Semiclassical transport with Berry curvature: Chambers formula and applications to systems with Fermi surface topological transitions.

Emmanouil K. Kokkinis,1,2 Garry Goldstein,3 Dmitri V. Efremov,1,4 and Joseph J. Betouras1

1Department of Physics, Loughborough University, Loughborough, LE11 3TU, UK
2Physics department, University of Crete, Heraklion 71003, Greece.
3IFW Leibniz Institute for Solid State and Materials Research, Dresden, Helmholtz str. 20, 01069 Dresden, Germany.

(Dated: April 7, 2022)

Starting with general semiclassical equations of motion for electrons in the presence of electric and magnetic fields, we extend the Chambers formula to include in addition to a magnetic field, time-dependent electric fields and bands with Berry curvature. We thereby compute the conductivity tensor \( \sigma_{\alpha\beta}(B, \omega) \) in the presence of magnetic field for bands in two (2D) and three (3D) dimensions with Berry curvature. We focus then on several applications to magnetotransport for metals with Fermi surface topological transitions in 2D. In particular, we consider a rectangular lattice and a model related to overdoped graphene, to investigate the signatures of different types of Fermi surface topological transitions in metals in the Hall coefficient. Hall conductivity \( \sigma_{xy} \) and longitudinal conductivity \( \sigma_{xx} \). The behavior of those quantities as a function of frequency, when the electric field is time dependent, is also investigated. As an example of non-zero Berry curvature, we study the magnetotransport of the Haldane model within this context. In addition, we provide the linear and nonlinear electric current formula to order \( E^2 \).

I. INTRODUCTION

The majority of metals are well described by the Fermi liquid theory. Within this formalism the classical Hall effect arises due to the electron path curvature in the presence of an external magnetic fields \([1]\). It is characterized by the Hall coefficient \( R_H = E_y/(j_x B_z) \), where \( j_x \) is the current density perpendicular to the applied magnetic field \( B_z \) and \( E_y \) is the induced electric field. In the case of a single band the Hall coefficient depends only on the sign of the charge and the density of the carriers. Therefore the Hall coefficient at zero temperature is widely used as a measure of the number of electrons or holes enclosed in a closed orbit \([1, 2]\). However, the Hall coefficient may deviate from the simple form of counting of the number of electrons or holes enclosed in the electron-and hole-like pockets. It happens, e.g. in multi-band systems and in the systems close to the Fermi-surface topological transitions (FSTTs), as they are connected to singularities in the density of states, called Van Hove singularities \([3, 4]\).

There have been extensive studies of Lifshitz transitions (pocket appearing/disappearing or neck formation/collapse) and associated Van Hove singularities in a variety of materials including cuprates, iron based superconductors, cobaltates, Sr2RuO4 and heavy fermions \([7, 10]\). Most of these materials show logarithmic-type Van Hove singularity, which corresponds to a logarithmic singularity of the density of states at the Lifshitz transition in 2D. However, there is a recent surge of interest in higher order Van Hove singularities, which manifest itself in the stronger than logarithmic singularity in the density of states. These are the result of more complicated FSTTs. A prominent example is Sr3Ru2O7, where a \( n = 4 \) Van Hove singularity connected to a more complicated FSST, is shown to exist in the presence of an external magnetic field \([17]\). Higher order Van Hove saddle points has been observed in highly overdoped graphene and twisted bilayer graphene \([18, 21]\) and may be highly relevant for the recently observed phases of Bernal bilayer graphene \([22, 23]\).

In this work, we study the behavior of the Hall coefficient across a FSTT that correspond to a high-order Van Hove singularities using the Boltzmann equation in the presence of a static magnetic field, a Berry curvature \( \Omega(k) \) and a low-frequency electric field \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) \exp(i \omega t) \). The solution of the Boltzmann equation is provided in form of the Chambers formula \([24]\), which is widely used in studies of the magnetotransport \([1, 2, 25-33]\). We extend the Chambers formula to the case of Berry curvature.

The equations of motion used in this work are correct to leading order in the electric field \( E \) and magnetic field \( B \), but remarkably, retain the same form in corrected to order \( E^2 \) and \( B^2 \) \([34, 35]\). However, using the methods presented here it is possible to study the Boltzmann equation for arbitrary magnetic fields and electric fields with sufficiently accurate equations of motion \([34]\). To illustrate this idea we use the leading order equations of motion to study the solution to the Boltzmann equation in the magnetic field \( B \) and to quadratic order in the electric field \( E^2 \) (within the leading order equations of motion). We ignore the Zeeman splitting and consider spinless fermions thereby reducing to the case of negligible spin orbit coupling \([34, 35]\), thus avoiding more complicated expressions \([39]\). We also neglect any changes to the chemical potential due to the magnetic field, these
can be incorporated straightforwardly \[34, 38\].

The main contributions of this work are: (i) short and clear derivations of the Boltzmann equation relevant to the semiclassical motion of Fermi liquids, including the Berry curvature for leading order equations of motion. (ii) explicit solution of that Boltzmann equation to all orders in the magnetic and electric fields (formally exact), which can be implemented numerically for leading order equations of motion. (iii) method for semi-analytical asymptotic expansions of the Boltzmann equation solution to all orders in magnetic field and to an arbitrary (finite) order in the electric field for any set of equations of motion. (iv) explicit expansion of Boltzmann equation solution to linear and quadratic orders in the electric field with Berry curvature using the developed method, thus obtaining linear and bilinear response (recently introduced in Ref. \[40, 41\]), using the leading order equations of motion.

It is worth emphasizing that the work is valid within the different Fermi surface topologies but away from the transition points. At the transition points, as the Fermi velocity approaches zero, quantum effects become important as well as possible out-of-equilibrium effects in a time-dependent electric field. In particular magnetic breakdown quantum description of field-induced tunneling between semiclassical orbits shall be considered carefully close to Van Hove singularities \[32, 43\]. We, therefore, restrict the work away of those points.

The rest of the paper is organized as follows: in Section II we review the semiclassical equations of motion in the presence of external fields and Berry curvature, as well as the Boltzmann equation that follows. In Section III we derive the new 2D and 3D version of the Chambers formula with Berry curvature and time dependent external electrical field. In Sections IV-VI we apply these results to specific classes of materials. In Section IV we discuss the case of a Lifshitz transition of the form neck formation/collapse verifying that the rapid change between electron like to hole like Fermi surfaces leads to a rapid change of the Hall coefficient, which we also study as a function of frequency. In Section V we present the application to FSTT that corresponds to a higher Van Hove saddle in graphene and show how different FS topologies lead to different responses in conductivities. In Section VI we study the Hall conductivity of the Haldane model, as a representative example of a system with Berry curvature. In Section VII we conclude.

II. SEMICLASSICAL EQUATIONS OF MOTION - BOLTZMANN EQUATION AND SOLUTION

The semiclassical equations of motion for electrons of a single particle Hamiltonian \( H(k) \) to leading order in electric and magnetic fields was discussed in numerous works \[34, 36, 38, 44, 48\] and can be written as:

\[
\frac{d\mathbf{r}}{dt} = D^{-1}(\mathbf{r}, k, t) \left[ \nabla_k \varepsilon_M(k) + e \mathbf{E}(\mathbf{r}, t) \times \Omega(k) \right. \\
+ \varepsilon (\Omega(k) \cdot \nabla_k \varepsilon_M(k)) \mathbf{B}(\mathbf{r}, t) \left. \right], 
\]

\[
\frac{dk}{dt} = -D^{-1}(\mathbf{r}, k, t) \left[ e \mathbf{E}(\mathbf{r}, t) + e \nabla_k \varepsilon_M(k) \times \mathbf{B}(\mathbf{r}, t) \right. \\
+ \varepsilon^2 (\mathbf{B}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t)) \Omega(k) \left. \right]. 
\]

Here we have introduced \( D^{-1}(\mathbf{r}, k, t) = \frac{1}{1 + e^{\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{k}}}. \) The Berry curvature \( \Omega(k) \) is defined as the pseudo-vector \( \Omega(k) = \nabla_k \times \mathbf{A}(k) \), where \( \mathbf{A}(k) = i \langle u(k) | \nabla_k | u(k) \rangle \) is the Berry connection and \( | u(k) \rangle \) an eigenstate of the Hamiltonian \( H(k) \). The electron dispersion up to the second order of the magnetic fields can be written as \[36, 37\]:

\[
\varepsilon_M(k) = \varepsilon(k) - \mathbf{m}(k) \cdot \mathbf{B} 
\]

with:

\[
\mathbf{m}(k) = -\frac{e}{2\hbar} \langle \nabla_k u | [H(k) - \varepsilon(k)] \nabla_k u \rangle. 
\]

For simplicity, we consider below two dimensional (2D) case with both the electric and magnetic fields being uniform. Furthermore we consider time-independent magnetic field along z-axis (which in 2D is out of plane). The electric field is taken weakly time-dependent, that we can neglect the induction of the magnetic field. In two dimensions (2D), the Berry curvature is perpendicular to the plane \( \Omega_z(k) = \partial_{k_x} A_y - \partial_{k_y} A_x = -2Im \langle \partial_{k_x} u | \partial_{k_y} u \rangle \) and \( \Omega(k) \cdot \nabla_k \varepsilon_M(k) = 0 \). This configuration considerably simplifies the set of Eqs. \[1, 2\].

