The two-dimensional electron self-energy: Long-range Coulomb interaction

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The electron self-energy for long-range Coulomb interactions plays a crucial role in understanding the many-body physics of interacting electron systems (e.g. in metals and semiconductors), and has been studied extensively for decades. In fact, it is among the oldest and the most-investigated many-body problems in physics. However, there is a lack of an analytical expression for the self-energy of an interacting system. The calculation of the self-energy on the mass shell for a two-dimensional electron system with Coulomb interactions in the high density limit is unknown for arbitrary values of \( \varepsilon/k_B T \) and low temperature \( k_B T < \varepsilon/\varepsilon' \). For this reason, we revisit the problem and calculate analytically the self-energy in the regimes \( \varepsilon < k_B T \) and \( \varepsilon > k_B T \). The higher-order terms have subtle and highly non-trivial compound logarithmic contributions from both \( \varepsilon \) and \( T \), explaining why they have never before been calculated.

I. INTRODUCTION

In Landau’s Fermi liquid theory, an interacting Fermi system, at low excitation energies and temperatures, is described by long-lived excitations called “quasiparticles” which evolve adiabatically from the corresponding excitations of the noninteracting Fermi gas as the interactions are turned on [1]. The quasiparticle is well defined only when the damping of the single particle state is small, which happens at low temperatures close to the Fermi surface. In other words, the imaginary part of the self-energy \( \Sigma^{(R)}(k, \varepsilon) \) should be much smaller compared with \( \varepsilon + Re \Sigma^{(R)}(k, \varepsilon) \) at low energy \( \varepsilon \) in order to have well-defined quasiparticles, satisfying the Landau Fermi liquid paradigm. The electron self-energy is a crucial quantity, which determines not only the life-time of the quasiparticles [2–16], but also their effective mass [17–25], the renormalization factor, and many other single particle properties [26–29]. It is well-established that for low \( T (\ll T_F) \) and \( \varepsilon (\ll E_F) \), \( \varepsilon'T \) and \( \varepsilon^2 \) (up to logarithmic corrections) in three-dimensional (3D) and two-dimensional (2D) Fermi systems, leading to the existence of well-defined 2D and 3D Landau Fermi liquids. By contrast, in 1D interacting Fermi systems, quasiparticles do not exist, and the one to one correspondence with the Fermi gas picture is destroyed even for infinitesimal interactions.

The calculation of the self-energy of an interacting electron system is a condensed matter problem that has been extensively studied for decades [1] [3] [23] [25] [30] [31]. In fact, this is among the oldest many-body problems in physics, dating back to the 1950s, when field theoretic Feynman diagram techniques were first used in calculating properties of simple metals within the 3D interacting electron liquid model [1] [34]. Later, similar many body techniques were used to study the properties of 2D interacting electron liquids in various artificial semiconductor structures [35]. Most of these calculations, where the inter-electron interaction is the long-range Coulomb coupling, are either completely numerical, dubbed “GW” approximation [36], or just leading order theories in \( r_s \) or \( T \). To the best of our knowledge, the expression for the self-energy \( \Sigma^{(R)}(k, \varepsilon) \) with arbitrary \( \varepsilon/k_B T \) is unknown for such an interacting system with Coulomb interactions. For this reason, we revisit the problem and calculate analytically the on-shell self-energy using the random phase approximation (RPA).

For the long-range Coulomb coupling, it is well-known that asymptotically exact many-body description for the interacting self-energy is available in the high-density limit, \( r_s \ll 1 \), where only the infinite series of polarization diagrams (see Fig. 1 involving the electron-hole “bubbles” (or “rings”) need to be kept in the theory. This bubble diagram description of the system is equivalent (see Fig. 1) to a theory involving the leading-order self-energy calculation in the dynamically screened Coulomb coupling (instead of the bare Coulomb coupling appearing in the Hamiltonian), where the dynamical screening is approximated by RPA. We therefore refer to the self-energy theory in this leading order dynamical screening approximation itself as the RPA self-energy. Such an RPA self-energy is exact to the leading-order in \( r_s \), and
is extensively used in materials physics, where, for historical reasons, it is universally called the “GW” approximation. Our goal is to analytically calculate the interacting 2D self-energy of the leading-order in \( r_s \) (where it is exact for the interacting problem), and to nonleading orders in \( \varepsilon/T \).

The rest of this paper is organized as follows. In Sec. II we present the general formulas for the self-energy of a 2D electron system with Coulomb interactions. Using these formulas, in Sec. III we derive analytically the expressions for the imaginary and real parts of the electron self-energy in the high density \( (r_s \ll 1) \), low temperature \( (k_B T/E_F \ll r_s) \) and low energy \( (|\varepsilon|/E_F \ll r_s) \) limit for arbitrary value of \( \varepsilon/k_B T \). The asymptotic expressions for \( |\varepsilon|/k_B T \ll 1 \) and \( |\varepsilon|/k_B T \gg 1 \) are also presented in this section. In Sec. IV we conclude with a brief discussion. Finally, appendices are devoted to some technical details: In Appendix A the self-energy formulas shown in Sec. II are derived using the Keldysh technique. We provide in Appendix B the evaluation of the momentum integrals appearing in the self-energy formulas, and in Appendix C the frequency integrals involving hyperbolic tanh and coth functions.

II. GENERAL FORMULAS

This section summarizes the general formulas for the electron self-energy. The results provided in this section are, in principle, known, but we give them here for the sake of completeness because we have not seen them written down anywhere in the literature in the precise form necessary for our calculations. In addition, this section provides a context and serves the useful purpose of explaining our notations and the actual calculations as well as the analytical results. We consider a clean 2D electron system with Coulomb interactions, a parabolic energy dispersion and a spin degeneracy factor of 2, and from now on adopt the units \( k_B = \hbar = 1 \). (Spin is an implicit variable since we only consider a paramagnetic situation with no explicit spin-dependent scattering—the only interaction in the problem is the long-range Coulomb coupling, which is spin-independent.) The detailed derivation of these formulas is presented in Appendix A while an alternative Matsubara approach can be found in Ref. [1].

To the lowest order in the dynamically screened interaction, the retarded electron self-energy is given by

\[
\Sigma^{(R)}(\mathbf{k}, \varepsilon) = \frac{i}{2} \int_{\Omega_{\omega}} \left\{ D^{(K)}(\mathbf{q}, \omega) G_0^{(R)}(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) + D^{(A)}(\mathbf{q}, \omega) G_0^{(K)}(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \right\},
\]

and the corresponding self-energy diagram in plotted in Fig. 1(a). Here we have employed the shorthand notation \( \int_{\mathbf{q}} \equiv \int d^2q/(2\pi)^2 \) and \( \int_\omega \equiv \int_0^{\infty} d\omega/2\pi \).

\( G_0 \) denotes the non-interacting fermionic Green’s function, and is represented diagrammatically by the black line in Fig. 1. Its retarded (advanced) component acquires the form

\[
G_0^{(R)/(A)}(\mathbf{k}, \varepsilon) = [\varepsilon - \xi_k \pm i\eta]^{-1}.
\]

\( \eta \) is a positive infinitesimal, and \( \xi_k \equiv k^2/2m - \mu \), with \( \mu \equiv E_F \), where \( E_F \) is the noninteracting Fermi energy, being the chemical potential. We use \( k_F \) for the Fermi momentum, defined by \( E_F = k_F^2/2m \). We note that the dimensionless Coulomb couluing (or the effective fine structure constant) is simply given by \( r_s = \sqrt{2e^2}/v_F \), where the Fermi velocity \( v_F = k_F/m \). The Keldysh Green’s function is related to its retarded and advanced counterparts through the fluctuation-dissipation theorem (FDT):

\[
G_0^{(K)}(\mathbf{k}, \varepsilon) = \left[G_0^{(R)}(\mathbf{k}, \varepsilon) - G_0^{(A)}(\mathbf{k}, \varepsilon)\right] \tanh(\varepsilon/2T),
\]

and, unlike the other two components, depends on the occupation number. Hereafter, superscripts \( (R) \), \( (A) \) and \( (K) \) stand for the retarded, advanced, and Keldysh components, respectively.

In Eq. 1, \( D \) indicates the RPA dynamically screened interaction, which is represented by the red wavy line with a solid dot in Fig. 1. It can be considered as the dressed Green’s function for the bosonic field that, through the Hubbard-Stratonovich (H.S.) transformation, decouples the interactions (see Appendix A for details). In Fig. 1(b), the RPA interaction \( D \) is defined diagrammatically by the infinite Dyson series of the polarization bubble diagrams where the red wavy line with an open dot stands for the 2D bare Coulomb potential \( V(q) = 2\pi e^2/q \), and the black bubble corresponds to the polarization operator whose retarded component is given by

\[
\Pi^{(R)}(\mathbf{q}, \omega) = -i \int_{\mathbf{k}, \varepsilon} \left[G_0^{(R)}(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) G_0^{(K)}(\mathbf{k}, \varepsilon) + G_0^{(K)}(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) G_0^{(A)}(\mathbf{k}, \varepsilon)\right].
\]
It is therefore straightforward to see that the retarded RPA dynamically screened interaction can be extracted from the following Dyson equation

$$D^{(R)}(q, \omega) = \left[ V^{-1}(q) - \Pi^{(R)}(q, \omega) \right]^{-1}, \tag{5}$$

while its advanced and Keldysh components are related to the retarded one through

$$D^{(A)}(q, \omega) = \left[ D^{(R)}(q, \omega) \right]^*, \tag{6a}$$

$$D^{(K)}(q, \omega) = \left[ D^{(R)}(q, \omega) - D^{(A)}(q, \omega) \right] \coth (\omega/2T), \tag{6b}$$

as expected for a bosonic propagator. Here the last equation constitutes the FDT relation between the components of the RPA interaction $D$.

