Statistical cluster point and statistical limit point sets of subsequences of a given sequence

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Abstract

J.A. Fridy [Statistical limit points, Proc. Amer. Math. Soc., 1993] considered statistical cluster points and statistical limit points of a given sequence \( x \). Here we show that almost all subsequences of \( x \) have the same statistical cluster point set as \( x \). Also, we show an analogous result for the statistical limit points of \( x \).

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1. Introduction

Fridy [1] has proven that \( \Gamma_x \), the set of statistical cluster points of \( x = (x_n) \), is always a closed set and \( \Gamma_x \) is non-empty if \( x \) is bounded. However \( \Lambda_x \), the set of statistical limit points of \( x \), need not be closed. In [2] H.I. Miller studied statistical convergence and relations between statistical convergence of a sequence \( x \) and statistical convergence of the subsequences of \( x \). In particular, in [2], it is shown that if \( L \) is the statistical limit of \( x \), then almost all subsequences of \( x \) have \( L \) as their statistical limit. Here we combine two notions, statistical cluster points and subsequences, showing that \( \Gamma_x \) is equal to the statistical cluster point set of almost all subsequences of \( x \). This is a continuation of the results in [3] that also combine statistical cluster points and subsequences. Namely, in [3] it is shown that if \( \Gamma_x \neq \emptyset \) and \( F \) is a non-empty closed subset of \( \Gamma_x \), then there exists a subsequence \( y \) of \( x \) such that \( \Gamma_y = F \). Additionally we show that \( \Lambda_x \) is equal to the statistical limit point set of almost all subsequences of \( x \). This is a continuation of the results in [4] that also combine statistical limit points and subsequences.

2. Preliminaries

If \( t \in (0,1) \), then \( t \) has a unique binary expansion \( t = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \), \( e_n \in \{0,1\} \), with infinitely many ones. Next if \( x = (x_n) \) is a sequence of reals, for each \( t \in (0,1) \), let \( x(t) \) denote the subsequence of \( x \) obtained by the following rule: \( x_n \) is in the subsequence if and only if \( e_n = 1 \). Clearly the mapping \( t \to x(t) \) is a one-to-one onto mapping between \( (0,1] \) and the collection of all subsequences of \( x \).
If $K$ is a subset of the positive integers $N$, then following Fridy [1], $K_n$ denotes the set \{\(k \in K : k \leq n\)\} and $|K_n|$ denotes the number of elements in $K_n$. The natural density of $K$ (see [5]) is given by $\delta(K) = \lim_{n \to \infty} n^{-1}|K_n|$, provided this limit exists. In the case that $\delta(K) = 0$ we say that $K$ is thin, and otherwise we say that $K$ is non-thin.

Statistical convergence of a sequence is defined as follows.

We say that $L$ is the statistical limit of the sequence $x$, if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$  

Statistical convergence and its connection to subsequences is studied in [2].

Statistical limit points and statistical cluster points of a sequence $x$ are defined as follows.

We say that a number $\lambda$ is a statistical limit point of a sequence of reals $x = (x_n)$ if

$$\lim_{n \to \infty} x_{n_k} = \lambda$$

for some non-thin subsequence of $(x_n)$.

We say that a number $\gamma$ is a statistical cluster point of a sequence of reals $(x_n)$ if for every $\epsilon > 0$ the set $\{k \in N : |x_k - \gamma| < \epsilon\}$ is non-thin.

In [1], given a sequence $x$, three sets are considered. $L_x$, the set of limit points of $x$; $\Lambda_x$, the set of statistical limit points of $x$, and $\Gamma_x$, the set of statistical cluster points of $x$.

Also, if $x$ is bounded, then $\Gamma_x$ is closed and non-empty.

In this paper we want to examine, $\Gamma_x$ and its relation to $\Gamma_x(t)$. Additionally we also consider $\Lambda_x$ and its relation to $\Lambda_x(t)$.

3. Results

Our main result is the following.

**Theorem 3.1.** If $x = (x_n)$ is a bounded sequence, then $\Gamma_x = \Gamma_x(t)$ for almost all $t \in (0, 1)$ (in the sense of Lebesgue measure).

**Proof.** Since $\Gamma_x$ is closed, it is either finite or separable, i.e. there is a countable subset of $\Gamma_x$, \{\(l_n : n \in N\)\} such that its closure is $\Gamma_x$. We consider only the second case, the proof in the first case is much simpler.

