Bethe states of the integrable spin-s chain with generic open boundaries

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Received 20 August 2015, revised 26 October 2015
Accepted for publication 2 November 2015
Published 24 November 2015

Abstract

Based on the inhomogeneous $T$–$Q$ relation and the associated Bethe ansatz equations obtained via the off-diagonal Bethe ansatz, we construct the Bethe-type eigenstates of the $SU(2)$-invariant spin-s chain with generic non-diagonal boundaries by employing certain orthogonal basis of the Hilbert space.

Keywords: Bethe ansatz, $T$ — $Q$ relation, integrable spin chain

(Some figures may appear in colour only in the online journal)

1. Introduction

The $SU(2)$-invariant spin-s Heisenberg chain has attracted great attention since its close relationship to the Wess–Zumino–Novikov–Witten models [1–4] and low-dimensional supersymmetric quantum field theory [5–8]. With the fusion techniques [9–13], the integrable high spin model can be constructed from the fundamental $s = 1/2$ representation of the Yang–Baxter equation [14, 15]. The model with periodic [16–19], anti-periodic [20] and diagonal open boundaries [21–23] has been extensively studied. However, the story for the spin chains with generic non-diagonal boundaries is quite different even their integrabilities were known [24] for a long time. For the spin-$\frac{1}{2}$ case, the exact solution was first given in [25] by the off-diagonal Bethe ansatz method (ODBA) [25–28] (for comprehensive introduction, see [29]). It is remarked that some other methods such as the q-Onsager algebra method [30–36], the
separation of variables (SoV) method [37–42] and the modified algebraic Bethe ansatz method [43–46] were also used to approach the spin-$\frac{1}{2}$ chain with generic integrable boundary conditions. We should note that the spin-$\frac{1}{2}$ chain with triangular boundary reflection matrix was studied by Belliard et al [47] and later by Pimenta and Lima-Santos [48]. Ribeiro, Martins and Galleas obtained the exact solution of the $SU(N)$-invariant high spin chain with generic toroidal boundary conditions [49]. For the $SU(2)$-invariant spin-$s$ chains (with generic $s$), the exact solutions for the non-diagonal boundaries were previously known only for some special cases [50–55]. Until very recently, exact spectrum of the model with generic boundary conditions was derived [56] in terms of an inhomogeneous $T$–$Q$ relation via the ODBA. However, its eigenstates are still missing.

Up to now Bethe states, which have well-defined homogeneous limits, of integrable models with generic open boundaries are only known for few cases [43, 45, 46, 57, 58]. A remarkable fact is that the method proposed in [57, 58] allows us to retrieve the eigenstates based on the inhomogeneous $T$–$Q$ relations obtained from the ODBA in a systematic way. In this paper, we adopt this method to derive the Bethe-type eigenstates of the integrable spin-$s$ chain with generic non-diagonal boundaries.

The paper is organized as follows. In sections 2, we briefly review the fusion procedure and the ODBA solutions of the integrable spin-$s$ chain with generic open boundary condition. In section 3, we introduce a gauge transformation and commutation relations, which are quite useful in the following derivations. Section 4 is devoted to the construction of an orthogonal basis of the Hilbert space. In section 5, we show that the scalar product between an eigenstate and a basis vector can be expressed in terms of the corresponding eigenvalues. A useful inner product is calculated in section 6. Section 7 is devoted to the construction of the Bethe-type eigenstates. We summarize our results in section 8.

2. The model and its spectrum

The $R$-matrix of the spin-$s$ Heisenberg spin chain is [9–11]

$$R_{1,2}^{(s,s)}(u) = \prod_{j=1}^{2s} (u - j\eta) \sum_{l=0}^{2s} \prod_{k=1}^{2s} \frac{u + k\eta}{u - k\eta} P_{1,2}^{(l)},$$

(2.1)

where $u$ is the spectral parameter, $\eta$ is the crossing parameter and $P_{1,2}^{(l)}$ projects the tensor space of two spin-$s$ into the irreducible subspace of spin-$l$

$$P_{1,2}^{(l)} = \prod_{j=0,j\neq l}^{2s} \left( \frac{\vec{s}_1 + \vec{s}_2}{l(l+1)} - j(j+1) \right).$$

(2.2)

The $R_{1,2}^{(s,s)}(u)$ acting on the $(2s + 1) \times (2s + 1)$-dimensional tensor space $V_1 \otimes V_2$ satisfies the properties:

Initial condition: $R_{1,2}^{(s,s)}(0) = (2s)!\eta^{2s} P_{1,2}$,

(2.3)

Antisymmetry:$R_{1,2}^{(s,s)}(-\eta) = (-1)^{2s}(2s + 1)!\eta^{2s} P_{1,2}^{(0,0)}$,

(2.4)

where $P_{1,2}$ is the permutation operator in the tensor space of two spin-$s$ spaces.

The $R$-matrix (2.1) of the spin-$s$ Heisenberg spin chain can be constructed by the fusion procedure [9–13]. The starting point is the fundamental spin-$\frac{s}{2}$-$R$-matrix
\(R_{\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2}, \frac{1}{2})}(u) = u + \eta P_{\frac{1}{2}, \frac{1}{2}},\)  
\((2.5)\)

where \(P_{\frac{1}{2}, \frac{1}{2}} = \frac{1}{2}(1 + \hat{\sigma}_1 \cdot \hat{\sigma}_2)\) is the permutation operator defined in the tensor space of spin-\(\frac{1}{2}\) spaces and \(\hat{\sigma}\) is the Pauli matrix. By taking the fusion in the quantum space, we obtain the spin-(\(\frac{1}{2}, s\)) \(R\)-matrix \(R_{\frac{1}{2}, s}^{(\frac{1}{2}, s)}(u)\) defined in the spin-\(\frac{1}{2}\) auxiliary space (two-dimensional) and the spin-\(s\) quantum space (2s + 1-dimensional) as

\[
R_{\frac{1}{2}, s}^{(\frac{1}{2}, s)}(u) = u + \frac{\eta}{2} + \eta \hat{\sigma}_1 \cdot \hat{S}_2,
\]

\[
= \left( \begin{array}{cc}
\frac{\eta}{2} + \eta S_2^+ & \eta S_2^-\\
\eta S_2^- & u + \frac{\eta}{2} - \eta S_2^+
\end{array} \right),
\]

\((2.6)\)

where \(\hat{S}\) is the spin-\(s\) operator and \(S^\pm = S^x \pm iS^y\). The \(R\)-matrix \((2.6)\) can also be expressed as

\[
R_{2, s}^{(\frac{1}{2}, 2s)}(u) = \frac{1}{\prod_{k=1}^{2s-1} (u + \left(\frac{1}{2} - s + k\right)\eta)} \times P_{(1, \ldots, 2s)}^{(+)} \prod_{k=1}^{2s-1} \left\{ \frac{1}{2} \sum_{l=1}^{k} P_{lk} \right\},
\]

