The amalgamation in Boolean, and non-Boolean algebras with operators

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Abstract. We study the connections of the global amalgamation property, and some of its variations, in the context of classes (not necessarily varieties) of boolean algebras with operators, to the local interpolation and the congruence extension property on the free algebras of the varieties generated by such classes.

1 Introduction

The amalgamation property (for classes of models), since its discovery, has played a dominant role in algebra and model theory. Algebraic logic is the natural interface between universal algebra and logic (in our present context a variant of first order logic). Indeed, in algebraic logic amalgamation properties in classes of algebras are proved to be equivalent to interpolation results in the corresponding logic. Pigozzi and Comer worked such equivalences for cylindric algebras, the latter for finite dimensions the former for infinite ones.

The principal context of 

The fact that the class of locally finite cylindric algebras has the strong amalgamation property, proved earlier by Diagneuault is equivalent to the fact that first order logic has the Craig interpolation property. The classes that Pigozzi deals with consist solely of algebras that are infinite dimensional and we assume, to simplify notation, that such classes of algebras are \( \omega \)-dimensional, where \( \omega \) is the least infinite ordinal. These classes include the class of \( \omega \)-dimensional locally finite algebras (\( Lf_\omega \)), the class of dimension complemented algebras (\( Dc_\omega \)), the class of \( \omega \)-dimensional diagonal algebras (exact definitions
will be recalled below), and the class of $\omega$ dimensional semisimple algebras. Here a semisimple algebra is a subdirect product of simple algebras. All of these classes consist exclusively of algebras that are representable, but unlike $\text{RCA}_\omega$, none of these classes is first order definable, least a variety. While the amalgamation property speaks about amalgamating algebras in such a way that the amalgam only agrees on the common subalgebra, the amalgamation is said to be strong if the common subalgebra is the only overlap between the two algebras in the amalgam. The positive results of section 2.2 in combination with the negative ones of section 2.3 of [6] answer most of the natural questions one could ask about amalgamation for cylindric algebras. In particular, Pigozzi proves that in the (strictly) increasing sequence

$$\text{Lf}_\omega \subset \text{Dc}_\omega \subset \text{Di}_\omega \subset \text{RCA}_\omega$$

the first and third classes have the amalgamation property while the second and fourth fail to have it. However, most questions concerning the strong amalgamation property for several classes of cylindric algebras were posed as open questions, and other closely related ones appeared after Pigozzi’s paper was published. In [3] all of Pigozzi’s questions are answered.

Here we carry out similar investigations in a much broader context, that of Boolean algebras with operators. As a by product of our investigations we obtain several new results concerning algebraisations of first order logic, other than cylindric algebras. We will also have occasion to weaken the Boolean structure, dealing with non-classical or many valued logics.

## 2 Amalgamation

We start by the relevant definitions:

**Definition 2.1.** (1) $K$ has the Amalgamation Property if for all $\mathfrak{A}_1, \mathfrak{A}_2 \in K$ and monomorphisms $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$, $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ there exist $\mathfrak{D} \in K$ and monomorphisms $m_1 : \mathfrak{A}_1 \to \mathfrak{D}$ and $m_2 : \mathfrak{A}_2 \to \mathfrak{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$.

(2) If in addition, $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$ where $\{j,k\} = \{1,2\}$, then we say that $K$ has the superamalgamation property ($\text{SUPAP}$).

**Definition 2.2.** An algebra $\mathfrak{A}$ has the strong interpolation theorem, $\text{SIP}$ for short, if for all $X_1, X_2 \subseteq A$, $a \in \text{Sg}^\mathfrak{A}X_1$, $c \in \text{Sg}^\mathfrak{A}X_2$ with $a \leq c$, there exist $b \in \text{Sg}^\mathfrak{A}(X_1 \cap X_2)$ such that $a \leq b \leq c$.

For an algebra $\mathfrak{A}$, $\text{Co}\mathfrak{A}$ denotes the set of congruences on $\mathfrak{A}$. 2
Definition 2.3. An algebra $A$ has the congruence extension property, or CP for short, if for any $X_1, X_2 \subseteq A$ if $R \in \text{CoSpec}^A X_1$ and $S \in \text{CoSpec}^A X_2$ and

$$R \cap ^2 \text{Spec}^A (X_1 \cap X_2) = S \cap ^2 \text{Spec}^A (X_1 \cap X_2),$$

then there exists a congruence $T$ on $A$ such that

$$T \cap ^2 \text{Spec}^A X_1 = R \text{ and } T \cap ^2 \text{Spec}^A (X_2) = S.$$

Theorems 2.5, 2.6, 2.7 to come, give a flavour of the interconnections between the local properties of CP and SIP (on free algebras) and the global property of superamalgamation (of the entire class). Maksimova and Madárasz proved that if interpolation holds in free algebras of a variety, then the variety has the superamalgamation property. Using a similar argument, we prove this implication in a slightly more general setting. But first an easy lemma:

Lemma 2.4. Let $K$ be a class of BAO’s. Let $A, B \in K$ with $B \subseteq A$. Let $M$ be an ideal of $B$. We then have:

1. $\text{Ig}^A M = \{ x \in A : x \leq b \text{ for some } b \in M \}$
2. $M = \text{Ig}^A M \cap B$
3. If $C \subseteq A$ and $N$ is an ideal of $C$, then $\text{Ig}^A (M \cup N) = \{ x \in A : x \leq b + c \text{ for some } b \in M \text{ and } c \in N \}$
4. For every ideal $N$ of $A$ such that $N \cap B \subseteq M$, there is an ideal $N'$ in $A$ such that $N \subseteq N'$ and $N \cap B = M$. Furthermore, if $M$ is a maximal ideal of $B$, then $N'$ can be taken to be a maximal ideal of $A$.

