Analytical Solution of the Voter Model on Disordered Networks

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Abstract. We present a mathematical description of the voter model dynamics on heterogeneous networks. When the average degree of the graph is $\mu \leq 2$ the system reaches complete order exponentially fast. For $\mu > 2$, a finite system falls, before it fully orders, in a quasistationary state in which the average density of active links (links between opposite-state nodes) in surviving runs is constant and equal to \( \frac{(\mu - 2)}{\mu(\mu - 1)} \), while an infinite large system stays ad infinitum in a partially ordered stationary active state. The mean life time of the quasistationary state is proportional to the mean time to reach the fully ordered state $T$, which scales as $T \sim \frac{(\mu - 1)\mu^2 N}{(\mu - 2)\mu_2}$, where $N$ is the number of nodes of the network, and $\mu_2$ is the second moment of the degree distribution. We find good agreement between these analytical results and numerical simulations on random networks with various degree distributions.

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1. Introduction

The voter model has become one of the most popular interacting particle systems \cite{1, 2} with applications to the study of diverse processes like opinion formation \cite{3, 4}, kinetics of heterogeneous catalysis \cite{5, 6}, and species competition \cite{7}. The general version of the model considers a network formed by nodes holding either spin 1 or -1. In a single event, a randomly chosen node adopts the spin of one of its neighbors, also chosen at random. Beyond this standard version, several variations of the model have been considered in the literature, to account for zealots or inhomogeneities (individuals that favor one of the states) \cite{8}, constrained interactions \cite{9}, non-equivalent states \cite{10}, asymmetric transitions or bias \cite{11}, noise \cite{12} and ecological diversity \cite{13}. It is also known that several models presenting a coarsening process without surface tension belong to the voter model universality class \cite{14}.

In a regular lattice, the mean magnetization, i.e., the normalized difference in the number of 1 and -1 spins, is conserved at each time step. Thus the magnetization is not a useful order parameter to study the ordering dynamics of the voter model. Instead, it is common in the physics literature to use as an order parameter the density of interfaces $\rho$, i.e., the fraction of links connecting neighbors with opposite spins. In a finite system, the only possible final state is the fully ordered state, in which all spins have the same value, either -1 or 1, and therefore all pair of neighbors are aligned ($\rho = 0$). These are absorbing configurations given that the system cannot escape from them once they are reached \cite{15}. Despite its non-trivial dynamics, an exact solution has been obtained for regular lattices of general dimension $d$ \cite{5, 6}, becoming one of the few non-equilibrium models which are exactly solvable in any dimension. Indeed, the correspondence between the voter model and a system of coalescing random walkers helps to solve analytically many features of the dynamics \cite{16, 17}. For $d \leq 2$, there is a coarsening process where the average size of ordered regions composed by sites holding the same spin continuously grows. In the thermodynamic limit, the approach to the final frozen configuration is characterized by the monotonic decrease in $\rho$, that decays as $\rho \sim t^{-1/2}$ in 1d and $\rho \sim (\ln t)^{-1}$ in 2d \cite{5}. For $d > 2$, the density of active interfaces behaves as $\rho(t) \sim a - bt^{-d/2}$ \cite{6}, thus $\rho(t)$ reaches a constant value in the long time limit where the system reaches a stationary active state with nodes continuously flipping their spins. That is to say, full order is never reached. We need to clarify that the last is only true for infinite large systems, given that fluctuations in finite size lattices make the system to ultimately reach complete order. The level of order in the stationary state is quantified by the two-spin correlation function $C_{ij} \equiv \langle S_iS_j \rangle$ between spins $i$ and $j$, that decays with their spatial separation $r = |i - j|$ as $C(r) \sim r^{(2-d)}$ \cite{18}, i.e., far apart spins become uncorrelated. Recent studies of the voter model on fractals with fractal dimension in the range $(1, 2)$, reveal that the system orders following $\rho(t) \sim t^{-\alpha}$, with the exponent $\alpha$ in the range $(0, 1)$ \cite{19, 20}.

The voter model has recently been investigated on complex networks \cite{21, 22, 23, 24, 25, 26, 27}, where its behavior seems to strongly depend on the topological characteristics
of the network. A peculiar aspect is that the dynamics can be slightly modified giving different dynamical scaling laws. For instance with *node update*, i.e., selecting first a node and then one of its neighbors, the conservation of the magnetization is not longer fulfilled. Instead the weighted magnetization is in this case conserved at each time step. With *link update*, where a link is selected at random and then one of its ends is updated according to the neighbor’s spin, the conservation of the magnetization is restored \[23\].

A striking feature of the voter model on several complex networks, including Small-World, Barabási-Albert, Erdös-Rényi, Exponential and Complete Graph is the lack of complete order in the thermodynamic limit. In this article, we provide an analytical insight of the incomplete ordering phenomenon in heterogeneous networks by studying the evolution and final state of the system using a simple mean-field approach. Despite that this approach is meant to work well in networks with arbitrary degree distributions but without node degree correlations, the qualitative results are rather general for many networks. We obtain analytical predictions for the density of active links (links connecting nodes with opposite spin) and the mean time to reach the ordered state as a function of the system size and the first and second moments of the degree distribution. These predictions explain numerical results reported in \[23\] \[25\] \[26\] and they agree with previous analytical results for ordering times \[24\].

