Strong and Weak Forms of $\mu$-Kc-Spaces

Nadia A. Nadhim$^1$, Haider J. Ali$^2$, Rasha N. Majeed$^3$

$^1$Department of Mathematics, Faculty of Education for Pure Sciences, University of AL-Anbar, AL-Anbar, Iraq
$^2$Department of Mathematics, College of Science, University of AL-Mustansiriya, Baghdad, Iraq
$^3$Department of Mathematics, Faculty of Education for Pure Sciences Abn AL-Haitham, University of Baghdad, Baghdad, Iraq

Received: 5/5/ 2019 Accepted: 21/9/2019

Abstract

In this paper, we provide some types of $\mu$-Kc-spaces, namely, $\mu$-$K(ac)$- (respectively, $\mu$-$aK(ac)$-) and $\mu$-$\theta K(c)$- spaces for minimal structure spaces which are denoted by (m-spaces). Some properties and examples are given. The relationships between a number of types of $\mu$-Kc-spaces and the other existing types of weaker and stronger forms of m-spaces are investigated. Finally, new types of open (respectively, closed) functions of m-spaces are introduced and some of their properties are studied.

Keywords: Kc-space, minimal structure spaces, $\mu$-Kc-space, $\alpha$-open, $\theta$-open.

1. Introduction

The concept of $Kc$-space was introduced by Wilansky [1], that is "A topological space $(X,T)$ is said to be $Kc$-space if every compact subset of $X$ is closed". Also, many important properties were provided by that study, e.g., "Every $Kc$-space is $T_1$-space" and "every $T_2$-space is $Kc$-space". In 1996, Maki [2] introduced the minimal structure spaces, shortly m-spaces, that is "A sub collection $\mu$ of $P(X)$ is called the minimal structure of $X$, if $\emptyset \in \mu$ and $X \in \mu$, $(X, \mu)$ is said to be m-structure space". The elements of $\mu$ are called $\mu$-open sets and their complements are $\mu$-closed sets, which is a generalization of topological spaces. Popa and Noiri [3] studied the m-spaces and defined the notion of continuous functions between them. In 2015, Ali et al. [4] defined the concept of $Kc$-space with

*Email: na8496292@gmail.com
respect to the \( m \)-space to obtain a new space which they called the \( \mu\)-\( Kc \)-space. A weaker and stronger form of open sets plays an important role in topological spaces. In 1965, Najastad [5] introduced the concept of \( \alpha \)-open sets as a generalization of open sets. That is, let \(( X, T )\) be a topological space and a nonempty subset \( \mathcal{A} \) of \( X \) is said to be \( \alpha \)-open set, if \( \mathcal{A} \subseteq \text{Int}(\text{Cl}(\text{Int}(\mathcal{A}))) \). In 2010, Min [6] generalized the concept of \( \alpha \)-open sets to \( m \)-spaces. On the other hand, in 1968, Velicko [7] introduced the concept of \( \theta \)-open sets. That is “Let \(( X, T )\) be a topological space, \( N \subseteq X \), a point \( b \in X \) is said to be an \( \theta \mu \)-adherent point for a subset \( N \) of \( X \), if \( N \cap \text{Cl}(G) \neq \emptyset \) for any open set \( G \) of \( X \) and \( b \in N \). The set of \( \theta \)-adherent point is said to be an \( \theta \)-closure of \( N \) which is denoted by \( \theta \text{Cl}(N) \). A subset \( N \) of \( X \) is called \( \theta \)-closed set if every point to \( N \) is an \( \theta \)-adherent point. Also, in 2018, Makki [8] defined \( \theta \)-open sets in \( m \)-space. The aim of the present paper is to introduce and study new type of \( \mu \)-\( Kc \)-spaces, namely, \( \mu\)-\( K(\alpha) \)- (resp. \( \mu\)-\( \alpha\mathcal{K}(c) \)- , \( \mu\)-\( \mathcal{K}(\alpha\mathcal{C}) \)- and \( \mu\)-\( \theta\mathcal{K}(c) \)-) spaces by using the concept of \( \alpha \)-open, respectively \( \theta \)-open sets, with respect to the \( m \)-space. We study the basic properties of each space and give the relationships between them. Also, we introduce new kinds of continuous, open (respectively closed) functions on \( m \)-spaces and investigate their properties.

2. Preliminaries

Let us recall the following definitions, properties and theorems which we need in this work.

**Definition 2.1** [3] Let \( X \) be a non-empty set and \( P(X) \) be the power set of \( X \). A sub collection \( \mu \) of \( P(X) \) is called the minimal structure of \( X \), if \( \emptyset \in \mu \) and \( X \in \mu \). \( (X, \mu) \) is said to be \( m \)-structure space (shortly, \( m \)-spaces). The elements of \( \mu \) are called \( \mu \)-open sets and their complements are \( \mu \)-closed sets.

For a subset \( \mathcal{B} \) in an \( m \)-space on \( (X, \mu) \), the interior (respectively, closure) of \( \mathcal{B} \) denoted by \( \mu\text{Int}(\mathcal{B}) \) (respectively, \( \mu\text{Cl}(\mathcal{B}) \)) is defined as follows:

\[
\mu\text{Int}(\mathcal{B}) = \bigcup \{U : U \subseteq \mathcal{B}, U \in \mu\} \quad \text{and} \quad \mu\text{Cl}(\mathcal{B}) = \bigcap \{F : F \subseteq \mathcal{F}, F^c \in \mu\}.
\]

**Remark 2.2** Note that according to a previous study [9], \( \mu\text{Int}(\mathcal{B}) \) (respectively, \( \mu\text{Cl}(\mathcal{B}) \)) is not necessarily \( \mu \)-open (respectively, \( \mu \)-closed), but if \( \mathcal{B} \) is \( \mu \)-open then \( \mathcal{B} = \mu\text{Int}(\mathcal{B}) \), respectively, and if \( \mathcal{B} \) is \( \mu \)-closed, then \( \mathcal{B} = \mu\text{Cl}(\mathcal{B}) \).

**Definition 2.3** [10] an \( m \)-space \((X, \mu)\) has a property \( \beta \) (respectively \( \gamma \)) if the union (respectively intersection) of any family (respectively finite subsets) of \( \mu \) also belongs to \( \mu \).

**Definition 2.4** [6] A subset \( A \) of an \( m \)-space \((X, \mu)\) is said to be an \( \alpha\mu \)-open, if \( A \subseteq \mu\text{Int}(\mu\text{Cl}(\mu\text{Int}(A))) \). The complement of \( \alpha\mu \)-open set is called \( \alpha\mu \)-closed set or, equivalently, \( \mu\text{Cl}(\mu\text{Int}(\mu\text{Cl}(A))) \subseteq A \).

**Definition 2.5** [6] An \( m \)-space \((X, \mu)\) has a property \( \alpha\gamma \), if the intersection of finite \( \alpha\mu \)-open sets is an \( \alpha\mu \)-open set in \( X \).

