1. Introduction

In [4] Khovanov studies a class of complexes of bimodules, known as Soergel bimodules, over a polynomial algebra $R$ in variables $x_1, \ldots, x_r$. These complexes are indexed on elements of the braid group on $r$-strands, and are defined if one is given a presentation of a braid word as a product of elementary braids and their inverses (see below). It is shown in [4] that the Hochschild homology of the bimodules in this complex, with coefficients in the bimodule $R$, is an invariant of the link given by closing the braid, and is closely related to link homology.

These results mentioned above assume characteristic zero. In this note, we will extend most of this structure to arbitrary characteristic. There is however a small subtlety at the prime two, indicated in remarks 2.1, 3.1. Unless otherwise stated, all cohomology is with integer coefficients.

Another goal we wish to achieve in this note is to endow Soergel bimodules with an action of the Landweber-Novikov algebra. This algebra is an algebra of cohomology operations in complex cobordism theory. It contains an important sub-algebra which can be seen as an integral form of the universal enveloping algebra of the Lie algebra generated by positive Virasoro operators $L_n$, for $n \geq 0$. In this note we use topological constructions to describe integral forms of Soergel bimodules. We then proceed to describe a natural
action of the above mentioned sub-algebra on this category of Soergel bimodules, and in particular, on their Hochschild homology.

This is part of a larger project on the action of cohomology operations on link homology. We were directly influenced by [6], and the author is indebted to M. Khovanov and L. Rozansky for their interest. The author would also like to thank Jack Morava for sharing his insights on the structure of the Landweber Novikov algebra.

We begin by describing our results in the form of four observations:

**The first observation:**

The first point to observe is that all algebraic constructions made in [4] can be enriched in the category of topological spaces. Furthermore, these spaces have torsion free (singular) cohomology that is evenly graded. More precisely, given a braid word written in the form \( \tilde{\sigma} = \sigma_{i_1}^{e_1} \sigma_{i_2}^{e_2} \cdots \sigma_{i_k}^{e_k} \), with \( e_i = \{-1, 1\} \), consider the following constructions of the complex \( F(\tilde{\sigma}) \) of Soergel bimodules in terms of two-term complexes \( F(\sigma_{i}^{e_i}) \) defined in [4]:

\[
F(\tilde{\sigma}) = F(\sigma_{i_1}^{e_1}) \otimes_R F(\sigma_{i_2}^{e_2}) \otimes_R \cdots \otimes_R F(\sigma_{i_k}^{e_k}).
\]

We observe that the terms of the complex \( F(\tilde{\sigma}) \) are free \( R \)-modules which can in fact be defined over the ground ring \( \mathbb{Z} \), and can be canonically identified with the cohomology of a complex \( F(\tilde{\sigma}) \) of topological spaces (up to degree shifts induced by vector bundles). Under this identification, \( R \) corresponds to the cohomology of the space \( BT \), where \( T \) is a torus of finite rank \( r \), with classifying space denoted by \( BT \). We will also show that the complex \( F(\tilde{\sigma}) \) depends only on the underlying braid group element and not on the presentation (see theorem 7.4). In addition, this independence of presentation is achieved by a zig-zag of maps induced by the Bott-Samelson resolutions.

One can find other enrichments of the category of Soergel bimodules in various topological categories (see [11]). The value of our enrichment however is best demonstrated in the following observations.

**The second observation:**

The second observation one would like to make is that there is a filtration of the spaces in the complex \( F(\tilde{\sigma}) \) mentioned above, so that the induced associated graded algebra of the complex cobordism of these spaces can be canonically identified with the singular cohomology ring of these spaces. In particular, the Landweber Novikov algebra, which is the ring of cohomology operations for complex cobordism, acts naturally on the singular cohomology (with coefficients in the cobordism ring) of the spaces in \( F(\tilde{\sigma}) \) in a graded fashion. This algebra of operations contains a sub-algebra, we shall denote by \( \mathcal{U}(\mathbb{Z}) \) which is an integral form of the algebra of differential operators on the formal line \( \mathbb{A} \). In particular, its rational form \( \mathcal{U}(\mathbb{Q}) \) is the universal enveloping algebra on the Witt algebra generated by the positive Virasoro operators \( L_n = x^{n+1} \frac{d}{dx} \) for \( n \geq 0 \). The sub algebra \( \mathcal{U}(\mathbb{Z}) \) will be shown to preserve cohomology with integer coefficients.

**The third observation:**

The third and most interesting observation we make is that the Koszul resolution that computes the Hochschild homology of the form:

\[
HH_*(F(\tilde{\sigma}), R),
\]
admits an action of a sub algebra (denoted by $\hat{U}(\mathbb{Z})$) of a super version of the Landweber-Novikov algebra introduced above. The sub algebra $\hat{U}(\mathbb{Z})$ is an integral form of the algebra of differential operators on the (odd) tangent bundle of the formal line $TA$. In particular, its rational form $\hat{U}(\mathbb{Q})$ is the universal enveloping algebra of the Witt algebra generated by the Virasoro operators $L_n$ (for $n \geq 0$), and certain odd differentials $G_n$ such that $G_0$ is the Koszul differential. The commutation relations are given by a super-Witt algebra:

$$[L_m, L_n] = (n - m) L_{m+n}, \quad [L_m, G_n] = (n + 1) G_{m+n}, \quad \{G_m, G_n\} = 0.$$ 

From these relations, one notices that the operators $L_m$ do not commute with the differentials $G_n$. However, one may twist these operators to obtain another family of operators $\tilde{L}_m$ that do indeed commute with the differential $G_0$. Unfortunately, the operators $G_m$ are trivial up to homotopy, and so we are left with the following relations up to homotopy with respect to the differential $G_0$:

$$[\tilde{L}_m, \tilde{L}_n] = (n - m) \tilde{L}_{m+n}, \quad G_m = 0.$$ 

One therefore obtains an induced (twisted) action of $U(\mathbb{Z})$ on the Hochschild homology $HH_*(F(\tilde{\sigma}), R)$. In particular, one may reduce mod-$p$ to get a twisted action of the $\mathbb{F}_p$-form $\hat{U}(\mathbb{F}_p)$ on the Hochschild homology with coefficients in the field $\mathbb{F}_p$. The algebras $U(\mathbb{F}_p)$, and their super versions $\hat{U}(\mathbb{Z})$ admit a lot of structure. A particularly interesting sub algebra of $U(\mathbb{F}_p)$ is described in the following observation:

**The final observation:**

The final observation is that for any odd prime $p$, the mod-$p$ Steenrod algebra $\mathcal{A}$ is a canonical sub-algebra of the $\mathbb{F}_p$ form $\hat{U}(\mathbb{F}_p)$, and in particular one gets an action of the Steenrod algebra on the Hochschild complex of $F(\tilde{\sigma})$ over the field $\mathbb{F}_p$. Similarly, $U(\mathbb{F}_p)$ contains the sub algebra of the Steenrod algebra generated by the reduced $p$-th powers $P^i$. Therefore, a twist of the reduced powers in the Steenrod algebra act on Hochschild homology of $F(\tilde{\sigma})$.

2. **Topological enrichment of Soergel modules**

**Assumptions 2.1.** In this note, we will assume that $G$ is a connected compact Lie group (or a Kac-Moody group) of adjoint type, and rank $r$. Let $i \leq r$ be any index, and let $G_i$ be the compact parabolic subgroup corresponding to the root $\alpha_i$. In what follows, we will assume that the integral cohomology of the classifying space: $BG_i$ is torsion free for all $i$ (this is true for example for $G = PU(n)$, $n > 2$). Failing that condition, the reader must assume that the prime two has been inverted throughout (note that the only possible torsion in $H^*(BG_i)$ is two torsion).

