On Musielak $N$-functions

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Abstract. In this paper, the concept of Musielak $N$-functions and Musielak-Orlicz spaces generated by them will be introduced. Facts and results of the measure theory will be applied to consider properties, calculus and basic approximation of Musielak $N$-functions and their Musielak-Orlicz spaces. Finally, the relationship between Musielak $N$-functions and Musielak-Orlicz functions and their Musielak-Orlicz spaces will be considered using facts and results of the measure theory too.

Keywords: $\mu$—almost everywhere property, supremum, infimum, limit, convergence, basic convergence, Musielak $N$-function, Musielak-Orlicz function, Musielak-Orlicz spaces, Luxemburg norm

1 Introduction

$N$-functions, Orlicz functions and Orlicz classes and Orlicz spaces generated by $N$-functions and Orlicz functions have been studied by many mathematicians as in [24],[17],[28],[18],[25],[3],[4],[5],[26],[11],[20],[9]. Musielak-Orlicz functions and Musielak-Orlicz spaces generated by Musielak-Orlicz functions have been originated and developed by [23],[22],[21] where $f \in L_{MO}(\Omega,\Sigma,\mu)$ if and only if $\int_{\Omega} MO(t,f(t))d\mu < \infty$. Their properties have been studied by [13],[16],[8],[14],[15],[16],[33] and their applications can be found in differential equations [7],[10], fluid dynamics [29],[31], statistical physics [1], integral equations [17], image processing [2],[6],[12] and many other applications [27]. So, because such increasingly importance to these concepts in the modeling of modern materials, we want to investigate properties, calculus and basic approximations of Musielak $N$-functions and Musielak-Orlicz functions and their Musielak-Orlicz spaces using the measure theory where this will help us to consider $\mu$—almost everywhere property, supremum, infimum, limit, convergence and basic convergence of Musielak $N$-functions, Musielak-Orlicz functions and Musielak-Orlicz spaces generated by them by functioning facts and results of the measure theory and getting advantages from that to consider these concepts. The concept of Musielak $N$-function $M(t,u)$ is a generalization to the concept of $N$-function $M(u)$, where $M(t,u)$ may vary with location in space, whereas the Musielak-Orlicz function $MO(t,u)$ is a generalization to the concept of Orlicz functions $O(u)$, where $MO(t,u)$ may vary with location in space. The Musielak-Orlicz function $MO(t,u)$ is defined on $\Omega \times [0,\infty)$ into $\Omega \times [0,\infty)$ where for $\mu$—a.e. $t \in \Omega, MO(t,\cdot)$ is Orlicz function of $u$ on $[0,\infty)$ and for each $u \in [0,\infty), MO(\cdot,u)$ is a $\mu$—measurable function of $t$ on $\Omega$ and $(\Omega,\Sigma,\mu)$ is a measure space [3],[15]. So, we are going to define the Musielak $N$-function $M(t,u)$ on $\Omega \times \mathbb{R}$ into $\mathbb{R}$ in similar way, that is, for $\mu$—a.e. $t \in \Omega, M(t,\cdot)$ is $N$-function of $u$ on $\mathbb{R}$ and for each $u \in \mathbb{R}, M(\cdot,u)$ is a $\mu$—measurable function of $t$ on $\Omega$ and $(\Omega,\Sigma,\mu)$ is a measure space. The novelty to define the Musielak $N$-function $M(t,u)$ by this way is to get benefits from the results of the measure theory and use them to consider properties, calculus, and basic convergence of Musielak $N$-functions and Musielak-Orlicz spaces generated by them and the relationship between Musielak $N$-functions and Musielak-Orlicz functions and their Musielak-Orlicz spaces generated by them where this will give us more flexibility to pick a suitable measurable set $\Omega$ and then the functional $\int_{\Omega} M(t,\|f(t)\|_{BS})d\mu$ defined on it as we will see in section 3. So, the paper is organized as follows. Definition of Musielak $N$-function, developing preliminaries results about the equivalent definition of Musielak $N$-function and studying continuity of Musielak $N$-function are
introduced in section 2. Definition of Musielak-Orlicz space generated by a Musielak N-function, and using facts and results of the measure theory to study properties, calculus and basic approximation of Musielak N-functions and Musielak-Orlicz spaces generated by them in section 3. The relationship between Musielak N-functions and Musielak-Orlicz functions and Musielak-Orlicz spaces generated by them respectively using facts and results of the measure theory are introduced also in section 4. Examples of Musielak N-functions and Musielak-Orlicz functions that are not Musielak N-functions will be in section 5. The conclusion will be in section 6.

2 Preliminary Results

In this section, we introduce the concept of the Musielak N-function and some results about the equivalent definition and the continuity of the Musielak N-functions.

Definition 2.1 (Musielak N-function). Let $(\Omega, \Sigma, \mu)$ be a measure space. A function $M : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is called a Musielak N-function if:

1. for $\mu$-a.e. $t \in \Omega, M(t, u)$ is even convex of $u$ on $\mathbb{R}$
2. for $\mu$-a.e. $t \in \Omega, M(t, u) > 0$ for any $u > 0$
3. for $\mu$-a.e. $t \in \Omega, \lim_{u \to 0} \frac{M(t, u)}{u} = 0$
4. for $\mu$-a.e. $t \in \Omega, \lim_{u \to +\infty} \frac{M(t, u)}{u} = +\infty$
5. for each $u \in \mathbb{R}, M(t, u)$ is a $\mu$-measurable function of $t$ on $\Omega$.

The following theorem is a generalization to theorem 1.1[17] which is necessary to proof the next theorem where it gives us the equivalent definition of Musielak N-function.

Theorem 2.1. For $\mu$-a.e. $t \in \Omega$, every convex function $M(t, u) : \Omega \times [a, b] \rightarrow \mathbb{R}$ of $u$ on $[a, b]$ which satisfies the condition $M(t, a) = 0$ can be represented in the form

$$M(t, u) = \int_a^u p(t, s)ds,$$

where $p(t, u) : \Omega \times [a, b] \rightarrow \mathbb{R}$ is, for $\mu$-a.e. $t \in \Omega$, a non-decreasing, right-continuous function of $u$ on $[a, b]$, $(\Omega, \Sigma, \mu)$ is a measure space and $[a, b]$ is any interval.

Proof. We have for $\mu$-a.e. $t \in \Omega$, for $u_1, u_2 \in [a, b], u_1 < u_2$ that

$$p_-(t, u_1) \leq p_+(t, u_1) \leq p_-(t, u_2),$$

where

$$p_-(t, u) = \lim_{u \uparrow u_0} \frac{M(t, u) - M(t, u_0)}{u - u_0}$$

and

$$p_+(t, u) = \lim_{u \downarrow u_0} \frac{M(t, u) - M(t, u_0)}{u - u_0}.$$
that is, for \( \mu \)-a.e. \( t \in \Omega, p_-(t,u) \) is monotonic and hence it is continuous almost everywhere of \( u \) on \([a,b]\) (see theorem 4.19 [30]). For \( \mu \)-a.e. \( t \in \Omega \), let \( u_1 \) be a continuity point of \( p_-(t,u) \) and by taking the limit in (2.1) as \( u_2 \to u_1 \), we get by the Squeeze theorem for functions that for \( \mu \)-a.e. \( t \in \Omega \),

\[
p_-(t,u_1) \leq p_+(t,u_1) \leq p_-(t,u_1)
\]

which means that for \( \mu \)-a.e. \( t \in \Omega, p_-(t,u_1) = p_+(t,u_1) \). Moreover, from (2.1) we have by the fundamental theorem of calculus that for \( \mu \)-a.e. \( t \in \Omega, \frac{\partial M(t,u)}{\partial u} = p(t,u) = p_+(t,u) \). Since for \( \mu \)-a.e. \( t \in \Omega \), the function \( M(t,u) \) is convex of \( u \) on \([a,b]\), then for \( \mu \)-a.e. \( t \in \Omega, M(t,u) \) is absolutely continuous of \( u \) on \([a,b]\) (see lemma 1.3 [17]) and that for \( \mu \)-a.e. \( t \in \Omega \), \( M(t,u) \) is indefinite integral of its derivative \( \frac{\partial M(t,u)}{\partial u} \) (see theorem 13.17 [32]), that is

\[
M(t,u) = M(t,a) + \int_a^u \frac{\partial M(t,s)}{\partial s} ds = \int_a^u p(t,s) ds
\]

for \( u \in [a,b] \).

\( \square \)

**Theorem 2.2.** The function \( M : \Omega \times \mathbb{R} \to \mathbb{R} \) is Musielak \( N \)-function if and only if it can be written as follows: for \( \mu \)-a.e. \( t \in \Omega \),

\[
M(t,u) = \int_0^{[u]} p(t,s) ds,
\]

where \( p : \Omega \times [0,\infty) \to \mathbb{R} \) is, for \( \mu \)-a.e. \( t \in \Omega \), a non-decreasing, right-continuous function of 0 on \([0,\infty)\) satisfies \( p(t,0) = 0, p(t,u) > 0 \) when \( u > 0 \) and \( \lim_{u \to \infty} p(t,u) = \infty \) and for each \( u \in \mathbb{R}, p(t,u) \) is a \( \mu \)-measurable function of \( t \) on \( \Omega \).

**Proof.** Given that \( M(t,u) \) is a Musielak \( N \)-function, then for \( \mu \)-a.e. \( t \in \Omega \), \( M(t,u) \) is convex of \( u \) on \( \mathbb{R} \) and \( M(t,0) = 0 \). By theorem 2.1 that for \( \mu \)-a.e. \( t \in \Omega \),

\[
M(t,u) = \int_0^{[u]} p_+(t,s) ds \leq p_+(t,u) \int_0^{[u]} ds = up_+(t,u),
\]

i.e.,

\[
\frac{M(t,u)}{u} \leq p_+(t,u),
\]

where \( p_+(t,u) : \Omega \times [0,\infty) \) is, for \( \mu \)-a.e. \( t \in \Omega \), a non-decreasing, right-continuous function of 0 on \([0,\infty)\). From (2) and (4) of the definition 2.1, we have for \( \mu \)-a.e. \( t \in \Omega \) that \( p_+(t,u) > 0 \) whenever \( u > 0 \) and \( \lim_{u \to \infty} p_+(t,u) = \infty \) respectively. Moreover, for \( \mu \)-a.e. \( t \in \Omega \) and for any \( u > 0 \) that

\[
M(t,2u) = \int_0^{2u} p_+(t,s) ds > \int_u^{2u} p_+(t,s) ds
\]

\[
> p_+(t,u) \int_u^{2u} ds = up_+(t,u),
\]

i.e,

\[
p_+(t,u) \leq \frac{M(t,2u)}{u},
\]

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Again from (3) of definition of Musielak $N$-function for $\mu$–a.e. $t \in \Omega$ that

$$p_+(t, 0) = \lim_{u \to +0} p_+(t, u) \leq \lim_{u \to +0} \frac{M(t, 2u)}{u}$$

$$= 2 \lim_{u \to +0} \frac{M(t, 2u)}{2u} = 0$$

and for $\mu$–a.e. $t \in \Omega$,

$$0 = \lim_{u \to +0} \frac{M(t, u)}{u} \leq \lim_{u \to +0} p_+(t, u) = p_+(t, 0).$$

Thus by the Squeeze theorem of functions that for $\mu$–a.e. $t \in \Omega, p_+(t, 0) = 0$. Since for $\mu$–a.e. $t \in \Omega$ that

$$p_+(t, u) = \lim_{u \to +0} \frac{M(t, u) - M(t, u_0)}{u - u_0}$$

and for each $u \in \mathbb{R}$ that $M(t, u)$ is a $\mu$–measurable function of $t$ on $\Omega$, then for each $u \in \mathbb{R}, p_+(t, u)$ is measurable function of $t$ on $\Omega$.

