Forward and Reverse Entropy Power Inequalities in Convex Geometry

Mokshay Madiman, James Melbourne, Peng Xu*

Abstract

The entropy power inequality, which plays a fundamental role in information theory and probability, may be seen as an analogue of the Brunn-Minkowski inequality. Motivated by this connection to Convex Geometry, we survey various recent developments on forward and reverse entropy power inequalities not just for the Shannon-Boltzmann entropy but also more generally for Rényi entropy. In the process, we discuss connections between the so-called functional (or integral) and probabilistic (or entropic) analogues of some classical inequalities in geometric functional analysis.

Contents

1 Introduction

2 Entropy Power Inequalities

2.1 Some Basic Observations

2.2 The Shannon-Stam EPI and its variants

2.2.1 The Basic EPI

2.2.2 Fancier versions of the EPI

2.3 Rényi Entropy Power inequalities

2.3.1 First Rényi interpolation of the EPI and BMI

2.3.2 Second Rényi interpolation of the EPI and BMI

2.3.3 A conjectured Rényi EPI

2.3.4 Other work on Rényi entropy power inequalities

2.4 An EPI for Rényi entropy of order $\infty$

3 Reverse Entropy Power Inequalities

3.1 $\kappa$-concave measures and functions

3.2 Positional Reverse EPI’s for Rényi entropies

3.3 Reverse $\infty$-EPI via a generalization of K. Ball’s bodies

3.3.1 Busemann’s theorem for convex bodies

3.3.2 A Busemann-type theorem for measures

3.3.3 Busemann-type theorems for other Rényi entropies

3.4 Reverse EPI via Rényi entropy comparisons

*All the authors are with the Department of Mathematical Sciences, University of Delaware. E-mail: madiman@udel.edu, jamesm@udel.edu, xpeng@udel.edu. This work was supported in part by the U.S. National Science Foundation through grants DMS-1409504 (CAREER) and CCF-1346564. Some of the new results described in Section 3 were announced at the 2016 IEEE International Symposium on Information Theory [166] in Barcelona.
1 Introduction

The Brunn-Minkowski inequality plays a fundamental role not just in Convex Geometry, where it originated over 125 years ago, but also as an indispensable tool in Functional Analysis, and– via its connections to the concentration of measure phenomenon– in Probability. The importance of this inequality, and the web of its tangled relationships with many other interesting and important inequalities, is beautifully elucidated in the landmark 2002 survey of Gardner [70]. Two of the parallels that Gardner discusses in his survey are the Prékopa-Leindler inequality and the Entropy Power Inequality; since the time that the survey was written, these two inequalities have become the foundation and prototypes for two different but related analytic “liftings” of Convex Geometry. While the resulting literature is too vast for us to attempt doing full justice to in this survey, we focus on one particular strain of research– namely, the development of reverse entropy power inequalities– and using that as a narrative thread, chart some of the work that has been done towards these “liftings”.

Let $A, B$ be any nonempty Borel sets in $\mathbb{R}^d$. Write $A + B = \{x + y : x \in A, y \in B\}$ for the Minkowski sum, and $|A|$ for the $d$-dimensional volume (or Lebesgue measure) of $A$. The Brunn-Minkowski inequality (BMI) says that

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}. \quad (1)$$

The BMI was proved in the late 19th century by Brunn for convex sets in low dimension ($d \leq 3$), and Minkowski for convex sets in $\mathbb{R}^d$; the reader may consult Kjeldsen [86, 87] for an interesting historical analysis of how the notion of convex sets in linear spaces emerged from these efforts (Minkowski’s in particular). The extension of the BMI to compact– and thence Borel-measurable– subsets of $\mathbb{R}^d$ was done by Lusternik [99]. Equality holds in the inequality (1) for sets $A$ and $B$ with positive volumes if and only if they are convex and homothetic (i.e., one is a scalar multiple of the other, up to translation), possibly with sets of measure zero removed from each one. As of today, there are a number of simple and elegant proofs known for the BMI.

In the last few decades, the BMI became the starting point of what is sometimes called the Brunn-Minkowski theory, which encompasses a large and growing range of geometric inequalities including the Alexandrov-Fenchel inequalities for mixed volumes, and which has even developed important offshoots such as the $L^p$-Brunn-Minkowski theory [100]. Already in the study of the geometry of convex bodies (i.e., convex compact sets with nonempty interior), the study of log-concave functions turns out to be fundamental. One way to see this is to observe that uniform measures on convex bodies are not closed under taking lower-dimensional marginals, but yield log-concave densities, which do have such a closure property– while the closure property of log-concave functions under marginalization goes back to Prékopa.
and Brascamp-Lieb [38], their consequent fundamental role in the geometry of convex bodies was first clearly recognized in the doctoral work of K. Ball [10] (see also [11, 126]). Since then, the realization has grown that it is both possible and natural to state many questions and theorems in Convex Geometry directly for the category of log-concave functions or measures rather than for the category of convex bodies—V. Milman calls this the “Geometrization of Probability” program [125], although one might equally well call it the “Probabilization of Convex Geometry” program. The present survey squarely falls within this program.

For the goal of embedding the geometry of convex sets in a more analytic setting, two approaches are possible:

1. **Functional (integral) lifting**: Replace sets by functions, and convex sets by log-concave or $s$-concave functions, and the volume functional by the integral. This is a natural extension because if we identify a convex body $K$ with its indicator function $1_K$ (defined as being 1 on the set and 0 on its complement), then the integral of $1_K$ is just the volume of $K$. The earlier survey of V. Milman [125] is entirely focused on this lifting of Convex Geometry; recent developments since then include the introduction and study of mixed integrals (analogous to mixed volumes) independently by Milman-Rotem [121, 120] and Bobkov-Colesanti-Fragala [31] (see also [17]). Colesanti [47] has an up-to-date survey of these developments in another chapter of this volume.

2. **Probabilistic (entropic) lifting**: Replace sets by random variables (or strictly speaking their distributions), and convex sets by random variables with log-concave or $s$-concave distributions, and the volume functional by the entropy functional (actually “entropy power”, which we will discuss shortly). This is a natural analogue because if we identify a convex body $K$ with the random variable $U_K$ whose distribution is uniform measure on $K$, then the entropy of $U_K$ is the logarithm of $|K|$. The parallels were observed early by Costa and Cover [51] (and perhaps also implicitly by Lieb [96]); subsequently this analogy has been studied by many other authors, including by Dembo-Cover-Thomas [56] and in two series of papers by Lutwak-Yang-Zhang (see, e.g., [102, 101]) and Bobkov-Madiman (see, e.g., [24, 26]).

While this paper is largely focused on the probabilistic (entropic) lifting, we will also discuss how it is related to the functional (integral) lifting.

It is instructive at this point to state the integral and entropic liftings of the Brunn-Minkowski inequality itself, which are known as the Prékopa-Leindler inequality and the Entropy Power Inequality respectively.

**Prékopa-Leindler inequality (PLI)**: The Prékopa-Leindler inequality (PLI) [131, 92, 132] states that if $f, g, h : \mathbb{R}^d \to [0, \infty)$ are integrable functions satisfying, for a given $\lambda \in (0, 1)$,

$$h(\lambda x + (1 - \lambda)y) \geq f^\lambda(x)g^{1-\lambda}(y)$$

for every $x, y \in \mathbb{R}^d$, then

$$\int h \geq \left( \int f \right)^\lambda \left( \int g \right)^{1-\lambda}.$$  \hspace{1cm} (2)

If one prefers, the PLI can also be written more explicitly as a kind of convolution inequality, as implicitly observed in [38] and explicitly in [88]. Indeed, if one defines the Asplund product
of two nonnegative functions by
\[(f \ast g)(x) = \sup_{x_1 + x_2 = x} f(x_1)g(x_2),\]
and the scaling \((\lambda \cdot f)(x) = f^\lambda(x/\lambda)\), then the left side of (2) can be replaced by the integral of \([\lambda \cdot f] \ast [(1 - \lambda) \cdot g]\).

To see the connection with the BMI, one simply has to observe that \(f = 1_A, g = 1_B\) and \(h = 1_{\lambda A + (1 - \lambda)B}\) satisfy the hypothesis, and in this case, the conclusion is precisely the BMI in its "geometric mean" form \(|\lambda A + (1 - \lambda)B| \geq |A|^\lambda |B|^{1-\lambda}\). The equivalence of this inequality to the BMI in the form (1) is just one aspect of a broader set of equivalences involving the BMI. To be precise, for the class of Borel-measurable subsets of \(\mathbb{R}^d\), the following are equivalent:

\[
|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}} \tag{3}
\]

\[
|\lambda A + (1 - \lambda)B| \geq \left(\lambda |A|^{\frac{1}{d}} + (1 - \lambda) |B|^{\frac{1}{d}}\right)^d \tag{4}
\]

\[
|\lambda A + (1 - \lambda)B| \geq |A|^\lambda |B|^{1-\lambda} \tag{5}
\]

\[
|\lambda A + (1 - \lambda)B| \geq \min\{ |A|, |B| \}. \tag{6}
\]

Let us indicate why the inequalities (3)–(6) are equivalent. Making use of the arithmetic mean-geometric mean inequality, we immediately have (4) \(\Rightarrow\) (5) \(\Rightarrow\) (6). Applying (3) to \(\tilde{A} = \lambda A, \tilde{B} = (1 - \lambda)B\) we have

\[
|\lambda A + (1 - \lambda)B| = |\tilde{A} + \tilde{B}|
\geq (|\tilde{A}|^{\frac{1}{d}} + |\tilde{B}|^{\frac{1}{d}})^d
\]

\[
= \left( |\lambda A|^{\frac{1}{d}} + |(1 - \lambda)B|^{\frac{1}{d}} \right)^d
\]

\[
= \left( \lambda |A|^{\frac{1}{d}} + (1 - \lambda) |B|^{\frac{1}{d}} \right)^d,
\]

where the last equality is by homogeneity of the Lebesgue measure. Thus (3) \(\Rightarrow\) (4). It remains to prove that (6) \(\Rightarrow\) (3). First notice that (6) is equivalent to

\[
|A + B| \geq \min\{ |A|/\lambda^d, |B|/(1 - \lambda)^d \}
\]

\[
= \min\{ |A|/\lambda^d, |B|/(1 - \lambda)^d \}.
\]

It is easy to see that the right hand side is maximized when \(|A|/\lambda^d = |B|/(1 - \lambda)^d\), or

\[
\lambda = \frac{|A|^{\frac{1}{d}}}{|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}},
\]

Inserting \(\lambda\) into the above yields (3).

**Entropy Power Inequality (EPI):** In order to state the Entropy Power Inequality (EPI), let us first explain what is meant by entropy power. When random variable \(X = (X_1, \ldots, X_d)\) has density \(f(x)\) on \(\mathbb{R}^d\), the entropy of \(X\) is

\[
h(X) = h(f) := -\int_{\mathbb{R}^d} f(x) \log f(x) dx = \mathbb{E}[-\log f(X)]. \tag{7}
\]
This quantity is sometimes called the Shannon-Boltzmann entropy or the differential entropy (to distinguish it from the discrete entropy functional that applies to probability distributions on a countable set). The entropy power of $X$ is $N(X) = e^{2h(X)/d}$. As is usual, we abuse notation and write $h(X)$ and $N(X)$, even though these are functionals depending only on the density of $X$ and not on its random realization. The entropy power $N(X) \in [0, \infty]$ can be thought of as a “measure of randomness”. It is an (inexact) analogue of volume: if $U_A$ is uniformly distributed on a bounded Borel set $A$, then it is easily checked that $h(U_A) = \log |A|$ and hence $N(U_A) = |A|^{2/d}$. The reason we don’t define entropy power by $e^{h(X)}$ (which would yield a value of $|A|$ for the entropy power of $U_A$) is that the “correct” comparison is not to uniforms but to Gaussians. This is because just as Euclidean balls are special among subsets of $\mathbb{R}^d$, Gaussians are special among distributions on $\mathbb{R}^d$. Indeed, the reason for the appearance of the functional $|A|^{1/d}$ in the BMI is because this functional is (up to a universal constant) the radius of the ball that has the same volume as $A$, i.e., $|A|^{1/d}$ may be thought of as (up to a universal constant) the “effective radius” of $A$. To develop the analogy for random variables, observe that when $Z \sim N(0, \sigma^2 I)$ (i.e., $Z$ has the Gaussian distribution with mean 0 and covariance matrix that is a multiple of the identity), the entropy power of $Z$ is $N(Z) = (2\pi \sigma^2)^{d/2}$. Thus the entropy power of $X$ is (up to a universal constant) the variance of the isotropic normal that has the same entropy as $X$, i.e., if $Z \sim N(0, \sigma_Z^2 I)$ and $h(Z) = h(X)$, then

$$N(X) = N(Z) = (2\pi \sigma_Z^2).$$

Looked at this way, entropy power is the “effective variance” of a random variable, exactly as volume raised to $1/d$ is the effective radius of a set.

The EPI states that for any two independent random vectors $X$ and $Y$ in $\mathbb{R}^d$ such that the entropies of $X, Y$ and $X + Y$ exist,

$$N(X + Y) \geq N(X) + N(Y).$$

The EPI was stated by Shannon [145] with an incomplete proof; the first complete proof was provided by Stam [148]. The EPI plays an extremely important role in the field of Information Theory, where it first arose and was used (first by Shannon, and later by many others) to prove statements about the fundamental limits of communication over various models of communication channels. Subsequently it has also been recognized as an extremely useful inequality in Probability Theory, with close connections to the logarithmic Sobolev inequality for the Gaussian distribution as well as to the Central Limit Theorem. We will not further discuss these other motivations for the study of the EPI in this paper, although we refer the interested reader to [85, 105] for more on the connections to central limit theorems.

It should be noted that one insightful way to compare the BMI and EPI is to think of the latter as a “99% analogue in high dimensions” of the former, in the sense that looking at most of the Minkowski sum of the supports of a large number of independent copies of the two random vectors effectively yields the EPI via a simple instance of the asymptotic equipartition property or Shannon-McMillan-Breiman theorem. A rigorous argument is given by Szarek and Voiculescu [153] (building on [152]), a short intuitive explanation of which can be found in an answer of Tao to a MathOverflow question. The key idea of [153] is to use not the usual BMI but a “restricted” version of it where it is the exponent $2/d$ rather than $1/d$ that

---

1See [http://mathoverflow.net/questions/167951/entropy-proof-of-brunn-minkowski-inequality](http://mathoverflow.net/questions/167951/entropy-proof-of-brunn-minkowski-inequality).
Rényi entropies. Unified proofs can be given of the EPI and the BMI in two different ways, both of which may be thought of as providing extensions of the EPI to Rényi entropy. We will discuss both of these later; for now, we only introduce the notion of Rényi entropy. For a $\mathbb{R}^d$-valued random variable $X$ with probability density function $f$, define its Rényi entropy of order $p$ (or simply $p$-Rényi entropy) by

$$h_p(X) = h_p(f) := \frac{1}{1-p} \log \left( \int_{\mathbb{R}^d} f^p(x) dx \right),$$

(8)

if $p \in (0,1) \cup (1, \infty)$. Observe that, defining $h_1$ “by continuity” and using l’Hospital’s rule, $h_1(X) = h(X)$ is the (Shannon-Boltzmann) entropy. Moreover, by taking limits,

$$h_0(X) = \log |\text{supp}(f)|,$$

$$h_\infty(X) = -\log \|f\|_\infty,$$

where $\text{supp}(f)$ is the support of $f$ (i.e., the smallest closed set such that $f$ is zero outside it), and $\|f\|_\infty$ is the usual $L^\infty$-norm of $f$ (i.e., the essential supremum with respect to Lebesgue measure). We also define the $p$-Rényi entropy power by $N_p(X) = e^{2h_p(X)}$, so that the usual entropy power $N(X) = N_1(X)$ and for a random variable $X$ whose support is $A$, $N_0(X) = |A|^{2/d}$.

Conventions. Throughout this paper, we assume that all random variables considered have densities with respect to Lebesgue measure. While the entropy of $X$ can be meaningfully set to $-\infty$ when the distribution of $X$ does not possess a density, for the most part we avoid discussing this case. Also, when $X$ has probability density function $f$, we write $X \sim f$.

For real-valued functions $A, B$ we will use the notation $A \lessapprox B$ when $A(z) \leq CB(z)$ for some positive constant $C$ independent of $z$. For our purposes this will be most interesting when $A$ and $B$ are in some way determined by dimension.

Organization. This survey is organized as follows. In Section 2, we review various statements and variants of the EPI, first for the usual Shannon-Boltzmann entropy in Section 2.2 and then for $p$-Rényi entropy in Section 2.3, focusing on the $\infty$-Rényi case in Section 2.4. In Section 3, we explore what can be said about inequalities that go the other way, under convexity constraints on the probability measures involved. We start by recalling the notions of $\kappa$-concave measures and functions in Section 3.1. In Section 3.2, we discuss reverse EPI’s that require invoking a linear transformation (analogous to the reverse Brunn-Minkowski inequality of V. Milman), and explicit choices of linear transformations that can be used are discussed in Section 3.6. The three intermediate subsections focus on three different approaches to reverse Rényi EPI’s that do not require invoking a linear transformation. Finally we discuss the relationship between integral and entropic liftings, in the context of the Blaschke-Santaló inequality in Section 4, and end with some concluding remarks on nonlinear and discrete analogs in Section 5.

---

2We mention in passing that Barthe [16] also proved a restricted version of the PLI. An analogue of “restriction” for the EPI would involve some kind of weak dependence between summands; some references to the literature on this topic are given later.
2 Entropy Power Inequalities

2.1 Some Basic Observations

Before we discuss more sophisticated results, let us recall some basic properties of Rényi entropy.

**Theorem 2.1.** For independent \( \mathbb{R}^d \)-valued random variables \( X \) and \( Y \), and any \( p \in [0, \infty] \),

\[
N_p(X + Y) \geq \max\{N_p(X), N_p(Y)\}.
\]

**Proof.** Let \( X \sim f \) and \( Y \sim g \). For \( p \in (1, \infty) \), we have the following with the inequality delivered by Jensen’s inequality:

\[
\int (f * g)^p(x)dx = \int \left( \int f(x - y)g(y)dy \right)^p dx 
\leq \int \int f^p(x - y)g(y)dydx 
= \int \left( \int f^p(x - y)dx \right) g(y)dy 
= \int f^p(x)dx.
\]

Inserting the inequality into the order reversing function \( \varphi(z) = z^{\frac{2}{p(1-p)}} \) we have our result.

The case that \( p \in (0,1) \) is similar, making note that now \( z^p \) is concave while \( z^{\frac{2}{p(1-p)}} \) is order preserving. For \( p = 1 \), we can give a probabilistic proof: applying the nonnegativity of mutual information, which in particular implies that conditioning reduces entropy (see, e.g., [54]),

\[
h(X + Y) \geq h(X + Y|Y) = h(X|Y) = h(X),
\]

where we used translation-invariance of entropy for the first equality and independence of \( X \) and \( Y \) for the second. For \( p = 0 \), the conclusion simply follows by the fact that \( |A + B| \geq \max\{|A|,|B|\} \) for any nonempty Borel sets \( A \) and \( B \); this may be seen by translating \( B \) so that it contains 0, which does not affect any of the volumes and in which case \( A + B \supset A \).

For \( p = \infty \), the conclusion follows from Hölder’s inequality:

\[
\int f(x - y)g(y)dy \leq \|g\|_\infty \|f\|_1 = \|g\|_\infty.
\]

Thus we have the theorem for all values of \( p \in [0, \infty] \). \( \square \)

We now observe that for any fixed random vector, the Rényi entropy of order \( p \) is non-increasing in \( p \).

**Lemma 2.2.** For a \( \mathbb{R}^d \)-valued random variable \( X \), and \( 0 \leq q < p \leq \infty \), we have

\[
N_q(X) \geq N_p(X).
\]

**Proof.** The result follows by expressing, for \( X \sim f \),

\[
h_p(X) = \frac{\log(\int f^p)}{1-p} = -\log \mathbb{E}\|f(X)\|_{p-1}
\]

and using the “increasingness” of \( p \)-norms on probability spaces, which is nothing but an instance of Hölder’s inequality. \( \square \)
**Definition 2.3.** A function \( f : \mathbb{R}^d \to [0, \infty) \) is said to be log-concave if

\[
f(\alpha x + (1 - \alpha)y) \geq f(x)^\alpha f(y)^{1-\alpha},
\]

for each \( x, y \in \mathbb{R}^d \) and each \( 0 \leq \alpha \leq 1 \).

