EQUIVARIANT SPLITTING OF THE HODGE–DE RHAM
EXACT SEQUENCE

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Abstract. Let $X$ be an algebraic curve with an action of a finite
group $G$ over a field $k$. We show that if the Hodge–de Rham short
exact sequence of $X$ splits $G$-equivariantly then the action of $G$ on $X$
is weakly ramified. In particular, this generalizes the result of Köck and
Tait for hyperelliptic curves. We discuss also converse statements and
tie this problem to lifting coverings of curves to the ring of Witt vectors
of length 2.

1. Introduction

Let $X$ be a smooth proper algebraic variety over a field $k$. Recall that its
de Rham cohomology may be computed in terms of Hodge cohomology via
the spectral sequence

\[ E^{ij}_1 = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{dR}(X/k). \]

Suppose that the spectral sequence (1.1) degenerates at the first page. This
is automatic if $\text{char } k = 0$. For a field of positive characteristic, this happens
for instance if $X$ is a smooth projective curve or an abelian variety, or (by
a celebrated result of Deligne and Illusie from [DI87]) if $\dim X > \text{char } k$ and
$X$ lifts to $W_2(k)$, the ring of Witt vectors of length 2. Under this assumption
we obtain the following exact sequence:

\[ 0 \to H^0(X, \Omega^1_{X/k}) \to H^1_{dR}(X/k) \to H^1(X, \mathcal{O}_X) \to 0. \]

If $X$ is equipped with an action of a finite group $G$, the terms of the se-
quence (1.2) become $k[G]$-modules. In case when $\text{char } k \nmid \# G$, Maschke the-
orem allows one to conclude that the sequence (1.2) splits equivariantly.
However, this might not be true in case when $\text{char } k = p > 0$ and $p \mid \# G$,
as shown in [KT18]. The goal of this article is to show that for curves the sequence (1.2) usually does not split equivariantly.

Let $X$ be a curve over an algebraically closed field of characteristic $p > 0$ with an action of a finite group $G$. For $P \in X$, denote by $G_{P,n}$ the $n$-th ramification group of $G$ at $P$. Let also:

$$n_P := \max\{n : G_{P,n} \neq 0\}.$$  

Following [K04], we say that the action of $G$ is weakly ramified if $n_P \leq 1$ for every $P \in X$.

**Main Theorem.** Suppose that $X$ is a smooth projective curve over an algebraically closed field $k$ of a finite characteristic $p > 2$ with an action of a finite group $G$. If the sequence (1.2) splits $G$-equivariantly, then the action of $G$ on $X$ is weakly ramified.

The example below is a direct generalization of results proven in [KT18].

**Example 1.1.** Suppose that $k$ is an algebraically closed field of characteristic $p$. Let $X/k$ be the smooth projective curve with the affine part given by the equation:

$$y^m = f(z^p - z),$$

where $f$ is a separable polynomial and $p \nmid m$. Denote by $\mathcal{P}$ the set of points of $X$ at infinity. One checks that $\#\mathcal{P} = \delta := \text{GCD}(m, \deg f)$ (cf. [Tow96, Section 1]). The group $G = \mathbb{Z}/p$ acts on $X$ via the automorphism $\varphi(z, y) = (z+1, y)$. In this case

$$(1.4) \quad n_P = \begin{cases} 
m/\delta, & \text{if } P \in \mathcal{P}, \\
0, & \text{otherwise.} \end{cases}$$

(cf. Example 4.3). Thus if the exact sequence (1.2) splits $G$-equivariantly, then by Main Theorem either $p = 2$, or $m \mid \deg f$.

The main idea of the proof of Main Theorem is to compare $H^1_{dR}(X/k)^G$ and $H^1_{dR}(Y/k)$, where $Y := X/G$. The discrepancy between those groups is measured by the sheafified version of group cohomology, introduced by Grothendieck in [Gro57]. This allows us to compute the ‘defect’

$$(1.5) \quad \delta(X, G) := \dim_k H^0(X, \Omega_{X/k})^G + \dim_k H^1(X, \mathcal{O}_X)^G - \dim_k H^1_{dR}(X/k)^G$$

in terms of some local terms connected to Galois cohomology (cf. Proposition 3.1). We compute these local terms in case of Artin-Schreier coverings, which leads to the following theorem.
Theorem 1.2. Suppose that $X$ is a smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$ with an action of the group $G = \mathbb{Z}/p$. Then:

$$\delta(X, G) = \sum_{P \in X} \left( \left( \frac{(p - 1) \cdot (n_P + 1)}{p} \right) - 1 - \left[ \frac{n_P - 1}{p} \right] \right).$$

Theorem 1.2 shows that if the group action of $G = \mathbb{Z}/p$ on a curve is not weakly ramified and $p > 2$ then $\delta(X, G) > 0$. This immediately implies Main Theorem for $G \cong \mathbb{Z}/p$. The general case may be easily derived from this special one.

The natural question arises: to which extent is the converse of Main Theorem true? We give some partial answers. In characteristic 2, we were able to produce a counterexample (cf. Subsection 5.1). We provide also some positive results. In particular, we prove the following theorem.

Theorem 1.3. If the action of $G$ on a smooth projective curve $X$ over an algebraically closed field $k$ is weakly ramified then the sequence

$$0 \to H^0(X, \Omega_{X/k})^G \to H^1_{dR}(X/k)^G \to H^1(X, \mathcal{O}_X)^G \to 0$$

is exact also on the right.

To derive Theorem 1.3 we use the method of proof of Main Theorem and a result of Köck from [K04]. We were also able to show the splitting of the Hodge–de Rham exact sequence of a curve with a weakly ramified group action under some additional assumptions.

Theorem 1.4. Keep the above notation. The sequence (1.2) splits provided that any of the following conditions holds:

1. the action of $G$ on $X$ is weakly ramified and the $p$-Sylow subgroup of $G$ is cyclic,
2. the action of $G$ on $X$ lifts to $W_2(k)$,
3. $X$ is ordinary.

Note in particular that by Main Theorem, the conditions (2) and (3) of Theorem 1.4 imply that the action of $G$ on $X$ is weakly ramified. Parts (1), (2), (3) of Theorem 1.4 are proven in Lemma 5.4, Theorem 5.5 and Corollary 5.7 respectively. In order to prove (1) we use a description of modular representations of cyclic groups. (2) and (3) are easy corollaries of the equivariant version of results of Deligne and Illusie from [DI87]. The connection of [DI87] with the splitting of the Hodge–de Rham exact sequence was observed by Piotr Achinger.
Notation. Throughout the paper we will use the following notation (unless stated otherwise):

- $k$ is an algebraically closed field of a finite characteristic $p$.
- $G$ is a finite group.
- $X$ is a smooth projective curve equipped with an action of $G$.
- $Y := X/G$ is the quotient curve, which is of genus $g_Y$.
- $\pi : X \to Y$ is the canonical projection.
- $R = \sum_{P \in X} d_P \cdot (P) \in \text{Div}(X)$ is the ramification divisor of $\pi$.
- $R' := \left[ \frac{\pi^* R}{\# G} \right] \in \text{Div}(Y)$, where for $\delta \in \text{Div}(Y) \otimes \mathbb{Z}/\mathbb{Q}$, we denote by $[\delta]$ the integral part taken coefficient by coefficient.
- $k(X), k(Y)$ are the function fields of $X$ and $Y$.
- $\text{ord}_Q(f)$ denotes the order of vanishing of a function $f$ at a point $Q$.
- $A_X$ denotes the constant sheaf on $X$ associated to a ring $A$.

Fix now a (closed) point $P$ in $X$. Denote:

- $G_{P,i}$ – the $i$th ramification group of $\pi$ at $P$, i.e.
  $G_{P,i} := \{ g \in G : g(f) \equiv f \pmod{m^{i+1}_{X,P}} \text{ for all } f \in \mathcal{O}_{X,P} \}$.
  Note in particular that (since $k$ is algebraically closed) the inertia group $G_{P,0}$ coincides with the decomposition group at $P$, i.e. the stabilizer of $P$ in $G$. Moreover, one has:
  $$d_P = \sum_{i \geq 0} (#G_{P,i} - 1).$$
- $e_P$ – the ramification index of $\pi$ at $P$, i.e. $e_P = #G_{P,0}$.
- $n_P$ is given by the formula (1.3).

Also, by abuse of notation, for $Q \in Y$ we write $e_Q := e_P, d_Q := d_P, n_Q := n_P$ for any $P \in \pi^{-1}(Q)$. Note that these quantities don’t depend on the choice of $P$.

Outline of the paper. Section 2 presents some preliminaries on the group cohomology of sheaves. We focus on the sheaves coming from Galois coverings of a curve. We use this theory to express the ’defect’ $\delta(X,G)$ as a sum of local terms coming from Galois cohomology of certain modules in Section 3. In Section 4 we compute these local terms for Artin-Schreier coverings, which allows us to prove of Main Theorem and Theorem 1.2. In the final section we discuss the converse statements to Main Theorem and its relation to the problem of lifting curves with a given group action. Also, we give an counterexample in characteristic 2. We include also Appendix, which allows to compute the dimensions of $H^0(X,\Omega_X)^G$, $H^1(X,\mathcal{O}_X)^G$ and $H^1_{dR}(X/k)^G$. 
2. Review of group cohomology

Recall that our goal is to compare $H^1_{dR}(X/k)^G$ and $H^1_{dR}(Y/k)$, where $Y := X/G$. To this end, we need to work in the $G$-equivariant setting.

