A new Federer-type characterization of sets of finite perimeter in metric spaces

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Abstract

Federer’s characterization states that a set $E \subset \mathbb{R}^n$ is of finite perimeter if and only if $\mathcal{H}^{n-1}(\partial^* E) < \infty$. Here the measure-theoretic boundary $\partial^* E$ consists of those points where both $E$ and its complement have positive upper density. We show that the characterization remains true if $\partial^* E$ is replaced by a smaller boundary consisting of those points where the lower densities of both $E$ and its complement are at least a given number. This result is new even in Euclidean spaces but we prove it in a more general complete metric space that is equipped with a doubling measure and supports a Poincaré inequality.

1 Introduction

Federer’s [8] characterization of sets of finite perimeter states that a set $E \subset \mathbb{R}^n$ is of finite perimeter if and only if $\mathcal{H}^{n-1}(\partial^* E) < \infty$, where $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure and $\partial^* E$ is the measure-theoretic boundary; see Section 2 for definitions. A similar characterization holds also in the abstract setting of complete metric spaces $(X, d, \mu)$ that are equipped with a doubling measure $\mu$ and support a Poincaré inequality; in such spaces one replaces the $n-1$-dimensional Hausdorff measure with the codimension one Hausdorff measure $\mathcal{H}$. The “only if” direction of the characterization was shown in metric spaces by Ambrosio [1], and the “if” direction was recently shown by the author [20].

Federer also showed that if a set $E \subset \mathbb{R}^n$ is of finite perimeter, then $\mathcal{H}^{n-1}(\partial^* E \setminus \Sigma_{1/2} E) = 0$, where the boundary $\Sigma_{1/2} E$ consists of those points where both $E$ and its complement have density exactly 1/2. In metric spaces we similarly have $\mathcal{H}(\partial^* E \setminus \Sigma_\gamma E) = 0$, where $0 < \gamma \leq 1/2$ is a suitable constant depending on the space and the strong boundary $\Sigma_\gamma E$ is defined by

$$\Sigma_\gamma E := \left\{ x \in X : \liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \geq \gamma \text{ and } \liminf_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \geq \gamma \right\}.$$
This raises the natural question of whether the condition $\mathcal{H}(\Sigma_\beta E) < \infty$ for some $\beta > 0$, which appears much weaker than $\mathcal{H}(\partial^* E) < \infty$, is already enough to imply that $E$ is of finite perimeter. Recently Chlebík [6] posed this question in Euclidean spaces and noted that the (positive) answer is known only when $n = 1$.

In the current paper we show that this characterization does indeed hold in every Euclidean space and even in the much more general metric spaces that we consider.

**Theorem 1.1.** Let $(X,d,\mu)$ be a complete metric space with $\mu$ doubling and supporting a $(1,1)$-Poincaré inequality. Let $\Omega \subset X$ be an open set and let $E \subset X$ be a $\mu$-measurable set with $\mathcal{H}(\Sigma_\beta E \cap \Omega) < \infty$, where $0 < \beta \leq 1/2$ only depends on the doubling constant of the measure and the constants in the Poincaré inequality. Then $P(E,\Omega) < \infty$.

Explicitly, in the Euclidean space $\mathbb{R}^n$ with $n \geq 2$, we can take (see (7.2))

$$\beta = \frac{n^{13n/2}}{2^{26n^2 + 64n + 15} \omega_n^{13}},$$

where $\omega_n$ is the volume of the Euclidean unit ball.

Our strategy is to show that if $\mathcal{H}(\Sigma_\beta E \cap \Omega) < \infty$, then $\mathcal{H}(\partial^* E \setminus \Sigma_\beta E \cap \Omega) = 0$ and so the result follows from the previously known Federer’s characterization. Our proof consists essentially of two steps. First in Section 3, we show that for every point in the measure-theoretic boundary $\partial^* E$, arbitrarily close there is a point in the strong boundary $\Sigma_\beta E$. Then, after some preliminary results concerning connected components of sets of finite perimeter as well as functions of least gradient in Sections 4 and 5, in Section 6 we show that there exists an open set $V$ containing a suitable part of $\Sigma_\beta E$ such that $X \setminus V$ is itself a metric space with rather good properties. Thus we can apply the first step in this space. In Section 7 we combine the two steps to prove Theorem 1.1.

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### 2 Notation and definitions

In this section we introduce the notation, definitions, and assumptions that are employed in the paper.

Throughout this paper, $(X,d,\mu)$ is a complete metric space that is equipped with a metric $d$ and a Borel regular outer measure $\mu$ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x,2r)) \leq C_d \mu(B(x,r)) < \infty$$
for every ball $B(x, r) := \{ y \in X : d(y, x) < r \}$, with $x \in X$ and $r > 0$. Closed balls are denoted by $\overline{B}(x, r) := \{ y \in X : d(y, x) \leq r \}$. By iterating the doubling condition, we obtain that for every $x \in X$ and $y \in B(x, R)$ with $0 < r \leq R < \infty$, we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C_d^2} \left( \frac{r}{R} \right)^s,$$

(2.1)

where $s > 1$ only depends on the doubling constant $C_d$. Given a ball $B = B(x, r)$ and $\beta > 0$, we sometimes abbreviate $\beta B := B(x, \beta r)$; note that in a metric space, a ball (as a set) does not necessarily have a unique center point and radius, but these will be prescribed for all the balls that we consider.

We assume that $X$ consists of at least 2 points. When we want to state that a constant $C$ depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$. When a property holds outside a set of $\mu$-measure zero, we say that it holds almost everywhere, abbreviated a.e.

All functions defined on $X$ or its subsets will take values in $[-\infty, \infty]$. As a complete metric space equipped with a doubling measure, $X$ is proper, that is, closed and bounded sets are compact. Since $X$ is proper, for any open set $\Omega \subset X$ we define $L^1_{\text{loc}}(\Omega)$ to be the space of functions that are in $L^1(\Omega')$ for every open $\Omega' \subset \Omega$. Here $\Omega' \subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of $\Omega$. Other local spaces of functions are defined analogously.

For any set $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension one is defined by

$$H_R(A) := \inf \left\{ \sum_{j \in I} \frac{\mu(B(x_j, r_j))}{r_j} : A \subset \bigcup_{j \in I} B(x_j, r_j), r_j \leq R, I \subset \mathbb{N} \right\}.$$

The codimension one Hausdorff measure of $A \subset X$ is then defined by

$$H(A) := \lim_{R \to 0} H_R(A).$$

In the Euclidean space $\mathbb{R}^n$ (equipped with the Euclidean metric and the $n$-dimensional Lebesgue measure) this is comparable to the $n - 1$-dimensional Hausdorff measure.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into $X$. The length of a curve $\gamma$ is denoted by $\ell_\gamma$. We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [10, Theorem 3.2]). A nonnegative Borel function $g$ on $X$ is an upper gradient of a function $u$ on $X$ if for all nonconstant curves $\gamma$, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds := \int_0^{\ell_\gamma} g(\gamma(s)) \, ds,$$

(2.2)

where $x$ and $y$ are the end points of $\gamma$. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|, |u(y)|$ is infinite. Upper gradients were originally introduced in [13].
The 1-modulus of a family of curves $\Gamma$ is defined by

$$\text{Mod}_1(\Gamma) := \inf \int_X \rho \, d\mu$$

where the infimum is taken over all nonnegative Borel functions $\rho$ such that $\int_\gamma \rho \, ds \geq 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for 1-a.e. curve if it fails only for a curve family with zero 1-modulus. If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.2) holds for 1-a.e. curve, we say that $g$ is a 1-weak upper gradient of $u$.

Given an open set $\Omega \subset X$, we let

$$\|u\|_{N^{1,1}(\Omega)} := \|u\|_{L^1(\Omega)} + \inf \|g\|_{L^1(\Omega)},$$

where the infimum is taken over all upper gradients $g$ of $u$ in $\Omega$. Then we define the Newton-Sobolev space

$$N^{1,1}(\Omega) := \{ u : \|u\|_{N^{1,1}(\Omega)} < \infty \}.$$ 

In $\mathbb{R}^n$ this coincides, up to a choice of pointwise representatives, with the usual Sobolev space $W^{1,1}(\Omega)$; this is shown in Theorem 4.5 of [26], where the Newton-Sobolev space was originally introduced.

We understand Newton-Sobolev functions to be defined at every point $x \in \Omega$ (even though $\| \cdot \|_{N^{1,1}(\Omega)}$ is then only a seminorm). It is known that for every $u \in N^{1,1}_{\text{loc}}(\Omega)$ there exists a minimal 1-weak upper gradient of $u$ in $\Omega$, always denoted by $g_u$, satisfying $g_u \leq g$ a.e. in $\Omega$ for any other 1-weak upper gradient $g \in L^1_{\text{loc}}(\Omega)$ of $u$ in $\Omega$, see [4, Theorem 2.25]. In $\mathbb{R}^n$, the minimal 1-weak upper gradient coincides (a.e.) with $|\nabla u|$, see [4, Corollary A.4].

We will assume throughout the paper that $X$ supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P \geq 1$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \int_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

As a complete metric space equipped with a doubling measure and supporting a Poincaré inequality, $X$ is quasiconvex, meaning that for every pair of points $x, y \in X$ there is a curve $\gamma$ with $\gamma(0) = x, \gamma(\ell_\gamma) = y$, and $\ell_\gamma \leq C d(x, y)$, where $C$ is a constant and only depends on $C_d$ and $C_P$, see e.g. [4, Theorem 4.32]. Thus a biLipschitz change in the metric gives a geodesic space (see [4, Section 4.7]). Since Theorem 1.1 is easily seen to be invariant under such a biLipschitz change in the metric, we can assume that $X$ is geodesic. By [4, Theorem 4.39], in the Poincaré inequality we can now choose $\lambda = 1$. 

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The 1-capacity of a set $A \subset X$ is defined by
\[ \text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)}, \]
where the infimum is taken over all functions $u \in N^{1,1}(X)$ satisfying $u \geq 1$ in $A$. The variational 1-capacity of a set $A \subset \Omega$ with respect to an open set $\Omega \subset X$ is defined by
\[ \text{cap}_1(A, \Omega) := \inf \int_X g_u \, d\mu, \]
where the infimum is taken over functions $u \in N^{1,1}(X)$ satisfying $u = 0$ in $X \setminus \Omega$ and $u \geq 1$ in $A$, and $g_u$ is the minimal 1-weak upper gradient of $u$ (in $X$). By truncation, we see that we can assume $0 \leq u \leq 1$ on $X$. The variational 1-capacity is an outer capacity in the sense that if $A \subsetneq \Omega$, then
\[ \text{cap}_1(A, \Omega) = \inf_{\substack{V \text{ open} \\ A \subseteq V \subseteq \Omega}} \text{cap}_1(V, \Omega); \quad (2.3) \]
see [4, Theorem 6.19(vii)]. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [4].

We say that a set $U \subset X$ is 1-quasiopean if for every $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $U \cup G$ is open.

Next we present the definition and basic properties of functions of bounded variation on metric spaces, following [23]. See also e.g. [2, 7, 8, 9, 27] for the classical theory in the Euclidean setting. Given an open set $\Omega \subset X$ and a function $u \in L^1_{\text{loc}}(\Omega)$, we define the total variation of $u$ in $\Omega$ by
\[ \|Du\|_{\Omega} := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in N^{1,1}_{\text{loc}}(\Omega), u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\}, \]
where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $\Omega$. In $\mathbb{R}^n$ this agrees with the usual Euclidean definition involving distributional derivatives, see e.g. [2, Proposition 3.6, Theorem 3.9]. (In [23], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.)

We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in \text{BV}(\Omega)$, if $\|Du\|_{\Omega} < \infty$. For an arbitrary set $A \subset X$, we define
\[ \|Du\|(A) := \inf \{ \|Du\|(W) : A \subset W, W \subset X \text{ is open} \}. \]

If $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|_{\Omega} < \infty$, then $\|Du\|_{\cdot}$ is a Borel regular outer measure on $\Omega$ by [23, Theorem 3.4]. A $\mu$-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|_{\Omega} < \infty$, where $\chi_E$ is the characteristic function of $E$. The perimeter of $E$ in $\Omega$ is also denoted by
\[ P(E, \Omega) := \|D\chi_E\|_{\Omega}. \]

The measure-theoretic interior of a set $E \subset X$ is defined by
\[ I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} = 0 \right\}, \quad (2.4) \]
and the measure-theoretic exterior by
\[ O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}. \]

The measure-theoretic boundary \( \partial^* E \) is defined as the set of points \( x \in X \) at which both \( E \) and its complement have nonzero upper density, i.e.
\[
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.
\]

Note that the space \( X \) is always partitioned into the disjoint sets \( I_E, O_E, \) and \( \partial^* E \).

