A REMARK ON A THEOREM BY C. AMIOT

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Abstract. C. Amiot has classified the connected triangulated \( k \)-categories with finitely many isoclasses of indecomposables satisfying suitable hypotheses. We remark that her proof shows that these triangulated categories are determined by their underlying \( k \)-linear categories. We observe that if the connectedness assumption is dropped, the triangulated categories are still determined by their underlying \( k \)-categories together with the action of the suspension functor on the set of isoclasses of indecomposables.

1. The connected case

We refer to [1] for unexplained notation and terminology. Let \( k \) be an algebraically closed field and \( T \) a \( k \)-linear Hom-finite triangulated category with split idempotents. Recall Theorem 7.2 of [1]:

**Theorem 1** (Amiot). Suppose \( T \) is connected, algebraic, standard and has only finitely many isoclasses of indecomposables. Then there exists a Dynkin quiver \( Q \) and a triangle autoequivalence \( \Phi \) of \( D^b(\text{mod} kQ) \) such that \( T \) is triangle equivalent to the triangulated \( [6] \) orbit category \( D^b(\text{mod} kQ)/\Phi \).

Our aim is to show that the proof of this theorem in [1] actually shows that a given \( k \)-linear equivalence \( D^b(\text{mod} kQ)/F \sim \rightarrow T \), where \( F \) is a \( k \)-linear autoequivalence, lifts to a triangle equivalence \( D^b(\text{mod} kQ)/\Phi \sim \rightarrow T \), where \( \Phi \) is a triangle autoequivalence lifting \( F \). Thus, we obtain the

**Corollary 2.** Under the hypotheses of the theorem, the \( k \)-linear structure of \( T \) determines its triangulated structure up to triangle equivalence.

**Proof.** The facts that \( T \) is connected, standard and has only finitely many isoclasses of indecomposables imply that there is a Dynkin quiver \( Q \), a \( k \)-linear autoequivalence \( F \) of \( D^b(\text{mod} kQ) \) and a \( k \)-linear equivalence

\[
G : D^b(\text{mod} kQ)/F \sim \rightarrow T.
\]

This follows from the work of Riedtmann [7], cf. section 6.1 of [1]. We will show that \( F \) lifts to an (algebraic) triangle autoequivalence of \( D^b(\text{mod} kQ) \) and \( G \) to an (algebraic) triangle equivalence \( \Gamma \). We give the details in the case of \( F \) which were omitted in [1]. Put \( D = D^b(\text{mod} kQ) \). Since \( D \) is triangulated, the category \( \text{mod} D \) of finitely presented functors \( \text{mod} D \rightarrow \text{mod} D \) is an exact Frobenius category and we have a canonical isomorphism of functors \( \text{mod} D \rightarrow \text{mod} D \)

\[
\Sigma_m^3 \sim \rightarrow \Sigma_D,
\]

where \( \Sigma_D : \text{mod} D \rightarrow \text{mod} D \) denotes the functor \( \text{mod} D \rightarrow \text{mod} D \) induced by \( \Sigma \) and \( \Sigma_m \) is the suspension functor of the stable category \( \text{mod} D \), cf. [4, 16.4]. Notice that \( \Sigma_m \)

1

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only depends on the underlying \( k \)-category of \( D \). Now if \( S_U \) denotes the simple \( D \)-module associated with an indecomposable object \( U \) of \( D \), we have

\[
S_{SU} = \Sigma_p S_U \cong \Sigma^3 m S_U
\]