2.1. Group cohomology of sheaves. Let $A$ be any commutative ring and $G$ a finite group. We define the $i$-th group cohomology, $H^i_A(G, -)$, as the $i$-th derived functor of the functor $(-)^G : A[G]\text{-mod} \to A\text{-mod}$, $M \mapsto M^G := \{m \in M : g \cdot m = m\}$.

One checks that if $A \to B$ is a morphism of rings and $M$ is a $B[G]$-module then $H^i_B(G, M)$ and $H^i_A(G, M)$ are isomorphic $A$-modules for all $i \geq 0$ (cf. [Sta16, Lemma 0DVD]). In particular, $H^i_A(G, M)$ is isomorphic as a $\mathbb{Z}$-module to the usual group cohomology ($H^i_Z(G, M)$ in our notation). Thus without ambiguity we will denote it by $H^i(G, M)$. For a future use we note the following properties of group cohomology:

- If $M = \text{Ind}^G_H N$ is an induced module (which for finite groups is equivalent to being a coinduced module) then
  \begin{equation}
  H^i(G, M) \cong H^i(H, N),
  \end{equation}
  (this property is known as Shapiro’s lemma, cf. [Ser79, Proposition VIII.2.1.1]).

- If $M$ is a $\mathbb{F}_p[G]$-module and $G$ has a normal $p$-Sylow subgroup $P$ then:
  \begin{equation}
  H^i(G, M) \cong H^i(P, M)
  \end{equation}
  (for a proof observe that $H^i(G/P, N)$ is killed by multiplication by $p$ for any $\mathbb{F}_p[G]$-module $N$ and use [Ser79, Theorem IX.2.4.] to obtain $H^i(G/P, N) = 0$ for $i \geq 1$. Then use Lyndon–Hochschild–Serre spectral sequence).

- Suppose that $A$ is a finitely generated algebra over a field $k$, which is a local ring with maximal ideal $m$. If $M$ is a finitely generated $A$-module then
  \begin{equation}
  H^i(G, M) \cong H^i(G, \widehat{M}_m),
  \end{equation}
  where $\widehat{M}_m$ denotes the completion of $M$ with respect to $m$ (see e.g. proof of [BM00, Lemme 3.3.1] for a brief justification).
The above theory extends to sheaves, as explained e.g. in [Gro57] and [BM00]. We briefly recall this theory. Let \((Y, \mathcal{O})\) be a ringed space and let \(G\) be a finite group. By an \(\mathcal{O}[G]\)-sheaf on \((Y, \mathcal{O})\) we understand a sheaf \(\mathcal{F}\) equipped with an \(\mathcal{O}\)-linear action of \(G\) on \(\mathcal{F}(U)\) for every open subset \(U \subset Y\), compatible with respect to the restrictions. The \(\mathcal{O}[G]\)-sheaves form a category \(\mathcal{O}[G]\)-mod, which is abelian and has enough injectives. For any \(\mathcal{O}[G]\)-sheaf \(\mathcal{F}\) one may define a sheaf \(\mathcal{F}^G\) by the formula
\[
U \mapsto \mathcal{F}(U)^G := \{ f \in \mathcal{F}(U) : \forall g \in G \ g \cdot f = f \}.
\]
We denote the \(i\)-th derived functor of \((-)^G : \mathcal{O}[G]\)-mod \(\to \mathcal{O}\)-mod by \(H^i(Y, \mathcal{O})^G \xrightarrow{} G, -\). Similarly as in the case of modules, one may neglect the dependence on the sheaf \(\mathcal{O}\) and write simply \(H^i(G, M)\).

In particular, group cohomology of a quasicoherent \(\mathcal{O}[G]\)-sheaf is a quasicoherent \(\mathcal{O}\)-module. Moreover for any \(Q \in Y\):
\[
H^i(G, \mathcal{F})_Q \cong H^i(G, \mathcal{F}_Q).
\]

The sheaf group cohomology may be also computed using Čech complex (cf. [BM00, section 3.1]). However, we will not use this fact in any way.

2.2. Galois coverings of curves. We now turn to the case of curves over a field \(k\). Let \(X/k\) be a smooth projective curve with an action of a finite group \(G\), i.e. a homomorphism \(G \to \text{Aut}_k(X)\). In this case one can define the quotient \(Y := X/G\) of \(X\) by the \(G\)-action. It is a smooth projective curve. Its underlying space is the topological quotient \(X/G\) and the structure sheaf is given by \(\pi^*_G(\mathcal{O}_X)\), where \(\pi : X \to Y\) is the quotient morphism. We say that \(X\) is a \(G\)-covering of \(Y\).

In this section we will investigate the \(G\)-sheaves on \(Y\) coming from its \(G\)-coverings. Suppose that \(\pi : X \to Y\) is a \(G\)-covering of \(Y\). Let \(\mathcal{F}\) be a \(\mathcal{O}_X\)-sheaf on \(X\) with a \(G\)-action lifting that on \(X\). Then \(\pi_*\mathcal{F}\) is an \(\mathcal{O}_Y[\mathcal{G}]\)-module. It is natural to try to relate the group cohomology of \(\pi_*\mathcal{F}\) to the ramification of \(\pi\). Suppose for a while that the action of \(G\) on \(X\) is free, i.e. that \(\pi : X \to Y\) is unramified. In this case the functors
\[
\mathcal{F} \mapsto \mathcal{F}^G
\]
\[
\mathcal{F} \mapsto \mathcal{F}^G
\]
are exact and provide an equivalence between the category of coherent $\mathcal{O}_Y$-modules and coherent $\mathcal{O}_X$-modules (cf. [Mum08, Proposition II.7.2, p. 70]). In particular, $\mathcal{H}^i(G, \pi_*\mathcal{F}) = 0$ for all $i \geq 1$ and every coherent $\mathcal{O}_X$-module $\mathcal{F}$. The following Proposition treats the general case.

**Proposition 2.1.** Keep the notation introduced in Section 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, which is $G$-equivariant. Then for every $i \geq 1$

$$\mathcal{H}^i(G, \pi_*\mathcal{F})$$

is a torsion sheaf, supported on the wild ramification locus of $\pi$.

To prove Proposition 2.1 we shall need the following lemma involving group cohomology of modules over Dedekind domains.

**Lemma 2.2.** Let $k$ be an algebraically closed field. Let $B$ be a finitely generated $k$-algebra, which is a Dedekind domain equipped with a $k$-linear action of the group $G$. Suppose that $A := B^G$ is a principal ideal domain with a maximal ideal $q$. Let $G_{p,i}$ denote the $i$-th higher ramification group of a prime ideal $p \in \text{Spec } B$ over $q$. Then for every $B$-module $M$ we have an isomorphism of $B$-modules:

$$H^i(G, M) \cong H^i(G_{p,1}, M_p)$$

(here $M_p$ denotes the localisation of $M$ at $p$).

**Proof.** One easily sees that we have an isomorphism of $B[G]$-modules

$$\tilde{M}_q \cong \text{Ind}^G_{G_{p,0}} \tilde{M}_p$$

and thus by (2.3) and (2.1) $H^i(G, M) \cong H^i(G_{p,0}, M_p)$. Moreover, $G_{p,1}$ is a normal $p$-Sylow subgroup of $G_{p,0}$ (cf. [Ser79, Corollary 4.2.3., p. 67]). Hence the proof follows by (2.2). \hfill \square

**Proof of Proposition 2.1.** Denote by $\xi$ the generic point of $Y$. Recall that by the Normal Base Theorem (cf. [Jac85, sec. 4.14]), $k(X) = \text{Ind}^G k(Y)$ is an induced $G$-module. Therefore $(\pi_*\mathcal{F})_\xi$ is also an induced $G$-module (since it is a $k(X)$-vector space of finite dimension) and by (2.1):

$$\mathcal{H}^i(G, \pi_*\mathcal{F})_\xi = H^i(G, (\pi_*\mathcal{F})_\xi) = 0.$$

Thus, since the sheaf $\mathcal{H}^i(G, \pi_*\mathcal{F})$ is coherent, it must be a torsion sheaf. Note that if a point $Q \in Y$ is tamely ramified then $G_{P,1} = 0$ for any $P \in \pi^{-1}(Q)$ and thus $\mathcal{H}^i(G, \pi_*\mathcal{F})_Q = 0$ by Lemma 2.2. This concludes the proof. \hfill \square

We will recall now a standard formula describing $G$-invariants of an $\mathcal{O}_Y[G]$-module coming from an invertible $\mathcal{O}_X$-module. For a reference see e.g. the proof of [BM00, Proposition 5.3.2].
Lemma 2.3. For any $G$-invariant divisor $D \in \text{Div}(X)$:

$$\pi_*^G(\mathcal{O}_X(D)) = \mathcal{O}_Y\left(\left\lceil \frac{\pi_*D}{\#G} \right\rceil\right),$$

where for $\delta \in \text{Div}(Y) \otimes \mathbb{Z} \mathbb{Q}$, we denote by $[\delta]$ the integral part taken coefficient by coefficient.

