Efficient and Reliable Probabilistic Interactive Learning with Structured Outputs

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Abstract

In this position paper, we study interactive learning for structured output spaces, with a focus on active learning, in which labels are unknown and must be acquired, and on skeptical learning, in which the labels are noisy and may need relabeling. These scenarios require expressive models that guarantee reliable and efficient computation of probabilistic quantities to measure uncertainty. We identify conditions under which a class of probabilistic models—which we denote CRISPS—meet all of these conditions, thus delivering tractable computation of the above quantities while preserving expressiveness. Building on prior work on tractable probabilistic circuits, we illustrate how CRISPs enable robust and efficient active and skeptical learning in large structured output spaces.

Introduction

All probabilistic models learned from data are wrong, in the sense that they can be arbitrarily far from the data distribution. Models that can reliably quantify uncertainties about the distribution they encode, however, are useful. Reliable inference with guarantees indeed is the key for enabling learning “in the wild”, especially when dealing with (unreliable) human annotators. This becomes evident in the context of interactive learning with large structured output spaces (Tsochantaridis et al. 2004). In the wild, misspecification errors can quickly pile up as the learner receives noisy supervision. In a classic active learning scenario, the predictive accuracy of the model depends chiefly on its ability to (reliably) know when it does not know “enough” about a sample, in which case it should opt to obtain the label from a user (Settles 2012; Gal, Islam, and Ghahramani 2017). As providing all labels for a single sample in a structured-output prediction (SOP) task might be infeasible or highly expensive in practice, an efficient probabilistic model should be able to identify informative subset of labels to be annotated by the user. Furthermore, we can improve uncertainty estimates even more if our probabilistic model is able to question if the user-provided labels are correct, a setting recently proposed under the name of skeptical learning (Zeni et al. 2019; Bontempelli et al. 2020).

In this position paper, we investigate when and how a probabilistic model can satisfy the aforementioned desiderata in these settings. We define CRISPs (Conditional Randomized Interactive Skeptical Probabilistic circuits), a class of tractable probabilistic models that support reliable and efficient active and skeptical learning with full and partial labels. CRISPs exploit recent advancements in the literature of probabilistic circuits (Choi, Vergari, and Van den Broeck 2020; Vergari et al. 2021), deep computational graphs encoding complex probability distributions that support exact computation of the uncertainties needed for incremental learning (Vergari et al. 2021).

CRISPs offer several benefits. First, they leverage the expressiveness of deep neural classifiers by implementing a conditional probabilistic circuit (Shao et al. 2020) that is able to encode intricate dependencies over the labels and deal with sub-symbolic inputs. As such, they can act as a drop-in replacement for the ubiquitous, but generally intractable, softmax/sigmoid last layer in deep neural networks. Second, by ensuring that CRISPs satisfy certain structural properties, we can always guarantee that these quantities can be computed exactly and in time linear with respect to the size of the computational graph that encodes the CRISP model. To circumvent the need to learn the structure of the computational graph, we propose to leverage a randomized construction approach (Peharz et al. 2020; Shih and Ermon 2020) that guarantees the structural properties we need.

To the best of our knowledge, this is the first paper to investigate the benefits of tractable probabilistic inference for reliable interactive learning in the wild and how to combine it in a principled way with the intractable, unreliable but largely used deep learning paradigm.

Probabilistic Interactive Learning: Problem Statement and Inference Tasks

We consider structured prediction tasks in which a machine learns to associate instances $x \in \mathbb{R}^d$ to $c$ interdependent concepts, identified by labels $y \in \{0,1\}^c$. Dependencies among labels are sometimes realized as correlations, as is the case in multi-label classification (Dembczynski et al. 2012). Other times, they occur as hard constraints and are expressed as logical formulas encoding, e.g., a hierarchy over the labels (Giunchiglia and Lukasiewicz 2020). We focus on two challenging sequential learning tasks, active learning and skeptical learning, and illustrate which
uncertainty-based queries are necessary to solve them.

