TIME-INCONSISTENT OPTIMAL CONTROL PROBLEMS WITH REGIME-SWITCHING

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Abstract. In this paper, a time-inconsistent optimal control problem is studied for diffusion processes modulated by a continuous-time Markov chain. In the performance functional, the running cost and terminal cost depend on not only the initial time, but also the initial state of the Markov chain. By modifying the method of multi-person game, we obtain an equilibrium Hamilton-Jacobi-Bellman equation under proper conditions. The well-posedness of this equilibrium HJB Equation is studied in the case where the diffusion term is independent of the control variable. Furthermore, a time-inconsistent linear-quadratic control problem is considered as a special case.

1. Introduction. It is well-known that Bellman’s optimality principle plays a key role in the classical optimal control theory. However, there are many examples, i.e. the so-called time-inconsistent control problems, in which this principle does not hold, such as optimal control problems with non-exponential discounting and mean-variance portfolio selection (see e.g. [7] and [1]). In the seminal work [18], Strotz studies a cake eating problem within a game theoretic framework where the players are the agent and his/her future selves, and seeks a subgame perfect Nash equilibrium point for this game. Strotz’s work has been pursued by many others, such as [14], [13], [11] and [12], among others.

In recent years, research on the time-inconsistent optimal control in stochastic continuous-time setting has attracted an increasing attention. In [7] and [8], the precise definition of the equilibrium concept in continuous time is provided for the first time. Following the notion of equilibrium strategy in the previous papers, [3] studies the time-inconsistent control problem in a general Markovian framework, and derives an extended HJB equation as well as the verification theorem. In [4], the Markowitz’s problem with state-dependent risk aversion is investigated by utilizing the extended HJB equation obtained in [3].

Another approach to the time-inconsistent control problem, i.e. the method of multi-person game, is developed by [21, 22]. In these papers, the running cost and terminal cost functions depend on the initial time in some general way. A brief description of the method of multi-person game is given as follows. Let $T > 0$ be the fixed time horizon. Take a partition $\Pi = \{t_k \mid 0 \leq k \leq N\}$ of...
the time interval $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_N = T$, and with the mesh size $\|\Pi\| = \max_{1 \leq k \leq N} (t_k - t_{k-1})$. Consider an $N$-person differential game: for $k = 1, 2, \cdots, N$, the $k$-th player controls the system on $[t_{k-1}, t_k)$, starting from the initial state $(t_{k-1}, X(t_{k-1}))$ which is the terminal state of the $(k-1)$-th player, and tries to minimize/maximize his/her own performance functional. Each player knows that the later players will do their best, and will modify their control systems as well as their cost functionals. In the performance functional, each player discounts the utility in his/her own way. Then for any given partition $\Pi$, a Nash equilibrium strategy is constructed to the corresponding $N$-person differential game. Finally, it can be shown that as the mesh size $\|\Pi\|$ approaches to zero, the Nash equilibrium strategy to the $N$-person differential game approaches to the desired time-consistent solution of the original time-inconsistent problem. By this method, [21] considers a deterministic time-inconsistent linear-quadratic control problem. Considering a controlled stochastic differential equation with deterministic coefficients, [22] investigates a time-inconsistent problem with a general cost functional and derives an equilibrium HJB equation. For more research following the method of multi-person game, we refer the readers to [23], [26] and [19].

In this paper, we study the time-inconsistent optimal control problem for diffusion processes with regime-switching. More specifically, the drift and diffusion terms of the diffusion process are modulated by an exogenous continuous-time finite-state Markov chain. This kind of process has been extensively used in finance and actuarial science to describe the volatile financial market in the long run, see [5], [27], [25], [17] and [20], among others. Similar to [22], the objective of this paper is to derive an equilibrium Hamilton-Jacobi-Bellman equation for the time-inconsistent control problem. However, the running cost and terminal cost in this paper are allowed to depend on both the initial time and the initial state of the Markov chain. To handle the time-inconsistency caused by the Markov chain, we need to modify the method of multi-person game as follows (see Section 3 for more details): in each time interval $[t_{k-1}, t_k)$, the $k$-th player only controls the system up to the first time the Markov chain jumps in $[t_{k-1}, t_k)$, and then some other player controls the system on the rest of time in $[t_{k-1}, t_k)$. Obviously, the second player in $[t_{k-1}, t_k)$ arrives randomly as the Markov chain may not jump during $[t_{k-1}, t_k)$. Thus, given a partition of the planning horizon, we study a multi-person game with random arrivals.

The remainder of this paper is organized as follows. Section 2 introduces the time-inconsistent optimal control problem. Section 3 studies the multi-person differential game and derives an equilibrium Hamilton-Jacobi-Bellman equation. Section 4 shows the well-posedness of the equilibrium HJB Equation in the special case where the diffusion term is independent on the control variable. Section 5 investigates a time-inconsistent linear-quadratic control problem with regime-switching.

2. The time-inconsistent optimal control problem. Let $T > 0$ be a fixed finite time horizon, $W(\cdot)$ be a 1-dimensional standard Brownian motion, and $\alpha(\cdot)$ be a homogeneous, irreducible continuous-time Markov chain taking values in a finite set $A = \{1, 2, \cdots, M\}$. Denote by $Q = (q_{ij})_{M \times M}$ the generator of the Markov chain, where $-q_{ii} = q_i > 0$ for $i \in A$. We assume that $\alpha(\cdot)$ is RCLL and independent of the Brownian motion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space on which $W(\cdot)$ and $\alpha(\cdot)$ are defined, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the augmentation under $\mathbb{P}$ of $\mathcal{F}^W_t := \sigma(W(s), \alpha(s), 0 \leq s \leq t), t \in [0, T]$. 


Consider the following controlled system:
\[
\begin{aligned}
    dX(s) &= b(s, X(s), \alpha(s), u(s)) \, ds + \sigma(s, X(s), \alpha(s), u(s)) \, dW(s), \quad s \in [t, T], \\
    X(t) &= x, \quad \alpha(t) = i,
\end{aligned}
\]
where \( b : [0, T] \times \mathbb{R}^n \times \mathbb{A} \times U \to \mathbb{R}^n \) and \( \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{A} \times U \to \mathbb{R}^{n \times d} \) are suitable deterministic maps with \( U \subseteq \mathbb{R}^m \) being a closed set. We define the set of all admissible controls by the following:
\[
    U[t, T] = \{ u : [t, T] \times \Omega \to U \mid u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable} \}.
\]
Under proper conditions, for any \( (t, x, i) \in [0, T] \times \mathbb{R}^n \times \mathbb{A} \) and \( u(\cdot) \in U[t, T] \), (1) admits a unique strong solution \( X(\cdot) = X(\cdot; t, x, i, u(\cdot)) \).

Let us introduce the performance cost functional:
\[
    J(t, x, i; u(\cdot)) = \mathbb{E}_t \left[ \int_t^T g(t, i, s, X(s), \alpha(s), u(s)) \, ds + h(t, i, X(T), \alpha(T)) \right], \tag{2}
\]
where \( \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t] \). Note that in (2) the maps \( g(\cdot) \) and \( h(\cdot) \) depend on the initial time \( t \) and initial state \( i \) of the Markov chain. For convenience, in the rest of the paper, we refer the first two variables \( (t, i) \) of \( g(\cdot) \) and \( h(\cdot) \) as the preference of the decision-maker, and say the decision-maker has preference-\( (t, i) \) (or preference-\( t \), preference-\( i \)). The optimal control problem is

**Problem (N).** For any given initial state \( (t, x, i) \in [0, T] \times \mathbb{R}^n \times \mathbb{A} \), find a \( \bar{u}(\cdot) \in U[t, T] \) such that
\[
    J(t, x, i; \bar{u}(\cdot)) = \inf_{u(\cdot) \in U[t, T]} J(t, x, i; u(\cdot)).
\]

It is well-known that Problem (N) is time-inconsistent in the sense that the classical Bellman’s optimal principle does not hold. Our object is to find a time-consistent (equilibrium) strategy and the corresponding (equilibrium) value function of the problem.

We end this section by introducing some notations and assumptions. Let \( D[0, T] = \{ (\tau, t) \in [0, T] \times [0, T] \mid 0 \leq \tau \leq t \leq T \} \).

Define the functional
\[
    \mathcal{J}(\tau, t, x, i; u(\cdot)) \equiv \mathbb{E}_t \left[ \int_\tau^T g(\tau, i, s, X(s), \alpha(s), u(s)) \, ds + h(\tau, i, X(T), \alpha(T)) \right],
\]
for \( (\tau, t) \in D[0, T], i \in \mathbb{A}, x \in \mathbb{R}^n \) and \( u(\cdot) \in U[0, T] \). Obviously, \( J(t, x, i; u(\cdot)) = \mathcal{J}(t, t, x, i; u(\cdot)) \).

The following standing assumptions are similar to [22].

**Assumption 2.1.** For any \( i \in \mathbb{A} \), the functions \( b(\cdot, \cdot, \cdot, \cdot) \) and \( \sigma(\cdot, \cdot, \cdot, \cdot) \) are continuous, and there exist constant \( L > 0 \) and \( p > 0 \) such that
\[
    \begin{align*}
        |b(t, x, i, u) - b(t, y, i, u)| &\leq L \left( 1 + (|x| + |y|)^p + |u| \right) |x - y|, \\
        (x - y, b(t, x, i, u) - b(t, y, i, u)) &\leq L |x - y|^2, \\
        |\sigma(t, x, i, u) - \sigma(t, y, i, u)| &\leq L |x - y|, \quad \forall i \in \mathbb{A}, (t, u) \in [0, T] \times U, x, y \in \mathbb{R}^n,
    \end{align*}
\]
where \( |x| \vee |y| = \max \{|x|, |y|\} \) and
\[
    |b(t, 0, i, u)| + |\sigma(t, 0, i, u)| \leq L(1 + |u|), \quad \forall (t, u, i) \in [0, T] \times U \times \mathbb{A}.
\]
Assumption 2.2. For any \( i, j \in \mathbb{A} \), the functions \( g(\cdot, j, \cdot, i, \cdot) \) and \( h(\cdot, j, \cdot, i) \) are continuous, and there exist constant \( L > 0 \) and \( p > 0 \) such that
\[
\begin{align*}
0 \leq g(\tau, j, t, x, i, u) & \leq L \left( 1 + |x|^p + |u|^p \right), \\
0 \leq h(\tau, j, x, i) & \leq L \left( 1 + |x|^p \right), \\
\forall i, j \in \mathbb{A}, (\tau, t, x, u) & \in D[0, T] \times \mathbb{R}^n \times U.
\end{align*}
\]

3. Time-consistent solution via multi-person games. In this section, we are going to find an time-consistent solution to Problem \((N)\) by the method of multi-person differential games proposed by \([21, 22]\).

Let \( \Pi = \{ t_k \mid 0 \leq k \leq N \} \) be a partition of \([0, T]\) with \( 0 = t_0 < t_1 < \cdots < t_N = T \), and with the mesh size \( \|\Pi\| = \max_{1 \leq k \leq N} \pi_k \), where \( \pi_k = t_k - t_{k-1} \). Denote by \( \mathcal{P}[0, T] \) the set of all partitions of \([0, T]\).

Let
\[
\xi_k = \inf \{ s > t_{k-1} : \alpha(s-) \neq \alpha(s) \} \wedge t_k.
\]
If \( \xi_k < t_k \), then it is the first time the Markov chain jumps in \((t_{k-1}, t_k)\). If \( \xi_k = t_k \), then there is no jump occurs in \((t_{k-1}, t_k)\) or there is a jump exactly at \( t_k \).

Before we proceed, let us give a remark on preference-\((t, i)\) in the performance functional \((2)\). Preference-\(t\) is deterministic and transient, i.e., the decision-maker only hold preference-\(t\) at time \( t \). However, this is not the case for preference-\(i\), as the decision-maker changes preference-\(i\) randomly and does keep preference-\(i\) for a while. To capture the randomness of preference-\(i\), we shall consider the following multi-person game: given \( \alpha(t_{k-1}) = i \). Player \((k)\), who holds preference-\((t_{k-1}, i)\), controls the system from \( t_{k-1} \) to \( \xi_k \). If \( \xi_k < t_k \) and \( \alpha(\xi_k) = i' \), then some other player with preference-\((t_{k-1}, i')\), denoted by Player \((k)\), controls the system on \([\xi_k, t_k]\). If \( \xi_k = t_k \), then Player \((k)\) controls the system on the whole time interval \([t_{k-1}, t_k]\).

3.1. Definition of time-consistent equilibrium strategy. We give the following definition of time-consistent equilibrium strategy of Problem \((N)\).

Definition 3.1. A map \( \Psi : [0, T] \times \mathbb{R}^n \times \mathbb{A} \to U \) is called a time-consistent equilibrium strategy of Problem \((N)\), if for any \( i \in \mathbb{A} \), \( \Psi(\cdot, i) \) is continuous, and for any \((x, i) \in \mathbb{R}^n \times \mathbb{A} \), the equation
\[
\begin{cases}
\d X(s) = b(s, X(s), \alpha(s), \Psi(s, X(s), \alpha(s))) ds \\
\quad + \sigma(s, X(s), \alpha(s), \Psi(s, X(s), \alpha(s))) dW(s), & s \in [0, T], \\
X(0) = x, \quad \alpha(0) = i,
\end{cases}
\]

admits a unique solution \( X(\cdot) \equiv X(\cdot; 0, x, i, \Psi(\cdot)) \), and there exists a family of partitions \( \mathcal{P}_0[0, T] \subseteq \mathcal{P}[0, T] \) with
\[
\inf_{\Pi \in \mathcal{P}_0[0, T]} \|\Pi\| = 0,
\]
and two families of maps \( \Psi^\Pi(\cdot, i), \tilde{\Psi}^\Pi(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \times \mathbb{A} \to U \) parameterized by partitions \( \Pi \in \mathcal{P}_0[0, T] \) such that the following properties hold:

(i) The limit
\[
\lim_{\|\Pi\| \to 0} \left[ \left| \Psi^\Pi(t, x, i) - \Psi(t, x, i) \right| + \left| \tilde{\Psi}^\Pi(t, x, i) - \Psi(t, x, i) \right| \right] = 0
\]
holds for any \( i \in \mathbb{A} \) and \((t, x)\) in any compact subset of \([0, T] \times \mathbb{R}^n\).
(ii) For any partition \( \Pi = \{ t_k \mid 0 \leq k \leq N \} \in \mathcal{P}_0[0, T] \), for any \((t, x, i) \in [0, T] \times \mathbb{R}^n \times \mathcal{X} \), the following initial value problem

\[
\begin{aligned}
dX^\Pi(s) &= b \left( s, X^\Pi(s), \alpha(s), \Psi^\Pi \left( s, X^\Pi(s), \alpha(s) \right) \right) ds \\
&\quad + \sigma \left( s, X^\Pi(s), \alpha(s), \Psi^\Pi \left( s, X^\Pi(s), \alpha(s) \right) \right) dW(s), \\
&\quad \text{for } s \in [t \vee t_{k-1}, \xi_k], k = 1, \ldots, N,
\end{aligned}
\]

admits a unique solution \( X^\Pi(\cdot) = X^\Pi \left( \cdot ; t, x, i, \Psi^\Pi(\cdot), \tilde{\Psi}^\Pi(\cdot) \right) \) such that for any \( x \in \mathbb{R}^n \),

\[
J \left( \xi_k ; 0, X^\Pi(\cdot) \big|_{[t_k, T]} \right) \leq J \left( \xi_k ; 0, X^\Pi(\cdot) \big|_{[t_k, T]} \right) \\
\forall \bar{u}^k(\cdot) \in \mathcal{U}[\xi_k, t_k],
\]

and

\[
J \left( \xi_k ; 0, X^\Pi(\cdot) \big|_{[t_k, T]} \right) \leq J \left( \xi_k ; 0, X^\Pi(\cdot) \big|_{[t_k, T]} \right) \\
\forall \bar{u}^k(\cdot) \in \mathcal{U}[\xi_k, t_k],
\]

where

\[
\begin{aligned}
\left[ u^k(\cdot) \oplus \left( \Psi^\Pi(\cdot), \tilde{\Psi}^\Pi(\cdot) \right) \big|_{[\xi_k, T]} \right](s) &= \\
&= \begin{cases} \\
\bar{u}^k(s), & s \in [t_k, \xi_k], \\
\Psi^\Pi \left( s, X^k(s), \alpha(s) \right), & s \in [t_{m-1}, \xi_m], m = k + 1, \ldots, N, \\
\tilde{\Psi}^\Pi \left( s, X^k(s), \alpha(s) \right), & s \in [\xi_m, t_m], m = k, \ldots, N,
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\left[ \bar{u}^k(\cdot) \oplus \left( \Psi^\Pi(\cdot), \tilde{\Psi}^\Pi(\cdot) \right) \big|_{[t_k, T]} \right](s) &= \\
&= \begin{cases} \\
\tilde{u}^k(s), & s \in [\xi_k, t_k], \\
\Psi^\Pi \left( s, \hat{X}^k(s), \alpha(s) \right), & s \in [t_m, \xi_{m+1}], m = k, \ldots, N, \\
\tilde{\Psi}^\Pi \left( s, \hat{X}^k(s), \alpha(s) \right), & s \in [\xi_{m+1}, t_{m+1}], m = k, \ldots, N,
\end{cases}
\end{aligned}
\]
(iii) The following limits hold:

\[
\begin{align*}
\lim_{||\|\rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X^{\Pi} (s; 0, x, i, \Psi^{\Pi}(), \check{\Psi}^{\Pi}()) - \check{X} (s; 0, x, i, \Psi()) \right|^2 \right] = 0.
\end{align*}
\]

We call \( \check{X}() \equiv \check{X}(0; 0, x, i, \Psi()) \) a time-consistent equilibrium state process, \( \tilde{u}(\cdot) \equiv \Psi(\cdot, \check{X}(\cdot), \alpha(\cdot)) \) a time-consistent equilibrium control for the initial state \((x, i)\), and \((\check{X}(\cdot), \tilde{u}(\cdot))\) a time-consistent pair.

