Infinitely many solutions for a class of sublinear fractional Schrödinger equations with indefinite potentials

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Abstract
In this paper, we consider the following sublinear fractional Schrödinger equation:

\[ (-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N, \]

where \( s, p \in (0, 1) \), \( N > 2s \), \((-\Delta)^s\) is a fractional Laplacian operator, and \( K, V \) both change sign in \( \mathbb{R}^N \). We prove that the problem has infinitely many solutions under appropriate assumptions on \( K, V \). The tool used in this paper is the symmetric mountain pass theorem.

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1 Introduction and main result
In this paper, we consider the following sublinear fractional Schrödinger equation:

\[ (-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N, \] (1.1)

where \( s, p \in (0, 1) \), \( N > 2s \), \((-\Delta)^s\) is a fractional Laplacian operator, \( K, V \) both change sign in \( \mathbb{R}^N \) and satisfy some conditions specified below.

Problem (1.1) gives the following nonlinear field equation:

\[ \frac{i \partial \psi}{\partial t} = (-\Delta)^s \psi + (1 + E)\psi - K(x)|\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N, t \in \mathbb{R}^+. \] (1.2)

The nonlinear field Eq. (1.2) reflects the stable diffusion process of Lévy particles in random field. Later, people found that this stable diffusion of Lévy process has also a very important application in the mechanical system, flame propagation, chemical reactions in the liquid, and the anomalous diffusion of physics in the plasma. For more details, readers can refer to [5, 25, 26, 45] and the references therein.
Problem (1.1) involves the fractional Laplacian $(-\Delta)^s$, which is a nonlocal operator. After this question was raised, it immediately aroused the interest of mathematicians (see [1, 4, 6–14, 16–22, 24, 27–29, 31, 33–44, 46–55] and the references therein).

For fractional equations on the whole space $\mathbb{R}^N$, the main difficulty one may face is that the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is not compact for $q \in [2, 2^*_s)$. To overcome this difficulty, some authors [8, 10, 24, 31, 38, 50] considered fractional equations with the potential $V(x)$.

Due to condition (V), the subspace of $H^s(\mathbb{R}^N)$ embeds compactly into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_s)$, which is crucial in their paper. In fact, condition (V) is certain coercive condition. In the case of coercive condition $\lim_{|v| \to +\infty} V(x) = +\infty$, some authors, for example [12, 33], considered fractional equations on the whole space $\mathbb{R}^N$.

To overcome the difficulties caused by the lack of compactness, on the other hand, some authors restricted the energy functional to a subspace for $H^s(\mathbb{R}^N)$ of radially symmetric functions, which embeds compactly into $L^q(\mathbb{R}^N)$, for example, [9, 21, 34, 44, 54].

However, in this paper, we do not need some conditions like (V) or radially symmetric. That is, our paper does not use any compact embedding on the whole space $\mathbb{R}^N$.

It is worth noting that, for fractional equations on the whole space $\mathbb{R}^N$, most results need condition $V(x) \geq 0$ (see [1, 8–10, 12, 13, 16, 18, 20–22, 24, 28, 33, 34, 36–38, 44, 50, 52–54], in which some results were obtained in case of $V(x) = 1$ [16, 18, 21, 28, 44]). To the best of our knowledge, there are few results on the existence of solutions for fractional equations with a sign-changing potential except [11, 51]. In fact, replaced $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$ with $\inf_{x \in \mathbb{R}^N} V(x) > -\infty$, condition similar to (V) is needed in [11]. In [51], Xu, Wei, and Dong considered the following $p$-Laplacian equation with positive nonlinearity:

$$(-\Delta)^s u + V(x)|u|^{p-2}u - \lambda |u|^{p-2}u = f(x, u) + g(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N,$$

where $N, p \geq 2, s \in (0, 1)$, $\lambda$ is a parameter, $(-\Delta)^s$ is the fractional $p$-Laplacian, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. In the case of $\lambda = 0$, they obtained the existence of a nontrivial solution to this equation. Furthermore, they proved that this equation has infinitely many nontrivial solutions when $\lambda \leq 0$ or $\lambda > 0$ is small enough.

