Distributed Source Coding for Correlated Memoryless Gaussian Sources

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Abstract—We consider a distributed source coding problem of $L$ correlated Gaussian observations $Y_i$, $i = 1, 2, \ldots, L$. We assume that the random vector $Y^L = (Y_1, Y_2, \ldots, Y_L)$ is an observation of the Gaussian random vector $X^K = (X_1, X_2, \ldots, X_K)$, having the form $Y^L = AX^K + N^L$, where $A$ is a $L \times K$ matrix and $N^L = (N_1, N_2, \ldots, N_L)$ is a vector of $L$ independent Gaussian random variables also independent of $X^K$. The estimation error on $X^K$ is measured by the distortion covariance matrix. The rate distortion region is defined by a set of all rate vectors for which the estimation error is upper bounded by an arbitrary prescribed covariance matrix in the meaning of positive semi definite. In this paper we derive explicit outer and inner bounds of the rate distortion region. This result provides a useful tool to study the direct and indirect source coding problems on this Gaussian distributed source coding system, which remain open in general.

Index Terms—Multiterminal source coding, rate-distortion region, CEO problem.

I. INTRODUCTION

Distributed source coding of correlated information sources are a form of communication system which is significant from both theoretical and practical points of view in multi-user source networks. The first fundamental theory in those coding systems was established by Slepian and Wolf [11]. They considered a distributed source coding system of two correlated information sources. Those two sources are separately encoded and sent to a single destination, where the decoder reconstruct the original sources.

In the above distributed source coding systems we can consider the case where the source outputs should be reconstructed with average distortions smaller than prescribed levels. Such a situation suggests the multiterminal rate distortion theory.

The rate distortion theory for the above distributed source coding system formulated by Slepian and Wolf has been studied by [2]-[9]. Wagner et al. [10] gave a complete solution to this problem in the case of Gaussian information sources and quadratic distortion by proving that sum rate part of the inner bound of Berger [4] and Tung [3] is tight. Wang et al. [11] gave a new alternative proof.

As a practical situation of distributed source coding systems, we can consider a case where the distributed encoders can not directly access to the source outputs but can access to their noisy observations. This situation was first studied by Yamamoto and Ito [12]. They call the investigated coding system the communication system with a remote source.

Subsequently, a similar distributed source coding system was studied by Flynn and R. M. Gray [13].

In this paper we consider a distributed source coding problem of $L$ correlated Gaussian sources $Y_i$, $i = 1, 2, \ldots, L$ which are noisy observations of $X_i$, $i = 1, 2, \ldots, K$. We assume that $Y^L = (Y_1, Y_2, \ldots, Y_L)$ is an observation of the source vector $X^K = (X_1, X_2, \ldots, X_K)$, having the form $Y^L = AX^K + N^L$, where $A$ is a $L \times K$ matrix and $N^L = (N_1, N_2, \ldots, N_L)$ is a vector of $L$ independent Gaussian random variables also independent of $X^K$.

We consider two distortion criteria based on the covariance matrix of the estimation error on $X^K$. One is the criterion called the vector distortion criterion. The vector distortion criterion is upper bounded by a prescribed level. The other is the sum distortion criterion where the trace of the covariance matrix is upper bounded by a prescribed level. For each of the above two distortion criteria we derive explicit inner and outer bounds of the rate distortion region. We also derive an explicit matching condition in the case of the sum distortion criterion.

When $K = 1$, the source coding system becomes that of the quadratic Gaussian CEO problem investigated by [11], [13]-[16]. The system in the case of $K = L$ and sum distortion criterion was studied by Pandya et al. [17]. They derived lower and upper bounds of the minimum sum rate in the rate distortion region. Several partial solutions in the case of $K = L$, $A = I_L$ and sum distortion criterion are obtained by [18]-[22]. The case of $K = L$, $A = I_L$ and vector distortion criterion is studied by [20].

The remote source coding problem treated in this paper is also referred to as the indirect distributed source coding problem. On the other hand, the multiterminal rate distortion problem in the frame work of distributed source coding is called the direct distributed source coding problem. As shown in the paper of Wagner et al. [10] and in the recent work by Wang et al. [11], we have a strong connection between the direct and indirect distributed source coding problems.

In this paper we also consider the multiterminal rate distortion problem, i.e., the direct distributed source coding problem for the Gaussian information source specified with $Y^L = X^L + N^L$, which corresponds to the case of $K = L$ and $A = I_L$. We shall derive a result which implies a strong connection between the remote source coding problem and the multiterminal rate distortion problem. This result states that all results on the rate distortion region of the remote source coding problem can be converted into those on the rate distortion region of the multiterminal source coding problem. Using this result, we drive several new partial solutions to the Gaussian
multiterminal rate distortion problem.

II. PROBLEM STATEMENT AND PREVIOUS RESULTS

A. Formal Statement of Problem

In this subsection we present a formal statement of problem. Throughout this paper all logarithms are taken to the base natural. Let $X_i, i = 1, 2, \ldots, K$ be correlated zero mean Gaussian random variables. For each $i = 1, 2, \ldots, K$, $X_i$ takes values in the real line $\mathbb{R}$. We write a $k$ dimensional random vector as $X^K = (X_1, X_2, \ldots, X_K)$. We denote the covariance matrix of $X^K$ by $\Sigma_{X^K}$. Let $Y^L \triangleq (Y_1, Y_2, \ldots, Y_L)$ be an observation of the source vector $X^K$, having the form $Y^L = AX^K + N^L$, where $A$ is a $L \times K$ matrix and $N^L = (N_1, N_2, \ldots, N_L)$ is a vector of $L$ independent zero mean Gaussian random variables also independent of $X^K$. For $i = 1, 2, \ldots, L$, $\sigma_i^2$ stands for the variance of $N_i$. Let $\{(X_1(t), X_2(t), \ldots, X_K(t))\}_{t=1}^{\infty}$ be a stationary memoryless multiple Gaussian source. For each $t = 1, 2, \ldots, L$, $X^K(t) \triangleq (X_1(t), X_2(t), \ldots, X_K(t))$ has the same distribution as $X^K$. A random vector consisting of $n$ independent copies of the random variable $X_i$ is denoted by $X_i^n \triangleq (X_i(1), X_i(2), \ldots, X_i(n)).$

For each $t = 1, 2, \ldots, L$, $Y_i(t), i = 1, 2, \ldots, L$ is a vector of $L$ correlated observations of $X^K(t)$, having the form $Y^L(t) = AX^K(t) + N^L(t)$, where $N^L(t), t = 1, 2, \ldots$, are independent identically distributed (i.i.d.) Gaussian random vector having the same distribution as $N^L$. We have no assumption on the number of observations $L$, which may be $L \geq K$ or $L < K$.

The distributed source coding system for $L$ correlated Gaussian observations treated in this paper is shown in Fig. 1. In this coding system the distributed encoder functions $\varphi_i, i = 1, 2, \ldots, L$ are defined by

$$\varphi_i^{(n)} : \mathbb{R}^n \to M_i \triangleq \{1, 2, \ldots, M_i\}.$$ 

For each $i = 1, 2, \ldots, L$, set $R_i^{(n)} \triangleq \frac{1}{n} \log M_i$, which stands for the transmission rate of the encoder function $\varphi_i^{(n)}$. The joint decoder function $\psi^{(n)} = (\psi_1^{(n)}, \psi_2^{(n)}, \ldots, \psi_K^{(n)})$ is defined by

$$\psi_i^{(n)} : M_1 \times \cdots \times M_L \to \hat{\mathbb{R}}_i^n, i = 1, 2, \ldots, K,$$

where $\hat{\mathbb{R}}_i^n$ is the real line in which a reconstructed random variable of $X_i$ takes values. For $X^K = (X_1, X_2, \ldots, X_K)$, set

$$\varphi^{(n)}(Y^L) \triangleq \varphi_1^{(n)}(Y_1), \varphi_2^{(n)}(Y_2), \ldots, \varphi_L^{(n)}(Y_L),$$

$$\hat{X}^K \triangleq \left[ \begin{array}{c} \hat{X}_1 \\ \vdots \\ \hat{X}_K \end{array} \right],$$

$$d_{ij} \triangleq E(|X_i - \hat{X}_j|)^2,$$

$$d_{ij} \triangleq E(X_i - \hat{X}_i, X_j - \hat{X}_j), 1 \leq i \neq j \leq K,$$

where $||a||$ stands for the Euclid norm of $n$ dimensional vector $a$ and $(a, b)$ stands for the inner product between $a$ and $b$. Let $\Sigma_{X^K - \hat{X}^K}$ be a covariance matrix with $d_{ij}$ in its $(i, j)$ entry. Let $\Sigma_d$ be a given $L \times L$ covariance matrix which serves as a distortion criterion. We call this matrix a distortion matrix.

For a given distortion matrix $\Sigma_d$, the rate vector $(R_1, R_2, \ldots, R_L)$ is $\Sigma_d$-admissible if there exists a sequence $\{((\varphi_1^{(n)}, \varphi_2^{(n)}, \ldots, \varphi_L^{(n)}, \psi^{(n)}))_{n=1}^{\infty}\}$ such that

$$\limsup_{n \to \infty} R_i^{(n)} \leq R_i, \text{ for } i = 1, 2, \ldots, L,$$

$$\limsup_{n \to \infty} \sum_{i=1}^{L} R_i^{(n)} \leq \Sigma_d,$$

where $A_1 \preceq A_2$ means that $A_2 - A_1$ is positive semi-definite matrix. Let $R_L(\Sigma_d|\Sigma_{X^K Y^L})$ denote the set of all $\Sigma_d$-admissible rate vectors. We often have a particular interest in the minimum sum rate part of the rate distortion region. To examine this quantity, we set

$$R_{\text{sum}, L}(\Sigma_d|\Sigma_{X^K Y^L}) \triangleq \min_{R = (R_1, R_2, \ldots, R_L) \in R_L(\Sigma_d|\Sigma_{X^K Y^L})} \left\{ \sum_{i=1}^{L} R_i \right\}.$$ 

We consider two types of distortion criterion. For each distortion criterion we define the determination problem of the rate distortion region.

Problem I. Vector Distortion Criterion: Fix $K \times K$ invertible matrix $\Gamma$ and positive vector $D^K = (D_1, D_2, \ldots, D_K)$. For given $\Gamma$ and $D^K$, the rate vector $(R_1, R_2, \ldots, R_L)$ is $(\Gamma, D^K)$-admissible if there exists a sequence $\{((\varphi_1^{(n)}, \varphi_2^{(n)}, \ldots, \varphi_L^{(n)}, \psi^{(n)}))_{n=1}^{\infty}\}$ such that

$$\limsup_{n \to \infty} R_i^{(n)} \leq R_i, \text{ for } i = 1, 2, \ldots, L,$$

$$\limsup_{n \to \infty} \left[ \frac{1}{n} \sum_{K} (\Gamma^{-1} \Sigma_{X^K - \hat{X}^K})_{ij} \right] \leq D_i, \text{ for } i = 1, 2, \ldots, K,$$

where $[C]_{ij}$ stands for the $(i, j)$ entry of the matrix $C$. Let $R_L(\Gamma, D^K|\Sigma_{X^K Y^L})$ denote the set of all $(\Gamma, D^K)$-admissible rate vectors. When $\Gamma$ is equal to the $K \times K$ identity matrix $I_K$, we omit $\Gamma$ in $R_L(\Gamma, D|\Sigma_{X^K Y^L})$ to simply write $R_L(D|\Sigma_{X^K Y^L})$. Similar notations are used for other sets or quantities. To examine the sum rate part of $R_L(\Gamma, D^K|\Sigma_{X^K Y^L})$, define

$$R_{\text{sum}, L}(\Gamma, D^K|\Sigma_{X^K Y^L}) \triangleq \min_{R = (R_1, R_2, \ldots, R_L) \in R_L(\Gamma, D^K|\Sigma_{X^K Y^L})} \left\{ \sum_{i=1}^{L} R_i \right\}.$$ 

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Problem I. Vector Distortion Criterion: Fix $K \times K$ invertible matrix $\Gamma$ and positive vector $D^K = (D_1, D_2, \ldots, D_K)$. For given $\Gamma$ and $D^K$, the rate vector $(R_1, R_2, \ldots, R_L)$ is $(\Gamma, D^K)$-admissible if there exists a sequence $\{((\varphi_1^{(n)}, \varphi_2^{(n)}, \ldots, \varphi_L^{(n)}, \psi^{(n)}))_{n=1}^{\infty}\}$ such that

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$$R_{\text{sum}, L}(\Gamma, D^K|\Sigma_{X^K Y^L}) \triangleq \min_{R = (R_1, R_2, \ldots, R_L) \in R_L(\Gamma, D^K|\Sigma_{X^K Y^L})} \left\{ \sum_{i=1}^{L} R_i \right\}.$$
Problem 2. Sum Distortion Criterion: Fix $K \times K$ positive definite invertible matrix $\Gamma$ and positive $D$. For given $\Gamma$ and $D$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $(\Gamma, D)$-admissible if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^{\infty}$ such that

\[
\limsup_{n \to \infty} \psi^{(n)} \leq R_i, \quad \text{for } i = 1, 2, \cdots, L, \\
\limsup_{n \to \infty} \text{tr} \left[ \Gamma \left( \frac{1}{n} \sum_{K} - \hat{X}_K \right)^{\dagger} \Gamma \right] \leq D.
\]

To examine the sum rate part of $R_L(\Gamma, D| \Sigma_{X|Y} L)$, define

\[
R_{\text{sum}, L}(\Gamma, D| \Sigma_{X|Y} L) \triangleq \min_{(R_1, R_2, \cdots, R_L) \in \mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)} \left\{ \sum_{i=1}^{L} R_i \right\}.
\]

Let $\mathcal{S}_K(D^K)$ be a set of all $K \times K$ covariance matrices whose $(i, i)$ entry do not exceed $D_i$ for $i = 1, 2, \cdots, K$. Then we have

\[
\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L) = \bigcup_{\Gamma \Sigma \in \mathcal{S}_K(D^K)} \mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L),
\]

where $\mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L)$ is defined as in (1) and (2).

Furthermore, we have

\[
\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L) = \bigcup_{\Gamma \Sigma \in \mathcal{S}_K(D^K) \cap \mathcal{D}_L(\Gamma)} \mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L).
\]

In this paper we establish explicit inner and outer bounds of $\mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L)$ using the above bounds and equations (1) and (2), we can obtain new outer bounds of $\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)$ and $\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)$.

B. Inner Bounds and Previous Results

In this subsection we present inner bounds of $\mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L)$, $\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)$, and $\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)$. Those inner bounds can be obtained by a standard technique developed in the field of multiterminal source coding.

Set $\Lambda \triangleq \{1, 2, \cdots, L\}$. For $i \in \Lambda$, let $U_i$ be a random variable taking values in the real line $U_i$. For any subset $S \subseteq \Lambda$, we introduce the notation $U_S = (U_i)_{i \in S}$. In particular $U_A = U_L = (U_1, U_2, \cdots, U_L)$.

Define

\[
\mathcal{G}(\Sigma, \Lambda) \triangleq \{ U^L : U^L \text{ is a Gaussian random vector that satisfies } \}.
\]

and set

\[
\mathcal{R}^\text{(in)}_L(\Sigma, \Lambda) \triangleq \text{conv} \left\{ R^L : \text{There exists a random vector } U^L \in \mathcal{G}(\Sigma) \text{ such that } \sum_{i \in S} R_i \geq I(U_S; Y_S|U_S) \right\}
\]

for any $S \subseteq \Lambda$.

We can show that $\mathcal{R}^\text{(in)}_L(\Sigma, \Lambda)$, $\mathcal{R}^\text{(in)}_L(\Gamma, D| \Sigma_{X|Y} L)$, and $\mathcal{R}^\text{(in)}_L(\Gamma, D| \Sigma_{X|Y} L)$ satisfy the following property.

Property 1:

a) The set $\mathcal{R}^\text{(in)}_L(\Sigma, \Lambda)$ is not void if and only if $\Sigma > \Sigma_{X|Y} L$.

b) The set $\mathcal{R}^\text{(in)}_L(\Gamma, D| \Sigma_{X|Y} L)$ is not void if and only if $D > D^K(\Gamma \Sigma_{X|Y} L \Gamma)$.

c) The set $\mathcal{R}^\text{(in)}_L(\Gamma, D| \Sigma_{X|Y} L)$ is not void if and only if $D > \text{tr}[\Gamma \Sigma_{X|Y} L \Gamma]$.

On inner bounds of $\mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L)$, $\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)$, and $\mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L)$, we have the following result.

