COHOMOLOGICAL DIMENSION IN PRO-\(p\) TOWERS

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Abstract. We give a proof without use of perfectoid geometry of Scholze’s vanishing theorem of étale cohomology with \(\mathbb{F}_p\)-coefficients beyond the dimension of projective varieties in a specific pro \(p\)-tower in characteristic not equal to \(p\).

1. Introduction

If \(X\) is a variety of finite type of dimension \(d\) defined over an algebraically closed field \(k\) of characteristic \(p > 0\), Artin-Schreier theory implies that the cohomological dimension of étale cohomology of \(X\) with \(\mathbb{F}_p\)-coefficients is \(d\), i.e. \(H^i(X, \mathbb{F}_p) = 0\) for \(i > d\). If \(k\) has characteristic not equal to \(p\), the cohomological dimension of étale cohomology of \(X\) with \(\mathbb{F}_p\)-coefficients is \(2d\) if \(X\) is proper, and \(d\) if \(X\) is affine by Artin’s vanishing theorem.

However, when \(X\) is projective, Peter Scholze showed that there is a specific tower of degree \(p\)-power covers of \(X\) which makes its cohomological dimension equal to \(d\) in the limit.

Let \(X \subset \mathbb{P}^n\) be a projective variety of dimension \(d\). We choose coordinates \((x_0 : \ldots : x_n)\) on \(\mathbb{P}^n\). With this choice of coordinates, we define the covers

\[
\phi_r^n : \mathbb{P}^n \to \mathbb{P}^n, \quad (x_0 : \ldots : x_n) \mapsto (x_0^{p^r} : \ldots : x_n^{p^r}).
\]

We define \(X_r\) as the inverse image of \(X\) by \(\phi_r^n\).

Theorem 1.1 (Scholze, [Sch14, Theorem 17.3]). If \(k\) is an algebraically closed of characteristic \(0\), for \(i > d\), one has \(\lim_r H^i(X_r, \mathbb{F}_p) = 0\).

Scholze obtains the theorem as a corollary of his theory of perfectoid spaces. He does not detail the proof in loc. cit., but his argument is documented in [Sch15]. By classical base change, we may assume that \(k = \hat{\mathbb{Q}}_p\). By the comparison theorem [Sch15, Thm.IV.2.1], \(\lim_r H^r(X_r, \mathbb{F}_p) \otimes \mathcal{O}_C/p\) is ‘almost’ equal to \(H^r(\mathcal{X}, \mathcal{O}^{+}_\mathcal{X}/p)\) where \(\mathcal{X}\) is a perfectoid space he constructs, associated to \(\lim_r X_r\), and \(C = \hat{\mathbb{Q}}_p\). By [Sche92, Thm. 4.5], the spectral space \(\mathcal{X}\) has cohomological dimension at most the Krull dimension of \(X\).

The aim of this short note is to give an elementary proof, as was asked for over \(\mathbb{C}\) in [Sch14, Section 17]. It turns out that the proof holds in characteristic not equal to \(p\) as well. One obtains

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Theorem 1.2. If $k$ is an algebraically closed of characteristic not equal to $p$, for $i > d$, one has

$$\lim_{r \to} H^i(X_r, \mathbb{F}_p) = 0.$$ 

The ingredients are constructibility and base change properties for relative étale cohomology with compact supports, functoriality, and some easy fact of representation theory of a cyclic group of $p$-power order.

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2. General reduction

As all cohomologies considered are étale cohomology with coefficients in $\mathbb{F}_p$, we drop $\mathbb{F}_p$ from the notation whenever it does not create confusion.

As étale cohomology only depends on the underlying reduced structure, we may assume that $X$ is reduced.

If $X$ has dimension 0, its cohomological dimension is 0 so there is nothing to prove. Let $X$ be of dimension $d \geq 1$. If $X$ has at least two components, then it is the union of $X_1$ and $X_2$ where $X_2$ is irreducible and is not contained in $X_1$. Then $X_1 \cap X_2$ had dimension $\leq (d - 1)$. The Mayer-Vietoris exact sequence

$$\to H^{i-1}(X_1 \cap X_2) \to H^i(X_1 \cup X_2) \to H^i(X_1) \oplus H^i(X_2) \to H^i(X_1 \cap X_2) \to$$

shows that $H^i(X_1) = H^i(X_2) = H^{i-1}(X_1 \cap X_2) = 0$ for $i > d$, implies $H^i(X_1 \cup X_2) = 0$ for $i > d$. So the theorem in its general form follows from the theorem in the case where $X$ is irreducible.

