ON THE FOURTH MOMENT IN
THE RANKIN-SELBERG PROBLEM

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Abstract. If
\[ \Delta(x) := \sum_{n \leq x} c_n - Cx \]
denotes the error term in the classical Rankin-Selberg problem, then it is proved that
\[ \int_0^X \Delta^4(x) \, dx \ll x^{3+\varepsilon}, \quad \int_0^X \Delta_1^4(x) \, dx \ll x^{11/2+\varepsilon}, \]
where \( \Delta_1(x) = \int_0^x \Delta(u) \, du \). The latter bound is, up to ‘\( \varepsilon \)’, best possible.

1. Introduction and statement of results

The classical Rankin-Selberg problem consists of the estimation of the error term function
\begin{equation}
\Delta(x) := \sum_{n \leq x} c_n - Cx,
\end{equation}
where the notation is as follows. Let \( \varphi(z) \) be a holomorphic cusp form of weight \( \kappa \) with respect to the full modular group \( SL(2, \mathbb{Z}) \), and denote by \( a(n) \) the \( n \)-th Fourier coefficient of \( \varphi(z) \). We suppose that \( \varphi(z) \) is a normalized eigenfunction for the Hecke operators \( T(n) \), that is, \( a(1) = 1 \) and \( T(n)\varphi = a(n)\varphi \) for every \( n \in \mathbb{N} \). The classical example is \( a(n) = \tau(n) \) (\( \kappa = 12 \)), the Ramanujan function defined by
\[ \sum_{n=1}^{\infty} \tau(n) x^n = x \{ (1 - x)(1 - x^2)(1 - x^3) \cdots \}^{24} \quad (|x| < 1). \]
The constant $C(>0)$ in (1) may be written down explicitly (see e.g., [7]), and $c_n$ is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right)\right|^2.$$ 

The classical Rankin-Selberg bound of 1939 is

$$\Delta(x) = O(x^{3/5}),$$

hitherto unimproved. In their works, done independently, R.A. Rankin [10] derives (1.2) from a general result of E. Landau [9], while A. Selberg [12] states the result with no proof. Although it seems very difficult at present to improve the bound in (1.2), recently there have been some results on mean square estimates in the Rankin-Selberg problem (see the author’s works [5], [6]). Namely, let as usual $\mu(\sigma)$ denote the Lindelöf function

$$\mu(\sigma) := \limsup_{t \to \infty} \frac{\log |\zeta(\sigma + it)|}{\log t} \quad (\sigma \in \mathbb{R}).$$

Then we have

$$\int_0^X \Delta^2(x) \, dx \ll \varepsilon X^{1+2\beta+\varepsilon}, \quad \beta = \frac{2}{5 - 2\mu(\frac{1}{2})}.$$ 

Here and later $\varepsilon$ denotes positive constants which may be arbitrarily small, but are not necessarily the same at each occurrence, while $\ll \varepsilon$ means that the $\ll$-constant depends on $\varepsilon$. Note that with the sharpest known result (see M.N. Huxley [2]) $\mu(\frac{1}{2}) \leq 32/205$ we obtain $\beta = 410/961 = 0.426638917\ldots$. The limit of (1.3) is the value $\beta = 2/5$ if the Lindelöf hypothesis (that $\mu(\frac{1}{2}) = 0$) is true.

We propose to contribute here to the subject of mean value results for $\Delta(x)$ by proving (unconditionally) the following results.

**THEOREM 1.** For any given $\varepsilon > 0$ we have

$$\int_0^X \Delta^4(x) \, dx \ll \varepsilon X^{3+\varepsilon}.$$ 

Note that (1.4) follows from (1.3) only if the Lindelöf hypothesis $\mu(\frac{1}{2}) = 0$ is true.

