V_2) = \mathcal{P} \).

**Proof.** The proof is totally same with that of [3 Proposition 3.8], we omit the details. \( \square \)
4.3. Markov selection. In the following, we are going to establish the existence of almost surely Markov selection to martingale problem \((1.1)-(1.6)\). Denote by \(\mathcal{P}_y\) be the solution to the martingale problem \((1.1)-(1.6)\) starting from initial data \(\delta_y\).

Theorem 4.1. Suppose that assumptions \(A_1-A_4\) hold, then there exists an almost surely Markov family \(\mathcal{P}_y\) associated to \((1.1)-(1.6)\).

Denote by \(\{\mathcal{C}(y)\}_{y \in \mathcal{Y}}\) be the set of probability measure \(\mathcal{P}_y\) be the solution to the martingale problem \((1.1)-(1.6)\) starting from initial data \(\delta_y\). From Proposition 4.2, the proof of above theorem boils down to show the family \(\{\mathcal{C}(y)\}_{y \in \mathcal{Y}} \subset \text{Pr}(\Omega)\) is an almost surely pre-Markov family.

Lemma 4.1. For each \(y \in \mathcal{Y}\), the set \(\mathcal{C}(y)\) is non-empty and convex.

Proof. Due to Theorem 3.1 and Proposition 4.3, for each \(y \in \mathcal{Y}\), a solution to martingale problem exists making sure the set is non-empty. The convexity follows immediately all properties in Definition 4.5 involving integration with respect to measure \(\mathcal{P}\).

Lemma 4.2. For each \(y \in \mathcal{Y}\), the set \(\mathcal{C}(y)\) is compact and the map \(\mathcal{C} : \mathcal{Y} \rightarrow \text{Comp(Pr}(\Omega))\) is Borel measurable.

Proof. We claim that both properties are direct consequence of the following argument: for the initial data sequence \(y_n \in \mathcal{Y}\) which converges to \(y\) and \(\mathcal{P}_n \in \mathcal{C}(y_n)\), then there exists a subsequence of \(\mathcal{P}_n\) which converges to \(\mathcal{P} \in \mathcal{C}(y)\) weakly in \(\text{Pr}(\Omega)\).

From Theorem 3.1, there exists a sequence martingale weak solutions \((S^n, u^n, \phi^n, \psi^n)\) in the sense of Definition 3.1. By a same argument as Parts 2,3 in section 3, we also have that the sequence \((u^n, \phi^n, \psi^n, \mu^n, K(\psi^n))\) has uniform priori estimates and the probability measures \(\mathcal{P}^n\) induced by the law of sequence \((u^n, \phi^n, \psi^n, V^n_1, V^n_2)\) is tight on path space \(\mathcal{Y} \times (W^{1,2}_tH_1)_w \times (W^{1,2}_tL^2(\Gamma))_w\). The Prokhorov theorem tells us there exist a subsequence \(\mathcal{P}^{nk}\) and a probability measure \(\mathcal{P}\) such that \(\mathcal{P}^{nk}\) converges weakly to \(\mathcal{P} \in \text{Pr}(\Omega)\). Next we show that \(\mathcal{P} \in \mathcal{C}(y)\).

Again, the Skorokhod representative theorem 5.2 gives that there exist a new probability space \((\Omega, \mathcal{F}, \mathcal{P})\), a new subsequence still denoted by \((\bar{u}^n, \bar{\phi}^n, \bar{\psi}^n, \bar{V}_1^n, \bar{V}_2^n, \bar{W}^n)\) converges to \((\bar{u}, \bar{\phi}, \bar{\psi}, \bar{V}_1, \bar{V}_2, \bar{W})\) in \(\mathcal{Y} \times (W^{1,2}_tH_1)_w \times (W^{1,2}_tL^2(\Gamma))_w\), \(\mathcal{P}\)-a.s. and \(\mathcal{L}(\bar{u}^n, \bar{\phi}^n, \bar{\psi}^n, \bar{V}_1^n, \bar{V}_2^n) = \mathcal{P}^n\), where \(Y\) is the space defined in Part 3. Following the same line of Parts 3,4 of section 3, we have that \((\mathcal{S}, \bar{u}, \bar{\phi}, \bar{\psi})\) is a global martingale weak solution to system \((1.1)-(1.6)\) in the sense of Definition 3.1. Furthermore, we have \(\mathcal{P} = \mathcal{L}(\bar{u}, \bar{\phi}, \bar{\psi}, \bar{V}_1, \bar{V}_2)\). Finally, Proposition 4.3 yields that \(\mathcal{P}\) is a solution to the martingale problem associated to \((1.1)-(1.6)\) with initial law \(\delta_y\). The fact \(\mathcal{P} \in \mathcal{C}(y)\) verified.

