Spin and orbital angular momentum propagation in anisotropic media: theory

Antonio Picón¹, Albert Benseny², Jordi Mompart² and Gabriel F Calvo³,⁴

¹ JILA and Department of Physics, University of Colorado at Boulder, Boulder, CO 80309-0440, USA
² Grup d’Òptica, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain
³ Departamento de Matemáticas, ETSI Caminos, Canales y Puertos, Universidad de Castilla-La Mancha, E-13071 Ciudad Real, Spain
⁴ IMACI—Instituto de Matemática Aplicada a la Ciencia y la Ingeniería, Universidad de Castilla-La Mancha, E-13071 Ciudad Real, Spain

E-mail: apicon@jilau1.colorado.edu

Received 29 September 2010, accepted for publication 26 November 2010
Published 28 April 2011
Online at stacks.iop.org/JOpt/13/064019

Abstract
This paper is devoted to a study of the propagation of light beams carrying orbital angular momentum in optically anisotropic media. We first review some properties of homogeneous anisotropic media, and describe how the paraxial formalism is modified in order to proceed with a new approach dealing with the general setting of paraxial propagation along uniaxial inhomogeneous media. This approach is suitable for describing space-variant optical-axis phase plates.

Keywords: spin angular momentum of light, orbital angular momentum of light, anisotropic media, Laguerre–Gaussian modes

1. Introduction
In the past few years the research field of generation, manipulation and characterization of helical beams has attracted considerable interest. Under suitable generation conditions, these beams can transport angular momentum (spin and orbital) along their direction of propagation [1, 2]. These beams have triggered a wide range of applications (see the other papers in this special issue). Here, we examine the interplay between spin angular momentum (SAM) and orbital angular momentum (OAM) of light upon propagation through anisotropic media, which is not only interesting from the fundamental viewpoint but also yields novel applications, as we will discuss in the following.

During paraxial propagation in isotropic homogeneous media, both SAM and OAM are conserved quantities [1, 2], and thereby no exchange of angular momentum between these two is expected. However, in anisotropic media, during its propagation the beam can exhibit a coupling between SAM and OAM, or even a lossless angular momentum transfer between both of them [3]. A proof-of-principle demonstration of spin-controlled changes in the OAM of circularly polarized Gaussian beams in the visible domain, using patterned nematic liquid crystals, was experimentally achieved in 2006 [4]. The strong optical (uniaxial) anisotropic properties displayed by nematic liquid crystals were exploited to achieve a controllable space-dependent anisotropy of the medium. Moreover, one can reorient the optical axis of these media by applying external forces [5]. Hence, one can construct optical elements exhibiting a space-variant optical axis, in the sense that the optical axis varies at each point thus allowing the medium to act on polarization differently for each position. A straightforward application of this type of medium is the creation of non-scalar helical waves, based on spatially non-uniform polarization transformations such as the attractive radial polarization or azimuthal polarization light beams ([6–8] and references therein). The other main application is found in the realm of quantum information and communication. Photons can carry information encoded in different degrees of freedom: polarization (SAM), spatial modes (OAM) and energy. The aim of quantum communication (QC) is to transfer and distribute quantum (entangled) states among distant sites (nodes) of networks where they are further processed [9, 10]. Devices allowing unitary (lossless) information transfer from one degree of freedom to another degree of freedom in the same photon can really improve the robustness of the QC
network [11]. Anisotropic media are promising tools for transferring the entanglement encoded in the polarization of photons to their spatial mode profile, and vice versa [12, 13]. Although previous works have achieved impressive results in this line, it is still an open question how to construct truly unitary transformations. More complicated space-variant optical-axis phase plates making use of the anisotropy along the radial axis will probably be necessary to achieve such unitary transformations, in the same way as was anticipated in [14].

So far, due to the interesting applications arising from space-variant optical-axis phase plates, it is worth briefly reviewing them. Space-variant optical-axis phase plates have traditionally been fabricated from subwavelength metal stripes [6] and used as polarizers. In these, when the period of the grating is smaller than the wavelength of the incident light beam, only the zeroth order is a propagating range of problems. The reader interested in the propagation of cylindrical partial waves along biaxial non isotropic media is referred to [24].

In anisotropic media the Helmholtz wave equation satisfied by the electric field \( \mathbf{E} = \mathbf{E}(\mathbf{r}, t) \), when considering a monochromatic wavepacket \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{-i\omega t} \), is:

\[
\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) + \frac{\omega^2}{c^2}\hat{\varepsilon}(\omega)\mathbf{E} = 0,
\]

