Large deviations of empirical measures of diffusions in fine topologies

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Abstract

We consider large deviations of empirical measures of diffusion processes. In a first part, we present conditions to obtain a large deviations principle (LDP) in a Wasserstein-like topology. In particular, we propose a precise class of unbounded functions for which the LDP holds. This provides an analogue to the standard Cramér condition in the context of diffusion processes, which turns out to be related to a spectral gap condition for a Witten–Schrödinger operator. Secondly, we study more precisely the properties of the Donsker–Varadhan rate functional associated with the LDP. We revisit and generalize some standard duality results as well as a more original decomposition of the rate functional with respect to the symmetric and antisymmetric parts of the dynamics. Finally, we apply our results to overdamped and underdamped Langevin dynamics, showing the applicability of our framework for degenerate diffusions in unbounded configuration spaces.

1 Introduction

Empirical averages of diffusion processes and their convergence are commonly studied in statistical mechanics, probability theory and machine learning. In statistical physics, an observable averaged along the trajectory of a diffusion typically converges to the expectation with respect to its stationary distribution, which provides some macroscopic information on the system [71, 81]. For reversible dynamics, this convergence is known to be characterized by an entropy functional [105] [7], which generalizes result for small fluctuations such as the central limit theorem [72] or Berry-Esseen type inequalities [88]. It has been shown for some time that the approach can be extended to nonequilibrium systems by considering generalized entropy and free energy functionals, as provided the theory of large deviations [28, 45, 103]. From a more computational perspective, studying the convergence of empirical averages is an important problem for the efficiency of Monte Carlo Markov Chain methods [1, 97, 95, 36].

Since its initiation by Cramér in the 30s [26, 105], large deviations theory has been given many extensions. The theory takes its origin in the study of fluctuations for sums of independent variables, leading to the celebrated Sanov theorem [29]. Interestingly, the necessity of Cramér’s exponential moment condition for the Sanov theorem to hold in a Wasserstein topology has been only recently proved [108].

Due to the above mentioned applications, it is natural to try to apply such a theory to diffusions, or more generally Markovian dynamics. This is useful for instance in statistical physics, when considering Gallavotti–Cohen fluctuation relations for irreversible systems [51, 76, 75], as well as for characterizing dynamical phase transitions in physical systems [52, 53, 54]. From a more computational perspective, studying the rate function associated with a given dynamics is interesting for designing better sampling strategies [10, 95, 96], which is important for instance in a Bayesian framework [11, 12] or for molecular dynamics [79, 80]. The approach can also be used for deriving concentration results such as Bernstein-type inequalities [52, 13] and uncertainty quantification bounds [70, 54].

However, proving a large deviations principle for correlated processes turns out to be a difficult task. A milestone in the theory is the series of papers by Donsker and Varadhan [31, 32, 34, 35] and the dual approach followed by Gärtner and Ellis [57, 14]. The strategy of the former works is to build explicitly lower and upper large deviations bounds from the Tchebychev inequality and the Girsanov theorem [109]. On the other hand, the
Gärtner–Ellis theorem relies on the existence and regularity of a free energy functional. This technique has been later related to optimal control problems through the so-called weak convergence approach [48, 69].

Whichever strategy is chosen, proving large deviations principles for empirical measures of diffusions in unbounded configuration spaces remains difficult. Indeed, studying the stability of unbounded Markov processes is already challenging, and often relies on Lyapunov function techniques [81, 82, 24, 55]. Such a Lyapunov function can be interpreted as an energy associated with the system, which decreases in average and provides a control on the excursions of the process far away from the origin. This technique can be used for proving LDPs, see for instance [106, Section 9] and [39, 111, 39]. However, the above mentioned works consider LDP in the so-called strong (resp. weak) topology, i.e. with respect to the topology on measures associated with the convergence of measurable bounded (resp. continuous bounded) functions. To the best of our knowledge, convergence in Wasserstein topologies (i.e. associated with unbounded functions) for diffusions has only been addressed in [73] and [111, Section 2.2]. Unfortunately, the nonlinear approach of [73] does not allow to characterize precisely the set of functions for which the LDP holds, while [111] considers a particular system (Langevin dynamics). In both cases, the rate function is not related to the standard Donsker–Varadhan theory [33]. Our first result is to derive the LDP in a Wasserstein topology under very natural conditions, and to express the rate function in duality with a free energy. From a practical point of view, this allows to compute the rate function from the free energy, a standard procedure [55, 103, 24, 85, 47].

Once a large deviations principle has been derived, providing alternative expressions of the rate function is an important problem. This can be useful for computing this function more efficiently, or for interpreting some key aspects of the dynamics (such as irreversibility for physical systems). Our first contribution in this direction is to derive a variational representation of the rate function similar to the Donsker–Varadhan formula [33]. This provides a variational representation of the principal eigenvalue for any non-symmetric linear second order differential operator associated with a diffusion, under confinement and regularity conditions. To the best of our knowledge, there is no such formula in an unbounded setting, a fortiori for unbounded functions and Wasserstein topologies on probability measures. Finally, it has been shown in a pioneering work [15] that (for a Langevin dynamics) the above mentioned duality allows to decompose the rate function into two parts: one corresponding to a “reversible” part and the other to an “irreversible part” of the dynamics. We extend these results to general diffusions by using Sobolev seminorms, a feature inspired by the small fluctuations framework developed in [72]. This decomposition turns out to be useful for various purposes. We apply it to study the rate function of the Langevin dynamics, in particular its dependence on the friction both in the Hamiltonian and overdamped limits.

We now sketch the main results of the paper, the precise setting being presented in Section 2.1. Consider a diffusion process $(X_t)_{t \geq 0}$ over a state space $\mathcal{X} \subset \mathbb{R}^d$ with generator $\mathcal{L}$ and invariant probability measure $\mu$, and the empirical measure

$$L_t := \frac{1}{t} \int_0^t \delta_{X_s} \, ds,$$

where $\delta_x$ is the Dirac mass at $x \in \mathcal{X}$. Our first contribution is to prove a large deviations principle with respect to $\mu$ for the empirical measure $(L_t)_{t \geq 0}$, in a Wasserstein topology associated with an unbounded function $\kappa : \mathcal{X} \to [1, +\infty)$. That is, we prove the following type of long time scaling: for $\Gamma \subset \mathcal{P}(\mathcal{X})$,

$$\mathbb{P}\left( L_t \in \Gamma \right) \asymp e^{-\inf_{\nu \in \Gamma} f(\nu)},$$

(2)

where $I$ is a rate function. Here, $\mathcal{P}(\mathcal{X})$ denotes the set of probability measures on $\mathcal{X}$, and the above scaling holds for the topology on measures associated with the weak convergence against functions $f$ satisfying

$$\|f\|_{\mathcal{B}^\infty} := \sup_{x \in \mathcal{X}} \frac{|f(x)|}{\kappa(x)} < +\infty.$$  

(3)

As is standard for a LDP on unbounded state spaces [106, 111], our result relies on the existence of a Lyapunov function $W : \mathcal{X} \to [1, +\infty)$ twice differentiable and such that

$$\Psi := -\frac{\mathcal{L}W}{W}$$

(4)

has compact level sets (in words, it goes to infinity at infinity). Contrarily to previous works where this condition implies [2] in a topology of bounded test functions [106, 39, 111], we show in Section 2 that the LDP holds for the
Wasserstein topology associated with any cost function \( \kappa \) controlled by \( \Psi \). Moreover, the associated rate function \( I : \mathcal{P}(X) \to [0, +\infty] \), also called entropy, reads

\[
\forall \nu \in \mathcal{P}(X), \quad I(\nu) = \sup_{f} \{ \nu(f) - \lambda(f) \},
\]

where

\[
\lambda(f) = \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\int_{0}^{t} f(X_s) \, ds} \right],
\]

is the cumulant or free energy function. We mention that our strategy relies on the Gärtner–Ellis theorem, according to which the existence and regularity of (9) implies the large deviations principle. We actually show that (9) is well-defined because it matches the principal eigenvalue of the Feynman–Kac operator

\[
\varphi \mapsto \mathbb{E} \left[ \varphi(X_t) e^{\int_{0}^{t} f(X_s) \, ds} \right].
\]

A key remark for defining the above operator is that it is a local martingale, as noted by Wu in [111]. This allows to define (9) for functions \( \varphi \) such that \( \| \varphi \|_{\mathcal{W}^{\infty}} < +\infty \), as soon as \( f \) is dominated by the function \( \Psi \) defined in (4). As a result, for any such \( f \), the operator (9) can be shown to be compact over the space of functions controlled by \( W \) (see [53] [54]), and the functional (5) is obtained as the largest eigenvalue of the operator (9) through a generalized Perron–Frobenius theorem (the Krein–Rutman theorem [27]).

The second part of our work consists in rewriting the large deviations functional \( I \). For this, we revisit [33] by showing that

\[
\forall \nu \in \mathcal{P}(X), \quad I(\nu) = \sup \left\{ - \int_{X} \frac{\mathcal{L} u}{u} \, d\nu, \ u \in \mathcal{D}^{+} \right\},
\]

where \( \mathcal{D}^{+} \) is an appropriate domain defined in Section 3 which differs from that of [33] through an additional growth condition. This result leads to a variational formula for the largest eigenvalue \( \lambda(f) \) of the operator \( \mathcal{L} + f \) defined on a suitable functional space through

\[
\lambda(f) = \sup_{\nu \in \mathcal{P}(X)} \{ \nu(f) - I(\nu) \}.
\]

We mention that proving (8) is easily realized thanks to the spectral problem associated with the Feynman–Kac operator [46], relying on the recent work [47].

Finally, the variational representation (8) allows to generalize the results of [33] by splitting \( I \) into two parts. More specifically, denoting by \( \mathcal{L} = \mathcal{L}_{S} + \mathcal{L}_{A} \) the decomposition into symmetric and antisymmetric parts of the generator considered on \( L^{2}(\mu) \), we obtain, for any \( \nu \ll \mu \):

\[
I(\nu) = \frac{1}{4} \left[ \log \frac{d\nu}{d\mu} \right]^{2}_{\mathcal{W}^{1}(\nu)} + \frac{1}{4} \left[ \mathcal{L}_{A} \left( \log \frac{d\nu}{d\mu} \right) \right]^{2}_{\mathcal{W}^{-1}(\nu)},
\]

where \( \cdot \big|_{\mathcal{W}^{1}(\nu)} \) and \( \cdot \big|_{\mathcal{W}^{-1}(\nu)} \) refer to the Sobolev seminorms defined in Section 2.1. Interestingly, the proof relies on a generalized Witten transform performed in the variational representation (9), which we may therefore call variational Witten transform. This shows that, for a given invariant measure, an irreversible dynamics \( (\mathcal{L}_{A} \neq 0) \) produces more entropy than a reversible one, in accordance with the second law of thermodynamics. This decomposition is useful for instance to study the entropy production of the Langevin dynamics, which is irreversible but has a particular structure which allows to naturally separate its reversible and irreversible parts.

The paper is organized as follows. Section 2 presents a large deviations principle in a Wasserstein topology for the empirical measure of diffusions, under Lyapunov and regularity conditions. Section 3 provides a rewriting of the rate functional, and its decomposition into symmetric and antisymmetric parts. Some examples of application are given in Section 4, in particular for overdamped and underdamped Langevin dynamics. Section 5 discusses possible extensions and connections with related works. Finally, most of the proofs are postponed to Section 6.
2 Large deviations principle

2.1 Setting

This section introduces the main notation used throughout the paper. We consider a diffusion process \((X_t)_{t \geq 0}\) evolving in \(\mathcal{X} \subset \mathbb{R}^d\) with \(d \in \mathbb{N}^*\), and satisfying the following stochastic differential equation (SDE):

\[
dX_t = b(X_t) dt + \sigma(X_t) dB_t,
\]

where \(b : \mathcal{X} \rightarrow \mathbb{R}^d\), \(\sigma : \mathcal{X} \rightarrow \mathbb{R}^{d \times m}\) and \((B_t)_{t \geq 0}\) is a \(m\)-dimensional Brownian motion for some \(m \in \mathbb{N}^*\). In order to avoid issues related to boundaries of \(\mathcal{X}\), we assume that either \(\mathcal{X} = \mathbb{R}^d\), \(\mathcal{X} = \mathbb{T}^d\) or \(\mathcal{X} = \mathbb{T}^d \times \mathbb{R}^d\), where \(\mathbb{T}^d\) is the \(d\)-dimensional torus (applications to other domains of \(\mathbb{R}^d\) can be treated upon appropriate modifications).

The last case is motivated by applications to the Langevin equation, where \(\mathcal{X} = \mathbb{R}^d\) and \(\mathbb{T}^d\) would be a bounded position space and \(\mathbb{R}^d\) the unbounded momentum space (see Section 12). The generator of the dynamics (9), denoted by \(\mathcal{L}\), reads

\[
\mathcal{L} = b \cdot \nabla + S \cdot \nabla^2, \quad \text{with} \quad S = \frac{\sigma \sigma^T}{2},
\]

where \(\sigma^T\) denotes the transpose of the matrix \(\sigma\) and \(\cdot\) is the scalar product on \(\mathbb{R}^d\). Moreover, \(\nabla^2\) denotes the Hessian matrix, while for two matrices \(A, B\) belonging to \(\mathbb{R}^{d \times d}\), we write \(A : B = \text{Tr}(A^T B)\). The domain of \(\mathcal{L}\) and the conditions on \(b\) and \(\sigma\) will be made precise in Section 2.2. The function \(S\) takes values in the set of symmetric positive matrices (not necessarily definite). We also introduce the carré du champ operator \(\mathcal{S}\) associated with \(\mathcal{L}\) defined by, for two regular functions \(\varphi, \psi\):

\[
\mathcal{S}(\varphi, \psi) = \frac{1}{2} \left( \mathcal{L}(\varphi \psi) - \varphi \mathcal{L}\psi - \psi \mathcal{L}\varphi \right) = \nabla \varphi \cdot S \nabla \psi.
\]

We will use the space of smooth functions growing at most polynomially, and whose derivatives grow at most polynomially:

\[
\mathcal{C} = \left\{ \varphi \in C^\infty(\mathcal{X}) \mid \forall \alpha \in \mathbb{N}^d, \exists N > 0 \text{ such that } \sup_{x \in \mathcal{X}} \frac{|\partial^\alpha \varphi(x)|}{(1 + |x|^2)^N} < +\infty \right\},
\]

as well as \(C_c^\infty(\mathcal{X})\) (resp. \(C_0(\mathcal{X})\)) the space of smooth functions with compact support (resp. continuous and bounded).

The space of bounded measurable functions, denoted by \(B^\infty(\mathcal{X})\), is endowed with the norm

\[
\|\varphi\|_{B^\infty} = \sup_{x \in \mathcal{X}} |\varphi(x)|.
\]

Moreover, we will need weighted function spaces and the corresponding probability measure spaces, which commonly appear in Markov chain theory \([83, 73, 58]\). For any measurable function \(W : \mathcal{X} \rightarrow [1, +\infty)\) we define

\[
B^S_W(\mathcal{X}) = \left\{ \varphi : \mathcal{X} \rightarrow \mathbb{R} \mid \|\varphi\|_{B^S_W} := \sup_{x \in \mathcal{X}} \frac{\varphi(x)}{W(x)} < +\infty \right\},
\]

and the corresponding space of probability measures (see \([99]\) Chapter 2) for duality results on measure spaces):

\[
\mathcal{P}_W(\mathcal{X}) = \left\{ \nu \in \mathcal{P}(\mathcal{X}) \mid \nu(W) < +\infty \right\}.
\]

The associated weighted total variation distance is (see for instance \([58]\)):

\[
\forall \nu, \eta \in \mathcal{P}_W(\mathcal{X}), \quad \text{d}_W(\nu, \eta) = \sup_{\|\varphi\|_{B^S_W} \leq 1} \left\{ \int_{\mathcal{X}} \varphi \, d\nu - \int_{\mathcal{X}} \varphi \, d\eta \right\} = \int_{\mathcal{X}} W(x)|\nu - \eta|(dx).
\]

Note that the spaces \([12]\) and \([13]\) are defined for an arbitrary measurable function \(W \geq 1\). It is possible to weaken the assumption \(W \geq 1\) but we will not need these refinements in this paper. We denote by \(\tau\)-topology the weak topology associated with the convergence of functions belonging to \(B^\infty(\mathcal{X})\). This means that for a
sequence \((\nu_n)_{n \in \mathbb{N}}\) in \(\mathcal{P}(\mathcal{X})\), \(\nu_n \to \nu\) in the \(\tau\)-topology if \(\nu_n(\phi) \to \nu(\phi)\) for any \(\phi \in B_1^\infty(\mathcal{X})\). When considering functions \(\phi \in B_1^\infty(\mathcal{X})\), we denote by \(\tau^\infty\) the associated topology \(\tau^\infty\) see also [10, Lemma 3.3.8] for details. When \(W(x) = 1 + |x|^\alpha\) for \(\alpha > 0\), this is nothing else than the topology induced by the Wasserstein distance associated with the cost \(x \mapsto |x|^\alpha\), see [102, Theorem 7.12].

We associate to the dynamics \((X_t)_{t \geq 0}\) the semigroup \((P_t)_{t \geq 0}\) defined through
\[
\forall \phi \in B_1^\infty(\mathcal{X}), \quad (P_t \phi)(x) = \mathbb{E}_x[\phi(X_t)], \tag{15}
\]
where \(\mathbb{E}_x\) stands for the expectation with respect to all realizations of the Brownian motion in \(\mathcal{B}\), for dynamics starting at \(x \in \mathcal{X}\). We say that \(\mu \in \mathcal{P}(\mathcal{X})\) is invariant with respect to the dynamics \((X_t)_{t \geq 0}\) if \((\mu P_t)(\phi) = \mu(\phi)\) for any \(\phi \in C_b(\mathcal{X})\), with the notation
\[
(\mu P_t)(\phi) = \mu(P_t \phi) = \int X \in \mathcal{X} \mathbb{E}_x[\phi(X_t)] \mu(dx).
\]
An equivalent condition is that \(\mu(\mathcal{L} \phi) = 0\) for \(\phi \in C_2^\infty(\mathcal{X})\).

We now follow the path of [72, Chapter 2] for defining other useful functional spaces. For any probability measure \(\mu \in \mathcal{P}(\mathcal{X})\), let
\[
L^2(\mu) = \left\{ \phi : \text{measurable} \mid \int X |\phi|^2 \mu < +\infty \right\}. \tag{16}
\]
For \(\phi \in C_1^\infty(\mathcal{X})\), we introduce the seminorm
\[
|\phi|_{\mathcal{H}^0}(\mu) = \int X \mathcal{Q}(\phi, \phi) \mu(dx), \tag{17}
\]
and the equivalence relation \(\sim_1\) through \(\phi \sim_1 \psi\) if and only if \(|\phi - \psi|_{\mathcal{H}^0}(\mu) = 0\). We denote by \(\mathcal{H}^1(\mu)\) the adherence of \(C_1^\infty(\mathcal{X})\) quotiented by \(\sim_1\), for the norm \(|\cdot|_{\mathcal{H}^0}(\mu)\). Note that \(\mathcal{H}^1(\mu)\) and \(L^2(\mu)\) are not subspaces of each other in general, but \(L^2(\mu) \subset \mathcal{H}^1(\mu)\) for instance if \(\mu\) satisfies a Poincaré inequality and \(S\) is positive definite. The difference between \(L^2(\mu)\) and \(\mathcal{H}^1(\mu)\) is however important for degenerate dynamics, see the application in Section 4.2. We now construct a space dual to \(\mathcal{H}^1(\mu)\) with the same density argument by introducing the seminorm: for \(\phi \in C_\infty^\infty(\mathcal{X})\),
\[
|\phi|_{\mathcal{H}^0}(\mu) = \sup \left\{ 2 \int X X \mathcal{Q}(\phi, \phi) d\mu \mid \phi \in C_1^\infty(\mathcal{X}) \right\}.
\]

We define the equivalence relation \(\sim_{-1}\) on \(C_{\infty}^\infty(\mathcal{X})\) by \(\phi \sim_{-1} \psi\) if and only if \(|\phi - \psi|_{\mathcal{H}^0}(\mu) = 0\). The space \(\mathcal{H}^{-1}(\mu)\) is then the adherence of \(C_{\infty}^\infty(\mathcal{X})\) quotiented by \(\sim_{-1}\), for the \(\mathcal{H}^{-1}(\mu)\)-norm. This is actually the dual space of \(\mathcal{H}^1(\mu)\), see [72, Section 2.2, Claim F].

Let us relate \(\mathcal{H}^1(\mu)\) to the more standard \(H^1(\mu)\) Sobolev space. If \(\mu\) is invariant with respect to \(\mathcal{L}\) it holds, for \(\phi \in C_{\infty}^\infty(\mathcal{X})\) (using \(\mathcal{L}(\phi^2) = 2\phi\mathcal{L}\phi + 2\mathcal{Q}(\phi, \phi)\)),
\[
|\phi|^2_{\mathcal{H}^0}(\mu) = 2 \int X \phi(-\mathcal{L}\phi) d\mu.
\]
In particular, when \(S = \text{Id}\) we have
\[
|\phi|^2_{\mathcal{H}^0}(\mu) = \int X |\nabla \phi|^2 d\mu.
\]
In this case, \(|\cdot|_{\mathcal{H}^0}(\mu)\) is the standard \(H^1(\mu)\) Sobolev seminorm [10]. For precisions on \(\mathcal{H}^1(\mu)\) and its use for proving central limit theorems for Markov processes, see [72, Chapter 2].

