Quantising Higher-spin String Theories

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Abstract

In this paper, we examine the conditions under which a higher-spin string theory can be quantised. The quantisability is crucially dependent on the way in which the matter currents are realised at the classical level. In particular, we construct classical realisations for the $W_{2,s}$ algebra, which is generated by a primary spin-$s$ current in addition to the energy-momentum tensor, and discuss the quantisation for $s \leq 8$. From these examples we see that quantum BRST operators can exist even when there is no quantum generalisation of the classical $W_{2,s}$ algebra. Moreover, we find that there can be several inequivalent ways of quantising a given classical theory, leading to different BRST operators with inequivalent cohomologies. We discuss their relation to certain minimal models. We also consider the hierarchical embeddings of string theories proposed recently by Berkovits and Vafa, and show how the already-known $W$ strings provide examples of this phenomenon. Attempts to find higher-spin fermionic generalisations lead us to examine the whether classical BRST operators for $W_{2,\frac{n}{2}}$ ($n$ odd) algebras can exist. We find that even though such fermionic algebras close up to null fields, one cannot build nilpotent BRST operators, at least of the standard form.
1 Introduction

One of the best ways of studying string theory and its generalisations is by using BRST methods. Traditionally in this approach, one begins with a classical theory with local gauge symmetries which are then gauge fixed, leading to the introduction of ghost fields. The gauge-fixed action including ghosts has a nilpotent symmetry generated by the classical BRST operator. To quantise the theory, one must renormalise the symmetry transformation rules and introduce counterterms, order by order in $\sqrt{\hbar}$, in order to preserve the BRST invariance of the effective action at the quantum level. The theory is quantisable if one can carry out the procedure in all orders of $\sqrt{\hbar}$. If such procedure is not possible, the theory then suffers from an anomaly.

In a bosonic string theory with 26 scalars, there is no need either to add quantum counterterms or to modify the transformation rules. This is because a central charge $c = -26$ from ghosts is cancelled by the contributions of the matter scalars. If instead there were $d \neq 26$ scalars in the theory, it would still be anomaly free after adding $\sqrt{\hbar}$ dependent counterterms and modifications of the transformation rules. These have the interpretation of background charges in the matter energy-momentum tensor, with the criticality condition $c = 26$ achieved by choosing appropriate background charges. In both cases, the matter energy-momentum tensor forms a quantum Virasoro algebra with $c = 26$. Thus in this case one can construct the quantum BRST operator directly from the quantum Virasoro algebra.

Another example is provided by the $W_3$ string. Here one begins with a theory with classical $W_3$ symmetry generated by currents $T, W$ of spin 2 and spin 3. The classical OPE (i.e. single contractions) of the primary current $W$ is given by

$$W(z) W(w) \sim \frac{2T^2}{(z-w)^2} + \frac{\partial(T^2)}{z-w}.$$  \hfill (1)

Despite the non-linearity, it is straightforward to obtain the classical BRST operator. One way to realise the classical algebra is in terms of a scalar field $\varphi$ and an arbitrary energy-momentum tensor $T_X$:

$$T = -\frac{i}{4} (\partial \varphi)^2 + T_X, \quad (2)$$

$$W = \frac{i}{\sqrt{2}} \left( \frac{i}{2} (\partial \varphi)^3 + 2 \partial \varphi T_X \right). \quad (3)$$

With this realisation, the theory can be quantised by adding counterterms and modifying the transformation rules. The corresponding quantum BRST operator is the same as the one that was constructed by Thierry-Mieg [1] from an abstract quantum $W_3$ algebra with critical central charge $c = 100$. Unlike the Virasoro algebra, the quantum modification of the classical $W_3$ algebra is not merely reflected by introducing a central charge. The (quantum) OPE of the primary current $W$ is given by

$$\hbar^{-1}W(z) W(w) \sim \frac{16}{(22 + 5c)} \left[ \frac{2(TT) - \hbar \partial^2T}{(z-w)^2} + \frac{\partial((TT) - \hbar \partial^2T)}{z-w} \right] + \hbar \left[ \frac{1}{z-w} + \frac{1}{(z-w)^2} \frac{\partial T}{z-w} + \frac{2T}{(z-w)^3} + \frac{2T}{(z-w)^4} \right] + \hbar^2 \frac{c/3}{(z-w)^6}. \quad (4)$$

\hfill (5)
The above considerations can be extended to more complicated $W$ algebras. A discussion of the classical BRST operators for the $W_N$ algebras, and the structure of the quantum BRST operators, may be found in [3, 4]. Detailed results for the quantum BRST operator for $W_4$ were obtained in [5, 6]. In general, the quantum $W_N$ BRST operator can be viewed as the appropriate quantum renormalisation of the classical operator that arises in an anomaly-free quantisation of the theory.

These examples of string theories lead to two intriguing questions:

1. Is the existence of a quantum algebra essential to the quantisability of the theory?
2. Can all classical string-like theories be quantised?

We address these two questions in section 2, by looking at specific examples. In particular, we construct classical realisations for all the classical $W_{2s}$ algebras, and obtain the corresponding classical BRST operators. We show by example that these BRST operators can be given a graded structure by performing canonical field redefinitions involving the ghost and the matter fields. We find that these graded classical BRST operators can be promoted to fully quantum-nilpotent operators by the addition of $\hbar$-dependent terms. In previous work results were obtained for $s \leq 7$ [9, 10]. We obtain new results in this paper for $s = 8$. In certain special cases, when $s = 3, 4$ or 6, these BRST operators are equivalent to the ones that could be obtained abstractly from the corresponding quantum $W_{2s}$ algebras. However in general, quantum generalisations of the $W_{2s}$ algebras do not exist. Nonetheless the associated classical theories are quantisable, i.e. quantum-nilpotent BRST operators exist. In fact, in general there can exist several inequivalent quantum BRST operators for the same classical theory. An interesting example is $W_{2,6}$, where there are four quantum BRST operators. Two of these are the ones that correspond to the abstract BRST operator for the $WG_2$ algebra, whilst the other two have no underlying quantum algebras. We conclude that the quantisability of a string theory does not have to depend on the existence of an underlying quantum algebra; in fact, it is quite possible that the existence of a quantum algebra does not play any rôle in the quantum theory. Thus although the Virasoro string and the $W_3$ string provide examples where a closed quantum symmetry algebra exists, there are other examples, such as certain higher-spin $W_{2s}$ strings, where a classical theory with a closed classical symmetry under spin-2 and spin-$s$ currents can be quantised even when there is no closed $W_{2s}$ algebra with critical central charge at the quantum level.

