On totally projective QTAG-modules

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ABSTRACT
A module $M$ over an associative ring $R$ with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules. In this paper we investigate the class of QTAG-modules whose every separable module is a direct sum of uniserial modules; such modules are called $\omega$-totally $\omega$-projective. We discuss an interesting characterization of this class and we show that the class of $(\omega + n)$-totally $(\omega + n)$-projective modules contains the class of $\omega$-totally $\omega$-projective modules.

1. Introduction

Many concepts for groups like purity, projectivity, injectivity, height etc. have been generalized for modules. When we generalize the results of groups for modules sometimes we find that they are not true for modules. Therefore we impose some conditions either on the module or on the underlying ring. Here we impose a condition on modules that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules while the rings are associative with unity. After these conditions many elegant results of groups can be proved for QTAG-modules which are not true in general. Many results of this paper are the generalization of the paper [1].

The study of QTAG-modules was initiated by Singh [2]. Mehdi et al. [3] worked a lot on these modules. They introduced many concepts and generalized different notions for these modules and contributed to the development of the theory of QTAG-modules. Yet there is much to explore.

All the rings $R$ considered here are associative with unity and right modules $M$ are unital QTAG-modules. An element $x \in M$ is uniform, if $xR$ is a non-zero uniform (hence uniserial) module and for any $R$-module $M$ with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \{d(yR/xR) \mid y \in M, x \in yR \text{ and } y\text{uniform}\}$ are the exponent and height of $x$ in $M$, respectively. $H_k(M)$ denotes the submodule of $M$ generated by the elements of height at least $k$ and $H^\omega(M)$ is the submodule of $M$ generated by the elements of exponents at most $k$ [4]. $H_\omega(M)$ denotes the first Ulm-submodule of a module $M$ consisting of all elements of infinite height, and $H_{\omega+n}(M) = H_\omega(H_\omega(M))$. $M$ is $h$-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is $h$-reduced if it does not contain any $h$-divisible submodule. In other words it is free from the elements of infinite height. A module $M$ is said to be bounded, if there exists an integer $n$ such that $H(x) \leq n$ for every uniform element $x \in M$. A module $M$ is pure-complete if for every submodule of $P \subseteq \text{Soc}(Q)$ there is a pure subgroup $R \subseteq Q$ such that $\text{Soc}(R) = P$. A family of nice submodules $\mathcal{N}$ of submodules of $M$ is called a nice system in $M$ if

(i) $0 \in \mathcal{N}$;
(ii) $0 \subseteq \mathcal{N}$.

A $h$-reduced QTAG-module $M$ is called totally projective if it has a nice system.

Several results which hold for TAG-modules also hold good for QTAG-modules [2]. Notations and terminology are follows from [6,7].

A module $M$ is said to be $\Sigma$-module [8], if every high submodule of $M$ is the direct sum of uniserial modules (where a submodule $N$ of a module $M$ is high if it is maximal with the property $N \cap H_\omega(M) = \{0\}$). In general, a submodule of a $\Sigma$-module is not necessarily a $\Sigma$-module. $M$ is said to be a totally $\Sigma$-module if every submodule of $M$ is also a $\Sigma$-module. We focus on the
several characterizations of this class. For example, $M$ is a totally $\Sigma$-module if and only if it is the direct sum of countably generated modules and a module which is direct sum of uniserial modules. Alternatively, we say that $M$ is $\omega$-totally $\omega$-projective if every separable submodule $N$ of $M$ is a direct sum of uniserial modules. The class of $\omega$-totally $\omega$-projective modules coincide with the the class of $\omega$-totally pure-complete modules, i.e. those modules whose every separable modules is pure complete. Precisely, this is the class of module whose every $(\omega + n)$-bounded submodule is a direct sum of countably generated modules.

If $K$ is any module, then there is a module $M$ such that $H_\omega(M) = K$ and $M/H_\omega(M)$ is a direct sum of uniserial modules. Since for any high submodule $P$ of $M$ there is an embedding $\varphi : P \to M/H_\omega(M)$, $P$ must be a direct sum of uniserial modules, so that $M$ will be a $\Sigma$-module containing $K$. On the other hand, if $K$ is not countably generated, then $M$ will not be a totally $\Sigma$-module.

