RANDOM VARIABLES IN A GRAPH $W^*$-PROBABILITY SPACE

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ABSTRACT. In [15], we constructed a $W^*$-probability space $(W^*(G), E)$ with amalgamation over a von Neumann algebra $D_G$, where $W^*(G)$ is a graph $W^*$-algebra induced by the countable directed graph $G$. In [15], we computed the $D_G$-valued moments and cumulants of arbitrary random variables in $(W^*(G), E)$ and we could characterize the $D_G$-freeness of generators of $W^*(G)$, by the so-called diagram-distinctness on $G$. In this paper, we will observe some special $D_G$-valued random variables in $(W^*(G), E)$, for instance, $D_G$-semicircular elements, $D_G$-even elements, $D_G$-valued R-diagonal elements and the generating operator of $W^*(G)$. In particular, we can get that (i) if $l$ is a loop in the graph $G$, then the random variable $L_l + L^*_l$ is $D_G$-semicircular, (ii) if $w$ is a finite path, then the random variable $L_w + L^*_w$ is $D_G$-even, (iii) if $w$ is a finite path, then the random variables $L_w$ and $L^*_w$ are $D_G$-valued R-diagonal.

In this paper, we construct the graph $W^*$-probability spaces. The graph $W^*$-probability theory is one of the good example of Speicher’s combinatorial free probability theory with amalgamation (See [16]). In this paper, we will observe how to compute the moment and cumulant of an arbitrary random variables in the graph $W^*$-probability space and the freeness on it with respect to the given conditional expectation. Also, we consider certain special random variables of the graph $W^*$-probability space, for example, semicircular elements, even elements and R-diagonal elements. This shows that the graph $W^*$-probability spaces contain the rich free probability objects.

In [10], Kribs and Power defined the free semigroupoid algebras and obtained some properties of them. Our work is highly motivated by [10]. Roughly speaking, graph $W^*$-algebras are $W^*$-topology closed version of free semigroupoid algebras. Throughout this paper, let $G$ be a countable directed graph and let $\mathcal{F}^+(G)$ be the free semigroupoid of $G$, in the sense of Kribs and Power. i.e., it is a collection of all vertices of the graph $G$ as units and all admissible finite paths, under the admissibility. As a set, the free semigroupoid $\mathcal{F}^+(G)$ can be decomposed by

$$\mathcal{F}^+(G) = V(G) \cup FP(G),$$

where $V(G)$ is the vertex set of the graph $G$ and $FP(G)$ is the set of all admissible finite paths. Trivially the edge set $E(G)$ of the graph $G$ is properly contained in

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$FP(G)$, since all edges of the graph can be regarded as finite paths with their length. We define a graph $W^*$-algebra of $G$ by

$$W^*(G) \overset{df}{=} \mathbb{C}[\{L_w, L^*_w : w \in F^+(G)\}]^w,$$

where $L_w$ and $L^*_w$ are creation operators and annihilation operators on the generalized Fock space $H_G = l^2(F^+(G))$ induced by the given graph $G$, respectively. Notice that the creation operators induced by vertices are projections and the creation operators induced by finite paths are partial isometries. We can define the $W^*$-subalgebra $D_G$ of $W^*(G)$, which is called the diagonal subalgebra by

$$D_G \overset{df}{=} \mathbb{C}[\{L_v : v \in V(G)\}]^v.$$

Then each element $a$ in the graph $W^*$-algebra $W^*(G)$ is expressed by

$$a = \sum_{w \in F^+(G:a), u_w \in \{1,*\}} p_w L_w^u \cdot \text{for } p_w \in \mathbb{C},$$

where $F^+(G:a)$ is a support of the element $a$, as a subset of the free semigroupoid $F^+(G)$. The above expression of the random variable $a$ is said to be the Fourier expansion of $a$. Since $F^+(G)$ is decomposed by the disjoint subsets $V(G)$ and $FP(G)$, the support $F^+(G:a)$ of $a$ is also decomposed by the following disjoint subsets,

$$V(G:a) = F^+(G:a) \cap V(G)$$

and

$$FP(G:a) = F^+(G:a) \cap FP(G).$$

Thus the operator $a$ can be re-expressed by

$$a = \sum_{v \in V(G:a)} p_v L_v + \sum_{w \in FP(G:a), u_w \in \{1,*\}} p_w L_w^u.$$

Notice that if $V(G:a) \neq \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Thus we have the canonical conditional expectation $E : W^*(G) \to D_G$, defined by

$$E(a) = \sum_{v \in V(G:a)} p_v L_v,$$

for all $a = \sum_{w \in F^+(G:a), u_w \in \{1,*\}} p_w L_w^u$ in $W^*(G)$. Then the algebraic pair $(W^*(G), E)$ is a $W^*$-probability space with amalgamation over $D_G$ (See [16]). It is easy to check that the conditional expectation $E$ is faithful in the sense that if $E(a^*a) = 0_{D_G}$, for $a \in W^*(G)$, then $a = 0_{D_G}$. 


