Orthogonal Basis Function Over the Unit Circle with the Minimax Property

Richard J. Mathar
Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany
(Dated: February 28, 2018)

We construct an orthogonal basis of functions defined over the unit circle as the product of the common sinusoidal functions of the azimuth angle by radial functions which are essentially sines of a polynomials of the radial distance to the origin. The main impetus of this approach is to generate basis functions where the minima and maxima along both coordinates, the azimuth and the distance \( r \) to the center, have the same amplitude, akin to the Chebyshev polynomial basis of the one-dimensional unit interval. The construction is based on numerical evaluation of the overlap integrals, which have the format of generalized Fresnel integrals.

PACS numbers: 42.15.Fr, 95.75.Pq
Keywords: Circle Polynomial, Orthogonal Basis, Chebyshev

I. AIM AND SCOPE

The celebrated Zernike polynomials define an orthogonal system of functions that are products \( M_m(\varphi)R_n^m(r) \) of azimuthal functions \( M_m \) over \( 0 \leq \varphi \leq 2\pi \) by radial polynomials \( R_n^m(r) \) depending on the distance to the circle center, \( 0 \leq r \leq 1 \) [8]. Some weakness of that system is that the polynomials start to oscillate with increasing amplitude near \( r \to 1 \) for large \( n \), such that the coefficients of the expansion of some function of interest over that basis do not convey well how important these functions are compared to the amplitudes contributed by the other bases concentrated nearer to the origin.

We shall replace these unevenly wiggly radial polynomials by functions \( Q_{n,m}(r) \) that have the minimax property of the Chebyshev Polynomials of the first kind [2], which means by functions where successive minima and maxima along \( r \) have equal absolute value. The difference to the Chebyshev Polynomials is of course the weight \( r \) in the orthogonality relation of the overlap integrals:

\[
\int_0^1 rQ_{n,m}(r)Q_{n',m}(r)dr = \delta_{n,n'}.
\]

As the azimuthal function are likewise orthogonal,

\[
\int_0^{2\pi} M_m(\varphi)M_{m'}(\varphi)d\varphi = \delta_{m,m'},
\]

\[
M_m(\varphi) = \sqrt{\epsilon_m/2\pi} \times \begin{cases} \cos(m\varphi), & m \text{ even;} \\ \sin(m\varphi), & m \text{ odd}, \end{cases}
\]

(with \( \epsilon_0 = 1 \) and \( \epsilon_{|m|\geq1} = 2 \)) the orthogonality of the \( Q \)-basis is only concerned with orthogonality with respect to the index \( n \).

Among various strategies to approach the minimax property, we select one possible route; we abandon the idea of expressing \( Q \) as polynomials—giving up some of the associated nice analytical results of the Zernike basis [7, 9, 16] or other orthogonal bases [14]— and enforce the minimax property by using sines and cosines also along the radial coordinate.

II. DESIGN OF CIRCLE FUNCTIONS

For odd \( m \) we construct radial functions

\[
Q_{n,m}(r) = N_{n,m} \sin\theta_{n,m}(r), \quad n = m, m+2, m+4, \ldots, \quad m \text{ odd}
\]

* http://www.mpia.de/~mathar
and for even \( m \) we construct radial functions

\[
Q_{n,m}(r) = \begin{cases} 
N_{n,m} \sin \theta_{n,m}(r), & n = m, m + 2, m + 4, \ldots, \ m > 0, \ m \text{ even} \\
N_{n,0} \cos \theta_{n,0}(r), & n = 0, 2, 4, \ldots, \ m = 0
\end{cases}
\] (5)

The cosine terms of \( m = 0 \) are chosen to yield nonzero values at the origin, representing forms of astigmatism. The parities are kept in accordance with the Zernike polynomials:

\[
\theta_{n,m}(-r) = (-)^m \theta_{n,m}(r),
\]

and therefore

\[
Q_{n,m}(-r) = (-)^m Q_{n,m}(r).
\] (7)

The normalization coefficients \( N_{n,m} \) are computed to comply with (5); the sign of \( N_{n,m} \) is chosen to keep \( Q_{n,m}(1) > 0 \) for compatibility with the Zernike radial functions.