By Lebesgue’s differentiation theorem (see e.g. [12, Chapter 1]), for a \( \mu \)-measurable set \( E \) we have \( \mu(E \Delta I_E) = 0 \), where \( \Delta \) is the symmetric difference.

Given a number \( 0 < \gamma \leq 1/2 \), we also define the strong boundary
\[
\Sigma_\gamma E := \left\{ x \in X : \liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \geq \gamma \quad \text{and} \quad \liminf_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \geq \gamma \right\}.
\]
(2.5)

For an open set \( \Omega \subset X \) and a \( \mu \)-measurable set \( E \subset X \) with \( P(E, \Omega) < \infty \), we have \( \mathcal{H}((\partial^* E \setminus \Sigma_\gamma E) \cap \Omega) = 0 \) for \( \gamma \in (0, 1/2] \) that only depends on \( C_d \) and \( C_P \), see [1, Theorem 5.4]. Moreover, for any Borel set \( A \subset \Omega \) we have
\[
P(E, A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H},
\]
(2.6)
where \( \theta_E : \Omega \to [\alpha, C_d] \) with \( \alpha = \alpha(C_d, C_P) > 0 \), see [1, Theorem 5.3] and [3, Theorem 4.6].

The following coarea formula is given in [23, Proposition 4.2]: if \( \Omega \subset X \) is an open set and \( u \in L^1_{\text{loc}}(\Omega) \), then
\[
\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt,
\]
(2.7)
where we abbreviate \( \{u > t\} := \{ x \in \Omega : u(x) > t \} \). If \( \|Du\|(\Omega) < \infty \), then (2.7) holds with \( \Omega \) replaced by any Borel set \( A \subset \Omega \).

We know that for an open set \( \Omega \subset X \), an arbitrary set \( A \subset \Omega \), and any \( \mu \)-measurable sets \( E_1, E_2 \subset X \), we have
\[
P(E_1 \cap E_2, A) + P(E_1 \cup E_2, A) \leq P(E_1, A) + P(E_2, A);
\]
(2.8)
for a proof in the case \( A = \Omega \) see [23, Proposition 4.7], and then the general case follows by approximation. Using this fact as well as the lower semicontinuity of the total variation with respect to \( L^1_{\text{loc}} \)-convergence in open sets, we have for any \( E_1, E_2, \ldots \subset X \) that
\[
P\left( \bigcup_{j=1}^{\infty} E_j, \Omega \right) \leq \sum_{j=1}^{\infty} P(E_j, \Omega).
\]
(2.9)
Applying the Poincaré inequality to sequences of approximating \(N^1_{loc}\)-functions in the definition of the total variation, we get the following BV version: for every ball \(B(x,r)\) and every \(u \in L^1_{loc}(X)\), we have
\[
\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \|Du\|(B(x,r)).
\]
Recall here and from now on that we take the constant \(\lambda\) to be 1, and so it does not appear in the inequalities. For a \(\mu\)-measurable set \(E \subset X\), by considering the two cases \((\chi_E)_{B(x,r)} \leq 1/2\) and \((\chi_E)_{B(x,r)} \geq 1/2\), from the above we get the relative isoperimetric inequality
\[
\min\{\mu(B(x,r) \cap E), \mu(B(x,r) \setminus E)\} \leq 2C_P r P(E,B(x,r)).
\] (2.10)
From the \((1,1)\)-Poincaré inequality, by [4, Theorem 4.21, Theorem 5.51] we also get the following Sobolev inequality: if \(x \in X\), 0 < \(r < \frac{1}{4}\) \(\text{diam} X\), and \(u \in N^{1,1}(X)\) with \(u = 0\) in \(X \setminus B(x,r)\), then
\[
\int_{B(x,r)} |u| \, d\mu \leq C_S r \int_{B(x,r)} g_u \, d\mu
\] (2.11)
for a constant \(C_S = C_S(C_d,C_P) \geq 1\). For any \(\mu\)-measurable set \(E \subset B(x,r)\), applying the Sobolev inequality to a suitable sequence approximating \(u\), we get the isoperimetric inequality
\[
\mu(E) \leq C_S r P(E,X).
\] (2.12)
The lower and upper approximate limits of a function \(u\) on an open set \(\Omega\) are defined respectively by
\[
u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u < t\})}{\mu(B(x,r))} = 0 \right\}
\] (2.13)
and
\[
u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u > t\})}{\mu(B(x,r))} = 0 \right\}
\] (2.14)
for \(x \in \Omega\). Unlike Newton-Sobolev functions, we understand BV functions to be equivalence classes of a.e. defined functions, but \(\nu^\wedge\) and \(\nu^\vee\) are pointwise defined.

The BV-capacity of a set \(A \subset X\) is defined by
\[
\text{Cap}_{BV}(A) := \inf \left( \|u\|_{L^1(X)} + \|Du\|(X) \right),
\]
where the infimum is taken over all \(u \in BV(X)\) with \(u \geq 1\) in a neighborhood of \(A\). By [11, Theorem 4.3] we know that for some constant \(C_{cap} = C_{cap}(C_d,C_P) \geq 1\) and every \(A \subset X\), we have
\[
\text{Cap}_1(A) \leq C_{cap} \text{Cap}_{BV}(A).
\] (2.15)
We also define a variational BV-capacity for any \(A \subset \Omega\), with \(\Omega \subset X\) open, by
\[
\text{cap}_{BV}(A,\Omega) := \inf \|Du\|(X),
\]
where the infimum is taken over functions $u \in BV(X)$ such that $u^\wedge = u^\vee = 0 \; \mathcal{H} \text{-a.e.}$ in $X \setminus \Omega$ and $u^\vee \geq 1 \; \mathcal{H} \text{-a.e.}$ in $A$. By [19, Theorem 5.7] we know that
\[
\text{cap}_1(A, \Omega) \leq C_r \text{cap}^\vee_{BV}(A, \Omega)
\] (2.16)
for a constant $C_r = C_r(C_d, C_P) \geq 1$.

**Standing assumptions:** In Section 3 we will consider a different metric space $Z$ (which will later be taken to be a subset of $X$), but in Sections 4 to 7 we will assume that $(X, d, \mu)$ is a complete, geodesic metric space that is equipped with the doubling Radon measure $\mu$ and supports a $(1,1)$-Poincaré inequality with $\lambda = 1$.

## 3 Strong boundary points

In this section we consider a complete metric space $(Z, \hat{d}, \mu)$ where $\mu$ is a Borel regular outer measure and doubling with constant $\hat{C}_d \geq 1$. We define the Mazurkiewicz metric
\[
\hat{d}_M(x, y) := \inf \{ \text{diam} F : F \subset Z \text{ is a continuum containing } x, y \}, \; x, y \in Z, \tag{3.1}
\]
and we assume the space to be “geodesic” in the sense that $\hat{d}_M = \hat{d}$. As usual, a continuum means a compact connected set.

**Definition 3.2.** We say that $(x_0, \ldots, x_m)$ is an $\varepsilon$-chain from $x_0$ to $x_m$ if $\hat{d}(x_j, x_{j+1}) < \varepsilon$ for all $j = 0, \ldots, m - 1$.

The following proposition gives the existence of a strong boundary point.

**Proposition 3.3.** Let $x_0 \in Z$, $R > 0$, and let $E \subset Z$ be a $\mu$-measurable set such that
\[
\frac{1}{2\hat{C}_d^2} \leq \frac{\mu(B(x_0, R) \cap E)}{\mu(B(x_0, R))} \leq 1 - \frac{1}{2\hat{C}_d^2} \tag{3.4}
\]
Then there exists a point $x \in B(x_0, 6R)$ such that
\[
\frac{1}{4\hat{C}_d^{12}} \leq \liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \leq \limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \leq 1 - \frac{1}{4\hat{C}_d^{12}}. \tag{3.5}
\]

**Proof.** The proof is by suitable iteration, where we consider two options.

**Case 1.** Suppose that
\[
\frac{\mu(B(x, 2^{-2}R) \cap E)}{\mu(B(x, 2^{-2}R))} < \frac{1}{2} \tag{3.6}
\]
for all $x \in B(x_0, R)$; the case “$>$” is considered analogously. Define a “bad” set
\[
P := \left\{ x \in B(x_0, R) : \frac{\mu(B(x, 2^{-2j}R) \cap E)}{\mu(B(x, 2^{-2j}R))} \leq \frac{1}{4\hat{C}_d^6} \right\}
\]
for some $j \in \mathbb{N}$. 

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For every \( x \in P \) there is a radius \( r_x \leq R/20 \leq R \) such that
\[
\frac{\mu(B(x, 5r_x) \cap E)}{\mu(B(x, 5r_x))} \leq \frac{1}{4C_d^6}.
\]
Thus \( \{B(x, r_x)\}_{x \in P} \) is a covering of \( P \). By the 5-covering theorem, pick a countable collection of pairwise disjoint balls \( \{B(x_j, r_j)\}_{j=1}^\infty \) such that \( P \subset \bigcup_{j=1}^\infty B(x_j, 5r_j) \).

Now
\[
\mu(P \cap E) \leq \sum_{j=1}^\infty \mu(B(x_j, 5r_j) \cap E) \leq \frac{1}{4C_d^6} \sum_{j=1}^\infty \mu(B(x_j, 5r_j))
\leq \frac{1}{4C_d^3} \sum_{j=1}^\infty \mu(B(x_j, r_j))
\leq \frac{1}{4C_d^3} \mu(B(x_0, 2R))
\leq \frac{1}{4C_d^2} \mu(B(x_0, R)).
\]

Thus
\[
\mu(P) = \mu(P \cap E) + \mu(P \setminus E)
\leq \frac{1}{4C_d^2} \mu(B(x_0, R)) + \mu(B(x_0, R) \setminus E)
\leq \frac{1}{4C_d^2} \mu(B(x_0, R)) + \left(1 - \frac{1}{2C_d^2}\right) \mu(B(x_0, R)) \quad \text{by (3.4)}
\leq \left(1 - \frac{1}{4C_d^2}\right) \mu(B(x_0, R)).
\]

In particular, there is a point \( y \in B(x_0, R) \setminus P \). Now there are two options.

**Case 1(a).** The first option is that for each \( j \in \mathbb{N} \), we have
\[
\frac{\mu(B(y, 2^{-2j} R) \cap E)}{\mu(B(y, 2^{-2j} R))} \leq \frac{1}{2}
\]
and then in fact
\[
\frac{1}{4C_d^6} \leq \frac{\mu(B(y, 2^{-2j} R) \cap E)}{\mu(B(y, 2^{-2j} R))} < \frac{1}{2},
\]
for all \( j \in \mathbb{N} \), since \( y \in B(x_0, R) \setminus P \). From this we easily find that (3.5) holds (with \( x = y \)).

