Dispersion Relations and Relativistic Causality

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Abstract

In this paper we show that if the refractive index, or rather, \( n(\omega) - 1 \) satisfies the dispersion relations then, it is implied by Titchmarsh’s theorem that, \( n(\omega) \rightarrow 1 \) as \( \omega \rightarrow \infty \). Any other limiting value for \( n(\omega) \) would violate relativistic causality, by which we mean not only that cause must precede effect but also that signals cannot travel faster–than–c, the velocity of light in a vacuum.

KEYWORDS: Dispersion relations, Kramers Kronig relations, Causality, Refractive index, faster–than–c signals

1 Introduction

There has been much interest in recent theory and experiment on superluminal light propagation in media [1, 2, 3, 4, 5, 6, 7]. These articles appear not to violate relativistic causality, by which we mean the front velocity cannot travel faster–than–c. The front velocity is the speed at which the very first extremely small vibrations of the wave will occur [8]. The front of the wave is composed of the highest uv frequency components (ie \( \omega \rightarrow \infty \)). Sommerfeld [9], predicted that the front velocity should travel at c. A brief history of events is given in Brillouin’s book [8]. Later Voigt [8], gave a physical explanation for Sommerfeld’s result, stating that the modern theory of dispersion uses the assumption of point like electrons with a finite mass. Inertia prevents the electrons from oscillating at the very start of the wave. Electron oscillation needs time to build; only after the wave has been in motion for some time can the electrons react back on the wave. The very early oscillations of the wave therefore pass though the dispersive medium as if through vacuum. Also, electrons having a finite mass, cannot oscillate at infinite frequencies without amassing infinite energy in the process. We will restrict our material media to less than infinite energy. If \( n(\omega) \rightarrow \beta \) as \( \omega \rightarrow \infty \) we take the front velocity to be \( v = c/\beta \), it is required that \( \beta = 1 \). In fact we will prove (via Titchmarsh’s theorem to be stated later) that in order that \( n(\omega) - 1 \) not violate causality it must obey the dispersion (Kramers Kronig) relations and as a consequence of this \( n(\omega) \rightarrow 1 \) as \( \omega \rightarrow \infty \).

This may appear as obvious and well known to some readers. However several well known text book accounts [10, 11, 12] on the subject invoke a physical model for refractive index to require \( n(\omega) \rightarrow 1 \) at high frequencies as we shall discuss later. It has not been made clear that the mathematical derivation of the dispersion relation for \( n(\omega) - 1 \) requires \( n(\omega) - 1 \rightarrow 0 \) as \( \omega \rightarrow \infty \) as part of the proof. The precise physical model is irrelevant. This is an important point which this article will clarify.

It appears not all recent articles agree that \( n(\omega) \rightarrow 1 \) for high frequency is a requirement for causality of \( n(\omega) - 1 \). It has been claimed that a signal can travel at \( v = c/n(\infty) \) where \( n(\infty) \leq 1 \), [13]. Furthermore, it was first suggested that the Scharnhorst effect [14, 15, 16, 17], relating to the normal propagation of light between two parallel mirrors in a vacuum might allow a signal velocity (more precisely a front velocity) to exceed c. In the upcoming section we briefly discuss the Scharnhorst effect and give arguments which show that the original derivation may be flawed. Also we show that the effect is extremely small and in practice unmeasurable. We then concentrate on physical media (non–vacuum) where experimental verification is possible. Our purpose is to show that in general signals travel at c for material media despite experimental claims to the contrary in the literature. We give physical arguments to show why \( n(\omega) \rightarrow 1 \) as \( \omega \rightarrow \infty \) for these physical media.
2 Scharnhorst Effect

The Scharnhorst effect relates to light propagating in the vacuum between two parallel mirrors. The vacuum modes are changed by the boundaries (in much the same way as in the Casimir effect) and the light experiences the vacuum as a dispersive birefringent medium. The real part of the refractive index in a direction parallel to the mirror surface is unity. The refractive index in a direction perpendicular to the mirror surface is found to be less than unity. These calculations are done using perturbation theory valid only for small frequencies \( \omega \ll m \) where \( m \) is the electron mass.

Scharnhorst [14], derived a refractive index (perpendicular to the mirror surface) for the vacuum based on low frequencies and showed \( n(0) < 1 \). (Note that \( n(0) \) implies the small frequency limit.) Combining this with the Kramers Kronig relations written for \( \text{Re} n(\omega) - \text{Re} n(\infty) \) (as opposed to \( \text{Re} [n(\omega) - 1] \)), and setting \( \omega = 0 \), Barton and Scharnhorst [16] showed that either (real part) \( \text{Re} n(\infty) < 1 \), which would imply signals moving at faster–than–c speed, or (imaginary part) \( \text{Im} n(\omega) < 0 \) for some frequency range, which would imply that the vacuum could amplify the light signal for some range of frequency. The paper [16], discusses both options but does not make a definite choice between the two.

The dispersion relations used by Barton and Scharnhorst [16] and later by Scharnhorst alone [17] are not complete in the sense that they have set a term \( \text{Im} n(\omega \to \infty) = 0 \), which in principle might change their end result. (See Eq. (41) in this paper). Barton and Scharnhorst say [16]:

“We have no conclusive proof that \( \text{Re} n \) converges and \( \text{Im} n \) vanishes as \( \omega \to \infty \) because [· · ·] their asymptotics are quite likely to be governed by non–perturbative effects that we cannot calculate.”

