Electric/magnetic Newton-Hooke and Carroll Jackiw-Teitelboim gravity

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ABSTRACT: We construct the electric and magnetic Newton-Hooke and Carroll Jackiw-Teitelboim gravity theories using the isomorphism of Newton-Hooke and (A-)dS Carroll algebras in (1+1)-spacetime dimensions. The starting point is the non-relativistic and Carroll version of Jackiw-Teitelboim gravity without restrictions on the geometry studied in [1].

KEYWORDS: 2D Gravity, Gauge Symmetry

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1 Introduction

There has recently been a surge in interest in non-Lorentzian theories and their associated geometries. The reasons are several: their applications in condensed matter physics [2, 3], non-relativistic holographic theories [4–8], Einstein gravity and black holes [9–15], and also in hydrodynamics [16–18].

Non-Lorentzian theories can arise from the non-relativistic (or Galilean, i.e. $c \to \infty$) and Carroll ($c \to 0$) limits of Lorentzian theories. In particular, for electromagnetism, Le Bellac and Lévy-Leblond showed that there exist two different Galilean limits [19]: the so-called “electric” and “magnetic” limits. Later, the Carroll analogue was derived in [20]. Also, it has been recently shown that the Hamiltonian formulation of general relativity admits at least two Carroll limits [21]. The so-called magnetic case is equivalent to the Carroll theory of gravity, defined through a gauging of the Carroll algebra [22].

Alternatively, non-Lorentzian theories can also be built from non-relativistic and Carroll groups using the tools of gauge theory without performing a limit process [23]. So far, it is not clear how to identify the “electric” and “magnetic” analogue theories in this context. However, based on the results obtained from the Lagrangian formulation of non-Lorentzian field theories [24], it is possible to notice that an “electric” non-relativistic field theory implies the inclusion of extra fields while the “magnetic” one does not. The opposite is true for the Carroll case; that is, we have a “magnetic” Carroll field theory when extra fields
are included in the theory and an “electric” Carroll theory when they are not. Here, the terminology magnetic and electric is based on this fact.

The relationship among the non-relativistic and Carroll theories [25, 26] in a new way has also been studied [27]. The main idea is that given a non-relativistic or Carroll theory, we can construct two more models with additional fields restricting the initial dynamics. This fact will play a very prominent role in this work.

In this note, we further analyze the relation between electric and magnetic gravities in (1 + 1)-dimensions. In other words, we construct the corresponding Jackiw-Teitelboim (JT) gravity theories with extra fields playing the role of Lagrange multipliers. In this case, the Newton-Hooke algebras and the corresponding Carroll ones are isomorphic. We start by considering the non-relativistic and Carroll BF gravities introduced in [1] and following the procedure of [27] we construct new non-relativistic and Carroll gravity theories with extra fields that further restrict the dynamics. We also build new gravity theories without introducing an U(1) gauge field.

2 Basics on non-relativistic JT gravity in the first order formalism

A first-order formulation of JT gravity can be defined by gauging the (A-)dS$_2$ symmetry and constructing a BF theory [28–30]. The gauge algebra is given by $\mathfrak{so}(1,2)$ in the A-dS case ($\tilde{\Lambda} < 0$) and $\mathfrak{so}(2,1)$ for dS ($\tilde{\Lambda} > 0$). The generators are based on the Lie algebra $\mathfrak{so}(1,2)$

\[ [\tilde{J}, \tilde{P}_a] = \epsilon_{ab} \tilde{P}_b, \quad [\tilde{P}_a, \tilde{P}_b] = -\tilde{\Lambda} \epsilon_{ab} \tilde{J}, \] (2.1)

where $\tilde{P}_a$ stand for translations, $\tilde{J}$ is the boost generator, $\tilde{\Lambda}$ is the cosmological constant, and $a = 0, 1$. This algebra admits the following most general non-degenerate invariant bilinear form

\[ \langle \tilde{J}, \tilde{J} \rangle = \tilde{\gamma}_0, \quad \langle \tilde{P}_a, \tilde{P}_b \rangle = -\tilde{\gamma}_0 \tilde{\Lambda} \eta_{ab}, \] (2.2)

where $\tilde{\gamma}_0 \neq 0$ is an arbitrary constant. In order to define a BF theory we consider a scalar field $B$, valued on the (A-)dS$_2$ algebra

\[ B = \Phi^a \tilde{P}_a + \Phi \tilde{J}, \] (2.3)

