-trees and laminations for free groups I: Algebraic laminations

Thierry Coulbois, Arnaud Hilion, Martin Lustig

To cite this version:

Thierry Coulbois, Arnaud Hilion, Martin Lustig. -trees and laminations for free groups I: Algebraic laminations. Proceedings of the London Mathematical Society, 2008, 78, pp.723-736. hal-00094735v2

HAL Id: hal-00094735
https://hal.science/hal-00094735v2
Submitted on 9 Jun 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
1 Introduction

This paper is the first of a sequence of three papers, where the concept of an \( \mathbb{R} \)-tree dual to (the lift to the universal covering of) a measured geodesic lamination \( \mathcal{L} \) in a hyperbolic surface \( S \) is generalized to arbitrary \( \mathbb{R} \)-trees provided with a (very small) action of the free group \( F_N \) of finite rank \( N \geq 2 \) by isometries.

In [CHL-II] to any such \( \mathbb{R} \)-tree \( T \) a dual algebraic lamination \( L^2(T) \) is associated in a meaningful way, and in [CHL-III] we consider invariant measures (called currents) \( \mu \) on \( L^2(T) \) and investigate the induced dual metric \( d_\mu \) on \( T \).

In this first paper we define and study the basic tools for the two subsequent papers: laminations in the free group \( F_N \). We will use three different approaches, algebraic laminations \( L^2 \), symbolic laminations \( L_A \), and laminary languages \( \mathcal{L} \). Each of them will be explained in detail, and each has its own virtues. Algebraic laminations do not need a specified basis of \( F_N \) and are hence of conceptual superiority. The other two objects are concretely defined in terms of infinite words (for symbolic laminations) or of finite words (for laminary languages) in a fixed basis \( A \). They are more practical for many tasks: Symbolic laminations are more suited for dynamical and laminary languages more for combinatorial purposes. The set of each of these three objects come naturally with a topology, a partial order, and an action by homeomorphisms of the group \( \text{Out}(F_N) \) of outer automorphisms of \( F_N \). We will prove that the three approaches are equivalent:

**Theorem 1.1.** Let \( F_N \) denote the free group of finite rank \( N \geq 2 \), and let \( A \) be a basis of \( F_N \). There are canonical \( \text{Out}(F_N) \)-equivariant, order preserving homeomorphisms

\[
\Lambda^2(F_N) \longmapsto \Lambda_A \longmapsto \Lambda_{\mathcal{L}}(A)
\]

between the space \( \Lambda^2(F_N) \) of algebraic laminations in \( F_N \), the space \( \Lambda_A \) of symbolic laminations in \( A^{\pm 1} \), and the space \( \Lambda_{\mathcal{L}}(A) \) of laminary languages in \( A^{\pm 1} \).

Symbolic laminations are subshifts (= symbolic flows) as classically used in symbolic dynamics, except that we work with the free group \( F_N = F(A) \) rather
than with the free monoid $A^*$. Similarly, laminary languages over the alphabet $A$ rather than $A^\pm = A \cup A^{-1}$ are already studied in combinatorics, compare for instance [Nar96].

As in the surface case, the subset $\Lambda_{\text{rat}} \subset \Lambda^2(F_N)$ of rational laminations, each corresponding to a finite collection of non-trivial conjugacy classes in $F_N$ (see §2), is of special interest. Contrary to the analogous statement for measured laminations on a surface, or for currents on $F_N$ (compare [Mar95]), we obtain in the setting of algebraic laminations:

**Theorem 1.2.** Rational laminations are not dense in $\Lambda^2(F_N)$. However, the closure $\overline{\Lambda_{\text{rat}}}$ contains all minimal laminations.

Algebraic laminations, as defined and studied in this paper, have three direct “ancestors”, all three of them inspired by geodesic laminations on surfaces: In [Lus92] *combinatorial laminations* are defined to study decomposable automorphisms of $F_N$, in [BFH00] an *attracting lamination* is associated to each exponential stratum of an automorphism of $F_N$ (see §2), and in [LL03] a kind of laminations is associated to certain $R$-tree actions of $F_N$.

This paper (as well as the subsequent ones [CHL-II] and [CHL-III]) is a further attempt to bridge the “cultural gap” between two mathematical communities: symbolic and combinatorial dynamics on one hand, and geometric group theory on the other. Notice that in geometric group theory the notion of an algebraic lamination extends naturally to the more general setting of word-hyperbolic groups.

We hope to have given enough detail to carry along the novice reader from the “other” mathematical subculture, and not too much to bore the expert reader from “this” one.

**Acknowledgements.** This paper originates from a workshop organized at the CIRM in April 05, and it has greatly benefited from the discussions started there and continued around the weekly Marseille seminar “Teichmüller” (partially supported by the FRUMAM).

## 2 Algebraic laminations

Let $F_N$ denote the free group of finite rank $N \geq 2$, and let $\partial F_N$ denote its Gromov boundary, as usual equipped with the action of $F_N$ (from the left) and with Gromov’s topology at infinity, which gives $\partial F_N$ the topology of a Cantor set. The choice of a basis $A$ of $F_N$ allows us to identify the elements of $F_N$ with reduced words $w = x_1 x_2 \ldots x_n$ (with $x_{i+1} \neq x_i^{-1}$) in $A \cup A^{-1}$, and thus defines in particular the length function $w \mapsto |w|_A = n$ on $F_N$. This length function induces the *word metric* $d_A(v,w) = |v^{-1}w|_A$ on $F_N$, which in turn defines a metric on $\partial F_N = \{x_1 x_2 x_3 \ldots x_i \in A^\pm: x_i \neq x_i^{-1}\}$, stated explicitly in §6.

Choosing another basis gives rise to a Lipschitz-equivalent metric on $F_N$ and to a Hölder-equivalent metric on $\partial F_N$ (compare [GdhH90]). As a consequence,
the topology on $F_N \cup \partial F_N$ induced by the word metric does not depend on the choice of the basis $A$. More details are given below in §8. Note that $F_N$ and $\partial F_N$ are compact spaces, and that every $F_N$-orbit in $\partial F_N$ is dense.

For any element $w \neq 1$ of $F_N$ we denote by $w^{+\infty}$ the limit in $\partial F_N$ of the sequence $(w^n)_{n \in \mathbb{N}}$ and by $w^{-\infty}$ that of $(w^{-n})_{n \in \mathbb{N}}$. If $w = x_1 \ldots x_p \cdot y_1 \ldots y_q \cdot x^{-1}_p \ldots x^{-1}_1$ is a reduced word in $A^\pm$, then

$$w^{+\infty} = x_1 \ldots x_p \cdot y_1 \ldots y_q \cdot y_1 \ldots y_q \cdot y_1 \ldots$$

Following standard notation (see for example [Kap04, Kap03]), we define $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \Delta$, where $\Delta$ denotes the diagonal in $\partial F_N \times \partial F_N$. It follows directly that $\partial^2 F_N$ inherits from $\partial F_N$ a topology and an $F_N$-action, given by $w(X, Y) = (wX, wY)$ for any $w \in F_N$ and any $X, Y \in \partial F_N$ with $X \neq Y$. The set $\partial^2 F_N$ admits also the flip involution $(X, Y) \mapsto (Y, X)$, which is an $F_N$-equivariant homeomorphism. Note that $\partial^2 F_N$ is not compact.

