Quantum vs classical computation: a proposal opening a new perspective

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We develop a classical model of computation (the S model) which captures some important features of quantum computation, and which allows to design fast algorithms for solving specific problems. In particular, we show that Deutsch’s problem can be treated within the S model of computation in the same way as within quantum computation; also Grover’s search problem of an unsorted database finds a surprisingly fast solution. The correct understanding of these results put into a new perspective the relationship between quantum and classical computation.

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I. INTRODUCTION

The intensive research of the past decade has shown that quantum computation can solve some problems more efficiently than classical computation: the two most striking example are Shor’s algorithm for factoring large integers, which is exponentially faster than any classical algorithm so far developed, and Grover’s search algorithm, which is quadratically faster than the corresponding classical one. The underlying reason for the power of quantum computers is still under debate, but the general feeling is that entanglement plays a fundamental role.

The procedure for translating a classical computational problem into a quantum mechanical one is well established and it goes as follows. Consider a problem of classical computation expressed by a function from \(n\) bits to \(m\) bits: \(f(x): B_n \rightarrow B_m\);

\[
f(x): B_n \rightarrow B_m;
\]

(1)

since quantum algorithms are necessarily given in terms of unitary — thus reversible — operations, one has first of all to rephrase the above problem in a reversible way. There is a standard procedure for doing so, which does not alter the complexity of the problem and consists in replacing the function \(f(x)\) with the new function:

\[
F(x, y): B_n \times B_m \rightarrow B_n \times B_m
\]

\[
(x, y) \mapsto (x, y \oplus f(x))
\]

(2)

(\(\times\) denotes the cartesian product between two sets and \(\oplus\) the bit–wise addition modulo 2). Note that the problem is the same as the original one, as the first \(n\) bits added to the output are equal to the corresponding input bits and thus give no extra information on the properties of the function \(f(x)\), whose precise form is determined only by the remaining last \(m\) bits; in particular, if \(y\) is initially set to \(00\ldots0\) (\(m\) times), then the output gives directly the value \(f(x)\).

The quantum mechanical translation of the function \(F(x, y)\) is now straightforward. Consider the unitary operator \(U\), defined on the computational basis of the tensor product Hilbert space \(H_n \otimes H_m\), in terms of the function \(F(x, y)\) as follows:

\[
U: H_n \otimes H_m \rightarrow H_n \otimes H_m
\]

\[
|x\rangle \oplus |y\rangle \rightarrow |x\rangle \oplus |y \oplus f(x)\rangle.
\]

(3)

Given the quantum circuit implementing the operator \(U\), one can search for quick algorithms for the solution of the problem.

The reason behind the success of algorithms such as those of Shor and Grover is the following: while, classically, we can input only one value at a time into the circuit implementing the function \(f(x)\), quantum mechanically we can do much better: by preparing the input state of the first \(n\) qubits in a superposition of all computational basis states \(|N\rangle = 2^n\):

\[
|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle,
\]

(4)

and setting (for simplicity) \(y = 0\), then, in virtue of the linear character of quantum operators, we get as the output of the quantum circuit:

\[
U|\psi\rangle \otimes |0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \otimes |f(x)\rangle.
\]

(5)

With a single step, we have been able to compute all values of the function \(f(x)\). Of course, we do not have direct

\[\text{2} \] The Hilbert space \(H_i\) has dimension equal to \(2^i\).
access to any precise state of a superposition like |5⟩; nevertheless, with appropriate manipulations, it is possible to extract the desired information in a rather quick way: this, basically, is the goal of quantum algorithm design.

In this paper, we briefly review how a computational problem is treated both within classical and quantum computation. We then formulate a new classical model of computation (the S model) which captures some features of quantum computation, in particular the possibility of inputting superposition (to be understood in an appropriate way) of states: thanks to this property, we will show that some classical problems can be solved in a surprisingly fast way. The correct understanding of this result yields a new perspective on the relationship between quantum and classical computation.

II. COMPUTATIONAL PROBLEMS

Within computational complexity theory [18], problems are formalized by resorting to the formal-language theory; for the purposes of this article, a much simpler approach is sufficient: we define a computational problem as that of finding a specific property of a given function

\[ f(x) : \mathcal{B}_n \rightarrow \mathcal{B}_m, \quad (6) \]

taking \( n \) bits into \( m \) bits. A typical example is the SAT problem: given any Boolean function (i.e. a function whose variables are connected only by \& (AND), \lor (OR), \neg (NOT) Boolean connectives) taking \( n \) bits into 1 bit, e.g.:

\[ f(x_1, x_2, x_3) = (\neg x_1 \lor x_2 \lor \neg x_2) \land (x_1 \lor \neg x_2 \lor x_3) \quad (7) \]

we have to find whether there is an input value whose output is 1.

A. Solving problems within classical computation

It is well known that any function of the type (6) is computable, i.e. there exists a circuit such that, given \( x \) as input, it outputs the value \( f(x) \). It is convenient to divide the procedure for finding a solution of a computational problem into the following two steps:

1) Given a function \( f(x) \) of the type (6), one first constructs the circuit implementing it.

2) Given the circuit, one works out appropriate algorithms for finding the solution of the problem.

A remarkable property of quantum computation is the following: it has been proved (see, e.g. [6]) that there is a general procedure for translating any classical circuit into the corresponding quantum circuit (in the sense of S), which is polynomial in the size of the circuit, i.e. in the number of elementary gates appearing in it. Accordingly, the complexity of the construction of a circuit is the same, within classical and within quantum computation. What marks the difference between the two theories is the possibility to work out quantum algorithms for solving specific problems which are faster than any known classical algorithm that solves the same problem.

B. Solving problems within quantum computation

In analogy with the classical situation, the procedure for solving a computational problem within quantum computation can be divided into two steps.