and a set of one-form gauge fields, $E^a = E^a_\mu dx^\mu$ and $\Omega = \Omega_{\mu} dx^\mu$, corresponding to the zweibein form and the dual spin connection $\Omega \equiv -\frac{1}{2} \epsilon_{ab} \Omega^{ab}$, respectively. Here the space-time indexes run as $\mu = 0, 1$. The gauge fields define the (A-)dS$_2$ connection one form

\[ A = E^a \tilde{P}_a + \Omega \tilde{J}, \] (2.4)

together with the corresponding curvature two-form $F = dA + [A, A]$, given by

\[ F = R^a(\tilde{P}) \tilde{P}_a + R(\tilde{J}) \tilde{J} \] (2.5)

\[ R^a(\tilde{P}) = dE^a - \epsilon_b \Omega E^b \] (2.6)

\[ R(\tilde{J}) = d\Omega - \frac{\tilde{\Lambda}}{2} \epsilon_{ab} E^a E^b. \] (2.7)

\[ \text{We use the conventions } \epsilon_{01} = -\epsilon_{10} = 1, \text{ and } \eta_{ab} = \text{diag}(-, +) \text{ is the two-dimensional Minkowski metric.} \]

\[ \text{Wedge product among differential forms is assumed, for instance } E^a E^b = E^a \wedge E^b = -E^b E^a. \]
The (A-)dS BF action then reads
\[ S_{[B,A]} = \int \left( \Phi R(J) - \hat{\Lambda} \Phi_a R^a(P) \right). \] (2.8)

The action (2.8) is invariant under infinitesimal gauge transformations
\[ \delta A = d\tilde{\chi} + [A, \tilde{\chi}], \quad \delta B = [B, \tilde{\chi}], \] (2.9)
where \( \tilde{\chi} = \tilde{\lambda}^a \tilde{P}_a + \tilde{\lambda} \tilde{J} \) is a gauge parameter valued in the (A-)dS\(_2\) algebra.

The theory (2.8) admits a second-order formulation. The proof works as follows: varying with respect to \( \Phi^a \) yields \( R^a(P) = 0 \), which allows solving for the spin connection in terms of the zweibein and its derivatives. On the other hand, varying with respect to \( \Omega \) gives the equation \( d\Phi - \hat{\Lambda} \epsilon_{ab} E^a \Phi^b = 0 \), which allows expressing \( \Phi^a \) in terms of \( \Phi \). Plugging back \( \Omega \) and \( \Phi^a \) into (2.8), the BF action takes the form of the JT gravity action. Since the fields \( \{ \Omega, \Phi^a \} \) were algebraically obtained from their own field equations, there is a dynamical equivalence between the first and the well-known second-order formulation of JT gravity, where \( \Phi \) is the dilaton field (see, for instance, [31]).

A finite non-relativistic (NR) limit of JT gravity in the first-order formulation can be defined starting from a BF theory with gauge algebra (A-)dS\(_2 \times \mathbb{R} \) [1, 32]. Thus we extend the (A-)dS\(_2\) symmetry (2.1) by including an Abelian generator \( \tilde{Y} \), and its gauge field \( X \). To carry out the contractions, we express the relativistic algebra generators with a linear combination of new generators that involves a dimensionless parameter \( \epsilon \). The NR contraction follows from defining NR one-form gauge fields \( \tau, e, \omega \) and \( m \), as
\[ \tau = \epsilon (E^0 + X), \quad e = E^1, \] (2.10a)
\[ m = \frac{1}{\epsilon} (E^0 - X), \quad \omega = \frac{1}{\epsilon} \Omega, \] (2.10b)
and also the definition of the NR scalar fields \( \{ \eta, \rho, \phi, \zeta \} \)
\[ \eta = \frac{\epsilon}{2} \left( \Phi^0 + \Psi \right), \quad \zeta = \frac{\epsilon}{1} \left( \Phi^0 - \Psi \right), \] (2.11a)
\[ \phi = \frac{\Phi}{\epsilon}, \quad \rho = \Phi^1. \] (2.11b)
Replacing eqs. (2.10) and eqs. (2.11) into the eq. (2.8), and defining the NR cosmological constant \( \Lambda_{\text{NR}} \) and the constant \( \gamma_0 \) as
\[ \Lambda_{\text{NR}} = \frac{1}{\epsilon^2} \tilde{\Lambda}, \quad \gamma_0 = \epsilon^2 \tilde{\gamma}_0, \] (2.12)
we find, in the limit \( \epsilon \to 0 \), the NR two-dimensional gravity theory
\[ S_{\text{NR}} = \gamma_0 \int \left( \phi R_{\text{NR}}(G) + \Lambda_{\text{NR}} \left( \eta R_{\text{NR}}(M) + \zeta R_{\text{NR}}(H) - \rho R_{\text{NR}}(P) \right) \right), \] (2.13)
where
\[ R_{\text{NR}}(H) = d\tau, \quad R_{\text{NR}}(P) = de + \omega \tau, \] (2.14a)
\[ R_{\text{NR}}(G) = d\omega - \Lambda_{\text{NR}} \tau e, \quad R_{\text{NR}}(M) = dm + \omega e. \] (2.14b)
The action (2.13) is invariant under the extended Newton-Hooke\textsuperscript{±,}\textsuperscript{3} algebra, given by