First we show that $\Gamma_x \subseteq \Gamma_x(t)$ for almost all $t$. It is sufficient to show that $m(B_n) = 1$ for $n = 1, 2, \ldots$, where $B_n = \{t \in (0, 1] : l_n \in \Gamma_x(t)\}$. This is true since in that case $m(B) = 1$ for $B = \bigcap_{n=1}^{\infty} B_n$ and then $\{l_n : n \in N\} \subseteq \Gamma_x(t)$ for all $t \in B$ and consequently $\Gamma_x \subseteq \Gamma_x(t)$ for all $t \in B$.

Since $l_n \in \Gamma_x$, then for every $\epsilon > 0$, $\{k \in N : |x_k - l_n| < \epsilon\}$ is non-thin . If $\epsilon = \frac{1}{p}$ we can denote the above set by \(\{k_1^j, k_2^j, k_3^j, \ldots\}\). Then, since it is non-thin there exists $\delta_j > 0$ such that

$$\frac{1}{p}|\{i : k_i^j \leq p\}| > \delta_j$$

for infinitely many $p$. We can assume that $p = k_i^j$ for infinitely many sufficiently large $M$. Now for each $j$, by the Law of Large Numbers, the limiting frequency of $x_{k_i^j} i = 1, 2, \ldots$ among the sequence $x(t)$ is $\frac{1}{2}$ for almost all $t \in (0, 1]$, i.e. if $t = \sum_{m=1}^{\infty} \frac{\theta_m}{2^m}$, then $\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} t_{k_i^j} = \frac{1}{2}$ for almost all $t \in (0, 1]$. That is, $m(D_j) = 1$, where

$$D_j = \{t \in (0, 1] : \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} t_{k_i^j} = \frac{1}{2}\}$$

for all $j$. Hence if $D = \bigcap_{j=1}^{\infty} D_j$, $m(D) = 1$. Now we will check that $l_n$ is a statistical cluster point for each $t$ in $D$.

To see this we will show that $\{i \in N : |x(t)_i - l_n| < \frac{1}{j}\}$ is non-thin for every $j \in N$ and every $t \in D_j$. 


Consider the earlier mentioned $p = k^j_M$ for $M$ large enough. Then the number of such $i \leq p$, with $|x_i - t_n| < \frac{1}{j}$ is greater than $p\delta_j$. Now take $t \in D_j$. By (3.1),
\[ \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^m t^j_{ki} = \frac{1}{2}. \]
So for large $M$, $p = k^j_M$, we have
\[ \frac{1}{p} |i \leq p : |x(t)_i - l_n| < \frac{1}{j}| > \frac{\delta_j}{4}, \]
i.e. this holds for infinitely many $p$, i.e. $\{ i \in N : |x(t)_i - l_n| < \frac{1}{j} \}$ is non-thin for every $j \in N$ and every $t \in D_j$. Hence $l_n$ is a statistical cluster point for every $t \in D$. This completes the proof that $\Gamma_x \subseteq \Gamma_{x(t)}$ for almost all $t$.

Next we show that $\Gamma_{x(t)} \subseteq \Gamma_x$ for almost all $t$. We will show that this inclusion holds for all normal $t \in (0,1]$, i.e. for all $t = \sum_{n=1}^\infty \frac{e_n}{n}$ for which $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n e_i = \frac{1}{2}$. It is well known that almost all $t \in (0,1]$ are normal (see [5]).

Suppose that $l$ is a statistical cluster point of $x(t)$ for some normal $t$. Then for any $\epsilon > 0$, $\{ i : |(x(t))_i - l| < \epsilon \}$ is non-thin, i.e. there exists $\delta_\epsilon > 0$ such that
\[ \frac{1}{n} |i \leq n : |(x(t))_i - l| < \epsilon| > 2\delta_\epsilon \]
for infinitely many $n$. This implies that
\[ \frac{1}{n} |i \leq n : |x_i - l| < \epsilon| > \frac{1}{2}\delta_\epsilon \]
for infinitely many $n$, and hence $l$ is a statistical cluster point of $x$. Therefore $\Gamma_{x(t)} \subseteq \Gamma_x$ for all normal $t$, and consequently for almost all $t \in (0,1]$. Therefore we conclude that $\Gamma_{x(t)} = \Gamma_x$ for almost all $t \in (0,1]$. $\square$

Next, we will prove an analogous result for the set of statistical limit points of $x$ and its subsequences. The set $\Lambda_x$ is not necessarily closed (see [4]). However the following useful theorem was proved by Kostyrko, Mačaj, Salát and Strauch [4].

