General collective dephasing for qubit-qutrit $(2 \otimes 3)$ quantum systems

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Abstract

Most studies of collective dephasing for bipartite as well as multipartite quantum systems have considered a very specific orientation of magnetic field, that is, z-orientation. However, in practical situations, there are always small fluctuations in stochastic field and it is necessary that more general orientations of fields should be considered. Here, we investigate qubit-qutrit systems such that the qubit part is exposed to a general orientation of magnetic field and the qutrit part has standard $\sigma_z$ dephasing operator. We study entanglement properties of various specific quantum states under this general collective dephasing. We examine sudden death of entanglement, time-invariant entanglement and freezing dynamics of entanglement. We also compare entanglement dynamics of random states for various orientations of stochastic fields. We find that for specific quantum states as well as for random states, freezing dynamics of entanglement is the dominant dynamics under general collective dephasing. We also analyze the asymptotic states and find the conditions for having either sudden death of entanglement or freezing dynamics. We believe our results are relevant for ion-trap experiments and can be verified with current setups.

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I. INTRODUCTION

Most realistic quantum systems are coupled to an environment that induces decoherence and dissipation. The study of such open quantum systems with different environments is an active area of research for both Markovian [1] and non Markovian scenarios [2]. The quantum part of correlations present in quantum states has potential applications, not limited to remote state preparation [3], entanglement distribution [4, 5], quantum metrology [6], quantum communication, and quantum computation [7], to name a few. In last two decades, considerable efforts have been devoted to develop a theoretical framework for quantification and characterization of quantum correlations [8, 9]. Another essential aspect is to analyze and simulate the effects of decoherence and dissipation on quantum correlations. Several authors have studied decoherence effects on quantum correlations for both bipartite and multipartite systems [10–23].

To realize protocols of quantum information and quantum computation, a large variety of physical systems have been proposed and investigated. Electronic excitations of atoms and molecules, ion-traps, nuclear magnetic resonances, quantum dots, superconducting quantum interference devices, etc. [24]. Of all these possibilities, ion-trap approach seems to be viable to realize quantum computer. In these experiments, ions/atoms are trapped in electromagnetic fields and quantum computations are performed by quantum logic gates, quantum measurements etc. [25–27]. The typical noise here is caused by intensity fluctuations of electromagnetic fields which leads to collective dephasing process. The description of collective dephasing normally take a special orientation of magnetic field, that is, along z-axes, such that Hamiltonian describing the interaction contain the corresponding dephasing operators. It is well known that decoherence degrades quantum correlations in general and entanglement in particular. Many authors have investigated the effects of such collective dephasing on entanglement for bipartite and multipartite quantum systems [28–38].

Recently, the specific z-oriented fluctuations in magnetic field has been extended to an arbitrary orientation for N noninteracting atomic qubits [39, 40]. The resulting decoherence process can be called as general collective dephasing. It was shown that entanglement of a specific two qubits state may first decay upto some numerical value before suddenly stop decaying and maintain this stationary entanglement at all times [39, 40]. This non-trivial feature of entanglement decay is named as freezing dynamics of entanglement [41],
where we found that specific quantum states of three and four qubits as well as most of the respective random states exhibit such dynamics under general collective dephasing. We mention here that normally there are no decoherence free subspaces (DFS) under general collective dephasing, whereas for z-oriented field such spaces are always there. Freezing dynamics of entanglement has also shown to be present for collective dephasing with DFS. Another non-trivial dynamics of entanglement present/observed under collective dephasing is so called time-invariant entanglement. Time-invariant entanglement does not necessarily mean that the quantum states live in DFS. In fact, quantum states may change at every instance whereas their entanglement remain constant throughout the dynamical process. This feature was first observed for qubit-qutrit systems [33] and then later on for qubit-qubit systems both in theory and in experiment [34]. Recently, we have investigated time-invariant phenomenon for genuine entanglement of three and four qubits. We have found no evidence of time-invariant for three qubits, whereas for four qubits we have demonstrated its presence explicitly [35]. More recently, we have explored the possibility of either time-invariant entanglement or freezing dynamics for qutrit-qutrit (3⊗3) systems [37]. We found no evidence for time-invariant entanglement, however we observed the exclusive evidence for freezing dynamics of entanglement. We have noticed that in most studies on time-invariant entanglement and freezing dynamics for a given Hilbert space, there exist either time-invariant entanglement or freezing dynamics. Interestingly, so far only for qubit-qutrit systems, we found both these features present [38]. We have shown that there are certain states which exhibit either time-invariant entanglement or sudden death of entanglement but never freezing dynamics. On the other hand, some other quantum states exhibit either freezing dynamics or sudden death but never time-invariant dynamics [38]. In this work, we extend general collective dephasing to qubit-qutrit system such that qubit is exposed to stochastic fields in an arbitrary orientation $\vec{n}$, whereas qutrit is described by standard dephasing operator $\sigma_z$. We solve the system completely and provide a master equation in differential form which can be solved straight forwardly for any arbitrary initial quantum state. We provide the most general solution of density matrix elements in terms of field and decoherence parameters. We study dynamics of entanglement for various orientations of magnetic field both for specific quantum states and random states. We also analyze quantum states at infinity and study entanglement properties of asymptotic states. We find the conditions such that we can either expect sudden death of entanglement or freezing
This paper is organized as follows. In section II, we briefly discuss our model of interest and obtain the master equation for the system alone whose most general solution for an arbitrary initial density matrix is straightforward. In section III, we review the idea of entanglement for qubit-qutrit systems and provide the details of how we can calculate the entanglement for any initial state. In section IV, we present our main results of the dynamics of entanglement for some specific quantum states also for random states. We analyze asymptotic quantum states in section V. Finally we conclude our work in section VI.

II. COLLECTIVE DEPHASING FOR QUBIT-QUTRIT SYSTEMS

Our physical model consists of a qubit and a qutrit (one two-level atom and one three-level atom for an example) A and B that are coupled to a noisy environment, collectively. The qutrit as an atom, can be realized in any configuration depending on experimental convenience. There are three well known configurations for a three level atom. In V-type energy level configuration, the transition among excited levels is forbidden. This means that first excited state will decay to ground level only and similarly the second excited level will also decay to ground level. In Λ-type configuration, the excited level \( |2\rangle \) can decay either to level \( |1\rangle \) or directly to level \( |0\rangle \). The transition from \( |1\rangle \) to \( |0\rangle \) is forbidden. The third type of configuration is called cascade configuration in which the energy level \( |2\rangle \) first decay to \( |1\rangle \) and then energy level \( |1\rangle \) decays to ground level \( |0\rangle \). For amplitude damping, it is very important to specify the configuration as the atomic transition operators for each configuration are different and hence the subsequent dynamics would be different. The atoms are sufficiently far apart and they do not interact with each other, so that we can treat them as independent. The collective dephasing refers to coupling of atoms to the same noisy environment, which can be stochastic magnetic field \( B(t) \). There are at least two approaches to write a Hamiltonian for such physical situations. First, the Hamiltonian could be time independent and we can write a unitary propagator \( U(t) = \exp(-iHt/\hbar) \). As there are fluctuations in magnetic field strength, the integration over it will induce a probability distribution \( p(w) \) of characteristic energy splitting. The time evolution of atom can be written as an integral over \( p(\omega) \) and unitary evolution, i.e., \( \rho(t) = \int p(\omega)U(t)\rho(0)U(t)^\dagger d\omega \). The form of \( p(\omega) \) will determine the nature of noise. Another approach is to take the
Hamiltonian as time dependent and embed the fluctuations of magnetic field in stochastic function $B(t)$, which already includes the information about characteristic function and so that the ensemble average over it introduce the decay parameter $\Gamma$. Both approaches are equivalent and generates the same dynamics. We point out that to our knowledge all previous works are restricted to a very specific orientation of magnetic field and the theory for a general description of magnetic fields in any arbitrary directions is not studied before for qubit-qutrit systems. To this aim, we extend previous studies by allowing an arbitrary orientation of magnetic field on qubit part only. The Hamiltonian of the quantum system (with $\hbar = 1$) can be written as

$$H(t) = -\frac{\mu}{2} \left[ B(t) \left( \bar{n} \cdot \vec{\sigma}^A + \sigma_z^B \right) \right],$$

$$= -\frac{\mu}{2} \left[ B(t) \left( n_x \sigma_x^A + n_y \sigma_y^A + n_z \sigma_z^A + \sigma_z^B \right) \right],$$

(1)

where $\mu$ is gyromagnetic ratio, $\bar{n} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$, is a unit vector such that $|n_x|^2 + |n_y|^2 + |n_z|^2 = 1$, $\sigma_i^A$ are standard Pauli matrices for qubit $A$ and $\sigma_z^B$ is the dephasing operator for qutrit $B$. We note that for $n_x = n_y = 0$, $n_z = 1$, we have a specific orientation (z-axes) of magnetic field. The stochastic magnetic fields refer to statistically independent classical Markov processes satisfying the conditions

$$\langle B(t) B(t') \rangle = \frac{\Gamma}{\mu^2} \delta(t - t'),$$

$$\langle B(t) \rangle = 0,$$

(2)

with $\langle \cdots \rangle$ as ensemble time average and $\Gamma$ denote the phase-damping rate for collective dephasing.