Corollary 2.4. Keep the notation of Section 2. Let:

$$R' = \left\lceil \frac{\pi_*R}{\#G} \right\rceil \in \text{Div}(Y).$$

Then:

$$\pi_*^G \Omega_{X/k} = \Omega_{Y/k} \otimes \mathcal{O}_Y(R').$$

In particular:

$$\dim_k H^0(X, \Omega^1_{X/k})^G = \begin{cases} g_Y, & \text{if } R' = 0, \\ g_Y - 1 + \deg R', & \text{otherwise}. \end{cases}$$

Proof. The first claim follows by Lemma 2.3 by taking $D$ to be the canonical divisor of $X$ and using the Riemann-Hurwitz formula. To prove the second claim, observe that

$$H^0(X, \Omega^1_{X/k}) = H^0(Y, \pi_*^G \Omega_{X/k}) = H^0(Y, \Omega_{Y/k} \otimes \mathcal{O}_Y(R'))$$

and apply the Riemann-Roch theorem (cf. [Har77, Theorem IV.1.3]). \qed

We end this section with one more elementary observation.

Lemma 2.5. $R'$ (given as above) vanishes if and only if the morphism $\pi : X \to Y$ is tamely ramified.

Proof. Recall that $R = \sum_{P \in X} d_P \cdot (P)$. Hence

$$R' = \sum_{Q \in Y} \left\lceil \frac{d_Q \cdot \# \pi^{-1}(Q)}{\#G} \right\rceil (Q) = \sum_{Q \in Y} \left\lceil \frac{d_Q}{e_Q} \right\rceil (Q).$$

Note however that $d_Q \geq e_Q - 1$ with an equality if and only if $\pi$ is tamely ramified at $Q$. This completes the proof. \qed
3. Computing the defect

The goal of this section is to prove the following Proposition.

**Proposition 3.1.** We follow the notation introduced in Section 1. Then:

\[ \delta(X, G) = \sum_{Q \in Y} \dim_k \text{im} \left( H^1(G, (\pi_* \mathcal{O}_X)_Q) \to H^1(G, (\pi_* \Omega_{X/k})_Q) \right), \]

where

\[ H^1(G, (\pi_* \mathcal{O}_X)_Q) \to H^1(G, (\pi_* \Omega_{X/k})_Q) \]

is the map induced by the differential \( \mathcal{O}_X \to \Omega_{X/k} \).

3.1. **Proof – preparation.** Recall that \( i \)-th hypercohomology \( \mathbb{H}^i(Y, -) \) is defined as the \( i \)-th derived functor of \( H^0(Y, -) : k_Y\text{-mod} \to k\text{-mod} \).

The hypercohomology may be computed in terms of the usual cohomology using the spectral sequence (3.1)

\[ E^{ij}_1 = H^j(Y, \mathcal{F}^i) \Rightarrow \mathbb{H}^{i+j}(Y, \mathcal{F}^\bullet). \]

The de Rham cohomology of \( X \) is defined as the hypercohomology of the de Rham complex \( \Omega^\bullet_{X/k} \). Note that \( \pi \) is an affine morphism. Therefore \( \pi_* \) is an exact functor on the category of quasi-coherent sheaves. Thus using the spectral sequence (3.1) we obtain:

\[ H^j_{dR}(X/k) = \mathbb{H}^j(X, \Omega^\bullet_{X/k}) = \mathbb{H}^j(Y, \pi_* \Omega^\bullet_{X/k}). \]

We start with the following observation.

**Lemma 3.2.** The spectral sequence

\[ E^{ij}_1 = H^j(Y, \pi_* ^{G} \Omega^i_{X/k}) \Rightarrow \mathbb{H}^{i+j}(Y, \pi_* ^{G} \Omega^\bullet_{X/k}) \]

degenerates at the first page.

**Proof.** We have a morphism of complexes \( \Omega^\bullet_{Y/k} \to \pi_* ^{G} \Omega^\bullet_{X/k} \), which is an isomorphism on the zeroth term. Thus for \( j = 0, 1 \) we obtain a commutative diagram:

\[
\begin{array}{ccc}
H^j(Y, \mathcal{O}_Y) & \cong & H^j(Y, \pi_* ^{G} \Omega_X) \\
\downarrow & & \downarrow \\
H^j(Y, \Omega_{Y/k}) & \to & H^j(Y, \pi_* ^{G} \Omega_{X/k})
\end{array}
\]
where the left arrow is zero and the upper arrow is an isomorphism. Therefore for \( j = 0, 1 \) the maps

\[
H^j(Y, \pi_*^G \Omega_{X/k}) \to H^j(Y, \pi_*^G \Omega_{X/k})
\]

are zero. This is the desired conclusion. \( \square \)

**Corollary 3.3.**

\[
\delta(X, G) = \left( \dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}) - \dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}) \right)
\]

\[
- \left( \dim_k H^1(Y, \pi_*^G \Omega_X) - \dim_k H^1(Y, \pi_* \Omega_X) \right).
\]

**Proof.** By Lemma 3.2 we obtain an exact sequence:

\[
0 \to H^0(Y, \pi_*^G \Omega_{X/k}) \to \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}) \to H^1(Y, \pi_*^G \Omega_{X/k}) \to 0.
\]

Hence:

\[
\delta(X, G) = \dim_k H^0(X, \Omega_{X/k})^G + \dim_k H^1(X, \Omega_X)^G - \dim_k H^1_{dR}(X/k)^G
\]

\[
= \left( \dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}) - \dim_k H^1(Y, \pi_*^G \Omega_X) \right)
\]

\[
+ \dim_k H^1(Y, \pi_*^G \Omega_X)^G - \dim_k H^1_{dR}(X/k)^G
\]

\[
= \left( \dim_k \mathbb{H}^1(Y, \pi_*^G \Omega_{X/k}) - \dim_k \mathbb{H}^1(Y, \pi_* \Omega_{X/k}) \right)
\]

\[
- \left( \dim_k H^1(Y, \pi_*^G \Omega_X) - \dim_k H^1(Y, \pi_* \Omega_X) \right).
\]

\( \square \)

Corollary 3.3 implies that we need to compare the hypercohomology groups

\[
\mathbb{H}^i(Y, (\mathcal{F}^\bullet)^G) \text{ and } \mathbb{H}^i(Y, (\mathcal{F}^\bullet)^G).
\]

for \( \mathcal{F}^\bullet = \pi_* \Omega_X \) (treated as a complex concentrated in degree 0) and \( \mathcal{F}^\bullet = \pi_* \Omega_{X/k}^\bullet \) (note that it is a complex of \( \mathcal{K}_Y[G]\)-modules rather than \( \mathcal{O}_Y\)-modules, since the differentials in the de Rham complex are not \( \mathcal{O}_Y\)-linear). Consider the commutative diagram:

\[
\begin{array}{c}
\mathcal{K}_Y[G] \text{-mod} \xrightarrow{(-)^G} \mathcal{K}_Y \text{-mod} \\
\downarrow \Gamma(Y,-) \quad \downarrow \Gamma(Y,-) \\
k[G] \text{-mod} \xrightarrow{(-)^G} k \text{-mod}.
\end{array}
\]

By applying Grothendieck spectral sequence to compositions of the functors in the diagram, we obtain two spectral sequences:

(3.2) \( I E_2^{ij} = \mathbb{H}^i(Y, \mathcal{H}^j(G, \mathcal{F}^\bullet)) \Rightarrow \mathbb{R}^{i+j} \Gamma^G(\mathcal{F}^\bullet) \)

(3.3) \( II E_2^{ij} = H^i(G, \mathbb{H}^j(Y, \mathcal{F}^\bullet)) \Rightarrow \mathbb{R}^{i+j} \Gamma^G(\mathcal{F}^\bullet), \)
(note that here $\mathcal{H}^j(G, F^\bullet)$ denotes a complex of $k_Y$-modules with $l$th term being $\mathcal{H}^j(G, F^l)$).

