PARTIAL REGULARITY FOR A LIOUVILLE SYSTEM

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. We prove that the singular set of any extremal solution of the system

$$\begin{align*}
-\Delta u &= \mu e^v, \\
-\Delta v &= \lambda e^u
\end{align*}$$

in $\Omega$, with $u = v = 0$ on $\partial \Omega$, $\mu, \lambda \geq 0$, has Hausdorff dimension at most $n - 10$.

1. Introduction

In this article we consider the issue of partial regularity of extremal solutions to the Liouville system

$$\begin{align*}
-\Delta u &= \mu e^v & \text{in } \Omega, \\
-\Delta v &= \lambda e^u & \text{in } \Omega, \\
& \quad u = v = 0 & \text{on } \partial \Omega,
\end{align*}$$

with $\Omega$ a bounded smooth open subset of $\mathbb{R}^n$, and $\lambda, \mu$ nonnegative parameters.

This system is a generalization of the equation

$$\begin{align*}
-\Delta u &= \lambda e^u & \text{in } \Omega, \\
& \quad u = 0 & \text{on } \partial \Omega
\end{align*}$$

where $\lambda$ denotes a positive parameter. It is well known that there is a maximal parameter $\lambda^* > 0$ for existence of solutions of (2) and for $0 < \lambda < \lambda^*$ there is a minimal solution $u_\lambda$. As $\lambda \to \lambda^*, \lambda < \lambda^*$ the solution $u_\lambda$ converges to the so-called extremal solution, which turns out to be smooth for $n \leq 9$, see [3, 11]. The interested reader may find in the book [7] the developments of the theory for the last six decades, with a particular focus on stable solutions.

Recently it was proved by K. Wang [13] that for $n \geq 10$ the extremal solution of (2) has a singular set of dimension at most $n - 10$. F. Da Lio [5] obtained partial regularity for any weak stationary solution in dimension 3 (not necessarily stable). See related results for the Lane-Emden equation in [14, 6].

Here we generalize the results of [13] to the system (1). For this system, M. Montenegro [12] proved the existence of a nonempty open set $\mathcal{U}$ in the quarter plane $\lambda, \mu > 0$ such that for a couple of parameters $(\mu, \lambda)$ in $\mathcal{U}$ there is a smooth minimal solution $(u, v)$ and no smooth solution exists if the couple is in the complement of

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$\mathcal{U}$. Minimality means $u \leq \tilde{u}$ and $v \leq \tilde{v}$ in $\Omega$ for any other smooth solution $(\tilde{u}, \tilde{v})$ for the same $(\mu, \lambda)$.

For each slope $m > 0$, $\mathcal{U}$ intersected with the line $\mu = m\lambda$ is a segment $\{(m\lambda, \lambda) : \lambda \in (0, \lambda^*(m))\}$ and at the extremal point $(m\lambda^*(m), \lambda^*(m)) \in \partial \mathcal{U}$ there is a solution, called the extremal solution. It is defined as the limit as $\lambda \uparrow \lambda^*(m)$ of the minimal solution with parameters $(m\lambda, \lambda)$ and it may be singular. In a recent work [8], L. Dupaigne, A. Farina and B. Sirakov proved that the extremal solutions for the Liouville system (1) are smooth if $n \leq 9$. C. Cowan [1] had obtained the same conclusion under the restrictions $3 \leq n \leq 9$ and $\frac{n-2}{8} \leq \frac{\mu}{\lambda} \leq \frac{8}{n-2}$. In higher dimensions this fails at least in the radial case and for $\lambda = \mu$, where (1) reduces to (2).

Let us recall that an extremal solution $(u, v)$ satisfies (1) in the sense that $u, v \in L^1(\Omega), e^u \text{dist}(\cdot, \partial \Omega), e^v \text{dist}(\cdot, \partial \Omega) \in L^1(\Omega)$, and

$$\int_{\Omega} u(-\Delta \varphi) = \int_{\Omega} \mu e^v \varphi, \quad \int_{\Omega} v(-\Delta \varphi) = \int_{\Omega} \lambda e^u \varphi,$$

for all $\varphi \in C^2(\Omega)$ with $\varphi = 0$ on $\partial \Omega$.