Furthermore, for \( t \in [0, T] \),

\[
\lim_{||\|\rightarrow 0} J \left( l_{\Pi}(t), X^{\Pi} \left( l_{\Pi}(t); 0, x, i, \Psi^{\Pi}(), \check{\Psi}^{\Pi}() \right), \alpha (l_{\Pi}(t)) ; \left( \Psi^{\Pi}(), \check{\Psi}^{\Pi}() \right) \big|_{l_{\Pi}(t), T} \right) = J \left( t, \tilde{X}(t; 0, x, i, \Psi()), \alpha(t); \Psi() \big|_{[t, T]} \right),
\]

where

\[
l_{\Pi}(t) = \sum_{k=1}^{N} t_{k-1} \mathbf{1}_{(t_{k-1}, t_{k})}(t), \quad s \in [0, T].
\]

A function \( V : [0, T] \times \mathbb{R}^n \times \mathbb{R} \) is called an equilibrium value function of Problem (N) if

\[
V (t, \tilde{X}(t; 0, x, i, \Psi()), \alpha(t)) = J \left( t, \tilde{X}(t; 0, x, i, \Psi()), \alpha(t); \Psi() \big|_{[t, T]} \right), \quad \text{for} \quad t \in [0, T].
\]

(3)

\[
V (t, \tilde{X}(t; 0, x, i, \Psi()), \alpha(t)) = J \left( t, \tilde{X}(t; 0, x, i, \Psi()), \alpha(t); \Psi() \big|_{[t, T]} \right),
\]

(3)
for all \((t, x, i) \in [0, T] \times \mathbb{R}^n \times A\).

### 3.2. Multi-person game

In this section, we introduce and solve a multi-person differential game associated with partition \(\Pi\). We first introduce some notations.

Let \(\mathbb{S}^n\) be the set of all \((n \times n)\) symmetric real matrices. For any \((\tau, t) \in D[0, T], j, i \in A\) and \((x, u, p, P) \in \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{S}^n\), let

\[
\mathbb{H}(\tau, j, t, x, i, u, p, P) = b(t, x, i, u, p) + \text{tr} \left[ \Sigma(t, x, i, u)P \right] + g(\tau, j, t, x, i, u),
\]

where \(\Sigma(t, x, i, u) = \frac{1}{2} \sigma(t, x, i, u)\sigma^T(t, x, i, u)\).

For any \((\tau, t) \in D[0, T], j, i \in A\) and \((x, p, P) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n\), define

\[
H(\tau, j, t, x, i, p, P) = \inf_{u \in U} \mathbb{H}(\tau, j, t, x, i, u, p, P).
\]

Similar to [22], we define the domain of \(H\) as

\[
D(H) = \{(\tau, j, t, x, i, p, P) \mid (\tau, t) \in D[0, T], j, i \in A, (x, p, P) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n, \\
\text{such that } H(\tau, j, t, x, i, p, P) > -\infty \}\.
\]

For any \((\tau, t) \in D[0, T], j, i \in A\) and \((x, p, P) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n\), let

\[
\arg\min \mathbb{H}(\tau, j, t, x, i, \cdot, p, P)
\]

\[
= \left\{ \bar{u} \in U \mid \mathbb{H}(\tau, j, t, x, i, \bar{u}, p, P) = \min_{u \in U} \mathbb{H}(\tau, j, t, x, i, u, p, P) \right\},
\]

which is a multi-valued map. Suppose we can find a map \(\psi : D(\psi) \subseteq D(H) \rightarrow U\) such that

\[
H(\tau, j, t, x, i, p, P) \equiv \mathbb{H}(\tau, j, t, x, i, \psi(\tau, j, t, x, i, p, P), p, P)
\]

\[
= \inf_{u \in U} \mathbb{H}(\tau, j, t, x, i, u, p, P) > \infty,
\]

for all \((\tau, t) \in D[0, T], j, i \in A\) and \((x, p, P) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n\). The set \(D(\psi)\) is the domain of \(\psi\), which consists of all points \((\tau, j, t, x, i, p, P) \in D(H)\) such that the infimum in the above equation is attained at \(\psi(\tau, j, t, x, i, p, P)\). Obviously,

\[
\psi(\tau, i, t, x, j, p, P) \in \arg\min \mathbb{H}(\tau, j, t, x, i, \cdot, p, P), \quad \forall (\tau, j, t, x, i, p, P) \in D(\psi).
\]

Furthermore, we impose the following assumption on \(\psi(\cdot)\).

**Assumption 3.1.** The map \(\psi(\cdot)\) is well-defined and has needed regularity.

#### 3.2.1. Player \((N)\)

Player \((N)\) controls the system on \([t_{N-1}, \xi_N]\). Given \(\alpha(t_{N-1}) = i\), the state process of Player \((N)\) is

\[
\frac{dX^{N,i}(s)}{ds} = b(s, X^{N,i}(s), i, u^{N,i}(s)) ds + \sigma(s, X^{N,i}(s), i, u^{N,i}(s)) dW(s), \quad s \in [t_{N-1}, \xi_N), \quad (4)
\]

\[
X^{N,i}(t_{N-1}) = x.
\]

We shall give the cost functional of Player \((N)\) later, and now consider the optimization problem for Player \((N)\) who controls the system on \([\xi_N, t_N]\). Let us first introduce the following control problem starting from some fixed initial time \(z \in [t_{N-1}, t_N]\). Given \(\alpha(z) = i'\), the objective is to find a strategy \(\tilde{u}^{N,i'}(\cdot)\) such that

\[
\tilde{J}^{N,i'}(z, x, i'; \tilde{u}^{N,i'}(\cdot)) \equiv \inf_{u^{N,i'}(\cdot) \in U(z, t_N]} \tilde{J}^{N,i'}(z, x, i'; u^{N,i'}(\cdot)), \quad (5)
\]
where
\[
\tilde{j}^{N,i'}(z, x, i'; u^{N,i'}(\cdot)) \equiv \mathbb{E}_z \left[ \int_z^{t_N} g \left( t_{N-1}, i', v, X^{N,i'}(v), \alpha(v), u^{N,i'}(v) \right) \mathrm{d}v 
+ h \left( t_{N-1}, i', X^{N,i'}(t_N), \alpha(t_N) \right) \right],
\]
(6)
with \( X^{N,i'}(\cdot) \equiv X^{N,i'}(\cdot; z, x, i', u^{N,i'}(\cdot)) \) being the unique solution to the SDE
\[
\begin{cases}
\mathrm{d}X^{N,i'}(v) = b \left( v, X^{N,i'}(v), \alpha(v), u^{N,i'}(v) \right) \mathrm{d}v \\
+ \sigma \left( v, X^{N,i'}(v), \alpha(v), u^{N,i'}(v) \right) \mathrm{d}W(v), \quad v \in [z, t_N],
\end{cases}
\]
(7)
Noting that \( i' \), the second variable of \( g(\cdot) \) and \( h(\cdot) \), is the state of the Markov chain at the initial time \( z \), the optimization problem given by (5-7) is time-inconsistent. However, we look for the pre-commitment optimal strategy (i.e., \( i' \) in the second variables of \( g(\cdot) \) and \( h(\cdot) \) is fixed as a parameter).

If the system of PDEs
\[
\begin{align*}
\dot{\tilde{V}}^{N,i'}_t(s, x, j') &+ \left\{ b \left( s, x, j', \psi \left( t_{N-1}, i', s, x, j', \tilde{V}^{N,i'}_t(s, x, j') \right), \tilde{V}^{N,i'}_t(s, x, j') \right) \right. \\
&\quad \left. + \sum_{\nu \in \mathbb{A}} q^{j',\nu} \tilde{V}^{N,i'}_t(s, x, j') \right) = 0, \quad (s, x, j') \in [z, t_N] \times \mathbb{R}^n \times \mathbb{A},
\end{align*}
\]

admits a unique classical solution \( \tilde{V}^{N,i'}(\cdot, \cdot, \cdot) \), by the similar arguments as in [24] and [9], and using the generalized Itô’s formula (see, e.g. [2]), one can verify (see, e.g. [16] and [6]) that the pre-commitment optimal strategy is given by
\[
\tilde{u}^{N,i'}(s) \equiv \psi \left( t_{N-1}, i', s, \tilde{X}^{N,i'}(s), \alpha(s), \tilde{V}^{N,i'}_t(s, \tilde{X}^{N,i'}(s), \alpha(s)) \right),
\]
\[
\tilde{V}^{N,i'}_t(s, \tilde{X}^{N,i'}(s), \alpha(s))
\]
for \( s \in [z, t_N] \), where \( \tilde{X}^{N,i'}(\cdot) \equiv X^{N,i'}(\cdot; z, x, i', \tilde{u}^{N,i'}(\cdot)) \). Furthermore, \( \tilde{V}^{N,i'}(z, x, i') = \tilde{j}^{N,i'}(z, x, i'; \tilde{u}^{N,i'}(\cdot)) \).

For \( s \in [z, t_N], \tau \in [0, s], i, j' \in \mathbb{A} \) and \( x \in \mathbb{R}^n \), define the function:
\[
\tilde{\Theta}^{N,i'}(\tau, i, s, x, j') \equiv \mathbb{E}_s \left[ \int_s^\tau g \left( \nu, \tilde{X}^{N,i'}(\nu), \alpha(\nu), \tilde{u}^{N,i'}(\nu) \right) \mathrm{d}\nu 
+ h \left( \tau, i, \tilde{X}^{N,i'}(t_N), \alpha(t_N) \right) \right],
\]
which can be regarded as the cost of the player with preference-\((\tau, i)\) on \([s, t_N]\) for \( s \in [z, t_N] \). Similar to [22, Section 4.1], by using the techniques of FBSDE, the
function $\tilde{\Theta}^{N,i}(\tau, i, \cdot, \cdot, \cdot)$ is given by the classical solution (if it exists) to

$$
\begin{align*}
&
\left(\tilde{\Theta}^{N,i}(\tau, i, s, x, j')
+ b \left( s, x, j', \psi \left( t_{N-1}, i', s, x, j', \tilde{V}_x^{N,i}, \tilde{V}_{xx}^{N,i} \right) \right) \right) \cdot \tilde{\Theta}^{N,i}(\tau, i, s, x, j')
+ \Delta \left( \sum \left( s, x, j', \psi \left( t_{N-1}, i', s, x, j', \tilde{V}_x^{N,i}, \tilde{V}_{xx}^{N,i} \right) \right) \right) \cdot \tilde{\Theta}^{N,i}(\tau, i, s, x, j')
+ g \left( \tau, i, s, x, j', \psi \left( t_{N-1}, i', s, x, j', \tilde{V}_x^{N,i}, \tilde{V}_{xx}^{N,i} \right) \right)
+ \sum_{j'' \in A} q_{j''} \cdot \tilde{\Theta}^{N,i}(\tau, i, s, x, j'' = 0, \tau \in [0, s], s \in [z, t_N], i, j' \in A, x \in \mathbb{R}^n),
\end{align*}
$$

where we have suppressed $(s, x, j')$ for $\tilde{V}_x^{N,i}(\cdot)$ and $\tilde{V}_{xx}^{N,i}(\cdot)$.

Given $F_{\xi_N}$, the optimization problem for Player $(N)$ is given by (5-7) with $(z, x, i')$ replaced by $(\xi_N, X^{N,i}(\xi_N), \alpha(\xi_N))$, and she/he seeks the pre-commitment optimal strategy. Therefore, conditioned on $F_{\xi_N}$, the cost of Player $(N)$ on $[\xi_N, t_N]$ is $\tilde{\Theta}^{N,\alpha(\xi_N)}(t_{N-1}, i, \xi_N, X^{N,i}(\xi_N), \alpha(\xi_N))$.

Thus, the cost of Player $(N)$ on $[t_{N-1}, t_N]$ is

$$
E_{t_{N-1}} \left[ \int_{t_{N-1}}^{t_N} g \left( t_{N-1}, i, s, X^{N,i}(s), i, u^{N,i}(s) \right) ds + \tilde{\Theta}^{N,\alpha(\xi_N)}(t_{N-1}, i, \xi_N, X^{N,i}(\xi_N), \alpha(\xi_N)) \right]
$$

$$
=E_{t_{N-1}} \left\{ 1_{\{\xi_N=t_N\}} \left[ \int_{t_{N-1}}^{t_N} g \left( t_{N-1}, i, s, X^{N,i}(s), i, u^{N,i}(s) \right) ds + h \left( t_{N-1}, i, X^{N,i}(t_N), i \right) \right] + 1_{\{\xi_N<t_N\}} \left[ \int_{t_{N-1}}^{\xi_N} g \left( t_{N-1}, i, s, X^{N,i}(s), i, u^{N,i}(s) \right) ds + \tilde{\Theta}^{N,\alpha(\xi_N)}(t_{N-1}, i, \xi_N, X^{N,i}(\xi_N), \alpha(\xi_N)) \right] \right\}
$$

$$
=E_{t_{N-1}} \left[ \int_{t_{N-1}}^{t_N} e^{-\eta_i(s-t_N)} g \left( t_{N-1}, i, s, X^{N,i}(s), i, u^{N,i}(s) \right) ds + \sum_{i'' \neq i} q_{i''} \cdot \tilde{\Theta}^{N,i''}(t_{N-1}, i, s, X^{N,i}(s), i'' = 0) \right] + e^{-\eta_i t_N} h \left( t_{N-1}, i, X^{N,i}(t_N), i \right) \right].
$$

(8)

Thus, we have the following optimization problem for Player $(N)$.

**Problem (C_N).** Given $\alpha(t_{N-1}) = i$, for any initial state $(t, x) \in [t_{N-1}, t_N] \times \mathbb{R}^n$, find a $\tilde{u}^{N,i}(\cdot) \in U[t, t_N]$ such that

$$
V^{N,i}(t, x) \equiv J^{N,i} \left( t, x; \tilde{u}^{N,i}(\cdot) \right) = \inf_{u^{N,i}(\cdot) \in U[t, t_N]} J^{N,i} \left( t, x, u^{N,i}(\cdot) \right),
$$

where

$$
J^{N,i} \left( t, x; u^{N,i}(\cdot) \right) = E_{t} \left[ \int_{t}^{t_N} e^{-\eta_i(s-t)} g \left( t_{N-1}, i, s, X^{N,i}(s), i, u^{N,i}(s) \right) ds \right]
$$

1Let $\xi = \inf \{ s > t_{N-1} : \alpha(s) \neq \alpha(s) \}$. Then $\{ \xi_N = t_N \} = \{ \xi > t_N \} \subset \{ \xi = t_N \}$. We know that conditioned on $\alpha(t_{N-1}) = i$, $\xi$ follows an exponential distribution and the probability of the event $\{ \xi = t_N \}$ is zero. Therefore, we may put $i$, instead of $\alpha(t_N)$, in the fourth variable of $h(\cdot)$ in the first equation of (8).
standard diffusion process without regime-switching.

\[
+ \int_{t}^{t_{n}} e^{-q_{i}(s-t)} \sum_{i' \neq i} q_{ii'} \tilde{\Theta}^{N,i'}(t_{n-1}, i, s, X^{N,i}(s), i') \, ds
\]

and \(X^{N,i}(\cdot)\) is given by (4) with \(\xi_{N}\) replaced by \(t_{N}\).

**Remark 3.1.** Although Player (N) controls the system only on \([t_{N-1}, \xi_{N}]\), after taking the expectation with respect to \(\xi_{N}\), his/her optimization problem is equivalent to Problem \((C_{N})\) which is defined on \([t_{N-1}, t_{N}]\), and the state process is a standard diffusion process without regime-switching.

Noting that Problem \((C_{N})\) is a standard optimal control problem, if the PDE

\[
\begin{aligned}
V_{t}^{N,i}(t, x) & = b(t, x, i, \psi(t_{N-1}, i, t, x, i, V_{x}^{N,i}(t, x), V_{xx}^{N,i}(t, x))) + \operatorname{tr} \left[ \sum (t, x, i, \psi(t_{N-1}, i, t, x, i, V_{x}^{N,i}(t, x), V_{xx}^{N,i}(t, x))) V_{xx}^{N,i}(t, x) \right] \\
& + g(t, x, i, \psi(t_{N-1}, i, t, x, i, V_{x}^{N,i}(t, x), V_{xx}^{N,i}(t, x))) \\
& + \sum_{i' \neq i} q_{ii'} \Theta^{N,i'}(t_{N-1}, i, t, x, i') - q_{i} V_{x}^{N,i}(t, x) = 0,
\end{aligned}
\]

(9)

admits a unique classical solution, then the optimal strategy of Player (N) is

\[
\tilde{u}^{N,i}(s) = \psi(t_{N-1}, i, s, \tilde{X}^{N,i}(s), i, V_{x}^{N,i}(s), V_{xx}^{N,i}(s)),
\]

for \(s \in [t_{N-1}, t_{N}]\), where \(\tilde{X}^{N,i}(\cdot) = X^{N,i}(\cdot; t, x, \tilde{u}^{N,i}(\cdot))\) is the optimal state process of Player (N).

