Do measurements of the one-point distribution of aperture-mass improve constraints on cosmology?

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ABSTRACT

We study the possibility of using the entire probability distribution function (PDF) of the aperture mass $M_{ap}$ and its related cumulative probability distribution function (CPDF) to obtain meaningful constraints on cosmological parameters. Deriving completely analytic expressions for the associated covariance matrices, we construct the Fisher matrix and use it to estimate the accuracy with which various cosmological parameters can be recovered from future surveys using such statistics. This formalism also includes the effect of various noises such as intrinsic ellipticity distribution of galaxies and finite survey size. The estimation errors are then compared with the ones derived from low order moments of the PDF (variance and skewness) to check how efficiently the high $M_{ap}$ tail can be used to constrain cosmological parameters such as $\Omega_m$, $\sigma_8$ and dark energy equation of state $w_{de}$. We find that for future surveys such as JDEM the full PDF does not bring significant tightening of constraints on cosmology beyond what is already achievable by the joint use of second and third order moments.

Key words: Cosmology: theory – gravitational lensing – large-scale structure of Universe Methods: analytical – Methods: statistical –Methods: numerical

1 INTRODUCTION

Weak lensing surveys can be used as a very efficient probe to constrain the background cosmology as well as contents of the Universe (see Munshi & Valageas 2005 and references therein). Typically low order quantities such as two- and three-point functions or their reduced forms, variance and skewness, or the entire PDF are used to obtain cosmological constraints from observational data (Bernardeau et al. 2004). In weak lensing studies, observables such as $\kappa$, $\gamma$ or more commonly $M_{ap}$ and their low order moments are extensively studied (e.g. Jarvis et al. (2004) for recent results for $M_{ap}$). Use of $M_{ap}$ has the additional advantage of being able to separate gravity induced shear signals (“Electric” modes) from the (“Magnetic”) modes due to various systematics such as point spread function. The one-point probability distribution function of $M_{ap}$ encodes information regarding non-Gaussianities at all orders (Bernardeau & Valageas 2002) and thus can be useful to pinpoint background cosmological parameters by breaking degeneracies which appear at the level of two-point correlation functions (Schneider et al. 2002). We develop an analytical formalism to study the covariance of binned PDF and employ Fisher formalism techniques to study covariance of estimation error of cosmological parameters from realistic weak lensing surveys. The purpose of this study is twofold. Firstly we check the estimation error associated with cosmological parameters while using binned PDF as the primary observable. Secondly we use these results to infer how much tightening of constraints, if any, can be achieved in general by using higher order informations regarding non-Gaussianities. This has been a topic of discussion in the recent years using the lowest order non-Gaussian statistics, e.g. bi-spectrum or related collapsed one-point skewness (Takada & Jain 2002, Kilbinger & Schneider 2004, Munshi & Valageas 2005). Thus our results based on PDF extend such studies to include all-orders of non-Gaussianity. This letter is organised as follows: in section 2 we discuss the analytical results concerning covariance structure of the PDF and CPDF. Borrowing results from Valageas et al. (2004) we show how the Fisher matrix can be constructed from PDF and CPDF data. In section 3 the numerical results are provided. Section 4 is left for discussion of our results.
2 ANALYTICAL RESULTS

2.1 Covariance of PDF of aperture-mass $M_{ap}$

We recall in this section the analytical results presented in Valageas et al. (2005). The aperture-mass $M_{ap}$ can be written as a function of the tangential shear $\gamma_t$ as (Kaiser et al. 1994; Schneider 1996):

$$M_{ap} = \int d\vartheta \, Q_{M_{ap}}(\vartheta) \, \gamma_t(\vartheta),$$

with (using the same filter as Schneider 1996):

$$Q_{M_{ap}}(\vartheta) = \frac{\Theta(\vartheta < \theta_s)}{\pi \sigma^2} 6 \left( \frac{\vartheta}{\theta_s} \right)^2 \left( 1 - \frac{\vartheta^2}{\theta_s^2} \right).$$

Then, in order to measure the aperture-mass $M_{ap}$ within a single circular field of angular radius $\theta_s$, in which $N$ galaxies are observed at positions $\vartheta_j$ with tangential ellipticity $\epsilon_{t,j}$, we can use the estimator $M$ defined by:

$$M = \frac{\theta_s^2}{N} \sum_{j=1}^{N} Q_{M_{ap}}(\vartheta_j) \, \epsilon_{t,j}. \quad (3)$$