**Case 1(b).** The second option is that there is a smallest index \( l \geq 2 \) such that
\[
\frac{\mu(B(y, 2^{-2l} R) \cap E)}{\mu(B(y, 2^{-2l} R))} \geq \frac{1}{2}.
\]
Then

\[ \frac{1}{2C_d^2} \leq \frac{\mu(B(y, 2^{-2l+2}R) \cap E)}{\mu(B(y, 2^{-2l+2}R))} < \frac{1}{2}, \]

and also

\[ \frac{1}{4C_d^6} \leq \frac{\mu(B(y, 2^{-2j}R) \cap E)}{\mu(B(y, 2^{-2j}R))} < \frac{1}{2} \quad \text{for all } j = 1, \ldots, l - 2. \]

Note that regardless of the direction of the inequality in (3.6), we get

\[ \frac{1}{2C_d^2} \leq \frac{\mu(B(y, 2^{-2l+2}R) \cap E)}{\mu(B(y, 2^{-2l+2}R))} < 1 - \frac{1}{2C_d^2} \]

and

\[ \frac{1}{4C_d^6} \leq \frac{\mu(B(y, 2^{-2j}R) \cap E)}{\mu(B(y, 2^{-2j}R))} < 1 - \frac{1}{4C_d^6} \quad \text{for all } j = 1, \ldots, l - 2. \quad (3.7) \]

**Case 2.** Alternatively, suppose that we find two points \( x, y \in B(x_0, R) \) such that

\[ \frac{\mu(B(x, 2^{-2}R) \cap E)}{\mu(B(x, 2^{-2}R))} \geq \frac{1}{2} \]

and

\[ \frac{\mu(B(y, 2^{-2}R) \cap E)}{\mu(B(y, 2^{-2}R))} \leq \frac{1}{2}. \]

Then, using the fact that \( \hat{d}_M = \hat{d} \), we find a continuum \( F \) that contains \( x \) and \( y \) and is contained in \( B(x_0, 3R) \). Since \( F \) is connected, for every \( \varepsilon > 0 \) there is an \( \varepsilon \)-chain in \( F \) from \( x \) to \( y \). In particular, we find an \( R/4 \)-chain in \( F \) from \( x \) to \( y \). Let \( z \) be the last point in the chain for which we have

\[ \frac{\mu(B(z, 2^{-2}R) \cap E)}{\mu(B(z, 2^{-2}R))} \geq \frac{1}{2}. \]

If \( z = y \), then we have

\[ \frac{\mu(B(z, 2^{-2}R) \cap E)}{\mu(B(z, 2^{-2}R))} = \frac{1}{2}. \]

Else there exists \( w \in F \) with \( \hat{d}(z, w) < R/4 \) and

\[ \frac{\mu(B(w, 2^{-2}R) \cap E)}{\mu(B(w, 2^{-2}R))} < \frac{1}{2} \quad \text{and thus} \quad \frac{\mu(B(w, 2^{-2}R) \setminus E)}{\mu(B(z, 2^{-1}R))} \geq \frac{1}{2C_d^2}. \]

Now

\[ \frac{\mu(B(z, 2^{-1}R) \cap E)}{\mu(B(z, 2^{-1}R))} = \frac{\mu(B(z, 2^{-1}R)) - \mu(B(z, 2^{-1}R) \setminus E)}{\mu(B(z, 2^{-1}R))} \leq \frac{\mu(B(z, 2^{-1}R)) - \mu(B(w, 2^{-2}R) \setminus E)}{\mu(B(z, 2^{-1}R))} \leq 1 - \frac{1}{2C_d^2}. \]
Conversely,
\[
\frac{\mu(B(z, 2^{-1}R) \cap E)}{\mu(B(z, 2^{-1}R))} \geq \frac{\mu(B(z, 2^{-2}R) \cap E)}{C_d \mu(B(z, 2^{-2}R))} \geq \frac{1}{2C_d}.
\]
In conclusion, there is \( z \in B(x_0, 3R) \) with
\[
\frac{1}{2C_d^2} \leq \frac{\mu(B(z, 2^{-1}R) \cap E)}{\mu(B(z, 2^{-1}R))} \leq 1 - \frac{1}{2C_d^2};
\]
note that this holds also in the case \( z = y \).

To summarize, in Case 1(a) we obtain infinitely many balls (and then we are done), in Case 1(b) we obtain the \( l + 1 \) new balls \( B(y, 2^{-2}R), \ldots, B(y, 2^{-2l+2}R) \), where \( B(y, 2^{-2l+2}R) \) satisfies (3.4), and in Case (2) we obtain one new ball satisfying (3.4).

By iterating the procedure and concatenating the new balls obtained in each step to the previous list of balls, we find a sequence of balls with center points \( x_k \in B(x_{k-1}, 3r_{k-1}) \) and radii \( r_k \) such that \( r_0 = R, \ r_k \in [r_{k-1}/4, r_{k-1}/2] \), and (recall (3.7))
\[
\frac{1}{4C_d^6} \leq \frac{\mu(B(x_k, r_k) \cap E)}{\mu(B(x_k, r_k))} \leq 1 - \frac{1}{4C_d^6}
\]
for all \( k \in \mathbb{N} \). (Note that several consecutive balls in this sequence will have the same center points if they are obtained from Case 1.) By completeness of the space we find \( x \in Z \) such that \( x_k \to x \). For each \( l = 0, 1, \ldots \) we have
\[
d(x, x_l) \leq \sum_{k=l}^{\infty} d(x_k, x_{k+1}) \leq 3 \sum_{k=l}^{\infty} r_k \leq 6r_l.
\]
In particular, \( d(x, x_0) \leq 6R \). Now \( B(x_l, r_l) \subset B(x, 7r_l) \subset B(x_l, 13r_l) \) for all \( l \in \mathbb{N} \), and so
\[
\frac{\mu(B(x, 7r_l) \cap E)}{\mu(B(x, 7r_l))} \geq \frac{\mu(B(x_l, r_l) \cap E)}{\mu(B(x_l, 13r_l))} \geq \frac{1}{C_d^4} \frac{\mu(B(x_l, r_l) \cap E)}{\mu(B(x_l, r_l))} \geq \frac{1}{4C_d^{10}}
\]
and similarly
\[
\frac{\mu(B(x, 7r_l) \setminus E)}{\mu(B(x, 7r_l))} \geq \frac{\mu(B(x_l, r_l) \setminus E)}{\mu(B(x_l, 13r_l))} \geq \frac{1}{C_d^4} \frac{\mu(B(x_l, r_l) \setminus E)}{\mu(B(x_l, r_l))} \geq \frac{1}{4C_d^{10}}.
\]
It follows that
\[
\liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \geq \frac{1}{4C_d^{12}}
\]
and
\[
\liminf_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \geq \frac{1}{4C_d^{12}}
\]
proving (3.5). \( \square \)
Corollary 3.8. Let \( x_0 \in Z, R > 0 \), and let \( E \subset Z \) be a \( \mu \)-measurable set such that
\[
0 < \mu(B(x_0,R) \cap E) < \mu(B(x_0,R)).
\]
Then there exists a point \( x \in B(x_0,9R) \) such that
\[
\frac{1}{4C^1_d} \leq \liminf_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \leq \limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \leq 1 - \frac{1}{4C^1_d}. \tag{3.9}
\]
Proof. Again consider two cases. The first is that we find two points \( y, z \in B(x_0, R) \) such that
\[
\frac{\mu(B(y,2^{-1}R) \cap E)}{\mu(B(y,2^{-1}R))} \geq \frac{1}{2} \quad \text{and} \quad \frac{\mu(B(z,2^{-1}R) \cap E)}{\mu(B(z,2^{-1}R))} \leq \frac{1}{2}.
\]
Then just as in the proof of Proposition 3.3 Case 2, we find \( w \in B(x_0,3R) \) with
\[
\frac{1}{2C^2_d} \leq \frac{\mu(B(w,R) \cap E)}{\mu(B(w,R))} \leq 1 - \frac{1}{2C^2_d}.
\]
Now Proposition 3.3 gives a point \( x \in B(w,6R) \subset B(x_0,9R) \) such that (3.9) holds.

The second possible case is that for all \( y \in B(x_0,R) \) we have
\[
\frac{\mu(B(y,2^{-1}R) \cap E)}{\mu(B(y,2^{-1}R))} < \frac{1}{2}
\]
(the case “>” being analogous). By Lebesgue’s differentiation theorem, we find a point \( y \in I_E \cap B(x_0,R) \) (recall (2.4)) and then it is easy to find a radius \( 0 < r \leq R/2 \) such that
\[
\frac{1}{2C^2_d} \leq \frac{\mu(B(y,r) \cap E)}{\mu(B(y,r))} < \frac{1}{2}.
\]
Now Proposition 3.3 again gives a point \( x \in B(y,6r) \subset B(x_0,4R) \) such that (3.9) holds. \( \square \)

4 Components of sets of finite perimeter

In Sections 4 to 7 we assume that \((X,d,\mu)\) is a complete, geodesic metric space that is equipped with the doubling measure \( \mu \) and supports a \((1,1)\)-Poincaré inequality.

In this section we consider connected components, or components for short, of sets of finite perimeter. The following is the main result of the section.

Proposition 4.1. Let \( B(x,R) \) be a ball with \( 0 < R < \frac{1}{4} \text{diam} X \) and let \( F \subset X \) be a closed set with \( P(F,X) < \infty \). Denote the components of \( F \cap \overline{B}(x,R) \) having nonzero \( \mu \)-measure by \( F_1, F_2, \ldots \). Then \( \mu \left( \overline{B}(x,R) \cap F \setminus \bigcup_{j=1}^{\infty} F_j \right) = 0 \), \( P(F_j, B(x,R)) < \infty \) for all \( j \in \mathbb{N} \), and for any sets \( A_j \subset F_j \) with \( P(A_j, B(x,R)) < \infty \) for all \( j \in \mathbb{N} \) we have
\[
P \left( \bigcup_{j=1}^{\infty} A_j, B(x,R) \right) = \sum_{j=1}^{\infty} P(A_j, B(x,R)).
\]
Of course, there may be only finitely many $F_j$’s, and so we will always understand that some $F_j$’s can be empty. In fact, supposing that $\mu(F \cap B(x, R)) > 0$, we will know only after Lemma 4.20 that any $F_j$’s are nonempty.

Next we gather a number of preliminary results. Recall the definition of 1-quasiopen sets from page 5.

**Proposition 4.2** ([18, Proposition 4.2]). Let $\Omega \subset X$ be open and let $F \subset X$ be $\mu$-measurable with $P(F, \Omega) < \infty$. Then the sets $I_F \cap \Omega$ and $O_F \cap \Omega$ are 1-quasiopen.

**Proposition 4.3.** Let $F \subset X$ with $P(F, X) < \infty$. Then for 1-a.e. curve $\gamma$, $\gamma^{-1}(I_F)$ and $\gamma^{-1}(O_F)$ are relatively open subsets of $[0, \ell_\gamma]$.

**Proof.** By Proposition 4.2, the sets $I_F$ and $O_F$ are 1-quasiopen. Then by [25, Remark 3.5], they are also 1-path open, meaning that for 1-a.e. curve $\gamma$ in $X$, the sets $\gamma^{-1}(I_F)$ and $\gamma^{-1}(O_F)$ are relatively open subsets of $[0, \ell_\gamma]$. \hfill $\Box$

For any set $A \subset X$, we define the measure-theoretic closure as

$$\overline{A}^m := I_A \cup \partial^* A.$$  \hfill (4.4)

**Lemma 4.5.** Let $B(x, R)$ be a ball with $0 < R < \frac{1}{4} \text{diam} \, X$ and let $E_1 \supset E_2 \supset \ldots$ such that $P(E_j, B(x, R)) < \infty$ for all $j \in \mathbb{N}$, and $\mu(E_j) \to 0$ and $P(E_j, B(x, R)) \to 0$ as $j \to \infty$. Let $0 < r < R$. Then

$$\text{Cap} (\overline{E_j}^m \cap B(x, r)) \to 0.$$

**Proof.** Take a cutoff function $\eta \in \text{Lip}_c(B(x, R))$ with $0 \leq \eta \leq 1$ on $X$, $\eta = 1$ in $B(x, r)$, and $g_\eta \leq 2/(R - r)$, where $g_\eta$ is the minimal 1-weak upper gradient of $\eta$. Then for all $j \in \mathbb{N}$, by a Leibniz rule (see [17, Proposition 4.2]) we have

$$\|D(\chi_{E_j} \eta)\|(X) = \|D(\chi_{E_j} \eta)\|(B(x, R)) \leq \frac{2\mu(E_j)}{R - r} + P(E_j, B(x, R)) \to 0$$

as $j \to \infty$. By (2.16) and the fact that $(\chi_{E_j} \eta)^\vee = 1$ in $\overline{E_j}^m \cap B(x, r)$, we get

$$\text{cap} (\overline{E_j}^m \cap B(x, r), B(x, R)) \leq C_r \text{cap}_{BV} (\overline{E_j}^m \cap B(x, r), B(x, R)) \leq \|D(\chi_{E_j} \eta)\|(X) \to 0 \quad \text{as } j \to \infty.$$

Then by the Sobolev inequality (2.11) we easily get

$$\text{Cap}_1 (\overline{E_j}^m \cap B(x, r)) \to 0.$$ \hfill $\Box$

The variation measure is always absolutely continuous with respect to the 1-capacity, in the following sense.

**Lemma 4.6** ([21, Lemma 3.8]). Let $\Omega \subset X$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ with $\text{Cap}_1(A) < \delta$, then $\|Du\|(A) < \varepsilon$.  

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Lemma 4.7. Let $\Omega \subset X$ be open, let $F_1 \subset F_2 \subset X$ with $P(F_1, \Omega) < \infty$ and $P(F_2, \Omega) < \infty$, and let $A \subset \Omega$ such that for all $x \in A$, we have

$$\lim_{r \to 0} \frac{\mu(B(x, r) \cap (F_2 \setminus F_1))}{\mu(B(x, r))} = 0.$$ 

Then $P(F_1, A) = P(F_2, A)$.

