Connections between Hilbert W*-modules and direct integrals

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Abstract

Investigating the direct integral decomposition of von Neumann algebras of bounded module operators on self-dual Hilbert W*-modules an equivalence principle is obtained which connects the theory of direct disintegration of von Neumann algebras on separable Hilbert spaces and the theory of von Neumann representations on self-dual Hilbert A-modules with countably generated A-pre-dual Hilbert A-module over commutative separable W*-algebras A. Examples show possibilities and bounds to find more general relations between these two theories, (cf. R. Schaflitzel’s results). As an application we prove a Weyl–Berg–Murphy type theorem: For each given commutative W*-algebra A with a special approximation property (*) every normal bounded A-linear operator on a self-dual Hilbert A-module with countably generated A-pre-dual Hilbert A-module is decomposable into the sum of a diagonalizable normal and of a "compact" bounded A-linear operator on that module.

The idea to investigate the subject treated in the present paper arose in discussions with K. Schmüdgen and J. Friedrich at the University of Leipzig. They suggested to the author that self-dual Hilbert W*-modules over commutative W*-algebras might be closely connected with direct integrals of measurable fields of Hilbert spaces or, respectively, with some topologically related objects. Moreover, von Neumann algebras of bounded module operators on these self-dual Hilbert W*-modules should be
decomposable into direct integrals of measurable fields of von Neumann algebras in a very easy way.

Following this line appropriate facts have been proved. One gets a new view on the nowadays well-known theory of direct integral decomposition of von Neumann algebras $\mathcal{M}$ on separable Hilbert spaces. This theory is shown to be equivalent to the theory of von Neumann representations of $W^*$-algebras $\mathcal{M}$ on self-dual Hilbert $W^*$-modules $\mathcal{H}$ over $W^*$-subalgebras $\mathcal{B}$ of the center of $\mathcal{M}$, where $\mathcal{B}$ has to be separable and the Hilbert $\mathcal{B}$-modules have to possess countably generated $\mathcal{B}$-pre-dual Hilbert $\mathcal{B}$-modules. The most interesting point is that the basic structures, Hilbert $W^*$-modules and direct integrals of Hilbert spaces, are quite different. However, this equivalence will not be preserved turning to direct integrals of von Neumann algebras on non-separable Hilbert spaces, in general. It would be interesting to make further considerations in this direction taking in account recent results of R. Schafitcel, P. Richter and other authors. Applying this equivalence principle, a new result is found generalizing theorems of H. Weyl, I. D. Berg and G. J. Murphy.

Last but not least one realizes that the forthcoming theory is closely related to the description of self-dual Hilbert $AW^*$-modules over commutative $AW^*$-algebras in terms of Boolean valued analysis and logic created by M. Ozawa and G. Takeuti during 1979-85. There are also relations to the work of H. Takemoto who has described similar phenomena in terms of continuous fields of Hilbert spaces. In the present more special case the mathematical terminology describing the situation is taken from measure theory.

The present paper is organized as follows: The first section is a short summary of facts from the theory of direct integrals of measurable fields of Hilbert spaces and of von Neumann algebras, at one side, and from the theory of Hilbert $W^*$-modules over commutative Hilbert $W^*$-algebras, at the other. We slightly modify the traditional denotations for our purposes and recall some necessary facts from the literature. The second section deals with the interrelation between self-dual Hilbert $W^*$-modules over commutative $W^*$-algebras $\mathcal{A}$ possessing a countably generated $\mathcal{A}$-pre-dual Hilbert $\mathcal{A}$-
module and special sets of mappings into measurable fields of Hilbert spaces, giving rise to isomorphisms. Considering von Neumann algebras of bounded module operators on those Hilbert W*-modules we obtain their direct integral decomposition. As an application for commutative W*-algebras A with a special property (*) we prove that on self-dual Hilbert A-modules which possess a countably generated A-pre-dual Hilbert A-module every normal bounded module operator T is decomposable into the sum of a normal, diagonalizable bounded module operator D and a "compact" bounded module operator K.

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1 Preliminaries

We start with some necessary informations about Hilbert $W^*$-modules. Throughout the present paper the symbol $A$ is denoting a $C^*$-algebra. We make the convention that all modules over $A$ are left modules by definition. Following W. L. Paschke [22] and other authors [2, 14, 20, 21, 39] we define a pre–Hilbert $A$-module over a certain $C^*$-algebra $A$ as an $A$-module $H$ equipped with a mapping $\langle \cdot, \cdot \rangle : H \times H \to A$ satisfying:

(i) $\lambda (ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in A$, $x \in H$.

(ii) $\langle x, x \rangle \geq 0$ for every $x \in H$.

(iii) $\langle x, x \rangle = 0$ if and only if $x = 0$.

(iv) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in H$.

(v) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for every $a, b \in A$, every $x, y, z \in H$.