**Remark 2.6** [6] From Definition 2.4, it is clear that every \( \mu \)-open (respectively \( \mu \)-closed) set is an \( \alpha\mu \)-open (respectively \( \alpha\mu \)-closed) set.

**Definition 2.7** [10] Let \((X, \mu)\) be an \( m \)-space. A point \( x \in X \) is called an \( \alpha\mu \)-adherent point of a set \( A \subseteq X \) if and only if \( G \cap A \neq \emptyset \) for all \( G \in \mu \) such that \( x \in G \). The set of all \( \alpha\mu \)-adherent points of a set \( A \) is denoted by \( \alpha\text{muCl}(A) \), where \( \alpha\mu\text{Cl}(A) = \bigcap \{F : A \subseteq F, F \in \mu\text{Cl}(F)\} \).

**Proposition 2.8** [6] A subset \( F \) of \( m \)-space \( X \) is \( \alpha\mu \)-closed set in \( X \) iff \( F = \alpha\mu\text{Cl}(F) \).

**Definition 2.9** [7] Let \((X, \mu)\) be an \( m \)-space, \( \mathcal{A} \subseteq X \). Then \( a \in X \) is said to be \( \alpha\mu \)-interior point to \( \mathcal{A} \) iff \( U \subseteq A \), for some \( \alpha\mu \)-open set \( U \) and \( x \in U \). The \( \alpha\mu \)-interior point of a set \( \mathcal{A} \) is all \( \alpha\mu \)-interior point of \( \mathcal{A} \) and denoted by \( \alpha\text{muInt}(\mathcal{A}) \), where \( \alpha\text{muInt}(\mathcal{A}) = \bigcup \{U : U \subseteq \mathcal{A}, U \in \alpha\mu \text{-open set}\} \).

**Proposition 2.10** [6] any subset of \( m \)-space \( X \) is \( \alpha\mu \)-open set iff every point in it is an \( \alpha\mu \)-interior point.

**Remark 2.11** [6] If \((X, \mu)\) is an \( m \)-space, then:
1. The union of any family of \( \alpha\mu \)-open sets is an \( \alpha\mu \)-open set.
2. The intersection of any two \( \alpha\mu \)-open sets may be not \( \alpha\mu \)-open set.

**Definition 2.12** [12] An \( m \)-space, \((X, \mu)\) is called \( \mu \)-compact if any \( \mu \)-open cover of \( X \) has a finite subcover. A subset \( H \) of an \( m \)-space is said to be \( \mu \)-compact in \( X \), if for any cover by \( \mu \)-open of \( X \), there is a finite subcover of \( H \).

**Proposition 2.13** [11] Every \( \mu \)-closed set in \( \mu \)-compact space is an \( \mu \)-compact set.
**Definition 2.14** [6] An \(m\)-space \((X, \mu)\) is said to be \(\alpha\mu\)-compact space if any \(\alpha\mu\)-open cover of \(X\) has a finite subcover. A subset \(B\) of \(m\)-space \(X\) is called \(\alpha\mu\)-compact, if any \(\alpha\mu\)-open set of \(X\) which covers \(B\) has a finite subcover of \(B\).

**Remark 2.15** Any \(\alpha\mu\)-compact is \(\mu\)-compact set. However the converse is not necessarily true as shown by the following example.

**Example 2.16** Let \(\mathcal{R}\) be the set of real numbers and \(X\) be a non-empty set such that \(X=\{x\} \cup \{r: r \in \mathcal{R}\}, \) where \(x \in X\). Also \(\mu=\{\emptyset, X, \{x\}\},\) then \(C=\{\{x, r\}: r \in \mathcal{R}\}\) is an \(\alpha\mu\)-open cover to \(X\). Since \(\{x, r\} \subseteq \mu\text{Int}\left(\mu\text{Cl}(\{x, r\})\right) = X,\) so \(\{x, r\}\) is an \(\alpha\mu\)-open set. Now, \(C\) is an \(\alpha\mu\)-open cover to \(X\), but it has no finite subcover to \(X\), since, if we remove \(\{x, 50\}\) then the reminder is not cover \(X\) (cover all \(X\) except 50), and it is infinite cover. Hence, \(X\) is not \(\alpha\mu\)-compact space and it is clear that \(X\) is \(\mu\)-compact space, since the only \(\mu\)-open cover of \(X\) is \(X\) itself, which is one set, that is, a finite open cover to \(X\).

**Definition 2.17** [10] An \(m\)-space is called an \(\mu-T_1\)-space, if for any two points \(a, b\) in \(X, a \neq b\) there is two \(\mu\)-open sets \(N, M\) such that \(a \in N, b \notin N\) and \(b \in M\) but \(a \notin M\).

**Proposition 2.18** [4] An \(m\)-space is \(\mu\)-closed space if and only if every singleton set is \(\mu\)-closed set, whenever \(X\) has \(\beta\) property.

**Definition 2.19** [10] An \(m\)-space is said to be \(\alpha\mu\)-T1-space, if for every two \(t\) points \(c, d\) in \(X,\) there are two \(\alpha\mu\)-open sets \(\mathcal{K}, \mathcal{H}\) with \(c \in \mathcal{K},\) but \(\mathcal{H} \notin \mathcal{H}\) and \(d \notin \mathcal{H}\) and \(b \notin \mathcal{K}\).

**Remark 2.20** [10] Every \(\mu\)-T1-space is \(\alpha\mu\)-T1-space.

**Definition 2.21** [10] An \(m\)-space \((X, \mu)\) is called \(\mu\)-T2-space (respectively \(\alpha\mu\)-T2-space), if for any two distinct points \(x, y\) in \(X,\) there are two \(\mu\)-open (respectively \(\alpha\mu\)-open) \(U, V\), such that \(x \in U, y \in V,\) and \(U \cap V = \emptyset\).

**Definition 2.22** [4] An \(m\)-space \((X, \mu)\) is said to be \(\mu\)-Kc-space if any \(\mu\)-compact subset of \(X\) is \(\mu\)-closed set.

**Example 2.23** Let \(\mathcal{R}\) be the real numbers, \((\mathcal{R}, \mu_{\mathcal{R}})\) is the usual \(\mu\)-space which is \(\mu\)-Kc-space.

**Proposition 2.24** [12] Every \(\mu\)-compact set in \(\mu\)-T2-space, that has the property \(\beta\) and \(Y,\) is \(\mu\)-closed set.

**Remark 2.25** [4]
1. Every \(\mu\)-Kc space is \(\mu\)-T1-space.
2. Every \(\mu\)-T2-space with the property \(\beta\) and \(Y\) is \(\mu\)-Kc-space.