We define the singular set $\Sigma$ of an extremal solution $(u, v)$ by $x \notin \Sigma$ if there is a neighborhood $W$ of $x$ such that $u, v$ are bounded in $W$. By elliptic regularity, $u, v$ are then smooth in this neighborhood.

**Theorem 1.1.** Assume $n \geq 10$ and let $(u, v)$ be an extremal solution of the Liouville system (1) and $\Sigma$ be its singular set. Then the Hausdorff dimension of $\Sigma$ is less or equal than $n - 10$.

The rest of the article is devoted to the proof of this theorem. We first recall a useful inequality which is valid for stable solutions of the system, obtained in C. Cowan, N. Ghoussoub [2] and L. Dupaigne, A. Farina, B. Sirakov [8]. We then state a comparison result between $u$ and $v$. Next, we perform a Moser iteration scheme to control the growth of some integrals of $e^u$ and $e^v$ on balls. The final step is an adaptation of an argument of K. Wang [13] using an $\varepsilon$-regularity result. The result in this paper is also closely related to the work of L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault [9] on stable solutions of $\Delta^2 u = e^u$ in a bounded domain or entire space.

2. Proof of Theorem 1.1

From [12] we know that for $(\mu, \lambda) \in \mathcal{U}$, the associated minimal solution $(u, v)$ of (1), which is smooth, is stable in the sense that there exist $\varphi, \psi : \Omega \to \mathbb{R}$, smooth and positive in $\Omega$, satisfying

$$\begin{cases}
-\Delta \varphi - \mu e^v \psi = \eta \varphi & \text{in } \Omega, \\
-\Delta \psi - \lambda e^u \varphi = \eta \psi & \text{in } \Omega, \\
\varphi = \psi = 0 & \text{on } \partial \Omega,
\end{cases}$$

for some $\eta > 0$. C. Cowan, N. Ghoussoub [2] and independently L. Dupaigne, A. Farina, B. Sirakov [8] have showed that this stability condition implies the following estimate.

**Lemma 2.1.** Let $(u, v)$ be a smooth stable solution of the system (1). For any $\varphi$ in $H^1_0(\Omega)$

$$\sqrt{\mu} \int_{\Omega} \exp\left(\frac{u + v}{2}\right) \varphi^2 \leq \int_{\Omega} |\nabla \varphi|^2.$$
2.1. Comparison. It will be useful later to have the following inequalities between the components of a solution of (1).

Lemma 2.2. Assume \( \lambda \geq \mu \). Then for any smooth solution to the Liouville system (1) we have:

\[
\frac{\mu}{\lambda} \leq \frac{\lambda}{\mu} + \log \lambda - \log \mu.
\]

Proof. Introduce \( w = v - u - \log \lambda + \log \mu \). Then \( w \leq 0 \) on \( \partial \Omega \). We have \(-\Delta w = \lambda e^u - \mu e^v = -\lambda e^u(e^u - 1)\), and then

\[
-\Delta w + \lambda e^u(e^u - 1)w = 0.
\]

Then due to the maximum principle \( w \leq 0 \) in \( \Omega \). For the first inequality in (4) introduce \( \tilde{w} = v - u \). Then \(-\Delta \tilde{w} = \lambda e^u - \mu e^v \geq \lambda(e^u - e^v) = -a(x)\tilde{w} \) where \( a(x) \geq 0 \). Then by the maximum principle \( \tilde{w} \geq 0 \) in \( \Omega \).

2.2. Reverse Hölder inequality. The following estimate is similar to the one obtained in [8] and [9], see also [4] for the scalar case. We assume that \((u, v)\) is a smooth stable solution of (1).

Lemma 2.3. For any \( 0 < \alpha < 4 \) there exists a constant \( C = C(n, \alpha, \lambda, \mu) \) such that for any \( \varphi \in C^\infty_c(\Omega) \) we have

\[
\|\nabla(e^{\alpha u})\varphi\|^2_{L^2(\Omega)} + \|\nabla(e^{\alpha v})\varphi\|^2_{L^2(\Omega)} \leq C \int_\Omega e^{\alpha u}(|\nabla \varphi|^2 + |\varphi \Delta \varphi|^2) + C \int_\Omega e^{\alpha v}(|\nabla \varphi|^2 + |\varphi \Delta \varphi|^2).
\]

Remark 1. Although the constant \( C \) depends on \( \mu, \lambda \) it remains bounded as \( (\mu, \lambda) \) approaches any extremal couple on \( \partial \Omega \).