with \( \omega \) being the frequency of the monochromatic wave, \( c \) the speed of light, and \( \hat{\varepsilon} \) the relative dielectric tensor. Note that we explicitly include the possibility of a frequency dependence for \( \hat{\varepsilon} \). In anisotropic media, the third term in equation (1) couples the field components, whereas the second term, which is often neglected in isotropic media, plays a relevant contribution here. When \( \hat{\varepsilon} \) is a symmetric tensor, we can always choose a reference frame where the electrical permittivity is expressed in a diagonal form \( \hat{\varepsilon} \equiv \text{diag}(\varepsilon_x, \varepsilon_y, \varepsilon_z) \). It is well known that two plane wave families exist that satisfy equation (1) for a given propagation direction, having different velocity phase and polarization. These two families are the so-called ordinary and extraordinary plane waves. Uniaxial media are materials with cylindrical symmetry \( \varepsilon \equiv \text{diag}(\varepsilon_\perp, \varepsilon_\parallel, \varepsilon_z) \) around the so-called optical axis (OA), corresponding to \( \varepsilon_\parallel \); in this case the OA is in the \( z \)-direction. Defining the wavevector of a plane wave as \( \mathbf{k} = \frac{\omega}{c}\mathbf{n} = \frac{\omega}{c}\hat{n} = k_\parallel\hat{n} \) (\( \mathbf{n} \) is a unitary vector pointing along the propagation direction, \( n \) is the so-called refractive index, and \( \mathbf{n} \) is the vector of refraction), equation (1) imposes two conditions on the plane wavevectors:

\[
n^2 = \varepsilon_\perp,
\]

\[
\frac{n_\parallel^2 + n_\perp^2}{\varepsilon_\parallel} + \frac{n_\parallel^2}{\varepsilon_\perp} = 1.
\]

In uniaxial media it is common to define \( \varepsilon_\perp \equiv n_\perp^2 \) and \( \varepsilon_\parallel \equiv n_\parallel^2 \), where \( n_\parallel \) and \( n_\perp \) are the so-called ordinary and extraordinary refractive indices. The dispersion relations (2) and (3) are the main equations describing uniaxial anisotropic media, which yield two solutions related to the refractive index. The first solution, which satisfies equation (2), does not depend on the direction of the wavevector (ordinary waves), at variance with the solution satisfying equation (3), which depends on the wave propagation direction (extraordinary waves). Equation (3) is
valid in the reference frame where the electrical permittivity is diagonal. In the general case, the OA points along an arbitrary direction \( \mathbf{u}_{\text{OA}} \equiv (\cos \psi \sin \theta, \sin \psi \sin \theta, \cos \theta) = \cos \psi \sin \theta \hat{i} + \sin \psi \sin \theta \hat{j} + \cos \theta \hat{k} \), with \( \hat{i}, \hat{j}, \hat{k} \) being unitary and orthogonal vectors associated with the laboratory frame axes. The dispersion relation (3) is modified and can be easily obtained considering a rotation of the laboratory frame in order to fix the \( z \)-axis along the OA direction, so that one can resort to equation (3) with just a simple coordinate transformation. Here we have chosen the \( \theta \) angle rotation around the \( y \)-direction (laboratory frame) and then a \( \varphi \) angle rotation around the \( z \)-direction (laboratory frame), obtaining the general dispersion relation

\[
k_z^2 = \frac{k_z^2}{n_z^2(\theta)} + \frac{2k_z \cos \theta \sin \theta}{\Delta n^2} (k_x \cos \varphi + k_y \sin \varphi)
\]

\[
+ \left( \frac{k_x \cos \varphi + k_y \sin \varphi}{n^2(\theta)} \right)^2 + \left( \frac{k_x \sin \varphi - k_y \cos \varphi}{n^2(\theta)} \right)^2,
\]

where we have defined

\[
\frac{1}{n^2(\theta)} = \frac{\sin^2 \theta}{n_x^2} + \frac{\cos^2 \theta}{n_y^2} + \frac{\sin^2 \theta}{n_z^2},
\]

When \( \theta = 0 \) (the OA is parallel to the \( z \)-axis) one recovers equation (3). The two plane wave families can be expressed as

\[
E_\theta^{(\pm)}(r) = \tilde{E}_\theta \exp[i \mathbf{q} \cdot \mathbf{r}_\perp \mp i(k_x^2 n_x^2 - q_x^2)^{1/2}z],
\]

\[
E_\varphi^{(\pm)}(r) = \tilde{E}_\varphi \exp[i \mathbf{q} \cdot \mathbf{r}_\perp \pm i(k_x \cos \varphi, k_y \sin \varphi)],
\]

where \( \mathbf{q} = k_x \hat{i} + k_y \hat{j} \) is the transverse wavevector, \( \mathbf{r}_\perp \equiv x \hat{i} + y \hat{j} = (r \cos \varphi, r \sin \varphi) \) is the transverse coordinate, the function \( k_x(k_x, k_y) \) is given by equation (4), \( \tilde{E}_\theta \) and \( \tilde{E}_\varphi \) are the ordinary and extraordinary polarization vectors (we will explain below how to calculate them). Then, a monochromatic plane wave is completely described by the transverse components \( k_x \) and \( k_y \). Any electric field propagating in a uniaxially anisotropic medium, which is a solution of the Helmholtz equation (1), can be written as a superposition of ordinary and extraordinary plane waves (6):

\[
\mathbf{E}(r) = \int d^2 q [\tilde{E}_\theta \exp[i \mathbf{q} \cdot \mathbf{r}_\perp \pm i(k_x^2 n_x^2 - q_x^2)^{1/2}z] + \tilde{E}_\varphi \exp[i \mathbf{q} \cdot \mathbf{r}_\perp \pm i k_x(k_x, k_y)z]].
\]