in the stable category \( \text{mod} D \). Since \( F \) is a \( k \)-linear autoequivalence, the functor it induces in \( \text{mod} D \) commutes with \( \Sigma_m \) and we have \( S_{F \Sigma U} \cong S_{SU} \) in \( \text{mod} D \) for each indecomposable \( U \) of \( D \). It follows that if \( S_U \) is not zero in \( \text{mod} D \), then we have an isomorphism \( F \Sigma U \cong \Sigma FU \) in \( D \). Now \( S_U \) is zero in \( \text{mod} D \) only if \( S_U \) is projective, which happens if and only if the canonical map \( P_U \to S_U \) is an isomorphism, where \( P_U = D(?, U) \) is the projective module associated with \( U \). This is the case only if no arrows arrive at \( U \) in the Auslander–Reiten quiver of \( D \) and this happens if and only if no arrows start or arrive at \( U \). The same then holds for the suspensions \( \Sigma^n U \), \( n \in \mathbb{Z} \). Since we have assumed that \( \mathcal{T} \) and hence \( D \) is connected, this case is impossible. Therefore, we have an isomorphism \( \Sigma FU \cong F \Sigma U \) for each indecomposable \( U \) of \( D \). It follows that \( T = F(kQ) \) is a tilting object of \( D \). By \cite{Riedtmann1982}, we can lift \( T \) to a \( kQ \)-bimodule complex \( Y \) which is even unique in the derived category of bimodules if we take the isomorphism \( kQ \cong \text{End}(F(kQ)) \) into account. Since the \( k \)-linear functors \( F \) and \( \Phi = ? \otimes_{kQ} Y \) are isomorphic when restricted to \( \text{add}(kQ) \), they are isomorphic as \( k \)-linear functors by Riedtmann’s knitting argument \cite{Riedtmann1982}. Since the triangulated category \( \mathcal{T} \) is algebraic, we may assume that it equals the perfect derived category \( \text{per} A \) of a small \( \text{dg} \) \( k \)-category \( A \). Using Riedtmann’s knitting argument again, it follows from the proof of Theorem 7.2 in \cite{Keller1994} that the composition

\[
D \xrightarrow{\pi} D / \Phi \xrightarrow{G} \mathcal{T} = \text{per} A
\]

lifts to a triangle functor \( L \otimes_{kQ} X \) for a \( kQ \)-\( A \)-bimodule \( X \). Moreover, it is shown there that this composition factors through an algebraic triangle equivalence

\[
\Gamma : D / \Phi \cong \mathcal{T} = \text{per} A.
\]

Since the compositions \( \Gamma \circ \pi \) and \( G \circ \pi \) are isomorphic as \( k \)-linear functors, the functors \( \Gamma \) and \( G \) are isomorphic as \( k \)-linear functors.

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2. THE NON CONNECTED CASE

Let \( k \) be an algebraically closed field and \( \mathcal{T} \) a \( k \)-linear \( \text{Hom} \)-finite triangulated category with split idempotents and finitely many isomorphism classes of indecomposables. We assume that \( \mathcal{T} \) is algebraic and standard but possibly non connected.

Assume first that \( \mathcal{T} \) is \( \Sigma \)-connected, i.e. that the \( k \)-linear orbit category \( \mathcal{T} / \Sigma \) is connected. Then the argument at the beginning of the above proof shows that either \( \mathcal{T} \) is connected or \( \mathcal{T} \) is \( k \)-linearly equivalent to \( D^b(\text{mod} kA_1) / F \) for a \( k \)-linear equivalence \( F \) of \( D^b(\text{mod} kA_1) \). Clearly \( F \) lifts to a triangle autoequivalence, namely a power \( \Sigma^N \), of the suspension functor of \( D^b(\text{mod} kA_1) \). We may assume \( N > 0 \) equals the number of isoclasses of indecomposables of \( \mathcal{T} \). Since the underlying \( k \)-category of \( \mathcal{T} \) is abelian and semi-simple, all triangles of \( \mathcal{T} \) split and \( \mathcal{T} \) is triangle equivalent to \( D^b(\text{mod} kA_1) / \Sigma^N \).

Let us now drop the \( \Sigma \)-connectedness assumption on \( \mathcal{T} \). Then clearly \( \mathcal{T} \) decomposes, as a triangulated category, into finitely many \( \Sigma \)-connected components (the pre-images of the connected components of \( \mathcal{T} / \Sigma \)). Each of these is either connected or triangle equivalent to \( D^b(\text{mod} kA_1) / \Sigma^N \) for some \( N > 0 \). Thus, the indecomposables of \( \mathcal{T} \) either lie in connected components or in \( \Sigma \)-connected non connected components and the triangle equivalence class of the latter is determined by the action of \( \Sigma \) on the isomorphism classes of indecomposables. We obtain the
Corollary 3. $\mathcal{T}$ is determined up to triangle equivalence by its underlying $k$-category and the action of $\Sigma$ on the set of isomorphism classes of indecomposables.

We refer to Theorem 6.5 of [3] for an analogous result concerning the $\Sigma$-finite triangulated categories $\mathcal{T}$ and to [2] for an application.

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