If a probability density function \( f \) is log-concave, we will also use the adjective “log-concave” for a random variable \( X \) distributed according to \( f \), and for the probability measure induced by it. Log-concavity has been deeply studied in probability, statistics, optimization and geometry, and is perhaps the most natural notion of convexity for probability density functions.

In general, the monotonicity of Lemma 2.2 relates two different Rényi entropies of the same distribution in one direction, but there is no reason for a bound to exist in the other direction. Remarkably, for log-concave random vectors, all Rényi entropies are comparable in both directions.

**Lemma 2.4.** ([115]) If a random variable \( X \) in \( \mathbb{R}^d \) has log-concave density \( f \), then for \( p \geq q > 0 \),

\[
h_q(f) - h_p(f) \leq d \log \frac{q}{q-1} - d \log \frac{p}{p-1},
\]

with equality if \( f(x) = e^{-\sum_{i=1}^{d} x_i} \) on the positive orthant and 0 elsewhere.

This lemma generalizes the following sharp inequality for log-concave distributions obtained in [25]:

\[
h(X) \leq d + h_{\infty}(X).
\]

In fact, Lemma 2.4 has an extension to the larger class (discussed later) of \( s \)-concave measures with \( s < 0 \); preliminary results in this direction are available in [25] and sharp results obtained in [22].

### 2.2 The Shannon-Stam EPI and its variants

#### 2.2.1 The Basic EPI

The EPI has several equivalent formulations; we collect these together with minimal conditions below.

**Theorem 2.5.** Suppose \( X \) and \( Y \) are independent \( \mathbb{R}^d \)-valued random variables such that \( h(X), h(Y) \) and \( h(X+Y) \) exist. Then the following statements, which are equivalent to each other, are true:

1. We have

\[
N(X + Y) \geq N(X) + N(Y).
\]

2. For any \( \lambda \in [0, 1] \),

\[
h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda h(X) + (1 - \lambda)h(Y).
\]
3. Denoting by $X^G$ and $Y^G$ independent, isotropic\(^3\), Gaussian random variables with $h(X^G) = h(X)$ and $h(Y^G) = h(Y)$, one has

$$h(X + Y) \geq h(X^G + Y^G).$$

(13)

In each case, equality holds if and only if $X$ and $Y$ are Gaussian random variables with proportional covariance matrices.

Proof. First let us show that we can assume $h(X), h(Y) \in (-\infty, \infty)$. By Theorem 2.1 we can immediately obtain $h(X + Y) \geq \max\{h(X), h(Y)\}$. It follows that all three inequalities hold immediately in the case that $\max\{h(X), h(Y)\} = \infty$. Now assume that neither $h(X)$ nor $h(Y)$ take the value $+\infty$ and consider $\min\{h(X), h(Y)\} = -\infty$. In this situation, the inequalities (11) and (12) are immediate. For (13), in the case that $h(X) = -\infty$ we interpret $X^G$ as a Dirac point mass, and hence $h(X^G + Y^G) = h(Y^G) = h(Y) \leq h(X + Y)$.

We now proceed to prove the equivalences.

(11) $\Rightarrow$ (12): Apply (11), substituting $X$ by $\sqrt{X}/X$ and $Y$ by $\sqrt{1 - X}Y$ and use the homogeneity of entropy power to obtain

$$N(\sqrt{X}/X + \sqrt{1 - X}Y) \geq \lambda N(X) + (1 - \lambda)N(Y).$$

Apply the AM-GM inequality to the right hand side and conclude by taking logarithms.

(12) $\Rightarrow$ (13): Applying (12) in its exponentiated form $N(\sqrt{X}/X + \sqrt{1 - X}Y) \geq N^\lambda(X)N^{1 - \lambda}(Y)$ after writing $X + Y = \sqrt{X}/X + \sqrt{1 - X}Y/\sqrt{1 - X}$ we obtain

$$N(X + Y) \geq \left(\frac{X}{\sqrt{X}}\right)^\lambda \left(\frac{Y}{\sqrt{1 - X}}\right)^{1 - \lambda}.$$  

Making use of the identity $N(X^G + Y^G) = N(X^G) + N(Y^G)$ and homogeneity again, we can evaluate the right hand side at $\lambda = N(X^G)/N(X^G + Y^G)$ to obtain exactly $N(X^G + Y^G)$, recovering the exponentiated version of (13).

(13) $\Rightarrow$ (11): Using the exponentiated version of (13),

$$N(X + Y) \geq N(X^G + Y^G) = N(X^G) + N(Y^G) = N(X) + N(Y).$$

Observe from the proof that a strict inequality in one statement implies a strict inequality in the rest.

What is left is to prove any of the 3 statements of the EPI when the entropies involved are finite. There are many proofs of this available in the literature (see, e.g., [148, 20, 96, 56, 153, 134]), and we will not detail any here, although we later sketch a proof via the sharp form of Young’s convolution inequality.

\(^3\)By isotropic here, we mean spherical symmetry, or equivalently, that the covariance matrix is taken to be a scalar multiple of the identity matrix.
not (with $X'$ an i.i.d. copy of $X$), necessarily $h(X) = -\infty$, so that it remains true that if the entropy exists and is a real number, then the entropy of the self-convolution also exists. They also have other interesting examples of the behavior of entropy on convolution: [30, Example 1] constructs a distribution with entropy $-\infty$ such that that the entropy of the self-convolution is a real number, and [30, Proposition 5] constructs a distribution with finite entropy such that its convolution with any distribution of finite entropy has infinite entropy.

### 2.2.2 Fancier versions of the EPI

Many generalizations and improvements of the EPI exist. For three or more independent random vectors $X_i$, the EPI trivially implies that

$$N(X_1 + \cdots + X_n) \geq \sum_{i=1}^{n} N(X_i), \tag{14}$$

with equality if and only if the random vectors are Gaussian and their covariance matrices are proportional to each other. In fact, it turns out that this can be refined, as shown by S. Artstein, K. Ball, Barthe and Naor [4]:

$$N \left( \sum_{i=1}^{n} X_i \right) \geq \frac{1}{n-1} \sum_{j=1}^{n} N \left( \sum_{i \neq j} X_i \right). \tag{15}$$

This implies the monotonicity of entropy in the Central Limit Theorem, which suggests that quantifying the Central Limit Theorem using entropy or relative entropy is a particularly natural approach. More precisely, if $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) square-integrable random vectors, then

$$h \left( \frac{X_1 + \cdots + X_n}{\sqrt{n}} \right) \leq h \left( \frac{X_1 + \cdots + X_{n-1}}{\sqrt{n-1}} \right). \tag{16}$$

Simpler proofs of (15) were given independently by [106, 146, 160]. Generalizations of (15) to arbitrary collections of subsets on the right side was given by [107, 108], and some further fine properties of the kinds of inequalities that hold for the entropy power of a sum of independent random variables were revealed in [109]. Let us mention a key result of this type due to [108]. For a collection $C$ of nonempty subsets of $[n] := \{1, 2, \ldots, n\}$, a function $\beta : C \to \mathbb{R}_+$ is called a fractional partition if for each $i \in [n]$, we have $\sum_{s \in C, i \in s} \beta_s = 1$. Then the entropy power of convolutions is fractionally superadditive, i.e., if $X_1, \ldots, X_n$ are independent $\mathbb{R}^d$-valued random variables, one has

$$N \left( \sum_{i=1}^{n} X_i \right) \geq \sum_{s \in C} \beta_s N \left( \sum_{i \in s} X_i \right).$$

This yields the usual EPI by taking $C$ to be the collection of all singletons and $\beta_s \equiv 1$, and the inequality (15) by taking $C$ to be the collection of all sets of size $n-1$ and $\beta_s \equiv \frac{1}{n-1}$.

For i.i.d. summands in dimension 1, [3] and [82] prove an upper bound of the relative entropy between the distribution of the normalized sum and that of a standard Gaussian random variable. To be precise, suppose $X_1, \ldots, X_n$ are independent copies of a random

---

4If there exists a fractional partition $\beta$ for $C$ that is $\{0, 1\}$-valued, then $\beta$ is the indicator function for a partition of the set $[n]$ using a subset of $C$; hence the terminology.
variable $X$ with $\text{Var}(X) = 1$, and the density of $X$ satisfies a Poincaré inequality with constant $c$, i.e., for every smooth function $s$,

$$c \text{Var}(s(X)) \leq E[(s'(X))^2].$$

Then, for every $a \in \mathbb{R}^n$ with $\sum_{i=1}^{n} a_i^2 = 1$ and $\alpha(a) := \sum_{i=1}^{n} a_i^4$,

$$h(G) - h\left(\sum_{i=1}^{n} a_iX_i\right) \leq \frac{\alpha(a)}{\frac{n}{2} + \left(1 - \frac{n}{2}\right) \alpha(a)} (h(G) - h(X)),$$

where $G$ is a standard Gaussian random variable. Observe that this refines the EPI since taking $c = 0$ in the inequality (17) gives the EPI in the second form of Theorem 2.5. On the other hand, specializing (17) to $n = 2$ with $a_1 = a_2 = \frac{1}{\sqrt{2}}$, one obtains a lower bound for $h\left(\frac{X_1 + X_2}{\sqrt{2}}\right) - h(X)$ in terms of the relative entropy $h(G) - h(X)$ of $X$ from Gaussianity. Ball and Nguyen [13] develop an extension of this latter inequality to general dimension under the additional assumption of log-concavity.

It is natural to ask if the EPI can be refined by introducing an error term that quantifies the gap between the two sides in terms of how non-Gaussian the summands are. Such estimates are referred to as “stability estimates” since they capture how stable the equality condition for the inequality is, i.e., whether closeness to Gaussianity is guaranteed for the summands if the two sides in the inequality are not exactly equal but close to each other.

For the EPI, the first stability estimates were given by Carlen and Soffer [42], but these are qualitative and not quantitative (i.e., they do not give numerical bounds on distance from Gaussianity of the summands when there is near-equality in the EPI, but they do assert that this distance must go to zero as the deficit in the inequality goes to zero). Recently Toscani [159] gave a quantitative stability estimate when the summands are restricted to have log-concave densities: For independent random vectors $X$ and $Y$ with log-concave densities,

$$N(X + Y) \geq (N(X) + N(Y)) R(X,Y),$$

where the quantity $R(X,Y) \geq 1$ is a somewhat complicated quantity that we do not define here and can be interpreted as a measure of non-Gaussianity of $X$ and $Y$. Indeed, [159] shows that $R(X,Y) = 1$ if and only if $X$ and $Y$ are Gaussian random vectors, but leaves open the question of whether $R(X,Y)$ can be related to some more familiar distance from Gaussianity. Even more recently, Courtade, Fathi and Pananjady [53] showed that if $X$ and $Y$ are uniformly log-concave (in the sense that the densities of both are of the form $e^{-V}$ with the Hessian of $V$ bounded from below by a positive multiple of the identity matrix), then the deficit in the EPI is controlled in terms of the quadratic Wasserstein distances between the distributions of $X$ and $Y$ and Gaussianity.

There are also strengthenings of the EPI when one of the summands is Gaussian. Set $X^{(t)} = X + \sqrt{t}Z$, with $Z$ a standard Gaussian random variable independent of $X$. Costa [50] showed that for any $t \in [0, 1]$,

$$N(X^{(t)}) \geq (1 - t)N(X) + tN(X + Z).$$

This may be rewritten as $N(X^{(t)} - N(X)) \geq t[N(X + Z) - N(X)] = N(\sqrt{t}X + \sqrt{t}Z) - N(\sqrt{t}X)$. Setting $\beta = \sqrt{t}$, we have for any $\beta \in [0, 1]$ that $N(X + \beta Z) - N(X) \geq N(\beta X + \beta Z) - N(\beta X)$, substituting $X$ by $\beta X$, we get

$$N(X + Z) - N(X) \geq N(\beta X + Z) - N(\beta X).$$
for any $\beta \in [0, 1]$. Therefore, for any $\beta, \beta' \in [0, 1]$ with $\beta > \beta'$, substitute $X$ by $\beta X$ and $\beta$ by $\beta' / \beta$ in (20), we have

$$N(\beta X + Z) - N(\beta X) \geq N(\beta' X + Z) - N(\beta' X).$$

In other words, Costa’s result states that if $A(\beta) = N(\beta X + Z) - N(\beta X)$, then $A(\beta)$ is a monotonically increasing function for $\beta \in [0, 1]$. To see that this is a refinement of the EPI in the special case when one summand is Gaussian, note that the EPI in this case is the statement that $A(1) \geq A(0)$. An alternative proof of Costa’s inequality was given by Villani [161]; for a generalization, see [128].

Very recently, a powerful extension of Costa’s inequality was developed by Courtade [52], applying to a system in which $X, X + Z, V$ form a Markov chain (i.e., $X$ and $V$ are conditionally independent given $X + Z$) and $Z$ is a Gaussian random vector independent of $X$. Courtade’s result specializes in the case where $V = X + Z + Y$ to the following: If $X, Y, Z$ be independent random vectors in $\mathbb{R}^d$ with $Z$ being Gaussian, then

$$N(X + Z)N(Y + Z) \geq N(X)N(Y) + N(X + Y + Z)N(Z). \tag{21}$$

Applying the inequality (21) to $X$, $\sqrt{1-t}Z'$ and $\sqrt{t}Z$ where $Z'$ is the independent copy of the standard normal distribution $Z$, we have

$$N(X(t))N(\sqrt{1-t}Z' + \sqrt{t}Z) \geq N(X)N(\sqrt{1-t}Z') + N(X + \sqrt{1-t}Z' + \sqrt{t}Z)N(\sqrt{t}Z).$$

By the fact that $\sqrt{1-t}Z' + \sqrt{t}Z$ has the same distribution as $Z$, and by the fact that $N(Z) = 1$, we have $N(X(t)) \geq (1-t)N(X) + tN(X + Z)$, which is Costa’s inequality (19).

Motivated by the desire to prove entropic central limit theorems for statistical physics models, some extensions of the EPI to dependent summands have also been considered (see, e.g., [42, 154, 155, 79, 80]), although the assumptions tend to be quite restrictive for such results.

Finally there is an extension of the EPI that applies not just to sums but to more general linear transformations applied to independent random variables. The main result of Zamir and Feder [171] asserts that if $X_1, \ldots, X_n$ are independent real-valued random variables, $Z_1, \ldots, Z_n$ are independent Gaussian random variables satisfying $h(Z_i) = h(X_i)$, and $A$ is any matrix, then $h(AX) \geq h(AZ)$ where $AX$ represents the left-multiplication of the vector $X$ by the matrix $A$. As explained in [171], for this result to be nontrivial, the $m \times n$ matrix $A$ must have $m < n$ and be of full rank. To see this, notice that if $m > n$ or if $A$ is not of full rank, the vector $AX$ does not have full support on $\mathbb{R}^m$ and $h(AZ) = -\infty$, while if $m = n$ and $A$ is invertible, $h(AX) = h(AZ)$ holds with equality because of the conditions determining $Z$ and the way entropy behaves under linear transformations.

### 2.3 Rényi Entropy Power inequalities

#### 2.3.1 First Rényi interpolation of the EPI and BMI

Unified proofs can be given of the EPI and the BMI in different ways, each of which may be thought of as providing extensions of the EPI to Rényi entropy.

The first unified approach is via Young’s inequality. Denote by $L^p$ the Banach space $L^p(\mathbb{R}, dx)$ of measurable functions defined on $\mathbb{R}$ whose $p$-th power is integrable with respect to Lebesgue measure $dx$. In 1912, Young [167] introduced the fundamental inequality

$$\|fg\|_r \leq \|f\|_p\|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 < p, q, r < +\infty, \tag{22}$$
for functions $f \in L^p$ and $g \in L^q$, which implies that if two functions are in (possibly different) $L^p$-spaces, then their convolution is contained in a third $L^q$-space. In 1972, Leindler [91] showed the so-called reverse Young inequality, referring to the fact that the inequality (22) is reversed when $0 < p, q, r < 1$. The best constant that can be put on the right side of (22) or its reverse was found by Beckner [18]: the best constant is $(C_pC_q/C_r)^d$, where

$$C_p^2 = \frac{p^d}{|p'|^d},$$

(23)

and for any $p \in (0, \infty]$, $p'$ is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$  

(24)

Note that $p'$ is positive for $p \in (1, \infty)$, and negative for $p \in (0, 1)$. Alternative proofs of both Young’s inequality and the reverse Young inequality with this sharp constant were given by Brascamp and Lieb [37], Barthe [15], and Cordero-Erausquin and Ledoux [48].

We state the sharp Young and reverse Young inequalities now for later reference.

**Theorem 2.6.** [18] Suppose $r \in (0, 1)$ and $p_i \in (0, 1)$ satisfy

$$\sum_{i=1}^n \frac{1}{p_i} = n - \frac{1}{r'}.$$  

(25)

Then, for any functions $f_j \in L^{p_j}$ ($j = 1, \ldots, n$),

$$\left\|\ast_{j \in [n]} f_j\right\|_r \geq \frac{1}{C_r^d} \prod_{j \in [n]} \left[ C_{p_j}^d \|f_j\|_{p_j}\right].$$  

(26)

The inequality is reversed if $r \in (1, \infty)$ and $p_i \in (1, \infty)$.

Dembo, Cover and Thomas [56] interpret the Young and reverse Young inequalities with sharp constant as EPI’s for the Rényi entropy. If $X_i$ are random vectors in $\mathbb{R}^d$ with densities $f_i$ respectively, taking the logarithm of (26) and rewriting the definition of the Rényi entropy power as $N_p(X) = \|f\|_p^{2p/d}$, we have

$$\frac{d}{2pr} \log N_r \left( \sum_{i \in [n]} X_i \right) \leq d \log C_r - d \sum_{i \in [n]} \log C_p + \sum_{i \in [n]} \frac{d}{2p'_i} \log N_{p'_i}(X_i).$$  

(27)

Introduce two discrete probability measures $\lambda$ and $\kappa$ on $[n]$, with probabilities proportional to $1/p'_i$ and $1/p_i$ respectively. Setting $L_r = rn - r + 1 = r(n - 1/r')$, the condition (25), allows us to write explicitly

$$\lambda_i = \left( \frac{r}{L_r} \right) \frac{1}{p_i},$$

$$\kappa_i = \frac{r'}{p'_i},$$

for each $i \in [n]$, also using $1/p_i + 1/p'_i = 1$ for the latter. Then (27) reduces to

$$h_r(Y_{[n]}) \geq \frac{dr'}{2} \log C_r^2 - \frac{dr'}{2} \sum_{i \in [n]} \log C_{p_i}^2 + \sum_{i \in [n]} \lambda_i h_{p_i}(X_i).$$

Now, some straightforward calculations show that if we take the limit as $p_i, r \to 0$ from above, we get the BMI, while if we take the limit as $p_i, r \to 1$, we get the EPI (this was originally observed by Lieb [96]).
2.3.2 Second Rényi interpolation of the EPI and BMI

Wang and Madiman [162] found a rearrangement-based refinement of the EPI that also applies to Rényi entropies. For a Borel set $A$, define its spherically decreasing symmetric rearrangement $A^*$ by

$$A^* := B(0, r),$$

where $B(0, r)$ stands for the open ball with radius $r$ centered at the origin and $r$ is determined by the condition that $B(0, r)$ has volume $|A|$. Here we use the convention that if $|A| = 0$ then $A^* = \emptyset$ and that if $|A| = \infty$ then $A^* = \mathbb{R}^d$. Now for a measurable non-negative function $f$, define its spherically decreasing symmetric rearrangement $f^*$ by

$$f^*(y) := \int_0^\infty 1_{\{y \in B_t^*\}} dt,$$

where $B_t := \{x : f(x) > t\}$. It is a classical fact (see, e.g., [39]) that rearrangement preserves $L^p$-norms, i.e., $\|f\|_p = \|f^*\|_p$. In particular, if $f$ is a probability density function, so is $f^*$. If $X \sim f$, denote by $X^*$ a random variable with density $f^*$; then the rearrangement-invariance of $L^p$-norms immediately implies that $h_p(X^*) = h_p(X)$ for each $p \in [0, \infty]$ (for $p = 1$, this is not done directly but via a limiting argument).

**Theorem 2.7.** [162] Let $X_1, \ldots, X_n$ be independent $\mathbb{R}^d$-valued random vectors. Then

$$h_p(X_1 + \ldots + X_n) \geq h_p(X_1^* + \ldots + X_n^*)$$

(28)

for any $p \in [0, \infty]$, provided the entropies exist.

