On Schwinger’s formula for pair production

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Abstract

We present some comments on Schwingers’s calculation of electron-positron production in a prescribed constant electric field. The range of validity of \(2 \text{Im} \mathcal{L}^{(1)}(E)\) is discussed thoroughly and limiting cases are provided.

1 Number of electron-positron pairs produced in a uniform electric field

Start with Schwinger’s expression for \(2 \text{Im} \mathcal{L}^{(1)}(E)\) [1]:

\[
2 \text{Im} \mathcal{L}^{(1)} = 2 \left(\frac{eE}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{e}{n\pi}\right)^2, \quad \beta = \frac{eE}{m^2}
\]

\[
= \text{Li}_2(e^{-\pi}), \text{Euler’s dilogarithm}
\]

\[\uparrow^2 \quad E = \text{const.} \]

\[\hbar = c = 1, \quad V = L^3\]

In general for spin \(s = \frac{1}{2}\) and \(s = 0\):

\[
2 \text{Im} \mathcal{L}^{(1)}(E) = (2s + 1) \left(\frac{eE}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} e^{-n\pi}
\]

\[
= \pm(2s + 1) \left(\frac{eE}{2\pi}\right)^2 \text{Li}_2(\pm e^{-\pi}). \quad (1.2)
\]
Continuous phase space integration, how to count states:

\[
\int d^3p \frac{V}{(2\pi)^3} \bigg/ = \int_{-\infty}^{+\infty} \frac{L}{2\pi} dp_1 \int_{-\infty}^{+\infty} \frac{L}{2\pi} dp_2 \int_{-\infty}^{+\infty} \frac{L}{2\pi} dp_3 \bigg/. \quad (1.3)
\]

\[t = \frac{eE}{\hbar}\] in const. \(E\)-field: \(dp_3 = eE dt, \quad 0 \leq t \leq T\) or \(\int_{-\infty}^{+\infty} dp_3 \to eET\).

Replace one factor \((eE)\) in (1.1) by \(\int_{-\infty}^{+\infty} dp_3\). Then

\[ImL^{(1)}(E)T = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dp_3 (eE) \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\pi \frac{n}{E}}. \quad (1.4)
\]

Rewrite the sum in (1.4):

\[
eE \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi}{E} n} = \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi}{E} n} \frac{1}{2} \left\{ \frac{1}{2 \pi \frac{n}{E}} \right\} = \frac{1}{\pi} \int_{0}^{\infty} dx e^{-ax}, \quad x = \frac{\pi n}{eE}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi}{E} n} \frac{1}{2} \int_{0}^{\infty} dp_{\perp} p_{\perp} \exp \left\{ -\pi \frac{m^2 + p_{\perp}^2}{eE} n \right\}
\]

\[
= \int_{0}^{\infty} dp_{\perp} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left\{ -\pi \frac{m^2 + p_{\perp}^2}{eE} n \right\} \Longrightarrow
\]

Use \(\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad |x| < 1, \quad x = \exp \left\{ -\pi \frac{m^2 + p_{\perp}^2}{eE} \right\}\)

\[
\Longrightarrow = \int_{0}^{\infty} dp_{\perp} \ln(1 - e^{-\pi \lambda p_{\perp}}) \Longrightarrow, \quad \lambda_p = \frac{m^2 + p_{\perp}^2}{eE} \quad (1.5)
\]

\[
e^{-\pi \lambda_p} = e^{-\pi \frac{m^2 + p_{\perp}^2}{eE}} = \bar{n}_{p}. \quad (1.6)
\]

This important expression - also probability for tunneling - relates the imaginary part of the Lagrangian of the field to the mean number \(\bar{n}_{p}\) of electron-positron...
pairs produced by the field in the state with given momentum and spin projection. \( \bar{n}_p \) is degenerate with respect to spin (two) and momentum \( p_3 \) with \( \Delta p_3 = eET \).