The ability to quantise a classical theory is in fact dependent also on the specific realisation of the underlying symmetry algebra. We find that a theory that is quantisable with one realisation of the matter currents can be anomalous with another realisation. An example of this can be found in the $W_3$ string. In addition to the standard multi-scalar classical realisations [3], there are four special classical realisations associated with the four exceptional Jordan algebras [7]. It had already been established that these realisations cannot be extended to realisations of the full quantum $W_3$ algebra by adding quantum corrections. However, in view of the above observations on the possibility of quantising a string theory even when no quantum algebra exists, one might suspect that quantisation of the “Jordan” $W_3$ strings might nevertheless be possible. However, as we shall discuss in section 3, we find evidence that this in fact not possible.

In section 4 we examine the quantum $W_{2s}$ BRST operators in more detail, and extend previous discussions of their cohomologies. In the case where $T_X$ is realised with two or more
scalars, the physical states take the form of effective Virasoro states, built from $X^\mu$, tensored with primary operators built from $\varphi$ and the higher-spin ghost fields. In many cases these primary operators can be associated with those of certain minimal models. There are, however, cases where the connections with minimal models are obscure.

The cohomologies of $W$ string theories, and their connection to minimal models, are indicative of a kind of hierarchical structure of string theories, which was first articulated in the case of supersymmetric extensions of string theories by Berkovits and Vafa\cite{12}. We examine the possibility of fermionic higher-spin extension of such hierarchical embedding structure in section 5. It turns out, however, that although the Jacobi identity for classical $W^2_s$, strings is satisfied up to null fields (which vanish with a specific realisation), the corresponding classical BRST operator does not exist.

2 $W_{2,s}$ strings

In this section, we shall investigate higher-spin string theories based on a classical symmetry algebra generated by currents $T$ and $W$ of spin 2 and spin $s$, where $s$ is an integer. Such a closed, non-linear, $W_{2,s}$ algebra exists classically for all $s \geq 3$. We shall find it convenient in this paper to present the Poisson brackets for the classical algebra in the form of “classical OPEs,” in the sense that only single contractions are to be taken in the operator products. Thus the classical algebra of the currents $T$ and $W$ is

\begin{align*}
T(z)T(w) & \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
T(z)W(w) & \sim \frac{sW(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}, \\
W(z)W(w) & \sim \frac{2T(w)^{s-1}}{(z-w)2} + \frac{\partial T(w)^{s-1}}{z-w}.
\end{align*}

It is straightforward to verify that this algebra satisfies the Jacobi identity at the classical level.

In the case of a linear algebra $[T_i, T_j] = f_{ij}^k T_k$, one knows that the BRST charge will have the form $Q = c^i T_i + \frac{1}{2} f_{ij}^k c^i c^j b_k$. In our case, we may interpret the non-linearity on the right-hand side of the OPE of $W$ with $W$ as $T$–dependent structure constants, leading to the expectation that the BRST current should have the form

\[ J = c (T + T_{\beta\gamma} + \frac{1}{2} T_{bc}) + \gamma W - \partial \gamma \gamma b T^{s-2} , \]

where the $(b, c)$ are the antighost and ghost for $T$, and $(\beta, \gamma)$ are the antighost and ghost for $W$. They are anticommuting, and have spins $(2, -1)$ and $(s, 1 - s)$ respectively. The ghost currents are given by

\begin{align*}
T_{bc} & = -2b \partial c - \partial b c , \\
T_{\beta\gamma} & = -s \beta \partial \gamma - (s - 1) \partial \beta \gamma .
\end{align*}

Performing the classical OPE, we find that (8) is indeed nilpotent (the coefficient $-1$ in the last term in (8) is determined by the nilpotency requirement).

\footnote{For even $s$ a generalisation seems possible by adding $2a T^{s/2-1} W$ to the second order pole in the OPE of $W$ with $W$ and $a \partial (T^{s/2-1} W)$ to the first order pole. However, one can always choose generators $T, W = W - \alpha/s^2 T^{s/2}$ such that $\alpha$ is zero for $WW$. In this form the algebra was called $W_{s/s-2}$ in [13].}
In order to construct a string theory based on the classical $W_{2s}$ symmetry, we need an explicit realisation for the matter currents. Such a realisation may be obtained in the following manner, in terms of a scalar field $\varphi$ and an arbitrary energy-momentum tensor $T_X$, which may itself be realised, for example, in terms of scalar fields $X^\mu$:

\begin{align*}
T &= -\frac{1}{2} (\partial \varphi)^2 + T_X, \\
W &= \sum_{n=0}^{N} g_n(s) (\partial \varphi)^{s-2n} T_X^n,
\end{align*}

where $N = [s/2]$. The constants $g_n(s)$ are determined by demanding that $W$ satisfy (8), and we find that they are given by

\begin{equation}
    g_n(s) = s^{-1} (-2)^{-s/2} 2^{n+1} \binom{s}{2n}.
\end{equation}

(Actually, as we shall discuss later, when $s$ is even there is also a second solution for the constants $g_n(s)$, which is, however, associated with a “trivial” string theory.)

In order to discuss the quantisation of the classical $W_{2s}$ string theories, the traditional procedure would be to undertake an order-by-order computation of the quantum effective action, introducing counterterms and corrections to the transformation rules in each order in the loop-counting parameter, which is $\sqrt{\hbar}$ in this case, in order to preserve BRST invariance. Such a procedure is cumbersome and error prone, but fortunately a more straightforward method is available to us here. We can simply parametrise all the possible quantum corrections to the BRST operator, and solve for the coefficients of these terms by demanding nilpotence at the full quantum level. By this means, we can take advantage of computer packages for calculating operator products [14]. Before carrying out this procedure, we shall first discuss a simplification of the structure of the BRST operator that can be achieved by performing a canonical redefinition involving the ghost and the matter fields.

For the case of $W_{2,3}$, this field redefinition was first described in [8]. At the classical level, the redefinition is given by

\begin{align*}
    c &\rightarrow c - b \partial \gamma \gamma + \sqrt{2} i \partial \varphi \gamma \\
b &\rightarrow b \\
\gamma &\rightarrow \gamma \\
\beta &\rightarrow \beta - \partial b b \gamma - \sqrt{2} i \partial \varphi b \\
\varphi &\rightarrow \varphi + \sqrt{2} i b \gamma \\
T_X &\rightarrow T_X.
\end{align*}