In particular,

$$N(X + Y) \geq N(X^* + Y^*),$$

(29)

where $X$ and $Y$ are independent random vectors with density functions $f$ and $g$ respectively and $X^*$ and $Y^*$ are independent random vectors with density function $f^*$ and $g^*$ respectively. Thanks to (29), we have effectively inserted an intermediate term in between the two sides of the formulation (13) of the EPI:

$$N(X + Y) \geq N(X^* + Y^*) \geq N(X^G + Y^G),$$

where the second inequality is by the fact that $h(X^G) = h(X^*) = h(X)$, combined with the third equivalent form of the EPI in Theorem 2.5. In fact, it is also shown in [162] that the EPI itself can be deduced from (29).

2.3.3 A conjectured Rényi EPI

Let us note that neither of the above unifications of BMI and EPI via Rényi entropy directly gives a sharp bound on $N_p(X + Y)$ in terms of $N_p(X)$ and $N_p(Y)$. The former approach relates Rényi entropy powers of different indices, while the latter refines the third formulation in Theorem 2.1 (but not the first, because the equivalence that held for Shannon-Boltzmann entropy does not work in the Rényi case). The question of finding a sharp direct relationship between $N_p(X + Y)$ with $N_p(X)$ and $N_p(Y)$ remains open, with some non-sharp results for the $p > 1$ case obtained by Bobkov and Chistyakov [30], whose argument and results were recently tightened by Ram and Sason [133].
Theorem 2.8. \[133\] For \( p \in (1, \infty) \) and independent random vectors \( X_i \) with densities in \( \mathbb{R}^d \),

\[
N_p(X_1 + \cdots + X_n) \geq c_p^{(n)} \sum_{i=1}^{n} N_p(X_i),
\]

where \( p' = p/(p-1) \) and

\[
c_p^{(n)} = p^{1/(p-1)} \left( 1 - \frac{1}{np'} \right)^{np'-1} \geq \frac{1}{e}.
\]

We now discuss a conjecture of Wang and Madiman \[162\] about extremal distributions for Rényi EPI’s of this sort. Consider the one-parameter family of distributions, indexed by a parameter \(-\infty < \beta \leq \frac{2}{d+2}\), of the following form:

\[
g_0 \text{ is the standard Gaussian density in } \mathbb{R}^d,
\[
\text{and for } \beta \neq 0, \quad g_\beta(x) = A_\beta \left( 1 - \frac{\beta}{2} \|x\|^2 \right)^{\frac{\beta - d}{2} - 1},
\]

where \( A_\beta \) is a normalizing constant (which can be written explicitly in terms of gamma functions). We call \( g_\beta \) the standard generalized Gaussian of order \( \beta \); any affine function of a standard generalized Gaussian yields a “generalized Gaussian”. The densities \( g_\beta \) (apart from the obviously special value \( \beta = 0 \)) are easily classified into two distinct ranges where they behave differently. First, for \( \beta < 0 \), the density is proportional to a negative power of \((1 + b\|x\|^2)\) for a positive constant \( b \), and therefore correspond to measures with full support on \( \mathbb{R}^d \) that are heavy-tailed. For \( \beta > 0 \), note that \((1 - b\|x\|^2)_+\) with positive \( b \) is non-zero only for \( \|x\| < b^{-\frac{1}{2}} \), and is concave in this region. Thus any density in the second class, corresponding to \( 0 < \beta \leq \frac{2}{d+2} \), is a positive power of \((1 - b\|x\|^2)_+\), and is thus a concave function supported on a centered Euclidean ball of finite radius. It is pertinent to note that although the first class includes many distributions from what one might call the “Cauchy family”, it excludes the standard Cauchy distribution; indeed, not only do all the generalized Gaussians defined above have finite variance, but in fact the form has been chosen so that, for \( Z \sim g_\beta \),

\[
\mathbb{E}[\|Z\|^2] = d
\]

for any \( \beta \). The generalized Gaussians have been called by different names in the literature, including Barenblatt profiles, or the Student-\( r \) distributions (\( \beta < 0 \)) and Student-\( t \) distributions (\( 0 < \beta \leq \frac{2}{d+2} \)).

For \( p > \frac{d}{d+2} \), define \( \beta_p \) by

\[
\frac{1}{\beta_p} = \frac{1}{p-1} + \frac{d + 2}{2},
\]

and write \( Z^{(p)} \) for a random vector drawn from \( g_{\beta_p} \). Note that \( \beta_p \) ranges from \(-\infty\) to \( \frac{2}{d+2} \) as \( p \) ranges from \( \frac{d}{d+2} \) to \( \infty \). The generalized Gaussians \( Z^{(p)} \) arise naturally as the maximizers of the Rényi entropy power of order \( p \) under a variance constraint, as independently observed by Costa, Hero and Vignat \[49\] and Lutwak, Yang and Zhang \[103\]. They play the starring role in the conjecture of Wang and Madiman \[162\].
Conjecture 2.9. [162] Let $X_1, \ldots, X_n$ be independent random vectors taking values in $\mathbb{R}^d$, and $p > \frac{d}{d+2}$. Suppose $Z_i$ are independent random vectors, each a scaled version of $Z^{(p)}$, such that $h_p(X_i) = h_p(Z_i)$. Then

$$N_p(X_1 + \ldots + X_n) \geq N_p(Z_1 + \ldots + Z_n).$$

Until very recently, this conjecture was only known to be true in the case where $p = 1$ (when it is the classical EPI) and the case where $p = \infty$ and $d = 1$ (which is due to Rogozin [137] and discussed in Section 2.4). In [114], we have very recently been able to prove Conjecture 2.9 for $p = \infty$ and any finite dimension $d$, generalizing Rogozin’s inequality. All other cases remain open.

2.3.4 Other work on Rényi entropy power inequalities

Johnson and Vignat [84] also demonstrated what they call an “entropy power inequality for Rényi entropy”, for any order $p \geq 1$. However, their inequality does not pertain to the usual convolution, but a new and somewhat complicated convolution operation (depending on $p$). This new operation reduces to the usual convolution for $p = 1$, and has the nice property that the convolution of affine transforms of independent copies of $Z^{(p)}$ is an affine transform of $Z^{(p)}$ (which fails for the usual convolution when $p > 1$).

As discussed earlier, Costa [50] proved a strengthening of the classical EPI when one of the summands is Gaussian. Savaré and Toscani [143] recently proposed a generalization of Costa’s result to Rényi entropy power, but the notion of concavity they use based on solutions of a nonlinear heat equation does not have obvious probabilistic meaning. Curiously, it turns out that the definition of Rényi entropy power appropriate for the framework of [143] has a different constant in the exponent $(\frac{2}{d} + p - 1)$ as opposed to $\frac{2}{d}$. Motivated by [143], Bobkov and Marsiglietti [27] very recently proved Rényi entropy power inequalities with non-standard exponents. Their main result may be stated as follows.

Theorem 2.10. [27] For $p \in (1, \infty)$ and independent random vectors $X_i$ with densities in $\mathbb{R}^d$,

$$\tilde{N}_p(X_1 + \ldots + X_n) \geq \sum_{i=1}^n \tilde{N}_p(X_i),$$

where

$$\tilde{N}_p(X) = e^{\frac{p+1}{d}h_p(X)}.$$

It would be interesting to know if Theorem 2.10 is true for $p \in [0, 1)$ (and hence all $p \geq 0$), since this would be a particularly nice interpolation between the BMI and EPI.

It is natural to look for Rényi entropy analogues of the refinements and generalizations of the EPI discussed in Section 2.2.2. While little has been done in this direction for general Rényi entropies (apart from the afore-mentioned work of [143]), the case of the Rényi entropy of order 0 (i.e., inequalities for volumes of sets)– which is, of course, of special interest– has attracted some study. For example, Zamir and Feder [172] demonstrated a nontrivial version of the BMI for sums of the form $v_1A_1 + \ldots v_kA_k$, where $A_i$ are unit length subsets of $\mathbb{R}$ and $v_i$ are vectors in $\mathbb{R}^d$, showing that the volume of the Minkowski sum is minimized when each $A_i$ is an interval (i.e., the sum is a zonotope). This result was motivated by analogy with the “matrix version” of the EPI discussed earlier.

Indeed, the strong parallels between the BMI and the EPI might lead to the belief that every volume inequality for Minkowski sums has an analogue for entropy of convolutions,
and vice versa. However, this turns out not to be the case. It was shown by Fradelizi and Marsiglietti [68] that the analogue of Costa’s result (19) on concavity of entropy power, namely the assertion that $t \mapsto |A + tB_2^d|^\frac{1}{d}$ is concave for positive $t$ and any given Borel set $A$, fails to hold\(^5\) even in dimension 2. Another conjecture in this spirit that was made independently by V. Milman (as a generalization of Bergström’s determinant inequality) and by Dembo, Cover and Thomas [56] (as an analogue of Stam’s Fisher information inequality, which is closely related to the EPI) was disproved by Fradelizi, Giannopoulos and Meyer [63]. In [32], it was conjectured that analogues of fractional EPI’s such as (15) hold for volumes, and it was observed that this is indeed the case for convex sets. If this conjecture were true for general compact sets, it would imply that for any compact set, the volumes of the Minkowski self-averages (obtained by taking the Minkowski sum of $k$ copies of the set, and scaling by $1/k$) are monotonically increasing\(^6\) in $k$. However, [65] showed that this conjecture does not hold\(^7\) in general– in fact, they showed that there exist many compact sets $A$ in $\mathbb{R}^d$ for any $d \geq 12$ such that $|A + A + A| < (\frac{3}{2})^d|A + A|$. Finally while volumes of Minkowski sums of convex sets in $\mathbb{R}^d$ are supermodular (as shown in [66]), entropy powers of convolutions of log-concave densities fail to be supermodular even in dimension 1 (as shown in [109]). Thus the parallels between volume inequalities and entropy inequalities are not exact.

Another direction that has seen considerable exploration in recent years is stability of the BMI. This direction began with stability estimates for the BMI in the case where the two summands are convex sets [57, 73, 61, 62, 144]\(^8\), asserting that near-equality in the BMI implies that the summands are nearly homothetic. For general Borel sets, qualitative stability (i.e., that closeness to equality entails closeness to extremizers) was shown by Christ [45, 44], with the first quantitative estimates recently developed by Figalli and Jerison [60]. Qualitative stability for the more general Young’s inequality has also been recently considered [43], but quantitative estimates are unknown to the extent of our knowledge.

### 2.4 An EPI for Rényi entropy of order $\infty$

In discussing Rényi entropy power inequalities, it is of particular interest to consider the case of $p = \infty$, because of close connections with the literature in probability theory on small ball estimates and the so-called Lévy concentration functions [127, 58], which in turn have applications to a number of areas including stochastic process theory [95] and random matrix theory [138, 158, 139].

Observe that by Theorem 2.1 we trivially have

$$N_\infty(X + Y) \geq \max\{N_\infty(X), N_\infty(Y)\} \geq \frac{1}{2}(N_\infty(X) + N_\infty(Y)). \quad (30)$$

In fact, the constant $\frac{1}{2}$ here is sharp, as uniform distributions on any symmetric convex set $K$ (i.e., $K$ is convex, and $x \in K$ if and only if $-x \in K$) of volume 1 are extremal: if $X$ and

---

\(^5\)They also showed some partial positive results– concavity holds in dimension 2 for connected sets, and in general dimension on a subinterval $[t_0, \infty)$ under some regularity conditions.

\(^6\)The significance of this arises from the fact that the Minkowski self-averages of any compact set converge in Hausdorff distance to the convex hull of the set, and furthermore, one also has convergence of the volumes if the original compact set had nonempty interior. Various versions of this fact were proved independently by Emerson and Greenleaf [59], and by Shapley, Folkmann and Starr [150]: a survey of such results including detailed historical remarks can be found in [66].

\(^7\)On the other hand, partial positive results quantifying the convexifying effect of Minkowski summation were obtained in [65, 66].

\(^8\)There is also a stream of work on stability estimates for other geometric inequalities related to the BMI, such as the isoperimetric inequality, but this would take us far afield.
are independently distributed according to $f = 1_K$, then denoting the density of $X - X'$ by $u$, we have

$$\|u\|_\infty = u(0) = \int f^2(x)dx = 1 = \|f\|_\infty,$$

so that $N_\infty(X + X') = N_\infty(X - X') = N_\infty(X) = \frac{1}{2}[N_\infty(X) + N_\infty(X')]$.

What is more, it is observed in [30] that when each $X_i$ is real-valued, $1/2$ is the optimal constant for any number of summations.

**Theorem 2.11.** [30] For independent, real-valued random variables $X_1, \ldots, X_n$,

$$N_\infty\left(\sum_{i=1}^n X_i\right) \geq \frac{1}{2} N_\infty(X_i).$$

The constant $1/2$ clearly cannot be improved upon (one can take $X_3, \ldots, X_n$ to be deterministic and the result follows from the $n = 2$ case). That one should have this sort of scaling in $n$ for the lower bound (namely, linear in $n$ when the summands are identically distributed with bounded densities) is not so obvious from the trivial maximum bound above. The proof of Theorem 2.11 draws on two theorems, the first due to Rogozin [137], which reduces the general case to the cube, and the second a geometric result on cube slicing due to K. Ball [9].

**Theorem 2.12.** [137] Let $X_1, \ldots, X_n$ be independent $\mathbb{R}$-valued random variables with bounded densities. Then

$$N_\infty(X_1 + \cdots + X_n) \geq N_\infty(Y_1 + \cdots + Y_n),$$

where $Y_1, \ldots, Y_n$ are a collection of independent random variables, with $Y_i$ chosen to be uniformly distributed on a symmetric interval such that $N_\infty(Y_i) = N_\infty(X_i)$.

**Theorem 2.13.** [9] Every section of the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ denoted $Q_d$ by an $(d - 1)$-dimensional subspace has volume bounded above by $\sqrt{2}$. This upper bound is attained iff the subspace contains a $(d - 2)$-dimensional face of $Q_d$.

**Proof of Theorem 2.11.** For $X_i$ independent and $\mathbb{R}$-valued, with $Y_i$ chosen as in Theorem 2.12,

$$N_\infty(X_1 + \cdots + X_n) \geq N_\infty(Y_1 + \cdots + Y_n).$$

Applying a sort of change of variables, and utilizing the degree 2 homogeneity of entropy powers, one can write

$$N_\infty(Y_1 + \cdots + Y_n) = \left(\sum_{i=1}^n N_\infty(Y_i)\right) N_\infty(\theta_1 U_1 + \cdots + \theta_n U_n),$$

where the $U_i$ are independent uniform on $[-\frac{1}{2}, \frac{1}{2}]$ and $\theta$ is a unit vector (to be explicit, take $\theta_i = \sqrt{N_\infty(Y_i)}/\sum_j N_\infty(Y_j)$ and the above can be verified). Then utilizing the symmetry of $\theta_1 U_1 + \cdots + \theta_n U_n$ and the BMI, we see that the maximum of its density must occur at 0, yielding

$$N_\infty(\theta_1 U_1 + \cdots + \theta_n U_n) = \left|Q_d \cap \theta^{-1}\right|_{d-1}^{-2} \geq \frac{1}{2}.$$

The result follows. \qed
Theorem 2.11 admits two natural generalizations. The first, also handled in [30] (and later recovered in [133] by taking the limit as \( p \to \infty \) in Theorem 2.8), is the following.

**Theorem 2.14.** [30] For independent random vectors \( X_1, \ldots, X_n \) in \( \mathbb{R}^d \),

\[
N_\infty(X_1 + \cdots + X_n) \geq \left(1 - \frac{1}{n}\right)^{n-1} \left[N_\infty(X_1) + \cdots + N_\infty(X_n)\right]
\]

\[
\geq \frac{1}{e} \left[N_\infty(X_1) + \cdots + N_\infty(X_n)\right].
\]

A second direction was pursued by Livshyts, Paouris and Pivovarov [97] in which the authors derive sharp bounds for the maxima of densities obtained as the projections of product measures. Specifically, [97, Theorem 1.1] shows that given probability density functions \( f_i \) on \( \mathbb{R} \) with \( \|f_i\|_\infty \leq 1 \), with joint product density \( f \) defined by

\[
f(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i),
\]

then

\[
\|\pi_E(f)\|_\infty \leq \min \left\{ \frac{n}{n-k} \left(1 - \frac{k}{n}\right)^{(n-k)/2}, 2^{k/2} \right\},
\]

where \( \pi_E(f) \) denotes the pushforward of the probability measure induced by \( f \) under orthogonal projection to a \( k \)-dimensional subspace \( E \), i.e., \( \pi_E(f)(x) = \int_{x+E^\perp} f(y) dy \). In addition, cubes are shown to be extremizers of the above inequality. In the language of information theory, this can be rewritten as follows.

**Theorem 2.15.** [97] Let \( X = (X_1, \ldots, X_n) \) where \( X_i \) are independent \( \mathbb{R} \)-valued random variables, and \( N_\infty(X_i) \geq 1 \). Then

\[
N_\infty(P_E X) \geq \max \left\{ \frac{1}{2}, \left(1 - \frac{k}{n}\right)^{(n-k)/2} \right\},
\]

where \( P_E \) denotes the orthogonal projection to a \( k \)-dimensional subspace \( E \), and equality can be achieved for \( X_i \) uniform on intervals.

In the \( k = 1 \) case, this implies Theorem 2.11 by applying the inequality (35) to

\[
Y_i = X_i/\sqrt{N_\infty(X_i)},
\]

and taking \( E \) to be the space spanned by the unit vector \( \theta_i = \sqrt{N_\infty(X_i)/\sum_j N_\infty(X_j)} \). The \( Y_i \) defined satisfy the hypothesis so we have \( N_\infty(P_E Y) \geq 1/2 \), but

\[
N_\infty(P_E Y) = N_\infty(\langle \theta, Y \rangle)
\]

\[
= \frac{X_1 + \cdots + X_n}{\sqrt{\sum_{j=1}^n N_\infty(X_j)}}
\]

\[
= \frac{N_\infty(X_1 + \cdots + X_n)}{\sum_{j=1}^n N_\infty(X_j)},
\]

and the implication follows.
Conversely, for the one-dimensional subspace $E$ spanned by the unit vector $\theta$, and $X_i$ satisfying $N_\infty(X_i) \geq 1$, if one applies Theorem 2.11 to $Y_i = \theta_i X_i$, we recover the one-dimensional case of the projection theorem as

$$N_\infty(P_E X) = N_\infty(Y_1 + \cdots + Y_n) \geq \frac{1}{2} (N_\infty(Y_1) + \cdots + N_\infty(Y_n)) = \frac{1}{2} (\theta_1^2 N_\infty(X_1) + \cdots + \theta_n^2 N_\infty(X_n)) \geq \frac{1}{2}.$$ 

Thus Theorem 2.15 can be seen as a $k$-dimensional generalization of the $\infty$-EPI for real random variables.

In recent work [114], we have obtained a generalization of Rogozin’s inequality that allows us to prove multidimensional versions of both Theorems 2.14 and 2.15. Indeed, our extension of Rogozin’s inequality reduces both the latter theorems to geometric inequalities about Cartesian products of Euclidean balls, allowing us to obtain sharp constants in Theorem 2.11 for any fixed dimension as well as to generalize Theorem 2.15 to the case where each $X_i$ is a random vector.

3 Reverse Entropy Power Inequalities

3.1 $\kappa$-concave measures and functions

$\kappa$-concave measures are measures that satisfy a generalized Brunn-Minkowski inequality, and were studied systematically by Borell [34, 35].

As a prerequisite, we define the $\kappa$-mean of two numbers, for $a, b \in (0, \infty)$, $t \in (0, 1)$ and $\kappa \in (-\infty, 0) \cup (0, \infty)$ define

$$M^t_\kappa(a, b) = ((1 - t)a^\kappa + tb^\kappa)^{\frac{1}{\kappa}}.$$ (36)

For $\kappa \in \{-\infty, 0, \infty\}$ define $M^t_\kappa(a, b) = \lim_{\kappa' \to \kappa} M^t_{\kappa'}(a, b)$ corresponding to

$$\{\min(a, b), a^{1-t}b^t, \max(a, b)\}$$

respectively. $M_\kappa$ can be extended to $a, b \in [0, \infty)$ via direct evaluation when $\kappa \geq 0$ and again by limits when $\kappa < 0$ so that $M_\kappa(a, b) = 0$ whenever $ab = 0$.

Definition 3.1. Fix $\kappa \in [-\infty, \frac{1}{d}]$. We say that a probability measure $\mu$ on $\mathbb{R}^d$ is $\kappa$-concave if the support of $\mu$ has non-empty interior$^9$, and

$$\mu((1 - t)A + tB) \geq M^t_\kappa(\mu(A), \mu(B))$$

for any Borel sets $A, B$, and any $t \in [0, 1]$.