Theorem 1 (Berger [4] and Tung [5]): For any $\Sigma > \Sigma_{X|Y} L$, we have

\[
\mathcal{R}^\text{(in)}_L(\Sigma, | \Sigma_{X|Y} L) \subseteq \mathcal{R}_L(\Sigma, | \Sigma_{X|Y} L).
\]

For any $\Gamma$ and any $D > D^K(\Gamma \Sigma_{X|Y} L \Gamma)$, we have

\[
\mathcal{R}^\text{(in)}_L(\Gamma, D| \Sigma_{X|Y} L) \subseteq \mathcal{R}_L(\Gamma, D^K| \Sigma_{X|Y} L).
\]

For any $\Gamma$ and any $D > \text{tr}[\Gamma \Sigma_{X|Y} L \Gamma]$, we have

\[
\mathcal{R}^\text{(in)}_L(\Gamma, D| \Sigma_{X|Y} L) \subseteq \mathcal{R}_L(\Gamma, D| \Sigma_{X|Y} L).
\]

The above three inner bounds can be regarded as variants of the inner bound which is well known as that of Berger [4] and Tung [5].

When $K = 1$ and $L \times 1$ column vector $A$ has the form $A = t[1 \cdots 1]$, the system considered here becomes the quadratic Gaussian CEO problem. This problem was first posed and
investigated by Viswanathan and Berger [14]. They further assumed \( \Sigma_{N^L} = \sigma^2 I_L \). Set \( \sigma^2_X \triangleq \Sigma_X \) and
\[
R_{\text{sum}}(D|\sigma^2_X, \sigma^2) \triangleq \liminf_{L \to \infty} R_{\text{sum},L}(D|\Sigma_{XY^L}).
\]
Viswanathan and Berger [14] studied an asymptotic form of \( R_{\text{sum}}(D|\sigma^2_X, \sigma^2) \) for small \( D \). Subsequently, Oohama [13] determined an exact form of \( R_{\text{sum}}(D|\sigma^2_X, \sigma^2) \). The region \( R_L(D|\Sigma_{XY^L}) \) was determined by Oohama [16].

In the case where \( K = L \) and \( \Gamma = A = I_L \), Oohama [18, 20] derived inner and outer bounds of \( R_L(D|\Sigma_{XY^L}) \). Oohama [19] also derived explicit sufficient conditions for inner and outer bounds to match and found examples of information sources for which rate distortion region are explicitly determined. In [20], Oohama derived explicit outer bounds of \( R_L(\Sigma_d|\Sigma_{XY^L}), R_L(D|\Sigma_{XY^L}), \) and \( R_L(D|\Sigma_{XY^L}) \).

Recently, Wagner et al. [10] have determined \( R_2(D|\Sigma_{XY^L}) \). Their result is as follows.

**Theorem 2 (Wagner et al. [10]):** For any \( D^2 > d^2([\Sigma_X^K]_{Y^L}) \), we have
\[
R_2(D^2|\Sigma_{XY^L}) = R_2^{(\text{in})}(D^2|\Sigma_{XY^L}).
\]

Their method for the proof depends heavily on the specific property of \( L = 2 \). It is hard to generalize it to the case of \( L \geq 3 \).

### III. MAIN RESULTS

#### A. Inner and Outer Bounds of the Rate Distortion Region

In this subsection we state our result on the characterization of \( R_L(\Sigma_d|\Sigma_{XY^L}), R_L(\Gamma, D^K|\Sigma_{XY^L}), \) and \( R_L(\Gamma, D|\Sigma_{XY^L}) \). To describe those results we define several functions and sets. For \( r_i \geq 0, i \in A \), let \( N_i(r_i) \), \( i \in A \) be \( L \) independent Gaussian random variables with mean 0 and variance \( \sigma^2_{N_i} / (1 - e^{-2r_i}) \). Let \( \Sigma_{N(r)} \) be a covariance matrix for the random vector \( N^L(r^L) \) . Fix nonnegative vector \( r^L \). For \( \theta > 0 \) and for \( S \subseteq A \), define
\[
\Sigma_{N^L(r)} \triangleq \Sigma_{N^L(r)} = \left[ \prod_i e^{2r_i} \right]^{\frac{1}{2}} \left[ \sum_{i \in S} \sum_{i \in L} - A \Sigma_{N^L(r)}^{-1} \right]^{\frac{1}{2}},
\]
\[
J(S, r^L|\Sigma_{N^L(r)}) \triangleq \frac{1}{2} \log \left[ \sum_{i \in S} \left( \sum_{i \in L} - A \Sigma_{N^L(r)}^{-1} \right) \right],
\]
\[
J_S(r^L|\Sigma_{N^L(r)}) \triangleq \frac{1}{2} \log \left[ \sum_{i \in S} \left( \sum_{i \in L} - A \Sigma_{N^L(r)}^{-1} \right) \right],
\]
where \( S^C = A \setminus S \) and \( \log^+ x \triangleq \max\{\log x, 0\} \). Set
\[
A_L(\Sigma_d) \triangleq \left\{ r^L \geq 0 : \left[ \sum_{i \in S} \left( \sum_{i \in L} - A \Sigma_{N^L(r)}^{-1} \right) \right]^{\frac{1}{2}} \leq \Sigma_d \right\}.
\]

We can show that for \( S \subseteq A \), \( J(S, r^L|\Sigma_{N^L(r)}) \) and \( J_S(r^L|\Sigma_{N^L(r)}) \) satisfy the following two properties.

**Property 2:**

a) If \( r^L \in A_L(\Sigma_d) \), then for any \( S \subseteq A \),
\[
J(S, r^L|\Sigma_{N^L(r)}) \leq J_S(r^L|\Sigma_{N^L(r)}).
\]

b) Suppose that \( r^L \in A_L(\Sigma_d) \). If \( r^L|_{S^C} = 0 \) still belongs to \( A_L(\Sigma_d) \), then
\[
J_S(|\Sigma_{d}, r^L|\Sigma_{N^L(r)})|_{r^L = 0} = J_S(r^L|\Sigma_{N^L(r)})|_{r^L = 0} = 0.
\]

**Property 3:** Fix \( r^L \in A_L(\Sigma_d) \). For \( S \subseteq A \), set
\[
f_S = J_S(r^L|\Sigma_{N^L(r)}) \triangleq J_S(|\Sigma_{d}, r^L|\Sigma_{N^L(r)})|_{r^L = 0}.
\]

By definition, it is obvious that \( f_S, S \subseteq A \) are nonnegative. We can show that \( f \triangleq \{f_S\}_{S \subseteq A} \) satisfies the following:

a) \( f_\emptyset = 0 \).

b) \( f_A \leq f_B \) for \( A \subseteq B \subseteq A \).

c) \( f_A + f_B \leq f_{A \cup B} + f_{A \cap B} \).

In general, \((\Lambda, f)\) is called a co-polymatroid if the nonnegative function \( f \) on \( 2^\Lambda \) satisfies the above three properties. Similarly, we set
\[
\bar{f}_S = f_S(r^L|\Sigma_{N^L(r)}) \triangleq J_S(r^L|\Sigma_{N^L(r)}), \quad \bar{f} = \{\bar{f}_S\}_{S \subseteq A}.
\]

Then \((\Lambda, \bar{f})\) also has the same three properties as those of \((\Lambda, f)\) and becomes a co-polymatroid.

To describe our result on \( R_L(\Sigma_d|\Sigma_{XY^L}) \), set
\[
R_L^{(\text{out})}(\theta, r^L|\Sigma_{XY^L}) \triangleq \left\{ r^L : \sum_i \theta_i \geq J_S(\theta, r^L|\Sigma_{N^L(r)}) \right\},
\]
where \( S \subseteq \Lambda \). From this theorem we can show that \( R_L^{(\text{out})}(\Sigma_d|\Sigma_{XY^L}) \) and \( R_L^{(\text{out})}(\Sigma_d|\Sigma_{XY^L}) \) satisfy the following property.

**Property 4:** The sets \( R_L^{(\text{in})}(\Sigma_d|\Sigma_{XY^L}) \) and \( R_L^{(\text{out})}(\Sigma_d|\Sigma_{XY^L}) \) are not void if and only if \( \Sigma_d \succ \Sigma_{XY^L} \). Our result on inner and outer bounds of \( R_L(\Sigma_d|\Sigma_{XY^L}) \) is as follows.

**Theorem 3:** For any \( \Sigma_d \succ \Sigma_{XY^L} \), we have
\[
R_L^{(\text{in})}(\Sigma_d|\Sigma_{XY^L}) \subseteq R_L^{(\text{out})}(\Sigma_d|\Sigma_{XY^L}) \subseteq R_L(\Sigma_d|\Sigma_{XY^L}) \subseteq R_L^{(\text{out})}(\Sigma_d|\Sigma_{XY^L}) .
\]

Proof of this theorem is given in Section V. This result includes the result of Oohama [20] as a special case by letting \( K = L \) and \( \Gamma = A = I_L \). From this theorem we can
derive outer and inner bounds of $\mathcal{R}_L(\Gamma, D^K|\Sigma_X Y L)$ and $\mathcal{R}_L(\Gamma, D|\Sigma_X Y L)$. To describe those bounds, set
\[
\mathcal{R}_L^{(\text{out})}(\Gamma, D^K|\Sigma_X Y L) \triangleq \bigcup_{\Gamma \subseteq \Sigma_d, d \in \Delta(\Gamma^K)} \mathcal{R}_L^{(\text{out})}(\Sigma_d|\Sigma_X Y L),
\]
\[
\mathcal{R}_L^{(\text{in})}(\Gamma, D^K|\Sigma_X Y L) \triangleq \text{conv} \left\{ \bigcup_{\Gamma \subseteq \Sigma_d, d \in \Delta(\Gamma^K)} \mathcal{R}_L^{(\text{in})}(\Sigma_d|\Sigma_X Y L) \right\},
\]
\[
\mathcal{R}_L^{(\text{out})}(\Gamma, D|\Sigma_X Y L) \triangleq \bigcup_{\Gamma \subseteq \Sigma_d, d \in \Delta(\Gamma^K)} \mathcal{R}_L^{(\text{out})}(\Sigma_d|\Sigma_X Y L),
\]
\[
\mathcal{R}_L^{(\text{in})}(\Gamma, D|\Sigma_X Y L) \triangleq \text{conv} \left\{ \bigcup_{\Gamma \subseteq \Sigma_d, d \in \Delta(\Gamma^K)} \mathcal{R}_L^{(\text{in})}(\Sigma_d|\Sigma_X Y L) \right\}
\]

Set $A(r_L) \triangleq \left\{ \Sigma_d : \Sigma_d \geq (\Sigma_X^{-1} + t A \Sigma_N^{-1}(\Gamma L)^{-1} A^{-1})^{-1} \right\}$,
\[
\theta(\Gamma, D^K, r_L) \triangleq \max_{\Sigma_d : \Sigma_d \in A(r_L), \Gamma \subseteq \Sigma d, d \in \Delta(\Gamma^K)} |\Sigma_d|,
\]
\[
\theta(\Gamma, D, r_L) \triangleq \max_{\Sigma_d : \Sigma_d \in A(r_L), \Gamma \subseteq \Sigma d, d \in \Delta(\Gamma^K)} |\Sigma_d|.
\]

Furthermore, set
\[
B_L(\Gamma, D^K) \triangleq \left\{ r_L \geq 0 : \Gamma (\Sigma_X^{-1} + t A \Sigma_N^{-1}(\Gamma L)^{-1} A^{-1})^{-1} \in \Delta(\Gamma^K) \right\},
\]
\[
B_L(\Gamma, D) \triangleq \left\{ r_L \geq 0 : \Gamma (\Sigma_X^{-1} + t A \Sigma_N^{-1}(\Gamma L)^{-1} A^{-1})^{-1} \leq D \right\}.
\]

It can easily be verified that $\mathcal{R}_L^{(\text{out})}(\Gamma, D^K|\Sigma_X Y L)$ and $\mathcal{R}_L^{(\text{in})}(\Gamma, D^K|\Sigma_X Y L)$ are not void if and only if $D^K > d^K(\Sigma_X \Gamma Y L \Gamma)$. The following result is obtained as a simple corollary from Theorem 3.

**Corollary 1:** For any $\Gamma$ and any $D^K > d^K(\Sigma_X \Gamma Y L \Gamma)$, we have
\[
\mathcal{R}_L^{(\text{in})}(\Gamma, D^K|\Sigma_X Y L) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D^K|\Sigma_X Y L) \subseteq \mathcal{R}_L(\Gamma, D^K|\Sigma_X Y L).
\]

For any $\Gamma$ and any $D > d(\Gamma Y L \Gamma)$, we have
\[
\mathcal{R}_L^{(\text{in})}(\Gamma, D|\Sigma_X Y L) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D|\Sigma_X Y L) \subseteq \mathcal{R}_L(\Gamma, D|\Sigma_X Y L).
\]

Those results include the result of Oohama [20] as a special case by letting $K = L$ and $\Gamma = A = I_L$. Next we compute $\theta(\Gamma, D, r_L)$ to derive a more explicit expression of $\mathcal{R}_L^{(\text{out})}(\Gamma, D|\Sigma_X Y L)$ to be tight. Let $\alpha_i = \alpha_i(r_L), i = 1, 2, \ldots, K$ be $K$ eigen values of the matrix
\[
\Gamma^{-1} \left( \Sigma_X^{-1} + t A \Sigma_N^{-1}(\Gamma L)^{-1} A^{-1} \right)^{-1} \Gamma^{-1}.
\]

Let $\xi$ be a nonnegative number that satisfy
\[
\sum_{i=1}^{K} \left\{ \xi - \alpha_i^{-1} \right\}^{-1} = D.
\]

Define
\[
\omega(\Gamma, D, r_L) \triangleq |\Gamma|^{-2} \prod_{i=1}^{K} \left\{ \xi - \alpha_i^{-1} \right\}^{-1}.
\]

The function $\omega(\Gamma, D, r_L)$ has an expression of the so-called water filling solution to the following optimization problem:
\[
\omega(\Gamma, D, r_L) = |\Gamma|^{-2} \max_{\xi, \alpha_i \geq 1, i \in A} \prod_{i=1}^{K} \xi_i.
\]

Then we have the following theorem.

**Theorem 4:** For any $\Gamma$ and any positive $D$, we have
\[
\theta(\Gamma, D, r_L) = \omega(\Gamma, D, r_L).
\]

A more explicit expression of $\mathcal{R}_L^{(\text{out})}(\Gamma, D|\Sigma_X Y L)$ is given by
\[
\mathcal{R}_L^{(\text{out})}(\Gamma, D|\Sigma_X Y L) \triangleq \bigcup_{r_L \in B_L(\Gamma, D^K)} \mathcal{R}_L^{(\text{out})}(\omega(\Gamma, D, r_L), r_L|\Sigma_X Y L).
\]

Proof of this theorem will be given in Section V. The above expression of the outer bound includes the result of Oohama [20] as a special case by letting $K = L$ and $\Gamma = A = I_L$. 

B. Matching Condition Analysis

For $L \geq 3$, we present a sufficient condition for $R_L^{(\text{out})} (\Gamma, D, \Sigma_{\mathcal{K} \times \mathcal{Y} \times L} \subseteq R_L^{(\text{in})} (D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L})$. We consider the following condition on $\theta (\Gamma, D, r^L)$.

**Condition:** For any $i \in \Lambda, e^{-2r_i} \theta (\Gamma, D, r^L)$ is a monotone decreasing function of $r_i \geq 0$.

We call this condition the MD condition. The following is a key lemma to derive the matching condition. This lemma is due to Oohama [19],[21].

**Lemma 1 (Oohama [19],[21]):** If $\theta (\Gamma, D, r^L)$ satisfies the MD condition on $B_L (\Gamma, D)$, then

\[
R_L^{(\text{in})} (\Gamma, D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L}) = R_L (\Gamma, D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L})
\]

Based on Lemma 1, we derive a sufficient condition for the MD condition to hold. The following is a key lemma to derive a sufficient condition for the MD condition to hold.

**Lemma 3:** If $\alpha_{\min}(u^L)$ and $\alpha_{\max}(u^L)$ satisfy

\[
\frac{1}{\alpha_{\min}(u^L)} - \frac{1}{\alpha_{\max}(u^L)} \leq \frac{1}{||\hat{a}_i||^2 \sigma_{\hat{a}_i} + \eta_i(u^L_{[i]})},
\]

then $\theta (\Gamma, D, u^L)$ satisfies the MD condition on $B_L (\Gamma, D)$.

