Poles of intégrale tritronquée and anharmonic oscillators. Asymptotic localization from WKB analysis

Davide Masoero

Mathematical Physics Sector, SISSA, via Bonomea 265, 34136, Trieste, Italy
E-mail: masoero@sissa.it

Received 3 March 2010, in final form 7 July 2010
Published 20 August 2010
Online at stacks.iop.org/Non/23/2501
Recommended by A R Its

Abstract

Poles of intégrale tritronquée are in bijection with cubic oscillators that admit the simultaneous solutions of two quantization conditions. We show that the poles are well approximated by solutions of a pair of Bohr–Sommerfeld quantization conditions (the Bohr–Sommerfeld–Boutroux (B–S–B) system): the distance between a pole and the corresponding solution of the B–S–B system vanishes asymptotically.

Mathematics Subject Classification: 33E17, 34E20, 34L20, 34M30, 34M50
(Some figures in this article are in colour only in the electronic version)

1. Statement of main result

In a previous paper [Mas10], the author studied the distribution of poles of solutions of the first Painlevé equation

\[ y'' = 6y^2 - z, \quad z \in \mathbb{C}, \]

with a particular attention to the poles of the intégrale tritronquée. This is the unique solution of P-I with the following asymptotic behaviour at infinity

\[ y(z) \sim -\sqrt{\frac{z}{6}}, \quad \text{if } |\arg z| < \frac{4\pi}{5}. \]

The problem of computing the poles of the tritronquée solution was mapped to a pair of spectral problems for the cubic anharmonic oscillator. More precisely, it was shown that the
Let $a \in \mathbb{C}$ be a pole of the tritronquée solution if and only if there exists $b \in \mathbb{C}$ such that the following Schrödinger equation
\[ \frac{d^2 \psi(\lambda)}{d\lambda^2} = V(\lambda; a, b) \psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b \] (1)
admits the simultaneous solutions of two different quantization conditions.

Using a suitable complex WKB method, the author studied this pair of quantization conditions. He derived a system of two equations, the Bohr–Sommerfeld–Boutroux (B–S–B) system, whose solutions describe approximately the distribution of the poles. We say that $(a, b)$ satisfies the B–S–B (for the precise definition see definition 5) system if
\[ \oint_{c^{-1}} \sqrt{V(\lambda; a, b)} d\lambda = i\pi (2n - 1), \]
\[ \oint_{c^{1}} \sqrt{V(\lambda; a, b)} d\lambda = i\pi (2m - 1). \] (2)
Here $m, n \in \mathbb{N} - 0$ are called quantum numbers and the cycles $c_{\pm 1}$ are depicted in figure 1.

For any pair of quantum numbers there is one and only one solution to the B–S–B system; this is proven for example in [Kap03].

Solutions of B–S–B system have naturally a multiplicative structure.

**Definition 1.** Let $(a^*, b^*)$ be a solution of the B–S–B system with quantum numbers $n, m$ such that $2n - 1$ and $2m - 1$ are coprime. We call $(a^*, b^*)$ a primitive solution of the system and denote it $(a^q, b^q)$, where $q = \frac{2n - 1}{2m - 1} \in \mathbb{Q}$. Due to lemma 4, we have that
\[ (a^q_k, b^q_k) = ((2k + 1)^2 a^q, (2k + 1)^2 b^q), \quad k \in \mathbb{N} \]
is another solution of the B–S–B system. We call it a descendant solution. We call $\{(a^q_k, b^q_k)\}_{k \in \mathbb{N}}$ the $q$-sequence of solutions.

In [Mas10] it is shown that the sequence of real solutions of B–S–B system is the 1-sequence of solutions. The real primitive solution is computed numerically as $a^{1} \cong -2, 34, b^{1} \cong -0, 064$.

In this paper we prove that any $q$-sequence approximates a sequence of poles of the tritronquée solution. The error between the pole and its WKB estimate is of order $(2k + 1)^{-\frac{4}{5}}$ (see theorem 1).

