Graphs in the 3–sphere with maximum symmetry

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Abstract

We consider the orientation preserving actions of finite groups $G$ on pairs $(S^3, \Gamma)$, where $\Gamma$ is a connected graph of genus $g > 1$, embedded in $S^3$. For each $g$ we give the maximum order $m_g$ of such $G$ acting on $(S^3, \Gamma)$ for all such $\Gamma \subset S^3$. Indeed we will classify all graphs $\Gamma \subset S^3$ which realize these $m_g$ in different levels: as abstract graphs and as spatial graphs, as well as their group actions.

Such maximum orders without the condition “orientation preserving” are also addressed.

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1 Introduction

We will study smooth, faithful actions of finite groups $G$ on pairs $(S^3, \Gamma)$, where $\Gamma$ denotes a finite connected graph with an embedding $e : \Gamma \to S^3$. We also call such a $G$-action on $\Gamma$ is extendable (w.r.t. $e$). Let $g(\Gamma)$ denote the genus of $\Gamma$, defined as the rank of its free fundamental group. We will always assume that $g > 1$ in the present paper.
Except in Section 5 we will only consider orientation-preserving finite group actions on $S^3$. Referring to the recent geometrization of finite group actions on $S^3$, we will consider only orthogonal actions of finite groups on $S^3$, i.e., finite subgroups $G$ of the orthogonal group $SO(4)$.

Let $m_g$ denote the maximum order of such a group $G$ acting on a pair $(S^3, \Gamma)$, for all embeddings of finite graphs $\Gamma$ of a fixed genus $g$ into $S^3$. In this paper we will determine $m_g$ and classify all finite graphs $\Gamma$ which realize the maximum order $m_g$.

A similar question for the pair $(S^3, \Sigma_g)$, where $\Sigma_g$ is the closed orientable surface of genus $g$, has been studied in [WWZZ], and the corresponding maximum order $\text{OE}_g$ of finite groups acting on $(S^3, \Sigma_g)$ for all possible embeddings $\Sigma_g \subset S^3$ is obtained.

Let $V_g$ denote the handlebody of genus $g$. Each graph $\Gamma \subset S^3$ of genus $g$ has a regular neighborhood $V_g$. Call $\Gamma$ unknotted if the complement of its regular neighborhood is also a handlebody, and otherwise knotted.

Our first result is a simple but significant observation, which claim $m_g = \text{OE}_g$, and can be considered as a version of [WWZZ, Theorem 2.2].

**Theorem 1.1.** For each $g > 1$, $m_g = \text{OE}_g$, therefore $m_g$ are given in the following table.

| $m_g$                     | $g$                      |
|---------------------------|--------------------------|
| $12(g-1)$                 | $2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601$ |
| $8(g-1)$                  | $7, 49, 73$              |
| $20(g-1)/3$               | $16, 19, 361$            |
| $6(g-1)$                  | $21, 481$                |
| $24(g-1)/5$               | $41$                     |
| $30(g-1)/7$               | $29, 841, 1681$          |
| $4(\sqrt{g}+1)^2$        | $g = k^2, k \neq 3, 5, 7, 11, 19, 41$ |
| $4(g+1)$                  | remaining numbers        |

Moreover $m_g$ is realized by unknotted graphs for all $g$ except $g = 21, 481$.

The difference between graph case and surface case is that, given a fixed $g$, there is a unique closed orientable surface of genus $g$, but there are infinitely many graphs of genus $g$. We will consider only minimal graphs, i.e., the graphs without free edges and without extra vertices: an edge is called free if one of its vertices has degree one; a vertex is extra if it has degree two and its stable subgroup (stabilizer in $G$) equals the stable subgroup of its each adjacent edge. Clearly there are only finitely many minimal graphs of genus $g$ for each $g > 1$. Note that this is really no restriction since we can delete all free edges and extra vertices (or, vice versa, add arbitrarily many free edges and extra vertices in a $G$-equivariant way) without changing the genus, and since the group actions we considered are in $SO(4)$ (see Proposition 3.1).

The major part of the paper is to classify all minimal graphs $\Gamma \subset S^3$ which realize these maximum orders $m_g$ in different levels. To be precise and brief, we need some definitions. Suppose $G$ acts on $(S^3, \Gamma)$ for a minimal graph $\Gamma$ embedded in $S^3$ such that $|G| = m_g$. We call $\Gamma$ an (abstract) MS
graph (of genus $g$), $(S^3, \Gamma)$ a spatial (MS) graph for $\Gamma$, and the $G$-action a MS action for $\Gamma$, and resp. for $(S^3, \Gamma)$, where MS represents “maximum symmetry”. In the above definitions, we may ignore the words “abstract”, “of genus $g$” and “MS” in the brackets when there is no confusion.

It is natural to ask: For given $g$, (i) what are the MS graphs? (ii) For a fixed MS graph $\Gamma$, what are the MS actions on $\Gamma$? (iii) What are the spatial MS graphs $(S^3, \Gamma)$ for $\Gamma$? (iv) For a fixed spatial MS graph $(S^3, \Gamma)$, what are the MS actions for $(S^3, \Gamma)$? We have the following theorem.

**Theorem 1.2.** (1) For each $g$, the number of abstract MS graphs of genus $g$ is four for $g = 11, 241$; is two for $g = 3, 5, 7, 17, 19, 29, 41, 97, 601, 841, 1681$; and is one for all the remaining $g$.

(2) For each abstract MS graph $\Gamma$ of genus $g$, the number of spatial MS graphs for $\Gamma$ is infinite for $\Gamma$ of genus 21, 481; is two for $\Gamma$ of genus 9, 121, 361 and is one $\Gamma$ of genus 11; and is one for all the remaining MS graphs.

(3) Each spatial MS graph has a unique MS action. This is also true for each (abstract) MS graph except for one graph of genus 29.

Actually Theorem 1.2 will be included in very precise results (Theorem 4.4 and also two appendices) in the coming text, which present all the MS graphs, as well as spatial MS graphs except $g = 21, 481$. Of course, the meaning of “present all abstract and spatial MS graphs” itself will be addressed soon.

Our approach relies on [WWZZ] which builds the connection between the study of $OE_g$ and orbifold theory. The paper is organized as below.

In Section 2, we will give a brief introduction to the orbifold theory, and introduce necessary terminologies to present Theorem 2.1, the main result of [WWZZ], which is a list of spherical orbifolds $O$ with marked (allowable) singular edges and dashed arcs. Indeed we also try to outline the ideas of [WWZZ].

In Section 3 by some quick arguments based on Section 2 we first prove Theorem 1.1. Then by picking information from Theorem 2.1 exactly related the maximum order $m_g$, and refining those information respect to the graph case, we present Theorem 3.3 which list all spatial MS graphs in the following sense: $\Gamma$ is a spatial MS graph if and only if $\Gamma = p^{-1}(a)$, where $a$ is a marked singular edge or dashed arc of a 3-orbifold $O$ in Theorem 3.3 and $p : S^3 \to O$ is the orbifold covering.

Note the information provided by Theorem 3.3 in terms of orbifolds does not tell us the following: Suppose $p^{-1}(a)$ and $p^{-1}(b)$ are two MS graphs of genus $g$ provided by Theorem 3.3. (1) Are they the same abstract graph? and if yes, are they the same spatial graph? And more naively: (2) Can we see $\Gamma = p^{-1}(a)$ as an abstract MS graph and as a MS spatial graph intuitively? At least graph theorist should ask what are the primary graph invariants of those graphs?

In Section 4, various methods are introduced to give the detailed classification result Theorem 4.4 which give a precise answer to Question (1).

The answer to Question (2) for abstract graphs is in Appendix A which is a table of all MS graphs with various invariants. Appendix B is devoted
to answer Question (2) for spatial graphs, where we try to visualize those \( \Gamma \subset S^3 \) by projecting them onto \( \mathbb{R}^3 \), at least for general cases and some other cases. Both Appendices A and B are produced by mathematics in Section 4 with assistance of computer programs.

Certainly the roles of the figures in those appendices are limited, since they are hard to see for large \( g \), specially for spatial graphs. An alternative intuition of those symmetries are given in the section “Intuitive view of large symmetries of \((S^3; \Sigma_g)\)” in [WWZZ1] via spherical tessellations, 4-dimensional polytons and equivalent Dehn surgeries.