The following two results are important for next proof of disintegration and reconstruction properties, which were given in [16] Proposition B.1, Proposition B.4).

Proposition 4.4. Let \(\mathcal{Y}\) and \(\mathcal{Q}\) be two real-valued continuous \(\mathcal{B}_t\)-adapted processes and \(s \geq 0\), the followings are equivalent:
(i) \((\mathcal{Y}_t, \mathcal{B}_t, \mathcal{P})\) is a square integrable martingale with quadratic variation \(Q_t\) for \(t \geq s\);
(ii) there is a \(\mathcal{P}\)-nullset \(\mathcal{N} \in \mathcal{B}_s\) such that for all \(\omega \notin \mathcal{N}\), process \((\mathcal{Y}_t, \mathcal{B}_t, \mathcal{P}_{\omega|\mathcal{B}_t})\) is a square integrable martingale with quadratic variation \(Q_t\) for \(t \geq s\) and \(\mathbb{E}^\mathcal{P}\left(\mathbb{E}^{\mathcal{P}|\mathcal{B}_t}[Q_t]\right) < \infty\).
Lemma 4.3. The disintegration property of Definition 4.4 holds for the family of properties. Once both properties hold, Theorem 4.1 follows from Proposition 4.2.

The proof is similar to [16, Lemma 4.4], we still give the details for reader’s convenience. For $\mathbb{P}$-nullset $N \in \mathcal{B}_s$ such that for all $\omega \notin N$, process $(\mathcal{Y}_t, \mathcal{B}_t, \mathbb{P}|_{\mathcal{B}_t})$ is an almost surely super-martingale and for all $t \geq s$

$$\mathbb{E}^\mathbb{P}(\alpha_t + \beta_t) < \infty;$$

(ii) there is a $\mathcal{P}$-nullset $N \in \mathcal{B}_s$ such that for all $\omega \notin N$, process $(\mathcal{Y}_t, \mathcal{B}_t, \mathbb{P}|_{\mathcal{B}_t})$ is an almost surely super-martingale and for all $t \geq s$

$$\mathbb{E}^\mathbb{P}|_{\mathcal{B}_t}(\alpha_t + \beta_t) < \infty,$$

and

$$\mathbb{E}^\mathbb{P}(\mathbb{E}^\mathbb{P}|_{\mathcal{B}_t}[^{\mathbb{P}|_{\mathcal{B}_t}}(\alpha_t + \beta_t)]) < \infty.$$ 

Next, we show the family $\{\mathcal{C}(y)\}_{y \in \mathcal{Y}}$ satisfies the disintegration and reconstruction properties. Once both properties hold, Theorem 4.1 follows from Proposition 4.2.

Lemma 4.3. The disintegration property of Definition 4.4 holds for the family $\{\mathcal{C}(y)\}_{y \in \mathcal{Y}}$.