Given the optimal pair \((\tilde{X}^{N,i}(\cdot), \tilde{u}^{N,i}(\cdot))\) of Player (N), we can define the function

\[
\Theta^{N}(\tau, j, t, x, i) \equiv E_{t} \left[ \int_{t}^{t_{N}} e^{-q_{i}(s-t)} g(\tau, j, s, \tilde{X}^{N,i}(s), i, \tilde{u}^{N,i}(s)) \, ds \right. \\
+ \left. \int_{t}^{t_{N}} e^{-q_{i}(s-t)} \sum_{i' \neq i} q_{ii'} \tilde{\Theta}^{N,i'}(\tau, j, s, \tilde{X}^{N,i}(s), i') \, ds \right. \\
+ e^{-q_{i}(t_{N}-t)} h(\tau, j, \tilde{X}^{N,i}(t_{N}), i) \right],
\]

\(\tau \in [0, t], t \in [t_{N-1}, t_{N}], i, j \in A, x \in \mathbb{R}^{n}\). (10)

Similarly, this function is the cost on \([t, t_{N}]\) of the player with preference-\((\tau, j)\). It can be given by the classical solution (if it exists) to the following PDE:

\[
\begin{aligned}
\Theta_{t}^{N}(\tau, j, t, x, i) & + \langle b(\tau, j, t, x, i, \psi(t_{N-1}, i, t, x, i, V_{x}^{N,i}(t, x), V_{xx}^{N,i}(t, x))), \Theta_{x}^{N}(\tau, j, t, x, i) \rangle \\
& + \operatorname{tr} \left[ \sum (\tau, j, t, x, i, \psi(t_{N-1}, i, t, x, i, V_{x}^{N,i}(t, x), V_{xx}^{N,i}(t, x))) \Theta_{xx}^{N}(\tau, j, t, x, i) \right] \\
& + g(\tau, j, t, x, i, \psi(t_{N-1}, i, t, x, i, V_{x}^{N,i}(t, x), V_{xx}^{N,i}(t, x))) \\
& + \sum_{i' \neq i} q_{ii'} \Theta^{N,i'}(\tau, j, t, x, i') - q_{i} \Theta^{N}(\tau, j, t, x, i) = 0, \\
& \tau \in [0, t], t \in [t_{N-1}, t_{N}], i, j \in A, x \in \mathbb{R}^{n}, \\
\Theta^{N}(\tau, j, t_{N}, x, i) & = h(\tau, j, x, i), \quad \tau \in [0, t_{N}], i, j \in A, x \in \mathbb{R}^{n}.
\end{aligned}
\]

(11)
3.2.2. Player \((N-1)\). Given \(\alpha(t_{N-2}) = i\), the state process for Player \((N-1)\) is

\[
\begin{aligned}
&dX_{N-1,i}(s) = b\left(s, X_{N-1,i}(s), i, u_{N-1,i}(s)\right)\,ds \\
&\quad + \sigma\left(s, X_{N-1,i}(s), i, u_{N-1,i}(s)\right)\,dW(s), \quad s \in [t_{N-2}, \xi_{N-1}], \\
&X_{N-1,i}(t_{N-2}) = x.
\end{aligned}
\]

Player \((N-1)\) controls the system on \([\xi_{N-1}, t_{N-1}]\). Similarly, to state the optimization problem for Player \((N-1)\), we consider the following control problem starting from some fixed initial time \(z \in [t_{N-2}, t_{N-1}]\). Given \(\alpha(z) = i'\), the objective is to find a strategy \(\hat{u}^{N-1,i'}(\cdot)\) such that

\[
\hat{j}^{N-1,i'}(z, x, i'; \hat{u}^{N-1,i'}(\cdot)) \equiv \inf_{u^{N-1,i'}(\cdot) \in \mathcal{U}[z, t_{N-1}]} \hat{j}^{N-1,i'}(z, x, i'; u^{N-1,i'}(\cdot)),
\]

where

\[
\begin{aligned}
\hat{j}^{N-1,i'}(z, x, i'; u^{N-1,i'}(\cdot)) = & E_z \left[ \int_{z}^{t_{N-1}} g\left(t_{N-2}, i', s, X_{N-1,i'}(s), \alpha(s), u_{N-1,i'}(s)\right)\,ds \\
&+ \Theta^N\left(t_{N-2}, i', t_{N-1}, X_{N-1,i'}(t_{N-1}), \alpha(t_{N-1})\right) \right],
\end{aligned}
\]

with \(X_{N-1,i'}(\cdot) \equiv X_{N-1,i'}(\cdot; z, x, i', u^{N-1,i'}(\cdot))\) being the unique solution to the SDE

\[
\begin{aligned}
&dX_{N-1,i'}(v) = b\left(v, X_{N-1,i'}(v), \alpha(v), u_{N-1,i'}(v)\right)\,dv \\
&\quad + \sigma\left(v, X_{N-1,i'}(v), \alpha(v), u_{N-1,i'}(v)\right)\,dW(v), \quad v \in [z, t_{N-1}], \\
&X_{N-1,i'}(z) = x, \quad \alpha(z) = i'.
\end{aligned}
\]

If the following system of PDEs (suppressing \((s, x, j')\) for \(\hat{V}_{z}^{N-1,j'}(\cdot)\) and \(\hat{V}_{xx}^{N-1,j'}(\cdot)\)) admits a unique classical solution \(\hat{V}_{z}^{N-1,j'}(\cdot, \cdot, \cdot)\), then the pre-commitment optimal strategy is given by

\[
\begin{aligned}
\hat{u}_{N-1,i'}(s) & \equiv \psi\left(t_{N-2}, i', s, \hat{X}_{z}^{N-1,i'}(s), \alpha(s), \hat{V}_{z}^{N-1,i'}(s, \hat{X}_{z}^{N-1,i'}(s), \alpha(s))\right), \\
\hat{V}_{xx}^{N-1,i'}(s, \hat{X}_{z}^{N-1,i'}(s), \alpha(s)) & \equiv \hat{V}_{z}^{N-1,i'}(s, \hat{X}_{z}^{N-1,i'}(s), \alpha(s)),
\end{aligned}
\]

for \(s \in [z, t_{N-1}]\), where \(\hat{X}_{z}^{N-1,i'}(\cdot) \equiv X_{N-1,i'}(\cdot; z, x, i', \hat{u}_{N-1,i'}(\cdot))\). Furthermore,

\[
\hat{V}_{z}^{N-1,i'}(z, x, i') = \hat{j}^{N-1,i'}(z, x, i'; \hat{u}_{N-1,i'}(\cdot)).
\]
Given the optimal pair \((\tilde{X}^{N-1,i'}, \tilde{u}^{N-1,i'}(\cdot))\), we can define the cost of the player with preference-\((\tau, i)\): for \(\tau \in [0, s], s \in [z, t_{N-1}], i, j' \in \mathcal{A} \) and \(x \in \mathbb{R}^n\),

\[
\Theta^{N-1,i'}(\tau, i, s, x, j') = \mathbb{E}_s \left[ \int_{\tau}^{t_{N-1}} g(\tau, i, \nu, \tilde{X}^{N-1,i'}(\nu), \alpha(\nu), \tilde{u}^{N-1,i'}(\nu)) \, d\nu + \Theta^N(\tau, i, t_{N-1}, \tilde{X}^{N-1,i'}(t_{N-1}), \alpha(t_{N-1})) \right],
\]

which is given by the classical solution (if it exists) to

\[
\begin{align*}
\hat{\Theta}^{N-1,i'}(\tau, i, s, x, j') &= \mathbb{E}_s \left[ \int_{\tau}^{t_{N-1}} g(\tau, i, \nu, \tilde{X}^{N-1,i'}(\nu), \alpha(\nu), \tilde{u}^{N-1,i'}(\nu)) \, d\nu + \Theta^N(\tau, i, t_{N-1}, \tilde{X}^{N-1,i'}(t_{N-1}), \alpha(t_{N-1})) \right], \\
+ \Theta^N(\tau, i, t_{N-1}, \tilde{X}^{N-1,i'}(t_{N-1}), \alpha(t_{N-1}))
\end{align*}
\]

where we have suppressed \((s, x, j')\) for \(\tilde{V}^{N-1,i'}(\cdot)\) and \(\tilde{V}^{N-1,i'}(\cdot)\).

Given \(\mathcal{F}_{\xi_{N-1}}\), the optimization problem for Player \((N-1)\) is given by (13-14) with \((z, x, i')\) replaced by \((\xi_{N-1}, X^{N-1,i}(\xi_{N-1}), \alpha(\xi_{N-1}))\), and she/he seeks the pre-commitment optimal strategy. Therefore, conditioned on \(\mathcal{F}_{\xi_{N-1}}\), the cost of Player \((N-1)\) on \([\xi_{N-1}, t_{N-1}]\) is \(\Theta^{N-1,0}(\xi_{N-1}) (t_{N-2}, i, t_{N-1}, X^{N-1,i}(\xi_{N-1}), \alpha(\xi_{N-1}))\).

Thus, the cost of Player \((N-1)\) on \([t_{N-2}, t_{N}]\) is

\[
\mathbb{E}_{t_{N-2}} \left[ \int_{t_{N-2}}^{\xi_{N-1}} g(t_{N-2}, i, s, X^{N-1,i}(s), i, u^{N-1,i}(s)) \, ds + \Theta^{N-1,0}(\xi_{N-1}) (t_{N-2}, i, \xi_{N-1}, X^{N-1,i}(\xi_{N-1}), \alpha(\xi_{N-1})) \right]
\]

\[
= \mathbb{E}_{t_{N-2}} \left\{ 1_{\xi_{N-1}=t_{N-1}} \left[ \int_{t_{N-2}}^{t_{N-1}} g(t_{N-2}, i, s, X^{N-1,i}(s), i, u^{N-1,i}(s)) \, ds + \Theta^N(t_{N-2}, i, t_{N-1}, X^{N-1,i}(t_{N-1}), i) \right] \\
+ 1_{\xi_{N-1}<t_{N-1}} \left[ \int_{t_{N-2}}^{\xi_{N-1}} g(t_{N-2}, i, s, X^{N-1,i}(s), i, u^{N-1,i}(s)) \, ds + \hat{\Theta}^{N-1,0}(\xi_{N-1}) (t_{N-2}, i, \xi_{N-1}, X^{N-1,i}(\xi_{N-1}), \alpha(\xi_{N-1})) \right] \right\}
\]

\[
= \mathbb{E}_{t_{N-2}} \left[ \int_{t_{N-2}}^{t_{N-1}} e^{-q_i(s-t_{N-2})} g(t_{N-2}, i, s, X^{N-1,i}(s), i, u^{N-1,i}(s)) \, ds \\
+ \int_{t_{N-2}}^{t_{N-1}} e^{-q_i(s-t_{N-2})} \sum_{i' \neq i} q_{ii'} \hat{\Theta}^{N-1,i'}(t_{N-2}, i, s, X^{N-1,i}(s), i') \, ds \right. \\
\left. + e^{-q_i \pi_{N-1}} \Theta^N(t_{N-2}, i, t_{N-1}, X^{N-1,i}(t_{N-1}), i) \right].
\]

Therefore, we have the following optimization problem for Player \((N-1)\).

**Problem \((C_{N-1})\).** Given \(\alpha(t_{N-2}) = i\), for any initial state \((t, x) \in [t_{N-2}, t_{N-1}] \times \mathbb{R}^n\), find a \(u^{N-1,i}(\cdot) \in \mathcal{U}[t, t_{N-1}]\) such that

\[
V^{N-1,i}(t, x) \equiv J^{N-1,i}(t, x; u^{N-1,i}(\cdot)) \equiv \inf_{u^{N-1,i}(\cdot) \in \mathcal{U}[t, t_{N-1}]} J^{N-1,i}(t, x; u^{N-1,i}(\cdot)),
\]
where
\[ J^{N-1,i}(t,x; u^{N-1,i}(\cdot)) \]

\[ = \mathbb{E}_t \left[ \int_t^{t_N-1} e^{-q(s-t)} g(t_{N-2}, i, s, X^{N-1,i}(s), i, u^{N-1,i}(s)) \, ds \right. \]

\[ + \frac{1}{N} \sum_{i' \neq i} \mathbb{E}_t \left[ \int_t^{t_N-1} e^{-q(s-t)} \sum_{i' \neq i} q_{ii'} \Theta^{N-1,i'}(t_{N-2}, i, s, X^{N-1,i}(s), i') \, ds \right] \]

\[ + e^{-q(t_{N-1}-t)} \Theta^N(t_{N-2}, i, t_{N-1}, X^{N-1,i}(t_{N-1}), i) \]

and \( X^{N-1,i}(\cdot) \) is given by (12) with \( t_{N-1} \) replaced by \( t_{N-1} \).

If the PDE
\[
\begin{cases}
V^{N-1,i}(t,x) = \Theta^N(t_{N-2}, i, t_{N-1}, x, i), & x \in \mathbb{R}^n, \\
+ \langle b(t, x, i, \psi(t_{N-2}, i, x, i, V^N, i^1), V^N, i^2, \psi(t_{N-2}, i, x, i, V^N, i^1), V^N, i^2) \rangle \}
\end{cases}
\]

admits a unique classical solution, then the optimal strategy of Player \((N - 1)\) is given by
\[ \bar{u}^{N-1,i}(s) \equiv \psi(t_{N-2}, i, s, \bar{X}^{N-1,i}(s), i, V^N, i^1(s)), \]
for \( s \in [t_{N-2}, t_{N-1}] \), where \( \bar{X}^{N-1,i}(\cdot) \equiv \bar{X}^{N-1,i}(\cdot; t, x, \bar{u}^{N-1,i}(\cdot)) \).

Given the optimal pair \( (\bar{X}^{N-1,i}(\cdot), \bar{u}^{N-1,i}(\cdot)) \) of Player \((N - 1)\), we define the function
\[ \Theta^{N-1}(\tau, j, t, x, i) \equiv \mathbb{E}_t \left[ \int_t^{t_N-1} e^{-q(s-t)} g(\tau, j, s, \bar{X}^{N-1,i}(s), i, \bar{u}^{N-1,i}(s)) \, ds \right. \]

\[ + \frac{1}{N} \sum_{i' \neq i} \mathbb{E}_t \left[ \int_t^{t_N-1} e^{-q(s-t)} \sum_{i' \neq i} q_{ii'} \Theta^{N-1,i'}(\tau, j, s, \bar{X}^{N-1,i}(s), i') \, ds \right] \]

\[ + e^{-q(t_{N-1}-t)} \Theta^N(\tau, j, t_{N-1}, \bar{X}^{N-1,i}(t_{N-1}), i), \]

\[ \tau \in [0, t], t \in [t_{N-2}, t_{N-1}], i, j \in \mathcal{A}, x \in \mathbb{R}^n. \]

It can be obtained by the classical solution (if it exists) to the PDE
\[
\begin{cases}
\Theta^N(t_{N-2}, i, t_{N-1}, x, i) = \Theta^N(t_{N-2}, i, t_{N-1}, x, i), & \tau \in [0, t_{N-1}], i, j \in \mathcal{A}, x \in \mathbb{R}^n.
\end{cases}
\]
3.2.3. Player \((k)\). By backward induction, we consider the optimization problems for Player \((k)\), \(k = 1, 2, \ldots, N - 2\).

Given \(\alpha(t_{k-1}) = i\), the state process of Player \((k)\) is

\[
\begin{align*}
\mathrm{d}X^{k,i}(s) &= b(s, X^{k,i}(s), i, u^{k,i}(s)) \, \mathrm{d}s + \sigma(s, X^{k,i}(s), i, u^{k,i}(s)) \, \mathrm{d}W(s), \\
X^{k,i}(t_{k-1}) &= x.
\end{align*}
\]

(16)

Player \((\tilde{k})\) controls the system on \([\xi_k, t_k]\). We consider an optimization problem starting from some fixed \(z \in [t_{k-1}, t_k]\). Given \(\alpha(z) = i'\), the objective is to find a strategy \(\tilde{u}^{k,i'}(\cdot)\) such that

\[
\tilde{j}^{k,i'}(z, x, i'; \tilde{u}^{k,i'}(\cdot)) \equiv \inf_{u^{k,i'}(\cdot) \in U(z, t_k)} \tilde{j}^{k,i'}(z, x, i'; u^{k,i'}(\cdot)),
\]

where

\[
\tilde{j}^{k,i'}(z, x, i'; u^{k,i'}(\cdot)) = E_z \left[ \int_z^{t_k} g(t_{k-1}, i', s, X^{k,i'}(s), \alpha(s), u^{k,i'}(s)) \, \mathrm{d}s + \Theta^{k+1}(t_{k-1}, i', t_k, X^{k,i'}(t_k), \alpha(t_k)) \right],
\]

(17)

with \(X^{k,i'}(\cdot) \equiv X^{k,i'}(z, x, i', u^{k,i'}(\cdot))\) being the unique solution to the SDE

\[
\begin{align*}
\mathrm{d}X^{k,i'}(v) &= b(v, X^{k,i'}(v), \alpha(v), u^{k,i'}(v)) \, \mathrm{d}v \\
&\quad + \sigma(v, X^{k,i'}(v), \alpha(v), u^{k,i'}(v)) \, \mathrm{d}W(v), \quad v \in [z, t_k], \\
X^{k,i'}(z) &= x, \quad \alpha(z) = i'.
\end{align*}
\]

If the system of PDEs

\[
\begin{align*}
\tilde{V}_t^{k,i'}(s, x, j') &= b(s, x, j', \psi(t_{k-1}, i', s, x, j', \tilde{V}^{k,i'}_{xx}(s, x, j'), \tilde{V}^{k,i'}(s, x, j'))) + \tilde{V}^{k,i'}_{xx}(s, x, j') \\
&\quad + \mathrm{tr} \left\{ \sum(s, x, j', \psi(t_{k-1}, i', s, x, j', \tilde{V}^{k,i'}_{xx}(s, x, j'), \tilde{V}^{k,i'}(s, x, j'))) \tilde{V}^{k,i'}_{xx}(s, x, j') \right\} \\
&\quad + g(t_{k-1}, i', s, x, j', \psi(t_{k-1}, i', s, x, j', \tilde{V}^{k,i'}_{xx}(s, x, j'), \tilde{V}^{k,i'}(s, x, j'))) \\
&\quad + \sum_{j'' \in A} q_{j'j''} \tilde{V}^{k,i'}(s, x, j'') = 0, \quad (s, x, j') \in [z, t_k] \times \mathbb{R}^n \times A,
\end{align*}
\]

admits a unique classical solution, then the pre-commitment strategy for the optimization problem (17-19) is given by

\[
\tilde{u}^{k,i'}(s) \equiv \psi(t_{k-1}, i', s, \tilde{X}^{k,i'}(s), \alpha(s), \tilde{V}^{k,i'}_{xx}(s, \tilde{X}^{k,i'}(s), \alpha(s))),
\]

\[
\tilde{V}^{k,i'}_{xx}(s, \tilde{X}^{k,i'}(s), \alpha(s)),
\]

for \(s \in [z, t_k]\), where \(\tilde{X}^{k,i'}(\cdot) \equiv X^{k,i'}(z, x, i', \tilde{u}^{k,i'}(\cdot))\). Furthermore, \(\tilde{V}^{k,i'}(z, x, i') = \tilde{j}^{k,i'}(z, x, i'; \tilde{u}^{k,i'}(\cdot))\).