For the fixed operator $a \in W^*(G)$, the support $F^+(G : a)$ of the operator $a$ is again decomposed by

$$F^+(G : a) = V(G : a) \cup FP_\circ(G : a) \cup FP_\ast(G : a),$$

with the decomposition of $FP(G : a)$,

$$FP(G : a) = FP_\circ(G : a) \cup FP_\ast(G : a),$$

where

$$FP_\circ(G : a) = \{ w \in FP(G : a) : \text{both } L_w \text{ and } L_w^\ast \text{ are summands of } a \}$$

and

$$FP_\ast(G : a) = FP(G : a) \setminus FP_\circ(G : a).$$

The above new expression plays a key role to find the $D_G$-valued moments of the random variable $a$. In fact, the summands $p_v L_v$’s and $p_w L_w + p_w^\ast L_w^\ast$, for $v \in V(G : a)$ and $w \in FP_\circ(G : a)$ act for the computation of $D_G$-valued moments of $a$. By using the above partition of the support of a random variable, we can compute the $D_G$-valued moments and $D_G$-valued cumulants of it via the lattice path model $LP_n$ and the lattice path model $LP_n^\ast$ satisfying the $*$-axis-property.

At a first glance, the computations of $D_G$-valued moments and cumulants look so abstract (See Chapter 3) and hence it looks useless. However, these computations, in particular the computation of $D_G$-valued cumulants, provides us how to figure out the $D_G$-freeness of random variables by making us compute the mixed cumulants. As applications, in the final chapter, we can compute the moment and cumulant of the operator that is the sum of $N$-free semicircular elements with their covariance $2$. If $a$ is the operator, then the $n$-th moment of $a$ is

$$\left\{ \begin{array}{ll} (2N)^{\frac{n}{2}} \cdot c_n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{array} \right.$$ 

and the $n$-th cumulant of $a$ is

$$\left\{ \begin{array}{ll} 2N & \text{if } n = 2 \\ 0 & \text{otherwise} \end{array} \right.$$ 

Based on the $D_G$-cumulant computation, we can characterize the $D_G$-freeness of generators of $W^*(G)$, by the so-called diagram-distinctness on the graph $G$. i.e., the random variables $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ if and only if $w_1$ and $w_2$ are diagram-distinct the sense that $w_1$ and $w_2$ have different diagrams on the graph $G$. Also, we could find the necessary condition for the $D_G$-freeness of two arbitrary random variables $a$ and $b$. i.e., if the supports $F^+(G : a)$ and $F^+(G : b)$ are diagram-distinct, in the sense that $w_1$ and $w_2$ are diagram distinct for all pairs
(w_1, w_2) \in \mathbb{F}^+(G : a) \times \mathbb{F}^+(G : b), \text{ then the random variables } a \text{ and } b \text{ are free over } D_G.

From Chapter 4 to Chapter 6, we will consider some special $D_G$-valued random variables in a graph $W^*$-probability space $(W^*(G), E)$. The those random variables are the basic objects to study Free Probability Theory. We can conclude that

(i) if $l$ is a loop, then $L_L + L_L^*$ is $D_G$-semicircular.

(ii) if $w$ is a finite path, then $L_L + L_L^*$ is $D_G$-even.

(iii) if $w$ is a finite path, then $L_L$ and $L_L^*$ are $D_G$-valued R-diagonal.

In Chapter 5, we consider the generating operator of the graph $W^*$-algebra $W^*(G)$. We compute the moments and cumulants of the generating operators of the one-vertex graph with $N$-edges and the circulant graph.