Proof. Only (iv) deserves attention. The special case when $n = \{0\}$ is straightforward. The general case follows from this one, by considering $A/N$, $B/(N \cap B)$ and $M/(N \cap B)$, in place of $A, B$ and $M$ respectively.

The previous lemma will be frequently used without being explicitly mentioned.

Theorem 2.5. Let $K$ be a class of BAO’s such that $HK = SK = K$. Assume that for all $A, B, C \in K$, inclusions $m : C \rightarrow A$, $n : C \rightarrow B$, there exist $\mathcal{D}$ with SIP and $h : \mathcal{D} \rightarrow C$, $h_1 : \mathcal{D} \rightarrow A$, $h_2 : \mathcal{D} \rightarrow B$ such that for $x \in h^{-1}(C)$,

$$h_1(x) = m \circ h(x) = n \circ h(x) = h_2(x).$$

Then $K$ has SUPAP.
Proof. Let $\mathcal{D}_1 = h_1^{-1}(\mathfrak{A})$ and $\mathcal{D}_2 = h_2^{-1}(\mathfrak{B})$. Then $h_1 : \mathcal{D}_1 \to \mathfrak{A}$, and $h_2 : \mathcal{D}_2 \to \mathfrak{B}$.

Let $M = kerh_1$ and $N = kerh_2$, and let $\tilde{h}_1 : \mathcal{D}_1/M \to \mathfrak{A}, \tilde{h}_2 : \mathcal{D}_2/N \to \mathfrak{B}$ be the induced isomorphisms.

Let $l_1 : h^{-1}(\mathfrak{C})/h^{-1}(\mathfrak{C}) \cap M \to \mathfrak{C}$ be defined via $\bar{x} \to h(x)$, and $l_2 : h^{-1}(\mathfrak{C})/h^{-1}(\mathfrak{C}) \cap N$ to $\mathfrak{C}$ be defined via $\bar{x} \to h(x)$. Then those are well defined, and hence $k^{-1}(\mathfrak{C}) \cap M = h^{-1}(\mathfrak{C}) \cap N$. Then we show that $\Psi = \mathfrak{Ig}(M \cup N)$ is a proper ideal and $\mathfrak{D}/\Psi$ is the desired algebra. Now let $x \in \mathfrak{Ig}(M \cup N) \cap \mathcal{D}_1$. Then there exist $b \in M$ and $c \in N$ such that $x \leq b + c$. Thus $x - b \leq c$. But $x - b \in \mathcal{D}_1$ and $c \in \mathcal{D}_2$, it follows that there exists an interpolant $d \in \mathcal{D}_1 \cap \mathcal{D}_2$ such that $x - b \leq d \leq c$. We have $d \in N$ therefore $d \in M$, and since $x \leq d + b$, therefore $x \in M$. It follows that $\mathfrak{Ig}(M \cup N) \cap \mathcal{D}_1 = M$ and similarly $\mathfrak{Ig}(M \cup N) \cap \mathcal{D}_2 = N$. In particular $P = \mathfrak{Ig}(M \cup N)$ is a proper ideal.

Let $k : \mathcal{D}_1/M \to \mathcal{D}/P$ be defined by $k(a/M) = a/P$ and $h : \mathcal{D}_2/N \to \mathcal{D}/P$ by $h(a/N) = a/P$. Then $k \circ m$ and $h \circ n$ are one to one and $k \circ m \circ f = h \circ n \circ g$. We now prove that $\mathcal{D}/P$ is actually a superamalgam. i.e we prove that $K$ has the superamalgamation property. Assume that $k \circ m(a) \leq h \circ n(b)$. There exists $x \in \mathcal{D}_1$ such that $x/P = k(m(a))$ and $m(a) = x/M$. Also there exists $z \in \mathcal{D}_2$ such that $z/P = h(n(b))$ and $n(b) = z/N$. Now $x/P \leq z/P$ hence $x - z \in P$. Therefore there is an $r \in M$ and an $s \in N$ such that $x - r \leq z + s$. Now $x - r \in \mathcal{D}_1$ and $z + s \in \mathcal{D}_2$, it follows that there is an interpolant $u \in \mathcal{D}_1 \cap \mathcal{D}_2$ such that $x - r \leq u \leq z + s$. Let $t \in \mathfrak{C}$ such that $m \circ f(t) = u/M$ and $n \circ g(t) = u/N$. We have $x/P \leq u/P \leq z/P$. Now $m(f(t)) = u/M \geq x/M = m(a)$. Thus $f(t) \geq a$. Similarly $n(g(t)) = u/N \leq z/N = n(b)$, hence $g(t) \leq b$. By total symmetry, we are done.

The intimate relationship between $CP$ on free algebras generating a certain variety and the $AP$ for such varieties, has been worked out extensively by Pigozzi for various classes of cylindric algebras. Here we prove an implication in one direction for $BAO$’s. Notice that we do not assume that our class is a variety.

**Theorem 2.6.** Let $K$ be such that $HK = SK = K$. If $K$ has the amalgamation property, then the $V(K)$ free algebras, on any set of generators, have $CP$.