Now we derive the Boltzmann equation for our case. The derivation is the same in 2D or 3D so the dimensionality will be denoted by \( d \) and we will distinguish the differences in the next subsections. Let us take a small volume in phase space then the total number of particles in the system satisfies the continuity equation \[34, 41, 49, 50\]:

\[
\frac{\partial [Df]}{\partial t} + \nabla \cdot (\mathbf{w} Df) = \frac{D}{\tau_S(k)} (f_0 - f),
\]

where \( f = f(k, \mathbf{r}, t) \) is the distribution of electrons while \( f_0 = f_0 (\varepsilon_M(k)) \) is the Fermi-Dirac distribution, \( \tau_S(k) \) is the relaxation time and \( \mathbf{w} = (\frac{\partial \varepsilon}{\partial k_x}, \frac{\partial \varepsilon}{\partial k_y}) \), \( \nabla = (\nabla_k, \nabla_{\mathbf{k}}) \) is the six velocity in phase space. Using the identity \[37, 49\]:

\[
\nabla \cdot \mathbf{w} = -\frac{d}{dt} \ln(D(k, \mathbf{r}, t)) \\
= -\frac{\partial}{\partial t} \ln(D(k, \mathbf{r}, t)) - \mathbf{w} \cdot \nabla \ln(D(k, \mathbf{r}, t)) 
\]

we obtain the Boltzmann equation for the distribution of electrons:

\[
\frac{\partial}{\partial t} [f(k, \mathbf{r}, t)] + \mathbf{w} \cdot \nabla [f(k, \mathbf{r}, t)] = \frac{f_0 (\varepsilon_M) - f(k, \mathbf{r}, t)}{\tau_S(k)}
\]
with $\frac{d f}{d t}$ and $\frac{d f}{d t} = \frac{d f}{d t} \frac{d f}{d t}$ being given by Eq. (2). By considering spatially uniform magnetic and electric fields, the equation become:

$$\frac{d f}{d t} + \frac{d f}{d t} \frac{d f}{d t} = -\frac{1}{\tau_S(k)} (f - f_0(\varepsilon_M)).$$

The solution to Eq. (7) is given by [51):

$$f(k, t) = \int_{t_0}^{t} \frac{d f_0(\varepsilon_M(k(s)))}{\tau_S(k(s))} \exp\left(-\int_{s}^{t} \frac{d t'}{\tau_S(k(t'))}\right).$$

In the above $f(k_0, t_0)$ corresponds to the initial conditions. The components of the current density are given by:

$$J_\alpha(t) = -e \int \frac{d^2 k}{(2\pi)^d} (\frac{d f}{d t}) f(k, t),$$

where the volume $V$ of the unit cell of the lattice is set to unity and $d = 2, 3$ is the dimensionality of the system. If the initial time is set to $t_0 \to -\infty$ then:

$$f(k, t) = \int_{-\infty}^{t} ds \frac{f_0(\varepsilon_M(k(s)))}{\tau_S(k(s))} \exp\left(-\int_{s}^{t} \frac{d t'}{\tau_S(k(t'))}\right).$$

and we note if we further linearize Eq. (10) we recover the linear response equations.

## III. GENERAL RESULTS

We first present here the general formulae, while the derivations are left for the Appendix [2]. The 2D and 3D cases are treated separately, due to the differences in the equations of motion Eq. (2), when the Berry curvature is taken into account. As stressed in the introduction, the formulae are exact within the accuracy of the leading order equations of motion in the fields $E$ and $B$.

### A. Three-dimensional case

To find $k(t)$, the equation of motion is given by Eq. (2). In the limit where $E(t)$ is small, the last term of Eq. (2) can be taken as a perturbation. In that spirit let us denote by $k_0(t)$ the solution to the equation:

$$\frac{d k_0}{d t} = -e D^{-1}(k_0) \nabla k \varepsilon_M(k(t)) \times B|_{k=k_0}$$

and write: $E(t) = \lambda E(t)$ with $\lambda = 1$. Then Eq. (2) becomes analytically dependent on the parameter $\lambda$ and its solutions can be written as an analytic asymptotic series of the form [52–54]:

$$k(t) = k_0(t) + \lambda k_1(t) + \lambda^2 k_2(t) + ...$$

with $k_1(t_0) = k_2(t_0) = k_3(t_0) = ... = 0$. These expansions and the method below are valid for any accuracy of the equations of motion. To order $\lambda$:

$$\frac{d k_1}{d t} = -e \sum_\beta k_{1\beta}(t) \left[ \frac{\partial}{\partial k}\left( D^{-1}(k) \nabla k \varepsilon_M(k(t)) \times B \right) \right]_{k=k_0} \tag{13}$$

$$e D^{-1}(k_0(t)) [E(t) + (\mathbf{B} \cdot \mathbf{E}(t)) \Omega(k_0(t))].$$

Taking into account that $\frac{d e(k(t))}{d t} = \nabla k \varepsilon(k(t)) \cdot \frac{d k}{d t}$, we can also write perturbatively:

$$\varepsilon_M(t) = \varepsilon_0(t) + \lambda \varepsilon_1(t) + \lambda^2 \varepsilon_2(t) + ...$$

with $\varepsilon_0(t_0) = \varepsilon_M(t_0)$ and $\varepsilon_1(t_0) = \varepsilon_2(t_0) = \varepsilon_3(t_0) = ... = 0$. Again this is valid for any accuracy of the equations of motion. Then to order $\lambda$ and for $t > t_0$:

$$\varepsilon_1(t) = e \int_{t_0}^{t} ds D^{-1}(k_0(s)) \times [\nabla k \varepsilon_M(k(s)) \cdot (\mathbf{E}(s) + (\mathbf{B} \cdot \mathbf{E}(s)) \Omega(k(s))]_{k=k_0}$$

and to order $\lambda^2$

$$\varepsilon_2(t) = e \int_{t_0}^{t} ds \sum_{\alpha \beta} E_{\beta}(s) k_{1\alpha}(s) \frac{\partial}{\partial k_{\alpha}} \left[ D^{-1}(k_0(s)) \right]$$

$$\times \left[ \frac{\partial}{\partial k_{\beta}} \varepsilon_M(k(s)) + e B_{\beta}(\nabla k \varepsilon_M(k(s)) \cdot \Omega(k(s))) \right]_{k=k_0}$$

We introduce for convenience the quantities:

$$u_0(k) \equiv D^{-1}(k) \nabla k \varepsilon_M(k) + e (\Omega(k) \cdot \nabla k \varepsilon_M(k)) B$$

$$\eta(t; t_0) \equiv \exp \left(-\int_{t_0}^{t} \frac{d s}{\tau_S(k_0(s))}\right)$$

As a result, we obtain the current to linear order in $E$:...
where the first term generalizes the Streda formula [36], while the second term generalizes the Chambers formula [24]. Next, to order $E^2$ we obtain:

\[ J^{(2)}_{\alpha}(t) = -e\int \frac{d^3k}{(2\pi)^3} D(k) u_{0\alpha}(k) \times \]
\[ \left[ e \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \sum_{\gamma} \sum_{\gamma} k_{1\gamma}(t') \frac{\partial}{\partial k_{\gamma}} \left[ D^{-1}(k_0(t')) \nabla_{k}(E'(t')) \cdot \left[ E(t') + e(\mathbf{B} \cdot E(t')) \Omega(k_0(t')) \right] \right] \eta(t';t') \right. \]
\[ + e^2 \left. \frac{\partial^2 f_0(\varepsilon(k_0(t)))}{\partial \varepsilon^2} \int_{-\infty}^{t} dt' \sum_{\gamma} k_{1\gamma}(t') \nabla_{k}(E'(t')) \cdot \left[ E(t') + e(\mathbf{B} \cdot E(t')) \Omega(k_0(t')) \right] \eta(t';t') \right. \]
\[ \times \left. \int_{t'}^{t} dt'' \sum_{\gamma} k_{1\gamma}(t'') \nabla_{k}(E'(t'')) \cdot \left[ E(t') + e(\mathbf{B} \cdot E(t')) \Omega(k_0(t')) \right] \eta(t';t'') \right. \]
\[ - e^3 \int \frac{d^3k}{(2\pi)^3} [E(t) \times \Omega(k)]_\alpha f_0(\varepsilon_M(k(t))) \int_{-\infty}^{t} dt' D^{-1}(k_0(t')) \nabla_{k}(E'(t')) \cdot \left[ E(t') + e(\mathbf{B} \cdot E(t')) \Omega(k_0(t')) \right] \eta(t';t') \]
\[ (16) \]

The first term is a shift of the Fermi-Dirac distribution due to the electric field, the second term is the quadratic shift, the third term is a novel one that corresponds to a non-constant relaxation time and the last term generalizes the Berry dipole introduced in [40, 41] to potentially any magnetic field and electric fields with arbitrary time dependence. As it is explained in Appendix D, it is important in order convergence to be guaranteed, that the eigenvalues of the matrix:

\[ M_{\alpha\beta}(s) = -e \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} B_{\gamma} \frac{\partial}{\partial k_{\gamma}} \left[ D^{-1}(k_0(s)) \right] \left[ \frac{\partial}{\partial k_{\gamma}} \varepsilon_M(k_0(s)) \right] \]

is smaller than $\frac{1}{\tau_S(k)}$.