**Active learning.** In incremental active learning (Hoi et al. 2021), aka selective sampling, the machine receives a sequence of instances $x_1, x_2, \ldots$, and for each one of them it has to decide whether to query a human supervisor—at a non-negligible cost—to obtain the corresponding complete ground truth label $y_i$. This decision depends crucially on how confident the model is in its own prediction $y_i = \text{argmax}_y p_\theta(y | x_i)$, where $p_\theta$ denotes the model’s conditional distribution. Ultimately, the model requests a ground-truth label $y_i$ if it holds that:

$$U_\theta(x_i) \geq T$$

where $U_\theta(x)$ quantifies how uncertain the model is about the label of instance $x$, and $T$ is a custom threshold.

A popular way of modelling $U_\theta(x)$ is to use the Shannon entropy of $p_\theta$, denoted as $H_\theta(x)$, which is defined as:

$$H_\theta(x) := -\sum_{y \in \{0,1\}} p_\theta(y | x) \log p_\theta(y | x). \quad (2)$$

Another option is to compute the probabilistic margin, defined as the distance of the mode of the distribution from absolute certainty:

$$M_\theta(x) := 1 - \max_y p_\theta(y | x). \quad (3)$$

Acquiring the label of uncertain instances is guaranteed to reduce the model’s total uncertainty. Unfortunately, evaluating Eqs. (2) and (3) is NP-hard for general probabilistic models (Vergari et al. 2021; Darwiche and Marquis 2002).

Popular solutions to this problem include using a sigmoid final classification layer in deep neural networks, which renders marginalization and maximization over the labels straightforward. This, however, amounts to assuming all labels to be independent given the features $x$. Clearly, this simplifying assumption discards precious information about the label dependencies, which is crucial in several SOP tasks such as multi-label classification (Ohamrabi and McCallum 2003). Furthermore, the sigmoid layer solution has been consistently reported to deliver poorly calibrated uncertainties in deep learning models (Gal, Islam, and Ghahramani 2017; Guo et al. 2017). Other common workarounds employing low-order approximate models (Behpour, Liu, and Ziebart 2019) fall prey of the same issues.

Solutions based on expected gradient length (EGL) define uncertainty in terms of the (square norm of the) expected change in model parameters $\theta$ after receiving the label of a query (Settles, Craven, and Rav 2007). The EGL can be computed rather cheaply and exactly for differentiable models, but it tends to perform sub-optimally. This is partly because the norm of the gradient depends on the encoding of the parameters. A standard solution is to compute the natural gradient, which however involves evaluating the Fisher information matrix (Kunstner, Hennig, and Balles 2019). This is computationally challenging for realistically sized models and involves approximating the computation, especially in human-in-the-loop settings in which low response times are key for keeping the user engaged in the interaction (Teso et al. 2021).

**Fine-grained active learning.** In applications with large output spaces, the cost of annotating a full label $y$ can be excessive. One remedy is to only ask the supervisor to annotate a (small) subset of output variables $Q^* \subseteq Y$, selected so to be maximally informative of the full label $y$, cf. Tong and Koller (2000); Small and Roth (2010). This finer-grained form of interaction requires careful handling of the uncertainty associated to partial predictions.

A principled solution is to select $Q^*$ such that the uncertainty over the whole label set $Y$ is maximally reduced by solving:

$$Q^* := \arg\max_{Q \subseteq Y} \mathbb{E}_q[U_\theta(Y \mid x) - U_\theta(Y \mid q, x)] \quad (4)$$

$$\text{s.t. } a(Q, x) \leq a_{\text{max}} \quad (5)$$

where $a(Q, x)$ of $Q$ denotes the cost of annotating $q$ and $a_{\text{max}}$ a limit to it. E.g., for uniform cost over all concepts in $Y$ the constraint simplifies to $|Q| \leq a_{\text{max}}$. Since $x$ is constant, the above objective can be simplified to:

$$\arg\min_{Q \subseteq Y} \mathbb{E}_q[U_\theta(Y \mid q, x)] \quad \text{s.t. } a(Q, x) \leq a_{\text{max}} \quad (6)$$

