Joint Replenishments Optimization for the (Rn, Sn) policy

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Abstract

This paper considers the periodic-review nonstationary stochastic joint replenishment problem (JRP) under Bookbinder and Tan’s static-dynamic uncertainty control policy. According to a static-dynamic uncertainty control rule, the decision maker fixes timing of replenishments once and for all at the beginning of the planning horizon, inventory position is then raised to a predefined order-up-to-position at the beginning of each replenish period. We present a mixed integer linear programming (MILP) model for approximating optimal static-dynamic uncertainty policy parameters. We further demonstrate that our MILP model can be used to approximate the optimal control rule for the JRP, also known as (σ, S) policy. An extensive computational study illustrates the effectiveness of our approach when compared to other competitor approaches in the literature.

Keywords: inventory, stochastic joint replenishment, \((R, S)\) policy, mixed integer linear programming

1. Introduction

The Joint Replenishment Problem (JRP) occurs when several items are ordered from the same supplier, or several products have the same means of transportation, or several products are processed on the same piece of equipment [Salameh et al. 2014]. Every time an order is placed, the group fixed ordering cost is incurred regardless the number of items replenished; in addition there are also item-specific fixed and variable ordering costs that are charged whenever an item is included in a replenishment order. The goal of the JRP is to determine the optimal inventory replenishment plan that minimises the cost of replenishing multiple items.

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Literature on JRP can be roughly categorised into deterministic and stochastic based on the nature of demand. In the deterministic joint replenishment inventory system demand for each individual item is known to be constant over an infinite time horizon and replenishments are made at equally spaced time intervals; the problem is to determine the length of replenishment cycles and the frequency of replenishing individual items, e.g., [Goyal & Belton, 1979; Kaspi & Rosenblatt, 1991; Viswanathan, 1996; Wildeman et al., 1997; Harga, 1994; Goyal & Deshmukh, 1993; Boctor et al., 2004; Nilsson et al., 2007]. In the stochastic joint replenishment inventory system the demand for individual items is unknown but follows a certain type of distributions; the problem is to decide the optimal parameters of a given inventory policy, e.g., [Balintfy, 1964; Atkins & Iyogun, 1988; Renberg & Planché, 1967; Kalpakam & Arivarignan, 1993; Viswanathan, 1997; Nielsen & Larsen, 2005; Özkaya et al., 2006]. Most literature still presents applications to constant and dynamic deterministic demands; however, the study regarding stochastic demand has received increasing attention due to its practical relevance [Bastos et al., 2017]. This work belongs to the growing literature on the stochastic joint replenishment.

This paper applies the static-dynamic strategy, proposed by Bookbinder & Tan (1988) for tackling single-item lot-sizing problems, in the context of a JRP system. The static-dynamic strategy, known as \((R, S)\), features two control parameters: \(R\), timing of replenishment, and \(S\), order-up-to-position. At each review period, the decision maker places an order so as to increase the inventory position (net inventory level + outstanding orders) to a given order-up-to-position. In the context of the JRP system, a periodic-review \((R, S)\) policy is adopted for each item. The \((R, S)\) policy is an appealing strategy since it eases the coordination between supply chain players [Kilic & Tarim, 2011], and facilitates managing joint replenishment [Silver et al., 1998].

Our goal is to tackle the periodic-review stochastic JRP under \((R, S)\) policy. We first present a mixed-integer linear programming (MILP) model for computing optimal policy parameters that optimise the expected total cost comprising group fixed ordering costs, item-specific group ordering costs, holding costs, and penalty costs over the planning horizon. Our model generalises Rossi et al. (2015), which discussed an MILP model for approximating optimal \((R, S)\) policy parameters for single-item lot-sizing problems. We further show that our MILP model can be used to approximate the optimal \((\sigma, \tilde{S})\) policies, which is known to be optimal for this class of problem [Liu & Esogbue, 2012]. Under this policy, decision makers order up to \(\tilde{S}\) if opening inventory positions fall in \(\sigma\) \((\sigma \subset \mathbb{R}^N, \tilde{S} \in \mathbb{R}^N, N \text{ represents the number of items})\) at the beginning of each time period. Numerical experiments illustrate the effectiveness of our models.

We contribute to the literature on the stochastic JPR as follows.

- We present an MILP model for tackling the nonstationary stochastic JRP under \((R, S)\) policy.
- We demonstrate that the MILP model can be used to approximate \((\sigma, \tilde{S})\) policies.
In an extensive computational study based on existing test beds drawn from the literature we demonstrate the effectiveness of our models when compared to other competing approaches in the literature.

This rest of this paper is organised as follows. Section 2 surveys relevant literature. Section 3 describes problem settings. Section 4 presents an MILP model for computing \((R, S)\) policy parameters. Section 5 extends the MILP model for approximating the optimal \((\sigma, \tilde{S})\) policy parameters. An extensive computational study is conducted in Section 6. We draw conclusions in Section 7.

2. Literature review

The problem of controlling the inventory of a multi-item system under joint replenishment has received increasing attention over the past several decades. For a thorough review of literature readers could refer to (Silver & Peterson, 1985; Goyal & Satir, 1989; Van Eijs et al., 1992; Khouja & Goyal, 2008; Bastos et al., 2017). In this section, we focus our attention on existing policies for tackling stochastic JRP. In particular, we survey control policies that have been considered in the literature.

\((\sigma, \tilde{S})\) policy. Since the landmark study Scarf (1960) proved the optimality for the single-item inventory problem, there have been few attempts to prove the optimality for multi-item inventory systems. Johnson (1967) proved the optimal policy in the stationary case is a \((\sigma, \tilde{S})\) policy, where \(\sigma \subset \mathbb{R}^N\) and \(\tilde{S} \in \mathbb{R}^N\), and one orders up to \(\tilde{S}\) if inventory levels \(\tilde{I} \in \sigma\) and \(\tilde{I} \leq \tilde{S}\) and one does not order if \(\tilde{I} \notin \sigma\). Kalin (1980) showed when \(\tilde{I} \in \sigma\) and \(\tilde{I} \nleq \tilde{S}\), there exists \(\tilde{S}(\tilde{I}) \geq \tilde{I}\) such that the optimal policy is to order up to \(\tilde{S}(\tilde{I})\), this policy is named \((\sigma, \tilde{S}(\cdot))\) policy. Ohno & Ishigaki (2001) proved the optimality of \((\sigma, \tilde{S}(\cdot))\) policy for continuous-time inventory problems with compound Poisson demands. Gallego & Sethi (2005) gave the general definition of \(K\)-convexity in \(\mathbb{R}^N\), which encompasses both the joint ordering and individual ordering case.