There are some more design choices for compatibility with the Zernike radial functions:

1. \( Q_{n,n}(r) \) rises monotonously from \( r = 0 \) to \( r = 1 \) with a single local maximum at the rim \( r = 1 \),
2. \( Q_{n,m-2}(r) \) has one more extremum than \( Q_{n,m}(r) \) in the range \( 0 \leq r \leq 1 \),
3. the lowest-order Taylor coefficient near \( r \to 0 \) is of degree

\[
Q_{n,m}(r) \sim r^m.
\] (8)

The final fixture of the \( Q \)-functions is to let the phase functions \( \theta_{n,m}(r) \) be polynomials of \( r \),

\[
\theta_{n,m}(r) = \sum_{i=m,m+2,\ldots,n} \beta_{n,m,i} r^i.
\] (9)

such that the integrals (1) are actually generalized Fresnel integrals of chirp functions which are sums or differences of the radial polynomials \( \vartheta(r) \) [10].

### III. RADIAL PHASE POLYNOMIALS

#### A. Monotonicity

The desired number of extrema is established by driving the phase of the sine-functions monotonously from 0 up to multiples of \( \pi/2 \) as \( r \) runs from 0 to 1:

\[
\theta_{n,n-2l}(1) = (1 + 2l)\frac{\pi}{2}; \quad m = n - 2l > 0
\] (10)

\[
\theta_{n,0}(1) = n\frac{\pi}{2}; \quad m = 0, n = 0, 2, 4, \ldots
\] (11)

This clamps the radial functions at the right end-point of the interval. We could express the polynomials also in an equivalent Bernstein polynomial basis (Appendix B):

\[
\theta_{n,m}(r) \equiv r^m \vartheta_{n,m}(1) \sum_{i=0}^{(n-m)/2} \alpha_{n,m,i} B_{i,(n-m)/2}(r^2).
\] (12)

Dropping a factor \( x^m \) and the constant \( \vartheta_{n,m}(1) \) reduces the information about the radial functions to the expansion coefficients \( \alpha \) of “reduced” radial phase polynomials \( \bar{\vartheta} \):

\[
\bar{\theta}_{n,m}(r) \equiv \sum_{i=0}^{(n-m)/2} \alpha_{n,m,i} B_{i,(n-m)/2}(r^2).
\] (13)

The aim is to find the expansion coefficients \( \beta_{n,m,i} \) or \( \alpha_{n,m,i} \). The procedure is to define \( Q_{m,m} \) first by finding \( \alpha_{m,m,i} \) and then to calculate recursively the \( Q_{m+2l,m} \) for \( l = 1, 2, \ldots \) by fixing their \( l \) coefficients \( \alpha_{m+2l,m,i} \) by enforcing the orthogonality to the already established values at smaller \( n \). The \( 1 + (n-m)/2 \) degrees of freedom embodied in the coefficients of (12) match the one degree to clamp the functions at \( r = 1 \) plus \( (n-m)/2 \) degrees to shape the functions to stay orthogonal with the already established functions of the same \( m \) but smaller \( n \).
B. Special Cases

The piston term at \( n = m = 0 \) has the same constant value as for the Zernike basis:

\[
\theta_{0,0}(r) = 0 \quad \therefore \cos \theta_{0,0}(r) = 1 \quad \therefore N_{0,0} = \sqrt{2} \quad \therefore Q_{0,0}(r) = \sqrt{2}.
\] (14)

