Non-bouncing solutions in loop quantum cosmology

Martin Bojowald

Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, PA 16802, U.S.A.

E-mail: bojowald@gravity.psu.edu

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Abstract. According to the Belinskii-Khalatnikov-Lifshitz scenario, a collapsing universe approaching a spacelike singularity can be approximated by homogeneous cosmological dynamics, but only if asymptotically small spatial regions are considered. It is shown here that the relevant small-volume behavior in solvable models of loop quantum cosmology is crucially different from the large-volume behavior exclusively studied so far. While bouncing solutions exist and may even be generic within a given quantum representation, they are not generic if quantization ambiguities such as choices of representations are taken into account. The analysis reveals an interesting interplay between sl(2, R)-representation theory and canonical effective theory.

Keywords: cosmic singularity, quantum cosmology

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1 Introduction

Claims such as “loop quantum cosmology replaces the big bang singularity by a bounce” have become commonplace in a large fraction of the literature on the subject. However, without any qualifications and the specification of assumptions, they are not supported by current results in this field. The available evidence is a mixture of numerical and analytical results which are interpreted as demonstrating the existence of what is referred to as a bounce, preventing space from approaching a degenerate geometry. However, these results demonstrate specific features which are realized for a simple non-singular bounce picture — a time-dependent volume with a single local minimum, reached when the energy density is about Planckian — but do not necessarily imply it. Quoting this collection of results as proof of singularity resolution via a generic bounce is therefore misleading.

For instance, numerical solutions for wave functions in simple models may show a bouncing trajectory of the volume expectation value as a function of internal time, bounded away from zero by a minimum reached close to the Planck density [1]. But since it is difficult to control a sufficiently large set of initial states based only on numerics, it is important to know analytical properties of solutions, at least within certain approximations. On this side, the main witness of bouncing behavior is usually summoned in the form of upper bounds on the eigenvalues or expectation values of density operators [2, 3]. Such upper bounds are close to the density at which numerical solutions reach their minimal volume. They have therefore
been interpreted as generic analytical evidence, strengthening the circumstantial evidence
provided by numerical investigations.

However, the case for a generic bounce constructed so far is incomplete due to several conceptual gaps in the arguments put forward. For instance, the existence of upper bounds of the expected densities does not imply that the expected geometry is never degenerate, if quantum fluctuations are taken into account: we will present explicit counter-examples in the form of solutions in which the triad expectation value crosses zero and the density expectation value always stays below a Planckian upper bound. If the expected triad vanishes, a fluctuating state is supported on positive and negative triads, such that the volume (related to the absolute value of the triad variable) remains positive and follows a bouncing trajectory. However, the geometry may still become degenerate. The unquestioned link between bounded densities, bouncing volume expectation values, and singularity avoidance, made commonly in arguments in favor of a generic bounce in loop quantum cosmology, therefore constitutes a conceptual gap.¹

A further conceptual gap consists in the assumption that only solutions describing the evolution of a large homogeneous region (a large averaging volume \( V_0 \)) need be considered because the late-time initial state should have large-scale homogeneity if it is to approximate our universe. The matter energy in such a region is large, even if the density is small. In an effective Friedmann equation, the classical matter term then dominates quantum fluctuations when the state is evolved toward the big bang, which simplifies arguments in favor of a bounce. However, as recently pointed out [4], the assumption of a large averaging volume fails to describe the approach to a spacelike singularity in a generic way: while the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [5] does show that homogeneous models may be used to understand the dynamics close to a spacelike singularity, such models do not describe an entire homogeneous space but only an asymptotically small region of decreasing size not bounded from below. Therefore, small averaging regions, rather than large ones, are relevant near a spacelike singularity of BKL type. (On approach to a spacelike singularity, the size of the averaging region must be reduced continually through infrared renormalization in order to maintain the approximation by a homogeneous model. A large averaging volume at late times is therefore consistent with a small averaging volume close to a spacelike singularity.) For small volumes, it is no longer clear whether the matter term always dominates quantum fluctuations, and the usual bounce arguments no longer apply. In fact, it was shown already in [6] that non-bouncing solutions may be possible in models of loop quantum cosmology if fluctuations are sufficiently large. In this paper, we present a more detailed analysis of the small-volume behavior, using further developments of the methods introduced in [6]. In a novel combination with \( \text{sl}(2, \mathbb{R}) \)-representation theory, we will be able to prove that bouncing solutions are generic within certain representations, including the one implicitly chosen in [2], but not within the set of all possible representations.

Models of loop quantum cosmology have been analyzed by a variety of methods, including algebraic ones and Hilbert-space techniques. While the former are more general because

¹See for instance the attempted contrast between Wheeler-DeWitt quantum cosmology and loop quantum cosmology in [2]: “Thus, for a generic state matter density diverges in the distant past (or distant future). In this sense the singularity is unavoidable in the WDW theory. In LQC by contrast, on a dense subspace the expectation value of the volume operator has a nonzero minimum and diverges both in the distant past and future. Thus, the density remains finite and undergoes a bounce. In this sense the quantum bounce is generic and not tied to semiclassical states.” While the technical comments about loop quantum cosmology are correct, they do not rule out a singularity in the sense of a degenerate spatial geometry being reached at finite time.
they are independent of the choice of representations and can often incorporate quantization
ambiguities in a more generic fashion, they have occasionally been claimed to be less rigorous
or even inequivalent to Hilbert-space results; see for instance [2]. Since our new results in the
present paper are based mainly on algebraic methods, we will begin by demonstrating that
the algebraic statements of [6] are equivalent to those derived with Hilbert-space techniques,
in particular those used in [2], up to choices of representations and factor-orderings. This
equivalence will serve different purposes in addition to demonstrating the validity of algebraic
results: it will show more clearly how an analysis of bounce claims is subject to quantization
ambiguities, and it will lead to new analog models in which the bouncing (or non-bouncing)
behavior is shown clearly and intuitively without being obscured by technical considerations
of unobservable features such as the specific form of a Hilbert-space representation.

By juxtaposing different outcomes for small-volume solutions, including bouncing and
non-bouncing ones depending on properties of quantum states as well as quantization ambi-
guities, this paper highlights specific tasks that remain to be completed before one can claim
the robustness or genericness of a bounce in loop quantum cosmology, if it is in fact realized.

2 Solvable models

Loop quantum cosmology [7, 8] is a canonical quantization of homogeneous models of
general relativity. Classically, the scale factor $a$ has a canonical momentum given by
$p_a = -(3/4\pi G)V_0 \dot{a}a$ where the time derivative is by proper time and $V_0$ is the coordinate
volume of the homogeneous region chosen to represent all of space. If space is compact,
$V_0$ may but need not be the entire coordinate volume. However, infrared renormalization
implies that $V_0$ must be small compared with the entire volume if the homogeneous model is
supposed to describe the geometry near a BKL-type singularity.

The canonical variables appear in phase-space functions, such as the Friedmann equation
or Hamiltonian constraint, which may be modified in order to model quantum effects. When
parameterizing quantization ambiguities, it is convenient to work with a more general set of
canonical variables given by

$$
Q = \frac{3(\ell_0 a)^{2(1-x)}}{8\pi G(1-x)}, \quad P = -\ell_0^{1+2x} a^{2x} \dot{a},
$$

where $\ell_0$ is such that $\ell_0^3 = V_0$ and the parameter $x \neq 1$ determines a 1-parameter family
of canonical pairs. (In several expressions derived below the limit $x \to 1$ can be taken and
then refers to canonical variables in which $Q = (3/4\pi G)\log a$.) The choice $x = -1/2$ is
particularly convenient because it leads to a momentum $P = -\dot{a}/a$ independent of $\ell_0$, while
$Q$ is then proportional to the geometrical volume, $V = \ell_0^3 a^3 = 4\pi G|Q|$. To facilitate a
comparison with [2], we will often highlight results obtained with this choice of $x$, but note
that it is far from being unique.\(^2\)

\(^2\)Most studies in loop quantum cosmology use a fixed value of $x = -1/2$ because it is the only choice
such that $P$ is proportional to the Hubble parameter and independent of $\ell_0$. It is then possible to implement
holonomy modifications as functions of $P$ without referring to the averaging volume of homogeneous models.
However, such arguments in favor of $x = -1/2$ ignore infrared renormalization, which shows that $\ell_0$ is a
variable scale; see also [9]. Parameters of an effective or averaged theory, such as a minisuperspace model,
should in general run with the scale, and $\ell_0$-independence of quantum modifications cannot be used as a
condition to select a specific value for such parameters. For this reason we keep $x$ as a free parameter
in our study of highly quantum, near-singular behavior. One would hope that further developments in loop
quantum gravity (not restricted to homogeneous models) will eventually make it possible to derive the running
parameters, but this stage has not yet been reached. The only indication at present is that the fundamental
value should, in fact, be close to $x = 0$ because loop quantum cosmology uses holonomies as functions of the
Ashtekar-Barbero connection, which in spatially flat isotropic geometries is proportional to $\dot{a}$ [10].
We take $Q$ to be a real number, extended from the definition (2.1) to negative values by identifying the sign of $Q$ with the orientation of a triad underlying the spatial geometry determined by $a$. Triads are, in fact, fundamental geometrical objects in loop quantum cosmology [10]. A real-valued $Q$, not restricted to be positive, therefore allows for a one-to-one correspondence with the basic geometrical variable of the theory.