\((2.7)\)

with the product in the order of increasing \(k\) from the left to the right, where \(P_{(1, \ldots, 2s)}^{(+)}\) is the symmetric projector given by

\[
P_{(1, \ldots, 2s)}^{(+)} = \frac{1}{(2s)!} \prod_{k=1}^{2s-1} \left( \sum_{l=1}^{k} P_{lk} \right).
\]

\((2.8)\)

Further more, taking the fusion in the auxiliary space, the spin-(\(j, s\)) \(R\)-matrix can be given by

\[
R_{(1, \ldots, 2j), (1, \ldots, 2s)}^{(j, s)}(u) = P_{(1, \ldots, 2j)}^{(+)} \prod_{k=1}^{2j} \left\{ R_{(1, \ldots, 2j)}^{(\frac{j}{2}, \frac{j}{2})} \left( u + \left( k - \frac{1}{2} \right)\eta \right) \right\} P_{(1, \ldots, 2s)}^{(+)},
\]

\((2.9)\)

The spin-(\(s_j, s_j\)) \(R\)-matrix \(R_{(s_j, s_j)}^{(s_j, s_j)}(u)\) acting on the \((2s_j + 1) \times (2s_j + 1)\)-dimensional tensor space \(V_j \otimes V_j\) satisfies the Yang–Baxter equation

\[
R_{1, 2}^{(s_j, s_j)}(u - v) R_{1, 3}^{(s_j, s_j)}(v) R_{2, 3}^{(s_j, s_j)}(v) = R_{2, 3}^{(s_j, s_j)}(v) R_{1, 3}^{(s_j, s_j)}(u) R_{1, 2}^{(s_j, s_j)}(u - v),
\]

\((2.10)\)

The reflection matrix \(K^{(-)}_{(\alpha)}\) of spin-s Heisenberg spin chain can also be obtained by the fusion procedure developed in \([21, 59, 60]\)

\[
K^{(-)}_{(\alpha)}(u) = P_{(\alpha)}^{(+)} \prod_{k=1}^{2s} \left\{ \prod_{l=1}^{k-1} R_{\alpha, \beta}^{(\frac{1}{2}, \frac{1}{2})} (2u + (k + l - 2s - 1)\eta) \right\} \times K^{(-)}_{\alpha} \left( u + \left( k - s - \frac{1}{2} \right)\eta \right) \times P_{(\alpha)}^{(+)},
\]

\((2.11)\)
which satisfies the reflection equation [24]
\[
R^{(j,s)}_{[a], [b]}(u - v)K^{-(j)}_{[a]}(u)R^{(j,s)}_{[b], [a]}(u + v)K^{-(j)}_{[b]}(v) = K^{-(j)}_{[b]}(v)R^{(j,s)}_{[a], [b]}(u + v)K^{-(j)}_{[a]}(u)R^{(j,s)}_{[b], [a]}(u - v)
\]  
(2.12)
and \(K^{(j,s)}_{0}(u)\) is the fundamental spin-\(\frac{1}{2}\) reflection matrix given by [61, 62]:
\[
K^{(j,s)}_{0}(u) = \left\{ \begin{array}{ll} 
\frac{p + u}{q} & a_j \\
\frac{p - u}{q} & b_j
\end{array} \right.
\]
(2.13)
where \(p\) and \(q\) are two generic boundary parameters. The corresponding dual reflection matrix \(K^{+,(j,s)}(u)\) is thus defined as
\[
K^{+,(j,s)}_{0}(u) = \frac{1}{f^{(j,s)}(u)}K^{-(j)}_{0}(u)\left( -u - \eta \right)_{[p, q, \ldots]},
\]  
(2.14)
where \(q\) and \(\xi\) are two generic boundary parameters and the normalization operator \(f^{(j,s)}(u)\) is
\[
f^{(j,s)}(u) = \frac{2^{l-1}}{l} \prod_{\ell=1}^{l} \left[ -\phi(2u + (l + k + 1 - 2\ell)\eta) \right],
\]  
(2.15)
\[\phi(u) = (u + \eta)(u - \eta).
\]  
(2.16)
The fundamental spin-\(\frac{1}{2}\) dual reflection matrix reads
\[
K^{+,(j,s)}_{0}(u) = \left\{ \begin{array}{ll} 
q - u - \eta & \xi(u + \eta) \\
\xi(u + \eta) & q + u + \eta
\end{array} \right.
\]
(2.17)
The one-row monodromy matrices for spin-\((j, s)\) are given by
\[
T^{(j,s)}_{[a], [b]}(u) = R^{(j,s)}_{[a], [b]}(u - \theta_N) \ldots R^{(j,s)}_{[a], [b]}(u - \theta_1),
\]  
(2.18)
\[
\hat{T}^{(j,s)}_{[a]}(u) = R^{(j,s)}_{[a], [b]}(u + \theta_N) \ldots R^{(j,s)}_{[a], [b]}(u + \theta_1),
\]  
(2.19)
which satisfy the Yang–Baxter relations
\[
R^{(j,s)}_{[0], [0]}(u - v)T^{(j,s)}_{0}(u)T^{(j,s)}_{0}(v) = T^{(j,s)}_{0}(v)T^{(j,s)}_{0}(u)R^{(j,s)}_{[0], [0]}(u - v),
\]  
(2.20)
\[
R^{(j,s)}_{[0], [0]}(u - v)\hat{T}^{(j,s)}_{0}(u)\hat{T}^{(j,s)}_{0}(v) = \hat{T}^{(j,s)}_{0}(v)\hat{T}^{(j,s)}_{0}(u)R^{(j,s)}_{[0], [0]}(u - v),
\]  
(2.21)
where \([\theta_j] = 1, \ldots, N\) are some generic inhomogeneity parameters and \(N\) is the number of sites. Accordingly, the double-row monodromy matrix for spin-\((j, s)\) is defined as
\[
\mathcal{T}^{(j,s)}_{[0], [0]}(u) = T^{(j,s)}_{0}(u)K^{-(j)}_{0}(u)\hat{T}^{(j,s)}_{0}(u),
\]  
(2.22)
which satisfies the reflection equation
\[
R^{(j,s)}_{[0], [0]}(u - v)\mathcal{T}^{(j,s)}_{0}(u)R^{(j,s)}_{[0], [0]}(u + v)\mathcal{T}^{(j,s)}_{0}(v) = \mathcal{T}^{(j,s)}_{0}(v)R^{(j,s)}_{[0], [0]}(u + v)\mathcal{T}^{(j,s)}_{0}(u)R^{(j,s)}_{[0], [0]}(u - v),
\]  
(2.23)
The spin-\((j, s)\) transfer matrix is thus defined as
\[
t^{(j,s)}_{[a]}(u) = \text{tr}_{[a]}\left\{ K^{+,(j,s)}_{0}(u)\mathcal{T}^{(j,s)}_{0}(u) \right\},
\]  
(2.24)
The corresponding Hamiltonian in terms of the transfer matrix \(t^{(\pm,\pm)}(u)\) is thus given by
\[
H = \frac{\partial}{\partial u} \left( \ln \left[ f^{(\pm)}(u) \right] t^{(\pm,\pm)}(u) \right) \bigg|_{u=0, s=0}.
\] (2.25)

