A NOTE ON FINITE LATTICES WITH MANY CONGRUENCES

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Abstract. By a twenty year old result of Ralph Freese, an \( n \)-element lattice \( L \) has at most \( 2^{n-1} \) congruences. We prove that if \( L \) has less than \( 2^{n-1} \) congruences, then it has at most \( 2^{n-2} \) congruences. Also, we describe the \( n \)-element lattices with exactly \( 2^{n-2} \) congruences.

1. Introduction and motivation

It follows from Lagrange’s Theorem that the size \(|S|\) of an arbitrary subgroup \( S \) of a finite group \( G \) is either \(|G|\), or it is at most the half of the maximum possible value, \(|G|/2\). Furthermore, if the size of \( S \) is the half of its maximum possible value, then \( S \) has some special property since it is normal. Our goal is to prove something similar on the size of the congruence lattice \( \text{Con}(L) \) of an \( n \)-element lattice \( L \).

For a finite lattice \( L \), the relation between \(|L|\) and \(|\text{Con}(L)|\) has been studied in some earlier papers, including Freese [3], Grätzer and Knapp [8], Grätzer, Lakser, and Schmidt [9], Grätzer, Rival, and Zaguia [10]. In particular, part (i) of Theorem 2.1 below is due to Freese [3]; note that we are going present a new proof of part (i). Although Czédli and Mureşan [2] and Mureşan [12] deal only with infinite lattices, they are also among the papers motivating the present one.

2. Our result and its proof

Mostly, we follow the terminology and notation of Grätzer [5]. In particular, the glued sum \( L_0 \dot{+} L_1 \) of finite lattices \( L_0 \) and \( L_1 \) is their Hall–Dilworth gluing along \( L_0 \cap L_1 = \{1_{L_0}\} = \{0_{L_1}\} \); see, for example, Grätzer [5, Section IV.2]. Note that \( \dot{+} \) is an associative operation. Our result is the following.

Theorem 2.1. If \( L \) is a finite lattice of size \( n = |L| \), then the following hold.

(i) \( L \) has at most \( 2^{n-1} \) many congruences. Furthermore, \(|\text{Con}(L)| = 2^{n-1}\) if and only if \( L \) is a chain.

(ii) If \( L \) has less than \( 2^{n-1} \) congruences, then it has at most \( 2^{n-1}/2 = 2^{n-2} \) congruences.

(iii) \(|\text{Con}(L)| = 2^{n-2}\) if and only if \( L \) is of the form \( C_1 \dot{+} B_2 \dot{+} C_2 \) such that \( C_1 \) and \( C_2 \) are chains and \( B_2 \) is the four-element Boolean lattice.

For \( n = 8 \), part (iii) of this theorem is illustrated in Figure 1. Note that part (i) of the theorem is due to Freese [3, page 3458]; however, our approach to Theorem 2.1 includes a new proof of part (i).
Proof of Theorem 2.1. We prove the theorem by induction on \( n = |L| \). Since the case \( n = 1 \) is clear, assume as an induction hypothesis that \( n > 1 \) is a natural number and all the three parts of the theorem hold for every lattice with size less than \( n \). Let \( L \) be a lattice with \( |L| = n \). For \( (a,b) \in L^2 \), the least congruence collapsing \( a \) and \( b \) will be denoted by \( \text{con}(a,b) \). A prime interval or an edge of \( L \) is an interval \([a,b]\) with \( a \prec b \). For later reference, note that
\[
(2.1) \quad \text{Con}(L) \text{ has an atom, and every of its atoms is of the form } \text{con}(a, b) \text{ for some prime interval } [a, b];
\]
this follows from the finiteness of \( \text{Con}(L) \) and from the fact that every congruence on \( L \) is the join of congruences generated by covering pairs of elements; see also Grätzer [7, page 39] for this folkloric fact.

Based on (2.1), pick a prime interval \([a, b]\) of \( L \) such that \( \Theta = \text{con}(a, b) \) is an atom in \( \text{Con}(L) \). Consider the map \( f: \text{Con}(L) \to \text{Con}(L) \) defined by \( \Psi \mapsto \Theta \lor \Psi \).

We claim that, with respect to \( f \),
\[
(2.2) \quad \text{every element of } f(\text{Con}(L)) \text{ has at most two preimages.}
\]
Suppose to the contrary that there are pairwise distinct \( \Psi_1, \Psi_2, \Psi_3 \in \text{Con}(L) \) with the same \( f \)-image. Since the \( \Theta \land \Psi_i \) belong to the two-element principal ideal \( \downarrow \Theta := \{ \Gamma \in \text{Con}(L) : \Gamma \leq \Theta \} \) of \( \text{Con}(L) \), at least two of these meets coincide. So we can assume that \( \Theta \land \Psi_1 = \Theta \land \Psi_2 \) and, of course, we have that \( \Theta \lor \Psi_1 = f(\Psi_1) = f(\Psi_2) = \Theta \lor \Psi_2 \). This means that both \( \Psi_1 \) and \( \Psi_2 \) are relative complements of \( \Theta \) in the interval \([\Theta \land \Psi_1, \Theta \lor \Psi_1]\). According to a classical result of Funayama and Nakayama [4], \( \text{Con}(L) \) is distributive. Since relative complements in distributive lattices are well-known to be unique, see, for example, Grätzer [5, Corollary 103], it follows that \( \Psi_1 = \Psi_2 \). This is a contradiction proving (2.2).