The rest of the article is organized as follows. In section 2, we define the model and its updating rule on a general network. We then develop in section 3 a mean-field approach for the time evolution of the density active links and the link magnetization. This approximation reveals a transition at a critical value of the average connectivity $\mu = 2$. When $\mu$ is smaller than 2, complete order is reached exponentially fast, whereas for $\mu > 2$, the system quickly settles in a quasistationary disordered state characterized by a constant density of active links whose value only depends on $\mu$, independent on the degree distribution. We find that $\rho$ is proportional to the product of the spin densities with a proportionality constant that depends on $\mu$. This relation allows us to derive an approximate Fokker-Planck equation for the magnetization in section 4. This equation is used in section 5 to study the relaxation of a finite system to the absorbing ordered state and in section 6 to obtain an expression for the survival probability of independent runs. The mean time to reach complete order, calculated in section 7 shows that the dependence of the results on the network topology enters through the first and the second moments of the degree distribution only. Convergence to the ordered state slows down as $\mu$ approaches 2, where ordering times seem to diverge faster than $N$. The summary and conclusions are provided in section 8. In the appendix we present some details of calculations.

2. The model

We consider a network composed by a set of $N$ nodes and the links connecting pair of nodes. We assume that the network has no degree correlations, i.e., the neighbors of each node are randomly selected from the entire set. We denote by $P_k$ the degree
distribution, which is the fraction of nodes with \( k \) links, subject to the normalization condition \( \sum_k P_k = 1 \). In the initial configuration, spins are assigned the values 1 or -1 with probabilities given by the initial densities \( \sigma_+ \) and \( \sigma_- \) respectively. In a single time step, a node \( i \) with spin \( S_i = s \) (\( s = 1 \) or \(-1\)) and degree \( k \) is chosen. We denote by \( n \) the number of active links connected to node \( i \) before the update. With probability \( n/k \) an active link (in this example \( i - j \)) is randomly chosen. Node \( i \) flips its state changing the state of its links from active to inert and vice-versa, and giving a local change of the number of active links \( \Delta n = k - 2n \) and a global density change \( \Delta \rho = \frac{2(k-2n)}{\mu N} \). Here \( \mu N/2 \) is the total number of links, \( \mu \equiv \langle k \rangle = \sum_k kP_k \) is the number of links per node or average degree. Assembling these factors, the change in the average density of active links is given by \( \Delta \rho = \frac{2(k-2n)}{\mu N} \).

3. Mean-Field theory

In order to obtain an insight about the time evolution of the system we develop a mean-field (MF) approach. There are two types of links in the system, links between nodes with different spin or active links and links between nodes with the same spin or inert links. Given that a single spin-flip update happens only when an active link is chosen, it seems natural to consider the global density of active links \( \rho \) as a parameter that measures the level of activity in the system.

In Fig. 1 we describe the possible changes in \( \rho \) and their probabilities in a time step, when a node \( i \) with spin \( S_i \) and one of its neighbors \( j \) with spin \( S_j \) are chosen at random. Then \( i \) adopts \( j \)'s spin \( (S_i \rightarrow S_i = S_j) \) (see Fig. 1). This step is repeated until the system reaches complete order and it cannot longer evolve.
links in a single time step of time interval $dt = 1/N$ is described by the master equation:

$$\frac{d\rho}{dt} = \sum_k P_k \frac{d\rho}{dt}|_k = \sum_k \frac{P_k}{1/N} \sum_{n=0}^k B(n, k) \frac{n}{k} \frac{2(k - 2n)}{\mu N},$$

(1)

where $B(n, k)$ is the probability that $n$ active links are connected to a node of degree $k$, and $\frac{d\rho}{dt}|_k$ denotes the average change in $\rho$ when a node of degree $k$ is chosen. Given that, during the evolution, the densities of $+$ and $-$ spins are not the same, we expect that $B(n, k)$ will depend on the spin of node $i$. For instance, when the system is about to reach the $+$ fully ordered state, we expect a configuration where most of the neighbors of a given node (independent on its spin) have $+$ spin, thus the probability that a link connected to a node with spin $+$ ($-$) is active will be close to zero (one). Therefore, we take $B(n, k)$ as the average probability over the two types of spins

$$B(n, k) = \sum_{s=\pm} \sigma_s B(n, k|s),$$

(2)

where $B(n, k|s)$ is the conditional probability that $n$ of the $k$ links connected to a node are active, given that the node has spin $s$. Replacing Eq. (2) into Eq. (1) we obtain

$$\frac{d\rho}{dt} = \frac{2}{\mu} \sum_k P_k \sum_{s=\pm} \sigma_s \sum_{n=0}^k B(n, k|s) \frac{n}{k} (k - 2n)$$

(3)

$$= \frac{2}{\mu} \sum_k P_k \sum_{s=\pm} \sigma_s \left[ \langle n \rangle_{k,s} - \frac{2}{k} \langle n^2 \rangle_{k,s} \right],$$

(4)

where $\langle n \rangle_{k,s}$ and $\langle n^2 \rangle_{k,s}$ are the first and the second moments of $B(n, k|s)$ respectively.

In order to calculate $B(n, k|s)$ we assume that only correlations between the states of first neighbors are relevant, neglecting second or higher neighbors correlations. Therefore, we consider the conditional probability $P(-s|s)$, that a neighbor of node $i$ has spin $-s$ given that $i$ has spin $s$, to be independent of the other neighbors of $i$. This is known in the lattice models literature with the name of pair approximation, and it is supposed to work only in networks without degree correlations. Thus, $B(n, k|s)$ becomes the binomial distribution with $P(-s|s)$ as the single event probability that a link connected to $i$ is active. $P(-s|s)$ can be calculated as the average fraction of neighbors with spin $-s$ to a node with spin $s$, i.e., the ratio between the total number $\rho \mu N/2$ of $s \rightarrow -s$ links and the total number $\mu \sigma s N$ of links connected to nodes with spin $s$. We have used the symmetry in the states of the voter model and assumed that the average degrees of nodes holding spin $1$ and $-1$ are the same and equal to $\mu$. We have numerically checked that the last is valid for the original voter model, but if the two states are not equivalent or a biased is introduced, the average degrees are different. Then, $P(-s|s) = \rho/2 \sigma_s$, and the first and the second moments of $B(n, k|s)$ are