We say that $\mu$ is a convex measure if it is $\kappa$-concave for some $\kappa \in [-\infty, \frac{1}{d}]$.

When the law of a random vector $X$ is a $\kappa$-concave measure, we will refer to $X$ as a $\kappa$-concave random vector.

$^9$We only assume this for simplicity of exposition— a more general theory not requiring absolute continuity of the measure $\mu$ with respect to Lebesgue measure on $\mathbb{R}^d$ is available in Borell’s papers. Note that while the support of $\mu$ having nonempty interior in general is a weaker condition than absolute continuity, the two conditions turn out to coincide in the presence of a $\kappa$-concavity assumption.
Thus, the $\kappa$-concave measures are those that distribute volume in such a way that the vector space average of two sets is larger than the $\kappa$-mean of their respective volumes. Let us state some preliminaries. First notice that by Jensen’s inequality $\mu$ being $\kappa$-concave implies $\mu$ is $\kappa'$-concave for $\kappa' \leq \kappa$. The support of a $\kappa$-concave measure is necessarily convex, and since we assumed that the support has nonempty interior, the dimension of the smallest affine subspace of $\mathbb{R}^d$ containing the support of $\mu$ is automatically $d$.

It is a nontrivial fact that concavity properties of a measure can equivalently be described pointwise in terms of its density.

**Theorem 3.2 ([34]).** A measure $\mu$ on $\mathbb{R}^d$ is $\kappa$-concave if and only if it has a density (with respect to the Lebesgue measure on its support) that is a $s_{\kappa,d}$-concave function, in the sense that

$$f((1-t)x + ty) \geq M_{s_{\kappa,d}}^t(f(x), f(y))$$

whenever $f(x)f(y) > 0$ and $t \in (0,1)$, and where

$$s_{\kappa,d} := \frac{\kappa}{1 - \kappa d}.$$

**Examples:**

1. If $X$ is the uniform distribution on a convex body $K$, it has an $\infty$-concave density function $f = |K|^{-1}1_K$ and thus the probability measure is $1/d$-concave. Let us note that by our requirement that $\mu$ is “full-dimensional” (i.e., has support with nonempty interior), the only $1/d$-concave probability measures on $\mathbb{R}^d$ are of this type.

2. A measure that is 0-concave is also called a log-concave measure. Since $s_{0,d} = 0$ for any positive integer $d$, Theorem 3.2 implies that an absolutely continuous measure $\mu$ is log-concave if and only if its density is a log-concave function (as defined in Definition 2.3).

3. If $X$ is log-normal distribution with density function

$$f(x) := \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

Then the density function of $X$ is $-\frac{\sigma}{4}$-concave, and for $\sigma < 4$, the probability measure is $\frac{1}{\max(\alpha,\beta)}$-concave.

4. If $X$ is a Beta distribution with density function

$$\frac{x^\alpha (1-x)^\beta}{B(\alpha, \beta)}$$

with shape parameters $\alpha \geq 1$ and $\beta \geq 1$, then the density function of $X$ is $\min(\frac{1}{\alpha-1}, \frac{1}{\beta-1})$-concave, and the probability measure is $\frac{1}{\max(\alpha,\beta)}$-concave.

5. If $X$ is a $d$-dimensional Student’s $t$-distribution with density function

$$f(x) := \frac{\Gamma(\frac{\nu+d}{2})}{\nu^{\frac{\nu+d}{2}}\pi^{\frac{d}{2}}\Gamma(\frac{\nu}{2})} \left(1 + \frac{|x|^2}{\nu}\right)^{-\frac{\nu+d}{2}}$$


with \( \nu > 0 \), then the density function of \( X \) is \(-\frac{1}{\nu+d}\)-concave, and the probability measure is \(-\frac{1}{\nu}\)-concave.

6. If \( X \) is a \( d \)-dimensional Pareto distribution of the first kind with density function

\[
f(x) := a(a+1) \cdots (a+d-1) \left( \prod_{i=1}^{d} \theta_i \right)^{-1} \left( \sum_{i=1}^{d} \frac{x_i}{\theta_i} - d + 1 \right)^{-(a+d)}
\]

for \( x_i > \theta_i > 0 \) with \( a > 0 \), then the density function of \( X \) is \(-\frac{1}{a+d}\)-concave, and the probability measure is \(-\frac{1}{a}\)-concave.

The optimal \( \kappa \) for the distributions above can be found through direct computation on densities, let us also remind the reader that \( \kappa \)-concavity is an affine invariant. In other words, if \( X \) is \( \kappa \)-concave and \( T \) is affine, then \( TX \) is \( \kappa \)-concave as well, which supplies further examples through modification of the examples above.

We will also find useful an extension of Lemma 2.4 to convex measures (this was obtained in [25] under an additional condition, which was removed in [22]).

**Lemma 3.3.** Let \( \kappa \in (-\infty, 0] \). If \( X \) is a \( \kappa \)-concave random vector in \( \mathbb{R}^d \), then

\[
h(X) - h_{\infty}(X) \leq \sum_{i=0}^{d-1} \frac{1 - \kappa d}{1 - \kappa i},
\]

with equality for the \( n \)-dimensional Pareto distribution.

To match notation with [25] notice that \( X \) being \( \kappa \)-concave is equivalent to \( X \) having a density function that can be expressed as \( \varphi^{-\beta} \), for \( \beta = d - \frac{1}{\kappa} \) and \( \varphi \) convex.

We now develop reverse Rényi entropy power inequalities for \( \kappa \)-concave measures, inspired by work on special cases (such as the log-concave case corresponding to \( \kappa = 0 \) in the terminology above, or the case of Shannon-Boltzmann entropy) in [26, 168, 33, 12].

### 3.2 Positional Reverse EPI’s for Rényi entropies

The reverse Brunn-Minkowski inequality (Reverse BMI) is a celebrated result in convex geometry discovered by V. Milman [122] (see also [123, 124, 130]). It states that given two convex bodies \( A \) and \( B \) in \( \mathbb{R}^d \), one can find a linear volume-preserving map \( u : \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that

\[
|u(A) + B|^{1/d} \leq C(|A|^{1/d} + |B|^{1/d}).
\]

The EPI may be formally strengthened by using the invariance of entropy under affine transformations of determinant \( \pm 1 \), i.e., \( N(u(X)) = N(X) \) whenever \( |\det(u)| = 1 \). Specifically,

\[
\inf_{u_1, u_2} N(u_1(X) + u_2(Y)) \geq N(X) + N(Y),
\]

where the maps \( u_i : \mathbb{R}^d \rightarrow \mathbb{R}^d \) range over all affine entropy-preserving transformations. It was shown in [24] that in exact analogy to the Reverse BMI, the inequality (39) can be reversed with a constant not depending on dimension if we restrict to log-concave distributions. To state such results compactly, we adopt the following terminology.
Definition 3.4. For each \( d \in \mathbb{N} \), let \( \mathcal{M}_d \) be a class of probability measures on \( \mathbb{R}^d \), and write \( \mathcal{M} = (\mathcal{M}_d : d \in \mathbb{N}) \). Suppose that for every pair of independent random variables \( X \) and \( Y \) whose distributions lie in \( \mathcal{M}_d \), there exist linear maps \( u_1, u_2 : \mathbb{R}^d \to \mathbb{R}^d \) of determinant 1 such that

\[
N_p(u_1(X) + u_2(Y)) \leq C_p (N_p(X) + N_p(Y)),
\]

where \( C_p \) is a constant that depends only on \( p \) (and not on \( d \) or the distributions of \( X \) and \( Y \)). Then we say that a Positional Reverse \( p \)-EPI holds for \( \mathcal{M} \).

**Theorem 3.5.** [24] Let \( \mathcal{M}_{d,LC} \) be the class of log-concave probability measures on \( \mathbb{R}^d \), and \( \mathcal{M}_{d,LC}^C = (\mathcal{M}_{d,LC}^C : d \in \mathbb{N}) \). A Positional Reverse 1-EPI holds for \( \mathcal{M}_{d,LC}^C \).

Specializing to uniform distributions on convex bodies, it is shown in [26] that Theorem 3.5 recovers the Reverse BMI. Thus one may think of Theorem 3.5 as completing in a reverse direction the already extensively discussed analogy between the BMI and EPI.

Furthermore, [26] found\(^{10}\) that Theorem 3.5 can be extended to larger subclasses of the class of convex measures.

**Theorem 3.6.** [26] For \( \beta_0 > 2 \), let \( \mathcal{M}_{d,\beta} \) be the class of probability measures whose densities of the form \( f(x) = V(x)^{-\beta} \) for \( x \in \mathbb{R}^d \), where \( V : \mathbb{R}^d \to (0, \infty) \) is a positive convex function and \( \beta \geq \beta_0 d \). Then a Positional Reverse 1-EPI holds for \( \mathcal{M}_{d,\beta} = (\mathcal{M}_{d,\beta} : d \in \mathbb{N}) \).

In [33], it is shown that a Reverse EPI is not possible over all convex measures.

**Theorem 3.7.** [33] For any constant \( C \), there is a convex probability distribution \( \mu \) on the real line with a finite entropy, such that

\[
\min\{N(X + Y), N(X - Y)\} \geq C N(X),
\]

where \( X \) and \( Y \) are independent random variables distributed according to \( \mu \).

We have the following positional reverse \( p \)-Rényi EPI for log-concave random vectors; this does not seem to have explicitly observed before.

**Theorem 3.8.** For any \( p \in (0, \infty) \), a Positional Reverse \( p \)-Rényi EPI holds for \( \mathcal{M}_{d,LC}^C \). Moreover, for \( p \geq 1 \), the constant \( C_{\mathcal{M}_p} \) in the corresponding inequality does not depend on \( p \).

**Proof.** For any pair of independent log-concave random vectors \( X \) and \( Y \), there exist linear maps \( u_1, u_2 : \mathbb{R}^d \to \mathbb{R}^d \) of determinant 1, such that for all \( p > 1 \), by Lemma 2.2, Theorem 3.5 and Lemma 2.4, one has

\[
N_p(u_1(X) + u_2(Y)) \leq N(u_1(X) + u_2(Y)) \lesssim N(X) + N(Y) \lesssim N_\infty(X) + N_\infty(Y) \leq N_p(X) + N_p(Y).
\]

For \( p < 1 \), by Lemma 2.4 and Lemma 2.2, there exist a constant \( C(p) \) depending solely on \( p \) such that

\[
N_p(u_1(X) + u_2(Y)) \leq C(p) N(u_1(X) + u_2(Y)) \leq C(p) (N(X) + N(Y)) \leq C(p) (N_p(X) + N_p(Y)),
\]

which provides the theorem. \( \square \)

Later we will show that Theorem 3.8 can be used to recover the functional version of the reverse Brunn-Minkowski inequality proposed by Klartag and V. Milman [88].

\(^{10}\) Actually [26] only proved this under the additional condition that \( \beta \geq 2d + 1 \), but it turns out that this condition can be dispensed with, as explained in [115].
3.3 Reverse ∞-EPI via a generalization of K. Ball’s bodies

3.3.1 Busemann’s theorem for convex bodies

We first consider Bobkov’s extension of K. Ball’s convex bodies associated to log-concave measures. In this direction we associate a star shaped body to a density function via a generalization of the Minkowski functional of a convex body.

**Definition 3.9.** For a probability density function $f$ on $\mathbb{R}^d$ with the origin in the interior of the support of $f$, and $p \in (0, \infty)$, define $\Lambda^p_f : \mathbb{R}^d \to [0, \infty]$ by

$$
\Lambda^p_f(v) = \left( \int_0^\infty f(rv) dr \right)^{-1/p}
$$

We will consider the class of densities $\mathcal{F}_p$ where $\Lambda^p_f(v) \in [0, \infty)$ for all $v \in \mathbb{R}^d$. For such densities, we can associate a body defined by

$$
K^p_f = \{ v \in \mathbb{R}^d : \Lambda^p_f(v) \leq 1 \}.
$$

We can now state Bobkov’s generalization [28] of the Ball-Busemann theorem.

**Theorem 3.10.** If $f$ is a $s$-concave density on $\mathbb{R}^d$, with $-\frac{1}{d} \leq s \leq 0$, then

$$
\Lambda^p_f((1-t)x + ty) \leq (1-t)\Lambda^p_f(x) + t\Lambda^p_f(y),
$$

for every $x, y \in \mathbb{R}^d$ and $t \in (0, 1)$, provided $0 < p \leq -1 - 1/s$.

**Remark 3.11.** Notice that, since $\Lambda^p_f$ is positive homogeneous and (by Theorem 3.10) convex, it necessarily satisfies the triangle inequality. If we add the assumption that $f$ is even, then $\Lambda^p_f$ defines a norm.

There is remarkable utility in this type of association. In [11], Ball used the fact that one can directly pass from log-concave probability measures to convex bodies using this method to derive an analog of Hensley’s theorem [76] for certain log-concave measures, demonstrating comparability of their slices by different hyperplanes. By generalizing this association to convex measures in [28], Bobkov derived analogs of Blaschke-Santalo inequalities, the Meyer-Reisner theorem [117] (this was proved independently in unpublished work, by Keith Ball, as discussed in [116]) for floating surfaces, and Hensley’s theorem for convex measures. Thus this association of convex bodies with convex measures may be seen as a way to “geometrize” said measures.

Another application of this association of bodies to measures is to the study of so-called intersection bodies.

**Definition 3.12.** For any compact set $K$ in $\mathbb{R}^d$ whose interior contains the origin, define $r : \mathbb{S}^{d-1} \to (0, \infty)$ by $r(\theta) = |K \cap \theta^\perp_{d-1}|$ (i.e., the volume of the $(d-1)$-dimensional slice of $K$ by the subspace orthogonal to $\theta$). The star-shaped body whose boundary is defined by the points $\theta r(\theta)$ is called the intersection body of $K$, and denoted $I(K)$.

The most important fact about intersection bodies is the classical theorem of Busemann [40].

**Theorem 3.13.** [40] If $K$ be a symmetric convex body in $\mathbb{R}^d$, then $I(K)$ is a symmetric convex body as well.
The symmetry is essential here; the intersection body of a non-symmetric convex body need not be convex\textsuperscript{11}. Busemann’s theorem is a fundamental result in convex geometry since it expresses a convexity property of volumes of central slices of a symmetric convex body, whereas Brunn’s theorem (an easy implication of the BMI) asserts a concavity property of volumes of slices that are perpendicular to a given direction.

Busemann’s theorem may be recast in terms of Rényi entropy, as implicitly recognized by K. Ball and explicitly described below.

**Theorem 3.14.** If $X$ is uniformly distributed on a symmetric convex body $K \subset \mathbb{R}^d$, then the mapping $M^X_{\infty} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$M^X_{\infty}(v) = \begin{cases} N^1_{\infty}(\langle v, X \rangle) & v \neq 0 \\ 0 & v = 0 \end{cases}$$

defines a norm on $\mathbb{R}^d$.

Before showing that Theorems 3.13 and 3.14 are equivalent, we need to recall the definition of the Minkowski functional.

**Definition 3.15.** For a convex body $L$ in $\mathbb{R}^d$ containing the origin, define $\rho_L : \mathbb{R}^d \to [0, \infty)$ by

$$\rho_L(x) = \inf\{t \in (0, \infty) : x \in tL\}.$$ 

It is straightforward that $\rho_L$ is positively homogeneous (i.e., $\rho_L(ax) = a\rho_L(x)$ for $a > 0$) and convex. When $L$ is assumed to be symmetric, $\rho_L$ defines a norm.

**Proof of Theorem 3.13 ⇔ Theorem 3.14.** Let $K$ be a symmetric convex body and without loss of generality take $|K| = 1$. Let $X = X_K$ denote a random variable distributed uniformly on $K$.

For a unit vector $\theta \in S^{d-1}$, as the pushforward of a symmetric log-concave measure under the linear map $x \mapsto \langle \theta, x \rangle$, the distribution of the real-valued random variable $\langle \theta, X \rangle$ is symmetric and log-concave. Denoting the symmetric, log-concave density of $\langle \theta, X \rangle$ by $f_\theta$, we see that the mode of $f_\theta$ is 0, and consequently,

$$N^1_{\infty}(\langle \theta, X \rangle) = \frac{1}{f_\theta(0)} = \frac{1}{|K \cap \theta^\perp|_{d-1}} = \frac{1}{r(\theta)}.$$ 

By the definition of $I(K)$, we have $\rho_{I(K)}(r(\theta)\theta) = 1$. Thus, for any $\theta \in S^{d-1}$,

$$\rho_{I(K)}(\theta) = \rho_{I(K)} \left( \frac{r(\theta)\theta}{r(\theta)} \right) = \frac{1}{r(\theta)} = M^X_{\infty}(\theta).$$

By homogeneity, this immediately extends to $\mathbb{R}^d$, establishing our result and also a pleasant duality; up to a constant factor, the Minkowski functional associated to $I(K)$ is a Rényi entropy power of the projections of $X_K$. \hfill \Box

\textsuperscript{11}There is a nontrivial way to extend the definition of intersection body to non-symmetric convex bodies so that the new definition results in a convex body; see [118] for details.
3.3.2 A Busemann-type theorem for measures

Theorem 3.14 is a statement about \( \infty \)-Rényi entropies associated to a \( 1/d \)-concave random vector \( X \) (see Example 1 after Theorem 3.2). It is natural to wonder if Busemann’s theorem can be extended to other \( p \)-Rényi entropies and more general classes of measures.

In [12], Ball-Nayar-Tkocz also give a simple argument, essentially going back to [11], that the information-theoretic statement of Busemann’s theorem (namely Theorem 3.14) extends to log-concave measures. Interpreting in the language of Bo rell’s \( \kappa \)-concave measures, [12] extends Theorem 3.14 to measures that are \( \kappa \)-concave with \( \kappa \geq 0 \). In what follows, we use the same argument as [12] to prove that Busemann’s theorem can in fact be extended to all convex measures by invoking Theorem 3.10.

**Theorem 3.16.** Let \( \kappa \in [-\infty, 1/2] \). If \( (U, V) \) is a symmetric \( \kappa \)-concave random vector in \( \mathbb{R}^2 \), then

\[
e^{h_\infty(U+V)} \leq e^{h_\infty(U)} + e^{h_\infty(V)}.
\]

**Proof.** It is enough to prove the result for the weakest hypothesis \( \kappa = -\infty \). We let \( \varphi \) denote the density function of \( (U, V) \) so that

\[
U + V \sim w(x) = \int_{\mathbb{R}} \varphi(x-t, t) dt \\
U \sim u(x) = \int_{\mathbb{R}} \varphi(x, t) dt \\
V \sim v(x) = \int_{\mathbb{R}} \varphi(t, x) dt.
\]

Since symmetry and the appropriate concavity properties of the densities forces the maxima of \( u, v, w \) to occur at 0,

\[
1 \left/ \|w\|_\infty \right. = 1 \left/ w(0) \right. = \left( \int_{\mathbb{R}} \varphi(-t, t) dt \right)^{-1} \\
= \left( 2 \int_{0}^{\infty} \varphi(t(e_2 - e_1)) dt \right)^{-1} \\
= \frac{1}{2} \Lambda_\varphi^{-1}(e_2 - e_1) \\
\leq \frac{1}{2} \left( \Lambda_\varphi^{-1}(e_2) + \Lambda_\varphi^{-1}(e_1) \right) \\
= \left( 2 \int_{0}^{\infty} \varphi(0, t) dt \right)^{-1} + \left( 2 \int_{0}^{\infty} \varphi(t, 0) dt \right)^{-1} \\
= \frac{1}{u(0)} + \frac{1}{v(0)} \\
= \frac{1}{\|u\|_\infty} + \frac{1}{\|v\|_\infty},
\]

where the only inequality follows from Theorem 3.10 with \( a = 1 \) and \( p = 1 = n - 1 - 1/\kappa \). By definition of \( h_\infty \), we have proved the desired inequality. \( \square \)
As a nearly immediate consequence we have Busemann’s theorem for convex measures.

**Corollary 3.17.** For $\kappa \in [-\infty, \frac{1}{d}]$, if $X$ is symmetric and $\kappa$-concave the function

$$M_X^\kappa(v) = \begin{cases} N_\infty^{\frac{1}{2}}(\langle v, X \rangle) & v \neq 0 \\ 0 & v = 0 \end{cases}$$

defines a norm.