We sharpen this observation by showing that any separable module $A$ is a module of length $\omega + 1$ which is a $\Sigma$-module. If $A$ is a class of modules and $\beta$ is an ordinal, we say that $M$ is $\beta$-totally $\beta$ if every $\beta$-submodule of $M$ is in $A$. $M$ is $\beta$-totally $\beta$ if and only if every submodule of $M$ has the property that all of its $\beta$-high submodules are in $A$. Similarly, a module $M$ is $\omega$-totally $\omega$-projective if and only if it is a direct sum of uniserial modules. Further, if $M_1$ and $M_2$ are $(\omega + n)$-projective, then $M_1$ and $M_2$ are isomorphic if and only if $H^0(M_1)$ and $H^0(M_2)$ are isomorphic, i.e. there is an isomorphism that preserves the height functions on the two modules.

As the results of this paper are motivated by Danchev and Keef [1], so we will state those results before their generalization. For the better understanding of the mentioned topic here one must go through the papers [9–11].

2. Main results

We start with the following:

[[1], Proposition 2.1.1.] If $G$ is a group, $\alpha$ is an ordinal and $C$ is a class of groups, then $G$ is $\alpha$-totally $C$ iff every subgroup $T \subseteq G$ has the property that every $p^\alpha$-high subgroup of $T$ is a member of $C$.

**Proposition 2.1:** If $A$ is a class of modules, $M$ is a module and $\beta$ is an ordinal, then $M$ is $\beta$-totally $A$ if and only if every submodule $N$ of $M$ has the property that every $\beta$-high submodule of $N$ is in $A$.

**Proof:** Suppose every $\beta$-high submodule of a submodule of $M$ is in $A$. If $B$ is any $\beta$-bounded submodule of $M$, then $B$ is a $\beta$-high submodule of itself. Therefore it must be in $A$, so that $M$ is $\beta$-totally $A$. Conversely, suppose that $M$ is $\beta$-totally $A$ and $N$ is an arbitrary submodule of $M$. If $B$ is a $\beta$-high submodule of $N$, then $B$ is $\beta$-bounded, so by hypothesis, $B \in A$, completing the proof.

[[1], Lemma 2.2.1] Suppose $G$ and $H$ are groups, $H$ has infinite cardinality $\kappa$, $P$ is a subgroup of $H$ such that $H/P$ is $\Sigma$-cyclic and there is an injective homomorphism $\phi : P \to G$ such that (1) for every $x \in P$, $ht_G(\phi(x)) \geq ht_H(x)$; and (2) for every $m < \omega$, $(p^mG)[p]/(p^mG \cap \phi(P))[p]$ has cardinality at least $\kappa$. Then $\phi$ extends to an injective homomorphism $\psi : H \to G$.

**Lemma 2.1:** Suppose $M$ and $S$ are modules where the cardinality of the generating set of $S$ is infinite $\kappa$, $Q$ is a submodule of $S$ such that $S/Q$ is a direct sum of uniserial modules and there is an injective homomorphism $\pi : Q \to M$ such that (i) for every $x \in Q$, $H_\kappa(\pi(x)) \geq H_\kappa(x)$ and (ii) for every $n > 0$, $\text{Soc}(H_n(M))/\text{Soc}(H_n(M) \cap \pi(Q))$ has cardinality at least $\kappa$. Then $\pi$ extends to an injective homomorphism $\psi : S \to M$.

**Proof:** Without loss of generality we can assume that $S/Q$ has cardinality $\kappa$ and let $S/Q \cong \bigoplus_{x \in x} (x + Q)R$, and for $y < \kappa$ let $S_y = \bigoplus_{x \in x} (x + yR)$, thus $S = \bigcup_{y < \kappa} S_y$. We inductively extend $\pi$ to an injection $\pi_y : S_y \to M$, so that $x < y$ implies that $\pi_y$ have the same value as $\pi_x$ on $S_x$. Assume we have constructed $\pi_x$ for all $x < \gamma$. If $\gamma$ is a limit ordinal, then we clearly need to take just unions. On the other hand if $\gamma$ is isolated and $(x_{\gamma+1} + R + Q)$ has exponent $n$ in $S/Q$, i.e. $\exists x_{\gamma+1}^0 R \subseteq Q$ and $\pi(x_{\gamma+1}^0)$ is defined. By condition (i), we have $H_\kappa(\pi(\iota_{x_{\gamma+1}} R)) \geq n$. Let $x \in M$ such that $x^0 = \pi(x_{\gamma+1}^0)$ where $d(vR/v^0R) = n. (vR + \pi_{y-1} S_{y-1})/\pi(Q)$ has rank $|\gamma| < \kappa$. By condition (ii), there is an element $u \in \text{Soc}(H_{\kappa-1}(M))/\pi R + \pi_{y-1} (S_{y-1})$. (*)