In terms of $Q$ and $P$, the spatially flat Friedmann equation can be written as

$$
\left( \frac{8\pi G|1-x|}{3}|Q| \right)^{(1+2x)/(x-1)} P^2 = \frac{8\pi G}{3} \rho 
$$

(2.2)

with the matter energy density $\rho$. Solvable models in different forms are obtained for a free massless scalar $\phi$ as the only matter choice, such that

$$
\rho = \frac{1}{2} \frac{P^2}{V^2} = \frac{1}{2} \left( \frac{8\pi G|1-x|}{3}|Q| \right)^{3/(x-1)} P^2 
$$

(2.3)

with the canonical momentum $p_\phi = \ell_0^3 a^3 \dot{\phi}$ of $\phi$. Therefore, imposing the Friedmann equation is equivalent to setting

$$
\begin{align*}
\rho & = \frac{16\pi G}{3} (1-x)^2 Q P^2 \\
\rho & = \frac{16\pi G}{3} (1-x)^2 Q P^2 
\end{align*}
$$

(2.4)

or

$$
\begin{align*}
p_\phi & = \pm \sqrt{\frac{16\pi G}{3} |1-x||QP|} \\
p_\phi & = \pm \sqrt{\frac{16\pi G}{3} |1-x||QP|} 
\end{align*}
$$

(2.5)

If the scalar $\phi$ is used as internal time, the momentum $p_\phi$ plays the role of a Hamiltonian, generating evolution equations for $Q(\phi)$ and $P(\phi)$. Up to the absolute value, this Hamiltonian is quadratic according to (2.5), suggesting a coherent quantum behavior. We simplify the expression for the Hamiltonian by introducing

$$
\lambda = \sqrt{16\pi G/3} |1-x| \phi 
$$

(2.6)

as internal time, which is canonically conjugate to

$$
p_\lambda = \sqrt{\frac{3}{16\pi G(1-x)^2}} P_\phi 
$$

(2.7)

and therefore implies the Hamiltonian

$$
H = p_\lambda = \pm |QP| .
$$

(2.8)

2.1 Holonomy modifications

Loop quantum cosmology suggests modifications of the Friedmann equation of two types, inverse-triad corrections [11] and holonomy modifications [12, 13]. The former are motivated by the fact that operator versions of $Q$ in loop quantum cosmology do not have densely defined inverses, such that there is no direct quantization of the density (2.3). Nevertheless, following methods of the full theory of loop quantum gravity [14], it is possible to construct operators which have an inverse power of $V$ or $Q$ as their classical limit, but have quantum corrections for small volume. These inverse-volume corrections imply that the Hamiltonian (2.8) should
be multiplied by a function that approaches one in the classical limit but does not identically equal one. For our considerations, inverse-volume corrections will only play a supporting role. Details will therefore be provided in a later section dedicated to their potential implications.

Models of loop quantum cosmology do not provide an operator version of \( P \). There are, rather, operators that quantize

\[
\hbar \delta = \exp(i \delta P)
\]

for any real \( \delta \) but are not continuous at \( \delta = 0 \), such that the would-be operator version of \( P = -i \hbar h_\delta / \delta |_{\delta = 0} \) does not exist. The Friedmann equation, therefore, can be quantized only such that the classical version is obtained approximately for small \( \delta P \), but with holonomy modifications when \( \delta P \) is of the order one. These modifications are crucial for possible bounces because a simple way of writing \( P \) in terms of \( \hbar \delta \) is given by the bounded function

\[
\frac{\hbar_\delta - \hbar_\delta^*}{2i \delta} = \frac{\sin(\delta P)}{\delta}
\]

which approaches \( P \) for \( \delta P \ll 1 \) but, unlike \( P \), is bounded. For \( x = -1/2 \) in (2.2), the modification implies that the energy density is always bounded.

Using this modification, the deparameterized Hamiltonian equals

\[
H_\delta = \pm \frac{|Q \sin(\delta P)|}{\delta}
\]

instead of (2.8). It is no longer quadratic, but still leads to linear equations of motion for the \( \text{sl}(2, \mathbb{R}) \)-variables \( Q, \text{Re}J \) and \( \text{Im}J \) with \( J = Q h_\delta = Q \exp(i \delta P) \) [6]. (See also [15]. A different \( \text{sl}(2, \mathbb{R}) \)-model of loop quantum cosmology has been investigated in [16–20].) The brackets

\[
\{ Q, \text{Re}J \} = -\delta \text{Im}J, \quad \{ Q, \text{Im}J \} = \delta \text{Re}J, \quad \{ \text{Re}J, \text{Im}J \} = \delta Q
\]

are linear, and

\[
H_\delta = \pm \frac{|\text{Im}J|}{\delta}
\]

is linear in the generators, up to the absolute value. The variables are subject to the condition

\[
Q^2 - |J|^2 = 0
\]

(implying that \( P \) is real) which is preserved by the evolution equations — the condition selects a specific value of the quadratic Casimir of \( \text{sl}(2, \mathbb{R}) \).

Quantum cosmology usually gives rise to a large number of quantization ambiguities which may influence potential physical outcomes. The reformulation of loop quantum cosmology with a free massless scalar in terms of a linear system with relations (2.12) can be expected to reduce the number of ambiguities, or at least make them classifiable in terms of representation theory of the underlying Lie algebra, \( \text{sl}(2, \mathbb{R}) \). However, as we will discuss in detail in what follows, several crucial ambiguities remain because \( \text{sl}(2, \mathbb{R}) \) has a very interesting representation theory with distinct series of representations in which the dynamics around the big-bang singularity, when applied to our model, would be markedly different.

In the next subsection, we will show that the representations implicitly used so far in loop quantum cosmology belong to the discrete series of \( \text{sl}(2, \mathbb{R}) \), while the continuous series has not been explored yet.

In order to prepare such an analysis, we will define a loop quantization of our system as a choice of \( \text{sl}(2, \mathbb{R}) \)-representation for the relations (2.12). As we will see in several explicit examples, this definition allows for a representation-theoretic classification of factor-ordering and other ambiguities.
2.2 Representations

The model can be quantized such that the linear sl(2, \(\mathbb{R}\)) relations are maintained for \(\hat{Q} - \frac{1}{2}\hbar\delta\) and \(\hat{J}\), ordering

\[
\hat{J} = \hat{Q}\hbar\delta
\]

with canonical commutation relations for \(\hat{Q}\) and \(\hat{P}\) contained in \(\hat{h}\delta\) via (2.9). The specific sl(2, \(\mathbb{R}\))-relations then read

\[
[\hat{Q}, \text{Re } \hat{J}] = -i\hbar\delta \text{Im } \hat{J}, \quad [\hat{Q}, \text{Im } \hat{J}] = i\hbar\delta \text{Re } \hat{J}, \quad [\text{Re } \hat{J}, \text{Im } \hat{J}] = i\hbar\delta \left(\hat{Q} - \frac{1}{2}\hbar\delta\right)
\]

(2.16)

and the quantum Hamiltonian appears in the factor ordering

\[
\hat{H}_\delta = \pm\frac{|\text{Im } \hat{J}|}{\delta} = \pm\frac{1}{2\delta} \left|\hat{Q}\exp(i\delta P) - \exp(-i\delta P)\hat{Q}\right|
\]

(2.17)

2.2.1 Inequivalent representations with fixed Casimir value

Imposing the quantum reality condition \(\hat{Q}^2 - \hat{J}\hat{J}^\dagger = 0\), such that \(\hat{P}\) is self-adjoint according to the ordering given in (2.15), shows that

\[
\hat{Q}^2 - (\text{Re } \hat{J})^2 - (\text{Im } \hat{J})^2 = \frac{1}{2} \left(\hat{Q}^2 - \exp(-i\delta P)\hat{Q}^2\exp(i\delta P)\right) = \delta\hbar \left(\hat{Q} - \frac{1}{2}\hbar\delta\right)
\]

(2.18)

The Casimir operator

\[
\left(\hat{Q} - \frac{1}{2}\hbar\delta\right)^2 - (\text{Re } \hat{J})^2 - (\text{Im } \hat{J})^2 = -\frac{1}{4}\delta^2\hbar^2
\]

(2.19)

therefore takes a value which happens to be contained in both the continuous series of sl(2, \(\mathbb{R}\))-representations — \(C_{1/4}^0\) in the notation of [21] — and in the discrete series — as a positive and negative version, \(D_{1/2}^+\) and \(D_{1/2}^-\), respectively, where \(D_{k}^\pm\), are discrete-series representations with Casimir \(k(k-1)\). (An irreducible discrete-series representation has \(\hat{Q}\)-eigenvalues of only one sign.) The value of the Casimir, on its own, therefore does not uniquely characterize the representation, and the representation is not guaranteed to be irreducible. If (2.15) is constructed from a standard Schrödinger representation of \(\hat{Q}\) and \(\hat{h}\delta\) (via \(\hat{P}\)), the resulting representation of \(\hat{J}\) as difference operators can easily be seen to correspond to \(D_{1/2}^+ \oplus D_{1/2}^-\), and is therefore indeed irreducible.