Let $|2\rangle$, $|1\rangle$, and $|0\rangle$ be the first excited state, second excited, and ground state of the qutrit, respectively. We choose the computational basis $\{ |0,0\rangle, |0,1\rangle, |0,2\rangle |1,0\rangle, |1,1\rangle, |1,2\rangle \}$, where we have dropped the subscripts $A$ and $B$ with the understanding that first basis represents qubit $A$ and second qutrit $B$. Also the notation $|0\rangle \otimes |0\rangle = |00\rangle$ has been adopted for simplicity. The time-dependent density matrix for the system is obtained by taking ensemble average over the noisy field, i.e., $\rho(t) = \langle \rho_{st}(t) \rangle$, where $\rho_{st}(t) = U(t)\rho(0)U^\dagger(t)$ and $U(t) = \exp[-i \int_0^t dt' H(t')]$. We assume that there are no initial correlations between the qubit-qutrit system and the stochastic field, that is, $\rho(0) = \rho_S \otimes \rho_R$, where $\rho_S$ is density matrix for an arbitrary quantum state of qubit-qutrit system and $\rho_R$ is the density matrix.
of environment. There are several ways to obtain the time evolved density matrix of the qubit-qutrit system. We prefer to solve the system using master equation approach.

According to general reservoir theory \cite{42}, we consider a qubit-qutrit system (S) interacting with a reservoir (R). The combined density operator for system can be written as $\rho_{SR}$. The reduced density operator $\rho_{S}$ for system (S) is calculated using the standard technique of taking partial trace over the reservoir degrees of freedom, that is, $\rho_{S} = Tr_{R}(\rho_{SR})$. In the interaction picture, the equation of motion can be written as

$$i \dot{\rho}_{SR}(t) = \left[ H(t), \rho_{SR}(t) \right].$$

(3)

We can simply integrate this equation to obtain

$$\rho_{SR}(t) = \rho_{SR}(t_{i}) - i \int_{t_{i}}^{t} \left[ H(t'), \rho_{SR}(t') \right] dt',$n

(4)

where $t_{i}$ is starting time of interaction. Substituting Eq.(4) back into Eq.(3), we get equation of motion

$$\dot{\rho}_{SR}(t) = -i \left[ H(t), \rho_{S}(t_{i}) \right]$$

$$- \int_{t_{i}}^{t} \left[ H(t), \left[ H(t'), \rho_{SR}(t') \right] \right] dt'.$n

(5)

As interaction between the system and reservoir is weak, we look for a solution of Eq.(5) of the form

$$\rho_{SR}(t) = \rho_{S}(t) \otimes \rho_{R}(t_{i}) + \rho_{c}(t),$$

(6)

where $\rho_{c}(t)$ is of higher order in $H(t)$. For consistancy, we require that $Tr_{R}(\rho_{c}(t)) = 0$. Substituting Eq.(6) into Eq.(5) and after some simplifications we get

$$\dot{\rho}_{S}(t) = -i Tr_{R} \left[ H(t), \rho_{S}(t_{i}) \otimes \rho_{R}(t_{i}) \right]$$

$$- Tr_{R} \int_{t_{i}}^{t} \left[ H(t), \left[ H(t'), \rho_{S}(t) \otimes \rho_{R}(t_{i}) \right] \right] dt'.n

(7)

In this equation, the reduced density operator $\rho_{S}(t)$ depends on past history from $t = t_{i}$ to $t'$. Typically, reservoir have many degrees of freedom, which leads to delta function $\delta(t - t')$. Under Markovian assumption, the system density matrix $\rho_{S}(t')$ can be replaced by $\rho_{S}(t)$, which is quite reasonable since damping destroys memory of past. We can write Eq.(7)

$$\dot{\rho}_{S}(t) = -i Tr_{R} \left[ H(t), \rho_{S}(t_{i}) \otimes \rho_{R}(t_{i}) \right]$$

$$- Tr_{R} \int_{t_{i}}^{t} \left[ H(t), \left[ H(t'), \rho_{S}(t) \otimes \rho_{R}(t_{i}) \right] \right] dt'.n

(8)
This is a valid master equation for a system $\rho_S$ interacting with a reservoir represented by $\rho_R$. Substituting the Hamiltonian Eq. \((1)\) in Eq. \((8)\), using the relations in Eq. \((2)\), and doing some algebra, we arrive at equation for system given as

$$\dot{\rho}(t) = -\frac{\Gamma}{4} \left\{ n_x^2 \sigma_x^A \sigma_x^A \rho(t) - n_y^2 \sigma_y^A \rho(t) \sigma_x^A + n_x n_y \sigma_y^A \sigma_x^A \rho(t) 
-n_x n_y \sigma_x^A \rho(t) \sigma_y^A + n_x n_z \sigma_z^A \sigma_x^A \rho(t) - n_x n_z \sigma_x^A \rho(t) \sigma_z^A 
+n_x \sigma_z^B \sigma_x^A \rho(t) - n_x \sigma_x^A \rho(t) \sigma_z^B + n_x n_y \sigma_z^A \sigma_x^A \rho(t) 
-n_x n_y \sigma_x^A \rho(t) \sigma_z^A + n_y^2 \sigma_y^A \sigma_y^A \rho(t) - n_y^2 \sigma_y^A \rho(t) \sigma_y^A 
+n_y n_z \sigma_z^A \sigma_y^A \rho(t) - n_y n_z \sigma_y^A \rho(t) \sigma_z^A + n_y \sigma_z^B \sigma_y^A \rho(t) 
-n_y \sigma_y^A \rho(t) \sigma_z^B + n_z n_y \sigma_z^A \sigma_z^A \rho(t) - n_z n_y \sigma_z^A \rho(t) \sigma_z^A + n_z^2 \sigma_z^A \sigma_z^A \rho(t) 
+n_z \sigma_z^B \sigma_z^A \rho(t) - n_z \sigma_z^A \rho(t) \sigma_z^B + n_y \sigma_z^B \sigma_z^A \rho(t) 
-n_y \sigma_z^B \rho(t) \sigma_z^A + n_z \sigma_z^A \sigma_z^B \rho(t) - n_z \sigma_z^B \rho(t) \sigma_z^A + n_z^2 \sigma_z^A \sigma_z^B \rho(t) 
+n_z \sigma_z^B \sigma_z^B \rho(t) - n_z \sigma_z^B \rho(t) \sigma_z^B + n_y \sigma_z^A \sigma_z^B \rho(t) 
-n_y \sigma_z^B \rho(t) \sigma_z^B + n_z \sigma_z^B \sigma_z^B \rho(t) - n_x \sigma_z^B \rho(t) \sigma_z^B + n_x \sigma_z^B \sigma_z^B \rho(t) 
+\sigma_z^B \sigma_z^B \rho(t) - \sigma_z^B \rho(t) \sigma_z^B + n_x^2 \sigma_x^A \rho(t) \sigma_x^A 
+n_x^2 \rho(t) \sigma_x^A \sigma_x^A - n_x n_y \sigma_y^A \rho(t) \sigma_x^A + n_x n_y \rho(t) \sigma_x^A \sigma_y^A 
-n_x n_z \sigma_z^A \rho(t) \sigma_x^A + n_x n_z \rho(t) \sigma_x^A \sigma_z^A - n_x \sigma_z^B \rho(t) \sigma_x^A 
+n_x \rho(t) \sigma_x^A \sigma_z^B - n_x n_y \sigma_x^A \rho(t) \sigma_y^A + n_x n_y \rho(t) \sigma_y^A \sigma_x^A 
-n_y^2 \sigma_y^A \rho(t) \sigma_y^A + n_y^2 \rho(t) \sigma_y^A \sigma_y^A - n_y n_z \sigma_z^A \rho(t) \sigma_y^A 
+n_y n_z \rho(t) \sigma_y^A \sigma_z^A - n_y \sigma_z^B \rho(t) \sigma_y^A + n_y \rho(t) \sigma_y^A \sigma_z^B 
-n_x n_z \sigma_z^A \rho(t) \sigma_y^A + n_x n_z \rho(t) \sigma_y^A \sigma_z^A - n_x n_z \sigma_y^A \rho(t) \sigma_z^A 
+n_x n_z \rho(t) \sigma_z^A \sigma_y^A - n_x^2 \sigma_z^A \rho(t) \sigma_y^A + n_x^2 \rho(t) \sigma_y^A \sigma_z^A 
-n_x \sigma_z^B \rho(t) \sigma_z^A + n_x \rho(t) \sigma_z^A \sigma_z^B - n_x \sigma_z^B \rho(t) \sigma_z^B 
+n_x \rho(t) \sigma_z^B \sigma_z^A - n_x \sigma_z^B \rho(t) \sigma_z^B + n_x \rho(t) \sigma_z^B \sigma_z^A 
-n_x \sigma_y^A \rho(t) \sigma_z^B + n_x \rho(t) \sigma_z^B \sigma_z^A - \sigma_z^B \rho(t) \sigma_z^B 
+\rho(t) \sigma_z^B \sigma_z^B \right\}. \quad (9)$$

We have dropped the subscript $S$ for system as there is no chance of confusion now. This is a simple differential equation, which can be straightforwardly solved to give us the most general solution of the system. We provide the general solution of the equation in
appendix A. However, we just mention that there are groups of four matrix elements coupled together, like \((\rho_{11}(t), \rho_{14}(t), \rho_{41}(t), \rho_{44}(t))\), another \((\rho_{22}(t), \rho_{25}(t), \rho_{52}(t), \rho_{55}(t))\) etc., as can also be observed from general solution. For general collective dephasing \((n_x \neq 0, n_y \neq 0,\) and \(n_z \neq 0)\), there are no decoherence free subspaces (DFS) \([28]\) in this system. Another interesting property of the dynamics is the fact that all initially zero matrix elements may not remain zero at later times.