The mapping $\langle \cdot, \cdot \rangle$ is the so called $A$-valued inner product on $H$. A pre–Hilbert $A$-module is called to be Hilbert if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|_{A}^{1/2}$. Two Hilbert $A$-modules $\{H_1, \langle \cdot, \cdot \rangle_1\}$, $\{H_2, \langle \cdot, \cdot \rangle_2\}$ over a certain $C^*$-algebra $A$ are isomorphic if there exists a bijective, $A$-linear, bounded mapping $T : H_1 \to H_2$ such that $\langle \cdot, \cdot \rangle_1 \equiv \langle T(\cdot), T(\cdot) \rangle_2$ on $H_1 \times H_1$. A Hilbert $A$-module $H$ is finitely generated if it is finitely generated as an $A$-module. It is countably generated if there exists a countable set of generators inside $H$ such that the set of all finite $A$-linear combinations of generators is norm-dense in $H$. A Hilbert $A$-module $\{H, \langle \cdot, \cdot \rangle\}$ over a $C^*$-algebra $A$ is faithful if the norm-closed $A$-linear hull of the range of the inner product, $\langle H, H \rangle$, is identical with $A$.

A central notion in the theory of Hilbert $C^*$-modules is the notion of self-duality since self-dual Hilbert $C^*$-modules form a proper subcategory of the category of Banach $C^*$-modules with advantageous properties, cf. [4, 5]. Denote by $H'$ the set of all bounded module maps $f : H \to A$. Following W. L. Paschke [22] a Hilbert $C^*$-module $\{H, \langle \cdot, \cdot \rangle\}$ is called to be self-dual if every map $r \in H'$ is of the form $\langle \cdot, a_r \rangle$. A Hilbert $C^*$-module $\{H, \langle \cdot, \cdot \rangle\}$ is called to be self-dual if every map $r \in H'$ is of the form $\langle \cdot, a_r \rangle$.
for a certain element \( a_r \in \mathcal{H} \). Let us remark, that a Hilbert \( AW^* \)-module over a commutative \( AW^* \)-algebra is self-dual if and only if it is Kaplansky-Hilbert.

In the following we direct our attention to Hilbert \( W^* \)-modules over commutative \( W^* \)-algebras. In that case the \( A \)-valued inner product on \( \{ \mathcal{H}, \langle ., . \rangle \} \) lifts to an \( A \)-valued inner product \( \langle ., . \rangle_D \) on the Banach \( A \)-module \( \mathcal{H}' \) turning \( \{ \mathcal{H}', \langle ., . \rangle_D \} \) into a self-dual Hilbert \( A \)-module. The equalities

\[
\langle \langle ., x \rangle, \langle ., y \rangle \rangle_D = \langle x, y \rangle \text{ for every } x, y \in \mathcal{H},
\]

\[
\langle \langle ., x \rangle r(\cdot) \rangle_D = r(x) \text{ for every } x \in \mathcal{H}, \text{ every } r \in \mathcal{H}'
\]

are satisfied, cf. [22, Th. 3.2]. Moreover, the following criterion for self-duality can be formulated:

**Proposition 1.1 (cf. [22, Th. 3.2, Th. 3.12])** Let \( A \) be a commutative \( W^* \)-algebra and let \( \{ \mathcal{H}, \langle ., . \rangle \} \) be a Hilbert \( A \)-module. Then the following two conditions are equivalent:

(i) \( \mathcal{H} \) is self-dual.

(ii) There exist an index set \( I \) and a collection of (not necessarily distinct) projections \( \{ p_\alpha : \alpha \in I \} \) of \( A \) indexed by \( I \) such that \( \mathcal{H} \) is isomorphic to the set of all \( I \)-tuples

\[
\tau - \Sigma \{ A p_\alpha : \alpha \in I \} = \{ \{ x_\alpha \} : x_\alpha \in A p_\alpha, \alpha \in I, \| \sum x_\alpha x_\alpha^* \|_A < +\infty \}
\]

equipped with the \( A \)-valued inner product

\[
\langle x, x \rangle = w^* - \lim_{S \in \mathcal{F}} \sum_{\alpha \in S} x_\alpha x_\alpha^*, x = \{ x_\alpha : \alpha \in I \},
\]

where \( \mathcal{F} \) is the net of all finite subsets of \( I \) being partially ordered by inclusion.

**Corollary 1.2** Let \( A \) be a commutative \( W^* \)-algebra, \( \mathcal{H} \) be a self-dual Hilbert \( A \)-module being representable as \( \tau - \Sigma \{ A p_\alpha : \alpha \in I \} \) for a countable set \( I \). Then there exists a countably generated Hilbert \( A \)-module \( \mathcal{K} \) such that the \( A \)-dual Banach \( A \)-module of \( \mathcal{K} \) is \( \mathcal{H} \).
Beside the Hilbert $A$-modules we would like to consider $A$-linear bounded operators $T$ on them. If the underlying Hilbert C*-module $\mathcal{H}$ is self-dual they always possess an adjoint operator $T^*$ being bounded and $A$-linear. The set of all such operators, $\text{End}_A(\mathcal{H})$, forms a C*-algebra in that situation. Moreover, $\text{End}_A(\mathcal{H})$ becomes a W*-algebra over self-dual Hilbert W*-modules, cf. [22]. An important subset of $\text{End}_A(\mathcal{H})$ is the set of "compact" operators $K_A(\mathcal{H})$ being defined as the norm-closed linear hull of the set

$$\{\theta_{a,b} \in \text{End}_A(\mathcal{H}) : \theta_{a,b}(c) = \langle c, a \rangle b \text{ for every } a, b, c \in \mathcal{H}\}.$$ 

It is a C*-subalgebra and a two-sided ideal of $\text{End}_A(\mathcal{H})$.