B. Two-dimensional case

We can now reduce all previous results to 2D. The equation of motion reads:

\[ \frac{dk}{dt} = -D^{-1}(k) (eE(t) + e\nabla_{k}\varepsilon_M(k) \times \mathbf{B}) \quad (17) \]

If we introduce for convenience the notation:

\[ u_{02D}(k) = D^{-1}(k) \nabla_{k}\varepsilon_M(k), \quad (18) \]

we arrive at a similar expression for the current as Eq. (15) and (16) with the only differences that $u_0$ is replaced by $u_{02D}$, the integration over all $k$’s is now a 2D integration and $\mathbf{B} \cdot E(t) \rightarrow 0$. The result for the linear response is:

\[ J^{(1)}_{\alpha}(t) = -e^2 \int \frac{d^2k}{(2\pi)^2} [E(t) \times \Omega(k)]_\alpha f_0(\varepsilon_M(k(t))) \int_{-\infty}^{t} dt' D(k) u_{02D}(k) \frac{\partial f_0(\varepsilon_M(k_0(t)))}{\partial \varepsilon} \times \]
\[ \times \int_{-\infty}^{t} D^{-1}(k_0(t')) \nabla_{k}(E'(t')) \cdot \left[ E(t') + e(\mathbf{B} \cdot E(t')) \Omega(k_0(t')) \right] \eta(t';t') dt' \quad (19) \]

For completeness we include the formula for the nonlinear response in 2D in the Appendix (B3) and we proceed to the scaling analysis below.

1. Scaling analysis in 2D

From the expression of the current we obtain the expression of the conductivity which is divided into a part that we call topological and a regular part.
\[
\sigma_{\alpha\beta}(e, \tau_S, T, B, \omega, \epsilon_M) = -e^2 \int \frac{d^2 k}{(2\pi)^2} \epsilon_{\alpha\beta} \Omega(k) f_{0T}(\epsilon_M(k)) - e \int \frac{d^2 k}{(2\pi)^2} \nabla_{k_\alpha} \epsilon_M(k) \frac{\partial f_{0T}(\epsilon_M(k_0(t)))}{\partial \epsilon} \times \\
\times \int_{-\infty}^{t} dt' D^{-1}(k_0(t)) \nabla_{k_\alpha} \epsilon_M(k_0(t')) \cdot \exp(i\omega(t' - t)) \exp(-(t - t')/\tau_S)
\]

with \(\sigma_{\alpha\beta}^{reg}(e, \tau_S, T, B, \omega, \epsilon_M) = -e^2 \int \frac{d^2 k}{(2\pi)^2} \epsilon_{\alpha\beta} \Omega(k) f_{0T}(\epsilon_M(k)) \) and

\[
\sigma_{\alpha\beta}^{reg}(e, \tau_S, T, B, \omega, \epsilon_M) = -e \int \frac{d^2 k}{(2\pi)^2} \nabla_{k_\alpha} \epsilon_M(k) \frac{\partial f_{0T}(\epsilon_M(k_0(t)))}{\partial \epsilon} \times \\
\times \int_{-\infty}^{t} dt' D^{-1}(k_0(t')) \nabla_{k_\alpha} \epsilon_M(k_0(t')) \cdot \exp(i\omega(t' - t)) \exp(-(t - t')/\tau_S)
\]

Where we assume a constant relaxation time \(\tau_S\). By introducing \(B = eB, \tau = (t' - t)/\tau_S, \epsilon_M = \tau_S \epsilon_M, \omega = \omega \tau_S\) and taking into account the equations of motion, we obtain for the regular part of the conductivity:

\[
\sigma_{\alpha\beta}^{reg}(e, \tau_S, T, B, \omega, \epsilon_M) = -e\tau_S^{-1} \int \frac{d^2 k}{(2\pi)^2} \nabla_{k_\alpha} E_M(k) \frac{\partial f_{0T}(E_M(k))}{\partial E} \times \\
\times \int_{-\infty}^{0} D^{-1}(k_0(\bar{t})) \nabla_{k_\alpha} E_M(k_0(\bar{t})) \cdot \exp(i\omega \bar{t}) \exp(\bar{t}) \, d\bar{t} = e\tau_S^{-1} \sigma_{\alpha\beta}^{reg}(1, 1, T \tau_S, eB, \omega \tau_S, \tau_S \epsilon_M)
\]

where in the above equation, we have used the notation:

\[
\frac{\partial f_{0T}(\epsilon_M(k))}{\partial \epsilon} = \frac{\partial}{\partial \epsilon_M} \left( \frac{1}{1 + \exp(\beta \epsilon_M)} \right) = \frac{\partial}{\partial \epsilon_M} \left( \frac{1}{1 + \exp(\beta \epsilon_M/\tau_S)} \right) = \frac{\partial f_{0T}(E_M(k))}{\partial E}
\]

This scaling relation will be useful for the numerical calculations in the following sections. In addition, as \(\sigma_{\alpha\beta}^{reg}(1, 1, T \tau_S, eB, \omega \tau_S, \tau_S \epsilon_M)\) is a power series in \(eB\), it is impossible to completely disentangle all the powers of \(e\) that enter the expression for the conductivity.

### IV. Lifshitz Transition of Neck Formation/Collapse on Rectangular Lattice

We use the frequency dependent Chambers formula to calculate the components of the conductivity tensor in the case of an energy dispersion of the form

\[
\epsilon(k) = -2t_x \cos(k_x) - 2t_y \cos(k_y) \quad (t_y > t_x)
\]

We define \(\mu_e = 2(t_y - t_x)\) and \(\mu_0 = 2(t_y + t_x)\). For this band we’ve got the following topologies:

\[
\begin{align*}
-\mu_0 < \mu < -\mu_e & \quad \text{electron pockets} \\
-\mu_e < \mu < \mu_e & \quad \text{open Fermi surface} \\
\mu_e < \mu < \mu_0 & \quad \text{hole pockets}
\end{align*}
\]

The components of \(\sigma\) for all three cases read (the details of all calculations in this section are left in Appendix C1):

\[
\sigma_{xx}^i = \frac{2\sigma_0}{K} \sum_n \frac{[1 + i\omega \tau_S] \text{sech}^2 \left( \frac{\pi \kappa'}{2K} \right)}{[1 + i\omega \tau_S]^2 + [\omega_c \tau_S]^2} \cdot \sin \left( \frac{\pi \mu_n}{2K} \right)
\]

\[
\sigma_{yy}^i = \frac{\sigma_0 \delta_{i,\alpha}}{K} \frac{1}{1 + i\omega \tau_S} + \\
+ \frac{2\sigma_0}{K} \sum_n \frac{[1 + i\omega \tau_S] \text{sech}^2 \left( \frac{\pi \kappa'}{2K} \right) \cos^2 \left( \frac{\pi \mu_n}{2K} \right)}{[1 + i\omega \tau_S]^2 + [\omega_c \tau_S]^2}
\]

\[
\sigma_{xy}^i = \left( \delta_{i,\alpha} + \delta_{i,e} - \delta_{i,h} \right) \frac{\sigma_0}{K} \times \\
\times \sum_n \frac{[\omega_c \tau_S] \text{sech}^2 \left( \frac{\pi \kappa'}{2K} \right)}{[1 + i\omega \tau_S]^2 + [\omega_c \tau_S]^2} \cdot \sin \left( \frac{\pi \mu_n}{2K} \right)
\]

where \(h\) stands for hole orbits, \(e\) stands for electron orbits and \(o\) stands for open orbits. Furthermore, we have defined that \(\kappa = \sqrt{\mu_e^2 - \mu_0^2}, \omega_0 = eB \sqrt{4t_x t_y}, m_0 = \frac{1}{\sqrt{4t_x t_y}}\), \(\sigma_0 = e^2 \tau_S \sqrt{4t_x t_y}\) and \(K(\kappa)\) is the complete elliptic integral of the first kind. In the case of closed surfaces \(K \equiv K(\kappa), K' \equiv K(\sqrt{1 - \kappa^2})\). For open surfaces we
substitute $K(\kappa) \rightarrow \frac{1}{\kappa} K(1/\kappa)$ and similarly for $K'$. In addition

$$\omega_c = \begin{cases} \frac{\pi \omega_h}{2K(\kappa)} & \text{closed orbits} \\ \frac{\pi \kappa \omega_0}{2K(1/\kappa)} & \text{open trajectories} \end{cases}$$  \hfill (29)

Finally $u_i$ and $u_o$ are defined via Jacobean elliptic functions

$$sn(u_c, \kappa) = \sqrt{\frac{\mu_0 - \mu_c}{\mu_0 - \mu}}$$  \hfill (30)
$$sn(u_h, \kappa) = \sqrt{\frac{\mu_0 - \mu_c}{\mu_0 + \mu}}$$  \hfill (31)
$$sn(\kappa u_o, 1/\kappa) = \sqrt{\frac{\mu_0 + \mu}{\mu_0 + \mu_c}}$$  \hfill (32)

For closed Fermi surfaces the sums are over positive odd integers while for open surfaces the sums are over positive even integers.