**Definition 2.26** Let \(f: (X, \mu) \rightarrow (Y, \mu')\) be a function. Then \(f\) is called:
1. \(m\)-continuous [15] iff for any \(\mu\)-open \(N\) in \(X,\) the inverse image \(f^{-1}(N)\) is an \(\mu\)-open set in \(X,\)
2. \(\alpha m\)-continuous [6] iff for any \(\mu\)-open set \(M\) in \(Y,\) the inverse image \(f^{-1}(M)\) is an \(\alpha\mu\)-open set in \(X,\)

**Proposition 2.27** [14] The \(m\)-continuous image of \(\mu\)-compact is \(\mu\)-compact.

**Definition 2.28** [4] A function \(f: (X, \mu) \rightarrow (Y, \mu')\) is said to be \(m\)-homeomorphism, if \(f\) is injective, surjective, continuous and \(f^{-1}\) continuous. If there exists an \(m\)-homeomorphism between \((X, \mu)\) and \((Y, \mu')\) then we say that \((X, \mu)\) \(\text{m-homeomorphic to} (Y, \mu').\)

**Definition 2.29** [13] Let \((X, \mu)\) be \(m\)-space, \(F\) be a subset of \(X\) and \(x \in X.\) A point \(x\) is called an \(\theta\mu\)-interior point of \(F\) if there is \(C \subseteq \mu\) such that \(x \in C\) and \(x \in \mu Cl(C) \subseteq F.\) And \(\theta\mu\)-interior set which is denoted by \(\theta\mu\text{Int}(F)\) is the set of all \(\theta\mu\)-interior points. A subset \(F\) of \(X\) is called an \(\theta\mu\)-open set if every point of \(F\) is an \(\theta\mu\)-interior point.

**Definition 2.30** [13] Let \((X, \mu)\) be \(m\)-space, \(H \subseteq X,\) a point \(b \in X\) is said to be an \(\theta\mu\)-adherent point for a subset \(H\) of \(X,\) if \(H \cap \mu Cl(G) \neq \emptyset\) for any \(\mu\)-open set \(G\) of \(X\) and \(b \in H.\) The set of \(\theta\mu\)-adherent point is said to be an \(\theta\mu\)-closure of \(H,\) which is denoted by \(\mu Cl(H).\) A subset \(H\) of \(X\) is called \(\theta\mu\)-closed set if every point to \(H\) is an \(\theta\mu\)-adherent point.

**Example 2.31** Any subset of a discrete \(m\)-space \((\mathcal{R}, \mu_{\mathcal{R}})\) on a real number \(\mathcal{R}\) is \(\theta\mu\)-closed set and \(\theta\mu\)-open set.

**Definition 2.32** [8] An \(m\)-space \((X, \mu)\) is said to have the property \(\theta Y\) (respectively \(\theta B\)) if the intersection (respectively union) of any finite number (respectively family) of \(\theta\mu\)-open sets is an \(\theta\mu\)-open set.

**Remark 2.33** [8] If an \(m\)-space \((X, \mu)\) has \(\theta Y\) property, then every \(\theta\mu\)-closed is an \(\mu\)-closed.
Definition 2.34 [8] Let \((X, \mu)\) be a \(m\)-space, \(X\) is said to be \(\theta \mu\)-compact if any \(\theta \mu\)-open cover of \(X\) has a finite subcover. A subset \(A\) of an \(m\)-space \((X, \mu)\) is said to be \(\theta \mu\)-compact if for any \(\theta \mu\)-open cover \(\{V_\alpha : \alpha \in I\}\) of \(X\) and cover \(A\) then there is a finite subset \(\{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n\}\) such that \(A \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}\).

Example 2.35 Let \((\mathcal{R}, \mu_{in})\) be an \(m\)-space where \(\mu_{in}\) be indiscrete \(m\)-space on a real number \(\mathcal{R}\), so is \(\theta \mu\)-compact.

Remark 2.36 [8] Every \(m\)-compact with the property \(\theta \beta\) is \(\theta \mu\)-compact.

Definition 2.37 [8] An \(m\)-space \((X, \mu)\) is called \(\theta \mu\)-T\(_2\) -space, if for every two points \(a, b\) that belong to \(X\), \(a \neq b\), there is \(\theta \mu\)-open sets \(M\) and \(N\) containing \(a\) and \(b\), respectively, such that \(M \cap N = \emptyset\).

Definition 2.38 [8] Let \((X, \mu)\) and \((Y, \mu')\) be two \(m\)-spaces and \(f: (X, \mu) \to (Y, \mu')\) be a function. Then \(f\) is called:

1. \(\theta m\)-continuous function if for any \(\mu'\)-closed \((\mu'\)-open\) subset \(K\) of \(Y\), the inverse image \(f^{-1}(K)\) is \(\theta \mu\)-closed \((\theta \mu\)-open\) set in \(X\).
2. \(\theta^* m\)-continuous function if for every \(\theta \mu'\)-closed \((\theta \mu'\)-open\) \(M\) subset of \(Y\), the inverse image \(f^{-1}(M)\) is \(\theta \mu\)-closed \((\theta \mu\)-open\) set in \(X\).
3. \(\theta^* m\)-continuous function if for any \(N \theta \mu'\)-closed \((\theta \mu'\)-open\) \(N\) subset of \(Y\), the inverse image \(f^{-1}(N)\) is \(\theta \mu\)-closed \((\theta \mu\)-open\) set in \(X\).
4. \(\theta m\)-closed function if \(f(F)\) is \(\theta \mu'\)-closed set in \(Y\) for each \(\mu\)-closed subset \(F\) of \(X\).
5. \(\theta^* m\)-closed function if \(f(F)\) is \(\mu'\)-closed set in \(Y\) for each \(\theta \mu\)-closed subset \(F\) of \(X\).

Proposition 2.39 [8] The \(\theta^* m\)-continuous image of \(\theta \mu\)-compact is \(\theta \mu'\)-compact.

Proposition 2.40 [8] If \(f: (X, \mu) \to (Y, \mu')\) is an \(m\)-homeomorphism and \(B\) is a \(\theta \mu'\)-compact set in \(Y\) then \(f^{-1}(B)\) is a \(\theta \mu\)-compact set in \(X\), with \(X\) has the property \(\theta \beta\).

3. Strong and weak forms of \(\mu\)-K\(_c\)-spaces

In this section, we provide some weak forms of \(\mu\)-K\(_c\)-space, namely \(\mu\)-K\((ac)\)-space, \(\mu\)-aK\((c)\)-space and \(\mu\)-aK\((ac)\)-space. In addition, we introduce \(\mu\)-\(\theta\)K\((c)\)-space as a strong form of \(\mu\)-K\(_c\)-space.

Definition 3.1 An \(m\)-space \((X, \mu)\) is said to be \(\mu\)-K\((ac)\)-space if every \(\mu\)-compact set in \(X\) is an \(\alpha \mu\)-closed set.

Now, we give some examples to explain the concept of \(\mu\)-K\((ac)\)-space.

Example 3.2 The discrete \(m\)-space \((X, \mu_D)\) is \(\mu\)-K\((ac)\)-space.