If we take $U_\theta$ to be the margin, then the objective involves evaluating and maximizing the conditional margin $M_\theta(q, x) = \max_y p_\theta(y \mid q, x)$. For the Shannon entropy $H_\theta$, and letting $M := Y \setminus Q$, the objective becomes:

$$\mathbb{E}_q[H_\theta(Y \mid q, x)] = \mathbb{E}_q[H_\theta(M \mid q, x)]$$

$$= H_\theta(M | Q, x) = H_\theta(Y \mid x) - H_\theta(Q | x) \quad (7)$$

where last step follows from the chain rule of entropies. Since $H_\theta(Y | x)$ is constant w.r.t. $Q$, this shows that Eq. (4) is equivalent to maximizing the conditional entropy of $Q$ given $X = x$, namely:

$$H_\theta(Q | x) = -\sum_y p_\theta(q \mid x) \log p_\theta(q | x). \quad (9)$$

Regardless, this formulation is appealing because the conditional entropy is submodular (Fujishige 2005), and therefore sports approximation guarantees (Nemhauser, Wolsey, and Fisher 1978; Sviridenko 2004) via iterative greedy maximization. However, this still requires computing the entropy reliably at each iteration for different subsets of labels, which may be intractable in general (Krause and Guestrin 2008) and lead to arbitrarily suboptimal results if the errors of approximating the entropy cannot be safely bounded.

**Skeptical learning.** In skeptical learning, the machine observes fully annotated examples $(x_1, \tilde{y}_1), (x_2, \tilde{y}_2), \ldots$, but the labels $\tilde{y}_i$ are noisy – due, for instance, to inattention on the annotator’s part – and the machine is allowed to ask the supervisor to double-check and relabel them (Zeni et al. 2019; Bontempeli et al. 2020). Existing skeptical approaches identify suspicious examples by comparing how much the machine trusts the user’s annotation $\tilde{y}_i$ versus how much it trusts its own prediction.
\( \hat{y}_t = \arg\max_y p_0(y \mid x_t) \). Most commonly, the model's \textit{suspiciousness} is defined as the difference in likelihood between the model's guess and the user's annotation, namely:

\[
S_0(x_t) := \max_y p_0(y \mid x_t) - p_0(\hat{y}_t \mid x_t).
\]

This measure has been adapted to different classes of models, including deep neural networks (Teso et al. 2021) and Gaussian Processes (Bontempi et al. 2020).

**Designing CRISPs: A Property-driven Approach for Guaranteed Reliable Inference**

In this section, we study how we can build a class of models that guarantee to reliably compute the measures of uncertainty we just introduced in the context of active and skeptical learning. We will focus on models that support exact computation—hence delivering the highest form of reliability—of these queries and that can do so tractably, that is, in time polynomial in their sizes.\(^3\)

We carry out our analysis within the framework of probabilistic circuits (PCs) (Vergari et al. 2020; Choi, Vergari, and Van den Broeck 2020). Expressive models encoded by computational graphs for which tractable computation of queries of interest can be guaranteed as long as the graph satisfies certain structural properties. Specifically, we aim at pinpointing which structural properties of circuits can enable or inhibit the exact computation of the uncertainty queries discussed in the previous section (Vergari et al. 2021). We denote the resulting model class as CRISPs.

**Probabilistic circuits.** We start with a brief overview of PCs. A PC \( p \) over variables \( X \) is a computational graph built out of input distribution, sum, and product units. Each input distribution unit \( n \) represents a base distribution \( p_n(\text{scope}(n)) \) over some variables \( \text{scope}(n) \subseteq X \). Each sum and product unit \( n \) receives inputs from some input units, denoted \( \text{in}(n) \), and computes their weighted sum \( \sum_{c \in \text{in}(n)} w_c p_c(\text{scope}(c)) \), with \( w_c \geq 0 \), or product \( \prod_{c \in \text{in}(n)} p_c(\text{scope}(c)) \), respectively. The last unit in the graph encodes the joint distribution \( p(X) \). The probability of an assignment \( p(x) \) is readily obtained in a single feedforward evaluation of the PC, and has a computational cost linear in the size of the PC, i.e., the number of edges, denoted as \( |p| \).

