A LIMITATION ON PROVING THE EXISTENCE OF SMALL GAPS BETWEEN ZETA-ZEROS

DANIEL A. GOLDSTON, TIMOTHY S. TRUDGIAN, AND CAROLINE L. TURNAGE-BUTTERBAUGH

Abstract. We assume the Riemann Hypothesis (RH) in this paper. The existence of Landau–Siegel zeros (or the Alternative Hypothesis) implies that there are long ranges where the zeros of the Riemann zeta-function are always spaced no closer than one half of the average spacing. However, numerical evidence strongly agrees with the GUE model where there are a positive proportion of consecutive zeros within any small multiple of the average spacing. Currently, assuming RH, the best result known produces infinitely many consecutive zeros within 0.515396 times the average spacing. This is obtained using the Montgomery–Odlyzko (M-O) method. It is also known that the M-O method fails to prove the existence of consecutive zeros closer than 1/2 times the average spacing. It is a tantalizing hope that the M-O method could still obtain infinitely many consecutive zeros arbitrarily close to 1/2 times the average spacing. We prove however that the M-O method can never find infinitely many consecutive zeros within 0.5042 times the average spacing.

1. Introduction

Write the nontrivial zeros of the Riemann zeta-function $\zeta(s)$ as $\rho = \beta + i\gamma$, where $\beta \in (0, 1)$ and $\gamma \in \mathbb{R}$. Let $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$ denote the ordinates of the nontrivial zeros of $\zeta(s)$ in the upper half-plane. Since $N(T) = \sum_{0 < \gamma \leq T} 1 \sim \frac{T}{2\pi} \log T$, it follows that the gap between consecutive zeros $\gamma_{n+1} - \gamma_n$ is $2\pi/\log \gamma_n$ on average. To examine how often gaps deviate from this average, we define

$$\mu = \liminf_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{2\pi/\log \gamma_n} \quad \text{and} \quad \lambda = \limsup_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{2\pi/\log \gamma_n}.$$ 

Trivially, we have that $\mu \leq 1 \leq \lambda$, and it is expected that $\mu = 0$ and $\lambda = \infty$. We refer the reader to [4, 8] for the history of this problem. The best current results under RH are: $\mu \leq 0.515396$ by Preobrazhenskiĭ [7] and $\lambda \geq 3.18$ by Bui and Milinovich [1].

The result of [7] is based on a method introduced by Montgomery and Odlyzko [6]. Define, for $T \geq 2$, $c > 0$, and $a_k$ a sequence of complex numbers,

$$h(c) = c - \frac{\Re \sum_{kn \leq y} a_k \overline{a_{kn}} g(n) \Lambda(n) / n^{1/2}}{\sum_{k \leq y} |a_k|^2},$$

(1)

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where
\[ g(n) = \frac{2 \sin \left( \frac{\pi c \log n}{\log T} \right)}{\pi \log n} , \]
and \( y = T^{1-\delta} \) for some small \( \delta > 0 \). Montgomery and Odlyzko proved that if \( h(c) > 1 \) for all sufficiently large \( T \) for some choice of \( a_k \)'s, \( c \), and a small \( \delta \), then assuming RH we have \( \mu \leq c \). To study large gaps, if we have \( h(c) < 1 \) then \( \lambda \geq c \). Conrey, Ghosh, and Gonek proved that, for any choice of \( a_k \)
\[
(2) \quad h(c) < 1 \quad \text{if} \quad c < 1/2 ,
\]
which shows that the Montgomery–Odlyzko method is unable to obtain \( \mu < 1/2 \). This barrier is important given the connection to Landau–Siegel zeros — see [3]. In this note we prove that the Montgomery–Odlyzko method falls well short of being able to prove \( \mu \leq 1/2 \).

**Theorem 1.** If \( c < 0.5042 \), then \( h(c) < 1 \).

We also mention the following information concerning limitations of the Montgomery-Odlyzko method for large gaps between zeros. Conrey, Ghosh, and Gonek [2, p. 423] showed that \( h(c) > 1 \) if \( c \geq 6.2 \), whence, the Montgomery-Odlyzko method cannot prove the existence of gaps at least 6.2 times the average spacing. In a note added in the proof stage of their paper, Conrey, Ghosh, and Gonek remark that 6.2 may be replaced by 3.74. Correcting for a misprint in their paper, their first result is based on the inequality
\[
(3) \quad h(c) \geq c - 2 \left( c \pi \int_0^1 \left| \frac{\sin \pi cv}{v} \right| dv \right)^{1/2} .
\]
Using Mathematica one finds that \( h(c) > 1 \) for \( c \geq 5.5602 \ldots \). Their second improvement result can be obtained from the inequality
\[
(4) \quad h(c) \geq c - 2 \left( c \pi \int_0^{\pi c} \left( \frac{\sin v}{v} \right)^2 dv \right)^{1/2}
\]
proved by a small change in the proof of the previous bound. One now finds with Mathematica that \( h(c) > 1 \) if \( c \geq 3.6747 \ldots \).