\((s, c, S)\) policy. Several works on stochastic JRP have focused on computing \((s, c, S)\) policies, introduced by Balintfy (1964). This policy features three control parameters: \(s\), reorder point; \(c\), can-order level; \(S\), order-up-to-position. Under this policy, decision makers order up to \(S\) when either at a demand epoch the inventory position drops to or below \(s\); or when at a special replenishment opportunity the inventory position is at or below \(c\). Under the assumption of Poisson-distributed demands, Ignall (1969) proved that the \((s, c, S)\) policy is not optimal even for two-item problems. Silver (1974) proposed the decomposition method to compute \((s, c, S)\) policy parameters, where the multi-item problem is decomposed into several single-item problems. This approximation technique was followed by Melchiors (2002), Johansen & Melchiors (2003), Kayis et al. (2008) modelled the two-item JRP problem as a semi-Markov decision model, and proposed an enumerative approach to approximate \((s, c, S)\) policies. In addition, Schaack & Silver (1972), Thompstone & Silver (1975), Silver (1981)
studied JRP with compound Poisson-distributed demands.

(R,T) policy. Atkins & Iyogun [1988] proposed two periodic-review (R,T)-type policies, namely periodic policy P and modified periodic policy MP, which differ only in the way the ordering periods T_i are determined. Under this policy, every T_i periods, the inventory position of item i is raised to R_i. Numerical experiments demonstrate that the MP policy performs consistently better than the (s,c,S) policy, and the P policy generally outperforms the (s,c,S) policy excepting problems involving small values of group fixed ordering cost.

(Q,S) policy. This policy was first proposed by Renberg & Planche [1967]. Under this policy, whenever the total inventory position drops to the group reorder point, an order is placed to raise inventory position of each item to item-specific order-up-to-position S_i. The combined order quantity is Q, and the group reorder point is reached when the combined usage reaches Q. Pantumsinchai [1992] evaluated the computational performance of the (Q,S) policy by comparing it against the (s,c,S) policy, P policy and MP policy on the basis of long-run total average costs. Computational experiments showed that the MP policy consistently outperforms the (s,c,S) policy on the test instances, and both MP and (Q,S) policy perform better as the group ordering cost increases. The study showed that the (Q,S) policy is appropriate for items for which the stock-out costs are low and the major set-up cost is high relative to the minor set-up cost.

P(s,S) policy. This policy was proposed by Viswanathan [1997] for periodic-review inventory systems, in which inventory position of each item is reviewed at every fixed and constant time interval. At each review time, the (s,S) policy is applied to each item, so that any item with inventory position at or below s is order up to S. For a fixed review period, the algorithm of Zheng & Federgruen [1991] is adopted to compute the optimal (s,S) policy parameters. Computational studies indicated that although the proposed policy requires more computational effort, it generally dominates the MP policy, and dominates (s,c,S) policy, and (Q,S) policy for most test instances.

Q(s,S) policy. Nielsen & Larsen [2005] combined features of (Q,S) policy and P(s,S) policy, and proposed the Q(s,S) policy. By operating under this policy, the total inventory position is continuously reviewed while the item-specific inventory positions are reviewed only when the total consumption since the last order reaches Q. Then every item with inventory position less than or equal to its respective reorder point s is order to S. An analytic solution is derived by using the Markov decision theory in Nielsen & Larsen [2005]. Computational study demonstrated that the Q(s,S) policy outperforms P(s,S) policy, and dominates (Q,S) policy in 17 of 18 test instances on the data set of Atkins & Iyogun [1988].

(Q,S,T) policy. This continuous-review policy was proposed by Ozkaya et al. [2006]. Decision makers raise the inventory position of each item i to its order-up-to-position S_i whenever a total of Q demands accumulated or T time units have elapsed, whichever occurs first. This policy is a hybrid of the continuous review (Q,S) policy, proposed by Renberg & Planche [1967], and
the periodic review \((R, T)\) policy, proposed by Atkins & Iyogun [1988]. Thus, it features benefits of two separate policies. The comprehensive numerical study indicates that the proposed policy dominates the \(P(s, S)\) policy, \((Q, s)\) policy, \(Q(s, S)\) policy, and \((s, c, S)\) policy in 100 of 139 instances.

\([R, S] \) policy. This policy is proposed by Bookbinder & Tan [1988] for controlling single-item inventory system. The policy requires decision makers to place an order at each replenish period to increase the inventory position to the order-up-to-position \(S\). This policy has been widely studied in stream of single-item lot-sizing problems. Tarim & Kingsman [2004, 2006] formulated a mixed integer programming (MIP) model for computing optimal \((R, S)\) policy parameters. Tarim et al. [2011] relaxed the MIP model, and solved it as a shortest path problem which does not require the use of any MIP or Constraint Programming (CP) commercial solver. In addition, Özen et al. [2012] showed a DP-based algorithm for solving small-size problems, and an approximation heuristic and a relaxation heuristic for tackling larger-size problems; Tunc et al. [2014] suggested a deterministic equivalent MIP model. Recently, Rossi et al. [2015] generalised the discussions above and developed a unified MILP model for approximating \((R, S)\) polices by adopting the piecewise linear approximation technique in Rossi et al. [2014]. Although various efficient modelling methods for computing \((R, S)\) policy parameters were proposed, they generally control the single-item inventory system. The main purpose of this work is to apply the \((R, S)\) policy for the multi-item inventory system. In the context of the JRP, a periodic-review \((R, S)\) policy is adopted for each item.

The stochastic JRP is an open research area for the development of more efficient computational methods and control policies. In this study, we apply the periodic review \((R, S)\) policy, originally proposed by Bookbinder & Tan [1988] for tackling single-item lot sizing problems, to JRP’s with stochastic demand and fixed lead time. In the context of the JRP system, a periodic review \((R, S)\) policy is adopted for each item. Note that when the demand is stationary stochastic, the \((R, S)\) policy is the same as the \(MP\) policy proposed by Atkins & Iyogun [1988], where every \(T_n\) periods, raising the inventory position of item \(n\) to the order-up-to-position \(R_n\). However, the \((R, S)\) policy also deals with non-stationary stochastic demands which was not addressed in Atkins & Iyogun [1988]. In this paper, we present an MILP approach for approximating \((R, S)\) policies under non-stationary stochastic demands. Nonlinear costs are approximated by leveraging technique introduced in Rossi et al. [2014]. Numerical experiments investigate the effectiveness of our approach against competing policies from the literature.

3. Problem description
Consider a periodic-review \(N\)-item inventory management system over a \(T\)-period planning horizon. We assume that demand \(d_{nt}\) of item \(n, n = 1, \ldots, N\), in period \(t, t = 1, \ldots, T\) are independently distributed random variables with known probability density function \(g_{nt}(\cdot)\), and cumulative distribution function \(G_{nt}(\cdot)\).
We assume that ordering decisions are made at the beginning of each time period. There is a group fixed ordering cost \( K \) and an item-specific fixed ordering cost \( k^n \). The group fixed ordering cost is incurred whenever an order is placed at a given time period, no matter which and how many items are included in this order. The item-specific fixed ordering cost is incurred whenever an order for item \( n \) is placed at a given time period, no matter how many items are included in this order.

