The cohomology ring of certain families of periodic virtually cyclic groups

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March 7, 2016

Abstract

Let $G$ be a virtually cyclic of the form $(\mathbb{Z}_a \rtimes \mathbb{Z}_b) \times \mathbb{Z}$ or $[\mathbb{Z}_a \rtimes (\mathbb{Z}_b \rtimes \mathbb{Q}_2)] \rtimes \mathbb{Z}$. We compute the integral cohomology ring of $G$, and then obtain the periodicity of the Farrell cohomology of these groups.

Keywords: periodic groups, cohomology ring, Farrell cohomology

2010 Mathematics Subject Classification: primary: 20J06; secondary: 20F50.

1 Introduction

Virtually cyclic groups can be considered the simplest family of groups which contains both the infinite groups and the finite ones. They play an important role in several subjects, like the Fibered Isomorphism conjecture by Farrell and Jones, see [6], and the study of space forms of infinite discrete groups. The space forms for finite groups have been widely studied and, in the similar problem for infinite discrete groups, the simplest case to consider is the one where the groups are virtually cyclic. The structure of the cohomology ring of these groups is an interesting question in its own right and very useful for the study of the space forms. In more detail, consider, for the infinite virtually cyclic groups, the problem of deciding which ones act on homotopy spheres, as well the classification of the homotopy type of the orbit spaces.

For results about space forms, mostly for finite groups and a few infinite groups, see for example [9], [10], [11], [12], [13], [14], [16], [17] and [8]. In particular, for the special question of which virtually cyclic groups act on homotopy spheres, the answer has been given in [8]. For the classification of the homotopy
type of the orbit spaces the cohomology ring of the group plays an important role.

The goal of this work is to describe the ring structure of two families of virtually cyclic groups, namely \((\mathbb{Z}_a \rtimes \mathbb{Z}_b) \rtimes \mathbb{Z}\) and \([\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times \mathbb{Q}_{2^i})] \rtimes \mathbb{Z}\).

The main results of this work are Theorems 3.3, 4.9 and 4.10. Besides describing the cohomology ring structure of these groups with integral coefficients, we also determine explicitly the cohomology class which determine the periodicity (in the sense of [3, Section X.6]).

This work contains 3 sections besides the introduction. In section 2 we present the basic tools about the Lyndon-Hochschild-Serre spectral sequence which is going to be used later. In section 3 we use section 2 for our first family of groups to obtain the main result which is Theorem 3.3. Section 4 we use section 2 for our second family of groups to obtain the main results which are Theorems 4.9 and 4.10.

This project is in part sponsored by by FAPESP — Fundação de Amparo a Pesquisa do Estado de São Paulo, Projetos Temáticos Topologia Algébrica, Geométrica e Diferencial — 2008/57607-6 and 2012/24454-8, and process 2013/07510-4.

2 Preliminaries

Given an exact sequence of groups

\[
1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1,
\]  

(1)

the Lyndon-Hochschild-Serre spectral sequence in cohomology is such that

\[
E_2^{p,q} = H^p(Q; H^q(H; M)) \Rightarrow H^{p+q}(G; M)
\]

for any \(G\)-module \(M\). In this paper our main interest is in groups of the form \(F \rtimes \theta \mathbb{Z}\), in which case the above spectral sequence gives us the following result for \(M = \mathbb{Z}\):

**Lemma 2.1.** For each positive integer \(n\), there is an exact sequence

\[
0 \longrightarrow H^{n-1}(F; \mathbb{Z})_\mathbb{Z} \longrightarrow H^n(F \rtimes_\theta \mathbb{Z}; \mathbb{Z}) \longrightarrow H^n(F; \mathbb{Z})_\mathbb{Z} \longrightarrow 0.
\]

**Proof.** Since \(H^n(\mathbb{Z}; M) = 0\) if \(n \geq 2\), the spectral sequence associated to the extension

\[
1 \longrightarrow F \longrightarrow F \rtimes_\theta \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1,
\]

is such that \(E_2^{p,q} = 0\) for \(p \geq 2\), hence \(E_2^{p,q} = E_\infty^{p,q}\) for all \(p, q\) and we get for each \(n \geq 1\) the exact sequence

\[
0 \longrightarrow E_2^{1,n-1} \longrightarrow H^n(F \rtimes_\theta \mathbb{Z}; \mathbb{Z}) \longrightarrow E_2^{0,n} \longrightarrow 0.
\]
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On the other hand,

\[ E_0^{0,n} = H^0(\mathbb{Z}; H^n(F; \mathbb{Z})) = H^n(F; \mathbb{Z}) \] and
\[ E_0^{1,n-1} = H^1(\mathbb{Z}; H^{n-1}(F; \mathbb{Z})) = H^{n-1}(F; \mathbb{Z}), \]

so the result follows.

**Corollary 2.2.** If \( F \) is a finite group with periodic cohomology, then

\[ H^n(F \rtimes_{\theta} \mathbb{Z}; \mathbb{Z}) \cong H^n(F; \mathbb{Z}) \] for even
and
\[ H^n(F \rtimes_{\theta} \mathbb{Z}; \mathbb{Z}) \cong H^{n-1}(F; \mathbb{Z}) \] for odd.

**Proof.** The result follows from the above Lemma and the fact that a finite group with periodic cohomology has trivial odd dimensional integral cohomology groups \([4, \text{Section VI.9, exercise 4, page 159}]\). □

The Lyndon-Hochschild-Serre spectral sequence can be constructed in a purely algebraic manner, but it can also be given a topological interpretation: the extension \([1]\) gives rise to a fibration sequence

\[ K(H, 1) \longrightarrow K(G, 1) \longrightarrow K(Q, 1) \]
and the resulting Leray-Serre spectral sequence is exactly the Lyndon-Hochschild-Serre sequence (see \([1]\)). Theorem 5.2 of \([19]\) can now be applied to show that, on the Lyndon-Hochschild-Serre spectral sequence, the cup product \( \cup \) and the product \( \cdot \) on \( E_2 \) are related by

\[ u \cdot v = (-1)^{pq} (u \cup v) \] for \( u \in E_2^{p,q} \) and \( v \in E_2^{p',q'} \). Depending on the properties of \( E_2 \), this equation may be enough to recover the cohomology ring \( H^*(G; \mathbb{Z}) \) from the Lyndon-Hochschild-Serre spectral sequence in cohomology. For instance, if \( F \) is a finite group with periodic cohomology, the Lyndon-Hochschild-Serre with \( \mathbb{Z} \) coefficients applied to the extension

\[ 1 \longrightarrow F \longrightarrow F \rtimes_{\theta} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1 \]
is such that \( E_2^{p,q} = 0 \) if \( p \geq 2 \) or if \( q \) is odd, so \( E_2^{p,q} = E_\infty^{p,q} \) and the product on \( E_2 \) gives us the cup product in \( H^*(F \rtimes_{\theta} \mathbb{Z}; \mathbb{Z}) \).

Now we make some remarks about the periodicity of the cohomology of \((F \rtimes_{\theta} \mathbb{Z})\): if \( F \) is a finite group with periodic cohomology, then \( F \rtimes_{\theta} \mathbb{Z} \) has periodic Farrell cohomology. This follows from \([3]\) Theorem X.6.7, item (iii)], since a finite subgroup of \( F \rtimes_{\theta} \mathbb{Z} \) is actually a subgroup of \( F \times \{0\} \cong F \). Moreover, \( \text{vcd}(F \rtimes_{\theta} \mathbb{Z}) = 1 \) implies \( H^i(F \rtimes_{\theta} \mathbb{Z}; M) \cong H^i(F \rtimes_{\theta} \mathbb{Z}; M) \) for all \( i \geq 2 \) and every \((F \rtimes_{\theta} \mathbb{Z})\)-module \( M \), where \( H^* \) denotes the Farrell cohomology. Finally, we know that the cup product in the Farrell cohomology is compatible with the regular cup product \([3]\) Section X.3], hence we have the following result:
Theorem 2.3. Let $F$ be a finite group with periodic cohomology. If $d$ is a positive integer and $u \in H^d(F \rtimes Z; Z)$ is a cohomology class such that

$$(u \sim \_)_i: H^i(F \rtimes Z; M) \to H^{i+d}(F \rtimes Z; M)$$

is an isomorphism for all $i \geq 2$ and every $(F \rtimes Z)$-module $M$, then

$$(\hat{u} \sim \_)_i: \hat{H}^i(F \rtimes Z; M) \to \hat{H}^{i+d}(F \rtimes Z; M)$$

is an isomorphism for all $i \in \mathbb{Z}$ and every $(F \rtimes Z)$-module $M$, where $\hat{u}$ denotes the image of $u$ under the canonical isomorphism $H^d(F \rtimes Z; M) \cong H^d(F \rtimes Z, Z; M)$.