PCs can be adapted to encode conditional distributions \( p(Y \mid X) \) in a number of ways. Shen, Choi, and Darwiche (2018) explicitly model the conditioning operations as a circuit. However, this can be done tractably only for discrete variables \( X \) with finite domains. As it explicitly requires to enumerate all possible conditioning states, this approach hardly scales to numbers of variables beyond tens. Instead, the amortized approach introduced in [Shao et al. 2020, 2022] requires only a single circuit structure over \( Y \), that is shared across all possible input states \( x \). Conditional dependence is realized by having the parameters of such a circuit being modeled by an external module, e.g., a regressor that takes \( X \) as input.

**Structural properties of PCs.** The key feature of PCs is that many complex functions of the distribution that they encode can be computed in polynomial time as long as the circuit satisfies certain structural properties over their scope or support. In the next sections, we will make use of the following structural properties: \textit{smoothness}, \textit{decomposability}, \textit{compatibility}, and \textit{determinism}. While we refer the reader to (Vergari et al. 2021; Choi, Vergari, and Van den Broeck 2020; Darwiche and Marquis 2002) for an in-depth treatment, the next propositions restate some important results connecting these properties to the tractability of core operations that will appear in the next sections: marginalization and maximization of circuits encoding a conditional distribution \( p(Y \mid X) \). We start with marginalization, which is a key operation when computing several uncertainty measures like entropies, as we would need to sum over all possible label configurations \( y \), cf. Eq. (2). Marginalization is tractable for all smooth and decomposable circuits.

**Proposition 1** (Tractable marginalization. Shao et al. 2020; Choi, Vergari, and Van den Broeck 2020). Let \( p(Z \mid x) \) be a circuit that is smooth and decomposable over \( Z \) with input functions that can be tractably marginalized out. Then for any variables \( Y \subseteq Z \) and their assignment \( y \), the marginalization \( \sum_{\tilde{y} \in \text{val}(Y)} p(y, \tilde{y} \mid x) \) can be computed exactly in time linear in the size of \( p \), where \( \tilde{Y} \) denotes \( Z \setminus Y \).

Maximization, also known as MAP inference (Koller and Friedman 2009) or most probable explanation (Darwiche 2009), is pivotal in the basic classification setting we are in. The key ingredient for it is determinism.

**Proposition 2** (Tractable maximization. Darwiche 2009; Choi, Vergari, and Van den Broeck 2020). Let \( p(Y \mid x) \) be a circuit that is deterministic and decomposable over \( Y \) with input functions that can be tractably maximized. Then the maximization problem \( \arg\max_y p(y \mid x) \) can be computed exactly in time linear in the size of \( p \).

**Expressive and fast randomized circuits.** Ensuring that a PC \( p_0(Y \mid X) \) that satisfies the above structural properties to be both compact and feature-high capacity is challenging. For CRISPs, we tackle this challenge by realizing an amortized conditional circuit as in (Shao et al. 2020) and using deep neural networks for its parameters. Conditional circuits in CRISPs are composed of two elements: a distribution \( p_{\theta}(X \mid Y) \) implemented as a PC and a deep gating function \( g(X) \), that, given an input \( x \), outputs the parameters \( \theta \) of the circuit. Intuitively, this means that a CRISP associates a different conditional distribution over the labels to each \( x \), for improved flexibility and compactness. One benefit of this approach is that the neural network can be obtained by replacing the top softmax/sigmoid layer of a pretrained network with a different learned gating function. Another benefit is that the whole model, including \( g \), can be trained (or just fine-tuned) end-to-end.

To avoid the need of learning the computational graph of \( p \), we propose to adopt a randomly structured circuit

\(^3\)For many of these queries the time will be \textit{linear} in the model size.
– modified to satisfy the structural properties we need. This reduces learning a CRISP to learning its parameters, which can be done by fast gradient-based optimizers (like Adam (Kingma and Ba, 2015)). This construction is convenient also because, by exploiting compact tensorized representations of PCs, it is possible to implement learning seamlessly and scalably using GPU-accelerated libraries such as pytorch (Peharz et al., 2020b,a).