Clearly, \( f \) is a retraction map onto the filter \( \uparrow \Theta \). It follows from (2.2) that \( |\uparrow \Theta| \geq |\text{Con}(L)|/2 \). Also, by the well-known Correspondence Theorem, see Burris and Sankappanawar [1, Theorem 6.20], or see Theorem 5.4 (under the name Second Isomorphism Theorem) in Nation [11], \( |\uparrow \Theta| = |\text{Con}(L/\Theta)| \) holds. Hence, it follows that
\[
(2.3) \quad |\text{Con}(L/\Theta)| \geq \frac{1}{2} \cdot |\text{Con}(L)|.
\]
Since \( \Theta \) collapses at least one pair of distinct elements, \( (a,b) \), we have that \( |L/\Theta| \leq n-1 \). Thus, it follows from part (i) of the induction hypothesis that \( |\text{Con}(L/\Theta)| \leq 2^{(n-1)-1} = 2^{n-2} \). Combining this inequality with (2.3), we obtain that \( |\text{Con}(L)| \leq 2 \cdot |\text{Con}(L/\Theta)| \leq 2^{n-1} \). This shows the first half of part (i).

If \( L \) is a chain, then \( \text{Con}(L) \) is known to be the \( 2^{n-1} \)-element boolean lattice; see, for example, Grätzer [7, Corollaries 3.11 and 3.12]. Hence, we have that \( |\text{Con}(L)| = \)
2^{n-1}$ if $L$ is a chain. Conversely, assume the validity of $|\text{Con}(L)| = 2^{n-1}$, and let $k = |L/\Theta|$. By the induction hypothesis, $|\text{Con}(L/\Theta)| \leq 2^{k-1}$. On the other hand, $|\text{Con}(L/\Theta)| \geq |\text{Con}(L)|/2 = 2^{n-2}$ holds by (2.3). These two inequalities and $k < n$ yield that $k = n-1$ and also that $|\text{Con}(L/\Theta)| = 2^{n-2} = 2^{k-1}$. Hence, the induction hypothesis implies that $L/\Theta$ is a chain. For the sake of contradiction, suppose that $L$ is not a chain, and pick a pair $(u, v)$ of incomparable elements of $L$. The $\Theta$-blocks $u/\Theta$ and $v/\Theta$ are comparable elements of the chain $L/\Theta$, whence we can assume that $u/\Theta \leq v/\Theta$. It follows that $u/\Theta = u/\Theta \wedge v/\Theta = (u \wedge v)/\Theta$ and, by duality, $v/\Theta = (u \lor v)/\Theta$. Thus, since $u, v, u \wedge v$ and $u \lor v$ are pairwise distinct elements of $L$ and $\Theta$ collapses both of the pairs $(u \wedge v, u)$ and $(v, u \lor v)$, we have that $k = |L/\Theta| < n - 2$, which is a contradiction. This proves part (i) of the theorem.

As usual, for a lattice $K$, let $J(K)$ and $M(K)$ denote the set of nonzero join-irreducible elements and the set of meet-irreducible elements distinct from 1, respectively. By a narrows we will mean a prime interval $[a, b]$ such that $a \in M(L)$ and $b \in J(L)$. Using Grätzer [6], it follows in a straightforward way that

$$\text{(2.4)}$$

- if $[a, b]$ is a narrows, then $\{a, b\}$ is the
- only non-singleton block of $\text{con}(a, b)$.

Now, in order to prove part (ii) of the theorem, assume that $|\text{Con}(L)| < 2^{n-1}$. By (2.1), we can pick a prime interval $[a, b]$ such that $\Theta := \text{con}(a, b)$ is an atom in $\text{Con}(L)$. There are two cases to consider depending on whether $[a, b]$ is a narrows or not; for later reference, some parts of the arguments for these two cases will be summarized in (2.5) and (2.6) redundantly. First, we deal with the case where $[a, b]$ is a narrows. We claim that

$$\text{(2.5)}$$

- if $|\text{Con}(L)| < 2^{n-1}$, $[a, b]$ is a narrows, and $\Theta = \text{con}(a, b)$
- is an atom in $\text{Con}(L)$, then $L/\Theta$ is not a chain.