Our standard reference sources for direct disintegration theory are the monographs [17, 24, 32] and the papers [3, 27, 28]. For recent developments in this area see [15, 16, 37, 40], [25, 26].

The following definition we would like to take as a basis ([32, Def. 8.9], [17, p. 206-207]): Let $X$ be a locally compact Hausdorff measure space with Borel measure $\mu$. A set $\{H_x : x \in X\}$ of Hilbert spaces indexed by $X$ is called to be a $\mu$-measurable field of Hilbert spaces if there exists a subspace $\mathcal{E}$ of the product space $\prod\{H_x : x \in X\}$ with the properties:

(i) For every $z \in \mathcal{E}$ the function $\|z(x)\|$ is an element of $L^\infty(X, \mu)$.

(ii) If for a certain $y \in \prod\{H_x : x \in X\}$ the function $\langle y(x), z(x) \rangle$ belongs to $L^\infty(X, \mu)$ for every $z \in \mathcal{E}$ then $y \in \mathcal{E}$.

(iii) There exists a countable subset $\{z_i : i \in \mathbb{N}\}$ of elements of $\mathcal{E}$ such that for every $x \in X$ the set $\{z_i(x) : i \in \mathbb{N}\}$ is a basis of the Hilbert space $H_x$.

Elements $\{h_x : x \in X\}$ of $\mathcal{E}$ are called to be $\mu$-measurable. We will specify the set $\mathcal{E}$ in further considerations since the structure of $\mathcal{E}$ is sometimes important for our purposes. Let us remark that (iii) implies the separability of the Hilbert spaces $H_x$, $x \in X$, of the $\mu$-measurable field $\{H_x : x \in X\}$. Moreover, the map $x \in X \to \dim(H_x) \in \mathbb{R}$ is $\mu$-measurable.
Denote by \( L^\infty(X, \mu, \{H_x : x \in X\}) \) the set of all rest classes of essentially bounded, \( \mu \)-measurable mappings of \( X \) into the \( \mu \)-measurable field of Hilbert spaces \( \{H_x : x \in X\}, (x \mapsto H_x) \), where the elements of one rest class differ only on subsets of \( \mu \)-measure zero. Analogously, define \( L^1(X, \mu, \{H_x : x \in X\}) \) as the set of all rest classes of mappings \( f : x \in X \mapsto H_x \in \{H_x : x \in X\} \) possessing a finite integral \( \int_X \|f(x)\|_{H_x} \, d\mu(x) \), where the elements of one rest class differ only on subsets of \( X \) of \( \mu \)-measure zero. Defining suitable operations, norms and other structural elements on \( L^\infty(X, \mu, \{H_x : x \in X\}) \) and \( L^1(X, \mu, \{H_x : x \in X\}) \) they become a faithful self-dual Hilbert \( L^\infty(X, \mu) \)-module with countably generated \( L^\infty(X, \mu) \)-pre-dual Hilbert \( L^\infty(X, \mu) \)-module and a Banach \( L^\infty(X, \mu) \)-module, respectively. The third structure needed in the following is the classical direct integral of the \( \mu \)-measurable field of Hilbert spaces \( \{H_x : x \in X\}, \int_X H_x \, d\mu(x) \).

Recall, that an operator \( T \) on \( \int_X H_x \, d\mu(x) \) is called to be \( \mu \)-measurable if the operator \( T(x) \) acts on \( H_x \) as a bounded linear operator for almost every \( x \in X \) and \( T(\mathcal{E}) \subseteq \mathcal{E} \). Now, following [32, Def. 7.7, Cor. 7.8, Def. 7.9] denote by \( \int_X \text{End}_C(H_x) \, d\mu(x) \) the set of decomposable operators on \( \int_X H_x \, d\mu(x) \), i.e. the set of all rest classes of essentially bounded, \( \mu \)-measurable fields of operators \( \{B_x : B_x \in \text{End}_C(H_x); x \in X\} \), where the elements of one rest class differ only on subsets of \( X \) of \( \mu \)-measure zero. Note, that the commutant with respect to \( \text{End}_C(\int_X H_x \, d\mu(x)) \) of the set \( \{a \cdot \text{id}_{L^2} : a \in L^\infty(X, \mu)\} \) of all diagonal operators on \( \int_X H_x \, d\mu(x) \) equals to \( \int_X \text{End}_C(H_x) \, d\mu(x) \) at least if the Hilbert spaces \( H_x \) are \( \mu \)-almost everywhere separable. This property is lost in certain cases when the Hilbert spaces \( H_x \) are taken to be non-separable, (cf. [25, 26]). With suitable chosen operations it is a normed \( * \)-algebra. Moreover, in the classical situation when \( \mu \)-almost all Hilbert spaces \( H_x \) are separable it is a \( W^* \)-algebra of type I.