1) Given a computable function of the type (6), one first constructs the quantum circuit implementing the unitary operator

\[ U : \mathcal{H}_n \otimes \mathcal{H}_m \rightarrow \mathcal{H}_n \otimes \mathcal{H}_m \]

\[ |x\rangle \otimes |y\rangle \rightarrow |x\rangle \otimes |y \oplus f(x)\rangle. \quad (8) \]

2) Given the quantum circuit, one works out appropriate algorithms for finding the solution of the problem.

In this section we define a new model of computation, which we call the S model: its building blocks are just three states, \(|0\rangle, |1\rangle\) and \(|s\rangle\). The state \(|s\rangle\) will formally play the role of the superposition, to be understood in an appropriate way, of states \(|0\rangle\) and \(|1\rangle\). In the subsequent sections we will analyze how the S model can be used to solve certain computational problems.

III. THE S MODEL OF COMPUTATION

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A. The sbit

Single sbit. Consider a set \( K_1 \) containing three elements \(|0\rangle, |1\rangle, |s\rangle\): such elements represent the possible (5) states of a single sbit. In \( K_1 \) we define a sum + according to the following rules:
It is easy to see that this sum is commutative and associative, and these are the only two properties which we are interested in: thanks to them we can write any sum of three or more elements without specifying the order in which the sum is performed.

We define the computational basis as the smallest subset of $\mathcal{K}_1$ such that every element of $\mathcal{K}_1$ can be written as a sum of elements of the computational basis: in the present case, the computational basis is simply \{0\}, \{1\}\ and has the same cardinality as $\mathcal{B}_1$.

**Multiple sbits.** The state space $\mathcal{K}_n$ of $n$ sbits is, like in classical computation, the cartesian product of $n$ sets $\mathcal{K}_1$, i.e. the set of all strings of the form:

$$|0\rangle |1\rangle |0\rangle |0\rangle |s\rangle \ldots |s\rangle \equiv |010s \ldots s\rangle. \quad (9)$$

$\mathcal{K}_n$ contains $3^n$ elements and, in it, we define a sum $+ \langle s\rangle$ as the sbit–wise sum of the two elements being added, like e.g.

$$|01s\rangle + |110\rangle = (|0\rangle + |1\rangle)(|1\rangle + |1\rangle)(|s\rangle + |0\rangle) = \sum_{x} x|s\rangle. \quad (10)$$

The sum identifies a computational basis in $\mathcal{K}_n$ defined, as before, as the smallest subset\(^3\) such that every element of $\mathcal{K}_n$ can be written as a sum of elements of the computational basis; it is not difficult to check that the computational basis states are only those which do not contain $|s\rangle$ sbits: there are $2^n$ states of this kind, which can be trivially set into a one–to–one correspondence with the bits of $\mathcal{B}_n$.

### B. Operations with sbits

In analogy with the quantum situation, one would be tempted to consider as “valid” operations only additive functions on $\mathcal{K}_n$, i.e. those which, given two elements $|a\rangle$ and $|b\rangle$ and their images, act as follows:

$$G(|a\rangle + |b\rangle) = G|a\rangle + G|b\rangle; \quad (11)$$

however this is not possible, as request leads to inconsistencies. As an example, consider an operator $G : \mathcal{K}_2 \to \mathcal{K}_1$ which acts on the computational basis as follows:

$$G|00\rangle = |0\rangle, \quad G|01\rangle = |1\rangle, \quad G|10\rangle = |0\rangle, \quad G|11\rangle = |0\rangle. \quad (12)$$

Since $|ss\rangle = |00\rangle + |01\rangle + |10\rangle + |11\rangle = |00\rangle + |10\rangle + |11\rangle = |00\rangle + |11\rangle$, if $G$ satisfied equation (11), we would have, at the same time, both $G|ss\rangle = |s\rangle$ and $G|ss\rangle = |0\rangle$, which is not consistent.

To avoid problems of this kind, we adopt the following strategy. Consider a generic sbit $|a\rangle \in \mathcal{K}_n$, and the set $\mathcal{A}$ defined as the biggest subset of $\mathcal{K}_n$, in which the elements of the subset, one takes into account only different elements) of $\mathcal{B}_n$ such that:\(^4\)

$$|a\rangle = \sum_{x \in \mathcal{A}} |x\rangle; \quad (13)$$

such a set exists, because of the way in which the computational basis is defined, and it is unique. It is easy to check that, given a multiple sbit $|a\rangle$, e.g. $|a\rangle = |010s \ldots s10\ldots\rangle$, for which $s$ appears $k$ times, the maximal set $\mathcal{A}$ associated to it contains precisely $2^k$ computational basis sbits, which are obtained by replacing in all possible ways the $s'$ of the ordered sequence $010s \ldots s10\ldots$ defining $|a\rangle$ by 0's and 1's, while the 0's and 1's appearing in original sequence are kept fixed.

We can now define the operations on sbits which are allowed as those that satisfy the requirement

$$G|a\rangle = \sum_{x \in \mathcal{A}} G|x\rangle, \quad (14)$$

for any state $|a\rangle$ belonging to $\mathcal{K}_n$, where $\mathcal{A}$ is the maximal set associated to $|a\rangle$. We call this condition weak–additivity, and the operations satisfying it will be called weakly–additive (w–additive)\(^5\).

Note that the number of terms appearing in the sum defining the w–additivity property — just as the number of terms appearing in equation (4) — in general grows exponentially with the number of input sbits. Anyway, as it will be clear from the examples of the following sections, there are w–additive circuit implementing w–additive operators, whose size (number of elementary gates) is polynomial in the number of input sbits, yet they compute sums of the type $\sum_{s}$, without involving loops or any kind of hidden exponential slowdown in the computational time.

We now introduce some elementary w–additive gates, starting with the simplest ones, which take one sbit into one sbit. There are only two interesting gates of this kind, the NOT and H gates; their action on the computational basis is:

\(^3\) One can prove that such a set exists and is unique.

\(^4\) In the following, it is understood that when we write a sum like $\sum_{s}$, the set $\mathcal{A}$ appearing in it is always the maximal set in the sense given above.