Proof. Multiply \(-\Delta u = \mu e^v\) by \( e^{\alpha u} \varphi^2 \) and integrate by parts to obtain

\[
\mu \int_\Omega e^{\alpha u} \varphi^2 = \int_\Omega \nabla u \nabla(e^{\alpha u} \varphi^2) = \frac{4}{\alpha} \int_\Omega \varphi^2 |\nabla(e^{\alpha u} \varphi)|^2 + \frac{1}{\alpha} \int_\Omega \nabla(e^{\alpha u} \varphi) \nabla \varphi^2.
\]

This reads also

\[
\mu \int_\Omega e^{\alpha u} \varphi^2 = \frac{4}{\alpha} \int_\Omega |\nabla(e^{\alpha u} \varphi)|^2 - \frac{2}{\alpha} \int_\Omega e^{\alpha u}(|\nabla \varphi|^2 - \varphi \Delta \varphi).
\]

A similar equality is valid replacing respectively \( u \) by \( v \) and \( \mu \) by \( \lambda \). Introducing \( X = \int_\Omega |\nabla(e^{\alpha u} \varphi)|^2 \), \( Y = \int_\Omega |\nabla(e^{\alpha v} \varphi)|^2 \), \( A = \frac{2}{\alpha} \int_\Omega e^{\alpha u}(|\nabla \varphi|^2 - \varphi \Delta \varphi) \), and \( B = \frac{2}{\alpha} \int_\Omega e^{\alpha v}(|\nabla \varphi|^2 - \varphi \Delta \varphi) \), we then have

\[
\frac{4}{\alpha} X = \mu \int_\Omega e^{\alpha u} \varphi^2 + A,
\]

\[
\frac{4}{\alpha} Y = \lambda \int_\Omega e^{\alpha v} \varphi^2 + B.
\]

We combine Hölder’s inequality and the stability estimate (3) to obtain

\[
\mu \int_\Omega e^{\alpha u} \varphi^2 \leq \mu (\int_\Omega e^{\frac{\alpha u}{2}} e^{\alpha u} \varphi^2)^{1-\frac{\alpha}{2}} (\int_\Omega e^{\frac{\alpha u}{2}} e^{\alpha v} \varphi^2)^{\frac{\alpha}{2}} \leq \left(\frac{\mu}{\lambda}\right)^{\frac{\alpha}{2}} X^{1-\frac{\alpha}{2}} Y^{\frac{\alpha}{2}}.
\]
Analogously, we have the same inequality replacing \( u \) by \( v \) and \( \mu \) by \( \lambda \). Hence we obtain

\[
\frac{4}{\alpha} X \leq \left( \frac{\mu}{\lambda} \right)^{\frac{1}{2}} X^{1-\frac{\alpha}{n}} Y^{\frac{\alpha}{n}} + A,
\]

\[
\frac{4}{\alpha} Y \leq \left( \frac{\lambda}{\mu} \right)^{\frac{1}{2}} Y^{1-\frac{\alpha}{n}} + B.
\]

Multiplying these inequalities leads to

\[
\left( \frac{16}{\alpha^2} - 1 \right) XY \leq \frac{A}{\delta} \left( 1 + \sqrt{1 + \delta} \right),
\]

or

\[
\frac{A}{\delta} \left( 1 + \sqrt{1 + \delta} \right)
\]

hold. Assuming that (8) is true and combining with (6) we get \( X \leq CA \). Using Young’s inequality in (7) we obtain \( Y \leq C(A + B) \) so that \( X + Y \leq C(A + B) \) holds, which is (5). Assuming the validity of (9) we obtain the same conclusion. \( \square \)

A consequence of the previous lemma is the following.

**Lemma 2.4.** Set \( 2^* = \frac{2n}{n-2} \). For any \( 0 < \alpha < \beta < 2(2^*) \), if \( B_2(x) \subset \Omega \) we have

\[
\left( \int_{B_r(x)} (e^{\beta u} + e^{\beta v}) \right)^{\alpha/\beta} \leq C r^{n-1} \int_{B_{2r}(x)} e^{\alpha u} + e^{\alpha v}
\]

**Proof.** Follows from repeated applications of Lemma 2.3, using Sobolev’s embedding and Hölder’s inequality. \( \square \)

**Remark 2.** Lemmas 2.3 and 2.4 are independent of the boundary conditions of \( u \) and \( v \), and do not use the comparison of \( u \) to \( v \) of Lemma 2.2.

