Factors and connected factors in tough graphs with high isolated toughness

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Abstract

Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$. Assume that for all $S \subseteq V(G)$,

$$\sum_{v \in I(G \setminus S)} f(v)(f(v) + 1) \leq |S|,$$

where $I(G \setminus S)$ denotes the set of isolated vertices of $G \setminus S$. In this paper, we show that if for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,$$

and $\sum_{v \in V(G)} f(v)$ is even, then $G$ has a factor $F$ such that for each vertex $v$, $d_F(v) = f(v)$, where $\omega(G \setminus S)$ denotes the number of components of $G \setminus S$. Moreover, we show that if for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \frac{1}{4}|S| + 1,$$

and $f \geq 2$, then $G$ has a connected factor $H$ such that for each vertex $v$, $d_H(v) \in \{f(v), f(v) + 1\}$.

Keywords:
Toughness; isolated toughness; regular factor; connected factor; $f$-factor.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let $G$ be a graph. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. We also denote by $iso(G)$, $odd(G)$, and $\omega(G)$ the number of isolated vertices of $G$, the number of components of $G$ with odd number of vertices, and the number of components of $G$, respectively. For a vertex set $S$ of $G$, we denote by $G[S]$ the induced subgraph of $G$ with the vertex set $S$ containing precisely those edges of $G$ whose ends lie in $S$. The vertex set $S$ is called an independent set, if there is no edge of $G$ connecting vertices in $S$. The maximum size of all independent sets of $G$ is denoted by $\alpha(G)$. Let $t$
be a positive real number, a graph $G$ is said to be $t$-tough, if $\omega(G \setminus S) \leq \max\{1, \frac{1}{t}|S|\}$ for all $S \subseteq V(G)$. Furthermore, $G$ is said to be $t$-iso-tough, if $iso(G \setminus S) \leq \frac{1}{t}|S|$ for all $S \subseteq V(G)$. This definition is a little different from [13, 14] for the sake of simplicity. Note that when $G$ is $t$-iso-tough, for each vertex $v$, the number of its neighbours must be at least $t$ and hence the conditions $d_G(v) \geq t$ and $V(G) \geq t + 1$ must automatically hold. More generally, when $t$ is a real function on $V(G)$, we say that $G$ is $t$-iso-tough, if for all $S \subseteq V(G)$, $\sum_{v \in I(G \setminus S)} t(v) \leq |S|$, where $I(G \setminus S)$ denotes the set of all isolated vertices of $G \setminus S$. We denote by $N_G(I)$ the set of all neighbours of vertices of $I$ in $G$. For a set $A$ of integers, an $A$-factor is a spanning subgraph with vertex degrees in $A$. Let $g$ and $f$ be two integer-valued functions on $V(G)$. A $(g,f)$-factor of $G$ is a spanning subgraph $F$ such that for each vertex $v$, $g(v) \leq d_F(v) \leq f(v)$. When $g = f - 1$, we call it a $\{g,g+1\}$-factor as well. An $f$-factor of $G$ refers to a spanning subgraph $F$ such that for each vertex $v$, $d_F(v) = f(v)$. A near $f$-factor refers to a spanning subgraph $F$ such that for each vertex $v$, $d_F(v) = f(v)$, except for at most one vertex $u$ with $d_F(u) = f(u) + 1$. Note that when the sum of all $f(v)$ taken over all vertices $v$ is even, $F$ is a near $f$-factor if and only if $F$ is an $f$-factor. Note that several theorems in graph theory for the existence of $f$-factors can be developed to a near $f$-factor version. This type of factors is useful when a factor is required for extending to connected factors with bounded degrees as the proof of Theorem 4.8. For example, see [3, 9]. For convenience, we write $\min f$ for $\min\{f(v) : v \in V(G)\}$ and write $\max f$ for $\max\{f(v) : v \in V(G)\}$. Let $A$ and $B$ be two disjoint vertex sets. We denote by $\omega_f(G,A,B)$ the number of components $C$ of $G \setminus (A \cup B)$ satisfying $\sum_{v \in V(C)} f(v) \geq \frac{2}{k} d_G(C,B)$, where $d_G(C,B)$ denotes the number of edges of $G$ with one end in $V(C)$ and the other one in $B$. Throughout this article, all variables $k$ are positive integers.