Now, this randomized construction enforces smoothness and decomposability, but it does violate determinism as well as compatibility with itself (a property also called structured-decomposability (Pipatsrisawat and Darwiche, 2008)), two properties that are needed to compute uncertainty queries tractably. To enforce determinism, we propose to apply a randomized construction as in Shih and Ermon (2020), where one simultaneously conditions on random sets of variables. To guarantee structured-decomposability, on the other hand, we need to constraint the way the random computational graph is built. Starting from the algorithm in Peharz et al. (2020b), we can systematically realize a “template” for decomposing scopes, also called a region graph (Poon and Domingos, 2011) that ensures a structured-decomposable circuit as output.

Lastly, CRISPs model learned in this way can naturally be made consistent with respect to given background knowledge K encoded as logical formulas over the input and output variables. This can be achieved by compiling the logical formulas into a compact circuit, for instance, an arithmetic or logical circuit (Darwiche and Marquis, 2002; Choi, Kisa, and Darwiche, 2013), that shares the same structural properties as circuit of the CRISP model. The remaining step is to ensure that the support of \( p_\theta(Y \mid X) \) avoids all infeasible output configurations, it is sufficient to multiply it with the circuit encoding \( K \), an operation that becomes tractable whenever the two circuits are compatible (Vergari et al., 2021).

### Active Learning with CRISPs

We start by showing how the commonly used measures of uncertainty \( U_\alpha(x) \) can be tractably computed with CRISPs. We begin with Shannon entropy.

**Proposition 3** (Tractable uncertainty with CRISPs). Let \( p_\theta(Y \mid X) \) be representable as a CRISP circuit, then computing its Shannon Entropy over \( Y \) for an input configuration \( x \), as defined in Eq. 2 or its margin, as defined in Eq. 3 can be done in time linear in \(| p |\).

**Proof.** The tractability of the margin follows directly from Proposition 2. As for the entropy, notice that for any input \( x \) the conditional circuit \( p_\theta(x)(Y) \) is equivalent to an unconditional circuit \( p_\theta(Y) \), with \( \theta = g(x) \). For CRISPs, this circuit is by construction smooth, decomposable, and deterministic. Therefore, they support the tractable computation of the Shannon entropy as proved in Vergari et al. (2021).

Let us now consider fine-grained active learning. In this case, computing the Shannon entropy or the margin, solving Eq. 4– either via combinatorial search or greedy maximization – involves marginalizing over an arbitrary set of variables in each step. Unfortunately, even CRISPs circuit can be intractable after marginalizing out some variables, unless we compromise on expressiveness, as shown by the following proposition.

**Proposition 4.** Let \( p_\theta(Y \mid X) \) be representable as a CRISP circuit. Then, computing the conditional Shannon entropy \( H_\theta(Q \mid x) \) or conditional margin \( M_\theta(Q \mid x) \) for all possible \( Q \subseteq Y \) is intractable in general, unless \( p_\theta \) encodes a fully-factorized distribution.

**Proof.** Again, this follows from reasoning over the unconditional circuit \( p_\theta(Y) \) with \( \theta = g(x) \). Recall that for CRISPs \( p_\theta \) is deterministic. For a fixed \( Q \subseteq Y \), we wish to compute \( H_\theta(Q \mid x) \) and \( M_\theta(Q \mid x) \). This can be done tractably if the circuit obtained after marginalizing out \( Y \setminus Q \) is itself deterministic. This property is called marginal determinism in Choi, Vergari, and Van den Broeck (2020). Furthermore, this must hold all possible subsets \( Q \). This restrictive condition can be trivially satisfied if \( p_\theta \) can be represented as a fully-factorized distribution (Choi, Vergari, and Van den Broeck, 2020, Section 8.4).

The proof of the above proposition suggests that: i) the computation of the conditional Shannon entropy and the conditional margin face the same tractability challenges, as both require marginal determinism in CRISPs; and ii) tractable and exact computation of uncertainty in the context of fine-grained active learning can be achieved if we manage to replace these uncertainty measures with an alternative that does not require marginal determinism to be tractably computed.