In this article, we are interested in the existence of infinitely many solutions for problem (1.1) with potential function $V(x)$ changing sign in $\mathbb{R}^N$. Moreover, nonlinearity can be allowed to change sign. To state our main result, we assume the following:

(V1) $V \in L^\infty(\mathbb{R}^N)$ and there exist $\alpha, R_0 > 0$ such that

$$V(x) \geq \alpha, \quad \forall |x| \geq R_0.$$

(V2) $\|V^+\|_S < \frac{1}{2}$, where $V^\pm(x) = \max(\pm V(x), 0)$ and $S$ is the constant of Sobolev:

$$\|u\|_{Q_2}^2 \leq S\|u\|^2_{H^s(\mathbb{R}^N)}, \quad \forall u \in H^s(\mathbb{R}^N),$$

where $Q_2 = \frac{2N}{N - 2s}$.

(K) $K \in L^\infty(\mathbb{R}^N)$ and there exist $\beta > 0$, $R_1 > R_2 > 0$, $y_0 = (y_1, \ldots, y_N) \in \mathbb{R}^N$ such that

$$K(x) \leq -\beta, \quad \forall |x| > R_1; \quad K(x) > 0, \quad \forall x \in B(y_0, R_2) \subset B(0, R_1).$$
Our main result of this paper can be stated as follows.

**Theorem 1.1** Assume \((V_1)-(V_2)\) and \((K)\) hold. Then problem (1.1) possesses infinitely many nontrivial solutions.

**Remark 1.1** The ideas in this article come from the paper [3], where Schrödinger equations were considered. However, our proof is nontrivial since we present a simplified proof for the PS condition by comparing to that in [3]. In fact, the PS condition was proved in [3] by concentration compactness principle. It is noticed that the PS condition plays important role in the proof of the main results in [3].

### 2 Notations and preliminaries

In this paper, we use the following notations. Let

\[ \|u\|_q = \left( \int_{\mathbb{R}^N} |u|^q \, dx \right)^{\frac{1}{q}} , \quad 1 \leq q < +\infty. \]

Let \(E\) be a Banach space and \(\psi : E \rightarrow \mathbb{R}\) be a functional of class \(C^1\). The Fréchet derivative of \(\psi\) at \(u\), \(\psi'(u)\) is an element of the dual space \(E^*\), and we denote \(\psi'(u)\) evaluated at \(v \in E\) by \(\langle \psi'(u), v \rangle\).

Let \(s \in (0,1)\), the fractional Sobolev space \(H^s(\mathbb{R}^N)\) is defined by

\[ H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2} + s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\} \]

and endowed with the natural norm

\[ \|u\|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{\frac{1}{2}}, \]

here

\[ [u]_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{\frac{1}{2}} \]

is the so-called Gagliardo (semi) norm of \(u\).

Using Fourier transform, the space \(H^s(\mathbb{R}^N)\) can also be defined by

\[ H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left( 1 + |\xi|^{2s} \right)|\mathcal{F} u|^2 \, d\xi < +\infty \right\}, \]

where \(\mathcal{F} u\) denotes the Fourier transform of \(u\).

Let \(\ell\) be the Schwartz space of rapidly decreasing \(C^\infty\) function on \(\mathbb{R}^N\), \(u \in \ell\), one has

\[ (-\triangle)^s u(x) = C(N,s) \text{PV} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} \, dy, \]

the symbol \(\text{PV}\) stands for the Cauchy value, and \(C(N,s)\) is a constant dependent only on the space dimension \(N\) and the order \(s\).
From the results of [15], we have

\[ (-\Delta)^{s} u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F} u)) \quad \text{for any } \xi \in \mathbb{R}^{N}. \]

Then, by Proposition 3.4 and Proposition 3.6 of [15], we have

\[
[u]_{H^s}^2 = \frac{2}{C(N,s)} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u|^2 \, d\xi = \frac{2}{C(N,s)} \|(-\Delta)^{s/2} u\|_{2^*}^2.
\]

From the above facts, the norms on $H^s(\mathbb{R}^N)$ defined as follows

\[
\|u\| \mapsto \left(\|u\|_{2}^2 + \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u|^2 \, d\xi\right)^{\frac{1}{2}},
\]

\[
u \mapsto \left(\|\nu\|_{2}^2 + \|(-\Delta)^{s/2} \nu\|_{2}^2\right)^{\frac{1}{2}},
\]

\[
\|u\| \mapsto \|u\|_{H^s(\mathbb{R}^N)}
\]

are all equivalent.