Let $T \subset G$ be the maximal torus. Since $G$ is of adjoint type, we use the simple roots to identify $T$ with a rank $r$-torus: $(S^1)^r$. Let $EG$ denote the free contractible $G$-complex that serves as the universal principal $G$-bundle. Notice that the space $EG$ serves as a model of $ET$ and also of $EG_i$. We define the space $F(\sigma_i)$ as the top left corner in the pullback diagram below, where the map $\rho$ is induced by the right action of $G_i$ on $EG$, and the map
Now, given a sequence $J = (j_1, \ldots, j_s)$, let us define $\mathcal{F}(\bar{\sigma}_{j_s}) = EG \times_T (G_{j_1} \times_T \cdots \times_T G_{j_s})/T$. Then we have a family pullback diagrams:

$$
\begin{array}{ccc}
\mathcal{F}(\bar{\sigma}_{j_s}) & \xrightarrow{\rho_{s-1}} & \mathcal{F}(\sigma_{j_s}) \\
\pi_s & & \pi \\
\mathcal{F}(\bar{\sigma}_{j_{s-1}}) & \xrightarrow{\rho_{s-1}} & EG/T,
\end{array}
$$

where $\rho_{s-1}$ is induced by the multiplication map on the first $(s - 1)$ factors:

$$
\rho_{s-1} : G_{j_1} \times_T \cdots \times_T G_{j_s}/T \to G \times_T G_{j_s}/T.
$$

By an induction argument using the above pullback, we easily see that:

**Corollary 2.3.** The bimodule $H^*(\mathcal{F}(\bar{\sigma}_{j_s}))$ is a free (left or right) $H^*(BT)$-module given by:

$$
H^*(\mathcal{F}(\bar{\sigma}_{j_s})) = H^*(\mathcal{F}(\sigma_{j_1}), \mathbb{Z}) \otimes_{H^*(BT)} H^*(\mathcal{F}(\sigma_{j_2})) \otimes_{H^*(BT)} \cdots \otimes_{H^*(BT)} H^*(\mathcal{F}(\sigma_{j_s})).
$$

Now observe that the fibration $\pi$ supports two canonical sections $s(0)$ and $s(\infty)$ corresponding to the $T$-fixed points of $G_i/T$ given by $[T]$ and $[\sigma_i T]$ respectively. The pullback of the map $\rho$ along $s(0)$ is the identity map on $BT$. On the other hand, the pullback of the map $\rho$ along $s(\infty)$ is the automorphisms of $BT$ induced by $\sigma_i$. Now if we collapse the section $s(\infty)$ to a point, we obtain the Thom space $BT^{\alpha_i}$ for the line bundle $L(\alpha_i)$ over $BT$ corresponding to the positive root $\alpha_i$.

**Claim 2.4.** Consider two topological maps given respectively by taking the topological quotient by the section $s(\infty)$, and by including the section $s(0)$:

$$
qs(\infty) : \mathcal{F}(\sigma_i) \to \mathcal{F}(\sigma_i)/s(\infty) = BT^{\alpha_i}, \quad s(0) : BT \to \mathcal{F}(\sigma_i).
$$

Then, these induce maps of bimodules over $H^*(BT)$:

$$
br_i : \tilde{H}^*(BT^{\alpha_i}) \to H^*(\mathcal{F}(\sigma_i)), \quad br_i : H^*(\mathcal{F}(\sigma_i)) \to H^*(BT).
$$
Proof. The only thing that is not obvious is the induced $H^*(BT)$ bimodule structure on $H^*(BT^{\alpha_i})$. Now, by restricting along the sections $s(0)$ and $s(\infty)$ one can easily verify that the image of the Thom class in $H^*(F(\sigma_i))$ is given by $\text{th}(\alpha_i) = \frac{1}{2}(\rho^*(\alpha_i) + \pi^*(\alpha_i))$. Here we have identified a character with its first Chern class. This class is in fact an integral class (as it restricts to an integral class on the base and fiber of $\pi$). Now by symmetry, $\text{th}(\alpha_i)$ satisfies the required property.

Definition 2.5. Let $F(\sigma_i)^{-\xi_i}$ denote the Thom spectrum of the (formal virtual) bundle defined by the equality $2\xi_i = \rho^*(L(\alpha_i)) \oplus \pi^*(L(\alpha_i))$, where $L(\alpha_i)$ is the line bundle corresponding to the character $\alpha_i$. This formal spectrum is well defined away from the prime two (see remark 3.1 below). Hence, we may desuspend $rb_i$ to a two-term complex: $F(\sigma_i)^{-1} : H^*(BT) \to \tilde{H}^*(F(\sigma_i)^{-\xi_i})$ graded so that $H^*(BT)$ is in degree zero. We also define $F(\sigma_i) : H^*(F(\sigma_i)) \to H^*(BT)$ to be the map $br_i$, graded as before. By tensoring these elementary complexes together, we may define complexes $F(\tilde{\sigma})$ for arbitrary words.

Remark 2.6. By construction, the maps in the complex of bimodules $F(\tilde{\sigma})$ are induced by maps between the spaces $F(\tilde{\sigma}_J)$, or Thom spectra of complex vector bundles over them. Notice also that our conventions are compatible with those in [9], and differ from the ones in [4].

3. The Atiyah-Hirzebruch filtration

Now we describe a natural filtration of the ring: $MU^*(F(\tilde{\sigma}_J))$, where $MU$ denotes complex cobordism theory with coefficient ring $\Omega$. The corresponding associated graded ring (defined as a direct sum of associated quotients) is isomorphic to $H^*(F(\tilde{\sigma}_J)) \otimes \Omega$.

The construction of this filtration and the isomorphism is a standard fact invoked in the construction of the Atiyah-Hirzebruch spectral sequence [1]. If one has a space (or spectrum) with the homotopy type of a CW complex of finite type, then the above filtration is given by the kernels of the restriction maps of the CW sub-skeleta. If the space in question is one with cells in even degrees (like our spaces $F(\tilde{\sigma}_J)$ or Thom spectra of complex vector bundles over such), its singular cohomology is torsion free and the spectral sequence collapses to yield a ring-isomorphism between singular cohomology with coefficients, and the associated graded object of the filtration.

From the above observation we notice that cohomology operations of complex cobordism acts on the rings of the form $H^*(F(\tilde{\sigma}_J)) \otimes \Omega$. These cohomology operations are known as the Landweber-Novikov algebra [1] (see the next section for details). This algebra is graded, and this grading is compatible with the action on the graded ring $H^*(F(\tilde{\sigma}_J)) \otimes \Omega$.