Proof. The proof is similar to [16, Lemma 4.4], we still give the details for reader’s convenience. For $y \in \mathcal{Y}$ and $\mathcal{P} \in \mathcal{C}(y)$, Let $T_\mathcal{P}$ be the set of exceptional times of $\mathcal{P}$ and for $t \notin T_\mathcal{P}$, denote by $\mathbb{P}|_{\mathcal{B}_t}$ the regular conditional probability distribution of $\mathcal{P}$ on $\mathcal{B}_t$. Our goal is to show that there exists a $\mathcal{P}$-nullset $N \in \mathcal{B}_t$ such that for all $\omega \notin N$, $\mathbb{P}|_{\mathcal{B}_t}^\omega \circ \theta_t \in \mathcal{C}(y)$. That is, we need to verify (M.1) – (M.5) hold for $\mathbb{P}|_{\mathcal{B}_t}^\omega \circ \theta_t$.

Define by the set

$$\sum_t = \left\{ \omega \in \tilde{\Omega} : \omega |_{[0,t]} \in L^\infty([0,t]; \mathcal{Y}_1) \cap L^2([0,t]; \mathcal{X}_1) \times W^{1,2}([0,t); \mathcal{H}^1 \times L^2(\Gamma)) \right\},$$

$$\sum_\omega = \left\{ \omega \in \tilde{\Omega} : \omega |_{[t,\infty]} \in L^\infty_{\text{loc}}([t,\infty); \mathcal{Y}_1) \cap L^2_{\text{loc}}([t,\infty); \mathcal{X}_1) \times W^{1,2}_{\text{loc}}([t,\infty); \mathcal{H}^1 \times L^2(\Gamma)) \right\}.$$ 

We have by (M.1) for $\mathcal{P}$ that

$$1 = \mathcal{P} \left( \sum_t \cap \sum_\omega \right) = \int_{\sum_t} \mathbb{P}|_{\mathcal{B}_t}^\omega \left( \sum_\omega \right) \mathcal{P}(d\omega),$$

following there exists a $\mathcal{P}$-nullset $N_1 \in \mathcal{B}_t$ such that for all $\omega \notin N_1$, $\mathbb{P}|_{\mathcal{B}_t}^\omega \left( \sum_\omega \right) = 1$.

Let $\{v_n\}_{n \geq 1}$ be the dense subset of $D(A_0)$, then we have by (M.2) for $\mathcal{P}$ that $(\mathbb{M}^n|_{v_n}, \mathcal{B}_t, \mathcal{P})$ is a square integrable martingale with quadratic variation

$$\langle \mathbb{M}^n|_{v_n} \rangle = \frac{1}{2} \int_0^t (h(g^1, \nabla g^2), v_n)^2 ds.$$ 

Proposition 4.3 gives that there exists a $\mathcal{P}$-nullset $N_2^n$ such that $(\mathbb{M}^n|_{v_n}, \mathcal{B}_t, \mathbb{P}|_{\mathcal{B}_t}^\omega)_{t \geq t}$ is a square integrable martingale with quadratic variation $(\mathbb{M}^n|_{v_n})_{t \geq t}$ for all $\omega \notin N_2^n$. Choosing $N_2 = \bigcup N_2^n$.

Denote by $\sum_t, \sum_\omega$

$$\sum_t := \left\{ \omega \in \tilde{\Omega} : (\tilde{g}^2, v_1)|_{t=0}^{r=s} + \int_0^s (\nabla \partial_t g^4, \nabla v_1) - B_2(g^1, v_1, g^2) dr = 0, s \leq t \right\},$$
and
\[ \tilde{\sum}^t := \left\{ \omega \in \tilde{\Omega}; (\tilde{g}_2^2, v_1) \right\}_{t=s}^{t=s} + \int_t^s (\nabla \partial_t \bar{g}_1, \nabla v_1) - B_2(g_1, v_1, \bar{g}_2)dr = 0, s \geq t \right\}. \]
Then (M.3) and (M.4) hold for \( P|_{\tilde{B}_t} \circ \theta_t \) a.s. \( \omega \notin N_3 \cup N_4 \) by a same argument as above.
Regarding (M.5), we set
\[ \alpha_t = \frac{1}{2} \varepsilon_t^2 + \int_0^t \mathcal{E}_t(\|\nabla g_1, \nabla \partial_t g_1\|^2_{L_2} + \|g_2^2 - g_3^2\|^2_{V_2} + \|\partial_t \bar{g}_2\|^2_{L_2(\Gamma)})ds \]
and
\[ \beta_t = \frac{1}{2} \varepsilon_0^2 + \int_0^t \mathcal{E}_t(h(g_1, \nabla \bar{g}_2)^2_{L_2(H, L_2^2)})ds + \int_0^t (h(g_1, \nabla \bar{g}_2), g_2^2)^2ds. \]
Note that \( \alpha_t \) is left lower semi-continuous and \( \beta_t \) is non-decreasing, so that \( G_t \) is also left lower semi-continuous. Since we have (M.5) for \( P \) and by Proposition 4.5, there exists a \( \mathcal{P} \)-nullset \( N_5 \), for all \( \omega \notin N_5 \) the result holds.
By same argument as M.1, there exists a \( \mathcal{P} \)-nullset \( N_6 \in \tilde{B}_t \) such that for all \( \omega \notin N_6 \), (M.6) holds for \( P|_{\tilde{B}_t} \) a.s.
Finally, letting \( \tilde{N} = N_1 \cup N_2 \cup N_3 \cup N_4 \cup N_5 \cup N_6 \), we complete the proof. \( \square \)