The reasons for assuming \( \text{Im} n(\omega \to \infty) = 0 \) are given in their paper [16] and we refer the reader there for further discussion. A further objection to the Scharnhorst effect is that it violates special relativity (SR). Even if you cannot measure the effect, to be discussed below, the fact that it can exist even in principle is objectionable.

Consider the case of a light clock, a pair of mirrors with a light pulse bouncing between them. This type of clock has been used to derive effects like time dilation and Lorentz contraction in undergraduate text book accounts of SR. For time dilation the clock moves in a direction parallel to the surface of the mirrors. Since the Scharnhorst effect predicts that the vacuum refractive index is equal to unity in that direction then we would expect the light clock (in a Scharnhorst type arrangement) to give much the same prediction as in SR. It should be noted however that the speed of light in the moving frame of the clock has altered due to the Scharnhorst effect, so predictions are not exactly the same as with no Scharnhorst effect. In the Lorentz contraction arrangement, the light clock is tilted on its side. The direction of motion of the clock is the same as the light bouncing between the mirrors, (ie. perpendicular to the mirror surface). It appears that several light clocks in relative motion would not calculate the same length scales if the Scharnhorst effect were operating between the mirrors. Observers in relative motion would not know what to use for \( c \) in the SR velocity addition formula. The change in the velocity of light, as predicted by the Scharnhorst effect, is inversely proportional to the fourth power of the distance between the mirrors, see \( \delta c \) below. The distance measured between the mirrors differs from one reference frame to another in relative motion due to Lorentz contraction. Hence, the measured value of the velocity of light, measured from one frame to another, must also change and this violates SR in a very fundamental way. The whole of SR is based on \( c \) being an invariant. Which invariant value of \( c \) do you use in this case? It appears SR does indeed break down if the Scharnhorst effect were real, even if you cannot measure the Scharnhorst change in \( c \) in practice.

Two papers have appeared both stating that measurement of faster–than–c signals between mirrors was impossible. The first was by Milonni and Svozil [18], which uses an argument based on the uncertainty relation for velocity and the uncertainty in time due to switching on a signal. Their argument can be summarized as follows; A measurement of velocity involves a distance and time measurement \( v = L/t \). The time is limited by a signal turn on time \( \delta t \approx 1/\omega \) where \( \omega \) is the frequency of the signal. We assume the shortest possible delay. Then \( \delta v = L\delta t/t^2 \geq c^2 \delta t/L \). Using \( c/\omega \approx \lambda \) we obtain \( \delta v = c\lambda/L \). The change in the
velocity of light predicted by the Scharnhorst effect is \( \delta c = k \alpha^2 (\lambda_c/L)^4 \) where \( \lambda_c = h/(mc) = 3.9 \times 10^{-11} \text{cm} \) is the Compton wavelength, \( \alpha = 1/137 \) and \( k \approx 10^{-2} \). Hence the ratio of

\[
\frac{\delta v}{\delta c} \geq \frac{1}{k \alpha^2} \left( \frac{L}{\lambda_c} \right)^3 = 1.5 \times 10^6 \left( \frac{L}{\lambda_c} \right)^3
\]  
(1)

where we have used \( \lambda \approx \lambda_c \). Thus the measured uncertainty in velocity, \( \delta v \), is much greater than the predicted change in the signal velocity of light, \( \delta c \), by many orders of magnitude. Thus the Scharnhorst effect is not useful as an experimental verification that faster-than-c speeds for signals are possible. Milonni and Svozil also conclude [18],

“... it is clear that the uncertainty in the measured propagation velocity will always be enormously larger than the correction to c associated with the Scharnhorst effect. We conclude, therefore, that no measurement of the faster-than-c velocity of light predicted by the Scharnhorst effect is possible.”

The second paper by Ben-Menahem [19], uses an argument based on the sharpness of the wavefront.

“... in order to observe faster-than-c propagation of the wavefront, it is necessary to sharpen the falloff of the fields at the wavefront to a length scale less than \( 1/m \) [where \( m \) is the electron mass]. This feat requires the inclusion in the packet of waves with \( \omega > m \), for which eq. (3) [the vacuum refractive index derived by Scharnhorst for \( \omega \ll m \)] is a bad approximation.”

In the following sections, we will concentrate on physical media (non-vacuum media which have only finite energies), in flat space-time, and we will derive the dispersion relations for \( n(\omega) - 1 \) and in the process prove that for causal propagation \( n(\omega) \rightarrow 1 \) as \( \omega \rightarrow \infty \). We will state Titchmarsh’s theorem and use it to prove our result. We conclude with a discussion that for non-violation of causality it is a necessary condition that for physical media \( n(\omega) \rightarrow 1 \) for \( \omega \rightarrow \infty \) and thus signals must travel at \( c \), the velocity of light in free space.