**Definition 2.** We denote $D_{\varepsilon}(a^\prime) = [a - a^\prime | < \varepsilon, \varepsilon \neq 0]$. 

---

**Figure 1.** Riemann surface $\mu^2 = V(\lambda; a, b)$. 

- branch cuts defining the square root of the potential
- Stokes line
- $\Sigma_{\pm i}$: Stokes sector
The main results of this paper is the following:

**Theorem 1 (Main theorem).** Let $\varepsilon$ be an arbitrary positive number. If $\frac{1}{5} < \alpha < \frac{6}{5}$, then it exists $K \in \mathbb{N}^*$ such that for any $k \geq K$ inside the disc $D_{k-\varepsilon}(a_k^0)$ there is one and only one pole of the intégrale tritronquée.

The rest of the paper is devoted to the proof of the theorem.

2. **Proof**

2.1. **Multidimensional Rouche theorem**

The main technical tool of the proof is the following generalization of the classical Rouche theorem.

**Theorem 2 ([AY83]).** Let $D, E$ be bounded domains in $\mathbb{C}^n$, $D \subset E$, and let $f(z), g(z)$ be holomorphic maps $E \to \mathbb{C}^n$ such that

- $f(z) \neq 0$, $\forall z \in \partial D$,
- $|g(z)| < |f(z)|$, $\forall z \in \partial D$,

then $w(z) = f(z) + g(z)$ and $f(z)$ have the same number (counted with multiplicities) of zeroes inside $D$. Here $|f(z)|$ is any norm on $\mathbb{C}^n$.

2.2. **Monodromy of Schrödinger equation**

Poles of intégrale tritronquée are in bijections with the simultaneous solutions of two eigenvalues problems for the cubic anharmonic oscillator [Mas10]. Below we recall the basics of anharmonic oscillators theory; all the details can be found in [Mas10].

Fix $k \in \mathbb{Z}_5 = \{-2, \ldots, 2\}$ and the branch of $\lambda^{\frac{1}{5}}$ in such a way that $\text{Re} \lambda^{\frac{1}{5}} \to +\infty$ as $|\lambda| \to \infty$, $\arg \lambda = \frac{2\pi k}{5}$. Then there exists a unique solution $\psi_k(\lambda)$ of equation (1) such that

$$
\lim_{\lambda \to \infty, |\lambda - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon} \lambda^\frac{1}{5} e^{\frac{1}{5} \lambda^{\frac{1}{5}} - \frac{1}{5} a \lambda^{\frac{1}{5}}} \psi_k(\lambda; a, b) = 1.
$$

(3)

For any pair of functions $\psi_1, \psi_{1+2}$, we call

$$
u_k(l, l + 2) = \lim_{\lambda \to \infty, |\arg \lambda - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon} \frac{\psi_1(\lambda)}{\psi_{1+2}(\lambda)} \in \mathbb{C} \cup \infty, \quad k \in \mathbb{Z}_5.
$$

(4)

the $k$th asymptotic value.

If $\psi_1$ and $\psi_{1+2}$ are linearly independent then $\nu_k(l, l + 2) = \nu_m(l, l + 2)$ if and only if $\psi_k$ and $\psi_m$ are linearly dependent.

**Definition 3.** Let $E$ be the (open) subset of the $(a, b)$ plane such that $\psi_0(\lambda; a, b)$ and $\psi_{1+2}(\lambda; a, b)$ are linearly independent (its complement in the $(a, b)$ plane is the union of two smooth surfaces [EG09]). On $E$ we define the following functions:

$$
\nu_2(a, b) = \frac{w_2(0, -2)}{w_1(0, -2)},
$$

(5)

$$
\nu_2(a, b) = \frac{w_{-2}(0, 2)}{w_1(0, 2)},
$$

(6)

$$
U(a, b) = \begin{pmatrix}
u_2(a, b) - 1 & 0 \\ 0 & \nu_{-2}(a, b) - 1
\end{pmatrix}.
$$

(7)
All the functions are well defined and holomorphic. Indeed, due to WKB theory we have that \( w_{l+1}(l, l+2) \) is always different from 0 and \( \infty \).