In 1947 Tutte introduced the following criterion for the existence of a perfect matching.

**Theorem 1.1.**([16]) A graph $G$ has a 1-factor if and only if for all $S \subseteq V(G)$, $\text{odd}(G \setminus S) \leq |S|$.

In 1978 Vergenas formulated a criterion for the existence of $(1,f)$-factors and showed that the criterion becomes simpler for the following special case.

**Theorem 1.2.**([11]) Let $G$ be a graph and let $f$ be an integer-valued function on $V(G)$ with $f \geq 2$. Then $G$ has a $(1,f)$-factor if and only if for all $S \subseteq V(G)$, $\text{iso}(G \setminus S) \leq \sum_{v \in S} f(v)$.

In 1985 Enomoto, Jackson, Katerinis, and Saito proved the following theorem on tough graphs, which was originally conjectured by Chvátal (1973) [5]. In 1990 Katerinis [10] generalized their result by replacing a weaker sufficient toughness condition for the existence of $[a,b]$-factors, provided that $a > b$.

**Theorem 1.3.**([8]) Every $k$-tough graph $G$ of order at least $k + 1$ with $k|V(G)|$ even has a $k$-factor.

In 2007 Ma and Yu strengthened Katerinis’ result by replacing isolated toughness condition as the following theorem. In this paper, we provide a supplement for their result by improving Theorem 1.3 for $(k+1)$-iso-tough graphs by showing that the needed toughness can be pushed down to the fixed number 1. In Section 5, we also establish another refined version in $k$-iso-tough graphs.
Theorem 1.4. ([13]) Every \((a - \frac{b-a}{b})\)-iso-tough graph has an \([a,b]\)-factor, when \(b > a \geq 1\).

In 1973 Chvátal [5] conjectured that there exists a positive real number \(t_0\) such that every \(t_0\)-tough graph of order at least three admits a Hamiltonian cycle. In 2000 Ellingham and Zha [7] confirmed a weaker version of this conjecture by proving that every 4-tough graph of order at least three admits a connected \(\{2,3\}\)-factor. Motivated by this result, one way ask whether higher toughness can guarantee the existence of connected \(\{k,k+1\}\)-factors. The following theorem shows that the answer is positive. In this paper, we show that the needed toughness of this theorem can be pushed down to the fixed number 3 but in \((k+1)\)-iso-tough graphs.

Theorem 1.5. ([6, 7], see [9]) Every \(k\)-tough graph of order at least \(k+1\) has a connected \(\{k,k+1\}\)-factor, where \(k \geq 3\).

In 1990 Katerinis formulated the following sufficient toughness condition for the existence of \(f\)-factors. In Section 4, we introduce some sufficient toughness conditions for the existence of \(f\)-factors and connected \(\{f,f+1\}\)-factors in graphs with high enough isolated toughness as mentioned in the abstract.

Theorem 1.6. ([10]) Let \(G\) be a graph and let \(f\) be a positive integer-valued function on \(V(G)\) satisfying \(a \leq f \leq b\), where \(a\) and \(b\) are two positive integers. If \(G\) is \(\frac{1}{b}(b^2 + 2(b-a) + 1)\)-tough and \(\sum_{v \in V(G)} f(v)\) is even, then \(G\) has an \(f\)-factor.

2 Tools and preliminary results

In this section, we shall provide some necessary tools for applying in the next sections. Before doing so, let us recall a theorem due to Tutte (1952) as the following version.

Theorem 2.1. ([17]) Let \(G\) be a general graph and let \(f\) be an integer-valued function on \(V(G)\). Then \(G\) has a near \(f\)-factor if and only if for all disjoint subsets \(A\) and \(B\) of \(V(G)\),

\[
\omega_f(G, A, B) \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d_{G \setminus A}(v) - f(v)) + 1.
\]

The following corollary is an application of Theorem 2.1, which is inspired by Lemma 4 in [10].