**Proof.** For $R \in Co\mathfrak{A}$ and $X \subseteq A$, by $(\mathfrak{A}/R)^{(X)}$ we understand the subalgebra of $\mathfrak{A}/R$ generated by $\{x/R : x \in X\}$. Let $\mathfrak{A}, X_1, X_2, R$ and $S$ be as specified in in the definition of $CP$. Define

$$\theta : \mathfrak{Gg}^{\mathfrak{A}}(X_1 \cap X_2) \to \mathfrak{Gg}^{\mathfrak{A}}(X_1)/R$$

by

$$a \mapsto a/R.$$
Then \( \ker \theta = R \cap Sg^A(X_1 \cap X_2) \) and \( \text{Im} \theta = (Sg^A(X_1)/R)^{(X_1 \cap X_2)} \). It follows that

\[
\bar{\theta} : Sg^A(X_1 \cap X_2)/R \cap Sg^A(X_1 \cap X_2) \to (Sg^A(X_1)/R)^{(X_1 \cap X_2)}
\]

defined by

\[
a/R \cap Sg^A(X_1 \cap X_2) \mapsto a/R
\]

is a well defined isomorphism. Similarly

\[
\bar{\psi} : Sg^A(X_1 \cap X_2)/S \cap Sg^A(X_1 \cap X_2) \to (Sg^A(X_2)/S)^{(X_1 \cap X_2)}
\]

defined by

\[
a/S \cap Sg^A(X_1 \cap X_2) \mapsto a/S
\]

is also a well defined isomorphism. But

\[
R \cap Sg^A(X_1 \cap X_2) = S \cap Sg^A(X_1 \cap X_2),
\]

Hence

\[
\phi : (Sg^A(X_1)/R)^{(X_1 \cap X_2)} \to (Sg^A(X_2)/S)^{(X_1 \cap X_2)}
\]

defined by

\[
a/R \mapsto a/S
\]

is a well defined isomorphism. Now \( (Sg^A(X_1)/R)^{(X_1 \cap X_2)} \) embeds into \( Sg^A(X_1)/R \) via the inclusion map; it also embeds in \( A^{(X_2)}/S \) via \( i \circ \phi \) where \( i \) is also the inclusion map. For brevity let \( A_0 = (Sg^A(X_1)/R)^{(X_1 \cap X_2)} \), \( A_1 = Sg^A(X_1)/R \) and \( A_2 = Sg^A(X_2)/S \) and \( j = i \circ \phi \). Then \( A_0 \) embeds in \( A_1 \) and \( A_2 \) via \( i \) and \( j \) respectively. Then there exists \( \mathcal{B} \in V \) and monomorphisms \( f \) and \( g \) from \( A_1 \) and \( A_2 \) respectively to \( \mathcal{B} \) such that \( f \circ i = g \circ j \). Let

\[
\bar{f} : Sg^A(X_1) \to \mathcal{B}
\]

be defined by

\[
a \mapsto f(a/R)
\]

and

\[
\bar{g} : Sg^A(X_2) \to \mathcal{B}
\]

be defined by

\[
a \mapsto g(a/R).
\]

Let \( \mathcal{B}' \) be the algebra generated by \( \text{Im} f \cup \text{Im} g \). Then \( \bar{f} \cup \bar{g} : X_1 \cup X_2 \to \mathcal{B}' \) is a function since \( \bar{f} \) and \( \bar{g} \) coincide on \( X_1 \cap X_2 \). By freeness of \( A \), there exists \( h : A \to \mathcal{B}' \) such that \( h|_{X_1 \cup X_2} = \bar{f} \cup \bar{g} \). Let \( T = \ker h \). Then it is not hard to check that

\[
T \cap Sg^A(X_1) = R \quad \text{and} \quad T \cap Sg^A(X_2) = S.
\]

\[\square\]
Finally we show that \( CP \) implies a weak form of interpolation.

**Theorem 2.7.** If an algebra \( \mathfrak{A} \) has \( CP \), then for \( X_1, X_2 \subseteq \mathfrak{A} \), if \( x \in \mathcal{S}^g(X_1) \) and \( z \in \mathcal{S}^g(X_2) \) are such that \( x \leq z \), then there exists \( y \in \mathcal{S}^g(X_1 \cap X_2) \), and a term \( \tau \) such that \( x \leq y \leq \tau(z) \). If \( Ig^{\mathfrak{A}}(z) = Ig^{\mathfrak{A}}(\{z\}) \), then \( \tau \) can be chosen to be the identity term. In particular, if \( z \) is closed, or \( \mathfrak{A} \) comes from a discriminator variety, then the latter case occurs.

**Proof.** Now let \( x \in \mathcal{S}^g(X_1) \), \( z \in \mathcal{S}^g(X_2) \) and assume that \( x \leq z \). Then

\[
x \in (\lg^g(z)) \cap \mathcal{S}^g(X_1).
\]

Let

\[
M = \lg^g(z) \quad \text{and} \quad N = \lg^g(X_2)(M \cap \mathcal{S}^g(X_1 \cap X_2)).
\]

Then

\[
M \cap \mathcal{S}^g(X_1 \cap X_2) = N \cap \mathcal{S}^g(X_1 \cap X_2).
\]

By identifying ideals with congruences, and using the congruence extension property, there is an ideal \( P \) of \( \mathfrak{A} \) such that

\[
P \cap \mathcal{S}^g(X_1) = N \quad \text{and} \quad P \cap \mathcal{S}^g(X_2) = M.
\]

It follows that

\[
\lg^g(N \cup M) \cap \mathcal{S}^g(X_1) \subseteq P \cap \mathcal{S}^g(X_1) = N.
\]

Hence

\[
(\lg^g(z)) \cap A^{(X_1)} \subseteq N.
\]

and we have

\[
x \in \lg^g(X_1) [\lg^g(X_2) \{z\} \cap \mathcal{S}^g(X_1 \cap X_2)].
\]

This implies that there is an element \( y \) such that

\[
x \leq y \in \mathcal{S}^g(X_1 \cap X_2)
\]

and \( y \in \lg^g(X_1) \}, \) hence the first required. The second required follows, also immediately, since \( y \leq z \), because \( \lg^g(z) = \mathfrak{R}_2 \mathfrak{A} \).

