Method for Solving the Many-Body Bound State Nuclear Problem

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Abstract

We present a method based on hyperspherical harmonics to solve the nuclear many-body problem. It is an extension of accurate methods used for studying few-body systems to many bodies and is based on the assumption that nucleons in nuclei interact mainly via pairwise forces. This leads to a two-variable integro-differential equation which is easy to solve. Unlike methods that utilize effective interactions, the present one employs directly nucleon-nucleon potentials and therefore nuclear correlations are included in an unambiguous way. Three body forces can also be included in the formalism. Details on how to obtain the various ingredients entering into the equation for the A-body system are given. Employing our formalism we calculated the binding energies for closed and open shell nuclei with central forces where the bound states are defined by a single hyperspherical harmonic. The results found are in agreement with those obtained by other methods.

Key words: Nuclear structure, Hyperspherical Harmonics, Integro–differential equations.

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1 Introduction

The discovery by Jensen and Goeppert-Mayer of the shell model structure of nuclei [1], was a great step forward in understanding the organization of nucleons in nuclei. Since the magic numbers occurring in the shell model can be explained in terms of harmonic oscillator quantum numbers, with the occurrence of a large spin-orbit force, it was taken for granted that the nucleons move

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inside an average one-body potential. Indeed, Goeppert-Mayer and Jensen [1] assumed that “⋅⋅⋅ each nucleon moves in an average field of force V(r), of spherical symmetry, and independent of the exact instantaneous position of all the other nucleons.” In order to explain the existence of magic numbers, this one body field was completed by the introduction of a strong spin-orbit force. Some interesting remarks, however, have been put forward by de Shalit and Talmi about the validity of this interpretation [2], namely, that if such a self-consistent central field is found, then one should consider the residual interaction, which is not accounted for, as a perturbation.

The shell model was introduced at the beginning of the fifties at the time where the nucleon-nucleon potential was not yet well known because the collision energy was not large enough to detect the strong repulsive core needed to reproduce the S-phase shifts beyond 300 MeV. This strong repulsive core generates important two-body correlations in nuclei in such a way that Hartree-Fock methods applied with actual realistic nucleon-nucleon potentials produce a very small binding energy. Thus, the residual interaction responsible for the largest part of the binding energy cannot be considered as a perturbation. Of course if one considers that the free nucleon-nucleon potential is not the true interaction between nucleons in nuclei, the previous remarks do not apply but then one has to state it explicitly and indicate what interaction should be used instead. For instance in the self-consistent mean-field models [3] a Hartree-Fock method is used with Skyrme interactions.

The shell model was first established for atoms but as pointed out by de Shalit and Talmi, there is an important difference between nuclei and atoms, namely, in atoms electrons have as a natural reference point the nucleus and “⋅⋅⋅ no such reference point exists in the nucleus, and at a first look it seems hard to understand why a central potential may form a good starting point for a nuclear model” [2].

Fifty years after the introduction of the shell model, the situation was still the same. After a lengthy discussion about the validity of the shell model, Talmi [4] pointed out that “⋅⋅⋅ Today, as in 1949, the best proof for the validity of the shell model is the good agreement of predictions with experiments”. The only well established property is that the quantum numbers associated with the magic numbers are those of the harmonic oscillator in a ground state corrected by a strong spin-orbit force.

In order to find an alternative explanation, to that given by the Independent Particle Model (IPM), to interpret the magic numbers, let us start from the harmonic oscillator (HO) model. For an A identical-particle system, the HO potential, neglecting the parameter defining the strength, is proportional to the sum of the squares of the linear coordinates of the particles, \( V_{HO}(r) \sim \sum_{i=1}^{A} x_i^2 \). This sum, when expressed in the center of mass frame in po-
lar coordinates in the $D = 3(A - 1)$ dimensional space spanned by the particle coordinates $\vec{x}_i$ ($i = 1, \cdots, A$), is nothing else but the radial coordinate, called hyperradius. The other space variables are the angular coordinates $\Omega$. Let $\Omega$ and $r$ be the polar coordinates fixing the position of the particles. For any potential $V(r)$ which depends only on the radial coordinate $r$, such as for the HO potential where $V(r) \sim r^2$, the rotational and vibrational motion described by a function of the radial coordinate are independent. The rotational motion in the physical three-dimensional space is described by spherical harmonics which are harmonic polynomials (HP) when the radial coordinate is equal to unity. In the $D$–dimensional space, the rotations are similarly described by hyperspherical harmonics which are HPs when the hyperradius is equal to one. These harmonics are independent of the shape of the potential $V(r)$ since the rotational and vibrational motions are independent. Nevertheless, the speed of the rotation enters into the radial equation through the repulsive centrifugal barrier $L(L + 1)/r^2$, $L = L + (D - 3)/2$, where $L$ is the degree of the HP defining the state in the $D$–dimensional space.

Whatever the potential $V(r)$ is, the ground state is obtained when the repulsive centrifugal barrier, i.e., the degree of the HP defining the state, is the smallest. For bosons where all particles can be in the same $S$-state, the degree is $L_m = 0$. For identical fermions where two particles cannot be in the same state, the wave function, which must be antisymmetric in the exchange of two particles, is the product of an antisymmetric HP and a function of the hyperradius $r$. This HP is a Slater determinant constructed from the individual polynomials $s^j_i t^j_i x_i^{2n_j + \ell_j} Y_{m_j}^{\pm \ell_j} (\omega_i)$ where $s$ and $t$ are the spin and isospin states and $i$ refers to the rows and $j$ to the columns for $i$ and $j$ running from 1, $\cdots$, $A$. The degree of this HP is the smallest when all the $m_j$ quantum numbers are used for the smallest possible values of $2n_j + \ell_j$. It corresponds, as we shall show in the next section, to a HO Slater determinant in ground state and, therefore, to the filling of the states giving the minimum energy for the HO potential leading to the shell model. From this analysis it turns out that the “average field of force $V(r)$ of spherical symmetry” Goeppert-Mayer and Jensen speak about [1], which “represents the action of the other nucleons” is that part of the nuclear potential, called hypercentral, which is invariant by rotation in the $D$-space and thus a function of the hyperradius only.

For the spin-orbit force one uses the property that a sum over all pairs of pairwise spin-orbit operators is equivalent to a sum over all particles of one–body spin-orbit operators [2]. When the spin-orbit interaction is the product of a hypercentral potential and a sum of pairwise spin-orbit operators, then this is equivalent to the product of a hypercentral potential and a sum of one-body spin-orbit operators [1], but this time the potential is not the one of an independent particle model but a function of the collective radial coordinate, i.e., the hyperradius. The solution obtained from the wave equation where the potential is purely hypercentral does not contain any correlations since $r$ is a
collective variable that describes only monopolar excitations when the radial
wave function has nodes. The wave function has therefore to be improved
when we have to deal with two-body potentials generating correlations.

In this paper a method for introducing the correlations generated by the nu-
clear potential is proposed. It leads to a system of coupled integro-differential
equations taking the various operators of the potential into account. It is an
extension to many bodies of accurate methods used for studying the few-body
systems.

In Sect. II we briefly describe the antisymmetric harmonic polynomials and in
Sect. III the handling of the nuclear problem that neglects two-body correla-
tions namely via the hypercentral approximation and the spin-orbit force. In
Sect. IV we discuss the Integro-differential equation formalism while in Sect.
V we describe in details the construction of the so-called weight and pseudo-
weight functions associated with exchange operators followed by a description
of the projection function in Sect. VI. The use of nuclear potentials is de-
scribed in Sect. VII while a brief discussion concerning spin-isospin exchange
generated elements is given in Sect. VIII. Our results are given in Sect. IX
and our conclusions in Sect. X. Finally, details on the coordinates, and on the
various coefficients needed in the formalism are given in appendices A, and B.

2 Antisymmetric Harmonic Polynomials

Any analytical function, including the wave function of a many-body system,
can be expanded in terms of HPs or equivalently in terms of Hyperspherical
Harmonics (HHs) multiplied by coefficients depending only on the hyperra-
dius. Any such expansion starts from a HP of minimal degree \( L_m \) called ground
polynomial. When we have to deal with a hypercentral potential, i.e., with a
potential invariant by rotation in the \( D \)-dimensional space spanned by the
coordinates of the particles of an \( A \)-body system in the center of mass frame,
this polynomial describes a ground state when the repulsive centrifugal bar-
rier is minimal. When the potential is not hypercentral, the product of the
potential with a ground polynomial generates new harmonic polynomials of
higher degree.

For the ground state and low excited states we expect, as was found for
medium light nuclei, that the ground polynomial brings the largest con tri-
bution to the HH expansion of the wave function of the state under investi-
gation. Since the other polynomials are obtained by starting from the ground
polynomial, we define the state by the quantum numbers defining the ground
polynomial. Ground polynomials for identical fermions must be antisymmetric
in any exchange of two particles. In order to construct a ground polynomial
suitable for describing a system of \( A \) fermions, we introduce homogeneous polynomials

\[
\phi_i(n, \ell, m) = x_i^{2n+\ell} Y_{\ell m}(\omega_i)
\]

(1)

of degree \( 2n + \ell \) in terms of the polar coordinates \( (x_i, \omega_i) \) of the \( i \)th particle where \( i = 1, \cdots, A \).

In nuclear physics each nucleon \( i \) has a spin \( s \) and an isospin \( t \) and thus to each \( \phi_i(n, \ell, m) \) polynomial a spin-isospin state \( st \) is associated. Let us construct a Slater determinant with the individual states \( \phi_i(n_j, \ell_j, m_j) s^j t^j \) where \( i \) defines the rows and \( j \) the columns for \( i \) and \( j \) running from 1 to \( A \)

\[
D = \| s^j t^j \phi_i(n_j, \ell_j, m_j) \|
\]

(2)

If for each independent state \( (s, t, \ell, m) \) the determinant is constructed by using all the \( n \) from \( n = 0 \) to \( n_{\text{max}}(s, t, \ell, m) \), where \( n_{\text{max}} \) is defined independently for each \( s, t, \ell, m \), then this determinant is a harmonic polynomial. Indeed, if the Laplace operator

\[
\Delta = \sum_{i=1}^{A} \Delta_i
\]

is applied to any column it decrease by 2 the degree of the homogeneous polynomial \( D \) without changing the quantum numbers \( (s, t, \ell, m) \). Since \( D \) has been constructed in such a way as to generate the lowest degree polynomial for the selected \( (s, t, \ell, m) \) quantum numbers, such a polynomial does not exist. Thus \( \Delta D = 0 \) and the antisymmetric homogeneous polynomial \( D \) is harmonic.

Substituting the Laguerre polynomial \( L^{\ell+1/2}_n(x_i^2) \) for \( x_i^{2n} \) in \( \phi_i \) one gets a new determinant

\[
D = \| s^j t^j x_i^j Y_{\ell_j m_j}(\omega_j) L^{\ell+1/2}_n(x_i^2) \|, \quad i, j = 1, \cdots, A
\]

If the same procedure of selecting the quantum numbers is applied, then according to the rule in which a determinant with proportional columns disappears, the new determinant, except for a normalization constant, is identical to the previous one. This new determinant multiplied by an exponential

\[
D \exp(-\sum_{i=1}^{A} x_i^2/2)
\]

is a harmonic oscillator Slater determinant [5]. This property will enable us to calculate two-body matrix elements by using HO Slater determinants.

Instead of using spin-isospin and HO quantum numbers \( n, \ell, m_\ell \) one can as well combine the spin \( s \) and angular momentum \( \ell \) to generate a total angular
momentum \( j = \ell \pm \frac{1}{2} \) defined for each particle. Then the state of each nucleon is defined by the isospin \( t \) and the quantum numbers \( n, \ell, j, m_j \) where \( m_j \) is the projection of the total angular momentum \( j \). This description will be used for open shell nuclei where the spin-orbit interaction contributes significantly to the binding energy.

The removal of excitations of the center of mass motion occurring in the product of individual wave functions, is a difficult problem in the case of an IPM. As discussed by Talmi ([4], p. 51) "\( \cdots \) it is only in the case of the HO potential that eigenstates can be written as product of an intrinsic function and a function of center of mass coordinates \( \cdots \). Removing the contribution of the center of mass motion can be done only approximately \( \cdots \). It is still important to realize that such corrections are the price we pay for the great convenience of using shell model wave functions of nucleons moving independently in a potential well." In the hyperspherical model the expansion of wave function starts from a ground HP translationally invariant. Indeed, by substitution of \( \vec{x}_i - \vec{X} \) (\( \vec{X} \) for the center of mass) for \( x_i \) in Eq. (1) and according to the properties of determinants, Eq. (2), constructed according to the rule explained above, the \( \vec{X} \) dependence disappears. Since the next terms to be included in the wave function are correlations described by functions of the relative coordinates \( \vec{r}_{ij} = \vec{x}_i - \vec{x}_j \) the overall wave function is translationally invariant and the center of mass problem is solved.

Ground states are defined by ground HP where all available nucleons eigenstates of the shells are occupied up to the last one which, except for closed shell nuclei, are not fully filled. Excited states where one or several nucleons of the last shell are raised to the next shell are still described by a ground HP. But deep hole states where a nucleon is extracted from the core and raised to a higher shell, does not generate a ground HP and therefore such states cannot be described by our scheme. This results agree with the remark of Talmi ([4], p. 7) that "\( \cdots \) simple shell model states, like single nucleon or single hole states, if they lie at sufficient high energy, are not pure single hole states of deep-lying orbits have high excitations and may no longer be accurately described in terms of a single hole wave functions, the deeper the state of the missing nucleon, the higher the excitation energy of the hole state. The fragmentation of single nucleon or single hole states at higher excitations impose practical limitations to the simple shell model picture."

### 3 The Hypercentral Approximation

Let us start the description of our formalism by considering the case where correlations are ignored in the study of the \( A \)-body system, \( i.e \), by discussing the hypercentral model first. The HO potential, neglecting the HO parameter,
is
\[
V_{HO}(\vec{x}) = \sum_{i=1}^{A} x_i^2 = \sum_{i=1}^{A} (\vec{x}_i - \vec{X})^2 + AX^2
\]  

where \( \vec{X} \) is the center of mass. The last term in the right-hand-side is irrelevant for isolated systems and therefore can be omitted. The other term is the sum of the square of the radial coordinates of the \( D = 3(A-1) \) dimensional space spanned by the coordinates \( \vec{x}_i, i = 1, \cdots, A \), of the \( A \)-particles in the center of mass frame expressed in polar coordinates \( (r, \Omega) \). This sum defines the hyperradius \( r \),
\[
r^2 = 2 \sum_{i=1}^{A} (\vec{x}_i - \vec{X})^2 = 2 \sum_{i,j=1}^{A} (\vec{x}_i - \vec{x}_j)^2,
\]  

where the factor 2 is introduced so that for two particles \( r^2 = (\vec{x}_1 - \vec{x}_2)^2 \).

The hyperradius is invariant by rotation in the \( D \)-dimensional space. Here ‘hyper’ refers to a \( D \)-dimensional space with \( D > 3 \). Since the HO potential is invariant by rotation in the \( D \)-space one may write the Schrödinger equation in polar coordinates \( (r, \Omega) \) where \( \Omega \) is the set of \( D-1 \) angular coordinates on the unit hypersphere \( r = 1 \) in the \( D \)-space which together with \( r \) fixes the coordinates \( \vec{x}(\vec{x}_1, \cdots, \vec{x}_A) \) of the particles. Let \( V(r) \) be a potential invariant by rotation in the \( D \)-space. The Schrödinger equation,
\[
\left\{ -\frac{\hbar^2}{2m} \sum_{i=1}^{A} \Delta_i + V(r) - E \right\} \Psi = 0,
\]  
in polar coordinate becomes
\[
\left\{ -\frac{\hbar^2}{m} \left[ \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{L^2(\Omega)}{r^2} \right] + V(r) - E \right\} \Psi = 0
\]  

where \( L^2(\Omega) \) is the square of the grand orbital operator which is a generalization of the orbital momentum \( \ell(\omega) \) in the 3-dimensional space to the \( D \)-dimensional space. It should be noted that in Eq. (6), the vibrational and rotational motions are independent.

Let \( H_L(\vec{x}) \) be a HP of degree \( L \). Since it is a homogeneous polynomial, the \( r \)-dependence can be factorized out, \( H_L(\vec{x}) = r^L H_L(\Omega) \). The \( H_L(\Omega) \) is a HH, i.e., the value of \( H_L(\vec{x}) \) on the unit hypersphere \( r = 1 \). From the assumption that \( H_L(\vec{x}) \) is a HP, with the definition of \( \Delta \) in terms of \( r \) and \( \Omega \) given in Eq.
\[ (6), \text{ one obtains} \]
\[ [L^2(\Omega) + L(L + D - 2)]H_L(\Omega) = 0. \]  
\[ (7) \]

The wave function \( \Psi \) can be written as a product of a HH and a hyper-radial function,
\[ \Psi = \frac{1}{r^{(D-1)/2}} u_L(r) H[u](\Omega) \]  
\[ (8) \]
where the hyper-radial function \( u_L(r) \) is a solution of
\[ \left\{ \frac{\hbar^2}{m} \left[ -\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L} + 1)}{r^2} \right] + V(r) - E \right\} u_L(r) = 0 \]  
\[ (9) \]

with \( \mathcal{L} = L + (D - 3)/2 \) and where \([L]\) stands for the set of quantum numbers including spin and isospin defining the state of grand orbital \( L \).

Whatever the potential \( V(r) \) is, the ground state is obtained when the repulsive central barrier is the smallest. It corresponds to a HP of minimum degree \( L_m \). For a system of bosons where all particles can be in the \( S \)-state this degree is \( L_m = 0 \). For fermions, \( L \) is the smallest when \( H_{L_m}(\vec{x}) \) is a Slater determinant constructed from the individual polynomials \( x_i 2^{n_j + \ell_j} Y_{\ell_j}^m(\omega_i) \), where \( i \) refers to the rows and \( j \) to the columns, when all the \( m_j \) are used for the smallest possible values of \( 2n_j + \ell_j \). As we explained in the previous section, this corresponds to a Slater determinant for a HO in a ground state. The minimum degree is the sum over the degree of the individual polynomials in \( H_{[L_m]}(\vec{x}) \) which is, as we have seen in the previous section, the sum over the radial and orbital quantum numbers of all the HO occupied states,
\[ L_m = \sum_{j=1}^{A} (2n_j + \ell_j). \]  
\[ (10) \]

The \( L_m \) is related only to the rotation in the \( D \)-space and is independent of the shape of the hypercentral potential \( V(r) \). It is thus clear that in the hyperspherical scheme the “average field of force \( V(r) \) of spherical symmetry” Mayer and Jensen speak about, which “represents the action of the other nucleons”, is the hypercentral potential generated by the two body nuclear potential \( V(\vec{r}_{ij}) \). This average potential is given by
\[ V(r) = \frac{A(A - 1)}{2} \int H_{[L_m]}^*(\Omega)V(\vec{r}_{ij})H_{[L_m]}(\Omega) \, d\Omega \]  
\[ (11) \]
where $\vec{r}_{ij} = \vec{x}_i - \vec{x}_j$ and $V(\vec{r}_{ij})$ contains the spin and isospin exchange operators and in general all the components of the nuclear interaction $V(\vec{r}_{ij})$ and where
\begin{equation}
\int |H_{[L,m]}(\Omega)|^2 \, d\Omega = 1. \tag{12}
\end{equation}

It depends on the collective variable $r$ and does not generate any correlations since when $r$ is fixed, a variation in the distance $\vec{r}_{ij}$ between two particles does not affect the hypercentral potential and the corresponding wave function.

To investigate the effect of a spin-orbit force we recall that it operates only on the last open shells. Following the derivation of de Shalit and Talmi we write for the spin-orbit operator for the pair $(i, j)$, $(\vec{\sigma}_i + \vec{\sigma}_j) \cdot \vec{\ell}_{ij}$ where

\begin{equation}
\vec{\ell}_{ij} = \frac{1}{2}(\vec{x}_i - \vec{x}_j) \times (\vec{p}_i - \vec{p}_j) = \frac{1}{2}(\vec{\ell}_i + \vec{\ell}_j) - \frac{1}{2}[\vec{x}_j \times \vec{p}_i + \vec{x}_i \times \vec{p}_j]
\end{equation}

After elimination of the center of mass, the sum over all pairs of the spin–orbit operator gives
\begin{equation}
\sum_{i,j>i}^{A} (\vec{\sigma}_i + \vec{\sigma}_j) \cdot \vec{\ell}_{ij} = A \sum_{i=1}^{A} \vec{\ell}_i \cdot \vec{s}_i + \vec{S} \cdot \vec{L} \tag{13}
\end{equation}

where
\begin{equation}
\vec{s}_i = \vec{\sigma}_i/2, \quad \vec{S} = \sum_{i=1}^{A} \vec{s}_i, \quad \vec{L} = \sum_{i=1}^{A} \vec{\ell}_i.
\end{equation}

The first term in the right hand side of Eq. (13) is the sum of one body spin-orbit operator occurring in the standard shell model and the last one, a collective spin-orbit operator, is a small perturbation.

In the $jj$-coupling scheme, the spin-orbit operator does not modify the structure of the Slater determinant defining the state and thus one has
\begin{equation}
\sum_{i,j>i}^{A} (\sigma_i + \sigma_j) \cdot \vec{\ell}_{ij} H_{[L]}(\Omega) = C_{LS} H_{[L]}(\Omega).
\end{equation}

The coefficient $C_{LS}$ is determined by the filling of the shell of the determinant in the $jj$-coupling.