The Landweber-Novikov algebra has a natural topology with respect to which it is complete. Topologically, it is generated by scalar multiplication operators $\Omega$, and an important sub-algebra $L$ which is an integral form of the universal enveloping algebra of the Witt algebra of positive Virasoro operators $L_n$ for $n \geq 0$ [12]. It is important to bear in mind that the scalar multiplication operators $\Omega$ are not central in the Landweber-Novikov algebra. In fact, the twisting given by the action of $L_n$ on $\Omega$ can be described via the Bosonic-Fock representation of the Virasoro algebra [7]. The sub-algebra $L$ is the algebra we called $U(\mathbb{Z})$ in the introduction. By the naturally of cohomology operations, we see that $L$ preserves the subring $H^*(F(\tilde{\sigma}_J))$ (see remark 3.1 below for details).
Remark 3.1. The action of the operators $L_n$ on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_r]$ is the standard action $\sum x_i^{n+1} \frac{\partial}{\partial x_i}$ (see next section). Now since $H^*(\mathcal{F}(\bar{\sigma}))$ is a quotient of a polynomial ring in $2r|J|$ variables, we have a complete handle on the action of $\mathcal{L}$ on the spaces $H^*(\mathcal{F}(\bar{\sigma}))$. The action of $\mathcal{L}$ on $\tilde{H}^*(\mathcal{F}(\sigma_i)^{-\zeta_i})$ is determined (via the derivation property), once we understand its action on the Thom class $Th(-\zeta_i)$. This action is given by (see 4.1):

$$L_n(Th(-\zeta_i)) = -\frac{1}{2} Th(-\zeta_i) \cup (\rho^*(\alpha_i)^n + \pi^*(\alpha_i)^n).$$

Recall that $\frac{1}{2}(\rho^*(\alpha_i) + \pi^*(\alpha_i))$ is an integral class. It follows from elementary arithmetic that the operators $L_n$ preserve integral cohomology. However, more general elements in $\mathcal{L}$ may introduce denominators given by powers of two (see [12] for explicit formulas).

Corollary 3.2. The subalgebra $\mathcal{L}$ of the Landweber-Novikov algebra acts naturally on the complex of Soergel bimodules: $H^*(\mathcal{F}(\bar{\sigma}))$ (with the above proviso).

4. The (Super) Landweber-Novikov Algebra

Before we discuss the super case, let us describe the usual Landweber-Novikov algebra in terms that is useful to us.

We define the algebra $\mathcal{O}(S) = \mathbb{Z}[s_0^{\pm1}, s_1, \ldots]$ as the commutative Hopf algebra of functions on the group $S$ of invertible formal power series under composition (i.e. formal diffeomorphisms of the affine line). Therefore, the $R$-valued points of $S$ are given by:

$$S(R) = \{ f(X) = \sum_{i \geq 0} b_i X^{i+1}, \quad b_i \in R, \quad b_0 \in R^\times \},$$

where $s_i$ is the co-ordinate function of $X^{i+1}$. Let $A = \Omega$ denote the cobordism ring. By results of Quillen, the ungraded ring underlying $A$ is isomorphic to the Lazard ring that represents the one dimensional formal group laws. Define $\Gamma := A \otimes \mathcal{O}(S)$. Then the pair $(A, \Gamma)$ is the Hopf algebroid that represents the groupoid of one dimensional formal group laws. This can be interpreted as saying that the pair $(Spec(A), Spec(\Gamma))$ is a groupoid. This is none other than the groupoid of formal group laws, and as such it is isomorphic to the co-operations in cobordism $MU, MU^{-1}$.

The groupoid $(Spec(A), Spec(\Gamma))$ acts on the universal formal group $\mathbb{G}$, as described by the diagram:

$$\begin{array}{ccc}
Spec(\Gamma) \times_{Spec(A)} \mathbb{G} & \overset{\mu}{\longrightarrow} & \mathbb{G} \\
\downarrow & & \downarrow \\
Spec(\Gamma) & \overset{\eta_R}{\longrightarrow} & Spec(A).
\end{array}$$

where $\eta_R$ represents the “target” map on the morphisms $Spec(\Gamma)$. Taking functions:

$$\mu^* : \mathcal{O}(\mathbb{G}) = A[[X]] \longrightarrow \Gamma[[X]] = A[s_0^{\pm1}, s_1, \ldots][[X]] = \mathcal{O}(Spec(\Gamma) \times_{Spec(A)} \mathbb{G}).$$

It is easy to see that:

$$\mu^*(X) = \sum_{i \geq 0} s_i X^{i+1}.$$
Define the ring $\mathcal{L}$ to be the sub-algebra of the Landweber-Novikov algebra given by the continuous $\mathbb{Z}$-linear dual of $\mathcal{O}(\mathbb{G})$: $\text{Hom}(\mathcal{O}(\mathbb{G}), \mathbb{Z})$. This algebra acts on the ring $\mathbb{Z}[X]$, with a dual monomial $(s^I)^* := (s_1^{I_1} s_2^{I_2} \cdots s_m^{I_m})^*$ acting on $X^k$ via the rule:

$$(s^I)^*(X^k) = \langle \mu^*(X)^k, (s^I)^* \rangle.$$ 

Using the above formula, it follows that elements of $\mathcal{L}$ given by $(s^I)^*$ act trivially on $X$ unless the multi-index $I$ has the property $\sum I_k = 1$. The corresponding primitive classes $s_m^* := L_m$ form a Lie sub-algebra of $\mathcal{L}$. Their action on $\mathcal{O}(\mathbb{G})$ is described as:

$L_m(X) = \langle \mu^*(X), s_m^* \rangle = X^{m+1} \Rightarrow L_m = X^{m+1} \frac{\partial}{\partial X}.$

This implies that one has a map from the universal enveloping algebra of the positive Virasoro $\mathbb{W} := \mathcal{U}(L_m, m \geq 0)$:

$$\zeta : \mathbb{W} \longrightarrow \mathcal{L}.$$ 

An easy dimensional count shows that $\zeta$ is a rational isomorphism. It follows that $\mathcal{L}$ is an integral form of the algebra of differential operators on the formal line that can be explicitly described [12].

The functions $\mathbb{Z}[X]$ on the affine line should be thought of as the cohomology of $\mathbb{C}P^\infty$ in the context of cohomology operations. Similarly, the ideal generated by $X$ in this ring should be considered as the cohomology of the Thom space of the universal line bundle $\gamma$ over $\mathbb{C}P^\infty$, which we denote by $\mathbb{C}P^\gamma$. The action of $\mathcal{L}$ on the Thom class is given by a generating function:

$$\sum_I (s^I)^* \text{Th}(\gamma) = \text{Th}(\gamma) \cup \langle (s^I)^*, \sum_{i \geq 0} s_i X^i \rangle = \text{Th}(\gamma) \cup \langle \sum_{i \geq 0} X^i \rangle.$$ 

Taking the (external) sum of the universal bundles $\gamma = \gamma_1 \boxplus \gamma_2$ over the product of two infinite complex projective spaces $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, one obtains a formula for the action of the algebra $\mathcal{L}$ on the cohomology of the Thom space of an external sum:

$$\sum_I (s^I)^* \text{Th}(\gamma) = \text{Th}(\gamma) \cup \langle \sum_I (s^I)^*, \sum_{i \geq 0} s_i X^i \rangle \cup \langle \sum_{j \geq 0} s_j Y^j \rangle.$$ 

Of course, one may extend the above formula to arbitrary external sums. Taking pullback along a suitable diagonal, one gets a Cartan formula for a $k$ fold Whitney sum of $\gamma$:

$$\sum_I (s^I)^* \text{Th}(k\gamma) = \text{Th}(k\gamma) \cup \langle \sum_I (s^I)^*, \sum_{i \geq 0} s_i X^i \rangle \cup \langle \sum_{j \geq 0} s_j Y^j \rangle \rangle^k.$$ 

Now consider the following formal expression:

$$\langle \sum_{i \geq 0} s_i X^i \rangle \cup \langle \sum_{j \geq 0} s_j Y^j \rangle \rangle^{\pm \frac{1}{k}}.$$ 

Using the invertibility of $s_0$, one sees that the above expression is well defined and belongs to the ring $\mathbb{Z}[\frac{1}{k}][s_0^\pm, s_1, \ldots][[X, Y]]$. This observation, along with the Cartan formula above motivates the following definition:

**Definition 4.1.** Let $\gamma = \gamma_1 \boxplus \gamma_2$ be as above, then the action of the algebra $\mathcal{L}$ on the Thom class of the formal virtual vector bundle $-\frac{\gamma}{2}$ is given by the formal generating function:

$$\sum_I (s^I)^* \text{Th}(-\frac{\gamma}{2}) = \text{Th}(-\frac{\gamma}{2}) \cup \langle \sum_I (s^I)^*, \sum_{i \geq 0} s_i X^i \rangle \cup \langle \sum_{j \geq 0} s_j Y^j \rangle \rangle^{-\frac{1}{k}}.$$ 

7
The super algebra:

We would now like to extend the above structure to describe a super algebra \( \hat{L} \) that acts on the functions on the (odd) tangent bundle of the affine line \( TA \) given by the \( \mathbb{Z}[X] \)-module: \( \mathbb{Z}[X, \epsilon]/\epsilon^2 \). Proceeding as before, this algebra is the continuous \( \hat{Z} \) dual of the algebra:

\[
\mathcal{O}(\hat{S}) = \Lambda(t_0, t_1, \ldots) \otimes \mathbb{Z}[s_0^\pm, s_1, \ldots],
\]

defined as the super-commutative Hopf algebra of functions on the super-group \( \hat{S} \) of invertible formal power series under composition that yield formal diffeomorphisms of the odd tangent bundle of the affine line. The \( R \)-points of \( \hat{S} \) are given by composable power series of the type:

\[
\hat{S}(R) = \{ f(X) = \sum_{i \geq 0} (a_i \theta + b_i)X^{i+1}, \ a_i, b_i \in R, \ b_0 \in R^\times \},
\]

where \( \theta \) is the odd variable in the tangent direction, and \( t_i \) and \( s_i \) are the co-ordinate functions on the coefficients of \( \theta X^{i+1} \) and \( X^{i+1} \) respectively. Therefore, the co-action of \( \mathcal{O}(\hat{S}) \) on \( \mathbb{Z}[X, \epsilon]/\epsilon^2 \) is induced by:

\[
\mu^*(X) = \sum_{i \geq 0} s_i X^{i+1}, \quad \mu^*(\epsilon) = \epsilon + \sum_{i \geq 0} t_i X^{i+1}.
\]

As before, it follows from the co-action above that elements of \( \hat{L} \) given by the dual monomials \( (s^l)^* := (s_1^l, s_2^l, \ldots, s_m^l)^* \) and \( (t^l)^* := (t_0^l, t_1^l, \ldots, t_m^l)^* \) act trivially on \( X \) or \( \epsilon \) unless the multi-index \( l \) has the property \( \sum I_k = 1 \). The corresponding primitive classes \( s_m^* := L_m, \) and \( t_m^* := G_m \) form a Lie sub-algebra of \( \hat{L} \). Their action on \( \mathbb{Z}[X, \epsilon]/\epsilon^2 \) is described as:

\[
L_m = X^{m+1} \frac{\partial}{\partial X}, \quad G_m = X^{m+1} \frac{\partial}{\partial \epsilon}.
\]

The following commutation relations can now be verified:

\[
[L_m, L_n] = (n - m) L_{m+n}, \quad [L_m, G_n] = (n + 1) G_{m+n}, \quad \{G_m, G_n\} = 0.
\]

**Corollary 4.2.** The super algebra described above: \( \hat{U}(\mathbb{Z}) := \hat{L} \) acts externally (see remark 4.3 below) on the homological Hochschild (bi) complex: \( \text{CHH}(H^* (\mathcal{F}(\hat{\sigma})), H^* (BT)) \) with \( G_0 \) being the Koszul differential.

**Remark 4.3.** Notice that the action of \( \hat{U}(\mathbb{Z}) \) described above does not commute with the Koszul differential. However, it does indeed commute with the external differential induced by the maps of topological spaces that define \( \mathcal{F}(\hat{\sigma}) \). We call such an action an external action. To simplify matters, we shall call \( \hat{L} \) the (super) Landweber-Novikov algebra, even though strictly speaking it is only a sub algebra of the usual Landweber-Novikov algebra.

**Remark 4.4.** In the example of the super Landweber-Novikov algebra discussed above, the super functions on the formal line can be interpreted as elements in the cohomology of the free loop space of \( \mathbb{C} P^\infty \): \( \hat{L} \mathbb{C} P^\infty \).

5. RELATION TO THE MOD-\( p \) STEENROD ALGEBRA

Let us assume in this section that \( \mathbb{F} = \mathbb{F}_p \) is the primary field of characteristic \( p \), for an odd prime \( p \) (see remark 5.2 for \( p = 2 \)). Let \( A \) denote the mod-\( p \) Steenrod algebra. In analogy with the super Landweber-Novikov algebra described above, the dual Steenrod algebra
can be expressed as: $\mathcal{A}^* = \mathcal{O} \text{Aut}(T G_a)$, where $\mathcal{O} \text{Aut}(T G_a)$ denotes the commutative Hopf algebra of functions on the group of automorphisms of the odd tangent bundle of the additive formal group $G_a$ (see [3] section 4).

We may explicitly describe the Hopf-algebra of functions as the (super) commutative ring:

$$\mathcal{O} \text{Aut}(T G_a) = \Lambda(\tau_0, \tau_1, \cdots) \otimes \mathbb{F}[\xi_0^\pm, \xi_1, \xi_2, \cdots]$$

with $\tau_i, \xi_i$ being the coordinate functions given by the coefficient of $\theta X^{p^i}$ and $X^{p^i}$ respectively, on the affine group scheme $\text{Aut}(T G_a)$ over $\mathbb{F}$. The $R$-points of $\text{Aut}(T G_a)$ are given by the group formal power series (under composition):

$$\text{Aut}(T G_a)(R) = \{ f(X) = \sum_{i \geq 0} (c_i \theta + d_i) X^{p^i}, \quad c_i, d_i \in R, \quad d_0 \in R^\times \}.$$ 

In particular, we have an injective group homomorphism between the group $\text{Aut}(T G_a)$ and the formal automorphisms of the tangent bundle of the affine line described in the previous section:

$$\text{Aut}(T G_a) \longrightarrow \hat{S}.$$ 

Taking functions and dualizing, we see that the Steenrod algebra is a sub algebra of the Landweber-Novikov algebra $\hat{L} \otimes \mathbb{F}$.

**Corollary 5.1.** For an odd prime $p$, the Steenrod algebra $\mathcal{A}$ acts externally (see remark 4.3) on the mod-$p$ homological Hochschild (bi) complex:

$$\text{CHH}(H^*(\mathcal{F}(\hat{\sigma}), \mathbb{F}), H^*(BT, \mathbb{F})),$$

with the Bockstein homomorphism $\beta \in \mathcal{A}^1$ being the Koszul differential.

**Remark 5.2.** For $p = 2$, the dual Steenrod algebra can be identified with functions on $\text{Aut}(G_a)$ ([2], [12]). Therefore $\mathcal{A}^* = \mathbb{F}[\xi_0^\pm, \xi_1, \cdots]$.