The BRST operator in (11) for the case $s = 3$ then becomes

\begin{align*}
Q &= Q_0 + Q_1 \\
Q_0 &= \oint c(T + T_\beta \gamma + \frac{1}{2} T_{bc}) \\
Q_1 &= \oint \gamma \left((\partial \varphi)^3 + \frac{3}{2} \partial \varphi \beta \partial \gamma\right).
\end{align*}
It is easy to check that $Q_0$ and $Q_1$ are graded in that $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. One would expect that similar canonical redefinition should be possible for all values of $s$. In particular, we found the canonical redefinition for $W_{2,4}$:

$$
c \rightarrow c - 2\beta \partial \gamma \gamma - \frac{2}{3} (\partial \phi)^2 \gamma + \frac{4}{3} (\partial \phi)^2 b \partial \gamma \gamma - \frac{1}{2} T_X \gamma - \frac{2}{3} T_X b \partial \gamma \gamma
$$

$$
b \rightarrow b
$$

$$
\gamma \rightarrow \gamma + 2b \partial \gamma \gamma
$$

$$
\beta \rightarrow \beta + 4b \beta \partial \gamma + 2b \partial \beta \gamma + \frac{2}{3} (\partial \phi)^2 b + \frac{4}{3} (\partial \phi)^2 \partial b \beta \gamma + \frac{2}{3} T_X b - \frac{2}{3} T_X b \partial b \gamma + 4\partial b \beta \partial \gamma \gamma + 2\partial b \beta \gamma
$$

$$
\varphi \rightarrow \varphi - \frac{2}{3} \partial \phi \beta \gamma
$$

$$
T_X \rightarrow T_X + T_X b \partial \gamma + T_X \partial b \gamma + \frac{2}{3} T_X \partial b \partial \gamma \gamma + \frac{2}{3} T_X \partial b \gamma .
$$

The field redefinition becomes more and more complicated with increasing $s$. However, we conjecture that the BRST operator in (9) can be transformed by canonical field redefinition into the following graded form:

$$
Q = Q_0 + Q_1
$$

$$
Q_0 = \oint c(T + T_{b\gamma} + \frac{1}{2} T_{bc})
$$

$$
Q_1 = \oint \gamma (\partial \phi)^s + \frac{s^2}{3} (\partial \phi)^s \beta \partial \gamma
$$

In the case of $s = 4$, we explicitly verified that the field redefinitions in (17) turn the field operator in (9) into this form. It is easy to verify that $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$ classically for all $s$.

It is worth mentioning that for $s = 2k$ there exists another solution for the realisation of $W$ given in (13), in which $W$ can be written as $\frac{1}{2} T^k$. In this case, there exists a canonical field redefinition under which the BRST operator in (9) becomes simply $Q = Q_0$. It is not surprising that the BRST operator with this realisation describes the ordinary bosonic string since in this realisation the constraint $W = 0$ is implied by the constraint $T = 0$. We shall not consider this case further.

To quantise the classical $W_{2,s}$ string and obtain the quantum BRST, we add $\sqrt{h}$-dependent counterterms to the classical BRST. In order to do this in a systematic way, it is useful to identify the $h$ dimensions of the quantum fields. An assignment that is consistent with the OPEs is $\{h, \sqrt{h}, h, 1, h^{s/2}, h^{1-s/2}\}$ for $\{T_X, \partial \phi, b, c, \beta, \gamma\}$. We shall make the assumption that the graded structure of the classical BRST operator is preserved at the quantum level. For $W_{2,3}$, this has been explicitly found to be true. For $s \geq 4$, there certainly exist quantum BRST operators with the graded structure, as we shall discuss below. Whether there could exist further quantum BRST operators that do not preserve the grading is an open question.

The quantum corrections that can be added to $Q_0$ simply take the form of background-charge terms for the scalar fields $\varphi$ and $X^\mu$ that appear in $T$. In $Q_1$, the possible quantum corrections amount to writing $Q_1 = \oint \gamma F(\varphi, \beta, \gamma)$, where $F(\varphi, \beta, \gamma)$ is the most general possible spin-$s$ operator with ghost number zero. Its leading-order (i.e. classical) terms are given in (21). The equations resulting from imposing nilpotence for such BRST operators were analysed in detail in (13) for $s = 4, 5$ and $6$, and in (14) for $s = 7$. It was found that there are two inequivalent BRST operators when $s = 4$, one for $s = 5$, four for $s = 6$ and one for $s = 7$. Later in this paper, we shall present some new results for $s = 8$. 

5
As we discussed earlier, the case \( s = 3 \) corresponds to the \( W_3 = W_{A_2} \) algebra, which exists as a closed quantum algebra for all values of the central charge, including, in particular, the critical value \( c = 100 \). For \( s = 4 \), it was shown in [10] that the two \( W_{2,4} \) quantum BRST operators correspond to BRST operators for the \( WB_2 \) algebra, which again exists at the quantum level for all values of the central charge. The reason why there are two inequivalent BRST operators in this case is that \( B_2 \) is not simply-laced and so there are two inequivalent choices for the background charges that give rise to the same critical value \( c = 172 \) for the central charge [10]. Two of the four \( W_{2,6} \) BRST operators can similarly be understood as corresponding to the existence of a closed quantum \( WG_2 \) algebra for all values of the central charge, including in particular the critical value \( c = 388 \) [10]. However, the remaining quantum \( W_{2,s} \) BRST operators cannot be associated with any closed quantum \( W_{2,s} \) algebras. For example, although a quantum \( W_{2,5} \) algebra exists, it is only consistent, in the sense of satisfying the Jacobi identity, for a discrete set of central-charge values, namely \( c = \{-7, 6/7, -350/11, 134 + 60\sqrt{5}\} \). Since none of these central charges includes the value \( c = 268 \) needed for criticality, we see that although the quantum \( W_{2,5} \) BRST operator can certainly be viewed as properly describing the quantised \( W_{2,5} \) string, it is not the case that there is a quantum \( W_{2,5} \) symmetry in the \( W_{2,5} \) string. This is an explicit example of the fact that a classical theory can be successfully quantised, without anomalies, even when a quantum version of the symmetry algebra does not exist. It appears that the existence of closed quantum \( W \) algebras is inessential for the existence of consistent \( W \)-string theories.

3 Jordan \( W_3 \) strings?

We have seen from the discussion in the previous section that the procedure for constructing a \( W \) string amounts to starting out from a classical theory having some local worldsheet symmetries corresponding to a classical \( W \) algebra, and then quantising the theory, ensuring, order by order in \( \sqrt{\hbar} \), that the BRST symmetry is preserved. This procedure, which, if successful, eventually leads to a BRST operator that is nilpotent at the quantum level, does not necessarily require that a closed quantum version of the original classical \( W \) algebra must exist.