From the Yang-Baxter equation (2.10), the reflection equation (2.12) and its dual version, one can check that the transfer matrix with different spectral parameters are mutually commutative for arbitrary \(j, j', s \in \{ \frac{1}{2}, 1, \frac{3}{2}, \ldots \}\)
\[
\left[ t^{(j, \pm)}, t^{(j', \mp)} \right] = 0,
\] (2.26)
which implies that they have common eigenstates. In fact, the transfer matrices \(t^{(j, \pm)}(u)\) satisfy the fusion hierarchy relation [59, 60]
\[
i\left(\frac{1}{2} \pm \frac{s}{2}\right) \left(\frac{1}{2} \pm \frac{s}{2} \right) = \delta \left(\frac{1}{2} + \frac{s}{2} \right) + \delta \left(\frac{1}{2} - \frac{s}{2} \right) + \delta \left(\frac{1}{2} + \frac{s}{2} \right) \left(\frac{1}{2} \pm \frac{s}{2} \right),
\] (2.27)
with \(\delta^{(\pm)}(u) = id\) and
\[
\delta^{(\pm)}(u) = \frac{(2u - 2\eta)(2u + 2\eta)}{(2u - \eta)(2u + \eta)} \left( \frac{1 + \zeta^2}{2} \right) u^2 - \left( \frac{1 + \zeta^2}{2} \right) u^2 - q^2 u^2
\]
\[
	imes \prod_{l=1}^N \left( u - \theta_l + \left( \frac{1}{2} + s \right) \eta \right) \left( u + \theta_l + \left( \frac{1}{2} + s \right) \eta \right)
\]
\[
	imes \prod_{l=1}^N \left( u - \theta_l - \left( \frac{1}{2} + s \right) \eta \right) \left( u + \theta_l - \left( \frac{1}{2} + s \right) \eta \right).
\]
With the initial condition (2.3) of \(R^{(\pm,\pm)}(u)\), the hierarchy relation (2.27) is closed at the inhomogeneity points [56]
\[
i\left(\frac{1}{2} \pm \frac{s}{2}\right) \left(\frac{1}{2} \pm \frac{s}{2} \right) = \delta \left(\frac{1}{2} + \frac{s}{2} \right) \left(\frac{1}{2} \pm \frac{s}{2} \right) \left(\frac{1}{2} + \frac{s}{2} \right) \left(\frac{1}{2} \pm \frac{s}{2} \right),
\] (2.28)
which together with the crossing symmetry \(i\left(\frac{1}{2} \pm \eta\right) = i\left(\frac{1}{2} \pm \eta\right)\) and the asymptotic behavior
\[
i\left(\frac{1}{2} \pm \frac{s}{2}\right) = 2(\xi - 1)u^{2N+2} \times id + \cdots,
\] (2.29)
i\left(\frac{1}{2} \pm \frac{s}{2}\right) = 2pq \prod_{l=1}^N \left( \theta_l + \left( \frac{1}{2} + s \right) \eta \right) \left( -\theta_l + \left( \frac{1}{2} + s \right) \eta \right) \times id,
\] (2.30)
allows us to express \(\Lambda^{(\pm,\pm)}(u)\), the eigenvalues of \(i\left(\frac{1}{2} \pm \eta\right)\), in the following inhomogeneous \(T-Q\) formalism [56]
\[
\Lambda^{(\pm,\pm)}(u) = d^{(\pm)}(u) \frac{Q(u - \eta)}{Q(u)} + d^{(\pm)}(u) \frac{Q(u + \eta)}{Q(u)} + cu(u + \eta) \frac{F^{(\pm)}(u)}{Q(u)},
\] (2.31)
where the functions \( a^{(s)}(u), d^{(s)}(u), F^{(s)}(u) \) and the constant \( c \) are given by

\[
d^{(s)}(u) = \frac{2u + 2\eta}{2u + \eta} \left( \sqrt{1 + \xi^2 u + q} \right) \left( \sqrt{1 + \xi^2 u + p} \right)
\times \prod_{l=1}^{N} \left( u - \theta_l + \left( \frac{1}{2} + s \right) \eta \right) \left( u + \theta_l + \left( \frac{1}{2} + s \right) \eta \right),
\]

(2.32)

\[
d^{(s)}(u) = a^{(s)}(-u - \eta).
\]

(2.33)

\[
F^{(s)}(u) = \prod_{l=1}^{N} \prod_{k=0}^{2u} \left( u - \theta_l + \left( \frac{1}{2} - s + k \right) \eta \right) \left( u + \theta_l + \left( \frac{1}{2} - s + k \right) \eta \right),
\]

(2.35)

\[
c = 2 \left( \xi^2 - 1 - \sqrt{1 + \xi^2} \sqrt{1 + \xi^2} \right).
\]

(2.36)

The \( Q \)-function is parameterized as

\[
Q(u) = \prod_{j=1}^{2sN} \left( u - \lambda_j \right) \left( u + \lambda_j + \eta \right)
\]

(2.37)

and the \( 2sN \) Bethe roots \( \{ \lambda_j | j = 1, \ldots, 2sN \} \) should satisfy the Bethe ansatz equations (BAEs)

\[
da^{(s)}(\lambda_j) Q(\lambda_j - \eta) + d^{(s)}(\lambda_j) Q(\lambda_j + \eta) + c \lambda_j(\lambda_j + \eta) F^{(s)}(\lambda_j) = 0,
\]

(2.38)

\[ j = 1, \ldots, 2sN. \]

### 3. Gauge transformation

Without losing generality, we put \( \varsigma = 0 \) in the following text. For convenience, we introduce the notations

\[
T_0^{(\frac{1}{2}, -\frac{1}{2})}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},
\]

(3.1)

\[
\tilde{T}_0^{(\frac{1}{2}, -\frac{1}{2})}(u) = (-1)^N \begin{pmatrix} D(-u - \eta) & -B(-u - \eta) \\ -C(-u - \eta) & A(-u - \eta) \end{pmatrix}.
\]

(3.2)

\[
\mathcal{R}_0^{(\frac{1}{2}, -\frac{1}{2})}(u) = T_0^{(\frac{1}{2}, -\frac{1}{2})}(u) K_0^{(-\frac{1}{2}, \frac{1}{2})}(u) \tilde{T}_0^{(\frac{1}{2}, -\frac{1}{2})}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}.
\]

(3.3)

Let us introduce the gauge matrix

\[
U_0 = \begin{pmatrix} \sqrt{1 + \xi^2} - 1 & \xi \\ -\sqrt{1 + \xi^2} - 1 & \xi \end{pmatrix},
\]