We note that all of the above results hold for \( MV \) algebras which satisfy all axioms of Boolean algebras except idempotency.
3 Sheaf theoretic duality and epimorphisms

Here we deal with non-classical logics; we review some known basic notions and concepts culminating in defining the algebras we shall deal with. Our work closely follows Comer, except that we deal with Zarski topologies rather than Stone topologies. We obtain an analogous representabilty theorem to the effect that every theory can be represented as the continuous sections of a Sheaf. We start with the origin of our algebras.

**Definition 3.1.** A t norm is a binary operation $\ast$ on $[0, 1]$, i.e $(t : [0, 1]^2 \to [0, 1])$ such that

(i) $\ast$ is commutative and associative, that is for all $x, y, z \in [0, 1]$,

$$x \ast y = y \ast x$$

$$(x \ast y) \ast z = x \ast (y \ast z).$$

(ii) $\ast$ is non decreasing in both arguments, that is

$$x_1 \leq x_2 \implies x_1 \ast y \leq x_2 \ast y$$

$$y_1 \leq y_2 \implies x \ast y_1 \leq x \ast y_2.$$  

(iii) $1 \ast x = x$ and $0 \ast x = 0$ for all $x \in [0, 1]$.

The following are the most important (known) examples of continuous t norms.

(i) Lukasiewicz t norm: $x \ast y = \max(0, x + y - 1)$

(ii) Godel t norm $x \ast y = \min(x, y)$

(iii) Product t norm $x \ast y = x \cdot y$

We have the following known result [?] lemma 2.1.6

**Theorem 3.2.** Let $\ast$ be a continuous t norm. Then there is a unique operation $x \implies y$ satisfying for all $x, y, z \in [0, 1]$, the condition $(x \ast z) \leq y$ iff $z \leq (x \implies y)$, namely $x \implies y = \max\{z : x \ast z \leq y\}$

The operation $x \implies y$ is called the residuam of the t norm. The residuam $\implies$ defines its corresponding unary operation of precomplement $(-)x = (x \implies 0)$. The Godel negation satisfies $(-)0 = 1$, $(-)x = 0$ for $x > 0$. Abstracting away from t norms, we get
Definition 3.3. A residuated lattice is an algebra
\[(L, \cup, \cap, *, \Rightarrow, 0, 1)\]
with four binary operations and two constants such that

(i) \((L, \cup, \cap, 0, 1)\) is a lattice with largest element 1 and the least element 0 (with respect to the lattice ordering defined the usual way: \(a \leq b \) iff \(a \cap b = a\)).

(ii) \((L, *, 1)\) is a commutative semigroup with largest element 1, that is \(\ast\) is commutative, associative, \(1 \ast x = x\) for all \(x\).

(iii) Letting \(\leq\) denote the usual lattice ordering, we have \(\ast\) and \(\Rightarrow\) form an adjoint pair, i.e. for all \(x, y, z\)
\[z \leq (x \Rightarrow y) \iff x \ast z \leq y.\]

\(BL\) algebras, introduced and studied by Hajek [?], are what is called \(MTL\) algebras satisfying the identity \(x \ast (x \Rightarrow y) = x \land y.\) Both are residuated lattices with extra conditions. The propositional logic \(MTL\) was introduced by Esteva and Godo [?]. It has three basic connectives \(\rightarrow, \land\) and \(\&.\) We say that \(\mathcal{L}\) is a core fuzzy logic if \(\mathcal{L}\) expands \(MTL, \mathcal{L}\) has the Local Deduction Theorem \((LDT)\), and \(\mathcal{L}\) satisfies \((\ast)\) \(\phi \equiv \psi \vdash \chi(\phi) \equiv \chi(\psi)\) for all formulas \(\phi, \psi, \chi.\) (Here \(\equiv\) is defined via \(\&\) and \(\Rightarrow\).) The \((LDT)\) says that for a theory \(T\) and a formula \(\phi,\) whenever \(T \cup \{\phi\} \vdash \psi,\) then there exists a natural number \(n\) such that \(T \vdash \phi^n \rightarrow \psi.\) Here \(\phi^n\) is defined inductively by \(\phi^1 = \phi\) and \(\phi^n = \phi^{n-1} \& \phi.\) Thus core fuzzy logics are axiomatic expansions of \(MTL\) having \(LDT\) and obeying the substitution rule \((\ast)\). The basic notions of evaluation, tautology and model for core fuzzy logics are defined the usual way. Let \(\mathcal{L}\) be a core fuzzy logic and \(I\) the set of additional connectives of \(\mathcal{L}\). An \(\mathcal{L}\) algebra is a structure \(\mathcal{B} = (B, \cup, \cap, *, \Rightarrow, c_B)_{c \in I}, 0, 1)\) such that \((B, \cup, \cap, *, \Rightarrow, 0, 1)\) is an \(MTL\) algebra and each additional axiom of \(\mathcal{L}\) is a tautology of \(\mathcal{B}\). Throughout the paper the operations of algebras are denoted by \(\cup, \cap, \Rightarrow, *\) and the corresponding logical operations by \(\lor, \land, \rightarrow, \&.\)

Generalizing a very nice result of Comer we represent \(BAO;\)s as the continuous sections of sheaves, the representation here is indeed a functor that is strongly invertible. We start from a concrete example adressing variants and extension first order logics. The following discussion applies to \(L_n, L_{\omega, \omega}, Dc,\) Keislers logics with and without equality, finitray logics of infinitary relations. It also applies to non classical ologics, whose Stone space is the Zarski topology.