For \(\tau \in [0, s], s \in [z, t_k], i', j' \in A, x \in \mathbb{R}^n\), define the function

\[
\tilde{\Theta}^{k,i'}(\tau, i, s, x, j') = E_s \left[ \int_{\tau}^{t_k} g(\tau, i, \nu, \tilde{X}^{k,i'}(\nu), \alpha(\nu), \tilde{u}^{k,i'}(\nu)) \, \mathrm{d}\nu \\
&\quad + \Theta^{k+1}(\tau, i, t_k, \tilde{X}^{k,i'}(t_k), \alpha(t_k)) \right],
\]
which is given by the classical solution (if it exists) to
\[
\begin{align*}
\hat{\Theta}_{i,j}^k (\tau, i, s, x, j') \\
+ \left\{ b \left( s, x, j', \psi \left( t_{k-1}, i', s, x, j', \bar{V}_{i,j}^k (s, x, j'), \bar{V}_{i,j}^{k'} (s, x, j') \right) \right), \hat{\Theta}_{i,j}^k (\tau, i, s, x, j') \right\} \\
+ \text{tr} \left( \Sigma \left( s, x, j', \psi \left( t_{k-1}, i', s, x, j', \bar{V}_{i,j}^k (s, x, j'), \bar{V}_{i,j}^{k'} (s, x, j') \right) \right) \right) \hat{\Theta}_{i,j}^k (\tau, i, s, x, j') \\
+ g \left( \tau, i, s, x, j', \psi \left( t_{k-1}, i', s, x, j', \bar{V}_{i,j}^k (s, x, j'), \bar{V}_{i,j}^{k'} (s, x, j') \right) \right) \hat{\Theta}_{i,j}^k (\tau, i, s, x, j') \\
+ \sum_{i'' \in \Theta} q_{i,j,i''} \hat{\Theta}_{j,i''}^k (\tau, i, s, x, j') = 0, \quad \tau \in [0, s], \, s \in [z, t_k], \, i, j' \in \mathbb{A}, \, x \in \mathbb{R}^n,
\end{align*}
\]

Given $F_{\xi_k}$, the optimization problem for Player ($k$) is given by (17-19) with $(z, x, i')$ replaced by $(\xi_k, X^{k,i}(\xi_k), \alpha(\xi_k))$, and she/he seeks the pre-commitment optimal strategy. Therefore, conditioned on $F_{\xi_k}$, the cost of Player $(k)$ on $[t_{k-1}, t_N]$ is $\hat{\Theta}_{i,j}^{k,\alpha(\xi_k)} (t_{k-1}, i, \xi_k, X^{k,i}(\xi_k), \alpha(\xi_k))$.

Thus, the cost of Player $(k)$ on $[t_{k-1}, t_N]$ is
\[
E_{t_{k-1}} \left[ \int_{t_{k-1}}^{\xi_k} g \left( t_{k-1}, i, s, X^{k,i}(s), i, u^{k,i}(s) \right) ds \right]
\]
\[
+ \hat{\Theta}_{i,j}^{k,\alpha(\xi_k)} (t_{k-1}, i, \xi_k, X^{k,i}(\xi_k), \alpha(\xi_k))
\]
\[
= E_{t_{k-1}} \left\{ 1_{\{\xi_k = t_k\}} \left[ \int_{t_{k-1}}^{t_k} g \left( t_{k-1}, i, s, X^{k,i}(s), i, u^{k,i}(s) \right) ds \right] \right. \\
+ \hat{\Theta}_{i,j}^{k+1} (t_{k-1}, i, t_k, X^{k,i}(t_k), i) \right\} \left[ \int_{t_{k-1}}^{\xi_k} g \left( t_{k-1}, i, s, X^{k,i}(s), i, u^{k,i}(s) \right) ds \right]
\]
\[
+ \hat{\Theta}_{i,j}^{k,\alpha(\xi_k)} (t_{k-1}, i, \xi_k, X^{k,i}(\xi_k), \alpha(\xi_k)) \right\}
\]
\[
= E_{t_{k-1}} \left[ \int_{t_{k-1}}^{t_k} e^{-q_i(s-t_{k-1})} g \left( t_{k-1}, i, s, X^{k,i}(s), i, u^{k,i}(s) \right) ds \right] \\
+ \int_{t_{k-1}}^{t_k} e^{-q_i(s-t_{k-1})} \sum_{i' \neq i} q_{i,i'} \hat{\Theta}_{i,j}^{k,i'} (t_{k-1}, i, s, X^{k,i}(s), i') ds \\
+ e^{-q_i(t_k-t_{k-1})} \hat{\Theta}_{i,j}^{k+1} (t_{k-1}, i, t_k, X^{k,i}(t_k), i) \right].
\]

Thus, we have the following optimization problem for Player $(k)$.

**Problem (C_k).** Given $\alpha(t_{k-1}) = i$, for any initial state $(t, x) \in [t_{k-1}, t_k] \times \mathbb{R}^n$, find a $u^{k,i}() \in \mathcal{U}[t, t_k]$ such that
\[
V_{k,i}(t, x) \equiv J^{k,i} (t, x; u^{k,i}()) \equiv \inf_{u^{k,i}() \in \mathcal{U}[t, t_k]} J^{k,i} (t, x; u^{k,i}()),
\]
where
\[
J^{k,i} (t, x; u^{k,i}()) \equiv E_t \left[ \int_t^{t_k} e^{-q_i(s-t)} g \left( t_{k-1}, i, s, X^{k,i}(s), i, u^{k,i}(s) \right) ds \right]
\]
\[
+ \int_t^{t_k} e^{-q_i(s-t)} \sum_{i' \neq i} q_{i,i'} \hat{\Theta}_{i,j}^{k,i'} (t_{k-1}, i, s, X^{k,i}(s), i') ds \\
+ e^{-q_i(t_k-t)} \hat{\Theta}_{i,j}^{k+1} (t_{k-1}, i, t_k, X^{k,i}(t_k), i) \right]
\]
and $X^{k,i}()$ is given by (16) with $\xi_k$ replaced by $t_k$. 
If the PDE
$$
\begin{align*}
V^{k,i}(t, x) &= \left\{ \begin{array}{l}
+ \langle b(t, x, i, \psi (t_{k-1}, i, t, x, i, V_x^{k,i}(t, x), V_{xx}^{k,i}(t, x))), V^{k,i}(t, x) \rangle \\
+ \text{tr} \left[ \sum (t, x, i, \psi (t_{k-1}, i, t, x, i, V_x^{k,i}(t, x), V_{xx}^{k,i}(t, x))), V_{xx}^{k,i}(t, x) \right] \\
+ g(t_{k-1}, i, t, x, i, \psi (t_{k-1}, i, t, x, i, V_x^{k,i}(t, x), V_{xx}^{k,i}(t, x))) \\
+ \sum_{i' \neq i} q_{ii'} \tilde{\Theta}^{k,i'}(t_{k-1}, i, t, x, i') - q_i V^{k,i}(t, x) = 0, \\
(t, x) \in [t_{k-1}, t_k] \times \mathbb{R}^n, \\
V^{k,i}(t_k, x) = \Theta^{k+1}(t_{k-1}, i, t_k, x, i), \quad x \in \mathbb{R}^n
\end{array} \right. \\
(21)
\end{align*}
$$

has a unique classical solution, then the optimal strategy of Player (k) is given by
$$
\bar{u}^{k,i}(s) = \psi (t_{k-1}, i, s, \bar{X}^{k,i}(s), i, V_x^{k,i}(s), V_{xx}^{k,i}(s)), \ V_x^{k,i}(s), \bar{X}^{k,i}(s)),
$$
for \( s \in [t_{k-1}, t_k] \), where \( \bar{X}^{k,i}(\cdot) \equiv X^{k,i}(\cdot, t, x, \bar{u}^{k,i}(\cdot)). \)

Finally, for \( \tau \in [0, t], t \in [t_{k-1}, t_k], i, j \in \mathbb{A}, x \in \mathbb{R}^n \), we define the function
$$
\Theta^k(\tau, j, t, x, i) = E^j \left[ \int_t^{t_k} e^{-q(s-t)} g(\tau, j, s, \bar{X}^{k,i}(s), i, \bar{u}^{k,i}(s)) ds \\
+ \int_t^{t_k} e^{-q(s-t)} \sum_{i' \neq i} q_{ii'} \tilde{\Theta}^{k,i'}(\tau, j, s, \bar{X}^{k,i}(s), i') ds \\
+ e^{-q(t_k-t)} \Theta^{k+1}(\tau, j, t_k, \bar{X}^{k,i}(t_k), i) \right].
$$

It is given by the classical solution (if it exists) to the PDE
$$
\begin{align*}
\Theta^k(\tau, j, t, x, i) &= \left\{ \begin{array}{l}
+ \langle b(t, x, i, \psi (t_{k-1}, i, t, x, i, V_x^{k,i}(t, x), V_{xx}^{k,i}(t, x))), \Theta^k(\tau, j, t, x, i) \rangle \\
+ \text{tr} \left[ \sum (t, x, i, \psi (t_{k-1}, i, t, x, i, V_x^{k,i}(t, x), V_{xx}^{k,i}(t, x))), \Theta^k_{xx}(\tau, j, t, x, i) \right] \\
+ g(\tau, j, t, x, i, \psi (t_{k-1}, i, t, x, i, V_x^{k,i}(t, x), V_{xx}^{k,i}(t, x))) \Theta^k_{xx}(\tau, j, t, x, i) \\
+ \sum_{i' \neq i} q_{ii'} \tilde{\Theta}^{k,i'}(\tau, j, t, x, i') - q_i \Theta^k(\tau, j, t, x, i) = 0, \\
\tau \in [0, t], t \in [t_{k-1}, t_k], j, i \in \mathbb{A}, x \in \mathbb{R}^n,
\end{array} \right. \\
(22)
\end{align*}
$$

3.3. Equilibrium HJB equation. This subsection is devoted to find the equation that can be used to characterize the time-consistent equilibrium strategy and equilibrium value function in the sense of Definition 3.1.

Define
$$
\begin{align*}
V^H(t, x, i) &= \sum_{k=1}^{N-1} V^{k,i}(t, x) 1_{[t_k, t_{k+1})}(t), \\
\Theta^H(\tau, j, t, x, i) &= \sum_{k=1}^{N-1} \Theta^{k+1}(\tau, j, t, x, i) 1_{[t_k, t_{k+1} \land \tau)}(\tau), \\
\tilde{V}^H(t, x, i) &= \sum_{k=1}^{N-1} \tilde{V}^{k,i}(t, x, i) 1_{[t_k, t_{k+1})}(t), \\
\tilde{\Theta}^H(t, x, i) &= \sum_{k=1}^{N-1} \tilde{\Theta}^{k+1,i}(\tau, j, t, x, i) 1_{[t_k, t_{k+1} \land \tau)}(\tau).
\end{align*}
$$
It is easy to check
\[
\Theta^\Pi(t, j, t, x, i) = \begin{cases} 
\langle b (t, x, i, \psi (l_t(t), i, t, x, i, V^\Pi_x(t, x, i)), V^\Pi_x(t, x, i)) \rangle , 
\Theta^\Pi(t, j, t, x, i) 
+ \langle \sum (s, x, i, \psi (l_t(t), i, t, x, i, V^\Pi_x(t, x, i)), V^\Pi_x(t, x, i)) \rangle \Theta^\Pi(t, j, t, x, i) 
+ g \left( \tau, j, s, x, i, \psi (l_t(t), i, t, x, i, V^\Pi_x(t, x, i)) \right) \Theta^\Pi(t, j, t, x, i) 
+ q_i \sum (s, x, i, \psi (l_t(t), i, t, x, i, V^\Pi_x(t, x, i)) \rangle \Theta^\Pi(t, j, t, x, i) 
\sum_{i' \neq i} q_i \Theta^\Pi(t, j, t, x, i) - q_i \Theta^\Pi(t, j, t, x, i) = 0,
(t, t) \in D[0, T], j, i \in \mathbb{A}, x \in \mathbb{R}^n,
\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qanda;
where \( \tilde{\psi}^n(s, x, i) \equiv \psi \left( \ln(s), i, s, x, i, \tilde{\Theta}^n(s, i, s, x, i), \tilde{\Theta}^n_{xx}(s, i, s, x, i) \right) \).

Note that for \( t \in [0, t], t \in [t_k, t_k], i, j \in \mathcal{A}, x \in \mathbb{R}^n, \)

\[
\Theta^n_\tau(\tau, j, t, x, i) = \int_{t_k}^{t_i} \left\{ \left\{ b(s, x, i, \psi^n(s, x, i)) \right\}, \Theta^n_{xx}(\tau, j, s, x, i) \right\} \xrightarrow{\tau \to 0} \Theta^n_\tau(\tau, j, t, x, i).
\]

Thus, for \( t \in [t_k, t_k], \tilde{\Theta}^{n, i}(\tau, j, t, x, i) \) and \( \Theta^n(\tau, j, t, x, i) \) have the same terminal values. Therefore, if all the coefficients are bounded and the equation is uniformly elliptic, then

\[
\left| \tilde{\Theta}^{n, i}(\tau, j, t, x, i) \right| \leq \left( \tilde{\Theta}^{n, i}(\tau, j, t, x, i) \right) - \tilde{\Theta}^{n, i}(\tau, j, t, x, i) \int_{t_k}^{t_i} \left\{ \left\{ b(s, x, i, \psi^n(s, x, i)) \right\}, \Theta^n_{xx}(\tau, j, s, x, i) \right\} \xrightarrow{\tau \to 0} \Theta^n_\tau(\tau, j, t, x, i).
\]

Letting \( \| \Pi \| \to 0 \) in (26), we get

\[
\lim_{\| \Pi \| \to 0} \left( \left| \tilde{\Theta}^{n, i}(\tau, j, t, x, i) \right| \right) = 0,
\]

uniformly for \( (\tau, t) \in D(0, T], i, j \in \mathcal{A}, x \in \mathbb{R}^n \).

Assume that there exists a constant \( K > 0 \) such that

\[
|g_x(\tau, j, t, x, i, u)| + |h_x(\tau, j, t, x, i)| \leq K, \forall (\tau, t) \in D(0, T], i, j \in \mathcal{A}, (x, u) \in \mathbb{R} \times U.
\]

By backward induction, it is easy to see that \( \Theta^n(\tau, j, t, x, i), \Theta^n_{xx}(\tau, j, t, x, i) \) are continuous with respective to \( \tau \).

Now, assume that there exists a function \( \Theta(\cdot, \cdot, \cdot, \cdot, \cdot) \) such that

\[
\lim_{\| \Pi \| \to 0} \left( \left| \Theta^n(\tau, j, t, x, i) \right| \right) = 0,
\]

uniformly any \( i, j \in \mathcal{A}, (\tau, t) \in D(0, T] \) and \( x \in \mathbb{R}^n \).

Letting \( \| \Pi \| \to 0 \) in (26), we get

\[
\Theta(\tau, j, t, x, i)
\]

\[
= \int_{t_k}^{t_i} \left\{ \left\{ b(s, x, i, \psi(s, i, s, x, i, \Theta_x(s, i, s, x, i), \Theta_{xxx}(s, i, s, x, i)) \right\}, \Theta_x(\tau, j, s, x, i) \right\} \xrightarrow{\tau \to 0} \Theta^n_\tau(\tau, j, t, x, i).
\]

Therefore, if all the coefficients are bounded and the equation is uniformly elliptic, then

\[
\Theta^n(\tau, j, t, x, i) \leq \Theta^n_\tau(\tau, j, t, x, i)
\]

uniformly for \( (\tau, t) \in D(0, T], i, j \in \mathcal{A}, x \in \mathbb{R}^n \), where \( \Theta^{N+1}(\tau, j, t, x, i) \equiv h(\tau, j, x, i) \).

Similarly, if \( h(\tau, j, x, i) \in C^2(\mathbb{R}^n) \), then we have

\[
\lim_{\| \Pi \| \to 0} \left( \left| \Theta^n(\tau, j, t, x, i) \right| \right) = 0,
\]

uniformly for \( (\tau, t) \in D(0, T], i, j \in \mathcal{A}, x \in \mathbb{R}^n \).

Assume that there exists a function \( \Theta(\cdot, \cdot, \cdot, \cdot, \cdot) \) such that

\[
\lim_{\| \Pi \| \to 0} \left( \left| \Theta^n(\tau, j, t, x, i) \right| \right) = 0,
\]

uniformly any \( i, j \in \mathcal{A}, (\tau, t) \in D(0, T] \) and \( x \in \mathbb{R}^n \).

Letting \( \| \Pi \| \to 0 \) in (26), we get

\[
\Theta(\tau, j, t, x, i)
\]

\[
= \int_{t_k}^{t_i} \left\{ \left\{ b(s, x, i, \psi(s, i, s, x, i, \Theta_x(s, i, s, x, i), \Theta_{xx}(s, i, s, x, i)) \right\), \Theta_x(\tau, j, s, x, i) \right\} \xrightarrow{\tau \to 0} \Theta^n_\tau(\tau, j, t, x, i).
\]
which is the integral form of the following differential equation:

\[
\begin{align*}
&\left\{ \Theta_t(\tau, j, t, x, i) \\
&\quad + \begin{pmatrix}
 b(t, x, i, \psi(t, i, t, x, i, \Theta_x(t, i, t, x, i), \Theta_{xx}(t, i, t, x, i)), \\
 \Theta_x(\tau, j, t, x, i), \\
 \Theta_{xx}(\tau, j, t, x, i)
\end{pmatrix}
\right. \\
&\quad + g(\tau, j, t, x, i, \psi(t, i, t, x, i, \Theta_x(t, i, t, x, i), \Theta_{xx}(t, i, t, x, i)))) \\
&\quad + \sum_{i' \in \mathcal{A}} q_{ii'} \Theta(\tau, j, t, x, i') \\
&\quad \in \mathcal{A}, \\
&\quad \Theta(\tau, j, T, x, i) = h(\tau, j, x, i), \\
&\quad \tau \in [0, T], j, i \in \mathcal{A}, x \in \mathbb{R}^n,
\end{align*}
\]

(32)

Similar to [22], we call (32) the equilibrium HJB equation. In the following, we show that if (32) has a unique classical solution, then we can obtain a time-consistent equilibrium strategy and the corresponding equilibrium value function.