**Remark 1.** The space \(\mathcal{H}^{-1}(\mu)\) can be thought of as a weaker version of the space \(L^2(\mu)\) of functions in \(L^2(\mu)\) with average zero with respect to \(\mu\). Indeed, assume for instance that \(\phi \in L^2(\mu)\) (so \(\phi \in L^1(\mu)\)), \(\int X \phi d\mu \geq 0\) (which is not restrictive upon considering \(-\phi\)) and \(|\phi|_{\mathcal{H}^{-1}} < +\infty\). We may choose \(\psi \in C_{\infty}^\infty(\mathcal{X})\) such that
\[
\psi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2, \end{cases}
\]
and set $\psi_n(x) = n\psi(x/n)$, so $|\psi_n|_{L^p(\mu)} \leq C$ for some constant $C > 0$ independent of $n$. The definition of $H^{-1}(\mu)$ shows that

$$|\varphi|_{H^{-1}(\mu)} \geq 2n \int_{|x| \leq n} \varphi \, d\mu - C.$$  

By the dominated convergence theorem it holds

$$\int_{|x| \leq n} \varphi \, d\mu \longrightarrow \int_X \varphi \, d\mu \geq 0.$$  

Since $|\varphi|_{H^{-1}(\mu)} < +\infty$, we obtain by letting $n \to +\infty$ that $\mu(\varphi) = 0$. Since the functions of $H^{-1}(\mu)$ may not belong to $L^2(\mu)$, this dual space generalizes $L_2^0(\mu)$.

We also introduce some notation concerning the growth of functions. A function $f = \mathcal{X} \to \mathbb{R}$ is said to have compact level sets if for any $M \in \mathbb{R}$, the set

$$\{x \in \mathcal{X} \mid f(x) \leq M\}$$

is compact (with the convention that $\emptyset$ is compact). A function $g$ is said to be negligible with respect to $f$ (denoted by $g \ll f$) if $f/g$ has compact level sets, and $g$ is said to be equivalent to $f$ (denoted by $g \sim f$) if there exist constants $c, c' > 0$ and $R, R' \in \mathbb{R}$ such that

$$\forall x \in \mathcal{X}, \quad c'g(x) - R' \leq f(x) \leq cg(x) + R.$$  

**Remark 2.** The above definitions are useful when the state space $\mathcal{X}$ is unbounded. A sufficient condition for $f$ to have compact level sets in this case is for this function to be lower semicontinuous and to go to infinity at infinity (i.e. to be coercive). If $\mathcal{X}$ is bounded, all these criteria are automatically met for smooth functions.

Finally, we denote by $\liminf$ and $\limsup$ the inferior and superior limits respectively, while for a subset $A \subset \mathcal{Y}$ of a topological space $\mathcal{Y}$, $\overline{A}$ and $\mathring{A}$ denote the interior and closure of $A$ for the chosen topology on $\mathcal{Y}$. The function $1_A$ denotes the indicator function of the set $A$, i.e. $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise. For a Banach space $E$, $B(E)$ refers to the Banach space of bounded linear operators over $E$ with the usual norm. Some elements of large deviations theory are reminded in Appendix A.

### 2.2 Statement of the main results

The large deviations principle relies on three standard assumptions: hypoellipticity of the generator, irreducibility of the dynamics, and a Lyapunov condition. We start with our hypoellipticity assumption (which could certainly be relaxed for particular applications, see for instance [111]). It will be useful for proving regularization properties of the Feynman–Kac semigroup in Lemma 5.

**Assumption 1 (Hypoellipticity).** The functions $b$ and $\sigma$ in (9) belong to $\mathcal{S}^d$ and $\mathcal{S}^{d \times m}$ respectively, and the generator $\mathcal{L}$ defined in (10) satisfies the hypoelliptic Hörmander condition. More precisely, $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_{i=1}^d A_i^\dagger A_i + A_0,$$

where $(A_i)_{i=0}^d$ are first order differential operators with coefficients belonging to $\mathcal{S}$ and such that the family

$$\{A_i\}_{i=1}^d \bigcup \{[A_i, A_j]\}_{i,j=0}^d \bigcup \{[[A_i, A_j], A_k]\}_{i,j,k=0}^d \cdots$$

spans $\mathbb{R}^d$ at any $x \in \mathcal{X}$ for a finite number of commutators $n_x \in \mathbb{N}$.

This assumption is natural in practical situations, as illustrated in the applications of Section 4 covering elliptic and hypoelliptic diffusions, see [83, 89, 114] for details. Note that excluding the operator $A_0$ from the first family means that, if $\mathcal{L}$ satisfies Assumption 1, $\partial_t + \mathcal{L}$ is hypoelliptic and the transition kernel of $(X_t)_{t \geq 0}$ has a smooth density for any $t > 0$. This regularity requirement comes together with a controllability condition (recall that $\sigma$ takes values in $\mathbb{R}^{d \times m}$).
Assumption 2 (Controllability). For any \( x, y \in X \) and \( T > 0 \), there exists a \( C^1 \)-control \((u_t)_{t \in [0,T]}\) in \( \mathbb{R}^m \) such that the path \((\phi_t)_{t \in [0,T]}\) in \( X \) defined as

\[
\begin{align*}
\phi_0 &= x \\
\dot{\phi}_t &= b(\phi_t) + \sigma(\phi_t)\dot{u}_t \\
\end{align*}
\]

(18)
is well-defined and satisfies \( \phi_T = y \).

Assumption 2 together with Assumption 1 implies that the process is irreducible, i.e. that the transition density of \((X_t)_{t \geq 0}\) is everywhere positive, which will be used in Lemma 6. Note that constructing a control \((u_t)_{t \in [0,T]}\) may be difficult in general [67]. However, for the overdamped and underdamped Langevin dynamics we are interested in, building such a control turns out to be genuinely feasible, see [83, 102, 94, 80, 82] and references therein. Let us mention that the above two assumptions are not specific to our problem of large deviations [94].

A typical idea for dealing with Markov chain stability and large deviations on an unbounded state space is to reduce the analysis to a compact set and to control the excursions of the dynamics out of this set with a Lyapunov function [84, 111]. Our Witten–Lyapunov condition for the dynamics reads as follows (for the terminology, see Remark 4 below).

Assumption 3 (Witten–Lyapunov condition). There exists a function \( W : X \rightarrow [1, +\infty) \) of class \( C^2(X) \), with compact level sets, and such that

\[
\Psi := -\frac{LW}{W}
\]

has compact level sets. Moreover, there exists a \( C^2(X) \) function \( \Upsilon : X \rightarrow [1, +\infty) \) with \( \Upsilon \ll W \) and such that, for some constants \( C_1 > 0, C_2 \in \mathbb{R} \),

\[
\Upsilon^2 \leq C_1 W, \quad \Psi \sim -\frac{LW}{W}, \quad -2\frac{LW}{W} \leq -\frac{LW}{W} + C_2.
\]

(20)

In all what follows, we consider an arbitrary continuous function \( \kappa : X \rightarrow [1; +\infty) \) such that:

* \( \kappa \ll \Psi \);
* \( \kappa \) is either bounded or has compact level sets.

Remark 3. Note that, since \( \kappa \ll \Psi \) and \( \Psi \sim -\frac{LW}{W} \), it holds \( \kappa \ll -\frac{LW}{W} \). This fact will be frequently used in the proofs. Moreover the conditions (20) are not restrictive for exponential-like Lyapunov function as shown in Proposition 4 below – the idea being that \( \Upsilon \) can be set to \( \sqrt{W} \). In practice, the auxiliary function \( \Upsilon \) is used to obtain some control in the proofs of Lemmas 3 and 5 (in particular to apply a Grönwall lemma), and it could certainly be phrased differently. The continuity condition on \( \kappa \) may be relaxed for instance by assuming this function to be lower semicontinuous and bounded on compact sets.

Although we stated Assumption 3 in order to fit standard conditions when considering large deviations on unbounded state spaces [105, 111], in practice it can be obtained from a non-linear Lyapunov condition in the spirit of [73] and [29, Condition 2.2]. This is the purpose of the next proposition, whose proof is postponed to Appendix B.

Proposition 1. Assume that there exists \( V \in \mathcal{V} \) such that:

* \( V \) has compact level sets;
* \( |\sigma^T \nabla V| \) has compact level sets;
* for any \( \theta \in (0,1) \),

\[
-\mathcal{L}V - \frac{\theta}{2} |\sigma^T \nabla V|^2 \sim |\sigma^T \nabla V|^2.
\]

(21)

Then Assumption 3 is satisfied with

\[
W(x) = e^{\theta V(x)}, \quad \Upsilon(x) = e^{\epsilon V(x)},
\]

for \( \theta \in (0,1) \) and \( \epsilon < \theta/2 \) small enough. In this case it holds

\[
\Psi \sim |\sigma^T \nabla V|^2.
\]
Note that (24) means that the term $-\mathcal{L}V$ coming from the dynamics must compensate the quadratic loss proportional to $|\sigma^T \nabla V|^2$. A first consequence of Assumptions 1 to 3 is the ergodicity of the dynamics.

**Proposition 2.** Under Assumptions 1, 2, and 3, there exists a global solution, and the process $(X_t)_{t \geq 0}$ admits a unique invariant probability measure $\mu \in \mathcal{P}_W(X)$. This measure has a positive $C^\infty(X)$-density with respect to the Lebesgue measure: there exists $\rho^\mu \in C^\infty(X)$ with $\rho^\mu > 0$ such that $\mu(dx) = \rho^\mu(x)dx$. Moreover, the dynamics is ergodic with respect to $\mu$: there exist $C, c > 0$ such that

$$\forall t > 0, \quad \forall \varphi \in B_B^{\infty}(X), \quad \|P_t \varphi - \mu(\varphi)\|_{B_B^{\infty}} \leq Ce^{ct} \|\varphi - \mu(\varphi)\|_{B_B^{\infty}}.$$  

Equivalently,

$$\forall t \geq 0, \quad \forall \nu \in \mathcal{P}_W(X), \quad d_W(\nu P_t, \mu) \leq Ce^{ct} d_W(\nu, \mu).$$

**Proof.** The existence of a unique local strong solution is standard when Assumption 1 holds, see [93, Chapter IX, Exercise (2.10)]. Assumption 3 then implies the existence of $a > 0$, $b \in \mathbb{R}$ such that

$$\mathcal{L}W \leq -aW + b,$$

and global existence can be deduced from the above Lyapunov inequality [93]. The end of the proof is a direct application of [94, Theorem 8.9] since Assumption 2 together with Assumption 1 ensures irreducibility.

We can now present the large deviations principle associated with the empirical measure of the process $(X_t)_{t \geq 0}$ with respect to its invariant measure $\mu$. Recall that the empirical measure of the process is defined by

$$L_t := \frac{1}{t} \int_0^t \delta_{X_s} \, ds,$$  

where $\delta_x$ denotes the Dirac mass at $x \in X$. When one considers large deviations principles for empirical averages of the form (22), the topology on probability measures has to be specified. As mentioned in the introduction, most of the proofs of LDPs consider topologies associated with bounded measurable functions (resp. continuous bounded), the so-called strong topology or $\tau$-topology (resp. weak topology). We now prove that, in our setting, a LDP holds in the $\tau$-topology defined in Section 2.1, for any function $\kappa$ satisfying Assumption 2. The proof of Theorem 1 is presented in Section 6.1. We recall that a rate function is said to be good if its level sets are compact.

**Theorem 1.** Suppose that Assumptions 1, 2, and 3 hold true, and consider a function $\kappa$ as in Assumption 3. Then, almost surely,

$$L_t \xrightarrow{t \to +\infty} \mu,$$  

in the $\tau$-topology. Moreover, the functional

$$f \in B_B^{\infty}(X) \mapsto \lambda(f) := \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E}_x \left[ e^{\int_0^t f(X_s) \, ds} \right]$$

is well-defined, convex and finite, and $(L_t)_{t \geq 0}$ satisfies a LDP in the $\tau$-topology with the good rate function defined by:

$$\forall \nu \in \mathcal{P}(X), \quad I(\nu) = \begin{cases} 
\sup_{f \in B_B^{\infty}} \{ \nu(f) - \lambda(f) \}, & \text{if } \nu \in \mathcal{P}_a(X) \text{ and } \nu \ll \mu, \\
+\infty, & \text{otherwise}.
\end{cases}$$

More precisely, for any $\tau$-measurable set $\Gamma \subset \mathcal{P}(X)$, it holds

$$-\inf_{\nu \in \Gamma} I(\nu) \leq \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{P}_x \left( L_t \in \Gamma \right) \leq \liminf_{t \to +\infty} \frac{1}{t} \log \mathbb{P}_x \left( L_t \in \Gamma \right) \leq -\inf_{\nu \in \Gamma} I(\nu),$$

where the interior and closure of $\Gamma$ are taken with respect to the $\tau$-topology. Finally, for any $\nu \in \mathcal{P}(X)$, it holds $I(\nu) = 0$ if and only if $\nu = \mu$. 

8
Our conclusion is in essence close to that of [73], but the conditions to reach it seem more natural to us and correspond to usual conditions for proving large deviations principles in an unbounded state space, see [111, 39] and [106] Section 9. In particular, they allow to derive the duality representation (25), and we do not need to consider non-linear operators. Our strategy (presented in Section 6.1) relies on the Gärtner–Ellis theorem [54, 44, 45, 28], for which the existence of the free-energy (24) is a key element. The originality of our work is to make use of the local martingale (27) introduced by Wu [111] in order to solve the spectral problem associated with the Feynman–Kac operator, which proves the existence of (24). This directly provides the LDP in the $\tau^n$-topology by duality. However, there may be cases in which a LDP holds although the conditions of the Gärtner–Ellis theorem are not satisfied, for instance in the framework of the Sanov theorem [108], so our conditions may not be necessary.

Another advantage of our approach is to characterize precisely the set of functions for which a LDP holds from the standard condition on $\Psi$ defined in (19), like in [31, 106]. This condition is also used in [111, Corollary 2.3] for proving a level 1 LDP for the Langevin dynamics. We present below a clear connection with a spectral gap condition for the Witten–Schrödinger operator in the reversible case. The comparison with Cramér’s condition for independent variables highlights the effect of correlations on fluctuations.

**Remark 4** (Reversible processes, Witten Laplacian and Cramér’s condition). Consider the following reversible diffusion

$$dX_t = -\nabla V(X_t)\, dt + \sqrt{2} dB_t,$$

where $V : X \to \mathbb{R}$ is a smooth potential with compact level sets. The generator of this dynamics is $\mathcal{L} = -\nabla V \cdot \nabla + \Delta$ and its invariant probability measure reads $\mu(dx) = Z^{-1} e^{-V(x)} dx$, where we assume that $Z = \int_X e^{-V(x)} dx < +\infty$.

Define

$$W_\theta(x) = e^{\theta V(x)},$$

for some $\theta \in (0, 1)$. This is a standard choice for obtaining compactness of the evolution operator [73, Section 8], and optimal control representations of rate functions [39], see also Proposition 1. An easy computation shows that

$$\Psi_\theta = -\frac{\mathcal{L}W_\theta}{W_\theta} = \theta (1 - \theta) |\nabla V|^2 - \theta \Delta V.$$  

However, we also know [109] that the generator $\mathcal{L}$ considered on $L^2(\mu)$ is unitarily equivalent to the operator

$$\tilde{\mathcal{L}} := e^{-\frac{V}{2}} \mathcal{L} (e^{\frac{V}{2}} \cdot),$$

defined on $L^2(dx)$ (a procedure also called symmetrization [104, Section 4.3]), which is actually the opposite of the Witten Laplacian [105, 60]:

$$\tilde{\mathcal{L}} = \Delta - \frac{1}{4} |\nabla V|^2 + \frac{1}{2} \Delta V = -\left(-\Delta + \Psi \right).$$

In this case, the condition for (27) to have compact level sets when $\theta = 1/2$ is actually equivalent to a confinement condition (or spectral gap condition [61]) for the Witten–Schrödinger operator $\tilde{\mathcal{L}}$ defined in (28). In that sense, Assumption 3 is a natural generalization of a spectral gap condition for the Witten Laplacian in the case of possibly non-reversible dynamics. This is why we call Assumption 3 a Witten–Lyapunov condition.

We now compare this Witten–Lyapunov condition to Cramér’s exponential moment condition in the case of independent variables of law $\mu$. Consider the case when $V(x) = |x|^q$ for $q > 1$, for which Assumption 3 is satisfied by application of Proposition 4. The standard Cramér condition in the case of independent variables states that the empirical measure

$$\frac{1}{n} \sum_{i=1}^n \delta_{X_i},$$

satisfies a large deviations principle in the $\tau^n$-topology if and only if [108] Theorem 1.1:

$$\forall \theta \in \mathbb{R}, \quad \int_X e^{\theta \kappa} \, d\mu < +\infty.$$
If $\mu(dx) \propto e^{-|x|^\alpha} dx$, a sufficient condition for the above condition to hold is to choose $\kappa(x) = 1 + |x|^\alpha$ with $0 \leq \alpha < q$. On the other hand, the Witten–Lyapunov potential \cite{21} reads in this case

$$
\Psi_\theta(x) = \theta(1 - \theta)q^2|x|^{2(q-1)} - \theta q(q-1)|x|^{q-2},
$$

so that we may choose $\kappa(x) = 1 + |x|^\alpha$ for $0 \leq \alpha < 2(q-1)$. When comparing the two conditions, we obtain the following different situations depending on $q$:

- $q > 2$ (super-Gaussian case): $2(q-1) > q$, the Witten–Lyapunov condition is less restrictive than Cramér’s condition;
- $q = 2$ (Gaussian case): $2(q-1) = q$, the two conditions are equivalent;
- $q \in (1,2)$ (sub-Gaussian case): $2(q-1) < q$, the Witten–Lyapunov condition is more restrictive than Cramér’s condition.

This simple example shows that considering a correlated system instead of independent variables has a non-trivial effect on the stability of the system. Depending on the confinement potential, the Witten–Lyapunov condition for \cite{21} to have compact level sets can be more or less restrictive than Cramér’s condition for independent variables distributed according to the invariant measure $\mu$. Finally, we remark that for $q \in (1,3/2)$, the process is heavy-tailed in the sense that $2(q-1) < 1$ and the observable $f(x) = x$ (assuming $d = 1$) does not satisfy a LDP. In other words, the average position of the process defined by

$$
\frac{1}{t} \int_0^t X_s \, ds
$$

cannot be shown to satisfy a large deviations principle at speed $t$ with our arguments.

We finally mention that, in the case where the observable $f$ grows faster at infinity than the potential $\Psi$, it seems possible to derive a level 1 large deviations principle at a speed smaller than $t$. We refer to \cite{21} for a recent account dealing with the case of an Ornstein–Uhlenbeck process, and to \cite{76, 2} for related issues.

We close this section with a practical corollary of Theorem 1 which generalizes the level 1 LDP proved in \cite{111, Corollary 2.3}.

**Corollary 1** (Level 1 large deviations principle). Suppose that Assumptions 1, 2 and 3 hold true and consider a function $f \in B^\infty_2(\mathcal{X})$. Then, the function

$$
\theta \in \mathbb{R} \mapsto \lambda_f(\theta) := \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E}_x \left[ e^{\theta \int_0^t f(X_s) \, ds} \right]
$$

is well-defined and differentiable. Moreover,

$$
L_t(f) := \frac{1}{t} \int_0^t f(X_s) \, ds \underset{t \to +\infty}{\longrightarrow} \int f \, d\mu
$$

almost surely, and $L_t(f)$ satisfies a large deviations principle in $\mathbb{R}$ at speed $t$ with good rate function given by

$$
\forall a \in \mathbb{R}, \quad I_f(a) = \inf \left\{ I(\nu), \ \nu \in \mathcal{P}(\mathcal{X}), \ \nu(f) = a \right\},
$$

where $I$ is defined in \cite{21}. Moreover, it holds

$$
I_f(a) = \sup_{\theta \in \mathbb{R}} \left\{ \theta a - \lambda_f(\theta) \right\}.
$$

Corollary 1 is useful for practical applications, since \cite{31} is a natural way to estimate the rate function $I_f$ associated with an observable $f$, see for instance \cite{55, 65, 101, 23, 17}.