Now given that $M(t, u)$ satisfies (2.3). So, for $\mu$–a.e. $t \in \Omega, M(t, u)$ is even and positive for any $u > 0$. By theorem 2.1, for $\mu$–a.e. $t \in \Omega, M(t, u)$ is convex of $u$ on $\mathbb{R}$ and by (2.3) for $\mu$–a.e $t \in \Omega$,

$$0 < \frac{M(t, u)}{u} \leq p_+(t, u) < \frac{M(t, 2u)}{u}$$

for any $u > 0$. Then

$$0 < \lim_{u \to +0} \frac{M(t, u)}{u} \leq \lim_{u \to +0} p_+(t, u) = 0.$$

By the Squeeze theorem for functions that for $\mu$–a.e. $t \in \Omega,$

$$\lim_{u \to +0} \frac{M(t, u)}{u} = 0$$

and

$$2 \lim_{2u \to +\infty} \frac{M(t, 2u)}{2u} > \lim_{u \to +\infty} \frac{M(t, u)}{u} = +\infty,$$

i.e,

$$\lim_{u \to +\infty} \frac{M(t, u)}{u} = +\infty.$$ 

Since for each $u \in \mathbb{R}, p_+(t, u)$ is measurable function of $t$ on $\Omega$ and for $\mu$–a.e. $t \in \Omega, M(t, u)$ satisfies (2.3), then for each $u \in \mathbb{R}, M(t, u)$ is a $\mu$–measurable function of $t$ on $\Omega$. Therefore, $M : \Omega \times \mathbb{R} \to \mathbb{R}$ is Musielak $N$-function.

\[\square\]

**Theorem 2.3.** Any Musielak $N$-function $M : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous from the right of 0 on $\mathbb{R}$ for $\mu$–a.e. $t \in \Omega$. 

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Proof. Given that $M : \Omega \times \mathbb{R} \to \mathbb{R}$ is Musielak $N$-function. By theorem 2.2, we have for $\mu$-a.e. $t \in \Omega$,

$$0 \leq \lim_{u \to +0} M(t, u) = \lim_{u \to +0} \int_0^{|u|} p(t, s) ds = \int_0^0 p(t, s) ds = \lim_{n \to \infty} \sum_{i=1}^{n} p(t, u_i) \Delta u_i = \lim_{n \to \infty} \sum_{i=1}^{n} p(t, u_i) 0 = 0$$

where $u_i = 0 + i \Delta u$ and $\Delta u = \frac{u}{n}, n \in \mathbb{N}$. By the Squeeze theorem for functions that for $\mu$-a.e. $t \in \Omega$, $\lim_{u \to +0} M(t, u) = 0$.

3 Properties, calculus and basic approximation of Musielak $N$-functions

In this section, we are going to define the Musielak-Orlicz space generated by a Musielak $N$-function which is similar to the one that generated by a Musielak-Orlicz function [21] and investigate properties, calculus and basic approximation of Musielak $N$-functions and the Musielak-Orlicz spaces generated by them using the facts and results of the measure theory.

Definition 3.1. Let $(\Omega, \Sigma, \mu)$ be a measure space and $M$ be a Musielak $N$-function. The Musielak-Orlicz space $L_M(\Omega, \Sigma, \mu)$ generated by $M$ is defined by

$$L_M(\Omega, \Sigma, \mu) = \{ f \in BS_\Omega : \int_\Omega M(t, \| f(t) \|_{BS}) d\mu < \infty \}$$

where $BS_\Omega$ is the set of all $\mu-$measurable functions from $\Omega$ to $BS$ and $(BS, \| \cdot \|_{BS})$ is a Banach space, with Luxemburg norm

$$\| f \|_M = \inf \{ \lambda > 0 : \int_\Omega M(t, \| f(t) \|_{BS}) \lambda d\mu \leq 1 \}$$

Remark 3.1. The Musielak-Orlicz space generated by a Musielak $N$-function $M$ on a measure space $(\Omega, \Sigma, \mu)$ is the Orlicz space generated by an $N$-function $\varphi$ on a measure space $(\Omega, \Sigma, \mu)$ whenever the Musielak $N$-function $M$ is the $N$-function $\varphi$. That is, $L_M(\Omega, \Sigma, \mu) = L_\varphi(\Omega, \Sigma, \mu)$ whenever for $\mu$-a.e. $t \in \Omega$ that $M(t, u) = \varphi(u)$ of $u$ on $\mathbb{R}$.

Remark 3.2. Every Musielak $N$-function $M$ is a Musielak-Orlicz function $MO$ with two additional conditions: for $\mu$-a.e. $t \in \Omega$,

$$\lim_{u \to 0} \frac{M(t, u)}{u} = 0, \lim_{u \to \infty} \frac{M(t, u)}{u} = \infty$$

so, the set of all Musielak $N$-functions

$$F_M = \{ M | M : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Musielak } N \text{ function} \}$$

is contained in the set of all Musielak-Orlicz functions

$$F_{MO} = \{ MO | MO : \Omega \times [0, \infty) \to [0, \infty) \text{ is a Musielak – Orlicz function} \}.$$ 

Remark 3.3. The Musielak-Orlicz space generated by a Musielak $N$-function $M$ with the luxemburg norm $(L_M(\Omega, \Sigma, \mu), \| \cdot \|_M)$ is a Banach space, since the Musielak-Orlicz space generated by a Musielak-Orlicz function with the luxemburg norm is a Banach space (see theorem 7.7[21]) and by remark 3.2 the Musielak $N$-function is a Musielak-Orlicz function with two additional conditions.
Theorem 3.1. Let $M_1$ and $M_2$ be two Musielak $N$-functions such that for $\mu$-a.e. $t \in \Omega, M_2(t,u) \leq rM_1(t,u)$ for some number $r > 0$ and all $u \geq u_0 > 0$. Then $L_{M_1}(\Omega, \Sigma, \mu) \subseteq L_{M_2}(\Omega, \Sigma, \mu)$.

Proof. Take $f \in L_{M_1}(\Omega, \Sigma, \mu)$, then for $\mu$-a.e. $t \in \Omega$,

$$\int_{\Omega} M_1(t, \|f(t)\|_{BS})d\mu < \infty.$$ 

From the assumption that for $\mu$-a.e. $t \in \Omega$, there exist $r > 0$ and $u_0 > 0$,

$$M_2(t,u) \leq rM_1(t,u), ~ u \geq u_0,$$

then for $\mu$-a.e. $t \in \Omega$, there exist $r > 0$ and $u_0 > 0$ such that,

$$\int_{\Omega} M_2(t, \|f(t)\|_{BS})d\mu \leq r \int_{\Omega} M_1(t, \|f(t)\|_{BS})d\mu < \infty, ~ f \neq 0$$

i.e., $f \in L_{M_2}(\Omega, \Sigma, \mu)$.

□

Corollary 3.1 Let $M_1$ and $M_2$ be two Musielak $N$-functions such that for $\mu$-a.e. $t \in \Omega$, $r_1 M_1(t,u) \leq M_2(t,u) \leq r_2 M_1(t,u)$ for some numbers $r_1, r_2 > 0$ and all $u \geq u_0 > 0$. Then $L_{M_1}(\Omega, \Sigma, \mu) = L_{M_2}(\Omega, \Sigma, \mu)$.

Proof. From the assumption there exist $r_1, r_2 > 0$ and $u_0 > 0$ such that for $\mu$-a.e. $t \in \Omega$, $M_2(t,u) \leq r_2 M_1(t,u)$ and $M_1(t,u) \leq r_3 M_2(t,u)$, where $r_3 = 1/r_1$. From theorem 3.1 we have $L_{M_1}(\Omega, \Sigma, \mu) \subseteq L_{M_2}(\Omega, \Sigma, \mu)$ and $L_{M_2}(\Omega, \Sigma, \mu) \subseteq L_{M_1}(\Omega, \Sigma, \mu)$.

□

Theorem 3.2. If $M_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Musielak $N$-function and $M_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that for $\mu$-a.e. $t \in \Omega, M_1(t,u) = M_2(t,u)$ of $u$ on $\mathbb{R}$, then $M_2$ is a Musielak $N$-function.

Proof. It is given that $M_1$ is a Musielak $N$-function and for $\mu$-a.e. $t \in \Omega, M_1(t,u) = M_2(t,u)$ of $u$ on $\mathbb{R}$, then for $\mu$-a.e. $t \in \Omega, M_2(t,u) = M_1(t,u)$ is even convex function of $u$ on $\mathbb{R}$, for $\mu$-a.e. $t \in \Omega, M_2(t,u) = M_1(t,u) > 0$ for any $u > 0$ and for $\mu$-a.e. $t \in \Omega$,

$$\lim_{u \to 0} \frac{M_2(t,u)}{u} = \lim_{u \to 0} \frac{M_1(t,u)}{u} = 0$$

and

$$\lim_{u \to \infty} \frac{M_2(t,u)}{u} = \lim_{u \to \infty} \frac{M_1(t,u)}{u} = \infty.$$ 

Moreover, for each $u \in \mathbb{R}$, $M_2(t,u)$ is a $\mu$-measurable function of $t$ on $\Omega$. Then, $M_2$ is a Musielak $N$-function on $\Omega \times \mathbb{R}$ according to definition 2.1.