We note that for z-orientation of magnetic field, that is, \(n_x = n_y = 0,\) and \(n_z = 1\), we recover the solution of system given as

\[
\rho(t) = \begin{pmatrix}
\rho_{11} & \xi \rho_{12} & \xi^4 \rho_{13} & \xi^4 \rho_{14} & \xi^9 \rho_{15} & \xi^{16} \rho_{16} \\
\xi \rho_{21} & \rho_{22} & \xi \rho_{23} & \xi^4 \rho_{24} & \xi^9 \rho_{25} & \xi^{16} \rho_{26} \\
\xi^4 \rho_{31} & \xi \rho_{32} & \rho_{33} & \rho_{34} & \xi \rho_{35} & \xi^4 \rho_{36} \\
\xi^4 \rho_{41} & \xi \rho_{42} & \rho_{43} & \rho_{44} & \xi \rho_{45} & \xi^4 \rho_{46} \\
\xi^9 \rho_{51} & \xi^4 \rho_{52} & \xi \rho_{53} & \rho_{54} & \rho_{55} & \xi \rho_{56} \\
\xi^{16} \rho_{61} & \xi^9 \rho_{62} & \xi^4 \rho_{63} & \xi \rho_{64} & \xi \rho_{65} & \rho_{66}
\end{pmatrix},
\]

where \(\xi = e^{-\Gamma t/4}\) \([38]\).

### III. Entanglement for \(2 \otimes 3\) Quantum Systems

The characterization and quantification of entanglement for qubit-qubit \((2 \otimes 2)\) and qubit-quatrit \((2 \otimes 3)\) quantum systems has been already worked out and well known. It is known that for these systems, if the partial transpose with respect of any one of the subsystem has all positive eigenvalues then the quantum state is separable or PPT \([43]\). If any eigenvalue of partial transposed matrix is negative then state is entangled or NPT. The condition of being NPT is necessary and sufficient for \(2 \otimes 2\) and \(2 \otimes 3\) systems to be entangled, whereas for larger dimensions of Hilbert space, there may exist PPT-entangled states (also called bound entangled states) \([44]\). There are many measures of entanglement defined in literature, like Entanglement of Formation \([45]\), Concurrence \([46]\), Schmidt number \([47]\), etc. The details of these and all other measures can be found in review article \([44]\). However, most of these measures have closed formulas only for qubit-qubit quantum system and in general it is not easy to calculate them for an arbitrary quantum mixed state of other dimension of Hilbert space. Negativity is a measure which is easy to compute and completely quantifies
entanglement of qubit-qutrit system, which is defined as the sum of absolute values of all possible negative eigenvalues \[48\]. Hence, negativity is defined as

\[
N(\rho) = 2 \left( \sum |\eta_i| \right),
\]

(11)

where \(\eta_i\) are possible negative eigenvalues and multiplication with 2 is for normalization so that for maximally entangled states, this measure should have numerical value of 1.

The problem to compute entanglement for our system is then to find negative eigenvalues of partially transposed matrix. This is not possible sometimes due to large number of parameters in density matrix. For example, in our current work we have two parameters related with orientation of magnetic field \((n_i)\), one or two parameters with initial quantum states and one parameter \(\Gamma t\) related with decoherence. To find analytical eigenvalues of a \(6 \times 6\) matrix with 4 or larger parameters is hard even with computer algebra systems (CAS). However, the solution to this problem exists already in literature. Recently, a powerful technique has been worked out to detect and characterize entanglement \[49\]. The method is to use positive partial transpose mixtures (PPT mixtures). This method was worked out primarily to detect genuine entanglement for multipartite systems, however for bipartite systems it is equal to negativity. The semidefinite programming has been utilized to compute the measure efficiently and the optimality of the solution can be certified. We have used the programs YALMIP and SDPT3, and a ready-to-use implementation, all of these are freely available, for details, see \[49\]. In fact the measure is shown to be an entanglement monotone for genuine multiparticle entanglement. For bipartite systems, this monotone is equivalent to the so-called negativity \[48\].

IV. ENTANGLEMENT DYNAMICS OF SPECIFIC AND RANDOM QUANTUM STATES

Being equipped with general solution for any arbitrary initial state and a way to compute their entanglement, we now proceed to study how some specific quantum states react to an arbitrary magnetic field as compared with a special z-orientation. We also generate random states and study their behavior for various settings of \(\vec{n}\).
A. Dynamics of specific quantum states

Example 1. First we take quantum states with two real parameters $\alpha$ and $\gamma$ defined for $2 \otimes d$ quantum systems [50]. For qubit-qutrit system, the states are given as

$$\rho_{\alpha,\gamma} = \alpha (|0 2\rangle \langle 0 2| + |1 2\rangle \langle 1 2|) + \beta (|\phi^+\rangle \langle \phi^+| + |\phi^-\rangle \langle \phi^-|$$

$$+ |\psi^+\rangle \langle \psi^+|) + \gamma |\psi^-\rangle \langle \psi^-|, \quad (12)$$

where

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|0 0\rangle \pm |1 1\rangle) \quad (13)$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}} (|0 1\rangle \pm |1 0\rangle), \quad (14)$$

and parameter $\beta$ is dependent on $\alpha$ and $\gamma$ by the unit trace condition,

$$2 \alpha + 3 \beta + \gamma = 1. \quad (15)$$

From Eq. (12) one can easily obtain the range of parameters as $0 \leq \alpha \leq 1/2$, $0 \leq \beta \leq 1/3$, and $0 \leq \gamma \leq 1$. We note that the states of the form $\rho_{0,\gamma}$ are equivalent to Werner states [51] in a $2 \otimes 2$ quantum systems. Moreover, the states $\rho_{\alpha,\gamma}$ have the property that their PPT (positive partial transpose) region is always separable [50]. It is also known that an arbitrary quantum state $\rho$ in $2 \otimes d$ can be transformed to $\rho_{\alpha,\gamma}$ with the help of local operations and classical communication (LOCC).

Figure (1) depicts dynamics of entanglement for state $\rho_{\alpha,\gamma}(t)$ with five settings of $\vec{n}$. We have set the following values of parameters, $\alpha = 0.1$, $\beta = 0.1$, and $\gamma = 0.5$. The solid (black) line is for $z$-orientation of magnetic field. As we can see that even in this case, the states loose their entanglement at $\Gamma t \approx 2.76$ hence undergoing sudden death. All other orientations of magnetic field have even more adverse effect on entanglement as sudden death is hastened to earlier times. In section (V), we analyze asymptotic states and clearly demonstrate that these states must exhibit sudden death of entanglement for an arbitrary orientation of magnetic field. Therefore, these states are quite fragile under general collective dephasing.

Example 2. We define a single parameter class of states as the mixture of two maximally entangled states. The states are given as

$$\rho_\alpha = \alpha |\psi_3\rangle \langle \psi_3| + (1 - \alpha) |\psi_2\rangle \langle \psi_2|, \quad (16)$$
where $0 \leq \alpha \leq 1$, the maximally entangled state $|\psi_2\rangle$ is defined as

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\ 1\rangle + |1\ 2\rangle),$$

and another maximally entangled state $|\psi_3\rangle$ is defined as

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} (|0\ 2\rangle + |1\ 0\rangle).$$

In Ref. [38], we have studied this state for only z-orientation of magnetic field and have found that in this mixture $|\psi_2\rangle$ decay, whereas $|\psi_3\rangle$ lives in decoherence free subspace. We have also observed freezing dynamics of entanglement for various values of parameter $\alpha$ [38]. Here, we aim to study entanglement properties of these states for more general orientations of magnetic field.

Figure (2) shows entanglement monotone plotted against parameter $\Gamma t$ for various settings of $\vec{n}$ with a specific value of parameter $\alpha = 0.3$. The solid (black) line denote entanglement for z-orientation of magnetic field as we have already observed in a previous study, leads to freezing dynamics of entanglement. The setting $\vec{n} = (1, 0, 0)$ turns out to be most destructive and leads to entanglement sudden death at $\Gamma t \approx 2.27$. As we demonstrate in section (V), solid (black) line, dashed-dotted (blue) line, and dashed (red) line corresponds to freezing dynamics of entanglement, whereas dotted (green) line leads to sudden death of
FIG. 2. Entanglement $E(\rho_\alpha(t))$ is plotted against parameter $\Gamma t$ with various settings of $\vec{n}$. We have taken a specific value of parameter $\alpha = 0.3$. We can see that entanglement is lost at finite time for $n_x = 1$, and $n_y = n_z = 0$. For other settings, read the text for details.

entanglement for the current choice of $\alpha = 0.3$. However, we find that if $\alpha > 0.337$, the dotted (green) curve can exhibit freezing dynamics of entanglement. Only for $\vec{n} = (1, 0, 0)$, the states exhibit sudden death of entanglement for all values of $\alpha$. Hence we observe that freezing dynamics is quite common phenomenon here.