### 6. A TWIST OF THE LANDWEBER-NOVIKOV ALGEBRA AND HOCHSCHILD HOMOLOGY

Consider the Koszul resolution of a polynomial algebra $\mathbb{Z}[X]$ as a bimodule over $\mathbb{Z}[X, Y]$ under the specialization $X = Y$:

$$\mathcal{C} = \mathbb{Z}[X, Y, \epsilon]/\epsilon^2, \quad d(\epsilon) = (X - Y).$$

This is clearly sub complex of the complex that resolves $\mathbb{Z}$ as a $\mathbb{Z}[X, Y]$ bimodule:

$$\mathcal{D} = \mathbb{Z}[X, Y, \epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2), \quad d(\epsilon_1) = X, \quad d(\epsilon_2) = Y.$$ 

In particular, the algebra $\hat{U}(\mathbb{Z})$ acts on $\mathcal{C}$ by restriction of its action on $\mathcal{D}$. Now let us define symmetric polynomials $\pi_k$ by (see [6] page 18):

$$\pi_k(X, Y) = \frac{X^{k+1} - Y^{k+1}}{X - Y} = X^k + X^{k-1}Y + \cdots + Y^k.$$ 

Then one may use these functions to deform the action of the super group $\hat{S}$ on $\mathbb{Z}[X, Y]$ defined in section 4, to obtain a new action of $\hat{S}$ uniquely defined by:

$$\mu^*(X) = \sum_{i \geq 0} s_i X^{i+1}, \quad \mu^*(Y) = \sum_{i \geq 0} s_i Y^{i+1}, \quad \mu^*(\epsilon) = \sum_{i \geq 0} \epsilon s_i \pi_i + \sum_{i \geq 0} t_i (X^{i+1} - Y^{i+1}).$$
In particular, we get the expressions for the super Lie algebra generators:

$$\tilde{L}_m = X^{m+1} \frac{\partial}{\partial X} + Y^{m+1} \frac{\partial}{\partial Y} + \epsilon \pi_m \frac{\partial}{\partial \epsilon}, \quad \tilde{G}_m = (X^{m+1} - Y^{m+1}) \frac{\partial}{\partial \epsilon}.$$

Let us define operators $\lambda_m$ and $\lambda_{m,n}$ as:

$$\lambda_m = \epsilon \pi_m \frac{\partial}{\partial \epsilon}, \quad \lambda_{m,n} = \epsilon \pi_m \pi_n \frac{\partial}{\partial \epsilon}.$$

It is not hard to verify the following equalities:

$$[\tilde{L}_m, \tilde{L}_n] = (n - m) \tilde{L}_{m+n}, \quad [\tilde{L}_m, G_n] = (n + 1) G_{m+n} - [G_0, \lambda_{m,n}], \quad G_m = [G_0, \lambda_m].$$

It follows that the operators $G_m$ are trivial up to homotopy, and the operators $\tilde{L}_k$ commute with the differential $G_0$, and satisfy the required commutation relations.

One may now extend the above twist in the obvious fashion to the Koszul resolution of $\mathbb{Z}[X_1, \ldots, X_r]$ over $\mathbb{Z}[X_1, Y_1, \ldots, X_r, Y_r]$ under the specialization $X_i = Y_i$. It is easy to see that the operators $\tilde{L}_k$ as well as the relations above, extend to the Hochschild chain complex obtained by tensoring the above Koszul resolution with any Soergel bimodule. By the commutation relations above, one recovers an integral lift of the twisted action of the Witt algebra $U(\mathbb{Q})$ on the Hochschild homology of Soergel bimodules that was shown to exist in [6]:

**Corollary 6.1.** One has a twisted action of $U(\mathbb{Z})$ on the homological Hochschild (bi) complex: $\text{CHH}(H^*(F(\tilde{\sigma})), H^*(BT))$ that descends to an action on $HH_*(H^*(F(\tilde{\sigma})), H^*(BT))$.

Reducing mod $p$, for any odd prime $p$, one obtains a (twisted) action of the reduced Steenrod powers $P^i$ on the (bi) complex $\text{CHH}(H^*(F(\tilde{\sigma}), \mathbb{F}_p), H^*(BT, \mathbb{F}_p))$ that descends to an action on the complex of mod-$p$ Hochschild homologies of Soergel bimodules: $HH_*(H^*(F(\tilde{\sigma}), \mathbb{F}_p), H^*(BT, \mathbb{F}_p))$.

### 7. Independence of presentation: The Braids relation and Bott-Samelson resolutions

So far we have worked with a fixed presentation of a braid word: $\tilde{\sigma} = \sigma_{i_1}^{e_1} \sigma_{i_2}^{e_2} \cdots \sigma_{i_k}^{e_k}$. One would like to know how much of the structure developed in the previous sections remains preserved if we change the presentation. In [6] the authors show that in characteristic zero, the twisted action of $U(\mathbb{Q})$ on the homology of the complex $HH_*(F(\tilde{\sigma}, \mathbb{Q}), H^*(BT, \mathbb{Q}))$ is independent of presentation (it is in fact, a link invariant!). In this section, we will establish independence of presentation using topological arguments, so that the action of the Landweber-Novikov algebra is preserved. We begin with:

Let $i, j$ be two elements in the set indexing the fundamental reflections in the Weyl group. Let us adopt the convention from [9] of denoting a general element of the set $\{i, j\}$ by the letter $\nu$, and the complement element by the letter $-\nu$. Consider a braid relation of the form:

$$\sigma := \cdots \sigma_{-\nu} \sigma_{-\nu} \sigma_{\nu} = \cdots \sigma_{-\nu} \sigma_{\nu} \sigma_{-\nu} \quad m_{ij} \text{ factors.}$$

For $n \leq m_{ij} < \infty$, we will denote by $\tilde{\sigma}(\nu, n)$ the abstract braid word:

$$\tilde{\sigma}(\nu, n) = \cdots \sigma_{\nu} \sigma_{-\nu} \sigma_{\nu} \quad n \text{ factors.}$$

Let $F(\nu, n)$ denote the complex of bi-modules corresponding to $\tilde{\sigma}(\nu, n)$. Recall that they consist of terms of the form $H^*(F(\tilde{\sigma}))$ for sub-words $\tilde{\sigma}$ (see 2.5).
In this section, we will use topological arguments to establish canonical zig-zag of equivalences between \( F(\nu, m_{ij}) \) and \( F(-\nu, m_{ij}) \). Consequently, we will establish a topologically induced equivalence between the complexes \( F(\tilde{\sigma}) \) corresponding to any two presentations of the same braid group element (see theorem 7.4 for a precise statement).

We begin by invoking some standard facts about the topology of flag varieties for Kac-Moody groups. A standard reference for these results is [8]. Let \( P \subseteq G \) denote the rank two parabolic corresponding to the reflections \( \sigma_\nu \) and \( \sigma_{-\nu} \). Let \( G_{\pm \nu} \) denote the rank one parabolics corresponding to the individual reflections \( \sigma_{\pm \nu} \). By \( X(\nu, n) \subseteq P/T \) we shall mean the Schubert variety given by the closure of all Shubert cells corresponding to elements \( \nu \) in the Weyl group so that \( \nu \) has a presentation given by a sub-word of \( \tilde{\sigma}(\nu, n) \). The space \( X(\nu, n) \) is known to be a CW complex of dimension \( 2n \), with \( T \)-invariant cells in even degree. For \( k < n \), the \( 2k \)-skeleton of \( X(\nu, n) \) is given by the union \( X(\nu, k) \cup X(-\nu, k) \).

We will use the notation \( X(\nu, n) \) to denote the spaces \( EG \times_T X(\nu, n) \).

The left \( H^*(BT) \)-module \( H^*(EG \times_T (P/T)) \) supports a Schubert basis denoted by \( \delta_{\nu, n} \) indexed by elements \( \tilde{\sigma}(\nu, n) \) for \( n \leq m_{ij} \). We set \( \delta_{\nu, 1} = \delta_\nu \). By definition, the Schubert basis pairs diagonally with the generators of \( H_{2k}(X(\nu, k)) \) (see [8] Ch. 9, §11). We will denote the restrictions of the Schubert basis to sub-skeleta by the same name.