In the static limit $\omega \ll k_F q/m$, $-\Pi^{(R)}(q, \omega)$ is well approximated by $\nu = m/\pi$, the density of states at the Fermi level. As a result, $D^{(R)/(A)}(q, \omega)$ is reduced to the static screened interaction

$$\tilde{V}(q) = \frac{1}{V^{-1}(q)} - \nu = \frac{1}{\nu q + k_{TF}}, \tag{7}$$

with $k_{TF} \equiv 2\pi e^2 \nu$ being the Thomas-Fermi screening wavenumber.

Inserting the explicit expression for the non-interacting electron Green’s function $G_0$ (Eq. (2)) into Eq. (1) and utilizing the FDT relation as well as the Kramers-Krönig relation (Eq. (A24)) for both $G_0$ and $D$, one finds that the imaginary and real parts of the electron self-energy are given by

$$\text{Im} \Sigma^{(R)}(k, \varepsilon) = \frac{m}{4\pi^2 k} \int_{-\infty}^{\infty} d\omega \left[ \coth \left( \frac{\omega + \varepsilon}{2T} \right) - \tanh \left( \frac{\omega + \varepsilon}{2T} \right) \right] \int_{q_{-}(-\omega)}^{q_{+}(\omega)} dq \frac{\text{Im} D^{(R)}(q, \omega)}{\sqrt{1 - \left[ \frac{m}{k^* q} (\omega + \Delta \varepsilon) \right]^2}}, \tag{8a}$$

$$\text{Re} \Sigma^{(R)}(k, \varepsilon) = \frac{m}{4\pi^2 k} \int_{-\infty}^{\infty} d\omega \tanh \left( \frac{\varepsilon + \omega}{2T} \right) \int_{q_{-}(-\omega)}^{q_{+}(\omega)} dq \frac{\text{Re} D^{(R)}(q, \omega)}{\sqrt{1 - \left[ \frac{m}{k^* q} (\omega + \Delta \varepsilon) \right]^2}} - \frac{m}{4\pi^2 k} \int_{-\infty}^{\infty} d\omega \coth \left( \frac{\omega}{2T} \right) \left( \int_{0}^{q_{-}(-\omega)} dq + \int_{q_{+}(\omega)}^{\infty} dq \right) \frac{\text{Im} D^{(R)}(q, \omega) \text{sgn}(\omega + \Delta \varepsilon)}{\sqrt{\left[ \frac{m}{k^* q} (\omega + \Delta \varepsilon) \right]^2 - 1}}. \tag{8b}$$

For simplicity, here we have defined

$$\Delta \varepsilon \equiv \varepsilon - \xi_k - q^2/2m, \tag{9a}$$

$$q_{\pm}(\varepsilon) \equiv \pm k + \sqrt{k^2 + 2m(\omega + \varepsilon - \xi_k)}. \tag{9b}$$

Equations (8a) and (8b) can be directly numerically calculated for arbitrary $r_s$, $\varepsilon$, and $T$ to provide the self-energy function of a 2D interacting electron liquid. Our goal is to obtain the analytical expressions for small values of $r_s$, $\varepsilon$, and $T$ as described in the next section. We note that formally Eqs. (8a) and (8b) appear to be 2D integrals over $\omega$ and $q$, but this is misleading since the screened interaction $D$ itself (see Eqs. (6) above) is formally a 3D integral. Thus, Eq. (8) in general defines a highly singular five-dimensional integral, which is not easy to handle directly numerically although its $T = 0$ version has been calculated numerically [12] and crude numerical calculations have also been performed for the temperature dependent 2D self-energy by approximating the dynamically screened interaction $D$ as a simple function with poles within the so-called plasmon pole approximation [37]. There had also been an early purely numerical attempt to calculate the 2D self-energy at very high ($T \gg E_F$) temperatures, where the static screening approximation was used to replace the dynamically screened interaction [38]. Such numerical self-energy calculations carried out with simplistic and uncontrolled approximations fail to provide any analytical insight into the low-temperature quasiparticle properties of the interacting 2D Fermi liquid, which is the goal of our study.

### III. RESULTS

In the previous section, the real and imaginary parts of the electron self-energy $\Sigma^{(R)}(k, \varepsilon)$ are expressed as two-variable integrals in terms of the retarded RPA interaction $D^{(R)}(q, \omega)$ given by Eq. (5). Having these formulas, we now calculate analytically the on-shell ($\varepsilon = \xi_k$) self-energy $\Sigma^{(R)}(k, \varepsilon)$ close to the Fermi surface ($k \approx k_F$) in the high density limit ($r_s \ll 1$), where the RPA ring diagram approximation should be exact for the Coulomb coupling. We work in the regime where $r_s^{3/2} E_F \ll \Delta \ll r_s E_F$, with $\Delta = \{ T, |\varepsilon| \}$, and obtain the electron self-energy up to the order of $(\Delta/E_F r_s)^3$ and the leading order in $r_s$. 
In the low temperature regime, the polarization operator $\Pi^{(R)}(q, \omega)$ can be approximated by its zero temperature result, whose explicit form has been calculated in Ref. [39] and may be expressed in terms of two dimensionless variables: $q/k_F$ and $m\omega/k_F q$ (see Eq. 8). Because of the RPA interaction and the thermal factors, the most significant contribution to the integrals in Eq. 8 comes from the region $q \lesssim k_T = \sqrt{2} r_s k_F$ and $\omega \lesssim \max(T, |\varepsilon|)$. As a result, terms of higher orders in $q/k_F$ and $m\omega/k_F q$ in the integrands lead to, respectively, higher order terms in $r_s$ and $\max(T, |\varepsilon|)/E_F r_s$ in the electron self-energy $\Sigma^{(R)}(k, \varepsilon)$, and are negligible in the high density ($r_s \ll 1$), low temperature ($T/E_F \ll r_s$) and small energy ($|\varepsilon|/E_F \ll r_s$) limit. It is therefore only necessary to keep the first few leading terms in $q/k_F$ and $m\omega/k_F q$ in the integrals to obtain the leading results of interest to us.

### A. Imaginary part of the self-energy

To calculate the imaginary part of the self-energy on the mass shell ($\varepsilon = \xi_k$), we insert the expression for the polarization operator [39] into Eq. 8a and set $\Delta \varepsilon = -q^2/2m$. After the momentum integration, this yields

\[
\text{Im} \Sigma^{(R)}(\varepsilon) = \int_0^\infty \frac{d\omega}{2\pi} \left[ 2 \coth \left( \frac{\omega}{2T} \right) - \tanh \left( \frac{\omega + \varepsilon}{2T} \right) - \tanh \left( \frac{\omega - \varepsilon}{2T} \right) \right] \text{Im} I(\omega),
\]

\[
\text{Im} I(\omega) = -\left\{ \frac{1}{4} \frac{|\varepsilon|}{E_F^2} \ln \left( \frac{2\sqrt{2} r_s E_F}{|\omega|} \right) \right\} + \frac{1}{2} \left( \frac{\omega}{E_F} \right)^2 \text{sgn} \omega.
\]

Here $I(\omega)$ is defined as the integral

\[
I(\omega) = \frac{m}{2\pi k} \int_{q_0(\omega)}^{q_1(\omega)} dq \frac{D^{(R)}(q, \omega)}{\sqrt{1 - \left[ \frac{m}{kq} \left( \omega - \frac{q^2}{2m} \right) \right]^2}}.
\]

Its detailed calculation is shown in Appendix [3].

The frequency integration in Eq. 10a can be done by expressing the hyperbolic functions as infinite exponential series:

\[
\tanh(x) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx}, \quad x > 0,
\]

\[
\coth(x) = 1 + 2 \sum_{k=1}^{\infty} e^{-2kx}, \quad x > 0.
\]

We evaluate integrals of such forms in Appendix [3] and use the results (Eq. [35]) to obtain the imaginary part of the self-energy on the mass shell

\[
\text{Im} \Sigma^{(R)}(\varepsilon) = -\frac{T^2}{E_F} \ln \left( \frac{\sqrt{2} r_s E_F}{T} \right) g_1 \left( \frac{\varepsilon}{T} \right) - \frac{T^2}{E_F} g_2 \left( \frac{\varepsilon}{T} \right) - \frac{T^3}{r_s E_F^2} g_3 \left( \frac{\varepsilon}{T} \right),
\]

where

\[
g_1 \left( \frac{\varepsilon}{T} \right) = \frac{1}{8\pi} \left( \pi^2 + \frac{\varepsilon^2}{T^2} \right),
\]

\[
g_2 \left( \frac{\varepsilon}{T} \right) = -\frac{\pi}{24} \left( 6 - \gamma_E - \ln \frac{2}{\pi^2} - 24 \ln A \right) - \frac{(2 - \gamma_E - \ln 2) \varepsilon^2}{8\pi T^2} + \frac{1}{4\pi} \left[ \partial_s \text{Li}_s(-e^{-\varepsilon/T}) + \partial_s \text{Li}_s(-e^{\varepsilon/T}) \right]_{s=2}.
\]

\[
g_3 \left( \frac{\varepsilon}{T} \right) = \frac{\sqrt{3}}{\pi} \left[ \zeta(3) - \frac{1}{2} \text{Li}_3(-e^{-\varepsilon/T}) - \frac{1}{2} \text{Li}_3(-e^{\varepsilon/T}) \right].
\]