We note that the work by Bui and Milinovich [1] uses a different method based on the work of Hall [5] and hence is not limited in this way.

2. **Proof of Theorem**

We take \( 0 < c < 1 \). Letting \( a_k = b_k k^{-1/2} \), we obtain from (1) that
\[
h(c) \leq c + \frac{S}{\sum_{k \leq y} \frac{|b_k|^2}{k}} , \quad \text{where} \quad S = \sum_{kn \leq y} \frac{|b_k||b_{kn}||g(n)||\Lambda(n)|}{kn} .
\]
For any \( \alpha, \beta > 0 \) with \( 4\alpha \beta \geq 1 \), we have \( |ab| \leq \alpha |a|^2 + \beta |b|^2 \), and therefore
\[
(5) \quad |S| \leq \alpha \sum_{kn \leq y} \frac{|b_k|^2}{k} |g(n)|\frac{\Lambda(n)}{n} + \beta \sum_{kn \leq y} \frac{|b_{kn}|^2}{kn} |g(n)||\Lambda(n)| := \alpha S_1 + \beta S_2 .
\]
Using $|\sin x| \leq |x|$, we have for $1 \leq k \leq y$ and $0 < c < 1$

\begin{equation}
0 < g(n) = \frac{2 \sin \left( \frac{\pi c \log n}{\log T} \right)}{\pi \log n} \leq \frac{2c}{\log T}.
\end{equation}

We evaluate $S_1$ by using partial summation with the prime number theorem in the form $\sum_{n \leq x} \Lambda(n) \sim x$. Thus

\begin{equation}
S_1 = \sum_{k \leq y} |b_k|^2 \sum_{n \leq y/k} \frac{2 \sin \left( \frac{\pi c \log n}{\log T} \right) \Lambda(n)}{\pi n \log n}
\end{equation}

\begin{equation}
= \left( \frac{2}{\pi} + o(1) \right) \sum_{k \leq y} \frac{|b_k|^2}{k} \int_{2}^{y/k} \frac{\sin(\pi c \log (u/\log T))}{u \log u} du
\end{equation}

\begin{equation}
= \left( \frac{2}{\pi} + o(1) \right) \sum_{k \leq y} \frac{|b_k|^2}{k} \int_{0}^{\frac{\pi c \log (y/k)}{\log T}} \frac{\sin v}{v} dv.
\end{equation}

For $S_2$ we use (6) and the elementary relation $\sum_{d \mid n} \Lambda(d) = \log n$, to obtain

\begin{equation}
S_2 \leq \frac{2c}{\log T} \sum_{k \leq y} |b_{kn}|^2 \Lambda(n) = \frac{2c}{\log T} \sum_{m \leq y} \frac{|b_m|^2}{m \sum_{n \mid m} \Lambda(n)} = \frac{2c}{\log T} \sum_{k \leq y} \frac{|b_k|^2 \log k}{k}.
\end{equation}

Hence from (5) we obtain

\begin{equation}
S \leq (1 + o(1)) \sum_{k \leq y} \frac{|b_k|^2}{k} \left( \frac{2\beta c \log k}{\log T} + \frac{2\alpha}{\pi} \int_{0}^{\frac{\pi c \log (y/k)}{\log T}} \frac{\sin v}{v} dv \right).
\end{equation}

Now we define, for $1 \leq w \leq y$,

\begin{equation}
G(w) = G(w, \alpha, \beta, c) := \frac{2\beta c \log w}{\log T} + \frac{2\alpha}{\pi} \int_{0}^{\frac{\pi c \log (y/w)}{\log T}} \frac{\sin v}{v} dv,
\end{equation}

and conclude

\begin{equation}
\frac{S}{\sum_{k \leq y} |b_k|^2 k} \leq (1 + o(1)) \max_{1 \leq w \leq y} G(w).
\end{equation}

To evaluate this maximum, we take $4\alpha \beta = 1$ and note

\begin{equation}
G'(w) = \frac{2c}{w \log T} \left( \beta - \alpha \operatorname{sinc} \left( \frac{\pi c \log (y/w)}{\log T} \right) \right), \quad \text{where} \quad \operatorname{sinc}(x) := \frac{\sin x}{x}.
\end{equation}

Consider first the easy case when $\beta \geq \alpha$. Since $\operatorname{sinc}(x) \leq 1$ and $\operatorname{sinc}(x) = 1$ only when $x = 0$, we see $G(w)$ is nondecreasing and max $G(w) = G(y) \leq 2\beta c$. Since the smallest value of $\beta$ is $\alpha = \beta = 1/2$, we see that we only recover (2) with this choice and do worse otherwise.