**Definition 2.1.** An **algebraic lamination** is a subset $L^2$ of $\partial^2 F_N$ which is non-empty, closed, symmetric (= flip invariant) and $F_N$-invariant. The set of all algebraic laminations is denoted by $\Lambda^2 = \Lambda^2(\mathcal{F}_N)$.

The set $\Lambda^2$ of algebraic laminations inherits naturally a Hausdorff topology from $\partial^2 F_N$ which we will discuss in §6.

In [BFH00], M. Bestvina, M. Feighn and M. Handel associate an attracting lamination to each exponential stratum of an automorphism of $F_N$. These laminations are laminations in our sense. However, in [BFH00] there is no topology introduced on the space of laminations but rather only on $\partial^2 F_N$, and even there, their topology differs slightly from ours.

An important special class of algebraic laminations are the **rational laminations**, which are finite unions of **minimal rational laminations** $L(w)$, defined for any $w \in F_N \setminus \{1\}$ by:

$$L(w) = \{(vw^{-\infty}, vw^{+\infty}) \mid v \in F_N\} \cup \{(vw^{+\infty}, vw^{-\infty}) \mid v \in F_N\}$$

Note that the lamination $L(w)$ depends only on the conjugacy class of $w$. We denote by $\Lambda_{rat}$ the subspace of rational laminations. The Hausdorff topology on $\Lambda^2$ is stronger than one might intuitively expect. In particular on obtains the following result, proved in §6:

**Proposition 2.2.** The subset $\Lambda_{rat}$ is not dense in $\Lambda^2$.

We observe that there is a natural (left) action of $\text{Out}(F_N)$ on $\Lambda^2$, induced by the action of $\text{Aut}(F_N)$ on $\partial F_N$. Indeed, an automorphism of $F_N$ is a bi-Lipschitz homeomorphism on $F_N$ and extends continuously to the boundary. Inner automorphisms act by left-multiplication on the boundary and thus trivially on the space $\Lambda^2$ of algebraic laminations (as the latter are $F_N$-invariant
subsets of $\partial^2 F_N$). More details about the $\text{Out}(F_N)$-action on $\Lambda^2$ will be given in §8.

Note that this action restricts to an action of $\text{Out}(F_N)$ on the space of rational laminations $\Lambda_{\text{rat}}$: If $\alpha$ is an automorphism of $F_N$ and $\hat{\alpha}$ its class in the outer automorphism group $\text{Out}(F_N)$ and, if $w$ is an element of $F_N$, $\hat{\alpha}(L(w)) = L(\alpha(w))$.

To stimulate the interest of the reader in these rather delicate matters we would like to pose here a question which is inspired by the thesis of R. Martin [Mar95]:

**Question 2.3.** Let $A$ be any basis of $F_N$, and fix $a \in A$ arbitrarily. Is the closure $\text{Out}(F_N)L(a)$ of the $\text{Out}(F_N)$-orbit of $L(a)$ a minimal closed $\text{Out}(F_N)$-invariant non-empty subset of $\Lambda^2$? If so, is it the unique such minimal set?

An answer to this question will be given in Proposition 8.2. Note that if $N = 2$ and $\{a, b\}$ is a basis of $F_2$ and $[a, b] = a^{-1}b^{-1}ab$, then it is well known that for any automorphism $\alpha$ of $F_N$, $\alpha([a, b])$ is conjugated to either $[a, b]$ or its inverse. Therefore $L([a, b])$ is a global fixed point of the action of $\text{Out}(F_N)$ on $\Lambda$.

### 3 Surface laminations

An important class of algebraic laminations comes from geodesic laminations on hyperbolic surfaces. The discussion started below, to compare algebraic laminations in general with laminations on surfaces, is carried further in [CHL-II] and [CHL-III]. Throughout this section we assume a certain familiarity of the reader with this subject; for background see for example [CB88] and [FLP91]. Note that this section can be skipped by the reader without loss on the intrinsic logics of the material presented in this paper.

Let $S$ be a hyperbolic surface with non-empty boundary and negative Euler characteristic, and fix an identification $\pi_1 S = F_N$. The surface $S$ is provided with a hyperbolic structure, given by an identification of the universal covering $\tilde{S}$ of $S$ by hyperbolic isometries. Let $\mathcal{L}$ be a geodesic lamination on $S$ and let $\tilde{\mathcal{L}}$ be the (full) lift of $\mathcal{L}$ to the universal covering $\tilde{S}$ of $S$. The induced identification (an $F_N$-equivariant homeomorphism) between $\partial F_N$ and the boundary at infinity $\partial \tilde{S}$ of $\tilde{S}$ defines for any leaf $l$ of $\tilde{\mathcal{L}}$ a pair of endpoints $(X, Y) \in \partial^2 F_N$, as well as its flipped pair $(Y, X)$. The set of all such pairs is easily seen to define (via the above identification $\partial F_N = \partial \tilde{S}$) an algebraic lamination $L^2(\mathcal{L}) \in \Lambda^2(F_N)$.

**Definition 3.1.** An algebraic lamination $L^2 \in \Lambda^2(F_N)$ is called an **algebraic surface lamination** if there exists a hyperbolic surface $\tilde{S}$ and an identification $\pi_1 \tilde{S} = F_N$ such that for some geodesic lamination $\mathcal{L}$ on $S$ one has:

$$L^2 = L^2(\mathcal{L})$$
At first guess it may seem that the space $\Lambda^2(F_N)$ is a rather weak analogue of the space of geodesic laminations in a surface. Notice however that, if $L^2 \in \Lambda^2(F_N)$ is an algebraic surface lamination with respect to an isomorphism $\pi_1 S_1 = F_N$ for some surface $S_1$, and if $S_2$ is a second surface with identification $\pi_1 S_2 = F_N$, then typically a biinfinite geodesic on $S_2$, which realises an element of $L^2$, will self-intersect: Thus $L^2$ does not admit a realization as geodesic lamination on $S_2$.