Let $H_a \subset \mathbb{P}^n$ denote the hyperplane defined by $x_a = 0$, and $Y_a = H_a \cap X$. If there is one $Y_a$ such that the dimension of $X \cap Y_a$ is $d$, then $X \cap Y_a = X$ as $X$ is irreducible, and one replaces in the statement and the proof $\mathbb{P}^n$ by $Y_a = \mathbb{P}^{n-1}$. So we may assume that $Y = \bigcup Y_a$ is a divisor on $X$. Let $U \subset X \setminus Y$ be open and dense, $Z = X \setminus U \supset Y$ be the boundary divisor. We let $U_r$ resp. $Y_r$ resp. $Z_r$ denote the pull-back of $U$, resp. $Y$, resp. $Z$ along $\phi^U_r$. Then $U_r = X_r \setminus Z_r$ is open dense, and $Z_r$ is a divisor on $X_r$. The morphism $\phi^U_r$ restricted to $U_r$ is proper and étale, thus the direct system $\lim_{r \to} H^i_c(U_r)$ of étale cohomology with compact supports and coefficients $\mathbb{F}_p$ is defined.

From the Gysin sequence

$$\to H^{i-1}(Z_r) \to H^i_c(U_r) \to H^i(X_r) \to H^i(Z_r) \to \ldots$$

and induction on the dimension, one deduces that the theorem is true if and only if

$$\lim_{r \to} H^i_c(U_r) = 0$$

for $i > d$. 

On the other hand, for \( d = n \) then \( X = \mathbb{P}^n \), \( H^{2i}(\mathbb{P}^n) = \mathbb{F}_p \cdot [L] \) where \( L \) is linear of codimension \( i \). Thus \( \phi^n_1[L] = \mathbb{F}_p \cdot p^i[L] \) which is equal to 0 as soon as \( i > 0 \).

So throughout the rest of the note, we make the general

**Assumption 2.1.** \( X \) is an irreducible, reduced projective variety of dimension \( d \) with \( 0 < d < n \) over an algebraically closed field \( k \) of characteristic not equal to \( p \).

And we want to draw the conclusion

\[
\lim_{r \to \infty} H^i_c(U_r) = 0 \text{ for all } i > d.
\]

### 3. Local systems

#### 3.1. Geometry preparation.

We define the torus \( T = \mathbb{P}^n \setminus \bigcup_{i=0}^{n} H_i \). It has coordinates \( (\frac{x_0}{x_1}, \ldots, \frac{x_n}{x_0}) \). In particular, the projection \( q : T \to \mathbb{G}_m \) to either of the factors has the property that \( q \circ \phi^n_r \) factors through \( \phi^1_r \). For \( U \subset T \cap X \) open dense, there is a projection \( \eta : U \to \mathbb{G}_m \) to one of the factors which is dominant. As \( U \) is irreducible, all the fibers of \( \eta \) have dimension \( \leq (d - 1) \).

One has the commutative diagram

\[
\begin{array}{ccc}
U_r & \xrightarrow{\phi^n_r} & U' \xrightarrow{q'} U' \xrightarrow{q} U \\
\downarrow{q_r} & & \downarrow{q} \\
(G_m), & \xrightarrow{\phi^1_r} & G_m
\end{array}
\]

defining \( q_r \), where \( (U'_r = U \times_{\mathbb{G}_m, \phi^1_r} \mathbb{G}_m, q'_r = q \times \phi^1_r) \).

#### 3.2. Constructibility.

Recall that \( R^j_q! \mathbb{F}_p \) is constructible, see [SGA4 Thm. 5.3.5]. As \( \phi^1_r \) is proper, for any \( j \in \mathbb{N} \), \( \phi^1_r*(R^j_q! \mathbb{F}_p) \) is constructible as well and one has a morphism

\[
\phi^1_r*(R^j_q! \mathbb{F}_p) \to R^j q_r! \mathbb{F}_p
\]

of constructible sheaves, which for any \( i \in \mathbb{N} \), induces a \( \mathbb{F}_p \)-linear map

\[
\phi^n_r* : H^i_c((\mathbb{G}_m, R^j_q! \mathbb{F}_p) \to H^i_c((\mathbb{G}_m)_r, \phi^1_r*(R^j_q! \mathbb{F}_p)) \to H^i_c((\mathbb{G}_m)_r, R^j_q_r! \mathbb{F}_p) = H^i_c((\mathbb{G}_m, \phi^1_r R^j q_r! \mathbb{F}_p)).
\]

Here the left map is defined by adjunction.