1. Graph $W^*$-Probability Spaces

Let $G$ be a countable directed graph and let $\mathbb{F}^+(G)$ be the free semigroupoid of $G$. i.e., the set $\mathbb{F}^+(G)$ is the collection of all vertices as units and all admissible finite paths of $G$. Let $w$ be a finite path with its source $s(w) = x$ and its range $r(w) = y$, where $x, y \in V(G)$. Then sometimes we will denote $w$ by $w = xwy$ to express the source and the range of $w$. We can define the graph Hilbert space $H_G$ by the Hilbert space $l^2(\mathbb{F}^+(G))$ generated by the elements in the free semigroupoid $\mathbb{F}^+(G)$, i.e., this Hilbert space has its Hilbert basis $B = \{\xi_w : w \in \mathbb{F}^+(G)\}$. Suppose that $w = e_1...e_k \in FP(G)$ is a finite path with $e_1,...,e_k \in E(G)$. Then we can regard $\xi_w$ as $\xi_{e_1} \otimes ... \otimes \xi_{e_k}$. So, in [10], Kribs and Power called this graph Hilbert space the generalized Fock space. Throughout this paper, we will call $H_G$ the graph Hilbert space to emphasize that this Hilbert space is induced by the graph.

Define the creation operator $L_w$, for $w \in \mathbb{F}^+(G)$, by the multiplication operator by $\xi_w$ on $H_G$. Then the creation operator $L$ on $H_G$ satisfies that

(i) $L_w = L_{xwy} = L_x L_w L_y$, for $w = xwy$ with $x, y \in V(G)$.

(ii) $L_{w_1} L_{w_2} = \begin{cases} L_{w_1 w_2} & \text{ if } w_1 w_2 \in \mathbb{F}^+(G) \\ 0 & \text{ if } w_1 w_2 \notin \mathbb{F}^+(G), \end{cases}$

for all $w_1, w_2 \in \mathbb{F}^+(G)$.

Now, define the annihilation operator $L_w^*$, for $w \in \mathbb{F}^+(G)$ by
The above definition is gotten by the following observation:

\[
< L_w \xi_h, \xi_h > = < \xi_h, \xi_h > = < \xi_h, L_w^* \xi_h > ,
\]

where \(<, >\) is the inner product on the graph Hilbert space \(H_G\). Of course, in the above formula we need the admissibility of \(w\) and \(h\) in \(\mathbb{F}^+(G)\). However, even though \(w\) and \(h\) are not admissible (i.e., \(wh \notin \mathbb{F}^+(G)\)), by the definition of \(L_w^*\), we have that

\[
< L_w \xi_h, \xi_h > = 0 = < \xi_h, 0 > = < \xi_h, L_w^* \xi_h > .
\]

Notice that the creation operator \(L\) and the annihilation operator \(L^*\) satisfy that

\[
(1.1) \quad L^*_w L_w = L_y \quad \text{and} \quad L_w L_w^* = L_x , \quad \text{for all} \quad w = xwy \in \mathbb{F}^+(G),
\]

under the weak topology, where \(x, y \in V(G)\). Remark that if we consider the von Neumann algebra \(W^*(\{L_w\})\) generated by \(L_w\) and \(L_w^*\) in \(B(H_G)\), then the projections \(L_y\) and \(L_x\) are Murray-von Neumann equivalent, because there exists a partial isometry \(L_w\) satisfying the relation (1.1). Indeed, if \(w = xwy\) in \(\mathbb{F}^+(G)\), with \(x, y \in V(G)\), then under the weak topology we have that

\[
(1.2) \quad L_w L_w^* L_w = L_w \quad \text{and} \quad L_w^* L_w L_w^* = L_w^*.
\]

So, the creation operator \(L_w\) is a partial isometry in \(W^*(\{L_w\})\) in \(B(H_G)\). Assume now that \(v \in V(G)\). Then we can regard \(v\) as \(v = vuv\). So,

\[
(1.3) \quad L_v^* L_v = L_v = L_v L_v^* = L_v^*.
\]

This relation shows that \(L_v\) is a projection in \(B(H_G)\) for all \(v \in V(G)\).

Define the graph \(W^*\)-algebra \(W^*(G)\) by

\[
W^*(G) \overset{\text{def}}{=} \mathbb{C}[\{L_w, L_w^* : w \in \mathbb{F}^+(G)\}]^w.
\]

Then all generators are either partial isometries or projections, by (1.2) and (1.3). So, this graph \(W^*\)-algebra contains a rich structure, as a von Neumann algebra. (This construction can be the generalization of that of group von Neumann algebra.)
Naturally, we can define a von Neumann subalgebra $D_G \subset W^*(G)$ generated by all projections $L_v, v \in V(G)$, i.e.

$$D_G \overset{def}{=} W^*(\{L_v : v \in V(G)\}).$$

We call this subalgebra the **diagonal subalgebra** of $W^*(G)$. Notice that $D_G = \Delta|_G \subset M|_G(\mathbb{C})$, where $\Delta|_G$ is the subalgebra of $M|_G(\mathbb{C})$ generated by all diagonal matrices. Also, notice that $1_{D_G} = \sum_{v \in V(G)} L_v = 1_{W^*(G)}$.

