STOCHASTIC OPTIMIZATION ON MATRICES AND A GRAPHON MCKEAN-VLASOV LIMIT

ZAID HARCHAOUI, SEWOONG OH, SOUMIK PAL, RAGHAV SOMANI, AND RAGHAVENDRA TRIPATHI

Abstract. We consider stochastic gradient descents on the space of large symmetric matrices of suitable functions that are invariant under permuting the rows and columns using the same permutation. We establish deterministic limits of these random curves as the dimensions of the matrices go to infinity while the entries remain bounded. Under a “small noise” assumption the limit is shown to be the gradient flow of functions on graphons whose existence was established in [OPST21]. We also consider limits of stochastic gradient descents with added properly scaled reflected Brownian noise. The limiting curve of graphons is characterized by a family of stochastic differential equations with reflections and can be thought of as an extension of the classical McKean-Vlasov limit for interacting diffusions to the graphon setting. The proofs introduce a family of infinite-dimensional exchangeable arrays of reflected diffusions and a novel notion of propagation of chaos for large matrices of diffusions converging to such arrays in a suitable sense.

1. Introduction

The study of particle systems under mean-field interaction is a classical topic in probability theory [Gär88]. It involves multidimensional diffusions that interact through their empirical distributions of the type

\[ \frac{d\bar{X}(t)}{dt} = \mu(\bar{X}(t)) \]  

1: Zaid Harchaoui, Department of Statistics, University of Washington, Seattle WA 98195, USA, Email: zaid@uw.edu

Sewoong Oh, Paul G. Allen School of Computer Science & Engineering, University of Washington, Seattle WA 98195, USA, Email: sewoong@cs.washington.edu

Soumik Pal, Department of Mathematics, University of Washington, Seattle WA 98195, USA, Email: soumik@uw.edu

Raghav Somani, Paul G. Allen School of Computer Science & Engineering, University of Washington, Seattle WA 98195, USA, Email: raghavs@cs.washington.edu

Raghavendra Tripathi, Department of Mathematics, University of Washington, Seattle WA 98195, USA, Email: raghavt@uw.edu

Date: May 29, 2024.

2020 Mathematics Subject Classification. 60K35,62L20,60G09,68W20.

Key words and phrases. gradient flows on graphons, stochastic approximations, stochastic gradient descent on matrices, exchangeable arrays of diffusions, reflected Brownian motions.

This research is partially supported by the following grants. All the authors are supported by NSF grant DMS-2134012. Additionally, Oh is supported by NSF grant CCF-2019844 and CNS-2002664; Pal is supported by NSF grant DMS-2052239 and DMS-2133244; Harchaoui is supported by NSF grant CCF-2019844, DMS-2023166, DMS-2133244. Thanks to PIMS Kantorovich Initiative for facilitating this collaboration. The authors are listed in alphabetical order.
of i.i.d. standard $d$-dimensional Brownian motions. Prominent examples of such particle systems include the diffusion given by the SDE

$$\begin{align*}
(2) \quad dX_i(t) &= -\nabla V (X_i(t)) \, dt - \frac{1}{N} \sum_{j=1}^{N} \nabla W (X_i(t) - X_j(t)) \, dt + dB_i(t), \quad t \in \mathbb{R}_+,
\end{align*}$$

for $i \in [N]$, where $V$ and $W$ are differentiable convex functions on $\mathbb{R}^d$. However, any drift that is symmetric in the coordinates ("mean-field interactions") can be represented as (1) for some suitable function $b$. Often, the SDE (1) includes a reflection term to constrain the coordinate process to a subset of the Euclidean space [Szn84]. The study of such systems originated from the probabilistic study of the Boltzmann and Vlasov equations due to Kac [Kac56], McKean [McK75], Dobrushin [Dob79], Tanaka [Tan79] and many others. For modern surveys, see Sznitman [Szn91], Villani [Vil12], Chaintron and Diez [CD22] and Jabin [Jab14].

Under suitable assumptions, as the number of particles go to infinity, it is known that the process of empirical distributions of the particle system converges to the solutions of families of well-known PDEs. For example, for the system (2), the random process $\hat{\mu}^N(t)$ converges weakly to the solution of granular media equation [CGM08], as $N \to \infty$. The convergence is often obtained via propagation of chaos where, in the large particle limit, a finite collection of randomly chosen particles evolve independently and identically distributed according to the McKean-Vlasov SDE [Gär88]: $dX(t) = b(X(t), \mu(t)) \, dt + dB(t), \quad t \in \mathbb{R}_+$, where $\mu(t)$ is the law of $X(t)$.

In this work, we study an analogous evolution of symmetric matrices where the coordinates interact via a suitably symmetric function. As an example, consider the function $R_n$ defined on $\mathcal{M}_n^0$, the set of all $n \times n$ symmetric matrices with entries in $[0,1]$, given by

$$R_n(A) := \frac{1}{2} \left( n^{-2} \sum_{i,j=1}^{n} A(i,j) - \epsilon \right)^2 + \frac{1}{2} (n^{-3} \text{Tr}[A^3] - \tau)^2 + \mathcal{E}_n(A),$$

where $\epsilon, \tau \in [0,1]$ are fixed and $\mathcal{E}_n(A) = n^{-2} \sum_{i,j=1}^{n} h(A(i,j))$ where $h: [0,1] \to \mathbb{R}$ is the convex entropy function defined as $h(p) := p \log p + (1 - p) \log(1 - p)$, if $p \in (0,1)$, and $h(0) = h(1) = 0$. The importance of this particular example will soon become apparent.

The function $R_n$ satisfies a permutation invariance property, that is, its value does not change if we permute the rows and columns of the matrix $A$ by the same permutation over $[n] := \{1,2,\ldots,n\}$. Consider the following diffusion on symmetric $n \times n$ matrices

$$\begin{align*}
(4) \quad dX_n(t) &= -n^2 \nabla R_n(X_n(t)) \, dt + \beta \, dB_n(t) + dL_n(t), \quad t \in \mathbb{R}_+,
\end{align*}$$

where $B_n$ is a system of $n \times n$ symmetric matrix-valued process of coordinatewise independent Brownian motions and $L_n$ is the coordinatewise bounded variation local time process that constrains each coordinate process to stay in the interval $[0,1]$ (see Section 2.3 for details). One may ask what is an appropriate notion of limit of such a process as $n \to \infty$? Does (4) exhibits propagation of chaos?

Note that the function $R_n$ in (4) is not covered by the classical McKean-Vlasov theory since $R_n(A)$ is not symmetric in the $n^2$ (up to symmetry) many entries of a matrix $A$. Therefore, $R_n$ cannot be expressed as a function of the empirical distribution of the entries of the argument matrix. The same is true for any arbitrary differentiable function over $n \times n$ symmetric matrices that is invariant under permuting the rows and the columns using the same permutation. Spectral functions, for example, satisfy such an invariance, as do functions on edge-weighted graphs (represented by their adjacency matrices) that are invariant under
vertex relabeling. This particular class of symmetry is captured, not by empirical measures but, by graphons. In other words, such functions can be thought of as functions on the space of graphons instead of measures. Analogous to the classical McKean-Vlasov theory, we show in this paper that, under suitable assumptions, (4) exhibits a propagation of chaos. Furthermore, in $n \to \infty$ limit, the coordinates of $X_n(t)$ become conditionally independent and the evolution of a randomly chosen coordinate can be described by a novel graphon valued McKean-Vlasov equation. The existence and uniqueness of such process is established in Proposition 4.5. Proposition 4.6 shows that the process $X_n(t)$ converges to a deterministic curve on the space of graphons, $\hat{\mathcal{W}}$ (see Section 2).

Recently various authors [DGL16, BBW19, Cop22, DM22] have investigated McKean-Vlasov limits for interacting particles systems on dense graphs. This is akin to equation (2) where particles interact only if they are neighbors in some underlying graph. In these work, the McKean-Vlasov system describes the evolution of random particle from an infinite ensemble where the underlying interaction is determined by a graph or graphon. Extensions to the sparse regime can be found in [LRW19, OR19, BCN20, BCW20, ORS20]. We note that our McKean-Vlasov limit describes the evolution of graphon itself, and not the distribution of any particle system. We borrow the name McKean-Vlasov to stress that each edge-weight evolves by an ensemble effect of all the other edge weights but that ensemble is a graphon and not the empirical distribution of any particle system as done in the papers cited above.

While the evolution of matrices described by (4) can arise in many different contexts, our primary motivation for studying such evolution comes from the stochastic gradient descent on large graphs. Consider the function $R_n$ defined in (3). Notice that if we regard $A \in \mathcal{M}_n^0$ as the adjacency matrix of a weighted graph, then $n^{-2} \sum_{i,j=1}^n A_{i,j}$ can be thought of as the edge-density of the graph while $n^{-3} \text{Tr}[A^3]$ can be regarded as the density of triangles in the graph. Consider the problem of minimizing $\mathcal{E}_n(A)$ over all $A \in \mathcal{M}_n^0$ subject to the condition that the edge-density and the triangle-density are $e$ and $\tau$, respectively. The non-convexity of this problem makes it very hard and indeed the minimizers in general are not unique [NRS23, KRRS17]. For certain feasible regime of $(e, \tau)$, this minimization problem has been studied in [NRS23]. In some regimes where the minimizer is known to be unique, the minimizer is characterized. Similar results were obtained in [KRRS17] when $\tau = e^3$. In general, however, determining the structure of minimizers is extremely hard and even reasonable guesses are not available in most cases. This problem has rich connections with extremal combinatorics [Raz08, PR17] and exponential random graph models [BCM21, CV11]. While minimizing $\mathcal{E}_n$ with such ‘hard constraints’ is difficult, notice that minimizing $R_n(A) := \left( n^{-2} \sum_{i,j=1}^n A(i,j) - e \right)^2 / 2 + (n^{-3} \text{Tr}[A^3] - \tau)^2 / 2 + \mathcal{E}_n(A)$ can be considered as a relaxation of this problem. Our method provides a numerical scheme to obtain minimizers of such problems. Notice that (4) arise as the limit of the projected stochastic gradient descent algorithm which is used in practice to optimize $R_n$. As mentioned above, we establish that the curves described by (4) converges to a deterministic curve on the space of graphons. Under appropriate assumptions on $R_n$ (see Section 5.3) and in zero-noise limit, the (deterministic) limiting curve on the space of graphons is a gradient flow and hence converges to the minimizer exponentially fast. Thus, the evolution (4) gives a way to numerically approximate the minimizer. More generally, the limiting curve converges to stationary points and thus (4) provides an algorithm to numerically approximate these stationary points that may be useful in obtaining reasonable guesses regarding the structure of the minimizers in
such problems. We describe the projected gradient descent and projected stochastic gradient descent algorithms in more detail in the following paragraphs.

Projected Gradient Descent (GD) based algorithms are the workhorse in optimizing such functions [Cau47, Bub15, BCN18]. However, in most cases, computing gradients can be computationally intensive. In practice, stochastic approximation algorithms based on projected Stochastic Gradient Descent (SGD) are instead used to minimize such functions since they are often faster to simulate [RM51, KW52]. The details of this common Markov chain are described later in the section, and the reader can refer to the monographs [Ben99, KY03, Bor09, MB11, KC12] for a detailed overview. Roughly, if the current state is a symmetric matrix $A$, one jumps to a new state by taking a small step along the negative Euclidean gradient $-\nabla R_n(A)$, and potentially adding independent, centered, and variance-bounded noise to each matrix entry (up to symmetry). Each matrix entry is then projected on to the interval $[0, 1]$ to satisfy the entrywise constraint.

Gradient descent (GD), with small step sizes, approximates the Euclidean gradient flow obtained as a solution to the Cauchy’s problem

$$\dot{A}_{i,j}(t) = -\nabla_{i,j} R_n(A(t)), \quad (i, j) \in [n]^2, \quad t \in \mathbb{R}_+,$$

in the interior of $\mathcal{M}_n^0$. Here $\mathbb{R}_+$ denotes the set of non-negative real numbers which is used to index time, $\nabla_{i,j}$ refers to the partial derivative with respect to the $(i, j)$-th matrix entry. It is therefore natural to understand a suitable scaling limit of SGD on the space of such matrices.

A previous work [OPST21, Theorem 4.17] showed that under suitable assumptions on $(R_n)_{n \in \mathbb{N}}$, the implicit Euler update scheme approximates a gradient flow curve, in an appropriate sense, over the space of graphons, $\widehat{\mathcal{W}}$, when the step size is taken to zero and $n$ grows to infinity. The reader is referred to [LS06, BCL+08, BCL+12] and Section 2.1 for the required exposition on graphons. In this work, we ask a similar question for SGD based algorithms. We show that under an appropriate “small noise” assumption and a consistency and other suitable assumptions on the functions $(R_n)_{n \in \mathbb{N}}$, the SGD iterations converge appropriately to a limiting deterministic curve (up to symmetry) on the space of such matrices. Moreover, when an extra Gaussian noise is added to each SGD iterate, the noisy SGD iterations also converge to a deterministic curve on graphons which admits a McKean-Vlasov description. Similar McKean-Vlasov system has been studied in [APST23], however, the focus of [APST23] is to study a particular Markov chain on large graphs, namely a version of the Metropolis Markov chain. These Markov chains are designed to mimic the gradient flow in the limit.

Very roughly, $\mathcal{W}$, the set of bounded symmetric measurable functions on $[0, 1]^2$ or kernels, is our limiting space for symmetric matrices. The set of graphons, $\widehat{\mathcal{W}}$, is obtained as a quotient of $\mathcal{W}$ where we identify two kernels to be the same if one can be obtained from the other by using the same measure-preserving transformation on its “rows” and “columns” (see Section 2.1). Thus, a function $R: \widehat{\mathcal{W}} \to \mathbb{R}$ over graphons naturally extends to a function over the set of kernels $\mathcal{W}$. For any $n \in \mathbb{N}$, the set of symmetric matrices $\mathcal{M}_n$, over which algorithms like GD and SGD operate on, can be naturally identified with a subset, finite dimensional kernels, $\mathcal{W}_n \subset \mathcal{W}$ of the kernels (see Section 2.1 for details). This identification/embedding will be denoted by $K$ (as in kernel) and its inverse will be denoted by $M_n$ (as in matrix). Using $K$, the restriction of the function $R$ to $\mathcal{W}_n$ can be viewed as a function $R_n$ on $\mathcal{M}_n$. 
Define the projection operator \( P : \mathbb{R} \to [-1, 1] \) as
\[
P(x) := \begin{cases} 
-1 & \text{if } x \in (-\infty, -1), \\
x & \text{if } x \in [-1, 1], \\
1 & \text{if } x \in (1, \infty).
\end{cases}
\]
The operator \( P \) can be used coordinatewise on matrices and kernels. For every \( n \in \mathbb{N} \), let \( \tau_n := (\tau_{n,k})_{k \in \mathbb{Z}_+} \) be a sequence of positive step sizes (also known as the learning rate). Here \( \mathbb{Z}_+ \) denotes the set of all non-negative integers. Given the step size sequence \( \tau_n \), we can define a monotonically increasing sequence of times \( (t_{n,k})_{k \in \mathbb{Z}_+} \), defined as a cumulative sum of \( \tau_n \), i.e., \( t_{n,0} = 0 \) and \( t_{n,k} := \sum_{j=0}^{k-1} \tau_{n,j} \) for any \( k \in \mathbb{N} \). We assume \( \tau_n \) to have a divergent sum so to cover the whole non-negative real line \( \mathbb{R}_+ \), i.e., to satisfy \( \lim_{k \to \infty} t_{n,k} = \infty \). We define the norm of the step size sequence \( \tau_n \) as \( |\tau_n| := \sup_{k \in \mathbb{Z}_+} \tau_{n,k} \), which is assumed to be finite. We now describe our first iterative scheme.

**Definition 1.1 (Projected GD).** Let \( n \in \mathbb{N} \) and let \( R_n : \mathcal{M}_n \to \mathbb{R} \) be a differentiable function. The projected GD iterates of \( R_n \) starting at \( V_{n,0} \in \mathcal{M}_n \) is defined to be a sequence of symmetric matrices \( (V_{n,k})_{k \in \mathbb{Z}_+} \) given iteratively as
\[
V_{n,k+1} = P\left(V_{n,k} - n^2 \tau_{n,k} \nabla R_n(V_{n,k})\right), \quad k \in \mathbb{Z}_+.
\]

There is a natural notion of gradient of functions defined on \( \hat{W} \) that we call Fréchet-like derivative (see Definition 2.4), and is related to the Euclidean gradients in finite dimensions by a scaling of \( n^2 \). Suppose \( R \) is such a function whose Fréchet-like derivative evaluation map is denoted by \( \phi \). If \( R_n \) is obtained from \( R \) by restricting \( R \) to \( \mathcal{M}_n \) and the function \( R_n \) is differentiable up to the boundary of \( \mathcal{M}_n \) for every \( n \in \mathbb{N} \), then it is shown in [OPST21, Lemma 4.16] show that
\[
n^2 \nabla R_n = M_n \circ \phi \circ K.
\]

Simply put, \( n^2 \) times the Euclidean gradient of \( R_n \) at a matrix argument \( A \) can be identified as the Fréchet-like derivative \( \phi \) of \( R \) at the kernel argument \( K(A) \). The time in the Euclidean gradient in Definition 1.1 is therefore scaled by \( n^2 \) following the relation (5). The PGD algorithm is essentially the explicit Euler iteration scheme up to the projection.

We now define the stochastic optimization setup for \( R_n \). In order to do so, we first fix some notations and make some assumptions on \( R \) and \( R_n \). Let \( (\xi_{k+1})_{k \in \mathbb{Z}_+} \) be an i.i.d. sequence of random variables with some distribution \( D \) over some arbitrary measurable space \( (\Omega, \mathcal{A}) \). Let \( g : W \times \Omega \to L^\infty([0, 1]^{(2)}) \) where \( L^\infty([0, 1]^{(2)}) \) is the set of all bounded measurable functions \( \phi : [0, 1]^2 \to \mathbb{R} \) such that \( \phi(x, y) = \phi(y, x) \). To emphasize that \( \phi \) is symmetric, we denote the domain by \( [0, 1]^{(2)} \) which denotes the set \( \{(x, y) \in [0, 1]^2 : x \leq y\} \). Define \( g_n \) on \( \mathcal{M}_n \times \Omega \) as \( g_n(A; \xi) = g(K(A); \xi) \) for every \( n \in \mathbb{N} \) and \( A \in \mathcal{M}_n \), and assume that
\[
\nabla R_n = \mathbb{E}_{\xi \sim D}[g_n(\cdot; \xi)].
\]

Under suitable assumptions (see Assumption 2) on the function \( g \), the function \( R \) is invariant under measure preserving transformations and hence defines a function on \( \hat{W} \). We are interested in stochastic analogues of the iteration scheme in Definition 1.1, for such a function \( R \), possibly with a noise at each iteration. In other words, our interest lies in noisy variations of projected GD iterations (see Definition 1.1). In this setting, we will consider two ways to introduce noise at each iteration.
(1) **Small noise:** We can replace the Euclidean derivative $\nabla R_n$ in equation (PGD) by its unbiased stochastic proxy $g_n(\cdot; \xi_{k+1})$. As a special case, $g$ can be obtained from a function $\ell: W \times \Omega \rightarrow \mathbb{R}$, as $g(\cdot; \xi) := (D_W^t) \ell(\cdot; \xi)$ for all $\xi \in \Omega$, where $(D_W^t) \ell(\cdot; \xi)$ is the Fréchet-like derivative (see Definition 2.4) of $\ell(\cdot; \xi)$. Such a stochastic approximation is known as Stochastic Gradient Descent (SGD).

(2) **Large noise:** We can add an additive noise to iterates in equation (PGD) before the projection, as we describe in Definition 1.2 below.

We can now define the noisy analogs of (PGD), that is, *projected (noisy) SGD*. We will use the operator $\circ$ over symmetric matrices to denote the Hadamard (elementwise) product.

**Definition 1.2 (Projected SGD with and without noise).** Let $n \in \mathbb{N}$. Starting at $W_{n,0} \in \mathcal{M}_n$, the projected (noisy) SGD algorithm produces a sequence of iterates $(W_{n,k})_{k \in \mathbb{Z}^+}$ defined as

$$(\text{PNSGD}) \quad W_{n,k+1} = P\left(W_{n,k} - n^2 \tau_{n,k} g_n(W_{n,k}; \xi_{k+1}) + \tau_{n,k}^{1/2} G_{n,k}\right), \quad k \in \mathbb{Z}^+.$$ 

Here $(G_{n,k})_{k \in \mathbb{Z}^+}$ is an $n \times n$ symmetric matrix valued martingale difference sequence independent of $(\xi_{k+1})_{k \in \mathbb{Z}^+}$. We only consider the noise $G_{n,k}$, for $k \in \mathbb{Z}_+$, of the form $G_{n,k} = \Sigma_n(W_{n,k}) \circ \xi_{n,k}$ for some $\Sigma_n$ that maps matrices in $\mathcal{M}_n$ to $n \times n$ symmetric matrices with non-negative entries and $(\xi_{n,k})_{k \in \mathbb{Z}_+}$ is a sequence of independent $n \times n$ symmetric random matrices with standard normal entries (up to matrix symmetry).