Hence, assuming that a (paraxial) beam is propagating along a well-defined direction, in this case the \( z \)-direction, this can be perfectly decomposed as in equation (7), but under the paraxial wave approximation \( k_z \gg k_x, k_y \) we can rewrite equation (7) in a simpler form that describes the propagation of any paraxial beam along the uniaxial medium. We therefore perform the paraxial wave approximation on equation (4) isolating \( k_x(k_x, k_y) \). Therefore we obtain

\[
k_x(k_x, k_y) = -q_x \cos \theta \sin \theta \frac{n_x^2(\theta)}{\Delta n^2} - k_0 n_x(\theta) \left[ 1 - \frac{n_x^2(\theta)}{2n_x^2n_x^2k_0^2} q_x^2 - \frac{1}{2n_x^2k_0^2} q_x^2 \right],
\]

where we have defined

\[
q_x \equiv k_x \cos \varphi + k_y \sin \varphi, \quad q_y \equiv k_x \sin \varphi - k_y \cos \varphi.
\]

Now, instead of \( k_x \) and \( k_y \), we use henceforth \( q_x \) and \( q_y \), which depend of the OA direction, in particular on the \( \varphi \) variable (which refers to rotations around the \( z \) axis).

Secondly, we need to calculate the polarization vectors \( \tilde{E}_\theta(q) \) and \( \tilde{E}_\varphi(q) \) in equation (7). For the sake of brevity we will only sketch how to derive them. The ordinary polarization is perpendicular to the wavevector and the OA, thus \( \tilde{E}_\theta(q) \propto \mathbf{k} \times \mathbf{u}_{\text{OA}} \). On the other hand, the extraordinary polarization is perpendicular to the Poynting vector and the ordinary polarization, \( \tilde{E}_\varphi(q) \propto \mathbf{E} \times \mathbf{S} \). The direction of the Poynting vector is parallel to the gradient of the wavevector surface given by equation (4). Knowing the direction of the two polarizations, we can finally obtain \( \tilde{E}_\theta(q) \) and \( \tilde{E}_\varphi(q) \) by imposing the boundary conditions; the Fourier transform of the electric field (7) at \( z = 0 \) is

\[
\tilde{E}(q) = \frac{1}{(2\pi)^2} \int d^2 q' \exp(-i \mathbf{q} \cdot \mathbf{r}_\perp') \mathbf{E}(\mathbf{r}_\perp', 0),
\]

which is equal to \( \tilde{E}(q) = \tilde{E}_\theta(q) + \tilde{E}_\varphi(q) \). Here we provide the results within the paraxial approximation, distinguishing two well-defined regimes; when \( \theta = 0 \) and \( \theta \neq 0 \). If \( \theta = 0 \) (the OA is parallel to the \( z \)-axis), the polarization vectors reduce to

\[
\tilde{E}_\theta(q) = \frac{1}{q^2} [(k_x^2 \tilde{E}_x - k_x k_y \tilde{E}_y) \hat{i} + (k_y^2 \tilde{E}_y - k_x k_y \tilde{E}_x) \hat{j}],
\]

\[
\tilde{E}_\varphi(q) = \frac{1}{q^2} [(k_x^2 \tilde{E}_x + k_x k_y \tilde{E}_y) \hat{i} + (k_y^2 \tilde{E}_y + k_x k_y \tilde{E}_x) \hat{j}],
\]

which are in agreement with previous results obtained in [22]. On the other hand, when \( \theta \neq 0 \), the polarization vectors become

\[
\tilde{E}_\theta(q) = (\cos \varphi \tilde{E}_x - \sin \varphi \tilde{E}_y)[- \sin \varphi \hat{i} + \cos \varphi \hat{j}],
\]

\[
\tilde{E}_\varphi(q) = (\cos \varphi \tilde{E}_x + \sin \varphi \tilde{E}_y)
\]

\[
\times \left[ \cos \varphi \hat{i} + \sin \varphi \hat{j} - \frac{\cos \theta \sin \theta}{\Delta n^2} \hat{k} \right],
\]

with an explicit dependence on the direction of the OA. In this case there is a \( \varphi \) component in the extraordinary part that cannot be neglected even in the paraxial approximation.