**Example 3.** As another example, we first define a single parameter class of states given as

$$\tilde{\rho}_\alpha = \alpha |\psi_1\rangle\langle\psi_1| + \frac{1 - \alpha}{6} \mathbb{I}_6,$$

where $\mathbb{I}_6$ is $6 \times 6$ identity matrix and $0 \leq \alpha \leq 1$. In this equation, the pure state $|\psi_1\rangle$ is another maximally entangled state as

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|0 0\rangle + |1 2\rangle).$$

Such states are called isotropic states and they are NPT for $1/4 < \alpha \leq 1$, and hence entangled. We can now define a two parameter family of states, which are mixture of isotropic states and $|\psi_3\rangle$, given as

$$\rho_{\alpha,\beta} = \beta |\psi_3\rangle\langle\psi_3| + (1 - \beta) \tilde{\rho}_\alpha,$$

(19)
where $0 \leq \beta \leq 1$. In Ref. [38], we have observed time-invariant entanglement for this state. Here we want to study how these states change their entanglement under more general collective dephasing.

In Figure (3), we plot entanglement monotone against decay parameter $\Gamma t$ for various choices of parameter $\vec{n}$. We have taken specific values of parameters $\alpha = 0.4$ and $\beta = 0.7$. For $n_x = n_y = 0$, and $n_z = 1$, solid (black) line clearly reflects time-invariant entanglement as we have observed earlier [38]. We observe that for $n_y = n_z = 0$, and $n_x = 1$ starred (pink) line leads to sudden death of entanglement at $\Gamma t \approx 0.9$. Once again this specific orientation of magnetic field seems to be most destructive for entanglement. Another interesting feature is the appearance of freezing dynamics of entanglement for other three settings of $\vec{n}$. We can see that dashed-dotted (blue) line, dashed (red) line, and dotted line lead to freezing dynamics of entanglement. We can analyze asymptotic quantum states as shown in section (V) and based on the eigenvalues of the partially transposed matrix, we can completely understand the behavior of entanglement dynamics for these states.
B. Dynamics of random pure states

In order to compare entanglement dynamics of specific quantum states with generic states, we first generate 100 random pure states for qubit-qutrit systems. A state vector for qubit-qutrit systems, randomly distributed according to the Haar measure can be generated in the following way \[52\]: First, we generate a vector such that both the real and the imaginary parts of the vector elements are Gaussian distributed random numbers with a zero mean and unit variance. Second we normalize the vector. It is easy to prove that the random vectors obtained this way are equally distributed on the unit sphere \[52\]. Note that the random pure states, which we generate in the global Hilbert space of dimension 6, so the unit sphere is not the Bloch ball.

After generation of 100 random pure states, we find their time-evolved density matrices interacting with general collective dephasing and compute negativity using PPT-mixture package \[49\], for each state against parameter $\Gamma t$. From this data we can also obtain an error estimate to indicate the reliability of the measure. This can, for instance, be defined as a confidence interval \[20\]

\[
CI = \mu \pm \sqrt{\delta},
\]  

(22)

where $\mu$ stands for mean value and $\delta$ for variance of quantity being measured. Note, however, that this is not a confidence interval in the mathematical sense.

In Figure (4), we plot negativity against parameter $\Gamma t$ for 100 initial random pure states. We have taken $n_x = 2/\sqrt{6}$ and $n_y = n_z = 1/\sqrt{6}$. We observe that very few states exhibit almost time-invariant entanglement feature, which is quite interesting as there are no decoherence free spaces for this choice of magnetic field. Another feature is the presence of entanglement sudden death, as some states become PPT at finite time. Finally, we also observe the freezing dynamics of entanglement as well. We observe that most of the states tends to stable their entanglement after long time. The thick dashed (red) line denote the average entanglement of 100 states, which is calculated by adding entanglement of these 100 states for each instance of time and divided by number 100. It is evident that average entanglement also tends to exhibit freezing dynamics. The thick dashed-dotted (blue) lines denote the confidence interval (CI) defined above. The top line represents as sum of the mean value and variance, whereas below line is for difference of mean value and variance.
FIG. 4. Entanglement monotone is plotted against parameter $\Gamma t$ for 100 random pure states. We have taken $n_x = 2/\sqrt{6}$, and $n_y = n_z = 1/\sqrt{6}$. As we observe, most of the states exhibit freezing dynamics of entanglement and some of the state exhibit sudden death of entanglement. See text for details.

We see that the error estimation is similar above and below the line for positive and negative sign in definition, respectively. This is quite good indicator of reliability of the measure.

In Figure (5), negativity is plotted against parameter $\Gamma t$ for 100 initial random states for another setting of magnetic field with $n_x = 1$, and $n_y = n_z = 0$. As we have seen above that for all specific examples of quantum states, this orientation of magnetic field is most destructive for entanglement. However, for random states we see that very few states exhibit sudden death of entanglement. Most of the states exhibit freezing dynamics of entanglement. The thick dashed (red) line is the average negativity for 100 random states and this curve also tends to freezing dynamics. The thick dashed-dotted (blue) lines are confidence intervals, with top line as sum of mean value and variance whereas below thick dashed-dotted (blue) line is for the difference between mean value and variance. This behavior is similar to what we observe in Figure (4) again providing the reliability of the measure.

In Figure (6), we plot entanglement monotone $E(\rho(t))$ for 100 initial random pure states against parameter $\Gamma t$. We have taken $n_x = 0$, and $n_y = n_z = 1/\sqrt{2}$. We observe that most of the states exhibit freezing dynamics of entanglement. About one fifth of the states also
FIG. 5. Entanglement monotone is plotted against parameter $\Gamma t$ for 100 random pure states. We have taken $n_x = 1$, and $n_y = n_z = 0$. As we observe, most of the states exhibit freezing dynamics of entanglement and some of the state exhibit sudden death of entanglement. See text for details.

FIG. 6. Entanglement monotone is plotted against parameter $\Gamma t$ for 100 random pure states. We have taken $n_x = 0$, and $n_y = n_z = 1/\sqrt{2}$. As we observe, most of the states exhibit freezing dynamics of entanglement and some of the state exhibit sudden death of entanglement. See text for details.
FIG. 7. Entanglement monotone is plotted against parameter $\Gamma t$ for 100 random pure states. We have taken $n_x = n_y = n_z = 1/\sqrt{3}$. As we observe, most of the states exhibit freezing dynamics of entanglement and some of the states exhibit sudden death of entanglement. See text for details.

As a final example with random states, we take the parameters $n_x = n_y = n_z = 1/\sqrt{3}$. Figure (7) shows entanglement monotone plotted against parameter $\Gamma t$ for 100 random pure states. Interestingly, we observe that at least one random state exhibit time-invariant entanglement, whereas two other states show almost time-invariant behavior. Most of the other states exhibit freezing dynamics of entanglement and some states undergo sudden death of entanglement. The thick dashed (red) line is the average value of entanglement. The mean value also show freezing dynamics of entanglement. The other two thick dashed-dotted (blue) lines represent confidence intervals with same details as mentioned above. We observe similar behavior as we observed in all previous cases.

We clarify that we have generated new random states each time to get the data for Figures (4-7). This means that the data presented here is actually for 400 different random
V. ASYMPTOTIC QUANTUM STATES

The most general solution given in appendix A suggests that we can analyze the quantum states at infinity. By taking $\Gamma t \to \infty$, we obtained asymptotic states given as

$$
\rho(\infty) = \begin{pmatrix}
\varrho_{11} & \varrho_{13} & \varrho_{14} & \varrho_{16} \\
0 & \varrho_{22} & 0 & \varrho_{25} \\
\varrho_{31} & 0 & \varrho_{33} & \varrho_{34} & \varrho_{36} \\
\varrho_{41} & 0 & \varrho_{43} & \varrho_{44} & 0 \\
0 & \varrho_{52} & 0 & \varrho_{55} & 0 \\
\varrho_{61} & 0 & \varrho_{63} & \varrho_{64} & \varrho_{66}
\end{pmatrix},
$$

(23)

where $\varrho_{ij}$ are function of parameters $n_k$ and initial density matrix elements $\rho_{qr}$. We provide the detailed expressions of $\varrho_{ij}$ in appendix B. We note that from these expressions, we can get the special case of $z$-orientation of field, that is, $n_x = n_y = 0$, and $n_z = 1$, we have all off-diagonal elements equal to zero except $\varrho_{34}$ and $\varrho_{43}$. Therefore we recover the earlier result. For two other special orientations, ($n_x = 1$, $n_y = n_z = 0$) and ($n_y = 1$, $n_x = n_z = 0$) and in general for $n_x \neq n_y \neq n_z \neq 0$, we have all $\varrho_{ij}$ as nonzeros. Hence there are no decoherence free subspaces for general collective dephasing.

Based on these observations, we can easily identify the set of entangled states which must exhibit sudden death of entanglement. For other entangled states, we may get either freezing dynamics or sudden death of entanglement. Let us first write the various entangled pure states in Schmidt decomposition

$$
|\Phi_1\rangle = \alpha_1 |0 0\rangle \pm \beta_1 |1 2\rangle,
$$

(24)

$$
|\Phi_2\rangle = \alpha_2 |0 0\rangle \pm \beta_2 |1 1\rangle,
$$

(25)

$$
|\Phi_3\rangle = \alpha_3 |0 1\rangle \pm \beta_3 |1 2\rangle,
$$

(26)

$$
|\Phi_4\rangle = \alpha_4 |0 1\rangle \pm \beta_4 |1 0\rangle,
$$

(27)

$$
|\Phi_5\rangle = \alpha_5 |0 2\rangle \pm \beta_5 |1 0\rangle,
$$

(28)

$$
|\Phi_6\rangle = \alpha_6 |0 2\rangle \pm \beta_6 |1 1\rangle.
$$

(29)
It should be mentioned here that an arbitrary pure state for qubit-qutrit system can be written as

\[ |\Phi\rangle = (U_A \otimes U_B) \left( \alpha |00\rangle + \sqrt{1 - \alpha^2} |11\rangle \right), \]  

(30)

where \( U_A \) and \( U_B \) denote transformations from the computational basis to the Schmidt basis on qubit and qutrit, respectively. A close examination of asymptotic states suggest that any mixed quantum state which has entangled states \(|\Phi_2\rangle\) (Eq. 25), \(|\Phi_3\rangle\) (Eq. 26), \(|\Phi_4\rangle\) (Eq. 27), and \(|\Phi_6\rangle\) (Eq. 29) as a dominant fraction in it, necessarily exhibit entanglement sudden death. Whereas mixed states with large fractions of states \(|\Phi_1\rangle\) (Eq. 24) and \(|\Phi_5\rangle\) (Eq. 28) can lead to either sudden death of entanglement or freezing dynamics of entanglement.