**Proof.** For $f \in B^\infty_2(\mathcal{X})$, the application $L_t \in \mathcal{P}_c(\mathcal{X}) \mapsto L_t(f) \in \mathbb{R}$ is continuous in the $\tau^*$-topology \cite{21} Lemma 3.3.8. Therefore, $L_t(f)$ obeys a large deviations principle in $\mathbb{R}$ by the contraction principle \cite{21} Theorem 4.2.1, with good rate function given by \cite{21}. Moreover, one can redo the proofs leading to Theorem 1 and show that $\lambda_f$ defined in \cite{21} is smooth and well-defined on $\mathbb{R}$. This implies that a LDP with good rate function \cite{31} holds through the Gärtner–Ellis theorem applied in $\mathbb{R}$. Since the rate function is unique, the expressions \cite{21} and \cite{31} coincide. \hfill \Box
3 Decomposition of the rate function

Our goal in this section is to rewrite $I$ in various ways, which is useful for theoretical understanding and practical purposes. In Section 3.1, we first show an extension of the standard Donsker–Varadhan formulation for $I$. This result is easily obtained by making use of the spectral analysis of the operator $L + f$ for $f \in B_κ^∞(X)$, which is presented in Section 6.1. We then apply this result to obtain a variational representation for the principal eigenvalue $λ(f)$ of the operator $L + f$. Next, in Section 3.2, we split the expression of the rate function according to the symmetric and antisymmetric parts of the dynamics, extending the work [15] to general diffusions. Such a decomposition will prove useful in Section 4 to compare the entropy of overdamped and underdamped Langevin dynamics. Most of the proofs of this section are postponed to Section 6.2.

3.1 Donsker–Varadhan variational formula

We start with the variational representation of the entropy. Our proof, which can be found in Section 6.2.1, is an adaptation of [30, Lemma 4.2.35] relying on the Feynman–Kac semigroup and its spectral elements.

Proposition 3. The rate function defined in \((25)\) admits the following representation:

$$∀ ν ∈ \mathcal{P}(X), \ I(ν) = \sup \left\{ -\int_X \frac{Lu}{u} \, dν, \ u ∈ \mathcal{D}^+ \right\}, \quad (32)$$

where

$$\mathcal{D}^+ = \left\{ u ∈ B_κ^∞(X) \mid u > 0, -\frac{Lu}{u} ∈ B_κ^∞(X) \right\}. \quad (33)$$

In particular, the functional defined in \((32)\) is equal to $+∞$ if $ν ∉ \mathcal{P}_κ(X)$ or $ν$ is not absolutely continuous with respect to $μ$.

This result is standard when $X$ is compact [33], but does not seem to be known for an unbounded space $X$ and for the $τ^κ$-topology we consider. In this situation the space $\mathcal{D}^+$ has to be designed with caution (in particular, $\mathcal{D}^+$ is not empty since it contains the functions of the form $u = e^ψ$ for $ψ ∈ C_κ^∞(X)$). Note also that the last statement of Proposition 3 is consistent with the Fenchel definition \((25)\) of the rate function. In order to get some intuition on the formula \((32)\), let us mention that the proof relies on replacing the maximum over functions $u ∈ \mathcal{D}^+$ by the supremum over eigenfunctions $h_f$ satisfying

$$(L + f)h_f = λ(f)h_f,$$

for $f ∈ B_κ^∞(X)$. The above equation rewrites, since $h_f > 0$,

$$-\frac{Lh_f}{h_f} = f - λ(f).$$

By integrating with respect to a measure $ν ∈ \mathcal{P}_κ(X)$ we find \((32)\) on the left hand side, and the Fenchel transform \((26)\) on the right hand side. The functional spaces associated with $f$ and $h_f$ motivate the choice of $\mathcal{D}^+$.

A natural consequence of Proposition 3 is the following variational representation for the principal eigenvalue. The proof, postponed to Section 6.2.2, relies on the convexity of the cumulant function to invert the Fenchel transform \((26)\).

Corollary 2. Suppose that Assumptions 7, 8, and 9 hold true, and consider $f ∈ B_κ^∞(X)$. Then, the principal eigenvalue $λ(f)$ associated with the operator $L + f$ over $B_κ^∞(X)$ is isolated and admits the following representation:

$$λ(f) = \sup_{ν ∈ \mathcal{P}_κ} \left\{ ν(f) - I(ν) \right\}, \quad (34)$$

where $I$ is defined in \((32)\).
Corollary 4 may seem anecdotal, but it provides a variational representation for the principal eigenvalue of nonsymmetric diffusion operators, as pioneered by Donsker and Varadhan in their seminal paper [33] for a compact space \(X\). To the best of our knowledge, this formula had not been shown in an unbounded setting, for which we need to introduce the “generalized domain” \(D^+\) defined in [34]. However, our set of assumptions implies that the largest eigenvalue \(\lambda(f)\) is isolated for any \(f\) (because of the compactness of the resolvent provided by Lemma 7), whereas in [33], [34] may be the supremum of the essential spectrum of the operator. This suggests that [34] holds under weaker assumptions. A possible approach for generalizing our results may be to consider different methods for studying the long time behaviour of unnormalized semigroups, see for instance [21] or [22] or to resort to more subtle spectral analysis tools [110, 112, 52, 13].

### 3.2 Entropy decomposition: symmetry and antisymmetry

Our goal is now to provide refined expressions for the rate function \(I\) in terms of symmetric and antisymmetric parts of the dynamics, inspired in particular by [114]. In the following, for any closed operator \(T\), we denote by \(T^*\) its adjoint on \(L^2(\mu)\), where \(\mu\) is the invariant probability measure of the process, as obtained in Proposition 2.

Considering the generator \(L\) of the diffusion \(\nu\), we can always decompose it into symmetric and antisymmetric parts with respect to \(\mu\) through

\[
L = L_S + L_A, \quad L_S = \frac{L + L^*}{2}, \quad L_A = \frac{L - L^*}{2}.
\]

(35)

It is important to note that \(L_A\) is a first order differential operator (and therefore obeys the chain rule of first order differentiation). The decomposition (35) then allows to separate the rate function (32) into two parts. This is the purpose of the next key result, whose proof can be read in Section 6.2.3. It is inspired by the computations in [15] Proposition 2], which we simplify and generalize here through a variational Witten transform and the use of the Sobolev spaces introduced in Section 2.1. The algebra of the proof also suggests to consider \(I(\nu)\) for probability measures \(\nu\) of the form \(d\nu = e^\psi d\mu\).

**Theorem 2.** Suppose that Assumptions 1, 2 and 3 hold true, consider a measure \(\nu \in \mathcal{P}_\kappa(X)\) such that \(\nu \ll \mu\) and \(d\nu = e^\psi d\mu\) with \(\psi \in \mathcal{H}^1(\nu)\) and \(L_A \nu \in \mathcal{H}^{1-}(\nu)\). Then, the rate function \(I\) defined in (32) admits the following decomposition:

\[
I(\nu) = I_S(\nu) + I_A(\nu),
\]

(36)

where

\[
I_S(\nu) = \frac{1}{4} |\nu|_{\mathcal{H}^1(\nu)}^2
\]

(37)

and

\[
I_A(\nu) = \frac{1}{4} |L_A \nu|_{\mathcal{H}^{1-}(\nu)}^2.
\]

(38)

Theorem 2 expresses the rate function as the sum of dual norms of the symmetric and antisymmetric parts of the dynamics. Note also that we consider a measure of the form \(d\nu = e^\psi d\mu\), that is the Radon–Nikodym derivative of \(\nu\) with respect to \(\mu\) is positive. However, we believe that we can consider more general measures \(\nu\), see Remark 7 in the proof. Since the measure \(\nu\) at hand appears both inside the norms and in the definition of the norms themselves, a possibly clearer rewriting is the following:

\[
I(\nu) = \frac{1}{4} \left| \log \frac{d\nu}{d\mu} \right|^2_{\mathcal{H}^1(\nu)} + \frac{1}{4} \left| L_A \left( \log \frac{d\nu}{d\mu} \right) \right|^2_{\mathcal{H}^{1-}(\nu)}.
\]

Moreover, the symmetric part of the rate function (37) can be written as a Fisher information for the invariant measure \(\mu\), a standard result [54], denoting by \(\rho = d\nu/d\mu\), it holds

\[
I_S(\nu) = \frac{1}{4} \int_X \nabla \rho \cdot S \nabla \rho \, d\mu.
\]

The next corollary builds upon 6.3 by rewriting \(I_A\) using a Poisson equation, which can be manipulated more easily. The proof can be found in Section 6.2.4.
Corollary 3. Suppose that Assumptions 1, 4 and 5 hold true, and consider κ as in Assumption 3. Consider a measure ν ∈ Pκ(aviors) such that dν = e^v dµ with v ∈ H^1(ν) and L_νν ∈ H^{-1}(ν). Then, the antisymmetric part of the rate function (39) reads

\[ I_\Lambda(ν) = \frac{1}{4} \int_X \mathcal{C}(ψ_ν, ψ_ν) dν, \]

where ψ_ν is the unique solution in H^1(ν) to the Poisson equation

\[ \bar{\nabla}(S∇ψ_ν) = L_\Lambda ν, \]

the symmetric matrix S being defined in (10) and \( \bar{\nabla} \) denoting the adjoint of the gradient operator in L^2(ν).

It has been known for a long time [33] that the rate function of a reversible process is a Fisher information as in (37). The antisymmetric part of the rate function has been less investigated, although an expression like (39) already appears in [54] (see also [95, 15]). However, our setting provides natural well-posedness conditions for both parts of the rate function to be finite. Moreover, the uniqueness of ψ_ν is a consequence of the definition of H^1(ν) through equivalence classes, see Section 2.1.

Interestingly, the solution ψ_ν of (10) can be formally represented through (30)

\[ ψ_ν = \int_0^{+\infty} e^{t\mathcal{L}_ν}(L_\Lambda ν) dt, \]

where \( \mathcal{L}_ν = -\bar{\nabla}(S∇). \) Then, the stochastic process \( (X^ν_t)_{t≥0} \) associated with \( \mathcal{L}_ν \) is reversible with respect to ν. Denoting by e^{-νc} the density of ν with respect to the Lebesgue measure, \( (X^ν_t)_{t≥0} \) is solution to the following SDE:

\[ dX^ν_t = -S∇V(X^ν_t) dt + \nabla \cdot S(X^ν_t) dt + \sigma(X^ν_t) dB_t. \]

Finally (40) takes the form

\[ I_\Lambda(ν) = \frac{1}{4} \int_0^{+\infty} \mathbb{E}_ν \left[ \mathcal{L}_\Lambda ν(X^ν_0)\mathcal{L}_\Lambda ν(X^ν_0) \right] dt. \]

The antisymmetric part of the entropy is therefore the autocorrelation of \( \mathcal{L}_\Lambda ν \) along a reversible process that realizes the fluctuation corresponding to the measure ν. From a mathematical point of view, it seems interesting to relate (11) to the so-called level 2.5 of large deviations [11, 25], since this approach consists in considering joint fluctuations of the empirical measure and the associated empirical current. In this case, the large deviations function is explicit: this reflects the fact that a Markov process is characterized entirely by its density and current. Exploring further the connection between (11) and level 2.5 large deviations is an interesting direction for future works.

Remark 5. It is also possible to consider the adjoint \( \mathcal{L}^* \) not with respect to the invariant measure µ, but instead a reference measure \( µ_{ref} \) such that

\[ \mathcal{L}^* = \mathcal{L}_S - L_\Lambda + ξ, \]

for a measurable function ξ. The measure \( µ_{ref} \) may be an equilibrium measure for systems subject to a small external nonequilibrium forcing. In [25], the technique is used to study atom chains in contact with an inhomogeneous heat bath, \( µ_{ref} \) being the Gibbs measure associated with a fixed temperature profile. This leads to an additional term \(-\int_\Lambda ξ dν \) in the expression of the rate function (40), as can be readily checked by a straightforward adaptation of the proof.

4 Applications

4.1 Overdamped Langevin dynamics

In this section, we come back to the setting of Remark 4 by considering a diffusion process over \( \mathcal{X} = \mathbb{R}^d \) subject to

\[ dX_t = b(X_t) dt + \sqrt{2} dB_t, \]

where \( b(X_t) \) is a measurable function, and \( dB_t \) is a Brownian motion.
where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function and $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion. This corresponds to (9) with $\sigma = \sqrt{2}$, in which case the generator reads

$$L = b \cdot \nabla + \Delta.$$ 

We will treat the reversible case where $b = -\nabla V$ for a smooth potential $V$, and $b = -\nabla V + F$ for a smooth function $F$ such that $\nabla \cdot (Fe^{-V}) = 0$. In both cases, the invariant probability measure $\mu$ of the process is (assuming $e^{-V} \in L^1(X)$)

$$\mu(dx) = Z^{-1}e^{-V(x)}dx, \quad Z = \int_X e^{-V} < +\infty.$$

(43)

The dynamics (42) is reversible (i.e. $L^* = L$, where $L^*$ denotes the adjoint of $L$ in $L^2(\mu)$) if and only if $b = -\nabla V$.

**Assumption 4.** The potential $V \in \mathcal{S}$ has compact level sets, satisfies $e^{-V} \in L^1(X)$ and, for any $\theta \in (0, 1)$, it holds

$$(1 - \theta)|\nabla V|^2 - \Delta V \xrightarrow{|x| \to +\infty} +\infty.$$ 

(44)

This assumption is satisfied for smooth potentials growing like $|x|^q$ for $q > 1$ at infinity, and it also implies that the invariant probability measure $\mu$ satisfies a Poincaré inequality [4]. Similar conditions are derived in [73] in the context of large deviations. The next proposition is a direct application of Propositions 1 and 2, Theorem 1 and Corollary 3.

**Proposition 4.** Under Assumption 4, the process (42) with $b = -\nabla V$ admits the function

$$W(x) = e^{\theta V(x)}$$

for any $\theta \in (0, 1)$ as a Lyapunov function in the sense of Assumption 3. For any fixed $\theta \in (0, 1)$, there exist $C, c > 0$ such that for any initial measure $\nu \in \mathcal{P}_W(X)$,

$$dW(\nu P_t, \mu) \leq C e^{-ct} dW(\nu, \mu).$$

Moreover,

$$\Psi = \frac{LW}{W} = \theta \left( (1 - \theta)|\nabla V|^2 - \Delta V \right)$$

(45)

has compact level sets and, for any $\kappa : X \to [1, +\infty)$ bounded, or with compact level sets and such that

$$\Psi(x) / \kappa(x) \xrightarrow{|x| \to +\infty} +\infty,$$

the empirical measure

$$L_t := \frac{1}{t} \int_0^t \delta X_s, ds$$

satisfies a large deviations principle in the $\tau^\kappa$-topology. The good rate function is defined by: for all $\nu \in \mathcal{P}_\kappa(X)$ with $d\nu = \rho d\mu = e^v d\mu$,

$$I(\nu) = \frac{1}{4} \int_X |\nabla v|^2 d\nu = \frac{1}{4} \int_X \frac{|\nabla \rho|^2}{\rho} d\mu,$$

(46)

and $I(\nu) = +\infty$ otherwise.

In this reversible example, we see that the rate function is only defined through its symmetric part [37], as shown in Theorem 2. We now consider a modification of this dynamics when a divergence-free drift is added. The next proposition is an extension of the examples proposed in [35] to the unbounded state space case.

**Proposition 5.** Suppose that Assumption 4 holds and consider the diffusion process:

$$dX_t = (-\nabla V(X_t) + F(X_t))dt + \sqrt{2} dB_t,$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function and $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion.
with $F$ a smooth vector field such that $\nabla \cdot (F e^{-V}) = 0$ and
\[
\frac{F \cdot \nabla V}{\Psi} \quad \text{near } |x| \to +\infty = 0,
\]
where $\Psi$ is defined in \[19\]. Then, with the notation of Section 7.2 it holds $L_S = -\nabla V \cdot \nabla + \Delta$ and $L_A = F \cdot \nabla$. Moreover
\[
\Psi_F := -\frac{L + F \cdot \nabla W}{W} = \theta \left( (1 - \theta) |\nabla V|^2 - \Delta V - F \cdot \nabla V \right) \sim \Psi,
\]
and $(X_t)_{t \geq 0}$ satisfies an LDP in the $\tau^\kappa$-topology for any function $\kappa$ bounded, or with compact level sets and such that
\[
\Psi_F := -\frac{L + F \cdot \nabla W}{W} = \theta \left( (1 - \theta) |\nabla V|^2 - \Delta V - F \cdot \nabla V \right) \sim \Psi,
\]
where $\psi_v$ is the unique $H^1(\nu)$-solution to $-\Delta \psi_v + \nabla (V - v) \cdot \nabla \psi = F \cdot \nabla v$.

Proposition 5 shows that in this simple case, the equilibrium and nonequilibrium dynamics admit a LDP for the same class of functions but with different rate functions, the irreversible dynamics producing more entropy. It is therefore an extension of the case treated in \[95, Theorem 2.2\]. As for this result, Proposition 5 can be used to design algorithms with accelerated convergence to equilibrium, see also \[64, 65, 37\]. A setting in which Proposition 5 typically applies is when $V(x) = |x|^q$ for some $q > 1$ and $F = A \nabla V$ with $A \in \mathbb{R}^{d \times d}$ such that $A^T = -A$ (see \[95\]).

4.2 Underdamped Langevin dynamics

We now apply our framework to the underdamped Langevin dynamics. A first nice feature of our results is that, compared to \[111\], we obtain a stronger result with similar assumptions — that is our LDP for the empirical measure holds for a finer topology than the one associated with bounded measurable functions. Note however that \[111, Corollary 2.3\] obtains results similar to ours for a contraction of the rate function. In addition, Theorem 2 and Corollary 3 allow to obtain fine results on the dependency of the rate function on the friction parameter $\gamma$.

We start by describing the Langevin equation in Section 4.2.1, before stating the large deviations principle in Section 4.2.2. Finally Section 4.2.3 provides asymptotics on the rate function depending on the friction.

4.2.1 Description of the dynamics

The dynamics is set on $\mathcal{X} = \mathbb{R}^d \times \mathbb{R}^d$, with $(X_t)_{t \geq 0} = (q_t, p_t)_{t \geq 0} \in \mathbb{R}^d \times \mathbb{R}^d$ evolving as
\[
\begin{cases}
\quad dq_t = p_t \, dt, \\
\quad dp_t = -\nabla \langle V(q_t) \rangle \, dt - \gamma p_t \, dt + \sqrt{2\gamma} \, dB_t,
\end{cases}
\]
where $\gamma > 0$ is a friction parameter, $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth potential, and $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion. We could also consider the easier case where the position space is bounded ($q \in \mathbb{T}^d$) but leave this easy modification to the reader. The generator of the dynamics is
\[
L_\gamma = L_{\text{ham}} + \gamma L_{\text{FD}},
\]
where
\[
L_{\text{ham}} = p \cdot \nabla q - \nabla \langle V \rangle \cdot \nabla p, \quad L_{\text{FD}} = -p \cdot \Delta p + \Delta p.
\]
The operator $\mathcal{L}_\gamma$ leaves invariant the measure
\[ \mu(dx) = \mu(dq dp) = \tilde{\mu}(dq) \omega(dp), \quad \tilde{\mu}(dq) = Z^{-1}_\gamma e^{-V(q)} dq, \quad \omega(dp) = (2\pi)^{-d/2} e^{-\frac{p^2}{2}} dp. \] (50)
The invariant measure \[ \mu(dq dp) = Z^{-1}_\gamma e^{-H(q,p)} dq dp, \] (51)
where
\[ H(q, p) = V(q) + \frac{p^2}{2} \] (52)
is the Hamiltonian of the system, and we assume that the normalization constant $Z$ in (51) is finite (which is indeed the case when $e^{-V} \in L^1(\mu)$). In (49), the Liouville operator $\mathcal{L}_{\text{ham}}$ corresponding to the Hamiltonian part of the dynamics is antisymmetric in $L^2(\mu)$. On the other hand, the fluctuation-dissipation part with generator $\mathcal{L}_\text{FD}$ is symmetric in $L^2(\mu)$, so that $\mathcal{L}_\Lambda = \mathcal{L}_{\text{ham}}$ and $\mathcal{L}_\delta = \gamma \mathcal{L}_\text{FD}$ with the notation of Section 3.2.

Before turning to the LDP associated with the Langevin dynamics (48), we give some intuition on the behaviour of the process as $\gamma$ varies. First, it is clear that in the small $\gamma$ limit, (45) becomes the Hamiltonian dynamics
\[ \begin{align*} dq_t &= p_t dt, \\
   dp_t &= -\nabla V(q_t) dt. \end{align*} \] (50)
To be more precise, we introduce the process $(Q_t^\gamma, P_t^\gamma) = (q_t, p_t)$ where $(q_t, p_t)_{t \geq 0}$ is solution to (48). It can then be shown that, in the limit $\gamma \to 0$, the Hamiltonian $H(Q_t^\gamma, P_t^\gamma)$ converges to an effective diffusion on a graph $\mathbb{R}^{d} \times \mathbb{R}^{d}$. In particular the relevant time scale in the underdamped limit is $\gamma^{-1} t$.

On the other hand, in the limit $\gamma \to +\infty$ and under an appropriate time rescaling, we recover the overdamped dynamics studied in Section 4.1. To see this, we integrate the second line in (48) to obtain
\[ P_t - P_0 = -\int_0^t \nabla V(q_s) ds - \gamma(q_t - q_0) + \sqrt{2\gamma} B_t. \]
By introducing now $Q_t^\gamma = q_{\gamma t}$ and $P_t^\gamma = p_{\gamma t}$, the latter equality becomes
\[ Q_t^\gamma - Q_0^\gamma = \frac{P_t^\gamma - P_0^\gamma}{\gamma} - \int_0^t \nabla V(Q_s^\gamma) ds + \sqrt{2\gamma} B_t. \]
When $\gamma \to +\infty$, we observe that $Q_t^\infty$ converges formally towards the solution of (49), see [90, Section 6.5]. The relevant time scale in the overdamped limit is therefore $\gamma t$. These remarks will be of interest when studying the rate function below.

