NUMERICAL RECOVERY OF MAGNETIC DIFFUSIVITY IN A THREE DIMENSIONAL SPHERICAL DYNAMO EQUATION

DJEMAA MESSAOUDI
School of Mathematics and Statistics, Central China Normal University
Wuhan 430079, China
Kasdi Merbah University Ouargla-Algeria

OSAMA SAID AHMED
School of Mathematics and Statistics, Central China Normal University
Wuhan 430079, China
Department of mathematics, University of EL-Imam El-mahdi.Kosti-Sudan

KOMIVI SOULEY AGBODJAN
School of Mathematics and Statistics, Central China Normal University
Wuhan 430079, China
University of Lome, TOGO

TING CHENG AND DAIJUN JIANG∗
School of Mathematics and Statistics, Central China Normal University
Wuhan 430079, China

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Abstract. This paper is concerned with the analysis on a numerical recovery of the magnetic diffusivity in a three dimensional (3D) spherical dynamo equation. We shall transform the ill-posed problem into an output least squares nonlinear minimization by an appropriately selected Tikhonov regularization, whose regularizing effects and mathematical properties are justified. The nonlinear optimization problem is approximated by a fully discrete finite element method and its convergence shall be rigorously established.

1. Introduction. It is well known that many astrophysical bodies have intrinsic magnetic fields. But only in the last few decades people begin to understand more about the origin of this field. So far a widely accepted theory is the so-called mean-field dynamo theory, which is modelled by the following nonlinear spherical dynamo
The noise level. We will reconstruct the magnetic diffusivity \( \beta \). The convergence analysis of the numerical solutions is carried out.

A method for approximating the continuous minimization problem is proposed and demonstrate its regularizing effects. In Section 3, a fully discrete finite element problem is considered inverse problem \( I \) to an optimization problem with Tikhonov regularization.

(1.2) \[
\begin{aligned}
\partial_t B + \nabla \times (\beta(x)\nabla \times B) = & R_\alpha \nabla \times (f(x,t) \vec{B}) + R_m \nabla \times (u \times B) \quad \text{in} \; \Omega \times (0,T), \\
\nabla \cdot B = & 0 \quad \text{in} \; \Omega \times (0,T), \\
B \cdot n = & 0, \; \nabla \times B \times n = 0 \quad \text{on} \; \partial \Omega \times (0,T), \\
B(x,0) = & B_0(x) \quad \text{in} \; \Omega,
\end{aligned}
\]

where \( \Omega = B_{r_o}(0) \backslash B_{r_i}(0) \subset \mathbb{R}^3 \), \( 0 < r_i < r_o < \infty \) is the physical domain of interest. Here \( B_{r_o} \) and \( B_{r_i} \) denote two circles with center at 0 and radius \( r_o \) and \( r_i \) respectively. \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \) denotes the boundary of \( \Omega \), which consists of the inner boundary \( \Gamma_1 \) and outer boundary \( \Gamma_2 \), and \( n \) denotes the unit outer normal vector to the boundary of \( \Omega \). The function \( B = B(x,t) \) and \( u = u(x,t) \) represent magnetic field and the fluid velocity field respectively, \( f(x,t) \) is a model-oriented function, \( R_\alpha \) is a dynamo parameter, \( R_m \) is the magnetic Reynolds number, \( \sigma \) is a constant and the parameter \( \beta(x) \) is the magnetic diffusivity.

When \( u, f, \sigma, \beta \) and \( B_0 \) are given, one can solve the system (1.1) to find the behavior of magnetic field \( B \) in \( \Omega \). This is usually called a direct dynamo problem. For the numerical simulations and mathematical theory analysis of the system (1.1), one may refer to [2, 3, 7, 10, 11, 12] and the references therein. And for the numerical analysis of some stochastic interface problems, we can refer to [1, 8] and the references therein.

In this work we shall consider the case when \( u, f, \sigma, \beta \) and \( B_0 \) are known, but the magnetic diffusivity \( \beta(x) \) is unavailable in \( \Omega \). In order to recover the magnetic diffusivity \( \beta(x) \), we need to have some extra measurement data from the magnetic field \( B \). We shall assume the measurement data \( B \) is available in some small subregion inside \( \Omega \) over the time interval \((0,T)\), which occurs the following inverse problem.

**Inverse Problem 1.** Let \( \omega \) be a subregion in \( \Omega \). Given the noisy measurement data

(1.2) \[
B(x,t) \approx z^\delta(x,t), \quad (x,t) \in \omega \times (0,T),
\]

we will reconstruct the magnetic diffusivity \( \beta(x) \) in the entire domain \( \Omega \). Here \( \delta \) is the noise level.

The rest of the paper is organized as follows. In Section 2, we transform the considered inverse problem I to an optimization problem with Tikhonov regularization, and demonstrate its regularizing effects. In Section 3, a fully discrete finite element method for approximating the continuous minimization problem is proposed and the convergence analysis of the numerical solutions is carried out.

We end this section with some useful notations and lemmas. For \( m \in \mathbb{R}, H^m(\Omega) \) is the usual Sobolev space, and we denote \( H^m(\Omega) \) and \( L^m(\Omega) \) by \( H^m(\Omega) \) and \( L^m(\Omega) \) respectively. We shall use \((\cdot, \cdot)\) and \( \| \cdot \|_{m, \Omega} \) to denote the scalar product in
\( L^2(\Omega) \) or \( L^2(\Omega) \) and the norm of \( H^m(\Omega) \) or \( H^m(\Omega) \) respectively. Throughout this work, \( C \) is often used for a generic positive constant.

Moreover, we introduce some useful Sobolev spaces for the subsequent analysis:

\[
H(\text{curl}, \text{div}; \Omega) = \{ C \in L^2(\Omega); \text{curl}C \in L^2(\Omega), \text{div}C \in L^2(\Omega) \},
\]

\[
H_0(\text{curl}, \text{div}; \Omega) = \{ C \in H(\text{curl}, \text{div}; \Omega); C \cdot n = 0 \text{ on } \partial \Omega \},
\]

\[
V = \{ C \in H_0(\text{curl}, \text{div}; \Omega); \text{div} C = 0 \},
\]

\[
L^2_0(\Omega) = \{ v \in L^2(\Omega); \int_\Omega v dx = 0 \}.
\]

As the spaces \( H(\text{curl}, \text{div}; \Omega) \) and \( H_0(\text{curl}, \text{div}; \Omega) \) will be frequently used, we shall write

\[
H = H(\text{curl}, \text{div}; \Omega) \quad \text{and} \quad H_0 = H_0(\text{curl}, \text{div}; \Omega),
\]

which are both equipped with the norm

\[
\|C\|_H = (\|C\|^2_{0,\Omega} + \|\nabla \times C\|^2_{0,\Omega} + \|\nabla \cdot C\|^2_{0,\Omega})^{\frac{1}{2}}.
\]

It has been shown that \( \|\cdot\|_H \) is equivalent \( \|\cdot\|_{1,\Omega} \) (see, e.g., [6]).

In the following, we shall give a lemma for our later use.

**Lemma 1.1.** (Aubin-Lions Lemma, [14], p.189) Let \( X_0, X \) be two Banach spaces and \( X_1 \) be a Hilbert space with \( X_0 \subset X \subset X_1 \), the injections being continuous and the injection of \( X_0 \) into \( X \) being compact. Then the injection of \( \mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1) \) into \( L^\alpha_\Theta(0,T;X) \) is compact for any finite number \( \alpha_0 > 1 \), where \( \mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1) = \{ v \in L^\alpha_\Theta(0,T;X_0); v' = \frac{dv}{dt} \in L^{\alpha_1}(0,T;X_1) \} \).

2. Tikhonov regularization and its regularizing effects. In this section we shall formulate the ill-posed Inverse Problem I stated in Section 1 as a stabilized minimization system and establish the existence of the solutions and stability with respect to the change in the error of the observation data.

Before considering Inverse Problem I, we refer to [3] and recall the equivalent variational problem of system 1.1 and its well-posedness.