If $a \in W^*(G)$ is an operator, then it has the following decomposition which is called the Fourier expansion of $a$:

$$a = \sum_{w \in \mathbb{F}^+(G:a), u_w \in \{1,\ast\}} p_w L_u^w,$$

where $p_w \in C$ and $\mathbb{F}^+(G:a)$ is the support of $a$ defined by

$$\mathbb{F}^+(G:a) = \{w \in \mathbb{F}^+(G) : p_w \neq 0\}.$$ 

Remark that the free semigroupoid $\mathbb{F}^+(G)$ has its partition $\{V(G), FP(G)\}$, as a set. i.e.,

$$\mathbb{F}^+(G) = V(G) \cup FP(G) \quad \text{and} \quad V(G) \cap FP(G) = \emptyset.$$ 

So, the support of $a$ is also partitioned by

$$\mathbb{F}^+(G:a) = V(G:a) \cup FP(G:a),$$

where

$$V(G:a) \overset{def}{=} V(G) \cap \mathbb{F}^+(G:a)$$

and

$$FP(G:a) \overset{def}{=} FP(G) \cap \mathbb{F}^+(G:a).$$

So, the above Fourier expansion (1.4) of the random variable $a$ can be re-expressed by

$$a = \sum_{v \in V(G:a)} p_v L_v + \sum_{w \in FP(G:a), u_w \in \{1,\ast\}} p_w L_u^w.$$

We can easily see that if $V(G:a) \neq \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Also, if $V(G:a) = \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v = 0_{D_G}$. So, we can define the following canonical conditional expectation $E : W^*(G) \to D_G$ by
(1.6) \( E(a) = E \left( \sum_{w \in F^*(G:a)} p_w L^u_w \right) \) \( \overset{\text{def}}{=} \sum_{v \in V(G:a)} p_v L_v \),

for all \( a \in W^*(G) \). Indeed, \( E \) is a well-determined conditional expectation; it is a bimodule map satisfying that

\[ E(d) = d, \text{ for all } d \in D_G. \]

And

\[
E \left( da d' \right) = E \left( d(a_d + a_0) d' \right) = E \left( da_d d' + da_0 d' \right)
= E \left( da_d d' \right) = d \left( E(a) \right) d',
\]

for all \( d, d' \in D_G \) and \( a = a_d + a_0 \in W^*(G) \), where

\[
a_d = \sum_{v \in V(G:a)} p_v L_v \quad \text{and} \quad a_0 = \sum_{w \in F^*(G:a), u_w \in \{1, \ast\}} p_w L^u_w.\]

Also,

\[
E (a^*) = E \left( (a_d + a_0)^* \right) = E \left( a_d^* + a_0^* \right) = a_d^* = E(a)^*,
\]

for all \( a \in W^*(G) \). Here, \( a_d^* = \left( \sum_{v \in V(G:a)} p_v L_v \right)^* = \sum_{v \in V(G:a)} p_v^* L_v \) in \( D_G \).

**Definition 1.1.** Let \( G \) be a countable directed graph and let \( W^*(G) \) be the graph \( W^* \)-algebra induced by \( G \). Let \( E: W^*(G) \to D_G \) be the conditional expectation defined above. Then we say that the algebraic pair \( (W^*(G), E) \) is the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \). By the very definition, it is one of the \( W^* \)-probability space with amalgamation over \( D_G \). All elements in \( (W^*(G), E) \) are called \( D_G \)-valued random variables.

We have a graph \( W^* \)-probability space \( (W^*(G), E) \) over its diagonal subalgebra \( D_G \). We will define the following free probability data of \( D_G \)-valued random variables.