We can characterize the poles of the intégrale tritronqué as the zeroes of \( U \).

**Theorem 3** ([Mas10]). The point \( a \in \mathbb{C} \) is a pole of the intégrale tritronqué if and only if there exists \( b \in \mathbb{C} \) such that \( (a, b) \) belongs to the domain of \( U \) and \( U(a, b) = 0 \). In other words \( \psi_1(\lambda; a, b) \) and \( \psi_2(\lambda; a, b) \) are linearly dependent and \( \psi_1(\lambda; a, b) \) and \( \psi_2(\lambda; a, b) \) are linearly dependent.

We remember that the complex number \( b \) in previous lemma is the coefficient of the quartic term in the Laurent expansion of the tritronqué solution around \( a \) (see section 2.2 in [Mas10]).

### 2.3. WKB Theory

Let \( V(\lambda; a, b) \) be the potential of equation (1). We call turning point any zero of \( V \). A Stokes line is any curve in the complex \( \lambda \) plane along which the real part of the action is constant, such that at least one turning point belongs to its boundary. The union of all the Stokes line and all turning points is called the Stokes complex of the potential.

A Stokes complex is naturally a graph embedded in the complex plane. The Stokes graphs have been classified topologically in [Mas10] and the graph of type ‘320’ (see figure 2) was shown to be crucial to the approximate description of the poles of the intégrale tritronqué.

**Definition 4.** Let \( (a^*, b^*) \) be a point such that the Stokes graph of \( V(\lambda; a, b) \) is of type ‘320’. On a sufficiently small neighbourhood of \( (a^*, b^*) \) we define the following analytic functions:

\[
\chi_{\pm 2}(a, b) = \oint_{c_{\pm 1}} \sqrt{V(\lambda; a, b)} \, d\lambda, \quad (8)
\]

\[
\tilde{u}_{\pm 2}(a, b) = -e^{\chi_{\pm 2}(a, b)}, \quad (9)
\]

\[
\tilde{U}(a, b) = \left( \frac{\tilde{u}_2(a, b) - 1}{\tilde{u}_{-2}(a, b) - 1} \right). \quad (10)
\]

The cycles \( c_{\pm 1} \) are depicted in figure 1 and the branch of \( \sqrt{V} \) is chosen such that \( \text{Re} \sqrt{V(\lambda)} \to +\infty \) as \( \lambda \to \infty \) along the positive semi-axis in the cut plane.
Definition 5. We say that \((a, b)\) satisfies the B–S–B system if the Stokes graph of \(V(\cdot; a, b)\) is of type ‘320’ and
\[
\chi_2(a, b) = \oint_{c_1} \frac{1}{\sqrt{V(\lambda; a, b)}} d\lambda = i\pi (2n - 1),
\]
\[
\chi_{-2}(a, b) = \oint_{c_1} \frac{1}{\sqrt{V(\lambda; a, b)}} d\lambda = i\pi (2m - 1).
\]
Here \(m, n \in \mathbb{N} - 0\) are called quantum numbers. Equivalently the B–S–B system can be written as \(\tilde{U}(a, b) = 0\).

In [Kap03] the following lemma was proven.

Lemma 1. For any pair of quantum numbers \(n, m \in \mathbb{N} - 0\) there exists one and only one solution of the B–S–B system.

After the results of [Mas10] section 4.3, we can compare the functions \(U\) and \(\tilde{U}\) defined above.

