On some quantum bounded symmetric domains

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Abstract

In the framework of quantum group theory we obtain a noncommutative analog for the algebra of functions in a bounded symmetric domain, endowed with a whole symmetry. Also we provide a construction for its faithfull irreducible representation and an invariant integral over the bounded symmetric domain.

1 Introduction

Recall some well-known facts on bounded symmetric domains. We focus on a series of bounded symmetric domains \( \mathbb{D} \) from the well-known Cartan list.

Let

\[
\mathfrak{a} = \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & -1 & 2 & -2 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2
\end{pmatrix}.
\]

The Lie algebra \( \mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C}) \) is isomorphic to the Lie algebra with generators \( e_i, f_i, h_i, i = 1, \ldots, n \) and relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,
\]

\[
[e_i, f_j] = \delta_{ij} h_i, \quad i, j = 1, \ldots, n,
\]

together with Serre’s relations. The linear span \( \mathfrak{h} \) of \( h_1, h_2, \ldots, h_n \) is a Cartan subalgebra, and the linear functionals \( \alpha_1, \alpha_2, \ldots, \alpha_n \) on \( \mathfrak{h} \) defined by

\[
\alpha_j(h_i) = a_{ij}, \quad i, j = 1, 2, \ldots, n
\]

form a system of simple roots for the Lie algebra \( \mathfrak{sp}_{2n}(\mathbb{C}) \).

Let \( h_0 \in \mathfrak{h} \) be the element given by

\[
\alpha_n(h_0) = 2, \quad \alpha_j(h_0) = 0, \quad j < n.
\]

It is easy to prove that \( h_0 = h_1 + 2h_2 + \ldots + nh_n. \)

Let \( \mathfrak{t} \subset \mathfrak{g} \) be the Lie subalgebra generated by

\( e_i, f_i, \quad i \neq n, \quad h_i, \quad i = 1, \ldots, n. \)
The pair \((\mathfrak{g}, \mathfrak{f})\) is Hermitian symmetric, i.e. \(\mathfrak{g}\) is equipped with a grading \(\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{f} \oplus \mathfrak{p}^+\), where

\[
\mathfrak{p}^\pm = \{ \xi \in \mathfrak{g} | [h_0, \xi] = \pm 2\xi \}, \\
\mathfrak{f} = \{ \xi \in \mathfrak{g} | [h_0, \xi] = 0 \}.
\]

Note that \(\mathfrak{p}^-\) is isomorphic to the normed vector space of symmetric complex \(n \times n\)-matrices with the operator norm. Harish-Chandra proved that an irreducible bounded symmetric domain \(D\) can be embedded into the normed vector space \(\mathfrak{p}^-\) as the unit ball.

In this paper we consider quantum analogs for the algebra \(\mathbb{C}[\mathfrak{p}^-]\) of holomorphic polynomials on \(\mathfrak{p}^-\) and the algebra \(\text{Pol}(\mathfrak{p}^-)\) of polynomials on \(\mathfrak{p}^-\) and give some results on representation theory on \(D\).

2 Algebras \(\mathbb{C}[\mathfrak{p}^-]_q\) and \(\text{Pol}(\mathfrak{p}^-)_q\)

In the sequel \(q \in (0, 1)\), \(\mathbb{C}\) is the ground field, and all the algebras are assumed associative and unital.

Let \(d_i, i = 1, \ldots, n\) be coprime numbers that symmetrize the Cartan matrix \(a\). One can check that \(d_i = 1\) for \(i = 1, \ldots, n - 1\) and \(d_n = 2\).

Denote by \(U_q\mathfrak{g} = U_q\mathfrak{sp}_{2n}\) a Hopf algebra with generators \(K_i, K_i^{-1}, E_i, F_i, i = 1, \ldots, n\), and relations

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,
\]

\[
K_i E_j = q^{d_i a_{ij}} E_j K_i, \quad K_i F_j = q^{-d_i a_{ij}} F_j K_i, \quad i, j = 1, \ldots, n
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},
\]

together with \(q\)-analogs of the well-known Serre relations [4].