**Lemma 2.1.** The equivalent variational problem of system 1.1: For a.e. \( t \in (0,T) \), find \( B(\cdot,t) \in H_0, \ p(\cdot,t) \in L^2_0(\Omega) \) such that \( B(\cdot,0) = B_0(\cdot) \) and

\[
\begin{cases}
\partial_t B + (\beta \nabla \times B, \nabla \times A) + \gamma (\nabla \cdot B, \nabla \cdot A) + (p, \nabla \cdot A) \\
= R_\alpha \left( \frac{f(x,t)}{1 + \sigma |B|^2} B, \nabla \times A \right) + R_m (u \times B, \nabla \times A) \quad \forall \ A \in H_0,
\end{cases}
\]

where \( p(x,t) \) is a Lagrange multiplier and \( \gamma \) is a constant. Moreover, we have the following stability estimate for the solution \( (B,p) \) to system (2.1):
Tikhonov regularization: ill-posed, we formulate it into a mathematically stabilized minimization system with to emphasize their dependence on $B$

\[
B_0 ∈ V, \ f ∈ H^1(0, T; L^∞(Ω)) \text{ and } u ∈ H^1(0, T; L^∞(Ω)).
\]

For convenience, we often write the solutions of the system (2.1) as $(B(β), p(β))$ to emphasize their dependence on $β$. In general, Inverse Problem I is mathematically ill-posed, we formulate it into a mathematically stabilized minimization system with Tikhonov regularization:

\[
\begin{align*}
\min_{β ∈ K} J(β) &= \frac{1}{2} \int_0^T \int_Ω |B(β) - z|^2 dxdt + \frac{λ}{2} \int_Ω |∇ β|^2 dx,
\end{align*}
\]

where the constraint set

\[
K = \{β(x) ∈ H^1(Ω) : 0 < β_1 ≤ β(x) ≤ β_2 \text{ a.e. in } Ω\}.
\]

Here $β_1$ and $β_2$ are two positive constants and $λ > 0$ is the regularization parameter.

We are now ready to justify the regularizing effects of the nonlinear optimization system (2.2) that it always has solutions and its solutions are stable with respect to the noise error in the observation data $z^δ$. The first theorem establishes the existence of solutions.

**Theorem 2.2.** There exists at least a minimizer to the optimization problem (2.2).

**Proof.** Since $J(β) ≥ 0$ for any $β ∈ K$, there exists a minimizing sequence $\{β_n\} ⊂ K$ such that

\[
\lim_{n→∞} J(β_n) = \inf_{β ∈ K} J(β).
\]

Then $|J(β_n)| ≤ C$, which implies that $|∇ β_n|₀,Ω ≤ C$. By the definition of $K$, $\{β_n(x)\}$ is bounded in $L^∞(Ω)$, then in $L^2(Ω)$. So $\{β_n(x)\}$ is bounded in $H^1(Ω)$ and there exists a subsequence of $\{β_n(x)\}$ denoted still by $\{β_n(x)\}$ and some $β^∗ ∈ H^1(Ω)$ such that

\[
β_n → β^∗ \text{ in } H^1(Ω), \text{ and } β_n → β^∗ \text{ in } L^2(Ω).
\]

As $K$ is a closed convex subset of $H^1(Ω)$, hence $K$ is weakly-closed and we have $β^∗ ∈ K$.

For convenience, let $(B^n, p^n) = (B(β_n), p(β_n))$. Due to Lemma 2.1, there exists a subsequence, still denoted by $\{B_n, p_n\}$ and some $(B^∗, p^*)$ such that

\[
B_n → B^* \text{ in } L^∞(0, T; H^1(Ω)), \quad B_n → B^* \text{ in } H^1(0, T; L^2(Ω)),
\]

\[
p_n → p^* \text{ in } L^2(0, T; L^0(Ω)).
\]

Next we shall show that $B^* = B(β^*)$ and $p^* = p(β^*)$. To do so, we multiply both sides of (2.1) $(B$ is replaced by $B^*$, $β$ is replaced by $β^*)$ by a function $η(t) ∈ C^1[0, T]$...
and get

\[
\int_0^T \int_\Omega \partial_t B^n \cdot A \eta dx dt + \int_0^T \int_\Omega \beta \nabla \times B^n \cdot (\nabla \times A) \eta(t) dx dt \\
+ \gamma \int_0^T \int_\Omega (\nabla \cdot B^n) \cdot (\nabla \cdot A) \eta(t) dx dt + \int_0^T \int_\Omega p^n (\nabla \cdot A) \eta(t) dx dt \\
= R_\alpha \int_0^T \int_\Omega f(x, t) \frac{B^n \cdot (\nabla \times A) \eta(t)}{1 + \sigma |B^n|^2} dx dt \\
+ R_m \int_0^T \int_\Omega u \times B^n \cdot (\nabla \times A) \eta(t) dx dt \\
- \int_0^T \int_\Omega (\beta_n - \beta)(\nabla \times B^n) \cdot (\nabla \times A) \eta(t) dx dt \quad \forall A \in H_0, \\
(2.7) \\
\int_0^T \int_\Omega (\nabla \cdot B^n) q \eta(t) dx dt = 0 \quad \forall q \in L^2_0(\Omega). \\
(2.8)
\]

We first claim that the last term in the right hand side of (2.7) tends to 0 as \( n \to \infty \). Indeed, by Cauchy Schwarz inequality and the fact that \( \|B^n\|_{L^\infty(0,T; H^1(\Omega))} \leq C \), we have

\[
\left| \int_0^T \int_\Omega (\beta_n - \beta)(\nabla \times B^n) \cdot (\nabla \times A) \eta(t) dx dt \right| \\
\leq \left( \int_0^T \int_\Omega |(\nabla \times B^n)|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega |(\beta_n - \beta)(\nabla \times A) \eta(t)|^2 dx dt \right)^{\frac{1}{2}} \\
\leq C \left( \int_0^T \int_\Omega |(\beta_n - \beta)(\nabla \times A) \eta(t)|^2 dx dt \right)^{\frac{1}{2}},
\]

which converges to zero as \( n \to \infty \) by (2.4) and the Lebesgue’s dominated convergence theorem.

Then we shall show that

\[
\lim_{n \to \infty} R_\alpha \int_0^T \int_\Omega \frac{f(x, t)}{1 + \sigma |B^n|^2} B^n \cdot (\nabla \times A) \eta(t) dx dt \\
= R_\alpha \int_0^T \int_\Omega \frac{f(x, t)}{1 + |\nabla \times A|^2} B^* \cdot (\nabla \times A) \eta(t) dx dt.
(2.9)
\]

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By direct computation, we get

\[
\left| \int_0^T \int_\Omega \left( \frac{f(x,t)}{1 + \sigma |B^0|^2} B^n \cdot (\nabla \times A) \eta(t) - \frac{f(x,t)}{1 + \sigma |B^0|^2} B^* \cdot (\nabla \times A) \eta(t) \right) dx dt \right| \\
\leq \int_0^T \int_\Omega \left| \frac{f B^0(1 + \sigma |B^0|^2) - f B^*(1 + \sigma |B^0|^2)}{(1 + \sigma |B^0|^2)(1 + \sigma |B^0|^2)} \cdot (\nabla \times A) \eta \right| dx dt \\
\leq \int_0^T \int_\Omega |B^n - B^*| \cdot |f| \cdot |\nabla \times A| \cdot |\eta| dx dt \\
+ \int_0^T \int_\Omega |f(\nabla \times A)| \cdot \left| \frac{\sigma |B^n - B^*| |B^0|^2 + \sigma B^*(|B^0|^2 - |B^n|^2)}{(1 + \sigma |B^0|^2)(1 + \sigma |B^0|^2)} \right| dx dt \\
\leq 2 \int_0^T \int_\Omega |B^n - B^*| \cdot |f(\nabla \times A)| \eta dx dt \\
+ \int_0^T \int_\Omega |f(\nabla \times A)| \eta \cdot \left| \frac{\sigma |B^n|(|B^0| + |B^n|)(|B^0| - |B^n|)}{(1 + \sigma |B^0|^2)(1 + \sigma |B^0|^2)} \right| dx dt \\
\leq 2 \int_0^T \int_\Omega |B^n - B^*| \cdot |f(\nabla \times A)| \eta dx dt \\
+ \int_0^T \int_\Omega |f(\nabla \times A)| \eta \cdot |B^* - B^n| \cdot \left| \frac{\sigma |B^0|^2 + \frac{\sigma}{2}(|B^0|^2 + |B^n|^2)}{(1 + \sigma |B^0|^2)(1 + \sigma |B^0|^2)} \right| dx dt \\
\leq 4 \int_0^T \int_\Omega |B^n - B^*| \cdot |f(\nabla \times A)| \eta dx dt \\
\leq 4 \left( \int_0^T \int_\Omega |B^n - B^*|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega |f(\nabla \times A)| \eta|^2 dx dt \right)^{\frac{1}{2}} \\
\leq C \Vert B^n - B^* \Vert_{L^2(0,T;L^2(\Omega))},
\]

which tends to 0 as \( n \to \infty \) if \( \Vert B^n - B^* \Vert_{L^2(0,T;L^2(\Omega))} \rightarrow 0 \) as \( n \to \infty \). Now we will prove

(2.10) \( B^n \to B^* \) in \( L^2(0,T;L^2(\Omega)) \).