The calculation of the Hall number for both high and low magnetic fields leads to expressions that are independent of the frequency and similar to the ones calculated in \[54\]. For completeness we present the formulae in the Appendix C1. In Figs (1) and (2) we present the results for two limiting values of $\omega_c \tau_S = 0.01 \ll 1$ and for $\approx 1$. For each of these values several different frequencies have been chosen. The magnetic field is $B = 0.01$ in units of $\hbar c/(a^2 e)$ where $a$ is the lattice constant (for simplicity we work with $a = h = c = 1$). Consistently with the fact that at $\mu/t_y = 0.25$, there is a regular (logarithmic) Van Hove singularity (in the language of Lifshitz transitions it is classified as a neck formation/collapse), the real part of the Hall coefficient changes sign from electron to hole-like. At values of $\mu/t_y$ close to $\pm 1$ the discontinuities signal the change from closed pockets to open Fermi surface. The real part do not show any significant frequency dependence. On the contrary, the imaginary part displays a strong frequency dependence with all features (change of sign at the Lifshitz transition and discontinuity at close-open Fermi surface transition) becoming more pronounced with increasing $\omega$. Away from these special values of $\mu/t_y$, the imaginary part of the Hall number is practically zero, reflecting that the dispersion relation is parabolic. Close to the Lifshitz transition quantum tunnelling must be taken into account for full quantitative analysis \[42\ \[43\].

V. SUPERMETAL: A SIMPLE MODEL FOR HIGHLY DOPED GRAPHENE

A. A relevant tight binding model

We will consider a Hamiltonian on hexagonal lattice which is relevant to recent studies of graphene that showed the emergence of a higher order Van Hove singularity \[20\ \[21\]. By considering third nearest neighbors tunnelling the simple Hamiltonian that generates higher order Van Hove singularities, as a consequence of FSTTs, reads:

$$H = \begin{pmatrix} 0 & f(k) \\ f^*(k) & 0 \end{pmatrix}$$  \hfill (33)

Where

$$f(k) = -t \left[ e^{-ik_y} + e^{i(\sqrt{3}k_x + k_y)} + e^{i(-\sqrt{3}k_x + k_y)} \right]$$
$$- e^{2ik_y} + e^{-i(\sqrt{3}k_x + k_y)} + e^{-i(-\sqrt{3}k_x + k_y)}$$  \hfill (34)

and energy dispersion $E_{\pm}(k) = \pm \sqrt{f^*(k)f(k)}$.

This leads to a higher order Van Hove saddle in both $E_{\pm}(k)$ at wave vector $G = \frac{2\pi}{a}(0,1)$ (for lattice constant $a = 1$) for $c = \frac{1}{2}$ while for $c = \frac{1}{3}$ it is a nodal point.
FIG. 2. Same as in Fig. 1 the real and imaginary part of the Hall number for four different frequencies and now for $\omega_c \tau_S = 1$. As before, the results are valid away from the Lifshitz transition.

FIG. 3. Dispersion for graphene with third nearest neighbors for $c/t = 1/4$ where the higher order Van Hove singularity (cusp, on the left) can be visualized at $G = \frac{2\pi}{3}(0,1)$. For $c/t = 1/3$ (on the right) it is a nodal point with ill-defined gradient. For the plots we take $t = 30$.

with closure of gap and ill defined gradient. It is rather instructive to present the contour plots of $E_+$ for both cases in Figure 3. The higher order saddle for $c = \frac{4}{3}$ is a cusp at $G = \frac{2\pi}{3}(0,1)$, based on the catastrophe theory classification [5].

FIG. 4. Real and imaginary part of Hall coefficient for $\omega_c \tau_S = 0.3$.

FIG. 5. Real and imaginary part of Hall conductivity for $\omega_c \tau_S = 0.3$ for the simple model of highly doped graphene. On the left figure, the Fermi surfaces in different regimes have been depicted, so the topological transition is evident. The shaded areas, around FSTTs, denote where magnetic breakdown phenomena should be taken into account for a full quantitative analysis.
B. Numerical results

The numerical results for the Hall coefficient as well as the conductivity components are presented in Figs (4-9). From the scaling formula Eq. (22) $T \tau_S \rightarrow 0$ and we take $eB \rightarrow 0.01$ for the calculations. In this case $\omega_c$ is defined as $\omega_c = \frac{eB}{m^* t}$ with the effective mass $m^* \propto t^{-1}$. We have chosen again two values of $\omega_c \tau_S$ (0.3 and 1) and for each one three frequencies $\omega$ for the time-dependent electric field. For $c/t=1/4$, the higher order Van Hove singularity is at $\mu/t = 0.25$. This explains the sharp features of the conductivities and the change of sign of the Hall coefficient at that value. In Fig. (5a) the Fermi surface topology is depicted across both sides of the discontinuous behavior. The maximum of the real part of the conductivity takes place when the area of the Fermi surface is largest. The frequency dependence is pronounced in the imaginary part of the Hall coefficient $R_H$ and around the value of $\mu/t = 0.25$. Away from $\mu/t = 0.25$ the imaginary part of $R_H$ is zero, while for the real part of $R_H$ the behavior is linear in $\mu/t$. This result is very well explained by the fact that away from the FSTT, the dispersion relation is very well approximated by a parabolic one with precisely this behavior as the imaginary part of $\rho_{xy}$ is 0 and for the real part $\rho_{xy} \propto$
B/(ue), where u is the density of electrons which is linear in \( \mu \). As a result \( R_{H} \) exhibits that behavior for \( \mu/t > 1 \). As the frequency is increased the region where there is appreciable effect of the singularity is larger.

VI. HALL CONDUCTIVITY OF THE HALDANE MODEL

We consider the Haldane Hamiltonian on a honeycomb lattice with \( H(\theta) = d_{0}(\theta) + d(\theta) \cdot \sigma \) where:

\[
\begin{align*}
    d_{0}(\theta) &= -2t_{2} \cos \theta_{1} \cos \theta_{2} + \cos(\theta_{1} + \theta_{2}) \cos \phi \\
    d_{x}(\theta) &= -t_{1}(1 + \cos \theta_{1} + \cos \theta_{2}) \\
    d_{y}(\theta) &= t_{1}(\sin \theta_{1} - \sin \theta_{2}) \\
    d_{z}(\theta) &= m - 2t_{2} \sin \theta_{1} \sin \theta_{2} - \sin(\theta_{1} + \theta_{2}) \sin \phi
\end{align*}
\]

and \( \sigma \) denotes the Pauli matrices. The energy eigenvalues are given by \( E_{\pm} = d_{0} \pm |d_{1}| \). For simplicity, we choose \( \phi = \pi/2 \), \( m = 0 \) and consider \( E_{+} \) for \( t_{1} = 50 \) and \( t_{2} = 25 \) with a nontrivial Chern number of -1. The energy contours of the model is presented in Fig. (10). We numerically calculate the Hall conductivity and present the results in Figs (11) - (14). The real part of the Hall conductivity shows different behavior in different regimes of the chemical potential reflecting the different Fermi surface topology. This topology of the Fermi surface is shown in the insets of Fig. (2), in each segment where the behavior is different. In the value of the total Hall conductivity a constant \( -\frac{\pi}{2} \) has been added in Fig. (2), due to the contribution of the lower band. The results are for \( \omega_{c} \tau_{S} = 0.3 \), where \( \omega_{c} = \frac{eB}{2m^{*}} \) and the effective mass \( m^{*} \propto t_{1}^{-1} \) and \( eB = 0.01 \). The contributions from the different terms of Eq. (19) are presented separately and the significant role of the second term of Eq. (19) is clearly demonstrated.