For motivation, suppose at first that the 'obstructions'

$$\mathcal{H}^i(G, F^l) \quad \text{and} \quad H^i(G, \mathbb{H}^l(Y, F^\bullet))$$

vanish for all $i \geq 1$ and $l \geq 0$ (this happens e.g. if char $k = 0$). Then the spectral sequences (3.2) and (3.3) lead us to the isomorphisms:

$$\mathbb{H}^i(Y, (F^\bullet)^G) \cong \mathbb{R}^1\Gamma^G(F^\bullet) \cong (\mathbb{H}^i(Y, F^\bullet))^G.$$

In general case the relation between $\mathbb{H}^i(Y, (F^\bullet)^G)$ and $\mathbb{H}^i(Y, F^\bullet)^G$ is more complicated. However, in the case of the first hypercohomology group, one can extract some information from the low-degree exact sequences of (3.2) and (3.3):

$$(3.4) \quad 0 \rightarrow \mathbb{H}^1(Y, (F^\bullet)^G) \rightarrow \mathbb{R}^1\Gamma^G(F^\bullet) \rightarrow$$

$$\rightarrow \mathbb{H}^0(Y, \mathcal{H}^1(G, F^\bullet)) \rightarrow \mathbb{H}^2(Y, (F^\bullet)^G) \rightarrow$$

$$\rightarrow \mathbb{R}^2\Gamma^G(F^\bullet)$$

and

$$(3.5) \quad 0 \rightarrow H^1(G, \mathbb{H}^0(Y, F^\bullet)) \rightarrow \mathbb{R}^1\Gamma^G(F^\bullet) \rightarrow$$

$$\rightarrow \mathbb{H}^1(Y, F^\bullet)^G \rightarrow H^2(G, \mathbb{H}^0(Y, F^\bullet)) \rightarrow$$

$$\rightarrow \mathbb{R}^2\Gamma^G(F^\bullet).$$

This will be done separately in the case of wild and tame ramification in the Subsections 3.2 and 3.3.

### 3.2. Proof — the wild case.

Consider first the case when $\pi$ is wildly ramified, i.e. by Lemma 2.5 when $R' \neq 0$. Then, as one easily sees by Lemma 3.2, Corollary 2.4 and Riemann–Roch theorem (cf. [Har77, Theorem IV.1.3]):

$$\mathbb{H}^2(Y, \pi_*^G\Omega^\bullet_{X/k}) = H^1(Y, \pi_*^G\Omega_{X/k}) = H^1(Y, \Omega_{Y/k} \otimes \mathcal{O}_Y(R')) = 0.$$

By (3.4) we see that

$$\dim_k \mathbb{R}^1\Gamma^G(\pi_*\Omega^\bullet_{X/k}) = \dim_k \mathbb{H}^1(Y, (\pi_*\Omega^\bullet_{X/k})^G) + \dim_k \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_*\Omega^\bullet_{X/k})).$$

On the other hand, (3.5) yields:

$$\dim_k \mathbb{R}^1\Gamma^G(\pi_*\Omega^\bullet_{X/k}) = \dim_k H^1(G, \mathbb{H}^0(Y, \pi_*\Omega^\bullet_{X/k})) + \dim_k \mathbb{H}^1(Y, \pi_*\Omega^\bullet_{X/k})^G - c_1,$$
where

\[ c_1 = \dim_k \ker \left( H^2(G, \mathbb{H}^0(Y, \pi_* \Omega^\bullet_{X/k})) \to \mathbb{R}^2 \Gamma^G(\pi_* \Omega^\bullet_{X/k}) \right). \]  

Thus by comparing (3.6) and (3.7):

\[ \dim_k H^1(Y, \pi_* \Omega^\bullet_{X/k})^G = \dim_k H^1(Y, (\pi_* \Omega^\bullet_{X/k})^G) \]

\[ + \quad \dim_k \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_* \Omega^\bullet_{X/k})) \]

\[ - \quad \dim_k H^1(G, \mathbb{H}^0(Y, \pi_* \Omega^\bullet_{X/k})) + c_1 \]

By repeating the same argument for \( \pi_* \mathcal{O}_X \), we obtain:

\[ \dim_k H^1(Y, \pi_* \mathcal{O}_X)^G = \dim_k H^1(Y, (\pi_* \mathcal{O}_X)^G) \]

\[ + \quad \dim_k \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_* \mathcal{O}_X)) \]

\[ - \quad \dim_k H^1(G, \mathbb{H}^0(Y, \pi_* \mathcal{O}_X)) + c_2, \]

where:

\[ c_2 = \dim_k \ker \left( H^2(G, \mathbb{H}^0(Y, \pi_* \mathcal{O}_X)) \to \mathbb{R}^2 \Gamma^G(\pi_* \mathcal{O}_X) \right). \]

By combining (3.9), (3.10) and Corollary 3.3 we obtain:

\[ \delta(X, G) = \dim_k \mathbb{H}^1(Y, \mathcal{H}^1(G, \pi_* \mathcal{O}_X)) \to \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_* \Omega^\bullet_{X/k})) \]

\[ + \quad (c_2 - c_1). \]

Note that since \( \mathcal{H}^1(G, \pi_* \mathcal{O}_X) \), \( \mathcal{H}^1(G, \pi_* \Omega^\bullet_{X/k}) \) are torsion sheaves, we can compute their sections by taking stalks and using (2.4):

\[ \dim_k \mathbb{H}^1(Y, \mathcal{H}^1(G, \pi_* \mathcal{O}_X)) \to \mathbb{H}^0(Y, \mathcal{H}^1(G, \pi_* \Omega^\bullet_{X/k})) \]

\[ = \sum_{Q \in Y} \dim_k \mathbb{H}^1(G, (\pi_* \mathcal{O}_X)_Q) \to \mathbb{H}^1(G, (\pi_* \Omega^\bullet_{X/k})_Q). \]

Thus we are left with showing that \( c_1 = c_2 \). This will be done in Subsection 3.4.

3.3. **Proof – the tame case.** Consider now the case of tame ramification, i.e. \( R' = 0 \). Then by Proposition 2.1 we see that \( \mathcal{H}^i(G, \pi_* \Omega^j_{X/k}) = 0 \) for \( i \geq 1, j \geq 0 \). Thus it is evident by (3.2) that

\[ \mathbb{R}^i \Gamma^G(\pi_* \Omega^\bullet_{X/k}) \cong \mathbb{H}^i(Y, (\pi_* \Omega^\bullet_{X/k})^G). \]

Therefore the exact sequence (3.5) implies that:

\[ \dim_k \mathbb{H}^1(Y, \pi_* \Omega^\bullet_{X/k})^G = \dim_k \mathbb{H}^1(Y, \pi_* \Omega^G_{X/k}) + \dim_k H^1(G, k) + c_1, \]

where \( c_1 \) is given by (3.8). One proceeds analogously as in the wildly ramified case to obtain:

\[ \delta(X, G) = (c_2 - c_1). \]
Again, it remains to prove that $c_1 = c_2$.

3.4. **Proof – the end.** Recall that in order to prove Proposition 3.1 we have to investigate the map

$$H^2(G, \mathbb{H}^0(Y, \mathcal{F}^*)) \to \mathbb{R}^2\Gamma^G(\mathcal{F}^*)$$

arising from the exact sequence (3.5).

**Lemma 3.4.** Let $\mathcal{F}^*$ be complex of $\mathcal{O}[G]$-sheaves on a ringed space $(Y, \mathcal{O})$, which is a noetherian topological space of dimension 1. Suppose that $\mathcal{F}^i = 0$ for $i \neq 0, 1$ and that the support of the sheaf $\mathcal{H}^i(G, \mathcal{F}^j)$ is a finite subset of $Y$ for $i \geq 1$. There exists a natural monomorphism

$$\mathbb{H}^0(Y, \mathcal{H}^2(G, \mathcal{F}^*)) \hookrightarrow \mathbb{R}^2\Gamma^G(\mathcal{F}^*).$$

It is an isomorphism, provided that $\mathcal{F}^*$ is a complex concentrated in degree 0.

**Proof.** Note that for $i \geq 2$, $j \geq 1$

$$iE^i_{2j} = \mathbb{H}^i(Y, \mathcal{H}^j(G, \mathcal{F}^*)) = 0.$$

Indeed, this follows by (3.1), since for every $l$, $H^i(Y, \mathcal{H}^j(G, \mathcal{F}^l)) = 0$ for $i, j \geq 1$ (cf. [Har77, Theorem III.2.7]), for $i \geq 2$ and for $j \geq 2$. Thus it is evident that there exists a natural monomorphism

$$\mathbb{H}^0(Y, \mathcal{H}^2(G, \mathcal{F}^*)) = iE^0_{22} = iE^0_{\infty} \hookrightarrow \mathbb{R}^2\Gamma^G(\mathcal{F}^*).$$

Suppose now that $\mathcal{F}^*$ is concentrated in degree 0. Then $iE^i_{2j} = 0$ for $i, j \geq 1$ and for $i \geq 2$. Therefore $iE^1_{\infty} = iE^1_2 = 0$ and $iE^{20}_{\infty} = iE^{20}_2 = 0$, which leads to the conclusion. □

**Corollary 3.5.** There exists a commutative diagram

$$\begin{array}{ccc}
H^2(G, \mathbb{H}^0(Y, \mathcal{F}^*)) & \longrightarrow & \mathbb{R}^2\Gamma^G(\mathcal{F}^*) \\
\downarrow & & \\
\mathbb{H}^0(Y, \mathcal{H}^2(G, \mathcal{F}^*)) & \longrightarrow & \mathbb{R}^2\Gamma^G(\mathcal{F}^*)
\end{array}$$

where the upper arrow is (3.12), and the lower arrow is as in Lemma 3.4.