As \( \{B^n\} \) is bounded in \( L^2(0,T;H^1(\Omega)) \), it suffices to show that \( \{\partial_t B^n\} \) is bounded in \( L^2(0,T;(H^1(\Omega))') \) by Lemma 1.1. For any \( A \in H_0 \), we have from the variational form (2.1) that for any \( t \in (0,T) \)

\[
\begin{align*}
|\langle \partial_t B^n, A \rangle| & \leq C \Vert \nabla \times B^n \Vert_{0,\Omega} \Vert \nabla \times A \Vert_{0,\Omega} \\
+ \Vert \nabla \cdot B^n \Vert_{0,\Omega} \Vert \nabla \cdot A \Vert_{0,\Omega} + \Vert p^n \Vert_{0,\Omega} \Vert \nabla \cdot A \Vert_{0,\Omega} \\
+ R_m \left( \int_\Omega \frac{f(x,t)}{1 + \sigma |B^n|^2} |B^n|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega \frac{f(x,t)}{1 + \sigma |B^n|^2} |\nabla \times A|^2 dx \right)^{\frac{1}{2}} \\
+ 2R_m \Vert u \Vert_{L^\infty(\Omega)} \Vert B^n \Vert_{0,\Omega} \Vert \nabla \times A \Vert_{0,\Omega} \\
\leq C \Vert A \Vert_{1,\Omega} \Vert B^n \Vert_{1,\Omega} + \Vert p^n \Vert_{0,\Omega} + |A|_{L^\infty(\Omega)} \Vert \nabla \times A \Vert_{0,\Omega} \\
+C \Vert u \Vert_{L^\infty(\Omega)} \Vert B^n \Vert_{0,\Omega} + \Vert f \Vert_{L^\infty(\Omega)} + \Vert u \Vert_{L^\infty(\Omega)} \Vert B^n \Vert_{0,\Omega}.
\end{align*}
\]
Further, we have

\[
\left| \int_0^T (\partial_t B^n, A) \, dt \right| \leq C\|A\|_{L^2(0,T;H^1(\Omega))}^2 \|B^n\|_{L^2(0,T;H^1(\Omega))} + \|B^n\|_{L^2(0,T;L^2(\Omega))} \|f\|_{H^1(0,T;L^\infty(\Omega))} + \int_0^T \|u\|_{L^\infty(\Omega)}^2 \|B^n\|^2 \, dt \leq C\|A\|_{L^2(0,T;H^1(\Omega))},
\]

which implies that \( \{\partial_t B^n\} \) is bounded.

Our next goal is to show that for any \( A \in H_0 \),

\[
\lim_{n \to \infty} R_m \int_0^T \int_{\Omega} u \times B^n \cdot (\nabla \times A) \eta(t) \, dx \, dt = R_m \int_0^T \int_{\Omega} u \times B^* \cdot (\nabla \times A) \eta(t) \, dx \, dt.
\]

(2.11)

Indeed, by direct computation and (2.5), we have

\[
\lim_{n \to \infty} \int_0^T \int_{\Omega} u \times B^n \cdot (\nabla \times A) \eta(t) \, dx \, dt = \lim_{n \to \infty} \int_0^T (\nabla \times A) \eta \times u \cdot B^n \, dx \, dt = \int_0^T \int_{\Omega} (\nabla \times A) \eta \times u \cdot B^* \, dx \, dt = \int_0^T \int_{\Omega} u \times B^* \cdot (\nabla \times A) \eta \, dx \, dt.
\]

Finally, passing to the limit on both sides of (2.7) and (2.8), and making use of (2.5)-(2.6), (2.9) and (2.11), we obtain that

\[
\int_0^T \int_{\Omega} \partial_t B^* \cdot A \eta(t) \, dx \, dt + \int_0^T \int_{\Omega} \beta \nabla \times B^* \cdot (\nabla \times A) \eta(t) \, dx \, dt + \int_0^T \int_{\Omega} \gamma (\nabla \cdot B^*) \cdot (\nabla \cdot A) \eta(t) \, dx \, dt + \int_0^T \int_{\Omega} \sigma (\nabla \cdot B^*) \eta(t) \, dx \, dt = R_m \int_0^T \int_{\Omega} u \times B^* \cdot (\nabla \times A) \eta(t) \, dx \, dt + R_m \int_0^T \int_{\Omega} (\nabla \cdot B^*) \eta \, dx \, dt = 0, \quad \forall \ A \in H_0, \ \eta \in C^1[0,T],
\]

Further, we shall prove \( B^*(x,0) = B_0(x) \), which together with the definition of \( (B^*(\beta^*), p(\beta^*)) \) implies that

(2.12) \( (B^*, p^*) = (B(\beta^*), p(\beta^*)) \).

Choosing \( \eta(t) \in C^1[0,T] \) with \( \eta(T) = 0 \), we have by integration by parts with respect to \( t \) that

\[
\int_0^T \int_{\Omega} \partial_t B^n \cdot A \eta(t) \, dx \, dt = -\int_0^T \int_{\Omega} B^n \cdot A \eta'(t) \, dx \, dt - \int_0^T B_0(x) \cdot A(x,0) \eta(0) \, dx.
\]
Letting $n \to \infty$ in the above equality and using (2.5), we have
\[
\int_0^T \int_\Omega \partial_t B^* \cdot A \eta(t) dx \, dt = -\int_0^T \int_\Omega B^* \cdot A \eta'(t) dx \, dt - \int_\Omega B_0(x) \cdot A(x, 0) \eta(0) dx.
\] (2.13)

On the other hand, by integration by parts with respect to $t$, we also have
\[
\int_0^T \int_\Omega \partial_t B^* \cdot A \eta(t) dx \, dt = -\int_0^T \int_\Omega B^* \cdot A \eta'(t) dx \, dt - \int_\Omega B^*(x, 0) \cdot A(x, 0) \eta(0) dx,
\]
which together with (2.13) implies $B^*(x, 0) = B_0(x)$.

Therefore, from (2.4), (2.5), (2.12) and the semi-continuity of the norm, we derive
\[
J(\beta^*) \leq \liminf_{n \to \infty} J(\beta_n) = \inf_{\beta \in K} J(\beta),
\]
which implies that $\beta^*$ is a minimizer to the optimization problem (2.2). \qed

The next theorem demonstrates that the minimization system (2.2) is indeed a stabilization of the ill-posed Inverse Problem I with respect to the changes of the observation errors. Similar stability results were established for inverse elliptic and parabolic problems in [4, 9, 17].

**Theorem 2.3.** Let $\{z_n\}$ be a sequence such that $z_n \to z^\delta$ in $L^2(0, T; L^2(\omega))$ as $n \to \infty$ and $\{\beta_n\}$ be the minimizers of problem (2.2) with $z^\delta$ replaced by $z_n$. Then there exists a subsequence of $\{\beta_n\}$ that converges in $H^1(\Omega)$, and the limit of every such convergent subsequence is a minimizer of (2.2).

**Proof.** By the definition of $\{\beta_n\}$, we have
\[
\frac{1}{2} \int_0^T \int_\omega |B(\beta_n) - z_n|^2 dx \, dt + \frac{1}{2} \lambda \int_\Omega |\nabla \beta_n|^2 dx \\
\leq \frac{1}{2} \int_0^T \int_\omega |B(\beta) - z_n|^2 dx \, dt + \frac{1}{2} \lambda \int_\Omega |\nabla \beta|^2 dx, \quad \forall \beta \in K,
\]
which with $\beta_n \in K$ implies that $\{\beta_n\}$ is bounded in $H^1(\Omega)$. Similar to the proof of Theorem 2.2, there exists a subsequence, denoted still by $\{\beta_n\}$, and some $\beta^* \in K$ such that
\[
\beta_n \to \beta^* \text{ in } H^1(\Omega), \quad \beta_n \to \beta^* \text{ in } L^2(\Omega),
\]

$$B(\beta_n) \to B(\beta^*) \text{ in } L^2(0, T; L^2(\Omega)).$$

Hence we have
\[
\lim_{n \to \infty} \int_0^T \int_\omega |B(\beta_n) - z_n|^2 dx \, dt = \int_0^T \int_\omega |B(\beta^*) - z^\delta|^2 dx \, dt.
\]
Then, using the lower semi-continuity of a norm, we deduce that

\[
J(\beta^*) = \frac{1}{2} \int_0^T \int_\omega |B(\beta^*) - z|^2 \, dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta^*|^2 \, dx
\]

\[
\leq \liminf_{n \to \infty} \left\{ \frac{1}{2} \int_0^T \int_\omega |B(\beta_n) - z_n|^2 \, dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta_n|^2 \, dx \right\}
\]

\[
\leq \limsup_{n \to \infty} \left\{ \frac{1}{2} \int_0^T \int_\omega |B(\beta) - z_n|^2 \, dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta|^2 \, dx \right\} \quad \forall \beta \in K
\]

\[
= \frac{1}{2} \int_0^T \int_\omega |B(\beta) - z|^2 \, dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta|^2 \, dx \quad \forall \beta \in K
\]

(2.16) \quad = J(\beta), \quad \forall \beta \in K.