VII. CONCLUSION

In this work a generalization of the celebrated Chambers formula has been introduced, relevant to time dependent electric fields and bands with Berry curvature. The nonlinear conductivity, to order \( E^{2} \) in the electric field has been also computed within the leading order equations of motion method. These general formulae have been used to study a number of examples. In particular, we studied bands where by changing the chemical potential a range of FSTTs become available. These FSTTs lead to Van Hove singularities at the Fermi surface where the high density of states but our work is valid away from the regions close to the topological transitions to avoid considering quantum Hall or out-of-equilibrium effects. Due to the change of the Fermi surface from hole like to electron like there is a jump in the Hall coefficient of the material at the Fermi surface topological transition, while a wealth of different features and signatures appear both in real and imaginary parts of the Hall coefficient and conductivities, especially pronounced at higher frequencies. Different types of FSTTs provide their signatures on the conductivities. Furthermore, the Hall conductivity of the Haldane model has been studied.
The main assumption of the general part of our work is that the system is a Fermi liquid and the effects of the interactions can be included in the lifetime and the effective mass. There is recent seminal work that takes into account strong interactions [56] and presents a formula that does not require the existence of quasiparticles. We leave as a future work how to bridge the two approaches. Given the enormous interest in the field, we believe that this work will stimulate significant future activity associated with applications of the extended, formally exact Chambers formulae presented here. This work is based on simplified semiclassical equations of motion that are commonly used but are only strictly valid to leading order in the electric and magnetic fields. As it has been already emphasized, close to Van Hove singularities, magnetic breakdown effects (quantum tunnelling), must be taken into account [42, 43]. These will be considered elsewhere.

Acknowledgments. We would like to thank Claudio Chamon, Anirudh Chandrasekaran, Mark Greenaway and Xenophon Zotos for useful discussions. The work has been supported by the EPSRC grants EP/P002811/1 and EP/T034351/1 (JJB). DVE was supported by the RSF-DFG grant 405940956.

Appendix A: Chambers formula

1. Boltzmann Equation

Following the Chambers original work [24] we write the current as:

\[ J_\alpha(t) = -e \int \frac{d^3k}{(2\pi)^3} \frac{d\alpha}{dt}(k) f(k,t) \]  

(A1)

with \( f(k,t) \) being the solution to the Boltzmann equation:

\[ \frac{\partial f(k,t)}{\partial t} + \frac{d}{dt} \frac{\partial f(k,t)}{\partial k} = \frac{1}{\tau_S(k)} (f_0(\varepsilon_M(k)) - f(k,t)) \]  

(A2)
In addition:

\[
\frac{d\mathbf{k}}{dt} = -e \nabla_k \varepsilon (\mathbf{k}) \times \mathbf{B} + \mathbf{E} (t) \quad (A3)
\]

The solution as is well known, is given by:

\[
\lambda
\]

We obtain:

\[
\text{to Eq. (A3) can be written as:}
\]

2. Momentum as a function of time

In the limit where \( \mathbf{E} (t) \) is small the second term on the right hand side of Eq. \( (A3) \) may be viewed as a perturbation. In that spirit let us denote by \( \mathbf{k}_0 (t) \) the solution to the equation:

\[
\frac{d\mathbf{k}_0 (t)}{dt} = -e \nabla_k \varepsilon (\mathbf{k}_0 (t)) \times \mathbf{B} \quad (A6)
\]

If we write: \( \mathbf{E} (t) = \lambda \mathbf{E} (t) \) with \( \lambda = 1 \), then the solution to Eq. \( (A3) \) can be written as:

\[
\mathbf{k} (t) = \mathbf{k}_0 (t) + \lambda \mathbf{k}_1 (t) + \lambda^2 \mathbf{k}_2 (t) + \ldots \quad (A7)
\]

with \( \mathbf{k}_1 (t_0) = \mathbf{k}_2 (t_0) = \mathbf{k}_3 (t_0) = \ldots = 0 \). Then, to order \( \lambda \) we obtain:

\[
\frac{d\mathbf{k}_1 (t)}{dt} = -e \left[ \sum_{\alpha} k_{1\alpha} (t) \frac{\partial}{\partial k_{\alpha}} \nabla_k \varepsilon (\mathbf{k}_0 (t)) \right] \times \mathbf{B} - e \mathbf{E} (t) \quad (A8)
\]

In addition:

\[
\frac{d\varepsilon (\mathbf{k} (t))}{dt} = \nabla_k \varepsilon (\mathbf{k} (t)) \cdot \frac{d\mathbf{k}}{dt} = -e \nabla_k \varepsilon (\mathbf{k} (t)) \cdot [\nabla_k \varepsilon (\mathbf{k}) \times \mathbf{B} + \mathbf{E} (t)]
\]

\[
= -e \nabla_k \varepsilon (\mathbf{k} (t)) \cdot \mathbf{E} (t) \quad (A9)
\]

We can now similarly expand in powers of \( \lambda \):

\[
\varepsilon_M (t) = \varepsilon_0 (t) + \lambda \varepsilon_1 (t) + \lambda^2 \varepsilon_2 (t) + \ldots \quad (A10)
\]

with \( \varepsilon_1 (t_0) = \varepsilon_2 (t_0) = \varepsilon_3 (t_0) = \ldots = 0 \). To order \( \lambda \) for \( t > t_0 \):

\[
\varepsilon_1 (t) = e \int_{t_0}^{t} \nabla_k \varepsilon (\mathbf{k}_0 (s)) \cdot \mathbf{E} (s) ds \quad (A11)
\]

and

\[
\varepsilon_2 (t) = e \sum_{\alpha\beta} \int_{t_0}^{t} E_\alpha (s) k_{1\alpha} (s) \frac{\partial^2}{\partial k_{\alpha} \partial k_{\beta}} \varepsilon (\mathbf{k}_0 (s)) ds
\]

3. Calculation of the current in linear response in 3D

We first note the identity:

\[
\int_{-\infty}^{t} \frac{dt'}{\tau_S (k (t'))} \exp \left( - \int_{t'}^{t} \frac{ds}{\tau_S (k (s))} \right) = 1, \quad (A13)
\]

therefore there is no need to expand the term \( \int_{-\infty}^{t} \frac{dt'}{\tau_S (k (t'))} \exp \left( - \int_{t'}^{t} \frac{ds}{\tau_S (k (s))} \right) \) in Eq. \( (A5) \). As a result we obtain:

\[
f (\mathbf{k}, t) = f_0 (\varepsilon (\mathbf{k}_0 (t))) + \frac{\partial f_0 (\varepsilon (\mathbf{k}_0 (t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1 (t')}{\tau_S (\mathbf{k}_0 (t'))} \eta (t; t')
\]

Noticing that the current is written as:

\[
J_\alpha (t) = \frac{\partial f_0 (\varepsilon (\mathbf{k}_0 (t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1 (t')}{\tau_S (\mathbf{k}_0 (t'))} \eta (t; t')
\]

we obtain the result to first order in the electric field (linear response), all terms \( \sim E^2 \) are neglected and a term is dropped due to the relation:

\[
- e \int_{(2\pi)^3} \nabla_k \varepsilon (\mathbf{k}) f_0 (\varepsilon (\mathbf{k}_0 (t))) = 0
\]

as there is no current without an electric field. Then:

\[
J_\alpha (t) = -e \int \frac{d^3 k}{(2\pi)^3} \nabla_k \varepsilon (\mathbf{k}) \cdot \frac{\partial f_0 (\varepsilon (\mathbf{k}_0 (t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1 (t')}{\tau_S (\mathbf{k}_0 (t'))} \eta (t; t')
\]

The first term in the bracket in Eq. \( (A15) \) is what Chambers calculated and it is given by \( \mathbf{24, 25} \), we can integrate it by parts using \( \frac{d}{dt} \varepsilon_1 (t) = -e \nabla_k \varepsilon (\mathbf{k}_0 (t)) \cdot \mathbf{E} (t) \) to obtain the final expression:

\[
J_\alpha (t) = -e^2 \int \frac{d^3 k}{(2\pi)^3} \nabla_k \varepsilon (\mathbf{k}) \frac{\partial f_0 (\varepsilon (\mathbf{k}_0 (t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \nabla_k \varepsilon (\mathbf{k}_0 (t')) \cdot \mathbf{E} (t) \eta (t; t')
\]
4. Calculation of nonlinear current to order $E^2$

