Cosmic structures via Bose–Einstein condensation and its collapse

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Abstract. We develop our novel model of cosmology based on Bose–Einstein condensation. This model unifies the dark energy and the dark matter, and predicts the multiple collapse of condensation, followed by the final acceleration regime of cosmic expansion. We first explore the generality of this model, especially the constraints on the boson mass and condensation conditions. We further argue the robustness of this model over a wide range of parameters of mass, self-coupling constant and the condensation rate. Then the dynamics of BEC collapse and the preferred scale of the collapse are studied. Finally, we describe possible observational tests of our model, especially the periodicity of the collapses and the gravitational wave associated with them.

Keywords: dark matter, dark energy theory
1. Introduction

After WMAP, fundamental parameters in astrophysics, $H_0, q, \Omega_M, \Omega_\Lambda$, etc, have become more accurate and mutually consistent. Although the standard $\Lambda$CDM model can explain most of the observations, consistent with these parameters, it still remains as a phenomenological model. In our previous paper [1], in order to improve this model, we discussed a cosmological model in the framework of relativistic Bose–Einstein condensate (BEC) and constructed a unified model of dark energy and dark matter. There, the quantitative study has been limited within the spatially uniform development of condensates. In the present paper, we develop this cosmological model to include spatially inhomogeneous modes to describe the instability of the BEC and to clarify the preferred scales of the collapsed objects. Furthermore, we establish the robustness of this model, especially showing a generality of the condensation strength.

In general, the BEC proceeds in a Bose gas of mass $m$ and number density $n$, when the thermal de Broglie wavelength $\lambda_{dB} \equiv \sqrt{2\pi \hbar^2/(mkT)}$ exceeds the mean interparticle distance $n^{1/3}$, and the wavepacket percolates in space:

$$kT < \frac{2\pi \hbar^2 n^{2/3}}{m}. \quad (1.1)$$
On the other hand, cosmic evolution has the same temperature dependence since the matter-dominant universe behaves, in an adiabatic process, as

$$\rho \propto T^{3/2}. \quad (1.2)$$

Hence, if the boson temperature is equal to radiation temperature at $z = 1000$, for example, we have the critical temperature at present $T_{\text{critical}} = 0.0027$ K, since $T_m \propto a^{-2}$ and therefore $T_r/T_m \propto a$ in an adiabatic evolution. Using the present energy density of the universe $\rho = 9.44 \times 10^{-30}$ g cm$^{-3}$, the BEC takes place provided that the boson mass satisfies

$$m < 1.87 \text{eV}. \quad (1.3)$$

(This constraint will be somewhat reduced later. See equation (3.8).)

Conventional BEC is described in terms of the mean field which obeys the Gross–Pitaevskii (GP) equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V \psi + g |\psi|^2 \psi. \quad (1.4)$$

Here $\psi(\vec{x}, t)$ is the condensate mean field and $V(\vec{x})$ is the potential. The coupling strength $g$ is related to the s-wave scattering length $a$ as

$$g = 4\pi \hbar^2 a/m, \quad (1.5)$$

and therefore implies attractive interaction for $a < 0$. Originally, only a positive value for $a$ (and therefore $g$) has been considered for the BEC, since a negative value for $a$ necessarily yields imaginary terms in the ground state energy and chemical potential [2]. However, the appearance of the imaginary part in the energy simply implies that the ground state is unstable but the BEC itself takes place, as a transient state, even in negative $g$ [3].

This instability of the BEC turns out to be crucial in the context of cosmology.

The above GP equation is apparently non-relativistic. In the context of cosmology, we need a relativistic GP equation. The relativistic GP equation has a form of the Klein–Gordon equation with self-interaction and the Lagrangian is given by

$$L = \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 \right). \quad (1.6)$$

We discuss the metric tensor given by

$$ds^2 = (1 + 2\Phi) \, dt^2 - a^2(1 - 2\Phi) \, d\vec{x}^2, \quad (1.7)$$

where $\Phi = \Phi(t, \vec{x})$ represents the gravitational potential. The instability of this field has already been studied in [4] disregarding the cosmic expansion. We consider the normal mass signature but the self-coupling is negative. The instability analysis is applicable for either signatures for $m^2$ and $\lambda$. We will discuss the difference of our formalism from the usual Higgs mechanism later.

In section 2, the essence of the BEC cosmology is briefly summarized. Scalar dark matter plays an essential role in our scenario. We show in section 3 that the BEC model

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6 On the other hand, the system has no instability against quantum tunneling since the field we consider is a classical mean field.
7 The uniqueness is not yet proved although this form is very natural. See the discussion just after equation (2.2).
is quite robust, especially the final accelerated-expansion regime is always realized in a wide range of parameters. In section 4, extending the phenomenological scenario reviewed in section 2, we argue the inhomogeneous modes and the instability of the BEC based on the microscopic Lagrangian equation (1.6) with the metric equation (1.7). Section 5 is devoted to the miscellaneous observational problems to probe the remnants of this BEC cosmological model.

2. Basics of the BEC cosmology

We briefly describe the basic scenario of the BEC cosmology developed in [1,5]. The backbones of this model are (1) a relativistic GP equation, (2) steady slow process of the BEC and (3) BEC instability which leads to the dark energy collapse.

2.1. Relativistic Gross–Pitaevski equation

From equation (1.6), the relativistic version of the GP equation, in the Minkowski space, will become

\[ \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi + \lambda (\phi^* \phi) \phi = 0, \]

with the potential

\[ V \equiv m^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2. \]

We briefly discuss the validity of this equation. The ordinary GP equation (1.4) has the form of a nonlinear Schrödinger equation. The ordinary Schrödinger equation is easily transformed into a relativistic form of the Klein–Gordon equation. This transformation and the calculation of conserved charge associated with these equations are standard textbook subjects. An addition of the nonlinear term to these equations as in the above is a straightforward operation. Therefore it is natural to think that equation (2.1) is a correct relativistic form of the GP equation describing the dynamics of the order parameter in the BEC. Actually, the decomposition \( \phi = e^{-i \Omega t} \psi(t, \vec{x}) \) and the long-time approximation \( \dot{\psi} / \psi \ll m \) correctly reproduce the ordinary non-relativistic GP equation.

This form of equation (2.1) is manifestly Lorenz covariant as it should be at the basic equation level. On the other hand, in general, an individual solution of it may have a preferred reference frame. For example, in the case of spatially uniform BEC, there seems to be a preferred reference frame on which the particles are at rest. However, this fact cannot be apparently described by the spatially uniform field \( \psi \). This does not mean the invalidity of equation (2.1) of course, but it simply represents the applicability limit of the mean field approximation which we are discussing, irrespective of relativistic or non-relativistic descriptions. In this mean field approximation, individual particles are described by the same mode function despite the fact that the individual particles are actually fluctuating. Therefore any individual property for each particle is erased and the continuous description remains. This mean field approximation is very effective and works fairly well despite the above discrepancy. We need to go beyond the mean field approximation.

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8 We set \( c = 1, \hbar = 1 \) hereafter.
approximation for a much more complete description of the BEC, which we will report in our future papers.