**Theorem 3.2.** For every bounded sequence $x$, the set $\Lambda_x$ is an $F_\sigma$-set in $R$.

In the proof of the above theorem, the authors show that
\[ \Lambda_x = \bigcup_{j=1}^\infty \Lambda(x, \frac{1}{j}) \]
where $\Lambda(x, \frac{1}{j}) = \{ l, \exists k_i, i = 1, 2, \ldots, \lim_{i \to \infty} x_{k_i} = l, \bar{\delta}(\{k_i\}) \geq \frac{1}{j} \}$ where $\bar{\delta}$ denotes the upper statistical density (i.e. $\bar{\delta}(\{k_i\}) = \lim \sup_{i \to \infty} \frac{k_i}{i}$) and $\Lambda(x, \frac{1}{j})$ is closed for all $j$.

Here is our second result.

**Theorem 3.3.** If $x = (x_n)$ is a bounded sequence, then $\Lambda_x = \Lambda_{x(t)}$ for almost all $t \in (0,1]$ (in the sense of Lebesgue measure).

**Proof.** We proceed in a similar manner as in the proof of Theorem 3.1.

First we show that $\Lambda_x \subseteq \Lambda_{x(t)}$ for almost all $t$.

As mentioned earlier, $\Lambda_x = \bigcup_{j=1}^\infty T_j$, where
\[ T_j = \Lambda(x, \frac{1}{j}) = \{ l, \exists k_i, i = 1, 2, \ldots, \lim_{i \to \infty} x_{k_i} = l, \bar{\delta}(\{k_i\}) \geq \frac{1}{j} \}. \]

Suppose $j \in N$ is fixed. Using the above notation (from [4]), $T_j$ is closed and separable so there exists a set $\{l_{ij} : i \in N \}$ such that its closure is $T_j$. Let $i \in N$. If $l = l_{ij}$, then by the Law of Large Numbers, $l \in \Lambda(x(t), \frac{1}{j})$, for all $t \in B_{ij}$, where $m(B_{ij}) = 1$. Let $B_j = \bigcap_{i=1}^\infty B_{ij}$. Then $m(B_j) = 1$. Hence $\{l_{ij} : i \in N \} \subseteq \Lambda(x(t), \frac{1}{j})$ for every $t \in B_j$. Now since $T_j$ and $\Lambda(x(t), \frac{1}{j})$ are both closed we get that $T_j \subseteq \Lambda(x(t), \frac{1}{j})$ for every $t \in B_j$. 

Therefore, \( \Lambda_x = \bigcup_{j=1}^{\infty} T_j \subseteq \bigcup_{j=1}^{\infty} \Lambda(x(t), \frac{1}{j}) = \Lambda_{x(t)} \) for all \( t \in \bigcap_{j=1}^{\infty} B_j \). Since \( m(\bigcap_{j=1}^{\infty} B_j) = 1 \), we have shown that \( \Lambda_x \subseteq \Lambda_{x(t)} \) for almost all \( t \).

Next we show that \( \Lambda_{x(t)} \subseteq \Lambda_x \) for almost all \( t \). Again we show that this inclusion holds for all normal \( t \in (0, 1] \). Suppose that \( l \) is a statistical limit point of \( x(t) \) for some normal \( t \). Then \( x(t) \) has a non-thin subsequence that converges to \( l \) (in the normal sense). It is easy to see that this subsequence \( x(t)_i = x_{k_i} \) is then also a non-thin subsequence of \( x \) and therefore \( l \) is also a statistical limit point of \( x \). This completes the proof.

4. Concluding remarks

We mentioned that \( m(\nu) = 1 \), where \( \nu \) is the set of normal numbers in \((0, 1]\). However \( \nu \) is a set of first Baire category. In light of this we suspect that a category analogue of our Theorem 3.1 is not true.

Also, one could examine possible analogues of our results using permutations rather than subsequences.

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