Confluent Heun equations: convergence of solutions in series of Coulomb wavefunctions

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Received 24 September 2012, in final form 9 January 2013
Published 6 February 2013
Online at stacks.iop.org/JPhysA/46/085203

Abstract

The Leaver solutions in series of Coulomb wavefunctions for the confluent Heun equation are given by two-sided infinite series, that is, by series where the summation index \( n \) runs from minus to plus infinity (Leaver 1986 J. Math. Phys. 27 1238). First we show that, in contrast to the D’Alembert test, under certain conditions the Raabe test ensures that the domains of convergence of these solutions include an additional singular point. We also consider solutions for a limit of the confluent Heun equation. For both equations, new solutions are generated by transformations of variables. Finally, we discuss the time dependence of the Klein–Gordon equation in two cosmological models and the spatial dependence of the Schrödinger equation to a family of quasi-exactly solvable potentials. For a subfamily of these potentials, we obtain infinite-series solutions which converge and are bounded for all values of the independent variable, in opposition to a common belief.

PACS numbers: 02.30.Hq, 03.65.--w, 03.65.Ge

1. Introduction

In 1986, Leaver [1] found two types of solutions in series of confluent hypergeometric functions for the confluent Heun equation (CHE) and presented a limit procedure to generate solutions for the double-confluent Heun equation (DCHE) out of solutions for the CHE. Later on, we found that there are two other physically relevant equations whose solutions can also be derived from the Leaver solutions for the CHE and DCHE by means of a procedure called Whittaker–Ince limit [2–4]. Furthermore, from the solutions of the CHE and/or DCHE, we can find the solutions for the Mathieu, Whittaker–Hill and spheroidal equations [1, 5].

In view of the above connections, from the beginning we establish the convergence properties of Leaver’s solutions. We consider only the expansions in series of Coulomb wavefunctions which are given by a set of three solutions, one in series of regular confluent hypergeometric functions and two in series of irregular functions. By redefining the Coulomb
functions, we avoid difficulties arising from the Leaver definitions and find that the convergence of the solutions for the CHE and its Whittaker–Ince limit follows from the Raabe test. Furthermore, we investigate the transformations of these solutions.

First we write the aforementioned equations, present the connections among them and call attention to the fact that there are three types of series expansions whose convergence requires different treatments. After that we introduce the D’Alembert and Raabe tests for convergence and outline the structure of the paper.

The Leaver form for the CHE is [1]

\[ z(z - z_0) \frac{d^2U}{dz^2} + (B_1 + B_2z) \frac{dU}{dz} + [B_3 - 2\eta \omega(z - z_0) + \omega^2z(z - z_0)]U = 0, \] (1)

where \( B_i, \eta \) and \( \omega \) are constants. The points \( z = 0 \) and \( z = z_0 \) (if \( z_0 \neq 0 \)) are regular singular points, whereas \( z = \infty \) is an irregular point. Since \( z_0 \) is free, by taking \( z_0 = 0 \) Leaver obtained the DCHE

\[ z^2 \frac{d^2U}{dz^2} + (B_1 + B_2z) \frac{dU}{dz} + (B_3 - 2\eta \omega z + \omega^2z^2)U = 0, \quad \{B_1 \neq 0, \ \omega \neq 0\} \] (2)

in which case \( z = 0 \) and \( z = \infty \) are both irregular singularities.

In addition, the CHE and the DCHE admit a (Whittaker–Ince) limit which changes the nature of the singularity at \( z = \infty \), keeping the other singular points unaltered. This limit is given by [2, 3]

\[ \omega \to 0, \ \eta \to \infty, \text{ such that } 2\eta \omega = -q \text{ (Whittaker–Ince limit)}, \] (3)

where \( q \) is a nonvanishing constant. The Whittaker–Ince limit of the CHE is

\[ z(z - z_0) \frac{d^2U}{dz^2} + (B_1 + B_2z) \frac{dU}{dz} + [B_3 + q(z - z_0)]U = 0, \quad \{q \neq 0\} \] (4)

(if \( q = 0 \), then this equation can be transformed into a hypergeometric equation), while the limit of the DCHE is

\[ z^2 \frac{d^2U}{dz^2} + (B_1 + B_2z) \frac{dU}{dz} + (B_3 + qz)U = 0, \quad \{q \neq 0, \ B_1 \neq 0\} \] (5)

(if \( q = 0 \) and/or \( B_1 = 0 \), then the equation degenerates into a confluent hypergeometric equation or simpler equations). This last equation also follows from equation (4) when \( z_0 = 0 \) (Leaver’s limit).

The Mathieu equation as well as the Whittaker–Hill and the spheroidal equations have been studied by themselves, but they are particular cases of the above equations. The Mathieu equation reads [6]

\[ \frac{d^2w}{da^2} + \sigma^2[a - 2k^2 \cos(2\sigma u)]w = 0 \quad \text{(Mathieu equation)}, \] (6)

where \( a \) and \( k \) are constants, while \( \sigma = 1 \) or \( \sigma = i \) for the Mathieu or the modified Mathieu equation, respectively. This equation is transformed into particular instances of equations (1), (2) and (4) by the substitutions of variables [3, 5]. The Whittaker–Hill equation (WHE) can be written in the form [7, 8]

\[ \frac{d^2W}{du^2} + \varsigma^2 \left[ \beta - \frac{1}{8} \xi^2 - (p + 1)\xi \cos(2\varsigma u) + \frac{1}{8} \xi^2 \cos(4\varsigma u) \right] W = 0, \quad \text{(WHE)}, \] (7)

where \( \beta, \xi \) and \( p \) are parameters; if \( u \) is a real variable, then this represents the WHE when \( \varsigma = 1 \) and the modified WHE when \( \varsigma = i \). The WHE reduces to the CHE (1) and DCHE (2) by the substitutions [3, 5]. Finally, the spheroidal equation reads [9]

\[ \frac{d}{dy} \left[ (1 - y^2) \frac{dS(y)}{dy} \right] + \left[ \lambda + y^2(1 - y^2) - \frac{\mu^2}{1 - y^2} \right] S(y) = 0, \] (8)
where $\gamma$, $\lambda$, and $\mu$ are constants. The substitutions (58a) transform this into a special case of the CHE (1).

On the other hand, in general, there are solutions given by three different types of series, called the two-sided infinite series, one-sided infinite series and finite series. These take, respectively, the forms

$$
\sum_n a_n f_n^\nu(z) := \sum_{n=-\infty}^{\infty} a_n f_n^\nu(z), \quad \sum_{n=0}^{N} b_n g_n(z), \quad \sum_{n=0}^{N} b_n g_n(z),
$$

(9)

where $f_n^\nu(z)$ and $g_n(z)$ are functions of the independent variable $z$, $N$ is a non-negative integer and $\nu$ is a parameter which does not appear in the differential equations. In the present case, the series coefficients $a_n$ and $b_n$ satisfy three-term recurrence relations. No finite-series solutions are known for the Mathieu equation (6) nor for the Whittaker–Ince limit (5) of the DCHE.

Two-sided infinite series, the only considered by Leaver, are necessary to ensure the convergence of solutions of equations in which there is no free parameter, as in scattering problems [2] or in some wave equations in curved spacetimes [10]. In such cases, the parameter $\nu$ must be determined as solutions of a transcendental (characteristic) equation. However, when truncated on the left ($n \geq 0$), the two-sided infinite series give the one-sided infinite series which are useful for equations with a free parameter; in turn, these lead to solutions given by the finite series if the parameters of the equation satisfy certain constraints.

Finite-series solutions are suitable for quasi-exactly solvable (QES) problems, that is, for quantum-mechanical problems where one part of the energy spectrum and the respective eigenfunctions can be computed explicitly [11, 12]. For QES problems obeying equations of the Heun family [3], that part of the spectrum may be derived from finite-series solutions if these are known. Indeed, a problem is also said to be QES if it admits solutions given by the finite series whose coefficients necessarily satisfy three-term or higher order recurrence relations [13], and is said to be exactly solvable if its solutions can be expressed by hypergeometric functions.

The convergence of the two-sided infinite series is obtained from the limits

$$
L_1(z) = \lim_{n \to \infty} \left| \frac{a_{n+1} f_{n+1}^\nu(z)}{a_n f_n^\nu(z)} \right|, \quad L_2(z) = \lim_{n \to -\infty} \left| \frac{a_{n-1} f_{n-1}^\nu(z)}{a_n f_n^\nu(z)} \right|.
$$

(10)

By the D’Alembert test, the series converges in the intersection of the regions for which $L_1 < 1$ and $L_2 < 1$, and diverges otherwise (if $L_1 = L_2 = 1$, the test is inconclusive). Leaver’s definitions for the Coulomb wavefunctions lead to ratios between terms presenting square roots (except if $\eta = 0$) which make it difficult to deal with the convergence tests. To avoid this problem, we use alternative definitions that, in addition, permit us to apply the Raabe test for the solutions of the CHEs. By the Raabe test [14, 15], if

$$
L_1(z) = 1 + \frac{A}{|n|}, \quad L_2(z) = 1 + \frac{B}{|n|},
$$

(11)

where $A$ and $B$ are constants, then the series converges in the region where $A < -1$ and $B < -1$, and diverges otherwise (if $A = B = -1$, the test is inconclusive). For the one-sided series, the convergence may be enhanced since we use only the limit $L_1$, while for the finite series the convergence must be decided from the behaviour of each term of the series.

Furthermore, by using transformations of variables we find four sets of two-sided solutions instead of one set as by Leaver. By the Raabe test, under certain conditions these solutions converge absolutely for $|z| \geq |z_0|$ or $|z - z_0| \geq |z_0|$ rather than for $|z| > |z_0|$ or $|z - z_0| > |z_0|$; the one-sided solutions given by the series of regular confluent hypergeometric functions converge for $|z| \geq 0$. Nevertheless, the behaviour of each solution for $z \to \infty$ must be analysed carefully because, in computing $L_1(z)$ and $L_2(z)$, we assume that $z$ is bounded. We
also have to examine the behaviour of the solutions at the finite singular points because the series appear multiplied by factors which may become unbounded at such points.

For brevity, here we do not consider all the above points. Indeed, we deal only with the two-sided solutions for the CHE and its Whittaker–Ince limit. However, for future reference, elsewhere [16] we provide the one-sided solutions as well as the solutions for the DCHE (2) and its limit (5).

In section 2.1, we discuss the two-sided infinite expansions for CHE (1), and in section 2.2 we consider the corresponding Whittaker–Hill limit. For the spheroidal equation, we obtain the Meixner solutions in series of Bessel functions [9] instead of the Chu and Stratton solutions [17] mentioned by Leaver. For the Mathieu equation, we recover known solutions, but now the convergence is improved by the Raabe test.

In section 3, we consider examples which illustrate the consequences of the Raabe test. In particular, we show that the Schrödinger equation for some QES potentials admits infinite-series solutions which are convergent and bounded for all values of the independent variable. Thus, in addition to the energy levels resulting from the finite series, in principle, it is possible to obtain additional energy levels as solutions of characteristic equations corresponding to the infinite series.

In section 4, there are some final considerations. Appendix A gives the normalization used for the Coulomb functions and takes the case \( \eta = 0 \) as a criterion to decide in favour of one of two possibilities for the ratio between successive Coulomb functions. The derivation of the recurrence relations for the series coefficients is given in appendix B.

## 2. The two-sided series expansions

In this section, we examine separately the two-sided expansions for the CHE and for its Whittaker–Ince limit. In this limit, the expansions in series of Coulomb functions give solutions in series of Bessel functions. The solutions of the CHE with \( \eta = 0 \) are also expressible by series of Bessel functions. In all cases, given an initial set of solutions, new sets are generated by transformations of variables which preserve the form of the differential equations. Note that the linear independence of the functions used as basis for the series expansions will impose restrictions on the characteristic parameter \( \nu \) and/or on some parameters of the differential equations.

In equations (60)–(62), we recover the Meixner solutions for the spheroidal equation as particular cases of the solutions for the CHE, while in equations (81a)–(81e), we recover the usual solutions in series of Bessel functions for the Mathieu equation as particular cases of the solutions for the Whittaker–Ince limit of the CHE.

### 2.1. Solutions for the CHE

The initial set of solutions, \( U_1(z) \), is reconstructed in appendix B. It reads

\[
U_1(z) = z^{-\frac{\nu}{2}} \sum_n b_n^1 U_{n+\nu}(\eta, \omega z), \quad U_{n+\nu}(\eta, \omega z) = (\phi_{n+\nu}, \psi_{n+\nu}')(\eta, \omega z), \tag{12}
\]

where \( \sum_n \) denotes the two-sided series, \( \phi_{n+\nu} \) and \( \psi_{n+\nu} \) represent the Coulomb wavefunctions defined in equations (A.12) and (A.13), and the coefficients \( b_n^1 \) satisfy three-term recurrence relations. So, we have a set of three expansions, one in series of regular confluent hypergeometric functions and two in series of irregular functions. This set corresponds to Leaver’s solutions who used the definitions (A.14) and (A.15) for the Coulomb functions. In addition, if \( U(z) = U(B_1, B_2, B_3; z_0, \omega, \eta; z) \) denotes an arbitrary solution of the CHE, we
can find other solutions by means of the transformations \( T_1, T_2, T_3 \) and \( T_4 \) which operate as \([3, 5]\)

\[
T_1 U(z) = z^{+B_1/z_0} U(C_1, C_2, C_3; z_0, \omega, \eta; z), \\
T_2 U(z) = (z - z_0)^{+B_2-B_3/z_0} U(B_1, B_2, B_3; z_0, \omega, \eta; z), \\
T_3 U(z) = U(B_1, B_2, B_3; z_0, -\omega, -\eta; z), \\
T_4 U(z) = U(-B_1 - B_2z_0, B_2, B_3 + 2\eta_0z_0; z_0, -\omega, \eta; z_0 - z),
\]

where

\[
C_1 = -B_1 - 2z_0, \quad C_2 = 2 + B_2 + \frac{2B_1}{z_0}, \quad C_3 = B_3 + \left(1 + \frac{B_1}{z_0}\right)\left(B_2 + \frac{B_1}{z_0}\right).
\]

These transformations allow constructing a group with four sets of the two-sided series \( U_i \) \((i = 1, \ldots, 4)\) where coefficients \( b_n^i \) satisfy recurrence relations having the form

\[
a_n^i b_n^i + b_n^i b_n^i + y_n^i b_{n-1}^i = 0, \quad [-\infty < n < \infty],
\]

where \( a_n^i, b_n^i \) and \( y_n^i \) depend on the parameters of the differential equation as well as on \( \nu \) and \( n \). These relations lead to transcendental (characteristic) equations given as a sum of two infinite continued fractions. By omitting the superscripts of \( a_n^i, b_n^i \) and \( y_n^i \), the characteristic equations read

\[
\beta_0 = \frac{\alpha_{-1} \gamma_0 \alpha_{-2} \gamma_1 \alpha_{-3} \gamma_2 \cdots + \alpha_0 \gamma_1 \alpha_1 \gamma_2 \cdots}{\beta_{-1} \gamma_0 \beta_{-2} \gamma_1 \beta_{-3} \gamma_2 \cdots}
\]

which are equivalent to the vanishing of the determinants of infinite tridiagonal matrices, as in equation (32). If the CHE has no free parameter, then equation (16) may be used to find the possible values of \( \nu \) (characteristic parameter); if the CHE has an arbitrary parameter, then equation (16) permits one to find the values of that parameter corresponding to suitable values of \( \nu \).