In Section 5 we will discuss maximum orders of extendable finite group actions on \((S^3, \Gamma)\) allowing orientation reversing elements based on the results in previous sections and [WWZ]. We will see differences between the maximum orders of arbitrary graphs and of minimal graphs: The former can be determined and the later are still unknown for some genus. Precisely let \( M_g \) be the general maximum orders of extendable group actions on minimal graphs of genus \( g \). Then \( M_g \) are given in the following table.

| \( M_g \)          | \( g \)               |
|--------------------|-----------------------|
| 24\((g - 1)\)      | 3, 4, 5, 6, 11, 17, 97, 601 |
| 16\((g - 1)\)      | 7, 9, 73              |
| 40\((g - 1)/3\)    | 16, 19                |
| 12\((g - 1)\)      | 2, 25, 121, 241       |
| 48\((g - 1)/5\)    | 41                    |
| 60\((g - 1)/7\)    | 29, 841, 1681         |
| 8\((\sqrt{g} + 1)^2\) | \( k^2 \), \( k \neq 11 \) |
| 8\((g + 1) > M_g \geq 4(g + 1)\) | remaining numbers |

Conjecture 1.3. Suppose \( g \) is neither a square number nor one of those finitely many \( g \) listed in the table above. Then \( M_g \) is \( 4(g + 1) \) for prime \( g \), and \( 4(p + 1)(q + 1) \) otherwise, where \( pq = g \), \( p \) is the smallest non trivial divisor of \( g \).

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2 3-orbifold and main results in [WWZZ]

For orbifold theory, see [Th], [Du1] or [BMP]. We give a brief introduction here for later use.

An n–orbifold we considered has the form \( M/H \). Here \( M \) is an orientable n–manifold and \( H \) is a finite group acting faithfully and orientation preservingly on \( M \). For each point \( x \in M \), denote its stable subgroup by \( St(x) \), its image in \( M/H \) by \( x' \). If \( |St(x)| > 1 \), \( x' \) is called a singular point with index \( |St(x)| \), otherwise it is called a regular point. If we forget the singular set we get its topological underlying space \( |M/H| \).

We can also define covering spaces and the fundamental group of an orbifold. There is an one to one correspondence between orbifold covering
spaces and conjugacy classes of subgroups of the fundamental group, and regular covering spaces correspond to normal subgroups. A Van-Kampen theorem is also valid, see [BMP]. In the following, automorphisms, covering spaces and fundamental groups always refer to the orbifold setting.

We call \( B^n / H (S^n / H, V_g / H) \) the discal(spherical, handlebody) orbifold. Here \( B^n (S^n) \) denotes the \( n \)-dimensional ball(sphere). By classical results, \( B^2 / H \) is a disk, possibly with one singular point; \( B^3 / H \) belongs to one of the five models in Figure 1 corresponding to the five classes of finite subgroups of \( SO(3) \). Here the labeled numbers denote indices of interior points of the corresponding edges. \( V_g / H \) can be obtained by pasting finitely many \( B^3 / H \) along some \( B^2 / H \) in their boundaries. It is easy to see the singular set of a 3-orbifold \( M/H \) is always a trivalent graph \( \Theta \).

Figure 1: Five models

Suppose \( G \) acts on \( (S^3, \Sigma_g) \). Call a 2-orbifold \( \mathcal{F} = \Sigma_g / G \subset \mathcal{O} = S^3 / G \) is allowable if \( |G| > 4(g - 1) \). A sequence of observations about allowable 2-orbifolds were made in [WWZZ] (Lemma 2.4, 2.7, 2.8, 2.9, 2.10), in particular: Suppose \( \mathcal{F} \subset \mathcal{O} \) is allowable, then (i) \( |\mathcal{O}| = S^3 \); (ii) \( \mathcal{F} \subset \mathcal{O} \) is \( \pi_1 \)-surjective; (iii) \( |\mathcal{F}| = S^2 \) with four singular points having one of the following types: \( (2, 2, 2, n) (n \geq 3) \), \( (2, 2, 3, 3) \), \( (2, 2, 3, 4) \), \( (2, 2, 3, 5) \); and very crucially (iv) \( \mathcal{F} \) bounds a handlebody orbifold, which is a regular neighborhood of either an edge of the singular set or a dashed arc, presented in (a) or (b) of Figure 2. Here labels are omitted in (a), and more description of (b) will be given later. (i) allows us to consider only Dunbar’s famous list in [Du1] of all spherical 3-orbifolds with underlying space \( S^3 \). Searching for all possible 2-suborbifolds that satisfy the conditions (ii), (iii) and (iv) by further analysis from topological, combinatoric, numerical, and group theoretical aspects leads to a list in Theorem 6.1 of [WWZZ], presented here as Theorem 2.1. And we are going to explain it.

Figure 2: Handlebody orbifolds

Since all the spherical 3-orbifolds we considered have underlying space \( S^3 \), they are determined by their labeled singular trivalent graphs. From now on, a singular edge always means an edge of \( \Theta \), the singular set of the orbifold, and following only singular edges with index 2 are not labeled; a
dashed arc is always a regular arc with two ends at two edges of $\Theta$ with indices 2 and 3 respectively as in Figure 2(b). An edge/dashed arc is allowable if the boundary of its regular neighborhood is an allowable 2-orbifold.

For each 3-orbifold $O$ in the list, the order of $\pi_1(O)$ is firstly given. Then singular edges/dashed arcs are listed, which are marked by letters $a, b, c, ...$ to denote the boundaries of their regular neighborhoods. Then singular types of the boundaries and genera of their pre-images in $S^3$ are given. When the singular type is $(2, 2, 3, 3)$, there are two subtypes denoted by I and II, corresponding to Figure 2(a) and Figure 2(b) (exactly the dashed arc case).

We say that an orientable separating 2-suborbifold (2-subsurface) $F$ in an orientable 3-orbifold (3-manifold) $O$ is unknotted or knotted, depending on whether it bounds handlebody orbifolds (handlebodies) on both sides. A singular edge/dashed arc is unknotted or knotted, depending on whether the boundary of its regular neighborhood is unknotted or knotted.

If a marked singular edge/dashed arc is knotted, then it has a foot notation ‘k’. If a marked dashed arc is unknotted, then there also exists a knotted one (indeed infinitely many) and it has a foot notation ‘uk’. Call two singular edges/dashed arcs are equivalent, if there is an orbifold automorphism sending one to the other, or the boundaries of their regular neighborhoods as 2-orbifolds are orbifold-isotopic.

**Theorem 2.1.** Up to equivalence, the following tables list all allowable singular edges/dashed arcs except those of type II. In the type II case, if there exists an allowable dashed arc in some $O$, then $O$ and one such arc in it are listed. The arc will be unknotted if there exists an unknotted one in $O$.

**Table 1: Fibred case: type is $(2, 2, 3, 3)$**

| $G$ | $|G|$ | $a_{uk}$ | $g$ |
|-----|-------|---------|-----|
| | | I | 2 |
| $a$ | $|G| = 6$ | $a_{uk}$: II, $g = 2$ |
| $b$ | $|G| = 6$ | $b_{uk}$: II, $g = 2$ |
| $a_1$ | $|G| = 18$ | $a_{uk}$: II, $g = 4$ |
| $a_2$ | $|G| = 18$ | $a_{uk}$: II, $g = 4$ |
| $a_3$ | $|G| = 18$ | $a_{uk}$: II, $g = 4$ |

| $G$ | $|G|$ | $a_{uk}$ | $g$ |
|-----|-------|---------|-----|
| | | I | 4 |
| $a$ | $|G| = 48$ | $a_{uk}$: II, $g = 9$ |
| $b$ | $|G| = 48$ | $b_{uk}$: II, $g = 9$ |
| $a_1$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
| $a_2$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
| $a_3$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |

| $G$ | $|G|$ | $a_{uk}$ | $g$ |
|-----|-------|---------|-----|
| | | I | 2 |
| $a$ | $|G| = 720$ | $a_{uk}$: II, $g = 121$ |
| $b$ | $|G| = 720$ | $b_{uk}$: II, $g = 121$ |
| $a_1$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
| $a_2$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
| $a_3$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |

| $G$ | $|G|$ | $a_{uk}$ | $g$ |
|-----|-------|---------|-----|
| | | I | 4 |
| $a$ | $|G| = 720$ | $a_{uk}$: II, $g = 121$ |
| $b$ | $|G| = 720$ | $b_{uk}$: II, $g = 121$ |
| $a_1$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
| $a_2$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
| $a_3$ | $|G| = 144$ | $a_{uk}$: II, $g = 25$ |
Table 2: Fibred case: type is not $2,2,3,3$
Table 3: Non-fibred case

| $|G|$ | Description | | $|G|$ | Description |
|---|---|---|---|---|
| 96 | $a: I, g = 17$ | 60 | $a: (2,2,2,3), g = 6$ |
| 576 | $a: (2,2,2,4), g = 73$ | 24 | $a: (2,2,2,3), g = 3$ |
| 48 | $a: (2,2,2,3), g = 5$ | 120 | $a: (2,2,2,3), g = 11$ |
| 192 | $a: (2,2,2,3), g = 17$ | 7200 | $a: (2,2,2,3), g = 601$ |
| 288 | No allowable 2-suborbifold | 24 | $a: I, g = 5$ |
| 1152 | $a: (2,2,2,3), g = 97$ | 120 | $b: (2,2,2,3), g = 11$ |
| 12 | $a: I, g = 3$ | 24 | $a: I, g = 5$ |
| 60 | $a: I, g = 11$ | 2880 | $a,b,c,d: (2,2,2,3)$ |
| 24 | $a: I, g = 11$ | 2880 | $a,b,c,d: (2,2,2,3)$ |