The hypercentral potential, including the spin-orbit force $V_{ls}(r_{ij}) (\vec{\sigma}_i + \vec{\sigma}_j) \cdot \vec{\ell}_{ij}$ operating on a Slater determinant in the $jj$-scheme, is $V(r) + C_{LS} V_{LS}(r)$. The ground state results from a competition between the repulsive centrifugal barrier $\hbar^2/m \mathcal{L}(\mathcal{L} + 1)/r^2$ and the potential including the effect of the spin-orbit force through the coefficient $C_{LS}$ fixed by the filling of the individual states in the last shell.
In summary the quantum numbers defining the ground state are those of the HO in ground state for which the repulsive central barrier is the smallest corrected by the effect of the spin–orbit force. It is the model proposed by Goeppert-Mayer and Jensen but this time the potential is not a one body potential in which all nucleons move but a potential of the collective symmetrical hyper-radial coordinate whose shape should be deduced from the realistic nucleon-nucleon potential. More information about the Hypercentral Approximation (HCA) where only the HH of minimal order $L_m$ is taken into account in the wave function can be found in Refs. [5–7]. We only note here that for a two-body potential $V(r_{ij})$, the HCA provides a ground state binding energy for nuclei a little stronger but nearly identical to the one given by a variational calculation using a HO Slater determinant where the strength parameter is adjusted to give the lowest eigen-energy [5]. The HCA variationally provides an upper bound to the eigen-energy. A further improvement in the solution can be achieved with the inclusion of the two-body correlations generated by the two-body potential.

4 Integro-differential Equations Approach

For nuclear potentials with a strong repulsive core, the Hartree-Fock method, which reduces the interaction to a sum of individual potentials leading to an IPM, gives very poor results because the residual interaction responsible for the correlations is large and is not taken into account in the procedure. A Jastrow function might be added to take care of the correlations. In the method developed in this paper, one operates in the $D$-dimensional space spanned by the Jacobi coordinates and uses a radial coordinate system $(r, \Omega)$ where $r$ is the hyperradius and $\Omega$ stands for the angular coordinates at the surface of the unit hypersphere $r = 1$.

When we have to deal with the one-body problem in polar coordinates and the potential in the physical three-dimensional space is not central, the wave function can be expanded in terms of spherical harmonics and the Schrödinger equation, projected on the spherical harmonics basis, generates a system of radial, coupled, second order, differential equations that have to be integrated to solve the problem. When we have to deal with many-body systems, treated in polar coordinates in the $D$-dimensional space spanned by the the coordinates of the particles, we have the same situation as for the one-body problem, except that in this case the space is larger. The interactions occurring in many body systems, for instance a sum of two-body potentials, are generally not invariant by rotation in the $D$-dimensional space, and thus we have to deal in this space with a deformed potential. The expansion method can be applied in which HH must be substituted for Spherical Harmonics. Like for one particle in a deformed potential, the Schrödinger equation is transformed into a
system of coupled, second order, differential equations in the hyperradius $r$. In this procedure we are facing a different problem namely that of degeneracy. While the $(2\ell + 1)$-degeneracy of the spherical harmonics for each orbital $\ell$ is moderate, leading to the use of a rather small number of significant terms in the expansion in the three-dimensional space, in contrast the degeneracy of the HH basis for a grand orbital $L$ increases rapidly with the dimension $D$ of the space leading to an intractable large number of significant coupled equations.

A partial solution to this problem has been obtained by selecting the HH describing only two-body correlations, namely, the Potential Harmonics (PH) [6,8]. With this restricted basis, a good approximation can be achieved for the solution of the few-body problem. The rate of convergence can be improved by introducing functions limiting the number of significant coupled equations [9]. But for more particles, the number of coupled equations to be solved to obtain a good accuracy becomes again too large [10].

To overcome this difficulty the Integro-Differential Equation Approach (IDEA) has been proposed [11–13,7,14,15] in which the two-body correlations are taken into account. In the IDEA method, the Schrödinger equation is transformed into two-variable integro-differential equations, whatever the number of particles is. The approximation is justified in saturated systems, like nuclei, by the rather low kinetic energy occurring between two particles because the volume occupied by the system increases proportionally to the number of particles. At this low kinetic energy, the pairs are mainly in $S$-state and many-body correlations are expected to be rather small. It should be noted that the IDEA equation is identical to the Faddeev equation for three-particles when the interaction operates on pairs in an $S$-state only.

The capability of the IDEA to solve the Schrödinger equations for nuclei arises from the large component of the potential which is invariant by rotation in the D-space, i.e., from the hypercentral potential. The contribution of this potential to the binding energy balances the one of the kinetic energy in such a way that the binding energy in nuclei is mainly provided by correlations when a strong repulsive core (that give rise to large correlations) exists in the potential. In other words, we agree with Talmi [4] that “... the relevance to nuclear many-body theory is in realizing that two-nucleon effective interactions are all that is necessary to calculate nuclear energies. The two-nucleon matrix elements obtained by this procedure determines the structure of nuclei in their ground state and at low excitations.” In what follows we shall describe the IDEA formalism that takes into account two-body correlations and discuss various aspects of it.
4.1 Two-Body Correlations

We assume that we are dealing with a system of \( A \) identical fermions of mass \( m \), in particular nucleons. Our aim is to solve the Schrödinger equation

\[
(T + V(\vec{x}) - E)\Psi(\alpha, \vec{x}) = 0
\]  

(14)

where \( \vec{x} = (\vec{x}_1, \cdots, \vec{x}_A) \) are the particle coordinates of an \( A \)-body system, \( \alpha \) denotes the space independent degrees of freedom like spin and isospin, \( T \) is the kinetic energy operator, \( V(\vec{x}) \) is the interaction potential, and \( E \) is the energy state of the system. The wave function can be expanded in terms of harmonic polynomials and the expansion begins with a ground polynomial \( H_{[L_m]}(\vec{x}) \) of degree \( L_m \) for the state under consideration,

\[
\Psi(\alpha, \vec{x}) = \sum_{L=L_m}^{\infty} \sum_{[L]} H_{[L]}(\vec{x}) u_{[L]}(r) = \sum_{L=L_m}^{\infty} \sum_{[L]} H_{[L]}(\Omega) r^L u_{[L]}(r)
\]  

(15)

where \([L]\) is the set of quantum numbers, including the space independent degrees of freedom \( \alpha \), defining \( H_{[L]}(\vec{x}) \). The notation \( H_{[L]}(\Omega) = H_{[L]}(\vec{x})/r^L \) is used for the associated Hyperspherical Harmonics (HH) which is the value of \( H_{[L]}(\vec{x}) \) on the unit hypersphere \( r = 1 \). Antisymmetric Harmonic polynomials are denoted by \( D_{[L]}(\vec{x}) \) (to remind us that they are constructed as Slater determinants) and the associated HH by \( D_{[L]}(\Omega) \).

To solve the Schrödinger Eq. (14) one needs to specify the properties of the interaction. When the interaction is a sum of one-body potentials \( V(\vec{x}) = \sum_{i=1}^{A} V(\vec{x}_i) \) we have to deal with an Independent Particle Model (IPM) and the wave function is a product of individual states eigenfunctions of the one-body problem. When the potential is a many-body one, various methods have been proposed to obtain the solution of Eq. (14). In the most popular one, the state is described by the product of a Slater determinant constructed with individual states adequately chosen and a Jastrow function which is the product of two-body functions \( \prod_{i,j>i} f(r_{ij}) \) for describing two-body correlations originating from a two-body potential. For most systems of identical particles like nuclei, the dominant part of the interaction is a sum of two-body potentials. The three-body potentials occurring next, can be reduced, except for a small contribution, to a sum of two-body potentials as well [16]. Thus in the present work we concentrate to the case where the interaction is a sum of two-body potentials only.

In order to understand the structure of the solution, one starts from the HCA (discussed in Sect. 3) where only the ground harmonic, \( i.e. \), only the first term in the HH expansion of the wave function is taken into account. In this case,
the wave function, in the center of mass coordinate, is the product of a HH and a radial function

\[ \Psi(\alpha, \vec{x}) = \Psi_0(r, \Omega) = D_{[L_m]}(\Omega) \frac{1}{r^{(D-1)/2}} u_0(r) \tag{16} \]

where \( D_{[L_m]}(\Omega) \) is the ground antisymmetric HH for the investigated state.

A projection of Eq. (14) on \( D_{[L_m]}(\Omega) \) for the wave function given by Eq. (16), results in the radial equation

\[ \left[ \frac{\hbar^2}{m} \left\{ -\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L} + 1)}{r^2} \right\} + V_{[L_m]}(r) - E \right] u_0(r) = 0 \tag{17} \]

where \( \mathcal{L} = L_m + (D - 3)/2. \) The hypercentral potential \( V_{[L_m]}(r) \) is given by the integral

\[ V_{[L_m]}(r) = \int D_{[L_m]}^*(\Omega)V(r, \Omega)D_{[L_m]}(\Omega) \, d\Omega \tag{18} \]

in terms of \((r, \Omega),\) the polar coordinates of \( \vec{x}, \) taken over the surface of the unit hypersphere \( r = 1. \) In the three-dimensional space it would be the central part of the potential. When \( V(r, \Omega) \) is a sum of two-body potentials over all pairs and \( D_{[L_m]}(\Omega) \equiv H_{[L_m]}(\Omega) \) is normalized according to Eq. (12) then Eq. (11) holds. For the infinite hyperspherical well, the Coulomb, and the Harmonic Oscillator potentials the radial solutions \( u_0(r) \) are known analytically \[5\]. In the last case the Schrödinger equation can be solved either as a collective or as an independent particle model.

To proceed beyond the HCA and obtain a solution of Eq. (14) with a better accuracy, one writes

\[ \Psi(\alpha, \vec{x}) = \Psi(\vec{x}) = \Psi_0(r, \Omega) + \Psi_1(r, \Omega) \tag{19} \]

where \( \Psi_1(r, \Omega) \) is the next improvement in the wave function which depends on the structure of the potential. Here we are interested for potentials which can be written as a sum of two-body interactions

\[ V(\vec{x}) = \sum_{i<j} V(\vec{r}_{ij}), \quad \vec{r}_{ij} = \vec{x}_i - \vec{x}_j \tag{20} \]

Using Eq. (19), Eq. (14) becomes

\[ (T - E)(\Psi_0 + \Psi_1) = \sum_{i<j} V(\vec{r}_{ij})(\Psi_0 + \Psi_1) \tag{21} \]
Omission of the correction term on the right hand side, implies a structure of \( \Psi_1 \) of the form

\[
\Psi_1 = \Psi_0 \sum_{i<j} F(\vec{r}_{ij}, r)
\] (22)

where \( F(\vec{r}_{ij}, r) \) is a two body amplitude. Thus, for nuclei with central potentials we may write

\[
\Psi(\vec{x}) = D_{[L_m]}(\Omega) \sum_{i<j} F(r_{ij}, r)
\] (23)

in terms of the two-body amplitude \( F(r_{ij}, r) \) where the pair is assumed to be in an \( S \)-state.

In order to find the amplitude \( F(r_{ij}, r) \) we have two options: Either to solve the amplitude equation

\[
(T - E)D_{[L_m]}(\Omega)F(r_{ij}, r) = -V(r_{ij})D_{[L_m]}(\Omega) \sum_{\ell, k<\ell} F(r_{kl}, r)
\] (24)

or to solve the Schrödinger equation

\[
(T - E)\Psi(\vec{x}) = -\sum_{j, i<j} V(r_{ij})\Psi(\vec{x})
\] (25)

where now \( \Psi \) is given by Eq (23). The first option, Eq. (24), where the sum over all pairs reproduces the Schrödinger equation, leads to the IDEA [11–13,7,14,15], while the second leads to the Variational Integro-Differential Equation (VIDE) [17,18]. It should be noted that for three-body the IDEA becomes a Faddeev-type equation while the VIDE does not have any counterpart. The solutions are not the same, as the VIDE is variational while the IDEA is not, but the latter is easier to solve. When the potential \( V(r_{ij}) \) operates only on pairs in \( S \)-state the Faddeev equation provides the exact solution but when the potential is local and operates on all orbitals, the VIDE gives far more accurate solutions. In order to obtain the same accuracy, a few coupled Faddeev equations have to be solved. In both cases the wave function has the structure of a sum of two-body amplitudes.

The correlations are described by a product when Jastrow functions are used. However, the product structure of the Jastrow function commonly used, does not have the correct structure of the solution for three or more particles with an \( S \)-projected interaction. In contrast, in the IDEA formalism we only assume that a sum of two-body amplitudes solution of Eq. (24) is sufficient to provide
us with a good solution although many-body correlations are not taken into account.

4.2 Reduction of the IDEA Equation

We assume that the wave function is given by Eq. (23), i.e., by a product of an antisymmetric HH and a symmetrized two-body amplitudes. Our aim is to solve Eq. (24) to obtain the two-body amplitude \( F(r_{ij}, r) \) associated with the ground polynomial \( D_{[L_m]}(\Omega) \) defined by the occupied states in a Slater determinant. In order to have an equation for \( r_{ij} \) only, for instance for a reference pair \( \vec{r}_{ij} = \vec{\xi}_N \) with \( N = A - 1, i = 1, \) and \( j = 2, \) we have to eliminate the part which in \( F(r_{k\ell}, r) \) depends on the other Jacobi variables \( \vec{\xi}_i, i < N, \) which are related to many-body correlations (For the A-particle coordinates in the center of mass frame, see Appendix A). For this purpose, one projects \( F(r_{ij}, r) \) for \((i, j) \neq (1, 2)\) on the pair \((1, 2)\) in an \( S\)-state. Let us call \( P_c \) the operator projecting the connected pairs like \((1, i), (2, i), \ldots, (A, i)\) and \( P_d \) the disconnected pairs like for \( i \) and \( j > 2. \) Then the total projection operator is

\[
P^0 = 2(A - 2)P^0_c + \frac{(A - 2)(A - 3)}{2}P^0_d
\]

(26)

where \( P^0_c \) operates on one connected pair, e.g. the pair \((2, 3)\), and \( P^0_d \) on one disconnected pair, e.g. the pair \((3, 4)\). Since our aim is to calculate correlations, we isolate on the left hand side the hypercentral part of the potential and in the right hand side only that part of the potential generating correlations. Then Eq. (24) becomes

\[
\left(T + \frac{A(A - 1)}{2}V_{[L_m]}(r) - E\right)D_{[L_m]}(\Omega)F(r_{12}, r) = -\left(V(r_{12}) - V_{[L_m]}(r)\right)
\times D_{[L_m]}(\Omega)\left\{F(r_{12}, r) + (A - 2)\left[2P^0_cF(r_{23}, r) + \frac{A - 3}{2}P^0_dF(r_{34}, r)\right]\right\}
\]

(27)

where the functions within the bracket \{ \} depend only on \( r_{12} \) and \( r. \) Eq. (27) is written for the pair \((i, j) = (1, 2)\). It is noted that by summing over all pairs \((i, j)\) one recovers the Schrödinger equation (25).

The two variables equation in \( r \) and \( z \) we are looking for, which determine the two-body amplitudes, is obtained by multiplying Eq. (27) at left by \( D^*_{[L_m]}(\Omega) \) and by integrating over the surface element \( d\Omega_{N-1} \) (see Appendix A). First we require the integral

\[
\int D^*_{[L_m]}(\Omega)D_{[L_m]}(\Omega) d\Omega_{N-1} = W_{[L_m]}^{(D)}(z, \omega),
\]

(28)
taken over all angular coordinates $\Omega_{N-1}$ excluding $z = \cos 2\phi$ with $r_{ij}/r = \cos \phi$ and $\omega$, the angular coordinate of $\vec{r}_{ij}$, and thus this integral is a function of $z$ and $\omega$. We may define the weight function $W_{[L_m]}(z, \omega)$ for the state $[L_m]$,

$$W_{[L_m]}(z, \omega) = W_0(z)W_{[L_m]}^{(D)}(z, \omega),$$  \hspace{1cm} (29)

where

$$W_0(z) = 2^{-D/2}(1-z)^{(D-5)/2}(1+z)^{1/2},$$  \hspace{1cm} (30)

in such a way that for normalized $D_{[L_m]}(\Omega)$ (see Eq. A.8)

$$\int D_{[L_m]}^{*}(\Omega)D_{[L_m]}(\Omega) d\Omega = \int W_{[L_m]}(z, \omega) dz d\omega = 1$$  \hspace{1cm} (31)

where $d\Omega$ is the surface element on the unit hypersphere given by $d\Omega = W_0(z) dz d\omega d\Omega_{N-1}$. The construction of $W_{[L_m]}(z, \omega)$ is discussed in the next section. We only note here that for bosons in ground state $[L_m] = 0$,

$$D_{[0]}(\Omega) = Y_{[0]} = \frac{\Gamma(D/2)}{2\pi^{D/2}},$$

and $W_{[0]}(z, \omega)$ reduces to $W_0(z)$ except for a normalization constant.

To proceed we must calculate the kinetic energy and the projection terms. For the former term we use the relation (A.12) with the center of mass at rest omitted,

$$T = -\frac{\hbar^2}{m} \Delta \xi = -\frac{\hbar^2}{m} \sum_{i=1}^{A-1} \nabla_{\xi_i}^2, \hspace{1cm} i = 1, \ldots, A - 1$$

and set

$$F(\xi_N, r) = \frac{1}{r^{(D-1)/2}} P(z, r).$$  \hspace{1cm} (32)

Therefore, we have to calculate

$$\Delta \left[ D_{L_m}(\Omega) \frac{1}{r^{(D-1)/2}} P(z, r) \right]$$

where

$$\Delta \equiv \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{L^2(\Omega)}{r^2}$$

with the grand orbital operator being written as

16
\[
L^2(\Omega) = \frac{4}{W_0(z)} \frac{\partial}{\partial z} (1 - z^2) W_0(z) \frac{\partial}{\partial z} + \frac{2\ell(\omega)}{1 + z} + \text{derivatives in } \Omega_{N-1} \text{ coordinates (33)}
\]

Thus, multiplying Eq. (27) from left by \(D^*_{Lm}(\Omega)\), integrating over the surface element \(d\Omega_{N-1}\), and taking into account the relation
\[
2D^*_{Lm}(\Omega) \frac{\partial}{\partial z} D_{Lm}(\Omega) = \frac{\partial}{\partial z} \left[ D^*_{Lm}(\Omega) D_{Lm}(\Omega) \right]
\]
one gets the basic equation [11–13]

\[
\left\{ \frac{\hbar^2}{m} \left[ -\frac{\partial^2}{\partial r^2} + \frac{\mathcal{L}(\mathcal{L} + 1)}{r^2} - \frac{4}{r^2 W_{Lm}(z)} \frac{\partial}{\partial z} (1 - z^2) W_{Lm}(z) \frac{\partial}{\partial z} \right] + \frac{A(A - 1)}{2} V_{Lm}(r) - E \right\} P(z, r) = -\left( V(r\sqrt{1 + z}/2) - V_{Lm}(r) \right) \left[ P(z, r) + \mathcal{P}^0 P(z, r) \right]
\]

(34)

where \(\mathcal{P}^0 P(z, r)\) is the projection of the two-body amplitudes for all the pairs \((i, j)\) for \((i, j) \neq (1, 2)\) on the reference pair \((1, 2)\) in S-state [13] (see forth Sect. 6).

Eq. (34) can be easily solved directly as a two-variable integro-differential equation to obtain the binding energy of the nucleus under consideration. However, it is desirable to use also the adiabatic and the more accurate uncoupled adiabatic approximations which can provide us not only reliable solutions but also the eigen-potentials from which further physical information, such as low energy scattering states, can be extracted.

### 4.3 The Extreme Adiabatic Approximation

In most systems the energy contained in the rotation in the \(D\)-dimensional space is very much larger than the one in the radial motion. Indeed, in nuclei the radial energy of the ground state is of the order of half the monopolar excitation energy (breathing mode), \(i.e\), about 10 MeV, compared to the several hundreds or thousands of MeV generated by the rotation which can be estimated from the Fermi gas model giving the average kinetic energy per particle \(E_{\text{kin}}/A = 28.7r_0^{-2}\) fm MeV, \(i.e\), \(\sim 23.7\) MeV for \(r_0 = 1.1\) fm . This means that the rotation and the vibration are nearly decoupled and thus one can use an adiabatic approximation in which one freezes the \(r\)-motion and solve the rotational motion equation to obtain for each \(r\) an eigenpotential that
is subsequently used in the radial equation [19]. More specifically, one writes

\[ P(z, r) = P_\lambda(z, r)u_\lambda(r) \]

and solves

\[
\left[ \frac{4 \hbar^2}{r^2 m} \left\{ \frac{1}{W[L_m](z)} \frac{\partial}{\partial z} (1 - z^2) W[L_m](z) \frac{\partial}{\partial z} \right\} + U_\lambda(r) \right] P_\lambda(z, r) = \left[ V(r \sqrt{1 + z}/2 - V[L_m](r)) \left[ P_\lambda(z, r) + P^0 P_\lambda(z, r) \right] \right]
\]

(35)

for fixed \( r \) to obtain the eigenpotential \( U_\lambda(r) \) which is then used in the radial equation

\[
\left\{ \frac{\hbar^2}{m} \left[ -\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L} + 1)}{r^2} \right] + \frac{A(A - 1)}{2} V[L_m](r) + U_\lambda(r) - E \right\} u_\lambda(r) = 0
\]

(36)

to obtain the total energy \( E \) and the radial function \( u_\lambda(r) \) [11]. The Eq. (35) and (36) constitute the so-called Extreme Adiabatic Approximation (EAA).