By (2.4), $|L/\Theta| = n - 1$. By the already proved part (i), $L$ is not a chain, whence there are $u, v \in L$ such that $u \parallel v$. We claim that $u/\Theta$ and $v/\Theta$ are incomparable elements of $L/\Theta$. Suppose the contrary. Since $u$ and $v$ play a symmetric role, we can assume that $u/\Theta \lor v/\Theta = v/\Theta$, i.e., $(u \lor v)/\Theta = v/\Theta$. But $u \lor v \neq v$ since $u \parallel v$, whereby (2.4) gives that $\{v, u \lor v\} = \{a, b\}$. Since $a < b$, this means that $v = a$ and $u \lor v = b$. Thus, $u \lor v \in J(L)$ since $[a, b]$ is a narrows. The membership $u \lor v \in J(L)$ gives that $u \lor v \in \{u, v\}$, contradicting $u \parallel v$. This shows that $u/\Theta \parallel v/\Theta$, whence $L/\Theta$ is not a chain. We have shown the validity of (2.5). Using part (i) and $|L/\Theta| = n - 1$, it follows that $|\text{Con}(L/\Theta)| < 2^{(n-1)-1}$. By the induction hypothesis, we can apply (ii) to $L/\Theta$ to conclude that $|\text{Con}(L/\Theta)| \leq 2^{(n-1)-2}$. This inequality and (2.3) yield that $|\text{Con}(L)| \leq 2 \cdot |\text{Con}(L/\Theta)| \leq 2^{n-2}$, as required.

Second, assume that $[a, b]$ is not a narrows. Our immediate plan is to show that

$$\text{(2.6)}$$

- if a prime interval $[a, b]$ of $L$ is not a narrows
- and $\Theta = \text{con}(a, b)$, then $|L/\Theta| \leq n - 2$.

By duality, we can assume that $a$ is meet-reducible. Hence, we can pick an element $c \in L$ such that $a < c$ and $c \neq b$. Clearly, $c \neq b \lor c$ and $\Theta = \text{con}(a, b)$ collapses both $(a, b)$ and $(c, b \lor c)$, which are distinct pairs. Thus, we obtain that $|L/\Theta| \leq n - 2$, proving (2.6). Hence, $\text{Con}(L/\Theta) \leq 2^{n-3}$ by part (i) of the induction hypothesis. Combining this inequality with (2.3), we obtain the required inequality $\text{Con}(L) \leq 2^{n-2}$. This completes the induction step for part (ii).

Next, in order to perform the induction step for part (iii), we assume that $|\text{Con}(L)| = 2^{n-2}$. Again, there are two cases to consider. First, we assume
that there exists a narrows \([a, b]\) in \(L\) such that \(\Theta := \text{con}(a, b)\) is an atom in \(\text{Con}(L)\). Then \(|L/\Theta| = n - 1\) by (2.4) and \(L/\Theta\) is not a chain by (2.5). By the induction hypothesis, parts (i) and (ii) hold for \(L/\Theta\), whereby we have that \(|\text{Con}(L/\Theta)| \leq 2^{(n-1)-2} = 2^{n-3}\). On the other hand, it follows from (2.3) that \(|\text{Con}(L/\Theta)| \geq |\text{Con}(L)|/2 = 2^{n-3}\). Hence, \(|\text{Con}(L/\Theta)| = 2^{n-3} = 2^{(L/\Theta)-2}\). By the induction hypothesis, \(L/\Theta\) is of the form \(C_1 + B_2 + C_2\). We know from (2.4) that \(\{a, b\} = [a, b]\) is the unique non-singleton \(\Theta\)-block. If this \(\Theta\)-block is outside \(B_2\), then \(L\) is obviously of the required form. If the \(\Theta\)-block \(\{a, b\}\) is in \(C_2 \cap B_2\), then \(L\) is of the required form simply because the situation on the left of Figure 2 would contradict the fact that \([a, b]\) is a narrows. A dual treatment applies for the case \(\{a, b\} \in C_1 \cap B_2\). If the \(\Theta\)-block \(\{a, b\}\) is in \(B_2 \setminus (C_1 \cup C_2)\), then \(L\) is of the form \(C_1 + N_5 + C_2\), where \(N_5\) is the “pentagon”; see the middle part of Figure 2. For an arbitrary bounded lattice \(K\) and the two-element chain \(2\), it is straightforward to see that

\[
\text{Con}(K + 2) \cong \text{Con}(2 + K) \cong \text{Con}(K) \times 2.
\]

A trivial induction based on (2.7) yields that \(|\text{Con}(C_1 + N_5 + C_2)|\) is divisible by \(5 = |\text{Con}(N_5)|\). But 5 does not divide \(|\text{Con}(L)| = 2^{n-2}\), ruling out the case that the \(\Theta\)-block \(\{a, b\}\) is in \(B_2 \setminus (C_1 \cup C_2)\). Hence, \(L\) is of the required form.