Here $\text{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ denotes the polylogarithm function and $\zeta(s) = \text{Li}_s(1)$ represents the Riemann zeta function. $\gamma_E \approx 0.577216$ is the Euler’s constant and $A \approx 1.28243$ is the Glaisher’s constant. Following a straightforward calculation, we find from Eq. 13 the asymptotic behavior of $\text{Im} \Sigma^{(R)}(\varepsilon)$ in the regime $|\varepsilon| \ll T$:

\[
\text{Im} \Sigma^{(R)}(|\varepsilon| \ll T) = -\frac{\pi}{8} \frac{T^2}{E_F} \ln \left( \frac{\sqrt{2} r_s E_F}{T} \right) + \frac{\pi}{24} \left( 6 + \ln 2 \pi^2 - 36 \ln A \right) \frac{T^2}{E_F} - \frac{7\zeta(3)}{2\sqrt{2} \pi} \frac{T^3}{r_s E_F^2}.
\]
This leads to the following asymptotic expression for $\text{Im } \Sigma^{(R)}(\varepsilon)$ when $|\varepsilon| \gg T$:

$$
\text{Im } \Sigma^{(R)}(|\varepsilon| \gg T) = -\frac{\varepsilon^2}{8\pi E_F} \ln \left( \frac{\sqrt{2} r_s T}{|\varepsilon|} \right) - \frac{\varepsilon^2}{8\pi E_F} \left( \ln 2 - \frac{1}{2} \right) - \frac{1}{6\sqrt{2} r_s T^3 |\varepsilon|^3}.
$$

We note that the leading order terms (i.e., the first terms on the right hand sides) in the asymptotic expressions Eqs. 15 and 18 are consistent with the results in Ref. [5], which obtained the correct leading order $T^2 \ln T$ and $\varepsilon^2 \ln |\varepsilon|$ forms for the 2D imaginary self-energy. We note that in spite of the additional logarithmic factors compared with the corresponding 3D self-energy [11][2], the imaginary self-energy has the quadratic dependence on $T$ and/or $\varepsilon$, indicating that the interacting 2D system is a Landau Fermi liquid at low temperatures and excitation energies. Our results in Eqs. 15 and 18 provide the full analytical form for the 2D imaginary self-energy including the next to leading order terms in the excitation energy and temperature.

In Fig. 2 panel (a)[(b)], we plot the imaginary part of the on-shell self-energy given in Eqs. 13 and 14 as a function of $\varepsilon/E_F (T/E_F)$ for different values of $T/E_F (\varepsilon/E_F)$. In Fig. 2 panel (a) [(b)], we plot the imaginary part of the on-shell self-energy given in Eqs. 13 and 14 as a function of $\varepsilon/E_F (T/E_F)$ for different values of $T/E_F (\varepsilon/E_F)$, together with the asymptotic expression for $T \ll \varepsilon (\varepsilon \ll T)$ given in Eqs. 15 and 18. The solid curves from top to bottom are associated with $\varepsilon/E_F (T/E_F)$ equals to 0.1, 0.2 and 0.5. As the value of $\varepsilon/E_F (T/E_F)$ decreases, the corresponding solid curve approaches the dashed one, which represents the analytical asymptotic result for $T \ll \varepsilon (\varepsilon \ll T)$ as given in Eqs. 15 and 18.

### B. Real part of the self-energy

Compared with the imaginary part of the self-energy, the calculation of the real part is much more difficult since it requires one more integration involving a branch cut. For the real part of the self-energy on the mass shell ($\varepsilon = \xi_k$), the second integral in Eq. 8b vanishes to the leading order in $r_s$, while the first integral, after the momentum integration (for details, see Appendix B), is further reduced to

$$
\text{Re } \Sigma^{(R)}(\varepsilon) = \frac{\pi}{2} \left[ \frac{\xi_k}{\varepsilon} \right] \left[ \text{Re } I(\omega) \right],
$$

$$
\text{Re } I(\omega) = \frac{\pi}{\sqrt{2}} \left[ \ln \left( \frac{2\sqrt{2}}{r_s} \right) - \frac{\pi |\omega|}{4\sqrt{2} r_s E_F} + \frac{5}{16 r_s^2 E_F} \ln \left( \frac{4 \sqrt{2} r_s E_F}{|\omega|} \right) - \frac{17}{32 r_s^2 E_F^2} \omega^2 \right].
$$
We then evaluate the $\omega$-integration in Eq. \[19\] by utilizing the exponential expansion of the hyperbolic function (Eq. \[12\]). The detailed calculation is relegated to Appendix C. From Eq. \[C6\] we find that the real part of the on-shell self-energy has the following asymptotic form for low energies and temperatures

$$\text{Re} \Sigma^{(R)}(\varepsilon) = h_0 \varepsilon + \frac{T \varepsilon}{E_F} h_1 \left( \frac{\varepsilon}{T} \right) + \frac{T^2 \varepsilon}{r_s E_F^2} \ln \left( \frac{r_s E_F}{T} \right) h_2 \left( \frac{\varepsilon}{T} \right) + \frac{T^2 \varepsilon}{r_s E_F^2} h_3 \left( \frac{\varepsilon}{T} \right),$$

(20)

where

$$h_0 = \frac{r_s}{\sqrt{2\pi}} \ln \left( \frac{2\sqrt{2}}{r_s} \right),$$

$$h_1 \left( \frac{\varepsilon}{T} \right) = -\frac{1}{8} \left[ \frac{T}{\varepsilon} \left( \text{Li}_2(-e^{-\frac{\pi}{\varepsilon}}) - \text{Li}_2(-e^{\frac{\pi}{\varepsilon}}) \right) \right],$$

$$h_2 \left( \frac{\varepsilon}{T} \right) = \frac{5}{48\sqrt{2\pi}} \left( \frac{\varepsilon^2}{T^2} + \frac{\varepsilon^2}{T^2} \right),$$

$$h_3 \left( \frac{\varepsilon}{T} \right) = -\frac{1}{96\sqrt{2\pi}} \left( 32 - 10\gamma_E - 25 \ln 2 \right) \left( \frac{\varepsilon^2}{T^2} + \frac{\varepsilon^2}{T^2} \right)$$

$$- \frac{5}{8\sqrt{2\pi}} \varepsilon \left[ \partial_s \text{Li}_s(-e^{-\frac{\pi}{\varepsilon}}) - \partial_s \text{Li}_s(-e^{\frac{\pi}{\varepsilon}}) \right] \bigg|_{s=3}.$$

(21)

The subleading term $T \varepsilon h_1(\frac{\varepsilon}{T})/E_F$ in Eq. \[20\] is of a form similar to the result obtained for a model with short-range interaction in Refs. \[21, 22\]. Our system is completely different, i.e., an electron system with the realistic long-range Coulomb interaction, so it is important that long-range and short-range interactions lead to similar subleading terms in the self-energy. We emphasize that we also obtain additional terms of higher orders in $T/E_F$ or $\varepsilon/E_F$, which was not done in Refs. \[21, 22\].

For $|\varepsilon| \ll T$, Eq. \[20\] becomes

$$\text{Re} \Sigma^{(R)}(|\varepsilon| \ll T) = \frac{r_s}{\sqrt{2\pi}} \ln \left( \frac{2\sqrt{2}}{r_s} \right) \varepsilon - \frac{\ln 2}{4} \frac{T \varepsilon}{E_F} + \frac{5\pi}{48\sqrt{2\pi}} \frac{T^2 \varepsilon}{r_s E_F^2} \ln \left( \frac{r_s E_F}{T} \right)$$

$$+ \frac{T^2 \varepsilon}{r_s E_F^2} \left[ -\frac{\pi}{96\sqrt{2}} (32 - 10\gamma_E - 25 \ln 2) 

- \frac{5}{8\sqrt{2\pi}} \left( c'(2) + \frac{\pi^2}{6} \ln 2 \right) \right].$$

(22)

We note that the quasiparticle effective mass obtained from this asymptotic expression is consistent with the result in Ref. \[17\].

We then rewrite Eq. \[20\] as

$$\text{Re} \Sigma^{(R)}(\varepsilon) = h_0 \varepsilon + \frac{\varepsilon}{E_F} h_1 \left( \frac{\varepsilon}{T} \right) + \frac{\varepsilon^3}{r_s E_F^2} \ln \left( \frac{r_s E_F}{\varepsilon} \right) h_2 \left( \frac{\varepsilon}{T} \right) + \frac{\varepsilon^3}{r_s E_F^2} h_3 \left( \frac{\varepsilon}{T} \right),$$

(23)

where

$$h_1 \left( \frac{\varepsilon}{T} \right) \equiv \frac{T}{\varepsilon} h_1 \left( \frac{\varepsilon}{T} \right),$$

$$h_2 \left( \frac{\varepsilon}{T} \right) \equiv \left( \frac{T}{\varepsilon} \right)^2 h_2 \left( \frac{\varepsilon}{T} \right),$$

$$h_3 \left( \frac{\varepsilon}{T} \right) \equiv \left( \frac{T}{\varepsilon} \right)^2 \ln \left( \frac{\varepsilon}{T} \right) h_2 \left( \frac{\varepsilon}{T} \right) + \left( \frac{T}{\varepsilon} \right)^2 h_3 \left( \frac{\varepsilon}{T} \right).$$

(24)