### 4.2.2 Large deviations

In order to obtain a large deviations principle for (48), let us make the following classical assumption on the growth of the potential $\{111, 102, 83\}$.

**Assumption 5.** The potential $V \in S$ has compact level sets, satisfies $e^{-V} \in L^1(\mathcal{X})$ and there exist $c_V > 0$, $C_V \in \mathbb{R}$ such that
\[ q \cdot \nabla V(q) \geq c_V |q|^2 - C_V. \]

We can now find a Lyapunov function for (48) by following e.g. [111, 112, 83], as made precise in Appendix C. Recall that the Hamiltonian $H$ is defined in (52).

**Lemma 1.** Suppose that $(X_t)_{t \geq 0} = (q_t, p_t)_{t \geq 0}$ solves (48) where $V$ satisfies Assumption 5. Then for any $\gamma > 0$ and $\theta \in (0, 1)$, there exists $\varepsilon > 0$ such that
\[ W(q, p) = e^\theta H(q, p) + \varepsilon \gamma p \] (53)
is a Lyapunov function in the sense of Assumption 4. More precisely, for any $\gamma > 0$ and $\theta \in (0, 1)$, there exist $\varepsilon > 0$ and $a, b, C > 0$ such that
\[ -\frac{\mathcal{L} W}{W} \geq a |q|^2 + b |p|^2 - C. \]
The Lyapunov function \(S\) can be adapted to cases where \(V\) has singularities, see [82] [82]. We can now deduce our main theorem on the Langevin dynamics, since Assumptions 1 and 2 are readily satisfied, see for instance [62].

**Theorem 3.** Assume that \((X_t)_{t\geq 0} = (q_t, p_t)_{t\geq 0}\) solves (13) where \(V\) satisfies Assumption 3 and consider \(\kappa(q, p) = 1 + |q|^\alpha + |p|^\beta\) with \(\alpha \in [0, 2), \beta \in [0, 2)\). Then \((X_t)_{t\geq 0}\) is ergodic with respect to the measure \(\mu\) in the sense of Proposition 3, with Lyapunov function defined in (53). Moreover, the empirical measure

\[
I_t := \frac{1}{t} \int_0^t \delta(q_t, p_t) \, ds
\]

satisfies a LDP in the \(\tau^\ast\)-topology. Finally, for any \(\nu \in \mathcal{P}_c(X)\) such that \(d\nu = e^\nu \, d\mu\) with \(v \in \mathcal{H}^1(\nu)\) and \(\mathcal{L}_{\text{ham}} \, v \in \mathcal{H}^\ast(\nu)\), the rate function reads

\[
I_\gamma(\nu) = \frac{\gamma}{4} \int_X |\nabla_p v|^2 \, d\nu + \frac{1}{4\gamma} \int_X |\nabla_p \psi|^2 \, d\nu,
\]

where \(\psi\) is the unique solution in \(\mathcal{H}^1(\nu)\) to the Poisson problem:

\[
- \Delta_p \psi + (p - \nabla_p \psi) \cdot \nabla_p \psi = \mathcal{L}_{\text{ham}} v.
\]

The proof of Theorem 3 is a direct application of the results of Sections 2 and 3. For the expression of the rate function, we use (40) and (41) together with the fact that in this case, the matrix \(S\) defined in Section 2.1 reads

\[
S = \gamma \begin{pmatrix} 0 & 0 \\ 0 & L_{d\times d} \end{pmatrix} \in \mathbb{R}^{2d \times 2d}.
\]

While \(\kappa\) can be chosen independently of the friction \(\gamma\), it is interesting to note the dependency of the rate function \(S\) with respect to this parameter. We discuss more precisely the scaling of the rate function with respect to \(\gamma\) in the next section, depending on the form of \(\nu\).

### 4.2.3 Low and large friction asymptotics of the rate function

The next corollary shows how the decomposition \(S\) allows to identify the most likely fluctuations in the overdamped and underdamped limits. By this we mean that, when \(\gamma \to 0\) or \(\gamma \to +\infty\), most fluctuations become exponentially rare in \(\gamma\), but some of them are associated with rate functions that vanish as \(\gamma \to 0\) and \(\gamma \to +\infty\). The expression of these typical fluctuations is motivated by the discussion on the overdamped and underdamped limits in Section 3.4, from which the scalings of the rate function appear natural. Recall the definition of the marginal in position \(\bar{\mu}\) in [60].

**Corollary 4.** Suppose that the assumptions of Theorem 3 hold true.

- **Overdamped limit \(\gamma \to +\infty\).** Take a measure \(\nu \in \mathcal{P}_c(X)\) with \(d\nu = e^\nu \, d\mu\) that has equilibrated in speed, i.e. such that \(v(q, p) = v(q)\) with \(v \in \mathcal{H}^1(\nu)\) and \(p \cdot \nabla_q v \in \mathcal{H}^\ast(\nu)\). Then, for any \(\gamma > 0\),

\[
I_\gamma(\nu) = \frac{1}{4\gamma} \int_{\mathbb{R}^d} |\nabla v(q)|^2 \, \bar{\nu}(dq),
\]

where \(\bar{\nu} = e^\nu \bar{\mu}\).

- **Hamiltonian limit \(\gamma \to 0\).** Consider a Hamiltonian fluctuation, i.e. \(d\nu = e^\nu \, d\mu\) with \(v(q, p) = g(H(q, p)) \in \mathcal{H}^1(\nu)\) for \(g \in C^1(\mathbb{R})\), where \(H\) is defined in [53]. Then, for any \(\gamma > 0\),

\[
I_\gamma(\nu) = \frac{\gamma}{4} \int_X \left| \frac{\partial g}{\partial q'} (H(q, p)) \right|^2 \nu(dq \, dp).
\]

The proof is an immediate consequence of [53].
Proof. Consider first the case where \( dv = e^v \, d\mu \) with \( v(q,p) = v(q) \). We have

\[
\frac{\gamma}{4} \int_X |\nabla v|^2 \, dv = 0.
\]

Next, (50) becomes

\[-(\Delta_p - p \cdot \nabla_p)\psi(q,p) = p \cdot \nabla_q \psi(q).\]

The solution to this equation is \( \psi(q,p) = -p \cdot \nabla_q \psi(q) \) which indeed belongs to \( \mathcal{H}^1(\nu) \) since \( \mathcal{L}_{\text{ham}} v \in \mathcal{H}^{-1}(\nu) \) (in fact we may add to \( \psi \) any function depending on \( q \) only but the solutions would be equal by definition of the space \( \mathcal{H}^1(\nu) \) in Section 2.1). Plugging this solution into (54) leads to (56).

Assume now that \( v(q,p) = g(H(q,p)) \) belongs to \( \mathcal{H}^1(\nu) \) with \( g \in C^1(\mathbb{R}) \). It holds

\[
\mathcal{L}_{\text{ham}} v(q,p) = g'(H(q,p)) \mathcal{L}_{\text{ham}} H(q,p) = 0.
\]

As a result, the solution \( \psi \) to (55) is \( \psi = 0 \) (again, up to a function of \( q \) only), from which (57) follows since \( v \in \mathcal{H}^1(\nu) \).

Corollary 5 characterizes the dominant fluctuations in the small and large friction regimes. In the overdamped limit \( \gamma \to +\infty \) the dominant fluctuations are in position only, and the rate function is actually that of the limiting overdamped dynamics (60) up to a time rescaling \( t \mapsto \gamma t \), with is coherent with the discussion on the overdamped limit in Section 4.2.1. On the other hand, in the Hamiltonian limit \( \gamma \to 0 \), the dominant fluctuations are Hamiltonian, with the inverse time rescaling \( t \mapsto \gamma^{-1} t \). This is consistent with the small temperature limit of Hamiltonian systems (43).

Although Corollary 5 provides interesting information, its structure is quite rigid. For instance, in the overdamped limit, we consider only position-dependent perturbations, which is not realistic. We now refine the asymptotics by considering the next order correction in \( \gamma \) for the perturbation in both regimes, which shows the robustness of the analysis.

**Corollary 5.** Suppose that the assumptions of Theorem 3 hold true.

- **Overdamped limit** \( \gamma \to +\infty \). Consider the measure \( \nu_\gamma \in \mathcal{P}_r(X) \) defined by \( \nu_\gamma = e^{\gamma v} \, d\mu \) with \( v(q,p) = v(q) + \gamma^{-1} \bar{v}(q,p) \) where \( \mathcal{L}_{\text{ham}} \bar{v} \in \mathcal{H}^{-1}(\nu) \), and \( \bar{v} \in \mathcal{H}^1(\nu) \) is bounded and satisfies \( \nabla q \bar{v} \cdot \nabla p \bar{v} \in \mathcal{H}^{-1}(\nu) \) and \( \mathcal{L}_{\text{ham}} \bar{v} \in \mathcal{H}^{-1}(\nu) \). Then

\[
\forall \gamma \geq 1, \quad I_\gamma(\nu_\gamma) = \frac{1}{4\gamma} \left( \int_X |\nabla q \bar{v}|^2 \, dv + \int_{\mathbb{R}^d} |\nabla q \bar{v}|^2 \, d\bar{v} \right) + O\left( \frac{1}{\gamma^2} \right),
\]

where \( \bar{v} = e^\gamma \bar{\mu} \).

- **Hamiltonian limit** \( \gamma \to 0 \). Consider \( \nu_\gamma = e^{\gamma v} \, d\mu \) with \( v(q,p) = g(H(q,p)) + \gamma \bar{v}(q,p) \), where \( g \in C^1(\mathbb{R}) \), \( g(H) \in \mathcal{H}^1(\nu) \), and \( \bar{v} \in \mathcal{H}^1(\nu) \) is bounded and satisfies \( \mathcal{L}_{\text{ham}} \bar{v} \in \mathcal{H}^{-1}(\nu) \). Then

\[
\forall \gamma \leq 1, \quad I_\gamma(\nu_\gamma) = \frac{\gamma}{4} \left( \int_X |p g'(H(q,p))|^2 \nu(\, dq \, dp) + \int_X |\nabla_p \bar{v}|^2 \, dv \right) + O(\gamma^2),
\]

where \( \bar{v} \) is the unique \( \mathcal{H}^1(\nu) \)-solution to

\[
- \Delta_p \bar{v} - (1 - g'(H(q,p))) p \cdot \nabla_p \bar{v} = \mathcal{L}_{\text{ham}} \bar{v}.
\]

We believe it is also instructive to mention the relation between the rate function (54) and the asymptotic variance of the Langevin dynamics. Indeed, when considering small perturbations of the invariant measure, Corollary 5 shows that

\[
I_\gamma \sim \min \left( \gamma, \frac{1}{\gamma} \right).
\]
On the other hand, the resolvent estimates in [70] Section 2.1 and [59] [60] show that the asymptotic variance $\sigma_\gamma^2$ scales like

$$\sigma_\gamma^2 \sim \max\left(\frac{\gamma}{\gamma}, \frac{1}{\gamma}\right).$$  \hfill (62)

Since we expect the asymptotic variance to be the inverse of the rate function around the invariant measure [29, 95], the scalings (62) and (63) are consistent. However, as [54] suggests, this scaling is no longer true for general fluctuations. We now present the proof of Corollary 4.

**Proof.** We first consider the overdamped limit $\gamma \to +\infty$. Since $\tilde{v}$ is bounded we have, for any $\gamma \geq 1$ and $\psi \in \mathcal{H}^1(\nu_\gamma)$,

$$e^{\frac{1}{4\gamma^2}}|\psi|^2_{\mathcal{H}^2(\nu_\gamma)} \leq |\psi|^2_{\mathcal{H}^1(\nu_\gamma)} \leq e^{\frac{1}{2\gamma^2}}|\psi|^2_{\mathcal{H}^2(\nu_\gamma)}. \hfill (63)$$

Thus, the norms $\mathcal{H}^1(\nu_\gamma)$ and $\mathcal{H}^1(\nu)$ are equivalent for any fixed $\gamma \geq 1$, and the functions of $\mathcal{H}^1(\nu_\gamma)$ and $\mathcal{H}^1(\nu)$ coincide (we repeatedly use this fact below, and we will use a similar argument when $\gamma \leq 1$). A similar conclusion holds for the corresponding dual norms. This consequence of the boundedness of $\tilde{v}$ makes the analysis simpler.

Recall that we consider $\nu_\gamma(q, p) = v(q) + \gamma^{-1}\tilde{v}(q, p)$ in the overdamped limit. The symmetric part of the rate function is easily computed since $v$ only depends on the position variable, namely

$$I_\nu(\nu_\gamma) = \frac{\gamma}{4} \int X \left|\nabla_p (v + \gamma^{-1}\tilde{v})\right|^2 e^{\gamma p + \tilde{v}} d\mu = \frac{1}{4\gamma} \int X \left|\nabla_p \tilde{v}\right|^2 dv + O\left(\frac{1}{\gamma}\right),$$

where we used that $\tilde{v}$ belongs to $\mathcal{H}^1(\nu)$ and is bounded to expand the exponential. For the antisymmetric part, by (55), we have to consider the solution $\psi_\gamma \in \mathcal{H}^1(\nu_\gamma)$ to

$$-\Delta_p \psi_\gamma + \left(p - \frac{1}{\gamma} \nabla_p \tilde{v}\right) \cdot \nabla_p \psi_\gamma = L_{\text{ham}} \psi_\gamma.$$  \hfill (49)

Corollary 4 suggests that at leading order in $\gamma$ it holds $\psi_\gamma = \psi + O(\gamma^{-1})$ where $\psi(q, p) = p \cdot \nabla v(q)$. In order to make this idea more precise we compute

$$\left(-\Delta_p + \left(p - \frac{1}{\gamma} \nabla_p \tilde{v}\right) \cdot \nabla_p\right) (\psi_\gamma - \psi) = \frac{1}{\gamma} (L_{\text{ham}} \tilde{v} + \nabla_q v \cdot \nabla_p \tilde{v}).$$

In what follows, we denote by $u = L_{\text{ham}} \tilde{v} + \nabla_q v \cdot \nabla_p \tilde{v}$ the right hand side of the above equation. Since $\nabla_q v \cdot \nabla_p \tilde{v} \in \mathcal{H}^-1(\nu_\gamma)$ and $L_{\text{ham}} \tilde{v} \in \mathcal{H}^-1(\nu_\gamma)$ by assumption, it holds $u \in \mathcal{H}^-1(\nu_\gamma)$. Thus, multiplying by $\psi_\gamma - \psi$ and integrating with respect to $\nu_\gamma$ we obtain

$$\int_X \left|\nabla_p (\psi_\gamma - \psi)\right|^2 d\nu_\gamma = -\frac{1}{\gamma} \int_X (\psi_\gamma - \psi) u d\nu_\gamma.$$  \hfill (50)

Using the duality between $\mathcal{H}^1(\nu_\gamma)$ and $\mathcal{H}^-1(\nu_\gamma)$ (see [72] Section 2.2 Claim F) and (51), we obtain

$$\forall \gamma \geq 1, \quad |\psi_\gamma - \psi|_{\mathcal{H}^2(\nu_\gamma)} \leq C \frac{1}{\gamma} |u|_{\mathcal{H}^-1(\nu_\gamma)},$$

where $C$ is some constant independent of $\gamma$. This shows that $\psi_\gamma = \psi + \gamma^{-1} \tilde{v}_\gamma$ with $|\tilde{v}_\gamma|_{\mathcal{H}^-1(\nu_\gamma)} \leq C'$ for a constant $C' > 0$ and all $\gamma \geq 1$. Plugging this estimate into (49) and using that $\nabla_p \psi = \nabla_q v$, we obtain the second term on the right hand side of (53).

The arguments to prove the limit $\gamma \to 0$ follow a similar path, so we only sketch the proof. First, the boundedness of $\tilde{v}$ allows again to compare the Sobolev norms associated with $\nu$ and $\nu_\gamma$ for any $\gamma \leq 1$ (by writing the counterpart of (63) in this regime). The first term on the right hand side of (50) is easily obtained as in Corollary 4 using that $g(H) \in \mathcal{H}^1(\nu)$ and $\tilde{v}$ is bounded. Concerning the antisymmetric part, (54) now reads

$$(-\Delta_p + (p - \nabla_p \psi_\gamma) \cdot \nabla_p) \psi_\gamma = \gamma L_{\text{ham}} \tilde{v},$$  \hfill (55)
since $\mathcal{L}_{\text{ham}} g(H(q, p)) = 0$. Because of the scaling in $\gamma$ on the right hand side of the above equation, the solution $\psi_\gamma$ can be expanded as $\psi_\gamma = \gamma \tilde{\psi} + O(\gamma^2)$ in $\mathcal{H}^1(\nu)$, where $\tilde{\psi}$ is solution to 

$$-\Delta_p \tilde{\psi} + (1 - g'(H(q, p))) p \cdot \nabla_p \tilde{\psi} = \mathcal{L}_{\text{ham}} \tilde{v}.$$ 

This reasoning can be made rigorous by a precise asymptotic analysis as above. Plugging this expansion into (54) provides the second term on the right hand side of (59).

5 Conclusion and perspectives

The goal of this paper was twofold. Our first aim was to provide, given a diffusion process, a precise class of unbounded functions for which a large deviations principle holds. This question is answered in Section 2 where we prove a LDP for the empirical measure in a topology associated with unbounded functions, in relation with a Witten–Lyapunov condition. In particular, a comparison with Cramér’s condition for independent variables shows the effect of correlations on the stability of the SDE at hand. These results extend in several directions and refine results from previous works [111, 73]. However, the necessity of our Lyapunov condition for a LDP to hold (as is known for the Sanov theorem [108]) is still an open problem. Our second concern was to provide finer expressions of the rate function governing the LDP, in particular in order to study Langevin dynamics which appear for instance in molecular simulation. We answer to this question in two ways in Section 3. We first provide an alternative variational formula for the rate function in Section 3.1, which gives as a by product a very general representation formula for the principle eigenvalue of second order differential operators, without symmetry assumption. This extends the important work of Donsker and Varadhan [33] in an unbounded setting. In Section 3.2, we show a general decomposition of the rate function into symmetric and antisymmetric parts of the dynamics based on the computations in [15]. Interestingly, the proof of the result relies on a Witten-like transform in the above mentioned variational representation of the rate function. These results allow us to describe precisely the rate function of an irreversible overdamped Langevin dynamics in Section 4.1 revisiting results from [95] in an unbounded setting. More interestingly in Section 4.2 we provide, for the Langevin dynamics, asymptotics of the rate function for the overdamped and the underdamped limits. We thus characterize the most likely fluctuations in both regimes with a natural physical interpretation. Considering piecewise deterministic processes [11, 41, 42] (which lack regularity) instead of the Langevin dynamics is also an interesting problem.

We would like to mention several interesting directions for future works. A first natural issue is to rephrase our results in the optimal control framework developed e.g. in [18, 38, 39]. This is particularly interesting for numerical purposes, since the optimal control representation can be learnt on the fly with stochastic approximation methods [17, 9, 10, 47]. We believe that such results can be obtained by harvesting the contraction principle provided by Corollary 8.

On a more theoretical ground, dual Sobolev norms have recently attracted attention in the optimal control community due to the so-called optimal matching problem, see for instance [77, 78] and the references therein. With these works in mind, the dual Sobolev norm in the antisymmetric part of the rate function described in Section 3.2 could be interpreted as an infinitesimal transport cost related to the antisymmetric part of the dynamics, which is an alluring interpretation of irreversibility. Note that the relations between optimal transport and large deviations theory have a fruitful history, see e.g. [57].

Then, it has been known for some time in the physics literature that the empirical density of a diffusion may not contain enough information to describe its fluctuations in an irreversible regime. It is actually more relevant to consider the fluctuations of both the empirical density and current, a procedure sometimes called level 2.5 large deviations [25, 7]. This framework can be used to provide a clear description of the rate function of irreversible dynamics. As shown in [7], such large deviations results can be derived by Krein–Rutman arguments like those used in the present paper. Therefore, we believe that our results can be extended to prove level 2.5 large deviations principles and characterize precisely the admissible currents.

Finally, it is important to understand the behaviour of observables which are not covered by our analysis. It has been recently shown [57] in the case of the Ornstein–Uhlenbeck process that observables growing too fast at infinity with respect to the confinement are characterized by a heavy tail behaviour. This leads to a level 1 large deviations principle at an anomalous speed with a localization in time of the fluctuation, and the Krein–Rutman
strategy developed in the present paper does not apply. We therefore believe there are several interesting open questions in this direction.

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6 Proofs

6.1 Proof of the large deviations principle

As mentioned after Theorem 1, our proof relies on the Gärtner–Ellis theorem [28], for which we need several preliminary results. The key object is the functional

\[
 f \in B^\infty_\kappa(\mathcal{X}) \mapsto \lambda(f) := \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E}_x \left[ e^{\int_0^t f(X_s) \, ds} \right].
\]

Roughly speaking, the Gärtner–Ellis theorem (Theorem 4 in Appendix A) states that if this functional is finite and Gateau-differentiable over \( B^\infty_\kappa(\mathcal{X}) \) and \((L_t)_{t \geq 0}\) defined in (2) is exponentially tight for the \( \tau^\kappa \)-topology, then \((L_t)_{t \geq 0}\) satisfies a LDP in the dual space of \( B^\infty_\kappa(\mathcal{X}) \). This theorem and some elements of analysis are reminded in Appendix A.