### 3 Pulse propagation through a medium

Consider an electric pulse \( E(z, t) \) incident on a thin dielectric medium of thickness \( \delta \). The following arguments can be found in the book by Nussenzveig [20]. The pulse is normally incident on the medium and has a transmission coefficient given by \( 2/(n(\omega) + 1) \) which we shall take to be approximately equal to unity. This turns out to have no influence on causality considerations. We assume that the medium has a complex refractive index \( n(\omega) = n_r(\omega) + in_i(\omega) \). We define the absorption coefficient \( \alpha = 2n_i(\omega)\omega/c \).

\[
E(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\text{in}}(z, \omega)e^{-i\omega t} d\omega
\]  
(2)

The pulse is made up of many Fourier components. The component of frequency \( \omega \) can be written as

\[
E_{\text{in}}(z, \omega) = E_{\text{in}}(0, \omega)e^{i\omega z/c}
\]  
(3)

If the pulse travels along the \( z \) axis and we place the left face of the medium at position \( z = 0 \) then the corresponding electric field component on the exit face of the medium is

\[
E_{\text{out}}(\delta, \omega) = E_{\text{in}}(0, \omega)e^{i\omega n(\omega)\delta/c}
\]  
(4)

We will assume that the output is a linear function of the input, so that the superposition principle applies. This shows that given input \( f(t) \) the output \( x(t) \) must satisfy

\[
x(t) = \int_{-\infty}^{\infty} g(t, t')f(t')dt'
\]  
(5)

Furthermore we assume time translation invariance so that if the input is delayed (or advanced) by a time interval \( \tau \) then the output will be delayed (or advanced) by the same time interval. It follows that the function
$g(t, t')$ must depend only on the time difference between the input and output pulses, $g(t, t') = g(t - t')$, so that

$$x(t) = \int_{-\infty}^{\infty} g(t - t') f(t')dt'$$  \hspace{1cm} (6)

This takes the form of a familiar convolution integral ("Faltung" integral in German, meaning folding integral). Equation (6) has the consequence that

$$X(\omega) = G(\omega)F(\omega)$$ \hspace{1cm} (7)

where $F(\omega), X(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t), x(t)$ and $g(t)$ respectively. Thus

$$G(\omega) = \int_{-\infty}^{\infty} g(\tau)e^{i\omega\tau}d\tau$$ \hspace{1cm} (8)

If we invoke a primitive causality condition; that the output cannot precede the input, so that if $f(t) = 0$ for $t < \tau$ then the same is true for $x(t)$. Hence $g(\tau) = 0$ for $\tau < 0$ and we can replace the lower bound in the above integral by zero.

$$G(\omega) = \int_{0}^{\infty} g(\tau)e^{i\omega\tau}d\tau$$ \hspace{1cm} (9)

This implies that $G(\omega)$ has a regular analytic continuation over the upper half of the complex plane, $I_+$. If we write $\omega = x + iy$ for $y \geq 0$ then

$$G(\omega) = \int_{0}^{\infty} g(\tau)e^{ix\tau}e^{-y\tau}d\tau$$ \hspace{1cm} (10)

the decay factor $e^{-y\tau}$ can only help convergence when $\tau \geq 0$. We still do not have sufficient conditions to derive the dispersion relations for $G(\omega)$. We need information regarding the limiting form of $G(\omega)$ as $\omega \to \infty$. In particular, we need to prove that it decreases rapidly at infinity, otherwise the Re $G(\omega)$ and Im $G(\omega)$ may be totally unrelated as in the case of a complex constant $a+ib$, [20]. We take $G(\omega)$ to be square integrable along the real axis, (in mathematical notation, $G(\omega) \in L^2(-\infty, \infty)$), so that

$$\int_{-\infty}^{\infty} |G(\omega)|^2d\omega < K$$ \hspace{1cm} (11)

where $K$ is a finite constant. If $F(\omega)$ and $G(\omega)$ are Fourier Transforms of $f(t)$ and $g(t)$ defined in a similar fashion to $E_{in}(t)$ then according to Parseval’s theorem

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$ \hspace{1cm} (12)

Since both $f(t)$ and $g(t) = 0$ for $t < 0$, when we set $f(t) = g(t)$ in the above integral equation we get,

$$\int_{0}^{\infty} |g(t)|^2dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2d\omega \leq \frac{K}{2\pi}$$ \hspace{1cm} (13)

Also, since $G(x + iy)$ is the Fourier Transform of $e^{-yt}g(t)$ from Eq.(10), then by Parseval’s theorem

$$\int_{-\infty}^{\infty} |G(x + iy)|^2dx = 2\pi \int_{0}^{\infty} e^{-2\mu|g(t)|^2}dt \leq 2\pi \int_{0}^{\infty} |g(t)|^2dt \leq 2\pi \int_{0}^{\infty} |g(t)|^2dt \leq K \hspace{1cm} (y \geq 0) \hspace{1cm} (14)$$
integrable along any line parallel to the real axis for $0 \leq y \leq \infty$. This is a very important result and leads to another condition that $|G(x + iy)| \to 0$ for $y \geq 0$ when $x \to \pm \infty$. This also forms part of Titchmarsh’s Theorem 93, Lemma [21], which we shall define in the upcoming section.