**Lemma 4.4.** The reconstruction property of Definition 4.4 holds for the family \( \{\mathcal{C}(y)\}_{y \in Y} \).

**Proof.** Let \( y \in Y \) and \( P \in \mathcal{C}(y) \). Let \( T_P \) be the set of exceptional times of \( P \). We need to prove that for a \( \tilde{B}_t \) measurable map \( Q_\omega : \omega \mapsto \text{Pr}(\Omega^t) \), there exists a \( \mathcal{P} \)-nullset \( N_7 \in \tilde{B}_t \) such that for all \( \omega \in N_7 \), \( Q_\omega \circ \theta_t \in \mathcal{C}(\omega(t)) \), then we have the probability measure \( \mathcal{P} \otimes_t Q \) given by Proposition 4.1 belongs to \( \mathcal{C}(y) \). That is, (M.1) – (M.5) need to be verified for \( \mathcal{P} \otimes_t Q \).

First, we have (M.1) holds for \( Q_\omega \), therefore \( Q_\omega \left( \sum_t \right) = 1 \), which yields
\[ \mathcal{P} \otimes_t Q \left( \sum_t \cap \sum_t' \right) = \int_{\sum_t} Q_\omega \left( \sum_t' \right) \mathcal{P}(d\omega) = \mathcal{P} \left( \sum_t \right) = 1, \]
where sets \( \sum_t, \sum_t' \) are same as Lemma 4.3.

Since (M.2) holds for \( Q_\omega \), so \( \left( \mathcal{M}_{t_0}^t \right)_{t \geq s} \) is a \( (\tilde{B}_t, \mathcal{Q}_t, Q_\omega) \) square integral martingale, together with the fact that \( Q_\omega \) is a regular conditional probability distribution of \( \mathcal{P} \otimes_s Q \) on \( \tilde{B}_s \), we have \( \left( \mathcal{M}_{t_0}^t \right)_{t \geq s} \) is a \( (\tilde{B}_t, \mathcal{P} \otimes_s Q) \) square integral martingale by Proposition 4.4.

Since \( \mathcal{P} \) and \( \mathcal{P} \otimes_t Q \) coincide on \( \tilde{B}_t \) and \( \left( \mathcal{M}_{t_0}^t, \mathcal{P}_t \right)_{t \leq s} \) is a square integrable martingale, we conclude that \( \left( \mathcal{M}_{t_0}^s, \tilde{B}_t, \mathcal{P} \otimes_s Q \right)_{s \geq 0} \) is a square integrable martingale.

Since \( Q_\omega \left( \sum_t' \right) = 1 \) by (M.3) for \( Q_\omega \), we have
\[ \mathcal{P} \otimes_t Q \left( \sum_t' \cap \sum_t' \right) = \int_{\sum_t} Q_\omega \left( \sum_t' \right) d\mathcal{P}(\omega) = \mathcal{P} \left( \sum_t \right) = 1. \]

Also, (M.4) holds for \( \mathcal{P} \otimes_t Q \) by a same argument.