The orthogonality

\[
\int_0^1 r \cos(n \frac{\pi}{2} r^2) \cos(n' \frac{\pi}{2} r^2) dr = \frac{1}{2} \int_0^1 \cos(n \frac{\pi}{2} r) \cos(n' \frac{\pi}{2} r) dr = \frac{1}{4} \delta_{n,n'}, \quad n - n' \text{ even, } (n,n') \neq (0,0)
\] (15)
yields simple phase polynomial coefficients of \( m = 0 \) and all (even) \( n \):

\[
\beta_{n,0,i} = \begin{cases} n \frac{\pi}{2}, & i = 2; \\ 0, & i \neq 2 \end{cases}
\] (16)

Likewise

\[
\int_0^1 r \sin(n \frac{\pi}{2} r^2) \sin(n' \frac{\pi}{2} r^2) dr = \frac{1}{2} \int_0^1 \sin(n \frac{\pi}{2} r) \sin(n' \frac{\pi}{2} r) dr = \frac{1}{4} \delta_{n,n'}, \quad n - n' \text{ even, } (n,n') \neq (0,0),
\] (18)
yields simple phase polynomial coefficients for \( m = 2 \) and all (even) \( n \):

\[
\beta_{n,2,i} = \begin{cases} (n - 1) \frac{\pi}{2}, & i = 2; \\ 0, & i \neq 2 \end{cases}
\] (19)

\[
N_{n,0} = (-1)^{n/2} \times 2.
\] (17)

\[
N_{n,2} = (-1)^{1+n/2} \times 2.
\] (20)

For \( n = m \) the design choice \( \theta \propto r^m \) (8) and the limit (10) on the circle rim enforces

\[
\theta_{n,n}(r) = \frac{\pi}{2} r^n = \frac{\pi}{2} B_{n,n}(r) \quad \therefore \beta_{n,n,i} = \begin{cases} \frac{\pi}{2}, & i = n; \\ 0, & i \neq n. \end{cases}
\] (21)

The normalization constant \( N_{n,n} \) allows a semi-analytical treatment:

\[
\int_0^1 r dr N_{n,n}^2 \sin^2(\frac{\pi}{2} r^n) = 1.
\] (22)

A power series is \([4, 1.412.1]\]

\[
\sin^2 x = \frac{1}{2} [1 - \cos(2x)] = \sum_{k \geq 1} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!} = x^2 - \frac{1}{3} x^4 + \frac{2}{45} x^6 - \frac{1}{315} x^8 + \frac{2}{14175} x^{10} - \frac{2}{467775} x^{12} + \cdots.
\] (23)

Insertion into the integrand gives

\[
r \sin \frac{\pi r^n}{2} = \sum_{k \geq 1} (-1)^{k+1} \frac{\pi^{2k} r^{2k+1}}{2(2k)!} = \frac{\pi^2}{2^2} r^{2n+1} - \frac{\pi^4}{3 \cdot 2^4} r^{4n+1} + \frac{2 \pi^6}{45 \cdot 2^6} r^{6n+1} - \frac{\pi^8}{315 \cdot 2^8} r^{8n+1} + \frac{2 \pi^{10}}{14175 \cdot 2^{10}} r^{10n+1} - \cdots.
\] (24)

According to (22), \( N_{n,n} \) is the inverse of the square root of the following hypergeometric function:

\[
\int_0^1 r \sin \frac{\pi r^n}{2} dr = \sum_{k \geq 1} (-1)^{k+1} \frac{\pi^{2k}}{2(2kn+2)(2k)!} \frac{1}{4} - \frac{1}{4} F_2 \left( \frac{1}{2}; 1 + \frac{1}{n}, \frac{1}{2}; \frac{\pi^2}{4} \right)
\]

\[
= \frac{\pi^2}{2^2(2n+2)} - \frac{\pi^4}{3 \cdot 2^4(4n+2)} + \frac{\pi^6}{45 \cdot 2^6(6n+2)} - \frac{\pi^8}{315 \cdot 2^8(8n+2)} + \frac{2 \pi^{10}}{14175 \cdot 2^{10}(10n+2)} - \cdots
\] (25)
In particular

\[
N_{1,1} = \frac{2\pi}{\sqrt{\pi^2 + 4}} \approx 1.68712721613; \quad (26)
\]
\[
N_{2,2} = 2; \quad (27)
\]
\[
N_{3,3} \approx 2.27799236632; \quad (28)
\]
\[
N_{4,4} \approx 2.52776603703; \quad (29)
\]
\[
N_{5,5} \approx 2.75587198375, \quad (30)
\]

which are useful indicators for error bars in the numerical results of Appendix A.