To proceed with the next order term, we note that:

$$f(k, t) \simeq \int_{-\infty}^{t} \frac{d}{dt} f_0 \left( \left[ \xi_0 + \xi_1 + \xi_2 \right] (t') \right) dt'$$

$$\simeq \int_{-\infty}^{t} \frac{d}{dt} f_0 \left( \xi_0(k_0(t')) \right) dt' + \frac{\partial f_0 \left( \xi_0(k_0(t)) \right)}{\partial \xi} \int_{-\infty}^{t} \xi_1(t') dt'$$

$$+ \frac{1}{2} \frac{\partial^2 f_0 \left( \xi_0(k_0(t)) \right)}{\partial \xi^2} \int_{-\infty}^{t} \xi_2(t') dt'$$

$$+ \frac{\partial f_0 \left( \xi_0(k_0(t)) \right)}{\partial \xi} \int_{-\infty}^{t} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} \left( \frac{\alpha_0 \xi_0 \xi_2}{\tau_S (k_0(t'))} \right) \eta(t'; t') dt'$$

where Eq. (A13) has been used. Integrating the above expression by parts and using the formula for the current:

$$J_{\alpha}^{(2)} (t) = -e^2 \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi} \cdot \mathbf{E}(t') \eta(t'; t') \right) dt'$$

$$- e \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \frac{\xi_2(t')}{\tau_S (k_0(t'))} \eta(t'; t') dt'$$

$$- \frac{1}{2} e \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \frac{\xi_2(t')}{\tau_S (k_0(t'))} \eta(t'; t') dt'$$

finally, after another integration by parts, we obtain the expression of the non-linear current:

$$J_{\alpha}^{(2)} (t) = -e^2 \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi} \cdot \mathbf{E}(t') \eta(t'; t') \right) dt'$$

$$- e^2 \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \frac{\partial^2 f_0 \left( \xi_0(k_0(t)) \right)}{\partial \xi^2} \eta(t'; t') dt'$$

$$- \frac{1}{2} e \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \frac{\partial^2 f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi^2} \eta(t'; t') dt'$$

$$- e^2 \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi} \cdot \mathbf{E}(t') \eta(t'; t') \right) dt'$$

$$- \frac{1}{2} e \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \frac{\partial^2 f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi^2} \eta(t'; t') dt'$$

$$- e^2 \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi} \cdot \mathbf{E}(t') \eta(t'; t') \right) dt'$$

$$- \frac{1}{2} e \int \frac{d^3k}{(2\pi)^3} \nabla_{\xi_0} \left( \frac{\partial f_0 \left( \xi_0(k) \right)}{\partial \xi} \right) \int_{-\infty}^{t} dt' \frac{\partial^2 f_0 \left( \xi_0(k_0(t')) \right)}{\partial \xi^2} \eta(t'; t') dt'$$

Appendix B: Inclusion of Berry curvature

1. Calculation of the current in 3D in linear response

The distribution function is written as:

$$f(k, t) \simeq \int_{-\infty}^{t} \frac{dt}{\tau_S (k_0(t))} f_0 (\xi (k_0(t))) \eta(t; t') + \frac{\partial f_0 (\xi (k_0(t)))}{\partial \xi} \int_{-\infty}^{t} \frac{dt'}{\tau_S (k_0(t'))} \eta(t; t')$$

Substituting Eq. (B1) into Eq. (10) we get that the current is:

$$J_\alpha (t) = -e \int \frac{d^3k}{(2\pi)^3} \frac{dr_\alpha}{dt} \frac{D(k)}{\tau_S (k_0(t))} f(k, t)$$

$$= -e \int \frac{d^3k}{(2\pi)^3} \left[ u_{\alpha k} (k) + D^{-1} (k) [\xi_0 (k) \times \Omega (k)] \right] \times$$

$$\left[ f_0 (\xi (k_0(t))) \int_{-\infty}^{t} \frac{dt'}{\tau_S (k_0(t'))} \eta(t; t') + \frac{\partial f_0 (\xi (k_0(t)))}{\partial \xi} \int_{-\infty}^{t} \frac{dt'}{\tau_S (k_0(t'))} \eta(t; t') \right]$$

Following similar steps as before to obtain the linear response, we get Eq. (15) of the main text.
2. Calculation of nonlinear current to order \( E^2 \) in 3D

In this case, the distribution function reads:

\[
\begin{align*}
 f(k, t) &\approx \int_{-\infty}^{t} \frac{f_0(\varepsilon(k_0(t)))}{\tau_S(k_0(t'))} \eta(t'; t) dt' + \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} \varepsilon_1(t') + \varepsilon_2(t') \frac{\eta(t'; t')}{\tau_S(k_0(t'))} dt' \\
 &+ \frac{1}{2} \frac{\partial^2 f_0(\varepsilon(k_0(t)))}{\partial \varepsilon^2} \int_{-\infty}^{t} \frac{\varepsilon_1^2(t')}{\tau_S(k_0(t'))} \eta(t'; t') dt' \\
 &- \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\eta(t'; t)} \int_{t'}^{t} \frac{\nabla_k \tau_S(k_0(l)) \cdot k_1(l)}{\tau_S^2(k_0(l))} dl 
\end{align*}
\]

(B2)

Then, we obtain for the current:

\[
J(\alpha(t) = -e \int \frac{d^3k}{(2\pi)^3} D(k) \left[ u_{\alpha 0}(k) + D^{-1}(k) \varepsilon [E(t) \times \Omega(k)]_\alpha \right] \times \\
\left[ \int_{-\infty}^{t} dt' \frac{f_0(\varepsilon(k_0(t)))}{\tau_S(k_0(t'))} \exp \left( -\int_{t'}^{t} \frac{ds}{\tau_S(k_0(s))} \right) + \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1(t') + \varepsilon_2(t')}{\tau_S(k_0(t'))} \eta(t'; t') \\
+ \frac{1}{2} \frac{\partial^2 f_0(\varepsilon(k_0(t)))}{\partial \varepsilon^2} \int_{-\infty}^{t} \frac{\varepsilon_1^2(t')}{\tau_S(k_0(t'))} \eta(t'; t') dt' - \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \frac{\varepsilon_1(t')}{\eta(t'; t)} \int_{t'}^{t} \frac{\nabla_k \tau_S(k_0(l)) \cdot k_1(l)}{\tau_S^2(k_0(l))} dl \right] 
\]

Focusing on terms \( \propto E^2 \) and using partial integration and some simplifications, finally Eq. (10) of the main text is obtained.

3. Calculation of the current in 2D

Having introduced \( u_{0 2D} \) in the main text and following an identical derivation to Section B2, the results in 2D for the nonlinear response reads:

\[
J(\alpha)_{2D}^{(2)}(t) = -e^2 \int \frac{d^2k}{(2\pi)^2} D(k) u_{0 2D}(k) \left[ \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' \sum_{\beta \gamma} E_{\beta}(t) k_{1 \gamma}(t') \frac{\partial}{\partial k_{\beta}} \left[ D^{-1}(k_0(t')) \frac{\partial}{\partial k_{\beta}} \varepsilon(k_0(t')) \right] \eta(t'; t') \\
+ \frac{\partial^2 f_0(\varepsilon(k_0(t)))}{\partial \varepsilon^2} \int_{-\infty}^{t} dt' D^{-1}(k_0(t')) \nabla_k \varepsilon(k_0(t)) \cdot E(t') \eta(t'; t') \int_{t'}^{t} D^{-1}(k_0(l)) \nabla_k \varepsilon(k_0(l)) \cdot E(l) dl \\
+ \frac{\partial f_0(\varepsilon(k_0(t)))}{\partial \varepsilon} \int_{-\infty}^{t} dt' D^{-1}(k_0(t')) \nabla_k \varepsilon(k_0(t')) \cdot E(t') \eta(t'; t') \int_{t'}^{t} \frac{\nabla_k \tau_S(k_0(l)) \cdot k_1(l)}{\tau_S^2(k_0(l))} dl \\
- e^3 \varepsilon_{\alpha \beta} \int \frac{d^2k}{(2\pi)^2} E_{\beta}(t) \Omega(k) \cdot \frac{\partial f_0(\varepsilon(k))}{\partial \varepsilon} \int_{-\infty}^{t} dt' D^{-1}(k_0(t')) \nabla_k \varepsilon(k_0(t')) \cdot E(t') \eta(t'; t') \right] 
\]

(B3)

Appendix C: Hall coefficient calculations for rectangular lattice

In Ref. [55] the authors have solved Boltzmann transport equations for this energy dispersion analytically. Then they calculated the velocities \( u_x \) and \( u_y \) and took their Fourier series expansions which are shown below:

\[
u_{0x}^i(t) = (1 - 2\delta_{i, h}) \frac{2\pi}{m_0 K(\kappa)} \sum_{n=1}^{\infty} \text{sech} \left[ \frac{2(n-1) \pi K'}{2K(\kappa)} \right] \sin \left[ \frac{2(n-1) \pi u_d}{2K(\kappa)} \right] \sin \left[ \frac{2(n-1) \pi \omega dt}{2K(\kappa)} \right] 
\]

(C1)

\[
u_{0y}^i(t) = \frac{2\pi}{m_0 K(\kappa)} \sum_{n=1}^{\infty} \text{sech} \left[ \frac{2(n-1) \pi K'}{2K(\kappa)} \right] \cos \left[ \frac{2(n-1) \pi u_d}{2K(\kappa)} \right] \cos \left[ \frac{2(n-1) \pi \omega dt}{2K(\kappa)} \right] 
\]

(C2)
where \( i = e \) for electrons and \( i = h \) for holes. For open surfaces the corresponding results are:

\[
\begin{align*}
 u_{0x}^o (t) &= \frac{2\pi \kappa}{m_0 K (1/\kappa)} \sum_{n=1}^{\infty} \text{sech} \left[ \frac{n\pi K'}{K(1/\kappa)} \right] \sin \left[ \frac{n\pi u_{0x}}{K(1/\kappa)} \right] \sin \left[ \frac{n\pi \kappa \omega_0 t}{K(1/\kappa)} \right] \\
u_{0y}^o (t) &= \frac{2\pi \kappa}{m_0 K (1/\kappa)} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \text{sech} \left[ \frac{n\pi K'}{K(1/\kappa)} \right] \cos \left[ \frac{n\pi u_{0x}}{K(1/\kappa)} \right] \cos \left[ \frac{n\pi \kappa \omega_0 t}{K(1/\kappa)} \right] \right\}
\end{align*}
\]  

(C3)  

(C4)

where the definitions of \( m_0, \kappa, \omega_0, K \) and \( K' \) are given in the main text.