Note: $g$ represents the order of the group element.
Remark 2.2. The way to list orbifolds in Theorem 2.1 is influenced by the lists of [Du1] and [Du2].

3 Edges in orbifolds provide \((S^3, \Gamma)\) with maximum symmetry

3.1 Maximum orders of symmetries on \((S^3, \Gamma)\)

We first prove Theorem 1.1. The following primary fact is used repeatedly and implicitly in the proof.

Proposition 3.1. Suppose \(G\) is a finite group of \(SO(4)\) acting on \((S^3, \Gamma)\). Here \(\Gamma\) is a polyhedron which can not be embedded into a circle. (Specially applies when \(\Gamma\) is either a surface, or a handlebody, or a graph with \(g > 1\)). Then the restriction of \(G\) on \(\Gamma\) is faithful.

Proof. Suppose \(g \in G\) and the restriction of its action on \(\Gamma\) is the identity. As an orientation preserving isometry, by a classical result its fixed point set \(\text{Fix}(g)\) is either the empty set, or a circle, or the whole \(S^3\). Since \(\Gamma \subset \text{Fix}(g)\) and \(\Gamma\) is not a subset of a circle, we have \(\text{Fix}(g) = S^3\), and hence \(g\) is the identity of \(G \subset SO(4)\).

Proof of Theorem 1.1. We can assume \(S^3\) has the standard spherical geometry and \(G \subset SO(4)\). Suppose \(G\) acts on \((S^3, \Gamma)\) for some graph \(\Gamma \subset S^3\) of genus \(g\). Let \(N_\epsilon(\Gamma) = \{x \in S^3 \mid \text{dist}(x, \Gamma) \leq \epsilon\}\) be the \(\epsilon\)-neighborhood of \(\Gamma\). When \(\epsilon > 0\) is sufficiently small, \(N_\epsilon(\Gamma)\) is a handlebody of genus \(g\). Since \(G\) acts isometrically, \(N_\epsilon(\Gamma)\) is invariant under the group action. Notice that generally \(\partial N_\epsilon(\Gamma)\) is not smooth. But we can choose a smaller equivariant neighborhood \(U_\epsilon\) of \(\Gamma\) such that \(\partial U_\epsilon\) is a smooth submanifold in \(S^3\) and \(\partial U_\epsilon \simeq \Sigma_g\) (see [WWZ, Remark 3.3]). Hence we have a \(G\)-action on \((S^3, \Sigma_g)\), and it follows that \(m_g \leq OE_g\).

Suppose \(G\) acts on \((S^3, \Sigma_g)\) for some \(\Sigma_g \subset S^3\) and \(|G| > 4(g-1)\). Then by [WWZZ], \(\Sigma_g\) bounds an handlebody \(V_g\) in \(S^3\). Since \(G\) acts orientation preservingly on both \(S^3\) and \(\Sigma_g\), \(V_g\) is invariant under the group action, and moreover \(V_g/G\) is a \(N(a)\), the regular neighborhood of an allowable singular edge/dashed arc in an orbifold \(O = S^3/G\) listed in Theorem 2.1. Let \(p : S^3 \to O\) be the branched covering. Then \(\Gamma = p^{-1}(a)\) is a connected graph which is invariant under the \(G\)-action. \(V_g\) is a regular neighborhood of \(\Gamma\), therefore the genus of \(\Gamma\) is \(g\). Since \(OE_g > 4(g-1)\), we have \(OE_g \leq m_g\).

Hence \(m_g = OE_g > 4(g-1)\). We have proved above that if \(|G| > 4(g-1)\), then \(m_g\) is realized by an unknotted (knotted) one if and only if \(OE_g\) is realized by an unknotted (knotted) one. Therefore we can copy [WWZZ, Theorem 2.2] as the remaining part of Theorem 1.1.
3.2 Edges in orbifolds provide \((S^3, \Gamma)\) with maximum symmetry

Let us have a brief recall of some facts about graph of groups before further discussions. For more details, see [Se]. Suppose we have an extendable \(G\)-action on \(\Gamma\) with respect to an embedding \(e : \Gamma \hookrightarrow S^3\). Then we have an embedding \(e/G : \Gamma/G \hookrightarrow S^3/G\). Here \(\Gamma/G\) can be thought as a graph of groups or a ‘graph orbifold’.

For a vertex \(v\) of \(\Gamma/G\), let \(v'\) be one of its pre-image in \(\Gamma\). Then the vertex group \(G_v\) can be identified to the stable subgroup of \(v'\). Similarly for an edge \(e\) of \(\Gamma/G\) we have the edge group \(G_e\). The index of \(v\)(or \(e\)) can be defined to be \(|G_v|\) or \(|G_e|\). If \(v\) is a vertex of \(e\), then there is a nature injection from \(G_e\) to \(G_v\). If we forget the groups and injections, we get the underlying graph of \(\Gamma/G\), denoted by \(|\Gamma/G|\).

We can define the Euler characteristic of \(\Gamma/G\) by

\[
\chi(\Gamma/G) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}. \tag{3.1}
\]

Here the first sum consists of all vertices of \(\Gamma/G\), and the second sum consists of all edges of \(\Gamma/G\). One can show that

\[
\chi(\Gamma/G) = \chi(\Gamma)/|G| \tag{3.2}
\]

**Lemma 3.2.** Suppose there is an extendable \(G\)-action on \(\Gamma\), \(\Gamma\) is minimal and \(|G| > 4(g - 1)\), then \(\Gamma/G\) is an allowable singular edge/dashed arc.

**Proof.** By (3.1), it is easy to derive that we have

\[
\chi(\Gamma/G) = \chi(|\Gamma/G|) - \sum (1 - \frac{1}{|G_v|}) + \sum (1 - \frac{1}{|G_e|}) \tag{3.3}
\]

Here the first sum consists of all vertices of \(\Gamma/G\) with indices bigger than 1; The second sum consists of all edges of \(\Gamma/G\) with indices bigger than 1.

Because in our case each \(G_v\) is a finite subgroup of \(SO(3)\), we have

\[
(1 - \frac{1}{|G_v|}) = \sum_v (1 - \frac{1}{|G_v|})/2 \tag{3.4}
\]

Here the sum consists of all edges of \(\Theta\) containing \(v\) as a vertex. Recall that \(\Theta\) is the singular trivalent graph of \(S^3/G\). Then combining (3.4) with (3.3) we have

\[
\chi(\Gamma/G) = \chi(|\Gamma/G|) - \sum' (1 - \frac{1}{|G_e|})/2. \tag{3.5}
\]

Here the sum consists of all edges of \(\Theta\) which are not edges of \(\Gamma/G\) but contain vertices of \(\Gamma/G\).

Since \(|G| > 4(g-1)\), applying (3.2) we have \(-1/4 < \chi(\Gamma/G) < 0\). Clearly every item in \(\sum'\) is not smaller than 1/4. Hence we must have \(\chi(|\Gamma/G|) = 1\) and the sum \(\sum'\) contains at most 4 items. Then we know that \(|\Gamma/G|\) is a tree. Since \(\Gamma\) contains no free edge, every leaf of \(\Gamma/G\) gives at least two items in \(\sum'\). Then \(\Gamma/G\) has exactly two leaves and all the other vertices have degree 2. Since \(\Gamma\) contains no extra vertices, every vertex of \(\Gamma/G\) other than leaves gives at least one item in \(\sum'\). Hence there is no such vertex in \(\Gamma/G\) and \(\Gamma/G\) contains only one edge. Clearly its boundary is allowable, then \(\Gamma/G\) is an allowable singular edge or dashed arc.
Hence by Theorem 1.1 and Lemma 3.2 to find all maximum symmetry of \((S^3, \Gamma)\) we have only to find all singular edges/dashed arcs. And this have been contained in the proof of Theorem 2.1 in [WWZZ]. In the graph case, call two singular edges/dashed arcs are equivalent, if there is an orbifold automorphism sending one to the other.