2.3. Integrability of solutions.

**Lemma 2.5.** Assume \((u, v)\) is a stable smooth solution of (1) with parameter \((\mu, \lambda)\) of the form \( \mu = m\lambda \) for some fixed \( m > 0 \). For \( 1 \leq \alpha < 5 \) there is \( C \) independent of \( \lambda \) such that

\[
\int_{\Omega} e^{\alpha u} + e^{\alpha v} \leq C.
\]

We note that \( C \) in general depends on the slope \( m \). In this lemma we need the inequalities between \( u \) and \( v \) of Lemma 2.2. For the proof, we refer to [8] where the following was proved.

**Lemma 2.6.** Assume \( \lambda \geq \mu \). If \((u, v)\) is a stable smooth solution of (1) with parameter \((\mu, \lambda)\) of the form \( \mu = m\lambda \) for some fixed \( m > 0 \), then for \( 1 \leq \alpha < 5 \) there is \( C \) independent of \( \lambda \) such that

\[
\int_{\Omega} e^{\alpha u} \leq C.
\]

Lemma 2.5 follows from Lemmas 2.6 and 2.2 in the case \( \lambda \geq \mu \). By a symmetric argument we obtain the same conclusion if \( \lambda \leq \mu \).
2.4. $\varepsilon$-regularity. A crucial step is the following $\varepsilon$-regularity result, whose version for stable solutions in the scalar case is due to K. Wang [13], see also [9] for a biharmonic equation with exponential nonlinearity.

**Lemma 2.7.** Let $(u, v)$ be an extremal solution of (1). Then there is $\varepsilon_2 > 0$ such that if for some $r_0 > 0$ with $B_{r_0}(x) \subset \Omega$ one has

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2$$

then there is a neighborhood of $x$ such that $u, v$ are smooth in this neighborhood.

For the proof we need the following key step, which is adapted from [13] in the scalar case.

**Lemma 2.8.** There exists $\varepsilon_0 > 0$ and $\theta > 0$ depending only on $n$ such that for any $0 < \varepsilon \leq \varepsilon_0$, if $(u, v)$ is a stable smooth solution of (1), $B_{r_0}(x) \subset \Omega$ and

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon$$

then

$$(\theta r_0)^{2-n} \int_{B_{\theta r_0}(x)} (e^u + e^v) \leq \varepsilon.$$

**Proof.** Let us assume that $x = 0$ by shifting coordinates. We rescale the functions by setting

$$\tilde{u}(x) = u(r_0 x) + 2 \log(r_0), \quad \tilde{v}(x) = v(r_0 x) + 2 \log(r_0),$$

and note that the new functions (where the $\tilde{\cdot}$ in the notation will be dropped) satisfy

$$-\Delta u = \mu e^v, \quad -\Delta v = \lambda e^u, \quad \text{in } B_1(0).$$

Let us decompose $u = u_1 + u_2$, $v = v_1 + v_2$ where

- $\Delta u_1 = 0$ in $B_{1/2}(0)$,
- $-\Delta u_2 = \mu e^v$ in $B_{1/2}(0)$,
- $\Delta v_1 = 0$ in $B_{1/2}(0)$,
- $-\Delta v_2 = \lambda e^u$ in $B_{1/2}(0)$,

Let $\gamma > 0, 0 < \theta < 1/4$ to be fixed later on and $\varepsilon > 0$. Let us estimate

$$\theta^{2-n} \int_{B_{\theta}(0)} e^u = \theta^{2-n} \int_{B_{\theta}(0) \cap [u_2 \leq \varepsilon]} e^{u_1 + u_2} + \theta^{2-n} \int_{B_{\theta}(0) \cap [u_2 > \varepsilon]} e^u.$$