**Lemma 2.1** ([15, 30, 34]) Let $0 < s < 1$ such that $2s < N$. Then there exists $C = C(n,s)$ such that

\[
\|u\|_{2^*} \leq C \|u\|_{H^s(\mathbb{R}^N)}
\]

for every $u \in H^s(\mathbb{R}^N)$. Moreover, the embedding $H^s(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2^*]$ and locally compact whenever $p \in [2, 2^*)$.

Let the homogeneous Sobolev space

\[
H^s_0(\mathbb{R}^N) = \{u \in L^{2s}(\mathbb{R}^N) : |\xi|^s \mathcal{F} u \in L^2(\mathbb{R}^N)\}.
\]

This space can be equivalently defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

\[
\|u\|_{H^s_0(\mathbb{R}^N)}^2 \triangleq \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u|^2 \, d\xi.
\]

The Sobolev space $E = H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ is endowed with the norm

\[
\|u\| = \|u\|_0 + \|u\|_{p+1}.
\]

Obviously, $E$ is a reflexive Banach space.

The energy functional $\varphi : E \to \mathbb{R}$ corresponding to problem (1.1) is defined by

\[
\varphi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u|^2 \, d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x) |u|^{p+1} \, dx.
\]

Under our conditions, $\varphi \in C^1(E)$ and its critical points are solutions of problem (1.1).
That is, there exists a sequence $u_k$ such that $0 \not\in A$ and $\gamma(A) \geq k$.

The following result is a version of the classical symmetric mountain pass theorem [2,32]. For the proof, please see [23].

**Theorem 2.1** ([23]) Let $E$ be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy:

(I₁) $I$ is even, bounded from below, $I(0) = 0$, and $I$ satisfies the Palais–Smale condition.

(I₂) For each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that

$$\sup_{u \in A_k} I(u) < 0.$$ 

Then either of the following two conditions holds:

(i) there exists a sequence $u_k$ such that $I(u_k) = 0, I(u_k) < 0$ and $u_k$ converges to zero; or

(ii) there exist two sequences $u_k$ and $v_k$ such that $I(u_k) = 0, I(u_k) = 0, u_k \neq 0$,

$$\lim_{k \to \infty} u_k = 0, I(v_k) = 0, I(v_k) < 0, \lim_{k \to \infty} I(v_k) = 0 \text{ and } v_k \text{ converges to a non-zero limit.}$$

**3 Proof of Theorem 1.1**

**Lemma 3.1** Suppose that $(V₁)$–$(V₂)$ and $(K)$ hold. Then any PS sequence of $\varphi$ is bounded in $E$.

**Proof** Let $(u_n) \subset E$ be such that

$$\varphi(u_n) \text{ is bounded } \quad \text{ and } \quad \varphi'(u_n) \to 0 \quad \text{ as } n \to \infty.$$ 

That is, there exists $C > 0$ such that $\varphi(u_n) \leq C$. So, according to Hölder’s inequality and Sobolev’s inequality, one has that

$$C \geq \varphi(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2^*} |\mathcal{F} u_n|^2 \, d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_n^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2^*} |\mathcal{F} u_n|^2 \, d\xi - \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_n^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} K^+(x) |u_n|^{p+1} \, dx$$

$$\geq \frac{1}{2} \|u_n\|_0^2 - \frac{1}{2} \left( \int_{\mathbb{R}^N} V^- \frac{N}{2} \, dx \right)^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} (|u_n|^2)^{\frac{2^*}{2}} \, dx \right)^{\frac{2}{2^*}}$$

$$- \frac{1}{p + 1} \int_{\mathbb{R}^N} K^+(x) |u_n|^{p+1} \, dx$$

$$\geq \left( \frac{1}{2} - \frac{S}{2} \left\| V^- \right\|_N \right) \|u_n\|_0^2 - \frac{S^{p+1}}{p + 1} \left\| K^+ \right\|_N^{\frac{2^*}{2(p+1)}} \|u_n\|_0^{p+1}.$$
Since $0 < p < 1$, there exists $\eta > 0$ such that
\[
\|u_n\|_0^2 \leq \eta, \quad \forall n \in \mathbb{N}. \quad (3.1)
\]