**Corollary.** For any given $\varepsilon > 0$ we have

$$\Delta(x) \ll \varepsilon x^{3/5+\varepsilon}.$$ 

The bound in (1.5) is only by an ‘$\varepsilon$’–factor weaker than the strongest known bound (1.2). To obtain (1.5) from (1.4) note that we have (see Lemma 1 below)

$$\Delta(X) = \frac{1}{2H} \int_{X-H}^{X+H} \Delta(x) \, dx + O(H) \quad (X^\varepsilon \leq H \leq \frac{1}{2}X).$$

It follows from (1.6) by Hölder’s inequality for integrals that

$$\Delta^4(X) \ll H^{-1} \int_{X-H}^{X+H} \Delta^4(x) \, dx + H^4 \ll_\varepsilon H^{-1}X^{3+\varepsilon} + H^4$$

by (1.4), and (1.2) follows with $H = X^{3/5}$. Note that if (1.4) holds with the exponent $\theta$ on the right-hand side of (1.4), then the above argument gives

$$\Delta(x) \ll x^{\theta/5},$$

and the best possible exponent $\theta$ must satisfy $\theta \geq 15/8$ since $\Delta(x) = \Omega_\pm(x^{3/8})$ (see the author’s work [4]).

THEOREM 2. If $\Delta_1(x) = \int_0^x \Delta(u) \, du$, then for any given $\varepsilon > 0$ we have

$$\int_0^X \Delta_1^4(x) \, dx \ll_\varepsilon X^{11/2+\varepsilon}.$$ 

Note that it was proved in [7] that

$$\int_0^X \Delta_1^2(x) \, dx = \frac{2}{13}(2\pi)^{-4} \left( \sum_{n=1}^\infty c_n^2 n^{-7/4} \right) X^{13/4} + O_\varepsilon(X^{3+\varepsilon}),$$

so that from (1.7) and the Cauchy-Schwarz inequality for integrals we obtain that

$$\int_0^X \Delta_1^4(x) \, dx \gg X^{11/2}.$$ 

This shows that, up to ‘$\varepsilon$’, the bound in (1.6) is best possible.
2. The necessary lemmas

In this section we shall state the lemmas necessary for the proof of our theorems.

**LEMMA 1.** For \( X^\varepsilon \leq H \leq \frac{1}{2} X \) we have

\[
\Delta(X) = \frac{1}{2H} \int_{X-H}^{X+H} \Delta(x) \, dx + O(H).
\]

**Proof.**

\[
\Delta(X) - \frac{1}{2H} \int_{X-H}^{X+H} \Delta(x) \, dx = \frac{1}{2H} \int_{X-H}^{X+H} (\Delta(X) - \Delta(x)) \, dx
\]

\[
= \frac{1}{2H} \int_{X-H}^{X+H} \left( \sum_{x<n \leq X} \frac{c_n + C(X-x)}{x} \right) \, dx
\]

\[
\ll \sum_{X-H \leq n \leq X+H} c_n + H \ll H,
\]

where in the last step a well-known result of P. Shiu on multiplicative functions [14] in short intervals was used (see also Lemma 4 of [7]).

The next two lemmas are the explicit, truncated formula of the Voronoï type for \( \Delta(x) \) and \( \Delta_1(x) \), respectively.

**LEMMA 2.** For \( 1 \ll K_0 \ll x \) a parameter we have

\[
\Delta(x) = \frac{x^{3/8}}{2\pi} \sum_{k \leq K_0} c_k k^{-5/8} \sin \left( 8\pi (kx)^{1/4} + \frac{3\pi}{4} \right) + O_{\varepsilon} \left( x^{3/4+\varepsilon} K_0^{-1/4} \right),
\]