Summarizing, we can expand any electric field into ordinary and extraordinary parts, as we did in equation (7). Hence, in the paraxial approximation (we still consider the paraxial wave propagating along \( z \)) the \( z \) and \( xy \) components
of the electric field are
\[ E_{\perp\circ}(r_{\perp}, z) = e^{i k_0 n_{\circ}} \int d^3 q \, e^{i q r_{\perp} - i q z} \hat{P}_o \hat{E}_{\perp}(q) \]
\[ = e^{i k_0 n_{\circ}} A_{\perp\circ}(r_{\perp}, z), \]
\[ E_{\circ\circ}(r_{\perp}, z) = 0, \]
\[ E_{\perp\circ}(r_{\perp}, z) = \frac{-i q z}{k_0 n_{\circ}} \int d^3 q \, e^{i q r_{\perp} - i q z} \hat{P}_o \hat{E}_{\perp}(q) \equiv e^{i k_0 n_{\circ}} A_{\perp\circ}(r_{\perp}, z), \]
\[ \times e^{-i \frac{q z}{2 n_{\circ}} \frac{q x}{2 n_{\circ}} \frac{q y}{2 n_{\circ}}} \hat{P}_e \hat{E}_{\perp}(q) \]
\[ \times \int d^3 q \, (\cos \varphi \hat{E}_x(q) + \sin \varphi \hat{E}_y(q)) \]
\[ \times e^{-i \frac{q z}{2 n_{\circ}} \frac{q x}{2 n_{\circ}} \frac{q y}{2 n_{\circ}}} \],

where the projector operators \( \hat{P}_o \) and \( \hat{P}_e \) are given below in equations (17). Recall that the \( q_x \) and \( q_y \) variables, defined in equations (9), depend on the OA direction. Moreover, we note again the presence of a \( z \) component in the extraordinary field (13) which cannot be neglected. However, in many systems, where the extraordinary refractive index does not differ much from the ordinary refractive index, the \( 1/\Delta n^2 \) term is quite small (see equations (5)), and we can neglect the \( z \) component of the electric field (when the OA is collinear or orthogonal to the wave propagation, it is a fairly good approximation). Let us examine more features. The ordinary part does not depend on the direction of the OA, as one would expect in uniaxial media. However, the extraordinary part has suffered a noticeable change. As the paraxial wave is composed of a plane wave times a slowly varying amplitude \( A_{\perp\circ}(r_{\perp}, z) \) (for ordinary waves this is \( A_{\perp\circ}(r_{\perp}, z) \)), we expect that its plane-wave-like part will depend on the propagation direction, with a refractive index given by \( n_{\circ}(\theta) \) (for ordinary waves \( n_{\circ} \)). The second main remark is related to the slowly varying amplitude \( A_{\perp\circ}(r_{\perp}, z) \), which does not obey the standard (isotropic-like) paraxial wave equation, due to the lack of symmetry in the transverse plane during the propagation. The slowly varying amplitudes obey
\[ \left( \frac{\partial}{\partial z} + \frac{1}{2 k_0 n_{\circ}} \nabla^2 \right) A_{\perp\circ} = 0, \]
\[ \left( \frac{\partial}{\partial z} + \frac{n_{\circ}^2(\theta)}{2 k_0 n_{\circ}} \frac{\partial}{\partial \hat{r}_{\perp}} + i \sin \theta \cos \theta \right) \frac{n_{\circ}^2(\theta)}{2 k_0 n_{\circ}} \frac{\partial}{\partial \hat{r}_{\perp}} \right) \]
\[ \times A_{\perp\circ} = 0, \]

where we have defined
\[ r_x \equiv x \cos \varphi + y \sin \varphi, \quad r_y \equiv x \sin \varphi - y \cos \varphi. \]

The rotated variables of equations (15) simply come from the rotated momentum variables defined in equations (9). Another interesting point is the occurrence of a new term in the phase of the extraordinary wave, proportional to \( q_z \) (originated by the walk-off effect). This term is responsible for a transverse translation of the extraordinary wave during propagation. But most surprising is the fact that projectors \( \hat{P}_o \) and \( \hat{P}_e \) are
\[ \hat{P}_o = \frac{1}{q^2} \left[ \begin{array}{cc} k_0^2 - k_x k_y & -k_x k_y \\ -k_x k_y & k_x^2 \end{array} \right], \]
\[ \hat{P}_e = \frac{1}{q^2} \left[ \begin{array}{cc} k_0^2 & k_x k_y \\ k_x k_y & k_x^2 \end{array} \right], \]

when \( \theta = 0 \), but in the case when \( \theta \neq 0 \) projectors collapse to
\[ \hat{P}_o = \left[ \begin{array}{cc} \sin^2 \varphi & -\sin \varphi \cos \varphi \\ -\sin \varphi \cos \varphi & \cos^2 \varphi \end{array} \right], \]
\[ \hat{P}_e = \left[ \begin{array}{cc} \cos^2 \varphi & \sin \varphi \cos \varphi \\ \sin \varphi \cos \varphi & \sin^2 \varphi \end{array} \right] = \frac{\hat{I} + \hat{R}_z(\varphi) \hat{\sigma}_z \hat{R}_z(-\varphi)}{2}, \]

where \( \hat{R}_z(\varphi) \) represents a \( \varphi \) angular rotation around the \( z \)-axis and \( \hat{\sigma}_z \) is the usual third component Pauli matrix
\[ \hat{R}_z(\varphi) = \left[ \begin{array}{cc} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{array} \right], \quad \hat{\sigma}_z = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]. \]

Notice that projectors (17) are independent of the \( \theta \) variable.

