HOMOGENEOUS HERMITIAN MANIFOLDS AND SPECIAL METRICS

FABIO PODESTÀ

ABSTRACT. We consider non-Kähler compact complex manifolds which are homogeneous under the action of a compact Lie group of biholomorphisms and we investigate the existence of special (invariant) Hermitian metrics on these spaces. We focus on a particular class of such manifolds comprising the case of Calabi-Eckmann manifolds and we prove the existence of an invariant Hermitian metric which is Chern-Einstein, namely whose second Ricci tensor of the associated Chern connection is a positive multiple of the metric itself. The uniqueness is also discussed.

1. INTRODUCTION

A generalized flag manifold, namely a simply connected homogeneous space $M = G/K$ where $G$ is a compact Lie group and $K$ is the centralizer of a torus in $G$, can be endowed with invariant complex structures and invariant Kähler metrics. Once we fix an invariant complex structure $J$ on $M$, it is well known that there exists precisely one invariant Kähler-Einstein metric which is somehow canonically associated to $(M, J)$. A simply connected complex homogeneous space $G/H$ where $G$ is compact and $H$ is a compact subgroup is Kähler precisely when $H$ coincides with the centralizer of a torus in $G$, while the problem of finding special (invariant) Hermitian metrics on non-Kähler $G/H$ is far from being obvious.

Given an Hermitian manifold $(M, J, g)$, there are several connections $D$ which leave both the metric $g$ and the complex structure $J$ parallel. Among them, the Chern connection is the only one with such a property and moreover the torsion $T$ being of type $(2,0)$. The curvature tensor $R$ of the Chern connection can be traced in two different ways yielding two different Ricci tensors $S^{(1)}$ and $S^{(2)}$ which are both Hermitian. While the $(1,1)$ form $\rho$ which can be associated to $S^{(1)}$ is closed and represents the first Chern class $c_1(M)$, the form associated to $S^{(2)}$ is not even closed and there are no obvious relations between these two tensors. When the second Chern-Ricci $S^{(2)}$ is positive definite (or at least nonnegative and positive at least at one point), then the Hodge numbers $h^{p\bar{q}} = 0$ and therefore the arithmetic genus $\chi(M, \mathcal{O}) = 1$ (see [15]). More recently (15) the second Chern-Ricci tensor has been involved in defining a Hermitian flow $\frac{dt}{t} h_t = -S^{(2)}(h_t)$ (called HCF in the sequel), which preserves the Hermitian structure, is strictly parabolic and coincides with the Kähler-Ricci flow whenever the initial metric is Kähler. This flow is actually a simplified version of the hermitian curvature flow introduced and studied by Streets and Tian ([16]) and in [19] it has been recently proved that on a compact Hermitian manifold the HCF preserves the Griffiths non-negativity of the Chern curvature.

From this point of view, Hermitian metrics $h$ which are Chern-Einstein, namely whose second Ricci tensor $S^{(2)}$ satisfies $S^{(2)} = \mu h$ for some $\mu \in C^\infty(M)$, are a distinguished class of metrics which has been first introduced in [8] and which deserves a special attention. It can be proved that on a generalized flag manifold there might exist several Chern-Einstein invariant metrics beyond the standard Kähler-Einstein metric, which on the other hand turns out to be the only one in some particular cases.

In this paper we start the investigation of the existence of special metrics on compact simply connected complex homogeneous spaces and in particular we focus on a special subclass $C$ of such homogeneous manifolds that includes the Calabi-Eckmann manifolds. A complex manifold in the class $C$ is a $T^2$-bundle over the product of two compact Hermitian symmetric spaces and can be endowed with a two-parameter family of inequivalent invariant complex structures. We prove that these manifolds, which are non-Kähler, do not satisfy the $\partial\bar{\partial}$-lemma, do not support any balanced nor SKT metrics, while for every invariant complex structure there exists an invariant Chern-Einstein metric with $\mu = 1$. We also prove that this special metric is unique whenever the complex structure belongs to a suitable neighborhood of the so called standard complex structure on the manifold.

In Section 2 we recall some basic facts about the compact complex manifolds which are homogeneous under the action of a compact Lie group of biholomorphisms. We discuss the $\partial\bar{\partial}$-lemma and we then focus on special homogeneous manifolds, called M-manifolds, and a special subclass $C$ which comprises the Calabi-Eckmann manifolds. We then discuss the existence of balanced metric on M-manifolds or their suitable products.

In Section 3 we review some basic notions about the Chern connection, we introduce the definition of Chern-Einstein metric and give some basic properties. We then state our main result, as Theorem (3.4), where we state that a manifold...
in the class $C$ carries an invariant Hermitian Chern-Einstein metric, but no balanced, nor SKT metric. We then describe
the Chern connection of an invariant metric and its curvature algebraically, providing then a proof of our main result.
We conclude with a remark on the behaviour of the HCF in a suitable neighborhood of a Chern-Einstein solution on a
particular manifold in $C$.

In Section 4 we discuss the existence of invariant balanced metrics on compact complex homogeneous spaces.

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### 2. Homogeneous Complex Manifolds

Let $M$ be a compact complex manifold with complex structure $J$ and let $G$ be a compact connected Lie group acting
almost effectively, transitively and holomorphically on $(M, J)$. We will write $M = G/L$ for some compact subgroup $L$.

The complexified group $G^\mathbb{C}$ acts holomorphically on $G/L$, so that $M = G^\mathbb{C}/U$ for some complex subgroup $U \subset G^\mathbb{C}$.
It is well known that the **Tits fibration** $\phi$ provides a holomorphic fibering of the homogeneous space $M$ onto a compact
total homogeneous space $Q := G^\mathbb{C}/P$, where the parabolic subgroup $P$ is in general defined to be the normalizer
$N_{G^\mathbb{C}}(U^\mathbb{C})$ of $U^\mathbb{C}$ (see [3]).

We will now suppose that $G$ is semisimple and that $M$ is supposed to be simply connected. Then $U$ (and $L$) is
connected and the fibres of $\phi$ are complex tori. The flag manifold $G^\mathbb{C}/P$ can be written as a $G$-
invariant complex structure $I$, where $H$ is the centralizer of some torus in $G$. Accordingly the Lie algebra $\mathfrak{g}$ can be
decomposed as

\[
\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{t} \oplus \mathfrak{n},
\]

where $\mathfrak{m}$ identifies with the tangent space $T_{eL}M$, $\mathfrak{t}$ with the tangent space of the fiber $T_{eL}F$, $F = \phi^{-1}([eH])$, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{t}$ and $\mathfrak{n}$ is an $\text{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. The fiber $F$ being a complex torus implies $[\mathfrak{t}, \mathfrak{t}] = \{0\}$. Moreover the algebra $\mathfrak{h}$ is contained in the normalizer of $\mathfrak{l}$ in $\mathfrak{g}$ by construction, hence $[\mathfrak{l}, \mathfrak{t}] \subset \mathfrak{l} \cap \mathfrak{t} = \{0\}$ and $\mathfrak{t}$ is in the center of $\mathfrak{h}$.