We define \( Q_t^n \) as the quantity of item \( n \) ordered in period \( t \), which will be received after lead time \( L^n \). Then, the ordering cost of item \( n \) in period \( t \) with ordering quantity \( Q_t^n \) can be written as,

\[
c_t^n(Q_t^n) = \begin{cases} k^n, & Q_t^n > 0, \\ 0, & Q_t^n = 0. \end{cases}
\]

(1)

Let \( c_t(\vec{Q}_t) \) denote the ordering cost of period \( t \) with ordering quantity vector \( \vec{Q}_t = (Q_t^1, \ldots, Q_t^N) \). \( c_t(\vec{Q}_t) \) has the following structure

\[
c_t(\vec{Q}_t) = \begin{cases} K + \sum_{n=1}^{N} c_t^n(Q_t^n), & \exists Q_t^n = 0, \quad \text{otherwise}. \end{cases}
\]

(2)

A penalty cost \( b^n \) is incurred for each unit of item \( n \) of backorder demand per period, and a holding cost \( h^n \) is charged for each unit of item \( n \) carried from one period to the next. The immediate penalty and holding cost of period \( t \) can be expressed as

\[
L_t(\vec{y}) = \sum_{n=1}^{N} \left( b^n \cdot \mathbb{E}[\max(d^n_t - y^n, 0)] + h^n \cdot \mathbb{E}[\max(y^n - d^n_t, 0)] \right),
\]

(3)

where vector \( \vec{y} = (y^1, \ldots, y^N) \) is the inventory level immediately after orders are received at the beginning of period \( t \), and “\( \mathbb{E} \)” denotes the expectation taken with respect to the random demand.

Let \( I_t^n \) denote the net inventory level of item \( n \) at the end of period \( t \), which is also the opening inventory level of period \( t + 1 \), and \( C_t(I_{t-1}) \) denote the expected total cost of an optimal policy over period \( t, \ldots, T \), given opening inventory level \( I_{t-1} = (I_{t-1}^1, \ldots, I_{t-1}^N) \) at the beginning of period \( t \). Note that there is no outstanding order at the beginning of the planning horizon. Then, \( C_t(I_{t-1}) \) can be written as,

\[
C_t(I_{t-1}) = \min_{\vec{Q}_t} \{ c_t(\vec{Q}_t) + L_t(I_{t-1} + \vec{Q}_{t-1} - \vec{D}_t) \} + E[C_{t+1}(I_{t-1} + \vec{Q}_{t-1} - \vec{D}_t)],
\]

(4)

where \( \vec{D}_t = (d_{t}^1, \ldots, d_{t}^N) \), \( \vec{L} = (L^1, \ldots, L^N) \), and

\[
C_T(I_{T-1}) = \begin{cases} \min_{\vec{Q}_T} \{ c_T(\vec{Q}_T) + L_T(I_{T-1} + \vec{Q}_{T-1}) \}, & t = \vec{L} + 1, \\ \min_{\vec{Q}_T} \{ c_T(\vec{Q}_T) + L_T(I_{T-1}) \}, & \text{otherwise}; \end{cases}
\]

(5)
represents the boundary condition. Moreover, let us define, $t = L^n + 1, \ldots, T$,

$$G_t(\vec{I}_{t-1}) = L_t(\vec{I}_{t-1} + \vec{Q}_{t-L}) + E[G_{t+1}(\vec{I}_{t-1} + \vec{Q}_{t-L} - \vec{D}_t)].$$  

(6)

**Example.** We consider an instance in which the group fixed ordering cost is $K = 10$, the item-specific ordering cost $k$ is 0, the holding cost is $h = 1$, the stock-out penalty cost is $b = 5$. We control inventory for two items over a planning horizon of $T = 4$ periods. We assume that the demand of item $n$ in period $t$ follows a Poisson distribution with rate $\lambda_{nt}$; where $\lambda_1 = \lambda_2 = \{3, 6, 9, 6\}$. For simplicity, we assume that the lead time is 0 for every item. The expected total cost, i.e. $C_1(\vec{I}_0)$, of an optimal policy, given initial inventory level $I_{01} = I_{02} = 0$, can be obtained via stochastic dynamic programming (SDP) and is equal to 65.4. In Fig. 1 we plot $G_1(\vec{I}_0)$ for $I_{01} \in [0, 14]$ and $I_{02} \in [0, 14]$.

![Figure 1: Expected total cost, i.e. $G_1(\vec{I}_0)$, contour plot for the two-item joint replenishment numerical example](image)

4. **An MILP model for approximating non-stationary stochastic (R, S) policies**

In this section, we formulate the stochastic JRP problem under the (R, S) policy as an MILP model. Under the (R, S) policy, the replenish periods and associated order-up-to-positions are fixed at the beginning of the planning horizon, while actual order quantities are decided at the beginning of each replenish period. Note that in the context of JRP, a periodic-review (R, S) policy is adopted for each item. We first introduce a stochastic programming formulation in Section 4.1 and then we reformulate it as an MILP model in Section 4.2.
4.1. A stochastic program

Consider the periodic-review N-item T-period JRP described in Section 3. We introduce binary variables \( \delta_t \), \( t = 1, \ldots, T \), and \( n = 1, \ldots, N \); \( \delta_t \) takes value 1 if a group order is made in period \( t \) no matter how many types of items involved, otherwise 0; \( y_{nt} \) is set to 1 if item \( n \) is replenished in period \( t \).

We further assume that the system is forced to place an order in period 1, and all orders should be received by the end of the planning horizon.

We reformulate the stochastic dynamic programming model in Section 3 as

The objective is to find the optimal replenish plan so to minimise the expected ordering costs, penalty costs, and holding costs of \( N \) items over the \( T \)-period planning horizon. Constraints (8) imply that if at least an item is ordered, then a group replenishment is issued. Constraints (9) force the system to replenish every item in period 1. Constraints (10) are inventory conservation constraints in periods 1, \ldots, \( L^n \): inventory level at the end of period \( t \) is equal to the initial inventory level, minus demands raised up to period \( t \). Constraints (11) ensure all replenishments are received by the end of the planning horizon. Constraints (12) are the inventory conservation constraints in periods \( 1 + L^n, \ldots, T \): inventory level at the end of period \( t \) is equal to the initial inventory level, plus all orders received before the end of period \( t \), demands raised up to period \( t \). Constraints (13)- (16) state domains of \( y_{nt}, Q^n, \delta_t, \) and \( I^n_t \).

4.2. An MILP model

The stochastic programming formulation in Fig. 2 can be reformulated into an MILP model via the piecewise approximation approach in (Rossi et al., 2014).

Figure 2: Stochastic programming formulation of the JRP.
In the rest of this paper, let \( \sim \) denote the expectation operator. We introduce the first order loss function
\[
\mathcal{L}(x, \omega) = \int_{-\infty}^{\infty} \max(t - x, 0) g_\omega(t) d(t)
\]
and its complementary function
\[
\hat{\mathcal{L}}(x, \omega) = \int_{-\infty}^{\infty} \max(x - t, 0) g_\omega(t) d(t),
\]
where \( \omega \) is a random variable with probability density function \( g_\omega(\cdot) \), and \( x \) is a scalar variable.