We also note that \(|\Phi_2\rangle\) and \(|\Phi_4\rangle\) are related with each other by a local switch on qutrit alone. Similarly relation is also between \(|\Phi_3\rangle\) and \(|\Phi_6\rangle\) and also between \(|\Phi_1\rangle\) and \(|\Phi_5\rangle\). We demonstrate below that under general collective dephasing asymptotic entangled states may exist.

As our specific example 1 has \(|\Phi_2\rangle\) (Eq. 25) and \(|\Phi_4\rangle\) (Eq. 27) in it with \( \alpha_i = \beta_i = 1/\sqrt{2} \), so we must have sudden death of entanglement for all orientations of magnetic field. This is precisely what we observed in Figure (1).

Our example 2 is defined as mixture of \(|\Phi_3\rangle\) and \(|\Phi_5\rangle\) with specific \( \alpha_i = \beta_i = 1/\sqrt{2} \) so we get either sudden death of entanglement or freezing dynamics depending upon different settings of \( \vec{n} \). For \( \vec{n} = (0, 1/\sqrt{2}, 1/\sqrt{2}) \), we have only one possible negative eigenvalue for asymptotic state, given as

\[ \eta_1 = \frac{1}{16} \left\{ (2 - 3\sqrt{2})\alpha - (2 + \sqrt{2}) \right\} \right) \]  

(31)

This eigenvalue is found to be negative for \( \alpha > 0.15 \). Therefore, dashed-dotted (blue) line in Figure (2) represents freezing dynamics of entanglement. For \( \vec{n} = 1/\sqrt{3}(1, 1, 1) \), the only possible negative eigenvalue at infinity is given as

\[ \lambda_1 = \frac{1}{24} \left\{ 3 + \sqrt{3} - 3\alpha(-1 + \sqrt{3}) \right\} - \sqrt{6(2 + \sqrt{3})\sqrt{1 - 2\alpha + 5\alpha^2}} \right) \]  

(32)

This eigenvalue is found to be negative for \( \alpha > 0.224 \). Hence the dashed (red) line in Figure (2) also show freezing dynamics of entanglement. For \( \vec{n} = 1/\sqrt{6}(2, 1, 1) \), we found
that the only possible negative eigenvalue at infinity is given as

$$\delta_1 = \frac{1}{48} \left\{ 6 + \sqrt{6} - 3 \alpha (-2 + \sqrt{6}) \right. \right.$$ 

$$\left. - \sqrt{6(7 + 2\sqrt{6})\sqrt{1 - 2 \alpha + 5 \alpha^2}} \right\} . \tag{33}$$

This eigenvalue is negative for $\alpha > 0.337$. As we have taken $\alpha = 0.3$ for Figure (2), so we note that dotted (green) line decays and represent sudden death of entanglement. Similiar for $\vec{n} = (1, 0, 0)$, we found no negative eigenvalues which means necessarily sudden death of entanglement for all values of $\alpha$. For $\vec{n} = (0, 0, 1)$, we have already observed freezing dynamics of entanglement [38].

Our specific example 3 is mixture of $|\Phi_1\rangle$ with $\alpha_1 = \beta_1 = 1/\sqrt{2}$ and $|\Phi_5\rangle$ with $\alpha_5 = \beta_5 = 1/\sqrt{2}$. Therefore, we may get either sudden death of entanglement or freezing dynamics depending upon orientation of magnetic field. We note that time-invariant entanglement can only occur for special orientation of $\vec{n} = (0, 0, 1)$. It is simple to check the possible negative eigenvalues for the respective asymptotic state and discuss the dynamical behavior. We can also find the eigenvalues for each random state at infinity and check their entanglement properties in a similar fashion.

VI. CONCLUSIONS

We have extended the previous studies on Markovian collective dephasing of qubit-qutrit systems from a specific orientation of magnetic field to an arbitrary orientation of magnetic field. The usual $z$-oriented magnetic field give rise to decoherence free subspaces (DFS), whereas the more general collective dephasing have no such subspaces. The extension from $z$-oriented field collective dephasing was recently done only for multipartite systems composed of qubits only [39]. We have considered the situation where our qubit is interacting with stochastic fields oriented in an arbitrary direction dictated by $\vec{n}$, however the field is oriented along $z$-axes for the qutrit part. We have obtained a master equation and its solution for any arbitrary initial quantum state. Using a computable entanglement monotone, it is possible for us to study dynamics of quantum entanglement for specific quantum states as well as for random states. Due to lengthy solutions for each density matrix element and also due to presence of two parameters related with arbitrary magnetic field and one decay parameter, it is not possible to obtain analytical eigenvalues for any specific quantum
state. As we studied recently, that entanglement in general has three non-trivial types of dynamics, namely, time-invariant entanglement, sudden death of entanglement, and freezing dynamics of entanglement. Previous studies on these features of entanglement dynamics for bipartite as well as for multipartite quantum systems gave the impression that we could not have all three features available for one specific dimension of Hilbert space under collective dephasing. In our recent study [38], we found counter examples to this impression for qubit-qutrit quantum systems. It was observed that for qubit-qubit systems interacting with general directions of magnetic fields [39], one can find freezing dynamics of entanglement instead of specific z-direction where we can also find time-invariant feature. In this work, we have seen that for all specific quantum states, general collective dephasing degrades entanglement more than specific z-oriented field. Even for quantum state exhibiting time-invariant entanglement under specific z-oriented magnetic field, we get either sudden death of entanglement or freezing dynamics of entanglement. We have also studied statistics of entanglement dynamics for random states for various settings of magnetic field $\vec{n}$. We have found that very few random states exhibit sudden death of entanglement for general orientations. In all orientations of magnetic field, majority of random states exhibit freezing dynamics of entanglement. We have been able to find most general asymptotic quantum states in terms of parameters $n_i$ and the initial density matrix elements. This knowledge can conclusively explain the dynamics of entanglement for any arbitrary orientation of magnetic field. These observations are quite positive regarding preservation of entanglement under this type of decoherence. As many experiments are already in quite advanced stage for ion-traps, where this kind of noise is dominant, we believe that our study is most relevant to such experiments and these observations can be readily demonstrated. Future avenues could be to develop theoretical models for general collective dephasing models for high dimensional quantum systems.
Appendix A

In this appendix we provide the most general solution of Eq. (9) for an arbitrary density matrix. The time-evolved density matrix elements are given as

\[
\rho_{11}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ \left( 1 + e^{\Gamma t} \right) \rho_{11} + \left( -1 + e^{\Gamma t} \right) \rho_{44} \right\} n_x^2 \\
+ \left( -1 + e^{\Gamma t} \right) n_z (\rho_{14} + \rho_{41}) n_x + 2 e^{\Gamma t} n_x^2 \rho_{11} \\
+ i \left( -1 + e^{\Gamma t} \right) n_y n_z (\rho_{14} - \rho_{41}) \\
+ n_y^2 \left( e^{\Gamma t} \rho_{11} + \rho_{11} + e^{\Gamma t} \rho_{44} - \rho_{44} \right) \}. \quad (A1)
\]

\[
\rho_{12}(t) = \frac{1}{4} e^{-9\Gamma t/4} \left\{ \left( 1 + e^{2\Gamma t} \right) (\rho_{15} - \rho_{42}) n_x^3 \\
+ \left\{ 3 e^{2\Gamma t} \rho_{12} + \rho_{12} + (-1 + e^{2\Gamma t}) \right\} \\
\left( i n_y (\rho_{15} + \rho_{42}) + \rho_{45} \right) \} n_x^2 + \left( -1 + e^{2\Gamma t} \right) \\
\left( (\rho_{15} - \rho_{42}) n_y^2 + n_z (n_z \rho_{15} + \rho_{15} - n_z \rho_{42} + \rho_{42}) \right) n_x \\
+ 4 e^{2\Gamma t} n_z^2 \rho_{12} + i \left( -1 + e^{2\Gamma t} \right) n_y^3 (\rho_{15} + \rho_{42}) \\
+ i \left( -1 + e^{2\Gamma t} \right) n_y n_z ((n_z + 1) \rho_{15} + (n_z - 1) \rho_{42}) \\
+ n_y^2 \left( 3 e^{2\Gamma t} \rho_{12} + \rho_{12} + (-1 + e^{2\Gamma t}) \rho_{45} \right) \}. \quad (A2)
\]