Due to the natural identification of $\mathcal{M}_n$ with $W_n$, the GD iterates $(V_{n,k})_{k \in \mathbb{Z}_+} \subset \mathcal{M}_n$ and the SGD iterates $(W_{n,k})_{k \in \mathbb{Z}_+} \subset \mathcal{M}_n$ in Definitions 1.1 and 1.2 respectively, can be viewed as kernel valued iterates $(V_{k}^{(n)})_{k \in \mathbb{Z}_+} \subset W_n$ and $(W_{k}^{(n)})_{k \in \mathbb{Z}_+} \subset W_n$, under the embeddings $V_{k}^{(n)} = K(V_{n,k})$ and $W_{k}^{(n)} = K(W_{n,k})$ respectively for $k \in \mathbb{Z}_+$. This allows us to interpret (PGD) and (PNSGD) as kernel-valued updates.

We consider piecewise constant interpolations of the iterates (see Definition 2.1) and in this paper, we establish the existence of the scaling limit of these curves. We also characterize the limit under the absence of “large noise”. Our limiting procedure takes two steps. First, for every fixed $n \in \mathbb{N}$, we take the step size, i.e., $|\tau_n| \rightarrow 0$ to obtain a limiting SDE on $\mathcal{M}_n$. We then characterize the limit of the SDEs as $n \rightarrow \infty$ as an absolutely continuous curve on the space of graphons.

**Theorem 1.3.** Let $n \in \mathbb{N}$ be fixed, and suppose Assumptions 1, 2 and 3 hold (see Section 2.2). Let $W_n : \mathbb{R}_+ \rightarrow \mathcal{M}_n$ be the piecewise constant interpolation (Definition 2.1) of noisy SGD iterates $(W_{n,k})_{k \in \mathbb{Z}_+}$ as defined in (PNSGD). Then, $W_n$ converges weakly in the space of càdlàg processes to $X_n$ as $|\tau_n| \rightarrow 0$ that satisfies the SDE:

$$(\text{RSDE}) \quad dX_n(t) = -n^2 \nabla R_n(X_n(t)) \, dt + \Sigma_n(X_n(t)) \circ dB_n(t) + dL^+_n(t) - dL^-_n(t),$$

for $t \in \mathbb{R}_+$, starting at $X_n(0) = W_{n,0}$. Here $B_n$ is an $n \times n$ symmetric matrix valued process with coordinatewise independent standard Brownian motions up to matrix symmetry, and $(X_n, L^+_n, L^-_n)$ solves the Skorokhod problem with respect to the set $\mathcal{M}_n$ (see Section 2.3).

Note that the diffusion coefficients in (RSDE) act diagonally on the Brownian increments for each coordinate of the matrix valued process. In practice it makes sense to consider non-diagonal diffusion coefficients as an approximation to SGD. See [LTE19] for a discussion. Practitioners also use variants of SGD under the “small noise” setup where instead of having a single unbiased stochastic proxy of the gradient, an average over independent batches of
stochastic gradients is used at every step. Authors in [MLPA22] derive weak SDE approximations of various popularly used stochastic optimization algorithms that use batches. However this existing literature does not cover SDEs with boundary terms.

Our main interest is in the limit of the kernel valued stochastic process \( X^{(n)}(\cdot) = K(X_n(\cdot)) \) (Theorem 1.3), as \( n \to \infty \). This limit is a deterministic curve in \( \tilde{W} \) that we now describe. Consider, for simplicity, the special case when each \( \Sigma_n \) is \( \beta \) times the identity matrix for some \( \beta > 0 \). On a probability space that supports a standard linear Brownian motion \( B_{1,2}(\cdot) \) and a pair of independent \( \text{Uni}[0,1] \) random variables \( (U_1, U_2) \) and given some \( W_0 \in \mathcal{W} \), one can construct a unique solution of the following family of one-dimensional reflected diffusions. Given \( (U_1, U_2) = (x, y) \), for some \( (x, y) \in [0,1]^2 \), let \( X_{1,2} \) be a diffusion with state space \([-1,1]\) with the initial condition \( X_{1,2}(0) = W_0(x, y) \), and satisfying

\[
(7) \quad dX_{1,2}(t) = -\phi(\Gamma(t)) (x, y) \, dt + \beta \, dB_{1,2}(t) + dL_{1,2}^-(t) - dL_{1,2}^+(t),
\]

for some \( \beta \in \mathbb{R}_+ \) and \( t \in \mathbb{R}_+ \). Here, \( \phi \) is the Fréchet-like derivative of \( R \) in (5), \( L_{1,2}^- \) and \( L_{1,2}^+ \) are the local time processes such that \( (X_{1,2}, L_{1,2}^-, L_{1,2}^+) \) solves the Skorokhod problem with respect to \([-1,1]\) (see Section 2.3). The kernel-valued process \( \Gamma : \mathbb{R}_+ \to \mathcal{W} \) is given by

\[
(8) \quad \Gamma(t)(u, v) := \mathbb{E}[X_{1,2}(t) \mid (U_1, U_2) = (u, v)], \quad \forall \ (u, v) \in [0,1]^2,
\]

and any \( t \in \mathbb{R}_+ \). In Proposition 4.5, we show that the coupled system \((X_{1,2}, \Gamma)\) exists in a strong sense and is pathwise unique and that the kernel-valued process \( X^{(n)} \) in Theorem 1.3 converges to the curve \( \Gamma \) in the following sense as \( n \to \infty \).

**Theorem 1.4.** Suppose Assumptions 1, 3, and 4 hold (see Section 2.2). Then, for any sequence of initial kernels \( (W_0^{(n)} \in \mathcal{W}_n)_{n \in \mathbb{N}} \) that converges in \( L^2([0,1]^2) \) norm \( \| \cdot \|_2 \), i.e.,

\[
(9) \quad \lim_{n \to \infty} \left\| W_0^{(n)} - W_0 \right\|_2 = 0,
\]

the process of random kernels \( (X^{(n)}(t) = K(X_n(t)))_{t \in \mathbb{R}_+} \) obtained from solutions of the SDE (RSDE), converges locally uniformly in the cut norm, in probability, to the curve \( \Gamma : \mathbb{R}_+ \to \mathcal{W} \), with \( \Gamma(0) = W_0 \), defined in equation (8) as \( n \to \infty \).

**Remark 1.5.** The assumption \( \left\| W_0^{(n)} - W_0 \right\|_2 \to 0 \) can not be weakened to \( \left\| W_0^{(n)} - W_0 \right\|_\square \to 0 \) as \( n \to \infty \). To see this, take \( \nabla R_n \equiv 0 \) and \( \Sigma \equiv 1 \) and let \( W_0 \equiv 0 \). It is clear that \( \Gamma(t) \equiv 0 \) for all \( t \geq 0 \).

On the other hand, let \( \xi \) be a random variable taking values \(-1/2\) and \(+1\) with probability \(2/3\) and \(1/3\) respectively. And, let \( W_0^{(n)} \) be the step-kernel corresponding to \( n \times n \) symmetric random matrix whose entries (on and above the diagonal) are i.i.d. and has the same distribution as \( \xi \). Then, \( \left\| W_0^{(n)} - W_0 \right\|_\square \to 0 \) almost surely. However, in this case, the coordinates of \( X_n \) are i.i.d. (up to the matrix symmetry) and have the same distribution as an RBM (reflected at \( \pm 1 \)) with initial distribution \( \xi \). In particular, \( K(X_n(t)) \) converges to \( W(t) \equiv \mathbb{E}[X_{n,1,2}(t)] \). It is therefore sufficient to show that \( \mathbb{E}[X_{n,1,2}(t)] \) is not identically 0 for a.e. \( t \in \mathbb{R}_+ \).

To see this, we argue by contradiction. If \( \mathbb{E}[X_{n,1,2}(t)] = 0 \) for all \( t \geq 0 \) then \( \frac{d}{dt} \mathbb{E}[X_{n,1,2}(t)] = 0 \). Using [RY04, Exercise 1.12, pg-407], we obtain that \( \frac{d}{dt} \mathbb{E}[X_{n,1,2}(t)] = \frac{2}{3}(p_t(-\frac{1}{2}) - p_t(\frac{1}{2})) + \frac{1}{3}(p_t(2) - 1) \neq 0 \), where \( p_t \) is the standard heat kernel at time \( t \). This yields a contradiction.
Remark 1.6. We should also remark that arranging for $W_0^{(n)}$ such that $\|W_0^{(n)} - W_0\|_2 \to 0$ as $n \to \infty$ is not difficult. For any $W_0$ and $n \in \mathbb{N}$, let $W_0^{(n)}$ be the $L^2([0,1]^{(2)})$ projection of $W_0$ on $\mathcal{W}_n$. Then $W_0^{(n)}$ satisfies this condition.

In Section 4 a more general statement with state-dependent diffusion has been proved (see Proposition 4.6). It is worth noting that presence of noise and the boundary $\{-1,1\}$ in our problem makes it non-trivial. To see this, consider (RSDE) for a constant function $R_n$ (i.e., $\nabla R_n \equiv 0$) and without the local times, say starting at $W_{n,0} \in \mathcal{M}_n$. The solution is a symmetric matrix of independent Brownian motions. It can be easily checked that, if $\lim_{n \to \infty} \| W_{n,0} - W_0 \|_\Box = 0$, then $\lim_{n \to \infty} \sup_{t \in [0,T]} \| X_n(t) - W_0 \|_\Box = 0$ for any finite $T > 0$. However, if we consider (RSDE) again with $\nabla R_n \equiv 0$ but with reflection at the boundary, the coordinate processes are independent reflected Brownian motions. In this case the cut limit of $X(n)(t)$ is also the cut limit of the kernel $\mathbb{E}[X(n)(t)]$. But reflecting Brownian motions do not have constant expectations in time due to boundary effect. Hence, the limit of $X(n)(t)$ is not constant in $t$. But, if this limit were a gradient flow, it would be a constant.

1.1. Scaling limit without added noise. When $\Sigma_n \equiv 0$, equation (RSDE) reduces to
\begin{equation}
    dX_n(t) = -n^2 \nabla R_n(X_n(t)) \, dt + dL_n^-(t) - dL_n^+(t), \quad t \in \mathbb{R}_+, \quad X_n(0) = W_{n,0},
\end{equation}
such that $(X_n, L_n^+, L_n^-)$ solves the Skorokhod problem on $\mathcal{M}_n$ (see Section 2.3 for details). Moreover, it is shown in Section 3 that the solution of (10) is the same as the solution of (11) given below. Furthermore, it is shown in [OPST21, Theorem 4.4, Theorem 4.14] that if the solution $X_n: \mathbb{R}_+ \to \mathcal{M}_n$ of
\begin{equation}
    dX_n(t) = -n^2 \nabla R_n(X_n(t)) \circ 1_{G_n(X_n(t))} \, dt, \quad t \in \mathbb{R}_+,
\end{equation}
exists, where $G_n(A)$ is the subset of $[n]^2$ (defined in equation (39) later in Section 3.1), then $X_n$ is a gradient flow on $\mathcal{M}_n$ in a suitable sense. Further, it is shown in [OPST21, Theorem 4.17] that under reasonable assumptions on $R$, the sequence of solutions $(X_n)_{n \in \mathbb{N}}$ of equation (11) obtained for all natural numbers $n \in \mathbb{N}$, converge to an absolutely continuous curve $W: \mathbb{R}_+ \to \mathcal{W}$ (appropriately in the cut metric (see Definition 2.2)), which is a curve of maximal slope [AGS08] (a.k.a. gradient flow) of $R$, as $n \to \infty$. This yields the following.

**Theorem 1.7.** Suppose Assumptions 1 and 2 hold (see Section 2.2). Let $R$ be continuous in the cut norm, and $\lambda$-semiconvex with respect to $\| \cdot \|_2$ for some $\lambda \in \mathbb{R}$ (see Section 2.1 for definitions). For every $n \in \mathbb{N}$, let $X_n: \mathbb{R}_+ \to \mathcal{M}_n$ be a gradient flow of $R_n$ staring at $X_n(0) = W_{n,0} = M_n(W_0^{(n)}) \in \mathcal{W}_n$, and satisfying equation (10). If $(W_0^{(n)})_{n \in \mathbb{N}}$ converges to $W_0 \in \mathcal{W}$ in the cut norm, then,
\[
    \lim_{n \to \infty} \sup_{s \in [0,T]} \| K(X_n(s)) - W(s) \|_\Box = 0,
\]
for any $T > 0$, where $W$ defined as $W(t) := W_0 - \int_0^t \phi(W(s))1_{G_W(s)}$ for $t \in \mathbb{R}_+$, is the gradient flow for $R$.

We should mention that our method allows us to also obtain a non-asymptotic rate of convergence. We refer the reader to Remark 4.10 for details.

As an example, consider the function $R_n$ considered at the beginning of Section 1. $R_n$ is the restriction to $\mathcal{M}_n^0$ of the function $R$ on $\mathcal{W}_0 :=$
\[ W \in \mathcal{W} \mid W(x, y) \in [0, 1] \text{ for a.e. } (x, y) \in [0, 1]^{(2)} \] given by
\[
R(W) = \frac{1}{2} (H_-(W) - e)^2 + \frac{1}{2} (H_\Delta(W) - \tau)^2 + \mathcal{E}(W),
\]
where \( \mathcal{E} := \int_0^1 \int_0^1 h(W(x, y)) \, dx \, dy \). The function \( H_F \) is the homomorphism density of \( F \) [OPST21, Section 5.1.2]. The function \( R \) satisfies all the assumptions of Theorem 1.7. See Section 5.3 for details.

1.2. SGD and permutation symmetries in Deep Neural Networks (DNNs). DNNs typically consist of a sequence of matrices that share row/column labels with their adjacent ones. Most modern DNNs possess permutation symmetries in their parametric representations. That is, their output is invariant under permutations applied to the rows/columns of the matrices appearing in DNN representation. The goal is to obtain the sequence of matrices that minimizes the risk function \( R_n \) for \( n \in \mathbb{N} \). Authors in [AHS23] empirically study the effectiveness of SGD in optimizing the non-convex DNN risk functions \( R_n \) for large \( n \in \mathbb{N} \). For simplicity, consider the special case when the DNN is parameterized through a single finite symmetric matrix and therefore does not involve shared labels. Let \((U_{n,k})_{k \in \mathbb{Z}^+} \) and \((V_{n,k})_{k \in \mathbb{Z}^+} \) be the SGD iterations, starting at two independent initializations, say, \( U_{n,0} \neq V_{n,0} \). Authors in [AHS23] observe that \((U_{n,k})_{k \in \mathbb{Z}^+} \) and \((V_{n,k})_{k \in \mathbb{Z}^+} \) can be “aligned” by optimizing over the set of all permutations. That is, for every \( k \in \mathbb{Z}^+ \), they solve for
\[
\pi_k^* \in \arg \min_{\pi_k \in S_n} \| U_{n,k} - V_{n,k} \|_{F}^2,
\]
where \( \| \cdot \|_F \) denotes the Frobenius norm, \( S_n \) is the set of all permutations of \([n]\), and \( V_{n,k}^{\pi_k^*} \) is the matrix \( V_{n,k} \) with rows and columns relabeled by the permutation \( \pi_k \in S_n \). The authors observe an emergent property of SGD called “linear mode connectivity” (LMC) [FDRC20]. This property essentially says that \( R_n \) does not fluctuate a lot on \( W_{n,k}(\lambda) \) for large \( k \in \mathbb{Z}^+ \), where
\[
W_{n,k}(\lambda) = (1 - \lambda) U_{n,k} + \lambda V_{n,k}^{\pi_k^*}, \quad \lambda \in [0, 1].
\]
Further, they observe that \( R_n(W_{n,k}(\lambda)) \) approaches a constant uniformly on \( \lambda \in [0, 1] \) as \( n \) goes to infinity. Authors in [BSM+22] observe through experiments that for a fixed and large enough \( k \in \mathbb{Z}^+ \setminus \{0\} \), the permutation \( \pi_k^* \) has negative convexity gap
\[
R_n \left( (1 - \lambda) U_{n,0} + \lambda V_{n,0}^{\pi_k^*} \right) - \left( (1 - \lambda) R_n(U_{n,0}) + \lambda R_n \left( V_{n,0}^{\pi_k^*} \right) \right).
\]

Following these empirical observations and the hypothesis made by the authors in [ESSN22], it makes sense to consider DNNs up to its permutation symmetries, and as a consequence study limiting behaviors of stochastic optimization algorithms over the space of graphons.

2. Background, Assumptions and Setup

Since we want to obtain continuous time scaling limits of the iterative schemes defined in Definition 1.1 and Definition 1.2, we will use piecewise constant interpolations.

Definition 2.1 (Piecewise constant interpolation). Given a sequence \((a_k)_{k \in \mathbb{Z}^+}\) over any domain, and a sequence of positive step sizes \( \tau = (\tau_k)_{k \in \mathbb{Z}^+} \), we can define a piecewise constant interpolation of \((a_k)_{k \in \mathbb{Z}^+}\) as a right-continuous curve \( a: \mathbb{R}^+ \to \{a_k\}_{k \in \mathbb{Z}^+} \) as
\[
a(t) := a_k, \quad \text{if} \quad t \in [t_k, t_{k+1}),
\]
for some $k \in \mathbb{Z}_+$, where $t_0 = 0$ and $t_k := \sum_{j=0}^{k-1} \tau_j$ for any $k \in \mathbb{N}$.

We now provide a background on graphons (see [Lov12, Jan13] for broader expositions).

2.1. **Background on Graphons.** Consider the set $\mathcal{S}$ of all bounded, Borel measurable function $W: [0,1]^2 \to \mathbb{R}$ such that $W(x,y) = W(y,x)$ for a.e. $(x,y) \in [0,1]^2$. For any function $W \in \mathcal{S}$ one can define the cut norm, $\| \cdot \|_\square: \mathcal{S} \to \mathbb{R}_+$ as

$$\|W\|_\square := \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} W(x,y) \, dx \, dy \right|, \quad W \in \mathcal{S},$$

where the supremum is taken over Borel measurable sets $S,T \subseteq [0,1]$. The cut norm was first introduced in [FK99] in the context of matrices and was later extended to $\mathcal{S}$ in [BCL+08]. In the following definitions, let $\mathcal{T}$ denote the set of all measure preserving transformations on $[0,1]$ equipped with the Lebesgue measure. We say $W_1 \cong W_2$ (i.e., $W_1$ and $W_2$ are weakly isomorphic) if there exists $W \in \mathcal{S}$ and measure preserving transformations $\varphi_1, \varphi_2 \in \mathcal{T}$ such that for $W^{\varphi_1} \in \mathcal{S}$ defined as $W^{\varphi_1}(x,y) := W(\varphi_1(x),\varphi_1(y))$ for a.e. $(x,y) \in [0,1]^2$ and $i \in [2]$, $W_1 = W^{\varphi_1}$, and $W_2 = W^{\varphi_2}$.

The cut norm $\| \cdot \|_\square$ induces a metric called the cut metric, denoted by $\delta_\square$, when restricted to the quotient space $\tilde{\mathcal{S}} := \mathcal{S}/\cong$. We denote the equivalence class of $W \in \mathcal{S}$ under weak isomorphism ($\cong$) as $[W] := \{ U \in \mathcal{S} | U \cong W \} \in \tilde{\mathcal{S}}$. We now define the cut metric.

**Definition 2.2** (Cut Metric [BCL+08, Section 3.2]). Let $[W_1], [W_2] \in \tilde{\mathcal{S}}$. Then,

$$\delta_\square([W_1],[W_2]) := \inf_{\varphi, \psi \in \mathcal{T}} \left\| W_1^\varphi - W_2^\psi \right\|_\square.$$

More generally, given any norm $\| \cdot \|$ on $\mathcal{S}$, one can define an induced metric $\delta_\| \cdot \|_\square$ on $\tilde{\mathcal{S}}$ as

$$\delta_\| \cdot \|([W_1],[W_2]) := \inf_{\varphi, \psi \in \mathcal{T}} \left\| W_1^\varphi - W_2^\psi \right\|_\| \cdot \||.$$

In particular, the induced metric due to the $L^2$ norm, $\| \cdot \|_2: L^2([0,1]^2) \to \mathbb{R}_+$, is called the invariant $L^2$ metric, $\delta_2$, and it would be used in our discussion.

As defined in Section 1, the set of kernels $\mathcal{W} \subset \mathcal{S}$ is the set of measurable, symmetric functions $W: [0,1]^2 \to [-1,1]$ and correspondingly $\tilde{\mathcal{W}} := \mathcal{W}/\cong$ is the set of graphons. For most of our discussion, we will be concerned only with the space of graphons equipped with either the cut metric $\delta_\square$ or the invariant $L^2$ metric $\delta_2$. The metrics on $\mathcal{S}$ induced by the norms $\| \cdot \|_\square$ and $\| \cdot \|_2$ with be denoted by $d_\square$ and $d_2$ respectively.

