The pair annihilation reaction $D + D \rightarrow \emptyset$ in disordered media and conformal invariance

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The raise and peel model describes the stochastic model of a fluctuating interface separating a substrate covered with clusters of matter of different sizes, and a rarefied gas of tiles. The stationary state is obtained when adsorption compensates the desorption of tiles. This model is generalized to an interface with defects ($D$). The defects are either adjacent or separated by a cluster. If a tile hits the end of a cluster with a defect nearby, the defect hops at the other end of the cluster changing its shape. If a tile hits two adjacent defects, the defect annihilate and are replaced by a small cluster. There are no defects in the stationary state. This model can be seen as describing the reaction $D + D \rightarrow \emptyset$, in which the particles (defects) $D$ hop at long distances changing the medium and annihilate. Between the hops the medium also changes (tiles hit clusters changing their shapes). Several properties of this model are presented and some exact results are obtained using the connection of our model with a conformal invariant quantum chain.

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I. INTRODUCTION

The raise and peel model [1 2] which is a stochastic model of a fluctuating interface is, to our knowledge, the first example of a stochastic model which has the space-time symmetry of conformal invariance. This implies that the dynamic critical exponent $z = 1$ and certain scaling properties of various correlation functions are known. This model was extended in order to take into account sources at the boundaries [3, 4, 5] keeping conformal invariance. In all these cases, the stationary states have magic combinatorial properties.

In the present paper we describe another extension of the raise and peel model keeping conformal invariance (see also Appendix A and B) by introducing defects on the interface. These defects hop at long distances in a medium which is changed by the hops. Between the hops the medium also changes. Finally when two defects touch, they can annihilate. The stationary state is the one as the original raise and peel model with no defects.

The whole process can be seen as a reaction $D + D \rightarrow \emptyset$, where $D$ is a defect, taking place in a disordered unquenched medium.

In Sec. II we describe the model. Like the raise and peel model [1], the present model comes from considering the action of a Hamiltonian expressed in terms of Temperley-Lieb generators on a vector space which is a left ideal of the Temperley-Lieb algebra. The ideal can be mapped on graphs which constitute the configuration space of the model. We shortly review in Appendix A the mathematical background of the model and refer for details to Refs. [4, 5].

In Sec. III, using Monte Carlo simulations, we describe the long range hopping of defects and give the Lévy flights probability distribution.

In Sec. IV, again using Monte Carlo simulations, starting with a configuration which consists only of defects, we study the variation in time $t$ of their density for a lattice of size $L$. We obtain the scaling function which gives the number of defects in terms of $t/L$ and show how conformal invariance gives some of its properties. In the thermodynamic limit the density decreases in time like $1/t$ as is expected since in a conformal invariant theory time and space are on equal footing. In Sec. V we present our conclusions.

II. THE RAISE AND PEEL MODEL WITH DEFECTS.

We consider an interface of a one-dimensional lattice with $L + 1$ sites. An interface is formed by attaching at each site a non-negative integer height $h_i$ ($i = 0, 1, \ldots, L$). We take $h_0 = h_L = 0$. If for two consecutive sites $j$ and $j + 1$ we have $h_j = h_{j+1} = 0$, on the link connecting the two sites we put an arrow called defect (see Fig. 1). For the remaining sites, the heights obey the restrict solid-on-solid (RSOS) rules:

$$h_{i+1} - h_i = \pm 1, \quad h_i \geq 0. \quad (1)$$

A domain in which the RSOS rules are obeyed \{ $h_j = h_l = 0, h_k > 0, j < k < l$ \} is called a cluster. There are 3 clusters and 3 defects in Fig. 1 ($L = 21$). There are
There are 3 defects (arrows on the links) and 3 clusters. Also shown are 5 tiles (tilted squares) (a)-(e) belonging to the gas. When a tile hits the surface the effect is different in the five cases.

\((\binom{L}{L/2})\) possible configurations of the interface (we denote by \([x]\) the integer part of \(x\)).