Proof of Lemma 3 will be stated in Section V. Let $\alpha^*_{\max}$ be the maximum eigen value of

\[
\Gamma^{-1}(\Sigma_{\mathcal{K} \times \mathcal{Y}})^{-1} + A \Sigma_{\mathcal{N} \times \mathcal{L}}^{-1} A
\]

From Lemmas 13 and an elementary computation we obtain the following.

**Theorem 5:** If we have

\[
\text{tr}[\Gamma_{\Sigma_{\mathcal{K}} | Y \times 1} \Gamma_i] < D \leq \frac{K+1}{\alpha^*_{\max}},
\]

then

\[
R_L^{(\text{out})} (\Gamma, D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L}) = R_L^{(\text{in})} (\Gamma, D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L})
\]

In particular,

\[
R_{\text{sum},L} (\Gamma, D) = \min_{r \in B_L (\Gamma, D)} \left\{ \sum_{i=1}^{L} r_i + \frac{1}{2} \log \frac{\sum_{i=1}^{L} \left| \gamma_i \right|^2 + t A \Sigma_{\mathcal{N} \times \mathcal{L}}^{-1} A}{\left| \gamma_i \right|^2 \sigma_{\gamma_i}} \right\}
\]

Proof of Theorem 5 will be stated in Section V. From this theorem, we can see that if the value of $D$ is very close to $\text{tr}[\Gamma_{\Sigma_{\mathcal{K}} | Y \times 1} \Gamma_i]$, $R_L^{(\text{in})} (\Gamma, D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L})$ and $R_L^{(\text{out})} (\Gamma, D|\Sigma_{\mathcal{K} \times \mathcal{Y} \times L})$ match.
Fig. 2. Distributed source coding system for $L$ correlated Gaussian sources

IV. APPLICATION TO THE MULTITERMINAL RATE DISTORTION PROBLEM

In this section we consider the multiterminal rate distortion problem for Gaussian information source specified with $Y^L$. We consider the case where $K = L$ and $A = I_L$. In this case we have $Y^L = X^L + N^L$. The Gaussian random variables $Y_{i}, i = 1, 2, \cdots, L$ are $L$-noisy components of random vector $X^L$. The Gaussian random vector $X^L$ can be regarded as a “hidden” information source of $Y^L$. Note that $(X^L, Y^L)$ satisfies $Y_S \rightarrow X^L \rightarrow Y_S^c$ for any $S \subseteq \Lambda$.

A. Problem Formulation and Previous Results

The distributed source coding system for $L$ correlated Gaussian source treated here is shown in Fig. 2. Definitions of encoder functions $\varphi_i, i = 1, 2, \cdots, L$ are the same as the previous definitions. The decoder function $\phi^{(n)} = (\varphi_{1}^{(n)}, \varphi_{2}^{(n)}, \cdots, \varphi_{L}^{(n)})$ is defined by

$$\phi^{(n)}: \mathcal{M}_1 \times \cdots \times \mathcal{M}_L \rightarrow \hat{Y}^n_i, i = 1, 2, \cdots, K,$$

where $\hat{Y}_i$ is the real line in which estimations of $Y_i$ take values. For $Y^L = (Y_1, Y_2, \cdots, Y_L)$, set

$$\hat{Y}^L = \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \vdots \\ \hat{Y}_L \end{bmatrix} = \begin{bmatrix} \varphi_{1}^{(n)}(\varphi_i^{(n)}(Y^L)) \\ \varphi_{2}^{(n)}(\varphi_i^{(n)}(Y^L)) \\ \vdots \\ \varphi_{L}^{(n)}(\varphi_i^{(n)}(Y^L)) \end{bmatrix},$$

$$\hat{d}_{ii} = E|Y_i - \hat{Y}_i|^2, \quad \hat{d}_{ij} = E(Y_i - \hat{Y}_i, Y_j - \hat{Y}_j), 1 \leq i \neq j \leq L.$$

Let $\Sigma_{Y^L - Y^L}$ be a covariance matrix with $\hat{d}_{ij}$ in its $(i, j)$ entry.

For a given $\Sigma_d$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $\Sigma_d$-admissible if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^{\infty}$ such that $\limsup_{n \rightarrow \infty} R_i^{(n)} \leq R_i$, for $i = 1, 2, \cdots, L$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \Sigma_{Y^L - Y^L} \leq \Sigma_d.$$

Let $R_L(\Sigma_d | \Sigma_{Y^L})$ denote the set of all $\Sigma_d$-admissible rate vectors.

We consider two types of distortion criterion. For each distortion criterion we define the determination problem of the rate distortion region.

**Problem 3. Vector Distortion Criterion:** For given $L \times L$ invertible matrix $\Gamma$ and $D > 0$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $(\Gamma, D^L)$-admissible if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^{\infty}$ such that

$$\limsup_{n \rightarrow \infty} R_i^{(n)} \leq R_i, \quad \text{for } i = 1, 2, \cdots, L,$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \Sigma_{Y^L - Y^L} \leq D,$$

Let $R_L(\Gamma, D^L | \Sigma_{Y^L})$ denote the set of all $(\Gamma, D^L)$-admissible rate vectors. The sum rate part of the rate distortion region is defined by

$$R_{\text{sum},L}(\Gamma, D^L | \Sigma_{Y^L}) \triangleq \min_{\left(\sum_{i=1}^{L} R_i \right)} \left\{ \sum_{i=1}^{L} R_i \right\}.$$

**Problem 4. Sum Distortion Criterion:** For given $L \times L$ invertible matrix $\Gamma$ and $D > 0$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $(\Gamma, D)$-admissible if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^{\infty}$ such that $\limsup_{n \rightarrow \infty} R_i^{(n)} \leq R_i, \quad \text{for } i = 1, 2, \cdots, L,

$$\limsup_{n \rightarrow \infty} \text{tr} \left[ \Gamma \left( \frac{1}{n} \Sigma_{Y^L - Y^L} \right)^{\dagger} \Gamma \right] \leq D.$$

Let $R_L(\Gamma, D | \Sigma_{Y^L})$ denote the set of all admissible rate vectors. The sum rate part of the rate distortion region is defined by

$$R_{\text{sum},L}(\Gamma, D | \Sigma_{Y^L}) \triangleq \min_{\left(\sum_{i=1}^{L} R_i \right)} \left\{ \sum_{i=1}^{L} R_i \right\}.$$

Relations between $R_L(\Sigma_d | \Sigma_{Y^L})$, $R_L(\Gamma, D^L | \Sigma_{Y^L})$, and $R_L(\Gamma, D | \Sigma_{Y^L})$ are as follows.

$$R_L(\Gamma, D^L | \Sigma_{Y^L}) = \bigcup_{\Gamma \Sigma_d \Gamma^T \in \mathcal{S}_L(D^L)} R_L(\Sigma_d | \Sigma_{Y^L}), \quad (6)$$

$$R_L(\Gamma, D | \Sigma_{Y^L}) = \bigcup_{\text{tr}[\Gamma \Sigma_d \Gamma^T] \leq D} R_L(\Sigma_d | \Sigma_{Y^L}). \quad (7)$$

Furthermore, we have

$$R_L(\Gamma, D | \Sigma_{Y^L}) = \bigcup_{\sum_{i=1}^{L} D_i \leq D} R_L(\Gamma, D^L | \Sigma_{Y^L}). \quad (8)$$

We first present inner bounds of $R_L(\Sigma_d | \Sigma_{Y^L})$, $R_L(\Gamma, D^L | \Sigma_{Y^L})$, and $R_L(\Gamma, D | \Sigma_{Y^L})$. Those inner bounds can be obtained by a standard technique of multiterminal source
coding. Define
\[ \hat{\mathcal{G}}(\Sigma_d) \triangleq \{ U^L : U^L \text{ is a Gaussian random vector that satisfies} \]
\[ U_S \rightarrow Y_S \rightarrow X^L \rightarrow Y_S' \rightarrow U_S', \]
\[ U^L \rightarrow Y^L \rightarrow X^L \]
\[ \text{for any } S \subset \Lambda \text{ and } \Sigma_{Y^L-\phi(U^L)} \subseteq \Sigma_d \]
\[ \text{for some linear mapping } \phi : U^L \rightarrow \hat{Y}^L. \}

and set
\[ \hat{\mathcal{R}}^{(in)}_{L}(\Sigma_d|\Sigma_{Y^L}) \triangleq \text{conv } \{ R^L : \text{there exists a random vector} \]
\[ U^L \in \hat{\mathcal{G}}(\Sigma_d) \text{ such that} \]
\[ \sum_{i \in S} R_i \geq I(U_S; Y_S|U_S') \]
\[ \text{for any } S \subseteq \Lambda. \} , \]
\[ \hat{\mathcal{R}}^{(in)}_{L}(\Gamma, D^L|\Sigma_{Y^L}) \triangleq \text{conv } \left\{ \sum_{\Gamma \subseteq S \subseteq \Lambda} \hat{\mathcal{R}}^{(in)}_{L}(\Sigma_d|\Sigma_{Y^L}) \right\} . \]

Then we have the following result.

**Theorem 6 (Berger [4] and Tung [5]):** For any positive definite \( \Sigma_d \), we have
\[ \hat{\mathcal{R}}^{(in)}_{L}(\Sigma_d|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}). \]

For any invertible \( \Gamma \) and any \( D^L > 0 \), we have
\[ \hat{\mathcal{R}}^{(in)}_{L}(\Gamma, D^L|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}). \]

The inner bound \( \hat{\mathcal{R}}^{(in)}_{L}(D^L|\Sigma_{Y^L}) \) for \( \Gamma = I_L \) is well known as the inner bound of Berger [4] and Tung [5]. The above three inner bounds are variants of this inner bound.

Optimality of \( \hat{\mathcal{R}}^{(in)}(D^2|\Sigma_{Y^2}) \) was first studied by Oohama [9]. Without loss of generality we may assume that
\[ \Sigma_{Y^2} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \quad \rho \in [0, 1). \]

For \( i = 1, 2 \), set
\[ \mathcal{R}_{i,2}(D_i|\Sigma_{Y^2}) \triangleq \bigcup_{D_{i-1} > 0} \mathcal{R}_2(D^2|\Sigma_{Y^2}). \]

Oohama [9] obtained the following result.

**Theorem 7 (Oohama [9]):** For \( i = 1, 2 \), we have
\[ \mathcal{R}_{i,2}(D_i|\Sigma_{Y^2}) = \mathcal{R}^*_{i,2}(D_i|\Sigma_{Y^2}), \]

where
\[ \mathcal{R}^*_{i,2}(D_i|\Sigma_{Y^2}) \triangleq \left\{ (R_1, R_2) : R_i \geq \frac{1}{2} \log \left( 1 - \rho^2 \right) \frac{1}{\rho^2} \left( 1 + \frac{s^2}{\rho^2} \right), \right. \]
\[ \left. R_{i-1} \geq \frac{1}{2} \log \left( \frac{s}{\rho} \right) \right\} \text{ for some } 0 < s \leq 1. \]

Since \( \mathcal{R}^*_{1,2}(D_1|\Sigma_{Y^2}), i = 1, 2 \) serve as outer bounds of \( \mathcal{R}_2(D^2|\Sigma_{Y^2}) \), we have
\[ \mathcal{R}_2(D^2|\Sigma_{Y^2}) \subseteq \mathcal{R}^*_{1,2}(D_1|\Sigma_{Y^2}) \cap \mathcal{R}^*_{2,2}(D_2|\Sigma_{Y^2}). \]  

(9)

Wagner et al. [10] derived the condition where the outer bound in the right hand side of (9) is tight. To describe their result set
\[ D \triangleq \{ (D_1, D_2) : \min\{D_1, D_2\} \leq \min\{1, \rho^2 \min\{D_1, D_2\} + 1 - \rho^2\} \}. \]

Wagner et al. [10] showed that if \( D^2 \notin \mathcal{D} \), we have
\[ \mathcal{R}_2(D^2|\Sigma_{Y^2}) = \mathcal{R}^*_{1,2}(D_1|\Sigma_{Y^2}) \cap \mathcal{R}^*_{2,2}(D_2|\Sigma_{Y^2}). \]

Next we consider the case of \( D^2 \in \mathcal{D} \). In this case by an elementary computation we can show that \( \hat{\mathcal{R}}^{(in)}_2(D^2|\Sigma_{Y^2}) \) has the following form:
\[ \hat{\mathcal{R}}^{(in)}_2(D^2|\Sigma_{Y^2}) = \mathcal{R}^*_{1,2}(D_1|\Sigma_{Y^2}) \cap \mathcal{R}^*_{2,2}(D_2|\Sigma_{Y^2}) \cap \mathcal{R}^*_{3,2}(D^2|\Sigma_{Y^2}) \]

where
\[ \mathcal{R}^*_{3,2}(D^2|\Sigma_{Y^2}) \triangleq \left\{ (R_1, R_2) : R_1 + R_2 \right. \]
\[ \geq \frac{1}{2} \log \left( 1 - \rho^2 \right) \frac{2}{\rho^2} \cdot \frac{D_1 D_2}{D_1 + D_2}, \]
\[ \beta^* \triangleq 1 + \sqrt{1 + \frac{4 \rho^2}{1 - \rho^2} \cdot D_1 D_2}. \]

The boundary of \( \hat{\mathcal{R}}^{(in)}_2(D^2|\Sigma_{Y^2}) \) consists of one straight line segment defined by the boundary of \( \mathcal{R}^*_{3,2}(D^2|\Sigma_{Y^2}) \) and two curved portions defined by the boundaries of \( \mathcal{R}^*_{1,2}(D_1|\Sigma_{Y^2}) \) and \( \mathcal{R}^*_{2,2}(D_2|\Sigma_{Y^2}) \). Accordingly, the inner bound established by Berger [4] and Tung [5] partially coincides with \( \mathcal{R}_2(D^2|\Sigma_{Y^2}) \) at two curved portions of its boundary. Recently, Wagner et al. [10] have completed the proof of the optimality of \( \hat{\mathcal{R}}^{(in)}_2(D^2|\Sigma_{Y^2}) \) by determining the sum rate part \( \mathcal{R}_{\text{sum,2}}(D^2|\Sigma_{Y^2}) \). Their result is as follows.

**Theorem 8 (Wagner et al. [10]):** For any \( D^2 \in \mathcal{D} \), we have
\[ \mathcal{R}_{\text{sum,2}}(D^2|\Sigma_{Y^2}) = \min_{(R_1, R_2) \in \mathcal{R}^*_{3,2}(D^2|\Sigma_{Y^2})} (R_1 + R_2) \]
\[ = \frac{1}{2} \log \left( 1 - \rho^2 \right) \frac{\beta^*}{2} \cdot \frac{1}{D_1 D_2}. \]

According to Wagner et al. [10], the results of Oohama [15] and [16] play an essential role in deriving their result. Their method for the proof depends heavily on the specific property of \( L = 2 \). It is hard to generalize it to the case of \( L \geq 3 \).
B. New Partial Solutions

In this subsection we state our results on the characterization of \( R_L(\Sigma_d|\Sigma_{Y^L}) \), \( R_L(\Gamma, D^L|\Sigma_{Y^L}) \), and \( R_L(\Gamma, D|\Sigma_{X^L+}) \). Before describing those results we derive an important relation between remote source coding problem and multiterminal rate distortion problem. We first observe that by an elementary computation we have

\[
X^L = \hat{A}Y^L + \tilde{N}^L,
\]

where \( \hat{A} = (\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1})^{-1} \Sigma_{N^L}^{-1} \) and \( \tilde{N}^L \) is a zero mean Gaussian random vector with covariance matrix \( \Sigma_{\tilde{N}^L} = (\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1})^{-1} \). The random vector \( \tilde{N}^L \) is independent of \( Y^L \). Set

\[
B \triangleq \hat{A}^{-1} \Sigma_{N^L} \hat{A}^{-1} = \Sigma_{N^L} + \Sigma_{N^L} \Sigma_{X^L}^{-1} \Sigma_{N^L},
\]

\[
\tilde{b}^L \triangleq \{ [B]_{11}, [B]_{22}, \ldots, [B]_{LL} \},
\]

\[
\tilde{B} = \Gamma B^T \Gamma,
\]

\[
\tilde{b}^L \triangleq \{ [\tilde{B}]_{11}, [\tilde{B}]_{22}, \ldots, [\tilde{B}]_{LL} \}.
\]

From (10), we have the following relation between \( X^L \) and \( Y^L \):

\[
X^L = \hat{A}Y^L + \tilde{N}^L,
\]

where \( \tilde{N}^L \) is a sequence of \( n \) independent copies of \( \tilde{N} \) and is independent of \( Y^L \). Now, we fix \( \{(\varphi^{(1)}_1, \varphi^{(1)}_2, \ldots, \varphi^{(1)}_L, \psi^{(1)}_n)n=1, \ldots, \} \), arbitrary. For each \( n = 1, 2, \ldots \), the estimation \( \hat{X}^L \) of \( X^L \) is given by

\[
\hat{X}^L = \begin{bmatrix}
\psi^{(n)}_1(\varphi^{(n)}_1(Y^L)) \\
\psi^{(n)}_2(\varphi^{(n)}_2(Y^L)) \\
\vdots \\
\psi^{(n)}_L(\varphi^{(n)}_L(Y^L))
\end{bmatrix}.
\]

Using this estimation, we construct an estimation \( \hat{Y}^L \) of \( Y^L \) by

\[
\hat{Y}^L = \hat{A}^{-1} \hat{X}^L,
\]

which is equivalent to

\[
\hat{X}^L = \hat{A} \hat{Y}^L.
\]

From (11) and (13), we have

\[
X^L - \hat{X}^L = \hat{A}(Y^L - \hat{Y}^L) + \tilde{N}^L.
\]

Since \( \hat{Y}^L \) is a function of \( Y^L \), \( Y^L - \hat{Y}^L \) is independent of \( \tilde{N}^L \). Computing \( \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} \) based on (14), we obtain

\[
\frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} = \hat{A} \left( \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} \right) \hat{A} + \Sigma_{\tilde{N}^L}.
\]

From (15), we have

\[
\frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} = \hat{A}^{-1} \left( \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} \right) \hat{A} + \Sigma_{\tilde{N}^L}.
\]

Conversely, we fix \( \{(\varphi^{(n)}_1, \varphi^{(n)}_2, \ldots, \varphi^{(n)}_L, \phi^{(n)}_n)\}_{n=1}^\infty \), arbitrary. For each \( n = 1, 2, \ldots \), using the estimation \( \tilde{Y}^L \) of \( Y^L \) given by

\[
\tilde{Y}^L = \begin{bmatrix}
\phi_1^{(n)}(\varphi^{(n)}_1(Y^L)) \\
\phi_2^{(n)}(\varphi^{(n)}_2(Y^L)) \\
\vdots \\
\phi_L^{(n)}(\varphi^{(n)}_L(Y^L))
\end{bmatrix},
\]

we construct an estimation \( \hat{X}^L \) of \( X^L \) by (13). Then using (1) and (13), we obtain (14). Hence we have the relation (15).