\[ [G, H] = P, \quad [G, P] = M, \quad [H, P] = -\Lambda_{\text{NR}} G, \]  

(2.15)

where \( H, P \) are generators of time and spatial translations, \( G \) the Galilean boost generator, and \( M \) is an additional abelian generator. It is worth mentioning that this action can be alternatively obtained as a BF theory based on the extended Newton-Hooke (NH) algebra (2.15), where the one-form gauge connection \( A = A_\mu dx^\mu \) and the \( B \) scalar field are given by

\[ A = \tau H + eP + \omega G + m M, \]
\[ B = \eta H + \rho P + \phi G + \zeta M. \]

(2.16)

By considering the NR gauge symmetry parameter \( \chi \) valued on the extended NH algebra as \( \chi = \lambda H + \xi P + \Theta G + \Gamma M \), the symmetry transformations of the gauge fields are given by

\[ \delta_{\text{NH}} \tau_\mu = \partial_\mu \lambda, \]
\[ \delta_{\text{NH}} e_\mu = \partial_\mu \xi - \tau_\mu \Theta + \omega_\mu \lambda, \]
\[ \delta_{\text{NH}} \omega_\mu = \partial_\mu \Theta + \Lambda_{\text{NR}} (e_\mu \lambda - \tau_\mu \xi), \]
\[ \delta_{\text{NH}} m_\mu = \partial_\mu \Gamma + \omega_\mu \xi - e_\mu \Theta, \]
\[ \delta_{\text{NH}} \eta = 0, \]
\[ \delta_{\text{NH}} \rho = \lambda \phi - \Theta \eta, \]
\[ \delta_{\text{NH}} \zeta = \xi \phi - \Theta \rho, \]
\[ \delta_{\text{NH}} \phi = \Lambda_{\text{NR}} (\lambda \rho - \xi \eta). \]

(2.17)

Like the relativistic case, we can move on to the second-order formulation of (2.13). Indeed, the field equations follow upon varying with \( \rho \) and \( \eta \), \( R_{\text{NR}}(P) = 0 \) and \( R_{\text{NR}}(M) = 0 \), allows to solve the NR spin-connection, we get

\[ \omega_\mu = 2 \tau^{[\alpha} e_{\beta]} \left( e_\mu \partial_\alpha e_\beta - \tau_\mu \partial_\alpha m_\beta \right). \]

(2.18)

Then, plugging (2.18) into (2.13), we get the magnetic extended NH gravity theory

\[ S_{\text{Mag. Ext. NH}} = \kappa \int d^2x \det (\tau e) \phi \left( R_{\text{NR}} - 2\Lambda_{\text{NR}} \right), \]

(2.19)

where \( \kappa = -\gamma_0/2, \) \( \det (\tau e) \equiv -\epsilon^{\mu\nu} \tau_\mu e_\nu, \) and

\[ R_{\text{NR}} = 8 \tau^{[\mu} e_{\nu]} \partial_\mu \left( \tau^\alpha e^{\beta]} \left( e_\nu \partial_\alpha e_\beta - \tau_\nu \partial_\alpha m_\beta \right) \right). \]

(2.20)

Note that theories described by (2.13) and (2.19) are not necessarily dynamically equivalent because the equations solved by (2.18) are not those obtained by varying with respect to \( \omega_\mu \). At any rate, in the sector \( d\tau = 0 \), the two systems have equivalent field equations.

\textsuperscript{3}The plus sign corresponds to the contraction of dS algebra while the minus sign corresponds to the contraction of AdS algebra. Note that, the space-time symmetries of the harmonic oscillator form the NH\textsuperscript{+} algebra, whereas the inverted harmonic oscillator has a NH\textsuperscript{-} symmetry algebra.
and the invariance of (2.19) under Newton-Hooke boost symmetry can be confirmed in the sector \( d\tau = 0 \).\(^4\) The action (2.19) corresponds to the NR second-order formulation of JT gravity [1].