$$\langle n \rangle_{k,s} = \frac{k \rho}{2 \sigma_s},$$

$$\langle n^2 \rangle_{k,s} = \frac{k \rho}{2 \sigma_s} + \frac{k(k - 1) \rho^2}{4 \sigma^2 s}.$$
Replacing these expressions for the moments in Eq. (4) and performing the sums we finally obtain

\[ \frac{d\rho}{dt} = \frac{2\rho}{\mu} \left[ \left( \mu - 1 \right) \left( 1 - \frac{\rho}{2\sigma_+(1 - \sigma_+)} \right) - 1 \right]. \] (5)

Equation (5) is the master equation for the time evolution of \( \rho \) as a function of the spin density \( \sigma_+(t) \). It has two stationary solutions, but depending on the value of \( \mu \), only one is stable. For \( \mu \leq 2 \), the stable solution \( \rho = 0 \) corresponds to a fully ordered frozen system. For \( \mu > 2 \), the stable solution is

\[ \rho(t) = 4\xi(\mu) \sigma_+(t) \left[ 1 - \sigma_+(t) \right], \] (6)

where we define

\[ \xi(\mu) \equiv \frac{(\mu - 2)}{2(\mu - 1)}, \] (7)

corresponding to a partially ordered system, composed by a fraction \( \rho > 0 \) of active links, as long as \( \sigma_+ \neq 0, 1 \).

In Fig. 2 we test Eq. (6) by plotting the time evolution of the ratio between \( \rho \) and \( \sigma_+ (1 - \sigma_+) \) in a single realization, for various values of \( \mu \). We observe that, even though the ratio varies over time, it fluctuates around the constant value \( 4\xi \) predicted by Eq. (6). It is worth noting that the behavior of the ratio is the same from times...
of order one to the end of the realization, where fluctuations increase in amplitude before the system reaches complete order. We also notice that fluctuations decrease as \( \mu \) increases, and they become zero in the complete graph case \( (\mu = N - 1) \), where we have \( \rho(t) = 2\sigma_+(t)[1 - \sigma_+(t)], \) for \( N \gg 1 \).

In infinite large systems, fluctuations in \( \sigma_+(t) \) vanish. Therefore, in a single realization we would see that \( \sigma_+(t) = \sigma_+(0) \) for all \( t > 0 \) and that \( \rho \) reaches an infinite long lived stationary state with \( \rho = 4\xi\sigma_+(0)[1 - \sigma_+(0)] \). Then, for networks with average degree \( \mu > 2 \), full order is never reached in the thermodynamic limit.

In finite size networks, fluctuations eventually drive the system to one of the two absorbing states, \( \sigma_+ = 1 \) or \( \sigma_+ = 0 \), characterized by the absence of active links \( (\rho = 0) \). Although the parameter \( \rho \) is useful for finding an absorbing state, it does not allow us to know which of the two states is reached. For this reason we introduce the link magnetization \( m = \rho_{++} - \rho_{--} \), where \( \rho_{++} (\rho_{--}) \) are the density of links connecting two nodes with spins 1 (-1). It measures the level of order in the system, \( m = 1 \) \( (m = -1) \) corresponding to the + (-) fully ordered absorbing state and \( m = 0 \) representing the totally mixed disordered state. Given that \( \rho \) becomes zero when \( m \) takes the values \( \pm 1 \), we guess that \( \rho \) should be proportional to \( 1 - m^2 \). To prove this, we first relate \( \sigma_s \) with \( \rho_{ss} \) by calculating the total number of links coming out from node \( s \) with spin \( s \). This number of links is \( \mu\sigma_s N \), from which \( \rho_{ss} \mu N/2 \) are \( s \rightarrow -s \) links, and \( \rho_{ss} \mu N \) are \( s \rightarrow s \) links. We arrive to

\[
\rho_{ss} = \sigma_s - \rho/2.
\]

Then, the link magnetization is simply the spin magnetization

\[
m = \rho_{++} - \rho_{--} = \sigma_+ - \sigma_- = 2\sigma_+ - 1. \tag{8}
\]

Combining Eqs. (6) and (8) we obtain that, neglecting fluctuations, \( \rho \) and \( m \) are related through the equation

\[
\rho(t) = \xi [1 - m^2(t)] \tag{9}.
\]

Fig. 3 shows \( \rho \) vs \( m \) in one realization with \( \mu = 4 \) and \( N = 10^4 \). The system starts with equal density of + and - spins \( (m = 0 \) and \( \rho = 1/2) \), and after an initial transient of order one, in which \( m \) stays close to zero and \( \rho \) decays to a value similar to \( \xi \), \( \rho \) fluctuates around the parabola described by Eq. (9). This particular trajectory ends at the \( (m = 1, \rho = 0) \) absorbing state.