Lemma 2. Let \((a, b)\) be such that the Stokes graph is of type ‘320’. There exist a neighbourhood of \((a, b)\) and two continuous positive functions \(\rho_{\pm 2}\) such that \(\chi_{\pm 2}\) are holomorphic and
\[
|\tilde{u}_{\pm 2} - u_{\pm 2}| \leq \frac{1}{2} (e^{2\rho_{\pm 2}} - 1).
\]
Moreover, if \(\rho_{\pm 2} < \ln \frac{3}{2}\) then \(\psi_{\pm 2}\) and \(\psi_{\pm 2}\) are linearly independent.

We remark that in [Mas10] \(\rho_{\pm 2}\) were denoted \(\rho_{0,\pm 2}\).

Using classical relations of the theory of elliptic functions we have the following.

Lemma 3. The map \(\tilde{U}\) defined in (10) is always locally invertible (hence its zeroes are always simple) and
\[
\frac{\partial \chi_2}{\partial a}(a, b) \frac{\partial \chi_{-2}}{\partial b}(a, b) - \frac{\partial \chi_{-2}}{\partial a}(a, b) \frac{\partial \chi_2}{\partial b}(a, b) = -28\pi i.
\]

Proof. On the compactified elliptic curve \(\mu^2 = V(\lambda; a, b)\), consider the differentials
\[
\omega_a = -\frac{1}{\mu} d\lambda, \quad \omega_b = -\frac{d\lambda}{\mu}.
\]
It is easily seen that
\[
\frac{\partial \chi_{\pm 2}}{\partial a}(a, b) = \oint_{c_{\pm 1}} \omega_a, \quad \frac{\partial \chi_{\pm 2}}{\partial b}(a, b) = 14 \oint_{c_{\pm 1}} \omega_b.
\]
Moreover, we have that
\[
J_{\tilde{U}} = \left( \frac{\partial \chi_2}{\partial a}(a, b) \frac{\partial \chi_{-2}}{\partial b}(a, b) - \frac{\partial \chi_{-2}}{\partial a}(a, b) \frac{\partial \chi_2}{\partial b}(a, b) \right) \tilde{u}_{\pm 2}.
\]
where \(J_{\tilde{U}}\) is the Jacobian of the map \(\tilde{U}\).

The statement of the lemma follows from the classical Legendre relation between complete elliptic periods of the first and second kind [EMOT53].

Our aim is to locate the zeroes of \(U\) (the poles of the intégrale tritronqué after theorem 3) knowing the location of zeroes of \(\tilde{U}\) (the solutions of the B–S–B system). We want to find a neighbourhood of a given solution of the B–S–B system inside which there is one and only one zero of \(U\). Due to estimate (11) and Rouché theorem, it is sufficient to find a domain on whose boundary the following inequality holds:
\[
\frac{1}{2} (e^{2\rho_{\pm 2}} - 1)|u_{\pm 2}| + \frac{1}{2} (e^{2\rho_{\pm 2}} - 1)|u_{-2}| < |1 - \tilde{u}_{\pm 2}| + |1 - \tilde{u}_{-2}|.
\]
2.3.1. Scaling Law. In order to analyse the important inequality (12), we take advantage of the following scaling behaviour that was proven in [Mas10, section 4.4].

Lemma 4. Let \((a^*, b^*)\) be such that the Stokes graph is of type ‘320’ and \(E\) be a neighbourhood of \((a^*, b^*)\) such that estimates (11) are satisfied. Then, for any real positive \(x\) the point \((x^2a^*, x^3b^*)\) is such that the Stokes graph is of type ‘320’ and in the neighbourhood \(E_X = \{(x^2a^*, x^3b^*) : (a, b) \in E\}\) estimates (11) are satisfied. Moreover, for any \((a, b) \in E\) the following scaling laws are valid:

- \(\chi_{\pm 2}(x^2a, x^3b) = x^2\chi_{\pm 2}(a, b)\),
- \(\frac{\partial (\pm \sigma)\chi_{\pm 2}}{\partial (\pm \sigma)}(x^2a, x^3b) = x^2\frac{\partial (\pm \sigma)\chi_{\pm 2}}{\partial (\pm \sigma)}(a, b)\),
- \(\rho_{\pm 2}(x^2a, x^3b) = x^{-2}\rho_{\pm 2}(a, b)\).