From this equation above, we arrive at the asymptotic form of \(\text{Re} \Sigma^{(R)}(\varepsilon)\) for $|\varepsilon| \gg T$:

$$\text{Re} \Sigma^{(R)}(|\varepsilon| \gg T) = \frac{r_s}{\sqrt{2\pi}} \ln \left( \frac{2\sqrt{2}}{r_s} \right) \varepsilon - \frac{\varepsilon}{16 E_F} + \frac{5}{48\sqrt{2\pi}} \frac{\varepsilon^3}{r_s E_F^2} \ln \left( \frac{r_s E_F}{\varepsilon} \right) + \frac{5}{288\sqrt{2\pi}} \frac{\varepsilon^3}{r_s E_F^2}.$$

(25)

We note that, although the leading-order dependence of the real part of the self-energy on the excitation energy and temperature manifests the expected linear-in-$\varepsilon$ behavior, the next-to-leading-order terms as shown in Eqs. \[22\] and \[24\] are nontrivial, and impossible to guess because of the logarithmic factors which disallow for a simple dimensional argument. The analytical result of the real part of the on-shell self-energy given by Eqs. \[20\] and \[21\] is presented in Fig. 3. Since the leading order term in $\text{Re} \Sigma^{(R)}(\varepsilon)$ scales as $\varepsilon$, here we plot the ratio $\text{Re} \Sigma^{(R)}(\varepsilon)/\varepsilon$ instead of $\text{Re} \Sigma^{(R)}(\varepsilon)$ itself. In Figs. \[3\] (b), $\text{Re} \Sigma^{(R)}(\varepsilon)/\varepsilon$ is shown as a function of $\varepsilon/E_F$ for different value of $T/E_F$ $(T/E_F)$, and is compared with the corresponding asymptotic expression for $T \ll \varepsilon$ $(\varepsilon \ll T)$ as given in Eqs. \[22\] and \[25\].

IV. CONCLUSION

In this paper, we present the calculation of the on-shell $\langle \varepsilon = \xi_k \rangle$ self-energy $\Sigma^{(R)}(k, \varepsilon)$ of a 2D electron system with Coulomb interactions in the high density limit $r_s \ll 1$. We work in the regime where the temperature $T$ and the energy $\varepsilon$ are arbitrary with respect to each other but both are small and satisfy $\varepsilon^3 \ll \Delta/E_F \ll r_s$, $\Delta = \{T, \varepsilon\}$. Thus, we are in the low-energy quasiparticle limit, but we keep the next-to-leading-order terms involving both energy and temperature together. We obtain analytically the real and imaginary parts of the self-energy for arbitrary values of $\varepsilon/T$ up to the order $(\min(|\varepsilon|, T)/r_s E_F)^3$ and to the leading order in $r_s$. The asymptotic behaviors of the electron self-energy in the
FIG. 3. Real part of the on-shell electron self-energy \( \text{Re} \Sigma^{(R)}(\epsilon, T) \). In panel (a) [(b)], \( \text{Re} \Sigma^{(R)}(\epsilon, T)/\epsilon \) given by Eq. 20 is plotted as a function of \( \epsilon/E_F \) (at various \( T/E_F \) which admits the values of 0.1, 0.2 and 0.5 (correspond to solid curves from top to bottom), and is compared with the asymptotic result for \( T \ll \epsilon \) \( (\epsilon \ll T) \) given by Eq. 25 (Eq. 22) represented by the dashed curve. \( r_s = 1 \) in both panels.

The imaginary part of the self-energy, therefore goes as \( \ln |\epsilon| \) or \( \epsilon^2 \ln |\epsilon| \) behavior, with the logarithmic part a special 2D feature not arising in 3D systems. Our work establishes the next-to-leading-order terms going as \( O(T^2) \) and \( O(T^3) \), respectively in the 2D imaginary self-energy for \( T \gg |\epsilon| \) or \( |\epsilon| \ll T \) as the case may be. No additional logarithms arise in these higher-order terms. On the Fermi surface, where \( \epsilon = 0 \), the quasiparticle broadening therefore goes as \( O(T^2) + O(T^3) \). The real part of the 2D self-energy is even more subtle in our theory because of the non-analytical contributions arising from the special form of the 2D polarization bubble (with a kink at \( k = k_F \)). In particular, the leading order results is the usual \( O(\epsilon) \) for \( T \gg |\epsilon| \) and \( T \gg |\epsilon| \), which provides the quasiparticle effective mass renormalization to the electronic specific heat. The next-to-the-leading order terms in the real 2D self-energy are \( O(\epsilon T^2 \ln T) + O(\epsilon T^3) \) for \( T \gg |\epsilon| \), indicating a linear-in-\( T \) correction to the usual specific heat coefficient in violation of the Sommerfeld expansion. For \( T \gg |\epsilon| \), the higher order self-energy corrections to the real part of the 2D self-energy go as \( O(\epsilon^2) + O(\epsilon^3 \ln |\epsilon|) + O(\epsilon^3) \)– the appearance of the log here is again special to 2D systems. We also note that the full RPA low-energy and low-temperature self-energy expression derived by us in this work does not suffer from the leading-order logarithmic corrections found in the Hartree-Fock theories 10–28. The full analytical expressions (when \( \epsilon \sim T \)) for the imaginary and real parts of the 2D self-energy, given in Eqs. 13 and 20 respectively, are very complex and do not allow for a simple discussion.
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APPENDIX A: Derivation of the general formulas for the Fermi liquid self-energy

1. Keldysh formalism for interacting electrons

In this appendix, we derive the self-energy formulas presented in Sec. II using the Keldysh technique. We start from the partition function of a 2D system of interacting electrons:

\[
Z \equiv \int \mathcal{D} (\bar{\psi}, \psi) \exp \left( i S_0 + i S_{\text{int}} \right),
\]

\[
S_0 = \int_{r, r', t, t'} \bar{\psi}(r, t) \hat{G}_0^{-1}(r, t; r', t') \psi(r', t'),
\]

\[
S_{\text{int}} = -\frac{1}{2} \sum_{a=1,2} \zeta_a \int_{r, r'} \bar{\psi}_{a,\sigma}(r, t) \psi_{a,\sigma'}(r', t) \times V(r - r') \psi_{a,\sigma'}(r', t) \psi_{a,\sigma}(r, t). 
\]

Here \(\psi_{a,\sigma}(r, t)\) is a Grassmann field that carries a spin index \(\sigma \in \{\uparrow, \downarrow\}\) and a Keldysh label \(a \in \{1, 2\}\). \(a = 1\) (2) indicates the forward (backward) part of the Keldysh contour, and the corresponding sign factor \(\zeta_a\) assumes the value of \(+1\) (\(-1\)). \(V(r) = e^2/r\) is the bare Coulomb interaction potential, and \(G_0\) denotes the non-interacting Green’s function on the Keldysh contour:

\[
\hat{G}_0(r, t; r', t') \equiv -i \left\langle T_e \psi(r, t) \bar{\psi}(r', t') \right\rangle_0. \tag{A2}
\]

\(T_e\) is the Keldysh contour ordering operator, and the angular bracket with subscript “0” stands for the functional averaging over the non-interacting action.

We then introduce an auxiliary bosonic field \(\phi = [\phi_{cl}, \phi_{\eta}]^T\) to Hubbard-Stratonovich (H.S.) decouple the interaction:

\[
e^{i S_{\text{int}}} = \int \mathcal{D} \phi \exp \left[ i \int_{q, \omega} \phi_{cl}(q, \omega) V^{-1}(q) \phi_{\eta}(-q, -\omega) \right. \\
- \frac{i}{\sqrt{2}} \int_{k, \epsilon, q, \omega} \phi_{cl}(q, \omega) \bar{\psi}(k + q, \epsilon + \omega) \hat{\tau}^3 \psi(k, \epsilon) \\
- \frac{i}{\sqrt{2}} \int_{k, \epsilon, q, \omega} \phi_{\eta}(q, \omega) \bar{\psi}(k + q, \epsilon + \omega) \psi(k, \epsilon). \tag{A3}
\]

\(\hat{\tau}\) here represents the Pauli matrix acting on the Keldysh space.

To further simplify the calculation, we apply the Keldysh rotation to the fermionic field

\[
\psi(k, \epsilon) \to \hat{\tau}^3 \hat{U}_K \psi(k, \epsilon), \quad \bar{\psi}(k, \epsilon) \to \bar{\psi}(k, \epsilon) \hat{U}_K^\dagger,
\]

\[
\hat{U}_K \equiv (1 + i \hat{\tau}^2)/\sqrt{2}. \tag{A4}
\]

After the rotation, the non-interacting fermionic Green’s function \(\hat{G}_0\) assumes the following structure in the Keldysh space

\[
\hat{G}_0(k, \epsilon) = \begin{bmatrix} G_0^{(R)}(k, \epsilon) & 0 \\
0 & G_0^{(A)}(k, \epsilon) \end{bmatrix}, \tag{A5}
\]

with the components given by Eqs. 2 and 3.