4. Master Equation for the link magnetization

In order to study the time evolution of the system we start by deriving a master equation for the probability \( P(m, t) \) that the system has link magnetization \( m \) at time \( t \). In a time step, a node with spin \( s \) and degree \( k \) flips its spin with probability \( \sigma_s P(-s|s) = \rho/2 \), after which the magnetization changes by \( \Delta m = s \delta_k \), with \( \delta_k = \frac{2k}{\mu N} \) (see Fig. 1), and with probability \( \sigma_s [1 - P(-s|s)] = \sigma_s (1 - \rho/2\sigma_s) \) its spin remains unchanged. We have used that the density of \( s \) spins and the conditional probability \( P(-s|s) \) in the subset
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Figure 3. Trajectory of the system in a single realization plotted on the active links-link magnetization ($\rho - m$) plane, for a network of size $N = 10^4$ and average degree $\mu = 4$. Insets: Time evolution of $m$ (left) and $\rho$ (right) for the same realization. We note that $\rho$ and $m$ are not independent but fluctuate in coupled manner, following a parabolic trajectory described by $\rho = \frac{1}{3}(1 - m^2)$ from Eq. (9) (solid line).

of nodes with degree $k$ is independent on $k$ and equal to the global density $\sigma_s$ (this was first noticed in [24] and [26]). Using Eq. (9) we can write the probabilities of the possible changes in $m$ due to the selection of a node of degree $k$ as

$$W_{m \to m - \delta_k} = \frac{\xi}{2} (1 - m^2) P_k$$
$$W_{m \to m + \delta_k} = \frac{\xi}{2} (1 - m^2) P_k$$
$$W_{m \to m} = \left[1 - \xi (1 - m^2)\right] P_k.$$  \tag{10}

Thus, the problem is reduced to the motion of a symmetric random walk in the $(-1, 1)$ interval, with absorbing boundaries at the ends and hopping distances and their probabilities that depend on the walker’s position $m$ and the degree distribution $P_k$. The time evolution of $P(m, t)$ is described by the master equation

$$P(m, t + \delta t) = \sum_k P_k \left\{ W_{m + \delta_k \to m} P(m + \delta_k, t) + W_{m - \delta_k \to m} P(m - \delta_k, t) \right\} + W_{m \to m} P(m, t)$$
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\[ \sum_k P_k \left\{ \frac{\xi}{2} \left[ 1 - (m + \delta_k)^2 \right] P(m + \delta_k, t) 
+ \frac{\xi}{2} \left[ 1 - (m - \delta_k)^2 \right] P(m - \delta_k, t) + \left[ 1 - \xi(1 - m^2) \right] P(m, t) \right\}, \]  

where \( \delta t = 1/N \) is the time step corresponding to a spin-flip attempt. In Eq. (11), the probability that the walker is at site \( m \) at time \( t + \delta t \) is written as the sum of the probabilities for all possible events that take the walker from a site \( m + \Delta \) to site \( m \), with \( \Delta = 0, \pm \delta_k \) and \( k \geq 0 \). The probability of a single event is the probability \( P(m + \Delta, t) \) of being at site \( m + \Delta \) at time \( t \) times the probability \( W_{m+\Delta \rightarrow m} \) of hopping to site \( m \).

Expanding Eq. (11) to second order in \( m \) and first order in \( t \) we obtain

\[ N \delta t \frac{\partial P}{\partial t} = \frac{2\xi}{\mu^2 N} \sum_k P_k k^2 \left\{ -2m \frac{\partial P}{\partial m} + (1 - m^2) \frac{\partial^2 P}{\partial m^2} \right\}. \]

Thus, in the continuum limit \((\delta t = 1/N \rightarrow 0 \text{ as } N \rightarrow \infty)\), we arrive to the Fokker-Planck equation

\[ \frac{\partial P(m, t')}{\partial t'} = \frac{\partial^2}{\partial m^2} \left[ (1 - m^2)P(m, t') \right], \]  

where \( t' \equiv t/\tau \) is a rescaled time,

\[ \tau \equiv \frac{\mu^2 N}{2\xi(\mu) \mu_2} = \frac{(\mu - 1)\mu^2 N}{(\mu - 2) \mu_2} \]

is an intrinsic time scale of the system and \( \mu_2 = \sum_k k^2 P_k \) is the second moment of the degree distribution. We shall see in section 7 that the time to reach the ordered state equals \( \tau \) times a function of the initial magnetization. Note that, in complete graph, the corresponding Fokker-Planck equation derived for instance in [28], has the same form as Eq. (12) with \( t' = t/N \), obtained as a particular case of a graph with distribution \( P_k = \delta_{k,\mu}, \mu = N - 1 \) and \( \mu_2 = \mu^2 \). The general solution to Eq. (12) is given by the series expansion [28, 29]

\[ P(m, t') = \sum_{l=0}^{\infty} A_l C_{l}^{3/2}(m) e^{-(l+1)(l+2) t'}, \]  

where \( A_l \) are coefficients determined by the initial condition and \( C_{l}^{3/2}(x) \) are the Gegenbauer polynomials [30] page 980. Equation (14) is of fundamental importance because it allows to find the two most relevant magnitudes in the voter model dynamics, namely, the average density of active links and the survival probability, as we shall see in sections 5 and 6 respectively.

5. Approach to the final frozen state

We are interested in how the average density of active links \( \langle \rho \rangle \) decays to zero, where \( \langle \cdot \rangle \) denotes an average over many independent realizations of the dynamics starting from the same initial spin densities. Using Eq. (11) we can write

\[ \langle \rho(t') \rangle = \xi \langle 1 - m^2(t') \rangle = \xi \int_{-1}^{1} dm \ (1 - m^2) \ P(m, t'), \]

\[ \sum_k P_k \left\{ \frac{\xi}{2} \left[ 1 - (m + \delta_k)^2 \right] P(m + \delta_k, t) 
+ \frac{\xi}{2} \left[ 1 - (m - \delta_k)^2 \right] P(m - \delta_k, t) + \left[ 1 - \xi(1 - m^2) \right] P(m, t) \right\}, \]
with \( P(m, t') \) given by Eq. (14). The solution to the above integral with an initial magnetization \( m_0 = 2\sigma_+(0) - 1 \) is (see appendix Appendix A)

\[
\langle \rho(t') \rangle = \xi(1 - m_0^2) e^{-2t'},
\]