Choosing \( K_0 = x^{3/5} \) and estimating trivially the sum in (2.2) we obtain again the bound \( \Delta(x) \ll_{\varepsilon} x^{3/5+\varepsilon} \). The formula (2.2) was proved (see [7, Lemma 2] with \( \rho = 0 \)) by Ivić, Matsumoto and Tanigawa, where the general (Riesz) sum \( \sum_{n \leq x} (x-n)^\rho c_n \) for fixed \( \rho \geq 0 \) is investigated, and a proof of (1.2) is given. A similar formula holds in general for Dirichlet series of degree four in the Selberg class (see e.g., the survey article [8] of Kaczorowski-Perelli on functions from the Selberg class \( S \), and Selberg’s original paper [13]). Thus the generalization of our results will hold for error terms associated to suitable Dirichlet series in \( S \). However, in some cases the special structure of the problem at hand allows for sharper results. For example, consider the estimation of \( \Delta_4(x) \), the error term in the asymptotic formula for the summatory function of the divisor function
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\[ d_4(n) = \sum_{abcd=n:a,b,c,d \in \mathbb{N}} 1. \] The generating function in this case is \( \zeta^4(s) \), and we have (see e.g., [3, Chapter 13])

\[
\int_0^X \Delta_4^2(x) \, dx \ll \varepsilon X^{7/4+\varepsilon},
\]

which is (up to \( \varepsilon \)) best possible in view of \( \Delta_4(x) = \Omega_+ (x^{3/8}) \) (see [4]). Since one has \( \Delta_4(x) \ll x^{1/2+\varepsilon} \), it trivially follows from (2.3) that

\[
\int_0^X \Delta_4(x) \, dx \ll \varepsilon X^{11/4+\varepsilon},
\]

and the exponent in (2.4) is better than the exponent in (1.4).

**Lemma 3.** For \( 1 \ll K_0 \ll x^2 \) a parameter we have

\[
\Delta_1(x) = \frac{x^{9/8}}{(2\pi)^2} \sum_{k \leq K_0} c_k k^{-7/8} \sin \left(8\pi(kx)^{1/4} + \frac{\pi}{4}\right) + O_\varepsilon \left(x^{1+\varepsilon} + x^{3/2+\varepsilon} K^{-1/2}_{0}\right).
\]

Lemma 3 is the case \( \rho = 1 \) of [7, Lemma 2].

The chief ingredient in the proof of Theorem 2 is the new result of O. Robert–P. Sargos [11], which is the following

**Lemma 4.** Let \( k \geq 2 \) be a fixed integer and \( \delta > 0 \) be given. Then the number of integers \( n_1, n_2, n_3, n_4 \) such that \( N < n_1, n_2, n_3, n_4 \leq 2N \) and

\[
|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}
\]

is, for any given \( \varepsilon > 0 \),

\[
\ll \varepsilon N^{\varepsilon} (N^4 \delta + N^2).
\]

**3. Proof of Theorem 1**

To prove (1.4), it is sufficient to prove the bound in question for the integral over the interval \([X, 2X]\). Let, for

\[
X^{1/2} \leq V \ll X^{3/5},
\]

\[
I(V, X) := \int_{X, V \leq |\Delta(x)| < 2V} \Delta_4(x) \, dx,
\]
where the upper bound in (3.1) holds because of (1.2). Clearly we have

\begin{equation}
\int_{X}^{2X} \Delta^4(x) \, dx \ll X^3 + \log X \max_{X^{1/2} \leq V \ll X^{3/5}} I(V, X).
\end{equation}

Hence the problem is reduced to the estimation of \( I(V, X) \) in (3.2). Thus we fix a value \( V = C2^{-j}X^{3/5} \) (\( j = 0, 1, 2, \ldots, C > 0 \)) and split the interval \([X, 2X]\) into subintervals of length \( H \left( X^{1/2} \leq H \leq X^{1-\varepsilon} \right) \), where the last of these intervals may be shorter. Suppose there are \( R = R(V) \) of these subintervals which contain a point \( x \) for which \( V \leq |\Delta(x)| < 2V \) holds. Further suppose that \( x_r \) is the point in the \( r \)-th of these intervals where the largest value of \(|\Delta(x)|\) is attained. To obtain the spacing condition