We pose the

**Induction hypothesis:** Given an irreducible subvariety \( X \subset \mathbb{P}^n \) of dimension \( d' \) and a Zariski open subvariety \( U \subset T \cap X \), such that \( X \) is the closure of \( U \), there is a natural number \( r_0 \), such that for all \( r \geq r_0 \), the map \( \phi^n_r* : H^i_c(U) \to H^i_c(U_r) \) vanishes for all \( i > d' \).
The induction hypothesis is trivially verified for \( d' = 0 \). In the sequel we assume it is verified for \( d' \leq d - 1 \).

**Lemma 3.1.** With the assumption \[ \text{3.1} \] on \( X \) and \((U, q)\) as in \[ \text{3.1} \] for \( j > d - 1 \), there is an \( r_1 \in \mathbb{N} \) such that for all \( r \geq r_1 \)

\[
\phi_r^{n*} : R^j q_1^F \mathbb{F}_p \to \phi_r^1 R^j q_{r!}^F \mathbb{F}_p
\]

vanishes.

**Proof.** By [SGA4, Thm. 5.2.8], \( R^j q_1^F \mathbb{F}_p \) verifies base change with stalks \((R^j q_1^F \mathbb{F}_p)_x = H^i_c(q^{-1}(x))\) on geometric points \( x \in \mathbb{G}_m \). The map

\[
\phi_r^{n*} : H^i_c(q^{-1}(x)) \to H^i_c(q^{-1}(\phi_r^1)^{-1}(x))
\]

vanishes by induction for \( r \geq r(x) \) large enough depending on \( x \). Taking \( x \) to be the geometric generic point \( \text{Spec}(k(\mathbb{G}_m)) \) defines \( r = r(x) \). If \( \mathcal{U} \subset (\mathbb{G}_m)_r \) is a dense open on which \( R^j q_{r!}^F \mathbb{F}_p \) is a local system, which is lying in the smooth locus \( \mathcal{U}^0 \) of \( \phi_r^1 \), then \( \cap_g \mathcal{U} \), for \( g \) in the Galois group \( \mathbb{Z}/p^r \) of \( \mathcal{U}^0/\phi_r^1(\mathcal{U}^0) \), is Galois invariant and dense in \((\mathbb{G}_m)_r \), thus of the shape \((\mathbb{G}_m)_r \), for some dense open \( \mathbb{G}_m^0 \subset \mathbb{G}_m \). Then for all closed points \( x \in \mathbb{G}_m^0 \), we may take \( r \) constant. We take \( r_1 \) greater or equal to \( r \) and to the finitely many \( r(x) \) for \( x \) closed in \( \mathbb{G}_m \setminus \mathbb{G}_m^0 \). This finishes the proof.

\[ \square \]

### 3.3. Representation theory.

With the assumption \[ \text{2.1} \] on \( X \) and \((U, q)\) as in \[ \text{3.1} \] we fix some \( j \) and consider the dense open \( \mathbb{G}_m^0 \subset \mathbb{G}_m \) over which \( R^j q_{r!}^F \mathbb{F}_p \) is a local system, \( q \) is smooth and the fibers \( q^{-1}(x) \) are irreducible. In particular the Gysin map

\[
H^2_c(\mathbb{G}_m^0, R^j q_1^F \mathbb{F}_p) \to H^2_c(\mathbb{G}_m, R^j q_{r!}^F \mathbb{F}_p)
\]

is an isomorphism.

**Proposition 3.2.** There is an \( r_2 \in \mathbb{N} \) such that for all \( j \in \mathbb{N} \), all \( r \geq r_2 \),

\[
\phi_r^{1*} : H^2_c(\mathbb{G}_m, R^j q_1^F \mathbb{F}_p) \to H^2_c((\mathbb{G}_m)_r, R^j q_{r!}^F \mathbb{F}_p)
\]

vanishes.