To analyse the properties of the solutions, we write explicitly each of the three solutions, instead of using the abbreviated form (12). Thus, we denote by \( U_1 = (U_1, U_1^+, U_1^-) \) the solutions associated, respectively, with the functions \((\phi_{\nu+}, \psi_{\nu+}, \psi_{\nu+})\). This gives the solutions (19) which, by the transformations (13), generate the three sets of solutions that have not been considered by Leaver. The four sets of the two-sided solutions are denoted by

\[
U_i(z) = U_i(z), U_i^+(z), U_i^-(z), \quad i = 1, \ldots, 4,
\]

if \( \eta \neq 0 \); if \( \eta = 0 \), the notation is given in equation (47). These solutions correspond to eight sets of one-sided solutions \( U_i \) which are denoted by

\[
\bar{U}_i(z) = \bar{U}_i(z), \bar{U}_i^+(z), \bar{U}_i^-(z), \quad i = 1, \ldots, 8,
\]

and do not depend on \( \nu \). In fact they are generated by expressing the parameter \( \nu \) of each \( U_i \) as two different functions of the parameters of the CHE. The convergence of the solutions \( \bar{U}_i \) is obtained by considering only the limits \( n \to \infty \) in the computations given in this paper.

2.1.1. The four sets of the solutions. Explicitly, the first set \( U_1 \), given in equation (12), reads

\[
\begin{align*}
U_1(z) &= e^{i\omega z} \sum_n b_n^1 \Phi \left[ n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2i\omega \right] \\
U_1^+(z) &= e^{i\omega z} \sum_n b_n^1 \Phi \left[ n + \nu + 1 - i\eta, 2n + 2\nu + 2; 2i\omega \right] \\
U_1^-(z) &= e^{-i\omega z} \sum_n b_n^1 \Phi \left[ n + \nu + 1 - i\eta, 2n + 2\nu + 2; -2i\omega \right].
\end{align*}
\]
In fact, from the explicit forms of these sets, one verifies that remaining transformations allow us to form a group constituted by four sets of solutions, \( \beta \) namely,

\[
\alpha_n^1 = \frac{2i\omega_0 [n + v + 2 - \frac{B_1}{2}]}{(2n + 2v + 2)(2n + 2v + 3)}, \quad \beta_n^1 = -B_3 - \eta\omega_0 - \left( n + v + 1 - \frac{B_2}{2} \right) \left( n + v + \frac{B_2}{2} \right) - \frac{\eta\omega_0 [B_2 - 2]}{(2n + 2v)(2n + 2v + 2)}, \quad (20)
\]

\[
\gamma_n^1 = -\frac{2i\omega_0 [n + v + \frac{B_2}{2} - 1]}{(2n + 2v - 1)(2n + 2v)}. \quad (21)
\]

By applying the transformation \( T_3 \) on \( U_1 \), we find the equivalence

\[
T_3[U_1(z), U_1^+(z), U_1^-(z)] \Leftrightarrow [U_1(z), U_1^+(z), U_1^-(z)]. \quad (22)
\]

Precisely, we find \( T_3 [U_1, U_1^+, U_1^-] = [\tilde{U}_1, \tilde{U}_1^+, \tilde{U}_1^-] \) with

\[
\tilde{U}_1(z) = e^{i\omega_0} \sum_n \frac{\tilde{b}_n^1 [-2i\omega_0 [n + 1 + i\eta, 2n + 2v + 2; -2i\omega_1] \Phi (n + v + 1 + i\eta, 2n + 2v + 2; -2i\omega_0),}{\Gamma[2n + 2v + 2]}
\]

\[
\tilde{U}_1^\pm(z) = e^{i\pm\omega_0} \sum_n \frac{\tilde{b}_n^1 [-2i\omega_0 [n + v + 1 \mp i\eta, 2n + 2v + 2; \mp2i\omega_1] \Psi (n + v + 1 \pm i\eta, 2n + 2v + 2; \mp2i\omega_1) ,}{\Gamma[n + v + 1 \mp i\eta]}
\]

where the recurrence relations for \( \tilde{b}_n^1 \) are

\[
-\alpha_n^1 \tilde{b}_{n+1}^1 + \beta_n^1 \tilde{b}_n^1 - \gamma_n^1 \tilde{b}_{n-1}^1 = 0.
\]

Up to a multiplicative constant independent of \( n \), we can set \( \tilde{b}_n^1 = (-1)^n b_n^1 \) in order to establish relation (21). Thus, the transformation \( T_3 \) is ineffective in the present case. The remaining transformations allow us to form a group constituted by four sets of solutions, namely,

\[
U_1(z), \quad U_2(z) = T_2 U_1(z); \quad U_3(z) = T_4 U_1(z), \quad U_4(z) = T_5 U_2(z) = T_1 U_3(z). \quad (23)
\]

In fact, from the explicit forms of these sets, one verifies that \( T_1 \) does not generate new solutions when applied on \( U_1 \) and \( U_2 \); similarly, \( T_5 \) has no effect on \( U_3 \) and \( U_4 \).

The three sets of solutions are written below by the preceding transformations are written below:

\[
U_2(z) = f_2(z) \frac{2i\omega_0 [n + v + 1 + i\eta, 2n + 2v + 2; -2i\omega_1]}{\Gamma[2n + 2v + 2]}
\]

\[
U_3^\pm(z) = f_2(z) \frac{2i\omega_0 [n + v + 1 \pm i\eta, 2n + 2v + 2; \mp2i\omega_1] \Psi (n + v + 1 \pm i\eta, 2n + 2v + 2; \pm2i\omega_1),}{\Gamma[n + v + 1 \mp i\eta]} \quad (24)
\]

where \( f_2 = f_2(z) = e^{i\omega_0 [z^n + (B_2/2)]} (z - z_0)^{1-B_2-(B_1/2)} \). The coefficients for the recurrence relations are given by

\[
\alpha_n^2 = \frac{2i\omega_0 [n + v + 1 + \frac{B_2}{2}]}{(2n + 2v + 2)(2n + 2v + 3)}, \quad \beta_n^2 = \beta_n^1,
\]

\[
\gamma_n^2 = -\frac{2i\omega_0 [n + v + 1 - \frac{B_2}{2}]}{(2n + 2v - 1)(2n + 2v)}. \quad (25)
\]
The transformation $T_4$ acting on $\mathbb{U}_1$ gives the set $\mathbb{U}_3 = \mathbb{U}_3(z)$, namely,

\begin{equation}
U_3 = f_2 e^{i\omega(z - z_0)} \sum_{n} b_n^2 \frac{[2i\omega(z-z_0)]^n}{\Gamma[2n + 2v + 2]} \Phi [n + v + 1 + i\eta, 2n + 2v + 2; -2i\omega(z - z_0)],
\end{equation}

\begin{equation}
U_3^{\pm} = f_2 e^{i\omega(z - z_0)} \sum_{n} b_n^2 \frac{[2i\omega(z-z_0)]^n}{\Gamma[n + v + 1 \mp i\eta]} \Psi [n + v + 1 \pm i\eta, 2n + 2v + 2; \mp 2i\omega(z - z_0)],
\end{equation}

where $f_2 = f_2(z) = (z - z_0)^v - 2 \nu + 1 - (B_1/2)$ and, in the recurrence relations,

\begin{equation}
\alpha_n^{\pm} = -\frac{2i\omega z_0 [n + v + 2 - \frac{B_1}{2}]}{(2n + 2v + 2)(2n + 2v + 3)}, \quad \beta_n^{\pm} = \beta_n^1,
\end{equation}

\begin{equation}
\gamma_n^{\pm} = \frac{2i\omega z_0 [n + v - 1 + \frac{B_1}{2}]}{(2n + 2v - 1)(2n + 2v)}. \quad (25)
\end{equation}

The fourth set, obtained by applying $T_1$ on $U_3$, reads

\begin{equation}
U_4 = f_4 e^{i\omega(z - z_0)} \sum_{n} b_n^2 \frac{[2i\omega(z-z_0)]^n}{\Gamma[2n + 2v + 2]} \Phi [n + v + 1 + i\eta, 2n + 2v + 2; -2i\omega(z - z_0)],
\end{equation}

\begin{equation}
U_4^{\pm} = f_4 e^{i\omega(z - z_0)} \sum_{n} b_n^2 \frac{[2i\omega(z-z_0)]^n}{\Gamma[n + v + 1 \mp i\eta]} \Psi [n + v + 1 \pm i\eta, 2n + 2v + 2; \mp 2i\omega(z - z_0)],
\end{equation}

where $f_4 = f_4(z) = z + (B_1/2) - (B_1/2)$ and, in recurrence relations for $b_n^4$,

\begin{equation}
\alpha_n^{4} = -\frac{2i\omega z_0 [n + v + \frac{B_1}{2}]}{(2n + 2v + 2)(2n + 2v + 3)}, \quad \beta_n^{4} = \beta_n^4,
\end{equation}

\begin{equation}
\gamma_n^{4} = \frac{2i\omega z_0 [n + v + 1 - \frac{B_1}{2}]}{(2n + 2v - 1)(2n + 2v)}. \quad (27)
\end{equation}

If there is no free parameter in the CHE, $\nu$ must be determined as solutions of a characteristic equation. However, by considering the form of the solutions and respective recurrence relations for the series coefficients, we find that

\begin{equation}
2v \neq \nu \mp i\eta \quad (29)
\end{equation}

cannot be integers for the two-sided series. The restriction $\nu \mp i\eta \neq \text{integer}$ ensures that factors 1/Γ $(n + v + 1 \pm i\eta)$ which appear in $U_3^{\pm}(z)$ are not zero for any value of $n$ as well as ensures that the factors $(n + v + i\eta)(n + v - i\eta)$ in $\gamma_n^{\pm}$ do not vanish for any $n$. In fact, such restriction is necessary to have two-sided infinite series for the three solutions in each of the four sets $\mathbb{U}_i$.

The condition $2v \neq \text{integer}$ is necessary in order to avoid two terms linearly dependent on the series of $U_3^{\pm}(z)$. Indeed, suppose that $2v = \text{integer}$ in the solutions $U_3^{\pm}(z)$. These are series expansions in terms of

\begin{equation}
R_n^{\pm}(z) = [-2i\omega]^{n+v+1} \Psi [n + v + 1 \pm i\eta, 2n + 2v + 2; \mp 2i\omega z].
\end{equation}

By setting $n = n_1$ and using (A.3), we find

\begin{equation}
R_n^{\pm}(z) = \pm (-1)^{v+y} [-2i\omega]^{-n_1-v} \Psi [-n_1 - v \pm i\eta, -2n_1 - 2v; \mp 2i\omega z].
\end{equation}

Hence, $R_n^{\pm}$ and $R_n^{\pm}$ are proportional to each other for some $n = n_2$ such that $n_1 + n_2 + 1 = -2v$. Similar results are found for the other solutions $U_i^{\pm}(z)$. On the other hand, by supposing that $2v \neq \text{integer}$, the functions $\Psi(a, c; y)$ which appear in $U_i(z)$ are well defined because the
parameter \( c = 2n + 2v + 2 \) cannot be a negative integer. Nevertheless, see the section containing equations (55) and (56) for some remarks concerning the case \( \eta = 0 \).

According to equations (A.4), if the conditions (29) are true, then the three hypergeometric functions are linearly independent and each one can be written as a combination of the others by means of (A.5). In this case, in a common region of validity, we can write one solution of a doubly infinite \(( -\infty < n < \infty )\) if, in addition to (29), \( v \) satisfies the restrictions that

\[
 v \pm \frac{B_2}{2} \quad \text{and} \quad v \pm \left( \frac{B_1}{z_0} + \frac{B_2}{2} \right) \quad \text{are not integers.} \tag{30}
\]

These conditions ensure that \( \alpha _n^{(i)} \) and \( \gamma _n^{(i)} \) do not vanish for any value of \( n \). In effect, if \( \alpha _n^{(i)} = 0 \) for some \( n = N_1 \), then the series should begin at \( n = N_1 + 1 \) in order to ensure the validity of the theory of the three-term recurrence relations; for the same reason, if \( \gamma _n^{(i)} = 0 \) for some \( n = N_2 \), then the series should terminate at \( n = N_2 - 1 \).

Note that, for two-sided solutions,

\[
 a_n^{(i)} = a_n^{(i)}, \quad \alpha _n^{(i)} = \alpha _n^{(i)} \quad [i = 2, 3, 4]. \tag{31}
\]

Thence, equation (16) implies that all the solutions \( \cup_i \) satisfy the same characteristic equation and, consequently, the parameter \( v \) takes the same values in all solutions. In addition, as noted by Leaver, the characteristic equations are periodic in \( v \) with period 1. In effect, in order to indicate that the coefficients depend on \( v \) we rewrite the recurrence relations as

\[
 \alpha _n^{(i)} b_{n+1} + \beta _n^{(i)} b_n + \gamma _n^{(i)} b_{n-1} = 0 \quad \text{or as the following tridiagonal matrix equation:}
\]

\[
 \begin{bmatrix}
 -\gamma _n^{(v)} & \beta _n^{(v)} & \alpha _n^{(v)} \\
 \beta _n^{(v)} & \gamma _{n+1}^{(v)} & \beta _{n+1}^{(v)} \\
 \alpha _n^{(v)} & \beta _{n+1}^{(v)} & \gamma _{n+2}^{(v)} \\
 \end{bmatrix} \begin{bmatrix}
 b_{n}^{(v)} \\
 b_{n+1}^{(v)} \\
 b_{n+2}^{(v)} \\
 \end{bmatrix} = 0 \quad [-\infty < n < \infty], \tag{32}
\]

where \( 0 \) denotes the null column vector. Thence, the values for \( v \) may be determined by requiring that the determinant of the above matrix vanishes. However, as

\[
 \gamma _n^{(v+1)} = \gamma _n^{(v)}, \quad \beta _n^{(v+1)} = \beta _n^{(v)}, \quad \alpha _n^{(v+1)} = \alpha _n^{(v)}, \quad \ldots
\]

and \(-\infty < n < \infty \), the matrix and its determinant are not modified by the replacement \( v \to v + 1 \) (or \( v \to v + N \), where \( N \) is any integer).

Furthermore, if (29) and (30) are fulfilled, all coefficients can be written in terms of \( b_n^{1} \).