We can choose a Cartan subalgebra $\mathfrak{s}$ of the form $\mathfrak{s} = \mathfrak{t}_0 \oplus \mathfrak{t}'$, where $\mathfrak{t}_0$ is a maximal abelian subalgebra of $\mathfrak{l}$. Denote by $R$ the corresponding root system of $\mathfrak{g}'$, by $R_0$ the subsystem relative to $\mathfrak{l}$ so that $R = \mathfrak{t}_0 \oplus \bigoplus_{\alpha \in R_0} \mathfrak{g}_\alpha$, and by $R_\mathfrak{h}$ the symmetric subset of $R$ such that $n^\mathbb{C} = \bigoplus_{\alpha \in R_\mathfrak{h}} \mathfrak{g}_\alpha$. The $G$-invariant complex structure $I$ induces an endomorphism of $n^\mathbb{C}$ that is $\text{Ad}(H)$-invariant and therefore the corresponding subspace $n^{1,0}$ is a sum of root spaces. The integrability of $I$ is equivalent to the condition

\[
[n^{1,0}, n^{1,0}]_{n^\mathbb{C}} \subseteq n^{1,0}
\]

and one can prove (see e.g. [3]) that there is a suitable ordering of $R_\mathfrak{h} = R^+_\mathfrak{h} \cup R^-_\mathfrak{h}$ such that

\[
n^{1,0} = \bigoplus_{\alpha \in R^+_\mathfrak{h}} \mathfrak{g}_\alpha, \quad n^{0,1} = \bigoplus_{\alpha \in R^-_\mathfrak{h}} \mathfrak{g}_\alpha.
\]

The $G$-invariant complex structure $J$ on $G/L$ induces an $\text{Ad}(L)$-invariant endomorphism, still denoted by $J$, of $m^\mathbb{C}$, where $m := \mathfrak{t} + \mathfrak{n}$. It leaves both $\mathfrak{t}$ and $\mathfrak{n}$ invariant with $J|_{\mathfrak{n}} = I$ and the integrability of $J$ is equivalent to the vanishing of the Nijenhuis tensor $N_J$, namely for $X, Y \in m$

\[
[JX, JY]|_m = [X, Y]|_m - J[JX, Y]|_m - J[X, JY]|_m = 0.
\]

Equation (2.2) is trivial for $X, Y \in \mathfrak{t}$ and with $X \in \mathfrak{t}$ and $Y \in \mathfrak{n}$ it reduces to the $\text{ad}(\mathfrak{t})$-invariance of $I$. When $X, Y \in \mathfrak{n}$, then (2.2) is the integrability of $I$ because $[n^{1,0}, n^{1,0}] \subseteq n^{1,0}$.

Viceversa, we start with a decomposition as in (2.1), where $\mathfrak{t} + \mathfrak{n} = \mathfrak{h}$ and $\mathfrak{h}$ is the centralizer of an abelian subalgebra.

If we fix an $\text{ad}(\mathfrak{h})$-invariant integrable complex structure $I$ on $\mathfrak{n}$ and we extend it by choosing an arbitrary complex structure $J_\mathfrak{t}$ on $\mathfrak{t}$, then $J_\mathfrak{t} + I$ will provide an integrable $L$-invariant complex structure $J$ on the homogeneous space $G/L$.

Note that $(G/L, J)$ is Kähler if and only if $\mathfrak{t} = \{0\}$, i.e. $L = H$. We are mainly interested in the non-Kähler case.

**Proposition 2.1.** The compact complex manifold $(G/L, J)$ does not satisfy the $\partial\bar{\partial}$-Lemma if it is not Kähler.

**Proof.** We fix a nonzero element $\xi \in \mathfrak{t}$ and consider the $\text{Ad}(L)$-invariant element $\xi^* \in \mathfrak{g}^*$ given by the dual of $\xi$ w.r.t. the $\text{Ad}(G)$-invariant inner product $B$ on $\mathfrak{g}$. Consider the 2-form $\omega = d\xi^*$, where we still denote by $\xi^*$ the $G$-invariant 1-form on $M$ determined by $\xi$. We claim that $\omega$ is a non zero $(1, 1)$-form and that it cannot be written as $\omega = \partial\bar{\partial} f$ for $f \in C^\infty(M)$. The last assertion is clear, because $\omega$ is $G$-invariant and the function $f$ can be chosen to be invariant as well, hence a constant. We show that $\omega$ is not trivial. Indeed, if $\alpha \in R_\mathfrak{h}$ we have

\[
\omega(E_\alpha, E_{-\alpha}) = -B(\xi, [E_\alpha, E_{-\alpha}]) = B(E_\alpha, \xi) = -\alpha(\xi)B(E_\alpha, E_{-\alpha}).
\]

We select a root $\alpha \in R_\mathfrak{h}$ so that $\alpha(\xi) \neq 0$ and our claim follows. In order to prove that $\omega$ is of type $(1, 1)$, we complexify it and observe that $\omega(X, Y) \neq 0$ if and only if $X \in n^{1,0}$ and $Y \in n^{0,1}$ and therefore $\omega(JX, JY) = \omega(X, Y)$ holds. □
While flag manifolds are Kähler and do have a special (invariant) metric which is represented by the unique Kähler-Einstein metric, for non-Kähler homogeneous spaces the question about the existence of special (invariant) Hermitian metrics is meaningful and deserves a special investigation.

Following [20], an interesting class of complex homogeneous spaces is provided by M-manifolds and their products. Given a compact simply connected Lie group $G$, a M-manifold is a $G$-homogeneous space of the form $G/L$ where $L$ is a subgroup of $G$ which coincides with the semisimple part of the centralizer of a torus in $G$. Using the fact that the semisimple $L$ has finite fundamental group we see that $G/L$ is simply connected and has finite second fundamental group. Moreover an even-dimensional M-manifold and the product of two odd-dimensional ones carry infinitely many non-equivalent $G$-invariant complex structures (see [20]). Simple examples of this situation is given by the Calabi-Eckmann manifolds, which can be described in group theoretic way as $SU(n_1) \times SU(n_2)/SU(n_1 - 1) \times SU(n_2 - 1) \cong S^{2n_1-1} \times S^{2n_2-1}$.

**Proposition 2.2.** The even-dimensional M-manifolds or the product of two odd-dimensional M-manifolds do not admit any balanced metric.

**Proof.** Indeed, let $M$ be such a manifold, which is the total space of a holomorphic toric fibration $\pi$ over a flag manifold $Q$. Since $M$ is simply connected and has finite second fundamental group, we see that $H_{2n-2}(M)$ ($n = \dim \mathbb{C} M$) is trivial. Suppose $M$ admits a balanced metric whose Kähler form therefore satisfies $\delta(\omega^{n-1}) = 0$. If $Z \subset Q$ is a codimension one compact submanifold of $Q$, then $\tilde{Z} = \pi^{-1}(Z)$ is a codimension one compact submanifold of $M$ that bounds and therefore $\int_{\tilde{Z}} \omega^{n-1} = 0$, a contradiction. \hfill $\square$

**Remark 2.3.** Note that the $G$-invariance of any balanced metric in the above proposition is not assumed - however, when a balanced metric exists, we can always find an invariant one (see [4]). Note also that this result has to be contrasted with the Kähler case. Indeed, any invariant Hermitian metric $h$ on a flag manifold $G/K$ is balanced, since the codifferential $\delta \omega$ of the Kähler form $\omega$ of $h$ is a $G$-invariant 1-form and a flag manifold supports no non-trivial invariant 1-forms.