Corollary 2.2. (see [10]) Let \(G\) be a general graph and let \(f\) be an integer-valued function on \(V(G)\). Then \(G\) has a near \(f\)-factor, if

\[
\omega(G \setminus (A \cup B)) \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d_{G \setminus A}(v) - f(v)) + 1,
\]

for all disjoint subsets \(A\) and \(B\) of \(V(G)\) satisfying \(d_{G \setminus B}(u) \leq f(u) - 2\) and \(d_{G \setminus A}(u) \leq 2f(u) - 1\) for each \(u \in B\).
Proof. Let us define $g = f$. We are going to show that the inequality holds for any two disjoint subsets $A$ and $B$ of $V(G)$ and so the proof follows from Theorem 2.1. By induction on $|B|$. Let $q(A, B)$ be the right-hand side of the inequality in the corollary. Assume that $B$ has a vertex $u$ with $d_{G[A]}(u) \geq g(u) - 1$ or $d_{G[B]}(u) \geq g(u)$. Define $B_u = B \setminus \{u\}$. If $d_{G[B]}(u) \geq g(u) - 1$, then
\[
\omega(G \setminus (A \cup B)) \leq \omega(G \setminus (A \cup B_u)) + d - 1 \leq q(A, B_u) + d - 1 = q(A, B) - d_{G[B]}(u) + g(u) - 1 \leq q(A, B),
\]
where $d$ denotes the number of edges of $G$ incident to $u$ with the other end in $V(G) \setminus (A \cup B)$. Also, if $d_{G[A]}(u) \geq g(u)$, then
\[
\omega(G \setminus (A \cup B)) = \omega(G \setminus (A_u \cup B_u)) \leq q(A_u, B_u) = q(A, B) - d_{G[A]}(u) + f(u) + g(u) \leq q(A, B),
\]
where $A_u = A \cup \{u\}$. Hence the lemma holds. \hfill \Box

The following theorem can generalize Lemma 1 in [10] and plays an essential role in this paper.

**Theorem 2.3.** Let $H$ be a graph. If $\varphi$ is a nonnegative real function on $V(H)$, then there is a maximal independent subset $I$ of $V(H)$ such that
\[
\sum_{v \in V(H)} \varphi(v) \leq \sum_{v \in I} \varphi(v)(d_H(v) + 1).
\]

**Proof.** Define $H_0 = H$. For every nonnegative integer $i$ with $|V(H_i)| \geq 1$, take $u_i$ to be a vertex of $H_i$ with the maximum $\varphi(u_i)$ and set $H_{i+1} = H_i \setminus (N(u_i) \cup \{u_i\})$, where $N(u_i)$ denotes the set of all neighbours of $u_i$ in $H_i$. Define $I$ to be the set of all selected vertices $u_i$. It is not hard to check that $I$ is a maximal independent set of $H$ and $\{N(u) \cup \{u\} : u \in I\}$ is a partition of $V(H)$. Since $\varphi(u) \geq 0$,
\[
\sum_{v \in V(H) \setminus I} \varphi(v) = \sum_{u \in I} \sum_{v \in N(u)} \varphi(v) \leq \sum_{u \in I} \varphi(u)d_H(u).
\]
This inequality can complete the proof. \hfill \Box

**Corollary 2.4.**([4, 15]) For every graph $H$, we have $\alpha(H) \geq \sum_{v \in V(H)} \frac{1}{1 + d_H(v)}$.

**Proof.** Apply Theorem 2.3 with $\varphi(v) = 1/(1 + d_H(v))$. \hfill \Box

The following corollary provides an equivalent version for Theorem 2.3.

**Corollary 2.5.** Let $H$ be a graph and let $\varphi$ and $d$ be two real functions on $V(H)$. If for each $v \in V(H)$, $\varphi(v) \geq d(v) \geq d_H(v)$, then there is a maximal independent subset $I$ of $V(H)$ such that
\[
\sum_{v \in V(H)} (\varphi(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(\varphi(v) - d(v)).
\]

**Proof.** Apply Theorem 2.3 with replacing $(\varphi(v) - d(v))$ instead of $\varphi(v)$. \hfill \Box
3 Isolated toughness and the existence of \( \{f, f+1\} \)-factors

Our aim in this section is to generalize Theorem 1.4 by giving isolated toughness conditions for existence of \((g, f)\)-factors, provided that \( g < f \). For this purpose, we need the following lemma due to Lovász (1970).