For every $n \in \mathbb{N}$, the set $\mathcal{M}_n$ can be naturally identified with a subset of $\mathcal{W}$. Let $\mathcal{V}_n := \{ V_i \}_{i \in [n]}$ be a partition of the interval $[0,1]$ into contiguous intervals of equal length (Lebesgue measure). We define the set of kernels $\mathcal{W}_n \subset \mathcal{W}$ which contain kernels which are constant a.e. over sets in $\mathcal{V}_n \times \mathcal{V}_n$.

We note some crucial properties of these metric spaces that will be frequently used throughout this paper even without explicitly mentioning.

1. Properties of $\delta_\square$:
   (a) The topology induced by the cut metric $\delta_\square$ on $\tilde{\mathcal{W}}$ is compact [LS07], [Lov12, Section 9.3].
   (b) Convergence in the cut metric is related to the convergence of homomorphism functions via the counting and the inverse counting lemmas [Lov12, Section 7.2, Lemma 10.23, Lemma 10.32].
(2) Properties of $\delta_2$:
    (a) The metric space $(\hat{W}, \delta_2)$ is a geodesic metric space [OPST21, Theorem 3.5].
    (b) The metric space $(\hat{W}, \delta_2)$ is complete and separable but not compact.
    (c) Convergence in $\delta_2$ implies convergence in $\delta_\square$, implying that the topology generated by $\delta_2$ is stronger that the one generated by $\delta_\square$ on $\hat{W}$.

As $(\hat{W}, \delta_2)$ is a geodesic metric space, it therefore makes sense to talk about geodesically convex or geodesically semiconvex functions.

**Definition 2.3** ($\lambda$-geodesic semiconvexity w.r.t. $\delta_2$). A function $R: \hat{W} \to \mathbb{R}$ is $\lambda$-geodesically semiconvex with respect to $\delta_2$, if for any $[W_0], [W_1] \in \hat{W}$ there exists a constant speed geodesic $\omega: [0, 1] \to \hat{W}$ w.r.t. $\delta_2$ with $\omega(0) = [W_0]$ and $\omega(1) = [W_1]$ such that $R$ is $\lambda$-semiconvex on $\omega$ with respect to $\delta_2$ for some $\lambda \in \mathbb{R}$. (See [OPST21, Definition 2.14-2.16]).

In Section 1, we noted in equation (5) that Euclidean gradient $\nabla R_n$ of $R_n$ is closely related to what we call the Fréchet-like derivative of $R: \mathcal{W} \to \mathbb{R}$. We state its definition below.

**Definition 2.4** (Fréchet-like derivative on $\mathcal{W}$). The Fréchet-like derivative of $R: \mathcal{W} \to \mathbb{R}$ at $V \in \mathcal{W}$ is given by $\phi(V) \in L^\infty([0, 1]^{(2)})$ that satisfies the following condition,

$$
\lim_{W \in \mathcal{W}, \|W-V\|_2 \to 0} \frac{R(W) - R(V) - (\langle \phi(V), W \rangle - \langle \phi(V), V \rangle)}{\|W-V\|_2} = 0,
$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $L^2([0, 1]^{(2)})$. If $R$ admits a Fréchet-like derivative at every $V \in \mathcal{W}$, we say that $R$ is Fréchet differentiable.

**Remark 2.5.** Note that here we define the Fréchet-like derivative for all functions if it exists, unlike as defined in [OPST21, Definition 4.6] where it is only defined for invariant functions. This is done so to allow $\ell(\cdot; \xi)$ (see item 1 in Section 1) to be Fréchet-differentiable for all $\xi \in \mathcal{D}$ despite it not necessarily being an invariant function.

The scaling limit as we obtain in Theorem 1.4, under certain assumptions can be shown to be absolutely continuous with respect to $d_2$ (see Proposition 4.7). We state its definition for the sake of completeness.

**Definition 2.6.** A curve $W: \mathbb{R}_+ \to \mathcal{W}$ is absolutely continuous with respect to $d_2$ if there exists $m \in L^1(\mathbb{R}_+)$ such that for all $0 \leq r < s < \infty$,

$$
d_2(W(r), W(s)) = \|W(r) - W(s)\|_2 \leq \int_r^s m(t) \, dt.
$$

The set of all absolutely continuous curves on $(\mathcal{W}, d_2)$ will be denoted by $\text{AC}(\mathcal{W}, d_2)$.

2.2. **Assumptions.** In this section we state all the required assumptions we need to prove our results (see Theorem 1.3 and Theorem 1.4).

**Assumption 1.** We make following assumptions on $R$, $g$ and $\phi$:

(1) For every $n \in \mathbb{N}$, the function $R_n$ is in $C^1(\mathcal{M}_n)$ up to the boundary of $\mathcal{M}_n$.
(2) The map $\phi$ is $\kappa_2$-Lipschitz with respect to $\|\cdot\|_2$, for some constant $\kappa_2 \in \mathbb{R}_+$. That is,

$$
\|\phi(W_1) - \phi(W_2)\|_2 \leq \kappa_2 \|W_1 - W_2\|_2, \quad \forall W_1, W_2 \in \mathcal{W}.
$$

(3) For every $n \in \mathbb{N}$, the function $g_n(\cdot; \xi) = g(\cdot; \xi) \circ K$ is in $C^0(\mathcal{M}_n)$ up to the boundary of $\mathcal{M}_n$ for all $\xi \in \Omega$. 

Assumption 2. We assume the following about the “small noise”.

(1) Law of the random variable \( g(W; \xi) \) for \( \xi \sim \mathcal{D} \) is invariant under measure preserving transformations for all \( W \in \mathcal{W} \), i.e., \( \text{Law}(g(W; \xi)) = \text{Law}(g(W^\varphi; \xi)) \) for all \( \varphi \in \mathcal{T} \).
(2) The random variable \( g(\cdot; \xi) \) for \( \xi \sim \mathcal{D} \) has uniformly bounded variance over all finite dimensional kernels. That is, there exists \( \sigma \geq 0 \) such that for all \( A \in \cup_{n \in \mathbb{N}} \mathcal{W}_n \),
\[
\mathbb{E}_{\xi \sim \mathcal{D}} \left[ \| g(A; \xi) - \phi(A) \|_2^2 \right] \leq \sigma^2.
\]

Assumption 3. We assume the following on the “large noise” for every \( n \in \mathbb{N} \).

(1) There exists a function \( \Sigma: \mathcal{W} \to L^\infty([0,1]^{[2]}) \) such that the diffusion coefficient functions \( (\Sigma_n)_{n \in \mathbb{N}} \) are restrictions of \( \Sigma \), i.e., for every \( n \in \mathbb{N} \), \( \Sigma_n = M_n \circ \Sigma \circ K \) on \( \mathcal{M}_n \).
(2) The map \( \Sigma: \mathcal{W} \to L^\infty([0,1]^{[2]}) \) is \( \kappa_2 \)-Lipschitz in \( \| \cdot \|_2 \) and uniformly bounded in \( \| \cdot \|_\infty \) by some constant \( M_\infty \in \mathbb{R}_+ \), i.e., for all \( U, V \in \mathcal{W} \),
\[
\| \Sigma(U) - \Sigma(V) \|_2 \leq \kappa_2 \| U - V \|_2, \quad \text{and} \quad \| \Sigma(U) \|_\infty \leq M_\infty.
\]

Assumption 4. There exists a constant \( \kappa_\square \in \mathbb{R}_+ \) such that, for almost every \( (x, y) \in [0,1]^{[2]} \), the map \( \phi_{x,y} := \phi(\cdot)(x, y) \) is \( \kappa_\square \)-Lipschitz in cut norm \( \| \cdot \|_\square \). That is, for every \( U, V \in \mathcal{W} \),
\[
| \phi_{x,y}(U) - \phi_{x,y}(V) | \leq \kappa_\square \| U - V \|_\square.
\]

2.3. System of reflected diffusions. For \( n \in \mathbb{N} \), consider the domain \( \mathcal{M}_n \). Notice that \( \mathcal{M}_n \) is a cube, and is closed with respect to the usual topology. Consider the SDE:
\[
\begin{align*}
\text{(15)} \quad dX_n(t) &= -n^2 \nabla R_n(X_n(t)) \, dt + \Sigma_n(X_n(t)) \circ dB_n(t) + dL_n^-(t) - dL_n^+(t),
\end{align*}
\]
for \( t \in [0, T] \) for some fixed \( T \in \mathbb{R}_+ \) and starting at \( X_n(0) = X_{n,0} \in \mathcal{M}_n \). Here \( \Sigma_n \) is a map from \( \mathcal{M}_n \) to the set of \( n \times n \) symmetric matrices with non-negative entries, \( B_n \) is a \( n \times n \) symmetric matrix valued process containing a set of standard Brownian motions \( (B_{n, (i,j)})_{(i,j) \in [n]^{[2]}} \), which are independent up to matrix symmetry, and the processes \( L_n^- \) and \( L_n^+ \) are local times at the boundary. More precisely, they satisfying the following conditions:

(1) The processes \( X_n, L_n^+ \) and \( L_n^- \) are adapted processes.
(2) The process \( L_n^- \) and \( L_n^+ \) are coordinatewise non decreasing processes a.e.
(3) For every \( (i, j) \in [n]^{[2]} \),
\[
\int_0^\infty 1 \{ X_{n,(i,j)}(t) > -1 \} \, dL_{n,(i,j)}^-(t) = 0, \quad \text{and} \quad \int_0^\infty 1 \{ X_{n,(i,j)}(t) < +1 \} \, dL_{n,(i,j)}^+(t) = 0.
\]

We say that \( (X_n, L_n^+, L_n^-) \) solves the Skorokhod problem with respect to the set \( \mathcal{M}_n \). Following [KLRS07, Definition 1.2], the strong solution \( (X_n, L_n^+, L_n^-) \) of the Skorokhod problem exists and is unique if \( n^2 \nabla R_n \) and \( \Sigma_n \) are Lipschitz with respect to \( \| \cdot \|_F \) (following Assumption 1, Assumption 3 and equation (5)).

2.3.1. The Lipschitz property of the Skorokhod map. Let \( Y_1 \) and \( Y_2 \) be two real valued stochastic processes. Let \( \Lambda_{[-1,1]} \) denote the Skorokhod map that maps the set of càdlàg functions on \([0, T]\) to itself. If \( (X_1 := \Lambda_{[-1,1]}(Y_1), L_1^+, L_1^-) \) and \( (X_2 := \Lambda_{[-1,1]}(Y_2), L_2^+, L_2^-) \) solve the Skorokhod problem with respect to the set \([-1, 1]\), then the Skorokhod map \( \Lambda_{[-1,1]} \) is 4-Lipschitz
under the uniform metric [KLRS07, Corollary 1.6], i.e.,
\begin{equation}
\sup_{t \in [0,T]} |X_1(t) - X_2(t)| \leq 4 \sup_{t \in [0,T]} |Y_1(t) - Y_2(t)|, \quad \forall T \in \mathbb{R}_+.
\end{equation}

3. Convergence of Projected Noisy Stochastic Gradient Descent

The goal of this section is to show that for each \( n \in \mathbb{N} \), the projected noisy SGD iterates, defined in (PNSGD), converges weakly to the strong solution of the SDE (RSDE) as \( |\tau_n| \to 0 \). This is done in two steps that we describe below.

Recall the projected noisy SGD iterates defined in Definition 1.2, starting from \( W_{n,0} \in \mathcal{M}_n \), rewritten for convenience:

\begin{equation}
\text{(PNSGD)} \quad W_{n,k+1} = P\left(W_{n,k} - n^2 \tau_{n,k} \nabla R_n(W_{n,k}) - \tau_{n,k} \Delta M_{n,k} + \tau_{n,k}^{1/2} G_{n,k}\right),
\end{equation}

for \( k \in \mathbb{R}_+ \), where \( (G_{n,k})_{k \in \mathbb{Z}_+} \) is any \( n \times n \) real symmetric matrix valued martingale difference sequence with each element containing centered and independent entries up to matrix symmetry, as defined in Section 1, and

\[ \Delta M_{n,k} := n^2 g_n(W_{n,k}; \xi_{k+1}) - n^2 \nabla R_n(W_{n,k}), \quad k \in \mathbb{Z}_+. \]

Observe that \( (\Delta M_{n,k})_{k \in \mathbb{Z}_+} \) is an \( n \times n \) symmetric matrix valued martingale difference sequence with respect to the filtration \( (\mathcal{F}_k)_{k \in \mathbb{Z}_+} \), where \( \mathcal{F}_k := \sigma\{\{W_{n,0}, \xi_{i+1}, G_{n,i}\}_{i \in \{0,1,\ldots,j\}} \cup \{\xi_{k+1}\}\} \) for \( k \in \mathbb{Z}_+ \). Without the martingale difference term \( \tau_{n,k} \Delta M_{n,k} \), equation (PNSGD) reduces to the projected GD iterates with additive noise, \( (V_{n,k})_{k \in \mathbb{Z}_+} \), starting at \( V_{n,0} = W_{n,0} \), described in (PNGD), re-written below

\begin{equation}
\text{(PNGD)} \quad V_{n,k+1} = P\left(V_{n,k} - n^2 \tau_{n,k} \nabla R_n(V_{n,k}) + \tau_{n,k}^{1/2} G_{n,k}\right), \quad k \in \mathbb{Z}_+.
\end{equation}

Let \( W_k^{(n)} := K(W_{n,k}) \) and \( V_k^{(n)} := K(V_{n,k}) \) for all \( k \in \mathbb{Z}_+ \), and let \( W^{(n)} \) and \( V^{(n)} \) be piecewise constant interpolations of \( (W_k^{(n)})_{k \in \mathbb{Z}_+} \) and \( (V_k^{(n)})_{k \in \mathbb{Z}_+} \) respectively with the step size sequence \( \tau_n \). Using Grönwall’s inequality and an obvious coupling between the processes (PNSGD) and (PNGD), we show in Lemma 3.1 that the two processes are close as \( |\tau_n| \to 0 \).

**Lemma 3.1.** Let \( R: \mathcal{W} \to \mathbb{R} \) be such that the Fréchet-like derivative \( \phi = D_{\mathcal{W}} R \) exists. Suppose Assumptions 1, and 2 hold. Let \( n \in \mathbb{N} \). Let \( W_n \) and \( V_n \) be the piecewise constant interpolations (see Definition (2.1)) of \( (W_{n,k})_{k \in \mathbb{Z}_+} \) and \( (V_{n,k})_{k \in \mathbb{Z}_+} \) respectively, as defined in (PNSGD) and (PNGD), with step size sequence \( \tau_n := (\tau_{n,k})_{k \in \mathbb{Z}_+} \). Then, there exists a universal constant \( C > 0 \) such that for any \( T > 0 \) we have

\[ \mathbb{E}\left[ \sup_{s \in [0,T]} \|W^{(n)}(s) - V^{(n)}(s)\|_2^2 \right] \leq C \sigma^2 T |\tau_n| \exp\left[C \kappa_2^2 T^2\right]. \]

**Proof.** Let \( W_n \) and \( V_n \) be the piecewise constant interpolations of \( (W_{n,j})_{j \in \mathbb{Z}_+} \) and \( (V_{n,j})_{j \in \mathbb{Z}_+} \) respectively as defined in Definition 2.1. Define \( \Delta: \mathbb{R}_+ \to \mathbb{R}_+ \) as

\begin{equation}
\Delta(t) := \mathbb{E}\left[ \sup_{s \in [0,t]} \|W_n(s) - V_n(s)\|_F^2 \right], \quad t \in \mathbb{R}_+.
\end{equation}
Let \( k \in \mathbb{Z}_+ \) be such that \( t \in [t_{n,k}, t_{n,k+1}) \). Then, using [Slo94, Theorem 1],

\[
\Delta(t) \leq C\mathbb{E} \left( \sum_{j=0}^{k-1} \tau_{n,j} \| n^2 \nabla R_n(W_{n,j}) - n^2 \nabla R_n(V_{n,j}) \|_F \right)^2 
\]

\[
+ C\mathbb{E} \left( \sum_{j=0}^{k-1} \tau_{n,j}^2 \| \Delta M_{n,j} \|_F^2 \right),
\]

where \( C > 0 \) is some universal constant. From Assumption 1, since \( \phi \) is \( \kappa_2 \)-Lipschitz as a map from \( L^2([0,1]^2) \) to \( L^2([0,1]^2) \), following equation (5) and the fact that \( \| A_n \|^2_F = n^2 \| K(A_n) \|^2_2 \) for all \( A_n \in \mathcal{M}_n \), we see that the map \( \nabla R_n : \mathcal{M}_n \to \mathbb{R}^{[n]^2} \) satisfies

\[
\| n^2 \nabla R_n(A_n) - n^2 \nabla R_n(B_n) \|_F^2 \leq \kappa_2^2 \| A_n - B_n \|_F^2, \quad \forall A_n, B_n \in \mathcal{M}_n.
\]

Using the Cauchy-Schwarz inequality, and equation (20), we first bound the second term in equation (19) as

\[
\mathbb{E} \left( \sum_{j=0}^{k-1} \tau_{n,j} \| n^2 \nabla R_n(W_{n,j}) - n^2 \nabla R_n(V_{n,j}) \|_F \right)^2 
\]

\[
\leq \mathbb{E} \left( \sum_{j=0}^{k-1} \tau_{n,j} \| n^2 \nabla R_n(W_{n,j}) - n^2 \nabla R_n(V_{n,j}) \|_F \right)^2 
\]

\[
\leq \kappa_2^2 t \mathbb{E} \left( \sum_{j=0}^{k-1} \tau_{n,j} \| W_{n,j} - V_{n,j} \|_F^2 \right) \leq \kappa_2^2 t \int_0^t \Delta(s) \, ds,
\]

where the last inequality follows by observing that if \( s \in [t_{n,j}, t_{n,j+1}) \) for some \( j \in \mathbb{Z}_+ \), then

\[
\mathbb{E} \left[ \| W_n(s) - V_n(s) \|^2_F \right] = \mathbb{E} \left[ \| W_{n,j} - V_{n,j} \|^2_F \right] \leq \Delta(s).
\]

Using Assumption 2, first note that

\[
\| \Delta M_{n,j} \|_F^2 = \| n^2 g_n(W_{n,k}; \xi_{k+1}) - n^2 \nabla R_n(W_{n,k}) \|^2_F 
\]

\[
= n^2 \| K(n^2 g_n(W_{n,k}; \xi_{k+1}) - n^2 \nabla R_n(W_{n,k})) \|^2_2 \leq n^2 \sigma^2.
\]

We use the above to bound the first term in equation (19) as

\[
\mathbb{E} \left( \sum_{j=0}^{k-1} \tau_{n,j}^2 \| \Delta M_{n,j} \|_F^2 \right) \leq n^2 \sigma^2 t |\tau_n|,
\]

where \( |\tau_n| \) is defined in Section 1 as \( \sup_{j \in \mathbb{Z}_+} \tau_{n,j} \).

Plugging back (21) and (23) in equation (19) we get

\[
\Delta(t) \leq C n^2 \sigma^2 t |\tau_n| + C \kappa_2^2 t \int_0^t \Delta(s) \, ds,
\]

and applying Grönwall’s inequality [Grö19], we obtain \( \Delta(t) \leq C n^2 \sigma^2 t |\tau_n| \exp[C \kappa_2^2 t^2] \). \qed
Our next step is to show that sequence of iterates defined in (PNGD) is close to the solution of the SDE (RSDE) which we reproduce below

\[
\begin{align*}
\text{d}X_n(t) &= -n^2 \nabla R_n(X_n(t)) + \Sigma_n(X_n(t)) \circ dB_n(t) \\
&\quad - dL_n^+(t) + dL_n^-(t), \quad t \in \mathbb{R}_+,
\end{align*}
\]

(RSDE)

where \( B_n \) is an \( n \times n \) symmetric matrix valued process whose entries are independent Brownian motions up to matrix symmetry, and \( X_n(0) = V_{n,0} = W_{n,0} \in \mathcal{M}_n \). The tuple \( (X_n, L_n^+, L_n^-) \) solves the Skorokhod problem with respect to the set \( \mathcal{M}_n \) (see Section 2.3).

In Lemma 3.2 we compare (PNGD) with a discretization of the SDE (RSDE). This is obtained by coupling the discrete noise in (PNGD) with the Brownian motion driving the SDE (RSDE). Combining these we conclude the convergence of (PNSGD) to the SDE (RSDE) as \( |\tau_n| \to 0 \).