(16)

and replacing back \( t' \) and \( \xi(\mu) \) we finally obtain

\[
\langle \rho(t) \rangle = \frac{\mu - 2}{2(\mu - 1)} (1 - m_0^2)e^{-2t/\tau}.
\]

(17)

We find that for \( \mu > 2 \), \( \langle \rho(t) \rangle \) has an exponential decay with a time constant \( \tau/2 \), whose inverse gives the rate at which \( \langle \rho \rangle \) decays. Given that \( \tau \) is proportional to \( N \) (Eq. (13)), the decay becomes slower for increasing system sizes. Eventually, in the limit of an infinite large network \( \langle \rho(t) \rangle \) remains at the constant value \( \xi(1 - m_0^2) \) as it was discussed in section 3, while in a finite network, \( \langle \rho(t) \rangle \) reaches zero in a time of order \( \tau \).

We have simulated the voter model on various types of random networks: degree-regular random graph (DR), Erdős-Rényi graph (ER), Exponential network (EN) and Barabási-Albert network (BA). In Fig. 4 we observe that the analytical prediction (Eq. (17)) is in good agreement with numerical simulations on these four networks. For a fix average degree \( \mu \) and system size \( N \), \( \tau \) is determined by the second moment \( \mu_2 \) of the network degree distribution \( P_k \). For these particular networks, \( \mu_2 \) can be written as a function of \( \mu \), because \( P_k \) only depends on \( \mu \) and \( k \). As a consequence of this, \( \tau(\mu, N) \) is only a function of \( \mu \) and \( N \). The values of \( \tau \) and \( \mu_2 \) in the large \( N \) limit are summarized in table 1. For the case of DR, ER and EN, \( \langle \rho \rangle \) is a function of \( t/N \) as it is shown in Fig. 4 and \( \mu_2 \) is finite and independent on \( N \). We have checked that the scaling works very well for networks of size \( N > 100 \). For BA networks, \( \mu_2 \) diverges with \( N \) (see calculation details in appendix Appendix B), thus we rescaled the x-axis by \( N/\mu_2(N) \) in order to obtain an overlap for the curves of different system sizes.

| Network | \( P_k \) | \( \mu_2 \) | \( \tau(\mu, N) \) |
|---------|---------|---------|----------------|
| DR      | \( \delta_{k,\mu} \) | \( \mu^2 \) | \( \frac{\mu(\mu-1)}{\mu(\mu-2)}N \) |
| ER      | \( e^{-\mu \mu^2/k!} \) | \( \mu(\mu + 1) \) | \( \frac{\mu(\mu-1)}{\mu+1(\mu-2)}N \) |
| EN      | \( \frac{2 e^{\mu}}{\mu} \exp \left( -\frac{2k}{\mu} \right) \) | \( \frac{5}{4}\mu^2 \) | \( \frac{4(\mu-1)}{5(\mu-2)}N \) |
| BA      | \( \frac{\mu(\mu+2)}{2k(k+1)(k+2)} \) | \( \frac{\mu(\mu+2)}{4} \ln \left( \frac{\mu(\mu+2)^4N}{(\mu+4)^4} \right) \) | \( \frac{4\mu(\mu-1)N(\mu^4-4)}{\ln \left( \frac{\mu(\mu+2)^4N}{(\mu+4)^4} \right)} \) |
| CG      | \( \delta_{k,N-1} \) | \( (N-1)^2 \) | \( N \) |

Table 1. Node degree distribution \( P_k \), its second moment \( \mu_2 \) and the decay time constant of the average density of active links \( \tau \), for different networks.

### 6. Survival probability

In the last section we found that the density of active links, when averaged over many runs, decays exponentially fast to zero. In doing this average at a particular time \( t \),
Figure 4. Time evolution of the average density of active links $⟨ρ(t)⟩$ for (a) DR, (b) ER, (c) EN and (d) BA networks with average degree $μ = 8$. The open symbols correspond to networks of different sizes: $N = 1000$ (circles), $N = 5000$ (squares) and $N = 10000$ (diamonds). Solid lines are the analytical predictions from Eq. (17). The average was taken over 1000 independent realizations, starting from a uniform distribution with magnetization $m_0 = 0$.

we are considering all runs, even those that die before $t$ and, therefore, contribute with $ρ = 0$ to the average. In order to gain an insight about the evolution of a single run [25], we consider the density of active links averaged only over surviving runs $⟨ρ^{\text{surv}}(t)⟩$. If we define the survival probability $S(t)$ as the probability that the system has not reached the fully ordered state up to time $t$, then we can write $⟨ρ(t)⟩ = S(t)⟨ρ^{\text{surv}}(t)⟩$.

In the 1d random walk mapping that we discussed in section 4, $S(t)$ corresponds to the probability that the RW is still alive at time $t$, that is to say, that it has not hit the absorbing boundaries $m = ±1$ up to time $t$. If at time $t = 0$, we launch many walkers from the same position $m_0$, each of which representing an individual run, then $S(t)$ can be calculated as the fraction of surviving walkers at time $t$

\[
S(t) = \int_{-1}^{1} dm \, P(m,t).
\]

The result of this integral for symmetric initial conditions ($m_0 = 0$) is given by the series
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Figure 5. Survival probability $S$ and average density of active links in surviving runs $\langle \rho_{\text{surv}} \rangle$ vs the rescaled time $t/N$ for DR networks with degree $\mu = 4$ and sizes $N = 100$ (circles), $N = 400$ (squares) and $N = 1600$ (diamonds). Top and bottom solid lines are the analytical solutions $S(t)$ and $\langle \rho_{\text{surv}} \rangle = \langle \rho(t) \rangle / S(t)$ respectively obtained using equations (19) and (17).