The Keldysh rotation is then followed by a distribution function dependent transformation

\[
\psi(k, \epsilon) \to \hat{M}_F(\epsilon) \psi(k, \epsilon), \quad \bar{\psi}(k, \epsilon) \to \bar{\psi}(k, \epsilon) \hat{M}_F(\epsilon),
\]

\[
\hat{M}_F(\epsilon) = \begin{bmatrix} 1 \tanh(\epsilon/2T) \\
0 & -1 \end{bmatrix}. \tag{A7}
\]

After the combined transformation, the bare fermionic Green’s function \(\hat{G}_0\) becomes diagonal in the Keldysh space and distribution function independent,

\[
\hat{G}_0(k, \epsilon) = \begin{bmatrix} G_0^{(R)}(k, \epsilon) & 0 \\
0 & G_0^{(A)}(k, \epsilon) \end{bmatrix}, \tag{A8}
\]

and the partition function is now given by

\[
Z = \int \mathcal{D} (\bar{\psi}, \psi) \mathcal{D} \phi \exp \left( i S_\psi + i S_\phi + i S_c \right),
\]

\[
S_\phi = \int_{q, \omega} \phi_{cl}(q, \omega) V^{-1}(q) \phi_{\eta}(-q, -\omega),
\]

\[
S_c = -\int_{k, \epsilon} \bar{\psi}(k, \epsilon) \left[ \epsilon - \xi_k + i \eta \hat{\tau}^3 \right] \psi(k, \epsilon),
\]

\[
S_{\text{cl}} = -\int_{k, k', \epsilon, \epsilon'} \frac{\phi_{cl}(k - k', \epsilon - \epsilon') \bar{\psi}(k, \epsilon) \hat{M}_F(\epsilon) \hat{M}_F(\epsilon') \psi(k', \epsilon')}{\sqrt{2}} \\
- \int_{k, k', \epsilon, \epsilon'} \frac{\phi_{\eta}(k - k', \epsilon - \epsilon') \bar{\psi}(k, \epsilon) \hat{M}_F(\epsilon) \hat{\tau}^3 \hat{M}_F(\epsilon') \psi(k', \epsilon')}{\sqrt{2}}. \tag{A9}
\]

2. Dressed propagator for the H.S. field

It is clear from Eq. A9 that the bare propagator of the H.S. field is

\[
\hat{D}_0(q, \omega) \equiv -i \langle \phi(q, \omega) \phi^\dagger(-q, -\omega) \rangle_0 = \begin{bmatrix} 0 & V(q) \\
V(q) & 0 \end{bmatrix}. \tag{A10}
\]
\( \langle \exp(iS_c) \rangle \equiv \int \mathcal{D}(\tilde{\psi}, \psi) \exp(iS_\psi + iS_c). \) (A11)

To the leading order in the cumulant expansion, which is equivalent to the random phase approximation (RPA), \( \ln \langle \exp(iS_c) \rangle \approx \frac{1}{2} \langle iS_c \rangle^2 \) and can be expressed as a quadratic form

\[
\frac{1}{2} \langle iS_c \rangle^2 = -i \int \frac{d^5q}{q_{\omega}} \phi^T(-q, -\omega) \hat{\Pi}(q, \omega) \phi(q, \omega),
\]

(A12)

where the kernel \( \hat{\Pi}(q, \omega) \) is the self-energy for the H.S. field \( \phi \), with components

\[
\Pi^{ab}(q, \omega) = -i \int \text{Tr}_{k, \varepsilon} \left\{ \left[ 1 + \frac{\omega_0}{2} + \frac{1 - \omega_0}{2} \right] \hat{M}_F(\varepsilon + \omega) G_0^R(k + q, \varepsilon + \omega) \hat{M}_F(\varepsilon) \right. \\
\left. \times \left[ 1 + \frac{\omega_0}{2} + \frac{1 - \omega_0}{2} \right] \hat{M}_F(\varepsilon) G_0(k, \varepsilon) \hat{M}_F(\varepsilon) \right\}.
\]

(A13)

Inserting Eqs. (A8) and (A7) into the equation above, and using the causality relation

\[
\int \text{Tr}_{k, \varepsilon} \left\{ \left[ 1 + \frac{\omega_0}{2} + \frac{1 - \omega_0}{2} \right] \hat{M}_F(\varepsilon + \omega) G_0^R(k + q, \varepsilon + \omega) \hat{M}_F(\varepsilon) \right. \\
\left. \times \left[ 1 + \frac{\omega_0}{2} + \frac{1 - \omega_0}{2} \right] \hat{M}_F(\varepsilon) G_0(k, \varepsilon) \hat{M}_F(\varepsilon) \right\} = 0,
\]

(A14)

we find that \( \hat{\Pi}(q, \omega) \) possesses the standard causality structure of a bosonic self-energy

\[
\hat{\Pi}(q, \omega) = \begin{bmatrix}
0 & \Pi^{(A)}(q, \omega) \\
\Pi^{(R)}(q, \omega) & \Pi^{(K)}(q, \omega)
\end{bmatrix}.
\]

(A15)

Its retarded component is given by Eq. (4) which can be further simplified to

\[
\Pi^{(R)}(q, \omega) = \int \frac{\tanh(\xi_k + q/2T) - \tanh(\xi_k/2T)}{\omega + \xi_k - \xi_{k+q} + i\eta},
\]

(A16)

and is related to its advanced and Keldysh components through:

\[
\Pi^{(A)}(q, \omega) = \left( \Pi^{(R)}(q, \omega) \right)^*,
\]

\[
\Pi^{(K)}(q, \omega) = \left[ \Pi^{(R)}(q, \omega) - \Pi^{(A)}(q, \omega) \right] \coth\left( \frac{\omega}{2T} \right).
\]

(A17)

We then combine the actions \( iS_\phi \) (Eq. A9) and \( \langle \frac{1}{2} iS_c \rangle \) (Eq. A12), and obtain the dressed propagator for the H.S. field \( \phi \)

\[
\hat{D}(q, \omega) \equiv \langle \phi^T(q, \omega) \phi(q, -\omega) \rangle = \left[ \hat{D}_0(q, \omega) - \hat{\Pi}(q, \omega) \right]^{-1}.
\]

(A18)

In Fig. 1(b), we show the diagrammatic representation of the Dyson equation above. Red wavy lines with open and solid dots are used to indicate, respectively, the bare propagator \( \hat{D}_0(q, \omega) \) and dressed propagator \( \hat{D}(q, \omega) \), while the black bubble represents the bosonic self-energy \( \hat{\Pi}(q, \omega) \).

Using Eqs. (A10) and (A15) one can easily see that \( \hat{D}(q, \omega) \) admits the following form in the Keldysh space

\[
\hat{D}(q, \omega) = \begin{bmatrix}
\Pi^{(K)}(q, \omega) & \Pi^{(R)}(q, \omega) \\
\Pi^{(A)}(q, \omega) & 0
\end{bmatrix},
\]

(A19)

in accordance with the causality structure for a bosonic propagator, and its components are given by Eqs. 5 and 6a.

### 3. Electron self-energy

The RPA self-energy diagram for the fermionic field \( \psi \) is plotted in Fig. 1(a), where the red wavy and black solid lines represent, respectively, the dressed H.S. propagator \( \hat{D} \) and the bare fermionic propagator \( \hat{G}_0 \). The corresponding self-energy expression is

\[
\hat{\Sigma}(k, \varepsilon) = \frac{i}{2} \int \mathcal{D}(q, \omega) \Pi^{(K)}(q, \omega) \hat{M}_F(\varepsilon + \omega) \hat{G}_0(k + q, \varepsilon + \omega) \hat{M}_F(\varepsilon) \\
+ \frac{i}{2} \int \mathcal{D}(q, \omega) \Pi^{(R)}(q, \omega) \hat{M}_F(\varepsilon) \hat{G}_0(k + q, \varepsilon + \omega) \hat{M}_F(\varepsilon + \omega) \hat{M}_F(\varepsilon) \\
+ \frac{i}{2} \int \mathcal{D}(q, \omega) \Pi^{(A)}(q, \omega) \hat{M}_F(\varepsilon) \hat{G}_0(k + q, \varepsilon + \omega) \hat{M}_F(\varepsilon + \omega) \hat{M}_F(\varepsilon).
\]

(A20)
Making use of the causality relation

$$\int_{q, \omega} G_0^{(R)}(k + q, \epsilon + \omega) D_0^{(R)}(q, \omega) = 0,$$  \hspace{1cm} (A21)

as well as the FDT relations Eqs. [3] and [12], we find that \( \Sigma(k, \epsilon) \)'s off-diagonal components vanishes in the Keldysh space,

$$\tilde{\Sigma}(k, \epsilon) = \left[ \begin{array}{cc} [\Sigma^{(R)}(k, \epsilon)] & 0 \\ 0 & [\Sigma^{(A)}(k, \epsilon)] \end{array} \right].$$  \hspace{1cm} (A22)

The retarded component \( \Sigma^{(R)}(q, \omega) = [\Sigma^{(A)}(k, \epsilon)]^* \) is given by Eq. [4] which may be rewritten as

$$\Sigma^{(R)}(k, \epsilon) = -2 \int_{q, \omega, \omega'} \text{Im} D^{(R)}(q, \omega) \text{Im} G_0^{(R)}(k + q, \epsilon + \omega') \times \frac{1}{\omega' - \omega - i\eta} \left[ \coth \left( \frac{\omega'}{2T} \right) - \tanh \left( \frac{\epsilon + \omega' - \Delta \epsilon}{2T} \right) \right].$$  \hspace{1cm} (A23)

Here we have employed the Kramers-Krönig relation

$$f^{(R)}(k, \epsilon) = \int_{-\infty}^{\infty} \frac{d\epsilon'}{\pi} \frac{\text{Im} f^{(R)}(k, \epsilon')}{\epsilon' - \epsilon - i\eta},$$  \hspace{1cm} (A24)

for both the bosonic propagator \( D \) and the fermionic propagator \( G_0 \).