Proof. First note that $P(F_2 \setminus F_1, \Omega) < \infty$ by (2.8), and then by (2.6) we have

$$P(F_2 \setminus F_1, A) = 0.$$ 

Using (2.8) again, we have

$$P(F_2, A) \leq P(F_1, A) + P(F_2 \setminus F_1, A) = P(F_1, A)$$ 

and

$$P(F_1, A) \leq P(F_2, A) + P(F_2 \setminus F_1, A) = P(F_2, A).$$

\[\square\]

The following lemma says that perimeter can always be controlled by the measure of a suitable “curve boundary”.

Lemma 4.8. Let $\Omega \subset X$ be open, let $E \subset X$ be closed, and let $A \subset \Omega$ be such that 1-a.e. curve $\gamma$ in $\Omega$ with $\gamma(0) \in I_E$ and $\gamma(\ell_\gamma) \in X \setminus E$ intersects $A$. Then $P(E, \Omega) \leq C_d \mathcal{H}(A)$.

Proof. We can assume that $\mathcal{H}(A) < \infty$. Fix $\varepsilon > 0$. We find a covering of $A$ by balls $\{B_j = B(x_j, r_j)\}_{j \in I}$, with $I \subset \mathbb{N}$, such that $r_j \leq \varepsilon$ and

$$\sum_{j \in I} \frac{\mu(B_j)}{r_j} \leq \mathcal{H}(A) + \varepsilon. \quad (4.9)$$

Denote the exceptional family of curves by $\Gamma$. Take a nonnegative Borel function $\rho$ such that $\|\rho\|_{L^1(\Omega)} < \varepsilon$ and $\int_\gamma \rho \, ds \geq 1$ for all $\gamma \in \Gamma$. Let

$$g := \sum_{j \in I} \frac{\chi_{2B_j}}{r_j} + \rho.$$

Then let

$$u(x) := \min \left\{ 1, \inf \int_\gamma g \, ds \right\},$$

where the infimum is taken over curves $\gamma$ (also constant curves) in $\Omega$ with $\gamma(0) = x$ and $\gamma(\ell_\gamma) \in \Omega \setminus \left(E \cup \bigcup_{j \in I} 2B_j\right)$. We know that $g$ is an upper gradient of $u$ in $\Omega$, see [4, Lemma 5.25]. Moreover, $u$ is $\mu$-measurable by [15, Theorem 1.11]; strictly speaking this result is written for functions defined on the whole space, but the proof clearly works also for functions defined in an open set such as $\Omega$. If $x \in$
with \( \gamma \) and for any sets \( A \) \( \subset \Omega \) \( \setminus \left( E \cup \bigcup_{j \in I} 2B_j \right) \), clearly \( u(x) = 0 \). If \( x \in I_E \setminus \bigcup_{j \in I} 2B_j \), consider any curve \( \gamma \) in \( \Omega \) with \( \gamma(0) = x \) and \( \gamma(\ell_\gamma) \in \Omega \setminus \left( E \cup \bigcup_{j \in I} 2B_j \right) \). Then either \( \int_\gamma \rho \, ds \geq 1 \) or there is \( t \) such that \( \gamma(t) \in A \). In the latter case, for some \( j \in I \) we have \( \gamma(t) \in B_j \). Then
\[
\int_\gamma g \, ds \geq \int_\gamma \frac{\chi_{2B_j}}{r_j} \, ds \geq 1.
\]
Thus \( u(x) = 1 \), and so by Lebesgue’s differentiation theorem we have \( u = \chi_E \) a.e. in \( \Omega \setminus \bigcup_{j \in I} 2B_j \). Thus
\[
\int_\Omega \left| u - \chi_E \right| d\mu \leq \int_\Omega \chi_{\bigcup_{j \in I} 2B_j} d\mu \leq \sum_{j \in I} \mu(2B_j) \leq \varepsilon \sum_{j \in I} \frac{\mu(B_j)}{r_j} \leq \varepsilon (C_d \mathcal{H}(A) + \varepsilon).
\]
Moreover, using (4.9) we get
\[
\int_\Omega g \, d\mu \leq \sum_{j \in I} \int_\Omega \frac{\chi_{2B_j}}{r_j} \, d\mu + \int_\Omega \rho \, d\mu \leq C_d \mathcal{H}(A) + C_d \varepsilon + \varepsilon.
\]
Now for each \( i \in \mathbb{N} \), use the above construction to obtain functions \( u_i \in N^{1,1}_{loc}(\Omega) \) and upper gradients \( g_i \in L^1(\Omega) \) corresponding to \( \varepsilon = 1/i \). We have
\[
\int_\Omega \left| u_i - \chi_E \right| d\mu \leq i^{-1} (C_d \mathcal{H}(A) + i^{-1}) \to 0 \quad \text{as } i \to \infty
\]
and thus
\[
P(E,\Omega) \leq \liminf_{i \to \infty} \int_\Omega g_i \, d\mu \leq \liminf_{i \to \infty} (C_d \mathcal{H}(A) + C_d^{-1} + i^{-1}) = C_d \mathcal{H}(A).
\]
\[ \Box \]

**Proposition 4.10.** Let \( B(x, R) \) be a ball with \( 0 < R < \frac{1}{4} \text{diam } X \) and let \( F \subset X \) be a closed set with \( P(F, X) < \infty \). Denote the components of \( F \cap \overline{B}(x, R) \) having nonzero \( \mu \)-measure by \( F_1, F_2, \ldots \). Then
\[
\sum_{j=1}^{\infty} P(F_j, B(x, R)) < \infty,
\]
and for any sets \( A_j \subset F_j \) with \( P(A_j, B(x, R)) < \infty \) for all \( j \in \mathbb{N} \) we have
\[
P \left( \bigcup_{j=1}^{\infty} A_j, B(x, R) \right) = \sum_{j=1}^{\infty} P(A_j, B(x, R)). \tag{4.11}
\]

**Proof.** Let \( \Gamma_b \) be the exceptional family of curves of Proposition 4.3; then \( \text{Mod}_1(\Gamma_b) = 0 \). Consider a component \( F_j \); it is a closed set. Consider a curve \( \gamma \notin \Gamma_b \) in \( B(x, R) \) with \( \gamma(0) \in I_{F_j} \) and \( \gamma(\ell_\gamma) \in X \setminus F_j \). Then \( \gamma(0) \in I_F \). Take
\[
t := \max \{ s \in [0, \ell_\gamma] : \gamma([0, s]) \subset F_j \}.
\]
Clearly $t < \ell_\gamma$. There cannot exist $\delta > 0$ such that $\gamma(s) \in F$ for all $s \in (t, t + \delta)$ because this would connect $F_j$ with at least one other component of $F \cap \overline{B}(x, R)$.

Thus there are points $s_j \in (t, t + \delta)$ with $\gamma(s_j) \in X \setminus F \subset O_F$. By Proposition 4.3, this implies that either $\gamma(t) \in \partial^* F$ or $\gamma(t) \in O_F$. In the latter case, there is a point $\tilde{t} \in (0, t)$ with $\gamma(\tilde{t}) \in \partial^* F$. In both cases, we have found $t$ such that $\gamma(t) \in \partial^* F \cap F_j$.

Thus by Lemma 4.8,

$$P(F_j, B(x, R)) \leq C_d \mathcal{H}(\partial^* F \cap F_j)$$

and so

$$\sum_{j=1}^{\infty} P(F_j, B(x, R)) \leq C_d \sum_{j=1}^{\infty} \mathcal{H}(\partial^* F \cap F_j)$$

$$\leq C_d \mathcal{H}(\partial^* F) \quad \leq C_d \alpha^{-1} P(F, X) \quad \text{by (2.6)}$$

$$< \infty,$$

as desired. Next note that one inequality in (4.11) follows from (2.9). To prove the other one, note that the sets $F_j$ are closed and then in fact compact, and so for any $\mu$-measurable sets $A_j \subset F_j$ with $P(A_j, B(x, R)) < \infty$ for all $j \in \mathbb{N}$, we have

$$\text{dist}(A_j, A_k) \geq \text{dist}(F_j, F_k) > 0 \quad \text{for all} \ j \neq k.$$

Take $N, M \in \mathbb{N}$ with $N \leq M$. We have (recall (4.4))

$$P\left(\bigcup_{j=1}^{\infty} A_j, B(x, R)\right) \geq P\left(\bigcup_{j=1}^{\infty} A_j, B(x, R) \setminus \bigcup_{j=M+1}^{\infty} A_j\right)$$

$$= P\left(\bigcup_{j=1}^{M} A_j, B(x, R) \setminus \bigcup_{j=M+1}^{\infty} A_j\right) \quad \text{by Lemma 4.7}$$

$$= \sum_{j=1}^{M} P\left(A_j, B(x, R) \setminus \bigcup_{j=M+1}^{\infty} A_j\right) \quad \text{by (4.13)}$$

By (2.9) and (4.12), we have

$$P\left(\bigcup_{j=M+1}^{\infty} F_j, B(x, R)\right) \leq \sum_{j=M+1}^{\infty} P(F_j, B(x, R)) \to 0 \quad \text{as} \ M \to \infty.$$

Then by Lemma 4.5 we have

$$\text{Cap}_1\left(\bigcup_{j=M+1}^{\infty} A_j \cap B(x, r)\right) \leq \text{Cap}_1\left(\bigcup_{j=M+1}^{\infty} F_j \cap B(x, r)\right) \to 0 \quad \text{as} \ M \to \infty.$$
for all $0 < r < R$. From (4.14) and Lemma 4.6 we now get

$$
P\left(\bigcup_{j=1}^{\infty} A_j, B(x, R)\right) \geq \sum_{j=1}^{N} P(A_j, B(x, r)).$$

Letting $r \nearrow R$ and $N \to \infty$, we get the conclusion. □

For any nonnegative $g \in L^1_{\text{loc}}(X)$, define the centered Hardy-Littlewood maximal function

$$
\mathcal{M}g(x) := \sup_{r > 0} \int_{B(x, r)} g \, d\mu, \quad x \in X.
$$

Recall the definition of the exponent $s > 1$ from (2.1). The argument in the following lemma was inspired by the study of the so-called MEC$_p$-property in [15].

**Lemma 4.15.** Let $B(x_0, r)$ be a ball and let $V \subset X$ be an open set with

$$
\text{Cap}_1(V \cap B(x_0, r)) < \frac{1}{20 \cdot 10^5 C_P C^5_d} \frac{\mu(B(x_0, r))}{r}.
$$

Then there is a connected subset of $\overline{B(x_0, r/2)} \setminus V$ with measure at least $\mu(B(x_0, r))/(4 \cdot 10^5 C^2_d)$.

**Proof.** Take $u \in N^{1,1}(X)$ with $u = 1$ in $V \cap B(x_0, r)$ and

$$
\|u\|_{N^{1,1}(X)} < \frac{1}{20 \cdot 10^5 C_P C^5_d} \frac{\mu(B(x_0, r))}{r}.
$$

Thus there is an upper gradient $g$ of $u$ with

$$
\|g\|_{L^1(X)} < \frac{1}{20 \cdot 10^5 C_P C^5_d} \frac{\mu(B(x_0, r))}{r}.
$$

By the Vitali-Carathéodory theorem (see e.g. [14, p. 108]) we can assume that $g$ is lower semicontinuous. We define

$$
A := \{\mathcal{M}g > (10 C_P C^2_d r)^{-1}\} \quad \text{and} \quad D := \{u \geq 1/2\}.
$$

Then by the weak $L^1$-boundedness of the maximal function (see e.g. [4, Lemma 3.12]) as well as (2.1), we estimate

$$
\mu(A) \leq 10 C_P C^5_d r \|g\|_{L^1(X)} \leq \frac{1}{2 \cdot 10^5 C^6_d} \mu(B(x_0, r)) \leq \frac{1}{2} \mu(B(x_0, r/10)).
$$

Similarly,

$$
\mu(D) \leq 2\|u\|_{L^1(X)} \leq \frac{1}{4} \mu(B(x_0, r/10)),
$$

and then

$$
\mu(B(x_0, r/10) \setminus (A \cup D)) \geq \frac{1}{4} \mu(B(x_0, r/10)) \geq \frac{\mu(B(x_0, r))}{4 \cdot 10^5 C^2_d}. \quad (4.16)
$$
In particular, we can fix \( x \in B(x_0, r/10) \setminus (A \cup D) \). Let \( \delta := (100 C_P C_d^2 r)^{-1} \). For every \( k \in \mathbb{N} \), let \( g_k := \min\{g, k\} \) and

\[
v_k(y) := \inf \int_{\gamma} (g_k + \delta) \, ds, \quad y \in B(x_0, r/2),
\]

where the infimum is taken over curves \( \gamma \) (also constant curves) in \( B(x_0, r/2) \) with \( \gamma(0) = x \) and \( \gamma(\ell_\gamma) = y \). Then \( g_k + \delta \leq g + \delta \) is an upper gradient of \( v_k \) in \( B(x_0, r/2) \) (see [4, Lemma 5.25]) and \( v_k \) is \( \mu \)-measurable by [15, Theorem 1.11]. Since the space is geodesic, each \( v_k \) is \((k + \delta)\)-Lipschitz in \( B(x_0, r/10) \) and thus all points in \( B(x_0, r/10) \) are Lebesgue points of \( v_k \). Define \( B_j := B(x, 2^{-j+1} r/10) \), for \( j = 0, 1, \ldots \).