□

Theorem 3.3. If $(L_{M_1}(\Omega, \Sigma, \mu), \| \cdot \|_{M_1})$ is a Musielak-Orlicz space generated by a Musielak $N$-function $M_1$ with the luxemburg norm and $M_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that for $\mu$-a.e. $t \in \Omega, M_1(t,u) = M_2(t,u)$ of $u$ on $\mathbb{R}$, then for $\mu$-a.e. $t \in \Omega, (L_{M_1(t,u)}(\Omega, \Sigma, \mu), \| \cdot \|_{M_1(t,u)}) = (L_{M_2(t,u)}(\Omega, \Sigma, \mu), \| \cdot \|_{M_2(t,u)})$ of $u$ on $\mathbb{R}$ and hence $(L_{M_2}(\Omega, \Sigma, \mu), \| \cdot \|_{M_2})$ is a Musielak-Orlicz space generated by $M_2$ with the luxemburg norm.
**Proof.** Since for $\mu$-a.e. $t \in \Omega$, $M_1(t, u) = M_2(t, u)$ of $u$ on $\mathbb{R}$, from theorem 3.2 we have that $M_2$ is a Musielak $N$-function and
\[
L_{M_2}(\Omega, \Sigma, \mu) = \left\{ f \in BS \Omega : \int_{\Omega} M_2(t, \|f(t)\|_{BS}) d\mu < \infty \right\}
\]
\[
= \left\{ f \in BS \Omega : \int_{\Omega} M_1(t, \|f(t)\|_{BS}) d\mu < \infty \right\}
\]
\[
= L_{M_1}(\Omega, \Sigma, \mu)
\]
and for $f \in L_{M_2}(\Omega, \Sigma, \mu) (= L_{M_2}(\Omega, \Sigma, \mu))$ that
\[
\|f\|_{M_1} = \inf \left\{ \lambda > 0 : \int_{\Omega} M_1 \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \int_{\Omega} M_2 \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \|f\|_{M_2}
\]
that is, for $\mu$-a.e. $t \in \Omega$, $\|f\|_{M_2}$ can satisfies the norm’s properties on $L_{M_2}(\Omega, \Sigma, \mu)$ of $u$ on $\mathbb{R}$. Then, for $\mu$-a.e. $t \in \Omega$, the equality hold in the assumption of $u$ on $\mathbb{R}$ and $(L_{M_2}(\Omega, \Sigma, \mu), \| \cdot \|_{M_2})$ is a Musielak-Orlicz space generated by $M_2$ with the luxemburg norm.

\[\square\]

**Theorem 3.4.** If $\{M_n : n \in \mathbb{N}\}$ is a sequence of Musielak $N$-functions $M_n : \Omega \times \mathbb{R} \to \mathbb{R}$, then
\[
\sup_{n \in \mathbb{N}} M_n, \inf_{n \in \mathbb{N}} M_n, \lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n, \lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n
\]
are Musielak $N$-functions on $\Omega \times \mathbb{R}$ into $\mathbb{R}$.

**Proof.** Since for all $n \in \mathbb{N}$ that $M_n$ is a Musielak $N$-function, we have for $\mu$-a.e. $t \in \Omega$, $\sup_{n \in \mathbb{N}} M_n(t, u)$ and $\inf_{n \in \mathbb{N}} M_n(t, u)$ are even convex of $u$ on $\mathbb{R}$; for $\mu$-a.e. $t \in \Omega$, $\sup_{n \in \mathbb{N}} M_n(t, u) > 0$ and $\inf_{n \in \mathbb{N}} M_n(t, u) > 0$ for any $u > 0$ and for $\mu$-a.e. $t \in \Omega$,
\[
\lim_{u \to 0} \frac{\sup_{n \in \mathbb{N}} M_n(t, u)}{u} = 0, \quad \lim_{u \to +\infty} \frac{\sup_{n \in \mathbb{N}} M_n(t, u)}{u} = +\infty
\]
and
\[
\lim_{u \to 0} \frac{\inf_{n \in \mathbb{N}} M_n(t, u)}{u} = 0, \quad \lim_{u \to +\infty} \frac{\inf_{n \in \mathbb{N}} M_n(t, u)}{u} = +\infty.
\]
Moreover, that for each $u \in \mathbb{R}$, $\sup_{n \in \mathbb{N}} M_n(t, u)$ and $\inf_{n \in \mathbb{N}} M_n(t, u)$ are $\mu$-measurable functions of $t$ on $\Omega$. So $\sup_{n \in \mathbb{N}} M_n$ and $\inf_{n \in \mathbb{N}} M_n$ are Musielak $N$-function on $\Omega \times \mathbb{R}$. Also, for $\mu$-a.e. $t \in \Omega$,
\[
\lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n(t, u) = \inf_{n \in \mathbb{N}} \sup_{n \in \mathbb{N}} M_n(t, u),
\]
\[
\lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n(t, u) = \sup_{n \in \mathbb{N}} \inf_{n \in \mathbb{N}} M_n(t, u)
\]
of $u$ on $\mathbb{R}$, it follows that $\lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n$ and $\lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n$ are Musielak $N$-functions on $\Omega \times \mathbb{R}$.

\[\square\]
Theorem 3.5. Let \( \{M_n : n \in \mathbb{N}\} \) be a sequence of Musielak \( N \)-functions \( M_n : \Omega \times \mathbb{R} \to \mathbb{R} \) such that for each \( u \in \mathbb{R}, M_n(t, u) \leq M_{n+1}(t, u) \) on \( \Omega \) for all \( n \in \mathbb{N} \). If \( \{(L_{M_n}(\Omega, S, \mu), \| \cdot \|_{M_n}) : n \in \mathbb{N}\} \) is a sequence of Musielak-Orlicz spaces generated by \( \{M_n : n \in \mathbb{N}\} \) with the Luxemburg norm respectively, then

\[
\lim_{n \to \infty} \sup_{n \in \mathbb{N}} (L_{M_n}(\Omega, S, \mu), \| \cdot \|_{M_n}) = (L_{\mu}(\Omega, S, \mu), \| \cdot \|_{\mu}) \quad \text{and} \quad \lim_{n \to \infty} \inf_{n \in \mathbb{N}} (L_{M_n}(\Omega, S, \mu), \| \cdot \|_{M_n}) = (L_{\mu}(\Omega, S, \mu), \| \cdot \|_{\mu})
\]

via \( \{M_n : n \in \mathbb{N}\} \), and hence \( (L_{\mu}(\Omega, S, \mu), \| \cdot \|_{\mu}), (L_{\mu}(\Omega, S, \mu), \| \cdot \|_{\mu}) \) and \( (L_{\mu}(\Omega, S, \mu), \| \cdot \|_{\mu}) \) are Musielak-Orlicz spaces generated by

\[
S = \sup_{n \in \mathbb{N}} M_n, I = \inf_{n \in \mathbb{N}} M_n, LS = \lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n \text{ and } LI = \lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n
\]

with the Luxemburg norm respectively.

Proof. We have from Theorem 3.4 that \( S, I, LS \) and \( LI \) are Musielak \( N \)-functions; so by the monotone convergence theorem that for \( \mu \)-a.e. \( t \in \Omega \),

\[
L_{M_S(t,u)}(\Omega, S, \mu) = \left\{ f \in BS_{\Omega} : \int_{\Omega} M_S(t, \| f(t) \|_{BS}) \, d\mu < \infty \right\}
\]

\[
= \left\{ f \in BS_{\Omega} : \sup_{n \in \mathbb{N}} \int_{\Omega} M_n(t, \| f(t) \|_{BS}) \, d\mu < \infty \right\}
\]

\[
= \left\{ f \in BS_{\Omega} : \inf_{n \in \mathbb{N}} \sup_{n \in \mathbb{N}} \int_{\Omega} M_n(t, \| f(t) \|_{BS}) \, d\mu < \infty \right\}
\]

and for \( f, g \in L_{M_S(t,u)}(\Omega, S, \mu) \),

\[
\|f\|_{M_S(t,u)} = \inf_{\lambda > 0} \left\{ \int_{\Omega} M_S(t, \frac{\| f(t) \|_{BS}}{\lambda}) \, d\mu \leq 1 \right\}
\]

\[
= \inf_{\lambda > 0} \left\{ \sup_{n \in \mathbb{N}} \int_{\Omega} M_n(t, \frac{\| f(t) \|_{BS}}{\lambda}) \, d\mu \leq 1 \right\}
\]

\[
= \sup_{n \in \mathbb{N}} \| f \|_{M_n(t,u)}
\]

so for \( \mu \)-a.e. \( t \in \Omega \), that \( \|f\|_{M_S(t,u)} = \sup_{n \in \mathbb{N}} \| f \|_{M_n(t,u)} \geq 0 \) and \( \|f\|_{M_I(t,u)} = \inf_{n \in \mathbb{N}} \| f \|_{M_n(t,u)} \geq 0 \) of \( u \) on \( \mathbb{R} \). For \( \mu \)-a.e. \( t \in \Omega \), \( \|f\|_{M_S(t,u)} = 0 \Rightarrow \sup_{n \in \mathbb{N}} \| f \|_{M_n(t,u)} = 0 \) if and only if \( f(t) = 0 \) and
because $\parallel L_u(t) \parallel \leq \parallel t \parallel$ and $\parallel g(t) \parallel \leq \parallel t \parallel$ \(\parallel M \parallel_2 \leq \parallel t \parallel \) and $\parallel f(t) \parallel \leq \parallel t \parallel$ \(\parallel M \parallel_2 \leq \parallel t \parallel \) and $\parallel f(t) \parallel \leq \parallel t \parallel$ of $u$ on $\mathbb{R}$. And since for all $n \in \mathbb{N}$, for $\mu$-a.e. $t \in \Omega$, $M_n(t,u)$ are convex functions of $u$ on $\mathbb{R}$, then for $\mu$-a.e. $t \in \Omega$ that

$$
\int_{\Omega} M_S \left( t, \frac{\parallel f(t) + g(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \right) \, d\mu = \int_{\Omega} \sup_{n \in \mathbb{N}} M_n \left( t, \frac{\parallel f(t) + g(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \right) \, d\mu
$$

$$
= \sup_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel f(t) + g(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \right) \, d\mu
$$

$$
\leq \frac{\parallel f(t) \parallel_{M_S}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \sup_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \right) \, d\mu
$$

$$
+ \frac{\parallel g(t) \parallel_{M_S}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \sup_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel g(t) \parallel_{M_S}}{\parallel g(t) \parallel_{M_S}} \right) \, d\mu
$$

$$
\leq \frac{\parallel f(t) \parallel_{M_S}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \sup_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \right) \, d\mu
$$

$$
+ \frac{\parallel g(t) \parallel_{M_S}}{\parallel f(t) \parallel_{M_S} + \parallel g(t) \parallel_{M_S}} \sup_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel g(t) \parallel_{M_S}}{\parallel g(t) \parallel_{M_S}} \right) \, d\mu
$$

$$
\leq 1,
$$

because $\int_{\Omega} M_n \left( t, \frac{\parallel f(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_S}} \right) \, d\mu \leq 1$ and $\int_{\Omega} M_n \left( t, \frac{\parallel g(t) \parallel_{BS}}{\parallel g(t) \parallel_{M_S}} \right) \, d\mu \leq 1$, so for $\mu$-a.e. $t \in \Omega$, $\parallel f(t) + g(t) \parallel_{M_S(t,u)} \leq \parallel f(t) \parallel_{M_S(t,u)} + \parallel g(t) \parallel_{M_S(t,u)}$ of $u$ on $\mathbb{R}$; and

$$
\int_{\Omega} M_I \left( t, \frac{\parallel f(t) + g(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \right) \, d\mu = \int_{\Omega} \inf_{n \in \mathbb{N}} M_n \left( t, \frac{\parallel f(t) + g(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \right) \, d\mu
$$

$$
= \inf_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel f(t) + g(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \right) \, d\mu
$$

$$
\leq \frac{\parallel f(t) \parallel_{M_I}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \inf_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \right) \, d\mu
$$

$$
+ \frac{\parallel g(t) \parallel_{M_I}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \inf_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel g(t) \parallel_{M_I}}{\parallel g(t) \parallel_{M_I}} \right) \, d\mu
$$

$$
\leq \frac{\parallel f(t) \parallel_{M_I}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \inf_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \right) \, d\mu
$$

$$
+ \frac{\parallel g(t) \parallel_{M_I}}{\parallel f(t) \parallel_{M_I} + \parallel g(t) \parallel_{M_I}} \inf_{n \in \mathbb{N}} \int_{\Omega} M_n \left( t, \frac{\parallel g(t) \parallel_{M_I}}{\parallel g(t) \parallel_{M_I}} \right) \, d\mu
$$