The following proposition provides an important strong connection between remote source coding problem and multiterminal rate distortion problem.

**Proposition 1:** For any positive definite \( \Sigma_d \), we have

\[
R_L(\Sigma_d|\Sigma_{Y^L}) = R_L(\hat{A}(\Sigma_d + B)^T \hat{A}) \Sigma_{X^L+Y^L}.
\]

For any invertible \( \Gamma \) and any \( D^L > 0 \), we have

\[
R_L(\Gamma, D^L|\Sigma_{Y^L}) = R_L(\Gamma \hat{A}^{-1}, D^L + \tilde{b}^L) \Sigma_{X^L+Y^L}.
\]

For any invertible \( \Gamma \) and any \( D > 0 \), we have

\[
R_L(\Gamma, D|\Sigma_{Y^L}) = R_L(\Gamma \hat{A}^{-1}, D + tr[\tilde{B}] \Sigma_{X^L+Y^L}).
\]

**Proof:** Suppose that \( \hat{R}^L \in R_L(\hat{A}(\Sigma_d + B)^T \hat{A}) \Sigma_{X^L+Y^L} \). Then there exists \( \{(\varphi^{(n)}_1, \varphi^{(n)}_2, \ldots, \varphi^{(n)}_L, \psi^{(n)}_n)\}_{n=1}^\infty \) such that

\[
\limsup_{n \to \infty} \hat{R}^L_n \leq \hat{R}_i, \text{ for } i = 1, 2, \ldots, L,
\]

\[
\limsup_{n \to \infty} \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} \preceq \hat{A}(\Sigma_d + B)^T \hat{A}^{-1} = B = \Sigma_d,
\]

which implies that \( \hat{R}^L \in R_L(\hat{A}(\Sigma_d + B)^T \hat{A}) \Sigma_{X^L+Y^L} \). Thus

\[
R_L(\Sigma_d|\Sigma_{Y^L}) \supset R_L(\Sigma_d + B)^T \hat{A} \Sigma_{X^L+Y^L}
\]

is proved. Next we prove the reverse inclusion. Suppose that \( \hat{R}^L \in R_L(\Sigma_d|\Sigma_{Y^L}) \). Then there exists \( \{(\varphi^{(n)}_1, \varphi^{(n)}_2, \ldots, \varphi^{(n)}_L, \phi^{(n)}_n)\}_{n=1}^\infty \) such that

\[
\limsup_{n \to \infty} \hat{R}^L_n \leq \hat{R}_i, \text{ for } i = 1, 2, \ldots, L,
\]

\[
\limsup_{n \to \infty} \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} \preceq \Sigma_d.
\]

Using \( \hat{Y}^L \), we construct an estimation \( \hat{X}^L \) of \( X^L \) by \( \hat{X}^L = \hat{A} \hat{Y}^L \). Then from (15), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} = \limsup_{n \to \infty} \frac{1}{n} \Sigma_{X^L - \hat{X}^L} \preceq \hat{A}^{-1} - B.
\]

Using \( \hat{Y}^L \), we construct an estimation \( \hat{X}^L \) of \( X^L \) by \( \hat{X}^L = \hat{A} \hat{Y}^L \). Then from (15), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \Sigma_{X^L - \hat{X}^L} = \limsup_{n \to \infty} \frac{1}{n} \Sigma_{Y^L - \hat{Y}^L} \preceq \hat{A}^{-1} - B.
\]

Thus

\[
\hat{A}^{-1} \hat{X}^L = \hat{A}^{-1} (\hat{A} \hat{Y}^L) = \hat{A}^{-1} \hat{A} \hat{Y}^L = \hat{Y}^L.
\]

Thus

\[
\hat{X}^L = \hat{A} \hat{Y}^L.
\]

Thus

\[
\hat{X}^L = \hat{A} \hat{Y}^L.
\]
which implies that $R_L \in R_L(\hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$. Thus,

$$R_L(\Sigma_d | \Sigma_{Y,L}) \subseteq R_L(\hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

is proved. Next we prove the second equality. We have the following chain of equalities:

$$R_L(\Gamma, D^L | \Sigma_{Y,L}) = \bigcup_{\Gamma \Sigma_d^t \Gamma \in S_L(D^L)} R_L(\Gamma \hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$= \bigcup_{\Gamma \Sigma_d^t \Gamma \in S_L(D^L)} R_L(\Gamma \hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$= \bigcup_{\Gamma \Sigma_d^t \Gamma \in S_L(D^L)} R_L(\hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$= \bigcup_{\Sigma_d = \hat{A}(\Sigma_d + B)^t \hat{A} \in \Sigma_{X,Y,L}} R_L(\Sigma_d^t \Gamma \Sigma_d | \Sigma_{X,Y,L})$$

Thus the second equality is proved. Finally we prove the third equality. We have the following chain of equalities:

$$R_L(\Gamma, D | \Sigma_{Y,L}) = \bigcup_{\Gamma \Sigma_d^t \Gamma \in S_L(D)} R_L(\Gamma \hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$= \bigcup_{\Gamma \Sigma_d^t \Gamma \in S_L(D)} R_L(\Gamma \hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$= \bigcup_{\Gamma \Sigma_d^t \Gamma \in S_L(D)} R_L(\hat{A}(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$= \bigcup_{\Sigma_d = \hat{A}(\Sigma_d + B)^t \hat{A} \in \Sigma_{X,Y,L}} R_L(\Sigma_d^t \Gamma \Sigma_d | \Sigma_{X,Y,L})$$

Thus the third equality is proved. 

Proposition 1 implies that all results on the rate distortion regions for the remote source coding problems can be converted into those on the multitermination source coding problems. In the following we derive inner and outer bounds of $R_L(\Sigma_d | \Sigma_{Y,L})$, $R_L(\Gamma, D^L | \Sigma_{Y,L})$, and $R_L(\Gamma, D | \Sigma_{Y,L})$ using Proposition 1.

We first derive inner and outer bounds of $R_L(\Sigma_d | \Sigma_{Y,L})$. For $r_i \geq 0$, $i \in \Lambda$, let $V_i(r_i)$, $i \in \Lambda$ be $L$ independent Gaussian random variables with mean 0 and variance $\sigma_i^2/\epsilon_{2r_i}$. Let $\Sigma_{Y,L}(r_i)$ be a covariance matrix for the random vector $V^L(r_i)$. Fix nonnegative vector $r$. For $\theta > 0$ and for $S \subseteq \Lambda$, define

$$\Sigma^{-1}_{V_S(r_S)} \triangleq \Sigma^{-1}_{V_S^{-1}} | r_S = 0$$

$$\tilde{J}_S(\theta, r_S | r_S) \triangleq \frac{1}{2} \log^+ \left[ \frac{|\Sigma_{Y,L} + B| \prod_{i=1}^L e^{2r_i}}{\theta |\Sigma_{Y,L} | \Sigma^{-1}_{V_S(r_S)} + \Sigma^{-1}_{V_S^{-1}}(r_S)} \right]$$

$$\tilde{J}_S(r_S | r_S) \triangleq \frac{1}{2} \log^+ \left[ \frac{|\Sigma_{Y,L} + B| \prod_{i=1}^L e^{2r_i}}{|\Sigma_{Y,L} + B| \prod_{i=1}^L e^{2r_i} + \Sigma^{-1}_{V_S(r_S)}} \right]$$

Set

$$\tilde{A}_L(\Sigma_d) \triangleq \left\{ r^L \geq 0 : \left[ \Sigma_{Y,L}^{-1} + \Sigma^{-1}_{V_S^{-1}}(r_S) \right]^{-1} \subseteq \Sigma_d \right\}$$

Define four regions by

$$R_L(\text{out})(\theta, r^L | \Sigma_{Y,L}) \triangleq \left\{ R_L : \sum_{i \in S} R_i \geq \tilde{J}_S(\theta, r_S | r_S) \right\}$$

for any $S \subseteq \Lambda$. 

$$R_L(\text{out})(\Sigma_d | \Sigma_{Y,L}) \triangleq \bigcup_{r^L \in \tilde{A}_L(\Sigma_d)} R_L(\text{out})(\Sigma_d | \Sigma_{Y,L})$$

$$R_L(\text{opt})(r^L | \Sigma_{Y,L}) \triangleq \left\{ R_L : \sum_{i \in S} R_i \geq J_S(r_S | r_S) \right\}$$

for any $S \subseteq \Lambda$. 

$$R_L(\text{opt})(\Sigma_d | \Sigma_{Y,L}) \triangleq \text{conv} \left\{ \bigcup_{r^L \in \tilde{A}_L(\Sigma_d)} R_L(\text{opt})(r^L | \Sigma_{Y,L}) \right\}$$

The functions and sets defined above have properties shown in the following.

**Property 6:**

a) For any positive definite $\Sigma_d$, $\tilde{G}(\Sigma_d) = \mathcal{G}(\hat{A}(\Sigma_d + B)^t \hat{A})$.

b) For any positive definite $\Sigma_d$, we have

$$R_L(\text{opt})(\Sigma_d | \Sigma_{Y,L}) = R_L(\tilde{A}_L(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

c) For any positive definite $\Sigma_d$ and any $S \subseteq \Lambda$, we have

$$\tilde{J}_S(\Sigma_d + B, r_S | r_S) = \tilde{J}_S(\hat{A}(\Sigma_d + B)^t \hat{A}, r_S | r_S)$$

$$J_S(r_S | r_S) = J_S(r_S | r_S)$$

d) For any positive definite $\Sigma_d$, $\tilde{A}_L(\Sigma_d) = A_L(\hat{A}(\Sigma_d + B)^t \hat{A})$.

e) For any positive definite $\Sigma_d$, we have

$$R_L(\text{out})(\Sigma_d | \Sigma_{Y,L}) = R_L(\text{out})(\tilde{A}_L(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

$$R_L(\text{opt})(\Sigma_d | \Sigma_{Y,L}) = R_L(\text{opt})(\tilde{A}_L(\Sigma_d + B)^t \hat{A} | \Sigma_{X,Y,L})$$

From Theorem 3, Proposition 1 and Property 6 we have the following.

**Theorem 9:** For any positive definite $\Sigma_d$, we have

$$R_L(\text{out})(\Sigma_d | \Sigma_{Y,L}) \subseteq R_L(\text{out})(\Sigma_d | \Sigma_{Y,L})$$

$$R_L(\Sigma_d | \Sigma_{Y,L}) \subseteq R_L(\text{out})(\Sigma_d | \Sigma_{Y,L})$$
Next, we derive inner and outer bounds of $\mathcal{R}_L(\Gamma, D^K | \Sigma_{Y^L})$ and $\mathcal{R}_L(\Gamma, D | \Sigma_{Y^L})$. Set

$$\hat{A}_L(r^L) \triangleq \{ \Lambda_d : \Lambda_d \geq (\Sigma_{Y^L}^{-1} + \Sigma_{V^L(r^L)}^{-1})^{-1} \},$$

$$\hat{\theta}(\Gamma, D^L, r^L) \triangleq \max_{\Lambda_d \in \hat{A}_L(r^L), \Gamma_x^L \in S_L(D^L)} |\Lambda_d + B|,$$

$$\hat{\theta}(\Gamma, D, r^L) \triangleq \max_{\Lambda_d \in \hat{A}_L(r^L), \Gamma_x^L \in S_L(D^L)} |\Lambda_d + B|.$$

Furthermore, set

$$\hat{B}_L(\Gamma, D^L) \triangleq \{ r^L \geq 0 : \Gamma_x^L \in S_L(D^L) \},$$

$$\hat{B}_L(\Gamma, D) \triangleq \{ r^L \geq 0 : \Gamma_x^L \in S_L(D^L) \}. $$

Define four regions by

$$\mathcal{R}_L^{(\text{out})}(\Gamma, D^L | \Sigma_{Y^L}) \triangleq \bigcup_{r^L \in \hat{B}_L(\Gamma, D^L)} \mathcal{R}_L^{(\text{out})}(\hat{\theta}(\Gamma, D^L, r^L), r^L | \Sigma_{Y^L}),$$

$$\mathcal{R}_L^{(\text{in})}(\Gamma, D^L | \Sigma_{Y^L}) \triangleq \mathcal{R}_L^{(\text{out})}(\Gamma, D^L | \Sigma_{Y^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}).$$

Next, we derive a matching condition for $\mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L})$ to coincide with $\mathcal{R}_L^{(\text{in})}(\Gamma, D | \Sigma_{Y^L})$. By Theorems 3 and 10, Proposition 1 and Property 7, we establish the following.

**Theorem 10:** For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\text{in})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\text{in})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}).$$

If we have

$$0 < D \leq (L + 1) \mu_{\text{min}} - \text{tr} \left[ \Gamma_x \Sigma_{X^{-1} X^L} \right],$$

then

$$\mathcal{R}_L^{(\text{in})}(\Gamma, D | \Sigma_{Y^L}) = \mathcal{R}_L^{(\text{out})}(\Gamma, D | \Sigma_{Y^L}).$$

We are particularly interested in the case where $\Gamma$ is the following diagonal matrix:

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 \\ & \ddots \\ & 0 & \gamma_L \end{pmatrix}, \quad \sum_{i=1}^{L} \gamma_i^{-2} = 1. $$

Let $\delta > 0$ be an arbitrary positive constant specified later. We choose $\Sigma_{N^L} = \delta^2 \Gamma^{-2}$. Set $\Sigma_{X^L} \triangleq \Gamma_x \Sigma_{X^L}$. Then, we have $B = \delta I_L + \delta^2 \Sigma_{X^L}^{-1}$. Hence we have

$$\mu_{\text{min}} \geq \delta.$$  

(18)

Let $\lambda_{\text{min}}$ be the minimum eigen value of $\Sigma_{X^L}$. Since $\Sigma_{X^L} \succeq \lambda_{\text{min}} I_L$, we have $\Sigma_{X^L}^{-1} \succeq \lambda_{\text{min}}^{-1} \Gamma^{-2}$. Hence we have

$$\text{tr} \left[ \Sigma_{X^L}^{-1} \right] \leq \lambda_{\text{min}} \text{tr} [\Gamma^{-2}] = \lambda_{\text{min}}^{-1}.$$  

(19)
where the last equality follows from the choice of \( \Gamma \) specified with (17). From (18) and (19), we have
\[
(L + 1) \mu^*_\min - \text{tr}[B] \geq (L + 1) \delta - \delta^2 \Sigma_{X,L}^{-1}
\]
\[
\geq (L + 1) \delta - L \delta - \delta^2 \lambda^\min_{\min}
\]
\[
= \delta - \delta^2 \lambda^\min_{\min}.
\]
Hence if
\[
0 < D \leq \delta - \delta^2 \lambda^\min_{\min},
\]
then the matching condition holds. The right member of (21) takes the maximum value \( \frac{1}{2} \lambda^\min_{\min} \) for \( \delta = \frac{1}{2} \lambda^\min_{\min} \). Summarizing the above argument, we establish the following corollary from Theorem 11.