**Lemma 3.2.** Let \( n \in \mathbb{N} \). Let \( B_n \) be an \( n \times n \) symmetric matrix valued process whose coordinates are i.i.d. Brownian motion (up to matrix symmetry) defined on some probability space. Let \( X_n \) be the strong solution of SDE (RSDE) with initial condition \( X_n(0) = V_{n,0} \) (see (PNGD)). Then, there exists a càdlàg process \( \tilde{V}_n \) on \( \mathcal{M}_n \), defined on the same probability space as \( B_n \), such that it has the same law as \( V_n \), the piecewise constant interpolation (see Definition 2.1) of \( (V_{n,k})_{k \in \mathbb{Z}_+} \) obtained from (PNGD). Moreover, for any \( T \in \mathbb{R}_+ \),

\[
\lim_{|\tau_n| \to 0} \mathbb{E} \left[ \sup_{s \in [0,T]} \left| K(X_n(s)) - K(\tilde{V}_n(s)) \right|^2 \right] = 0.
\]

**Proof.** Let \( B_n \) be as given in the assumption and let \( X_n \) be the strong solution of the SDE (RSDE). Since the discrete noise in (PNGD) is Gaussian (see Assumption 3), there is an obvious way to couple it with the Brownian motion driving the SDE in (RSDE). Given \( B_n \) and the step size sequence \( \tau_n = (\tau_{n,k} > 0)_{k \in \mathbb{Z}_+} \), define the discrete time \( n \times n \) symmetric matrix valued martingale difference sequence \( \left(\tilde{Z}_{n,k}\right)_{k \in \mathbb{Z}_+} \) as

\[
\tilde{Z}_{n,k} := \tau_n^{-1/2}(B_n(t_{n,k}+1) - B_n(t_{n,k})), \quad k \in \mathbb{Z}_+.
\]

Note that the entries in \( \tilde{Z}_{n,k} \) are distributed as \( N(0,1) \) up to matrix symmetry for every \( k \in \mathbb{Z}_+ \). Starting from \( \tilde{V}_{n,0} = V_{n,0} \), we now define an auxiliary process \( \left(\tilde{V}_{n,k}\right)_{k \in \mathbb{Z}_+} \) on the same probability space as \( B_n \), iteratively as

\[
\tilde{V}_{n,k+1} = P\left(\tilde{V}_{n,k} - n^2 \tau_n \nabla R_n(\tilde{V}_{n,k}) + \tau_n^{1/2} \Sigma_n(\tilde{V}_{n,k}) \circ \tilde{Z}_{n,k}\right), \quad k \in \mathbb{Z}_+.
\]

Following Assumption 3, \( \tilde{V}_{n,k} \) has the same law as \( V_{n,k} \) for each \( k \in \mathbb{Z}_+ \). Let \( \tilde{V}_n : \mathbb{R}_+ \to \mathcal{M}_n \) be piecewise constant interpolation of \( \left(\tilde{V}_{n,k}\right)_{k \in \mathbb{Z}_+} \). The particular choice of \( \left(\tilde{Z}_{n,k}\right)_{k \in \mathbb{Z}_+} \) in equation (25) allows us to couple \( \tilde{V}_n \) with the strong solution of the SDE (RSDE). Let \( \tilde{G}_{n,j} := \Sigma_n(\tilde{V}_{n,j}) \circ \tilde{Z}_{n,j} \) for all \( j \in \mathbb{Z}_+ \). The curve \( \tilde{V}_n \) can be written as

\[
\tilde{V}_n(t) = \tilde{V}_{n,0} - \sum_{j=0}^{k-1} n^2 \tau_{n,j} \nabla R_n(\tilde{V}_{n,j}) + \sum_{j=0}^{k-1} \tau_{n,j}^{1/2} \tilde{G}_{n,j} + \sum_{j=0}^{k-1} \tau_{n,j} (L_{n,j}^- - L_{n,j}^+), \quad 0 \leq t \leq k \tau_n.
\]
for $t \in [t_{n,k}, t_{n,k+1})$. Here $(L_{n,j}^{\pm})_{j \in \mathbb{Z}^+}$ is chosen so that the piecewise constant interpolation (see Definition 2.1) of $(V_{n,k}, L^{-}_{n,k}, L^{+}_{n,k})_{k \in \mathbb{Z}^+}$ solves the Skorokhod problem with respect to $\mathcal{M}_n$ (see Section 2.3).

Also consider three auxiliary processes $Y_n$, $\bar{Y}_n$, and $\hat{Y}_n$ taking values over $n \times n$ real symmetric matrices, defined as

\begin{align}
Y_n(t) &:= X_n(0) - \int_0^t n^2 \nabla R_n(X_n(s)) \, ds + \int_0^t \Sigma_n(X_n(s)) \circ dB_n(s), \\
\hat{Y}_n(t) &:= X_n(0) - \int_0^t n^2 \nabla R_n\left(\hat{V}_n(s)\right) \, ds + \int_0^t \Sigma_n(\hat{V}_n(s)) \circ dB_n(s), \\
\bar{Y}_n(t) &:= X_n(0) - \sum_{j=0}^{k-1} n^2 \tau_{n,j} \nabla R_n(\hat{V}_{n,j}) + \sum_{j=0}^{k-1} \frac{1}{2} \hat{G}_{n,j},
\end{align}

for every $k \in \mathbb{Z}^+$ and all $t \in [t_{n,k}, t_{n,k+1})$. Observe that the curves $X_n$ and $\hat{V}_n$ can be obtained by applying the Skorokhod map to the curves $Y_n$ and $\bar{Y}_n$ pointwise respectively. Let $\hat{V}_n: \mathbb{R}^+ \to \mathcal{M}_n$ be obtained from $\hat{Y}_n$ by applying the Skorokhod map. First observe that using the Lipschitzness of the Skorokhod map, $\phi$ and $\Sigma_n$ (see Assumption 1, Assumption 3, Section 2.3 and equation (20)), we obtain

\begin{align}
\mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \hat{V}_n(t) - X_n(t) \right\|_F^2 \right] &\leq 16 \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \hat{Y}_n(t) - Y_n(t) \right\|_F^2 \right] \\
&\leq 16 \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t n^2 \nabla R_n(X_n(s)) - n^2 \nabla R_n\left(\hat{V}_n(s)\right) \, ds \right\|_F^2 \right] \\
&\quad + 16 \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t \left( \Sigma_n(X_n(s)) - \Sigma_n(\hat{V}_n(s)) \right) \circ dB_n(s) \right\|_F^2 \right] \\
&\leq 16 \kappa_2^2 \mathbb{E} \left[ \int_0^T \left\| X_n(s) - \hat{V}_n(s) \right\|_F^2 \, ds \right] \\
&\quad + 64 \mathbb{E} \left[ \int_0^T \left\| \Sigma_n(X_n(s)) - \Sigma_n(\hat{V}_n(s)) \right\|_F^2 \, ds \right] \\
&\leq 80 \kappa_2^2 \int_0^T \mathbb{E} \left[ \sup_{s \in [0,t]} \left\| X_n(s) - \hat{V}_n(s) \right\|_F^2 \right] \, ds,
\end{align}

where the second last inequality follows from Doob’s maximal inequality \cite[Theorem 3.8.iv]{KS91} and the fact that for all $A_n \in \mathcal{M}_n$, $\|A_n\|_F^2 = n^2 \|K(A_n)\|_2^2$. For any $t \in [0,T]$, define $k_t := \arg \min_{j \in \mathbb{Z}^+} \left\{ t \geq t_{n,j} \right\}$. Using the Lipschitzness of Skorokhod map (see
Section 2.3) we obtain
\[
\mathbb{E}\left[ \sup_{s \in [0,T]} \left\| \tilde{V}_n(t) - \tilde{V}_n(t) \right\|_F^2 \right] \leq 16 \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \tilde{Y}_n(t) - \tilde{V}_n(t) \right\|_F^2 \right]
\]
\[
\leq 32 \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \int_0^t n^2 \nabla R_n(\tilde{V}_n(s)) \, ds - \sum_{j=0}^{k_t-1} n^2 \tau_{nj} \nabla R_n(\tilde{V}_{n,j}) \right\|_F^2 \right]
\]
\[
+ 32 \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \sum_{j=0}^{k_t-1} \tau_{nj}^{1/2} \Sigma_n(\tilde{V}_{n,j}) \circ \tilde{Z}_{n,j} - \int_0^t \Sigma_n(\tilde{V}_n(s)) \circ dB_n(s) \right\|_F^2 \right],
\]
where the last inequality follows from Assumption 3.

We now bound the first term from the above inequality (32). To this end observe that
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \int_0^t n^2 \nabla R_n(\tilde{V}_n(s)) \, ds - \sum_{j=0}^{k_t-1} n^2 \tau_{nj} \nabla R_n(\tilde{V}_{n,j}) \right\|_F^2 \right]
\]
\[
= \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| n^2 (t - t_{n,k1}) \nabla R_n(\tilde{V}_{n,k}) \right\|_F^2 \right] \leq |\tau_n|^2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| n^2 \nabla R_n(\tilde{V}_{n,k}) \right\|_F^2 \right]
\]
\[
n^2 |\tau_n|^2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \phi(\tilde{V}^{(n)}(t)) \right\|_2^2 \right] \leq n^2 |\tau_n|^2 M_2^2,
\]
for some constant \( M_2 \in \mathbb{R}_+ \) by Assumption 1.

We now bound the second term in the inequality (32). Using the coupling defined in (25) and noting that \( \tilde{V}(s) = \tilde{V}_{n,j} \) for \( s \in [t_{n,j}, t_{n,j+1}) \) (see Definition 2.1), we obtain that
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \sum_{j=0}^{k_t-1} \tau_{nj}^{1/2} \Sigma_n(\tilde{V}_{n,j}) \circ \tilde{Z}_{n,j} - \int_0^t \Sigma_n(\tilde{V}_n(s)) \circ dB_n(s) \right\|_F^2 \right]
\]
\[
= \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \Sigma_n(\tilde{V}_{n,k}) \circ (B_n(t) - B_n(t_{n,k1})) \right\|_F^2 \right] \leq M_2^2 n^2 C_{1,T} |\tau_n| \log \frac{1}{|\tau_n|},
\]
where the last inequality follows from Assumption 3 and [Slo01, Lemma A.4] for \( C_{1,T} \in \mathbb{R}_+ \).

Now define \( \Delta: \mathbb{R}_+ \to \mathbb{R}_+ \) as
\[
\Delta(t) := \mathbb{E}\left[ \sup_{s \in [0,t]} \left\| X_n(s) - \tilde{V}_n(s) \right\|_F^2 \right], \quad t \in \mathbb{R}_+.
\]

Using the triangle inequality by combining equations (31), (32), (33) and (34), we get
\[
\Delta(T) \leq 32n^2 |\tau_n|^2 M_2^2 + 32n^2 M_2^2 C_{1,T} |\tau_n| \log \frac{1}{|\tau_n|} + 80 \kappa_2 \int_0^T \Delta(t) \, dt.
\]
Applying Grönwall’s inequality [Grö19], we get
\[
\Delta(T) \leq 32n^2 \left( |\tau_n|^2 M_2^2 + M_2^2 C_{1,T} |\tau_n| \log \frac{1}{|\tau_n|} \right) \exp\left[ 80 \kappa_2^2 T \right].
\]
Taking limit as \( |\tau_n| \to 0 \) on the above bound, completes the proof. \( \square \)
We combine Lemma 3.1 and 3.2 to conclude the proof of Theorem 1.3. Moreover, we also the following non-asymptotic error rate
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} \| W^{(n)}(s) - K(X_n)(s) \|^2_2 \right] \leq C n^2 (M + \sigma^2 T) |\tau_n| \log \frac{1}{|\tau_n|} \exp[C n^2 T]
\]
for some constants \(C, M < \infty\).

### 3.1. Convergence of Projected Stochastic Gradient Descent

In the absence of “large noise” (i.e., when \(\Sigma_n \equiv 0\)), the SDE (RSDE) reduces to the SDE
\[(37) \quad dX_n(t) = -n^2 \nabla R_n(X_n(t)) \, dt + dL_n^-(t) - dL_n^+(t), \quad X_n(0) = W_{n,0},\]
As we describe in Section 1.1, it is show in [OPST21, Theorem 4.4, Theorem 4.14] that if the solution of
\[(38) \quad dX_n(t) = -n^2 \nabla R_n(X_n(t)) \circ 1_{G_n(X_n(t))} \, dt,\]
exists, where \(G_n(A)\) is the subset of \([n]^2\) defined as
\[G_n(A) := \{(i, j) \in [n]^2 \mid |A(i, j)| < 1\} \cup \{(i, j) \in [n]^2 \mid A(i, j) = 1, \partial_{i,j} R_n(A) > 0\} \cup \{(i, j) \in [n]^2 \mid A(i, j) = -1, \partial_{i,j} R_n(A) < 0\},\]
for all \(A \in \mathcal{M}_n\), then the solution \(X_n\) is a gradient flow on \(\mathcal{M}_n\) in a suitable sense. In this section, we will argue that the solutions \(X_n\) of equation (37) and (38) are equal. To this end, we define processes \(L_n^\pm\) as
\[(40) \quad L_n^+(t) := -\int_0^t n^2 \nabla R_n(X_n(s)) \circ 1_{\{X_n(s) = +1, \nabla R_n(X_n(s)) < 0\}} \, ds,
L_n^-(t) := +\int_0^t n^2 \nabla R_n(X_n(s)) \circ 1_{\{X_n(s) = -1, \nabla R_n(X_n(s)) > 0\}} \, ds,
\]
for \(t \in \mathbb{R}_+\), and equation (38) can be rewritten as
\[(41) \quad dX_n(t) = -n^2 \nabla R_n(X_n(t)) \circ 1_{G_n(X_n(t))} + dL_n^-(t) - dL_n^+(t),\]
and the processes \(L_n^+\) and \(L_n^-\) satisfy the following conditions:

1. The processes \(X_n, L_n^+\) and \(L_n^-\) are adapted processes.
2. The processes \(L_n^-\) and \(L_n^+\) are non-decreasing processes.
3. For every \((i, j) \in [n]^2\),
\[
\int_0^\infty 1\{X_n(i, j)(t) > -1\} \, dL_n^-(i, j)(t) = 0, \quad \text{and}
\int_0^\infty 1\{X_n(i, j)(t) < +1\} \, dL_n^+(i, j)(t) = 0.
\]

Following Section 2.3, these conditions ensure that the processes \(L_n^+\) and \(L_n^-\) are unique and \((X_n, L_n^+, L_n^-)\) solves the Skorokhod problem with respect to the set \(\mathcal{M}_n\). This proves Theorem 1.7.
4. Convergence of the finite dimensional SDEs

4.1. The limit at infinity: infinite exchangeable array of diffusions. Let $\mathcal{E}$ be a standard Borel space. The sets $[n]^{(2)}$ and $\mathbb{N}^{(2)}$ will refer to the set of natural number pairs $(i, j)$ in $\mathbb{N}^2$ and $[n]^2$ respectively, such that $i < j$. Recall that an $\mathcal{E}$-valued exchangeable (symmetric) array refers to a doubly indexed collection of random elements $(\zeta_{i,j} := \zeta(i,j) \in \mathcal{E})_{(i,j)\in\mathbb{N}^{(2)}}$ that remain invariant in law under finite permutations of natural numbers $\mathbb{N}$. Two special cases of $\mathcal{E}$ that are important to us are $\mathcal{E} = [-1,1]$ and $\mathcal{E} = C[0,\infty)$ with the usual Borel topology. The Aldous-Hoover representation theorem [Ald85, Hoo79, Hoo82] says that given any exchangeable array as above, there exists a measurable function $f : [0,1] \times [0,1]^{(2)} \times [0,1] \rightarrow \mathcal{E}$ such that $\zeta_{i,j} = f(U, U_i, U_j, U_{i,j})$ for $(i,j) \in \mathbb{N}^{(2)}$, where $U$, $(U_i)_{i\in\mathbb{N}}$, $(U_{i,j} = U(i,j))_{(i,j)\in\mathbb{N}^{(2)}}$ are i.i.d. Uni$[0,1]$ random variables. The function $f$ is typically not unique. Following [Aus08], we say that $\zeta$ is directed by $f$.

The relationship between exchangeable arrays and graphons follows from the Aldous-Hoover representation [DJ08]. Assume that $\zeta_{i,j}$s are real valued and take values in the closed interval $[-1, 1]$. An infinite exchangeable array gives rise to a random graphon reminiscent of the de Finetti representation theorem for exchangeable sequences of random variables. Although we believe that the following result is well-known, we could not find a statement to this effect in the literature. However, it inspires our later constructions.

**Lemma 4.1.** Let $\zeta \in [-1,1]^{[n]^{(2)}}$ be an infinite exchangeable array directed by $f$. Consider the family of symmetric kernels $(g_u, u \in [0,1])$ defined by

$$
(42) \quad g_u(x, y) := \mathbb{E}[f(u, x, y, V)], \quad u \in [0,1], \quad (x, y) \in [0,1]^{(2)},
$$

where the above expectation is with respect to a Uni$[0,1]$ random variable $V$. Then, for $u \in [0,1]$, given $\{U = u\},$

$$
(43) \quad \lim_{n \to \infty} \delta_{\mathbb{E}}\left(K\left((\zeta_{i,j} = f(u, U_i, U_j, U_{i,j}))_{(i,j)\in[n]^{(2)}}\right), [g_u]\right) = 0, \quad a.s.
$$

**Proof.** Fix $(i,j) \in \mathbb{N}^{(2)}$ and note that $f(U, U_i, U_j, U_{i,j}) = f(U, U_j, U_i, U_{i,j})$ since $\zeta_{i,j} = \zeta_{j,i}$ and $U_{i,j} = U_{j,i}$. Therefore, $\mathbb{E}[f(U, U_i, U_j, U_{i,j}) \mid U, U_i, U_j] = \mathbb{E}[f(U, U_j, U_i, U_{i,j}) \mid U, U_i, U_j]$, and,

$$
\begin{align*}
\quad g_u(x, y) &= \mathbb{E}[f(U, U_i, U_j, U_{i,j}) \mid U = u, U_i = x, U_j = y] \\
&= \mathbb{E}[f(U, U_j, U_i, U_{i,j}) \mid U = u, U_i = x, U_j = y] = g_u(y, x),
\end{align*}
$$

for a.e. $(x, y) \in [0,1]^{(2)}$. Since the maps $f$, $\mathbb{E}$ and $[\cdot]$ are all measurable, their composition is also measurable. Because $U$ is a random variable, $[g_U]$ is also a random variable obtained as a composition of measurable maps.

To see (43), start with the Aldous-Hoover representation $\zeta_{i,j} = f(U, U_i, U_j, U_{i,j})$ for every $(i,j) \in \mathbb{N}^{(2)}$. Condition on $\{U = u\}$ throughout for $u \in [0,1]$. For any finite simple graph $F$, with $k$ vertices,

$$
(44) \quad h_F\left(K\left((\zeta_{i,j})_{(i,j)\in[n]^{(2)}}\right)\right) = \frac{1}{n^{kk}} \sum_{i_1, i_2, \ldots, i_k} \prod_{\{j,l\} \in E(F)} \zeta_{i_ji_l} = \frac{1}{n^{kk}} \sum_{i_1, i_2, \ldots, i_k} \prod_{\{j,l\} \in E(F)} f(u, U_{i_j}, U_{i_l}, U_{i_ji_l}),
$$
where the summation runs over the \( n^k := n!/(n-k)! \) many injections from \([k]\) to \([n]\), and 
\[ h_F: \mathcal{W} \to \mathbb{R} \] is the homomorphism density function of \( F \) \cite[Section 7.2]{Lov12}. Notice that 
\[
\mathbb{E}\left[ h_F\left( K\left( (\zeta_{i,j})_{(i,j)\in [n]^2} \right) \right) \right] = \int_{[0,1]^k} \prod_{\{i,j\} \in E(F)} \mathbb{E}[f(u, u_j, u_i, V)] \, du_1 \cdots du_k = h_F(g_u),
\]
where \( g_u \) is defined in (42). Hence, the lemma will be true if we show that the strong law of large numbers holds. That the weak law of large numbers holds, can be seen by a variance computation. That the convergence is a.e. follows from Borel-Cantelli lemma \cite[Theorem 4.18]{Kal21}. We skip the standard argument. The conclusion holds following the inverse counting lemma \cite[Lemma 10.32]{Lov12}.

**Remark 4.2.** As a corollary of the previous result, although the function \( f \) is not unique in the Aldous-Hoover representation, the law of the random graphon \([g_U]\) is indeed unique.

Consider \((C[0, \infty))^\mathbb{N}^{(2)}\) with the natural filtration generated by the coordinate process. Enlarge the filtration by expanding the probability space to accommodate the countably many i.i.d. \( \text{Un}[0,1] \) random variables \((U_i)_{i \in \mathbb{N}}\) and including the sigma algebra generated by them in the sigma algebra at time zero. Endow this filtered probability space with a probability measure \( P^\infty \) that denote the joint law of \((U_i)_{i \in \mathbb{N}}\) and that of an independent array of countably many independent Brownian motions (BMs) \( \{B_{i,j} = B_{(i,j)}\}_{(i,j) \in \mathbb{N}^{(2)}}\). Finally we turn the natural filtration to one that is right-continuous and complete, thereby satisfying the so-called usual conditions and denote it by \( \mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+} \). All our processes will be adapted to this filtration associated with this set-up. Note that all uniform random variables \((U_i)_{i \in \mathbb{N}}\) are measurable with respect to \( \mathcal{F}_0 \).