Now we are prepared for further considerations.
2 An equivalence principle

Let $A = L^\infty(X, \mu)$ be a commutative $W^*$-algebra, $X$ be a suitable chosen locally compact, Hausdorff measure space with Borel measure $\mu$. The purpose of the considerations below is, first, to show that each self-dual Hilbert $A$-module $\mathcal{H}$ possessing a countably generated $A$-pre-dual Hilbert $A$-module is isomorphic to a certain Hilbert $L^\infty(X, \mu)$-module of type $L^\infty(X, \mu, \{H_x : x \in X\})$ for a suitable chosen $\mu$-measurable field of separable Hilbert spaces $\{H_x : x \in X\}$ on $(X, \mu)$, and secondly, to derive the direct integral decomposition of the operator algebra $\text{End}_A(\mathcal{H})$. Moreover, we will look for possibilities and bounds of generalization of these equivalence relations we get. Formulating the theorems below we enclose two results of I. E. Segal [27, 28] for completeness.

**Theorem 2.1 (existence of isomorphisms)**

Let $A$ be a commutative $W^*$-algebra and $\mathcal{H}$ be a self-dual Hilbert $A$-module being the $A$-dual Banach $A$-module of a countably generated Hilbert $A$-module. Then there exists a locally compact, Hausdorff measure space $X$ with a Borel measure $\mu$ and a $\mu$-measurable field of Hilbert spaces $\{H_x : x \in X\}$ such that:

(i) $A$ is (isometricly) $*$-isomorphic to $L^\infty(X, \mu)$.

(ii) $\mathcal{H}$ is isometricly isomorphic to $L^\infty(X, \mu, \{H_x : x \in X\})$ as a Hilbert $A$-module.

(iii) $\text{End}_A(\mathcal{H})$ is (isometricly) $*$-isomorphic to the $W^*$-algebra $\int_X \text{End}_C(H_x) d\mu(x)$ on the Hilbert space $\int_X H_x d\mu(x)$.

(iv) The pre-dual of $\mathcal{H}$ is isometricly isomorphic to $L^1(X, \mu, \{H_x : x \in X\})$.

(v) The pre-dual of $\text{End}_A(\mathcal{H})$ is isometricly isomorphic to $L^1(X, \mu, \{[\text{End}_C(H_x)]_* : x \in X\})$. 
Theorem 2.2 (uniqueness of isomorphisms)

If under the assumptions of the previous theorem the $W^*$-algebra $A$ is faithfully and normally representable on a separable Hilbert space then one has:

(i) If there exist two locally compact, second countable, Hausdorff measure spaces $X_1, X_2$ equipped with the $\sigma$-finite Borel measures $\mu_1, \mu_2$, respectively, such that $A$ is $\ast$-isomorphic to both $L^{\infty}(X_1, \mu_1)$ and $L^{\infty}(X_2, \mu_2)$ then there exist two null sets $N_1 \subset X_1, N_2 \subset X_2$, a Borel isomorphism $\phi : X_2 \setminus N_2 \to X_1 \setminus N_1$ and a $\ast$-isomorphism $\pi : L^{\infty}(X_1, \mu_1) \to L^{\infty}(X_2, \mu_2)$ such that $\mu_1$ and $\phi(\mu_2)$ are equivalent in the sense of absolute continuity on $X_1 \setminus N_1$, and the equality $\pi(a)(x) = a(\phi(x))$ holds for every $a \in L^{\infty}(X_1, \mu_1)$ and for every $x \in X_2 \setminus N_2$.

(ii) If there are, additional, two different $\mu_1, \mu_2$-measurable fields of Hilbert spaces $\{H^{(1)}_x : x \in X\}, \{H^{(2)}_x : x \in X\}$ satisfying condition (ii) of the previous theorem then there exist two null sets $Y_1 \subset X_1, Y_2 \subset X_2$, a Borel isomorphism $\psi : X_2 \setminus Y_2 \to X_1 \setminus Y_1$ and a $\mu_1$-$\mu_2$-measurable field of unitary operators $\{U_x : H^{(1)}_x \to H^{(2)}_{\psi^{-1}(x)} : x \in X_1 \setminus Y_1\}$ such that $\mu_1$ and $\psi(\mu_2)$ are equivalent in the sense of absolute continuity on $X_1 \setminus Y_1$, and that $U_x \text{End}_C(H^{(1)}_x)U_x^* = \text{End}_C(H^{(2)}_{\psi^{-1}(x)})$ for every $x \in X_1 \setminus Y_1$.