(see appendix Appendix C)

$$S(t) = \sum_{l=0}^{\infty} \frac{(-1)^l(4l + 3)(2l - 1)!!}{(2l + 2)!!} \exp \left( -\frac{2(2l + 1)(l + 1) t}{\tau(\mu, N)} \right).$$

(19)

As we observe in Fig. 5, there are two regimes. For $t \ll N$, is $S(t) \approx 1$. For $t \geq N/4$, only the first term corresponding to the lowest $l$ ($l = 0$) gives a significant contribution to the series, thus neglecting the terms with $l > 0$ gives $S(t) \approx \frac{3}{2} \exp \left( -\frac{t}{\tau(\mu, N)} \right)$. For a general initial condition $m_0$, we obtain that the survival probability decays as

$$S(t) \approx \frac{3}{2} (1 - m_0^2) \exp \left( -\frac{2(\mu - 2) \mu_2 t}{(\mu - 1) \mu^2 N} \right) \text{ for } t > N.$$  

(20)

Using Eqs. (17) and (20) we finally obtain that the density of active links in surviving runs is

$$\langle \rho_{\text{surv}}(t) \rangle \approx \begin{cases} \frac{(\mu - 2)}{2(\mu - 1)} (1 - m_0^2)e^{-2t/\tau} & \text{for } t \ll N; \\ \frac{\mu - 2}{3(\mu - 1)} & \text{for } t \geq N. \end{cases}$$

(21)
We find that the system reaches in a time of order \( N \) a partially ordered steady state, in which the average density of active links is

\[
\frac{2}{3} \xi(\mu) = \frac{(\mu - 2)}{3(\mu - 1)}.
\]  

(22)

In fig. 6 we plot the average height of the plateau as a function of \( \mu \) obtained from numerical simulations on a Barabási-Albert network and a degree-regular random graph. As Eq. (22) shows, the average plateau value \( 2 \xi/3 \) is only a function of the first moment of the distribution, as long as the network is random. The plateau is also independent on the initial condition \( m_0 \), and the system size \( N \) for \( N \) large.

A natural question is about the typical size of spin domains in the stationary state, where we use the term domain to identify a set of connected nodes with the same spin. Numerical simulations reveal that the system is always composed by two large domains with opposite spin until by fluctuations one of them takes over and the system freezes. This can be explained using percolation transition arguments on random graphs. Two connected nodes belong to the same domain if the link that connects them is inert, and this happens with probability \( q = 1 - \rho \). Then, a domain that spans the system exists if \( q > q_c = \frac{1}{\kappa - 1} \), with \( \kappa = \frac{\mu_2}{\mu} \) [31]. This gives a critical density

\[
\rho_c = \frac{\mu_2 - 2\mu}{\mu_2 - \mu}.
\]  

(23)

Given that \( \mu_2 \geq \mu^2 \), we have \( \rho_c \geq \frac{\mu^2 - 2}{\mu - 1} = 2 \xi \), and because the density of active links in one realization is equal or smaller than \( \xi \) (see Fig. 3), the system remains in the
“percolated phase”, i.e., most of the nodes with the same spin are connected forming a giant domain of the order of the system size.

7. Ordering time in finite systems

A quantity of interest in the study of the voter model is the mean time to reach the fully ordered state when initially the system has magnetization \( m \). In the random walk terminology of section 4, this is equivalent to the mean exit time \( T(m) \), i.e., the time that the walker takes to reach either absorbing boundary \( m = \pm 1 \) by the first time, starting from the position \( m \). \( T(m) \) obeys the following recursion formula:

\[
T(m) = \sum_k P_k \left\{ \frac{\xi}{2} (1 - m^2) [T(m + \delta_k) + \delta t] \\
+ \frac{\xi}{2} (1 - m^2) [T(m - \delta_k) + \delta t] + [1 - \xi (1 - m^2)] [T(m) + \delta t] \right\},
\]

with boundary conditions

\[ T(-1) = T(1) = 0. \] (24)

The mean exit time starting from site \( m \) equals the probability of taking a step to a site \( m + \Delta \) times the exit time starting from this site. We then have to sum over all possible steps \( \Delta = 0, \pm \delta_k \) and add the time interval \( \delta t \) of a single step. In the continuum limit \( (\delta_k, \delta t \to 0 \text{ as } N \to \infty) \) this equation becomes

\[
\frac{d^2 T(m)}{dm^2} = -\frac{\tau}{(1 - m^2)}, \quad (25)
\]

where \( \tau \) is defined in Eq. (13). The solution to this equation is

\[
T(m) = \tau \left[ \frac{1 + m}{2} \ln \left( \frac{1 + m}{2} \right) + \frac{1 - m}{2} \ln \left( \frac{1 - m}{2} \right) \right],
\]

or, in terms of the initial density of + spins \( \sigma_+ = (1 + m)/2 \)

\[
T(\sigma_+) = -\frac{(\mu - 1)\mu^2}{(\mu - 2) \mu_2} N \left[ \sigma_+ \ln \sigma_+ + (1 - \sigma_+) \ln (1 - \sigma_+) \right]. \quad (26)
\]

This expression differs with the one obtained in work [24] by a prefactor of \( \frac{\mu - 1}{\mu - 2} \). However this factor does not seem to change the scaling of \( T(m) \) with the system size \( N \), that was found to be in good agreement with numerical simulations. In Fig. [17] we show the ordering time \( t(\sigma_+) \) as a function of the initial density of + spins, for a BA network with \( \mu = 20 \), ER and DR networks with \( \mu = 6 \).