Choose $b \in M$ such that $d(br/uR) = n - 1$. We let $\pi_y$ agree with $\pi_{y-1}$ on $S_{y-1}$ and $\pi_y(x_{\gamma+1}) = v + b$. To show $\pi_y$ is an injection, suppose $z \in S_y$ and $\pi_y(z) = 0$. Let $z = c + tx_{\gamma+1}$, where $c \in S_{y-1}$ and $t$ is an integer. Clearly, $n|t$, if this failed, then for some integer $l$
we would have \(lt = n - 1\) modulo the exponent of \(b\). Therefore
\[
0 = \pi_y(lz) = \pi_y(lc + ltx_{y-1}) = \pi_{y-1}(lc) + ltv + ub = \pi_{y-1}(lc) + ltv + ul 
\]
implies
\[
u = -ltv - \pi_{y-1}(lc),
\]
which contradicts *. Therefore \(n|t\), so that \(tx_{y-1} \in Q\), and hence \(z \in S_{y-1}\). Since \(\pi_{y-1}\) is injective, we have \(z = 0\), as required. Setting \(\psi = \bigcup_{y < \omega} \pi_y\), which completes the proof. ■

Recall that \(M\) is \(\omega\)-totally \((\omega + n)\)-projective means that every separable submodule of \(M\) is \((\omega + n)\)-projective.

[[1], Proposition 2.3.1] If \(n < \omega\) and \(G\) is \(\omega\)-totally \(p^{n+n}\)-projective, then \(p^{n+n}G\) is countable.

**Proposition 2.2:** If \(n > 0\) and \(M\) is \(\omega\)-totally \((\omega + n)\)-projective, then \(H_{\omega+n}(M)\) is countably generated.

**Proof:** Suppose on the contrary that \(H_{\omega+n}(M)\) is not countably generated. Let \(Q\) be a separable module of cardinality \(\aleph_1\) which is \((\omega + n + 1)\)-projective but not \((\omega + n)\)-projective. Let \(S\) be a submodule of \(Q\) with \(H_{\omega+n}(S) = (0)\) and \(Q/S\) is a direct sum of uniserial modules. Since \(H_{\omega+n}(M)\) is not countably generated, there is a submodule \(S'\) of \(H_{\omega}(M)\) which is isomorphic to \(S\) such that \(\text{Soc}(H_{\omega}(M))/\text{Soc}(S')\) is not countably generated. By Lemma 2.1, the isomorphism of \(S\) and \(S'\) extends to an embedding \(Q \rightarrow M\). Since \(Q\) is separable and not \((\omega + n)\)-projective, we can conclude that \(M\) is not \(\omega\)-totally \((\omega + n)\)-projective, contrary to our assumption. ■

Since an \((\omega + n)\)-totally \((\omega + n)\)-projective module is \(\omega\)-totally \((\omega + n)\)-projective, we have the following:

**Corollary 2.1:** If \(n > 0\), and \(M\) is \((\omega + n)\)-totally \((\omega + n)\)-projective, then \(H_{\omega+n}(M)\) is countably generated.

**Corollary 2.2:** If \(n > 0\), and \(M\) is \(\omega\)-totally \((\omega + n)\)-projective, then \(M\) is a direct sum of countably generated modules if and only if \(M/H_{\omega}(M)\) is a direct sum of uniserial modules.