Up to multiplicative constants independent of \( n \), we have

\[
 \begin{align*}
 b_2 &= \frac{\Gamma [n + v + 2 - \frac{B_1}{2}]}{\Gamma [n + v + 2]} \frac{\Gamma [n + v + 1 - \frac{B_1}{2}]}{\Gamma [n + v + 1]} b_1^{1}, \\
 b_3 &= \frac{(-1)^n \Gamma [n + v + 1 - \frac{B_1}{2}]}{\Gamma [n + v + 1 + \frac{B_1}{2}]} b_1^{1}, \\
 b_4 &= \frac{(-1)^n \Gamma [n + v + 2 - \frac{B_1}{2}]}{\Gamma [n + v + 2 + \frac{B_1}{2}]} b_1^{1}, \\
 \end{align*}
\]

(33)

As an example, we consider the solutions \( W (u) \) for the WHE (7). These may be obtained from the solutions \( U (z) \) of the CHE (1) by taking

\[
 \begin{align*}
 W (u) &= U (z), \quad z = \cos^2 (\varsigma u), \quad [\varsigma = 1, i] \quad \Rightarrow \quad z_0 = 1, \\
 B_1 &= -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = \left( \frac{p + 1}{\omega} - \frac{\partial}{4} \right), \quad i\omega = \frac{\xi}{2}, \quad i\eta = \frac{p + 1}{2} \quad \text{WHE as a CHE.} \tag{34}
\end{align*}
\]
Thus, the solutions \( U_i = (U_i, U_i^+, U_i^-) \) lead to four sets of solutions
\[
\mathcal{W}_i(u) = [W_i(u) = U_i(z), W_i^+(u) = U_i^+(z), W_i^-(u) = U_i^-(z)]. \tag{35a}
\]
In this case, the coefficients of the recurrence relations for \( b_n^i \) simplify to
\[
\alpha_n^i = \frac{i\omega}{2}, \quad \beta_n^i = -B_3 - \eta \omega \left\lfloor n + v + \frac{1}{2} \right\rfloor^2, \quad \gamma_n^i = -\frac{i\omega}{2} [n + v + i\eta][n + v - i\eta],
\]
whereas equations (33) reduce to
\[
b_o^i = (n + v + \frac{1}{2}) b_n^1, \quad b_o^i = (-1)^n b_n^1, \quad b_o^i = (-1)^n (n + v + \frac{1}{2}) b_n^1. \tag{35c}
\]

2.1.2. Convergence and asymptotic behaviours. The D’Alembert test implies two subgroups of solutions since \( U_1 \) and \( U_2 \) converge for any finite \( z \) such that \( |z| > |z_0| \), whereas \( U_3 \) and \( U_4 \) converge for \( |z - z_0| > |z_0| \). However, by the Raabe test they may converge also at \( |z| = |z_0| \) and \( |z - z_0| = |z_0| \) under the conditions
\[
|z| > |z_0| \quad \text{if} \quad \begin{cases} \text{Re} \left[ \frac{B_2 + B_1}{z_0} \right] < 1 \quad \text{in} \quad U_1, \\
|z - z_0| > |z_0| \quad \text{if} \quad \begin{cases} \text{Re} \left[ \frac{B_2}{z_0} \right] > -1 \quad \text{in} \quad U_3, \\
\text{Re} \left[ \frac{B_2 + B_1}{z_0} \right] > 1 \quad \text{in} \quad U_2; \\
\text{Re} \left[ \frac{B_1}{z_0} \right] < -1 \quad \text{in} \quad U_4. \end{cases}
\end{cases} \tag{36}
\]
where the restrictions on parameters of the equation are necessary only to ensure convergence at \( |z| = |z_0| \) or \( |z - z_0| = |z_0| \). In particular, for the solutions of the WHEs we find
\[
|\cos(\zeta u)| \geq 1 \quad \text{in} \quad \mathcal{W}_1, \quad |\cos(\zeta u)| > 1 \quad \text{in} \quad \mathcal{W}_2,
\]
\[
|\sin(\zeta u)| \geq 1 \quad \text{in} \quad \mathcal{W}_3, \quad |\sin(\zeta u)| > 1 \quad \text{in} \quad \mathcal{W}_4. \tag{37}
\]
Thence the two-sided solutions are useless for the WHE (that is, for \( \zeta = 1 \) and \( u = \text{real} \)), but may be useful for the modified WHE (\( \zeta = i, u = \text{real} \)). If \( \text{Re}[B_2 + (B_1/z_0)] = 1 \) and \( \text{Re}[B_1/z_0] = -1 \) in (36), the Raabe test becomes inconclusive in the sense that the solutions may converge or diverge at \( |z| = |z_0| \) or \( |z - z_0| = |z_0| \).

From conditions (36), we obtain the behaviour of Leaver’s solutions on the boundary of the regions of convergence, \( z = z_0 \) and \( z = 0 \). We find
\[
\lim_{z \to z_0} U_1(z) \propto \sum_{n=-\infty}^{\infty} \cdots, \quad \lim_{z \to z_0} U_2(z) \propto [z - z_0]^{1-B_2+\frac{B_1}{2}} \sum_{n=-\infty}^{\infty} \cdots,
\]
where the convergence of the series depends on \( 1 - B_2 - B_1/z_0 \) according to the first part of (36). On the other hand, for series expansions \( S(z) \) in powers of \( z - z_0 \) we find
\[
\lim_{z \to z_0} S(z) = b_0, \quad \lim_{z \to z_0} S(z) = \tilde{b}_0 [z - z_0]^{1-B_2-\frac{B_1}{2}},
\]
where now \( b_0 \) and \( \tilde{b}_0 \) are constants independent of the characteristic exponent \( 1 - B_2 - B_1/z_0 \).

So, if \( \text{Re}[1 - B_2 - B_1/z_0] > 0 \), then the two last expressions and \( U_1(z) \) are bounded and convergent at \( z = z_0 \), while the series in \( U_2(z) \) diverges at \( z = z_0 \). This fact is sufficient to reveal that the two types of series have different properties at \( z_0 \). The same conclusion is obtained from
\[
\lim_{z \to 0} U_3(z) \propto \sum_{n=-\infty}^{\infty} \cdots, \quad \lim_{z \to 0} U_4(z) \propto z^{B_2} \sum_{n=-\infty}^{\infty} \cdots,
\]
where the convergence of the series depends on \( 1 + B_1/z_0 \) according to the second part of (36); in contrast to expansions \( S(z) \) in powers of \( z \),
\[
\lim_{z \to 0} S(z) = c_0, \quad \lim_{z \to 0} S(z) = \tilde{c}_0 z^{1+\frac{B_1}{2}}.
\]
In addition, we mention that sometimes the relevant thing is not the behaviour of the solutions at \( z = 0 \) or \( z = z_0 \), but the behaviour of the product of such solutions by a function of the variable \( z \), as the product given in equation (97).

To obtain the conditions (36), it is sufficient to consider the convergence of the first set of solutions. The results for the other sets arise from transformations (13) applied in the order given in (22). Thus, by using the form (12) for the first set, the domains of convergence follow from the ratios

\[
\lim_{n \to \infty} \frac{b_{n+1}^i \varphi_{n+1}^{\pm} (\eta, \omega \zeta)}{b_n^i \varphi_{n+1}^{\pm} (\eta, \omega \zeta)} \quad \text{and} \quad \lim_{n \to -\infty} \frac{b_{n-1}^i \varphi_{n+1}^{\pm} (\eta, \omega \zeta)}{b_n^i \varphi_{n+1}^{\pm} (\eta, \omega \zeta)}.
\]

The ratios \( b_{n+1}^i/b_n^i \) and \( b_{n-1}^i/b_n^i \) come from the relations

\[
\alpha_n^i b_{n+1}^i + \beta_n^i b_n^i + \gamma_{n-1}^i b_{n-1}^i = 0
\]

which, when \( n \to \pm \infty \), yield

\[
i \omega \zeta \left[ 1 - \frac{1}{n} \left( B_2 + \frac{B_1}{z_0} - \frac{3}{2} \right) \right] = \frac{b_{n+1}^i}{b_n^i} \sim \frac{2i}{\omega \zeta} \left[ 1 - \frac{1}{n} \left( B_2 + \frac{B_1}{z_0} - \frac{3}{2} \right) \right],
\]

and the minimal solution for \( b_{n-1}^i/b_n^i \) when \( n \to -\infty \) is

\[
\frac{b_{n-1}^i}{b_n^i} \sim \frac{2n^2}{i \omega \zeta} \left[ 1 + \frac{1}{n} \left( 2B_2 + \frac{1}{z_0} + \frac{B_1}{z_0} \right) \right].
\]

On the other hand, from relations (A.19) and (A.20) we find that, for finite values of \( z \),

\[
n \to \infty : \quad \frac{\psi_{n+1}^{\pm}}{\psi_{n+1}^{\pm}} \sim \frac{i \omega \zeta}{2n^2} \left[ 1 - \frac{1}{n} \left( 2B_2 + \frac{1}{z_0} + \frac{B_1}{z_0} \right) \right],
\]

\[
n \to -\infty : \quad \frac{\psi_{n+1}^{\pm}}{\psi_{n+1}^{\pm}} \sim \frac{2n^2}{i \omega \zeta} \left[ 1 + \frac{1}{n} \left( 2B_2 + \frac{1}{z_0} + \frac{B_1}{z_0} \right) \right],
\]

Thence, by means of (39a), we find

\[
n \to \infty : \quad \frac{b_{n+1}^i \psi_{n+1}^{\pm}}{b_n^i \psi_{n+1}^{\pm}} \sim \frac{\omega^2 z_0}{4n^2} \left[ 1 + \frac{1}{n} \left( B_2 + \frac{B_1}{z_0} - 2B_2 - 4 \right) \right],
\]

and, by means of (39b),

\[
n \to -\infty : \quad \frac{b_{n-1}^i \psi_{n+1}^{\pm}}{b_n^i \psi_{n+1}^{\pm}} \sim \frac{z_0}{z} \left[ 1 + \frac{1}{n} \left( B_2 - 2 + \frac{B_1}{z_0} \right) \right],
\]

From these limits, we obtain

\[
\lim_{n \to \infty} \frac{b_{n+1}^i \phi_{n+1}}{b_n^i \phi_{n+1}} = \frac{\omega^2 z_0}{4n^2} \quad \text{and} \quad \lim_{n \to -\infty} \frac{b_{n-1}^i \phi_{n+1}}{b_n^i \phi_{n+1}} = \frac{|z_0|}{|z|} \left[ 1 + \frac{1}{n} \right] \left( B_2 - 2 + \frac{B_1}{z_0} \right).
\]
and
\[
\lim_{n \to \infty} \frac{\beta_{n+1} j_{\nu+v+1}}{\beta_n j_{\nu+v}} = \lim_{n \to \infty} \frac{\beta_{n-1} j_{\nu+v-1}}{\beta_n j_{\nu+v}} = \frac{|z_0|}{|z|} \left[ 1 + \frac{1}{n} \text{Re} \left( \frac{B_2 - 2 + \frac{B_1}{z_0}}{B_2} \right) \right].
\]

(43)

So, by the D’Alembert test the series converge absolutely for \(|z| > |z_0|\) because the right-hand sides of (42) and (43) are < 1. However, if \(|z| = |z_0|\), by expressions (11) for the Raabe test, the series converge even for \(|z| = |z_0|\) provided that the numerators of \(|n|\) in (42) and (43) are < -1, that is,

\[
\text{if } \text{Re} \left( B_2 + \frac{B_1}{z_0} \right) < 1, \text{ then the series in } U_1(z) \text{ converge for } |z| > |z_0|.
\]

If \(\text{Re} \left( B_2 + \frac{B_1}{z_0} \right) > 1\), the series diverge and, if \(\text{Re} \left( B_2 + \frac{B_1}{z_0} \right) = 1\), the test is inconclusive. The convergence regions (36) for the other sets of solutions are obtained by transforming the parameters and the variable \(z\) of \(U_1\) in accordance with equations (22). Only the limit \(n \to \infty\) is relevant for the one-sided series \((n \geq 0)\) and, then, the solutions \(U_i\) converge for any finite value \(z\) in virtue of the first limit given in (42). The convergence of \(U_i^\pm\) is similar to that of \(U_1^\pm\).

Since the previous regions of convergence were derived by supposing that \(z\) is finite, now we consider the behaviour of the solutions at \(z = \infty\). By using (A.7) we find that, when \(z \to \infty\),

\[
U_1(z) \sim e^{i\omega z} [i \omega z]^{i\nu+\frac{1}{2}} \sum \frac{b_n}{\Gamma[n + v + 1 - i\eta]} + e^{-i\omega z} [-2i\omega z]^{i\nu+\frac{1}{2}} \sum \frac{(-1)^n e^{-v-1+i\eta} b_n}{\Gamma[n + v + 1 + i\eta]}
\]

(44)

Thus, \(U_i\) may be unbounded by virtue of the exponential factors. This is consistent with the fact that, if conditions (29) and (30) are satisfied, then \(U_1(z)\) can be written as a linear combination of \(U_i^+(z)\) and \(U_i^-(z)\). In fact, when \(z \to \infty\), equation (A.6) gives

\[
U_i^+(z) \sim e^{i\omega z} [-2i\omega z]^{i\nu+\frac{1}{2}} \sum \frac{b_n}{\Gamma[n + v + 1 - i\eta]} - \frac{3\pi}{2} < \text{arg} (-2i\omega z) < \frac{3\pi}{2},
\]

\[
U_i^-(z) \sim e^{-i\omega z} [2i\omega z]^{i\nu+\frac{1}{2}} \sum \frac{(-1)^n e^{-v-1+i\eta} b_n}{\Gamma[n + v + 1 + i\eta]} - \frac{3\pi}{2} < \text{arg} (2i\omega z) < \frac{3\pi}{2}.
\]

(45)

Thus, the series in \(U_i^\pm\) converge at \(z = \infty\) but one of them may be unbounded depending on the exponential factors. For instance, if \(\text{Re}(i\omega z) \to \infty\), \(U_i^+ \to \infty\) but \(U_i^-\) is bounded.