3 The cohomology ring of $(\mathbb{Z}_a \rtimes \mathbb{Z}_b) \rtimes \mathbb{Z}$

Let $a$ and $b$ be relatively prime integers and let $G$ be the group $(\mathbb{Z}_a \rtimes \mathbb{Z}_b) \rtimes \mathbb{Z}$, where $\alpha: \mathbb{Z}_b \to \text{Aut}(\mathbb{Z}_a)$ is given by $\alpha(1_b)(1_a) = r \cdot 1_a$ with $\text{gcd}(a, (r-1)b) = 1$ and $r^b \equiv 1 \pmod{a}$, and $\theta: \mathbb{Z} \to \text{Aut}(\mathbb{Z}_a \rtimes \mathbb{Z}_b)$ is given by $\theta(1) = c_a \cdot 1_a$ and $\theta(1) = c \cdot 1_a + c_b \cdot 1_b$. See [11] for more details on the integers $c, c_a$ and $c_b$.

The Lyndon-Hochschild-Serre spectral sequence in cohomology (with $\mathbb{Z}$ coefficients) associated to the exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_a \longrightarrow (\mathbb{Z}_a \rtimes \mathbb{Z}_b) \longrightarrow \mathbb{Z}_b \longrightarrow 1$$

is such that $E^{p, q}_2 = H^p(\mathbb{Z}_b; H^q(\mathbb{Z}_a; Z))$. The group $\mathbb{Z}_b$ acts trivially on $H^0(\mathbb{Z}_a; Z)$ and acts on $H^2(\mathbb{Z}_a; Z)$ by multiplication by $r$. Since the cohomology ring $H^*(\mathbb{Z}_a; Z)$ is given by

$$\frac{\mathbb{Z}[\alpha_2]}{(a \alpha_2 = 0)}$$

with $\dim(\alpha_2) = 2$, it follows that $\mathbb{Z}_b$ acts on $H^2(\mathbb{Z}_a; Z)$ by multiplication by $r^j$. Hence $E^{p, q}_2$ is possibly non-null only if $p$ and $q$ are even and $pq = 0$ and it follows that

$$E^{p, q}_\infty = E^{p, q}_2 = \begin{cases} 
\mathbb{Z}, & \text{if } p = q = 0, \\
\mathbb{Z}_{\delta_i}, & \text{if } p = 0 \text{ and } q = 2i > 0, \\
\mathbb{Z}_b, & \text{if } q = 0 \text{ and } p > 0 \text{ is even,} \\
0, & \text{otherwise},
\end{cases}$$

where $\delta_i = \text{gcd}(r^i - 1, a)$. Since $a$ and $b$ are relatively prime, so are $\delta_i$ and $b$, and we have proved the following:

Proposition 3.1. The integral cohomology groups of $(\mathbb{Z}_a \rtimes \mathbb{Z}_b)$, with $\alpha(1_b) = r \cdot 1_a$, are given by

$$H^n(\mathbb{Z}_a \rtimes \mathbb{Z}_b; Z) = \begin{cases} 
\mathbb{Z}, & \text{if } n = 0, \\
\mathbb{Z}_{\delta_i} \oplus \mathbb{Z}_b \cong \mathbb{Z}_{\delta_i b}, & \text{if } n = 2i > 0, \\
0, & \text{if } n \text{ is odd.}
\end{cases}$$
We will now compute the cohomology groups $H^*(G; \mathbb{Z})$, and in order to do that we need some simple facts and some notation.

If a cyclic group $H$ acts on the finite cyclic group $\mathbb{Z}_k$, then it is straightforward to check that $(\mathbb{Z}_k)^H \cong (\mathbb{Z}_k)_H$, hence $H^n(G; \mathbb{Z}) \cong H^{n+1}(G; \mathbb{Z})$ if $n \geq 2$ is even.

The action of $\mathbb{Z}$ on $H^0(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z})$ is trivial, so $H^0(G; \mathbb{Z}) \cong H^1(G; \mathbb{Z}) \cong \mathbb{Z}$.

Also, noticing that $\text{Aut}(\mathbb{Z})$ is even, we have $H^0(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z})$ is trivial, so $H^1(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z}) \cong \mathbb{Z}$.

In order to determine the multiplicative structure in $H^*(G; \mathbb{Z})$, we need representatives for the generating classes of each cohomology group $H^n(G; \mathbb{Z})$.

For $n = 0$, let us denote the generating class by $1$. For $n = 1$, a representative for the generating class of

$$H^1(\mathbb{Z}; H^0(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z})) \cong \mathbb{Z}$$

is the map $\eta: \mathbb{Z}[\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b] \to \mathbb{Z}$ of $\mathbb{Z}[\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b]$-modules given by $\eta(1) = 1$. More precisely, if $\tilde{\eta}: \mathbb{Z}[\mathbb{Z}] \to H^0(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z})$ is the $\mathbb{Z}[\mathbb{Z}]$-homomorphism defined by $\tilde{\eta}(1) = [\eta]$, then $[\tilde{\eta}]$ generates $H^1(\mathbb{Z}; H^0(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z}))$ and $[\eta]$ is the identity in $H^*(\mathbb{Z}_a \rtimes^\alpha \mathbb{Z}_b; \mathbb{Z})$.

For $n \geq 2$, we need representatives for the generating classes of the following groups (where $i > 0$):
1. $H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) = H^0(Z_b; H^{2i}(Z_a; Z))$
2. $H^0(Z; H^{2i}(Z_b; H^0(Z_a; Z))) = H^{2i}(Z_b; H^0(Z_a; Z))$
3. $H^1(Z; H^0(Z_b; H^{2i}(Z_a; Z))) = H^0(Z_b; H^{2i}(Z_a; Z))_Z$
4. $H^1(Z; H^{2i}(Z_b; H^0(Z_a; Z))) = H^{2i}(Z_b; H^0(Z_a; Z))_Z$

We start with $H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) = H^0(Z_b; H^{2i}(Z_a; Z))$. We know that $Z_b$ acts on $H^{2i}(Z_a; Z)$ by multiplication by $r^i$, so $H^0(Z_b; H^{2i}(Z_a; Z))_Z \cong (Z_{\delta_i})_Z$, where

$$Z_{\delta_i} = \{ x \in Z_a : r^i x \equiv x \pmod{a} \} = \{ x \in Z_a : x \equiv 0 \pmod{a/\delta_i} \} = \{ x \in Z_a : x = (ka) / \delta_i, k \in Z \}.$$

Since $Z$ acts on $Z_{\delta_i}$ by multiplication by $c_i^a$, we obtain

$$(Z_{\delta_i})_Z = \{ x \in Z_{\delta_i} : (c_i^a - 1)x \equiv 0 \pmod{a} \},$$

and the fact that $x = (ka) / \delta_i$ yields the equation $\frac{(c_i^a - 1)ka}{\delta_i} \equiv 0 \pmod{a}$, which has the solution

$$k \equiv 0 \pmod{\gcd(a, c_i^a - 1)} \iff k \equiv 0 \pmod{\frac{a}{\delta_i} \gcd(\delta_i, c_i^a - 1)}.$$

Hence a representative for the generating class of

$$H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) \cong (Z_{\delta_i})_Z$$

is the map $u^i_a : Z[Z_a] \rightarrow Z$ given by

$$u^i_a(1) = \frac{a}{\delta_i} \cdot \frac{\delta_i}{\gcd(\delta_i, (c_i^a - 1))} = \frac{a}{\gcd(\delta_i, c_i^a - 1)}.$$
and we define $\varphi^i_0: \mathbb{Z}[\mathbb{Z}_b] \to H^0(\mathbb{Z}_a; \mathbb{Z})$ and $\varphi^i_1: \mathbb{Z}[\mathbb{Z}] \to H^2(\mathbb{Z}_b; H^0(\mathbb{Z}_a; \mathbb{Z}))$ in a way similar to $\varphi^i_0$ and $\varphi^i_1$.

The same reasoning applies to determine a representative for the generating class of

$$H^1(\mathbb{Z}; H^0(\mathbb{Z}_b; H^2(\mathbb{Z}_a; \mathbb{Z}))) = H^0(\mathbb{Z}_b; H^2(\mathbb{Z}_a; \mathbb{Z}))\mathbb{Z} \cong (\mathbb{Z}_{\delta_i})\mathbb{Z},$$

this representative being the map $v^i_1: \mathbb{Z}_a \to \mathbb{Z}$ defined by

$$v^i_1(1) = \frac{a_i}{\delta_i}.$$

Finally, a representative for the generating class of

$$H^1(\mathbb{Z}; H^2(\mathbb{Z}_b; H^0(\mathbb{Z}_a; \mathbb{Z}))) = H^2(\mathbb{Z}_b; H^0(\mathbb{Z}_a; \mathbb{Z}))\mathbb{Z} = (\mathbb{Z}_b)\mathbb{Z}$$

is the map $v^i_2: \mathbb{Z}[\mathbb{Z}_a] \to \mathbb{Z}$ given by

$$v^i_2(1) = 1.$$

The maps $\psi^i_0$, $\psi^i_1$, $\psi^i_2$ and $\psi^i_3$ are defined similarly to $\varphi^i_0$, $\varphi^i_1$, $\varphi^i_2$ and $\varphi^i_3$.