Since (M.5) holds for \( Q_\omega \), \( (\mathcal{G}_t, \tilde{B}_t, Q_\omega)_{t \geq s} \) is an almost surely supermartingale. Proposition 4.5 gives that \( (\mathcal{G}_t, \tilde{B}_t, \mathcal{P} \otimes_s Q)_{t \geq s} \) is an almost surely supermartingale. Since \( \mathcal{P} \) and \( \mathcal{P} \otimes_t Q \) coincide on \( \tilde{B}_t \) and \( (\mathcal{G}_t, B_t, \mathcal{P})_{t \leq s} \) is an almost surely supermartingale, we obtain that \( (\mathcal{G}_t, \tilde{B}_t, \mathcal{P} \otimes_s Q)_{t \geq 0} \) is an almost surely supermartingale.
As the proof of (M.1), we have
\[ \mathcal{P} \otimes_t Q(\partial_t \varrho^4 = -\Delta \varrho^2 + f(\varrho^2)) \]
\[ = \mathcal{P} \otimes_t Q(\partial_t \varrho^4|_0 = -\Delta \varrho^2 + f(\varrho^2)|_0, \partial_t \varrho^4|_t = -\Delta \varrho^2 + f(\varrho^2)|_t) \]
\[ = \int_{\{\partial_t \varrho^4|_0 = -\Delta \varrho^2 + f(\varrho^2)|_0\}} Q_\omega(\partial_t \varrho^4|_t = -\Delta \varrho^2 + f(\varrho^2)|_t) d\mathcal{P}(\omega) \]
\[ = \int_{\{\partial_t \varrho^4|_0 = -\Delta \varrho^2 + f(\varrho^2)|_0\}} d\mathcal{P}(\omega) = 1. \]
Similarly, we also have \( \mathcal{P} \otimes_t Q(\partial_t \varrho^3 = A_t \varrho^3 + \partial_n \varrho^2 + \varrho^3 + g(\varrho^3)) = 1. \)
This completes the proof. \( \square \)

5. Appendix

At the beginning, we introduce some lemmas used frequently for improving the regularity and obtaining the required estimates.

**Lemma 5.1.** [11, Theorem 1.1.1] Let \( \mathcal{L} : L^{p_1}(0, T) \to L^{p_2}(\mathcal{D}) \) and \( L^{q_1}(0, T) \to L^{q_2}(\mathcal{D}) \) be a linear operator with \( q_1 > p_1 \) and \( q_2 < p_2 \). Then, for any \( s \in (0, 1) \), the operator \( \mathcal{L} : L^r(0, T) \to L^s(\mathcal{D}), \) where \( r_1 = \frac{1}{s/p_1+(1-s)/q_1}, \) \( r_2 = \frac{1}{s/q_2+(1-s)/p_2}. \)

**Lemma 5.2.** [20, Lemmas 4.2, 4.3] For all \( \kappa \in (0, 1) \), there exists \( \alpha > 0 \) such that
\[ \|u\|_{L^2(\Gamma)}^2 \leq \kappa \|
abla u\|_{L^2}^2 + \kappa^{-\alpha} \|u\|_{L^2}^2, \]
as a consequence, we have
\[ \|u_r\|_{L^2(\Gamma)}^2 \leq \kappa \|
abla u\|_{L^2}^2 + \kappa^{-\alpha} \|u\|_{L^2}^2. \]

**Lemma 5.3.** [40, Lemma 3] Let \( r^* = \left\{ \begin{array}{ll} \frac{dr}{d-r}, & \text{if } r < d, \\ \infty, & \text{if } r > d, \end{array} \right. \)
where \( d = 2, 3 \) is the dimension. For \( 1 \leq r \leq \infty, 1 \leq s \leq \infty, \) if \( \frac{1}{r} + \frac{1}{s} = 1, \) and \( \frac{1}{r} + \frac{1}{s} = \frac{1}{r} \), \( f \in H^{1,r} \) and \( g \in H^{-1,s} \), then \( fg \in H^{-1,t} \), that is,
\[ \|fg\|_{H^{-1,t}} \leq \|f\|_{H^{1,r}} \|g\|_{H^{-1,s}}. \]