Example 3.3 Let \(X = \{1, 2, 3\}\) and let \(\mu = \{\emptyset, X, \{1\}\}\). Then \((X, \mu)\) is not \(\mu\)-K\((ac)\)-space, since \(Y\) exists an \(\mu\)-compact set \(\{1, 2\}\) in \(X\) but it is not \(\alpha \mu\)-closed.

To show that Definition 3.1 is well defined, we give the following example to illustrate that there is no relation between the concepts of \(\mu\)-K\(_c\)-space and \(\alpha \mu\)-closed set.

Example 3.4

1. In the discrete \(m\)-space \((\mathcal{R}, \mu_D)\) where \(\mathcal{R}\) is a real number, \(\mathcal{Q}\) is the rational numbers subset of \(\mathcal{R}\), \(\mathcal{Q}\) is \(\alpha \mu\)-closed but not \(\alpha \mu\)-compact set.
2. In the indiscrete \(m\)-space \((\mathcal{R}, \mu_{ind})\), \(\mathcal{Q}\) is \(\mu\)-compact but not \(\alpha \mu\)-closed set.

Remark 3.5

1. Every \(\mu\)-K\(_c\) space is \(\mu\)-K\((ac)\)-space.
2. In discrete \(m\)-space, the two definitions of \(\mu\)-K\(_c\)-space and \(\mu\)-K\((ac)\)-spaces are satisfied.

The following example indicates that the converse of Remark 3.5 part (1) is not necessarily hold.

Example 3.6 Let \((X, \mu)\) be an \(m\)-space, \(X = \{a, b, c\}, \mu = \{\emptyset, X, \{a\}\}\), so \(\{c\}\) is \(\mu\)-compact since \(\{c\}\) is finite set. Also it is \(\alpha \mu\)-closed set since \(\mu Cl(\mu \{\mu \{\mu Cl\{c\}\}\}) = \emptyset \subseteq \{c\}\), so \(X\) is \(\mu\)-K\((ac)\)-space, but not \(\mu\)-K\(_c\) space since \(\{c\}\) is not \(\mu\)-closed set.

Proposition 3.7 An \(\alpha \mu\)-compact subset of \(\alpha \mu\)-T\(_2\)-space is \(\alpha \mu\)-closed, whenever \(X\) has \(\alpha \mu\) property.

Proof: Let \(B\) be \(\alpha \mu\)-compact in \(\alpha \mu\)-T\(_2\)-space. To show that \(B\) is \(\alpha \mu\)-closed, let \(p \in B^c\), since \(X\) is \(\alpha \mu\)-T\(_2\)-space. So for every \(q \in B\), \(p sq\), there exist \(\alpha \mu\)-open sets \(G, H\) with \(p \in H\), \(q \in G\), such that \(G \cap H = \emptyset\). Now the collection \(\{G_q; q \in B, i \in I\}\) is \(\alpha \mu\)-open cover of \(B\). Since \(B\) is \(\alpha \mu\)-compact set, then there is a finite subcover of \(B\), so \(B \subseteq \bigcup_{i=1}^{m} G_{q_i}\). Let \(H^* = \bigcap_{i=1}^{m} H_{q_i}(p)\) and \(G^* = \bigcup_{i=1}^{m} G_{q_i}\), then \(H^*\) is an \(\alpha \mu\)-open set \(p \in H^*\) (since \(X\) has property \(\alpha \mu\)). Claim that \(G^* \cap H^* = \emptyset\), let \(x \in G^*\), then \(x \in G_{q_i}\) for some \(q_i\) and suppose that \(x \in H^*\), \(B \cap H^* \neq \emptyset\). This is a contradiction, then \(p \in H^* \subseteq B^c\), so \(B^c\) is \(\alpha \mu\)-open set in \(X\), hence \(B\) is \(\alpha \mu\)-closed set.

Theorem 3.8 Every \(\alpha \mu\)-closed set in \(\alpha \mu\)-compact space is \(\alpha \mu\)-compact set.

1083
Proof: Let $(X, \mu)$ be $\alpha\mu$-compact, $A$ is $\alpha\mu$-closed in $X$, and $(V_\alpha)_{\alpha \in 1}$ is an $\alpha\mu$-open cover of $A$, that is, $A \subseteq \bigcup_{\alpha \in 1} V_\alpha$, where $V_\alpha$ is $\alpha\mu$-open in $X$. For $\alpha \in 1$, since $X = A \cup A^C \subseteq \bigcup_{\alpha \in 1} V_\alpha \cup A^C$, also $A^C$ is $\alpha\mu$-open (since $A$ is $\alpha\mu$-closed in $X$). So $\bigcup_{\alpha \in 1} V_\alpha \cup A^C$ is $\alpha\mu$-open cover for $X$ which is $\alpha\mu$-compact space, then there exists $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ such that $X \subseteq \bigcup_{i=1}^n V_{\alpha_i} \cup A^C$, so $A \subseteq \bigcup_{i=1}^n V_{\alpha_i}$. Therefore $A$ is $\alpha\mu$-compact set.

Remark 3.9 In the above theorem, if we replace the $\alpha\mu$-compact by $\mu$-compact, the theorem will not be true.

Now, we introduce the weak form of $\mu$-$K(ac)$-space which was introduced in Definition 3.1.

Definition 3.10 A space $X$ is said to be $\mu$-$\alpha K(ac)$-space if any $\alpha\mu$-compact subset of $X$ is $\alpha\mu$-closed set.

Example 3.11 Let $(R, \mu_D)$ be a discrete $m$-space where $R$ is a real number. Let $Q$ is $\alpha\mu$-compact subset of $R$, then $Q$ is $\mu$-compact in $R$ from Remark 2.15, and $Q$ is $\alpha\mu$-closed by Remark 2.6. Hence $(R, \mu_D)$ is $\mu$-$\alpha K(ac)$-space.

Proposition 3.12 Every $\mu$-$\alpha K(c)$-space is $\mu$-$\alpha K(ac)$-space.

Proof: Let $(X, \mu)$ be $m$-space and $K$ be $\alpha\mu$-compact subset of $X$, which is $\mu$-$\alpha K(c)$-space, so $K$ is $\mu$-closed subset of $X$ and, by Remark 2.6, $K$ is $\alpha\mu$-closed subset. Hence $X$ is $\mu$-$\alpha K(ac)$-space.

Theorem 3.13 $(X, \mu)$ is $\alpha\mu-T_1$-space iff $\{x\}$ is $\alpha\mu$-closed subset of $X$ for all $x \in X$.

Proof: Let $\{x\}$ be $\alpha\mu$-closed set $V \subseteq X$. Let $a, d \in X$ with $a \neq d$, and $\{a\}$ and $\{d\}$ are $\alpha\mu$-closed sets, then $\{a\}^C$ is $\alpha\mu$-open subset of $X$, with $d \in \{a\}^C$ and $a \notin \{a\}^C$. Also $\{d\}^C$ is $\alpha\mu$-open subset of $X$, with $a \in \{d\}^C$ and $d \notin \{d\}^C$, so $X$ is $\alpha\mu-T_1$-space.