3. Paraxial propagation in inhomogeneous uniaxial media

In section 2 we examined the propagation of a paraxial wave along an anisotropic medium, considering the OA constant in the entire medium; in other words, the medium was homogeneous and the OA has the same direction at any position of the medium. However, what happens if the medium is inhomogeneous (i.e. the OA depends on the position)? We will solve this problem under certain conditions.

Before addressing the inhomogeneous problem, it is convenient to rewrite equations (13) in a suitable form, which will be essential for further calculations. Therefore, recalling the Fourier transform (10), we can substitute it into the transverse electric part of equations (13), obtaining
\[ E_{\perp\circ}(r_{\perp}, z) = \frac{k_0 n_{\circ}}{2 \pi i c} e^{i k_0 n_{\circ}} \int d^3 r_{\perp}' \hat{P}_e(\hat{r}_{\perp}') \hat{E}_{\perp}(r_{\perp}') \]
\[ = \frac{k_0 n_{\circ}}{2 \pi i c} n_{\circ}^2(\theta) \frac{e^{i k_0 n_{\circ}}}{2 \pi i n_{\circ}^2(\theta)} \int d^3 r_{\perp}' \hat{P}_e(\hat{r}_{\perp}') \hat{E}_{\perp}(r_{\perp}') \]
\[ \times \left[ e_{\perp\circ}(\hat{r}_{\perp}') \sin \theta \cos \theta \frac{d^2}{d^2 \varphi_{\perp\circ}'} \left( r_{\perp}' - r_{\perp} \right) \right] + \left( r_{\perp} - r_{\perp}' \right), \]

where we have used the definitions of the rotated position variables (15). Notice the similarity of equations (19) with the Fresnel integral in the isotropic case. The Fresnel integral describes the evolution of an input paraxial wave, taking into account its initial boundary conditions. It is basically composed of the product of the input paraxial wave with the Fresnel kernel (propagator). In anisotropic media, integrals (19), analogously to the Fresnel integral, describe the evolution of the paraxial wave. Hence, performing the integration with respect to the variable \( r_{\perp}' \), which corresponds
to the transverse plane in \( z = 0 \), we can determine the evolution of the wave for any \( z \). First of all, projectors \( P_o \) and \( P_e \) select the ordinary and extraordinary part of the input electric field, depending on its polarization. After that, each part evolves independently with its corresponding ordinary and extraordinary kernel. Notice that the ordinary kernel is invariant under reference system rotations, like the usual Fresnel kernel. However, the extraordinary kernel depends on the OA direction, thus affecting the wave propagation. We can introduce the following new functions:

\[
F_o = e^{i k_o n_1 \beta z} e^{i k_o n_1 |r_1 - r_o|^2}
\]

\[
F_e = e^{i k_o n_2 \beta z} e^{i k_o n_2 \beta z} \left[ \frac{n_1^2}{n_2^2} \left( F_o + F_e \right) \right] + \left[ \frac{n_1^2}{n_2^2} F_e - F_o \right]
\]

which, together with projectors (17) (we are not considering the projectors (16), i.e. the case \( \theta = 0 \), as the Fourier transform cannot be cast in the same analytical compact expression), allow us to simplify expressions (19) as

\[
E_{LE}(r_1, z) = E_{LO}(r_1, z) + E_{LE}(r_1, z)
\]

\[
= \frac{k_0 h_o}{2 \pi i z} \int d^2 r_o \left\{ \left[ \frac{n_1^2}{n_2^2} F_o + F_e \right] \frac{\hat{R}_0}{2} + \left[ \frac{n_1^2}{n_2^2} F_e - F_o \right] \frac{\hat{R}_0}{2} \right\} E_{LE}(r_1')
\]

Equation (21) has two main terms. The first one is a factor \( [n_1^2/n_2^2(\theta)]F_o + F_e \) times the identity matrix, which does not change the input polarization. However, it couples both extraordinary and ordinary parts. On the other hand, the second term, \( [n_1^2/n_2^2(\theta)]F_e - F_o \) times a rotation matrix, also combines the ordinary and the extraordinary parts but, at the same time, it varies the input polarization. In the isotropic limit, when \( n_o \) tends to \( n_o \), we find that from \( [n_1^2/n_2^2(\theta)]F_e + F_o \) one recovers the standard Fresnel kernel, whereas \( [n_1^2/n_2^2(\theta)]F_e - F_o \) vanishes.