Instead of representing \(\hat{J}\) through a standard quantization of \(Q\) and \(P\), a quantum system can be defined directly by selecting a specific representation of sl(2, \(\mathbb{R}\)) and using the corresponding representation space as a Hilbert space. The existence of a continuous-series representation \(C_{1/4}^0\) with the same value if the Casimir as in (2.19) therefore implies a quantization ambiguity because this choice would lead to an inequivalent representation. The availability of two inequivalent representations for the given Casimir has dynamical implications that will play an important role in our discussions later on. The spectrum of \(\hat{Q}\) is discrete in both cases, but in the reducible case of \(D_{1/2}^+ \oplus D_{1/2}^-\) the two subspaces of fixed \(\text{sgn}Q\) are left invariant by any sl(2, \(\mathbb{R}\))-element, including the Hamiltonian. It is therefore impossible for a state supported only on \(Q > 0\), say, to evolve into a state with some support on \(Q < 0\).
Such a representation, unlike the irreducible option of $C^0_{1/4}$, therefore makes it more likely for initial states to bounce, even though the possibility of $\langle \hat{Q} \rangle$ approaching zero asymptotically is not ruled out. However, unless one can show that the reducible representation is somehow distinguished, ensuring bouncing solutions by a choice of representation would be ad-hoc. The algebraic treatment is clearly of advantage here because it highlights possible choices that may be obscured by constructions that start with the choice of a specific Hilbert space, such as the kinematical one used in [2]. As we will see below, the algebraic approach also allows us to derive representation-independent statements about solutions.

2.2.2 Different Casimir values

As we have just seen, it is straightforward, although ambiguous, to represent the basic operators $Q$ and $J$ as well as the Hamiltonian $H_\delta$ on a Hilbert space. Since we have already chosen an internal time, such a representation amounts to deparameterized quantization on a physical Hilbert space.\footnote{Deparameterization restricts the available matter ingredients and may affect the physical outcome [22–26], although in certain regimes it can be shown to approximate more realistic procedures based on local time [27]. Here, we can ignore the possibly restrictive nature of deparameterization by studying quantization ambiguities in a single class of models with fixed matter content. Since, as we will see, even in this restricted setting there are a host of quantization ambiguities, many of which have not been recognized before, our results will be maintained if one were to broaden the scope of models under consideration.}

Alternatively, one may represent $Q$, $J$ (or $h_\delta$) as well as $\phi$ and $p_\phi$ on a kinematical Hilbert space and then impose the quantized Friedmann equation via a constraint. Dirac quantization then leads to the physical Hilbert space. In [2], group averaging has been applied to complete this procedure, implying a representation that at first sight looks rather different from what one would expect for a quantization of (2.13). (See also [28] with additional results about non-uniqueness of scalar products in a Hilbert-space representation.) If $P$ is restricted to the $2\pi/\delta$-periodicity of (2.13), $\hat{Q}$ has a discrete spectrum $h\delta\mathbb{Z}$. The inner product of two wave functions, $\psi_1(Q)$ and $\psi_2(Q)$, on this Hilbert space is given by

$$ (\psi_1, \psi_2) = \sum_{Q \in h\delta\mathbb{Z}} \frac{\psi_1(Q)^* \psi_2(Q)}{|Q|} $$

(2.20)

and states obey the evolution equation

$$ -\frac{\partial^2 \psi(Q, \lambda)}{\partial \lambda^2} = |\hat{Q}| \frac{\sin(\delta P)}{\delta} |Q| \frac{\sin(\delta P)}{\delta} \psi(Q, \lambda) $$

(2.21)

(adapted to our notation and correcting a sign mistake in [2]).

There is no obvious sl(2, $\mathbb{R}$)-structure in these equations. Nevertheless, we show that this representation is closely related to a quantization of (2.13), but with a different value of the sl(2, $\mathbb{R}$)-Casimir compared with (2.19). The existence of this representation is therefore an example of another quantization ambiguity, given by the choice of the Casimir value.

First, we can transform to a standard $\ell^2$ inner product by applying a unitary transformation from $\psi(Q)$ to $\chi(Q) = \psi(Q)/\sqrt{|Q|}$, such that

$$ (\chi_1, \chi_2) = \sum_{Q \in h\delta\mathbb{Z}} \chi_1(Q)^* \chi_2(Q). $$

(2.22)
This transformation to a standard inner product facilitates a comparison with the algebraic treatment. The evolution equation for $\chi$ is then

$$-rac{\partial^2 \chi(Q, \lambda)}{\partial \lambda^2} = \sqrt{|\hat{Q}| \frac{\sin(\delta P)}{\delta}} |\hat{Q}| \frac{\sin(\delta P)}{\delta} \sqrt{|\hat{Q}|} \chi(Q, \lambda)$$

(2.23)

$$= \left( \sqrt{|\hat{Q}| \frac{\sin(\delta P)}{\delta}} \sqrt{|\hat{Q}|} \right)^2 \chi(Q, \lambda).$$

(2.24)

Solutions of this second-order equation are superpositions of solutions of the Schrödinger-like equation

$$i \frac{\partial \chi(Q, \lambda)}{\partial \lambda} = \pm \sqrt{|\hat{Q}| \frac{\sin(\delta P)}{\delta}} |\hat{Q}| \chi(Q, \lambda)$$

(2.25)

in which the Hamiltonian

$$\hat{H}'_\delta = \pm \left( \sqrt{|\hat{Q}| \frac{\sin(\delta P)}{\delta}} \sqrt{|\hat{Q}|} \right) = \pm \frac{1}{2\delta} \left( \sqrt{|\hat{Q}|(\hat{Q} + \hbar \delta)} \right) \left( \exp(i\delta P) - \exp(-i\delta P) \right) \sqrt{|\hat{Q}|}$$

(2.26)

is clearly a quantization of (2.13) in a specific factor ordering different from (2.17). The quantization given in [2] could therefore have been derived by postulating (2.26) as a specific ordering choice in (2.13), followed by retracing the steps just described. In particular, the quantization constructed in [2] is just a different factor ordering of the quantization of the same model given earlier in [6].

In order to see whether this ordering may imply qualitatively new features, we should relate (2.17) to (2.26). Using basic relationships such as $\hat{A} \hat{B} = \sqrt{|\hat{A}| \frac{\sin(\delta P)}{\delta}} \sqrt{|\hat{B}|}$, we write

$$\sqrt{|\hat{Q}| \frac{\sin(\delta P)}{\delta}} \sqrt{|\hat{Q}|} = \frac{1}{2i} \left( \sqrt{|\hat{Q}|(\hat{Q} + \hbar \delta)} \right) \left( \exp(i\delta P) - \exp(-i\delta P) \right) \sqrt{|\hat{Q}|}$$

(2.27)

The Hamiltonian (2.26) can therefore be written in the form (2.17), but using

$$\hat{K} = \sqrt{|\hat{Q}|(\hat{Q} + \hbar \delta)} \exp(i\delta P)$$

(2.28)

instead of

$$\hat{J} = \hat{Q} \exp(i\delta P).$$

(2.29)

The reality condition

$$\hat{J} \hat{J}^\dagger = \hat{Q}^2$$

(2.30)

is replaced by

$$\hat{K} \hat{K}^\dagger = |\hat{Q}(\hat{Q} + \hbar \delta)|.$$

(2.31)

With this new ordering, the brackets of $\text{sl}(2, \mathbb{R})$ may be violated, but only for small $Q$: while the brackets $[\hat{Q}, \text{Re} \hat{K}]$ and $[\hat{Q}, \text{Im} \hat{K}]$ are of the correct form, for the remaining bracket we obtain

$$[\hat{K}, \hat{K}^\dagger] = \left[ \sqrt{|\hat{Q}|(\hat{Q} + \hbar \delta)} \exp(i\delta P), \exp(-i\delta P) \sqrt{|\hat{Q}|(\hat{Q} + \hbar \delta)} \right]$$

$$= \frac{2\hbar \delta \hat{Q}}{2|\hat{Q}|^2 \text{sgn} \hat{Q}} |\hat{Q} + \hbar \delta| \right| \begin{cases} 2\hbar \delta \hat{Q} & \text{if } |\hat{Q}| \geq \hbar \delta \\ 2\hat{Q}^2 \text{sgn} \hat{Q} & \text{if } |\hat{Q}| < \hbar \delta \end{cases}$$

(2.32)
where the inequalities for \(|Q|\) correspond to the support of a state in the \(Q\)-representation on which the commutator acts.