In Theorem 2.1 if the represented 2-suborbifold is unknotted, then the two handlebody orbifolds separated by it are always regular neighborhoods of some singular edges/dashed arcs. Then we call these two edges/arc are dual to each other.

In the study of maximum symmetry of \((S^3, \Sigma_g)\), it is reasonable to call two dual edges are equivalent, since they produce the same allowable 2-orbifold. But it is not good to call them equivalent in the study of maximum symmetry of \((S^3, \Gamma)\), since they may be different graph orbifolds or correspond to different graphs.

Then by a routine checking of Theorem 2.1 as well as Lemma 3.2 we have the following Theorem 3.3 for our further study of the realizations of the maximum symmetry of \((S^3, \Gamma)\) in the following sense: (1) we only pick information from Theorem 2.1 related \(m_g\) (so among 40 orbifolds listed in Theorem 2.1 only 17 orbifolds appear in Theorem 3.3); and list the orbifolds according to the sizes of \(m_g\) (so some orbifold in Theorem 2.1 becomes several orbifolds in Theorem 3.3, say 15E and 27). (2) We mark a pair of edges by \(a\) and \(a'\) and so on if they are dual.

**Theorem 3.3.** 1. If \(g \neq 21, 481\), then \(\Gamma\) is a MS-graph if and only if \(\Gamma/G \rightarrow S^3/G\) belongs to the following table, labeled by \(a, a', b, c, \ldots\) (up to automorphisms of \(S^3/G\)).

2. When \(g = 21, 481\), \(\Gamma\) is a MS-graph if and only if \(\Gamma/G\) is an allowable dashed arc in orbifolds listed in \(|G| = 6(g-1)\), and we just give one possible \(\Gamma/G\).

| Table 4: Allowable edges/arcs corresponding to MS-graphs |
|-------------------------------------------------------|
| \(|G| = 12(g-1)\)                                      |
| ![Graph 1](image1) \(g = 2\)                         |
| ![Graph 2](image2) \(g = 3\)                         |
| ![Graph 3](image3) \(g = 4\)                         |
| ![Graph 4](image4) \(g = 5\)                         |
| ![Graph 5](image5) \(g = 6\)                         |
| ![Graph 6](image6) \(g = 9\)                         |
Remark 3.4. 1. Theorem 1.1 is also valid for the handlebody case, just considering the handlebody $U_e(\Gamma)$ and handlebody orbifold $U_e(\Gamma)/G$ defined in the proof of Theorem 1.1.

2. In graph case or handlebody case, we can also consider maximum order problem for all the unknotted (knotted) embeddings, as in [WWZZ]. The results will be completely the same as in the surface case. For $g > 1$, the maximum order of unknotted surface case and knotted surface case are presented as Theorem 1.2 and Theorem 1.3 in [WWZZ] respectively.
4 Abstract and spatial MS-graphs

To give the detailed classification of MS graphs, MS spatial graphs and the group actions, we will use some notations: let $O_N$ denote the number $N$ orbifold in Theorem 3.3. If $\gamma$ denotes an allowable singular edge/dashed arc in $O_N$, let $\Gamma_{\gamma}^N$ denote the $(G$-invariant MS) graph in $S^3$ which is the pre-image of $\gamma$. We often put more information on $\Gamma_{\gamma}^N$, denoted by $\Gamma_{\gamma}^N(g)$ if $\Gamma_{\gamma}^N$ has genus $g$, or $\Gamma_{\gamma}^N(g,k)$ if $\Gamma_{\gamma}^N$ is also knotted.

4.1 To picture and to identify/distinguish MS-graphs

This subsection is prepared for the proof of our detailed classification in next subsection.

To picture MS-graphs: Suppose there is an extendable $G$-action on $\Gamma$, such that $\Gamma/G$ is a singular edge/dashed arc in $S^3/G$. Then $\pi_1(\Gamma/G) \cong \pi(U_\epsilon/G)$ and the pre-image of $U_\epsilon/G$ in $S^3$ is connected. Hence the embedding $U_\epsilon/G \hookrightarrow S^3/G$ induces a surjection on fundamental groups, see [WWZZ]. And we have a representation $\phi : \pi_1(\Gamma/G) \rightarrow G$. Notice that $\phi$ can be given by Wirtinger presentations, see [BMP] or [WWZZ].

Claim. $\Gamma$ can be reconstructed from $\phi$ as following.

Given $\Gamma/G$ a direction, denote the start point by $A$, the terminal point by $B$ and the edge by $e$. Denote the corresponding vertex groups and edge group by $G_A$, $G_B$ and $G_e$. Then $\pi_1(\Gamma/G) \cong G_A \ast_{G_e} G_B$, and $\phi$ is injective on $G_A$, $G_B$ and $G_e$. Hence $\phi(G_A) \cong G_A$, $\phi(G_B) \cong G_B$, $\phi(G_e) \cong G_e$, and $\phi(G_e)$ lies in the intersection of $\phi(G_A)$ and $\phi(G_B)$.

Choose $|G|$ directed edges each of which has a label in $G$. We identify the edges and the group elements. Then $\phi(G_A)$ ($\phi(G_B)$, $\phi(G_e)$) acts on the $|G|$ elements by left multiplication. For the elements in the same orbit, we identify their start points (their terminal points, the elements themselves). Since $\phi$ is surjective, we get a connected graph which is the ‘covering’ of $\Gamma/G$. On it there is a $G$-action via the right multiplication.

We can use the above construction to write a computer program for [GAP] and [Mathematica]. Then the graph can be pictured.

Example 4.1. Using computer to picture $\Gamma_{34^+}^0(11)$.

A presentation of $O_{44}$ is given in the proof of Theorem 4.4. Firstly run the following codes in [GAP], we can get a list of arrows. Even numbers correspond to the start points and odd numbers correspond to the terminal points.

```
f := FreeGroup (" x ", " y ", " z ");
x := f . 1;
y := f . 2;
z := f . 3;
G := f / \{ x \hat{2}, y \hat{3}, z \hat{2}, (z * y)^\hat{2}, (y * x * z)^\hat{2}, (y * x * z * x)^\hat{3}\};
x := G . 1; # now x is an element in G
y := G . 2;
z := G . 3;
GA := GroupWithGenerators (\{ x, z * y \}); # vertex group
```
GB := GroupWithGenerators ([z*y, y]);
Ge := GroupWithGenerators ([z*y]);  # edge group
l := RightCosets (G, GA);  # right cosets of vertex groups
r := RightCosets (G, GB);
for i in [1..Size(l)]
do  for j in [1..Size(r)]
doi f (Size (Intersection2 (l[i], r[j])) <> 0) then
   for k in [1..Size (Intersection2 (l[i], r[j])) / Size (Ge)]
doi f Print (" (", 2*i - 1," - > ",2*j, ") , ");
fi;  # the union of cosets of the edge group
od;
fi;
Then copy the list to Mathematica, and run the following codes.
<< Combinatorica ' Needs ["GraphUtilities "]
O34a' = {(1 -> 2), (1 -> 4), (3 -> 2), (3 -> 6), (5 -> 2), (5 -> 8), (7 -> 4),
(7 -> 10), (9 -> 4), (9 -> 12), (11 -> 6), (11 -> 14), (13 -> 8), (13 -> 16),
(15 -> 6), (15 -> 18), (17 -> 8), (17 -> 20), (19 -> 10), (19 -> 22),
(21 -> 12), (21 -> 24), (23 -> 10), (23 -> 26), (25 -> 12), (25 -> 28),
(27 -> 14), (27 -> 30), (29 -> 16), (29 -> 32), (31 -> 18), (31 -> 28),
(33 -> 20), (33 -> 30), (35 -> 14), (35 -> 22), (37 -> 16), (37 -> 24),
(39 -> 18), (39 -> 32), (41 -> 20), (41 -> 30), (43 -> 22), (43 -> 34),
(45 -> 24), (45 -> 36), (47 -> 26), (47 -> 36), (49 -> 28), (49 -> 34),
(51 -> 30), (51 -> 38), (53 -> 32), (53 -> 38), (55 -> 34), (55 -> 40),
(57 -> 36), (57 -> 40), (59 -> 38), (59 -> 40)}
G34a' = SetGraphOptions [ToCombinatoricaGraph [O34a'],
   EdgeDirection -> False]
GraphPlot [G34a']

Finally we will get the picture as in Figure 3.