**Corollary 2:** If the minimum eigen value \( \lambda^\min_{\min} \) of \( \Sigma_{X,L} \) satisfies
\[
0 < D \leq \frac{1}{4} \lambda^\min_{\min},
\]
then for any diagonal matrix \( \Gamma \) specified with (17) we have
\[
\mathcal{R}_L^{(in)}(\Gamma, D|\Sigma_{Y,L}) = \mathcal{R}_L^{(in)}(\Gamma, D|\Sigma_{Y,L})
\]
\[
= \mathcal{R}_L(\Gamma, D|\Sigma_{Y,L}) = \mathcal{R}_L^{(out)}(\Gamma, D|\Sigma_{Y,L}).
\]

**C. Sum Rate Characterization for the Cyclic Shift Invariant Source**

In this subsection we further examine an explicit characterization of \( \mathcal{R}_{\text{sum},L}(D|\Sigma_{Y,L}) \) when the source has a certain symmetrical property. Let
\[
\tau = \left( \begin{array}{cccccc}
1 & 2 & \cdots & i & \cdots & L
\end{array} \right)
\]
be a cyclic shift on \( \Lambda \), that is,
\[
\tau(1) = 2, \tau(2) = 3, \ldots, \tau(L - 1) = L, \tau(L) = 1.
\]
Let \( p_{X_L}(x_L) = p_{X_1 X_2 \cdots X_L}(x_1, x_2, \cdots, x_L) \) be a probability density function of \( X^L \). The source \( X^L \) is said to be cyclic shift invariant if we have
\[
p_{X,L}(x_{\tau(\Lambda)}) = p_{X_1 X_2 \cdots X_L}(x_1, x_2, \cdots, x_L, x_1)
\]
for any \( x_1, x_2, \cdots, x_L \in X^L \). In the following argument we assume that \( X^L \) satisfies the cyclic shift invariant property. We further assume that \( N_i, i \in \Lambda \) are i.i.d. Gaussian random variables with mean 0 and variance \( \epsilon \). Then, the observation \( Y^L = X^L + N^L \) also satisfies the cyclic shift invariant property. We assume that the covariance matrix \( \Sigma_{N,L} \) of \( N^L \) is given by \( \epsilon I_L \). Then \( \tilde{A} \) and \( B \) are given by
\[
\tilde{A} = \left( \epsilon \Sigma_{X,L}^{-1} + I_L \right)^{-1},
\]
\[
B = \epsilon (I_L + \epsilon \Sigma_{X,L}^{-1}).
\]
Fix \( r > 0 \), let \( N_i(r) \), \( i \in \Lambda \) be \( L \) i.i.d. Gaussian random variables with mean 0 and variance \( \epsilon/(1 - e^{-2r}) \). The covariance matrix \( \Sigma_{N,L}(r) \) for the random vector \( N^L(r) \) is given by
\[
\Sigma_{N,L}(r) = \frac{1 - e^{-2r}}{\epsilon} I_L.
\]
Let \( \lambda_i, i \in \Lambda \) be \( L \) eigen values of the matrix \( \Sigma_{X,L} \) and let \( \beta_i = \beta_i(r), i \in \Lambda \) be \( L \) eigen values of the matrix
\[
\tilde{A}^{-1} \left( \frac{1 - e^{-2r}}{\epsilon} I_L \right) \tilde{A}.
\]
Using the eigen values of \( \Sigma_{X,L} \), \( \beta_i(r), i \in \Lambda \) can be written as
\[
\beta_i(r) = \frac{1}{\epsilon} \left[ \frac{\lambda_i}{\lambda_i + \epsilon} - \left( \frac{\lambda_i}{\lambda_i + \epsilon} \right)^2 e^{-2r} \right].
\]
Let \( \xi \) be a nonnegative number that satisfies
\[
\sum_{i=1}^{L} \left\{ [\xi - \beta_i^{-1}] + \beta_i^{-1} \right\} = D + \text{tr}[B].
\]
Define
\[
\hat{\omega}(D, r) \triangleq \prod_{i=1}^{L} \left\{ [\xi - \beta_i^{-1}] + \beta_i^{-1} \right\}.
\]
The function \( \hat{\omega}(D, r) \) has an expression of the so-called water filling solution to the following optimization problem:
\[
\hat{\omega}(D, r) = \max_{\sum_{i=1}^{L} \xi_i \leq D + \text{tr}[B]} \prod_{i=1}^{L} \xi_i.
\]
Set
\[
\tilde{J}(D, r) \triangleq \frac{1}{2} \log \left[ \frac{e^{2r \xi} |\Sigma_{X,L} + B|}{\omega(D, r)} \right],
\]
\[
\zeta(r) \triangleq \text{tr} \left[ \left( \Sigma_{X,L}^{-1} + \frac{1 - e^{-2r}}{\epsilon} I_L \right)^{-1} \tilde{A}^{-1} \right].
\]
By definition we have
\[
\zeta(r) = \sum_{i=1}^{L} \beta_i(r).
\]
Since \( \zeta(r) \) is a monotone decreasing function of \( r \), there exists a unique \( r \) such that \( \zeta(r) = D + \text{tr}[B] \), we denote it by \( r^*(D + \text{tr}[B]) \). Note that
\[
(r, r, \cdots, r) \in B_L(\tilde{A}^{-1}, D + \text{tr}[B])
\]
\[
\iff \zeta(r) \leq D + \text{tr}[B] \iff r \geq r^*(D + \text{tr}[B]),
\]
\[
\hat{\omega}(D, r^*) = |\tilde{A}|^{-2} \left| \Sigma_{X,L}^{-1} + \frac{1 - e^{-2r}}{\epsilon} I_L \right|^{-1}.
\]
Set
\[
R_{\text{sum},L}(D|\Sigma_{Y,L}) \triangleq \min_{r \geq r^*(D + \text{tr}[B])} \tilde{J}(D, r).
\]
Then, we have the following.

**Theorem 12:** Assume that the source \( X \) and its noisy version \( Y = X + N \) are cyclic shift invariant. Then, we have
\[
R_{\text{sum},L}(D|\Sigma_{Y,L}) \geq R_{\text{sum},L}^{(1)}(D|\Sigma_{Y,L}).
\]
Proof of this theorem will be stated in Section V.

Next, we examine a sufficient condition for \( R_{\text{sum},L}^{(1)}(D|\Sigma_{Y,L}) \) to coincide with \( R_{\text{sum},L}(D|\Sigma_{Y,L}) \). It is obvious from the definition of \( \tilde{J}(D, r) \) that when \( e^{-2Lr} \hat{\omega}(D, r) \) is a
monotone decreasing function of \( r \in [r^*(D + \text{tr}[B]), +\infty) \), we have \( R_{\text{sum},L}(D|\Sigma_{Y^L}) = R_{\text{sum},L}(D|\Sigma_{Y^L}) \). Let \( \lambda \) be the maximum eigen value of \( \Sigma_{Y^L} \). Set
\[
\beta_i(r) \triangleq \min_{1 \leq i \leq L} \beta_i(r), \beta_i(r) \triangleq \max_{1 \leq i \leq L} \beta_i(r).
\]
Then we have the following two lemmas.

**Lemma 4:** If
\[
\beta_i(r) - \beta_i(0) \leq e^{2\sigma_i} \cdot \frac{L}{L - 1} \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 (\beta_i(0))^2
\]
or equivalent to
\[
\left( \frac{\lambda_{i1}}{\lambda_{i0} + \epsilon} - \frac{\lambda_{i0}}{\lambda_{i1} + \epsilon} \right)
\leq \frac{L}{L - 1} \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 \left( \frac{\lambda_{i0}}{\lambda_{i1} + \epsilon} \right)^2 \left( \frac{\lambda_{i1}}{\lambda_{i0} + \epsilon} \right)^2
\]
holds for \( r \geq r^*(D + \text{tr}[B]) \), then \( e^{-2\sigma_i}(D, r) \) is a monotone decreasing function of \( r \in [r^*(D + \text{tr}[B]), +\infty) \).

**Lemma 5:** If we have
\[
\frac{\lambda_{i1} - \lambda_{i0}}{\lambda_{i1} + \lambda_{i0}} \leq \frac{4L}{L - 1} \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 \left( \frac{\lambda_{i0}}{\lambda_{i1} + \epsilon} \right) \frac{\lambda_{i1}}{\lambda_{i0} + \epsilon} \left( \frac{\lambda_{i0}}{\lambda_{i1} + \epsilon} \right)^2 \left( \frac{\lambda_{i1}}{\lambda_{i0} + \epsilon} \right)^2
\]
then the sufficient condition (24) in Lemma 4 holds for any nonnegative \( r \).

Proofs of Lemmas 4 and 5 will be given in Section V. If we take \( \epsilon \) sufficiently small in (25) in Lemma 5 then the left hand side of this inequality becomes close to zero. On the other hand, the right hand side of (25) becomes close to \( \frac{4L}{L - 1} \). Hence if we choose \( \epsilon \) sufficiently small, then the inequality (24) in Lemma 4 always holds. Next we suppose that the Gaussian source \( Y^L \) satisfies the cyclic shift invariant property. It is obvious that for arbitrarily prescribed small positive \( \epsilon \), we can always choose a Gaussian random vector \( N^L \) so that \( \Sigma_{N^L} = \epsilon I_L \) and \( Y^L = X^L + N^L \). For the above choice of \( N^L \), the Gaussian remote source \( X^L \) also satisfies the cyclic shift invariant property. Summarizing those arguments we obtain the following theorem.

**Theorem 13:** If \( Y^L \) is cyclic shift invariant, then \( R_{\text{sum},L}(D|\Sigma_{Y^L}) = R_{\text{sum},L}(D|\Sigma_{Y^L}) \). Furthermore, the curve \( R = R_{\text{sum},L}(D|\Sigma_{Y^L}) \) has the following parametric form:
\[
R = \frac{1}{2} \log \left[ \Sigma_{Y^L} + B \exp\{2\sigma_i(r) \prod_{i=1}^L \beta_i(r) \} \right],
\]
\[
D = \sum_{i=1}^L \beta_i(r) - \text{tr}[B].
\]

**V. PROOFS OF THE RESULTS**

**A. Derivation of the Outer Bounds**

In this subsection we prove the results on outer bounds of the rate distortion region. We first state two important lemmas which are mathematical cores of the converse coding theorem. For \( i = 1, 2, \cdots, L \), set
\[
W_i = \phi_i(Y_i), r_i^{(n)} = \frac{1}{n} I(Y_i; W_i|X^K).
\]
For \( S \subseteq \Lambda \), let \( Q_S \) be a unitary matrix which transforms \( X^K \) into \( Z^K = QX^K \). For
\[
X^K = (X^K(1), \cdots, X^K(n))
\]
we set
\[
Z^K = QX^K = (QX^K(1), \cdots, QX^K(n)).
\]
Furthermore, for \( \hat{X}^K = (\hat{X}^K(1), \cdots, \hat{X}^K(n)) \), we set
\[
\hat{Z}^K = Q\hat{X}^K = (Q\hat{X}^K(1), \cdots, Q\hat{X}^K(n)).
\]
We have the following two lemmas.

**Lemma 6:** For any \( i = 1, 2, \cdots, K \), we have
\[
h(Z_i | Z_{[i]}^K W_L) \leq h(Z_i - Z_{[i]} | Z_{[i]} - \hat{Z}_{[i]}^K) \leq \frac{n}{2} \log \left( 2\pi e \left[ \left( \frac{1}{n} \Sigma^{-1}_{X^K} - \Sigma^{-1}_{\hat{X}^K} \right) + Q^{-1}Q \right]_{ii} \right).
\]
where \( h(\cdot) \) stands for the differential entropy.

**Lemma 7:** For any \( i = 1, 2, \cdots, K \), we have
\[
h(Z_i | Z_{[i]}^K W_L) \leq \frac{n}{2} \log \left( 2\pi e \left[ \left( \frac{1}{n} \Sigma^{-1}_{X^K} - \Sigma^{-1}_{\hat{X}^K} \right) + Q^{-1}Q \right]_{ii} \right).
\]
Proofs of Lemmas 6 and 7 will be stated in Appendices A and B, respectively. The following corollary immediately follows from Lemmas 6 and 7.

**Corollary 3:** For any \( \Sigma_{X^K Y^L} \) and any \( (\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi_1^{(n)}, \psi_2^{(n)}) \), we have
\[
1/\frac{n}{2} \Sigma^{-1}_{X^K} - \Sigma^{-1}_{\hat{X}^K} \leq 1/\frac{n}{2} A \Sigma^{-1}_{N_{X}(r_{(n)})} A^T.
\]
From Lemma 7 we obtain the following corollary.

**Corollary 4:** For any \( S \subseteq \Lambda \), we have
\[
I(X^K; W_S) \leq \frac{n}{2} \log \left[ 1 + 1/\Sigma^{-1}_{X^K} + 1/\Sigma^{-1}_{N_{X}(r_{(n)})} A^T \right].
\]

**Proof:** For each \( i \in \Lambda - S \), we choose \( W_i \) so that it takes a constant value. In this case we have \( r_i^{(n)} = 0 \) for \( i \in \Lambda - S \). Then by Lemma 7 for any \( i = 1, 2, \cdots, K \), we have
\[
h(Z_i | Z_{[i]}^K W_S) \geq \frac{n}{2} \log \left( 2\pi e \left[ \left( \frac{1}{n} \Sigma^{-1}_{X^K} + A \Sigma^{-1}_{N_{X}(r_{(n)})} A^T \right) + Q^{-1}Q \right]_{ii} \right). \]
We choose a unitary matrix \( Q \) so that
\[
Q^{-1} \left( \frac{1}{n} \Sigma^{-1}_{X^K} + A \Sigma^{-1}_{N_{X}(r_{(n)})} A^T \right) Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ & \ddots \end{pmatrix}
\]
becomes the following diagonal matrix:
\[
Q^{-1} \left( \frac{1}{n} \Sigma^{-1}_{X^K} + A \Sigma^{-1}_{N_{X}(r_{(n)})} A^T \right) Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ & \ddots \end{pmatrix}
\]
where \( \lambda_{i1} \leq \lambda_{i0} \leq \lambda_{i1} + \epsilon \), \( \lambda_{i0} \leq \lambda_{i1} \leq \lambda_{i0} + \epsilon \), and \( \lambda_{i0} \leq \lambda_{i1} \leq \lambda_{i0} + \epsilon \) for any nonnegative \( r \).
Then we have the following chain of inequalities:

\[ I(X^K; W_S) = h(X^K) - h(X^K|W_S) \]

\[ \geq h(X^K) - h(Z^K|W_S) \]

\[ \leq h(X^K) - \sum_{i=1}^{K} h(Z_i|Z^K_i|W_S) \]

\[ \leq \left( \frac{n}{2} \right) \log \left( \frac{1}{2\pi e} \right) |\Sigma_{X^K}| \]

\[ + \sum_{i=1}^{K} \frac{n}{2} \log \left( \frac{1}{2\pi e} \right) Q\left( \Sigma^{-1}_{X^K} + \frac{4}{n} A \Sigma^{-1}_{N_{S}(r_{S})} A \right)^{1/2} \]

\[ = \left( \frac{n}{2} \right) \log |\Sigma_{X^K}| + \sum_{i=1}^{K} \frac{n}{2} \log \lambda_i \]

\[ = \left( \frac{n}{2} \right) \log |\Sigma_{X^K}| + \frac{n}{2} \log \left| \Sigma^{-1}_{X^K} + \frac{4}{n} A \Sigma^{-1}_{N_{S}(r_{S})} A \right| \]

\[ \leq \left( \frac{n}{2} \right) \log \left( \frac{1}{2\pi e} \right) |\Sigma_{X^K}| \]

where steps (a), (b) and (c) follow from \([30]\). We estimate a lower bound of \(I(X^K; W_S|W_{S'})\). Observe that

\[ I(X^K; W_S|W_{S'}) = I(X^K; W_{S'}) - I(X^K; W_S|W_{S'}) \quad (34) \]