On the other hand, we have that
\[
C + \frac{\|u_n\|}{2} \geq \psi(u_n) - \frac{1}{2}\langle \psi'(u_n), u_n \rangle \\
\geq \left( \frac{1}{2} - \frac{1}{p + 1} \right) \int_{\mathbb{R}^N} K(x)|u_n|^{p+1} \, dx \\
= \left( \frac{1}{2} - \frac{1}{p + 1} \right) \int_{\mathbb{R}^N} K^+(x)|u_n|^{p+1} \, dx + \left( \frac{1}{p + 1} \right) \int_{\mathbb{R}^N} K^-(x)|u_n|^{p+1} \, dx \\
= \left( \frac{1}{2} - \frac{1}{p + 1} \right) \int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))|u_n|^{p+1} \, dx \\
+ \left( \frac{1}{p + 1} - \frac{1}{2} \right) \int_{\mathbb{R}^N} (K^-(x) + \chi_{B(0,R_1)}(x))|u_n|^{p+1} \, dx,
\]
where $\| \cdot \|$ denotes the norm in $E$.

Thanks to $(K)$, we have that
\[
K^+(x) = 0 \quad \text{for all } |x| > R_1.
\]

Then, by $K \in L^\infty(\mathbb{R}^N)$, we get
\[
\int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x)) \left| \frac{2}{2 - (p+1)} \right| \, dx = \int_{B(0,R_1)} (K^+ + \chi_{B(0,R_1)}(x)) \left| \frac{2}{2 - (p+1)} \right| < \infty.
\]

Hence, by Hölder’s inequality and Sobolev’s inequality, we have that
\[
\int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))|u_n|^{p+1} \, dx \\
\leq \left( \int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x)) \left| \frac{2}{2 - (p+1)} \right| \, dx \right)^{\frac{2}{2 - (p+1)}} \times \left( \int_{\mathbb{R}^N} (|u_n|^{p+1}) \left| \frac{2}{2 - (p+1)} \right| \, dx \right)^{\frac{2 - (p+1)}{2}} \\
\leq \frac{S_{2(x)}}{2} \| K^+ + \chi_{B(0,R_1)} \|_{L^\infty} \| u_n \|_0^{p+1}. \quad (3.2)
\]

Using $(K)$ again, we know that $K^-(x) \geq \beta$ for all $|x| > R_1$. Then we have that
\[
\int_{\mathbb{R}^N} (K^-(x) + \chi_{B(0,R_1)}(x))|u_n|^{p+1} \, dx \geq \min(\beta, 1) \| u_n \|_0^{p+1}. \quad (3.3)
\]

According to (3.1), (3.2), and (3.3), there exists a constant $C_1 > 0$ such that
\[
\| u_n \|_0^{p+1} \leq C_1 + C_1 \| u_n \|_0^{p+1} \quad \text{for all } n \in \mathbb{N}.
\]

Since $0 < p < 1$, there exists a constant $C_2 > 0$ such that
\[
\| u_n \|_p^{p+1} \leq C_2, \quad \forall n \in \mathbb{N}. \quad (3.4)
\]
Hence, it follows from (3.1) and (3.4) that \( \{u_n\} \) is bounded in \( E \). \( \square \)

**Lemma 3.2** Suppose that \((V_1)\)–\((V_2)\) and \((K)\) hold. Then \( \varphi \) satisfies the PS condition on \( E \).