Now to apply these results to our situation, first we note that we are dealing with electric fields whose intensity must be finite, hence both the input and output can be taken to be square integrable. Mathematically,

$$\int_{-\infty}^{\infty} |E(\omega)|^2 d\omega < \infty$$  \hspace{1cm} (15)$$

which holds for both the input $E_{\text{in}}$ and output $E_{\text{out}}$. Considering the frequency dependent terms, we note that, Eq.(4) is of the same form as Eq.(7). Here $X(\omega) \equiv E_{\text{out}}(\delta, \omega)$, $F(\omega) \equiv E_{\text{in}}(0, \omega)$ and

$$G(\omega) \equiv e^{i\omega n(\omega)\delta/c}$$  \hspace{1cm} (16)$$

Hence

$$g(\delta, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega n(\omega)\delta/c} e^{-i\omega \tau} d\omega$$  \hspace{1cm} (17)$$

where $\tau = t - t'$ and

$$E_{\text{out}}(\delta, t) = \int_{-\infty}^{\infty} g(\delta, t - t') E_{\text{in}}(0, t') dt'$$  \hspace{1cm} (18)$$

The integral for $G(\omega)$ is defined for negative and well as positive frequency. The refractive index is usually defined for positive frequency only. Since $E_{\text{in}}$ and $E_{\text{out}}$ are both real, $g$ must be real hence

$$n(-\omega) = n^*(\omega)$$  \hspace{1cm} (19)$$

for real $\omega$ which extends the definition of $n(\omega)$ for the negative frequencies. Due to the spatial separation $\delta$ between input and output we need to invoke the relativistic causality condition rather than the “primitive” causality condition quoted earlier. We need to restrict our output signal so that no signal can propagate with velocity greater than the velocity of light, $c$. This implies that $E_{\text{out}}(\delta, t)$ can only depend on $E_{\text{in}}(0, t')$ when $t' < t - \delta/c$, since the output time $t = t' + \delta/c$. The signal takes a time $\delta/c$ to traverse the medium. Hence $g(\delta, \tau) = 0$ when $\tau < \delta/c$. Furthermore, from Eq.(16),

$$G(\omega) = e^{i\omega n(\omega)\delta/c} = \int_{\delta/c}^{\infty} g(\delta, \tau)e^{i\omega \tau} d\tau$$  \hspace{1cm} (20)$$

Note that usually $|G(\omega)| \leq 1$ even without the transmission coefficient since the complex refractive index results in exponential decay. It is usually the case that the output signal intensity is less than or equal to the input signal intensity. The above equation can be written as,

$$G(\omega) = e^{\omega [n(\omega) - 1]\delta/c} = \int_{0}^{\infty} g(\delta, t' + \delta/c)e^{i\omega t'} dt'$$  \hspace{1cm} (21)$$

The right hand side of this equation has a regular analytic continuation in $I_+$ (the upper half of the complex plane), this implies that the left hand side of the equation does also [20, 21]. Titchmarsh [21], gives a very clear criterion for analytic continuation of a function which is zero for all negative values of its argument.

His theorem 93 has been restated by Toll [22] as;

“A function of integrable square is zero for all negative values of its argument if and only if its Fourier transform is a causal transform”.

Thus the causality condition that $g(\delta, t' + \delta/c) = 0$ for negative $t'$ is equivalent to the requirement that $G(\omega)$ be a causal transform.
4 Titchmarsh’s Theorem

Theorems 93–95 of Titchmarsh can be restated as follows:
Let \( G(\omega) \) be an analytic and square integrable function along the real axis. (ie. \( G(\omega) \in L^2(-\infty, \infty) \)). Then if \( G(\omega) \) obeys one of the three conditions below, it obeys all three of them.

(i) No output before the input, causality condition: If \( g(\tau) \) is the inverse Fourier transform of \( G(\omega) \), then

\[
g(\tau) = 0 \quad (\tau < 0)
\]

this means \( G(\omega) \) is a causal transform.
(ii) \( G(x) \) is, for “almost all” \( x \), the limit as \( y \to 0 \) of an analytic function \( G(x+iy) \) that is regular (holomorphic) in \( I_+ \) (the upper half of the complex plane) and square integrable over any line parallel to the real axis.

Mathematically this means,

\[
\int_{-\infty}^{\infty} |G(x+iy)|^2 dy < K \quad (y \geq 0)
\]

where \( K \) is a finite real constant. See our derivation of Eq.(14) above. It follows that

\[
\lim_{x \to \pm \infty} G(x+iy) = 0 \quad (y \geq 0)
\]

The above limiting result comes from the following consideration; Cauchy’s theorem for \( y \geq 0 \) gives,

\[
G(\omega) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\nu)}{\nu - \omega} d\nu \quad (\omega, \nu \text{ complex})
\]

where the frequency is treated as real \( \omega = \omega_r \). Each of the above expressions implies the other. Here \( P \) indicates the principal part of the integral, (sometimes denoted by a bar through the integral sign) to be evaluated at \( \nu = \omega_r \). These relations between the real and imaginary parts of \( G(\omega) \) are the dispersion relations also known as Plemelj formulas.