**Proof.** The morphism $\mathcal{F}^* \to \mathcal{F}^0$ (where we treat $\mathcal{F}^0$ as a complex concentrated in degree 0) yields by functoriality the commutative diagram:

$$\begin{array}{ccc}
H^2(G, \mathbb{H}^0(Y, \mathcal{F}^*)) & \longrightarrow & \mathbb{R}^2\Gamma^G(\mathcal{F}^*) \\
\downarrow & & \\
H^2(G, \mathbb{H}^0(Y, \mathcal{F}^0)) & \longrightarrow & \mathbb{R}^2\Gamma^G(\mathcal{F}^0) \cong \mathbb{H}^0(Y, \mathcal{H}^2(G, \mathcal{F}^0))
\end{array}$$
By composing the maps from the diagram we obtain a map
\[(3.13) \quad H^2(G, \mathbb{H}^0(Y, \mathcal{F}^*)) \to H^2(G, H^0(Y, \mathcal{F}^0)) \to R^2\Gamma^G(\mathcal{F}^0) \cong H^0(Y, \mathcal{H}^2(G, \mathcal{F}^0)).\]
One easily checks that the image of the map (3.13) lies in the image of
\[H^0(Y, \mathcal{H}^2(G, \mathcal{F}^*)) \to H^0(Y, \mathcal{H}^2(G, \mathcal{F}^0)).\]
This clearly completes the proof. \[\square\]

We are now ready to finish the proof of Proposition 3.1. Recall that we are left with showing that \(c_1 = c_2\) (where \(c_1\) and \(c_2\) are given by (3.8) and (3.11) respectively). By using Corollary 3.5 for \(\mathcal{F}^* = \pi_*\Omega^*_{X/k}\), Lemma 3.4 and the equality
\[H^0(Y, \pi_*\Omega^*_{X/k}) = H^0(Y, \pi_*\mathcal{O}_X) = k\]
we obtain:
\[c_1 = \dim_k \ker \left( H^2(G, \mathbb{H}^0(Y, \pi_*\Omega^*_{X/k})) \to \mathbb{R}^2\Gamma^G(\pi_*\Omega^*_{X/k}) \right) \]
\[= \dim_k \ker \left( H^2(G, \mathbb{H}^0(Y, \pi_*\Omega^*_{X/k})) \to \mathbb{H}^0(Y, \mathcal{H}^2(G, \pi_*\Omega^*_{X/k})) \right) \]
\[= \dim_k \ker \left( H^2(G, \mathbb{H}^0(Y, \pi_*\mathcal{O}_X)) \to H^0(Y, \mathcal{H}^2(G, \pi_*\mathcal{O}_X)) \right) \]
\[= \dim_k \ker \left( H^2(G, \mathbb{H}^0(Y, \pi_*\mathcal{O}_X)) \to \mathbb{R}^2\Gamma^G(\pi_*\mathcal{O}_X) \right) \]
\[= c_2.\]

4. Computation of local terms

4.1. Proofs of main results. The main goal of this section is to compute the local terms occurring in Proposition 3.1. This is achieved in the following proposition.

Proposition 4.1. Keep the notation introduced in Section 1 and suppose that \(G \cong \mathbb{Z}/p\). Then for any \(Q \in Y\) the dimension of
\[\text{im} \left( H^1(G, (\pi_*\mathcal{O}_X)_Q) \to H^1(G, (\pi_*\Omega^*_{X/k})_Q) \right)\]
equals
\[
\left\lfloor \frac{(n_Q + 1) \cdot (p - 1)}{p} \right\rfloor - 1 - \left\lfloor \frac{n_Q - 1}{p} \right\rfloor.
\]
Proposition 4.1 will be proven in the Subsection 4.2. We now show how the Proposition 4.1 implies the Theorems announced in the Introduction.

Proof of Theorem 1.2. Theorem 1.2 follows by combining Propositions 3.1 and 4.1. \[\square\]
Proof of Main Theorem. We consider first the case \( G = \mathbb{Z}/p \). An easy computation shows that for any \( n \geq 1, p \geq 3 \) one has:

\[
\left[ \frac{(p-1) \cdot (n+1)}{p} \right] \geq 1 + \left[ \frac{n-1}{p} \right]
\]

with an equality only for \( n = 1 \) (here is where we use the assumption \( p > 2 \)). Thus by Theorem 1.2, \( \delta(X, G) = 0 \) holds if and only if \( \pi \) is weakly ramified.

Suppose now that \( G \) is arbitrary and \( \Gamma_{P, 2} \neq 0 \) for some \( P \in X \). Note that \( \Gamma_{P, 2} \) is a \( p \)-group (cf. [Ser79, Corollary 4.2.3., p. 67]) and thus contains a subgroup \( H \) of order \( p \). Observe that \( \pi : X \to X/H \) is an Artin-Schreier covering and it is non-weakly ramified, since \( H_{P, 2} = H \neq 0 \). Therefore by the first paragraph of the proof, the sequence (1.2) does not split \( H \)-equivariantly and therefore it cannot split as a sequence of \( k[G] \)-modules. \( \square \)

4.2. Galois cohomology of sheaves on Artin-Schreier coverings. We start by recalling the most important facts concerning Artin–Schreier coverings. For a reference see e.g. [PZ12, sec. 2.2]. Let \( X \) be a smooth algebraic curve with an action of \( G = \mathbb{Z}/p \) over an algebraically closed field \( k \) of characteristic \( p \). By Artin–Schreier theory, the function field of \( X \) is given by the equation

\[
z^p - z = f
\]

for some \( f \in k(Y) \), where \( Y := X/G \). The action of \( G = \langle \sigma \rangle \cong \mathbb{Z}/p \) is then given by \( \sigma(z) := z + 1 \). Let \( \mathcal{P} \subset Y \) denote the set of points at which \( \pi \) is ramified. Note that \( \mathcal{P} \) is contained in the set of poles of \( f \) and moreover for any \( Q \in Y \):

\[
\#\pi^{-1}(Q) = \begin{cases} p, & \text{for } Q \notin \mathcal{P}, \\ 1, & \text{otherwise.} \end{cases}
\]

Lemma 4.2. Keep the above setting. Fix a point \( Q \in \mathcal{P} \) and let \( \pi^{-1}(Q) = \{P\} \). Suppose that \( p \nmid n := \text{ord}_Q(f) \). Then for some \( t \in \widehat{O}_{X, P} \) and \( x \in \widehat{O}_{Y, Q} \):

- \( \widehat{O}_{X, P} = k[[t]], \widehat{O}_{Y, Q} = k[[x]] \),
- \( t^{-np} - t^{-n} = x^{-n} \),
- the action of \( G \cong \mathbb{Z}/p \) on \( t \) is given by an automorphism:

\[
\sigma(t) = \frac{t}{(1 + t^n)^{1/n}} = t - \frac{1}{n} t^{n+1} + \text{(terms of order } \geq n + 2).\]

In particular, \( n \) is equal to \( n_Q \) as defined by (1.4).

Proof. Let \( x, t \) be arbitrary uniformizers at \( Q \) and \( P \) respectively. Then \( \widehat{O}_{Y, Q} = k[[x]] \) and \( \widehat{O}_{X, P} = k[[t]] \). Before the proof observe that an equation \( u^m = h(x) \) has a solution \( u \in k[[x]] \), whenever \( p \nmid m \) and \( m|\text{ord}(h) \) (this
follows easily from Hensel’s lemma). We will denote any solution by \( h(x)^{1/m} \).

Note that:

\[
1 - 1/z^{1-p} = z^{1-p} - z^{1-p}. 
\]

By comparing the valuations we see that \( \text{ord}_P(z) = -n \). Thus we may replace \( t \) by \( z^{-1/n} \) to ensure that \( z = t^{-n} \). Then:

\[
\sigma(t)^n = \sigma(t^n) = \sigma\left( \frac{1}{z} \right) = \frac{1}{z+1} = \frac{1}{t^{-n} + 1} = \frac{t^n}{1 + t^n} 
\]

and thus we can assume without loss of generality (by replacing \( \sigma \) by its power if necessary) that \( \sigma(t) = (1 + t^n)^{1/n}. \) Finally, we replace \( x \) by \( f(x)^{-1/n} \) to ensure that \( t^{-np} - t^{-n} = x^{-n} \).

\[\square\]

**Example 4.3.** Let \( X \) be the curve considered in Example 1.1. Then \( X \) is a \( \mathbb{Z}/p \)-covering of a curve \( Y \) with the affine equation:

\[ y^m = f(x). \]

The function field of \( X \) is given by the equation \( z^p - z = x \). As proven in [Tow96] the function \( x \in k(Y) \) has \( \delta := \text{GCD}(m, \deg f) \) poles, each of them of order \( m/\delta \). This establishes the formula (1.4).

**Remark 4.4.** Suppose that \( \pi : X \to Y \) is an Artin-Schreier covering. For every point \( Q \in P \) we can find functions \( f_Q \in k(Y), z_Q \in k(X) \) such that the function field of \( X \) is given by the equation \( z^p_Q - z_Q = f_Q \) and either \( f_Q \in \mathcal{O}_{Y,Q} \), or \( p \nmid \text{ord}_Q(f_Q) \). Indeed, in order to obtain \( f_Q \) one can repeatedly subtract from \( f \) a function of the form \( h^p - h \), where \( h \) is a power of a uniformizer at \( Q \).