In Section 4 we will give a more detailed description of invariant balanced metrics.

We will now focus on a particular class $C$ of homogeneous non-Kähler complex spaces given by a product of two M-manifolds. This class $C$ comprises the Calabi-Eckmann manifolds. We first describe them as a homogeneous space and then we study special hermitian invariant metrics on the manifold.

We consider two irreducible compact Hermitian symmetric spaces $G_1/T^1 \cdot H_1, G_2/T^2 \cdot H_2$, where $G_1, G_2$ are two compact simply connected simple Lie groups. The product of the corresponding M-manifolds provides a homogeneous manifold

$$M := (G_1/H_1) \times (G_2/H_2)$$

which can be endowed with a family of invariant complex structures, already considered in [13] (see also the more recent results in [17] concerning also non invariant complex structures). Indeed, we consider the Cartan decompositions

$$g_i = \mathbb{R} \cdot Z_i \oplus h_i^* \oplus n_i, \quad [n_i, n_i] \subseteq h_i, \quad i = 1, 2,$$

where $h_i^*$ denotes the simple part of $h_i$ and $Z_i$ in the center of $h_i$ determines the complex structure $I_i$ on $n_i$ by $I_i = \text{ad}(Z_i)$. Therefore we have $t := h_1^* \oplus h_2^*$ and $t$ is spanned by $Z_1, Z_2$. The complex structure $I_1 \in \text{End}(t)$ can be represented by the matrix $\left( \begin{array}{cc} 0 & -i \bar{a} \\ i & 0 \end{array} \right)$ w.r.t. the basis $\{Z_1, Z_2\}$, where $a, b \in \mathbb{R}, b \neq 0$. The complex structure $J_0$ with $a = 0, b = 1$ will be called standard.

### 3. The Chern connection and the main theorem

Given a Hermitian manifold $(M, h, J)$, the associated Chern connection is the unique hermitian connection, i.e. which leaves $h$ and $J$ parallel, and such that its torsion tensor $T$ is of type $(2,0)$, namely

$$T(JX, Y) = JT(X, Y) \quad (3.3)$$

for every vector fields $X, Y$ on $M$. Since the torsion of any hermitian connection on a complex manifold has vanishing $(0,2)$-component, (3.3) is equivalent to (see [7])

$$T(JX, JY) = -T(X, Y) \quad (3.4)$$

for every $X, Y$. This in turn is equivalent to saying that $T(Z, W) = 0$ for sections $Z, W \in \Gamma(T^{10}M)$. The curvature $R$ is a section of $\Lambda^{1,1}(T^*M) \otimes u(TM)$ and in local holomorphic coordinates it has the expression

$$R_{ijkl} = -\frac{\partial h_{k\bar{l}}}{\partial z_j} + h^{a\bar{l}} \frac{\partial h_{k\bar{a}}}{\partial z_i} \frac{\partial h_{a\bar{l}}}{\partial z_j}. \quad (3.3)$$
The curvature $R$ can be traced in two different ways. The first Ricci tensor $S^{(1)}$ is defined by tracing the endomorphism part, namely
\[ S^{(1)}_{ij} = h^{p\bar{q}} R_{ijp\bar{q}} = -\frac{\partial^2 \log(\det(h))}{\partial z_i \partial \bar{z}_j} \]
and its associated $(1, 1)$ form is closed and represents the first Chern class $c_1(M)$. The second Ricci tensor is given by the trace
\[ S^{(2)}_{ij} = h^{p\bar{q}} R_{pqij} \]
and still there exists an associated $(1, 1)$ form, which is not necessarily closed. The two Ricci tensors differ by a term which depends on the covariant derivative of the torsion (see e.g. Lemma 2.4 in [16]). It is known (see [15], [14]) that when $S^{(2)}$ is positive definite (negative definite resp.) the Hodge numbers $h^{p,0} = 0$ for $p = 1, \ldots, \dim M$ (has no holomorphic vector fields resp.). We are led to the following definition, which was first considered in [8]

**Definition 3.1.** A Hermitian metric $h$ is called Chern-Einstein if there exists $\mu \in C^\infty(M)$ so that
\[ S^{(2)} = \mu \cdot h. \]

**Remark 3.2.** It follows using general formulas (see e.g. [9], p. 501) that a metric conformal to a Kähler-Einstein metric turns out to be Chern-Einstein. For this reason the original definition in [8] included the hypothesis that the metric is Gauduchon, namely $\partial \bar{\partial}(\omega^{n-1}) = 0$, in order to exclude this less significant situation. In case of homogeneous manifold, it is clear that $\mu$ has to be constant and indeed every invariant Hermitian metric is Gauduchon. In [15] and [8] it is shown that the canonical metric on the Hopf manifold $S^{2n+1} \times S^3$ is Chern-Einstein with $\mu > 0$. This also shows that, in contrast with the Kähler-Einstein case, the existence of a positive Chern-Einstein metric does not imply the simply connectedness of the manifold. Nevertheless, a compact complex manifold $M$ with finite fundamental group is simply connected when it carries a positive Chern-Einstein metric. This indeed follows from the fact the arithmetic genus $\chi(M, O) = 1$ and this invariant is actually multiplicative with finite coverings.

**Remark 3.3.** We also recall that there exists a third Ricci tensor $S^{(3)}$ which is defined as $S^{(3)}_{\alpha\beta} = h^{p\bar{q}} R_{\alpha pq\beta}$. The Einstein condition $S^{(1)} = \mu h$ or $S^{(3)} = \mu h$ for some constant $\mu \neq 0$ is easily seen to imply $h$ to be Kähler (see [3]).

We may now state our main result

**Theorem 3.4.** Let $M$ be a manifold in the class $\mathcal{C}$ endowed with an invariant complex structure $J$. Then $M$ is simply connected, non Kähler and
\begin{itemize}
  \item[a)] $M$ does not admit any balanced or SKT Hermitian metric;
  \item[b)] $M$ admits an invariant Hermitian metric $\bar{g}$ which is Chern-Einstein with $S^{(2)}(\bar{g}) = \bar{g}$. Moreover, if $J$ belongs to a suitable neighborhood of the standard complex structure, the metric $\bar{g}$ is the only invariant Chern-Einstein metric satisfying $S^{(2)}(\bar{g}) = \bar{g}$;
  \item[c)] $c_1(M) \geq 0$.
\end{itemize}

Before starting with the proof of the main Theorem, we describe the Chern connection of an invariant Hermitian metric and prove some basic facts.

Given an invariant Hermitian metric $h$ on a complex homogeneous space $G/L$, we will describe its associated Chern connection $\nabla$.