Another singular limit that can be carried out on the relativistic theory (2.13) is the Carroll limit. This contraction is obtained from the observation that the Carroll (A-)dS\(_2\) algebra admits a central extension in the commutator of the Galilean boost and momentum generator,\(^5\) we name this algebra "extended Carroll (A-)dS\(_2\) algebra." One can see that this symmetry follows from the extended NH\(_2\) algebra by interchanging the generators \( H \) and \( P \) and changing the sign of the cosmological constant \( \Lambda_{\text{NR}} \). This fact allows one to pass from Newton-Hooke to Carroll symmetries. The isomorphism reads as

\[
\begin{align*}
\text{Extended NH}_2^+ & \leftrightarrow \text{Extended Carroll AdS}_2 \\
\text{Extended NH}_2^- & \leftrightarrow \text{Extended Carroll dS}_2.
\end{align*}
\]

Then, the extended (A-)dS\(_2\) Carroll algebra is defined as

\[
\begin{align*}
[G, P] &= H, \\
[G, H] &= M, \\
[H, P] &= -\Lambda_C G,
\end{align*}
\]

where \( \Lambda_C = -\Lambda_{\text{NR}} \). The gravity theories invariant under these two algebras are classically related through the interchange of its fields: \( \tau \leftrightarrow e \) and \( \eta \leftrightarrow \rho \). Hence, the Carroll action can be obtained from the NR one (2.13), yielding

\[
S_C = \gamma_0 \int \left( \phi R_C(G) + \Lambda_C \left( \rho R_C(M) + \zeta R_C(P) - \eta R_C(H) \right) \right),
\]

with the Carroll curvature two-forms given by

\[
\begin{align*}
R_C(H) &= d\tau + \omega e, \\
R_C(P) &= de, \\
R_C(G) &= d\omega - \Lambda_C \tau e, \\
R_C(M) &= dm + \omega \tau.
\end{align*}
\]

The action (2.23) is invariant under infinitesimal Carroll gauge transformations

\[
\begin{align*}
\delta_C \tau_\mu &= \partial_\mu \lambda - \Theta e_\mu + \omega_\mu \xi, \\
\delta_C e_\mu &= \partial_\mu \xi, \\
\delta_C \omega_\mu &= \partial_\mu \Theta + \Lambda_C (e_\mu \lambda - \tau_\mu \xi), \\
\delta_C m_\mu &= \partial_\mu \Gamma + \omega_\mu \lambda - \tau_\mu \Theta, \\
\delta_C \eta &= \xi \phi - \Theta \rho, \\
\delta_C \rho &= 0, \\
\delta_C \zeta &= \lambda \phi - \Theta \eta, \\
\delta_C \phi &= \Lambda_C (\lambda \rho - \xi \eta).
\end{align*}
\]

\(^4\)This is similar to the case of first-order gravity in four dimensions \( S = \kappa \int E \wedge E \wedge R(\Omega) \) which is invariant under local translations only when the torsion vanishes.

\(^5\)The existence of this extension is a unique feature of the two-dimensional case since, unlike the Galilean case, the Carroll algebra does not admit a non-trivial central extension in four dimensions.
The action (2.23) admits a second-order formulation in the sector \( R_C(H) = de = 0 \). We refer to it as the electric extended Carroll gravity theory

\[
S_{\text{Elec. Ext. C}} = \kappa \int d^2x \det(\tau e) \phi \left( R_C + 2\Lambda_C \right),
\]

(2.27)

where we have defined

\[
R_C = 8e[^\mu~^\tau^\nu] \partial_\mu \left( e[^\alpha~^\tau^\beta] (\tau_\alpha \partial_\tau^\beta - e_\alpha \partial_\tau^m_\beta) \right).
\]

(2.28)

Unlike the relativistic case, we emphasize that the NR second-order actions (2.19) and (2.27) are not dynamically equivalent to their first-order versions. This is because the equivalence holds only in the \( d\tau = 0 \) and \( de = 0 \) sectors.

3 Electric and magnetic (extended) versions of non-relativistic and Carroll JT gravity

Recent results in non-Lorentzian field theories reveal that one can derive two different NR/Carroll field theories invariant under the same symmetry group \([21, 24]\). Following the ideas for the flat field theories of \([27]\), we show a way to obtain the so-called electric/magnetic gravity theories for NR and Carroll JT gravity. By considering the second-order formalism, the derivation of these gravity theories suggests the inclusion of Lagrange multipliers to constrain the geometry and matter.