However, studying the range of functions \( f \) for which the functional \( \lambda \) is finite and Gateau-differentiable is not an easy task. Our strategy is to prove that \( r(f) \), the largest eigenvalue in modulus of the Feynman–Kac operator \( L + f \), is real for any function \( f \in B^\infty_\kappa(\mathcal{X}) \), and to show that it is actually equal to the cumulant function \( \lambda(f) \) defined in (24). This amounts to showing the well-posedness and regularity of a family of spectral problems. For this, we use several ideas from [46], which shows that under Lyapunov and irreducibility conditions, the eigenvalue problem to which \( \lambda \) is associated is well defined. The seminal paper by Gärtner [54, Section 3] provides useful technical tools, as well as [44, 111].

In all this section, we suppose that Assumptions 1, 2 and 3 hold true and consider a function \( \kappa : \mathcal{X} \to [1, +\infty) \) as in Assumption 3, i.e. such that \( \kappa \ll \Psi \) and either \( \kappa \) has compact level sets or is bounded. We repeatedly use that \( \kappa \ll -L \) in view of (20). We start with important properties of key martingales that appear regularly in the proofs of the required technical results.

Lemma 2. If \((X_t)_{t \geq 0}\) is solution to (3), then the stochastic processes defined by

\[
 M_t = W(X_t) e^{-\int_0^t \frac{1}{\kappa(X_s)} \, ds} \quad \text{and} \quad \mathcal{M}_t = \mathcal{W}(X_t) e^{-\int_0^t \frac{1}{\kappa(X_s)} \, ds}
\]

are continuous non-negative local martingales, hence supermartingales. Moreover, it holds almost surely

\[
 \mathcal{M}_t^2 \leq C_1 e^{C_2 M_t},
\]

where \( C_1 > 0 \) and \( C_2 \in \mathbb{R} \) are the constants from Assumption 3.

Proof. First, Itô formula shows that

\[
 dM_t = e^{-\int_0^t \frac{1}{\kappa(X_s)} \, ds} \nabla W(X_t) \cdot \sigma(X_t) \, dB_t.
\]
Since $W$ is $C^2(X)$ and $\sigma$ is continuous, $M_t$ is a continuous local martingale \cite{65}. Since it is non-negative, it is a supermartingale by Fatou's lemma, and the same conclusion holds for $M_t$. On the other hand, \cite{71} shows that for any continuous path $(X_t)_{t \geq 0}$ we have

$$M_t^2 = \mathcal{W}(X_1)^2 \int_0^t e^{-2\mathcal{W}(X_s)} ds \leq C_1 W(X_t) \exp \left[ \int_0^t \left( -\frac{\mathcal{L}W}{W}(X_s) + C_2 \right) ds \right] \leq C_1 e^{C_2 t} M_t,$$

which concludes the proof. \qed

The use of the martingale $M_t$ is inspired by \cite{114} where it is considered to control return times to compact sets. Here, it allows to define the Feynman–Kac semigroup associated with the dynamics $(X_t)_{t \geq 0}$ with weight function $f \in B_{\infty}^{\infty}(X)$.

**Lemma 3.** For $f \in B_{\infty}^{\infty}(X)$, consider the Feynman–Kac operator defined as

$$(P_t f)(x) := \mathbb{E}_x \left[ f(X_t) e^{\int_0^t f(X_s) ds} \right].$$

Then for any $f \in B_{\infty}^{\infty}(X)$, any $t > 0$ and any $a > 0$, there exist $c_{a,t} \geq 0$ and a compact subset $K_{a,t} \subset X$ such that

$$\forall x \in X, \quad (P_t f)(x) \leq e^{-at} W(x) + c_{a,t} 1_{K_{a,t}}(x).$$

Moreover, for any $f \in B_{\infty}^{\infty}(X)$ and $t > 0$ it holds $P_t f \in \mathcal{B}(B_{\infty}^{\infty}(X))$, and $(P_t f)_{t \geq 0}$ is a semigroup of bounded operators over $B_{\infty}^{\infty}(X)$. It has generator $\mathcal{L} + f$ defined on the domain

$$\mathcal{D}_{\mathcal{L},f} = \left\{ \varphi \in B_{\infty}^{\infty}(X) \mid (\mathcal{L} + f)\varphi \in B_{\infty}^{\infty}(X) \right\}.$$  

Note that for any $f \in B_{\infty}^{\infty}(X)$ the domain $\mathcal{D}_{\mathcal{L},f}$ contains $C_c^\infty(X)$, but $C_c^\infty(X)$ is not dense in $B_{\infty}^{\infty}(X)$ for the associated norm. However $(P_t f)_{t \geq 0}$ generates a semigroup of bounded operators, so $\mathcal{L} + f$ can be defined on $B_{\infty}^{\infty}(X)$ with domain $\mathcal{D}_{\mathcal{L},f}$.

**Proof.** We first show that for any $f \in B_{\infty}^{\infty}(X)$, $(P_t f)_{t \geq 0}$ is a semigroup of bounded operators on $B_{\infty}^{\infty}(X)$, before turning to the proof of \cite{72}. For a fixed $f \in B_{\infty}^{\infty}(X)$, since $\kappa \ll \Psi$ there is $c > 0$ such that, for any $t > 0$, it holds

$$P_t f W(x) = \mathbb{E}_x \left[ W(X_t) e^{\int_0^t f(X_s) ds} \right] \leq e^{ct} \mathbb{E}_x \left[ W(X_t) e^{-\int_0^t \mathcal{L}W(X_s) ds} \right].$$

Using Lemma \cite{2} the supermartingale property leads to

$$P_t f W(x) \leq e^{ct} \mathbb{E}_x |M_t| \leq e^{ct} W(x).$$

This shows that, for all $\varphi \in B_{\infty}^{\infty}(X)$, it holds

$$\forall x \in X, \quad |P_t f(\varphi)(x)| \leq P_t |\varphi|(x) \leq \|\varphi\|_{B_{\infty}^{\infty}} P_t f W(x),$$

and then

$$\|P_t f\|_{B_{\infty}^{\infty}} \leq e^{ct} \|\varphi\|_{B_{\infty}^{\infty}}.$$

Thus $(P_t f)_{t \geq 0}$ is a semigroup of bounded operators over $B_{\infty}^{\infty}(X)$. This semigroup has generator $\mathcal{L} + f$, which can be shown (although $f$ is not regular) by using a stochastic integration by part, see \cite{93} Chapter VIII, Proposition 3.10.

The domain of this generator is then defined by \cite{58}.

We next prove \cite{67} for a fixed $f \in B_{\infty}^{\infty}(X)$, which we assume non-zero without loss of generality. Note that

$$\frac{\mathcal{L}W}{W} + f \leq -\Psi + \|f\|_{B_{\infty}^{\infty}} \kappa = -\left(1 - \|f\|_{B_{\infty}^{\infty}} \frac{\kappa}{\Psi}\right) \Psi.$$
Since $\Psi$ has compact level sets and $\kappa \ll \Psi$, for any $a > 0$ there exists a compact set $K_a \subset \mathcal{X}$ and a constant $b_{a,a}$ such that

$$\frac{\mathcal{L}W}{W} + f \leq -(a + \alpha) + b_{a,a} \mathbb{I}_{K_a},$$

where $\alpha > 0$ is a constant to be chosen later on. This implies that

$$(\mathcal{L} + f)W \leq -(a + \alpha)W + b_a \mathbb{I}_{K_a},$$

with $b_a = b_{a,a} \sup_K W < +\infty$ since $W \in C^2(\mathcal{X})$. Therefore,

$$\frac{d}{dt} \left( e^{(a+\alpha)t} P_t^f W \right) = e^{(a+\alpha)t} P_t^f \left( (a + \alpha)W + (\mathcal{L} + f)W \right) \leq b_a e^{(a+\alpha)t} P_t^f \mathbb{I}_{K_a} \leq b_a e^{(a+\alpha)t} P_t^f \mathbb{I}.$$

(69)

We can now bound the right hand side of the above equation with a technique similar to the one used in Section 2.3. Indeed, for any $x \in \mathcal{X}$,

$$\left( P_t^f \mathbb{I} \right)(x) = \mathbb{E}_x \left[ e^{\int_0^t f(X_s) \, ds} \right] \leq \mathbb{E}_x \left[ e^{\|f\|_{B^\infty} \int_0^t \kappa(X_s) \, ds} \right].$$

(70)

Since $\kappa \ll -\frac{\mathcal{L}W}{W}$, there exists a constant $c > 0$ depending on $f$ such that

$$\kappa \leq \frac{1}{\|f\|_{B^\infty}} \left( -\frac{\mathcal{L}W}{W} \right) + c.$$

Plugging this estimate into (70) and using that $\mathcal{W} \geq 1$ leads to

$$\left( P_t^f \mathbb{I} \right)(x) \leq e^{ct} \mathbb{E}_x \left[ \mathcal{W}(X_t) e^{\int_0^t \frac{\mathcal{L}W}{W}(X_s) \, ds} \right] = e^{ct} \mathbb{E}_x \left[ \mathcal{W} \right] \leq e^{ct} \mathcal{W}(x),$$

where the last bound comes from Lemma 2. Therefore, using this estimate to bound the right hand side of (69), we end up with

$$\frac{d}{dt} \left( e^{(a+\alpha)t} P_t^f W \right) \leq b_a e^{(a+\alpha+c)t} \mathcal{W}.$$

Integrating with respect to time leads to

$$\left( P_t^f W \right)(x) \leq e^{-t(a+\alpha)W}(x) + b_a \mathcal{W}(x), \quad \tilde{b}_a = \frac{b_a}{a + \alpha + c} e^{ct}.$$

Since $\mathcal{W} \ll W$, there exists a compact set $K_{a,t} \subset \mathcal{X}$ such that $\tilde{b}_a \mathcal{W} \leq e^{-(a+\alpha)t}W$ outside $K_{a,t}$, so that we have

$$\forall x \in \mathcal{X}, \quad \left( P_t^f W \right)(x) \leq 2 e^{-(a+\alpha)t}W(x) + \left( \tilde{b}_a \sup_{K_{a,t}} \mathcal{W} \right) \mathbb{I}_{K_{a,t}}(x).$$

We can now assume that we chose from the beginning $\alpha > \log(2)/t$ (recall that $t$ is fixed). Setting $c_{a,t} = \tilde{b}_a \sup_{K_{a,t}} \mathcal{W}$, this leads to

$$\forall x \in \mathcal{X}, \quad \left( P_t^f W \right)(x) \leq e^{-ct}W(x) + c_{a,t} \mathbb{I}_{K_{a,t}}(x),$$

which proves 4.}

Lemma 3 proves crucial to obtain the compactness of the evolution operator $P_t^f$, as noted in 16 (a result inspired by 22 Theorem 8.9]). Another key ingredient is the regularization property of the evolution. The following bound on the Feynman-Kac semigroup depending on the weight function $f$ is one element in this direction.

**Lemma 4.** Suppose that Assumptions 2, 3 and 4 hold true, and fix $f, g \in B^\infty(\mathcal{X})$. Then, for any $t > 0$, any $\varphi \in B^\infty(\mathcal{X})$, and any $x \in \mathcal{X}$, it holds

$$\left| P_t^f \varphi(x) - P_t^g \varphi(x) \right| \leq \| \varphi \|_{B^\infty} \mathbb{E}_x \left[ W(X_t) \left( \int_0^t |f(X_s) - g(X_s)| \, ds \right) e^{\|f\|_{B^\infty} + \|g\|_{B^\infty} \int_0^t \kappa(X_s) \, ds} \right].$$

(71)
We use Assumption 1 to revisit [54, pages 34-35] in an unbounded setting and with a hypoelliptic flavour. Proof. For any $u \in \mathcal{C}^{\infty}_c(\mathcal{X})$, the theorem ensures that $\parallel u \parallel_{\mathcal{C}^{\infty}_c(\mathcal{X})}$ converges uniformly over any compact $K \subset \mathcal{X}$, by proving that upper bounds of continuous functions is continuous. We introduce to this end the events

$$\forall m \geq 1, \quad \delta_m = \left\{ \frac{1}{t} \int_0^t \Psi(X_s) \, ds \leq m \right\},$$

and fix a compact set $K' \subset \mathcal{X}$. The right hand side of (72) can then be split into two terms

$$(A) = \mathbb{E}_x \left[ \mathbb{I}_K(X_t) \mathbb{I}_{\sigma_m} W(X_t) \left( \int_0^t |f(X_s) - f_n(X_s)| \, ds \right) e^{\delta \int_0^t \kappa(X_s) \, ds} \right],$$

$$(B) = \mathbb{E}_x \left[ \mathbb{I}_K(X_t) \mathbb{I}_{\sigma_m} W(X_t) \left( \int_0^t |f(X_s) - f_n(X_s)| \, ds \right) e^{\delta \int_0^t \kappa(X_s) \, ds} \right],$$

Our goal is now to show that $P^f_t(\varphi \mathbb{1}_K)$ converges uniformly over any compact $K'$ to $P^f_t(\varphi \mathbb{1}_K)$, by proving that the right hand side of (72) goes uniformly to 0 over $K'$. This will conclude the proof since a uniform limit of continuous functions is continuous.
for which we show convergence to 0, uniformly for $x \in K'$, starting with (A). Since $\kappa \ll -L/\|W\|$, there exists $c > 0$ such that

$$2\delta \kappa \ll -L/\|W\| + c.$$ 

Moreover, $\|f_n\|_{B_r} \leq \|f\|_{B_r}^\infty$, $a \leq e^a$, and $\|W\| \geq 1$, so that

$$(A) \leq e^{(m+c)\|W\|} \mathbb{E}_x \left[ \mathbb{1}_{K}(X_0) \mathbb{1}_{\|W\|} W(X_0) e^{\int_0^t -\frac{\kappa}{2m}(X_s) \, ds} \right].$$ 

By definition of $\mathcal{M}$ in (64) we have

$$(A) \leq e^{(m+c)\|W\|} \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right].$$ 

The Cauchy-Schwarz inequality then shows that

$$(A) \leq e^{(m+c)\|W\|} \left( \mathbb{E}_x \left[ \mathcal{M}_0^2 \right] \right)^{1/2} \mathbb{E}_x \left[ \mathcal{M}_0 \right].$$ 

By (65) it holds $\mathbb{E}_x \left[ \mathcal{M}_0^2 \right] \leq \sqrt{C_1} e^{C_2 t/2} \sqrt{W(x)}$. Next, by Tchebychev's inequality and since $W \geq 1$,

$$\mathbb{P}_x \left( \int_0^t \Psi(X_s) \, ds > mt \right) \leq e^{-mt} \mathbb{E}_x \left[ e^{\int_0^t \Psi(X_s) \, ds} \right] \leq e^{-mt} \mathbb{E}_x \left[ W(X_t) e^{-\int_0^t \frac{\kappa}{2m}(X_s) \, ds} \right] \leq e^{-mt} W(x).$$ 

As a result, we obtain

$$(A) \leq e^{-mt} \left( \mathbb{E}_x \left[ \mathcal{M}_0 \right] \right)^{1/2} \mathbb{E}_x \left[ \mathcal{M}_0 \right] \sqrt{C_1} e^{C_2 t/2}. $$ 

Therefore, for any $\varepsilon > 0$, we can choose $m > 0$ such that $(A) \leq \varepsilon$.

Let us now control (B), introducing $g_n = |f - f_n|$. Since $\kappa \ll \Psi$, it holds for some $c' > 0$,

$$\delta \kappa \ll \Psi + c'.$$

Using the definition (73) we have

$$(B) \leq e^{(m+c')t} \left( \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right] \right)^{1/2} \left( \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right] \right)^{1/2} \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right],$$

where $B_R$ is the ball of center 0 and radius $R > 0$. Let us first bound $(B')$, which retains only the parts of the trajectories performing excursions out of $B_R$. Using $\kappa \ll \Psi$, for $\varepsilon > 0$ and $m > 0$ as fixed above, there exist $R > 0$, $C_R > 0$ such that

$$\kappa \ll \varepsilon \frac{e^{-(m+c')t}}{tm(\mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \right] \|f\|_{B_r}^\infty)} \Psi + C_R \mathbb{1}_{B_R}. $$

We fix $R > 0$ and $C_R > 0$ such that the above inequality holds true. Using again $g_n \ll \|f\|_{B_r}^\infty \kappa$, we are led to

$$\left( B' \right) \leq e^{(m+c')t} \left( \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right] \right)^{1/2} \left( \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right] \right)^{1/2} \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right],$$

where

$$\mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right] \leq \mathbb{E}_x \left[ \mathbb{1}_{\|W\|} \mathcal{M}_0 \right] \leq \varepsilon.$$
where the last line follows from the definition (73) of $\mathcal{E}_m$. Therefore, once $m$ is fixed, there exists $R > 0$ such that for any $n \geq 1$ and $x \in K'$, it holds $(B') \leq \varepsilon$. It remains to control $(B'')$ in order to obtain the uniform convergence to zero of $(B)$ over $K'$ as $n \to +\infty$. In fact,

\[
(B'') \leq e^{(m+\varepsilon')t} \left( \sup_{K} W \right) \int_0^t \mathbb{E}_x \left[ g_n(X_s)1_{B_R}(X_s) \right] ds = e^{(m+\varepsilon')t} \left( \sup_{K} W \right) \int_0^t P_s(g_n 1_{B_R})(x) ds,
\]

where $(P_s)_{s \geq 0}$ is the evolution semigroup defined in (15). Since $(1_{B_R} g_n)_{n \geq 1}$ is a sequence of bounded functions converging almost everywhere to zero and the transition kernel $P_t$ has a smooth density for $s > 0$, it follows that $(P_s(g_n 1_{B_R}))_{n \geq 1}$ goes uniformly to zero over compact sets for any $s > 0$ as $n \to +\infty$, see e.g. [54, 51]. Moreover, it can be shown that

\[
\delta \mapsto \int_0^\delta P_s(g_n 1_{B_R}) ds
\]

goes to zero when $\delta \to 0$, uniformly in $x \in K'$ and $n \in \mathbb{N}$. Therefore, for $\varepsilon > 0$, $R > 0$ and $m \geq 0$ fixed as above, there exist $\delta > 0$, and $n' \in \mathbb{N}$ such that for all $n \geq n'$ and $x \in K'$,

\[
0 \leq \int_0^\delta P_s(g_n 1_{B_R})(x) ds = \int_0^\delta P_s(g_n 1_{B_R})(x) ds + \int_\delta^\delta P_s(g_n 1_{B_R})(x) ds \leq \varepsilon e^{-(m+\varepsilon')t} \sup_{K} W'.
\]

Then, for any $n \geq n'$, $x \in K'$, it holds

\[
(B'') \leq \varepsilon.
\]

Let us summarize the various approximations: for any $\varepsilon > 0$, we first fix $m \geq 0$ so that $(A) \leq \varepsilon$. Then, we choose $R > 0$ large enough so that $(B') \leq \varepsilon$. Finally, we take $\delta$ small enough and $n$ large enough in (73) so that $(B'') \leq \varepsilon$ for $n \geq n'$. As a result, for any $\varepsilon > 0$ there is $n' \geq 0$ such that for all $n \geq n'$ and $x \in K'$, it holds $(A) + (B) \leq 3\varepsilon$.

In conclusion, the right hand side of (72) goes to zero uniformly as $n \to +\infty$ over any compact set $K'$. Therefore $P_t^n(\varphi 1_{K})$ is continuous and converges uniformly over $K'$ to $P_t^1(\varphi 1_{K})$, which is therefore continuous over $K'$. Since the compact $K' \subset X$ is arbitrary, $P_t^1(\varphi 1_{K})$ is continuous over $X$, which concludes the proof. \hfill \qed

Before presenting the main result concerning the spectral properties of the semigroup $P_t^f$ and its consequences on the definition of the cumulant function $\lambda(f)$, we need the following “irreducibility” lemma, which relies on Assumption 2.

**Lemma 6.** For any time $t > 0$, $x \in X$ and any set $A \subset X$ with non-empty interior, it holds

\[
(P_t^f 1_A)(x) > 0.
\]

**Proof.** Take $x \in X$ and $y \in \hat{A}$ (which is possible since $A$ has non-empty interior). By Assumption 2, there exists a $C^1$-path $(\phi_s)_{s \in [0,t]}$ solving (18) such that $\phi_0 = x$ and $\phi_t = y$. We can then use the proof of the Stroock–Varadhan support theorem, see [51] Theorem 6.1 for an overview. In particular, Assumption 2 implies that [100, Eq. (5.5)] is satisfied. Therefore, [100, Eq. (5.6)] ensures that, for any $\varepsilon > 0$,

\[
P_x \left( \sup_{0 \leq s \leq t} |X_s - \phi_s| \leq \varepsilon \right) > 0.
\]

Moreover, since $\phi_t = y \in \hat{A}$ and upon reducing $\varepsilon > 0$ we may assume that $B(y, \varepsilon) \subset A$, where $B(y, \varepsilon)$ denotes the ball of center $y$ and radius $\varepsilon > 0$. Recalling that $f \in B_b^\infty(X)$, we then obtain

\[
(P_t^f 1_A)(x) = \mathbb{E}_x \left[ 1_{(X_t \in A)} e^{-\int_0^t f(X_s) du} \right] \geq \mathbb{E}_x \left[ 1_{(\sup_{0 \leq s \leq t} |X_s - \phi_s| \leq \varepsilon)} e^{-\|f\|_{B_b^\infty} \int_0^t \kappa(X_s) du} \right]
\]

\[
\geq \exp \left( -t\|f\|_{B_b^\infty} \sup_{\phi \in \hat{A}} \kappa \right) P_x \left( \sup_{0 \leq s \leq t} |X_s - \phi_s| \leq \varepsilon \right),
\]

26
where we denote by $S_{\phi, \epsilon}$ the $\epsilon$-tube around the path $(\phi_s)_{s \in [0, t]}$, namely $S_{\phi, \epsilon} = \{ x \in \mathcal{X} \mid \exists s \in [0, t] \text{ with } |\phi_s - x| \leq \epsilon \}$. Since $S_{\phi, \epsilon}$ is a bounded set and $\kappa$ is continuous over $\mathcal{X}$, it holds
\[ \sup_{S_{\phi, \epsilon}} \kappa < +\infty. \]

The combination of (80) and (81) leads to the desired result (85). \qed

At this stage, we can adapt the spectral analysis developed in [46] to our situation. However, we trade the minorization condition on which [46] relies for the irreducibility granted by Lemma 6.