3. Proof of the main theorem

From lemma 4 we can extract the leading behaviour of \(\tilde{U}\) around solutions of the B–S–B system.

Lemma 5. Let \(z = (2k + 1)^{\alpha}(a - a_k^0), \ c_{\pm 2} = \frac{\partial \chi_{\pm 2}(a^0, b^0)}{\partial \sigma}(a^0, b^0), \ w = (2k + 1)^{\beta}(b - b_k^0)\) and \(d_{\pm 2} = \frac{\partial \chi_{\pm 2}(a^0, b^0)}{\partial b}(a^0, b^0)\). If \(\alpha > \frac{1}{2}\) and \(\beta > -\frac{1}{2}\), then

\[
\tilde{u}_2(z, w) = 1 + c_2(2k + 1)^{\frac{1}{2} - \alpha}z + d_2(2k + 1)^{-\frac{1}{2} - \beta}w + O((2k + 1)^{-\gamma}),
\]

\[
\tilde{u}_{-2}(z, w) = 1 + c_{-2}(2k + 1)^{\frac{1}{2} - \alpha}z + d_{-2}(2k + 1)^{-\frac{1}{2} - \beta}w + O((2k + 1)^{-\gamma}),
\]

\(\gamma' > -\frac{1}{2} + \alpha, \gamma'' > \frac{1}{2} + \beta\). \hspace{1cm} (13)

Proof. It follows from lemma 4. \(\square\)

Definition 6. We denote \(D_{k, \delta}^{(a^0, b^0)} = \{|a - a'| < \varepsilon, |b - b'| < \delta, \varepsilon, \delta \neq 0\}\) the polydisc centred at \((a', b')\).

We have collected all the elements for proving the following:

Lemma 6. Let \(\varepsilon, \delta\) be arbitrary positive numbers. If \(\frac{1}{2} < \sigma < \frac{6}{7}, -\frac{1}{2} < \beta < \frac{4}{5}\), then there exists a \(\tilde{K} \in \mathbb{N}^+\) such that for any \(k \geq \tilde{K}\), \(U\) and \(\tilde{U}\) are well defined and holomorphic on \(D_{k, \varepsilon, \delta}^{(a^0, b^0)}\) and the following inequality holds true:

\[
|U(a, b) - \tilde{U}(a, b)| < |\tilde{U}(a, b)|, \quad \forall (a, b) \in \partial D_{k, \varepsilon, \delta}^{(a^0, b^0)}. \hspace{1cm} (14)
\]

Proof. The polydisc \(D_{k, \varepsilon, \delta}^{(a^0, b^0)}\) is the image under rescaling \(a \rightarrow (2k + 1)^{\frac{1}{4}}a, b \rightarrow (2k + 1)^{\frac{3}{4}}b\) of a shrinking polydisc centred at \((a^0, b^0)\); call it \(\tilde{D}_k\). Hence due to lemma 2, for \(k \geq \tilde{K}\) \(\tilde{D}_k\) is such that \(\rho_{\pm 2}\) are bounded, \(\chi_{\pm 2}\) are holomorphic and estimates (11) hold. Call \(\rho^*\) the supremum of \(\rho_{\pm 2}\) on \(\tilde{D}_K\). Due to scaling property, for all \(k \geq \tilde{K}\) \(\rho_{\pm 2}\) is bounded from above by \((2k + 1)^{-\frac{1}{2}}\rho^*\) on \(D_{k, \varepsilon, \delta}^{(a^0, b^0)}\), such a bound is eventually smaller than \(\frac{\ln 3}{\gamma}\).