[[1], Theorem 2.6.] If \(G\) is a group, then the following are equivalent:

(a) \(G\) is a totally \(\Sigma\)-group;
(b) \(G\) is \(\omega\)-totally \(\Sigma\)-cyclic;
(c) \(G\) is a \(\Sigma\)-group and \(p^nG\) is countable;
(d) \(G/p^nG\) is \(\Sigma\)-cyclic and \(p^nG\) is countable;
(e) \(G \subseteq C \oplus M\), where \(C\) is countable and \(M\) is \(\Sigma\)-cyclic;
(f) \(G\) is \(\omega\)-totally pure-complete;
(g) For all \(n < \omega\), \(G\) is an \((\omega + n)\)-totally dsc-grup;
(h) For some \(n < \omega\), \(G\) is an \((\omega + n)\)-totally dsc-grup;

Now we discuss the main result of the paper, which characterizes the class of modules that are \(\omega\)-totally \(\omega\)-projective.

**Theorem 2.1:** If \(M\) is a module, then the following statements are equivalent:

(i) \(M\) is a totally \(\Sigma\)-module;
(ii) \(M\) is \(\omega\)-totally \(\omega\)-projective;
(iii) \(M\) is a \(\Sigma\)-module and \(H_{\omega}(M)\) is countably generated;
(iv) \(M/H_{\omega}(M)\) is a direct sum of uniserial modules and \(H_{\omega}(M)\) is countably generated;
(v) \(M \cong H \oplus K\), where \(H\) is countably generated and \(K\) is a direct sum of uniserial modules;
(vi) \(M\) is \(\omega\)-totally pure-complete;
(vii) For all \(n < \omega\), \(M\) is an \((\omega + n)\)-totally direct sum of countably generated modules;
(viii) For some \(n < \omega\), \(M\) is an \((\omega + n)\)-totally direct sum of countably generated modules.

**Proof:** The equivalence of (i) and (ii) follows from Proposition 2.1. Now suppose that \(M\) is a totally \(\Sigma\)-module. Clearly, if \(M\) is a totally \(\Sigma\)-module, then it is a \(\Sigma\)-module. Therefore \(H_{\omega}(M)\) is countably generated, by Proposition 2.2, which is (iii). Next, suppose \(M\) is a \(\Sigma\)-module with \(H_{\omega}(M)\) is countably generated and we show that \(M/H_{\omega}(M)\) is a direct sum of uniserial modules. Suppose \(L\) is a \(h\)-high submodule of \(M\), so \(L\) is a direct sum of uniserial modules. Since \(H_{\omega}(M)\) embeds as an essential submodule of \(M/L\), it gives that \(M/L\) is countably generated. As \(M/L \rightarrow M/Soc(L + H_{\omega}(M))\) is a surjection, so \(M/Soc(L + H_{\omega}(M))\) is also countably generated. Since there exists a short exact sequence
\[
0 \rightarrow L \rightarrow M/H_{\omega}(M) \rightarrow M/Soc(L + H_{\omega}(M)) \rightarrow 0
\]
it gives that \(M/H_{\omega}(M)\) is a direct sum of uniserial modules which is (iv). The equivalence of (iv) and (v) are elementary. Now suppose \(M\) satisfies (iv) and (v) and we show that (ii) holds. If \(P\) is any separable submodule of \(M\), then \(P/(P \cap H_{\omega}(M))\) embeds \(M/H_{\omega}(M)\), and since \(M/H_{\omega}(M)\) is a direct sum of uniserial modules, it follows that \(P/(P \cap H_{\omega}(M))\) is a direct sum of uniserial modules. As \(P \cap H_{\omega}(M)\) is countably generated, implying that \(P\) is a direct sum of uniserial modules, as required.

Now we will show the equivalence of (ii) and (vi). For this, note that if \(M\) is \(\omega\)-totally \(\omega\)-projective and \(Q\) is a separable submodule of \(M\), then \(Q\) must be a direct sum of uniserial modules. Since any module which is a direct sum of uniserial modules, is pure-complete, it follows that \(M\) is \(\omega\)-totally pure-complete. For the reverse implication we will use the method of contrapositive proof. Let \(M\) be not \(\omega\)-totally \(\omega\)-projective, so it has a separable submodule \(P\) which is not a direct sum of uniserial modules. By virtue of the “core class property” from [6], one may infer that \(P\) contains a submodule \(Q\) which is \((\omega + 1)\)-projective but not a direct sum of uniserial modules. Since a pure-complete \((\omega + 1)\)-projective module
must be a direct sum of uniserial modules, therefore $P$ is not pure-complete implying that $M$ is not $\omega$-totally pure-complete.