**Definition 7.1.** Define a (negativity graded) complex of bimodules: \( X_\bullet(\nu, n) \) with:

\[
X_{-k}(\nu, n) := H^*(X(\nu, k)) \oplus H^*(X(-\nu, k)), \quad \text{if } 0 < k < n,
\]

\[
X_{-n}(\nu, n) := H^*(X(\nu, n)), \quad X_0(\nu, n) := H^*(BT).
\]

The differentials \( d_k : H^*(X_{-k}(\nu, n)) \to H^*(X_{1-k}(\nu, n)) \) are defined by cellular restrictions with appropriate signs as:

\[
d_k(\alpha \oplus \beta) = (\alpha + (-1)^{k-1}\beta) \oplus (\beta + (-1)^{k-1}\alpha).
\]

**Remark 7.2.** Since all boundary maps in degree bigger than \(-n\) are surjective, it is easy to see that the homology of the complex \( X_\bullet(\nu, n) \) is a free rank one left (or right) \( H^*(BT) \)-module in degree \(-n\), generated by the schubert cell \( \delta_{\nu, n} \). Notice also that the braid relation implies that there is a canonical isomorphism:

\[
X_\bullet(\nu, m_{ij}) = X_\bullet(-\nu, m_{ij}).
\]

Consider the map induced by multiplication in \( P \), known as the Bott-Samelson resolution:

\[
BS(\nu, n) : F(\nu, n) \to X(\nu, n).
\]

**Claim 7.3.** The map \( BS(\nu, n) \) induces a map of complexes of bimodules:

\[
BS_\bullet(\nu, n) : X_\bullet(\nu, n) \to F(\nu, n).
\]

**Proof.** We shall construct the required map \( BS_\bullet(\nu, n) \) by induction, starting with a map:

\[
\mu_\bullet : X_\bullet(\nu, n) \to X_\bullet(-\nu, n-1) \otimes_{H^*(BT)} F(\sigma_\nu).
\]

Topologically, the map \( \mu_\bullet \) will be realized inside a pair of pullback diagrams:

\[
\begin{array}{ccc}
EG \times_T (X(-\nu, n-1) \times_T G_\nu/T) & \xrightarrow{\tilde{\mu}} & X(\nu, n) \\
\pi & & \pi_{\nu} \\
X(-\nu, n-1) & \xrightarrow{\kappa} & EG \times_T (\tilde{Y}(\nu, n)/G_\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X(\nu, n) & \xrightarrow{\rho} & EG/T \\
\pi & & j \\
X(-\nu, n-1) & \xrightarrow{\kappa} & EG \times_T (\tilde{Y}(\nu, n)/G_\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
EG \times_T (X(-\nu, n-1) \times_T G_\nu/T) & \xrightarrow{\tilde{\mu}} & X(\nu, n) \\
\pi & & \pi_{\nu} \\
X(-\nu, n-1) & \xrightarrow{\kappa} & EG \times_T (\tilde{Y}(\nu, n)/G_\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X(\nu, n) & \xrightarrow{\rho} & EG/T \\
\pi & & j \\
X(-\nu, n-1) & \xrightarrow{\kappa} & EG \times_T (\tilde{Y}(\nu, n)/G_\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
EG \times_T (X(-\nu, n-1) \times_T G_\nu/T) & \xrightarrow{\tilde{\mu}} & X(\nu, n) \\
\pi & & \pi_{\nu} \\
X(-\nu, n-1) & \xrightarrow{\kappa} & EG \times_T (\tilde{Y}(\nu, n)/G_\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X(\nu, n) & \xrightarrow{\rho} & EG/T \\
\pi & & j \\
X(-\nu, n-1) & \xrightarrow{\kappa} & EG \times_T (\tilde{Y}(\nu, n)/G_\nu) \\
\end{array}
\]
where $\mu$ is induced by group multiplication in $P$, and factors the Bott-Samelson resolution. The space $\tilde{X}(\nu, n)$ is the subspace of $P$ given by the pre image of $X(\nu, n) \subseteq P/T$. The space $\tilde{Y}(\nu, n)$ is similarly defined by as the pre image of $Y(\nu, n) \subseteq P/G_{\nu}$, with $Y(\nu, n)$ being the Shubert variety in $P/G_{\nu}$, given by the image of $fX(\nu, n)$. The map $\kappa$ above is given by the restriction of $\pi_{\nu}$ to the subspace $X(\nu, n - 1) \subset X(\nu, n)$.

Let $\mathcal{X}(\pm \nu, n - 1, \nu)$ denote the space $EG \times_T (\tilde{X}(\pm \nu, n - 1) \times_T G_{\nu}/T)$. Using the Eilenberg-Moore spectral sequence, as in corollary 2.3, we see that:

$$H^*(\mathcal{X}(\pm \nu, n - 1, \nu)) = H^*(\mathcal{X}(\pm \nu, n - 1)) \otimes_{H^*(BT)} H^*(\sigma_{\nu}) = H^*(\mathcal{X}(\pm \nu, n - 1)) \otimes F_{-1}(\sigma_{\nu}).$$

Under the above identification, we extend the map $\mu^*$ to $\mu_\bullet$ on the level of complexes:

For an element $\gamma \in H^*(\mathcal{X}(\nu, k)) \subset \mathcal{X}_{-k}(\nu, n)$, define $\mu_{-k}$ by:

$$\mu_{-k}(\gamma) = \mu^*(\gamma) \in H^*(\mathcal{X}(-\nu, k - 1)) \otimes F_{-1}(\sigma_{\nu}).$$

For an element $\beta \in H^*(\mathcal{X}(-\nu, k)) \subset \mathcal{X}_{-k}(\nu, n)$, we define:

$$\mu_{-k}(\beta) = -\mu^*(\beta) \oplus \beta \in H^*(\mathcal{X}(\nu, k - 1)) \otimes F_{-1}(\sigma_{\nu}) \oplus H^*(\mathcal{X}(-\nu, k)),$$

where we have also used the letter $\mu$ to denote the (degenerate) Bott-Samelson map:

$$\mu : \mathcal{X}(\nu, k - 1, \nu) \longrightarrow \mathcal{X}(\nu, k - 1) \subset \mathcal{X}(-\nu, k).$$

It is straightforward (albeit tedious) to check that $\mu_\bullet$ commutes with the differential. \qed

Theorem 7.4. Let $A = (a_{ij})$ be the Cartan matrix for $G$. If all primes $p \leq a_{ij}a_{ji}$ have been inverted, then $BS_\bullet(\nu, n)$ is a homotopy equivalence, and consequently, the homotopy type of $F(\tilde{\sigma})$ is independent of presentation away from such primes. Over the integers, the Bott-Samelson map always induces a quasi-isomorphism:

$$BS_\bullet(\nu, n) : \mathcal{X}_\bullet(\nu, n) \longrightarrow F(\nu, n).$$

In particular, the quasi-isomorphism type of $F(\tilde{\sigma})$ is always independent of presentation.

Proof. Let us first prove that the Bott-Samelson map is always a quasi-isomorphism. Recall from 7.2 that the homology of $\mathcal{X}_\bullet(-\nu, n - 1)$ is concentrated in degree $1 - n$, and is generated by the Schubert class $\delta_{-\nu,n-1}$. Each step in the inductive construction of $BS_\bullet(\nu, n)$ is easily seen to preserve the homology generator, and is therefore a quasi-isomorphism. This inductive construction also shows that the cokernel of $BS_\bullet(\nu, n)$ is a complex of bi-modules that has trivial homology, and is free as a complex of left (or right) $H^*(BT)$-modules. Hence, tensoring the map $BS_\bullet(\nu, n)$ on the left by a complex of the form $F(\tilde{\sigma}')$ and on the right by another complex of the form $F(\tilde{\sigma}'')$ preserves the fact that it is a quasi-isomorphism. Consequently, replacing $F(\nu, m_{ij})$ with $F(-\nu, m_{ij})$ in a presentation for $F(\tilde{\sigma})$ preserves the quasi-isomorphism type.