This yields that \( \beta^* \) is a minimizer to system (2.2).

Next we shall prove \( \nabla \beta_n \to \nabla \beta^* \) in \( L^2(\Omega) \), and then \( \beta_n \to \beta^* \) in \( H^1(\Omega) \). Since (2.16) holds for any \( \beta \in K \), we take \( \beta = \beta^* \) and obtain that

\[
\lim_{n \to \infty} \left\{ \frac{1}{2} \int_0^T \int_\omega |B(\beta_n) - z_n|^2 \, dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta_n|^2 \, dx \right\}
\]

\[
= \frac{1}{2} \int_0^T \int_\omega |B(\beta^*) - z|^2 \, dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta^*|^2 \, dx.
\]

Combining this with (2.15), we get

\[
\lim_{n \to \infty} \int_\Omega |\nabla \beta_n|^2 \, dx = \int_\Omega |\nabla \beta^*|^2 \, dx,
\]

which with \( \nabla \beta_n \to \nabla \beta^* \) in \( L^2(\Omega) \) by (2.14), we have \( \nabla \beta_n \to \nabla \beta^* \) in \( L^2(\Omega) \). \( \square \)

3. Finite element approximation and its convergence. In this section, we shall propose a fully discretized finite element approximation for solving the continuous minimization problem (2.2).

For the space discretization, we consider a shape regular triangulation \( T_h \) of \( \Omega \) with a mesh size \( h \), consisting of tetrahedral elements. Then we introduce some finite element spaces, which were proposed in [3]:

\[
H_h = \{ w \in C(\bar{\Omega})^3 : w|_A \in P_2(A)^3, \forall A \in T_h \},
\]

\[
H_{0h} = \{ w \in H_h : w \cdot n_F = 0, \forall F \in \mathcal{F}_h \cap \partial \Omega \},
\]

\[
Q_h = \{ w \in C(\bar{\Omega}) : w|_A \in P_1(A), \forall A \in T_h \},
\]

\[
Q_{0h} = \{ w \in Q_h : \int_\Omega w \, dx = 0 \},
\]

\[
V_h = \{ w \in H^1(\Omega) : w|_A \in P_1(A), \forall A \in T_h \},
\]

where \( \mathcal{F}_h \) is the set of all faces of elements in \( T_h \) and \( n_F \) is the unit normal vector of a face \( F \in \mathcal{F}_h \), \( P_1(A) \) and \( P_2(A) \) are the spaces of piecewise linear and quadratic polynomials on \( A \) respectively. We will approximate the magnetic field \( B \) and Lagrange multiplier \( p \) by \( H_{0h} \) and \( Q_{0h} \) respectively. Moreover, the constrained subset \( K \) is approximated by \( K_h = K \cap V_h \).

To fully discretize system (2.7)-(2.8), we also need the time discretization. To do so, we divide the time interval \([0, T]\) into \( M \) equally spaced subintervals using
nodal points
\begin{equation}
0 = t_0 < t_1 < \cdots < t_M = T
\end{equation}
with \( t_n = n\tau, \quad \tau = \frac{T}{M} \). For a continuous mapping \( u : [0, T] \rightarrow L^2(\Omega) \), we define \( u^n = u(\cdot, t_n) \) for \( 0 \leq n \leq M \). For a given sequence \( \{u^n\}_{n=0}^M \subset L^2(\Omega) \), we define its first-order backward finite differences and average values as follows:
\[
\begin{align*}
\partial_t u^n &= \frac{u^n - u^{n-1}}{\tau}, \quad n = 1, 2, \ldots, M, \\
\bar{u}^n &= \frac{1}{\tau} \int_{t_{n-1}}^{t_n} u(\cdot, t)dt, \quad n = 1, 2, \ldots, M, \text{ and } \bar{u}^0 = u(\cdot, 0).
\end{align*}
\]

Then we introduce two important operators for later use. The first one is the so-called modified Scott-Zhang interpolation operator \( S_h : (\text{see } [3] \text{ or } [13]) \), which preserves the boundary condition in \( H_0 \); for any \( B \in H_0 \), we have \( S_h B \in H_{0h} \) and it has the following properties:

**Lemma 3.1.** Let \( u \in H^1(\Omega) \), then there exists a constant \( C \), independent of \( h \), such that
\[
\|S_h u\|_{1, \Omega} \leq C\|u\|_{1, \Omega}, \quad \|u - S_h u\|_{0, \Omega} \leq Ch\|u\|_{1, \Omega},
\]
and
\[
\lim_{h \to 0} \|u - S_h u\|_{1, \Omega} = 0.
\]
Moreover, if \( u \in H^2(\Omega) \), we have
\[
\|u - S_h u\|_{1, \Omega} \leq Ch\|u\|_{2, \Omega}.
\]

The second operator is the \( L^2 \) quasi-projection operator \( \Pi_h : L^2(\Omega) \rightarrow Q_h \), which has the following properties (see [16]):

**Lemma 3.2.** For \( w \in L^2(\Omega) \), we have
\[
\|\Pi_h w\|_{0, \Omega} \leq C\|w\|_{0, \Omega}, \quad \lim_{h \to 0} \|w - \Pi_h w\|_{0, \Omega} \to 0.
\]
Moreover, if \( w \in H^1(\Omega) \), we have
\[
\|\Pi_h w\|_{1, \Omega} \leq C\|w\|_{1, \Omega}, \quad \lim_{h \to 0} \|w - \Pi_h w\|_{1, \Omega} \to 0.
\]

Now we are ready to formulate the finite element approximation of the continuous minimization (2.2) as follows:
\begin{equation}
\begin{align*}
\min_{\beta_h \in K_h} J_{h, \tau}(\beta_h) &= \frac{\tau}{2} \sum_{n=0}^{M} \alpha_n \int_{\omega} (B^n_h - z^{h,n})^2 dx + \frac{\lambda}{2} \int_{\Omega} |\nabla \beta_h|^2 dx, \\
&\quad + \gamma \int_{\Omega} (\nabla \cdot B^n_h)(\nabla \cdot A_h) dx + \int_{\Omega} p^n_h (\nabla \cdot A_h) dx \\
&= R_\alpha \int_{\Omega} \frac{1}{1 + |B^n_h|^2} B^n_h \cdot (\nabla \times A_h) dx \\
&\quad + R_m \int_{\Omega} (\bar{u}^n \times B^n_h) \cdot (\nabla \times A_h) dx, \\
&\quad + \int_{\Omega} (\nabla \cdot B^n_h) q_h dx = 0,
\end{align*}
\end{equation}
for all \((A_h, q_h) \in H_{0h} \times Q_{0h}, n = 1, 2, \ldots, M\). Here \(\ddot{u}^n \in L^\infty(\Omega)\) and \(\ddot{f}^n \in L^\infty(\Omega)\), and \(\{a_n\}\) are the coefficients of the composite trapezoidal rule, i.e., \(a_0 = a_M = \frac{1}{2}\) and \(a_n = 1\) for \(n \neq 0, M\).

Before analyzing the convergence, we refer to \([3]\) and present the well-posedness and stability estimates for the solutions to the discrete system (3.3).

**Lemma 3.3.** There exists a unique solution \((B^n_h, p^n_h)\) to the discrete system (3.3) for each fixed \(n (1 \leq n \leq M)\) and the sequence \(\{(B^n_h, p^n_h)\}_{n=0}^M\) has the following stability estimates:

\[
\max_{1 \leq n \leq M} \|B^n_h\|_{0,\Omega}^2 + \tau \sum_{n=1}^M (\|\nabla \times B^n_h\|_{0,\Omega}^2 + \|\nabla \cdot B^n_h\|_{0,\Omega}^2) \leq C\|B^0_h\|_{0,\Omega}^2,
\]

\[
\max_{1 \leq n \leq M} \|\nabla \times B^n_h\|_{0,\Omega}^2 + \tau \sum_{n=1}^M \|\nabla \cdot B^n_h\|_{0,\Omega}^2 + \tau \sum_{n=1}^M \|\partial_\tau B^n_h\|_{0,\Omega}^2 \leq C\|B^0_h\|_{0,\Omega}^2.
\]

**Theorem 3.4.** There exists at least a minimizer to the discrete minimization problem (3.2).

**Proof.** Due to the stability estimates in Lemma 3.3, we could get the existence of the minimizer to (3.2) by the similar technique in the proof of Theorem 2.2. \(\square\)

Now we will consider the convergence of the minimizer of the discrete system (3.2) to the minimizer of the continuous problem (2.2). For the purpose, we first give the following classical approximation result \([15] [18]\).