Let \( V(t, x, i) = \Theta(t, i, t, x, i) \). Then by (25), (27) and (29-31), it holds that

\[
\begin{align*}
\lim_{||\Pi|| \to 0} \left( ||V^\Pi(t, x, i) - V(t, x, i)|| + |V_x^\Pi(t, x, i) - V_x(t, x, i)| \right) \\
&\quad + |V_{xx}^\Pi(t, x, i) - V_{xx}(t, x, i)| = 0, \quad (\tau, t) \in D[0, T], j, i \in \mathcal{A}, x \in \mathbb{R}^n,
\end{align*}
\]

(33)

uniformly for any \( i \in \mathcal{A} \) and \((t, x)\) in any compact sets. For \((t, x, i, j) \in [0, T] \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{A}\), define the functions

\[
\begin{align*}
\Psi(t, x, i) &= \psi(t, i, t, x, i, V_x(t, x, i), V_{xx}(t, x, i)), \\
\Psi^\Pi(t, x, i) &= \psi(\Pi(t), i, t, x, i, V_x^\Pi(t, x, i), V_{xx}^\Pi(t, x, i)), \\
\hat{\Psi}^\Pi(t, x, i) &= \psi(\Pi(t), j, t, x, i, V_x^\Pi(t, x, i), V_{xx}^\Pi(t, x, i)).
\end{align*}
\]

Then, by Assumption 3.1, (33) and (34), we have

\[
\begin{align*}
\lim_{||\Pi|| \to 0} \left( ||\Psi^\Pi(t, x, i) - \Psi(t, x, i)|| + |\hat{\Psi}^\Pi(t, x, i) - \Psi(t, x, i)| \right) = 0, \quad (\tau, t) \in D[0, T], j, i \in \mathcal{A}, x \in \mathbb{R}^n.
\end{align*}
\]

(35)

uniformly for any \( i \in \mathcal{A} \) and \((t, x)\) in any compact sets.

In the following, for simplicity, we assume that \( b(\cdot) \) and \( \sigma(\cdot) \) are Lipschitz continuous with respect to \( x \) and \( u \), \( \Psi(\cdot) \), \( \Psi^\Pi(\cdot) \) and \( \hat{\Psi}^\Pi(\cdot) \) are Lipschitz continuous with respect to \( x \), and there exists a constant \( L > 0 \) independent of \( \Pi \), such that

\[
|\Psi(t, x, i)| + |\Psi^\Pi(t, x, i)| + |\hat{\Psi}^\Pi(t, x, i)| \leq L(1 + |x|).
\]

Let \( \tilde{X}(\cdot) \) and \( X^\Pi(\cdot) \) be the solutions to the SDEs

\[
\begin{align*}
&\left\{ \begin{array}{ll}
 d\tilde{X}(s) = b(s, \tilde{X}(s), \alpha(s), \bar{u}(s)) \, ds + \sigma(s, \tilde{X}(s), \alpha(s), \bar{u}(s)) \, dW(s), s \in [0, T], \\
 \tilde{X}(0) = x, \quad \alpha(0) = i,
\end{array} \right.
\end{align*}
\]

(36)
and
\[
\begin{align*}
\begin{cases}
    dX^\Pi(s) = b(s, X^\Pi(s), \alpha(s), u^\Pi(s)) \, ds + \sigma(s, X^\Pi(s), \alpha(s), u^\Pi(s)) \, dW(s), \\
    X^\Pi(0) = x, \quad \alpha(0) = i,
\end{cases}
\end{align*}
\]  

respectively, where \( \bar{u}(s) = \Psi(s, \bar{X}(s), \alpha(s)) \), and
\[
u^\Pi(s) = \sum_{k=1}^N \left[ \Psi^\Pi(s, X^\Pi(s), \alpha(s)) \mathbf{1}_{[t_{k-1}, t_k)}(s) \\
+ \bar{\Psi}^\Pi,\alpha(\xi_k)(s, X^\Pi(s), \alpha(s)) \mathbf{1}_{[\xi_k, t_k)}(s) \right], \quad s \in [0, T].
\]

Under Assumptions 2.1 and 3.1, one can get
\[
\|X(\cdot)\|_{L^2_\mathcal{F}(\Omega; C([0, T]; \mathbb{R}^n))} \leq K \left(1 + |x|^2\right),
\]
where
\[
L^2_\mathcal{F}(\Omega; C([0, T]; \mathbb{R}^n)) = \left\{ X : [0, T] \times \Omega \to \mathbb{R}^n \mid X(\cdot) \text{ has continuous paths, such that } \mathbb{E}\left[ \sup_{t \in [0, T]} |X(t)|^2 \right] < \infty \right\}.
\]

Recalling (35) and Assumption 3.1, if \( \|\Pi\| \) is small enough, we also have
\[
\|X^\Pi(\cdot)\|_{L^2_\mathcal{F}(\Omega; C([0, T]; \mathbb{R}^n))} \leq K \left(1 + |x|^2\right),
\]
where the constant \( K > 0 \) is independent of \( \Pi \).

Similarly, we have \( \|\bar{u}(\cdot)\|_{L^2_\mathcal{F}[0, T]}, \|u^\Pi(\cdot)\|_{L^2_\mathcal{F}[0, T]} \leq K \), where \( K \) is a constant independent of \( \Pi \) (but depends on \( T \) and \( x \)), and
\[
\mathcal{U}^2[t, T] = \left\{ u : [t, T] \times \Omega \to U \mid u(\cdot) \text{ is } \mathcal{F}\text{-progressively measurable such that } \mathbb{E}\left[ \int_t^T |u(s)|^2 \, ds \right] < \infty \right\}.
\]

Noting that the for any \( k = 1, \cdots, N \), the probability of the Markov chain jumping two or more times in \([t_{k-1}, t_k]\) is \( o(\|\Pi\|) \), by the similar arguments as in [15, Chapter V], we can show that
\[
\lim_{\|\Pi\| \to 0} \left\| X^\Pi(\cdot) - \bar{X}(\cdot) \right\|_{L^2_\mathcal{F}(\Omega; C([0, T]; \mathbb{R}^n))} = 0,
\]
and
\[
\lim_{\|\Pi\| \to 0} \left\| u^\Pi(\cdot) - \bar{u}(\cdot) \right\|_{L^2_\mathcal{F}[0, T]} = 0.
\]

Furthermore, for \( t \in [0, T] \), we have
\[
J \left( \Pi(t), X^\Pi(\Pi(t)), \alpha(\Pi(t)) ; u^\Pi(\cdot) \right) = V^\Pi \left( \Pi(t), X^\Pi(\Pi(t)), \alpha(\Pi(t)) \right).
\]

Thus, passing to the limits, we have (3). Hence, by Definition 3.1, \( \Psi(\cdot, \cdot, \cdot) \) is a time-consistent equilibrium strategy and, \( V(\cdot, \cdot, \cdot) \) is the corresponding equilibrium value function of Problem (N).
Remark 3.2. Recall that in the multi-person game, we consider only the first change of preference-i in each small time interval. This is enough for deriving the equilibrium HJB equation, though there may be many jumps in \([t_{k-1}, t_k], k = 1, 2, \ldots, N\). For example, let us consider the first two changes of preference-1. Let \(\xi_k = \inf \{s > \xi_k: \alpha(s- \neq \alpha(s)\} \wedge t_k\). Given \(\alpha(t_{k-1}) = i\), the first player, denoted by Player \((k1)\), has preference-\((t_{k-1}, i)\) and controls the system from \([t_{k-1}, \xi_k)\); the second player, denoted by Player \((k2)\), has preference-\((t_{k-1}, \alpha(\xi_k))\) and controls the system from \([\xi_k, \xi_k)\); the third player, denoted by Player \((k3)\), has preference-\((t_{k-1}, \alpha(\xi_k))\) and controls the system from \([\xi_k, t_k)\). Note that Player \((k2)\) and Player \((k3)\) play the similar roles of Player \((k)\) and Player \((\bar{k})\), respectively. Thus, by the similar arguments in Subsection 3.2.1, we can show that the total cost of Player \((N1)\) on \([t_{N-1}, t_N)\) is given by

\[
E_{t_{N-1}} \left[ \int_{t_{N-1}}^{t_N} e^{-g_i(s-t_{N-1})} g \left( t_{N-1}, i, s, X^{N,i}(s), i, u^{N,i}(s) \right) ds + \int_{t_{N-1}}^{t_N} e^{-g_i(s-t_{N-1})} \sum_{i' \neq i} q_{ii'} \Theta^N \left( t_{N-1}, i, s, X^{N,i}(s), i' \right) ds + e^{-g_{iiN}} h \left( t_{N-1}, i, X^{N,i}(t_N, i) \right) \right].
\]

(42)

Then we can propose a new Problem \((C_N)\) based on (42), say Problem \((\hat{C}_N)\). Denote by \(V^N, \hat{V}^N, \ddot{V}^N\) the value function of Problem \((C_N)\) and define \(\hat{\Theta}^N, \ddot{\Theta}^N\) similarly to (10). By induction, we can define \(\hat{V}^{k,i}\) and \(\hat{\Theta}^k\), \(k = 1, \ldots, N - 1\). Obviously, \(\hat{V}^{k,i}(t, x)\) satisfies (21) with \(\hat{\Theta}^{k,i'}(t_{k-1}, i, t, x, i')\) and \(\hat{\Theta}^{k+1}(t_{k-1}, i, t, x, i)\) replaced by \(\hat{\Theta}^k(t_{k-1}, i, t, x, i)\) and \(\hat{\Theta}^{k+1}(t_{k-1}, i, t, x, i)\), respectively; \(\hat{\Theta}^k(t, j, t, x, i)\) satisfies (22) with \(V^{k,i}(t, x), \hat{\Theta}^{k,j'}(t, j, t, x, i')\) and \(\hat{\Theta}^{k+1}(t, j, t, x, i)\) replaced by \(\hat{V}^{k,i}(t, x), \hat{\Theta}^k(t, j, t, x, i)\) and \(\hat{\Theta}^{k+1}(t, j, t, x, i)\), respectively. Similarly, we can define \(\hat{\Theta}^\Pi\) and \(\hat{\Theta}^\Pi\). From (29) we can see that the pairs \(\{\hat{\Theta}^\Pi, \hat{\Theta}^\Pi\}\) and \(\{\hat{\Theta}^\Pi, \hat{\Theta}^\Pi\}\) lead to the same equilibrium HJB equation.

4. Well-posedness of the equilibrium HJB equation. In this section, we discuss the well-posedness of the equilibrium HJB equation (32). Similar to (22), we assume that the control does not enter the diffusion of the state equation, i.e.,

\[
\sigma(t, x, i, u) = \sigma(t, x, i), \quad (t, x, i, u) \in [0, T] \times \mathbb{R}^n \times \mathcal{A} \times U.
\]

(43)

In this case the equilibrium HJB equation becomes

\[
\begin{cases}
\Theta^j(t, x) + \text{tr} \left[ \Sigma^j(t, x) \Theta_x^j(t, x) \right] + \left\langle b^j(t, x, \Theta_x^j(t, x)), \Theta^j(t, x) \right\rangle + g^j(t, x, \Theta_x^j(t, x)) + \sum_{i' \in \mathcal{A}} q_{ii'} \Theta^{j'}(t, x) = 0, \\
(t, x) \in D[0, T], j, i \in \mathcal{A}, x \in \mathbb{R}^n,
\end{cases}
\]

\[
\Theta^j(T, x) = h^j(t, x), \quad t \in [0, T], j, i \in \mathcal{A}, x \in \mathbb{R}^n.
\]

(44)

where

\[
\begin{cases}
\Theta^j(t, x, i) = \Theta(t, j, x, i), \quad \Sigma^j(t, x) = \Sigma(t, x) \\
b^j(t, x, p) = b(t, x, i, \psi(t, i, x, i, p)), h^j(t, x) = h(t, j, x, i), \\
g^j(t, x, p) = g(t, j, x, i, \psi(t, i, x, i, p)).
\end{cases}
\]

for \((t, x) \in D[0, T], j, i \in \mathcal{A}, x, p \in \mathbb{R}^n\).
To study (44), we shall need the following notations. Let $C^0(\mathbb{R}^n)$ be the space of all continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that
\[ \|\varphi(\cdot)\|_{C^0(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |\varphi(x)| < \infty, \]
and $C^\alpha(\mathbb{R}^n), 0 < \alpha < 1,$ be the space of all continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that
\[ \|\varphi(\cdot)\|_{C^\alpha(\mathbb{R}^n)} \equiv \|\varphi(\cdot)\|_{C^0(\mathbb{R}^n)} + \|\varphi_\alpha(\cdot)\|_{C^0(\mathbb{R}^n)} + [\varphi]_\alpha < \infty, \]
where
\[ [\varphi]_\alpha = \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}. \]

Furthermore, let $C^{1+\alpha}(\mathbb{R}^n)$ and $C^{2+\alpha}(\mathbb{R}^n)$ be the spaces of all functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that
\[ \|\varphi(\cdot)\|_{C^{1+\alpha}(\mathbb{R}^n)} \equiv \|\varphi(\cdot)\|_{C^0(\mathbb{R}^n)} + \|\varphi(x)(\cdot)\|_{C^0(\mathbb{R}^n)} + [\varphi_x]_\alpha < \infty, \]
and
\[ \|\varphi(\cdot)\|_{C^{2+\alpha}(\mathbb{R}^n)} \equiv \|\varphi(\cdot)\|_{C^0(\mathbb{R}^n)} + \|\varphi(x)(\cdot)\|_{C^0(\mathbb{R}^n)} + \|\varphi_{xx}(\cdot)\|_{C^0(\mathbb{R}^n)} + [\varphi_{xx}]_\alpha < \infty, \]
respectively.

Let $\Pi = \{t_k \mid 0 \leq k \leq N\}$ be a partition of $[0, T]$. Denote by $C^0([t_{k-1}, t_k]; C^\alpha(\mathbb{R}^n))$ be the set of all continuous functions $f : [t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}$ such that for each $t \in [t_{k-1}, t_k], f(t, \cdot) \in C^\alpha(\mathbb{R}^n)$ and
\[ \|f(\cdot, \cdot)\|_{C([t_{k-1}, t_k]; C^\alpha(\mathbb{R}^n))} \equiv \sup_{t \in [t_{k-1}, t_k]} \|f(t, \cdot)\|_{C^\alpha(\mathbb{R}^n)} < \infty. \]

Let $C^0(D[t_{k-1}, t_k]; C^\alpha(\mathbb{R}^n))$ be the set of all continuous functions $f : D[t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}$ such that for each $(\tau, t) \in D[t_{k-1}, t_k], f(\tau, t, \cdot) \in C^\alpha(\mathbb{R}^n)$ and
\[ \|f(\cdot, \cdot, \cdot)\|_{C([D[t_{k-1}, t_k], C^\alpha(\mathbb{R}^n))} \equiv \sup_{(\tau, t) \in D[t_{k-1}, t_k]} \|f(\tau, t, t)\|_{C^\alpha(\mathbb{R}^n)} < \infty, \]
Similarly, we define $C^0([t_{k-1}, t_k]; C^{m+\alpha}(\mathbb{R}^n))$ and $C^0(D[t_{k-1}, t_k]; C^{m+\alpha}(\mathbb{R}^n))$, respectively, for $m = 1, 2$.

Furthermore, let $C^\omega(\mathbb{R}^n)$ be the space of all matrix functions $\varphi(\cdot) \equiv (\varphi_{ij}(\cdot))_{M \times M}$ such that $\varphi_{ij}(\cdot) \in C^\omega(\mathbb{R}^n)$ for all $i, j \in \mathcal{A}$. The norm on $C^\omega(\mathbb{R}^n)$ is defined as
\[ \|\varphi(\cdot)\|_{C^\omega(\mathbb{R}^n)} \equiv \sum_{i,j \in \mathcal{A}} \|\varphi_{ij}(\cdot)\|_{C^\alpha(\mathbb{R}^n)}. \quad (45) \]

Similarly, for $m = 0, 1, 2$, we define the spaces $C^{m+\alpha}(\mathbb{R}^n)$, $C^0([t_{k-1}, t_k]; C^{m+\alpha}(\mathbb{R}^n))$ and $C^0(D[t_{k-1}, t_k]; C^{m+\alpha}(\mathbb{R}^n))$. The norms on these spaces, denoted by $\|\cdot\|_S$ with $S$ being the corresponding spaces, are defined similar to (45).

We make the following hypotheses for the equation (44).

Assumption 4.1. For any $i, j \in \mathcal{A}$, the maps $\Sigma_i : [0, T] \times \mathbb{R}^n \to \mathbb{S}^n$, $b_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g_{ij} : D[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h_{ij} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and bounded. Moreover, there exists a constant $L > 0$ such that
\[ |\Sigma^i_x(t, x)| + |b^i_x(t, x, p)| + |g^i_{xx}(t, t, x, p)| + |h^i_{xx}(t, x)| + |b^i_p(t, x, p)| + |g^i_p(t, t, x, p)| \leq L, \quad (46) \]
for all $(\tau, t, x, p) \in D[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $i, j \in \mathcal{A}$. Furthermore, $(\Sigma^i_x(t, x))^{-1}$ exists for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $i \in \mathcal{A}$, and there exist constants $\lambda_0, \lambda_1 > 0$ such that
\[ \lambda_0 I \preceq (\Sigma^i_x(t, x))^{-1} \preceq \lambda_1 I, \forall (t, x) \in [0, T] \times \mathbb{R}^n, i \in \mathcal{A}. \quad (47) \]
We first give a prior estimate of a solution to (44).

**Proposition 4.1.** Let Assumption 4.1 hold and, \( \Theta(\cdot, \cdot, \cdot) \equiv (\Theta^{ji}(\cdot, \cdot, \cdot))_{M \times M} \) be a solution to (44). Then it holds that

\[
\| \Theta(\cdot, \cdot, \cdot) \|_{C(D[0, T]; C^1(\mathbb{R}^n))} \leq K \left( 1 + \| h(\cdot, \cdot) \|_{C([0, T]; C^1(\mathbb{R}^n))} \right),
\]

where \( K > 0 \) is a constant.