**Proof.** As we have observed $M = M_X^\kappa$ is homogeneous. To prove the triangle inequality take vectors $u, v \in \mathbb{R}^d$ and define $(U, V) = (\langle X, u \rangle, \langle X, v \rangle)$, so that $U + V = \langle X, u + v \rangle$. Notice that $(U, V)$ is clearly symmetric and as the affine pushforward of a $\kappa$-concave measure, is thus $\kappa$-concave as well. Thus by Theorem 3.16 we have

$$e^{h_\infty(U + V)} \leq e^{h_\infty(U)} + e^{h_\infty(V)}.$$

But this is exactly

$$N_\infty^{\frac{1}{2}}(\langle X, u + v \rangle) \leq N_\infty^{\frac{1}{2}}(\langle X, u \rangle) + N_\infty^{\frac{1}{2}}(\langle X, v \rangle),$$

which is what we sought to prove. \qed

### 3.3.3 Busemann-type theorems for other Rényi entropies

While the above extension deals with general measures, a further natural question relates to more general entropies. Ball-Nayar-Tkocz [12] conjecture that the Shannon entropy version holds for log-concave measures.

**Conjecture 3.18.** [12] When $X$ is a symmetric log-concave vector in $\mathbb{R}^d$ then the function

$$M_X^1(v) = \begin{cases} N_1^{\frac{1}{2}}(\langle v, X \rangle) & v \neq 0 \\ 0 & v = 0 \end{cases}$$

defines a norm on $\mathbb{R}^d$.

As the homogeneity of $M$ is immediate, the veracity of the conjecture depends on proving the triangle inequality

$$e^{h_1(U + V)} \leq e^{h_1(U)} + e^{h_1(V)} + e^{h_1((u, X))},$$

which is easily seen to be equivalent to the following modified Reverse EPI for symmetric log-concave measures on $\mathbb{R}^2$.

**Conjecture 3.19.** [12] For a symmetric log-concave random vector in $\mathbb{R}^2$, with coordinates $(U, V)$,

$$N_1^{\frac{1}{2}}(U + V) \leq N_1^{\frac{1}{2}}(U) + N_1^{\frac{1}{2}}(V).$$

Towards this conjecture, it is proved in [12] that $e^{\alpha h_1(U + V)} \leq e^{\alpha h_1(U)} + e^{\alpha h_1(V)}$ when $\alpha = 1/5$. By extending the approach used by [12], we can obtain a family of Busemann-type results for $p$-Rényi entropies.
Theorem 3.20. Fix \( p \in [1, \infty] \). There exists a constant \( \alpha_p > 0 \) which depends only on the parameter \( p \), such that for a symmetric log-concave random vector \( X \) in \( \mathbb{R}^d \) and two vectors \( u, v \in \mathbb{R}^d \), we have
\[
e^{\alpha_p h_p((u+v,X))} \leq e^{\alpha_p h_p(u,X)} + e^{\alpha_p h_p(v,X)}.
\]
Equivalently, for a symmetric log-concave random vector \( (X,Y) \) in \( \mathbb{R}^2 \) we have
\[
e^{\alpha_p h_p(X+Y)} \leq e^{\alpha_p h_p(X)} + e^{\alpha_p h_p(Y)}.
\]
In fact, if \( p \in [1, \infty) \), one can take \( \alpha_p \) above to be the unique positive solution \( \alpha \) of
\[
p^{\alpha-1} = \theta_p^\alpha + (1 - \theta_p)^\alpha,
\]
where
\[
\theta_p := \left( \frac{\log p}{p - 1} \right) \frac{1}{2(e + 1)[2pe^2 + (4p + 1)e + 1]},
\]
with the understanding that the \( p = 1 \) case is understood by continuity (i.e., the left side of equation (42) is \( e^\alpha \) in this case, and the pre-factor \( \frac{\log p}{p - 1} \) in \( \theta_p \) is replaced by \( 1 \)).

Remark 3.21. If \( p < \infty \), then \( \theta_p > 0 \), and on the other hand, trivially \( \theta_p < \frac{1}{2(1+e)} < 1 \). Denote the left and right sides of the equation (42) by \( L_p(\alpha) \) and \( R_p(\alpha) \) respectively. Then \( 1 = L_p(0) < R_p(0) = 2 \), and since \( p^{1/(p-1)} > 1 \) for \( p \in (1, \infty) \), we also have \( \infty = \lim_{\alpha \to \infty} L_p(\alpha) > \lim_{\alpha \to \infty} R_p(\alpha) = 0 \). Since \( L_p \) and \( R_p \) are continuous functions of \( \alpha \), equation (42) must have a positive solution \( \alpha_p \). Moreover, since \( L_p \) is an increasing function and \( R_p \) is a decreasing function, there must be a unique positive solution \( \alpha_p \). In particular, easy simulation gives \( \alpha_1 \approx 0.240789 > 1/5 \), and simulation also shows that the unique solution \( \alpha_p \) is non-decreasing in \( p \). Consequently it appears that for any \( p \), one can replace \( \alpha_p \) in the above theorem by \( 1/p \).

Since Theorem 3.20 is not sharp, and the proof involves some tedious and unenlightening calculations, we do not include its details. We merely mention some analogues of the steps used by [12] to prove the case \( p = 1 \). As done there, one can “linearize” the desired inequality to obtain the following equivalent form: if \( (X,Y) \) is a symmetric log-concave vector in \( \mathbb{R}^2 \) with \( h_p(X) = h_p(Y) \), then for every \( \theta \in [0, 1] \),
\[
h_p(\theta X + (1 - \theta)Y) \leq h_p(X) + \frac{1}{\alpha_p} \log \left( \theta^\alpha p + (1 - \theta)^\alpha p \right).
\]
To prove this form of the theorem, it is convenient as in [12] to divide into cases where \( \theta \) is “small” and “large”. For the latter case, the bound
\[
e^{h_p(X+Y)} \leq e^{h_\infty(X+Y)} + \frac{\log p}{p^{1/(p-1)} \left( e^{h_\infty(X)} + e^{h_\infty(Y)} \right) \leq p^{1/(p-1)} \left( e^{h_p(X)} + e^{h_p(Y)} \right),
\]
easily obtained by combining Lemmata 2.2 and 2.4, suffices. The former case is more involved and relies on proving the following extension of [12, Lemma 1]: If \( w : \mathbb{R}^2 \to \mathbb{R}_+ \) is a symmetric log-concave density, and we define \( f(x) := \int w(x,y)dy \) and \( \gamma = \int w(0,y)dy/ \int w(x,0)dx \), then
\[
\frac{\int \int -f(x)^{p-2}f'(x)y w(x,y)dx dy}{\int f(x)^{p}dx} \leq \left( 2e(e + 2) + \frac{e + 1}{p} \right) \gamma.
\]
Staring at Theorem 3.16 and Conjecture 3.19, and given that one would expect to be able to interpolate between the \( p = 1 \) and \( p = \infty \) cases, it is natural to pose the following conjecture that would subsume all of the results and conjectures discussed in this section.
Conjecture 3.22. Fix $\kappa \in [-\infty, \frac{1}{d}]$. For a symmetric $\kappa$-concave random vector in $\mathbb{R}^2$, with coordinates $(U, V)$, it holds for any $p \in [1, \infty]$ that

$$N_p^{1/2}(U + V) \leq N_p^{1/2}(U) + N_p^{1/2}(V),$$

whenever all these quantities are finite. Equivalently, when $X$ is a symmetric $\kappa$-concave random vector in $\mathbb{R}^d$, then for any given $p \in [1, \infty]$, the function

$$M_p^X(v) = \begin{cases} N_p^{1/2}(\langle v, X \rangle) & v \neq 0 \\ 0 & v = 0 \end{cases}$$

defines a norm on $\mathbb{R}^d$ when it is finite everywhere.

Given the close connection of the $p = \infty$ case with intersection bodies and Busemann’s theorem, one wonders if there is a connection between the unit balls of the conjectured norms $M_p^X$ in Conjecture 3.22 on the one hand, and the so-called $L_p$-intersection bodies that arise in the dual $L_p$ Brunn-Minkowski theory (see, e.g., Haberl [74]) on the other.

After the first version of this survey was released, Jiange Li (personal communication) has verified that Conjecture 3.22 is true when $p = 0$ (with arbitrary $\kappa$) and when $p = 2$ (with $\kappa = 0$, i.e., in the log-concave case).

3.4 Reverse EPI via Rényi entropy comparisons

The Rogers-Shephard inequality [136] is a classical and influential inequality in Convex Geometry. It states that for any convex body $K$ in $\mathbb{R}^d$,

$$|K - K| \leq \left(\frac{2d}{d}\right)\text{Vol}(K)$$

where $K - K := \{x - y : x, y \in K\}$. Since $\left(\frac{2d}{d}\right) < 4^d$, this implies that $|K - K|^{1/d} < 4|K|^{1/d}$, complementing the fact that $|K - K|^{1/d} \geq 2|K|^{1/d}$ by the BMI. In particular, the Rogers-Shephard inequality may be thought of as a Reverse BMI. In this section, we discuss integral and entropic liftings of the Rogers-Shephard inequality.

An integral lifting of the Rogers-Shephard inequality was developed by Colesanti [46] (see also [2, 5]). For a real non-negative function $f$ defined in $\mathbb{R}^d$, define the difference function $\Delta f$ of $f$,

$$\Delta f(z) := \sup\{\sqrt{f(x)f(-y)} : x, y \in \mathbb{R}^d, \frac{1}{2}(x + y) = z\}$$

It is proved in [46] that if $f : \mathbb{R}^d \to [0, \infty)$ is a log-concave function, then

$$\int_{\mathbb{R}^d} \Delta f(z)dz \leq 2^d \int_{\mathbb{R}^d} f(x)dx,$$

where the equality is attained by multi-dimensional exponential distribution.

On the other hand, an entropic lifting of the Rogers-Shephard inequality was developed by [33]. We develop an extension of their argument and result here. In order to state it, we need to recall the notion of relative entropy between two distributions: if $X, Y$ have densities $f, g$ respectively, then

$$D(X\|Y) = D(f\|g) := \int_{\mathbb{R}^d} f(x) \log \frac{f(x)}{g(x)} dx$$
is the relative entropy between $X$ and $Y$. By Jensen’s inequality, $D(X\|Y) \geq 0$, with equality if and only if the two distributions are identical.

**Lemma 3.23.** Suppose $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ has a $\kappa$-concave distribution, with $\kappa < 0$. If $X$ and $Y$ are independent, then

$$h(X - Y) \leq \min\{h(X) + D(X\|Y), h(Y) + D(Y\|X)\} + \sum_{i=0}^{d-1} \frac{1 - \kappa d}{1 - \kappa i}.$$  

**Proof.** By affine invariance, the distribution of $X - Y$ is $\kappa$-concave, so that one can apply Lemma 3.3 to obtain

$$h(X - Y) \leq \log \|f\|^{-1} + \sum_{i=0}^{d-1} \frac{1 - \kappa d}{1 - \kappa i}.$$

Denoting the marginal densities of $X$ and $Y$ by $f_1$ and $f_2$ respectively, we have $f(0) = \int_{\mathbb{R}^d} f_1(x)f_2(x)dx$, and hence

$$h(X - Y) \leq -\log \int_{\mathbb{R}^d} f_1(x)f_2(x)dx + \sum_{i=0}^{d-1} \frac{1 - \kappa d}{1 - \kappa i}$$

$$\leq \int_{\mathbb{R}^d} f_1(x)[-\log f_2(x)]dx + \sum_{i=0}^{d-1} \frac{1 - \kappa d}{1 - \kappa i}$$

$$= h(X) + D(X\|Y) + \sum_{i=0}^{d-1} \frac{1 - \kappa d}{1 - \kappa i}.$$  

Clearly the roles of $X$ and $Y$ here are interchangeable, yielding the desired bound. \hfill \Box

In the case where the marginal distributions are the same, Lemma 3.23 reduces as follows.

**Theorem 3.24.** Suppose $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ has a $\kappa$-concave distribution, with $\kappa < 0$. If $X$ and $Y$ are independent and identically distributed, then

$$N(X - Y) \leq C_\kappa N(X),$$  

where

$$C_\kappa = \exp\left\{\frac{2}{d(1 - d\kappa)} \sum_{j=0}^{d-1} \frac{1}{1 - j\kappa}\right\}.$$

As $\kappa \to 0$, we recover the fact, obtained in [33], that $N(X - Y) \leq e^2 N(X)$ for $X, Y$ i.i.d. with log-concave marginals. We believe that this statement can be tightened, even in dimension 1. Indeed, it is conjectured in [111] that for $X, Y$ i.i.d. with log-concave marginals,

$$N(X - Y) \leq 4N(X)$$

is the tight entropic version of Rogers-Shepard in one dimension, with equality for the one-sided exponential distribution.
3.5 Reverse Rényi EPI via Convex Ordering

3.5.1 Convex ordering and entropy maximization

In this section, we build on an elegant approach of Y. Yu [168], who obtained inequalities for Rényi entropy of order $p \in (0, 1]$ for i.i.d. log-concave measures under stochastic ordering assumptions. In particular, we achieve extensions to $\kappa$-concave measures with $\kappa < 0$ and impose weaker distributional symmetry assumptions, and observe that the resulting inequalities may be interpreted as Reverse EPI’s.

**Lemma 3.25.** Let $X \sim f, Y \sim g$ be random vectors on $\mathbb{R}^d$. In order to prove $h_p(X) \geq h_p(Y),$

it suffices to prove

\[ \mathbb{E} f^{p-1}(X) \geq \mathbb{E} f^{p-1}(Y), \quad \text{if } p \in (0, 1), \quad (46) \]
\[ \mathbb{E} f^{p-1}(X) \leq \mathbb{E} f^{p-1}(Y), \quad \text{if } p \in (1, \infty), \quad (47) \]
\[ -\mathbb{E} \log f(X) \geq -\mathbb{E} \log f(Y), \quad \text{if } p = 1. \quad (48) \]

**Proof.** Notice that the expressions in the hypothesis for $p \neq 1$ can be re-written as $\mathbb{E} f^{p-1}(X) = \int_{\mathbb{R}^d} f^{p-1}(x) f(x) dx$ and $\mathbb{E} f^{p-1}(Y) = \int_{\mathbb{R}^d} f^{p-1}(x) g(x) dx$. For $p \in (0, 1),$

\[ \int f^p dx = \left( \int f^{p-1} f \right)^p \left( \int f^p \right)^{1-p} \geq (a) \left( \int f^{p-1} g \right)^p \left( \int f^p \right)^{1-p} \geq (b) \int g^p dx, \]

where (a) is from applying the hypothesis and (b) is by Hölder’s inequality (applied in the probability space $(\mathbb{R}^d, g dx)$). Inequality (46) follows from the fact that $(1 - p)^{-1} \log x$ is order-preserving for $p \in (0, 1)$.

When $p \in (1, \infty),$

\[ \int f^p dx = \left( \int f^{p-1} f \right)^p \left( \int f^p \right)^{1-p} \leq (c) \left( \int f^{p-1} g \right)^p \left( \int f^p \right)^{1-p} \leq \int g^p dx, \]

where Hölder’s inequality is reversed for $p \in (1, \infty)$ accounting for (c). Inequality (47) follows since $(1 - p)^{-1} \log x$ is order-reversing for such $p$.

In the case $p = 1$, we use the hypothesis and then Jensen’s inequality to obtain,

\[ h(X) = -\mathbb{E} \log f(X) \geq -\mathbb{E} \log f(Y) \geq -\mathbb{E} \log g(Y) = h(Y), \]

which yields inequality (48) and completes the proof of the lemma. \qed
Of the observations in Lemma 3.25, (46) and (48) were used in [168]; we add (47), which is relevant to Reverse EPI’s for \( \kappa \)-concave measures with \( \kappa > 0 \).

We recall the notion of convex ordering for random vectors.

**Definition 3.26.** For random variables \( X, Y \) taking values in a linear space \( V \), we say that \( X \) dominates \( Y \) in the convex order, written \( X \geq_{cx} Y \), if \( \mathbb{E}\varphi(X) \geq \mathbb{E}\varphi(Y) \) for every convex and continuous function \( \varphi : V \to \mathbb{R} \).

We need a basic lemma relating supports of distributions comparable in the convex ordering.

**Lemma 3.27.** Given random vectors \( X \sim f \) and \( Y \sim g \) such that \( Y \leq_{cx} X \), if \( \text{supp}(f) \) is a convex set, then \( \text{supp}(g) \subset \text{supp}(f) \).

**Proof.** Take \( \rho \) to be the Minkowski functional (Definition 3.15) associated to \( \text{supp}(f) \) and then define \( \varphi(x) = \max\{\rho(x) - 1, 0\} \).

As the maximum of two convex functions, \( \varphi \) is convex. Also observe that \( \varphi \) is identically zero on \( \text{supp}(f) \) while strictly positive on the complement. By the ordering assumption

\[
0 \leq \mathbb{E}(\varphi(Y)) \leq \mathbb{E}(\varphi(X)) = 0.
\]

Thus \( \mathbb{E}(\varphi(Y)) = 0 \), which implies the claim. \( \square \)

We can now use convex ordering as a criterion to obtain a maximum entropy property of convex measures under certain conditions.

**Theorem 3.28.** Let \( X \) and \( Y \) be random vectors in \( \mathbb{R}^d \), with \( X \) being \( \kappa \)-concave for some \( \kappa \in (-\infty, 1/d] \). If \( X \geq_{cx} Y \), then

\[
h_p(X) \geq h_p(Y)
\]

for \( 0 \leq p \leq \kappa/(1 - d\kappa) + 1 \).

**Proof.** Recall that \( X \) is \( \kappa \)-concave if and only if it admits a \( s_{\kappa,d} \)-concave density \( f \) on its support, with \( s_{\kappa,d} = \kappa/(1 - d\kappa) \). Thus it follows that for \( a \leq s_{\kappa,d} \), \( f^a \) is a convex function, (resp. concave) for \( a > 0 \) (resp. \( a > 0 \)). Our hypothesis is simply that that \( p - 1 \leq s_{\kappa,d} \).

For \( p < 1 \) we can apply the convex ordering to necessarily convex function \( f^{p-1} \), as \( \mathbb{E}f^{p-1}(X) \geq \mathbb{E}f^{p-1}(Y) \) and apply Lemma 3.25 under the hypothesis (46).

When \( p > 1 \) the proof is the same as the application of convex ordering to the concave function \( f^{p-1} \) will reverse the inequality to attain \( \mathbb{E}f^{p-1}(X) \leq \mathbb{E}f^{p-1}(Y) \) and then invoking Lemma 3.25 under hypothesis (47) will yield the result.

To consider \( p = 1 \), \( X \) must be at least log-concave, in which case we can follow [168] exactly. This amounts to applying convex ordering to \( -\log f \) and Lemma 3.25 a final time.

After recalling that the support of a \( \kappa \)-concave measure is a convex set, the \( p = 0 \) case follows from Lemma 3.27. \( \square \)

Theorem 3.28 extends a result of Yu [168], who shows that for \( X \) log-concave, \( h_p(X) \geq h_p(Y) \) for \( 0 < p \leq 1 \) when \( X \geq_{cx} Y \). Observe that as \( \kappa \) approaches \( 1/d \), the upper limit of the range of \( p \) for which Theorem 3.28 applies approaches \( \infty \).
Some care should be taken to interpret Theorem 3.28 and the entropy inequalities to come. For example, the $t$-distribution (see Example 4 after Theorem 3.2) does not have finite $p$-Rényi entropy when $p \leq \frac{1}{\nu + d}$ and hence the theorem only yields non-trivial results on the interval $(\frac{d}{\nu + d}, 1 - \frac{1}{\nu + d})$. Notice that in the important special case where $X$ is Cauchy, corresponding to $\nu = 1$, this interval is empty; thus Theorem 3.28 fails to give a maximum entropy characterization of the Cauchy distribution (which is of interest from the point of view of entropic limit theorems).

**Definition 3.29.** We say that a family of random vectors $\{X_1, \ldots, X_n\}$ is exchangeable when $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ and $(X_1, \ldots, X_n)$ are identically distributed for any permutation $\sigma$ of $\{1, \ldots, n\}$.

**3.5.2 Results under an exchangeability condition**

Let us also remind the reader of the notion of majorization for $a, b \in \mathbb{R}^n$. First we recall that a square matrix is doubly stochastic if its row sums and column sums are all equal to 1.

**Definition 3.30.** For vectors $a, b \in \mathbb{R}^n$, we will write $b \prec a$ (and say that $b$ is majorized by $a$) if there exists a doubly stochastic matrix $M$ such that $Ma = b$.