\[ \text{Lemma 3.1.} \quad \text{Let } G \text{ be a graph and let } g \text{ and } f \text{ be two integer-valued functions on } V(G) \text{ with } g < f. \text{ Then } G \text{ has a } (g, f) \text{-factor, if and only if for all disjoint subsets } A \text{ and } B \text{ of } V(G), \]

\[ 0 \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d_{G \setminus A}(v) - g(v)). \]

The following theorem provides a common generalization for both of Theorems 2 and 3 in [13].

\[ \text{Theorem 3.2.} \quad \text{Let } G \text{ be a graph and let } g \text{ and } f \text{ be two nonnegative integer-valued functions on } V(G) \text{ with } g < f. \text{ Let } a \text{ be a positive real number with } a \leq f. \text{ Then } G \text{ has a } (g, f) \text{-factor, if it } \text{\( t \)-iso-tough, where for each vertex } v, \]

\[ t(v) = \begin{cases} g(v)(1 + \frac{1}{a}) - 1, & \text{when } g(v) \leq a + 2; \\ \frac{1}{a}((g + a + 1)^2 - \varepsilon_0(v)) - 1, & \text{otherwise}, \end{cases} \]

\[ \text{in which } \varepsilon_0(v) \in \{0, 1\} \text{ such that } \varepsilon_0(v) = 1 \text{ if and only if } g(v) \text{ and } a \text{ are integers with the same parity.} \]

\[ \text{Proof.} \quad \text{Let } A \text{ and } B \text{ be two disjoint subsets of } V(G). \text{ In order to apply Lemma 3.1, we should prove the inequality } 0 \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d(v) - g(v)), \text{ where } d(v) = d_{G \setminus A}(v). \text{ For this purpose, we may assume that for each } v \in B, d(v) \leq g(v) - 1. \text{ By applying Corollary 2.5 with } \varphi = g, \text{ the graph } G[B] \text{ has an independent set } I \text{ such that} \]

\[ \sum_{v \in B} (g(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(g(v) - d(v)). \]

Since \( G \) is \( t \)-iso-tough, we have \( \sum_{v \in I} t(v) \leq |A \cup N_G(I)| \leq |A| + \sum_{v \in I} d(v). \) Since \( d(v) \) is integer, we must have

\[ (d(v) + 1)(g(v) - d(v)) + a d(v) = (d(v) + 1)(g(v) + a - d(v)) - a \leq a t(v), \]

regardless of \( g(v) - 1 \leq (g(v) + a)/2 \) or not. This implies that

\[ \sum_{v \in B} (g(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(g(v) - d(v)) \leq \sum_{v \in I} (a t(v) - a d(v)) \leq a |A|. \]

Therefore,

\[ 0 \leq a |A| + \sum_{v \in B} (d(v) - g(v)) \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d(v) - g(v)). \]

Hence the assertion follows from Lemma 3.1. \( \Box \)

When we consider the special case \( \max g < \min f \), the theorem becomes simpler as the following result.
Corollary 3.3. Let $G$ be a graph and let $g$ and $f$ be two nonnegative integer-valued functions on $V(G)$ with $\max g < \min f$. If $G$ is $(g - 1 + \frac{a}{\min f})$-iso-tough, then it has a $(g, f)$-factor.

Proof. Apply Theorem 3.2 with $a = \min f$. □

Corollary 3.4. Every $f(f + 1)$-iso-tough graph $G$ has an $\{f, f + 1\}$-factor, where $f$ is a nonnegative integer-valued function on $V(G)$.

Proof. Apply Theorem 3.2 with setting and $g' = f, f' = f + 1$, and $a = 1$. □

4 Toughness, isolated toughness, and the existence of $f$-factors

In this section, we are going to present some sufficient toughness conditions for the existence of $f$-factors in graphs with high enough isolated toughness.

4.1 Regular factors in 1-tough graphs

The following theorem significantly improves the needed toughness in Theorem 1.3 in graphs with a bit higher isolated toughness.