Inserting the explicit expression for \( G_0 \) (Eq. [2]) into Eq. [A23] and performing the angular integration, we arrive at

$$\Sigma^{(R)}(k, \epsilon) = \frac{m}{2\pi k} \int_{\omega, \omega'} dq \Theta \left( 1 - \frac{|m\omega'|}{kq} \right) \frac{\text{Im} D^{(R)}(q, \omega)}{\sqrt{1 - \left( \frac{m\omega'}{kq} \right)^2}} \times \frac{1}{\omega' - \Delta \epsilon - \omega - i\eta} \left[ \coth \left( \frac{\omega'}{2T} \right) - \tanh \left( \frac{\epsilon + \omega' - \Delta \epsilon}{2T} \right) \right],$$  \hspace{1cm} (A25)

where \( \Delta \epsilon \) is defined in Eq. [9a] Eq. [8a] which gives \( \text{Im} \Sigma^{(R)}(k, \epsilon) \) can be deduced directly from Eq. [A25].

To obtain \( \text{Re} \Sigma^{(R)}(k, \epsilon) \), we first rewrite Eq. [A25] as

$$\Sigma^{(R)}(k, \epsilon) = \frac{m}{4\pi^2 k} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dw' \tan \left( \frac{\epsilon + \omega' - \Delta \epsilon}{2T} \right) \times \frac{1}{\sqrt{1 - \left( \frac{m\omega'}{kq} \right)^2}} \frac{D^{(A)}(q, \omega' - \Delta \epsilon - \omega - i\eta) \Theta \left( 1 - \frac{|m\omega'|}{kq} \right)}{2T},$$  \hspace{1cm} (A26)

with the help of the Kramers-Krönig relation (Eq. [A24]).

It is then straightforward to see that \( \text{Re} \Sigma^{(R)}(k, \epsilon) \) equals the principal integral of Eq. [A26] and is given by Eq. [8b].

**APPENDIX B: Momentum integrals**

This appendix is devoted to the evaluation of the integral defined in Eq. [11]. For convenience, we introduce the following dimensionless quantities:

$$\delta = \frac{\omega}{4E_F}, \quad x = \frac{q}{2k_F}, \quad \alpha = \frac{r_s}{\sqrt{2}}.$$  \hspace{1cm} (B1)

The integral \( I(\omega) \) then reduced to the form

$$I = \frac{m}{\pi} \int_{x_1}^{x_2} \frac{dx}{x^{1/2}} D^{(R)}(x, \delta) \left[ 1 - \left( \frac{x - \delta}{x} \right)^2 \right]^{-1/2},$$

$$= \frac{m}{\pi} \int_{x_1}^{x_2} \frac{dx}{x^{1/2}} D^{(R)}(x, \delta) x \left[ (x_2^2 - x_1^2)(x_2^2 - x_1^2) \right]^{-1/2},$$  \hspace{1cm} (B2)

where \( x_{1,2} \equiv q_{1,2}/2k_F \) satisfies the equation \( (x - \delta/x)^2 = 1 \), and is given by:

$$x_1 = \frac{1}{2} \left| 1 - \sqrt{1 - 4\delta} \right| \approx |\delta| - \delta^2 \text{sgn} \delta + O(\delta^3),$$

$$x_2 = \frac{1}{2} \left( 1 + \sqrt{1 + 4\delta} \right) \approx 1 + \delta + O(\delta^2).$$  \hspace{1cm} (B3)

Using the variables defined in Eq. [B1], \( D^{(R)} \) is given by:

$$D^{(R)}(x, \delta) = \nu^{-1} \alpha \left( x - \alpha \text{Re} \Pi_0 \nu^{-1} \right) + i(\alpha \text{Im} \Pi_0 \nu^{-1}) \left( x - \alpha \text{Re} \Pi_0 \nu^{-1} \right)^2 + (\alpha \text{Im} \Pi_0 \nu^{-1})^2.$$  \hspace{1cm} (B4)

Here \( \Pi_0 \) is the zero-temperature polarization bubble \[39\], and is given by:

$$\text{Re} \Pi_0 \nu^{-1} = -1$$

$$+ \frac{1}{2\pi^2} \text{sgn} \left( 1 - \frac{\delta}{x^2} \right) \text{Re} \sqrt{-(x_2^2 - x_1^2)(x_2^2 - x_1^2)}$$

$$+ \frac{1}{2\pi^2} \text{sgn} \left( 1 + \frac{\delta}{x^2} \right) \text{Re} \sqrt{4\delta x_2^2 - (x_2^2 - x_1^2)(x_2^2 - x_1^2)},$$  \hspace{1cm} (B5a)

$$\text{Im} \Pi_0 \nu^{-1} = -\frac{1}{2\pi^2} \text{Re} \sqrt{(x_2^2 - x_1^2)(x_2^2 - x_1^2)}$$

$$+ \frac{1}{2\pi^2} \text{Re} \sqrt{(x_2^2 - x_1^2)(x_2^2 - x_1^2)} - 4\delta x_2^2.$$  \hspace{1cm} (B5b)

For this calculation, we will work in the regime where \( |\delta| \ll \alpha \ll 1 \). Additionally, we will assume \( \alpha \ll (|\delta|/\alpha)^2 \), and therefore, we will calculate results to leading order in \( \alpha \), but to several orders in \( |\delta|/\alpha \). Specifically, we note that since \( |\delta| \) is smaller than \( \alpha \), we need only keep leading order terms in \( \delta \). We wish to evaluate integral [B2] as an
expansion in $\alpha$ and $|\delta|/\alpha$. To do so, it is necessary to expand the integrand in terms of $x/\alpha$, $\alpha/x$, $x_1/x$ or $x/x_2$. These ratios are all small in some regimes of integration, but become large in other regimes, and therefore the integrand cannot be expanded in any of these ratios across the entire region of integration. To proceed, we must divide the region of integration into three subregions, and then expand in terms of the appropriate ratios in each subregion. Thus we define the boundaries between the subregions $l_1$ and $l_2$ such that $x_1 < l_1 < \alpha < l_2 < x_2$. Since we will be dividing up the region of integration, for convenience define $I(a, b)$ to be the contribution to the integral $I$ from the interval $(a, b)$. This lack of a single small parameter over the whole integration region is the key technical difficulty hindering the analytical evaluation of the self-energy in the next to the leading order, explaining why it has not been achieved in spite of the long history of the subject.

1. Simplification of $\Pi_0$

Inside the region of integration, the second term of Eq. (B5a) vanishes, since the radical is purely imaginary for $x$ between $x_1$ and $x_2$. Likewise, the third term also vanishes so long as $x$ is not too close to the endpoints, specifically if $x_1[1 + 2\delta + O(\delta^2)] < x < x_2[1 - 2\delta + O(\delta^2)]$. In order to simplify $\Re \Pi_0$, we must exclude the portion of the region of integration that lies outside this interval. Additionally, as we shall show later, in order to simplify $\Im \Pi_0$, we will need to exclude a slightly larger region around $x_1$. Thus for some $\beta$ between 0 and 1, we define $a_1 = x_1[1 + O(|\delta|^{1-\beta})]$ and $a_2 = x_2[1 - O(|\delta|)]$. We will show that the contribution to the integral from the excluded regions near the endpoints is higher order in $|\delta|$ than the rest of the integral.

Consider the contribution to the integral from the region $x_1 < x < a_1$. For this region, we introduce the change of variables $z^2 = x^2 - x_1^2$. The upper bound of integration becomes $\sqrt{a_1^2 - x_1^2} = O(|\delta|^{1-\beta/2})$, and the lower bound is zero. We then need to find an upper bound for $|D^{(R)}|$ in this region. The points where $|D^{(R)}|$ becomes the largest are $z \rightarrow 0$ and $z \sim |\delta|^{1/2}$. When $z \rightarrow 0$, $\Im \Pi_0 \nu^{-1} \rightarrow 0$, and $\Re \Pi_0 \nu^{-1} \rightarrow -1 + \Re \sqrt{\delta}/x_1$. For $z \sim |\delta|^{1/2}$, there is a point where $(x - \alpha \Re \Pi_0 \nu^{-1}) \rightarrow 0$ if $\delta > 0$. At this point, $\Im \Pi_0 \nu^{-1} = O(|\delta|^{1/2})$. For all cases, $|D^{(R)}| \leq \nu^{-1} \alpha O(\alpha^{-1})$. The integral then becomes:

$$I(x_1, a_1) = \int_0^{O(|\delta|^{1/2})} dz \ D^{(R)}(z) \left[ x_2^2 - z^2 - x_1^2 \right]^{-1/2} = O(|\delta|^{1/2}) \nu^{-1} \alpha O(\alpha^{-1})$$

(B6)

Now consider the region $a_2 < x < x_2$. For this region, we introduce the change of variables $u^2 = x_2^2 - x^2$. With this transformation, the limits of integration become 0 to $O(|\delta|^{1/2})$. It is easy to see that in this region, $|\Pi_0 \nu^{-1}| = O(|\delta|^{1/2})$, and thus $|D^{(R)}| = \nu^{-1} O(\alpha)$. Then the integral becomes:

$$I(a_2, x_2) = \int_0^{O(|\delta|^{1/2})} du \ D^{(R)}(u) \left[ x_2^2 - u^2 - x_1^2 \right]^{-1/2} = \alpha O(|\delta|^{1/2})$$

(B7)

For $a_1 < x < a_2$, both the second and the third term of Eq. (B5a) vanish, and so in this region we have exactly:

$$\Re \Pi_0 \nu^{-1} = -1$$

(B8)

