On Quantum Mechanics with a Magnetic Field on $\mathbb{R}^n$ and on a Torus $\mathbb{T}^n$, and Their Relation

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Abstract We show in elementary terms the equivalence in a general gauge of a $U(1)$-gauge theory of a scalar charged particle on a torus $\mathbb{T}^n = \mathbb{R}^n / \Lambda$ to the analogous theory on $\mathbb{R}^n$ constrained by quasiperiodicity under translations in the lattice $\Lambda$. The latter theory provides a global description of the former: the quasiperiodic wavefunctions $\psi$ defined on $\mathbb{R}^n$ play the role of sections of the associated hermitean line bundle $E$ on $\mathbb{T}^n$, since also $E$ admits a global description as a quotient. The components of the covariant derivatives corresponding to a constant (necessarily integral) magnetic field $B = dA$ generate a Lie algebra $g_Q$ and together with the periodic functions the algebra of observables $\mathcal{O}_Q$. The non-abelian part of $g_Q$ is a Heisenberg Lie algebra with the electric charge operator $Q$ as the central generator; the corresponding Lie group $G_Q$ acts on the Hilbert space as the translation group up to phase factors. Also the space of sections of $E$ is mapped into itself by $g \in G_Q$. We identify the so-called magnetic translation group as a subgroup of the observables' group $Y_Q$. We determine the unitary irreducible representations of $\mathcal{O}_Q, Y_Q$ corresponding to integer charges and for each of them an associated orthonormal basis explicitly in configuration space. We also clarify how in the $n = 2m$ case a holomorphic structure and Theta functions arise on the associated complex torus.

These results apply equally well to the physics of charged scalar particles on $\mathbb{R}^n$ and on $\mathbb{T}^n$ in the presence of periodic magnetic field $B$ and scalar potential. They are also necessary preliminary steps for the application to these theories of the deformation procedure induced by Drinfel’d twists.

Keywords Bloch theory with magnetic field · Fiber bundles · Gauge symmetry · Quantization on manifolds
1 Introduction

Tori are among the simplest manifolds with nontrivial topology. Gauge theories on them have deserved a lot of attention for both mathematical and physical reasons. As known, any $n$-dimensional (real) torus $\mathbb{T}^n$ can be described in a global way (i.e. without introducing local trivializations) as the quotient $\mathbb{T}^n = \mathbb{R}^n / \Lambda$ of $\mathbb{R}^n$ (the universal cover of the torus) over a lattice $\Lambda \subset \mathbb{R}^n$ of rank $n$. The physical and mathematical communities seem not equally aware that also gauge theories on tori (i.e. fiber bundles on $\mathbb{T}^n$) can be described in a global way as quotients of gauge theories with trivial topology (i.e. quotients of trivial fiber bundles) on $\mathbb{R}^n$. The first purpose of this paper is to present this fact in elementary terms sticking for simplicity to the $U(1)$-gauge theory of a scalar quantum particle (but keeping $n$ and the gauge generic): we show that on $\mathbb{T}^n$ such a theory is in fact equivalent to the analogous (and much simpler) theory on $\mathbb{R}^n$ constrained by the requirement that under translations by a $\lambda \in \Lambda$ the wavefunctions $\psi(x)$ be quasiperiodic, i.e. periodic up to a $x$-dependent factor $V$; in more mathematical terms, we show that the space of sections of a hermitean line bundle $E$ on $\mathbb{T}^n$ can be described as a quasiperiodic subspace $\mathcal{X}^V \subset C^\infty(\mathbb{R}^n)$ of the space of sections of the (topologically trivial) hermitean line bundle on $\mathbb{R}^n$, modulo gauge transformations on $\mathbb{R}^n$. The second purpose is to exploit this equivalence to determine the symmetries and give a rather explicit and detailed description of the irreducible unitary representations of these theories.

Scalar $U(1)$-gauge theory are physically interesting in themselves, beside being propaedetical to gauge theories with more complicated gauge groups, particles with spin, etc. The Schrödinger equation for a scalar particle with electric charge $q$

$$H\psi = i\partial_t \psi, \quad H := \frac{1}{2m} \nabla_a \nabla_a + V, \quad \nabla_a := -i\partial_a + qA_a, \quad (1)$$

(in units such that $\hbar = 1$) when considered on $\mathbb{T}^2$ can describe the dynamics of such a particle confined on a very thin domain (within the physical space $\mathbb{R}^3$) modelled as a 2-torus (a preliminary step to study e.g. the Quantum Hall effect on it). Formulating (1) on $\mathbb{T}^3$ instead of $\mathbb{R}^3$ is a way to obtain a relatively easy finite-volume theory before e.g. taking the infinite-volume limit. Similarly, $S = (\psi, H\psi)$ appears as the Euclidean action in the path-integral quantization of a charged scalar field in an electromagnetic background and one may wish to compactify the 4-dimensional Euclidean spacetime into a torus $\mathbb{T}^4$ as an infrared cutoff. In string and Kaluza-Klein theories [9, 19], with or without D-branes, the same action on tori $\mathbb{T}^n$ of various (usually even) dimensions resulting from the compactification of the extra-dimensions may appear as part of the total action. On the other hand, the equivalent quasiperiodic theories on $\mathbb{R}^n$ are characterized by periodic $B$ and scalar potential $V$, and correspondingly $H$ becomes the Hamiltonian of Bloch electrons in a magnetic field $B$. The corresponding dynamical evolution maps $\mathcal{X}^V$ into itself.

The plan of the work is as follows. In Sect. 2 we introduce quasiperiodicity factors $V$, spaces $\mathcal{X}^V$ and show that a compatible connection $A$ on $\mathbb{R}^n$ necessarily yields a periodic curvature 2-form $B = dA$ with (up to a factor) integer-valued fluxes $\phi_{ab}$ through the basic plaquettes of the lattice (on $\mathbb{T}^n$ these integers will represent the Chern numbers). In the proof one has to use a self-consistency condition for $V$; on $\mathbb{T}^n$ the latter becomes the cocycle condition for the transition functions of a hermitean line bundle (Sect. 4). $V$ is determined by the $\phi_{ab}$ up to gauge transformations, and is necessarily a phase factor. One can decompose the covariant derivative as $\nabla = \nabla^{(0)} + QA'$, where $\nabla^{(0)}$ yields a constant $QB \propto [\nabla^{(0)}, \nabla^{(0)}]$ and $A'$ is periodic. The components $p_{ab} \equiv \nabla_a^{(0)}$ generate a Lie algebra $\mathfrak{g}_O$ and together with the periodic functions the algebra of observables $\mathcal{O}_G$, whose elements map $\mathcal{X}^V$ into itself.
In particular, $H, \nabla_a \in O_Q$. The non-abelian part of $g_Q$ is a Heisenberg Lie algebra with the electric charge operator $Q$ as the central generator. The corresponding Lie group $G_Q$ acts unitarily on $X^V$ and its Hilbert space completion $H^V$ as the translation group up to phase factors; the latter are trivial if $V = 1$, i.e. if the $q_{\alpha \beta}$ vanish. For these reasons we call $G_Q$ the projective translation group. We also introduce a larger group $Y_Q$, which we call observables’ group, of unitary operators on $H^V$ including the functions $e^{il \cdot x}$ ($l \in \mathbb{Z}^n$) as well. In Sect. 3 we replace $Q \rightarrow q \in \mathbb{Z}$ (integer charge) and identify the magnetic translation group $M$ of $[26–29]$ as the discrete subgroup of $Y_Q$ commuting with $G_Q$, the discrete part of the centre of $Y_Q$ as a subgroup of $M$. Then we classify the unitary irreducible representations of $O_Q, Y_Q$ with integer charge and determine them rather explicitly in configuration space (we write down orthonormal bases of eigenfunctions of a complete set of commuting observables in $O_Q$); for fixed Chern numbers they are parametrized by a point $\tilde{a}$ in the reciprocal torus of $\mathbb{R}^n$ (the ‘Jacobi torus’, or ‘Brillouin zone’). Each representation of $Y_Q$ extends to the Hilbert space completion $H^V$ of $X^V$; $H^V$ carries also the associated representation of the group algebra $\mathcal{P}_Q$ of $Y_Q$ (a $C^*$-algebra). We also clarify how in the $n = 2m$ case a holomorphic structure and Theta functions arise on the associated complex torus. In Sect. 4 we show how to realize a hermitean line bundle $E \rightarrow \mathbb{T}^n$ as a quotient $E = (\mathbb{R}^n \times \mathbb{C})/\mathbb{Z}^n$, where the free abelian action of $\mathbb{Z}^n$ on $\mathbb{R}^n \times \mathbb{C}$ involves the quasiperiodicity factor $V$; we then construct a one-to-one correspondence between $(X^V, V)$ and pairs $(\Gamma(\mathbb{T}^n, E), \nabla)$, where $\Gamma(\mathbb{T}^n, E)$ stands for the space of sections of $E$ and $\nabla$ the associated covariant derivative. In general globally defined gauge transformations on $\mathbb{R}^n$ induce locally defined gauge transformations on $\mathbb{T}^n$, $\Gamma(\mathbb{T}^n, E)$ becomes a $G_Q$-module upon lifting of the action from $X^V$. In Sect. 5 we summarize the main conclusions of the work.

The subject is not new. That unitary representations of the covering group of the universal cover of a manifold transform trivial bundles on the former and their sections into bundles on the latter and their sections is explained e.g. in [10, 20]. This is applied to the case of constant magnetic field on $\mathbb{T}^2, \mathbb{R}^2$ and $\mathbb{T}^n, \mathbb{R}^n$ resp. in [3, 22, 23]. In [1, 10] it is shown (for $n = 2$ and general $n$, respectively) that the Hilbert space of states of the theory on $\mathbb{R}^n$ with a given periodic magnetic field $B$ is the direct integral of the Hilbert spaces of all the inequivalent theories on $\mathbb{T}^n$ characterized by the same $B$. In [8] it is shown that $U(1)$ gauge covariant quantum mechanics on a torus is—in the sense of geometric quantization [13, 24]—an example of a full quantization of a symplectic manifold, and not only a pre-quantization. Several results in Sects. 2, 4 are the analog in the smooth framework of known results in the holomorphic one (see e.g. [2, 18]), but our treatment is based only on the basic definition of a real torus; additional structures such as holomorphic ones on complex tori (which can arise only for $n = 2m$), complex abelian varieties, etc., are not necessary. The clear definition and decomposition of the algebra/group of observables $O_Q, Y_Q$, the physical identification of the central generator $Q$ as the electric charge operator, the complete classification of the irreducible unitary representations of $Y_Q$, as well as an explicit determination of the latter in configuration space, up to our knowledge are new. Moreover, our treatment is valid not only for all $n$ but also for all $\alpha$ gauges. This is not the case in [22, 23] and is not evident e.g. in [2, 18], where the presence of the gauge group is hidden by the fact that automophy factors, that play a role similar to our quasiperiodicity factors $V$, may take values in all of $\mathbb{C} \setminus \{0\}$ rather than in the gauge group $U(1)$.

Finally, the above results provide a setting which allows to apply the so-called twist-induced deformation procedure to the quantum physics of charged scalar particles on $\mathbb{R}^n$ and on $\mathbb{T}^n$ in the presence of periodic magnetic field $B$ and scalar potential: in fact the Drinfel’d twist $\mathcal{E}[4, 5]$, which is the fundamental object on which the deformation is based, has to be a formal power series in the deformation parameter $\lambda$ with coefficients in $Ug_Q \otimes Ug_Q$, so that
their action is globally defined on the bundles, rather than in the abelian algebra \( Ug_0 \otimes Ug_0 \) (as would be suggested by a naive extension of the Groenewold-Moyal-Weyl deformation of \( \mathbb{R}^n \)), which in the most general form are defined only locally on the bundles. This will allow to extend the deformation procedure of Ref. [6] from Drinfel’d twists of abelian type to Drinfel’d twists \( \mathcal{F} \in (Ug_0 \otimes Ug_0)[[\lambda]] \) of general type (even the ones leading to strictly quasitriangular Hopf algebras).