The wave function, within this approximation, is obtained from \( P_\lambda(z, r) \) and \( u_\lambda(r) \),

\[
\Psi_{EAA}(\vec{x}) = \frac{1}{r^{3A/2 - 2}} u_\lambda(r) \sum_{i<j \leq A} P_\lambda(2r^2ij/r^2 - 1, r).
\]

(37)

When the hypercentral potential \( V[L_m](r) \) is ignored, in Eqs. (34-36) for \( A = 3 \), one finds the Faddeev equation for \( S \)-state projected potentials [20,21]. For \( A > 3 \) it will be called \( S \)-State Integro-Differential Equation (SIDE) and should be applied when we are dealing with \( S \)-state projected potentials since \( P^0 \) in Eqs. (34-36) projects any other pair \((i, j)\) on the pair \((1, 2)\) in \( S \)-state. In the IDEA only the residual interaction generating correlations operates on pairs in \( S \)-state.

4.4 The Uncoupled Adiabatic Approximation

In order to separate the two-variable integrodifferential equation (34) into two, one-variable, equations (35 and 36) we assumed that the amplitude can be written as a product \( P(z, r) = P_\lambda(z, r)u_\lambda(r) \) and the EAA is obtained when the variation of \( P_\lambda(z, r) \) with respect to \( r \) is neglected. It provides a lower bound to the eigen-energy [22,21]. A further improvement in the binding can be achieved if such a variation is taken into account.

The new equation is obtained by substitution of \( P_\lambda(z, r)u_\lambda(r) \) for \( u(r) \) in (36) and by taking the derivatives of \( P_\lambda(z, r) \) with respect to \( r \) into account. One
first normalizes the $P_\lambda(z, r)$ according to

$$
\langle P_\lambda | P_\lambda \rangle \equiv \int_{-1}^{+1} P^2_\lambda(z, r) W_{[L_m]}(z) \, dz = 1
$$

which leads to the derivatives

$$
\langle P_\lambda | \frac{dP_\lambda}{dr} \rangle = 0, \quad \langle P_\lambda | \frac{d^2P_\lambda}{dr^2} \rangle = -\langle \frac{dP_\lambda}{dr} | \frac{dP_\lambda}{dr} \rangle
$$

Multiplying Eq. (36) with $P_\lambda(z, r) W_{[L_m]}(z)$ from right and integrating over $z$, we obtain the new equation for $u_\lambda$

$$
\left\{ \frac{\hbar^2}{m} \left[ -\frac{d^2}{dr^2} + \mathcal{L} \mathcal{L} + 1 \right] + \frac{A(A-1)}{2} V_{[L_m]}(r) \right. \\
\left. + U_\lambda(r) + \frac{\hbar^2}{m} \langle \frac{dP_\lambda}{dr} | \frac{dP_\lambda}{dr} \rangle - E \right\} u_\lambda(r) = 0.
$$

The solution thus obtained is called the Uncoupled Adiabatic Approximation (UAA). We note that the extra term introduced in (40) is always negative and the new effective potential

$$
V_{\text{eff}}(r) \equiv \frac{A(A-1)}{2} V_{[L_m]}(r) + U_\lambda(r) + \frac{\hbar^2}{m} \langle \frac{dP_\lambda}{dr} | \frac{dP_\lambda}{dr} \rangle
$$

always provides an upper bound to the eigen-energy.

\subsection*{4.5 A Variational Equation for the IDEA}

The IDEA Equation (34) is a two-variable integro-differential equation where the integral part comes from the projection $\mathcal{P}^0$ of all pairs $(i, j)$ on the reference pair $(1, 2)$. The two-body amplitude $P(z, r)$ is a solution of an integro-differential equation which must fulfill certain asymptotic conditions for $z = \pm 1$ and $r = 0$ and $r = \infty$. A variational equation can also be obtained in the adiabatic approximation by multiplying Eq. (35) by the weight function, normalized to unity, and by integrating over $z$ in the range $[-1, +1]$. The normalization is

$$
\int_{-1}^{+1} P_\lambda(z, r) W_{[L_m]}(z) \, dz = 1
$$
implying that the hypercentral part of \( P_\lambda(z, r) \) is taken to be 1. It is the dominant contribution in the HH expansion of \( P_\lambda(z, r) \). The kinetic energy term in the left hand side disappears and the equation for \( U_\lambda(r) \) becomes

\[
U_\lambda(r) = \int_{-1}^{+1} \left[ V(r\sqrt{(1 + z)/2} - V_{[L_m]}(r) \right]
\left[ P(z, r) + \mathcal{P}^0 P(z, r) \right] W_{[L_m]}(z) \, dz
\]

Thus one may search for the solution \( P_\lambda(z, r) \) for which \( U_\lambda(r) \) is a minimum which is then introduced in the radial equation (36) to calculate the ground state energy. One can also control the quality of \( P_\lambda(z, r) \) solution of Eq. (35) by calculating the eigen-potential \( U_\lambda(r) \) from (43). This should be identical (within numerics) to the one obtained from Eq. (35).

5 Construction of the Weight Function

The HO wave function can be written either in the \( D \)-space as a product of a HP \( D_{[L_m]}(\vec{x}) \) and an exponential in the hyperradius or as a Slater determinant constructed from individual HO eigenfunctions. The weight function can then be obtained by identification of the Fourier transform in the relative coordinate \( \vec{r}_{ij} \) of \( |D_{[L]}(\vec{x})|^2 \) calculated first from the HP representation in the \( D \)-space and then from the standard HO Slater determinant.

Let \( D^{HO}_{[L]}(\vec{x}) \) be the normalized Slater determinant describing the states of an \( A \)-nucleon system

\[
D^{HO}_{[L]}(\vec{x}) = \frac{1}{\sqrt{A!}} ||s_j^{i_j} t_j^{i_j} \psi_{n_j, \ell_j, m_j}(\vec{x}_j)||
\]

where \( i \) refers to the rows, \( j \) to the columns (\( i, j = 1, \ldots, A \)) and where \( s \) and \( t \) denote the spin and isospin states. The normalized HO eigenfunction for one particle in polar coordinates \( (x, \omega) \) is

\[
\psi_{n, \ell, m}(\vec{x}) = \left[ \frac{2n!}{b^3 \Gamma(n + \ell + \frac{3}{2})} \right]^{1/2} V^{m}_{\ell}(\omega)(x/b)^\ell L_{n}^{\ell+1/2} \left( (x/b)^2 \right) e^{-(x/b)^2/2}
\]

where \( b \) is a parameter related to the size of the system. The determinant describing the HO ground state of the \( A \)-nucleon system is constructed in such a way that for a given set of quantum numbers \( s^j, t^j, \ell_j, m_j \) all the quantum numbers \( n_j \), for \( n_j \) running from 0 to a maximum value \( n_j(s^j, t^j, \ell_j, m_j) \)

\[
20
\]
chosen independently for each set, are filled. According to the properties of
determinants it can also be written as

\[ D_{[L]}^{HO}(\vec{x}) = C_L \parallel s_i^j t_i^j x_i^{2n_j+\ell_j} Y_{\ell_j}^{m_j}(\omega_i) \parallel e^{-(\sum_i x_i^2)/g_0^2} \]  \hspace{1cm} (46)

where \( g_0^2 = 2b^2 \), \( \sum_i x_i^2 = r^2/2 + AX^2 \), \( \bar{X} \) being the center of mass coordinate, and \( C_L \) is a normalization constant. This determinant is a translationally invariant ground HP of degree \( L = \sum_j (2n_j + \ell_j) \).

One may factorize \( r^L \) and rewrite Eq. (46) as

\[ D_{[L]}^{HO}(\vec{x}) = D_{[L]}^{[L]}(\Omega) \left[ \frac{2r^{2L} e^{-r^2/\rho^2}}{g_0^{2L+D} \Gamma(L+D/2)} \right]^{1/2} \left[ \frac{2e^{-R^2/b^2}}{b^D \Gamma(3/2)} \right]^{1/2} \]  \hspace{1cm} (47)

where \( R^2 = AX^2 \) and \( D_{[L]}(\Omega) \) is a normalized HH of grand orbital \( L \). Indeed, by integrating \( |D_{[L]}^{HO}(\vec{x})|^2 \) over the 3A–dimensional space with

\[ d^{3A} x = d^3 x_1 \cdots d^3 x_A = d\Omega r^{D-1} dr d^3 R \]  \hspace{1cm} (48)

one finds

\[ \int |D_{[L]}^{HO}(\vec{x})|^2 d^{3A} x = \int |D_{[L]}(\Omega)|^2 d\Omega = 1. \]  \hspace{1cm} (49)

The integral \( \int |D_{[L]}(\Omega)|^2 d\Omega_{N-1} \) (occurring in the weight function) can be obtained from the Fourier transform

\[ \mathcal{D}(y, \omega_k) \equiv \langle D_{[L]}^{HO}(\vec{x}) | e^{ik \cdot \vec{\xi}_N} | D_{[L]}^{HO}(\vec{x}) \rangle, \quad \vec{\xi}_N = \vec{r}_{12} \]  \hspace{1cm} (50)

calculated by two different methods. Since \( r^L D_{[L]}(\Omega) \) is a polynomial of degree \( L \) homogeneous in the Jacobi coordinates \( \vec{\xi}_i, \ i = 1, \cdots, N(= A - 1) \), then \( |r^L D_{[L]}(\Omega)|^2 \) is an even homogeneous polynomial and the integral over \( \Omega_{N-1} \) in Eq. (28) is a homogeneous polynomial in the \( \vec{\xi}_N \) and \( \rho^2 = r^2 - \bar{X}_N^2 \) variables

\[ \int |r^L D_{[L]}(\Omega)|^2 d\Omega_{N-1} = \sum_{n, \ell \ even} \langle [L]|n, \ell \rangle Y_\ell^0(\omega) \xi_N^{2n+\ell} \rho^{2Lm-(2n+\ell)} \]  \hspace{1cm} (51)

where \( \langle [L]|n, \ell \rangle \) are coefficients that have to be defined. The volume element for the coordinate system \( (\Omega_{N-1}, \rho, \vec{\xi}_N) \) in the \( D = 3N = 3(A-1) \)–dimensional space is

\[ d^{3N} \xi = d\Omega_{N-1} \rho^{D-4} d\rho \xi_N^2 d\xi_N d\omega \]  \hspace{1cm} (52)
where \( \omega \) is for the angular coordinates of \( \vec{\xi}_N \). Since the operator \( \exp[i\vec{k} \cdot \vec{r}_{12}] \) is independent of the center of mass, the integral over \( \vec{R} \) in (50) gives 1 and the Fourier transform becomes

\[
\mathcal{D}(y, \omega_k) = \frac{2}{\Gamma(L + D/2)} \sum_{n, \ell \text{ even}} \langle [L]|n, \ell \rangle \int e^{i\vec{k} \cdot \vec{\xi}_N} Y^{0}_\ell(\omega) \\
\times d\omega \frac{e^{-(\vec{\xi}_N + \rho^2)/\theta_0}}{\theta_0} \frac{2^{(n+1)+\ell}}{\sqrt{kx}} J_{\ell+1/2}(kx)
\]

One substitutes for the plane wave \( e^{i\vec{k} \cdot \vec{x}} \), \( \vec{x} = \vec{\xi}_N \), the expansion in the polar coordinates \( (\omega, x) \) of \( \vec{x} \),

\[
e^{i\vec{k} \cdot \vec{x}} = (2\pi)^{3/2} \sum_{\ell, m} Y^m_\ell(\omega_k) Y^{m*}_\ell(\omega) \frac{1}{\sqrt{kx}} J_{\ell+1/2}(kx).
\]

where \( (k, \omega_k) \) are the polar coordinates of \( \vec{k} \). After integration over the \( (\rho, \vec{\xi}_N) \) variables, the Fourier transform becomes

\[
\mathcal{D}(y, \omega_k) = \frac{\pi^{3/2}}{\Gamma(L + D/2)} \sum_{n, \ell} (-1)^{\ell/2} \langle [L]|n, \ell \rangle n! \\
\times \Gamma(L + (D - 3)/2 - n - \ell/2) Y^{0}_\ell(\omega_k) y^{\ell} L_n^{\ell+1/2}(y^2) e^{-y^2}
\]

where \( y = kb/\sqrt{2} = k\theta_0/2 \).

The Laguerre polynomials \( L_n^\alpha(x) \), \( x = y^2 \), constitute a complete orthogonal polynomial basis associated with the weight function \( x^\alpha e^{-x} \). Thus, multiplying Eq. (55) by \( Y^{0}_\ell(\omega_k) y^{\ell} L_n^{\ell+1/2}(y^2) e^{-y^2} \) and integrating over \( y \) in the range \( 0 \leq y \leq \infty \) and over \( \omega_k \) we obtain the expansion coefficient occurring in Eq. (51)

\[
\langle [L]|n, \ell \rangle = (-1)^{\ell/2} \frac{2}{\pi^{3/2} \Gamma(L + (D - 3)/2 - n + \ell/2) \Gamma(n + \ell + 3/2)} \\
\times \int \mathcal{D}(y, \omega_k) Y^{0}_\ell(\omega_k) y^{\ell} L_n^{\ell+1/2}(y^2) y^2 \, dy \, d\omega_k
\]

According to Eq. (55), the Fourier transform is a polynomial in \( Y^{0}_\ell(\omega_k) y^{2\nu+\ell} \) which can also be written as

\[
\mathcal{D}(y, \omega_k) = \sqrt{4\pi} \sum_{\nu, \lambda} (-1)^{\lambda/2} \langle [L]|\nu, \lambda \rangle Y^{0}_\lambda(\omega_k) y^{2\nu+\lambda} e^{-y^2}
\]
where \(([L]|\nu, \lambda)\) are new coefficients to be determined. Introducing (57) in (56) and using the analytical expression

\[
\int_0^\infty y^{2(\nu+\ell)} L_n^{\ell+1/2}(y^2) e^{-y^2} \, dy = \frac{(-1)^n}{2} \binom{\nu}{n} \Gamma(\nu + \ell + 3/2)
\]  

we find the relation between the two kinds of coefficients \(\langle [L]|n, \ell \rangle\) and \(( [L]|\nu, \ell \rangle\)

\[
\langle [L]|n, \ell \rangle = 2 \pi \frac{\Gamma(L + D/2)}{\Gamma(L + (D - 3)/2 - n - \ell/2)} \times \sum_{\nu \geq n} \binom{\nu}{n} \Gamma(\nu + \ell + 3/2) ([L]|\nu, \ell \rangle .
\]  

Introducing (59) in (51) with \(\xi_N = r \cos \phi, \rho = r \sin \phi\), and \(z = \cos 2\phi\), dividing by \(r^{2L}\), and using the definition of the Jacobi polynomials \(P^{\alpha, \beta}_n(x)\) [23] one obtains

\[
\int |D_{[L]}(\Omega)|^2 \, d\Omega_{N-1} = \frac{2}{\pi} \frac{\Gamma(L + D/2)}{\Gamma(L + (D - 3)/2 - n - \ell/2)} \sum_{n, \ell \ even} \frac{(-1)^n n!}{\Gamma(L + (D - 3 - \ell)/2)} \times \langle [L]|n, \ell \rangle Y_0^0(\omega)(\cos \phi)^\ell (\sin \phi)^{2L-2n-\ell} P^{\alpha-n-\ell/2, \beta}(\cos 2\phi)
\]  

where \(\alpha = L + (D - 5)/2\) and \(\beta = \ell + 1/2\).

Since \(\ell\) is even while \(\sin^2 \phi = (1 - z)/2\) and \(\cos^2 \phi = (1 + z)/2\) the sum over \(\ell\) and \(n\) generates a polynomial in \(z\). We have now two expressions at our disposal for \(W_{[L_m]}^{(D)}(z, \omega)\) occurring in the weight function (29) according to whether one uses \(\langle [L_m]|n, \ell \rangle\) or \(( [L_m]|n, \ell \rangle\) coefficients. In the first case

\[
W_{[L_m]}^{(D)}(z, \omega) = \frac{1}{2^{L_m}} \sum_{n, \ell} \langle [L_m]|n, \ell \rangle Y_0^0(\omega)(1 + z)^{n+\ell/2} (1 - z)^{L_m-n-\ell/2}
\]  

and in the second

\[
W_{[L_m]}^{(D)}(z, \omega) = \frac{1}{2^{L_m} \pi} \sum_{n, \ell} (-2)^n n! \frac{\Gamma(L_m + D/2)}{\Gamma(L_m + (D - 3 - \ell)/2)} \langle [L_m]|n, \ell \rangle
\]

\[
\times (1 - z)^{L_m-n-\ell/2} (1 + z)^{\ell/2} Y_0^0(\omega) P^{\alpha-n-\ell/2, \beta+1/2}(z)
\]  

In order to find explicit expressions for the coefficients occurring in either (61) or (62) we have to calculate the Fourier transform, Eq. (50), where \(D_{[L]}^{\text{HO}}(\vec{x})\) is the IPM representation of the Harmonic Polynomial \(D_{[L]}(\vec{x})\), Eq. (44). For
this purpose we first write
\[ e^{i\vec{k} \cdot \vec{x}_N} = e^{i\vec{k} \cdot \vec{x}_1} e^{-i\vec{k} \cdot \vec{x}_2} \]
and then develop the normalized Slater determinant \( D^{\text{HO}}_{[L_m]}(\vec{x}) \) in Eq. (44) with respect to the first two rows for \( \vec{x}_1 \), and \( \vec{x}_2 \)
\[ D^{\text{HO}}_{[L]}(\vec{x}) = \frac{1}{A(A-1)} \sum_{i,j<i} d_{ij}(\vec{x}_1, \vec{x}_2) D^{\text{HO}}_{ij}(\vec{x}_3, \cdots, \vec{x}_A) \] (63)

where
\[ d_{ij}(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} s^i_1 t^i_1 \psi_i(\vec{x}_1), & s^j_1 t^j_1 \psi_j(\vec{x}_1) \\ s^i_2 t^i_2 \psi_i(\vec{x}_2), & s^j_2 t^j_2 \psi_j(\vec{x}_2) \end{vmatrix} \]
Since \( D^{\text{HO}}_{ij} \) and \( D^{\text{HO}}_{k\ell} \) do not contain the same HO individual state except for \( i = k \) and \( j = \ell \) we have
\[ \langle D^{\text{HO}}_{ij} | D^{\text{HO}}_{k\ell} \rangle = \delta_{ik} \delta_{j\ell} \] (64)

Integration over all \( \vec{x}_i \) for \( i > 2 \) leads to
\[ \int |D^{\text{HO}}_{[L]}(x)|^2 \, d^3x_3 \cdots d^3x_A = \frac{1}{A(A-1)} \sum_{i,j>i} |d_{ij}(\vec{x}_1, \vec{x}_2)|^2 \] (65)

where
\[ d_{ij}(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} |i_1\rangle & |j_1\rangle \\ |i_2\rangle & |j_2\rangle \end{vmatrix} \]
and
\[ \int |d_{ij}(\vec{x}_1, \vec{x}_2)|^2 \, d^3x_1 \, d^3x_2 = 2 \cdot \]
Therefore,
\[ \int |D^{\text{HO}}_{[L]}(\vec{x})|^2 \, d^3x_1 \cdots d^3x_A = 1 \cdot \]
The \( d_{ii} \) contains identical columns and thus it vanishes. Further, since \( d_{ij}^* d_{ij} \) is invariant by exchange of \( i \) and \( j \), the sum over \( i \) and \( j > i \) can be transformed into a sum over all states independently of their position in the determinant \( D^{\text{HO}}_{[L]}(\vec{x}) \)
\[ \sum_{i,j>i} |d_{ij}(\vec{x}_1, \vec{x}_2)|^2 = \frac{1}{2} \sum_{i,j} |d_{ij}(\vec{x}_1, \vec{x}_2)|^2 \] (66)

For simplicity, we introduce a single set of quantum numbers \( \zeta \) for spin and
isospin states, labeled according to

\[ \zeta = +\frac{3}{2} \] for proton spin up
\[ \zeta = +\frac{1}{2} \] for proton spin down
\[ \zeta = -\frac{1}{2} \] for neutron spin up
\[ \zeta = -\frac{3}{2} \] for neutron spin down

where we count 1 or -1 for proton and neutron respectively and \( \frac{1}{2}, -\frac{1}{2} \) for spin up or down and we add the values to obtain \( \zeta \). In order to calculate the Fourier transform we do not have anymore to refer to the columns but only to the spin-isospin and space occupied states. Then a sum over all pairs of occupied states is substituted for the sum over \( i \) and \( j > i \) in Eq. (65).

Let us consider the pair of states \( \zeta_\alpha |\alpha\rangle \) and \( \zeta_\beta |\beta\rangle \) where \( |\alpha\rangle \) and \( |\beta\rangle \) stand for the HO eigenfunctions (45) with quantum numbers \( (n_\alpha, \ell_\alpha, m_\alpha) \) and \( (n_\beta, \ell_\beta, m_\beta) \). Then one may distinguish two cases in the calculation of the Fourier transform

\[ F_{\alpha\beta} = \langle d_{\alpha\beta}^*(12) | e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} | d_{\alpha\beta}(12) \rangle . \] (67)

In the first case one defines the direct term

\[ D_{\alpha\beta}(\vec{k}) = \langle \alpha | e^{i\vec{k} \cdot \vec{x}} | \alpha \rangle \langle \beta | e^{-i\vec{k} \cdot \vec{x}} | \beta \rangle + \{ \vec{k} \rightarrow -\vec{k} \} \] (68)

and the exchange term

\[ E_{\alpha\beta}(\vec{k}) = \langle \alpha | e^{i\vec{k} \cdot \vec{x}} | \beta \rangle \langle \beta | e^{-i\vec{k} \cdot \vec{x}} | \alpha \rangle + \{ \vec{k} \rightarrow -\vec{k} \} \] (69)

where \( \vec{k} \rightarrow -\vec{k} \) means the first term in which \( \vec{k} \) is changed to \( -\vec{k} \). These Fourier transforms are real and even functions of \( \vec{k} \) and thus they contain \( Y_\ell^0(\omega) \) for even \( \ell \) only, where \( \ell > 0 \) is related to the deformation of nuclei.