Substituting the decomposition of the classical mean field \( \phi = A e^{iS} \) and defining the momentum \( p_\mu = -\partial_\mu S = (\epsilon, -\vec{p}) \), where \( \vec{p} = m\gamma \vec{v}, \gamma = (1 - \vec{v}^2)^{-1/2} \), the relativistic GP equation reduces to the Euler equation for fluids:

\[
\epsilon \frac{\partial \vec{v}}{\partial t} + \nabla \left( \frac{\gamma v^2}{2} + \frac{\lambda}{12m} A^2 + \frac{\hbar^2}{2Am^2} \partial^2_\mu A \right) = 0.
\] (2.3)

The energy–momentum tensor associated with equation (1.6) becomes

\[
T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = 2\partial_\mu \phi^* \partial_\nu \phi - g_{\mu\nu} \left( \partial \phi^* \partial \phi - m^2 \phi^* \phi - \frac{1}{2} (\phi^* \phi)^2 \right).
\] (2.4)

For the isotropic relativistic fluid, it reduces to

\[
T_{\mu\nu} = \text{diag}(\rho, p, p, p),
\] (2.5)

in the local rest frame. Here, the condensate parts of \( \rho \) and \( p \) are given by

\[
\rho = T^{00} = \dot{\phi}^* \dot{\phi} + m^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 = \dot{\phi}^* \phi + V,
\] (2.6)

and

\[
p = T^{11} = T^{22} = T^{33} = \dot{\phi}^* \dot{\phi} - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 = \dot{\phi}^* \phi - V.
\] (2.7)

### 2.2. Steady slow process of the BEC

We consider the cosmic evolution of various energy densities on average in our model, leaving aside the dynamics of inhomogeneous modes, derived from the microscopic Lagrangian (1.6) to section 4. Further, here we discuss the BEC cosmology phenomenologically, leaving aside the generality of the parameters to section 3. The whole evolution is given by the following set of equations [1]:

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} (\rho_g + \rho_\phi + \rho_l),
\]

\[
\dot{\rho}_g = -3H\rho_g - \Gamma \rho_g,
\]

\[
\dot{\rho}_\phi = -6H (\rho_\phi - V) + \Gamma \rho_g - \Gamma' \rho_\phi,
\]

\[
\dot{\rho}_l = -3H\rho_l + \Gamma' \rho_\phi.
\] (2.8)

Here \( \Gamma \) is the decay rate of the boson gas (i.e. uniform DM) into the BEC and \( \Gamma' \) is the decay rate of the BEC into collapsed BEC (i.e. localized DM). The former \( \Gamma \) is a constant, but the latter \( \Gamma' \) appears only when the BEC satisfies the instability condition. These rates are transport coefficients which characterize the BEC phase transition although they should be, in principle, calculated from the Lagrangian equation (1.6) and the environmental information. Here we fix these values phenomenologically in the present stage of our model\(^9\). As is easily seen from equation (2.8), DE(\( \rho_\phi \)) and DM(\( \rho_g + \rho_l \)) are intimately related with each other in our model; the original uniform DM(\( \rho_g \)) condensates

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\(^9\) This is somewhat generalized later in section 3.2.
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Figure 1. Schematic diagram of the evolution of the BEC. (a) Over-hill regime and (b) inflationary regime are depicted. The inflation appears as a result of the balance between the condensation and the potential force, $V' = \Gamma \rho_g / \phi$.

into DE($\rho_\phi$), and it collapses into localized DM($\rho_l$), which eventually becomes a dominant component in the total DM($\rho_g + \rho_l$).

There are two relevant regimes of solutions to equation (2.8). One is (a) the over-hill regime, and the other is (b) the inflationary regime. The former appears when the condensation strength is high and the condensation overshoots the potential barrier, and the latter when it is low. A general evolution is a mixture of them; several regimes of (a) finally followed by the regime (b). Let us now examine these regimes.

(a) Over-hill regime. This regime generally appears when the condensation strength is faster than the potential force and the condensation overshoots the potential barrier, especially in the earlier stage of cosmic evolution. Actually, this regime is a fixed point of the first three equations in equation (2.8):

$$
\phi \to \infty, \quad \rho_\phi \to 0, \quad \rho_g \to 0, \quad H \to 0, \quad a \to a_*. \quad (2.9)
$$

The field goes over the hill of the potential, as in figure 1(a).

The condensation speed $\Gamma$ is fast at the first stage, and the Bose gas density is simply reduced $\rho_g \propto e^{-\Gamma t}$ and the cosmic friction term becomes negligible. Then, the third equation in equation (2.8) yields $\dot{\phi} \approx -V'$ and $\phi$ reaches singularity within a finite time. Since $\dot{\phi}$ increases rapidly in the last stage of the fall, the BEC reduction rate $-6H(\rho_\phi - V) \propto H\dot{\phi}^2$ dominates the BEC increase rate $\Gamma \rho_g$. Thus, we have eventually $\rho_\phi \to 0$ and $H \to 0$. However, actually, this virtual singularity is avoided by invoking the last equation in equation (2.8).

(b) Inflationary regime. This regime appears when the condensation strength is weaker than the potential force, especially in the later stage of cosmic evolution. This regime turns out to be a stable fixed point of equation (2.8):

$$
\phi \to \phi_*, \quad H \to H_*, \quad \rho \to 0, \quad \dot{\phi} \to 0. \quad (2.10)
$$

The BEC condensed field stops and stays at an intermediate position of the potential hill for ever, as in figure 1(b). This mechanism is a novel type of inflation, which is supported by the balance of the condensation speed ($\Gamma \rho_g$), and the potential force ($\dot{\phi}V'$):

$$
\dot{V} = \Gamma \rho_g. \quad (2.11)
$$
Figure 2. (a) Time evolution of various cosmic densities as a function of redshift $z$. This is a numerical solution of equation (2.8). The red line is $\rho_g$, the green line is $\rho_\phi$ and the blue line is $\rho_l$. Here we have set $\bar{m}^2 \simeq 0.01, \lambda = -0.1, \tilde{\Gamma} = 0.4$. The variables with a tilde are dimensionless ones defined in the text. Several BEC collapses take place, which are finally followed by a phase with constant energy density (accelerating expansion). (b) Time evolution of the $w$ parameters. The green line represents $w_\phi \equiv p_\phi/\rho_\phi$ as a function of $z$. It is apparent that BEC behaves as an ideal gas in the early stage and as a cosmological constant in the later stage. Black solid line represents $w$ of the whole system.

Though both sides of equation (2.11) exponentially decay to zero, the balance itself is automatically maintained\(^\text{10}\).

In the actual universe, the above two kinds of regimes are realized successively. First, the over-hill regime repeats multiple times in general until the Bose gas density is consumed and the condensation speed decreases. Eventually the condensation force balances with the potential force, and the final inflationary regime follows.

In figure 2(a), numerical results for the evolution of the cosmic energy densities are plotted. Red, green and blue curves represent, respectively in this order, the cosmic energy densities of the Bose gas ($\rho_g$), BEC ($\rho_\phi$) and the localized objects ($\rho_l$). In this example, four over-hill regimes are finally followed by the inflationary regime.

Further, in figure 2(b), the evolution of the corresponding EOS parameter $w \equiv p/\rho$ is shown. It is clear that the BEC ($\rho_\phi$) acquires the genuine DE property (i.e. $w \approx -1$) only recently $z < 3$. For $z > 3$, $\rho_\phi$ behaves as an ideal gas (i.e. $w \approx 1$). This is because the field $\phi$ is in the stage of condensation and moving. Therefore it possesses kinetic energy and positive pressure. Thus, a genuine DE with $w \approx -1$ only appears for $z > 3$, despite that we often call $\rho_\phi$ as DE and $\rho_g + \rho_l$ as DM in this paper. Thus, our model cannot be distinguished from the standard model with a cosmological constant, as far as we observe the cosmic evolution $a(t)$.