\(^5\) Obviously, w–additivity does not imply additivity; as an example, consider once more the operator defined in (12), and assume it to be w–additive: then, we have that $|ss\rangle = |00\rangle + |11\rangle$, but $G|ss\rangle = |s\rangle \neq G|00\rangle + G|11\rangle = |0\rangle$. 

The NOT gates simply act as its classical counterpart, while the H gate in some sense mimics the quantum Hadamard gate, since it takes both computational basis states in $|s\rangle$, which can be seen as the sum of $|0\rangle$ and $|1\rangle$.

In a similar manner, we can introduce gates such as the w–additive AND, OR, FANOUT, which are defined on the computational basis of the appropriate domains as their classical counterparts. In appendix 1, it is proved that there exists a universal set of w–additive gates, i.e. a fixed number of elementary w–additive operations which can be used to compute any other w–additive operation. Appendix 2 shows other important elementary gates; we call in particular the attention of the reader on the two w–additive gates $C_0$ and $C_1$, for the role they will play in what follows: $C_0$ outputs the sbit $|0\rangle$ for any input value, while $C_1$ outputs always the sbit $|1\rangle$.

It is important to stress that, while within quantum computation the product of two unitary operators is always a unitary operator, it is not always true that the successive application of two w–additive operations still gives a w–additive operation; accordingly, when we combine w–additive gates to implement a function, we have to check that the so obtained circuit is w–additive. This represents the most serious limitation of the S model of computation but, as it will become clear in what follows, this limitation does not make the model useless. Actually, in various interesting cases it is possible to show how to combine elementary w–additive gates to automatically obtain w–additive circuits which can be used to solve interesting instances of important computational problems. As a matter of fact, it is possible to prove some theorems stating sufficient conditions for a circuit to be w–additive: here we propose two of them (the proof of the first theorem is given in the appendix; the proof of the second one is left to the reader).

**Theorem 1.** Let $G_1 : \mathcal{K}_n \to \mathcal{K}_1$ and $G_2 : \mathcal{K}_m \to \mathcal{K}_1$ be two w–additive operations. Then $G : \mathcal{K}_n \times \mathcal{K}_m \to \mathcal{K}_1$ defined as $G \equiv \text{AND} \ [G_1 \times G_2]$ is also w–additive.

**Theorem 2.** Let $G_1 : \mathcal{K}_n \to \mathcal{K}_1$ be a w–additive operation. Then $G : \mathcal{K}_n \to \mathcal{K}_1$ defined as $G \equiv \text{NOT} \ [G_1]$ is also w–additive.

The above two theorems have an important consequence: given a classical circuit composed of AND and NOT gates (thus also OR gates, but not FANOUTs), the corresponding w–additive circuit (i.e. the w–additive circuit which, on the computational basis, acts as the classical one) can be obtained by substituting to every classical elementary gate the corresponding w–additive gate. This procedure, of course, is linear in the number of elementary gates of the circuit.

When FANOUT gates are present, the situation becomes more delicate. There are particular, but important, cases in which, given a circuit containing one (or more) FANOUT, it is easy to construct the corresponding w–additive circuit. Two such cases are presented in Fig. 3 and 4 when, in a classical circuit, a bit is copied with a FANOUT, one of the two copies goes through a NOT gate and, subsequently, the two copies are jointed in an AND gate, then the w–additive gate corresponding to this piece of the classical circuit is the $C_0$ gate. If, instead of the AND gate, an OR gate is present, then the corresponding w–additive gate is the $C_1$ gate.

We introduce now the following definition, which will play a crucial role for the following discussion.

**Definition:** a classical circuit is said to be convertible if there is an efficient (i.e. polynomial in the number of elementary gates) procedure for converting it into the corresponding w–additive circuit.

The previous analysis has shown that any circuit composed of AND, OR, NOT gates and FANOUT gates appearing in a configuration like that of Figs. 3 or 4 is convertible. Of course, the class of convertible circuits is much bigger: it is an open question to ascertain how big it is.
We conclude this section by giving a simple example of a convertible circuit. Consider the classical circuit depicted in Fig. 5. It is composed of AND, NOT, OR gates, and the only FANOUT appears in a configuration like that depicted in Fig. 4 accordingly, the circuit is convertible and the corresponding \( \omega \)-additive circuit, obtained by substituting to every elementary part of the circuit the corresponding \( \omega \)-additive one, is shown in Fig. 6.

We stress once more that:
1) The number of steps needed to build up the \( \omega \)-additive circuit is \emph{proportional} to the number of elementary gates, since the procedures require, basically, to substitute every component of the classical circuit with the corresponding \( \omega \)-additive component.
2) The resulting \( \omega \)-additive circuit acts on the computational basis of \( K_3 \) as its classical counterpart. This property follows automatically from the way it has been constructed.

2) Being \( \omega \)-additive, the new circuit admits as input states also “superpositions” of the type \(|ss0\rangle\rangle\): like in the quantum case, the output for such input states is computed automatically by the \( \omega \)-additive circuit with just one single query.

To summarize, we have defined a consistent computational model which tries to capture some features of quantum computation: the building block is the sbit which has two computational basis states plus a third state which can be seen as the superposition of the two basis states. Operations on sbits are defined on the computational basis and the requirement of weak additivity (the analog of linearity within quantum computation) defines in a unique way their action for all other states; although the combination of two \( \omega \)-additive operations is not always \( \omega \)-additive, all convertible classical circuits can be easily turned into the corresponding \( \omega \)-additive circuits. Finally, there exists a universal set of \( \omega \)-additive gates.

\section*{IV. Algorithms with SBITS}

We now analyze some computational problems which have particular relevance for quantum computation. As we did in the classical and quantum case, we divide the discussion into two parts: the construction of the (\( \omega \)-additive) circuit which implements the function defining the problem, and the formulation of the algorithm for solving the problem. In this section we discuss this second part, i.e. the formulation of the algorithm. We will tackle the problem of the construction of the correspond-
ing circuits in the next section.