\[
\rho_{13}(t) = \frac{1}{4} e^{-4\Gamma t} \left\{ \left( 1 + e^{4\Gamma t} \right) (\rho_{16} - \rho_{43}) n_x^3 \\
+ \left\{ (1 + 2 e^{3\Gamma t} + e^{4\Gamma t}) \rho_{13} + i \left( -1 + e^{4\Gamma t} \right) n_y \right\} (\rho_{16} + \rho_{43}) - \left( 1 - 2 e^{3\Gamma t} + e^{4\Gamma t} \right) \rho_{46} \} n_x^2 \\
+ \left\{ ( -1 + e^{4\Gamma t} ) \right\} n_y^2 (\rho_{16} - \rho_{43}) - n_z \left\{ (1 - e^{4\Gamma t}) \\
\left( n_z (\rho_{16} - \rho_{43}) + (1 - 2 e^{3\Gamma t} + e^{4\Gamma t}) (\rho_{16} + \rho_{43}) \right) \} n_x \\
+ 4 e^{3\Gamma t} n_z^2 \rho_{13} + i \left( -1 + e^{4\Gamma t} \right) n_y^3 (\rho_{16} + \rho_{43}) \\
+ i n_y n_z \left\{ (-1 + 2 e^{3\Gamma t} - e^{4\Gamma t}) \right\} (\rho_{16} - \rho_{43}) \\
+ (-1 + e^{4\Gamma t}) n_z (\rho_{16} + \rho_{43}) \} + n_y^2 \{ \\
\left( 1 + 2 e^{3\Gamma t} + e^{4\Gamma t} \right) \rho_{13} - \left( 1 - 2 e^{3\Gamma t} + e^{4\Gamma t} \right) \rho_{46} \} \}. \quad (A3)
\]
\[ \rho_{14}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ \left( (1 + e^{\Gamma t}) \rho_{14} + (-1 + e^{\Gamma t}) \rho_{41} \right) n_x^2 \\
+ (-1 + e^{\Gamma t}) (n_z (\rho_{11} - \rho_{44}) - 2i n_y \rho_{41}) n_x \\
+ 2 n_z^2 \rho_{14} + n_y^2 \left( e^{\Gamma t} \rho_{14} + \rho_{14} - e^{\Gamma t} \rho_{41} + \rho_{41} \right) \\
- i \left( -1 + e^{\Gamma t} \right) n_y n_z (\rho_{11} - \rho_{44}) \right\}. \quad (A4) \]

\[ \rho_{15}(t) = \frac{1}{4} e^{-9 \Gamma t/4} \left\{ \left( -1 + e^{2 \Gamma t} \right) (\rho_{12} - \rho_{45}) n_x^3 \\
+ \{ (e^{2 \Gamma t} (3 - 2 n_z) + 2 n_z + 1) \rho_{15} \\
+ (-1 + e^{2 \Gamma t}) \rho_{42} - i \left( -1 + e^{2 \Gamma t} \right) n_y \rho_{42} \} n_x^2 \\
+ (-1 + e^{2 \Gamma t}) \{ -2i \rho_{42} n_y \rho_{42} \\
+ (\rho_{12} - \rho_{45}) n_y^2 + n_z (n_z + 1) (\rho_{12} - \rho_{45}) \} n_x \\
-2 \left( e^{2 \Gamma t} (n_z - 1) - n_z - 1 \right) n_z^2 \rho_{15} + n_y^2 \left\{ e^{2 \Gamma t} (3 - 2 n_z) + 2 n_z + 1 \right\} \rho_{15} - e^{2 \Gamma t} \rho_{42} \\\n+ \rho_{42} \} - i \left( -1 + e^{2 \Gamma t} \right) n_y^3 (\rho_{12} - \rho_{45}) \\
- i \left( -1 + e^{2 \Gamma t} \right) n_y n_z (n_z + 1) (\rho_{12} - \rho_{45}) \right\}. \quad (A5) \]

\[ \rho_{16}(t) = \frac{1}{4} e^{-4 \Gamma t} \left\{ \left( -1 + e^{4 \Gamma t} \right) (\rho_{13} - \rho_{46}) n_x^3 \\
+ \{ (e^{4 \Gamma t} (1 - 2 n_z) + 2 e^{3 \Gamma t} + 2 n_z + 1) \rho_{16} \\
- (1 - 2 e^{3 \Gamma t} + e^{4 \Gamma t}) \rho_{43} - i \left( -1 + e^{4 \Gamma t} \right) \\
\} n_y (\rho_{13} - \rho_{46}) \} n_x^2 + \{ n_y^2 (-1 + e^{4 \Gamma t}) \\
(\rho_{13} - \rho_{46}) + 2i \rho_{43} n_y (1 - 2 e^{3 \Gamma t} + e^{4 \Gamma t}) \\
- n_z (-e^{4 \Gamma t} (n_z - 1) - 2 e^{3 \Gamma t} + n_z + 1) \\
(\rho_{13} - \rho_{46}) \} n_x + 2 n_z^2 \{ -e^{4 \Gamma t} (n_z - 1) \\
+ n_z + 1 \} \rho_{16} + n_y^2 \left\{ e^{4 \Gamma t} (1 - 2 n_z) \\
+ 2 e^{3 \Gamma t} + 2 n_z + 1 \right\} \rho_{16} + \rho_{43} \\\n\} \left( 1 - 2 e^{3 \Gamma t} + e^{4 \Gamma t} \right) \} - i \left( -1 + e^{4 \Gamma t} \right) \\
\} n_y (\rho_{13} - \rho_{46}) + i n_y n_z \{ -e^{4 \Gamma t} (n_z - 1) \\
- 2 e^{3 \Gamma t} + n_z + 1 \} (\rho_{13} - \rho_{46}) \right\}. \quad (A6) \]
\[ \rho_{22}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ (1 + e^{\Gamma t}) \rho_{22} + (-1 + e^{\Gamma t}) \rho_{55} \right\} n_x^2 \\
+ (-1 + e^{\Gamma t}) n_z (\rho_{25} + \rho_{52}) n_x + 2 e^{\Gamma t} n_x^2 \rho_{22} \\
+i (-1 + e^{\Gamma t}) n_y n_z (\rho_{25} - \rho_{52}) \\
+n_y^2 (e^{\Gamma t} \rho_{22} + \rho_{22} + e^{\Gamma t} \rho_{55} - \rho_{55}) \right\}. \quad (A7) \]

\[ \rho_{23}(t) = \frac{1}{4} e^{-\frac{9}{4} \Gamma t} \left\{ (-1 + e^{2\Gamma t}) (\rho_{26} - \rho_{53}) n_x^3 \\
+\{3 e^{2\Gamma t} \rho_{23} + \rho_{23} + (-1 + e^{2\Gamma t}) \\
(i n_y (\rho_{26} + \rho_{53}) + \rho_{56}) \} n_x^2 + (-1 + e^{2\Gamma t}) \\
\{ (\rho_{26} - \rho_{53}) n_y^2 + n_z (n_z \rho_{26} + \rho_{26} + \rho_{53} \\
- n_z \rho_{53}) \} n_x + 4 e^{2\Gamma t} n_x^2 \rho_{23} + i n_y^3 \\
(-1 + e^{2\Gamma t}) (\rho_{26} + \rho_{53}) + i (-1 + e^{2\Gamma t}) \\
n_y n_z ((n_z + 1) \rho_{26} + (n_z - 1) \rho_{53}) \\
+n_y^2 (3 e^{2\Gamma t} \rho_{23} + \rho_{23} + (-1 + e^{2\Gamma t}) \rho_{56}) \right\}. \quad (A8) \]

\[ \rho_{24}(t) = \frac{1}{4} e^{-\frac{9}{4} \Gamma t} \left\{ -(-1 + e^{2\Gamma t}) (\rho_{21} - \rho_{54}) n_x^3 \\
+\{-2 n_z + e^{2\Gamma t}(2 n_z + 3) + 1 \} \rho_{24} + \\
(-1 + e^{2\Gamma t}) \rho_{51} + i (-1 + e^{2\Gamma t}) n_y \\
\rho_{21} - \rho_{54} \} n_x^2 - (-1 + e^{2\Gamma t}) \\
\{ (\rho_{21} - \rho_{54}) n_y^2 + 2 i \rho_{51} n_y + (n_z - 1) n_z \\
(\rho_{21} - \rho_{54}) \} n_x + 2 n_x^2 \{-n_z + 1 + \\
e^{2\Gamma t}(n_z + 1) \} \rho_{24} + n_y^2 \{-2 n_z + 1 + \\
e^{2\Gamma t}(2 n_z + 3) \} \rho_{24} - e^{2\Gamma t} \rho_{51} + \rho_{51} \} \\
+i (-1 + e^{2\Gamma t}) n_y (\rho_{21} - \rho_{54}) + i \\
(-1 + e^{2\Gamma t}) n_y (n_z - 1) n_z (\rho_{21} - \rho_{54}) \right\}. \quad (A9) \]
\[
\rho_{25}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ (1 + e^{\Gamma t}) \rho_{25} + (-1 + e^{\Gamma t}) \rho_{52} \right\} n_x^2 \\
+ (-1 + e^{\Gamma t}) (n_z (\rho_{22} - \rho_{55}) - 2 i n_y \rho_{52}) n_x \\
+ 2 n_x^2 \rho_{25} + n_y^2 (e^{\Gamma t} \rho_{25} + \rho_{25} - e^{\Gamma t} \rho_{52} + \rho_{52}) \\
- i (-1 + e^{\Gamma t}) n_y n_z (\rho_{22} - \rho_{55}) \right\}. \tag{A10}
\]