For electron and hole pockets we can write for simplicity:

\[
\begin{align*}
 u_{0x} (t) &= \tilde{u}_x \sum_{n=1}^{\infty} a_n^x \sin \left[ \frac{(2n-1)\pi \omega_0 t}{2K (\kappa)} \right] \\
u_{0y} (t) &= \tilde{u}_y \sum_{n=1}^{\infty} a_n^y \cos \left[ \frac{(2n-1)\pi \omega_0 t}{2K (\kappa)} \right]
\end{align*}
\]  

(C5)  

(C6)

Eq. (17) is also true for open surfaces. But in the case of \( u_y (t) \) we should write

\[
\begin{align*}
 u_x^o (t) &= \tilde{u}_x \sum_{n=1}^{\infty} a_n^x \sin \left[ \frac{n\pi \kappa \omega_0 t}{K(1/\kappa)} \right] \\
u_y^o (t) &= \tilde{u}_y \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} a_n^y \cos \left[ \frac{n\pi \kappa \omega_0 t}{K(1/\kappa)} \right] \right\}
\end{align*}
\]  

(C7)  

(C8)

Both for holes and electrons the results read for closed Fermi surfaces:

\[
\begin{align*}
\sigma_{xy} &= \frac{e^3 B}{(2\pi)^2} \int_0^{4K/\omega_0} \tilde{u}_x \sum_{n=1}^{\infty} a_n^x \sin \left[ \frac{(2n-1)\pi \omega_0 t}{2K (\kappa)} \right] \int_{-\infty}^{t} \tilde{u}_y \sum_{m=1}^{\infty} a_m^y \cos \left[ \frac{(2m-1)\pi \omega_0 t'}{2K (\kappa)} \right] \exp \left( -\frac{1}{\tau_S + i\omega} (t-t') \right) dt' \\
&= \frac{e^3 B}{4\pi} \tilde{u}_x \tilde{u}_y \sum_{n=1}^{\infty} a_n^x a_n^y \frac{2n-1}{\left[ 1/\tau_S + i\omega \right]^2 + \left[ (2n-1)\pi \omega_0 / 2K (\kappa) \right]^2}
\end{align*}
\]  

(C9)

\[
\begin{align*}
\sigma_{xx} &= \frac{e^3 B}{(2\pi)^2} \int_0^{4K/\omega_0} \tilde{u}_x \sum_{n=1}^{\infty} a_n^x \sin \left[ \frac{(2n-1)\pi \omega_0 t}{2K (\kappa)} \right] \int_{-\infty}^{t} \tilde{u}_m \sum_{m=1}^{\infty} a_m^x \sin \left[ \frac{(2m-1)\pi \omega_0 t'}{2K (\kappa)} \right] \exp \left( -\frac{1}{\tau_S + i\omega} (t-t') \right) dt' \\
&= \frac{e^3 B}{(2\pi)^2} \tilde{u}_x^2 \sum_{n=1}^{\infty} \left( a_n^x \right)^2 \frac{\left[ 1/\tau_S + i\omega \right]}{\left[ 1/\tau_S + i\omega \right]^2 + \left[ (2n-1)\pi \omega_0 / 2K (\kappa) \right]^2} \frac{2K (\kappa)}{\omega_0}
\end{align*}
\]  

(C10)

\[
\begin{align*}
\sigma_{yy} &= \frac{e^3 B}{(2\pi)^2} \int_0^{4K/\omega_0} \tilde{u}_y \sum_{n=1}^{\infty} a_n^y \cos \left[ \frac{(2n-1)\pi \omega_0 t}{2K (\kappa)} \right] \int_{-\infty}^{t} \tilde{u}_m \sum_{m=1}^{\infty} a_m^y \cos \left[ \frac{(2m-1)\pi \omega_0 t'}{2K (\kappa)} \right] \exp \left( -\frac{1}{\tau_S + i\omega} (t-t') \right) dt' \\
&= \frac{e^3 B}{(2\pi)^2} \tilde{u}_y^2 \sum_{n=1}^{\infty} \left( a_n^y \right)^2 \frac{\left[ 1/\tau_S + i\omega \right]}{\left[ 1/\tau_S + i\omega \right]^2 + \left[ (2n-1)\pi \omega_0 / 2K (\kappa) \right]^2} \frac{2K (\kappa)}{\omega_0}
\end{align*}
\]  

(C11)

For open Fermi surfaces:
\[ \sigma_{xx} = \frac{e^3 B}{(2\pi)^2} \int_0^\infty \tilde{u}_x \sum_{n=1}^\infty a_n^x \sin \left( \frac{n\pi\kappa_0 t}{K(1/\kappa)} \right) dt \int_0^t \tilde{u}_y \sum_{m=1}^\infty a_m^y \sin \left( \frac{m\pi\kappa_0 t}{K(1/\kappa)} \right) \exp \left( -\frac{1}{\tau S} + i\omega \right) (t - t') dt' \]

\[ = \frac{e^3 B}{(2\pi)^2} \tilde{u}_x^2 \sum_{n=1}^\infty \left( a_n^x \right)^2 \frac{\frac{1}{\tau S} + i\omega}{[\frac{1}{\tau S} + i\omega]^2 + \left( \frac{n\pi\kappa_0}{K(1/\kappa)} \right)^2} \kappa_0 \]  

\[ \sigma_{xy} = \frac{e^3 B}{(2\pi)^2} \int_0^\infty \tilde{u}_x \sum_{n=1}^\infty a_n^x \sin \left( \frac{n\pi\kappa_0 t}{K(1/\kappa)} \right) dt \int_0^t \tilde{u}_y \left\{ \frac{1}{2} + \sum_{m=1}^\infty a_m^y \cos \left( \frac{m\pi\kappa_0 t}{K(1/\kappa)} \right) \right\} \exp \left( -\frac{1}{\tau S} + i\omega \right) (t - t') dt' \]

\[ = \frac{e^3 B}{(2\pi)^2} \tilde{u}_x \tilde{u}_y \sum_{n=1}^\infty a_n^x a_n^y \frac{n\pi}{[\frac{1}{\tau S} + i\omega]^2 + \left( \frac{n\pi\kappa_0}{K(1/\kappa)} \right)^2} \]  

\[ \sigma_{yy} = \frac{e^3 B}{(2\pi)^2} \int_0^\infty \tilde{u}_y \left\{ \frac{1}{2} + \sum_{n=1}^\infty a_n^y \cos \left( \frac{n\pi\kappa_0 t}{K(1/\kappa)} \right) \right\} dt \int_0^t \tilde{u}_y \left\{ \frac{1}{2} + \sum_{m=1}^\infty a_m^y \cos \left( \frac{m\pi\kappa_0 t}{K(1/\kappa)} \right) \right\} \exp \left( -\frac{1}{\tau S} + i\omega \right) (t - t') dt' \]

\[ = \frac{e^3 B}{(2\pi)^2} \tilde{u}_y^2 K(1/\kappa) + \frac{e^3 B}{(2\pi)^2} \tilde{u}_y^2 \sum_{n=1}^\infty \left( a_n^y \right)^2 \frac{\frac{1}{\tau S} + i\omega}{[\frac{1}{\tau S} + i\omega]^2 + \left( \frac{n\pi\kappa_0}{K(1/\kappa)} \right)^2} \kappa_0 \]  

From which if \( \tilde{u}_x, \tilde{u}_y, a_n^x, a_n^y \) are substituted, Eqs. (26), (27) and (28) follow. As we emphasized in the main text, the sums are over positive odd integers while for open surfaces the sums are over positive even integers.