Point (i) implies that routines commonly used to approximate marginal MAP queries can be exploited to approximate the marginal Shannon entropy. These include search algorithms and bound propagation schemes (Xue et al., 2016; Cheng et al., 2012; Maua and De Campos, 2012; Choi, Friedman, and Van den Broeck, 2021) or sampling (Krause and Guestrin, 2005). The price to pay is that the resulting computation is no longer exact. Even worse, the uncertainty would be approximated in each step of the search, leading to the accumulation of approximation errors and leading to query subsets that are arbitrarily far away from the optimum.

Point (ii), however, hints at an alternative strategy that does not involve approximating the uncertainty and that can still be tractably computed with CRISPs. We propose to quantify uncertainty using the Rényi entropy, a generalization of Shannon entropy defined as follows:

\[
R_\alpha^* (Y \mid x) = \frac{1}{1 - \alpha} \log \left( \sum_{y \in \{0,1\}^c} p_\theta(y \mid x)^\alpha \right)
\]

for all \( \alpha > 0, \alpha \neq 1 \). In the following, we let the conditional Rényi entropy \( R_\alpha^*(Q \mid x) \) for a subset of labels \( Q \subseteq Y \) be the quantity obtained by applying Eq. 11 to the conditional distribution \( p_\theta(Q \mid x) \).
The Rényi entropy is a perfectly valid measure of uncertainty and it converges to the Shannon entropy as \( \alpha \rightarrow 1 \). More interestingly, it acts as a lower bound of \( H_\alpha(x) \) for all \( \alpha > 1 \), i.e., \( R^\beta_\alpha(Q \mid x) \leq H_\alpha(Q \mid x) \). This suggests that, if we can compute it exactly and tractably for all possible label subsets, we can safely maximize it in place of the Shannon entropy in Eq. 6. The next proposition shows that for CRISPS this is indeed possible.

**Proposition 5 (Tractable Rényi Entropy of CRISPS).** Let \( p_0(Y \mid X) \) be representable as a CRISP, then computing the its Rényi Entropy over \( Q \) for an arbitrary label subset \( Q \subseteq Y \) and for \( \alpha \in \mathbb{N}, \alpha > 1 \) and an input configuration \( x \) can be done in time \( \mathcal{O}(|p|^{\alpha}) \).

**Proof.** The proof follows from turning \( p_0(y|x_i) \) into an unconditional circuit \( p_\theta \), which by construction of CRISPS is structured-decomposable and smooth. The idea is to first marginalize out the labels \( Y \setminus Q \), which is doable in time linear in the size of \( p_\theta \) (Proposition 4), and then compute the unconditional Rényi entropy by computing the \( \alpha \)-power of the distribution in \( |p|^{\alpha} \), cf. Vergari et al. (2021).

The \( \alpha \)-power circuit of \( p_0(y|x_i) \) can be materialized (as a tensorized circuit) just once, and then reused for all subsequent computations. This makes it possible to compute the Rényi entropy for different choices of \( Q \) efficiently and exactly. In turn, this makes it straightforward to find a high-quality uncertain label subset using a branch-and-bound procedure leveraging the tensorized circuit to reliably evaluate the uncertainty associated to each partial configuration.

**Skeptical Learning with CRISPS**

Skeptical learning, a realistic setting in which the machine monitors for incoming examples \((x_i, y_i)\) that may be mislabeled (Zeni et al. 2019). As we mentioned, the machine suspiciousness can be modeled by tracking the margin:

\[
S_\theta(x_i) := \max_Y p_\theta(y \mid x_i) - p_\theta(y_i \mid x_i)
\]  

(12)

between the user’s annotation \( y_i \) and the machine’s prediction \( \tilde{y}_i = \argmax_y p_\theta(y \mid x_i) \). It is easy to see that CRISPS also enable tractable computation of the model’s suspiciousness. This follows directly from Proposition 5 by noticing that \( p_\theta(y_i \mid x_i) \) is constant with respect to \( y \). Naturally, as for active learning, this result holds even under hard constraints between the output labels, a feature that is not supported by any other skeptical learning approach (see the Related Work below for a more detailed breakdown).