4 Symbolic laminations

To a basis $\mathcal{A}$ there is naturally associated the space $\Sigma_{\mathcal{A}}$ of biinfinite reduced words $Z$ in $\mathcal{A} \cup \mathcal{A}^{-1}$ with letters indexed by $\mathbb{Z}$:

$$\Sigma_{\mathcal{A}} = \{ Z = \ldots z_{i-1} z_i z_{i+1} \ldots | z_i \in \mathcal{A} \cup \mathcal{A}^{-1}, z_i \neq z_{i+1}^{-1} \text{ for all } i \in \mathbb{Z} \}.$$

We want to stress that in this paper a biinfinite word comes always with a $\mathbb{Z}$-indexing, i.e. formally speaking, a biinfinite word is a map $Z : \mathbb{Z} \to \mathcal{A} \cup \mathcal{A}^{-1}$. For example, the non-indexed “biinfinite word” 

$$\ldots abab \ldots$$

becomes a biinfinite word $Z$ only after specifying $z_1 = a$ or $z_1 = b$, which we indicate notationally by writing $Z = \ldots bab \cdot aba \ldots$ or $Z = \ldots aba \cdot bab \ldots$ respectively.

As usual, $\Sigma_{\mathcal{A}}$ comes with a canonical infinite cartesian product topology that makes it a Cantor set, and with a shift operator $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, given by

$$\sigma(Z) = Z',$$

where $Z = \ldots z_{i-1} z_i z_{i+1} \ldots$ and $Z' = \ldots z_i' z_i' z_{i+1}' \ldots$ with $z_i' = z_{i+1}$. Of course, $\sigma$ is a homeomorphism.

For each biinfinite word $Z = \ldots z_{i-1} z_i z_{i+1} \ldots$ we denote its inverse by

$$Z^{-1} = \ldots z_i' z_i' z_{i+1}' \ldots,$$

where $z_i' = (z_{i-1})^{-1}$. Again, the inversion map $\Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, $Z \mapsto Z^{-1}$ is easily seen to be a homeomorphism. A subset $L$ of $\Sigma_{\mathcal{A}}$ is called symmetric if $L = L^{-1}$.

**Definition 4.1.** A symbolic lamination in $\mathcal{A}^{\pm 1}$ is a non-empty subset $L_A \subset \Sigma_{\mathcal{A}}$ which is closed, symmetric and $\sigma$-invariant. Together with the restriction of $\sigma$ to $L_A$ (which we continue to call $\sigma$) it is a symbolic flow. The elements of a symbolic lamination are sometimes called the leaves of the lamination. We denote the set of symbolic laminations in $\mathcal{A}^{\pm 1}$ by $\Lambda_{\mathcal{A}}$.

In symbolic dynamist’s terminology, any symbolic lamination is a subshift of the subshift of finite type on the alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$ which consists of all biinfinite reduced words.

As $\Sigma_{\mathcal{A}}$ is compact and symbolic laminations are closed, we get:
Lemma 4.2. The intersection of a decreasing sequence

\[ L_A \supset L'_A \supset L''_A \supset \ldots \]

of symbolic laminations is a symbolic lamination. In particular it is non-empty.
\[ \square \]

Once the basis \( A \) is fixed, every boundary point \( X \in \partial F_N \) corresponds canonically to a reduced, (one-sided) infinite word \( X = x_1 x_2 \ldots \) with letters in \( A^{\pm 1} \). For such a (one-sided) infinite word \( X \) we denote by \( X_n \) its prefix (= initial subword) of length \( n \). For every pair \((X, Y) \in \partial^2 F_N\) we define a biinfinite reduced word

\[ X^{-1}Y = \ldots x_{k+2}^{-1}\cdots x_{k+1}^{-1}y_{k+1}y_{k+2} \ldots , \]

where \( X_k = x_1 x_2 \ldots x_k = y_1 y_2 \ldots y_k = Y_k \) is the longest common prefix of \( X \) and \( Y \).

There is a subtlety in the last definition which we would like to point out: Although for any \( X \neq Y \in \partial F_N \) the biinfinite (indexed) word \( X^{-1}Y \) is well defined by our above definition, this particular way to associate the indices from \( Z \) to the non-indexed “biinfinite word” \( \ldots x_{k+2}^{-1}\cdots x_{k+1}^{-1}y_{k+1}y_{k+2} \ldots \) is really in no way canonical, and often it does not behave quite naturally, in particular with respect to the action of \( \text{Aut} F_N \). Indeed, a biinfinite symbol sequence, contrary to a finite or a one-sided infinite one, doesn’t really come by nature with a canonical indexing, but rather corresponds to the whole \( \sigma \)-orbit of a biinfinite word in \( \Sigma_A \). Nevertheless one obtains as direct consequence of the definitions:

Remark 4.3. The map \( \rho_A : \partial^2 F_N \rightarrow \Sigma_A \) \( (X, Y) \mapsto X^{-1}Y \) is continuous.

We note that the biinfinite indexed word from \( \Sigma_A \) associated via \( \rho_A \) to \( w(X, Y) \), for any \( w \in F_N \), can differ from the indexed word \( X^{-1}Y \) only by an index shift. Conversely, for the pair \((X, Y) \in \partial^2 F_N\) with maximal common initial subword \( X_k = Y_k \) as above, the map \( \rho_A \) associates the biinfinite indexed word \( \sigma^m(X^{-1}Y) \) to the pair \( Y_{k+m}^{-1}(X, Y) \) for \( m \geq 0 \), and to \( X_{k-1-m}^{-1}(X, Y) \) for \( m \leq 0 \).

Hence the map \( \rho_A \) maps every \( F_N \)-orbit in \( \partial^2 F_N \) onto a \( \sigma \)-orbit in \( \Sigma_A \), and thus induces a well defined map from \( F_N \)-orbits in \( \partial^2 F_N \) to \( \sigma \)-orbits in \( \Sigma_A \). It is easy to see that this map between orbits is bijective, and that, moreover, this bijection respects the topology on both sides: Closed sets of \( F_N \)-orbits are mapped to closed sets of \( \sigma \)-orbits, and conversely. Finally, we note that the flip on \( \partial^2 F_N \) corresponds to the inversion of biinfinite words in \( \Sigma_A \).

Thus, given \( L^2 \in \Lambda^2 \), we can define a symbolic lamination \( L_A \) by

\[ L_A = \rho_A(L^2) = \{ X^{-1}Y \mid (X, Y) \in L^2 \} . \]

Conversely, given a symbolic lamination \( L_A \) as above, one obtains an algebraic lamination \( L^2 = \rho_A^{-1}(L_A) \) which consists of all pairs \( w(Z_-, Z_+) \), for all
$w \in F_N$, and all $Z = \ldots z_{i-1}z_i z_{i+1} \ldots \in L_A$ with associated right-infinite words $Z_-=z_0^{-1} z_{-1}^{-1} z_{-2}^{-1} \ldots$ and $Z_+=z_1 z_2 \ldots$.

