Positivity constraints on LECs of $\chi$PT lagrangian at $\mathcal{O}(p^6)$ level

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Positivity constraints on the LECs of $\mathcal{O}(p^6)$ $\chi$PT lagrangian are discussed. We demonstrate that the constraints are automatically satisfied inside the Mandelstam triangle for $\pi\pi$ scatterings, when $N_C$ is large. Numerical tests are made in the $N_C = 3$ case, and it is found that these constraints are also well respected.

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The concept of effective field theory plays one of the central role in modern particle physics. A well-known example is chiral perturbation theory ($\chi$PT) which is crucial in the study of low energy hadron physics [1]. It describes the interaction between pseudo-Goldstone bosons of QCD and its lagrangian is constructed based on the expansion of the external momentum and the mass of the pseudo-Goldstone bosons. In the leading order, only two parameters are involved: the pion decay constant and the pion mass (In this letter, we only focus on $SU(2) \chi$PT). When stepping into higher orders, there appears a number of low energy constants (LECs), which are free parameters of the chiral lagrangian and are not fixed by chiral symmetry requirement. Nevertheless, it is possible to obtain certain constraints on these LECs, using general properties that a quantum field theory has to obey, like analyticity, unitarity and positivity. Efforts have been made in the literature to understand these LECs along this line. In Ref. [2] positivity constraints on LECs are carefully studied at $\mathcal{O}(p^4)$ level, and it is found that these constraints are well obeyed in
general, for those LECs determined from phenomenology. The constraints on scattering lengths are discussed in Ref. [3, 4] at $\mathcal{O}(p^4)$. In Ref. [5] the positivity constraints for the full amplitudes are discussed at $\mathcal{O}(p^6)$. Recently, these positivity constraints on the LECs are carefully reinvestigated in Ref. [6]. In most of the previous investigations, only the $\mathcal{O}(p^4)$ lagrangian is examined, because the large uncertainties existed for those $\mathcal{O}(p^6)$ coefficients. On the other side, studies to the $\mathcal{O}(p^6)$ LECs have been recently extended [7], comparing with the previous estimation [8]. In order to have an understanding on the positivity constraints in a more transparent way and to test the newly determined $\mathcal{O}(p^6)$ LECs, it is worthwhile to re-investigate the positivity constraints at $\mathcal{O}(p^6)$ level. We in this note firstly study in the leading order of $1/N_C$ expansion which enables us to obtain simple analytic expressions. We find that, at leading order of $1/N_C$, these constraints are automatically satisfied, owing to the positivity of mass and the positivity of decay width of a resonance. We also investigate those positivity constraints in the case of $N_C = 3$, using the expressions of $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ LECs derived in Ref. [7, 9] and find that they are well respected as well.

The $\pi\pi$ scattering amplitude is determined by the function $A(s, t, u)$,

\[ A \left[ \pi^a(p_1) + \pi^b(p_2) \to \pi^c(p_3) + \pi^d(p_4) \right] = \delta^{ab}\delta^{cd}A(s, t, u) + \delta^{ac}\delta^{bd}A(t, u, s) + \delta^{ad}\delta^{bc}A(u, t, s). \]

We express the amplitude $A(s, t, u)$ explicitly in terms of LECs (independent of the pseudo-Goldstone masses), momenta and pseudo-Goldstone masses:

\[ A(s, t, u)_{\chi PT} = \frac{s - m_\pi^2}{f^2} + \frac{m_\pi^4}{f^4} \left( 8l_1 + 2l_3 \right) - \frac{8m_\pi^2s}{f^4}l_1 + \frac{s^2}{f^4} \left( 2l_1 + \frac{l_2}{2} \right) + \frac{(t - u)^2}{2f^4}l_2 - \frac{8m_\pi^6}{f^6}l_3^2 + \frac{m_\pi^6}{f^6} \left( r_1 + 2r_f \right) + \frac{m_\pi^4s}{f^6} \left( r_2 - 2r_f \right) + \frac{m_\pi^2s^2}{f^6}r_3 + \frac{m_\pi^2(t - u)^2}{f^6}r_4 + \frac{s^3}{f^6}r_5 + \frac{s(t - u)^2}{f^6}r_6, \]