By the Poincaré inequality,

\[
|v_k(x) - (v_k)_{B_0}| \leq \sum_{j=0}^{\infty} |(v_k)_{B_{j+1}} - (v_k)_{B_j}| \leq C_d \sum_{j=0}^{\infty} \int_{B_j} |v_k - (v_k)_{B_j}| \, d\mu \\
\leq C_d C_P \sum_{j=0}^{\infty} 2^{-j+1} r/10 \int_{B_j} (g + \delta) \, d\mu \tag{4.17}
\]

\[
\leq C_d C_P r (\mathcal{M} g(x) + \delta) \\
\leq 1/8.
\]

Similarly, for every \( y \in B(x_0, r/10) \setminus (A \cup D) \) we have

\[
|v_k(y) - (v_k)_{B(y, r/5)}| \leq 1/8 \tag{4.18}
\]

and

\[
|(v_k)_{B(x, r/5)} - (v_k)_{B(y, r/5)}| \leq 2 C_d^2 \int_{B(x, 2r/5)} |v_k - (v_k)_{B(x, 2r/5)}| \, d\mu \\
\leq 2 C_d^2 C_P r \int_{B(x, 2r/5)} (g + \delta) \, d\mu \tag{4.19}
\]

\[
\leq 2 C_d^2 C_P r (\mathcal{M} g(x) + \delta) \\
\leq 1/4.
\]

Combining (4.17), (4.18), and (4.19), we get

\[
v_k(y) = |v_k(x) - v_k(y)| \leq 1/2.
\]

This means that there is a curve \( \gamma_k \) in \( B(x_0, r/2) \) with \( \gamma_k(0) = x \), \( \gamma_k(\ell_{\gamma_k}) = y \), and \( \int_{\gamma_k} (g_k + \delta) \, ds \leq 1/2 \), for every \( k \in \mathbb{N} \). Note that

\[
\ell_{\gamma_k} \leq \frac{1}{\delta} \int_{\gamma_k} (g_k + \delta) \, ds \leq \frac{1}{2\delta}.
\]

Consider the reparametrizations \( \tilde{\gamma}_k(t) := \gamma_k(t \ell_{\gamma_k}), \ t \in [0, 1] \). By the Arzela-Ascoli theorem (see e.g. [24, p. 169]), passing to a subsequence (not relabeled) we find \( \tilde{\gamma} : [0, 1] \to X \) such that \( \tilde{\gamma}_k \to \tilde{\gamma} \) uniformly. It is straightforward to check that \( \tilde{\gamma} \) is
continuous and rectifiable. Let $\gamma$ be the parametrization of $\tilde{\gamma}$ by arc-length; then $\gamma(0) = x$ and $\gamma(\ell_\gamma) = y$, and by [15, Lemma 2.2], we have for every $k_0 \in \mathbb{N}$ that

$$\int_\gamma g_{k_0} \, ds \leq \liminf_{k \to \infty} \int_{\gamma_k} g_{k_0} \, ds \leq \liminf_{k \to \infty} \int_{\gamma_k} g_k \, ds \leq 1/2.$$  

Letting $k_0 \to \infty$, we obtain

$$\int_\gamma g \, ds \leq 1/2.$$ 

Note that if $\gamma$ intersected a point $z \in V$, then we would have

$$\int_\gamma g \, ds \geq |u(x) - u(z)| > |1/2 - 1| = 1/2,$$

so this is not possible. Thus $\gamma$ is in $\overline{B}(x_0, r/2) \setminus V$; let us denote this curve, and also its image, by $\gamma_y$. Define the desired connected set as the union

$$\bigcup_{y \in B(x_0, r/10) \setminus (A \cup D)} \gamma_y.$$ 

By (4.16) this has measure at least $\mu(B(x_0, r))/(4 \cdot 10^8 C_d^2)$.

**Lemma 4.20.** Let $B(x, R)$ be a ball with $0 < R < 1/4 \text{diam} X$ and let $F \subset X$ be a closed set with $P(F, X) < \infty$. Denote the components of $F \cap \overline{B}(x, R)$ having nonzero $\mu$-measure by $F_1, F_2, \ldots$, and $H := \overline{B}(x, R) \cap F \setminus \bigcup_{j=1}^\infty F_j$. Then $\mu(H) = 0$.

**Proof.** It follows from Proposition 4.10 that $P \left( \bigcup_{j=1}^\infty F_j, B(x, R) \right) < \infty$, and then by (2.8) also $P(H, B(x, R)) < \infty$. By (2.6) and a standard covering argument (see e.g. the proof of [17, Lemma 2.6]), we find that

$$\lim_{r \to 0} \frac{P \left( \bigcup_{j=1}^\infty F_j, B(y, r) \right)}{\mu(B(y, r))} = 0$$

for all $y \in B(x, R) \setminus \left( \partial^* \left( \bigcup_{j=1}^\infty F_j \right) \cup N \right)$, with $\mathcal{H}(N) = 0$, in particular for all $y \in B(x, R) \cap I_H \setminus N$.

Take $y \in B(x, R) \cap I_H \setminus N$ (if it exists). We find arbitrarily small $r > 0$ such that $B(y, r) \subset B(x, R)$ and

$$\frac{\mu(B(y, r) \setminus H)}{\mu(B(y, r))} \leq \frac{1}{80 \cdot 10^8 C_p C_d^8 C_{\text{cap}}}$$  \hspace{1cm} (4.21)

and

$$\frac{P \left( \bigcup_{j=1}^\infty F_j, B(y, r) \right)}{\mu(B(y, r))} \leq \frac{1}{80 \cdot 10^8 C_p C_d^8 C_{\text{cap}}}.$$ 

Now suppose that

$$P(H, B(y, r)) \leq \frac{1}{80 \cdot 10^8 C_p C_d^8 C_{\text{cap}}} \frac{\mu(B(y, r))}{r}.$$
Then since \( H \cup \bigcup_{j=1}^{\infty} F_j = F \cap B(x, R) \), by (2.8) we get

\[
P(F, B(y, r)) \leq P(H, B(y, r)) + P \left( \bigcup_{j=1}^{\infty} F_j, B(y, r) \right)
\leq \frac{1}{40 \cdot 10^s C_p C_d^8 C_{\text{cap}}} \frac{\mu(B(y, r))}{r}.
\]

Define the Lipschitz function

\[
\eta := \max \left\{ 0, 1 - \frac{\text{dist}(\cdot, B(y, r/2))}{r/2} \right\},
\]

so that \( 0 \leq \eta \leq 1 \) on \( X \), \( \eta = 1 \) in \( B(y, r/2) \), \( \eta = 0 \) in \( X \setminus B(y, r) \), and \( g_\eta \leq \frac{2}{r} \chi_{B(y, r)} \) (see [4, Corollary 2.21]). Then by a Leibniz rule (see [17, Proposition 4.2]), we have

\[
\text{Cap}_{BV}(B(y, r/2) \setminus F) \leq \|D(\eta \chi_{X \setminus F})\|(X)
\leq P(F, B(y, r)) + 2 \frac{\mu(B(y, r) \setminus F)}{r}
\leq P(F, B(y, r)) + 2 \frac{\mu(B(y, r) \setminus H)}{r}
\leq \frac{1}{20 \cdot 10^s C_p C_d^8 C_{\text{cap}}} \frac{\mu(B(y, r))}{r}.
\]

Then by (2.15),

\[
\text{Cap}_1(B(y, r/2) \setminus F) \leq \frac{1}{20 \cdot 10^s C_p C_d^8} \frac{\mu(B(y, r))}{r} < \frac{1}{20 \cdot 10^s C_p C_d^8} \frac{\mu(B(y, r/2))}{r/2}.
\]

Then by Lemma 4.15, there is a connected subset of \( F \cap \overline{B(y, r/4)} \) with measure at least

\[
\frac{\mu(B(y, r/2))}{4 \cdot 10^s C_d^2} \geq \frac{\mu(B(y, r))}{4 \cdot 10^s C_d^2}.
\]

By (4.21) this must be (partially) contained in \( H \), a contradiction since \( H \) contains no components of nonzero measure. Thus for all \( y \in I_H \cap B(x, R) \setminus N \), we have

\[
\limsup_{r \to 0} r \frac{P(H, B(y, r))}{\mu(B(y, r))} \geq \frac{1}{80 \cdot 10^s C_p C_d^8 C_{\text{cap}}}.
\]

By a simple covering argument, it follows that

\[
\mu(I_H \cap B(x, R) \setminus N) \leq \varepsilon \cdot 80 \cdot 10^s C_p C_d^{11} C_{\text{cap}} P(H, B(x, R))
\]

for every \( \varepsilon > 0 \). Thus \( \mu(H \cap B(x, R) \setminus N) = 0 \) and so \( \mu(H \cap B(x, R)) = 0 \). Since the space \( X \) is geodesic, by [5, Corollary 2.2] we know that \( \mu(\{y \in X : d(y, x) = R\}) = 0 \) and so in fact \( \mu(H) = 0 \).

**Proof of Proposition 4.1.** This follows from Proposition 4.10 and Lemma 4.20. \( \square \)
5 Functions of least gradient

In this section we consider functions of least gradient, or more precisely superminimizers and solutions of obstacle problems in the case $p = 1$. We will follow the definitions and theory developed in [22]. Throughout this section the symbol $\Omega$ will always denote a nonempty open subset of $X$. We denote by $\text{BV}_c(\Omega)$ the class of functions $\varphi \in \text{BV}(\Omega)$ with compact support in $\Omega$, that is, $\text{spt}\, \varphi \subseteq \Omega$.

**Definition 5.1.** We say that $u \in \text{BV}_{\text{loc}}(\Omega)$ is a 1-minimizer in $\Omega$ (often called function of least gradient) if for all $\varphi \in \text{BV}_c(\Omega)$, we have

$$\|Du\|(\text{spt}\, \varphi) \leq \|D(u + \varphi)\|(\text{spt}\, \varphi).$$  \hspace{1cm} (5.2)

We say that $u \in \text{BV}_{\text{loc}}(\Omega)$ is a 1-superminimizer in $\Omega$ if (5.2) holds for all nonnegative $\varphi \in \text{BV}_c(\Omega)$. We say that $u \in \text{BV}_{\text{loc}}(\Omega)$ is a 1-subminimizer in $\Omega$ if (5.2) holds for all nonpositive $\varphi \in \text{BV}_c(\Omega)$, or equivalently if $-u$ is a 1-superminimizer in $\Omega$.

Equivalently, we can replace $\text{spt}\, \varphi$ by any set $A \subset \Omega$ containing $\text{spt}\, \varphi$ in the above definitions.

If $\Omega$ is bounded, and $\psi : \Omega \to \mathbb{R}$ and $f \in L^1_{\text{loc}}(X)$ with $\|Df\|(X) < \infty$, we define the class of admissible functions

$$K_{\psi,f}(\Omega) := \{ u \in \text{BV}_{\text{loc}}(X) : u \geq \psi \text{ in } \Omega \text{ and } u = f \text{ in } X \setminus \Omega \}.$$  

The (in)equalities above are understood in the a.e. sense. For brevity, we sometimes write $K_{\psi,f}$ instead of $K_{\psi,f}(\Omega)$.

**Definition 5.3.** We say that $u \in K_{\psi,f}(\Omega)$ is a solution of the $K_{\psi,f}$-obstacle problem if $\|Du\|(X) \leq \|Dv\|(X)$ for all $v \in K_{\psi,f}(\Omega)$.

Whenever the characteristic function of a set $E$ is a solution of an obstacle problem, for simplicity we will call $E$ a solution as well. Similarly, if $\psi = \chi_A$ for some $A \subset X$, we let $K_{A,f} := K_{\psi,f}$.

Now we list some properties of superminimizers and solutions of obstacle problems derived mostly in [22].