Theorem 4.1. Let $k$ be a positive integer and let $n$ be a real number with $n \geq 1$. If $G$ is a $(k+1/n)$-iso-tough graph and for all $S \subseteq V(G)$,

$$\omega(G \setminus S) < \frac{1}{n}|S| + 2,$$

then $G$ has a near $k$-factor.

Proof. Let $A$ and $B$ be two disjoint subsets of $V(G)$ such that for each $v \in B$, $d_{G[B]}(v) \leq k - 2$. By applying a greedy coloring, one can decompose $B$ into $k - 1$ independent vertex sets $B_1, \ldots, B_{k-1}$. Let $S_i = A \cup N_G(B_i)$. By the assumption, we must have

$$(k + \frac{1}{n})|B_i| \leq (k + \frac{1}{n}) iso(G \setminus S_i) \leq |S_i| \leq |A| + \sum_{v \in B_i} d_{G \setminus A}(v),$$

which implies that

$$(k + \frac{1}{n})|B| = \sum_{1 \leq i < k} (k + \frac{1}{n})|B_i| \leq (k - 1)|A| + \sum_{1 \leq i < k} \sum_{v \in B_i} d_{G \setminus A}(v) = (k - 1)|A| + \sum_{v \in B} d_{G \setminus A}(v).$$

By the assumption, we must also have

$$\omega(G \setminus A \cup B) < \frac{1}{n}(|A| + |B|) + 2.$$
Therefore,
\[
\omega(G \setminus (A \cup B)) < (k - 1 + \frac{1}{n})|A| + \sum_{v \in B} (d_{G \setminus A}(v) - k) + 2 \leq k|A| + \sum_{v \in B} (d_{G \setminus A}(v) - k) + 2.
\]

Thus the assertion follows from Corollary 2.2. \( \square \)

**Remark 4.2.** Note that when \( G \) has no complete subgraphs of order \( k - 1 \) and \( k \geq 5 \), independent sets \( B_i \) could be chosen such that \( B_{k-1} = \emptyset \) using Brooks’ Theorem [2]. This fact allows us to reduce the lower bound on \( n \) to \( 1/2 \).

### 4.2 Graphs with toughness less than 1

The following theorem gives a sufficient toughness condition for the existence of \( f \)-factors.

**Theorem 4.3.** Let \( \varepsilon \) be a real number with \( 0 < \varepsilon \leq 1 \). Let \( G \) be a graph and let \( f \) be a positive integer-valued function on \( V(G) \). If \( G \) is \( f(f + 1)/\varepsilon \)-iso-tough and for all \( S \subseteq V(G) \),
\[
\omega(G \setminus S) < \sum_{v \in S} (f(v) - \varepsilon) + 2,
\]
then \( G \) has a near \( f \)-factor.

**Proof.** Let \( A \) and \( B \) be two disjoint subsets of \( V(G) \). We may assume that \( d(v) \leq 2f(v) - 1 \) for each \( v \in B \), where \( d(v) = d_{G \setminus A}(v) \). For each \( v \in B \), define \( \varphi(v) = 2f(v) - \varepsilon \) so that \( \varphi(v) \geq d(v) \). By Corollary 2.5, the graph \( G[B] \) has an independent set \( I \) such that
\[
\sum_{v \in B} (\varphi(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(\varphi(v) - d(v)).
\]

For each vertex \( v \), define \( t(v) = f(v)(f(v) + 1)/\varepsilon - 1 \). Since \( G \) is \( t \)-iso-tough, we have \( \sum_{v \in I} t(v) \leq |A \cup N_G(I)| \leq |A| + \sum_{v \in I} d(v) \). Since \( d(v) \) is integer, we must have
\[
(d(v) + 1)(\varphi(v) - d(v)) + \varepsilon d(v) = (d(v) + 1)(2f(v) - d(v)) - \varepsilon \leq \varepsilon t(v),
\]
which implies that
\[
\sum_{v \in B} (\varphi(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(\varphi(v) - d(v)) \leq \sum_{v \in I} (\varepsilon t(v) - \varepsilon d(v)) \leq \varepsilon |A|.
\]

On the other hand, by the assumption,
\[
\omega(G \setminus (A \cup B)) < \sum_{v \in A \cup B} (f(v) - \varepsilon) + 2 = \sum_{v \in A} (f(v) - \varepsilon) + \sum_{v \in B} (f(v) - \varepsilon) + 2.
\]

Therefore,
\[
\omega(G \setminus (A \cup B)) \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d(v) - f(v)) + 1.
\]

Hence the assertion follows from Corollary 2.2. \( \square \)
Corollary 4.4. Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$. If $G$ is $(f + 1)$-iso-tough and for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,$$

then $G$ has a near $f$-factor.