Then for a sufficiently large \(k\), \(D_{k, \varepsilon, \delta}^{(a^0, b^0)}\) is a subset of the domain of \(U\) and inside it \(U\) and \(\tilde{U}\) satisfy (11) and (13).
We divide the boundary in two subsets: ∂D_{k-\alpha, k-\beta \delta} = D_0 \cup D_1,
\begin{align*}
D_0 &= \{ |a - a_k^0| = k^{-\alpha} \varepsilon; |b - b_k^0| \leq k^{-\beta} \delta \}, \\
D_1 &= \{ |a - a_k| \leq k^{-\alpha} \varepsilon; |b - b_k| = k^{-\beta} \delta \}.
\end{align*}

Inequality (14) will be analysed separately on $D_0$ and $D_1$.

If $|d_{2}| \leq |d_{-2}|$, denote $d_{2} = d, d_{-2} = D, c = c_{2}, C = c_{-2}$; in the opposite case $|d_{2}| > |d_{-2}|$, denote $d_{-2} = d, d_{2} = D, c = c_{-2}, C = c_{2}$. By the triangle inequality and expansion (13), we have that

$$|	ilde{U}(a, b)| \geq (2k + 1)^{1-\alpha \varepsilon} \left| c - \frac{Cd}{D} \right| + \text{higher order terms}, \quad (a, b) \in D_0.$$  

Similarly, if $|c_{2}| \leq |c_{-2}|$ denote $d_{2} = d, d_{-2} = D, c = c_{2}, C = c_{-2}$; in the opposite case $|c_{2}| > |c_{-2}|$, denote $d_{-2} = d, d_{2} = D, c = c_{-2}, C = c_{2}$. By the triangle inequality and expansion (13), we have that

$$|	ilde{U}(a, b)| \geq (2k + 1)^{1-\beta \delta} \left| d - \frac{Dc}{C} \right| + \text{higher order terms}, \quad (a, b) \in D_1.$$  

We observe that $(c - \frac{Cd}{D}) \neq 0$ and $(d - \frac{Dc}{C}) \neq 0$, since (see lemma 3) $c_{2}d_{-2} - c_{-2}d_{2} = -28\pi i$. By hypothesis $-1 < \frac{1}{2} - \alpha < 0$ and $-1 < -\frac{1}{2} - \beta < 0$.

Conversely, $|U(a, b) - \tilde{U}(a, b)| \leq \frac{\sigma}{2\pi i} + \text{higher order terms}$, for all $(a, b) \in D_0 \cup D_1$.

The lemma is proven.

As a corollary of lemma 6 and of Rouché theorem, we obtain the following theorem which implies theorem 1.

**Theorem 4.** Let $\varepsilon, \delta$ be arbitrary positive numbers. If $\frac{1}{5} < \alpha < \frac{6}{5}, -\frac{1}{2} < \beta < \frac{4}{5}$, then it exists a $K \in \mathbb{N}^*$ such that for any $k \geq K$, inside the polydisc $D_{k-\alpha, k-\beta \delta}$ there is one and only one solution of the system $U(a, b) = 0$.

**Acknowledgments**

The author is indebted to his advisor Professor B Dubrovin who constantly gave him suggestions and advice.

**References**

[AY83] Aizenberg I A and Yuzhakov A P 1983 Integral Representations and Residues in Multidimensional Complex Analysis (Translations of Mathematical Monographs vol 58) (Providence, RI: American Mathematical Society)

[EG09] Eremenko A and Gabrielov A 2009 Analytic continuation of eigenvalues of a quartic oscillator Commun. Math. Phys. **287** 431–57

[EMOT53] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F 1953 Higher Transcendental Functions vols I and II (New York: McGraw-Hill)

[Kap03] Kapoč A 2003 Monodromy approach to scaling limits in isomonodromic systems Teor. Mat. Fiz. **137** 393–407

[Mas10] Masoero D 2010 Poles of intégrale tritronquée and anharmonic oscillators. A WKB approach J. Phys. A: Math. Theor. **43** 095201