A. Deutsch–Jozsa problem

The Deutsch–Jozsa problem \[3, 19, 20, 21\] has a special role within quantum computation, first because it was the first problem which was proven to be easily solvable by quantum computers, second because, being rather easy, it has become the paradigmatic example of the way in which quantum computers work.

The problem is the following one: we are given a function \( f(x) : \mathcal{B}_n \rightarrow \mathcal{B}_1 \) which is either constant or balanced; we have to decide whether \( f(x) \) is constant or balanced.

Any classical deterministic algorithm requires, in the worst case, \( 2^n - 1 \) computations of the function \( f(x) \) — i.e. queries to the circuit implementing it — to check whether it is constant or balanced; in contrast, the quantum Deutsch–Jozsa algorithm gives the correct solution of the problem with just one query. Without repeating the argument for the quantum case, we pass directly to discuss the situation within the sbit model of computation.

Sbit computational problem. In analogy with the quantum situation, consider the w–additive circuit \( G : \mathcal{K}_n \rightarrow \mathcal{K}_1 \), mapping the computational basis state \( |x\rangle \) of \( \mathcal{K}_n \) into \( |f(x)\rangle \). The algorithm for determining whether \( f \) is constant or balanced works as follows.

Step 1. Prepare \( n \) sbits, all in state \( |0\rangle \).
Step 2. Apply a H gate to each sbit:
\[
|000...0\rangle \longrightarrow |sss...s\rangle = \sum_{x} |x\rangle. \tag{15}
\]
Step 3. Make a call to the circuit:
\[
\sum_{x} |x\rangle \longrightarrow \sum_{x} G |x\rangle = \begin{cases}
|0\rangle & \text{if } f(x) = 0 \forall x, \\
|1\rangle & \text{if } f(x) = 1 \forall x, \\
|s\rangle & \text{if } f \text{ is balanced}.
\end{cases} \tag{16}
\]
Step 4. Make a measurement: the outcome will immediately reveal whether the function \( f \) is constant or balanced and, if it is constant, whether it is equal to 0 or 1.

With just a single query to the w–additive circuit, we get the solution of the problem. Note that, once given the circuit implementing \( G \), the resources required by the algorithm are polynomial in the size of the problem, since only \( n \) sbits, \( n \) H gates and one measurement are needed. Note also that the above algorithm can be employed to solve the more general and harder problem of checking whether a given function is constant or not: also in this case, a single query to the w–additive circuit is sufficient to distinguish a constant from a non constant function since, if \( f(x) \) is non constant, in the second sum of equation \(16\) both terms \(|0\rangle\) and \(|1\rangle\) appear and the output is \(|s\rangle\).

B. Grover’s search problem

Consider a function which is constant everywhere in its domain, except for one point \( a \):
\[
f(x) : \mathcal{B}_n \longrightarrow \mathcal{B}_1 \quad f(x) = \begin{cases} 0 & \text{if } x \neq a, \\ 1 & \text{if } x = a; \end{cases} \tag{17}
\]
the problem is to identify \( a \).

We know that, within classical computation, the problem cannot be solved, on the average, with less than \( N/2 \) computations of the circuit implementing the function \( f \); quantum mechanically we can do better, as Grover proved that only \( \sim \sqrt{N} \) applications of the quantum oracle \( U \) associated to the function \( f(x) \) defined in \(17\) are needed; we now show how the problem can be solved within the sbit computational model.

Sbit computational problem. The sbit circuit, corresponding to the classical circuit, implements the w–additive operation \( G : \mathcal{K}_n \rightarrow \mathcal{K}_1 \) which maps the computational basis state \( |x\rangle \) of \( \mathcal{K}_n \) into \( |f(x)\rangle \), where \( f(x) \) is the function defined in \(17\). We now describe the algorithm for solving the search problem.

Step 1. Initialize each sbit in the \(|0\rangle\) state.
Step 2. Apply a H gate to all sbits except to the last one:
\[
|000...00\rangle \longrightarrow |sss...s0\rangle = \sum_{x} |x\rangle, \tag{18}
\]
where \( A \) is the maximal set relative to \(|sss...s0\rangle\), i.e. the set of all \( n \)–bit–long strings whose last bit is equal to 0.

Step 3. Make a call to the circuit:
\[
\sum_{x} |x\rangle \longrightarrow \sum_{x} G |x\rangle = \begin{cases} |s\rangle & \text{if } a \in A, \\ |0\rangle & \text{otherwise} \end{cases} \tag{19}
\]
Step 4. Make a measurement; if the output is \(|s\rangle\), then the last bit of the binary expression of \( a \) is equal to 0; otherwise it is equal to 1: with just one call to the circuit we have been able to find out the last digit of \( a \).

Step 5. Repeat steps 2, 3 and 4 making the following change: in step 2, apply an H gate to each input sbit except to the last but one (the last but two, ...) A measurement of the output of the circuit will reveal the value of the last but one (last but two, ...) bit of the binary expansion of \( a \).

The above algorithm reaches the solution to the search problem with just \( n \) computations of the circuit, much faster that Grover’s algorithm.
As in the previous example, the resources needed to implement the algorithm are polynomial in the size of the problem: only \( n \) sbits, \( n(n-1) \) elementary gates and \( n \) measurements are necessary; accordingly, our procedure is not subject to the criticisms \cite{22} raised against recent proposals\cite{23,24} aiming at implementing Grover’s algorithm by resorting to classical mechanical systems.

V. IMPLEMENTATION OF THE W–ADDITIVE CIRCUITS

In this section we face the problem of constructing the w–additive circuit necessary for implementing the algorithms previously discussed.

A. Deutsch–Jozsa problem

For simplicity we consider the simplest situation, in which we have only one input bit (this is the original Deutsch problem\cite{2}). In this case, there are two constant functions:

\[
\begin{align*}
  f_1(x) &= x \land \neg x \equiv 0 \\
  f_2(x) &= x \lor \neg x \equiv 1,
\end{align*}
\]

and two balanced functions:

\[
\begin{align*}
  f_3(x) &= x \\
  f_4(x) &= \neg x;
\end{align*}
\]

the classical circuits implementing \( f_1(x) \) and \( f_2(x) \) are depicted in Fig. 7 and those implementing \( f_3(x) \) and \( f_4(x) \) are shown in Fig. 8.