Proof. Apply Theorem 4.3 with $\varepsilon = 1$. \hfill $\Box$

The isolated toughness needed in Corollary 4.4 can be improved by a coefficient for graphs with higher toughness as the next theorem, provided that $\min f$ is sufficiently large.

Theorem 4.5. Let $G$ be a graph and let $f$ be a positive integer-valued function on $V(G)$, and let $a$ be a positive real number with $f \geq a \geq 1$. If $G$ is $\frac{1}{a}(f + a/2)^2$-iso-tough and for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - a) + 2,$$

then $G$ has a near $f$-factor.

Proof. Let $A$ and $B$ be two disjoint subsets of $V(G)$. We may assume that $d(v) \leq 2f(v) - 1$ for each $v \in B$, where $d(v) = d_{G \setminus A}(v)$. For each $v \in B$, define $\varphi(v) = 2f(v) - 1$ so that $\varphi(v) \geq d(v)$. By Corollary 2.5, the graph $G[B]$ has an independent set $I$ such that

$$\sum_{v \in B} (\varphi(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(\varphi(v) - d(v)).$$

For each vertex $v$, define $t(v) = \frac{1}{a}((f(v) + a/2)^2) - 1$. Since $G$ is $t$-iso-tough, we have $\sum_{v \in I} t(v) \leq |A \cup N_{G}(I)| \leq |A| + \sum_{v \in I} d(v)$. In addition, we must have

$$(d(v) + 1)(\varphi(v) - d(v)) + a d(v) = (d(v) + 1)(2f(v) - 1 + a - d(v)) - a \leq a t(v),$$

which implies that

$$\sum_{v \in B} (\varphi(v) - d(v)) \leq \sum_{v \in I} (d(v) + 1)(\varphi(v) - d(v)) \leq \sum_{v \in I} (a t(v) - ad(v)) \leq a|A|.$$

On the other hand, by the assumption,

$$\omega(G \setminus (A \cup B)) < \sum_{v \in A \cup B} (f(v) - a) + 2 = \sum_{v \in A} (f(v) - a) + \sum_{v \in B} (f(v) - a) + 2.$$

Therefore,

$$\omega(G \setminus (A \cup B)) \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d(v) - f(v)) + 1.$$

Hence the assertion follows from Corollary 2.2. \hfill $\Box$
4.3 Applications to the existence of connected \( \{f, f + 1\} \)-factors

The following lemma is a useful tool for extending factors to connected factors by inserting a matching.

**Lemma 4.6.** ([7], see [9]) Let \( \varepsilon \) be a real number with \( 0 < \varepsilon \leq 2 \). Let \( G \) be a simple graph and let \( F \) be a factor of \( G \) with minimum degree at least \( 2/\varepsilon + 1 \). If for all \( S \subseteq V(G) \),

\[
\omega(G \setminus S) \leq \frac{1}{2 + \varepsilon} |S| + 1,
\]

then \( G \) has a connected factor \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \in \{d_F(v), d_F(v) + 1\} \), and also \( d_H(u) = d_F(u) \) for an arbitrary given vertex \( u \).

The following result shows an application of Lemma 4.6 and Theorem 4.1.

**Theorem 4.7.** Every \( 3 \)-tough \((k + 1/3)\)-iso-tough graph has a connected \( \{k, k + 1\} \)-factor, where \( k \geq 3 \).

**Proof.** We may assume that \( G \) simple, by deleting multiple edges from \( G \) (if necessary). By Theorem 4.1, the graph \( G \) has a near \( k \)-factor \( F \) so that for all vertices \( v \), \( d_F(v) = k \), except for at most one vertex \( u \) with \( d_F(u) = k + 1 \). By applying Lemma 4.6 with \( \varepsilon = 1 \), the graph \( G \) has a connected factor \( H \) such that for each vertex \( v \), \( d_H(v) \in \{d_F(v), d_F(v) + 1\} \), and also \( d_H(u) = d_F(u) \). This implies that \( H \) is a connected \( \{k, k + 1\} \)-factor. □

The next result shows an application of Lemma 4.6 and Corollary 4.4.