**Definition 1.2.** Let \( W^*(G) \) be the graph \( W^* \)-algebra induced by \( G \) and let \( a \in W^*(G) \). Define the \( n \)-th (\( D_G \)-valued) moment of \( a \) by

\[
E(d_1 a d_2 a \ldots d_n a), \text{ for all } n \in \mathbb{N},
\]

where \( d_1, \ldots, d_n \in D_G \). Also, define the \( n \)-th (\( D_G \)-valued) cumulant of \( a \) by

\[
k_n(d_1 a, d_2 a, \ldots, d_n a) = C^{(n)}(d_1 a \otimes d_2 a \otimes \ldots \otimes d_n a),
\]
for all \( n \in \mathbb{N} \), and for \( d_1, \ldots, d_n \in D_G \), where \( \hat{C} = (C^{(n)})_{n=1}^{\infty} \in I^c (W^*(G), D_G) \) is the cumulant multiplicative bimodule map induced by the conditional expectation \( E \), in the sense of Speicher. We define the \( n \)-th trivial moment of \( a \) and the \( n \)-th trivial cumulant of \( a \) by

\[
E(a^n) \quad \text{and} \quad k_n \left( a, a, \ldots, a \right) \}_{n\text{-times}} = C^{(n)} \left( a \otimes a \otimes \ldots \otimes a \right),
\]

respectively, for all \( n \in \mathbb{N} \).

To compute the \( D_G \)-valued moments and cumulants of the \( D_G \)-valued random variable \( a \), we need to introduce the following new definition:

**Definition 1.3.** Let \((W^*(G), E)\) be a graph \( W^* \)-probability space over \( D_G \) and let \( a \in (W^*(G), E) \) be a random variable. Define the subset \( FP_\ast(G : a) \) in \( FP(G : a) \) by

\[
FP_\ast(G : a) \overset{def}{=} \{ w \in \mathbb{F}^+(G : a) : \text{both} \; L_w \; \text{and} \; L_w^* \; \text{are summands of} \; a \}.
\]

And let \( FP_\ast^\prime(G : a) \overset{def}{=} FP(G : a) \setminus FP_\ast(G : a) \).

We already observed that if \( a \in (W^*(G), E) \) is a \( D_G \)-valued random variable, then \( a \) has its Fourier expansion \( a_d + a_0 \), where

\[
a_d = \sum_{v \in V(G:a)} p_v L_v
\]

and

\[
a_0 = \sum_{w \in FP(G:a), u \in \{1,\ast\}} p_w L_w^u.
\]

By the previous definition, the set \( FP(G : a) \) is partitioned by

\[
FP(G : a) = FP_\ast(G : a) \cup FP_\ast^\prime(G : a),
\]

for the fixed random variable \( a \) in \((W^*(G), E)\). So, the summand \( a_0 \), in the Fourier expansion of \( a = a_d + a_0 \), has the following decomposition:

\[
a_0 = a_\ast + a_{(\text{non-\ast})},
\]

where

\[
a_\ast = \sum_{l \in FP_\ast(G:a)} (p_l L_l + p_l^* L_l^*)
\]

and
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\[ a_{(non-*)} = \sum_{w \in FP^*(G:a), u_w \in \{\ast\}} p_w L_{uw}^w, \]

where $p_l$ is the coefficient of $L^*_l$ depending on $l \in FP^*(G:a)$. (There is no special meaning for the complex number $p_l$.) In general, i.e. $a_\ast = \sum_{l_1 \in FP^*(G:a)} p_{l_1} L_{l_1} + \sum_{l_2 \in FP^*(G:a)} p_{l_2} L^*_l$. But for the convenience of using notation, we will use the notation $p_l$, for the coefficient of $L^*$. For instance, let $V(G:a) = \{v_1, v_2\}$ and $FP(G:a) = \{w_1, w_2\}$ and let the random variable $a$ in $(W^*(G), E)$ be

\[ a = L_{v_1} + L_{v_2} + L^*_{w_1} + L_{w_1} + L^*_{w_2}, \]

Then we have that $a_d = L_{v_1} + L_{v_2}$, $a_\ast = L^*_{w_1} + L_{w_1}$ and $a_{(non-*)} = L^*_{w_2}$. By definition, $a_0 = a_\ast + a_{(non-*)}$.

2. $D_G$-Moments and $D_G$-Cumulants of Random Variables

Throughout this chapter, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. In this chapter, we will compute the $D_G$-valued moments and the $D_G$-valued cumulants of arbitrary random variable

\[ a = \sum_{w \in FP^*(G:a), u_w \in \{\ast\}} p_w L_{uw}^w \]

in the graph $W^*$-probability space $(W^*(G), E)$.

2.1. Lattice Path Model.

Throughout this section, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. Let $w_1, \ldots, w_n \in \mathbb{F}^+(G)$ and let $L_{w_1} \ldots L_{w_n} \in (W^*(G), E)$ be a $D_G$-valued random variable. In this section, we will define a lattice path model for the random variable $L_{w_1} \ldots L_{w_n}$. Recall that if $w = e_1 \ldots e_k \in FP(G)$ with $e_1, \ldots, e_k \in E(G)$, then we can define the length $|w|$ of $w$ by $k$. i.e., the length $|w|$ of $w$ is the cardinality $k$ of the admissible edges $e_1, \ldots, e_k$. 