**Related Work**

**PC learning.** Starting from Lowd and Domingos (2005), learning the structure and parameters of tractable models that can be represented as PCs has been an active research field. These include arithmetic circuits (Darwiche 2009), probabilistic sentential decision diagrams (Kisa et al. 2014), sum-product networks (Poon and Domingos 2011), and cutset networks (Rahman, Kothalkar, and Gogate 2014). For a survey, see Vergari et al. (2020). The vast majority of these structure learning algorithms, however, are non-differentiable and hence cannot be seamlessly integrated with deep neural networks. Randomized approaches (Di Mauro et al. 2017; Di Mauro, Vergari, and Basili 2015; Pehrzh et al. 2020) (Di Mauro et al. 2021; Ventola et al. 2020; Mauro et al. 2017) alleviate this issue. An orthogonal direction is to employ ensembles to boost model accuracy (Vergari, Di Mauro, and Eposito 2015; Dang, Vergari, and Van den Broeck 2022; Di Mauro et al. 2017; Rahman and Gogate 2016) and it is potentially applicable to CRISPS.

**Active learning.** Most work on active learning from sequential data focuses on simple shallow classifiers (e.g., linear separators) and neglects the issues of learning in the wild (Hoi et al. 2021). Active learning with partial labels has been studied in the context of Bayesian networks (Tong and Koller 2000), max-margin predictors (Roth and Small 2006; Small and Roth 2010), and other models (Sun, Laddha, and Batra 2015; Mo, Scott, and Batra 2016; Liu and Ferrari 2017; Khodabandeh et al. 2017; Hu et al. 2018; Behpour, Liu, and Ziebart 2019; Ning et al. 2019; Nakano, Cerri, and Vens 2020). Closest to our work, Platanios, Kapoor, and Horvitz (2017) also develop selection heuristics for picking informative example/sub-label pairs based on entropy reduction. Luo, Schwing, and Urtasun (2013), instead, look at pool-based active learning for structured-output prediction with high marginal entropy. These approaches do not sport reliable subset computation, however. Krishnamurthy et al. (2017) look at online learning for cost-sensitive multi-class classification problems and propose a theoretically well-founded algorithm, which however does not translate to high-capacity models. None of the above approaches considers sequential settings, nor provide a unified framework to design models supporting the tractable computation of the uncertainty queries of interest.

**Skeptical learning.** Existing approaches to skeptical learning (Zeni et al. 2019; Bontempelli et al. 2020) address human-in-the-loop learning tasks with noisy examples. Another close line of work is learning from weak annotations, which consider a similar setup but in a pool-based setting (Urner, Ben-David, and Shalmir 2012; Kremser, Sha, and Igel 2018). In contrast to CRISPS, these approaches are either restricted to multi-class classification or offer no support for representation learning. For instance, Zeni et al. (2019) tackle hierarchical classification, but they do so using a custom structured-output classifier based on random forests that offers rather sub-optimal uncertainty estimates (Bontempelli et al. 2020). On the other hand, Teso et al. (2020) combine skeptical learning with explanation-based interaction in the context of deep neural networks (Schramowski et al. 2020), but are unconcerned with structured output spaces.
Conclusion and Outlook

We have introduced CRISP s, a novel class of PCs designed specifically for tasks involving interaction with (unreliable) human agents. Our key contribution is identifying structural properties of probabilistic circuits that ensure reliable computation of uncertainty in the context of active and skeptical learning. These include the margin and several entropy formulations. Then, within this framework, we have shown how CRISP s can tackle the reliable computation of query label subsets for fine-grained active learning by relying on the Rényi entropy. We plan to evaluate CRISP s on real-world active learning benchmarks for deep learning (Gal, Islam, and Ghahramani 2017) as well as SOP tasks such as hierarchical multi-label classification (Giunchiglia and Lukasiewicz 2020).

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