$$
\leq 1,
$$

because $\int_{\Omega} M_n \left( t, \frac{\parallel f(t) \parallel_{BS}}{\parallel f(t) \parallel_{M_I}} \right) \, d\mu \leq 1$ and $\int_{\Omega} M_n \left( t, \frac{\parallel g(t) \parallel_{BS}}{\parallel g(t) \parallel_{M_I}} \right) \, d\mu \leq 1$, so for $\mu$-a.e. $t \in \Omega$, $\parallel f(t) + g(t) \parallel_{M_I(t,u)} \leq \parallel f(t) \parallel_{M_I(t,u)} + \parallel g(t) \parallel_{M_I(t,u)}$ of $u$ on $\mathbb{R}$; that is, for $\mu$-a.e. $t \in \Omega$, $\parallel f(t) \parallel_{M_S(t,u)}$ and $\parallel f(t) \parallel_{M_I(t,u)}$ are norms on $L_{M_S(t,u)}(\Omega, \Sigma, \mu)$ and $L_{M_I(t,u)}(\Omega, \Sigma, \mu)$ of $u$ on $\mathbb{R}$ respectively. So, for $\mu$-a.e. $t \in \Omega$, the equality hold in the assumption of $u$ on $\mathbb{R}$ and $(L_{M_S}(\Omega, \Sigma, \mu), \parallel \cdot \parallel_{M_S})$ and $(L_{M_I}(\Omega, \Sigma, \mu), \parallel \cdot \parallel_{M_I})$ are Musielak-Orlicz spaces generated by $M_S$ and $M_I$ with the luxemburg norm respectively.
Now, again by the monotone convergence theorem that for \( \mu \)-a.e. \( t \in \Omega \),
\[
L_{M_{LS}(t,u)} (\Omega, \Sigma, \mu) = \left\{ f \in BS_\Omega : \int_\Omega M_{LS}(t, \|f(t)\|_{BS})d\mu < \infty \right\}
\]
\[
= \left\{ f \in BS_\Omega : \lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n(t, \|f(t)\|_{BS})d\mu < \infty \right\}
\]
\[
= \left\{ f \in BS_\Omega : \lim_{n \to \infty} \inf_{n \in \mathbb{N}} \int_\Omega M_n(t, \|f(t)\|_{BS})d\mu < \infty \right\}
\]
\[
= \lim_{n \to \infty} \sup_{n \in \mathbb{N}} L_{M_k(t,u)} (\Omega, \Sigma, \mu),
\]
\[
L_{M_{LI}(t,u)} (\Omega, \Sigma, \mu) = \left\{ f \in BS_\Omega : \int_\Omega M_{LI}(t, \|f(t)\|_{BS})d\mu < \infty \right\}
\]
\[
= \left\{ f \in BS_\Omega : \lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n(t, \|f(t)\|_{BS})d\mu < \infty \right\}
\]
\[
= \lim_{n \to \infty} \inf_{n \in \mathbb{N}} L_{M_k(t,u)} (\Omega, \Sigma, \mu);
\]
and for \( f \in L_{M_{LS}(t,u)}(\Omega, \Sigma, \mu) \) \( (f \in L_{M_{LI}(t,u)}(\Omega, \Sigma, \mu)) \)
\[
\|f\|_{M_{LS}(t,u)} = \inf \left\{ \lambda > 0 : \int_\Omega M_{LS} \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \int_\Omega \lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \int_\Omega M_n \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \|f\|_{M_n(t,u)},
\]
\[
\|f\|_{M_{LI}(t,u)} = \inf \left\{ \lambda > 0 : \int_\Omega M_{LI} \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \int_\Omega \lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \inf \left\{ \lambda > 0 : \lim_{n \to \infty} \inf_{n \in \mathbb{N}} \int_\Omega M_n \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]
\[
= \lim_{n \to \infty} \inf_{n \in \mathbb{N}} \|f\|_{M_n(t,u)};
\]
that is, for \( \mu \)-a.e. \( t \in \Omega \), \( \|f\|_{M_{LS}(t,u)} \) and \( \|f\|_{M_{LI}(t,u)} \) can satisfy the norm’s properties on \( L_{M_{LS}(t,u)}(\Omega, \Sigma, \mu) \) and \( L_{M_{LI}(t,u)}(\Omega, \Sigma, \mu) \) of \( u \) on \( \mathbb{R} \) respectively. Therefore, for \( \mu \)-a.e. \( t \in \Omega \), we get the equality in the assumption of \( u \) on \( \mathbb{R} \) and \( (L_{M_{LS}(\Omega, \Sigma, \mu), \| \cdot \|_{M_{LS}}}) \) and \( (L_{M_{LI}(\Omega, \Sigma, \mu), \| \cdot \|_{M_{LI}}}) \) are Musielak-Orlicz spaces generated by \( M_{LS} \) and \( M_{LI} \) with the luxemburg norm respectively.

\[\square\]

**Theorem 3.6.** If \( \{ M_n : n \in \mathbb{N} \} \) is a sequence of Musielak \( N \)-functions that satisfy the \( \Delta_2 \)-condition \( M_n : \Omega \times \mathbb{R} \to \mathbb{R} \), and \( M_n \to M, M : \Omega \times \mathbb{R} \to \mathbb{R} \) pointwisely as \( n \to \infty \), then \( M \) is a Musielak \( N \)-function and satisfy the \( \Delta_2 \)-condition.

**Proof.** Since the convergence of \( M_n \) to \( M \) is pointwisely, then for \( \mu \)-a.e. \( t \in \Omega \) that
\[
M(t,u) = \lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_n(t,u) = \lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_n(t,u)
\]
of \( u \) on \( \mathbb{R} \), and the \( \Delta_2 \)-condition is clear to satisfy.
Theorem 3.7. Let \( \{M_n : n \in \mathbb{N}\} \) be a sequence of Musielak-Orlicz spaces \( \mathcal{M}_n : \Omega \times \mathbb{R} \to \mathbb{R} \) such that for each \( u \in \mathbb{R} \), \( M_n(t,u) \to M(t,u) \) on \( \Omega \) as \( n \to \infty \) and \( |M_n(t,u)| \leq G(t,u) \), where for each \( u \in \mathbb{R} \), \( G \) is absolutely integrable on \( \Omega \). If \( \{L(M_n(\Omega, \Sigma, \mu), \| \cdot \|_{M_n}) : n \in \mathbb{N}\} \) is a sequence of Musielak-Orlicz spaces generated by \( \{M_n : n \in \mathbb{N}\} \) with the Luxemburg norm, then \( \{L(M_n(\Omega, \Sigma, \mu), \| \cdot \|_{M_n}) \to (L(M(\Omega, \Sigma, \mu), \| \cdot \|_M) \) via \( \{M_n : n \in \mathbb{N}\} \) as \( n \to \infty \) and hence \( L(M(\Omega, \Sigma, \mu), \| \cdot \|_M) \) is a Musielak-Orlicz space generated by \( M \) with the Luxemburg norm.

Proof. From the assumptions and from Theorem 3.6 that \( M \) is a Musielak-Orlicz space and so by the Lebesgue’s dominated convergence theorem that for \( \mu \)-a.e. \( t \in \Omega \),

\[
L_M(t,u)(\Omega, \Sigma, \mu) = \begin{cases} f \in BS_{\Sigma} : \int_{\Omega} M(t, \|f(t)\|_{BS})d\mu < \infty \end{cases}
\]

\[
= \begin{cases} f \in BS_{\Sigma} : \int_{\Omega} \lim_{n \to \infty} M_n(t, \|f(t)\|_{BS})d\mu < \infty \end{cases}
\]

\[
= \begin{cases} f \in BS_{\Sigma} : \lim_{n \to \infty} \int_{\Omega} M_n(t, \|f(t)\|_{BS})d\mu < \infty \end{cases}
\]

\[
= \lim_{n \to \infty} L_{M_n(t,u)}(\Omega, \Sigma, \mu);
\]

and for \( f, g \in L_M(t,u)(\Omega, \Sigma, \mu) \),

\[
\|f\|_{L_M(t,u)} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \int_{\Omega} \lim_{n \to \infty} M_n\left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \lim_{n \to \infty} \int_{\Omega} M_n\left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) d\mu \leq 1 \right\}
\]

\[
= \lim_{n \to \infty} \|f\|_{M_n(t,u)};
\]

that is, for \( \mu \)-a.e. \( t \in \Omega \), that \( \|f\|_{L_M(t,u)} = \lim_{n \to \infty} \|f\|_{M_n(t,u)} \geq 0 \) of \( u \) on \( \mathbb{R} \). For \( \mu \)-a.e. \( t \in \Omega \), \( \|f\|_{L_M(t,u)} = 0 \) if and only if \( f(t) = 0 \) for an arbitrary \( \lambda \). For \( \mu \)-a.e. \( t \in \Omega \), for any scalar \( \alpha \), \( \|\alpha f\|_M = \lim_{n \to \infty} \|\alpha f\|_{M_n} = |\alpha| \lim_{n \to \infty} \|f\|_{M_n} = |\alpha| \|f\|_M \) of \( u \) on \( \mathbb{R} \). And since for all \( n \in \mathbb{N} \), for \( \mu \)-a.e. \( t \in \Omega \), \( M_n(t,u) \) are convex of \( u \) on \( \mathbb{R} \), then for \( f, g \in L_M(\Omega, \Sigma, \mu) \) we have that

\[
\int_{\Omega} M\left( t, \frac{\|f(t) + g(t)\|_{BS}}{\|f(t)\|_M + \|g(t)\|_M} \right) d\mu = \int_{\Omega} \lim_{n \to \infty} M_n\left( t, \frac{\|f(t) + g(t)\|_{BS}}{\|f(t)\|_M + \|g(t)\|_M} \right) d\mu
\]

\[
\leq \frac{\|f(t)\|_M}{\|f(t)\|_M + \|g(t)\|_M} \lim_{n \to \infty} \int_{\Omega} M_n\left( t, \frac{\|f(t)\|_{BS}}{\|f(t)\|_M + \|g(t)\|_M} \right) d\mu
\]

\[
+ \frac{\|g(t)\|_M}{\|f(t)\|_M + \|g(t)\|_M} \lim_{n \to \infty} \int_{\Omega} M_n\left( t, \frac{\|g(t)\|_{BS}}{\|f(t)\|_M + \|g(t)\|_M} \right) d\mu
\]

\[
\leq \frac{\|f(t)\|_M}{\|f(t)\|_M + \|g(t)\|_M} \lim_{n \to \infty} \int_{\Omega} M_n\left( t, \frac{\|f(t)\|_{BS}}{\|f(t)\|_M + \|g(t)\|_M} \right) d\mu
\]

\[
+ \frac{\|g(t)\|_M}{\|f(t)\|_M + \|g(t)\|_M} \lim_{n \to \infty} \int_{\Omega} M_n\left( t, \frac{\|g(t)\|_{BS}}{\|f(t)\|_M + \|g(t)\|_M} \right) d\mu
\]

\[
\leq 1,
\]

because \( \int_{\Omega} M_n\left( t, \frac{\|f(t)\|_{BS}}{\|f(t)\|_M} \right) d\mu \leq 1 \) and \( \int_{\Omega} M_n\left( t, \frac{\|g(t)\|_{BS}}{\|g(t)\|_M} \right) d\mu \leq 1 \), so for \( \mu \)-a.e. \( t \in \Omega \), \( \|f(t) + g(t)\|_{M(t,u)} \leq \|f(t)\|_{M(t,u)} + \|g(t)\|_{M(t,u)} \) of \( u \) on \( \mathbb{R} \); that is, for \( \mu \)-a.e. \( t \in \Omega \), \( \|f(t)\|_{M(t,u)} \) is norm on \( L_M(t,u)(\Omega, \Sigma, \mu) \) of
proof. From above assumptions and according to theorem 3.1 we have \( f_n \rightarrow f \) under the Luxemburg norm \( \| f \|_M \).