**Lemma 3.5.** Let \(X\) be a Banach space. For a given function \(f \in C([0, T]; X)\), we define a step function approximation of \(f\):

\[
S_\Delta f(x, t) = \sum_{n=1}^M \chi_n(t) f(x, t_n),
\]

where \(\chi_n(t)\) is the characteristic function on the interval \((t_{n-1}, t_n)\). Then we have

\[
\lim_{\tau \to 0} \int_0^T \|S_\Delta f(\cdot, t) - f(\cdot, t)\|^2_X dt = 0.
\]

Further, we define some interpolations based on \(\{B^n_h\}\) and \(\{p^n_h\}\) as follows: for any \((x, t) \in \Omega \times (t_{n-1}, t_n)\), let

\[
B_{h, \tau}(x, t) = \frac{t - t_{n-1}}{\tau} B^n_h(x) + \frac{t_n - t}{\tau} B^{n-1}_h(x),
\]

\[
\dot{B}_{h, \tau}(x, t) = \sum_{n=1}^M \chi_n(t) B^n_h(x) \quad \text{and} \quad \dot{p}_{h, \tau}(x, t) = \sum_{n=1}^M \chi_n(t) p^n_h(x).
\]
Lemma 3.6. The following results hold:

\[ \| \hat{B}_{h, \tau} \|_{L^2(0, T; H^1(\Omega))}^2 = \tau \sum_{n=1}^{M} \| B_{h_n}^n \|_{L^2(\Omega)}^2, \]

\[ \| \frac{\partial}{\partial t} B_{h, \tau} \|_{L^2(0, T; (H^1(\Omega))')}^2 = \tau \sum_{n=1}^{M} \| \partial_t B_{h_n}^n \|_{(H^1(\Omega))'}^2, \]

\[ \| \hat{p}_{h, \tau} \|_{L^2(0, T; L^2(\Omega))}^2 = \tau \sum_{n=1}^{M} \| p_{h_n}^n \|_{L^2(\Omega)}^2, \]

\[ \| B_{h, \tau} \|_{L^2(0, T; H^1(\Omega))}^2 \leq \tau \sum_{n=1}^{M} \| B_{h_n}^n \|_{L^2(\Omega)}^2. \]

Proof. We first prove the first three equalities. By direct computation, it is easy to see that

\[ \| \hat{B}_{h, \tau} \|_{L^2(0, T; H^1(\Omega))}^2 = \sum_{n=1}^{M} \left( \int_{t_{n-1}}^{t_n} \int_{\Omega} \sum_{n=1}^{M} \chi_n(t) B_{h_n}^n(x) \| B_{h_n}^n(x) \|^2 \, dx \, dt + \int_{t_{n-1}}^{t_n} \int_{\Omega} \sum_{n=1}^{M} \chi_n(t) \nabla B_{h_n}^n(x) \| \nabla B_{h_n}^n(x) \|^2 \, dx \, dt \right) = \tau \sum_{n=1}^{M} \| B_{h_n}^n \|_{L^2(\Omega)}^2; \]

\[ \| \frac{\partial}{\partial t} B_{h, \tau} \|_{L^2(0, T; (H^1(\Omega))')}^2 = \int_{0}^{T} \| \frac{\partial}{\partial t} B_{h_n}^n \|_{(H^1(\Omega))'}^2 \, dt = \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \| \frac{\partial}{\partial t} \left( \frac{t - t_{n-1}}{\tau} B_{h_n}^n(x) + \frac{t_n - t}{\tau} B_{h_n}^{n-1}(x) \right) \|_{(H^1(\Omega))'}^2 \, dt = \tau \sum_{n=1}^{M} \| \partial_t B_{h_n}^n \|_{(H^1(\Omega))'}^2; \]

\[ \| \hat{p}_{h, \tau} \|_{L^2(0, T; L^2(\Omega))}^2 = \sum_{n=1}^{M} \left( \int_{t_{n-1}}^{t_n} \int_{\Omega} \sum_{n=1}^{M} \chi_n(t) p_{h_n}^n(x) \| p_{h_n}^n(x) \|^2 \, dx \, dt \right) = \tau \sum_{n=1}^{M} \| p_{h_n}^n \|_{L^2(\Omega)}^2. \]
Then we show the last inequality:

\[
\|B_{h,\tau}\|_{L^2(0,T;H^1(\Omega))}^2 = \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \left\{ \int_{\Omega} \left( \frac{t-t_{n-1}}{\tau} B_n^h(x) + \frac{t_{n-1}}{\tau} B_{h-1}^n(x) \right)^2 dx \right. \\
+ \left. \int_{\Omega} \left( \frac{t-t_{n-1}}{\tau} \nabla B_n^h(x) + \frac{t_{n-1}}{\tau} \nabla B_{h-1}^n(x) \right)^2 dx \right\} \, dt \\
= \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \int_{\Omega} \left[ \left( \frac{t-t_{n-1}}{\tau} \right)^2 |B_n^h|^2 + \left( \frac{t_{n-1}}{\tau} \right)^2 |B_{h-1}^n|^2 \right. \\
+ \left. 2 \left( \frac{t-t_{n-1}}{\tau} \right) \left( \frac{t_{n-1}}{\tau} \right) B_n^h \cdot B_{h-1}^n \right] \, dx \, dt \\
+ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \int_{\Omega} \left[ \left( \frac{t-t_{n-1}}{\tau} \right)^2 |\nabla B_n^h|^2 + \left( \frac{t_{n-1}}{\tau} \right)^2 |\nabla B_{h-1}^n|^2 \right. \\
+ \left. 2 \left( \frac{t-t_{n-1}}{\tau} \right) \left( \frac{t_{n-1}}{\tau} \right) \nabla B_n^h \cdot \nabla B_{h-1}^n \right] \, dx \, dt \\
= \frac{\tau}{3} \sum_{n=1}^{M} \int_{\Omega} \left[ |B_n^h|^2 + |B_{h-1}^n|^2 + B_n^h \cdot B_{h-1}^n \right. \\
+ |\nabla B_n^h|^2 + |\nabla B_{h-1}^n|^2 + \left. \nabla B_n^h \cdot \nabla B_{h-1}^n \right] \, dx \\
\leq \tau \sum_{n=0}^{M} \|B_n^h\|^2_{1,\Omega}. \]

\[
\square
\]

Next, for any \(\varphi(x) \in H_0\) and \(\psi(t) \in C^\infty(0,T)\), let \(\phi(x,t) = \varphi(x)\psi(t)\) and \(\phi_{h,\tau}(x,t) = \sum_{n=1}^{M} \chi_n(t) S_h \phi(x,t_n)\). We have by the triangle inequality, (3.4) and Lemma 3.1 that

\[
\int_0^T \|\phi(\cdot,t) - \phi_{h,\tau}(\cdot,t)\|^2_{1,\Omega} \, dt \\
\leq 2 \int_0^T \|\phi(\cdot,t) - S_\Delta \phi(\cdot,t)\|^2_{1,\Omega} \, dt + 2T \max_{0 \leq t \leq T} |\psi(t)| \|\varphi(\cdot) - S_h \varphi(\cdot)\|^2_{1,\Omega} \\
\to 0 \quad \text{as} \quad h, \tau \to 0.\]
Lemma 3.7. Direct computations give us the following equalities:

\[
\begin{align*}
\int_0^T \int_\Omega \frac{\partial}{\partial t} B_{h,t}(x,t) \phi_{h,t}(x,t) \, dx \, dt &= \tau \sum_{n=1}^M \int_\Omega \tau B_n^* S_h \phi(x,t_n) \, dx; \\
\int_0^T \int_\Omega \gamma \nabla \cdot (\dot{B}_{h,t}) (\nabla \cdot \phi_{h,t}) \, dx \, dt &= \tau \sum_{n=1}^M \int_\Omega \gamma (\nabla \cdot B_n^*) (\nabla \cdot S_h \phi(x,t_n)) \, dx; \\
\int_0^T \int_\Omega \beta_h \nabla \times \dot{B}_{h,t}(x,t) \cdot \nabla \times \phi_{h,t}(x,t) \, dx \, dt \\
&= \tau \sum_{n=1}^M \int_\Omega \beta_h \nabla \times B_n^* \cdot \nabla \times S_h \phi(x,t_n) \, dx; \\
\int_0^T \int_\Omega \frac{\dot{p}_{h,t} \nabla \cdot \phi_{h,t}}{1 + \sigma |B_{h,t}|^2} \hat{B}_{h,t} \cdot \nabla \times \phi_{h,t}(x,t) \, dx \, dt \\
&= \tau \sum_{n=1}^M \int_\Omega \frac{\dot{p}_n \nabla \cdot B_n^*}{1 + \sigma |B_n^*|^2} B_n^* \cdot \nabla \times S_h \phi(x,t_n) \, dx; \\
\int_0^T \int_\Omega u \times B_{h,t} \cdot \nabla \times \phi_{h,t} \, dx \, dt &= \tau \sum_{n=1}^M \int_\Omega \bar{u}^n \times B_n^* \cdot \nabla \times S_h \phi(x,t_n) \, dx.
\end{align*}
\]