In Nuclear interactions spin and isospin can be exchanged between two nucleons \( i \) and \( j \). The exchange operators are \( P^{(\epsilon)}_{ij} \) where \( \epsilon = 0 \) without exchange while \( \epsilon = \sigma, \tau, \sigma\tau \) where spin, isospin or spin and isospin are exchanged respectively. These operators are traditionally known as Wigner (\( \epsilon = 0 \), Bartlett (\( \epsilon = \sigma \)), Heisenberg (\( P^{H}_{ij} = -P^{\tau}_{ij} \)), and Majorana (\( P^{M}_{ij} = -P^{\sigma\tau}_{ij} \)), exchanged operators. In the above, the Fourier transform has been calculated without exchanged operators i.e for \( \epsilon = 0 \) only. This case corresponds to the Wigner force for which the weight function is given by Eq. (29) with (61) or (62). Therefore, we have still to calculate the so-called pseudo-weight functions associated with the operators \( P^{(\epsilon)}_{ij} \), where \( \epsilon = \sigma, \tau, \) or \( \sigma\tau \).

To evaluate the contribution brought by the direct and exchange terms \( D_{\alpha\beta} \)
and $E_{\alpha\beta}$, to the Fourier transform

$$F^{(\epsilon)}_{\alpha\beta} = \langle d_{\alpha\beta}^* (12) | e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} P^{(\epsilon)}_{12} | d_{\alpha\beta} (12) \rangle ,$$

including the exchange operator $P^{(\epsilon)}_{12}$ for $\epsilon = \sigma, \tau, \sigma\tau$, we have to distinguish four cases:

1) The spin-isospin states $\zeta_{\alpha}$ and $\zeta_{\beta}$ are the same, i.e. $\zeta_{\alpha} - \zeta_{\beta} = 0$.
2) The spin-isospin states are different, i.e. $\zeta_{\alpha} \neq \zeta_{\beta}$ but the $P^{(\epsilon)}_{ij}$ operator does not modify the spin-isospin states.
3) The operator $P^{(\epsilon)}_{ij}$ exchanges the two spin-isospin states.
4) The operator $P^{(\epsilon)}_{ij}$ generates new spin-isospin states orthogonal to the original $\zeta_{\alpha}$ and $\zeta_{\beta}$ states.

In terms of the Kronecker symbol $\delta_{ab} = 1$ for $a = b$ and zero otherwise, the contribution in the first case is

$$\delta_{\zeta_{\alpha}\zeta_{\beta}} (D_{\alpha\beta} - E_{\alpha\beta}) ,$$

in the second

$$(1 - \delta_{\zeta_{\alpha}\zeta_{\beta}}) D_{\alpha\beta} ,$$

in the third

$$-(1 - \delta_{\zeta_{\alpha}\zeta_{\beta}}) E_{\alpha\beta} ,$$

while in the last is zero. Applying these rules one finds that for the Wigner operator $P^{(0)}_{12} = 1$ we have the Fourier transform

$$F^{(0)}_{\alpha\beta} = D_{\alpha\beta} - \delta_{\zeta_{\alpha}\zeta_{\beta}} E_{\alpha\beta} .$$

Instead of considering the $P^{\sigma}_{12}$ and $P^{\tau}_{12}$ operators separately, it is more convenient to use the combinations $1/2(P^{\sigma}_{12} \pm P^{\tau}_{12})$ leading to the expression

$$F^{(\epsilon)}_{\alpha\beta} = \frac{1}{2} (1 - \delta_{(\zeta_{\alpha} + \zeta_{\beta})0}) \left\{ (1 + \delta_{\zeta_{\alpha}\zeta_{\beta}}) \frac{1}{2} (\delta_{\epsilon\sigma} + \delta_{\epsilon\tau}) \left[ D_{\alpha\beta} - E_{\alpha\beta} \right] ight. \\
- \left.(-1)^{\zeta_{\alpha} + \zeta_{\beta}} (1 - \delta_{\zeta_{\alpha}\zeta_{\beta}}) \frac{1}{2} (\delta_{\epsilon\sigma} - \delta_{\epsilon\tau}) \left[ D_{\alpha\beta} + E_{\alpha\beta} \right] \right\},$$

where $\epsilon = \sigma$ or $\epsilon = \tau$. Finally, for the spin-isospin exchange operator $\sigma\tau$ we have

$$F^{(\sigma\tau)}_{\alpha\beta} = \delta_{\zeta_{\alpha}\zeta_{\beta}} D_{\alpha\beta} - E_{\alpha\beta} .$$

From the above analysis we see that in order to calculate the weight and pseudo-weight functions associated with the operators occurring in realistic
nuclear potentials one needs to know the Fourier transform of the product of
two individual HO eigenfunctions. Indeed, since we have to deal with a two-
body weight function, which depends on \( z = 2r_{ij}^2/r^2 - 1 \) (here \((i, j) = (1, 2))
the Fourier transform of the product of two individual HO eigenfunctions
occur twice for the states \(|\alpha>\) and \(|\beta>\) in the calculation of the direct and
exchange terms in Eqs. (68) and (69).

5.1 Weight and Pseudo-Weight Functions: General Case

Let us define the Fourier transform

\[
\int \psi_{n_\alpha,\ell_\alpha,m_\alpha}^* e^{i\vec{k} \cdot \vec{x}} \psi_{n_\beta,\ell_\beta,m_\beta} d^3x = \langle \alpha | e^{i\vec{k} \cdot \vec{x}} | \beta \rangle
\]

where \(|\alpha>\) and \(|\beta>\) refer, respectively, to the HO eigenstates with quantum
numbers \(n_\alpha, \ell_\alpha, m_\alpha\) and \(n_\beta, \ell_\beta, m_\beta\) in the HO wave function (45). The Fourier
transform is the sum over the quantum numbers \(n_\alpha, \ell_\alpha, m_\alpha\)

\[
\langle \alpha | e^{i\vec{k} \cdot \vec{x}} | \beta \rangle = \sqrt{\frac{4}{\pi n \ell}} \sum_{m} \langle \alpha|n_\ell,m \beta \rangle Y_{m}^{\ell}(\omega_k) y^{2n+\ell} e^{-y^2/2}
\]

where \(m = m_\beta - m_\alpha\). The coefficients \((\alpha|n_\ell,m \beta \rangle\), normalized according to
\((\alpha|0_0,0 \beta \rangle = \delta_{\alpha\beta}\), are given in the appendix. For \(\beta = \alpha\) the simplified notation
\((\alpha|n_\ell,m_\alpha \rangle = (\alpha|n_\ell, \alpha \rangle\) is also used with \((\alpha|00) = 1\).

The calculation of the Fourier transform (57) depends on the direct and ex-
change terms \(D_{\alpha\beta}(\vec{k})\) and \(E_{\alpha\beta}(\vec{k})\). However, the question of normalization must
be addressed first. In Ref. [24] it was shown that for closed shell nuclei the
coefficient \((|L|\nu_\lambda \rangle\) occurring in (57) and in the weight function written as in
Eq. (62) are simple and most of them integers and the normalization in Eq.
(49) is \(A(A-1)/2\) instead of 1. To obtain afterwards the normalization to 1
obviously one divides by \(A(A-1)/2\). To obtain these coefficients one must
calculate the Fourier transform for the expression

\[
\mathcal{F} = \frac{1}{4} \sum_{i,j} \langle d_{ij} | e^{i\vec{k} \cdot (\vec{x}_1-\vec{x}_2)} | d_{ij} \rangle
\]

where the factor 4 stems from the normalization to \(A(A-1)/2\) of this expres-
sion for \(k = 0\) and from the fact that \(d_{ii} = 0\) \(d_{ij} = -d_{ji}\) with \(\langle d_{ij} | d_{ij} \rangle = 2\).

Using Eq. (75) and the expansion of the product of two spherical harmonics

\[
Y_{\ell_1}^{m_1}(\omega_k)Y_{\ell_2}^{m_2}(\omega_k) = \sum_{\lambda} \langle Y_{\ell_1}^{m_1} | Y_{\ell_2}^{m_2} \rangle Y_{\lambda}^{\mu}(\omega_k),
\]

27
with $\mu = m_1 + m_2$, the direct term, Eq. (68), for the couple of states $|\alpha> \text{ and } |\beta>$ with $m_1 = m_2 = 0$ becomes

$$D_{\alpha\beta}(\vec{k}) = \sqrt{4\pi} \sum_{\lambda,\nu} (-1)^{\lambda/2} d(\alpha, \beta|\nu, \lambda) Y^0_\lambda(\omega_k)y^{2\nu+\lambda} e^{-y^2}$$  \hspace{0.5cm} (78)$$

where the direct coefficient

$$d(\alpha, \beta|\nu, \lambda) = \sqrt{4\pi} \sum_{n_1 \ell_1 \atop n_2 \ell_2} (-1)^{(\ell_1 + \ell_2 - \lambda)/2} (\alpha|n_1, \ell_1)(\beta|n_2, \ell_2) \langle Y^0_\lambda|Y^0_{\ell_1}|Y^0_{\ell_2}\rangle$$  \hspace{0.5cm} (79)$$

for $\nu = n_1 + n_2 + (\ell_1 + \ell_2 - \lambda)/2$ where $\ell_1, \ell_2$, and $\lambda$ are even and $|\ell_1 - \ell_2| \leq \lambda \leq \ell_1 + \ell_2$. A factor of 2, stemming from the two terms in (68) for $\vec{k}$ and $\vec{k} \rightarrow -\vec{k}$, has also been taken into account. One notices that for $n_1 = n_2 = \ell_1 = \ell_2 = 0$ for which $\nu = \lambda = 0$ the normalization $d(\alpha, \beta|00) = 1$ leads to $D_{\alpha\beta}(\vec{k} = 0) = 1$.

A similar procedure is applied to obtain the exchange term

$$E_{\alpha\beta}(\vec{k}) = \sqrt{4\pi} \sum_{\lambda,\nu} (-1)^{\lambda/2} e(\alpha, \beta|\nu, \lambda) Y^0_\lambda(\omega_k)y^{2\nu+\lambda} e^{-y^2}$$  \hspace{0.5cm} (80)$$

where the exchange coefficient

$$e(\alpha, \beta|\nu, \lambda) = \sqrt{4\pi} \sum_{n_1 \ell_1 \atop n_2 \ell_2} (-1)^{(\ell_1 + \ell_2 - \lambda)/2} (\alpha|n_1, \ell_1)(\beta|n_2, \ell_2) \langle Y^0_\lambda|Y^m_{\ell_1}|Y^m_{\ell_2}\rangle$$  \hspace{0.5cm} (81)$$

for $\nu = n_1 + n_2 + (\ell_1 + \ell_2 - \lambda)/2$, $\ell$ even, $m_1 = m_\beta - m_\alpha$, $m_2 = -m_1$ and the normalization $e(\alpha, \beta|0, 0) = \delta_{\alpha\beta}$

Let us come back to the Fourier transform, Eqs. (67) and

$$\int \mathcal{D}^{HO\ast}_L(x) e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} D^{HO}_L(\vec{x}) d^3A_x = \frac{1}{4} \sum_{\text{all occupied states}} F_{\alpha\beta}$$  \hspace{0.5cm} (82)$$

for the normalization (76) where the wave function is normalized to $A(A-1)/2$ as explained above. Using (78) and (80) one finds, by identification with (57), the coefficient for $\varepsilon = 0$ i.e for $P^{(0)}_{ij} = 1$ in Eq. (71),

$$([L]|\nu, \lambda)^{(0)} = \frac{1}{2} \sum_{\text{all occupied states}} \left\{ d(\alpha, \beta|\nu, \lambda) - \delta_{\alpha\zeta, \beta} e(\alpha, \beta|\nu, \lambda) \right\}$$  \hspace{0.5cm} (83)$$
where the sum is taken independently over all HO occupied states in the Slater determinant $D_{[L]}^{HO}(\vec{x})$ defining the state under consideration. This coefficient, introduced in Eq. (62), defines the weight function

$$W_{[L_m]}(z, \omega) \equiv W_{[L_m]}^{(0)}(z, \omega) = W_0(z)W_{[L_m]}^{(D)}(z, \omega)$$

according to Eq. (29). The upper index (0) is to recall that we are dealing with a Wigner potential.

For the pseudo-weight function corresponding to the spin, isospin, and spin-isospin exchange operators $P_{ij}^{\epsilon}$ respectively for $\epsilon = \sigma$, $\tau$, $\sigma\tau$ one uses Eq. (72) and (73) leading to the new spin and isospin coefficients

$$([L]|\nu, \lambda)^{\sigma}(\epsilon) = \frac{1}{2} \sum_{\text{all occupied states}} \frac{1}{2}(1 - \delta_{\zeta_\alpha + \zeta_\beta}) \left\{ (1 + \delta_{\zeta_\alpha \zeta_\beta}) \frac{1}{2}(\delta_{\epsilon\sigma} + \delta_{\epsilon\tau}) \right. \times \left\{ (1 + \delta_{\zeta_\alpha \zeta_\beta}) \frac{1}{2}(\delta_{\epsilon\sigma} + \delta_{\epsilon\tau}) \left[ d(\alpha, \beta|\nu, \lambda) - e(\alpha, \beta|\nu, \lambda) \right] ight. 
\left. - (-1)^{\zeta_\alpha + \zeta_\beta}(1 - \delta_{\zeta_\alpha \zeta_\beta}) \frac{1}{2}(\delta_{\epsilon\sigma} - \delta_{\epsilon\tau}) \left[ d(\alpha, \beta|\nu, \lambda) + e(\alpha, \beta|\nu, \lambda) \right] \right\} \right) \quad (84)$$

for the Bartlett ($\epsilon = \sigma$) and Heisenberg ($\epsilon = \tau$) forces and to

$$([L]|\nu, \lambda)^{\sigma\tau} = \frac{1}{2} \sum_{\text{all occupied states}} \left\{ \delta_{\zeta_\alpha \zeta_\beta} \times d(\alpha, \beta|\nu, \lambda) - e(\alpha, \beta|\nu, \lambda) \right\} \quad (85)$$

for the Majorana force ($\epsilon = \sigma\tau$). The weight and pseudo-weight functions are thus defined by

$$W^{(\epsilon)}_{[L_m]}(z, \omega) = W_0(z)W_{[L_m]}^{(D,\epsilon)}(z, \omega), \quad \epsilon = 0, \sigma, \tau, \sigma\tau \quad (86)$$

where in Eq. (62) one uses $([L_m]|n, \ell)^{\epsilon}$ instead of $([L_m]|n, \ell)$ and with normalization

$$\int |D_{[L_m]}^{HO}(x)|^2 dx^{3A} = \int W_{[L_m]}^{(0)} dz d\omega = \frac{A(A - 1)}{2} \quad (87)$$

### 5.2 Weight Function for Coulomb Potential

The Coulomb interaction is of Wigner type i.e., it does not contain any spin-isospin exchange operator, the interaction being considered only between charged
particles, in our case the protons. The projection operator applied to \( d_{\alpha\beta}(1, 2) \) is

\[
P_{12}^{(c)} d_{\alpha\beta}(1, 2) = \frac{1}{4} (1 + \text{sgn} \, \zeta_{\alpha})(1 + \text{sgn} \, \zeta_{\beta}) d_{\alpha\beta}(1, 2)
\]

which cancels the neutron states.

In the Fourier transform (70) with \( \varepsilon = c \) (for Coulomb) only the first two cases contribute. Using the same procedure as for the other projection operators \( P_{ij}^{(c)} \) the two-body coefficients are given by

\[
\langle [L]|\nu, \lambda \rangle^{c} = \frac{1}{8} \sum_{\text{all occupied states}} (1 + \text{sgn}(\zeta_{\alpha}))(1 + \text{sgn}(\zeta_{\beta}))
\]

\[
\times \left\{ d(\alpha, \beta|\nu, \lambda) - \delta_{\zeta_{\alpha}\zeta_{\beta}} e(\alpha, \beta|\nu, \lambda) \right\}
\]

(89)

The hypercentral potential for the Coulomb interaction can be obtained directly from the Fourier transform of the Coulomb weight function [24] with

\[
v(k) = \frac{e^2}{2\pi^2 k^2}
\]

(90)

the result being

\[
V_{[L_{m}]}^{c}(r) = \frac{2}{\pi^2} \frac{\Gamma(L_{m} + D/2)}{\Gamma(L_{m} + (D - 1)/2)} \sum_{n=0}^{2\ell_{m}} \Gamma(n + 1/2)(2I_{0}(n) - I_{\alpha\beta}(n))
\]

(91)

for closed shell nuclei. The coefficients \( I_{0}(n) \) and \( I_{\alpha\beta}(n) \) [24] are given in appendix B.

5.3 Weight and Pseudo-Weight Functions for Closed Shell Nuclei

The various ingredients occurring in the calculation of the weight function \( W^{(c)}(z, \omega) \) have been defined in terms of the spin-isospin and the eigenstates occupied in the state determinant \( D_{L_{m}}^{HO}(\vec{x}) \) describing the ground harmonic defining the state of the nucleus. The \( \langle \alpha|n, \ell|\beta \rangle \) coefficients are known analytically. Therefore, it is only a matter of computer programming to get the the needed \( W^{(c)}(z, \omega) \) functions. Nevertheless, analytical calculations can be pushed further when we have to deal with closed shell or sub-shell nuclei by using the summation over the spherical harmonics. The analytical expressions are interesting because the nucleus is constituted by a core of nucleons in
closed shells and by other nucleons outside the core. The core is spherical with a total angular momentum $J = 0$. The particular state of the nucleus is defined by the configuration of the last open shell. The analytical derivation of the coefficients for closed shells are given in appendix B.

Since closed shell nuclei are spherical only terms with $\ell = 0$ appear in (61) and (62). A shell $\Lambda_\alpha$ is defined by the value $\Lambda_\alpha = 2n_\alpha + \ell_\alpha$ in terms of the HO quantum numbers of the occupied state. The contribution to the direct term in Eqs. (83-86) for each spin-isospin state when all the shells for which $\Lambda_\alpha = 2n_\alpha + \ell_\alpha$ are filled from $\Lambda_\alpha = 0$ to the last shell $\Lambda_\alpha = \ell_m$ is [24]

$$I_0(\ell_m, n) = \frac{(-1)^n}{n!} \sum_{p=0}^{n} \binom{n}{p} \binom{\ell_m + 3}{p + 3} \binom{\ell_m + 3}{n + 3 - p}$$

(92)

By taking the four spin-isospin states and the normalization (87) into account, the contribution of the direct term in Eqs. (83-86) for closed shell $N = Z$ nuclei is [24]

$$([L_m]|\nu, 0)_d^e = 2I_0(\ell_m, n) \left[ 4\delta_{0\epsilon} + 2(\delta_{\sigma\epsilon} + \delta_{\tau\epsilon}) + \delta_{(\sigma\tau)\epsilon} \right]$$

(93)

where $\epsilon = 0, \sigma, \tau, \sigma\tau$.

The contribution coming for the exchange term, denoted by $I_{\alpha\beta}$, cannot be obtained analytically. However, the procedure for its computation is explained in the appendix. For closed shells with $N = Z$ spherical nuclei one has the relation

$$([L_m]|\nu, 0)_{\alpha\beta}^e = 2 \left\{ I_0(\ell_m, \nu) \left[ 4\delta_{0\epsilon} + 2(\delta_{\sigma\epsilon} + \delta_{\tau\epsilon}) + \delta_{(\sigma\tau)\epsilon} \right] ight. - I_{\alpha\beta}(\ell_m, \nu) \left[ \delta_{0\epsilon} + 2(\delta_{\sigma\epsilon} + \delta_{\tau\epsilon}) + 4\delta_{(\sigma\tau)\epsilon} \right] \right\}$$

(94)

for the normalization (87).

For the Coulomb case, where only the proton states are taken into account in the calculation for $I_0$ and $I_{\alpha\beta}$, we have the relation

$$([L]|\nu, \lambda)^e = 2I_0(\nu) - I_{\alpha\beta}(\nu)$$

(95)

A more compact expression for the weight functions $W^{(e)}(z)$ can be obtained by using the Rodrigues’ formula (Ref. [23], Vol II p. 169)

$$(-2)^n n! P_n^{a,b}(x) = (1 - x)^{-a}(1 + x)^{-b} D_x^n \left[ (1 - x)^{a+n}(1 + x)^{b+n} \right]$$

(96)
where \( D^n_x \) denotes the operator \( D^n_x = d^n / dx^n \), \( a = L_m + (D - 5)/2 - n \), and \( b = 1/2 \). Then, for \( \ell = 0 \) Eq. (86) becomes, after integrating Eq. (62) over \( \omega \),

\[
W^{(e)}_{[L_m]}(z) = \frac{2^{-(L_m + D/2 - 2)}}{\sqrt{\pi}} \frac{\Gamma(L_m + D/2)}{\Gamma(L_m + (D - 3)/2)} \times \sum_{n=0}^{2\ell_m} ([L_m]|n, 0)^{(e)} D^n_z [(1 - z)^{L_m + (D - 5)/2} (1 + z)^{n + 1/2}] .
\]  

(97)

Alternatively we may write

\[
W^{(e)}(z) = \frac{4}{\sqrt{\pi}} \left(\frac{1}{2}\right)^{L_m + D/2} \frac{\Gamma(L_m + D/2)}{\Gamma(L_m + (D - 3)/2)} (1 - z)^{\alpha}(1 + z)^{\beta} \times \sum_{n=0}^{2\ell_m} (-2)^n n! ([L_m]|n, 0)^{(e)} (1 - z)^{2\ell_m - n} P_{n}^{L_m + (D - 5)/2 - n, \beta}(z)
\]  

(98)

The weight function can thus be written as a product

\[
W^{(e)}_{[L_m]}(z) = (1 - z)^{\alpha}(1 + z)^{\beta} \rho_{[L_m]}^{(e)}(z)
\]  

(99)

where now \( \alpha = L_m + (D - 5)/2 - 2\ell_m \), \( \beta = 1/2 \), \( \ell_m \) refers to the sum \( 2n + \ell \) of radial and orbital quantum numbers in the last shell, and \( \rho_{[L_m]}^{(e)}(z) \) is a polynomial of degree \( 2\ell_m \),

\[
\rho_{[L_m]}^{(e)}(z) = C^{(e)} \sum_{n=0}^{2\ell_m} (-2)^n n! ([L_m]|n, 0)^{(e)} (1 - z)^{2\ell_m - n} P_{n}^{L_m + (D - 5)/2 - n, \beta}(z)
\]  

(100)

The normalization of (97) is given by the first term, i.e. for \( n = 0 \),

\[
\int_{-1}^{+1} W^{(e)}_{[L_m]}(z) \, dz = ([L_m]|0, 0)^{(e)}
\]  

(101)

where we used the fact that for \( n > 0 \)

\[
\int_{-1}^{+1} D^n_z [(1 - z)^{\alpha}(1 + z)^{n + 1/2}] \, dz = 0 .
\]

The \( \rho_{[L_m]}^{(e)}(z) \), being a polynomial of degree \( 2\ell_m \), can be expressed in terms of its roots

\[
\rho_{[L_m]}^{(e)}(z) = C^{(e)} \prod_{n=1}^{2\ell_m} (z - z_m)
\]  

(102)
which can simplify the numerical calculations.