The last necessary ingredients we need to calculate the cup products in $H^*(G, \mathbb{Z})$ are the diagonal approximations for the known free resolutions of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}]$ and over $\mathbb{Z}[\mathbb{Z}_m]$ for $m > 1$.

For $\mathbb{Z} = \langle s \rangle$, a free resolution $F$ of $\mathbb{Z}$ over $\mathbb{Z}G$ is given by

$$0 \longrightarrow F_1 \xrightarrow{s-1} F_0 \xrightarrow{e} \mathbb{Z} \longrightarrow 0,$$

where $F_1 = F_0 = \mathbb{Z}[\mathbb{Z}]$, and a direct verification shows that $\Delta_n: F_n \to (F \otimes F)_n$ defined by

$$\begin{align*}
\Delta_0: F_0 & \to F_0 \otimes F_0 \\
\Delta_0(1) & = 1 \otimes 1,
\end{align*}$$

$$\begin{align*}
\Delta_1: F_1 & \to (F_1 \otimes F_0) \oplus (F_0 \otimes F_1) \\
\Delta_1(1) & = 1 \otimes s + 1 \otimes 1 \\
& \quad \text{if } F_1 \otimes F_0 + F_0 \otimes F_1
\end{align*}$$

(3)

is a diagonal approximation for the resolution $F$.

For the cyclic group $\mathbb{Z}_m = \langle t \mid t^m = 1 \rangle$, a free resolution $P$ of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}_m]$ and a diagonal approximation for $P$ are found in [4]. The resolution $P$ is given by

$$\cdots \xrightarrow{N} \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}_m] \xrightarrow{e} \mathbb{Z} \longrightarrow 0,$$

where $P_n = \mathbb{Z}[\mathbb{Z}_m]$ for all $n \geq 0$ and $N = 1 + t + \cdots + t^{m-1}$. A diagonal approximation $\Delta$ for $P$ has components $\Delta_{pq}: P_{p+q} \to P_p \otimes P_q$ given by

$$\begin{align*}
\Delta_{pq}(1) & = \begin{cases} 
1 \otimes 1, & \text{if } p \text{ is even,} \\
1 \otimes t, & \text{if } p \text{ is odd, } q \text{ is even,} \\
\sum_{0 \leq i < j \leq m-1} t^i \otimes t^j, & \text{if } p \text{ and } q \text{ are odd.}
\end{cases}
\end{align*}$$

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Now let \( i, j > 0 \). We begin by considering the product

\[
H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) \otimes H^0(Z; H^0(Z_b; H^{2j}(Z_a; Z))) \to
\]

\[
\to H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) \otimes H^0(Z_b; H^{2j}(Z_a; Z)) \to
\]

\[
\to H^0(Z; H^0(Z_b; H^{2i+j}(Z_a; Z))),
\]

where in the above composition we evaluate the cup products in \( H^*(Z, \_\_\_\_) \), \( H^*(Z_b, \_\_\_) \) and \( H^*(Z_a, \_\_\_) \), and make the identification \( Z \otimes Z \cong Z \). Using the diagonal approximations \( 3 \) and 4, we obtain

\[
(\tilde{\varphi}_a^i \sim \tilde{\varphi}_a^j)(1) = \tilde{\varphi}_a^i(1) \otimes \tilde{\varphi}_a^j(1) = [\varphi_a^i] \otimes [\varphi_a^j],
\]

\[
(\varphi_a^i \sim \varphi_a^j)(1) = \varphi_a^i(1) \otimes \varphi_a^j(1) = [u_a^i] \otimes [u_a^j],
\]

\[
(u_a^i \sim u_a^j)(1) = u_a^i(1) \otimes u_a^j(1) \mapsto \frac{a^2}{\gcd(\delta_i, c_a^i - 1) \gcd(\delta_j, c_a^j - 1)},
\]

from where it follows that

\[
[\tilde{\varphi}_a^i] - [\tilde{\varphi}_a^j] = \frac{a \gcd(\delta_i+j, c_a^{i+j} - 1)}{\gcd(\delta_i, c_a^i - 1) \gcd(\delta_j, c_a^j - 1)}[\tilde{\varphi}_a^{i+j}].
\]

Similarly, the product

\[
H^0(Z; H^{2i}(Z_b; H^0(Z_a; Z))) \otimes H^0(Z; H^{2j}(Z_b; H^0(Z_a; Z))) \to
\]

\[
\to H^0(Z; H^{2i}(Z_b; H^0(Z_a; Z))) \otimes H^{2j}(Z_b; H^0(Z_a; Z)) \to
\]

\[
\to H^0(Z; H^{2i+j}(Z_b; H^0(Z_a; Z))) \to
\]

is such that

\[
[\tilde{\varphi}_b^i] - [\tilde{\varphi}_b^j] = \frac{b \gcd(b, c_b^{i+j} - 1)}{\gcd(b, c_b^i - 1) \gcd(b, c_b^j - 1)}[\tilde{\varphi}_b^{i+j}].
\]

As to the product

\[
H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) \otimes H^0(Z; H^{2j}(Z_b; H^0(Z_a; Z))) \to
\]

\[
\to H^0(Z; H^0(Z_b; H^{2i}(Z_a; Z))) \otimes H^2(Z_b; H^0(Z_a; Z)) \to
\]

\[
\to H^0(Z; H^{2j}(Z_b; H^{2i}(Z_a; Z))) \to
\]

\[
\to H^0(Z; H^2(Z_b; H^{2j}(Z_a; Z))) = 0,
\]

we can only have

\[
[\tilde{\varphi}_a^i] - [\tilde{\varphi}_b^j] = 0.
\]
Similarly, the compositions
\[ H^0(Z; H^0(Z_b; H^2(Z_a; Z))) \otimes H^1(Z; H^2(Z_b; H^0(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^0(Z_b; H^2(Z_a; Z))) \otimes H^2(Z_b; H^0(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^0(Z_a; Z))) \otimes H^0(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^2(Z_a; Z))) = 0 \]
and
\[ H^0(Z; H^2(Z_b; H^0(Z_a; Z))) \otimes H^1(Z; H^0(Z_b; H^2(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^0(Z_b; H^2(Z_a; Z))) \otimes H^0(Z_b; H^2(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^0(Z_a; Z))) \otimes H^2(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^2(Z_a; Z))) = 0 \]
show us that
\[ [\tilde{\varphi}_a^i] \sim [\tilde{\psi}_a^j] = 0, \quad [\tilde{\varphi}_b^i] \sim [\tilde{\psi}_b^j] = 0. \]

The product
\[ H^0(Z; H^0(Z_b; H^2(Z_a; Z))) \otimes H^1(Z; H^0(Z_b; H^2(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^0(Z_b; H^2(Z_a; Z))) \otimes H^0(Z_b; H^2(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^0(Z_b; H^2(Z_a; Z))) \otimes H^2(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^0(Z_b; H^2(Z_a; Z))) = 0 \]
is such that
\[ (\tilde{\varphi}_a^i \sim \tilde{\psi}_a^j)(1) = \tilde{\varphi}_a^i(1) \otimes \tilde{\psi}_a^j(1) = [\varphi_a^i] \otimes [\psi_a^j], \]
\[ (\varphi_a^i \sim \psi_a^j)(1) = \varphi_a^i(1) \otimes \psi_a^j(1) = [u_a^i] \otimes [v_a^j], \]
\[ (u_a^i \sim v_a^j)(1) = u_a^i(1) \otimes v_a^j(1) \rightarrow \frac{a^2}{\delta_i \gcd(c_a^i, c_b^i - 1)}. \]

hence
\[ [\tilde{\varphi}_a^i] \sim [\tilde{\psi}_a^j] = \frac{a \delta_i c_b^i}{\delta_j \gcd(c_a^i, c_b^i - 1)}[\tilde{\psi}_b^{i+j}]. \]

Similarly, the product
\[ H^0(Z; H^2(Z_b; H^0(Z_a; Z))) \otimes H^1(Z; H^2(Z_b; H^0(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^0(Z_a; Z))) \otimes H^2(Z_b; H^0(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^0(Z_a; Z))) \otimes H^0(Z_a; Z))) \rightarrow \]
\[ \rightarrow H^1(Z; H^2(Z_b; H^0(Z_a; Z))) = 0 \]
is such that
\[ [\tilde{\varphi}_b^i] \sim [\tilde{\psi}_b^j] = \frac{b}{\gcd(b, c_b^i - 1)}[\tilde{\psi}_b^{i+j}]. \]
Finally, observing that $[\eta]$ is the identity in $H^*(\mathbb{Z}_a \rtimes_{\alpha} \mathbb{Z}_b; \mathbb{Z})$, we also have

$$[\bar{\varphi}_a^i] \sim [\eta] = \frac{\delta_i}{\gcd(\delta_i, c_a^i - 1)} [\bar{\psi}_a^i]$$

and

$$[\bar{\varphi}_b^i] \sim [\eta] = \frac{b}{\gcd(b, c_b^i - 1)} [\bar{\psi}_b^i].$$