The coproduct \(\Delta\), the counit \(\varepsilon\) and the antipod \(S\) are defined as follows:

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,
\]

\[
S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1},
\]

\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.
\]

Denote by \(U_q\mathfrak{f} \subset U_q\mathfrak{g}\) a Hopf subalgebra generated by

\[
E_j, F_j, \quad j < n \quad \text{and} \quad K_i^\pm, \quad i = 1, \ldots, n.
\]

\(U_q\mathfrak{g}\) can be equipped with the involution \(*\) given by

\[
(K_j^\pm)^* = K_j^\mp, \quad j = 1, \ldots, n.
\]

\[
E_j^* = \begin{cases} K_j F_j, & j < n, \\ -K_j F_j, & j = n \end{cases}, \quad F_j^* = \begin{cases} E_j K_j^{-1}, & j < n, \\ -E_j K_j^{-1}, & j = n \end{cases}
\]

The \(*\)-Hopf algebra \((U_q\mathfrak{g}, *)\) is a quantum analog for the universal enveloping algebra of \(\mathfrak{sp}_{2n}(\mathbb{R})\).
Recall a general notion \[1\]. Let \( F \) be an algebra which is also a module over a Hopf algebra \( A \). \( F \) is called an \( A \)-module algebra if the multiplication \( m : F \otimes F \to F \) is a morphism of \( A \)-modules and the unit \( 1 \in F \) is \( A \)-invariant.

In \[1\], L.Vaksman with his collaborators introduced a \( U_q \mathfrak{g} \)-module algebra \( \mathbb{C}[p^-]_q \), a quantum analog of the algebra of holomorphic polynomials on \( p^- \). They followed V.Drinfeld’s approach to producing quantum analogs of function algebras by duality. The next two theorems give an explicit description of the \( U_q \mathfrak{g} \)-module algebra \( \mathbb{C}[p^-]_q \) in terms of generators and relations.

**Theorem 1** \( \mathbb{C}[p^-]_q \) is isomorphic to the algebra generated by \( z_{ij}, 1 \leq j \leq i \leq n \), whose defining relations are the following:

\[
\begin{align*}
\text{(1)} & \quad z_{ij}z_{kl} = q^2 z_{kl}z_{ij}, & i = j = l < k \\
\text{(2)} & \quad z_{ij}z_{kl} = q^2 z_{kl}z_{ij}, & j < i = k = l \\
\text{(3)} & \quad z_{ij}z_{kl} = qz_{kl}z_{ij}, & j < l < i = k \\
\text{(4)} & \quad z_{kl}z_{ij} = z_{ij}z_{kl}, & j < l < k < i \\
\text{(5)} & \quad z_{ij}z_{kl} = z_{kl}z_{ij} + q(q^2 - q^{-2})z_{ij}z_{kl}, & i = j < k = l \\
\text{(6)} & \quad z_{ij}z_{kl} = z_{kl}z_{ij} + (q^2 - q^{-2})z_{ij}z_{kl}, & i = j < l < k \\
\text{(7)} & \quad z_{ij}z_{kl} = z_{kl}z_{ij} + (q - q^{-1})z_{ij}z_{kl}, & j < i < k < l \\
\text{(8)} & \quad z_{ij}z_{kl} = z_{kl}z_{ij} + (q - q^{-1})z_{ij}z_{kl}, & j < l < k < i \\
\text{(9)} & \quad z_{ij}z_{kl} = z_{kl}z_{ij} + (q - q^{-1})z_{ij}z_{kl}, & j < i = l < k \\
\end{align*}
\]

**Theorem 2** \( \mathbb{C}[p^-]_q \) is equipped with the structure of \( U_q \mathfrak{g} \)-module algebra defined on generators as follows: for \( k < n \)

\[
K_z = \begin{cases} 
q^2z_{ij}, & i = j = k \\
q^{-2}z_{ij}, & i = j = k + 1 \\
z_{ij}, & i = k > j \text{ or } i - 1 > k = j \\
q^{-1}z_{ij}, & i - 1 = k > j \text{ or } i > k + 1 = j \\
z_{ij}, & \text{otherwise.}
\end{cases}
\]

\[
E_z = \begin{cases} 
(q + q^{-1})z_{i-1j}, & i = j = k + 1 \\
z_{i-1j}, & i = k + 1 > j \\
z_{ij}, & i > k + 1 = j \\
0, & \text{otherwise,}
\end{cases}
\]

\[
F_z = \begin{cases} 
(q + q^{-1})z_{i+1j}, & i = j = k \\
z_{i+1j}, & i = k > j \\
z_{ij}, & i > k = j \\
0, & \text{otherwise.}
\end{cases}
\]

and

\[
K_nz_{ij} = \begin{cases} 
q^2z_{ij}, & i = j = n \\
z_{ij}, & i = n > j \\
z_{ij}, & \text{otherwise.}
\end{cases}
\]

\[
F_nz_{ij} = \begin{cases} 
z, & i = j = n \\
z_{ij}, & \text{otherwise.}
\end{cases}
\]

\[
E_nz_{ij} = -\begin{cases} 
qz_{ij}, & i = n > j \\
z_{ij}, & \text{otherwise.}
\end{cases}
\]
Note, that the algebra $\mathbb{C}[p^-]_q$ was obtained in the Kamita’s paper [3], but only as a $U_q\mathfrak{g}$-module algebra.