The new commutation relation (2.32) together with (2.31) shows that
\[
|\hat{Q}(\hat{Q} + \hbar \delta)| - (\text{Re} \hat{K})^2 - (\text{Im} \hat{K})^2 = |\hat{Q}(\hat{Q} + \hbar \delta)| - \frac{1}{2} (\hat{K} \hat{K}^\dagger + \hat{K}^\dagger \hat{K}) = \frac{1}{2} [\hat{K}, \hat{K}^\dagger]
\]
\[
= \begin{cases} 
\hbar \delta \hat{Q} & \text{if } |Q| \geq \hbar \delta \text{ or } |Q| = 0 \\
\hat{Q}^2 \text{sgn} \hat{Q} & \text{if } |Q| < \hbar \delta 
\end{cases}.
\]

(2.33)

Therefore, restricted to the subspace on which \([\hat{K}, \hat{K}^\dagger]\) is linear in \(\hat{Q}\), we have an \(\text{sl}(2, \mathbb{R})\)-representation with Casimir operator
\[
\hat{Q}^2 - (\text{Re} \hat{K})^2 - (\text{Im} \hat{K})^2 = 0,
\]
which is inequivalent to the representation given by (2.19). This representation, chosen implicitly in [2], is also a reducible combination of discrete-series representations, but given by \(D^+_1 \oplus D^-_1\) instead of \(D^+_1/2 \oplus D^-_1/2\) in section 2.2.1.

If one uses a representation in which \(\hat{Q}\) has discrete spectrum \(\hbar \delta \mathbb{Z}\), only a 1-dimensional subspace is contained in the range subject to the formally non-linear relation in (2.32). Since this subspace is annihilated by \(\hat{Q}\), there is in fact no non-linearity in such a representation. However, the non-linearity exhibited in (2.32) may be relevant for other representations, for instance if one follows [29] and uses a non-separable Hilbert space on which \(\hat{Q}\) has a discrete spectrum containing all real numbers, including non-zero eigenvalues in the range between \(-\hbar \delta\) and \(\hbar \delta\). Even then, the deviation from linear behavior is rather weak, and affects only wave functions that have a small expectation value \(|\langle \hat{Q} \rangle| < \hbar \delta\). For such solutions, which are relevant if infrared renormalization is taken into account, quantum back-reaction of fluctuations on expectation values is more pronounced than in the complete \(\text{sl}(2, \mathbb{R})\)-model (2.17), and any identities between expectation values and moments derived from properties of \(\text{sl}(2, \mathbb{R})\) may be violated. The latter consequence may have implications for non-bouncing solutions, to which we will turn in the next section.

2.2.3 Loop quantum cosmology from the point of view of representation theory

As a brief summary of the present section, we recall that there are two distinct sources of general quantization ambiguities in the specific system of loop quantum cosmology under consideration, related to the Lie algebra \(\text{sl}(2, \mathbb{R})\). The same system can be quantized with different choices of representations that give rise to crucially different properties of the range of volume eigenvalues, which may or may not allow transitions through zero volume. In the language of representation theory, the usual unwieldy choices of factor orderings of dynamical operators can easily be classified in two broad categories: (i) Different choices of the Casimir value, and (ii) different inequivalent representations (irreducible or reducible) for a given Casimir. We have presented three specific examples in this section, based on reinterpretations of previous constructions in terms of representation theory: two inequivalent representations, \(D^+_1/2 \oplus D^-_1/2\) and \(C^0_{1/4}\) with the same Casimir, and \(D^+_1 \oplus D^-_1\) with a different Casimir. In the remainder of this paper, we will provide additional properties of these representations related to dynamical solutions. However, it would not be justified to insist that one of these three representations must be chosen. In general discussions of potential dynamical properties, we will therefore use the understanding that a loop quantization of the model under consideration may be given by any non-trivial \(\text{sl}(2, \mathbb{R})\)-representation with a Casimir.
value $Q$ that approaches zero in the continuum limit ($\delta \to 0$) or in the semiclassical limit ($\hbar \to 0$); see (2.14). These conditions implement the minimal relations (2.12) with the definition (2.15) required for the desired dynamics to result from a Hamiltonian defined via (2.13) in the given representation. The value of the Casimir is therefore a continuous ambiguity parameter, but it does not constitute the only ambiguity because inequivalent representations exist even when the Casimir value is fixed.

3 Solutions

Both models, the original one using the Hamiltonian (2.17) in [6] and the later one using the Hamiltonian (2.26) in [2] have been called “solvable.” However, the degree of solvability as demonstrated so far is quite different. Since (2.17) is part of a completely linear system, there is no quantum back-reaction of fluctuations and higher moments of a state on expectation values, leading to dynamical coherent behavior. Using (2.26), it so happens that one can find analytic solutions for evolving wave functions, but this is a formal kind of solvability that does not need to imply any physical properties. In fact, as shown by (2.32), this choice of factor orderings may lead to non-linear brackets and therefore quantum back-reaction, depending on how $\hat{Q}$ is represented. Dynamical states that get close to small volume in this model do not fully maintain coherence. However, if one uses a representation on which $\hat{Q}$ has discrete spectrum $\hbar \delta Z$, our result (2.32) demonstrates that the model of [2] has a hidden form of solvability equivalent to the explicit solvability of the model given in [6].

Even in the ordering (2.17) or in a restriction of (2.26) to the subspace on which (2.32) is linear, the solvable nature of the dynamics could be challenged by the absolute value in (2.13), which is not a linear function. However, as pointed out already in [30], one can eliminate the absolute value if one works with states that are supported only on either the positive or negative part of the spectrum of $\text{Im} \hat{J}$. For such states, the system with the ordering (2.17) is fully linear. One may wonder whether restricting the support of states could limit the size of quantum fluctuations of $Q$ or $P$, which will play an important role in our subsequent classification of bouncing and non-bouncing solutions. A separate paper demonstrates that this is not the case [31].

3.1 Fluctuations

Given a linear algebra of basic operators $\{\hat{Q}, \hat{J}\}$ and the Hamiltonian $\hat{H}_d$ in (2.17), Heisenberg’s equations of motion for operators or Ehrenfest’s equations for expectation values are linear:

$$\frac{d\langle \hat{Q} \rangle}{d\lambda} = \text{Re}\langle \hat{J} \rangle$$

$$\frac{d\text{Re}\langle \hat{J} \rangle}{d\lambda} = \langle \hat{Q} \rangle - \frac{1}{2} \hbar \delta$$

$$\frac{d\text{Im}\langle \hat{J} \rangle}{d\lambda} = 0.$$  \hspace{1cm} (3.1) \hspace{1cm} (3.2) \hspace{1cm} (3.3)

(The shift by $-\frac{1}{2} \hbar \delta$ is absent if (2.26) is used as Hamiltonian.) They can easily be solved by

$$\langle \hat{Q} \rangle(\lambda) = \frac{1}{2} \hbar \delta + \frac{1}{2} A \exp(\lambda) + \frac{1}{2} B \exp(-\lambda), \quad \text{Re}\langle \hat{J} \rangle(\lambda) = \frac{1}{2} A \exp(\lambda) - \frac{1}{2} B \exp(-\lambda)$$

with two constants $A$ and $B$, while $\text{Im}\langle \hat{J} \rangle(\lambda) = \delta \rho \lambda$ is constant.
The expectation values are subject to a quantum version of the classical reality condition, which selects the Casimir of $sl(2, \mathbb{R})$. Taking an expectation value of the identity (2.30), we obtain the condition
\[
(\langle \hat{Q} \rangle - \frac{1}{2} \hbar \delta)^2 + (\Delta Q)^2 - (\text{Re}(\langle \hat{J} \rangle))^2 - (\text{Re}J)^2 - (\text{Im}(\langle \hat{J} \rangle))^2 - (\Delta \text{Im}J)^2 = -\frac{1}{4} \hbar^2 \delta^2, \tag{3.5}
\]
or
\[
(\langle \hat{Q} \rangle - \frac{1}{2} \hbar \delta)^2 - (\text{Re}(\langle \hat{J} \rangle))^2 = \delta^2 (p_\lambda^2 - \hbar^2/4) + \delta^2 (\Delta p_\lambda)^2 + (\text{Re}J)^2 - (\Delta Q)^2. \tag{3.6}
\]
(If (2.26) and (2.34) are used, the latter condition reads
\[
\langle \hat{Q} \rangle^2 - (\text{Re}(\langle \hat{J} \rangle))^2 = \delta^2 p_\lambda^2 + \delta^2 (\Delta p_\lambda)^2 + (\text{Re}J)^2 - (\Delta Q)^2. \tag{3.7}
\]
The shift of $\langle \hat{Q} \rangle$ by $-\frac{1}{2} \hbar \delta$ on the left-hand side of (3.6), as well as the negative term $-\frac{1}{4} \hbar^2 \delta^2$ on the right-hand side of this equation, are subject to the main two quantization ambiguities, as we will see in section 5.1.)

For fluctuations $\Delta Q$ smaller than $\delta \sqrt{p_\lambda - \hbar^2/4}$, the right-hand side of (3.6) is guaranteed to be positive. The resulting condition ($\langle \hat{Q} \rangle - \frac{1}{2} \hbar \delta)^2 - (\text{Re}(\langle \hat{J} \rangle))^2 = AB > 0$ can then be fulfilled for the $\lambda$-dependent solutions only if $AB > 0$ in (3.4). By adjusting the zero value of $\lambda$, we can always assume that $B = |A|$ unless $B = 0$ or $A = 0$. For $AB > 0$, $B = A > 0$, such that
\[
\langle \hat{Q} \rangle(\lambda) = \frac{1}{2} \hbar \delta + A \cosh(\lambda) \tag{3.8}
\]
follows a bouncing trajectory.