2.1.3. The case \(\eta = 0\), the spheroidal and Mathieu equations. Taking \(\eta = 0\) and keeping the other parameters fixed, the previous solutions are rewritten in series of Bessel functions of the first kind, \(J_\nu(y)\), and in series of the first and the second Hankel functions, \(H^{(1)}_\nu(y)\) and \(H^{(2)}_\nu(y)\). We also include the Bessel functions \(Y_\nu(y)\) of the second kind. These four functions are denoted by \(Z^{(1)}_\nu(y)\) — or by \(\psi^{(1)}_\nu(y)\) — according to [7, 18]

\[
Z^{(1)}_\nu(y) = J_\nu(y), \quad Z^{(2)}_\nu(y) = Y_\nu(y), \quad Z^{(3)}_\nu(y) = H^{(1)}_\nu(y), \quad Z^{(4)}_\nu(y) = H^{(2)}_\nu(y).
\]

(46)

There are connections among these functions. For example, the relation \(Y_\nu = \left[H^{(1)}_\nu - H^{(2)}_\nu\right]/(2i)\) permits one to obtain the expansion in series of \(Y_\nu\) as a linear combination of the expansions in series of Hankel functions. Thus, we get four sets of solutions, each containing four solutions. These sets are written as

\[
U^{(i)}_j(z) = \left[U^{(i1)}_j(z), U^{(i2)}_j(z), U^{(i3)}_j(z), U^{(i4)}_j(z)\right], \quad i = 1, 2, 3, 4,
\]

(47)
where the right-hand side corresponds to the Bessel functions \(46\). For the one-sided series there are eight sets \(U^{(j)}\). The solutions \(U_1\) lead to \(U^{(j)}\) which, in turn, give the other sets by means of the transformations \(22\), that is, \n\]

\[
U^{(j)}_1(z) = T_2 U^{(j)}_1(z) ; \quad U^{(j)}_2(z) = T_1 U^{(j)}_1(z) , \quad U^{(j)}_4(z) = T_1 U^{(j)}_1(z) . \quad (48)
\]

Thus, we put \(\eta = 0\) in \(U_1\) given in \((19)\), use relations \((A.22)\) together with \((18)\)

\[
\Gamma(2z) = 2^{2z-1}\Gamma(z) [z + (1/2)]/\sqrt{\pi},
\]

and redefine the coefficients as \(a^{(j)}_n = i^j b^{(j)}_n/\Gamma(n + v + 1)\). So, we find

\[
U^{(j)}_1(z) = \left(1 \pm \frac{\beta_2}{2}\right) \sum_a \left[ a_n \mathcal{Z}^{(j)}_{\alpha} (\omega z) \right], \quad [2v \neq 0, \pm 1, \pm 2, \ldots ] \quad (49a)
\]

In the recurrence relations \((15)\) for \(a^{(j)}_n\), we have

\[
a^{(j)}_1 = \frac{\omega z_0 \left[n + v + 2 - \frac{\beta_2}{2}\right]}{\left(2n + 2v + 3\right)} \left[n + v + 1 - \frac{\beta_2}{2}\right],
\]

\[
\beta^{(j)}_n = - \left(n + v + 1 - \frac{\beta_2}{2}\right) \left(n + v + \frac{\beta_2}{2}\right) - B_3,
\]

\[
\gamma^{(j)}_n = - \frac{\omega z_0 \left[n + v + \frac{\beta_2}{2} - 1\right]}{\left(2n + 2v + 1\right)} \left[n + v + \frac{\beta_2}{2} + \frac{\beta_2}{2}\right], 
\]

\(\text{The other sets are given by } (\beta^{(j)}_n = \beta^{(j)}_n, \ 2v \neq 0, \pm 1, \pm 2, \ldots): \)

\[
U^{(j)}_2(z) = \left(1 \pm \frac{\beta_2}{2}\right) \sum_a \left[ a_n \mathcal{Z}^{(j)}_{\alpha} (\omega z) \right],
\]

\[
a^{(j)}_n = \frac{\omega z_0 \left[n + v + 2 - \frac{\beta_2}{2}\right]}{\left(2n + 2v + 3\right)} \left[n + v + 1 + \frac{\beta_2}{2} + \frac{\beta_2}{2}\right], 
\]

\[
\gamma^{(j)}_n = - \frac{\omega z_0 \left[n + v - \frac{\beta_2}{2}\right]}{\left(2n + 2v - 1\right)} \left[n + v - \frac{\beta_2}{2} - \frac{\beta_2}{2}\right]. 
\]

\[
U^{(j)}_3(z) = \left(1 \pm \frac{\beta_2}{2}\right) \sum_a \left[ a_n \mathcal{Z}^{(j)}_{\alpha} (\omega z - z_0) \right],
\]

\[
a^{(j)}_n = - \frac{\omega z_0 \left[n + v + 2 - \frac{\beta_2}{2}\right]}{\left(2n + 2v + 3\right)} \left[n + v + 1 + \frac{\beta_2}{2} + \frac{\beta_2}{2}\right],
\]

\[
\gamma^{(j)}_n = - \frac{\omega z_0 \left[n + v - \frac{\beta_2}{2}\right]}{\left(2n + 2v - 1\right)} \left[n + v - \frac{\beta_2}{2} - \frac{\beta_2}{2}\right].
\]

\[
U^{(j)}_4(z) = \left(1 \pm \frac{\beta_2}{2}\right) \sum_a \left[ a_n \mathcal{Z}^{(j)}_{\alpha} (\omega z - z_0) \right],
\]

\[
a^{(j)}_n = - \frac{\omega z_0 \left[n + v + \frac{\beta_2}{2}\right]}{\left(2n + 2v + 3\right)} \left[n + v + 1 - \frac{\beta_2}{2} - \frac{\beta_2}{2}\right],
\]

\[
\gamma^{(j)}_n = - \frac{\omega z_0 \left[n + v + 1 - \frac{\beta_2}{2}\right]}{\left(2n + 2v - 1\right)} \left[n + v + \frac{\beta_2}{2} + \frac{\beta_2}{2}\right].
\]
By using relations [18]

\[ J_\kappa(ye^{i\vartheta}) = e^{i\kappa \vartheta} J_\kappa(y), \quad Y_\kappa(ye^{i\vartheta}) = e^{-i\kappa \vartheta} Y_\kappa(y) + 2i \cos(\pi \kappa) J_\kappa(y), \]

\[ H^{(1)}_\kappa(ye^{i\vartheta}) = -e^{-i\kappa \vartheta} H^{(1)}_\kappa(y), \quad H^{(2)}_\kappa(ye^{i\vartheta}) = e^{i\kappa \vartheta} H^{(1)}_\kappa(y) + 2 \cos(\pi \kappa) H^{(2)}_\kappa(y), \]  

(53)

with \( \kappa = \nu + 1 + (1/2) \), we find that the change \( \omega \to -\omega \) does not lead to new independent solutions. In this sense, once more the transformation \( T_3 \) is ineffective.

Also in the present case (\( \eta = 0 \)) conditions (30), that is,

\[ v \pm \frac{B_1}{2} \quad \text{and} \quad \nu \pm \frac{B_1}{2} + \frac{B_2}{2} \quad \text{are not integers} \]  

(see relations 30)  

(54a)

are necessary in order to have the two-sided infinite series, and relations (33) hold for the coefficients \( a_\mu^j \). On the other hand, the restrictions (29) are replaced by conditions \( 2\nu \neq 0, \pm 1, \pm 2, \ldots \) which ensure the independence of the Bessel function:

\[ 2\nu \neq 0, \pm 1, \pm 2, \ldots \quad \text{(independence of Bessel functions).} \]  

(54b)

In fact, it is necessary that \( v \neq \pm 1/2, \pm 3/2, \ldots \) in order to avoid two linearly dependent functions of integer order, like \( Z^{(1)}_\nu(y) \) and \( Z^{(1)}_{\nu+1}(y) \) \([Z_\ell = (-1)^\ell \ell Z_\ell], \text{where } \ell \text{ is zero or positive integer. In addition, } v \neq 0, \pm 1, \pm 2, \ldots \text{ ensures the independence of the Hankel functions in the same series: in contrast, we would have functions like } H_{\ell+1/2} \text{ and } H_{-\ell-1/2} \text{ which are proportional to each other since} [19] \]

\[ H^{(1)}_{-\ell+1/2}(y) = i(-1)^\ell H^{(1)}_{\ell+1/2}(y), \quad H^{(2)}_{-\ell+1/2}(y) = -i(-1)^\ell H^{(2)}_{\ell+1/2}(y). \]  

(55)

However, for series of Bessel functions of the first and second kinds, we have

\[ J_{-\ell-1/2}(y) = (-1)^\ell J^{(1)}_{\ell+1/2}(y), \quad Y^{(2)}_{-\ell-1/2}(y) = (-1)^\ell Y^{(2)}_{\ell+1/2}(y), \]  

(56)

that is, for \( v = 0, \pm 1, \pm 2, \ldots \) the functions \( J_{\ell+1/2} \) and \( J_{-\ell-1/2} \) (or, \( Y_{\ell+1/2} \) and \( Y_{-\ell-1/2} \)) are linearly independent. In spite of this, by assuming that \( v \neq 0, \pm 1, \pm 2, \ldots \) also for \( J \) and \( Y \), we guarantee that all of the solutions (60) and (61) for the spheroidal equation are two-sided since \( a_\nu^j \) and \( y_\nu^j \) do not vanish for \(-\infty < n < \infty \).

On the other hand, for the two-sided series the domains of convergence are again given by (36) with \( U^{(1)}_\nu \) or \( U^{(2)}_\nu \) as in the first kind, converge for any finite \( z \). The behaviour of the solutions at \( z = \infty \) can be found from the fact that, for \( \kappa \) fixed and \( |y| \to \infty \) [18],

\[ J_\kappa(y) \sim \frac{2}{\pi y} \cos \left[ y - \frac{\kappa \pi}{2} - \frac{\pi}{4} \right], \quad Y_\kappa(y) \sim \frac{2}{\pi y} \sin \left[ y - \frac{\kappa \pi}{2} - \frac{\pi}{4} \right]; \quad |\arg y| < \pi; \]

\[ H^{(1)}_\kappa(y) \sim \frac{2}{\pi y} e^{y - \frac{\kappa \pi}{2} - \frac{\pi}{4}}; \quad -\pi < \arg y < 2\pi; \]

\[ H^{(2)}_\kappa(y) \sim \frac{2}{\pi y} e^{-y - \frac{\kappa \pi}{2} - \frac{\pi}{4}}; \quad -2\pi < \arg y < \pi. \]  

(57)

Now we consider the Meixner solutions. The substitutions

\[ y = 1 - 2z, \quad S(y) = z^2 \left[ (z - 1)^2 U(z) \right] \leftrightarrow S(y) \propto |y^2 - 1|^2 U \left( z = \frac{1}{2} \right) \]  

(58a)

transform the spheroidal wave equation (8) into

\[ z(z - 1) \frac{d^2 U}{dz^2} + [-\nu(\nu + 1) + (2\mu + 2)z] \frac{dU}{dz} + [\mu(\mu + 1) - \lambda + 4y^2 z(z - 1)] U = 0, \]

which is the CHE (1) with the parameters

\[ z_0 = 1, \quad B_1 = -\mu - 1, \quad B_2 = 2\mu + 2, \quad B_3 = \mu(\mu + 1) - \lambda, \quad \omega = \pm 2y, \quad \eta = 0. \]  

(58b)
Instead of $Z_j^1(v)$, Meixner used the functions $\psi_j^{(i)}(v)$ which are given by [9]

$$
\psi_j^{(i)}(v) = \sqrt{\pi / (2v)} Z_j^{(i)}(v),
$$

(59)
in analogy with the definitions of the spherical Bessel functions $j_\ell$, $y_\ell$, $h_\ell^1$ and $h_\ell^2$ [19]. So, by taking $\omega = -2y$ and using this notation, we obtain

$$
S_1^{(j)}(\mu, y) = \left[ \frac{y + 1}{y - 1} \right]^j \sum a_n \psi_{n+1}^{(j)}(y - 1)], \quad S_2^{(j)}(\mu, y) = S_1^{(j)}(-\mu, y),
$$

$$
\alpha_n^{(j)} = \frac{2\gamma(n + v + 1 - \mu)(n + v + 1)}{(2n + 2v + 3)}, \quad \beta_n^{(j)} = (n + v)(n + v + 1) - \lambda,
$$

(60)

and

$$
S_3^{(j)}(\mu, y) = \left[ \frac{y - 1}{y + 1} \right]^j \sum (-1)^n a_n \psi_{n+1}^{(j)}(y + 1)], \quad S_4^{(j)}(\mu, y) = S_3^{(j)}(-\mu, y).
$$

(61)

For these solutions, conditions (54a) and (54b) reduce to

$$
2v \neq 0, \pm 1, \pm 2, \ldots, \quad v \pm (\mu + 1) \neq \text{integer}.
$$

The Meixner solutions are given by $S_2^{(j)}(\mu, y)$ and $S_4^{(j)}(\mu, y)$. By the D’Alembert test $S_i^{(j)}$ converge for $|y - 1| > 2$ (if $i = 1, 2$) or for $|y + 1| > 2$ (if $i = 3, 4$), as stated in [9, 20]. However, by the Raabe test they may converge at $|y - 1| = 2$ or $|y + 1| = 2$ because relations (36) and (58b) yield

$$
|y - 1| \geq 2, \text{ if } \begin{cases} \Re(\mu) < 0 \text{ in } S_1^{(j)}, \\ \Re(\mu) > 0 \text{ in } S_3^{(j)}; \end{cases} \quad |y + 1| \geq 2, \text{ if } \begin{cases} \Re(\mu) < 0 \text{ in } S_3^{(j)}, \\ \Re(\mu) > 0 \text{ in } S_4^{(j)}; \end{cases}
$$

(62)

(if $\Re(\mu) = 0$, the test is inconclusive).

On the other hand, the solutions $w(u)$ for the Mathieu equation (6), considered as a particular case of the CHE, may be obtained by setting

$$
w(u) = U(z), \quad z = \cos^{(\eta)}(\sigma u), \quad \sigma = 1, i, \quad z_0 = 1, \quad \mathbf{B}_1 = -1/2, \quad \mathbf{B}_2 = 1, \quad \mathbf{B}_3 = 2k^2 - a, \quad \omega = 4k, \quad \eta = 0\quad \text{as a CHE},
$$

(63)

where $U(z)$ are the solutions for CHE with $\eta = 0$. Thus, from the previous solutions $U_{\ell}^{(i)}(z)$ we obtain four sets of two-sided solutions $w_{\ell}^{(i)}(u)$:

$$
w_{\ell}^{(1)}(u) = \sum_n a_n Z_{n+\ell+\frac{1}{2}}^\ell \left[ 4k \cos^2 \frac{\sigma u}{2} \right], \quad |\cos \frac{\sigma u}{2}| > 1,
$$

(64a)

$$
w_{\ell}^{(2)}(u) = \tan \frac{\sigma u}{2} \sum_n \left[ n + \ell + \frac{1}{2} \right] a_n Z_{n+\ell+\frac{1}{2}}^\ell \left[ 4k \cos^2 \frac{\sigma u}{2} \right], \quad |\cos \frac{\sigma u}{2}| > 1,
$$

(64b)

$$
w_{\ell}^{(3)}(u) = \sum_n (-1)^n a_n Z_{n+\ell+\frac{1}{2}}^\ell \left[ -4k \sin^2 \frac{\sigma u}{2} \right], \quad |\sin \frac{\sigma u}{2}| > 1,
$$

(64c)

$$
w_{\ell}^{(4)}(u) = \cot \frac{\sigma u}{2} \sum_n (-1)^n \left[ n + \ell + \frac{1}{2} \right] a_n Z_{n+\ell+\frac{1}{2}}^\ell \left[ -4k \sin^2 \frac{\sigma u}{2} \right], \quad |\sin \frac{\sigma u}{2}| > 1,
$$

(64d)

where the coefficients $a_n$ satisfy the relations

$$
2k(n + v + 1)a_{n+1} + \left[ a - 2k \right] a_{n+1} = 2k(n + v)a_{n+1} = 0.
$$

(64e)

For any solutions the behaviour when $z = \cos^{\eta}(\sigma u/2) \to \infty$ must be determined by using (57). If $\sigma = 1$ and $u$ is real (Mathieu equation), the previous solutions are useless. Note, however, that the one-sided solutions $w_{\ell}^{(1)}$ in series of Bessel functions of the first kind are convergent for all finite values of $z$. 