Now assume that all primes $p \leq a_{ij}a_{ji}$ have been inverted, then we will show in the appendix that one may split off trivial summands from $F(\nu, n)$ in the same was as done in [9] (proof of proposition 9.2). In particular, one obtains a retraction from $F(\nu, n)$ onto the image of $BS_\bullet(\nu, n)$. This establishes the homotopy equivalence of the Bott-Samelson map away from the primes $p \leq a_{ij}a_{ji}$, where $A = (a_{ij})$ is the Cartan matrix of $G$. \qed

12
8. Independence of Presentation: The Identity: $F(\sigma_i) \otimes_R F(\sigma_i^{-1}) \simeq R$

Given an index $i$, recall that $F(\sigma_i) \otimes_R F(\sigma_i^{-1})$ is represented by the following complex in degrees $(-1, 0, 1)$, with maps induced by the generating maps 2.5:

$$\mathcal{C} : H^*(F(\sigma_i)) \rightarrow H^*(BT) \bigoplus H^*(F(\sigma_i)) \otimes_{H^*(BT)} \tilde{H}^*(F(\sigma_i)^{-\zeta_i}) \rightarrow \tilde{H}^*(F(\sigma_i)^{-\zeta_i}).$$

Now Rouquier shows in [9] that this complex is (abstractly) homotopy equivalent to the complex of bimodules with a single non-zero term $H^*(BT)$ placed in degree zero. In particular, the complex above is homotopy equivalent to its homology as a bimodule. This homology is straightforward to compute:

$$H^*(\mathcal{C}) = \frac{\delta_i \otimes \tilde{H}^*(F(\sigma_i)^{-\zeta_i})}{\delta_i \otimes \text{th}(\alpha_i)} \subset \frac{H^*(F(\sigma_i)) \otimes \tilde{H}^*(F(\sigma_i)^{-\zeta_i})}{\text{Image } H^*(F(\sigma_i))},$$

where $\text{th}(\alpha_i)$ is the image in cohomology of the Thom class of the bundle obtained by collapsing the section $s(\infty)$ in $F(\sigma_i)$ (see claim 2.4). Similarly, $\delta_i$ is the Schubert basis element corresponding to $\sigma_i$, which can be identified with the image of the Thom class of the bundle obtained by collapsing the section $s(0)$. One may write $\delta_i$ explicitly as $\frac{1}{2}(\rho^*(\alpha_i) - \pi^*(\alpha_i)).$ In this $H^*(BT)$-bimodule, the left and right actions agree, and the generator is given by the class $\delta_i \otimes \text{Th}(-\zeta_i).$ It is straightforward to verify using 3.1 that the action of the operators $L_n$ fixes this generator and hence the isomorphism between $H^*(\mathcal{C})$ and $H^*(BT)$ preserves the action of the algebra $\mathcal{L}$.

Remark 8.1. A similar argument to the one given above establishes the opposite isomorphism $F(\sigma_i^{-1}) \otimes_R F(\sigma_i) = R.$ We leave it to the reader to formulate it.

9. Appendix

In this appendix we describe a topological analog of the fundamental lemma of Rouquier [9](Lemma 9.1). Let $A = (a_{ij})$ be the Cartan matrix of $G.$ We will deduce, based on these results, that the Bott-Samelson map $BS_{\bullet}(\nu, n)$ is a homotopy equivalence away from all primes $p \leq a_{ij}a_{ji}$ and $n \leq m_{ij}.$ We begin by assuming the conventions of section 7, and we let $k$ be any integer such that $0 \leq k < m_{ij} - 1.$

Lemma 9.1. There exist topologically induced decompositions of chain complexes of bimodules:

(A) The complex obtained by tensoring $F(\sigma_\nu)$ with $H^*(\mathcal{X}(\nu, k))$ is isomorphic to a sum of the following two complexes:

$$\text{Id} : H^*(\mathcal{X}(\nu, k)) \rightarrow H^*(\mathcal{X}(\nu, k)), \quad H^*(\mathcal{X}(\nu, k)) \otimes \delta_\nu \rightarrow 0,$$

where $H^*(\mathcal{X}(\nu, k)) \otimes \delta_\nu$ is the ideal in $H^*(\mathcal{X}(\nu, k)) \otimes_{H^*(BT)} H^*(F(\sigma_\nu))$ generated by the Schubert basis element $\delta_\nu$. It is easy to see that this ideal is a sub-(bi)module.

(B) Also consider the chain complex induced by tensoring the restriction map with $H^*(F(\sigma_\nu))$:

$$\mathcal{C} : H^*(\mathcal{X}(-\nu, k + 1)) \otimes_{H^*(BT)} H^*(F(\sigma_\nu)) \rightarrow H^*(\mathcal{X}(\nu, k)) \otimes_{H^*(BT)} H^*(F(\sigma_\nu)).$$

Then the Bott-Samelson maps $\mu^*$ induces a short exact sequence of chain complexes of bimodules:

$$\{H^*(\mathcal{X}(\nu, k + 2)) \rightarrow H^*(\mathcal{X}(\nu, k))\} \rightarrow \mathcal{C} \rightarrow \{H^*(\mathcal{X}(\nu, k)) \otimes \delta_\nu \rightarrow H^*(\mathcal{X}(\nu, k)) \otimes \delta_\nu\},$$

where the kernel is the complex given by cellular restriction, and the cokernel is the trivial complex given by the identity map.
Proof. Consider the (degenerate) Bott-Samelson map:

$$\mu : \mathcal{X}(\nu, k, \nu) \rightarrow \mathcal{X}(\nu, k).$$

Recall that $$H^* (\mathcal{X}(\nu, k, \nu)) = H^* (\mathcal{X}(\nu, k)) \otimes_{H^*(BT)} H^*(\mathcal{F}(\sigma_\nu)).$$ Furthermore, the map $$\mu$$ admits a zero-section, whose kernel is given by $$H^* (\mathcal{X}(\nu, k)) \otimes \delta_\nu.$$ The first part of the lemma follows.

For the second part, consider the following commutative diagram, with the horizontal maps induced by cellular inclusion and the vertical maps by multiplication:

$$\begin{array}{ccc}
\mathcal{X}(\nu, k, \nu) & \xrightarrow{\mu_L} & \mathcal{X}(-\nu, k+1, \nu) \\
\downarrow & & \downarrow \\
\mathcal{X}(\nu, k) & \xrightarrow{\mu_R} & \mathcal{X}(\nu, k+2).
\end{array}$$

In cohomology, this induces a diagram with injective vertical maps:

$$\begin{array}{ccc}
H^* (\mathcal{X}(\nu, k + 2)) & \xrightarrow{\mu_R^*} & H^* (\mathcal{X}(\nu, k)) \\
\downarrow & & \downarrow \\
H^* (\mathcal{X}(-\nu, k + 1)) \otimes_{H^*(BT)} H^*(\mathcal{F}(\sigma_\nu)) & \xrightarrow{\mu_L^*} & H^* (\mathcal{X}(\nu, k)) \otimes_{H^*(BT)} H^*(\mathcal{F}(\sigma_\nu)).
\end{array}$$

It is easy to see that the cokernel of $$\mu_L^*$$ and $$\mu_R^*$$ are isomorphic. Now, by the previous part, the cokernel of $$\mu_L^*$$ can be identified with the bimodule: $$H^* (\mathcal{X}(\nu, k)) \otimes \delta_\nu.$$ The result follows.

\[\square\]

**Lemma 9.2.** Assume that one has inverted all primes $$p$$ such that $$p \leq a_\nu a_{ji},$$ then the extension of the complex $$\mathcal{C}$$ described in part (B) of the previous lemma, splits as complexes of bimodules.