Definition 2.1. Let $G$ be a countable directed graph and $F^+(G)$, the free semigroupoid. If $w \in F^+(G)$, then $L_w$ is the corresponding $D_G$-valued random variable in $(W^+(G), E)$. We define the lattice path $l_w$ of $L_w$ and the lattice path $l_w^{-1}$ of $L_w$ by the lattice paths satisfying that:

(i) the lattice path $l_w$ starts from $* = (0, 0)$ on the $\mathbb{R}^2$-plane.

(ii) if $w \in V(G)$, then $l_w$ has its end point $(0, 1)$.

(iii) if $w \in E(G)$, then $l_w$ has its end point $(1, 1)$.

(iv) if $w \in E(G)$, then $l_w^{-1}$ has its end point $(-1, -1)$.

(v) if $w \in FP(G)$ with $|w| = k$, then $l_w$ has its end point $(k, k)$.

(vi) if $w \in FP(G)$ with $|w| = k$, then $l_w^{-1}$ has its end point $(-k, -k)$.

Assume that finite paths $w_1, ..., w_s$ in $FP(G)$ satisfy that $w_1...w_s \in FP(G)$. Define the lattice path $l_{w_1,...,w_s}$ by the connected lattice path of the lattice paths $l_{w_1}, ..., l_{w_s}$, i.e., $l_{w_2}$ starts from $(k_{w_1}, k_{w_1}) \in \mathbb{R}^+$ and ends at $(k_{w_1} + k_{w_2}, k_{w_1} + k_{w_2})$, where $|w_1| = k_{w_1}$ and $|w_2| = k_{w_2}$. Similarly, we can define the lattice path $l_{w_1,...,w_s}^{-1}$ as the connected path of $l_{w_s}^{-1}, l_{w_{s-1}}^{-1}, ..., l_{w_1}^{-1}$.

Definition 2.2. Let $G$ be a countable directed graph and assume that $L_{w_1}, ..., L_{w_n}$ are generators of $(W^+(G), E)$. Then we have the lattice paths $l_{w_1}, ..., l_{w_n}$ of $L_{w_1}, ..., L_{w_n}$, respectively in $\mathbb{R}^2$. Suppose that $L_{w_1}^{u_1} ... L_{w_n}^{u_n} \neq 0_{D_G}$ in $(W^+(G), E)$, where $u_1, ..., u_n \in \{1, *, \}$. Define the lattice path $l_{w_1, ..., w_n}^{u_1, ..., u_n}$ of nonzero $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$ by the connected lattice path of $l_{w_1}^{u_1}, ..., l_{w_n}^{u_n}$, where $t_{w_j} = 1$ if $u_{w_j} = 1$ and $t_{w_j} = -1$ if $u_{w_j} = *$. Assume that $L_{w_1}^{u_1} ... L_{w_n}^{u_n} = 0_{D_G}$. Then the empty set $\emptyset$ in $\mathbb{R}^2$ is the lattice path of it. We call it the empty lattice path. By $LP_n$, we will denote the set of all lattice paths of the $D_G$-valued random variables having their forms of $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$, including empty lattice path.

Also, we will define the following important property on the set of all lattice paths:

Definition 2.3. Let $l_{w_1, ..., w_n}^{u_1, ..., u_n} \neq \emptyset$ be a lattice path of $L_{w_1}^{u_1} ... L_{w_n}^{u_n} \neq 0_{D_G}$ in $LP_n$. If the lattice path $l_{w_1, ..., w_n}^{u_1, ..., u_n}$ starts from $*$ and ends on the $*$-axis in $\mathbb{R}^+$, then we say that the lattice path $l_{w_1, ..., w_n}^{u_1, ..., u_n}$ has the $*$-axis-property. By $LP_n^*$, we will denote the set of all lattice paths having their forms of $l_{w_1, ..., w_n}^{u_1, ..., u_n}$ which have the $*$-axis-property. By little abuse of notation, sometimes, we will say that the $D_G$-valued random variable $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$ satisfies the $*$-axis-property if the lattice path $l_{w_1, ..., w_n}^{u_1, ..., u_n}$ of it has the $*$-axis-property.