In order to show that a bounce happens generically in this model, one should demonstrate that $Q$-fluctuations can never be so large that the right-hand side of (3.6) is no longer positive. Semiclassical situations cannot lead to a non-bouncing scenario because semiclassical fluctuations $\Delta Q$ cannot overcome the large $\delta p_\lambda$. However, as we approach a BKL-type singularity, infrared renormalization implies that the averaging volume $V_0$ gets smaller and smaller, without any non-zero lower bound in the classical theory. Moreover, because the energy density $p_\lambda^2/(2V^2)$ is independent of $V_0$, $p_\lambda$ decreases with decreasing $V_0$. Since $p_\lambda$ decreases by a classical effect, unrelated to $\hbar$, it is not inconceivable that $\delta^2 (p_\lambda^2 - \hbar^2/4)$ could be smaller than
\[
\Delta = (\Delta Q)^2 - \delta^2 (\Delta p_\lambda)^2 - (\Delta \text{Re}J)^2. \tag{3.9}
\]
Note that products of fluctuations such as $\Delta Q \Delta P$ are bounded from below by uncertainty relations independent of $V_0$. Therefore, they are not suppressed as much by infrared renormalization and can remain significant even as $Q$ gets smaller and smaller; see [32] for a detailed discussion.

If the fluctuations collected in $\Delta$ are so large that they cancel out the term $\delta^2 (p_\lambda^2 - \hbar^2/4)$, the constants in our solutions (3.4) have to obey $AB = 0$, such that $A = 0$ or $B = 0$, in which case we have a non-bouncing
\[
\langle \hat{Q} \rangle(\lambda) = \frac{1}{2} \hbar \delta + \frac{1}{2} A \exp(\pm \lambda) \tag{3.10}
\]
that resembles the singular classical solutions. If fluctuations are just slightly larger, $AB < 0$ implies that we can adjust the zero value of $\lambda$ such that $B = -A$, in which case
\[
\langle \hat{Q} \rangle(\lambda) = \frac{1}{2} \hbar \delta \pm A \sinh(\lambda) \tag{3.11}
\]
follows a non-classical non-bouncing trajectory. These solutions, together with the bouncing one (3.8), are illustrated in figure 1.

The relationship between fluctuations and expectation values, or detailed knowledge of the quantum state close to a singularity, is therefore required to see whether a bounce happens generically. The non-trivial nature of this behavior is shown by a result of [33]: for a state Gaussian in $Q$, $\Delta$ is always negative, even if the state is squeezed. Expectation values of such a state always bounce, provided that $p_\lambda > \frac{1}{4} \hbar$. For a state which is still Gaussian and possibly squeezed, but in $\log |Q|$ rather than $Q$, $\Delta = 0$ and only the small (infrared-renormalized) $\delta^2 p^2_\lambda$ remains, in addition to $-\frac{\hbar^2}{4}$. This contribution may still be positive, so that these states could bounce as well, but only if $p_\lambda$ is above a minimal value which need not be respected by infrared renormalization. Since the model of [2] implies the condition (3.7) in which the term $-\frac{1}{4} \hbar^2 \delta^2$ is absent, any non-zero $p_\lambda$, however small, would lead to bouncing solutions even if $\Delta = 0$. In this sense, [2], compared with [6], makes bouncing solutions more likely.

So far the possibility has not been ruled out that a non-Gaussian state might lead to $\Delta > 0$, such that even the positive $\delta^2 p^2_\lambda$ could be overcome. No such state has been found yet, but not much of the state space has been explored beyond Gaussian ones, and going beyond Gaussian wave functions in a systematic way requires a tedious analysis. At present it is therefore impossible to conclude, based on such methods, that a bounce happens generically. However, we are now able to demonstrate the genericness of bouncing solutions within the model of [2] (but not within loop quantum cosmology in general) using our identification of this model with the reducible representation of $\text{sl}(2, \mathbb{R})$ derived in section 2.2.2.
3.2 Representation theory

There is an interesting relationship between the possibility of non-bouncing solutions and representation theory of \( \text{sl}(2, \mathbb{R}) \). In the two examples with Casimirs (2.19) and (2.34), respectively, it is possible to use a reducible representation which is a direct sum of two irreducible ones, one such that \( Q > 0 \) and one such that \( Q < 0 \) in terms of \( \hat{Q} \)-eigenvalues; see also [15] in the context of bouncing solutions. Therefore, an evolving state supported on \( Q > 0 \) initially will always be supported on \( Q > 0 \) if evolution is generated by an \( \text{sl}(2, \mathbb{R}) \)-element in the same representation. Time-dependent expectation values such as (3.11) are then impossible, and \( Q = 0 \) will never be crossed. The non-bouncing possibility (3.10) with \( Q = 0 \) approached asymptotically is not ruled out by this statement. However, such a solution, with \( AB = 0 \) in (3.4), requires a specific value of the fluctuation parameter \( \Delta \) in (3.9) for fixed \( p_\lambda \), and would therefore not be considered generic. Within a given model of this form, bouncing solutions are therefore generic, but we should examine the possibility of other \( \text{sl}(2, \mathbb{R}) \)-representations, in addition to what has serendipitously been selected in [6] or [2], before we can tell whether bouncing solutions are generic within loop quantum cosmology, understood as any quantum model with holonomy-modified dynamics as defined in section 2.2.3.

The reducible representations implicitly used in [6] and [2] make use of the discrete series, on which the Casimir \( R \) has to respect the inequality \( R \geq -\frac{1}{4} \hbar^2 \delta^2 \) [21]. The limiting value is realized in (2.19), while (2.34) makes a more advantageous choice of the Casimir, increasing the likelihood of bouncing solutions. For general \( R \), \( \delta^2(p_\lambda^2 - \hbar^2/4) \) in (3.6) is replaced by \( \delta^2 p_\lambda^2 + R \). (Moreover, as shown in (3.7), the shift of \( \langle \hat{Q} \rangle \) by \( \frac{1}{2} \hbar \delta \) in (3.6) is absent if the derivation is repeated in the model of [2] where according to (2.32) an unshifted \( \hat{Q} \) is one of the algebra generators.) Compared with the limiting value of (2.19), larger fluctuations are therefore necessary for \( AB = 0 \) in (3.4) to result from imposing the reality condition, and non-bouncing solutions are less likely. Choosing \( R = 0 \), as implicitly done in [2], therefore makes bounces more likely than choosing \( R = -\frac{1}{4} \hbar^2 \delta^2 \) as in [6], and representations with \( R > 0 \) that have not been studied yet would further enhance this likelihood.

Representations in the continuous series, which so far have not been studied in this context either, respect the inequality \( R < 0 \), all of which decrease the likelihood of a bounce. Moreover, in this case a representation containing both positive and negative eigenvalues of \( Q \) is irreducible, such that a Hamiltonian constructed in such a representation is able to map a state supported on \( Q > 0 \) to a state with some support on \( Q < 0 \). The non-bouncing possibility of (3.11) is therefore not ruled out, and bouncing solutions do not appear to be generic. Our argument using fluctuations \( \Delta \) in (3.6) shows that non-bouncing solutions should indeed be more likely precisely for a negative range of \( R \), in which case we have irreducible representations in which \( Q > 0 \) can be mapped to \( Q < 0 \). From a fundamental perspective, see for instance [34], one may prefer an irreducible representation of the dynamical algebra, which supports the possibility of non-bouncing solutions.

3.3 Analog harmonic oscillators

The role of fluctuations can nicely be illustrated by analog models using upside-down harmonic oscillators. The classical Hamiltonian \( QP \) is equivalent to such an oscillator based on the canonical transformation \( \sqrt{2} q = Q - P \), \( \sqrt{2} p = Q + P \) such that \( QP = \frac{1}{2}(p^2 - q^2) \).

However, for a generalization to the holonomy-modified model, it is more convenient to proceed without applying a canonical transformation. In the classical case, we can derive an
analog model of the Hamiltonian system generated by \( H = QP \) by rewriting the first-order equations it generates, \( \dot{Q} = Q \) and \( \dot{P} = -P \), in terms of a second-order equation for \( Q \). If \( \dot{Q} = Q \), we have \( \dot{Q} = \dot{Q} = Q \), which is the second-order equation of motion generated by the upside-down harmonic Hamiltonian

\[
H_{\text{analog}} = \frac{1}{2}(\pi^2 - Q^2) \tag{3.12}
\]

with momentum \( \pi = \dot{Q} \). Again using \( \dot{Q} = Q \), we obtain \( \pi = Q \) or \( E = H_{\text{analog}} = 0 \) on solutions we are interested in. The first-order equations generated by \( QP \) are therefore equivalent to equations of motion generated by \( H_{\text{analog}} \) together with the condition that the energy be zero. Standard knowledge about the upside-down oscillator then immediately shows that we have solutions of the form (3.10) in which \( Q = 0 \) is approached asymptotically.