For $a_1 < x < l_2$, we again use the variable transformation $z^2 = x^2 - x_1^2$. The upper bound of this region becomes $\sqrt{l_2^2 - x_1^2} = l_2[1 + O(\delta^2/l_2^2)]$, and the lower bound is $O(|\delta|^{13/2-\beta})$, as discussed previously. Specifically, we note that $\delta x_1^2/z^2$ is of order $O(|\delta|^{1/2})$. Then we simplify $\Im \Pi_0$ as follows:

$$\Im \Pi_0 \nu^{-1} = -\frac{zz_2}{2(z^2 + z_1^2)} \left[ 1 - \sqrt{1 - \frac{4\delta(z^2 + z_1^2)}{z^2z_2^2}} + O\left(\frac{z_2^2}{z^2}\right) \right] = -\frac{zz_2}{2(z^2 + z_1^2)} \times \frac{4\delta(z^2 + z_1^2)}{2z^2z_2^2} \left[ 1 + O\left(|\delta|\right) \right] = -\frac{\delta}{2} \left[ 1 + O\left(|\delta|\right) \right].$$

(B9)

For $l_2 < x < a_2$, all that is required for the computation is the following, which can be shown easily:

$$\Im \Pi_0 \nu^{-1} = O\left(|\delta|^{1/2}, \frac{\delta}{l_2}\right).$$

(B10)

2. The $\Re I$ integral

For $x$ between $a_1$ and $l_1$, we again use $z^2 = x^2 - x_1^2$, and $\Re D^{(R)}$ is given by:

$$\Re D^{(R)} = \nu^{-1} \frac{1 + \frac{x}{\alpha}}{\left(1 + \frac{x}{\alpha}\right)^2 + \frac{\delta^2}{z^2} + O\left(|\delta|^{13/2}\right)} = \nu^{-1} \frac{1 + \frac{x}{\alpha}}{1 + \frac{\delta^2}{z^2}} \left[ 1 - \frac{2x}{\alpha} \frac{\delta^2}{z^2} + \frac{\alpha^{-2}}{\left(1 + \frac{\delta^2}{z^2}\right)^2} + O\left(\frac{x^3}{\alpha^3}\right) \right].$$

(B11)

This can be integrated, giving:

$$\Re I(a_1, l_1) = \nu \int_{O(\delta^{1/2})}^{\delta^{-1/2} + (2\delta)} dz \ \Re D^{(R)}(z) \left[ 1 + O(l_1^2) \right] = \alpha \left[ \frac{z}{\alpha} - \frac{z(5\delta^2 + z^2)}{2\alpha^2\delta^2 + z^2} - \frac{\delta}{\alpha} \arctan \frac{z}{\delta} \right]$$
\[
+ \frac{5\delta^2}{2\alpha^2} \log \left( z + \sqrt{\delta^2 + z^2} \right) \bigg|_{l_1 - \delta^2/(2\alpha)}^{l_1 - \delta^2/(2\alpha)} \\
= \alpha \left[ -\frac{\pi\delta}{2\alpha} + \frac{l_1}{\alpha} - \frac{7\delta^2}{4\alpha^2} + \frac{5\delta^2}{2\alpha^2} \log \frac{l_1}{\alpha} + \frac{\delta^2}{2\alpha l_1} \\
- \frac{l_1^2}{2\alpha^2} + O\left( \frac{l_l^3}{\alpha^3}, \frac{\delta^3}{l_l}, \frac{\delta^3}{\alpha}, |\delta|^{3/2} \right) \right].
\] (B12)

For \( x \) between \( l_1 \) and \( l_2 \), we can expand in \( |\delta|/x \) and \( x \), but can no longer expand \( \Re \Pi_0 \) in terms of \( x/\alpha \), thus we simply use:

\[
\Re D^{(R)} = \nu^{-1} \left[ 1 + \frac{x}{\nu} \right] \left[ 1 + \frac{\delta^2}{x^2} + O\left( \frac{\delta^3}{x^2}, x \right) \right].
\] (B13)

Integrating gives:

\[
\Re I(l_1, l_2) = \nu \int_{l_1}^{l_2} dx \, \Re D^{(R)}(x) \left[ 1 + \frac{\delta^2}{x^2} + O\left( \frac{\delta^3}{x^2}, x \right) \right] \\
= \alpha \left[ -\frac{l_1}{\alpha} - \frac{5\delta^2}{2\alpha^2} + \frac{5\delta^2}{2\alpha^2} \log \frac{l_1}{\alpha} - \frac{\delta^2}{2\alpha l_1} + \frac{l_1^2}{2\alpha^2} + \log \frac{l_2}{\alpha} \\
+ O\left( \frac{l_l^3}{\alpha^3}, \frac{\delta^3}{l_l}, \frac{\delta^3}{\alpha}, l_l \right) \right].
\] (B14)

For \( x \) between \( l_2 \) and \( 1 \), we need only the leading order term in \( \alpha/x \), and thus we can simply use:

\[
D^{(R)}(x) = \nu^{-1} \left[ \frac{\alpha}{x} + O\left( \frac{\alpha^2}{x^2} \right) \right].
\] (B15)

Then integrating, we obtain:

\[
\Re I(l_2, a_2) = \int_{l_2}^{1+O(\delta)} dx \, \left[ \frac{\alpha}{x} + O\left( \frac{\alpha^2}{x^2} \right) \right] (1 - x^2)^{-1/2} \\
= \alpha \left[ \log \frac{2}{l_2} + O\left( \frac{1}{l_2}, l_2 \right) \right].
\] (B16)

Therefore, combining these results, we have:

\[
\Re I = \alpha \left[ \log \frac{2}{\alpha} - \frac{\pi\delta}{2\alpha} + \frac{\delta^2}{\alpha^2} \left( \frac{17}{4} + \frac{5}{2} \log \frac{2\alpha}{|\delta|} \right) \\
+ O\left( \frac{\delta^3}{\alpha}, |\delta|^{3/2}, |\delta|^{3/2} \right) \right].
\] (B17)

which is equivalent to Eq. (19b).

Once we have \( \Re I \), we can proceed to calculate the real part of self-energy using

\[
\Re \Sigma^{(R)}(\varepsilon) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tanh \left( \frac{\omega + \varepsilon}{2T} \right) \Re I(\omega),
\] (B18)

which can be rewritten as Eq. (19a) after a change of variable \( \omega \to -\omega \) for negative \( \omega \). However, we note that it might seem at first that Eqs. (19a) and (B18) give rise to different results. Consider as an example the leading order term in \( \Re \Sigma^{(R)} \) which is proportional to the following integral

\[
\int_{-\infty}^{\infty} d\omega \tanh \left[ (\omega + \varepsilon)/2T \right].
\] (B19)

Applying the transformation \( \omega \to \omega - \varepsilon \) leads to 0, while rewriting the integral as

\[
\int_{0}^{\infty} d\omega \left\{ \tanh \left[ (\omega + \varepsilon)/2T \right] - \tanh \left[ (\omega - \varepsilon)/2T \right] \right\},
\] (B20)

yields nonvanishing result. This discrepancy is due to the fact that integral Eq. (B19) is not well defined, and the accurate way to carry out the integration is to rewrite it as Eq. (B20) instead of making the shift for the following reasons. First of all, \( \tanh \left[ (\omega + \varepsilon)/2T \right] \) has an effective discontinuity at \( \omega = -\varepsilon \), which can not be simply transferred to \( \omega = 0 \) since “\( \varepsilon \)” has a physical meaning. Furthermore, when deriving Eq. (B18) we performed a small \( \omega \) expansion and kept only the leading order terms. This means that the integration possesses a cutoff which leads to a nonzero result after the shift. In other words, the small \( \omega \) (or \( \delta \)) expansion is only justified because of the factor \( \tanh \left[ (\omega + \varepsilon)/2T \right] - \tanh \left[ (\omega - \varepsilon)/2T \right] \) which restricts the integration to the region \( |\omega| < \max(|\varepsilon|, T) \).

3. The Im I integral

We now consider the imaginary part of \( I \). For \( a_1 < x < l_1 \), we use the same transformation \( z^2 = x^2 - \delta^2 \) as above. Then we have (to the same order as in the previous section):

\[
\Im I(a_1, l_1) = -\int_{l_1}^{l_1 - \delta^2/(2\alpha)} dz \frac{\delta}{z} \frac{\delta}{z} \left( \frac{\sqrt{z^2 + \delta^2}}{z} + 1 \right)^2 + \left( \frac{\delta}{z} \right)^2 \\
= -\int_{l_1 - \delta^2/(2\alpha)}^{l_1} dz \frac{\delta}{z} \left( \frac{\sqrt{z^2 + \delta^2}}{z} + 1 \right)^2 \times \\
\left[ 1 - \frac{2\sqrt{z^2 + \delta^2}}{\alpha} \left( \frac{1}{\alpha} + \left( \frac{\sqrt{z^2 + \delta^2}}{z} \right)^{-1} \right) \right] \\
= -\alpha \left[ \frac{\delta}{\alpha} \log \frac{l_1}{|\delta|} + 4 \frac{\delta^2}{\alpha^2} \text{sgn} \delta - 2 \frac{l_1 \delta}{\alpha^2} \right].
\] (B21)

For \( l_1 < x < l_2 \):

\[
\Im I(l_1, l_2) = -\int_{l_1}^{l_2} dx \frac{\delta}{x(\alpha + x)^2} \\
= -\alpha \left[ -\frac{\delta}{\alpha} + \frac{\delta}{\alpha} \log \frac{l_1}{|\delta|} + 2 \frac{l_1 \delta}{\alpha^2} \right].
\] (B22)