**Proof** Let \( \{u_n\} \subset E \) be such that

\[
\varphi(u_n) \text{ is bounded} \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{as} \ n \to \infty.
\]

By Lemma 3.1, \( \{u_n\} \) is bounded in \( E \). Going if necessary to a subsequence, from Lemma 2.1 we can assume that

\[
u_n \to u \quad \text{in} \quad E; \quad u_n \to u \quad \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^N), \quad 2 \leq q < 2^*_s; \quad u_n \to u \text{ a.e in } \mathbb{R}^N. \tag{3.5}
\]

So, \( \forall \psi \in C_0^\infty(\mathbb{R}^N) \), we have

\[
\int_{\mathbb{R}^N} |\xi|^{2s} F u_n F \psi \, d\xi + \int_{\mathbb{R}^N} V(x) u_n \psi \, dx \to \int_{\mathbb{R}^N} |\xi|^{2s} F u F \psi \, d\xi + \int_{\mathbb{R}^N} V(x) u \psi \, dx.
\]

By \( u_n \to u \) in \( L^{p+1}(\text{supp}(\psi)) \) \([15, 30]\) and Lebesgue’s dominated convergence theorem, one has that

\[
\int_{\mathbb{R}^N} K(x) |u_n|^{p-1} u_n \psi \, dx \to \int_{\mathbb{R}^N} K(x) |u|^{p-1} u \psi \, dx.
\]

Hence, we have

\[
0 = \lim_{n \to +\infty} \langle \varphi'(u_n), \psi \rangle = \langle \varphi'(u), \psi \rangle, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N).
\]

Then

\[
\langle \varphi'(u), u \rangle = 0.
\]

Let \( v_n = u_n - u \), then \( u_n = v_n + u \), we have that

\[
\langle \varphi'(u_n), u_n \rangle = \int_{\mathbb{R}^N} |\xi|^{2s} (F u_n)^2 \, d\xi + \int_{\mathbb{R}^N} V(x) u_n^2 \, dx - \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx
\]

\[
= \int_{\mathbb{R}^N} |\xi|^{2s} (|F v_n|^2 + |F u|^2 + 2F v_n F u) \, d\xi
\]

\[
+ \int_{\mathbb{R}^N} (V(x)v_n^2 + V(x)u^2 + 2V(x)v_n u) \, dx
\]

\[
- \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx + \int_{\mathbb{R}^N} K(x) |u|^{p+1} \, dx - \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx
\]

\[
= \langle \varphi'(u), u \rangle + \int_{\mathbb{R}^N} |\xi|^{2s} |F v_n|^2 \, d\xi + \int_{\mathbb{R}^N} V(x)v_n^2 \, dx
\]

\[
- \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx + \int_{\mathbb{R}^N} K(x) |u|^{p+1} \, dx + o_n(1)
\]

\[
\geq \int_{\mathbb{R}^N} |\xi|^{2s} |F v_n|^2 \, d\xi - \int_{\mathbb{R}^N} V^{-}(x)v_n^2 \, dx
\]

\[
- \int_{\mathbb{R}^N} K(x) (|u_n|^{p+1} - |u|^{p+1}) \, dx + o_n(1).
\]
Thanks to (3.5) and Lemma 4.2 in [3], we have that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x) \left[ |u_n|^{p+1} - |u|^{|p+1}| \right] dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x) |v_n|^{p+1} dx.
\]
So, we have that
\[
\langle \phi'(u_n), u_n \rangle \geq \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{F} v_n|^2 d\xi - \int_{\mathbb{R}^N} V^-(x)v_n^2 dx \\
- \int_{\mathbb{R}^N} K(x)|v_n|^{p+1} dx + o_n(1)
\]
\[
= \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{F} v_n|^2 d\xi - \int_{\mathbb{R}^N} V^-(x)v_n^2 dx \\
- \int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))|v_n|^{p+1} dx \\
+ \int_{\mathbb{R}^N} (K^-(x) + \chi_{B(0,R_1)}(x))|v_n|^{p+1} dx + o_n(1). \tag{3.6}
\]

**Claim 1** \( \int_{\mathbb{R}^N} V^-(x)v_n^2 dx \to 0 \) as \( n \to +\infty \).

In fact, by \((V_1)\), we have that \( V^-(x) = 0 \) for all \( |x| \geq R_0 \). So, from \( v_n \to 0 \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \), \( 2 \leq q < 2^*_s \), and \( V \in L^\infty(\mathbb{R}^N) \), we obtain \( \int_{\mathbb{R}^N} V^-(x)v_n^2 dx \to 0 \) as \( n \to +\infty \).