Therefore, we have the following result:

**Theorem 3.3.** Let $G = (\mathbb{Z}_a \rtimes_{\alpha} \mathbb{Z}_b) \rtimes_{\theta} \mathbb{Z}$, where $\gcd(a, b) = 1$, and with $\alpha$ and $\theta$ defined by

$$\alpha: \mathbb{Z}_b \to \text{Aut}(\mathbb{Z}_a),$$

$$\alpha(1_b)(1_a) = r \cdot 1_a$$

and

$$\theta: \mathbb{Z} \to \text{Aut}(\mathbb{Z}_a \rtimes_{\alpha} \mathbb{Z}_b),$$

$$\theta(1)(1_b) = c_a \cdot 1_a,$$

$$\theta(1)(1_b) = c \cdot 1_a + c_b \cdot 1_b.$$

The cohomology groups $H^*(G; \mathbb{Z})$ are given by

$$H^n(G; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0 \text{ or } n = 1, \\ \mathbb{Z}_{A_j} \oplus \mathbb{Z}_{B_j}, & \text{if } n = 2j \text{ or } n = 2j + 1 \text{ (and } j > 0), \end{cases}$$

where $A_j = \gcd(c_a^j - 1, \delta_j)$, $\delta_j = \gcd(r^j - 1, a)$ and $B_j = \gcd(c_b^j - 1, b)$. Moreover, there is a generator $[\bar{\eta}]$ of $H^1(G; \mathbb{Z})$, there are generators $[\bar{\varphi}_a^i]$ and $[\bar{\varphi}_b^i]$ of $H^2(G; \mathbb{Z})$ for $i > 0$ such that $\langle [\bar{\varphi}_a^i] \rangle \cong \mathbb{Z}_{A_i}$ and $\langle [\bar{\varphi}_b^i] \rangle \cong \mathbb{Z}_{B_i}$, and there are generators $[\bar{\psi}_a^i]$ and $[\bar{\psi}_b^i]$ of $H^{2i+1}(G; \mathbb{Z})$ for $i > 0$ such that $\langle [\bar{\psi}_a^i] \rangle \cong \mathbb{Z}_{A_i}$ and
Finally, if we let $d, d_a$, and $d_b$ be the orders of $r, c_a$ and $c_b$ in $\mathcal{U}(\mathbb{Z}_a)$, respectively, and $p = \text{lcm}(d, d_a, d_b)$, then

$$(\tilde{\varphi}_a^p + [\tilde{\varphi}_b^p]) \sim H^n(G; \mathbb{Z}) \to H^{n+2p}(G; \mathbb{Z})$$

is an isomorphism for all $n \geq 2$.

Proof. We only need to prove the statement concerning the periodicity of $H^*(G; \mathbb{Z})$. Noticing that $r^p \equiv 1 \pmod{a}$, $c_a^p \equiv 1 \pmod{a}$ and $c_b^p \equiv 1 \pmod{b}$, we have

$$\delta_p = a,$$
$$\delta_{p+j} = \delta_j,$$
$$\gcd(\delta_{p+j}, c_a^{p+j} - 1) = \gcd(\delta_j, c_a^j - 1),$$
$$\gcd(b, c_b^{p+j} - 1) = \gcd(b, c_b^j - 1).$$

Therefore, using the formulas we already have for the cup products show us
that, for all $n \geq 2$,
\[
(\tilde{\phi}_p^a + \tilde{\phi}_p^b) \sim [\tilde{\phi}_a^j],
\]
\[
(\tilde{\phi}_p^a + \tilde{\phi}_p^b) \sim [\tilde{\phi}_b^j],
\]
\[
(\tilde{\phi}_p^a + \tilde{\phi}_p^b) \sim [\tilde{\psi}_a^j],
\]
\[
(\tilde{\phi}_p^a + \tilde{\phi}_p^b) \sim [\tilde{\psi}_b^j],
\]
and that proves that
\[
(\tilde{\phi}_p^a + \tilde{\phi}_p^b) \sim \cdots \colon H^n(G; \mathbb{Z}) \cong H^{n+2p}(G; \mathbb{Z})
\]
is an isomorphism for $n \geq 2$.

**Corollary 3.4.** If $[\tilde{\phi}_p^a + \tilde{\phi}_p^b] \in H^{2p}(G; \mathbb{Z})$ is the cohomology class described in the previous theorem, then
\[
(\tilde{\phi}_p^a + \tilde{\phi}_p^b) \sim \cdots : \hat{H}^n(G; \mathbb{Z}) \cong \hat{H}^{n+2p}(G; \mathbb{Z})
\]
is an isomorphism for all $n \in \mathbb{Z}$.

## 4 The cohomology ring of $[\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times Q_{2^i})] \rtimes \mathbb{Z}$

Let $i \geq 3$ be an integer and $Q_{2^i} = (x, y \mid x^{2^i-2} = y^2, xyx = y)$. The group $G = [\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times Q_{2^i})] \rtimes \mathbb{Z}$ for $\gcd(a, b) = \gcd(ab, 2) = 1$ can be obtained as a sequence of extensions, and we will follow through this sequence in order to calculate its integral cohomology ring.

First, we note that the integral cohomology ring of the groups $Q_{2^i}$ (actually, $Q_{4n}$) has been determined in [29].

**Theorem 4.1** (Tomoda and Zvengrowski, [29]). The cohomology ring $H^*(Q_{2^i}; \mathbb{Z})$ has the following presentation for $i \geq 3$:

\[
H^*(Q_{2^i}; \mathbb{Z}) = \begin{cases}
\mathbb{Z}[\gamma_2, \gamma'_2, \delta_4], & \text{if } i > 3; \\
\mathbb{Z}[\gamma_2, \gamma'_2, \delta_4] / \langle 2\gamma_2 = 2\gamma'_2 = 2\delta_4 = 0, \gamma'_2 = 0, \gamma_2^{2i-1} \delta_4 = 0, \gamma_2^2 = 8\delta_4 = 0, \gamma_2^2 = 0, 2\gamma_2 = 4\delta_4 = 0 \rangle, & \text{if } i = 3;
\end{cases}
\]

where $\dim(\gamma_2) = \dim(\gamma'_2) = 2$, $\dim(\delta_4) = 4$.

Let’s consider the extension
\[
1 \longrightarrow \mathbb{Z}_b \longrightarrow \mathbb{Z}_b \times Q_{2^i} \longrightarrow Q_{2^i} \longrightarrow 1
\]
and the associated spectral sequence \( E \) with 
\[
E_2^{p,q} = H^p(Q_2; H^q(Z_b; \mathbb{Z})) \Rightarrow H^{p+q}(Z_b \times Q_{2'}; \mathbb{Z}).
\]

The group \( Q_2 \) acts trivially on \( Z_b \), hence \( Q_2 \) also acts trivially on \( H^*(Z_b; \mathbb{Z}) \). It follows now from [5, section XII.7] and the fact that \( b \) is odd that 
\[
E_2^{p,q} = \begin{cases} 
\mathbb{Z}, & \text{if } p = q = 0; \\
Z_b, & \text{if } p = 0 \text{ and } q \geq 2 \text{ is even}; \\
Z_b^2, & \text{if } q = 0 \text{ and } p \equiv 2 \pmod{4}; \\
Z_{2'}, & \text{if } q = 0 \text{ and } p \equiv 0 \pmod{4} \text{ for } p \geq 4; \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore \( E_2^{p,q} = E_\infty^{p,q} \), we have no extension problems to consider since \( b \) is odd, and the induced product on \( E_2 \) gives us the cohomology ring of \( Z_b \times Q_{2'} \):

writing \( H^*(Z_b; \mathbb{Z}) = \frac{\mathbb{Z}[\beta_2]}{(b \beta_2 = 0)} \) with \( \dim(\beta_2) = 2 \), we have \( \beta_2 \gamma_2 = \beta_2 \gamma_2' = \beta_2 \delta_4 = 0 \) in \( H^*(Z_b \times Q_{2'}; \mathbb{Z}) \), plus the relations among \( \gamma_2, \gamma_2' \), and \( \delta_4 \) that already hold in \( H^*(Q_{2'}; \mathbb{Z}) \). Thus we have proved the following:

**Theorem 4.2.** The cohomology ring \( H^*(Z_b \times Q_{2'}; \mathbb{Z}) \) has the following presentation for \( i \geq 3 \):

\[
H^*(Z_b \times Q_{2'}; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}[\beta_2, \gamma_2, \gamma_2', \delta_4], & \text{if } i > 3; \\
\frac{\mathbb{Z}[\beta_2, \gamma_2, \gamma_2', \delta_4]}{\begin{cases} 
b \beta_2 = 2 \gamma_2 = 2 \gamma_2' = 2 \gamma_4 = 0 \\
\beta_2 \gamma_2 = \beta_2 \gamma_2' = \beta_2 \delta_4 = 0 \\
\gamma_2^2 = 0, \gamma_2'^2 = \gamma_2 \gamma_2' = 2^{i-1} \delta_4 \\
\mathbb{Z}[\gamma_2, \gamma_2', \delta_4] \end{cases}}, & \text{if } i = 3;
\end{cases}
\]

where \( \dim(\beta_2) = \dim(\gamma_2) = \dim(\gamma_2') = 2, \dim(\delta_4) = 4 \).