**Lemma 7.** For any $f \in B^\infty(\mathcal{X})$ the operator $\mathcal{L} + f$ considered over $B^\infty(\mathcal{X})$ has a real largest eigenvalue $r(f)$ with eigenspace of dimension one, and an associated continuous eigenvector $h_f \in D_{\mathcal{L}, f}$ such that $h_f(x) > 0$ for any $x \in \mathcal{X}$, and
\[ (\mathcal{L} + f)h_f = r(f)h_f \quad \text{and} \quad \frac{\mathcal{L}h_f}{h_f} \in B^\infty(\mathcal{X}). \]

Moreover, $h_f$ is the only positive eigenvector of $\mathcal{L} + f$ (up to multiplication by a positive constant). Finally, $r(f)$ is equal to the cumulant function defined in (23):
\[ r(f) = \lambda(f). \] (79)

The result of Lemma 7 is twofold: it entails the well-posedness of the principal eigenproblem associated with $\mathcal{L} + f$ for any $f \in B^\infty(\mathcal{X})$, and then identifies this principal eigenvalue with the free energy function (24).

**Proof.** We closely follow the path of [46] and split the proof into several steps.

**Step 1: Compactness of the evolution operator.** We first show that, for given $t > 0$ and $f \in B^\infty(\mathcal{X})$, the operator $P^t_f$ defined in Lemma 5 is compact when considered on $B^\infty(\mathcal{X})$. For any $\epsilon < t/2$ and any compact set $K \subset \mathcal{X}$ we have the decomposition
\[ P^t_f = P^{t-2\epsilon}_f 1_K P^t_f 1_K P^t_f 1_K + P^{t-2\epsilon}_f 1_K f P^t_f 1_K + P^{t-2\epsilon}_f 1_K P^t_f 1_K + P^{t-2\epsilon}_f 1_K P^t_f 1_K. \]

We first consider the compact sets $K_a$ from (77) for $a > 0$ (omitting the dependence on $t$ in the notation since the time is fixed here) and note that $1_{K_a} P^t_f$ converges to 0 in operator norm as $a \to +\infty$. Indeed, for any $f \in B^\infty(\mathcal{X})$, (77) leads to
\[ \| 1_{K_a} P^t_f f \|_{B^\infty_W} \leq \| f \|_{B^\infty_W} e^{-at}, \] (81)
so that $\| 1_{K_a} P^t_f \|_{B(B^\infty_W)} \to 0$ when $a \to +\infty$.

Next we show that $P^{t-2\epsilon}_f 1_K P^t_f 1_K$ is compact over $B^\infty(\mathcal{X})$ for any compact set $K \subset \mathcal{X}$. Consider a sequence $(\varphi_k)_{k \in \mathbb{N}}$ bounded in $B^\infty(\mathcal{X})$. Following the first step of the proof of [46] Lemma 2 and using our strong Feller result, Lemma 5, we see that $P^{t-2\epsilon}_f 1_K$ and $P^t_f 1_K$ are strong Feller operators, so $P^{t-2\epsilon}_f 1_K P^t_f 1_K$ is ultra-Feller (see [46] Lemma 6). This means that the operator $P^{t-2\epsilon}_f 1_K P^t_f 1_K$ is continuous in total variation norm, so that the family $(P^{t-2\epsilon}_f 1_K P^t_f 1_K)_{k \in \mathbb{N}}$ is uniformly equicontinuous. We used here that since $f \in B^\infty(\mathcal{X})$ and $W$ is continuous, it holds $1_K \varphi \in B^\infty(\mathcal{X})$. The sequence $(P^{t-2\epsilon}_f 1_K P_f 1_K \varphi_k)_{k \in \mathbb{N}}$ therefore converges in $B^\infty(\mathcal{X})$ up to extraction by the Ascoli theorem [92] Theorem 11.28, and in $B^\infty(\mathcal{X})$ since $W \geq 1$. Therefore, the operator $P^{t-2\epsilon}_f 1_K P_f 1_K$ sends a bounded sequence into a convergent one (up to extraction), so it is compact in $B^\infty(\mathcal{X})$ [72].

The decomposition (80) and the bound (81) then show that $P^t_f$ is the limit in operator norm of the compact operators $P^{t-2\epsilon}_f 1_K \varphi_k P^t_f 1_K$ as $a \to +\infty$, so it is compact in $B^\infty(\mathcal{X})$ (see e.g. [72] Theorem VI.12)).
Step 2: Existence of the principal eigenvalue. We can now use the Krein–Rutman theorem on the (closed) total cone $\mathbb{K}_W = \{ \varphi \in B_{W}^{+} | \varphi \geq 0 \}$ (see [27, 39] for definitions). It is clear that $P_t^f$ leaves this cone invariant. We next show that $P_t^f$ has a non-zero spectral radius

$$R_t(f) = \lim_{n \to +\infty} \left\| (P_t^f)^n \right\|_{B(B_{W}^{+})}.$$

To this end, fix a compact set $K$ with non-empty interior. We have shown in Lemma 9 that

$$\forall x \in K, \quad \left( P_t^f 1_K \right)(x) > 0.$$

Since $P_t^f 1_K$ is continuous by Lemma 5, this shows that

$$\alpha_K := \min_{x \in K} \left( P_t^f 1_K \right)(x) > 0. \quad (82)$$

Therefore, for any $x \in K$,

$$\left[ \left( P_t^f \right)^2 1_K \right](x) = \mathbb{E}_x \left[ (P_t^f 1_K)(X_t) e^{\int_0^t f(X_s) \, ds} \right] \geq \alpha_K \mathbb{E}_x \left[ 1_K(X_t) e^{\int_0^t f(X_s) \, ds} \right] = \alpha_K \left( P_t^f 1_K \right)(x) \geq \alpha_K^2,$$

so that $1_K(x) (P_t^f)^2 1_K(x) \geq \alpha_K 1_K(x)$ for $x \in \mathcal{X}$. Iterating the procedure for any $n \geq 1$ we get

$$\left\| (P_t^f)^n \right\|_{B(B_{W}^{+})} \geq \left\| 1_K (P_t^f)^n 1_K \right\|_{B_{W}^{+}} \geq \frac{\alpha_K^n}{\sup_K W}.$$

As a result, since $1 \leq \sup_K W < +\infty$, we obtain in the large $n$ limit the following lower bound for the spectral radius:

$$R_t(f) \geq \alpha_K > 0,$$

which shows that $R_t(f)$ is positive. Since $P_t^f$ is compact, [27, Theorem 19.2] ensures that $R_t(f)$ is a real eigenvalue of $P_t^f$ with associated eigenvector $h_f \in \mathbb{K}_W$ (in particular, $h_f \geq 0$). Using the semigroup property of $P_t^f$ and standard arguments (see [31, Theorem 2.4]), we can show that $R_t(f) = e^{r(f)t}$ where $r(f)$ is the largest eigenvalue of $\mathcal{L} + f$, and $h_f \in \mathcal{D}_{\mathcal{L}, f}$ satisfies

$$(\mathcal{L} + f) h_f = r(f) h_f,$$

as well as

$$P_t^f h_f = e^{r(f)t} h_f. \quad (83)$$

Step 3: Properties of $h_f$. For the remainder of the proof, we write for simplicity $r := r(f)$ and $h := h_f$ (the function $f$ being fixed). We show here that $h$ is continuous and positive. For any compact $K \subset \mathcal{X}$, [39] leads to

$$\left| P_t^f 1_K h - e^{rt} h \right| = \left| P_t^f 1_K h - P_t^f h \right| = \left| P_t^f \left( 1_K e^{-rt} P_t^f h \right) \right| \leq e^{-rt} \left\| h \right\|_{B_{W}^{+}} \left\| P_t^f \right\|_{B(B_{W}^{+})} \left\| 1_K \right\|_{B(B_{W}^{+})}.$$

Using Lemma 3 we obtain that, for any $a > 0$, there exists a compact set $K_a$ such that

$$\left\| e^{-rt} P_t^f 1_K h - h \right\|_{B_{W}^{+}} \leq Ce^{-at} \quad \text{with} \quad C = e^{-2at} \left\| h \right\|_{B_{W}^{+}} \left\| P_t^f \right\|_{B(B_{W}^{+})},$$

so that $h$ is continuous as the uniform limit of continuous functions (since $P_t^f 1_K h$ is continuous by Lemma 4). Finally, since $h \geq 0$ and $h$ is not identically equal to 0, there exists $x_0 \in \mathcal{X}$ such that $h(x_0) > 0$. Moreover $h$ is continuous, so there is $\varepsilon > 0$ for which $h > 0$ on $B(x_0, \varepsilon)$. By [39] it holds, for any $x \in \mathcal{X}$,

$$e^{rt} h(x) = (P_t^f h)(x) \geq P_t^f \left( h 1_{B(x_0, \varepsilon)} \right)(x) \geq \inf_{B(x_0, \varepsilon)} h \left( P_t^f 1_{B(x_0, \varepsilon)} \right)(x).$$

Since $h > 0$ on $B(x_0, \varepsilon)$ and $h$ is continuous, $\inf_{B(x_0, \varepsilon)} h > 0$. Moreover $(P_t^f 1_{B(x_0, \varepsilon)})(x) > 0$ for any $x \in \mathcal{X}$ by Lemma 6, so the previous lower bound shows that $h(x) > 0$ for all $x \in \mathcal{X}$.
Step 4: Properties of eigenspaces and eigenfunctions. We now show that the eigenspace associated with \( h \) is of dimension one, and that any other eigenvector vanishes somewhere in \( X \). For this, we introduce the so called \( h \)-transform [73, 93, 94, 96]. A key element here is the fact that \( h(x) > 0 \) for all \( x \in X \), which allows to define the following Markov operator, for an arbitrary time \( t > 0 \):

\[
Q_h \varphi = e^{-rt}h^{-1}P_t(h\varphi),
\]

where \( h, h^{-1} \) refer here to the multiplication operators by the functions \( h \) and \( h^{-1} \) respectively. We now prove that \( Q_h \) is ergodic by first noting that \( Q_h \) admits \( Wh^{-1} \) as a Lyapunov function (using [67] and the normalization \( \|h\|_{\infty}^1 = 1 \) which implies that \( Wh^{-1} \geq 1 \)). Moreover, we can prove that \( Q_h \) satisfies a minorization condition on any compact set. For this, consider \( K \subset X \) compact with non-empty interior and denote by \( \eta_K \) the uniform Lebesgue measure on \( K \). By [92], for any \( t > 0 \) there is \( \alpha_K > 0 \) such that, for any measurable set \( A \subset X \),

\[
\forall x \in K, \quad (P_t^I1_A)(x) \geq (P_t^I1_{K \cap A})(x) \geq \alpha_K \eta_K(A).
\]

Since \( h \) is continuous, this implies that, for any measurable \( \varphi \geq 0 \),

\[
\forall x \in K, \quad (Q_h\varphi)(x) \geq \frac{|K|\alpha_K \min_{K \cap A} \eta_K(\varphi)}{\max_K h^{\varphi}},
\]

where both the minimum and maximum above are finite and non-zero (recall that \( |K| > 0 \) is the Lebesgue measure of \( K \)). This shows that \( Q_h \) satisfies a minorization condition [58] over any compact set. Using Assumption [3], we can also show that \( Wh^{-1} \) has compact level sets, see [46, Appendix E] for details. Therefore, the Markovian dynamics with kernel \( Q_h \) admits a unique invariant probability measure \( \mu_h \), with respect to which it is ergodic in \( B_{Wh^{-1}}^\infty(X) \). By this we mean that (in view of [58, Theorem 1.2]) there exist \( \bar{a} > 0 \) and \( C > 0 \) such that for any \( \varphi \in B_{Wh^{-1}}^\infty(X) \),

\[
\forall n \geq 1, \quad \|(Q_h)^n \varphi - \mu_h(\varphi)\|_{B_{Wh^{-1}}^\infty} \leq C e^{-\bar{a}n} \|\varphi - \mu_h(\varphi)\|_{B_{Wh^{-1}}^\infty}, \tag{84}
\]

and it holds \( \mu_h(W/h) < +\infty \).

We can now use this ergodic behaviour to show that the eigenspace associated with \( r \) has dimension one and that \( P_t^I \) cannot have another positive eigenvector with norm 1 in \( B_{Wh}^\infty(X) \). Indeed, if there were another eigenvector \( \tilde{h} \in B_{Wh}^\infty(X) \) associated with \( r \), then the fact that \( h/\tilde{h} \in B_{Wh^{-1}}^\infty(X) \) together with [54] ensure that

\[
(Q_h)^n \left( \frac{h}{\tilde{h}} \right) = \frac{h}{\tilde{h}} \xrightarrow{n \to +\infty} \mu_h \left( \frac{h}{\tilde{h}} \right).
\]

This shows that \( h \) and \( \tilde{h} \) would be proportional, and answers the claim that the eigenspace associated with \( r \) has dimension one. Assume now that there is another real eigenvalue \( \tilde{r} < r \) with real eigenvector \( \tilde{h} \in B_{Wh}^\infty(X) \) such that \( \tilde{h}(x) > 0 \) for all \( x \in X \). Noting again that \( h/\tilde{h} \in B_{Wh^{-1}}^\infty(X) \) and since \( h > 0 \), [54] shows that, for any \( x \in X \),

\[
(Q_h)^n \left( \frac{h}{\tilde{h}} \right)(x) \xrightarrow{n \to +\infty} \mu_h \left( \frac{h}{\tilde{h}} \right) > 0. \tag{85}
\]

However it now holds, for any \( x \in X \),

\[
(Q_h)^n \left( \frac{h}{\tilde{h}} \right)(x) = e^{(\tilde{r}-r)tn} \frac{h}{\tilde{h}}(x) \xrightarrow{n \to +\infty} 0,
\]

where we used that \( h > 0 \) and \( \tilde{r} < r \). Combining the two equations above shows that

\[
\mu_h \left( \frac{h}{\tilde{h}} \right) = 0,
\]

which contradicts [55]. As a result, there cannot be another eigenvalue with a positive eigenvector.
Step 5: The principal eigenvalue is the cumulant function. Proving (79) now follows by a simple rewriting. For \( x \in \mathcal{X} \) and \( t_0 > 0 \) fixed, it holds for any \( n \in \mathbb{N}^* \),
\[
E_x \left[ e^{\int_0^{nt_0} f(X_s) \, ds} \right] = \left[ (P_{t_0}^f)^n 1 \right](x) = e^{rnt_0} [h(Q_n)^n h^{-1}](x),
\]
so that
\[
\frac{1}{nt_0} \log E_x \left[ e^{\int_0^{nt_0} f(X_s) \, ds} \right] = \frac{1}{nt_0} \log [e^{rnt_0} h(Q_n)^n h^{-1}(x)] = r + \frac{1}{nt_0} \log [h(Q_n)^n h^{-1}(x)].
\]
By (84) (since \( h^{-1} \in B_{W_{h^{-1}}}^\infty(\mathcal{X}) \)), we see that \( h(Q_n)^n h^{-1}(x) \) converges to a constant, so that
\[
r(f) = \lim_{n \to +\infty} \frac{1}{nt_0} \log E_x \left[ e^{\int_0^{nt_0} f(X_s) \, ds} \right].
\]
We have chosen to work with an arbitrary time \( t_0 > 0 \) for convenience, so a priori the above limit depends on \( t_0 \). Showing that the limit actually does not depend on this \( t_0 \) and that
\[
r(f) = \lim_{t \to +\infty} \frac{1}{t} \log E_x \left[ e^{\int_0^{t} f(X_s) \, ds} \right]
\]
follows by standard arguments not reproduced here (see e.g. [62, 69]), which concludes the proof. \( \square \)

An important ingredient for the lower bound of the LDP is the Gateau-differentiability of the cumulant functional, which we prove below.

Lemma 8. The functional
\[
f \in B^\infty_\kappa(\mathcal{X}) \mapsto \lambda(f) = \lim_{t \to +\infty} \frac{1}{t} \log E_x \left[ e^{\int_0^{t} f(X_s) \, ds} \right]
\]
is convex and Gateau-differentiable.

Proof. The convexity of \( \lambda \) is a standard consequence of Hölder’s inequality. Concerning Gateau-differentiability, we follow the strategy of [54, Section 3] for a compact state space, relying on results of Kato [69]. For this, we interpret the cumulant function (86) as the largest eigenvalue of the tilted generator \( r(f) \) as defined in Lemma 7. More precisely, for \( f, g \in B^\infty_\kappa(\mathcal{X}) \) and \( \alpha \in \mathbb{R} \), \( \lambda(f + \alpha g) \) is associated with the largest eigenvalue of the operator \( P_t^{f+\alpha g} \) in \( B^\infty_\kappa(\mathcal{X}) \) through
\[
P_t^{f+\alpha g} h_{f+\alpha g} = e^{\lambda(f+\alpha g)} h_{f+\alpha g},
\]
so that derivability in \( \alpha \) can be shown through the differentiability of the spectrum of a bounded operator. We thus show that the operator \( P_t^{f+\alpha g} \) is differentiable in operator norm. To this end, we fix \( C > 0 \) and prove that for \( |\alpha| \leq C \) it holds
\[
P_t^{f+\alpha g} \varphi = P_t^f \varphi + \alpha Q_t^{f,g,(1), \alpha} \varphi + \alpha^2 Q_t^{f,g,(2), \alpha} \varphi,
\]
where \( Q_t^{f,g,(1)} \) and \( Q_t^{f,g,(2), \alpha} \) are bounded operators on \( B_\kappa^\infty(\mathcal{X}) \) and \( |\alpha_0| \leq C \).

For this we first define
\[
Q_t^{f,g,(1)} : \varphi \in B_\kappa^\infty(\mathcal{X}) \mapsto E_x \left[ \varphi(X_t) \left( \int_0^t g(X_s) \, ds \right) e^{\int_0^t f(X_s) \, ds} \right].
\]
This operator is bounded in \( B_\kappa^\infty(\mathcal{X}) \) by the same martingale estimate used to prove Lemma 4. In the same way, the second order operator reads
\[
Q_t^{f,g,(2), \alpha} : \varphi \in B_\kappa^\infty(\mathcal{X}) \mapsto \frac{1}{2} E_x \left[ \varphi(X_t) \left( \int_0^t g(X_s) \, ds \right)^2 e^{\int_0^t (f(X_s) + \alpha g(X_s)) \, ds} \right].
\]
This operator is also bounded in $B^\Phi_t(\mathcal{X})$ since, for $a \geq 0$ it holds $a^2/2 \leq e^a$ so that, for $\varphi \in B^\Phi_t(\mathcal{X})$ and $x \in \mathcal{X}$ we obtain
\[
\left| Q^{f,g,(2),\alpha}_t \varphi(x) \right| \leq \|\varphi\|_{B^\Phi_t(\mathcal{X})} \|g\|_{B^\Phi_t(\mathcal{X})}^2 \mathbb{E}_x \left[ W(X_1) \frac{1}{2} \left( \int_0^t \kappa(X_s) \, ds \right)^2 \right. \\
\left. \times \left( \|f\|_{B^\Phi_t(\mathcal{X})} \|g\|_{B^\Phi_t(\mathcal{X})} \right)^2 \int_0^t \kappa(X_s) \, ds \right] \\
\leq \|\varphi\|_{B^\Phi_t(\mathcal{X})} \|g\|_{B^\Phi_t(\mathcal{X})}^2 \mathbb{E}_x \left[ W(X_1) e^{\|f\|_{B^\Phi_t(\mathcal{X})} + |\alpha| \|g\|_{B^\Phi_t(\mathcal{X})}} \int_0^t \kappa(X_s) \, ds \right] \\
\leq \|\varphi\|_{B^\Phi_t(\mathcal{X})} \|g\|_{B^\Phi_t(\mathcal{X})}^2 e^{c W(x)},
\]
for some constant $c > 0$ depending on $\|f\|_{B^\Phi_t(\mathcal{X})}$, $\|g\|_{B^\Phi_t(\mathcal{X})}$ and $\alpha$. Next, it suffices to note that
\[
Q^{f,g,(1)}_t = \frac{d}{d\alpha} P^{f+\alpha g}_t, \quad Q^{f,g,(2),\alpha}_t = \frac{1}{2} \frac{d^2}{d\alpha^2} P^{f+\alpha g}_t,
\]
to obtain (S7) through a Taylor expansion, where $|\alpha_0| \leq C$.