Example 3.4. Let \(\mathcal{L}\) be a logic, and \(T\) be a theory in \(\mathcal{Fm}_L.\) Let \(\mathcal{Gn}_L\) denote the set of sentences, that is formulas with no free variables. We assume that
$T \in S_n$ has no free variable, let $X_T = \{ \Delta \subseteq S_n : \Delta \text{ is complete} \}$. This is simply the stone space of $S_n$. For each $\Delta \in X_T$ let $Fm_\Delta$ be the corresponding Tarski-Lindenbaum algebra. Let $\delta T$ be the following disjoint union

$$\bigcup_{\Delta \in X_T} \{ \Delta \} \times Fm_\Delta.$$ 

Define the following topologies, on $X_T$ and $\delta T$, respectively.

On $X_T$ the Stone topology, and on $\delta T$ the topology with base $B_{\psi,\phi} = \{ \Delta, [\phi]_\Delta, \psi \in \Delta, \Delta \in \Delta_T \}$. Then $(X_T, \delta T)$ is a sheaf, and its dual consisting of continuous sections, $\Gamma(T, \Delta) \cong Fm_T$.

Example 3.5. It also applies to non classical logic. Let $L$ be a predicate language for $BL$ algebras (This for example incudes MV algebras). Let $X_T$ be the Zariski topology on $S_n$ based on $\{ \Delta \in \text{Max} : a \notin \Delta \}$. Let $\delta T = \bigcup_{\Delta \in X_T} \{ \Delta \} \times Fm_\Delta$.

Definition 3.6. Let $B$ be an algebra. A filter of $B$ is a nonempty subset $F \subseteq A$ such that for all $a, b \in B$,

(i) $a, b \in F$ implies $a \ast b \in F$.

(ii) $a \in F$ and $a \leq b$ imply $b \in F$.

It easy to check that if $F$ is a filter on $A$ then $1 \in F$ and whenever $a, a \implies b \in F$ then $b \in F$. Also $a \ast b \in F$ if and only if $a \cap b \in F$ iff $a \in F$ and $b \in F$. A filter $F$ is proper if $F \neq A$ and it is easy to see that a filter $F$ is proper iff $0 \notin F$.

Definition 3.7. A filter $P$ of $A$ is prime provided that it is a prime filter of the underlying lattice $L(B)$ of $B$, that is $a \cup b \in P$ implies $a \in P$ or $b \in P$. This is equivalent to the statement that for all $a, b \in B$, $a \iff b \in P$ or $b \iff a \in P$. A proper filter $F$ is maximal if it is not properly contained in any other proper filter.

We let $Max(B)$ denote the set of maximal filters and $Spec(B)$ the family of prime filters. Then it is not hard to actually show that $Max(B) \subseteq Spec(B)$ [?]. For a set $X \subseteq B$, $\frak{F}^B X$ denotes the filter generated by $X$. A filter $F$ is called principal, if $F = \frak{F}\{a\} = \{ x \in B : x \geq a \}$. The following notions are taken from [?]. Proofs are also found in [?]. Let $B$ be a non-trivial algebra. For each $X \subseteq B$, we set

$$V(X) = \{ P \in Spec(X) : X \subseteq P \}.$$ 

Then the family $\{ V(X) \}_{X \subseteq B}$ of subsets of $\text{spec}(B)$ satisfies the axioms for closed sets in a topological space. The resulting topology is called the Zariski
topology, and the resulting topological space is called the prime spectrum of \( \mathcal{B} \). We write \( V(a) \) for the more cumbersome \( V(\{a\}) \). For any \( X \subseteq \mathcal{B} \), let

\[
D(X) = \{ P \in \text{Spec}(X) : X \not\subseteq P \}
\]

Then \( \{D(X)\}_{X \subseteq A} \) is the family of open sets of the Zariski topology. We write \( D(a) \) for \( D(\{a\}) \). The minimal spectrum of \( \mathcal{B} \) is the topology induced by the Zariski topology on \( \text{Max}(\mathcal{B}) \). For \( X \subseteq \mathcal{B} \) and \( a \in \mathcal{B} \), let

\[
V_M(X) = V(X) \cap \text{Max}(\mathcal{B}) \quad \text{and} \quad D_M(a) = D(a) \cap \text{Max}(\mathcal{B}).
\]

In other words,

\[
V_M(a) = \{ F \in \text{Max}(\mathcal{B}) : a \in F \}
\]

and

\[
D_M(a) = \{ F \in \text{Max}(\mathcal{B}) : a \notin F \}.
\]

**Lemma 3.8.** Let \( \mathcal{B} \) be an algebra. Let \( a, b \in \mathcal{B} \). Then the following hold:

(i) \( D_M(a) \cap D_M(b) = D_M(a \cup b) \).

(ii) \( D_M(a) \cup D_M(b) = D_M(a \cap b) = D_M(a * b) \).

(iii) \( D_M(X) = \text{Max}(\mathcal{B}) \) iff \( \mathcal{B}^\mathbb{R} X = \mathcal{B} \).

(iv) \( D_M(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} D_M(X_i) \).

(v) \( V_M(a) \cap V_M(b) = V_M(a \cap b) \).

(vi) \( a \leq b \) if and only if \( V_M(a) \subseteq V_M(b) \).

**Proof.** [?] proposition 2.8. We only prove one side of the last item, since it is not mentioned in [?]. Assume that \( V_a \subseteq V_b \). If it is not the case that \( a \leq b \), then we may assume that \( a \cap (b \implies 0) \) is not 0. Hence there is a proper maximal filter \( F \), such that \( a \cap (b \implies 0) \in F \). Hence \( a \in F \) and \( b \rightarrow 0 \) is in \( F \). But this implies that \( b \notin F \) lest \( 0 \in F \). Hence \( F \in V_a \) and \( F \notin V_b \). This is a contradiction, and the required is proved.