So we can continue to write (1.5):

\[
\Rightarrow - \int_0^\infty dp_\perp p_\perp \ln(1 - \bar{n}_p), \quad \int_0^\infty dp_\perp 2\pi p_\perp/\pi = \int_{-\infty}^\infty dp_1 dp_2/\pi.
\]

Here then is the relation between \( Im\mathcal{L}^{(1)}(E) \) and \( \bar{n}_p \) (insert \( \hbar \) and \( V = L^3 \)):

\[
\frac{2}{\hbar} Im\mathcal{L}^{(1)}VT = -2 \int d^3p \frac{V}{(2\pi)^3} \ln(1 - \bar{n}_p) \quad \text{(1.7)}
\]

\[
\bar{n}_p = \exp \left\{ -\pi \frac{m^2 + p_\perp^2}{eE} \right\} \quad \text{(1.8)}
\]

This is Nikishov’s virial representation of the imaginary part of \( \mathcal{L}^{(1)}(E) \) [2].

With the aid of (1.8) let us prove Nikishov’s result for the mean number of pairs in four-volume \( VT \) by counting states:

\[
\bar{n} = 2 \int_{-\infty}^{+\infty} dp_1 \frac{L}{(2\pi)} \int_{-\infty}^{+\infty} dp_2 \frac{L}{(2\pi)} \int_{-\infty}^{eEL} dp_3 \frac{T}{(2\pi)} \bar{n}_p \Rightarrow \]

\[
= \frac{2L^2}{(2\pi)^2} 2\pi \int_0^\infty p_\perp dp_\perp \frac{T}{2\pi} eELe^{-\pi\lambda_p} \]

\[
= \frac{(eE)^2}{(2\pi)^2} VT \int_0^\infty dp_\perp e^{-\pi \frac{p_\perp^2}{eE}} e^{-\frac{E}{1/2}} = \frac{eE}{(2\pi)^2} VT \frac{1}{\pi eE} e^{-\frac{E}{2}}.
\]

Therefore

\[
\bar{n} = 2 \frac{(eE)^2}{(2\pi)^3} VTe^{-\frac{E}{2}}, \quad \beta = \frac{eE}{m^2}, \quad \text{(1.9)}
\]

which is Nikishov’s result for the mean number of pairs produced in volume \( V = L^3 \) during time \( T \). For Bose particles the factor 2 is suppressed.

Introducing

\[
\xi = \exp \left\{ -\pi \frac{m^2}{eE} \right\}, \quad \gamma = VT \frac{(eE)^2}{4\pi^3} \quad \text{(1.10)}
\]

we also can write

\[
\bar{n} = \gamma \xi. \quad \text{(1.11)}
\]
Formula (1.9) is an approximation of the following expression for $n = 1$:

$$\int d^3p \frac{V}{(2\pi)^3} \sum_{n=1}^{\infty} \bar{n}_p^n.$$  \hfill (1.12)

Furthermore the well-known vacuum persistence probability $|\langle O_+|O_- \rangle|^2$ is given by

$$|\langle O_+|O_- \rangle|^2 = p_0 = \prod_{s,p} (1-e^{-\pi \lambda_p})$$

$$= \prod_{s,p} \left( 1-e^{-\pi \frac{n^2 + p^2}{eE}} \right) = e^{-2Im\mathcal{L}^{(1)}(E)VT}$$

$$= \exp \left\{ -VT \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi n^2}{eE}} \right\}$$

$$= \exp \{ -\gamma Li_2(\xi) \}. \hfill (1.13)$$

Schwinger writes in his brilliant article [1] as well as in his 2nd volume on “sources, particles and fields”: “We recognize in $2\Im L(1)$ a measure in the probability, per unit time and unit spatial volume, that an electron-positron pair has been created.” This statement is only true for very weak fields ($eE \ll m^2$) in which case the contributions of the $n = 2, 3, \ldots$ terms in the sum of (1.1) can be neglected:

$$2Im\mathcal{L}^{(1)}(eE \ll m^2) = 2 \frac{(eE)^2}{(2\pi)^3} e^{-\frac{\pi}{eE}} = \frac{\bar{n}}{VT} = \frac{\gamma \xi}{VT} \simeq \frac{\gamma L}{VT}; \quad L = -\ln(1-\xi),$$

which is identical to Nikishov’s result. Here, in order to save at least part of Schwinger’s statement, we are being a bit casual since $\bar{n}$ is not a probability but an average number. To be more specific let us start with (1.12):

$$2 \int d^3p \frac{V}{(2\pi)^3} \sum_{n=1}^{\infty} \bar{n}_p^n, \quad \int dp_3 = TeE$$