There are several equivalent formulations of this notion that are well studied (see, e.g., [147]), but we will not have use for them. Note that if $1$ is the vector with all coordinates equal to 1, then $1^T (Ma) = (1^T M)a = 1^T a$, implying that $b \prec a$ can only hold if the sum of coordinates of $a$ equals the sum of coordinates of $b$.

**Lemma 3.31.** Let $X_1, \ldots, X_n$ be exchangeable random variables taking values in a real vector space $V$, and let $\varphi : V^n \to \mathbb{R}$ be a convex function symmetric in its coordinates. If $b \prec a$,

$$E \varphi(a_1 X_1, \ldots, a_n X_n) \geq E \varphi(b_1 X_1, \ldots, b_n X_n).$$

**Proof.** Since every doubly stochastic matrix can be written as the convex combination of permutation matrices by the Birkhoff von-Neumann theorem (see, e.g., [147]), we can write $b \prec a$ as $b = (\sum \lambda_\sigma P_\sigma)a$ where $\lambda_i \in [0, 1]$ with $\sum \lambda_\sigma = 1$ and $P_\sigma$ is a permutation matrix. We compute

$$E \varphi(b_1 X_1, \ldots, b_n X_n) = E \varphi\left(\left(\sum_\sigma \lambda_\sigma P_\sigma a\right)_1 X_1, \ldots, \left(\sum_\sigma \lambda_\sigma P_\sigma a\right)_n X_n\right)$$

$$\leq \sum_\sigma \lambda_\sigma E \varphi(a_{\sigma(1)} X_1, \ldots, a_{\sigma(n)} X_n)$$

$$= \sum_\sigma \lambda_\sigma E \varphi(a_{\sigma(1)} X_{\sigma(1)}, \ldots, a_{\sigma(n)} X_{\sigma(n)})$$

$$= \sum_\sigma \lambda_\sigma E \varphi(a_1 X_1, \ldots, a_n X_n)$$

$$= E \varphi(a_1 X_1, \ldots, a_n X_n),$$

where the steps are justified– in order– by definition, convexity, exchangeability, coordinate symmetry, and then algebra. \qed
Theorem 3.32. Let $X = (X_1, \ldots, X_n)$ be an exchangeable collection of $d$-dimensional random vectors. Suppose $b < a$ and that $a_1X_1 + \cdots + a_nX_n$ has a $s$-concave density. Then for any $p \in [0, s + 1)$,

$$h_p(b_1X_1 + \cdots + b_nX_n) \leq h_p(a_1X_1 + \cdots + a_nX_n).$$

Proof. Let $f$ denote the $s$-concave density function of $a_1X_1 + \cdots + a_nX_n$. Thus for $p < 1$ (resp. $p > 1$) the function

$$\varphi(x_1, \ldots, x_n) = f^{p-1}(x_1 + \cdots + x_n)$$

is convex (resp. concave) and clearly symmetric in its coordinates, hence by Lemma 3.31

$$\mathbb{E}\varphi(b_1X_1, \ldots, b_nX_n) \leq \mathbb{E}\varphi(a_1X_1, \ldots, a_nX_n),$$

( resp. $\mathbb{E}\varphi(b_1X_1, \ldots, b_nX_n) \geq \mathbb{E}\varphi(a_1X_1, \ldots, a_nX_n)$).

But this is exactly,

$$\mathbb{E}f^{p-1}(b_1X_1, \ldots, b_nX_n) \leq \mathbb{E}f^{p-1}(a_1X_1, \ldots, a_nX_n),$$

( resp. $\mathbb{E}f^{p-1}(b_1X_1, \ldots, b_nX_n) \geq \mathbb{E}f^{p-1}(a_1X_1, \ldots, a_nX_n)$),

and thus by Lemma 3.25,

$$h_p(b_1X_1 + \cdots + b_nX_n) \leq h_p(a_1X_1 + \cdots + a_nX_n).$$

The case $p = 1$ is similar by setting

$$\varphi(x_1, \ldots, x_n) = -\log f(x_1 + \cdots + x_n),$$

and applying Lemma 3.31 and Lemma 3.25. \qed

Definition 3.33. For $\Omega \subseteq \mathbb{R}^d$, we define a function $\varphi : \Omega \rightarrow \mathbb{R}$ to be Schur-convex in the case that for any $x, y \in \Omega$ with $x < y$ we have $\varphi(x) \leq \varphi(y)$.

Corollary 3.34. Suppose $X = (X_1, \ldots, X_n)$ is an exchangeable collection of random vectors in $\mathbb{R}^d$, with $X$ being $\kappa$-concave. Let $\Delta_n = \{\theta \in [0, 1]^n : \sum_{i=1}^n \theta_i = 1\}$ be the standard simplex, and define the function $\Phi_{X,p} : \Delta_n \rightarrow \mathbb{R}$ by

$$\Phi_{X,p}(\theta) = h_p(\theta_1X_1 + \cdots + \theta_nX_n).$$

For $p \in [0, s_{\kappa,d} + 1]$, $\Phi_{X,p}$ is a Schur-convex function. In particular, $\Phi_{X,p}$ is maximized by the standard basis elements $e_i$, and minimized by $(\frac{1}{n}, \ldots, \frac{1}{n})$.

Proof. If $X$ is $\kappa$-concave, then by affine invariance $\theta_1X_1 + \cdots + \theta_nX_n$ is $\kappa$-concave, and hence Theorem 3.32 applies. The extremizers are identified by observing that for any $\theta$ in the simplex

$$(1/n, \ldots, 1/n) < \theta < e_i,$$

and the corollary follows. \qed

Of course, using the standard simplex is only a matter of normalization; analogous results are easily obtained by setting $\sum \theta_i$ to be any positive constant.

Let us remark that when the coordinates of $X_i$ are assumed to be independent, then $X$ is log-concave if and only if each $X_i$ each log-concave. As a consequence we recover in the $\kappa = 0$ and $p \leq 1$ case, the theorem of Yu in [168].

Theorem 3.35. [168] Let $X_1, \ldots, X_n$ be i.i.d. log-concave random vectors in $\mathbb{R}^d$. Then the function $a \mapsto h_p(a_1X_1 + \cdots + a_nX_n)$ is Schur-convex on the simplex for $p \in (0,1]$. 

34
3.5.3 Results under an assumption of identical marginals

We now show that the exchangeability hypothesis can be loosened in Corollary 3.34.

**Theorem 3.36.** Let \( X = (X_1, \ldots, X_n) \) be a collection of \( d \)-dimensional random vectors with \( X_i \) identically distributed and \( \kappa \)-concave. For \( p \in [0, s_{\kappa,d} + 1] \), the function \( \Phi_{X,p} \) defined in Corollary 3.34 satisfies

\[
\Phi_{X,p}(a) \leq \Phi_{X,p}(e_i).
\]

Stated explicitly, for \( a \in \Delta_n \), we have

\[
h_{p}(a_1X_1 + \cdots + a_nX_n) \leq h_{p}(X_1).
\]

**Proof.** Let \( f \) be the density function of \( X_1 \) and \( a \in \Delta_n \). If \( p < 1 \), by Lemma 3.25, it suffices to prove that

\[
E f_{p-1}(a_1X_1 + \cdots + a_nX_n) \leq E f_{p-1}(X_1).
\]

Since \( f \) is a \( s_{\kappa,d} \)-concave function and \( p - 1 \leq s_{\kappa,d} \), \( f \) is also \( (p - 1) \)-concave, which means that \( f_{p-1} \) is convex. Consequently, we have

\[
E f_{p-1}(a_1X_1 + \cdots + a_nX_n) \leq a_1E f_{p-1}(X_1) + \cdots + a_nE f_{p-1}(X_n) = E f_{p-1}(X_1),
\]

where the equality is by the fact that \( X_i \) are identically distributed. The cases of \( p > 1 \) and \( p = 1 \) follow similarly. \( \square \)

**Corollary 3.37.** Suppose \( X_1, X_2, \ldots, X_n \) are identically distributed and \( \kappa \)-concave. If \( p \in [0, s_{\kappa,d} + 1] \), we have the triangle inequality

\[
N_{p}^{1/2} \left( \sum_{i=1}^{n} X_i \right) \leq \sum_{i=1}^{n} N_{p}^{1/2} (X_i).
\]

Moreover, for any \( p > s_{\kappa,d} + 1 \),

\[
N_{p}^{1/2} \left( \sum_{i=1}^{n} X_i \right) \leq \frac{(s_{\kappa,d} + 1)^{1/s_{\kappa,d}}}{p^{1/(p-1)}} \sum_{i=1}^{n} N_{p}^{1/2} (X_i).
\]

**Proof.** We have, by Theorem 3.36, for \( p \in [0, s_{\kappa,d} + 1] \),

\[
N_{p}^{1/2} \left( \sum_{i=1}^{n} X_i \right) \leq N_{p}^{1/2} (nX_1) = \sum_{i=1}^{n} N_{p}^{1/2} (X_i).
\]

The second inequality can be derived from Lemma 3.3, combined with Theorem 3.36 and the
monotonicity of Rényi entropies:

\[
N_p^{1/2} \left( \sum_{i=1}^{n} X_i \right) \leq N_{s_{n,d}+1}^{1/2} \left( \sum_{i=1}^{n} X_i \right) \\
\leq N_{s_{n,d}+1}^{1/2} (nX_1) \\
= \exp \left( h_{s_{n,d}+1}(X_1)/d + \log n \right) \\
\leq \exp \left( h_p(X_1)/d + \left[ \log n + \frac{\log(s_{n,d}+1)}{s_{n,d}} - \frac{\log p}{p-1} \right] \right) \\
= \left( \frac{s_{n,d}+1}{p^{1/(p-1)}} \right)^{1/s_{n,d}} \sum_{i=1}^{n} N_p^{1/2}(X_i).
\]

□

Observe that Corollary 3.37 is very reminiscent of Conjectures 3.18 and 3.22; the main difference is that here we have the assumption of identical marginals as opposed to central symmetry of the joint distribution.

We state the next corollary as a direct application of Corollary 3.37 for the log-concave case.

**Corollary 3.38.** Suppose \( X_1, X_2, \cdots, X_n \) are identically distributed log-concave random vectors in \( \mathbb{R}^d \). Then

\[
N_p \left( \sum_{i=1}^{n} X_i \right) \leq n^2 N_p(X_1) \quad \text{for } p \in [0, 1], \tag{49}
\]

\[
N_p \left( \sum_{i=1}^{n} X_i \right) \leq e^{2\beta_p^2/(1-p)}n^2 N_p(X_1) \leq e^{2n^2 N_p(X_1)} \quad \text{for } p \in (1, \infty]. \tag{50}
\]

In particular, if \( X \) and \( X' \) are identically distributed log-concave random vectors, then

\[
N_p(X + X') \leq 4N_p(X) \quad \text{for } p \in [0, 1],
\]

\[
N_p(X + X') \leq 4e^{2\beta_p^2/(1-p)}N_p(X) \leq 4e^{2n^2 N_p(X)} \quad \text{for } p \in (1, \infty].
\]

Cover and Zhang [55] proved the remarkable fact that if \( X \) and \( X' \) (possibly dependent) have the same log-concave distribution on \( \mathbb{R} \), then \( h(X + X') \leq h(2X) \) (in fact, they also showed a converse of this fact). As observed by [111], their method also works in the multivariate setting, where it implies that \( N(X + X') \leq 4N(X) \) for real-valued, i.i.d. log-concave \( X, X' \). This fact is recovered by the previous corollary.

Let us finally remark that if we are not interested in an explicit constant, then a version of this inequality already follows from the Reverse EPI of [26]. Indeed,

\[
N(X + X') \leq CN(X),
\]

since the same unit-determinant affine transformation must put both \( X \) and \( X' \) in \( M \)-position. However, the advantage of the methods we have explored is that we can obtain explicit constants.
3.6 Remarks on special positions that yield reverse EPI’s

Let us recall the definition of isotropic bodies and measures in the convex geometric sense.

**Definition 3.39.** A convex body \( K \) in \( \mathbb{R}^d \) is called isotropic if there exists a constant \( L_K \) such that
\[
\frac{1}{|K|^{1 + \frac{d}{2}}} \int_K \langle x, \theta \rangle^2 dx = L_K^2,
\]
for all unit vectors \( \theta \in S^{d-1} \). More generally, a probability measure \( \mu \) on \( \mathbb{R}^d \) is called isotropic if there exists a constant \( L_K \) such that
\[
\int_{\mathbb{R}^d} \langle x, \theta \rangle^2 \mu(dx) = L_K^2,
\]
for all unit vectors \( \theta \in S^{d-1} \).

The notion of \( M \)-position (i.e., a position or choice of affine transformation applied to convex bodies for which a reverse Brunn-Minkowski inequality holds) was first introduced by V. Milman [122]. Alternative approaches to proving the existence of such a position were developed in [124, 130, 72]. It was shown by Bobkov [29] that if the standard Gaussian measure conditioned to lie in a convex body \( K \) is isotropic, then the body is in \( M \)-position and the reverse BMI applies. The notion of \( M \)-position was extended from convex bodies to log-concave measures in [24], and further to convex measures in [26]. Using this extension, together with the sufficient condition obtained in [29], one can give an explicit description of a position for which a reverse EPI applies with a universal—but not explicit—constant.

Nonetheless there are other explicit positions for which one can get reverse EPI’s with explicit (but not dimension-independent) constants. One instance of such is obtained from an extension to convex measures obtained by Bobkov [28] for Hensley’s theorem (which had earlier been extended from convex sets to log-concave functions by Ball [9]).

**Theorem 3.40.** [28] For a symmetric, convex probability measure \( \mu \) on \( \mathbb{R}^d \) with density \( f \) such that the body \( \Lambda_{d-k}^f \) is isotropic, we have for any linear two subspaces \( H_1, H_2 \) of codimension \( k \),
\[
\int_{H_1} f dx \leq C_k \int_{H_2} f dx.
\]

What is more, \( C_k < \left( \frac{1}{2} e^2 \pi k \right)^{\frac{1}{k}} \).

As a consequence we have the following reverse \( \infty \)-Rényi EPI in the isotropic context.

**Corollary 3.41.** Suppose the joint distribution of the random vector \( (X, Y) \in \mathbb{R}^d \times \mathbb{R}^d \) is symmetric and convex, with density \( f = f(x, y) \). If the body \( \Lambda^f \) is isotropic, then
\[
N_\infty(X + Y) \leq \pi e^2 d \min\{N_\infty(X), N_\infty(Y)\}.
\]

**Proof.** Define two \( d \)-dimensional subspaces of \( \mathbb{R}^d \): \( H_1 := \{x = 0\} \), \( H_2 := \{x + y = 0\} \).
Computing directly and applying Theorem 3.40 we have our result as follows,

\[ \frac{N_\infty(X + Y)}{N_\infty(X)} = \left( \frac{\|f_X\|_\infty}{\|f_{X+Y}\|_\infty} \right)^\frac{2}{d} \]

\[ = \left( \frac{\int_{\mathbb{R}^d} f(0, z) dz}{\int_{\mathbb{R}^d} f(z, -z) dz} \right)^\frac{2}{d} \]

\[ = \left( \frac{2^\frac{d}{2} \int_{H_1} f}{\int_{H_2} f} \right)^\frac{2}{d} \]

\[ \leq \pi e^{2d}. \]

\[ \square \]

4 The relationship between functional and entropic liftings

In this section, we observe that the integral lifting of an inequality in Convex Geometry may sometimes be seen as a Rényi entropic lifting.

We start by considering integral and entropic liftings of a classical inequality in Convex Geometry, namely the Blaschke-Santaló inequality. For a convex body \( K \subset \mathbb{R}^d \) with \( 0 \in \text{int}(K) \), the polar \( K^\circ \) of \( K \) is defined as

\[ K^\circ = \{ y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in K \}, \]

and, more generally, the polar \( K^z \) with respect to \( z \in \text{int}(K) \) by \((K - z)^\circ\). There is a unique point \( s \in \text{int}(K) \), called the Santaló point of \( K \), such that the volume product \( |K| |K^s| \) is minimal—it turns out that this point is such that the barycenter of \( K^s \) is 0. The Blaschke-Santaló inequality states that

\[ |K| |K^s| \leq |B_2|^d, \]

with equality if and only if \( K \) is an ellipsoid. In particular, the volume product \( |K| |K^\circ| \) of a centrally symmetric convex body \( K \) is maximized by the Euclidean ball. This inequality was proved by Blaschke [21] in dimensions 2 and 3, and by Santaló [142] in general dimension; the equality conditions were settled by Petty [129]. There have been many subsequent proofs; see [19] for a recent Fourier analytic proof as well as a discussion of the earlier literature.

More generally, if \( K, L \) are compact sets in \( \mathbb{R}^d \), then

\[ |K| \cdot |L| \leq \omega_d^2 \max_{x \in K, y \in L} \langle x, y \rangle^d. \]

(51)

The inequality (51) implies the Blaschke-Santaló inequality by taking \( K \) to be a symmetric convex body, and \( L \) to be the polar of \( K \).

Let us now describe an integral lifting of the inequality (51), which was proved by Lehec [89, 90] building on earlier work of Ball [11], Artstein-Klartag-Milman [6], and Fradelizi-Meyer [69].

Let \( f \) and \( g \) be non-negative Borel functions on \( \mathbb{R}^d \) satisfying the duality relation

\[ \forall x, y \in \mathbb{R}^d, f(x)g(y) \leq e^{-\langle x, y \rangle}. \]
If \( f \) (or \( g \)) has its barycenter (defined as \( (\int f)^{-1} \int xf(x)dx \)) at 0 then
\[
\int_{\mathbb{R}^d} f(x)dx \int_{\mathbb{R}^d} g(y)dy \leq (2\pi)^d.
\]
The inequality (51) also has an entropic lifting. For any two independent random vectors \( X \) and \( Y \) in \( \mathbb{R}^d \), Lutwak-Yang-Zhang [102] showed that
\[
N(X) \cdot N(Y) \leq \frac{4\pi^2 e^2}{d} \mathbb{E}[\langle X, Y \rangle^2],
\]
with equality achieved for Gaussians. They also have an even more general (and still sharp) statement that bounds \([N_p(X)N_p(Y)]^{p/2}\) in terms of \( \mathbb{E}[\langle X, Y \rangle^p]\), with extremizers being certain generalized Gaussian distributions. As \( p \to \infty \), the expression \( (\mathbb{E}[\langle X, Y \rangle^p])^{1/p} \) approaches the essential supremum of \( |\langle X, Y \rangle| \), which in the case that \( X \) and \( Y \) are uniformly distributed on convex bodies is just the maximum that appears in the right side of inequality (51). Thus the Blaschke-Santaló inequality appears as the \( L_\infty \) instance of the family of inequalities proved by Lutwak-Yang-Zhang [102], whereas the entropic lifting (52) is the \( L_2 \) instance of the same family. This perspective of entropy inequalities as being tied to an \( L_2 \)-analogue of the Brunn-Minkowski theory is greatly developed in a series of papers by Lutwak, Yang, Zhang, sometimes with additional coauthors (see, e.g., [101] and references therein), but this is beyond the scope of this survey.

For a function \( V: \mathbb{R}^d \to \mathbb{R} \), its Legendre transform \( \mathcal{L}V \) is defined by
\[
\mathcal{L}V(x) = \sup_y \left[ \langle x, y \rangle - V(y) \right].
\]
For \( f = e^{-V} \) log-concave, following Klartag and V. Milman [88], we define its polar by
\[
f^\circ = e^{-\mathcal{L}V}.
\]
Some basic properties of the polar are collected below.

**Lemma 4.1.** Let \( f \) be a non-negative function on \( \mathbb{R}^d \).

1. If \( f \) is log-concave, then
\[
(f^\circ)^\circ = f.
\]
2. If \( g \) is also a non-negative function on \( \mathbb{R}^d \), and the “Asplund product” of \( f \) and \( g \) is defined by \( f \ast g(x) = \sup_{x_1 + x_2 = x} f(x_1)g(x_2) \), then
\[
(f \ast g)^\circ = f^\circ g^\circ.
\]
3. For any linear map \( u: \mathbb{R}^d \to \mathbb{R}^d \) with full rank, we have the composition identity
\[
f^\circ \circ u = (f \circ u^{-T})^\circ,
\]
where \( u^{-T} \) is the inverse of the adjoint of \( u \).
4. If \( f(x) \) takes its maximum value at \( x = 0 \), one has
\[
\sup f^\circ = \frac{1}{\sup f}.
\]
5. For any $p > 0$,

$$(f^o)^p(x) = (f^p)^o(px).$$

Proof. Write $f := e^{-V}$ for a function $V : \mathbb{R}^d \to \mathbb{R}$. The first two properties are left as an exercise for the reader– these are also standard facts about the Legendre transform and its relation to the infimal convolution of convex functions (see, e.g., [135]). For the third, we have

$$(f^o \circ u)(x) = e^{-\sup_y [(ux, y) - V(y)]} = e^{-\sup_y [(x, u^T y) - V(y)]} = (f \circ u^{-T})^o(x),$$

which proves the property.