Consider a partition of the support \( \Omega \) of \( \omega \) into \( W \) disjoint subregions \( \Omega_1, \ldots, \Omega_W \), the probability mass \( p_i = \Pr\{\omega \in \Omega_i\} = \frac{1}{p_i} \int_{\Omega_i} g_\omega(t) dt \), \( i = 1, \ldots, W \). By applying Jensen’s lower bound \footnote{Similarly, the Edmundson-Madansky upper bound can be applied for approximating the expected excess inventory and back-orders as well, for further details refer to (Rossi et al., 2014).} \( L(x, \omega) \) and \( \hat{\mathcal{L}}(x, \omega) \) can be approximated as piecewise linear functions, as presented in the following lemma.

**Lemma 4.1.** For the first order loss function and its complementary function, the lower bounds
\[
\mathcal{L}(x, \omega) \geq \mathcal{L}_{lb}(x, \omega) = x \sum_{k=1}^{i} p_k + \sum_{k=1}^{i} p_k E[\omega|\Omega_k] + (x - \tilde{\omega}),
\]
\[
\hat{\mathcal{L}}(x, \omega) \geq \hat{\mathcal{L}}_{lb}(x, \omega) = x \sum_{k=1}^{i} p_k + \sum_{k=1}^{i} p_k E[\omega|\Omega_k]
\]
are piecewise linear functions with \( W + 1 \) segments.

We introduce two sets of variables \( \tilde{B}^n_t \geq 0 \) and \( \tilde{H}^n_t \geq 0 \) represent lower bounds of \( E[\max(-I^n_t, 0)] \) and \( E[\max(I^n_t, 0)] \), \( t = 1, \ldots, T, n = 1, \ldots, N \). Then, the objective function \footnote{Similarly, the Edmundson-Madansky upper bound can be applied for approximating the expected excess inventory and back-orders as well, for further details refer to (Rossi et al., 2014).} in Fig. 2 can be rewritten as
\[
\min \sum_{t=1}^{T} \left( K \cdot \delta_t + \sum_{n=1}^{N} \left( k^n \cdot y^n_t + b^n \tilde{B}^n_t + h^n \tilde{H}^n_t \right) \right).
\]
applying Lemma 4.1, \( \hat{B}_t^n \) and \( \hat{H}_t^n \) can be written as follows, \( t = 1, \ldots, L^n, n = 1, \ldots, N, i = 1, \ldots, W, \)

\[
\hat{B}_t^n = -I_t^n + \sum_{k=1}^i p_k I_0^n - \sum_{k=1}^i p_k E[d_{i,t}] | \Omega_i |, \\
\hat{H}_t^n = \sum_{k=1}^i p_k I_0^n - \sum_{k=1}^i p_k E[d_{i,t}] | \Omega_i |. 
\]

(20)

(21)

Additionally, constraints [10] in Fig. 2 can be rewritten as,

\[
\tilde{I}_t^n + \tilde{d}_t^n - \hat{I}_{t-1}^n = 0, \quad t = 1, \ldots, L^n. 
\]

(22)

The second part involves periods \( 1 + L^n, \ldots, T, n = 1, \ldots, N \). Consider a single cycle of item \( n \) over periods \( i, \ldots, j \), in which a single order is received at the beginning of period \( i \), and the next order will be received at the beginning of period \( j + 1 \). Since the lead time of item \( n \) is \( L^n \), the order that arrives in period \( i \) must be issued in period \( i - L^n \) with order-up-to-position \( S_{i-L^n}^n \). Thus, \( I_t^n \), \( t = \{i, \ldots, j\} \), must equal to the order-up-to-position \( S_{i-L^n}^n \), minus the demand convolution over periods \( i - L^n, \ldots, t, i.e., I_t^n = S_{i-L^n}^n - d_{i-L^n,t}^n \).

We introduce a binary variable \( P_{jt}^n \) which is set to one if the most recent order received before period \( t \) arrived in period \( j \), where \( j \leq t, j = 1 + L^n, \ldots, t, t = 1 + L^n, \ldots, T \), and \( n = 1, \ldots, N \); and we introduce the following constraints, \( t = 1 + L^n, \ldots, T, n = 1, \ldots, N, \)

\[
\sum_{j=1+L^n}^{t} P_{jt}^n = 1, \\
P_{jt}^n \geq y_{j-L^n}^n - \sum_{k=j-L^n+1}^{t-L^n} y_k^n, \quad j = 1 + L^n, \ldots, t. 
\]

(23)

(24)

Constraints [23] indicate that the most recent order received before period \( t \) arrived in period \( j \). Constraints [24] identify uniquely the period in which the most recent order received before period \( t \) has been received. Therefore, the inventory level \( I_t^n = \sum_{j=1+L^n}^{t} (S_{j-L^n}^n - d_{j-L^n,t}^n) P_{jt}^n \) where \( t = 1 + L^n, \ldots, T, \) and \( S_{j-L^n}^n \) represents the order-up-to-position of item \( n \) in period \( j - L^n \). We write the back-orders and excess inventory as the first order loss function and its complementary, \( \sum_{j=1+L^n}^{t} \mathcal{L}(S_{j-L^n}^n, d_{j-L^n,t}^n) P_{jt}^n \) and \( \sum_{j=1+L^n}^{t} \hat{\mathcal{L}}(S_{j-L^n}^n, d_{j-L^n,t}^n) P_{jt}^n \). By applying Lemma 4.1, \( \hat{B}_t^n \) and \( \hat{H}_t^n \) can be written as, \( t = 1 + L^n, \ldots, T, n = \)
in the piecewise-linear approximations of $B^n$, $C^n$, $H^n$ costs, penalty costs, and holding costs of minimise the expected group fixed ordering costs, item-specific fixed ordering costs in period 8 could not be received by the end of the planning horizon. Additionally, item 1 is expected to be ordered every two periods with the same

\[ \sum_{t=1+L^n}^{T} \tilde{d}^n_{j-L^n,t} P^n_{j,t} \sum_{k=1}^{i} p_k - \sum_{j=1+L^n}^{T} \sum_{k=1}^{i} p_k E[d^n_{j-L^n,t} | \Omega_l] P^n_{j,t} , \]

\[ \tilde{H}^n_t \geq (\tilde{I}^n_t + \sum_{j=1+L^n}^{T} \tilde{d}^n_{j-L^n,t} P^n_{j,t}) \sum_{k=1}^{i} p_k - \sum_{j=1+L^n}^{T} \sum_{k=1}^{i} p_k E[d^n_{j-L^n,t} | \Omega_l] P^n_{j,t} . \quad (26) \]