**Proof.** Let

\[
\Gamma^i(t, x; s, y) = \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} e^{-\frac{(x-y)^2}{4(s-t)}},
\]

which satisfies

\[
\begin{cases}
|\Gamma^i(t, x; s, y)| \leq \frac{K}{(s-t)^{\frac{n}{2}}} e^{-\frac{\lambda|x-y|^2}{4(s-t)}},
|\Gamma^i_x(t, x; s, y)| \leq \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-y|^2}{4(s-t)}},
\lambda < \lambda_0.
\end{cases}
\]

In the following, the constant \( K \) may be different from line to line. If \( \Theta(\cdot, \cdot, \cdot) \equiv (\Theta^{ji}(\cdot, \cdot, \cdot))_{M \times M} \) is a solution to (44), then for \((\tau, t, x) \in D[0, T] \times \mathbb{R}^n\),

\[
\Theta^{ji}(\tau, t, x) = \int_{\mathbb{R}^n} \Gamma^i(t, x; t_N, y) h^{ji}(\tau, y) dy + \int_t^{t_N} \int_{\mathbb{R}^n} \Gamma^i(t, x; s, y) \left[ b^i(s, y, \Theta^u(s, s, y)), \Theta^{ji}(\tau, s, y) \right]
+ g^{ji}(\tau, t, s, y) \Theta^{ji}(\tau, s, y) + \sum_{i' \in A} q_{i'i} \Theta^{ji}(\tau, s, y) dy ds.
\]

Similar to (5.25) in [22], we have

\[
\left| \Theta^{ji}_x(\tau, t, x) \right|
\leq \int_t^{t_N} \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-y|^2}{4(s-t)}} \left( \left| \Theta^{ji}_x(\tau, s, y) \right| + \sum_{i' \in A} \left| \Theta^{ji}(\tau, s, y) \right| \right) dy ds
+ K \left( 1 + \| h^{ji}(\tau, \cdot) \|_{C^1(\mathbb{R}^n)} \right).
\]

Equation (50) can be iterated as

\[
\left| \Theta^{ji}_x(\tau, t, x) \right|
\leq \int_t^{t_N} \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-y|^2}{4(s-t)}} \int_s^{t_N} \frac{K}{(r-s)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-z|^2}{4(r-s)}}
\times \left( \left| \Theta^{ji}_x(\tau, r, z) \right| + \sum_{i' \in A} \left| \Theta^{ji}(\tau, r, z) \right| \right) dz dv dy ds
+ \int_t^{t_N} \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-y|^2}{4(s-t)}} \left( 1 + \| h^{ji}(\tau, \cdot) \|_{C^1(\mathbb{R}^n)} \right)
+ \sum_{i' \in A} \left| \Theta^{ji}(\tau, s, y) \right| \right) dy ds + K \left( 1 + \| h^{ji}(\tau, \cdot) \|_{C^1(\mathbb{R}^n)} \right)
\]

\[
\leq \int_t^{t_N} \int_{\mathbb{R}^n} \left( \int_t^{t_N} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-y|^2}{4(s-t)}} \int_s^{t_N} \frac{K}{(r-s)^{\frac{n+1}{2}}} e^{-\frac{\lambda|x-z|^2}{4(r-s)}} dy ds \right) dy ds,
\]
From (49) and (52), we have
\[
\Theta_{ij}(\tau, r, z) \leq \int_t^T \left( \Theta_{ij}^H(\tau, r, z) + \sum_{i' \in A} \Theta_{ij'}^H(\tau, r, z) \right) \, dz \, dr \\
+ \int_t^T \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda (r-z)^2}{4(s-t)}} \sum_{i' \in A} \Theta_{ji'}^H(\tau, s, y) \, dy \, ds \\
+ K \left( 1 + \|h_{ji}(\tau, \cdot)\|_{C^1(\mathbb{R}^n)} \right)
\]
where the third inequality follows from Lemma 3 of [10, Chapter 1]. Thus,
\[
\|\Theta_{ij}(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \\
\leq \int_t^T K \left( \|\Theta_{ij}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} + \sum_{i' \in A} \|\Theta_{ij'}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \right) \, dr \\
+ \int_t^T \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda (r-t)^2}{4(s-t)}} \sum_{i' \in A} \|\Theta_{ji'}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, dy \, ds \\
+ K \left( 1 + \|h_{ji}(\tau, \cdot)\|_{C^1(\mathbb{R}^n)} \right)
\]
which, together with Gronwall’s inequality, implies that
\[
\|\Theta_{ij}(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \leq \int_t^T K \sum_{i' \in A} \|\Theta_{ij'}(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, dr \\
+ \int_t^T \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda (r-t)^2}{4(s-t)}} \sum_{i' \in A} \|\Theta_{ji'}(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, dy \, ds \\
+ K \left( 1 + \|h_{ji}(\tau, \cdot)\|_{C^1(\mathbb{R}^n)} \right)
\]
From (49) and (52), we have
\[
\|\Theta_{ij}(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \\
= \int_t^T \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda (r-t)^2}{4(s-t)}} \|\Theta_{ij}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, dy \, ds \\
+ K \left[ \int_t^T \sum_{i' \in A} \|\Theta_{ij'}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, ds + 1 + \|h_{ji}(\tau, \cdot)\|_{C^1(\mathbb{R}^n)} \right] \\
\leq \int_t^T \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} e^{-\frac{\lambda (r-t)^2}{4(s-t)}} \sum_{i' \in A} \|\Theta_{ji'}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, dy \, ds \\
+ \int_t^T \int_{\mathbb{R}^n} \frac{K}{(s-t)^{\frac{n+1}{2}}} \int_s^T \int_{\mathbb{R}^n} e^{-\frac{\lambda (r-s)^2}{4(t-s)}} \times \sum_{i' \in A} \|\Theta_{ji'}^H(\tau, t, \cdot)\|_{C^0(\mathbb{R}^n)} \, dz \, dr \, dy \, ds
Using Gronwall’s inequality again, we have sufficiently small. We prove the result by backward induction: Give a partition \( \Pi = \{ t_k \mid 0 \leq k \leq N \} \) of \([0, T]\) with the mesh size \(|\Pi|\) sufficiently small. We prove the result by backward induction:

**Step 1:** Show the existence and uniqueness of \( \Theta(t, t, x) \) on \( D[t_{N-1}, t_N] \times \mathbb{R}^n \);

**Step 2:** Show the existence and uniqueness of \( \Theta(t, t, x) \) on \( [t_0, t_{N-1}] \times \mathbb{R}^n \).

**Step 3:** For \( k = 1, \cdots, N-1 \), show the existence and uniqueness of \( \Theta(t, t, x) \) for \( (t, t, x) \in D[t_{k-1}, t_k] \times \mathbb{R}^n \), and then for \( (t, x) \in [t_0, t_{k-1}] \times \mathbb{R}^n \).

The proof of this step is similar to Steps 1 and 2.

Let us consider Step 2 first. Since we have got \( \Theta(t, t, x) \) for \( (t, x) \in [t_{N-1}, t_N] \times \mathbb{R}^n \) in Step 1, \( (44) \) becomes a standard linear parabolic system (parameterized by \( (\tau, j) \)) which admits a unique solution under Assumption 4.1, see, e.g. [10, Chapter 9].

In the following, we only prove Step 1. First, we show that if \( \pi_N \) is small enough, a solution to \( (44) \) is in the space

\[
S(\delta) \equiv \left\{ \Theta(\cdot, \cdot, \cdot) \in C(D[0,T]; C^1(\mathbb{R}^n)) : \Theta(\cdot, \cdot, \cdot) \in C(D[t_{N-1}, t_N]; C^1(\mathbb{R}^n)) \right\}
\]
where $\eta > 0$ is a given constant. From (48) and (49), we have

$$|\Theta^{ji}(\tau, t, x) - h^{ji}(\tau, x)|$$

$$\leq \left| \int_{\mathbb{R}^n} \Gamma^i(t, x; t_N, y) h^{ji}(\tau, y) dy - h^{ji}(\tau, x) \right|$$

$$+ \int_t^{t_N} \int_{\mathbb{R}^n} \left| \Gamma^i(t, x; s, y) \right| \left| \langle b^i(s, y, \Theta_x^{ji}(s, s, y)), \Theta_x^{ji}(\tau, s, y) \rangle \right| dy ds$$

$$+ g^{ji}(\tau, s, x, \Theta_x^{ji}(s, s, y)) + \sum_{i' \in \mathbb{A}} q_{i'i} \Theta_x^{ji}(\tau, s, y) \bigg| dy ds$$

$$\leq \int_{\mathbb{R}^n} \Gamma^i(t, x; t_N, y) h^{ji}(\tau, y) dy - h^{ji}(\tau, x) + K \pi_N,$$

where $K > 0$ is a constant independent of $\Pi$. By Proposition 5.1 (ii) in [22] (or Theorem 1 in [10, Chapter 1]), we have

$$|\Theta^{ji}(\tau, t, x) - h^{ji}(\tau, x)| \leq \frac{1}{M^2 \eta}, \quad \forall (\tau, t, x) \in D[t_{N-1}, t_N] \times \mathbb{R}^n,$$

provided that $\pi_N$ is sufficiently small. Thus, $\Theta(\cdot, \cdot, \cdot) \in S(\eta)$. Consequently, to show (44) admits a unique solution on $C(D[t_{N-1}, t_N]; C^1(\mathbb{R}^n))$, it is sufficient to show the existence and uniqueness on $S(\eta)$.

Now, we show there exists a unique solution to (44) in $S(\eta)$. Given $\Theta(\cdot, \cdot, \cdot) \equiv (\theta^{ji}(\cdot, \cdot, \cdot))_{\mathbb{A} \times M} \in S(\eta)$, let us consider the equation

$$\begin{cases}
\Theta^{ji}_t(\tau, t, x) + \text{tr} \left[ \sum_i (t, x) \Theta_x^{ji}(\tau, t, x) \right] + \langle b^i(t, x, \Theta_x^{ji}(t, t, x)), \Theta_x^{ji}(\tau, t, x) \rangle \\
\quad + g^{ji}(\tau, t, x, \Theta_x^{ji}(t, t, x)) + \sum_{i' \in \mathbb{A}} q_{i'i} \Theta_x^{ji}(\tau, t, x) = 0, \\
\Theta^{ji}(\tau, T, x) = h^{ji}(\tau, x), \quad \tau \in [t_{N-1}, t_N], j, i \in \mathbb{A}, x \in \mathbb{R}^n.
\end{cases} \quad (54)$$

Note that for given any $i, j \in \mathbb{A}$, (54) is Equation (5.20) studied in [22]. The unique classical solution $\Theta^{ji}(\cdot, \cdot, \cdot)$ to (54) satisfies

$$\Theta^{ji}(\tau, t, x) = \int_{\mathbb{R}^n} \Gamma^i(t, x; t_N, y) h^{ji}(\tau, y) dy$$

$$+ \int_t^{t_N} \int_{\mathbb{R}^n} \Gamma^i(t, x; s, y) \left[ \langle b^i(s, y, \Theta_x^{ji}(s, s, y)), \Theta_x^{ji}(\tau, s, y) \rangle \right] dy ds$$

$$+ g^{ji}(\tau, s, x, \Theta_x^{ji}(s, s, y)) + \sum_{i' \in \mathbb{A}} q_{i'i} \Theta_x^{ji}(\tau, s, y) \bigg] dy ds. \quad (55)$$

Similar to (52), it holds that

$$\|\Theta_x^{ji}(\cdot, \cdot, \cdot)\|_{C(D[t_{N-1}, t_N]; C^0(\mathbb{R}^n))}$$

$$\leq K \left( 1 + \|h^{ji}(\cdot, \cdot)\|_{C(D[t_{N-1}, t_N]; C^1(\mathbb{R}^n))} + \|\Theta(\cdot, \cdot, \cdot)\|_{C(D[t_{N-1}, t_N]; C^0(\mathbb{R}^n))} \right)$$

$$\leq K \left( 1 + \|h(\cdot, \cdot)\|_{C(D[t_{N-1}, t_N]; C^1(\mathbb{R}^n))} \right),$$

where the last inequality follows from $\Theta(\cdot, \cdot, \cdot) \in S(\eta)$. Thus, by (55) and choosing $\pi_N$ sufficiently small, we have

$$|\Theta^{ji}(\tau, t, x) - h^{ji}(\tau, x)|$$
\[
\begin{align*}
\leq & \left| \int_{\mathbb{R}_+} \Gamma^i(t, x; t_N, y) h^{ji}(\tau, y) dy - h^{ji}(\tau, x) \right| \\
& + \int_t^{t_N} \int_{\mathbb{R}_+} \left| \Gamma^i(t, x; s, y) \right| \left| \left\langle b^i(s, y, \theta_x^{ji}(s, s, y)) \right\rangle, \Theta_x^{ji}(\tau, s, y) \right\rangle \\
& + g^{ji}(\tau, s, x, \theta_x^{ji}(s, s, y)) + \sum_{i' \in H} q_{ii'} \Theta_x^{ji}(\tau, s, y) \\& \leq \int_{\mathbb{R}_+} \Gamma^i(t, x; t_N, y) h^{ji}(\tau, y) dy - h^{ji}(\tau, x) \\
& + \pi_N K \left( 1 + \|h(\cdot, \cdot)\|_{C([t_{N-1}, t_N]; C^1(\mathbb{R}_+))} \right) \\
& \leq \frac{1}{M^2} \eta,
\end{align*}
\]

which implies that \( \Theta(\cdot, \cdot, \cdot) \equiv (\Theta^{ji}(\cdot, \cdot, \cdot)) \in \mathcal{S}(\eta) \). Note that \( K \) in (56) is independent of \( \theta(\cdot, \cdot, \cdot) \), which implies that we can choose \( \pi_N \) sufficiently small and independent of \( \theta(\cdot, \cdot, \cdot) \).

Now, let \( \theta^{(m)}(\cdot, \cdot, \cdot) = (\theta^{(m)ji}(\cdot, \cdot, \cdot)) \in \mathcal{S}(\eta), m = 1, 2 \). Let \( \Theta^{(m)}(\cdot, \cdot, \cdot) = (\Theta^{(m)ji}(\cdot, \cdot, \cdot)) \in \mathcal{S}(\eta) \) be the corresponding solutions to (54). Thus,

\[
\begin{align*}
\Theta^{(1)ji}(\tau, t, x) - \Theta^{(2)ji}(\tau, t, x) \\
= & \int_t^{t_N} \int_{\mathbb{R}_+} \Gamma^i(t, x; s, y) \left[ \left\langle b^i(s, y, \theta_x^{(1)ji}(s, s, y)) \right\rangle, \Theta_x^{(1)ji}(\tau, s, y) - \Theta_x^{(2)ji}(\tau, s, y) \right\rangle \\
& + \left\langle b^i(s, y, \theta_x^{(1)ji}(s, s, y)) \right\rangle, \Theta_x^{(2)ji}(\tau, s, y) \right\rangle - \left\langle b^i(s, y, \theta_x^{(2)ji}(s, s, y)) \right\rangle, \Theta_x^{(2)ji}(\tau, s, y) \right\rangle \\
& + g^{ji}(\tau, s, x, \theta_x^{(1)ji}(s, s, y)) - g^{ji}(\tau, s, x, \theta_x^{(2)ji}(s, s, y)) \\
& + \sum_{i' \in H} q_{ii'} \left( \theta^{(1)ji'}(\tau, s, y) - \theta^{(2)ji'}(\tau, s, y) \right) \right] dy ds,
\end{align*}
\]  

and

\[
\begin{align*}
\Theta_x^{(1)ji}(\tau, t, x) - \Theta_x^{(2)ji}(\tau, t, x) \\
= & \int_t^{t_N} \int_{\mathbb{R}_+} \Gamma^i(t, x; s, y) \left[ \left\langle b^i(s, y, \theta_x^{(1)ji}(s, s, y)) \right\rangle, \Theta_x^{(1)ji}(\tau, s, y) - \Theta_x^{(2)ji}(\tau, s, y) \right\rangle \\
& + \left\langle b^i(s, y, \theta_x^{(1)ji}(s, s, y)) \right\rangle, \Theta_x^{(2)ji}(\tau, s, y) \right\rangle - \left\langle b^i(s, y, \theta_x^{(2)ji}(s, s, y)) \right\rangle, \Theta_x^{(2)ji}(\tau, s, y) \right\rangle \\
& + g^{ji}(\tau, s, x, \theta_x^{(1)ji}(s, s, y)) - g^{ji}(\tau, s, x, \theta_x^{(2)ji}(s, s, y)) \\
& + \sum_{i' \in H} q_{ii'} \left( \theta^{(1)ji'}(\tau, s, y) - \theta^{(2)ji'}(\tau, s, y) \right) \right] dy ds.
\end{align*}
\]

Hence,

\[
\left| \Theta_x^{(1)ji}(\tau, t, x) - \Theta_x^{(2)ji}(\tau, t, x) \right|
\]
By iterating (58) (using similar arguments in the proof of Proposition 4.1), we have

\[
\begin{align*}
&\leq \int_t^{t_N} \int_{\mathbb{R}^n} \frac{K}{(s-t)^{n+\varepsilon}} e^{-\frac{K|\theta|^2}{(s-t)^{2\varepsilon}}} \left[ \| \Theta^{(1)ji}(\tau,s,y) - \Theta^{(2)ji}(\tau,s,y) \| + \| \theta^{(1)ji}(s,s,y) - \theta^{(2)ji}(s,s,y) \| \right) \| (1 + \Theta^{(2)ji}(\tau,s,y)) \| \\
&\quad + \sum_{i' \in \mathcal{A}} \left[ \theta^{(1)ji'}(\tau,s,y) - \theta^{(2)ji'}(\tau,s,y) \right] \right] dy ds \\
&\leq \int_t^{t_N} \int_{\mathbb{R}^n} \frac{K}{(s-t)^{n+\varepsilon}} e^{-\frac{K|\theta|^2}{(s-t)^{2\varepsilon}}} \left[ \| \Theta^{(1)ji}(\tau,s,y) - \Theta^{(2)ji}(\tau,s,y) \| \right] dy ds \\
&\quad + K \pi_n^{-\frac{1}{2}} \left( 1 + \| h(\cdot,\cdot) \|_{C([t_N-1,t_N]; C^1(\mathbb{R}^n))} \right) \| \theta^{(1)}(\cdot,\cdot,\cdot) - \theta^{(2)}(\cdot,\cdot,\cdot) \|_{C([t_N-1,t_N]; C^1(\mathbb{R}^n))}.
\end{align*}
\]  

(58)

By choosing \( \pi_N \) small, we get a contraction mapping \( \Theta(\cdot,\cdot,\cdot) \mapsto \Theta(\cdot,\cdot,\cdot) \) on \( \mathcal{S}(\eta) \), which means that there exists a unique solution to (44) on \( \mathcal{S}(\eta) \). Furthermore, replacing \( \theta(\cdot,\cdot,\cdot) \) in (55) by \( \Theta(\cdot,\cdot,\cdot) \) and by the regularity of \( \Gamma'(\cdot,\cdot,\cdot,\cdot) \), we know that \( \Theta(\cdot,\cdot,\cdot) \) is twice continuously differentiable in \( x \).

In Section 3.3, by assuming the convergence of \( \Theta^{ii}(\cdot,\cdot,\cdot,\cdot) \), we get the equilibrium HJB equation for \( \Theta(\cdot,\cdot,\cdot,\cdot) \). We now show under Assumption 4.1, we do have the expected convergence.