3.1 Electric extended Newton-Hooke gravity theory

Let us start with the extended Carroll version of second-order JT gravity (2.27). We will now show how this Carroll gravity theory, which is invariant in the torsionless sector, can be modified to contain NH symmetry. The corresponding is given by

\[
S_{\text{Elec. Ext. NH}} = \kappa \int d^2x \left( \det(\tau e) \phi \left( R_C + 2\Lambda_C \right) + \partial_\mu \pi[^\mu[^\nu\mu]_\nu] a_\nu \right),
\]

(3.1)

where \( \pi[^\mu[^\nu\mu]_\nu] \equiv 4 \det(\tau e) \phi e[^\mu[^\nu\mu]) \), and \( a_\mu \) the auxiliary field enforcing the Newton-Hooke symmetry. This extra field is a convenient manner to extend the Carroll version of the second-order JT gravity to contain the Newton-Hooke symmetry. We call (3.1) the electric extended Newton-Hooke gravity theory. The variation of the action (2.27) under NH boosts yields a term proportional to an NH invariant piece. Due to this, one can also apply the procedure proposed in \([27]\). Indeed, by using the NH gauge transformations for the vector fields

\[
\delta_{\text{NH}} \tau^\mu = e^\mu \Theta - \left( e^\mu \partial_\nu \xi + e^\mu \omega_\nu \lambda + \tau^\mu \partial_\nu \lambda \right) \tau^\nu,
\]

(3.2a)

\[
\delta_{\text{NH}} e^\mu = -(e^\mu \partial_\nu \xi + e^\mu \omega_\nu \lambda + \tau^\mu \partial_\nu \lambda) e^\nu,
\]

(3.2b)

one can easily show that the term \( \pi[^\mu[^\nu\mu]_\nu] \) is an extended NH invariant. It is manifest that the field \( a_\mu \) acts as a Lagrange multiplier enforcing the following conditions for the zweibein \( e \), the clock one-form \( \tau \) and the scalar field \( \phi \)

\[
\partial_\mu \pi[^\mu[^\nu\mu]_\nu] = 0,
\]

(3.3)
which are obtained varying (3.1) with respect to $a_\nu$. One can verify that the last NR gravity theory (3.1) is invariant under boosts of the extended NH algebra if the auxiliary field $a_\mu$ transforms as
\begin{equation}
\delta_{\text{NH}} a_\mu = 2e^{[\alpha \tau \beta]}(\partial_\alpha m_\beta \tau_\mu + \partial_\alpha e_\beta e_\mu)\Theta - \tau^\alpha \partial_\alpha \Theta e_\mu.
\end{equation}

It is also possible to modify (2.19) to be Carroll invariant by including Lagrange multipliers. In this way, one arrives at
\begin{equation}
S_{\text{Mag. Ext. } C} = \kappa \int d^2 x \left( \det(e) \phi \left( R_{\text{NR}} - 2\Lambda_{\text{NR}} \right) - \partial_\mu \pi^{[\mu \nu]} b_\nu \right),
\end{equation}
where we have introduced the Lagrange multiplier $b_\mu$ transforming under boosts of the extended Carroll algebra as
\begin{equation}
\delta_{C} b_\mu = 2\tau^{[\alpha e \beta]}(\partial_\alpha m_\beta \tau_\mu + \partial_\alpha e_\beta e_\mu)\Theta - e^\alpha \partial_\alpha \Theta \tau_\mu.
\end{equation}
We call (3.5) the magnetic extended Carroll gravity theory. Note that using the isomorphism between extended NH and (A-)dS Carroll algebras, one can easily obtain (3.5) from (3.1) by interchanging $\tau \leftrightarrow e$. In the same way, the gauge transformations of the Lagrange multipliers $a_\mu$ and $b_\mu$ are related through the same mapping. This new action is invariant under the extended Carroll algebra.