Figure 3: The MS graph $\Gamma_{34}^{a'}(11)$

To identify/distinguish MS-graphs:
A main method to identify graphs is the following.
Suppose that $\gamma_i$ is a singular edge/dashed arc in $S^3/G_i$, and a representation $\phi_i : \pi_1(\gamma_i) \cong G_{A_i} *_{G_{e_i}} G_{B_i} \to G_i$ is induced by the orbifold embedding, and $\Gamma^i = p^{-1}(\gamma_i)$, $i = 1, 2$. 
If we have two isomorphisms $\eta$ and $\psi$ in the diagram above to make it commutative, where $\eta$ maps $G_A_1$ to $G_A_2$, $G_B_1$ to $G_B_2$, and $G_e_1$ to $G_e_2$, then clearly $\Gamma^1$ and $\Gamma^2$ are $G$-equivariant (abstract) graphs.

Practically we will first give the presentations of the groups and then give the map between group generators and check if the map gives us a required group isomorphism.

We distinguish graphs by computing the graph invariants such as the number of vertices or edges, the degree of a vertex, the diameter of a graph and the girth (the length of a minimal loop) of a graph.

For complicated cases, we need computer programs to determine if maps are isomorphisms and to compute graph invariants.

**Example 4.2.** The map $(\overline{u}, \overline{u}_l, \overline{u}_r) \mapsto (\overline{v}, \overline{v}_l, \overline{v}_r)$ in the proof of Theorem 4.4 is an isomorphism from $\pi_1(O_{28})$ to $\pi_1(O_{34}).$

```gap
f := FreeGroup("x","y","z");
x := f.1;
y := f.2;
z := f.3;
O28 := f / [x^5, y^2, z^2, (x*z)^3, (x*y)^2, (y*z*(1))^2];
O34 := f / [x^2, y^3, z^2, (z*y)^2, (y*x*z)^2, (y*x*z*x)^3];
x := O28.1; # x is an element in O28
y := O28.2;
z := O28.3;
r := O34.1; # r is an element in O34
s := O34.2;
t := O34.3;
iso28 := IsomorphismPermGroup(O28); # pass to permutation group
iso34 := IsomorphismPermGroup(O34);
G28 := Image(iso28);
G34 := Image(iso34);
u := Image(iso28, x*y*z*(1)*x*(1));
u1 := Image(iso28, x*z*x*(1));
ur := Image(iso28, x*z);
v := Image(iso34, r*(1));
v1 := Image(iso34, s*(1)*t*(1));
vr := Image(iso34, t*(1)*r*(1)*s*(1)*r);
GroupHomomorphismByImages([u, u1, ur], [v, v1, vr]);
```

Run the above codes in [GAP]. If it is an isomorphism, then [GAP] will give the correspondence between elements of the two groups. Otherwise the output will be “fail”. In this example the map is an isomorphism.

**Example 4.3.** Compute the diameter and girth of $\Gamma_{34}(11)$. 

```
```
Diameter [G34a’] 
Girth [G34a’]

In Example 4.1, after input the list into Mathematica, we just add the above two sentences. Then the computer will show that the diameter is 10 and the girth is 12.

4.2 Detailed classifications

Theorem 4.4. (1) For each $g$, the MS graphs and their spatial graphs are:

- $g = 11$. Four MS graphs, three with a unique spatial graph: $\Gamma_{28}^a(11)$, $\Gamma_{34}^a(11)$, $\Gamma_{34}^d(11)$; one with two spatial graphs $\Gamma_{28}^a(11)$ and $\Gamma_{34}^b(11, k)$;
- $g = 241$. Four MS graphs with a unique spatial graph: $\Gamma_{38}^a(241)$, $\Gamma_{38}^b(241)$, $\Gamma_{38}(241, k)$, $\Gamma_{38}^b(241, k)$;
- $g = 3, 5, 7, 17, 19, 29, 41, 97, 601, 1681$. Two MS graphs with a unique (unknotted) spatial graph for each $g$ (coming from pairs of dual edges of orbifolds of the corresponding genus in Theorem 3.2);
- $g = 841$. Two MS graphs with a unique spatial graph: $\Gamma_{22D}^a(841)$, $\Gamma_{19}^a(841)$;
- $g = 9, 121, 361$. One MS graph with two spatial graphs for each genus: $\Gamma_{20C}^a(9)$ and $\Gamma_{20C}^b(9, k)$, $\Gamma_{22B}^a(121)$ and $\Gamma_{22B}^b(121, k)$, $\Gamma_{22C}^a(361)$ and $\Gamma_{22C}^b(361, k)$;
- $g = 21, 481$. One MS graph in each case, with infinitely many different (all knotted) spatial graphs: $\Gamma_{28}^d(21, k) \cong \Gamma_{34}^d(21, k)$, $\Gamma_{38}^d(481, k)$;
- For all other values of $g$, there is just one MS graph with a unique (unknotted) spatial graph.

(2) For each spatial MS graph $\Gamma$, there is a unique MS action. This is also true for each (abstract) MS graph $\Gamma$ except $\Gamma_{28}^c(29) \cong \Gamma_{15E}^a(29)$, where the actions correspond to $\Gamma_{28}^c(29)$ and $\Gamma_{15E}^a(29)$ are different.

Proof. For each minimal graph $\Gamma$, denote the set of invariants by

$$\Lambda(\Gamma) = \{d_k(\Gamma), E(\Gamma), D(\Gamma), G(\Gamma)\}$$

where $d_k$, $E$, $D$ and $G$ donate the number of vertices of degree $k$, the number of edges; the diameter, and the girth of $\Gamma$ respectively.

For a given $g$, $\gamma$ in $O_N$ realizing $m_\gamma$ are listed in Theorem 3.3, and $\Gamma_N^{\gamma}(g)$ are listed in Theorem 4.3. We will be able to picture all those graphs $\Gamma_N^{\gamma}(g)$ and to compute their invariants $\Lambda$ by the methods in last subsection, which are shown as in Appendix A. As a result, graphs in the discussion have the same invariants $\Lambda$ are listed as blow:

(i) $\Lambda(\Gamma_{28}^c(29)) = \Lambda(\Gamma_{15E}^a(29))$,
(ii) $\Lambda(\Gamma_{34}^d(11)) = \Lambda(\Gamma_{34}^d(11, k))$,
(iii) $\Lambda(\Gamma_{20C}^a(9)) = \Lambda(\Gamma_{20C}^b(9, k))$. 

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(iv) \( \Lambda(\Gamma_{22B}^c(121)) = \Lambda(\Gamma_{22B}^c(121, k)) \),
(v) \( \Lambda(\Gamma_{22C}^a(361)) = \Lambda(\Gamma_{22C}^a(361, k)) \),
(vi) \( \Lambda(\Gamma_{28}^c(21, k)) = \Lambda(\Gamma_{34}^c(21, k)) \) for any knotted dashed arcs \( d \) and \( c \),
(vii) \( \Lambda(\Gamma_{38}^b(481, k)) \) is constant for all knotted dashed arcs \( d \).

Therefore for the genus not appeared in the list (i)-(vii), Theorem 4.4 is proved, and for genus 29, 11, 9, 121, 361, 21, 481, we need only to address the graphs appeared in (i)-(vii).

It is clearly that \( \Gamma_{28}^c(29) \) and \( \Gamma_{15E}^a(29) \) are the same as abstract graphs, which is isomorphic to \( K_{(2,30)} \), but their corresponding group actions are different: one group is \( D_{30} \times \mathbb{Z}_2 \) and the other is \( A_5 \times \mathbb{Z}_2 \). There is no diffeomorphism sending \( (S^3, \Gamma_{28}^c(29)) \) to \( (S^3, \Gamma_{15E}^a(29)) \) since at the vertex of degree 30 in \( (S^3, \Gamma_{28}^c(29)) \) all edges are tangent to some plane, and this is not the situation for \( (S^3, \Gamma_{15E}^a(29)) \). So \( \Gamma_{28}^c(29) \) and \( \Gamma_{15E}^a(29) \) are not the same spatial graphs.