We summarize the above discussion:

**Proposition 4.4.** For any basis $\mathcal{A}$ of the free group $F_N$, the maps $L^2 \mapsto L_A = \rho_A(L^2)$ and $L_A \mapsto L^2 = \rho_A^{-1}(L_A)$ define a bijection

$$\rho_A^2 : \Lambda^2(F_N) \to \Lambda_A$$

between the set $\Lambda^2(F_N)$ of algebraic laminations $L^2$ and the set $\Lambda_A$ of symbolic laminations $L_A$ in $\mathcal{A}^{\pm 1}$.

The map $\rho_A^2$ respects the partial order given on algebraic or symbolic laminations by the inclusion as subsets of $\partial^2 F_N$ or $\Sigma_A$ respectively. In particular, a minimal lamination $L_A$ (or $L^2$) with respect to this partial order is precisely given by the analogous property that characterizes classically minimal symbolic flows: Every $< \sigma, (\cdot)^{-1} >$-orbit (or $< F_N, \text{flip} >$-orbit, respectively) is dense in the lamination. Moreover, we note that Lemma 4.2 holds for algebraic laminations.

In order to connect the content (and also the notations) introduced in this section to the already existing notions in symbolic dynamics, we note:

A symbolic flow $\sigma : \Sigma_0 \to \Sigma_0$ in the “classical sense”, i.e. a symbolic flow only on the letters of $\mathcal{A}$ (and not of $\mathcal{A}^{-1}$), gives directly rise to a symbolic lamination $L_A(\Sigma_0) = \Sigma_0 \cup \Sigma_0^{-1} \in \Lambda_A$. Conversely, a symbolic lamination $L_A \in \Lambda_A$ or a symbolic flow $\sigma : L_A \to L_A$ is called orientable if $L$ can be written as disjoint union $L_A = L_+ \cup L_+^{-1}$ of two $\sigma$-invariant closed subsets $L_+$ and $L_+$ that are inverses of each other, and it is called positive if one of them, say $L_+$, only uses letters from $\mathcal{A}$ (and not from $\mathcal{A}^{-1}$).

**Remark 4.5.** The fact that the laminations considered are positive is crucial for many of the traditional approaches and methods of symbolic dynamics. Similarly, for laminations (or foliations) on surfaces, almost always one first considers the orientable case and later tries to pass to the general situation via branched coverings. Note that in the context of free groups considered here any such attempt would miss most of the typical phenomena, and that hence struggling with the general kind of non-orientable laminations seems unavoidable. For an interesting case of such an encounter of the free group environment with the “already existing culture” in the context of the Rauzy fractal see [ABHS05].

## 5 Laminary languages

As before, we fix a basis $\mathcal{A}$ of $F_N$, and we denote by $F(\mathcal{A})$ the set of reduced words in $\mathcal{A}^{\pm 1}$. Although there is a canonical identification between $F_N$ and $F(\mathcal{A})$, it is helpful in the context of this section to think of the elements of $F(\mathcal{A})$ as words and not as group elements.
**Definition 5.1.** Let \( S \) be any (finite or infinite) set of finite, one-sided infinite or biinfinite reduced words in \( A^{\pm 1} \). We denote by \( L(S) \subset F(A) \) the language generated by \( S \), i.e., the set of all finite subwords (\( = \) factors) of any element of \( S \). Moreover, for any integer \( n \) we denote by \( L_n(S) \) the subset of \( L(S) \) consisting of words of length smaller or equal to \( n \).

We specially have in mind the language associated to a (symbolic) lamination. We thus abstractly define laminary languages which are in one-to-one correspondence with (symbolic) laminations.

**Definition 5.2.** A non-empty set \( \mathcal{L} \subset F(A) \) of finite reduced words in \( A^{\pm 1} \) is a laminary language if it is (i) symmetric, (ii) factorial and (iii) bi-extendable. By this we mean that it is closed with respect to (i) inversion, (ii) passing to subwords, and (iii) that for any word \( u \in \mathcal{L} \) there exists a word \( v \in \mathcal{L} \) in which \( u \) occurs as subword other than as prefix or as suffix: \( v = uwv' \) is a reduced product, with nontrivial \( w, w' \in F(A) \). We denote by \( \Lambda = \Lambda(\mathcal{L}) \) the set of laminary languages over a fixed basis \( A \).

It is obvious from the definition that the set \( \Lambda \) is closed under (possibly infinite) unions in \( F(A) \), and also under nested intersections (compare with Lemma 4.2). Note that the analogy of the former statement, for symbolic laminations rather than laminary languages, is false: An infinite union of symbolic laminations will in general not be a symbolic lamination; one first needs to take again the closure in \( \Sigma_A \). Note also that for any symbolic lamination \( L_A \subset \Sigma_A \) the language \( L(L_A) \) is laminary.

For an infinite language \( \mathcal{L} \subset F(A) \), we denote by \( L(\mathcal{L}) \) the set of all bi-infinite words from \( \Sigma_A \) whose finite subwords are subwords of elements from \( \mathcal{L} \cup \mathcal{L}^{-1} \). As \( \mathcal{L} \) is infinite (hence in particular, if \( \mathcal{L} \) is a laminary language), the definition enforces that \( L(\mathcal{L}) \) is not empty. It follows directly that \( L(\mathcal{L}) \) is indeed a symbolic lamination. We thus obtain a one-to-one correspondence between symbolic laminations and laminary languages (always for a fixed basis \( A \) of \( F_N \)): For any symbolic lamination \( L_A \) one has

\[
L(L(L_A)) = L_A.
\]

and conversely, for any laminary language \( \mathcal{L} \) one has

\[
\mathcal{L}(L(\mathcal{L})) = \mathcal{L}.
\]

Moreover, a language \( \mathcal{L} \) is laminary if and only if it is infinite, and if the last equation holds. For any set \( S \) of finite, one-sided infinite or biinfinite reduced words in \( A^{\pm 1} \), where we assume that \( S \) is infinite in case \( S \subset F(A) \), we observe that \( L(L(\mathcal{L}(S))) \) is the largest laminary language contained in \( \mathcal{L}(S) \). We call \( L(L(\mathcal{L}(S))) \) the symbolic lamination and \( L(L(\mathcal{L}(S))) \) the laminary language generated by \( S \). We summarize this discussion:

**Proposition 5.3.** For any finite alphabet \( A \) the maps \( L_A \mapsto L(L_A) \) and \( \mathcal{L} \mapsto L(\mathcal{L}) \) define a bijection

\[
\rho_A^L : \Lambda_A \rightarrow \Lambda_L
\]
between the set $\Lambda_A$ of symbolic laminations $L_A$ and the set $\Lambda_L$ of laminary languages $L$ in $A^\pm 1$. □

As in Proposition 4.4, the bijection $\rho^A_L : \Lambda_A \rightarrow \Lambda_L$ respects the partial order given by the inclusion.