The following theorem shows that finding $E (L_{w_1}^{u_1} ... L_{w_n}^{u_n})$ is checking the $*$-axis-property of $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$.
Theorem 2.1. Let $L_{w_1 \ldots w_n}^{u_1 \ldots u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable, where $u_1, \ldots, u_n \in \{1, *\}$. Then $E(L_{w_1 \ldots w_n}^{u_1 \ldots u_n}) \neq 0_{D_G}$ if and only if $L_{w_1 \ldots w_n}^{u_1 \ldots u_n}$ has the $*$-axis-property (i.e., the corresponding lattice path $l_{w_1 \ldots w_n}^{u_1 \ldots u_n}$ of $L_{w_1 \ldots w_n}^{u_1 \ldots u_n}$ is contained in $LP_n^*$. Notice that $\emptyset \notin LP_n^*$.)

Proof. $(\Leftarrow)$ Let $l = l_{w_1 \ldots w_n}^{u_1 \ldots u_n} \in LP_n^*$. Suppose that $w_1 = vv_1$ and $w_n = v_n w_n v_n'$, for $v_1, v_n, v'_n \in V(G)$. If $l$ is in $LP_n^*$, then

\begin{equation}
(2.1.1) \begin{cases}
v_1 = v_n' & \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = 1 \\
v_1 = v_n & \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = * \\
v_n' = v_n' & \text{if } u_{w_1} = * \text{ and } u_{w_n} = 1 \\
v_1' = v_n & \text{if } u_{w_1} = * \text{ and } u_{w_n} = *.
\end{cases}
\end{equation}

By the definition of lattice paths having the $*$-axis-property and by (2.1.1), if $l_{w_1 \ldots w_n}^{u_1 \ldots u_n} \in LP_n^*$, then there exists $v \in V(G)$ such that

$L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} = L_v$

where

\begin{equation}
(2.1.2) \begin{cases}
v = v_1 = v_n' & \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = 1 \\
v = v_1 = v_n & \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = * \\
v = v_n' = v_n' & \text{if } u_{w_1} = * \text{ and } u_{w_n} = 1 \\
v = v_n' = v_n & \text{if } u_{w_1} = * \text{ and } u_{w_n} = *.
\end{cases}
\end{equation}

This shows that $E(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) = L_v \neq 0_{D_G}$.

$(\Rightarrow)$ Assume that $E(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) \neq 0_{D_G}$. This means that there exists $L_v$, with $v \in V(G)$, such that

\begin{equation}
(2.1.3) \quad E(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) = L_v.
\end{equation}

Equivalently, we have that $l_{w_1 \ldots w_n}^{u_1 \ldots u_n} \in L_v$ in $W^*(G)$. Let $l = l_{w_1 \ldots w_n}^{u_1 \ldots u_n} \in LP_n$ be the lattice path of the $D_G$-valued random variable $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$. By (2.1.3), trivially, $l \neq \emptyset$, since $l$ should be the connected lattice path. Assume that the nonempty lattice path $l$ is contained in $LP_n \setminus LP_n^*$. Then, under the same conditions of (2.1.1), we have that
Therefore, by (2.1.2), there is no vertex $v$ satisfying $L_{w_1}^{n_1}...L_{w_n}^{n_n} = L_v$. This contradicts our assumption. 

By the previous theorem, we can conclude that $E(L_{w_1}^{n_1}...L_{w_n}^{n_n}) = L_v$, for some $v \in V(G)$ if and only if the lattice path $l_{w_1}^{n_1}...l_{w_n}^{n_n}$ has the $*$-axis-property (i.e., $l_{w_1}^{n_1}...l_{w_n}^{n_n} \in LP_n$).

### 2.2. $D_G$-Valued Moments and Cumulants of Random Variables.

Let $w_1, ..., w_n \in F^+(G), u_1, ..., u_n \in \{1, *\}$ and let $L_{w_1}^{u_1}...L_{w_n}^{u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable. Recall that, in the previous section, we observed that the $D_G$-valued random variable $L_{w_1}^{u_1}...L_{w_n}^{u_n} = L_v \in (W^*(G), E)$ with $v \in V(G)$ if and only if the lattice path $l_{w_1}^{u_1}...l_{w_n}^{u_n}$ of $L_{w_1}^{u_1}...L_{w_n}^{u_n}$ has the $*$-axis-property (equivalently, $l_{w_1}^{u_1}...l_{w_n}^{u_n} \in LP_n$). Throughout this section, fix a $D_G$-valued random variable $a \in (W^*(G), E)$. Then the $D_G$-valued random variable $a$ has the following Fourier expansion,

$$a = \sum_{v \in V(G; a)} p_v L_v + \sum_{l \in FP(G; a)} (p_l L_l + p_l^* L_l) + \sum_{w \in FP^*(G; a), u \in \{1, *\}} p_w L_{w}^{u_n}.$$

