On the Exponential Decay of the n-Point Correlation Functions and the Analyticity of the Pressure.

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Abstract

The goal of this paper is to provide estimates leading to a direct proof of the exponential decay of the n-point correlation functions for certain unbounded models of Kac type. The methods are based on estimating higher order derivatives of the solution of the Witten Laplacian equation on one forms associated with the Hamiltonian of the system. We also provide a formula for the Taylor coefficients of the pressure that is suitable for a direct proof of the analyticity.

1 Introduction

In recent publications [66] we have given a generalization to the higher dimensional case of the exponential decay of the two-point correlation functions for models of Kac type. In this paper, we shall establish a weak exponential decay of the n-point correlation functions, and provide an exact formula suitable for a direct proof of the analyticity.

Let $\Lambda$ be a finite subset of $\mathbb{Z}^d$, and consider a Hamiltonian $\Phi$ of the phase space $\mathbb{R}^\Lambda$. We shall focus on the case where $\Phi = \Phi_\Lambda$ is given by

$$\Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x),$$

under suitable assumptions on $\Psi$.

Recall that if $\langle f \rangle$ denote the mean value of $f$ with respect to the Gibbs measure

$$e^{-\Phi(x)}\,dx,$$

the covariance of two functions $g$ and $h$ is defined by

$$\text{cov}(g, h) = \langle (g - \langle g \rangle)(h - \langle h \rangle) \rangle.$$  \hfill (2)

If one wants to have an expression of the covariance in the form

$$\text{cov}(g, h) = \langle \nabla h \cdot \mathbf{w} \rangle_{L^2(\mathbb{R}^\Lambda; \mathbb{R}^\Lambda; e^{-\Phi}dx)},$$

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for a suitable vector field \( w \) we get, after observing that
\[
\nabla h = \nabla (h - \langle h \rangle),
\]
and integrating by parts,
\[
\text{cov}(g, h) = \int (h - \langle h \rangle)(\nabla \Phi \cdot w) e^{-\Phi(x)} dx. \tag{4}
\]
(Here we have assumed that \( g \) and \( h \) are functions of polynomial growth).

This leads to the question of solving the equation
\[
g - \langle g \rangle = (\nabla \Phi - \nabla) \cdot w. \tag{5}
\]
Now, trying to solve this above equation with \( w = \nabla f \), we obtain the equation
\[
g - \langle g \rangle = \left\{ \begin{array}{l}
-\Delta + \nabla \Phi \cdot \nabla f \\
\langle f \rangle = 0
\end{array} \right\} \tag{6}
\]
The existence and smoothness of the solution of this equation were established in [8] (see also [66]) under certain assumptions on \( \Phi \). Now taking gradient on both sides of (6), we get
\[
\nabla g = [(-\Delta + \nabla \Phi \cdot \nabla) \otimes \text{Id} + \text{Hess} \Phi] \nabla f. \tag{7}
\]
We then obtain the emergence of two differential operators:
\[
A_{\Phi}^{(0)} := -\Delta + \nabla \Phi \cdot \nabla \tag{8}
\]
and
\[
A_{\Phi}^{(1)} := A_{\Phi}^{(0)} \otimes \text{Id} + \text{Hess} \Phi. \tag{9}
\]
Here the tensor notation means that \( A_{\Phi}^{(0)} \) acts diagonally on the vector field solution to produce a system of equations.

Thus
\[
\text{cov}(g, h) = \int \left( A_{\Phi}^{(1)}^{-1} \nabla g \cdot \nabla h \right) e^{-\Phi(x)} dx. \tag{10}
\]
The operators \( A_{\Phi}^{(0)} \) and \( A_{\Phi}^{(1)} \) are called the Helffer-Sjöstrand’s operators. These are unbounded operators acting on the weighted spaces
\[
L^2(\mathbb{R}^\Lambda, e^{-\Phi} dx) \text{ and } L^2(\mathbb{R}^\Lambda, \mathbb{R}^\Lambda, e^{-\Phi} dx)
\]
respectively.

The formula (10) was introduced by Helffer and Sjöstrand and in some sense is a generalization of Brascamp-Lieb inequality as already pointed out in [1].

The unitary transformation
\[
U_\Phi : L^2(\mathbb{R}^\Lambda) \rightarrow L^2(\mathbb{R}^\Lambda, e^{-\Phi} dx)
\]
\[
u \mapsto e^{\frac{\Phi}{2}} u
\]

\[2\]
will allow us to work with the unweighted spaces $L^2(\mathbb{R}^\Lambda)$ and $L^2(\mathbb{R}^\Lambda, \mathbb{R}^\Lambda)$ by converting the operators $A^{(0)}_\Phi$ and $A^{(1)}_\Phi$ into equivalent operators

$$W^{(0)}_\Phi = -\Delta + \frac{\nabla \Phi^2}{4} - \frac{\Delta \Phi}{2}$$

and

$$W^{(1)}_\Phi = \left(-\Delta + \frac{\nabla \Phi^2}{4} - \frac{\Delta \Phi}{2}\right) \otimes I + \text{Hess}\Phi.$$ 

respectively.

The equivalence can be seen by observing that

$$W^{(\cdot)}_\Phi = e^{-\Phi/2} \circ A^{(\cdot)}_\Phi \circ e^{\Phi/2}. \quad (13)$$

The operators $W^{(0)}_\Phi$ and $W^{(1)}_\Phi$ are unbounded operators acting on $L^2(\mathbb{R}^\Lambda)$ and $L^2(\mathbb{R}^\Lambda, \mathbb{R}^\Lambda)$ respectively. These are in fact, the euclidean versions of the Laplacians on zero and one forms respectively, already introduced by E. Witten [18] in the context of Morse theory.

The equivalence between the operators $A^{(\cdot)}_\Phi$ and Witten’s Laplacians was first observed by J. Sjöstrand [13] in 1996.

2 Higher Order Exponential Estimates

We shall consider a Hamiltonian of the form

$$\Phi(x) = \Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^\Lambda.$$

where

$$|\partial^\alpha \nabla \Psi| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{[\Lambda]}.$$ \hspace{1cm} (14)

$g$ will denote a smooth function on $\mathbb{R}^\Gamma$ with lattice support $S_g = \Gamma(\mathbb{Z} \Lambda)$. We shall identify $g$ with $\tilde{g}$ defined on $\mathbb{R}^\Lambda$ and shall assume that

$$|\partial^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{[\Gamma]}.$$ \hspace{1cm} (15)

As in [66], we shall momentarily assume that $\Psi$ is compactly supported in $\mathbb{R}^\Lambda$ and $g$ is compactly supported in $\mathbb{R}^\Gamma$ but these assumptions will be relaxed later on.

Let $M$ be the diagonal matrix

$$M = (\delta_{ij}\rho(i))_{i,j \in \Lambda}$$

where $\rho$ is a weight function on $\Lambda$ satisfying

$$e^{-\lambda} \leq \frac{\rho(i)}{\rho(j)} \leq e^\lambda, \quad \text{if} \ i \sim j \ \text{for some} \ \lambda > 0.$$ \hspace{1cm} (16)
Assume also that for every $M$ as above, there exists $\delta_0 \in (0, 1)$ such that
\[
\langle M^{-1} \text{Hess} \Phi(x) Ma, a \rangle \geq \delta_0 a^2, \quad \forall x \in \mathbb{R}^\Lambda, \forall a \in \mathbb{R}^\Lambda.
\] (17)

For instance, the $d$–dimensional nearest neighbor Kac model
\[
\Phi_\Lambda(x) = \frac{x^2}{2} - 2 \sum_{i \sim j} \ln \cosh \left( \sqrt{\nu} (x_i + x_j) \right).
\]
satisfies this assumption for $\nu$ small enough. See [66] for details.