Proof. By direct computation, we have the following equalities:

\[
\begin{align*}
\int_0^T \int_\Omega \frac{\partial}{\partial t} B_{h,t}(x,t) \phi_{h,t}(x,t) \, dx \, dt &= \int_0^T \int_\Omega \frac{\partial}{\partial t} B_n^* \phi_{h,t}(x,t) \, dx \, dt \\
&= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \int_\Omega \frac{\partial}{\partial t} B_n^* \sum_{n=1}^M \chi_n(t) S_h \phi(x,t_n) \, dx \, dt = \tau \sum_{n=1}^M \int_\Omega \frac{\partial}{\partial t} B_n^* S_h \phi(x,t_n) \, dx;
\end{align*}
\]

\[
\begin{align*}
\int_0^T \int_\Omega \gamma \nabla \cdot (\dot{B}_{h,t}) (\nabla \cdot \phi_{h,t}) \, dx \, dt \\
&= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \int_\Omega \gamma \nabla \cdot \left( \sum_{n=1}^M \chi_n(t) B_n^*(x) \right) \nabla \cdot \left( \sum_{n=1}^M \chi_n(t) S_h \phi(x,t_n) \right) \, dx \, dt \\
&= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \int_\Omega \gamma \left( \nabla \cdot B_n^* \right) \nabla \cdot S_h \phi(x,t_n) \, dx \, dt \\
&= \tau \sum_{n=1}^M \int_\Omega \gamma \left( \nabla \cdot B_n^* \right) \nabla \cdot S_h \phi(x,t_n) \, dx;
\end{align*}
\]
\begin{align*}
\int_0^T \int_{\Omega} \beta_h \nabla \times \hat{B}_{h,\tau}(x,t) \cdot \nabla \times \phi_{h,\tau}(x,t) \, dx \, dt \\
= \int_0^T \int_{\Omega} \beta_h \nabla \times \sum_{n=1}^M \chi_n(t) B^n_h(x) \cdot \nabla \times \sum_{n=1}^M \chi_n(t) S_h \phi(x,t_n) \, dx \, dt \\
= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \beta_h \nabla \times B^n_h(x) \cdot \nabla \times S_h \phi(x,t_n) \, dx dt \\
= \tau \sum_{n=1}^M \int_{\Omega} \beta_h \nabla \times B^n_h \cdot \nabla \times S_h \phi(x,t_n) \, dx ;
\end{align*}

\begin{align*}
\int_0^T \int_{\Omega} \hat{p}_{h,\tau}(\nabla \cdot \phi_{h,\tau}) \, dx \, dt \\
= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \int_{\Omega} \left( \sum_{n=1}^M \chi_n(t) p^n_h(x) \right) (\nabla \cdot \sum_{n=1}^M \chi_n(t) S_h \phi(x,t_n)) \, dx \, dt \\
= \tau \sum_{n=1}^M \int_{\Omega} p^n_h(x) (\nabla \cdot S_h \phi(x,t_n)) \, dx ;
\end{align*}

\begin{align*}
\int_0^T \int_{\Omega} \frac{f}{1 + |\beta_h,\tau|^2} \hat{B}_{h,\tau} \cdot \nabla \times \phi_{h,\tau}(x,t) \, dx \, dt \\
= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \int_{\Omega} \frac{f}{1 + |\beta_h^n(x)|^2} B^n_h(x) \cdot \nabla \times S_h \phi(x,t_n) \, dx \, dt \\
= \tau \sum_{n=1}^M \int_{\Omega} \frac{f^n}{1 + |\beta^n_h|^2} B^n_h \cdot \nabla \times S_h \phi(x,t_n) \, dx ,
\end{align*}

\begin{align*}
\int_0^T \int_{\Omega} \hat{u} \times \hat{B}_{h,\tau} \cdot \nabla \times \phi_{h,\tau} \, dx \, dt \\
= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \int_{\Omega} \hat{u} \times \sum_{n=1}^M \chi_n(t) B^n_h(x) \cdot \nabla \times \sum_{n=1}^M \chi_n(t) S_h \phi(x,t_n) \, dx dt \\
= \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \hat{u} dt \times B^n_h \cdot \nabla \times S_h \phi(x,t_n) \, dx \\
= \tau \sum_{n=1}^M \int_{\Omega} \hat{u}^n \times B^n_h \cdot \nabla \times S_h \phi(x,t_n) \, dx .
\end{align*}

We then derive some important convergence results.

**Lemma 3.8.** For any \( \beta_h \in K_h, \beta \in K, \ U_{h,\tau}, U \in L^2(0,T;H^1(\Omega)) \) and \( V_{h,\tau}, V \in L^2(0,T;L^2(\Omega)) \), if \( \beta_h \to \beta \) in \( L^2(\Omega) \) as \( h \to 0 \), \( U_{h,\tau} \rightharpoonup U \) in \( L^2(0,T;H^1(\Omega)) \) and

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as \( h, \tau \to 0 \), we have the following convergence results:

\[
\lim_{h,\tau \to 0} \int_0^T \int_\Omega \beta_h \nabla \times U_{h,\tau} \cdot V_{h,\tau} \, dx \, dt = \int_0^T \int_\Omega \beta \nabla \times U \cdot V \, dx \, dt,
\]

(3.6)

\[
\lim_{h,\tau \to 0} \int_0^T \int_\Omega \frac{f}{1 + \sigma |U_{h,\tau}|^2} U_{h,\tau} \cdot V_{h,\tau} \, dx \, dt = \int_0^T \int_\Omega \frac{f}{1 + \sigma |U|^2} U \cdot V \, dx \, dt,
\]

(3.7)

\[
\lim_{h,\tau \to 0} \int_0^T \int_\Omega \mathbf{u} \times U_{h,\tau} \cdot V_{h,\tau} \, dx \, dt = \int_0^T \int_\Omega \mathbf{u} \times U \cdot V \, dx \, dt.
\]

(3.8)

**Proof.** We first prove (3.6). By the triangle inequality, we have

\[
| \int_0^T \int_\Omega (\beta_h \nabla \times U_{h,\tau} - \beta \nabla \times U) \cdot V \, dx \, dt |
\]

\[\leq | \int_0^T \int_\Omega (\beta_h - \beta) \nabla \times U_{h,\tau} \cdot (V_{h,\tau} - V) \, dx \, dt |
\]

\[+ | \int_0^T \int_\Omega (\beta_h - \beta) \nabla \times U_{h,\tau} \cdot V \, dx \, dt |
\]

\[+ | \int_0^T \int_\Omega \beta (\nabla \times U_{h,\tau} \cdot V_{h,\tau} - \nabla \times U \cdot V) \, dx \, dt |
\]

\[\leq I + II + III.
\]

To estimate I, it is readily to see that

\[I = | \int_0^T \int_\Omega (\beta_h - \beta) \nabla \times U_{h,\tau} \cdot (V_{h,\tau} - V) \, dx \, dt |
\]

\[\leq 2\beta_2 \| \nabla \times U_{h,\tau} \|_{L^2(0,T;L^2(\Omega))} \| V_{h,\tau} - V \|_{L^2(0,T;L^2(\Omega))} \to 0,
\]

as \( h, \tau \to 0 \) due to the fact that \( V_{h,\tau} \to V \) in \( L^2(0,T;L^2(\Omega)) \).

Then we start to analyze II:

\[| \int_\Omega (\beta_h - \beta) \nabla \times U_{h,\tau} \cdot V \, dx \, dt |
\]

\[\leq ( \int_\Omega |\beta_h - \beta| \| \nabla \times U_{h,\tau} \|^2 \, dx \, dt )^{1/2} ( \int_\Omega |\beta_h - \beta| \| V \|^2 \, dx \, dt )^{1/2}
\]

\[\leq C( \int_\Omega |\beta_h - \beta| \| V \|^2 \, dx \, dt )^{1/2} \to 0
\]

as \( h, \tau \to 0 \) by Lebesgue dominated convergence theorem.