A similar expression can be obtained in terms of the variable $X = (1 + z)/2 = r^2_{ij}/r^2$, $(i, j) = (1, 2)$, for $0 \leq X \leq 1$. By substitution one gets

$$W^{(e)}_{[L_m]}(z) \, dz = \frac{2}{\sqrt{\pi}} \frac{\Gamma(L_m + D/2)}{\Gamma(L_m + (D - 3)/2)} \times \sum_n ([L_m] | n, 0)^{(e)} D^n_X ((1 - X)^{L_m + (D - 5)/2} X^{n+1/2}) \, dX \quad (103)$$

Analytical formulas for the hyperradial part of the two-body potentials written in terms of $X$ can be obtained from (103) since

$$\int_0^1 f(X) D^n_X \left[ (1 - X)^{\alpha} X^{n+1/2} \right] \, dX = (-1)^n \int_0^1 (1 - X)^{\alpha} X^{n+1/2} \frac{d^n f(X)}{dX^n} \, dX \quad (104)$$

The integral can be obtained analytically when $f(X)$ is either a polynomial or an exponential or the product of both.

6 The Projection Function

We have seen that in order to have an equation for the calculation of the two-body correlations, using Eq. (24) one must extract from the amplitude $F(r_{k\ell}, r)$ where the pair particles $(k, \ell)$ is not the reference pair $(i, j)$, the part of $F(r_{k\ell}, r)$ which, nevertheless, depends on $r_{ij}$. There are two cases: Either one of the $(k, \ell)$ is either $i$ or $j$, in which case we have a connected pairs like $(i, k)$ or $(j, k)$ where $k \neq i$ or $j$; or $k$ and $\ell$ are neither $i$ nor $j$ and we have disconnected pairs. The procedure is the following i) Expand $F(r_{k\ell}, r)$ in terms of HPs which are polynomials in $r^2_{k\ell}$ and $r^2$ (which contains $r^2_{ij}$). ii) Write $r^2_{k\ell}$ in terms of the kinematical rotation vector (A.3) [6,25]. iii) Extract from each polynomial the part which depends on $r^2_{ij}$ and ignore the residual part which contains many body correlations [8].

Practically since the two-body amplitude in Eq. (34) is expressed in terms of the $z = \cos 2\phi = 2r^2_{ij}/r^2 - 1$ and $r^2$ (for $(i, j) = (1, 2)$), the expansion is done for any pair in terms of Potential Harmonics (PH) for pairs in $S$-state. They have been designed in order to provide a complete expansion basis for pairs of particles in $S$-state. For bosons in ground state the PH are the Jacobi polynomials $P_K^{(D-5)/2,1/2}(z)$ associated with the weight function $W_0(z)$ (see Eq. (A.8)).

Let $z = \cos 2\phi = 2r^2_{ij}/r^2 - 1$ for the reference pair $(i = 1$, and $j = 2$ in Eq.
(27)) and \( z_{k\ell} = 2r_{k\ell}^2/r^2 - 1 \) for another pair. The two-body amplitude can be expanded as

\[
P(z_{k\ell}, r) = \sum_{K=0}^{\infty} \left\{ \frac{1}{h^\alpha,\beta_K} \int_{-1}^{+1} (1 - z')^\alpha (1 + z')^\beta P^\alpha,\beta_K(z') P^\alpha,\beta(z_{k\ell}) \right\} P(z', r) \, dz' \tag{105}
\]

where \( P^\alpha,\beta_K(z) \) are Jacobi polynomials associated with the weight function \((1 - z)^\alpha (1 + z)^\beta\) with \( \alpha = (D - 5)/2 \) and \( \beta = 1/2 \). The \( h^\alpha,\beta_K \) is the normalization constant for the Jacobi polynomials

\[
h^\alpha,\beta_K = \frac{2^{\alpha+\beta+1} \Gamma(K+\alpha+1) \Gamma(K+\beta+1)}{\Gamma(2K+\alpha+\beta+1) K! \Gamma(K+\alpha+\beta+1)} \tag{106}
\]

The expression in the braces in (105) is the \( \delta \)-function \( \delta(z' - z_{k\ell}) \).

It was shown in Ref. [6] that for the connected pairs where \( z_c = z_{ik} \) or \( z_c = z_{jk} \), \( k \neq i \) or \( j \), and for equal mass particles \( P^\alpha,\beta_K(z_c) \) can be separated into two terms

\[
P^\alpha,\beta_K(z_c) = \frac{P^\alpha,\beta_K(-1/2)}{P^\alpha,\beta_K(+1)} P^\alpha,\beta(z) + \text{other terms} \tag{107}
\]

and for disconnected pairs \( z_d = z_{k\ell}, k, \ell \neq i, j \),

\[
P^\alpha,\beta_K(z_d) = \frac{P^\alpha,\beta_K(-1)}{P^\alpha,\beta_K(+1)} P^\alpha,\beta(z) + \text{other terms} \tag{108}
\]

Since \( P^\alpha,\beta_K(z) \) is a HH we have \( \Delta r^{2K} P^\alpha,\beta_K(z) = 0 \) whatever the pair is. Therefore, \( \Delta [r^{2K} \times \{ \text{other terms} \}] = 0 \) and the residual part is also a HH orthogonal to \( P^\alpha,\beta_K(z) \). Since only \( P^\alpha,\beta_K(z) \) contains two-body correlations, the other terms are related to many-body correlations that we neglect. Finally, the projection function for pairs in \( S \)-states is the sum of the projection functions of the \( 2(A - 2) \) connected pairs and the \( (A - 2)(A - 3)/2 \) disconnected pairs. Thus we may write

\[
\mathcal{P}^0 P(z, r) = \int_{-1}^{+1} f_{0|}(z, z') P(z', r) \, dz' \tag{109}
\]

with
\[ f_{[0]}(z, z') = (1 - z')^\alpha (1 + z')^\beta (A - 2) \sum_{K=0}^{\infty} \frac{2P_K^{\alpha,\beta}(-1/2) + \frac{A-3}{2}P_K^{\alpha,\beta}(-1)}{P_K^{\alpha,\beta}(1)} \times P_K^{\alpha,\beta}(z)P_K^{\alpha,\beta}(z')/h_K^{\alpha,\beta} \]  

(110)

An analytical expression of the projection function has been derived [8,13].

The projection of a potential harmonic for a pair \((k, \ell)\) on a reference pair \((i, j)\), both in \(S\)-state, is given by [6]

\[ P_0^{\alpha,\beta}(\cos 2\delta) \]

where \(\delta = 2\pi/3\) for connected pairs and \(\delta = \pi/2\) for disconnected pairs. The projection function in terms of the angular parameter \(\delta\) is for one pair

\[ f_{[0]}(z, z', \delta) = (1 - z')^\alpha (1 + z')^\beta \sum_{K=0}^{\infty} \frac{P_K^{\alpha,\beta}(\cos 2\delta)}{P_K^{\alpha,\beta}(1)}P_K^{\alpha,\beta}(z)P_K^{\alpha,\beta}(z')/h_K^{\alpha,\beta} \]  

(111)

The sum over the series, Eq. (111), can be carried out analytically [8,13] providing the projection function in terms of \(\cos \delta\) which is related to the choice of the pair to be projected where \(\delta = 2\pi/3\) or \(\delta = \pi/2\) for connected or disconnected pairs of equal mass particle. In terms of \(\cos \phi = r_{12}/r = \sqrt{(1 + z)/2}\) for the reference pair

\[
\int_{-1}^{+1} f_{[0]}(z, z', \cos 2\delta)P(z', r) \, dz' = \frac{4}{\sqrt{\pi}} \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda)} \frac{1}{\sin 2\delta \sin 2\phi} \\
\times \left[ \frac{1}{\sin \delta \sin \phi} \right]^{2(\lambda-1)} \int_{a}^{b} [(u - a)(b - u)]^{(\lambda-1)} P(2u^2 - 1) \, u \, du
\]

(112)

where \(a = \cos(\phi + \delta), b = \cos(\phi - \delta),\) and \(\lambda = \alpha + 1/2 = D/2 - 2\). For disconnected pairs since \(\sin 2\delta \to 0\) we have to take the limit of Eq. (112) for \(\delta \to \pi/2\) the result being

\[
\int_{-1}^{+1} f_{[0]}(z, z', -1)P(z', r) \, dz' = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda - 1)} (1 - z)^{1/2-\lambda} \\
\times \int_{-1}^{z} [-z + z')]^{\lambda-2}(1 + z')^{1/2} P(z', r) \, dz'
\]

(113)

For fermions and in particular for nuclei one follows the same procedure as for bosons in \(S\)-state. One starts from the weight function \(W_{[L_m]}(z)\) to find the
associated polynomials $P_{K}^{[L_m]}(z)$ which fulfill the normalization condition

$$
\int_{-1}^{+1} P_{K}^{[L_m]}(z) P_{K'}^{[L_m]}(z) W_{[L_m]}(z) \, dz = \delta_{K,K'}
$$

(114)

For bosons in ground states we have seen that they are the normalized Jacobi polynomials $P_{K}^{\alpha,\beta}(z)/\sqrt{P_{K}^{\alpha,\beta}}$. From $P_{K}^{[L_m]}(z)$ one generates the projection function

$$
f_{[L_m]}(z, z') = (A - 2) \sum_{K=0}^{\infty} \frac{2P_{K}^{[L_m]}(-\frac{1}{2}) + \frac{D-2}{2}P_{K}^{[L_m]}(-1)}{P_{K}^{[L_m]}(1)} \times P_{K}^{[L_m]}(z) P_{K}^{[L_m]}(z') W_{[L_m]}(z')
$$

(115)

where, whatever the normalization of $W_{[L_m]}(z)$ is, the polynomials $P_{K}^{[L_m]}(z)$ are normalized according to (114).

Let be $z_{1}, \ldots, z_{2\ell_m}$ the zeros of $\rho_{[L_m]}(z)$. According to the Christoffel’s formula [23], the polynomial $P_{K}^{[L_m]}(z)$ is given by the determinant

$$
P_{K}^{[L_m]}(z) = N_{K}^{[L_m]} / \rho_{[L_m]}(z)
$$

(116)

$$
\begin{vmatrix}
P_{K}^{\alpha,\beta}(z), & P_{K+1}^{\alpha,\beta}(z), & \cdots, & P_{K+n}^{\alpha,\beta}(z) \\
P_{K}^{\alpha,\beta}(z_{1}), & P_{K+1}^{\alpha,\beta}(z_{1}), & \cdots, & P_{K+n}^{\alpha,\beta}(z_{1}) \\
\vdots & \vdots & \vdots & \vdots \\
P_{K}^{\alpha,\beta}(z_{n}), & P_{K+1}^{\alpha,\beta}(z_{n}), & \cdots, & P_{K+n}^{\alpha,\beta}(z_{n})
\end{vmatrix}
$$

where $N_{K}^{[L_m]}$ is a normalization constant fixed by Eq. (114) and $P_{K}^{\alpha,\beta}(z)$ are Jacobi polynomials. Obviously for bosons in ground state where $L_m = 0$ one recovers the Jacobi polynomials for $\alpha = (D - 5)/2$ quoted in Eq. (105).

The expression of $P_{K}^{[L_m]}(z)$ in terms of a determinant, Eq. (116), presupposes that the roots of the polynomial $\rho_{[L_m]}(z) (\epsilon = 0)$ are known. An alternative way to obtain $P_{K}^{[L_m]}(z)$ is to use the moments formula [23]

$$
P_{K}^{[L_m]}(z) = C_{[L_m]}^{K}
$$

(117)

$$
\begin{vmatrix}
1, & (1 + z)/2, & \cdots, & ((1 + z)/2)^{K} \\
C_{0}, & C_{1}, & \cdots, & C_{K} \\
C_{1} & C_{2}, & \cdots, & C_{K+1} \\
\vdots & \vdots & \vdots & \vdots \\
C_{K-1} & C_{K}, & \cdots, & C_{2K-1}
\end{vmatrix}
$$
where the elements $C_K$ are given by

$$C_K = \int_{-1}^{+1} \left( \frac{1 + z}{2} \right)^K W_{[L_m]}(z) \, dz \quad (118)$$

and $C_{K[L_m]}$ is the normalization constant.

### 7 The Effective Nuclear Potential

The Eq. (34) for IDEA has been obtained from (27) by multiplying at left by $D^\ast_{[L_m]}(\Omega)$ and integrating over all angular coordinates $\Omega_{N-1}$ associated with the Jacobi coordinates $\xi_i$ for $i < N$, $(\xi_N = \bar{r}_{12})$ and then by dividing by the weight function $W_{[L_m]}(z)$. Eq. (34) thus obtained is valid for Wigner-type potentials, i.e., without taking into account the exchange operators. In general, however, nuclear potentials include exchange operators that give rise to pseudo-weight functions as well. Thus, the right hand side of Eq. (27), after integration over $d\Omega_{N-1}$ gives $\sum_\epsilon W^{(\epsilon)}_{[L_m]}(z, \omega) V^{(\epsilon)}(\xi_N)$ with $\xi_N = r_{12} = r \sqrt{(1+z)/2}$. For spherical nuclei the weight function does not contain the angular coordinates $\omega$ of $\xi_N$ and we may define

$$W^{(\epsilon)}_{[L_m]}(z) = \int W^{(\epsilon)}_{[L_m]}(z, \omega) \, d\omega \quad (119)$$

and the effective potential in the right hand side of Eq. (34) reads

$$V(\xi_N) \equiv V_{\text{eff}}(\xi_N) = \sum_{\epsilon=0,\sigma,\tau,\sigma\tau} V^{(\epsilon)}(\xi_N) W_{[L_m]}^{(\epsilon)}(z) / W_{[L_m]}^{(0)}(z) \quad (120)$$

while the effective hypercentral potential in Eq. (34) is

$$V_{[L_m]}(r) = \int_{-1}^{+1} W_{[L_m]}^{(0)}(z) V_{\text{eff}}(\xi_N) \, dz = \int_{-1}^{+1} \sum_{\epsilon} W_{[L_m]}^{(\epsilon)}(z) V^{(\epsilon)}(\xi_N) \, dz \quad (121)$$

The nuclear potentials are given in terms of the triplet or singlet for even or odd states. The following relations hold between the $V^{(\epsilon)}$, where $\epsilon = 0, \sigma, \tau, \sigma\tau$, and the $V^{3+}$, $V^{1+}$, $V^{3-}$, and the $V^{1-}$ potentials

$$V^0 = \frac{1}{4} \left[ V^{1+} + V^{3+} + V^{1-} + V^{3-} \right] \quad (122)$$
\[ V^\sigma = \frac{1}{4} \left[ -V^{1+} + V^{3+} - V^{1-} + V^{3-} \right] \quad (123) \]
\[ V^\tau = \frac{1}{4} \left[ V^{1+} - V^{3+} - V^{1-} + V^{3-} \right] \quad (124) \]
\[ V^{\sigma \tau} = \frac{1}{4} \left[ -V^{1+} + V^{3+} + V^{1-} + V^{3-} \right] \quad (125) \]

In terms of Wigner, Bartlett, Heisenberg and Majorana potentials, the following relations hold

\[ V^W = V^0, \quad V^B = V^\sigma, \quad V^H = -V^\tau, \quad V^M = -V^{\sigma \tau} \quad (126) \]

For the \( N = Z \) nuclei constructed from spin and isospin saturated HO states, like the \( \alpha \)-particle, the relation (94) still is valid and the effective potentials associated with the direct and exchange terms \( I_0 \) and \( I_{\alpha \beta} \) are

\[ V_{I_0} = \frac{3}{2} \left( V^{1+} + V^{3+} \right) + \frac{1}{2} V^{1-} + \frac{9}{2} V^{3-} \quad (127) \]

for \( I_0 \) and

\[ V_{I_{\alpha \beta}} = -\frac{3}{2} \left( V^{1+} + V^{3+} \right) + \frac{1}{2} V^{1-} + \frac{9}{2} V^{3-} \quad (128) \]

for \( I_{\alpha \beta} \).

The effective potential can be alternatively defined in terms of the polynomials

\[ \rho_{[L_m]}^{(D)}(z) = \sum_n 2^n n! I_0(n)(z - 1)^{2\ell_m - n} P_n^{\alpha-n,1/2}(z) \quad (129) \]

with \( \alpha = L_m + (D - 5)/2 \) and

\[ \rho_{[L_m]}^{(E)}(z) = \sum_n 2^n n! I_{\alpha \beta}(n)(z - 1)^{2\ell_m - n} P_n^{\alpha-n,1/2}(z) \quad (130) \]

the result being

\[ V_{\text{eff}}(r_{ij}) = \frac{1}{4} \frac{\rho_{[L_m]}^{(D)}(z)V_D(r_{ij}) - \rho_{[L_m]}^{(E)}(z)V_E(r_{ij})}{4\rho^{(D)}_{[L_m]}(z) - \rho^{(E)}_{[L_m]}(z)} \quad (131) \]

where \( V_D = 2V_{I_0} \) and \( V_E = 2V_{I_{\alpha \beta}} \) and where the polynomial occurring in the weight function is

\[ \rho^0_{[L_m]}(z) = 2 \left[ 4\rho^{(D)}_{[L_m]}(z) - \rho^{(E)}_{[L_m]}(z) \right] \quad (132) \]
It is interesting to note that the one pion exchange potential (OPEP) defined by
\[ V_{\text{OPEP}}(r_{ij}) = (\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{\tau}_i \cdot \vec{\tau}_j)Y(r_{ij}) \] (133)
where
\[ Y(r_{ij}) = V_0 e^{-\mu r_{ij}} \]
for \( 3V_0 \sim 10\text{MeV} \) and \( \mu \sim 0.7\text{fm}^{-1} \) does not contribute to the direct term. Indeed, since
\[ V_{\text{OPEP}}^{1+} = V_{\text{OPEP}}^{3+} = -3Y(r_{ij}) \]
\[ V_{\text{OPEP}}^{1-} = 9Y(r_{ij}) \]
\[ V_{\text{OPEP}}^{3-} = Y(r_{ij}) \]
the contribution in the direct term disappears and amounts to \( 18Y(r_{ij}) \) in the exchange term, Eq. (128).

Since the contribution to the effective potential of the exchange term decreases rapidly for increasing \( A \), \( I_0(0) = (A/4)^2 \), while \( I_{\alpha\beta}(0) = A/4 \) the contribution of the OPEP, which is the dominant term in the description of the long range part of the nucleon-nucleon potential, fades away for large nuclei.

7.1 Spurious component

In the Schrödinger equation, Eq. (25), the sum over all pairs of the two-body potential contributes while only one component \( V(r_{ij}) \) associated with the reference pair \((i, j)\) appears in the amplitude of Eq. (24). The potential can be expanded on the complete potential basis associated with the weight function. The polynomials of degree one
\[ P_1(z) = \left( \frac{1 + z}{2} - \frac{1}{A - 1} \right) / N = \left( \frac{r_{ij}^2}{r^2} - \frac{1}{A - 1} \right) / N \]
where \( N \) is the normalization constant given by
\[ N^{-2} = \int_{-1}^{+1} \left( \frac{1 + z}{2} - \frac{1}{A - 1} \right)^2 W_{[L_m]}(z) dz, \]
is independent of \([L_m]\) and disappears when the sum is taken over all pairs, \( \sum_{i,j>i} r_{ij}^2 / r^2 = A/2 \). Therefore one term occurring in the expansion of the potential disappears in the Schrödinger equation.
The occurrence of this spurious component in the potential has been discussed in Ref. [18,27]. This component contributes in the SIDE, *i.e. when* \( V_{[L_m]}(r) \) in Eq. (34) is set to zero since in the right hand side \( P(z,r) \) is an amplitude where the reference pair \((i,j)\) is in \(S\)-state while \(V(r_{ij})P^0\) operates as an \(S\)-state projected potential that vanishes when in the other amplitudes expanded in terms of the Jacobi coordinates, Eq. (A.1), the reference pair is not in \(S\)-state [26].