The next step is to consider the extension 
\[
1 \longrightarrow Z_a \longrightarrow Z_a \times \beta (Z_b \times Q_{2'}) \longrightarrow Z_b \times Q_{2'} \longrightarrow 1,
\]

where \( \beta : Z_b \times Q_{2'} \rightarrow \text{Aut}(Z_a) \) is given by 
\[
\beta(1_b)(1_a) = r \cdot 1_a, \\
\beta(1_b)(x) = r_x \cdot 1_a, \\
\beta(1_b)(y) = r_y \cdot 1_a,
\]

with \( r, r_x \), and \( r_y \) satisfying \( r^b \equiv r_x^b \equiv r_y^b \pmod{a} \). The Lyndon-Hochschild-Serre sequence associated to the above extension is such that 
\[
E_2^{p,q} = H^p(Z_b \times Q_{2'}; H^q(Z_a; \mathbb{Z})) \Rightarrow H^{p+q}(Z_a \times \beta (Z_b \times Q_{2'}); \mathbb{Z}).
\]
Since the ring $H^*(\mathbb{Z}_a; \mathbb{Z})$ is generated by an element of dimension 2, we have

$$E_2^{p,q} = \begin{cases} H^p(\mathbb{Z}_b \times Q_{2^j}; \mathbb{Z}), & \text{if } q = 0; \\ H^p(\mathbb{Z}_b \times Q_{2^j}; \hat{\mathbb{Z}}_a), & \text{if } q \geq 2 \text{ is even}; \\ 0, & \text{if } q \text{ is odd}; \end{cases}$$

(6)

where $\hat{\mathbb{Z}}_a = H^2(\mathbb{Z}_a; \mathbb{Z})$ represents the $(\mathbb{Z}_b \times Q_{2^j})$-module $\mathbb{Z}_a$ with the appropriate action of $(\mathbb{Z}_b \times Q_{2^j})$. For $j > 0$ and the given $\beta: \mathbb{Z}_b \times Q_{2^j} \to \text{Aut}(\mathbb{Z}_a)$, the action of $1_b \in \mathbb{Z}_b$ on $H^2(\mathbb{Z}_a; \mathbb{Z}) = \hat{\mathbb{Z}}_a$ is the multiplication by $r^j$, the action of $x \in Q_{2^j}$ is the multiplication by $r^j_y$ and the action of $y \in Q_{2^j}$ is the multiplication by $r^j_y$.

If we now consider the spectral sequence $\hat{E}$ associated to the extension

$$1 \longrightarrow \mathbb{Z}_b \longrightarrow \mathbb{Z}_b \times Q_{2^j} \longrightarrow Q_{2^j} \longrightarrow 1$$

with coefficients in $\hat{\mathbb{Z}}_a$, we have

$$\hat{E}_2^{p,q} = H^p(Q_{2^j}; H^q(\mathbb{Z}_b; \hat{\mathbb{Z}}_a)) \Rightarrow H^{p+q}(\mathbb{Z}_b \times Q_{2^j}; \hat{\mathbb{Z}}_a),$$

and from this we actually get

$$\hat{E}_2^{p,q} = \begin{cases} H^p(Q_{2^j}; \mathbb{Z}_{\delta_j}), & \text{if } q = 0; \\ 0, & \text{if } q > 0; \end{cases}$$

with $\delta_j = \gcd(a, r^j - 1)$. The spectral sequence therefore collapses on $\hat{E}_2$ and, since $\gcd(\delta_j, 2^j) = 1$, we have

$$H^p(\mathbb{Z}_b \times Q_{2^j}; \hat{\mathbb{Z}}_a) = H^p(Q_{2^j}; \mathbb{Z}_{\delta_j}) = \begin{cases} \mathbb{Z}_{\varepsilon_j}, & \text{if } p = 0; \\ 0, & \text{if } p > 0; \end{cases}$$

(7)

where $\varepsilon_j = \gcd(a, r^j - 1, r_y^j - 1)$. Note that $\varepsilon_j = \delta_j$ when $j$ is even, since $r^j_x = r^j_y \equiv 1 \pmod{a}$.

Plugging these results back into (6), we get $E_\infty^{p,q} = E_2^{p,q}$ and, since $\gcd(a, b) = \gcd(ab, 2) = 1$, the following result is then proved:

**Theorem 4.3.** The integral cohomology groups of $\mathbb{Z}_a \times \beta(\mathbb{Z}_b \times Q_{2^j})$, for $\beta: \mathbb{Z}_b \times Q_{2^j} \to \text{Aut}(\mathbb{Z}_a)$ defined by

$$\beta(1_b)(1_a) = r \cdot 1_a, \\
\beta(1_b)(x) = r_x \cdot 1_a, \\
\beta(1_b)(y) = r_y \cdot 1_a,$$

are given by

$$H^n(\mathbb{Z}_a \times \beta(\mathbb{Z}_b \times Q_{2^j}); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \text{ is odd}; \\ \mathbb{Z}_{\varepsilon_j} \oplus \mathbb{Z}_b \oplus \mathbb{Z}_2^j, & \text{if } n = 4j + 2; \\ \mathbb{Z}_{\delta_j} \oplus \mathbb{Z}_b \oplus \mathbb{Z}_{2^j}, & \text{if } n = 4j \text{ (and } j > 0). \end{cases}$$
Our work above actually allows us to determine the integral cohomology ring of $\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r})$, and not just its cohomology groups.

**Theorem 4.4.** The cohomology ring $H^*(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z})$ is generated by the elements

\[
\left( \frac{a}{\epsilon_j} \right) \alpha_j^2, \beta_2, \gamma_2, \gamma_2', \text{ and } \delta_4,
\]

where $j > 0$, $\epsilon_j = \gcd(a, r_j^0 - 1, r_j^1 - 1, r_j^2 - 1)$, $\dim \left( \frac{a}{\epsilon_j} \alpha_j^2 \right) = 2j$, $\dim(\beta_2) = \dim(\gamma_2) = \dim(\gamma_2') = 2$ and $\dim(\delta_4) = 4$. We have $\epsilon_j \left( \frac{a}{\epsilon_j} \alpha_j^2 \right) = 0$, the relations among $\beta_2, \gamma_2, \gamma_2'$ and $\delta_4$ are exactly the ones described in Theorem 4.2, and

\[
\left( \frac{a}{\epsilon_i} \alpha_i^2 \right) \sim \left( \frac{a}{\epsilon_j} \alpha_j^2 \right) \sim \left( \frac{a}{\epsilon_i \epsilon_j} \alpha_i \alpha_j \right) \sim \left( \frac{a}{\epsilon_{i+j}} \alpha_{i+j} \right),
\]

\[
\left( \frac{a}{\epsilon_i} \alpha_i^2 \right) \sim \beta_2 = \left( \frac{a}{\epsilon_i} \alpha_i^2 \right) \sim \gamma_2 = \left( \frac{a}{\epsilon_i} \alpha_i^2 \right) \sim \gamma_2' = \left( \frac{a}{\epsilon_i} \alpha_i^2 \right) \sim \delta_4 = 0.
\]

Moreover, let $d$ be the order of $r$ in $U(\mathbb{Z}_n)$. If $d$ is even,

\[
(\alpha_2^d + \beta_2^d + \delta_4^d/2) \sim : H^n(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z}) \to H^{n+2d}(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z})
\]

is an isomorphism for $n > 0$. If $d$ is odd, then

\[
(\alpha_2^d + \beta_2^d + \delta_4^d) \sim : H^n(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z}) \to H^{n+4d}(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z})
\]

is an isomorphism for $n > 0$.