5 Dispersion Relations

Returning to our specific example, and Eq.(21), since \( \delta \) is very small, but otherwise arbitrary, \( \omega [n(\omega) - 1] \) must have the same property (as \( G(\omega) \) of analytic continuation in \( I_+ \)) [20]. Nussenzveig also shows that the refractive index is regular in the upper half of the complex plane. However, this does not exclude the possibility of singularities on the real axis. This condition is also not sufficient to show that the real and imaginary parts of \( [n(\omega) - 1] \) must be conjugate functions, or equivalently, that they obey dispersion relations and hence imply causality. We must also show that \( [n(\omega) - 1] \to 0 \) as \( \omega \to \infty \), which is part of condition (ii) in Titchmarsh’s theorem above. This is in general not easy to do without resorting to a physical model for the refractive index.
5.1 Using a Model for refractive index

The usual approach in the literature, taken by both Nussenzveig [20] and several textbooks [10, 11, 12], is to assume that for very high driving frequency, the frequency will be much higher than any binding frequency of the electrons in the medium, and so the electrons will behave as though they are free. The Lorentz equation of motion becomes

\[ m \ddot{x} = eE(\omega)e^{-i\omega t} \tag{27} \]

which would involve a trial solution of the form \( x(t) = x(\omega)\exp(-i\omega t) \) and hence a result \( x(\omega) = -eE(\omega)/(m\omega^2) \).

Now since the polarization of the medium is given by

\[ \chi = \frac{1}{4\pi} \left[ n^2(\omega) - 1 \right] = -\frac{Ne^2}{m\omega^2} \quad (\omega \to \infty) \tag{29} \]

Hence, \( n(\omega) \to 1 \) as \( \omega \to \infty \) from physical considerations. This shows that \( [n^2(\omega) - 1] \in L^2(-\infty, \infty) \), is square integrable along the real axis. Here, \( \chi(\omega) \) plays the role of \( G(\omega) \). The causality condition, that you get no polarization before the electric field is applied, implies that \( \chi(\omega) = 0 \) for \( \omega < 0 \), so that condition (i) of Titchmarsh’s theorem is satisfied. Hence,

\[ \int_{-\infty}^{\infty} |n^2(x + iy) - 1|^2 dy < K \quad (y \geq 0) \tag{30} \]

where \( K \) is a finite real constant. Thus, from physical considerations, \( [n(x + iy) - 1] \), which is analytic in \( I_+ \), from the above discussion, must approach zero as \( x \to \pm \infty \) when \( y \geq 0 \). This implies,

\[ \int_{-\infty}^{\infty} |n(x + iy) - 1|^2 dy < K' \quad (y \geq 0) \tag{31} \]

where \( K' \) is a real, finite constant. Since this inequality satisfies condition (ii), then \( |n(\omega) - 1| \) must also satisfy the dispersion relations and be a causal transform.

5.2 No physical model for refractive index

We have yet to make clear that the physical model is not essential to the derivation of the dispersion relations. We shall now obtain the same results as in section 5.1 without invoking the physical model for refractive index. We begin by considering the polarization Eq. (28). This is clearly of the form Eq. (7), where the polarization \( \mathcal{P}(\omega) \equiv X(\omega) \), \( E(\omega) \equiv F(\omega) \) and \( \chi(\omega) \equiv G(\omega) \). The intensities of the electric fields must remain finite so it is reasonable to assume these quantities are square integrable along the real axis. \( \chi(\omega) \) may also be square integrable or it may satisfy the weaker restriction of being bounded \( |\chi(\omega)|^2 \leq K_0 \), where \( K_0 \) is a constant. In this case it is possible to form a new function by the method of subtraction (see discussions section 6.) which is square integrable. For physical media (of finite energy) it is reasonable to assume that the function \( \chi(\omega) \) is decreasing and that it tends to zero for high frequency in all physically realizable situations. Electrons in a real media cannot oscillate at infinite frequency which requires infinite energy and thus the medium cannot become polarized at infinite frequency. Furthermore, the causality condition, that you get no polarization before the electric field impinges on the medium, requires that \( \chi(\omega) \) is also causal. Since \( \chi(\omega) \) is proportional to \( n^2(\omega) - 1 \) (see Eq. (29) ) then \( n^2(\omega) - 1 \) is causal, and also square integrable. Now it appears that \( n^2(\omega) - 1 \) satisfies Titchmarsh’s condition (i), so that implies it satisfies all conditions.

We can therefore write down a dispersion relation for \( n^2(\omega) - 1 \) and in doing so we assume that (by condition (ii) of Titchmarsh’s theorem) \( n^2(\omega) - 1 \to 0 \) as \( \omega \to \infty \). This in turn implies that \( n(\omega) \to 1 \) as \( \omega \to \infty \) note without having invoked any physical model for refractive index.

Returning to our earlier discussion, in section 3. and at the beginning of section 5., we had found that \( n(\omega) - 1 \) could be extended into \( I_+ \). We may assume that \( n(\omega) - 1 \) would be at least bounded. The remaining requirement was that \( n(\omega) - 1 \to 0 \) as \( \omega \to \infty \) in order to satisfy Titchmarsh’s condition (ii). In considering \( n^2(\omega) - 1 \) we have found that indeed \( n(\omega) - 1 \) as \( \omega \to \infty \). Now we have satisfied condition (ii) of Titchmarsh’s theorem for \( n(\omega) - 1 \) and thus we may write down the dispersion relations for \( n(\omega) - 1 \).
5.3 Calculation of the Dispersion Relations

