Graphs with domination roots in the right half-plane

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\textbf{ABSTRACT}

Let $G$ be a simple graph of order $n$. The domination polynomial of $G$ is the polynomial
$D(G, \lambda) = \sum_{i=0}^{n} d(G, i)\lambda^{i}$, where $d(G, i)$ is the number of dominating sets of $G$ of size $i$.

Every root of $D(G, \lambda)$ is called the domination root of $G$. In this paper we study some families of graphs whose complex domination roots have positive real part.

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\section{Introduction}

Let $G$ be a simple graph. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. An $i$-subset of $V(G)$ is a

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subset of $V(G)$ of cardinality $i$. Let $D(G,i)$ be the family of dominating sets of $G$ which are $i$-subsets and let $d(G,i) = |D(G,i)|$. The polynomial $D(G,x) = \sum_{i=0}^{\frac{|V(G)|}{2}} d(G,i)x^i$ is defined as domination polynomial of $G$ ([3, 4]). A root of $D(G,x)$ is called a domination root of $G$. We denote the set of all roots of $D(G,x)$ by $Z(D(G,x))$. For more information and motivation of domination polynomial and domination roots refer to [3, 6, 7, 8, 9].

In Section 2 we obtain the domination polynomial of Dutch Windmill graphs. We show that no nonzero real number is domination root of these kind of graphs. Also we see that these kind of graphs have domination roots in the right half-plane. In Section 3 we study the domination polynomial of other classes of graphs related to Dutch Windmill graphs.

2 Graphs with domination roots in the right half-plane

The roots of domination polynomial was studied recently by several authors, see [2, 3, 4]. It is clear that $(0, \infty)$ is zero free interval for domination polynomial of a graph. It is interesting that to investigate graphs which have complex domination roots with positive real parts.

Here we consider the graphs obtained by selecting one vertex in each of $n$ triangles and identifying them. Some call them Dutch Windmill Graphs [13]. See Figure 1. We denote these graphs by $G^3_n$. Note that these graphs also called friendship graphs. We obtain domination polynomial of these graphs and show that there are some of these graphs whose have complex domination roots with positive real parts.

To obtain the domination polynomial of $G^3_n$ we need some preliminaries.

Theorem 1. ([3]) For every $n \in \mathbb{N}$

$$D(K_{1,n},x) = x^n + x(1+x)^n.$$
Theorem 2. (3) Let $G_1$ and $G_2$ be graphs of orders $n_1$ and $n_2$, respectively. Then

$$D(G_1 + G_2, x) = ((1 + x)^{n_1} - 1)((1 + x)^{n_2} - 1) + D(G_1, x) + D(G_2, x).$$

The vertex contraction $G/v$ of a graph $G$ by a vertex $v$ is the operation under which all vertices in $N(v)$ are joined to each other and then $v$ is deleted (see [12]).

The following result also appear in [11] but as stated in [11] were proved independently.

Theorem 3. (5, 10) For any vertex $v$ in a graph $G$ we have

$$D(G, x) = xD(G/v, x) + D(G - v, x) + xD(G - N[v], x) - (x + 1)p_v(G, x)$$

where $p_v(G, x)$ is the polynomial counting those dominating sets for $G - N[v]$ which additionally dominate the vertices of $N(v)$ in $G$.

Theorem 3 can be used to give a recurrence relation which removes triangles. Define a new operation on edges incident to a vertex $u$: we denote by $G \odot u$ the graph obtained from $G$ by the removal of all edges between any pair of neighbors of $u$. Note $u$ is not removed from the graph. The following recurrence relation is useful on graphs which have many triangles.
Theorem 4. Let $G = (V, E)$ be a graph and $u \in V$. Then

$$D(G, x) = D(G - u, x) + D(G \odot u, x) - D(G \odot u - u, x)$$

Theorem 5. For every $n \in \mathbb{N}$,

$$D(G^m_3, x) = (2x + x^2)^n + x(1 + x)^{2n}.$$ 

Proof. Suppose that $u$ is center vertex of $G^m_3$. By above definition $G^m_3 \odot u = K_{1,2n}$. Also $G^m_3 - u = \bigcup_{i=1}^{2n} K_2$ and $G^m_3 \odot u - u = \bigcup_{i=1}^{2n} K_1$. By Theorems 1 and 4 we have the result. 

Remark. Note that another approach for computing the domination polynomial of $G^m_3$ is Theorem 2.

In [2] authors stated the following problem:

Problem. Characterize all graphs with no real domination root except zero.

One of the family with no nonzero real domination roots is $K_{n,n}$ for even $n$:

Theorem 6. For every even $n$, no nonzero real numbers is domination root of $K_{n,n}$.

Proof. It is easy to see that

$$D(K_{n,n}, x) = \left((1 + x)^n - 1\right)^2 + 2x^n.$$ 

If $D(K_{n,n}, x) = 0$, then $\left((1 + x)^n - 1\right)^2 = -2x^n$. Obviously this equation does not have real nonzero solution for even $n$.

Here we prove that Dutch windmill graph $G^m_3$ have no real roots except zero.
**Theorem 7.** For every natural $n$, no nonzero real numbers is domination root of $G_3^n$.

**Proof.** By Theorem 5, for every $n \in \mathbb{N}$, 

$$D(G_3^n, x) = (2x + x^2)^n + x(1 + x)^{2n}.$$ 

If $D(G_3^n, x) = 0$, then we have 

$$x = -\left(1 - \frac{1}{(1 + x)^2}\right)^n.$$ 

First suppose that $x \geq 0$. Obviously the above equality is true just for real number 0, since for nonzero real number the left side of equality is positive but the right side is negative. Now suppose that $x < -1$. In this case the left side is less than $-1$ and the right side $-\left(1 - \frac{1}{(1 + x)^2}\right)^n$ is greater than $-1$, a contradiction. Finally we shall consider $-1 < x < 0$. This case is similar to the second case when we substitute $x$ with $\frac{1}{x}$. \[\square\]

**Remark.** Using Maple we observed that the domination polynomial of $G_3^n$ for $n \geq 6$ have complex roots with positive real parts. For example $D(G_3^6, x)$ has complex root with real part 0.0003550296365. See Figure 2.