Now we explain the detail of our numerical calculations above and how to obtain physical scales from them. In equation (2.8) we simultaneously use parameters of quite different orders of magnitude, as a result of bridging microphysics to the macrophysics, such as the Planck mass $m_{Pl}$, cosmic expansion rate $H$, the boson mass $m$, etc. First we normalize dimensional observables by two mass scales; $m_s$ for space–time and $m_0$ for

\(^{10}\) This exponentially decreasing amplitude of the balance may lead to the instability of the inflationary regime and the autonomous termination of this regime, given some small external perturbations.
energy, whose orders are fixed later:

\[ \tau = m_* t, \quad \tilde{H} = \frac{\dot{H}}{m_*}, \quad \tilde{\rho} = \frac{\rho}{m_0^2}, \quad \tilde{m} = \frac{m}{m_0}, \]

where \( m_* \) and \( m_0 \) are related through the gravitational constant \( G \) or Planck mass \( m_{\text{pl}} \):

\[ \frac{8\pi G}{3c^2 m_*^2} = \frac{1}{m_0^4} \quad \text{or} \quad m_0^2 = m_{\text{pl}} m_*^2. \tag{2.13} \]

Then equation (2.8) is rewritten as

\[
\begin{align*}
\tilde{H}^2 &= \tilde{\rho}_g + \tilde{\rho}_\phi + \tilde{\rho}_l \\
\tilde{\rho}_g' &= -3\tilde{H}\tilde{\rho}_g - \tilde{\Gamma}\tilde{\rho}_g \\
\tilde{\rho}_\phi' &= -3\tilde{H} \left( \frac{m_*}{m_0} \right)^2 \tilde{\phi}^* \tilde{\phi}^\prime + \tilde{\Gamma}\tilde{\rho}_g - \tilde{\Gamma}'\tilde{\rho}_\phi \\
\tilde{\rho}_l' &= -3\tilde{H}\tilde{\rho}_l + \tilde{\Gamma}'\tilde{\rho}_\phi,
\end{align*}
\tag{2.14}
\]

where the prime means the derivative w.r.t. \( \tau \). The ratio of the mass scales \( m_* / m_0 \) appears only in the first term on the RHS of the third equation of equation (2.14), and this term does not contribute since this factor is extremely small, \( m_* / m_0 \approx m_0 / m_{\text{pl}} \ll 1 \).

The scales \( m_0 \) and therefore \( m_* \) can be fixed, from our numerical calculation, as follows. The energy density of the BEC is normalized as

\[ \tilde{\rho}_\phi = \left( \frac{m_*}{m_0} \right)^2 \tilde{\phi}^* \tilde{\phi}^\prime + \tilde{m}^2 \tilde{\dot{\phi}}^2 + \frac{\lambda}{2} (\tilde{\phi}^* \tilde{\phi})^2. \tag{2.15} \]

We have started our calculation, in the above example of figure 2, with \( \tilde{m} = 0.1 \) and obtained the numerical value for \( \tilde{\rho}_\phi \) at present, \( \tilde{\rho}_\phi = 0.00015 \). We identify \( \rho_{\text{cr}} = 0.73 \rho_{\text{cr0}} \) with \( \rho_{\text{cr0}} = 9.44 \times 10^{-30} \text{ g cm}^{-3} \), from which it follows that

\[ m_0 \approx 0.030 \text{ eV}, \quad m = \tilde{m} m_0 = 0.0030 \text{ eV}, \quad \text{and} \quad m_* = 2.09 \times 10^{-31} \text{ eV}. \tag{2.16} \]

As we have mentioned after equation (2.8), DE(\( \rho_\phi \)) and DM(\( \rho_g + \rho_l \)) are intimately related to each other in our model; DM(\( \rho_g \)) condensates into DE(\( \rho_\phi \)), and it collapses into localized DM(\( \rho_l \)). Therefore, it is natural to expect that the amounts of DM and DE are almost the same. This is a kind of self-organized criticality [7] (SOC). Finally dominant DM(\( \rho_l \)) is composed from the quantity which had been DE(\( \rho_\phi \)) before.

2.3. BEC collapse

There is no degenerate pressure for bosons, unlike fermions. Therefore, only the quantum pressure, expressed in the last term on the left-hand side of equation (2.3), induced from the Heisenberg uncertainty principle can prevent the BEC from collapse. However, there is a maximum mass for this mechanism to work. The Compton wavelength of the object \( \lambda_{\text{compton}} = 2\pi \hbar/(mc) \) must be larger than the Schwarzschild black hole radius.
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or its innermost stable radius $3(2GM/c^2)$. This condition defines a characteristic mass scale

$$M_{\text{critical}} \approx m_{\text{pl}}^2/m \equiv M_{K_{\Lambda UP}}$$

(2.17)

only below which a stable configuration is possible for the BEC. This structure is well known as a boson star [8]. For example, the boson mass $m = 10^{-3}$ eV gives $M_{\text{critical}} = 10^{-7}M_\odot$, which is almost the mass of Mercury. If the object’s mass exceeds this critical value, black holes are inevitably produced. There is no limit for the black hole mass. Thus, DE black holes of any size are naturally produced in our model. However, actually, shock waves would also be naturally produced in the collapse process of the BEC. They convert the potential energy to heat and yield huge pressures, which may prevent the collapse to black holes. We will study the initial linear instability of the BEC in the following sections. In any case, it will be true that many compact clumps (boson stars, black holes, hot clusters) are rapidly formed after the collapse of the BEC. This fascinating scenario, early formation of black holes and clumps, has been extensively discussed in [6] in other contexts.

3. Robustness of the BEC cosmology

After a brief introduction of the BEC cosmology in the above, we would like to argue first the robustness and generality of the BEC model. In particular, we would like to clarify the condition under which the BEC phase is possible in the universe. This argument is deeply related with the basic parameters of the model: mass $m$ of the boson and the condensation rate $\Gamma$.

3.1. Mass constraints and the BEC condition

Let us now consider how the BEC is possible in the expanding universe. In general, the charge density $n$ of the boson gas is expressed as the sum of two contributions; the particle with positive signature and the antiparticle with negative signature:

$$n = \int \frac{dp^3}{(2\pi)^3} \left[ \frac{1}{e^{\beta(\omega+\mu)} - 1} - \frac{1}{e^{\beta(\omega-\mu)} - 1} \right],$$

(3.1)

which is a function of temperature $1/\beta$ and the chemical potential $\mu$, and energy $\omega = \sqrt{p^2 + m^2}$, in units of $\hbar = k_B = c = 1$. The BEC takes place when $\mu = m$ and the critical temperature $T_c$ or the critical density $n_c$ is determined by setting so in the above.

In the non-relativistic regime, i.e. $1/\beta \ll m$, the above form yields

$$n_c = \zeta (3/2) \left( \frac{mT}{2\pi} \right)^{3/2}$$

(3.2)

or $T_c = (2\pi/m)(n/\zeta(3/2))^{2/3}$. Below $T_c$ or above $n_c$, the wavefunctions of individual particles begin to overlap with each other, i.e. the thermal de Broglie length exceeds the mean separation of particles:

$$\lambda_{\text{dB}} \equiv \left( \frac{2\pi}{mkT} \right)^{1/2} > r \equiv n^{-1/3}.$$

(3.3)
In this regime, the cosmic energy density of the non-relativistic dark matter has the same dependence on the temperature:

\[ n = n_0 \left( \frac{T}{T_0} \right)^{3/2}, \quad (3.4) \]

if we assume the entropy is conserved during the expansion and therefore \( \rho \propto a^{-3} \propto T^{3/2}. \) Thus, provided that the dark matter gas temperature had been once below the critical temperature at some moment in the non-relativistic regime, the universe would always be below the critical temperature and the BEC can initiate all the time.