Now that we have established that \( n(\omega) - 1 \) satisfies Titchmarsh’s theorem, whether we invoke a physical model for refractive index or not, we may calculate the dispersion relation. Cauchy’s theorem expresses \( [n(\omega) - 1] \) at any point in \( I_+ \), in terms of a contour integral, for example a large semi–circle in the upper half plane. Here we define frequency as a complex quantity \( \omega = \omega_r + i\omega_i \). Since we have shown that the refractive index (for real frequency) \( n(\omega_r) \to 1 \) as \( \omega_r \to \infty \) the contour integral reduces to an integral over the real axis, as indicated by condition (ii) in Titchmarsh’s theorem above, and derived for a rectangular integral in the Appendix. Hence we have,

\[
[n(\omega) - 1] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[n(\nu) - 1]}{\nu - \omega} \, d\nu \quad (\text{Im } \omega \geq 0).
\] (32)

Taking the limit as \( \omega \) approaches the real axis, we set complex \( \omega \to \omega_r + i\epsilon \) where \( \epsilon \) is a small term, the radius of a small semi–circle in the upper half plane about the real frequency \( \omega_r \). We shall assume that \( \omega_r \) represents a pole on the real axis. We obtain,

\[
[n(\omega) - 1] = 1 + \frac{2P}{\pi} \int_{-\infty}^{\infty} \frac{[n(\nu) - 1]}{\nu - \omega_r - i\epsilon} \, d\nu
\] (33)

The denominator can be written as

\[
\frac{1}{\nu - \omega_r - i\epsilon} = P\left(\frac{1}{\nu - \omega_r}\right) + i\pi\delta(\nu - \omega_r)
\] (34)

hence, in the limit where \( \epsilon \to 0 \) we have,

\[
2[n(\omega_r) - 1] = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{[n(\nu) - 1]}{\nu - \omega_r} \, d\nu + [n(\omega_r) - 1]
\]

\[
n(\omega_r) - 1 = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{[n(\nu) - 1](\nu + \omega_r)}{(\nu - \omega_r)(\nu + \omega_r)} \, d\nu
\] (35)

Using our symmetry relation Eq.(19) we find \( n_r(\omega) \) is an even function and \( n_i(\omega) \) is an odd function. We ignore the odd integrands which give zero upon integration. Thus we can write the real and imaginary parts of \( n(\omega) \) where the frequency \( \omega_r = \omega \) is now purely real, as

\[
\text{Re}[n(\omega)] = 1 + \frac{2P}{\pi} \int_{0}^{\infty} \frac{\nu\text{Im}[n(\nu)]}{\nu^2 - \omega^2} \, d\nu
\]

\[
= 1 + \frac{cP}{\pi} \int_{0}^{\infty} \frac{\alpha(\nu)d\nu}{\nu^2 - \omega^2}
\] (36)

\[
\text{Im}[n(\omega)] = -\frac{2P}{\pi} \int_{0}^{\infty} \omega\text{Re}[n(\nu) - 1] \, d\nu
\]

\[
= -\frac{2P}{\pi} \int_{0}^{\infty} \frac{\omega\text{Re}[n(\nu) - 1] \, d\nu}{\nu^2 - \omega^2}
\] (37)

where \( \alpha \) is the absorption coefficient defined earlier. These results are consistent with the findings of the text books [10, 11, 12] and also Nussenzveig [20], Toll [22], Landau and Lifshitz [23] and Rauch and Rohrlich [24].

6 Discussion and Conclusions

We have shown in section 2, that although the Scharnhorst effect suggested that faster–than–c signals were in principle possible, in practice it would be impossible to detect any such increase in \( c \). We further suggested that the original calculations of the effect may be flawed, and that the effect implies a violation of SR. Later sections of this paper relate to the case of physical media, by which we mean non–vacuum, finite energy media in non–curved spacetime. We show that physically a result \( n(\omega) \to 1 \) as \( \omega \to \infty \) is the only reliable result based on electron energies being less than infinity.
Historically, the result of Eq.(36) was first derived by Kronig (1926) [25] and the equivalent result for the dielectric constant was treated by Kramers in (1927) [26]. Kronig was interested in the physical model behind the derivation of the refractive index and how many atoms are required before you can sensibly use the refractive index idea for a bulk material. Kramers first employed Cauchy integrals to derive the above dispersion relations. However, neither paper puts any emphasis on causality. This comes in much later by Kronig (1942) [27]. More recent tutorial accounts of the dispersion relations which include a discussion on causality can be found, for example see [28, 29]. Text book accounts [10, 11, 12], will mention that \( n(\omega) \to 1 \) as \( \omega \to \infty \) but will only argue based on the physical model for the refractive index. We have shown above that if \( [n(\omega) - 1] \) satisfies the dispersion relations then, regardless of how it was derived, Titchmarsh’s theorem proves that \( [n(\omega) - 1] \) obeys causality and must also therefore have the limiting result that \( n(\omega) \to 1 \) for \( \omega \to \infty \) by condition (ii) of Titchmarsh’s theorem. This, to the authors knowledge, has not been stressed in the literature. No formal proof of \( n(\omega) \to 1 \) as \( \omega \to \infty \) has been previously given, it has always been tacitly assumed from a physical model. I would like to stress that the causality condition (or the dispersion relations) requires this to be true regardless of the model in use. No model for refractive index can claim to have a different limiting value on \( n(\omega) \) as \( \omega \to \infty \) without violating causality for \( n(\omega) - 1 \).