\[
\rho_{26}(t) = \frac{1}{4} e^{-9\Gamma t/4} \left\{ (-1 + e^{2\Gamma t}) (\rho_{23} - \rho_{56}) n_x^3 \\
+ \left\{ (e^{2\Gamma t}(3 - 2 n_z) + 2 n_z + 1) \rho_{26} + \\
(-1 + e^{2\Gamma t}) \rho_{53} - i (-1 + e^{2\Gamma t}) n_y \\
(\rho_{23} - \rho_{56}) \right\} n_x^2 + (-1 + e^{2\Gamma t}) n_x \right\} \\
(\rho_{23} - \rho_{56}) n_y^2 - 2 i \rho_{53} n_y + n_z (n_z + 1) \\
(\rho_{23} - \rho_{56}) \right\} - 2 (e^{2\Gamma t}(n_z - 1) - n_z - 1) \\
n_x^2 \rho_{26} + n_y^2 \left\{ (e^{2\Gamma t}(3 - 2 n_z) + 2 n_z + 1) \right. \\
\left. \rho_{26} - e^{2\Gamma t} \rho_{53} + \rho_{53} \right\} - i (-1 + e^{2\Gamma t}) \\
n_y^3 (\rho_{23} - \rho_{56}) - i (-1 + e^{2\Gamma t}) n_y n_z \\
(n_z + 1) (\rho_{23} - \rho_{56}) \right\}. \tag{A11}
\]

\[
\rho_{33}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ (1 + e^{\Gamma t}) \rho_{33} + (-1 + e^{\Gamma t}) \rho_{66} \right\} n_x^2 \\
+ (-1 + e^{\Gamma t}) n_z (\rho_{36} + \rho_{63}) n_x + 2 e^{\Gamma t} n_x^2 \rho_{33} \\
+ i (-1 + e^{\Gamma t}) n_y n_z (\rho_{36} - \rho_{63}) \\
+ n_y^2 (e^{\Gamma t} \rho_{33} + \rho_{33} + e^{\Gamma t} \rho_{66} - \rho_{66}) \right\}. \tag{A12}
\]
\[
\rho_{34}(t) = \frac{1}{4} e^{-4 \Gamma t} \left\{ -(-1 + e^{4 \Gamma t}) \left( \rho_{31} - \rho_{64} \right) n_x^3 \right. \\
+ \left\{ (-2 n_z + 2 e^{3 \Gamma t} + e^{4 \Gamma t}(2 n_z + 1) + 1) \right. \\
- (1 - 2 e^{3 \Gamma t} + e^{4 \Gamma t}) \left. \rho_{61} + i (-1 + e^{4 \Gamma t}) n_y \\
(\rho_{31} - \rho_{64}) \right\} n_x^2 + \left\{ (-1 + e^{4 \Gamma t}) (\rho_{31} - \rho_{64}) n_y^2 \\
+ 2 i (1 - 2 e^{3 \Gamma t} + e^{4 \Gamma t}) \rho_{61} n_y - n_z \right\} - n_z \\
- 2 e^{3 \Gamma t} + e^{4 \Gamma t}(n_z + 1) + 1 \right\} (\rho_{31} - \rho_{64}) \} n_x \\
+ 2 n_x^2 \left( -n_z + e^{4 \Gamma t}(n_z + 1) + 1 \right) \rho_{34} \\
+ n_y^2 \left\{ (-2 n_z + 2 e^{3 \Gamma t} + e^{4 \Gamma t}(2 n_z + 1) + 1) \right. \\
+ (1 - 2 e^{3 \Gamma t} + e^{4 \Gamma t}) \left. \rho_{61} \right\} + i (-1 + e^{4 \Gamma t}) n_y^3 \\
(\rho_{31} - \rho_{64}) + i n_y n_z \left\{ -n_z - 2 e^{3 \Gamma t} \right. \\
+ e^{4 \Gamma t}(n_z + 1) + 1 \right\} (\rho_{31} - \rho_{64}) \right\}. \tag{A13}
\]

\[
\rho_{35}(t) = \frac{1}{4} e^{-9 \Gamma t/4} \left\{ -(-1 + e^{2 \Gamma t}) \left( \rho_{32} - \rho_{65} \right) n_x^3 \right. \\
+ n_x^2 \left\{ (-2 n_z + e^{2 \Gamma t}(2 n_z + 3) + 1) \right. \\
+ (-1 + e^{2 \Gamma t}) \left. \rho_{62} + i (-1 + e^{2 \Gamma t}) n_y \\
(\rho_{32} - \rho_{65}) \right\} - (-1 + e^{2 \Gamma t}) \left\{ (\rho_{32} - \rho_{65}) \\
n_x^2 + 2 i n_x n_y (n_z - 1) n_z (\rho_{32} - \rho_{65}) \right\} \\
n_x + 2 n_x^2 \left( -n_z + e^{2 \Gamma t}(n_z + 1) + 1 \right) \rho_{35} \\
+ n_y^2 \left\{ (-2 n_z + e^{2 \Gamma t}(2 n_z + 3) + 1) \right. \\
- e^{2 \Gamma t} \rho_{62} + \rho_{62} \right\} + i (-1 + e^{2 \Gamma t}) \left. n_y^3 \\
(\rho_{32} - \rho_{65}) + i (-1 + e^{2 \Gamma t}) n_y \\
(n_z - 1) n_z (\rho_{32} - \rho_{65}) \right\}. \tag{A14}
\]

\[
\rho_{36}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ \left( (1 + e^{\Gamma t}) \rho_{36} + (-1 + e^{\Gamma t}) \rho_{65} \right) n_x^2 \\
+ (-1 + e^{\Gamma t}) (n_z (\rho_{33} - \rho_{66}) - 2 i n_y \rho_{63}) n_x \\
+ 2 n_x^2 \rho_{36} + n_y^2 \left\{ e^{\Gamma t} \rho_{36} + \rho_{36} - e^{\Gamma t} \rho_{63} + \rho_{63} \right\} \\
- i (-1 + e^{\Gamma t}) n_y n_z (\rho_{33} - \rho_{66}) \right\}. \tag{A15}
\]
\[
\rho_{44}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ \left( -1 + e^{\Gamma t} \right) \rho_{11} + \left( 1 + e^{\Gamma t} \right) \rho_{44} \right\} n_x^2 \\
- \left( -1 + e^{\Gamma t} \right) n_z (\rho_{14} + \rho_{41}) n_x - i \left( -1 + e^{\Gamma t} \right) n_y n_z (\rho_{14} - \rho_{41}) + 2 e^{\Gamma t} n_z^2 \rho_{44} + n_y^2 \\
\left( e^{\Gamma t} \rho_{11} - \rho_{11} + e^{\Gamma t} \rho_{44} + \rho_{44} \right) \right\}. \quad (A16)
\]

\[
\rho_{45}(t) = \frac{1}{4} e^{-9\Gamma t/4} \left\{ \left( 1 - e^{2\Gamma t} \right) (\rho_{15} - \rho_{42}) n_x^3 \\
+ \left\{ \left( -1 + e^{2\Gamma t} \right) \rho_{12} - i \left( -1 + e^{2\Gamma t} \right) n_y \right. \\
\left( \rho_{15} + \rho_{42} \right) + 3 e^{2\Gamma t} \rho_{45} + \rho_{45} \} n_x^2 \\
- \left( -1 + e^{2\Gamma t} \right) \left\{ (\rho_{15} - \rho_{42}) n_y^2 + \\
n_z (n_z \rho_{15} + \rho_{15} - n_z \rho_{42} + \rho_{42}) \right\} n_x \\
- i \left\{ \left( -1 + e^{2\Gamma t} \right) n_x^3 (\rho_{15} + \rho_{42}) - i n_y n_z \\
\left( -1 + e^{2\Gamma t} \right) \left\{ (n_x + 1) \rho_{15} + (n_x - 1) \rho_{42} \right\} \\
+ 4 e^{2\Gamma t} n_z^2 \rho_{45} + n_y^2 \left\{ \left( -1 + e^{2\Gamma t} \right) \\
\rho_{12} + 3 e^{2\Gamma t} \rho_{45} + \rho_{45} \right\} \right\}. \quad (A17)
\]

\[
\rho_{46}(t) = \frac{1}{4} e^{-4\Gamma t} \left\{ \left( 1 - e^{4\Gamma t} \right) (\rho_{16} - \rho_{43}) n_x^3 \\
+ \left\{ \left( -1 + 2 e^{3\Gamma t} - e^{4\Gamma t} \right) \rho_{13} - i \left( -1 + e^{4\Gamma t} \right) \\
n_y (\rho_{16} + \rho_{43}) + (1 + 2 e^{3\Gamma t} + e^{4\Gamma t}) \rho_{46} \} n_x^2 \\
+ n_x \left\{ \left( 1 - e^{4\Gamma t} \right) (\rho_{16} - \rho_{43}) n_y^2 + n_z \left\{ \left( 1 - e^{4\Gamma t} \right) \\
n_z (\rho_{16} - \rho_{43}) + (1 - 2 e^{3\Gamma t} + e^{4\Gamma t}) (\rho_{16} + \rho_{43}) \left\} \right\} \\
- i \left\{ \left( -1 + e^{4\Gamma t} \right) n_y^3 (\rho_{16} + \rho_{43}) + 4 e^{3\Gamma t} n_z^2 \rho_{46} + i \\
n_y n_z \left\{ \left( 1 - 2 e^{3\Gamma t} + e^{4\Gamma t} \right) (\rho_{16} - \rho_{43}) - \\
\left( -1 + e^{4\Gamma t} \right) n_z (\rho_{16} + \rho_{43}) \right\} - n_y^2 \left\{ \left( 1 - 2 e^{3\Gamma t} \\
+ e^{4\Gamma t} \right) \rho_{13} - \left( 1 + 2 e^{3\Gamma t} + e^{4\Gamma t} \right) \rho_{46} \right\} \right\}. \quad (A18)
\]
\[ \rho_{55}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ \left( -1 + e^{\Gamma t} \right) \rho_{22} + \left( 1 + e^{\Gamma t} \right) \rho_{55} \right\} n_x^2 \\
- \left( -1 + e^{\Gamma t} \right) n_z (\rho_{25} + \rho_{52}) n_x - i \left( -1 + e^{\Gamma t} \right) n_y n_z (\rho_{25} - \rho_{52}) + 2 e^{\Gamma t} n_z^2 \rho_{55} + \right. \\
\left. n_y^2 \left( e^{\Gamma t} \rho_{22} - \rho_{22} + e^{\Gamma t} \rho_{55} + \rho_{55} \right) \right\}. \quad (A19) \]