The following theorem has been proved in [66]:

**Theorem 1 (A. Lo [66])** Let $g$ be a smooth function with compact support on $\mathbb{R}^\Gamma$ satisfying
\[
|\partial^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{|\Gamma|}
\] (18)
and $\Phi$ is as above. If $f$ is the unique $C^\infty$–solution of the equation
\[
\begin{cases}
-\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle \\
\langle f \rangle_{L^2(\mu)} = 0,
\end{cases}
\]
then
\[
\sum_{i \in \Lambda} f_i^2(x)e^{2\kappa d(i, S_g)} \leq C \quad \forall x \in \mathbb{R}^\Lambda.
\]
$k$ and $C$ are positive constants. $C$ could possibly depend on the size of the support of $g$ but does not depend on $\Lambda$ and $f$.

We now propose to generalize this theorem to higher order derivatives.

**Proposition 2** If in addition to the assumptions of theorem 1, $\Phi$ satisfies the following growth condition: for $\kappa > 0$ as above,
\[
\sum_{j, i_1, \ldots, i_k \in \Lambda} \Phi_{j, i_1, \ldots, i_k}^2(x)e^{2\kappa d(i_1, \ldots, i_k), S_g} \leq C_k \quad \forall x \in \mathbb{R}^\Lambda, \text{ for } k \geq 2
\] (19)
for some $C_k > 0$ not dependent on $\Lambda$ and $f$, then for any $k \geq 1$, $f$ satisfies
\[
\sum_{i_1, \ldots, i_k \in \Lambda} f_{i_1, \ldots, i_k}^2(x)e^{2\kappa d(i_1, \ldots, i_k), S_g} \leq C_{k,g} \quad \forall x \in \mathbb{R}^\Lambda
\] (20)
where $C_{k,g} > 0$ is a constant that depends on the size of the support of $g$ but not on $\Lambda$ and $f$.

**Proof.**

The case $k = 1$ being theorem 1, we assume for induction that the result is true when $k$ is replaced by $k < k$ with $k \geq 2$. 
For $k \geq 2$ (see [8] for details), we have
\[
\langle \nabla^k g, t_1 \otimes \ldots \otimes t_k \rangle = (\nabla \Phi \cdot \nabla - \Delta) \langle \nabla^k f, t_1 \otimes \ldots \otimes t_k \rangle \\
+ \sum_{j=1}^{k} \langle \nabla^k f, t_1 \otimes \ldots \otimes \text{Hess} \Phi t_j \otimes \ldots \otimes t_k \rangle \\
+ \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset \atop \# B \leq k-2} \langle t_A(\partial_x) \nabla \Phi, t_B(\partial_x) \nabla f \rangle.
\]

In the right hand side of this last above equality, we have used the notation
\[
t_J(u) := \langle \nabla^{\# J} u, t_1 \otimes \ldots \otimes t_{\# J} \rangle.
\]

Now fix $i_2, \ldots, i_k \in \Lambda$. Because $\nabla^k f(x) \to 0$ as $|x| \to \infty$ (see [66]), we consider $x_o \in \mathbb{R}^\Lambda$ that maximizes
\[
x \mapsto -\sum_{i_1} f_{x_{i_1}}^2 \ldots x_{i_k} \rho^2(i_1, \ldots, i_k)
\]
where
\[
\rho(i_1, \ldots, i_k) = e^{\kappa d(i_1, \ldots, i_k), S_u}.
\]
Observe here that $x_o$ could possibly depend on $i_2, \ldots, i_k \in \Lambda$.

Choose
\[
t_1 = \left( \rho(i_1, \ldots, i_k) f_{x_{i_1}} \ldots x_{i_k}(x_o) \right)_{i_1 \in \Lambda}
\]
and
\[
t_j = e_{i_j} \quad \text{if} \quad j = 2, \ldots, k
\]
Let $M_1$ be the diagonal matrix
\[
M_1 = (\delta_{si_1, \rho(i_1, \ldots, i_k)})_{si_1}
\]
and
\[
M_j = I \quad \text{if} \quad j \neq 1 \tag{21}
\]
in particular, we have
\[
\langle \nabla^k g, M_1 t_1 \otimes \ldots \otimes M_k t_k \rangle \\
= (\nabla \Phi \cdot \nabla - \Delta) \langle \nabla^k f, M_1 t_1 \otimes \ldots \otimes M_k t_k \rangle \\
+ \sum_{j=1}^{k} \langle \nabla^k f, M_1 t_1 \otimes \ldots \otimes \text{Hess} \Phi M_j t_j \otimes \ldots \otimes M_k t_k \rangle \\
+ \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset \atop \# B \leq k-2} \langle t_{M_A}(\partial_x) \nabla \Phi, t_{M_B}(\partial_x) \nabla f \rangle.
\[ t_{MA}(\partial_x)u := \langle \nabla^A f, M_{t_{i_1}} \otimes \ldots \otimes M_{\# A t_{i_{\# A}}} \rangle, \quad j_i \in A. \]

As in [66], the function

\[ x \mapsto \langle \nabla^k f(x), M_{t_1} \otimes \ldots \otimes M_{t_k} \rangle \]

achieves its maximum at \( x_o \). Using the notation \( \Phi_{x_{i_A}} = \Phi_{x_{i_{t_1}} \ldots x_{i_{t_r}}} \) if \( A = \{\ell_1, \ldots, \ell_r\} \subset \{1, \ldots, k\} \), we therefore have

\[
\sum_{i_1 \in A} g_{x_{i_1} \ldots x_{i_k}}(x_o)\rho(i_1, \ldots, i_k)^2 f_{x_{i_1} \ldots x_{i_k}}(x_o)
\geq \sum_{s \in \Lambda} \sum_{i_1 \in A} f_{x_{i_1} \ldots x_{i_k}}(x_o)\rho(i_1, \ldots, i_k)^2 \Phi_{x_{i_1} \ldots x_{i_k}}(x_o)
\]

\[
+ \sum_{j=2}^{k} \sum_{i_1 \in A} \sum_{s \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{j_1} \ldots x_{j_k}}(x_o) \rho(i_1, \ldots, i_k)^2 \Phi_{x_{i_1} \ldots x_{i_k}}(x_o)
\]

\[
+ \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset \atop \# B \leq k-2} \sum_{i_1 \in A} \sum_{\mathbf{1} \in B} \left\langle \sum_{i_1 \in A} \nabla \Phi_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2, \nabla f_{x_{i_1} \ldots x_{i_k}}(x_o) \right\rangle
\]

\[
+ \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset \atop \# B \leq k-2} \sum_{i_1 \in A} \sum_{i_2 \in B} \nabla \Phi_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2.
\]

Equivalently

\[
\sum_{i_1 \in A} g_{x_{i_1} \ldots x_{i_k}}(x_o)\rho(i_1, \ldots, i_k)^2 f_{x_{i_1} \ldots x_{i_k}}(x_o)
\geq \sum_{s \in \Lambda} \sum_{i_1 \in A} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{j_1} \ldots x_{j_k}}(x_o) \rho(i_1, \ldots, i_k)^2 \Phi_{x_{i_1} \ldots x_{i_k}}(x_o)
\]

\[
+ \sum_{j=2}^{k} \sum_{i_1 \in A} \sum_{s \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{j_1} \ldots x_{j_k}}(x_o) \rho(i_1, \ldots, i_k)^2 \Phi_{x_{i_1} \ldots x_{i_k}}(x_o)
\]

\[
+ \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset \atop \# B \leq k-2} \sum_{i_1 \in A} \Phi_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2 f_{x_{i_1} \ldots x_{i_k}}(x_o)
\]