**Proof.** We have just shown that the spectral sequence with $E_2$ page given by (6) is such that $E_2^{p,q} = E_{\infty}^{p,q}$ and $E_2^{p,q} = 0$ if $p$ and $q$ are both positive and also if one of $p$ or $q$ is odd. Hence the product on $E_2$ induces the cup product on $H^*(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z})$. The derivation of equation (7) shows that one generator for $E_2^{0,2} = \mathbb{Z}_{\epsilon_j}$ is $\left( \frac{a}{\epsilon_j} \right) \alpha_j^2$, where $\alpha_2$ is one generator of $H^2(\mathbb{Z}_n; \mathbb{Z})$ such that $H^*(\mathbb{Z}_n; \mathbb{Z}) = \mathbb{Z}^2[\alpha_2] = (\alpha_2 = 0)$. If $d$, the order of $r$ in $U(\mathbb{Z}_n)$, is even, then $\alpha_2^d$ is one generator of $E_2^{0,2d}$ and the cup product with $\alpha_2^d$ defines an isomorphism

\[
E_2^{0,q} \xrightarrow{\sim} E_2^{0,2d+q}
\]

for all $q > 0$. Also, the cup product with $\beta_2^d + \delta_4$ defines an isomorphism

\[
E_2^{p,0} \xrightarrow{\sim} E_2^{p+4,0}
\]

for $p > 0$, as can be seen from Theorem 4.2. Therefore, if $d$ is even, the cup product with $(\alpha_2^d + \beta_2^d + \delta_4^d/2) \in H^{2d}(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z})$ defines an isomorphism

\[
H^n(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z}) \xrightarrow{\sim} H^{n+2d}(\mathbb{Z}_n \rtimes \beta (\mathbb{Z}_b \times \mathbb{Q}_{2^r}); \mathbb{Z})
\]
for \( n > 0 \). If \( d \) is odd, then the cup product with \((\alpha z_2^{2d} + \beta z_2^{2d} + \delta z_4^{d})\) in \( H^{4d}(H_\epsilon \times H_\delta; \mathbb{Z}) \) defines an isomorphism

\[
H^n(H_\epsilon \times H_\delta; \mathbb{Z}) \cong H^{n+4d}(H_\epsilon \times H_\delta; \mathbb{Z})
\]

for \( n > 0 \).

Finally, let \( G = [\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})] \rtimes \theta \mathbb{Z} \) for some \( \theta : \mathbb{Z} \to \text{Aut}(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})) \) and let’s consider the extension

\[
1 \longrightarrow \mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}) \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1.
\]

Corollary 2.2 says that

\[
H^n([\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})] \rtimes \theta \mathbb{Z}; \mathbb{Z}) = \begin{cases} H^n(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}); \mathbb{Z}) & \text{if } n \text{ is even;} \\ H^{n-1}(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}); \mathbb{Z}) & \text{if } n \text{ is odd;} \end{cases}
\]

so we will study now the action of \( \mathbb{Z} \) on \( H^*(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}); \mathbb{Z}) \). The action on \( H^0(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}); \mathbb{Z}) \) is trivial, hence \( H^n([\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})] \rtimes \theta \mathbb{Z}; \mathbb{Z}) = \mathbb{Z} \) for \( n = 0 \) and \( n = 1 \). For \( j > 0 \), we have

\[
\text{Aut}(\mathbb{Z}_{\epsilon_j} \rtimes \mathbb{Z}_b \rtimes \mathbb{Z}_2^2) \cong \text{Aut}(\mathbb{Z}_{\epsilon_j}) \oplus \text{Aut}(\mathbb{Z}_b) \oplus \text{Aut}(\mathbb{Z}_2^2),
\]

\[
\text{Aut}(\mathbb{Z}_{\epsilon_j} \rtimes \mathbb{Z}_b \rtimes \mathbb{Z}_2^2) \cong \text{Aut}(\mathbb{Z}_{\epsilon_j}) \oplus \text{Aut}(\mathbb{Z}_b) \oplus \text{Aut}(\mathbb{Z}_2^2),
\]

since \( \epsilon_j, b \) and 2 are pairwise coprime. Therefore, if \( j > 0 \), in order to study the action of \( \mathbb{Z} \) on

\[
H^{2j}(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}); \mathbb{Z}) = \begin{cases} \mathbb{Z}_{\epsilon_j} \oplus \mathbb{Z}_b \oplus \mathbb{Z}_2^2 & \text{if } j \text{ is odd;} \\ \mathbb{Z}_{\epsilon_j} \oplus \mathbb{Z}_b \oplus \mathbb{Z}_2^2 & \text{if } j \text{ is even;} \end{cases}
\]

we must study the \( \mathbb{Z} \)-action on each of the summands

\[
\mathbb{Z}_{\epsilon_j} \cong H^0(Q_{2^j}; H^0(\mathbb{Z}_b; H^{2j}(\mathbb{Z}_{\epsilon_j}; \mathbb{Z}))),
\]

\[
\mathbb{Z}_b \cong H^0(Q_{2^j}; H^{2j}(\mathbb{Z}_b; H^0(\mathbb{Z}_{\epsilon_j}; \mathbb{Z}))),
\]

\[
\mathbb{Z}_2^2 \cong H^{2j}(Q_{2^j}; H^0(\mathbb{Z}_b; H^0(\mathbb{Z}_{\epsilon_j}; \mathbb{Z}))), \quad (j \text{ odd})
\]

\[
\mathbb{Z}_2^2 \cong H^{2j}(Q_{2^j}; H^0(\mathbb{Z}_b; H^0(\mathbb{Z}_{\epsilon_j}; \mathbb{Z}))), \quad (j \text{ even})
\]

Let \( \theta : \mathbb{Z} \to \text{Aut}(\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})) \) be defined by

\[
\theta(1)(a) = c_a \cdot 1_a,
\]

\[
\theta(1)(b) = c \cdot 1_a + c_b \cdot 1_b, \quad (9)
\]

\[
\theta(1)(x) = c_x \cdot 1_a + x^k, \quad (k \text{ odd})
\]

\[
\theta(1)(y) = c_y \cdot 1_a + x^\ell y, \quad (\ell \in \{0, 1, \ldots, 2^{i-1} - 1\})
\]
where $c, c_a, c_b, c_z$ and $c_y$ are chosen according to [11]. Let’s denote by $\theta_a$ the map $\theta(1)|_{\mathbb{Z}_a} : \mathbb{Z}_a \to \mathbb{Z}_a$, by $\theta_b$ the map $\theta(1)|_{\mathbb{Z}_b} : \mathbb{Z}_b \to \mathbb{Z}_b$, where $\theta(1) : \mathbb{Z}_b \times Q_{2i} \to \mathbb{Z}_b \times Q_{2i}$ is the induced map on the quotient, and by $\theta_Q$ the map $\theta(1) : Q_{2i} \to Q_{2i}$. With this notation, we have

$$\theta_a(1_a) = c_a \cdot 1_a, \quad \theta_b(1_b) = c_b \cdot 1_b, \quad \theta_Q(x) = x^k, \quad \theta_Q(y) = x^\ell y.$$ 

Now the map $H^*(\theta_Q; H^*(\theta_b; H^*(\theta_a; \mathbb{Z})))$ can be used to calculate the $\mathbb{Z}$-action on each of the summands described in Equation (8). We get that the $\mathbb{Z}$-action on $H^0(Q_{2i}; H^0(\mathbb{Z}_b; H^2(\mathbb{Z}_a; \mathbb{Z})))$ is the multiplication by $c'_a$ on $\mathbb{Z}_{c_i}$, and the $\mathbb{Z}$-action on $H^0(Q_{2i}; H^2(\mathbb{Z}_b; H^0(\mathbb{Z}_a; \mathbb{Z})))$ is the multiplication by $c'_b$ on $\mathbb{Z}_b$. As $\text{Aut}(\mathbb{Q}_4)$ doesn’t follow the pattern of $\text{Aut}(\mathbb{Q}_{2i})$ for $i > 3$, the analysis of the $\mathbb{Z}$-action on $H^0(Q_{2i}; H^2(\mathbb{Z}_b; H^0(\mathbb{Z}_a; \mathbb{Z})))$ will be broken in the cases $i = 3$ and $i > 3$.

**Lemma 4.5.** Let $i > 3$ and let $\theta$ be the $\mathbb{Z}$-action on $Q_{2i}$ given by $\theta(1)(x) = x^k$, with $k$ odd, and $\theta(1)(y) = x^\ell y$, with $\ell \in \{0, 1, \ldots, 2^i - 1\}$. The induced $\mathbb{Z}$-action $\theta^{(2)}$ on $H^2(Q_{2i}; \mathbb{Z})$ is the trivial action if $\ell$ is even. If $\ell$ is odd, $\theta^{(2)}$ is determined by the automorphism $\theta^{(2)}(1)$ given by $\theta^{(2)}(1)(\gamma_2) = \gamma_2$, $\theta^{(2)}(1)(\gamma_2') = \gamma_2 + \gamma_2'$, where $\gamma_2$ and $\gamma_2'$ are the generators of $H^*(Q_{2i}; \mathbb{Z})$ given by Theorem 4.7.