To enforce the link between symbolic laminations and their laminary languages we introduce the following notation and state the following lemma, which will be used in the sequel: For any integer $k \geq 0$ and any reduced word $w = x_1 x_2 \ldots x_n \in F(A)$ denote by $w \dagger k$ ("$w$ chop $k$") the word

(a) $w \dagger k = 1$, if $|w| \leq 2k$, and

(b) $w \dagger k = x_{k+1} x_{k+2} \ldots x_{n-k}$, if $|w| > 2k$.

Similarly, for any integer $k \geq 0$ and any language $L$ we denote by $L \dagger k$ ("$L$ chop $k$") the language obtained from $L$ by performing, in the given order:

1. replace every $w \in L$ by $w \dagger k$, and

2. add all subwords (= factors) to the language.

The following properties of (laminary) languages are rather useful; they follow directly from the definition.

Lemma 5.4. (a) Every laminary language $L$ satisfies, for every integer $k \geq 0$, the equality $L = L \dagger k$.

(b) For every infinite language $L$ and for every integer $k$, $L(L \dagger k) = L(L)$ and $L(L(L)) = \cap_{k \in \mathbb{N}} L \dagger k$. □

Recall that a symbolic lamination $L \in \Lambda_A$ is minimal if $L$ is equal to the closure of any of its orbits, with respect to both, shift and inversion. This is equivalent to saying that $L$ does not contain a proper sublamination. One can easily characterize laminary languages of such a minimal lamination:

Definition 5.5. A language $L$ has the bounded gap property if for any word $u$ in $L$ there exists an integer $n = n(u) \in \mathbb{N}$ such that any word $w \in L$ of length greater than $n$ contains $u$ or $u^{-1}$ as a subword.

The following is part of symbolic dynamics folklore [Fog02]:

Proposition 5.6. A (symbolic) lamination is minimal if and only if its laminary language has the bounded gap property. □

Note that, if in addition the lamination is non-orientable, then for $n$ big enough any word $w$ of the laminary language will contain both, $u$ and $u^{-1}$. 

9
6 Metrics and topology on the set of laminations

For any laminar languages $\mathcal{L}, \mathcal{L}' \in \Lambda_\mathcal{L}$ we define:

$$d(\mathcal{L}, \mathcal{L}') = \exp(-\max\{\{n \geq 0 \mid \mathcal{L}_{2n+1} = \mathcal{L}'_{2n+1}\} \cup \{0\})).$$

This defines a distance on $\Lambda_\mathcal{L}$ which is easily seen to be ultra-metric, and it is clear that $\Lambda_\mathcal{L}$ is a compact Hausdorff totally disconnected perfect metric space: a Cantor set.

Similarly, one can define on the set $\Sigma_A$ of biinfinite reduced words in $A^{\pm 1}$ a metric, by defining for any $Z, Z' \in \Sigma_A$ the distance

$$d(Z, Z') = \exp(-\max\{\{n \geq 0 \mid Z_n = Z'_n\} \cup \{0\})),
$$

where for any reduced biinfinite word $Z = \ldots z_{i-1}z_iz_{i+1}\ldots$ we denote the central subword of length $2n+1$ by $Z_n = z_{-n}z_{-n+1}\ldots z_n$.

From these definitions and the shift-invariance of a symbolic lamination we obtain directly that a symbolic lamination $\mathcal{L}_A$ is contained in the $\varepsilon$-neighborhood in $\Sigma_A$ of a second symbolic lamination $\mathcal{L}'_A$ if and only if $\mathcal{L}_{2n+1}(\mathcal{L}_A)$ is a subset of $\mathcal{L}_{2n+1}(\mathcal{L}'_A)$, for $\varepsilon = e^{-n}$. This metric on $\Sigma_A$ induces a Hausdorff metric on the set $\Lambda_\mathcal{A}$ of symbolic laminations in $A^{\pm 1}$. We obtain directly:

**Proposition 6.1.** The bijection $\rho_A^\mathcal{L} : \Lambda_\mathcal{A} \rightarrow \Lambda_\mathcal{L}$ given by $\mathcal{L}_A \mapsto \mathcal{L}(\mathcal{L}_A)$ is an isometry with respect to the above defined metrics:

$$d(L_A, L'_A) \leq e^{-n} \iff \mathcal{L}_{2n+1}(\mathcal{L}_A) = \mathcal{L}_{2n+1}(\mathcal{L}'_A)$$

As indicated in §2, the choice of a basis $A$ of the free group $F_N$ defines a word metric on $F_N$ and also a (ultra-)metric at infinity on $\partial F_N$, by specifying for any $X, Y \in \partial F_N$, with prefixes $X_n$ and $Y_n$ respectively, the distance

$$d_A(X, Y) = \exp(-\max\{n \geq 0 \mid X_n = Y_n\}).$$

In a similar vein as above for $\Sigma_A$, this distance can be used to define a distance on $\partial^2 F_N$, and we can define a Hausdorff metric $d_\mathcal{A}$ on $\Lambda^2(F_N)$. With a little care we can show that this makes the bijection $\rho_A^\mathcal{A} : \Lambda^2(F_N) \rightarrow \Lambda_\mathcal{A}$ from Proposition 4.4 an isometry. However, contrary to the case of $\Lambda_\mathcal{A}$ and $\Lambda_\mathcal{L}$, the choice of a basis in $F_N$ and hence of the metric on $\partial F_N$ is not really natural, so that we prefer for $\Lambda^2(F_N)$ only to consider the topology induced by these metrics. Whenever a basis is specified, it is in any case more convenient to pass directly to $\Lambda_\mathcal{A}$ or to $\Lambda_\mathcal{L}$. It is well known (and can easily be derived from the material presented in §7 below) that different bases of $F_N$ induce Hölder-equivalent metrics on $\partial F_N$ and on $\partial^2 F_N$, and thus also on $\Lambda^2(F_N)$. Thus we obtain:

$$\square$$
Proposition 6.2. The canonical bijections

\[ \Lambda^2(F_N) \xrightarrow{\rho^2_A} \Lambda_A \xrightarrow{\rho^2_L} \Lambda_L \]

are homeomorphisms. They also preserve the partial order structure defined on each of them by the inclusion as subsets. \(\square\)

The topology on the space of laminations is explicitly encapsulated in the following:

Remark 6.3. A sequence \((L^2_k)_{k \in \mathbb{N}}\) of algebraic laminations converges to an algebraic lamination \(L^2\) if and only if, for some (and hence any) basis \(A\) of \(F_N\), the sequence of corresponding symbolic laminations \(L_k = \rho^2_A(L^2_k)\) and their presumed limit \(L = \rho^2_A(L^2)\) satisfy the following:

Convergence criterion: For any integer \(n \geq 1\) there exists a constant \(K(n) \geq 1\) such that for all \(k \geq K(n)\) one has:

\[ \mathcal{L}_n(L_k) = \mathcal{L}_n(L) \]

The following lemma will be used in [CHL-III].