**Lemma 5.4.** [36, Lemma A.1] Let \((\phi, \psi)\) be the solution to the following elliptic boundary problem
\[ \begin{cases} -\Delta \phi = l_1, & \text{in } \mathcal{D}, \\ -\Delta_t \psi + \partial_n \phi + \psi = l_2, & \text{on } \Gamma. \end{cases} \]
If \((l_1, l_2) \in L^2_x \times L^2(\Gamma)\), then, it holds for some constant \( C \)
\[ \|((\phi, \psi))|_{H^2 \times H^2(\Gamma)} \leq C \|(l_1, l_2)\|_{L^2_x \times L^2(\Gamma)}. \]

**Lemma 5.5.** [42] Let \( \mathcal{D} \) be a sufficient smooth boundary domain and \( s \geq 0 \). For every \( \epsilon > 0 \), the mollifier operator \( \mathcal{J}_\epsilon \) maps \( H^s(\mathcal{D}) \) into \( H^{s'}(\mathcal{D}) \) where \( s' > s \) and has the following properties.
(1) The collection \( \{\mathcal{J}_\epsilon\}_{\epsilon > 0} \) is uniformly bounded in \( H^s(\mathcal{D}) \) independence of \( \epsilon \), i.e. there exists a positive constant \( C=C(s) \) such that,
\[ \|\mathcal{J}_\epsilon f\|_{H^s} \leq C \|f\|_{H^s}, \quad f \in H^s(\mathcal{D}). \]
(2) For every \( \varepsilon > 0 \), if \( s \geq 1 \) then for \( f \in H^s(\mathcal{D}) \),
\[
\|J_\varepsilon f\|_{H^{s'}} \leq \frac{C}{\varepsilon^{s'-s}}\|f\|_{H^s}.
\]

(3) Sequence \( J_\varepsilon f \) converges to \( f \), for \( f \in H^s(\mathcal{D}) \), that is,
\[
\lim_{\varepsilon \to 0}\|J_\varepsilon f - f\|_{H^s} = 0.
\]

Remark 5.1. The mollifier operator \( J_\varepsilon \) could be defined as \( J_\varepsilon u = R\tilde{J}_\varepsilon Eu \), where \( \tilde{J}_\varepsilon \) is a standard Friedrich’s mollifier on \( \tilde{\mathcal{D}} \). \( \tilde{\mathcal{D}} \) is the un-boundary extend of smooth domain \( \mathcal{D} \), and here \( E \) is an extension operator from \( H^s(\mathcal{D}) \) into \( H^s(\tilde{\mathcal{D}}) \), \( R \) is a restriction operator from \( H^s(\tilde{\mathcal{D}}) \) into \( H^s(\mathcal{D}) \), see [13, Chapter 5].

In order to establish the tightness of a family of solutions, the following Aubin-Lions lemma and Skorokhod representative theorem are commonly used. For more backgrounds we recommend Corollary 5 of [39] and Theorem 2.4 of [7]. The Vitali convergence theorem is applied to identify the limit, see for example, Chapter 3 of [30].

**Lemma 5.6.** (The Aubin-Lions lemma) Suppose that \( X_1 \subset X_0 \subset X_2 \) are Banach spaces and \( X_1 \) and \( X_2 \) are reflexive and the embedding of \( X_1 \) into \( X_0 \) is compact. Then for any \( 1 \leq p < \infty \), \( 0 < \alpha < 1 \), the embedding
\[
L^p_t X_1 \cap C^\alpha_t X_2 \hookrightarrow L^p_t X_0,
\]
is compact.