The above fact sets the upper limit on the boson mass, for the BEC to take place at the present universe. Let us suppose the boson gas was in thermal equilibrium with radiation in the far past. Suppose the boson had transformed from relativistic to non-relativistic at time \( t_* \) with the scale factor \( a_* = a(t_*) \). The boson temperature \( T_B(t) \) at this time \( t_* \) will be \( T_B(t_*) = m = T_\gamma(t_*) \). After that time, the boson temperature reduces as \( \propto a^{-2} \), and the photon temperature \( T_\gamma(t) \) as \( \propto a^{-1} \). Therefore, for \( t > t_* \),

\[ T_B(t) = \left( \frac{a_*}{a(t)} \right)^2 m, \quad T_\gamma(t) = \left( \frac{a_0}{a(t)} \right) T_\gamma \quad (3.5) \]

Putting the present value of radiation temperature \( T_\gamma = 2.73 \text{ K} \) into the above equation, we can estimate the present temperature of the boson gas \( T_B(t_0) = T_{B0} \) as

\[ T_{B0} = \left( \frac{T_\gamma^0}{m} \right) T_\gamma \quad (3.6) \]

The present value of the critical temperature can be estimated from the present energy density \( \rho_0 = 9.44 \times 10^{-30} \text{ g cm}^{-3} \). The ratio of them is

\[ \frac{T_{B0}}{T_\gamma^0} = \left( \frac{3}{2} \right)^{2/3} \frac{T_\gamma^2 m^{2/3}}{2 \pi \rho_0^{2/3}} \quad (3.7) \]

The requirement that this ratio is smaller than 1 yields the upper limit of the boson mass:

\[ m < \frac{(2\pi)^{3/2} \rho_0}{T_\gamma^0 \zeta (3/2)} \approx 19 \text{ eV} \quad (3.8) \]

(This value is manifestly larger than the previous estimate in equation (1.3). This is because we now set the condensation condition at the epoch when the boson becomes non-relativistic in general, while previously at the recombination epoch at \( z = 1000 \) disregarding the individual mass.) It must be noted that this upper limit does not apply to the boson which has not been in thermal equilibrium with radiation in the past, or the boson composed from the pair of fermions.

On the other hand, in the ultra-relativistic regime, i.e. \( T \gg m \), the critical density becomes

\[ n_c = \frac{mT^2}{3} \quad (3.9) \]
Figure 3. Schematic diagram of the critical temperature $T_{cr}$ and the cosmic dark matter temperature $T_{\text{universe}}$. In the non-relativistic regime ($m > T$), $T_{cr} \propto \rho^{2/3}$ and $T_{\text{universe}} \propto \rho^{2/3}$. In the relativistic regime ($m < T$), $T_{cr} \propto \rho^{1/2}$ and $T_{\text{universe}} \propto \rho^{1/4}$. The DM temperature evolution line for the universe is the same as the adiabatic expansion.

and the cosmic energy density of the ultra-relativistic matter behaves as

$$n = n_0 \left( \frac{T}{T_0} \right)^4.$$  \hspace{1cm} (3.10)

Therefore, contrary to the non-relativistic regime, even if the cosmic temperature had been once below the critical temperature at some moment in the ultra-relativistic regime, the boson temperature in the universe would eventually exceed the critical temperature and the BEC would melt into a thermal boson gas at that time. This trend is depicted in figure 3.

In the above argument, it was essential that the temperature of the bosons scales as $T \propto 1/a^2$. This scaling generally holds for cold dark matter although we only considered, in the above, the interacting case in which bosons are kept in thermal equilibrium. On the other hand, the neutrinos, of similar mass, scale as $T \propto 1/a$. This striking difference simply reflects that our bosons are assumed to be cold dark matter (CDM) and the neutrinos are typical hot dark matter (HDM). When they decouple, the former was already non-relativistic and the latter was still relativistic, by definition. The distribution function will be simply frozen at the decoupling epoch in the FRW universe provided that the dark matter cannot maintain their thermal equilibrium by themselves later. Thus, after the decoupling, the momentum $p$ simply scales as $p \propto 1/a$, where $a$ is the scale factor normalized at the decoupling time. Therefore, the CDM distribution function behaves as $f(p) = \text{const}[\exp(E_d/T_d) \pm 1]^{-1}$, which leads to the quasi-temperature for CDM as $T = T_d/a^2$. On the other hand the HDM distribution function behaves as $f(p) \approx \text{const}[\exp(E_d/T_d) \pm 1]^{-1}$, which leads to the quasi-temperature for HDM as $T = T_d/a$. The neutrino is a typical example of this class (see [9]).

3.2. Time-dependent $\Gamma$ and the robustness of the BEC model

In the previous calculations [1, 5], we assumed the condensation speed $\Gamma$ is a constant parameter. However, this quantity $\Gamma$ is not a simple term appearing in a Lagrangian, but a transport coefficient, which includes all the information of the many-body environment, e.g. temperature, density, fluctuations, etc. It should be calculated from the quantum
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Figure 4. Time evolution of various cosmic densities as a function of redshift $z$. Same as figure 2(a), but with time-dependent condensation rate $\Gamma(t)$, where $\alpha = -1, -2, -3$ and +1, respectively for (a)–(d). Other parameters are set as $\bar{m}^2 \simeq 0.01, \lambda = -0.1$.

field theory of finite temperature and density in the expanding universe, though such a theory does not exist at present. Therefore we take the second best method, i.e. we try all possible time-dependent $\Gamma$. Although this does not specify $\Gamma$, we may establish some robustness of the BEC cosmological model.

General transport coefficients would depend on the temperature and the density of the environment. Time dependence of such global parameters can be represented by a scale factor in the uniformly expanding universe. Therefore, possible time dependences of the parameter $\Gamma$ will be exhausted by the inclusion of the scale factor, which normally depends on the form of power. Thus we assume

$$\Gamma = \bar{\Gamma} a(t)^{\alpha}$$

where $\bar{\Gamma}$ and $\alpha$ are constants. Even in this case, the basic mechanism of the BEC cosmology does not change. Actually, the insertion of the expression equation (3.11) in equation (2.8) is equivalent to recasting the behavior of the boson gas density, which is the source of condensation, as

$$\rho_g \propto a(t)^{-3} \rightarrow \rho_g \propto a(t)^{\alpha-3}$$

while $\Gamma \rightarrow \bar{\Gamma}$ is still a constant. Intuition tells us that such a change of the source gas density does not alter the scenario in essence.

In order to check this intuition, we have performed several demonstrations using numerical methods. Results are given in figure 4.
In general, the reaction rate reduces when the temperature decreases. Therefore it will be natural to choose negative values for the parameter $\alpha$. In all calculations with $\alpha = -1, -2, -3$, the qualitative behavior of the model, i.e. multiple BEC collapses followed by an inflation, does not change. Quantitative changes are the total number of BEC collapses and the identification of the present time. These results could be easily foreseen from the fact that the change equation (3.12) is equivalent to equation (3.11). We have also performed positive values for $\alpha$ which may be less relevant in the actual universe. In this case, only $\alpha = 1$ yields the qualitatively similar behavior, but the cases for $\alpha > 1$ are not clear within our calculations, probably due to numerical error.

3.3. Numerical $\Gamma$ and $m$-robustness of the BEC

Here we show the robustness of the BEC over the relatively wide range of numerical values of condensation rate $\Gamma$, boson mass $m$, etc.