For the loop model, the first-order equations generated by (2.17) are given in (3.1). Ignoring the constant shift of \( Q \) by \(-\frac{1}{2}\hbar \delta\), which is irrelevant here, we still have the second-order equation \( \ddot{Q} = d\text{Re}J/d\lambda = Q \), corresponding to the same analog Hamiltonian \( H_{\text{analog}} \) used for the unmodified dynamics in (3.12). However, \( \pi = \dot{Q} = \text{Re}J \neq Q \) in general, such that we are looking for solutions with non-zero energy. Again, standard knowledge about the upside-down oscillator shows that negative energies give rise to bouncing solutions (3.8), while positive energies imply non-bouncing solutions (3.11).

In the loop model, the energy of analog solutions is determined by quantum fluctuations. Since \( \pi = \text{Re}J \), we can derive its relationship with \( Q \) using the reality condition

\[
-2E = Q^2 - (\text{Re}J)^2 = \delta^2(p^2 - \hbar^2/4) - \Delta. \tag{3.13}
\]

(See (3.6) and (3.9).) Therefore, we have \( E < 0 \) and bouncing solutions if \( \Delta \) is sufficiently small or negative. However, we may have non-bouncing solutions of the form (3.10) if \( E = 0 \), or of the form (3.11) if \( E > 0 \). In the latter two cases, \( p_\lambda \) must be such that \( \delta^2(p^2 - \hbar^2/4) < \Delta \).

Factor-ordering choices affect the potential of an analog model. For instance, the non-linear relation (2.32) for \( |Q| < \hbar \delta \) implies that \( \dot{Q} \) is proportional to \( Q^2 \text{sgn}Q \) instead of \( Q \), which requires an anharmonic potential proportional to \(-|Q|^3\). This potential still vanishes at \( Q = 0 \) where it has a local maximum. Qualitatively, the behavior of solutions with different energies is therefore similar to the harmonic analog system, but the anharmonic potential likely leads to more significant changes of fluctuations, or \( \Delta \), during evolution.

In these analog models, the question of whether there is a bounce is a matter of initial values rather than the dynamics. Initial values, in turn, are determined by the quantum state relevant in the small-volume regime.

4 Bounded densities

Different versions of upper bounds on the energy density have been derived in Hilbert-space representations of models of loop quantum cosmology [2, 3]. At first sight, the possibility of non-bouncing solutions for \( Q \) approaching or crossing zero seems to be inconsistent with such bounds, a conclusion which is often suggested in the literature. However, we can explicitly demonstrate that there is no inconsistency. After all, the density \( 1/(2\hat{p}^2/V^2) \), or as a simpler substitute the positive expressions \( \hat{Q}^2 \) or \( \hat{Q}^{-2} \), may have bounded expectation values even if \( \langle \hat{Q} \rangle \) is zero, simply because \( \langle \hat{Q}^2 \rangle = \langle \hat{Q} \rangle^2 + (\Delta Q)^2 \) contains a contribution from fluctuations.

A more refined argument that suggests a strict relationship between density bounds and bouncing solutions refers to the Planckian value of the upper bound on energy densities
derived in [2, 3], which agrees with the Planckian density usually obtained at the bounce point of bouncing solutions in models of loop quantum cosmology. However, even such a quantitative relationship does not imply that general statements about energy bounds imply bouncing solutions. In order to demonstrate this perhaps subtle statement, we use the algebraic model to derive a bound on $\langle \hat{Q}^2 \rangle$, which we can then analyze in the bouncing case $AB > 0$ in (3.4) and in the non-bouncing one, $AB \leq 0$. For these cases, we summarize our previous solutions (3.8), (3.10) and (3.11) for expectation values as

$$
\langle \hat{Q} \rangle (\lambda) = \begin{cases} 
\frac{1}{2} \delta h + A \cosh(\lambda) & \text{if } AB > 0 \\
\frac{1}{2} \delta h \pm \frac{1}{2} A \exp(\lambda) & \text{if } AB = 0 \\
\frac{1}{2} \delta h \pm A \sinh(\lambda) & \text{if } AB < 0
\end{cases}
$$

(4.1)

where $A > 0$ and $|B| = A$ unless $AB = 0$; see also figure 1.

To be specific, we first assume sl(2, $\mathbb{R}$)-relations such that there is no shift of $Q$ and a zero Casimir, corresponding to the model of [2]. Ehrenfest’s equations of motion can be derived in the linear model not only for expectation values but also for fluctuations. Unlike expectation values, fluctuations are subject to uncertainty relations, which bound possible initial values for their equations of motion. Fluctuations and expectation values therefore have different positivity properties, which is relevant for the existence of local minima.

Volume fluctuations in the linear model always obey the relation [30]

$$
(\Delta Q)^2(\lambda) = \frac{1}{2} (c_3 \exp(-2\lambda) + c_4 \exp(2\lambda)) - \frac{1}{4} (c_1 + c_2)
$$

(4.2)

with constants $c_i$, where

$$
c_1 = -\Delta = AB - \delta^2 p_\lambda^2
$$

(4.3)

and

$$
c_1 - c_2 = 2\delta^2 (\Delta p_\lambda)^2.
$$

(4.4)

Since $(\Delta Q)^2(\lambda) \geq 0$ for all $\lambda$, $c_3$ and $c_4$ cannot be negative. Moreover, as shown in [35], uncertainty relations imply that

$$
c_3 c_4 \geq \hbar^2 \delta^2 p_\lambda^2 + \frac{1}{4} (c_1 + c_2)^2 > 0.
$$

(4.5)

Therefore, $c_3$ and $c_4$ are strictly positive for all states, such that $(\Delta Q)^2(\lambda)$ always has a local minimum. It is located at $\lambda = \frac{1}{4} \log(c_3/c_4)$, at which time we have the minimal fluctuations

$$
(\Delta Q)^2_{\text{min}} = \sqrt{c_3 c_4} - \frac{1}{4} (c_1 + c_2).
$$

(4.6)

For a dynamical coherent state [30], for instance,

$$
(\Delta Q)^2_{\text{min}} = \frac{1}{2} \sqrt{\hbar^2 \delta^2 p_\lambda^2 + \delta^4 (\Delta p_\lambda)^4 + \Delta^2 + 2\Delta \delta^2 (\Delta p_\lambda)^2} + \frac{1}{2} \delta^2 (\Delta p_\lambda)^2 + \frac{1}{2} \Delta
$$

(4.7)

in which $\Delta$ is the only parameter characterizing the state that does not refer to the matter ingredients. It is easy to see that $(\Delta Q)^2_{\text{min}}$, as a function of $\Delta$, is monotonically increasing, with $\lim_{\Delta \to \infty} (\Delta Q)^2_{\text{min}} = \infty$ and $\lim_{\Delta \to -\infty} (\Delta Q)^2_{\text{min}} = 0$. For large $\Delta > 0$, we have the set
of states that may not bounce because $\Delta$ can overcome $\delta^2 p_\lambda^2$. However, in this range of $\Delta$, $(\Delta Q)^2_{\min}$ is large, which explains why $\langle \hat{Q}^2 \rangle = \langle \hat{Q} \rangle^2 + (\Delta Q)^2$ can be bounded even if $\langle \hat{Q} \rangle$ approaches or crosses zero.

In order to demonstrate a strict bound for $\langle \hat{Q}^2 \rangle$ as well as its Planckian nature, we combine $\langle \hat{Q} \rangle(\lambda)$ from (3.4) with $(\Delta Q)^2(\lambda)$ from (4.2):

$$\langle \hat{Q}^2 \rangle(\lambda) = \langle \hat{Q} \rangle(\lambda)^2 + (\Delta Q)^2(\lambda) = \frac{1}{4} ((2c_3 + A^2) \exp(-2\lambda) + (2c_4 + B^2) \exp(2\lambda)) + \frac{1}{2} AB - \frac{1}{4} (c_1 + c_2).$$

Because both $c_3$ and $c_4$ are positive, we have $(2c_3 + A^2)(2c_4 + B^2) > 0$ and $\langle \hat{Q}^2 \rangle(\lambda)$, unlike $\langle \hat{Q} \rangle(\lambda)$, always has a local minimum, even if $AB \leq 0$:

$$\langle \hat{Q}^2 \rangle_{\min} = \frac{1}{2} \sqrt{(2c_3 + A^2)(2c_4 + B^2)} + \frac{1}{2} AB - \frac{1}{4} (c_1 + c_2).$$

Since the square root is always positive, this minimum is bounded from below by

$$\langle \hat{Q}^2 \rangle_{\min} > \frac{1}{2} AB - \frac{1}{4} (c_1 + c_2) = \frac{1}{2} \delta^2 p_\lambda^2 + \frac{1}{4} (c_1 - c_2) = \frac{1}{2} \delta^2 (\lambda^2 + (\Delta p_\lambda)^2) = \frac{1}{2} \delta^2 (\rho_{\Delta p_\lambda}^2),$$

using (4.3) and (4.4).