Proofs of the theorems: The assertion (i) of the first theorem was proved by I.E.Segal ([27, 28]) in the early fifties and can be found at [24, Th. 3.4.4], whereas item (i) of the second theorem can be derived from [32, Lemma 8.22, Th. 8.23] as a special case. Therefore, one can identify $A$ with $L^{\infty}(X_K, \mu_K)$ for a special locally compact, Hausdorff measure space $X_K$ with Borel measure $\mu_K$ being constructed from the compact Hausdorff space $K$ realizing the $\ast$-isomorphy $A = C(K)$ along the line of [32, p.110]; i.e., taking $X_K$ as the union of the support sets $\Gamma_\alpha \subseteq K$ of a maximal family of positive normal measures $\mu_\alpha$ on $K$ with disjoint supports, and defining $\mu_K$ on $X_K$ by the formula $\mu_K(f) = \sum_\alpha \mu_\alpha(f)$ for each continuous function $f$ on $X_K$ with compact support. Finally, one has a bijection between continuous functions on $K$ and rest classes of $\mu_K$-measurable, essentially bounded functions of $L^{\infty}(X_K, \mu_K)$. Now, according to the isometric isomorphy of the Hilbert $A$-modules
\[ \mathcal{H} = \tau - \Sigma \{ A p_i : i \in \mathbb{N} \} \] one gets assertion (ii) of the first theorem identifying \( A \) with \( C(K) \) and considering the Hilbert spaces \( H_x = \tau - \Sigma \{ f(x) \cdot p_i(x) : f \in C(K), i \in \mathbb{N} \} \) for each \( x \in X_K \subseteq K \). They form a \( \mu_K \)-measurable field of separable Hilbert spaces on \( X_K \), where \( \mathcal{E} \) is the subset of all square-integrable elements of \( L^\infty(X_K, \mu_K, \{ H_x : x \in X_K \}) \). Note, that \( \mathcal{E} \) is norm-dense in \( \int_{X_K} \mathcal{H}_x \, d\mu(x) \) by definition and \( \tau_1 \)-dense in \( L^\infty(X_K, \mu_K, \{ H_x : x \in X_K \}) \). Turning to \( A = L^\infty(X_K, \mu_K) \) and recalling the definition of \( \tau_1 - \Sigma \) type Hilbert \( C^* \)-modules one finishes.

Now consider the set \( \text{End}_A(\mathcal{H}) \) of all bounded, \( A \)-linear operators on \( \mathcal{H} \). The boundedness and the \( A \)-linearity of these operators guarantee the invariance of the Hilbert spaces \( H_x \) under the action of them. Moreover, \( \text{End}_A(\mathcal{H}) \) acts on each Hilbert space \( H_x \), \( (x \in X_K) \), like \( \text{End}_C(H_x) \) and, globally, preserves \( \mathcal{E} \) and the \( \mu_K \)-measurability of the field \( \{ H_x : x \in X_K \} \). That is, \( \text{End}_A(\mathcal{H}) \) is embeddable into \( \int_X \text{End}_C(H_x) \, d\mu(x) \) in the sense of the coincidence of these operators on \( \mathcal{E} \). Vice versa, every essentially bounded, \( \mu_K \)-measurable map from \( X_K \) onto \( \{ \text{End}_C(H_x) : x \in X_K \} \) induces a bounded, \( A \)-linear operator on \( \mathcal{E} \) and, hence, on \( \mathcal{H} \). So one has shown the (isometric) \( * \)-isomorphy of these \( W^* \)-algebras, i.e. assertion (iii) of the first theorem.

Item (ii) of the second theorem can be derived from the isometric Hilbert \( A \)-module isomorphism of \( L^\infty(X_1, \mu_1, \{ H^{(1)}_x : x \in X_1 \}) \) and \( L^\infty(X_2, \mu_2, \{ H^{(2)}_x : x \in X_2 \}) \), from the commutativity of \( A \) and from (i) of both the theorems, whereas the facts (iv) and (v) of the first theorem follow from the self-duality of Hilbert spaces and from [24, p.70, Prop.] or from [32, Prop. 8.38], respectively.

**Example 2.3** For a fixed \( W^* \)-algebra \( A \) use the denotations

\[
\begin{align*}
    l_2(A) &= \left\{ \{ a_i \}_{i \in \mathbb{N}} : a_i \in A, \sum_i a_i a_i^* \text{ converges in } \| \cdot \|_A \right\} \\
    l_2(A)' &= \left\{ \{ a_i \}_{i \in \mathbb{N}} : a_i \in A, \left\| \sum_i a_i a_i^* \right\|_A \text{ converges } \right\}
\end{align*}
\]

for the standard countably generated Hilbert \( A \)-module and its \( A \)-dual Banach \( A \)-module.

a) Let \( A = l^\infty \) and \( \mathcal{H} = (l^2(l^\infty))' \). The \( W^* \)-algebra \( l^\infty \) is faithfully representable as the von Neumann algebra of all bounded, diagonal operators on the separable
Hilbert space $l^2$. According to the theorems above one has

$$H = (l^2(l^\infty))' = L^\infty(N, \nu, \{l^2_i : i \in N\})$$

$$\text{End}_A(H) = \int_N \text{End}_C(H_x) d\nu(x),$$

where $\nu$ denotes a discrete measure on the set of natural numbers $N$.

b) For $A = L^\infty([0, 1], \lambda)$ and $H = (l^2(L^\infty([0, 1], \lambda)))'$ one has

$$H = L^\infty([0, 1], \lambda, \{l^2_i : i \in [0, 1]\})$$

$$\text{End}_A(H) = \int_{[0,1]} \text{End}_C(H_x) d\lambda(x),$$

where $\lambda$ denotes the Lebesgue measure on the unit interval.