Note that $S^n_{j-L^n} = \tilde{I}^n_t + \tilde{d}^n_{j-L^n,t}$. In addition, constraints (12)-(14) in Fig. 2 can be reformulated as follows,

\[ y^n_{i-L^n} = 0 \rightarrow \tilde{I}^n_t + \tilde{d}^n_t - \tilde{I}^n_{t-1} = 0, \quad t = 1 + L^n, \ldots, T, \quad (27) \]
\[ \tilde{I}^n_t + \tilde{d}^n_t - \tilde{I}^n_{t-1} \geq 0, \quad t = 1 + L^n, \ldots, T. \quad (28) \]

We now present the overall model in Fig. 3. The objective function (29) minimise the expected group fixed ordering costs, item-specific fixed ordering costs, penalty costs, and holding costs of $N$-item over the $T$-period planning horizon. Constraints (30) imply an individual item can only be included in a group replenishment if that replenishment is made. Constraints (31) - (32) assume that the first order is issued at the beginning of period 1, and there is no outstanding replenishment at the beginning of the planning horizon. Constraints (33) - (34) represent the expected back-orders and on-hand stocks of item $n$ over periods 1, $L^n$. Constraints (35) state all orders are received by the end of the planning horizon. Constraints (36) - (37) are inventory balance constraints. Constraints (38) - (39) ensure the most recent replenishment that has arrived before period $t$ was received in period $j$. Constraints (40) - (41) represent the expected back-orders and on-hand stocks of item $n$ over periods $1 + L^n, \ldots, T$. Constraints (42) - (44) indicate domains of binary variables $d^n_t$, $\tilde{y}^n_t$, and $P^n_{j,t}$.

By solving the model in Fig. 3, the optimal replenishment plan including group replenish periods $\delta_t$, and item-specific replenish periods $\tilde{y}^n_t$, and the item-specific order-up-to-positions $S^n_t = \tilde{I}^n_{t+L^n} + \tilde{d}^n_{t,t+L^n}$ are obtained, for $t = 1, \ldots, T$, and $n = 1, \ldots, N$.

**Example.** We demonstrate the modelling strategy behind the MILP model on a 5-item 10-period example. It is assumed that the demand is Poisson-distributed with rate $\lambda^n_i$ presented in Table 1. The initial inventory level is taken as zero. Other parameters are: $K = 500$, $b = 10$, $h = 2$, $k^n = \{120, 100, 80, 120, 150\}$, and $L^n = \{1, 2, 3, 1, 3\}$. We employ eleven segments in the piecewise-linear approximations of $B^n_t$ and $H^n_t$ (for $n = 1, \ldots, 5$, and $t = 1, \ldots, 10$).

The resulting expected total cost is 14236.24. Replenishment plans of each item are presented in Fig. 4. Items 1, 2 and 4 are replenished in periods 1, 3, 5, and 8; while item 3 and 5 are replenished only in periods 1, 3, and 5 since orders in period 8 could not be received by the end of the planning horizon. Additionally, item 1 is expected to be ordered every two periods with the same
\[
\min \sum_{t=1}^{T} \left( K \cdot s_t + \sum_{n=1}^{N} \left( k^{n} \cdot y^{n}_t + h^{n} \tilde{H}^{n}_t + h^{n} \tilde{B}^{n}_t \right) \right)
\]

Subject to, \( n = 1, \ldots, N \)

\[
\delta_t \geq y^{n}_t \quad t = 1, \ldots, T
\]

\[
y^{n}_t = 1 \quad t = 1, \ldots, T
\]

\[
l^{n}_t + d^{n}_t - l^{n}_{t-1} = 0 \quad t = 1, \ldots, T
\]

\[
B^{n}_t \geq -l^{n}_t + \sum_{k=1}^{t} \sum_{k=1}^{t} p_k E[d_{k,t}^{n} | \Omega_k] \quad t = 1, \ldots, T
\]

\[
H^{n}_t \geq \sum_{k=1}^{t} \sum_{k=1}^{t} p_k E[d_{k,t}^{n} | \Omega_k] \quad t = 1, \ldots, T
\]

\[
y^{n}_1 = 0 \quad t = 1, \ldots, T
\]

\[
l^{n}_t + d^{n}_t - l^{n}_{t-1} \geq 0 \quad t = 1, \ldots, T
\]

\[
y^{n}_L - L^n = 0 \rightarrow l^{n}_t + d^{n}_t - l^{n}_{t-1} = 0 \quad t = 1 + L^n, \ldots, T
\]

\[
\sum_{j=1+L^n}^{P^n_{jt}} y^{n}_j - \sum_{k=1+L^n}^{P^n_{jt}} y^{n}_k \quad t = 1 + L^n, \ldots, T, j = 1, \ldots, t
\]

\[
B^{n}_t \geq -l^{n}_t + \sum_{j=1+L^n}^{t} \sum_{j=1+L^n}^{t} d_{j-L^n, t}^{n} i_{j}^{n} \sum_{k=1}^{t} \sum_{k=1}^{t} p_k E[d_{k,t}^{n} | \Omega_k] L^n i_{j}^{n} \quad t = 1 + L^n, \ldots, T, i = 1, \ldots, W
\]

\[
H^{n}_t \geq \sum_{j=1+L^n}^{t} \sum_{j=1+L^n}^{t} d_{j-L^n, t}^{n} i_{j}^{n} \sum_{k=1}^{t} \sum_{k=1}^{t} p_k E[d_{k,t}^{n} | \Omega_k] L^n i_{j}^{n} \quad t = 1 + L^n, \ldots, T, i = 1, \ldots, W
\]

\[
\delta_t = (0, 1) \quad t = 1, \ldots, T
\]

\[
y^{n}_t = (0, 1) \quad t = 1, \ldots, T
\]

\[
P^n_{jt} = (0, 1) \quad t = 1 + L^n, \ldots, T, j = 1 + L^n, \ldots, t
\]

Figure 3: MILP model for approximating \((R, S)\) policies

| \( \lambda^n_t \) | period | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------------------|-------|---|---|---|---|---|---|---|---|---|----|
| item              |       |   |   |   |   |   |   |   |   |   |    |
| 1                 | 40    | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 | 40 |    |
| 2                 | 40    | 56 | 29 | 54 | 70 | 50 | 50 | 45 | 45 | 45 |    |
| 3                 | 40    | 55 | 72 | 86 | 78 | 51 | 42 | 38 | 38 | 30 | 26 |
| 4                 | 41    | 58 | 75 | 63 | 40 | 35 | 33 | 18 | 29 | 39 |    |
| 5                 | 45    | 40 | 22 | 31 | 38 | 46 | 59 | 62 | 62 | 62 | 40 |

Table 1: Demand rates \( \lambda^n_t \) of the 5-item 10-period example
order-up-to-position 123 by the nature of stationary demand, while it is ordered up to a higher position 164 in period 5 to cover demands in the next 3 periods in order to coordinate with other items.