Let \( \phi \) and \( \Sigma \) be two functions from \( \mathcal{W} \) to \( L^\infty([0,1]^{(2)}) \) that are both \( \kappa_2 \)-Lipschitz functions on kernels with respect to the the \( L^2 \) norm \( \|\cdot\|_2 \) (Assumption 1 and 3). Our goal is to construct, on the above probability space with filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), an exchangeable array of reflected diffusions satisfying
\[
dX_{i,j}(t) = -\phi(\Gamma(t))(U_i, U_j) \, dt + \Sigma(\Gamma(t))(U_i, U_j) \, dB_{i,j}(t) + dL^-_{i,j}(t) - dL^+_{i,j}(t),
\]
with the initial condition \( X_{i,j}(0) = W_0(U_i, U_j) \) for all \((i,j) \in \mathbb{N}^{(2)}\), for some \( W_0 \in \mathcal{W} \) and 
\( \Gamma(t)(x, y) = \mathbb{E}[X_{1,2}(t) \mid U_1 = x, U_2 = y] \).

We construct a diffusion with more general drift as follows. Let \( b: [-1, 1] \times \mathcal{W} \to L^\infty([0,1]^{(2)}) \) be satisfy Assumption 5. Given \( W_0 \in \mathcal{W} \), let \( X := (X_{i,j} := X_{(i,j)})_{(i,j) \in \mathbb{N}^{(2)}} \) be the solution of the following system of SDE taking values in \([-1, 1]^{N^{(2)}}\) with the initial condition \( (X_{i,j}(0) = W_0(U_i, U_j))_{(i,j) \in \mathbb{N}^{(2)}} \), and satisfying
\[
dX_{i,j}(t) = b(X_{i,j}(t), \Gamma(t))(U_i, U_j) \, dt + \Sigma(\Gamma(t))(U_i, U_j) \, dB_{i,j}(t) + dL^-_{i,j}(t) - dL^+_{i,j}(t),
\]
for \((i,j) \in \mathbb{N}^{(2)}\) and \( t \in \mathbb{R}_+ \). The processes \( L^-_{i,j} \) and \( L^+_{i,j} \) are such that \( (X_{i,j}, L^-_{i,j}, L^+_{i,j}) \) solves the Skorokhod problem with respect to \([-1,1]\) (see Section 2.3), i.e., \( L^-_{i,j} \) and \( L^+_{i,j} \) are non-decreasing processes that keep the processes \( X_{i,j} \) in the closed interval \([-1, 1]\). The kernel valued process \( \Gamma: \mathbb{R}_+ \to \mathcal{W} \) is adapted to the sigma algebra generated by the uniform random variables \((U_i)_{i \in \mathbb{N}}\) and the independent BMs \((B_{i,j})_{(i,j) \in \mathbb{N}^{(2)}}\), and given by
\[
\Gamma(t)(x, y) := \mathbb{E}[X_{1,2}(t) \mid U_1 = x, U_2 = y],
\]
for \((x, y) \in [0, 1]^{(2)}\) and \(t \in \mathbb{R}_+\). Note that if the solution \(X\) of the system of SDEs (46) exists, then conditioned over the sigma algebra \(\mathcal{F}_0\), the coordinate processes of \(X\) are all independent but not necessarily identically distributed. In particular, taking \(b(z, W)(x, y) = -\phi(W)(x, y)\), we recover the system of diffusions in (45).

It is not obvious if an infinite-dimensional stochastic process satisfying (46) and (47) exists, although it is obvious that such a process, if it exists, will be an infinite-dimensional Brownian motion (RBMs) with Lipschitz (but time-varying) drifts and diffusion. In particular, the kernel \(\Gamma\) is indeed well-defined. As will be made clear in Proposition 4.6, the limiting object \(\Gamma\) is the counterpart to the measure-valued solution of the McKean-Vlasov equation, while every \(X_{i,j}\) for \((i, j) \in \mathbb{N}^{(2)}\) is the counterpart to the non-linear evolution of a randomly chosen particle evolving in the McKean-Vlasov interacting system. It should be noted that the particles in this McKean-Vlasov interaction correspond to the edges of the graphs not the vertices. The McKean-Vlasov equation here describes how the graphon itself evolves in time and it is different from the McKean-Vlasov system described in the introduction where the McKean-Vlasov equation describes the evolution of particles which may possibly depend on some underlying graphon.

**Assumption 5.** For a.e. \((x, y) \in [0, 1]^{(2)}\), \(W_1, W_2 \in \mathcal{W}\) and \(z_1, z_2 \in [-1, 1]\), the drift function \(b: [-1, 1] \times \mathcal{W} \to L^\infty([0, 1]^{(2)})\) satisfies

1. There exists \(L \in \mathbb{R}_+\) such that \(\sup_{W \in \mathcal{W}} |b(z_1, W)(x, y) - b(z_2, W)(x, y)| \leq L|z_1 - z_2|\).
2. There exists \(\kappa \in \mathbb{R}_+\) such that \(\sup_{z \in [-1, 1]} \|b(z, W_1) - b(z, W_2)\|_2 \leq \kappa \|W_1 - W_2\|_2\).

Observe that Assumption 5 implies Assumption 1(2) for \(\kappa^2_2 = 2(L^2 + \kappa^2)\) and that \(\|b(z, W)\|_\infty \leq C\) uniformly over all \(z \in [-1, 1]\) and \(W \in \mathcal{W}\).

To argue about the existence of a unique solution of the system of SDEs (46), we construct a sequence of stochastic processes \((X^{(k)}, \Gamma^{(k)})_{k \in \mathbb{Z}_+}\) on \(C([0, \infty), [-1, 1]^{\mathbb{N}^{(2)}} \times \mathcal{W})\) iteratively. Start by defining \((X^{(0)}, \Gamma^{(0)})\) as \(X_{i,j}^{(0)}(t) \equiv W_0(U_i, U_j), \Gamma^{(0)}(t) \equiv W_0,\) for all \((i, j) \in \mathbb{N}^{(2)}\), and \(t \in \mathbb{R}_+\). The induction proceeds by showing that whenever \((X^{(k)}, \Gamma^{(k)})\) for \(k \in \mathbb{Z}_+\) is well defined, \(X^{(k)}\) is an infinite exchangeable array (Lemma 4.3 below) and, \(\Gamma^{(k)}\) is a deterministic process of kernels (Lemma 4.4). Note that these claims are clearly true for \(k = 0\). Then, inductively, define the process \(X^{(k+1)}\) as the strong solution to the coordinate wise reflected SDE:

\[
dX^{(k+1)}_{i,j}(t) = b\left(X^{(k)}_{i,j}(t), \Gamma^{(k)}(t)\right)(U_i, U_j)\, dt + \Sigma\left(\Gamma^{(k)}(t)\right)(U_i, U_j)\, dB_{i,j}(t) \\
+ dL^{(k+1)-}_{i,j}(t) - dL^{(k+1)+}_{i,j}(t),
\]

for \(t \in \mathbb{R}_+\), with the same initial condition \(X^{(k+1)}_{i,j}(0) = W_0(U_i, U_j)\) for all \((i, j) \in \mathbb{N}^{(2)}\).

As usual, \(L^{(k+1)-}_{i,j}\) and \(L^{(k+1)+}_{i,j}\) are processes such that \((X^{(k+1)}_{i,j}, L^{(k+1)-}_{i,j}, L^{(k+1)+}_{i,j})\) solves the Skorokhod problem with respect to \([-1, 1]\) (see Section 2.3) for every \((i, j) \in \mathbb{N}^{(2)}\). Since the drift and diffusion functions \(\phi\) and \(\Sigma\) are deterministic and Lipschitz (Assumption 1), given \(\mathcal{F}_0\), every process \(X^{(k)}\) for \(k \in \mathbb{N}\) exists uniquely in the strong sense.

In fact, given \(\mathcal{F}_0\), the entries of the array \(X^{(k+1)}\) are independent and distributed as reflected Brownian motions (RBMs) with Lipschitz (but time-varying) drifts and diffusion coefficients. In particular, the kernel \(\Gamma^{(k+1)}\) is constructed from the array \(X^{(k+1)}\) (which over the entire probability space is exchangeable, as we show next in Lemma 4.3) as described in
equation (42) in Lemma 4.1, and is therefore defined as
\begin{equation}
\Gamma^{(k+1)}(t)(x, y) := \mathbb{E}\left[ X_{1,2}^{(k+1)}(t) \mid U_1 = x, U_2 = y \right], \quad t \in \mathbb{R}_+.
\end{equation}
The kernel \( \Gamma^{(k+1)}(t) \) is well-defined for a.e. \((x, y) \in [0, 1]^2\) and all \(t \in \mathbb{R}_+\). The induction hence continues.

**Lemma 4.3.** Suppose that, for some \(k \in \mathbb{Z}_+\), there is a unique in law solution to the SDE (48) for \(X^{(k+1)}(t)\) and that \(\Gamma^{(k+1)}(t)\) is a deterministic process of kernels. Then the process \(X^{(k+1)}(t)\) is an infinite exchangeable array taking values in \(\mathcal{E} = C[0, \infty)\), equipped with the usual locally uniform metric.

**Proof.** To argue the exchangeability, let \(\sigma: \mathbb{N} \to \mathbb{N}\) be a finite permutation of the natural numbers \(\mathbb{N}\). Note that \(\sigma\) fixes every large enough natural number. We need to argue that \(\left(X^{(k+1)}_{i,j}\right)_{(i,j) \in \mathbb{N}^2}\) has the same law as \(\left(X^{(k+1)}_{\sigma(i),\sigma(j)}\right)_{(i,j) \in \mathbb{N}^2}\) in the sense of equality of the two probability measures on \((C[0, \infty))^\mathbb{N}^2\).

Let \(\tilde{U}_i := U_{\sigma(i)}\), for all \(i \in \mathbb{N}\). Then \((\tilde{U}_i)_{i \in \mathbb{N}}\) is again a sequence of i.i.d. \(\text{Uni}[0, 1]\) random variables. Let \(Y^{(k+1)}_{i,j} \equiv X^{(k+1)}_{\sigma(i),\sigma(j)}\) for every \((i, j) \in \mathbb{N}^2\). Since \(Y^{(k+1)}_{i,j}(0) = W_0(U_{\sigma(i)}, U_{\sigma(j)}) := W_0(\tilde{U}_i, \tilde{U}_j)\). It follows that \(\left(Y^{(k+1)}_{i,j}(0)\right)_{(i,j) \in \mathbb{N}^2}\) has the same distribution as \(\left(X^{(k+1)}_{i,j}(0)\right)_{(i,j) \in \mathbb{N}^2}\). Moreover, for every \((i, j) \in \mathbb{N}^2\), the process \(Y^{(k+1)}\) satisfies the SDEs
\begin{align*}
dY^{(k+1)}_{i,j}(t) &= b\left(X^{(k)}_{\sigma(i),\sigma(j)}(t), \Gamma^{(k)}(t)\right)(U_{\sigma(i)}, U_{\sigma(j)}) dt + \sum \left(\frac{\partial}{\partial x}(\Gamma^{(k)}(t))\right)(U_{\sigma(i)}, U_{\sigma(j)}) dB_{\sigma(i),\sigma(j)}(t) \\
&\quad + dL_{\sigma(i),\sigma(j)}^{(k+1)-}(t) - dL_{\sigma(i),\sigma(j)}^{(k+1)+}(t) \\
&= b\left(Y^{(k)}_{i,j}(t), \Gamma^{(k)}(t)\right)(\tilde{U}_i, \tilde{U}_j) dt + \sum \left(\frac{\partial}{\partial x}(\Gamma^{(k)}(t))\right)(\tilde{U}_i, \tilde{U}_j) dB_{\sigma(i),\sigma(j)}(t) \\
&\quad + dL_{\sigma(i),\sigma(j)}^{(k+1)-}(t) - dL_{\sigma(i),\sigma(j)}^{(k+1)+}(t),
\end{align*}
for \((i, j) \in \mathbb{N}^2\) and \(t \in \mathbb{R}_+\). Note that, \(\Gamma^{(k)}\) does not get affected by the permutation \(\sigma\).

Relabeling \(\tilde{B}_{i,j} := B_{\sigma(i),\sigma(j)}, \tilde{L}_{i,j}^{(k+1)-} := L_{\sigma(i),\sigma(j)}^{(k+1)-}\) and \(\tilde{L}_{i,j}^{(k+1)+} := L_{\sigma(i),\sigma(j)}^{(k+1)+}\) for every \((i, j) \in \mathbb{N}^2\), leaves their joint law unchanged, and we get
\begin{align*}
dY^{(k+1)}_{i,j}(t) &= b\left(Y^{(k)}_{i,j}(t), \Gamma^{(k)}(t)\right)(\tilde{U}_i, \tilde{U}_j) dt + \sum \left(\frac{\partial}{\partial x}(\Gamma^{(k)}(t))\right)(\tilde{U}_i, \tilde{U}_j) d\tilde{B}_{i,j}(t) \\
&\quad + d\tilde{L}_{i,j}^{(k+1)-}(t) - d\tilde{L}_{i,j}^{(k+1)+}(t),
\end{align*}
for every \((i, j) \in \mathbb{N}^2\) and \(t \in \mathbb{R}_+\). Since \(X^{(k+1)}(t)\) and \(Y^{(k+1)}(t)\) follow the same system of recursive SDEs (48), their equivalence in law follows from the uniqueness in law of the SDE.

**Lemma 4.4.** Under the same assumption as in Lemma 4.3 and Assumption 5, the kernel-valued map \(t \mapsto \Gamma^{(k)}(t)\), is deterministic and absolutely continuous. Moreover, for each \(t \in \mathbb{R}_+\), we have
\begin{equation}
\lim_{n \to \infty} \delta_\mathcal{O} \left( K\left( \left(X^{(k)}_{i,j}(t)\right)_{(i,j) \in [n]^2}\right) \right) = 0, \quad \text{a.s.}
\end{equation}

**Proof.** By definition, for \((x, y) \in [0, 1]^2\), and \(t \in \mathbb{R}_+\), \(\Gamma^{(k)}(t)(x, y) := \mathbb{E}\left[ X^{(k)}_{1,2}(t) \mid U_1 = x, U_2 = y \right]\). This is a deterministic kernel for every \(t \in \mathbb{R}_+\). To see (50), repeat the proof of Lemma 4.1. Notice that, there is no random variable \(U\) as in Lemma 4.1.
(also see Remark 4.2). This is now a consequence of Kolmogorov’s zero-one law [Kal21, Theorem 4.13]. For \( n \in \mathbb{N} \), let \( \mathcal{G}_n \) be the sigma algebra generated by \( U_n \) and the i.i.d. standard Brownian motions \( B_{i,j} \)s for the set of indices \( \{(i, j) \in \mathbb{N}^{(2)} \mid j = n \} \). This is a sequence of independent sigma algebras. Consider its tail sigma algebra \( \mathcal{T} := \cap_{n \in \mathbb{N}} \cap_{\ell \geq n} \mathcal{G}_\ell \). This is a trivial sigma algebra by the Kolmogorov zero-one law.

Consider, for any finite simple graph \( F \) and \( t \in \mathbb{R}_+ \), the limiting homomorphism densities \( \lim_{n \to \infty} h_F(K((X_{i,j}^{(k)}(t))_{(i,j) \in [n]^{(2)}})) \), as in equation (44). These limiting homomorphism densities do not depend on finitely many elements in \( \{X_{i,j}^{(k)}(t)\}_{(i,j) \in \mathbb{N}^{(2)}} \) or \( \{U_i\}_{i \in \mathbb{N}} \). In particular, such limits are measurable with respect to the tail sigma algebra \( \mathcal{T} \). Exactly as in the proof of Lemma 4.1, it follows that

\[
\lim_{n \to \infty} \delta_{\mathcal{T}} \left( \left[ K \left( (X_{i,j}^{(k)}(t))_{(i,j) \in [n]^{(2)}} \right) \right], \left[ \Gamma^{(k)}(t) \right] \right) = 0.
\]

In particular, the graphon \( \left[ \Gamma^{(k)}(t) \right] \) is measurable with respect to \( \mathcal{T} \), and thus constant a.e.

Finally, the absolute continuity of \( t \mapsto \Gamma(t) \) follows from the path continuity of the process \( X_{1,2}^{(k)} \) and our assumptions on \( b \) and \( \Sigma \).

**Proposition 4.5.** Assume that the drift functions \( b: [-1, 1] \times \mathcal{W} \to L^\infty([0, 1]^{(2)}) \) satisfies Assumption 5, and the diffusion coefficient function \( \Sigma: \mathcal{W} \to L^\infty([0, 1]^{(2)}) \) is bounded and \( \kappa_2 \)-Lipschitz in \( \| \cdot \|_2 \) (Assumption 3). Then the sequence of processes taking values in \( C([-1, 1]) \) given by \( \left( (X_{1,2}^{(k)}(t), \Gamma^{(k)}(t))_{t \in \mathbb{R}_+} \right)_{k \in \mathbb{Z}_+} \) converges locally uniformly in the 2-product metric of \([−1, 1] \) and \( (\mathcal{W}, d_2) \), to a pathwise unique process \( (X_{1,2}(t), \Gamma(t))_{t \in \mathbb{R}_+} \) starting from \( \Gamma(0) = W_0 \in \mathcal{W} \) and \( X_{1,2}(0) = W_0(U_1, U_2) \). That is, for every \( t \in \mathbb{R}_+ \),

\[
\lim_{k \to \infty} \sup_{s \in [0,t]} \left[ \left\| X_{1,2}^{(k)}(s) - X_{1,2}(s) \right\|^2 + \left\| \Gamma^{(k)}(s) - \Gamma(s) \right\|^2 \right] = 0, \quad \text{a.s.}
\]

In particular, the limiting processes \( X_{1,2} \) is continuous and \( \Gamma \) is absolutely continuous and deterministic.

**Proof.** The proof is a standard Picard iteration based proof of existence of solutions of SDEs. See, for example, the proof of [KS91, Theorem 2.9, page 289]. Hence, we will skip some of the details and refer the reader to the above cited reference.

We will take \( k \to \infty \) and produce a limit. Start by noticing that the process \( X_{1,2}^{(k+1)}: \mathbb{R}_+ \to [-1, 1] \) is the result of applying the Skorokhod map [KLRS07] pathwise to the “noise before reflection” process \( Y_{1,2}^{(k+1)} \) obtained as the unique strong solution to the SDE:

\[
\lim_{k \to \infty} \sup_{s \in [0,t]} \left[ \left\| X_{1,2}^{(k)}(s) - X_{1,2}(s) \right\|^2 + \left\| \Gamma^{(k)}(s) - \Gamma(s) \right\|^2 \right] = 0, \quad \text{a.s.}
\]

(52)

\[
dY_{1,2}^{(k+1)}(t) = b\left( X_{1,2}^{(k)}(t), \Gamma^{(k)}(t) \right)(U_1, U_2) \, dt + \Sigma \left( \Gamma^{(k)}(t) \right)(U_1, U_2) \, dB_{1,2}(t),
\]

for \( t \in \mathbb{R}_+ \), with initial conditions \( Y_{1,2}^{(k+1)}(0) = X_{1,2}^{(k+1)}(0) = W_0(U_1, U_2) \) for all \( k \in \mathbb{Z}_+ \).