The boson mass turns out to be 0.0024, 0.0030, 0.0041, 0.0033, 0.0019 and 0.0024 eV, respectively, for the parameters (a)-(f) in figure 5. The condensation rate $\Gamma$ turns out to be $1.411 \times 10^{-32}$, $8.37 \times 10^{-32}$, $4.050 \times 10^{-32}$, $1.0521 \times 10^{-31}$, $8.593 \times 10^{-33}$ and $2.272 \times 10^{-31}$ eV in the same order. The qualitative feature of our model did not change under the above variations of parameters. Further variation will be possible in principle, but our numerical code at present does not yield reliable results. This applies especially in the late stage of the accelerating expansion, due to the exponentially reducing energy density.

From the above results, we notice that the physical quantities $m$ and $\Gamma$ are not simply related to parameters $\tilde{m}$ and $\tilde{\Gamma}$. Actually, the boson mass $m$ only changes by about a factor of 2 while the parameter mass $\tilde{m}$ changes by about a factor of 16. On the other hand, the condensation rate $\Gamma$ changes by about a factor of 20 while the parameter $\tilde{\Gamma}$ only changes by about a factor of 4.

Robustness of the boson mass value $0.003 \pm 0.001$ eV may be somewhat interesting. However, we have not yet resolved the origin of this robustness; it may be due to the intrinsic nature of our model, or it may simply represent that our numerical range is too limited. Therefore, we will not specify the boson mass in the following arguments, but leave some possible range of mass values open for further discussions.

Some related comments are in order. There is a possibility that the condensation rate depends on time in an exponentially reducing form. This corresponds to the case that the condensation takes place in a finite epoch in cosmic history. The fate of the universe then becomes subtle and will require detailed analysis. It is important to observe that the condensation rate enters the set of equations (equation (2.8)) with the combination $\Gamma \rho_g$, and the final accelerated expansion continues even if the gas density $\rho_g$ reduces exponentially. Recent observations from WMAP and large scale galaxy distributions indicate that the radiation-matter equality epoch should be around $z = 3000$. On the other hand, the DE/DM ratio at present must be of order one. The former and latter requirements yield, respectively, the upper and lower limits of the value for the condensation rate $\Gamma$. Although we believe, from the above robustness of the BEC model, that there is a wide range in between these upper and lower limits, a careful consistency check with observations will still be necessary for the calculated values of $\Gamma$ in the future.
BEC instability and the collapse

BEC collapse is a very complicated process. During the collapse of the BEC, gravitational potential energy, $GM^2/R$, is released, where $M$ and $R$ are the typical mass and scale of the collapsing region. If this collapse were free from shock waves, the collapse were spherically symmetric and the collapsing BEC exceeded the critical mass, then most of the collapsing BEC would turn into black holes. However, such ideal conditions would never be fulfilled.

If the BEC collapse takes place smoothly, and the process is adiabatic, then the temperature of the BEC is given by

$$T \approx R^{3(1-\gamma)} \approx M^{1-\gamma} \rho^{\gamma-1},$$

(3.13)

which is the same, if $\gamma = 5/3$, as the behavior of the adiabatic universe in figure 3, but in the opposite time direction. The boson temperature is always below the critical temperature, $T \propto \rho^{2/3}$, and the BEC is maintained.
If the BEC collapse takes place violently, and the gravitational energy always turns into uniform thermal energy, then the process is in virtual equilibrium, and the temperature of the boson is given by

\[ T \approx \frac{GMm}{R} \approx GM^{2/3}m^{1/3} \rho^{1/3}, \tag{3.14} \]

which becomes \( T \approx GMm/R \to m \), in the limit that the system size approaches the Schwarzschild radius \( R \approx 2GM \). This means that the boson becomes relativistic. Even in this case, the boson temperature is always below the critical temperature, and the BEC is maintained. See figure 3.

However, in the real universe, shock waves are inevitably produced and fluctuations associated with the collapse would be enormous. As a result, some small portion of the collapsed BEC turns into black holes and the remainder would become a normal boson gas. In any case, the universe becomes very clumpy at small scales.

The collapsed BEC will gravitationally attract baryons to form a cluster, as in the standard CDM model. By contrast, at large scales, the potential is not affected and is the same as that in the standard ΛCDM model. This point is further clarified below.

We would like to restrict our study to the linear instability analysis in this paper. This is the subject in the next section.

4. Instability of the BEC and large scale structure

After examining robustness of the BEC cosmological model in the above, we will now argue how the instability of the BEC, which is extending to the entire universe, can manifest to form localized structures. This instability has been phenomenologically treated, by introducing the decay rate \( \Gamma' \) in section 2. Since this instability has a very short timescale compared with the condensation timescale, \( \Gamma' \gg \Gamma \), we simply neglect the latter effect in this section.

The BEC collapse inevitably takes place in the over-hill regime in our model. These collapsed components form localized objects and become direct seeds of the structures in the universe. Our special interest is the preferred scale of these structures. Though the full dynamics of the BEC collapse would require involved numerical calculations, linear instability analysis is always tractable [4], within semi-analytical calculations, to which we devote this paper.

The metric is chosen as

\[ ds^2 = (1 + 2\Phi)dt^2 - a^2(1 - 2\Phi)dx^2, \tag{4.1} \]

where \( \Phi = \Phi(t, \vec{x}) \) represents the gravitational potential and \( a = a(t) \) the cosmic scale factor. The Lagrangian for the BEC condensate mean field \( \phi \) becomes, on this metric without a source term:

\[
\begin{align*}
L &= a(t)^3(1 - 2\Phi)((1 + 2\Phi)^{-1}|\dot{\phi}|^2 - a(t)^{-2}(1 - 2\Phi)^{-1}(\nabla \phi) \cdot (\nabla \phi^\dagger) \\
&\quad - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 + L_g) \\
&\approx a(t)^3(1 - 4\Phi)|\dot{\phi}|^2 - a(t)(\nabla \phi)^2 - m^2a(t)^3(1 - 2\Phi)\phi^2 \\
&\quad - \frac{\lambda}{2}a(t)^3(1 - 2\Phi)\phi^4 + (1 - 2\Phi)a(t)^3L_g, \tag{4.2}
\end{align*}
\]
where the last line is the linearized form. The source term can be negligible in our analysis, since the BEC collapse takes place very rapidly compared to the slow steady condensation timescale. The equation of motion for the condensate becomes, up to the first order in $\Phi$,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2 \phi + m^2 (1 + 2\Phi) \phi + \lambda(1 + 2\Phi)|\phi|^2 \phi = 0. \quad (4.3)$$

The factor $1 + 2\Phi$ in the last term, which was not in [4], plays an essential role. The associated Poisson equation becomes

$$\nabla^2 \Phi = 4\pi Ga^2 \left\{ \phi^\dagger \phi + \nabla \phi \cdot \nabla \phi^\dagger + m^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4 - \rho_0 \right\}, \quad (4.4)$$

where $\rho_0$ is the uniform background energy density.