There is a simple bijection between the configurations of interfaces and defects, where Fig. 1 is an example, and ballot paths \([3]\). A ballot path is obtained if one follow the RSOS rules \([1]\), take \(h_L = 0\) but let \(h_0\) free \((0 \leq h_L \leq L)\). This fact was used in \([3]\) to define another stochastic model than the one described below. In the case \(L = 4\), the six possible configurations are shown in Fig. 2. The configuration shown in Fig. 2b has 2 defects and one cluster, while there are no defects in Fig. 2f.

We consider the interface separating a film of tiles (clusters with defects) from a gas of tiles (tilted squares). The evolution of the system (Monte Carlo steps) is given by the following rules. With a probability \(P_i = \frac{1}{L-1}\) a tile from the gas hits site \(i\) \((i = 1, \ldots, L - 1)\). As a result of this hit, the following effects can take place:

a) The tile hits a local maximum of a cluster (“a” in Fig. 1). The tile is reflected.

b) The tile hits a local minimum of a cluster (“b” in Fig. 1). The tile is adsorbed.

c) The tile hits a cluster and the slope is positive \((h_{i+1} > h_i > h_{i-1})\) (“c” in Fig. 1). The tile is reflected after triggering the desorption of a layer of tiles from the segment \((h_j > h_i = h_{i+b}, j = i + 1, \ldots, i + b - 1)\), i.e., \(h_j \rightarrow h_j - 2, j = i + 1, \ldots, i + b - 1\). The layer contains \(b - 1\) tiles (this is an odd number). Similarly, if the slope is negative \((h_{i+1} < h_i < h_{i-1})\), the tile is reflected after triggering the desorption of a layer of tiles belonging to the segment \((h_j > h_i = h_{i-b}, j + i - b + 1, \ldots, i - 1)\).

d) The tile hits the right end of a cluster \(h_j > h(i - c) = h(i) = 0\) \((j = i - c + 1, \ldots, i - 1)\) and \(h(i + 1) = 0\). The link \((i, i + 1)\) contains a defect (“d” in Fig. 1). The defect hops on the link \((c, c + 1)\) after triggering the desorption of a layer of tiles \((h_j \rightarrow h_j - 2, j = i - c + 1, \ldots, i - 1)\) and the tile is adsorbed producing a new small cluster \((h_{i-1} = h_{i+1} = 0, h_i = 1)\) (see Fig. 3). If the defect is at the left end of a cluster, the rules are similar, the defect hops to the right after peeling the cluster, and a new small cluster appears at the end of the old one.

e) The tile hits a site between two defects \((h_{i-1} = h_i = h_{i+1} = 0)\). This is the case “e” in Fig. 1. The two defects annihilate and in their place appears a small cluster \((h_{i-1} = h_{i+1} = 0, h_i = 1)\). See Fig. 4.

To sum up. The defects \((D)\) hop non-locally in a disordered (not quenched) medium which changes between successive hops (local adsorption and nonlocal desorption take place in the clusters). During the hop the defect peels the cluster and therefore also changes the medium. The annihilation reaction \(D + D \rightarrow \emptyset\) is local. If one starts the stochastic process with a certain configuration (for example, only defects as in Fig. 3\), due to the annihilation process, for \(L\) even all the defects disappear and in the stationary state one has only clusters (RSOS configurations). The properties of the stationary states have been studied elsewhere \([1, 2]\). In the case \(L\) odd,
in the stationary states one has one defect. In the next section we are going to see how this defect hops and will observe that the defect behaves like a random walker performing Lévy flights. This will help us understand the annihilation process \( D + D \rightarrow \emptyset \) described in Sec. IV. The rules described above were obtained by using a representation of the Temperley-Lieb algebra in a certain ideal \([3, 4, 5]\) (see Appendix A). The finite-size scaling of the Hamiltonian eigenspectrum is known from conformal field theory (see Appendix B), therefore the physical properties of the model can be traced back to conformal invariance.