Lemma 6.4. For any given algebraic lamination \(L^2\) the set \(\delta(L^2)\) of sublaminations of \(L^2\) is a compact subset of \(\Lambda^2\).

Proof. Since \(\Lambda^2\) is compact, it suffices to show that \(\delta(L^2)\) is closed. Any sublamination of \(L^2\) has as laminary language a sublanguage of the laminary language \(\mathcal{L}(L^2)\) defined by \(L^2\), and conversely. Moreover, for laminary languages the analogous statement as given by the lemma is trivially true, as follows directly from the above Convergence criterion. \(\square\)

We would like to point the reader’s attention to the fact that the space \(\Lambda^2\) is rather large, and for some purposes perhaps too large: it contains more objects than one would naturally think of as analogues of surface laminations. Of particular interest seems to be the natural subspace of \(\Lambda^2\) given by the closure \(\overline{\Lambda}_{\text{rat}} = \overline{\Lambda}_{\text{rat}}(F_N)\) of the the space \(\Lambda_{\text{rat}}\) of rational laminations (compare §2). We can now restate and prove Proposition 2.2:

Proposition 6.5. The inclusion \(\overline{\Lambda}_{\text{rat}} \subset \Lambda^2(F_N)\), for \(N \geq 2\), is not an equality.

Proof. For \(a\) and \(b\) in \(A\) consider the symbolic lamination \(L(\mathcal{L}(Z))\) generated by the biinfinite word \(Z = \ldots a\cdot a\cdot a\ldots\). It consists precisely of the \(\sigma\)-orbit of \(Z\) and of the two periodic words \(\ldots a\cdot a\cdot a\ldots\) and \(\ldots b\cdot b\cdot b\ldots\), together with all of their inverses. The laminary language \(\mathcal{L}_n(Z)\) consists of the words \(a^n, a^{n-1}b, a^{n-2}b^2, \ldots, ab^{n-1}, b^n\) and their inverses. However, every rational lamination \(L\), with the property that the corresponding laminary language contains these words, must contain the rational sublamination \(L(w)\) for some \(w \in F(a,b)\) that contains both letters, \(a\) and \(b\), or their inverses. But then \(\mathcal{L}_n(L)\) must also contain the word \(bx\) in \(\mathcal{L}_2(L)\), for some \(x \in A \cup A^{-1} \setminus \{b, b^{-1}\}\). This contradicts the above Convergence criterion from Remark 6.3, for any \(L_k = L\) as above. \(\square\)
On the other hand, the closure of the rational laminations seems to be a reasonable subspace of \( \Lambda^2 \), as shown by the following:

**Proposition 6.6.** \( \mathcal{X}_{\text{rat}} \) contains all minimal algebraic laminations.

**Proof.** We prove the proposition for non-orientable minimal laminations, where \( F_N \)-orbits and \( < F_N \), flip \( > \)-orbits agree, and leave the generalization for orientable laminations to the reader.

Let \( L^2 \) be a minimal algebraic lamination and \( A \) a basis of \( F \). Let \( L_A = \rho_A^2(L^2) \) be the symbolic lamination and \( L = \rho_L^2(L_A) \) the laminary language canonically associated to \( L^2 \). By minimality of \( L^2 \) the language \( L \) has the bounded gap property (see Proposition 5.6): For any integer \( n \) there exists a bound \( K = K(n) \) such that for any words \( u \) and \( w \) of \( L \) where the length of \( u \) is smaller than \( n \) and the length of \( w \) is greater than \( K \), \( u \) occurs as a subword of \( w \).

This proves that for any word \( w \) of \( L \) of length greater than \( K \) we have \( L_n(w) = L_n(L^2) \). If moreover \( w \) is cyclically reduced, we obtain:

\[
L_n(L(w)) \supset L_n(w) = L_n(L^2)
\]

Now let \( u \) be any word of \( L \) of length \( n \) and \( v \) another word of \( L \) of length \( 3K \). Write \( v = w_1w_2w_3 \) where \( w_1, w_2, w_3 \) are all of length \( K \). The product \( w_1w_2w_3 \) is reduced, and each \( w_i \) is a subword of \( v \). Now \( u \) must be a subword of both, \( w_1 \) and \( w_3 \): We can write the corresponding reduced products \( w_1 = w'_1uw'_1' \) and \( w_3 = w'_3uw'_3' \), and we define:

\[
v' = uww'w_3'
\]

Since \( v' \) contains \( w_2 \) as subword, its length is bigger than \( K \), and hence the previous equality applies: \( L_n(v') = L_n(L^2) \). Moreover, since \( w'_1u \) is a subword of the reduced word \( w_3 \), it follows that \( v' \) is cyclically reduced, and hence \( L_n(L(v')) \supset L_n(L^2) \). Finally, since \( u \) has length \( n \), any subword of length \( n \) of the reduced biinfinite word \( \ldots v'v'v' \ldots \) that is not a subword of \( v' \) is necessarily a subword of \( w_2w_3' \), and hence of \( v' \). Hence we get \( L_n(L(v')) \subset L_n(L^2) \) and thus

\[
L_n(L(v')) = L_n(L^2).
\]

Thus, for any integer \( n \) we found a word \( v' = v'(n) \in F(A) \) such that the rational lamination \( L(v'(n)) \) satisfies \( L_n(L(v'(n))) = L_n(L^2) \). Hence the Convergence criterion of Remark 6.3 gives directly that \( L(v'(n)) \xrightarrow{n \to \infty} L^2 \). \( \square \)

The two previous propositions imply directly Theorem 1.2.

### 7 Bounded cancellation

An important tool when dealing with more than one basis in a free group \( F_N \) is Cooper’s cancellation bound [Coo87]. We denote by \( |w|_A \) the length of the element \( w \in F_N \) when written as reduced word in a basis \( A \) of \( F_N \).
Lemma 7.1. Let \( \alpha \) be an automorphism of a free group \( F_N \) and let \( \mathcal{A} \) be a basis of \( F_N \). Then there exists a constant \( C \geq 0 \) such that, for any elements \( u, v \in F_N \) with

\[
|u|_{\mathcal{A}} + |v|_{\mathcal{A}} = |uv|_{\mathcal{A}}
\]

(i.e. there is no cancellation in the product \( uv \) of the reduced words \( u \) and \( v \)) one has

\[
0 \leq |\alpha(u)|_{\mathcal{A}} + |\alpha(v)|_{\mathcal{A}} - |\alpha(uv)|_{\mathcal{A}} \leq 2C
\]

As any second base \( \mathcal{B} \) is the preimage of \( \mathcal{A} \) under some \( \alpha \in \text{Aut}(F_N) \), the last line of the above statement can equivalently be replaced by

\[
0 \leq |u|_{\mathcal{B}} + |v|_{\mathcal{B}} - |uv|_{\mathcal{B}} \leq 2C
\]

We denote by \( \text{BBT}(\mathcal{A}, \alpha) \) or \( \text{BBT}(\mathcal{A}, \mathcal{B}) \) the smallest such constant \( C \).