5. MILP model for approximating the optimal \((\sigma, \vec{S})\) policies

Since the landmark study of Scarf (1960) which proved the optimality for the single-item inventory system, there have been few attempts to prove the optimality for multi-item inventory systems, e.g.: (Johnson, 1967; Kalin, 1980; Ohno & Ishigaki, 2001; Gallego & Sethi, 2005). In this section we show how the MILP model proposed in Section 4.2 can be used to approximate the optimal replenish plan under \((\sigma, \vec{S})\) policy for the JRP.

**Definition 5.1.** Function \(G(\cdot) : \mathbb{R}^N \rightarrow \mathbb{R}\) is \(K\)-convex if
\[
G(ax + (1 - a)z) \leq aG(x) + (1 - a)[G(z) + K\delta(z - x)],
\]
where \(\delta(0) = 0, \delta(i) = 1\) for \(i > 0\), \(x \leq z\), and \(a \in [0, 1]\).

Gallego & Sethi (2005) showed the optimal policy for the joint setup cost case by studying function
\[
G_t(\vec{y}) = L_t(\vec{y}) + C_{t+1}(\vec{y} - \vec{d}_t). \tag{45}
\]

Consider a continuous \(K\)-convex function \(G_t(\cdot)\), then it has global minimum at \(S_t\). Define set \(\Sigma = \{\vec{I}_{t-1} \leq S_t | G_t(\vec{I}_{t-1}) \leq G_t(S_t) + K\}\), and set \(\sigma = \{\vec{I}_{t-1} \leq S_t | \vec{I}_{t-1} \notin \Sigma\}\). Lemma 5.1 shows that the optimal replenish plan is to order up to \(S_t\) if opening inventory levels \(\vec{I}_{t-1} \in \sigma\) and \(\vec{I}_{t-1} \leq S_t\); and not to order, otherwise.

**Lemma 5.1 (Gallego & Sethi (2005)).** If \(G\) is continuous \(K\)-convex, continuous and coercive, then

\begin{itemize}
  \item \(\vec{I} \in \Sigma \Rightarrow G(\vec{I}) \leq K + G(S)\),
  \item \(\vec{I} \in \sigma \Rightarrow G(\vec{I}) > K + G(S)\).
\end{itemize}

We next show that the MILP model in Fig. 3 can be adjusted to approximate set \(\sigma\) and \(S\). Since the \((\sigma, S)\) optimises group fixed ordering costs, holding costs and penalty costs. We first drop the item-specific fixed ordering cost, i.e., \(K^n \cdot y_n^n\), in the objective function (29). We then set the lead time of all items to 0, i.e.: \(L^n = 0, n = 1, \ldots, N\). Since orders are delivered immediately, we drop constraints (31) - (35).

Due to the complexity of \(\sigma\), it is impractical to derive a closed form expression for it. Alternatively, one may propose a strategy to determine whether given initial inventory levels \(\vec{I}_0 \in \sigma\). By solving our modified MILP model over planning horizon \(k, \ldots, T\), we observe the minimised expected total cost
Figure 4: Replenish plans of the 5-item 10-period example
$G_k(\tilde{S}_k)$, order-up-to-levels $\tilde{S}_k$, and the first period order decision $\delta_k$. If $\delta_k = 1$, then $I_{k-1} \in \sigma$; otherwise, $I_{k-1} \in \Sigma$. Therefore our MILP model can be used to determine whether given initial inventory levels $\tilde{I}_0 \in \sigma$. Moreover, by repeating this procedure, one can approximate the optimal replenish strategy for every period $k = 1, \ldots, T$.

**Example.** We illustrate the concept introduced on the 2-item 4-period example presented in Section 3. Assuming the initial inventory level $\tilde{I}_0 \in [0, \ldots, 20]$, we plot the expected total cost contours, obtained via the modified MILP in Fig. 5(a). Note that there are two similar minima, which is expected since the ordering cost is relatively small and the demand variance is large. We plot set $\sigma$ and $\tilde{S}$ obtained via the modified MILP model, and compare them with that obtained via stochastic dynamic programming in Fig. 5(b). The optimal policy is to place an order whenever inventory levels $\tilde{I}_0 = (I_0^1, I_0^2)$ fall in set $\sigma$, and not to place an order if $\tilde{I}_0$ fall in $\Sigma$. We observe that set $\sigma$ and $\tilde{S}$ obtained via the modified MILP model neatly approximate those obtained via stochastic dynamic programming.

![Expected total cost contour plot obtained via MILP approximation](image)

(a) Expected total cost contour plot obtained via MILP approximation

![Plot of expected total costs obtained via MILP and SDP](image)

(b) Plot of expected total costs obtained via MILP and SDP

Figure 5: Plot of expected total costs for the two-item joint replenishment numerical example

### 6. Computational Experiments

In this section we assess the cost performance of the $(R, S)$ policy by comparing its cost performance against $(Q, S, T)$ policy (Ozkaya et al., 2006), $Q(s, S)$ policy (Nielsen & Larsen, 2005), $P(s, S)$ policy (Viswanathan, 1997), $(Q, S)$ policy (Pantumsinchai, 1992), $MP$ policy (Atkins & Iyogun, 1988), $(s, c, S)_M$ policy (Melchiors, 2002), and $(s, c, S)_F$ policy (Federgruen et al., 1984), on data sets of Atkins & Iyogun (1988) and Viswanathan (1997). These data sets consider stationary demand over an infinite horizon. Unfortunately, computing $(R, S)$ policy parameters for infinite horizon JRP via our MILP model is computationally expensive; however, since demand is stationary, it is possible to
derive an efficient shortest path reformulation, which we present in Appendix A and we use in our computational study.

Computational experiments are conducted by using IBM ILOG CPLEX Optimization Studio 12.7 and Matlab R2016a on a 3.20 GHz Intel Core i5-6500 CPU with 16.0 GB RAM, 64 bit machine.

Since the shortest path reformulation operates over a finite horizon, in order to compare the cost performance of the \((R, S)\) policy with continuous-review \((s, c, S)\), \((Q, S)\), and \((Q, S, T)\) policy, we discretize each time period into 20 small periods. We consider a planning horizon length of 6.6 periods for a total of 132 small periods. For each test instance, we first obtain the optimal replenishment plan by solving the shortest path reformulation presented in Appendix A. The computational time is limited to 5 minutes, if a timeout occurs, the best solution available is adopted. Next, we simulate the expected average cost of each test instance via Monte Carlo Simulation (100,000 replications). Finally, we compare the average cost per small period against the average cost under existing policies.