\[
+ \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset \atop \# B \leq k-2} \sum_{i_1 \in A} \sum_{i_2 \in B} \Phi_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2.
\]
Now taking summation over $i_2, \ldots, i_k$, we get
\[
\sum_{i_2, \ldots, i_k \in \Lambda} \sum_{i_1 \in \Lambda} g_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k) f_{x_{i_1} \ldots x_{i_k}}(x_o) \geq
\sum_{i_2, \ldots, i_k \in \Lambda} \sum_{s \in \Lambda} \sum_{i_1 \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k) \Phi_{x_{i_1}x_{i_1}}(x_o)
\]
\[
+ \sum_{i_2, \ldots, i_k \in \Lambda} \sum_{s \in \Lambda} \sum_{i_1 \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_2} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k) \Phi_{x_{i_2}x_{i_2}}(x_o)
\]
\[
+ \sum_{i_2, \ldots, i_k \in \Lambda} \sum_{s \in \Lambda} \sum_{i_1 \in \Lambda} \sum_{1 \leq B \leq k-2} \sum_{i_{B+1} \in \Lambda} \Phi_{x_{i_1}x_{i_B}x_{i_{B+1}}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2 f_{x_{i_{B+1}}x_{i_{B+1}}}(x_o)
\]
\[
+ \sum_{i_2, \ldots, i_k \in \Lambda} \sum_{s \in \Lambda} \sum_{i_1 \in \Lambda} \sum_{1 \leq B \leq k-2} \sum_{i_{B+1} \in \Lambda} \Phi_{x_{i_1}x_{i_B}x_{i_{B+1}}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_{B+1}}} \rho(i_1, \ldots, i_k)^2.
\]

Next, we propose to estimate each term of the right hand side of this above inequality.
\[
\sum_{i_2, \ldots, i_k \in \Lambda} \sum_{s \in \Lambda} \sum_{i_1 \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_1} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2 \Phi_{x_{i_1}x_{i_1}}(x_o)
\]
\[
= \sum_{i_2, \ldots, i_k \in \Lambda} \left\langle \nabla f_{x_{i_2} \ldots x_{i_k}}(x_o), \text{Hess} \Phi M_1 t_1 \right\rangle
\]
\[
= \sum_{i_2, \ldots, i_k \in \Lambda} \left\langle M_1 \nabla f_{x_{i_2} \ldots x_{i_k}}(x_o), M_1^{-1} \text{Hess} \Phi M_1 t_1 \right\rangle
\]
\[
= \sum_{i_2, \ldots, i_k \in \Lambda} \left\langle t_1, M_1^{-1} \text{Hess} \Phi M_1 t_1 \right\rangle
\]
\[
\geq \delta_0 \sum_{i_2, \ldots, i_k \in \Lambda} \|t_1\|^2
\]
\[
= \delta_0 \sum_{i_1, \ldots, i_k \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o)^2 \rho(i_1, \ldots, i_k)^2.
\]

Similarly, it is easy to see that
\[
\sum_{i_2, \ldots, i_k \in \Lambda} \sum_{j=2}^{k} \sum_{i_1 \in \Lambda} \sum_{s \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o) f_{x_{i_2} \ldots x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2 \Phi_{x_{i_2}x_{i_2}}(x_o)
\]
\[
\geq (k-1) \delta_0 \sum_{i_1, \ldots, i_k \in \Lambda} f_{x_{i_1} \ldots x_{i_k}}(x_o)^2 \rho(i_1, \ldots, i_k)^2
\]
To estimate the last two terms, we have

\[
\sum_{i_2, \ldots, i_k \in \Lambda} \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} \sum_{s \in \Lambda} \left| \Phi_{x_{i_2}, \ldots, x_{i_k}}(x_o) \rho(i_1, \ldots, i_k)^2 f_{x_{i_2} \ldots x_k, x_o}(x_o) \right|
\]

\[
\leq \left[ \sum_{i_2, \ldots, i_k \in \Lambda} \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} f_{x_{i_2} \ldots x_k}(x_o) \rho(i_1, \ldots, i_k)^2 \right]^{1/2} \times \left[ \sum_{i_2, \ldots, i_k \in \Lambda} \left( \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} \sum_{s \in \Lambda} \Phi_{x_{i_2} \ldots x_k}(x_o) \rho(i_1, \ldots, i_k) f_{x_{i_2} \ldots x_k, x_o}(x_o) \right) \right]^{1/2}. 
\]

To estimate the second factor of the right hand side of this last above inequality, we have

\[
\sum_{i_1, \ldots, i_k \in \Lambda} \left( \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} \sum_{s \in \Lambda} \Phi_{x_{i_2} \ldots x_k}(x_o) \rho(i_1, \ldots, i_k) f_{x_{i_2} \ldots x_k, x_o}(x_o) \right)^2
\]

\[
\leq C_k \sum_{i_1, \ldots, i_k \in \Lambda} \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} \left( \sum_{s \in \Lambda} \Phi_{x_{i_2} \ldots x_k}(x_o) \rho(i_1, \ldots, i_k) f_{x_{i_2} \ldots x_k, x_o}(x_o) \right)^2
\]

\[
\leq C_k \sum_{i_1, \ldots, i_k \in \Lambda} \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} \left( \sum_{s \in \Lambda} \Phi_{x_{i_2} \ldots x_k}(x_o) e^{2\kappa d(i, j \in A, S_o)} \right) \times 
\]

\[
\left( \sum_{s \in \Lambda} \rho^2(i_1, \ldots, i_k) f^2_{x_{i_2} \ldots x_k, x_o}(x_o) \right)
\]

\[
\leq C_k \sum_{i_1, \ldots, i_k \in \Lambda} \sum_{A \cup B = \{1, \ldots, k\}, A \cap B = \emptyset} \left( \sum_{s \in \Lambda} \Phi_{x_{i_2} \ldots x_k}(x_o) e^{2\kappa d(i, j \in A, S_o)} \right) \times 
\]

\[
\left( \sum_{s \in \Lambda} e^{2\kappa d(i, j \in B \cup \{s\}, S_o)} f^2_{x_{i_2} \ldots x_k, x_o}(x_o) \right)
\]

\[
\leq C_k.
\]

This last inequality above follows from the induction assumption and that of \( \Phi \).
Thus,

\[
\sum_{i_2,\ldots,i_k \in A \cup B = \{1,\ldots,k\}, A \cap B = \emptyset} \sum_{i \in \Lambda} \sum_{s \in \Lambda} \Phi_{x_A x_s} f_{x_i} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 f_{x_i} f_{x_{i_1} \ldots x_{i_k}} (x_o)
\]

\[
\geq -C_k \left[ \sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 \right]^{1/2}.
\]

Similarly, we have

\[
\sum_{i_2,\ldots,i_k \in A \cup B = \{1,\ldots,k\}, A \cap B = \emptyset} \sum_{i \in \Lambda} \sum_{s \in \Lambda} \Phi_{x_A x_s} f_{x_i} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2
\]

\[
\geq -C_k \left[ \sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 \right]^{1/2}.
\]

We then finally get

\[
\sum_{i_1,\ldots,i_k \in A} g_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 f_{x_{i_1} \ldots x_{i_k}} (x_o)
\]

\[
\geq k \delta_o \sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2
\]

\[
-C_k \left[ \sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 \right]^{1/2}.
\]

If

\[
\sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 = 0
\]

then there is nothing to prove, otherwise we have, after using Cauchy-Schwartz and dividing by

\[
\sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2,
\]

\[
\left( \sum_{i_1,\ldots,i_k \in A} f_{x_{i_1} \ldots x_{i_k}} (x_o) \rho(i_1, \ldots, i_k)^2 \right)^{1/2}
\]

\[
\leq \frac{1}{k \delta_o} \left( \sum_{i_1,\ldots,i_k \in A} g_{x_{i_1} \ldots x_{i_k}} (x_o) \right)^{1/2} + C_{k,g}
\]

\[
\leq C_{k,g}.
\]
3 Relaxing the Assumptions of Compact Support

As in [8], we consider the family cutoff functions

\[ \chi = \chi_\varepsilon \]