For the first term we proceed by noting that $e^{u_1}$ is subharmonic in $B_{1/2}(0)$ and $u_2 \geq 0$, so

$$\theta^{2-n} \int_{B_{\theta}(0) \cap [u_2 \leq \varepsilon]} e^{u_1 + u_2} \leq \theta^{2-n} e^{\varepsilon \gamma} \int_{B_{\theta}(0) \cap [u_2 \leq \varepsilon]} e^{u_1} \leq \theta^{2-n} e^{\varepsilon \gamma} \int_{B_{\theta}(0)} e^{u_1} \leq C \theta^2 e^{\varepsilon \gamma} \int_{B_{1/2}(0)} e^{u_1} \leq C \theta^2 e^{\varepsilon \gamma} \varepsilon.$$
where we have used (11). For the second term in (14) we have

\[
\theta^{2-n} \int_{B_{\theta}(0) \cap [u_2 > \varepsilon^\gamma]} e^u \leq \theta^{2-n} \varepsilon^{-\gamma} \int_{B_{\theta}(0) \cap [u_2 > \varepsilon^\gamma]} u_2 e^u \\
\leq \theta^{2-n} \varepsilon^{-\gamma} \int_{B_{1/2}(0)} u_2 e^u \\
\leq \theta^{2-n} \varepsilon^{-\gamma} \| u_2 \|_{L^2(B_{1/2}(0))} \| e^u \|_{L^2(B_{1/2}(0))}.
\]

(16)

To estimate \( \| e^u \|_{L^2(B_{1/2}(0))} \) we apply (10) with \( \alpha = 1, \beta = 2 \) to get

\[
\| e^u \|_{L^2(B_{1/2}(0))} \leq C \varepsilon^{1/2}.
\]

(17)

For \( \| u_2 \|_{L^2(B_{1/2}(0))} \), first note that

\[
\| e^v \|_{L^2(B_{1/2}(0))} \leq C \varepsilon^{1/2}.
\]

Hence by \( L^2 \) regularity theory

\[
\| u_2 \|_{W^{2,2}(B_{1/2}(0))} \leq C \varepsilon^{1/2}.
\]

By using the Sobolev embedding \( W^{2,2} \subset L^{\frac{2n}{n-4}} \) we get

\[
\| u_2 \|_{L^{\frac{2n}{n-4}}(B_{1/2}(0))} \leq C \varepsilon^{1/2}.
\]

(18)

By interpolation

\[
\| u_2 \|_{L^2(B_{1/2}(0))} \leq \varepsilon^{m} \| e^v \|_{L^1(B_{1/2}(0))} \| u_2 \|^{1-m}_{L^{\frac{2n}{n-4}}(B_{1/2}(0))}
\]

(19)

where \( m = \frac{4}{n+4} \in (0, 1) \). But

\[
\| u_2 \|_{L^1(B_{1/2}(0))} \leq C \varepsilon \| e^v \|_{L^1(B_{1/2}(0))} \leq C \varepsilon,
\]

so (19) combined with (18) and (20) yields

\[
\| u_2 \|_{L^2(B_{1/2}(0))} \leq C \varepsilon^{m} \varepsilon^{(1-m)/2} = C \varepsilon^{\frac{1+m}{2}}.
\]

(21)

Therefore, using (16), (17) and (21) we find

\[
\theta^{2-n} \int_{B_{\theta}(0) \cap [u_2 > \varepsilon^\gamma]} e^u \leq C \theta^{2-n} \varepsilon^{1+m/2-\gamma}.
\]

Combining this and (15) we obtain

\[
\theta^{2-n} \int_{B_{\theta}(0)} e^u \leq C \theta^{2} \varepsilon^{\gamma} \varepsilon + C \theta^{2-n} \varepsilon^{1+m/2-\gamma}.
\]

Since \( m > 0 \) we may choose \( 0 < \gamma < m/2 \). Then fix \( \theta > 0 \) so that \( C \varepsilon^{\theta^{2}} \leq 1/2 \) and then choose \( \varepsilon_0 > 0 \) sufficiently small so that \( C \theta^{2-n} \varepsilon_0^{m/2-\gamma} \leq 1/2 \). It follows that for any \( 0 < \varepsilon \leq \varepsilon_0 \)

\[
\theta^{2-n} \int_{B_{\theta}(0)} e^u \leq \varepsilon.
\]

A similar argument yields the corresponding estimate for \( e^v \). Rescaling back we obtain (12).