(3.4)
with which $K_0^{+(\frac{1}{2})}$-matrix can be diagonalized as
\[
\hat{K}_0^{+(\frac{1}{2})}(u) = U_0 K_0^{+(\frac{1}{2})}(u) U_0^{-1} = \begin{pmatrix} q + \sqrt{1 + \xi^2} (u + \eta) & 0 \\ 0 & q - \sqrt{1 + \xi^2} (u + \eta) \end{pmatrix},
\]
and the gauged $K^{-(\frac{1}{2})}$-matrix $\hat{K}_0^{-(\frac{1}{2})}(u)$ becomes
\[
\hat{K}_0^{-(\frac{1}{2})}(u) = U_0 K_0^{-(\frac{1}{2})}(u) U_0^{-1},
\]
\[
= \begin{pmatrix} p - \frac{u}{\sqrt{1 + \xi^2}} & -\frac{\sqrt{1 + \xi^2} - 1}{\sqrt{1 + \xi^2}} u \\ -\frac{\sqrt{1 + \xi^2} + 1}{\sqrt{1 + \xi^2}} u & p + \frac{u}{\sqrt{1 + \xi^2}} \end{pmatrix},
\]
\[
= \begin{pmatrix} \hat{K}_{11}(u) & \hat{K}_{12}(u) \\ \hat{K}_{21}(u) & \hat{K}_{22}(u) \end{pmatrix}.
\]
Accordingly, the one-row monodromy matrices under the above gauge transformation read
\[
T_0^{+(\frac{1}{2})}(u) = U_0 T_0^{+(\frac{1}{2})}(u) U_0^{-1} = \begin{pmatrix} \hat{A}(u) & \hat{B}(u) \\ \hat{C}(u) & \hat{D}(u) \end{pmatrix},
\]
\[
\hat{T}_0^{+(\frac{1}{2})}(u) = U_0 T_0^{+(\frac{1}{2})}(u) U_0^{-1} = (-1)^N \begin{pmatrix} \hat{D}(-u - \eta) & -\hat{B}(-u - \eta) \\ -\hat{C}(-u - \eta) & \hat{A}(-u - \eta) \end{pmatrix}.
\]
The double-row monodromy matrix $\tilde{\hat{M}}_0^{+(\frac{1}{2})}(u)$ is gauged to
\[
\tilde{\hat{M}}_0^{+(\frac{1}{2})}(u) = U_0 T_0^{+(\frac{1}{2})}(u) \hat{K}_0^{+(\frac{1}{2})}(u) T_0^{+(\frac{1}{2})}(u) U_0^{-1},
\]
\[
= T_0^{+(\frac{1}{2})}(u) \hat{K}_0^{+(\frac{1}{2})}(u) T_0^{+(\frac{1}{2})}(u) U_0^{-1} \tilde{\hat{M}}_0^{+(\frac{1}{2})}(u) = \begin{pmatrix} \hat{\mathcal{J}}(u) & \hat{\mathcal{B}}(u) \\ \hat{\mathcal{C}}(u) & \hat{\mathcal{D}}(u) \end{pmatrix},
\]
which gives the following relations
\[
\hat{\mathcal{J}}(u) = (-1)^N \left\{ \hat{K}_{11}(u) \hat{A}(u) \hat{D}(-u - \eta) + \hat{K}_{21}(u) \hat{B}(u) \hat{D}(-u - \eta) - \hat{K}_{12}(u) \hat{A}(u) \hat{C}(-u - \eta) - \hat{K}_{22}(u) \hat{B}(u) \hat{C}(-u - \eta) \right\},
\]
\[
\hat{\mathcal{B}}(u) = (-1)^N \left\{ -\hat{K}_{11}(u) \hat{A}(u) \hat{B}(-u - \eta) - \hat{K}_{21}(u) \hat{B}(u) \hat{B}(-u - \eta) + \hat{K}_{12}(u) \hat{A}(u) \hat{A}(-u - \eta) + \hat{K}_{22}(u) \hat{B}(u) \hat{A}(-u - \eta) \right\},
\]
\[
\hat{\mathcal{C}}(u) = (-1)^N \left\{ \hat{K}_{11}(u) \hat{C}(u) \hat{D}(-u - \eta) + \hat{K}_{21}(u) \hat{D}(u) \hat{D}(-u - \eta) - \hat{K}_{12}(u) \hat{C}(u) \hat{C}(-u - \eta) - \hat{K}_{22}(u) \hat{D}(u) \hat{C}(-u - \eta) \right\},
\]
\[ \mathcal{G}(u) = (-1)^N \left\{ -K_{11}(u) \mathcal{C}(u) \mathcal{B}(-u - \eta) - K_{22}(u) \mathcal{D}(u) \mathcal{B}(-u - \eta) + K_{12}(u) \mathcal{C}(u) \mathcal{A}(-u - \eta) + K_{21}(u) \mathcal{D}(u) \mathcal{A}(-u - \eta) \right\}. \]  

(3.12)

The transfer matrix \( t^\pm(u) \) can be expressed as

\[ t^\pm(u) = t^0 \left( \mathcal{K}_0^{-\frac{1}{2}}(u) \mathcal{Y}_0^0(\frac{1}{2}) \right)(u) = \mathcal{K}_{11}(u) \mathcal{G}(u) + \mathcal{K}_{22}(u) \mathcal{G}(u). \]  

(3.13)

Thanks to the SU\((2)\)-invariance of the R-matrix, the gauged one-row monodromy matrix also satisfies the relation

\[ R_{(\frac{1}{2},\frac{1}{2})}^{(\frac{1}{2},\frac{1}{2})} (u - v) T_0^{(\frac{1}{2},\frac{1}{2})} (u) T_0^{(\frac{1}{2},\frac{1}{2})} (v) = T_0^{(\frac{1}{2},\frac{1}{2})} (v) T_0^{(\frac{1}{2},\frac{1}{2})} (u) R_{0,0}^{(\frac{1}{2},\frac{1}{2})} (u - v), \]

which gives rise to the following commutation relations

\[ \mathcal{A}(u) \mathcal{B}(v) = \frac{u - v - \eta}{u - v} \mathcal{B}(v) \mathcal{A}(u) + \frac{\eta}{u - v} \mathcal{B}(u) \mathcal{A}(v), \]  

(3.14)

\[ \mathcal{D}(u) \mathcal{B}(v) = \frac{u - v + \eta}{u - v} \mathcal{B}(v) \mathcal{D}(u) - \frac{\eta}{u - v} \mathcal{B}(u) \mathcal{D}(v), \]  

(3.15)

\[ \mathcal{B}(u) \mathcal{D}(v) = \frac{u - v + \eta}{u - v} \mathcal{D}(v) \mathcal{B}(u) - \frac{\eta}{u - v} \mathcal{D}(u) \mathcal{B}(v), \]  

(3.16)

\[ \mathcal{C}(u) \mathcal{A}(v) = \frac{u - v + \eta}{u - v} \mathcal{A}(v) \mathcal{C}(u) - \frac{\eta}{u - v} \mathcal{A}(u) \mathcal{C}(v), \]  

(3.17)

\[ \mathcal{C}(u) \mathcal{D}(v) = \frac{u - v - \eta}{u - v} \mathcal{D}(v) \mathcal{C}(u) + \frac{\eta}{u - v} \mathcal{D}(u) \mathcal{C}(v), \]  