**Lemma 3.9.** If \( F \) is a maximal filter in a BL algebra \( \mathfrak{A} \), then for any \( a \in A \) either \( a \) or \( a \rightarrow 0 \) is in \( F \).

**Proof.** Let \( \mathfrak{A} \in BL \). Assume that both \( a \) and \( -a = a \rightarrow 0 \) are not in \( F \). Then, by maximality, the filter generated by \( F \cup \{ -a \} \) is the whole algebra \( \mathfrak{A} \). Then \( a \geq x \cap -a \), for some \( x \in F \). Hence \( 0 = a \cap x \cap -a = x \cap -a \). But then \( x \leq a \) and \( a \in F \) after all.
Theorem 3.10. Let \( \mathcal{B} \) be an algebra.

(i) \( \{D_M(a)\}_{a \in \mathcal{B}} \) is a basis for a compact Hausdorff topology on \( \text{Max}(\mathcal{B}) \).

(ii) Furthermore if \( a = \bigvee a_i \), then \( V_M(a) \sim \bigcup V_M(a_i) \) is a nowhere dense subset of \( \text{Max}(\mathcal{B}) \). Similarly if \( a = \bigwedge a_i \), then \( \bigcap V_M(a_i) \sim V_M(a) \) is nowhere dense.

(iii) If \( \mathcal{B} \) is countable, then \( \text{Max}(\mathcal{B}) \) is a Polish space.

Proof.

(i) We include the proof for self completeness and also because the ‘nowhere density’ part is completely new, and as we shall see in a while it will play a pivotal role in the proof of the omitting types theorem. That \( \text{Max}(\mathcal{B}) \) is compact and Hausdorff is proved in [?], theorem 2.9, the proof goes as follows: Assume that \( \text{Max}(\mathcal{B}) = \bigcup_{i \in I} D_M(a_i) = D_M(\bigcup_{i \in I} a_i) \).

Then \( \mathcal{B} = \mathfrak{F}\{\bigcup_{i \in I} a_i\} \), hence \( 0 \in \mathfrak{F}\{\bigcup_{i \in I} a_i\} \). There is an \( n \geq 1 \) and \( i_1, \ldots, i_n \in I \) such that \( a_{i_1} \ast \ldots a_{i_n} = 0 \). But

\[ \text{Max}(\mathcal{B}) = D_M(0) = D_M(a_{i_1} \ast \ldots a_{i_n}) = D_M(a_{i_1}) \cup \ldots D_M(a_{i_n}). \]

Hence every cover is reducible to a finite subcover. Hence the space is compact. Now we show that it is Hausdorff. Let \( M, N \) be distinct maximal filters. Let \( x \in M \sim N \) and \( y \in N \sim M \). Let \( a = x \implies y \) and \( b = y \implies x \). Then \( a \notin M \) and \( b \notin N \). Hence \( M \in D_M(a) \) and \( N \in D_M(b) \). Also \( D_M(a) \cap D_M(b) = D_M(a \lor b) = D_M(1) = \emptyset \). We have proved that the space is Hausdorff.

(ii) Now assume that \( a = \bigvee a_i \) and \( V_M(a) \sim \bigcup V_M(a_i) \) is not nowhere dense. Then there exists \( d \) such that \( D_M(d) \subseteq V_M(a) \sim V_M(a_i) \). Hence

\[ V_M(a_i) \subseteq V_M(a) \sim D_M(d) = V_M(a) \cap V_M(d) = V_M(a \cap d). \]

It follows that \( a \cap d = a \) so \( a \leq d \). Then \( D_M(d) \subseteq D_M(a) \). So we have, \( D_M(d) \subseteq D_M(a) \cap V_M(a) = \emptyset \) contradiction. Conversely assume that \( a = \bigwedge a_i \) and assume that

\[ D_M(d) \subseteq \bigcap V_M(a_i) \sim V_M(a). \]

Let \( e = d \to 0 \). Then \( V_M(e) = D_M(d) \). Now we have

\[ V_M(e) \subseteq \bigcap V_M(a_i) \sim V_M(a). \]
Taking complements twice, we get
\[ V_M(e) \subseteq D_M(a) \sim \bigcup D_M(a_i) \]
Then \( V_M(e) \subseteq D_M(a) \sim D_M(a_i) \). So
\[ D_M(a_i) \subseteq D_M(a) \sim V_M(e) = D_M(a) \cap D_M(e) = D_M(a \cup e). \]
Hence \( V_M(a \cup e) \subseteq V_M(a_i) \). So \( a \cup e \leq a \) for each \( i \). Thus \( a \cup e = a \) from which we get that \( e \leq a \). Hence \( V_M(e) \subseteq V_M(a) \). But \( V_M(e) \subseteq D_M(a) \) it follows that \( V_M(e) = \emptyset \). But \( V_M(e) = D_M(d) \) and we are done.

(iii) If \( \mathcal{B} \) is countable, then \( \text{Max}\mathcal{B} \) is second countable, so the required follows from (i) together with theorem ??.

We consider a class \( \mathcal{K} \) of \( BL \) algebras with operators (\( BLOs \)). If \( \mathfrak{A} \in \mathcal{K}_\alpha \) and \( X \subseteq A \), then \( \text{lg}^\alpha X \) denotes the ideal generated by \( X \). For \( x \in A \), we define \( \Delta x = \{ i \in I : f_i x \neq x \} \). We assume that \( \Delta x = \Delta(-x), \Delta(x \cap y) \subseteq \Delta x \cap \Delta y \).

\( 3d\mathfrak{A} \) denotes the Boolean algebra. That is \( Zd\mathfrak{A} = \{ x \in \mathfrak{A} : f_i x = x, \forall i \in \alpha \} \). If \( \mathfrak{A} \) is a locally finite cylindric algebra of formulas, then \( 3d\mathfrak{A} \) is the Boolean algebra of sentences.