Now consider $\beta < \alpha$. Thus $\beta < 1/2$, and we substitute $\alpha = 1/(4\beta)$. Since $\operatorname{sinc} \left( \frac{\pi c \log (y/w)}{\log T} \right)$ increases on $w \in [1, y]$, we see $G'(w)$ decreases through the interval and $G'(y) = 2c(\beta - \alpha)/(y \log T) < 0$. Thus $G(w)$ has at most one value $w_0$ of $G'(w) = 0$ in this interval. Such

\footnote{See the last section for comments on how this approach differs from that of [2].}
a critical point is a relative maximum, and $G(w_0)$ will be the maximum of $G$ in the entire interval. Finally, it is clear that if $\beta$ is close enough to $\alpha$ then such a critical point exists.

Defining $\phi_0$ by $T^{\phi_0} := y/w_0$, we may rewrite (8) as

$$S \leq (1 + o(1))G(T^{1-\phi_0-\delta}).$$

Note that given a choice of $\beta$ we can recover the corresponding value of $\phi_0$ via the identity

$$\text{sinc}(\pi c\phi_0) = \frac{\beta}{\alpha} = 4\beta^2, \quad 0 < \beta < 1/2.$$

Using Mathematica it is easy to compute the largest $c$ obtainable from (9) for which $h(c) < 1$. We start with an initial choice of $c = c_1 = 0.5$. Searching with a grid of values of $\beta$ we determine their corresponding values of $\phi$ from (10). We then select a smaller range of $\beta$ containing the maximum of $G(T^{1-\phi_0})$ found, and then replace $c_1$ by a larger value $c_2$ and repeat. This quickly converges. We can stop this process whenever we attain as many digits of accuracy as we desire, at which point we have found values $\beta_n$, $c_n$, $w_n$, and $\phi_n$. We now can check separately that the maximum of $G(w) = G(w, 1/4\beta_n, c_n, w_n)$ is close to the value $G(w_n)$. Finally, we can take $\delta$ in (10) smaller than the accuracy of the calculations. In this way we found $c_0 = 0.5042$, $\beta_0 = 0.476$, $\phi_0 = .5197462443\ldots$, and $h(c_0) \leq 0.999993501\ldots$.

### 3. A COMMENT ON THE APPROACH

In the previous section we recovered the result (2) of [2] in the simple case that $\beta \geq \alpha$. The proof of (2) in [2], however is different, which we describe here for the interested reader. There the authors use (6) and the elementary relation $\sum_{n \leq x} \Lambda(n)/n = \log x + O(1)$ to obtain

$$S \leq \frac{2c}{\log T} \sum_{k \leq y} \frac{|b_k|^2}{k} \sum_{n \leq y/k} \Lambda(n)/n = \frac{2c}{\log T} \sum_{k \leq y} \frac{|b_k|^2}{k} \left(\log(y/k) + O(1)\right).$$

Thus, in place of (7) they obtain

$$S \leq (1 + o(1)) \frac{2c}{\log T} \sum_{k \leq y} \frac{|b_k|^2}{k} \left(\alpha \log(y/k) + \beta \log k\right).$$

Letting $f(u) = \alpha \log(y/u) + \beta \log u$, one finds that $f'(u) = \alpha \log y, f'(y) = \beta \log y$, and $f'(u) = \frac{\beta - \alpha}{u}$, and thus $f(u) \leq \max(\alpha, \beta) \log y$ for $1 \leq u \leq y$. The optimal bound is obtained by taking $\alpha = \beta = 1/2$, and with this choice

$$S \leq (c + o(1)) \frac{\log y}{\log T} \sum_{k \leq y} \frac{|b_k|^2}{k} \leq (c + o(1)) \sum_{k \leq y} \frac{|b_k|^2}{k}. $$

Substituting into (5) the authors obtain $h(c) \leq 2c + o(1)$, and thus (2). Actually in [2] the usual choice $\alpha = \beta = 1/2$ was used in the argument, which we now see is also the optimal choice when using (11).

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San Jose State University
*Email address:* daniel.goldston@sjsu.edu

The University of New South Wales
*Email address:* t.trudgian@adfa.edu.au

Carleton College
*Email address:* cturnageb@carleton.edu