Conversely, we must prove that $\{x\}$ is $\alpha\mu$-closed subset of $X$, that is $\alpha\mu Cl(\{x\}) = \{x\}$, since $\{x\} \subseteq \alpha\mu Cl(\{x\})$. Let $y \in \alpha\mu Cl(\{x\})$ and $y \notin \{x\}$, so $x \neq y$, but $X$ is $\mu$-$\alpha\mu-T_1$-space, so there exist two $\alpha\mu$-open sets $U_x$ and $V_y$ containing $x$ and $y$, respectively, with $y \notin U_x$ and $x \notin V_y$. Then $V_y$ containing $y$, so $y$ is not $\alpha\mu$-adherent point to $\{x\}$, that is $y \notin \alpha\mu Cl(\{x\})$, and this is contradiction. Therefore, $y \notin \{x\}$ and $\alpha\mu Cl(\{x\}) \subseteq \{x\}$, so by (1) and (2) we get $\alpha\mu Cl(\{x\}) = \{x\}$, and by Proposition 2.8, $\{x\}$ is $\alpha\mu$-closed subset of $X$.

Proposition 3.14 Every $\mu$-$\alpha K(ac)$-space is $\mu$-$\alpha\mu-T_1$-space.

Proof: Let $x \in X$ and $\{x\}$ be $\alpha\mu$-compact set in $X$, since $X$ is $\mu$-$\alpha K(ac)$-space, hence $\{x\}$ is $\alpha\mu$-closed set, so $X$ is $\mu$-$\alpha\mu-T_1$-space by Theorem 2.18.

The next example shows that the converse of Proposition 3.14 is not true.

Example 3.15 Let $(R, \mu_{cof})$ be a co-finite $m$-space on a real number $R$ which is $\mu$-$\alpha\mu-T_1$-space, if we take $Q \subseteq R$ as $\alpha\mu$-compact (since there exists one $\alpha\mu$-open cover of $Q$ which is $R$), but $Q$ is not $\alpha\mu$-closed in $R$ (since $\mu Cl(\mu Int(\mu Cl(\{Q\}))) = R \not\subseteq Q$.

Proposition 3.16 Every $\alpha\mu$-$\alpha\mu$-space is $\mu$-$\alpha K(ac)$-space, whenever $X$ has $\alpha Y$ property.

Proof: Let $(X, \mu)$ be an $m$-space and $P$ be an $\alpha\mu$-compact subset in $X$. Also $X$ is $\alpha\mu$-$\alpha\mu$-space, so $P$ is an $\alpha\mu$-closed set from Proposition 3.7. Therefore, $X$ is $\mu$-$\alpha K(ac)$-space.

The converse of Proposition 3.16 may not be hold. The following example shows that.

Example 3.17 Let $(R, \mu_{coc})$ be a co-countable $m$-space on a real number $R$, which is $\mu$-$\alpha K(ac)$-space, but not $\alpha\mu$-$\alpha\mu$-space, since the $\mu$-compact set in it are just the finite set, if we $\mu$-compact set then it is finite, so, it is countable, then it is $\mu$-closed since in $\mu_{coc}$ the closed take sets are $\emptyset, R$ and countable sets. Now suppose that it is $\mu$-$\alpha\mu$-$\alpha\mu$-space, $\forall x, y \in R, x \neq y$, there are $U_x, V_y$ as two $\alpha\mu$-open sets such that $x \in U_x, y \in V_y$ and $U_x \cap V_y = \emptyset$, then $U_x = U_x^C, (U_x \cap V_y)^C = \emptyset$, so $(U_x^C \cup (V_y)^C = R$, but this is a contradiction. Since $U_x$ and $V_y$ are countable, the union also countable, but $R$ is not countable so it is not $\alpha\mu$-$\alpha\mu$-space. Therefore $(R, \mu_{coc})$ are $\mu$-$Kc$-, $\mu$-$K(ac)$- and $\mu$-$\alpha K(ac)$-spaces.

Proposition 3.18 A subset $F$ of an $m$-space $X$ is $\alpha\mu$-closed set in $X$ if and only if there exists an $\alpha$-closed set $M$ such that $\mu Cl(\mu Int(M)) \subseteq F \subseteq M$.

Proof: Suppose that $F$ is $\alpha\mu$-closed set in $X$, so $\mu Cl(\mu Int(\mu Cl(F))) \subseteq F$, by Definition 2.3, and $F \subseteq \mu Cl(F)$, then $\mu Cl(\mu Int(\mu Cl(F))) \subseteq F \subseteq \mu Cl(F)$, put $\mu Cl(F) = M$, so $\mu Cl(\mu Int(M)) \subseteq F \subseteq M$. 

1084
Conversely, suppose that $\mu Cl(\mu Int(M)) \subseteq \mathcal{F} \subseteq M$. To prove that $\mathcal{F}$ is $\alpha\mu$-closed set whenever $M$ is $\mu$-closed set, $\mu Cl(\mu Cl(\mu Int(M))) \subseteq \mu Cl(\mathcal{F}) \subseteq \mu Cl(M) = M$, then $\mu Cl(\mu Int(M)) \subseteq \mu Cl(\mathcal{F}) \subseteq M$, and $\mu Int(\mu Cl(\mu Int(M))) \subseteq \mu Int(\mu Cl(\mathcal{F})) \subseteq \mu Int(M)$, by hypothesis $\mu Cl(\mu Int(M)) \subseteq \mathcal{F} \subseteq M$, we get $\mu Cl(\mu Cl(\mathcal{F})) \subseteq \mathcal{F}$. Therefore $\mathcal{F}$ is $\alpha\mu$-closed set.

**Definition 3.19** An $m$-space $\mathcal{X}$ is called $\mu$-$\alpha K(c)$-space if any $\alpha\mu$-compact subset in $\mathcal{X}$ is $\mu$-closed set.

**Example 3.20** Let $(\mathcal{R}, \mu_D)$ be a discrete $m$-space on any space $\mathcal{X}$, it is $\mu$-$\alpha K(c)$-space.

**Remark 3.21**
1. Every $\mu$-$Kc$-space is $\mu$-$\alpha K(c)$-space.
2. Every $\mu$-$\alpha K(c)$-space is $\mu$-$\alpha K(\alpha c)$-space.
3. Every $\mu T_2$-space is $\mu$-$\alpha K(c)$-space.
4. Every $\mu$-$\alpha K(c)$-space is $\alpha\mu T_1$-space.

Now, we define a strong form of $\mu$-$Kc$-space which is $\mu$-$\theta K(c)$-space.

**Definition 3.22** An $m$-space $(\mathcal{X}, \mu)$ is called $\mu$-$\theta K(c)$-space, if every $\theta\mu$-compact of $\mathcal{X}$ is $\mu$-closed set.