Two graphs in each pair of (ii), (iii), (iv), (v) are different spatial graphs, since one is knotted and the other is unknotted. We will prove these two graphs are \( G \)-equivalent by the method discussed in last subsection, therefore finish the proof of Theorem 4.4 of genus 11, 9, 121, 361.

When \( g = 11 \), we will prove that the representations induced by \( a' \) in \( \mathcal{O}_{28} \) and \( b \) in \( \mathcal{O}_{34} \) are equivalent. Using the Wirtinger presentation, we have the following.

\[
\begin{align*}
\pi_1(\mathcal{O}_{28}) &= \langle x, y, z \mid x^2, y^2, (xz)^3, (xy)^2, (yz)^2, (xyz)^3 \rangle, \\
\pi_1(\mathcal{O}_{34}) &= \langle x, y, z \mid x^2, y^3, z^2, (zy)^2, (yxz)^2 \rangle.
\end{align*}
\]

\( \pi_1(a') \cong \pi_1(b) \cong D_2 \ast_{\mathbb{Z}_2} D_3 \). We choose three generators for \( \pi_1(a') \): \( u \) is the generator of \( D_2 \cap D_3 \cong \mathbb{Z}_2 \), \( u_l \) is an order 2 element in \( D_2 \) different from \( u, u_r \) is an order 3 element in \( D_3 \). Similarly choose generators \( v, v_l, v_r \) for \( \pi_1(b) \). Then the equivalence is given by:

\[
\begin{align*}
\pi_1(a') &\rightarrow \pi_1(\mathcal{O}_{28}) : (u, u_l, u_r) \mapsto (xyz^{-1}x^{-1}, xzx^{-1}, xz), \\
\pi_1(b) &\rightarrow \pi_1(\mathcal{O}_{34}) : (v, v_l, v_r) \mapsto (x^{-1}, y^{-1}z^{-1}, z^{-1}x^{-1}y^{-1}x), \\
\pi_1(a') &\rightarrow \pi_1(b) : (u, u_l, u_r) \mapsto (v, v_l, v_r), \\
\pi_1(\mathcal{O}_{28}) &\rightarrow \pi_1(\mathcal{O}_{34}) : (\overline{u}, \overline{v}_l, \overline{v}_r) \mapsto (\overline{u}, \overline{v}_l, \overline{v}_r).
\end{align*}
\]

When \( g = 9, 121, 361 \), we need to prove that for \( N = 20C, 22B, 22C \) the representations induced by \( a \) and \( b \) are equivalent.

\( N = 20C \):

\[
\begin{align*}
\pi_1(\mathcal{O}_{28}) &= \langle x, y, z \mid x^2, y^2, (xz)^3, (xy)^2, (yz)^2, (xyz)^3 \rangle, \\
\pi_1(\mathcal{O}_{34}) &= \langle x, y, z \mid x^2, y^3, z^2, (zy)^2, (yxz)^2 \rangle.
\end{align*}
\]
\( \pi_1(\mathcal{O}_{20C}) = \langle x, y, z \mid x^2, y^3, z^2, (y^{-1}x)^2, (y^{-1}(xz)^3x)^2, (y^{-1}xz^{-1})^2 \rangle. \)

\( \pi_1(a) \cong \pi_1(b) \cong D_2 \ast_{\mathbb{Z}_2} D_3. \) We choose three generators for \( \pi_1(a) \): \( u \) is the generator of \( D_2 \cap D_3 \cong \mathbb{Z}_2 \), \( u_t \) is an order 2 element in \( D_2 \) different from \( u \), \( u_r \) is an order 3 element in \( D_3 \). Similarly choose generators \( v, v_l, v_r \) for \( \pi_1(b) \). Then the equivalence is given by:

\[
\begin{align*}
\pi_1(a) &\to \pi_1(\mathcal{O}_{20C}) : (u, u_l, u_r) \mapsto (y^{-1}(xz)^3x, y^{-1}xz^{-1}, y), \\
\pi_1(b) &\to \pi_1(\mathcal{O}_{20C}) : (v, v_l, v_r) \mapsto ((xz)^2x, y^{-1}(xz)^3x, xzxyxzx), \\
\pi_1(a) &\to \pi_1(b) : (u, u_l, u_r) \mapsto (v, v_l, v_r).
\end{align*}
\]

\( N = 22B: \)

\[
\pi_1(\mathcal{O}_{22B}) = \langle x, y, z \mid x^2, y^3, z^2, (y^{-1}x)^2, (y^{-1}(xz)^5x)^2, (y^{-1}xz^{-1})^2 \rangle.
\]

\( \pi_1(a) \cong \pi_1(b) \cong D_2 \ast_{\mathbb{Z}_2} D_3. \) We choose three generators for \( \pi_1(a) \): \( u \) is the generator of \( D_2 \cap D_3 \cong \mathbb{Z}_2 \), \( u_t \) is an order 2 element in \( D_2 \) different from \( u \), \( u_r \) is an order 3 element in \( D_3 \). Similarly choose generators \( v, v_l, v_r \) for \( \pi_1(b) \). Then the equivalence is given by:

\[
\begin{align*}
\pi_1(a) &\to \pi_1(\mathcal{O}_{22B}) : (u, u_l, u_r) \mapsto (y^{-1}(xz)^5x, y^{-1}xz^{-1}, y), \\
\pi_1(b) &\to \pi_1(\mathcal{O}_{22B}) : (v, v_l, v_r) \mapsto ((xz)^4x, y^{-1}xz^{-1}, (xzxz)y(xz)xz^{-1}), \\
\pi_1(a) &\to \pi_1(b) : (u, u_l, u_r) \mapsto (v, v_l, v_r).
\end{align*}
\]

\( N = 22C: \)
The alternating permutation group of 5 elements. Clear separately. The notations are explained as following.

isometry group a regular icosahedron (or a regular dodecahedron) and allowable dashed arcs are equivalent. orbifolds groups are isomorphic to that for $N = 34$ and for $N = 38$ all the representations induced by allowable dashed arcs are equivalent.

In the last two case (vi) and (vii), that is $g = 21,481$, we need to prove that for $N = 28,34$ and for $N = 38$ all the representations induced by allowable dashed arcs are equivalent.

Notice that in these cases $\Gamma / G$ are all dashed arcs whose fundamental groups are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$. By [Du2], the fundamental groups of orbifolds $O_{28}$, $O_{34}$ and $O_{38}$ are isomorphic to $I \times \mathbb{Z}_2$, $A_5 \times \mathbb{Z}_2$ and $I \times O$ separately. The notations are explained as following.

$O$ is the isometry group of a regular octahedron (or a cube), $I$ is the isometry group a regular icosahedron (or a regular dodecahedron) and $A_5$ is the alternating permutation group of 5 elements. Clear $O$ and $I$ belong to $SO(3)$. And $I \times O$ is the two sheet coverings of $I \times O$ under the two to one map $SO(4) \to SO(3) \times SO(3)$. $I$ has a lift in $SO(4)$ and can acts on $S^3$. $A_5$ can act on a regular 4-simplex, which has 5 vertices, as the orientation preserving isometry group. Hence it also belongs to $SO(4)$ and can acts on $S^3$. The $\mathbb{Z}_2$ summand acts on $S^3$ as an antipodal map which commutes with the action given by $I$ or $A_5$.

Then the results can be derived from Lemma 4.6 below.

**Remark 4.5.** In [Du2], $J$ is used to denote $I$ and notations $J \times J$, $J \times J$ and $J \times O$ are used to denote the fundamental groups of $O_{28}$, $O_{34}$ and $O_{38}$.

**Lemma 4.6.** Suppose $G$ is one of $I \times \mathbb{Z}_2$, $A_5 \times \mathbb{Z}_2$ and $I \times O$. If $\{x, y\}$ and $\{z, w\}$ both generate $G$, $x, z$ have order 2 and $y, w$ have order 3. Then the map $x \mapsto z, y \mapsto w$ gives an automorphism of $G$.

**Proof.** Notice that $I \cong A_5$ and $O \cong S_4$, hence the first two groups are isomorphic to $A_5 \times \mathbb{Z}_2$ and $I \times O$ is a two sheet covering of $A_5 \times S_4$. Using the
permutation representation, we may assume that $A_5$ acts on $\{1, 2, 3, 4, 5\}$, $Z_2$ acts on $\{6, 7\}$, and $S_4$ acts on $\{6, 7, 8, 9\}$.