**Theorem 4.8.** Let \( \varepsilon \) be a real number with \( 0 < \varepsilon \leq 2 \). Let \( G \) be graph and let \( f \) be a positive integer-valued function on \( V(G) \) with \( f \geq 2/\varepsilon + 1 \). If \( G \) is \( f(f + 1) \)-iso-tough and for all \( S \subseteq V(G) \),

\[
\omega(G \setminus S) \leq \frac{1}{2 + \varepsilon} |S| + 1,
\]

then \( G \) has a connected \( \{f, f + 1\} \)-factor.

**Proof.** We may assume that \( G \) simple, by deleting multiple edges from \( G \) (if necessary). By Corollary 4.4, the graph \( G \) has a near \( f \)-factor \( F \) so that for all vertices \( v \), \( d_F(v) = f(v) \), except for at most one vertex \( u \) with \( d_F(u) = f(u) + 1 \). By applying Lemma 4.6, the graph \( G \) has a connected factor \( H \) such that for each vertex \( v \), \( d_H(v) \in \{d_F(v), d_F(v) + 1\} \), and also \( d_H(u) = d_F(u) \). This implies that \( H \) is a connected \( \{f, f + 1\} \)-factor. □

**Corollary 4.9.** Every \( 3 \)-tough \( f(f + 1) \)-iso-tough graph \( G \) has a connected \( \{f, f + 1\} \)-factor, where \( f \) is a positive integer-valued function on \( V(G) \) with \( f \geq 3 \).

**Proof.** Apply Theorem 4.8 with \( \varepsilon = 1 \). □
5 Graphs with higher toughness

Our in this section is to provide another improvement for Theorem 1.3. Before doing so, let us refine Theorem 4.5 slightly for graphs with higher toughness.

**Theorem 5.1.** Let $G$ be a graph, let $f$ be a positive integer-valued function on $V(G)$, and let $a$ be a real number with $f \geq a > 1$. For each vertex $v$, let $\varepsilon_0(v) \in \{0, 1\}$ such that $\varepsilon_0(v) = 1$ if only if $f(v)$ and $a$ are integers with the same parity. Then $G$ has a near $f$-factor, if $G$ is $\frac{1}{4(a-1)}((f(v) + a - 1)^2 - \varepsilon_0(v))$-iso-tough and for all $S \subseteq V(G)$,

$$\omega(G \setminus S) + \sum_{v \in I_+(G \setminus S)} (f(v) - 1) \leq |S| + 1,$$

where $I_+(G \setminus S)$ is the set of center vertices of the star components of $G \setminus S$ in which for stars with one edge $xy$ the vertex $x$ is center whenever $f(x) \geq f(y)$.

**Proof.** The proof presented here is inspired by the proof of Theorem 1 in [10]. Let $A$ and $B$ be two disjoint subsets of $V(G)$. For notational simplicity, we write $d(v)$ for $d_{G \setminus A}(v)$. Define $B_0 = \{v \in B : d(v) < f(v)\}$. By Corollary 2.5, the graph $G[B_0]$ has an independent set $I_0$ such that

$$\sum_{v \in B_0} (f(v) - d(v)) \leq \sum_{v \in I_0} (d(v) + 1)(f(v) - d(v)). \tag{1}$$

For each vertex $v$, let $t(v) = \frac{1}{4(a-1)}((f(v) + a - 1)^2 - \varepsilon_0(v))$. Since $G$ is $t$-iso-tough, we have $\sum_{v \in I_0} t(v) \leq |A \cup N_G(I_0)| \leq |A| + \sum_{v \in I_0} d(v)$. Since $d(v)$ is integer, we must have

$$(d(v) + 1)(f(v) - d(v)) + a d(v) - f(v) = d(v)(f(v) + a - 1 - d(v)) \leq (a - 1)t(v),$$

which implies that

$$\sum_{v \in I_0} (d(v) + 1)(f(v) - d(v)) - \sum_{v \in I_0} f(v) \leq \sum_{v \in I_0} ((a - 1)t(v) - a d(v)) \leq (a - 1)|A| - \sum_{v \in I_0} d(v). \tag{2}$$