We now decompose the variables into the background component, with suffixes 0, and the linearly perturbed component, with suffixes 1, as

$$\phi = \phi_0 + \phi_1, \quad (4.5)$$

$$\Phi = 0 + \Phi_1. \quad (4.6)$$

The background is spatially uniform and only depends on time, while the perturbation is time-and space-dependent. The background solution satisfies, from equations (4.3) and (4.4),

$$\ddot{\phi}_0 + 3\frac{\dot{a}}{a}\dot{\phi}_0 + \left(m^2 + \lambda|\phi_0|^2\right) \phi_0 = 0, \quad (4.7)$$

$$\dot{\phi}_0^\dagger \phi_0 + \left(m^2 + \frac{\lambda}{4}|\phi_0|^2\right) |\phi_0|^2 - \rho_0 = 0. \quad (4.8)$$

In these equations (4.7) and (4.8), the variable $\phi_0$ can be assumed to be real without generality since all the coefficients are real. The equations of motion for the perturbations are

$$0 = \ddot{\phi}_1 + 3\frac{\dot{a}}{a}\dot{\phi}_1 - \frac{1}{a^2}\nabla^2 \phi_1 + m^2 (\phi_1 + 2\Phi_1 \phi_0) + \lambda \left(\phi_0^2 \phi_1^\dagger + 2\phi_0^\dagger \phi_1 + 2\Phi_1 \phi_0^\dagger \phi_1^\dagger\right), \quad (4.9)$$

$$\frac{1}{4\pi Ga^2}\nabla^2 \Phi_1 = \dot{\phi}_0 (\phi_1 + \phi_1^\dagger) + \left(m^2 + \lambda|\phi_0|^2\right) \phi_0 (\phi_1 + \phi_1^\dagger).$$

The variable $\phi_1$ is complex and $\Phi_1$ is real. Therefore we replace them by three real functions:

$$\phi_1 = x + iy, \quad (4.10)$$

$$\Phi_1 = z, \quad (4.11)$$

where $x = x(t, \vec{r})$, $y = y(t, \vec{r})$ and $z = z(t, \vec{r})$. These new functions $x, y, z$ are decomposed into Fourier modes:

$$x = \tilde{x} \exp \left(\Omega t + i\vec{k} \cdot \vec{r}\right), \quad (4.12)$$

$$y = \tilde{y} \exp \left(\Omega t + i\vec{k} \cdot \vec{r}\right), \quad (4.13)$$

$$z = \tilde{z} \exp \left(\Omega t + i\vec{k} \cdot \vec{r}\right), \quad (4.14)$$
where $\bar{x}, \bar{y}, \bar{z}, \Omega, \vec{k}$ are constants. Putting these into equations (4.9), we have
\begin{align}
\left(\Omega^2 + \frac{3\dot{\alpha}}{a} \frac{\Omega}{a^2} + \frac{k^2}{a^2} + m^2 + 3\lambda\phi_0^2\right)\bar{x} + 2(m^2 + \lambda\phi_0^2)\phi_0 \bar{z} &= 0, \\
\left(\Omega^2 + \frac{3\dot{\alpha}}{a} \frac{\Omega}{a^2} + \frac{k^2}{a^2} + m^2 + \lambda\phi_0^2\right)\bar{y} &= 0, \\
2\dot{\phi}_0\Omega + 2(m^2 + \lambda\phi_0^2)\phi_0 \bar{x} + \frac{k^2}{4\pi G a^2} \bar{z} &= 0.
\end{align}

Existence of a non-trivial solution $\bar{x}, \bar{y}, \bar{z}$ requires that the above set of linear equations is dependent on each other. Thus we have the condition
\begin{align}
\left[\left(\Omega^2 + \frac{3\dot{\alpha}}{a} \frac{\Omega}{a^2} + \frac{k^2}{a^2} + m^2 + 3\lambda\phi_0^2\right) \frac{k^2}{4\pi G a^2} \right. \\
- 2(m^2 + \lambda\phi_0^2)\phi_0 \left(2\dot{\phi}_0\Omega + 2(m^2 + \lambda\phi_0^2)\phi_0\right) \times \left(\Omega^2 + \frac{3\dot{\alpha}}{a} \frac{\Omega}{a^2} + \frac{k^2}{a^2} + m^2 + \lambda\phi_0^2\right) &= 0,
\end{align}

which determines the instability parameter $\Omega$ as a function of the wavenumber $k$. If one of the solutions $\Omega$ in this equation becomes positive for some $k$, such a mode becomes unstable with the timescale $\Omega^{-1}$. Since the shortest timescale for the structure of scale $l \equiv a/k$ to form is $l/c$ from causality, such structure formation would be actually possible only if $\Omega^{-1} > l$. Thus the structure of linear scale $l$ is possible only if the condition $k/a > \Omega$ is satisfied. More precisely, the first structure formation takes place at the shortest timescale. This condition, by setting $\alpha$ as a positive constant smaller than unity,
\begin{align}
\alpha \frac{k}{a} = \Omega \quad \text{with} \ 0 < \alpha < 1,
\end{align}

uniquely determines the preferred linear scale $a/k_*$ of the structure formed after the BEC collapse.

The expression for $\alpha k/a = \Omega$ is solved for $k$, choosing the value associated with the most unstable mode among four solutions of $\Omega$ for equation (4.18), as
\begin{align}
\frac{k_*}{a} = \left(\frac{-m_{\text{eff}}^2 + \sqrt{m_{\text{eff}}^4 + 64\pi G (1 + \alpha^2) (m^4\phi_0^2 - 2\kappa m^2\phi_0^4 + \kappa^2\phi_0^6)}}{2(1 + \alpha^2)}\right)^{1/2}
\end{align}

where the adiabatic approximation $H = 0, \dot{\phi}_0 = 0$ is utilized since the collapse timescale is much smaller than cosmic and condensation timescales. In equation (4.20), we consider the regime so that $m_{\text{eff}}^2 \equiv m^2 - 3\kappa\phi_0^2 > 0$, i.e. $\phi_0$ smaller than the value at the inflection point of the potential $V(\phi)$. This regime is first realized in the BEC condensation process.

\footnote{It should be remarked that even in the simple case $a = a_0 = 1, \phi_0 = \text{const}, \Omega^2$ has positive real roots and is unstable for the case of $m^2 < 0, \lambda > 0$, unlike the statement in [4].}
The above expression, equation (4.20), can be further reduced to

$$\frac{k_*}{a} \approx \frac{8m^2\phi_0}{m_{\text{eff}}} \sqrt{\frac{\pi G}{6}} \left(1 - 2\kappa\left(\frac{\phi_0}{m}\right)^2 + \kappa^2\left(\frac{\phi_0}{m}\right)^4\right),$$  \hspace{1cm} (4.21)

since, in general, the present mass scales, i.e. of order eV, are negligibly small in comparison with the Planck mass $10^{28}$ eV: $m_{\text{eff}}^2 \approx \kappa\phi_0^2 \approx m^2 \ll m_{\text{pl}}^2 \approx G^{-1}$.

A rough estimate of equation (4.21) and of the preferred scale is possible. Setting $m_{\text{eff}}^2 \approx \phi_0^2 \approx m^2$, equation (4.21) yields the linear scale $l_* \equiv a/k_*:

$$l_* \approx \left(\frac{m_{\text{pl}}}{m}\right)^{\frac{1}{2}} \approx 10^{23} \text{ cm} \approx 30 \left(\frac{\text{eV}}{m}\right)^2 \text{ kpc.}$$  \hspace{1cm} (4.22)

The resultant mass scale associated with the BEC collapse, which took place at redshift $z$, would be

$$M_* = \rho_0 z^3 \frac{4\pi}{3} l_*^3 \approx 1.6 \times 10^{11} \left(\frac{z}{20}\right)^3 \left(\frac{m}{\text{eV}}\right)^{-6} M_\odot,$$  \hspace{1cm} (4.23)

in which strong mass dependence is apparent. Actually there are several different scenarios for the formation of localized structures, depending on the mass $m$ of the boson. We suppose the first BEC collapses at about redshift $z \approx 20$.