Fix \( t \in \mathbb{R}_+ \) and consider \( \sup_{s \in [0,t]} \left| X_{1,2}^{(k+1)}(s) - X_{1,2}^{(k)}(s) \right| \) for any \( k \in \mathbb{N} \). Since the Skorokhod map is 4-Lipschitz in the local uniform norm (see Section 2.3), the above distance is bounded
by 
\[ 4 \sup_{s \in [0, t]} \left| Y_{1,2}^{(k+1)}(s) - Y_{1,2}^{(k)}(s) \right| . \]
Now for every fixed \( k \in \mathbb{N} \), from equation (52) we have
\[
Y_{1,2}^{(k+1)}(t) - Y_{1,2}^{(k)}(t) = \int_0^t \left( b \left( X_{1,2}^{(k-1)}(t), \Gamma^{(k-1)}(t) \right) (U_1, U_2) - b \left( X_{1,2}^{(k)}(t), \Gamma^{(k)}(t) \right) (U_1, U_2) \right) \, ds \\
- \int_0^t \left( \Sigma \left( \Gamma^{(k-1)} \right)(U_1, U_2) - \Sigma \left( \Gamma^{(k)} \right)(U_1, U_2) \right) \, dB_{1,2}(s). \tag{53}
\]

Define \( \Delta, M : \mathbb{R}_+ \to \mathbb{R} \) for \( t \in \mathbb{R}_+ \) as
\[
\Delta(t) := \int_0^t \left( b \left( X_{1,2}^{(k-1)}(t), \Gamma^{(k-1)}(t) \right) (U_1, U_2) - b \left( X_{1,2}^{(k)}(t), \Gamma^{(k)}(t) \right) (U_1, U_2) \right) \, ds, \\
M(t) := \int_0^t \left( \Sigma \left( \Gamma^{(k-1)} \right)(U_1, U_2) - \Sigma \left( \Gamma^{(k)} \right)(U_1, U_2) \right) \, dB_{1,2}(s).
\]

Note that, for a kernel \( A \in \mathcal{W} \), we have \( \| A \|_2^2 = \mathbb{E}[A^2(U_1, U_2)] \), for \( U_1, U_2 \) i.i.d. as \( \text{Uni}[0, 1] \).
Using Jensen’s inequality and interchanging expectation with integral and Assumption 5,
\[
\mathbb{E} \left[ \sup_{s \in [0, t]} \Delta^2(s) \right] \\
\leq t \mathbb{E} \left[ \int_0^t \left| b \left( X_{1,2}^{(k-1)}(t), \Gamma^{(k-1)}(t) \right) (U_1, U_2) - b \left( X_{1,2}^{(k)}(t), \Gamma^{(k)}(t) \right) (U_1, U_2) \right|^2 \, ds \right] \\
= t \int_0^t \left\| b \left( X_{1,2}^{(k-1)}(t), \Gamma^{(k-1)}(t) \right) - b \left( X_{1,2}^{(k)}(t), \Gamma^{(k)}(t) \right) \right\|_2^2 \, ds \\
\leq 2\kappa^2 t \int_0^t \left\| \Gamma^{(k-1)}(s) - \Gamma^{(k)}(s) \right\|_2^2 \, ds + 2L^2 t \int_0^t \mathbb{E} \left[ \left| X^{(k-1)}(s) - X^{(k)}(s) \right|^2 \right] \, ds. \tag{54}
\]

For \( M \), we use the fact that it is a stochastic integral of a bounded integrand with respect to a Brownian motion, and hence a continuous martingale. By an application of Doob’s maximal inequality [KS91, Theorem 3.8.iv, page 14], we get that,
\[
\mathbb{E} \left[ \sup_{s \in [0, t]} M^2(s) \right] \leq 4 \int_0^t \mathbb{E} \left[ \left| \Sigma \left( \Gamma^{(k-1)}(s) \right)(U_1, U_2) - \Sigma \left( \Gamma^{(k)}(s) \right)(U_1, U_2) \right|^2 \right] \, ds.
\]

Using the assumption that \( \Sigma \) is \( \kappa_2 \)-Lipschitz in \( \| \cdot \|_2 \) and the same argument as above,
\[
\mathbb{E} \left[ \sup_{s \in [0, t]} M^2(s) \right] \leq 4\kappa^2 t \int_0^t \left\| \Gamma^{(k-1)}(s) - \Gamma^{(k)}(s) \right\|_2^2 \, ds. \tag{55}
\]
Now, taking absolute values on both sides on (53), we immediately get,

\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \left| X_{1,2}^{(k+1)}(s) - X_{1,2}^{(k)}(s) \right|^2 \right] \\
\leq 16 \mathbb{E} \left[ \sup_{s \in [0,t]} \left| Y_{1,2}^{(k+1)}(s) - Y_{1,2}^{(k)}(s) \right|^2 \right] \leq 32 \mathbb{E} \left[ \sup_{s \in [0,t]} \Delta^2(s) + \sup_{s \in [0,t]} M^2(s) \right] \\
\leq 64 (\kappa^2 t + 2\kappa^2_S) \int_0^t \| \Gamma^{(k+1)}(s) - \Gamma^{(k)}(s) \|^2_2 \, ds \\
+ 64L^2 t \int_0^t \mathbb{E} \left[ \left| X^{(k-1)}(s) - X^{(k)}(s) \right|^2 \right] \, ds.
\]

Using the fact that the operator \( \Gamma \), given by a conditional expectation (49), and, therefore, must have a smaller \( L^2 \) norm

\[
\sup_{s \in [0,t]} \| \Gamma^{(k+1)}(s) - \Gamma^{(k)}(s) \|^2_2 \leq \mathbb{E} \left[ \sup_{s \in [0,t]} \left| X_{1,2}^{(k+1)}(s) - X_{1,2}^{(k)}(s) \right|^2 \right].
\]

Combining the last two bounds above, one gets the recursive bound

\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \left| X_{1,2}^{(k+1)}(s) - X_{1,2}^{(k)}(s) \right|^2 \right] \\
\leq 128((\kappa^2 + L^2)t + 4\kappa^2_S) \int_0^t \mathbb{E} \left[ \left| X^{(k-1)}(s) - X^{(k)}(s) \right|^2 \right] \, ds.
\]

The rest of the argument follows exactly as in [KS91, page 290] by applications of Grönwall’s lemma [Grö19] and the Borel-Cantelli lemma [Kal21, Theorem 4.18]. We skip the similar argument for pathwise uniqueness. See the proof of [KS91, Proposition 2.13, page 291]. □

**Proposition 4.6.** Suppose the assumptions in Proposition 4.5 holds. Given any kernel \( W_0 \in \mathcal{W} \), there exists a pathwise unique strong solution to the coupled system (46) and (47) in the following sense. In any probability space supporting countably many i.i.d. \( \text{Uni}[0,1] \) random variables \( (U_i)_{i \in \mathbb{N}} \) and an independent infinite (symmetric) array of i.i.d. standard Brownian motions \( (B_{i,j})_{(i,j) \in \mathbb{N}^2} \), one can construct an infinite exchangeable array of reflected diffusions \( (X_{i,j})_{(i,j) \in \mathbb{N}^2} \) that satisfy (46) and (47) and every \( X_{i,j} \) is pathwise unique.

Moreover, for every \( t \in \mathbb{R}_+ \), \( \Gamma(t) \) can be recovered as the \( \delta_{\square} \) limit of the sequence of graphons \( \big( \left[ K((X_{i,j}(t))_{(i,j) \in [n]^2}) \right] \big)_{n \in \mathbb{N}} \) locally uniformly in time. That is, for any \( t \in \mathbb{R}_+ \),

\[
\lim_{n \to \infty} \sup_{s \in [0,t]} \left( \delta_{\square} \left( \left[ K((X_{i,j}(s))_{(i,j) \in [n]^2}) \right], \Gamma(s) \right) \right) = 0, \quad \text{a.s.}
\]

**Proof.** Start with the countably many i.i.d. \( \text{Uni}[0,1] \) random variables \( (U_i)_{i \in \mathbb{N}} \) and an independent infinite (symmetric) array of i.i.d. standard Brownian motions \( (B_{i,j})_{(i,j) \in \mathbb{N}^2} \) and construct the deterministic process \( \Gamma \) in Proposition 4.5.

Given \( \Gamma \) and \( (U_i)_{i \in \mathbb{N}} \) and following the system of SDEs (46), the diffusions \( X_{i,j} \)’s are independent (but not identically distributed) reflected Brownian motions with deterministic bounded time-dependent drifts for \( (i, j) \in \mathbb{N}^2 \). So, they exist in a pathwise or strong sense exactly as the process \( X_{1,2} \) does in Proposition 4.5 and satisfies the constraint (46) since \( \Gamma \) is a fixed point of the Picard iterations.
It is obvious from the symmetry of the construction that the infinite array \((X_{i,j})_{(i,j) \in \mathbb{N}^2}\) is exchangeable in the sense of Section 4.1 with \(\mathcal{E} = C[0, \infty)\), the set of continuous functions from \([0, \infty)\) to \(\mathbb{R}\).

For the limit (57) we will make use of the following result from [Lov12, Proposition 8.12], which states that for any \(V \in \mathcal{W}\),

\[
\|V\|_4 \leq h_{C_4}(V) \leq 4\|V\|_2.
\]

Here \(C_4\) is the cyclic graph with four vertices and \(h_{C_4}(V)\) is the homomorphism density function of the simple graph \(C_4\). We will apply this for the choice of \(V_n(t) := K((X_{i,j}(t))_{(i,j) \in [n]^2}) - K((\Gamma(t)(U_i, U_j))_{(i,j) \in [n]^2})\). Thus,

\[
H_n(t) := h_{C_4}(V_n(t)) = \frac{1}{n^{14}} \sum_{i_1, i_2, \ldots, i_4} \prod_{l=1}^{4} (X_{i_l, i_{l+1}}(t) - \Gamma(t)(U_{i_l}, U_{i_{l+1}}))
\]

\[
= \frac{1}{n^{14}} \sum_{i_1, i_2, \ldots, i_4} \prod_{l=1}^{4} (X_{i_l, i_{l+1}}(t) - \mathbb{E}[X_{i_l, i_{l+1}}(t) \mid \mathcal{F}_0]),
\]

with the convention that, when \(l = 4\), \(l + 1 \equiv 1\). The above sum is over all injections in \([n]^4\).

Notice that \(H_n(0) = 0\). The fact that for each \(t \in \mathbb{R}_+\), \(\lim_{n \to \infty} H_n(t) = 0\) almost surely follows similarly to the proof of Lemma 4.1. We now show that \(t \mapsto H_n(t)\) is equicontinuous. From which, using a standard argument, we can show that almost surely, \(H_n(t) \to 0\) for each \(t \in \mathbb{R}_+\), that is,

\[
\lim_{n \to \infty} \delta_2\left(\left[K((X_{i,j}(s))_{(i,j) \in [n]^2})\right], [\Gamma(s)]\right) = 0, \quad \text{a.s.} \quad \forall \ s \in [0, t].
\]

To show that \((H_n)_{n \in \mathbb{N}}\) is equicontinuous, we first observe that for any \(s_1, s_2 \in [0, t]\),

\[
|H_n(s_2) - H_n(s_1)| \leq 16\left\|K((X_{i,j}(s_2))_{(i,j) \in [n]^2}) - K((X_{i,j}(s_1))_{(i,j) \in [n]^2})\right\|_2
\]

\[
+ 16\left\|\Gamma(s_2) - \Gamma(s_1)\right\|_2,
\]

where the inequality follows by an application of the counting lemma [Lov12, Lemma 10.23, Exercise 10.27], the triangle inequality and using the fact that the cut norm \(\| \cdot \|_2\) is upper bounded by the \(L^2\) norm \(\| \cdot \|_2\).
Using the Lipschitzness of the Skorokhod map (see equation (17)), we therefore obtain

\[
\left\| K\left((X_{i,j}(s_2))_{(i,j)\in[n]^2}\right) - K\left((X_{i,j}(s_1))_{(i,j)\in[n]^2}\right) \right\|^2_2 \\
\leq \frac{2^4}{n^2} \sum_{(i,j)\in[n]^2} |Y_{i,j}(s_2) - Y_{i,j}(s_1)|^2 \\
\leq \frac{2^5}{n^2} \sum_{(i,j)\in[n]^2} \left| \int_{s_1}^{s_2} b(X_{1,j}(u), \Gamma(u))(U_i, U_j) \, du \right|^2 \\
\quad + \frac{2^5}{n^2} \sum_{(i,j)\in[n]^2} \left| \int_{s_1}^{s_2} \Sigma(\Gamma(u))(U_i, U_j) \, dB_{i,j}(u) \right|^2 \\
\leq 2^5 M_{\infty}^2 |s_2 - s_1|^2 + \frac{2^5}{n^2} \sum_{(i,j)\in[n]^2} \left| \int_{s_1}^{s_2} \Sigma(\Gamma(u))(U_i, U_j) \, dB_{i,j}(u) \right|^2.
\]

(60)

Now let \(|s_2 - s_1| \leq \delta\) for some \(\delta > 0\). Set for all \((i, j)\in[n]^2\),

\[
\eta_{i,j} := \sup_{s_1, s_2 \in [0, t], |s_2 - s_1| \leq \delta} \left| \int_{s_1}^{s_2} \Sigma(\Gamma(u))(U_i, U_j) \, dB_{i,j}(u) \right|^2.
\]

From [Slo01, Lemma A.4], there exist constants \(C_{1,t}, C_{2,t} \in \mathbb{R}_+\) depending on \(t\), such that for all \((i, j) \in [n]^2\),

\[
\mathbb{E}[\eta_{i,j}] \leq M_{\infty}^2 C_{1,t} \log \frac{1}{\delta}, \quad \text{and} \quad \mathbb{E}[\eta_{i,j}^2] \leq M_{\infty}^4 C_{2,t}^2 \delta^2 \log^2 \frac{1}{\delta}.
\]

Since, \(\eta_{i,j}\)'s are independent and have finite variance, it follows from the Chebyshev's inequality [Kal21, Lemma 5.1] that

\[
\mathbb{P}\left\{ \left| \frac{1}{n^2} \sum_{(i,j)\in[n]^2} \eta_{i,j} - \mathbb{E}[\eta_{i,j}] \right| \geq \max_{(i,j)\in[n]^2} \text{Var}^{1/2}(\eta_{i,j}) \right\} \leq \frac{1}{n^2}.
\]

Using the Borel-Cantelli lemma [Kal21, Theorem 4.18], it follows that almost surely,

\[
\frac{1}{n^2} \sum_{(i,j)\in[n]^2} \eta_{i,j} \leq M_{\infty}^2 (C_{1,t} + C_{2,t}) \delta \left| \log \frac{1}{\delta} \right|
\]

(62)

for all \(n \in \mathbb{N}\), sufficiently large. Combining equations (59) and (62), we obtain that almost surely, for all \(n \in \mathbb{N}\) sufficiently large, we have

\[
\sup_{s_1, s_2 \in [0, t], |s_2 - s_1| \leq \delta} |H_n(s_2) - H_n(s_1)| \leq 2^8 M_{\infty} \left( \delta + (C_{1,t} + C_{2,t})^{1/2} \delta^{1/2} \log^{1/2} \frac{1}{\delta} \right) + 16 \omega(\delta),
\]

where \(\omega(\delta) := \sup_{s_1, s_2 \in [0, t], |s_2 - s_1| \leq \delta} \|\Gamma(s_2) - \Gamma(s_1)\|_2\) is the modulus of continuity of the curve \(t \mapsto \Gamma(t)\). Since \(s \mapsto \Gamma(s)\) is continuous in \((\mathcal{W}, d_2)\) (and independent of \(n\)), it follows that, almost surely, \((H_n)_{n \in \mathbb{N}}\) is equicontinuous. Since \((H_n)_{n \in \mathbb{N}}\) is equicontinuous uniformly bounded almost surely, the proof is complete by a standard application of Arzelà-Ascoli theorem [Mun00, Theorem 47.1]. \(\square\)
Proposition 4.7. Suppose that $\Sigma \equiv \beta > 0$ and $b(z, W) = -\phi(W)$. Then, the limiting curve $\Gamma$ in Proposition 4.6 has a velocity
\[
\dot{\Gamma}(t) = -\phi(\Gamma(t)) - \left[p_{\beta t}^{(\pm 1)}(W_0, \phi \circ \Gamma, \beta) - p_{\beta t}^{(-1)}(W_0, \phi \circ \Gamma, \beta)\right],
\]
where $p_{\beta t}^{(\pm 1)}(W_0, \phi \circ \Gamma, \beta)(x, y)$ is the density of the real-valued reflected Brownian motion $Z$ at $\pm 1$, at time $s \in \mathbb{R}_+$, starting at $Z(0) = W_0(x, y)$, satisfying
\[
dZ(s) = -\frac{1}{\beta^2} \phi(\Gamma(s/\beta^2))(x, y) \, ds + dB(s) + dL^-(s) - dL^+(s), \quad s \in \mathbb{R}_+,
\]
where $(Z, L^+, L^-)$ solves the Skorokhod problem with respect to the set $[-1, 1]$ (see Section 2.3).

Proof. Given $(U_1, U_2) = (x, y)$, the process $X_{1,2}$ is a diffusion with a Lipschitz drift and a constant diffusion coefficient. Using (47) and Itô's formula, we get
\[
\frac{d}{dt} \Gamma(t)(x, y) = -\frac{d}{dt} \phi(\Gamma(t))(x, y)
\]
\[
+ \frac{d}{dt} \mathbb{E}[L_{1,2}^-(t) \mid U_1 = x, U_2 = y] - \frac{d}{dt} \mathbb{E}[L_{1,2}^+(t) \mid U_1 = x, U_2 = y].
\]

Now consider the reflecting diffusion $Z$ which solves the SDE
\[
dZ(s) = \Psi(s; \beta) \, ds + dB(s) + dL^-(s) - dL^+(s), \quad s \in \mathbb{R}_+,
\]
starting at $Z(0) = W_0(x, y)$, such that $(Z, L^+, L^-)$ solves the Skorokhod problem with respect to the set $[-1, 1]$, and $\Psi(s; \beta) := \frac{1}{\beta^2} b(\Gamma(s/\beta^2))(x, y)$ for all $s \in \mathbb{R}_+$ (see Section 2.3). By reparametrizing $s = \beta^2 t$ and setting $Z(s) = X_{1,2}(t)$, we get back our reflected diffusion $X_{1,2}$ in law following
\[
dZ(\beta^2 t^2) = -\frac{1}{\beta^2} \phi(\Gamma(t))(x, y) \, d(\beta^2 t) + dB(\beta^2 t) + dL^-(\beta^2 t) - dL^+(\beta^2 t),
\]
\[
\Rightarrow X_{1,2}(t) = -\phi(\Gamma(t)) \, dt + \beta dB(t) + dL^-(\beta^2 t) - dL^+(\beta^2 t), \quad t \in \mathbb{R}_+,
\]
where the processes $(L^+(\beta^2 t))_{t \in \mathbb{R}_+}$ and $(L^-(\beta^2 t))_{t \in \mathbb{R}_+}$ constrain the process $X_{1,2}$ in the interval $[-1, 1]$ (see Section 2.3). Here the equality is in law. We use the fact that the solution of both the above SDEs agree in law since the distribution of $B(\beta^2 t)$ and $\beta B(t)$ coincide for all $\beta \in \mathbb{R}_+$. Let $p_{\beta t}^{(\pm 1)}(W_0, \phi \circ \Gamma, \beta)(x, y)$ denote the transition density of the solution of SDE (65) at time $s \in \mathbb{R}_+$ at the boundary $\pm 1$, then the transition density of the process $X_{1,2}$ at time $t$ at the boundary $\pm 1$ is $p_{\beta t}^{(\pm 1)}(W_0, \phi \circ \Gamma, \beta)(x, y)$.

Using [RY04, Exercise (1.12), page 407] and equation (64), we deduce that
\[
\frac{d}{dt} \mathbb{E}[L_{1,2}^+(t)] = p_{\beta t}^{(\pm 1)}(W_0, b \circ (X_{1,2}; \Gamma), \beta)(x, y),
\]
which gives us the desired result. \qed

Remark 4.8. Note that the (pointwise) velocity of the curve $\Gamma$ at time $t \in \mathbb{R}_+$ is not $-(\phi \circ \Gamma)(t)$ when $\beta > 0$. That is, $\Gamma$ is not a gradient flow of the function $R$ when $\beta > 0$, and the effect of the boundary $\{-1, 1\}$, as seen in (63), is qualitatively different from that when $\beta = 0$ (see Section 1.1).
4.2. Convergence of the finite dimensional processes. Consider now the finite dimensional SDE (RSDE):

\[ dX_n(t) = -n^2 \nabla R_n(X_n(t)) \, dt + \sum_n(X_n(t)) \circ dB_n(t) + dL_n^-(t) - dL_n^+(t). \]

The Fréchet-like derivative of \( R \) is a symmetric kernel-valued map from \( \mathcal{W} \to L^\infty([0, 1]) \). Thus, for \((x, y) \in [0, 1]^2\), there is a real-valued map \( \phi_{x,y}: \mathcal{W} \to \mathbb{R} \) given by \( \phi_{x,y}(V) = \phi(V)(x, y) \) for all \( V \in \mathcal{W} \). This is the same map that we get when we replace \((x, y)\) by \((y, x)\).

To show that the finite dimensional processes converge as \( n \to \infty \), we will need to put further assumptions on the drift and diffusion functions.

**Assumption 6.** There exists a constant \( \kappa_\square \in \mathbb{R}_+ \) such that for all \( W_1, W_2 \in \mathcal{W} \), the drift function \( b: [-1, 1] \times \mathcal{W} \to L^\infty([0, 1]^2) \) and the diffusion coefficient function \( \Sigma: \mathcal{W} \to L^\infty([0, 1]^2) \) satisfy

\[
\sup_{(x, y) \in [0, 1]^2} \sup_{z \in [-1, 1]} |b(z, W_1)(x, y) - b(z, W_2)(x, y)| \leq \kappa_\square \|W_1 - W_2\|_\square, \quad \text{and}
\sup_{(x, y) \in [0, 1]^2} |\Sigma(W_1)(x, y) - \Sigma(W_2)(x, y)| \leq \kappa_\square \|W_1 - W_2\|_\square.
\]

**Proposition 4.9.** Suppose the assumptions in Proposition 4.5 and Assumption 6 hold. Then, for any sequence of initial kernels \( (W_0^{(n)}) \in \mathcal{W}_n \) that converges to \( W_0 \in \mathcal{W} \) in the \( L^2([0, 1]^2) \) norm \( \| \cdot \|_2 \), i.e., whenever

\[ \lim_{n \to \infty} \|W_0^{(n)} - W_0\|_2 = 0, \]

the process of random kernels \( (K(X_n(t)))_{t \in \mathbb{R}_+} \) obtained from solutions of the SDEs (67), converges locally uniformly in the cut norm as \( n \to \infty \), in probability, to the limiting process \( \Gamma: \mathbb{R}_+ \to \mathcal{W} \), with \( \Gamma(0) = W_0 \), established in Proposition 4.6.

**Proof.** Consider a probability space satisfying the assumptions of Proposition 4.6 and an infinite exchangeable array of diffusions \( (X_{i,j})_{(i,j) \in \mathbb{N}^2} \) on it. For \( k \in [n] \) and any \( t \in \mathbb{R}_+ \), consider the sampled \( k \times k \) symmetric matrix \( \Gamma(t)[k] \) whose \((i, j)\)-th element is \( \Gamma(t)(U_i, U_j) \), \((i, j) \in [k]^2\). Consider also the corresponding \( k \times k \) matrix of diffusions \( X^{(k)}(t) \) := \( (X_{(i,j)})_{(i,j) \in [k]^2} \).