\[ \rho_{56}(t) = \frac{1}{4} e^{-9 \Gamma t/4} \left\{ \left( 1 - e^{2 \Gamma t} \right) (\rho_{26} - \rho_{53}) n_x^3 \\
+ \left\{ \left( -1 + e^{2 \Gamma t} \right) \rho_{23} - i \left( -1 + e^{2 \Gamma t} \right) n_y (\rho_{26} + \rho_{53}) + 3 e^{2 \Gamma t} \rho_{56} \right\} n_x^2 \\
- \left( -1 + e^{2 \Gamma t} \right) \left\{ ((\rho_{26} - \rho_{53}) n_y^2 + n_z \right. \\
\left. (n_x \rho_{26} + \rho_{26} - n_z \rho_{53} + \rho_{53}) \right\} n_x - i \left( -1 + e^{2 \Gamma t} \right) \\
\left. \left( -1 + e^{2 \Gamma t} \right) n_y^2 (\rho_{26} + \rho_{53}) - i \left( -1 + e^{2 \Gamma t} \right) n_y n_z ((n_z + 1) \rho_{26} + (n_z - 1) \rho_{53}) + 4 e^{2 \Gamma t} n_z^2 \rho_{56} \\
+ n_y^2 \left( -1 + e^{2 \Gamma t} \right) \rho_{23} + 3 e^{2 \Gamma t} \rho_{56} + \rho_{56} \right\}. \quad (A20) \]

\[ \rho_{66}(t) = \frac{1}{2} e^{-\Gamma t} \left\{ \left( -1 + e^{\Gamma t} \right) \rho_{33} + \left( 1 + e^{\Gamma t} \right) \rho_{66} \right\} n_x^2 \\
- \left( -1 + e^{\Gamma t} \right) n_z (\rho_{36} + \rho_{63}) n_x - i \left( -1 + e^{\Gamma t} \right) n_y n_z (\rho_{36} - \rho_{63}) + 2 e^{\Gamma t} n_z^2 \rho_{66} + n_y^2 \\
\left( e^{\Gamma t} \rho_{33} - \rho_{33} + e^{\Gamma t} \rho_{66} + \rho_{66} \right). \quad (A21) \]

Appendix B: Density matrix elements at infinity

In this appendix, we provide density matrix elements at infinity in terms of parameters \(n_i\) and initial density matrix elements \(\rho_{ij}\).

\[ \rho_{11} = \frac{1}{2} \left\{ 2 \rho_{11} n_z^2 + i n_y (\rho_{14} - \rho_{41}) n_z + n_x n_z \right. \\
\left. (\rho_{14} + \rho_{41}) + (n_z^2 + n_y^2) (\rho_{11} + \rho_{14}) \right\}. \quad (B1) \]
\[ g_{13} = \frac{1}{4} \left\{ (\rho_{13} + i (n_y (\rho_{16} + \rho_{43}) + i \rho_{46}) ) n_x^2 \ight. \\
+ (\rho_{16} - \rho_{43}) n_x^3 + \left\{ (\rho_{16} - \rho_{43}) n_y^2 + \\n+ n_z ((n_z - 1) \rho_{16} - (n_z + 1) \rho_{43}) \right\} n_x \\
+ i n_y \left\{ (\rho_{16} + \rho_{43}) n_y^2 - i (\rho_{13} - \rho_{46}) \\
+ n_y + n_z ((n_z - 1) \rho_{16} + (n_z + 1) \rho_{43}) \right\} \right\} \quad (B2) \]

\[ g_{14} = \frac{1}{2} (n_x - i n_y) \left\{ i n_y (\rho_{14} - \rho_{41}) \\
+ n_x (\rho_{14} + \rho_{41}) + n_z (\rho_{11} - \rho_{44}) \right\} \quad (B3) \]

\[ g_{16} = \frac{1}{4} \left\{ (\rho_{13} - \rho_{46}) n_x^3 + \left\{ -2 n_z \rho_{16} + \rho_{16} - \rho_{43} \\
- i n_y (\rho_{13} - \rho_{46}) \right\} n_x^2 + \left\{ (\rho_{13} - \rho_{46}) n_y^2 \\
+ 2 i \rho_{43} n_y + (n_z - 1) n_z (\rho_{13} - \rho_{46}) \right\} n_x \\
- 2 (n_z - 1) n_x^2 \rho_{16} + n_y^2 (-2 n_z \rho_{16} + \rho_{16} + \rho_{43}) \\
- i n_y^3 (\rho_{13} - \rho_{46}) - i n_y (n_z - 1) n_z (\rho_{13} - \rho_{46}) \right\} \quad (B4) \]

\[ g_{22} = \frac{1}{2} \left\{ 2 \rho_{22} n_x^2 + i n_y (\rho_{25} - \rho_{52}) n_x + n_x n_z \\
(\rho_{25} + \rho_{52}) + (n_x^2 + n_y^2) (\rho_{22} + \rho_{55}) \right\} \quad (B5) \]

\[ g_{25} = \frac{1}{2} (n_x - i n_y) \left\{ i n_y (\rho_{25} - \rho_{52}) + n_x \\
(\rho_{25} + \rho_{52}) + n_z (\rho_{22} - \rho_{55}) \right\} \quad (B6) \]

\[ g_{33} = \frac{1}{2} \left\{ (\rho_{33} + \rho_{66}) n_x^2 + n_z (\rho_{36} + \rho_{63}) n_x + \\
2 n_z^2 \rho_{33} + i n_x n_z (\rho_{36} - \rho_{63}) + n_y^2 (\rho_{33} + \rho_{66}) \right\} \quad (B7) \]

\[ g_{34} = \frac{1}{4} \left\{ (\rho_{64} - \rho_{31}) n_x^3 + \{ 2 n_z \rho_{34} + \rho_{34} - \rho_{61} \\
+ i n_y (\rho_{31} - \rho_{64}) \} n_x^2 + \{ (\rho_{64} - \rho_{31}) n_y^2 \\
+ 2 i \rho_{61} n_y - n_z (n_z + 1) (\rho_{31} - \rho_{64}) \} n_x \\
+ 2 n_z^2 (n_z + 1) \rho_{34} + n_z^2 (2 n_z \rho_{34} + \rho_{34} + \rho_{61}) \\
+ i n_y^3 (\rho_{31} - \rho_{64}) + i n_y n_z (n_z + 1) (\rho_{31} - \rho_{64}) \right\} \quad (B8) \]
\[ g_{36} = \frac{1}{2} (n_x - in_y) \{ in_y (\rho_{36} - \rho_{63}) + n_x \\
(\rho_{36} + \rho_{63}) + n_z (\rho_{33} - \rho_{66}) \} \]  
(B9)

\[ g_{44} = \frac{1}{2} \left\{ 2 \rho_{44} n_z^2 - i n_y n_z (\rho_{14} - \rho_{41}) - n_x n_z \\
(\rho_{14} + \rho_{41}) + (n_z^2 + n_y^2) (\rho_{11} + \rho_{44}) \right\} \]  
(B10)

\[ g_{46} = \frac{1}{4} \left\{ - (\rho_{13} + i(n_y (\rho_{16} + \rho_{43}) + i \rho_{46})) n_x^2 \\
+ (\rho_{43} - \rho_{16}) n_z^3 + \{(\rho_{43} - \rho_{16}) n_y^2 + n_z \\
- n_z \rho_{16} + n_z \rho_{43} + \rho_{43} \} n_x \\
- i n_y \{(\rho_{16} + \rho_{43}) n_y^2 - i(\rho_{13} - \rho_{46}) n_y \\
+ n_z ((n_z - 1) \rho_{16} + (n_z + 1) \rho_{43}) \} \right\} \]  
(B11)

\[ g_{55} = \frac{1}{2} \left\{ (\rho_{22} + \rho_{55}) n_z^2 - n_z n_x (\rho_{25} + \rho_{52}) \\
n - i n_y n_x (\rho_{25} - \rho_{52}) + 2 n_z^2 \rho_{55} + n_y^2 (\rho_{22} + \rho_{55}) \right\} \]  
(B12)

\[ g_{66} = \frac{1}{2} \left\{ (2 \rho_{66} n_z^2 - i n_y n_z (\rho_{36} - \rho_{63}) - n_x n_z \\
(\rho_{36} + \rho_{63}) + (n_z^2 + n_y^2) (\rho_{33} + \rho_{66}) \right\} \]  
(B13)

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