C. Numerical Synthesis

The numerical analysis fixes \( Q_{n,n}(r) \) via (21) and rewrites in a loop over \( m = n - 2, n - 4, \ldots \) the orthogonality requirements (1) and the limiting value (10) by the following system of equations (for \( m > 0 \)):

\[
\int_0^1 r dr \sin \left[ \sum_{i=m,m+2,\ldots,n} \beta_{n,m,i} r^i \right] \sin \left[ \sum_{i=m,m+2,\ldots,n-2} \beta_{n-2,m,i} r^i \right] = 0; \quad (31)
\]
\[
\int_0^1 r dr \sin \left[ \sum_{i=m,m+2,\ldots,n} \beta_{n,m,i} r^i \right] \sin \left[ \sum_{i=m,m+2,\ldots,n-4} \beta_{n-4,m,i} r^i \right] = 0; \quad (32)
\]
\[
\ldots = 0; \quad (33)
\]
\[
\int_0^1 r dr \sin \left[ \sum_{i=m,m+2,\ldots,n} \beta_{n,m,i} r^i \right] \sin \left[ \sum_{i=m,m+2,\ldots,n-4} \beta_{n-4,m,i} r^i \right] = 0; \quad (34)
\]
\[
\sum_{i=m,m+2,\ldots,n} \beta_{n,m,i} = (1 + n - m) \frac{\pi}{2}. \quad (35)
\]

The \( 1 + (n - m)/2 \) unknown values are \( \beta_{n,m,m}, \beta_{n,m,m+2}, \ldots, \beta_{n,m,n} \). The last equation is used to eliminate \( \beta_{n,m,n} \) in all earlier ones and we end up with a system of \( (n - m)/2 \) nonlinear equations

\[
\int_0^1 r dr \sin \left[ \frac{1 + n - m}{2} r^n + \sum_{i=m,m+2,\ldots,n-2} \beta_{n,m,i} (r^i - r^n) \right] \sin \left[ \sum_{i=m,m+2,\ldots,n-2} \beta_{n-2,m,i} r^i \right] = 0; \quad (36)
\]
\[
\int_0^1 r dr \sin \left[ \frac{1 + n - m}{2} r^n + \sum_{i=m,m+2,\ldots,n-2} \beta_{n,m,i} (r^i - r^n) \right] \sin \left[ \sum_{i=m,m+2,\ldots,n-4} \beta_{n-4,m,i} r^i \right] = 0; \quad (37)
\]
\[
\ldots = 0; \quad (38)
\]
\[
\int_0^1 r dr \sin \left[ \frac{1 + n - m}{2} r^n + \sum_{i=m,m+2,\ldots,n-2} \beta_{n,m,i} (r^i - r^n) \right] \sin \left[ \sum_{i=m,m+2,\ldots,n-4} \beta_{n,m,i} r^i \right] = 0. \quad (39)
\]

This system is solved recursively by the common multivariate Newton-method: insert a set of initial guesses \( \{ \beta_{n,m,i} \} \), evaluate the deviations of the right hand side from zero, compute the matrix of derivatives of the left hand side (the Jacobian) with respect to the unknowns (which essentially means replacing \( r \to r(r^i - r^n) \) and replacing the first sine by a cosine), and solve the associated linear system of equations to calculate updates of the unknown parameters. All the integrals are evaluated numerically.