Given (43), the differential forms of (26) and (28) become

\[
\Theta^{ii}(\tau,j,t,x,i) + tr \left[ \sum(t,x,i) \Theta^{ii xx}(\tau,j,t,x,i) \right] \\
+ \left( b(t,x,i,\psi(l(t),i,t,x,i)) - q_i(\tau,j,t,x,i) \right), \Theta^{ii}(\tau,j,t,x,i) \\
+ g_{i'}(\tau,j,t,x,i,\psi(l(t),i,t,x,i)) - q_i(\tau,j,t,x,i) = 0, \\
(\tau,t) \in D[0,T], j \in A, x \in \mathbb{R}^n,
\]

(59)

**Proof.**
and
\[
\hat{\Theta}^{H,i}(\tau, j, t, x, i) + \text{tr} \left[ \Sigma(t, x, i) \hat{\Theta}^{H,i}(\tau, j, t, x, i) \right] \\
+ b \left( t, x, i, \psi \left( l_1(t), i, t, x, i, \hat{\Theta}^{H,i}(l_1(t), i, t, x, i) \right) \right) \cdot \hat{\Theta}^{H,i}(\tau, j, t, x, i) \\
+ g \left( \tau, j, t, x, i, \psi \left( l_1(t), i, t, x, i, \hat{\Theta}^{H,i}(l_1(t), i, t, x, i) \right) \right) \\
+ \sum_{i' \in A} q_{i'} \hat{\Theta}^{H,i}(\tau, j, t, x, i') = 0,
\]
\[
\hat{\Theta}^{H}(\tau, j, t_k, x, i) = \Theta^{k+1}(\tau, j, t_k, x, i), \quad \tau \in [0, t], t \in [t_{k-1}, t_k], i, j \in A, x \in \mathbb{R}^n,
\]
respectively.

By Proposition 5.1 in [22], we have
\[
\Theta^H(\tau, j, t, x, i) - \Theta(\tau, j, t, x, i) \\
= \int_0^T \int_{\mathbb{R}^n} \Gamma^i(t, x, s, y) \\
\times \left[ \left( b \left( s, y, i, \psi \left( l_1(s), i, s, y, i, \Theta^H(l_1(s), i, s, y, i) \right) \right) \right) \cdot \Theta^H(\tau, j, s, y, i) \right] \\
- \left( b \left( s, y, i, \psi \left( s, i, s, y, i, \Theta_x(s, i, s, y, i) \right) \right) \right) \cdot \Theta_x(\tau, j, s, y, i) \\
+ g \left( \tau, j, s, y, i, \psi \left( l_1(s), i, s, y, i, \Theta^H(l_1(s), i, s, y, i) \right) \right) \\
- g \left( \tau, j, s, x, i, \psi \left( s, i, s, x, i, \Theta_x(s, i, s, y, i) \right) \right) \\
+ \sum_{i' \neq i} q_{i'} \hat{\Theta}^{H,i'}(\tau, j, s, y, i') - q_i \Theta^H(\tau, j, s, y, i) - \sum_{i' \in A} q_{i'} \Theta(\tau, j, s, x, i') \right] \\
dyds
\]
\[
= \int_0^T \int_{\mathbb{R}^n} \Gamma^i(t, x, s, y) \\
\times \left[ \left( b \left( s, y, i, \psi \left( l_1(s), i, s, y, i, \Theta^H(l_1(s), i, s, y, i) \right) \right) \right) \cdot \Theta^H(\tau, j, s, y, i) - \Theta_x(\tau, j, s, y, i) \right] \\
+ \left( b \left( s, y, i, \psi \left( l_1(s), i, s, y, i, \Theta^H(l_1(s), i, s, y, i) \right) \right) \right) \cdot \Theta_x(\tau, j, s, y, i) \\
- b \left( s, y, i, \psi \left( s, i, s, y, i, \Theta_x(s, i, s, y, i) \right) \right) \cdot \Theta_x(\tau, j, s, y, i) \\
+ g \left( \tau, j, s, y, i, \psi \left( l_1(s), i, s, y, i, \Theta^H(l_1(s), i, s, y, i) \right) \right) \\
- g \left( \tau, j, s, x, i, \psi \left( s, i, s, x, i, \Theta_x(s, i, s, y, i) \right) \right) \\
+ \sum_{i' \neq i} q_{i'} \hat{\Theta}^{H,i'}(\tau, j, s, y, i') - q_i \Theta^H(\tau, j, s, y, i) - \sum_{i' \in A} q_{i'} \Theta(\tau, j, s, x, i') \right] \\
dyds.
\]
Consequently,
\[
|\Theta^H_x(\tau, j, t, x, i) - \Theta_x(\tau, j, t, x, i)| \\
\leq \int_0^T \int_{\mathbb{R}^n} \frac{K e^{-\frac{(t-s)}{(t-s)}^2}}{(t-s) \frac{1}{2}} \left[ |\Theta^H_x(\tau, j, s, y, i) - \Theta_x(\tau, j, s, y, i)| \\
+ |l_1(s) - s| + |\Theta^H_x(l_1(s), i, s, y, i) - \Theta_x(s, i, s, y, i)| \\
+ \sum_{i' \neq i} |\hat{\Theta}^{H,i}(\tau, j, s, y, i') - \Theta^H(\tau, j, s, y, i')| \\
+ \sum_{i' \in A} |\Theta^H(\tau, j, s, y, i') - \Theta(\tau, j, s, y, i')| \right] \\
dyds.
\[
\leq \int_t^T \int_{\mathbb{R}^n} Ke^{-\frac{\lambda(s-y)^2}{(r-t)^2}} \left[ \|\Theta^\Pi_x(\tau, j, s, y, i) - \Theta_x(\tau, j, s, y, i)\| + \\
+ |I_\Pi(s) - s| + \|\Theta^\Pi_x(I_\Pi(s), i, s, y, i) - \Theta_x(I_\Pi(s), i, s, y, i)\| + \\
+ \|\Theta_x(I_\Pi(s), i, s, y, i) - \Theta_x(s, i, s, y, i)\| + \\
+ \sum_{i' \neq i} |\Theta^\Pi(\tau, j, s, y, i') - \Theta^\Pi(\tau, j, s, y', i')| + \\
+ \sum_{i' \neq i} |\Theta^\Pi(\tau, j, s, y', i') - \Theta(\tau, j, s, y', i')| \right] \ dgyds,
\]

and

\[
\sup_{\tau \in [0,t]} \|\Theta^\Pi_x(\tau, j, t, \cdot, i) - \Theta_x(\tau, j, t, \cdot, i)\|_{C^0(\mathbb{R}^n)}
\leq \int_t^T \int_{\mathbb{R}^n} Ke^{-\frac{\lambda(s-y)^2}{(r-t)^2}} \sup_{\tau \in [0,s]} \|\Theta^\Pi_x(\tau, j, s, \cdot, i) - \Theta_x(\tau, j, s, \cdot, i)\|_{C^0(\mathbb{R}^n)} \ dgyds
\]

\[
+ \int_t^T \int_{\mathbb{R}^n} Ke^{-\frac{\lambda(s-y)^2}{(r-t)^2}} \sum_{i' \in \mathbb{A}} \sup_{\tau \in [0,s]} \|\Theta^\Pi(\tau, j, s, \cdot, i') - \Theta(\tau, j, s, \cdot, i')\|_{C^0(\mathbb{R}^n)} \ dgyds
\]

\[
+ K \sup_{s \in [0,T]} \|\Theta_x(I_\Pi(s), i, s, \cdot, i) - \Theta_x(s, i, s, \cdot, i)\|_{C^0(\mathbb{R}^n)}
\]

\[
+ K \sup_{s \in [0,T]} \sum_{i' \neq i} \|\Theta^\Pi(\tau, j, s, \cdot, i') - \Theta^\Pi(\tau, j, s, \cdot, i')\|_{C^0(\mathbb{R}^n)} + K \|\Pi\|.
\]

Similar to (52), we have

\[
\sup_{\tau \in [0,t]} \|\Theta^\Pi_x(\tau, j, t, \cdot, i) - \Theta_x(\tau, j, t, \cdot, i)\|_{C^0(\mathbb{R}^n)}
\leq \int_t^T \sum_{i' \in \mathbb{A}} \sup_{\tau \in [0,r]} \|\Theta^\Pi(\tau, j, r, \cdot, i') - \Theta(\tau, j, r, \cdot, i')\|_{C^0(\mathbb{R}^n)} \ dr
\]

\[
+ \int_t^T \int_{\mathbb{R}^n} Ke^{-\frac{\lambda(s-y)^2}{(r-t)^2}} \sum_{i' \in \mathbb{A}} \sup_{\tau \in [0,r]} \|\Theta^\Pi(\tau, j, r, \cdot, i') - \Theta(\tau, j, r, \cdot, i')\|_{C^0(\mathbb{R}^n)} \ dzdr
\]

\[
+ K \sup_{s \in [0,T]} \|\Theta_x(I_\Pi(s), i, s, \cdot, i) - \Theta_x(s, i, s, \cdot, i)\|_{C^0(\mathbb{R}^n)}
\]

\[
+ K \sup_{s \in [0,T]} \sum_{i' \neq i} \|\Theta^\Pi(\tau, j, s, \cdot, i') - \Theta^\Pi(\tau, j, s, \cdot, i')\|_{C^0(\mathbb{R}^n)} + K \|\Pi\|.
\]

Furthermore, it follows from (60) that

\[
\sup_{\tau \in [0,t]} \|\Theta^\Pi(\tau, j, t, \cdot, i) - \Theta(\tau, j, t, \cdot, i)\|_{C^0(\mathbb{R}^n)}
\leq \int_t^T \int_{\mathbb{R}^n} Ke^{-\frac{\lambda(s-y)^2}{(r-t)^2}} \left[ \sup_{\tau \in [0,s]} \|\Theta^\Pi_x(\tau, j, s, \cdot, i) - \Theta_x(\tau, j, s, \cdot, i)\|_{C^0(\mathbb{R}^n)} + \\
+ |I_\Pi(s) - s| + \sup_{s \in [t,T]} \|\Theta_x(I_\Pi(s), i, s, \cdot, i) - \Theta_x(s, i, s, \cdot, i)\|_{C^0(\mathbb{R}^n)} + \\
+ \sup_{s \in [t,T]} \sum_{i' \neq i} \|\Theta^\Pi(\tau, j, s, \cdot, i') - \Theta^\Pi(\tau, j, s, \cdot, i')\|_{C^0(\mathbb{R}^n)} \right] \ dgyds.
\]
Similar to (53), we have

\[ \begin{aligned}
& + \sum_{i' \in A} \sup_{\tau \in [0,s)} \| \Theta^H(\tau, j, s, \cdot, i') - \Theta(\tau, j, s, \cdot, i) \|_{C^{0}(\mathbb{R}^n)} d\gamma d\sigma \\
\leq & \int_{t}^{T} \int_{\mathbb{R}^n} Ke^{-\frac{(x-y)^2}{2(t-s)^2}} \left[ \sup_{\tau \in [0,s]} \| \Theta^H(\tau, j, s, \cdot, \cdot) - \Theta(\tau, j, s, \cdot, i) \|_{C^{0}(\mathbb{R}^n)} \\
& + \sum_{i' \in A} \sup_{\tau \in [0,s]} \| \Theta^H(\tau, j, s, \cdot, i') - \Theta(\tau, j, s, \cdot, i) \|_{C^{0}(\mathbb{R}^n)} \right] d\gamma d\sigma \\
& + K \sup_{s \in [0,T]} \sum_{i' \neq i} \| \Theta^H(\tau, j, s, \cdot, i') - \Theta^H(\tau, j, s, \cdot, i) \|_{C^{0}(\mathbb{R}^n)} + K \| \Pi \|. 
\end{aligned} \]

Similar to (53), we have

\[ \sum_{i,j \in A} \| \Theta^H(\cdot, j, \cdot, \cdot, i) - \Theta(\cdot, j, \cdot, \cdot, i) \|_{C(D[0,T];C^{0}(\mathbb{R}^n))} \leq K \left[ \| \Pi \| + \sum_{i \in A} \sup_{s \in [0,T]} \| \Theta_x(l\Pi(s), i, s, \cdot) - \Theta_x(s, i, s, \cdot, i) \|_{C^{0}(\mathbb{R}^n)} \\
+ \sup_{s \in [0,T]} \sum_{i' \neq i} \| \Theta^H(\tau, j, s, \cdot, i') - \Theta^H(\tau, j, s, \cdot, i) \|_{C^{0}(\mathbb{R}^n)} \right]. \] (62)

The convergence follows from (61) and (62).

5. A time-inconsistent linear-quadratic problem. Let us consider the linear-quadratic problem in this section. For any initial state \((t, x, i) \in [0, T) \times \mathbb{R}^n \times A\), the state equation is

\[ \begin{aligned}
& dX(s) = [A(s, \alpha(s))X(s) + B(s, \alpha(s))u(s)] ds \\
& + [A_1(s, \alpha(s))X(s) + B_1(s, \alpha(s))u(s)] dW(s), \quad s \in [t, T], \\
X(t) = x, \quad & \alpha(t) = i,
\end{aligned} \] (63)

and the cost functional is

\[ J(t, x, i; u(\cdot)) = \frac{1}{2} \mathbb{E}_t \int_{t}^{T} \left( \langle Q(t, i, s, \alpha(s)), X(s), X(s) \rangle + \langle R(t, i, s, \alpha(s)), u(s), u(s) \rangle \right) ds \\
+ \langle G(t, i, \alpha(T)), X(T) \rangle. \] (64)

Then

\[ \mathbb{H}(\tau, j, t, x, i, u, p, P) = \langle p, A(t, i)x \rangle + \frac{1}{2} \left( [A_1^T(t, i)PA_1(t, i) + Q(\tau, j, t, i)] x, x \right) \\
+ \frac{1}{2} \left( [R(t, j, t, i) + B_1^T(t, i)PB_1(t, i)] u, u \right) \\
+ \langle u, B^T(t, i)p + B_1^T(t, i)PA_1(t, i)x \rangle. \]

It yields that

\[ \psi(\tau, j, t, x, i, p, P) = - \left( R(t, j, t, i) + B_1^T(t, i)PB_1(t, i) \right)^{-1} \left( B^T(t, i)p + B_1^T(t, i)PA_1(t, i)x \right). \]
To simplify the notation, we suppressing \( (t, i) \) of \( A(\cdot), B(\cdot), A_1(\cdot) \) and \( B_1(\cdot) \) in the rest of this section. Therefore,

\[
\mathbb{H}(\tau, j, t, x, i, \psi(t, i, t, x, i, \bar{p}, \bar{P}), p, P) = \langle p, Ax \rangle + \frac{1}{2} \left\langle [A_1^T PA_1 + Q(\tau, j, t, i)] x, x \right\rangle \\
+ \frac{1}{2} \left\langle [R(\tau, j, t, i) + B_1^T P B_1] \psi(t, i, t, x, i, \bar{p}, \bar{P}), \psi(t, i, t, x, i, \bar{p}, \bar{P}) \right\rangle \\
+ \langle \psi(t, i, t, x, i, \bar{p}, \bar{P}), B^T p + B_1^T PA_1 x \rangle
\]

\[
= \langle p, Ax \rangle + \frac{1}{2} \left\langle [A_1^T PA_1 + Q(\tau, j, t, i)] x, x \right\rangle \\
+ \frac{1}{2} \left\langle [R(\tau, j, t, i) + B_1^T P B_1] \left[ R(t, i, t, i) + B_1^T P B_1 \right]^{-1} [B^T \bar{p} + B_1^T PA_1 x], \right. \\
\left. [R(t, i, t, i) + B_1^T P B_1]^{-1} [B^T \bar{p} + B_1^T PA_1 x] \right\rangle \\
- \left\langle [R(t, i, t, i) + B_1^T P B_1]^{-1} [B^T \bar{p} + B_1^T PA_1 x], B^T p + B_1^T PA_1 x \right\rangle.
\]

Thus, the equilibrium HJB equation becomes (suppressing \( (t, i, t, i) \) of \( R(\cdot) \))

\[
\begin{aligned}
\Theta_1(\tau, j, t, x, i) + (\Theta_1x(\tau, j, t, x, i), Ax) \\
+ \frac{1}{2} \left\langle [A_1^T \Theta_{xx}(\tau, j, t, x, i), A_1] + Q(\tau, j, t, i) \right\rangle x, x \\
+ \frac{1}{2} \left\langle [R(\tau, j, t, i) + B_1^T \Theta_{xx}(\tau, j, t, x, i)B_1] \left[ R(t, i, t, i) + B_1^T \Theta_{xx}(t, i, t, x, i)B_1 \right]^{-1} \\
\cdot [B^T \Theta_x(t, i, t, x, i) + B_1^T \Theta_{xx}(t, i, t, x, i)A_1 x], \\
\left. [R + B_1^T \Theta_{xx}(t, i, t, x, i), B_1]^{-1} [B^T \Theta_x(t, i, t, x, i) + B_1^T \Theta_{xx}(t, i, t, x, i)A_1 x] \right\rangle \\
- \langle [R + B_1^T \Theta_{xx}(t, i, t, x, i), B_1]^{-1} [B^T \Theta_x(t, i, t, x, i) + B_1^T \Theta_{xx}(t, i, t, x, i)A_1 x], \\
B^T \Theta_x(t, j, t, x, i) + B_1^T \Theta_{xx}(t, j, t, x, i)A_1 x \rangle \\
+ \sum_{i' \in \mathcal{A}} q_{i'i} \Theta(\tau, j, t, x, i') = 0, \quad (\tau, t) \in D[0, T], j, i \in \mathcal{A}, x \in \mathbb{R}^n,
\end{aligned}
\]

\[
\Theta(\tau, j, T, x, i) = \frac{1}{2} \langle G(\tau, j, i)x, x \rangle, \quad \tau \in [0, T], j, i \in \mathcal{A},
\]

(65)

Similar to [22, Section 6.1], let

\[
\Theta(\tau, j, t, x, i) = \frac{1}{2} \langle P(\tau, j, t, i)x, x \rangle, \quad (\tau, t, x, i, j) \in D[0, T] \times \mathbb{R} \times \mathcal{A} \times \mathcal{A},
\]