**Corollary 3.2.** Under theorem 3.7’s assumptions with \( |M_n(t,u)| \leq M(t,u) \) and \( M_n \) satisfies the \( \Delta_2 \)-condition for all \( n \in \mathbb{N} \), if \( f \in (L_M(\Omega, \Sigma, \mu), \| f \|_M) \) then there exists a sequence of functions \( \{f_n : n \in \mathbb{N}\} \) such that \( f_n \in (L_{M_n}(\Omega, \Sigma, \mu), \| f \|_{M_n}) \) for all \( n \in \mathbb{N} \) and \( f_n \rightarrow f \) under the Luxemburg norm \( \| f \|_M \).

**Proof.** From above assumptions and according to theorem 3.1 we have \( L_M(\Omega, \Sigma, \mu) \subseteq L_{M_n}(\Omega, \Sigma, \mu) \) for all \( n \in \mathbb{N} \); that is \( L_M(\Omega, \Sigma, \mu) = \bigcap_{n=1}^{\infty} L_{M_n}(\Omega, \Sigma, \mu) \). So, if \( f \in L_M(\Omega, \Sigma, \mu) \), then \( f \in L_{M_n}(\Omega, \Sigma, \mu) \) for all \( n \in \mathbb{N} \). Fix \( n_0 \in \mathbb{N} \), since \( (L_{M_{n_0}}(\Omega, \Sigma, \mu), \| f \|_{M_{n_0}}) \) is separable, because \( M_{n_0} \) satisfies the \( \Delta_2 \)-condition[18], there exists \( f_{n_0} \in (L_{M_{n_0}}(\Omega, \Sigma, \mu), \| f \|_{M_{n_0}}) \) such that \( \| f_{n_0} - f \|_{M_{n_0}} < \frac{1}{n_0} \). Then,

\[
0 \leq \| f_{n_0} - f \|_M = \lim_{n \rightarrow \infty} \| f_{n_0} - f \|_{M_n} < \frac{1}{n_0} \lim_{n \rightarrow \infty} \| f_{n_0} - f \|_{M_{n_0}}.
\]

Letting \( n_0 \rightarrow \infty \), we get \( \lim_{n \rightarrow \infty} \| f_{n_0} - f \|_M = 0 \) by the Squeeze theorem of functions.

**Theorem 3.8.** If \( M_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \), \( i = 1, 2 \) are Musielak \( N \)-functions, then for \( r \in \mathbb{R}_+ \) that \( rM_1 \) and \( M_1 + M_2 \) are Musielak \( N \)-functions.

**Proof.** Since \( M_i \), \( i = 1, 2 \) are Musielak \( N \)-functions, then for \( r \in \mathbb{R}_+ \), for \( \mu \)-a.e. \( t \in \Omega \), \( rM_1(t,u) \) and \( (M_1 + M_2)(t,u) \) are even convex of \( u \) on \( \mathbb{R} \); for \( \mu \)-a.e. \( t \in \Omega \), \( rM_1(t,u) > 0 \) and \( (M_1 + M_2)(t,u) > 0 \) for any \( u > 0 \) and for \( \mu \)-a.e. \( t \in \Omega \),

\[
\begin{align*}
\lim_{u \rightarrow 0} \frac{rM_1(t,u)}{u} &= 0, & \lim_{u \rightarrow +\infty} \frac{rM_1(t,u)}{u} &= +\infty \\
\lim_{u \rightarrow 0} \frac{(M_1 + M_2)(t,u)}{u} &= 0, & \lim_{u \rightarrow +\infty} \frac{(M_1 + M_2)(t,u)}{u} &= +\infty
\end{align*}
\]

Moreover, that for each \( u \in \mathbb{R} \), \( rM_1(t,u) \) and \( (M_1 + M_2)(t,u) \) are \( \mu \)-measurable functions of \( t \) on \( \Omega \). So, \( \Omega \), \( rM_1 \) and \( M_1 + M_2 \) are Musielak \( N \)-functions on \( \Omega \times \mathbb{R} \).

**Theorem 3.9.** If \( (L_M(\Omega, \Sigma, \mu), \| \cdot \|_M) : i = 1, 2 \) are Musielak-Orlicz spaces generated by Musielak \( N \)-functions \( M_i \), \( i = 1, 2 \) with the Luxemburg norm respectively, then \( (L_{rM_1}(\Omega, \Sigma, \mu), \| \cdot \|_{rM_1}), r \geq 1 \) and \( (L_{M_1+M_2}(\Omega, \Sigma, \mu), \| \cdot \|_{M_1+M_2}) \) are Musielak-Orlicz spaces generated by Musielak \( N \)-functions \( rM_1 \) and \( M_1 + M_2 \) with the Luxemburg norm respectively.

**Proof.** We have from theorem 3.8 for \( r \geq 1 \) that \( rM_1 \) and \( M_1 + M_2 \) are Musielak \( N \)-functions, then for \( \mu \)-a.e. \( t \in \Omega \),

\[
\begin{align*}
L_{rM_1(t,u)}(\Omega, \Sigma, \mu) &= \left\{ f \in BS_\Omega : \int_\Omega rM_1(t, f(t))d\mu < \infty \right\}, \\
L_{M_1(t,u)}(\Omega, \Sigma, \mu) &= L_{M_1(t,u)}(\Omega, \Sigma, \mu), \\
L_{M_1+M_2(t,u)}(\Omega, \Sigma, \mu) &= \left\{ f \in BS_\Omega : \int_\Omega (M_1 + M_2)(t, f(t))d\mu < \infty \right\};
\end{align*}
\]
and for \( f, g \in L_{r,M_1(t,u)}(\Omega, \Sigma, \mu) \) \( f, g \in L_{(M_1+M_2)(t,u)}(\Omega, \Sigma, \mu) \),

\[
\|f\|_{rM_1(t,u)} = \inf \left\{ \lambda > 0 : \int_{\Omega} rM_1 \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) \, d\mu \leq 1 \right\} = \inf \left\{ \lambda > 0 : \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|f(t)\|_{BS}}{\lambda} \right) \, d\mu \leq 1 \right\}.
\]

It is clear that \( \|f\|_{rM_1(t,u)} \) is a norm on \( L_{rM_1(t,u)}(\Omega, \Sigma, \mu) \). Now, for \( \mu \)-a.e. \( t \in \Omega \), \( \|f\|_{(M_1+M_2)(t,u)} \geq 0 \) of \( u \) on \( \mathbb{R} \). For \( \mu \)-a.e. \( t \in \Omega \), \( \|f\|_{(M_1+M_2)(t,u)} = 0 \) if and only if \( f(t) = 0 \) for an arbitrary \( \lambda \). For \( \mu \)-a.e. \( t \in \Omega \), any scalar \( \alpha \), \( \|\alpha f\|_{(M_1+M_2)(t,u)} = |\alpha|\|f\|_{(M_1+M_2)(t,u)} \) of \( u \) on \( \mathbb{R} \); and since for \( \mu \)-a.e. \( t \in \Omega \), \((M_1+M_2)(t,u)\) is a convex function of \( u \) on \( \mathbb{R} \), let \( S_n(M_1+M_2) = \|f(t)\|_{(M_1+M_2)} + \|g(t)\|_{(M_1+M_2)} \), then

\[
\int_{\Omega} (M_1 + M_2) \left( t, \frac{\|f(t) + g(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu \leq \frac{\|f(t)\|_{(M_1+M_2)}}{S_n(M_1+M_2)} \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|f(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu + \frac{\|g(t)\|_{(M_1+M_2)}}{S_n(M_1+M_2)} \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|g(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu \leq \frac{\|f(t)\|_{(M_1+M_2)}}{S_n(M_1+M_2)} \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|f(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu + \frac{\|g(t)\|_{(M_1+M_2)}}{S_n(M_1+M_2)} \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|g(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu \leq 1,
\]

because \( \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|f(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu \leq 1 \) and \( \int_{\Omega} (M_1 + M_2) \left( t, \frac{\|g(t)\|_{BS}}{S_n(M_1+M_2)} \right) \, d\mu \leq 1 \), so for \( \mu \)-a.e. \( t \in \Omega \), \( \|f(t) + g(t)\|_{(M_1+M_2)(t,u)} \leq \|f(t)\|_{(M_1+M_2)(t,u)} + \|g(t)\|_{(M_1+M_2)(t,u)} \) of \( u \) on \( \mathbb{R} \); that is, for \( \mu \)-a.e. \( t \in \Omega \), \( \|f\|_{(M_1+M_2)(t,u)} \) is norm on \( L_{(M_1+M_2)(t,u)}(\Omega, \Sigma, \mu) \) of \( u \) on \( \mathbb{R} \). Thus, for \( r \geq 1 \) that \((L_{rM_1}(\Omega, \Sigma, \mu), \| \cdot \|_{rM_1}) \) and \((L_{(M_1+M_2)(\Omega, \Sigma, \mu), \| \cdot \|_{(M_1+M_2)}) \) are Musielak-Orlicz spaces generated by \( rM_1 \) and \( M_1 + M_2 \) with the Luxemburg norm respectively.

Remark 3.4 If \( M_i : \Omega \times \mathbb{R} \to \mathbb{R}, i = 1, 2 \) are Musielak \( N \)-functions, then the subtraction in \( M_1 - M_2 \) does not preserve the positivity and the convexity of the Musielak \( N \)-functions, so \( M_1 - M_2 \) is not necessary to be a Musielak \( N \)-function and if so, it would be as \( M_1 + M_2 \) and the \((L_{M_1-M_2}(\Omega, \Sigma, \mu), \| \cdot \|_{M_1-M_2}) \) would be as \((L_{M_1+M_2}(\Omega, \Sigma, \mu), \| \cdot \|_{M_1+M_2}) \). Moreover, \( M_1 + k, k \in \mathbb{R} \setminus \{0\} \), \( M_1, M_2^n, n \in \mathbb{N} \) and \( M_1/M_2 \) are not Musielak \( N \)-functions by theorem 2.2, where for \( \mu \)-a.e. \( t \in \Omega \) that \((M_1+k)(t,0) \neq 0, (M_1M_2)(t,0) \neq 0, M_1^t(t,0) \neq 0 \) and \( M_1^t(t,0) \neq 0 \).

Theorem 3.10. If \( M : \Omega \times \mathbb{R} \to \mathbb{R} \) is a bounded Musielak \( N \)-function, then there exists a sequence of Musielak \( N \)-function \( \{ \varphi_n : n \in \mathbb{N} \}, \varphi_n : \Omega \times \mathbb{R} \to \mathbb{R} \) such that \( \varphi_n \to M \) on \( \Omega \times \mathbb{R} \).