In all four cases, the classical circuits are convertible, i.e. there is a procedure — requiring a number of steps proportional to the number of elementary gates — for constructing the corresponding w–additive circuits, which consists in replacing every piece of the classical circuits by the corresponding w–additive one. The w–additive circuits so obtained, for completeness, are shown in Figs. 9 and 10.

The crucial thing to analyze is how the w–additive circuits scale when the domain of the constant/balanced functions increases. Let us start with the constant functions. The two constant functions taking \( n \) bits into 1 bit can be written as follows:

\[
\begin{align*}
  f(x_1, x_2, \ldots, x_n) &= f_1(x_1) \land f_1(x_2) \land \ldots \land f_1(x_n) \\
  g(x_1, x_2, \ldots, x_n) &= f_2(x_1) \land f_2(x_2) \land \ldots \land f_2(x_n).
\end{align*}
\]

\footnote{See also refs. \cite{22,23}, for experimental realizations.}
The function \( f \) is identically equal to 0, while \( g \) is equal to 1; \( f_1 \) and \( f_2 \) are the functions defined in (20).

It is easy to see that \( f \) and \( g \) are implemented by circuits which are convertible. Such circuits contain a number of elementary gates which is proportional to \( n \), the number of input bits. This means that the procedure necessary to convert the classical circuits into \( w \)-additive circuits requires a number of steps proportional to \( n \), and produces \( w \)-additive circuits whose size is, again, proportional to \( n \). Moreover, by construction, such circuit act on the computational basis like their classical counterparts.

In the case of balanced functions, the situation is more complex since the number of possible functions increases with \( n \). One way to study the scaling problem is to recognize that the function \( f_3(x) \) previously defined is a particular example of a “projection” function. A projection \( \pi_j \) is a function from \( n \) bits into 1 bit, which gives as the output the value of the \( j \)-th bit\(^9\); e.g. \( \pi_3(x_1, x_2, x_3, x_4) = x_3 \).

In an analogous way, the function \( f_4(x) \) can be seen as the “NOT” of a projection function. An example of a projection defined in terms of Boolean connectives is the following:

\[
\pi_1(x_1, x_2, \ldots, x_n) = (x_1 \lor ((x_2 \land \neg x_2) \land (x_3 \land \neg x_3) \land \ldots (x_n \land \neg x_n))).
\]

The function \( \pi_1 \) gives as the output the first bit \( x_1 \) and, clearly, it is balanced.

As in the case of constant functions, the classical circuit implementing a balanced function of the type \( f_4 \) is convertible and its size is proportional to \( n \): this means that, once more, the procedure leading to the corresponding \( w \)-additive circuit is also proportional to \( n \) and produces a circuit whose size is \( n \).

This analysis shows that both the procedure leading from the classical circuits to the corresponding \( w \)-additive circuits and the size of the \( w \)-additive circuits scale linearly with \( n \), the number of input bits.

Of course, we stress once more that the great limitation of the \( S \) model of computation is that we do not know whether all circuits implementing balanced functions are convertible; anyway the previous analysis shows that, within a precise mathematical framework, some important features of quantum computation can be recovered classically.

### B. Grover’s search problem

Grover’s algorithm has found two important applications: it can speed up the research through an unsorted database and it can speedup the solution of \( NP \)-complete problems. Let us start by considering the search through an unsorted database.

**Search through an unsorted database.** Suppose we have an unsorted database (e.g., a list of names) containing \( N \) elements, one of which is marked: we have to find that element. In the literature, this search problem has been modelled as follows: consider a quantum circuit implementing the unitary operator \( U \) which, on the computational basis of \( \mathcal{H}_n \otimes \mathcal{H}_1 \), maps \( |x \rangle \otimes |y \rangle \) into \( |x \rangle \otimes |y \oplus f(x) \rangle \), where the function \( f(x) \) has been defined in equation (17). Once given the circuit, Grover’s procedure allows one to find item \( a \) quadratically faster than any classical algorithm. Needless to say, no exponential growth is hidden into the construction of the quantum circuit.

In ref. 22 a classical model mimicking Grover’s algorithm has been proposed. The authors consider a mechanical system of coupled harmonic oscillators, all of which (but one) are equal: each harmonic oscillator corresponds to an element of a database, and the oscillator which is different from the others corresponds to the item to be found. The authors show that one can resort to Grover’s algorithm for finding the desired oscillator, thus obtaining a quadratic speedup over usual classical algorithms.

Is then possible to reproduce classically Grover’s quantum search? In ref. 22 it has been argued that this is not the case. In fact, if the database contains \( N \) elements (thus its size, expressed in bits, is equal to \( n = \log_2 N \)), the corresponding mechanical system contains \( 2^n \) oscillators. As a consequence, even if the search algorithm is as fast as the quantum one, the overall procedure cannot be considered satisfactory from the point of view of complexity theory, since the construction of the circuit requires an exponential growth of physical resources.

By resorting to the \( S \) model of computation, we have proposed an algorithm which is even faster than Grover’s. What about the size of the \( w \)-additive circuit? Here we show that no exponential growth of physical resources is hidden in it. An example of \( w \)-additive circuit taking \( n \) bits into 1 bit, which maps the state \( |x \rangle \rangle \) of the computational basis of \( K_n \) into \( |f(x) \rangle \rangle \), is shown in Fig. 11. The circuit contains \( 2n - 1 \) elementary gates, thus its size grows linearly with the number of bits encoding the size of the problem. Moreover, the circuit does not contain loops or any other trick hiding an exponential growth, e.g., in the time it requires to perform a computation: the number of physical resources is genuinely proportional to \( n \).