3 The domination polynomial of other classes of graphs

In this section we use Theorem 4 to study the domination polynomial of other classes of graphs.

(1) Let $G_n$ be the Dutch Windmill graph with an extra edge $vu$. The three graphs $G - u$, $G \odot u$ and $G \odot u - u$ in Theorem 4 are (i) $K_2$, $n$ times and $K_1$; (ii) $K_{1, 2n+1}$; and (iii) $P_1$, $2n + 1$ times, respectively. So, by Theorem 4 we have

$$D(G_n, x) = x(2x + x^2)^n + x^{2n+1} + x(1 + x)^{2n+1} - x^{2n+1}$$

$$= x((x^2 + 2x)^n + (x + 1)^{2n+1}).$$

See (A213658).
Figure 2: Domination roots of $G_n^3$ for $2 \leq n \leq 40$.

It is interesting that the graph $G_{30}$ has domination roots in the right-half plane (Figure 3). Also for these kind of family of graphs we have the following theorem:

**Theorem 8.** For every even natural $n$, no nonzero real numbers is domination root of $G_n$, and for odd $n$ there is one nonzero real root in $(-1, 0)$.

**Proof.** Suppose that $\alpha$ is a root of $D(G_n, x)$. So we have $\left(1 + \frac{1}{(\alpha + 1)^2 - 1}\right)^n = -\frac{1}{\alpha + 1}$.

Obviously this equation does not hold for positive $\alpha$ and $\alpha \leq -1$. Let $\alpha \in (-1, 0)$. So equality is not true for even $n$. If $n$ is odd, by substituting $\alpha + 1 = t$, we shall have $t^3 + t^2 - 1 = 0$. Therefore this equation and so $D(G_n, x)$ have only one real root in $(-1, 0)$. $\Box$

As you can see in the Figure 3, the graph $G_{30}$ has domination roots in the right-half plane.
The fan graph. A fan graph $F_{m,n}$ is defined as the graph join $\overline{K}_m + P_n$, where $\overline{K}_m$ is the empty graph on $m$ vertices and is $P_n$ the path graph on $n$ vertices. First we consider $F_{2,n}$. To obtain $F_{2,n}$ take an edge $uv$ and join each of $n$ vertices $1, 2, ..., n$ to both $u$ and $v$ ($n$ triangles with a common edge). Taking our $u$ to be the $u$ in Theorem 4 the three graphs are: (i) $K_{1,n}$; (ii) $K_{1,n+1}$; (iii) $P_1$, $(n+1)$-times. So,

$$D(F_{2,n}, x) = x^n + x(1 + x)^n + x^{n+1} + x(1 + x)^{n+1} - x^{n+1}$$

$$= x^n + x(1 + x)^n(2 + x).$$

See (A213657).

The domination roots of graph $F_{2,60}$ has shown in Figure 3.

![Figure 3: Domination roots of graphs $G_{30}$ and $F_{2,60}$, respectively.](image)

Using Theorem 2 we have the following corollary:

**Corollary 1.** For every natural $m, n \in \mathbb{N},$

$$D(F_{m,n}, x) = ((1 + x)^m - 1)((1 + x)^n - 1) + x^m + D(P_n, x).$$

The Gem graph $G$. Consider the path $P_{n+1}$ and an additional vertex $u$; join $u$ to each vertex of the path. The three graphs in Theorem 4 are: (i) $P_{n+1}$; (ii) the star
Figure 4: Domination roots of graphs $G'_70$ and $W_{70}$, respectively.

$K_{1,n+1}$; (iii) $P_1$, $(n + 1)$ times. So,

$$D(G, x) = D(P_{n+1}, x) + x^{n+1} + x(1 + x)^{n+1} - x^{n+1} = D(P_{n+1}, x) + x(1 + x)^{n+1}$$

See (A213662).

(4) Let $G'_n$ be the Gem graph with an extra edge $vu$. The three graphs in Theorem 4 are (i) $P_{n+1}$ and $P_1$; (ii) $K_{1,n+2}$; (iii) $P_1$, $(n + 2)$ times. So,

$$D(G', x) = xD(P_{n+1}, x) + x^{n+2} + x(1 + x)^{n+2} - x^{n+2} = x(D(P_{n+1}, x) + (1 + x)^{n+2})$$

(5) Join a vertex $u$ with two consecutive vertices of the cycle $C_n$ (i.e. a triangle placed on an edge of $C_n$). Let to denote this graph by $G$. The three graphs in Theorem 4 are: (i) $C_n$; (ii) $C_{n+1}$; (iii) $P_n$. So,

$$D(G, x) = D(C_n, x) + D(C_{n+1}, x) - D(P_n, x).$$

(A213664)

(6) The wheel graph $W_n$. The three graphs in Theorem 4 are (i) $C(n - 1)$; (ii) $K_{1,n-1}$; (iii) $P_1$, $n - 1$ times. So,

$$D(W_n, x) = D(C_{n-1}, x) + x^{n-1} + x(1 + x)^{n-1} - x^{n-1} = D(C_{n-1}, x) + x(1 + x)^{n-1}.$$
Remark. As you see in the Figure 4, there are graphs in the families of $G'_n$ and $W_n$ which their domination roots are in the right half-plane.

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