For simplicity we shall assume \( \Lambda = 2\pi \mathbb{Z}^n \), whence the reciprocal lattice is \( \mathbb{Z}^n \). This is no loss of generality as far as we are concerned, since \( \Lambda \) can be always transformed into \( 2\pi \mathbb{Z}^n \) by a linear transformation of \( \mathbb{R}^n \), \( x \mapsto gx \), \( g \in GL(n) \).\(^1\) We shall use the following abbreviations. \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \); \( K^t \) stands for the transpose of a matrix \( K \); elements \( h, k \in \mathbb{C}^n \) are considered as columns; \( h \cdot k := h^*k \) (at the rhs the product is row by column); \( u^l_i := e^{il \cdot x} \) for all \( l := (l_1, \ldots, l_m) \in \mathbb{Z}^n \); \( U(1) \) stands for the group of complex numbers of modulus 1; we denote as \( \mathcal{Z}(A) \) the center of a group or an algebra \( A \). We denote as \( \mathcal{X} \) the subalgebra of \( C^\infty(\mathbb{R}^n) \) consisting of periodic functions \( f \), namely such that \( f(x + 2\pi l) = f(x) \) for all \( l \in \mathbb{Z}^n \), or equivalently of functions (Laurent series) of \( u \equiv (u^1, \ldots, u^n) \equiv (e^{ix_1}, \ldots, e^{ix_n}) \) only. \( \mathcal{X} \) can be identified also with \( C^\infty(\mathbb{T}^n) \). We denote as \( \{e_1, \ldots, e_n\} \) the canonical basis of \( \mathbb{R}^n \): \( e_1 := (1, 0, \ldots, 0) \), etc.

2 Quasiperiodic Wavefunctions and Related Connections on \( \mathbb{R}^n \)

The simplest way to impose invariance of the particle probability density under discrete translations \( \lambda \in \Lambda \) is to impose it on the wavefunction \( \psi \), i.e. to require it to be periodic. But this is not necessary; it suffices to require \( |\psi| \) to be periodic, that is \( \psi \) to be quasiperiodic, i.e. invariant up to a phase factor \( V \). A set of quasiperiodicity conditions of the form

\[
\psi(x + 2\pi l) = V(l, x)\psi(x) \quad \forall x \in \mathbb{R}^n, \quad l \in \mathbb{Z}^n
\]

relates the values of \( \psi \) in any two points \( x, x + 2\pi l \) of the lattice \( x + 2\pi \mathbb{Z}^n \) through a phase factor \( V(l, x) \).\(^2\) For a Bloch electron in a crystal in absence of magnetic field a factor depending only on \( l \) is enough: \( V = e^{ik \cdot l} \), where \( k \) is the quasimomentum of the electron. But one can allow \( V \) to depend also on \( x \). In any case nontrivial solutions \( \psi \) of (2) may exist only if the factors relating three generic points \( x, x + 2\pi l, x + 2\pi (l + l') \) of the lattice are consistent with each other, i.e.

\[
V(l + l', x) = V(l, x + 2\pi l')V(l', x), \quad \forall l, l' \in \mathbb{Z}^n.
\]

Note that this implies \( V(0, x) \equiv 1 \) and \( [V(l, x)]^{-1} = V(-l, x + 2\pi l) \). We introduce an auxiliary Hilbert space \( \mathcal{H}_Q \) with an orthonormal basis \( \{|q\rangle\}_{q \in \mathbb{Z}} \) and on \( \mathcal{H}_Q \) a self-adjoint operator \( Q \) defined by \( Q|q\rangle = q|q\rangle \). We regard a smooth wavefunction \( \psi \) of a particle with electric charge \( q \) (in e units) as an element of \( C^\infty(\mathbb{R}^n) \otimes |q\rangle \). As the latter is an eigenspace with eigenvalue \( q \) of \( 1 \otimes Q \), we shall adopt \( 1 \otimes Q \) as the electric charge operator. We give the covariant derivative a form independent of \( q \) through \( \nabla := -(i)d \otimes 1 + A(x) \otimes Q \);

\(^1\)As the holomorphic structure w.r.t. the complex variables \( z^l = x^l + i\alpha^{m+l} \) is not invariant under \( x \mapsto gx \) for generic \( g \in GL(n) \), the choice \( \Lambda = 2\pi \mathbb{Z}^n \) would be a loss of generality if \( n = 2m \) and we were concerned with holomorphic line bundles on the complex \( m \)-torus \( \mathbb{T}^m = \mathbb{C}^m/\Lambda \). See also the end of Sect. 3.

\(^2\)Actually it is not necessary to assume from the start that \( V \in U(1) \); assuming just that it is nonvanishing complex, \( V \in U(1) \) will follow from the reality of \( A_n \), see below.
here $d$ stands for the exterior derivative. When we risk no confusion we shall abbreviate $\nabla = -id + A(x)Q$, $\psi \in C^\infty(\mathbb{R}^n|q)$, etc. Given a smooth function $V : \mathbb{Z}^n \times \mathbb{R}^n \mapsto \mathbb{C} \setminus \{0\}$ fulfilling (3) consider the space

$$\mathcal{X}^V := \{ \psi \in C^\infty(\mathbb{R}^n) \otimes |q| \mid \psi(x + 2\pi l) = V(l, x)\psi(x) \forall x \in \mathbb{R}^n, l \in \mathbb{Z}^n \}. \quad (4)$$

We look for covariant derivatives $\nabla$ such that their components map $\mathcal{X}^V$ into itself,

$$\nabla_a : \mathcal{X}^V \mapsto \mathcal{X}^V. \quad (5)$$

Given such a $\nabla$, also $QB_{ab}(x)\psi(x) = \{ \frac{i}{2} [\nabla_a, \nabla_b]\psi \}(x)$ fulfills (2), implying that all the $B_{ab} = \frac{i}{2}(\partial_a A_b - \partial_b A_a)$ are periodic functions. From the Fourier expansions it follows

$$B_{ab}(x) = \beta_{ab}^A + \sum_{l \neq 0} \beta_{ab}^l e^{ilx} \Rightarrow A_a(x) = x^b \beta_{ba}^A + \alpha_a + \sum_{l \neq 0} \alpha_a^l e^{ilx} \quad (6)$$

up to a gauge transformation, where the periodic function $A'(x)$ is such that $B' = dA'$. A dependence of $\psi$, $A'$, $V$ also on the time variable $t$ is not excluded, but we will not write the argument $t$ explicitly. The reasons why we have isolated the constant parts $\beta_{ab}^A, \alpha_a \in \mathbb{R}$ in the expansions of $B_{ab}, A_a$ will become clear below. We decompose the covariant derivative in a gauge-independent part $A'_a Q$ and a gauge-dependent part $p_a$:

$$\nabla_a := -i \partial_a + A_a Q = p_a + A'_a Q, \quad A'_a \in \mathcal{X}; \quad (7)$$

$p_a = -i \partial_a + x^b \beta_{ba}^A Q + \alpha_a Q$ up to a gauge transformation. Going back to (5), $\nabla_a \psi$ will fulfill (2) iff also $p_a \psi$ does, by the periodicity of $A'_a(x)$; up to a gauge transformation this yields the first formula in

$$V(l, x) \equiv V^A(l, x) = e^{-iq2\pi l^b \beta^A_{ba}}, \quad p_a = -i \partial_a + x^b \beta_{ba}^A Q + \alpha_a Q, \quad (8)$$

which is consistent with (3) for all eigenvalues $q \in \mathbb{Z}$ of $Q$ iff the quantization conditions

$$v_{ab} \in \mathbb{Z}, \quad v_{ab} := 2\pi \beta_{ab}^A \quad (9)$$

for all $a, b$ are satisfied. (This in particular excludes a dependence of $\beta^A$ on $t$). For $q = 0$ or $\beta^A = 0$ we find $V \equiv 1$ and $\mathcal{X}^1 = \mathcal{X} \otimes |0| \simeq \mathcal{X}$. Otherwise (2), (8)$_1$ do not admit solutions of the form $\psi(x) = f(x), \quad f \in \mathcal{X}$. Introducing fundamental $k$-dimensional cells $C_{a_1 \cdots a_k}^y$ for $k \leq n$ and $a_1 < a_2 < \cdots < a_k$ by

$$C_{a_1 \cdots a_k}^y := \{ x \in \mathbb{R}^n \mid x^{a_h} \in [y^{a_h}, y^{a_h} + 2\pi], \quad h = 1, \ldots, k; \quad x^a = y^a \text{ otherwise} \}; \quad (10)$$

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3. $0 \equiv (p_a \psi)(x + 2\pi l) - V(l, x)(p_a \psi)(x) = (-i \partial_a + x^b \beta_{ba}^A q + 2\pi l_b \beta_{ba}^A q + \alpha_a q) \psi(x) - (-i \partial_a + x^b \beta_{ba}^A q + \alpha_a q) V(l, x)\psi(x) \quad \Rightarrow \quad (8)$.\footnote{As $V^{-1}(l + l', x) V(l, x + 2\pi l') V(l', x) = e^{-iq2\pi l' \beta^A_{l'b}}$, (3) amounts to $q 2\pi l' \beta^A_{l'} \in \mathbb{Z}$ for all $q \in \mathbb{Z}$ and $l, l' \in \mathbb{Z}^n$, i.e. to (9).}
one easily finds that the flux \( \phi_{ab} \) of \( B = B_{ab}dx^a dx^b \) through a plaquette \( C_{ab}^y \) equals that of \( \beta^A = \beta^A_{ab} dx^a dx^b \)

\[
\phi_{ab} = \int_{C_{ab}^y} B = \int_{C_{ab}^y} \beta^A = 2\pi v_{ab}
\]

(11)

and more generally\(^5\)

\[
\int_{C_{a_1 \cdots a_{2m}}} B^m = \int_{C_{a_1 \cdots a_{2m}}} (\beta^A)^m = (2\pi)^m v_{a_1a_2v_{a_{2m-1}a_{2m}}}
\]

(12)

for all \( m \leq n/2 \) and \( a_1 < a_2 < \cdots < a_{2m} \); the square bracket denotes antisymmetrization w.r.t. the indices \( a_1a_2 \cdots a_{2m} \). The results are independent of \( y \). From (8)–(9) we easily find that

\[
\frac{i}{4\pi} \log \left[ \frac{V(l, x + 2\pi l') V(l', x)}{V(l', x + 2\pi l) V(l, x)} \right] = qll' =: \tilde{v}(l, l')
\]

(13)

is \( x \)-independent and defines an antisymmetric bilinear \( \mathbb{Z} \)-valued form \( \tilde{v} \) on \( \Lambda = \mathbb{Z}^n \); later we shall see that it is also gauge-independent. Hence \( 2\pi \beta^A = \tilde{v} \) can be recovered from the quasiperiodicity factor \( V \), in any gauge.