**Corollary 2.4 (cf. [32, Cor. 8.20])** For $A = L^\infty(X, \mu)$, $H = (l^2(A))'$ one has

$$\text{End}_A(H) = \text{End}_C(l^2)\otimes A,$$

where $\otimes$ denotes the $w^*$-tensor product.

**Remark 2.5** If the locally compact measure space $X$ is not second countable and the Borel measure $\mu$ on $X$ is not $\sigma$-finite then the statement of Theorem 2.2 is not longer true, in general. If one omits the separability condition to the Hilbert spaces $H_x$ then Theorem 2.1 fails to be true, in general.

For example, consider the $W^*$-algebra $A = L^\infty([0, 1], \lambda)$, where $\lambda$ denotes the Lebesgue measure, and the Hilbert $A$-module $H = \tau - \Sigma\{A_{(\alpha)} : \alpha \in \{[0, 1], \mu\}\}$, with $\nu$ a discrete measure on $[0, 1]$. Following the idea of Theorem 2.1,(ii) one should like to compare $H$ with $L^\infty([0, 1], \lambda, \{l^2_{(\alpha)}([0, 1]) : \alpha \in \{[0, 1], \mu\}\})$. (For the more complicated general definition of a $\mu$-measurable field of non-separable Hilbert spaces see R. Schafitizel [25, 26] e.g.) However, the special mapping

$$\{[0, 1], \lambda\} \longrightarrow l^2([0, 1])$$

$$x \longrightarrow \{\delta_x, t(t) : t \in \{[0, 1], \nu\}\}$$

belongs to $L^\infty([0, 1], \lambda, \{l^2_{(\alpha)}([0, 1]) : \alpha \in \{[0, 1], \mu\}\})$ as a non-zero element, whereas its reflection inside $H = \tau - \Sigma\{A_{(\alpha)} : \alpha \in \{[0, 1], \nu\}\}$ gives the zero element.
Beside this, from a result of R. Schaflitzel [25], [26, Lemma 6] there follows that under the assumption of the continuum-hypothesis it may happen that the algebra of decomposable operators is not the commutant of the algebra of diagonalizable operators on direct integrals of certain \( \mu \)-measurable fields of Hilbert spaces \( \{ H_x : x \in X \} \) on \( X \) with \( \dim(H_x) \geq c \), \( \text{card}(X) \geq c \) and an almost pointwise orthogonal generating set \( \Gamma_o \) of the corresponding direct integral of the Hilbert spaces \( H_x \) with \( \text{card}(\Gamma_o) > c \). As a concrete example he considered \( X = [0,1], \lambda \) - the Lebesgue measure and \( H_x = l^2([0,1]) \), i.e. the same situation as above.

Now we are interested in a direct integral decomposition of von Neumann algebras \( M \) of operators on self-dual Hilbert \( W^* \)-modules \( \mathcal{H} \) over commutative \( W^* \)-algebras \( A \). Conversely, we ask for which \( W^* \)-subalgebras \( B \) of the centre of a given \( W^* \)-algebra \( M \) there exists a self-dual Hilbert \( B \)-module \( \mathcal{H} \) such that \( M \) is faithfully \( \ast \)-representable as a von Neumann subalgebra of \( \text{End}_B(\mathcal{H}) \). The answer can be derived from the direct disintegration theory, especially from [32, Th. 8.22, Th. 8.23].

**Corollary 2.6** Let \( A \) be a commutative \( W^* \)-algebra and \( \mathcal{H} \) be a self-dual Hilbert \( A \)-module possessing a countably generated \( A \)-pre-dual Hilbert \( A \)-module. Let \( M \in \text{End}_A(\mathcal{H}) \) be a von Neumann subalgebra.

If \( A = L^\infty(X,\mu), \mathcal{H} = L^\infty(X,\mu,\{H_x : x \in X\}) \) in the sense of Theorem 2.1,(i),(ii) then there exists a \( \mu \)-measurable field of von Neumann algebras \( \{ M_x : x \in X \} \) on the \( \mu \)-measurable field of Hilbert spaces \( \{ H_x : x \in X \} \) such that \( M \) is (isometricly) \( \ast \)-isomorphic to \( \int_X M_x d\mu(x) \).

**Corollary 2.7** Let \( M \) be a \( W^* \)-algebra possessing a normal, faithful representation on a separable Hilbert space. Let \( B \) be a \( W^* \)-subalgebra of the centre of \( M \) and let \( B \) be \( \ast \)-isomorphic to \( L^\infty(X,\mu) \) for a certain locally compact, Hausdorff, second countable measure space \( X \) with \( \sigma \)-finite Borel measure \( \mu \). Then for \( \mu \)-almost every \( x \in X \) there exist a Hilbert subspace \( H_x \subseteq H \) and a von Neumann algebra \( M_x \subseteq \text{End}_C(H_x) \) such that:
(i) The set of Hilbert spaces \( \{ H_x : x \in X \} \) is a \( \mu \)-measurable field of Hilbert spaces, and \( \int_X H_x \, d\mu(x) = H \).