In the case of weak lensing the observed complex ellipticity $\epsilon$ is related to the shear $\gamma$ by: $\epsilon = \gamma + i \epsilon_s$, where $\epsilon_s$ is the intrinsic ellipticity of the galaxy. Assuming that the intrinsic ellipticities of different galaxies are uncorrelated random Gaussian variables, the cumulant of order $p$ of $M$ is:

$$(M^p)_c = (M_{ap}^p)_c \left( 1 + \frac{\delta_{p,2}}{\rho} \right) \text{ with } \rho = \frac{5N(M_{ap}^2)}{3\sigma^2}, \quad (4)$$

where $\delta_{p,2}$ is the Kronecker symbol and $\sigma^2 = \langle \epsilon^2 \rangle$ is the dispersion of the intrinsic ellipticity of galaxies. Since the intrinsic ellipticities are Gaussian and we neglected any cross-correlation with the density field they only contribute to the variance of the estimator $M$ (note that $M^2$ is a biased estimator of $(M_{ap}^2)$ because of this additional term). The quantity $\rho$ measures the relative importance of the galaxy intrinsic ellipticities in the signal. They can be neglected if $\rho \gg 1$. Any Gaussian white noise associated with the detector can also be incorporated into the expression (4) by adding a relevant correction to $\sigma^2$. Finally, from eq. (4) we obtain for the generating function $\varphi_M$ of the estimator $M$:

$$\varphi_M(y) = \frac{1 + \rho}{\rho} \varphi_{M_{ap}} \left( \frac{\rho}{1 + \rho} - \frac{1}{1 + \rho} \frac{y^2}{2} \right), \quad (5)$$

where we defined as usual the characteristic function $\varphi_{M_{ap}}$ of any random variable $M$ by the logarithm of the Laplace transform of its PDF $P(M)$:

$$P(M) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dy}{2\pi i(M^2)} e^{\varphi_{M_{ap}}(y)/M^2}. \quad (6)$$

Thus, the PDF of the estimator $P(M)$ is related to the PDF of the aperture-mass $P(M_{ap})$ through eq. (4). Of course, for small $\rho$ we recover the Gaussian (i.e. $\varphi_{M}(y) = -y^2/2$) as $M$ is dominated by the galaxy intrinsic ellipticities, whereas for large $\rho$ we recover $\varphi_{M_{ap}}(y)$ as $M$ is dominated by the weak lensing signal ($M \simeq M_{ap}$).

Thus, each circular field of angular radius $\theta_s$ yields a particular value for the quantity $M$ defined in eq. (4). If the survey contains $N_c$ such cells on the sky, we can estimate the PDF $P(M)$ through the estimators $\bar{F}_j$ or $\bar{F}_j$ defined by:

$$P_j = \frac{1}{N_c \Delta_j} \sum_{n=1}^{N_c} I_j(n), \quad F_j = \Delta_j P_j = \frac{1}{N_c} \sum_{n=1}^{N_c} I_j(n), \quad (7)$$

where $I_j(n)$ is the characteristic function of the interval $I_j = [M_j, M_{j+1}]$ of width $\Delta_j$, applied to the value $M(n)$ of $M$ measured in the cell $n$:

$$I_j(n) = 1 \text{ if } M_j \leq M(n) < M_{j+1}, \quad I_j(n) = 0 \text{ otherwise.} \quad (8)$$

Here we restrict ourselves to non-overlapping intervals $I_j$. The estimators $F_j$ are slightly more convenient than the $P_j$ as they do not explicitly involve the widths $\Delta_j$ which can vary with $j$ or even be infinite (on each side of a binning of $[M_{min}, M_{max}]$). Then, from the sets $\{P_j\}$ or $\{F_j\}$ we obtain an histogram which provides an approximation to $P(M)$:

$$(F_j) = \bar{F}_j \quad \text{with} \quad \bar{F}_j = \int_{M_j}^{M_{j+1}} dM \, P(M). \quad (9)$$

Thus, for small enough $\Delta_j$ we have $\bar{F}_j = F_j / \Delta_j \simeq P([M_j + M_{j+1}]/2)$. Next, assuming that different cells are well separated so as to be uncorrelated the covariance matrix $C_{ij}^{FP}$ of estimators $\{F_j\}$ is simply:

$$C_{ij}^{FP} = \langle F_i F_j \rangle - \langle F_i \rangle \langle F_j \rangle = \frac{\delta_{ij} F_j - \bar{F}_j \bar{F}_j}{N_c}, \quad (10)$$