As side-remarks we note that: 1. Relation (8)\(_1\) implies that \( V(l, x) \) could not be an arbitrary complex number: it must be \( V(l, x) \in U(1) \) in the gauge under consideration and, by (26)\(_3\), in all gauges. 2. The \( V^A \beta^A \) make up a (discrete) abelian group upon pointwise multiplication \( V^A \beta^A V^\beta^A = V^\beta^A + \beta^A \), and similarly the \( \psi^A \), but we will not use this fact here. 3. Rational values for the expressions \( 2\pi \beta^A_{ab} = \frac{i}{2\pi} \phi_{ab} \) could be obtained imposing quasiperiodicity on some \( k \)-fold enlarged lattice (\( k \in \mathbb{N} \)), as done e.g. in [10, 17].\(^6\)

Summing up, the (left) action of \( Q, p_a \) and multiplication (from either side) by any \( f \in \mathcal{X} \) map \( \mathcal{X}^V \) into itself. In other words, \( \mathcal{X}^V \) is a \( \mathcal{X} \)-bimodule and a (left) module of the \( * \)-algebra \( \mathcal{O}_Q \) of smooth differential operators that are polynomials in \( Q, p_1, \ldots, p_m \) with coefficients \( f \) in \( \mathcal{X} \), constrained by

\[
[p_a, p_b] = -i2\beta^A_{ab} Q, \quad [Q, \cdot] = 0, \quad [p_a, f] = -i(\partial_a f), \quad f^*(x) = \overline{f(x)}, \quad p_a^* = p_a, \quad Q^* = Q.
\]

(14)

\(^5\)In fact,

\[
\int_{C_{a_1 \cdots a_{2m}}} B^m = \int_{C_{a_1 \cdots a_{2m}}} (\beta^A + d\Lambda) B^m = \int_{C_{a_1 \cdots a_{2m}}} \beta^A B^{m-1} + \int_{C_{a_1 \cdots a_{2m}}} d(\Lambda'B^{m-1})
\]

\[
= \int_{C_{a_1 \cdots a_{2m}}} \beta^A B^{m-1} + \int_{a_{C_{a_1 \cdots a_{2m}}} \Lambda'B^{m-1}} = \int_{C_{a_1 \cdots a_{2m}}} \beta^A B^{m-1} = \cdots
\]

\[
= \int_{C_{a_1 \cdots a_{2m}}} (\beta^A)^m = \int_{C_{a_1 \cdots a_{2m}}} \beta^A \cdots \beta^A_{a_{2m-1}a_{2m}} dx^{a_1} \cdots dx^{a_{2m}}
\]

\[
= (2\pi)^m v_{a_1a_2v_{a_{2m-1}a_{2m}}}
\]

(9)

The second equality holds because \( dB = 0 \), the third by Stokes theorem, the fourth by the periodicity of \( \Lambda', \beta \), which makes the border integral vanish.

\(^6\)For instance, imposing the conditions (2)–(3) for all \( q \in \mathbb{Z} \) and only for \( l, l' \) such that \( l_1 = kh_1, l'_1 = kh'_1 \) (with some \( h_1, h'_1 \in \mathbb{Z} \)) would lead to \( kv_{Ib}, kv_{b1} \in \mathbb{Z} \) i.e. \( v_{Ib}, v_{b1} \in Q \), and again to \( v_{ab} \in \mathbb{Z} \) for \( a, b > 1 \). The corresponding \( \psi \)'s could be interpreted as \( k \)-component wavefunctions, i.e. sections of a \( \mathbb{C}^k \)-vector bundle.
Note that these defining relations of $\mathcal{O}_Q$ depend on the $A_a$ only through the $\beta^{A}_{ab}$ of (9), so are gauge-independent. In particular $\nabla_a, H \in \mathcal{O}_Q$.

$Q, p_a$ generate a real Lie algebra $\mathfrak{g}_Q$ which is a (non-abelian) central extension of the abelian Lie algebra $\mathbb{R}^n$ spanned by the $-i\partial_j$; the $-2\beta^{A}_{ab}$ play the role of structure constants. $\mathcal{O}_Q$ and $\mathcal{X}$ are $U\mathfrak{g}_Q$-module $*$-algebras under the action

$$
p_a \triangleright p_b = -i2\beta^{A}_{ab} Q, \quad p_a \triangleright f = -i(\partial_a f), \quad Q \triangleright f = 0, \quad Q \triangleright p_a = 0,
$$

(15)

for all $f \in \mathcal{X}$, and $\mathcal{X}^V$ is a left $U\mathfrak{g}_Q$-equivariant $\mathcal{O}_Q$-module and $\mathcal{X}$-bimodule (but not an algebra, unless $V = 1$); this means that all these structures are compatible with each other and the Leibniz rule. 7 ‘Exponentiating’ $\mathfrak{g}_Q$ we obtain a Lie group $G_Q$ of transformations $g_z : \mathcal{X}^V \mapsto \mathcal{X}^V$, $z \equiv (z^0, z) \in \mathbb{R}^{n+1}$:

$$
g_z = e^{i(p \cdot z + Qz^0)}; \quad g_z g_z' = g_{z + z'}, \quad \tilde{z} + \tilde{z}' := \tilde{z} + \tilde{z}' - (z^I \beta^A z', 0, \ldots, 0).
$$

(16)

This group law formally follows from the Baker-Campbell-Hausdorff formula. 8 In the gauge (8) the actions of $Q, p_a$ and $g_z$ read

$$
Q \triangleright \psi = q\psi, \quad p_a \triangleright \psi = (-i \partial + q x^I \beta^A + qa) \psi, \quad [g_z \triangleright \psi](x) = e^{iq(x^0 + x^I \beta^A z + a^I z)} \psi(x + z);
$$

(18)

as a consistency check one can verify the group law (16) and that $g_z \triangleright \psi$ indeed fulfills (2). We shall call $G_Q$ the projective translation group because it acts as the abelian group $G_0$ of translations $\psi(x) \mapsto \psi(x + z)$ followed by multiplication by a phase factor, which is necessary to obtain again a wavefunction in $\mathcal{X}^V$. The phase factor reduces to 1 only for $\beta^A = 0$ or on carrier spaces characterized by $q = 0$, i.e. on $C^\infty(\mathbb{R}^n), \mathcal{X}$;

Let $L(n, \mathbb{Z})$ be the group of matrices with integer entries and determinant equal to $\pm 1$, $r$ the integer $r := 1/2 \text{rank}(\beta^A)$ (we recall that the rank of an antisymmetric matrix is always even); clearly $0 \leq 2r \leq n$. By the Frobenius theorem (see e.g. [11], p. 71) there exists a $S \in L(n, \mathbb{Z})$ such that

$$
\beta^A = S \tilde{\beta^A} S, \quad \tilde{\beta^A} := \begin{pmatrix} b & -b \\ 0_{n-2r} \end{pmatrix}, \quad b := \text{diag}(b_1, \ldots, b_r),
$$

(19)

where $(v_1, \ldots, v_r) := (2\pi b_1, \ldots, 2\pi b_r)$ is a sequence of positive (or negative) integers such that each $v_{j+1}$ is an integer multiple of $v_j$ [e.g. $(v_1, \ldots, v_4) = (3, 6, 18, 18)$]. $0_{n-2r}$ is the zero $(n - 2r) \times (n - 2r)$ matrix and all the missing blocks are zero matrices of the appropriate sizes. Therefore, after the change of generators

$$
p_a \mapsto (S^I p)_a, \quad x^I \mapsto (S^{-1} x)^I \Rightarrow u^I \mapsto u^{(S^{-1})^I},
$$

(20)

7Namely, for all $c \in \mathcal{O}_Q, \psi \in \mathcal{X}^V, f \in \mathcal{X}, g \in \mathfrak{g}_Q$ $g \triangleright (c \psi f) = (g \triangleright c) \psi f + c(g \triangleright \psi) f + c\psi (g \triangleright f).

8$$
e^{R, S} e^{R + S} e^{-1/2[R, S]}, \quad \text{if } [R, S] \text{ commutes with } R, S.
$$

(17)
resp. in \( \mathfrak{g}_Q \), \( C^\infty(\mathbb{R}^n) \) and \( \mathcal{X} \), the relations (14) will involve the matrix \( \tilde{\beta}^A \) instead of the original \( \beta^A \); more explicitly, (14)\(_1\) will become the commutation relations

\[
[p_j, p_{r+j}] = i2b_j Q \quad j = 1, \ldots, r, \quad [p_a, p_b] = 0 \quad \text{otherwise.} \tag{21}
\]

This shows that

\[
\mathfrak{g}_Q \simeq \mathfrak{h}_{2r+1} \oplus \mathbb{R}^{n-2r}, \quad G_Q \simeq H_{2r+1} \times \mathbb{R}^{n-2r}; \tag{22}
\]

\( \mathfrak{h}_{Qm}, H_{Qm} \) denote the Heisenberg Lie algebra, group of dimension \( m \) and central generator \( Q \).

The Weyl forms of \( x^a = x^a \), (14) and its consequence \( [p_a, x^b] = -i\partial_b^a \), are easily determined with the help of (17) and synthetically read

\[
e^{i(h\cdot p + y\cdot Q^0)} = e^{i[(h\cdot k)\cdot x + p\cdot (y\cdot z) + Q^0\cdot Q^0]} e^{-\frac{i}{2}[k\cdot y - h\cdot z + 2Q^0\cdot Q']}
\]

\[
\left[e^{i(h\cdot p + y\cdot Q^0)}\right]^* = e^{-i(h\cdot p + y\cdot Q^0)}
\]

for any \( h, k \in \mathbb{R}^n \) and \( (y^0, y), (z^0, z) \in \mathbb{R}^{n+1} \). We define groups \( Y_Q, G_Q, L \) by

\[
Y_Q := \left\{ e^{i(l\cdot x + p\cdot z + h\cdot Q^0)} \mid h^0 \in \mathbb{R}, l \in \mathbb{Z}^n, (z^0, z) \in \mathbb{R}^{n+1} \right\},
\]

\[
G_Q := \left\{ e^{i(p\cdot z + Q^0)} \mid (z^0, z) \in \mathbb{R}^{n+1} \right\},
\]

\[
L := \left\{ e^{i(l\cdot h + h\cdot Q^0)} \mid l \in \mathbb{Z}^n, h^0 \in \mathbb{R} \right\};
\]

the group laws can be read off (23) and depend only on \( \beta^A \). \( L \) is isomorphic to \( \mathbb{Z}^n \times U(1) \) and is a normal subgroup of \( Y_Q \); actually \( Y_Q \) is the semidirect product \( Y_Q = G_Q \rtimes L \). All elements of \( Y_Q \) are unitary w.r.t. the involution (23)\(_2\). We shall call \( Y_Q \) the observables' group because its elements are observables, i.e. map \( \mathcal{X} \) onto itself; the \( G_Q \) part acts as in (18), the \( L \) part acts by multiplication. Moreover, we shall call \( Y_Q \) the group algebra of \( Y_Q \); it is a \( C^* \)-algebra.

By (2) \( \psi^* \psi \) is periodic for all \( \psi', \psi \in \mathcal{X}^V \), and the formula

\[
(\psi', \psi) := \int_{\mathcal{C}_{1\ldots n}^{\mathcal{Y}}} d^n x \psi'^*(x)\psi(x),
\]

where \( \mathcal{C}_{1\ldots n}^{\mathcal{Y}} \) is any fundamental \( n \)-dimensional cell (10) [e.g. \( \mathcal{C}_{1\ldots n}^{\mathcal{O}} = ([0, 2\pi]^{n}) \), defines a hermitean structure in \( \mathcal{X}^V \) making the latter a pre-Hilbert space. As \( p_a \triangleright (\psi^* \psi) = p_a(\psi^* \psi) = -i\partial_a(\psi^* \psi) \), which has a vanishing integral, by the Leibniz rule the \( p_a \) are essentially self-adjoint. We shall call \( \mathcal{H}^V \) the Hilbert space completion of \( \mathcal{X}^V \). \( Y_Q \) extends as a group of unitary transformations of \( \mathcal{H}^V \); the \( L \) part still acts by multiplication, and the \( G_Q \) part in the above gauge acts still as in (18)\(_3\).