For a fixed \( N \), Eq. (26) predicts that \( T(m) \) diverges at \( \mu = 2 \), but ordering times in the voter model are finite for finite sizes. To analyze this point, we numerically calculated \( T \) for an Erdős-Rényi network as function of \( \mu \) for initial densities \( \sigma_+ = \sigma_- = 1/2 \) (see Fig. [8]). For low values of \( \mu \), there is a fraction of nodes with zero degree that have no dynamics, thus we normalized \( T \) by the number of nodes \( N \) with degree larger than zero. As we observe in Fig. [8] when \( \mu \) decreases the analytical solution given by Eq. (26) with \( \mu_2 = \mu (\mu + 1) \) start to diverge from the numerical solution. This
disagreement might be due to the fact that our mean-field approach assumes that the system is homogeneous, and neglects every sort of fluctuations, which are important in networks with low connectivity. However, we still find that $T$ reaches a maximum at $\mu \simeq 2$, where it seems to grow faster than $N$.

8. Summary and conclusions

In this article we have presented a mean-field approach over the density of active links that provides a description of the time evolution and final states of the voter model on heterogenous networks in both infinite and finite systems. The theory gives analytical results that are in good agreement with simulations of the model and also shows the connection between previous numerical and analytical results. The relation between the density of active links $\rho$ and the density of $+$ spins $\sigma_+$ expressed in Eq. (6) allows to treat random graphs as complete graphs, and to find expressions for $\rho$ and the mean ordering time in finite systems. For large average degree values, Eq. (6) reduces to the expression
for the density of active links in complete graph. Therefore, this work confirms that disordered networks with large enough connectivity are mean-field in character for the dynamics of the voter model.

We find that when the average degree $\mu$ is smaller than 2, the system orders, while for $\mu > 2$, the average density of active links in surviving runs reaches a plateau of height $\frac{(\mu-2)}{3(\mu-1)}$. Due to fluctuations, a finite system always falls into an absorbing, fully-ordered state. The relaxation time $T$ to the final absorbing state scales with the system size $N$ and the first and second moments, $\mu$ and $\mu_2$ respectively, of the degree distribution, as $T \sim \frac{(\mu-1)\mu_2 N}{(\mu-2)\mu_2}$.

Plateaus are also found on correlated networks with some level of node degree correlations, like for instance on small-world [21, 26], even though the plateau is lower than the one predicted by our theory. It might be interesting to modified the mean-field approach to account for degree correlations that correctly reproduce the behavior in very general networks.

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Appendix A. Average density of active links

To integrate Eq. (15), we use the series expansion Eq. (14) for \( P(m, t') \) and write

\[
\langle \rho(t') \rangle = \xi \sum_{l=0}^{\infty} A_l D_l e^{-(l+1)(l+2)t'} ,
\]

(A.1)

where we define the coefficient

\[
D_l \equiv \int_{-1}^{1} dm \, (1-m^2) C_{l}^{3/2}(m).
\]

To obtain the coefficients \( A_l \), we assume that the initial magnetization is \( m(t=0) = m_0 \), i.e., \( P(m, t=0) = \delta(m - m_0) \), from where the expansion for \( P(m, t') \) becomes

\[
\sum_{l=0}^{\infty} A_l C_{l}^{3/2}(m) = \delta(m - m_0).
\]

Multiplying both sides of the above equation by \( (1-m^2) C_{l'}^{3/2}(m) \) and integrating over \( m \) gives

\[
\sum_{l=0}^{\infty} \frac{2(l+1)(l+2)}{(2l+3)} A_l \delta_{l,l'} = (1-m_0^2) C_{l'}^{3/2}(m_0)
\]

(A.2)

where we used the orthogonality relation for the Gegenbauer polynomials Eq. MS 5.3.2 (8) in page 983 of [30] with \( \lambda = 3/2 \)

\[
\int_{-1}^{1} dm \, C_{l}^{3/2}(m) C_{l'}^{3/2}(m) (1-m^2) = \frac{\pi \Gamma(l+3)}{4 l! (l+3/2) \Gamma(3/2)^2} \delta_{l,l'}
\]

(A.3)

and the identities \( \Gamma(l) = (l-1)! \), \( \Gamma(l+1) = l \Gamma(l) \) and \( \Gamma(1/2) = \sqrt{\pi} \). Then, from Eq. (A.2) we obtain

\[
A_l = \frac{(2l+3)(1-m_0^2) C_{l}^{3/2}(m_0)}{2(l+1)(l+2)}.
\]

(A.4)

To find \( D_l \), we use that the zeroth order polynomial is \( C_0^{3/2}(m) = 1 \), together with the orthogonality relation Eq. (A.3):

\[
D_l = \int_{-1}^{1} dm \, C_{l}^{3/2}(m) C_0^{3/2}(m) (1-m^2) = \frac{\pi \Gamma(l+3)}{4 l! (l+3/2) \Gamma(3/2)^2} \delta_{l,0}
\]

(A.5)

Then, using Eqns. (A.4) and (A.5) we find that the coefficients \( A_l \) and \( D_l \) are related by \( A_l D_l = (1-m_0^2) C_{l}^{3/2}(m_0) \delta_{l,0} \). Replacing this relation in Eq. (A.1) and performing the summation we finally obtain

\[
\langle \rho(t') \rangle = \xi (1-m_0^2) e^{-2t'},
\]

as quoted in Eq. (16).
Appendix B. Calculation of $\mu_2$ for Barabási-Albert networks