But then we arrive to a paradox: how can the \( S \) model of computation, which is essentially classical, do better than classical computation? The answer is simple: in order to construct the \( w \)-additive circuit, we have to load the classical database (i.e., the list of names) into the hypothetical \( w \)-additive computer, and this operation requires a number of steps equal to \( N = 2^n \). Thus, there is no contradiction with well known classical results. Note anyway that the loading procedure is necessary also

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\(^9\) In our case, \( f_3(x) \equiv \pi_1(x) \).
FIG. 11: Circuit implementing the w–additive oracle for Grover’s search problem; the theorems of the previous section ensure that the circuit is w–additive. The gate \( G_i \) applied to sbit \( i \) is the identity gate if \( a_i = 1 \), otherwise it is a NOT gate: there are precisely \( 2^n \) ways to arrange identity and NOT gates in the circuit, each configuration corresponding to a different oracle, i.e. to a different function of the type (17).

within quantum computation, in order to construct the quantum counterpart of the classical database.

Accordingly, if one does not take into account the loading procedure, then the S model of computation provides a fast way of searching through a database, and no hidden exponential growth of physical resources or time or energy is present. The only exponential slowdown appears in the loading procedure: but this is common to other situations, like quantum computation.

NP–complete problems. Let us consider again the SAT problem: given a classical Boolean circuit taking \( n \) bits into one bit, we have do determined whether or not there is an input value whose output is 1. As already pointed out, the procedure for building the quantum analog of a classical circuit is polynomial in the size of the problem (i.e. in the number \( n \) of elementary gates forming the classical circuit); once one has the quantum circuit, he can use (a slight modification of) Grover’s algorithm for finding the solution of the problem.

Within the S model of computation the situation is different. As we have seen in the previous section, once given the appropriate w–additive circuit, there is a fast algorithm for finding a solution of the SAT problem. Anyway, in general we do not know of a general efficient procedure for transforming a classical circuit into the corresponding w–additive one, to which the algorithm can be applied. Only when the classical circuit is convertible, like the one depicted in Fig. 15, we can easily construct the corresponding w–additive circuit and apply the search algorithm to it, and the overall procedure is efficient. But, as already said, it is an open problem to determine which classical circuits are convertible.

VI. PHYSICAL IMPLEMENTATION OF THE SBIT MODEL OF COMPUTATION

Sbit–computation can be physically implemented in various ways: here we propose a simple one. The state of a single sbit is associated to 2 bits, according to the rules:

\[ |0\rangle \rightarrow 10 \]
\[ |1\rangle \rightarrow 01 \]
\[ |s\rangle \rightarrow 11; \]

the state 00 is not taken into account. Thus, physically, a sbit is realized in terms of two wires with current passing through them. In a similar way, the state of \( n \) sbits is associated to \( n \) couples of classical bits, and is physically realized by \( n \) couples of wires, according to the previous rules.

W–additive gates are easily implemented in terms of classical gates: as an example, Figs. 12, 13 and 14 show the classical circuits for the w–additive NOT, OR, and AND gates.

According to the previous rules, any w–additive circuit with \( n \) input sbits and composed of \( m \) elementary w–additive gates can be implemented by a classical circuit
FIG. 14: Circuit for the w–additive AND gate: a classical swap gate is applied to the last two bits, followed by a classical AND gate between the second and the third bit and a classical OR gate between the first and fourth bit.

working on 2n bits and made up of k · m elementary classical gates, k being a fixed constant: this means that the sbit model of computation is polynomially reducible to the classical circuit model of computation, and thus equivalent to it.

VII. DISCUSSION AND PERSPECTIVES

In the previous sections we have developed the S model of computation: this is a well defined computational model which presents some of the characteristic traits of quantum computation; in particular, there are states which are superpositions (to be understood in the appropriate way) of the computational basis states. Among the possible gates, the only allowed ones are those which satisfy the w–additivity condition. Thanks to this property (which is the analog of quantum linearity), one needs to define and control operators only on the computational basis of their domain, and w–additivity automatically defines their action on all other states.

We have divided a computational problem into two parts: the construction of the circuit which implements the function f(x) defining a problem, and the identification of the algorithm for finding the solution of the problem. We have shown that the S model allows the working out of algorithms which are faster than the corresponding classical and quantum algorithms; on the other hand, no general procedure for transforming a classical circuit into the corresponding w–additive circuit is known (while any classical circuit can be efficiently transformed into a quantum circuit): only convertible circuit admit — by definition — easy procedures for constructing the corresponding w–additive circuits.

From the previous analysis we can make the following comments on quantum computation:

1. The original Deutsch problem [3] of distinguishing between the two constant function [20] and the two balanced functions [21] can be solved also classically with a single query to the appropriate w–additive circuit. Moreover, the global procedure scales polynomially with the size of the problem.

2. Similarly, the search problem of an unsorted database can be solved efficiently by the S (classical) model of computation, modulo the loading procedure of the database.

3. Since the problems one is interested in solving are, in general, classical problems defined in terms of classical functions, any discussion on the efficiency of a quantum algorithm must take into account also the procedure for constructing the quantum circuit implementing the classical function. In particular, all discussions about the power of quantum computation for problems involving oracles should be made more precise by analyzing in detail the procedure necessary for converting a classical oracle into a quantum one.

This last point is of particular relevance. In the literature, classical algorithms involving an oracle are compared with quantum algorithms involving the corresponding quantum oracle. In the light of the previous analysis, one could also consider w–additive algorithms which resort to w–additive oracles: there would be no surprise to find out that, relative to an oracle, many other w–additive algorithm exist which are faster than the corresponding classical and quantum ones. This means that, in order to make the discussion clear and rigorous, one must always take into account the procedure needed to construct the quantum (or w–additive) oracles out of the classical one.

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VIII. APPENDIX 1: PROOF OF THEOREMS

Theorem: there exists a universal set of elementary w–additive gates.