It would be of interest to see if there are any other realisations of the \( W_3 \) algebra that could give rise to quantum consistent string theories. We shall restrict our attention to realisations built purely from free scalar fields. At the classical level, the form of the currents will be

\[
T = -\frac{1}{4} \partial \phi_i \partial \phi_i , \quad W = \frac{1}{4} d_{ijk} \partial \phi_i \partial \phi_j \partial \phi_k ,
\]

(21)

where \( d_{ijk} \) is a constant symmetric tensor. Closure at the classical level implies that \( d_{ijk} \) must satisfy

\[
d_{ij}^m d_{k\ell}^m = \lambda \delta_{ij}^k \delta_{\ell}^m ,
\]

(22)

where \( \lambda \) is a constant. It was shown in [7] that the solutions to this equation are either of the form

\[
d_{111} = -1 , \quad d_{1\mu\nu} = \delta_{\mu\nu} ,
\]

(23)

4Strictly speaking, the Jacobi identity is only satisfied identically in the case of the two irrational values for the central charge. For the other values listed, the Jacobi identity is satisfied modulo terms involving null fields built from the spin–2 and spin–\( s \) currents. These null fields are purely quantum in origin, in the sense that they do not arise in the Jacobi identity at the classical level. We shall encounter a very different situation later, when we look at an algebra of spin–2 and spin–\( \frac{5}{2} \) currents.
where we split the indices as \( i = \{1, \mu\} \), etc., or else \( d_{ijk} \) is the set of structure constants for one of the four exceptional Jordan algebras, over the real, complex, quaternionic or octonionic fields. The case (23) corresponds to the realisations (3) that we discussed previously.

The classical realisations defined by (23) can, as we have discussed previously, be extended to quantum realisations of the quantum \( W_3 \) algebra, by adding \( \hbar \)-dependent corrections to \( T \) and \( W \). On the other hand, it has been shown [7, 16, 17] that the classical Jordan realisations cannot be extended to full quantum realisations. This does not, \textit{a priori}, preclude the possibility that it might nevertheless be possible to build quantum-consistent \( W_3 \) string theories based on these classical realisations of the symmetry. In other words the possibility \textit{a priori} exists that one could still find quantum nilpotent BRST operators having, as their classical limits, the classical BRST operators built from the Jordan realisations.

One way to test this possibility is by starting from the classical \( W_3 \) BRST operator (11), with \( T \) and \( W \) given by one of the Jordan realisations, and then parametrise all possible quantum corrections. Demanding then that the resulting BRST operator \( Q \) be nilpotent at the quantum level will give a system of equations for the parameters. If a solution exists, then a consistent quantisation of the associated Jordan \( W_3 \) string is possible. The numbers of scalar fields \( \varphi_i \) for the real, complex, quaternionic and octonionic Jordan realisations are 5, 8, 14 and 26 respectively. For the simplest case of the real Jordan algebra, we have carried out the above procedure, and we find that no solution exists that can give rise to a nilpotent BRST operator at the quantum level. Although we have not examined the remaining three cases, there seems to be no particular reason to expect that solutions will exist there either. Thus it appears that one cannot consistently quantise \( W_3 \) strings based on the classical Jordan realisations of the \( W_3 \) algebra. This result was obtained by Vandoren et al. using the Batalin-Vilkovisky quantisation scheme [18].

Another example of a classical theory with local classical \( W_3 \) symmetry that cannot be quantised is provided by taking a one-scalar matter realisation. This corresponds to truncating out \( T_X \) in the multi-scalar realisation of \( W_3 \). As in the case of the Jordan realisations discussed above, it turns out that one cannot find quantum corrections to the classical BRST such that nilpotence is achieved at the quantum level.

4 Minimal models and \( W_{2,s} \) strings

It has been known for some time that there is a close connection between the spectra of physical states in \( W \)-string theories, and certain Virasoro or \( W \) minimal models. This connection first came to light in the case of the \( W_3 \) string [13, 21, 22], where it was found that the physical states in a multi-scalar realisation can be viewed as the states of Virasoro-type bosonic strings with central charge \( c_X = 25 \) and intercepts \( \Delta = \{1, \frac{15}{1}, \frac{1}{4}\} \). These quantities are dual to the central charge \( c_{\min} = \frac{1}{2} \) and weights \( h = \{0, \frac{1}{16}, \frac{1}{2}\} \) for the \((p,q) = (3,4)\) Virasoro minimal model, the Ising model, in the sense that \( 26 = c_X + c_{\min} \), and \( 1 = \Delta + h \). In fact, the physical operators of the multi-scalar \( W \) string have the form

\[
V = cU(\varphi, \beta, \gamma) V_X ,
\]

where \( V_X \) are the effective-spacetime physical operators of the Virasoro theory with energy-momentum tensor \( T_X \), and \( U(\varphi, \beta, \gamma) \) are primary operators of the minimal model with energy-momentum tensor \( T_{\min} = T_\varphi + T_{\beta\gamma} \).
If one were to look at the multi-scalar $W_N$ string, one would expect that analogously the physical states would be of the form of effective Virasoro string states for a $c_X = 26 - \frac{6}{N(N+1)}$ theory, tensored with operators $U(\varphi, \beta, \gamma)$ that are primaries of the $c_{\text{min}} = 1 - \frac{6}{N(N+1)}$ Virasoro minimal model, i.e. the $(p, q) = (N, N + 1)$ unitary model. Here, $\varphi$ denotes the set of $(N - 2)$ special scalars which, together with the $X^\mu$ appearing in $T_X$, provide the multi-scalar realisation of the $W_N$ algebra. Similarly, $\beta$ and $\gamma$ denote the $(N - 2)$ sets of antighosts and ghosts for the spin $3, 4, 5, \ldots, N$ currents. The rapid growth of the complexity of the $W_N$ algebras with increasing $N$ means that only incomplete results are available for $N \geq 4$, but partial results and general arguments have provided supporting evidence for the above connection.

A simpler case to consider is a $W_{2,s}$ string, corresponding to the quantisation of one of the classical theories described in section 2. The quantum BRST operators for $W_{2,s}$ theories with $s = 4, 5$ and $6$ were constructed in [1], and the results were extended to $s = 7$ in [12]. Here, we shall present some new results for the case $s = 8$. The conclusion of these various investigations is that there exists at least one quantum BRST operator for each value of $s$. If $s$ is odd, then there is exactly one BRST operator. If $s$ is even, then there are two or more inequivalent quantum BRST operators. One of these is a natural generalisation to even $s$ of the unique odd-$s$ sequence of BRST operators. This sequence of BRST operators, which we shall call the “regular sequence”, has the feature that the associated minimal model, with energy-momentum tensor $T = T_\varphi + T_\beta \gamma$, has central charge

$$c_{\text{min}} = \frac{2(s - 2)}{(s + 1)} .$$

This is the central charge of the lowest unitary $W_{s-1}$ minimal model, and in fact it was shown explicitly in [11] for $s = 4, 5$ and $6$ that the operators $U(\varphi, \beta, \gamma)$ appearing in the physical states include the expected spin $3, \ldots, s - 1$ currents of the associated $W_{s-1}$ algebra.

