A Compactness Theorem for Homogenization of Parabolic Partial Differential Equations

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Abstract

In order to have a better description of homogenization for parabolic partial differential equations with periodic coefficients, we define the notion of parametric two-scale convergence. A compactness theorem is proved to justify this notion.

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1 Introduction

It is well known that the modeling of physical processes in strongly inhomogeneous media leads to the study of differential equations with rapidly varying coefficients. Regarding coefficients as periodic functions, many attempts for getting approximate solutions are accomplished, and some of successful ones are G-convergence by Spagnolo, H-convergence by Tartar and Γ-convergence by De Giorgi. Another common way is to use formal asymptotic expansion - we first guess by a formal expansion what the limit should be and then justify it by energy method. Two-scale convergence defined by G. Allaire ([1]) is an efficient way of combining these two procedures - two-scale convergence guarantees the least amount of convergent degree, which is stronger than weak convergence and weaker than norm convergence. But it is restricted to the elliptic cases. For studying parabolic differential equations, the convergent nature is not clear and even two-scale convergence can not be applied directly. Even though stationary problems corresponding to the given parabolic problems may be considered for homogenization process, the convergent relationship between stationary problems and non-stationary problems is not justified by the two-scale convergence defined by G. Allaire. In this note, we present a new
approach which is a modification of the two-scale convergence technique to explain the homogenization of parabolic partial differential equations with periodic coefficients, and this also describes the convergence nature of parabolic equations on inhomogeneous media. This notion is justified by a compactness theorem which is the main result of this paper.

2 Parametric two-scale convergence

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $Y \equiv [0,1]^n$. We denote various spaces of periodic functions by a subscript $\sharp$. For example, $C^\infty_0(Y)$ denotes the space of infinitely differentiable functions on $\mathbb{R}^n$ that are periodic of period $Y$.

We define the parametric two-scale convergence for parabolic differential equations with periodic coefficients following the lead of Nguetseng [4].

**Definition 1** A sequence of functions $u^\varepsilon(t, x) \in L^\infty([0, \infty); L^2(\Omega))$ is said to parametric two-scale converge to a limit $U_0(t, x, y) \in L^\infty([0, \infty); L^2(\Omega \times Y))$ if for any test function $\Psi(t, x, y) \in L^1([0, \infty); C^\infty_0(\Omega; C^\infty_\sharp(Y)))$, we have

$$\lim_{\varepsilon \to 0} \int_0^\infty \int_\Omega u^\varepsilon(t, x) \Psi\left(t, x, \frac{x}{\varepsilon}\right) dx dt = \int_0^\infty \int_Y \int_\Omega U_0(t, x, y) \Psi(t, x, y) dx dy dt.$$ 

Roughly speaking, the convergence is a kind of weak*-convergence with respect to $L^\infty(0, \infty)$-norm and two-scale with respect to $L^2(\Omega)$-norm.

**Remark 2** In [4], we can find the following; For any $\Psi(x, y) \in L^2(\Omega; C^\infty_\sharp(Y))$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega \Psi\left(x, \frac{x}{\varepsilon}\right)^2 dx = \int_\Omega \int_Y \Psi(x, y)^2 dx dy.$$ 

We prove a compactness theorem for the parametric two-scale convergence.

**Theorem 3** Any bounded sequence $u^\varepsilon$ in $L^\infty([0, \infty); L^2(\Omega))$ has a parametric two-scale convergent subsequence.
Proof: Suppose that \( u^\varepsilon \) is a bounded sequence in \( L^\infty([0, \infty); L^2(\Omega)) \).
We want to show that there is a subsequence \( u^{\varepsilon_j} \) of \( u^\varepsilon \) and a function
\( U_0(t, x, y) \in L^\infty([0, \infty); L^2(\Omega \times Y)) \) such that for any \( \Psi(t, x, y) \) in \( L^1([0, \infty); C^\infty_0(\Omega; C^\infty_2(Y))) \)
\[
\lim_{\varepsilon_j \to 0} \int_0^\infty \int_{\Omega} u^{\varepsilon_j}(t, x) \Psi \left( t, x, \frac{x}{\varepsilon_j} \right) \, dx dt = \int_0^\infty \int_Y U_0(t, x, y) \Psi(t, x, y) \, dx dy dt.
\]

Let \( \mathcal{F}_\varepsilon(\Psi) \equiv \int_0^\infty \int_{\Omega} u^\varepsilon(t, x) \Psi(t, x, \frac{x}{\varepsilon}) \, dx dtdt \) and \( \mathcal{D} \equiv L^2(\Omega; C^2_\varepsilon(Y)) \). There is a positive constant \( C \) such that \( \|u^\varepsilon\|_{L^\infty([0, \infty); L^2(\Omega))} \leq C \). Since
\[
|\mathcal{F}_\varepsilon(\Psi)| \leq \int_0^\infty \|u^\varepsilon(t)\|_{L^2(\Omega)} \left\| \Psi \left( t, x, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} \, dt \leq C \int_0^\infty \max_{y \in Y} \|\Psi(t, x, y)\|_{L^2(\Omega)} \, dt = C \int_0^\infty \|\Psi(t)\|_D \, dt
\]
and \( \Psi \in L^1([0, \infty); \mathcal{D}) \). We have \( \mathcal{F}_\varepsilon \in (L^1([0, \infty); \mathcal{D}))' \). So a subsequence \( \mathcal{F}_{\varepsilon_j} \) is weak*-convergent to some \( U_0 \in (L^1([0, \infty); \mathcal{D}))' \). Hence for any \( \Psi \) in \( L^1([0, \infty); \mathcal{D}) \), we have
\[
\left\| \Psi \left( t, x, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} \leq \|\Psi(t)\|_D,
\]
so by Lebesgue Dominated Convergence Theorem,
\[
|U_0(\Psi)| = \lim_{\varepsilon_j \to 0} \left| \mathcal{F}_{\varepsilon_j}(\Psi) \right| \leq \limsup_{\varepsilon_j \to 0} \int_0^\infty \|u^{\varepsilon_j}(t)\|_{L^2(\Omega)} \left\| \Psi \left( t, x, \frac{x}{\varepsilon_j} \right) \right\|_{L^2(\Omega)} \, dt \leq C \int_0^\infty \limsup_{\varepsilon_j \to 0} \left\| \Psi \left( t, x, \frac{x}{\varepsilon_j} \right) \right\|_{L^2(\Omega)} \, dt = C \int_0^\infty \|\Psi(t)\|_{L^2(\Omega \times Y)} \, dt.
\]
Since \( L^1([0, \infty); \mathcal{D}) \) is dense in \( L^1([0, \infty); L^2(\Omega \times Y)) \), it follows that \( U_0 \) is in \( (L^1([0, \infty); L^2(\Omega \times Y)))' \). By Riesz Representation Theorem, we have
\[
U_0(\Psi) = \int_0^\infty \langle U_0(t), \Psi(t) \rangle_{L^2(\Omega \times Y)} \, dt
\]
for some \( U_0 \in L^\infty([0, \infty); L^2(\Omega \times Y)) \). Therefore
\[
\lim_{\varepsilon_j \to 0} \int_0^\infty \int_{\Omega} u^{\varepsilon_j}(t, x) \Psi \left( t, x, \frac{x}{\varepsilon_j} \right) \, dx dt \equiv \lim_{\varepsilon_j \to 0} \mathcal{F}_{\varepsilon_j}(\Psi) = U_0(\Psi) = \int_0^\infty \int_Y U_0(t, x, y) \Psi(t, x, y) \, dx dy dt.
\]
\[\square\]
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