We see $h$ as an $Ad(L)$-invariant inner product $h$ on the $Ad(L)$-invariant complement $m$ with $g = 1 \oplus m$. Moreover $h$ is supposed to be Hermitian w.r.t. the invariant complex structure $J$ on $m$. Being $G$-invariant, the torsion $T$ can be seen as an element of $\Lambda^2 m \otimes m$ and after complexification, the condition (3.4) is equivalent to
\[ T(m^{10}, m^{01}) = 0. \]

Since $\nabla$ is an invariant connection on $G/L$, it is well known that it is completely determined by a map $\Lambda \in Hom(m, End(m))$, where the correspondence can be described as follows. If $X \in \mathfrak{g}$ we denote by $X^*$ the corresponding vector field on $M = G/L$ and observe that the map $m \ni X \mapsto (X^*)|_{eL} \in T_{eL}M$ is an isomorphism. Then for $X, Y \in m$ and $p = [eL] \in M$,
\[ (\Lambda(X)Y)^*|_p = (\nabla_X Y^* - [X^*, Y^*])|_p. \]

**Lemma 3.5.** The condition $\nabla J = 0$ implies $[\Lambda(X), J] = 0$.

**Proof.** Using the definition of $J$ as an endomorphism of $m$ (namely $(JX)^*|_p = JX^*|_p$) we see that
\[ (\nabla_X JY)^*|_p = \nabla_X (JY)^* = \nabla(JY), X^* + [X^*, (JY)^*] + T(X^*, (JY)^*)|_p = (\nabla_J Y, X^* + [X^*, (JY)^*] + T(X^*, (JY)^*))|_p = (J\nabla_X Y^* + [JY^*, X^*] + T(JY^*, X^*) + [X^*, (JY)^*] + T(X^*, (JY)^*))|_p = \]

$$
\[ (J\nabla_X Y^* - J[X^*, Y^*])_p + [X^*, (JY)^*]_p = J(\Lambda(XY)^*)_p + [X^*, (JY)^*]_p \]

and our claim follows. \[ \square \]

Therefore when we extend \( \Lambda \) to an element \( \Lambda \in \text{Hom}(m^c, \text{End}(m^c)) \), we have that \( \Lambda(\cdot)(m^{10}) \subseteq m^{10} \). The torsion is given by (see [15], p.192)

\[ T(X, Y) = \Lambda(X)Y - \Lambda(Y)X - [X, Y]_m \]

and condition [15] implies for \( A, B \in m^{10} \)

\[ 0 = (\Lambda(A)B - \Lambda(B)A - [A, B])^{10} = (-\Lambda(B)A - [A, B])^{10}, \]

i.e.

\[ (3.6) \]

\[ \Lambda(B)A = (\Lambda(B)A)^{10} = [B, A]^{10}. \]

Conjugation yields

\[ (3.7) \]

\[ \Lambda(A)B = [A, B]^{01}. \]

3.1. \textbf{The proof of the main Theorem.} We already know that any manifold \( M \) in \( \mathcal{C} \) does not admit any balanced metric. We start here with some generalities in order to prove our main result.

Using the same notations as above, we consider the Cartan subalgebra \( a_i \) in \( \mathfrak{h}_n^c \) given by \( t_i \oplus s_i \), where \( t_i = \mathbb{R} \cdot Z_i \) and \( s_i \) is a Cartan subalgebra of \( (\mathfrak{h}_n^c)^i \) for \( i = 1, 2 \). We denote by \( R^{(i)} \) the corresponding root systems, which are then endowed with an invariant ordering corresponding to the invariant complex structure \( I_i \) on \( n_i \) (\( i = 1, 2 \)). Then \( R^{(1)} = R_{n_1} \cup R_{n_2}^* \cup R_{n_2}^* \). In the sequel we will extend each root \( \alpha \in R^{(i)} \) to a functional on \( a_1 \oplus a_2 \) by putting \( \alpha|_{a_j} \equiv 0 \) if \( i \neq j \).

We are interested in studying special Hermitian metrics on these manifolds. We recall that a Hermitian invariant metric \( h \) on \( M \) is given by an Hermitian \( \text{Ad}(L) \)-invariant inner product on \( m = t + n \), where \( n = n_1 \oplus n_2 \). Moreover, whenever \( \dim h_i > 1 \), \( i = 1, 2 \), the tangent space \( m \) splits as the sum of three inequivalent, and therefore \( h \)-orthogonal, submodules \( t, n_1, n_2 \). We put \( n_i := \dim h_i \) for \( i = 1, 2 \).

Moreover, by \( L \)-irreducibility, the metric \( h \) on each \( n_i \) is a negative multiple of the restriction of the Cartan Killing form \( B_t \) of \( g \) on \( n_i \) (\( i = 1, 2 \)). We will also denote by \( H \) a non zero element of \( t^{10} \), say \( H = Z_1 - iJ_1Z_1 \), and we put \( h_\alpha := h(H, \bar{H}) \), where \( \alpha \neq 0 \).

Note that if \( \alpha, \beta \) are positive roots in \( R_0 \) then by the \( \text{Ad}(L) \)-invariance for every \( v \in s_i \)

\[ 0 = h([v, E_\alpha], E_{-\beta}) + h(E_\beta, [v, E_{-\alpha}]) = (\alpha - \beta)(v) \cdot h(E_\alpha, E_{-\beta}). \]

Since \( \alpha \neq \beta \) implies that \( \alpha \neq \beta \) on \( s_i \), we see that \( h(E_\alpha, E_{-\beta}) \neq 0 \) only when \( \alpha = \beta \). In this case \( h(E_\alpha, E_\alpha) := g_i \) when we use the normalized root vectors \( \{E_\alpha\} \) given by a Chevalley basis (recall also that \( E_\alpha = -E_{-\alpha} \)).

**Lemma 3.6.** \( M \) does not admit any SKT metric.

**Proof.** Suppose \( h \) is a SKT metric, which can be supposed to be invariant using the compactness of the group \( G \). The SKT condition amounts to say that \( dd^c \omega = 0 \), where \( \omega \) is the corresponding Kähler form. We will use the Koszul formula for the differential of an invariant \( q \)-form, where for \( v_0, \ldots, v_q \in m \)

\[ d\phi(v_0, \ldots, v_q) = \sum_{i<j} (-1)^{i+j} \phi([v_i, v_j]_m, v_0, \ldots, \check{v}_i, \ldots, \check{v}_j, \ldots, v_q), \]

where \( \check{\ } \) indicates that the corresponding vector does not appear. Select \( \alpha, \beta \in R_{n_1} \) and we compute

\[ d(d^c \omega)(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = -d^c \omega((H_\alpha)_m, E_\beta, E_{-\beta}) - d^c \omega((H_\beta)_m, E_\alpha, E_{-\alpha}) = -d\omega(J(H_\alpha)_m, E_\beta, E_{-\beta}) - d\omega(J(H_\beta)_m, E_\alpha, E_{-\alpha}). \]

Now

\[ d\omega(J(H_\alpha)_m, E_\beta, E_{-\beta}) = -\omega(\beta(J(H_\alpha)_m)E_\beta, E_{-\beta}) - \omega(J(H_\alpha)_m)E_\beta, E_{-\beta}) - \omega((H_\beta)_m, J(H_\alpha)_m) = -h((H_\alpha)_m, (H_\beta)_m) \]

is symmetric in \( \alpha, \beta \) and therefore we get

\[ 0 = d(d^c \omega)(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = 2g((H_\alpha)_m, (H_\beta)_m). \]