**Example 4.5.** It might not be possible to find a function \( f \) such the function field of \( X \) is given by (4.1) and for any pole \( Q \) of \( f \) one has \( p \nmid \text{ord}_Q(f) \). Take for example an ordinary elliptic curve \( X/\mathbb{F}_p \). Let \( \tau \in \text{Aut}(X) \) be a translation by a \( p \)-torsion point. Consider the action of \( G = \langle \tau \rangle \cong \mathbb{Z}/p \) on \( X \). This group action is free and hence \( np = 0 \) for all \( P \in X \). Thus, if \( k(X) \) would have an equation of the form \( z^p - z = f \), where \( p \nmid \text{ord}_Q(f) \) for all \( Q \in P \), then \( f \) would have no poles. This easily leads to a contradiction.

Keep the notation of Lemma 4.2. Fix an integer \( a \in \mathbb{Z} \) and denote:

- \( B := \hat{\mathcal{O}}_{Y,Q} = k[[t]], L := k((t)), I := t^aB, \)
- \( A := \hat{\mathcal{O}}_{X,P} = k[[x]], K := k((x)) \).

In the below Lemma we will compute \( H^1(G, I) \). The dimension of \( H^1(G, I) \) is computed also in [BM00, Théorème 4.1.1] (see also [Kon07, formula (18)]). However, we need an explicit description of a basis of \( H^1(G, I) \).
Lemma 4.6.  (1) $H^1(G, I)$ may be identified with

\[ M := \text{coker}(L^G \to (L/I)^G). \]

(2) A basis of $H^1(G, I)$ is given by the images of the elements $(t^i)_{i \in J}$ in $M$, where

\[ J := \{a - n \leq i \leq a - 1 : p \nmid i \}. \]

(3) $\dim_k H^1(G, I) = n - \left\lfloor \frac{a - 1}{p} \right\rfloor + \left\lfloor \frac{a - 1 - n}{p} \right\rfloor$.

(4) The images of the elements:

\[ t^i \quad \text{for} \quad a - n \leq i \leq a - 1, \quad p \mid i \]

are zero in $M$.

Proof. For any $h \in L$, we will denote its images in $L/I$ and $M$ by $[h]_{L/I}$ and $[h]_M$ respectively.

(1) The proof follows by taking the long exact sequence of cohomology for the short exact sequence of $k[G]$-modules:

\[ 0 \to I \to L \to L/I \to 0 \]

and using the Normal Base Theorem (cf. [Jac85, sec. 4.14]).

(2) Note that for any $a - n \leq i \leq a - 1$, we have $[t^i]_{L/I} \in (L/I)^G$, since

\[ \sigma([t^i]_{L/I}) = [\sigma(t^i)]_{L/I} = [(t - \frac{1}{n}t^{n+1} + O(t^{2n}))^i]_{L/I} = [t^i - \frac{i}{n}t^{i+n} + O(t^{2n})]_{L/I} = [t^i]_{L/I}. \]

We’ll show now that the set $\{([t^i]_M)_{i \in J}\}$ spans $M$. Note that $L^G = K$. Therefore it suffices to show that for any $[h]_{L/I} \in (L/I)^G$, one has

\[ h \in K + \bigoplus_{i \in J} k \cdot t^i. \]

Let $h = \sum_{i=N}^{a-1} a_it^i \in L$, where $a_N \neq 0$. Observe that if $p \mid j$ and $a_j \neq 0$, then we may replace $h$ by $h - c \cdot \chi_j^{1/p}$ for a suitable constant $c \in k$, since valuation of $x$ in $L$ equals $p$. Thus without loss of generality we may assume that $a_j = 0$ for $p \mid j$ and that $p \nmid N$. The equality $\sigma([h]_{L/I}) = [h]_{L/I}$ is equivalent to

\[ \sum_{i=N}^{a-1} a_i \sigma(t)^i = \sum_{i=N}^{a-1} a_it^i + \sum_{i=a}^{\infty} b_it^i \]

for some $b_a, b_{a+1}, \ldots \in k$. By using equality (4.2) this implies:

\[ \sum_{i=N}^{a-1} a_it^i \cdot \left( 1 - \frac{i}{n}t^n + O(t^{2n}) \right) = \sum_{i=N}^{a-1} a_it^i + \sum_{i=a}^{\infty} b_it^i. \]
By comparing coefficients of $t^{N+n}$, we see that either $N + n \geq a$, or

$$a_N \cdot \left( - \frac{N}{n} \right) + a_{N+n} = a_{N+n}.$$ 

The second possibility easily leads to a contradiction. This proves (4.3).

We check now linear independence of the considered elements. Suppose that for some $a_i \in k$ not all equal to zero:

$$\sum_{i \in J} a_i [t^i]_M = 0$$

or equivalently,

$$\sum_{i \in J} a_i t^i = \sum_{j \geq N} b_j x^j + \sum_{j \geq a} c_j t^j$$

for some $b_j, c_j \in k$, $b_N \neq 0$. Consider the coefficient of $t^pN$ in (4.4).

Observe that $x = t^p + O(t^{p+1})$, since $\text{ord}_P(x) = p$. We see that either $pN \geq a$ (which is impossible, since $\sum_{i \in J} a_i t^i \not\in I$) or $0 = b_N + 0$, which also leads to a contradiction. This ends the proof.

(3) Follows immediately by (2).

(4) Note that

$$x = \frac{1}{(t^{-np} - t^{-n})^{1/n}} = \frac{t^p}{(1 - t^{n(p-1)})^{1/n}}$$

and thus for any $a - n \leq i \leq a - 1$, $p|i$:

$$x^{i/p} = t^i \cdot (1 + O(t^{n(p-1)})) = t^i + O(t^a),$$

and $[t^i]_{L/I} = [x^{i/p}]_{L/I}$, which shows that $[t^i]_M = 0$. \(\square\)

**Proof of Proposition 4.1** Fix a point $Q \in \mathcal{P}$ and keep the above notation. Note that $(\pi_* \Omega_X)_Q \cong B$, $\pi_* \Omega_{X/k} = B dt$. Moreover, note that $\frac{dt}{t^{n+1}}$ is a $G$-invariant form, since from the equation $t^{-np} - t^{-n} = x^{-n}$ one obtains:

$$\frac{dt}{t^{n+1}} = - \frac{dx}{x^{n+1}}.$$ 

Thus we have the following isomorphism of $B[G]$-modules:

$$B dt \longrightarrow t^{n+1}B$$

$$h(t) dt = t^{n+1} h(t) \cdot \frac{dt}{t^{n+1}} \longrightarrow t^{n+1} h(t).$$

(cf. [Kon07] proof of Lemma 1.11.) for the "dual" version of this isomorphism). Lemma 4.6 implies that $H^1(G, B)$ and $H^1(G, B dt)$ may be identified with

$$M_1 := \text{coker}(L^G \rightarrow (L/B)^G) \quad \text{and} \quad M_2 := \text{coker}((L dt)^G \rightarrow (L dt/B dt)^G)$$
respectively. One easily checks that the morphism \( d : H^1(G, B) \to H^1(G, B \ dt) \) corresponds to
\[
d : M_1 \to M_2, \quad d([h(t)]_{M_1}) = [dh(t)]_{M_2} = [h'(t) \ dt]_{M_2}.
\]
By using Lemma 4.6 (2), (4) for \( a = 0 \) and \( a = n + 1 \) we see that the basis of \( \text{im}(d : M_1 \to M_2) \) is
\[
[dt^i]_{M_2} = [it^{i-1} dt]_{M_2} \quad \text{for } i = -n, -n+1, \ldots, -1, \quad p \nmid i, \quad i+n \not\equiv 0 \pmod{p}.
\]
An elementary calculation allows one to compute the dimension of this space.

\[\square\]

5. Converse results

This section will be devoted to proving various converse statements to Main Theorem.

5.1. A counterexample. We start this section by giving an example of an elliptic curve over a field of characteristic 2 with a weakly ramified group action, for which the sequence (1.2) does not split equivariantly. It remains unclear whether similar counterexamples will arise over fields of different characteristic.

Consider the elliptic curve \( X \) over the field \( k := \mathbb{F}_2 \) with the affine part \( U_0 \) given by the equation:
\[
y^2 + y = x^3.
\]
Note that \( X \setminus U_0 = \{O\} \), where \( O \) is the point at infinity. The group \( G \) of automorphisms of \( X \) that fix \( O \) is of order 24 and is isomorphic to \( \text{SL}_2(\mathbb{F}_5) \).

In particular its 2-Sylow subgroup is isomorphic to the quaternion group \( Q_8 \).