Proof. It is given that \( M \) is a bounded Musielak \( N \)-function, so for each \( u \in \mathbb{R}, M(t,u) \) is bounded and \( \mu \)-measurable function of \( t \) on \( \Omega \); so, there exists a sequence of simple functions \( \{ \varphi_n : n \in \mathbb{N} \}, \varphi_n : \Omega \times \mathbb{R} \to \mathbb{R} \) such that for each \( u \in \mathbb{R}, \) for all \( \varepsilon > 0, \exists N \in \mathbb{N}, |M(t,u) - \varphi_n(t,u)| < \varepsilon \) for all \( n \geq N \), for all \( t \in \Omega \) by the basic approximation, then such convergence is uniform on \( \Omega \) and pointwise on \( \mathbb{R} \). Then, for all \( n \in \mathbb{N}, \exists N \in \mathbb{N} \) such that for \( \mu \)-a.e. \( t \in \Omega, \varphi_n(t,u) \) can satisfy the conditions (1-4) of definition 2.1, and for each \( u \in \mathbb{R}, \varphi_n(t,u) \) is \( \mu \)-measurable function of \( t \) on \( \Omega \) for all \( n \geq N \), that is, these simple functions \( \varphi_n, n \geq N \) are Musielak \( N \)-functions converge to \( M \) on \( \Omega \times \mathbb{R} \) as \( n \to \infty \).
Theorem 3.11. If \( (L_M(\Omega, \Sigma, \mu), \| \cdot \|_M) \) is a Musielak-Orlicz space generated by a bounded Musielak \( N \)-function \( M : \Omega \times \mathbb{R} \to \mathbb{R} \), then there exists a sequence of Musielak-Orlicz spaces \( \{ (L_{\varphi_n}(\Omega, \Sigma, \mu), \| \cdot \|_{\varphi_n}), n \in \mathbb{N} \} \), generated by a sequence of Musielak \( N \)-function \( \{ \varphi_n : n \in \mathbb{N} \}, \varphi_n : \Omega \times \mathbb{R} \to \mathbb{R} \) respectively, such that \( (L_{\varphi_n}(\Omega, \Sigma, \mu), \| \cdot \|_{\varphi_n}) \to (L_M(\Omega, \Sigma, \mu), \| \cdot \|_M) \) via \( \{ \varphi_n : n \in \mathbb{N} \} \) as \( n \to \infty \).

Proof. It is given that \( M : \Omega \times \mathbb{R} \to \mathbb{R} \) is a bounded Musielak \( N \)-function. By theorem 3.10, there exists a sequence of Musielak \( N \)-functions \( \{ \varphi_n : n \in \mathbb{N} \}, \varphi_n : \Omega \times \mathbb{R} \to \mathbb{R} \) such that \( \varphi_n \) converge to \( M \) uniformly on \( \Omega \) and pointwisely on \( \mathbb{R} \) as \( n \to \infty \); so \( \{ \varphi_n : n \in \mathbb{N} \} \) is uniformly bounded on \( \Omega \). By the Lebesgue’s dominated convergence theorem, for \( \mu \)-a.e. \( t \in \Omega \),

\[
L_{M(t,u)}(\Omega, \Sigma, \mu) = \begin{cases} f \in BS_\Omega : & \int_\Omega M(t, \| f(t) \|_{BS}) \, d\mu < \infty \\
= & \begin{cases} f \in BS_\Omega : & \int_\Omega \lim_{n \to \infty} \varphi_n(t, \| f(t) \|_{BS}) \, d\mu < \infty \\
= & \begin{cases} f \in BS_\Omega : & \lim_{n \to \infty} \int_\Omega \varphi_n(t, \| f(t) \|_{BS}) \, d\mu < \infty \\
= & \lim_{n \to \infty} L_{\varphi_n(t,u)}(\Omega, \Sigma, \mu),
\end{cases}\end{cases}\]

and for \( f \in L_{M(t,u)}(\Omega, \Sigma, \mu) \) that

\[
\| f \|_{M(t,u)} = \inf \left\{ \lambda > 0 : \int_\Omega M(t, \frac{\| f(t) \|_{BS}}{\lambda}) \, d\mu \leq 1 \right\} = \inf \left\{ \lambda > 0 : \int_\Omega \lim_{n \to \infty} \varphi_n(t, \frac{\| f(t) \|_{BS}}{\lambda}) \, d\mu < 1 \right\} = \inf \left\{ \lambda > 0 : \lim_{n \to \infty} \int_\Omega \varphi_n(t, \frac{\| f(t) \|_{BS}}{\lambda}) \, d\mu < 1 \right\} = \lim_{n \to \infty} \| f \|_{\varphi_n(t,u)};
\]

that is, for all \( n \in \mathbb{N} \), for \( \mu \)-a.e. \( t \in \Omega \), \( \| f \|_{\varphi_n(t,u)} \) can satisfy the norm’s properties on \( L_{\varphi_n(t,u)}(\Omega, \Sigma, \mu) \) of \( u \) on \( \mathbb{R} \). Therefore, for all \( n \in \mathbb{N}, \exists N \in \mathbb{N}, (L_{\varphi_n}(\Omega, \Sigma, \mu), \| \cdot \|_{\varphi_n}) \) is a Musielak-Orlicz spaces generated by a simple function \( \varphi_n \) with the luxemburg norm for all \( n \geq N \) respectively and \( (L_{\varphi_n}(\Omega, \Sigma, \mu), \| \cdot \|_{\varphi_n}) \to (L_M(\Omega, \Sigma, \mu), \| \cdot \|_M) \) as \( n \to \infty \).

\[\Box\]

4 Properties, calculus and basic approximation of Musielak-Orlicz functions

In this section we going to study properties, calculus and basic approximation of Musielak-Orlicz functions and the Musielak-Orlicz space generated by them and the relation between the Musielak \( N \)-function and the Musielak-Orlicz function and their Musielak-Orlicz spaces. Some theorems will be left without proof because they are a generalization to the ones that we did in section 3, so we can follow the similar way to proof them by consider the conditions of the Musielak-Orlicz function.

Theorem 4.1. Let \( M \) be a Musielak \( N \)-function and \( MO \) be a Musielak-Orlicz function such that for \( \mu \)-a.e. \( t \in \Omega \),

\[
\lim_{u \to 0} \frac{MO(t,u)}{u} \neq 0,
\]

then \( L_M(\Omega, \Sigma, \mu) \neq L_{MO}(\Omega, \Sigma, \mu) \).
Proof. Assume \( L_M(\Omega, \Sigma, \mu) = L_{MO}(\Omega, \Sigma, \mu) \), this means \( L_M(\Omega, \Sigma, \mu) \subseteq L_{MO}(\Omega, \Sigma, \mu) \) and \( L_{MO}(\Omega, \Sigma, \mu) \subseteq L_M(\Omega, \Sigma, \mu) \), so there exist two positive constants \( r_1 \) and \( r_2 \) and \( u_0 > 0 \) such that for \( f \in L_M(\Omega, \Sigma, \mu) (= L_{MO}(\Omega, \Sigma, \mu)) \), for \( \mu \)-a.e. \( t \in \Omega \),

\[
    r_1 \int_{\Omega} M_{MO}(t, \|f(t)\|_{BS})d\mu \leq \int_{\Omega} M(t, \|f(t)\|_{BS})d\mu < \infty, \quad f \neq 0
\]

and

\[
    r_2 \int_{\Omega} M(t, \|f(t)\|_{BS})d\mu \leq \int_{\Omega} MO(t, \|f(t)\|_{BS})d\mu < \infty, \quad f \neq 0.
\]

So for \( \mu \)-a.e. \( t \in \Omega \),

\[
    r_1 MO(t, u) \leq M(t, u), \quad u \geq u_0
\]

and

\[
    r_2 M(t, u) \leq MO(t, u), \quad u \geq u_0.
\]

Taking the limit \( u \to 0 \) we get for \( \mu \)-a.e. \( t \in \Omega \),

\[
    r_1 \lim_{u \to 0} \frac{MO(t, u)}{u} \leq \lim_{u \to 0} \frac{M(t, u)}{u} = 0
\]

and

\[
    0 = r_2 \lim_{u \to 0} \frac{M(t, u)}{u} \leq \lim_{u \to 0} \frac{MO(t, u)}{u}.
\]

By the Squeez theorem for functions, we get for \( \mu \)-a.e. \( t \in \Omega \),

\[
    \lim_{u \to 0} \frac{MO(t, u)}{u} = 0
\]

which contradicts the assumption. \( \square \)

Theorem 4.2. Let \( M \) be a Musielak \( N \)-function and \( MO \) be a Musielak-Orlicz function such that for \( \mu \)-a.e. \( t \in \Omega \),

\[
    \lim_{u \to \infty} \frac{MO(t, u)}{u} \neq \infty,
\]

then \( L_M(\Omega, \Sigma, \mu) \neq L_{MO}(\Omega, \Sigma, \mu) \).

Proof. Following the similar way of theorem 4.1’s proof and letting \( u \) go to \( \infty \), we get for \( \mu \)-a.e. \( t \in \Omega \),

\[
    r_1 \lim_{u \to \infty} \frac{MO(t, u)}{u} \leq \lim_{u \to \infty} \frac{M(t, u)}{u} = \infty
\]

and

\[
    \infty = r_2 \lim_{u \to \infty} \frac{M(t, u)}{u} \leq \lim_{u \to \infty} \frac{MO(t, u)}{u}.
\]

By the Squeez theorem for functions, we get for \( \mu \)-a.e. \( t \in \Omega \),

\[
    \lim_{u \to \infty} \frac{MO(t, u)}{u} = \infty
\]

which contradicts the assumption. \( \square \)
Theorem 4.3. If $MO_1 : \Omega \times [0, \infty) \to [0, \infty)$ is a Musielak-Orlicz function such that $\lim_{u \to 0} \frac{MO_1(t, u)}{u} \neq 0$ or $\lim_{u \to \infty} \frac{MO_1(t, u)}{u} \neq \infty$ and $MO_2 : \Omega \times [0, \infty) \to [0, \infty)$ is a function such that for $\mu$-a.e. $t \in \Omega$, $MO_1(t, u) = MO_2(t, u)$ of $u$ on $[0, \infty)$, then $MO_2$ is a Musielak-Orlicz function not Musielak $N$-function.

Proof. Since $MO_1$ is a Musielak-Orlicz function and for $\mu$-a.e. $t \in \Omega$, $MO_1(t, u) = MO_2(t, u)$ of $u$ on $[0, \infty)$, then for $\mu$-a.e. $t \in \Omega$, $MO_2(t, u)$ is convex function of $u$ on $[0, \infty)$; $MO_2(t, 0) = MO_1(t, 0) = 0$ and $MO_2(t, u) = MO_1(t, u) > 0$ for $u \neq 0$. Moreover, for each $u \in [0, \infty)$, $MO_2(t, u)$ is $\mu$-measurable function of $t$ on $\Omega$. So, $MO_2$ is a Musielak-Orlicz function on $\Omega \times [0, \infty)$. Now, assume that $MO_2$ is a Musielak $N$-function, then for $\mu$-a.e. $t \in \Omega$,

$$0 \neq \lim_{u \to 0} \frac{MO_1(t, u)}{u} = \lim_{u \to 0} \frac{MO_2(t, u)}{u} = 0,$$

or

$$\infty \neq \lim_{u \to \infty} \frac{MO_1(t, u)}{u} = \lim_{u \to \infty} \frac{MO_2(t, u)}{u} = \infty,$$

which is a contradiction.