**Theorem 5.1.** (The Vitali convergence theorem) Let \( p \geq 1 \), \( \{X_n\}_{n \geq 1} \in L^p \) and \( X_n \to X \) in probability. Then, the following are equivalent:
(1) \( X_n \to X \) in \( L^p \);
(2) the sequence \( |X_n|^p \) is uniformly integrable;
(3) \( E|X|^p \to E|X|^p \).

**Theorem 5.2.** (The Skorokhod representative theorem) Let \( X \) be a topological space. If the set of probability measures \( \{\nu_n\}_{n \geq 1} \) on \( B(X) \) is tight, then there exists a probability space \( (\Omega, \mathcal{F}, P) \) and a sequence of random variables \( u_n, u \) such that their laws are \( \nu_n, \nu \) and \( u_n \to u \), \( P \)-a.s. as \( n \to \infty \).

We need the following result taking from [8, Lemma 2.1] for passing the limit of the sequence of stochastic integrals,

**Lemma 5.7.** Assume that \( G_\varepsilon \) is a sequence of \( X \)-valued \( \mathcal{F}_t^\varepsilon \) predictable process such that
\[
G_\varepsilon \to G, \text{ in probability in } L^2_t L^2(\mathcal{H}; X),
\]
and the cylindrical wiener process sequence \( W_\varepsilon \) satisfies
\[
W_\varepsilon \to W, \text{ in probability in } C_t \mathcal{H}_0,
\]
then,
\[
\int_0^t G_\varepsilon dW_\varepsilon \to \int_0^t GdW, \text{ in probability in } L^2_t X.
\]
6. Auxiliary result

We are going to provide two well-posedness results of the stochastic Stokes equation with the Navier boundary condition, and the Cahn-Hilliard equation with the dynamic boundary condition.

The specific form of the stochastic Stokes equation with additive noise reads as

\[
\begin{cases}
du - \text{div}(2D(u))dt + \nabla pdt = h_0dt + hdW, \\
\nabla \cdot u = 0,
\end{cases}
\]

with the boundary conditions

\[
u \cdot n = 0, \quad 2(D(u) \cdot n)_\tau + u_\tau = \ell,
\]

and the initial data \(u|_{t=0} = u_0\).

**Proposition 6.1.** Assume that \(u_0 \in L^p_\omega L^2_x\) is \(F_0\)-measurable random variable, operator \(h \in L^p_\omega L^2_t L^2(H; L^2_x)\), functions \(h_0 \in L^p_\omega L^2_t L^2_x\) and \(\ell \in L^p_\omega L^4_t \Gamma\), then system (6.1)-(6.2) has a unique global weak solution \(u\) such that

\[
u \in L^p_\omega(\cap L^\infty_t L^2_x \cap L^2_t H^1),
\]

satisfying

\[
(u, v) + \int_0^t a_0(u, v)ds = (u_0, v) + \int_0^t (h, v) dW + \int_0^t (\ell, v)_\Gamma ds + \int_0^t (h_0, v)ds,
\]

for all \(t \geq 0, \ p \geq 2\) and \(v \in H^1\).

**Proof.** The proof could be implemented by the classical Galerkin approximate scheme, stochastic compactness and identifying the limit. Define space \(\mathbb{H}_n = \text{span}\{e_1, \ldots, e_k\}\) where the sequence \(\{e_k\}_{k \geq 1}\) is the orthonormal basis for \(L^2_x\) of the Stokes operator \(A_0\). Define

\[
u_n = P_n u = \sum_{k=1}^n (u, e_k)e_k, \quad \text{for all } u \in L^2_x,
\]

where the operator \(P_n : L^2_x \to \mathbb{H}_n\). Then, there exists a unique approximate solutions sequence to the following finite-dimensional equation

\[
(u_n, v) + \int_0^t a_0(u_n, v)ds = (u_0, v) + \int_0^t (h, v) dW + \int_0^t (\ell, v)_\Gamma ds + \int_0^t (h_0, v)ds,
\]

for any \(v \in \mathbb{H}_n\). Using Itô’s formula to the function \(\frac{1}{2}||u_n||^2_{L^2_x}\), we have

\[
\frac{1}{2}||u_n||^2_{L^2_x} + \int_0^t a_0(u_n, u_n)ds = \frac{1}{2}||u_0||^2_{L^2} + \int_0^t (h, u_n)dW + \int_0^t (\ell, u_n)_\Gamma ds
+ \frac{1}{2} \int_0^t ||Ph||^2_{L^2(H; L^2_x)}ds + \int_0^t (h_0, u_n)ds.
\]
Moreover, the mass is conserved, i.e.,

\[ \ell(\bar{H}) \]