For $A_5 \times Z_2$, if an order 2 element and an order 3 element generate the group, then the two generators have a form like $(12)(34)(67)$ and $(135)$. It is not hard to see the map between two such generating sets is an isomorphism.

For $A_5 \times S_4$, if an order 2 element and an order 3 element generate the group, then the two generators have a form like $(12)(34)(69)$ and $(135)(678)$. Also the map between two such generating sets is an isomorphism, and $A_5 \times S_4$ has the property stated in the lemma.

Now if an order 2 element $x$ and an order 3 element $y$ generate $I \times O$. Then the quotient $\overline{x}$ and $\overline{y}$ in $A_5 \times S_4$ have the above form. $\overline{y}$ has two preimages, one is $y$, the other has order 6. Hence all such $y$’s are conjugate in $I \times O$. Hence if $\{z, w\}$ satisfies the condition in the lemma, we can assume $w = y$. Then considering the permutation representation and the two to one covering, the possible $z$ has 36 choices.

Then using [GAP], one can clarify that all the 36 choices give equivalence generator pairs.

5 Maximum order of orientation-reversing case

Now we can consider maximum order problems for extendable actions on handlebodies and graphs for general case, which allow group elements acting orientation reversely.

Suppose $G$ acts on $(S^3, U_\varepsilon(\Gamma))$. Then for any $g \in G$, $g$ will preserve or reverse the orientation of $U_\varepsilon(\Gamma)$ and $S^3$ simultaneously, therefore preserve or reverse the orientation of $\Sigma_g = \partial U_\varepsilon(\Gamma)$ and $S^3$ simultaneously. That is to say, in graph and handlebody cases if there is an orientation reversing element in $G$, $G$ must be of type $(-, -)$ on $(S^3, \partial U_\varepsilon(\Gamma))$ as defined in [WWZ]. Then the maximum order of extendable actions for orientation reversing case will be $E_g(-, -)$, which is presented as Proposition 4.7 in [WWZ]. By definition, the action $G$ realizing $E_g(-, -)$ is faithful on $\partial U_\varepsilon(\Gamma)$, therefore is faithful on the handlebody $U_\varepsilon(\Gamma)$, but may not be faithful on the graph $\Gamma$.

**Proposition 5.1.** (Proposition 4.7 of [WWZ] and its proof, also refer proof of Theorem 5.3 for the “Moreover” part) $E_g(-, -)$ are given as below:

| $E_g(-, -)$ | $g$ |
|-----------|-----|
| $24(g - 1)$ | $3, 5, 6, 11, 17, 97, 601$ |
| $16(g - 1)$ | $7, 73$ |
| $40(g - 1)/3$ | $19$ |
| $48(g - 1)/5$ | $41$ |
| $60(g - 1)/7$ | $1681$ |
| $8(\sqrt{g} + 1)^2$ | $k^2$, $k > 1$ |
| $8(g + 1)$ | the remaining numbers |

Moreover for each $g$, $E_g(-, -)$ is realized (in the orbifold level) by a singular edge/dashed arc $\gamma$ of a 3-orbifold $O_N$ shown as (a) or (b) in Figure 8, where a “reflection” of $O_N$ fixes $\gamma$. We omit the labels here for convenience.
Similar to Proposition 3.1, we have the following.

**Proposition 5.2.** Suppose $G \subset O(4)$ acts on $(S^3, \Gamma)$, $g \in G$ is not the identity of $G$ and its action on $\Gamma$ is the identity. Then $g$ must be a reflection about a geodesic 2-sphere $S^2$ in $S^3$, and $\Gamma \subset S^2$.

**Proof.** By Proposition 3.1, we may assume $g$ is orientation reversing. By classical results in topology, $\text{Fix}(g)$ is either the empty set, or a pair of points, or a geodesic 2-sphere $S^2 \subset S^3$. Since $\Gamma \subset \text{Fix}(g)$, $\text{Fix}(g)$ must be a geodesic $S^2(\subset \Gamma)$ and $g$ interchanges two 3-balls separated by $S^2$. $\square$

Suppose the $G$-action on $\Gamma$ is not faithful. Then it is easy to see if we $G$-equivalently add some free edges to $\Gamma$ perpendicular to the $g$-geodesic $S^2$ containing $\Gamma$ to get $\Gamma^*$, then $G$ acts faithfully on $\Gamma^*$.

Now let $E(V_g)$, $M_g^*$ and $M_g$ be the general maximum orders of extendable group actions on handlebodies, arbitrary graphs, and minimal graphs, of genus $g$ respectively. Then we have

**Theorem 5.3.**

1. $m_g = O E_g \leq M_g \leq E(V_g) = M_g^*$.
2. $E(V_g) = M_g^*$ are given in the following table.

| $E(V_g) = M_g^*$ | $g$ |
|------------------|-----|
| 24$(g - 1)$      | 2, 3, 4, 5, 6, 11, 17, 97, 601 |
| 16$(g - 1)$      | 7, 9, 73 |
| 40$(g - 1)/3$    | 16, 19 |
| 12$(g - 1)$      | 25, 121, 241 |
| 48$(g - 1)/5$    | 41 |
| 60$(g - 1)/7$    | 29, 841, 1681 |
| 8$(\sqrt{g} + 1)^2$ | $k^2$, $k \neq 11$ |
| 8$(g + 1)$       | remaining numbers |

3. $M_g$ are given in the following table.

| $M_g$ | $g$ |
|-------|-----|
| 24$(g - 1)$ | 3, 4, 5, 6, 11, 17, 97, 601 |
| 16$(g - 1)$ | 7, 9, 73 |
| 40$(g - 1)/3$ | 16, 19 |
| 12$(g - 1)$ | 2, 25, 121, 241 |
| 48$(g - 1)/5$ | 41 |
| 60$(g - 1)/7$ | 29, 841, 1681 |
| 8$(\sqrt{g} + 1)^2$ | $k^2$, $k \neq 11$ |
| 8$(g + 1)$ | remaining numbers |

Figure 8: Orbifolds with reflections
Proof. (1) follows from the definitions, Theorem 1.1 and discussions around Proposition 5.2.

(2) Suppose $G$ acts on $V_g$ realizing $E(V_g)$. Then $G$ is either orientation preserving or of the type $(-,-)$. To get the table in (2), we need only to compare tables in Theorem 1.1 (see also Remark 3.4) and in Proposition 5.1 where $E(V_g)$ are chosen from the former for $g = 25, 121, 241$, and from the later for the remaining $g$. Note for table in (2) we can put 4, 9, 16, 25, 841 into the line of $k^2$, and 29 into the bottom line, then it has the form in the table of Proposition 5.1.

(3) By (2) and its proof, there is nothing to be further verified for $g = 25, 121, 241$, and to prove (3), we need to verify the following:

Claim for all remaining $g$, $M_g = E_g(-,-)$ if and only if $g$ is not in the bottom line and $g \neq 2$, and further more $M_2 = 12$.

(*) Note first that the “Moreover” part of Proposition 5.1 can be interpreted as that, for each $g$, $E_g(-,-)$ is realized by the orientation preserving maximum symmetry of $\Gamma^+(g)$ and reflections about geodesic 2-spheres which keep $\Gamma^+(g)$ invariant. Therefore, if $\Gamma^+(g)$ does not stay in any geodesic 2-sphere, then the group action realizing $E_g(-,-)$ acts faithfully on $\Gamma^+(g)$.

(**) Note then that $\Gamma^+(g) \subseteq S^3$ does not stay in any geodesic 2-sphere in $S^3$ if a vertex $v$ of $\gamma$ in the singular set of $O_N$ has three adjacent edges with index $(2,3,q)(g = 3,4,5)$, since for any vertex $v'$ of $\Gamma^+(g)$ in $p^{-1}(v)$, where $p : S^3 \to O_N$ is the orbifold covering, the action of the stabilizer $St(v')$ on a neighborhood of $v'$ is the same as the isometry of some regular polyhedron, and adjacent edges of $v'$ can not lie in any geodesic two sphere.

Now we are going to verify the “if” part of the claim by finding $\Gamma^+(g)$ realizing $E_g(-,-)$ so that either $\gamma$ in $O_N$ meets (**), or more directly $\Gamma^+(g)$ is non-planer. These will be carried in (i) and (ii) below respectively. Therefore the group action $G$ realizing $E_g(-,-)$ acts faithfully on $\Gamma^+(g)$.