Second, we assume that no narrows in \(L\) generates an atom of \(\text{Con}(L)\). By (2.1), we can pick a prime interval \([a, b]\) such that \(\Theta := \text{con}(a, b)\) is an atom of \(\text{Con}(L)\). Since \([a, b]\) is not a narrows, (2.6) gives that \(|L/\Theta| \leq n - 2\). We claim that we have equality here, that is, \(|L/\Theta| = n - 2\). Suppose to the contrary that \(|L/\Theta| \leq n - 3\). Then part (i) and (2.3) yield that

\[2^{n-2} = |\text{Con}(L)| \leq 2 \cdot |\text{Con}(L/\Theta)| \leq 2 \cdot 2^{(n-3)-1} = 2^{n-3},\]

which is a contradiction. Hence, \(|L/\Theta| = n - 2\). Thus, we obtain from by part (i) that \(|\text{Con}(L/\Theta)| \leq 2^{n-3}\). On the other hand, (2.3) yields that \(|\text{Con}(L/\Theta)| \geq |\text{Con}(L)|/2 = 2^{n-3}\), whence \(|\text{Con}(L/\Theta)| = 2^{n-3} = 2^{(L/\Theta)-1}\), and it follows by part (i) that \(L/\Theta\) is a chain. Now, we have to look at the prime interval \([a, b]\) closely. It is not a narrows, whereby duality allows us to assume that \(b\) is not the only cover of \(a\). So we can pick an element \(c \in L \setminus \{b\}\) such that \(a \prec c\), and let \(d := b \lor c\); see on the right of Figure 2. Since \((c, d) = \{c \lor a, c \lor b\} \in \text{con}(a, b) = \Theta\), any two elements of \([c, d]\) is collapsed by \(\Theta\). Using \((a, b) \in \Theta\), \((c, d) \in \Theta\), and \(|L/\Theta| = n - 2 = |L| - 2\), it follows that there is no “internal element” in the interval \([c, d]\), that is, \(c \prec d\).

Furthermore, \([a, b] = \{a, b\}\) and \([c, d] = \{c, d\}\) are the only non-singleton blocks of

![Figure 2. Illustrations for the proof](image-url)
Θ. In order to show that \( b < d \), suppose to the contrary that \( b < e < d \) holds for some \( e \in L \). Since \( d = b \lor c \leq e \lor c \leq d \), we have that \( e \lor c = d \), implying \( c \not\preceq e \).

Hence, \( c \land e < e \). Since \( (c \land e, e) \subseteq (c \land e, d \land e) \subseteq \Theta \), the \( \Theta \)-block of \( e \) is not a singleton. This contradicts the fact that \( \{a, b\} \) and \( \{c, d\} \) are the only non-singleton \( \Theta \)-blocks, whereby we conclude that \( b < d \). The covering relations established so far show that \( S := \{a = b \lor c, b, c, d = b \lor c\} \) is a covering square in \( L \). We know that both non-singleton \( \Theta \)-blocks are subsets of \( S \) and \( L/\Theta \) is a chain. Consequently, \( L \setminus S \) is also a chain.

Hence, to complete the analysis of the second case when \( \{a, b\} \) is not a narrows, it suffices to show that for every \( x \in L \setminus S \), we have that either \( x \leq a \) or \( x \geq d \).

So, assume that \( x \in L \setminus S \). Since \( L/\Theta \) is a chain, \( \{a, b\} \) and \( \{x\} \) are comparable in \( L/\Theta \). If \( \{x\} < \{a, b\} \), then \( \{x\} \lor \{a, b\} = \{a, b\} \) gives that \( x \lor a \in \{a, b\} \). If \( x \lor a \) happens to equal \( b \), then \( x \not\preceq a \) leads to \( x \land a < x \) and \( \langle x \land a, x \land b \rangle \in \Theta \), contradicting the fact the \( \{a, b\} \) and \( \{c, d\} \) are the only non-singleton \( \Theta \)-blocks. So if \( \{x\} < \{a, b\} \), then \( x \lor a = a \) and \( x < a \), as required. Thus, we can assume that \( \{x\} > \{a, b\} \). If \( \{x\} > \{c, d\} \), then the dual of the easy argument just completed shows that \( x \geq d \). So, we are left with the case \( \{a, b\} < \{x\} < \{c, d\} \). Then the equalities \( \{a, b\} \lor \{x\} = \{x\} \) and \( \{x\} = \{x\} \land \{c, d\} \) give that \( b \lor x = x = d \lor x \), that is, \( b \leq x \leq d \). But \( x \notin S \), so \( b < x < d \), contradicting \( b < d \). This completes the second case of the induction step for part (iii) and the proof of Theorem 2.1. □

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