The Barabási-Albert network is generated by starting with a number $m$ of nodes, and adding, at each time step, a new node with $m$ links that connect to $m$ different nodes in the network. When the number of nodes in the system is $N$, the total number of links is $mN$, and therefore the average degree is $\mu = 2m$. The expression for the resulting degree distribution, calculated for instance in [32], as a function of $\mu$ is

$$P(k) = \frac{\mu(\mu + 2)}{2k(k + 1)(k + 2)},$$

and its second moment is

$$\mu_2 = \int_{\mu/2}^{k_{\text{max}}} k^2 P(k) dk = \frac{\mu(\mu + 2)}{2} \int_{\mu/2}^{k_{\text{max}}} \frac{k dk}{(k + 1)(k + 2)}$$

$$= \frac{\mu(\mu + 2)}{2} \ln \left[ \frac{2(k_{\text{max}} + 2)^2(\mu + 2)}{(k_{\text{max}} + 1)(\mu + 4)^2} \right].$$

The lower limit $\mu/2$ of the above integrals correspond to the lowest possible degree $m$, since nodes already have $m$ links when they are added to the network. The reason for an upper limit $k_{\text{max}}$ is that the contribution to $\mu_2$ from large degree terms is important due to the slow asymptotic decay $P(k) \sim k^{-3}$, unlike for instance in Erdős-Rényi or Exponential networks where $P(k)$ decays faster than $k^{-3}$, thus high degree terms become irrelevant. $k_{\text{max}}$ is estimated as the degree for which the number of nodes with degree larger than $k_{\text{max}}$ is less than one. Then

$$\frac{1}{N} = \frac{\mu(\mu + 2)}{2} \int_{k_{\text{max}}}^{\infty} \frac{dk}{k(k + 1)(k + 2)} = \frac{\mu(\mu + 2)}{4} \ln \left( \frac{(k_{\text{max}} + 1)^2}{k_{\text{max}}(k_{\text{max}} + 2)} \right).$$

Assuming $k_{\text{max}} \gg 1$, the expansion of the logarithm to first order in $1/k_{\text{max}}$ is $1/k_{\text{max}}$.

Then, solving for $k_{\text{max}}$, we obtain

$$k_{\text{max}} \simeq \sqrt{u(u + 2)/4N^{1/2}},$$

i.e, the maximum degree diverges with the system size.

Taking $k_{\text{max}} \gg 1$ in Eq. (B.2) and replacing the value of $k_{\text{max}}$ from Eq. (B.3) gives the expression quoted in table 1 for the second moment of a BA network

$$\mu_2 = \frac{\mu(\mu + 2)}{4} \ln \left( \frac{\mu(\mu + 2)^3 N}{(\mu + 4)^4} \right).$$

Appendix C. Survival probability

By using the series representation Eq. (14), the survival probability quoted in Eq. (18) can be written as

$$S(t) = \sum_{l=0}^{\infty} A_l B_l e^{-(l+1)(l+2)t'},$$

where we define

$$B_l \equiv \int_{-1}^{1} dm C_l^{3/2}(m).$$
To obtain the coefficients $B_l$, we use the derivative identity $C_{l+1}^{3/2}(m) = \frac{d}{dm}C_{l+1}^{1/2}(m)$ derived from Eq. MS 5.3.2 (1) in page 983 of [30] with $\lambda = 3/2$. Then

$$B_l = C_{l+1}^{1/2}(1) - C_{l+1}^{1/2}(-1) = 1 - (-1)^{l+1} = \begin{cases} 0 & l \text{ odd} \\ 2 & l \text{ even} \end{cases}$$ (C.3)

where we have used the relations $C_{l}^{1/2}(1) = 1 \quad \forall l$ and $C_{l}^{1/2}(-1) = (-1)^l$ that follow from Eq. MO 98 (4) (page 983) and the parity of the polynomials (page 980) of [30] respectively.

An explicit function for the coefficients $A_l$ of Eq. (A.4) can only be found for the $m_0 = 0$ case, given that for $m_0 \neq 0$ it seems that a closed expression for the polynomials $C_{l}^{3/2}(m_0)$ cannot be obtained. To obtain the coefficients $C_{l}^{3/2}(0)$ we use the recursion relation Eq. Mo 98 (4) (page 981) of [30] for $m \equiv x = 0$ and $\lambda = 3/2$, together with the values of the zeroth and first order polynomials $C_0^{3/2}(0) = 1$ and $C_1^{3/2}(0) = 0$. Then

$$C_{l}^{3/2}(0) = -\frac{(l+1)}{l} C_{l-2}^{3/2}(0) = \begin{cases} 0 & l \text{ odd} \\ (-1)^{l/2} \frac{(l+1)!!}{l!!} & l \text{ even} \end{cases}$$ (C.4)

Plugging the above expression into Eq. (A.4) gives $A_l = 0$ for $l$ odd and $A_l = \frac{(-1)^{l/2}(2l+3)(l-1)!!}{2(l+2)!!}$ for $l$ even.

Then, using Eq. (C.3), the product $A_l B_l$ can be written as

$$A_l B_l = \begin{cases} 0 & l \text{ odd} \\ \frac{(-1)^{l/2}(2l+3)(l-1)!!}{(l+2)!!} & l \text{ even} \end{cases}$$ (C.5)

Finally, making the variable change $l \rightarrow 2l$, Eq. (C.1) becomes

$$S(t') = \sum_{l=0}^{\infty} \frac{(-1)^l(4l+3)(2l-1)!!}{(2l+2)!!} e^{-2(2l+1)(l+1)t'}.$$ (C.6)

Replacing $t'$ by $t/\tau(\mu, N)$, we obtain the expression quoted in Eq. (19).