where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2 = 4m_\pi^2 - s - t$; $f$ is the pion decay constant in the chiral limit and the chiral expansion of the pion decay constant $f_\pi$ up to $\mathcal{O}(p^6)$ has been used,

\[ f_\pi = f \left[ 1 + \frac{l_4m_\pi^2}{f^2} + \left( -2l_5l_4 + r_f \right) \frac{m_\pi^4}{f^4} + \mathcal{O}(m_\pi^6) \right]. \]

In both expressions given above, only leading terms in the $1/N_C$ expansion are kept, comparing with the original expression [8]. Comparing with the analysis in Ref. [5], instead of
using the $b_i$ parameters [5, 8], which are combinations of the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ LECs, we have reexpressed the amplitudes explicitly with the $\mathcal{O}(p^4)$ LECs $l_i$ and the $\mathcal{O}(p^6)$ LECs $r_i$, which can make the analysis order by order within the chiral expansion in a more transparent way [9].

There are three positivity constraints on the full $\pi\pi$ scattering amplitudes obtainable using forward dispersion relations:

\begin{align}
\frac{d^2}{ds^2} T(\pi^0\pi^0 \to \pi^0\pi^0)[s,t] &> 0, \\
\frac{d^2}{ds^2} T(\pi^+\pi^0 \to \pi^+\pi^0)[s,t] &> 0, \\
\frac{d^2}{ds^2} T(\pi^+\pi^+ \to \pi^+\pi^+)[s,t] &> 0,
\end{align}

which are valid in a region of the Mandelstam plane defined by $0 \leq t \leq 4m_{\pi}^2$, $s \leq 4m_{\pi}^2$, $s + t \geq 0$ (herewith called as extended Mandelstam triangle) [5, 6]. Notice that this region is larger than the conventional Mandelstam triangle defined by $0 \leq s, t, u < 4m_{\pi}^2$. These inequalities lead respectively to the following positivity constraints on the LECs:

\begin{align}
(l_1 + l_2) + \frac{m_{\pi}^2}{2f^2}(r_3 + 3r_4 + 6r_5 - 2r_6) - \frac{3t}{4f^2}(r_5 - 3r_6) &> 0, \\
(l_2 + 2) + \frac{2m_{\pi}^2}{f^2}r_4 + \frac{2t}{f^2}r_6 &> 0, \\
(2l_1 + 3l_2) + \frac{m_{\pi}^2}{f^2}(r_3 + 5r_4 + 12r_5 + 4r_6) - \frac{3s}{f^2}(r_5 + r_6) - \frac{t}{f^2}(3r_5 - 5r_6) &> 0.
\end{align}

It is known that, at level of $\mathcal{O}(p^4)$, there are only two independent constraints in the large $N_C$ limit. However, as seen from above equations, the three constraints are not degenerate in general. It is easy to check that merely in the special case when $2s + t = 4m_{\pi}^2$ only two of the three constraints given above are independent.

Positivity constraints are also obtainable in a simpler way by applying optical theorem to forward dispersion relations, which corresponds to a special case of taking $t = 0$ in the above analysis. In this way one gets,

\begin{align}
(l_1 + l_2) + \frac{m_{\pi}^2}{2f^2}(r_3 + 3r_4 + 6r_5 - 2r_6) &> 0, \\
(l_2 + 2) + \frac{2m_{\pi}^2}{f^2}r_4 &> 0, \\
(2l_1 + 3l_2) + \frac{m_{\pi}^2}{f^2}(r_3 + 5r_4 + 12r_5 + 4r_6) - \frac{3s}{f^2}(r_5 + r_6) &> 0.
\end{align}
Notice that the first two equations in above are the $O(p^6)$ extensions of those obtained in Ref. [10].