Landau and Lifshitz [30], have a footnote in their book relating to the dispersion relation for the dielectric constant \( \epsilon \).

“The property \( \epsilon \to 1 \) as \( \omega \to \infty \) is not important; if the limit \( \epsilon(\infty) \) were other than unity, we should simply take \( \epsilon(\omega) - \epsilon(\infty) \) in place of \( \epsilon(\omega) - 1 \) ...”

The same argument would apply to \( n(\omega) - 1 \), since it is \( n(\omega) - 1 \to 0 \) as \( \omega \to \infty \) which is implied by Titchmarsh’s theorem to be a condition for the dispersion relation to hold. If in a case when \( n(\infty) \neq 1 \) we used instead \( n(\omega) - n(\infty) \) then you would regain a valid dispersion relation in a sense that effect would not precede cause. It would be necessary to replace \( n(\omega) - 1 \) by \( n(\omega) - n(\infty) \) everywhere in the dispersion relations, which is equivalent to a dispersion relation with one subtraction at \( n(\infty) \), to be discussed below. Landau and Lifshitz were clearly not concerned with relativistic causality. The dispersion relations require primitive causality only, effect cannot precede cause. The author believes the above footnote to be somewhat misleading, especially now when physicists are concerned with signals traveling at faster-than-c velocity. The value \( n(\infty) = 1 \) is certainly important. It is required that \( n(\infty) = 1 \) to show that a front velocity of a signal propagates at \( c/1 \) and not \( c/n(\infty) \). Pulse propagation is governed by Maxwell’s equations which are certainly relativistically causal. Recent articles [13], on superluminal signal propagation based on unphysical models which allow \( n(\omega) \to \beta \) when \( \omega \to \infty \) where \( \beta < 1 \) must therefore be dismissed as a violation of relativistically causal behavior.

Finally, we would like to mention a paper by Weaver and Pao, [31]. They derive dispersion relations (for the wavenumber \( k(\omega) \)) for a general class of linear homogeneous or inhomogeneous media. The proof proceeds without a priori knowledge of the dispersion equation for the medium, or rather without knowledge of the phase velocity \( c_\infty = \lim (\omega/k(\omega)) \) as \( \omega \to \infty \). They represent \( k(\omega) \) as a Herglotz function. A Herglotz function \( k(\omega) \) has the following properties; \( k(\omega) \) is regular in \( I_+ \) and \( \text{Im} \ k(\omega) \geq 0 \), (see Appendix C page 393 in Nusseizenig’s book [20]). The Herglotz function has the desirable feature that its imaginary part is non-negative. The medium is assumed to be passive, no energy can be given to the wave (\(|G(\omega)| \leq 1\) which unfortunately negates the possibility of an amplifying medium, which may also be causal although its square integrability may be harder to ascertain. The author finds the use of Herglotz functions unnecessary (although mathematically valid). The required proof stems from Titchmarsh’s theorem 93, Lemma, which Weaver and Pao seem aware of. Weaver and Pao have also gone to great lengths to explain the use of subtractions in the dispersion relations, also explained in detail by Nusseizenig [20]. For example, if the square integrability condition on \( G(\omega) \) cannot be satisfied, but the weaker condition that \( G(\omega) \) is bounded, \(|G(\omega)|^2 \leq K_0\), can be satisfied, then we may construct a new function

\[
H(\omega) = \frac{G(\omega) - G(\omega_0)}{\omega - \omega_0} \quad \text{Im} \ \omega_0 \geq 0
\]

(38)

where \( H(\omega) \) is square integrable and has no poles in \( I_+ \) and hence satisfies the dispersion relations.

\[
H(\omega) = \frac{P}{i\pi} \int_{-\infty}^{\infty} \frac{H(\nu)}{\nu - \omega} d\nu \quad \text{real } \omega
\]

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\[ G(\omega) = G(\omega_0) + \frac{\omega - \omega_0}{i\pi} P \int_{-\infty}^{\infty} \frac{G(\nu) - G(\omega_0)}{\nu - \omega_0} d\nu \] (39)

where \( G(\omega_0) \) is called the subtraction constant. Taking the real part we find,

\[ \text{Re} \ G(\omega) = \text{Re} \ G(\omega_0) + \frac{\omega - \omega_0}{\pi} P \int_{-\infty}^{\infty} \text{Im} \left[ \frac{G(\nu) - G(\omega_0)}{\nu - \omega} \right] d\nu \] (40)

This is known as a dispersion relation for \( G(\omega) \) with one subtraction. Often, subtractions will occur at \( \omega_0 = 0 \) or \( \omega_0 = \infty \). More than one subtraction is allowed. Using the \( \omega = \infty \) subtraction for \( n(\omega) - 1 \) we find,

\[ \text{Re} \ n(\omega) = \text{Re} \ n(\infty) + \lim_{\omega_0 \to \infty} \left[ \frac{\omega - \omega_0}{\pi} P \int_{-\infty}^{\infty} \text{Im} \left[ \frac{n(\nu) - n(\omega_0)}{\nu - \omega_0} \right] d\nu \right] \]

\[ = \text{Re} \ n(\infty) + \frac{P}{\pi} \int_{-\infty}^{\infty} \text{Im} \left[ n(\nu) - n(\infty) \right] d\nu \]

\[ = \text{Re} \ n(\infty) + \frac{2P}{\pi} \int_{0}^{\infty} \text{Im} \left[ \nu n(\nu) - \omega \text{Im} n(\infty) \right] \frac{d\nu}{\nu^2 - \omega^2} \] (41)

where we have used the fact that \( \text{Im} n(\nu) \) is odd. This is the same as the result implied by Landau and Lifshitz [23, 30], but it only requires primitive causality. For relativistic causality we also require that \( n(\infty) = 1 \) so that the front velocity of a signal travels at \( c/1 \) and not some arbitrary \( c/n(\infty) \).