Fixed any function \( \psi_0 \in \mathcal{X}^V \) vanishing nowhere, \( \psi\psi_0^{-1} \) is well-defined and periodic, i.e. in \( \mathcal{X} \), for all \( \psi \in \mathcal{X}^V \), whence the decomposition \( \mathcal{X}^V = \mathcal{X} \psi_0 \).

Given a \( \ast \)-representation \( \rho(\mathcal{O}) \), \( \mathcal{X}^V \) of \( \mathcal{O} \) as a \( \ast \)-algebra of operators on the pre-Hilbert space \( \mathcal{X}^V \) (by definition \( \sigma \triangleright \psi := \rho(\sigma) \psi \) for all \( \sigma \in \mathcal{O} \)), and \( \psi \in \mathcal{X}^V \), a unitary equivalent one is obtained through a smooth gauge transformation \( U = e^{i\psi \phi} \) \( \phi(x) \) smooth and real valued) acting as the unitary transformation \( \rho(\mathcal{O}), \mathcal{X}^V \) \( \mapsto \rho(U(\mathcal{O}), \mathcal{X}^V U) \) defined by

\[
\rho_U(\sigma) = U \rho(\sigma) U^{-1}, \quad \psi^U = U \psi, \quad V^U(l, x) = U(x + 2\pi l) V(l, x) U^{-1}(x). \tag{26}
\]
With $o \in Y_Q$ and $\psi \in \mathcal{H}^V$ (26) defines also a unitary transformation $(\rho(Y_Q), \mathcal{H}^V) \mapsto (\rho^U(Y_Q), \mathcal{H}^{V^U})$. Although the realization of the $p_a$ is gauge-dependent, all the relations (2)–(5), (6) (7), (9)–(16), (19)–(25) remain valid; in particular, the $\mathbb{Z}$-valued 2-form $\tilde{v}$ (13) is gauge-invariant and allows to recover form the quasiperiodicity factors the integers $2\pi \beta^A = \nu$. Calling $(\rho(O_Q), \mathcal{X}^V)$ the representation [based on (8)] that we have used so far, choosing $U(x) = e^{i x^2 \beta^S x}$ and setting $\beta := \beta^A + \beta^S$, we find an equivalent representation $(\rho^U(O_Q), \mathcal{X}^{V^U})$ characterized by

$$V^U(l,x) = e^{-i q 2\pi l \beta(x+\iota n)}, \quad p_a = -i \partial_a + x^b \beta_{ba} Q + \alpha_a Q$$

(27) [for $U(x) \equiv 1$, i.e. $\beta = \beta^A$, we recover the original gauge (8)]. We shall adopt the shorter notations $\mathcal{X}^\beta \equiv \mathcal{X}^{V^U}$, $\mathcal{H}^\beta \equiv \mathcal{H}^{V^U}$, etc. for the spaces of complex functions fulfilling (2) with $V^U$ given by (27). Performing a change (20) and choosing $\beta^S$ so that $\beta$ becomes lower-triangular we find

$$\beta \equiv \bar{\beta} = \begin{pmatrix} 2b & 0_r \\ 0_{n-2r} & \end{pmatrix}$$

(28) and Eq. (2) becomes

$$\psi(x + 2\pi l) = e^{-i 2q \sum_{j=1}^r v_j l_{r+j} x^j} \psi(x) \quad \forall x \in \mathbb{R}^n, \quad l \in \mathbb{Z}^n.$$  

(29)

Choosing $l_{r+1} = \cdots = l_{2r} = 0$ one finds that $\psi(x)$ is periodic in $x^1, \ldots, x^r, x^{2r+1}, \ldots, x^n$. It is straightforward to check that on the Fourier decomposition of $\psi$ w.r.t. $x^1, \ldots, x^r$

$$\psi(x) = \sum_{k \in \mathbb{Z}^r} e^{i \sum_{j=1}^r k_j x^j} \psi_k(x^{r+1}, \ldots, x^n),$$

condition (29) for arbitrary $l$ reduces to the recurrence relations

$$\psi_{k+2q(v_1 l_{r+1}, \ldots, v_r l_2)}(x^{r+1}, \ldots, x^n) = \psi_k(x^{r+1} + 2\pi l_{r+1}, \ldots, x^{2r} + 2\pi l_{2r}, x^{2r+1}, \ldots, x^n).$$

By the latter one can express all $\psi_k$ in terms of those with $k \in K$, where

$$K := \{0, 1, \ldots, |2q v_1| - 1\} \times \cdots \times \{0, 1, \ldots, |2q v_r| - 1\} \subset \mathbb{Z}^r. \quad (30)$$

Therefore the most general solution of (29) reads

$$\psi(x) = \sum_{k \in K} \sum_{l \in \mathbb{Z}^r} e^{i \sum_{j=1}^r (k_j + 2q v_j l) x^j} \psi_k(x^{r+1} + 2\pi l_1, \ldots, x^{2r} + 2\pi l_r, x^{2r+1}, \ldots, x^n). \quad (31)$$

Replacing in (25) we find

$$(\psi', \psi) = (2\pi)^r \sum_{k \in K} \int_{\mathbb{R}} dx^{r+1} \cdots \int_{\mathbb{R}} dx^{2r} \int_0^{2\pi} dx^{2r+1} \cdots \int_0^{2\pi} dx^n \psi_k^* \psi_k.$$  

(32)

Hence $\psi \in \mathcal{H}^\beta$ iff $\psi_k \in L^2(\mathbb{R}^r \times \mathbb{T}^{n-2r})$, $\psi \in \mathcal{X}^\beta$ iff $\psi_k \in S(\mathbb{R}^r \times \mathbb{T}^{n-2r})$, for all $k \in K$, and

$$\mathcal{H}^\beta = \bigoplus_{k \in K} \mathcal{H}^k, \quad \mathcal{X}^\beta = \bigoplus_{k \in K} \mathcal{X}^k$$

(33)

where we have denoted as $\mathcal{X}^\beta \subset \mathcal{X}^\beta$, $\mathcal{H}^k \subset \mathcal{H}^\beta$ the subspaces characterized by $\psi_k' \equiv 0$ for $k' \in K \setminus \{k\}$ and $\psi_k$ belonging to $L^2(\mathbb{R}^r \times \mathbb{T}^{n-2r})$, $S(\mathbb{R}^r \times \mathbb{T}^{n-2r})$ respectively. The
correspondence $\psi \leftrightarrow \{\psi_k\}_{k \in K}$ is the generalized Weil-Brezin-Zak transform, see [30] and 1.10 in [7]. One can easily check that each $H^k$ is mapped onto itself by $G$ (resp. each $X^k$ is mapped into itself by $Ug$). We shall show this fact in next subsection while presenting bases of $X^k, \mathcal{X}^b$.

3 Decomposition and Irreducible Representations of the Observables’ Group for Integer Charge

Let us denote as $Y, \mathcal{Y}, G, g, \mathcal{C}, h_a, H_m$ the groups/algebras obtained from $Y_Q, \mathcal{Y}_Q, G_Q, g_Q, \mathcal{C}_Q, h_{Q_m}, H_{Q_m}$ replacing the central element $Q$ in (21)–(24) by some $q \in \mathbb{Z}$. In this subsection we decompose them and identify their unitary irreducible representations. We abbreviate $\tilde{A} := qA$, and similarly for all the derived objects: $\tilde{\alpha} := q\alpha$, etc. Note that $q, \beta^A$ enter the commutation relations only through their product $q\beta^A$.

In (24) we can set $\zeta^0 = 0$, as we can reabsorb $q\zeta^0$ into a redefinition of $h^0$. Define $Z^{(n)} := \{\xi^l := e^{i2\pi l'(p+2\beta^A)x} \mid l \in \mathbb{Z}^n\}, \quad \zeta_a := e^{i2\pi (pa+2\beta^A)x^b}$.

Using (23) it is easy to check that $[\xi^l, e^{i(l'x+pz+h^0)}] = 0$ for all $l, l' \in \mathbb{Z}^n, (h^0, z) \in \mathbb{R}^{n+1}$. Hence $Z^{(n)}$ is a discrete subgroup of $Z(Y)$, isomorphic to $\mathbb{Z}^n$; it is generated by the $\zeta_a$.

Let $r := \frac{1}{2} \text{rank}(\tilde{B}^A)$. We first perform a change (20) (chosen so that $\tilde{v}_j = 2\pi \tilde{b}_j \in \mathbb{N}$, i.e. are positive) and then define $\tilde{p}_j := 2\tilde{b}_j x^l + p_{r+j}, \quad \tilde{x}^j := \frac{1}{2b_j} p_j - x^{r+j}, \quad \tilde{x}^{r+j} := \frac{1}{2b_j} p_j,$

$\tilde{p}_{r+j} := p_{r+j}, \quad j = 1, \ldots, r \quad \tilde{x}^a := x^a, \quad \tilde{p}_a := p_a \quad a > 2r$

one obtains objects fulfilling the canonical commutation relations $[\tilde{x}^a, \tilde{p}_b] = i\delta^a_b, [\tilde{x}^a, \tilde{x}^b] = 0 = [\tilde{p}_a, \tilde{p}_b]$ for $a, b = 1, \ldots, n$. The set $\{1, \tilde{x}^{r+1}, \ldots, \tilde{x}^{2r}, \tilde{p}_{r+1}, \ldots, \tilde{p}_n\}$ is a new basis of $\mathfrak{g}$. The elements $m_j := e^{i\tilde{x}^j} = [u^{r+j}]^{-1} e^{i\tilde{v}_j p_j}, \quad m_{r+j} := e^{i\tilde{v}_j p_j} = u^j e^{i\tilde{v}_j p_{r+j}}$

commute with $G, \mathfrak{g}$ (and hence also with $Ug$) and fulfill $m_j m_{r+j} = m_{r+j} m_j e^{i\tilde{z}^j}, \quad j = 1, \ldots, r, \quad m_a m_b = m_b m_a \quad \text{otherwise.} \quad (37)$

From $l \cdot x + p \cdot z = \sum_{j=1}^r \left[ \frac{l_j}{2b_j} \tilde{p}_j - l_{r+j} \tilde{x}^j + p_j \left( \tilde{z}^j + \frac{l_{r+j}}{2b_j} \right) + p_{r+j} \left( \tilde{z}^{r+j} - \frac{l_j}{2b_j} \right) \right]$

$+ \sum_{a=2r+1}^n (l_a x^a + p_a z^a)$

and (17) we obtain the following decomposition of the generic element of $Y$:

$e^{i(l \cdot x + p \cdot z + h^0)} = \prod_{j=1}^r \left[ m_j^{-l_{r+j}} m_{r+j}^{l_j} e^{i\tilde{v}_j p_{r+j}} \right] \left[ e^{ib^0} \prod_{j=1}^r e^{ip_j (\tilde{z}^j + l_{r+j}/2b_j)} e^{ip_{r+j} (\tilde{z}^{r+j} - l_j/2b_j)} \right]$

$\times \left[ \prod_{a=2r+1}^n e^{i(l_a x^a + p_a z^a)} \right]. \quad (38)$
For any hermitian q, p fulfilling [q, p] = i denote as $H_3(q, p) := \{e^{i(h_0+qp)} \mid (h_0, y, z) \in \mathbb{R}^3\}$ the associated 3-dimensional Heisenberg Lie group and $Y(q, p) := \{e^{i(qp+h_0)} \mid (l, h_0, z) \in \mathbb{Z} \times \mathbb{R}^2\}$ (this group plays a role in quantum mechanics on the circle, see below).