When $s$ is even, there are further “exceptional” BRST operators in addition to the regular one described above. When $s = 4$, there is one exceptional case, with $c_{\text{min}} = -\frac{2}{3}$. This is the central charge of the $(p, q) = (3, 5)$ Virasoro minimal model, and in fact it was found in [10] that the physical states of the multi-scalar realisation do indeed have $U(\varphi, \beta, \gamma)$ operators that are primaries of the $(3, 5)$ Virasoro minimal model, with conformal weights $h = \{-\frac{1}{27}, 0, \frac{2}{7}, \frac{13}{7}\}$. Of course the occurrence of a negative weight for $U(\varphi, \beta, \gamma)$ implies correspondingly an intercept value $\Delta > 1$ for the effective spacetime Virasoro string, and hence the existence of some non-unitary physical states.

When $s = 6$, there are three further BRST operators in addition to the regular one [9]. These exceptional cases have $c_{\text{min}} = -\frac{4}{9}, -\frac{4}{3}, \text{ and } 0$ respectively. The first of these is the central charge of the $(p, q) = (4, 7)$ Virasoro minimal model. It was found in [10] that the physical states for this theory do in fact have $U(\varphi, \beta, \gamma)$ operators that are the primaries of the $(4, 7)$ Virasoro minimal model. The second case has $c_{\text{min}} = -\frac{4}{3}$, which is the central charge of the $(p, q) = (7, 12)$ Virasoro minimal model. In this case, it turns out that the physical states have $U(\varphi, \beta, \gamma)$ operators that describe a subset of the primaries of the $(7, 12)$ Virasoro minimal model. In fact, $-\frac{4}{3}$ is also the central charge of a $WB_2$ minimal model, and the operators $U(\varphi, \beta, \gamma)$ of the physical states are actually the primaries, and $W$ descendants, of this $WB_2$ minimal model [10].

The third exceptional case for $s = 6$ has $c_{\text{min}} = 0$. Here, we find from numerous examples that the conformal weights of the operators $U(\varphi, \beta, \gamma)$ in the physical states are given by $h =
\{-\frac{1}{25}, 0, \frac{4}{25}, \frac{9}{25}, \frac{14}{25}, \frac{19}{25}, \frac{24}{25}, 1, \frac{44}{25}, \ldots\}.\) These weights can be described by \(h = \frac{(3n+2)(n-1)}{50}\) and \(h = \frac{(3n+1)(n+2)}{50}\), for \(n \geq 0\). There is no obvious model with \(c_{\text{min}} = 0\) that would give rise to this set of conformal weights. Possibly one should look for some product of models with cancelling positive and negative central charges.

We now turn to the case \(s = 8\), which has not previously been studied. Here, we find that there are four exceptional quantum BRST operators, in addition to the regular case with \(c_{\text{min}} = \frac{1}{4}\). The central charges for the exceptional cases are \(c_{\text{min}} = -\frac{23}{5}, -\frac{13}{2}, -\frac{7}{2}, \) and \(-\frac{3}{2}\). The BRST operators are rather too complicated to be able to present explicitly here, with the operator \(F(\phi, \beta, \gamma)\) in \(Q_1 = \oint \gamma F(\phi, \beta, \gamma)\) having 75 terms (the two classical terms given by (24), plus 73 \(h\)-dependent quantum corrections). As usual, all the \(s = 8\) BRST operators have identical classical terms, and differ only in the detailed coefficients of the quantum corrections.

The regular \(s = 8\) BRST operator, with \(c_{\text{min}} = \frac{1}{4}\), gives, as expected, operators \(U(\phi, \beta, \gamma)\) in physical states whose conformal weights coincide with the conformal weights of the primary fields of the lowest unitary \(W_7\) minimal model. The exceptional BRST operator with \(c_{\text{min}} = -\frac{23}{5}\) appears to give rise to operators \(U(\phi, \beta, \gamma)\) that are primaries of the \((p, q) = (17, 30)\) Virasoro minimal model. The exceptional case with \(c_{\text{min}} = -\frac{13}{2}\) has a central charge that does not coincide with that for any Virasoro minimal model. On the other hand, it does coincide with the allowed central charges for certain \(W_G, W_B\) and \(W_C\) minimal models. Comparing with the conformal weights of the operators \(U(\phi, \beta, \gamma)\) for physical states in this case, we find that this BRST operator appears to describe a theory related to the \((p, q) = (10, 9)\) \(W_B\) minimal model.

The remaining exceptional \(s = 8\) BRST operators seem to be more difficult to characterise. The one with \(c_{\text{min}} = -\frac{3}{2}\) might, a priori, be expected to be related to the \((p, q) = (3, 5)\) Virasoro minimal model, or the \((10, 7)\) \(W_B\) minimal model, or the \((5, 7)\) \(W_C\) minimal model. However, we find that the conformal weights of the \(U(\phi, \beta, \gamma)\) operators in physical states have weights including \(h = \{-\frac{47}{20}, -\frac{5}{12}, -\frac{1}{6}, 0, \frac{1}{50}, \ldots\}\). Although the weights \(h = \{-\frac{3}{20}, 0, \frac{1}{5}, \frac{2}{5}, 1\}\) of the \(c_{\text{min}} = -\frac{3}{2}\) Virasoro minimal model are included in this list, the other weights seem to bear little relation to any minimal model. The other exceptional case, with \(c_{\text{min}} = \frac{47}{20}\), has a central charge that is not equal to that of any \(W\) minimal model. We find that the conformal weights of the operators \(U(\phi, \beta, \gamma)\) in this case all appear to have the form \(h = \frac{4n-3}{92}\), or \(h = \frac{n}{23}\), where \(n \geq 0\). This example seems to be analogous to the \(c_{\text{min}} = 0\) BRST operator for \(s = 6\), in that there is no apparent connection with any minimal model.

5 Hierarchies of string embeddings

It was proposed recently [12] that as part of the general programme of looking for unifying principles in string theory, one should look for ways in which string theories with smaller worldsheet symmetries could be embedded into string theories with larger symmetries. In particular, it was shown in [13] that the bosonic string could be embedded in the \(N = 1\) superstring, and that in turn, the \(N = 1\) string could be embedded in the \(N = 2\) superstring. In subsequent papers, it was shown by various methods that the cohomologies of the resulting theories were precisely those of the embedded theories themselves [22, 23].