Now \( (H_\alpha)_m = \frac{\beta_1(H_\alpha, Z_1)}{\beta_1(Z_1, Z_1)} Z_1 = -\frac{\beta_1}{\beta_1} Z_1 \), so that last equation implies \( h(Z_1, Z_1) = 0 \), a contradiction. \[ \square \]

In order to prove the existence of a Chern-Einstein metric, we prove the following

**Lemma 3.7.** Given \( \alpha, \beta \in R^+_0 \), \( \gamma \in R^+_0 \) with \( i \neq j \), we have
a) \( \Lambda(E_\alpha) E_\beta = 0 \), \( \Lambda(E_\alpha) E_{\pm \gamma} = 0 \) and \( \Lambda(E_\alpha) H = 0 \);

b) \( \Lambda(E_\alpha) E_{-\beta} = 0 \) for \( \beta \neq \alpha \); \( \Lambda(E_\alpha) E_{-\alpha} = -\sqrt{-1}(Z_\alpha)^{01} \);

c) \( \Lambda(E_\alpha) H = \frac{\lambda(H, H)_{\alpha}}{\alpha} E_{\alpha} \);

d) \( \Lambda(v) = \text{ad}(v) \) for \( v \in \mathfrak{t} \).

**Proof.** a) Given \( A \in \mathfrak{n}^\dagger \) we have by (3.7) and by \([n_i, n_i] \subseteq \mathfrak{h}_i \)

\[
h(\Lambda(E_\alpha) E_{\beta}, \bar{A}) = -h(E_{\beta}, \Lambda(E_\alpha) \bar{A}) = -h(E_{\beta}, [E_\alpha, \bar{A}]) = 0.
\]

Moreover,

\[
h(\Lambda(E_\alpha) E_{\beta}, \bar{H}) = -h(E_{\beta}, \Lambda(E_\alpha) \bar{H}) = -h(E_{\beta}, [E_\alpha, \bar{H}]) = h(E_{\beta}, \alpha(\bar{H}) E_\alpha) = 0.
\]

The same kind of arguments shows the second assertion. Finally by (3.7), \( \Lambda(E_\alpha) \bar{H} = [E_\alpha, \bar{H}]^{01} = \alpha(\bar{H}) E_\alpha^{01} = 0. \)

b) We have \( \Lambda(E_\alpha) E_{-\beta} = [E_\alpha, E_{-\beta}] \neq 0 \) only if \( \alpha = \beta \) and in this case \( \Lambda(E_\alpha) E_{-\alpha} = [E_\alpha, E_{-\alpha}]^{01} = [H_\alpha]^{01} \). Now

\[
[H_\alpha]^{01} = \frac{B_i(H_\alpha, Z_i) Z_i^{01}}{B_i(Z_i, Z_i)} = -\frac{\sqrt{-1}}{2n_i} Z_i^{01}.
\]

c) We have \( h(\Lambda(E_\alpha) H, E_{-\alpha}) = -h(H, H_{\alpha}^{01}) = -h(H, H_\alpha) \) and our claim follows.

d) First note that \( \Lambda(H) \bar{H} = [H, \bar{H}]^{01} = 0 \) and \( \Lambda(H) E_{-\alpha} = -\alpha(H) E_{-\alpha} \) by (3.7). From this we see that \( \Lambda(H) H = 0 \). Now

\[
h(\Lambda(H) E_\alpha, \bar{H}) = -h(\Lambda(H) \bar{H}, E_\alpha) = 0,
\]

\[
h(\Lambda(H) E_\alpha, E_{-\beta}) = -h(E_\alpha, -\beta(H) E_{-\beta}) = \delta_{\alpha, \beta} \alpha(H) h(E_\alpha, E_{-\alpha}),
\]

so that \( \Lambda(H) E_\alpha = \alpha(H) E_\alpha \). From this we see that \( \Lambda(H) E_\alpha = -\alpha(H) E_\alpha \). Since \( \alpha_i \in \mathbb{R} \), we see that \( \alpha(v) = -\alpha_i(v) \) for \( v \in \mathfrak{t}_i \), hence \( \Lambda(H) E_\alpha = \alpha(H) E_\alpha \). Similarly for \( \Lambda(H) E_{-\alpha} = -\alpha(H) E_{-\alpha} \). \( \square \)

In order to compute the second Ricci tensor \( S^{(2)} \) (for brevity throughout the following), we compute the curvature. We use the general formula for the curvature of an invariant metric (see e.g. [13], p. 192) for \( v, w \in \mathfrak{m} \)

\[
R(v, w) = [\Lambda(v), \Lambda(w)] - \Lambda([v, w]_\mathfrak{m}) = \text{ad}([v, w]_\mathfrak{m}).
\]

Given \( \alpha, \beta \in R_\mathfrak{n}^\dagger \) we have

\[
R(E_\alpha, E_{\beta}) = [\Lambda(E_\alpha), \Lambda(E_{\beta})] E_\beta - \Lambda((H_\alpha) E_\alpha) E_\beta - [(H_\alpha) v, E_\beta] = \\
= \Lambda(E_\alpha) \Lambda(E_{\beta}) E_\beta - \Lambda((H_\alpha) E_\beta) E_\beta - [(H_\alpha) v, E_\beta] = \\
= \Lambda(E_\alpha) \Lambda(E_{\beta}) E_\beta - \beta(H_\alpha) E_\beta,
\]

where we have used Lemma 3.7 (a)(d). Note also that \( R(H, H) E_\alpha = 0 \).

Using Lemma 3.7 we see that for \( \beta \in R_\mathfrak{n}^\dagger \)

\[
S(E_\alpha, E_{\beta}) = \sum_{\alpha \in R_\mathfrak{n}^\dagger} \frac{1}{g_\alpha} h(R(E_\alpha, E_{\beta}) E_{\beta}, \bar{E}_\beta) = \\
= \sum_{\alpha \in R_\mathfrak{n}^\dagger} \frac{1}{g_\alpha} h(\Lambda(E_{\alpha}) E_{\beta}, \Lambda(E_\alpha) E_{\beta}) - \sum_{\alpha \in R_\mathfrak{n}^\dagger} \beta(H_\alpha) = \\
= \frac{1}{g_\alpha} h(\Lambda(E_{\beta}) E_{\beta}, \Lambda(E_\alpha) E_{\beta}) - \sum_{\alpha \in R_\mathfrak{n}^\dagger} \beta(H_\alpha) = \\
= \frac{1}{g_\alpha} h((H_\beta) E_{\beta}) - \sum_{\alpha \in R_\mathfrak{n}^\dagger} \beta(H_\alpha) = \\
= \frac{1}{g_\alpha} h(H_\beta^{01}, H_{\beta}^{01}) - \sum_{\alpha \in R_\mathfrak{n}^\dagger} \beta(H_\alpha) = \\
= \frac{1}{g_\alpha} h(Z_\beta^{01}, Z_\beta^{01}) - \sum_{\alpha \in R_\mathfrak{n}^\dagger} \beta(H_\alpha) = \\
= \frac{1}{4g_\alpha} h(Z_\alpha^{01}, Z_\alpha^{01}) - \sum_{\alpha \in R_\mathfrak{n}^\dagger} \beta(H_\alpha) = -\frac{1}{8g_\alpha} h(Z_i, Z_i) = \frac{1}{2},
\]

where we have used that \( \sum_{\alpha \in R_\mathfrak{n}^\dagger} H_\alpha = -\frac{\sqrt{-1}}{2} Z_i \) and \( \beta(Z_i) = \sqrt{-1} \). Now it is immediate to see that

\[
h(Z_1, Z_1) = \frac{1}{2} h_o, \quad h(Z_2, Z_2) = \frac{1}{2} h_o, \quad h(Z_1, Z_2) = \frac{1}{2} h_o.
\]
so that