The data set of Atkins & Iyogun (1988) assumes that the demand of each item follows stationary Poisson distribution with rate \(\lambda^n\), \(n = 1, \ldots, 12\). The item-specific fixed ordering cost \(K^n\), expected demand \(\lambda^n\), and lead time \(L^n\) are displayed in Table 2. Items share the same penalty cost \(b = 30\), holding cost \(h \in \{2, 6, 20\}\), and group fixed ordering cost \(K \in \{20, 50, 100, 150, 500\}\).

| items | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|
| \(K^n\) | 10 | 10 | 20 | 20 | 40 | 20 | 40 | 40 | 60 | 60 | 80 | 80 |
| \(\lambda^n\) | 40 | 35 | 40 | 40 | 40 | 20 | 20 | 28 | 20 | 20 | 20 | 20 |
| \(L^n\) | 0.2 | 0.5 | 0.2 | 0.1 | 0.2 | 1.5 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |

Table 2: \(K^n\), \(\lambda^n\), and \(L^n\) of data set Atkins & Iyogun (1988)

The data set of Atkins & Iyogun (1988) contains some unusual lot sizing instances; more specifically, instances for which the group as well as item fixed ordering costs become negligible in comparison to holding costs. In the lot-sizing literature the fixed ordering cost is commonly assumed to be greater than the holding cost (see Axsäter 2010, p. 62, Property 2); moreover, the penalty cost should not be smaller than the holding cost. Additionally, we observe that it is meaningless when the fixed ordering cost is greater than the penalty cost since in such a case the inventory system tends to place orders in every period instead of penalising backorders. To focus on meaningful lot sizing instances — instances in which a trade off between fixed ordering and holding/penalty cost is sought — we filter test instances of the data set of Atkins & Iyogun (1988) by using the following conditions: \(K > b \geq h\). We also check the order frequency in each period and we discard instances in which orders are issued too frequently — i.e. instance in which a replenishment is issued more than twice per time period, as it turns out that for these instances order coordination is straightforward due to negligible item fixed ordering costs: if a group order is placed, all items are ordered. We present computational results in Table 3.

Let \(\Delta\%\) denote the percentage gap between the expected average cost of
existing policies and that of the proposed \((R,S)\) policy, over the expected averaged cost of the \((R,S)\) policy. By definition, a positive \(\Delta\%\) represents the \((R,S)\) policy outperforms existing policy. Note that expected average costs under \((Q,S,T)\), \(Q(s,S)\), \(P(s,S)\), \((Q,S)\), and \((s,c,S)_M\) policy are obtained from \citet{ozkaya2006}, that of \((s,c,S)_F\) policy is obtained from \citet{melchiors2002}, and that of \(MP\) policy is obtained from \citet{viswanathan1997}.

We observe that the \((R,S)\) policy fully dominates all policies in 2 of 9 test instances; \((Q,S,T)\) is the best policy in 2 instances; \(Q(s,S)\) is the best policy in 4 instances; \(P(s,S)\) is the best policy in 1 instance. Moreover, the \((R,S)\) policy outperforms the \((Q,S)\) and \((s,c,S)_F\) policy, and no dominant policy on all test instances. The average cost improvement \(\Delta\%\) increases with the increase of group fixed ordering cost, and decreases with the increase of holding cost compared with \((s,c,S)_M\) and \((s,c,S)_F\) policy. That means an increase in group fixed ordering cost or a decrease in holding cost improves the cost performance of \((R,S)\) policy. It is difficult to make a general remark with respect to group fixed ordering cost and holding cost compared with \((Q,S,T)\), \(Q(s,S)\), \(P(s,S)\), \((Q,S)\), and \(MP\) policy. On average, the \((R,S)\) policy performs better than \(Q(s,S)\), \((Q,S)\), \(MP\), \((s,c,S)_M\), and \((s,c,S)_F\) policy with an average improvement of 0.07\%, 1.74\%, 0.89\%, 2.84\%, and 7.02\%, respectively; however, the \((Q,S,T)\) and \(P(s,S)\) policies performs slightly better than the \((R,S)\) policy with an average improvement of 0.09\% and 0.14\%, respectively.

The data set of \citet{viswanathan1997} adopts the same parameters as the data set of \citet{atkins1988}, except \(b \in \{10,50,100,200,1000,5000,10000,20000\}, h \in \{2,6,10,200,600,1000\}\), and \(K \in \{20,50,100,200,500\}\).

We filter the computational results by using the same conditions previously adopted. We present computational results of the \((R,S)\) policy on the data set of \citet{viswanathan1997} in Table 3. We observe that the \((R,S)\) policy dominates 13 of 31 test instances; \((Q,S,T)\) is the best policy in 13 instances; \(Q(s,S)\) is the best policy in 9 instances; \(P(s,S)\) is the best policy in 1 instances. There is no dominant policy on all test instances. Regarding the comparison with other policies, the average cost improvement \(\Delta\%\) decreases as the penalty cost increase; while there is no obvious trend with respect to the group fixed ordering cost, and penalty cost. On average, the \((R,S)\) policy performs better than \(Q(s,S), P(s,S), (Q,S), MP,\) and \((s,c,S)_F\) policy with average cost improve-
ments of 0.37%, 0.37%, 1.81%, 1.41%, and 1.67%; while the \((Q,S,T)\) policy performs slightly better than the \((R,S)\) policy with average cost improvement 0.19%.

| \(K\) | \(b\) | \(h\) | \((R,S)\) | \((Q,S,T)\) | \((Q,S)\) | \((Q,S)\) | \((Q,S)\) | \(MP\) | \((\sigma,\vec{S})\) | \(\Delta\%\) |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 10 | 2 | 772.25 | -0.03 | 0.48 | 0.76 | 8.30 | 1.79 | 1.80 |
| 50 | 10 | 2 | 813.94 | -0.48 | 0.12 | 0.62 | 4.07 | 1.64 | 1.74 |
| 100 | 10 | 2 | 861.05 | 0.23 | 0.70 | 1.17 | 3.68 | 2.20 | 2.38 |
| 200 | 10 | 2 | 932.86 | 1.62 | 1.83 | 2.38 | 2.88 | 3.42 | 3.73 |
| 500 | 10 | 2 | 1131.42 | 0.14 | 0.14 | 0.59 | 1.18 | 1.60 | 2.12 |
| 20 | 10 | 6 | 1166.06 | 0.85 | 2.84 | 0.01 | 7.99 | 1.08 | 1.04 |
| 50 | 10 | 6 | 1222.82 | -0.15 | 1.83 | 0.62 | 0.47 | 1.64 | 1.74 |
| 100 | 10 | 6 | 1283.92 | 1.33 | 2.50 | 1.26 | 4.49 | 2.90 | 2.97 |
| 200 | 10 | 6 | 1348.52 | 1.62 | 1.83 | 2.38 | 2.88 | 3.42 | 3.73 |
| 500 | 10 | 6 | 1413.72 | 0.30 | 1.23 | 1.02 | 2.12 | 2.10 | 2.33 |

Table 4: Computational results on the data set of Viswanathan (1997)

Even though the \((R,S)\) policy does not fully dominate other competing policies, it presents a key advantage: in contrast to all other policies in the literature, it is able to tackle stationary as well as nonstationary demand.