$$= \int dp_1 \int dp_2 \sum_{n=1}^{\infty} \bar{n}_p^n, \quad \int dp_1 \int dp_2 = 2\pi \int_0^{\infty} dp_\perp$$

$$= \int \frac{(eE)}{2\pi^2} \sum_{n=1}^{\infty} e^{-\pi \lambda_p n}, \quad \bar{n}_p = e^{-\pi \lambda_p}, \lambda_p = \frac{m^2 + p^2_\perp}{eE}, \beta = \frac{eE}{m^2}$$

$$= \int \frac{(eE)}{2\pi^2} \sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{eE}} \frac{1}{2 \pi} \int_0^{\infty} dp_\perp e^{-\frac{\pi n^2}{eE}}$$

$$= \int \frac{(eE)}{2\pi^2} \sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{eE}} = VT \frac{(eE)^2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi n^2}{eE}}$$

$$= VT \frac{(eE)^2}{4\pi^3} \left[ -\ln(1-e^{-\frac{\pi n^2}{eE}}) \right] = \frac{p_1}{p_0} = \gamma L.$$
Finally

\[ p_1 = VT \left( \frac{(eE)^2}{4\pi^3} \left[ -\ln \left( 1 - e^{-\frac{m^2}{eE}} \right) \right] \right) \cdot \exp \left\{ -VT \left( \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n \frac{m^2}{eE}} \right) \right\} \]

\[ p_1 = \gamma L p_0, \quad L = -\ln(1 - \xi), \quad p_0 = \exp \{ -\gamma L i_2(\xi) \}. \quad (1.14) \]

So far we have

\[ p_0 = \exp \{ -\gamma L i_2(\xi) \}, \quad \text{Schwinger’s vacuum persistence probability} \]

\[ p_1 = \gamma L p_0, \quad \bar{n} = \gamma \xi. \]

Let’s denote by \( \alpha = (p, s) \) the quantum numbers of the electron states. Then we can write our vacuum-to-vacuum probability as

\[ p_0 = \prod_\alpha (1 - \bar{n}_\alpha). \quad (1.15) \]

In an electric field any number of pairs can be produced, so the probability that there are \( n = 0, 1, 2, \ldots \) electron-positron pairs shows up in the series

\[ \sum_{n=0}^{\infty} p_n = \prod_\alpha (1 - \bar{n}_\alpha) + \sum_\alpha \bar{n}_\alpha \prod_{\beta \neq \alpha} (1 - \bar{n}_\beta) \]

\[ + \frac{1}{2!} \sum_{\alpha \neq \beta} \bar{n}_\alpha \bar{n}_\beta \prod_{\gamma \neq \alpha, \beta} (1 - \bar{n}_\gamma) + \ldots = 1. \quad (1.16) \]

In our case, each of the quantities \( p_n \) describes the probability for the number \( n = 0, 1, 2, \ldots \) electron-positron pairs in four-volume. The first terms were calculated above:

\[ p_0 = \exp \{ -\gamma L i_2(\xi) \} \]

\[ p_1 = \gamma L p_0. \quad (1.17) \]

The next and followers for the numbers \( n \) were calculated by Krivoruchenko:

\[ p_2 = \frac{\gamma}{2} \left( \gamma L^2 + L - \frac{\xi}{1 - \xi} \right) p_0 \quad \text{etc.} \quad (1.18) \]

It is highly interesting to follow Krivoruchenko’s paper [3] and find out that in electric fields of supercritical strength \( |eE| > \frac{\pi m^2}{\ln 2} \), the unitary condition (1.16), \( \sum_{n=0}^{\infty} p_n = 1 \), changes into an asymptotic divergence, i.e. the positive definite-ness of the probability is violated. This divergence indicates a failure of the continuum limit approximation, i.e. by the replacement of the discrete sum by the integral over the phase space:

\[ \sum_\alpha \ldots \rightarrow 2 \int \frac{V}{(2\pi)^3} d^3 p \ldots = VT |eE| \int \frac{2d^2 p_\bot}{(2\pi)^3} \ldots. \quad (1.19) \]
2 Schwinger’s formula for $\text{Im}\mathcal{L}^{(1)}(E)$ the long way, i.e., without using the residue theorem.