This shows that $\alpha \mapsto P^{f+\alpha g}_t$ is differentiable in operator norm. Thus, the principal eigenvalue $\lambda(f + \alpha g)$, which is always isolated, is differentiable, see [69 Chapter II, Theorem 5.4] and [69 Chapter IV, Theorem 3.5]. This concludes the proof of Gateau-differentiability.

**Remark 6.** By pursuing further the Taylor expansion (S7) in the proof of Lemma 5 we can actually show that, for any $f, g \in B^\Phi_t(\mathcal{X})$, the function
\[ \alpha \in \mathbb{C} \mapsto \lambda(f + \alpha g) \]
is analytic (this analyticity was already proven in [73] using a different argument that can be simplified with our tools). This relies on the simple inequality $a^n/n! \leq e^a$ for any $a \geq 0$, together with the series expansion of the exponential and martingale estimates as in the proof of Lemma 8. Indeed, our proof, based on martingales, shows that for any $t > 0$, the function
\[ \alpha \mapsto \frac{1}{t} \log \mathbb{E}_x \left[ e^{\int_0^t \left( f(X_s) + \alpha g(X_s) \right) \, ds} \right] \]
is analytic. Moreover, it is finite on $\mathbb{R}$ and converges pointwise to a finite valued function as $t \to +\infty$, as shown in Lemma 7. Therefore, the convergence holds uniformly on any compact as $t \to +\infty$ (see [73, Theorem VI.3.3]). Now a locally uniform limit of analytic functions is analytic (see [73, Theorem 10.28]). Therefore, $\alpha \mapsto \lambda(f + \alpha g)$ is analytic.

The last step before proving the large deviations principle itself is an exponential tightness result, see [28, Section 1.2]. At this stage, the finiteness of $\lambda(f)$ together with the Gateau-differentiability of $f \in B^\Phi_t(\mathcal{X}) \mapsto \lambda(f)$ already provides the upper bound over compact sets and the lower bound in (28). In order to extend the upper bound to all closed sets, we prove exponential tightness in the $\tau^\kappa$-topology, see Appendix A for some definitions (this exponential tightness is not explicitly stated in [73]).

**Lemma 9.** The family of probability measures $t \mapsto \mathbb{P}_x(L_t \in \cdot)$ over $\mathcal{P}(\mathcal{X})$ is exponentially tight in the $\tau^\kappa$-topology.

**Proof.** We adapt the strategy of [111 Corollary 2.3] and [118 Section 2.2] by introducing the family of sets
\[ \Gamma_N = \left\{ \nu \in \mathcal{P}(\mathcal{X}) \mid \nu(\Psi) \leq N \right\}, \quad N > 0. \]
For $N > 0$, the sets $\Gamma_N$ are subsets of $\mathcal{P}_\kappa(\mathcal{X})$ since $\kappa \ll \Psi$. We show that they are actually precompact in the $\tau^\kappa$-topology.

Let us first show that $\Gamma_N$ is precompact in the usual weak topology for any $N > 0$. Consider for this the compact sets $K_\beta = \{ x \in \mathcal{X} \mid \Psi(x) \leq \beta \} \subset \mathcal{X}$ for $\beta > 0$ (recall that $\Psi$ has compact level sets). Then, for any $\nu \in \Gamma_N$, we have
\[ \beta \nu(K_\beta) + \nu(\Psi 1_{K_\beta}) \leq \nu(\Psi 1_{K_\beta}) + \nu(\Psi 1_{K_\beta}) = \nu(\Psi) \leq N. \]
This shows that for any $\beta > 0$ and any $\nu \in \Gamma_N$,
\[ \nu(K_\beta) \leq \frac{N}{\beta}, \]
31
hence (upon choosing $\beta$ sufficiently large) for any $N > 0$ the family of measures $\Gamma_N$ is tight, so it is precompact for the weak topology by the Prohorov theorem [12]. Now, if $\kappa$ is bounded, $\Gamma_N$ is tight for the $\kappa^\tau$-topology and the theorem is shown, so we may assume that $\kappa$ has compact level sets (see Assumption [3]). For proving compactness in our finer topology, we show that $\kappa$ is uniformly integrable over $\Gamma_N$ in order to use [1177] Theorem 7.12. Since $\kappa \ll \Psi$, the set

$$A_n = \left\{ x \in \mathcal{X} \mid \frac{\Psi(x)}{\kappa(x)} \leq n \right\}$$

is compact for any $n \geq 1$. Moreover, since we assume $\kappa$ to be continuous with compact level sets, for any $n \geq 1$ there exists $m_n \geq n$ such that

$$\{ \Psi/\kappa \leq n \} \subset \{ \kappa \leq m_n \},$$

with $m_n \to +\infty$ when $n \to +\infty$. Therefore, for any $\nu \in \Gamma_N$ and $n \geq 1$,

$$\int_{\{ \kappa \leq m_n \}} \kappa \, d\nu \leq \int_{A_n^\kappa} \kappa \, d\nu = \frac{1}{m} \int_{A_n^\kappa} m \kappa \, d\nu \leq \frac{1}{n} \int \Psi \, d\nu = \frac{1}{n} \nu(\Psi) \leq \frac{N}{n}.$$

Taking the supremum over $\nu \in \Gamma_N$ in the above equation and recalling that $m_n \to +\infty$ when $n \to +\infty$ we obtain

$$\lim_{n \to +\infty} \sup_{\nu \in \Gamma_N} \int_{\{ \kappa \leq m_n \}} \kappa \, d\nu = 0. \quad (88)$$

We can then conclude that $\Gamma_N$ is compact for the $\kappa^\tau$-topology. Consider indeed a sequence $(\nu_n)_{n \in \mathbb{N}} \subset \Gamma_N$. By Prohorov’s theorem, $(\nu_n)_{n \in \mathbb{N}}$ has a subsequence weakly converging towards a measure $\nu$, i.e. $\nu_n(\varphi) \to \nu(\varphi)$ for any $\varphi \in C_b(\mathcal{X})$. Then, by [107, Theorem 7.12], [98] ensures that $\nu \in \mathcal{P}_c(\mathcal{X})$ and for any $f \in B_b^\kappa(\mathcal{X})$, $\nu_n(f) \to \nu(f)$ as $n \to +\infty$. In other words, $\Gamma_N$ is precompact for the $\kappa^\tau$-topology.

We can now prove the $\kappa^\tau$-exponential tightness of the empirical distribution $(L_t)_{t \geq 0}$ in $\mathcal{P}(\mathcal{X})$. Indeed, for any $N, t > 0$, Chebyshev’s inequality leads to

$$\mathbb{P}_x \left( L_t \in \Gamma_N^\kappa \right) = \mathbb{P}_x \left( \int_0^t \Psi(X_s) \, ds > N t \right) \leq e^{-N t} \mathbb{E}_x \left[ e^{\int_0^t \Psi(X_s) \, ds} \right] = e^{-N t} P_t^\Psi \mathbbm{1}(x).$$

Renormalizing at log scale and using (24) leads to

$$\lim_{t \to +\infty} \frac{1}{t} \log \mathbb{P}_x \left( L_t \in \Gamma_N^\kappa \right) \leq -N + \lim_{t \to +\infty} \frac{1}{t} \log \left[ P_t^\Psi \mathbbm{1}(x) \right]. \quad (89)$$

The right hand side of the above quantity may look infinite since $\Psi$ grows faster than $\kappa$. However, using again the martingale $M_t$ defined in Lemma [2] we obtain, for any $t > 0$,

$$\mathbb{E}_x \left[ e^{\int_0^t \Psi(X_s) \, ds} \right] \leq \mathbb{E}_x \left[ W(X_t) e^{-\int_0^t \frac{\Psi}{\kappa}(X_s) \, ds} \right] = \mathbb{E}_x [M_t] \leq W(x).$$

Thus it holds

$$\lim_{t \to +\infty} \frac{1}{t} \log \left[ P_t^\Psi \mathbbm{1}(x) \right] \leq \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E}_x \left[ W(X_t) e^{\int_0^t \Psi(X_s) \, ds} \right] \leq \lim_{t \to +\infty} \frac{1}{t} \log W(x) \leq 0.$$

As a result, (89) becomes

$$\lim_{t \to +\infty} \frac{1}{t} \log \mathbb{P}_x \left( L_t \in \Gamma_N^\kappa \right) \leq -N.$$

Since $\Gamma_N$ is precompact in the $\kappa^\tau$-topology for any $N > 0$, and $N$ can be chosen arbitrarily large, this proves the exponential tightness of the family of empirical distributions in the $\kappa^\tau$-topology. \qed

We are now in position to prove Theorem [4].
Proof of Theorem 1. We assemble the previous lemmas to check that the assumptions of the Gärtner–Ellis theorem reminded in Appendix A are fulfilled. For this, we note that the cumulant function
\[ \lambda : f \in B_0^\infty(\mathcal{X}) \mapsto \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E}_x \left[ e^{\int_0^t f(X_s) \, ds} \right] \]
can be identified to the function \( \Lambda \) in Theorem 3. The topological dual of \((\mathcal{M}_s(\mathcal{X}), \tau^*)\) is \( B_0^\infty(\mathcal{X}) \), where \( \mathcal{M}_s(\mathcal{X}) \) is the set of measures over \( \mathcal{X} \) integrating \( \kappa \) (see [28, Appendix B] and [30, Lemma 3.3.8] for details). We have proved that \( \lambda \) is well defined, Gateau differentiable, and that the family of measures
\[ t \mapsto \pi_t(\cdot) := \mathbb{P}_t(L_t \in \cdot), \]
is exponentially tight in the \( \tau^* \)-topology. Therefore, \((\pi_t)_{t \geq 0}\) satisfies a large deviations principle in the \( \tau^* \)-topology with good rate function given by
\[ \forall \nu \in \mathcal{M}(\mathcal{X}), \quad I(\nu) = \sup_{f \in B_0^\infty} \left\{ \nu(f) - \lambda(f) \right\}. \]  
(90)

We observe that \( I(\nu) = +\infty \) if \( \nu \) is not normalized to 1 (take \( f \) to be constant in the supremum (90)), so we may consider \( I \) over \( \mathcal{P}(\mathcal{X}) \). Moreover, choosing \( f = \kappa \) in (90) and noting that \( \lambda(\kappa) < +\infty \), we get \( I(\nu) = +\infty \) if \( \nu \notin \mathcal{P}_s(\mathcal{X}) \). If \( \nu \) is not absolutely continuous with respect to \( \mu \), there exists \( A \subset \mathcal{X} \) such that \( \mu(A) = 0 \) and \( \nu(A) > 0 \). Since \( \mu \) has a positive Lebesgue density, this means that \( A \) has zero Lebesgue measure. Consider then \( f_\alpha = a \mathbb{1}_A \in B_0^\infty(\mathcal{X}) \) for \( a \in \mathbb{R} \). Since \( A \) has zero Lebesgue measure and \((X_t)_{t \geq 0}\) has a smooth density for all \( t > 0 \) (as a consequence of Assumption B) it holds, for all \( t > 0 \),
\[ \mathbb{E}_x[f_\alpha(X_t)] = a \mathbb{P}_x(X_t \in A) = 0. \]

Therefore, the process
\[ Z_t = \int_0^t f_\alpha(X_s) \, ds, \]
satisfies \( \mathbb{E}_x[Z_t] = 0 \) for all \( t > 0 \). Since \( Z_t \geq 0 \), it holds \( Z_t = 0 \) almost surely, for any \( t > 0 \). As a consequence we obtain
\[ \forall t > 0, \quad \frac{1}{t} \log \mathbb{E}_x \left[ e^{\int_0^t f_\alpha(X_s) \, ds} \right] = \frac{1}{t} \log \mathbb{E}_x \left[ e^{Z_t} \right] = 0. \]

This shows that \( \lambda(f_\alpha) = 0 \), so that from (90) we obtain
\[ I(\nu) \geq a \nu(A), \]
with \( \nu(A) > 0 \). By letting \( a \to +\infty \) we are led to \( I(\nu) = +\infty \).

Finally, we show that \( I(\nu) = 0 \) if and only if \( \nu = \mu \), and that \((L_t)_{t \geq 0}\) converges almost surely to \( \mu \) in the \( \tau^* \)-topology (see [25, Appendix B] for definitions). For this, we introduce the set of minimizers
\[ \mathcal{J} = \left\{ \nu \in \mathcal{P}(\mathcal{X}) \right| I(\nu) = \inf_{\mathcal{P}(\mathcal{X})} I \right\}. \]

Since \( I \) has compact level sets, \( \mathcal{J} \) is a non-empty closed subset of \( \mathcal{P}(\mathcal{X}) \) for the \( \tau^* \)-topology. Moreover, in order for the LDP upper bound to make sense, it holds \( \inf_{\mathcal{P}(\mathcal{X})} I > 0 \). If \( \mathcal{J}_\delta \) denotes an open neighborhood of \( \mathcal{J} \), the lower semicontinuity of \( I \) implies that
\[ \inf_{\mathcal{J}_\delta} I > 0. \]

Therefore, by the large deviations upper bound we have, for \( t \geq 0 \),
\[ \mathbb{P}_x(L_t \notin \mathcal{J}_\delta) = \mathbb{P}_x(L_t \in \mathcal{J}_\delta^c) \leq C \exp \left( -t \inf_{\mathcal{J}_\delta} I \right), \]
(91)
for some constant $C > 0$. We next introduce the random subset of $\mathbb{R}_+$ of indices $t \geq 0$ for which $L_t$ does not belong to $\mathcal{F}_s$, namely $T = \{ t \geq 0 \mid L_t \notin \mathcal{F}_s \}$. Since
\[
P_s \{ L_t \notin \mathcal{F}_s \} = \mathbb{E}_s [ \mathbf{1}_{\{ L_t \notin \mathcal{F}_s \}} ],
\]
we have, by the Fubini theorem, for any $t > 0$,
\[
\int_0^t P_s \{ L_s \notin \mathcal{F}_s \} ds = \mathbb{E}_s \left[ \int_0^t \mathbf{1}_{\{ L_s \notin \mathcal{F}_s \}} ds \right] = \mathbb{E}_s [ \min(T, t) ].
\]
By using (11) and the dominated convergence theorem, we obtain
\[
\mathbb{E}_s \lbrack |T| \rbrack = \int_0^{+\infty} P_s \{ L_t \notin \mathcal{F}_s \} dt < +\infty.
\]
As a result, $|T| < +\infty$ almost surely. This means that, for any neighborhood $\mathcal{F}_s$ of $\mathcal{F}$ in the $\tau^s$-topology, the empirical measure $(L_{t})_{t > 0}$ almost surely spends a finite Lebesgue measure time outside $\mathcal{F}_s$. In other words, $(L_{t})_{t > 0}$ converges almost surely in the $\tau^s$-topology to the set $\mathcal{F}$ (by definition of convergence in a topological space [23, Appendix B]). However, we know by Proposition 2 that the only possible limit for $(L_{t})_{t > 0}$ is $\mu$, hence $\mathcal{F} = \{ \mu \}$ and $(L_{t})_{t > 0}$ almost surely converges to $\mu$, which concludes the proof.

### 6.2 Proofs of Section 3

#### 6.2.1 Proof of Proposition 3

For the proof, which is partly inspired by [20, Lemma 4.1.36], we denote by $I_F$ the rate function given by the Fenchel transform in (25) and $I_V$ for the Varadhan functional on the right hand side of (32). We repeatedly use the results of Lemma 7 below.

We first show that $I_V(\nu) = +\infty$ if $\nu$ is not absolutely continuous with respect to $\mu$ or does not belong to $\mathcal{P}_s(\mathcal{X})$. Assume first that $\nu \ll \mu$ does not hold: there exists a set $A \subset \mathcal{X}$ such that $\nu(A) > 0$ and $\mu(A) = 0$. For any $a \in \mathbb{R}$ we introduce $f_a = a \mathbb{1}_A$ and denote by $h_a$ the eigenvector associated to the principal eigenvalue $\lambda(f_a)$ (which is continuous and positive by Lemma 4). As shown in the proof of Theorem 4 it holds $\lambda(f_a) = 0$, so
\[
\forall a \in \mathbb{R}, \quad (\mathcal{L} + f_a) h_a = 0.
\]
This leads to
\[
- \frac{\mathcal{L} h_a}{h_a} = a \mathbb{1}_A.
\]
Therefore, since $h_a \in B_0^\infty(\mathcal{X})$ is positive, it holds $h_a \in \mathcal{D}^+$ and
\[
I_V(\nu) \geq \int_X - \frac{\mathcal{L} h_a}{h_a} d\nu = a \nu(A) > 0.
\]
By letting $a \to +\infty$, we conclude that $I_V(\nu) = +\infty$ when $\nu$ is not absolutely continuous with respect to $\mu$. Next, if $\nu \notin \mathcal{P}_s(\mathcal{X})$, since $\kappa > 1$ it holds $\nu(\kappa) = +\infty$. We may then choose $f = \kappa \in B_0^\infty(\mathcal{X})$. By Lemma 4 the principal eigenvector $h_\kappa$ of $\mathcal{L} + \kappa$ belongs to $\mathcal{D}^+$ with $\lambda(\kappa) < +\infty$, so we have
\[
I_V(\nu) \geq \int_X - \frac{\mathcal{L} h_\kappa}{h_\kappa} d\nu = \int_X \kappa d\nu - \lambda(\kappa) = +\infty,
\]
i.e. $I_V(\nu) = +\infty$ if $\nu \notin \mathcal{P}_s(\mathcal{X})$. This shows that $I_F(\nu) = I_V(\nu)$ when $\nu$ is not absolutely continuous with respect to $\mu$ or $\nu \notin \mathcal{P}_s(\mathcal{X})$. We next show that $I_F = I_V$ when $\nu \ll \mu$ and $\nu \in \mathcal{P}_s(\mathcal{X})$, which we assume until the end of the proof.

Let us first show that $I_F \geq I_V$. For this, we consider $u \in \mathcal{D}^+$ and introduce
\[
f_u = \frac{\mathcal{L} u}{u}
\]
Because of the definition \([33]\) of \(D^+\), we know that \(f_a \in B_c^\infty(\mathcal{X})\), so we can write, since \(\nu \in \mathcal{P}_a(\mathcal{X})\),

\[ I_{\nu}(\nu) \geq \nu(f_a) - \lambda(f_a). \tag{92} \]

Moreover,

\[ (\mathcal{L} + f_a)u = \mathcal{L}u - \left( \frac{\mathcal{L}u}{u} \right) u = 0. \tag{93} \]

As a result, \(u > 0\) is an eigenvector of \(\mathcal{L} + f_a\) associated with the eigenvalue 0 (and hence it is an eigenvector of \(P_t^{f_a}\) with eigenvalue 1). But we know from Lemma \([7]\) that a positive eigenvector can only be associated with the principal eigenvalue \(\lambda(f_a)\), so that \(\lambda(f_a) = 0\) by \([33]\). Therefore, \(92\) leads to

\[ I_{\nu}(\nu) \geq \nu(f_a) - \lambda(f_a) = \int_X -\frac{\mathcal{L}u}{u} \, d\nu. \]

Since \(u \in D^+\) is arbitrary, taking the supremum shows that \(I_{\nu}(\nu) \geq I_{\nu}(\nu)\) for any \(\nu \in \mathcal{P}_a(\mathcal{X})\) with \(\nu \ll \mu\).

We now turn to the inequality \(I_{\nu} \leq I_{\nu}\). We again draw elements from \([33]\) Lemma 4.1.36, but we use simpler arguments based on the spectral analysis of the operator \(\mathcal{L} + f\). Consider for any arbitrary \(f \in B_c^\infty(\mathcal{X})\) the associated eigenvector \(h_f \in B_c^\infty(\mathcal{X})\) defined in Lemma \([7]\). It then holds:

- \(h_f \in B_c^\infty(\mathcal{X})\);
- \(h_f > 0\);
- by Lemma \([4]\) \(-\frac{\mathcal{L}h_f}{h_f} = f - \lambda(f) \in B_c^\infty(\mathcal{X})\);

Thus \(h_f \in D^+\). As a result we have, since \(\nu \in \mathcal{P}_a(\mathcal{X})\),

\[ I_{\nu}(\nu) \geq \int_X -\frac{\mathcal{L}h_f}{h_f} \, d\nu = \nu(f) - \lambda(f). \]

Given that, in the above equation, \(f\) is an arbitrary function belonging to \(B_c^\infty(\mathcal{X})\), taking the supremum leads to

\[ I_{\nu}(\nu) \geq \sup_{f \in B_c^\infty} \left\{ \nu(f) - \lambda(f) \right\}. \]

This finally shows that \(I_{\nu}(\nu) = I_{\nu}(\nu)\) for all \(\nu \in \mathcal{P}_a(\mathcal{X})\) with \(\nu \ll \mu\), which concludes the proof.