This group action may be given explicitly, cf. [Sil09, Appendix A] or [KST17, Section 3]. Let:
\[
A := \{(u, r, t) : u \in \mathbb{F}_4^\times, \quad t \in \mathbb{F}_4, \quad t^2 + t + r^3 = 0\}.
\]
Define for any \((u, r, t) \in A\) an automorphism \( g_{u,r,t} \in \text{Aut}(X)\) by:
\[
g_{u,r,t} \cdot (x, y) := (u^2 x + r, y + u^2 r^2 x + t).
\]
We’ll compute \( H^1_{dR}(X/k) \) using Æech cohomology. Recall that if a curve \( X \) may be covered by affine subsets \( U_0, U_\infty \), then:
\[
H^1_{dR}(X/k) \cong \frac{\{(\omega_0, \omega_\infty, f_{0\infty}) : df_{0\infty} = \omega_0 - \omega_\infty \}}{\{(df_0, df_\infty, f_0 - f_\infty) : f_i \in \mathcal{O}_X(U_i)\}},
\]
where we take \( \omega_i \in \Omega_X(U_i) \) for \( i = 0, \infty \) and \( f_{0\infty} \in \mathcal{O}_X(U_0 \cap U_\infty) \). The natural homomorphism \( \gamma : H^0(X, \Omega_X/k) \to H^1_{dR}(X/k) \) is then given by:
\[
\omega \mapsto [(\omega, \omega, 0)]
\]
(see [KT18 Section 2] for a reference). In our case, we may take \( U_0 \) as above and \( U_\infty = X \cap \{ x \neq 0 \} \). Then, by [KT18 Theorem 2.2.], one sees that \( H^0(X, \Omega_{X/k}) = k \cdot dx \) and that \( H^1_{dR}(X/k) \) is a \( k \)-vector space of dimension 2, generated by \( v_1 := \gamma(dx) \) and \( v_2 := [(x \, dx, y \, dx - x^2, y)] \).

Lemma 5.1. In the above situation:

(a) the exact sequence (1.2) does not split equivariantly with respect to \( G \),

(b) the action of \( G \) on \( X \) is weakly ramified.

Proof. (1) Suppose to the contrary, that

\[
P : H^1_{dR}(X/k) \to H^0(X, \Omega_{X/k})
\]

is \( k[G] \)-linear and that \( P \circ \gamma = \text{id} \). Let \( P(v_2) := \alpha v_1 \). Note that for \( g = g_{u,r,t} \):

\[
\begin{align*}
gv_1 &= u^2 v_1 \\
gv_2 &= u^2 tv_1 + uv_2.
\end{align*}
\]

Then:

\[
\begin{align*}
g \cdot P(v_2) &= g \cdot \alpha v_1 = u^2 \alpha v_1 \\
P(g \cdot v_2) &= P(u^2 tv_1 + uv_2) = (u^2 t + u\alpha)v_1.
\end{align*}
\]

By equating right hand sides of both equations we obtain

\[
(1 - u) \cdot \alpha = ut.
\]

The last equality is however impossible to hold for all \((u, r, t) \in A\) and a fixed \( \alpha \in k \). Indeed, one can take e.g. \((u, r, t) = (1, 0, 1), (1, 1, \zeta)\) for any \( \zeta \in \mathbb{F}_4 \setminus \mathbb{F}_2 \) to obtain a desired contradiction.

(2) One easily sees that if \( gP = P \) and \( g \neq \text{id} \) then \( P = \mathcal{O} \). Thus we are left with showing that \( G_{\mathcal{O}, 2} = 0 \). Observe that \( \text{ord}_\mathcal{O}(x) = -2 \) and \( \text{ord}_\mathcal{O}(y) = -3 \). Hence the function \( t := \frac{x}{y} \) is the uniformizer at \( \mathcal{O} \).

For \( g = g_{u,r,t} \) one has:

\[
g(t) - t = \frac{(u^2 + 1)c \cdot xy + u^2 r^2 \cdot x^2 + r \cdot y + t \cdot x}{y \cdot (y + u^2 r^2 x + t)}
\]

and

\[
\text{ord}_\mathcal{O}(g(t) - t) = \begin{cases} 2, & \text{if } u = 1, \\ 1, & \text{if } u \neq 1. \end{cases}
\]

Therefore \( G_{\mathcal{O}, 2} = 0 \) and

\[
G_{\mathcal{O}, 1} = \{ g_{1, r, s} : (1, r, s) \in A \} \cong Q_8. \quad \square
\]
5.2. The $G$-fixed subspaces. The methods used throughout the article seem to be insufficient to obtain a positive result regarding splitting of the exact sequence (1.2). However, we may say something about the $G$-fixed subspaces in the sequence (1.2).

Proof of Theorem 1.3. By Proposition 3.1 it is sufficient to show that the map
\[ H^1(G, (\pi_* O_X)_{\pi(P)}) \to H^1(G, (\pi_* \Omega_{X/k})_{\pi(P)}) \]
is zero for every $P \in X$. Just as in the proof of Proposition A.1 we observe that
\[ H^1(G, (\pi_* O_X)_{\pi(P)}) \cong H^1(G, k) \oplus H^1(G_P, \mathfrak{m}_{X,P}). \]
However, the map $d : k \to \Omega_{X/k}$ is zero and thus the induced map
\[ d : H^1(G_P, k) \to H^1(G, (\pi_* \Omega_{X/k})_{\pi(P)}) \]
is also zero. Moreover, since $\pi$ is weakly ramified, by a result of Köck (cf. [K04, Theorem 1.1]), $H^1(G_P, \mathfrak{m}_{X,P}) = 0$. This ends the proof. \[\square\]

Note that if an action of a finite group $G$ on $X$ is weakly ramified then the action of any subgroup of $G$ on $X$ is also weakly ramified. Therefore the condition imposed by Theorem 1.3 on the Hodge–de Rham exact sequence of $X$ seems to be strong from the group theoretical point of view. This raises the following question:

**Question 5.2.** Suppose that $k$ is a field of characteristic $p > 0$ and $G$ is a finite group. Let also
\begin{equation}
0 \to A \to B \to C \to 0
\end{equation}
be an exact sequence of $k[G]$-modules of finite dimension over $k$. Assume that for every subgroup $H \leq G$ the sequence
\[ 0 \to A^H \to B^H \to C^H \to 0 \]
is exact also on the right. Does it follow that the exact sequence (5.1) splits $G$-equivariantly?

The results of the Subsection 5.1 show that the answer to the Question 5.2 is negative for $\text{char } k = 2$. The following lemma reduces the Question 5.2 to the case of $p$-groups.

**Lemma 5.3.** Let $k$ be a field of characteristic $p > 0$ and let $G$ be a finite group with a $p$-Sylow subgroup $P$. Suppose that
\begin{equation}
0 \to A \to B \to C \to 0
\end{equation}
is an exact sequence of $k[G]$-modules. Then (5.2) splits as an exact sequence of $k[G]$-modules if and only if it splits as an exact sequence of $k[P]$-modules.
Proof. The proof is adapted from the proof of Maschke theorem. Suppose that \( s : C \to B \) is a \( k[P]\)-equivariant section of the map \( B \to C \). Let \( P \setminus G = \{ Pg_1, \ldots, Pg_m \} \), where \( p \nmid m = [G : P] \). Then, as one easily checks

\[
\tilde{s} : C \to B, \quad \tilde{s}(x) := \frac{1}{m} \sum_{i=1}^{m} g_i^{-1} s(g_i x)
\]

is a \( k[G]\)-equivariant section of \( B \to C \). \( \square \)

Unfortunately we are able to answer Question 5.2 only for the class of groups that have 'tame' modular representation theory, i.e. groups with a cyclic \( p \)-Sylow subgroup.

**Lemma 5.4.** Suppose that \( k \) is a field of characteristic \( p > 0 \) and \( G \) is a finite group with a cyclic \( p \)-Sylow subgroup. Let

\[
0 \to A \to B \to C \to 0
\]

be an exact sequence of \( k[G]\)-modules. If the sequence

\[
0 \to A^G \to B^G \to C^G \to 0
\]

is exact on the right then the exact sequence (5.3) splits \( G \)-equivariantly.

**Proof.** Without loss of generality we can assume that \( G = \mathbb{Z}/p^n \) is a cyclic \( p \)-group (by Lemma 5.3). Note that \( k[\mathbb{Z}/p^n] \cong k[x]/(x-1)^p \). The classification theorem of finitely generated modules over the principal ideal domain \( k[x] \) (cf. [DF04, Theorem 12.1.5]) implies that every finitely generated indecomposable \( k[\mathbb{Z}/p^n] \)-module is of the form:

\[
J_i = k[x]/(x-1)^i \quad \text{for some } i = 1, \ldots, p^n.
\]

Denote also \( J_0 := 0 \). Using Smith’s Normal Form theorem (cf. [DF04, Theorem 12.1.4]) we obtain a commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{1} & B \\
\downarrow{1} & & \downarrow{1} \\
\bigoplus_{i=1}^{l} J_{a_i} & \to & \bigoplus_{i=1}^{m} J_{b_i}
\end{array}
\]

where \( l \leq m, a_i \leq b_i \) and \( J_{a_i} \hookrightarrow J_{b_i} \) is the natural inclusion. Hence we are reduced to proving the claim for the exact sequence:

\[
0 \to J_a \to J_b \to J_c \to 0,
\]

where \( a + b = c, \ 0 \leq a, b, c \leq p^n \). However, the equality

\[
\dim_k J_s^G = \begin{cases} 
1, \quad & \text{if } s \neq 0 \\
0, \quad & \text{otherwise.}
\end{cases}
\]

makes it obvious that \( a = 0 \) or \( c = 0 \). This finishes the proof. \( \square \)
5.3. **Relation to the problem of lifting coverings.** Let \( X \) be a smooth algebraic variety over \( k \) equipped with an action of a finite group \( G \). We say that the pair \((X, G)\) lifts to \( W_2(k) \), if there exists a smooth scheme \( X \) over \( W_2(k) \) and a homomorphism \( G \to \text{Aut}_{W_2(k)}(X) \) such that
\[
(X, G \to \text{Aut}_{W_2(k)}(X)) \times_{W_2(k)} k = (X, G \to \text{Aut}_k(X))
\]
The following theorem is a \( G \)-equivariant version of the main result of [DI87] and follows immediately from the functoriality of the result of Deligne and Illusie.