We are now ready to obtain an upper bound for the energy density $\rho = \rho_\phi / (2V^2)$, which we define as $\langle \hat{p}_\phi \rangle^2 / (2V^2)$ on quantum states. Because the energy density is a composite, non-polynomial expression in terms of the generators of our $\text{sl}(2, \mathbb{R})$-model, there is no unique operator that could quantize it. The expression used here may be considered an effective energy density that combines the relevant expectation values in classical fashion. Other definitions, such as an expectation value of $\frac{1}{2} V^{-1} \hat{p}_\phi^2 V^{-1}$, are related to our choice through additional fluctuation terms, which may change upper bounds. Notice that $\hat{p}_\phi$ does not commute with $\hat{V}$ on the solution space of the Hamiltonian constraint. In our derivations below, the relevant property of an energy density will be that it depends on the volume through $V \propto |Q|$ or $Q^2$, but not on the sign of the oriented volume, $Q$. The former expressions have lower bounds, while the latter does not; see figure 2 for an illustration.

If $\langle \hat{Q}^2 \rangle$ is bounded from below, $\rho$ is bounded from above for fixed $p_\phi$. Using the relationships $|Q| = 4\pi GV$ and $p_\phi = \sqrt{12\pi G p_\lambda}$ for $x = -1/2$, we obtain

$$\rho = \frac{3}{8\pi G} \frac{\rho_\phi^2}{Q^2} \leq \frac{3}{4\pi G \delta^2} = 2\rho_{QG},$$

where $\rho_{QG}$ is the energy density at which large-volume solutions (3.8) bounce, such that $\rho_{QG} \approx \rho_\phi$ if $\delta \approx \ell_\rho$ for $x = -1/2$. Therefore, even if we do not have a bouncing solution, that is if $A$ and $B$ are such that the volume is of the form (3.11) or (3.10), the energy density is bounded from above by a fixed multiple of the Planck density. Similarly, for general $x$ we have the $Q$-dependent density bound

$$\rho = \frac{3}{8\pi G} \frac{\rho_\phi^2}{Q^2((8\pi G/3)(1 - x)|Q|^{(1+2x)/(1-x)} \leq \frac{3}{4\pi G \delta^2((8\pi G/3)(1 - x)|Q|^{(1+2x)/(1-x)} = \frac{2\rho_{QG}}{((8\pi G/3)(1 - x)|Q|^{(1+2x)/(1-x)}.}$$

(4.12)
It is interesting to contrast the limiting case $AB = 0$ with the Wheeler-DeWitt model, that is, the quantized model without holonomy modifications. For $\hat{H} = \pm \frac{1}{2}(\hat{Q}\hat{P} + \hat{P}\hat{Q})$, Ehrenfest’s equation

$$\frac{d\langle \hat{Q} \rangle}{d\lambda} = \pm \langle \hat{Q} \rangle$$

(4.13)

is solved by $\langle \hat{Q} \rangle(\lambda) = \frac{1}{2}A \exp(\pm \lambda)$, which is of the form (3.10). However, fluctuations in this model are crucially different from (4.2): for fluctuations, Ehrenfest’s equations imply

$$\frac{d(\Delta Q)^2}{d\lambda} = \pm 2(\Delta Q)^2,$$

(4.14)

such that $\Delta Q \propto \exp(\pm \lambda)$. Although this solution (squared) is formally of the form (4.2), it would require $c_3c_4 = 0$ which in the holonomy-modified model is ruled out by uncertainty relations. For this reason, the Wheeler-DeWitt model does not obey density bounds: our derivation cannot be applied to this model because $(\Delta Q)^2(\lambda)$ then does not have a local minimum. The solution (3.10), obtained with $AB = 0$ in the loop model, therefore describes a different state compared with the Wheeler-DeWitt solution, even though it has an identical behavior of $\langle \hat{Q} \rangle(\lambda)$.

### 5 Quantization ambiguities

We have already seen one example of a factor ordering choice, using $\hat{K}$ in (2.32) instead of $\hat{J}$ as implicitly done in [2], that affects the dynamics of states, in particular at small volume. Such ambiguities are therefore relevant for the question of whether there may be a generic bounce. Other ordering ambiguities can be formulated within the completely linear model.
5.1 Representations

Quantizing the solvable model amounts to the choice of an irreducible representation of sl(2, \(\mathbb{R}\)). Inequivalent representations are classified by the value \(R\) of the Casimir operator \(\hat{Q}^2 - (\text{Re}\hat{J})^2 - (\text{Im}\hat{J})^2 = R\). As we have seen, the interpretation \(|\text{Im}\hat{J}| = \delta p_\lambda\) in a cosmological model means that expectation values of the volume variable \(Q\) and its time derivative related to \(\text{Re}\hat{J}\) obey the relation

\[
\langle \hat{Q} \rangle^2 - (\text{Re}\hat{J})^2 = \delta^2 p_\lambda^2 + R - \Delta
\]

with a fluctuation term \(\Delta\); see (3.6). If the right-hand side is positive, \(\langle \hat{Q} \rangle(x)\) is cosh-like and bounces, while it is exponential or sinh-like, and therefore non-bouncing, if the right-hand side is zero or positive, respectively. (Incidentally, note eq. (5.1) suggests an interpretation of the Casimir value \(R\) as a state-independent off-set of quantum fluctuations. There is no classical interpretation of \(R\) because it is required to have a vanishing classical limit.)

For unitary irreducible representations of \(\text{sl}(2, \mathbb{R})\) on which \(\hat{Q}\) has positive and negative eigenvalues, we must use the continuous series, on which the Casimir is restricted by the inequality \(R < 0\). Bouncing solutions are most likely for small \(\delta p_\lambda\). However, choosing these representations, which are reducible and such that a range of relationships as before, we derive

\[
\langle \hat{Q} \rangle^2 - (\text{Re}\hat{J})^2 = \delta^2 p_\lambda^2 + R - \Delta
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The linear nature of the model is preserved, but in our previous relations $\hat{Q}$ is shifted by a constant to $\hat{Q} + \hbar \delta_1$. In particular, an arbitrary constant $\hbar \delta_1$ can be subtracted from a bouncing solution (3.8), possibly pushing the local minimum to negative values, accompanied by two zeros of $\langle \hat{Q}(\lambda) \rangle$. For small-volume solutions (after infrared renormalization), even a small $\hbar \delta_1$ can lead to this singular behavior. Large fluctuations are not required in this case. With a positive $\delta_1$, this type of quantization ambiguity can lead to singular solutions even in quantizations based on reducible representations using the discrete series: even though the expectation value of the corresponding sl(2, $\mathbb{R}$)-generator $\hat{L}_0 = \hat{Q} + \hbar \delta_1$ would never cross zero based on our previous arguments, the expectation value of the triad operator $\hat{Q} = \hat{L}_0 - \hbar \delta_1$ can cross zero if $\delta p_\lambda$ is sufficiently small.

5.2 Inverse-volume corrections

Another source of quantization ambiguities is given by inverse-volume corrections [11], which we have not considered in detail so far. These corrections do not preserve the linearity of the model and can therefore lead to further differences compared with a simple bouncing solution such as (3.8). An obvious place for inverse-volume corrections to appear is the matter Hamiltonian, $\frac{1}{2} p_\phi^2 / V$ for a free massless scalar. If this Hamiltonian is multiplied by a correction function that approaches zero at small $V$, the small-volume behavior is clearly modified. The general small-volume behavior of inverse-volume corrections is of the power-law form $f(Q) \sim f_0 Q^n$ with a positive integer $n > 0$, where both $n$ and $f_0$ depend on quantization ambiguities [36, 37].

A second, less obvious place for inverse-volume corrections is on the gravitational side of the Hamiltonian constraint or the Friedmann equation (2.2). Even though the classical contribution $V P^2$ does not contain an inverse $V$, any embedding of this isotropic term in an anisotropic model requires inverse triad components [38, 39]. Therefore, both sides of the Friedmann equation can receive independent correction functions of the form $f(Q) \sim f_0 Q^n$, with different $f_0$ and $n$ in each case. We can then combine these two functions by bringing both of them to one side, say the gravitational one. The ratio of these two functions is again of the form $f(Q) \sim f_0 Q^n$, but the integer $n$ is no longer restricted to be positive.

Inserting such a function in the full Hamiltonian constraint and solving for $p_\phi$ implies that the previous Hamiltonian $QP$ is replaced by the non-quadratic $|Q|^{1-n/2}P$, or the non-linear $|Q|^{-n/2}{\text{Im}}J/\delta$. In terms of analog models, $\ddot{Q} = (1 - n/2)|Q|^{-n/2}\dot{Q}\text{sgn}Q = (1 - n/2)|Q|^{1-n}\text{sgn}Q$ requires a potential

$$W(Q) = -\frac{1}{2} |Q|^{2-n}. \quad (5.5)$$

For $n \leq 2$, the qualitative behavior of solutions for given energy values is the same as in the harmonic case. If $n > 0$, $Q = 0$ may be reached even for negative-energy solutions. Inverse-volume corrections therefore make it more likely that small-volume solutions do not bounce. Their detailed role in this context remains to be explored.

6 Implications for signature change

The possibility of various bouncing or non-bouncing solutions (3.8), (3.10), or (3.11) implies an interesting behavior regarding signature change. This phenomenon has so far been considered only for bouncing solutions (3.8). But modified space-time structures that could
give rise to signature change are most likely realized at small volume where non-bouncing solutions are possible as well.