**Proposition 1** $Y$ decomposes into a product of commuting subgroups as follows:

$$Y = M^1 \cdots M^r H^1_3 \cdots H^1_r Y^{2r+1} \cdots Y^n.$$  \hspace{1cm} (39)

The elements of $M^1, \ldots, M^r, H^1_3, \ldots, H^1_r, Y^{2r+1}, \ldots, Y^n$ are those ranging resp. in the square brackets of the 1st, 2nd, 3rd block at the rhs (38), for all the possible values of $l, z, h_0$.

Namely,

- $M^j$ is discrete, generated by $m_j, m_{r+j}, e^{i\tau_j}$ fulfilling (37) and by the inverses $m_j^{-1}, m_{r+j}^{-1}$;
- $H^1_j := \{e^{i(h_0+wp_j+zpr_j)} \mid (h_0, w, z) \in \mathbb{R}^3\} = H_3(p_{\tau_j}, p_{r+j})$;
- $Y^a := Y(x^a, p^a)$.

Moreover, $Z(Y) = Z^{(n)}(U(1))$; if we adopt the $x, p$ variables at the rhs (20) then $\zeta_j = (m_j)^{2\tau_j}, \zeta_{r+j} = (m_{r+j})^{2\tau_j}$ ($j = 1, \ldots, r$).

**Proof** The rest of the proof is as follows. The elements of the different subgroups at the rhs’s of the relations (39) commute with each other by the relations $2\delta_j = \{i\zeta_j, p_{\tau_j}\}$. Clearly $Z(H^1_3 \cdots H^1_r) = Z(Y^{2r+1} \cdots Y^n) = \{e^{ih_0} \mid h_0 \in \mathbb{R}\} = U(1)$, whence $Z(Y) = Z^{(n)}(U(1))$, as claimed.

The center $Z(Y)$ of the $C^*$-algebra $Y$ is generated by the $\zeta^a$. The irreducible unitary representations (briefly irreps) of $Y$ are those of $Y$. We now study them, starting form the lowest $n$’s.

**Irreducible Unitary Representations of $Y$ for n = 1** \hspace{1cm} $\beta^A = 0, r = 0, Y \equiv Y^1 \simeq Y$ and $O$ is the algebra of observables of quantum mechanics on a circle. The formulae

$$\rho_\alpha\left[e^{i(lx+h_0)}\right]\psi(x) = e^{i(lx+h_0)}\psi(x), \quad \rho_\alpha\left(e^{izp}\right)\psi(x) = e^{i\tilde{\alpha}z+iz\alpha}\psi(x) = e^{i\tilde{\alpha}z}\psi(x+z), \hspace{1cm} (40)$$

where $\tilde{\alpha} \in \mathbb{R}$ and $\psi \in L^2(S^1)$, define an irrep $(\rho_\alpha, \mathcal{H}_\alpha)$ of $Y$, with $\mathcal{H}_\alpha = L^2(S^1)$. The formulæ

$$\rho_\alpha\left[e^{ilx}\right]\psi(x) = e^{ilx}\psi(x), \quad \rho_\alpha\left(p\right)\psi(x) = (\tilde{\alpha} - i\partial_x)\psi(x) \hspace{1cm} (41)$$

define the associated irrep $(\rho_\alpha, X_\alpha)$ of $O$ on the dense subspace $X_\alpha = C^\infty(S^1) \subset L^2(S^1)$.

If $\alpha' = \alpha + l$ then $\rho_\alpha$ and $\rho_{\alpha'}$ are related by the unitary transformation $\psi(x) \mapsto e^{ilx}\psi(x)$ and therefore are equivalent. This is in agreement with the fact that the casimir $\zeta = e^{i2\pi p}$ has the same eigenvalue $e^{i2\pi \tilde{\alpha}}$ in both $\rho_\alpha, \rho_{\alpha'}$, by (40) with $z = 2\pi$ and the periodicity $\psi(x+2\pi) = \psi(x)$. In fact the inequivalent irreps of $Y$ are parametrized by $\tilde{\alpha} \in \mathbb{R}/\mathbb{Z}$ and are defined by (40) up to equivalences (see [15, 16] or e.g. the more recent [12, 21]). $\{e^{ilx}/\sqrt{2\pi}\}_{l \in \mathbb{Z}} \subset C^\infty(S^1)$ is an orthonormal basis of $L^2(S^1)$ consisting of eigenvectors of $p$: $\rho_\alpha(p)e^{ilx} = (l+\tilde{\alpha})e^{ilx}$. 

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Irreducible Unitary Representations of $Y$ for $n = 2$, $\tilde{\mathcal{P}}^A \neq 0$. It is $r = 1$ and $\{p_1, p_2\} = i \tilde{b}$, where $\tilde{\mathcal{P}}^A = \begin{pmatrix} 0 & -\tilde{b} \\ \tilde{b} & 0 \end{pmatrix}$ and $\tilde{v} = 2\pi \tilde{b} \in \mathbb{N}$. The decomposition (39) becomes $Y = MH_3$ where $H_3 \sim G$ and $M$ is generated by $m_1, m_2$ fulfilling

$$m_1 m_2 = e^{i \pi} m_2 m_1.$$ 

By the Von Neumann theorem all irreps of $H_3$ are equivalent to the Schrödinger representation on $L^2(\mathbb{R})$. On the other hand, the eigenvalues of the unitary casimirs $\xi_a$ ($a = 1, 2$) of $M$ necessarily have the form $e^{i 2\pi \alpha a}$, $\alpha \in \mathbb{R}$; hence the pairs $(e^{i 2\pi \alpha a}, e^{i 2\pi \beta a})$, or equivalently the $\tilde{\alpha} \in \mathbb{R}^2/\mathbb{Z}^2$, identify the classes of inequivalent irreps $(\rho_{\tilde{\alpha}}, \mathcal{H}_{\tilde{\alpha}})$ of $M$, and therefore of $Y$. An irrep $(\rho_{\tilde{\alpha}}, \mathcal{X}_{\tilde{\alpha}})$ of $\mathcal{O}$ is obtained on the subspace $\mathcal{X}_{\tilde{\alpha}} := \mathcal{H}_{\tilde{\alpha}} \cap C^\infty(\mathbb{R}^2)$. The $\rho_{\tilde{\alpha}}$ are essentially self-adjoint. When $\tilde{\alpha} = 0$ $m_1, m_2$ are sometimes called ‘clock’ and ‘shift’.

From $\xi_a = (m_a)^{2\tilde{v}}$ and (37) it also follows that each $m_a$ has eigenvalues $e^{i \tilde{v}(\tilde{\alpha} a + k)}$, $k = 0, 1, \ldots, 2\tilde{v} - 1$. The decomposition (33) takes the form $\mathcal{H}_{\tilde{\alpha}} = \bigoplus_{k=0}^{2\tilde{v}-1} \mathcal{X}_{\tilde{\alpha}}^k$ (resp. $\mathcal{X}_{\tilde{\alpha}}^k = \bigoplus_{\tilde{\alpha}=0}^{2\tilde{v}-1} \mathcal{X}_{\tilde{\alpha}}^k$), and it is easy to check that $\mathcal{H}_{\tilde{\alpha}}^k$ (resp. $\mathcal{X}_{\tilde{\alpha}}^k$) is the eigenspace of $m_1$ with eigenvalue $e^{i \tilde{v}(\tilde{\alpha} a + k)}$; $m_1$ and the whole $G \sim H_3$ map each $\mathcal{H}_{\tilde{\alpha}}^k$ (resp. $\mathcal{X}_{\tilde{\alpha}}^k$) into itself; only $m_2$ maps it outside, into $\mathcal{H}_{\tilde{\alpha}}^{k+1}$ (resp. $\mathcal{X}_{\tilde{\alpha}}^{k+1}$). [We identify $\mathcal{H}_{\tilde{\alpha}}^{k+1}$ with $\mathcal{H}_{\tilde{\alpha}}^{k+1}$ and (37) it also follows that each $\rho_{\tilde{\alpha}}$ takes the form $e^{i \tilde{v}(\tilde{\alpha} a + k)}$.]

Setting

$$a := \frac{p_1 + i p_2}{\sqrt{4b}}, \quad a^* := \frac{p_1 - i p_2}{\sqrt{4b}}, \quad n := a^* a,$$

we find $[a, a^*] = 1$. An orthonormal basis of the carrier spaces $\mathcal{X}_{\tilde{\alpha}}^k, \mathcal{H}_{\tilde{\alpha}}^k$ consists of eigenvectors $\psi_{n,k}$ ($n \in \mathbb{N}_0$, $k = 0, 1, \ldots, 2\tilde{v} - 1$) of the complete set of commuting observables $\{m_1, n\}$:

$$\rho_{\tilde{\alpha}}(m_1) \psi_{n,k} = e^{i \tilde{v}(\tilde{\alpha} a + k)} \psi_{n,k}, \quad \rho_{\tilde{\alpha}}(n) \psi_{n,k} = n \psi_{n,k}. \quad (42)$$

Note that $n \psi_{0,k} = 0$. The $\psi_{n,k}$ with $n \in \mathbb{N}_0$ and fixed $k$ make up an orthonormal basis of $\mathcal{X}_{\tilde{\alpha}}^k, \mathcal{H}_{\tilde{\alpha}}^k$. Clearly the $\psi_{n,k}$ are also eigenvectors of the ‘Bochner Laplacian’ with constant magnetic field, $H_0$:

$$H_0 := \frac{1}{2} \left[ (p_1)^2 + (p_2)^2 \right] = \frac{\tilde{v}}{\pi} \left( n + \frac{1}{2} \right), \quad \rho_{\tilde{\alpha}}(H) \psi_{n,k} = E_n \psi_{n,k},$$

$$E_n = \frac{\tilde{v}}{\pi} \left( n + \frac{1}{2} \right). \quad (43)$$

The degeneracy of each “energy level” $n$ is thus $2\tilde{v}$; it is related to the commutation relation $[H_0, M] = 0$, which in the usual treatment (see e.g. [22, 26]) is used as a condition defining the magnetic translation group $M$, whereas here follows from the more general one $[U_g, M] = 0$. This finite degeneracy is well-known and in contrast with the infinite degeneracy of Landau levels in the absence of quasiperiodicity.