**Proof.** Recall that we defined the space $$Y(\nu, k)$$ to be the image of the Schubert variety $$\mathcal{X}(\nu, k)$$ in the homogeneous space $$P/G_\nu.$$ Let $$\mathcal{Y}(\nu, k)$$ denote the space $$EG \times_T Y(\nu, k).$$ Recall also the pair of pullback diagrams considered earlier:

$$\begin{array}{ccc}
\mathcal{X}(-\nu, k + 1, \nu) & \xrightarrow{\mu_R} & \mathcal{X}(\nu, k + 2) \\
\downarrow & & \downarrow \\
\mathcal{X}(-\nu, k + 1) & \xrightarrow{\kappa} & \mathcal{Y}(\nu, k + 2).
\end{array}$$

By the Eilenberg-Moore spectral sequence we notice that one gets:

$$H^* (\mathcal{X}(-\nu, k + 1, \nu)) = H^* (\mathcal{X}(-\nu, k + 1)) \otimes_{H^*(\mathcal{Y}(\nu,k+2))} H^*(\mathcal{X}(\nu, k+2)).$$

$$H^* (\mathcal{X}(\nu, k + 2)) = H^* (\mathcal{Y}(\nu, k + 2)) \otimes_{H^*(BG_\nu)} H^*(BT).$$

To construct a splitting of $$\mathcal{C},$$ we require to establish a splitting to $$\mu_R^*$$ as bimodules. This is equivalent to splitting the map $$\kappa$$ in cohomology, in the category of modules over $$H^*(BT) \otimes H^*(BG_\nu).$$

Let us work in the cohomology ring $$H^* (\mathcal{X}(\nu, k+2)).$$ Recall that by [4](11.3.17) there exists constants $$a, d$$ and elements $$b, c \in H^2(BT)$$ that satisfy the following relation among the Schubert basis elements in this ring:

$$\delta_{-\nu,k} \delta_\nu = \delta_{\nu,k+1} - b \delta_{-\nu,k} + d \delta_{-\nu,k+1}, \quad \delta_{-\nu} \delta_{-\nu,k} = a \delta_{-\nu,k+1} + c \delta_{-\nu,k}.$$
Furthermore, the constant $a$ is non-zero, and its prime divisors $p$ are no bigger than $a_{ij}a_{ji}$. This can easily be checked by reducing to non-equivariant cohomology (see [5], §10).

Now consider the class $e \in H^2(\mathcal{X}(\nu, k + 2))$ given by:

$$
e := \delta_{\nu} + y \delta_{-\nu} + x, \quad y = -\frac{d}{a}, \quad x = \frac{ab + cd}{a}.
$$

We claim that the restriction to $H^*(\mathcal{X}(-\nu, k+1))$ of the $H^*(BT) \otimes H^*(\mathcal{Y}(\nu, k+2))$-submodule generated by the class $e$ is the complementary summand generating the splitting we seek. To prove this, consider the Serre spectral sequence for the fibration $\pi_\nu$ above. It follows from this spectral sequence that $H^*(\mathcal{X}(\nu, k + 2))$ is a rank two free $H^*(BT) \otimes H^*(\mathcal{Y}(\nu, k + 2))$-module on the classes $\{1, e\}$.

Let us consider the action on the class $e$ of the individual Schubert basis elements $\delta_{-\nu,i}$ $i \leq k + 1$ that generate $H^*(\mathcal{Y}(\nu, k + 2))$. It is easy to see that the classes $\{e \delta_{-\nu,i}, 0 \leq i < k\}$ restrict to a free $H^*(BT)$-submodule in $H^*(\mathcal{X}(-\nu, k + 1))$ that is a complement to the image of $H^*(\mathcal{Y}(\nu, k + 2))$ as an $H^*(BT)$-submodule. It remains to show that this submodule is closed under the action of $H^*(\mathcal{Y}(\nu, k + 2))$.

Now, by the choice of our constants, we observe that $e \delta_{-\nu,k} = \delta_{\nu,k+1}$. In particular, $e \delta_{-\nu,k}$ is zero in $H^*(\mathcal{X}(-\nu, k + 1))$, since $\delta_{\nu,k+1}$ restricts to zero in this ring. Recalling the relation: $\delta_{-\nu} \delta_{-\nu,k} = a \delta_{-\nu,k+1} + c \delta_{-\nu,k}$, with $a \neq 0$, it also follows that $e \delta_{-\nu,k+1} = 0$.

The above argument shows that the $H^*(BT) \otimes H^*(\mathcal{Y}(\nu, k + 2))$-submodule generated by the class $e$ is a summand in $H^*(\mathcal{X}(-\nu, k + 1))$ that is a complement to $H^*(\mathcal{Y}(\nu, k + 2))$. This is what we wanted to prove.

\[\square\]

Remark 9.3. Via the above splitting, and using the argument in [9](see proof of proposition 9.2) it now follows that away from primes $p$ no bigger than the off-diagonal products in the Cartan matrix: $a_{ij}a_{ji}$, the Bott–Samelson map $BS_\nu(\nu, n)$ is a homotopy equivalence. So for example, in the case $G = PU(n)$, the condition is vacuous. For arbitrary $G$, the only possible problematic primes are 2 and 3. In an earlier draft of this paper, the author incorrectly assumed that the splitting of $C$ was true integrally for arbitrary $G$.

\section*{References}

1. J. F. Adams, \textit{Stable Homotopy and Generalised Homology}, Univ. of Chicago Press, 1974.
2. M. Atiyah, F. Hirzebruch, \textit{Cohomologie-Operationen und charakteristische Klassen}, Math. Z. 77, 149–187, 1961.
3. M. Inoue, \textit{Odd primary Steenrod algebra, additive formal group laws and modular invariants}, J. Math. Soc. Japan, Vol. 58, No. 2, 311–333, 2006.
4. M. Khovanov, \textit{Triply-graded link homology and Hochschild homology of Soergel bimodules}, International Journal of Math., vol. 18, no.8, 869–885, 2007, arXiv:math.GT/0510265.
5. N. Kitchloo, \textit{On the topology of Kac-Moody groups}, to appear in Math. Zeit. arXiv:0810.0851v2.
6. M. Khovanov, L. Rozansky, \textit{Positive half of the Witt algebra acts on triply graded Link homology}, arXiv:1305.1642.
7. J. Morava, \textit{On the complex cobordism ring as a Fock representation}, Homotopy theory and related topics (Kinosaki, 1988), 184–204, Lecture Notes in Math., 1418, Springer, Berlin, 1990.
8. S. Kumar, \textit{Kac-Moody Groups, their Flag Varieties, and Representation Theory}, Vol. 204, Progress in Math., Birkhäuser, 2002.
9. R. Rouquier, \textit{Categorification of $sl_2$ the braid groups}, Contemporary Math. Vol 406, 137–167, 2006, arXiv:0409593v1.
10. L. Smith, *Lectures on the Eilenberg–Moore spectral sequence*, Lecture Notes in Mathematics 134, Berlin, New York, 1970.
11. B. Webster, G. Williamson, *A geometric model for Hochschild homology of Soergel bimodules*, arXiv:0707.2003v2.
12. R. M. W. Wood, *Differential Operators and the Steenrod Algebra*, Proc. London Math. Soc. 194–220, 1997.