In addition to the $F_j$ which measure the PDF we have also considered the estimators $U_j$ (upper) and $L_j$ (lower) associated with the CPDF:

$$U_j = \sum_{i \geq j} F_i, \quad \bar{U}_j = \int_{M_{max}}^{M_{Mj}} dM \, P(M), \quad (11)$$

$$L_j = \sum_{i \leq j} F_i, \quad \bar{L}_j = \int_{M_{min}}^{M_{Mj}} dM \, P(M). \quad (12)$$

Their covariance matrices are simply:

$$C_{ij}^{U} = \frac{\bar{U}_{max}(i,j) - \bar{U}_{max}(j)}{N_c}, \quad C_{ij}^{LL} = \frac{\bar{L}_{min}(i,j) - \bar{L}_{min}(j)}{N_c}. \quad (13)$$

The second (product) term is of rank one but the presence of the first term makes the matrices non-degenerate. It is also possible to construct joint data vectors but this will not be pursued here as they all carry the same information as the set $\{F_j\}$. Note that if the $N_P$ intervals run over $[-\infty, \infty]$ so that $\sum_{j} F_j = 1$ the covariance matrices are degenerate as there are only $N_P - 1$ independent variables. Thus, in practice we consider the set $\{F_j\}$ with $j = 1, \ldots, N_P - 1$.

In this case we also have $\bar{L}_j + \bar{U}_j = 1$ which yields $C_{ij}^{LL} = C_{ij}^{LU}$.

2.2 Parameter estimation using the noisy PDF

To obtain the estimation error for cosmological parameters $\Theta_\alpha$ measured from the observables $F_j, U_j$ or $L_j$, which we generically denote by $X_j$, we employ the Fisher matrix formalism commonly used in the literature (e.g. Tegmark et al. 1997). The information regarding estimation errors $(\Delta \Theta_\alpha)^{1/2}$ and their cross-correlations $\langle \Delta \Theta_\alpha \Delta \Theta_\beta \rangle$ are encoded in the inverse of the Fisher matrix $F_{\alpha\beta}$ which can be constructed from the covariance matrix $C$ of the data vector and its derivative w.r.t. cosmological parameters $\Theta_\alpha$:

$$F_{\alpha\beta} = \sum_{i,j} \frac{dX_i}{d\Theta_\alpha} C_{ij} \frac{dX_j}{d\Theta_\beta}. \quad (14)$$
Here we ignored the next order correction terms which involve the derivatives of the covariance matrices (Munshi & Valageas 2005) as they are much smaller. The cosmological parameters that we consider for extraction from the data are \( \Omega_m, \sigma_8 \) and \( w_{de} \). In this letter we concentrate on one angular scale \( \theta_s = 2' \). Use of other scales would not change much the constraints as nearby scales are highly correlated and non-gaussianities measured at very small or very large angular scales are too noisy (e.g. Munshi & Valageas 2005).

### 3 NUMERICAL RESULTS

#### 3.1 Covariance matrix and the derivative vector

For all numerical computations we have adopted a SNAP like survey strategy and a concordance \( \Lambda \)CDM cosmology as the fiducial model as in Munshi & Valageas (2005).

In Fig. 1 we plot the covariance matrix \( C_{ij}^{\text{PF}} \) for \( \{ F_j \} \) using \( N_P = 91 \) bins from \( M = -0.05 \) up to \( M = 0.08 \).

For comparison we plot in the right panel the covariance matrix \( C_{ij}^{\text{LL}} \) for the PDF (here \( C_{LL} = C^{UU} \)). Note that being integrated quantities the covariance matrix of \( \{ U_j \} \) or \( \{ L_j \} \) is more narrowly focussed along the diagonal but we found that in practice the covariance matrix \( C_{ij}^{\text{PF}} \) is no more difficult to invert than \( C_{ij}^{\text{LL}} \).

The PDF \( \mathcal{P}(M) \) is shown in the upper-left panel of Fig. 3 along with the derivatives of the data vector \( \{ F_j \} \) with respect to the three cosmological parameters \( w_{de}, \Omega_m \) and \( \sigma_8 \). The absolute value of the derivatives shows a well pronounced maxima near \( M \approx 0 \) and two wings on each side. The symmetry is broken by inclusion of non-Gaussian effects as shown by the solid curves. Derivative vectors asymptotically reach zero for high and low values of \( M \), in bins where the PDF becomes small and noise dominated and no useful cosmological information can be extracted. We can already see that the dependence on \( w_{de} \) is much smaller than the sensitivity to \( \Omega_m \) or \( \sigma_8 \) which implies that errorbars for \( w_{de} \) will be larger, as will be checked in Fig. 3 below.