(66)

where the map \( P(\cdot) \) is to be determined. Plugging (66) into (65), we have

\[
\begin{aligned}
P_t(\tau, j, t, i) + P(\tau, j, t, i)A + A^T P(\tau, j, t, i) + A_1^T P(\tau, j, t, i)A_1 + Q(\tau, j, t, i) \\
+ [P(t, i, t, i)B + A_1^T P(t, i, t, i)B_1] \left[ R + B_1^T P(t, i, t, i)B_1 \right]^{-1} \\
\cdot [R(\tau, j, t, i) + B_1^T P(\tau, j, t, i)B_1] \left[ R + B_1^T P(t, i, t, i)B_1 \right]^{-1} \\
\cdot [B^T P(t, i, t, i) + B_1^T P(t, i, t, i)A_1] \\
- [P(t, i, t, i) + A_1^T P(t, i, t, i)B_1] \left[ R + B_1^T P(t, i, t, i)B_1 \right]^{-1} \\
\cdot [B^T P(\tau, j, t, i) + B_1^T P(\tau, j, t, i)A_1] \\
- [P(\tau, j, t, i) + A_1^T P(\tau, j, t, i)B_1] \left[ R + B_1^T P(t, i, t, i)B_1 \right]^{-1} \\
\cdot [B^T P(t, i, t, i) + B_1^T P(t, i, t, i)A_1] \\
+ \sum_{i'' \in \mathcal{A}} q_{i'i''} P(\tau, j, t, i'') = 0, \quad (\tau, t) \in D[0, T], j, i \in \mathcal{A},
\end{aligned}
\]

\[
P(\tau, j, T, i) = G(\tau, j, i), \quad \tau \in [0, T], j, i \in \mathcal{A},
\]

(67)
Denote
\[
\begin{aligned}
\Gamma(t, i) &= \left[ R(t, i, t, i) + B_{1}^{T}(t, i)P(t, i, t, i)B_{1}(t, i) \right]^{-1} \\
\cdot \left[ B_{1}^{T}(t, i)P(t, i, t, i) + B_{1}^{T}(t, i)P(t, i, t, i)A_{1}(t, i) \right], \\
\dot{A}(t, i) &= A(t, i) - B(t, i)\Gamma(t, i), \\
\dot{A}_{1}(t, i) &= A_{1}(t, i) - B_{1}(t, i)\Gamma(t, i), \\
\dot{Q}(\tau, j, t, i) &= Q(\tau, j, t, i) + \Gamma^{T}(t, i)R(\tau, j, t, i)\Gamma(t, i).
\end{aligned}
\]

Then (67) can be rewritten as
\[
\begin{cases}
P_{2}(\tau, j, t, i) + P(\tau, j, t, i)\dot{A}(t, i) + \dot{A}_{1}^{T}(t, i)P(\tau, j, t, i)\dot{A}_{1}(t, i) \\
+ \dot{Q}(\tau, j, t, i) + \sum_{i' \in \mathbb{A}} q_{ij}P(\tau, j, t, i') = 0, \quad (\tau, t) \in D[0, T], j, i \in \mathbb{A}, \\
P(\tau, j, T, i) = G(\tau, j, i), \quad \tau \in [0, T], j, i \in \mathbb{A}.
\end{cases}
\]

For \( i \in \mathbb{A} \), let \( \Phi(\cdot, \cdot, i) \) be the solution of the following forward stochastic differential equation:
\[
\begin{cases}
d\Phi(s, t, i) = \dot{A}(s, \alpha(s, t, i))\Phi(s, t, i)ds + \dot{A}_{1}(s, \alpha(s, t, i))\Phi(s, t, i)dW(s), \quad s \in [t, T], \\
\Phi(t, t, i) = I, \quad \alpha(t, t, i) = i.
\end{cases}
\]

Applying the generalized Itô's formula (see, e.g. [2]) to
\[
s \mapsto \langle P(\tau, j, s, \alpha(s, t, i))\Phi(s, t, i)x, \Phi(s, t, i)x \rangle
\]
on \([t, T]\), we have
\[
\begin{align*}
\langle \Phi^{T}(T, t, i)P(\tau, j, T, \alpha(T; t, i))\Phi(T, t, i)x, x \rangle &- \langle P(\tau, j, t, i)x, x \rangle \\
= \langle \Phi^{T}(T, t, i)G(\tau, j, \alpha(T; t, i))\Phi(T, t, i)x, x \rangle - \langle P(\tau, j, t, i)x, x \rangle \\
= \int_{t}^{T} - \langle \Phi^{T}(s, t, i)\dot{Q}(\tau, j, s, \alpha(s, t, i))\Phi(s, t, i)x, x \rangle ds \\
+ \int_{t}^{T} \langle \Phi^{T}(s, t, i) \left[ P(\tau, j, s, \alpha(s, t, i))\dot{A}_{1}(s, \alpha(s, t, i)) \\
+ \dot{A}_{1}^{T}(s, \alpha(s, t, i))P(\tau, j, s, \alpha(s, t, i)) \right] \Phi(s, t, i)x, x \rangle dW(s) \\
+ \int_{t}^{T} \sum_{i' \in \mathbb{A}} \langle \Phi^{T}(s, t, i)P(\tau, j, s, i')\Phi(s, t, i)x, x \rangle d\mathcal{M}^{i'}(s),
\end{align*}
\]
where \( \mathcal{M}^{i'}(\cdot), i \in \mathbb{A} \) are martingales. Thus, for \((\tau, t) \in D[0, T] \) and \( j, i \in \mathbb{A}, \)
\[
P(\tau, j, t, i) = \Phi^{T}(T, t, i)G(\tau, j, \alpha(T; t, i))\Phi(T, t, i) \\
+ \int_{t}^{T} \Phi^{T}(s, t, i)\dot{Q}(\tau, j, s, \alpha(s, t, i))\Phi(s, t, i)ds \\
- \int_{t}^{T} \Phi^{T}(s, t, i) \left[ P(\tau, j, s, \alpha(s, t, i))\dot{A}_{1}(s, \alpha(s, t, i)) \\
+ \dot{A}_{1}^{T}(s, \alpha(s, t, i))P(\tau, j, s, \alpha(s, t, i)) \right] \Phi(s, t, i)dW(s) \\
+ \int_{t}^{T} \sum_{i' \in \mathbb{A}} \Phi^{T}(s, t, i)P(\tau, j, s, i')\Phi(s, t, i)d\mathcal{M}^{i'}(s),
\]
which gives a stochastic representation of $P(\tau, j, t, i)$. From (71), we have
\[
P(\tau, j, t, i) = E_t \left[ \Phi^T(T, t, i) G(\tau, j, \alpha(T; t, i)) \Phi(T, t, i) \right.
\]
\[+ \int_t^T \Phi^T(s, t, i) \tilde{Q}(\tau, j, s, \alpha(s; t, i)) \Phi(s, t, i) ds \right].
\]  
(72)

Letting $\tau = t$ and $j = i$ in (72) and denoting $P(t, i) = P(t, i, t, i)$, it yields that
\[
P(t, i) = E_t \left[ \Phi^T(T, t, i) G(t, i, \alpha(T; t, i)) \Phi(T, t, i) \right.
\]
\[+ \int_t^T \Phi^T(s, t, i) [Q(t, i, s, \alpha(s; t, i))
\]
\[+ \Gamma^T(s, \alpha(s; t, i)) R(t, i, s, \alpha(s; t, i)) \Gamma(s, \alpha(s; t, i))] \Phi(s, t, i) ds \right].
\]  
(73)

In the following, we write $\alpha(\cdot)$ for $\alpha(\cdot; t, i)$. In summary, we obtained the following system for the function $P(\cdot, \cdot)$:
\[
\begin{aligned}
P(t, i) &= E_t \left[ \Phi^T(T, t, i) G(t, i, \alpha(T)) \Phi(T, t, i) + \int_t^T \Phi^T(s, t, i) [Q(t, i, s, \alpha(s))
\]
\[+ \Gamma^T(s, \alpha(s)) R(t, i, s, \alpha(s)) \Gamma(s, \alpha(s))] \Phi(s, t, i) ds \right], \quad t \in [0, T],
\end{aligned}
\]
\[
\begin{aligned}
\Phi(s, t, i) &= I + \int_t^s [A(r, \alpha(r)) - B(r, \alpha(r)) \Gamma(r, \alpha(r))] \Phi(r, t, i) dr
\end{aligned}
\]
\[
\begin{aligned}
&+ \int_t^s \left[ A_1(r, \alpha(r)) - B_1(r, \alpha(r)) \Gamma(r, \alpha(r)) \right] \Phi(r, t, i) dW(r), \quad (t, s) \in D[0, T],
\end{aligned}
\]
(74)

The above is actually a system of coupled forward-backward stochastic Volterra integral equations with regime-switching. If the system (74) admits a solution $(\Phi(\cdot, \cdot), P(\cdot, \cdot))$, then the time-consistent equilibrium control is given by
\[
\vec{u}(t) = -\Gamma(t, \alpha(t)) \tilde{X}(t), \quad t \in [0, T].
\]  
(75)

The following result is a extension of Proposition 6.1 of [22].

**Proposition 5.1.** Suppose for any $j, i \in \mathbb{A}$,
\[
\begin{aligned}
A(\cdot, i), A_1(\cdot, i) &\in C ([0, T]; \mathbb{R}^{n \times n}), \quad B(\cdot, i), B_1(\cdot, i) \in C ([0, T]; \mathbb{R}^{n \times m}),
\end{aligned}
\]
\[
\begin{aligned}
Q(j, \cdot, i) &\in C (D[0, T]; \mathbb{S}^n_+), \quad R(\cdot, j, \cdot, i) \in C (D[0, T]; \mathbb{S}^m_+),
\end{aligned}
\]
\[
\begin{aligned}
G(\cdot, j, i) &\in C ([0, T]; \mathbb{S}^m_+),
\end{aligned}
\]
and for some constant $0 < L < \infty$,
\[
\max_{i \in \mathbb{A}} \sup_{t \in [0, T]} \left| R(t, i, t, i) + B_1^T(t, i) P B_1(t, i) \right|^{-1} \left[ B^T(t, i) P + B_1^T(t, i) P A_1(t, i) \right] \leq L.
\]  
(76)

Then (75) admits a unique solution.

**Proof.** Let
\[
\mathcal{X}(\tau, T) = \{ p(\cdot, \cdot) : [0, T] \times \mathbb{A} \rightarrow \mathbb{S}^n_+ \mid p(\cdot, i) \in C ([0, T]; \mathbb{S}^n_+), \text{ for all } i \in \mathbb{A} \},
\]
and $\| p(\cdot, \cdot) \|_{\mathcal{X}(\tau, T)} = \sum_{i \in \mathbb{A}} \| p(\cdot, i) \|_{C([0, T]; \mathbb{S}^n_+)}$ be the norm in $\mathcal{X}(\tau, T).$
Furthermore (suppressing 

By (76), we have \[ \sum_{i \in \mathcal{A}} |\Gamma(t, i; p(\cdot, \cdot))| \leq K, (t, i) \in [0, T] \times \mathcal{A}, \] with the bound \( K \) independent of \( p(\cdot, \cdot) \in \mathcal{X}(0, T) \). Let \( \Phi(\cdot, \cdot; p(\cdot, \cdot)) \) be the solution to the following:

\[
\Phi(s, t, i; p(\cdot, \cdot)) = I + \int_t^s \left[ A(r, \alpha(r)) - B(r, \alpha(r)) \Gamma(r, \alpha(r); p(\cdot, \cdot)) \right] \Phi(r, t, i; p(\cdot, \cdot)) dr \\
+ \int_t^s \left[ A_1(r, \alpha(r)) - B_1(r, \alpha(r)) \Gamma(r, \alpha(r); p(\cdot, \cdot)) \right] \Phi(r, t, i; p(\cdot, \cdot)) dW(r),
\]

for \((t, s) \in D[0, T]\). We have \[ \mathbb{E} \left[ \sum_{i \in \mathcal{A}} \sup_{(t, s) \in D[0, T]} |\Phi(s, t, i; p(\cdot, \cdot))|^2 \right] \leq K \] for some constant \( K \) independent of \( p(\cdot, \cdot) \in \mathcal{X}(0, T) \). For \((t, i) \in (0, T] \times \mathcal{A}, \) define

\[
P(t, i; p(\cdot, \cdot)) = \mathbb{E}_t \left\{ \Phi^\top(T, t, i; p(\cdot, \cdot)) G(t, i, \alpha(T)) \Phi(T, t, i; p(\cdot, \cdot)) \\
+ \int_t^T \Phi^\top(s, t, i; p(\cdot, \cdot)) \left[ Q(s, t, i, s, \alpha(s)) \\
+ \Gamma(s, \alpha(s); p(\cdot, \cdot)) R(s, t, i, s, \alpha(s)) \Gamma(s, \alpha(s); p(\cdot, \cdot)) \right] \Phi(s, t, i; p(\cdot, \cdot)) ds \right\}. \tag{77}
\]

Clearly, we have \( P(\cdot, \cdot) \in \mathcal{X}[0, T] \). In the following, we are going to show that \( p(\cdot, \cdot) \mapsto P(\cdot, \cdot; p(\cdot, \cdot)) \) admits a unique fixed point. Let

\[
\Gamma_k(\cdot, \cdot) = \Gamma(\cdot, \cdot; p_k(\cdot, \cdot)), \quad \Phi_k(\cdot, \cdot) = \Phi(\cdot, \cdot; p_k(\cdot, \cdot)), \quad P_k(\cdot, \cdot) = P(\cdot, \cdot; p_k(\cdot, \cdot)),
\]

where \( p_k(\cdot, \cdot) \in \mathcal{X}[0, T] \) and \( k = 1, 2 \).

Then (suppressing \((t, i)\)),

\[
|\Gamma_1(t, i) - \Gamma_2(t, i)| \\
\leq \left| R + B_1^\top p_1 B_1 \right|^{-1} \left| B_2^\top (p_1 - p_2) + B_1^\top (p_1 - p_2) A_1 \right| \\
+ \left| \left( \left| R + B_1^\top p_1 B_1 \right|^{-1} - \left| R + B_1^\top p_2 B_1 \right|^{-1} \right) B_2^\top p_2 + B_1^\top p_2 A_1 \right| \\
\leq K |p_1 - p_2| \\
+ \left| R + B_1^\top p_1 B_1 \right|^{-1} B_2^\top (p_1 - p_2) B_1 \left| R + B_1^\top p_2 B_1 \right|^{-1} B_2^\top p_2 + B_1^\top p_2 A_1 \right| \\
\leq K |p_1(t, i) - p_2(t, i)|. \tag{78}
\]

Furthermore (suppressing \((r, \alpha(r))\)),

\[
\Phi_1(s, t, i) - \Phi_2(s, t, i) \\
= \int_t^s \left\{ (A - B \Gamma_1) \left[ \Phi_1(r, t, i) - \Phi_2(r, t, i) \right] - B \left( \Gamma_1 - \Gamma_2 \right) \Phi_1(r, t, i) \right\} dr \\
+ \int_t^s \left\{ (A_1 - B_1 \Gamma_2) \left[ \Phi_1(r, t, i) - \Phi_2(r, t, i) \right] - B_1 \left( \Gamma_1 - \Gamma_2 \right) \Phi_1(r, t, i) \right\} dW(r).
\]

Thus,

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |\Phi_1(s, t, i) - \Phi_2(s, t, i)|^2 \right] \leq K \sum_{j \in \mathcal{A}} \int_t^s |\Gamma_1(r, j) - \Gamma_2(r, j)|^2 dr
\]
\[ \leq K \sum_{j \in A} \int_{t}^{s} |p_1(r, j) - p_2(r, j)|^2 \, dr. \]  

(79)

Therefore, by (77-79), we have (suppressing \((s, \alpha(s))\) for \(\Gamma_1(\cdot), \Gamma_2(\cdot), Q(\cdot)\) and \(R(\cdot)\)),

\[
|P_1(t, i) - P_2(t, i)| \\
\leq K \mathbb{E}_t \left\{ |\Phi_1(T, t, i) - \Phi_2(T, t, i)| + \int_{t}^{T} |\Gamma_1 - \Gamma_2| + |\Phi_1(s, t, i) - \Phi_2(s, t, i)| \, ds \right\} \\
\leq K \sum_{j \in A} \left( \int_{t}^{T} |p_1(r, j) - p_2(r, j)|^2 \, dr \right)^{\frac{1}{2}},
\]

where \(K > 0\) is a absolute constant. Hence, we have

\[
\sum_{i \in A} \sup_{t \in [T - \delta, T]} |P_1(t, i) - P_2(t, i)| \leq K \delta^{\frac{1}{2}} \sum_{i \in A} \sup_{t \in [T - \delta, T]} |p_1(t, i) - p_2(t, i)|.
\]

Therefore, \(p(\cdot, \cdot) \mapsto P(\cdot, \cdot)\) is a contractive map on \(X[T - \delta, T]\) as long as \(\delta > 0\) is small enough. Similarly, we have the same result on \([T - 2\delta, T - \delta]\), etc. Thus, there is a unique fixed point of \(p(\cdot, \cdot) \mapsto P(\cdot, \cdot)\) on \(X[0, T]\). \(\square\)

To illustrate our results, we assume that \(X(\cdot)\) is one-dimensional, the Markov chain has two states and the coefficients in the model only depend on the Markov. In this case, \(P(\cdot)\) satisfying (69) is independent of \(\tau\). Let \(T = 10, q_1 = 0.3, q_2 = 0.5, \) and the values of the other coefficients be given by 

\[
A = \begin{pmatrix} 0.3 \\ 0.1 \end{pmatrix}, B = \begin{pmatrix} 0.3 \\ 0.8 \end{pmatrix}, A_1 = \begin{pmatrix} 0.2 \\ 0.4 \end{pmatrix}, B_1 = \begin{pmatrix} 0.1 \\ 0.5 \end{pmatrix}, Q = \begin{pmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{pmatrix}, R = \begin{pmatrix} 0.2 & 0.5 \\ 0.4 & 0.7 \end{pmatrix}, G = \begin{pmatrix} 0.3 & 0.5 \\ 0.5 & 0.8 \end{pmatrix}.
\]

Here, for convenience, we write the coefficients in the matrix form, for example \(G(1, 1) = G_{11} = 0.3\). The solutions for for \(P(1, t, 1)\) and \(P(2, t, 2)\), which are needed for the time-consistent equilibrium strategy, are plotted in Figure 1.

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![Figure 1](image-url)  
**Figure 1.** The solutions for \(P(1, t, 1)\) and \(P(2, t, 2)\)
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