Equation (21) still describes homogeneous anisotropic media, with an OA direction given by \( (\theta, \varphi) \). However, in an inhomogeneous medium, the OA direction can depend on the position. Thus, both \( \theta = \theta(r') \) and \( \varphi = \varphi(r') \) are functions of the position. Inserting this dependence into equation (21), and remembering that the function \( F_e \) also depends on the direction of the OA, we can calculate the evolution of a paraxial wave in uniaxial anisotropic inhomogeneous media. Of course, we consider that \( \theta(r') \) and \( \varphi(r') \) are sufficiently smooth functions.

The approach developed here is quite general and allows one to deal with paraxial beams propagating along inhomogeneous anisotropic media by using the integral propagator (21). In particular, the integral propagator (21) can be used for beams with a certain spatial structure carrying OAM. One can easily apply this integral propagator to describe the spin-to-orbital angular momentum switching observed experimentally in [4] by using space-variant optical-axis phase plates. For the sake of completeness, we will address this problem with our formalism in section 4.

4. Application: propagation of a Laguerre-Gaussian mode along a \( q \)-plate

In this section we consider a specific example, the propagation of a Laguerre-Gaussian (LG) beam through a space-variant optical-axis medium, whose OA is always perpendicular to the propagation direction (this means \( \theta(r') = \pi/2 \); see section 3. Here we will focus on the so-called \( q \)-plates, which have the following linear OA dependence: \( \varphi(r') = q \phi' + \alpha_0 \) (\( q \) and \( \alpha_0 \) are constants).

Hence, the initial electric field in equation (21) is written as \( E_L(r', \phi', z = 0) = (a \hat{u}_+ + b \hat{u}_-) LG_{l', p}(r', \phi', \varphi') \), where \( \hat{u}_o \equiv (\hat{i} + i a \hat{j})/\sqrt{2} \) are the circular polarization vectors and \( \sigma = \pm 1 \) for right- and left-hand circularly polarization, respectively. The functions \( LG_{l', p}(r', \phi', \varphi') \) denote the LG modes [1, 2], which comprise two terms \( LG_{l', p}(r', \phi', \varphi') = e^{i \phi'} R_{l', p}(r', \varphi') \), where \( R_{l', p}(r', \varphi') \) is a purely radial function depending on the two indices \( l \) and \( p \) which account for the orbital angular momentum (topological charge) and the radial node number of the spatial mode, respectively [1, 2].

Using equation (21) we can calculate the propagation of the Laguerre-Gaussian beam along the \( q \)-plate. The electric field can be separated into two parts \( E^{(1)}_{LE}(r_1, z) = E^{(1)}_{LO}(r_1, z) + E^{(1)}_{LE}(r_1, z) \) (22), where \( E^{(1)}_{LE}(r_1, z) \) preserves the initial polarization while that of \( E^{(1)}_{LO}(r_1, z) \) changes:

\[
E^{(1)}_{LO}(r_1, z) = \frac{k_0 h_o}{4 \pi i z} \int d^2 r_o \left( F_o + F_e \right) \left( a \hat{u}_+ + b \hat{u}_- \right)
\]

\[
\times e^{i \phi'} R_{l', p}(r')
\]

\[
E^{(2)}_{LO}(r_1, z) = \frac{k_0 h_o}{4 \pi i z} \int d^2 r_o \left( F_e - F_o \right) \left( b e^{-i \phi'} \hat{u}_+ + a e^{i \phi'} \hat{u}_- \right)
\]

\[
\times e^{i \phi'} R_{l', p}(r')
\]

Note that the second term that changes the polarization is also introducing exponential terms that include the OA dependence \( \varphi(r', \phi', \alpha_0) = q \phi' + \alpha_0 \). In this case propagators \( F_o \) and \( F_e \) (20) can be expressed as

\[
F_o \equiv e^{i k_o n_1 \beta z} e^{i k_o n_1 |r_1 - r_o|^2}
\]

\[
F_e \equiv e^{i k_o n_2 \beta z} e^{i k_o n_2 \beta z} \left[ \frac{n_1^2}{n_2^2} \left( F_o + F_e \right) \right] + \left[ \frac{n_1^2}{n_2^2} F_e - F_o \right]
\]

where the last approximation in the extraordinary propagator can be done as long as the birefringence is not too large, that is \( |n_2^2 - n_1^2| \ll n_2^2 + n_1^2 \). The neglected part accounts for a small astigmatism in the extraordinary part of the electric field. We will focus on the angular part of the integrals (22) and (23) in order to analyze the spin and angular momenta of the beam along the \( q \)-plate. All the integrals in equations (22) and (23) have the form

\[
\frac{k_0 h_o}{4 \pi i z} \int_0^{2 \pi} \int_0^{\infty} r'^2 e^{-i k_o n_1 (r'^2 + r'^2)} R_{l', p}(r')
\]

\[
\times \int_0^{2 \pi} d \phi' e^{-i \frac{\pi}{2} (r'^2 \cos^2(\phi' - \phi) - r'^2)} e^{i \phi'}
\]