Take the formula (5.27) or equivalently (6.33) of the “Lecture Notes 220” on “Effective Lagrangians in QED” by Dittrich and Reuter [4, 5]:

$$\mathcal{L}^{(1)}(B) = -\frac{1}{32\pi^2} \left\{ (2m^4 - 4m^2(eB) + \frac{4}{3}(eB)^2) \left[ 1 + \ln \left( \frac{m^2}{2eB} \right) \right] + 4m^2(eB) - 3m^4 - (4eB)^2 \zeta'\left(-1, \frac{m}{2eB}\right) \right\}. \quad (2.1)$$

This can also be written in the form

$$\mathcal{L}^{(1)}(B) = -\frac{1}{32\pi^2} \left\{ -3m^4 + 4(eB)^2 \left( \frac{1}{3} - 4\zeta'(-1) \right) + 4m^2(eB) \ln 2\pi - 1 \right\} - 2m^4 \ln \frac{2eB}{m^2} - 4m^2(eB) \ln \frac{2eB}{m^2} - \frac{4}{3}(eB)^2 \ln \frac{2eB}{m^2}$$

$$-16(eB)^2 \int_1^{\frac{1}{1+\frac{m^2}{2eB}}} dx \ln \Gamma(x) \right\}. \quad (2.2)$$

Introducing the critical field strength $B_{er} = \frac{m^2}{e}$ and measuring the magnetic field in this unit, we can rewrite the last expression as

$$\mathcal{L}^{(1)}(B) = -\frac{1}{2\pi^2} \left\{ \frac{3}{4} - B(\ln(2\pi) - 1) - B^2 \left( \frac{1}{3} - 4\zeta'(-1) \right) \right\}$$

$$+ \left( \frac{1}{2} + B + \frac{1}{3}B^2 \right) \ln(2B) + 4B^2 \int_1^{1+\frac{m^2}{2eB}} \ln(\Gamma(x)) dx \right\}. \quad (2.3)$$

For a pure electric field the Lagrangian is likewise given by

$$\mathcal{L}^{(1)}(E) = -\frac{1}{32\pi^2} \left\{ \left( 2m^4 + 4im^2 eE - \frac{4}{3}e^2E^2 \right) \left( \ln \left( i \frac{m^2}{2eE} \right) + 1 \right) - 3m^4 - 4im^2 eE + 16e^2E^2 \zeta'\left(-1, i \frac{m^2}{2eE}\right) \right\}. \quad (2.4)$$
It takes a little practice to separate this formula into its real and imaginary part:

\[
\mathcal{L}^{(1)}(E) = \frac{\alpha}{2\pi} \left\{ \frac{3}{4} + \frac{1}{2} \ln(2E) - \frac{\pi}{2} E + E^2 \left( \frac{1}{3} - 4\zeta'(-1) \right) - \frac{1}{3} E^2 \ln(2E) \right. \\
+ 4 E^2 \int_0^{1/2E} \text{Im} \ln(\Gamma(1 + y)) dy \left. \right\} \\
+ i \frac{\alpha}{2\pi} \left\{ - \frac{\pi}{4} - E \ln(2E) + E(\ln(2\pi) - 1) \\
+ \frac{\pi}{6} E^2 - 4 E^2 \int_0^{1/2E} \text{Re} \ln(\Gamma(1 + iy)) dy \right\}. 
\] (2.5)

With the aid of the relation (Gradshteyn / Ryzhik)

\[
\text{Re} \ln(\Gamma(1 + iy)) = \ln |\Gamma(1 + iy)| = \frac{1}{2} \ln |\Gamma(1 + iy)\Gamma(1 - iy)| \\
= -\frac{1}{2} \ln \frac{\sinh(\pi y)}{\pi y}
\]
and an integration by parts

\[
\int_0^{1/2E} 1 \cdot \ln \frac{\sinh(\pi y)}{\pi y} dy = \frac{1}{2E} \ln \left[ \frac{2E}{\pi} \sinh \left( \frac{\pi}{2E} \right) \right] - \int_0^{1/2E} [\pi y \coth(\pi y) - 1] dy,
\]
we obtain

\[
\text{Im} \mathcal{L}^{(1)}(E) = \frac{\alpha}{2\pi} \left\{ \frac{\pi}{4} + E \ln 2 + \frac{\pi}{6} E^2 + E \ln \left( \sinh \left( \frac{\pi}{2E} \right) \right) \\
- 2 E^2 \int_0^{1/2E} \pi y \coth(\pi y) dy \right\}. 
\] (2.6)