Choosing the gauge (27) with $\beta$ as in (28), the quasiperiodicity factor and the decomposition (31) of $\mathcal{X}_{\tilde{\alpha}}, \mathcal{H}_{\tilde{\alpha}}$ take the form

$$V(x) = e^{-i 2\tilde{v} l x_1}, \quad \psi(x) = \sum_{k=0}^{2\tilde{v}-1} \sum_{l \in \mathbb{Z}} e^{i (k+2\tilde{v} l) x_1} \psi_k(x^2 + 2\pi l); \quad (44)$$

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the subspaces \( \mathcal{X}_\alpha^k, \mathcal{H}_\alpha^k \) are characterized by \( \psi_k \equiv 0 \) for \( k' \in K \setminus \{k\} \). The formulae

\[
\begin{align*}
p_1 &= -i \partial_1 + 2 \bar{x}^2 + \bar{\alpha}_1, \\
p_2 &= -i \partial_2 + \bar{\alpha}_2, \\
a &= \frac{\partial_2 - i \partial_1 + 2 \bar{x}^2 + \bar{\alpha}_1 + i \bar{\alpha}_2}{\sqrt{4b}}, \\
a^* &= \frac{-\partial_2 - i \partial_1 + 2 \bar{x}^2 + \bar{\alpha}_1 - i \bar{\alpha}_2}{\sqrt{4b}}, \\
m_1 &= e^{\frac{\pi}{4} (i \bar{\alpha}_1 + \bar{\eta})}, \\
m_2 &= e^{i x_1 + \frac{\pi}{4} (i \bar{\alpha}_2 + \bar{\eta})}, \\
\zeta_1 &= e^{2 \pi (i \bar{\alpha}_1 + \bar{\eta})}, \\
\zeta_2 &= e^{i (2 \bar{x}_1 + \pi \bar{\alpha}_2) + 2 \pi \bar{\eta}}, \\
\psi_{0,0}(x; \bar{\alpha}, \bar{\nu}) &= \mathcal{N} \sum_{l \in \mathbb{Z}} e^{i 2 \bar{x}_l} e^{-\frac{\pi}{2 \nu} (\bar{z}^2 + 2 \pi l + i \bar{\alpha}_1 + i \bar{\alpha}_2)} |g\rangle, \\
\psi_{n,k}(x; \bar{\alpha}, \bar{\nu}) &= \mathcal{N} \sum_{l \in \mathbb{Z}} e^{i \pi l (2z + i \bar{\alpha})} |g\rangle.
\end{align*}
\]

(45)

give the explicit representation of \( \rho_\alpha \) and of the above basis in this gauge; one easily determines the normalization constant to be \( \mathcal{N} = \frac{1}{\sqrt{2 \pi e^{-\frac{\pi}{4} (i \bar{\alpha}_1 + \bar{\eta})}}} \). \( \psi_{0,0} \) is cyclic. Introducing the complex variables \( z = \frac{\bar{z}}{\pi} (x^1 + i x^2), \bar{z} = \frac{\nu}{\pi} (x^1 - i x^2) \) we find

\[
\begin{align*}
a &= \sqrt{\frac{\pi}{2 \nu}} \left[ i \frac{1}{2} (\bar{z} - z) + i \frac{2 \bar{\nu}}{\pi} \partial_z + \bar{\alpha}_1 + i \bar{\alpha}_2 \right], \\
a^* &= \sqrt{\frac{\pi}{2 \nu}} \left[ i \frac{1}{2} (\bar{z} - z) + i \frac{2 \bar{\nu}}{\pi} \partial_z + \bar{\alpha}_1 - i \bar{\alpha}_2 \right], \\
\psi_{0,k}(x; \bar{\alpha}, \bar{\nu}) &= g(x^2) e^{\frac{\pi}{2 \nu} (i(kz + i \bar{\alpha}_1 - \bar{\alpha}_2) - \frac{1}{4} z^2)} \vartheta[z + i \bar{\alpha}_1 + i k - \bar{\alpha}_2, i 2 \bar{\nu}] |q\rangle
\end{align*}
\]

(46)

where we have set \( g(x^2) := Ne^{-\frac{\pi}{4\nu} (z^2 + i \bar{\alpha}_1 + i \bar{\alpha}_2)^2} \) and used Jacobi Theta function

\[
\vartheta(z, \tau) := \sum_{l \in \mathbb{Z}} e^{i \pi l (2z + i \tau)}.
\]

(47)

Hence, up to the gaussian factor \( g(x^2) \) the \( \psi_{0,k} \) are analytic (actually entire) functions of \( z \).\(^9\)

Applying the transformation

\[
\psi \mapsto \psi' := g^{-1} \psi, \quad V(l, x) \mapsto V'(l, x) = g^{-1}(x + 2 \pi l) V(l, x) g(x),
\]

(48)

to the \( \psi_{0,k} \) one obtains \( \psi_{0,k} \) depending on \( x^1, x^2 \) only through the complex variables \( z, w.r.t. \) which they are holomorphic. The hermitean structure (25) and the action of \( \mathcal{O}, \mathcal{Y} \) are resp. transformed as follows: and:

\[
dx^1 dx^2 \mapsto dx^1 dx^2 |g|^2, \quad w \mapsto w' := g^{-1} w g, \quad \forall w \in \mathcal{O}, \mathcal{Y}.
\]

(49)

In particular, \( \alpha' = i \sqrt{\frac{2 \nu}{\pi}} \partial_z, a' = \sqrt{\frac{\pi}{2 \nu}} (\bar{z} - z) - i \sqrt{\frac{2 \nu}{\pi}} \partial_z \); moreover, if \( \bar{\alpha} = 0 \) one finds

\[
\begin{align*}
m'_1 &= e^{\frac{\pi}{4 \nu} \partial_z}, \\
m'_2 &= e^{\frac{\pi}{4 \nu} (iz - \frac{1}{2} + \bar{\alpha}_2)}, \\
\psi_{0,k}' &= e^{\frac{\pi}{4 \nu} (iz - \frac{1}{2})} \vartheta[z + i k, i 2 \bar{\nu}] |q\rangle.
\end{align*}
\]

(50)

\(^9\)Equation (46) is verified by direct inspection for \( (n, k) = (0, 0) \), using the relation \( m_2 f(x^1, x^2) = e^{i x^1 + \frac{\pi}{4} \bar{\alpha}_2} f(x^1, x^2 + \frac{\pi}{4 \nu}) \) [valid for all \( f \in C^\infty(\mathbb{R}) \)] and (45).
This shows the link with the holomorphic framework (see e.g. [2, 18]) on (the universal cover of) a complex torus with parameter \( \tau = i2\nu \): \( V' \) is called an automorphy factor and takes values in \( \mathbb{C} \setminus \{0\} \).

Irreducible Unitary Representations of \( Y \) for General \( n \) We are now ready for tori of general dimension \( n \). Using the transformation (20), the decomposition (39), the results for \( n = 1, 2 \), the equivalence (by Von Neumann theorem) of all irreps of \( H_{2r+1} \) to the Schrödinger representation on \( \mathbb{C}^2(\mathbb{R}^r) \), we find the following

**Proposition 2** The sets of joint eigenvalues \( \zeta_a = e^{i2\pi\tilde{\alpha}_a} \) of the unitary Casimirs, or equivalently the \( \tilde{\alpha} \in \mathbb{T}^r_\nu := \mathbb{R}^r/\mathbb{Z}^n \), identify up to unitary equivalences the irreducible unitary representations (irreps) \( (\rho_{\tilde{\alpha}}, \mathcal{H}_{\tilde{\alpha}}) \) of \( Y \) and \( \mathcal{Y} \). An irrep \( (\rho_{\tilde{\alpha}}, \mathcal{X}_{\tilde{\alpha}}) \) of \( \mathcal{O} \) is obtained by restriction on the subspace \( \mathcal{X}_{\tilde{\alpha}} := \mathcal{H}_{\tilde{\alpha}} \cap \mathcal{C}^\infty(\mathbb{R}^n) \). The \( p_a \) are essentially self-adjoint.

For \( j = 1, \ldots, r \), each \( m_j \) has eigenvalues \( e^{i\nu_j(\tilde{\alpha}_j + k_j)} \), \( k_j = 0, 1, \ldots, 2\nu_j - 1 \). The decomposition (33) takes the form \( \mathcal{H}_{\tilde{\alpha}} = \bigoplus_{k \in K} \mathcal{H}_{\tilde{\alpha}}^k \) (resp. \( \mathcal{X}_{\tilde{\alpha}} = \bigoplus_{k \in K} \mathcal{X}_{\tilde{\alpha}}^k \)), where \( \mathcal{H}_{\tilde{\alpha}}^k \) (resp. \( \mathcal{X}_{\tilde{\alpha}}^k \)) is the eigenspace with eigenvalues \( e^{i\frac{\pi}{\nu_j}(\tilde{\alpha}_j + k_j)} \), \( \ldots, e^{i\nu_j(\tilde{\alpha}_j + k_j)} \) of \( m_1, \ldots, m_r \). The latter operators and the whole \( G \) (resp. \( U \)) map each \( \mathcal{H}_{\tilde{\alpha}}^k \) (resp. \( \mathcal{X}_{\tilde{\alpha}}^k \)) into itself. The \( m_{r+j} \), map it into \( \mathcal{H}_{\tilde{\alpha}}^{k+\nu_r} \) (resp. \( \mathcal{X}_{\tilde{\alpha}}^{k+\nu_r} \)). [We identify \( \mathcal{H}_{\tilde{\alpha}}^{k+\nu_r} = \mathcal{H}_{\tilde{\alpha}}^k \), \( \mathcal{X}_{\tilde{\alpha}}^{k+\nu_r} = \mathcal{X}_{\tilde{\alpha}}^k \)]. Setting \( r := \frac{1}{2} \text{rank}(\tilde{\beta}^A) \leq \frac{1}{2} n \) and performing first a transformation (20) leading to \( (v_1, \ldots, v_r) \in \mathbb{N}^r \) with \( v_{j+1}/v_j \in \mathbb{N} \), and then defining

\[
\tilde{b}_j := \frac{v_j}{2\pi}, \quad a_j := \frac{p_j + i\nu_{r+j}}{\sqrt{2b_j}}, \quad a_j^* := \frac{p_j - i\nu_{r+j}}{\sqrt{2b_j}}, \quad n_j := a_j^* a_j
\]

for \( j = 1, \ldots, r \) we find that \( [a_i, a_j^*] = 1\delta_{ij} \). An orthonormal basis \( \mathcal{B} \) of the carrier Hilbert space \( \mathcal{H}_{\tilde{\alpha}} \) consists of joint eigenvectors \( \psi_{n,k,l} \) of the complete subset of commuting observables \( \{n_1, \ldots, n_r, m_1, \ldots, m_r, p_{2r+1}, \ldots, p_n\} \subset \mathcal{O} \):

\[
n \in \mathbb{N}_0^n, \quad k \in K := \{0, \ldots, 2\nu_1 - 1\} \times \cdots \times \{0, \ldots, 2\nu_r - 1\}, \quad l \in \mathbb{Z}^{n-2r}
\]

\[
n_j \psi_{n,k,l} = n_j \psi_{n,k,l}, \quad m_j \psi_{n,k,l} = e^{i\nu_j(\tilde{\alpha}_j + k_j)} \psi_{n,k,l}, \quad p_a \psi_{n,k,l} = (l_a + \tilde{\alpha}_a) \psi_{n,k,l},
\]

\( a > 2r \). It is \( a_j \psi_{0,k,l} = 0 \). An orthonormal basis of the subspace \( \mathcal{H}_{\tilde{\alpha}}^k, \mathcal{X}_{\tilde{\alpha}}^k \) consists of the \( \psi_{n,k,l} \) with that \( k \) only.