Proof. We have to show that, by resorting to a fixed number of w–additive gates, we can compute any w–additive operation

\[ G : \mathcal{K}_n \rightarrow \mathcal{K}_m. \]  

(24)

Since a operator with a m–sbit value is equivalent to m operators with one–sbit value each, it suffices to show that a fixed number of gates is sufficient to compute all w–additive operations

\[ G : \mathcal{K}_n \rightarrow \mathcal{K}_1. \]  

(25)

The proof is by induction on the “dimension” n of the domain. For n = 1, there are 9 w–additive gates: the
identity gate (which does nothing on the input), the NOT and H gates, and the following w–additive gates:

| \( C_0 \) | input | output |
|---------|-------|--------|
| \(|0\rangle\) | \(|0\rangle\) |
| \(|1\rangle\) | \(|0\rangle\) |

| \( C_1 \) | input | output |
|---------|-------|--------|
| \(|0\rangle\) | \(|1\rangle\) |
| \(|1\rangle\) | \(|1\rangle\) |

| \( S_0 \) | input | output |
|---------|-------|--------|
| \(|0\rangle\) | \(|s\rangle\) |
| \(|1\rangle\) | \(|0\rangle\) |

| \( S_1 \) | input | output |
|---------|-------|--------|
| \(|0\rangle\) | \(|s\rangle\) |
| \(|1\rangle\) | \(|1\rangle\) |

All the above mentioned gates can be expressed in terms of NOT, AND, OR and \( S_0 \) gates. As a matter of fact, \( C_0 \) can be implemented using an AND gate with the second sbit set equal to \(|0\rangle\); \( C_1 \) is equivalent to a OR gate with the second sbit equal to \(|1\rangle\); \( S_0 \) is equivalent to a NOT gate followed by a \( S_0 \) gate; \( S_1 \) is equivalent to a NOT gate followed by a \( S_1 \) gate, while \( S_1 \) is equivalent to a \( S_0 \) gate followed by a NOT gate; finally, \( H \) is equivalent to a \( S_1 \) gate followed by a \( S_1 \) gate.

Assume now that any w–additive operator on \( n \) sbits can be computed by a circuit consisting of w–additive elementary gates, and consider the w–additive operator

\[ G : \mathcal{K}_{n+1} \rightarrow \mathcal{K}_1, \tag{26} \]

having \( n+1 \) sbits as input. Let us introduce the following two operations taking \( n \) sbits into one sbit:

\[ G_0 |a_1 a_2 \ldots a_n\rangle = G_0 |0a_1 a_2 \ldots a_n\rangle \tag{27} \]
\[ G_1 |a_1 a_2 \ldots a_n\rangle = G_1 |1a_1 a_2 \ldots a_n\rangle. \tag{28} \]

It is easy to show that they are w–additive, so by the inductive hypothesis there exists w–additive circuits computing them. The following relation follows from \( \text{(27)} \) and \( \text{(28)} \):

\[ G |sa_1 a_2 \ldots a_n\rangle = G_0 |a_1 a_2 \ldots a_n\rangle + G_1 |a_1 a_2 \ldots a_n\rangle, \tag{29} \]

for any state \( |a_1 a_2 \ldots a_n\rangle \in \mathcal{K}_n \).

Let us now consider the w–additive T gate, taking three sbits into one sbit, which is defined on the computational basis in the following way:

| \( T \) | input | output |
|-------|-------|--------|
| \(|000\rangle\) | \(|0\rangle\) |
| \(|001\rangle\) | \(|0\rangle\) |
| \(|010\rangle\) | \(|1\rangle\) |
| \(|011\rangle\) | \(|1\rangle\) |
| \(|100\rangle\) | \(|0\rangle\) |
| \(|101\rangle\) | \(|1\rangle\) |
| \(|110\rangle\) | \(|0\rangle\) |
| \(|111\rangle\) | \(|1\rangle\) |

T has one control sbit (the first) and two target sbits: if the control sbit is set to \(|0\rangle\), then the output is equal to the second sbit of the row; if on the other hand the control sbit is equal to \(|1\rangle\), the output is equal to the second target sbit. It is easy to check that the w–additivity condition preserves the above property, e．g．\(|0s1\rangle\) is mapped into \(|s\rangle\) and \(|1s1\rangle\) into \(|1\rangle\); moreover, if the control sbit is equal to \(|s\rangle\), the output is equal to the sum of the two target sbits.

With the help of the T gate, it is easy to devise a circuit that computes G (see Fig. 15). The circuit computes both \( G_0 \) and \( G_1 \) on the last \( n \) sbits; then, depending on whether the first sbit is equal to \(|0\rangle\), \(|1\rangle\) or \(|s\rangle\), the T gate outputs \( G_0 \), \( G_1 \) or \( G_0 + G_1 \), which is the desired outcome, according to Eqs. \( \text{27} \) and \( \text{28} \)

The above proof shows that we can identify the following set of universal gates: NOT, \( S_0 \), FANOUT, AND, OR and T gate.

**Theorem:** let \( G_1 : \mathcal{K}_n \rightarrow \mathcal{K}_1 \) and \( G_2 : \mathcal{K}_m \rightarrow \mathcal{K}_1 \) be two w–additive operations. Then \( G : \mathcal{K}_n \times \mathcal{K}_m \rightarrow \mathcal{K}_1 \) defined as \( G \equiv \text{AND} \ G_1 \times G_2 \) is also w–additive.

**Proof.** Let us consider a generic \(|a\rangle \in \mathcal{K}_n \times \mathcal{K}_m \),

\[ |a\rangle = \sum_{x \in A} |x\rangle, \tag{30} \]

\(|a\rangle \) can also be written in the following way:

\[ |a\rangle = |a_1\rangle |a_2\rangle, \quad |a_1\rangle = \sum_{x_1 \in A_1} |x_1\rangle, \tag{31} \]
\[ |a_2\rangle = \sum_{x_2 \in A_2} |x_2\rangle, \]

with \(|a_1\rangle, |x_1\rangle \in \mathcal{K}_n \) and \(|a_2\rangle, |x_2\rangle \in \mathcal{K}_m \). It is easy to prove that the set \( A \) is the direct product of the two sets \( A_1 \) and \( A_2 \); accordingly, any element \(|x\rangle \in A \) can be written as \(|x\rangle = |x_1\rangle |x_2\rangle \), with \(|x_1\rangle \in A_1 \) and \(|x_2\rangle \in A_2 \) and vice-versa. We recall that \( A, A_1 \) and \( A_2 \) are the maximal sets associated to \(|a\rangle, |a_1\rangle \) and \(|a_2\rangle \), respectively.