Since an upper bound of \(I(X^K; W_{S'})\) is derived by Corollary \([4]\) it suffices to estimate a lower bound of \(I(X^K; W_{S''})\). We have the following chain of inequalities:

\[ I(X^K; W_{S''}) = h(X^K) - h(X^K|W_{S''}) \]

\[ \geq h(X^K) - h(X^K|\hat{X}^{K}) \]

\[ \geq h(X^K) - h(X^K - \hat{X}^{K}) \]

\[ \geq \left( \frac{n}{2} \right) \log \left( \frac{1}{2\pi e} \right) |\Sigma_{X^K - \hat{X}^{K}}| \]

Combining \((34), (35),\) and Corollary \([4]\) we have

\[ I(X^K; W_S|W_{S'}) + n \sum_{i\in S} r_{i}^{(n)} \]

\[ \geq \left( \frac{n}{2} \right) \log \left( \frac{1}{\prod_{i\in S} e^{r_{i}^{(n)}} |\Sigma_{X^K}|} \right) \]

\[ \geq \left( \frac{n}{2} \right) \log \left( \frac{1}{\prod_{i\in S} e^{r_{i}^{(n)}} \Sigma^{-1}_{X^K - \hat{X}^{K}} A |\hat{X}^{K}} \right) \]

\[ \geq \left( \frac{n}{2} \right) \log \left( \frac{1}{\prod_{i\in S} e^{r_{i}^{(n)}} \Sigma^{-1}_{X^K - \hat{X}^{K}} A |\hat{X}^{K}} \right) \]

Note here that \(I(X^K; W_S|W_{S'}) + n \sum_{i\in S} r_{i}^{(n)}\) is nonnegative. Hence, we have

\[ I(X^K; W_{S'|S}) + n \sum_{i\in S} r_{i}^{(n)} \]

\[ \geq \left( \frac{n}{2} \right) \log \left( \frac{1}{\prod_{i\in S} e^{r_{i}^{(n)}} |\Sigma_{X^K}|} \right) \]

\[ \geq \left( \frac{n}{2} \right) \log \left( \frac{1}{\prod_{i\in S} e^{r_{i}^{(n)}} \Sigma^{-1}_{X^K - \hat{X}^{K}} A |\hat{X}^{K}} \right) \]

Combining \((33)\) and \((36),\) we obtain

\[ \sum_{i\in S} R_{i}^{(n)} \geq \left( \frac{1}{n} \right) \sum_{i\in S} r_{i}^{(n)} \]

for \(S \subseteq \Lambda.\) On the other hand, by Corollary \([3]\) we have

\[ \Sigma^{-1}_{X^K} + \frac{4}{n} A \Sigma^{-1}_{N_{S}(r_{S})} A \geq \frac{1}{n} \Sigma^{-1}_{X^K} \]

By letting \(n \to \infty\) in \((37)\) and \((38)\) and taking \((31)\) into account, we have for any \(S \subseteq \Lambda\)

\[ \sum_{i\in S} R_{i} \geq \left( \frac{1}{n} \right) \sum_{i\in S} r_{i}^{(n)} \]

and

\[ \Sigma^{-1}_{X^K} + \frac{4}{n} A \Sigma^{-1}_{N_{S}(r_{S})} A \geq \frac{1}{n} \Sigma^{-1}_{X^K} \]

From \((39)\) and \((40),\) \(R_{S}|\Sigma_{X^K Y} \subseteq R_{S}^{(out)}(\Sigma_{X^K}, Y)\) is concluded.
Proof of Theorem 4. We choose a unitary matrix $Q$ so that
\[
Q\Gamma^{-1} \left( \Sigma_{X,K}^{-1} + t^A \Sigma_{N,L}^{-1}(r^L) A \right)^{-1} \Gamma^{-1} t^A \Gamma^{-1} Q
= \begin{bmatrix}
\alpha_1 & 0 \\
\alpha_2 & \ddots \\
0 & \ddots & \alpha_K
\end{bmatrix}.
\]
Then we have
\[
Q\Gamma \left( \Sigma_{X,K}^{-1} + t^A \Sigma_{N,L}^{-1}(r^L) A \right)^{-1} \Gamma^{1/2} Q
= \begin{bmatrix}
\alpha_1^{-1} & 0 \\
\alpha_2^{-1} & \ddots \\
0 & \ddots & \alpha_K^{-1}
\end{bmatrix}.
\]

For $\Sigma_d \in A(r^{L})$, set
\[
\tilde{\Sigma}_d \triangleq Q\Gamma \Sigma_d \Gamma^{1/2} Q, \quad \xi_i \triangleq \left[ \tilde{\Sigma}_d \right]_{ii}.
\]
Since
\[
\Gamma \Sigma_d \Gamma \geq \Gamma \left( \Sigma_{X,K}^{-1} + t^A \Sigma_{N,L}^{-1}(r^L) A \right)^{-1} \Gamma
\]
and $\text{tr}[\Gamma \Sigma_d \Gamma] \leq D$, we have
\[
\xi_i \geq \alpha_i^{-1}, \quad \text{for } i = 1, 2, \cdots, K,
\]
\[
\sum_{i=1}^{K} \xi_i = \text{tr} \left[ \tilde{\Sigma}_d \right] = \text{tr} \left[ \Gamma \Sigma_d \Gamma \right] \leq D.
\]
Furthermore, by Hadamard's inequality we have
\[
\left| \Sigma_d \right| \leq |\Gamma|^{-2} \left| \tilde{\Sigma}_d \right| \leq \left| \Gamma \right|^{-2} \prod_{i=1}^{K} \xi_i.
\]
Combining (42) and (43), we obtain
\[
\theta(\Gamma, D, r^{L}) = \max_{\Sigma_d : \Sigma_d \in A(r^{L}), \quad \text{tr}[\Gamma \Sigma_d \Gamma] \leq D} |\Sigma_d|
\leq |\Gamma|^{-2} \max_{\xi_i \geq 0, i = 1, 2, \cdots, K, \sum_{i=1}^{K} \xi_i \leq D} \prod_{i=1}^{K} \xi_i = \omega(\Gamma, D, r^{L}).
\]
The equality holds when $\Sigma_d$ is a diagonal matrix.

Proof of Theorem 3. Assume that $(R_1, R_2, \cdots, R_L) \in R_L(D\Sigma_{Y,L})$. Then, there exists a sequence $\{(\varphi_{11}^{(n)}, \varphi_{12}^{(n)}, \cdots, \varphi_{L}^{(n)}, \phi(n))_{n=1}^{\infty}\}$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} \text{tr}[\Sigma_{Y\lambda}(\phi(n))_{n=1}^{\infty} \Sigma_{X \lambda} = \Sigma_{d} \text{, tr}[\Sigma_{d}] \leq D}
\]
for some $\Sigma_{d}$.

For each $l = 0, 1, \cdots, L - 1$, we use $(\varphi_{r(L,1)}^{(n)}, \varphi_{r(L,2)}^{(n)}, \cdots, \varphi_{r(L,L)}^{(n)})$ for the encoding of $(Y_1, Y_2, \cdots, Y_L)$. For $i \in \Lambda$ and for $l = 0, 1, \cdots, L - 1$, set
\[
W_{i,l} \triangleq \varphi_{r(i)}(Y_{i,l}), \quad Y_{i,l} \triangleq \varphi_{r(i)}(\varphi_{r(i)}(Y_{i,l})),
\]
\[
r_{i,l}^{(n)} \triangleq \frac{1}{n} I(Y_{i,l}; W_{i,l} | X_{i,l}^{L}).
\]
In particular,
\[
r_{0,l}^{(n)} = r_{i}^{(n)} = \frac{1}{n} I(Y_{i,l}; W_{i,l} | X_{i,l}^{L}), \quad \text{for } i \in \Lambda.
\]
Furthermore, set
\[
r_{r(A)} \triangleq \left( r_{r(A)}^{(n)} = r_{r(A)}^{(n)} = \cdots, r_{r(L)}^{(n)} \right), \quad \text{for } l = 0, 1, \cdots, L - 1,
\]
\[
r_{r(n)} \triangleq \frac{1}{L} \sum_{i=1}^{L} r_{i,l}^{(n)}.
\]
By the cyclic shift invariant property of $X_{\lambda}$ and $Y_{\lambda}$, we have for $l = 0, 1, \cdots, L - 1$,
\[
\frac{1}{L} \sum_{i=1}^{L} r_{i,l}^{(n)} = \frac{1}{L} \sum_{i=1}^{L} r_{i,l}^{(n)} = r_{r(n)}.
\]
For $\Sigma_{d} = [d_{ij}]$, set
\[
r_{r}^{(i)}(\Sigma_{d}) \triangleq [d_{r(i),r(j)}], \quad \Sigma_{d} \triangleq \frac{1}{L} \sum_{i=0}^{L-1} r_{r}^{(i)}(\Sigma_{d}).
\]
Then, we have
\[
\limsup_{n \to \infty} \frac{1}{L} \sum_{i=0}^{L-1} \frac{1}{n} \Sigma_{Y_{\lambda}} - \Sigma_{Y_{\lambda}}(\Sigma_{d})
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{L} \sum_{i=0}^{L-1} \frac{1}{n} \Sigma_{Y_{\lambda}} - \Sigma_{Y_{\lambda}}(\Sigma_{d})
\]
\[
\leq \frac{1}{L} \sum_{i=0}^{L-1} \tau_{r}^{(i)}(\Sigma_{d}) = \Sigma_{d}.
\]

Step (a) follows from the cyclic shift invariant property of $Y_{\lambda}$. Step (b) follows from (44). Step (c) follows from the definition of $\Sigma_{d}$. From $Y_{\lambda}$, we construct an estimation $X_{\lambda}$ of $X_{\lambda}$ by $X_{\lambda} = \hat{A} Y_{\lambda}$. Then for $l = 0, 1, \cdots, L - 1$, we have the following.
\[
\Sigma_{X_{\lambda}}^{-1} + \Sigma_{N_{r(A)}}^{-1}(r_{r(A)}^{(n)}) \overset{(a)}{=} \Sigma_{X_{\lambda}}^{-1} + \Sigma_{N_{r(A)}}^{-1}(r_{r(A)}^{(n)})
\]
\[
\overset{(b)}{=} \frac{1}{n} \Sigma_{X_{\lambda}}^{-1} - X_{r(A)} + \frac{1}{n} \Sigma_{X_{\lambda}}^{-1} - X_{r(A)}
\]
\[
= \left[ \hat{A} \left( \frac{1}{n} \Sigma_{Y_{\lambda}} - Y_{\lambda} \right) \right]^{-1} + \Sigma_{X_{\lambda}(Y_{\lambda})}^{-1}.
\]
From (47), we have

\[
\frac{1}{L} \sum_{l=0}^{L-1} \left[ \sum_{X_A} - \sum_{\pi_{l}}^{r_{l}(\pi_{l})} \right] 
\geq \frac{1}{L} \sum_{l=0}^{L-1} \left[ \hat{A} \left( \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) \right)^{t} \hat{A} + \Sigma_{X_A} |Y_{\lambda} \right]^{-1} 
\geq \left[ \hat{A} \left( \frac{L}{l} \sum_{l=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) \right)^{t} \hat{A} + \Sigma_{X_A} |Y_{\lambda} \right]^{-1} 
\geq \left[ \hat{A} \left( \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) + B \right)^{t} \hat{A} \right]^{-1} .
\]

Step (a) follows from that \((\hat{A}^{\dagger} + \Sigma_{X_A} |Y_{\lambda} \right)\) is convex with respect to \(\Sigma\). On the other hand, we have

\[
\sum_{i=0}^{L-1} (\Sigma_{X_A} + \Sigma_{\pi_{l}}^{r_{l}(\pi_{l})}) = \Sigma_{X_A} + \frac{1}{L} \left( \sum_{i=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) \right) I_{L} 
= (\Sigma_{X_A} - \frac{1}{L} \left( \sum_{i=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) \right)) I_{L} .
\]

Step (a) follows from that \(- e^{-2a} = \text{concave function of } a\). Combining (48) and (49), we obtain

\[
\Sigma_{X_A}^{n} + \left( \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) \right) I_{L} 
\geq \left[ \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) \right] \left[ \hat{A} \left( \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) + B \right)^{t} \hat{A} \right]^{-1} .
\]

Next we derive a lower bound of the sum rate part. For each \(l = 0, 1, \ldots, L - 1\), we have the following chain of inequalities.

\[
\sum_{i=0}^{L-1} n R_{l}(\pi_{l}) \geq \sum_{i=0}^{L-1} \log M_{i} \geq \sum_{i=0}^{L-1} H(W_{l}, l) \geq H(W_{l}(\pi_{l})) 
= I(X_{A}; W_{l}(\pi_{l})) + H(W_{l}(\pi_{l}) | X_{A}) 
\geq \left[ \hat{A} \left( \frac{1}{L} \sum_{l=0}^{L-1} \frac{1}{n} \sum Y_{X_A} - Y_{l}(\pi_{l}) + B \right)^{t} \hat{A} \right]^{-1} .
\]

Set

\[
\hat{\xi}_{d} \triangleq Q \Sigma_{d}^{t} Q, \hat{B}_{d} \triangleq Q B^{t} Q, \xi_{i} \triangleq \left[ \Sigma_{d} + \hat{B} \right]_{ii} .
\]

From (54) and (55) we have

\[
\xi_{i} \geq \beta_{i}^{-1}(r), i \in \Lambda, 
\sum_{i=1}^{l} \xi_{i} = \log \left( \hat{\Sigma}_{d} + \hat{B} \right) = \log \left( \Sigma_{d} + B \right) \leq D + \log |B| .
\]

Furthermore, by Hadamard’s inequality we have

\[
|\Sigma_{d} + B| = |\Sigma_{d} + B| \leq L \prod_{i=1}^{l} \xi_{i} .
\]
Combining (56) and (58), we obtain
\[
|\Sigma_d + B| \leq \max_{\xi_i \in \Lambda} \xi_i \sum_{i=1}^L \xi_i = \hat{\omega}(D, r) . \tag{59}
\]

Hence, from (53), (57), and (59) we have
\[
\sum_{i=1}^L R_i \geq \min_{r > r^R(D, r)} \frac{1}{2} \log \left[ \frac{e^{Lr} \| \Sigma_Y + B \|}{\hat{\omega}(D, r)} \right] = \min_{r > r^R(D, r)} \frac{1}{2} \log \left( \frac{\Sigma_Y(\hat{\Sigma}_1) + \hat{\Sigma}_1^{-1}}{\omega(D, r)} \right) = R_{\text{sum}, L}(D|\Sigma_{Y^L}) ,
\]
completing the proof.

\[ \blacksquare \]

**C. Proofs of the Results on Matching Conditions**

We first observe that the condition
\[
\text{tr} \left[ \Gamma \left( \Sigma_{X^L}^{-1} + A \Sigma_{N^L}^{-1} \right) \right] \leq D
\]
is equivalent to
\[
\sum_{j=1}^K \frac{1}{\alpha_j(r^L)} \leq D . \tag{64}
\]

**Proof of Lemma 3** Let \( \hat{\Lambda} = \{1, 2, \ldots, K\} \) and let \( S \subseteq \hat{\Lambda} \) be a set of integers that satisfies \( \alpha_i^{-1} \geq \xi \) in the definition of \( \theta(\Gamma, D, u^L) \). Then, \( \theta(\Gamma, D, u^L) \) is computed as
\[
\theta(\Gamma, D, u^L) = \frac{1}{(K - |S|)^{-K - |S|}} \left( \prod_{i \in S} \frac{1}{\alpha_i} \right) \left( D - \sum_{i \in S} \frac{1}{\alpha_i} \right) .
\]

Fix \( i \in \Lambda \) arbitrary. For simplicity of notation we set \( \chi_i = ||\hat{\alpha}_i||^2 \frac{1}{\sigma_{N_i}} + \eta_i(u^L_i) \) and set
\[
\Psi \triangleq \text{log} \frac{1}{\sigma_{N_i} - u_i} - \log \theta(\Gamma, D, u^L) .
\]

Computing the partial derivative of \( \Psi \) by \( u_i \), we obtain
\[
\frac{\partial \Psi}{\partial u_i} = \sum_{j \in S} \left( \frac{\partial \alpha_j}{\partial u_i} \right) \left[ \frac{1}{\alpha_j} - \frac{K - |S|}{D - \sum_{j \in S} \frac{1}{\alpha_j}} \right] + \frac{1}{\sigma_{N_i} - u_i} .
\]

From Lemma 2 and (65), we obtain
\[
\frac{\partial \Psi}{\partial u_i} \geq \sum_{j \in S} \left( \frac{\partial \alpha_j}{\partial u_i} \right) \left[ \frac{1}{\alpha_j} - \frac{K - |S|}{D - \sum_{j \in S} \frac{1}{\alpha_j}} \right] + \frac{1}{\chi_i - \alpha_{\min}} .
\]