### 6.2.2 Proof of Corollary \([2]\)

Since \(I\) is the Fenchel transform of \(\lambda\), the result follows if we can show that \(\lambda\) defined on \(B_c^\infty(\mathcal{X})\) is stable by bi-Fenchel conjugacy. The convexity and finiteness of \(\lambda\) show that a (necessary and) sufficient condition for \(\lambda\) to be bi-Fenchel stable is for the functional \(f \mapsto \lambda(f)\) to be lower-semicontinuous (see \([5]\) Theorem 2.22). We show below that it is actually continuous: for any sequence \((f_n)_{n>0}\) in \(B_c^\infty(\mathcal{X})\) such that \(\|f_n - f\|_{B_c^\infty} \to 0\) for some \(f \in B_c^\infty(\mathcal{X})\), it holds \(\lambda(f_n) \to \lambda(f)\) as \(n \to +\infty\). We shall use for this a stability result from \([23]\).

Consider a sequence \((f_n)_{n>0}\) converging to \(f\) in \(B_c^\infty(\mathcal{X})\). Using Lemma \([4]\) for any \(\varphi \in B_c^\infty(\mathcal{X})\), \(t > 0\), \(x \in \mathcal{X}\) and \(n \in \mathbb{N}\), it holds (using again the inequality \(a \leq e^a\) for \(a \geq 0\))

\[
\begin{align*}
\left| (P^t_t^f \varphi) - (P^t_n^f \varphi) (x) \right| & \leq \|\varphi\|_{B_c^\infty} \left[ W(X_t) \left( \int_0^t |f(X_s) - f_n(X_s)| \, ds \right) e^{(f\|B_c^\infty + f_n\|_{B_c^\infty}) \int_0^t |X_s| \, ds} \right] \\
& \leq \|\varphi\|_{B_c^\infty} \left[ f - f_n \|B_c^\infty \left[ W(X_t) e^{(f\|B_c^\infty + f_n\|_{B_c^\infty}) \int_0^t |X_s| \, ds} \right] \right] \\
& \leq C\|\varphi\|_{B_c^\infty} \left[ f - f_n \|B_c^\infty \left[ M_t \right] \right] \\
& \leq C\|\varphi\|_{B_c^\infty} \left[ f - f_n \|B_c^\infty W(x), \right]
\end{align*}
\]
for some constant $C > 0$ depending on $\|f\|_{B^\infty}$ and $t > 0$. We used Lemma 2 and the supermartingale property of $M_t$ to obtain the last line. This leads to

$$\left\| P^f_t - P^{f_n}_t \right\|_{B(B^\infty)} \leq C\|f - f_n\|_{B^\infty} \quad \text{as} \quad n \to +\infty. \quad (94)$$

We know by Lemma 7 that $\lambda(f)$ and $\lambda(f_n)$ are associated with the isolated largest eigenvalue of the operators $P^f_t$ and $P^{f_n}_t$ respectively. Therefore, Section 2.2 and Proposition 2.11 show that the approximation is strongly stable (we refer to Proposition 2.2 in particular the definition in Section 2.2 and Proposition 2.11), so Proposition 2.2 ensures that $\lambda(f_n) \to \lambda(f)$ as $n \to +\infty$. This shows that the function $\lambda : B^\infty_0(\mathcal{X}) \to \mathbb{R}$ is continuous and concludes the proof.

### 6.2.3 Proof of Theorem 2

The proof, inspired by [114], relies on two ideas: performing a Witten transform inside the variational representation [60] and separating the symmetric and antisymmetric parts of the generator $\mathcal{L}$. We write $d\nu = \rho \, d\mu = e^\nu \, d\mu$ and consider first that $v \in C^\infty_c(\mathcal{X})$ instead of $\mathcal{F}^3(\nu)$. Starting from (32), we consider a function $u$ of the form

$$u = e^\frac{\nu}{\sqrt{\rho}}, \quad \psi \in C^\infty_c(\mathcal{X}). \quad (95)$$

We call this choice “variational Witten transform” for its similarity with the standard Witten transform [109, 15] and its use in the variational formula (32) defining $I$. Since $u = e^{\frac{\nu}{\sqrt{\rho}}}$ with $v, \psi \in C^\infty_c(\mathcal{X})$ it is clear that $u \in \mathcal{D}^+$. This follows by noting that, using the shorthand notation $w = \psi/2 + v/2 \in C^\infty_c(\mathcal{X})$, we have

$$-\frac{\mathcal{L}u}{u} = -e^{-w} \mathcal{L}e^w = -\mathcal{L}w - \frac{1}{2} |\sigma^T \nabla w|^2 \in C^\infty_c(\mathcal{X}) \subset B^\infty_0(\mathcal{X}).$$

Moreover, it holds $u = e^\nu > 0$ and $u$ is constant outside a compact set, so $u \in B^\infty_0(\mathcal{X})$ and it holds $u \in \mathcal{D}^+$. We now rewrite the expression in (32) for $u$ given by (95), using again the notation $w = \psi/2 + v/2$:

$$\begin{aligned}
-\int \frac{\mathcal{L}u}{u} \, d\nu &= -\int \mathcal{L}w \, d\nu - \frac{1}{2} \int |\sigma^T \nabla w|^2 \, d\nu.
\end{aligned}$$

Recalling that $S = \sigma \sigma^T$ and expanding $w = \psi/2 + v/2$, we obtain

$$\begin{aligned}
-\int \frac{\mathcal{L}u}{u} \, d\nu &= -\frac{1}{2} \int \mathcal{L}\psi \, d\nu - \frac{1}{2} \int \mathcal{L}v \, d\nu - \frac{1}{2} \int \nabla \psi \cdot S \nabla \psi \, d\nu - \frac{1}{2} \int \nabla v \cdot S \nabla v \, d\nu - \frac{1}{2} \int \nabla v \cdot S \nabla v \, d\nu.
\end{aligned} \quad (96)$$

We now decompose $\mathcal{L}$ into symmetric and antisymmetric parts. First, it holds

$$\begin{aligned}
-\frac{1}{2} \int \mathcal{L}\psi \, d\nu &= -\frac{1}{2} \int (\mathcal{L}\psi + \sigma^T \sigma \mathcal{L}\psi) \, d\nu - \frac{1}{2} \int (\mathcal{L}\psi - \sigma^T \sigma \mathcal{L}\psi) \, d\nu + \frac{1}{2} \int \nabla \psi \cdot S \nabla \psi \, d\nu - \frac{1}{2} \int \nabla v \cdot S \nabla v \, d\nu.
\end{aligned} \quad (97)$$

On the other hand, using that $\mathcal{L}_\Lambda$ is a first order differential operator satisfying $\mathcal{L}_\Lambda 1 = 0$, we obtain

$$\begin{aligned}
\int (\mathcal{L}_\Lambda \psi) e^\nu \, d\mu &= \int (\mathcal{L}_\Lambda e^\psi) \, d\mu = \int (\mathcal{L}_\Lambda e^\psi - \mathcal{L}_\Lambda \psi) \, d\mu = 0.
\end{aligned}$$

As a result

$$\begin{aligned}
-\frac{1}{2} \int \mathcal{L}v \, d\nu &= -\frac{1}{2} \int (\mathcal{L}v + \sigma^T \sigma \mathcal{L}v) \, d\mu - \frac{1}{2} \int (\mathcal{L}v - \sigma^T \sigma \mathcal{L}v) \, d\mu + \frac{1}{2} \int \nabla v \cdot S \nabla v \, d\nu.
\end{aligned} \quad (98)$$

By plugging (97) into (98), we obtain

$$\begin{aligned}
-\int \frac{\mathcal{L}u}{u} \, d\nu &= \frac{1}{4} \int \nabla v \cdot S \nabla v \, d\nu - \frac{1}{4} \int (\mathcal{L}_\Lambda \psi) \, d\nu - \frac{1}{4} \int \nabla \psi \cdot S \nabla \psi \, d\nu.
\end{aligned} \quad (99)$$

The first term in the above equation reads (recalling that $\rho = e^\nu$)

$$\frac{1}{4} \int \nabla v \cdot S \nabla v \, d\nu = \int \nabla (\sqrt{\rho}) \cdot S \nabla (\sqrt{\rho}) \, d\mu.$$
By density of \( C_c^\infty(\mathcal{X}) \) in \( \mathcal{H}^1(\mu) \), the above expression is valid for any \( \rho \) such that \( \sqrt{\rho} \in \mathcal{H}^1(\mu) \). The above computation shows that this condition is equivalent to assuming that \( v \in \mathcal{H}^2(\mu) \), and

\[
\frac{1}{4} \int_X \nabla v \cdot S \nabla v \, d\nu = \frac{1}{4} |v|_{\mathcal{H}^2(\mu)}^2,
\]

which does not involve the function \( \psi \in C_c^\infty(\mathcal{X}) \). Moreover, since \( \mathcal{L}_A \) is a first order differential operator, antisymmetric in \( L \), it holds

\[
\int_X (\mathcal{L}_A \psi) \, d\nu = -\int_X (\mathcal{L}_A v) \psi \, d\mu = -\int_X (\mathcal{L}_A v) \psi \, d\nu.
\]

As a result, (100) rewrites

\[
-\int_X \frac{\mathcal{L}_A u}{u} \, d\nu = \frac{1}{4} |v|_{\mathcal{H}^1(\mu)}^2 + \frac{1}{2} \int_X (\mathcal{L}_A v) \psi \, d\nu - \frac{1}{4} |\psi|_{\mathcal{H}^2(\mu)}^2,
\]

and this expression is finite for any \( \psi \in C_c^\infty(\mathcal{X}) \).

Our goal is now to take the supremum over functions \( \psi \in C_c^\infty(\mathcal{X}) \) in (100), and prove that this is enough to obtain the supremum over \( D^+ \). We consider for this the terms depending on \( \psi \) in (100) and, using the duality between \( \mathcal{H}^1(\nu) \) and \( \mathcal{H}^{1-}(\nu) \) (see [72, Section 2, Claim F]) we obtain

\[
\frac{1}{2} \int_X (\mathcal{L}_A v) \psi \, d\nu - \frac{1}{2} \int_X \nabla \psi \cdot S \nabla \psi \, d\nu \leq \frac{1}{2} |\mathcal{L}_A v|_{\mathcal{H}^{1-}(\nu)} |\psi|_{\mathcal{H}^1(\nu)} - \frac{1}{4} |\psi|_{\mathcal{H}^2(\mu)}^2
\]

\[
\leq \frac{1}{4\varepsilon} |\mathcal{L}_A v|_{\mathcal{H}^{1-}(\nu)}^2 - \frac{1}{4}(1 - \varepsilon) |\psi|_{\mathcal{H}^1(\nu)}^2,
\]

where we used Young's inequality with \( \varepsilon < 1 \) to obtain the second line. Since \( \mathcal{L}_A v \in \mathcal{H}^{1-}(\nu) \), the supremum over the functions \( \psi \in C_c^\infty(\mathcal{X}) \) takes the value \( -\infty \) when \( \psi \notin \mathcal{H}^1(\nu) \). Therefore, by density of \( C_c^\infty(\mathcal{X}) \) in \( \mathcal{H}^1(\nu) \), the supremum over the functions of the form (103) for \( \psi \in C_c^\infty(\mathcal{X}) \) recovers the supremum over \( D^+ \) and it holds

\[
\text{by definition of the } \mathcal{H}^{1-}(\nu)\text{-norm in Section 2.4, which concludes the proof.}
\]

**Remark 7.** We have proved our result for measures of the form \( d\nu = e^{\rho} \, d\mu \). Considering more general measures \( \nu \ll \mu \) is made difficult because the Radon–Nikodym derivative \( \rho = dv/d\mu \) may vanish on some region of \( \mathcal{X} \), hence the definition of \( \mathcal{L}_A(\log \rho) \) is not clear. Given (101), we see that we can give a sense to our computations provided \( \mathcal{L}_A(\log \rho) \) defines a linear form on \( \mathcal{H}^1(\nu) \), namely: there exists \( C > 0 \) such that

\[
\forall \psi \in \mathcal{H}^1(\nu), \quad \left| \int_X \psi \mathcal{L}_A(\log \rho) \, d\nu \right| \leq C \|\psi\|_{\mathcal{H}^1(\nu)}.
\]

We find it however clearer to work directly with exponential perturbations of the invariant measure \( \mu \).

**6.2.4 Proof of Corollary 3**

The proof follows from the variational formulation of Theorem 3. Indeed, let us rewrite (88) as

\[
I_\lambda(\nu) = -\frac{1}{2} \inf_{\psi \in \mathcal{H}^1(\nu)} I_\nu(\psi),
\]

where \( \nu \) is fixed and satisfies the assumptions of the theorem, and

\[
I_\nu(\psi) = \frac{1}{2} \int_X \mathcal{L}_\nu(\psi, \psi) \, d\nu - \int_X \psi(\mathcal{L}_A v) \, d\nu.
\]

37
By \cite[Section 2, Claim F]{72}, we can identify $\mathcal{H}^{-1}(\nu)$ with the dual of $\mathcal{H}^1(\nu)$, so that $\mathcal{I}_v$ reads
\[
\forall \psi \in \mathcal{H}^1(\nu), \quad \mathcal{I}_v(\psi) = \frac{1}{2} \|\psi\|_{\mathcal{H}^1(\nu)}^2 - \langle L\nu, \psi \rangle_{\mathcal{H}^{-1}(\nu), \mathcal{H}^1(\nu)}.
\]

Denoting by $\tilde{\nabla}$ the adjoint of the gradient operator in $L^2(\nu)$, the Lax-Milgram theorem \cite[Corollary V.8]{20}, whose conditions are readily fulfilled, shows that the minimum is attained at a unique $\psi_v \in \mathcal{H}^1(\nu)$ solution to
\[
\tilde{\nabla}(S\nabla \psi_v) = L\nu. \tag{104}
\]

Inserting $\psi_v$ solution to \eqref{104} in \eqref{103} leads to
\[
I_\lambda(\nu) = \frac{1}{4} \int_X \mathcal{C}(\psi_v, \psi_v) \, d\nu, \tag{105}
\]
which concludes the proof.

## A Tools for large deviations principle

In this section, we remind some large deviations concepts. For a Polish space $\mathcal{Y}$, we denote by $\mathcal{Y}'$ its topological dual (the set of continuous linear functionals over $\mathcal{Y}$). We first recall the definition of an exponentially tight family of measures. A family of measures $(\pi_t)_{t \geq 0}$ over a Polish space $\mathcal{Y}$ is called exponentially tight if for any $N < +\infty$, there exists a (pre)compact set $\Gamma_N \subset \mathcal{Y}$ such that
\[
\lim_{t \to +\infty} \frac{1}{t} \log \pi_t(\Gamma_N) < -N.
\]

In words, exponential tightness means that the measures $(\pi_t)_{t \geq 0}$ concentrate exponentially fast over compact sets. This is used in large deviations to turn an upper bound over compact sets into an upper bound over all closed sets.

We now define the cumulant function. Consider a family of measures $(\pi_t)_{t \geq 0}$ over a Polish space $\mathcal{Y}$. The logarithmic moment generating function is defined as in \cite[Section 4.5]{28}: for any $t \geq 0$, $f \in \mathcal{Y}'$ and a random variable $Z_t$ distributed according to $\pi_t$,
\[
\Lambda_t(f) = \log \mathbb{E} \left[ e^{\langle f, Z_t \rangle_{\mathcal{Y}', \mathcal{Y}}} \right] = \log \int_{\mathcal{Y}} e^{\langle f, y \rangle_{\mathcal{Y}'}, \mathcal{Y}} \pi_t(dy). \tag{106}
\]

The scaled cumulant generating function is defined by
\[
\bar{\Lambda}_t(f) = \frac{1}{t} \Lambda_t(tf). \tag{107}
\]

Let us relate this quantity with the objects introduced in Section 4. In our situation, we consider fluctuations of the empirical measure $L_t \in \mathcal{M}(\mathcal{X})$ (where $\mathcal{M}(\mathcal{X})$ is the space of measures with finite mass), so $\mathcal{Y} = \mathcal{M}(\mathcal{X})$ and for $\Gamma \in \mathcal{M}(\mathcal{X})$,
\[
\pi_t(\Gamma) = \mathbb{P}_t (L_t \in \Gamma).
\]

On the other hand, $f$ belongs to a space of functions, typically $\mathcal{Y}' = \mathcal{M}(\mathcal{X})' = B^\infty(\mathcal{X})$. In practice we may restrict ourselves to probability measures because the rate function is infinite otherwise. We see that considering $L_t \in \mathcal{P}_\pi(\mathcal{X})$ leads to choosing $f \in B^\infty(\mathcal{X})$. In any case the duality relation \eqref{105} reads in this case
\[
\Lambda_t(f) = \log \int_{\mathcal{P}_\pi(\mathcal{X})} e^{\langle f, L_t \rangle_{\mathcal{Y}'}, \mathcal{Y}} \mathbb{P}_t (L_t \in dy) = \log \mathbb{E}_x \left[ e^{\langle f, L_t \rangle} \right] = \log \mathbb{E}_x \left[ e^{\frac{1}{t} \int_0^t f(x_s) \, ds} \right],
\]
so that $\bar{\Lambda}_t(f)$ coincides with the argument of the limit in \cite{24}. With these preliminaries, we are in position to state the key theorem for the results in this work, which goes back to \cite{54} [45] and is presented for instance in \cite[Corollary 4.6.14]{28}. We recall that a rate function is said to be good if its level sets are compact for the considered topology.

38
Theorem 4 (Projective limit - Gärtner–Ellis). Let \((\pi_t)_{t \geq 0}\) be an exponentially tight family of probability measures on a Polish space \(\mathcal{Y}\). Assume that
\[
\Lambda(\cdot) = \lim_{t \to +\infty} \bar{\Lambda}_t(\cdot)
\]
is finite valued over \(\mathcal{Y}'\) and Gateau-differentiable. Then \((\pi_t)_{t \geq 0}\) satisfies a large deviations principle over \(\mathcal{Y}\) with good rate function \(\Lambda^*\), the Legendre-Fenchel transform of \(\Lambda\).

B Proof of Proposition 1

The proposition is a consequence of the equality
\[
\Psi = -\hat{\mathcal{W}} \hat{\mathcal{W}} = \theta \left( -\mathcal{L} V - \frac{\theta}{2} |\sigma^T \nabla V|^2 \right).
\]
Since \(|\sigma^T \nabla V|\) has compact level sets and \(\Psi \sim |\sigma^T \nabla V|^2\) by (21), \(\Psi\) has compact level sets. Since \(V\) has compact level sets, for \(\varepsilon < \theta/2\) it holds \(\mathcal{W} \ll \mathcal{W}\) and \(\mathcal{W}^2 \leq C_1 \mathcal{W}\) for some constant \(C_1 > 0\). Moreover, outside a compact set, the function
\[
\hat{\mathcal{L}} \hat{\mathcal{W}} = \varepsilon \left( \mathcal{W} - \frac{\varepsilon}{2} |\sigma^T \nabla V|^2 \right)
\]
is bounded above and below since the numerator and denominator are both equivalent to \(|\sigma^T \nabla V|^2\), so the second condition in (20) holds. Finally,
\[
-2 \hat{\mathcal{L}} \hat{\mathcal{W}} = 2 \varepsilon \left( \mathcal{W} - \frac{\varepsilon}{2} |\sigma^T \nabla V|^2 \right) = 2 \varepsilon \Psi + \varepsilon (\theta - \varepsilon) |\sigma^T \nabla V|^2.
\]
Since \(\Psi \sim |\sigma^T \nabla V|^2\), we may choose \(\varepsilon\) small enough so as to obtain
\[
-2 \hat{\mathcal{L}} \hat{\mathcal{W}} \leq \Psi + C_2,
\]
for some constant \(C_2 \in \mathbb{R}\). This proves the third item of (20).

C Proof of Lemma 1

The proof relies on manipulations similar to those of [83]. A simple computation shows that
\[
-\hat{\mathcal{L}} \hat{\mathcal{W}}(q,p) = \varepsilon q \cdot \nabla V - \gamma \varepsilon^2 |q|^2 + \gamma (1 - 2\theta) \varepsilon (1 - \theta) |p|^2 - \varepsilon |p|^2 - \theta \gamma d.
\]
For any \(\eta > 0\) it holds
\[
p \cdot q \geq -\frac{|q|^2}{2} - \frac{|p|^2}{2\eta}.
\]
As a result, Assumption 5 leads to
\[
-\hat{\mathcal{L}} \hat{\mathcal{W}}(q,p) \geq |q|^2 \left( c \varepsilon - \gamma \varepsilon^2 \right) + |p|^2 \left( \theta \gamma - \gamma \varepsilon^2 - \frac{\gamma \varepsilon}{2\eta} (1 - 2\theta) \right) - \theta \gamma d - \varepsilon C_V.
\]
Since \(\theta > 0\), it holds
\[
-\hat{\mathcal{L}} \hat{\mathcal{W}}(q,p) \geq a |q|^2 + b |p|^2 - C,
\]
with
\[
a = \varepsilon \left( c - \frac{\eta \gamma}{2} \right) - \gamma \varepsilon^2, \quad b = \theta \gamma (1 - \theta) - \varepsilon - \frac{\gamma \varepsilon}{2\eta}, \quad C = \theta \gamma d + \varepsilon C_V.
\]
The claim follows for \(\theta \in (0,1)\) by choosing \(\eta < 2c/\gamma\) and \(\varepsilon\) sufficiently small.
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