For completing the proof of the theorem, it remains to show the equivalence of (vii) and (viii) with the help of other conditions. Suppose that $M$ is $\omega$-totally $\omega$-projective. It gives that every submodule of $M$ must be $\omega$-totally $\omega$-projective, and hence a countably generated module. In particular, for every $n < \omega$, every $(\omega + n)$-bounded submodule of $M$ is a direct sum of countably generated modules, i.e. $M$ is an $(\omega + n)$-totally direct sum of countably generated module and hence (vii) follows. Clearly (vii) implies (viii). Suppose (viii) holds for some $n > 0$. It follows that every separable submodule of $M$ is separable and countably generated module, i.e. every separable submodule of $M$ is a direct sum of uniserial modules, which is (iii), completing the proof. □

Corollary 2.3: If $M$ is $\omega$-totally $\omega$-projective, then for $n > 0$ $M$ is $(\omega + n)$-totally $(\omega + n)$-projective.

Proof: If $M$ is $\omega$-totally $\omega$-projective, then it is an $(\omega + n)$-totally direct sum of countably generated modules, and since a $(\omega + n)$-bounded direct sum of countably generated module is $(\omega + n)$-projective, $M$ must be $(\omega + n)$-totally $(\omega + n)$-projective. □

[[1], Proposition 2.9.] If $S$ is any separable group, then there is a $\Sigma$-group $G$ of length $\omega + 1$ containing $S$ as a subgroup.

Proposition 2.3: If $P$ is any separable module, then there is a $\Sigma$-module $M$ of length $\omega + 1$ containing $P$ as a submodule.

Proof: Suppose $Q$ is any direct sum of countably generated module of length $\omega + 1$ for which there is an isomorphism $\psi : H_\omega(Q) \to \text{Soc}(P)$. Let

$$K = \{ (x, \psi(x)) : x \in H_\omega(Q) \} \subseteq Q \oplus P,$$

and

$$M = (Q + P)/K.$$

Since $Q \cong ((Q \oplus \{0\}) + K)/K \subseteq M$ and $P \cong ((\{0\} \oplus P) + K)/K \subseteq M$, we may treat $P$ and $Q$ as submodules of $M$ such that $M = P + Q$ and $H_\omega(Q) = \text{Soc}(P) = P \cap Q$. As $H_\omega(Q) \subseteq H_\omega(M)$ and

$$M/H_\omega(Q) \cong (P + Q)/(P \cap Q),$$

is separable, it gives that $H_\omega(M) = H_\omega(Q)$, so that $M$ has length $\omega + 1$. If $L$ is a high submodule of $M$, then $L \cap \text{Soc}(P) = L \cap H_\omega(Q) = \{0\}$, so that $L \cap P = \{0\}$, as well.

Since $L \cap P = \{0\}$ is the kernel of the composite homomorphism

$$L \hookrightarrow M = P + Q \to (P + Q)/P \cong Q/(P \cap Q)$$

it follows that this is an embedding. However, since $Q/H_\omega(Q)$ is direct sum of uniserial modules, we have that $M$ is also a direct sum of uniserial modules, so that $M$ is a $\Sigma$-module. □

It is interesting to note that in the above Proposition if $P$ is not a direct sum of uniserial modules, then $M$ is a $\Sigma$-module which is not a totally $\Sigma$-module.

[[1], Lemma 2.10.] Suppose $G$ is a group such that $G/p^nG$ is $p^{\omega + 1}$-projective. Then the following are equivalent:

(a) $G$ is a direct group decomposition $G = H \oplus M$ where $H$ is separable and $M$ is $\Sigma$-cyclic;
(b) $G/p^{\omega + 1}G$ is $p^{\omega + 1}$-projective;
(c) $K_0(G) \subseteq K(G)$ contains a cofree subspace of $K(G)$.