For three-bodies this equation is known as the Faddeev equation for \(S\)-states projected potentials. In the IDEA we assume that the potential is local and we isolate the hypercentral part of the potential, \(V_{[L_m]}(r)\), leaving only the residual potential to operate, in the r.h.s of Eq. (34), on pairs in \(S\)-states. When the sum in Eq. (24) is performed over all pairs \((i,j)\), one obtains the Schrödinger equation Eq. (25).

Since the component \(P_1(z)\) in the expansion of \(V(r_{ij})\) disappears in the sum \(\sum_{i,j>i} V(r_{ij})\) occurring in the Schrödinger equation, it must be canceled in \(V(r_{ij})\) for local potentials. Therefore, in order to avoid taking spurious parts of the potential operating in IDEA, one should consider instead

\[
\tilde{V}(r_{ij}) = V(r_{ij}) - V_1(r)P_1(z)
\]  

(134)

where

\[
V_1(r) = \int_{-1}^{+1} V(r\sqrt{(1 + z)/2})P_1(z)W_{[L_m]}(z) \, dz
\]

(135)

Since

\[
\sum_{\ell,k>\ell} \left( \frac{r_{k\ell}^2}{r^2} - \frac{1}{A-1} \right) = 0
\]

and for the reference pair

\[
P^0 P_1(z) = P_1(z),
\]

the projection on this pair is

\[
P^0 \sum_{\ell,k>\ell \neq i,j} P_1(z_{k,\ell}) = -P_1(z)
\]

which is in agreement with the coefficient

\[
(A - 2) \left[ 2P_1(-1/2) + \frac{A - 3}{2} P_1(-1) \right] / P_1(1) = -1
\]

occurring in the projection function, Eq. (110).
8 Spin-isospin Exchange Generated Elements of the State

Once the problem for the closed shell nuclei in ground state has been solved, with the inclusion of two-body correlations, one may argue that at the level of accuracy where many-body correlations are neglected, the solution obtained is complete. However, the exchange operators occurring in the potential generate new states of grand orbital $L_m + 2$ which can not be reached by solving the single IDEA equation, Eq. (34).

To see how to include these states, let us begin with the $\alpha$-particle where the problem is well known (the same situation holds for three nucleons in ground state). For three- and four-body systems, besides the space symmetric state associated with the spin-isospin antisymmetric state, the spin and isospin exchanged operators can generate also mixed symmetry states which must be taken into consideration in constructing the fully antisymmetric state. In such a case we have a sum of space and spin-isospin states products each state of the product having a definite symmetry in the exchange of two selected particles. To understand how such states are generated let

$$D_\alpha = ||\alpha p \beta p \alpha n \beta n||$$

be the spin and isospin antisymmetric Slater determinant associated with the fully-symmetric space state of $^4\text{He}$ in ground state, where $\alpha (\beta)$ denotes the spin-up(down) while $p$ denotes the proton and $n$ the neutron. The ground polynomial, from which the expansion of the $^4\text{He}$ in ground state starts, is $D_\alpha Y_{[0]}$ where $Y_{[0]}$ is a constant, in fact, a HH of order zero, i.e., with $L_m = 0$. Applying, for example, the Bartlett spin-exchange operator $P^\sigma_{ij}$ on $D_\alpha$ one obtains either 1 when the two spins are the same or -1 when the exchanged spins between two identical particles (protons or neutrons) are opposite, or 0 when the exchange is between two different spins belonging to two different particles. Indeed, in the last case the determinant has two pairs of identical columns and thus disappears, i.e., one gets that $\sum_{ij>1} P^\sigma_{ij} D_\alpha = 0$. Similarly, if the isospin is exchange between either the first and the last column or between the second and third column for which $\zeta_\alpha + \zeta_\beta = 0$, one gets a determinant with two pairs of identical columns and thus also disappears.

If now one considers the scalar operator $\sum_{ij>1} r_{ij}^2 P^\sigma_{ij}$ the exchange between the same columns generates new HPs, namely,

$$||\beta_1 p_1 \vec{x}_1 \beta_1 p_1 \alpha_1 n_1 \alpha_1 n_1 \vec{x}_1||$$

and

$$||\alpha_1 p_1 \alpha_1 p_1 \vec{x}_1 \beta_1 n_1 \beta_1 n_1 \vec{x}_1||$$

(137)

(138)
where in the expansion of the determinant with respect to the first two rows the scalar product $\vec{x}_1 \cdot \vec{x}_2$ appears. These two determinants are HP of degree two which cannot be generated by the Wigner or Majorana potentials and should be included in the description of $^4$He. These are the so-called mixed symmetry states which are coupled to the fully symmetric $S$-state through the spin-exchange (Bartlett) potential.

Apart from the scalar spin-exchange operator $r_{ij}^2 P_{ij}^\sigma$ one can apply also the scalar isospin-exchange operator $r_{ij}^2 P_{ij}^\tau$. The generated determinants

$$||\alpha n \vec{x} \beta p\ an \ \beta p \vec{x}||$$ (139)

and

$$||\alpha p \beta n \vec{x} \alpha p \vec{x} \beta n||$$ (140)

are HPs of opposite sign to those generated by the spin-exchange operator, i.e.

$$r_{ij}^2 P_{ij}^\tau D_\alpha = -r_{ij}^2 P_{ij}^\sigma D_\alpha,$$ (141)

and thus one can consider only HPs generated by the spin-exchange operator and coupled to the space symmetric state $D_\alpha$ through the potential

$$\left(V^\sigma (r_{ij}) P_{ij}^\sigma + V^\tau (r_{ij}) P_{ij}^\tau\right) D_\alpha = \frac{1}{2} \left(V^{3+} (r_{ij}) - V^{1+} (r_{ij})\right) P_{ij}^\sigma D_\alpha$$ (142)

where $V^{3+}$ and $V^{1+}$ are the triplet even and singlet even potentials respectively. It is clear that the HPs generated by the operator $r_{ij}^2 P_{ij}^\sigma D_\alpha$ cannot be neglected. They constitute the mixed symmetry component of the $^4$He wave function coupled, through the spin- and isospin-exchange operators, to the space symmetric component of the wave function.

Similar components are generated by the exchange operators in nuclei. They should be taken into account explicitly as they bring a non-negligible contribution to the binding energy. The potentials associated with the spin- or the isospin-exchange operators depend on the difference between the triplet $V^{3+}$ and singlet $V^{1+}$ even potentials which, for realistic potentials, is rather weak. Indeed, the contribution to the binding energy of the mixed symmetry state in three- and four-body bound states does not exceed a few MeV. In contrast to the Bartlett potentials, however, the Majorana potential depends on the sum of these potentials which is large. It is responsible, together with the Wigner potential, to nearly all the binding energy in few-body systems. The spin-isospin exchange operator does not generate mixed symmetry states in
the tri-nucleon or the alpha-particle system but in nuclei it does and therefore it can not be neglected.

Let us note by $D_\alpha | n, \ell, m >$ the part of a system of particles where four nucleons are in the same space HO state $| n, \ell, m >$ and the spin and isospin states are saturated as in $D_\alpha$ (see Eq. (136)). For instance, for $^{16}$O the HO wave function would be written as

$$D_{[L_m]}^{HO} = D_\alpha | 0, 0, 0 \rangle D_\alpha | 0, 1, -1 \rangle D_\alpha | 0, 1, 0 \rangle D_\alpha | 0, 1, +1 \rangle$$

where $D_\alpha | n, \ell, m >$ means that each of the four spin-isospin states in $D_\alpha$ is in the same space $| n, \ell, m >$ state. Let us further consider the operator

$$\mathcal{O} = \sum_{i,j} (1 - \delta_\zeta_\zeta) P_{ij}^{\sigma \tau}$$

which exchanges the spin-isospin states $\zeta_i$ and $\zeta_j \neq \zeta_i$. When the exchange is between spin-isospin states in different states $| n, \ell, m >$ and $| n', \ell', m' >$ it generates a determinant with two identical pairs of columns and then disappears. Proceeding, as for for spin and isospin, the scalar operator

$$\sum_{i,j} r_{ij}^2 (1 - \delta_\zeta_\zeta) (1 - \delta_\alpha_\alpha) P_{ij}^{\sigma \tau}$$

where

$$\delta_\alpha_\alpha' = \begin{cases} 1 & \text{when } n_\alpha = n_\alpha', \ell_\alpha = \ell_\alpha', m_\alpha = m_\alpha' \\ 0 & \text{otherwise} \end{cases}$$

applied to different space states $| \alpha >$ and $| \alpha' >$ generates HPs of degree $L_m + 2$ which should be included in the description of the wave function. They should much contribute to the binding energy of the nucleus since they are coupled to the main component of the wave function through the Majorana potential which contains the sum of the triplet and singlet even potentials.

It has already been shown, by solving the $^6$Li problem in the hypercentral $L_m$-approximation [28], that the contribution to the binding energy of the mixed symmetry state generated by the Majorana exchange operator is of the order of a few MeV. For a HO Serber force which is fitted in order to give the experimental size and binding energy of $^4$He, the eigen-energy calculated for $^6$Li without the inclusion of the mixed symmetry state is above the one of $^4$He [28]. An exact calculation could not provide, for potentials vanishing at infinity, an eigen-energy above the one of $^4$He since one should at least obtain the $^4$He binding energy for $^6$Li when the two nucleons in the $p$-shell are in zero energy scattering state.

When the operator (144) operates on closed shell it generates two pairs of
identical columns. Therefore, the operator (145) in order to remove the cancellation of the determinant must change the space state of the incriminated columns. It means that only the scalar product $\vec{x}_i \cdot \vec{x}_j$ in $r_{ij}^2$ must be taken into account. But the scalar product written as

$$\vec{x}_i \cdot \vec{x}_j = x_i x_j P_1(\cos \varphi) = x_i x_j \frac{4\pi}{3} \sum_{m=-1}^{+1} Y_1^m(\omega_i) Y_1^m(\omega_j), \quad (146)$$

where $\varphi$ is the angle between the directions $\omega_i$ and $\omega_j$ and $P_1(\cos \varphi)$ is a Legendre polynomial, is a harmonic polynomial of degree two as the product of two harmonic polynomials of degree one in the variable $\vec{x}_i$ and $\vec{x}_j$.

When the operator

$$Q^{\sigma \tau} = \sum_{i, j \geq i} \vec{x}_i \cdot \vec{x}_j [1 - \delta_{\zeta \zeta_j}](1 - \delta_{\alpha \alpha_j}) P_{ij}^{\sigma \tau} \quad (147)$$

is applied to the state $D_{[L_m]}^{HO}(\vec{x})$ it generates a sum of new HO states $D_{[L_m+2]}^{HO}(\vec{x})$ which are HP of degree $L_m + 2$. The new polynomials are indeed constructed according to the procedure described in Sect. II for HPs. Let us call

$$D_{[L_m+2]}^{\sigma \tau}(\vec{x}) = Q^{\sigma \tau} D_{[L_m]}^{HO}(\vec{x})$$

the sum of the HO Slater determinant generated by the $Q^{\sigma \tau}$ operator. Each one is constituted by the original $D_{[L_m]}^{HO}(\vec{x})$ Slater determinant where two HO individual states in the last occupied shell have been raised to the next shell. The $D_{[L_m+2]}^{HO}(\vec{x})$ must be normalized and the weight and pseudo-weight functions and projection function must be calculated. Then the coupling between the $D_{[L_m]}^{HO}(\vec{x})$ and $D_{[L_m+2]}^{HO}(\vec{x})$ states generated by the spin-isospin exchange operator can be introduced through the exchange pseudo-weight function

$$\int D_{[L_m+2]}^{\sigma \tau}(\vec{x}) P_{ij}^{\sigma \tau} D_{[L_m]}^{HO}(\vec{x}) d\Omega_{N-1}. \quad (148)$$

9 Applications

To solve the basic equation (34) for a specific nucleus and a specific filling of the HO shells, one has first to evaluate the coefficients $([L_m]|n, \ell)^{(e)}$ from which the weight functions (99) and the density functions $\rho_{[L_m]}(z)$, Eq. (100), or equivalently their roots (102) can be obtained. These coefficients are evaluated from Eqs. (83), (84), and (85) as well as from Eq. (89) when Coulomb forces are included. The results for the closed shell nuclei $A = 16$ and $A = 40$ and for the closed shell neutron systems $N = 8$ and $N = 20$ are given in table 1 while
the corresponding roots for $\rho^{(e)}_{[L_m]}(z)$ are given in table 2 for the $^{16}$O and $^{40}$Ca nuclei and in table 3 for the closed shell 8-neutron and 20-neutron systems. It is noted here that only the relative normalization constants $C^{(e)}/C^{(0)}$ are required to construct the effective potential defined by

$$V_{\text{eff}}(r_{ij}) = \sum_{\epsilon=0,\sigma,\tau,\sigma\tau} V^{(e)}(r_{ij}) W^{(e)}_{[L_m]}(z)/W^{(0)}_{[L_m]}(z)$$

$$\equiv \sum_{\epsilon=0,\sigma,\tau,\sigma\tau} V^{(e)}(r_{ij}) \rho^{(e)}_{[L_m]}(z)/\rho^{(0)}_{[L_m]}(z) \quad (148)$$

where the nuclear potentials $V^{(e)}$ are given by Eqs. (122–125) in terms of the singlet or triplet, even or odd potentials. The extracted ratios $C^{(e)}/C^{(0)}$ are also given in tables 2 and 3. It is further noted that the weight function for the $^4$He system, is given by (99) with $\rho^{(e)}(z) = 1$ for $L_m = 0$.

The next task is the choice of the potential. Since we are testing here the suitability of our method in nuclear structure calculations, we employed three interactions with different characteristics and for which results by other competing methods are available. The first potential employed is the widely used in nuclear structure calculations Brink and Boeker B1 effective soft core potential [29] which is of a rather long range and of soft core. Although this potential is not realistic, since it does not fit the scattering N-N phase shifts, there exist in the literature a lot of results obtained with it and therefore its use is warranted for comparison purposes. For convenience, we recall here this potential which has only Wigner and Majorana components

$$V_W = 595.55 \exp(-2.041 r^2) - 72.212 \exp(-0.512 r^2),$$

$$V_M = -206.04 \exp(-2.041 r^2) - 68.388 \exp(-0.512 r^2).$$

The second potential used is the Afnan and Tang [30] S3 potential. The original potential was adjusted to the static properties of $^4$He nucleus. However, in order to extend its applicability to heavier nuclei, Guardiola and collaborators [31] added a repulsive part in the singlet- and triplet-odd components, the modified potential thus obtained (known as MS3 potential) being

$$V_W = -5.75 \exp(-0.4 r^2) - 10.75 \exp(-0.6 r^2) - 41.5 \exp(-0.8 r^2) - 81.675 \exp(-1.05 r^2) + 1000 \exp(-3 r^2)$$

$$V_M = 5.75 \exp(-0.4 r^2) + 10.75 \exp(-0.6 r^2) + 41.5 \exp(-0.8 r^2)$$

$$V_B = -V_H = 5.75 \exp(-0.4 r^2) - 10.75 \exp(-0.6 r^2) + 41.5 \exp(-0.8 r^2)$$

$$\quad + 41.5 \exp(-0.8 r^2) - 81.675 \exp(-1.05 r^2)$$

This potential has a very strong repulsive core which give rise to strong two-body correlations. As a third potential we employed the more realistic soft
core interaction of Gogny, Pires, and de Tourreil (GPDT) [32] which was also used in the past in nuclear structure calculations.

Using the aforementioned potentials we firstly investigated whether 8 and 20 neutrons, forming a closed shell system, can sustain a bound state. It was found that, at least with the potentials employed, no bound state exists. Even with the soft B1 potential a bound state can only be generated by modifying the potential by more than 20%.

The ground state energies $E_g$ and root mean square radii $r_{\text{rms}}$ for the closed shell nuclei $^4\text{He}$, $^{16}\text{O}$, and $^{40}\text{Ca}$ are given in tables 4, 5, and 6. The results were obtained in the extreme and uncoupled adiabatic approximations and are compared with other results obtained with cluster expansion method (FAHT) [31], Bruekner-Hartree-Fock type (BHF) [31], Fermi-Hyper-Netted-Chains (FNHC) [33], variational Monte Carlo (VMC) [34,35] as well with results obtained with hyperspherical harmonics expansion methods (HHE) [10]. We refer also to the work of Guardiola et al. [36] where more relevant results are compiled using various techniques. The importance of the correlations, stemming mainly from the short range repulsion of the two-body force, is also inferred from these tables by comparing the IDEA results with those of hypercentral approximation.

It should be noted that the results obtained by our method do not involve any adjustable parameter unlike, for example, the FHNC method whose results depend on the Jastrow ansatz, and in general on the model wave function employed. This can be seen, for instance, in table 5 where the use of different ansatzs in the FHNC resulted in $\sim 13 \text{ MeV}$ difference in the ground state energy with B1 potential.

The inclusion of the Coulomb potential is of utmost importance when one is dealing with nuclei involved in reactions of astrophysical interest. Its inclusion, however, with the exact integral, differential, or integro-differential methods employed in Few-Body calculations is non-trivial and thus it is usually omitted. However, the incorporation of Coulomb forces in our formalism is straightforward. The extra repulsion generated by the Coulomb potential is also given in tables 4, 5, and 6. As expected, the smaller the rms radius, the higher the eigen-energy is.

We calculated the EAA and UAA giving respectively a lower- and upper-bound to the eigen energy. The difference decreases from $\sim 0.9 \text{ MeV}$ to $\sim 0.1 \text{ MeV}$ for A growing from A=4 to A=40. By taking the average for the eigen energy, the difference with respect to the exact value is less than 0.5 MeV for $^4\text{He}$ and becomes negligible for increasing A. The values obtained are in agreement with the spectrum of those obtained by other methods except for $^{40}\text{Ca}$ with the MS3 potential where we got a significant lower binding but unfortunately variational values are not available for comparison. We notice
that the strongest the core of the potential, the larger is the increase of binding energy brought by the correlations. Since most realistic potentials have strong repulsive core, we expect that the binding in nuclei originates from the correlations and that the effect of the hypercentral potential is to balance the kinetic energy only.

As an example for the applicability of our formalism to open shell nuclei we consider the $^{10}$B nucleus. The ground state of this nucleus is known to be $J^\pi = 3^+, T = 0$. Two likely ground configurations with this state are the

$$\psi_1 = (4s_{1/2})(4p_{1/2})(2p_{3/2})$$

configuration where the $p_{1/2}$ nucleons have the quantum numbers $J = 0$ and $T = 0$ and the

$$\psi_2 = (4s_{1/2})(6p_{3/2})$$

configuration where the two holes in the, otherwise, full $p_{3/2}$ subshell are also coupled to $J^\pi = 3^+, T = 0$. The form of the density function $\rho_{[\ell_m]}^{(e)}(z)$ in this case is given by [24,5,37],

$$\rho_{[\ell_m]}^{(e)}(z) = \sum_{n=0}^{2\ell_m} \langle [\ell_m]|n, \ell \rangle^{(e)} (1 - z)^{L_m - n} (1 + z)^n$$  \hspace{1cm} (149)$$

where the expansion coefficients are given by

$$\langle [\ell_m]|n, \ell \rangle^{(e)} = \frac{\Gamma(L_m + D/2)}{\Gamma(L_m + D/2 - n - 3/2)\Gamma(n - 3/2)} A_n^{\ell}.$$  \hspace{1cm} (150)$$

The $A_n^{\ell}$ are calculated from the shell structure of the nuclear state under discussion by means of the Talmi-Moshinsky or Gogny coefficients [38,39]. The explicit values for these coefficients are given in table 7 while the roots and the relative normalization constants for the polynomial $\rho_{[\ell_m]}^{(e)}$ are given in table 8. The corresponding binding energy results for the two configurations are given in table 9.

10 Conclusions

When the structure of nuclei has been identified as associated to the HO quantum numbers corrected by the effect of a strong spin-orbit force, it was taken for granted that nuclei can be described as an IPM. In the present work we have shown that the HO is both an IPM and a collective model and that the HO potential is a hypercentral potential invariant by rotation in the D-dimensional space spanned by the particle coordinates in the center of
mass frame. This leads to another interpretation where the most significant part of the interaction is hypercentral in such a way that in the ground state the repulsive centrifugal barrier must be minimum. This interpretation also leads to the appearance of the so-called magic numbers in nuclei whatever the nuclear potential is.

We have further shown that the next improvement after the HCA, where the wave function is described by the product of an antisymmetric harmonic polynomial and a function of the hyperradius, is obtained by introducing the two-body correlations generated by the nuclear two-body potential. This results to an integro-differential equation (IDEA) for the A-body bound system. In order to test the quality of this approximation where only one harmonic polynomial of minimal degree and two-body correlations are taken into account in the wave function, we compared the binding energies obtained with the IDEA to those computed for central forces by other methods. We found values in agreement with the spectrum of those available in the literature, the largest differences being in the case of $^{40}$Ca with S3 potential which has a strong repulsive core. Unfortunately, no variational results are available in the latter case.

Our formalism is based of course on the assumption that the NN force is the dominant one inside nuclei. It is well known, however, that three-body forces are also essential in the description of nuclear dynamics and their introduction might explain at least a part of the underbinding of nucleons in nuclei interacting by two-body forces only. Their employment in our basic equation (34) is straightforward when they are symmetrical and separable,

$$W(\vec{x}) = \sum_{i<j<k \leq A} V^{(3)}(r_{ij})V^{(3)}(r_{jk}),$$

as described in Ref. [13,16].

Up until now it was possible to solve three- and four-body problems with accuracy by using the Faddeev and Faddeev-Yakubovsky equations respectively. For more nucleons it is customary to assume that nucleons in nuclei form clusters, and then to construct effective inter-cluster interactions for the sake of solving the problem as a three- or four-body. The construction of the effective interactions, however, is a formidable task with all sort of ambiguities naturally creeping in due mainly to insufficient scattering data required in the construction of the nuclear forces involved.