**Example 3.23** Let $(\mathcal{R}, \mu_{cof})$ be a co-finite $m$-space on a real line $\mathcal{R}$. Then $(\mathcal{R}, \mu_{cof})$ is an $\alpha\mu$-$\theta K(c)$-space.

**Proposition 3.24** Every $\theta\mu$-compact subset of $\theta\mu T_2$-space is $\theta\mu$-closed, whenever that space has $\theta Y$ property.

**Proof:** Let $A$ be a $\theta\mu$-compact set in $\mathcal{X}$. Let $p \notin A$, so for each $q \in A$ then $p \neq q$. But $\mathcal{X}$ is $\theta\mu T_2$-space, so there exist two $\theta\mu$-open sets $U$ and $V$ containing $q$ and $p$, respectively, then $A = \bigcup_{a \in \mathcal{I}} \{U_{q_a}\}$. But $A$ is $\theta\mu$-compact, so $A = \bigcup_{i=1}^{n} \{U_{q_{a_i}}\} = U^*$ and $V^* = \bigcap_{i=1}^{n} V_{a_i}(p)$ is $\theta\mu$-open (since $\mathcal{X}$ has $\theta Y$ property). Claim that $U^* \cap V^* = \emptyset$, and suppose that $U^* \cap V^* \neq \emptyset$, since $p \in V^*$, let $p \in U^*$, that is $p \in A$, but this is a contradiction. So $U^* \cap V^* = \emptyset$ and then there exists $V^*$ containing $p$ and $V^* \subseteq A^c$, that is $p \in \mu Int(A^c)$, then $A^c$ is $\theta\mu$-open, by Proposition 2.10, so $A$ is $\theta\mu$-closed.

**Proposition 3.25** If an $m$-space has $\theta Y$ property, then every $\theta\mu T_2$-space is $\mu$-$\theta K(c)$-space.

**Proof:** Let $H$ be an $\theta\mu$-compact subset of $\mathcal{X}$. To prove that $H$ is $\mu$-closed set, since $\mathcal{X}$ is $\theta\mu T_2$-space, so by proposition 3.24, we get $H$ is $\theta\mu$-closed set and by Remark 2.33, we get $H$ is $\mu$-closed, hence $\mathcal{X}$ is $\mu$-$\theta K(c)$-space.

**Proposition 3.26** If an $m$-space has $\theta\beta$ property, then every $\mu$-$\theta K(c)$-space is $\mu$-$kc$-space.

**Proof:** Let $(\mathcal{X}, \mu)$ be $m$-space, $A$ be $\mu$-compact of $\mathcal{X}$ by Remark 2.36, $A$ is $\theta\mu$-compact and since $\mathcal{X}$ is $\mu$-$\theta K(c)$-space, so $\mathcal{X}$ is $\mu$-closed subset of $\mathcal{X}$, hence $\mathcal{X}$ is $\mu$-$kc$-space.

**Remark 3.27** The following diagram shows the relationships between the stronger and weaker forms of $\mu$-$kc$-space.
4-Some types of continuous, open (closed) function on \( m \)-spaces.

**Definition 4.1** Let \( f: (X, \mu) \rightarrow (Y, \mu') \) be a function, then \( f \) is called:
1. \( m \)-open (respectively \( m \)-closed) function \([2]\), if \( f(H) \) is an \( \mu' \)-open respectively \( \mu' \)-closed set in \( Y \) for any \( m \)-open (respectively \( m \)-closed) \( H \) in \( X \).
2. \( \alpha m \)-open (respectively \( \alpha m \)-closed) function \([6]\), if \( f(A) \) is an \( \alpha m' \)-open respectively \( \alpha m' \)-closed set in \( Y \) for every \( m \)-open (respectively \( m \)-closed) \( A \) in \( X \).
3. \( \alpha' m \)-open (respectively \( \alpha' m \)-closed) function, if \( f(K) \) is an \( \mu' \)-open (respectively \( \mu' \)-closed) set in \( Y \) for any \( \alpha m \)-open (respectively \( \alpha m \)-closed) subset \( K \) of \( X \).
4. \( \alpha' m \)-open (respectively \( \alpha' m \)-closed) function, if \( f(N) \) is an \( \alpha m' \)-open respectively \( \alpha m' \)-closed subset of \( Y \) for any \( \alpha' m \)-open (respectively \( \alpha' m \)-closed) set \( N \) in \( X \).
5. \( \alpha' m \)-continuous iff for any \( \alpha m' \)-open set \( A \) in \( Y \), the inverse image \( f^{-1}(A) \) is \( m \)-open set in \( X \).
6. \( \alpha' m \)-continuous for every \( \alpha m' \)-open set \( B \) in \( Y \), the inverse image \( f^{-1}(B) \) is \( m \)-open set in \( X \).

**Example 4.2** Let \( X = Y = \{a, b, c\} \), \( \mu = \mu' = \{\emptyset, X, \{a\}\} \) and \( f: (X, \mu) \rightarrow (Y, \mu') \) defined by \( f(a) = f(b) = a \) and \( f(c) = c \). Then \( f \) is \( \mu \)-open, \( \alpha m \)-open and \( \alpha' m \)-open but it is not \( \alpha' m \)-open function (where \( \alpha m \)-open in set \( \mu \) and \( \mu' \) are \( \{\emptyset, X, \{a, b\}, \{a, c\}\} \)).

Next, we introduce a proposition about \( \alpha' m \)-closed function. But before that we need to introduce the following proposition:

**Proposition 4.3** Let \( f: (X, \mu) \rightarrow (Y, \mu') \) be a function. Then for every subset \( A \) of \( X \):
1. \( f \) is \( m \)-homeomorphism iff \( \mu Cl(f(A)) = f(\mu Cl(A)) \).
2. \( f \) is \( m \)-homeomorphism iff \( \mu Int(f(A)) = f(\mu Int(A)) \).

**Proof:** The proof follows directly from the Definition 2.26 part (1) and Definition 4.1 part (1).

**Theorem 4.4** If \( f: (X, \mu) \rightarrow (Y, \mu') \) is \( m \)-homeomorphism, then \( f \) is \( \alpha' m \)-closed function.

**Proof:** Let \( \mathcal{F} \) be \( \alpha m \)-closed subset of \( X \), by Proposition 3.18, there exists \( m \)-closed set \( M \) such that \( \mu Cl(\mu Int(M)) \subseteq \mathcal{F} \subseteq M \). Now, by taking the image, we get \( f(\mu Cl(\mu Int(M))) \subseteq f(\mathcal{F}) \subseteq f(M) \).

But \( f \) is \( m \)-homeomorphism, so

\[
\mu Cl(f(\mu Int(M))) \subseteq f(\mathcal{F}) \subseteq f(M) \quad (1).
\]

Also from Proposition 4.3

\[
f(\mu Int(M)) = \mu Int(f(M)),
\]

hence

\[
\mu Cl(f(\mu Int(M))) = \mu Cl(\mu Int(f(M))) \quad (2).
\]

Now, from (1) and (2) we have,

\[
\mu Cl(\mu Int(f(M))) \subseteq f(\mathcal{F}) \subseteq f(M).
\]

Therefore, \( f(\mathcal{F}) \) is \( \alpha m \)-closed subset of \( Y \).