The role of the Bernstein basis for the radial phase polynomials is that the \( \alpha \)-coefficients are smaller than the \( \beta \)-coefficients and essentially of equal sign, so cancellation of digits is a lesser problem. Instead of (31)-(35) we may
solve
\[
\int_0^1 r dr \sin \left[ r^m \theta_{n,m}(1) \sum_{i=0}^{n-m} \alpha_{n,m,i} B_{i,(n-m)/2}(r^2) \right] \sin \left[ r^m \theta_{n-2,m}(1) \sum_{i=0}^{n-m-2} \alpha_{n-2,m,i} B_{i,(n-m-2)/2}(r^2) \right] = 0; \quad (40)
\]
\[
\int_0^1 r dr \sin \left[ r^m \theta_{n,m}(1) \sum_{i=0}^{n-m} \alpha_{n,m,i} B_{i,(n-m)/2}(r^2) \right] \sin \left[ r^m \theta_{n-4,m}(1) \sum_{i=0}^{n-m-4} \alpha_{n-4,m,i} B_{i,(n-m-4)/2}(r^2) \right] = 0; \quad (41)
\]
\[
\int_0^1 r dr \sin \left[ r^m \theta_{n,m}(1) \sum_{i=0}^{n-m} \alpha_{n,m,i} B_{i,(n-m)/2}(r^2) \right] \sin \left[ r^m \theta_{m,m}(1) \sum_{i=0}^{n-m} \alpha_{m,m,i} B_{i,0}(r^2) \right] = 0; \quad (43)
\]
\[
\alpha_{n,m,(n-m)/2} = 1; \quad (44)
\]

The last equation is inserted into the initial set of equations, so the orthogonality requirements are:
\[
\int_0^1 r dr \sin \left[ r^n \theta_{n,m}(1) + r^m \theta_{n,m}(1) \sum_{i=0}^{n-m-1} \alpha_{n,m,i} B_{i,n-m}(r^2) \right] \sin \left[ r^m \theta_{n-2,m}(1) \sum_{i=0}^{n-m-2} \alpha_{n-2,m,i} B_{i,2-n-2}(r^2) \right] = 0; \quad (45)
\]
\[
\int_0^1 r dr \sin \left[ r^n \theta_{n,m}(1) + r^m \theta_{n,m}(1) \sum_{i=0}^{n-m-1} \alpha_{n,m,i} B_{i,n-m}(r^2) \right] \sin \left[ r^m \theta_{n-4,m}(1) \sum_{i=0}^{n-m-4} \alpha_{n-4,m,i} B_{i,4-n-4}(r^2) \right] = 0; \quad (46)
\]
\[
\int_0^1 r dr \sin \left[ r^n \theta_{n,m}(1) + r^m \theta_{n,m}(1) \sum_{i=0}^{n-m-1} \alpha_{n,m,i} B_{i,n-m}(r^2) \right] \sin \left[ r^m \theta_{m,m}(1) \sum_{i=0}^{n-m} \alpha_{m,m,i} B_{i,0}(r^2) \right] = 0; \quad (48)
\]
The unknown values \(\alpha_{n,m,0}, \alpha_{n,m,1}, \ldots, \alpha_{n,m,(n-m-2)/2}\) are extracted with again with the multivariate Newton method.

\[\text{D. Initial values}\]

A numerical problem here is that for most guesses for the \(\beta_{n,m,i}\) this Newton procedure branches into parameter sets that yield non-monotonous \(Q_{n,m}(r)\), which fulfill the orthogonality relations but define radial functions that violate the minimax principle. Two rules of thumb to start with guesses in the attractor region are:

1. A good initial guess proposed by heuristics of the expansion coefficients of the Bernstein polynomials is
\[
\alpha_{n,n-2,0} \approx 3 - \pi + \left( \frac{\pi}{4} - \frac{1}{2} \right)n \approx - \frac{1}{6} + \frac{7}{24} n,
\]
so
\[
\beta_{n,n-2,n-2} = \frac{3}{2} \pi \alpha_{n,n-2,0}, \quad \beta_{n,n-2,n} = \frac{3}{2} \pi (1 - \alpha_{n,n-2,0}).
\]