4 Electric and magnetic non-relativistic and Carroll JT gravity

4.1 NH electric version

In (1 + 1)-spacetime dimensions, the Carroll algebra without a central extension admits a degenerate invariant bilinear form $\langle P, P \rangle = \gamma_0 \Lambda_C$. The BF theory that one can built from the two-dimensional Carroll algebra is
\begin{equation}
S_C[\rho, e] = \int \langle B, F \rangle = \gamma_0 \Lambda_C \int \rho R_C(P),
\end{equation}
with $B = \eta H + \rho P + \phi G, F = dA + A^2, A = \tau H + eP + \omega G$, with $H, P, G$ satisfying the commutation relations (2.22). As in the previous section, we can build an action invariant under the complete NH algebra from (4.1). Demanding this condition, it is necessary to introduce specific auxiliary fields ensuring that the last action satisfies this requirement. This action takes the form
\begin{equation}
S_{\text{NH}} = S_C + \gamma_0 \int \left( \phi R_{\text{NR}}(G) + \Lambda_{\text{NR}} \left( \alpha \eta + \beta R_{\text{NR}}(H) - \rho \omega \right) \right),
\end{equation}
where the NR curvatures $R_{\text{NR}}(H), R_{\text{NR}}(P),$ and $R_{\text{NR}}(G)$, are defined in eq. (2.14). The two-form $\alpha$ and the scalar $\beta$ are new auxiliary fields. The invariance of this new action under the unextended NH algebra requires choosing $\Lambda_C = -\Lambda_{\text{NR}}$, and the following symmetry transformations for the auxiliary fields
\begin{align}
\delta_{\text{NH}} \alpha &= \xi R_{\text{NR}}(G) - \Theta R_{\text{NR}}(P) \label{eq:delta_alpha} \\
\delta_{\text{NH}} \beta &= \xi \phi - \Theta \rho. \label{eq:delta_beta}
\end{align}
We stress that (4.3) and (4.4) are not the transformation laws of a BF gauge theory. They imply that $\alpha$ and $\beta$ are not part of a gauge connection that allows the construction of a gauge theory.

Unlike in the previous section, for Carroll gravities without the field $m$, we cannot solve algebraically for $\omega$ in terms of the remaining fields and their derivatives since the field equations of (4.2) do not fix a specific component of $\omega$. Then, we can solve partially and analytically the spin connection $\omega$ from the field equation $R_{\text{NH}}(P) = 0$ by projecting along the $\tau$ direction; we get

$$\omega_\mu = \omega_{\perp \mu} + \omega_{\parallel \mu}, \quad (4.5)$$

where

$$\omega_{\perp \mu} = \left(2\tau^{[\alpha} e^{\beta]} \partial_\alpha e_\beta\right) e_\mu, \quad (4.6)$$

which is clearly orthogonal to $\tau^\mu$ since $e_\mu \tau^\mu = 0$. In the second order formulation, we will interpret the indeterminate part $\omega_{\parallel \mu}$ as a Lagrange multiplier [23]. In the torsionless sector $d\tau = 0$ with $\eta = 0$, and integrating by parts, we find the electric NH gravity theory

$$S_{\text{Elec. NH}} = \hat{S}_C + \kappa \int d^2 x \, \partial_\mu \pi^{[\mu\nu]} \omega_{\parallel \nu}, \quad (4.7)$$

with

$$\hat{S}_C = \kappa \int d^2 x \, \det(\tau e) \phi \left(\hat{R}_C + 2\Lambda_C\right), \quad (4.8)$$

where we introduced the Carroll-invariant Ricci scalar $\hat{R}_C$, written in terms of $\omega_{\perp \mu}$

$$\hat{R}_C := 4\tau^{[\mu} e^{\nu]} \partial_\mu \omega_{\perp \nu} = 8\tau^{[\mu} e^{\nu]} \partial_\mu \left(\tau^{[\alpha} e^{\beta]} \partial_\alpha e_\beta e_\nu\right). \quad (4.9)$$

The action (4.7) written in this form allows for efficiently checking the invariance of each term. In fact, the first term $\hat{S}_C$ is Carroll-boost invariant. We refer to (4.8) as the electric Carroll gravity action. This theory coincides with (2.19) for $m_\mu = 0$ and identifying $\Lambda_C = -\Lambda_{\text{NR}}$. Therefore, there exists an equivalence between magnetic extended NH theory and electric Carroll theory when the extra $\text{U}(1)$ field vanishes.

On the other hand, as in the previous section, the term $\pi^{[\mu\nu]} = 4 \det(\tau e) e^{[\mu} \tau^{\nu]}$ is Carroll-boost invariant. The invariance of $S_{\text{Elec. NH}}$ under the NH symmetries implies to that the indeterminate part $\omega_{\parallel \mu}$ of the spin connection transforms as

$$\delta_{\text{NH}} \omega_{\parallel \mu} = e^\alpha e_\mu \partial_\alpha \Theta - 2\tau^{[\alpha} e^{\beta]} (e_\mu \partial_\alpha \tau_\beta + \tau_\mu \partial_\alpha e_\beta) \Theta. \quad (4.10)$$

We can interpret $\omega_{\parallel \mu}$ as a Lagrange multiplier constraining the geometry and matter of the theory. The variation of the action $S_{\text{Elec. NH}}$ with respect to $\omega_{\parallel \mu}$ implies the constraint (3.3) previously obtained.