Proposition 6.2. [20, Theorem 4.2] Using the embedding \( H^1(D) \hookrightarrow H^2(\Gamma) \hookrightarrow L^4(\Gamma) \) and Hölder’s inequality, we get

\[
\int_0^t (\ell, u_n)_\tau \, dt \leq \int_0^t \|\ell\|_{L^4(D)} \|u_n\|_{L^4(\Gamma)} \, dt \leq \|\ell\|_{L^4(D)} \|u_n\|_{L^4(\Gamma)} \leq C.
\]

The last term in (6.3) could be easily controlled by using Hölder’s inequality. Taking supremum about \( t \) in (6.3), power \( p \) and then expectation, using above estimates, and conditions on \( l, h, \) we have

\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|u_n\|_{L^p_x}^p \right) + \mathbb{E} \left( \int_0^T a_0(u_n, u_n) \, dt \right) \leq C,
\]

where constant \( C \) is independent of \( n \). Then, we could pass the limit using a stochastic argument. Note that system (6.1) is a linear stochastic evolution equation driven by the additive noise. Therefore, the procedure of identifying the limit could be achieved directly by passing the limit in approximate equation. The specific procedure is even simpler than the argument of Part 4 in section 3. Here, we omit the details. The uniqueness could be achieved easily due to the linearity of the system. We complete the proof.

We proceed to introduce the well-posedness result of the Cahn-Hilliard equation with the dynamic boundary condition which reads as

\[
\begin{cases}
\partial_t \phi - \Delta \bar{\mu} = \ell_1, \\
\partial_n \bar{\mu} = 0,
\end{cases}
\]

with

\[
\partial_t \psi + \bar{K}(\psi) = \ell_3, \quad \text{on } \Gamma \times (0, \infty),
\]

where \( \bar{\mu} := -\Delta \phi + \delta \partial_t \phi + \ell_2 \) for any \( \delta \in (0, 1) \) and \( \bar{K}(\psi) := A_r \psi + \partial_n \phi + \psi \). We focus on the initial data \( (\phi, \psi)|_{t=0} = (\phi_0, \psi_0) \).

Proposition 6.2. [20] Theorem 4.2] Suppose that \( \ell_1 \in L^2_t(H^1)' \) with \( \langle \ell_1 \rangle = 0 \), and \( \ell_2 \in L^2_t L^2_x, \ell_3 \in L^2_t L^2(\Gamma) \). Moreover, the initial data \( (\phi_0, \psi_0) \in V^1 \), then system (6.2) has a unique global solution \( (\phi, \psi) \) such that

\[
(\phi, \psi) \in L^\infty_t V^1 \cap L^2_t V^2 \cap W^{1,2}_t (L^2_x \times L^2(\Gamma)).
\]

Moreover, the mass is conserved, i.e., \( \langle \phi(t) \rangle = \langle \phi_0 \rangle \), and the following estimate holds:

\[
\begin{align*}
\sup_{t \in [0,T]} \|\phi_{x,\delta}, \psi_{x,\delta}\|_{L^2_t (H^1)^2} &+ \|\partial_t (\phi_{x,\delta}, \psi_{x,\delta})\|_{L^2_t (H^1)^2} \times L^2(\Gamma)} \leq \|\phi_0\|_{V^1}^2 + \|\ell_1\|_{L^2_t (H^1)^2}^2 + \|\ell_2\|_{L^2_t L^2_x}^2 + \|\ell_3\|_{L^2_t L^2(\Gamma)}^2.
\end{align*}
\]
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