For III, we have

\[III = | \int_0^T \int_\Omega \beta \nabla \times U_{h,\tau} \cdot (V_{h,\tau} - V) + \beta \nabla \times (U_{h,\tau} - U) \cdot V \, dx \, dt |
\]

\[\leq \beta_2 \| \nabla \times U_{h,\tau} \|_{L^2(0,T;L^2(\Omega))} \| V_{h,\tau} - V \|_{L^2(0,T;L^2(\Omega))}
\]

\[+ | \int_0^T \int_\Omega \nabla \times (U_{h,\tau} - U) \cdot \beta V \, dx \, dt |
\]

\[\to 0 \text{ as } h, \tau \to 0.
\]
Next, we shall show (3.7).

\[
\left| \int_0^T \int_\Omega \frac{f}{1 + \sigma |U|} U_{h,\tau} \cdot V_{h,\tau} \, dx \, dt \right| \leq \left| \int_0^T \int_\Omega U_{h,\tau} \cdot (V_{h,\tau} - V) \, dx \, dt \right|
\]

\[
\pm \int_0^T \int_\Omega \frac{f (1 + \sigma |U|)}{1 + \sigma |U|} \, dx \, dt
\]

\[
\leq C \left( \|f\|_{H^1(0,T,L^\infty(\Omega))} \|U_{h,\tau}\|_{L^2(0,T;L^2(\Omega))} \|V_{h,\tau} - V\|_{L^2(0,T;L^2(\Omega))} + \|f\|_{H^1(0,T,L^\infty(\Omega))} \|U_{h,\tau} - U\|_{L^2(0,T;L^2(\Omega))} \|V\|_{L^2(0,T;L^2(\Omega))} \right)
\]

\[
\leq C \left( \|V_{h,\tau} - V\|_{L^2(0,T;L^2(\Omega))} + \|U_{h,\tau} - U\|_{L^2(0,T;L^2(\Omega))} \right)
\]

\[
\to 0 \quad \text{as } h, \tau \to 0.
\]

Finally, we shall prove the last equation (3.8).

\[
\left| \int_0^T \int_\Omega u \times U_{h,\tau} \cdot V_{h,\tau} \, dx \, dt \right| \leq \left| \int_0^T \int_\Omega u \times U \cdot V \, dx \, dt \right|
\]

\[
\to 0 \quad \text{as } h, \tau \to 0.
\]

In the following, we prove a crucial lemma for our purpose.

**Lemma 3.9.** For the sequence \( \{\beta_h\}_{h>0} \subset K_h \), if \( \{\beta_h\}_{h>0} \) converges to some \( \beta \in K \) in \( L^2(\Omega) \) strongly, suppose \( z^\delta \in C(0,T;L^2(\omega)) \), then there exists a subsequence, also denoted by \( \{\beta_h\}_{h>0} \), such that

\[
\lim_{h, \tau \to 0} \tau \sum_{n=0}^M \alpha_n \int_\omega (B^n_{h,\tau}(\beta_h) - z^{\delta,n})^2 \, dx = \int_0^T \int_\omega |B(\beta) - z^{\delta}|^2 \, dx \, dt.
\]

**Proof.** For \( 1 \leq n \leq M \), we denote by \( B^n_{h,\tau} = B^n_{h}(\beta_h) \), \( B^n = B(\beta)(\cdot, t_n) \). Making use of (3.4), we find that

\[
\lim_{\tau \to 0} \tau \sum_{n=0}^M \alpha_n \int_\omega (B^n - z^{\delta,n})^2 \, dx = \int_0^T \int_\omega (B(\beta) - z^{\delta})^2 \, dx \, dt.
\]

So it suffices to show that

\[
\lim_{h, \tau \to 0} \tau \sum_{n=0}^M \alpha_n \int_\omega (B^n_{h} - B^n)^2 \, dx = 0.
\]
From Lemma 3.6 and Lemma 3.3, we conclude that \( \{B_{h,\tau}\} \) and \( \{\hat{B}_{h,\tau}\} \) are bounded in \( L^2(0, T; H^1(\Omega)) \), \( \{\frac{\partial}{\partial t} B_{h,\tau}\} \) is bounded in \( L^2(0, T; (H^1(\Omega))') \) and \( \{\hat{p}_{h,\tau}\} \) is bounded in \( L^2(0, T; L^2(\Omega)) \). Hence there exists a subsequence of \( \{B_{h,\tau}\} \) such that

\[
B_{h,\tau} \to ^* B \text{ in } L^2(0, T; H^1(\Omega)),
\]

\[
\hat{B}_{h,\tau} \to ^* B \text{ in } L^2(0, T; L^2(\Omega)),
\]

\[
\frac{\partial}{\partial t} B_{h,\tau} \to ^* C \text{ in } L^2(0, T; (H^1(\Omega))')
\]

and a subsequence of \( \{\hat{B}_{h,\tau}\} \) and a subsequence of \( \{\hat{p}_{h,\tau}\} \) such that

\[
B_{h,\tau} \to ^* B \text{ in } L^2(0, T; H^1(\Omega)),
\]

\[
\hat{p}_{h,\tau} \to ^* p \text{ in } L^2(0, T; L^2(\Omega)),
\]

for some \( ^* B, ^* B \in L^2(0, T; H^1(\Omega)), \) \( ^* C \in L^2(0, T; (H^1(\Omega))') \) and \( ^* p \in L^2(0, T; L^2(\Omega)) \).

Next, we show \( ^* B = ^* B \) and \( ^* C(x, t) = \frac{\partial}{\partial t} ^* B(x, t) \). Firstly, by (3.12) we have for any \( \varphi(x) \in H^1(\Omega) \) and \( \psi(t) \in C^\infty_0(0, T) \) that

\[
\lim_{h,\tau \to 0} \int_0^T \int_\Omega \frac{\partial}{\partial t} B_{h,\tau} \cdot \varphi(x) \psi(t) dx dt = \int_0^T \int_\Omega C^*(x, t) \cdot \varphi(x) \psi(t) dx dt.
\]

On the other hand, by integration by parts with respect to \( t \) and using (3.10), we get

\[
\lim_{h,\tau \to 0} \int_0^T \int_\Omega \frac{\partial}{\partial t} B_{h,\tau} \cdot \varphi(x) \psi(t) dx dt = \lim_{h,\tau \to 0} \int_0^T \int_\Omega B_{h,\tau} \cdot \varphi(x) \psi'(t) dx dt
\]

\[
= -\int_0^T \int_\Omega ^* B \cdot \varphi(x) \psi'(t) dx dt,
\]

which, together with (3.15), gives

\[
C^*(x, t) = \frac{\partial}{\partial t} ^* B(x, t).
\]

Then taking any \( \varphi(x) \in H^1(\Omega) \) and \( \psi(t) \in C^1[0, T] \) with \( \psi(T) = 0 \), integrating by parts with respect to \( t \) to both sides of (3.15) and using (3.16) and (3.10), we have

\[
\lim_{h,\tau \to 0} \left\{ -\int_\Omega S_h B_0(x) \cdot \varphi(x) \psi(0) dx - \int_0^T \int_\Omega B_{h,\tau} \cdot \varphi(x) \psi'(t) dx dt \right\}
\]

\[
= -\int_\Omega ^* B(x, 0) \cdot \varphi(x) \psi(0) dx - \int_0^T \int_\Omega ^* B(x, t) \cdot \varphi(x) \psi'(t) dx dt.
\]

By (3.10) and Lemma 3.1 we derive that \( ^* B(x, 0) = B_0(x) \).

Now we will show that \( ^* B(x, t) = ^* B^**(x, t) \). By direct computation and using Lemma 3.1, we have

\[
\int_0^T \|B_{h,\tau}(\cdot, t) - \hat{B}_{h,\tau}(\cdot, t)\|_{0, \Omega}^2 dt = \frac{\tau^3}{3} \sum_{n=1}^M \|\partial_n B_n^\tau\|_{0, \Omega}^2 \leq C \tau^2 \to 0 \text{ as } h, \tau \to 0,
\]
which, together with (3.11), implies

\[ B_{h,\tau} \to B^* \text{ in } L^2(0,T; L^2(\Omega)). \]

Then from (3.13) and the uniqueness of the limits, we get \( B^* = B^{**} \).