**Corollary 4.5** If \( f: (X, \mu) \rightarrow (Y, \mu') \) is \( m \)-homeomorphism, then \( f \) is \( \alpha' m \)-open function.

**Proof:** Let \( K \) be an \( \alpha m \)-open set in \( X \). To prove that \( f(K) \) is \( \alpha m \)-open set in \( Y \). Now, \( K^c \) is \( \alpha m \)-closed set in \( X \), and since \( f \) is \( m \)-homeomorphism. From Theorem 4.4, \( f(K^c) \) is \( \alpha m \)-closed set in \( Y \). But \( f \) is surjective, so \( f(K^c) = f(K)^c \), which means that \( f(K) \) is \( \alpha m \)-open set in \( Y \). Hence \( f \) is \( \alpha' m \)-open function.

**Theorem 4.6** Let \( f: (X, \mu) \rightarrow (Y, \mu') \) be \( \alpha' m \)-continuous. Then \( f(M) \) is \( \alpha m \)-compact in \( Y \), whenever \( M \) is \( \alpha m \)-compact in \( X \).

**Proof:** Let \( \mathcal{M} \) be an \( \alpha m \)-compact in \( X \). To prove that \( f(\mathcal{M}) \) is \( \alpha m \)-compact in \( Y \), let \( \{V_a; \alpha \in I\} \) be a family of \( \alpha m \)-open cover of \( f(\mathcal{M}) \). That is \( \{V_a; \alpha \in I\} \subseteq \mathcal{M} \), so \( f^{-1}(V_a) \) is \( \alpha m \)-open cover of \( \mathcal{M} \). Also, since \( \mathcal{M} \) is \( \alpha m \)-compact in \( X \), there exist \( \alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \) such that \( V_a = \bigcup_{i=1}^n f^{-1}(V_{\alpha_i}) \). Therefore, \( f(\mathcal{M}) \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i}) \).

**Theorem 4.7** Let \( f: (X, \mu) \rightarrow (Y, \mu') \) be \( \alpha' m \)-continuous function. Then \( f(N) \) is \( \mu \)-compact in \( Y \), whenever \( N \) is \( \alpha m \)-compact in \( X \).

**Proof:** Let \( \mathcal{N} \) be an \( \alpha m \)-compact in \( X \). To prove that \( f(\mathcal{N}) \) is \( \mu \)-compact in \( Y \), let \( \{V_a; \alpha \in I\} \) be a family of \( \mu \)-open cover of \( f(\mathcal{N}) \). That is \( \{V_a; \alpha \in I\} \subseteq \mathcal{N} \), so \( f^{-1}(V_a) \) is an \( \alpha m \)-open cover of \( \mathcal{N} \). Also, since \( \mathcal{N} \) is \( \alpha m \)-compact in \( X \), then \( \mathcal{N} \subseteq \bigcup_{i=1}^m f^{-1}(V_{\alpha_i}) \). This implies that \( f(\mathcal{N}) \subseteq \bigcup_{i=1}^m f^{-1}(V_{\alpha_i}) \).

**Theorem 4.8** Let \( f: (X, \mu) \rightarrow (Y, \mu') \) be \( \alpha' m \)-continuous function. If a space \( X \) is \( \alpha m \)-compact and a space \( Y \) is \( \alpha m \)-compact, then the function \( f \) is \( \alpha' m \)-closed, whenever \( X \) has \( \alpha Y \) property.
Proof: Let $H$ be an $\alpha\mu$-closed set in $X$. Since $X$ is $\alpha\mu$-compact, then $H$ is $\alpha\mu$-compact in $X$ by Theorem 3.8 and the function $f$ is $\alpha^*\mu$-continuous. Then $f(H)$ is $\alpha\mu'$-compact subset of $\mathcal{Y}$ from Theorem 4.6, and since $\mathcal{Y}$ is $\alpha\mu$-$T_2$-space, so $f(H)$ is $\alpha\mu'$-closed set of $\mathcal{Y}$ by proposition 3.7. Therefore $f$ is $\alpha^*\mu$-closed function.

Theorem 4.9 Let $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ be a $\alpha^*\mu$-continuous function, from $\alpha\mu$-compact space $X$ into $\mu$-$Kc$-space $\mathcal{Y}$, then $f$ is $\alpha^*$-$\mu$-closed function.

Proof: Let $B$ be an $\alpha\mu$-closed set in $X$ which is $\alpha\mu$-compact, so $B$ is $\alpha\mu$-compact in $X$ from Theorem 3.8. Also, from the hypotheses, $f$ is $\alpha\mu'$-continuous, then $f(B)$ is $\mu$-compact in $\mathcal{Y}$ by Theorem 4.7. But $\mathcal{Y}$ is $\mu$-$Kc$-space, hence $f(B)$ is $\mu'$-closed set of $\mathcal{Y}$. Therefore, $f$ is $\alpha\mu'$-closed function.

Proposition 4.10 Let the function $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ be $m$-continuous. If $(X, \mu)$ is $\mu$-compact and $(\mathcal{Y}, \mu')$ is $\mu$-$Kc$-space, then $f$ is $\alpha\mu'$-closed function.

Proof: Let $S$ be an $\mu$-closed set in $X$, also $X$ is $\mu$-compact, then $S$ is $\mu$-compact subset of $X$ from Proposition 2.13, and $f$ is $m$-continuous function, then $f(S)$ is $\mu$-compact set in $\mathcal{Y}$ from Proposition 2.27. Also $\mathcal{Y}$ is $\mu$-$Kc$-space, so $f(S)$ is $\alpha\mu'$-closed in $\mathcal{Y}$, therefore $f$ is $\alpha\mu'$-closed.

Proposition 4.11 If the function $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ is $\alpha^*\mu$-continuous, $(X, \mu)$ is $\alpha\mu$-compact and $(\mathcal{Y}, \mu')$ is $\alpha$-$K(\alpha)c$-space, then $f$ is $\alpha^*\mu$-closed function.

Proof: Let $F$ be an $\alpha\mu$-closed set of $X$, since $X$ is am-$\mu$-compact, so by Theorem 3.8, $F$ is $\alpha\mu$-compact in $X$ and $f$ is $\alpha^*\mu$-continuous. Then $f(F)$ is $\alpha\mu$-compact in $\mathcal{Y}$. Also by Theorem 4.6, $\mathcal{Y}$ is $\alpha$-$K(\alpha)c$-space, hence $f(F)$ is $\alpha^*$-$\mu$-closed in $\mathcal{Y}$. Therefore, $f$ is $\alpha^*\mu$-closed.