Therefore, Relations (1) and (2) can deduce that

$$\sum_{v \in B_0} (f(v) - d(v)) \leq (a - 1)|A| - \sum_{v \in I_0} (d(v) - f(v)). \tag{3}$$

Let $I$ be a maximal independent set in $G[B]$ containing the vertices of $I_0$ so that $B \setminus I \subseteq N_G(I)$. Denote by $x_1$ the number of components $C$ of $G \setminus (A \cup B)$ such that $d_G(v, I) = 1$ for each $v \in V(C)$. For every such a component $C$, select a vertex $z$ with $d_G(z, I) = 1$. Define $Z$ to be the set of all selected vertices. Also, denote by $x_2$ the number of components $C$ of $G \setminus (A \cup B)$ such that $d_G(v, I) \geq 1$ for each $v \in V(C)$, and $d_G(u, I) \geq 2$ for at least one vertex $u \in V(C)$. Set $S = A \cup (N_G(I) \setminus Z)$. According to this definition, it is not difficult to show that

$$|S| \leq |A| + \sum_{v \in I} d(v) - x_1 - x_2,$$
and
\[ \omega(G \setminus (A \cup B)) \leq \omega(G \setminus S) + x_1 + x_2 - |I|. \]

On the other hand, by the assumption,
\[ \omega(G \setminus S) + \sum_{v \in I} (f(v) - 1) \leq |S| + 1, \]
which implies that
\[ \omega(G \setminus (A \cup B)) \leq |A| + \sum_{v \in I} (d(v) - f(v)) + 1. \]

Since \( d(v) \geq f(v) \) for each \( v \in B \setminus (B_0 \cup I) \), we must have
\[ \omega(G \setminus (A \cup B)) \leq |A| + \sum_{v \in I_0} (d(v) - f(v)) + 1 \leq \sum_{v \in A} f(v) + \sum_{v \in B \setminus B_0} (d(v) - f(v)) + 1. \]

Therefore, Relations (3) and (4) can conclude that
\[ \omega(G \setminus (A \cup B)) \leq a|A| + \sum_{v \in B} (d(v) - f(v)) + 1 \leq \sum_{v \in A} f(v) + \sum_{v \in B} (d(v) - f(v)) + 1. \]

Hence the assertion follows from Corollary 2.2. \( \square \)

The following corollary is an improved version of Theorem 1.3.

**Corollary 5.2.** A graph \( G \) has a near \( k \)-factor, if for all \( S \subseteq V(G) \), \( iso(G \setminus S) \leq \frac{1}{k} |S| \), and
\[ \omega(G \setminus S) + (k - 1) w_*(G \setminus S) \leq |S| + 1, \]
where \( w_*(G \setminus S) \) denotes the number of star components of \( G \setminus S \).

**Proof.** Apply Theorem 5.1 with setting \( f(v) = a = k \) when \( k > 1 \). For the special case \( k = 1 \), one can apply Theorem 1.1 directly. \( \square \)

**Corollary 5.3.** (8) Every \( k \)-tough graph \( G \) of order at least \( k + 1 \) has a near \( k \)-factor.

**Proof.** We may assume that \( |V(G)| \geq k + 2 \). Let \( S \) be a subset of \( V(G) \). If \( |S| < k \), then \( \omega(G \setminus S) = 1 \) and also \( iso(G \setminus S) = 0 \). Since \( |V(G)| \geq k + 2 \), by the assumption, one can conclude that each vertex of \( G \) contains at least \( k + 1 \) neighbours. Hence \( w_*(G \setminus S) = 0 \). If \( |S| \geq k \), then we have \( iso(G \setminus S) \leq |S|/k \) and
\[ \omega(G \setminus S) + (k - 1) w_*(G \setminus S) \leq k \omega(G \setminus S) \leq |S|. \]
Now, it is enough to apply Corollary 5.2. \( \square \)

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