2. For systems with 3 or more unknown values one may “lift” an already known coefficient set of a smaller \(n\) as
\[
\beta_{n,m,m} \approx \pi + \beta_{n-2,m,m};
\]
\[
\beta_{n,m,i} \approx \beta_{n-2,m,i}, \quad m + 2 \leq i \leq n - 2;
\]
\[
\beta_{n,m,n} \approx 0.
\]

Although these guesses are far off the final, converged values, they seem to steer towards the correct phase polynomials if sufficient damping is included in the Newton updates.

The first few radial functions for azimuthal quantum numbers \(m = 1, 4\) and 5 are plotted in Figures 1–4.
FIG. 1. Radial functions $Q_{n,1}(r)$ and associated reduced polynomials $\theta_{n,1}(r)$. The “tip-tilt” representative $Q_{1,1}$ is no longer a straight line as it used to be in the Zernike case.

FIG. 2. Radial functions $Q_{n,2}(r)$ in comparison with normalized Zernike polynomials $\sqrt{2n+2}R_{n}^{2}(r)$.

Appendix A: Expansion Coefficients

The results of the numerical procedure are summarized complete for $n \leq 22$ in the following table for all $0 < m \leq n$. The coefficients for $m = 0$ are precisely known by Eqs. (16) and (17) and not listed. The coefficients for $m = 2$ are precisely known by Eqs. (19) and (20) but are run through the same numerical procedure to produce indicators of error bars.

The estimated accuracy of the numerical coefficients is of the order of four digits, although the values are printed to much larger apparent accuracy. They have been computed by evaluating the integrals (45)–(48) with a Romberg rule over 4096 sampling points in the interval $0 \leq r \leq 1$. Superior Gauss-Legendre methods \[5, 6, 15\] have not been employed.

An entry contains three types of lines.

1. A value of $n$, a value of $m$, the letter $N$ and the signed normalization constant $N_{n,m}$;
2. A value of $n$, a value of $m$, a value of $i$, the letter $\tau$ and the coefficient $\beta_{n,m,i}$;
3. A value of $n$, a value of $m$, a value of $i$, the letter $B$ and the coefficient $\alpha_{n,m,i}$. 
The fields in the line are separated by a blank. To save some paper, three columns of these entries have been printed in a single line of the print.

1  1  N  1.6871271565051231  1  1  0  B  1.0000000000000000  1  1  t  1.5707963267948966
2  2  N  1.99999905588837  2  2  0  B  1.0000000000000000  2  2  t  1.5707963267948966
3  3  N  2.277922195323444  3  1  N  -1.9479336816837157  3  1  0  B  0.6943837797470318  3  1  0  B  1.0000000000000000  3  1  t  1.5707963267948966
3  3  t  3.1707963267948966  3  1  B  1.0000000000000000  3  1  t  1.5707963267948966
4  4  N  2.5277658364697498  4  2  N  -1.9999990558849104  4  2  0  B  1.0000000000000000  4  2  0  B  1.0000000000000000  4  2  t  1.5707963267948966
4  4  t  3.1707963267948966  4  2  B  1.0000000000000000  4  2  t  1.5707963267948966
5  5  N  2.7558717238509236  5  3  1  B  1.0000000000000000  5  1  t  1.5707963267948966
5  5  t  3.1707963267948966  5  3  t  1.9479336816837157  5  3  t  1.9479336816837157
5  5  t  6.1094671372212544  5  5  t  6.1094671372212544  5  5  t  6.1094671372212544
5  5  t  1.9479336816837157  5  5  t  1.9479336816837157  5  5  t  1.9479336816837157
5  5  t  -1.3970781568365356  5  5  t  -1.3970781568365356  5  5  t  -1.3970781568365356
5  5  t  1.9479336816837157  5  5  t  1.9479336816837157  5  5  t  1.9479336816837157
5  3  t  -2.0854065890250237  5  1  N  1.9683580365897798  5  1  t  1.5707963267948966
5  3  t  1.2964691927283318  5  1  B  0.5430811859252311  5  1  t  -0.9652495150182099

FIG. 3. Radial functions $Q_{n,4}(r)$ and associated reduced phase polynomials $\bar{\theta}_{n,4}(r)$.