For the fourth, observe that we have, for any $x \in \mathbb{R}^d$,

$$\mathcal{L}V(x) = \sup_y [(x, y) - V(y)] \geq -V(0).$$

On the other hand,

$$\mathcal{L}V(0) = \sup_y [-V(y)] = -V(0).$$

Thus we have proved that $\inf \mathcal{L}V = -V(0)$, which is equivalent to the desired property.

The last property is checked by writing $(f^o)^p(x) = e^{-\sup_y [(px, y) - pV(y)]}$.

Bourgain and V. Milman [36] proved a reverse form of the Blaschke-Santaló inequality, which asserts that there is a universal positive constant $c$ such that

$$|K| \cdot |K^o| \geq c^d,$$

for any symmetric convex body $K$ in $\mathbb{R}^d$, for any dimension $d$. Klartag and V. Milman [88] obtained a functional lifting of this reverse inequality.

**Theorem 4.2.** [88] There exists a universal constant $c > 0$ such that for any dimension $d$ and for any log-concave function $f : \mathbb{R}^d \to [0, \infty)$ centered at the origin (in the sense that $f(0)$ is the maximum value of $f$) with $0 < \int_{\mathbb{R}^d} f < \infty$,

$$c^d < \left( \int_{\mathbb{R}^d} f \right) \left( \int_{\mathbb{R}^d} f^o \right) < (2\pi)^d.$$ 

Note that the upper bound here is just a special case of the integral lifting of the Blaschke-Santaló inequality discussed earlier.

We observe that Theorem 4.2 can be thought of in information-theoretic terms, namely as a type of certainty/uncertainty principle.

**Theorem 4.3.** Let $X \sim f$ be a log-concave random vector in $\mathbb{R}^d$, which is centered at the origin in the sense that $f$ is maximized there. Let $X^o$ be a random vector in $\mathbb{R}^d$ drawn from the density $f^o / \int_{\mathbb{R}^d} f^o$. Define the constants

$$A_{p,d} := \frac{d(\log 2\pi - \log p - p \log c)}{1 - p},$$

$$B_{p,d} := \frac{d(\log c - \log p - p \log 2\pi)}{1 - p},$$

40
where the constant $c$ is the same as in Theorem 4.2. Then, for $p > 1$, we have
\[
\max\{d \log c, A_{p,d}\} \leq h_p(X) + h_p(X^\circ) \leq \min\{d(\log 2\pi + 2), B_{p,d}\},
\]  
and for $p < 1$, we have
\[
\max\{d \log c, B_{p,d}\} \leq h_p(X) + h_p(X^\circ) \leq \min\left\{d \left(\frac{2\log p}{p-1} + \log 2\pi\right), A_{p,d}\right\}.
\]  
In particular, if $p = \infty$,
\[
d \log c \leq h_\infty(X) + h_\infty(X^\circ) \leq d \log 2\pi,
\]  
and for $p = 1$,
\[
d \log c \leq h(X) + h(X^\circ) \leq d(\log 2\pi + 2).
\]  
Proof. We have
\[
h_p(X) + h_p(X^\circ) = \frac{\log \left[\int f^p \left(\int f^p\right)^p\right] - p \log \int f^p}{1 - p}.
\]  
By property (57), we have $\int (f^p)^p = \frac{1}{p^p} \int (f^p)^\circ$. So by (62):
\[
h_p(X) + h_p(X^\circ) = \frac{\log \left[\int f^p \left(\int f^p\right)^\circ\right] - d \log p - p \log \int f^\circ}{1 - p}.
\]  
Thus, by applying Theorem 4.2 twice, if $p > 1$:
\[
h_p(X) + h_p(X^\circ) \geq \frac{d \log 2\pi - d \log p - p \log \int f^\circ}{1 - p} \geq A_{p,d}.
\]  
On the other hand,
\[
h_p(X) + h_p(X^\circ) \leq \frac{d \log c - d \log p - p \log \int f^\circ}{1 - p} \leq B_{p,d}.
\]  
Therefore we have
\[
A_{p,d} \leq h_p(X) + h_p(X^\circ) \leq B_{p,d}.
\]  
A similar argument for $p < 1$ gives
\[
B_{p,d} \leq h_p(X) + h_p(X^\circ) \leq A_{p,d}.
\]  
Letting $p \to \infty$, we have (60). For $p = 1$, by Lemma 2.4 and (60),
\[
n \log c \leq h_\infty(X) + h_\infty(X^\circ) \leq h(X) + h(X^\circ) \leq h_\infty(X) + h_\infty(X^\circ) + 2n \leq n(\log 2\pi + 2),
\]  
which provides (61). Thus for $p > 1$, by (60), (61) and Lemma 2.2, we also have
\[
n \log c \leq h_\infty(X) + h_\infty(X^\circ) \leq h_p(X) + h_p(X^\circ) \leq h(X) + h(X^\circ) \leq n(\log 2\pi + 2).
\]  
Combining with (63) provides (58), which provides the theorem. For $p < 1$, we have, by (61) and Lemma 2.2, we have
\[
d \log c \leq h(X) + h(X^\circ) \leq h_p(X) + h_p(X^\circ).
\]  
Combining this with (64) provides the left most inequality of (59). And by applying Lemma 2.4 on $h_p(f) - h_\infty(f)$ and by (60), we have
\[
h_p(X) + h_p(X^\circ) \leq \frac{2d \log p}{p-1} + h_\infty(X) + h_\infty(X^\circ) \leq \frac{2d \log p}{p-1} + d \log 2\pi.
\]  
Combining this with (64) gives (59).
Klartag and Milman [88] prove a reverse Prékopa-Leindler inequality (Reverse PLI).

**Theorem 4.4.** [88] Given $f, g: \mathbb{R}^d \to [0, \infty)$ be even log-concave functions with $f(0) = g(0) = 1$, then there exist $u_f, u_g$ in $SL(d)$ such that $\tilde{f} = f \circ u_f, \tilde{g} = g \circ u_g$ satisfy

$$\left( \int \tilde{f} \star \tilde{g} \right)^{\frac{1}{d}} \leq C \left( \left( \int \tilde{f} \right)^{\frac{1}{d}} + \left( \int \tilde{g} \right)^{\frac{1}{d}} \right),$$

where $C > 0$ is a universal constant, $u_f$ depends solely on $f$, and $u_g$ depends solely on $g$.

We observe that the Reverse PLI can be proved from the Positional Reverse Rényi EPI we proved earlier, modulo the reverse functional Blaschke-Santaló inequality of Klartag-Milman.

**Proposition 4.5.** Theorems 3.8 and 4.2 together imply Theorem 4.4.

**Proof.** Let $f, g: \mathbb{R}^d \to [0, \infty)$ be even log-concave functions with $f(0) = g(0) = 1$. Now by property (56), $\|f^o\|_\infty = 1$ as well. Now apply reversed $\infty$-EPI on a pair of independent random vectors $X$ and $Y$ with density functions $f^o/\int f^o$ and $g^o/\int g^o$ respectively, there exist linear maps $u_1, u_2 \in SL(d)$ depending solely on $f$ and $g$ respectively, such that

$$\left( \int \frac{(f^o \circ u_1(x)) \cdot (g^o \circ u_2(x))}{\int f^o \cdot \int g^o} \right)^{-\frac{2}{d}} = N_\infty(u_1(X) + u_2(Y)) \lesssim N_\infty(X) + N_\infty(Y) = \left\| \frac{f^o}{\int f^o} \right\|_\infty^{-\frac{2}{d}} + \left\| \frac{g^o}{\int g^o} \right\|_\infty^{-\frac{2}{d}} = \left( \int f^o \right)^{-\frac{2}{d}} + \left( \int g^o \right)^{-\frac{2}{d}}.$$

Therefore we have

$$\left( \int (f^o \circ u_1(x)) \cdot (g^o \circ u_2(x)) \right)^{-\frac{2}{d}} \lesssim \left( \int f^o \right)^{-\frac{2}{d}} + \left( \int g^o \right)^{-\frac{2}{d}}.$$

Thus by Theorem 4.2, we have the right hand side of (65) is

$$\left( \int f^o \right)^{-\frac{2}{d}} + \left( \int g^o \right)^{-\frac{2}{d}} \lesssim \left( \int f^o \right)^{\frac{2}{d}} + \left( \int g^o \right)^{\frac{2}{d}}.$$

On the other hand, by properties (53), (54) and (55), we have the right hand side of (65):

$$\left( \int (f^o \circ u_1(x)) \cdot (g^o \circ u_2(x)) \right)^{-\frac{2}{d}} \gtrsim \left( \int (f \circ u_1^{-t}) \star (g \circ u_2^{-t}) \right)^{\frac{2}{d}}.$$

Denote $u_f := u_1^{-t}, u_g := u_2^{-t}; \tilde{f} := f \circ u_f, \tilde{g} := g \circ u_g$, and combining (65) (66) and (67) provides Theorem 4.4. \qed

## 5 Concluding remarks

One productive point of view put forward by Lutwak, Yang and Zhang is that the correct analogy is between entropy inequalities and the inequalities of the $L^2$-Brunn-Minkowski theory rather than the standard Brunn-Minkowski theory. While we did not have space to pursue this direction in our survey apart from a brief discussion in Section 4, we refer to [101] and references therein for details.
A central question when considering integral or entropic liftings of Convex Geometry is whether there exist integral and entropic analogues of mixed volumes. Recent work of Bobkov-Colesanti-Fragala \[31\] has shown that an integral lifting of intrinsic volumes does exist, and Milman-Rotem \[121, 120\] independently showed this as well as an integral lifting of mixed volumes more generally. A fully satisfactory theory of “intrinsic entropies” or “mixed entropies” is yet to emerge, although some promising preliminary results in this vein can be found in \[78\].

It is also natural to explore nonlinear generalizations, to ambient spaces that are manifolds or groups. Log-concave (and convex) measures can be put into an even broader context by viewing them as instances of curvature in metric measure spaces. Indeed, thanks to path-breaking work of \[151, 98\], it was realized that one can give meaning (synthetically) to the notion of a lower bound on Ricci curvature for a metric space \((X, d)\) equipped with a measure \(\mu\) (thus allowing for geometry beyond the traditional setting of Riemannian manifolds). In particular, they extended the celebrated Curvature-Dimension condition \(CD(K, N)\) of Bakry and Émery \[8\] to metric measure spaces \((X, d, \mu)\): the simplest case \(CD(K, \infty)\) is defined by a “displacement convexity” (or convexity along optimal transport paths) property of the relative entropy functional \(D(\cdot \| \mu)\). For Riemannian manifolds, the \(CD(K, N)\) condition is satisfied if and only if the manifold has dimension at most \(N\) and Ricci curvature at least \(K\), while Euclidean space \(\mathbb{R}^d\) equipped with a log-concave measure may be thought of as having non-negative Ricci curvature in the sense that it satisfies \(CD(0, d)\). Moreover, \(\mathbb{R}^d\) equipped with a convex measure may be interpreted as a \(CD(K, N)\) space with effective dimension \(N\) being negative (other examples can be found in \[119\]). In these more general settings (where there may not be a group structure), it is not entirely clear whether there are natural formulations of entropy power inequalities. Even for the case of Lie groups, almost nothing seems to be known.

One may also seek discrete analogs of the phenomena studied in this survey, which are closely related to investigations in additive combinatorics. In discrete settings, additive structure plays a role as or more important than that of convexity. The Cauchy-Davenport inequality is an analog of the Brunn-Minkowski inequality in cyclic groups of prime or infinite order, with arithmetic progressions being the extremal objects (see, e.g., \[157\]); extensions to the integer lattice are also known \[140, 71, 149\]. A probabilistic lifting of the Cauchy-Davenport inequality for the integers is presented in \[163\]. Sharp lower bounds on entropies of sums in terms of those of summands are still not known for most countable groups; partial results in this direction may be found in \[156, 75, 77, 165\]. Such bounds are also relevant to the study of information-theoretic approaches to discrete limit theorems, such as those that involve distributional convergence to the Poisson or compound Poisson distributions of sums of random variables taking values in the nonnegative integers; we refer the interested reader to \[81, 83, 169, 170, 14\] for further details. Probabilistic liftings of other “sumset inequalities” from additive combinatorics can be found in \[104, 141, 112, 113, 156, 110, 1, 111, 94\].

There are other connections between notions of entropy and convex geometry that we have not discussed in this paper (see, e.g., \[23, 7, 164, 41, 67, 64, 93\]).

Acknowledgement

The authors are grateful to Eric Carlen, Bernardo González Merino, Igal Sason, Tomasz Tkocz, Elisabeth Werner, and an anonymous reviewer for useful comments and references.
References

[1] E. Abbe, J. Li, and M. Madiman. Entropies of weighted sums in cyclic groups and an application to polar codes. Preprint, \texttt{arXiv:1512.00135}, 2015.

[2] D. Alonso-Gutiérrez, B. González Merino, C. H. Jiménez, and R. Villa. Rogers–Shephard inequality for log-concave functions. \textit{J. Funct. Anal.}, 271(11):3269–3299, 2016.

[3] S. Artstein, K. M. Ball, F. Barthe, and A. Naor. On the rate of convergence in the entropic central limit theorem. \textit{Probab. Theory Related Fields}, 129(3):381–390, 2004.

[4] S. Artstein, K. M. Ball, F. Barthe, and A. Naor. Solution of Shannon’s problem on the monotonicity of entropy. \textit{J. Amer. Math. Soc.}, 17(4):975–982 (electronic), 2004.

[5] S. Artstein-Avidan, K. Einhorn, D. I. Florentin, and Y. Ostrover. On Godbersen’s conjecture. \textit{Geom. Dedicata}, 178:337–350, 2015.

[6] S. Artstein-Avidan, B. Klartag, and V. Milman. The Santaló point of a function, and a functional form of the Santaló inequality. \textit{Mathematika}, 51(1-2):33–48 (2005), 2004.

[7] S. Artstein-Avidan, B. Klartag, C. Schütt, and E. Werner. Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality. \textit{J. Funct. Anal.}, 262(9):4181–4204, 2012.

[8] D. Bakry and M. Émery. Diffusions hypercontractives. In \textit{Séminaire de probabilités, XIX, 1983/84}, volume 1123 of \textit{Lecture Notes in Math.}, pages 177–206. Springer, Berlin, 1985.

[9] K. Ball. Cube slicing in \( \mathbb{R}^n \). \textit{Proc. Amer. Math. Soc.}, 97(3):465–473, 1986.

[10] K. Ball. \textit{Isometric problems in} \( \ell^p \) \textit{and sections of convex sets}. PhD thesis, University of Cambridge, UK, 1986.

[11] K. Ball. Logarithmically concave functions and sections of convex sets in \( \mathbb{R}^n \). \textit{Studia Math.}, 88(1):69–84, 1988.

[12] K. Ball, P. Nayar, and T. Tkocz. A reverse entropy power inequality for log-concave random vectors. \textit{Preprint}, \texttt{arXiv:1509.05926}, 2015.

[13] K. Ball and V. H. Nguyen. Entropy jumps for isotropic log-concave random vectors and spectral gap. \textit{Studia Math.}, 213(1):81–96, 2012.

[14] A. D. Barbour, O. Johnson, I. Kontoyiannis, and M. Madiman. Compound Poisson approximation via information functionals. \textit{Electron. J. Probab.}, 15(42):1344–1368, 2010.

[15] F. Barthe. Optimal Young’s inequality and its converse: a simple proof. \textit{Geom. Funct. Anal.}, 8(2):234–242, 1998.

[16] F. Barthe. Restricted Prékopa-Leindler inequality. \textit{Pacific J. Math.}, 189(2):211–222, 1999.
[17] Y. Baryshnikov, R. Ghrist, and M. Wright. Hadwiger’s Theorem for definable functions. *Adv. Math.*, 245:573–586, 2013.

[18] W. Beckner. Inequalities in Fourier analysis. *Ann. of Math. (2)*, 102(1):159–182, 1975.

[19] G. Bianchi and M. Kelly. A Fourier analytic proof of the Blaschke-Santaló Inequality. *Proc. Amer. Math. Soc.*, 143(11):4901–4912, 2015.

[20] N.M. Blachman. The convolution inequality for entropy powers. *IEEE Trans. Information Theory*, IT-11:267–271, 1965.

[21] W. Blaschke. Über affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid. *Ber. Verh. Sächs. Akad. Wiss., Math. Phys. Kl.*, 69:412–420, 1917.

[22] S. Bobkov, M. Fradelizi, J. Li, and M. Madiman. When can one invert Hölder’s inequality? (and why one may want to). *Preprint*, 2016.

[23] S. Bobkov and M. Madiman. Concentration of the information in data with log-concave distributions. *Ann. Probab.*, 39(4):1528–1543, 2011.

[24] S. Bobkov and M. Madiman. Dimensional behaviour of entropy and information. *C. R. Acad. Sci. Paris Sér. I Math.*, 349:201–204, Février 2011.

[25] S. Bobkov and M. Madiman. The entropy per coordinate of a random vector is highly constrained under convexity conditions. *IEEE Trans. Inform. Theory*, 57(8):4940–4954, August 2011.

[26] S. Bobkov and M. Madiman. Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures. *J. Funct. Anal.*, 262:3309–3339, 2012.

[27] S. Bobkov and A. Marsiglietti. Variants of entropy power inequality. *Preprint*, arXiv:1609.04897, 2016.

[28] S. G. Bobkov. Convex bodies and norms associated to convex measures. *Probab. Theory Related Fields*, 147(1-2):303–332, 2010.

[29] S. G. Bobkov. On Milman’s ellipsoids and M-position of convex bodies. In C. Houdré, M. Ledoux, E. Milman, and M. Milman, editors, *Concentration, Functional Inequalities and Isoperimetry*, volume 545 of *Contemp. Math.*, pages 23–33. Amer. Math. Soc., 2011.

[30] S. G. Bobkov and G. P. Chistyakov. Entropy power inequality for the Rényi entropy. *IEEE Trans. Inform. Theory*, 61(2):708–714, February 2015.

[31] S. G. Bobkov, A. Colesanti, and I. Fragalà. Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities. *Manuscripta Math.*, 143(1-2):131–169, 2014.

[32] S. G. Bobkov, M. Madiman, and L. Wang. Fractional generalizations of Young and Brunn-Minkowski inequalities. In C. Houdré, M. Ledoux, E. Milman, and M. Milman, editors, *Concentration, Functional Inequalities and Isoperimetry*, volume 545 of *Contemp. Math.*, pages 35–53. Amer. Math. Soc., 2011.
[33] S. G. Bobkov and M. M. Madiman. On the problem of reversibility of the entropy power inequality. In Limit theorems in probability, statistics and number theory, volume 42 of *Springer Proc. Math. Stat.*, pages 61–74. Springer, Heidelberg, 2013. Available online at arXiv:1111.6807.

[34] C. Borell. Convex measures on locally convex spaces. *Ark. Mat.*, 12:239–252, 1974.

[35] C. Borell. Convex set functions in d-space. *Period. Math. Hungar.*, 6(2):111–136, 1975.

[36] J. Bourgain and V. D. Milman. New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$. *Invent. Math.*, 88(2):319–340, 1987.

[37] H. J. Brascamp and E. H. Lieb. Best constants in Young’s inequality, its converse, and its generalization to more than three functions. *Advances in Math.*, 20(2):151–173, 1976.

[38] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.

[39] A. Burchard. A short course on rearrangement inequalities. Available online at http://www.math.utoronto.ca/almut/rearrange.pdf, June 2009.

[40] H. Busemann. A theorem on convex bodies of the Brunn-Minkowski type. *Proc. Nat. Acad. Sci. U. S. A.*, 35:27–31, 1949.

[41] U. Caglar and E. M. Werner. Divergence for s-concave and log concave functions. *Adv. Math.*, 257:219–247, 2014.

[42] E. A. Carlen and A. Soffer. Entropy production by block variable summation and central limit theorems. *Comm. Math. Phys.*, 140, 1991.

[43] M. Christ. Near-extremizers of Young’s inequality for $\mathbb{R}^d$. Preprint, arXiv:1112.4875, 2011.