Now consider \( K(X_n(t)) \) from a solution of SDEs (67). One may construct a sampled \( k \times k \) matrix from this kernel as well. We estimate the cut distance of this sampled matrix from \( \Gamma(t)[k] \) by coupling this sampled matrix with \( K(X^{(k)}) \) in a particular way.

Notice that, for any \((i, j) \in [k]^2\) and \((m_i, m_j) \in [n]^2\), if \( U_i \in ((m_i - 1)/n, m_i/n] \) and \( U_j \in ((m_j - 1)/n, m_j/n] \), then \( K(X(t))(U_i, U_j) \equiv X_{n,m_i,m_j}(t) \). Let \( E_k(n) \) denote the event that that no two \( U_i, U_j \) for distinct \( i, j \in [k]^2 \), falls in the same interval \(((m - 1)/n, m/n]\). Under this event every entry of the sampled diffusions will be run by independent standard Brownian motions. Before we use this property to proceed with our coupling, let us show that \( E_k(n) \) happens with high probability as \( k \) is fixed and \( n \to \infty \). Order the uniform random variables as \( U_{(1)} < U_{(2)} < \ldots < U_{(k)} \). Clearly \( E_k(n) \) implies that there is at least one pair \((U_{(i)}, U_{(i+1)})\) for \( i \in [k - 1] \), such that \( U_{(i+1)} - U_{(i)} \leq 1/n \). Hence \( \mathbb{P}(E_k(n)) \leq \mathbb{P}(\min_{i \in [k-1]}(U_{(i+1)} - U_{(i)}) \leq \frac{1}{k-1}) \). But \( \min_{i \in [k-1]}(U_{(i+1)} - U_{(i)}) \) has a density at zero and hence the above probability is \( O(1/n) \), which goes to zero as \( n \to \infty \). Thus \( \lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}(E_k(n)) = 1 \).
On the event $E_k(n)$, every $m_i$, $i \in [k]$, is distinct. Consider the corresponding independent Brownian motion $B_{i,j}$ from the diffusion $X_{i,j}$ from equation (46). Since (67) admits a strong solution, construct a solution where the entry processes $X_{n,m_i,m_j}(\cdot)$ is driven by $B_{i,j}$, $(i, j) \in [k]^2$, while the rest of the entries of $X_n$ are driven by a disjoint subset of $(B_{i,j})_{(i,j) \in \mathbb{N}^2}$. Thus, one couples $K(X_n)(\cdot)(U_i, U_j)$ with $X_{i,j}$ which are both driven by the same Brownian motion and having a starting value of $W_0^{(n)}(U_i, U_j)$ and $W_0(U_i, U_j)$, respectively. Our subsequent analysis will be on the event $E_k(n)$ and it is unimportant how the coupling is done on $E_k(n)$.

Define, $	ilde{X}_{n,i,j}(t) := K(X_n(t))(U_i, U_j)$, $(i, j) \in [k]^2$. The evolution of $	ilde{X}_{n,1,2}$, for example, can be described by the SDE

\[
\begin{align*}
\frac{d\tilde{X}_{n,1,2}(t)}{dt} &= b\left(\tilde{X}_{n,1,2}(t), K(X_n(t))\right)(U_1, U_2) dt + \Sigma(K(X_n(t)))(U_1, U_2) dB_{1,2}(t) \\
&\quad + dL^-_{n,1,2}(t) - dL^+_{n,1,2}(t),
\end{align*}
\]

with the initial condition $\tilde{X}_{n,1,2}(0) = W_0^{(n)}(U_1, U_2)$. Since $X_{1,2}$ is also driven by the same Brownian motion, by using the Lipschitz property of the Skorokhod map and the triangle inequality, it follows that for any $(U_1, U_2) = (u_1, u_2)$ on the event $E_k(n)$, $\sup_{s \in [0,t]}|\tilde{X}_{n,1,2}(s) - X_{1,2}(s)|^2$ is at most

\[
48 \int_0^t \left| b\left(X_{1,2}(s), \Gamma(s)\right)(u_1, u_2) - b\left(\tilde{X}_{n,1,2}(s), K(X_n(s))\right)(u_1, u_2) \right|^2 ds \\
+ 48 \sup_{s \in [0,t]} \left| \int_0^s \left( \Sigma(\Gamma(r))(u_1, u_2) - \Sigma(K(X_n(r))(u_1, u_2)) \right) dB_{1,2}(r) \right|^2 \\
+ 48 \left| \tilde{X}_{n,1,2}(0) - X_{1,2}(0) \right|^2.
\]

We can now use Assumption 5 and 6 on the first term in (69) to get

\[
\begin{align*}
\left| b\left(X_{1,2}(s), \Gamma(s)\right)(u_1, u_2) - b\left(\tilde{X}_{n,1,2}(s), K(X_n(s))\right)(u_1, u_2) \right|^2 \\
\leq 2L^2 \left| X_{1,2}(s) - \tilde{X}_{n,1,2}(s) \right|^2 + 2\kappa^2 \left| \Gamma(s) - K(X_n(s)) \right|^2, \quad s \in \mathbb{R}_+.
\end{align*}
\]

Define for $s \in [0, t]$,

\[
M^{(n)}(s) := \int_0^s \left( \Sigma(\Gamma(r))(u_1, u_2) - \Sigma(K(X_n(r))(u_1, u_2)) \right) dB_{1,2}(r),
\]

which makes the second term in (69) equal to $48 \sup_{s \in [0,t]} M^2(s)$. Using Markov’s inequality followed by Doob’s maximal inequality [KS91, page 14, Theorem 3.8.iv], we obtain

\[
\mathbb{P} \left\{ \sup_{s \in [0,t]} M^{(n)}(s)^2 \geq 2\lambda_k \mathbb{E} \left[ M^{(n)}(t)^2 \right] \right\} \leq (2\lambda_k \mathbb{E} \left[ M^{(n)}(t)^2 \right])^{-1} \mathbb{E} \left[ \sup_{s \in [0,t]} M^{(n)}(s)^2 \right]
\]

\[
(71)
\leq (2\lambda_k \mathbb{E} \left[ M^{(n)}(t)^2 \right])^{-1} \mathbb{E} \left[ M^{(n)}(t)^2 \right] = 2\lambda_k^{-1},
\]

for every $\lambda_k > 0$. Let $(\lambda_k)_{k \in \mathbb{N}}$ satisfy $\lim_{k \to \infty} \lambda_k = \infty$. The choice of $\lambda_k$ will be made later.
Therefore, with probability at least $1 - 2\lambda_k^{-1}$,
\[
\sup_{s \in [0,t]} M^{(n)}(s)^2 \leq 2\lambda_k \mathbb{E}[M^{(n)}(t)^2]
\]
\[
= 2\lambda_k \int_0^t \| \Sigma(\Gamma(s))(u_1, u_2) - \Sigma(\Gamma(\mathbb{X}_n(s)))(u_1, u_2) \|^2 \, ds
\]
\[
\leq 2\lambda_k \kappa^2 \int_0^t \| \Gamma(s) - \Gamma(\mathbb{X}_n(s)) \|^2 \, ds.
\]

By the abuse of notation, we redefine the event $E_k(n)$ to intersect with the event where the above bound holds. By a union bound, we still have $\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}\{E_k(n)\} = 1$.

Using equations (70) and (72) in equation (69) we get
\[
\sup_{s \in [0,t]} \left| \mathbb{X}_{n,1,2}(s) - \mathbb{X}_{1,2}(s) \right|^2 \leq 48 \left| W_0^{(n)}(U_1, U_2) - W_0(U_1, U_2) \right|^2
\]
\[
+ 96\kappa^2(\lambda_k + 1) \int_0^t \| \Gamma(s) - \Gamma(\mathbb{X}_n(s)) \|^2 \, ds
\]
\[
+ 96L^2 \int_0^t \left| \mathbb{X}_{1,2}(s) - \mathbb{X}_{n,1,2}(s) \right|^2 \, ds.
\]

Replacing the role of $(1, 2)$ by any other $(i, j) \in [k]^{(2)}$, and summing over, we get
\[
\sup_{s \in [0,t]} \frac{1}{k^2} \sum_{(i,j) \in [k]^{(2)}} \left| \mathbb{X}_{n,i,j}(s) - \mathbb{X}_{i,j}(s) \right|^2
\]
\[
\leq \frac{48}{k^2} \sum_{(i,j) \in [k]^{(2)}} \left| W_0^{(n)}(U_i, U_j) - W_0(U_i, U_j) \right|^2
\]
\[
+ 96\kappa^2(\lambda_k + 1) \int_0^t \| \Gamma(s) - \Gamma(\mathbb{X}_n(s)) \|^2 \, ds
\]
\[
+ 96L^2 \int_0^t \frac{1}{k^2} \sum_{(i,j) \in [k]^{(2)}} \left| \mathbb{X}_{i,j}(s) - \mathbb{X}_{n,i,j}(s) \right|^2 \, ds.
\]

By the triangle inequality,
\[
\sup_{s \in [0,t]} \left\| K\left( \mathbb{X}_{n,i,j}(s) \right)_{(i,j) \in [k]^{(2)}} - K\left( \Gamma(s)(U_i, U_j) \right)_{(i,j) \in [k]^{(2)}} \right\| \leq 2 \sup_{s \in [0,t]} \left\| K\left( \mathbb{X}_{n,i,j}(s) \right)_{(i,j) \in [k]^{(2)}} - K\left( \mathbb{X}_{i,j}(s) \right)_{(i,j) \in [k]^{(2)}} \right\|^2
\]
\[
+ 2 \sup_{s \in [0,t]} \left\| K\left( \Gamma(s)(U_i, U_j) \right)_{(i,j) \in [k]^{(2)}} - K\left( \Gamma(s)(U_i, U_j) \right)_{(i,j) \in [k]^{(2)}} \right\|^2.
\]

Then notice that the kernel
\[
\frac{1}{2} K\left( \mathbb{X}_{n,i,j}(s) \right)_{(i,j) \in [k]^{(2)}} - \frac{1}{2} K\left( \Gamma(s)(U_i, U_j) \right)_{(i,j) \in [k]^{(2)}}
\]
has entries in \([-1, 1]\) and is sampled from the kernel \(\frac{1}{2} K(X_n(s)) - \frac{1}{2} \Gamma(s)\). By [Lov12, Lemma 10.6], the difference

\[
\left\| K\left(\left(\tilde{X}_{n,i,j}(s)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(\Gamma(s)(U_i, U_j)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2 - \left\| K(X_n(s)) - \Gamma(s)\right\|_2^2
\]

lies in the interval \([-24/k - 36/k^2, 64k^{-1/4} + 256k^{-1/2}]\) with probability at least \(1 - 4e^{-k^{1/2}/10}\), for all \(n \geq k\). Using this in (75) we get

\[
\sup_{s \in [0,t]} \left\| K\left(\left(\tilde{X}_{n,i,j}(s)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2 \geq \frac{1}{2} \left\| K(X_n(s)) - \Gamma(s)\right\|_2^2 - 320k^{-1/4} - \sup_{s \in [0,t]} \left\| K\left(\left(\Gamma(s)(U_i, U_j)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2.
\]

(76)

with probability at least \(1 - 4e^{-k^{1/2}/10}\). By an abuse of notation, we redefine the event \(E_k(n)\) to intersect with the event where the above bound holds. We still have \(\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}\{E_k(n)\} = 1\).

We first lower bound twice the left hand side of equation (74) using equation (76) as

\[
2 \sup_{s \in [0,t]} \left\| K\left(\left(\tilde{X}_{n,i,j}(s)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2 \geq \sup_{s \in [0,t]} \left\| K\left(\left(\tilde{X}_{n,i,j}(s)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2 + \sup_{s \in [0,t]} \left\| K\left(\left(\tilde{X}_{n,i,j}(s)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2
\]

(77)

\[
\geq \sup_{s \in [0,t]} \left\| K\left(\left(\tilde{X}_{n,i,j}(s)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2 + \frac{1}{2} \left\| K(X_n(s)) - \Gamma(s)\right\|_2^2 - 320k^{-1/4} - \sup_{s \in [0,t]} \left\| K\left(\left(\Gamma(s)(U_i, U_j)\right)_{(i,j)\in[k]^2}\right) - K\left(\left(X_{i,j}(s)\right)_{(i,j)\in[k]^2}\right)\right\|_2^2.
\]

Here we used the fact that the \(L^2\) norm is lower bounded by the cut norm. Using equation (77) back in equation (74) (multiplied by 2), and rearranging terms we get
\[
\sup_{s \in [0,t]} \left\| K\left( (\tilde{X}_{n,i,j}(s))_{(i,j) \in [k]^2} \right) - K\left( (X_{i,j}(s))_{(i,j) \in [k]^2} \right) \right\|_2^2
\]
\[
+ \frac{1}{2} \sup_{s \in [0,t]} \| K(X_n(s)) - \Gamma(s) \|_2^2
\]
\[
\leq \sup_{s \in [0,t]} \left\| K\left( (\Gamma(s)(U_i, U_j))_{(i,j) \in [k]^2} \right) - K\left( (X_{i,j}(s))_{(i,j) \in [k]^2} \right) \right\|_2^2
\]
\[
+ 320k^{-1/4} + \frac{96}{k^2} \sum_{(i,j) \in [k]^2} \left| W_0^{(n)}(U_i, U_j) - W_0(U_i, U_j) \right|^2
\]
\[
\leq 192L^2 \int_0^t \left\| K\left( (\tilde{X}_{n,i,j}(s))_{(i,j) \in [k]^2} \right) - K\left( (X_{i,j}(s))_{(i,j) \in [k]^2} \right) \right\|_2^2 ds
\]
\[
+ 192k^2(\lambda_k + 1) \int_0^t \left\| \Gamma(s) - K(X_n(s)) \right\|_2^2 ds.
\]

Now let
\[
A_k := \sup_{s \in [0,t]} \left\| K\left( (\Gamma(s)(U_i, U_j))_{(i,j) \in [k]^2} \right) - K\left( (X_{i,j}(s))_{(i,j) \in [k]^2} \right) \right\|_2^2,
\]
\[
B_k(n) := \frac{96}{k^2} \sum_{(i,j) \in [k]^2} \left| W_0^{(n)}(U_i, U_j) - W_0(U_i, U_j) \right|^2 + 320k^{-1/4}.
\]

Applying Grönwall’s inequality [Grö19] and noticing that the first term on the left of equation (78) is always non-negative, gives us that on the event \(E_k(n)\),
\[
\sup_{s \in [0,t]} \left\| K\left( (\tilde{X}_{n,i,j}(s))_{(i,j) \in [k]^2} \right) - K\left( (X_{i,j}(s))_{(i,j) \in [k]^2} \right) \right\|_2^2
\]
\[
+ \sup_{s \in [0,t]} \| K(X_n(s)) - \Gamma(s) \|_2^2 \leq 2(A_k + B_k(n)) \exp\left(192(L^2 + 2\kappa^2(\lambda_k + 1))t\right),
\]
for every \(n \geq k\). Note that
\[
\mathbb{E} \left[ \left| W_0^{(n)}(U_i, U_j) - W_0(U_i, U_j) \right|^2 \right] = \left| W_0^{(n)} - W_0 \right|_2^2 \to 0,
\]
as \(n \to \infty\), by assumption (68). By a variance bound it follows that
\[
\lim_{k \to \infty} \lim_{n \to \infty} B_k(n) = 0,
\]
in probability. Also, \(\lim_{k \to \infty} A_k = 0\) by Proposition 4.6. Since \(\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}\{E_k(n)\} = 1\),
\[
\lim_{n \to \infty} \sup_{s \in [0,t]} \| K(X_n(s)) - \Gamma(s) \|_2 = 0,
\]
and
\[
\lim_{k \to \infty} \lim_{n \to \infty} \sup_{s \in [0,t]} \frac{1}{k^2} \left\| \left( K(X_n(s))(U_i, U_j) \right)_{(i,j) \in [k]^2} - (X_{i,j}(s))_{(i,j) \in [k]^2} \right\|_F^2 = 0,
\]
in probability, by choosing \((\lambda_k)_{k \in \mathbb{N}}\) (depending on \((A_k, \lim_{n \to \infty} B_k(n))_{k \in \mathbb{N}}\)) that increases sufficiently slowly to infinity as \(k \to \infty\). This proves our claim. \(\square\)
Remark 4.10. To get a non-asymptotic error rate, we need to control on $A_k$ and $B_k(n)$. Observe that $B_k(n)$ depends on the initial condition and in general it can be arbitrarily slow. However, assuming that the initial condition is i.i.d., one can use Chebyshev’s inequality to obtain $P\{B_k(n) \geq 69k^{-1/4}\} \leq k^{-3/2}$.

On the other hand, it follows from the arguments in Proposition 4.6 that there exists a constant $M_1$ (depending only on $t$) such that for any $\delta > 0$ we have $P\{A_k \geq M_1(\delta \log(1/\delta))^{1/4}\} \leq k^{-2} + t^2 \delta^{-6}e^{-\lambda k}\log(1/\delta)/2$.

In particular, choosing $\delta = 64\sqrt{k^{-1}\log k}$ and $\lambda_k = \log(k)/(16 \cdot 384t(L^2 + 2\kappa_k^2))$, we have the left hand side of (79) bounded by $M_1k^{-1/16}\log^{3/2}k$ with probability at least $1 - \frac{k^2}{n} - 4k^{-\frac{1}{20}} - 2te^{-\sqrt{k}/20} - k^{-3/2}$, where $\kappa = 32\sqrt{6}(L^2 + 2\kappa_k^2)^{1/2}$. Since $t$ is fixed, we can choose $k$ to be a suitable function of $n$, say $k = n^{2/7}$, to get a non-asymptotic rate of convergence. Moreover, using the remark after the proof of Lemma 3.2, we can get a non-asymptotic rate of convergence with finite $n$ and $|\tau_n|$.

5. Examples

In this section we will verify our assumptions for a class of functions introduced as linear functions in [OPST21, Section 5.1]. Let $\{Z_t\}_{t \in [n]}$ be i.i.d. Uni[0, 1]. For any kernel $W \in \mathcal{W}$ and any $n \in \mathbb{N}$, sample a random matrix $G_n[W]$ as $G_n[W] := (W(Z_i, Z_j))_{(i,j) \in [n]^2} \in \mathcal{M}_n$. Let $\rho_n([W])$ denote its law, i.e., $\text{Law}(G_n[W]) = \rho_n([W])$. Now let $R: \mathcal{W} \to \mathbb{R}$ be defined as a linear function, i.e.,

$$R(W) := \int_{\mathcal{M}_n} R_n(z)\rho_n([W])(dz), \quad \forall W \in \mathcal{W},$$

Let $(\Omega, \mathcal{A})$ be the standard measurable space on $[0, 1]^n$. Let $\ell: \mathcal{W} \times \Omega$ be the function defined as

$$\ell(W, Z) := R_n\left((W(Z_i, Z_j))_{(i,j) \in [n]^2}\right).$$

Let $R_n$ satisfy Assumption 1(1) and let $R$ admit a Fréchet-like derivative evaluation map $\phi: \mathcal{W} \to L^\infty([0, 1]^{(2)})$ (see [OPST21, Section 5] for conditions). The map $\phi$ then satisfies

$$\phi(W)(x, y) = \sum_{(i,j) \in [n]^2} \mathbb{E}\left[\nabla R_n\left((W(Z_p, Z_q))_{(p,q) \in [n]^2}\right)\big| (Z_i, Z_j) = (x, y)\right],$$

and $D_\mathcal{W}\ell(\cdot; Z)$ for $Z \in [0, 1]^n$ satisfies

$$D_\mathcal{W}\ell(\cdot; Z)(W)(x, y) = \sum_{(i,j) \in [n]^2} \nabla R_n\left((W(Z_p, Z_q))_{(p,q) \in [n]^2}\big| (Z_i, Z_j) = (x, y)\right),$$

for $W \in \mathcal{W}$ and $(x, y) \in [0, 1]^{(2)}$.