Theorem 4.4. If $(L_{MO_1}(\Omega, \Sigma, \mu), \| \cdot \|_{MO_1})$ is a Musielak-Orlicz space generated by a Musielak-Orlicz function $M_1$ with the luxemburg norm and $MO_2 : \Omega \times [0, \infty) \to [0, \infty)$ is a function such that for $\mu$-a.e. $t \in \Omega$, $M_1(t, u) = M_2(t, u)$ of $u$ on $\mathbb{R}$, then for $\mu$-a.e. $t \in \Omega$, $(L_{MO_2}(\Omega, \Sigma, \mu), \| \cdot \|_{MO_2})$ is a Musielak-Orlicz space generated by $MO_2$ with the luxemburg norm.

Theorem 4.5. If $\{MO_n : n \in \mathbb{N}\}$ is a sequence of Musielak-Orlicz functions $MO_n : \Omega \times [0, \infty) \to [0, \infty)$ such that for all $n \in \mathbb{N}$, $\lim_{u \to 0} \frac{MO_n(t, u)}{u} \neq 0$ or $\lim_{u \to \infty} \frac{MO_n(t, u)}{u} \neq \infty$, then

$$\sup_{n \in \mathbb{N}} \inf MO_n, \limsup_{n \to \infty} \inf MO_n, \liminf_{n \to \infty} MO_n$$

are Musielak-Orlicz functions not Musielak $N$-functions on $\Omega \times [0, \infty)$.

Proof. Since for all $n \in \mathbb{N}$, $MO_n$ is a Musielak-Orlicz function, then for $\mu$-a.e. $t \in \Omega$, $\sup_{n \in \mathbb{N}} MO_n(t, u)$ and $\inf_{n \in \mathbb{N}} MO_n(t, u)$ are convex of $u$ on $[0, \infty)$; for $\mu$-a.e. $t \in \Omega$, $\sup_{n \in \mathbb{N}} MO_n(t, 0) = 0$ and $\inf_{n \in \mathbb{N}} MO_n(t, 0) = 0$; and for $\mu$-a.e. $t \in \Omega$, $\sup_{n \in \mathbb{N}} MO_n(t, u) > 0$ and $\inf_{n \in \mathbb{N}} MO_n(t, u) > 0$ for $u \neq 0$. Moreover, for each $u \in [0, \infty)$, $\sup_{n \in \mathbb{N}} MO_n(t, u)$ and $\inf_{n \in \mathbb{N}} MO_n(t, u)$ are $\mu$-measurable function of $t$ on $\Omega$. So, $\sup_{n \in \mathbb{N}} MO_n$ and $\inf_{n \in \mathbb{N}} MO_n$ are Musielak-Orlicz functions on $\Omega \times [0, \infty)$. Now, assume that $\sup_{n \in \mathbb{N}} MO_n$ and $\inf_{n \in \mathbb{N}} MO_n$ are Musielak $N$-functions, then for $\mu$-a.e. $t \in \Omega$,

$$\lim_{u \to 0} \frac{\sup_{n \in \mathbb{N}} MO_n(t, u)}{u} = 0 \iff \lim_{u \to 0} \frac{MO_n(t, u)}{u} = 0,$$

$$\lim_{u \to 0} \frac{\inf_{n \in \mathbb{N}} MO_n(t, u)}{u} = 0 \iff \lim_{u \to 0} \frac{MO_n(t, u)}{u} = 0,$$

or

$$\lim_{u \to \infty} \frac{\sup_{n \in \mathbb{N}} MO_n(t, u)}{u} = \infty \iff \lim_{u \to \infty} \frac{MO_n(t, u)}{u} = \infty,$$

$$\lim_{u \to \infty} \frac{\inf_{n \in \mathbb{N}} MO_n(t, u)}{u} = \infty \iff \lim_{u \to \infty} \frac{MO_n(t, u)}{u} = \infty,$$
which is a contradiction to the assumptions. Also, for \( \mu \)-a.e. \( t \in \Omega \),
\[
\lim_{n \to \infty} \sup_{n \in \mathbb{N}} M_{n}(t, u) = \inf_{n \in \mathbb{N}} \sup_{k \geq n} M_{n}(t, u)
\]
\[
\lim_{n \to \infty} \inf_{n \in \mathbb{N}} M_{n}(t, u) = \sup_{n \in \mathbb{N}} \inf_{k \geq n} M_{k}(t, u),
\]
of \( u \) on \([0, \infty)\), it follows that \( \lim_{n \to \infty} \sup \nabla_{n} \) and \( \lim_{n \to \infty} \inf \nabla_{n} \) are Musielak-Orlicz functions not Musielak \( N \)-functions on \( \Omega \times [0, \infty) \).

**Theorem 4.6.** Let \( \{\nabla_{n} : n \in \mathbb{N}\} \) be a sequence of Musielak-Orlicz functions \( \nabla_{n} : \Omega \times [0, \infty) \to [0, \infty) \) such that for each \( u \in [0, \infty) \), \( \nabla_{n}(t, u) \leq \nabla_{n+1}(t, u) \) on \( \Omega \) for all \( n \in \mathbb{N} \). If \( \{M_{\nabla_{n}}(t, \Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{n}} \} : n \in \mathbb{N}\) is a sequence of Musielak-Orlicz spaces generated by \( \{\nabla_{n} : n \in \mathbb{N}\} \) with the luxemburg norm of \( n \in \mathbb{N} \) respectively, then \( \sup_{n \in \mathbb{N}}(L_{\nabla_{n}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{n}}) = (L_{\nabla_{1}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{1}}) \) and \( \inf_{n \to \infty}(L_{\nabla_{n}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{n}}) = (L_{\nabla_{\infty}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{\infty}}) \) via \( \{\nabla_{n} : n \in \mathbb{N}\} \), and hence \( (L_{\nabla_{1}}(\Omega, \Sigma, \mu), \| \cdot \|_{1}) \), \( (L_{\nabla_{\infty}}(\Omega, \Sigma, \mu), \| \cdot \|_{\infty}) \) are Musielak-Orlicz spaces generated by
\[
S = \sup_{n \in \mathbb{N}} \nabla_{n}, I = \inf_{n \in \mathbb{N}} \nabla_{n}, LI = \lim_{n \to \infty} \sup \nabla_{n} \text{ and } LI = \lim_{n \to \infty} \inf \nabla_{n}
\]
with the luxemburg norm respectively.

**Theorem 4.7.** If \( \{\nabla_{n} : n \in \mathbb{N}\} \) is a sequence of Musielak-Orlicz functions \( \nabla_{n} : \Omega \times [0, \infty) \to [0, \infty) \) such that \( \lim_{u \to 0} \nabla_{n}(t, u) \neq 0 \) or \( \lim_{u \to \infty} \nabla_{n}(t, u) \neq \infty \) and \( \nabla_{n} \to \nabla_{n} \) pointwisely as \( n \to \infty \), then \( \nabla \) is a Musielak-Orlicz function not Musielak \( N \)-function and satisfy the \( \Delta_{2} \)-condition.

**Proof.** Since the convergence of \( \nabla_{n} \) to \( \nabla \) is pointwisely, then for \( \mu \)-a.e. \( t \in \Omega \) that
\[
\nabla(t, u) = \lim_{n \to \infty} \inf_{n \in \mathbb{N}} \nabla_{n}(t, u) = \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \nabla_{n}(t, u)
\]
of \( u \) on \( \mathbb{R} \). Now, assume that \( \nabla \) is a Musielak \( N \)-function, then for \( \mu \)-a.e. \( t \in \Omega \),
\[
0 \neq \lim_{n \to \infty} \lim_{u \to 0} \nabla_{n}(t, u) = \lim_{u \to 0} \lim_{n \to \infty} \nabla_{n}(t, u) = \lim_{u \to 0} \nabla(t, u) = 0,
\]
or
\[
\infty \neq \lim_{n \to \infty} \lim_{u \to 0} \nabla_{n}(t, u) = \lim_{u \to 0} \lim_{n \to \infty} \nabla_{n}(t, u) = \lim_{u \to 0} \nabla(t, u) = \infty,
\]
which is a contradiction. So, \( \nabla \) is a Musielak-Orlicz function not Musielak \( N \)-function on \( \Omega \times [0, \infty) \) according to theorem 4.5; and the \( \Delta_{2} \)-condition is clear to satisfy.

**Theorem 4.8.** Let \( \{\nabla_{n} : n \in \mathbb{N}\} \) be a sequence of Musielak-Orlicz functions \( \nabla_{n} : \Omega \times [0, \infty) \to [0, \infty) \) such that for each \( u \in [0, \infty) \), \( \nabla_{n}(t, u) \to \nabla(t, u) \) on \( \Omega \) as \( n \to \infty \) and \( |\nabla_{n}(t, u)| \leq G(t, u) \), where for each \( u \in [0, \infty) \), \( G \) is absolutely integrable on \( \Omega \). If \( \{L_{\nabla_{n}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{n}} \} : n \in \mathbb{N}\) is a sequence of Musielak-Orlicz spaces generated by \( \{\nabla_{n} : n \in \mathbb{N}\} \) with the luxemburg norm, then \( L_{\nabla_{n}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{n}} \) via \( \{\nabla_{n} : n \in \mathbb{N}\} \) as \( n \to \infty \) and \( L_{\nabla_{n}}(\Omega, \Sigma, \mu), \| \cdot \|_{\nabla_{n}} \) is a Musielak-Orlicz space generated by \( \nabla \) with the luxemburg norm.

**Corollary 4.1.** Under theorem 4.8's assumptions with \(|\nabla_{n}(t, u)| \leq \nabla(t, u)\) and \( \nabla_{n} \) satisfies the \( \Delta_{2} \)-condition for all \( n \in \mathbb{N} \), if \( f \in \{L_{\nabla_{n}}(\Sigma, \mu), \| \cdot \|_{\nabla_{n}}\} \) then there exists a sequence of functions \( \{f_{n} : n \in \mathbb{N}\} \) such that \( f_{n} \in \{L_{\nabla_{n}}(\Sigma, \mu), \| \cdot \|_{\nabla_{n}}\} \) for all \( n \in \mathbb{N} \) and \( f_{n} \to f \) under the luxemburg norm \( \| \cdot \|_{\nabla_{n}} \).
Theorem 4.9. If $MO : \Omega \times [0, \infty) \to [0, \infty)$ is a Musielak-Orlicz function such that $\lim_{u \to 0} \frac{MO(t, u)}{u} \neq 0$ and $M : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Musielak N-function, then for $r \in \mathbb{R}^+$ that $rMO$ and $MO + M$ are Musielak-Orlicz functions not Musielak N-functions.