III. THE RANDOM WALK OF A DEFECT

Before discussing the annihilation reaction of defects, it is useful to understand how defects hop. The simplest way to study the behavior of defects is to take the stationary states in the case \( L \) odd when we have only a single defect. Although there is a lot of information about these stationary states coming from combinatorics \([3, 4]\) and Monte Carlo simulations \([5]\), the results we present here are new.

One asks what is the probability \( P(s) \) for a defect to hop, at one Monte-Carlo step, at a distance \( s \) (we assume \( L \) very large). We first see if on physical grounds, one can’t guess the result. Let us assume that the defect behaves like a random walker and that \( P(s) \) describes Lévy flights \([3, 8, 9, 10]\). This implies that for large values of \( s \) we have:

\[
P(s) \sim \frac{1}{|s|^{1+\sigma}}.
\]

If the random walker starts at a point \( x = 0 \) (for example in the middle of the lattice), at large values of \( t \), the dispersion is \( 10 \):

\[
\langle x^2 \rangle \sim t^{2\sigma}.
\]

In a conformal invariant model, one has no other scales but the size of the system, space and time are on equal footing therefore one has to have \( \sigma = 1 \).

In Fig. 5 we show \( P(s) \) as obtained from Monte Carlo simulations for systems of different sizes. One notices a data collapse for a large domain of \( s \). A fit to the data for the largest lattice \((L = 4095)\) gives, for large \( s \):

\[
P(s) \approx \frac{2.25}{|s|^{2.06}},
\]

in agreement with what we expected.

IV. THE DENSITY OF DEFECTS AT LARGE TIMES

We are now going to study the number of defects \( N_d(t, L) \) as a function of time and lattice size taking at

\[
\ln[P(s)] = 0.81 - 2.06 \ln(s/|L|).
\]

In Fig. 6, we show \( N_d(t, L) \) for various lattice sizes \((L \) odd). One sees a nice data collapse except for very small values of \( t/L \) where the convergence is slower. A similar (but not identical!) function is obtained for \( L \) even.

We first discuss the behavior of \( N_d \) for large values of \( t/L \) (see Fig. 7). A fit to the data gives \((L \) odd):

\[
N_d(t/L) = f(L) = A_1^{(o)} e^{-\lambda_1^{(o)} t/L} + A_2^{(o)} e^{-\lambda_2^{(o)} t/L} + \cdots,
\]

where

\[
A_1^{(o)} = 6.75, \quad A_2^{(o)} = 17.27,
\]

\[
\lambda_1^{(o)} = 8.21, \quad \lambda_2^{(o)} = 26.48.
\]

We can now compare the data obtained from the fit with the finite-size scaling spectrum of the Hamiltonian (see Appendix B, Eqs. \([18, 19]\)):

\[
\lambda_1^{(o)} = \frac{3\sqrt{3}}{2} \approx 8.1620971 \cdots,
\]

\[
\lambda_2^{(o)} = \frac{3\sqrt{3}}{3} \approx 27.20699 \cdots.
\]

No prediction can be made about \( A_1^{(o)} \) or \( A_2^{(o)} \) since they are not universal, they depend on the initial conditions. Notice that \( A_2^{(o)} > A_1^{(o)} \) as it should be since the expansion should diverge for short times where we expect

\[
N_d \sim \frac{L}{t}.
\]
FIG. 6: (Color online) The number of defects $N_d(t, L)$ as a function of $t/L$ for several lattice sizes ($L$ odd). At $t = 0$, $N_d(t = 0, L) = L$. The error bars are also shown. The fitted linear curve shows that the density decreases as the inverse of time.