\( S(E_{\beta}, \bar{E}_{\beta}) = -\frac{h_o}{16g_1n_1^2} + \frac{1}{2}, \quad \beta \in R_{a_1}, \)

(3.10)

\( S(E_{\beta}, \bar{E}_{\beta}) = -\frac{(1 + a^2)h_o}{16b^2g_2n_2^2} + \frac{1}{2}, \quad \beta \in R_{a_2}, \)

(3.11)

We now compute

\[
S(H, \bar{H}) = \sum_{i=1}^{2} \sum_{\alpha \in K_{a_1}^i} \frac{1}{g_i} h(R(E_{\alpha}, \bar{E}_{\alpha})H, \bar{H}) = \sum_{i=1}^{2} \sum_{\alpha \in K_{a_1}^i} \frac{1}{g_i} h(\Lambda(E_{\alpha})H, \Lambda(\bar{E}_{\alpha})\bar{H}) = \sum_{i=1}^{2} \sum_{\alpha \in K_{a_1}^i} \frac{1}{g_i} |h(H, H_{\alpha})|^2.
\]

Now it is immediate to see that

\[
h(H, H_\alpha) = -\frac{\sqrt{-1}}{2n_1} h(Z_1, Z_1), \quad \alpha \in R_{a_1}^+,
\]

(3.14)

\[
h(H, H_\alpha) = \frac{\sqrt{-1}}{2n_2} \left( \frac{a}{b} - \frac{\sqrt{-1}}{b} \right) h(Z_1, Z_1), \quad \alpha \in R_{a_2}^+
\]

so that

\[
S(H, \bar{H}) = \frac{h_o^2}{16} \left( \frac{1}{g_1^2n_1} + \frac{1 + a^2}{b^2g_2^2n_2^2} \right).
\]

Summing up the Hermitian Einstein equations are

\[
\begin{align*}
\frac{1}{n_1} \left( \frac{1}{n_1} \frac{1}{g_1^2} + \frac{1 + a^2}{b^2g_2^2n_2^2} \right) &= 1 \\
-\frac{16a^2 g_1}{16g_1^2} + \frac{1}{2} &= g_1 \\
-\frac{16n_2^2 g_2}{16g_2^2n_2^2} + \frac{1}{2} &= g_2.
\end{align*}
\]

(3.12)

Putting \( x := 1/g_1, \ y := 1/g_2 \) and \( z := 16/h_o \), the system (3.15) can be written as

\[
\begin{align*}
z(x - 2) &= \frac{2(1 + a^2)}{b^2g_2} y^2 \\
z(y - 2) &= \frac{n_1}{2n_2} y^2
\end{align*}
\]

(3.13)

which is equivalent to

\[
\begin{align*}
z(x - 2) &= \frac{n_1}{n_1} x^2 + \frac{1 + a^2}{b^2g_2} y^2 \\
z(y - 2) &= \frac{n_1}{n_1} x^2
\end{align*}
\]

(3.14)

We have an admissible solution \( x, y, z \in \mathbb{R}^+ \) if and only if there is a solution \( x \) of the polynomial equation

\[
\phi(x) := \left[ \frac{1}{n_1} x^2 + \frac{1 + a^2}{b^2g_2} \left( \frac{2n_1 + 2n_2 - n_1 x}{n_2} \right) \right] \cdot (x - 2) - \frac{2}{n_1^2} x^2 = 0
\]

(3.15)

satisfying the conditions

\[
x > 0, \quad x < \frac{2n_1 + 2n_2 + 2}{n_1}.
\]

This follows immediately from the fact that

\[
\phi(0) < 0, \quad \phi\left(\frac{2n_1 + 2n_2 + 2}{n_1}\right) = \frac{8n_2(n_1 + n_2 + 1)^2}{n_1^4} > 0.
\]

We now put \( \frac{1 + a^2}{b^2g_2} = 1 \) throughout the following and prove the uniqueness of the Hermitian Einstein metric.

**Lemma 3.8.** Any solution of the equation \( \phi(x) = 0 \) satisfies \( x \in [2, 2 + \frac{4}{n_1}] \). If there are two distinct solutions \( x_1 < x_2 \), then the equation \( \phi'(x) = 0 \) has two distinct solutions \( y_1 < y_2 \) in \([2, 2 + \frac{4}{n_1}]\).
which has a Chern-Einstein metric as an equilibrium point. It is known that there exists a solution for some interval \( t \in [0, T) \) for any initial metric \( \bar{h} \). Moreover any solution \( x \) satisfies
\[
0 < \frac{2}{n_1^2} x^2 = \left[ \frac{1}{n_1^2} x^2 + \left( \frac{2 n_1 + 2n_2 + 2 - n_1 x}{n_2} \right)^2 \right] \cdot (x - 2),
\]
hence \( x \in [2, 2 + \frac{2}{n_1}] \). The second claim follows immediately from the fact that \( \phi' \) is a polynomial of degree \( 2 \).

As a last remark, we consider the Chern-Ricci flow
\[
(3.18)
\]
general formula (3.8) and Lemma 3.7, (d), we see that for \( \alpha \) that can be rewritten as
\[
(3.16)
\]
Now we observe that if, say, \( n_1 = 1 \), then \( n_2 \leq 2n_2 + 2 \), giving \( n_2 = 1 \) and similarly if \( n_2 = 1 \) we get \( n_1 = 1 \). The cubic equation \( \phi(x) = 0 \) with \( n_1 = n_2 = 1 \) can be easily checked to have only one solution, so that we can suppose \( n_1, n_2 \geq 2 \). By (3.16) we see that
\[
(3.17)
\]
Now the discriminant \( d \) of the equation \( \phi'(x) = 0 \) is given by
\[
0 < d := n_1^6 + n_2^6 + 2n_1^6n_2 + 2n_1^5n_1 + n_1^5n_2^2 + n_1^3n_2^3 + (2n_1^2n_2^2 + 3n_1^2n_2^3 + 3n_1^2n_2^4 + 2n_1n_2^5 + 2n_1^2n_2^5)
\]
which is a symmetric expression in \( n_1, n_2 \). We can suppose \( n_2 \leq n_1 \) and using \( n_2 \geq \sqrt{2n_2} \) by (3.17), we see that
\[
0 < d \leq 2n_1^6 + 4n_1^5 + 3n_1^6 - ((2 + \frac{9}{4\sqrt{2}})n_1^5 + (\frac{1}{\sqrt{2}} + \frac{1}{2})n_1^4) =
\]
\[
= -n_1^6 \left( \frac{9}{4\sqrt{2}}n_1^2 - \frac{7}{2} - \frac{1}{\sqrt{2}}n_1 - 3 \right) < 0
\]
for \( n_1 \geq 3 \). So we are left with the case \( n_1 = n_2 = 2 \). In general for \( n_1 = n_2 \) the equation admits only one solution, which is explicitely given by