For \( l_2 < x < a_2 \), there is no contribution to zeroth order in \( \alpha \), since the integrand is of order \( \alpha \). Then we have:

\[
\Im I = -\alpha \left[ \frac{\delta}{\alpha} \left( -1 + \log \frac{\alpha}{|\delta|} \right) + \frac{4 \delta^2}{\alpha^2} \text{sgn} \delta \right],
\] (B23)

which leads to Eq. (10).
4. Higher order terms in $\alpha$

We also calculate Re $I$ to first order in $\alpha$, but to only first order in $\delta/\alpha$. This means that we must keep terms of order $|\delta|$. Additionally, we also keep second order terms in $\alpha$, but zeroth order in $\delta/\alpha$. In other words, we now consider the region where $T/E$ and $|\varepsilon|/E$ are of the order of $r_2^2$, unlike in the previous sections. Then $I(x_1, a_1)$ is still of higher order; however, we do need to include $I(a_2, x_2)$. Again using $u^2 = x_2^2 - x^2$, we find:

$$\text{Re } I(a_2, x_2) = \nu \int_0^{\sqrt{x_2^2 - a_2^2}} du \ D^{(R)}(u) \left[ x_2^2 - u^2 - x_1^2 \right]^{-1/2}$$

$$= \alpha \left[ \sqrt{x_2^2 - a_2^2} + O(\alpha |\delta|^{1/2}) \right].$$

To first order in $\delta/\alpha$, there are no additional terms in Re $I(a_1, l_1)$, and thus we use the result from Eq. (B12) above:

$$\text{Re } I(a_1, l_1) = \alpha \left[ -\frac{\pi |\delta|}{2\alpha} + l_1 \frac{1}{\alpha} + O\left( \frac{l_2^2}{\alpha^2}, \frac{\delta^2}{l_1^2}, \frac{|\delta|^{3/2}}{\alpha^3}, \frac{|\delta|^{3/2}}{l_2^2} \right) \right].$$

For Re $I(l_1, l_2)$, we find:

$$\text{Re } I(l_1, l_2) = \alpha \int_{l_1}^{l_2} dx \frac{1}{x} + \alpha \left[ \frac{x^2}{2x_2^2} + O\left( \frac{\delta^2}{x_2^2}, x^3 \right) \right]$$

$$= \alpha \left[ -\frac{l_1}{\alpha} + \frac{\alpha}{l_2} + (1 - \delta) \log \frac{l_2}{l_1} - \frac{\alpha^2}{2l_2^2} - \frac{\alpha l_2}{2} + \frac{l_2^2}{4} \right]$$

$$+ \frac{\alpha^2}{2} \log l_2 + O\left( \frac{l_1^2}{\alpha^3}, \frac{\delta^2}{l_1^2}, \frac{3}{2}, \frac{l_2^2}{l_1^2} \right).$$

Finally, we calculate Re $I(l_2, a_2)$:

$$\text{Re } I(l_2, a_2) = \alpha \int_{l_2}^{a_2} dx \frac{1}{x} \left[ 1 - \frac{\alpha}{x} + O\left( \frac{\alpha^2}{x^2} \right) \right] (x_2^2 - x^2)^{-1/2}$$

$$= \alpha \left[ -\frac{\alpha}{l_2} + (1 - \delta) \log \frac{2(1 + \delta)}{l_2} - \frac{\alpha^2}{4} \log \frac{l_2^2}{4} \right]$$

$$+ \frac{\alpha^2}{2l_2^2} + \frac{\alpha l_2}{2} - \frac{l_2^2}{4} + \frac{\alpha^2}{2} \log \frac{l_2}{2} + O\left( \frac{l_2^3}{l_1^2}, \frac{l_2^3}{l_1^2} \right).$$

Thus, including our result in the previous section, to combined order in $\alpha$ and $\delta/\alpha$ no more than 2, Re $I$ is given by:

$$\text{Re } I = \alpha \left[ \log \frac{2}{\alpha} - \frac{\pi |\delta|}{2\alpha} + \delta \left( 1 - \log \frac{2}{\alpha} \right) \right]$$

$$+ \frac{\alpha^2}{4} \left( -1 + 2 \log 2 \right) + \frac{\delta^2}{\alpha^2} \left( -\frac{17}{4} + \frac{5}{2} \log \frac{2\alpha}{|\delta|} \right).$$

We emphasize that this is only the higher-order in $\alpha$ (i.e. in $r_s$) contribution arising from the ring diagrams (and in fact, just from the Re $I$ integral). There are other contributions to the higher-order terms in $\alpha$ which are beyond the scope of the current work where our interest is to get the exact leading order result in $\alpha$ (i.e. $r_s$). We in fact expect the last term in Eq. (B28) to be canceled by a contribution coming from one of our neglected effects. We show the result in Eq. (B28) for the sake of completeness in providing the structure of Re $I$ only, and do not use this form in the main part of our paper where our interest is the leading-order in $r_s$ exact theory.

To obtain the real part of the electron self-energy $\text{Re } \Sigma^{(R)}(\varepsilon, T)$ when $T/E_F$ and $|\varepsilon|/E_F$ are of the same order as $r_2^2$, we need to insert Eq. (B28) into the first term in Eq. (29) and carry out the frequency integration. We also emphasize that the second term in Eq. (29) might be non-vanishing in this case and contributes to $\text{Re } \Sigma^{(R)}(\varepsilon, T)$ as well. This is clearly beyond the scope of the current work.

APPENDIX C: Integrals with hyperbolic functions tanh and coth

In this appendix, we evaluate the integrals which appear in the calculation of the electron self-energy, and involve the hyperbolic functions tanh($x$) and coth($x$).

We first consider an integral of the following form

$$I_1(a) = \int_0^\infty dx f(x) \times \left[ 2 \coth(x) - \tanh(x + a) - \tanh(x - a) \right].$$

Expressing the hyperbolic functions in terms of exponential series as in Eq. (12) separately for the regimes $a > x \geq 0$ and $x \geq a$, we arrive at

$$I_1(a) = 2 \sum_{k=1}^\infty \int_0^\infty dx f(x) e^{-2kx} \left[ 2 - (-1)^k e^{-2ka} - (-1)^k e^{2ka} \right]$$

$$+ 2 \sum_{k=1}^\infty \int_0^a dx f(x) (-1)^k \cosh(2kx - 2ka) + 2 \int_0^a dx f(x)$$

$$= 2 \sum_{k=1}^\infty \int_0^\infty dx f(x) e^{-2kx} \left[ 2 - (-1)^k e^{-2ka} - (-1)^k e^{2ka} \right]$$

$$+ 4 \int_0^a dx f(x) (-1)^k \cosh(2kx - 2ka) + 2 \int_0^a dx f(x)$$

$$= 2 \int_0^\infty dx f(x) e^{-2kx} \left[ 2 - (-1)^k e^{-2ka} - (-1)^k e^{2ka} \right]$$

where in the second equality we have used the fact that $\sum_{k=1}^\infty (-1)^k \cosh(2kx - 2ka) = -1/2$.

We then evaluate the integral $\int_0^\infty dx f(x) e^{-2kx}$ for different $f(x)$, insert the result back into Eq. (C2) and then perform the summation. This leads to

$$\int_0^\infty dx \left[ 2 \coth(x) - \tanh(x + a) - \tanh(x - a) \right] x$$

$$= \frac{\pi^2}{4} + a^2,$$
\[
\int_0^\infty dx \left[ 2 \coth (x) - \tanh (x) - \tanh (x - a) \right] x \ln x
\]
\[
= (1 - \gamma_E - \ln A) a^2 + \frac{\pi^2}{12} \left( 3 - \gamma_E - \ln \frac{2}{\pi^2} - 24 \ln A \right)
\]
\[
- \frac{1}{2} \left[ \partial_s \text{Li}_s (-e^{-2a}) + \partial_s \text{Li}_s (-e^{-2a}) \right] \bigg|_{s=2},
\]
\[
\int_0^\infty dx \left[ 2 \coth (x) - \tanh (x + a) - \tanh (x - a) \right] x^2
\]
\[
= \zeta (3) - \frac{1}{2} \left[ \text{Li}_3 (-e^{-2a}) + \text{Li}_3 (-e^{-2a}) \right].
\]  
(C3)

Here \(\gamma_E\) and \(A\) represent, respectively, the Euler’s constant and Glaisher’s constant. \(\text{Li}_s (z)\) and \(\zeta (z)\) denote the polylogarithm function and the Riemann zeta function.

Similarly, making use of the expansion in Eq. (12), the integral

\[
I_2 (a) = \int_0^\infty dx f(x) \left[ \tanh (x + a) - \tanh (x - a) \right],
\]

(C4)

can be rewritten in the form

\[
I_2 (a) = 2 \sum_{k=1}^\infty \int_0^\infty dx f(x) e^{-2ka} (-1)^k \left[ e^{-2ka} - e^{2ka} \right].
\]

(C5)

It is then straightforward to show that

\[
\int_0^\infty dx \left[ \tanh (x + a) - \tanh (x - a) \right]
\]
\[
= 2a,
\]
\[
\int_0^\infty dx \left[ \tanh (x + a) - \tanh (x - a) \right] x
\]
\[
= \frac{1}{2} \left[ \text{Li}_3 (-e^{-2a}) - \text{Li}_3 (-e^{2a}) \right],
\]
\[
\int_0^\infty dx \left[ \tanh (x + a) - \tanh (x - a) \right] x^2
\]
\[
= \frac{2}{3} a^3 + \frac{\pi^2}{6} a,
\]
\[
\int_0^\infty dx \left[ \tanh (x + a) - \tanh (x - a) \right] x^2 \ln x
\]
\[
= (3 - 2\gamma_E - \ln 4) \left( \frac{2}{3} a^3 + \frac{\pi^2}{12} a \right)
\]
\[
+ \frac{1}{2} \left[ \partial_s \text{Li}_s (-e^{-2a}) - \partial_s \text{Li}_s (-e^{2a}) \right] \bigg|_{s=3}.
\]

(C6)

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