The existence or not of bound and of resonance states in three- and four-neutron has been the subject of numerous theoretical and experimental works [40,41]. Going beyond the four-neutron system [42] is also important as nuclear matter exhibits a much richer phase structure than light nuclei because of stronger correlations between nucleons. However, our results for the 8- and
20-neutron systems indicate that realistic nucleon-nucleon central forces alone are not sufficient to generate closed shell bound systems.

The huge differences between the results obtained with the three different two-body forces used, indicates that one has to employ a realistic two-nucleon potential supplemented with a three-body force that accounts for the missing binding energy. The use of effective forces, either at the two-body level, or cluster-cluster potentials, or optical potentials in processes where the details of the interior part of the wave function plays an important role, could provide misleading results. Such processes are, for example, the electromagnetic reactions, in which the transitions are guided by strict selection rules and depend on the details of the wave functions in each channel and their characteristics in the interior region. Therefore, the inclusion of the underlying correlations in an unambiguous way is of crucial importance.

A Jacobi Coordinates

Let us define the Jacobi coordinates for equal mass particles

\[
\vec{\xi}_N = (\vec{x}_2 - \vec{x}_1), \\
\vec{\xi}_{N-1} = \sqrt{3}(\vec{x}_3 - \bar{X}_3) \\
\vdots \\
\vec{\xi}_{N-i+1} = \sqrt{\frac{2i}{i+1}}(\vec{x}_{i+1} - \bar{X}_i) \\
\vdots \\
\vec{\xi}_1 = \sqrt{\frac{2A}{A-1}}(\bar{x}_A - \bar{X})
\]

(A.1)

with \( N = A - 1 \), \( \bar{X}_i = \frac{1}{i} \sum_{p=1}^{i} \vec{x}_p \), and

\[
r^2 = \sum_{i=1}^{N} \xi_i^2 = 2 \sum_{i=1}^{A} (\vec{x}_i - \bar{X})^2 = \frac{2}{A} \sum_{i,j} (\vec{x}_i - \vec{x}_j)^2
\]

(A.2)

and where \( \bar{X} \equiv \bar{X}_A \) corresponds to the center of mass. The normalization is such that for \( i = N \) one gets the vector \( \vec{r}_{12} \).
For describing the coordinates one introduces the kinematical rotation vector [6,25]

\[ \mathbf{r}(\delta, \varphi) = \cos \delta \mathbf{\xi}_N + \sin \cos \varphi \mathbf{\xi}_{N-1} + \text{(terms in } \mathbf{\xi}_j, \ j < N - 1) \quad (A.3) \]

where \( \delta \) and \( \varphi \) are angular parameters, \( \mathbf{\xi}_N = \mathbf{x}_i - \mathbf{x}_j \), and \( \mathbf{\xi}_{N-1} = \sqrt{3}(\mathbf{x}_k - 1/3(\mathbf{x}_i + \mathbf{x}_j + \mathbf{x}_k)). \) For \( \delta = 0, \) \( \mathbf{r}(0) = \mathbf{x}_i - \mathbf{x}_j \) is the reference pair, for \( \delta = 2\pi/3, \) \( \varphi = 0 \) \( \mathbf{r}(2\pi/3) = \mathbf{x}_k - \mathbf{x}_i \) is a connected pair, while for \( \delta = \pi/2, \) \( \mathbf{r}(\pi/2) \) is a disconnected pair.

One may introduce also the hyperspherical coordinates of Zernike and Brinkman [43]

\[ \xi \equiv \xi_N = r \cos \phi_N, \quad \rho = r \sin \phi_N \quad (A.4) \]

with

\[ \xi^2 + \rho^2 = r^2, \quad \rho^2 = \sum_{j=2}^{A-1} \xi_j^2. \quad (A.5) \]

For the other Jacobi coordinates we have

\[ \xi_{N-1} = r \sin \phi_N \cos \phi_{N-1} \]

\[ \vdots \]

\[ \xi_j = r \sin \phi_N \cdots \sin \phi_{j+1} \cos \phi_j, \quad (A.6) \]

\[ \vdots \]

\[ \xi_2 = r \sin \phi_N \cdots \sin \phi_3 \cos \phi_2, \]

\[ \xi_1 = r \sin \phi_N \cdots \sin \phi_2 \]

where \( \phi_1 = 0. \) It is clear that in the Zernike-Brinkman system of coordinates one has, apart from the hyperradius \( r, \) \( N - 1 \) angles \( \phi_i \) and \( 2N \) angles \( \omega_i, \) the volume element being

\[ d\Omega_{i+1} = d\Omega_i \sin^{(I-4)} \phi_{i+1} \cos^2 \phi_{i+1} d\phi_{i+1} d\omega_{i+1} \quad (A.7) \]

where \( I = 3(i + 1). \) The \( \Omega \) coordinates are separated into two parts, first the \( z = \cos 2\phi, \) \( (\phi = \phi_N), \) and \( \omega = \omega_N, \) the angular coordinates of \( \xi_N, \) and second the \( \Omega_{N-1} \) for the other hyperspherical coordinates \( (\phi_i, \omega_i), \ i < N \) where \( \omega_i \) are the angular coordinates of \( \xi_i. \) Let \( d\Omega_{N-1} \) be the surface element of the unit hypersphere \( \rho = 1 \) in the \( D - 3 \) dimensional space spanned by the Jacobi
coordinates $\vec{\xi}_i, i < N$. The volume element in the $D$-dimensional space is then given by

\[
d^{3N} \xi = r^{D-1} dr \, d\Omega
\]

\[
d\Omega = (\sin \phi)^{D-4} \cos^2 \phi \, d\phi \, d\omega \, d\Omega_{N-1}
\]

\[
= \frac{1}{2D/2} (1 - z)^{(D-5)/2} (1 + z)^{1/2} \, dz \, d\omega \, d\Omega_{N-1}
\]

\[
= W_0(z) \, dz \, d\omega \, d\Omega_{N-1}
\]  

(A.8)

where

\[
\cos \phi = \frac{\vec{r}_1 \cdot \vec{r}_2}{r} = \frac{\xi}{r} \quad z = \cos 2\phi = 2 \frac{\vec{r}_1 \cdot \vec{r}_2}{r^2} - 1 .
\]

In general the surface element $d\Omega_i$ is defined to be the part of the surface $d\Omega$ which contains the coordinates $\omega_j$ and $z_j$ for $j \leq i$, i.e

\[
d\Omega_i = d\omega_i \prod_{j=2}^{i} 2^{-3j/2} w_j(z_j) dz_j d\omega_j
\]  

(A.9)

where $w_j(z_j)$ is defined by

\[
w_j(z_j) = (1 - z_j)^{(3j-5)/2} (1 + z_j)^{1/2}
\]  

(A.10)

Knowing the element $d\Omega_j$ we may construct the element of $d\Omega_{j+1}$ via

\[
d\Omega_{j+1} = d\Omega_j (\sin \phi_{j+1})^{D-4} \cos \phi_{j+1}^2 d\phi_{j+1} d\omega_{j+1}
\]  

(A.11)

where $D = 3(j + 1)$.

The kinetic energy operator $T$, for equal mass particles, is given by

\[
T = -\frac{\hbar^2}{m} \sum_{j=1}^{A-1} \nabla_{\xi_j}^2 - \frac{\hbar^2}{2M} \nabla_{\chi}^2 .
\]  

(A.12)

We are interested to express the translationally invariant part of this equation in terms of the $\{\omega_i, z_i\}$. Due to the product structure of $d\Omega$ it can be written as

\[
T = -\frac{\hbar^2}{m} \left[ \frac{\partial^2}{\partial r^2} + \frac{3A - 4}{r} \frac{\partial}{\partial r} + \frac{L^2(\Omega)}{r^2} \right]
\]  

(A.13)
where \( L^2(\Omega) \) is the grand orbital operator. Let \( L^2_i(\Omega_1) \) be the operator associated with the first vectors \( \{\xi_1, \cdots, \xi_i\} \). It can be written in terms of the one for the vectors \( \{\xi_1, \cdots, \xi_{i-1}\} \) as follows

\[
L^2_i(\Omega_i) = D(z_i) + 2\ell^2(\omega_i) + \frac{2}{1 - z_i} L^2_{i-1}(\Omega_{i-1})
\]  

(A.14)

where

\[
D(z_i) = \frac{4}{w_i(z_i)} \frac{\partial}{\partial z_i}(1 - z_i^2)w_i(z_i) \frac{\partial}{\partial z_i}
\]  

(A.15)

and

\[
L^2(\Omega) \equiv L^2_N(\Omega).
\]

The \( \ell^2(\omega_i) \) is the orbital operator normalized according to

\[
[\ell^2(\omega) + \ell(\ell + 1)] Y_{\ell m}(\omega_i) = 0.
\]  

(A.16)

B Various Coefficients

B.1 The 3L-Coefficients

The 3L-coefficients are defined by the expansion of a product of two HO eigenfunctions [44]

\[
\psi^*_{\alpha}(\vec{x}) \psi_{\beta}(\vec{x}) = \frac{2}{\pi^{1/4}} \sum n_{\ell,m} \left[ n_{\alpha} n_{\beta} n \right] \ell_{\alpha} \ell_{\beta} \ell \times \langle Y_{\ell_{\alpha}}^m | Y_{\ell_{\beta}}^{m*} \rangle e^{-x^2/2} \psi_{n,\ell,m}(\vec{x})
\]

where \( \alpha \) and \( \beta \) stand for the HO quantum numbers \( n_\alpha, \ell_\alpha, m_\alpha \) and \( n_\beta, \ell_\beta, m_\beta \).

By integrating over the \( \vec{x} \) space for the HO quantum numbers \( \alpha, \beta \) with \( n = 0, \ell = 0, m = 0 \), and

\[
\psi_{0,0,0}(\vec{x}) = \frac{2}{\pi^{1/4}} e^{-x^2/2} Y_0^0,
\]

(B.2)

one finds the normalization

\[
\left[ \begin{array}{ccc} n_\alpha & n_\beta & 0 \\ \ell_\alpha & \ell_\beta & 0 \end{array} \right] = \delta_{\alpha\beta}.
\]  

(B.3)
Multiplying Eq. (B.1) by
\[
\left( \frac{n! \Gamma(3/2)}{\Gamma(n + \ell + 3/2)} \right)^{1/2} Y^{m*}_\ell(x) L^{\ell + 1/2}_n(x^2)
\]
and integrating over \(x\) one gets [45]
\[
\begin{bmatrix}
n \alpha & n \beta & n \\
\ell \alpha & \ell \beta & \ell
\end{bmatrix} = \sqrt{2 \pi^{1/2}} \left[ \frac{n!}{\Gamma(n + \ell + 3/2)} \right]^{1/2} \times \int_0^\infty x^{\ell + \ell + \ell}/2(x^2)L^{\ell + 1/2}_n(x^2)L^{\ell + 1/2}_m(x^2) e^{-x^2} x^2 dx. \quad (B.4)
\]
For \(n = \ell = 0\) one finds the normalization (B.3). Using the expansion for the Laguerre polynomials for \(\alpha\) and \(\beta\) gives for the integral in Eq. (B.4)
\[
I = \sum_{m,m'} (-1)^{m+m'} \frac{m! m!}{m! m!} \left( \frac{n_\alpha + \ell_\alpha + 1/2}{n_\alpha - m} \right) \left( \frac{n_\beta + \ell_\beta + 1/2}{n_\beta - m'} \right) \times \int_0^\infty x^{2(m+m'+(\ell_\alpha + \ell_\beta - \ell)/2)} x^2 L^{\ell + 1/2}_n(x^2) dx.
\]
This last integral is given, according to Eq. (58), by
\[
\int_0^\infty x^{2(\nu + \ell)} L^{\ell + 1/2}_n(x^2) dx = \left( \frac{-1}{2} \right)^n \left( \frac{\nu}{n} \right) \Gamma(\nu + \ell + 3/2) \quad (B.5)
\]
where here \(\nu = m + m' + (\ell_\alpha + \ell_\beta - \ell)/2\).

The 3L-coefficients are thus given by
\[
\begin{bmatrix}
n \alpha & n \beta & n \\
\ell \alpha & \ell \beta & \ell
\end{bmatrix} = (-1)^n \sqrt{\frac{n!}{\Gamma(n + \ell + 3/2)} \Gamma(n_\alpha + \ell_\alpha + 3/2) \Gamma(n_\beta + \ell_\beta + 3/2) \Gamma(n + \ell + 3/2)} \times \sum_{m,m'} (-1)^{m+m'} \frac{m!}{m!} \left( \frac{n_\alpha + \ell_\alpha + 1/2}{n_\alpha - m} \right) \left( \frac{n_\beta + \ell_\beta + 1/2}{n_\beta - m'} \right) \times \left( \frac{m + m' + (\ell_\alpha + \ell_\beta - \ell)/2}{n} \right) \Gamma(m + m' + \frac{\ell_\alpha + \ell_\beta + \ell + 3}{2}) \quad (B.6)
\]
We recall that \((\alpha \atop \beta)\) denotes the binomial coefficients, and that \(0 \leq m \leq n_{\alpha},\)
\(0 \leq m' \leq n_{\beta},\) and \(n \leq m + m' + (\ell_{\alpha} + \ell_{\beta} - \ell)/2.\)

### B.2 The \((\alpha \atop n, \ell \atop \beta)\) coefficients

To obtain the coefficients \((\alpha \atop n, \ell \atop \beta)\) which appear in the Fourier transform (74), we expand the plane wave

\[ e^{i\vec{k} \cdot \vec{x}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i^{\ell} j_{\ell}(k r) Y_{\ell m}^m(\omega) Y_{\ell m}^m(\omega_k) \]  

(B.7)

where \((k, \omega)\) and \((r, \omega)\) are the polar coordinates of \(\vec{k}\) and \(\vec{r}\) respectively and \(j_{\ell}\) is the spherical Bessel function, \(j_{\ell}(\rho) = \sqrt{\pi/2} \rho J_{\ell+1/2}(\rho).\) Then, using the integral

\[ \int_0^{\infty} \left(\frac{x}{b}\right)^{\ell} I_n^{\ell+1/2}(x^2/b^2) J_{\ell+1/2}(k x) e^{-(x/b)^2} x^2 \, dx = \frac{b^{\ell+1}}{2n+\ell+3} \frac{\pi^{n+\ell}}{n!} \text{e}^{-y^2/2} \]  

(B.8)

where \(y = kb/\sqrt{2},\) one obtains

\[ (\alpha \atop n, \ell \atop \beta) = \left[ \frac{2\pi^{3/2}}{2n+\ell+3} \frac{\pi^{n+\ell}}{n!} \right]^{1/2} \langle Y_{\ell\alpha}^m | Y_{\ell\beta}^m | Y_{\ell\beta}^m \rangle \left[ \frac{n_{\alpha} n_{\beta} n}{\ell_{\alpha} \ell_{\beta} \ell} \right] \]  

(B.9)

The \(\langle Y_{\ell\alpha}^m | Y_{\ell\beta}^m | Y_{\ell\beta}^m \rangle\) can be easily expressed in terms of the Clebsch-Gordan coefficients [46].

For \(n_{\alpha} = n_{\beta} = 0,\) as for instance for the first \(p-\) and \(d-\)shells,

\[ (0, \ell_{\alpha}, m_{\alpha} \atop n, \ell, m | 0, \ell_{\beta}, m_{\beta}) = \frac{\pi (-1)^n}{2n+\ell+3} \frac{\Gamma(n+\ell+3/2)}{\Gamma(n+\ell+3/2)} \frac{\Gamma((n+\ell+3/2))}{\Gamma(n+\ell+3/2)} \]  

(B.10)

with \(m = m_{\beta} - m_{\alpha},\) and \(|\ell_{\alpha} - \ell_{\beta}| \leq 2n + \ell \leq \ell_{\alpha} + \ell_{\beta}.\) The coefficient (B.9) is the product of two terms where only the first depends on the azimuthal numbers \(m_{\alpha}\) and \(m_{\beta}\) but not on \(n_{\alpha}\) and \(n_{\beta}.)
B.3 The $I_0(\ell_m, n)$ coefficient

For closed–shell nuclei the coefficient $I_0(\ell_m, n)$ can be calculated from (92). Table 10 gives the first 11 values of $I_0$ for $\ell_m = 0, \cdots, 5$. The rest can be easily calculated either from (92) or from the general relation [24]

$$I_0(\ell_m, n) = \sum_{n_1, n_2, \ell_1} \left[ \sum_{\alpha} (\alpha|n_1, \ell_1) \right] \left[ \sum_{\alpha} (\alpha|n_2, \ell_1) \right]$$  \hspace{1cm} (B.11)

with $n = n_1 + n_2 + \ell_1$.

B.4 The $I_{\alpha\beta}(\ell_m, n)$ coefficient

They are given by

$$I_{\alpha\beta}(\ell_m; n, 0) = \sum_{n_1, n_2, \ell_1} \sum_{\alpha\beta} (\alpha|n_1\ell_1|\beta) (\beta|n_2\ell_1|\alpha)$$  \hspace{1cm} (B.12)

where the sum over $\alpha$ and $\beta$ means a sum over all H.O. states (each must be taken once only either for $\alpha$ or $\beta$). The calculation of this coefficient is tedious. However, for closed-shell nuclei and for small values of $n$ it reduces to the following analytical expressions:

$$I_{\alpha\beta}(\ell_m, 0) = \frac{1}{3!} \frac{(\ell_m + 3)!}{\ell_m!} = \frac{A}{4},$$  

$$I_{\alpha\beta}(\ell_m, 1) = 0,$$  

$$I_{\alpha\beta}(\ell_m, 2) = \frac{1}{4!} \frac{(\ell_m + 3)!}{(\ell_m - 1)!} = \frac{A}{16} \ell_m,$$  

$$I_{\alpha\beta}(\ell_m, 3) = \frac{1}{5!} \frac{(\ell_m + 3)!}{(\ell_m - 2)!},$$  

$$I_{\alpha\beta}(\ell_m, 4) = \frac{1}{6!} \frac{5\ell_m - 1}{6} \frac{(\ell_m + 3)!}{(\ell_m - 2)!},$$  

$$I_{\alpha\beta}(\ell_m, 5) = \frac{1}{6!} \frac{\ell_m + 1}{12} \frac{(\ell_m + 3)!}{(\ell_m - 3)!},$$  

$$I_{\alpha\beta}(\ell_m, 6) = \frac{1}{5!} \frac{\ell_m^3 - 3\ell_m - 32}{(\ell_m + 1)} I_{\alpha\beta}(\ell_m, 5),$$  

$$I_{\alpha\beta}(\ell_m, 2\ell_m) = \left( \frac{1}{\ell_m^2} \right)^2,$$  

55
\[ I_{\alpha\beta}(\ell_m, 2\ell_m - 1) = -\frac{2}{(\ell_m - 2)!\ell_m!}, \]
\[ I_{\alpha\beta}(\ell_m, n > 2\ell_m) = 0. \]

Values of these coefficients of up to \( \ell_m = 7 \) are given in Table 11

\section*{B.5 Coefficients in the jj-coupling scheme}

For describing the state of the nucleus, the Slater determinant (2) has been filled with individual HO states each associated with its spin-isospin state. Instead, one can combine the spin \( \vec{s} \) and angular momentum \( \vec{\ell} \) to generate the total angular momentum \( \vec{j} \) of projection \( m \) and describe the individual states in terms of the isospin \( |t > \) and the \( |n, \ell, j, m > \) states. The total angular momentum is constructed as usual,

\[ \psi_{t j m}(\omega) = \sum_{m', m_s} \langle \ell, 1/2, m', m_s| j, m \rangle Y_{\ell}^{m'}(\omega) \chi_{m_s}, \]  

(B.14)

where \( \chi_{m_s} \) is the spin variable with \( m_s = \pm 1/2 \) and the bracket is a Clebsch-Gordan coefficient.

The description of the individual states in terms of angular \( \ell \) and total angular momentum \( j \) is well adapted to nuclei because the two-body spin-orbit operator is actually a sum over individual spin-orbit operators (see Eq. (13). In the \( jj \)-coupling scheme we have to calculate the new coefficient \( (\alpha|n, \ell|\beta) \) where this time \( |\alpha > \) stands for \( |n_\alpha, \ell_\alpha, j_\alpha, m_\alpha > \) and \( |\beta > \) for \( |n_\beta, \ell_\beta, j_\beta, m_\beta > \). The HO Slater determinant is now

\[ D_{[L_m]}^{HO}(\vec{x}) = ||t_\ell^{j_\ell}|n, \ell, j, m_j(\vec{x}_i)|| \]  

(B.15)

where

\[ \psi_{n, \ell, j, m}(\vec{x}) = \left[ \frac{2n!}{b^4 \Gamma(n + \ell + 3/2)} \right]^{1/2} \psi_{\ell, j, m}(\omega) y^\ell L_n^{\ell+1/2}(y^2) e^{-y^2/2} \]  

(B.16)

where \( y = x/b \).