Proof. It is given that $MO$ is a Musielak-Orlicz function and $M$ is a Musielak N-function, then both $MO$ and $M$ are non-negative convex functions on $[0, \infty)$. So, for $r \in \mathbb{R}^+$, for $\mu$-a.e. $t \in \Omega$, $rMO(t, u)$ and $(MO + M)(t, u)$ are convex of $u$ on $\mathbb{R}$; for $\mu$-a.e. $t \in \Omega$, $rMO(t, 0) = 0$ and $(MO + M)(t, 0) = 0$; and for $\mu$-a.e. $t \in \Omega$, $rMO(t, u) > 0$ and $(MO + M)(t, u) > 0$ for $u \neq 0$. Moreover, for each $u \in [0, \infty)$, $rMO(t, u)$ and $(MO + M)(t, u)$ are $\mu$-measurable and $\mu$-measurable functions of $t$ on $\Omega$. So, $rMO$ and $MO + M$ are Musielak-Orlicz functions on $\Omega \times [0, \infty)$. Now, assume that $rMO$ and $MO + M$ are Musielak N-functions, then for $\mu$-a.e. $t \in \Omega$,

$$0 = \lim_{u \to 0} \frac{rMO(t, u)}{u} = \lim_{u \to 0} \frac{MO(t, u)}{u} \neq 0$$

and

$$0 = \lim_{u \to 0} \frac{(MO + M)(t, u)}{u} = \lim_{u \to 0} \frac{MO(t, u)}{u} + \lim_{u \to 0} \frac{M(t, u)}{u} \neq 0;$$

so, both make a contradiction.

\[\square\]

Theorem 4.10. If $(L_{MO}(\Omega, \Sigma, \mu), \| \cdot \|_{MO})$ is a Musielak-Orlicz function $MO$ and $(L_{M}(\Omega, \Sigma, \mu), \| \cdot \|_{M})$ generated by a Musielak N-function $M$ with the luxemburg norm, then $(L_{rMO}(\Omega, \Sigma, \mu), \| \cdot \|_{rMO}), r \geq 1$ and $(L_{MO + M}(\Omega, \Sigma, \mu), \| \cdot \|_{MO + M})$ are Musielak-Orlicz spaces generated by Musielak-Orlicz functions $rM$ and $MO + M$ with the luxemburg norm respectively.

Remark 4.1. If $MO_i : \Omega \times [0, \infty) \to [0, \infty), i = 1, 2$ are Musielak-Orlicz functions such that $\lim_{u \to 0} \frac{MO_i(t, u)}{u} \neq 0, i = 1, 2$ and $M : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Musielak N-function, then $MO_i + k, k \in \mathbb{R}\{0\}, MO_1MO_2, MO_1^2, n \in \mathbb{N}, \frac{MO_i}{MO_1}, \frac{MO_i}{MO_2}$ and $\frac{M}{MO_1}$ are neither Musielak-Orlicz functions nor Musielak N-functions because for $\mu$-a.e. $t \in \Omega$, $(MO_1 + k)(t, 0) = 0, (MO_1MO_2)(t, 0) = 0, (MO_1^2)(t, 0) = 0, \frac{MO_1}{MO_2}(t, 0) \neq 0, \frac{MO_1}{MO_1(t, 0)} \neq 0$; and $(MO_1MO_2)(t, 0) \neq 0$. Also, $MO_1 = MO_2, MO_1 = M$ and $M = MO_1$ are not necessary to be Musielak-Orlicz functions because the subtraction in them do not preserve the positivity and the convexity of the Musielak-Orlicz functions and if so, they would be as $MO_1 + MO_2$ and $MO_1 + M$ and $(L_{rMO_1-MO_2}(\Omega, \Sigma, \mu), \| \cdot \|_{rMO_1-MO_2})$, $(L_{MO_1-M}(\Omega, \Sigma, \mu), \| \cdot \|_{MO_1-M})$ and $(L_{M-NO_1}(\Omega, \Sigma, \mu), \| \cdot \|_{M-NO_1})$ would be as $(L_{MO_1+MO_2}(\Omega, \Sigma, \mu), \| \cdot \|_{MO_1+MO_2})$ and $(L_{MO_1+M}(\Omega, \Sigma, \mu), \| \cdot \|_{MO_1+M})$.

Theorem 4.11. If $MO : \Omega \times [0, \infty) \to [0, \infty)$ is a bounded Musielak-Orlicz function, then there exists a sequence of Musielak-Orlicz function $\{\varphi_n : n \in \mathbb{N}\}$, $\varphi_n : \Omega \times [0, \infty) \to [0, \infty)$ such that $\lim_{u \to 0} \frac{\varphi_n(t, u)}{u} \neq 0, \lim_{u \to \infty} \frac{\varphi_n(t, u)}{u} \neq \infty$ and $\varphi_n \to M$ on $\Omega \times [0, \infty)$.

Proof. It is given that $MO$ is a bounded Musielak-Orlicz function, so for each $u \in [0, \infty), MO(t, u)$ is bounded and $\mu$-measurable function of $t$ on $\Omega$; so there exists a sequence of simple functions $\{\varphi_n : n \in \mathbb{N}\}$, $\varphi_n : \Omega \times [0, \infty) \to [0, \infty)$ such that for each $u \in [0, \infty)$, for all $\varepsilon > 0, \exists N \in \mathbb{N}, |MO(t, u) - \varphi_n(t, u)| < \varepsilon$ for all $n \geq N$, for all $t \in \Omega$ by the basic approximation, then such convergence is uniform on $\Omega$ and pointwise on $[0, \infty)$. Then, for all $n \in \mathbb{N}, \exists N \in \mathbb{N}$ such that for $\mu$-a.e. $t \in \Omega$, $\varphi_n(t, u)$ can satisfy the conditions of Orlicz function on $[0, \infty)$, and for each $u \in [0, \infty)$, $\varphi_n(t, u)$ is $\mu$-measurable function of $t$ on $\Omega$ for all $n \geq N$, then these simple functions $\varphi_n, n \geq N$ are Musielak-Orlicz functions converge to $MO$ on $\Omega \times [0, \infty)$ as $n \to \infty$. Now, assume $\{\varphi_n : n \in \mathbb{N}\}$ are Musielak N-functions, then
0 = \lim_{n \to \infty} \lim_{u \to 0} \frac{\varphi_n(t, u)}{u} = \lim_{u \to 0} \lim_{n \to \infty} \frac{\varphi_n(t, u)}{u} = \lim_{u \to 0} \frac{MO(t, u)}{u} \neq 0

or

\infty = \lim_{n \to \infty} \lim_{u \to \infty} \frac{\varphi_n(t, u)}{u} = \lim_{u \to \infty} \lim_{n \to \infty} \frac{\varphi_n(t, u)}{u} = \lim_{u \to \infty} \frac{MO(t, u)}{u} \neq \infty

which is a contradiction.

\[\square\]

**Theorem 4.12.** If \((L_{MO}(\Omega, \Sigma, \mu), \| \cdot \|_{MO})\) is a bounded Musielak-Orlicz space generated by a Musielak-Orlicz function \(MO : \Omega \times [0, \infty) \to [0, \infty)\), then there exists a sequence of Musielak-Orlicz spaces \(\{(L_{\varphi_n}(\Omega, \Sigma, \mu), \| \cdot \|_{\varphi_n}) : n \in \mathbb{N}\}\), generated by a sequence of Musielak-Orlicz functions \(\varphi_n : \Omega \times [0, \infty) \to [0, \infty)\) respectively, such that \((L_{\varphi_n}(\Omega, \Sigma, \mu), \| \cdot \|_{\varphi_n}) \to (L_{MO}(\Omega, \Sigma, \mu), \| \cdot \|_{MO})\) via \(\{M_{\varphi_n} : n \in \mathbb{N}\}\) as \(n \to \infty\).

**5 Examples**

**5.1 Examples of Musielak N-functions.**

1. Every \(N\)-function is a Musielak \(N\)-function.

2. \(M : \mathbb{R} \times \mathbb{R} \to [0, \infty), M(t, u) = (tu)^2\) is Musielak \(N\)-function, where
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, M(t, u)\) is even convex because \(M(t, -u) = M(t, u)\) and \(\frac{\partial^2 M(t, u)}{\partial u^2} = 2t^2 \geq 0\) for all \(u \in \mathbb{R}\)
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, M(t, u) = (tu)^2 > 0\) for any \(u > 0\)
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, \lim_{u \to 0} \frac{M(t, u)}{u} = \lim_{u \to 0} t^2 u = 0\)
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, \lim_{u \to \infty} \frac{M(t, u)}{u} = \lim_{u \to \infty} t^2 u = \infty\)
   - for each \(u \in \mathbb{R}, M(t, u) = (tu)^2\) is a \(\mu\)-measurable function of \(t\) on \(\mathbb{R}\) since it is continuous on measurable set \(\mathbb{R}\).

3. \(M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, M(t, u) = \exp(|u| + |t|) - |u| - |t|\) is Musielak \(N\)-function, where
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, M(t, u)\) is even convex because \(M(t, -u) = M(t, u)\) and \(\frac{\partial^2 M(t, u)}{\partial u^2} = \exp(|u| + |t|) - |u| - |t| > 0\) for all \(u \in \mathbb{R}\)
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, M(t, u) = \exp(|u| + |t|) - |u| - |t| > 0\) for any \(u > 0\)
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, \lim_{u \to 0} \frac{M(t, u)}{u} = \lim_{u \to 0} \frac{\exp(|u| + |t|) - |u| - |t|}{u} = 0\)
   - for \(\mu\)-a.e. \(t \in \mathbb{R}, \lim_{u \to \infty} \frac{M(t, u)}{u} = \lim_{u \to \infty} \frac{\exp(|u| + |t|) - |u| - |t|}{u} = \infty\)
   - for each \(u \in \mathbb{R}, M(t, u) = \exp(|u| + |t|) - |u| - |t|\) is a \(\mu\)-measurable function of \(t\) on \(\mathbb{R}\) since it is continuous on measurable set \(\mathbb{R}\).

**5.2 Examples of Musielak-Orlicz functions that are not Musielak \(N\)-functions.**

1. \(M : [0, \infty) \times [0, \infty) \to [0, \infty), M(t, u) = a^{tu} - 1, a > 1;\)
   - for \(\mu\)-a.e. \(t \in [0, \infty), a^{tu} - 1 > 0\) on \((0, \infty)\) and \(a^{tu} - 1 = 0\) whenever \(u = 0\)
   - for \(\mu\)-a.e. \(t \in [0, \infty), M(t, u) = a^{tu} - 1\) is convex function on \([0, \infty)\) since \(\frac{\partial^2 M(t, u)}{\partial u^2} = a^{tu}(t \log a)^2 > 0\) for all \(u \in [0, \infty)\)
The concept of Musielak-Orlicz function can be considered using facts and results of the measure theory. Also, the relationship between $\mu$-functions and Musielak-Orlicz functions and Musielak-Orlicz spaces generated by them have been studied according to facts and results of the measure theory.

**6 Conclusion**

The concept of $N$-function can be generalized to Musielak $N$-function as the concept of Orlicz function is generalized to Musielak-Orlicz function. $\mu$—almost everywhere property, supremum, infimum, limit, convergence and basic convergence of a sequence of Musielak $N$–functions and Musielak-Orlicz spaces generated by them can be considered using facts and results of the measure theory. Also, the relationship between Musielak $N$-functions and Musielak-Orlicz functions and Musielak-Orlicz spaces generated by them have been studied according to facts and results of the measure theory.
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