7. Conclusion

In this paper, we presented a mathematical programming approach for controlling the multi-item inventory system with joint replenishment under the \((R,S)\) policy. We first present an MILP-based model for approximating optimal \((R,S)\) policies, which is built upon the piecewise-linear approximation technique proposed by (Rossi et al., 2014). We further demonstrate that the MILP model can be used to approximate the \((\sigma,\vec{S})\) policy.
We conducted an extensive computational study comprising 55 instances. We first evaluated our approach on the data set of Atkins & Iyogun (1988). This evaluation demonstrates that the \((R, S)\) policy fully dominates other competing policies in the literature in 2 out of 10 test instances considered. The \((R, S)\) policy performs better than \((Q(s, S), (Q, S), MP, (s, c, S)_M, and (s, c, S)_F\) policies with an average improvement of 0.17%, 2.61%, 0.33%, 1.24%, and 4.60%, respectively; however, the \((Q, S, T)\) and \(P(s, S)\) policies performs slightly better than the \((R, S)\) policy with an average improvement of 1.01% and 0.71%.

Computational experiments on the data set of Viswanathan (1997) indicates that \((R, S)\) is the best policy in 13 out of 45 test instances. \((R, S)\) performs better than \((Q, S), MP, and (s, c, S)_F\) policies with average cost improvements of 1.81%, 0.62%, 0.82%; while \((Q, S, T), Q(s, S), \) and \(P(s, S)\) policies perform slightly better than it with an average cost improvement 1.11%, 0.64%, 0.40%. Most importantly, the \((R, S)\) policy comes with the additional advantage of being able to tackle stationary and nonstationary demand. Future research may focus on investigating the cost performance of \((R, S)\) policy in a rolling horizon setting.

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**Appendix A. Shortest path reformulation for approximating stationary stochastic (R, S) policies**

In this section we present an efficient shortest path reformulation for computing stationary (R, S) policies.

Consider a network \( G = (N, A) \) with nodes \( N = \{1, \ldots, T\} \) representing time periods, and arc \((i, j)\) between each pair of \((i, j)\) representing a possible decision to issue an order in period \(i\) to satisfy demands in periods \(i, \ldots, j\). Assigning a cost to this arc, solving the optimisation problem in Fig. 4 is equivalent to finding the shortest path between nodes 1 and \(T\) in the network \(G\). In the rest of this section, we first present how to compute the cost of each arc, and then present the shortest path reformulation.

Consider a replenishment cycle \(i, \ldots, j\), where the only order is issued in period \(i\) with order-up-to-position \(S^n_{i,j}\), and the next order is issued in period \(j + 1\), for \(i = 1, \ldots, T\), \(j = i, \ldots, T\), \(n = 1, \ldots, N\). We assume \(d^n\) follows Poisson distribution with rate \(\lambda^n\). Then, \(S^n_{i,j}\) is calculated by Askin (1981),

\[
\sum_{t=i}^{j} G_{d^n_{i,t+l,n}}(S^n_{i,j}) = \frac{(j - i + 1) \cdot h^n}{h^n + b^n}.
\]

Note that the order-up-to-position \(S^n_{i,j}\) actually accounts for demand variances over periods \(i, \ldots, j + L^n\), which is reflected on the cumulative distribution function \(G_{d^n_{i,t+l,n}}(\cdot)\) on the left-hand-side of Eq. (A.1).

Since the demand of item \(n\) follows Poisson distribution with rate \(\lambda^n\), we could approximate the cost of the replenishment cycle \(i, \ldots, j\) by that of the cycle \(i + L^n, \ldots, j + L^n\) as shown in Fig. A.6. As a result, the cycle cost \(c^n_{i,j}\) can be calculated as follows,

\[
c^n_{i,j} = k^n + h^n \sum_{t=i}^{j} \hat{L}(S^n_{i,j} - L^n \lambda^n, d^n_t) + b^n \sum_{t=i}^{j} \hat{L}(S^n_{i,j} - L^n \lambda^n, d^n_t).
\]

At the beginning of the planning horizon, the initial inventory level is \(I^n_0\). We check the cost of not issuing an order in period 1, \(\bar{c}^n_{1,j}\), and update \(c^n_{i,j}\) with \(\bar{c}^n_{i,j}\) if \(\bar{c}^n_{1,j} \leq c^n_{1,j}\), for \(j = 1, \ldots, T\).

\[
\bar{c}^n_{1,j} = h^n \cdot \sum_{t=1}^{j} \hat{L}(I^n_0, d^n_t) + b^n \cdot \sum_{t=1}^{j} \hat{L}(I^n_0, d^n_t).
\]

Additionally, we introduce an auxiliary binary variable \(P^n_j\), which is equal to 1 if an order is placed in period 1 to satisfy demands in cycle 1, \ldots, \(j\), otherwise 0.
We now present the shortest path reformulation in Fig. A.7. Let binary variable $Y_{nij}$ equal to 1 if an order is issued in period $i$ to cover demands in periods $i, \ldots, j$, otherwise 0. The objective is to find the optimal replenishment plan that minimizing the expected group fixed order costs, item-specific fixed order costs, holding costs and penalty costs over periods $1, \ldots, T$ for items $1, \ldots, N$.

\[
\min \sum_{i=1}^{T} K \cdot \delta_i + \sum_{n=1}^{N} \sum_{j=1}^{T} \sum_{i=1}^{T} c_{nj} \cdot Y_{nij} \quad (A.4)
\]

subject to, $n = 1, \ldots, N$,

\[
\delta_1 \geq \sum_{j=1}^{T} Y_{1j} \cdot P^n_j \quad (A.5)
\]

\[
d_i \geq \sum_{j=1}^{T} Y_{ij} \quad i = 2, \ldots, T \quad (A.6)
\]

\[
\sum_{j=1}^{T} Y_{1j} = 1 \quad (A.7)
\]

\[
\sum_{j=1}^{T} Y_{ij} - \sum_{k=1}^{i-1} Y_{ik} = 0 \quad i = 2, \ldots, T - 1 \quad (A.8)
\]

\[
\sum_{i=1}^{T} Y_{nP} = 1 \quad (A.9)
\]

Recall that $P^n_j$ represents the item-specific first period purchase decision, which is set to 1 if an order is issued in period 1, otherwise 0. Therefore, Constraints (A.5) guarantee the group fixed order cost in period 1 is properly counted. Constraints (A.6) ensure that the group fixed order cost is encountered whenever any item is replenished in period 2, \ldots, $T$. Constraints (A.7) ensure
that there is no more than one outgoing arc from period 1. Constraints (A.8) are flow balance equations. Constraints (A.9) guarantee that period $T$ is included in a replenishment cycle. By solving Fig. A.7, the group order decision $\delta^n_t$ and item-specific order decision $y^n_i$ are obtained\(^2\) for $t = 1, \ldots, T$, $n = 1, \ldots, N$.

\(^2\)This can be obtained by adding constraints $y^n_i = \sum_{j=1}^T Y^n_{ij} P^n_j$ and $y^n_i = \sum_{j=2}^T Y^n_{ij}$, $i = 2, \ldots, T$, to Fig. A.7.