(\varepsilon \in [0, 1]) in \( C_0^\infty(\mathbb{R}) \) with value in \([0, 1]\) such that

\[
\begin{cases}
\chi = 1 & \text{for } |t| \leq \varepsilon^{-1} \\
|\chi^{(k)}(t)| & \leq C_k \frac{\varepsilon}{|t|^k} & \text{for } k \in \mathbb{N}.
\end{cases}
\]

We then introduce

\[ \Psi_\varepsilon(x) = \chi_\varepsilon(|x|) \Psi(x) \quad x \in \mathbb{R}^\Lambda \]

and

\[ g_\varepsilon(x) = \chi_\varepsilon(|x|)g(x) \quad x \in \mathbb{R}^\Gamma. \]

A straightforward computation (see [66]) shows that \( \Psi_\varepsilon(x) \) and \( g_\varepsilon(x) \) satisfy

\[ |\partial^\alpha \nabla \Psi_\varepsilon| \leq C_\alpha + O_\alpha, \Lambda(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{\Lambda}. \]

and

\[ |\partial^\alpha \nabla g_\varepsilon| \leq C_\alpha + O_\alpha, \Lambda(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{\Gamma}, \]

and that

\[ M^{-1} \text{Hess} \Phi_\varepsilon(x) M \geq \delta', \quad 0 < \delta' < 1. \quad \text{in the sense of (17)} \]

It then only remains to check that

\[ \sum_{j, i_1, \ldots, i_k \in \Lambda} \Psi^2_{\varepsilon \varepsilon, x_{i_1} \ldots x_{i_k}}(x) e^{2\varepsilon d(i_1, \ldots, i_k), S_\varepsilon} \leq C_k + O_{k, \Lambda}(\varepsilon) \quad \forall x \in \mathbb{R}^\Lambda, \forall k \geq 2 \]

where \( C_k \) is a positive constant that does not depend on \( f \) and \( \Lambda \).

\[ \Psi_\varepsilon(x) = \chi_\varepsilon(r) \Psi(x) \]

Let \( \alpha \) be such that \( |\alpha| \geq 3 \). Using Leibniz’s formula, we have

\[ |\partial^\alpha \Psi_\varepsilon| \leq \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) |\partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} \Psi| \leq |\partial^\alpha \chi_\varepsilon(r) \Psi| + |\partial^\alpha \Psi| + \sum_{\beta < \alpha} \left( \frac{\alpha}{\beta} \right) |\partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} \Psi|. \]

Assuming that \( \Psi(0) = 0 \) and write

\[ \Psi(x) = \int_0^1 x \cdot \nabla \Psi(sx) ds \]
\[ |\partial^\alpha \chi_\varepsilon(r) \Psi(x)| \leq \sum_{j_1 \in \Lambda} \int_0^1 |x_{j_1} \partial^\alpha \chi_\varepsilon(r) \Psi_{x_{j_1}}(sx)| \, ds \leq C |r \partial^\alpha \chi_\varepsilon(r)|. \]

Now using the fact that
\[ r \partial^\alpha \chi_\varepsilon(r) = O_\alpha(\varepsilon), \]
we have
\[ |\partial^\alpha \chi_\varepsilon(r) \Psi(x)| = O_{\alpha, \Lambda}(\varepsilon). \]

Finally, using the fact that
\[ \partial^\beta \chi_\varepsilon(r) = O_\beta(\varepsilon) \text{ for every } |\beta| \geq 1, \quad (31) \]
it is then easy to see that
\[ \left( \sum_{\beta < \alpha \atop \beta \neq 0} \binom{\alpha}{\beta} |\partial^\beta \chi_\varepsilon(r) \partial^{\alpha - \beta} \Psi| \right)^2 = O_{\alpha, \Lambda}(\varepsilon). \quad (32) \]

Thus
\[ \sum_{j, i_1, \ldots, i_k \in \Lambda} \Psi^2_{x_j x_{i_1} \ldots x_{i_k}}(x) e^{2\kappa d(\{i_1, \ldots, i_k\}, S_k)} \leq C_{k, g} + O_{k, \Lambda}(\varepsilon) \quad \forall x \in \mathbb{R}^\Lambda, \forall k \geq 2. \quad (33) \]

Now using the arguments developed in [8] (see also [66]) about the convergence of the corresponding solutions as \( \varepsilon \to 0 \), we obtain:

**Proposition 3** If \( g(0) = \Psi(0) = 0 \), then Proposition 2 holds without the assumptions of compact support on \( \Psi \) and \( g \).

### 4 The n-Point Correlation Functions

The higher order correlation is defined as
\[ \langle g_1, \ldots, g_k \rangle := \langle (g_1 - \langle g_1 \rangle) \ldots (g_k - \langle g_k \rangle) \rangle. \quad (34) \]

For simplicity we shall take \( k = 3 \) and \( \Phi \) is as in proposition 2.

Let \( g_1, g_2, \) and \( g_3 \) be smooth functions satisfying (15) and \( f_i, i = 1, 2, 3 \) shall denote the unique solution of the system
\[ \begin{cases} -\Delta f_i + \nabla \Phi \cdot \nabla f_i = g_i - \langle g_i \rangle_{L^2(\mu)} \\ \langle f_i \rangle_{L^2(\mu)} = 0. \end{cases} \quad (35) \]

Recall that
\[ \nabla f_i = A^{(1)}_{\Phi} \nabla g_i. \]
For an arbitrary smooth function $c$, it is easy to see that

$$\langle c(x) (g_i - \langle g_i \rangle) \rangle = \langle \nabla f_i \cdot \nabla c \rangle.$$ 

A direct computation shows that

$$\langle g_1, g_2, g_3 \rangle = \langle \nabla f_3 \cdot (\text{Hess} f_1) \nabla g_2 \rangle + \langle \nabla f_3 \cdot (\text{Hess} g_2) \nabla f_1 \rangle + \langle \nabla f_2 \cdot (\text{Hess} f_1) \nabla g_3 \rangle + \langle \nabla f_2 \cdot (\text{Hess} g_3) \nabla f_1 \rangle.$$ 

Let us now estimate each term of the right and side of this equality.

Using Cauchy-Schwartz, and proposition 2, it is easy to see that

$$|\langle \nabla f_3 \cdot (\text{Hess} f_1) \nabla g_2 \rangle| \leq Ce^{-\kappa_1 d(S_{g_2}, S_{g_1})}$$

$$|\langle \nabla f_3 \cdot (\text{Hess} g_2) \nabla f_1 \rangle| \leq Ce^{-\kappa_1 d(S_{g_2}, S_{g_1})},$$

$$|\langle \nabla f_2 \cdot (\text{Hess} f_1) \nabla g_3 \rangle| \leq Ce^{-\kappa_1 d(S_{g_3}, S_{g_1})}$$

and

$$|\langle \nabla f_2 \cdot (\text{Hess} g_3) \nabla f_1 \rangle| \leq Ce^{-\kappa_1 d(S_{g_3}, S_{g_1})}$$

Here the constants $C$ only depends on the size of the support of the $g_i$’s. and $\kappa_1 > 0$.

Thus

$$|\langle g_1, g_2, g_3 \rangle| \leq C \left[ e^{-\kappa_1 d(S_{g_2}, S_{g_1})} + e^{-\kappa_1 d(S_{g_3}, S_{g_1})} \right]$$

If $g_1 = x_i, g_2 = x_j,$ and $g_3 = x_k$, we obtain

$$|\langle (x_i - \langle x_i \rangle) (x_j - \langle x_j \rangle) (x_k - \langle x_k \rangle) \rangle| \leq C \left[ e^{-\kappa_1 d(i,j)} + e^{-\kappa_1 d(i,k)} \right].$$

Thus if $d > 1$, we obtain this weak exponential decay of the truncated correlations in the sense that the exponential decay occurs as you simultaneously pull the spins away from a fixed one. Note that in the one dimensional case, we obtain a stronger exponential decay due to the fact that

$$i \leq j \leq k \implies d(i,k) = d(i,j) + d(j,k).$$

This was already pointed out in [8].