**Claim 2** \( \int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))|v_n|^{p+1} dx \to 0 \) as \( n \to +\infty \).

In fact, thanks to \((K)\), we have that \( K^+(x) = 0 \) for all \( |x| > R_1 \). So, by \( K \in L^\infty(\mathbb{R}^N) \) and \( v_n \to 0 \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \), \( 2 \leq q < 2^*_s \), we get
\[
\int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))|v_n|^{p+1} dx \to 0
\]
as \( n \to +\infty \).

From Claim 1, Claim 2, (3.3), and (3.6), we obtain that
\[
0 = \lim_{n \to +\infty} \left( \|v_n\|_0^2 + \min(\beta, 1)\|v_n\|_{p+1}^{p+1} \right).
\]
That is, \( v_n \to 0 \) in \( E \). The proof is complete. \( \square \)

**Lemma 3.3** Assume that \((V_1)–(V_2)\) and \((K)\) hold. Then, for each \( k \in \mathbb{N} \), there exists \( A_k \in \Gamma_k \) such that
\[
\sup_{u \in A_k} \psi(u) < 0.
\]

**Proof** The proof is based on some ideas of Kajikiya [23] and is very similar to the one contained in [3]. For readers' convenience, we give the proof. Let \( R_2 \) and \( y_0 \) be fixed as in \((K)\) and denote
\[
D(R_2) = \{ (x_1, \ldots, x_n) \in \mathbb{R}^N : |x_i - y_i| < R_2, 1 \leq i \leq N \}.
\]
Let $k \in \mathbb{N}$ be an arbitrary number and define $n = \min \{n \in \mathbb{N} : n^N \geq k \}$. By planes parallel to each face of $D(R_2)$, let $D(R_2)$ be equally divided into $n^N$ small parts $D_i$ with $1 \leq i \leq n^N$. In fact, the length $a$ of the edge $D_i$ is $\frac{D_i}{n}$. Let $F_i \subset D_i$ be new cubes such that $F_i$ has the same center as that of $D_i$. The faces of $F_i$ and $D_i$ are parallel, and the length of the edge of $F_i$ is $\frac{a}{2}$. Let $\phi_i, 1 \leq i \leq k$, satisfy: $\text{supp}(\phi_i) \subset D_i$; $\text{supp}(\phi_i) \cap \text{supp}(\phi_j) = \emptyset (i \neq j)$; $\phi_i(x) = 1$ for $x \in F_i$; $0 \leq \phi_i(x) \leq 1$, for all $x \in \mathbb{R}^N$. Let

\begin{align}
S^{k-1} &= \left\{(t_1, \ldots, t_k) \in \mathbb{R}^k : \max_{1 \leq i \leq k} |t_i| = 1 \right\}, \\
W_k &= \left\{ \sum_{i=1}^{k} t_i \phi_i(x) : (t_1, \ldots, t_k) \in S^{k-1} \right\} \subset E.
\end{align}

According to the fact that the mapping $(t_1, \ldots, t_k) \rightarrow \sum_{i=1}^{k} t_i \phi_i$ from $S^{k-1}$ to $W_k$ is odd and homeomorphic, so $\gamma(W_k) = \gamma(S^{k-1}) = k$. Since $W_k$ is compact in $E$, then $\exists \alpha_k > 0$ such that

$$
\|u\|^2 \leq \alpha_k, \quad \forall u \in W_k.
$$

On the other hand, by Hölder’s inequality and Sobolev’s embedding, we have that

$$
\|u\|_2 \leq c\|u\|^r_\infty \leq c\|u\|,
$$

where $r = \frac{2(1-p)}{2(p-1)}$.