More explicitly, in the gauge (27) with \( \beta \) as in (28) we obtain

\[
p_j = -i\partial_j + 2\tilde{b}_j x^{r+j} + \tilde{\alpha}_j, \quad p_a = -i\partial_a + \tilde{\alpha}_a, \quad a = r + 1, \ldots, n,
\]

\[
a_j = \frac{\partial_{r+j} - i\partial_j + 2\tilde{b}_j x^{r+j} + \tilde{\alpha}_j + i\tilde{\alpha}_{r+j}}{\sqrt{4\tilde{b}_j}}
\]

\[
a_j^* = \frac{-\partial_{r+j} - i\partial_j + 2\tilde{b}_j x^{r+j} + \tilde{\alpha}_j - i\tilde{\alpha}_{r+j}}{\sqrt{4\tilde{b}_j}}
\]

\[
m_j = e^{i\nu_j(\tilde{\alpha}_j + k_j)}, \quad m_{r+j} = e^{i\nu_{r+j}(\tilde{\alpha}_{r+j} + k_j)}
\]

\[
\zeta_j = e^{2\pi i(\tilde{\alpha}_j + k_j)}, \quad \zeta_{r+j} = e^{2\pi i x^{r+j} + 2\pi i(\tilde{\alpha}_{r+j} + k_j)}
\]
\[ \psi_{0,0,0}(x; \tilde{\alpha}, \tilde{\nu}) = N' \sum_{l \in \mathbb{Z}^r} \exp \left\{ \sum_{j=1}^{r} \left[ i 2 \tilde{\nu}_j l_j x^j - \frac{\pi}{2 \tilde{\nu}_j} \left( \frac{\tilde{\nu}_j}{\pi} x^{r+j} + 2 \tilde{\nu}_j l_j + \tilde{\alpha}_j + i \tilde{\alpha}_{r+j} \right)^2 \right] \right\} |q \rangle, \]

\[ \psi_{n,k,l} = \prod_{a=2r+1}^{n} e^{i l_a x^a} \left[ \prod_{j=1}^{r} \frac{(a^+_j)^{n_j}}{\sqrt{n_j!}} (m_{r+j})^{k_j} \right] \psi_{0,0,0}, \]

\[ a_j \psi_{n,k,l} = \sqrt{n_j} \psi_{n-e_j,k}, \quad a^*_j \psi_{n,k,l} = \sqrt{n_j+1} \psi_{n+e_j,k,l}; \]

the normalization constant is given by \( |N'| = \frac{1}{(2\pi)^{n/2}} \prod_{j=1}^{r} (\sqrt{2 \tilde{\nu}_j}) \exp\left[-\frac{\pi (\tilde{\alpha}_j)^2}{2 \tilde{\nu}_j}\right]. \) By gauge transformations (26) one explicitly obtains the other equivalent representations.

**Bibliographical Note**

\[ M := M^1 \cdots M^r \] defined in (39) is the “group of magnetic translations”, in the sense of Zak [26]; it is a subgroup of the commutant \( \tilde{M} \) of \( G \) within \( Y. ^{10} \) In the literature the torus \( \mathbb{T}^2 \) to which \( \tilde{\alpha} \) belongs, is sometimes called ‘Brillouin zone’, or ‘Jacobi torus’.

Clearly, \( \psi_{n,k,l} \) are also eigenvectors of the ‘Bochner Laplacian’ with constant \( B, H_{\beta A} \):

\[ H_{\beta A} := \sum_{a=1}^{n} (p_a)^2 = \sum_{j=1}^{r} \frac{\tilde{\nu}_j}{\pi} \left( a^+_j a_j + \frac{1}{2} \right) + \sum_{a=2r+1}^{n} p_a^2, \]

\[ H \psi_{n,k,l} = E_{n,l} \psi_{n,k,l}, \quad E_{n,l} = \sum_{j=1}^{r} \frac{\tilde{\nu}_j}{\pi} \left( n_j + \frac{1}{2} \right) + \sum_{a=2r+1}^{n} (l_a + \tilde{\alpha}_a)^2. \]

Again, the finite degeneracy \( \prod_{j=1}^{r} (2\tilde{\nu}_j) \) of the energy level \( E_{n,l} \) is related to the commutation relation \( [H_{\beta A}, M] = 0 \), which in the usual treatment (see e.g. [22, 26]) is used as a condition defining the magnetic translation group \( M \), whereas here follows from the more general one \( [Ug, M] = 0 \).

The \( \psi_{0,k,0} \) are related to Jacobi Theta function \( \vartheta \) and Riemann Theta function

\[ \vartheta(z, \tau) := \sum_{l \in \mathbb{Z}^r} e^{i \pi (2l^2 \tau + l^t \tau)}, \]

\[ z \equiv (z^1, \ldots, z^r) \in \mathbb{C}^r, \quad \tau \in \mathbb{H}_r := \{ \tau \in M_r(\mathbb{C}) \mid \tau^t = \tau, \Im(\tau) > 0 \} \]

\( ^{10} \) The whole commutant (centralizer) \( \tilde{M} \) of \( G \) within \( Y \) is the subgroup

\[ \tilde{M} = MM', \quad M' := \left\{ \exp \left[ ih^0 + i \sum_{a=2r+1}^{n} p_a z^a \right] \mid (h^0, z^{2r+1}, \ldots, z^n) \in \mathbb{R}^{n-2r+1} \right\} \]

Equivalently, \( [m, Ug] = 0 \), i.e. \( [m, p_a] = 0 \), for all \( m \in M \).
(\mathbb{H}_r \text{ is Siegel upper half space}) by

\[
\psi_{0,k,0}(x; \tilde{\alpha}, \tilde{\nu}) = G(x) \left[ \prod_{j=1}^{r} e^{\frac{\pi}{2}\theta(z_{k,a}^j, i2\tilde{\nu})} \right] |q\rangle
\]

\[
= G(x) \left[ \prod_{j=1}^{r} e^{\frac{\pi}{2}\theta(z_{k,a}^j, i2\tilde{\nu})} \right] \theta(z_{k,a}^j, i2\tilde{\nu}) |q\rangle
\]

\[G(x) := N' \exp \left[ -\sum_{j=1}^{r} \frac{\pi}{2\tilde{\nu}} \left( \frac{\tilde{\nu}}{\pi} x^{r+j} + \alpha_{r+j} + i\alpha_j \right)^2 \right],\]

\[z_{k,a}^j := z^j + ik_j - \tilde{\alpha}_{r+j} + i\alpha_j, \quad z^j := \frac{\tilde{\nu}}{\pi} (x^j + ix^{r+j}), \quad \tilde{\nu} := \text{diag}(\tilde{\nu}_1, \ldots, \tilde{\nu}_r).\]

(55)

Hence, up to the gaussian factor \(G(x)\) the \(\psi_{0,k,0}\) are analytic (actually entire) functions of the \(z^j, j = 1, \ldots, r\).

The above results, propaedeutical for the torus, are useful also for the physics of a scalar charged particle on \(\mathbb{R}^n\). The Hilbert space of states is \([1, 10]\) the direct integral over \(\tilde{\alpha} \in \mathbb{T}_n\) of the Hilbert spaces \(\mathcal{H}_\tilde{V}\) of the inequivalent representations of \(\mathcal{O}_\tilde{V}\) parametrized by \(\tilde{\alpha} \in \mathbb{T}_n\). In the presence of a periodic scalar potential \(V\) and a periodic magnetic field \(B\) with fixed fluxes (9) the Hamiltonian \(H\) of (1) belongs to \(\mathcal{O}_\tilde{V}\), and the corresponding evolution is such that \(\psi(t_0) \in \mathcal{H}_\tilde{V}\) implies \(\psi(t) \in \mathcal{H}_\tilde{V}\) for all \(t\).

4 Mapping \(X^V \sim T(\mathbb{T}^n, E)\)

The 1-particle wavefunction describing a charged scalar particle on an orientable Riemannian manifold \(X\) in the presence of a magnetic field 2-form \(B = B_{ab} dx^a dx^b\) is a \(L^2\)-section of a hermitean line bundle \(E \rightarrow X\) with connection having field strength \(B\) (see e.g. \([14, 24, 25]\)); if \(X\) is compact the fluxes of \(B\) are necessarily quantized. We now show that if \(X = \mathbb{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n\) then also \(E\), beside \(X\), can be realized as a quotient, and its \(L^2\)-sections are determined by the quasiperiodic wavefunctions \(\psi(x)\) on \(\mathbb{R}^n\) introduced in the earlier sections. We partly mimic arguments used for complex tori (see e.g. Appendix B in \([2]\)).

The formula \(T_l : x \in \mathbb{R}^n \mapsto x + 2\pi l \in \mathbb{R}^n (l \in \mathbb{Z}^n)\) defines an action of the abelian group \(\mathbb{Z}^n\) on \(\mathbb{R}^n\). The action is free, in that \(T_l(x) = x\) for some \(x\) implies that \(T_l = \text{id} = T_0\), and properly discontinuous, in that the inverse image of any compact subset is compact. Clearly \(T_l \circ T_l = T_{l+l'}\) and \(T_l^{-1} = T_{-l}\). Introducing in \(\mathbb{R}^n\) the equivalence relation \(x \sim x'\) iff \(x' = T_l(x)\) for some \(l \in \mathbb{Z}^n\), the definition of the torus \(\mathbb{T}^n\) as a quotient \(\mathbb{R}^n/2\pi \mathbb{Z}^n\) amounts to

\[
\mathbb{T}^n = \mathbb{R}^n / \sim;
\]

in other words, an element of \(\mathbb{T}^n\) is an equivalence class \([x] = \{T_l(x), l \in \mathbb{Z}^n\}\). The universal cover map is defined as \(P : x \in \mathbb{R}^n \mapsto [x] \in \mathbb{T}^n\). The fundamental cells \(C_{a_1\cdots a_k}\) defined in (10) are mapped by \(P\) onto fundamental k-cycles \(\tilde{C}_{[y]} \subset \mathbb{T}^n\) through \([y]\). The \(dx^a\) generate the exterior algebra \(\wedge^n\) not only on \(\mathbb{R}^n\) but also on \(\mathbb{T}^n\), and by multiplication by \(f \in X\) and linearity the algebra \(\Omega^*(\mathbb{T}^n)\) of differential forms on \(\mathbb{T}^n\). We can consider \(\Omega^*(\mathbb{T}^n)\) as a subspace of \(\Omega^*(\mathbb{R}^n)\), and the exterior derivative \(d\) on \(\Omega^*(\mathbb{R}^n)\) as the restriction of \(d\) on \(\Omega^*(\mathbb{T}^n)\).
Similarly, given a smooth phase factor $V : \mathbb{Z}^n \times \mathbb{R}^n \mapsto U(1)$ fulfilling (3), the formula

$$
\chi_l^V : (x, c) \in \mathbb{R}^n \times \mathbb{C} \mapsto (x + 2\pi l, V(l, x)c), \quad l \in \mathbb{Z}^n
$$

(57)
defines an action of the abelian group $\mathbb{Z}^n$ on $\mathbb{R}^n \times \mathbb{C}$. The action is free, in that $\chi_l^V[(x, c)] = (x, c)$ for some $(x, c)$ implies that $\chi_l^V = \text{id} = \chi_0^V$, and again properly discontinuous. By (3), $\chi_l^V \circ \chi_l^V = \chi_{2l}^V$ and $(\chi_l^V)^{-1} = \chi_{-l}^V$. In $\mathbb{R}^n \times \mathbb{C}$ we introduce an equivalence relation $\sim_V$ by setting $(x, c) \sim_V (x', c')$ iff $(x', c') = \chi_l^V[(x, c)]$ for some $l \in \mathbb{Z}^n$; we correspondingly define

$$
E = (\mathbb{R}^n \times \mathbb{C})/ \sim_V;
$$

(58)
in other words, an element of $E$ is an equivalence class $[(x, c)] = \{\chi_l^V[(x, c)], l \in \mathbb{Z}^n\}$. The projection $\pi : E \mapsto \mathbb{T}^n$ is defined by $\pi([(x, c)]) = [x]$. $E$ is trivial (i.e. $E = \mathbb{T}^n \times \mathbb{C}$) if $V$ is trivial [i.e. $V(l, x) \equiv 1$].