14
2.2. Solutions for the Whittaker–Ince limit of the CHE

To obtain solutions to the Whittaker–Ince limit of the CHE (4), we apply [21]

\[
\lim_{a \to \infty} \Phi \left( a, c; -\frac{y}{a} \right) = \Gamma(c) y^{(1-c)/2} J_{c-1}(2\sqrt{y}),
\]

\[
\lim_{a \to \infty} \left[ \Gamma(a + 1 - c) \Psi \left( a, c; -\frac{y}{a} \right) \right] = \left\{ \begin{array}{ll}
-i\pi e^{i\pi(c-1)/2} H_{c-1}^{(1)}(2\sqrt{y}), & \text{Im } y > 0, \\
-i\pi e^{-i\pi(c-1)/2} H_{c-1}^{(2)}(2\sqrt{y}), & \text{Im } y < 0,
\end{array} \right.
\]  \hspace{1cm} (65)

on the hypergeometric functions used as basis for the expansions of the solutions for the CHE. For this it is necessary to rewrite the latter solutions in a suitable form and keep \( n \) fixed. Expansions in series of \( Y_{c-1} \) are obtained as a linear combination of the expansions in series of Hankel functions. In this manner, from the first set (19), we obtain a set of four solutions for equation (4). These are again denoted by \( U_1^{(j)} \) \((j = 1, \ldots, 4)\). In fact, we will compute only the limit of \( U_1(z) \); the other solutions follow from the fact that the four Bessel functions satisfy the same differential and difference equations.

On the other hand, if \( U(z) = U(B_1, B_2, B_3; z_0, q; z) \) represents an arbitrary solution for equation (4), then other solutions are generated by the transformations \( \mathcal{F}_1, \mathcal{F}_2 \) and \( \mathcal{F}_3 \) given by [3]

\[
\mathcal{F}_1 U(z) = z^{1+B_1/n} U(C_1, C_2, C_3; z_0, q; z),
\]

\[
\mathcal{F}_2 U(z) = (z - z_0)^{1-B_1/n} U(B_1, D_2, D_3; z_0, q; z),
\]

\[
\mathcal{F}_3 U(z) = U(-B_1 - B_2 z_0, B_2, B_3 - q z_0; z_0, -q; z_0 - z),
\]  \hspace{1cm} (66)

where \( C_i \) and \( D_i \) are defined in equations (14). Thus, it is sufficient to take the limit of the first set of solutions (19) and of the coefficients (20). The other sets are obtained from \( U_1^{(j)} \) through

\[
U_2^{(j)} = \mathcal{F}_2 U_1^{(j)}, \quad U_3^{(j)} = \mathcal{F}_3 U_2^{(j)}, \quad U_4^{(j)} = \mathcal{F}_3 U_3^{(j)}. \hspace{1cm} (67)
\]

First we find the four sets of solutions and use the Raabe test to study their convergence. Then we write the solution for the Mathieu equation.

2.2.1. The four sets of solutions. To find the limit of \( U_1(z) \) given in (19), we rewrite that solution as

\[
U_1(z) = e^{i\omega z} \sum_n (-1)^n c_n [q(z)]^{n+1+1-(B_2/2)} \Phi \left[ n + v + 1 + i\eta, 2n + 2v + 2; -\frac{q z^2}{in} \right], \hspace{1cm} (68a)
\]

where \( c_n = b^1_{n-1} [-i\eta]^{-n} \) and \( q = -2i\omega \). From \( \alpha_n^1 b_{n+1}^1 + \beta_n^1 b_{n+1}^1 + \gamma_n^1 b_{n+1}^1 = 0 \), we obtain

\[
-i\eta \alpha_n^1 c_{n+1} + \beta_n^1 c_n + (-i\eta)^{-1} \gamma_n^1 c_{n-1} = 0, \hspace{1cm} (68b)
\]

where \( \alpha_n^1, \beta_n^1 \) and \( \gamma_n^1 \) are given in (20). By supposing that \( n \) is fixed and using (65), we obtain the solution \( U_1^{(1)} \) in series of Bessel functions of first kind. In fact, we may verify directly that the four solutions \( U_1^{(j)} \) given below satisfy equation (5).

Then, the first set is given by

\[
U_1^{(j)}(z) = z^{(1-B_2)/2} \sum_n (-1)^n c_n^j Z_{2n+2v+1}^j (2\sqrt{qz}), \hspace{1cm} (69a)
\]
where, in the recurrence relations \(\alpha_n^1 c_{n+1}^1 + \beta_n^1 c_{n+1}^1 + \gamma_n^1 c_{n-1}^1 = 0\), we have

\[
\alpha_n^1 = qz_0 [n + v + 2 - \frac{B_1}{2}] \left[ n + v + 1 - \frac{B_1}{2n_0} - \frac{B_2}{2} \right], \\
(2n + 2v + 2)(2n + 2v + 3),
\]

\[
\beta_n^1 = B_3 - qz_0 \left( n + v + 1 - \frac{B_2}{2} \right) \left( n + v + \frac{B_2}{2} \right) \frac{qz_0 [B_2 - 2] [B_2 + \frac{B_1}{2}]}{2(2n + 2v)(2n + 2v + 2)},
\]

\[
\gamma_n^1 = qz_0 \left( n + v + 1 + \frac{B_1}{2} \right) \left[ n + v + \frac{B_1}{2n_0} + \frac{B_2}{2} \right], \\
(2n + 2v - 1)(2n + 2v).
\]

The transformations (66), applied on \(U_1^{(j)}\) according to (67), generate the other sets, that is, \((\beta_n^j = \beta_1^j)\):

\[
U_2^{(j)}(z) = (z - z_0)^{1 - \frac{1}{2}} \sum_n (-1)^n c_n^j Z_{2n+2v+1}(2\sqrt{qz}).
\]

\[
\alpha_n^2 = \frac{qz_0 \left[ n + v + \frac{B_2}{2} \right] \left[ n + v + 1 + \frac{B_1}{2n_0} + \frac{B_2}{2} \right]}{(2n + 2v + 2)(2n + 2v + 3)},
\]

\[
\gamma_n^2 = \frac{qz_0 \left[ n + v + 1 - \frac{B_1}{2} \right] \left[ n + v - \frac{B_1}{2n_0} - \frac{B_2}{2} \right]}{(2n + 2v - 1)(2n + 2v)};
\]

\[
U_3^{(j)}(z) = (z - z_0)^{1 - \frac{3}{2}} \sum_n (-1)^n c_n^j Z_{2n+2v+1}[2\sqrt{q(z - z_0)}],
\]

\[
\alpha_n^3 = -\frac{qz_0 \left[ n + v + 2 - \frac{B_1}{2} \right] \left[ n + v + 1 + \frac{B_1}{2n_0} + \frac{B_2}{2} \right]}{(2n + 2v + 2)(2n + 2v + 3)},
\]

\[
\gamma_n^3 = -\frac{qz_0 \left[ n + v - 1 + \frac{B_1}{2} \right] \left[ n + v - \frac{B_1}{2n_0} - \frac{B_2}{2} \right]}{(2n + 2v - 1)(2n + 2v)};
\]

\[
U_4^{(j)}(z) = z^{1 + \frac{B_1}{2}}(z - z_0)^{-\frac{1}{2}} \sum_n (-1)^n c_n^j Z_{2n+2v+1}[2\sqrt{q(z - z_0)}],
\]

\[
\alpha_n^4 = -\frac{qz_0 \left[ n + v + \frac{B_1}{2} \right] \left[ n + v + 1 - \frac{B_1}{2} - \frac{B_1}{2n_0} \right]}{(2n + 2v + 2)(2n + 2v + 3)},
\]

\[
\gamma_n^4 = -\frac{qz_0 \left[ n + v + 1 - \frac{B_1}{2} \right] \left[ n + v + \frac{B_1}{2n_0} + \frac{B_2}{2} \right]}{(2n + 2v - 1)(2n + 2v)}.
\]

For these four sets of solutions, the conditions (29) and (30) are replaced by

\[
2v, \ v \pm \frac{B_2}{2} \text{ and } v \pm \left( \frac{B_1}{2n_0} + \frac{B_2}{2} \right) \text{ are not integers.}
\]

while the relations (33) remain valid for the coefficients \(c_n^j\).

The previous list completes the list given in [2] where the expansions \(U_i^{(1,2)}\) in series of \(J_n\) and \(Y_n\) have not been taken into account, whereas the expansions \(U_i^{(3,4)}\) have been written in terms of the modified Bessel functions \(K_{2n+2v+1}[\pm 2i\sqrt{q}]\) and \(K_{2n+2v+1}[\pm 2i\sqrt{q}(z - z_0)]\). Moreover, now the regions of convergence are modified by the use of the Raabe test.
2.2.2. Convergence of the solutions. As in the case of the CHE, by the D’Alembert test the two-sided expansions $U_1^{(j)}$ and $U_4^{(j)}$ converge absolutely for $|z| > |z_0|$, while $U_1^{(j)}$ and $U_4^{(j)}$ converge for $|z - z_0| > |z_0|$. However, by the Raabe test the solutions also converge at $|z| = |z_0|$ and $|z - z_0| = |z_0|$ under the conditions similar to (36), that is,

$$
\begin{align*}
|z| & \geq |z_0| \text{ if } \left| \frac{\text{Re} \left[ B_2 + B_1 \right]}{z_0} \right| < 1 \text{ in } U_1^{(j)}, \\
|z - z_0| & \geq |z_0| \text{ if } \left| \frac{\text{Re} \left[ B_2 + B_1 \right]}{z_0} \right| > 1 \text{ in } U_4^{(j)},
\end{align*}
$$

(74)

The test does not ensure convergence at $z = \infty$ and, so, the behaviour at $z = \infty$ again deserves special attention. We will find that the one-sided infinite series $U_1^{(j)}$ in series of Bessel functions of the first kind converge for any finite value of $z$.

Relations (74) correspond to (36) with the replacements $U_1^{(j)} \leftrightarrow U_j$. In fact, a few modifications in the previous example lead to (74). Thus, if $n \to \pm \infty$, we find

$$
qz_0 \left[ 1 - \frac{1}{n} \left( B_2 + B_1 \frac{1}{z_0} - \frac{1}{2} \right) \right] c_n + [4n(n + 2\nu + 1)]
$$

$$
+ qz_0 \left[ 1 + \frac{1}{n} \left( B_2 + B_1 \frac{1}{z_0} - \frac{1}{2} \right) \right] c_{n+1} = 0.
$$

When $n \to \infty$ the minimal solution for $c_{n+1}/c_n$ is

$$
n \to \infty: \frac{c_{n+1}}{c_n} \sim \frac{qz_0}{n} \left[ 1 - \frac{1}{n} \left( 2v - B_2 - B_1 \frac{1}{z_0} + \frac{7}{2} \right) \right]
$$

$$
\Rightarrow \frac{c_{n+1}}{c_n} \sim \frac{4n^2}{qz_0} \left[ 1 + \frac{1}{n} \left( 2v - B_2 - B_1 \frac{1}{z_0} + \frac{3}{2} \right) \right],
$$

(75)

while the minimal solution for $c_{n-1}/c_n$, when $n \to -\infty$, is

$$
n \to -\infty: \frac{c_{n-1}}{c_n} \sim \frac{-qz_0}{n} \left[ 1 - \frac{1}{n} \left( 2v + B_2 + B_1 \frac{1}{z_0} - \frac{3}{2} \right) \right]
$$

$$
\Rightarrow \frac{c_{n-1}}{c_n} \sim \frac{4n^2}{qz_0} \left[ 1 + \frac{1}{n} \left( 2v + B_2 + B_1 \frac{1}{z_0} + \frac{1}{2} \right) \right].
$$

(76)

On the other hand, the behaviours (A.24) and (A.25) for the Bessel functions lead to

$$
\lim_{n \to \infty} \frac{J_{2n+2\nu+3}}{J_{2n+2\nu+1}} = \frac{qz_0}{n^2} \left[ 1 - \frac{1}{n} \left( 2v + \frac{5}{2} \right) \right],
$$

$$
\lim_{n \to \infty} \frac{Z_{2n+2\nu+3}}{Z_{2n+2\nu+1}} = \frac{4n^2}{qz_0} \left[ 1 + \frac{1}{n} \left( 2v + \frac{3}{2} \right) \right],
$$

$$
\lim_{n \to -\infty} \frac{Z_{2n+2\nu-1}}{Z_{2n+2\nu+1}} = \frac{4n^2}{qz_0} \left[ 1 + \frac{1}{n} \left( 2v + \frac{1}{2} \right) \right], \quad [j = 1, 2, 3, 4].
$$

(77)

Thus, when $n$ tends to $+\infty$,

$$
\lim_{n \to \infty} \left[ \frac{c_{n+1}^{(j)} J_{2n+2\nu+3}}{c_n^{(j)} J_{2n+2\nu+1}} \right] = -\frac{qz_0}{16n^3} \left[ 1 - \frac{1}{n} \left( 4v + 6 - B_2 - B_1 \frac{1}{z_0} \right) \right],
$$

$$
\lim_{n \to \infty} \left[ \frac{c_{n+1}^{(j)} Z_{2n+2\nu+3}}{c_n^{(j)} Z_{2n+2\nu+1}} \right] = -\frac{z_0}{z} \left[ 1 - \frac{1}{n} \left( 2 - B_2 - B_1 \frac{1}{z_0} \right) \right], \quad [j = 2, 3, 4],
$$

(78)

and, when $n \to -\infty$,

$$
\lim_{n \to -\infty} \left[ \frac{c_{n+1}^{(j)} Z_{2n+2\nu-1}}{c_n^{(j)} Z_{2n+2\nu+1}} \right] = -\frac{z_0}{z} \left[ 1 + \frac{1}{n} \left( 2 - B_2 - B_1 \frac{1}{z_0} \right) \right], \quad [j = 1, 2, 3, 4].
$$

(79)
Hence, by the Raabe test the expansions \( U_{1}^{(j)} \) are convergent for \(|z| > |z_0|\) as indicated in (74). For the other sets of solutions, the domains of convergence follow from the transformations (66) applied on \( U_{1}^{(j)} \) according to (67). Moreover, from the first limit given in (78) we see that the one-sided infinite series \( U_{1}^{(j)} \) converge for all \( z \) except probably the point \( z = \infty \). The behaviour of any solution when \( z \to \infty \) must be determined by using (57).