(3.18)

\[ \left[ \mathcal{C}(u), \mathcal{B}(v) \right] = \frac{\eta}{u - v} \left[ \mathcal{D}(u) \mathcal{A}(v) - \mathcal{D}(v) \mathcal{A}(u) \right]. \]  

(3.19)

Similarly, the gauged double-row monodromy matrix satisfies

\[ R_{0,0}^{(\frac{1}{2},\frac{1}{2})} (u - v) T_0^{(\frac{1}{2},\frac{1}{2})} (u) T_0^{(\frac{1}{2},\frac{1}{2})} (v) = T_0^{(\frac{1}{2},\frac{1}{2})} (v) T_0^{(\frac{1}{2},\frac{1}{2})} (u) R_{0,0}^{(\frac{1}{2},\frac{1}{2})} (u - v), \]

which leads to the following commutation relations

\[ \mathcal{E}(u) \mathcal{F}(v) = \frac{(u + v)(u - v + \eta)}{(u - v)(u + v + \eta)} \mathcal{F}(v) \mathcal{E}(u) - \frac{\eta}{u + v + \eta} \mathcal{E}(u) \mathcal{F}(v) \]  

(3.21)

\[ \mathcal{F}(v) \mathcal{E}(u) = \frac{(u + v)(u - v + \eta)}{(u - v)(u + v + \eta)} \mathcal{E}(u) \mathcal{F}(v) - \frac{\eta}{u + v + \eta} \mathcal{F}(v) \mathcal{E}(u) \]  

(3.22)
\begin{equation}
\mathcal{J}(v)\mathcal{J}(v) = \mathcal{J}(v)\mathcal{J}(v) + \frac{\eta}{u+v+\eta} \mathcal{H}(v)\mathcal{H}(v),
\end{equation}
\begin{equation}
\mathcal{J}(v)\mathcal{J}(v) = \mathcal{J}(v)\mathcal{J}(v) + \frac{\eta}{u+v+\eta} \mathcal{H}(v)\mathcal{H}(v),
\end{equation}
\begin{equation}
\mathcal{J}(v)\mathcal{J}(v) = \mathcal{J}(v)\mathcal{J}(v) - \frac{\eta(u+v+2\eta)}{(u-v)(u+v+\eta)} \mathcal{H}(v)\mathcal{H}(v)
+ \frac{\eta(u+v+2\eta)}{(u-v)(u+v+\eta)} \mathcal{H}(v)\mathcal{H}(v),
\end{equation}
\begin{equation}
[\mathcal{J}(v), \mathcal{J}(v)] = [\mathcal{H}(v), \mathcal{H}(v)] = 0.
\end{equation}

4. Orthogonal basis

In order to obtain the orthogonal basis of the Hilbert space, we first introduce the reference state. For general spin-$s$ cases, the gauged $R_{0,n}^{(1)}$ is
\begin{equation}
R_{0,n}^{(1)}(u) = U_0 R_{0,n}^{(1)}(u) U_0^{-1} \equiv \left( \bar{r}_{11}(u) \bar{r}_{12}(u) \bar{r}_{21}(u) \bar{r}_{22}(u) \right),
\end{equation}
where
\begin{equation}
\bar{r}_{21}(u) = - \frac{1}{2\xi \sqrt{1 + \xi^2}} \left[ 2\xi \left( \sqrt{1 + \xi^2} + 1 \right) \eta S_n^- - \left( \sqrt{1 + \xi^2} + 1 \right)^2 \eta S_n^+ - \xi^2 \eta S_n^+ \right],
\end{equation}
\begin{equation}
\bar{r}_{12}(u) = - \frac{1}{2\xi \sqrt{1 + \xi^2}} \left[ 2\xi \left( \sqrt{1 + \xi^2} - 1 \right) \eta S_n^- - \left( \sqrt{1 + \xi^2} - 1 \right)^2 \eta S_n^+ - \xi^2 \eta S_n^+ \right].
\end{equation}

We introduce a set of local states $\{|s_n\rangle_n = \sum_a c^{(s)}_{n} |k\rangle_n, a = 1, \ldots, 2s + 1, k = -s, \ldots, s, n = 1, \ldots, N\}$, where $\{|k\rangle_n, k = -s, \ldots, s\}$ form the eigenstates of $S_n^z$, i.e., $S_n^z |k\rangle_n = k |k\rangle_n$. The coefficients $\{c_{n}^{(s)}\}$ are determined by the constraint

$$\bar{r}_{21}|s_n\rangle_n = 0,$$

which gives the coefficients of $|s_n\rangle_n$ as
\begin{equation}
c_{n}^{(s+1)} = \frac{\sqrt{2s(2s-1)\ldots(2s-j+1)}}{\sqrt{2} \left( \sqrt{1 + \xi^2} + 1 \right)^{2s-j}} \xi^j, \quad j = 0, \ldots, 2s,
\end{equation}
The coefficients \(c_{2s+j}^{(2s+1)}\), \(a = 2, \ldots, 2s + 1\) are determined by the condition
\[
|\tilde{b}_n\rangle = f(\eta)|\tilde{b}_{n-1}\rangle, \quad (\text{4.5})
\]
which gives rise to the values of \(c_{2s+j}^{(2s+1)}\) as \((f(\eta))\) is an irrelevant normalization factor
\[
c_{2s+j}^{(2s+1)} = (-1)^j \frac{\sqrt{2s(2s-1) \ldots (2s-j+1)}}{\sqrt{j}} \frac{1}{1 + (\xi i)^{j/2}}, \quad j = 0, \ldots, 2s. \quad (\text{4.6})
\]

The reference states \(|\tilde{b}_n\rangle\), \(n = 1, \ldots, N\) satisfy the following orthogonal relations
\[
j\{\tilde{b}_a | \tilde{b}_b\} = \delta_{ab}, \quad a, b = 1, 2, \ldots, 2s + 1, \quad j = 1, \ldots, N. \quad (\text{4.7})
\]

We introduce the product state \(|\Omega\rangle = \otimes_{n=1}^N |\tilde{b}_n\rangle\) and \(|\tilde{\Omega}\rangle = \otimes_{n=1}^N |\tilde{b}_{2s+1}\rangle\), which are the eigenstates of the operators \(\tilde{A}(u)\) and \(\tilde{D}(u)\)
\[
\tilde{A}(u)|\Omega\rangle = a(u)|\Omega\rangle, \quad \tilde{D}(u)|\Omega\rangle = d(u)|\Omega\rangle, \quad \tilde{C}(u)|\Omega\rangle = 0, \quad \langle \tilde{\Omega}|\tilde{A}(u)|\Omega\rangle = 0, \quad \langle \tilde{\Omega}|\tilde{D}(u)|\Omega\rangle = a(u)|\Omega\rangle, \quad \langle \tilde{\Omega}|\tilde{C}(u)|\Omega\rangle = 0, \quad (\text{4.8})
\]
with the corresponding eigenvalues
\[
a(u) = \prod_{l=1}^N \left( u - \theta_l + \left( \frac{1}{2} + s \right)\eta \right), \quad d(u) = \prod_{l=1}^N \left( u - \theta_l + \left( \frac{1}{2} - s \right)\eta \right). \quad (\text{4.10})
\]

Denoting \(\beta'_l \equiv \theta_l - \left( \frac{1}{2} + s \right)\eta\) and \(\beta_l \equiv \theta_l - \left( \frac{1}{2} - s \right)\eta\), we have \(a(\beta'_l) = 0\) and \(d(\beta_l) = 0\).