5 The Analyticity of the Pressure

In this section, we attempt to study a direct method for the analyticity of the pressure for certain classical convex unbounded spin systems. It is central in Statistical Mechanics to study the differentiability or even the analyticity of the pressure with respect to some distinguished thermodynamic parameters such as temperature, chemical potential or external field. In fact the analytic behavior of the pressure is the classical thermodynamic indicator for the absence or existence of phase transition. The most famous result on the analyticity of the pressure
is the circle theorem of Lee and Yang [28]. This theorem asserts the following: consider a \( \{-1, 1\} \) -valued spin system with ferromagnetic pair interaction and external field \( h \) and regard the quantity \( z = e^{ih} \) as a complex parameter, then all zeroes of all partition functions (with free boundary condition), considered as functions of \( z \) lie in the complex unit circle. This theorem readily implies that the pressure is an analytic function of \( h \) in the region \( h > 0 \) and \( h < 0 \). Heilmann [29] showed that the assumption of pair interaction is necessary. A transparent approach to the circle theorem was found by Asano [30] and developed further by Ruelle [31],[32], Slawny [33], and Gruber et al [34]. Griffiths [35] and Griffiths-Simon [36] found a method of extending the Lee-Yang theorem to real-valued spin systems with a particular type of a priori measure. Newman [37] proved the Lee-Yang theorem for every a priori measure which satisfies this theorem in the particular case of no interaction. Dunlop [38],[39] studied the zeroes of the partition functions for the plane rotor model. A general Lee-Yang theorem for multicomponent systems was finally proved by Lieb and Sokal [40]. For further references see Glimm and Jaffe [41].

The Lee-Yang theorem and its variants depend on the ferromagnetic character of the interaction. There are various other way of proving the infinite differentiability or the analyticity of the pressure for (ferromagnetic and non ferromagnetic) systems at high temperatures, or at low temperatures, or at large external fields. Most of these take advantage of a sufficiently rapid decay of correlations and/or cluster expansion methods. Here is a small sample of relevant references. Bricmont, Lebowitz and Pfister [42], Dobroshin [43], Dobroshin and Sholsman [44],[45], Duneau et al [46],[47],[48], Glimm and Jaffe [41],[49], Israel [50], Kotecky and Preiss [51], Kunz [52], Lebowitz [53],[54], Malyshev [55], Malychev and Milnos [56] and Prakash [57]. M. Kac and J.M. Luttinger [58] obtained a formula for the pressure in terms of irreducible distribution functions.

We propose a new way of analyzing the analyticity of the pressure for certain unbounded models through a representation by means of the Witten Laplacians of the coefficients in the Taylor series expansion. The methods known up to now rely on complicated indirect arguments.

6 Towards the analyticity of the Pressure

Let \( \Lambda \) be a finite domain in \( \mathbb{Z}^d \) (\( d \geq 1 \)) and consider the Hamiltonian of the phase space given by,

\[
\Phi(x) = \Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^\Lambda. \tag{36}
\]

where

\[
|\partial^\alpha \nabla \Psi| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{37}
\]

\[
\text{Hess}\Phi(x) \geq \delta_0, \quad 0 \leq \delta_0 < 1. \tag{38}
\]

Let \( g \) is a smooth function on \( \mathbb{R}^\Gamma \) with lattice support \( S_g = \Gamma \). We identified with \( \tilde{g} \) defined on \( \mathbb{R}^\Lambda \) by

\[
\tilde{g}(x) = g(x_\Gamma) \quad \text{where} \quad x = (x_i)_{i \in \Lambda} \quad \text{and} \quad x_\Gamma = (x_i)_{i \in \Gamma}. \tag{39}
\]
and satisfying
\[ |\partial^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{[\Gamma]} \]  
(40)

Under the additional assumptions that \( \Psi \) is compactly supported in \( \mathbb{R}^\Lambda \) and \( g \) is compactly supported in \( \mathbb{R}^\Gamma \), it was proved in [66] (see also [8]) that the equation

\[
\begin{cases}
-\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle \\
(f)_{L^2(\mu)} = 0
\end{cases}
\]

has a unique smooth solution satisfying \( \nabla^k f(x) \to 0 \) as \( |x| \to \infty \) for every \( k \geq 1 \).

Recall also that \( \nabla f \) is a solution of the system

\[
(-\Delta + \nabla \Phi \cdot \nabla) \nabla f + \text{Hess} \Phi \nabla f = \nabla g \quad \text{in} \quad \mathbb{R}^\Lambda. \]  
(41)

As in [66] and [8], these assumptions will be relaxed later on.

Let

\[
\Phi^t_\Lambda(x) = \Phi(x) - tg(x),
\]

(42)

where \( x = (x_i)_{i \in \Lambda} \), and assume additionally that \( g \) satisfies

\[
\text{Hess} g \leq C. \]  
(43)

We consider the following perturbation

\[
\theta_\Lambda(t) = \log \left[ \int \, dx \, e^{-\Phi^t_\Lambda(x)} \right].
\]  
(44)

Denote by

\[
Z_t = \int \, dx \, e^{-\Phi^t_\Lambda(x)}
\]  
(45)

and

\[
\langle \cdot \rangle_{t,\Lambda} = \frac{\int \, dx \, e^{-\Phi^t_\Lambda(x)} \cdot}{Z_t}.
\]  
(46)

7 Parameter Dependency of the Solution

From the assumptions made on \( \Phi \) and \( g \), it is easy to see that there exists \( T > 0 \) such that for every \( t \in [0, T) \), \( \Phi^t_\Lambda(x) \) satisfies all the assumptions required for the solvability, regularity and asymptotic behavior of the solution \( f(t) \) associated with the potential \( \Phi^t_\Lambda(x) \). Thus, each \( t \in [0, T) \) is associated with a unique \( C^\infty \)-solution, \( f(t) \) of the equation

\[
\begin{cases}
A^{(0)}_{\Phi^t_\Lambda} f(t) = g - \langle g \rangle_{L^2(\mu)} \\
(f(t))_{L^2(\mu)} = 0.
\end{cases}
\]

Hence,

\[
A^{(1)}_{\Phi^t_\Lambda} \nabla(t) = \nabla g
\]  
(47)
where \( \mathbf{v}(t) = \nabla f(t) \). Notice that the map
\[
t \mapsto \mathbf{v}(t)
\]
is well defined and
\[
\{ \mathbf{v}(t) : t \in [0, T) \}
\]
is a family of smooth solutions on \( \mathbb{R}^N \) satisfying
\[
\partial^\alpha \mathbf{v}(t) \to 0 \quad \text{as} \quad |x| \to \infty \quad \forall \alpha \in \mathbb{N}^{|\Lambda|} \quad \text{and for each} \quad t \in [0, T)
\]
and corresponding to the family of potential
\[
\{ \Phi^t : t \in [0, T) \}.
\]
(48)

Let us now verify that \( \mathbf{v} \) is a smooth function of \( t \in (0, T) \). We need to prove that for each \( t \in (0, T) \), the limit
\[
\lim_{\varepsilon \to 0} \mathbf{v}(t + \varepsilon) - \mathbf{v}(t) \]
exists. Let
\[
\mathbf{v}^\varepsilon(t) = \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon}.
\]
We use a technique based on regularity estimates to get a uniform control of \( \mathbf{v}^\varepsilon(t) \) with respect to \( \varepsilon \).