(ii) The set of von Neumann algebras \( \{ M_x : x \in X \} \) is a \( \mu \)-measurable field and \( M \) is \( \ast \)-isomorphic to \( \int_X M_x \, d\mu(x) \).

(iii) \( M \) is faithfully representable as a von Neumann algebra of bounded module operators on the self-dual Hilbert \( B \)-module \( \mathcal{H} = L^\infty(X,\mu,\{H_x : x \in X\}) \) being the \( B \)-dual Banach \( B \)-module of a countably generated Hilbert \( B \)-module.

Analysing these statements one concludes that the theory of direct integral decomposition of W*-algebras possessing a normal, faithful representation on a separable Hilbert space is one-to-one translatable to the theory of von Neumann subalgebras of the W*-algebras of bounded module operators on self-dual Hilbert \( A \)-modules with countably generated \( A \)-pre-dual Hilbert \( A \)-module over separable commutative W*-algebras \( A \).

3 An application

In the present section we like to generalize the following theorem of H. Weyl and I.D. Berg ([38] 1909 and [1] 1971):

**Proposition 3.1** (Berg’s and Weyl’s theorem) Every linear bounded normal operator \( T \) on a separable Hilbert space is decomposable into the sum of a normal, diagonalizable and a compact operator, \( D \) and \( K \). If \( T \) is self-adjoint then for every given \( \varepsilon > 0 \) one can even choose self-adjoint \( D \) and \( K \) such that \( \|K\| < \varepsilon \).

In 1970 P. R. Halmos has shown by some examples that Weyl’s theorem can not be generalized for self-adjoint operators on non-separable Hilbert spaces, ([4]). Nevertheless, the two theorems of the previous paragraph and a result of G. J. Murphy ([18] Th. 9) suggest to us another way of generalization weakening the notion of compactness.
Let $A$ be a commutative $\text{W}^*$-algebra and $H$ be a self-dual Hilbert $A$-module with countably generated $A$-pre-dual Hilbert $A$-module. Let us call an operator $T \in \text{End}_A(H)$ to be diagonalizable if and only if there exist a sequence of pairwise orthogonal projections $\{P_i : i \in \mathbb{N}\}$ of $K_A(H)$ and a sequence of elements $\{a_i : i \in \mathbb{N}\}$ of $A$ such that $T = \sum_{i \in \mathbb{N}} a_i P_i$ in the sense of w*-convergence. Furthermore, we say that the commutative $W^*$-algebra $A$ has property (*) if and only if the set of all normal states $f$ on $A$ with range projection $p_f$, for which the norm completion of the pre–Hilbert space $\{A p_f, f(\langle ., . \rangle_A)\}$ is separable, separates the elements of $A$. Finally, we call a locally compact Hausdorff measure space $X$ to be locally second countable if for every $x \in X$ there exists a clopen subset $Y \subseteq X$ containing $x$ and being second countable with respect to the measure $\mu$. We get the following result using assertions of R. V. Kadison [11, 12] and of K. Grove, G. K. Pedersen [6] in the proof:

**Theorem 3.2** Let $A$ be a commutative $W^*$-algebra with property (*). Let $H$ be a self-dual Hilbert $A$-module possessing a countably generated $A$-pre-dual Hilbert $A$-module. Let $T$ be an $A$-linear bounded normal operator on $H$. Then $T$ is decomposable into the sum of a $A$-linear bounded normal diagonalizable operator $D$ on $H$ and a $A$-linear bounded "compact" operator $K$ on $H$. If $T$ is self-adjoint then for every $\varepsilon > 0$ the operators $D$ and $K$ can be chosen to be self-adjoint and such that $\|K\| < \varepsilon$.

**Corollary 3.3** Let $X$ be a locally compact, locally second countable Hausdorff measure space with Borel measure $\mu$. Let $\{H_x : x \in X\}$ be a $\mu$-measurable field of Hilbert spaces on $X$. Then each normal decomposable bounded linear operator $T$ on the (non-separable, in general) Hilbert space $H = L^2(x, \mu, \{H_x : x \in X\})$ can be decomposed into the sum of a normal diagonalizable decomposable bounded linear operator $D$ on $H$ and a decomposable bounded linear operator $K$ on $H$, $K$ being compact on every subspace $L^2(Y, \mu, \{H_x : x \in Y\}) \subseteq H$ with $Y \subseteq X$ being clopen and second countable.

**Proof:** Choose a representation of the commutative $W^*$-algebra $A$ as $L^\infty(X, \mu)$ for a certain locally compact Hausdorff measure space $X$ with Borel measure $\mu$. By
the assumptions $X$ is locally second countable. Since $X$ is the union of a family of second countable clopen subsets $Y_\alpha$ with pairwise empty intersection one can suppose without loss of generality that $X$ is second countable and, consequently, $\mu$ is $\sigma$-finite.