\begin{equation}
|x_r - x_s| \geq H \quad (r \neq s; \ r, s = 1, \ldots, R)
\end{equation}

we consider separately the points with even and odd indices and, with a slight abuse of notation, each of these two systems of points is again denoted by \( \{x_r\}_{r=1}^{R} \). Therefore we have \(|\Delta(x_r)| \geq V (r = 1, 2 \ldots)\), and observe that

\[ \frac{x^{3/8}}{2\pi} \sum_{k \leq \delta X^{1/3}} c_k k^{-5/8} \sin \left( 8\pi(kx)^{1/4} + \frac{3\pi}{4} \right) \ll \delta^{3/8} X^{1/2} \]

for \( x = x_r \in [X, 2X] \), \( \delta > 0 \). Thus for \( \delta \) small enough it follows from (2.1) and (2.2) (changing \( K_0 \) to \( X^3 H^{-4} \) and recalling that \( e(z) = e^{2\pi iz} \)), that for \( r = 1, \ldots, R \)

\begin{equation}
V \ll \frac{X^{3/8}}{H} \int_{x_r - H/3}^{x_r + H/3} \sum_{\delta X^{1/3} \leq k \leq X^3 H^{-4}} c_k k^{-5/8} e\left( 4\pi (kx)^{1/4} \right) \, dx + HX^\varepsilon,
\end{equation}

where all the intervals \([x_r - H/3, x_r + H/3]\) are disjoint in view of (3.4). We take in (3.5)

\[ H = VX^{-2\varepsilon}, \]

square, use the Cauchy-Schwarz inequality for integrals and sum the resulting expressions. We obtain

\[ R \ll_{\varepsilon} \max_{\delta X^{1/3} \leq K \leq X^{3+8\varepsilon} V^{-4}} X^{3/4+2\varepsilon} V^{-3} \int_{X/2}^{5X/2} \sum_{K < k \leq 2K} c_k k^{-5/8} e\left( 4\pi (kx)^{1/4} \right) \, dx. \]

To evaluate the integral on the right-hand side we square out the sum and use the first derivative test (i.e., Lemma 3.1 of [1] or Lemma 2.1 of [3]). It follows that,
since \( c_n \ll \varepsilon n^\varepsilon \),

\[
\int_{X/2}^{5X/2} \left| \sum_{K<k \leq 2K} c_k k^{-5/8} e(4\pi (kx)^{1/4}) \right|^2 \, dx \\
= \sum_{K<k, k_2 \leq 2K} c_{k_1} c_{k_2} (k_1 k_2)^{-5/8} \int_{X/2}^{5X/2} e\left(4\pi x^{1/4}(k_1^{1/4} - k_2^{1/4})\right) \, dx \\
\ll \varepsilon X^{1+\varepsilon} K^{-1/4} + K^{-5/4} \sum_{K<k_1 \neq k_2 \leq 2K} \frac{X^{3/4+\varepsilon}}{|k_1^{1/4} - k_2^{1/4}|} \\
\ll \varepsilon X^{1+\varepsilon} K^{-1/4} + K^{-1/2} X^{3/4+\varepsilon} \sum_{K<k_1 \leq 2K} \sum_{k_2 \leq 2K, k_2 \neq k_1} \frac{1}{|k_2 - k_1|} \\
\ll \varepsilon X^{1+\varepsilon} K^{-1/4} + X^{3/4+\varepsilon} K^{1/2} \ll \varepsilon X^{3/4+\varepsilon} K^{1/2}
\]

for \( K \gg X^{1/3} \), which is the case in (3.5). Consequently

\[
(3.6) \quad R \ll \varepsilon \max_{\delta X^{1/3} \leq K \leq X^{3+8\varepsilon} \varepsilon} X^{3/4+2\varepsilon} V^{-3} X^{3/4+\varepsilon} K^{1/2} \ll \varepsilon X^{3+\varepsilon} V^{-5}.
\]