#### 3.2 Parameter constraints using the PDF

From the the covariance and the derivative of the data vector \( \{ F_j \} \) we compute the Fisher matrix to study parameter degeneracies. We consider two parameter pairs, \( \{ \Omega_m, \sigma_8 \} \)
where parameters are negatively correlated and $\{\Omega_m, w_{de}\}$ where they are positively correlated. We also use the two lowest order moments (variance and skewness) as data vectors to construct independent Fisher matrices. The latter are obtained from the unbiased estimators $M_p$ defined as:

$$
M_p = \frac{(\sigma^2_p)^p}{(N)^p} \sum_{(j_1, \ldots, j_p)} Q_{M_{\theta p}}(\vec{\theta}_{j_1}) \ldots Q_{M_{\theta p}}(\vec{\theta}_{j_p}) \epsilon_{1j_1} \ldots \epsilon_{pj_p} \tag{15}
$$

where the sum runs over all sets of $p$ different galaxies among the $N$ galaxies enclosed in the angular radius $\theta$, and $(N) = N(N-1) \ldots (N-p+1)$. This ensures that $\langle M_p \rangle$ is estimated from the PDF $P(M)$. The procedure is outlined in Munshi & Valageas (2005) and will not be repeated here. As shown by the comparison of Figs. 4 and 5 we find that the constraints from the whole PDF are only slightly tighter than the ones constructed from these two lowest order moments. The similarity in shape is expected since our model for the non-linear PDF is actually fully parameterized by the variance and skewness of the underlying density field (see Munshi et al. 2004). Hence the PDF and the two lowest order moments carry the same information within our model. The good agreement of our analytical predictions with numerical simulations (Munshi et al. 2004) shows that this must be true to a large extent for any realistic model. In a similar fashion, for the Gaussian approximation we noticed that the parameter constraints from only variance are in good agreement with the ones derived from PDF which is then fully defined by its variance. However, it was not obvious that the size of the contour area would be so close since the low-order estimators $\{M_2, M_3\}$ of eq. (15) might have been more or less noisy than the data vector $\{F\}$. Indeed, the set $\{F\}$ contains many more points than the two-point data vector $\{M_2, M_3\}$ but it is based on the estimator $M$ which is slightly more affected by the galaxy intrinsic ellipticities (e.g. although the variance of $M^2$ is equal to the variance of the estimator $M_2$ the variance of $M^3$ is somewhat larger than for $M_3$). In particular, we checked that using as data vector $\{\langle M^2 \rangle, \langle M^3 \rangle\}$ (i.e. the two lowest order moments of the PDF $P(M)$) yields confidence areas which are somewhat larger for $\{M_2, M_3\}$. Thus, it appears that although the full data vector $\{F\}$ is able to yield as good constraints as those obtained from $\{M_2, M_3\}$ the tails of the PDF are too noisy to provide significantly tighter constraints. Note that a drawback of $P(M)$ is that it is more difficult to evaluate in surveys which contain many holes while low order moments may be recovered by integrating the measured correlation functions.

We have also investigated the amount of information which can be extracted from the restriction of the PDF to $M < 0$ (dotted lines in Fig. 4) or $M > 0$ (dashed lines). We can note that the negative $M$ sector of the PDF generates ellipses which are more closely aligned with those from the variance analysis alone whereas the positive $M$ sector of the PDF produces ellipses which are more closely aligned with those from the skewness analysis. This is due to the fact that the skewness is mostly sensitive to the large positive tail of $P(M_{\theta})$ which is broader than the negative tail (e.g. Munshi et al. 2004). On the other hand, the positive sector $M > 0$ alone already provides reasonable constraints on

![Figure 4](image1.png)

**Figure 4.** The contribution to the error-ellipses from various segments of the estimated PDF is shown for the cosmological parameter pairs $\{\Omega_m, w_{de}\}$ (left panel) and $\{\Omega_m, \sigma_8\}$ (right panel). The dotted line corresponds to the $M < 0$ sector of the PDF and the dashed line to the $M > 0$ sector. The solid line corresponds to the entire range of $\{M_j\}$. All other parameters are assumed to be perfectly known. All contours in these panels as well as the next plots represent $3\sigma$ contours.

![Figure 5](image2.png)

**Figure 5.** The contribution to the error-ellipses from low-order moments are presented for $\{\Omega_m, w_{de}\}$ (left panel) and $\{\Omega_m, \sigma_8\}$ (right panel). The dotted lines represent constraints from variance only whereas the dashed lines represent constraints from skewness only. The solid lines show the joint constraints.