**Lemma 5.4 ([22, Lemma 3.6]).** If $x \in X$, $0 < r < R < \frac{1}{8}\text{diam } X$, and $A \subset B(x, r)$, then there exists $E \subset X$ that is a solution of the $K_{A,0}(B(x, R))$-obstacle problem with $P(E, X) \leq \text{cap}_1(A, B(x, R))$.

**Proposition 5.5 ([22, Proposition 3.7]).** If $u \in K_{\psi,f}(\Omega)$ is a solution of the $K_{\psi,f}$-obstacle problem, then $u$ is a 1-superminimizer in $\Omega$.

The following fact and its proof are similar to [16, Lemma 3.2].

**Lemma 5.6.** Let $F \subset X$ with $P(F, \Omega) < \infty$ and suppose that for every $H \in \Omega$, we have $P(F, \Omega) \leq P(F \setminus H, \Omega)$. Then $\chi_F$ is a 1-subminimizer in $\Omega$.
Proof. Take a nonnegative $\varphi \in BV_c(\Omega)$. Observe that for every $0 < s < 1$, we have $\supp\{\varphi \geq s\} \subset \Omega$. Thus by the coarea formula (2.7),

$$\|D(\chi_F - \varphi)\|(\supp \varphi) \geq \int_0^1 P(\{\chi_F - \varphi > t\}, \supp \varphi) \, dt$$

$$= \int_0^1 P(F \setminus \{\varphi \geq 1 - t\}, \supp \varphi) \, dt$$

$$\geq \int_0^1 P(F, \supp \varphi) \, dt = \|D\chi_F\|(\supp \varphi).$$

□

**Proposition 5.7.** Let $B(x, R)$ be a ball and let $F \subset X$ be a closed set with $P(F, X) < \infty$ and such that $\chi_F$ is a 1-subminimizer in $B(x, R)$. Denote the components of $F \cap \overline{B}(x, R)$ with nonzero $\mu$-measure by $F_1, F_2, \ldots$. Then each $\chi_{F_k}$ is a 1-subminimizer in $B(x, R)$.

Proof. Fix $k \in \mathbb{N}$ and take $H \subset B(x, R)$. We can assume that $H \subset F_k$ and that $P(F_k \setminus H, B(x, R)) < \infty$. Now

$$\sum_{j \in \mathbb{N}} P(F_j, B(x, R)) + P(F_k, B(x, R)) = \sum_{j=1}^{\infty} P(F_j, B(x, R))$$

$$= P(F, B(x, R)) \quad \text{by Proposition 4.1}$$

$$\leq P(F \setminus H, B(x, R))$$

$$= \sum_{j=1}^{\infty} P(F_j \setminus H, B(x, R)) \quad \text{by Proposition 4.1}$$

$$= \sum_{j \in \mathbb{N}} P(F_j, B(x, R)) + P(F_k \setminus H, B(x, R)).$$

Note that since $\sum_{j=1}^{\infty} P(F_j, B(x, R)) = P(F, B(x, R)) < \infty$, we now get

$$P(F_k, B(x, R)) \leq P(F_k \setminus H, B(x, R)).$$

By Lemma 5.6, $\chi_{F_k}$ is a 1-subminimizer in $B(x, R)$.

□

We have the following weak Harnack inequality. We denote the positive part of a function by $u_+ := \max\{u, 0\}$.

**Theorem 5.8 ([22, Theorem 3.10]).** Suppose $k \in \mathbb{R}$ and $0 < R < \frac{1}{2} \diam X$ with $B(x, R) \subset \Omega$, and assume either that

(a) $u$ is a 1-subminimizer in $\Omega$, or

(b) $\Omega$ is bounded, $u$ is a solution of the $K_{\psi, f}(\Omega)$-obstacle problem, and $\psi \leq k$ a.e. in $B(x, R)$.
Then for any $0 < r < R$ and some constant $C_1 = C_1(C_d, C_P)$,
\[
\text{ess sup}_{B(x,r)} u \leq C_1 \left( \frac{R}{R-r} \right)^s \int_{B(x,R)} (u - k)_+ d\mu + k.
\]

For later reference, let us note that a close look at the proof of the above theorem reveals that we can take
\[
C_1 = 2^{(s+1)^2} (6\tilde{C}S C_d)^s,
\]
where $\tilde{C}$ is the constant from an $(s/(s-1), 1)$-Sobolev inequality with zero boundary values.

**Corollary 5.10.** Suppose $k \in \mathbb{R}$, $x \in X$, $0 < R < \frac{1}{4} \text{diam } X$, and assume that $\chi_F$ is a 1-subminimizer in $B(x, R)$ with $\mu(F \cap B(x, R/2)) > 0$. Then
\[
\frac{\mu(B(x, R) \cap F)}{\mu(B(x, R))} \geq (2^s C_1)^{-1}.
\]

**Proof.** Let $0 < \varepsilon < R/2$. Applying Theorem 5.8(i) with $\Omega = B(x, R)$, $u = \chi_F$, $k = 0$, and $R/2, R - \varepsilon$ in place of $r, R$, we get
\[
1 \leq C_1 \left( \frac{R - \varepsilon}{R - \varepsilon - R/2} \right)^s \frac{\mu(B(x, R - \varepsilon) \cap F)}{\mu(B(x, R - \varepsilon))}.
\]
Letting $\varepsilon \to 0$, we get the result. \hfill \Box

Recall the definitions of the lower and upper approximate limits $u^\wedge$ and $u^\vee$ from (2.13) and (2.14).

**Theorem 5.11 ([22, Theorem 3.11]).** Let $u$ be a 1-superminimizer in $\Omega$. Then $u^\wedge : \Omega \to (-\infty, \infty]$ is lower semicontinuous.

**Lemma 5.12.** Let $B = B(x, R)$ be a ball with $0 < R < \frac{1}{32} \text{diam } X$, and suppose that $W \subset B$. Let $V \subset 4B$ be a solution of the $K_{W,0}(4B)$-obstacle problem (as guaranteed by Lemma 5.4). Then for all $y \in 3B \setminus 2B$,
\[
\chi_V^\vee(y) \leq C_2 R \frac{\text{cap}_1(W, 4B)}{\mu(B)}
\]
for some constant $C_2 = C_2(C_d, C_P)$.

**Proof.** By Lemma 5.4 we know that
\[
P(V, X) \leq \text{cap}_1(W, 4B),
\]
and thus by the isoperimetric inequality (2.12),
\[
\mu(V) \leq 4C_SR P(V, X) \leq 4C_SR \text{cap}_1(W, 4B).
\]

(5.13)
For any $z \in 3B \setminus 2B$ we have $B(z, R) \subset 4B \setminus B$. Since now $W \cap B(z, R) = \emptyset$, we can apply Theorem 5.8(b) with $k = 0$ to get

\[
\sup_{B(z, R/2)} \chi_V^Y \leq \operatorname{ess sup}_{B(z, R/2)} \chi_V^Y \leq C_1 \left( \frac{R}{R - R/2} \right)^s \int_{B(z, R)} (\chi_V^Y)_+ d\mu \\
= \frac{2^s C_1}{\mu(B(z, R))} \int_{B(z, R)} (\chi_V^Y)_+ d\mu \\
\leq \frac{2^s C_1 C_2^2}{\mu(B)} \mu(V) \\
\leq 2^{s+2} C_1 C_2^2 C_S R \frac{\operatorname{cap}_1(W, 4B)}{\mu(B)} \text{ by (5.13)}.
\]

Thus we can choose $C_2 = 2^{s+2} C_1 C_2^2 C_S$. \qed

### 6 Constructing a “geodesic” space

In this section we construct a suitable space where the Mazurkiewicz metric agrees with the ordinary one; this space will be needed in the proof of the main result.

Recall that in Section 3, in the space $(Z, \hat{d}, \mu)$ we defined the Mazurkiewicz metric $\hat{d}_M$; given a set $V \subset X$ we now define

\[
d^Y_M(x, y) := \inf \{ \text{diam} K : K \subset X \setminus V \text{ is a continuum containing } x, y \}, \quad x, y \in X \setminus V.
\]

If $V = \emptyset$, we leave it out of the notation, consistent with (3.1).

**Lemma 6.1.** Let $V \subset X$ be a bounded open set and let $B(x_0, R_0)$ be a ball such that $V \subset B(x_0, R_0)$, and $\overline{B}(x_0, R_0) \setminus V$ is connected. Moreover, suppose there is $R > 0$ such that for every $x \in X \setminus V$ and $0 < r \leq R$, the connected components of $\overline{B}(x, r) \setminus V$ intersecting $B(x, r/2)$ are finite in number.

Then $d^Y_M$ is a metric on $X \setminus V$ such that $d \leq d^Y_M$, $d_M^Y$ induces the same topology on $X \setminus V$ as $d$, $(d_M^Y)_M = d_M^Y$, and $(X \setminus V, d_M^Y)$ is complete.

Note that explicitly, for $x, y \in X \setminus V$,

\[
(d_M^Y)_M(x, y) = \inf \{ \text{diam} K' : K' \subset X \setminus V \text{ is a } d_M^Y \text{-continuum containing } x, y \}.
\]

**Proof.** Since $V \subset B(x_0, R_0)$ and $\overline{B}(x_0, R_0) \setminus V$ is connected, also every $\overline{B}(x_0, r) \setminus V$ with $r \geq R_0$ is connected, by the fact that $X$ is geodesic. Thus we have for all $x, y \in X \setminus V$

\[
d_M^Y(x, y) \leq 2 \max \{ R_0, d(x, x_0), d(y, x_0) \} < \infty.
\]

Obviously $d \leq d^Y_M$ and $d^Y_M(x, x) = 0$ for all $x \in X \setminus V$. If $d^Y_M(x, y) = 0$ then $d(x, y) = 0$ and so $x = y$. Obviously also $d_M^Y(x, y) = d_M^Y(y, x)$ for all $x, y \in X \setminus V$. Finally, take $x, y, z \in X \setminus V$. Take a continuum $K_1 \subset X \setminus V$ containing $x, y$ and
a continuum $K_2 \subset X \setminus V$ containing $y, z$. Then $K_1 \cup K_2 \subset X \setminus V$ is a continuum containing $x, z$ and so

$$d^V_M(x, z) \leq \text{diam}(K_1 \cup K_2) \leq \text{diam}(K_1) + \text{diam}(K_2).$$

Taking infimum over $K_1$ and $K_2$, we conclude that the triangle inequality holds. Hence $d^V_M$ is a metric on $X \setminus V$.

To show that the topologies induced on $X \setminus V$ by $d$ and $d^V_M$ are the same, take a sequence $x_j \to x$ with respect to $d$ in $X \setminus V$. Fix $\varepsilon \in (0, R)$. Consider the components of $\overline{B}(x, \varepsilon/2) \setminus V$ intersecting $B(x, \varepsilon/4)$. By assumption there are only finitely many. Each of them not containing $x$ is at a nonzero distance from $x$ and so for large $j$, every $x_j$ belongs to the component containing $x$; denote it $F_1$. For such $j$, we have

$$d^V_M(x_j, x) \leq \text{diam} F_1 \leq \varepsilon.$$

We conclude that $x_j \to x$ also with respect to $d^V_M$. Since we had $d \leq d^V_M$, it follows that the topologies are the same.

If $x, y \in X \setminus V$, and $\varepsilon > 0$, we can take a continuum $K$ containing $x$ and $y$, with $\text{diam} K < d^V_M(x, y) + \varepsilon$. The set $K$ is still a continuum in the metric space $(X \setminus V, d^V_M)$, and for every $z, w \in K$,

$$d^V_M(z, w) \leq \text{diam} K < d^V_M(x, y) + \varepsilon.$$

It follows that $\text{diam}_{d^V_M} K \leq d^V_M(x, y) + \varepsilon$, and so $(d^V_M)_M(x, y) \leq d^V_M(x, y) + \varepsilon$, showing that $(d^V_M)_M = d^V_M$.

Finally let $(x_j)$ be a Cauchy sequence in $(X \setminus V, d^V_M)$. Since $d \leq d^V_M$, it is also a Cauchy sequence in $(X, d)$, and so $x_j \to x \in X \setminus V$ with respect to $d$. But as we showed before, this implies that $x_j \to x$ with respect to $d^V_M$.

Let $B$ be a ball and let $B_1, B_2 \subset B$ be two other balls, and let $u \in L^1(B)$ such that $u = 1$ in $B_1$ and $u = 0$ in $B_2$. Then we have

$$\int_B |u - u_B| d\mu \geq \frac{1}{2} \min\{\mu(B_1), \mu(B_2)\}; \quad (6.2)$$

this follows easily by considering the cases $u_B \leq 1/2$ and $u_B \geq 1/2$.