Let’s change the variable \( x = \pi y \) and evaluate the integral on the right-hand side with the use of [Wolfram Mathematica online integrator]

\[
\int x \coth(x) dx = \frac{1}{2} \left( x(x + 2 \ln(1 - e^{-2x})) - Li_2(e^{-2x}) \right) \\
= \frac{1}{2} \left( x(x + 2 \ln(2e^{-x} \sinh(x)) - Li_2(e^{-2x}) \right). 
\] (2.7)

With integration limits we arrive at

\[
\frac{1}{\pi} \int_0^{\frac{\pi}{2E}} x \coth(x) dx = \frac{1}{2\pi} \left[ \frac{\pi}{2E} \left( \frac{\pi}{2E} + 2 \ln 2 - \frac{\pi}{E} + 2 \ln \left( \sinh \left( \frac{\pi}{2E} \right) \right) \right) \\
- Li_2 (e^{-\pi}) + \frac{\pi^2}{6} \right].
\] (2.8)
We substitute this in our last expression for $\text{Im} \mathcal{L}^{(1)}(E)$ and obtain

$$\text{Im} \mathcal{L}^{(1)}(E) = \frac{\alpha}{2\pi} \left\{ -\frac{\pi}{4} + E \ln 2 + \frac{\pi}{6} E^2 + E \ln \left( \sinh \left( \frac{\pi}{2E} \right) \right) + \frac{\pi}{4} - E \ln 2 - E \ln \left( \sinh \left( \frac{\pi}{2E} \right) \right) + \frac{E^2}{\pi} \text{Li}_2 \left( e^{-\frac{\pi}{4E}} \right) - \frac{\pi}{6} E^2 \right\}$$

$$= \frac{\alpha E^2}{2\pi} \text{Li}_2 \left( e^{-\frac{\pi}{4E}} \right)$$

$$\left( \text{Li}_2 \left( e^{-\frac{\pi}{4E}} \right) \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{\pi}{4E} n},$$

in units of $E_{cr} = \frac{m^2}{e}$: $\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{m^2}{e n}}$.

Finally we obtain J.S.’s famous result ($\alpha = \frac{e^2}{4\pi}$):

$$\text{Im} \mathcal{L}^{(1)}(E) = \frac{\alpha}{2\pi} E^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{m^2}{e n}}.$$  \hspace{1cm} (2.9)

At last we might add the result for the real part [6]:

$$\text{Re} \mathcal{L}^{(1)}(E) = -\frac{e^2 E^2}{4\pi^4} \left( C + \ln \frac{\pi m^2}{e E} \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \cosh \left\{ n \frac{\pi m^2}{e E} \right\}$$

$$-\frac{e^2 E^2}{4\pi^4} \ln \frac{n}{n^2} \cosh \left\{ n \frac{\pi m^2}{e E} \right\},$$  \hspace{1cm} (2.11)

where $C$ is the Euler-Mascheroni constant.

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References

[1] J. Schwinger, Phys. Rev. 82, 664 (1951).

Particles, Sources, and Fields, Volume II, p. 134, Addison-Wesley Publishing Company 1989

[2] Sov. Phys. JETP 30, 660 (1970)

[3] M. I. Krivoruchenko, Phys. Rev. D 86, 027704 (2012) [arXiv:1206.3836 [hep-ph]].
[4] W. Dittrich, J. Phys. A Vol. 9, No. 7, 1171 (1976)
Replace in this reference in (3.25) \( \zeta' \) by \(-\frac{1}{2} \zeta'\) and \(\ln \frac{m^2}{eH}\) by \(\ln \frac{m^2}{2eH}\).

[5] W. Dittrich and M. Reuter, Effective Lagrangians in Quantum Electrodynamics (Springer, New York, 1985)

[6] R. Soldati, J. Phys. A 44, 305401 (2011) [arXiv:1104.0468 [hep-th]].