From the equation (3.11), we find that the product state \(|\Omega\rangle\) and \(|\tilde{\Omega}\rangle\) are also the eigenstates of the operator \(\tilde{\mathcal{C}}(u)\)
\[
\tilde{\mathcal{C}}(u)|\Omega\rangle = (-1)^N \tilde{K}_2(u)d(u)d(-u-\eta)|\Omega\rangle, \quad (\text{4.11})
\]
\[
\langle \tilde{\Omega}|\tilde{\mathcal{C}}(u)|\Omega\rangle = (-1)^N \tilde{K}_2(u)a(u)a(-u-\eta)|\tilde{\Omega}\rangle. \quad (\text{4.12})
\]

Noting the fact that \([\tilde{\mathcal{C}}(u), \tilde{\mathcal{C}}(v)] = 0\), the eigenstates of \(\tilde{\mathcal{C}}(u)\) can form a basis of the Hilbert space in the sense of Sklyanin’s SoV [63–65]. Let us introduce the following states
\[
\left| \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\rangle = \bigotimes_{j=1}^N \prod_{k=0}^{\alpha_j-1} \tilde{\mathcal{O}}(\beta_j - k_j\eta)|\Omega\rangle, \quad \alpha_j = 0, 1, \ldots, 2s, \quad (\text{4.13})
\]
\[
\left\{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\} = \left\{ \tilde{\Omega} \bigotimes_{j=1}^N \prod_{k=0}^{\alpha_j-1} \mathcal{O}(-\beta_j' - (k_j + 1)\eta) \right\}, \quad \alpha_j = 0, 1, \ldots, 2s. \quad (\text{4.14})
\]

It should be noted that the products of \(\tilde{\mathcal{O}}(\beta_j - k_j\eta)\) in equation (4.13) are ordered by decreasing \(k_j\) while \(\mathcal{O}(-\beta_j' - (k_j + 1)\eta)\) in (4.14) are ordered by increasing \(k_j\) from left to right. Using the commutation relations (3.21)–(3.25), we conclude that equations (4.13) and (4.14) are eigenstates of \(\tilde{\mathcal{C}}(u)\)
\[
\tilde{\mathcal{C}}(u)|\beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)}\rangle = h\left( u, \left\{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\} \right)|\beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)}\rangle, \quad (\text{4.15})
\]
\[
\left\{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\} \tilde{\mathcal{C}}(u) = h\left( u, \left\{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\} \right)\left\{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\}. \quad (\text{4.16})
\]
with the eigenvalues

\[ h \left( u, \{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \} \right) = (-1)^N K_{21}(u) \frac{d(-u - \eta)d(u)}{u - \beta_j + \alpha_j \eta} \]

(4.17)

\[ \hat{h} \left( u, \{ \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \} \right) = (-1)^N K_{21}(u) a(-u - \eta)a(u) \]

\[ \times \prod_{j=1}^N \frac{(u - \beta_j + \alpha_j \eta)(u + \beta_j + \eta - \alpha_j \eta)}{(u - \beta_j)(u + \beta_j + \eta)}. \]

(4.18)

By using the commutation relations (3.21)–(3.25) and equations (4.15)–(4.16), we can prove that the order of the product of \( \mathcal{H}(\mathcal{D}) \) with respect to different \( \beta_j(\beta_j') \) in equation (4.13) (equation (4.14)) is changeable, while the order of that with the same \( \beta(\beta') \) cannot be changed. The right states given by equation (4.13) (the left states given by equation (4.14)) form a complete and orthogonal basis of the Hilbert space. Therefore, the eigenstates of the transfer matrices can be decomposed as a unique linear combination of the basis vectors.

5. The scalar product

For convenience, we introduce

\[ \mathcal{D}(u) = \frac{\eta}{2u + \eta} \mathcal{H}(u). \]

(5.1)

The transfer matrix \( \mathcal{I}(\frac{1}{2}) ) \( (u) \) can be expressed as

\[ \mathcal{I}(\frac{1}{2}) ) \( (u) = \left[ K_{11}(u) + \frac{\eta}{2u + \eta} K_{22}(u) \right] \mathcal{H}(u) + K_{22}(u) \mathcal{D}(u). \]

(5.2)

Let \( \langle \Psi | \) be an eigenstate of the transfer matrix of \( \mathcal{I}(\frac{1}{2}) ) \( (u) \), namely

\[ \langle \Psi | \mathcal{I}(\frac{1}{2}) ) \( (u) = \langle \Psi | \Lambda(\frac{1}{2}) ) \( (u), \]

(5.3)

where the eigenvalue \( \Lambda(\frac{1}{2}) ) \( (u) is given by the inhomogeneous \( T-Q \) relation (2.31).

Now let us evaluate the scalar product

\[ F(\alpha_1, \ldots, \alpha_N) = \langle \Psi \left| \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \right\rangle, \]

(5.4)

by calculating the quantity \( \langle \Psi \left| \left(\beta_n - m\eta\right) \beta_n^{(\alpha_n - m)} \ldots, \beta_N^{(\alpha_N)} \right\rangle \). Acting \( \mathcal{I}(\frac{1}{2}) ) \( (\beta_n - m\eta) to the left and to the right alternately, we obtain
\[
\Lambda \left( \frac{1}{\tau^i} \right) (\beta_n - m\eta) F(\alpha_1, \ldots, \alpha_n = m, \ldots, \alpha_N) \\
= \left[ \tilde{K}_{11}^+ (\beta_n - m\eta) + \frac{\eta \tilde{K}_{22}^+ (\beta_n - m\eta)}{2\beta_n - (2m - 1)\eta} \right] F(\alpha_1, \ldots, \alpha_n = m + 1, \ldots, \alpha_N) \\
+ \tilde{K}_{22}^+ (\beta_n - m\eta) \left\{ \Psi \tilde{D} (\beta_n - m\eta) \right\} \beta^{(\alpha_1)}, \ldots, \beta^{(\alpha_n = m)}, \ldots, \beta^{(\alpha_N = 1)} \right\}. 
\]

(5.5)

From equations (3.9) and (3.12), we have the following relations
\[
\tilde{A}(u)|\Omega\rangle = (-1)^N \left\{ \tilde{K}_{11}(u)a(u)d(-u - \eta)|\Omega\rangle + \tilde{K}_{22}(u)d(-u - \eta)\tilde{B}(u)|\Omega\rangle \right\}, 
\]