A similar fit, done for $L$ even (the data are shown in Fig. 8), gives:

$$N_d(t/L) = A_1^{(e)} e^{-\lambda_1^{(e)} t/L} + A_2^{(e)} e^{-\lambda_2^{(e)} t/L} + \cdots,$$

with

$$A_1^{(e)} = 2.83, \quad A_2^{(e)} = 6.93,$$
$$\lambda_1^{(e)} = 2.71, \quad \lambda_2^{(e)} = 16.64.$$  

(10)

We can again use the predictions of conformal invariance (see (B8) and (B9)) and get

$$\lambda_1^{(e)} = \frac{3\pi \sqrt{3}}{2} \frac{1}{3} = 2.72069 \cdots,$$

$$\lambda_2^{(e)} = \frac{3\pi \sqrt{3}}{2} = 16.324194 \cdots.$$  

(11)

In the small $t/L$ domain we get for $L$ even and odd:

$$\rho = \frac{N_d}{L} \approx \frac{0.322}{t}.$$  

(13)

In order to find the correction term in (13), we have computed $N_d t^2/L$ as shown in Fig. 9. We have obtained a straight line from which we get:

$$\rho = \frac{0.322}{t} + \frac{0.334}{t^2} + \cdots.$$  

(14)

This last result is the same for $L$ even and odd. Notice that the correction term in (13) is not given by the scaling function (5).

We have also computed the fluctuation of the density as a function of time and got:

$$\langle \rho^2 \rangle - \langle \rho \rangle^2 \approx \frac{0.237}{t^{1.06}}.$$  

(15)

We would like to compare our results with known results obtained for diffusion and annihilation reactions ($A + A \rightarrow \emptyset$) with Lévy flights [11, 12, 13, 14, 15]. In one dimension, for Lévy flights given by Eq. (2), one gets [12]:

$$\rho \sim \begin{cases} t^{-\frac{1}{\sigma}} & \text{for } \sigma > 1 \\ \ln t & \text{for } \sigma = 1 \\ t^{-1} & \text{for } \sigma < 1 \end{cases}$$  

(16)

the critical dimension being $d_c = \sigma$.

If one compares (10) for $\sigma = 1$, as obtained in Sec. III and (14), one notices the absence of the ln $t$ correction. Such a term, if present, could have been seen in...
of this model is that conformal invariance is preserved. The model mimics a system in which particles move in a disordered unquenched medium doing Lévy flights and changing the medium during the flights. Upon contact the defects annihilate. The properties of the system are simple and could be guessed on simple grounds based on conformal invariance. Conformal field theory enters in the description of the scaling function \( N_d = f(t/L) \) \( (N_d \) is the number of defects, \( L \) the size of the system and \( t \) is the time).

The original raise and peel model\(^2\) (this is the present model with the defects absent) depends on a parameter \( w \) which is the ratio of the desorption and adsorption rates. If \( w = 1 \), one has conformal invariance and the dynamic critical exponent \( z = 1 \). If one takes \( 0 < w < 1 \), in the disordered medium one has less clusters and \( z \) varies continuously in the interval \( 0 < z < 1 \). One can add defects to the model and repeat the exercise done in this paper for all values of \( w \). In this case one expects to find defects making Lévy flights with a probability distribution function:

\[
P(s) \sim \frac{1}{s^{1+z}}.
\]

(V) CONCLUSIONS

We have presented an extension of the raise and peel model taking into account defects. The main property

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APPENDIX A: THE CONNECTION BETWEEN TEMPERLEY-LIEB STOCHASTIC PROCESSES AND THE RAISE AND PEEL MODEL

We shortly review this connection, for details see [4,18] and [3].

Consider the Temperley-Lieb semigroup algebra $G$ defined by $L - 1$ generators $e_j$ ($j = 1, \ldots, L - 1$) and the relations:

$$e_j^2 = e_j, \quad e_j e_j \pm 1 e_j = e_j, \quad e_j e_k = e_k e_j \quad \text{for} \quad |j - k| > 1,$$

(A1)

and the Hamiltonian

$$H = \sum_{j=1}^{L-1} (1 - e_j).$$

(A2)