We have not commented on situations which relate to highly curved spacetime. These extreme cases of general relativity may allow for faster–than–c signals, but we have not studied those situations here and will not confirm or deny any claims made in the literature.

**Appendix**

In this section we would like to show that a causal transform of the form

\[ G(\omega) = \int_{0}^{\infty} g(\tau)e^{i\omega\tau} d\tau \quad (g(\tau) = 0 \text{ when } \tau < 0) \] (42)

which is square integrable along the real axis is square integrable along any line parallel to the real axis in the upper half of the complex plane, \( I_+ \). This condition further implies that

\[ \lim_{x \to \pm \infty} G(x + iy) = 0 \quad (y \geq 0) \] (43)

The following mathematical appendix is a summary from Titchmarsh’s book [21], Theorem’s 93 with Lemma and Theorem 95 which is related. These arguments can also be found in the book by Nussenzveig [20], page 22.

Consider the analytic continuation of \( G(\omega) \) into \( I_+ \). Let \( \nu = x + iy \) and let \( \omega = x_0 + iy_0 \) and consider the following Cauchy integral,

\[ G(\omega) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\nu)d\nu}{\nu - \omega} \] (44)

We take the contour \( \Gamma \) to be a rectangle with corners at \( \pm R \) and \( \pm R + iS \), where both \( R \) and \( S \) shall tend to infinity. As we shall show, the integrals of the vertical sides of the contour are both positive and finite and do not cancel out. They must therefore both be zero. The magnitude of the line integral along the right hand side of the contour is given by,

\[ \left| \int_{0}^{S} \frac{G(R + iy)dy}{R + iy - x_0 - iy_0} \right| \leq \max_{0 \leq y \leq S} |G(R + iy)| \int_{0}^{S} \frac{dy}{[(R-x_0)^2 + (y-y_0)^2]^{1/2}} \]

\[ \leq \max_{0 \leq y \leq S} |G(R + iy)| \ln \left| \frac{(S-y_0) + [(R-x_0)^2 + (S-y_0)^2]^{1/2}}{[(R-x_0)^2 + y_0^2]^{1/2} - y_0} \right| \] (45)
As $R \to \infty$ and $S \to \infty$ this implies that

$$\max_{0 \leq y \leq S} |G(R + iy)| \to 0 \quad (46)$$

since the natural log term tends to $\ln(1 + \sqrt{2})$ as both $R$ and $S$ tend to infinity. Similarly the magnitude of the integral on the left hand side of the contour is,

$$\left| - \int_0^S \frac{G(-R + iy)dy}{-R + iy - x_0 - iy_0} \right| = \left| \int_0^S \frac{G(-R + iy)dy}{(R + x_0) - i(y - y_0)} \right|$$

$$\leq \max_{0 \leq y \leq S} |G(-R + iy)| \int_0^S \frac{dy}{[(R + x_0)^2 + (y - y_0)^2]^{1/2}}$$

$$\leq \max_{0 \leq y \leq S} |G(-R + iy)| \ln \left| \frac{(S - y_0) + [(R + x_0)^2 + (S - y_0)^2]^{1/2}}{[(R + x_0)^2 + y_0^2]^{1/2} - y_0} \right| \quad (47)$$

The natural log has the same limit as above in the case when $R$ and $S \to \infty$. This leads to the same result as Eq.(46), now written for the limit $R \to -\infty$. Hence,

$$\lim_{R \to \pm\infty} |G(R + iy)| \to 0 \quad (y \geq 0) \quad (48)$$

What remains of the contour integral are the bottom and top line integrals which can be written, for $R \to \infty$, as;

$$G(\omega) = \frac{1}{2\pi i} \left[ \int_{-\infty}^{\infty} \frac{G(x)dx}{x - x_0 - iy_0} - \int_{-\infty}^{\infty} \frac{G(x + iS)dx}{x + iS - x_0 - iy_0} \right]. \quad (49)$$

According to Schwarz’s inequality,

$$\left| \int_{-\infty}^{\infty} \frac{G(x + iS)dx}{x + iS - x_0 - iy_0} \right|^2 \leq \int_{-\infty}^{\infty} |G(x + iS)|^2 dx \int_{-\infty}^{\infty} \frac{dx}{(x - x_0)^2 + (y - y_0)^2}$$

$$\leq \frac{\pi K}{S - y_0} \quad (50)$$

where we have used Titchmarsh’s theorem, condition (ii) in the last step. As $S \to \infty$ we see that the above expression tends to zero. Hence,

$$G(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G(\nu)d\nu}{\nu - \omega} \quad (\text{Im} \ \omega \geq 0) \quad (51)$$

which expresses the value of $G(\omega)$ at any point in the upper half of the complex plane in terms of values on the real axis.

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