![Figure 6](image3.png)

**Figure 6.** The estimation error $\Delta \Theta_\alpha$ for two cosmological parameter pairs are shown, $\{\Omega_m, w_{de}\}$ in right panel and $\{\Omega_m, \sigma_8\}$ in left panel (hence this corresponds to $1\sigma$ errorbars). The lines which reach constant asymptotes for high positive values of $M_j$ are for the estimator $L_j$ and the ones which reach constant asymptotes for low negative values of $M_j$ correspond to the estimator $U_j$. These asymptotes match exactly with the results based on the full estimator set $\{F_j\}$, as CPDF and PDF carry equivalent information.
The estimation error \( \Delta \Theta \) panel and bins is used for each of these points to reduce the scatter in \( \Delta \Theta \) of its lower order moments which is more prevalent in the literature. We find that for a SNAP like survey the PDF does not play any significant role and for a SNAP like survey can safely be ignored as far as estimation of cosmological parameters is concerned. Since the CPDF data \( U_j \) and \( L_j \) are derived from the original data vector \( \{ F_j \} \) they cannot provide additional cosmological constraints. However, they can be used as a useful consistency check. We have checked that they give exactly the same results as the PDF data vector \( \{ F_j \} \).

Finally, we display the estimation error \( \Delta \Theta \) for the pairs \( \{ \Omega_m, \sigma_8 \} \) and \( \{ \Omega_m, w_{de} \} \) associated with different parts of the CPDF (Fig. 7) or of the PDF (Fig. 7). The “U-shape” in Fig. 7 is consistent with the upper-left panel of Fig. 6, the tails of the PDF are too noisy to bring significant cosmological information. On the other hand, the small decrease of \( \Delta \Theta \) near \( M_j \approx 0.02 \) for \( P(< M) \) (solid lines in Fig. 7) is related to the secondary peak near \( M_j \approx 0.02 \) in the derivatives shown in Fig. 7. This feature contains significant cosmological information which moreover goes beyond the mere variance as its shape differs from the one obtained within a Gaussian approximation.

4 DISCUSSION

We have checked the possibility of using the entire \( M_{ap} \) PDF to constrain cosmological parameters as opposed to the use of its lower order moments which is more prevalent in the literature. We find that for a SNAP like survey the PDF does not yield significantly tighter constraints than those derived from the variance and skewness alone, despite the low-noise space based survey strategy. This also implies that higher order moments would not contribute either to further tightening of error ellipses. However, much larger surveys with high level of source galaxy distributions might still benefit from measurement of kurtosis, but higher number density requires increasing the depth of the survey which in turn makes the PDF itself more Gaussian. Our results only take into account the volume averages of higher order correlation functions and did not propose to quantify non-Gaussianity beyond collapsed one point objects. It is possible to extend our study by taking into account redshift binning or tomography which we will report elsewhere, as well as combining several angular scales. However, as seen in Munshi & Valageas (2005) for a space based survey such as SNAP with reasonably good sky coverage and high number density of source galaxies most of the useful information can in effect be obtained by studying one particular angular scale (e.g. \( \theta_s \approx 2^\circ \) in case of SNAP, compare Fig. 8 with Figs. 14, 15 in Munshi & Valageas 2005). Indeed, nearby scales are highly correlated and do not provide additional information whereas very large and very small angular scales are more affected by noise (due to the intrinsic ellipticity distribution of galaxies or the finite size of the survey).

The results provided here are for a monolithic survey strategy but real surveys will have more complicated topology. However our results can provide clues as to what extent the surveys can probe non-Gaussianity to extract meaningful constraints on cosmological parameters by using not only the first few lower order moments but the entire PDF.

We have not considered a Wiener filter based approach to reconstruct the PDF from noisy data as suggested by Zhang & Pen (2005) for convergence \( \kappa \) maps but such an approach can very easily be implemented by using the covariance matrices we have constructed. A complete Wiener filter based approach using compensated filter \( M_{ap} \) as presented here will be investigated elsewhere. In a different context shape of the non-Gaussian PDF was used by Huffenberger & Seljak (2005) to separate the kinetic-SZ contributions from primordial CMB. Similar approach can be useful in separating gravity generated \( E \) mode using \( M_{ap} \) PDF from various systematics.

Techniques presented here can also be applied to the case of weak lensing of diffuse background such as CMB (Kesden et al. 2002) and high-redshift 21 centimeter radiation from neutral hydrogen during the era of reionization (e.g. Cooray 2004). Results of such analysis will be reported elsewhere.

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