Applying the previous lemma we can prove
Lemma 2.9. There exists $\varepsilon_1 > 0$ and $\theta > 0$ depending only on $n$ such that for any $0 < \varepsilon \leq \varepsilon_1$, if $(u, v)$ is a stable smooth solution of (1), $B_{r_0}(x) \subset \Omega$ and
\[
\int_{B_{r_0}(x)} \frac{1}{2}(e^{f} + e^{g}) \leq \varepsilon
\]
then
\[
\int_{B_{r_0}(x)} \frac{1}{2}(e^{f} + e^{g}) \leq 2^n \theta^{2-n} \varepsilon
\]
for any $y \in B_{r_0/2}(x)$ and any $0 < r \leq r_0/2$.

Proof. By shifting coordinates we can assume that $x = 0$ and by the scaling (13) that $r_0 = 1$. Let $\varepsilon_0$, $\theta$ be the constants of Lemma 2.7. We choose $\varepsilon_1$ so that $2^{n-2}\varepsilon_1 = \varepsilon_0$. Then, for any $y \in B_{1/2}(0)$ and $0 < \varepsilon \leq \varepsilon_1$ we have
\[
\int_{B_{1/2}(y)} \frac{1}{2}(e^{f} + e^{g}) \leq 2^n \theta^{2-n} \int_{B_1(0)} (e^{f} + e^{g}) \leq 2^n \theta^{2-n} \leq \varepsilon_0.
\]
Applying inductively Lemma 2.7, for any integer $k \geq 1$ we have
\[
(\theta^k)^{2-n} \int_{B_{\theta^k}(y)} (e^{f} + e^{g}) \leq 2^n \theta^{2-n}.\]
If $0 < r \leq 1/2$ is arbitrary we select $k \geq 1$ an integer such that $\theta^{k+1} \leq r \leq \theta^k$. Then
\[
\int_{B_{r}(y)} (e^{f} + e^{g}) \leq (\theta^k)^{2-n} \int_{B_{\theta^k}(y)} (e^{f} + e^{g}) \leq 2^n \theta^{2-n} \varepsilon.
\]
\]

Proof of Lemma 2.7. The result of Lemma 2.9 holds also for any extremal solution. This can be proved by approximating an extremal solution $(u, v)$ of parameters $(m\lambda^*(m), \lambda^*(m)) \in \partial \mathcal{U}$ by minimal solutions with parameters $(m\lambda, \lambda)$ and $\lambda \uparrow \lambda^*(m)$. In this process, the constants appearing in the estimates remain bounded, see Remark 1.

Let $\varepsilon_1, \theta$ be the constants of Lemma 2.9. We take $0 < \varepsilon_2 < \varepsilon_1$ to be fixed later on. By the change of variables (13) we can assume that $x = 0$ and $r_0 = 1$, so now the hypothesis is
\[
\int_{B_1(0)} (e^{f} + e^{g}) \leq \varepsilon_2.
\]
Then by Lemma 2.9 we have
\[
\int_{B_{r}(y)} (e^{f} + e^{g}) \leq 2^n \theta^{2-n} \varepsilon_2
\]
for any $y \in B_{1/2}(0)$ and any $0 < r \leq 1/2$. This says that $e^{f}$, $e^{g}$ are in the Morrey space $M_{n/2}(B_{1/2}(0))$ and
\[
\|e^{f}\|_{M_{n/2}} + \|e^{g}\|_{M_{n/2}} \leq 2^n \theta^{2-n} \varepsilon_2.
\]
Let $\tilde{u}, \tilde{v}$ be the Newtonian potentials of $e^{f}\chi_{B_{1/2}(0)}$ and $e^{g}\chi_{B_{1/2}(0)}$ respectively. Then by [10] Lemma 7.20 we have
\[
\int_{B_1(0)} e^{\beta|\tilde{u}|} + e^{\beta|\tilde{v}|} \leq C_2
\]
for $\beta \leq \min\left(\frac{C_1}{\|e_u\|_{M_n/2}}, \frac{C_2}{\|e_v\|_{M_n/2}}\right)$ where $C_1, C_2 > 0$ depend only on dimension. By (22), choosing $\varepsilon_2 > 0$ small, we obtain that (23) holds for some $\beta > n/2$. Then $e^u, e^v \in L^\beta(B_{1/4}(0))$ for some $\beta > n/2$. By standard $L^p$ regularity $u, v \in L^\infty(B_{1/8}(0))$. Scaling back we have the conclusion. □