An elementary proof of the above lemma can be given inductively, by decomposing the given automorphism (or basis change) into elementary Nielsen transformations. In modern geometric group theory language, one can restate the lemma as a special case of the fact that any two word metrics on a group \( G \) based on two different finite generating systems give rise to a quasi-isometry which realizes the identity on \( G \).

This lemma has been interpreted and generalized in term of maps between trees in [GJLL98]. We describe now this interpretation; a generalization is given in [CHL-II].

Let \( T_{\mathcal{A}} \) and \( T_{\mathcal{B}} \) be the metric realizations (with constant edge length 1) of the Cayley graphs of \( F_N \) with respect to \( \mathcal{A} \) and \( \mathcal{B} \). Let \( i = i_{\mathcal{A}, \mathcal{B}} \) the equivariant map from \( T_{\mathcal{A}} \) to \( T_{\mathcal{B}} \) which is the identity on vertices and which is linear (and thus locally injective) on edges. Then Cooper’s cancellation lemma 7.1 can be rephrased as:

Lemma 7.2. For any (possibly infinite) geodesic \([P, Q]\) in \( T_{\mathcal{A}} \) the image \( i([P, Q]) \) lies in the \( C \)-neighborhood in \( T_{\mathcal{B}} \) of \([i(P), i(Q)]\), for some \( C > 0 \) (in particular for \( C = \text{BBT}(\mathcal{A}, \mathcal{B}) \) as above) independent on the choice of \( P, Q \in T_{\mathcal{A}} \).

Finally, we state the following lemma that is used in [CHL-II]:

Lemma 7.3. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two bases of \( F_N \). Any element \( w \) of \( F_N \) which is cyclically reduced with respect to the basis \( \mathcal{A} \) is “almost cyclically reduced with respect to \( \mathcal{B} \)”. More specifically, if

\[
w = y_1 \cdots y_r y_{r+1} \cdots y_n y_{r}^{-1} \cdots y_1^{-1}
\]

with \( y_i \in \mathcal{B}^{\pm 1} \) is a reduced word (in particular with \( y_{r+1} \neq y_{r}^{-1} \) and \( y_n \neq y_{r} \)), then one has \( r \leq \text{BBT}(\mathcal{A}, \mathcal{B}) \).

Proof. Apply Lemma 7.1 to \( w^2 \).

\( \square \)
8 The Out($F_N$)-action on laminations and laminary languages

In §2 we briefly mentioned that there is a natural action by any automorphism of $F_N$ as homeomorphism on the boundary $\partial F_N$, and thus on $\Lambda^2$. This is a well known result in geometric group theory: Indeed the very fact that the boundary of a free group can be defined without any reference to a given basis is exactly equivalent to that statement. The key fact here is that a basis change in $F_N$ (or, equivalently, an automorphism of $F_N$) induces a change of the metric on $F_N$ (see §6) in a Lipschitz equivalent way. Therefore it changes the induced metric on the boundary (viewed as the set of one-sided infinite reduced words, see §6) in a Hölder equivalent way.

A more direct combinatorial way to define the action of Out($F_N$) on languages is given as follows: Notice first that the elementwise image $\alpha(L)$ of a laminary language $L$ under an automorphism $\alpha \in \text{Aut}(F_N)$ is in general not a laminary language. By Lemma 7.1, for $C = \text{BBT}(A, \alpha)$ the language $\alpha(L)_{|C}$ is laminary, and by Lemma 5.4 we have $L(\alpha(L)) = L(\alpha(L)_{|C})$. Thus, if we consider the outer automorphism $\hat{\alpha} \in \text{Out}(F_N)$ defined by $\alpha$, we can define:

$$\hat{\alpha}(L) = \alpha(L)_{|C} = L(L(\alpha(L)))$$

It follows directly from the second equality that this does not depend on the choice of the automorphism $\alpha$ in the class $\hat{\alpha}$. It also follows directly from our definitions that this action of $\hat{\alpha}$ is in fact a homeomorphism of the space $\Lambda_L$ of laminary languages in $\mathcal{A}^{\pm 1}$.

Similarly, for any symbolic lamination $L_A$ we define

$$\hat{\alpha}(L_A) = L(L(\alpha(L))).$$

From these definitions we see directly that the actions of $\hat{\alpha}$ commute with the (bijective) map $\rho_A^2 : \Lambda_A \rightarrow \Lambda_L$ given in Proposition 5.3.

If $\beta$ is a second automorphism of $F_N$ and $C' = \text{BBT}(A, \beta)$, one gets from Lemma 7.1 that

$$\alpha(\beta(L))_{|C'} = (\alpha \beta)(L)_{|C''},$$

with $C'' = |\alpha|_A C' + C$ and $|\alpha|_A = \max\{|\alpha(x)|_A : x \in A\}$. This shows that the definitions above give an action of Out($F_N$) on $\Lambda_L$ and on $\Lambda_A$.

Applying Lemma 7.1 again, we get that, if $(X, X')$ is a leaf of an algebraic lamination $L^2$, then any subword of $\rho_A(\alpha(X), \alpha(X'))$ is a word in $\alpha(L(X^{-1}X'))_{|C'}$. This proves that $\rho_A^2$ is Out($F_N$)-equivariant and thus concludes the proof of Theorem 1.1.