(5.6)

\[
\tilde{D}(u)|\Omega\rangle = (-1)^N \left\{ \frac{(2u + \eta)\tilde{K}_{22}(u) - \eta\tilde{K}_{11}(u)}{2u + \eta}d(u)a(-u - \eta)|\Omega\rangle \\
- \frac{2u + 2\eta}{2u + \eta} \tilde{K}_{22}(u)d(u)\tilde{B}(-u - \eta)|\Omega\rangle \right\}. 
\]

(5.7)

It is easy to check
\[
\tilde{D}(\beta_j)|\Omega\rangle = 0, \quad j = 1, \ldots, N, 
\]

(5.8)

which allows us to write \( F(\alpha_1, \ldots, \alpha_n = 1, \ldots, \alpha_N) \) as
\[
F(\alpha_1, \ldots, \alpha_n = 1, \ldots, \alpha_N), \\
= \frac{(2\beta_n + \eta)\Lambda \left( \frac{1}{\tau^i} \right) (\beta_n)}{(2\beta_n + \eta)\tilde{K}_{11}^+(\beta_n) + \eta\tilde{K}_{22}^+(\beta_n)} F(\alpha_1, \ldots, \alpha_n = 0, \ldots, \alpha_N), \\
= (-1)^N (p + \beta_n)a(\beta_n)d(-\beta_n - \eta) \frac{Q(\beta_n - \eta)}{Q(\beta_n)} F(\alpha_1, \ldots, \alpha_n = 0, \ldots, \alpha_N). 
\]

(5.9)

Based on the properties of quantum determinant [66] (for a detailed description, see [29]),
\[
\text{Det}_q \left\{ \tilde{T} \left( \frac{1}{\tau^i} \right) (u) \right\} = \tilde{A}(u - \eta)\tilde{D}(u) - \tilde{C}(u - \eta)\tilde{B}(u), \\
= \tilde{D}(u - \eta)\tilde{A}(u) - \tilde{B}(u - \eta)\tilde{C}(u), 
\]

(5.10)

\[
\text{Det}_q \left\{ \tilde{T} \left( \frac{1}{\tau^i} \right) (u) \right\} = \tilde{A}(u - \eta)\tilde{D}(u) - \tilde{B}(u - \eta)\tilde{C}(u), \\
= \tilde{D}(u - \eta)\tilde{A}(u) - \tilde{C}(u)\tilde{B}(u - \eta). 
\]

(5.11)

and the commutation relations
\[
\tilde{A}(u)\tilde{B}(u - \eta) = \tilde{B}(u)\tilde{A}(u - \eta), \quad \tilde{C}(u)\tilde{D}(u - \eta) = \tilde{D}(u)\tilde{C}(u - \eta), 
\]

(5.12)

\[
\tilde{D}(u - \eta)\tilde{B}(u) = \tilde{B}(u - \eta)\tilde{D}(u), \quad \tilde{A}(u - \eta)\tilde{C}(u) = \tilde{C}(u - \eta)\tilde{A}(u), 
\]

(5.13)

we find that the following relation holds
\[
\tilde{D}(u - \eta)\tilde{A}(u) - \frac{2u}{2u - \eta}\tilde{B}(u - \eta)\tilde{C}(u) 
\]
According to equations (4.15) and (4.17), we know

\( \mathcal{G}(\beta_n - \alpha_n \eta) \left\{ \beta_n^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)}, \ldots, \beta_n^{(\alpha_N)} \right\} = 0. \)  

(5.15)

Using the relations (5.14) and (5.15), we obtain

\[
\mathcal{D}(\beta_n - mn) \left\{ \beta_n^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_{m-1})}, \beta_n^{(\alpha_m)} \right\} = \frac{2\beta_n - 2mn}{2\beta_n - (2m - 1)\eta} \left\{ p^2 - \left[ \beta_n - (m - 1)\eta \right]^2 \right\} \times a(\beta_n - (m - 1)\eta) d(-\beta_n + (m - 2)\eta) \\
\times a(-\beta_n + (m - 1)\eta) d(\beta_n - mn) \beta_n^{(\alpha_m)} \beta_n^{(\alpha_{m-1})} \beta_n^{(\alpha_m)} \beta_n^{(\alpha_{m-1})}, \quad m = 1, \ldots, 2s.
\]

(5.16)

Substituting equation (5.16) into (5.5), we obtain the recursive relations about \( F(\alpha_1, \ldots, \alpha_N) \)

\[
\Lambda(\mathbf{T}) (\beta_n - mn) F(\alpha_1, \ldots, \alpha_n = m, \ldots, \alpha_N) = \left[ K_{11}(\beta_n - mn) + \frac{\eta K_{22}(\beta_n - mn)}{2\beta_n - 2mn + \eta} \right] F(\alpha_1, \ldots, \alpha_n = m + 1, \ldots, \alpha_N) \\
+ \frac{2\beta_n - 2mn}{2\beta_n - (2m - 1)\eta} K_{22}(\beta_n - mn) \left\{ p^2 - \left[ \beta_n - (m - 1)\eta \right]^2 \right\} a(\beta_n - (m - 1)\eta) \\
\times d(-\beta_n + (m - 2)\eta) a(-\beta_n + (m - 1)\eta) d(\beta_n - mn) \times F(\alpha_1, \ldots, \alpha_n = m - 1, \ldots, \alpha_N), \quad m = 1, \ldots, 2s - 1.
\]

(5.17)

The initial condition (5.9) and the recursive relations (5.17) give rise to

\[
F(\alpha_1, \ldots, \alpha_N) = \prod_{j=1}^{N} \prod_{k=0}^{\alpha_j - 1} (-1)^N (p + \beta_j - k_j \eta) \\
\times a(\beta_j - k_j \eta) d(-\beta_j + (k_j - 1)\eta) \frac{Q(\beta_j - (k_j + 1)\eta)}{Q(\beta_j - k_j \eta)} F_0,
\]

(5.18)

where \( F_0 = \langle \Psi | \Omega \rangle \) is an overall scalar factor.

6. The inner product \( \langle 0 | \beta_1^{(\alpha_1)}, \ldots, \beta_N^{(\alpha_N)} \rangle \)

The definition of the one-row monodromy matrix \( T_0(u) \) implies

\[
\langle 0 | A(u) = a(u) \langle 0 |, \quad \langle 0 | D(u) = d(u) \langle 0 |, \quad \langle 0 | B(u) = 0,
\]

(6.1)

where the functions \( a(u) \) and \( d(u) \) are given by equation (4.10), \( \langle 0 | \Lambda \rangle \subseteq \langle s | \otimes \ldots \otimes_N \langle s | \). The double-row monodromy matrix (3.3) acting on the state \( \langle 0 | \) gives
\[ \langle 0 \rangle \mathcal{A}(u) = (-1)^N K_1(u) a(u) d(-u - \eta) \langle 0 \rangle, \]  
(6.2)

\[ \langle 0 \rangle \mathcal{D}(u) = (-1)^N \frac{-\eta}{2u + \eta} K_1(u) a(u) d(-u - \eta) \langle 0 \rangle + (-1)^N \frac{2u + \eta}{2u + \eta} K_2(u) - \eta K_1(u) a(-u - \eta) d(u) \langle 0 \rangle, \]  
(6.3)

\[ \langle 0 \rangle \mathcal{B}(u) = 0, \]  
(6.4)

\[ \langle 0 \rangle \mathcal{E}(u) = (-1)^N \frac{2u}{2u + \eta} - \eta K_1(u) d(-u - \eta) \langle 0 \rangle C(u) + (-1)^N \frac{-(2u + \eta) K_2(u) + \eta K_1(u)}{2u + \eta} d(u) \langle 0 \rangle C(-u - \eta). \]  
(6.5)