Taking $t = 4m_\pi^2$, which is used in [6], leads to the following results:

\begin{align}
(l_1 + l_2) + \frac{m_\pi^2}{2f^2}(r_3 + 3r_4 + 16r_6) &> 0, \quad (13) \\
l_2 + \frac{2m_\pi^2}{f^2}(r_4 + 4r_6) &> 0, \quad (14) \\
(2l_1 + 3l_2) + \frac{m_\pi^2}{f^2}(r_3 + 5r_4 + 24r_6) - \frac{3s}{f^2}(r_5 + r_6) &> 0. \quad (15)
\end{align}

Since the factors $m_\pi^2/2f^2$ and $m_\pi^2/4f^2$ are numerically of $O(1)$, one has to verify whether the $O(p^6)$ LECs $r_i$ play an important role numerically in above $O(p^6)$ relations. Before making numerical analysis we notice that the above relations can be rewritten in another form. In Ref. [9], using the partial wave dispersion relations and large $N_C$ technique, the LECs can be reexpressed in terms of mass and decay width of resonances without relying on any explicit resonance lagrangian:

\begin{align}
l_1 &= \frac{16\pi f^4}{3} \left( \frac{\Gamma_S}{M_S} - \frac{9\Gamma_V}{M_V} \right), \\
l_2 &= 48\pi f^4 \frac{\Gamma_V}{M_V},
\end{align}

\begin{align}
r_2 - 2r_f &= \frac{64\pi f^6\Gamma_S}{3M_S^2} \left( 1 + \frac{\beta_S}{3} + \frac{\gamma_S}{6} \right) + \frac{\pi f^6\Gamma_V}{M_V^2} (7584 + 1248\beta_V + 144\gamma_V), \\
r_3 &= \frac{64\pi f^6\Gamma_S}{3M_S^2} \left( 1 + \frac{\beta_S}{2} \right) - \frac{768\pi f^6\Gamma_V}{M_V^2} (1 + \frac{3\beta_V}{32}), \\
r_4 &= \frac{192\pi f^6\Gamma_V}{M_V^2} \left( 1 + \frac{\beta_V}{8} \right), \\
r_5 &= \frac{32\pi f^6\Gamma_S}{3M_S^2} + \frac{36\pi f^6\Gamma_V}{M_V^2}, \\
r_6 &= \frac{12\pi f^6\Gamma_V}{M_V^2},
\end{align}

where subscripts $V$ and $S$ denote vector and scalar resonances, respectively; $\Gamma_R$ and $M_R$ stand, respectively, for the value of the $R$ resonance’s width and mass in the chiral limit; the $O(m_\pi^2)$ corrections are reflected in coefficients $\alpha, \beta, \gamma$, which are defined as

\begin{equation}
\frac{\Gamma_R}{M_R^2} = \frac{\Gamma_R}{M_R^2} \left[ 1 + \beta_R \frac{m_\pi^2}{M_R^2} + O(m_\pi^4) \right],
\end{equation}
\[
\frac{\Gamma_R}{M_R^3} = \frac{\Gamma_R}{M_R^3} \left[ 1 + \alpha_R \frac{m_\pi^2}{M_R^2} + \gamma_R \frac{m_\pi^4}{M_R^4} + \mathcal{O}(m_\pi^6) \right].
\]

Substituting Eqs. (16) and (17) into Eqs. (7–9), we can translate the positivity constraints on LECs into the following simple form:

- from \( \pi^0\pi^0 \to \pi^0\pi^0 \),
  \[
  \overline{M}_S^2 + (\beta_S + 8)m_\pi^2 - \frac{3t}{2} > 0 ,
  \]
- from \( \pi^+\pi^0 \to \pi^+\pi^0 \),
  \[
  \overline{M}_V^2 + (\beta_V + 8)m_\pi^2 + \frac{t}{2} > 0 ,
  \]
- from \( \pi^+\pi^+ \to \pi^+\pi^+ \),
  \[
  \frac{(\beta_S + 14)m_\pi^2 - 3s - 3t + \overline{M}_S^2}{9\overline{M}_S^4} \Gamma_S + \frac{(\beta_V + 14)m_\pi^2 - 3s - t + \overline{M}_V^2}{2\overline{M}_V^4} \Gamma_V > 0 .
  \]