The \( (\alpha|n, \ell|\beta) \) coefficients in \( jj \)-coupling must be constructed from the HO individual functions (B.16). One starts from the matrix elements where the integral is performed over \( d\omega \).
\[
\langle \ell_a, j_a, m_a | Y_{\ell}^{m*} | \ell_\beta, j_\beta, m_\beta \rangle = \frac{1}{\sqrt{4\pi}} (-1)^{m_\beta-1/2} \left[ \hat{\ell}_\alpha \hat{\ell}_\beta \hat{j}_\alpha \hat{j}_\beta \right]^{1/2} \\
\times \left( \begin{array}{ccc} \ell_\alpha & \ell_\beta & \ell \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} j_\beta & j_\alpha & \ell \\ -m_\beta & m_\alpha & m \end{array} \right) \left( \begin{array}{c} \ell_\alpha \ell_\beta \ell \\ j_\beta j_\alpha 1/2 \end{array} \right) \right]^{1/2}
\]

(B.17)

in terms of the 3\( j \) and 6\( j \) coefficients. Here \( \hat{x} = 2x + 1 \), The one-body coefficient \((\alpha | n, \ell | \beta)\) in \( jj\)-coupling is obtained by substitution of (B.17) for \( \langle Y_{\ell_a}^{m_a} | Y_{\ell}^{m*} | Y_{\ell_\beta}^{m_\beta} \rangle \) in Eq. (B.9)

\[
(\alpha | n, \ell | \beta) = \left[ \frac{2\pi^{3/2}}{2^{2n+\ell n!\Gamma(n + \ell + 3/2)}} \right]^{1/2} \\
\times \langle \ell_a, j_a, m_a | Y_{\ell}^{m*} | \ell_\beta, j_\beta, m_\beta \rangle \left[ \begin{array}{ccc} n_\alpha & n_\beta & n \\ \ell_\alpha & \ell_\beta & \ell \end{array} \right] \] 

(B.18)

When \( \ell = 0 \) then

\[
\langle \ell_a, j_a, m_a | Y_0^0 | \ell_\beta, j_\beta, m_\beta \rangle = \frac{1}{\sqrt{4\pi}} \delta_{\alpha\beta}
\]

(B.19)

and

\[
(\alpha | n, 0 | \beta) = \left( \frac{\pi}{4} \right)^{1/4} \left[ 2^{2n} n! \Gamma(n + 3/2) \right]^{-1/2} \left[ \begin{array}{ccc} n_\alpha & n_\beta & n \\ \ell_\alpha & \ell_\beta & 0 \end{array} \right] \delta_{\epsilon_\alpha \epsilon_\beta} \delta_{j_\alpha j_\beta} \delta_{m_\alpha m_\beta}
\]

(B.20)

In the direct term the sum over \( -j_\alpha \leq m_\alpha \leq j_\alpha \) for a complete subshell \( n_\alpha, \ell_\alpha, j_\alpha \) gives

\[
\sum_{m_\alpha=-j_\alpha}^{j_\alpha} (\alpha | n, 0) = (2j_\alpha + 1) \left( \frac{\pi}{4} \right)^{1/4} \left[ 2^{2n} n! \Gamma(n + 3/2) \right]^{-1/2} \left[ \begin{array}{ccc} n_\alpha & n_\alpha & n \\ \ell_\alpha & \ell_\alpha & 0 \end{array} \right]
\]

(B.21)

In the \( jj\)-coupling scheme the Slater determinant is constructed according to (B.15). Instead of the four spin-isospin cases occurring in Eq. (70) we have to consider only two cases for \( d_{ij}(\vec{x}_1, \vec{x}_2) \) in Eq. (63): Either the two nucleons are the same or we have to deal with the neutron-proton pair. In the first case \( t^i - t^j = 0 \). For the Wigner (\( \epsilon = 0 \)) and isospin exchange \( \epsilon = \tau \) weight and pseudo-weight functions, the matrix element (B.18) must be used in Eqs. (79) and (81) to obtain the direct and exchange coefficients. The coefficients \( (|L|\nu, \lambda)^{\epsilon} \) are given by Eq. (83) for the weight function (\( \epsilon = 0 \)) and by Eq. (85) for the isospin exchange (\( \epsilon = \tau \)) pseudo-weight function where \( t \) is substituted for \( \zeta \).
The spin exchange operator generates new HO states \( \psi_{\alpha'}(\vec{x}_i)\psi_{\beta'}(\vec{x}_j) \) with

\[
P_{ij}^{\sigma} \psi_{\alpha}(\vec{x}_i)\psi_{\beta}(\vec{x}_j) = \sum_{m_{\alpha'}, m_{\beta'}, M\atop j_{\alpha'}, j_{\beta'}} \langle \alpha', \beta'| P_{ij}^{\sigma}\rangle |\alpha, \beta\rangle \langle \alpha, \beta| \psi_{\alpha'}(\vec{x}_i)\psi_{\beta'}(\vec{x}_j) \tag{B.22}\]

where

\[
\langle \alpha', \beta'| P_{ij}^{\sigma}| \alpha, \beta\rangle = -\left[\hat{j}_{\alpha}\hat{j}_{\beta}\hat{j}_{\alpha'}\hat{j}_{\beta'}\right] (-1)^{j_{\beta}+j_{\beta'}}
\times \langle j_{\alpha}j_{\beta}m_{\alpha}m_{\beta}|JM\rangle \langle j_{\alpha'}j_{\beta'}m_{\alpha'}m_{\beta'}|JM\rangle \left\{ \begin{array}{c} \ell_{\alpha} \quad 1/2 \quad j_{\alpha} \\ 1/2 \quad \ell_{\beta} \quad j_{\beta} \\ j_{\alpha'} \quad j_{\beta'} \quad J \end{array} \right\} \tag{B.23}\]

where \( \hat{j} = 2j + 1 \), \( \alpha' = |n_{\alpha'}\ell_{\alpha'}j_{\alpha'}m_{\alpha'}\rangle \), \( \beta' = |n_{\beta'}\ell_{\beta'}j_{\beta'}m_{\beta'}\rangle \), the brackets are Clebsch-Gordan coefficients, while the brace is a 9\( j \) coefficient.

When the states \( |\alpha\rangle \) and \( |\beta\rangle \) are coupled to give a definite total angular momentum \( J \) of projection \( M \), like for two nucleons outside a closed shell (e.g. \( ^6\text{Li} \)), the effect of the spin-isospin exchange operator is

\[
P_{ij}^{\sigma}[\psi_{\alpha}(\vec{x}_i) \otimes \psi_{\beta}(\vec{x}_j)]^M = -\left[\hat{j}_{\alpha}\hat{j}_{\beta}\right]^{1/2}
\times \sum_{j_{\alpha'}, j_{\beta'}} (-1)^{j_{\beta}+j_{\beta'}} \left[\hat{j}_{\alpha}\hat{j}_{\beta}\right]^{1/2} \left\{ \begin{array}{c} \ell_{\alpha} \quad 1/2 \quad j_{\alpha} \\ 1/2 \quad \ell_{\beta} \quad j_{\beta} \\ j_{\alpha'} \quad j_{\beta'} \quad J \end{array} \right\} [\psi_{\alpha'}(\vec{x}_i) \otimes \psi_{\beta'}(\vec{x}_j)]^M \tag{B.24}\]

When the spin exchange occurs, the direct coefficient is given by Eq. (79) where

\[
\langle \alpha, \beta|n_1, \ell_1, n_2, \ell_2|\alpha\beta\rangle^{(0)} = \sum_{m_{\alpha'}, m_{\beta'}, M\atop j_{\alpha'}, j_{\beta'}} \langle \alpha', \beta'| P_{ij}^{\sigma}| \alpha, \beta\rangle \langle \alpha|n_1, \ell_1|\alpha\rangle \langle \beta|n_2, \ell_2|\beta\rangle \tag{B.24}\]

where \( \langle \alpha|n, \ell|\alpha\rangle \) etc. given by (B.18), is substituted for \( \langle \alpha|n_1, \ell_1|\beta\rangle \) for \( \langle \beta|n_2, \ell_2\rangle \).

The exchange coefficient is obtained by using \( \langle \beta\alpha|n_1\ell_1n_2\ell_2|\alpha\beta\rangle^{(\sigma)} \) instead of \( \langle \alpha|n_1\ell_1|\beta\rangle^{(\sigma)} \) in Eq. (81). The spin \( (\epsilon = \sigma) \) and isospin \( (\epsilon = \sigma\tau) \) exchange coefficient \( \langle [L]|\nu, \lambda\rangle^{(\sigma)} \) and \( \langle [L]|\nu, \lambda\rangle^{(\sigma\tau)} \) are given respectively by (83) and (85) where \( t \) is substituted for \( \zeta \).
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Table 1
The coefficients $(L_m|\nu, \ell)(\epsilon)$ for the closed shell nuclei A = 16 and A=40 and for the A=8 and A=20 closed shell neutron systems.

| $\epsilon$ | $^{16}\text{O}$ | $^{40}\text{Ca}$ | 8n | 20n |
|-----------|----------------|----------------|-----|-----|
| WIGNER    | 120. 780. 28. 190.0 | -64. -800. -16. -200.0 | 6. 270. 1. 65.0 | -38. -9.0 | 1.5 0.25 |
| BARTLETT  | 48. 360. 8. 80.0 | -32. -400. -8 -100.0 | 0. 120. -1. 25.0 | -16. -3.0 | 0 0.250 |
| HEISENBERG | 48. 360. 28. 190.0 | -32. -400. -16. -200.0 | 0. 120. 1. 65.0 | -16. -9.0 | 0 0.25 |
| MAJORANA  | 0. 120. 8. 80.0 | -16 . -200. -8 . -100.0 | -6. 30. -1. 25.0 | -2. -3.0 | -1.5 0.25 |
| COULOMB   | 28. 190. 0 0 | -16. -200. 0 0 | 1. 65. 0 0 | -9. | 0.25 |

62
Table 2
Roots and relative normalization constants $C^{(e)}/C^{(0)}$ for $^{16}$O and $^{40}$Ca systems.

| $\epsilon$ | $^{16}$O | $^{40}$Ca |
|-------------|----------|----------|
|             | Roots    | $C^{(e)}/C^{(0)}$ | Roots | $C^{(e)}/C^{(0)}$ |
| **WIGNER**  | -1.2590158014, -1.1462382460 | 1.0 | -1.1554853491, -1.0152702740 | 1.0 |
|              | (-0.9828722545, 0.0692920115) |         | (-0.9828722545, -0.0692920115) |         |
| **BARTLET** | -1.0, 1.0 | -0.2085327360 | +1.0, -1.0 | -1.036460531 |
|              | (-0.9723433872, 0.0576076251) |         | (-0.9723433872, -0.0576076251) |         |
| **HEISENBERG** | -1.0, 1.0 | -0.2085327360 | +1.0, -1.0 | -1.036460531 |
|              | (-0.9723433872, 0.0576076251) |         | (-0.9723433872, -0.0576076251) |         |
| **MAJORANA** | -0.9283077625, -0.6527109110 | -1.5213318399 | -0.9879721459, -0.7766080532 | -1.2591151327 |
|              | (-0.9604317388, 0.0448314378) |         | (-0.9604317388, -0.0448314378) |         |
| **COULOMB** | -1.9906725214, -1.0483085670 | 0.1319112107 | -1.4266427288, -1.0070162824 | 0.1493923245 |
|              | (-0.9781410398, 0.0634856828) |         | (-0.9781410398, -0.0634856828) |         |
Table 3
Roots and relative normalization constants $C^{(e)}/C^{(0)}$ for 8n and 20n systems.

| $\epsilon$ | 8n | 20n |
|------------|----|-----|
|            | Roots | $C^{(e)}/C^{(0)}$ | Roots | $C^{(e)}/C^{(0)}$ |
| WIGNER     | -8.6346734966 | 1.0 | -2.1965713866 | 1.0 |
|            | -1.1135782516 |    | -1.0145962721 |    |
|            | (-0.9502197308,0.1301817431) |   | (-0.9502197308,-0.1301817431) |   |
| BARTLET    | -0.9146656586 | 10.3986013985 | -0.9870485512 | -2.4763777976 |
|            | -0.0227923104 |    | -0.3424370373 |    |
|            | (-0.9502197308, 0.1037682674) |   | (-0.9502197308,-0.1037682674) |   |
| HEISENBERG | -8.6346734966 | 1.0 | -2.1965713866 | 1.0 |
|            | -1.1135782516 |    | -1.0145962721 |    |
|            | (-0.9502197308,0.1301817431) |   | (-0.9502197308,-0.1301817431) |   |
| MAJORANA   | -0.9146656586 | 10.3986013985 | -0.9870485512 | -2.4763777976 |
|            | -0.0227923104 |    | -0.3424370373 |    |
|            | (-0.9270670083, 0.1037682674) |   | (-0.9270670083,-0.1037682674) |   |
Table 4
Ground state energies $E_g$ (in MeV) and root mean square radius $r_{rms}$ (in fm) for $^4$He. The IDEA results are those of the uncoupled adiabatic approximation (UAA). The results $a$ in FHNC are with Gaussian and $b$ with Euler correlations. $E_C$ is the increase in the binding energy due to the Coulomb interaction.

| Potential | HCA    | HCA+Coul | EAA | UAA | $E_C$ | Other Methods |
|-----------|--------|----------|-----|-----|-------|---------------|
| B1[29]    | 29.292 | 28.473   | 39.169 | 38.162 | 0.810 | FHNC/1$^a$ [33] | 37.7 |
|           | (1.497) | (1.502)  | (1.420) | (1.422) | (1.425) | FHNC/1$^b$ [33] | 37.9 |
|           |        |          |       |      |       | FAHT/III [31]  | 36.6 |
|           |        |          |       |      |       | BHF [31]       | 36.9 |
|           |        |          |       |      |       | VMC [34,35]    | 36.4 |
| MS3[30,31]| 7.177  | 6.516    | 28.070 | 26.760 | 0.805 | FHNC/1$^a$ [33] | 24.7 |
|           | (1.894) | (1.910)  | (1.443) | (1.454) | (1.458) | FAHT/III [31]  | 24.2 |
|           |        |          |       |      |       | BHF [31]       | 25.0 |
|           |        |          |       |      |       | VMC [34,35]    | 23.9-26.5 |
| GPDT[32]  | 14.199 | 13.436   | 18.848 | 18.175 | 0.757 |               |     |
|           | (1.651) | (1.661)  | (1.593) | (1.603) | (1.608) |               |     |
Table 5
Same as Table 4 but for $^{16}$O

| Potential | HCA      | HCA+Coul | EAA     | UAA     | $E_C$ | Other Methods                  |
|-----------|----------|----------|---------|---------|-------|--------------------------------|
| B1[29]    | 106.529  | 93.135   | 164.777 | 164.332 | 13.727| FHNC/1$^a$ [33] 150.4         |
|           | (2.604)  | (2.628)  | (2.559) | (2.566) | (2.587)| FHNC/1$^b$ [33] 152.4         |
|           |          |          |         |         |       | FAHT/III [31] 163.7            |
|           |          |          |         |         |       | BHF [31] 163.7                 |
|           |          |          |         |         |       | VMC [34,35] 150.9± 0.3         |
|           |          |          |         |         |       | IDEA[47] 165.2                 |
|           |          |          |         |         |       | HHE[10] 152.10                |
|           |          |          |         |         |       | $E_C$[10] 13.78                |
| MS3[30,31]| 12.745   | 1.443    | 103.261 | 102.793 | 13.843| FHNC/1$^b$ [33] 105.3          |
|           | (3.079)  | (3.167)  | (2.539) | (2.535) | (2.563)| FAHT/III [31] 107.7            |
|           |          |          |         |         |       | BHF [31] 118.6                 |
|           |          |          |         |         |       | IDEA [47] 103.2                |
| GPDT[32]  | 61.867   | 47.740   | 100.796 | 100.688 | 14.305| HHE[10] 94.63                 |
|           | (2.4791) | (2.513)  | (2.484) | (2.484) | (2.528)| $E_C$ [10] 14.71              |
| Potential   | HCA    | HCA+Coul | EAA    | UAA    | $E_C$ | Other Methods                  |
|-------------|--------|----------|--------|--------|------|--------------------------------|
| B1[29]      | 323.355| 250.844  | 475.381| 475.345| 75.461| FHNC/1$^a$ [33] 471.0          |
|             | (3.341)| (3.397)  | (3.251)| (3.251)| (3.300)| FHNC/1$^b$ [33] 482.0          |
|             |        |          |        |        |      | FAHT/III [31] 478.0            |
|             |        |          |        |        |      | BHF [31] 507.2                 |
|             |        |          |        |        |      | VMC [34,35] 483.0± 0.4         |
|             |        |          |        |        |      | WFA [47] 447.8                 |
|             |        |          |        |        |      | IDEA [47] 483.0                |
|             |        |          |        |        |      | HHE [10] 468.14                |
|             |        |          |        |        |      | $E_C$ [10] 76.0                |
| MS3[30,31]  | 48.419 | no bound | 259.447| 259.328| 72.149| FHNC/1$^b$ [33] 350.0          |
|             | (3.868)|          | (3.375)| (3.372)| (3.457)| FAHT/III [31] 335.6            |
|             |        |          |        |        |      | BHF [31] 354.0                 |
|             |        |          |        |        |      | IDEA [47] 272.54               |
| GPDT[32]    | 255.952| 171.852  | 376.331| 376.110| 86.31| HHE [10] 363.53                |
|             | (2.868)| (2.951)  | (2.842)|(2.842) | (2.912)| $E_C$ [10] 86.0               |
Table 7
The coefficients $A'_n$, Eq. (150) for the two configurations considered for the $^{10}$B nucleus.

| $\epsilon$       | $(4s_{1/2})(4p_{1/2})(2p_{3/2})$ | $(4s_{1/2})(6p_{3/2})$ |
|------------------|----------------------------------|------------------------|
|                  | $A'_0$  $A'_1$  $A'_2$           | $A'_0$  $A'_1$  $A'_2$ |
| WIGNER           | 41/2    19     11/2              | 41/2    19     11/2    |
| BARTLET          | 1/2     46/3   1/2               | 1/2     40/3   1/2    |
| HEISENBERG       | -1/2    12     -1/2              | -1/2    12     -1/2    |
| MAJORANA         | -171/8  323/12 -51/8             | -156/8  284/12 -36/8  |
Table 8
Roots and relative normalization constants $C^{(e)}/C^{(0)}$ for the $^{10}$B nucleus.

| $\epsilon$   | $(4s_{1/2})(4p_{1/2})(2p_{3/2})$ | $(4s_{1/2})(6p_{3/2})$ |
|--------------|----------------------------------|------------------------|
|              | Roots                            | $C^{(e)}/C^{(0)}$      | Roots                              | $C^{(e)}/C^{(0)}$      |
| WIGNER       | $-2.4267116231$                  | 1.0                    | $-2.4267116231$                    | 1.0                    |
|              | $-1.2816012954$                  |                        | $-1.2816012954$                    |                        |
| BARTLET      | $-1.0057745600$                  | $-0.6712940280$        | $-1.0066449150$                    | $-0.5602373564$        |
|              | $1.5278733354$                   |                        | $1.6322400952$                     |                        |
| HEISENBERG   | $-0.9926668226$                  | $-0.8464804834$        | $-0.9926668226$                    | $-0.8464804834$        |
|              | $0.5786209032$                   |                        | $0.5786209032$                     |                        |
| MAJORANA     | $-0.8743416859$                  | $-0.5924519571$        | $-0.8590657779$                    | $-2.1204747128$        |
|              | $4.6270236827$                   |                        | $0.2787982623$                     |                        |
Table 9
Binding energies and rms radii for the $^{10}\text{B}$ nucleus

| Potential | $(4s_{1/2})(4p_{1/2})(2p_{3/2})$ | $(4s_{1/2})(6p_{3/2})$ |
|-----------|----------------------------------|------------------------|
|           | EAA     | UAA     | EAA     | UAA     |
| B1[29]    | 54.787  | 54.542  | 41.955  | 41.615  |
|          | (2.699) | (2.703) | (2.753) | (2.758) |
| MS3[30,31]| 33.475  | 32.941  | 22.504  | 22.252  |
|          | (2.648) | (2.665) | (2.756) | (2.764) |
| GPDT[32]  | 17.928  | 17.827  | 9.350   | 9.269   |
|          | (3.139) | (3.144) | (3.361) | (3.369) |
Table 10  
The $I_0(\ell_m, n)$ coefficient

| $n$ | 0   | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 16  | 100 | 400 | 1225| 3136|
| 1   | -8  | -100| -600| -2450| -7840|
| 2   | 1   | 35  | 345 | 1960| 8036|
| 3   | -5  | -290/3| -2450/3| -13328/3|
| 4   | 1/4 | 14  | 1169/6| 4424/3|
| 5   | -1  | -329/12| -4634/15|
| 6   | 1/36| 161/72| 749/18|
| 7   | -7/72| -161/45|
| 8   | 1/576| 17/90|
| 9   | -1/180|
| 10  | 1/14400 |
Table 11
The $I_{\alpha \beta}(\ell_m, n)$ coefficient. The symbol $[-n]$ denotes the exponential $10^{-n}$.

| $n$ | $\ell_m$ | 0 | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-----|----------|----|----|----|----|----|----|----|----|
| 0   |          | 1  | 4  | 10 | 20 | 35 | 56 | 84 | 120 |
| 1   |          | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2   |          | 1  | 5  | 15 | 35 | 70 | 126| 210|    |
| 3   |          | -1 | -6 | -21| -56| -126| 252|    |    |
| 4   |          | 1/4| 7/3| 133/12| 112/3| 203/2| 238|    |    |
| 5   |          | -1/3| -35/12| -14| -49| -140|    |    |    |
| 6   |          | 1/36| 35/72 | 623/180| 15.9833| 56.583|    |    |    |
| 7   |          | -1/24| -8/15 | -3.4833| -15.833|    |    |    |    |
| 8   |          | 1/576| 5.1389[-2]| 0.51458| 3.1268|    |    |    |    |
| 9   |          | 6.9444[-5]| 3.1713[-3]| 4.2774[-2]|    |    |    |    |    |
| 10  |          | 1.9290[-6]| 1.2676[-4]|    |    |    |    |    |    |
| 11  |          | -3.3069[-6]|    |    |    |    |    |    |    |
| 12  |          | 3.9367[-8]|    |    |    |    |    |    |    |