or

\[
\frac{k_0 h_o}{4 \pi i z} \int_0^{2 \pi} \int_0^{\infty} r'^2 e^{-i k_o n_2 (r'^2 + r'^2)} R_{l', p}(r')
\]

\[
\times \int_0^{2 \pi} d \phi' e^{-i \frac{\pi}{2} (r'^2 \cos^2(\phi' - \phi) - r'^2)} e^{i (\phi' \pm \alpha_0)}
\]
where \( n_1 \) and \( n_2 \) are refractive indices denoting \( n_{\text{o}} \), \( n_{\text{z}} \), or \((n_{\text{o}}^2 + n_{\text{z}}^2)/(2n_{\text{c}})\). By resorting to the Jacobi–Anger expansion we can easily deal with the angular factor

\[
e^{-i\frac{k\ell}{z}rr'\cos(\phi-\phi')} = \sum_{k=-\infty}^{\infty} (-i)^k J_k \left( \frac{k\ell}{z}rr' \right) e^{i(k\phi-\phi')}. \tag{27}
\]

Therefore, by inserting the expansion (27) in the integrals (25) and (26), and integrating with respect to the variable \( \phi' \) (considering the \( \psi(r') \) dependence in the \( q \)-plates), we can reduce the integrals to

\[
(-i)^{\ell+1} \frac{k\ell n_0}{2z} e^{ik\ell_1\phi} e^{i\phi} \int_0^\infty r' \, dr' \, e^{i\frac{k\ell_2}{z}(r'^2 + r^2)} R_{\ell,p}(r') J_{\ell-1}(k\ell_2 r') \tag{28}
\]

or

\[
(-i)^{\ell+2q+1} \frac{k\ell n_0}{2z} e^{ik\ell_1\phi} e^{i\phi} \int_0^\infty r' \, dr' \, e^{i\frac{k\ell_2}{z}(r'^2 + r^2)} R_{\ell,p}(r') J_{\ell+2q}(k\ell_2 r'). \tag{29}
\]

By using integrals (28) and (29) we can analyze the propagation of the Laguerre–Gaussian beam along the \( q \)-plate given by equations (22) and (23). The part of the propagation that does not modify the polarization of the initial beam is not changing the angular structure of the beam either. Thus, the propagator (22) is preserving the polarization and the angular structure, in other words, the spin and angular momenta of the paraxial beam remain intact. This explains why, in [4], there was an experimentally observed part of the beam which exhibited unchanged polarization and angular structure. On the other hand, the propagator (23) does indeed affect the polarization and alters the angular structure too. Therefore, a right- or left-handed circularly polarized LG beam changes its polarization to left- or right-handed circularly polarized, respectively, and its topological charge from \( \ell \) to \( \ell + 2q \) in the case of left-handed circular polarization or to \( \ell - 2q \) in the case of right-handed circular polarization. In other words, the variation of SAM and OAM is \(-2\hbar \) and \(2q\hbar\), respectively, for an initial right-circularly polarized LG beam, resulting in a total variation of angular momentum of \(2(q-1)\hbar\). For an initial left-circularly polarized LG beam the total variation of angular momentum is \(-2(q-1)\hbar\), in complete agreement with the experiment performed in [4]. Moreover we can obtain the radial part of the LG beam propagating along the \( q \)-plate, which in general will be a superposition of LG modes with different radial number \( p \) (the radial part of the LG mode is not preserved). Furthermore, this formalism is not only able to explain the spin-to-orbital angular momentum switching, one can also analytically calculate the angular momentum of light before and after the space-variant-optical-axis phase plates, as we first proved in [3].

In this simple example we have showed how resourceful the developed formalism can be for addressing paraxial propagation in inhomogeneous media.

5. Conclusions

To conclude, we have developed an approach, based on the vectorial paraxial propagation of helical beams, to deal with homogeneous and inhomogeneous anisotropic media. Within this approach, one can calculate the propagation of a paraxial wave through common anisotropic media, such as polarizers, and more complex optical elements, such as space-variant phase plates which have shown promising applications. This approach also allows us to calculate the angular momentum of light, suitable for describing the OAM changes of the beam during propagation [3]. We have to remark that alternative approaches have been developed for some specific cases. Karimi et al [27] tried to avoid the approximation that \( \psi(r') \) and \( \varphi(r') \) are very smooth functions in the case of a specific space-variant phase plate, the so-called \( q \)-plate. However, in order to do so, they neglected the term \( \mathbf{V} \cdot \mathbf{E} \) in the anisotropic Helmholtz wave equation (1). This term is preserved in our approach. In a subsequent work, Vaveliuk [28] showed that the term \( \mathbf{V} \cdot \mathbf{E} \) cannot be neglected, providing a non-paraxial solution of such a complicated anisotropic wave equation. Unfortunately such solutions are only valid for \( q \)-plates with \( q = 1 \).

Acknowledgment

GFC wishes to thank Junta de Castilla-La Mancha for financial support via project PCI08-0093-6563.