We describe a functor that associates to each \( BLO \) a pair of topological spaces space \( (X(\mathfrak{A}), \delta(\mathfrak{A})) = \mathfrak{A}^d \), where \( \delta(\mathfrak{A}) \) has an algebraic structure, as well; in fact it is a subdirect product of algebras, that are simple under favourable circumstances, in which case \( \delta(\mathfrak{A}) \) is a semisimple algebra carrying a product topology. This pair is called the dual space of \( \mathfrak{A} \).

\( X(\mathfrak{A}) \) is the Zariski topology of \( 3d\mathfrak{A} \), defined on the prime spectrum.

Now we turn to defining the second component; this is more involved. For \( x \in X(\mathfrak{A}) \), let \( G_x = \mathfrak{A}/\text{lg}^\alpha x \) (the stalk over \( x \)) and
\[ \delta(\mathfrak{A}) = \bigcup \{ G_x : x \in X(\mathfrak{A}) \}. \]

This is clearly a disjoint union, and hence it can also be regarded as the following product \( \prod_{x \in \mathfrak{A}} G_x \) of algebras. This is not semi-simple, because \( x \) is only maximal in \( 3d\mathfrak{A} \). But the semisimple case will deserve special attention.

The projection \( \pi : \delta(\mathfrak{A}) \to X(\mathfrak{A}) \) is defined for \( s \in G_x \) by \( \pi(s) = x \). For \( a \in A \), we define a function \( \sigma_a : X(\mathfrak{A}) \to \delta(\mathfrak{A}) \) by \( \sigma_a(x) = a/\text{lg}^\alpha x \in G_x \).

Now we define the topology on \( \delta(\mathfrak{A}) \). It is the smallest topology for which all these functions are open, so \( \delta(\mathfrak{A}) \) has both an algebraic structure and a topological one, and they are compatible.

We can turn the glass around. Having such a space we associate an algebra in \( \mathcal{K} \). Let \( \pi : G \to X \) denote the projection associated with the space \( (X, G) \),
built on $\mathfrak{A}$. A function $\sigma : X \to G$ is a section of $(X, G)$ if $\pi \circ \sigma$ is the identity on $X$.

Dually, the construction of the corresponding $BAO$ from a reduced space, uses the sectional functor. The set $\Gamma(X, G)$ of all continuous sections of $(X, G)$ becomes a $BAO$ by defining the operations pointwise, recall that $G = \prod G_x$ is a $BLO$. The mapping $\eta : \mathfrak{A} \to \Gamma(X(\mathfrak{A}), \delta(\mathfrak{A}))$ defined by $\eta(a) = \sigma_a$ is as easily checked an isomorphism, completing the invertibility of the functor.

Note that under this map an element in $Zd\mathfrak{A}$ corresponds with the characteristic function $\sigma_N \in \Gamma(X, \delta)$ of the clopen set $N_a$.

Given two spaces $(Y, G)$ and $(X, \mathfrak{L})$ a sheaf morphism $H : (Y, G) \to (X, \mathfrak{L})$ is a pair $(\lambda, \mu)$ where $\lambda : Y \to X$ is a continuous map and $\mu$ is a continuous map $Y +_\lambda \mathfrak{L} \to G$ such that $\mu_y = \mu(y, -)$ is a homomorphism of $\mathfrak{L}_{\lambda(y)}$ into $G_y$. We consider $Y +_\lambda \mathfrak{L} = \{(y, t) \in Y \times \mathfrak{L} : \lambda(y) = \pi(t)\}$ as a subspace of $Y \times \mathfrak{L}$. That is, it inherits its topology from the product topology on $Y \times \mathfrak{L}$.

A sheaf morphism $(\lambda, \mu) = H : (Y, G) \to (X, \mathfrak{L})$ produces a $BAO$ homomorphism $\Gamma(H) : \Gamma(X, \mathfrak{L}) \to \Gamma(Y, G)$ the natural way: for $\sigma \in \Gamma(X, \mathfrak{L})$ define $\Gamma(H)\sigma$ by $(\Gamma(H)\sigma)(y) = \mu(y, \sigma(\lambda y))$ for all $y \in Y$. A sheaf morphism $h^d : \mathfrak{B}^d \to \mathfrak{A}^d$ can also be associated with a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$. Define $h^d = (h^*, h^o)$ where for $y \in X(\mathfrak{B})$, $h^*(y) = h^{-1} \cap Zd\mathfrak{A}$ and for $y \in X(\mathfrak{B})$ and $a \in A$

$$h^0(h, a/\lg a^\mathfrak{A}h^*(y)) = h(a)/\lg a^\mathfrak{B}y.$$  

**Definition.** An algebra $\mathfrak{A}$ is nice if whenever $x$ is a prime ideal in $Zd\mathfrak{A}$, then $\lg x^\mathfrak{A}$ is a maximal ideal in $\mathfrak{A}$.

It is easy to see that locally finite algebras are nice. For a class of algebras $K$ we say that $K$ has ES if epimorphisms (in the categorial sense) are surjective.

**Theorem.** Let $V$ be a class of algebras, such that the simple algebras in $V$ have the amalgamation property. Assume that there exist nice algebras $\mathfrak{A}, \mathfrak{B} \in V$ and an epimorphism $f : \mathfrak{A} \to \mathfrak{B}$ that is not onto. Then ES fails in the class of simple algebras defined above, some are cylindric-like, other are not.