In [1], a $(U_q\mathfrak{g}, \ast)$-module $\ast$-algebra $\text{Pol}(p^-)_q$, (i.e. we have $(\xi f)^\ast = S(\xi)^\ast f^\ast$ for all $\xi \in U_q\mathfrak{g}, f \in \text{Pol}(p^-)_q$) was introduced. It is considered as a quantum analog of the algebra of polynomials on $p^-$. Also the existence and uniqueness of the faithfull irreducible $\ast$-representation of $\text{Pol}(p^-)_q$ and the $(U_q\mathfrak{g}, \ast)$-invariant integral over the bounded symmetric domain $\mathbb{D}$ were proved (see [2, chap.2]).

**Proposition 1** A list of defining relations for the $\ast$-algebra $\text{Pol}(p^-)_q$ consists of the relations (11)-(14) and

\[
\begin{align*}
    z_{ij}^{\ast}z_{kl} &= z_{kl}z_{ij}^\ast, & j \neq k, l & \& i \neq k, l, \\
z_{ij}^{\ast}z_{kl} &= qz_{kl}z_{ij}^\ast - (q^{-1} - q)\sum_{m>k} z_{ml}z_{mj}^\ast, & i = k > j > l, \\
z_{ij}^{\ast}z_{kl} &= qz_{kl}z_{ij}^\ast - q(q^{-1} - q)\left(\sum_{i \geq m > k} z_{ml}z_{im}^\ast + q\sum_{m > i} z_{ml}z_{mi}^\ast\right), & i > k = j > l, \\
z_{ij}^{\ast}z_{kl} &= qz_{kl}z_{ij}^\ast - (q^{-1} - q)\left(\sum_{k \geq m > l} z_{km}z_{im}^\ast + q\sum_{m > i} z_{ml}z_{mi}^\ast\right), & i > k > j = l, \\
z_{ij}^{\ast}z_{kl} &= q^2z_{kl}z_{ij}^\ast - (1 + q^2)(q^{-1} - q)\left(\sum_{i \geq m > k} z_{mk}z_{im}^\ast + q\sum_{m > i} z_{mk}z_{mi}^\ast\right), & i > j = k > l, \\
z_{ij}^{\ast}z_{kl} &= q^2z_{kl}z_{ij}^\ast - (1 + q^2)(q^{-1} - q)\sum_{m > k} z_{ml}z_{mi}^\ast, & i = j > k > l, \\
z_{ij}^{\ast}z_{kl} &= q^2z_{kl}z_{ij}^\ast - q(q^{-1} - q)\left(\sum_{i \geq k' > j} z_{kk'}z_{ik'}^\ast + \sum_{k' > i} z_{k'l}z_{kj}^\ast + q^2\sum_{k' > i} z_{k'k}z_{k'i}^\ast\right) \\
&+ (q^{-1} - q)^2\sum_{k' > i, l' > j} z_{kk'}z_{k'l'}^\ast + (q^{-1} - q)^2\sum_{k' > i} z_{k'l}z_{kj}^\ast + (q^{-1} - q)^2\sum_{k' > j} z_{k'l}z_{kj}^\ast + 1 - q^2, & i = k > j = l, \\
z_{ij}^{\ast}z_{kl} &= q^4z_{kl}z_{ij}^\ast - q(q^{-1} - q)(1 + q^2)^2\sum_{k' > i} z_{k'l}z_{kj}^\ast \\
&+ (q^{-1} - q)^2\sum_{k' > i} z_{k'k}z_{k'j}^\ast + (q^{-1} - q)^2(1 + q^2)^2\sum_{k' > j} z_{k'j}z_{k'j}^\ast + 1 - q^4, & i = j = k = l.
\end{align*}
\]

together with relations which can be obtained from the above due to obvious involution properties (the involution $\ast$ is defined naturally $*: z_{ij} \mapsto z_{ij}^\ast$).

### 3 The faithfull representation and the invariant integral

In this section we outline some results which were obtained in [2, chap.2].