In the basis $\{w_c\}$ of the words of $G$ (the regular representation of $G$), $H$ is a matrix satisfying $H_{a,b} \leq 0$ for $a \neq b$ and $\sum_b H_{a,b} = 0$. Such a matrix is an intensity matrix and defines a Markov process in continuum time given by the master equation

$$\frac{d}{dt} P_a(t) = -\sum_b H_{a,b}P_b(t),$$

(A3)

where $P_a(t)$ is the (unnormalized) probability to find the system in the state $|a\rangle$ at time $t$ and the rate for the transition $|b\rangle \rightarrow |a\rangle$ is given by $-H_{a,b}$, which is non-negative. The Hamiltonian (A2) has an eigenvalue equal to zero. The corresponding left eigenvector $\langle 0 |$ is trivial, the right eigenvector $|0\rangle$ gives the probabilities in the stationary state:

$$\langle 0 | H = 0, \quad \langle 0 | = \sum_a \langle a |,$$

$$H|0\rangle = 0, \quad |0\rangle = \sum_a P_a |a\rangle, \quad P_a = \lim_{t \to \infty} P_a(t).$$

(A4)

$H$ defined by (A2) gives a Markov process not only if it acts in the vector space of the regular representation but also if it acts in the vector space of a left ideal $I$ because the generators $e_j$ map the left ideal into the left ideal:

$$e_j I = I.$$

(A5)

An easy way to define the left ideal in which we are interested and the action of the generators on this ideal is to use the language of graphs.

The generators $e_j$ can be pictorially represented by

$$e_j = \begin{bmatrix} \cdots & 1 & 2 & \cdots \\ j-1 & j & j+1 & \cdots \\ \cdots & L-1 & L \end{bmatrix}.$$  

(A6)

The elements of the ideal can be represented by links-defects diagrams. They can be obtained in the following way (see Fig. 10 for $L = 4$): take $L$ sites. If a site is not connected to another one, draw a vertical arrow. Two sites can be connected by a link. The links don’t cross each other and the arrows can’t cross the links. For a given $L$ the number of diagrams with $m$ defects is:

$$C_{L,m} = \left( \begin{array}{c} L \\ \frac{L-1}{2} \end{array} \right) - \left( \begin{array}{c} L - m - 1 \\ \frac{L-1}{2} \end{array} \right)$$

(A7)

and the total number of diagrams is

$$\sum_{s=0}^{L/2} C_{L,2s+L \mod 2} = \left( \begin{array}{c} L \\ \frac{L}{2} \end{array} \right),$$

(A8)

where $[x]$ is the integer part of $x$.

FIG. 10: The six links-defects diagrams for $L = 4$. The diagrams a-f correspond to the a-f RSOS configurations of Fig. [2].

The action of $e_j$ on a links-defects diagram is given by placing the graph of $e_j$ underneath the first diagram, removing the closed loops and the intermediate dashed line. Next one contracts the links in the composite picture. In Fig. 11 we show the action of $e_2$ on the diagram $b$ of figure [10].

FIG. 11: The action of $e_2$ on the diagram $b$ of Fig. [10].

The action of the Hamiltonian (A2) in the vector space given in Fig. 11 is

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. $$

(A9)

Notice that $H$ has a block triangular form. The stationary state $\langle 0 | = 2 \langle e \rangle + \langle f \rangle$ contains only the two states
without arrows (defects) $|e⟩$ and $|f⟩$. The various transition rates can be obtained from the matrix elements of $H$.

In Appendix B we are going to use a $2^L$ dimensional representation of the $L-1$ generators $e_i$, and of the Hamiltonian (A2). In this representation, the Hamiltonian describes a spin 1/2 quantum chain. Where can we find the eigenvalues of the left ideal (their number is given by (A5)), among the $2^k$ eigenvalues of the quantum chain? We are going to give an "almost correct" explanation. We take again the case $L = 4$ as an example. If on each of the 4 sites of the chain one takes a spin 1/2 representation of $sl(2)$, one finds the representation with spin 0 (2 times), spin 1 (3 times) and spin 2 (one time). If for each representation containing $2s + 1$ states ($s$ is the spin) one takes only the highest weight states, one gets precisely 6 states (the vertical arrows in Fig. 10 corresponding to up spins).