[44] M. Christ. Near equality in the Brunn-Minkowski inequality. Preprint, arXiv:1207.5062, 2012.

[45] M. Christ. Near equality in the two-dimensional Brunn-Minkowski inequality. Preprint, arXiv:1206.1965, 2012.

[46] A. Colesanti. Functional inequalities related to the Rogers-Shephard inequality. *Mathematika*, 53(1):81–101 (2007), 2006.

[47] A. Colesanti. Log concave functions. Preprint, 2016.

[48] D. Cordero-Erausquin and M. Ledoux. The geometry of Euclidean convolution inequalities and entropy. *Proc. Amer. Math. Soc.*, 138(8):2755–2769, 2010.

[49] J. Costa, A. Hero, and C. Vignat. On solutions to multivariate maximum alpha-entropy problems. *Lecture Notes in Computer Science*, 2683(EMMCVPR 2003, Lisbon, 7-9 July 2003):211–228, 2003.
[50] M.H.M. Costa. A new entropy power inequality. *IEEE Trans. Inform. Theory*, 31(6):751–760, 1985.

[51] M.H.M. Costa and T.M. Cover. On the similarity of the entropy power inequality and the Brunn-Minkowski inequality. *IEEE Trans. Inform. Theory*, 30(6):837–839, 1984.

[52] T. Courtade. Strengthening the entropy power inequality. *Preprint*, arXiv:1602.03033, 2016.

[53] T. Courtade, M. Fathi, and A. Pananjady. Wasserstein Stability of the Entropy Power Inequality for Log-Concave Densities. *Preprint*, arXiv:1610.07969, 2016.

[54] T. M. Cover and J. A. Thomas. *Elements of information theory*. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2006.

[55] T. M. Cover and Z. Zhang. On the maximum entropy of the sum of two dependent random variables. *IEEE Trans. Inform. Theory*, 40(4):1244–1246, 1994.

[56] A. Dembo, T.M. Cover, and J.A. Thomas. Information-theoretic inequalities. *IEEE Trans. Inform. Theory*, 37(6):1501–1518, 1991.

[57] V. I. Diskant. Stability of the solution of a Minkowski equation. *Sibirsk. Mat. Ž.*, 14:669–673, 696, 1973.

[58] Yu. S. Eliseeva, F. Götze, and A. Yu. Zaitsev. Arak inequalities for concentration functions and the Littlewood–Offord problem. *Preprint*, arXiv:1506.09034, 2015.

[59] W. R. Emerson and F. P. Greenleaf. Asymptotic behavior of products $C^p = C + \cdots + C$ in locally compact abelian groups. *Trans. Amer. Math. Soc.*, 145:171–204, 1969.

[60] A. Figalli and D. Jerison. Quantitative stability for sumsets in $\mathbb{R}^n$. *J. Eur. Math. Soc. (JEMS)*, 17(5):1079–1106, 2015.

[61] A. Figalli, F. Maggi, and A. Pratelli. A refined Brunn-Minkowski inequality for convex sets. In *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*. Elsevier, 2009.

[62] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.

[63] M. Fradelizi, A. Giannopoulos, and M. Meyer. Some inequalities about mixed volumes. *Israel J. Math.*, 135:157–179, 2003.

[64] M. Fradelizi, J. Li, and M. Madiman. Concentration of information content for convex measures. *Preprint*, arXiv:1512.01490, 2015.

[65] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. Do Minkowski averages get progressively more convex? *C. R. Acad. Sci. Paris Sér. I Math.*, 354(2):185–189, February 2016.

[66] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. On the monotonicity of Minkowski sums towards convexity. *Preprint*, 2016.
67] M. Fradelizi, M. Madiman, and L. Wang. Optimal concentration of information content for log-concave densities. In C. Houdré, D. Mason, P. Reynaud-Bouret, and J. Rosinski, editors, *High Dimensional Probability VII: The Cargèse Volume*, Progress in Probability. Birkhäuser, Basel, 2016. Available online at arXiv:1508.04093.

68] M. Fradelizi and A. Marsiglietti. On the analogue of the concavity of entropy power in the Brunn-Minkowski theory. *Adv. in Appl. Math.*, 57:1–20, 2014.

69] M. Fradelizi and M. Meyer. Some functional forms of Blaschke-Santaló inequality. *Math. Z.*, 256(2):379–395, 2007.

70] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)*, 39(3):355–405 (electronic), 2002.

71] R. J. Gardner and P. Gronchi. A Brunn-Minkowski inequality for the integer lattice. *Trans. Amer. Math. Soc.*, 353(10):3995–4024 (electronic), 2001.

72] A. Giannopoulos, G. Paouris, and B.-H. Vritsiou. The isotropic position and the reverse Santaló inequality. *Israel J. Math.*, 203(1):1–22, 2014.

73] H. Groemer. On the Brunn-Minkowski theorem. *Geom. Dedicata*, 27(3):357–371, 1988.

74] C. Haberl. $L_p$ intersection bodies. *Adv. Math.*, 217(6):2599–2624, 2008.

75] S. Haghighatshoar, E. Abbe, and E. Telatar. A new entropy power inequality for integer-valued random variables. *IEEE Trans. Inform. Th.*, 60(7):3787–3796, July 2014.

76] D. Hensley. Slicing convex bodies—bounds for slice area in terms of the body’s covariance. *Proc. Amer. Math. Soc.*, 79(4):619–625, 1980.

77] V. Jog and V. Anantharam. The entropy power inequality and Mrs. Gerber’s lemma for groups of order $2^n$. *IEEE Trans. Inform. Theory*, 60(7):3773–3786, 2014.

78] V. Jog and V. Anantharam. On the geometry of convex typical sets. In *Proc. IEEE Intl. Symp. Inform. Theory*, Hong Kong, China, June 2015.

79] O. Johnson. A conditional entropy power inequality for dependent variables. *IEEE Trans. Inform. Theory*, 50(8):1581–1583, 2004.

80] O. Johnson. An information-theoretic central limit theorem for finitely susceptible FKG systems. *Teor. Veroyatn. Primen.*, 50(2):331–343, 2005.

81] O. Johnson. Log-concavity and the maximum entropy property of the Poisson distribution. *Stochastic Process. Appl.*, 117(6):791–802, 2007.

82] O. Johnson and A.R. Barron. Fisher information inequalities and the central limit theorem. *Probab. Theory Related Fields*, 129(3):391–409, 2004.

83] O. Johnson, I. Kontoyiannis, and M. Madiman. Log-concavity, ultra-log-concavity, and a maximum entropy property of discrete compound Poisson measures. *Discrete Appl. Math.*, 161:1232–1250, 2013. DOI: 10.1016/j.dam.2011.08.025.

84] O. Johnson and C. Vignat. Some results concerning maximum Rényi entropy distributions. *Ann. Inst. H. Poincaré Probab. Statist.*, 43(3):339–351, 2007.
[85] Oliver Johnson. *Information theory and the central limit theorem*. Imperial College Press, London, 2004.

[86] T. H. Kjeldsen. From measuring tool to geometrical object: Minkowski’s development of the concept of convex bodies. *Arch. Hist. Exact Sci.*, 62(1):59–89, 2008.

[87] T. H. Kjeldsen. Egg-forms and measure-bodies: different mathematical practices in the early history of the modern theory of convexity. *Sci. Context*, 22(1):85–113, 2009.

[88] B. Klartag and V. D. Milman. Geometry of log-concave functions and measures. *Geom. Dedicata*, 112:169–182, 2005.

[89] J. Lehec. A direct proof of the functional Santaló inequality. *C. R. Math. Acad. Sci. Paris*, 347(1-2):55–58, 2009.

[90] J. Lehec. Partitions and functional Santaló inequalities. *Arch. Math. (Basel)*, 92(1):89–94, 2009.

[91] L. Leindler. On a certain converse of Hölder’s inequality. In *Linear operators and approximation (Proc. Conf., Oberwolfach, 1971)*, pages 182–184. Internat. Ser. Numer. Math., Vol. 20. Birkhäuser, Basel, 1972.

[92] L. Leindler. On a certain converse of Hölder’s inequality. II. *Acta Sci. Math. (Szeged)*, 33(3-4):217–223, 1972.

[93] J. Li, M. Fradelizi, and M. Madiman. Information concentration for convex measures. In *Proc. IEEE Intl. Symp. Inform. Theory*, Barcelona, Spain, July 2016.

[94] J. Li and M. Madiman. A combinatorial approach to small ball inequalities for sums and differences. *Preprint, arXiv:1601.03927*, 2016.

[95] W. V. Li and Q.-M. Shao. Gaussian processes: inequalities, small ball probabilities and applications. In *Stochastic processes: theory and methods*, volume 19 of *Handbook of Statist.*. pages 533–597. North-Holland, Amsterdam, 2001.

[96] E. H. Lieb. Proof of an entropy conjecture of Wehrl. *Comm. Math. Phys.*, 62(1):35–41, 1978.

[97] G. Livshyts, G. Paouris, and P. Pivovarov. On sharp bounds for marginal densities of product measures. *Preprint, arXiv:1507.07949*, 2015.

[98] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[99] L. A. Lusternik. Die Brunn-Minkowskische ungleichung fur beliebige messbare mengen. *C. R. (Doklady) Acad. Sci. URSS*, 8:55–58, 1935.

[100] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.*, 38(1):131–150, 1993.

[101] E. Lutwak, S. Lv, D. Yang, and G. Zhang. Affine moments of a random vector. *IEEE Trans. Inform. Theory*, 59(9):5592–5599, September 2013.
E. Lutwak, D. Yang, and G. Zhang. Moment-entropy inequalities. *Ann. Probab.*, 32(1B):757–774, 2004.

E. Lutwak, D. Yang, and G. Zhang. Moment-entropy inequalities for a random vector. *IEEE Trans. Inform. Theory*, 53(4):1603–1607, 2007.

M. Madiman. On the entropy of sums. In *Proc. IEEE Inform. Theory Workshop*, pages 303–307. Porto, Portugal, 2008.

M. Madiman. A primer on entropic limit theorems. *Preprint*, 2017.

M. Madiman and A.R. Barron. The monotonicity of information in the central limit theorem and entropy power inequalities. In *Proc. IEEE Intl. Symp. Inform. Theory*, pages 1021–1025. Seattle, July 2006.

M. Madiman and A.R. Barron. Generalized entropy power inequalities and monotonicity properties of information. *IEEE Trans. Inform. Theory*, 53(7):2317–2329, July 2007.

M. Madiman and F. Ghassemi. The entropy power of sums is fractionally superadditive. In *Proc. IEEE Intl. Symp. Inform. Theory*, pages 295–298. Seoul, Korea, 2009.

M. Madiman and F. Ghassemi. Combinatorial entropy power inequalities: A preliminary study of the Stam region. *Preprint*, 2016.

M. Madiman and I. Kontoyiannis. The entropies of the sum and the difference of two IID random variables are not too different. In *Proc. IEEE Intl. Symp. Inform. Theory*, Austin, Texas, June 2010.

M. Madiman and I. Kontoyiannis. Entropy bounds on abelian groups and the Ruzsa divergence. *Preprint*, arXiv:1508.04089, 2015.

M. Madiman, A. Marcus, and P. Tetali. Information-theoretic inequalities in additive combinatorics. In *Proc. IEEE Inform. Theory Workshop*, Cairo, Egypt, January 2010.

M. Madiman, A. Marcus, and P. Tetali. Entropy and set cardinality inequalities for partition-determined functions. *Random Struct. Alg.*, 40:399–424, 2012.

M. Madiman, J. Melbourne, and P. Xu. Rogozin’s convolution inequality for locally compact groups. *Preprint*, 2016.

M. Madiman, L. Wang, and S. Bobkov. Some applications of the nonasymptotic equipartition property of log-concave distributions. *Preprint*, 2016.

M. Meyer and S. Reisner. Characterizations of affinely-rotation-invariant log-concave measures by section-centroid location. In *Geometric aspects of functional analysis (1989–90)*, volume 1469 of *Lecture Notes in Math.*, pages 145–152. Springer, Berlin, 1991.

M. Meyer and S. Reisner. A geometric property of the boundary of symmetric convex bodies and convexity of flotation surfaces. *Geom. Dedicata*, 37(3):327–337, 1991.

M. Meyer and S. Reisner. The convex intersection body of a convex body. *Glasg. Math. J.*, 53(3):523–534, 2011.
[119] E. Milman. Sharp isoperimetric inequalities and model spaces for the curvature-dimension-diameter condition. *J. Eur. Math. Soc. (JEMS)*, 17(5):1041–1078, 2015.

[120] V. Milman and L. Rotem. $\alpha$-concave functions and a functional extension of mixed volumes. *Electron. Res. Announc. Math. Sci.*, 20:1–11, 2013.

[121] V. Milman and L. Rotem. Mixed integrals and related inequalities. *J. Funct. Anal.*, 264(2):570–604, 2013.

[122] V. D. Milman. Inégalité de Brunn-Minkowski inverse et applications à la théorie locale des espaces normés. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(1):25–28, 1986.

[123] V. D. Milman. Entropy point of view on some geometric inequalities. *C. R. Acad. Sci. Paris Sér. I Math.*, 306(14):611–615, 1988.

[124] V. D. Milman. Isomorphic symmetrizations and geometric inequalities. In *Geometric aspects of functional analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 107–131. Springer, Berlin, 1988.

[125] V. D. Milman. Geometrization of probability. In *Geometry and dynamics of groups and spaces*, volume 265 of *Progr. Math.*, pages 647–667. Birkhäuser, Basel, 2008.

[126] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 64–104. Springer, Berlin, 1989.

[127] H. H. Nguyen and V. H. Vu. Small ball probability, inverse theorems, and applications. In *Erdős centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 409–463. János Bolyai Math. Soc., Budapest, 2013.

[128] M. Payaró and D. P. Palomar. Hessian and concavity of mutual information, differential entropy, and entropy power in linear vector Gaussian channels. *IEEE Trans. Inform. Theory*, 55(8):3613–3628, 2009.

[129] C. M. Petty. Affine isoperimetric problems. In *Discrete geometry and convexity (New York, 1982)*, volume 440 of *Ann. New York Acad. Sci.*, pages 113–127. New York Acad. Sci., New York, 1985.

[130] G. Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[131] A. Prékopa. Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math. (Szeged)*, 32:301–316, 1971.

[132] A. Prékopa. On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged)*, 34:335–343, 1973.

[133] E. Ram and I. Sason. On Rényi Entropy Power Inequalities. *Preprint*, 2016.

[134] O. Rioul. Information theoretic proofs of entropy power inequalities. *IEEE Trans. Inform. Theory*, 57(1):33–55, 2011.
[135] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[136] C. A. Rogers and G. C. Shephard. The difference body of a convex body. *Arch. Math. (Basel)*, 8:220–233, 1957.

[137] B. A. Rogozin. An estimate for the maximum of the convolution of bounded densities. *Teor. Veroyatnost. i Primenen.*, 32(1):53–61, 1987.

[138] M. Rudelson and R. Vershynin. Smallest singular value of a random rectangular matrix. *Comm. Pure Appl. Math.*, 62(12):1707–1739, 2009.

[139] M. Rudelson and R. Vershynin. Non-asymptotic theory of random matrices: extreme singular values. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1576–1602. Hindustan Book Agency, New Delhi, 2010.

[140] I. Z. Ruzsa. Generalized arithmetical progressions and sumsets. *Acta Math. Hung.*, 65(4):379–388, 1994.

[141] I. Z. Ruzsa. Entropy and sumsets. *Random Struct. Alg.*, 34:1–10, 2009.

[142] L. A. Santaló. An affine invariant for convex bodies of $n$-dimensional space. *Portugaliae Math.*, 8:155–161, 1949.

[143] G. Savaré and G. Toscani. The concavity of Rényi entropy power. *IEEE Trans. Inform. Theory*, 60(5):2687–2693, May 2014.

[144] A. Segal. Remark on stability of Brunn-Minkowski and isoperimetric inequalities for convex bodies. In *Geometric aspects of functional analysis*, volume 2050 of *Lecture Notes in Math.*, pages 381–391. Springer, Heidelberg, 2012.

[145] C.E. Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423, 623–656, 1948.

[146] D. Shlyakhtenko. Shannon’s monotonicity problem for free and classical entropy. *Proc. Natl. Acad. Sci. USA*, 104(39):15254–15258 (electronic), 2007. With an appendix by Hanne Schultz.

[147] B. Simon. *Convexity: An analytic viewpoint*, volume 187 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011.

[148] A.J. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Information and Control*, 2:101–112, 1959.

[149] Y. V. Stanchescu. An upper bound for $d$-dimensional difference sets. *Combinatorica*, 21(4):591–595, 2001.

[150] R. M. Starr. Quasi-equilibria in markets with non-convex preferences. *Econometrica*, 37(1):25–38, January 1969.

[151] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.
[152] S. J. Szarek and D. Voiculescu. Volumes of restricted Minkowski sums and the free analogue of the entropy power inequality. *Comm. Math. Phys.*, 178(3):563–570, 1996.

[153] S. J. Szarek and D. Voiculescu. Shannon’s entropy power inequality via restricted Minkowski sums. In *Geometric aspects of functional analysis*, volume 1745 of *Lecture Notes in Math.*, pages 257–262. Springer, Berlin, 2000.

[154] S. Takano. The inequalities of Fisher information and entropy power for dependent variables. In *Probability theory and mathematical statistics (Tokyo, 1995)*, pages 460–470. World Sci. Publ., River Edge, NJ, 1996.

[155] S. Takano. Entropy and a limit theorem for some dependent variables. In *Proceedings of Prague Stochastics ’98*, volume 2, pages 549–552. Union of Czech Mathematicians and Physicists, 1998.

[156] T. Tao. Sumset and inverse sumset theory for Shannon entropy. *Combin. Probab. Comput.*, 19(4):603–639, 2010.

[157] T. Tao and V. Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[158] T. Tao and V. Vu. From the Littlewood-Offord problem to the circular law: universality of the spectral distribution of random matrices. *Bull. Amer. Math. Soc. (N.S.)*, 46(3):377–396, 2009.

[159] G. Toscani. A Strengthened Entropy Power Inequality for Log-Concave Densities. *IEEE Trans. Inform. Theory*, 61(12):6550–6559, 2015.

[160] A. M. Tulino and S. Verdú. Monotonic decrease of the non-gaussianness of the sum of independent random variables: A simple proof. *IEEE Trans. Inform. Theory*, 52(9):4295–7, September 2006.

[161] C. Villani. A short proof of the “concavity of entropy power”. *IEEE Trans. Inform. Theory*, 46(4):1695–1696, 2000.

[162] L. Wang and M. Madiman. Beyond the entropy power inequality, via rearrangements. *IEEE Trans. Inform. Theory*, 60(9):5116–5137, September 2014.

[163] L. Wang, J. O. Woo, and M. Madiman. A lower bound on the Rényi entropy of convolutions in the integers. In *Proc. IEEE Intl. Symp. Inform. Theory*, pages 2829–2833. Honolulu, Hawaii, July 2014.

[164] E. M. Werner. Rényi divergence and $L_p$-affine surface area for convex bodies. *Adv. Math.*, 230(3):1040–1059, 2012.

[165] J. O. Woo and M. Madiman. A discrete entropy power inequality for uniform distributions. In *Proc. IEEE Intl. Symp. Inform. Theory*, Hong Kong, China, June 2015.

[166] P. Xu, J. Melbourne, and M. Madiman. Reverse entropy power inequalities for $s$-concave densities. In *Proc. IEEE Intl. Symp. Inform. Theory*, pages 2284–2288, Barcelona, Spain, July 2016.

[167] W. H. Young. On the multiplication of successions of Fourier constants. *Proc. Roy. Soc. Lond. Series A*, 87:331—339, 1912.
[168] Y. Yu. Letter to the editor: On an inequality of Karlin and Rinott concerning weighted sums of i.i.d. random variables. *Adv. in Appl. Probab.*, 40(4):1223–1226, 2008.

[169] Y. Yu. Monotonic convergence in an information-theoretic law of small numbers. *IEEE Trans. Inform. Theory*, 55(12):5412–5422, 2009.

[170] Y. Yu. On the entropy of compound distributions on nonnegative integers. *IEEE Trans. Inform. Theory*, 55(8):3645–3650, 2009.

[171] R. Zamir and M. Feder. A generalization of the entropy power inequality with applications. *IEEE Trans. Inform. Theory*, 39(5):1723–1728, 1993.

[172] R. Zamir and M. Feder. On the volume of the Minkowski sum of line sets and the entropy-power inequality. *IEEE Trans. Inform. Theory*, 44(7):3039–3063, 1998.