Given a smooth function $\psi : \mathbb{R}^n \mapsto \mathbb{C}$ fulfilling (2) we can define a smooth map

$$
\psi : [x] \in \mathbb{T}^n \mapsto [(x, \psi(x))] = \{\chi_l^V[(x, \psi(x))], l \in \mathbb{Z}^n\}
$$

(2)

$$
\equiv \{\{x + 2\pi l, \psi(x + 2\pi l)\}, l \in \mathbb{Z}^n\} \in E,
$$

i.e. a (global) section $\psi \in \Gamma(\mathbb{T}^n, E)$. The correspondence $\phi : \psi \in \mathcal{X}^V \mapsto \psi \in \Gamma(\mathbb{T}^n, E)$ is one-to-one. Given $\psi, \psi' \in \Gamma(\mathbb{T}^n, E)$, then $\overline{\psi' \psi}([x]) := \overline{\psi'(x)\psi(x)}$ defines a function $\overline{\psi' \psi} \in \mathcal{X}$, and

$$
(\psi', \psi)_\Gamma := \int_{\mathbb{T}^n} d^n x \overline{\psi' \psi} = \int_{\mathcal{X}^{[y]}} d^n x \overline{\psi'(x)\psi(x)} \quad \text{(25)}
$$

(59)
a hermitean structure $(, )_\Gamma$ equal to that of $\mathcal{X}^V$ and making $\Gamma(\mathbb{T}^n, E)$ a hermitean line bundle. A compatible covariant derivative is defined by

$$
\nabla := \phi \circ \nabla \circ \phi^{-1}, \quad \nabla : \Omega^p(\mathbb{T}^n) \otimes \chi \Gamma(\mathbb{T}^n, E) \mapsto \Omega^{p+1}(\mathbb{T}^n) \otimes \chi \Gamma(\mathbb{T}^n, E).
$$

(60)
The curvature map $\nabla^2 : \Gamma(\mathbb{T}^n, E) \mapsto \Omega^2(\mathbb{T}^n) \otimes \chi \Gamma(\mathbb{T}^n, E)$ determines the field strength 2-form $B$ on $\mathbb{T}^n$ through the formula $\nabla^2 \psi = -2i\eta B \otimes \chi \psi$. Relations (11)–(12) become

$$
\phi_{ab} = \int_{\mathcal{X}^{[y]}} B = 2\pi v_{ab}, \quad \int_{\mathcal{X}^{[y]}} B_{ab} = (2\pi)^m v_{a_1 a_2 \ldots a_{2m-1} a_{2m}}
$$

(61)
(the result is independent of $[y]$). In particular $\phi_{ab}$ is the flux of $B$ through a 2-cycle $\mathcal{X}_{ab}$, therefore $E$ is characterized by the Chern numbers (9). The Hilbert space completion of $\Gamma(\mathbb{T}^n, E)$ with hermitean structure (59) is isomorphic to $\mathcal{H}^V$. The actions of $\mathcal{O}, \mathcal{g}, \mathcal{Y}, \mathcal{G}$ are lifted from $\mathcal{X}^V, \mathcal{H}^V$ to $\Gamma(\mathbb{T}^n, E)$ and the completion of the latter by replacing $\rho(\phi) \mapsto \phi \circ \rho(\phi) \circ \phi^{-1}$, where $\phi \in \mathcal{O}, \mathcal{g}, \mathcal{Y}, \mathcal{G}$ respectively. For instance we find

$$
[g \psi](\{x\}) := \{\{x, [g \psi](x)\}\}.
$$

(62)

Given any smooth gauge transformation $U(x)$ on $\mathbb{R}^n$, the replacements $(\mathcal{V}, \psi, \nabla) \mapsto (\mathcal{V}U, \psi U, U\nabla U^{-1})$ [see (26)] result into a new hermitean line bundle $E^U = (\mathbb{R}^n \times \mathbb{C})/ \sim_{VU}$ isomorphic to $E$, a section $\psi^U \in \Gamma(\mathbb{T}^n, E^U)$ isomorphic to $\psi \in \Gamma(\mathbb{T}^n, E)$ and a covariant
derivative $\nabla^U$ on $\Gamma(\mathbb{T}^n, E^U)$ isomorphic to $\nabla$ on $\Gamma(\mathbb{T}^n, E)$; in other words, they result into a gauge transformation on $\mathbb{T}^n$. The $m$-th Chern class of $E$ is given by the gauge-invariant

$$\text{Ch}_m = \left[ \frac{B^m}{m! (2\pi)^m} \right] = \left[ \frac{(\beta^A)^m}{m! (2\pi)^m} \right],$$

(63)

where $[\omega]$ stands for the de Rham cohomology class containing the closed $p$-form $\omega$.

The above data determine also trivializations of $E$, $\Gamma(\mathbb{T}^n, E)$, $\nabla$ in a canonical way. For each set $X_i$ of a (finite) open cover $\{X_i\}_{i \in \mathcal{I}}$ of $\mathbb{T}^n$ let $W_i$ be a subset of $\mathbb{R}^n$ such that the restriction $P_i : W_i \mapsto X_i$ is invertible. For $u \in X_i$ let

$$\psi_i(u) := \psi \left[ P_i^{-1}(u) \right], \quad A_{i\alpha}(u) := A_{\alpha} \left[ P_i^{-1}(u) \right], \quad \nabla_i := -id + q A_i.$$  \hspace{1cm} (64)

As a consequence of (2) we find in $X_i \cap X_j$ \footnote{The points $x \in W_j, x' \in W_i$ such that $u = P_j x = P_l x'$ are related by $x' = x + 2\pi l$, with some $l \in \mathbb{Z}^n$. One has just to replace the arguments $l, x$ of $V$ in (2) resp. by $P_l^{-1}(u) - P_j^{-1}(u), P_j^{-1}(u)$.}

$$\psi_j = t_{ij} \psi_j, \quad \nabla_i = t_{ij} \nabla_j t_{ij}, \quad t_{ij} : u \mapsto \sqrt{\frac{1}{2\pi}} \left[ P_j^{-1}(u) - P_j^{-1}(u), P_j^{-1}(u) \right]$$

(65)

Condition (3) becomes \footnote{The points $x \in U_k, x' \in U_i, x'' \in U_l$ such that $[x] = [x'] = u = P_k x'' = P_l x' = P_k x$ are related by $x' = x + 2\pi l', x'' = x' + 2\pi l'$ with some $l, l' \in \mathbb{Z}^n$. One has to replace $x, x + 2\pi l', l, l'$ in (3) resp. by $P_k^{-1}(u), P_j^{-1}(u), P_j^{-1}(u) - P_j^{-1}(u), P_j^{-1}(u) - P_j^{-1}(u)$, and use the above definition of $t_{ij}$.}

$$t_{ik} = t_{ij} t_{jk} \quad \text{in } X_i \cap X_j \cap X_k.$$ \hspace{1cm} (66)

This can be interpreted as the (Čech cohomology) cocycle condition for the transition functions $t_{ij}$ of the hermitean line $E$ defined in (58). The set $\{(X_i, \psi_i, \nabla_i)\}_{i \in \mathcal{I}}$ defines a trivialization of the section $\psi \in \Gamma(\mathbb{T}^n, E)$ of $E$ and of the compatible covariant derivative $\nabla$. The hermitean structure can be expressed in terms of the trivialization as $(\psi^U, \bar{\psi}_{\bar{U}}) := \sum_{i \in \mathcal{I}} \int_{X_i} d^n x \psi_i(x)$, where $\{X_i\}_{i \in \mathcal{I}}$ is a partition of $\mathbb{T}^n$ such that for all $\tau \in T$ it is $X_{i(\tau)} \subset X_i$ for some $i(\tau) \in \mathcal{I}$. The advantage of the definition (59) is that we don’t have to bother about patch-dependent $\psi_i$. Setting $U_i(u) := U[\bar{P}_i^{-1}(u)]$ for $u \in X_i$, the set $\{(X_i, U_i)\}_{i \in \mathcal{I}}$ defines the trivialization of a gauge transformation:

$$\psi_i \mapsto \psi_i^U = U_i \psi_i, \quad t^U_{ij} = U_i t_{ij} U_j^{-1}, \quad \nabla_i \mapsto \nabla_i^U = U_i \nabla_i U_j^{-1};$$

(67)

if $U(x)$ is periodic the transformation is \textit{globally defined}, $U_i(u) = U_j(u)$, and $t^U_{ij} = t_{ij}$.

For fixed cover $\{X_i\}_{i \in \mathcal{I}}$, a different choice $\{\bar{W}_i\}_{i \in \mathcal{I}}$ of the $\{W_i\}_{i \in \mathcal{I}}$ in the above construction amounts to a gauge transformation $\{(X_i, \bar{U}_i)\}_{i \in \mathcal{I}}$, \footnote{It must be $\bar{W}_i = W_i + 2\pi l_i$ for some $l_i \in \mathbb{Z}^n$, whence $\bar{\psi}_i(u) = \psi[\bar{P}_i^{-1}(u)] = \psi[P_i^{-1}(u) + 2\pi l_i] = V[l_i, P_i^{-1}(u)]$ where $V[l_i, P_i^{-1}(u)] = V[l_i] P_i^{-1}(u)$.} in agreement with the fact that it leads to the same bundle $E$.

For any $[z] \in \mathbb{T}^n$ let $T_{[z]} : \mathbb{T}^n \mapsto \mathbb{T}^n$ be the translation operator $T_{[z]}[x] := [x + z]$. In terms of the trivialization $\{(X_i, [g_z \psi_i])\}_{i \in \mathcal{I}}$ (62) reads

$$[g_z \psi_i]_{\bar{U}_i}(u) := [g_z \psi]_{\bar{U}_i}[P_i^{-1}(u)] = e^{i q z \cdot a(u) + 2\pi z \cdot [P_i^{-1}(u)]} \psi_i(T_{[z]}[u].)$$

(68)
where \( X_j \) is such that \( T_j u \in X_j \), and the second equality holds in the gauge (8) only. Equation (68) could have been hardly guessed without this lifting procedure, as it is non-local (the result for the \( i \)-th component involves other components).

5 Conclusions

Starting from the basic notion of quasiperiodicity factors \( V : \mathbb{Z}^n \times \mathbb{R}^n \mapsto U(1) \), in the first three sections we have introduced in a systematic way tools and general results regarding quantum mechanics of a charged scalar particle on \( \mathbb{R}^n \) in the presence of a magnetic field and a scalar potential periodic under discrete translations in a lattice \( \Lambda \) of maximal rank.

In Sect. 4 we have shown how these tools and results can be reinterpreted and re-used for the same theory on the torus \( T^n = \mathbb{R}^n / \Lambda \), using the one-to-one correspondences between smooth quasiperiodicity factors \( V \) and hermitean line bundles \( E \mapsto \mathcal{T}^n \), pre-Hilbert spaces \( \mathcal{X}^V \) of the type (2)–(3) and pre-Hilbert spaces \( \mathcal{H}^V \) of smooth sections of \( E \), and between the covariant derivatives, algebras/groups of observables acting on \( \mathcal{X}^V \), \( \mathcal{H}^V \) and those acting on \( \mathcal{T}^n \) and the Hilbert space completion of the latter. Working on the former is an elegant and convenient way to avoid the bothering work with local trivializations.

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14As \( P(x+z) = T_{\{z\}} u \in X_j \), then \( x + z = P_{j-1}^{-1}(T_{\{z\}} u) \), whereas \( x = P_{j-1}^{-1}(u) \in X_j \); replacing these formulæ in (18) we obtain the second equality in (68). As a consistency check, it is straightforward to verify that the conditions \( \{g_z \psi\}_i = t_{ij} \{g_z \psi\}_j \) are satisfied.
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