By the first theorem of the previous section there exists a $\mu$-measurable field of Hilbert spaces $\{H_x : x \in X\}$ such that $\mathcal{H}$ is isometrically isomorphic to $L^\infty(X, \mu, \{H_x : x \in X\})$ and that $\text{End}_A(\mathcal{H})$ is *-isomorphic to $\int_X \text{End}_C(H_x) \, d\mu(x)$. The latter can be interpreted as a C*-subalgebra of the set of all bounded linear operators $\text{End}_C(H)$ on the separable Hilbert space $H = \int_X H_x \, d\mu(x)$ by (iii) of that theorem. Consequently, one can apply Murphy’s theorem ([18, Theorem 9]) to $T \in \text{End}_A(\mathcal{H}) \equiv \int_X \text{End}_C(H_x) \, d\mu(x)$ under that point of view, and one gets a diagonalizable normal operator $D \in \int_X \text{End}_C(H_x) \, d\mu(x) \equiv \text{End}_A(\mathcal{H})$ and a compact on $H$ operator $K \in \int_X K_C(H_x) \, d\mu(x) \equiv K_A(\mathcal{H})$ such that $T = D + K$. Pay attention, that the diagonalizability of $D \in \int_X \text{End}_C(H_x) \, d\mu(x)$ on $H$ means that there exist a countably number of complex eigen-values $\{\lambda_n : n \in \mathbb{N}\}$ and a countably number of projections $\{P_n\} \in \int_X \text{End}_C(H_x) \, d\mu(x) \equiv \text{End}_A(\mathcal{H})$ such that $D = \sum_n \lambda_n P_n$ on $H$ and on $\mathcal{H}$ simultaneously, i.e. $D$ is diagonalizable on $\mathcal{H}$, too. The nature of the eigen-vectors does not matter. Moreover, if $T$ is self-adjoint then for every $\varepsilon > 0$ one can choose $D$ and $K$ in such a way that they are self-adjoint and $\|K\| < \varepsilon$.

Remark, that if for certain projections $p \in A$ the Hilbert $pA$-module $p\mathcal{H}$ is finitely generated (or, equivalently, $p\text{End}_A(\mathcal{H}) = pK_A(\mathcal{H})$) then $pK = 0$ and $pT$ is diagonalizable by the results of R. V. Kadison and K. Grove, G. K. Pedersen cited above.

The corollary can be derived from the theorem using item (iii) of Theorem 2.1 and Murphy’s theorem.

**Remark 3.4** Unfortunately, we are not able to say anything about the possibly validity of the theorem without assuming $A$ to have property (*).

Beside this, a generalization to the case of $A$ being a commutative AW*-algebra with a similar property like property (*) in the W*-case seems to be possible using e. g. a transfer principle developed by G. Takeuti and M. Ozawa [20, 19, 33, 34, 35, 36, 19] between the theory of self-dual Hilbert AW*-modules over commutative AW*-
algebras and its description in terms of Boolean valued analysis and logic. However, in the light of results of K. Grove and G. K. Pedersen much more general commutative C*-algebras than arbitrary AW*-algebras can not appear. A result of R. V. Kadison who proved that each normal element of the W*-algebra $M_n(A) = \text{End}_A(A^n)$ with $A$ being a W*-algebra is diagonalizable for every natural number $n$ encourages to check the non-commutative case. But all that remains for further research.

After this paper has circulated as a preprint the author had fruitful discussions with R. Schaflitzel about possibilities of application of the obtained equivalence principle to get an alternative definition of generalized direct integrals (i.e., the non-separable case). Let $I$ be an index set of non-countable cardinality $\text{card}(I)$. The self-dual Hilbert $L^\infty(X, \mu)$-module $M_{\text{card}(I)} = \tau - \Sigma\{L^\infty(X, \mu)_\alpha : \alpha \in I\}$ is $\text{card}(I)$-homogenous for each cardinality $\text{card}(I)$. Moreover, the cardinality $\text{card}(I)$ of the generating set $I$ of $M_{\text{card}(I)}$ is uniquely defined up to isomorphy of Hilbert C*-modules, (cf. [13, §10, Th. 4]). Now, the principal idea is to use the existing isomorphy $M_{\text{card}(I)} \cong L^\infty(X, \mu, \{H_x : x \in X\})$ (where $\{H_x : x \in X\}$ is a certain $\mu$-measurable field of Hilbert spaces $H_x$ of dimension $\text{card}(I)$) to define a standard direct integral of non-separable Hilbert spaces. Simply, take the subset of all square-integrable elements of $L^\infty(X, \mu, \{H_x : x \in X\})$ and close it up with respect to the direct integral norm. What turns out? One gets the smallest (non-separable, by construction) Hilbert space $H$ satisfying a generalized definition for direct integrals (cf. §1) and containing the constant mappings $x \in X \rightarrow h = \text{const.} \in H_x$. That is, $H$ equals to the direct integral norm closure of the set of such elements $h \in \prod_{x \in X} H_x$ satisfying two properties:

(i) The mapping $x \in X \rightarrow \|h(x)\|^2$ is integrable.

(ii) There is a subset $N \subseteq X$ of $\mu$-measure zero such that the set $\{h(x) : x \in X\}$ generates a separable subspace of $H_x$.

Of course, since $H$ is the smallest direct integral in a certain sense $H$ is unique up to isomorphy.

Consequently, one could not expect to get much more information about genera-
lized direct integrals in the non-separable case using only the described equivalence.

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