One has the following observations from the above inequalities:

- In the leading order of chiral expansion, these positivity constraints become automatic, owing to the positivity of mass and width of resonances. This can also be clearly seen, for example, by substituting Eq. (16) into Eqs. (10) – (12).

- Inside the Mandelstam triangle, these inequalities are even automatically hold at \( \mathcal{O}(p^6) \) level. Notice that positivity of the resonance width and mass, i.e., Eq. (18), requires 
  \( 1 + \beta_R \frac{m_\pi^2}{M_R} > 0 \). This condition enforces that the three constraints Eqs. (20) – (22) are unconditionally satisfied inside the Mandelstam triangle.

- In the extended region of Mandelstam triangle, Eqs. (20) and (21) are still automatically satisfied, but Eq. (22) is no longer the case. Set for example \( s = t = 4m_\pi^2 \), one gets,
  \[
  \frac{(\beta_S - 10)m_\pi^2 + \overline{M}_S^2}{9\overline{M}_S^4} \Gamma_S + \frac{(\beta_V - 2)m_\pi^2 + \overline{M}_V^2}{2\overline{M}_V^4} \Gamma_V > 0 .
  \]
  Therefore this analysis shows that to discuss the positivity condition in the enlarged region is useful in the sense that it indeed provides stronger constraints. Nevertheless, from the values given in Ref. [7], i.e., \( \beta_S = 2 \pm 8 \) and \( \beta_V = -7.7 \pm 0.3 \), \( \overline{\Gamma}_V = 177.8 \pm 2.5 \text{ MeV}, \overline{\Gamma}_S = 600 \pm 300 \text{ MeV}, \overline{M}_V = 764.3 \pm 1.1 \text{ MeV and } \overline{M}_S = 980 \pm 40 \text{ MeV} \), that the Eq. (22) is still satisfied very well numerically.
• In Ref. [6], $O(p^4)$ amplitudes with chiral loops are analyzed and it is concluded that the most stringent bounds are always found at $t = 4m^2_\pi$. In the $O(p^6)$ case at the leading order of $1/N_C$ expansion, the situation can be somewhat different. For example, by taking $t = 4m^2_\pi$ for Eqs. (20) and (22), one finds out that the constraints are stronger than taking $t = 0$, but for Eq. (21) is instead weaker.

Positivity constraint on partial waves are also discussed in the literature [2]. The $D$ wave projection of the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude is

$$T^D(\pi^0\pi^0 \to \pi^0\pi^0) = \frac{(s - 4m^2_\pi)^2}{1920\pi f^6} \times$$

$$\left[ 4f^2(l_1 + l_2) + 2m^2_\pi(r_3 + 3r_4 + 6r_5 - 2r_6) - 3(r_5 - 3r_6)s \right]. \quad (23)$$

Positivity requirement leads to

$$(l_1 + l_2) + \frac{m^2_\pi}{2f^2}(r_3 + 3r_4 + 6r_5 - 2r_6) - \frac{3s}{4f^2}(r_5 - 3r_6) > 0. \quad (24)$$

We find the constraint from the $D$ wave amplitude of $\pi^0\pi^0 \to \pi^0\pi^0$ is the same as the one from the full amplitude constraint given in Eq. (7).

In above analysis, we have made it clear that the positivity constraints are well satisfied in the large $N_C$ limit, and in most cases they are automatically obeyed, especially inside the region of Mandelstam triangle. The reason behind may be explained as, when chiral perturbation theory is embedded into resonance chiral theory, it has a genuine high energy behavior. The possible factors that could lead to the violation of positivity conditions as emphasized in Ref. [2] are hence no longer worrisome.