It is time to show that \( B^* = B(\beta), \ p^* = p(\beta) \). Using Lemma 3.7 and system (3.3), we get

\[
\int_0^T \int_\Omega \frac{\partial}{\partial t} B_{h,\tau}(x,t) \phi_{h,\tau}(x,t) dx dt + \int_0^T \int_\Omega \beta_h \nabla \times \hat{B}_{h,\tau} \cdot \nabla \times \phi_{h,\tau} dx dt \\
+ \gamma \int_0^T \int_\Omega \nabla \cdot (\hat{B}_{h,\tau}) (\nabla \cdot \phi_{h,\tau}) dx dt + \int_0^T \int_\Omega \hat{p}_{h,\tau} (\nabla \cdot \phi_{h,\tau}) dx dt
\]

(3.17) = \( R_\alpha \int_0^T \int_\Omega \frac{f}{1 + \sigma |B_{h,\tau}|^2} B_{h,\tau} \cdot \nabla \times \phi_{h,\tau}(x,t) dx dt \\
+ R_m \int_0^T \int_\Omega u \times \hat{B}_{h,\tau} \cdot \nabla \times \phi_{h,\tau} dx dt.
\]

Letting \( h, \tau \to 0 \) in the above equation and making use of (3.10)-(3.14) and (3.6)-(3.8), we have

\[
\int_0^T \int_\Omega \frac{\partial}{\partial t} B^* \cdot \varphi(x) \psi(t) dx dt + \int_0^T \int_\Omega \beta (\nabla \times B^*) \cdot (\nabla \times \varphi) \psi(t) dx dt \\
+ \gamma \int_0^T \int_\Omega (\nabla \cdot B^*) (\nabla \cdot \varphi) \psi(t) dx dt + \int_0^T \int_\Omega p^* (\nabla \cdot \varphi) \psi(t) dx dt
\]

(3.18) = \( R_\alpha \int_0^T \int_\Omega \frac{f}{1 + \sigma |B^*|^2} B^* \cdot (\nabla \times \varphi) \psi(t) dx dt \\
+ R_m \int_0^T \int_\Omega u \times B^* \cdot (\nabla \times \varphi) \psi(t) dx dt.
\]

Further, we shall prove

(3.19) \( \int_0^T \int_\Omega (\nabla \cdot B^*) \varphi(x) \psi(t) dx dt = 0, \ \forall \ \varphi \in (L^2(\Omega))^3, \ \psi \in C^\infty(0,T), \)

which together with (3.18) and the definitions of \( B(\beta) \) and \( p(\beta) \) in (2.1) yields that

(3.20) \( B^* = B(\beta) \) and \( p^* = p(\beta) \).

Indeed, for any \( \varphi \in L^2(\Omega)^3 \) and \( \psi \in C^\infty(0,T) \), let \( \tilde{q}_{h} = \Pi_h \varphi - \frac{1}{|\Omega|} \int_\Omega \Pi_h \varphi dx \).

Then \( \tilde{q}_{h} \in Q_{0h} \) and we get by (3.3) and the divergence theorem that

\[ \int_\Omega (\nabla \cdot B^h) \Pi_h \varphi dx = \int_\Omega (\nabla \cdot B^h_n) \tilde{q}_{h} dx + \frac{1}{|\Omega|} \int_\Omega \Pi_h \varphi dx \int_\Omega \nabla \cdot B^h dx = 0. \]

We can also derive

\[ \int_0^T \int_\Omega (\nabla \cdot \hat{B}_{h,\tau}) \Pi_h \varphi \psi(t) dx dt = \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \left( \int_\Omega (\nabla \cdot B^h_n) \Pi_h \varphi dx \right) \psi(t) dt = 0. \]

Hence (3.19) immediately holds by taking \( h, \tau \to 0 \) in the above equation and making use of Lemma 3.2 and (3.13).
Now we will prove (3.9). Indeed, noting that $B_{h,τ}(\cdot, t_n) = B_h^n$ by the definition of $B_{h,τ}$ in (3.5), we have

\[
\tau \sum_{n=1}^{M} \int_{\Omega} (B_h^n - B^n)^2 \, dx - \int_{0}^{T} \|B_{h,τ}(\cdot, t) - B(\cdot, t)\|^2_{\Omega} \, dt \\
= \sum_{n=1}^{M} \int_{t_{n-1}}^{T} \int_{\Omega} \{(B_{h,τ}(\cdot, t_n) - B^n)^2 - (B_{h,τ}(\cdot, t) - B(\cdot, t))^2\} \, dx \, dt \\
\leq C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|B - B^n\|^2_{\Omega} \right\}^{1/2} + C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|t_n - t\| \partial_{\tau} B_h^n\|^2_{\Omega} \right\}^{1/2} \\
\leq C \left\{ \int_{0}^{T} \|B - B^n\|^2_{\Omega} \right\}^{1/2} + C \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|t_n - t\|^2 \partial_{\tau} B_h^n\|^2_{\Omega} \right\}^{1/2} \\
\leq C \left\{ \int_{0}^{T} \|B - B^n\|^2_{\Omega} \right\}^{1/2} + C\tau \sum_{n=1}^{M} \|\partial_{\tau} B_h^n\|^2_{\Omega} \right\}^{1/2}.
\]

This together with Lemma 3.3, Lemma 3.5 and (3.11) implies that

\[
\tau \sum_{n=1}^{M} \int_{\Omega} (B_h^n - B^n)^2 \, dx \leq \int_{0}^{T} \|B_{h,τ}(\cdot, t) - B(\cdot, t)\|^2_{\Omega} \, dt \\
+ C \left\{ \int_{0}^{T} \|B - B^n\|^2_{\Omega} \right\}^{1/2} + C\tau \sum_{n=1}^{M} \|\partial_{\tau} B_h^n\|^2_{\Omega} \right\}^{1/2} \\
\rightarrow 0 \quad \text{as} \quad h, \tau \rightarrow 0,
\]

which completes the proof. \hfill \Box

Finally, we are ready to establish the main convergence theorem.

**Theorem 3.10.** Let $\{\beta_h^*\}_{h>0}$ be a sequence of minimizers to the discrete minimization problem (3.2) and suppose $z^* \in C(0, T; L^2(\omega))$, then as $h$ and $\tau$ tend to 0, each sequence of $\{\beta_h^*\}_{h>0}$ has a subsequence converging in $L^2(\Omega)$ to a minimizer of the continuous optimization problem (2.2).

**Proof.** The uniform boundedness of the sequence $\{\beta_h^*\}_{h>0}$ in $K_h$ implies that there exists a subsequence, still denoted by $\{\beta_h^*\}_{h>0}$, and some element $\beta^* \in K$ such that

$\beta_h^* \rightarrow \beta^*$ in $H^1(\Omega)$ and $\beta_h^* \rightarrow \beta^*$ in $L^2(\Omega)$ as $h, \tau \rightarrow 0$.

Next we will show $\beta^*$ is a minimizer of the continuous optimization problem (2.2). To do so, for any $\beta \in K$, we define $\beta_h = \Pi_h \beta$, then we know (cf. [16]) that $\beta_h \in K_h$ and

$\beta_h \rightarrow \beta$ in $H^1(\Omega)$ as $h, \tau \rightarrow 0$.
Therefore we can deduce by Lemma 3.9 and the lower semi-continuity of a norm that
\[
J(\beta^*) = \frac{1}{2} \int_0^T \int_\Omega |B(\beta^*) - z^\delta|^2 dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta^*|^2 dx
\]
\[
\leq \lim_{h,\tau \to 0} \frac{T}{2} \sum_{n=0}^M \alpha_n \int_\omega |B_h^n(\beta_h^*) - z^\delta_n|^2 dx + \liminf_{h \to 0} \frac{\lambda}{2} \int_\Omega |\nabla \beta_h^*|^2 dx
\]
\[
\leq \liminf_{h,\tau \to 0} J_{h,\tau}(\beta_h^*) \leq \liminf_{h,\tau \to 0} J_{h,\tau}(\beta_h)
\]
\[
= \liminf_{h,\tau \to 0} \left\{ \frac{T}{2} \sum_{n=0}^M \alpha_n \int_\omega |B_h^n(\beta_h) - z^\delta_n|^2 dx + \frac{\lambda}{2} \int_\Omega |\nabla \beta_h|^2 dx \right\}
\]
\[
= \frac{1}{2} \int_0^T \int_\omega |B(\beta) - z^\delta|^2 dx \, dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta|^2 dx
\]
\[
= J(\beta).
\]
This yields that $\beta^*$ is a minimizer of the continuous problem (2.2).

**Concluding remarks.** In this paper, we have considered the inverse problem of recovering the magnetic diffusivity for a 3D spherical dynamo equation. The highly ill-posed inverse problem has been transformed into a stable minimization problem by using Tikhonov regularization and the existence and stability of the minimizers to the minimization problem has also been verified. Then the finite element approximation and its convergence have investigated.

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E-mail address: messaoudidjemaa10@mail.ccnu.edu.cn
E-mail address: osama101@yahoo.com
E-mail address: fsouley@rocketmail.com
E-mail address: tcheng@mail.ccnu.edu.cn
E-mail address: jiangdaijun@mail.ccnu.edu.cn