Notice that the following relations hold
\[
\tilde{\mathcal{A}}(u) = \frac{1}{2 \xi \sqrt{1 + \xi^2}} \left\{ \xi \left( \sqrt{1 + \xi^2} - 1 \right) \mathcal{A}(u) + \xi^2 \mathcal{E}(u) \right. \\
+ \left. \xi^2 \mathcal{B}(u) + \xi \left( 1 + \sqrt{1 + \xi^2} \right) \mathcal{D}(u) \right\}, \]  
(6.6)

\[
\tilde{\mathcal{E}}(u) = \frac{1}{2 \xi \sqrt{1 + \xi^2}} \left\{ -\xi \left( 1 + \sqrt{1 + \xi^2} \right) \mathcal{A}(u) - \left( 1 + \sqrt{1 + \xi^2} \right)^2 \mathcal{B}(u) \right. \\
+ \left. \xi^2 \mathcal{E}(u) + \xi \left( 1 + \sqrt{1 + \xi^2} \right) \mathcal{D}(u) \right\}, \]  
(6.7)

\[
\tilde{\mathcal{D}}(u) = \frac{1}{2 \xi \sqrt{1 + \xi^2}} \left\{ \xi \left( 1 + \sqrt{1 + \xi^2} \right) \mathcal{A}(u) - \xi^2 \mathcal{E}(u) \right. \\
- \left. \xi^2 \mathcal{B}(u) + \xi \left( -1 + \sqrt{1 + \xi^2} \right) \mathcal{D}(u) \right\}. \]  
(6.8)

The relation \[ \langle 0 \rangle \tilde{\mathcal{E}}(\beta_n - (m - 1)\eta) \left| \beta_1^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)} \right\rangle = 0 \] gives rise to
\[ \langle 0 \rangle \tilde{\mathcal{E}}(\beta_n - (m - 1)\eta) \left| \beta_1^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)} \right\rangle = 0 \]  
(6.9)

With the help of equation (6.9), we have
\[ \langle 0 \rangle \left| \beta_1^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)} \right\rangle \]
\[ \langle 0 \rangle \left| \beta_1^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)} \right\rangle \]
\[ = \langle 0 \rangle \mathcal{A}(\beta_n - (m - 1)\eta) \left| \beta_1^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)} \right\rangle, \]
\[ = (-1)^N K_1(\beta_n - (m - 1)\eta) a(\beta_n - (m - 1)\eta) d(-\beta_n + (m - 2)\eta) \]
\[ \times \langle 0 \rangle \left| \beta_1^{(\alpha_1)}, \ldots, \beta_n^{(\alpha_m)} \right\rangle, \]
which induces the solution
\[
\langle 0 | \beta_N^{(\alpha_1), \ldots, \beta_N^{(\alpha_N)}} \rangle = \prod_{j=1}^{N} \prod_{k_j=0}^{\alpha_j-1} (-1)^{N_j} K_{ij}(\beta_j - k_j \eta) \\
\times a(\beta_j - k_j \eta)d(-\beta_j + (k_j - 1) \eta) \langle 0 | \Omega \rangle.
\] (6.10)

7. Bethe states

We introduce the following left Bethe states
\[
\langle \lambda_1, \ldots, \lambda_{2sN} | = \left\langle 0 \right| \prod_{j=1}^{2sN} \frac{\tilde{q}(\lambda_j)}{(-1)^N K_{2N}(\lambda_j)d(-\lambda_j - \eta)} \right\rangle.
\] (7.1)

The relations (4.15) and (6.10) imply that
\[
\left\langle \lambda_1, \ldots, \lambda_{2sN} | \beta_N^{(\alpha_1), \ldots, \beta_N^{(\alpha_N)}} \right\rangle = \prod_{j=1}^{N} \prod_{k_j=0}^{\alpha_j-1} (-1)^{N_j} (p + \beta_j - k_j \eta)a(\beta_j - k_j \eta)
\times d(-\beta_j + (k_j - 1) \eta) \frac{Q(-\beta_j + (k_j + 1) \eta)}{Q(-\beta_j - k_j \eta)} \langle 0 | \Omega \rangle,
\]
which is consistent with equation (5.18). Therefore, we conclude that the Bethe states given by equation (7.1) are the eigenstates of the transfer matrix \( \frac{1}{2} (\tau - Q) \), provided that the Bethe roots \( \{ \lambda_j | j = 1, \ldots, 2sN \} \) satisfy the BAEs (2.38). With a similar procedure, we can construct the right Bethe states of the transfer matrices as
\[
\left| \lambda_1, \ldots, \lambda_{2sN} \right\rangle = \left\{ \prod_{j=1}^{2sN} \frac{\tilde{R}(\lambda_j)}{(-1)^N K_{2N}(\lambda_j)a(-\lambda_j - \eta)} \right\} \langle 0 | \right\rangle,
\] (7.2)
with \( \langle 0 | = | s \rangle \otimes \ldots \otimes | s \rangle \).

From the definitions (3.4) of the gauge matrix, it is clear that both the reference state \( \langle 0 | \) (or \( \langle 0 | \)) and the generator \( \tilde{R}(u) \) (or \( \tilde{q}(u) \)) have well-defined homogeneous limits: \( \{ \theta_j \rightarrow 0 \} \). This implies that the homogeneous limit of the Bethe state (7.2) exactly gives rise to the corresponding Bethe state of the homogeneous spin-\( s \) chain with generic open boundaries, where the associated \( T - Q \) relation and BAEs are given by (2.31) and (2.38) with \( \{ \theta_j = 0 \} \).

8. Conclusions

In conclusion, the Bethe-type eigenstates of the integrable spin-\( s \) Heisenberg chain with generic open boundary condition are constructed based on the inhomogeneous \( T - Q \) relation. It is shown that the resulting Bethe states have well-defined homogeneous limits. The method developed in this paper provides a possible way to construct Bethe-type eigenstates of high-level integrable models with generic boundary conditions. It should be remarked that a generic scalar product \( \langle \Psi | \prod_{j=1}^{d} \tilde{R}(u_j) | 0 \rangle \), which is relevant to the form factors, can be expressed easily as a linear combination of \( F(\alpha_1, \ldots, \alpha_N) \).
Acknowledgments

The financial supports from the National Natural Science Foundation of China (Grant Nos. 11375141, 11374334, 11434013 and 11425522), the National Program for Basic Research of MOST (973 project under grant No 2011CB921700), BCMIIS and the Strategic Priority Research Program of the CAS are gratefully acknowledged.

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