(i) For $g = 3, 5, 11, 6, 17, 97, 601, 7, 73, 19, 41, 1681, 29$, (we follow the order of their appearance in Theorem 3.3), we choose $\Gamma^+(g)$ to be $\Gamma^{a}_2(3), \Gamma^{a}_{27}(5), \Gamma^{a}_{29}(11), \Gamma^{a}_{21}(6), \Gamma^{a}_{29}(17), \Gamma^{a}_{33}(97), \Gamma^{a}_{30}(601), \Gamma^{b}_{27}(7), \Gamma^{b}_{25}(73), \Gamma^{b}_{25}(19), \Gamma^{b}_{29}(41), \Gamma^{b}_{30}(1681), \Gamma^{b}_{28}(29)$.

(ii) For all the squares $g = k^2$, we choose $\Gamma^+(g)$ to be $\Gamma^{a}_{19}(k^2)$, where the parameter $n$ in $O_{19}$ is chosen to be $n = k + 1$. As an abstract graph $\Gamma^{a}_{19}(k^2)$ is isomorphic to $K_{(k+1,k+1)}$, which is a well-known non-planer graph, hence can not lie in a geodesic sphere.

For $g = 2$ and the remaining genus, the graph realizing $E_g(-,-)$ is $\Gamma^{a}_{15E}(g)$, which is abstractly isomorphic to $K_{(2,g+1)}$, indeed lies in a geodesic 2-sphere, and the action on $\Gamma^{a}_{15E}(g)$ realizing $E_g(-,-)$ is not faithful (see also Appendix B).

By Theorem 2.1 and the proof of Theorem 1.1, the second biggest order of orientation preserving extendable action on a genus 2 graph is 6 (corresponds to ‘a’ in $O_{01}$). Hence $M_2 \leq 12$, and we have $M_2 = 12$. □

Example 5.4. Suppose $g = pq$, $p$ is the smallest non trivial divisor of $g$. As our construction for $g = k^2$ for general case we can get a two parted graph $K_{(p+1,q+1)} \subset S^3$ which has genus $g$ and on it there is an extendable group.
action with order \(4(p + 1)(q + 1)\) which is bigger than \(4(g + 1)\). Clearly \(K_{(p+1,q+1)}\) is non-planar.

**Conjecture 5.5.** Suppose \(g\) is neither a square number nor one of those finitely many \(g\) listed in the table above. Then \(M_g\) is \(4(g + 1)\) for prime \(g\), and \(4(p + 1)(q + 1)\) otherwise, where \(pq = g\), \(p\) is the smallest non trivial divisor of \(g\).

**Remark 5.6.** (1) The maximum order of finite group action on minimal graphs of genus \(g\) is \(2^g g!\) if \(g > 2\) and is 12 if \(g = 2\) [WZ].

(2) Any faithful action of finite group \(G\) on a minimal graph \(\Gamma\) of genus \(g\) provides an embedding of \(G\) into the out-automorphism group of the free group of rank \(g\) for \(g > 1\).

## A Table of MS graphs with invariants

This appendix contains basic invariants of all the MS graphs. In following table, we denote by \(d_k\) the number of vertices of degree \(k\); \(E\) the number of edges of a graph; \(D\) the diameter of a graph; \(G\) the girth (or the length of a minimal loop) of a graph.

\[
|G| = 12(g - 1)
\]

\[
\begin{align*}
\Gamma^{a}_{15E} & \quad (2) \\
d_2 & = 3 \\
d_3 & = 2 \\
E & = 6 \\
D & = 2 \\
G & = 4 \\
\Gamma^{a}_{26} & \quad (3) \\
d_2 & = 4 \\
d_4 & = 2 \\
E & = 8 \\
D & = 2 \\
G & = 4 \\
\Gamma^{a'}_{26} & \quad (3) \\
d_2 & = 6 \\
d_3 & = 4 \\
E & = 12 \\
D & = 4 \\
G & = 6 \\
\Gamma^{a}_{19} & \quad (4) \\
d_2 & = 9 \\
d_3 & = 6 \\
E & = 18 \\
D & = 4 \\
G & = 8 \\
\Gamma^{a'}_{27} & \quad (5) \\
d_2 & = 12 \\
d_3 & = 8 \\
E & = 24 \\
D & = 6 \\
G & = 8 \\
\Gamma^{a}_{24} & \quad (6) \\
d_2 & = 10 \\
d_4 & = 5 \\
E & = 20 \\
D & = 4 \\
G & = 6 \\
\Gamma^{a}_{20C} & \quad (9), \quad \Gamma^{b}_{20C} (9, k) \\
d_2 & = 24 \\
d_3 & = 16 \\
E & = 48 \\
D & = 8 \\
G & = 12
\end{align*}
\]
\[ |G| = 8(g - 1) \]

\[ |G| = 20(g - 1)/3 \]
$|G| = 6(g - 1)$

$|G| = 24(g - 1)/5$ and $30(g - 1)/7$

$|G| = 4(\sqrt{g} + 1)^2$, here $g = k^2$, $k \neq 3, 5, 7, 11, 19, 41$
Remaining graphs are $K_{\sqrt{g}+1,\sqrt{g}+1}$.

$$d_2 = (\sqrt{g} + 1)^2, \ d_\sqrt{g}+1 = 2(\sqrt{g} + 1), \ E = 2(\sqrt{g} + 1)^2, \ D = 4, \ G = 8.$$ 

$$|G| = 4(g + 1), \text{ for remaining } g$$

$\Gamma_{28}^c (29)$
$\Gamma_{15E}^a (29)$
$d_2 = 30$
$d_{30} = 2$
$E = 60$
$D = 2$
$G = 4$

Remaining graphs are $K_{(2,g+1)}$.

$$d_2 = g + 1, \ d_{g+1} = 2, \ E = 2g + 2, \ D = 2, \ G = 4.$$

**B Spatial MS graphs for most genera**

This appendix contains most spatial MS graphs. These graphs are all pictured with [Mathematica](#). With one point removed, the unit sphere $S^3 \subset E^4$ will be finally mapped to $\mathbb{R}^3$ by a projection. Each MS graph in $S^3$ consists of geodesic segments as edges, which are mapped to circles in $\mathbb{R}^3$.

Many cases come from regular polyhedrons. In some complicated cases, we first picture a part of the graph, then applying the group action to it we can get the whole graph. “A part of the graph” refers to arcs in a pre-fundamental domain, see [Du2](#). After identifying $E^4$ with $\mathbb{H}$, the group elements can be represented by pairs of quaternion numbers via the two to one map $S^3 \times S^3 \to SO(4)$, and all things are computable. For more details of finite groups in $SO(4)$, one can see [CS](#) and [Du2](#).

Following Watkins’ notation, the generalized Petersen graph $G(n, k)$ is a graph with vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid i = 0, \ldots, n - 1\}$ where subscripts are to be read modulo $n$ and $k < n/2$.

A crown graph $S_n^0$ on 2n vertices is an undirected graph with two sets of vertices $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{u_i v_j \mid i \neq j\}$.

Following names or descriptions may (or may not) refer to the MS graph with degree two vertices removed.

$\Gamma_{15E}^a (8)$: Complete bipartite graph $K_{(2,g+1)}(g = 8)$ or the dipole graph $D_{g+1}(g = 8)$. 
\( \Gamma_{19}^a(16) \): Complete bipartite graph \( K_{(\sqrt{g+1}, \sqrt{g+1})} (g = 16) \).

\( \Gamma_{26}^a(25) \): 1-skeleton of a regular tetrahedron or the complete graph \( K_4 \) or the wheel graph \( W_4 \).

\( \Gamma_{27}^a(5) \): 1-skeleton of a cube or \( G(4, 1) \).
$\Gamma^a_{24}(6)$: 1-skeleton of a regular 4-simplex or the complete graph $K_5$.

$\Gamma^a_{20C}(9)$ and $\Gamma^b_{20C}(9, k)$: The generalized Petersen graph $G(8, 3)$ or the Möbius–Kantor graph.

$\Gamma^a_{28}(11)$ and $\Gamma^b_{34}(11, k)$: 1-skeleton of a regular dodecahedron or $G(10, 2)$. 
$\Gamma_{34}(11)$: Crown graph $S_5^0$.

$\Gamma'_{34}(11)$: The generalized Petersen graph $G(10,3)$ or the Desargues graph.
$\Gamma'_{20}(17)$: 1-skeleton of a regular 4-cube.

$\Gamma'_{30}(601)$: 1-skeleton of a regular 120-cell.

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