To examine signs of contents of the above summation we set
\[
\Phi_i \triangleq \left\{ \left. \begin{array}{ll}
D - \sum_{j \in S} \frac{1}{\alpha_j} - K - |S| \alpha_j \right| \\
\chi_i - \alpha_{\min}
\end{array} \right\} \left( \chi_i - \alpha_{\min} \right)
\]
and
\[
\Psi_i \triangleq \left\{ \left. \begin{array}{ll}
D - \sum_{j \in S} \frac{1}{\alpha_j} \right| \\
\chi_i - \alpha_{\min}
\end{array} \right\} \left( \chi_i - \alpha_{\min} \right)
\]
If \(|S| = K\), \(\Phi_j \geq 0, j \in \Lambda\) is obvious. We hereafter assume \(|S| \leq K - 1\). Computing \(\Phi_j\), we obtain
\[
\Phi_j = \chi_i \left( D - \sum_{j \in S} \frac{1}{\alpha_j} - \frac{K - |S|}{\alpha_j} \cdot (\chi_i - \alpha_{\min}) \right) + (\alpha_j - \alpha_{\min}) \left( D - \sum_{j \in S} \frac{1}{\alpha_j} \right) \geq \chi_i \left( D - \sum_{j \in S} \frac{1}{\alpha_j} - \frac{K - |S|}{\alpha_j} \cdot (\chi_i - \alpha_{\min}) \right).
\]

Thus, if we have \(D\alpha_{\min} - 1 \leq K\) or equivalent to \(D \leq (K + 1)/\alpha_{\max}\), we have \(\Phi_j \geq 0\) for \(j \in S\).

\[\Phi_j = \chi_i \left( D - \sum_{j \in S} \frac{1}{\alpha_j} - \frac{K - |S|}{\alpha_j} \cdot (\chi_i - \alpha_{\min}) \right) + (\alpha_j - \alpha_{\min}) \left( D - \sum_{j \in S} \frac{1}{\alpha_j} \right) \geq \chi_i \left( D - \sum_{j \in S} \frac{1}{\alpha_j} - \frac{K - |S|}{\alpha_j} \cdot (\chi_i - \alpha_{\min}) \right) \geq \chi_i (K - |S|) \left( \frac{1}{\alpha_{\max}} - \frac{1}{\alpha_{\min}} + \frac{1}{\chi_i} \right).
\]

Step (a) follows from the inequality (64). From (66), we can see that if
\[
\frac{1}{\alpha_{\min}(r_L)} - \frac{1}{\alpha_{\max}(r_L)} \leq 1
\]
for \(i \in \Lambda\), then, \(\Phi_j \geq 0\) for \(j \in S\).

**Proof of Theorem 5** By (64), we have
\[
\frac{1}{\alpha_{\min}(r_L)} \leq D - \frac{1 - K}{\alpha_{\max}(r_L)} = \frac{1}{\alpha_{\max}(r_L)} + D - \frac{K}{\alpha_{\max}(r_L)}.
\]

Hence, if
\[
D - \frac{K}{\alpha_{\max}(r_L)} \leq \frac{1}{\chi_i},
\]
or equivalent to
\[
\left( D - \frac{1}{\chi_i} \right) \alpha_{\max}(r_L) \leq K \quad (67)
\]
holds for \(r_L \in B_L(\Gamma, D)\) and \(i \in \Lambda\), the condition on \(\alpha_{\min}\) and \(\alpha_{\max}\) in Lemma 3 holds. By Lemma 2, we have
\[
\alpha_{\max}(r_L) \leq \alpha_{\max}^* \quad (68)
\]
It can be seen from (67) and (68) that
\[
\left( D - \frac{1}{\chi_i} \right) \alpha_{\max}^* \leq K \quad (69)
\]
is a sufficient condition for (67) to hold. By Lemma 2 we have
\[
\chi_i = ||\alpha_i||^2 \frac{1}{\sigma_{N_i}} + \eta_i(u_i^L) \leq \lim_{u_i \to \sigma_{N_i}} \alpha_{\max}(u_L) \leq \alpha_{\max}^* \quad \text{for} \ i \in \Lambda.
\]
From which we have
\[
\left( D - \frac{1}{\chi_i} \right) \alpha_{\max}^* \leq D\alpha_{\max}^* - 1.
\]

Thus, if we have \(D\alpha_{\max}^* - 1 \leq K\) or equivalent to \(D \leq (K + 1)/\alpha_{\max}^*\), we have (69).

**Proof of Lemma 4** We first derive expression of \(\hat{\omega}(D, r)\) using \(\beta_i = \beta_i(r)\), \(i \in \Lambda\). Let \(S\) be a set of integers that satisfies \(\beta_i^{-1} \geq \xi\) in the definition of \(\hat{\omega}(D, r)\). Then \(\hat{\omega}(D, r)\) is computed as
\[
\hat{\omega}(D, r) = \frac{1}{(L - |S|)^{1 - \epsilon}} \left( \prod_{k \in S} \frac{1}{\beta_k} \right) \left( D - \sum_{k \in S} \frac{1}{\beta_k} \right)^{L - |S|}.
\]

Fix \(i \in \Lambda\) arbitrary and set
\[
\Psi \triangleq Lr - \log \hat{\omega}(D, r).
\]
Computing the derivative of \(\Psi\) by \(r\), we obtain
\[
\frac{d\Psi}{dr} = \sum_{k \in S} \frac{e^{-2r}}{\epsilon} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \left[ 1 - \frac{L - |S|}{\beta_k} \right] + L = \sum_{k \in S} \left\{ \frac{e^{-2r}}{\epsilon} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \left[ 1 - \frac{L - |S|}{\beta_k} \right] + L \right\}.
\]

To examine signs of contents of the above summation we set
\[
\Phi_k \triangleq \frac{e^{-2r}|S|}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \left( D - \sum_{k \in S} \frac{1}{\beta_k} - \frac{L - |S|}{\beta_k} \right) + \beta_k \left( D - \sum_{k \in S} \frac{1}{\beta_k} \right).
\]

If \(|S| = L\), \(\Phi_k \geq 0\). If \(k \in \Lambda\) is obvious. We hereafter assume \(|S| \leq L - 1\). Computing \(\Phi_k\), we obtain
\[
\Phi_k = \left\{ \frac{e^{-2r}|S|}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 + \beta_k \right\} \left( D - \sum_{k \in S} \frac{1}{\beta_k} \right) - \frac{e^{-2r}|S|}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \frac{L - |S|}{\beta_k} \geq \left\{ \frac{e^{-2r}|S|}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 + \beta_k \right\} \left( \sum_{k \in \Lambda - S} \frac{1}{\beta_k} \right) - \frac{e^{-2r}|S|}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \frac{L - |S|}{\beta_k} \geq \left\{ \frac{e^{-2r}|S|}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 + \beta_k \right\} \left( \sum_{k \in \Lambda - S} \frac{1}{\beta_k} \right). \quad (70)
\]

Step (a) follows from
\[
D - \sum_{k=1}^{L} \frac{1}{\beta_k} \geq 0 \Leftrightarrow D - \sum_{k \in S} \frac{1}{\beta_k} \geq \sum_{k \in \Lambda - S} \frac{1}{\beta_k}.
\]
From (70), we can see that if

\[
\left\{ \frac{e^{-2r|S|}}{\epsilon L} \frac{\lambda_k}{\lambda_k + \epsilon} + \beta_{i_0} \right\} \frac{1}{\beta_{i_1}} - \frac{e^{-2r|S|}}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \cdot \frac{1}{\beta_{i_1}} = \frac{\beta_{i_0}}{\beta_{i_1}} - \frac{e^{-2r|S|}}{\epsilon L} \left( \frac{\lambda_k}{\lambda_k + \epsilon} \right)^2 \left( \frac{1}{\beta_{i_0}} - \frac{1}{\beta_{i_1}} \right) \geq 0, \quad (71)
\]

then \( \Phi_k \geq 0 \) for \( k \in \Lambda \). The inequality (71) is equivalent to

\[
\frac{1}{\beta_{i_0}} - \frac{1}{\beta_{i_1}} \leq \left( \frac{\lambda_k + \epsilon}{\lambda_k} \right)^2 \frac{e^{2rL} \beta_{i_0}}{|S| \beta_{i_1}}.
\]

Hence

\[
\frac{1}{\beta_{i_0}} - \frac{1}{\beta_{i_1}} \leq \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 \frac{e^{2rL} \beta_{i_0}}{L - 1 \beta_{i_1}} \quad (72)
\]

is a sufficient condition for \( \Phi_k \geq 0, \; k \in \Lambda \). The condition (72) is equivalent to

\[
\beta_{i_1}(r) - \beta_{i_0}(r) \leq \frac{e^{2rL}}{L - 1} \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 \left( \beta_{i_0}(r) \right)^2,
\]

completing the proof.

**Proof of Lemma 4**

Set

\[
F(r) \triangleq \left[ e^{2r} - \left( \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} + \frac{\lambda_{i_1}}{\lambda_{i_1} + \epsilon} \right) \right]^{-1} \left( e^{2r} - \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} \right)^2.
\]

Then, the sufficient condition stated in Lemma 4 is equivalent to

\[
\frac{\lambda_{i_2}}{\lambda_{i_1} + \epsilon} - \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} \leq \frac{L}{L - 1} \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 \left( \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} \right)^2 \cdot F(r).
\]

(73)

To derive an explicit sufficient condition for (73) to hold, we estimate a lower bound of \( F(r) \). Set

\[
T(r) \triangleq e^{2r} - \left( \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} + \frac{\lambda_{i_1}}{\lambda_{i_1} + \epsilon} \right), \quad P \triangleq \frac{\lambda_{i_2}}{\lambda_{i_1} + \epsilon}.
\]

Then

\[
F(r) = [T(r)]^{-1} [T(r) + P]^2 = T(r) + \frac{P^2}{T(r)} + 2P \geq 4P = \frac{4\lambda_{i_1}}{\lambda_{i_1} + \epsilon}.
\]

Hence,

\[
\frac{\lambda_{i_1}}{\lambda_{i_1} + \epsilon} - \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} \leq \frac{4L}{L - 1} \left( \frac{\lambda_{\max} + \epsilon}{\lambda_{\max}} \right)^2 \left( \frac{\lambda_{i_0}}{\lambda_{i_0} + \epsilon} \right)^2 \left( \frac{\lambda_{i_1}}{\lambda_{i_1} + \epsilon} \right)
\]

is a sufficient condition for (73) to hold.

**APPENDIX**

**A. Proof of Lemma 6**

In this appendix we prove Lemma 6. To prove this lemma we need some preparations. For \( i \in \Lambda \), set

\[
F_i(\Sigma | \{Q\}) \triangleq \sup_{p_{\hat{X}_K | X^K } : \Sigma_{X^K, X^K} \geq \Sigma} h(Z_i - \hat{Z}_i | Z_{[i]} | \hat{Z}_K^{\dagger}, Z_{[i]}^{\dagger}).
\]

To compute \( F_i(\Sigma | \{Q\}) \), define two random variables by

\[
\tilde{X}_K \triangleq X^K - \hat{X}_K, \quad \hat{Z}_K \triangleq Z_K - \hat{Z}_K.
\]

Note that by definition we have \( \tilde{Z}_K = Q \tilde{X}_K \). Let \( p_{\hat{X}_K | X^K } (\tilde{X}_K, \hat{Z}_K) \) be a density function of \( (\tilde{X}_K, \hat{Z}_K) \). Let \( q_{\tilde{Z}_K | \hat{Z}_K } (\tilde{Z}_K, \hat{Z}_K) \) be a density function of \( (\tilde{Z}_K, \hat{Z}_K) \) induced by the unitary matrix \( Q \), that is,

\[
q_{\tilde{Z}_K | \hat{Z}_K } (\tilde{Z}_K, \hat{Z}_K) \triangleq p_{\hat{X}_K | X^K } (\tilde{X}_K, \hat{Z}_K) (Q \tilde{X}_K, Q \hat{Z}_K).
\]

Expression of \( F_i(\Sigma | \{Q\}) \) using the above density functions is the following.

\[
F_i(\Sigma | \{Q\}) = \sup_{p_{\hat{X}_K | X^K } : \Sigma_{X^K, X^K} \geq \Sigma} h(Z_i - \hat{Z}_i | Z_{[i]} | \hat{Z}_K^{\dagger}, Z_{[i]}^{\dagger}) = \sup_{p_{\hat{X}_K | X^K } : \Sigma_{X^K, X^K} \geq \Sigma} - \int q_{\tilde{Z}_K | \hat{Z}_K } (\tilde{Z}_K, \hat{Z}_K) \log q_{\tilde{Z}_K | \hat{Z}_K } (\tilde{Z}_K, \hat{Z}_K) d(\tilde{Z}_K, \hat{Z}_K).
\]

The following two properties on \( F_i(\Sigma | \{Q\}) \) are useful for the proof of Lemma 6.

**Lemma 8:** \( F_i(\Sigma | \{Q\}) \) is concave with respect to \( \Sigma \).

**Lemma 9:**

\[
F_i(\Sigma | \{Q\}) = \frac{1}{2} \log \left\{ (2\pi e) \left[ Q \Sigma_i Q^\dagger \right]^{-1} \right\}.
\]

We first prove Lemma 6 using those two lemmas and next prove Lemmas 8 and 9.

**Proof of Lemma 4**

We have the following chain of inequalities:

\[
h(Z_i | Z_i^{K_i}) W^{K_i} \leq h(Z_i - \hat{Z}_i | Z_i^{K_i} - \hat{Z}_i^{K_i}) \leq \sum_{t=1}^n h(Z_i(t) - \hat{Z}_i(t) | Z_i^{K_i}(t) - \hat{Z}_i^{K_i}(t)) \leq \sum_{t=1}^n \left( F_i(\Sigma_{X^K(t) - \hat{X}_K^{\dagger}(t)} | Q) \right) \leq n F_i \left( \frac{1}{n} \sum_{t=1}^n \Sigma_{X^K(t) - \hat{X}_K^{\dagger}(t)} | Q \right) \leq n F_i \left( \frac{1}{n} \Sigma_{X^K - \hat{X}_K^{\dagger}} | Q \right) \leq \frac{n}{2} \log \left\{ (2\pi e) \left[ Q \left( \frac{1}{n} \Sigma_{X^K - \hat{X}_K^{\dagger}} \right)^{-1} Q \right]^{-1} \right\}.
\]
Step (a) follows from the definition of $F_i(S|Q)$. Step (b) follows from Lemma [8]. Step (c) follows from Lemma [9].