Space-time structure cannot be derived within a homogeneous model but rather requires an embedding in perturbative or some other form of inhomogeneity. The perturbative [40, 41] and midisuperspace case [42–44] have been studied in some detail, indicating a generic modification of the space-time structure as a consequence of holonomy corrections [45]: whenever an inhomogeneous Hamiltonian constraint \( H[N] \) is holonomy-modified by extending the replacement of \( P \) with \( \sin(\delta P)/\delta \) to inhomogeneity, its Poisson bracket

\[
\{H[N_1], H[N_2]\} = D[\beta(P)q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)]
\]

(6.1)
differs from the classical bracket (\( \beta = 1 \)) by a function

\[
\beta(P) = \cos(2\delta P).
\]

(6.2)

Via Dirac’s hypersurface deformations [46], any \( \beta \neq 1 \) demonstrates a modified space-time structure. In particular, if \( \beta \) can be negative for certain \( P \), such as around a local maximum of \( \sin(\delta P) \) in the case of (6.2), space-time has Euclidean signature [47–51].

For bouncing solutions (3.8), signature change happens around the local minimum of \( \langle \hat{Q}\rangle(\lambda) \), implying that the bounce, even if it occurs, is not deterministic. (The initial-value problem is not well-posed in Euclidean signature.) This result can be rederived for our solutions, where, as a new feature, we take into account the quantization ambiguity \( \delta_1 \). In order to express \( \beta \) in terms of our solutions (3.4), we use \( \text{Re}J = Q \cos(\delta P) \) and write

\[
\beta = \cos(2\delta P) = 2\cos^2(\delta P) - 1 = 2\left(\frac{\text{Re}J}{Q}\right)^2 - 1.
\]

(6.3)
The bouncing solution (3.8) is obtained for \( AB > 0 \), such that we can assume \( A = B > 0 \) by choosing a suitable zero value of \( \lambda \). Therefore,

\[
\text{Re}J(\lambda) = A\sinh(\lambda), \quad Q(\lambda) = A\cosh(\lambda) - \hbar \delta_1
\]

(6.4)

implies a non-constant

\[
\beta(\lambda) = 1 - 2\frac{1 - 2\hbar \delta_1 A^{-1} \cosh(\lambda) + \hbar^2 \delta_1^2 / A^2}{(\cosh(\lambda) - \hbar \delta_1 / A)^2}.
\]

(6.5)

While \( \beta \to 1 \) for large \( \lambda \), \( \beta(0) = -1 \) at the local minimum of \( Q(\lambda) \). Around the bounce, space-time is therefore Euclidean.

The situation is rather different for our new, non-bouncing solutions. The case of \( AB = 0 \), or (3.10), is interesting because it implies that \( \text{Re}J = Q + \hbar \delta \). In this case,

\[
\beta = 2\left(1 + \frac{\hbar \delta_1}{Q}\right)^2 - 1 = 1 + 4\frac{\hbar \delta_1}{Q} + 2\frac{\hbar^2 \delta_1^2}{Q^2}.
\]

(6.6)

Therefore, \( \beta = 1 \) of \( \delta_1 = 0 \), and the classical space-time structure is realized even at small volume. For \( AB < 0 \), we have

\[
\text{Re}J(\lambda) = A\cosh(\lambda), \quad Q(\lambda) = A\sinh(\lambda) - \hbar \delta_1
\]

(6.7)
and
\[ \beta(\lambda) = 1 + 2 \frac{h \delta_1 A^{-1} \sinh(\lambda) - \hbar^2 \delta_1^2 / A^2}{(\sinh(\lambda) - h \delta_1 / A)^2} = -1 + 2 \frac{1 + \sinh^2(\lambda)}{(\sinh(\lambda) - h \delta_1 / A)^2}. \] (6.8)
This function is negative for \( \sinh(\lambda) \) between
\[ s_\pm = -\frac{h \delta_1}{A} \pm \sqrt{2} \sqrt{\hbar^2 \delta_1^2 / A^2 - 1}. \] (6.9)
For \( \delta_1 = 0 \), \( \beta \) is always positive, such that there is no signature change even though the space-time structure is non-classical (\( \beta > 1 \)).

7 Conclusions

We have derived several new results related to algebraic properties of solvable models of loop quantum cosmology, relevant for the question of whether bouncing solutions are generic. A copious amount of quantization ambiguities has been illustrated by an explicit relationship between the algebraic and Hilbert-space treatments of such models in section 2.2.

There are two main independent types of ambiguities, related to choices of factor orderings and inequivalent representations through the value of a Casimir variable. In section 5.1, we have parameterized their outcomes in solutions of dynamical equations by the shift \( \delta_1 \) of the volume expectation value and the Casimir \( R \) which appears in the reality condition and determines implications of quantum fluctuations. This result has revealed that the Hilbert-space treatment given in [2] is far from being unique. Although the final representation (on a physical Hilbert space) used in this context has been derived from a representation on a kinematical Hilbert space, the latter is subject to choices and assumptions, for instance regarding inner products, which are difficult to classify. The algebraic treatment, by contrast, can build on the representation theory of \( \text{sl}(2, \mathbb{R}) \) in order to determine possible quantization choices, and to relate them to physical outcomes. As a consequence, it can be seen that [2] implicitly made several specific choices for ambiguous objects that increase the likelihood of bouncing solutions, corresponding to the values \( R = 0 \) and \( \delta_1 = 0 \) in our classification of section 5.1.

Our analysis in section 2.2 led us to an identification of the quantization given in [2] with a specific representation of \( \text{sl}(2, \mathbb{R}) \). This representation is reducible, given by the direct sum of two irreducible representations in the discrete series. Each of these two irreducible representations is such that only eigenstates of the triad operator \( \hat{Q} \) of a specific sign are included. The representation used in [2] therefore does not include an operator that would map a state supported on \( Q > 0 \) to a state with some support on \( Q < 0 \). This observation, based on representation theory, allowed us to prove, for the first time, that bouncing solutions are generic in the model constructed in [2]; see section 3.2. However, this conclusion has a sobering side too: within the set of all possible representations of \( \text{sl}(2, \mathbb{R}) \), which all amount to quantizations of holonomy-modified cosmological dynamics and therefore constitute loop quantum cosmology, choosing a specific representation, as implicitly done in the construction of [2], appears to be rather ad-hoc. Moreover, within the chosen representation, [2] also implicitly selected a factor ordering such that the triad operator \( \hat{Q} \) is a generator of \( \text{sl}(2, \mathbb{R}) \). More generally, there can be a constant shift determined by a quantization ambiguity \( \delta_1 \) in section 5.1, such that the generator \( \hat{L}_0 \) is given by \( \hat{Q} \) shifted by a constant. Each of the two irreducible representations then has a fixed sign of \( \hat{L}_0 \)-eigenvalues, and for a suitable sign of
the shift constant, $\langle \hat{Q} \rangle$ can cross zero and change sign even if $\langle \hat{L}_0 \rangle$ never does so in the given representation.

From this perspective, bouncing solutions are generic in the model of [2] only because the dynamics is formulated such that $\text{sgn} Q$ is fixed on an irreducible subrepresentation, which in combination with the discrete spectrum of $\hat{Q}$ implies that $Q = 0$ cannot be reached. There are irreducible representations of $\text{sl}(2, \mathbb{R})$, each of which includes states with positive as well as negative eigenvalues of $\hat{Q}$. All representations in the continuous series are of this form, and they may be preferred on fundamental grounds because they provide irreducible representations of the dynamical algebra. While [2] made a serendipitous choice of a representation that leads to generic bouncing solutions, bounces are not generic within loop quantum cosmology in general.

While choices can be made, explicitly or implicitly, that increase the likelihood of bouncing solutions, they still leave room for non-bouncing solutions and therefore do not suffice to prove that bounces are generic in loop quantum cosmology, in particular in the case of small-volume relevant near a BKL-type singularity. Within the set of representations of $\text{sl}(2, \mathbb{R})$, there is a large number of possibilities for non-bouncing solutions which asymptotically approach or cross zero triad expectation values. While evolution then remains meaningful in isotropic models, the formulation of inhomogeneous cosmological modes on such degenerate background geometries would be problematic. Throughout this discussion, we have seen several detailed relationships between the representation theory of $\text{sl}(2, \mathbb{R})$ and moment equations derived using canonical effective theory.

In section 4, we have shown that, perhaps surprisingly, non-bouncing solutions may be consistent with Planckian upper bounds on energy densities, but again the specific outcome depends on quantization ambiguities. Even if the triad expectation value crosses zero, quantum fluctuations prevent the volume expectation value from being zero. As a consequence, such upper bounds strongly depend on the form of relevant quantum states as well as the definition of a density operator. Also here, [2] has managed to implicitly choose a quantization that is beneficial in producing such a bound. Inverse-volume corrections do not change this qualitative picture, as shown in section 5.2 mainly using a new set of useful analog models introduced in section 3.3. An interesting feature of non-bouncing solutions is that they lead to novel examples of quantum space-time structures, possibly avoiding signature change.

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