### 4.2 Carroll magnetic version

An analogue procedure can be followed to obtain the magnetic version of Carroll gravity theory via the mapping between NH and Carroll algebras. Starting with the BF theory based on the degenerate bilinear form $\langle H, H \rangle = -\gamma_0 \Lambda_{\text{NR}}$ of the NH algebra, a new Carroll-invariant theory can be constructed.
The magnetic Carroll gravity theory, obtained by interchanging $\tau$ and $e$ in (4.7) is given by

$$S_{\text{Mag}, C} = S_{\text{NH}} - \kappa \int d^2x \, \partial_\mu \pi^{[\mu\nu]} \omega_{\nu|\nu}, \quad (4.11)$$

where the magnetic NH gravity theory is defined as

$$S_{\text{NH}} = \kappa \int d^2x \, \det(\tau e) \phi(\bar{\mathcal{R}}_{\text{NR}} - 2\Lambda_{\text{NR}}), \quad (4.12)$$

and the NH invariant Ricci scalar is

$$\bar{\mathcal{R}}_{\text{NR}} := 8e^{[\mu\tau\nu]}\partial_\mu(e^{[\alpha\tau\beta]}\partial_\alpha\tau_\beta\tau_\nu). \quad (4.13)$$

The magnetic terminology for $S_{\text{NH}}$ is justified because it does not present extra fields. Again, we find a coincidence between the electric extended Carroll theory and magnetic NH theory for $m_\mu = 0$ and $\Lambda_{\text{NR}} = -\Lambda_C$. For this case, the invariance of (4.11) under Carroll symmetry implies the following transformation for the indeterminate part of the spin connection (orthogonal to $\tau^\mu$)

$$\delta_{\text{C}}\omega_{||\mu} = \tau^\alpha\tau_\mu\partial_\alpha\Theta - 2e^{[\alpha\tau\beta]}(\tau_\mu\partial_\alpha e_\beta + e_\mu\partial_\alpha\tau_\beta)\Theta. \quad (4.14)$$

Let us mention that the symmetry transformations of the gravity theories found here do not correspond to the ones found in the previous section imposing $m_\mu = 0$.

5 Conclusions and outlook

The mapping between non-relativistic and Carroll field theories [27] has been extended to the case of JT gravity. Here, we have shown that the second order non-relativistic and Carroll JT gravity (depending on the vielbein variables $\tau, e$, and the $U(1)$ vector field $m$ introduced in [1]) can be deformed with two more theories introducing Lagrange multipliers that restrict the dynamics of the previous theories. However, they have the same symmetry algebras; the only difference with the gravity theories introduced in [1], is the transformation law for the Lagrange multipliers. The basis of this construction is due to the isomorphism between NH and the Carroll algebras.

It will be interesting to know the number of physical degrees of freedom of the eight theories we have constructed.

Our results suggest a deep connection between the physics of non-relativistic and Carroll gravities at least in $(1 + 1)$-dimensions. The extension of our results to the gravity theories in more dimensions is an interesting path to follow. The applications to other physical theories would also be useful.

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A Symmetry of electric Newton-Hooke JT gravity

In this appendix, we explicitly check the invariance of the action (4.7) under the extended Newton-Hooke boosts. To do this, we should note that $\hat{S}_C$ can be rewritten, up to boundary terms, as

$$
\hat{S}_C = \kappa \int d^2 x \det(\tau e) \phi \left( \hat{R}_C + 2\Lambda_C \right)
= \kappa \int d^2 x \det(\tau e) \phi \left( 4\pi^{[\mu}_\nu] \partial_\mu \omega_{\bot \nu} + 2\Lambda_C \right)
= \kappa \int d^2 x \partial_\mu \pi^{[\mu}_\nu] \omega_{\bot \nu} + 2\Lambda_C \kappa \int d^2 x \det(\tau e) \phi.
$$

(A.1)

Then, the invariance of $\hat{S}_C$ is seen as follows: the second term in the last equation is invariant under NH boost symmetry ($\delta_{\text{NH}}$) since $\delta_{\text{NH}} \phi = 0$ and $\delta_{\text{NH}} \det(\tau e) = 0$. Also, from the transformation (3.2) and $\delta_{\text{NH}} \phi = 0$ one can show

$$
\delta_{\text{NH}} \left( \partial_\mu \pi^{[\mu}_\nu \right) = 0.
$$

(A.2)

So, under NH boosts of $\hat{S}_C$ transforms as

$$
\delta_{\text{NH}} \hat{S}_C = \kappa \int d^2 x \partial_\mu \pi^{[\mu}_\nu \delta_{\text{NH}} \omega_{\bot \nu}.
$$

(A.3)

Therefore, the invariance of the action (4.7) yields

$$
\delta_{\text{NH}} S_{\text{Elec. NH}} = \kappa \int d^2 x \partial_\mu \pi^{[\mu}_\nu \delta_{\text{NH}} \omega_{\bot \nu} + \kappa \int d^2 x \partial_\mu \pi^{[\mu}_\nu \delta_{\text{NH}} \omega_{\| \nu},
$$

which is satisfied if

$$
\delta_{\text{NH}} \omega_{\bot} = -\delta_{\text{NH}} \omega_{\|}.
$$

(A.4)

The same procedure can be applied to the other gravity theories.