The above analysis are confined to the case of leading order of $1/N_C$ expansion. In the following, the effect of the $1/N_C$ corrections will be discussed. Discussions at $O(p^6)$ have been partly made in Ref. [5, 6] and the conclusion is that the constraints are in general very well satisfied for realistic value of LECs [5]. Here we will take another point of view to check the effect of the sub-leading order of $1/N_C$. In the large $N_C$ limit although one can predict the \chiPT LECs by integrating out the heavy resonances, such as the expressions given in Eqs. (16) and (17) or the ones in Ref. [12], it is however not clear at which scale these expressions apply. The scale dependence of the \chiPT LECs is of higher order effect in $1/N_C$ expansion. At $O(p^4)$ level, it is demonstrated the resonance saturation works pretty well at the scale of $\mu = M_\rho$ [12]. However there is no strict proof that the resonance saturation must
happen exactly at the scale of the resonance mass $M_R$. Instead of making the constraints on the explicit value of the LECs $[5, 6]$, we will investigate the positivity constraints on the saturation scale $\mu$ by taking into account the loop contributions given in [12]. We assume the renormalized LECs $l_3^\nu$ and $r_4^\nu$ in the $\pi\pi$ scattering amplitudes $[8]$ are provided by Eqs.(16) and (17). For $l_3^\nu, l_4^\nu$, we use the results from Ref. [12]

$$
{l_3^\nu} = 4\frac{c_m(c_m - c_d)}{M_S^2}, \quad {l_4^\nu} = 4\frac{c_mc_d}{M_S^2},
$$

(25)

where the values of $c_d = (26 \pm 7)$MeV, $c_m = (80 \pm 21)$MeV will be taken from Ref. [7].

Since we have fixed the renormalized LECs at an unknown scale $\mu$, the $\pi\pi$ scattering amplitudes given in [8] will be explicitly dependent on $\mu$. In this way, the positivity constraints on the $\pi\pi$ scattering amplitudes are translated into the constraints on the saturation scale $\mu$. We find the positivity constraints are all well satisfied at $\mu = 770$ MeV for the three channels within the Mandelstam triangle, and inside the extended region as well. The most stringent constraint on $\mu$ we find appears in $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ channel at $s = 0, t = 4m_{\pi}^2$: $\mu \gtrsim 245$MeV. The reason behind can be explained as that the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ process is only contributed by the scalar resonances and indeed scalar resonances get significant contribution from the sub-leading order of $1/N_C$ expansion. Since the value of $\mu = 245$MeV seems to be too small to be realistic, one can safely conclude that the positivity constraints are indeed very well satisfied in reality, at $\mathcal{O}(p^6)$ level. In Fig. 1, we plot the value of various amplitudes in the Mandelstam triangle and in the extended region, for $\mu = 770$MeV. As it has already been mentioned in [6] that the scalar one loop two point function is not smooth at threshold, we also find the uneven behavior of the amplitudes near the thresholds in Fig. 1.

The violation of positivity constraints signals the break down of effective theory. In this note, we extend the previous study on positivity constraints to $\mathcal{O}(p^6)$ and find the current determination of $l_i [11]$ and $r_i [7, 9]$ well satisfies the positivity relations given in the large $N_C$ limit, and also in reality.

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FIG. 1: Left: scattering amplitudes in the Mandelstam triangle; right: in the extended region. The red curve corresponds to the case $\pi^0\pi^0 \rightarrow \pi^0\pi^0$, green one corresponds to $\pi^+\pi^+ \rightarrow \pi^+\pi^+$, blue one corresponds to, $\pi^+\pi^0 \rightarrow \pi^+\pi^0$, respectively. Scale $\mu = 770\text{MeV}$. The amplitudes are given in unit of $m^4_{\pi}$ and $s, t$ are given in unit of $m^2_{\pi}$.

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