\[
g_1 = g_2 = \frac{n_1}{2n_1 + 1}, \quad \bar{h}_0 = \frac{8n_1^2}{(2n_1 + 1)^2}.
\]
In order to prove (c) in Theorem 3.4, we compute the first Chern-Ricci tensor \( \rho \) of an invariant metric \( \bar{h} \). Using the general formula (3.8) and Lemma 3.7, (d), we see that for \( \alpha \in R_{n_1} \),
\[
\rho(E_\alpha, \bar{E}_\alpha) = -\sum_{\beta \in R_{n_1}} \beta(H_\alpha) = \frac{1}{2}.
\]
Similarly for \( \alpha \in R_{n_2} \), we see that \( \rho(E_\alpha, \bar{E}_\alpha) = \frac{1}{2} \). Since \( R(H, \bar{H}) = 0 \), we have \( \rho(H, \bar{H}) = 0 \) and therefore \( \rho \geq 0 \).

The Chern-Ricci flow. As a last remark, we consider the Chern-Ricci flow
\[
(3.18)
\]
which has a Chern-Einstein metric as an equilibrium point. It is known that there exists a solution for some interval \( t \in [0, T) \) for any initial metric \( \bar{h} \). Moreover it is immediate to observe that the solution \( h_t \) still has the full group \( G \) acting by isometric biholomorphisms. Using a special case given by some manifold \( M \in C \) with \( n_1 = n_2 = 2 \), we see numerically that the long time existence is not guaranteed and that even when the initial metric \( \bar{h} \) has positive Ricci tensor \( S \), the flow does not necessarily converge to the Chern-Einstein metric.
We keep the same notations as in the previous sections and we consider a complex homogeneous space $M = G/L$ of complex dimension $n$ as in Section 2. We like to study the existence of invariant balanced metrics. We recall that a Hermitian metric $h$ is called balanced if $\omega^n = 0$ where $\omega$ denotes the Kähler form. This definition is actually equivalent to requiring that $\delta \omega = 0$, where $\delta$ denotes the co-differential w.r.t. the metric $h$.

We also recall that if a balanced metric exists, then an invariant balanced metric exists too (see [10]). We now focus on the possible construction of adapted balanced metrics on $G/L$, namely metrics which submerge an invariant Hermitian metric on the corresponding flag manifold $G/H$ with $t$ and $n$ being orthogonal (note that any invariant metric on $G/L$ is of this form whenever $\mathfrak{h}$ coincides with the centralizer in $\mathfrak{g}$ of its semisimple part). The condition of being adapted balanced has been already investigated in [10], Lemma 2; here we give a direct proof using some standard computations on the Levi Civita connection, which might be useful for further research. We start proving the following Lemma, where we denote by $D$ the Levi Civita connection of $h$.

**Lemma 4.1.** The metric $h$ is balanced if and only if, given $\{e_i\}_{i=1,\ldots,2n}$ an orthonormal basis of the tangent space $\mathfrak{m} \cong T_{[e_i]}M$ we have

$$\sum_i JD_{e_i} e_i - D_{Je_i} e_i = 0.$$ 

**Proof.** We know that $\delta \omega(v) = -\sum_i (D_{e_i} \omega) (e_i, v)$ for $v \in \mathfrak{m}$. We extend any element of $\mathfrak{m}$ to the corresponding Killing vector field which will be denoted by the same letter with *. We have

$$-(D_{e_i} \omega)(e_i^*, v^*) = -e_i^* \omega(e_i^*, v^*) + \omega(D_{e_i} e_i^*, v^*) + \omega(e_i^*, D_{e_i} v^*) =$$

$$= -\omega(e_i^*, [e_i^*, v^*]) + \omega(D_{e_i} e_i^*, v^*) + \omega(e_i^*, D_{e_i} v^*) =$$

$$= \omega(D_{e_i} e_i^*, v^*) + h(J e_i^*, D_{e_i} e_i^*) = h(JD_{e_i} e_i^*, v^*) - h(v^*, JD_{e_i} e_i^*)$$

and our claim follows. □

We compute the Levi Civita connection using the standard formula (see e.g. [13]) for $v, w, z \in \mathfrak{m}$

$$2h(D_v w, z) = h([v, w]_m, z) + h([z, v]_m, w) + h([z, w]_m, v).$$

We immediately see that for every $v, w \in \mathfrak{t}$ we have $D_v w = 0$ because $ad(v)(n) \subseteq n$ and $h(t, n) = 0$. For every $\alpha \in R_t^+$ we consider the vectors $e_\alpha := \frac{E_{\alpha} - E_{-\alpha}}{\sqrt{2g_\alpha}}$ so that $\{e_\alpha, Je_\alpha\}_{\alpha \in R_t^+}$ gives an orthonormal basis of $n$. A simple computation shows that

$$D_{e_\alpha} e_\alpha + D_{Je_\alpha} Je_\alpha = -\frac{1}{g_\alpha} [D_{E_\alpha} E_{-\alpha} + D_{E_{-\alpha}} E_\alpha], \quad D_{Je_\alpha} e_\alpha - D_{e_\alpha} Je_\alpha = \frac{i}{g_\alpha} [D_{E_{-\alpha}} E_\alpha - D_{E_\alpha} E_{-\alpha}].$$

Now, using (4.19) we see that for every root $\beta \in R_g, \gamma \in \Gamma$ we have

$$h(D_{E_\alpha} E_{-\alpha}, E_\beta) = 0, \quad h(D_{E_\alpha} E_{-\alpha}, v) = \frac{1}{2} h(H_\alpha, v),$$

so that

$$D_{E_\alpha} E_{-\alpha} = \frac{1}{2} (H_\alpha)_e, \quad D_{E_{-\alpha}} E_\alpha = D_{E_\alpha} E_{-\alpha} = -\frac{1}{2} (H_\alpha)_e.$$ 

Therefore $\sum_{i=1}^{2\alpha} D_{e_i} e_i = 0$ by (4.20) and the condition in Lemma 4.1 becomes $\sum_{i=1}^{2\alpha} D_{Je_i} e_i = 0$. Therefore by (4.20), $h$ is balanced if and only if

$$\sum_{\alpha \in R_t^+} \frac{1}{g_\alpha} H_\alpha |_e = 0 \quad \text{or equivalently} \quad \sum_{\alpha \in R_t^+} \frac{1}{g_\alpha} H_\alpha \in \sqrt{-1} \Pi$$

We define the vector

$$\delta_h := \sum_{\alpha \in R_t^+} \frac{1}{g_\alpha} H_\alpha$$

and note that $\delta_h$ is a slight modification of the standard Koszul element $\delta_\alpha := \frac{1}{2} \sum_{\alpha \in R_t^+} H_\alpha$ which lies in $\sqrt{-1} \mathfrak{h}$, where $\mathfrak{h}$ is the center of $\mathfrak{g}$. Indeed, we can prove that $\delta_h$ is a non zero vector in $\sqrt{-1} \mathfrak{h}$ by the following arguments. First of all we decompose $\mathfrak{n}^{1,0} = \bigoplus_{j=1}^{s} \mathfrak{q}_j$ as a sum of irreducible $\mathfrak{h}$-modules $\mathfrak{q}_j, j = 1, \ldots, s$. Note that there exist $R_j \subset R_t^+$ with $q_j = \bigoplus_{\alpha \in R_j} \mathfrak{g}_\alpha, j = 1, \ldots, s$. We prove the following