References

[1] Allen L, Beijersbergen M W, Spreeuw J C and Woerdman J P 1992 Orbital angular momentum of light and the transformation of Laguerre–Gaussian laser modes Phys. Rev. A 45 8185

[2] Calvo G F, Picón A and Bagán E 2006 Quantum field theory of photons with orbital angular momentum Phys. Rev. A 73 013805

[3] Calvo G F and Picón A 2007 Spin-induced angular momentum switching Opt. Lett. 32 838–40

[4] Marrucci L, Manzo C and Paparo D 2006 Optical spin-to-orbital angular momentum conversion in inhomogeneous anisotropic media Phys. Rev. Lett. 96 163905

[5] Goldstein D H 2003 Polarized Light (New York: Edward Collett)

[6] Bomszon A, Biener G, Kleiner V and Hasman E 2002 Radially and azimuthally polarized beams generated by space-variant dielectric subwavelength gratings Opt. Lett. 27 285–7

[7] Machavariani G, Lumer Y, Moshe I and Jackel S 2007 Effect of the spiral phase element on the radial-polarization (0, 1)’LG beam Opt. Commun. 271 190

[8] Kawauchi H, Kozawa Y, Sato S, Sato T and Kawakami S 2008 Simultaneous generation of helical beams with linear and radial polarization by use of a segmented half-wave plate Opt. Lett. 33 399–401

[9] Duan L M, Lukin M D, Cirac J I and Zoller P 2001 Long-distance quantum communication with atomic ensembles and linear optics Nature 414 413–8

[10] Zoller P et al 2005 Quantum information processing and communication Eur. Phys. J. D 36 203–28
[11] Deng L P, Haibo W and Wang K 2007 Quantum CNOT gates with orbital angular momentum and polarization of single-photon quantum logic J. Opt. Soc. Am. B 24 2517–20

[12] Nagali E, Sciarrino F, De Martini F, Marrucci L, Piccirillo B, Karimi E and Santamato E 2009 Quantum information transfer from spin to orbital angular momentum of photons Phys. Rev. Lett. 103 013601

[13] Nagali E, Sciarrino F, De Martini F, Piccirillo B, Karimi E, Marrucci L and Santamato E 2009 Polarization control of single photon quantum orbital angular momentum states Opt. Express 17 18745–59

[14] Calvo G F and Picón A 2008 Manipulation of single-photon states encoded in transverse spatial modes: possible and impossible tasks Phys. Rev. A 77 012302

[15] Lopez A G and Craighead H G 1998 Wave-plate polarizing beam splitter based on a form-birefringent multilayer grating Opt. Lett. 23 1627–9

[16] Berry M V 1987 The adiabatic phase and Pancharatnam’s phase for polarized light J. Mod. Opt. 34 1401

[17] Bomzon Z, Kleiner V and Hasman E 2001 Pancharatnam–Berry phase in space-variant polarization-state manipulations with subwavelength gratings Opt. Lett. 26 1424–6

[18] Zhan Q 2006 Properties of circularly polarized vortex beams Opt. Lett. 31 867–9

[19] Chen L, Zheng G, Xu J, Zhang B and She W 2006 Electrically controlled transfer of spin angular momentum of light in an optically active medium Opt. Lett. 31 3474–6

[20] Zhao Y, Edgar J S, Jeffries G D M, McGloin D and Chiu D T 2007 Spin-to-orbital angular momentum conversion in a strongly focused optical beam Phys. Rev. Lett. 99 073901

[21] Chen L and She W 2008 Electro-optically forbidden or enhanced spin-to-orbital angular momentum conversion in a focused light beam Opt. Lett. 33 696–8

[22] Ciattoni A, Crosignani B and Di Porto P 2001 Vectorial theory of propagation in uniaxially anisotropic media J. Opt. Soc. Am. A 18 1656–61

[23] Ciattoni A and Palma C 2003 Optical propagation in uniaxial crystals orthogonal to the optical axis: paraxial theory and beyond J. Opt. Soc. Am. A 20 2163–71

[24] Novitsky A V and Barkovsky L M 2006 Vector beams as the superposition of cylindrical partial waves in bianisotropic media J. Phys. A: Math. Gen. 39 13355–69

[25] Chen L and She W 2009 Electrically tunable and spin-dependent integer or noninteger orbital angular momentum generator Opt. Lett. 34 178–80

[26] Vaveliuk P, Moraes F, Fumeron S, Matos O M and Calvo M L 2010 Structure of the dielectric tensor in nematic liquid crystals with topological charge J. Opt. Soc. Am. A 27 1466–72

[27] Karimi E, Piccirillo B, Marrucci L and Santamato E 2009 Light propagation in a birefringent plate with topological charge Opt. Lett. 34 1225–7

[28] Vaveliuk P 2009 Nondiffracting wave properties in radially and azimuthally symmetric optical axis phase plates Opt. Lett. 34 3641–3