**Theorem 5.5 ([DI87]).** Suppose that the pair \((X, G)\) lifts to \( W_2(k) \). Then the exact sequence \((1.2)\) of \( k[G] \)-modules splits.

The following is an immediate consequence of Main Theorem and Theorem 5.5.

**Corollary 5.6.** Suppose that \( p > 2 \), \( X \) is a smooth projective curve over \( k \) and the pair \((X, G)\) lifts to \( W_2(k) \). Then the action of \( G \) on \( X \) is weakly ramified.

Note that it was known previously that non-weakly ramified actions on curves do not lift to \( W(k) \) (cf. [Nak86, Corollary, Sec. 4]).

The problem of lifting Galois coverings of curves from characteristic \( p \) to characteristic \( p^2 \) or 0 has been studied extensively in the literature, cf. e.g. [Pop14] for the case \( G = \mathbb{Z}/p \). In particular, it is possible to classify all weakly ramified group actions into liftable and non-liftable ones (cf. [BM00, Section 4.2] and [CK03, Section 4.1]). However, we weren’t able to extract any information that would help us to understand the behaviour of the sequence \((1.2)\) for curves with weakly ramified group action.

**Corollary 5.7.** Suppose that a finite group \( G \) acts on an ordinary curve \( X \). Then the exact sequence \((1.2)\) splits \( G \)-equivariantly.

**Proof.** Let \( A \) be the Jacobian variety of \( X \) and let \( A/W_2(k) \) be its Serre-Tate canonical lift (cf. [Mes72], p. 172-173, Theorem 3.3 for the definition). Then the action of \( G \) on \( A \) lifts to an action on \( A \). Hence by Theorem 5.5 the exact sequence \((1.2)\) for \( A \) splits \( G \)-equivariantly. It suffices to note that the Abel-Jacobi map induces an isomorphism between the Hodge–de Rham sequences of a curve and its Jacobian variety (cf. [Mil08], Proposition III.2.1, Lemma III.9.5.).

**Remark 5.8.** As observed by Piotr Achinger, in the case of Corollary 5.7 one may explicitly describe the \( G \)-equivariant section of the sequence \((1.2)\) as
\[
\iota \circ \mathcal{C} \circ (F^*_X)^{-1},
\]
where:
- \( F_X : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \) is the \( p \)-linear homomorphism induced by the absolute Frobenius on \( X \) (note that it is a bijection, since \( X \) is ordinary),
- \( C \) is the isomorphism
  \[
  H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{H}^0(\mathcal{O}_{X/k}^*)),
  \]
  \( (\mathcal{H}^1(\mathcal{O}_{X/k}^*)) \) denotes the quotient of the sheaf of exact differentials by the sheaf of closed differentials) induced by the Cartier isomorphism (cf. [DI87, Théoréme 1.2]),
- \( \iota : H^1(X, \mathcal{H}^0(\mathcal{O}_{X/k}^*)) \to H^1_{dR}(X/k) \) is the map arising from the degeneration of the conjugate spectral sequence
  \[
  E^{ij} = H^i(X, \mathcal{H}^j(\mathcal{O}_{X/k}^*)) \Rightarrow H^{i+j}_{dR}(X/k)
  \]
  (cf. [Wei94 5.7.9]).

Note in particular that Corollary 5.7 implies that ordinary curves admit only weakly ramified group actions. This follows also from the Deuring-Shafarevich formula (cf. [Sub75]).

Appendix A. Computing the Dimension of \( H^1(X, \mathcal{O}_X)^G \)

For completeness we include also the following proposition, which allows in many situations to compute dimensions of \( H^0(X, \mathcal{O}_{X/k})^G, H^1(X, \mathcal{O}_X)^G \) and \( H^1_{dR}(X/k)^G \) in terms of invariants of \( Y \) and group cohomology of sheaves. Note that by Corollary 2.3 and Proposition 3.1 we are left with computing the dimension of \( H^1(X, \mathcal{O}_X)^G \).

Proposition A.1. In the notation of Section 1 suppose that there exists \( Q_0 \in Y \) such that
\[
p \nmid \# \pi^{-1}(Q_0).
\]
Then:
\[
\dim_k H^1(X, \mathcal{O}_X)^G = g_Y + \sum_{Q \in Y} H^1(G, (\pi_* \mathcal{O}_X)_Q) - \dim_k H^1(G, k).
\]

Proof. By substituting \( F^0 = \pi_* \mathcal{O}_X \) in the formula 3.10 and using Lemma 3.4 and Corollary 3.5 it suffices to prove that the natural map
\[
A.1 \quad H^2(G, k) \cong H^2(G, H^0(Y, \pi_* \mathcal{O}_X)) \to H^0(Y, \mathcal{H}^2(G, \pi_* \mathcal{O}_X))
\]
is injective. One easily sees that
\[
\mathcal{H}^2(G, \pi_* \mathcal{O}_X) \cong \bigoplus_{Q \in Y} i_{Q,*} \left( H^2(G, (\pi_* \mathcal{O}_X)_Q) \right)
\]
is a direct sum of skyscraper sheaves. Choose any $P_0 \in \pi^{-1}(Q_0)$. Observe that by Lemma 2.2 we have:

$$H^2(G_1, (\pi_*(\mathcal{O}_X))_Q) \cong H^2(G_{P_0,1}, \mathcal{O}_{X,P_0}).$$

But $\mathcal{O}_{X,P_0} \cong k \oplus m_{X,P_0}$ as a $k[G_{P_0,1}]$-module and therefore

$$H^2(G_{P_0,1}, \mathcal{O}_{X,P_0}) \cong H^2(G_{P_0,1}, k) \oplus H^2(G_{P_0,1}, m_{X,P_0}).$$

One easily sees that the map (A.1) factors as

$$H^2(G, k) \rightarrow H^2(G_{P_0,1}, k) \hookrightarrow H^2(G_{P_0,1}, k) \oplus H^2(G_{P_0,1}, m_{X,P_0}),$$

where the first map is the restriction map $\text{res}_{G_{P_0,1}}$. Now note that $p \nmid \#\pi^{-1}(Q_0) = [G : G_{P_0}]$ and thus $G_{P_0,1}$ is a $p$-Sylow subgroup of $G$ by [Ser79, Corollary 4.2.3., p. 67]. Thus by (2.2) $\text{res}_{G_{P_0,1}}$ is an isomorphism. This ends the proof. \hfill \Box

**Example A.2.** Keep the notation introduced in Section 7 and suppose that $G \cong \mathbb{Z}/p$. Then by Lemma 4.2, one has $d_Q = (n_Q + 1) \cdot p$ for all $Q \in Y$ and therefore:

(A.2) \quad $R' = \sum_{Q \in Y} \left( \frac{(n_Q + 1) \cdot (p - 1)}{p} \right)(Q)$. 

Moreover, by Lemma 4.6

(A.3) \quad $\dim_k H^1(G, (\pi_*(\mathcal{O}_X))_Q) = \left[ \frac{(n_Q + 1) \cdot (p - 1)}{p} \right]$. 

Suppose that the action of $G$ on $X$ is not free. Then by Corollary 2.4, Proposition A.1 and (A.2) we obtain:

$$\dim_k H^0(X, \Omega_{X/k})^G = \dim_k H^1(X, \mathcal{O}_X)^G = g_Y - 1 + \sum_{Q \in Y} \left[ \frac{(n_Q + 1) \cdot (p - 1)}{p} \right].$$

Moreover, by previous computations and by Proposition 3.1 we obtain:

$$\dim_k H^d_{dR}(X/k)^G = 2(g_Y - 1) + \sum_{Q \in Y} \left( \left[ \frac{(n_Q + 1) \cdot (p - 1)}{p} \right] + 1 + \left[ \frac{n_Q - 1}{p} \right] \right).$$

If the action of $G$ is free, then a similar reasoning leads to the formulas:

$$\dim_k H^0(X, \Omega_{X/k})^G = \dim_k H^1(X, \mathcal{O}_X)^G = g_Y$$

$$\dim_k H^d_{dR}(X/k)^G = 2g_Y.$$
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