It is convenient to distinguish the following three cases.

First case: \( \forall x_1 \in A_1, G_1 |x_1\rangle = |0\rangle \) or \( \forall x_2 \in A_2, G_2 |x_2\rangle = |0\rangle \); suppose that the first situation is
true. Resorting to the w-additivity of $G_1$, we have:

$$ G|x| \equiv \text{AND} [G_1|x_1| G_2|x_2|] = |0| \ \forall \ x \in \mathcal{A}, $$
$$ G|a| \equiv \text{AND} [G_1|a_1| G_2|a_2|] = |0|; $$

(32)

this means the $G$ is w–additive.

Second case: $\forall \ x_1 \in \mathcal{A}_1, \ G_1|x_1| = |1| \text{ and } \forall \ x_2 \in \mathcal{A}_2, \ G_2|x_2| = |1|; \text{ since both } G_1 \text{ and } G_2 \text{ are w–additive operators, we get:}$

$$ G|x| \equiv \text{AND} [G_1|x_1| G_2|x_2|] = |1| \ \forall \ x \in \mathcal{A}, $$
$$ G|a| \equiv \text{AND} [G_1|a_1| G_2|a_2|] = |1|; $$

(33)

also in this case $G$ is w–additive.

Third case: $(\exists \exists \mathfrak{T}_1 \in \mathcal{A}_1, \exists \mathfrak{T}_2 \in \mathcal{A}_2 \text{ such that } G_1|\mathfrak{T}_1| \neq |0|, G_2|\mathfrak{T}_2| \neq |0|)$ and $(\exists \exists \mathfrak{T}_1 \in \mathcal{A}_1 \text{ such that } G_1|\mathfrak{T}_1| \neq |1| \text{ or } \exists \exists \mathfrak{T}_2 \in \mathcal{A}_2 \text{ such that } G_2|\mathfrak{T}_2| \neq |1|)$, i.e. both case 1 and case 2 are excluded. The above conditions imply that both $G_1|a_1| \neq |0|$ and $G_2|a_2| \neq |0|$, and $G_1|a_1| \neq |1|$ or $G_2|a_2| \neq |1|$; which in turn implies that $G|a| = |s|$

Assume now that $G$ is not w–additive,

$$ G|a| = |s| \neq \sum_{x \in \mathcal{A}} G|x|, $$

(34)

which means that either $G|x| = |1| \ \forall \ x \in \mathcal{A}$ or $G|x| = |0| \ \forall \ x \in \mathcal{A}$. Let us consider the case in which $G|x| = |1| \ \forall \ x \in \mathcal{A}$; then we have that $\forall \ x \in \mathcal{A}, \ |x| = |x_1| |x_2|):$ AND $[G_1|x_1| G_2|x_2|] = |1|$, which implies that both $G_1|a_1| = |1| \ \forall \ x_1 \in \mathcal{A}_1$ and $G_2|x_2| = |1| \ \forall \ x_2 \in \mathcal{A}_2$. But this cannot happen since such a situation (corresponding to case 2) has been excluded.

If on the other hand $G|x| = |0| \ \forall \ x \in \mathcal{A}$, we have that for any couple $x_1 \in \mathcal{A}_1$, $x_2 \in \mathcal{A}_2$: $G_1|x_1| = |0|$ or $G_2|x_2| = |0|$. This necessarily implies that $G_1$ is constant and equal to $|0|$ or that $G_2$ is constant and equal to $|0|$ (in fact, if this were not true we would have $(\exists \exists \mathfrak{T}_1 \in \mathcal{A}_1, \exists \mathfrak{T}_2 \in \mathcal{A}_2 \text{ such that both } G_1|\mathfrak{T}_1| \neq |0| \text{ and } G_2|\mathfrak{T}_2| \neq |0|$ which negates the previous statement) but, again, this situation (corresponding to case 1) has been excluded. The conclusion is that also for case 3 the operator $G$ must be w–additive. This completes the proof.

## IX. APPENDIX 2: GATES

We list all the w–additive gates which have been introduced in the paper, writing explicitly their action on the entire domain.

**One–sbit gates:**

| Input | Output |
|-------|--------|
| $|0|$  | $|0|$  |
| $|1|$  | $|1|$  |
| $|s|$  | $|s|$  |

**NOT:**

| Input | Output |
|-------|--------|
| $|0|$  | $|1|$  |
| $|1|$  | $|0|$  |
| $|s|$  | $|s|$  |

**Two–sbit gates:**

| AND | OR | XOR |
|-----|-----|-----|
| $|00|$ | $|0|$  | $|00|$ | $|0|$ |
| $|01|$ | $|0|$  | $|01|$ | $|1|$ |
| $|10|$ | $|0|$  | $|10|$ | $|1|$ |
| $|11|$ | $|1|$  | $|11|$ | $|0|$ |

**The T gate:**

| Input | Output |
|-------|--------|
| $|000|$ | $|0|$  |
| $|001|$ | $|0|$  |
| $|00s|$ | $|0|$  |
| $|010|$ | $|1|$  |
| $|011|$ | $|1|$  |
| $|01s|$ | $|1|$  |
| $|0s0|$ | $|s|$  |
| $|ss0|$ | $|s|$  |
| $|s01|$ | $|s|$  |
| $|s0s|$ | $|s|$  |
| $|s10|$ | $|s|$  |
| $|s11|$ | $|s|$  |
| $|ss1|$ | $|s|$  |
| $|s1s|$ | $|s|$  |
| $|ss0|$ | $|s|$  |
| $|ss1|$ | $|s|$  |
| $|sss|$ | $|s|$  |
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