**Proof.** On \( \mathbb{G}_m^0 \), we denote by \( \mathcal{V} \) the local system of \( \mathbb{F}_p \)-vector spaces dual to \( R^j q_1^F \mathbb{F}_p \). Then \( H^2_c(\mathbb{G}_m, R^j q_1^F \mathbb{F}_p) \) is dual to \( H^0((\mathbb{G}_m)_r, \mathcal{V}) \). On the other hand, on \((\mathbb{G}_m)_r \), one has \( \phi_r^{1*} R^j q_1^F \mathbb{F}_p = R^j q_{r!}^F \mathbb{F}_p \) thus \( \phi_r^{1*} \mathcal{V} \) is the local system dual to \( R^j q_{r!}^F \mathbb{F}_p \). Thus

\[
\phi_r^{1*} : H^2_c(\mathbb{G}_m, R^j q_1^F \mathbb{F}_p) \to H^2_c((\mathbb{G}_m)_r, R^j q_{r!}^F \mathbb{F}_p)
\]

is dual to the trace map

\[
\text{Tr}(\phi_r^1) : H^0((\mathbb{G}_m)_r, \phi_r^{1*} \mathcal{V}) \to H^0((\mathbb{G}_m)_r, \mathcal{V})
\]

from which we show now that it vanishes for \( r \) large. As \( \mathcal{V} \) is a local system, the dimension of \( H^0((\mathbb{G}_m)_r, \phi_r^{1*} \mathcal{V}) \) as an \( \mathbb{F}_p \)-vector space is bounded above by the rank of \( R^j q_1 \mathbb{F}_p \) and thus does not depend on \( r \). For \( N \) a natural number, in \( GL(N, \mathbb{F}_p) \) the order of a \( p \)-power element is bounded by a constant depending on \( N \) and \( p \). Thus for \( r \) large, the representation \( \rho \) of
the Galois group \( \mathbb{Z}/p^s \) of \( \phi_r^1 \) on \( H^0(\mathbb{G}_m^0, \phi_r^1 \mathcal V) \) can not be faithful. Thus \( \rho \) factors as \( \bar{\rho} \) through \( \mathbb{Z}/p^s \) for some \( s < r \). This implies that for any \( v \in H^0(\mathbb{G}_m^0, \phi_r^1 \mathcal V) \)

\[
\text{Tr}(\phi_r^1)(v) = \sum_{i=0}^{p^r-1} \rho(i)(v) = \sum_{i \in \mathbb{Z}/p^s} \sum_{j=0}^{p^r-s-1} \rho(i + jp^s)(v) = p^{r-s}\left( \sum_{i \in \mathbb{Z}/p^s} \bar{\rho}(i)(v) \right) = 0.
\]

In the formula, \( i \in \mathbb{Z}/p^s \) and maps to \( \bar{i} \in \mathbb{Z}/p^s \). This finishes the proof. \( \square \)

**Corollary 3.3.** With \( r_2 \) as in Proposition 3.2, for all \( r \geq r_2 \),

\[
\phi_r^{n*}: H^2_c(\mathbb{G}_m, R^j q^t \mathbb{F}_p) \to H^2_c(\mathbb{G}_m^0, R^j q^t \mathbb{F}_p)
\]

vanishes.

**Proof.** One has a factorization

\[
\phi_r^{n*}: H^2_c(\mathbb{G}_m, R^j q^t \mathbb{F}_p) \xrightarrow{\phi_r^{1*}} H^2_c(\mathbb{G}_m, R^j q^t \mathbb{F}_p) = H^2_c(\mathbb{G}_m, \phi_r^{1*} R^j q^t \mathbb{F}_p) \\
\to H^2_c(\mathbb{G}_m, R^j q^t \mathbb{F}_p)
\]

where the first map is the one considered in Proposition 3.2 and the second one comes by functoriality \( R^j q^t \mathbb{F}_p \to R^j q^t \mathbb{F}_p \). This finishes the proof. \( \square \)

4. **Proof of Theorem 1.2**

We argue by induction on \( d \), starting with \( d = 0 \). We use the notations of the previous sections, make the assumption \( \mathcal X \) on \( X \) and take \( (U, q) \) as in 3.4.