We have the following linear local connectedness; versions of this property have been proved before e.g. in [13], but they assume certain growth bounds on the measure, which we do not want to assume.

**Lemma 6.3.** Let $B(x_0, R)$ be a ball and let $V \subset B(x_0, 2R)$ with

$$\text{cap}_1(V, B(x_0, 3R)) \leq \frac{1}{12C_pC^3_d} \frac{\mu(B(x_0, R))}{R}. \quad (6.4)$$

Then every pair of points $y, z \in B(x_0, 5R) \setminus B(x_0, 4R)$ can be joined by a curve in $B(x_0, 6R) \setminus V$.  

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Proof. If \( d(y, z) \leq 2R \), then the result is clear since the space is geodesic. Thus assume that \( d(y, z) > 2R \). Consider the disjoint balls \( B_1 := B(y, R) \) and \( B_2 := B(z, R) \) which both belong to \( B(x_0, 6R) \setminus B(x_0, 3R) \). Denote by \( \Gamma \) the family of curves \( \gamma \) in \( B(x_0, 6R) \) with \( \gamma(0) \in B_1 \) and \( \gamma(\ell_\gamma) \in B_2 \). Note that \( \text{Mod}_1(\Gamma) < \infty \) since \( \text{dist}(B_1, B_2) > 0 \). Let \( \varepsilon > 0 \). Let \( g \in L^1(B(x_0, 6R)) \) such that \( \int_\gamma g \, ds \geq 1 \) for all \( \gamma \in \Gamma \) and
\[
\int_{B(x_0, 6R)} g \, d\mu < \text{Mod}_1(\Gamma) + \varepsilon.
\]
Let
\[
u(x) := \min \left\{ 1, \inf_{\gamma} \int_\gamma g \, ds \right\}, \quad x \in B(x_0, 6R),
\]
where the infimum is taken over curves \( \gamma \) (also constant curves) in \( B(x_0, 6R) \) with \( \gamma(0) = x \) and \( \gamma(\ell_\gamma) \in B_1 \). Then \( u = 1 \) in \( B_2 \). Moreover, \( g \) is an upper gradient of \( u \) in \( B(x_0, 6R) \), see \([4, \text{Lemma 5.25}]\), and \( u \) is \( \mu \)-measurable by \([15, \text{Theorem 1.11}]\). In total, \( u \in N^{1,1}(B(x_0, 6R)) \) with \( u = 0 \) in \( B_1 \) and \( u = 1 \) in \( B_2 \). Thus using the Poincaré inequality,
\[
\text{Mod}_1(\Gamma) > \int_{B(x_0, 6R)} g \, d\mu - \varepsilon
\]
\[
\geq \frac{1}{6C_\mu R} \int_{B(x_0, 6R)} |u - u_{B(x_0, 6R)}| \, d\mu - \varepsilon
\]
\[
\geq \frac{1}{12C_\mu R} \min\{\mu(B_1), \mu(B_2)\} - \varepsilon \quad \text{by (6.2)}
\]
\[
\geq \frac{1}{12C_\mu C^3_d R} \mu(B(x_0, R)) - \varepsilon
\]
and so
\[
\text{Mod}_1(\Gamma) \geq \frac{1}{12C_\mu C^3_d R} \mu(B(x_0, R)).
\]
On the other hand, by (6.4) we find a function \( v \in N^{1,1}(X) \) such that \( v = 1 \) in \( V \), \( v = 0 \) in \( X \setminus B(x_0, 3R) \), and \( v \) has an upper gradient \( \tilde{g} \) satisfying
\[
\int_X \tilde{g} \, d\mu < \frac{1}{12C_\mu C^3_d} \frac{\mu(B(x_0, R))}{R}.
\]
Denote the family of all curves intersecting \( V \) by \( \Gamma_V \). Now \( \int_\gamma \tilde{g} \, ds \geq 1 \) for all \( \gamma \in \Gamma \cap \Gamma_V \), and so
\[
\text{Mod}_1(\Gamma \cap \Gamma_V) \leq \int_X \tilde{g} \, d\mu < \frac{1}{12C_\mu C^3_d} \frac{\mu(B(x_0, R))}{R}.
\]

Thus \( \Gamma \setminus \Gamma_V \) is nonempty. Take a curve \( \gamma \in \Gamma \setminus \Gamma_V \). Now we get the required curve by concatenating three curves: the first going from \( y \) to \( \gamma(0) \) inside \( B(y, R) \) (using the fact that the space is geodesic), the second \( \gamma \), and the third going from \( \gamma(\ell_\gamma) \) to \( z \) inside \( B(z, R) \). \( \square \)
By using an argument involving Lipschitz cutoff functions, it is easy to see that for any ball \(B(x, r)\) and any set \(A \subset B(x, r)\), we have

\[
\text{cap}_1(A, B(x, 3r)) \leq C_d \mathcal{H}(A). \tag{6.5}
\]

In the following proposition we construct the space in which the metric and Mazurkiewicz metric agree.

**Proposition 6.6.** Let \(B = B(x, R)\) be a ball with \(0 < R < \frac{1}{32} \text{diam } X\), and let \(A \subset B\) with

\[
\mathcal{H}(A) \leq \frac{1}{24CPSC_2C_4d} \frac{\mu(B)}{R}. \tag{6.6}
\]

Let \(\epsilon > 0\). Then we find an open set \(V\) with \(A \subset V \subset 2B\) and

\[
P(V, X) \leq C_d \mathcal{H}(A) + \epsilon,
\]

and such that the following hold: the space \((Z, d_V^M, \mu)\) with \(Z = X \setminus V\) is a complete metric space with \((d_M^V)_{|M} = d_M^V\), \(\mu\) in \(Z\) is a Borel regular outer measure and doubling with constant \(2^sC_1C_2d\), and for every \(y \in X \setminus V\) and \(r > 0\) we have

\[
\frac{\mu(B_Z(y, r))}{\mu(B(y, r))} \geq (2^sC_1C_2d)^{-1}
\]

where \(B_Z(y, r)\) denotes an open ball in \(Z\), defined with respect to the metric \(d_M^V\).

**Proof.** Using the fact that \(\text{cap}_1\) is an outer capacity in the sense of (2.3), as well as (6.5), we find an open set \(W\), with \(A \subset W \subset B\), such that (note that the first inequality is obvious)

\[
\text{cap}_1(W, 4B) \leq \text{cap}_1(W, 3B) \leq \text{cap}_1(A, 3B) + \epsilon \leq C_d \mathcal{H}(A) + \epsilon.
\]

We can assume that

\[
\epsilon < \frac{1}{24CPSC_2C_4d} \frac{\mu(B)}{R}.
\]

Take a solution \(V\) of the \(K_{W,0}(4B)\)-obstacle problem. By Lemma 5.4, we have

\[
P(V, X) \leq \text{cap}_1(W, 4B) \leq C_d \mathcal{H}(A) + \epsilon.
\]

By Theorem 5.11, the function \(\chi_V^\wedge\) is lower semicontinuous, and by redefining \(V\) in a set of measure zero, we get \(\chi_V = \chi_V^\wedge\) and so \(V\) is open. By Lemma 5.12 we know that for all \(y \in 3B \setminus 2B\),

\[
\chi_V^\wedge(y) \leq C_2R \frac{\text{cap}_1(W, 4B)}{\mu(B)} \leq C_2R \frac{C_d \mathcal{H}(A) + \epsilon}{\mu(B)} < 1
\]

and so \(\chi_V^\wedge = 0\) in \(3B \setminus 2B\). Then in fact \(\chi_V = \chi_V^\wedge = 0\) in \(4B \setminus 2B\), that is, \(V \subset 2B\), because else we could remove the parts of \(V\) inside \(4B \setminus 3B\) to decrease \(P(V, X)\).
By the isoperimetric inequality (2.12),
\[ \mu(V) \leq 2C_SRP(V, X) \leq 2C_SCdRH(A) + 2C_SR\varepsilon \leq \frac{\mu(B)}{2C_d^2}. \]  
(6.7)

Moreover, by (2.16) we get
\[ \text{cap}_1(V, 3B) \leq C_R \text{cap}^\vee_{BV}(V, 3B) \]
\[ \leq C_R P(V, X) \leq C_R C_d H(A) + C_R \varepsilon < \frac{1}{12C_pC_d^3} \mu(B). \]

By Lemma 6.3, \( 5B \setminus 4B \) belongs to one component of \( 6\overline{B} \setminus V \). Since the space is geodesic, in fact \( 6\overline{B} \setminus 4B \) belongs to one component of \( 6\overline{B} \setminus V \). Call this component \( F_1 \). Moreover, denote \( F := X \setminus V; F \) is a closed set with \( P(F, X) = P(V, X) < \infty \). Consider all components of \( F \cap 6\overline{B} \). Suppose there is another component \( F_2 \) with nonzero \( \mu \)-measure. Denote by \( F_1, F_2, \ldots \) all the components with nonzero \( \mu \)-measure (as usual, some of these may be empty). By the relative isoperimetric inequality (2.10), we have
\[ P(F_2, 6B) > 0. \]  
(6.8)

Now the set \( \tilde{V} := V \cup \bigcup_{j=2}^\infty F_j \subset 4B \) is admissible for the \( K_{W,0}(4B) \)-obstacle problem, with
\[ P(\tilde{V}, X) = P(\tilde{V}, 6B) \]
\[ = P\left( X \setminus \left( V \cup \bigcup_{j=2}^\infty F_j \right), 6B \right) \]
\[ = P\left( F \setminus \bigcup_{j=2}^\infty F_j, 6B \right) \]
\[ = P(F, 6B) - \sum_{j=2}^\infty P(F_j, 6B) \quad \text{by Proposition 4.1} \]
\[ < P(F, 6B) \quad \text{by (6.8)} \]
\[ = P(V, 6B) = P(V, X). \]

This is a contradiction with the fact that \( V \) is a solution of the \( K_{W,0}(4B) \)-obstacle problem. Thus by Proposition 4.1, \( F \cap 6\overline{B} \) is the union of \( F_1 \) and a set of measure zero \( N \). Suppose
\[ y \in 6\overline{B} \cap F \setminus F_1 = 4B \cap F \setminus F_1. \]

Now \( y \) is at a nonzero distance from \( F_1 \). Thus for small \( \delta > 0 \),
\[ \mu(B(y, \delta) \cap F) \leq \mu(N) = 0. \]

Note that since we had \( \chi_V = \chi_V^\wedge \), it follows that \( \chi_F = \chi_F^\vee \). Thus in fact such \( y \) cannot exist and \( F \cap 6\overline{B} = F_1 \) is connected.
If \( y \in F \setminus B(x, 3R) \) and \( 0 < r \leq R \), then \( \overline{B}(y, r) \cap F = \overline{B}(y, r) \) is connected since the space is geodesic. If \( y \in F \cap B(x, 3R) \) and \( 0 < r \leq R \), by Proposition 4.1 we know that \( F \cap \overline{B}(y, r) \) consists of at most countably many components \( F_1, F_2, \ldots \) and a set of measure zero \( \tilde{N} \). By Proposition 5.5 we know that \( \chi_F \) is a 1-subminimizer in \( B(x, 4R) \), and then also in \( B(y, r) \subset B(x, 4R) \). Then each \( \chi_{F_j} \) is a 1-subminimizer in \( B(y, r) \) by Proposition 5.7. By Corollary 5.10 we get for each \( F_j \) with \( \mu(B(y, r/2) \cap F_j) > 0 \) that

\[
\frac{\mu(F_j \cap B(y, r))}{\mu(B(y, r))} \geq (2^sC_1)^{-1}.
\]

(6.9)

Thus there are less than \( 2^sC_1 + 1 \) such components, which we can relabel \( F_1, \ldots, F_M \). Suppose

\[
z \in B(y, r/2) \cap \tilde{N} \setminus \bigcup_{j=1}^{M} F_j.
\]

This is at nonzero distance from all \( F_1, \ldots, F_M \). Thus for small \( \delta > 0 \),

\[
\mu(B(z, \delta) \cap F) \leq \mu(\tilde{N}) + \sum_{j=M+1}^{\infty} \mu(F_j \cap B(y, r/2)) = 0.
\]

As before, we have \( \chi_F = \chi_F^V \). Thus in fact such \( z \) cannot exist and

\[
F \cap B(y, r/2) = B(y, r/2) \cap \bigcup_{j=1}^{M} F_j.
\]

Now Lemma 6.1 gives that \((Z, d^V_M, \mu)\), with \( Z = X \setminus V \), is a complete metric space, \( d \leq d^V_M \), the topologies induced by \( d \) and \( d^V_M \) are the same, and \((d^V_M)_M = d^V_M \). Note that \( \mu \) restricted to the subsets of \( X \setminus V \) is still a Borel regular outer measure, see [14, Lemma 3.3.11]. Since the topologies induced by \( d \) and \( d^V_M \) are the same, \( \mu \) remains a Borel regular outer measure in \( Z \). (Note that as sets, we have \( X \setminus V = F = Z \).)