With \( \varepsilon \) small enough, we have
\[
0 = -\Delta \left[ \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon} \right] + \nabla \Phi^{t+\varepsilon} \cdot \nabla \mathbf{v}(t + \varepsilon) - \nabla \Phi^t \cdot \nabla \mathbf{v}(t) + \frac{\text{Hess} \Phi^{t+\varepsilon} \mathbf{v}(t + \varepsilon) - \text{Hess} \Phi^t \mathbf{v}(t)}{\varepsilon}.
\]
Equivalently,
\[
-\Delta \left[ \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon} \right] + \frac{\nabla \Phi^{t+\varepsilon} \cdot \nabla [\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)] \varepsilon}{\varepsilon} + \text{Hess} \Phi^{t+\varepsilon} \left( \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon} \right)
\]
\[
= - \left( \text{Hess} \Phi^{t+\varepsilon} - \text{Hess} \Phi^t \right) \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon}
\]
\[
= \text{Hess} g \mathbf{v}(t) + \nabla g \cdot \nabla \mathbf{v}(t)
\]
and
\[
-\Delta \mathbf{v}^\varepsilon(t) + \nabla \Phi^{t+\varepsilon} \cdot \nabla \mathbf{v}^\varepsilon(t) + \text{Hess} \Phi^{t+\varepsilon} \mathbf{v}^\varepsilon(t)
\]
\[
= \text{Hess} g \mathbf{v}(t) + \nabla g \cdot \nabla \mathbf{v}(t)
\]
Let $w(t)$ be the unique $C^\infty$-solution of the system

$$-\Delta w(t) + \nabla \Phi \cdot \nabla w(t) + \text{Hess} \Phi w(t) = \text{Hess} g(v) + \nabla g \cdot \nabla v(t).$$  \hspace{1cm} (49)$$

Recall that the unitary transformation $U_{\Phi t+\varepsilon}$ allows us to reduce

$$-\Delta \varepsilon(t) + \nabla \Phi t+\varepsilon \cdot \nabla \varepsilon(t) + \text{Hess} \Phi t+\varepsilon \varepsilon(t) = \text{Hess} g(v) + \nabla g \cdot \nabla v(t)$$  \hspace{1cm} (50)$$

into

$$\left(-\Delta + \frac{|\nabla \Phi t+\varepsilon|^2}{4} - \frac{\Delta \Phi t+\varepsilon}{2}\right) \varepsilon + \text{Hess} \Phi t+\varepsilon \varepsilon = \text{Hess} g(v) + \nabla g \cdot \nabla v(t) e^{-\Phi t+\varepsilon/2}$$  \hspace{1cm} (51)$$

where $\varepsilon = v(t)e^{-\Phi t+\varepsilon/2}$.

**Remark 4** This unitary transformation already mentioned in the introduction was introduced in the proof of the existence of solution (see [66]) to avoid working with the weighted spaces $L^2(\mathbb{R}^\Lambda, \mathbb{R}^\Lambda, e^{-\Phi}dx)$. The proof was based on Hilbert space method. The method consists of determining an appropriate function space and an operator which is a natural realization of the problem. In this particular problem, the function spaces to be considered are the Sobolev spaces $B^k_\Phi(\mathbb{R}^\Lambda)$ defined by

$$B^k_\Phi(\mathbb{R}^\Lambda) = \{ u \in L^2(\mathbb{R}^\Lambda) : Z^\ell_\Phi \partial^\alpha u \in L^2(\mathbb{R}^\Lambda) \forall \ell + |\alpha| \leq k \}.$$  \hspace{1cm} (52)$$

where

$$Z_\Phi = \frac{|\nabla \Phi|}{2}$$

These are subspaces of the well known Sobolev spaces $W^{k,2}(\mathbb{R}^\Lambda)$, $k \in \mathbb{N}$.

Taking scalar product with $\varepsilon$ on both sides of (51), we get

$$\left\| \nabla \Phi t+\varepsilon \varepsilon \right\|_{L^2}^2 + \int \text{Hess} \Phi t+\varepsilon \varepsilon \cdot \varepsilon dx = \int [\text{Hess} g(v) + \nabla g \cdot \nabla v(t)] e^{-\Phi t+\varepsilon/2} \cdot \varepsilon dx.$$  \hspace{1cm} (53)$$

Now using the uniform strict convexity on the left hand side and Cauchy-Schwartz on the right hand side, we obtain

$$\| \varepsilon \|^2_{B^0} \leq C_{\varepsilon} \quad \text{for small enough } \varepsilon.$$  \hspace{1cm} (54)$$

We then deduce that

$$\left(-\Delta + \frac{|\nabla \Phi t+\varepsilon|^2}{4}\right) \varepsilon = \tilde{q}_\varepsilon$$  \hspace{1cm} (55)$$
where
\[ \tilde{q}_\varepsilon = [\text{Hess} g v(t) + \nabla g \cdot \nabla v(t)] e^{-\Phi^{t+\varepsilon}/2} + \frac{\Delta \Phi^{t+\varepsilon}}{2} V^\varepsilon - \text{Hess} \Phi^{t+\varepsilon} V^\varepsilon \] (56)
is bounded in \( B^0 \) uniformly with respect to \( \varepsilon \) for \( \varepsilon \) small enough.

Taking again scalar product with \( V^\varepsilon \) on both sides of (55) and integrating by parts, we obtain
\[ \| \nabla V^\varepsilon \|^2_{L^2} + \| \frac{\nabla \Phi^{t+\varepsilon}}{2} V^\varepsilon \|^2_{L^2} \leq \| \tilde{q}_\varepsilon \|_{L^2} \| V^\varepsilon \|_{L^2} \] (57)
It follows that \( V^\varepsilon \) is uniformly bounded with respect to \( \varepsilon \) in \( B^1_0 \) for \( \varepsilon \) small enough.

Next, observe that
\[ (-\Delta + \frac{\| \nabla \Phi^t \|^2}{4}) V^\varepsilon = \tilde{q}_\varepsilon \] (58)
where
\[ \tilde{q}_\varepsilon = \tilde{q}_\varepsilon - \frac{\| \nabla \Phi^{t+\varepsilon} - \nabla \Phi^t \|^2}{2} V^\varepsilon + \frac{\nabla \Phi^{t+\varepsilon} - \nabla \Phi^t \cdot \nabla \Phi^t}{2} V^\varepsilon \] (59)
\[ = \tilde{q}_\varepsilon - \frac{\varepsilon^2}{4} \frac{\| \nabla g \|^2}{2} V^\varepsilon - \frac{\varepsilon}{2} \nabla g \cdot \nabla \Phi^t V^\varepsilon \] (60)
is uniformly bounded in \( B^0 \) with respect to \( \varepsilon \) for small enough \( \varepsilon \). Using regularity, it follows that for small enough \( \varepsilon \), \( V^\varepsilon \) is uniformly bounded in \( B^1_0 \) with respect to \( \varepsilon \). This implies that \( \tilde{q}_\varepsilon \) is uniformly bounded in \( B^1_0 \) for \( \varepsilon \) small enough. Again, we can continue by a bootstrap argument to consequently get that for \( \varepsilon \) small enough, \( V^\varepsilon \) is uniformly bounded in \( B^k_0 \) with respect to \( \varepsilon \) for any \( k \).

Let
\[ V = w(t)e^{-\Phi^t/2}. \]
We have
\[ (-\Delta + \frac{\| \nabla \Phi^t \|^2}{4} - \frac{\Delta \Phi^t}{2}) V + \text{Hess} \Phi^t V = [\text{Hess} g v(t) + \nabla g \cdot \nabla v(t)] e^{-\Phi^t/2} \] (61)
Now combining this equation with (51), we obtain
\[ \left(-\Delta + \frac{\| \nabla \Phi^t \|^2}{4} - \frac{\Delta \Phi^t}{2}\right) (V^\varepsilon - V) + \text{Hess} \Phi^t (V^\varepsilon - V) = -[\text{Hess} g v(t) + \nabla g \cdot \nabla v(t)] e^{-\Phi^t/2} + [\text{Hess} g v(t) + \nabla g \cdot \nabla v(t)] e^{-\Phi^{t+\varepsilon}/2} \]
\[ + \left(\frac{\| \nabla \Phi^t \|^2}{4} - \frac{\| \nabla \Phi^{t+\varepsilon} \|^2}{4}\right) V^\varepsilon - \left(\frac{\Delta \Phi^t}{2} - \frac{\Delta \Phi^{t+\varepsilon}}{2}\right) V^\varepsilon \]
\[ + (\text{Hess} \Phi^t - \text{Hess} \Phi^{t+\varepsilon}) V^\varepsilon. \] (62)
Now let us check that for small enough \( \varepsilon \), the right hand side of (62) is \( O(\varepsilon) \) in \( B^0 \).