(a) If the boson mass is about 1 eV, then the typical mass of the structure will be

$$M_* \approx 1.6 \times 10^{11} M_\odot$$  \hspace{1cm} (4.24)

which is of the order of a galaxy. Some fraction of this mass turns into a black hole and the remaining bosons become a hot gas surrounding the black hole. This is the typical structure expected from the BEC collapse in the present cosmological model for this boson mass. The detail of the mass of such black holes necessitates elaborate numerical calculations, on which we will report in a separate paper in the future.

(b) If the boson mass is about $10^{-3}$ eV, then the typical scale well exceeds the size of the horizon, as easily observed from the power in equation (4.23). Thus, a structure is not formed in the early stage of the BEC condensation while the condition $m_{\text{eff}}^2 \equiv m^2 - 3\kappa\phi_0^2 > 0$ holds. In this case, BEC condensation further proceeds and crosses over the inflection point, beyond where $m_{\text{eff}}^2$ becomes negative and we can no longer use the approximation $m_{\text{eff}}^2 \approx \phi_0^2 \approx m^2$. Then, we have to go back to equation (4.20), which yields the solution

$$\frac{k_*}{a} \approx \frac{|m_{\text{eff}}|}{\sqrt{1 + \alpha^2}}.$$  \hspace{1cm} (4.25)

It means that the strong instability initiates immediately after the mean field crosses over the inflection point. Thus we define the time $\tau$ as the elapsed time after crossing the inflection point. A structure of scale $a/k$ is formed at around $\tau = a/(\alpha k)$. Since the BEC is very unstable and the timescale is short, we can expand $\phi(\tau) = \phi_{\text{inf}} + \dot{\phi}_{\text{inf}}\tau + O(\tau^2)$, where $\phi_{\text{inf}}$ is the value of the condensation at the inflection point, i.e. $m^2 = 3\kappa\phi_{\text{inf}}^2$. Utilizing the relations $(k_*/a)^2 \approx |m^2 - 3\kappa\phi^2|/(1 + \alpha^2) = 6\kappa\phi_{\text{inf}}\dot{\phi}_{\text{inf}}\tau^2/(1 + \alpha^2)$, we have $\tau^{-1} = (6\kappa\phi_{\text{inf}}\phi_{\text{inf}}\alpha^2/(1 + \alpha^2))^{1/3}$, and therefore $k/a = (\alpha\tau)^{-1} = (6\kappa\phi_{\text{inf}}\phi_{\text{inf}}/(\alpha + \alpha^3))^{1/3}$. 

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Putting approximate values \( \dot{\rho}_0 = -6H(\rho_0 - V) + \Gamma \rho_g \approx \Gamma \rho_g \), and \( \rho_g \approx \rho_0 \approx m^2 \phi^2 \), we have

\[
\frac{k_*}{a} \approx (6\kappa \phi^2 \Gamma/(\alpha + \alpha^3))^{1/3} \approx (m^2 \Gamma)^{1/3} \approx ((10^{-3} \text{ eV})^2 10^{-32} \text{ eV})^{1/3} \approx 10^{-13} \text{ eV},
\]

which corresponds to the size \( l_* \approx 10^3 \text{ km} \) and \( M_* \approx 10^{-1} \text{ g} \). Therefore, the BEC collapse cannot form macroscopic cosmological structures. These clumps would work as the ordinary DM and the scenario of large structure formation reduces to the standard model.

Thus the above situations can be summarized as follows. We have two Jeans wavenumbers: one is \( m^2/m_{pl} \), which appears for the condensation \( \phi_0 \) smaller than the inflection point, and \( (m^2 \Gamma)^{1/3} \), which appears for \( \phi_0 \) larger than the inflection point. For case (a) \( m \approx 1 \text{ eV} \), the former instability takes place and the latter has no chance to appear. For case (b) \( m \approx 10^{-3} \text{ eV} \), the former instability is not sufficient and the latter strong instability sets in to make the BEC collapse.

(c) If the boson mass is far below \( 10^{-3} \text{ eV} \), then we can estimate the preferred mass scale utilizing the above scaling relation \( M \propto m^{-2} \). For example, the boson of mass \( 10^{-22} \text{ eV} \) would yield an object of the size of a galaxy and \( 10^{-24} \text{ eV} \) an object the size of a cluster. Because the mass is ultra-low, the boson would always be in the condensed phase. Therefore the BEC has a chance to form DM directly. This consideration naturally brings us to a popular idea that the DM around a galaxy or a cluster is formed from a scalar field with ultra-low mass \([10]–[12]\). We will leave this possibility open in this paper and proceed to the next subject; radiation of gravitational waves.

5. Observational remnants of the BEC cosmology

We now turn our attention to possible observational remnants of the BEC cosmological history, especially in the context of BEC collapses. The most prominent effect would be the emission of gravitational waves, which may be remaining as a fossil in our present universe.

5.1. Gravitational wave associated with the BEC decay

In our model, universe necessarily repeats violent decay of BEC into localized objects. Associated with this process, gravitational waves are expected to be produced. The energy emission rate of the gravitational wave from the moving body can be calculated from the formula

\[
\frac{dE}{dt} = \frac{G}{45c^5} \left( \frac{d^3 D_{\alpha \beta}}{dt^3} \right)^2
\]

where

\[
D_{\alpha \beta} = \int \rho \left( 3x_\alpha x_\beta - r^2 \delta_{\alpha \beta} \right) dV
\]

is the quadrupole of the whole mass distribution. Suppose the object of linear size \( R \) collapses with typical speed \( v \). Then the total energy emitted would be the integration

\[
\int \frac{dE}{dt} dt = \frac{G}{45c^5} \left( \frac{d^3 D_{\alpha \beta}}{dt^3} \right)^2
\]
of the above formula during the collapsing timescale:

\[ E \approx \frac{dE}{dt} \Delta t \approx \left( \frac{GM^2}{R^2} \frac{v^6}{c^2} \right) \left( \frac{R}{v} \right) = \frac{GM^2}{R} \left( \frac{v}{c} \right)^5. \]  (5.3)

This is roughly the gravitational potential energy of the extended object multiplied by the 'efficiency' \((v/c)^5\). We now estimate the possible remnant gravitational waves in the present background sky, for several cases below.

(a) If we adopt the mass of the boson as about 1 eV, then from equation (4.24) the preferred scale is \( R = 10^{23} \) cm, \( M = 1.6 \times 10^{11} M_\odot \). If we tentatively assume \( v = c/10 \), then

\[ E \approx 10^{53} \text{ erg} \]  (5.4)

which should be compared with the total rest energy of a star: \( M_\odot = 10^{54} \) erg. If we further assume that the first BEC collapse took place at redshift \( z \approx 20 \), then the present energy density of the gravitational wave becomes

\[ \rho_{gr,z=0} = \rho_{gr,z=20} \left( \frac{20}{R} \right)^4 = \frac{E}{R^3} \left( \frac{20}{R} \right)^4 = 10^{-21} \frac{\text{erg}}{\text{cm}^3}. \]  (5.5)

which should be compared with the total energy density at present, \( \rho_{cr} = 10^{-29} \) g cm\(^{-3} \) = \( 10^{-8} \) erg cm\(^{-3} \). Thus

\[ \Omega_{gr,z=0} = 10^{-13}. \]  (5.6)

Since the strain \( h \) associated with the gravitational wave is related to

\[ \Omega_{gw} = \frac{\omega^2 h^2}{12 H_0^2}, \]  (5.7)

where \( \omega \) is the frequency of the wave, we have

\[ h \approx 10^{-11} \quad \text{for} \ \omega \approx (30 \text{ kpc})^{-1} = 10^{-12} \text{ Hz}, \]  (5.8)

\[ h \approx 10^{-26} \quad \text{for} \ \omega \approx 10^3 \text{ Hz}, \]  (5.9)

where the frequency is estimated by naive extrapolation. This should be compared with the present limit of the gravitational wave \( h \approx 10^{-21} \) for \( \omega \approx 10^8 \) Hz.