Denoting by \( F_1 \) the component of \( F \cap \overline{B}(y, r) \) containing \( y \), by (6.9) we have for \( y \in F \cap B(x, 3R) \) and \( 0 < r \leq R \) that

\[
\frac{\mu(B(y, r) \cap F_1)}{\mu(B(y, r))} \geq (2^sC_1)^{-1}.
\]

(6.10)

Recall that if \( y \in F \setminus B(x, 3R) \), then \( F_1 = \overline{B}(y, r) \) and so (6.10) holds. Eq. (6.10) is easily seen to hold also for all \( x \in F \) and \( r > R \) by (6.7). It follows that for all \( y \in F \) and \( r > 0 \), we have

\[
\frac{\mu(B_Z(y, 2r))}{\mu(B(y, r))} \geq (2^sC_1)^{-1}
\]

and so in fact

\[
\frac{\mu(B_Z(y, r))}{\mu(B(y, r))} \geq (2^sC_1C_d)^{-1} \quad \text{for all } y \in Z \text{ and } r > 0,
\]

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as desired. Thus
\[
\frac{\mu(B_Z(y,2r))}{\mu(B_Z(y,r))} \leq 2^s C_1 C_d \frac{\mu(B(y,2r))}{\mu(B(y,r))} \leq 2^s C_1 C_d^2.
\]
Thus in the space \((Z,d^V_M,\mu)\), the measure \(\mu\) is doubling with constant \(2^s C_1 C_d^2\). \(\square\)

7 Proof of the main result

In this section we prove the main result of the paper, Theorem 1.1.

First note that with the choice \(\hat{C}_d = 2^s C_1 C_d^2\), the constant appearing in Corollary 3.8 becomes
\[
\frac{1}{4C_d^{12}} = \frac{1}{4(2^s C_1 C_d^2)^{12}} =: \beta_0.
\]
Recall from (5.9) that we can take \(C_1 = 2^{(s+1)^2}(6\tilde{C}_S C_d)^s\). Define
\[
\beta := \frac{\beta_0}{2^s C_1 C_d} = \frac{1}{2^{2+s} C_1 C_d (2^s C_1 C_d^2)^{12}} = \frac{1}{2^{13s+2}(2^{(s+1)^2}(6\tilde{C}_S C_d)^s)^{13} C_d^{25}}
\]
\[
= \frac{1}{2^{13s^2+52s+153 s^{13} C_d^{13s+25}}.\tag{7.1}
\]
Note that in the Euclidean space \(\mathbb{R}^n, n \geq 2\), we can take \(C_d = 2^n\), \(s = n\), and \(\tilde{C}_S = 2^{-1} n^{-1/2} \omega_n^{1/n}\), where \(\omega_n\) is the volume of the Euclidean unit ball, and then
\[
\beta = 2^{-26n^2 - 64n - 153 - 13n^{13/2} \omega_n^{-13}}.\tag{7.2}
\]
Recall the definition of the strong boundary from (2.5).

Theorem 7.3. Let \(\Omega \subset X\) be open and let \(E \subset X\) be \(\mu\)-measurable with \(\mathcal{H}(\Sigma_\beta E \cap \Omega) < \infty\). Then \(\mathcal{H}(\partial^* E \setminus \Sigma_\beta E \cap \Omega) = 0\).

Proof. By a standard covering argument (see e.g. the proof of [17, Lemma 2.6]), we find that
\[
\lim_{r \to 0} r \frac{\mathcal{H}(\Sigma_\beta E \cap B(x,r))}{\mu(B(x,r))} = 0
\]
for all \(x \in \Omega \setminus (\Sigma_\beta E \cup N)\), with \(\mathcal{H}(N) = 0\). We will show that \(\partial^* E \cap \Omega \subset (\Sigma_\beta E \cup N) \cap \Omega\) and thereby prove the claim.

Suppose instead that there exists \(x \in \Omega \cap \partial^* E \setminus (\Sigma_\beta E \cup N)\). Then
\[
\lim_{r \to 0} r \frac{\mathcal{H}(\Sigma_\beta E \cap B(x,r))}{\mu(B(x,r))} = 0
\]
and
\[
\limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > 0.
\]
Thus for some \(0 < a < (2C_d^2)^{-1}\) we have
\[
\limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} > C_d a \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > C_d a.
\]
Now we can choose $0 < R_0 < \frac{1}{32} \text{diam } X$ such that

$$\frac{\mu(B(x, 40^{-1}R_0) \cap E)}{\mu(B(x, 40^{-1}R_0))} > a$$

and

$$R \frac{\mathcal{H}(\Sigma_\beta E \cap B(x, r))}{\mu(B(x, r))} < \frac{a}{24 \cdot 2^s C_P C_S C_1 C_2 C^8_d}$$

for all $0 < r \leq R_0$. Choose the smallest $j = 0, 1, \ldots$ such that for some $r \in (2^{-j-1}R_0, 2^{-j}R_0]$ we have

$$\frac{\mu(B(x, 40^{-1}r) \setminus E)}{\mu(B(x, 40^{-1}r))} > C_d a$$

and thus

$$\frac{\mu(B(x, 40^{-1-2j}R_0) \setminus E)}{\mu(B(x, 40^{-1-2j}R_0))} > a.$$

Let $R := 2^{-j}R_0$. If $j \geq 1$, then

$$\frac{\mu(B(x, 20^{-1}R) \setminus E)}{\mu(B(x, 20^{-1}R))} \leq C_d a$$

and so

$$\frac{\mu(B(x, 40^{-1}R) \cap E)}{\mu(B(x, 40^{-1}R))} \geq \frac{\mu(B(x, 40^{-1}R) - \mu(B(x, 20^{-1}R) \setminus E))}{\mu(B(x, 40^{-1}R))}$$

$$\geq 1 - C_d \frac{\mu(B(x, 20^{-1}R) \setminus E)}{\mu(B(x, 20^{-1}R))}$$

$$\geq 1 - C_d^2 a \geq 1 - C_d^2 \frac{1}{2C_d^2} = \frac{1}{2} > a.$$  

Thus

$$a < \frac{\mu(B(x, 40^{-1}R) \cap E)}{\mu(B(x, 40^{-1}R))} < 1 - a,$$  

which holds clearly also if $j = 0$, and

$$R \frac{\mathcal{H}(\Sigma_\beta E \cap B(x, R))}{\mu(B(x, R))} < \frac{a}{24 \cdot 2^s C_P C_S C_1 C_2 C^8_d}.$$  

Let $A := \Sigma_\beta E \cap B(x, R)$. By Proposition 6.6 we find an open set $V$ with $A \subset V \subset B(x, 2R)$ and such that denoting $Z = X \setminus V$, the space $(Z, d^M_M, \mu)$ is a complete metric space with $d \leq d^M_M = (d^M_M)_{M \in Z}$ in $Z$, $\mu$ in $Z$ is a Borel regular outer measure and doubling with constant $C_d = 2^s C_1 C^2_d$, and for every $y \in Z$ and $r > 0$ we have

$$\frac{\mu(B_Z(y, r))}{\mu(B(y, r))} \geq (2^s C_1 C_d)^{-1}.$$  

Moreover, by choosing a suitably small $\varepsilon > 0$,

$$P(V, X) \leq C_d \mathcal{H}(A) + \varepsilon < \frac{a}{2^{s+1} C_P C_S C_1 C_d} \frac{\mu(B(x, R))}{R}.$$  

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Thus by the isoperimetric inequality (2.12),
\[ \mu(V) \leq 2CSR P(V, X) < \frac{1}{C_d} \mu(B(x, R)) \leq \mu(B(x, 40^{-1} R)). \]

Thus we can choose \( y \in B(x, 40^{-1} R) \setminus V \). Denote \( F := X \setminus V \). Let \( F_1 \) be the component of \( B(y, 20^{-1} R) \setminus V \) containing \( y \). By (6.10) (and the comments after it) we know that
\[ \mu(F_1) \geq (2^s C_1)^{-1} \mu(B(y, 20^{-1} R)). \]

Since \( \mu(\{ z \in X : d(z, y) = 20^{-1} R \}) = 0 \) (see [5, Corollary 2.2]), now also
\[ \mu(B(y, 20^{-1} R) \cap F_1) \geq (2^s C_1)^{-1} \mu(B(y, 20^{-1} R)). \]

Suppose that
\[ \mu(B(y, 20^{-1} R) \setminus F_1) \geq \frac{a}{2^s C_1 C_d^2} \mu(B(y, 20^{-1} R)). \]

Then
\[ P(V, B(y, 20^{-1} R)) = P(F, B(y, 20^{-1} R)) \geq P(F_1, B(y, 20^{-1} R)) \text{ by Proposition 4.1} \]
\[ \geq \frac{a}{2 \cdot 2^s C_p C_1 C_d^2 20^{-1} R} \mu(B(y, 20^{-1} R)) \text{ by (2.10)} \]
\[ \geq \frac{a}{2^{s+1} C_p C_1 C_d^2} \mu(B(x, R)). \]

This contradicts (7.6), and so necessarily
\[ \mu(B(y, 20^{-1} R) \setminus F_1) < \frac{a}{2^s C_1 C_d^2} \mu(B(y, 20^{-1} R)) \leq \frac{a}{C_d^2} \mu(B(y, 20^{-1} R)). \] (7.7)

Now
\[ C_d \frac{\mu(B_Z(y, 10^{-1} R) \cap E)}{\mu(B(y, 10^{-1} R))} \geq \frac{\mu(B(y, 20^{-1} R) \setminus E \setminus F_1)}{\mu(B(y, 20^{-1} R))} \]
\[ \geq \frac{\mu(B(y, 20^{-1} R) \cap E)}{\mu(B(y, 20^{-1} R))} - \frac{a}{C_d^2} \text{ by (7.7)} \]
\[ \geq \frac{1}{C_d^2} \frac{\mu(B(x, 40^{-1} R) \cap E)}{\mu(B(x, 40^{-1} R))} - \frac{a}{C_d^2} \]
\[ > \frac{a}{C_d^2} - \frac{a}{C_d^2} = 0 \text{ by (7.4)}. \]

The same string of inequalities holds with \( E \) replaced by \( X \setminus E \). It follows that
\[ 0 < \mu(B_Z(y, 10^{-1} R) \cap E) < \mu(B_Z(y, 10^{-1} R)). \]

Denoting by \( \Sigma_{\beta_0}^Z E \) the strong boundary defined in the space \((Z, d_M^V, \mu)\), by Corollary 3.8 we find a point
\[ z \in \Sigma_{\beta_0}^Z E \cap B_Z(y, 9R/10) \subset \Sigma_{\beta_0}^Z E \cap B(y, 9R/10) \setminus V \subset \Sigma_{\beta_0}^Z E \cap B(x, R) \setminus V. \]
Now using (7.5), we get
\[
\liminf_{r \to 0} \frac{\mu(B(z,r) \cap E)}{\mu(B(z,r))} \geq \liminf_{r \to 0} \frac{\mu(B^*_Z(z,r) \cap E)}{\mu(B^*_Z(z,r))} \mu(B(z,r)) \geq \frac{1}{2^s C_1 C_d} = \beta,
\]
and analogously for \(X \setminus E\). Thus \(z \in \Sigma_\beta E \cap B(x,R) \setminus V\), a contradiction. \(\square\)

Recall the usual version of Federer’s characterization in metric spaces.

**Theorem 7.8** ([20, Theorem 1.1]). Let \(\Omega \subset X\) be an open set, let \(E \subset X\) be a \(\mu\)-measurable set, and suppose that \(\mathcal{H}(\partial^* E \cap \Omega) < \infty\). Then \(P(E,\Omega) < \infty\).

Now we can prove our main result; recall from the discussion on page 4 that one can assume the space to be geodesic, as we have done in most of the paper. (However, the constant \(\beta\), which is defined explicitly in geodesic spaces in (7.1), will have a different form in the original space considered in Theorem 1.1.)

**Proof of Theorem 1.1.** By Theorem 7.3 we get \(\mathcal{H}(\partial^* E \cap \Omega) < \infty\), and then Theorem 7.8 gives \(P(E,\Omega) < \infty\). \(\square\)

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