For the first term, we have

\[-|\text{Hess}v(t) + \nabla g \cdot \nabla v(t)| e^{-\Phi(t)/2} + |\text{Hess}g(t) + \nabla g \cdot \nabla v(t)| e^{-\Phi(t)/2}\]

\[= |\text{Hess}g(t) + \nabla g \cdot \nabla v(t)| e^{-\Phi(t)/2} (e^{\varepsilon g/2} - 1)\]

\[\sim \frac{\varepsilon}{2} |\text{Hess}g(t) + \nabla g \cdot \nabla v(t)| e^{-\Phi(t)/2}\]

Thus for \( \varepsilon \) small enough

\[\left\| -|\text{Hess}g(t) + \nabla g \cdot \nabla v(t)| e^{-\Phi(t)/2} + |\text{Hess}g(t) + \nabla g \cdot \nabla v(t)| e^{-\Phi(t)/2}\right\|_{B^0} \leq C \varepsilon.\]

For the second term, we have

\[\left| \frac{|\nabla \Phi(t)|}{4} - \frac{|\nabla \Phi(t) + \varepsilon|}{4} \right|\]

\[= \frac{1}{4} \left( |\nabla \Phi(t)| + |\nabla \Phi(t) + \varepsilon| \right) \left( |\nabla \Phi(t)| + |\nabla \Phi(t) + \varepsilon| \right)\]

\[\leq \frac{\varepsilon}{2} |\nabla g| \left( 2 |\nabla \Phi(t)| + \varepsilon |\nabla g| \right)\]

Using now the fact that \( V^\varepsilon \) is uniformly bounded in \( B^2_0 \), with respect to \( \varepsilon \) for any \( k \), we see that the second term of the right hand side of (62) is \( O(\varepsilon) \) in \( B^0_\Phi \). The last two terms of the right hand side of (62) are obviously \( O(\varepsilon) \) in \( B^0_\Phi \).

From the same regularity argument as above, we get that \( V^\varepsilon - V \) is \( O(\varepsilon) \) in \( B^2_\Phi \). Again iterating the regularity argument, we obtain that for small enough \( \varepsilon \), \( V^\varepsilon - V \) is \( O(\varepsilon) \) in \( B^k_\Phi \), for every \( k \). We have proved:

**Proposition 5** Under the above assumptions on \( \Phi \) and \( g \), there exists \( T > 0 \) so that for each \( t \in (0, T) \), \( v^\varepsilon(t) \) converges to \( w(t) \) in \( C^\infty \).

**Remark 6** The proposition establishes that \( v(t) \) is differentiable in \( t \) and \( \frac{d}{dt} v(t) \) is given by the unique \( C^\infty \)-solution \( w(t) \) of the system

\[-\Delta w(t) + \nabla \Phi(t) \cdot \nabla w(t) + \text{Hess} \Phi(t) w(t) = \text{Hess}g(t) - \nabla g \cdot \nabla v(t) \tag{63}\]

Iterating this argument, we easily get that, \( v(t) \) is smooth in \( t \in (0, T) \).

Now we are ready for the following:

**8 A Formula for the Taylor Coefficients**

First observe that for an arbitrary suitable function \( f(t) = f(t, w) \)

\[\frac{\partial}{\partial t} < f(t) >_{t, \Lambda} = < f'(t) >_{t, \Lambda} + \text{cov}(f, g). \tag{64}\]
Hence,
\[ \frac{\partial}{\partial t} < f(t) >_{t,\Lambda} = < f'(t) >_{t,\Lambda} + < A_{f_{\infty}}^{(1)} \cdot (\nabla f) \cdot \nabla g >_{t,\Lambda} . \]  
(65)

Let
\[ A_g f := A_{f_{\infty}}^{(1)} (\nabla f) \cdot \nabla g. \]  
(66)

Thus,
\[ \frac{\partial}{\partial t} < f(t) >_{t,\Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right) f >_{t,\Lambda} . \]  
(67)

The linear operator \( \frac{\partial}{\partial t} + A_g \) will be denoted by \( H_g \).

To obtain a formula for the coefficients in the Taylor expansion of
\[ \theta_{\Lambda}(t) = \log \left[ \int dx e^{-\Phi(t)} \right] , \]  
(68)
we first the derivatives of \( \theta_{\Lambda}(t) \) in terms of \( H_g \).

\[ \theta'_{\Lambda}(t) = < g >_{t,\Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right) g >_{t,\Lambda} = < H_g^0 g >_{t,\Lambda}; \]
\[ \theta''_{\Lambda}(t) = \frac{\partial}{\partial t} < g >_{t,\Lambda} = < A_{f_{\infty}}^{(1)} (\nabla g) \cdot \nabla g >_{t,\Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right) g >_{t,\Lambda}; \]
\[ \theta'''_{\Lambda}(t) = \frac{\partial}{\partial t} < A_{f_{\infty}}^{(1)} (\nabla g) \cdot \nabla g >_{t,\Lambda} + < \left( A_{f_{\infty}}^{(1)} (\nabla g) \right) \cdot \nabla g >_{t,\Lambda}
= < \left( \frac{\partial}{\partial t} + A_g \right) g >_{t,\Lambda} ; \]

By induction it is easy to see that
\[ \theta^{(n)}_{\Lambda}(t) = < \left( \frac{\partial}{\partial t} + A_g \right)^{n-1} g >_{t,\Lambda} = < H_g^{(n-1)} g >_{t,\Lambda} \quad (\forall n \geq 1) \]

Next, we propose to find a simpler formula for \( \theta^{(n)}_{\Lambda}(t) \) that only involves \( A_g \).

\[ H_g g = A_{f_{\infty}}^{(1)} (\nabla g) \cdot \nabla g \]
\[ = A_g g \]
\[ H_g^2 g = \frac{\partial}{\partial t} \nabla f \cdot \nabla g + \left( A_{f_{\infty}}^{(1)} \nabla \left( A_{f_{\infty}}^{(1)} (\nabla g) \cdot \nabla g \right) \right) \cdot \nabla g \]  
(69)

where \( f \) satisfies the equation
\[ \nabla f = A_{f_{\infty}}^{(1)} (\nabla g) . \]  
(70)
With \( \mathbf{v}(t) = \nabla f \), as before, we get

\[
\frac{\partial}{\partial t} \nabla f \cdot \nabla g = A_{\phi t}^{(1)-1} (\text{Hess} \mathbf{v}(t) + \nabla g \cdot \nabla \mathbf{v}(t)) \cdot \nabla g
\]

and \( H_g^2 \) becomes

\[
H_g^2 = A_{\phi t}^{(1)-1} \left[ (\text{Hess} \mathbf{v}(t) + \nabla g \cdot \nabla \mathbf{v}(t)) + \nabla \left( A_{\phi t}^{(1)} (\nabla g) \cdot \nabla g \right) \right] \cdot \nabla g
\]

\[
= A_{\phi t}^{(1)-2} 2 \nabla (A_g g) \cdot \nabla g
\]

\[
= 2 A_g^2 g.
\]