According to the above facts, there exists $c_k > 0$ such that

$$
\|u\|^2 \leq c_k \quad \text{for all } u \in W_k.
$$

Let $t > 0$ and $v = \sum_{i=1}^{k} t_i \phi_i(x) \in W_k$.

\begin{align}
\varphi(tu) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\xi|^{2|\alpha|} |\mathcal{F} u|^2 d\xi + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{p+1} \sum_{i=1}^{k} \int_{D_i} K(x)|tt_i \phi_i|^{p+1} dx \\
&\leq \frac{t^2}{2} \alpha_k + \frac{t^2}{2} \|V\|_\infty c_k - \frac{1}{p+1} \sum_{i=1}^{k} \int_{D_i} K(x)|tt_i \phi_i|^{p+1} dx.
\end{align}

From (3.7), there exists $j \in [1, k]$ such that $|t_j| = 1$ and $|t_i| \leq 1$ for $i \neq j$. So

\begin{align}
\sum_{i=1}^{k} \int_{D_i} K(x)|tt_i \phi_i|^{p+1} dx &= \int_{F_j} K(x)|tt_j \phi_j|^{p+1} dx \\
+ \int_{D_j \setminus F_j} K(x)|tt_j \phi_j(x)|^{p+1} dx + \sum_{i \neq j} \int_{D_i} K(x)|tt_i \phi_i|^{p+1} dx.
\end{align}

According to $\phi_j(x) = 1$ for $x \in F_j$ and $|t_j| = 1$, one has that

$$
\int_{F_j} K(x)|tt_j \phi_j|^{p+1} dx = |t|^{p+1} \int_{F_j} K(x) dx.
$$
By \((K)\), one has that
\[
\int_{D_j \setminus F_j} K(x)|tt_j \phi_j(x)|^{p+1} \, dx + \sum_{i \neq j} \int_{D_i} K(x)|tt_i \phi_i|^{p+1} \, dx \geq 0. \tag{3.11}
\]

According to (3.8), (3.9), (3.10), and (3.11), we have that
\[
\frac{\varphi(tu)}{t^2} \leq \frac{1}{2} \alpha_k + \frac{1}{2} \|V\|_{\infty} \epsilon_k - \frac{|t|^{p+1}}{(p+1)t^2} \inf_{1 \leq i \leq k} \left( \int_{D_i} K(x) \, dx \right).
\]

So,
\[
\lim_{t \to 0} \sup_{u \in W_k} \frac{\varphi(tu)}{t^2} = -\infty.
\]

Hence, we can fix \(t\) small enough such that \(\sup\{\varphi(u), u \in A_k\} < 0\), where \(A_k = tW_k \in \Gamma_k\). □

**Lemma 3.4** Assume that \((V_1)\)–\((V_2)\) and \((K)\) hold. Then \(\varphi\) is bounded from below.

**Proof** By \((K)\), Hölder’s inequality and Sobolev’s embedding, as in the proof of Lemma 3.1, we have that
\[
\varphi(u) = \frac{1}{2} \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u|^2 \, d\xi + \int_{\mathbb{R}^N} V(x)u^2 \, dx \right) - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x)|u|^{p+1} \, dx
\]
\[
\geq \frac{1}{2} \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u|^2 \, d\xi - \int_{\mathbb{R}^N} V^-(x)u^2 \, dx \right) - \frac{1}{p+1} \int_{\mathbb{R}^N} K^+(x)|u|^{p+1} \, dx
\]
\[
\geq \left( \frac{1}{2} - \frac{S^2}{2} \right) \|u\|_0^2 - \frac{S^{p+1}}{p+1} \|K^+\|_0 \|u\|_0^{p+1} \|u\|_0^{p+1}.
\]

Since \(0 < p < 1\), we conclude the proof. □

**Proof of Theorem 1.1** In fact, \(\varphi(0) = 0\) and \(\varphi\) is an even functional. Then by Lemmas 3.2, 3.3, and 3.4, conditions \((I_1)\) and \((I_2)\) of Theorem 2.1 are satisfied. Therefore, by Theorem 2.1, problem (1.1) possesses infinitely many nontrivial solutions converging to 0 with negative energy. □

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**Authors’ contributions**
All the authors have the same contribution. All authors read and approved the final manuscript.

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