Lemma 2.2: Suppose $M$ is a module such that $M/H_\omega(M)$ and $M/H_{\omega + 1}(M)$ are $(\omega + 1)$-projective, then there is a module decomposition $M = K \oplus L$, where $K$ is separable and $L/H_\omega(L)$ is direct sum of countably generated modules.

Proof: Suppose $M$ is a module such that $M/H_\omega(M)$ and $M/H_{\omega + 1}(M)$ are $(\omega + 1)$-projective. So that there is a decomposition $M/H_{\omega + 1}(M) = K \oplus N$ where $K$ is separable and $L$ is a direct sum of countably generated modules. Suppose $P, Q$ be submodules of $M$ such that $H_{\omega + 1}(M) = P \cap Q, P/H_{\omega + 1}(M) = K$ and $Q/H_{\omega + 1}(M) = N$. Clearly, $M/Q \cong K$ is separable, so that $H_\omega(M) \cong Q$ implying that for every $x \in H_{\omega + 1}(M)$, there is an element $y \in H_\omega(M) \subseteq Q$ such that $d(yRxR) = 1$. We will show that $H_\omega(M) \subseteq H_\omega(Q)$ by induction on $n$. Trivially it holds for $n = 0$. Next, suppose it holds for $n$ and let $z \in H_\omega(M)$. Considering $M/H_{\omega + 1}(M) \cong (P/H_{\omega + 1}(M)) \oplus (Q/H_{\omega + 1}(M))$, there is an element $q \in Q$ such that $x_1 = H_{\omega + 1}(q) - z \in H_{\omega + 1}(M)$. This means that $d(y_1Rx_1) = 1$, for some $y_1 \in H_\omega(M) \subseteq H_\omega(Q)$. Therefore, $d(uRx_1) = n$ for some $u \in Q$, so that $z = H_{\omega + 1}(q) - x_1 = H_{\omega + 1}(q) - H(y_1) = H_{\omega + 1}(q - u) \in H_{\omega + 1}(M)$, as required. Therefore we infer that $H_\omega(M) \subseteq H_{\omega + 1}(Q) \subseteq H_{\omega + 1}(M)$, so that $H_\omega(M) = H_\omega(Q)$, and hence $H_{\omega + 1}(M) = H_{\omega + 1}(Q)$. Therefore we have a commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & H_{\omega + 1}(Q) & \longrightarrow & P & \longrightarrow & K & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

with bottom row as $(\omega + 1)$-pure, and since $K$ is $(\omega + 1)$-projective, we have $M \cong K \oplus Q$. Finally, $Q/H_\omega(Q) \cong N/H_\omega(N)$ is direct sum of uniserial modules. □


As an immediate consequence we have the following:

**Corollary 2.4:** Suppose \( M \) is a module such that \( M/H_{\omega+1}(M) \) is \((\omega + 1)\)-projective.

(i) If \( H_\omega(M) \) is countably generated, then \( M \) is the direct sum of a separable \((\omega + 1)\)-projective module and a countably generated module.

(ii) If \( H_{\omega+1}(M) \) is countably generated, then \( M \) is the direct sum of a \((\omega + 1)\)-projective module and a countably generated module.

**Proof:** \( M/H_{\omega+1}(M) \) is \((\omega + 1)\)-projective implies that \( M/H_\omega(M) \) is \((\omega + 1)\)-projective. By Lemma 2.2, \( M \) can be expressed as \( M \cong K \oplus Q \), where \( K \) is separable \((\omega + 1)\)-projective module and \( Q \) is a module such that \( Q/H_\omega(Q) \) is direct sum of uniserial modules.

(i) Since \( H_\omega(Q) \) is countably generated, it follows that \( Q \) can be expressed as a direct sum of countably generated module and a module which is direct sum of uniserial modules, which is (i).

(ii) Clearly \( Q/H_{\omega+1}(Q) \) is a direct sum of countably generated modules and \( H_{\omega+1}(Q) \) is countably generated. Thus, \( Q \) is itself a direct sum of countably generated modules, which can be written as a direct sum of countably generated module and a direct sum of uniserial modules of length \( \omega + 1 \), which is \((\omega + 1)\)-projective. \(\blacksquare\)

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