2.2.3. Heine’s solutions for the Mathieu equation. From the previous solutions we recover the usual solutions in series of Bessel functions for Mathieu’s equation (called Heine’s solutions [22]) by means of the substitutions

\[
\begin{align*}
&v(u) = U(z), \quad z = \cos^2(\sigma u) \quad \Rightarrow \quad z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = k^2 - \frac{a}{4}, \quad q = k^2 \quad \text{Mathieu equation as Whittaker–Ince limit of the CHE.} \\
&\text{Relations (33), with } b_n \text{ replaced by } c_n, \text{ yield} \quad c_n^2 = \left(n + \frac{1}{2}\right)c_n^1, \quad c_3^3 = (-1)^{n}c_n^1, \quad c_n^4 = (-1)^{n}\left(n + \frac{1}{2}\right)c_n^1.
\end{align*}
\]

Then, by writing \( c_n = c_n^1, \ w_{i}^{(j)}(u) = U_{i}^{(j)}(z) \) and setting \( \sqrt{k^2} = k \), we find

\[
\begin{align*}
&w_{1}^{(j)}(u) = \sum_{n} (-1)^{n}c_nZ_{2n+2v+1}^{(j)}[2k\cos(\sigma u)], \quad |\cos(\sigma u)| \geq 1, \quad (81a) \\
&w_{2}^{(j)}(u) = \cot(\sigma u)\sum_{n} (-1)^{n}\left(n + \frac{1}{2}\right)c_nZ_{2n+2v+1}^{(j)}[2k\sin(\sigma u)], \quad |\sin(\sigma u)| \geq 1, \quad (81b) \\
&w_{3}^{(j)}(u) = \sum_{n} c_nZ_{2n+2v+1}^{(j)}[2ki\sin(\sigma u)], \quad |\sin(\sigma u)| \geq 1, \quad (81c) \\
&w_{4}^{(j)}(u) = \cot(\sigma u)\sum_{n} \left(n + \frac{1}{2}\right)c_nZ_{2n+2v+1}^{(j)}[2ki\sin(\sigma u)], \quad |\sin(\sigma u)| > 1, \quad (81d)
\end{align*}
\]

where the coefficients \( c_n \) satisfy the relations

\[
k^2c_{n+1} + ((2n + 2v + 1)^2 - a)c_n + k^2c_{n-1} = 0. \quad (81e)
\]

As in the case of the two-sided solutions (64a) and (64c), obtained from the CHE, the above solutions are useless for \( \sigma = 1 \) and \( u \) =real (Mathieu equation).

The conditions (73) reduce to \( 2v \notin \mathbb{Z} \) and ensure the linear independence of the terms in a given series. In addition, the above notation for the solutions of the Mathieu equation is similar to the one used by Erdélyi [22]. However, in [18, 19] the coefficients \( (c_{n+1}, c_n, c_{n-1}) \) are replaced by \( (c_{2n+2}, c_{2n}, c_{2n-2}) \) and the Bessel functions \( Z_{2n+2v+1} \) are written as \( Z_{2n+\nu} \); this is equivalent to writing \( 2v + 1 = \nu \) with \( \nu \notin \mathbb{Z} \). The above domains of convergence may be compared with those of solutions (28.23.2)–(28.23.5) of [19].

By the Raabe test, the two-sided solutions \( w_{1}^{(j)}(u) \) and \( w_{4}^{(j)}(u) \) are absolutely convergent also at \( |\cos(\sigma u)| = 1 \) and \( |\sin(\sigma u)| = 1 \), respectively. In [7, 18, 20, 23] these points are not included in the domains of convergence due to the use of the D’Alembert test. The one-sided solutions \( w_{1}^{(j)}(\nu) \) converge for any finite \( u \); in [19] it is stated that this property holds also for two-sided solutions (a misprint, we are certain).
3. Possible applications

In this section, we consider two examples which use solutions for the CHE. In the first example, we discuss solutions for the Klein–Gordon equation for a scalar test field $\Phi_1$ in the gravitational background of a singular and a non-singular spacetime. As the time dependence of $\Phi_1$ obeys Mathieu equations without any arbitrary constant, we have to use two-sided series solutions. The parameter $\nu$ must be determined from the characteristic equation. Then, the Raabe test ensures that in both the cases, the solutions of the Mathieu equations are bounded and convergent for all values of the time variable. However, the full wavefunction is bounded everywhere only for the nonsingular spacetime.

The second example deals with the one-dimensional Schrödinger equation for a family of QES potentials. In addition to the expected solutions given by the finite series, for a subfamily of the potentials we find infinite-series solutions which, due to the Raabe test, are bounded and convergent for all values of the independent variable.

3.1. Klein–Gordon equation in curved spacetimes

In its conformally static form, the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ for nonflat Friedmann–Robertson–Walker spacetimes is written as \[24\]

\[ ds^2 = [A(\tau)]^2 \left[ d\tau^2 - d\chi^2 - \frac{\sin^2(\sqrt{\epsilon}\chi)}{\epsilon}(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad x^\mu = (\tau, \chi, \theta, \phi) \tag{82} \]

where $\epsilon = \pm 1$ is the spatial curvature, $\tau$ is the time variable, whereas $\chi$, $\theta$ and $\phi$ are the spatial coordinates. The Klein–Gordon equation for a field $\Phi_1$ with mass $M (\hbar = c = 1)$ is

\[ \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi_1 \right) + \sqrt{-g} \left( M^2 + \frac{\rho R}{\Phi_1} \right) = 0, \quad \partial_\mu = \partial / \partial x^\mu, \]

where $g$ is the determinant associated with $g_{\mu\nu}$, $R$ is the Ricci scalar, $\rho = 1/6$ for conformal coupling and $\rho = 0$ for minimal coupling. By performing the separation of variables $\Phi_1(\chi, \theta, \phi, \tau) = [A(\tau)]^{-1} T(\tau) X(\chi) \Theta(\theta) e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{83} \]

one finds that $X$ and $\Theta$ are given by the same special functions for any scale factor $A(\tau)$ \[24\], while $T$ obeys the equation

\[ \frac{d^2 T}{d\tau^2} + \left[ \kappa^2 + M^2 A^2 + (6\rho - 1) \left( \frac{1}{A} \frac{d^2 A}{A d\tau^2} + \epsilon \right) \right] T = 0. \tag{84} \]

The constant of separation $\kappa$, determined from the spatial dependence of $\Phi_1$, is given by

\[ \kappa = 1, 2, 3, \ldots \text{ if } \epsilon = 1, \quad \text{and } 0 < \kappa < \infty \text{ if } \epsilon = -1. \]

For a nonsingular model of universe and for (singular) radiation-dominated models, equation (84) reduces to Mathieu equations.

3.1.1. Nonsingular metric. For the nonsingular case, the scale factor $A(\tau)$ given by \[25\]

\[ A(\tau) = a_0 \cosh \tau, \quad \epsilon = -1, \quad -\infty < \tau < \infty, \]

where $a_0$ is a positive constant, leads to the modified Mathieu equation

\[ \frac{d^2 T}{d\tau^2} + \left[ \kappa^2 + \frac{1}{2} M^2 a_0^2 + \frac{1}{2} M^2 a_0^2 \cosh(2\tau) \right] T = 0. \]

So, in equation (6) we have

\[ \sigma = i, \quad a = -\kappa^2 - \left( M^2 a_0^2 / 2 \right), \quad k = Ma_0 / 2, \quad u = \tau. \]
Solutions for this problem have already been given in [26] where the convergence at \( \tau = 0 \) is not discussed. Here this question is solved by using the Raabe test.

From the Heine-type solutions, only \( w^{(j)}_1 \) given in equation (81a) afford convergent and bounded wavefunctions for all \( \tau \in (-\infty, \infty) \). We find the solutions

\[
T^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n c_n Z_{2n+2\nu+1}^{(j)} (M a_0 \cosh \tau) , \quad [2\nu \notin \mathbb{Z}] , \quad (85a)
\]

where the recurrence relations for \( c_n \) are

\[
M^2 a_0^2 c_{n+1} + [ (4n + 4\nu + 2) + 4\kappa^2 + 2M^2 a_0^2 ] c_n + M^2 a_0^2 c_{n-1} = 0 . \quad (85b)
\]

The relations among the Bessel functions [19] imply that only two of the four solutions (85a) are linearly independent. Similar results are found by treating the Mathieu equation as a CHE. In effect, by using \( w^{(j)}_1 \) given in (64a) we obtain

\[
T^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} a_n Z_{n+\nu+1/2}^{(j)} [2M a_0 \cosh^2 (\tau/2)] , \quad [2\nu \notin \mathbb{Z}] , \quad (86a)
\]

where the recurrence relations for \( a_n \) are

\[
M a_0 [n + \nu + 1] a_{n+1} = \left[ \left( n + \nu + \frac{1}{2} \right)^2 + \kappa^2 + 2M^2 a_0^2 \right] a_n + M a_0 [n + \nu] a_{n-1} = 0 . \quad (86b)
\]

Thus, the Raabe test ensures that \( T^{(j)} \) and \( T^{(j)}_{(\nu)} \), as well as the corresponding wavefunctions (83), are bounded and convergent for all \( \tau \in (-\infty, \infty) \).

### 3.1.2. Singular metric.

For radiation-dominated spacetimes, \( A(\tau) = a_0 \sin (\epsilon \tau) / \sqrt{\tau} (\tau \geq 0) \) and, so,

\[
\frac{d^2 T}{d\tau^2} + \left[ \kappa^2 + \frac{\epsilon}{2} M^2 a_0^2 - \frac{\epsilon}{2} M^2 a_0^2 \cos (2\sqrt{\epsilon} \tau) \right] T = 0 . \quad (87)
\]

We consider only the case \( \epsilon = -1 \). We take \( \sigma = i, \nu = (M^2 a_0^2 / 2) - \kappa^2, k = M a_0 / 2 \) and \( \alpha = \tau \).

Then the Heine solutions \( w^{(j)}_1 (u) \) given in equation (81a) lead to

\[
T^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n c_n Z_{2n+2\nu+1}^{(j)} (M a_0 \cosh \tau) , \quad [2\nu \notin \mathbb{Z}] , \quad (88a)
\]

where the recurrence relations are

\[
M^2 a_0^2 c_{n+1} + [ (4n + 4\nu + 2) + 4\kappa^2 - 2M^2 a_0^2 ] c_n + M^2 a_0^2 c_{n-1} = 0 . \quad (88b)
\]

On the other hand, from the solutions \( w^{(j)}_1 (u) \) given in (64a) (CHE), we find

\[
T^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} a_n Z_{n+\nu+1/2}^{(j)} [2M a_0 \cosh^2 (\tau/2)] , \quad [2\nu \notin \mathbb{Z}] , \quad (89a)
\]

where the relations for \( a_n \) are

\[
M a_0 [n + \nu + 1] a_{n+1} = \left[ \left( n + \nu + \frac{1}{2} \right)^2 + \kappa^2 \right] a_n + M a_0 [n + \nu] a_{n-1} = 0 . \quad (89b)
\]

Once more, \( T^{(j)} \) and \( T^{(j)}_{(\nu)} \) are convergent and bounded for all \( \tau \geq 0 \) but now the wavefunctions (83) become unbounded at \( \tau = 0 \) due to the factor \( 1/A(\tau) = 1/(a_0 \sinh \tau) \).

Therefore, if \( \epsilon = -1 \), the solutions for the modified Mathieu obtained from the CHE and its Whittaker–Hill limit are suitable for the singular and nonsingular metrics. For the singular metric, the unboundedness of the solutions (83) at \( \tau = 0 \) is expected since at this point there is a physical singularity in the sense that the pressure and density energy diverge.
3.2. Schrödinger equation for quasi-exactly solvable potentials

Now we consider problems involving solutions given by the finite and infinite series for the CHE. To this end, we write the one-dimensional stationary Schrödinger equation for a particle with mass \( M \) and energy \( E \) as

\[
\frac{d^2 \psi}{du^2} + \left[ \mathcal{E} - \mathcal{V}(u) \right] \psi = 0, \quad u = ax, \quad \mathcal{E} = \frac{2M}{\hbar^2 a^2} E, \quad \mathcal{V}(u) = \frac{2M}{\hbar^2 a^2} V(x),
\]

where \( a \) is a constant with inverse-length dimension, \( h \) is the Planck constant divided by \( 2\pi \), \( x \) is the spatial coordinate and \( V(x) \) is the potential. For \( \mathcal{V}(u) \) we choose the Uschveridze quasi-exact solvable potential [12]

\[
\mathcal{V}(u) = 4\beta^2 \sinh^4 u + 4\beta \left[ \beta - 2(\gamma + \delta) - 2\ell \right] \sinh^2 u + 4 \left[ \delta - \frac{1}{4} \right] \left[ \delta - \frac{3}{4} \right] \frac{1}{\cosh^2 u},
\]

\[
-4 \left[ \gamma - \frac{1}{4} \right] \left[ \gamma - \frac{3}{4} \right] \frac{1}{\cosh^2 u}, \quad [\ell = 0, 1, 2, \ldots],
\]

where \( \beta, \gamma, \text{ and } \delta \) are real constants with \( \beta < 0 \) and \( \delta > 1/4 \).

When \( \delta \geq 1/4 \) and \( \ell \) is zero or a natural number, the above family of potentials is QES because it admits bounded wavefunctions given by the finite series which allow us to determine only a finite number of energy levels. However, for \( 1/4 < \delta < 1/2 \) and \( 1/2 < \delta \leq 3/4 \), we also find infinite-series solutions which are convergent and bounded for all the values of the independent variable: this suggests the possibility of determining the remaining part of the energy spectra as solutions of a characteristic equation. For \( \delta > 3/4 \) we find no solutions like these.

Note that Uschveridze supposed that \( \ell = 0, 1, 2, \ldots \). However we will see that, for

\[
(\gamma, \delta) = \left\{ \left( \frac{1}{4}, \frac{1}{2} \right), \left( \frac{1}{4}, \frac{1}{4} \right), \left( \frac{1}{4}, \frac{3}{4} \right), \left( \frac{3}{4}, \frac{1}{4} \right), \left( \frac{3}{4}, \frac{3}{4} \right) \right\},
\]

the potential is quasi-solvable even when \( \ell \) is a positive half-integer. In addition, Uschveridze supposed that \( u \in (-\infty, \infty) \). However, we obtain

\[
\lim_{u \to \pm \infty} \mathcal{V}(u) = \infty, \quad \lim_{u \to 0} \mathcal{V}(u) = \begin{cases} 
-4 \left[ \gamma - \frac{1}{4} \right] \left[ \gamma - \frac{3}{4} \right], & \text{if } \delta = \frac{1}{4} \text{ or } \delta = \frac{3}{4}; \\
-\infty, & \text{if } \delta \in \left( \frac{1}{4}, \frac{3}{4} \right); \\
+\infty, & \text{if } \delta \notin \left( \frac{1}{4}, \frac{3}{4} \right).
\end{cases}
\]

Hence, for \( \delta \notin \left[ 1/4, 3/4 \right] \) there is an infinite barrier at \( u = 0 \) and, so, we can suppose that \( u \geq 0 \) or \( u \leq 0 \).

3.2.1. Wavefunctions for the Whittaker–Hill equation. If \( \gamma \) and \( \delta \) take the values (92), then the potential (91) reduces to

\[
\mathcal{V}(u) = 4\beta^2 \sinh^4 u + 4\beta \left[ \beta - 2(\gamma + \delta) - 2\ell \right] \sinh^2 u + 4 \left[ \delta - \frac{1}{4} \right] \left[ \delta - \frac{3}{4} \right] \frac{1}{\cosh^2 u}, \quad [\ell = 0, 1, 2, \ldots],
\]

where \( u \in (-\infty, \infty) \). Thence, by using \( \sinh^2 u = [\cosh(2u) - 1]/2 \) and \( \sinh^4 u = [\cosh(4u) - 4 \cosh(2u) + 3]/8 \), equation (90) becomes a modified WHE (7) with the parameters

\[
\sigma = i, \quad \vartheta = -\mathcal{E} + 4\beta(\ell + \gamma + \delta), \quad p + 1 = 2(\ell + \gamma + \delta), \quad \xi = 2\beta.
\]

In fact, the WHE also occurs in the cases of the Razavy potential [27] and the symmetric double-Morse potential considered by Zaslavskii and Ulyanov [28].