### 1. Hall coefficient for rectangular lattice at limiting cases

For the high-field limit:

\[ \frac{1}{R_H} = \lim_{B \to 0} \frac{1}{B} \frac{\sigma_{xy}^2 + \sigma_{yx}^2}{\sigma_{xx} \sigma_{yy}} \]  

Using the above equations, we obtain:

\[ R^e_H = -\frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{(n-\frac{1}{2})} \text{sech}^2 \left( n - \frac{1}{2} \right) \frac{\pi K^\prime (\kappa)}{K(\kappa)} \sin \left( 2n - 1 \right) \frac{\pi u_x}{K(1/\kappa)} \]  

\[ R^h_H = \frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{(n-\frac{1}{2})} \text{sech}^2 \left( n - \frac{1}{2} \right) \frac{\pi K^\prime (\kappa)}{K(\kappa)} \sin \left( 2n - 1 \right) \frac{\pi u_h}{K(1/\kappa)} \]  

\[ R^o_H = -\frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{2n} \text{sech}^2 \left( n \pi K^\prime (1/\kappa) \right) \sin \left( \frac{n\pi u_o}{K(1/\kappa)} \right) - \frac{1}{\pi} \sum_{n=1}^\infty a_n^{sech^2} \]  

\[ = -\frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{2n} \text{sech}^2 \left( n \pi K^\prime (1/\kappa) \right) \sin \left( \frac{2n\pi u_o}{K(1/\kappa)} \right) \]  

and we observe that in the high-field case the Hall number does not depend on the frequency. Similarly, for the low-field case the formulae are:

\[ \frac{1}{R_H} = \lim_{B \to 0} \frac{1}{B} \frac{\sigma_{xy}^2 + \sigma_{yx}^2}{\sigma_{xx} \sigma_{yy}} \]
similarly we get:

\[ R^R_{HL} = - \frac{4}{\pi} \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \sin^2 \left( n \frac{\pi u_0}{K} \right) \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \cos^2 \left( n \frac{\pi u_0}{K} \right) \]

\[ R^M_{HL} = \frac{4}{\pi} \left[ \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \sin^2 \left( n \frac{\pi u_0}{K} \right) \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \cos^2 \left( n \frac{\pi u_0}{K} \right) \right] \frac{1}{\pi} \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \sin \left( \frac{2n - 1}{2} \pi u_0 \right) \]

\[ R^E_{HL} = \frac{4}{\pi} \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \sin^2 \left( n \frac{\pi u_0}{K} \right) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \cos^2 \left( n \frac{\pi u_0}{K} \right) \right] \frac{1}{\pi} \sum_{n=1}^{\infty} \text{sech}^2 \left( n \frac{\pi K'}{K} \right) \sin \left( \frac{2n - 1}{2} \pi u_0 \right) \]

**Appendix D: Solution to relevant ordinary differential equations (ODEs)**

1. General setup

Eq. (13) is of the general form:

\[ \frac{dX}{dt} = M(t) X + V(t) \quad (D1) \]

First we need to solve for

\[ \frac{dX_0}{dt} = M(t) X_0 \quad (D2) \]

Where

\[ X_0(t) = T \left( \exp \left( \int_{t_0}^{t} M(s) \, ds \right) \right) X_0(t_0) \quad (D3) \]

Here \( T \) stands for time ordering. Then, for Eq. (D1), we look for solutions of the form

\[ X(t) = T \left( \exp \left( \int_{t_0}^{t} M(s) \, ds \right) \right) Y(t) \]

\[ \frac{dX(t)}{dt} = M(t) X(t) + T \left( \exp \left( \int_{t_0}^{t} M(s) \, ds \right) \right) \frac{dY(t)}{dt} \]

\[ \equiv M(t) X + V(t) \]

\[ \Rightarrow \frac{dY(t)}{dt} = T \left( \exp \left( \int_{t_0}^{t} M(s) \, ds \right) \right)^{-1} V(t) \]

\[ Y(t) = Y_0 + \int_{t_0}^{t} T \left( \exp \left( \int_{t_0}^{\tau} M(s) \, ds \right) \right)^{-1} V(t') \, dt' \quad (D4) \]

2. Solution of Eq. (A8) in 3D

We use the results of Appendix D. In our case \( Y_0 = 0 \) and

\[ k_1(t) = \int_{t_0}^{t} T \left( \exp \left( \int_{t'}^{t} M(s) \, ds \right) \right) V(t') \, dt' \quad (D5) \]

where \( V(t') = -eE(t') \) and

\[ M_{\alpha\beta}(s) = -e \sum_{\gamma\delta} \varepsilon_{\alpha\gamma\delta} B_{\delta} \frac{\partial^2}{\partial k_{\beta} \partial k_{\gamma}} \varepsilon(k_0(s)) \quad (D6) \]

Then, for \( t > t_0 \):

\[ k_1(t) = -\int_{t_0}^{t} T \left( \exp \left( \int_{t'}^{t} M(s) \, ds \right) \right)^{-1} V(t') \, dt' \]

3. Solution of Eq. (13) in 3D

Then \( Y_0 = 0 \) and for \( t > t_0 \)

\[ k_1(t) = \int_{t_0}^{t} T \left( \exp \left( \int_{t'}^{t} M(s) \, ds \right) \right) V(t') \, dt' \quad (D7) \]

In addition:

\[ V(t') = -D^{-1}(k_0(t')) \left[ eE(t') + e^2 \left( B \cdot E(t') \right) \Omega(k_0(t')) \right] \quad (D8) \]

and

\[ M_{\alpha\beta}(s) = -e \sum_{\gamma\delta} \varepsilon_{\alpha\gamma\delta} B_{\delta} \frac{\partial}{\partial k_{\beta}} \left[ D^{-1}(k_0(s)) \left[ \frac{\partial}{\partial k_{\gamma}} \varepsilon_M(k_0(s)) \right] \right] \quad (D9) \]

therefore for \( t > t_0 \)

\[ k_1(t) = -\int_{t_0}^{t} T \left( \exp \left( \int_{t'}^{t} M(s) \, ds \right) \right)^{-1} V(t') \, dt' \quad (D10) \]
We note that the perturbative correction in Eq. (A17), \( k_1 (t) \) is linear in the external electric field as can be seen by Eqs. (D8), (D9) and (D10). As a result, these terms are well convergent for small electric fields. Due to the time ordered exponential in Eq. (D9) though, there may be limitation to how large the eigenvalues of the matrix time ordered exponential in Eq. (D9) though, there may be well convergent for small electric fields. Due to the by Eqs. (D8), (D9) and (D10). As a result, these terms lead to divergences. Therefore, it is important that the damping term is greater than the divergent one and as such the eigenvalues of the matrix in Eq. (D9) need to be smaller than \( \frac{1}{\tau_0 (k)} \).

4. Solution of equations in 2D

In a similar way we’ve got \( Y_0 = 0 \) and for \( t > t_0 \)
\[
k_1 (t) = \int_{t_0}^{t} T \left( \exp \left( \int_{t'}^{t} M (s) ds \right) \right) \mathbf{V} (t') dt' \quad (D11)
\]
We also have that: \( \mathbf{V} (t') = -eD \cdot (k_0 (t')) \mathbf{E} (t') \) and:
\[
M_{\alpha \beta} (s) = -e \sum_\gamma \varepsilon_{\alpha \gamma} B \frac{\partial}{\partial k_{\beta}} \left[ D^{-1} (k_0 (s)) \left( \frac{\partial}{\partial k_\gamma} \varepsilon_M (k_0 (s)) \right) \right]
\]
(12) Therefore for \( t > t_0 \) the solution reads:
\[
k_1 (t) = -\int_{t_0}^{t} T \left( \exp \left( \int_{t'}^{t} M (s) ds \right) \right)^{-1} \mathbf{V} (t') dt' \quad (D13)
\]
Topological Insulators. (Cambridge University Press, 2018, UK)

[40] I. Sodemann, and L. Fu, Phys. Rev. Lett. 115, 216806 (2015).
[41] J.I. Facio, D.V. Efremov, K. Koepernik, J.-S. You, I. Sodemann, J. van den Brink, Phys. Rev. Lett. 121, 246403 (2018).
[42] L. M. Falicov and P. R. Sievert Phys. Rev. 138, A88 (1965).
[43] A. Alexandradinata and L. Glazman, Phys. Rev. Lett. 119, 256601 (2017).
[44] E. Deyo, L. E. Golub, E. L. Ivchenko, and B. Spivak, arXiv 0904.1917.
[45] G. Sundaram and Q. Niu, Phys. Rev. B 59, 14915 (1999).
[46] M. C. Chang, and Q. Niu, Phys. Rev. Lett. 75, 1348 (1995).
[47] M. C. Chang, and Q. Niu, Phys. Rev. B 53, 7010 (1996).
[48] M. C. Chang, and Q. Niu, J. Phys. Cond. Matt. 20, 193202 (2008).
[49] D. Arovas, Lecture notes in condensed matter physics (unpublished).
[50] K-S Kim, H-J Kim, and M. Sasaki, Phys. Rev. B 89, 195137 (2014).
[51] C. MacCallum, Phys. Rev. 132, 930 (1963).
[52] W. R. Young, Pertubation theory (Unpublished).
[53] S. M. Bauer, S. Fillipov, A. L. Smirnov, P. E. Tovstik, R. Villancourt, Asymptotic methods in mechanics of solids (Springer International Publishing, 2015, Switzerland.).
[54] A. H. Nayfeh, Pertubation Methods (John-Wiley & Sons Inc, 1973, Germany)
[55] A. V. Maharaj, I. Esterlis, Y. Zhang, B. J. Ramshaw, and S. A. Kivelson, Phys. Rev B 96, 045132 (2017).
[56] A. Auerbach, Phys. Rev. Lett. 121, 066601 (2018).