The essential ingredient in the embeddings discussed in [12] is that a realisation for the currents of the more symmetric theory can be found in terms of the currents of the less symmetric theory, together with some additional matter fields whose eventual rôle for the cohomology is to
supply degrees of freedom that are cancelled by the additional ghosts of the larger theory. For example, the $N = 1$ superconformal algebra, at critical central charge $c = 15$, can be realised in terms of a $c = 26$ energy-momentum tensor $T_M$ as

$$
T = T_M - \frac{1}{2} b_1 \partial c_1 - \frac{1}{2} \partial b_1 c_1 + \frac{1}{2} \partial^2 (c_1 \partial c_1),
$$

$$
G = b_1 + c_1 (T_M + \partial c_1 b_1) + \frac{1}{2} \partial^2 c_1,
$$

(26)

where $b_1$ and $c_1$ are ghost-like spin $(\frac{1}{2}, -\frac{1}{2})$ anticommuting matter fields. The cohomology of the BRST operator for the $N = 1$ superstring, with this realisation of the $N = 1$ superconformal algebra, is precisely that of the usual bosonic string $[12, 22, 23]$. This is most easily seen using the method of $[23]$, where a unitary canonical transformation $Q \rightarrow e^R Q e^{-R}$ is applied to the $N = 1$ BRST operator, transforming it into the BRST operator for the bosonic string plus a purely topological BRST operator. In effect, the degrees of freedom of $b_1$ and $c_1$ are cancelled out by the degrees of freedom of the commuting spin $(\frac{1}{2}, -\frac{1}{2})$ ghosts for the spin–$\frac{1}{2}$ current $G$. The central charge of the energy-momentum tensor for $(b_1, c_1)$ is $c = 11$, which precisely cancels the $c = -11$ central charge for the spin–$\frac{1}{2}$ ghost system for the spin–$\frac{1}{4}$ current $G$.

It is natural to enquire whether some analogous sequence of embeddings for $W$ strings might exist, with, for example, the usual Virasoro string contained within the $W_3$ string, which in turn is contained in the $W_4$ string, and so on $[12]$. In fact, as was observed in $[13]$, such sequences of embeddings are already well known for $W$ strings. The simplest example is provided by the $W_3$ string, where the $W_3$ currents $T$ and $W$ are realised in terms of an energy-momentum tensor $T_X$, and a scalar field $\varphi$. The $\varphi$ field here plays a rôle analogous to the $(b_1, c_1)$ matter fields in the embedding of the bosonic string in the $N = 1$ superstring. Here, however, the central charge $c = 74\frac{1}{2}$ for the energy-momentum tensor of $\varphi$ does not quite cancel the central charge $c = -74$ of the $(\beta, \gamma)$ ghosts for the spin–3 current $W$, and so the nilpotence of the $W_3$ BRST operator requires that $T_X$ have central charge $c = 25\frac{1}{2}$ rather than $c = 26$. The $\varphi$ field has no associated continuous degrees of freedom in physical states, and the cohomology of the $W_3$ string is just that of a $c = 25\frac{1}{2}$ Virasoro string tensored with the Ising model.

In all of the $W$-string theories that have been constructed, the $W$ currents are realised in terms of an energy-momentum tensor $T_X$ together with some additional scalar fields that carry no continuous degrees of freedom in physical states. In view of our previous discussion in section 2, it should be emphasised that the important point in order to be able to view the bosonic string as being embedded in a particular $W$ string is that the classical currents that realise the classical $W$ algebra should be expressible in terms of $T_X$ plus the additional scalar fields.

It has also been suggested that one might be able to embed the $c = 26$ Virasoro string into, for example, the $W_3$ string. However, it would, perhaps, be surprising if it were possible to embed the Virasoro string into the $W_3$ string in two different ways, both for $c_X = 25\frac{1}{2}$ and also for $c_X = 26$. Indeed, there is no known way of realising the currents of the $W_3$ algebra, with the central charge $c = 100$ needed for nilpotence of the BRST operator, in terms of a $c = 26$ energy-momentum tensor plus other fields that would contribute no continuous degrees of freedom in physical states.

A very different approach was proposed in $[24]$, where it was shown that by performing a sequence of canonical transformations on the BRST operator of the $W_3$ string, it could be transformed into the BRST operator of an ordinary $c = 26$ bosonic string plus a purely topological BRST operator. However, as was shown in $[14]$, and subsequently reiterated in $[25]$, one step in the sequence of canonical transformations involved a non-local transformation...
that reduced the original $W_3$ BRST operator to one with completely trivial cohomology. A later step in the sequence then involved another non-local transformation that caused the usual cohomology of the bosonic string to grow out of the previous trivial cohomology. In effect one is gluing two trivialised theories back to back, and so the physical spectra of the two theories prior to trivialisation are disconnected from one another. This situation is therefore quite distinct from the kind of embedding proposed in [12], where a realisation of the Virasoro algebra is embedded in the larger algebra, no cohomology-changing non-local transformations are performed, and the physical states of the bosonic string arise directly in the cohomology of the $N = 1$ superstring for this realisation. The already-existing $W$-string theories, with their realisations involving $T_X$ plus extra scalars, are in fact the natural $W$ generalisations of the embedding discussed in [12].

An interesting possibility for generalising the ideas in [12] is to consider the case where the bosonic string is embedded in a fermionic higher-spin string theory. The simplest such example would be provided by looking at a theory with a spin–5/2 current in addition to the energy-momentum tensor. In order to present some results on this example, it is useful first to recast the $N = 1$ superstring, with the matter currents realised as in (26), in a simpler form. We do this by performing a canonical redefinition involving the spin–2 ghosts ($b,c$), the spin–3/2 ghosts ($r,s$), and the ghost-like matter fields ($b_1,c_1$) (which we shall refer to as pseudo-ghosts). If we transform these according to

$$
c \rightarrow c - sc_1,

r \rightarrow r - bc_1,

b_1 \rightarrow b_1 + bs,

$$

(with $b$, $s$ and $c_1$ suffering no transformation), then the BRST operator assumes the graded form

$$
Q_0 = \oint c \left( T_M + T_{b_1c_1} + T_{rs} + \frac{1}{2} T_{bc} + x \partial^2 (\partial c_1 c_1) \right),

Q_1 = \oint s \left( b_1 - x b_1 \partial c_1 c_1 + 3 x r \partial s c_1 + x \partial r s c_1 + 2 x^2 \partial^2 c_1 c_1 \right).

$$

Here $x$ is a free constant which actually takes the value $-\frac{1}{2}$ when one transforms (26) according to (27), but can be made arbitrary by performing a constant OPE-preserving rescaling of $b_1$ and $c_1$. The reason for introducing $x$ is that it can be viewed as a power-counting parameter for a second grading of $Q_0$ and $Q_1$, under the $(b_1,c_1)$ pseudo-ghost number. Thus $Q_0$ has terms of pseudo-ghost degrees $0$ and $2$, whilst $Q_1$ has terms of pseudo-ghost degrees $-1$, $1$ and $3$. (We have dropped an overall $x^{-1}$ factor from $Q_1$ for convenience. We are free to do this owing to the first grading under $(r,s)$ degree, which implies that $Q_0^2 = Q_1^2 = \{Q_0,Q_1\} = 0$.)