On the other hand, the substitutions

\[
z = \cosh^2 u, \quad \psi(u) = \psi[u(z)] = U(z), \quad [z \geq 1]
\]

transform the Schrödinger equation for the preceding potential into the CHE (1) with

\[
z_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = 1, \quad B_3 = \frac{\mathcal{E}}{4}, \quad i\omega = \pm \beta, \quad i\eta = \pm (\ell + \gamma + \delta),
\]

\[21\]
where the plus or minus sign must be used throughout. Thus, we can attempt to solve the problem by using known solutions for the CHE. For example, from the Baber–Hassé expansions in power series, the Hylleraas solutions or the Jaffé solutions [1, 29–31], we obtain even and odd finite-series solutions bounded for $z \geq 1$: such solutions allow us to find only a finite number of energy levels. There are also infinite-series solutions which, however, must be discarded because they are not bounded for any admissible value of $z$.

On the other hand, if we use one-sided series solutions $\bar{U}_i(z)$ in terms of Coulomb wavefunctions, we may find [16]

- even and odd finite-series solutions which are convergent and bounded for all $z \geq 1$,
- even infinite-series solutions which, due to the Raabe test, are convergent and bounded for all $z \geq 1$,
- odd infinite-series solutions which converge and are bounded only for finite values of $z$. To cover the entire interval $z \geq 1$, it is necessary to consider two such solutions.

Therefore, in this case we could find additional energy levels by solving a transcendental characteristic equation. This conclusion also follows from the two-sided infinite series solutions given in equations (100a) and (101a).

3.2.2. Wavefunctions for the cases $1/4 \leq \delta < 1/2$ and $1/2 < \delta \leq 3/4$. For the Ushveridze potential (91), the substitutions

$$z = \cosh^2 u, \quad \psi(u) = \psi[u(z)] = z^{\ell-1/2}(z-1)^{1/2}U(z), \quad [z \geq 1]$$

transform the Schrödinger equation (90) into a confluent Heun equation with

$$z_0 = 1, \quad B_1 = -2\gamma, \quad B_2 = 2\gamma + 2\delta, \quad B_3 = \frac{\xi}{4} + \left(\gamma + \delta - \frac{1}{2}\right)^2,$$

$$io\omega = \pm\beta, \quad in = \pm(\ell + \delta + \gamma).$$

Now we exclude the cases (92) and suppose that $\ell$ is a non-negative integer. We select

$$io\omega = -\beta, \quad in = -\ell - \gamma - \delta.$$

Then, by using for $U(z)$ the solutions given in equations (29a) and (29b) of [3], we find

$$\psi_1^{\mathrm{Baber}}[u(z)] = e^{-\beta z}z^{\ell-1/2}(z-1)^{1/2}\sum_{n=0}^{\ell} a_n (z-1)^n, \quad [\ell = 0, 1, 2, \ldots],$$

where the series coefficients satisfy ($a_{-1} = 0$),

$$(n + 1)(n + 2\delta)a_{n+1} + \frac{n(n + 2\gamma + 2\delta - 1 - 2\beta)}{4} a_n + \frac{n}{2} (\gamma + \delta - \frac{1}{2})^2 - 2\beta\delta a_n + -2\beta(n - \ell) a_{n-1} = 0.$$  

According to the theory of three-term recurrence relations [7], the series in (99a) ends at $n = \ell$ because the coefficient of $a_{n-1}$ in (99b) vanishes when $n = \ell + 1$. Since $\beta > 0$, the previous eigenfunctions are bounded for all $z \geq 1$ provided that $\delta \geq 1/4$. In fact, $\psi_1^{\mathrm{Baber}}$ represents $\ell + 1$ distinct solutions, each one with a different energy [7].

On the other hand, we find cases for which there are infinite-series solutions appropriate for any $z \geq 1$. For this we insert into (97) the two-sided solutions $U^+_n$ and $U^-_n$ given in (19) and (23), respectively, and use the Raabe test along with the limit (A.6). Thence by ascribing convenient values to the parameter $\nu$, we find the solutions $\psi_1^+$ and $\psi_2^+$ with the following properties.
The solutions \( \psi_1^+ \) are convergent and bounded for all \( z \geq 1 \) if \( 1/4 \leq \delta < 1/2 \).

The solutions \( \psi_2^+ \) are convergent and bounded for all \( z \geq 1 \) if \( 1/2 < \delta \leq 3/4 \).

If \( \delta = 1/2 \), then \( \psi_1^+ = \psi_2^+ \). For this case the Raabe test is inconclusive as to the convergence at \( z = 1 \).

In effect, for \( U(z) = U_1^+(z) \) solutions (97) yield

\[
\psi_1^+(z) = e^{-\beta z} z^{-\delta+\frac{1}{2}} (z-1)^{\delta-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{b_n}{\Gamma(n+v+1+\ell+\gamma+\delta)} [2\beta z]^n \\
\times \Psi [n+v+1-\ell-\gamma-\delta, 2n+2\nu+2; 2\beta z], \quad 1/4 \leq \delta < 1/2, \tag{100a}
\]

where in the recurrence relations (15) for \( b_n \) we have

\[
\alpha_n = -\frac{2\beta [n+v+2-\gamma-\delta][n+v+1+\gamma-\delta]}{(2n+2\nu+2)(2n+2\nu+3)}, \\
\beta_n = -\frac{\epsilon + \beta (\ell + \gamma + \delta)}{2} - \frac{\beta [\ell + \gamma + \delta][\gamma + \delta - 1][\gamma - \delta]}{[n+v][n+v+1]}, \tag{100b}
\]

\[
\gamma_n = \frac{2\beta [n+v-1+\gamma+\delta][n+v-\gamma+\delta][n+v+\ell+\gamma+\delta][n+v-\ell-\gamma-\delta]}{[2n+2\nu-1][2n+2\nu]}
\]

By the Raabe test the condition \( \delta < 1/2 \) ensures that the series converge at \( z = 1 \), while the condition \( \delta \geq 1/4 \) ensures that the factor \( (z-1)^{\delta-(1/4)} \) is bounded at \( z = 1 \). Similarly, for \( U(z) = U_2^+(z) \), we obtain

\[
\psi_2^+(z) = e^{-\beta z} z^{\delta-\frac{1}{2}} (z-1)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{b_n}{\Gamma(n+v+1+\ell+\gamma+\delta)} [2\beta z]^n \\
\times \Psi [n+v+1-\ell-\gamma-\delta, 2n+2\nu+2; 2\beta z], \quad 1/2 < \delta \leq 4/3, \tag{101a}
\]

where, in the recurrence relations (15) for \( b_n^2 \)

\[
\alpha_n^2 = -\frac{2\beta [n+v+\gamma+\delta][n+v+1-\gamma-\delta]}{(2n+2\nu+2)(2n+2\nu+3)}, \quad \beta_n^2 = \beta_n, \\
\gamma_n^2 = \frac{2\beta [n+v+1-\gamma-\delta][n+v+\gamma-\delta][n+v+\ell+\gamma+\delta][n+v-\ell-\gamma-\delta]}{[2n+2\nu-1][2n+2\nu]}, \tag{101b}
\]

where \( \beta_n^2 = \beta_n^1 \) is a functional identity; in fact, \( \beta_n^1 \) and \( \beta_n^2 \) are different from each other because they hold for distinct intervals of \( \delta \).

To ensure that all the terms of the series are linearly independent and that the summation extends from minus to plus infinity, the parameter \( v \) must be chosen such that

\[
v, \quad v \pm (\gamma + \delta) \quad \text{and} \quad v \pm (\gamma - \delta) \quad \text{are not integers,} \tag{102}
\]

where the values for \( \delta \) are different for solutions (100a) and (101a). The linear independence is ensured by requiring that \( 2\nu \) is not an integer, without any restrictions on the parameters of the potential. Thus, for fixed values of \( \gamma \) and \( \delta \), we can choose for \( \nu \) any value in the open interval \((0, 1/2)\) convenient to satisfy the above conditions. The use of one-sided series would lead to restrictions on \( \gamma \) and \( \delta \).

4. Conclusion

We have dealt with the convergence of Leaver’s expansions in series of Coulomb wavefunctions for solutions of the CHE. By redefining the Coulomb functions, we have completed the proof of convergence delineated by Leaver and, in addition, have found that the Raabe test improves...
the regions of convergence for solutions of the CHE and its Whittaker–Ince limit (4) if certain conditions are fulfilled. It is worth noting that in using the convergence tests we suppose that the independent variable $z$ is finite. So, when $z$ tends to infinity, the behaviour of each solution must be analysed carefully.

We have used transformations of variables which lead to solutions with different domains of convergence and/or different behaviours at the singular points. By this procedure, we have recovered all of the Meixner solutions [9] for the spheroidal equation (8) and the Heine solutions [19] for the Mathieu equation (6). In both cases these expansions are given by series of Bessel functions whose convergence regions may be improved by the Raabe test.

We have considered only two-sided solutions for the CHE and its Whittaker–Ince limit. Despite this, due to the validity of the Klein–Gordon equation in a non-singular model of the universe has solutions bounded and convergent for all values of the time variable. In section 3, we have also regarded the Schrödinger equation for a class of quasi-exactly solvable (QES) potentials.

If the real parameter $\delta$ satisfies $\delta \geq 1/4$, a part of the energy spectrum of the Schrödinger equation can be computed from finite-series solutions, as expected for any QES potential. However, the remarkable fact is the existence of infinite-series solutions which (by the Raabe test) converge and are bounded for all values of the independent variable if $1/4 \leq \delta \leq 3/4$. These are the solutions which, in principle, permit one to find new energy levels as solutions of a transcendental equation. However, here we have not found infinite-series wavefunctions appropriate for $\delta > 3/4$.

Finally, some comments on the solutions for the DCHE as well as on the one-sided solutions for the CHE and DCHE—for details see [16]. First, the Raabe test is useless for solutions of the DCHE. Second, we find two subgroups of solutions for the DCHE: one is obtained from solutions of the CHE when $z_0 \to 0$; the other follows from that subgroup by a transformation of the DCHE and cannot be derived as limit of expansions in series of Coulomb functions. Thus, we may seek solutions for the CHE which yield such subgroups for the DCHE when $z_0 \to 0$.

On the other hand, to obtain one-sided series solutions we restrict the summation of the two-sided series to $n \geq 0$ by writing the parameter $\nu$ as a function of the parameters of the differential equations. Thus, each of the four sets $\mathcal{U}_i(z)$ for the CHE gives two expressions for $\nu$ and, accordingly, eight sets $\tilde{\mathcal{U}}_i(z)$ of one-sided series solutions. For special values of the parameters, these solutions are given by finite series. It must be noted that: if $\eta \neq 0$, there are three possible types of recurrence relations for the series coefficients, and if $\eta = 0$, only two types. In addition, the form of these relations depends on the normalization used for the Coulomb functions.

Appendix A. Confluent hypergeometric and coulomb functions

Here we write some useful formulas concerning the confluent hypergeometric functions and, in equations (A.12) and (A.13), redefine the Coulomb wavefunctions. At the end, we obtain relations (A.21) which are important for applying the convergence tests for infinite-series solutions of the CHE.

The regular and irregular confluent hypergeometric functions, $\Phi(a, c; u)$ and $\Psi(a, c; u)$, are the solutions of the confluent hypergeometric equation [21]

$$y \frac{d^2 \varphi}{dy^2} + (c - y) \frac{d \varphi}{dy} - a \varphi = 0.$$ (A.1)
The functions $\Phi(a, c; y)$ and $\Psi(a, c; y)$ are also denoted by $M(a, c, y)$ and $U(a, c, y)$, respectively [18]. In fact, the following four solutions for equation (A.1)

$$
\begin{align*}
\psi^{(1)}(y) &= \Phi(a, c; y), \\
\psi^{(2)}(y) &= \Psi(a, c; y), \\
\psi^{(3)}(y) &= e^{y}\Psi(-a, -c; -y), \\
\psi^{(4)}(y) &= y^{1-c}\Phi(1+a-c, 2-c; y),
\end{align*}
$$

(A.2)

are all defined and distinct only if $c$ is not an integer [21]. Different forms for $\psi^{(i)}$ follow from the Kummer transformations

$$
\Phi(a, c; y) = e^{y}\Phi(-a, -c; -y), \quad \Psi(a, c; y) = y^{1-c}\Phi(1+a-c, 2-c; y).
$$

(A.3)

In this paper, we use only $\psi^{(1)}$, $\psi^{(2)}$ and $\psi^{(3)}$. Their Wronskians are [21]

$$
\begin{align*}
\mathcal{W}[\psi^{(1)}, \psi^{(2)}] &= \mathcal{W}[\Phi(a, c; y), \Psi(a, c; y)] = -\frac{\Gamma(c)}{\Gamma(a)} y^{-c} e^{y}, \\
\mathcal{W}[\psi^{(1)}, \psi^{(3)}] &= \mathcal{W}[\Phi(a, c; y), e^{y}\Psi(-a, -c; -y)] = \frac{\Gamma(c)}{\Gamma(c-a)} e^{2\pi i c} y^{-c} e^{y}, \\
\mathcal{W}[\psi^{(2)}, \psi^{(3)}] &= \mathcal{W}[\Psi(a, c; y), e^{y}\Psi(-a, -c; -y)] = e^{2\pi i (a-c)} y^{-c} e^{y}.
\end{align*}
$$

Therefore, if $a, c$ and $c-a$ are not zero or negative integers, then the three solutions are well defined and any two of them form a fundamental system of solutions for confluent hypergeometric equation [21]. If $c$ is not zero or a negative integer, the solutions are connected by [21]

$$
\frac{\Phi(a, c; y)}{\Gamma(c)} + \frac{e^{\pi i a}}{\Gamma(c-a)} \Psi(a, c; y) + \frac{e^{\pi i (c-a)}}{\Gamma(a)} e^{y} \Psi(-a, -c; -y) = 0.
$$

(A.5)

When $y \to \infty$, the behaviour of $\Psi(a, c; y)$ is given by [18]

$$
\Psi(a, c; y) \sim y^{-a} \sum_{m=0}^{\infty} \frac{(a)_{m}(a-c+1)_{m} (-y)^{-m}}{m!}, \quad -\frac{3\pi}{2} < \arg y < \frac{3\pi}{2},
$$