Introduce an irreducible $\ast$-representation of $\text{Pol}(p^-)_q$. Let $\mathcal{H}$ be a $\text{Pol}(p^-)_q$-module with a single generator $v_0$ and the relations

\[
z_{ij}^\ast v_0 = 0, \quad 1 \leq j \leq i \leq n.
\]

**Proposition 2** [2, sect. 2.2] 1. $\mathcal{H} = \mathbb{C}[p^-]_q v_0$. 


2. There exists a unique sesquilinear form $(\cdot,\cdot)$ on $H$ with the properties: i) $(v_0, v_0) = 1$; ii) $(fv, w) = (v, f^*w)$ for all $v, w \in H$, $f \in \text{Pol}(p^-)$.  
3. The form $(\cdot,\cdot)$ is positive definite on $H$.

Denote by $T_F$ the representation of $\text{Pol}(p^-)$:

$$T_F(f)v = fv, \quad f \in \text{Pol}(p^-), v \in H.$$

**Theorem 3** [sect. 2.2] 1. $T_F$ is a faithful irreducible $\ast$-representation.

2. A $\text{Pol}(p^-)$-representation with such properties is unique up to unitary equivalence.

Let $d\nu$ be an invariant measure on the irreducible bounded symmetric domain $D$. Note that

$$\int_D fd\nu = \infty, \quad f \in \text{Pol}(p^-), f \neq 0.$$

So we have to construct a quantum analog of the algebra of smooth functions on $D$ with compact supports and to define an invariant integral on it.

The algebra $\mathbb{C}[p^-]$ is equipped with a natural grading

$$\mathbb{C}[p^-] = \bigoplus_{k=0}^{\infty} \mathbb{C}[p^-]_{q,k}, \quad \text{deg } z_{ij} = 1.$$

Extend the $\ast$-algebra $\text{Pol}(p^-)$ by attaching an element $f_0$ which satisfies the following relations:

$$f_0^2 = f_0, \quad f_0^* = f_0, \quad \psi^* f_0 = f_0 \psi = 0, \quad \psi \in \mathbb{C}[p^-]_{q,1}.$$

The two-sided ideal of the extended algebra

$$\mathcal{D}(\mathbb{D})_q \overset{\text{def}}{=} \text{Pol}(p^-) \cdot f_0 \cdot \text{Pol}(p^-)$$

is treated as a quantum analog of the space of smooth functions with compact supports on $\mathbb{D}$. $\mathcal{D}(\mathbb{D})_q$ is equipped with a $(U_qg, \ast)$-module algebra structure via

$$F_j f_0 = \begin{cases} -\frac{q^2}{1-q^2} f_0 z_{nn}^*, & j = n, \\ 0, & j \neq n, \end{cases} \quad E_j f_0 = \begin{cases} -\frac{q}{1-q^2} z_{nn} f_0, & j = n, \\ 0, & j \neq n. \end{cases}$$

Let $\mathcal{H}_F \overset{\text{def}}{=} \mathbb{C}[p^-]_{q,f_0}$.

$\mathcal{H}_F$ is a $\mathcal{D}(\mathbb{D})_q$-module, a $\text{Pol}(p^-)_q$-module, and a $U_qg$-module. Denote by $T_F$ the corresponding representations of $\mathcal{D}(\mathbb{D})_q$ and $\text{Pol}(p^-)_q$ in the vector space $\mathcal{H}_F$ (these representations are related with their faithful irreducible $\ast$-representations) and by $\Gamma$ the representation of $U_qg$.

Define a linear functional on $\mathcal{D}(\mathbb{D})_q$:

$$\int_{\mathbb{D}} fd\nu = (1 - q^4)^{(n+1)/2} \text{tr}(T_F(f) \Gamma(K^{-1})), \quad f \in \mathcal{D}(\mathbb{D})_q,$$

with $K = K_1^{2n} K_2^{2(n-1)} \cdots K_{n-1}^{(n-1)(n+2)} K_n^{n(n+1)/2}$. 

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Theorem 4 \[2\ sect. 2.2\] 1. The above integral is $U_q\mathfrak{g}$-invariant and positive, i.e.

$$\int_{D_q} (\xi \cdot f) d\nu = \varepsilon(\xi) \int_{D_q} f d\nu, \quad \xi \in U_q\mathfrak{g}, f \in \mathcal{D}(\mathbb{D})_q,$$

and

$$\int_{D_q} (f^* f) d\nu > 0, \quad f \in \mathcal{D}(\mathbb{D})_q, f \neq 0.$$

2. A positive $U_q\mathfrak{g}$-invariant integral on $\mathcal{D}(\mathbb{D})_q$ is unique up to a constant multiple.

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**References**

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