**Proof.** Suppose, to the contrary that ES holds for simple algebras. Let $f^* : \mathfrak{A} \to \mathfrak{B}$ be the given epimorphism that is not onto. We assume that $A^d = (X, \mathfrak{L})$ and $B^d = (Y, G)$ are the corresponding dual sheaves over the Boolean spaces $X$ and $Y$ and by duality that $(h, k) = H : (Y, G) \to (X, \mathfrak{L})$ is a monomorphism. Recall that $X$ is the set of maximal ideals in $Zd\mathfrak{A}$, and similarly for $Y$. We shall first prove

(i) $h$ is one to one.
(ii) for each \( y \) a maximal ideal in \( \mathfrak{A}d \mathfrak{B} \), \( k(y, -) \) is a surjection of the stalk over \( h(y) \) onto the stalk over \( y \).

Suppose that \( h(x) = h(y) \) for some \( x, y \in Y \). Then \( G_x, G_y \) and \( \mathcal{L}_{hx} \) are simple algebra, so there exists a simple \( \mathfrak{D} \in V \) and monomorphism \( f_x : G_x \to \mathfrak{D} \) and \( f_y : G_y \to \mathfrak{D} \) such that

\[
f_x \circ k_x = f_y \circ k_y.
\]

Here we are using that the algebras considered are nice, and that the simple algebras have \( AP \). Consider the sheaf \((1, D)\) over the one point space \( \{0\} = 1 \) and sheaf morphisms \( H_x : (\lambda_x, \mu) : (1, D) \to (Y, G) \) and \( H_y = (\lambda_y, v) : (1, D) \to (Y, G) \) where \( \lambda_x(0) = x \) \( \lambda_y(0) = y \) \( \mu_0 = f_x \) and \( v_0 = f_y \). The sheaf \((1, \mathfrak{D})\) is the space dual to \( \mathfrak{D} \in V \) and we have \( H \circ H_x = H \circ H_y \). Since \( H \) is a monomorphism \( H_x = H_y \) that is \( x = y \). We have shown that \( h \) is one to one. Fix \( x \in Y \). Since, we are assuming that \( ES \) holds for simple algebras of \( V \), in order to show that \( k_x : \mathcal{L}_{hx} \to G_x \) is onto, it suffices to show that \( k_x \) is an epimorphism. Hence suppose that \( f_0 : G_x \to \mathfrak{D} \) and \( f_1 : G_x \to \mathfrak{D} \) for some simple \( \mathfrak{D} \) such that \( f_0 \circ k_x = f_1 \circ k_x \). Introduce sheaf morphisms \( H_0 : (\lambda, \mu) : (1, \mathfrak{D}) \to (Y, G) \) and \( H_1 = (\lambda, v) : (1, \mathfrak{D}) \to (Y, G) \) where \( \lambda(0) = x \) \( \mu_0 = f_0 \) and \( v_0 = f_1 \). Then \( H \circ H_0 = H \circ H_1 \), but \( H \) is a monomorphism, so we have \( H_0 = H_1 \) from which we infer that \( f_0 = f_1 \).

We now show that (i) and (ii) implies that \( f^* \) is onto, which is a contradiction. Let \( \mathfrak{A}d = (X, \mathfrak{L}) \) and \( \mathfrak{B}d = (Y, G) \). It suffices to show that \( \Gamma((f^*)d) \) is onto (Here we are taking a double dual). So suppose \( \sigma \in \Gamma(Y, G) \). For each \( x \in Y \), \( k(x, -) \) is onto so \( k(x, t) = \sigma(x) \) for some \( t \in \mathcal{L}_{h(x)} \). That is \( t = \tau_x(h(x)) \) for some \( \tau_x \in \Gamma(X, G) \). Hence there is a clopen neighborhood \( N_x \) of \( x \) such that \( \Gamma(\tau_x)(y) = \sigma(y) \) for all \( y \in N_x \). Since \( h \) is one to one and \( X, Y \) are Boolean spaces, we get that \( h(N_x) \) is clopen in \( h(Y) \) and there is a clopen set \( M_x \) in \( X \) such that \( h(N_x) = M_x \cap h(Y) \). Using compactness, there exists a partition of \( X \) into clopen subsets \( M_0 \ldots M_{k-1} \) and sections \( \tau_i \in \Gamma(M_i, L) \) such that

\[
k(y, \tau_i(h(y))) = \sigma(y)
\]

wherever \( h(x) \in M_i \) for \( i < k \). Defining \( \tau \) by \( \tau(z) = \tau_i(z) \) whenever \( z \in M_i \) \( i < k \), it follows that \( \tau \in \Gamma(X, \mathfrak{L}) \) and \( \Gamma((f^*)d) \tau = \sigma \). Thus \( \Gamma((f^*)d) \) is onto \( \Gamma(\mathfrak{B}d) \), and we are done.

4 Logical application

4.1 Beth definability

Here by algebra, we mean either cylindric, Pinter, quasipolyadic, or quasipolyadic equality algebra. The next theorem, whose proof will be omitted, will help us obtain two new results.
Lemma 4.1. 

(1) Let \( A \) be semisimple simple, then there exists a unique \( \delta(A) \in D_{c_{\alpha+\omega}} \) \( i : A \rightarrow \mathcal{N_{\alpha}}\delta(A) \). The algebra \( \delta(A) \) is called an \( \omega \) dilation of \( A \). Furthermore, if \( A \cong B \), then this isomorphism lifts to \( \delta(A) \cong \delta(B) \).

(2) Semisimple algebras have AP with respect to the representables.

(3) Simple algebras have AP.

In an unpublished manuscript of the author two nice BL algebras with opeartors were constructed such that the inclusion is an epimorphism that is not surjective. For all cylindric-algebras, infinite dimensional nice algebras as in the statement of the theorem were constructed by Madarazs. We readily obtain that in all these varieties epimorphisms are not surjective even in simple algebras, because by the last theorem simple algebras have AP.

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