B Equivalence between first and second order formalism for JT gravity

In this appendix, we show an explicit derivation of the equivalence between the first and second order formulation for JT gravity. Let us start considering the BF action

$$
S[B, A] = \int \langle B, F \rangle = \bar{\gamma}_0 \int \left( \Phi R(\hat{J}) - \hat{\Lambda} \Phi R^a(\hat{P}) \right),
$$

(B.1)
whose field equations are given by
\[
\begin{align*}
\delta \Phi^a & : \quad R^a(\tilde{P}) = 0, \quad (B.2a) \\
\delta \Phi & : \quad R(\tilde{J}) = 0, \quad (B.2b) \\
\delta E^a & : \quad d\Phi^a - \epsilon^a_b \left( \Phi E^b + \Omega \Phi^b \right) = 0, \quad (B.2c) \\
\delta \Omega & : \quad d\Phi - \tilde{\Lambda} \epsilon_{ab} E^a \Phi^b = 0. \quad (B.2d)
\end{align*}
\]

From equation (B.2a) we can solve the spin connection in terms of the zweibein as
\[
\Omega^\mu(E) = -E^{-1} \epsilon^{\alpha\beta} \partial_\alpha E^a_\mu E_a^b, \quad (B.3)
\]
where
\[
E \equiv \det(E^a_\mu) = -\frac{1}{2} \epsilon^{\mu\nu} \epsilon_{ab} E^a_\mu E^b_\nu. \quad (B.4)
\]
Using this result, one can evaluate the Riemann tensor
\[
\mathcal{R}(E)^a_{b\rho\sigma} = E^a_\mu E^b_\nu \mathcal{R}(E)^\mu_{\rho\sigma} = -\epsilon^a_b \left( \partial_\rho \Omega(E) - \partial_\sigma \Omega(E) \right), \quad (B.5)
\]
which indicates that the field equation (B.2b) can be rewritten in terms of the Ricci scalar
\[
\mathcal{R}_g = g^{\mu\nu} \mathcal{R}_\mu^\rho \rho^\nu \quad \text{as}
\]
\[
\epsilon^{\mu\nu} \left( \partial_\mu \Omega(E) - \frac{\tilde{\Lambda}}{2} \epsilon_{ab} E^a_\mu E^b_\nu \right) = -\frac{E}{2} \left( \mathcal{R}(E) - 2\tilde{\Lambda} \right) = 0, \quad (B.6)
\]
and reproduces the field equation
\[
\mathcal{R}_g - 2\tilde{\Lambda} = 0, \quad (B.7)
\]
with \(\mathcal{R}_g\) stands for the Ricci scalar constructed with the metric \(g_{\mu\nu}\) and Christoffel symbols. The Ricci scalars \(\mathcal{R}(E)\) and \(\mathcal{R}_g\) are equivalent when one uses the formula \(g_{\mu\nu} = E^a_\mu E^b_\nu \eta_{ab}\). This last formula corresponds to one of the field equations of JT gravity in the second order formalism.

On the other hand, from (B.2d) we can express \(\Phi^a\) in terms of \(\Phi\) as
\[
E^a_\mu \Phi^a = \frac{E^{-1}}{2} \epsilon^{\mu\nu} \partial_\nu \Phi. \quad (B.8)
\]
Using this equation in (B.2d) yields
\[
\nabla_\mu \nabla_\nu \Phi - \tilde{\Lambda} g_{\mu\nu} \Phi = 0, \quad (B.9)
\]
where the space-time metric is defined in the usual way. The formula (B.9) corresponds to the field equation for the scalar field \(\Phi\) of JT gravity in the second order formalism.

Finally, using (B.2a) and (B.6), the BF action (2.8) takes the form of the JT action
\[
S = -\frac{\tilde{\gamma}_0}{2} \int d^2x \Phi(\mathcal{R}(E) - 2\tilde{\Lambda}). \quad (B.10)
\]
Therefore, this derivation demonstrates the equivalence between the first and second order formulations of JT gravity.
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