We consider the Leray spectral sequence for \( q \) and \( H^i_c(U) \) for \( i > d \). One first has the corner map

\[
H^i(U) \to H^0(\mathbb{G}_m, R^j q^t \mathbb{F}_p).
\]

As \( i > d > d - 1 \) we apply Lemma 3.1. Thus there is a \( r_1 \) such that for all \( r \geq r_1 \), \( \phi_r^{n*} H^i(U) \) maps to a subquotient of

\[
H^1_c(\mathbb{G}_m^0, R^{i-1} q^t \mathbb{F}_p).
\]

As \( i - 1 > d - 1 \) the same argument show that there is a \( r_1' \geq r_1 \) such that for all \( r \geq r_1' \), \( \phi_r^{n*} H^i(U) \) maps to a quotient of

\[
H^2_c(\mathbb{G}_m^0, R^{i-2} q^t \mathbb{F}_p).
\]

If \( i > d + 1 \) then \( i - 2 > d - 1 \) and one applies again the same argument which finishes the proof. If \( i = d + 1 \), then \( i - 2 = d - 1 \), Corollary 3.3 implies that there is a \( r_2 \geq r_1' \) such that for all \( r \geq r_2 \), \( \phi_r^{n*} H^i(U) \) maps to 0. This finishes the proof of Theorem 1.2.
5. Degree estimates

Even if the proof of Theorem 1.2 does not use the degree estimates, it is perhaps helpful to write them. Lemma 5.1 implies for $X$ irreducible Theorem 1.2 for $i = 2d$, so in particular for $d = 1$, as $H^{2d}(X) = \mathcal{F} \cdot [x]$ where $[x]$ is the cycle class of a rational point.

Let $\delta$ be the degree of $X$ with respect to the projective embedding $X \subset \mathbb{P}^n$. By definition,

$$\delta = X \cdot H^{1\ldots d}$$

where $H^{1\ldots d} = H \cdot \ldots \cdot H$, and where $H = H_1$. Let $X_{r,a}, a = 1, \ldots, N(r)$, be the irreducible components of $X_r$, and let $D_a$ be the degree of $\phi_r^n$ restricted to $X_{r,a}$.

Lemma 5.1. With the assumption 2.1 on $X$, $D_a = D$ does not depend on $a = 1, \ldots, N(r)$ and there is an $r_0 \in \mathbb{N}$ depending on $\delta, n$ and $p$ such that for all $r \geq r_0$, $D = p^{m'}$ for some natural number $m' \geq 1$.

Proof. The degree $\delta_r$ of $X_r$ is computed by projection formula

$$\delta_r = X_r \cdot H^{1\ldots d} = \deg(\phi_r^n|_{H^{1\ldots d}}) X_r \cdot H^{1\ldots d} = p^{r(n-d)} \delta.$$

On the other hand, $\phi_r^n|_{X_r} : X_r \to X$ has degree $p^{nr}$. As $p^{nr} (p^{(n-d)} \delta)^{-1} = \delta^{-1} p^{dr} > 1$ for $r$ large, we conclude that $X_r$ can not be the union of $p^{nr}$-components, each of which of degree 1 over $X$. On the other hand $\phi_r^n|_{X_{r,a}}$ is finite and of degree $D_a$. As $\phi_r^n|_{U_r}$ is finite Galois étale, $D_a$ does not depend on $a$. Thus $D_a = D > 1$. The equality $N(r)D = p^{nr}$ concludes the proof. \Box

6. Remarks

1) If $Z \subset \mathbb{P}^n$ is any locally closed subscheme, applying again the Gysin argument one sees that Theorem 1.1 implies (and in fact is equivalent to)

$$\lim_{r \to \infty} \mathcal{H}^i_c(Z_r) = 0$$

for all $i > d$, where $Z_r = (\phi_r^n)^{-1}(Z)$.

2) It may happen that in Lemma 5.1 the minimum possible $r_0$ is $\geq 2$. For example if $p = 2$, and $X$ is a smooth conic in $\mathbb{P}^2$ such that the $H_i$, $i = 0, 1, 2$ are tangent to $X$, then $X_1$ splits entirely into the union of fours lines. So the minimum $r$ which kills the whole cohomology $H^i(X)$, $i > d$ is perhaps an intriguing geometric invariant of the triple

$$(X, \mathbb{P}^n, (x_0 : \ldots : x_n))$$

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