2.5. Proof of Theorem 1.1.

Proof. Let $1 \leq \alpha < 5$. We claim that

$$\Sigma \subset \left\{ x \in \Omega : \limsup_{r \to 0} r^{2\alpha-n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) > 0 \right\}.$$

Indeed, if $x \in \Omega$ and

$$\lim_{r \to 0} r^{2\alpha-n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) = 0$$

then by Hölder’s inequality also

$$\lim_{r \to 0} r^{2-n} \int_{B_r(x) \cap \Omega} (e^u + e^v) = 0.$$

Therefore for some $r_0 > 0$ so that $B_{r_0}(x) \subset \Omega$ we have

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2$$

where $\varepsilon_2 > 0$ is the constant from Lemma 2.7. Then by the same lemma $u, v$ are bounded in a neighborhood of $x$ and hence $x \notin \Sigma$. Since $e^{\alpha u} + e^{\alpha v} \in L^1(\Omega)$ by Lemma 2.5, we obtain that $\mathcal{H}^{n-2\alpha}(\Sigma) = 0$, see e.g. [7, Theorem 5.3.4]. Letting $\alpha \uparrow 5$ we deduce that the Hausdorff dimension of $\Sigma$ is less or equal than $n - 10$. □

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References

[1] (MR2840001) C. Cowan, Regularity of the extremal solutions in a Gelfand system problem, Adv. Nonlinear Stud., 11 (2011), 695–700.

[2] [10.1007/s00526-012-0582-4] C. Cowan and N. Ghoussoub, Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains, Calc. Var. and PDEs, (2012).

[3] (MR0382848) [10.1007/BF00280741] M. G. Crandall and P. H. Rabinowitz Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Rational Mech. Anal., 58 (1975), 207–218.

[4] (MR2465656) [10.1090/S0002-9939-08-09772-4] E. N. Dancer and A. Farina, On the classification of solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$: stability outside a compact set and applications, Proc. Amer. Math. Soc., 137 (2009), 1333–1338.

[5] (MR2475323) [10.1080/03605300802402625] F. Da Lio, Partial regularity for stationary solutions to Liouville-type equation in dimension 3, Comm. Partial Differential Equations, 33 (2008), 1890–1910.
[6] (MR2785899) [10.1016/j.jfa.2010.12.028] J. Dávila, L. Dupaigne and A. Farina, Partial regularity of finite Morse index solutions to the Lane-Emden equation, *J. Funct. Anal.*, **261** (2011), 218–232.

[7] (MR2779463) [10.1201/b10892] L. Dupaigne, *Stable Solutions of Elliptic Partial Differential Equations*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 143, Chapman & Hall/CRC, Boca Raton, FL, 2011.

[8] L. Dupaigne, A. Farina and B. Sirakov, Regularity of the extremal solution for the Liouville system, to appear in *Proceedings of the ERC Workshop on Geometric Partial Differential Equations*, Ed. Scuola Normale Superiore di Pisa.

[9] (MR3048594) [10.1007/s00205-013-0613-0] L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault, The Gel’fand for the biharmonic operator, *Arch. Ration. Mech. Anal.*, **208** (2013), 725–752.

[10] (MR1814364) D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[11] (MR383604) [10.1080/03005300008820155] F. Mignot and J.-P. Puel, Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe, *Comm. Partial Differential Equations*, **5** (1980), 791–836.

[12] (MR2131395) [10.1112/S0024609305002488] M. Montenegro, Minimal solutions for a class of elliptic systems, *Bull. London Math. Soc.*, **37** (2005), 405–416.

[13] (MR2915334) [10.1007/s00526-011-0446-3] K. Wang, Partial regularity of stable solutions to the Emden equation, *Calc. Var. Partial Differential Equations*, **44** (2012), 601–610.

[14] (MR2927586) [10.1016/j.na.2012.04.041] K. Wang, Partial regularity of stable solutions to the supercritical equations and its applications, *Nonlinear Anal.*, **75** (2012), 5238–5260.

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