Before moving on to the generalisation to higher spins, it is useful to present the unitary canonical transformation of ref. [23] in this language, which maps the BRST operator into that of the bosonic string plus a topological term. Thus we find that the charge

$$
R = \oint c_1 \left( - c \partial r - \frac{1}{2} \partial c r - x r s \partial c_1 \right)

$$

acts on the BRST operator $Q = Q_0 + Q_1$ to give

$$
e^R Q e^{-R} = \oint c (T_M - b \partial c) + \oint s b_1 .

$$

11
The first term on the right-hand side is the usual BRST operator of the bosonic string, and the second term is purely topological, with no serious cohomology.

We may now seek a spin \((2, \frac{3}{2})\) generalisation of this spin \((2, \frac{1}{2})\) theory. Thus we now consider commuting ghosts \((r, s)\) of spins \((\frac{1}{2}, -\frac{1}{2})\) for a spin–\(\frac{1}{2}\) current, and anticommuting pseudo-ghosts \((b_1, c_1)\) of spins \((\frac{1}{2}, -\frac{3}{2})\). We find that a graded BRST operator \(Q = Q_0 + Q_1\) again exists, where \(Q_0\) contains terms with pseudo-ghost degrees 0, 2 and 4, whilst \(Q_1\) has terms of pseudo-ghost degrees \(-1, 1, 3\) and 5. The coefficients of the various possible structures in \(Q_0\) and \(Q_1\) are determined by the nilpotency conditions \(Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0\). \(Q_0\) takes the form

\[
Q_0 = \oint c \left( T_M + T_{b_1 c_1} + T_{r_1} + \frac{1}{2} T_{bc} + x \partial^2 (3 \partial^3 c_1 c_1 + 7 \partial^2 c_1 \partial c_1) + y \partial^2 (\partial^3 c_1 \partial^2 c_1 \partial c_1 c_1) \right),
\]

(32)

where \(x\) and \(y\) are arbitrary constants associated with the terms in \(Q_0\) of pseudo-ghost degree 2 and 4 respectively. The form of \(Q_1\) is quite complicated;

\[
Q_1 = \oint s \left( b_1 - 6x b_1 \partial^2 c_1 \partial c_1 - 4x b_1 \partial^3 c_1 c_1 - 6x \partial b_1 \partial^2 c_1 c_1 - 2x \partial^2 b_1 \partial c_1 c_1 + \cdots \\
+ x \left( \frac{26}{7} x^2 + \frac{25}{6} y \right) \partial^4 c_1 \partial^3 c_1 \partial^2 c_1 \partial c_1 c_1 \right),
\]

(33)

where the ellipsis represents the 13 terms of pseudo-ghost degree 3.

One may again look for a charge \(R\) that acts unitarily and canonically on the BRST operator to give it a simpler form. We find that the required charge is given by

\[
R = \oint c \left( - c \partial r - \frac{3}{2} \partial c r - x c \partial^2 c_1 \partial r - \frac{4}{3} x \partial c \partial^2 c_1 \partial c_1 r \\
- 2x \partial c_1 \partial^2 r s - 6x \partial^2 c_1 \partial r s + 2x \partial^2 c_1 \partial r s - \frac{4}{3} y \partial^3 c_1 \partial^2 c_1 \partial c_1 r s \right).
\]

(34)

Acting on the BRST operator \(Q = Q_0 + Q_1\), this gives

\[
e^R Q e^{-R} = \oint c \left( T_M - b \partial c \right) + \oint s b_1,
\]

(35)

which shows that this theory is again simply equivalent to the bosonic string.

Although the spin \((2, \frac{3}{2})\) theory that we have described above has a BRST operator that is a natural generalisation of the \(N = 1\) superconformal BRST operator with the realisation \((29)\) for the matter currents, there is one important respect in which it differs. From the graded \((2, \frac{3}{2})\) BRST operator given by \((29)\), one can invert the canonical transformation \((27)\) and get back to a form in which one can replace the specific realisation \((28)\) of the superconformal currents by an abstract realisation in terms of currents \(T\) and \(G\). In this sense, one can say that the realisation \((28)\) describes an embedding of the bosonic string in the \(N = 1\) superstring. In the \((2, \frac{3}{2})\) case, on the other hand, where we started with the already-graded BRST operator given by \((32)\) and \((33)\), there is no canonical transformation that will map the BRST operator into a form where spin–2 and spin–\(\frac{3}{2}\) matter currents can be identified, and replaced by abstract spin–2 and spin–\(\frac{3}{2}\) currents. The underlying reason for this is that the Jacobi identity for the classical algebra of spin–2 and spin–\(\frac{3}{2}\) currents is not identically satisfied (but only up to null fields), and this implies that a nilpotent classical BRST operator of the kind that we are considering here does not exist. (See below for a further discussion of this point.) Thus it seems that there is no sense in which one could say that the theory described by the BRST operator \((32)\) and \((33)\)
is an embedding of the bosonic string in a $W_{2,\frac{5}{2}}$ string, as we expect that such a string theory is described by a different type of BRST operator.

We have explicitly checked for all higher half-integer values of spin, and we find that again a classical $W_{2,n/2}$ algebra does not identically satisfy the Jacobi identity. Thus again, although we expect that higher-spin generalisations of the BRST operator (32) and (33) exist, they would not be associated with any higher-spin string theory of the kind we are considering here.

The above situation is very different from that for the integer-spin $W_{2,s}$ strings. In that case the theories were again originally constructed by generalising the graded BRST operator structure $Q = Q_0 + Q_1$ of the $W_3 = W_{2,3}$ string to arbitrary $s$ [9], and a priori one had no particular reason to expect that the resulting BRST operators could be viewed as describing string theories with spin–2 and spin–$s$ currents. It was really only by using arguments of the kind presented in section 2, where we showed that the $W_{2,s}$ BRST operators describe the quantisation of classical theories with classical $W_{2,s}$ algebras as local symmetries, that their identification as string theories could be made.