Theorem 4.12 If $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ is $m$-closed, $\alpha^*\mu$-open bijective function and $(X, \mu)$ is $\mu$-$\alpha$-$K(\alpha)c$-space, then $(\mathcal{Y}, \mu')$ is $\alpha$-$K(\alpha)c$-space.

Proof: Let $K$ be an $\alpha\mu$-open cover of $\mathcal{Y}$ and $\{V_\alpha : \alpha \in I\}$ be an $\alpha\mu$-open cover of $f^{-1}(K)$ in $X$, that is $f^{-1}(K) \subseteq \bigcup_{\alpha \in I} V_\alpha$. Since $f$ is $\alpha\mu$-open, so $K = f(f^{-1}(K)) \subseteq \bigcup_{\alpha \in I} f(V_\alpha)$. And $f$ is $\alpha^*\mu$-open function, so $\bigcup_{\alpha \in I} f(V_\alpha)$ is $\alpha\mu'$-open in $\mathcal{Y}$, for each $\alpha \in I$. Also, $K$ is $\alpha^*\mu$-compact in $X$, so $K \subseteq \bigcup_{i=1}^n f(V_{\alpha_i})$. This implies that $f^{-1}(K) \subseteq \bigcup_{i=1}^n f^{-1}(f(V_{\alpha_i})) = \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$, so $f^{-1}(K)$ is $\alpha\mu$-compact in $X$, which is $\mu$-$K(\alpha)$-space, so $f^{-1}(K)$ is $\mu$-closed. Also, since $f$ is $m$-closed function, therefore $f(f^{-1}(K)) = K$ is $\mu$-closed in $\mathcal{Y}$. Hence $f$ is $\mu$-$K(\alpha)$-space.

Theorem 4.13 Let the injective function $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ be $m$-continuous and $\alpha^*\mu$m-continuous. Then $(X, \mu)$ is $\mu$-$K(\alpha)c$-space whenever $(\mathcal{Y}, \mu')$ is $\mu$-$K(\alpha)c$-space.

Proof: Let $A$ be $\alpha\mu$-compact in $X$, so $f(A)$ is $\alpha\mu$-compact in $\mathcal{Y}$ by Theorem 4.6. And since $\mathcal{Y}$ is $\mu$-$\alpha$-$K(\alpha)c$-space, so that $f(A)$ is $\alpha\mu'$-closed subset of $\mathcal{Y}$ and $f^{-1}(f(A)) = A$ (if $f$ is injective), so $A$ is $\alpha\mu'$-closed subset of $X$ since $f$ is $\alpha^*\mu$-continuous function. Therefore, $X$ is $\mu$-$\alpha\mu$-$K(\alpha)c$-space.

Proposition 4.15 If $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ is $m$-continuous function, $X$ is $\mu$-compact space and $\mathcal{Y}$ is $\mu$-$\theta\kappa(c)$-space, then $f$ is $\theta\mu'$-closed function, whenever $X$ has $\theta\mathcal{Y}$ property.

Proof: Let $N$ be $\theta\mu$-closed subset of $X$, so that $N$ is $\mu$-closed in $X$ by Remark 2.33. And since $X$ is $\mu$-compact, then $N$ is $\mu$-compact by Proposition 2.13. Also $f$ is $m$-continuous function, so by Proposition 2.27, $f(N)$ is $\mu$-compact, hence from Remark 2.36, $f(N)$ is $\theta\mu$-compact in $\mathcal{Y}$ which is $\mu$-$\theta\kappa(c)$-space. Therefore $f(N)$ is $\mu'$-closed. That is $f$ is $\theta^*\mu'$-closed function.

Proposition 4.16 Let $f : (X, \mu) \rightarrow (\mathcal{Y}, \mu')$ be $m$-homeomorphism function. Then $(\mathcal{Y}, \mu')$ is $\mu$-$\kappa(c)$-space, whenever $(X, \mu)$ is $\mu$-$\kappa(c)$-space which has $\theta\beta$ property.
Proof: Let $\mathcal{H}$ be an $\theta\mu$-compact set in $\mathcal{Y}$, by Proposition 2.40, $f^{-1}(\mathcal{H})$ is $\theta\mu$-compact in $\mathcal{X}$ which is $\mu-\theta k(c)$-space. So $f^{-1}(\mathcal{H})$ is $\mu$-closed set in $\mathcal{X}$ and $f(f^{-1}(\mathcal{H})) = \mathcal{H}$ is $\mu'$-closed set in $\mathcal{Y}$. Therefore, $(\mathcal{Y}, \mu')$ is $\mu-\theta k(c)$-space.

References
1. Wilansky, A. 1967. $T_1$ and $T_2$, Amer, Math Monthly, 74: 261-266.
2. Maki, H. 1996. On generalizing semi-open and preopen sets, Report for Meeting on topological spaces theory and its applications, August, Yaatsus Hiro College of Technology, 13-18.
3. Popa, V. and Noiri, T. 2000. On $m$-continuous functions, Anal. Univ. Dunarea de Jos Galati. Ser. Mat. Fiz. Mec. Teor. Fasci., 18(23): 31-41.
4. Ali, H. J. and Dahham, M. M. 2017. When $m$-compact sets are $m_x$-semi closed. International Journal of Mathematical Archive, 8(4): 116-120.
5. Najasted, O. 1964. On some classes of nearly open sets, Pacific journal of mathematics, 3: 961-970.
6. Won, K. M. 2010. $\alpha m$-open sets and $\alpha m$-continuous functions, Commun. Korean Math. Soc. 25 (2): 251-256.
7. Velicko, N. V. 1968. H-closed topological spaces, Trans. Amer.Math. Soc. Transl. 78 (1968): 102-118.
8. Das, P. 1973. Note on some application on semi open set, Progress of Math. 7: 33-44.
9. Ali, H. J and Harith, M. 2014. Some types of $m$-compact functions. Al Mustansiriyah J. Sci. 25(4): 65-74.
10. Carpintero, C., Rosas, E. and Salas, M. 2007. Minimal structure and separation properties, International Journal of Pure and Applied Mathematics, 34(4): 473-488.
11. Muthana, H. A and Ali, H. J. 2014. Some type of $\mu$-compact functions, Journal of sci. Al Mustansiriyah university, 25(4).
12. Ali, H. J. 2010. Strong and weak form of $m$-lindelof space, Editorial board of Zenco J. for pure and Apple. Sciences, Salahaddin university, Howler –Iraqi Kurdistan Region, special issue 22: 60-64.
13. Hader, J. A. and Marwa, M. D. 2018. When $m$-Lindelof sets are $m_x$-semi closed, Journal of Physics: Conference series.1003012044.
14. Popa, V. and Noiri, T. 2000. On M-continuous functions. Anal. Univ. “Dunarea de Jos” Galati. Ser. Mat. Fiz. Mec. Teor. Fasc. II, 18(23): 31-41.
15. Popa, V. and Noiri, T. 2001. On the definition of some generalized form of continuity under minimal conditions. Men. Fac. Ser. Kochi. Univ. Ser. Math. 22: 9-19.