(A.6)

while the behaviour of $\Phi(a, c; y)$ is given by

$$
\frac{\Phi(a, c; y)}{\Gamma(c)} \sim e^{\gamma a-c} \frac{1}{\Gamma(a)} \sum_{m=0}^{\infty} \frac{(1-a)_{m}(a-c)_{m} y^{-m}}{m!} + \frac{e^{\pi i a} y^{-a}}{\Gamma(c-a)} \sum_{m=0}^{\infty} \frac{(a)_{m}(a-c+1)_{m} (-y)^{-m}}{m!},
$$

$$
a \neq 0, -1, \ldots, c-a \neq 0, -1, \ldots,
$$

(A.7)

where the upper sign holds for $-\pi/2 < \arg y < 3\pi/2$ and the lower sign for $-3\pi/2 < \arg y \leq -\pi/2$. In these limits, $(x)_{m}$ denotes the Pochhammer symbol whose definition is

$$(x)_{0} = 1, \quad (x)_{1} = x, \quad (x)_{m} = x(x+1)(x+2) \cdots (x+m-1) = \Gamma(x+m)/\Gamma(x).
$$

Using these definitions, the function $\Phi(a, c : y)$ is written as

$$
\Phi(a, c; y) = \sum_{n=0}^{\infty} \frac{(a)_{n} y^{n}}{(c)_{n} n!} = 1 + y + \frac{a}{c} y + \frac{a(a+1)}{c(c+1)} \frac{y^{2}}{2!} + \cdots.
$$

(A.8)

In addition, we have the integral representations [21]

$$
\Phi(a, c; y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_{0}^{1} e^{u a^{-1}} (1-u)^{c-a-1} du, \quad \Re(c) > \Re(a) > 0,
$$

(A.9)

and

$$
\Psi(a, c; y) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-yu a^{-1}} (1+u)^{c-a-1} du, \quad \Re(a) > 0.
$$

(A.10)
On the other hand, the Coulomb wavefunctions are solutions of the equation
\[
d^2\mathcal{U}_{n+v}/dy^2 + \left[1 - \frac{2\eta}{y} \right] \frac{(n+v)(n+v+1)}{y^2} \mathcal{U}_{n+v} = 0.
\] (A.11)

If \( \eta = 0 \), this can be written in the usual form of the Bessel equation by a substitution of variable. If \( \eta \neq 0 \), the solutions \( \mathcal{U}_{n+v}(y) = \mathcal{U}_{n+v}^\pm(y, \eta) \) are written in terms of one regular confluent hypergeometric function \( \Phi \) and two irregular functions \( \Psi \), that is,
\[
\mathcal{U}_{n+v}^\pm(n, y) = [\phi_{n+v}^\pm(n, \eta), \psi_{n+v}^+(n, \eta), \psi_{n+v}^-(n, \eta)],
\] (A.12)

where, by definition, we take
\[
\phi_{n+v}^\pm(n, \eta) = \frac{e^{iy}}{\Gamma[2n+2\nu+2]} \left[2iy\right]^{n+\nu+1} \Phi[n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2iy],
\]
\[
\psi_{n+v}^\pm(n, y) = \frac{\pm2ie^{i\pi} e^{iy}}{\Gamma[n + \nu + 1 \mp i\eta]} \left[-2iy\right]^{n+\nu+1} \Psi[n + \nu + 1 \pm i\eta, 2n + 2\nu + 2; \mp2iy].
\] (A.13)

In \( \psi_{n+v}^\pm \), the irrelevant factors \( \pm2i \exp(i\pi) \) are maintained just to connect the above definitions with the ones used by Leaver. In fact, for \( \mathcal{U}_{n+v} \), Leaver used the functions \( F_{n+v}(\eta, y) \) and \( G_{n+v}(\eta, y) \) defined as
\[
F_{n+v}(\eta, y) = \frac{\left[\Gamma(n + \nu + 1 + i\eta)\Gamma(n + \nu + 1 - i\eta)\right]^{1/2}}{2\pi^{\nu+2/2}(2n + 2\nu + 2)} \times e^{i(2y)^{n+\nu+1}} \Phi[n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2iy],
\] (A.14)
\[
G_{n+v}(\eta, y) \mp iF_{n+v}(\eta, y) = e^{\pi/2} e^{i\pi(n+\nu+1/2)} \left[\Gamma(n + \nu + 1 \mp i\eta)\right]^{1/2} \times e^{i(2y)^{n+\nu+1}} \Psi[n + \nu + 1 \pm i\eta, 2n + 2\nu + 2; \mp2iy].
\] (A.15)

Thus, \( \phi_{n+v} \) and \( \psi_{n+v}^\pm \) are obtained by dividing the above expressions by \( \Gamma_n \), defined as
\[
\Gamma_n = (1/2) e^{-\pi/2}(-1)^{n+\nu+1}[\Gamma(n + \nu + 1 + i\eta)\Gamma(n + \nu + 1 - i\eta)]^{1/2}.
\] (A.16)

Inversely, when \( \phi_{n+v} \) and \( \psi_{n+v}^\pm \) are multiplied by \( \Gamma_n \), we recover the Leaver normalization.

From the properties of the functions \( F_{n+v}(\eta, y) \) and \( G_{n+v}(\eta, y) \) given in equations (126) and (125) of Leaver’s paper, we find that the functions (A.12) satisfy the equations
\[
d\mathcal{U}_{n+v}/dy = \frac{i(n + \nu)(n + \nu + 1 + i\eta)(n + \nu + 1 - i\eta)}{(n + \nu + 1)(2n + 2\nu + 1)} \mathcal{U}_{n+v+1} + \frac{i(n + \nu + 1)}{(n + \nu)(n + \nu + 1)} \mathcal{U}_{n+v-1} - \frac{\eta}{(n + \nu)(n + \nu + 1)} \mathcal{U}_{n+v+1} + \frac{i(n + \nu + 1)}{(n + \nu)(2n + 2\nu + 1)} \mathcal{U}_{n+v-1} = 0.
\] (A.17)

and
\[
\frac{(n + \nu)(n + \nu + 1 + i\eta)(n + \nu + 1 - i\eta)}{(2n + 2\nu + 1)} \mathcal{U}_{n+v+1} - i\left[\frac{(n + \nu)(n + \nu + 1)}{y} + \frac{\eta}{y}\right] \mathcal{U}_{n+v} - \frac{(n + \nu + 1)}{(2n + 2\nu + 1)} \mathcal{U}_{n+v-1} = 0.
\] (A.18)

For \( \nu \) and \( \eta \) fixed, by dividing all terms of (A.18) by \( (n^2/2) \mathcal{U}_{n+v} \) and letting \( n \to \pm \infty \) we find
\[
\left[1 + \frac{1}{n}\left(2\nu + 3/2\right)\right] \mathcal{U}_{n+v+1}/\mathcal{U}_{n+v} - \frac{2i}{y}\left[1 + \frac{1}{n}\left(2\nu + 1\right)\right] - \frac{1}{n^2}\left[1 + \frac{1}{2n}\left(2\nu + 1\right)\right] \frac{\mathcal{U}_{n+v-1}}{\mathcal{U}_{n+v}} = 0,
\]
whose solutions are
\[
\frac{\mathcal{U}_{n+v+1}}{\mathcal{U}_{n+v}} \sim \frac{i}{2n^2} \left[1 - \frac{1}{n}\left(2\nu + 5/2\right)\right] \iff \frac{\mathcal{U}_{n+v-1}}{\mathcal{U}_{n+v}} \sim -\frac{2in^2}{y}\left[1 + \frac{1}{n}\left(2\nu + 1/2\right)\right].
\] (A.19)
and
\[
\frac{\mathcal{U}_{n+2}}{\mathcal{U}_{n+v}} \sim \frac{2i}{y} \left[ 1 - \frac{1}{2n} \right] \Rightarrow \frac{\mathcal{U}_{n+1}}{\mathcal{U}_{n+v}} \sim \frac{y}{2i} \left[ 1 + \frac{1}{2n} \right].
\] (A.20)
provided that \(y/n^2 = 0\) when \(n \to \pm\infty\) (this condition is satisfied if \(y\) is finite). Thus, there are two possibilities for the ratios between successive Coulomb functions. By demanding that these relations are valid also for \(\eta = 0\), we find only one ratio: (i) the first expressions in (A.19) and (A.20) hold, respectively, for \(\phi_{n+v}\) and \(\psi^\pm_{n+v}\) when \(n \to \infty\), (ii) the second expression in (A.19) is valid for the three functions when \(n \to -\infty\). In other words,
\[
\frac{\phi_{n+v+1}}{\psi_{n+v}} \sim \frac{iy}{2n^2} \left[ 1 - \frac{1}{n} \left( 2v + \frac{5}{2} \right) \right], \quad \frac{\psi^\pm_{n+v+1}}{\psi^\pm_{n+v}} \sim \frac{2i}{y} \left[ 1 - \frac{1}{2n} \right], \quad [n \to \infty],
\]
\[
\frac{\mathcal{U}_{n+1}}{\mathcal{U}_{n+v}} \sim -\frac{2in^2}{y} \left[ 1 + \frac{1}{n} \left( 2v + \frac{1}{2} \right) \right], \quad \frac{\mathcal{U}_{n+v}}{\mathcal{U}_{n+v}} = \left( \phi_{n+v}, \psi^\pm_{n+v} \right), \quad [n \to -\infty]. \quad \text{(A.21)}
\]

The above conclusions are obtained as follows. In the first place, if \(\eta = 0\), the functions \(\phi_{n+v}\) and \(\psi^\pm_{n+v}\) can be rewritten in terms of Bessel functions since [21]
\[
\Phi(n + v + 1, 2n + 2v + 2; -2iy) = \Gamma(n + v + (3/2)) \left[ y/2 \right]^{-n-v-\frac{1}{2}} e^{-iy} J^n_{n+v+\frac{1}{2}}(y),
\]
\[
\Psi(n + v + 1, 2n + 2v + 2; -2iy) = \frac{i\sqrt{\pi}}{2} e^{-iy+i\pi(n+v+\frac{1}{2})} (2y)^{-n-v-\frac{1}{2}} H^{(1)}_{n+v+\frac{1}{2}}(y),
\]
\[
\Psi(n + v + 1, 2n + 2v + 2; +2iy) = -\frac{i\sqrt{\pi}}{2} e^{iy-i\pi(n+v+\frac{1}{2})} (2y)^{-n-v-\frac{1}{2}} H^{(2)}_{n+v+\frac{1}{2}}(y),
\]
where \(J^n\) is the Bessel function of the first kind, and \(H^{(1)}\) and \(H^{(2)}\) are the first and the second Hankel functions. Hence
\[
\phi_{n+v}(0, y) = \frac{\Gamma^\pm C}{\Gamma[n + v + 1]} \sqrt{\mathcal{U}_{n+v+\frac{1}{2}}(y)}, \quad \psi^\pm_{n+v}(0, y) = \frac{\Gamma^\pm C}{\Gamma[n + v + 1]} \sqrt{\mathcal{U}_{n+v+\frac{1}{2}}(y)}, \quad \text{(A.23)}
\]
where the constants \(C^\pm\) do not depend on \(n\). In the second place, if \(y\) is bounded and \(\kappa \to \infty\) [7],
\[
J^n(y) \sim \frac{1}{\Gamma(n + v + 1)} \left( \frac{y}{2} \right)^n, \quad H^{(1)}(y) \sim -H^{(2)}(y) \sim -\frac{i}{\pi} \Gamma(n + v + 1) \left( \frac{y}{2} \right)^n. \quad \text{(A.24)}
\]
Combining (A.23) with (A.24), we establish (A.21) for \(\kappa = n + v + (1/2)\) when \(n \to \infty\) \((\eta = 0)\). On the other hand, if \(\kappa \to -\infty\), we use the previous relations for \(H^{(1,2)}\) in conjunction with [19]
\[
H^{(1)}_{\kappa}(y) = e^{i\pi \kappa} H^{(1)}_{\kappa}(y), \quad H^{(2)}_{\kappa}(y) = e^{-i\pi \kappa} H^{(2)}_{\kappa}(y). \quad \text{(A.25)}
\]
Thus, we find (A.21) for \(\mathcal{U}_{n+v} = \psi^\pm_{n+v}\) when \(\kappa = n + v + (1/2)\) with \(n \to -\infty\) \((\eta = 0)\). For \(\mathcal{U}_{n+v} = \phi_{n+v}\), if \(y\) is bounded and \(\kappa \to -\infty\), once more we use the relation given in (A.24) for \(J^n(y)\) since
\[
J^n(y) = \left( \frac{y}{2} \right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n + m + 1)} \left( \frac{y}{2} \right)^{2m}
\]
\[
= \left( \frac{y}{2} \right)^n \left[ \frac{1}{\Gamma(n + 1)} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m! \Gamma(n + m + 1)} \left( \frac{y}{2} \right)^{2m} \right]. \quad \text{(A.26)}
\]
In this manner, we establish the ratio (A.21) for the three Coulomb functions when \(n \to -\infty\).
Appendix B. Recurrence relations for the series coefficients

Now we present the derivation of the recurrence relations for the series coefficients of the two-sided solutions. The relations for the other sets of solutions may be obtained from these by transformations of variables. Note that the derivation is formal in the sense that, in each series, we suppose linear independence of all Coulomb wavefunctions.

The Leaver substitutions \[ y(z) = z^{-B_2/2} H(y), \quad y = \omega z \] transform the CHE (1) into

\[
y(y - \omega z_0) \frac{d^2 H}{dy^2} + \left( 1 - \frac{2 \eta}{y} \right) H + C_1 \frac{d H}{dy} + \left[ C_2 + \frac{C_3 \omega}{y} \right] H = 0, \quad \text{where}
\]

\[
C_1 = B_1 + B_2 z_0, \quad C_2 = B_3 - \frac{B_2}{2} \left[ 1 + \frac{B_2}{B_2 z_0} \right], \quad C_3 = \frac{B_2 z_0}{2} \left[ 1 + \frac{B_2}{2} + \frac{B_1}{z_0} \right]. \tag{B.2}
\]

Expanding \( H(y) \) as

\[
H(y) = \sum_{n=-\infty}^{\infty} b_n^1 \mathcal{U}_{n} (\eta, y) \quad \Leftrightarrow \quad U_1(z) = z^{a_1} \sum_{n=-\infty}^{\infty} b_n^1 \mathcal{U}_{n} (\eta, y) \tag{B.3}
\]

and using equations (A.11), (A.17) and (A.18), we find

\[
\sum_{n=-\infty}^{\infty} b_n^1 \left[ a_n a_{n-1} \mathcal{U}_{n+1} (\eta, y) + \beta_n b_n \mathcal{U}_{n} (\eta, y) + \gamma_n b_n \mathcal{U}_{n-1} (\eta, y) \right] = 0. \tag{B.4}
\]

where \( a_n^{(1)}, \beta_n^{(1)} \) and \( \gamma_n^{(1)} \) are defined in equations (20).

If \( v \) is such that the summation runs from minus to plus infinity, then the preceding equation takes the form

\[
\sum_{n=-\infty}^{\infty} \left[ a_n b_n^{1} + \beta_n b_n^{1} + \gamma_n b_{n-1}^{1} \right] \mathcal{U}_{n+1} (\eta, y) = 0, \quad \text{for} \quad v > 0. \tag{B.5}
\]

which is satisfied by the three-term recurrence relations (15) provided that all the functions \( \mathcal{U}_{n+1} (\eta, y) \) are linearly independent.

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