It is worthwhile to examine in greater detail the issue of algebras where the Jacobi identity is satisfied only modulo null fields. In particular, such a null field will vanish if one has an explicit realisation of the currents that generate the algebra. Let us consider the $W_{2,\frac{5}{2}}$ algebra in more detail. Classically, the primary spin–$\frac{5}{2}$ current $G$ satisfies the OPE

$$G(z)G(w) \sim \frac{T^2}{z-w}. \quad (36)$$

The Jacobi identity is satisfied modulo a classical “null field”

$$N_1 \equiv 4T \partial G - 5\partial T \, G. \quad (37)$$

As we mentioned above, no classical BRST charge can be found of the type $cT + \gamma G + \ldots$, where the ellipsis denotes terms with $T, G$ and (anti)ghosts. To clarify this point, we look for a realisation of the $W_{2,\frac{5}{2}}$ algebra. We found the following realisation:

$$T = \frac{1}{2} \partial \psi \bar{\psi} - \frac{1}{2} \partial \bar{\psi} \psi,$n

$$G = \frac{i}{4}(\psi + \bar{\psi})T, \quad (38)$$

where $\psi$ is a complex fermion satisfying the OPE $\psi(z)\bar{\psi}(w) \sim 1/(z-w)$. One can easily verify for this realisation that the null field $N_1$ vanishes. It is now straightforward to write down the most general possible structure for the classical BRST operator for this realisation, and try solving for the coefficients by demanding nilpotence at the classical level. It turns out that no solution is possible. Indeed, in a realisation of the kind we are considering in this example, where the vanishing of the $G$ current is implied by the vanishing of the $T$ current, one can expect to need ghost–for–ghost terms in a full BRST analysis [27]. However, in the similar case of classical $W_{2,2k}$ algebras trivially realised in terms of $T$ and $W = \frac{1}{k}T^k$ that we mentioned earlier, a nilpotent BRST operator could be found even if one neglected this point. This suggests that in the present example, it is indeed the failure to satisfy the Jacobi identity that is responsible for the non-existence of a classical BRST operator for $W_{2,\frac{5}{2}}$.

The reason why the vanishing of the null field $N_1$ is not sufficient to ensure the nilpotence of the BRST operator can be understood in the following way. In the mode language, we may write the non-linear algebra as $[K_i, K_j] = f_{ij}^k K_k$, where the structure constants are field dependent (i.e. $K$ dependent). The Jacobi identity takes the form $[K_i, [K_j, K_k]] + [K_j, [K_k, K_i]] +$
\([K_k, [K_i, K_j]] = 3 f_{ij}^m f_{k|m}^\ell K_\ell\). When no null fields are present the fact that the Jacobi identities vanish, implies that \(f_{ij}^m f_{k|m}^\ell\) is zero (as is the case for the linear algebras). However, in general it is possible that \(f_{ij}^m f_{k|m}^\ell\) is non-zero (and not null), and yet the product of this (field-dependent) expression with \(K_\ell\) is null (this is exactly what happens in the \(W_{2,\frac{5}{2}}\) example above). The classical BRST operator has the general form \(Q = c_i K_i + f_{ijk} c_i c_j b_k + \ldots\) involving 7 or more ghosts \([28]\). We can see that nilpotence will require that \(f_{ij}^m f_{k|m}^\ell K_\ell\) be null and consequently vanishing in the specific matter realisation that one is using. (The calculation of \(Q^2\) does not generate any terms of the form \(f_{ij}^m f_{k|m}^\ell\), and so the fact that this expression might be null, and hence vanish in the specific realisation, has, of itself, no bearing on whether \(Q\) is nilpotent.) This argument shows that a classical BRST charge of this type does not exist.

The fact that the BRST charge for the realisation \([28]\) contains ghosts–for–ghosts indicates that it is necessary to introduce ghost–for–ghosts in the case of the abstract algebra. Indeed, one can view a null field as a relation between the “constraints” \(T, G\). In the case of \(W_{2,\frac{5}{2}}\), the null field \(N_1\) is not the only one. We can check by repeatedly computing Poisson brackets with \(N_1\) that there is an ideal in the Poisson algebra of \(T\) and \(G\), generated by \(N_1\) and \(N_2 \equiv 4 T^3 - 30 \partial G G\). (39)

More precisely, all other null fields can be written as:

\[ f_1(T, G) N_1 + f_2(T, G) N_2 + f_3(T, G) \partial N_1 + \ldots, \]  

with \(f_i(T, G)\) differential polynomials in \(T\) and \(G\). We see that the phase space of the Poisson algebra is not simply the space of differential polynomials in \(T\) and \(G\), but the additional constraints \(N_1 = N_2 = 0\) have to be taken into account. In such a case, the ordinary procedure of constructing a (classical) BRST charge does not work. Indeed, one should use the BRST-formalism appropriate for reducible constraints, and ghosts–for–ghosts have to be introduced \([27]\). This clearly explains why no “ordinary” BRST charge exists for this system.

Thus, it seems that the \(W_{2,\frac{5}{2}}\) string is of a very different type than other strings considered up to now. It remains to be seen if the resulting BRST-charge is in any way related to the one we constructed above, eqs. (32,33).

### 6 Conclusion

In this paper we have looked at the quantisation of \(W\)-string theories based on the classical \(W_{2,s}\) higher-spin algebras. One of the more noteworthy features of these theories is that anomaly-free quantisation is possible even when there does not exist a closed quantum extension of the classical \(W_{2,s}\) algebra at the critical central charge. Indeed, it can happen that there are several inequivalent quantum theories that arise from the same classical theory, corresponding to different possible choices for the coefficients of the quantum corrections to the classical BRST operator. We have studied this on a case-by-case basis up to \(s = 8\), but it would be interesting to have a more systematic and general understanding of how many different possibilities should arise for each value of \(s\).

In a multi-scalar realisation, the spectrum of physical states for a \(W_{2,s}\) string turns out to be described by the tensor product of sets of bosonic-string states in the effective spacetime times.
certain primary operator built from the \((\varphi, \beta, \gamma)\) fields. In most cases these primary fields can be recognised as those of some Virasoro or \(W\) minimal model. For example, the regular sequence of \(W_{2s}\) BRST operators, which exist for all \(s\), corresponds to the lowest unitary \(W_{s-1}\) minimal model, with \(c_{\text{min}} = 2(s - 2)/(s + 1)\). The other BRST operators, which seem only to arise when \(s\) is even, are associated with models for which there is no current systematic understanding. It would be interesting to develop a more comprehensive understanding of which models should arise for each value of \(s\).

We have looked also at string theories based on classical algebras involving a higher-spin fermionic current in addition to the energy-momentum tensor. These classical algebras do not satisfy the Jacobi identity identically, but only modulo null fields. When there exists a classical realisation, these null fields are identically zero. Nevertheless, it turns out not to be possible to build a classical nilpotent BRST operator, contrary to one’s intuitive expectation, based on experience with linear algebras, that nilpotence of the BRST operator should be guaranteed by the closure of the algebra. The reason for this can be traced back to occurrence of the null fields, which is a new feature that does not arise for linear algebras.

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Note Added

After this paper was completed, we received a paper that also demonstrated the non-existence of Jordan \(W_3\) strings [29].

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