**Lemma 4.2.** Given $\zeta_j := \sum_{\alpha \in R_j} H_\alpha$, then $\sqrt{-1} \zeta_j \in \mathfrak{h}$. 
Proof. We fix $\gamma \in R_\delta$ and for every $\beta \in R_\beta$ we consider the maximal $\gamma$-string $\{\beta + k\gamma, \ p \leq k \leq q\}$. Note that $(R_\beta + \gamma) \cap R \subseteq R_\beta$ by the $\text{Ad}(H)$-invariance of $\gamma$. This means that the whole $\gamma$-string belongs to $R_\beta$. Moreover

$$\sum_{p}^{q} (\beta + k\gamma, \gamma) = (q - p + 1)(\beta, \gamma) + \frac{1}{2}q(q + 1) + p(1 - p)||\gamma||^2 = \frac{||\gamma||^2}{2}[-(p + q)(q - p + 1) + q(q + 1) + p(1 - p)] = 0.$$ 

Since the whole $R_\beta$ splits up as the disjoint union of $\gamma$-strings, we can sum up all the scalar products with $\gamma$ and we get that $(\gamma, H_\gamma) = 0$. Since $\sqrt{-1}C_j$ belongs to the Cartan subalgebra of $\mathfrak{h}$ and is orthogonal to every $H_\gamma, \gamma \in R_0$, it lies in the center of $\mathfrak{h}$. \hfill \Box

Now it is clear that for every $j = 1, \ldots, s$ and for every $\alpha, \beta \in R_\beta$ we have $g_\alpha = g_\beta$ and this common value will be called $g_j$. We can write $\delta_h = \sum_{j=1}^{s} \frac{1}{2} \zeta_j$ and therefore it lies in $\sqrt{-1}C_3$. Moreover, since $(\delta_h, H_\alpha) > 0$ for every $\alpha \in R_\delta^+$ (see e.g. [5]), we see that $(\delta_h, \delta_h) > 0$ and therefore $\delta_h \neq 0$. Our result is the following

**Theorem 4.3.** Let $G$ be a compact connected semisimple Lie group.

i) Let $M = G/L$ be a compact simply connected complex homogeneous space. Then an adapted $G$-invariant Hermitian metric $h$ on $M$ is balanced if and only if $\sqrt{-1}\delta_h$ lies in the center of $L$.

ii) Let $Q := G/H$ be a flag manifold with $b_2(Q) \geq 3$. Then there exists a complex homogeneous space $G/L$ with Tits fibration $G/L \to Q$, which admits a balanced metric.

In order to prove (ii), we recall some standard facts about flag manifolds and $T$-roots (see e.g. [11]). It is known that there exists a system of simple roots $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_t\}$ for $\mathfrak{g}$ such that $\{\alpha_1, \ldots, \alpha_p\}$ is a system of simple roots for $\mathfrak{h}_\mathfrak{s}$ and $\beta_1, \ldots, \beta_t \in R_\delta^+$. Moreover we can reorder the modules $\zeta_j$ so that $\beta_j \in R_j$ for $j = 1, \ldots, t$. This implies that $\{\zeta_1, \ldots, \zeta_t\}$ is a basis of $\sqrt{-1}C_3$ and there exist non negative integers $n_{ij}$ with $\zeta_i = \sum_{j=1}^{t} n_{ij} \zeta_j$ for $i = t + 1, \ldots, s$. Now let $\Lambda$ be the integral lattice in $J$ given by the kernel of the exponential map. We can find $v \in \Lambda$ so that $\sqrt{-1}v = \sum_{j=1}^{t} c_j \zeta_j$ with $c_j > 0$ for $j = 1, \ldots, t$ and, up to a suitable scaling by a positive real number, we can suppose that $c_j > \sum_{k=t+1}^{s} n_{kj}$ for $j = 1, \ldots, s$. We now put $g_i = 1$ for $i = t + 1, \ldots, s$ and $\frac{1}{n_{ij}} = c_j - \sum_{k=t+1}^{s} n_{kj} > 0$, defining an invariant metric $h$ on $G/H$. The corresponding $\sqrt{-1}\delta_h$ will therefore generate a one-dimensional line in $J$ which integrates to a closed one-dimensional torus $T$ by construction. Since $b_2(Q) = \dim J \geq 3$, we can find a torus $\tilde{T}$ (of dimension 1 or 2) with $T \subseteq \tilde{T} \subset Z(H)$ so that the codimension of $\tilde{T}$ in $Z(H)$ is even and this gives the isotropy $L = \tilde{T} \cdot H_\mathfrak{s}$.

As a final remark, we note that the case when $b_2(Q) = 2$ has been already treated in full generality in Proposition 2.2.
REFERENCES

[1] D.V. Alekseevsky, Flag manifolds, Yugoslav Geometrical Seminar, Divcihare, (1996), 3–35
[2] D.V. Alekseevsky and A.M. Perelomov, Invariant Kähler-Einstein metrics on compact homogeneous spaces, Funct. Anal. Applic., 20 (1986), 171–182
[3] D. Akhiezer, Lie Group Actions in Complex Analysis, Aspects in Math. vol E27 Vieweg 1995
[4] A. Balas, Compact Hermitian manifolds of constant holomorphic sectional curvature, Math. Z. 189 (1985), 193–210
[5] M. Bordemann, M. Forger and H. Römer, Homogeneous Kähler Manifolds: paving the way towards new supersymmetric Sigma Models, Comm. Math. Phys. 102 (1986), 605–647
[6] A. Balas, Compact Hermitian manifolds of constant holomorphic sectional curvature, Math. Z. 189 (1985), 193–210
[7] P. Gauduchon, La topologie d’une surface hermitienne d’Einstein C.R. Acad.Sc. Paris t. 290 (1980), 509–512
[8] P. Gauduchon, La 1-forme de torsion d’une variété hermitienne compacte. Math. Ann. 267 (1984), 495-518
[9] D. Grantcharov, Y.S. Poon, Calabi-Yau connections with torsion on toric bundles, J. Differential Geom. 78 (2008), 13–32
[10] S. Helgason, Differential Geometry, Lie groups, and Symmetric spaces, Academic Press, Inc (1978)
[11] S. Kobayashi, K. Nomizu, Foundations of differential geometry. Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II John Wiley Sons, Inc., New York-London-Sydney 1969
[12] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[13] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[14] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[15] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[16] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[17] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[18] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[19] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4
[20] S. Kobayashi, H. Wu, On holomorphic sections of certain Hermitian vector bundles, Math. Ann. 189 (1970), 1–4

DIPARTIMENTO DI MATEMATICA E INFORMATICA "ULISSE DINI", UNIVERSITÀ DI FIRENZE, V.LE MORGAGNI 67/A, 50100 FIRENZE, ITALY
E-mail address: podesta@unifi.it