**Proof.** We have $H^2(Q_{2i}; \mathbb{Z}) \cong H_1(Q_{2i}; \mathbb{Z}) \cong (Q_{2i})_{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with generators $\gamma_2 = \mathfrak{c} e \gamma_2' = \mathfrak{c}$. Since $\theta(1)(x) = x^k$ and $k$ is odd, $\theta^{(2)}(1)(\alpha_2) = \alpha_2$, and $\theta(1)(y) = x^\ell y$ implies $\theta^{(2)}(1)(\gamma_2') = \gamma_2 + \gamma_2'$, if $\ell$ is odd and $\theta^{(2)}(1)(\gamma_2') = \gamma_2'$ if $\ell$ is even. □

By [11] and [20], the induced $\mathbb{Z}$-action on $H^2(Q_{2i}; ; H^0(\mathbb{Z}_b; H^0(\mathbb{Z}_a; \mathbb{Z})))$ for $j > 0$ even is the multiplication by $k^j$. From that, the following result is obtained:

**Proposition 4.6.** Let $\theta$ be the $\mathbb{Z}$-action on $Q_{2i}$ of the previous Lemma. The $\mathbb{Z}$-action $\theta^{(4m+2)}$ induced by $\theta$ on $H^{4m+2}(Q_{2i}; \mathbb{Z})$ coincides with $\theta^{(2)}$ induced on $H^2(Q_{2i}; \mathbb{Z})$.

**Proof.** By Theorem 4.7, the ring $H^*(Q_{2i}; \mathbb{Z})$ is given by

$$\mathbb{Z}[\gamma_2, \gamma_2', \delta_4] / (2\gamma_2 = 2\gamma_2' = 2^i \delta_4 = 0, \gamma_2^2 = 0, \gamma_2'^2 = \gamma_2 \gamma_2' = 2^{i-1} \delta_4).$$

The generators of $H^{4m+2}(Q_{2i}; \mathbb{Z})$ are $\delta_4^m \gamma_2$ and $\delta_4^m \gamma_2'$. If the induced $\mathbb{Z}$-action on $H^2(Q_{2i}; \mathbb{Z})$ is the trivial one, it is clear that the induced action on $H^{4m+2}(Q_{2i}; \mathbb{Z})$ is also trivial. If the action on $H^2(Q_{2i}; \mathbb{Z})$ is not trivial, then

$$\theta^{(4m+2)}(1)(\delta_4^m \gamma_2) = k^m \delta_4^m \gamma_2 = \delta_4^m k^m \gamma_2 = \delta_4^m \gamma_2$$

and

$$\theta^{(4m+2)}(1)(\delta_4^m \gamma_2') = \delta_4^m k^m \gamma_2 + k^m \delta_4^m \gamma_2' = \delta_4^m k^m \gamma_2 + \delta_4^m \delta_4^m \gamma_2' = \delta_4^m \gamma_2 + \delta_4^m \gamma_2',$$

since $k^m$ is odd, and the result follows. □
Lemma 4.7. For $Q_8 = \langle x, y \mid x^2 = y^2, xyx = y \rangle$ any action of $\mathbb{Z}$ on $H^2(Q_8; \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ can be realized as an induced action of $\mathbb{Z}$ on $Q_8$.

Proof. As in Lemma 4.3, the generators of $H^2(Q_8) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are $\gamma_2 = \pi$ and $\gamma'_2 = \bar{y}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_2)$ be the matrix that represents some $\mathbb{Z}$-action on $H^2(Q_8; \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $a, b, c, d$ are integers and $\pi, \bar{b}, \bar{c}, \bar{d}$ represent classes modulo 2. One action of $\mathbb{Z}$ on $Q_8$ that induces the desired action on $H^2(Q_8; \mathbb{Z})$ is $\pi(1)(x) = x^a y^b$ and $\pi(1)(y) = x^c y^d$.

Since any action of $\mathbb{Z}$ on $Q_8$ induces the trivial action on $H^4(Q_8; \mathbb{Z})$, a result analogue to Proposition 4.8 holds:

Proposition 4.8. Let $\theta$ be a $\mathbb{Z}$-action on $Q_8$. The induced $\mathbb{Z}$-action $\theta^{(4m+2)}$ on $H^{4m+2}(Q_8; \mathbb{Z})$ coincides with the induced action $\theta^{(2)}$ on $H^2(Q_8; \mathbb{Z})$.

Let $A_j = \gcd(c_n, 1, \varepsilon_j)$, $B_j = \gcd(c_n, b_j)$ and $C_j = \gcd(k^j - 1, 2^j)$, and recall that $\theta_q(x) = x^k$ and $\theta_q(y) = x^h y$. From our work so far in this section we can compute the cohomology groups $H^n([\mathbb{Z}_n \times \beta (\mathbb{Z}_n \times Q_2)] \times_{\theta} \mathbb{Z}; \mathbb{Z})$.

Theorem 4.9. Let $G = [\mathbb{Z}_n \times \beta (\mathbb{Z}_n \times Q_2)] \times_{\theta} \mathbb{Z}$, where $\gcd(a, b) = \gcd(ab, 2) = 1$, $i \geq 3$, and $\beta$ and $\theta$ are given by equations (2) and (9), respectively. If $i > 3$ and $\ell$ is even, then

$$H^n(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \text{ or } n = 1, \\ \mathbb{Z}_{A_{2j}} \oplus \mathbb{Z}_{B_{2j}} \oplus \mathbb{Z}_{C_{2j}}, & \text{if } n = 4j \text{ or } n = 4j + 1 \text{ (and } j > 0), \\ \mathbb{Z}_{A_{2j+1}} \oplus \mathbb{Z}_{B_{2j+1}} \oplus \mathbb{Z}_2^2, & \text{if } n = 4j + 2 \text{ or } n = 4j + 3. \end{cases}$$

If $i > 3$ and $\ell$ is odd, then

$$H^n(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \text{ or } n = 1, \\ \mathbb{Z}_{A_{2j}} \oplus \mathbb{Z}_{B_{2j}} \oplus \mathbb{Z}_{C_{2j}}, & \text{if } n = 4j \text{ or } n = 4j + 1 \text{ (and } j > 0), \\ \mathbb{Z}_{A_{2j+1}} \oplus \mathbb{Z}_{B_{2j+1}} \oplus \mathbb{Z}_2, & \text{if } n = 4j + 2 \text{ or } n = 4j + 3. \end{cases}$$

If $i = 3$ and the induced action $\theta^{(2)}$ on $H^2(Q_8; \mathbb{Z})$ is trivial, then

$$H^n(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \text{ or } n = 1, \\ \mathbb{Z}_{A_{2j}} \oplus \mathbb{Z}_{B_{2j}} \oplus \mathbb{Z}_8, & \text{if } n = 4j \text{ or } n = 4j + 1 \text{ (and } j > 0), \\ \mathbb{Z}_{A_{2j+1}} \oplus \mathbb{Z}_{B_{2j+1}} \oplus \mathbb{Z}_2^2, & \text{if } n = 4j + 2 \text{ or } n = 4j + 3. \end{cases}$$

If $i = 3$ and the induced action $\theta^{(2)}$ on $H^2(Q_8; \mathbb{Z})$ is non-trivial, then

$$H^n(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \text{ or } n = 1, \\ \mathbb{Z}_{A_{2j}} \oplus \mathbb{Z}_{B_{2j}} \oplus \mathbb{Z}_8, & \text{if } n = 4j \text{ or } n = 4j + 1 \text{ (and } j > 0), \\ \mathbb{Z}_{A_{2j+1}} \oplus \mathbb{Z}_{B_{2j+1}} \oplus \mathbb{Z}_2, & \text{if } n = 4j + 2 \text{ or } n = 4j + 3. \end{cases}$$
Cohomology of virtually cyclic groups