We give now the correspondence between the links-defects diagrams and the RSOS configurations considered in section 2. For a links-defects diagram with $L$ sites ($i = 1, 2, \ldots, L$) take the dual lattice with $L+1$ sites (on each bond between the sites $i$ and $i+1$ of the links-defects diagram you take the site $i$ on the dual lattice. On the dual lattice we have the sites $j$ ($j = 0, 1, \ldots, L$). An arrow (defect) on the site $i$ on the links-defects diagram, stays unchanged on the dual lattice (it is on the bonds of the dual lattice). For the links, one proceeds as follows: one takes a site on the dual lattice and a vertical line on this site. One counts how many links are cut by the vertical line and one takes a vertex with a height $h$ equal to the number of intersections. Figs. 2 and 10 illustrate the rules.

APPENDIX B: THE FINITE-SIZE SCALING LIMIT OF THE HAMILTONIAN EIGENSPECTRUM. RESULTS FROM CONFORMAL FIELD THEORY.

We are going to give a brief description of the time evolution operator of the stochastic model described in Sec. 2. Firstly we consider the spin 1/2 quantum chain defined by the Hamiltonian

$$H = \sum_{i=1}^{L-1} (1-e_i),$$

where

$$e_i = \frac{1}{2} [\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} - \frac{1}{2} \sigma^z_i \sigma^z_{i+1} + i \frac{\sqrt{3}}{2} (\sigma^x_i - \sigma^x_{i+1})],$$

and $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices. The Hamiltonian (B1) commutes with

$$S^z = \frac{1}{2} \sum_{i=1}^L \sigma^z_i.$$  

In the continuous time limit, the evolution of the system is given by a Hamiltonian $H^e$ which corresponds to the subspace of highest weight $U_q(sl(2))$ representations $(q = \exp i\pi/3)$ [19]. There are $(\frac{L}{2})^L$ states in these two sectors ([x] is the integer part of x). If we denote by $E_r$ ($r = 0, 1, \ldots$) the energy levels in non decreasing order: $E_0 = 0 < E_1 \leq E_2 \leq \cdots$, the partition function giving the finite-size scaling limit of the spectrum of $H^e$ is defined as follows:

$$Z(q) = \lim_{L \to \infty} \sum \sum_{n} q^{LE_n/\pi v_s},$$

where $v_s = 3\sqrt{3}/2$. One can show [20] that $Z(q)$ has the expression

$$Z(q) = \sum_s \zeta_s(q).$$

Here $s$ is the spin, taking the values $s = 0, 1, 2, \ldots$ for $L$ even and $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ for $L$ odd, and

$$\zeta_s(q) = q^{\Delta_s} (1-q^{2s+1}) \prod_{n=1}^\infty (1-q^n)^{-1},$$

where

$$\Delta_s = \frac{s(2s-1)}{3},$$

Moreover, for large lattice sizes, the energies are (see [21] and [B6])

$$E = \frac{3\pi \sqrt{3}}{2L} (\Delta_s + k),$$

where $k$ is an integer.

The Hamiltonian $H^e$ has a block diagonal form. The states with no defects ($L$ even) and those with one defect ($L$ odd) are in one block. This is the $s = 0$ ($s = \frac{1}{2}$) part of [B5]. The states with defects ($L$ even) and more than one defect ($L$ odd) correspond to higher spins. In Sec. 3 we found that the following values of $\Delta_s$ were useful:

$$\Delta_1 = \frac{1}{3}, \Delta_2 = 2 \quad (L \text{ even})$$

$$\Delta_{\frac{3}{2}} = 1, \Delta_{\frac{5}{2}} = \frac{10}{3} \quad (L \text{ odd})$$

$$\Delta_{\frac{5}{2}} = 1, \Delta_{\frac{7}{2}} = \frac{10}{3} \quad (L \text{ odd})$$