**Proposition 7** If

\[
\theta_\Lambda(t) = \log \left[ \int dx e^{-\Phi^t(x)} \right]
\]

where

\[
\Phi^t(x) = \Phi_\Lambda(x) - tg(x)
\]

is as above then \( \theta_\Lambda^{(n)}(t) \), the \( n \)th derivative of \( \theta_\Lambda(t) \) is given by the formula

\[
\theta_\Lambda^{(n)}(t) = \frac{(n)}{g > t, \Lambda},
\]

and for \( n \geq 1 \)

\[
\theta_\Lambda^{(n)}(t) = (n-1)! < A_{g}^{n-1} g > t, \Lambda.
\]

**Proof.** We have already established that

\[
\theta_\Lambda^{(n)}(t) = < H_n^{n-1} g > t, \Lambda \quad \text{for } n \geq 1.
\]

It then only remains to prove that

\[
H_n^{n-1} g = (n-1)! A_{g}^{n-1} g \quad \text{for } n \geq 1.
\]

The result is already established above for \( n = 1, 2, 3 \),. By induction, assume that

\[
H_n^{n-1} g = (n-1)! A_{g}^{n-1} g.
\]

if \( n \) is replaced by \( \tilde{n} \leq n \).

\[
H_{\tilde{n}}^{\tilde{n}} g = \left( \frac{\partial}{\partial t} + A_g \right) \left( (n-1)! A_{g}^{n-1} g \right)
\]

\[
= (n-1)! \left( \frac{\partial}{\partial t} A_{g}^{n-1} g + A_{g}^{n} g \right).
\]

Now

\[
A_{g}^{n-1} g = \left[ A_{\phi t}^{(1)-1} \nabla (A_{g}^{n-2} g) \right] \cdot \nabla g
\]

\[
= \nabla \varphi_n \cdot \nabla g
\]

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where
\[ \nabla \varphi_n = \left[ A_{q_t}^{(1)} \nabla \left( A^n g \right) \right]. \]

We obtain,
\[ \frac{\partial}{\partial t} \nabla \varphi_n = A_{q_t}^{(1)} \left( \frac{\partial}{\partial t} A^n g + \operatorname{Hess} g \nabla \varphi_n + \nabla g \cdot \nabla (\nabla \varphi_n) \right). \]

We then have
\[ \frac{\partial}{\partial t} A_{g}^{n-1} g = \frac{\partial}{\partial t} \nabla \varphi_n \cdot \nabla g \]
\[ = \left[ A_{q_t}^{(1)} \left( \frac{\partial}{\partial t} A^n g + \operatorname{Hess} g \nabla \varphi_n + \nabla \varphi_n \cdot \nabla (\nabla \varphi_n) \right) \right] \cdot \nabla g \]
\[ = A_g \left[ \frac{\partial}{\partial t} A^n g + A_g (A^n g) \right] \]
\[ = A_g \left( A^n g \right) \cdot \nabla g \]
\[ = A_g \left( \frac{1}{(n-2)!} H_g^{(n-2)} g \right) \quad \text{from the induction hypothesis} \]
\[ = \frac{1}{(n-2)!} A_g H_g^{(n-1)} g \]
\[ = \frac{1}{(n-2)!} A_g \left( (n-1)! A^n g \right) \quad \text{still by the induction hypothesis} \]
\[ = (n-1)! A^n g. \]

Thus,
\[ H^n g = (n - 1)! (n - 1 + 1) A^n g = n! A^n g. \]

\[ \textbf{Proposition 8} \quad \text{If } g(0) = 0, \text{ then the formula} \]
\[ \theta_{\lambda}^{(n)} (t) = (n-1)! < A^n g >_{\lambda, t}, \quad n \geq 2 \]
\[ \text{still holds if we no longer require } \Psi \text{ and } g \text{ to be compactly supported in } \mathbb{R}^\Lambda. \]

\[ \textbf{Proof.} \text{ As in } [8], \text{ consider the family cutoff functions} \]
\[ \chi = \chi_{\varepsilon} \quad \text{(71)} \]
\[ (\varepsilon \in [0, 1]) \text{ in } C^\infty (\mathbb{R}) \text{ with value in } [0, 1] \text{ such that} \]
\[ \left\{ \begin{array}{ll}
|\chi^{(k)} (t)| \leq C_k \varepsilon^k / |t|^k & \text{for } |t| \leq \varepsilon^{-1} \\
\end{array} \right. \text{ for } k \in \mathbb{N} \]

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We could take for instance
\[ \chi_\varepsilon(t) = f(\varepsilon \ln |t|) \]
for a suitable \( f \).

We then introduce
\[ \Psi_\varepsilon(x) = \chi_\varepsilon(|x|) \Psi, \quad x \in \mathbb{R}^\Lambda \tag{72} \]
and
\[ g_\varepsilon(x) = \chi_\varepsilon(|x|) g \quad x \in \mathbb{R}^\Gamma \tag{73} \]
One can check that both \( \Psi_\varepsilon(x) \) and \( g_\varepsilon(x) \) satisfies the assumptions made above on \( \Psi \) and \( g \). Now consider the equation
\[ -\Delta f_\varepsilon + \nabla \Phi^t \cdot \nabla f_\varepsilon = g_\varepsilon - < g_\varepsilon >_{\varepsilon, \Lambda}. \tag{74} \]
which implies
\[ (-\Delta + \nabla \Phi^t \cdot \nabla) \otimes v_\varepsilon + \text{Hess} \Phi^t v_\varepsilon = \nabla g_\varepsilon \tag{75} \]
where
\[ v_\varepsilon = \nabla f_\varepsilon \]
It was proved in [8] that \( v_\varepsilon = A_{\Phi^{-1}} \nabla g_\varepsilon \) converges in \( C^\infty \) to \( A_{\Phi^{-1}} \nabla g \) as \( \varepsilon \to 0 \).  

\textbf{Proposition 9} Let
\[ P_\Lambda(t) = \frac{\theta_\Lambda(t)}{|\Lambda|} \]
be the finite volume pressure.
Denote by \( a_n \) (\( n \geq 2 \)) the \( n \)th Taylor coefficient. We have
\[ a_n = \frac{< A_{g^{-1}} >_\Lambda}{n |\Lambda|} \]

\textbf{Remark 10} This formula for \( a_n \) gives a direction towards proving the analyticity of the pressure in the thermodynamic limit. In fact one only needs to provide a suitable \( C^n \) estimate for \( < A_{g^{-1}} >_\Lambda \).

\section{9 Some Consequences of the Formula for nth Derivative of the Pressure.}

In the following, we shall additionally assume that
\[ \nabla g(0) = 0, \quad \text{and} \]
\[ \nabla \Phi^t_{\Lambda}(0) = 0 \quad \text{for all } t \in [0, T). \]
When $n = 1$, we recall that $A^{0}_g g = g,$
\[
\theta^t_A = \langle g \rangle_{t,A}
\]
and if
\[
\nu(t) = \nabla f = A^{(1)}_{g}^{-1} \nabla g,
\]
then we have
\[
(-\Delta + \nabla \Phi^t_A \cdot \nabla) \otimes \nu(t) + \text{Hess} \Phi^t_A \nu(t) = \nabla g
\]
Again the tensor notation means that $(-\Delta + \nabla \Phi^t_A \cdot \nabla)$ acts diagonally on the components of $\nu(t)$.
As in [8] $\nu(t)$ is a solution of the equation
\[
g = \langle g \rangle_{t,A} + \nu(t) \cdot \Phi^t_A + \text{div} \nu(t).
\]
Using the assumptions above, we have
\[
\theta^t_A = \langle g \rangle_{t,A} + \text{div} \nu(t)(0).
\]
Similarly, the formula
\[
\theta^{(n)}_A = (n-1)! \langle A^{n-1}_g g \rangle_{t,A},
\]
implies that
\[
\theta^{(n)}_A = (n-1)! \text{div} \nu_n(t)(0),
\]
where
\[
\nu_n(t) = A^{(1)}_{g}^{-1} \nabla (A^{n-1}_g g).
\]

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