We finish by making some remarks about the cohomology ring \( H^*(G; \mathbb{Z}) \). Let \( d_k \) be the order of \( k \) in \( \mathcal{U}(\mathbb{Z}_2) \), \( d_c \) be the order of \( c \) in \( \mathcal{U}(\mathbb{Z}_2) \), \( d_c \) be the order of \( c \) in \( \mathcal{U}(\mathbb{Z}_3) \), and \( d \) be the order of \( r \) in \( \mathcal{U}(\mathbb{Z}_3) \). When \( i > 3 \) and \( \ell \) is even, generators for the cyclic subgroups \( \mathbb{Z}_{A_{2j}}, \mathbb{Z}_{C_{2j}}, \) and \( \mathbb{Z}_{C_{rj}} \) of \( H^{4j}(G; \mathbb{Z}) \) (for \( j > 0 \)) are given by \( \left( \frac{a}{A_{2j}} \right) \frac{2^j}{\mathbb{Z}_{2j}} \) and \( \left( \frac{2}{C_{2j}} \right) \frac{2^j}{\delta_4^j} \), respectively. Similarly, the generators for the subgroups \( \mathbb{Z}_{A_{2j+1}}, \mathbb{Z}_{B_{2j+1}}, \) and \( \mathbb{Z}_{C_{2j+1}} \) of \( H^{4j+2}(G; \mathbb{Z}) \) are given by \( \left( \frac{a}{A_{2j+1}} \right) \alpha_2^{j+1}, \left( \frac{b}{B_{2j+1}} \right) \beta_2^{j+1}, \) and \( \left( \frac{2}{C_{2j+1}} \right) \delta_4^{j+1} \), respectively. Also, for \( n = 4j + 1 \) (when \( j > 0 \)) and \( n = 4j + 3 \), the generators of \( H^n(G; \mathbb{Z}) \) are just the classes of the elements that generate the respective cohomology groups \( H^*(\mathbb{Z}_a \times \beta (\mathbb{Z}_b \times \mathbb{Z}_Q_2): \mathbb{Z}) \) that are given by Theorem 4.4. We can apply a similar reasoning to the one used to prove Theorem 4.3 and rewrite down all the cup products in \( H^n(G; \mathbb{Z}) \) if we so desire, and we can also compute the period in \( H^*(G; \mathbb{Z}) \): if \( p = \text{lcm}(d, d_c, d, d_k) \), then we have \( A_{2j} = a, B_{2j} = b \) and \( C_{2j} = 2 \) if, and only if, \( p \mid 2j \). It now follows from equation (3) and Theorem 4.4 that, if \( p \) is even,

\[
(\alpha_p^2 + \beta_2^2 + \delta_4^{j/2}) \sim: H^n(G; \mathbb{Z}) \to H^{n+2p}(G; \mathbb{Z})
\]

is an isomorphism for \( n \geq 2 \), and, if \( p \) is odd,

\[
(\alpha_2^{2p} + \beta_2^{2p} + \delta_4^{j}) \sim: H^n(G; \mathbb{Z}) \to H^{n+4p}(G; \mathbb{Z})
\]

is an isomorphism for \( n \geq 2 \). The other cases are similar, and we get the following result:

**Theorem 4.10.** Let \( G \) be as in Theorem 4.9, and let \( d_k \) be the order of \( k \) in \( \mathcal{U}(\mathbb{Z}_2) \), \( d_c \) be the order of \( c \) in \( \mathcal{U}(\mathbb{Z}_2) \), \( d_c \) be the order of \( c \) in \( \mathcal{U}(\mathbb{Z}_3) \), and \( d \) be the order of \( r \) in \( \mathcal{U}(\mathbb{Z}_3) \). Finally, let \( p = \text{lcm}(d, d_c, d, d_k) \). If \( p \) is even,

\[
(\alpha_p^2 + \beta_2^2 + \delta_4^{j/2}) \sim: H^n(G; \mathbb{Z}) \to H^{n+2p}(G; \mathbb{Z})
\]

is an isomorphism for \( n \geq 2 \), and, if \( p \) is odd,

\[
(\alpha_2^{2p} + \beta_2^{2p} + \delta_4^{j}) \sim: H^n(G; \mathbb{Z}) \to H^{n+4p}(G; \mathbb{Z})
\]

is an isomorphism for \( n \geq 2 \).

**Corollary 4.11.** Using the same notation of the previous theorem, if \( p \) is even, then

\[
(\alpha_p^2 + \beta_2^2 + \delta_4^{j/2}) \sim: \hat{H}^n(G; \mathbb{Z}) \to \hat{H}^{n+2p}(G; \mathbb{Z})
\]

is an isomorphism for all \( n \in \mathbb{Z} \), and, if \( p \) is odd,

\[
(\alpha_2^{2p} + \beta_2^{2p} + \delta_4^{j}) \sim: \hat{H}^n(G; \mathbb{Z}) \to \hat{H}^{n+4p}(G; \mathbb{Z})
\]

is an isomorphism for all \( n \in \mathbb{Z} \).
References

[1] Alejandro Adem and R. James Milgram. *Cohomology of finite groups*, volume 309 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2004.

[2] Alejandro Adem and Jeff H. Smith. Periodic complexes and group actions. *Ann. of Math. (2)*, 154(2):407–435, 2001.

[3] Glen E. Bredon. *Introduction to compact transformation groups*.

[4] Kenneth S. Brown. *Cohomology of groups*, volume 87 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

[5] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[6] F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic $K$-theory. *J. Amer. Math. Soc.*, 6(2):249–297, 1993.

[7] E. E. Floyd. On periodic maps and the Euler characteristics of associated spaces. *Trans. Amer. Math. Soc.*, 72:138–147, 1952.

[8] Marek Golasiński and Daciberg Lima Gonçalves. Free and properly discontinuous actions of groups $G \times \mathbb{Z}^n$ and $G_1 *_{C_n} G_2$. Accepted for publication in the *JHRS*, special vol. dedicated to R. Brown.

[9] Marek Golasiński and Daciberg Lima Gonçalves. Homotopy spherical space forms—a numerical bound for homotopy types. *Hiroshima Math. J.*, 31(1):107–116, 2001.

[10] Marek Golasiński and Daciberg Lima Gonçalves. Spherical space forms—homotopy types and self-equivalences. 215:153–165, 2004.

[11] Marek Golasiński and Daciberg Lima Gonçalves. Spherical space forms—homotopy types and self-equivalences for the groups $\mathbb{Z}_a \times \mathbb{Z}_b$ and $\mathbb{Z}_a \times (\mathbb{Z}_b \times \mathbb{Q}_2)$). *Topology Appl.*, 146/147:451–470, 2005.

[12] Marek Golasiński and Daciberg Lima Gonçalves. Homotopy types of orbit spaces with their self-equivalences for the periodic groups $\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times T_n^*)$ and $\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times O_n^*)$. *J. Homotopy Relat. Struct.*, 1(1):29–45, 2006.

[13] Marek Golasiński and Daciberg Lima Gonçalves. Spherical space forms: homotopy types and self-equivalences for the group $(\mathbb{Z}_a \rtimes \mathbb{Z}_b) \rtimes SL_2(\mathbb{F}_p)$. *Canad. Math. Bull.*, 50(2):206–214, 2007.

[14] Marek Golasiński and Daciberg Lima Gonçalves. Spherical space forms—homotopy self-equivalences and homotopy types, the case of the groups $\mathbb{Z}/a \rtimes (\mathbb{Z}/b \times TL_2(\mathbb{F}_p))$. *Topology Appl.*, 156(17):2726–2734, 2009.
[15] Marek Golasiński and Daciberg Lima Gonçalves. Automorphism groups of generalized (binary) icosahedral, tetrahedral and octahedral groups. *Algebra Colloq.*, 18(3):385–396, 2011.

[16] Marek Golasiński, Daciberg Lima Gonçalves, and Rolando Jimenez. Free and properly discontinuous actions of groups on homotopy 2n-spheres. Accepted for publication in the *Proceedings of the Edinburgh Mathematical Society* — PEIM.

[17] Marek Golasiński, Daciberg Lima Gonçalves, and Rolando Jimenez. Free and properly discontinuous actions of discrete groups on homotopy circles. *Russ. J. Math. Phys.*, 22(3):307–327, 2015.

[18] Daciberg Lima Gonçalves and John Guaschi. Classification of the virtually cyclic subgroups of the pure braid groups of the projective plane. *J. Group Theory*, 13(2):277–294, 2010.

[19] John McCleary. *A user’s guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[20] Morris Newman. *Integral matrices*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 45.

[21] Donald S. Passman. *The algebraic structure of group rings*. Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1985. Reprint of the 1977 original.

[22] C. A. Robinson. Moore-Postnikov systems for non-simple fibrations. *Illinois J. Math.*, 16:234–242, 1972.

[23] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[24] J. C. Su. Periodic transformations on the product of two spheres. *Trans. Amer. Math. Soc.*, 112:369–380, 1964.

[25] Richard G. Swan. Periodic resolutions for finite groups. *Ann. of Math. (2)*, 72:267–291, 1960.

[26] C. B. Thomas. The oriented homotopy type of compact 3-manifolds. *Proc. London Math. Soc. (3)*, 19:31–44, 1969.

[27] C. B. Thomas and C. T. C. Wall. The topological spherical space form problem. I. *Compositio Math.*, 23:101–114, 1971.

[28] Satoshi Tomoda. *Cohomology Rings of Certain 4-Periodic Finite Groups*. PhD thesis, University of Calgary, 2005.

[29] Satoshi Tomoda and Peter Zvengrowski. Remarks on the cohomology of finite fundamental groups of 3-manifolds. 14:519–556, 2008.
[30] C. T. C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.

[31] Joseph A. Wolf. *Spaces of constant curvature*. Publish or Perish, Inc., Houston, TX, fifth edition, 1984.