Each of the above two versions of the Out($F_N$)-actions has its own virtues: Surprisingly, the action on laminary languages generalizes much more directly to more general homomorphisms $\varphi : F_N \rightarrow F_M$ of free groups. It is noteworthy in this context that non-injective substitutions on biinfinite sequences are treated
classically in symbolic dynamics in a similar vein as injective ones, while from a geometric group theory standpoint it is impossible to extend a non-injective map $\varphi$ as above in any meaningful way to a map $\partial \varphi : \partial F_N \to \partial F_M$. The more common injective case, however, is easy to understand even from the geometric group theory standpoint:

**Remark 8.1.** It is well known that every finitely generated subgroup of a free group is quasi-convex. Thus an embedding $\varphi : F_M \subset F_N$ induces canonically an embedding $\partial \varphi : \partial F_M \subset \partial F_N$, see [GdlH90]. Clearly, this extends to an embedding $\partial \varphi^2 : \partial^2 F_M \subset \partial^2 F_N$, but since the image $\partial \varphi^2(\partial^2 F_M) \subset \partial^2 F_N$ is in general not $F_N$-invariant, an algebraic lamination $L^2 \subset \partial^2 F_M$ is mapped by $\partial \varphi^2$ to a set $\partial \varphi^2(L^2) \subset \partial^2 F_N$ that is in general not an algebraic lamination. By taking the closure of $\partial \varphi^2(L^2)$ with respect to the topology, the $F_N$-action, and the flip map, one obtains however a well defined algebraic lamination, which we denote by $\varphi_L(L^2)$, thus defining a natural map:

$$\varphi_L : \Lambda^2(F_M) \to \Lambda^2(F_N)$$

However, it has to be noted immediately that this map $\varphi_L$ does not have to be injective: It suffices that the embedding $\varphi$ maps elements $v, w \in F_M$ which are not conjugate in $F_M$ to elements $\varphi(v), \varphi(w)$ that are conjugate in $F_N$: Then the associated rational laminations satisfy

$$L^2(v) \neq L^2(w) \in \Lambda^2(F_M),$$

but also

$$\varphi_L(L^2(v)) = L^2(\varphi(v)) = L^2(\varphi(w)) = \varphi_L(L^2(w)) \in \Lambda^2(F_N).$$

On the other hand, we note that if $F_M$ is a free factor of $F_N$, then the lamination space $\Lambda^2(F_M)$ is canonically embedded into $\Lambda^2(F_N)$: it suffices to consider a basis of $F_N$ which contains as a subset a basis of $F_M$.

It seems to be an interesting question of when precisely the map $\varphi_L : \Lambda^2(F_M) \to \Lambda^2(F_N)$ induced by an embedding $\varphi : F_M \subset F_N$ is injective, and in particular, if this is the case if and only if the subgroup $F_M$ is malnormal in $F_N$.

We finish this paper with an answer to the question we posed in §2.

**Proposition 8.2.** Let $A$ be a basis of $F_N$, and let $a$ be an element of $A$. Then, for any $N \geq 2$, the closure of the $\text{Out}(F_N)$-orbit of the rational lamination $L(a)$ in $\Lambda^2$ is not the only non-empty minimal closed $\text{Out}(F_N)$-invariant subspace of $\Lambda^2$.

**Proof.** Let $a$ be as above, and let $b$ be another element of $A$. Consider the rational lamination $L([a,b])$. Then for any outer automorphism $\alpha$ of $F_N$ and any automorphism $\alpha$ representing it, one has

$$\alpha(L([a,b])) = L(\alpha([a,b])).$$


As the derived subgroup is characteristic, the $\text{Out}(F_N)$-orbit of $L([a, b])$ consists of some minimal rational laminations associated to cyclically reduced words of the derived subgroup. Now any cyclically reduced word of the derived subgroup contains a subword of the form $xy$, where $x, y$ are distinct elements of $A^{\pm 1}$ with $x \neq y^{-1}$. This proves that for any outer automorphism $\hat{\alpha}$, the laminatory language $L(\hat{\alpha}(L([a, b])))$ contains a reduced word of the form $xy$. It follows from the Convergence criterion in Remark 6.3 that $L(a)$ is not in the closure of the $\text{Out}(F_N)$-orbit of $L([a, b])$. $\square$

References

[ABHS05] P. Arnoux, V. Berthé, A. Hilion and A. Siegel. Fractal representation of the attractive lamination of an automorphism of the free group. *Ann. Inst. Fourier* **56**, 2161-2212 (2006)

[BFH00] M. Bestvina, M. Feighn and M. Handel. The Tits alternative for $\text{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms. *Ann. of Math.* **151**, 517–623 (2000).

[CB88] A.. Casson and S. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*. London Mathematical Society Student Texts **9**. Cambridge University Press., 1988.

[CHL-II] T. Coulbois, A. Hilion and M. Lustig. $\mathbb{R}$-trees and laminations for free groups II: The lamination associated to an $\mathbb{R}$-tree. Preprint 2006.

[CHL-III] T. Coulbois, A. Hilion and M. Lustig. $\mathbb{R}$-trees and laminations for free groups III: Currents and dual $\mathbb{R}$-tree metrics. Preprint 2006.

[Coo87] D. Cooper. Automorphisms of free groups have finitely generated fixed point sets. *J. Algebra* **111**, 453–456 (1987).

[FLP91] A. Fathi, F. Laudenbach and V. Poénaru, editors *Travaux de Thurston sur les surfaces*. Séminaire Orsay, Astérisque **66-67**, 1991. Reprint of *Travaux de Thurston sur les surfaces*, Soc. Math. France, Paris, 1979.

[Fog02] N. Pytheas Fogg. Substitutions in Dynamics, Arithmetics and Combinatorics. In *Introduction to Automata and Substitution Dynamical Systems*, V. Berthé, S. Ferenczi, Ch. Mauduit and A. Siegel (eds.) Springer, 2002.

[GJLL98] D. Gaboriau, A. Jaeger, G. Levitt and M. Lustig. An index for counting fixed points of automorphisms of free groups. *Duke Math. J.* **93**, 425–452 (1998).

[GdlH90] E. Ghys and P. de la Harpe. *Sur les groupes hyperboliques d’après Mikhael Gromov*. Progress in Mathematics **83**. Birkhäuser, Boston 1990.
[Kap03] I. Kapovich. The frequency space of a free group. To appear in *Internat. J. Algebra Comput.*, available on ArXiv GR/0311053.

[Kap04] I. Kapovich. Currents on free groups. Preprint 2004 Arxiv GR/0412128.

[LL03] G. Levitt and M. Lustig. Irreducible automorphisms of $F_n$ have north-south dynamics on compactified outer space. *J. Inst. Math. Jussieu* 2, 59–72 (2003).

[Lus92] M. Lustig. *Automorphismen von freien Gruppen*. Habilitationsschrift, Bochum 1992.

[Mar95] R. Martin. *Non-Uniquely Ergodic Foliations of Thin Type, Measured Currents and Automorphisms of Free Groups*. Ph.D.-thesis, UCLA, 1995.

[Nar96] Ph. Narbel. The boundary of iterated morphisms on free semigroups. *Internat. J. Algebra Comput.* 6, 229–260 (1996).

Thierry Coulbois, Arnaud Hilion and Martin Lustig
Mathématiques (LATP)
Université Paul Cézanne – Aix-Marseille III
av. escadrille Normandie-Niémen
13397 Marseille 20
France
Thierry.Coulbois@univ-cezanne.fr
Arnaud.Hilion@univ-cezanne.fr
Martin.Lustig@univ-cezanne.fr