FIG. 4. Radial functions $Q_{n,5}(r)$ and associated reduced phase polynomials $\bar{\theta}_{n,5}(r)$. 
FIG. 5. Radial functions $Q_{n,10}(r)$ in comparison with normalized Zernike polynomials $\sqrt{2n+2}R_n^{10}(r)$.
Appendix B: Bernstein Polynomials

The Bernstein Polynomials are defined over the unit interval \(0 \leq x \leq 1\) as [1, 11]:

\[
B_{i,j}(x) \equiv \binom{j}{i} x^i (1-x)^{j-i} = \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^k x^{i+k} = \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^k x^{i+k} = \binom{j}{i} \sum_{l=i}^{j} \binom{j-i}{l-i} (-1)^{l-i} x^l.
\]  

(B1)

Starting from the second term of the right hand side

\[
\frac{1}{(l)} B_{i,j}(x) = \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^k x^{i+k};
\]  

(B2)

and setting \(j - i - k \equiv l\) and then \(j - i \equiv n\) yields

\[
\frac{1}{(n)} B_{j-n,j}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} x^{j-k}.
\]  

(B3)

The well-known inversion of the binomial series expands the monomials in finite series of Bernstein Polynomials [13, Table 2.1][3, 12]:

\[
x^{j-n} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(l)} B_{j-k,j}(x).
\]  

(B4)

\[
x^i = \sum_{k=0}^{j-i} \binom{j-i}{k} \frac{1}{(l)} B_{j-k,j}(x) = \frac{1}{(l)} \sum_{k=0}^{j-i} \binom{j-i}{k} B_{j-k,j}(x) = \frac{1}{(l)} \sum_{s=i}^{j} \binom{s}{i} B_{s,j}(x).
\]  

(B5)

[1] Farouki, R. T. (2012), Comp. Aid. Geom. Design 29, 379.
[2] Fraser, W. (1965), J. ACM 12 (3), 295.
[3] Garloff, J., and A. P. Smith (2001), “Solution of systems of polynomial equations by using Bernstein polynomials,” in Symbolic Algebraic Methods and Verification Methods (Springer) pp. 87–97.
[4] Gradstein, I., and I. Ryshik (1981), Summen-, Produkt- und Integraltafeln, 1st ed. (Harri Deutsch, Thun).
[5] Hale, N., and A. Townsend (2012), Fast and accurate computation of Gauss-Legendre and Gauss-Jacobi quadrature nodes and weights, Tech. Rep. 12/79 (OCCAM).
[6] Hale, N., and A. Townsend (2013), SIAM J. Sci. Comput 35 (2), A652.
[7] Janssen, A. J. E. M. (2011), J. Enr. Opt. Soc. 6, 11028.
[8] Lakshminarayanan, V., and A. Fleck (2011), J. Mod. Opt. 58 (7), 545.
[9] Mathar, R. J. (2009), Serb. Astr. J. 179, 107.
[10] Mathar, R. J. (2012), arXiv:1211.3963.
[11] Rababah, A. (2003), Comput. Meth. Appl. Math. 3 (4), 608.
[12] Riordan, J. (1964), Amer. Math. Monthly 71 (5), 485.
[13] Riordan, J. (1968), Combinatorial Identities (John Wiley, New York).
[14] Sheng, Y., and L. Shen (1994), J. Opt. Soc. Am. A 11 (6), 1748.
[15] Swartztrauber, P. N. (2002), SIAM J. Sci. Comput 24 (3), 945.
[16] Tango, W. J. (1977), Appl. Phys. A 13 (4), 327.