From (3.6) we obtain that

\[
I(V, X) \ll \varepsilon X^{\varepsilon} V \cdot V^4 R \ll \varepsilon X^{3+\varepsilon},
\]

and (3.3) gives

\[
\int_X^{2X} \Delta^4(x) \, dx \ll \varepsilon X^{3+\varepsilon}.
\]

Theorem 1 follows if we replace \( X \) by \( X^{2-j} \) in the above bound and sum over \( j = 1, 2, \ldots \). An alternative proof of (1.4) may be obtained by going through the appropriate modification of the proof of Theorem 13.8 in [3], taking \( k = 4 \) and \( R = R_0, T = T_0 \) in (13.65). The choice \((k, \lambda) = (\frac{1}{2}, \frac{1}{2})\) will provide again the bound in (3.6) for the relevant range.

4. PROOF OF THEOREM 2

To prove Theorem 2 we use (2.5) of Lemma 3 with \( K_0 = X \) for \( X/2 \leq x \leq 5X/2 \). This gives

\[
(4.1) \quad \int_X^{2X} \Delta^4(x) \, dx \ll \varepsilon X^{9/2} \log X \max_{K \leq X} \int_{X/2}^{5X/2} \varphi(x) \left| \sum_{K<k \leq 2K} c_k k^{-7/8} e(4(xk)^{1/4}) \right|^4 \, dx \\
+ X^{5+\varepsilon},
\]
where \( \varphi(x) \geq 0 \) is a smooth function supported in \([X/2, 5X/2]\) such that \( \varphi(x) = 1 \) for \( X \leq x \leq 2X \) and \( \varphi^{(r)}(x) \ll X^{-r} \) \((r = 0, 1, \ldots)\). If we set

\[
\Delta := k^{1/4} + \ell^{1/4} - m^{1/4} - n^{1/4} \quad (k, \ell, m, n \in \mathbb{N}),
\]

then

\[
\int_{X/2}^{5X/2} \varphi(x) \cdots \, dx = \sum_{K<k, \ell, m, n \leq 2K} c_k c_\ell c_m c_n (k\ell mn)^{-7/8} \int_{X/2}^{5X/2} \varphi(x)e(4\pi \Delta x^{1/4}) \, dx.
\]

But integration by parts shows that

\[
\int_{X/2}^{5X/2} \varphi(x)e(4\pi \Delta x^{1/4}) \, dx = -\int_{X/2}^{5X/2} \frac{e(4\pi \Delta x^{1/4})}{2\pi i \Delta} \left(x^{3/4} \varphi'(x)\right) \, dx.
\]

Thus the exponential factor remained the same, but the order of the integrand has decreased by \( \Delta^{-1}X^{-1/4} \), provided that \( \Delta \neq 0 \). Since this will be repeated after every integration by parts, then it follows that the contribution of quadruples \((k, \ell, m, n)\) will be negligible if \( \Delta \geq X^{\varepsilon-1/4} \) for any given \( \varepsilon > 0 \). The contribution of the quadruples satisfying \( \Delta \leq X^{\varepsilon-1/4} \) is estimated by Lemma 4, where in (2.6) we take \( k = 4, \delta \asymp \Delta K^{-1/4} \). The ensuing integral is estimated trivially (using \( c_n \ll \varepsilon n^{\varepsilon} \)), and we obtain

\[
\int_X^{2X} \Delta^4(x) \, dx \ll \varepsilon X^{9/2+\varepsilon} \max_{K \ll X} K^{-7/2} \left(K^4(KX)^{-1/4} + K^2\right) + X^{5+\varepsilon}
\]

\[
\ll \varepsilon X^{21/4+\varepsilon} \max_{K \ll X} K^{1/4} + X^{11/2+\varepsilon} \ll \varepsilon X^{11/2+\varepsilon}.
\]

Theorem 2 follows if we replace \( X \) by \( X2^{-j} \) in the above bound and sum over \( j = 1, 2, \ldots \).
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