The gravitational wave background formed during inflation is estimated [15] as

\[ \Omega_{gw} = (H/m_{pl})^2 \Omega_r \approx \left( 10^{-5} \right)^2 \left( 10^{-4} \right) = 10^{-14}, \]

and the associated strain is

\[ h \approx 10^{-27} \quad \text{for} \ \omega \approx 10^3 \text{ Hz}. \]

The gravitational background formed during the oscillation of the cosmic string [15] is

\[ \Omega_{gw} = 100 \left( G \mu/c^2 \right) \Omega_r \approx 100 \times 10^{-5} \times 10^{-4} = 10^{-7}. \]

(b) If we adopt the mass of the boson as about \( 10^{-3} \) eV, then from equation (4.26) the preferred scale is \( R = 10^8 \) cm, \( M = 10^{-1} \) g. Then the present energy density of the gravitational wave becomes

\[ \rho_{gr,z=0} = 10^{-51} \frac{\text{erg}}{\text{cm}^3}, \quad \Omega_{gr,z=0} = 10^{-43}. \]  (5.10)
mainly at frequency $\omega \approx (10^8 \text{ cm})^{-1} \approx 10^3 \text{ Hz}$, which is totally small and would be never be detected.

The overall amount of energy density, as estimated in equation (5.6), will not affect the present standard cosmology. However, the strain, as estimated in equation (5.8), may have a chance to be detected as well as the case of gravitational waves formed in the inflationary stage.

### 5.2. Log-$z$ periodicity

The BEC collapses do not take place randomly but they are periodic events in the logarithm of cosmic time. As argued in section 2, the Bose gas density is simply reduced as $\rho_g \propto e^{-\Gamma t}$ in the over-hill regime since the condensation speed $\Gamma$ is faster than the cosmic dilution timescale. Therefore $\rho_g$ is simply transformed into $\rho_\phi$. Just after each collapse, new BE condensation always begins from $\phi = 0$ until it reaches some critical value $\rho_{\phi}^{cr} = O(1)V_{max} \approx m^4/(-\lambda)$. Therefore the condensation energy density behaves as $[\rho_g(t_0) - \rho_g(t)] \mod \rho_{\phi}^{cr}$ and

$$\rho_\phi (t) \approx [\rho_g (t_0) - \rho_g (t)] \mod \rho_{\phi}^{cr} \approx [\rho_g (t_0) \left(1 - e^{-\Gamma t}\right)] \mod \rho_{\phi}^{cr},$$

where $t_0$ is the time when the first condensation begins.

Accordingly, we expect that each BEC collapse takes place after the time interval $\Delta t$ from the preceding collapse at time $t$, where $\Delta t$ is determined by the condition

$$\rho_{\phi}^{cr} = \rho_g (t_0) \left(e^{-\Gamma t} - e^{-\Gamma(t+\Delta t)}\right).$$

This implies that the occurrence of the BEC collapse is periodic in the logarithm of time, $\log(t)$. If the cosmic expansion is a power law in time, i.e. $a(t) \propto t^{\text{const}}$, then $\log(t)$ periodicity directly implies $\log(a)$ and $\log(z)$ periodicities. For example, in the typical numerical calculation in figure 2, BEC collapse takes place at $z = \{33.2, 20.2, 13.3, 8.89, 5.50\}$, which is almost log-periodic.

This log($z$) periodicity is a general consequence of our model, provided that the over-hill regime and the BEC repeats several times. Furthermore, this periodic BEC collapse may leave its trace in the nonlinear regime in forms such as the discrete scale invariance or the hierarchical structure in the universe, provided the scale is appropriately chosen.

As was discussed in the previous section, the preferred scale of BEC collapse only depends on the basic parameters of the model; $l_\ast \approx m_{pl}/m^2$ for $m > 1 \text{ eV}$ and $l_\ast \approx (m^2\Gamma)^{1/3}$ for $m < 10^{-3} \text{ eV}$. Then the sequence of the cluster produced through BEC collapses has the series of mass proportional to $z^3$, where $z$ is the redshift of the collapse. Thus, the cluster mass also has the log periodicity, and the largest cluster is formed at the first BEC collapse. The most interesting case would be when the boson mass is about 1 eV. Then the hierarchy of galaxies is formed, for example in figure 2, $M/M_\odot = \{1.6 \times 10^{11}, 3.6 \times 10^{10}, 1.0 \times 10^{10}, 3.1 \times 10^9, 7.2 \times 10^8\}$. Details of the arguments on the observational size are a future problem. If the boson mass is about $10^{-3} \text{ eV}$, then the hierarchy would yield no interesting scales with respect to the large scale structure formation.
6. Summary

We have developed the cosmological model based on the Bose–Einstein condensation (BEC) from various points of view. This BEC cosmology is characterized by (1) the unification of DE and DM, (2) their mutual conversion, (3) quantum mechanical condensation as a novel phase of DE, (4) violent collapse of DE, (5) log \( z \) periodicity of the DE collapses, (6) black hole formation from DE, (7) formation of localized objects in the high redshift regime, (8) inevitable final phase of accelerated expansion, etc. These have been briefly explained in section 2. We have examined this model in detail especially with respect to its robustness and instability in this paper.

We have first examined the robustness of the model in section 3. By assuming thermal equilibrium of the boson field and the radiation in the early universe, we set the upper limit of the mass for the boson, which turned out to be of the order of eV. Thus we have shown that the BEC takes place very naturally in the universe. Moreover, if the boson has not been in equilibrium with radiation in the past, then even any value of mass is allowed. Next we demonstrated that the BEC takes place in the wide range of microscopic parameters of boson mass \( m \), self-coupling \( \lambda \) and condensation rate \( \Gamma \). Furthermore, we have revealed that the time dependence of \( \Gamma \) does not qualitatively affect the BEC model. Thus we have been able to show the robustness and the naturalness of the cosmological BEC model.

Then we have examined the instability of the BEC in section 4. We have calculated a preferred scale of the structure formed after the BEC collapse associated with this instability. This scale turns out to be quite sensitive to the mass of the boson. If the boson mass is about 1 eV, the preferred scale is \( l_\ast \approx m_{pl}/m^2 \) and it is about a galaxy size. If the boson mass is about \( 10^{-3} \) eV, the scale is \( l_\ast \approx (m^2 \Gamma)^{1/3} \) and it is about a gram. If the boson mass is much smaller, there is a possibility that DM is formed as the BEC, and the preferred scale can be of galaxy or cluster size. We have also estimated the amount of the remnant gravitational wave associated with the BEC collapse. It turns out to be marginally observable in the near future if the parameters of our model are most optimized; otherwise it is simply too small.

The present results, as a whole, suggest that the boson mass is probably of order \( 10^{-3} \) to 1 eV. These mass scales are so tiny that no empirical evidence for such bosons has been found yet. However, it is potentially interesting that these mass scales are the same order as the neutrino masses. Therefore it would be natural to consider that the boson particle is a composite of neutrino–neutrino (or neutrino–antineutrino) pairs \([13,14]\) though this requires further investigations including the problem of how the Fermi surface can be stable for such a tiny mass. We would like to further extend the cosmological BEC model in our next paper.

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