5.1. Scalar Entropy and Homomorphism density. Examples like the scalar entropy and the homomorphism density functions considered in [OPST21, Section 5.1-5.2], all satisfy Assumption 1 for some $\kappa_2 \in \mathbb{R}_+$ since $\|Hess(R_n)\|_{op}$ exists and is bounded uniformly in the domain. Specifically, for homomorphism density function $R = H_F$ for a simple graph $F$ with $n$ vertices and $m$ edges $\{e_i\}_{i=1}^m$, the constants $\kappa_2 = mn(n-1)$, and for scalar entropy $R = \mathcal{E}$, the constant $\kappa_2 = 2\epsilon^{-1}(1-\epsilon)^{-1}$ on its domain $\mathcal{W}_\epsilon := \{W \in \mathcal{W} \mid \epsilon \leq W \leq 1-\epsilon\}$ where $\epsilon \in (0, 1/2)$. Since this implies that there exists $M_\infty \in \mathbb{R}_+$ such that $\|\phi(W)\|_{\infty} \leq M_\infty$ for all $W$ in the domain, these example also satisfy Assumption 2 for $\sigma = M_\infty$. 
In the following, we define \( b: [-1,1] \times \mathcal{W} \to L^\infty([0,1]^{(2)}) \) as \( b(W(x,y),W(x,y)) = -\phi(W)(x,y) \) for all \( W \in \mathcal{W} \) and a.e. \((x,y) \in [0,1]^{(2)}\). We will now verify Assumption 6 when \( R \) is the sum of scalar entropy and some homomorphism density \( H_F \) for a simple graph \( F \) with \( n \) vertices and \( m \) edges. Note that for this example, we have

\[
(82) \quad b(z,W)(x,y) = \log \frac{z}{1-z} + \phi_{H_F}(W)(x,y), \quad z = W(x,y) \in [\epsilon, 1-\epsilon],
\]

for a.e. \((x,y) \in [0,1]^{(2)}\) where from [OPST21, Equation 113],

\[
\phi_{H_F}(W)(x,y) = \sum_{l=1}^m \mathbb{E}\left[ \prod_{r=1, r \neq l}^m W(Z_{e_r}) \bigg| Z_{e_l} = (x,y) \right].
\]

\[=: \sum_{l=1}^m t_{x,y}(F_{e_l}, W), \quad (x,y) \in [0,1],\]

\( Z_e = (Z_{e(1)}, Z_{e(2)}) \) and \( F_{e_l} \) is the simple graph obtained from \( F \) by removing the edge \( e_l \). It is shown in [OPST21, Section 5.1.2] that the map \( W \mapsto t_{c,e}(F_e, W) \) is continuous as a map from \((\mathcal{W}, d_\square)\) to \((L^\infty([0,1]^{(2)}), d_\square)\). To show that \( \phi_{H_F}(\cdot)(x,y) \) is Lipschitz in the cut norm for every \((x,y) \in [0,1]^{(2)}\), it is sufficient to show that \( t_{x,y}(F_e, \cdot) \) is Lipschitz in the cut norm for \( e \in \{e_l\}_{l=1}^m \). For \( W_1, W_2 \in \mathcal{W} \), note that

\[
t_{x,y}(F_e, W_1) - t_{x,y}(F_e, W_2) = \sum_{\{p,q\} \in E(F_e)} I_{p,q},
\]

where for any \( \{p,q\} \in E(F_e) \),

\[
(83) \quad I_{p,q} := \int_{[0,1]^{n-2}} (W_1(x_p, x_q) - W_2(x_p, x_q)) \prod_{(i,j) \in E(F_e) \setminus \{p,q\}} W_1(x_i, x_j) \prod_{v \in V(F_e) \setminus e} \text{d}x_v.
\]

Following the proof in [Lov12, Lemma 10.24], we get \(|I_{p,q}| \leq \|W_1 - W_2\|_\square\), which yields

\[
(84) \quad |t_{x,y}(F_e, W_1) - t_{x,y}(F_e, W_2)| \leq (m-1)\|W_1 - W_2\|_\square,
\]

i.e., the Lipschitz constant of \( t_{x,y}(F_e, \cdot) \) for every \( e \in E(F) \) is \( m - 1 \). This implies that the Lipschitz constant of \( \phi(\cdot)(x,y) \) with respect to \( \|\cdot\|_\square \) is \( m(m-1) \). Therefore, for \( b \) as in equation (82), we have

\[
|b(z,W_1)(x,y) - b(z,W_2)(x,y)| = |\phi_{H_F}(W_1)(x,y) - \phi_{H_F}(W_1)(x,y)|
\]

\[
\leq m(m-1)\|W_1 - W_2\|_\square.
\]

Therefore \( b \) (as in equation (82)) satisfies Assumption 6 with \( \kappa_\square = m(m-1) \).

5.2. Quadratic functions of homomorphism density. More generally, let \( k \in \mathbb{N} \) and let \( \{F^1, \ldots, F^k\} \) be a family of finite simple graphs. Let \( c_1, \ldots, c_k \in [0,1] \) be fixed constants. Define a function \( R: \mathcal{W} \to \mathbb{R} \) as

\[
R(W) := \frac{1}{2} \sum_{\alpha=1}^k (H_{F^\alpha}(W) - c_\alpha)^2.
\]

Note that a lower bound on \( R \) is achieved if \( H_{F^\alpha} \equiv c_\alpha \) for all \( \alpha \in [k] \). We note that \( R \) being a sum of squares of \( k \) many functions satisfies Assumption 1(2).
Moreover, let $\phi: \mathcal{W} \rightarrow L^\infty([0, 1]^{(2)})$ denote the Fréchet-like derivative evaluation map of $R$. It follows from chain-rule that

$$\phi(W)(x, y) = \sum_{\alpha=1}^{k} (H_{F^\alpha}(W) - c_\alpha)\phi_{H_{F^\alpha}}(W)(x, y).$$

Note that $W \mapsto \phi_{H_{F^\alpha}}(W)$ satisfies Assumption 1(2) with $\kappa_{2,\alpha} = m_\alpha(m_\alpha - 1)$ where $m_\alpha$ is the number of edges in $F^\alpha$. Further note that for any finite graph $F$ and $U, V \in \mathcal{W}$ we have $|H_F(U) - H_F(V)| \leq |E(F)||U - V|_L \leq |E(F)||U - V|_H$. A simple calculation using the fact that $|(H_{F^\alpha}(W) - c_\alpha)| \leq 1$ for all $W$ and that $\|\phi_{H_F}(W)\|_2 \leq |E(F)|$, we obtain that $\phi$ satisfies Assumption 1(2) with

$$\kappa_2 \leq \sum_{\alpha=1}^{k} (m_{\alpha}^2 + \kappa_{2,\alpha}) \leq km^2,$$

where $m = \max_{\alpha\in[n]} m_\alpha$.

Similarly, for any edge $e$ in a finite simple graph $F$, note $W \mapsto t_{x,y}(F, e, W)$ is $(m - 1)$-Lipschitz in cut norm for every $(x, y) \in [0, 1]^{(2)}$ and $W \mapsto H_F(W)$ is $m$-Lipschitz in cut norm where $m$ is the number of edges in $F$. Using the fact that $\|\phi_{H_F}(W)\|_{\infty} \leq m$ and $H_F(W) \in [0, 1]$ for every $W \in \mathcal{W}_0$, we conclude that $\phi(\cdot)(x, y)$ is $km^2$-Lipschitz with respect to $\|\cdot\|_H$ for a.e. $(x, y) \in [0, 1]^{(2)}$ and hence $\phi$ satisfies Assumption 6.

5.3. Entropy minimization with edge-triangle constraints. We conclude with the discussion of the example mentioned in the Introduction. Recall the problem of minimizing the scalar entropy $E$ over $\tilde{W}_0$ with prescribed edge density $H_-(\cdot) = e \in [0, 1]$ and triangle density $H_\triangle(\cdot) = \tau \in [0, 1]$. As mentioned in [NRS, 2023], in general this problem does not admit unique minimizer.

Let us consider a relaxation of this problem. Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing convex function such that $\psi'(−\log(2)) =: A > 1$. Consider minimizing the function

$$W \mapsto R(W) := \frac{1}{2} ((H_-(W) - e)^2 + (H_\triangle(W) - \tau)^2) + \psi(E(W)).$$

Since $\psi$ is non-decreasing, minimizing $E$ is equivalent to minimizing $\psi \circ E$. On the other hand, the term $\frac{1}{2} ((H_-(W) - e)^2 + (H_\triangle(W) - \tau)^2)$ penalizes any deviation from the marginal constraint on the edge and triangle densities.

It follows from the previous discussion that $W \mapsto \frac{1}{2}(H_-(W) - e)^2 + \frac{1}{2}(H_\triangle(W) - \tau)^2$ is $\lambda$-semiconvex with $\lambda = -8$. On the other hand, $E$ is 4-semiconvex and therefore $\psi \circ E$ is $4A$-semiconvex. In particular, if $A > 2$ then $R$ is strongly convex and hence admits a unique minimizer and the gradient flow converges exponentially fast to the minimizer of $R$. In this case, the gradient flow of $R$ converges exponentially fast to the minimizer.

For instance, take $\psi = 4id$ and consider the optimization algorithm described in Definition 1.2. For every $n \in \mathbb{N}$, $X_n \in \mathcal{M}_n$, and $(i, j) \in [n]^{(2)}$, we can evaluate $g_{n, (i, j)}(X_n; \xi)$ as

$$g_{n, (i, j)}(X_n; \xi) := 4\log\left(\frac{X_n(i, j)}{1 - X_n(i, j)}\right) + (X_n(i_1, i_2) - e) + (X_n(i_3, i_4)X_n(i_4, i_5)X_n(i_5, i_3) - \tau)X_n(i, i_6)X_n(i_6, j),$$

where $\xi = (i_2)_{x \in [0]} \sim id \sim \text{Uni}([n])^6$. Notice that $E_\xi[g_n(X_n; \xi)] = \nabla R_n(X_n)$, and Assumption 2 is satisfied. Theorem 1.3 and Theorem 1.7 tell us that the (PNSGD) algorithm in the absence
of large noise, converges to the minimizer of $R$ as the step size of the algorithm goes to zero, and $n \to \infty$.

If one takes $\psi = \text{id}$ then the function $R$ is not guaranteed to be convex. Therefore, there may be multiple minimizers of $R$ as mentioned in [NRS23]. Since $R$ is not strictly convex, the gradient flow may not converge to the minimizer, however, it does converge to a stationary point with polynomial rate.

References

[AGS08] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Gradient Flows: In Metric Spaces and in the Space of Probability Measures. Second Edition. Lectures in Mathematics. ETH Zürich. Birkhäuser Verlag AG, 2008. 8

[AHS23] Samuel Ainsworth, Jonathan Hayase, and Siddhartha Srinivasa. Git re-basin: Merging models modulo permutation symmetries. In The Eleventh International Conference on Learning Representations, 2023. 9

[Ald85] David J. Aldous. Exchangeability and related topics. In École d’Été de Probabilités de Saint-Flour, XIII–1983. Lecture Notes in Math., volume 1117, pages 1–198. Springer-Berlin, 1985. 19

[APST23] Siva Athreya, Soumik Pal, Raghav Somani, and Raghavendra Tripathi. Path convergence of markov chains on large graphs. 2023. 4

[Aus08] Tim Austin. On exchangeable random variables and the statistics of large graphs and hypergraphs. Probability Surveys, 5:80–145, 2008. 19

[BBW19] Shankar Bhamidi, Amarjit Budhiraja, and Ruoyu Wu. Weakly interacting particle systems on inhomogeneous random graphs. Stochastic Processes and their Applications, 129(6):2174–2206, 2019. 3

[BCL+08] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. Advances in Mathematics, 219(6):1801–1851, 2008. 4, 10

[BCL+12] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs II. multiway cuts and statistical physics. Annals of Mathematics, pages 151–219, 2012. 4

[BCM1] Alessandra Bianchi, Francesca Collet, and Elena Magnanini. Limit theorems for exponential random graphs. arXiv preprint arXiv:2105.06312, 2021. 3

[BCN18] Léon Bottou, Frank E. Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning. SIAM Review, 60(2):223–311, 2018. 4

[BCN20] Gianmarco Bet, Fabio Coppini, and Francesca R Nardi. Weakly interacting oscillators on dense random graphs. arXiv preprint arXiv:2006.07670, 2020. 3

[BCW20] Erhan Bayraktar, Suman Chakraborty, and Ruoyu Wu. Graphon mean field systems. arXiv preprint arXiv:2003.13180, 2020. 3

[Ben99] Michel Benaïm. Dynamics of stochastic approximation algorithms. In Séminaire de probabilités XXXIII, pages 1–68. Springer, 1999. 4

[Bor09] Vivek S. Borkar. Stochastic approximation: a dynamical systems viewpoint, volume 48. Springer, 2009. 4

[BSM+22] Frederik Benzing, Simon Schug, Robert Meier, Johannes Von Oswald, Yassir Akram, Nicolas Zuchet, Laurence Aitchison, and Angelika Steger. Random initialisations performing above chance and how to find them. In OPT 2022: Optimization for Machine Learning (NeurIPS 2022 Workshop), 2022. 9

[Bub15] Sébastien Bubeck. Convex Optimization: Algorithms and Complexity. Foundations and Trends® in Machine Learning, 8(3-4):231–357, 2015. 4

[Cau47] Augustin-Louis Cauchy. Méthode générale pour la résolution des systemes d’équations simultanées. Comptes Rendus de l’Académie des Science, 25:536–538, 1847. 4

[CD22] Louis-Pierre Chaintron and Antoine Diez. Propagation of chaos: A review of models, methods and applications. Kinetic and Related Models, 15(6):1017–1173, 2022. 2
[CGM08] Patrick Cattiaux, Arnaud Guillin, and Florent Malrieu. Probabilistic approach for granular media equations in the non uniformly convex case. *Probability Theory and Related Fields*, 140(1-2):19–40, 2008. 2

[Cop22] Fabio Coppini. A note on Fokker–Planck equations and graphons. *Journal of Statistical Physics*, 187(2):1–12, 2022. 3

[CV11] Sourav Chatterjee and SR Srinivasa Varadhan. The large deviation principle for the Erdős–Rényi random graph. *European Journal of Combinatorics*, 32(7):1000–1017, 2011. 3

[DGL16] Sylvain Delattre, Giambattista Giacomin, and Eric Luçon. A note on dynamical models on random graphs and fokker–planck equations. *Journal of Statistical Physics*, 165(4):785–798, 2016. 3

[DJ08] Persi Diaconis and Svante Janson. Graph limits and exchangeable random graphs. *Rendiconti di Matematica e delle sue Applicazioni*, 28(1):33–61, 2008. 19

[DM22] Paul Dupuis and Georgi S Medvedev. The large deviation principle for interacting dynamical systems on random graphs. *Communications in Mathematical Physics*, 390(2):545–575, 2022. 3

[Dob79] L. Dobrushin, R. Vlasov equations. *Functional Analysis and its applications*, 13:115–123, 1979. 2

[ESSN22] Rahim Entezari, Hanie Sedghi, Olga Saukh, and Behnam Neyshabur. The Role of Permutation Invariance in Linear Mode Connectivity of Neural Networks. In *International Conference on Learning Representations*, 2022. 9

[FDRC20] Jonathan Frankle, Gintare Karolina Dziugaite, Daniel Roy, and Michael Carbin. Linear Mode Connectivity and the Lottery Ticket Hypothesis. In Hal Daumé III and Aarti Singh, editors, *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pages 3259–3269. PMLR, 13–18 Jul 2020. 9

[FK99] Alan Frieze and Raviv Kannan. Quick approximation to matrices and applications. *Combinatorica*, 19(2):175–220, 1999. 10

[Gär88] J. Gärtner. On the McKean-Vlasov limit for interacting diffusions. *Math. Nachr.*, 137:197–248, 1988. 1, 2

[Grö91] Thomas Hakon Grönwall. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Annals of Mathematics*, pages 292–296, 1919. 14, 17, 25, 33

[Hoo79] D. N. Hoover. Relations on probability spaces and arrays of random variables, 1979. Preprint. Institute for Advanced Studies.

[Hoo82] D. N. Hoover. Row-column exchangeability and a generalized model for probability. *Exchangeability in probability and statistics (Rome, 1981)*, pages 281–291, 1982. 19

[Jab14] Pierre-Emmanuel Jabin. A review of the mean field limits for vlasov equations. *Kinetic and Related Models*, 7(4):661–711, 2014. 2

[Jan13] Svante Janson. Graphons, cut norm and distance, couplings and rearrangements. *NYJM Monographs*, 4, 2013. 10

[Kac56] Mark Kac. Foundations of kinetic theory. In *Proceedings of The third Berkeley symposium on mathematical statistics and probability*, volume 3, pages 171–197, 1956. 2

[Kal21] O. Kallenberg. *Foundations of Modern Probability*. Probability Theory and Stochastic Modelling. Springer International Publishing, 2021. 20, 23, 25, 27

[KC12] Harold Joseph Kushner and Dean S. Clark. *Stochastic approximation methods for constrained and unconstrained systems*, volume 26. Springer Science & Business Media, 2012. 4

[KLRS07] Łukasz Kruk, John Lehoczky, Kavita Ramanan, and Steven Shreve. An explicit formula for the Skorokhod map on [0, a]. *The Annals of Probability*, 35(5):1740 – 1768, 2007. 12, 13, 23

[KRRS17] Richard Kenyon, Charles Radin, Kui Ren, and Lorenzo Sadun. Multipodal structure and phase transitions in large constrained graphs. *Journal of Statistical Physics*, 168:233–258, 2017. 3

[KS91] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer, second edition, 1991. 16, 23, 24, 25, 30

[KW52] J. Kiefer and J. Wolfowitz. Stochastic estimation of the maximum of a regression function. *The Annals of Mathematical Statistics*, 23(3):462–466, 1952. 4

[KY03] Harold Kushner and G. George Yin. *Stochastic approximation and recursive algorithms and applications*, volume 35. Springer Science & Business Media, 2003. 4

[Lov12] László Lovász. *Large Networks and Graph Limits*, volume 60 of *Colloquium publications*. American Mathematical Society, 2012. 10, 20, 26, 32, 35
[LRW19] Daniel Lacker, Kavita Ramanan, and Ruoyu Wu. Local weak convergence for sparse networks of interacting processes. *arXiv preprint arXiv:1904.02585*, 2019.

[LS06] László Lovász and Balázs Szegedy. Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96(6):933–957, 2006.

[LS07] László Lovász and Balázs Szegedy. Szemerédi’s lemma for the analyst. *Geometric And Functional Analysis*, 17:252–270, 2007.

[LTE19] Qianxiao Li, Cheng Tai, and Weinan E. Stochastic Modified Equations and Dynamics of Stochastic Gradient Algorithms I: Mathematical Foundations. *Journal of Machine Learning Research*, 20(40):1–47, 2019.

[MB11] Eric Moulines and Francis Bach. Non-asymptotic analysis of stochastic approximation algorithms for machine learning. In J. Shawe-Taylor, R. Zemel, P. Bartlett, F. Pereira, and K.Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 24. Curran Associates, Inc., 2011.

[McK75] H. P. McKean. Fluctuations in the kinetic theory of gases. *Communications on Pure and Applied Mathematics*, 28(4):435–455, 1975.

[MLPA22] Sadhika Malladi, Kai Feng Lyu, Abhishek Panigrahi, and Sanjeev Arora. On the SDEs and scaling rules for adaptive gradient algorithms. In Alice H. Oh, Alekh Agarwal, Danielle Belgrave, and Kyunghyun Cho, editors, *Advances in Neural Information Processing Systems*, 2022.

[Mun00] James R Munkres. *Topology*. Prentice Hall Upper Saddle River, 2000.

[NRS23] Joe Neeman, Charles Radin, and Lorenzo Sadun. Typical large graphs with given edge and triangle densities. *Probability Theory and Related Fields*, pages 1–57, 2023.

[OPST21] Sewoong Oh, Soumik Pal, Raghav Somani, and Ragahvendra Tripathi. Gradient flows on graphons: existence, convergence, continuity equations. *arXiv preprint arXiv:2111.09459*, 2021.

[OPST23] Sewoong Oh, Soumik Pal, Raghav Somani, and Ragahvendra Tripathi. Gradient flows on graphons: Existence, convergence, continuity equations. *Journal of Theoretical Probability*, Jul 2023.

[OR19] Roberto I Oliveira and Guilherme H Reis. Interacting diffusions on random graphs with diverging average degrees: Hydrodynamics and large deviations. *Journal of Statistical Physics*, 176(5):1057–1087, 2019.

[ORS20] Roberto I Oliveira, Guilherme H Reis, and Lucas M Stolerman. Interacting diffusions on sparse graphs: hydrodynamics from local weak limits. *Electronic Journal of Probability*, 25:1–35, 2020.

[PR17] Oleg Pikhurko and Alexander Razborov. Asymptotic structure of graphs with the minimum number of triangles. *Combinatorics, Probability and Computing*, 26(1):138–160, 2017.

[Raz08] Alexander A Razborov. On the minimal density of triangles in graphs. *Combinatorics, Probability and Computing*, 17(4):603–618, 2008.

[RM51] Herbert Robbins and Sutton Monro. A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, 22(3):400 – 407, 1951.

[RY04] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2004.

[Slo94] Leszek Słomiński. On approximation of solutions of multidimensional SDE’s with reflecting boundary conditions. *Stochastic processes and their Applications*, 50(2):197–219, 1994.

[Slo01] Leszek Słomiński. Euler’s approximations of solutions of SDEs with reflecting boundary. *Stochastic processes and their applications*, 94(2):317–337, 2001.

[Szn84] Alain-Sol Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *Journal of Functional Analysis*, 56(3):311–336, 1984.

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In Paul-Louis Hennequin, editor, *Ecole d’Éte de Probabilités de Saint-Flour XIX — 1989*, pages 165–251, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.

[Tan79] H. Tanaka. Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete*, 46(1):67–105, 1978/79.

[Vil12] C. Villani. Optimal transportation, dissipative PDE’s and functional inequalities. Unpublished lecture notes. Accessed from...