**Proof of Lemma [8]** For given covariance matrices $\Sigma^{(0)}$ and $\Sigma^{(1)}$, let $p^{(0)}_{X^K|X^K}$ and $p^{(1)}_{X^K|X^K}$ be conditional densities achieving $F_i(\Sigma^{(0)}|Q)$ and $F_i(\Sigma^{(1)}|Q)$, respectively. For $0 \leq \alpha \leq 1$, define a conditional density parameterized with $\alpha$ by

$$p^{(\alpha)}_{X^K|X^K} = (1 - \alpha)p^{(0)}_{X^K|X^K} + \alpha p^{(1)}_{X^K|X^K}.$$ 

Let $p^{(\alpha)}_{X^K, \tilde{X}^K}$ be a density function of $(X^K, \tilde{X}^K)$ defined by $(p^{(\alpha)}_{X^K|X^K}, p^{(\alpha)}_{X^K}).$ Let $\Sigma^{(\alpha)}_X$ be a covariance matrix computed from the density $p^{(\alpha)}_{X^K}$. Since

$$\Sigma^{(\alpha)}_X = (1 - \alpha)\Sigma^{(0)}_X + \alpha \Sigma^{(1)}_X \leq (1 - \alpha)\Sigma^{(0)}_X + \alpha \Sigma^{(1)}_X.$$ 

Let $q^{(\alpha)}_{Z^K, \tilde{Z}^K}$ be a density function of $(Z^K, \tilde{Z}^K)$ induced by the unitary matrix $Q$, that is,

$$q^{(\alpha)}_{Z^K, \tilde{Z}^K}(z^K, \tilde{z}^K) \triangleq p^{(\alpha)}_{QZ^K, Q\tilde{Z}^K}(Qz^K, Q\tilde{z}^K).$$

By definition it is obvious that

$$q^{(\alpha)}_{Z^K} = (1 - \alpha)q^{(0)}_{Z^K} + \alpha q^{(1)}_{Z^K}.$$ 

Then we have

$$(1 - \alpha)F_i(S^{(0)}|Q) + \alpha F_i(S^{(1)}|Q) = -(1 - \alpha)\int q^{(0)}_{Z^K}(z^K) \log \frac{q^{(0)}_{Z^K}(z^K)}{q^{(\alpha)}_{Z^K}(z^K)} \, dz^K - \alpha \int q^{(1)}_{Z^K}(z^K) \log \frac{q^{(1)}_{Z^K}(z^K)}{q^{(\alpha)}_{Z^K}(z^K)} \, dz^K \leq (1 - \alpha)\int q^{(\alpha)}_{Z^K}(z^K) \log \frac{q^{(\alpha)}_{Z^K}(z^K)}{q^{(\alpha)}_{Z^K}(z^K)} \, dz^K = -\int q^{(\alpha)}_{Z^K}(z^K) \log q^{(\alpha)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i.$$ 

Step (a) follows from the fact that $q^{(G)}_Z$ and $q^{(G)}_{\hat{Z}}$ yield the same moments of the quadratic form $\log q^{(G)}_Z$. Step (b) is a well known formula on the determinant of matrix. Step (c) follows from $\Sigma_\tilde{X} \preceq \Sigma$. Thus

$$F_i(S|Q) \leq \frac{1}{2} \log \left( \frac{\Sigma^{(0)}_X}{\Sigma^{(1)}_X} \right) \leq \frac{1}{2} \log \left( \frac{\Sigma^{-1}}{\Sigma^{-1}} \right) = \frac{1}{2} \log \left( \frac{\Sigma^{-1}}{\Sigma^{-1}} \right) = \frac{1}{2} \log \left( \frac{\Sigma^{-1}}{\Sigma^{-1}} \right).$$

**B. Proof of Lemma [7]**

In this appendix we prove Lemma [7]. We write a unitary matrix $Q$ as $Q = [q_{ij}]$, where $q_{ij}$ stands for the $(i, j)$ entry of $Q$. The unitary matrix $Q$ transforms $X^K$ into $Z^K = QX^K$. Set $\hat{Q} = Q^\dagger A$ and let $\hat{q}_{ij}$ be the $(i, j)$ entry of $Q^\dagger A$. The following lemma states an important property on the distribution of Gaussian random vector $Z^K$. This lemma is a basis of the proof of Lemma [7].

**Lemma 10:** For any $i = 1, 2, \cdots, K$, we have the following.

$$Z_i = -\frac{1}{g_{ii}} \sum_{j \neq i} \nu_{ij} Z_j + \frac{1}{g_{ii}} \sum_{j=1}^{L} \frac{\hat{q}_{ij}}{\sigma_N} Y_j + \hat{N}_i,$$ \hspace{1cm} (76)

be a conditional density function induced by $q^{(G)}_{Z^K}(\cdot)$. We first observe that

$$\int q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i \geq 0. \hspace{1cm} (75)$$

From (75), we have the following chain of inequalities:

$$h(\tilde{Z}_i|\tilde{Z}^i) = -\int q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i \leq -\int q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i \leq -\int q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i + q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i \leq -\int q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i + q^{(G)}_{Z_i, Z_i^i}(z_i, z_i^i) \, dz_i \leq \frac{1}{2} \log \left( \frac{\Sigma^{(0)}_X}{\Sigma^{(1)}_X} \right) \leq \frac{1}{2} \log \left( \frac{\Sigma^{-1}}{\Sigma^{-1}} \right) = \frac{1}{2} \log \left( \frac{\Sigma^{-1}}{\Sigma^{-1}} \right).$$

Step (a) follows from the fact that $q^{(G)}_Z$ and $q^{(G)}_{\hat{Z}}$ yield the same moments of the quadratic form $\log q^{(G)}_Z$. Step (b) is a well known formula on the determinant of matrix. Step (c) follows from $\Sigma_\tilde{X} \preceq \Sigma$. Thus

$$F_i(S|Q) \leq \frac{1}{2} \log \left( \frac{\Sigma^{-1}}{\Sigma^{-1}} \right)$$

is concluded. Reverse inequality holds by letting $p^{(\alpha)}_{X^K|X^K}$ be Gaussian with covariance matrix $\Sigma$. ■
where
\[ g_{ii} = \left[ Q\Sigma^{-1}_X X^t Q \right]_{ii} + \sum_{j=1}^L \frac{\hat{q}_{ij}^2}{\sigma^2_{N_j}}. \] (77)

\( \nu_{ij}, j \in \{1, 2, \ldots, K\} - \{i\} \) are suitable constants and \( \tilde{N}_i \) is a zero mean Gaussian random variable with variance \( \frac{1}{g_{ii}} \). For each \( i \in S, \tilde{N}_i \) is independent of \( Z_j, j \in \{1, 2, \ldots, K\} - \{i\} \) and \( Y_j, j \in A \).

**Proof:** Without loss of generality we may assume \( i = 1 \). Since \( Y^L = AX^K + N^L \), we have
\[ \Sigma_{X^K Y^L} = \begin{bmatrix} \Sigma_{X^K} & \Sigma_{X^K A} \Sigma_{N^L} \\ \Sigma_{X^K A}^t & \Sigma_{N^L} \end{bmatrix}. \]

Since \( Z^K = QX^K \), we have
\[ \Sigma_{Z^K Y^L} = \begin{bmatrix} \Sigma_{Z^K} & \Sigma_{Z^K A} \Sigma_{N^L} \\ \Sigma_{Z^K A}^t & \Sigma_{N^L} \end{bmatrix}. \]

The density function \( p_{Z^K Y^L}(z^K, y^L) \) of \((Z^K, Y^L)\) is given by
\[
p_{Z^K Y^L}(z^K, y^L) = \frac{1}{(2\pi)^{\frac{K+L}{2}} |\Sigma_{Z^K Y^L}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} [z^K - \mu_{Z^K}]^t \Sigma_{Z^K Y^L}^{-1} [z^K - \mu_{Z^K}]ight),
\]
where \( \Sigma_{Z^K Y^L}^{-1} \) has the following form:
\[
\Sigma_{Z^K Y^L}^{-1} = \begin{bmatrix} Q(\Sigma_{X^K} + \text{t}A\Sigma_{N^L} A^t)Q - Q^tA\Sigma_{N^L}^{-1} & -Q^tA\Sigma_{N^L}^{-1} \\ -\Sigma_{N^L}^{-1}A^tQ & \Sigma_{N^L}^{-1} \end{bmatrix}.
\]

Set
\[
\nu_{ij} \triangleq \frac{Q(\Sigma_{X^K} + \text{t}A\Sigma_{N^L} A^t)Q}{\Sigma_{N^L}} [Q]_{ij} + \sum_{k=1}^L \frac{\hat{q}_{ik} \hat{q}_{jk}}{\sigma^2_{N_k}} \]
and
\[
\beta_{ij} \triangleq -Q^tA\Sigma_{N^L}^{-1} [Q]_{ij} = -\frac{\hat{q}_{ij}}{\sigma^2_{N_j}}.
\]

Now, we consider the following partition of \( \Sigma_{Z^K Y^L}^{-1} \):
\[
\Sigma_{Z^K Y^L}^{-1} = \begin{bmatrix} Q(\Sigma_{X^K} + \text{t}A\Sigma_{N^L} A^t)Q - Q^tA\Sigma_{N^L}^{-1} & -Q^tA\Sigma_{N^L}^{-1} \\ -\Sigma_{N^L}^{-1}A^tQ & \Sigma_{N^L}^{-1} \end{bmatrix}.
\]

\[
g_{11} = \nu_{11} = \left[ Q\Sigma^{-1}_X X^t Q \right]_{11} + \sum_{k=1}^L \frac{\hat{q}_{1k}^2}{\sigma^2_{N_k}},
\]
\[
g_{12} = \left[ Q\Sigma^{-1}_X X^t Q \right]_{12} + \sum_{k=1}^L \frac{\hat{q}_{1k} \hat{q}_{2k}}{\sigma^2_{N_k}}.
\]

It is well known that \( \Sigma_{Z^K Y^L}^{-1} \) has the following expression:
\[
\Sigma_{Z^K Y^L}^{-1} = \begin{bmatrix} 1 & g_{12} G_{2L} \end{bmatrix} \begin{bmatrix} 1 & g_{12} G_{2L} \\ g_{12} G_{2L} & G_{22} \end{bmatrix} \begin{bmatrix} 1 & g_{12} G_{2L} \\ g_{12} G_{2L} & G_{22} \end{bmatrix}^{-1} \begin{bmatrix} 1 & g_{12} G_{2L} \\ g_{12} G_{2L} & G_{22} \end{bmatrix} \begin{bmatrix} 1 & g_{12} G_{2L} \\ g_{12} G_{2L} & G_{22} \end{bmatrix}^{-1}.
\]

Set
\[
\hat{n}_i \triangleq \frac{1}{g_{ii}^2} \frac{\sum_{j \neq i} \nu_{ij} Z_j + \sum_{j=1}^L \frac{\hat{q}_{ij}^2}{\sigma^2_{N_j}} Y_j + \tilde{N}_i}{\sigma^2_{N_i}},
\]
then, we have
\[
\begin{align*}
& t[z^K Y^L] \Sigma_{Z^K Y^L}^{-1} \left[ z^K \right] \\
& = \frac{1}{g_{ii}^2} \frac{\sum_{j \neq i} \nu_{ij} z_j + \sum_{j=1}^L \frac{\hat{q}_{ij}^2}{\sigma^2_{N_j}} y_j + \tilde{n}_i}{\sigma^2_{N_i}}.
\end{align*}
\]

From (78)-(80), we have
\[
\hat{n}_i = z_1 + \frac{1}{g_{ii}^2} \sum_{j=2}^L \nu_{ij} z_j + \frac{1}{g_{ii}^2} \sum_{j=1}^L \beta_{ij} y_j = z_1 + \frac{1}{g_{ii}^2} \sum_{j=2}^L \nu_{ij} z_j - \frac{1}{g_{ii}^2} \sum_{j=1}^L \frac{\hat{q}_{ij}}{\sigma^2_{N_j}} y_j. \tag{82}
\]

It can be seen from (81) and (82) that the random variable \( \hat{N}_i \) defined by
\[
\hat{N}_i = Z_1 + \frac{1}{g_{ii}^2} \sum_{j=2}^L \nu_{ij} Z_j - \frac{1}{g_{ii}^2} \sum_{j=1}^L \hat{q}_{ij} Y_j
\]
is a zero mean Gaussian random variable with variance \( \frac{1}{g_{ii}^2} \) and is independent of \( Z_{[i]}^K \) and \( Y^L \). This completes the proof of Lemma [10].

The followings are two variants of the entropy power inequality.

**Lemma 11:** Let \( U_i, i = 1, 2, 3 \) be \( n \) dimensional random vectors with densities and let \( T \) be a random variable taking values in a finite set. We assume that \( U_3 \) is independent of \( U_1, U_2, T \). Then, we have
\[
\frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_2|U_1T)} + \frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_3|U_2T)} \geq \frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_1|U_2T)} + \frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_2|U_3T)}.
\]

**Lemma 12:** Let \( U_i, i = 1, 2, 3 \) be \( n \) random vectors with densities. Let \( T_1, T_2 \) be random variables taking values in finite sets. We assume that those five random variables form a Markov chain \((T_1, U_1) \rightarrow U_3 \rightarrow (T_2, U_2)\) in this order. Then, we have
\[
\frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_1+U_2|U_3T_1T_2)} \geq \frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_1|U_3T_1T_2)} + \frac{1}{2\pi e^2} e^{\frac{1}{8}\hat{h}(U_2|U_3T_1T_2)}.
\]

**Proof of Lemma 2** By Lemma 10 we have
\[
Z_i = -\frac{1}{g_{ii}^2} \sum_{j \neq i} \nu_{ij} Z_j + \frac{1}{g_{ii}^2} \sum_{j=1}^L \frac{\hat{q}_{ij}}{\sigma^2_{N_j}} Y_j + \tilde{N}_i,
\]
where \( \tilde{N}_i \) is a vector of \( n \) independent copies of zero mean Gaussian random variables with variance \( \frac{1}{g_{ii}^2} \). For each \( i \in A, \tilde{N}_i \) is independent of \( Z_j, j \in \{1, 2, \ldots, K\} - \{i\} \) and \( Y_j, j \in A \). Set
\[
h^{(n)} \triangleq \frac{1}{n} \hat{h}(Z_i^K, Z_i^L),
\]
Furthermore, for $k \in \Lambda$, define
\[
S_k \triangleq \{k, k + 1, \ldots, L\}, \quad \Psi_k = \Psi_k(Y_{S_k}) \triangleq \sum_{j=k}^{L} \frac{q_{ij}}{\sigma^2_{N_j}} Y_j.
\]
Applying Lemma 11 to (83), we have
\[
e^{2h(n)} \geq \frac{1}{(g_{ii})^2} e^{\frac{1}{2\pi e} h(\Psi_1^i|Z_{[i]}^K, W^L)} + \frac{1}{g_{ii}}, \tag{84}
\]
On the quantity $h(\Psi_1^i|Z_{[i]}^K, W^L)$ in the right member of (84), we have the following chain of equalities:
\[
h(\Psi_1^i|Z_{[i]}^K, W^L) = I(\Psi_1^i; X^K|Z_{[i]}^K, W^L) + h(\Psi_1^i|X^K, Z_{[i]}^K, W^L)
\]
\[
= I(\Psi_1^i; X^K|Z_{[i]}^K, W^L) + h(\Psi_1^i|X^K, W^L)
\]
\[
= I(\Psi_1^i; Z_{[i]}^K|X^K, W^L) + h(\Psi_1^i|X^K, W^L)
\]
\[
= h(Z_{[i]}^K|X^K, W^L) - h(Z_{[i]}|\Psi_1^i, X^K, W^L)
\]
\[
+ h(\Psi_1^i|X^K, W^L).
\]

Step (a) follows from that $Z^K_i$ can be obtained from $X^K$ by the invertible matrix $Q$. Step (b) follows from the Markov chain $Z_i \rightarrow (\Psi_1, Z_{[i]}^K) \rightarrow Y^L \rightarrow W^L$.

From (85), we have
\[
e^{2h(n)} \geq \frac{1}{2\pi e} \frac{e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)}}{g_{ii}} e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)} + \frac{1}{g_{ii}}. \tag{86}
\]
Substituting (86) into (84), we obtain
\[
e^{2h(n)} \geq \frac{1}{2\pi e} \frac{e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)}}{g_{ii}} e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)} + \frac{1}{g_{ii}} e^{-2r(n)} \tag{87}
\]
Solving (87) with respect to $e^{2h(n)}$, we obtain
\[
e^{2h(n)} \geq \left( g_{ii} - \frac{1}{2\pi e} e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)} \right)^{-1}. \tag{88}
\]
Next, we evaluate a lower bound of $e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)}$. Note that for $j = 1, 2, \ldots, s - 1$ we have the following Markov chain:
\[
(W_{S_{j+1}}, \Psi_{j+1}(Y_{S_{j+1}})) \rightarrow X^K \rightarrow (W_j, \frac{q_{ij}}{\sigma^2_{N_j}} Y_j). \tag{89}
\]
Based on (89), we apply Lemma 12 to $e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)}$ for $j = 1, 2, \ldots, s - 1$. Then, for $j = 1, 2, \ldots, s - 1$, we have the following chains of inequalities:
\[
e^{\frac{1}{2\pi e} h(\Psi_1^i|X^K, W^L)} = \frac{1}{2\pi e} e^{\frac{1}{2\pi e} h(\Psi_1^i, \Psi_{j+1}^i, Y_{S_{j+1}}, W_j)}
\]
\[
\geq \frac{1}{2\pi e} e^{\frac{1}{2\pi e} h(\Psi_1^i, Y_{S_{j+1}}, W_j)} + \frac{1}{2\pi e} e^{\frac{1}{2\pi e} h(\Psi_{j+1}^i, Y_{S_{j+1}}, W_j)}
\]
\[
= \frac{1}{2\pi e} e^{\frac{1}{2\pi e} h(\Psi_1^i, Y_{S_{j+1}}, W_j)} + \frac{q_{ij}}{\sigma^2_{N_j}} e^{-2r(n)} \tag{90}
\]
Using (90) iteratively for $j = 1, 2, \ldots, s - 1$, we have
\[
\frac{1}{2\pi e} e^{\sum_{j=1}^{s} q_{ij} e^{-2r(n)}} \tag{91}
\]
Combining (77), (88), and (91), we have
\[
e^{2h(n)} \geq \left[ Q^{-1} \sum_{j=1}^{s} \frac{q_{ij}}{\sigma^2_{N_j}} e^{-2r(n)} \right]^{-1}
\]
completing the proof.

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