Let’s observe the new $D_G$-valued random variable $d_1 a d_2 a...d_n a \in (W^*(G), E)$, where $d_1, ..., d_n \in D_G$ and $a \in W^*(G)$ is given. Put

$$d_j = \sum_{v_j \in V(G; d_j)} q_{v_j} L_{v_j} \in D_G, \text{ for } j = 1, ..., n.$$

Notice that $V(G : d_j) = F^+(G : d_j)$, since $d_j \in D_G \leftrightarrow W^*(G)$. Then

$$d_1 a d_2 a...d_n a$$

$$= \left( \sum_{v_1 \in V(G; d_1)} q_{v_1} L_{v_1} \right) \left( \sum_{u_1 \in F^+(G; a), u \in \{1, *\}} p_{u_1} L_{u_1}^{u_n} \right)$$

$$\cdot \cdots \left( \sum_{v_n \in V(G; d_n)} q_{v_n} L_{v_n} \right) \left( \sum_{u_n \in F^+(G; a), u \in \{1, *\}} p_{u_n} L_{u_n}^{u_n} \right)$$

$$= \sum_{(v_1, ..., v_n) \in \Pi_{j=1}^n V(G; d_j)} (q_{v_1} ... q_{v_n})$$
\[ (L_{v_1} \left( \sum_{w_1 \in \mathbb{F}^+(G;\alpha), u_{w_1} \in \{1, *\}} p_{w_1} L_{u_{w_1}}^{w_1} \right) \] 
\[ \cdots L_{v_n} \left( \sum_{w_n \in \mathbb{F}^+(G;\alpha), u_{w_n} \in \{1, *\}} p_{w_n} L_{u_{w_n}}^{w_n} \right) ) \] 
\]

(1.2.1)

\[ = \sum_{(v_1, \ldots, v_n) \in \Pi_{j=1}^n V(G; d_j)} (q_{v_1} \cdots q_{v_n}) \] 
\[ \sum_{(w_1, \ldots, w_n) \in \mathbb{F}^+(G;\alpha)^n, u_{w_j} \in \{1, *\}} (p_{w_1} \cdots p_{w_n}) L_{v_1} L_{u_{w_1}}^{w_1} \cdots L_{v_n} L_{u_{w_n}}^{w_n}. \] 

Now, consider the random variable \( L_{v_1} L_{u_{w_1}}^{w_1} \cdots L_{v_n} L_{u_{w_n}}^{w_n} \) in the formula (1.2.1). Suppose that \( w_j = x_j w_j y_j \), with \( x_j, y_j \in V(G) \), for all \( j = 1, \ldots, n \). Then

\[ L_{v_1} L_{u_{w_1}}^{w_1} \cdots L_{v_n} L_{u_{w_n}}^{w_n} = \delta(v_1, x_1, y_1; u_{w_1}) \cdots \delta(v_n, x_n, y_n; u_{w_n}) \left( L_{v_1} L_{w_1}^{u_{w_1}} \cdots L_{v_n} L_{w_n}^{u_{w_n}} \right), \]

where

\[ \delta(v_j, x_j, y_j; u_{w_j}) = \begin{cases} 
\delta v_j, x_j & \text{if } u_{w_j} = 1 \\
\delta v_j, y_j & \text{if } u_{w_j} = * 
\end{cases} \]

for all \( j = 1, \ldots, n \), where \( \delta \) in the right-hand side is the Kronecker delta. So, the left-hand side can be understood as a (conditional) Kronecker delta depending on \( \{1, *\} \).

By (1.2.1) and (1.2.2), the \( n \)-th moment of \( \alpha \) is

\[ E(d_1 a \cdots d_n a) \]
\[ = E\left( \sum_{(v_1, \ldots, v_n) \in \Pi_{j=1}^n V(G; d_j)} \left( \Pi_{j=1}^n q_{v_j} \right) \sum_{(w_1, \ldots, w_n) \in \mathbb{F}^+(G;\alpha)^n, w_j = x_j w_j y_j, u_{w_j} \in \{1, *\}} \left( \Pi_{j=1}^n p_{w_j} \right) \left( \Pi_{j=1}^n \delta(v_j, x_j, y_j; u_{w_j}) \right) \left( L_{v_1} L_{w_1}^{u_{w_1}} \cdots L_{v_n} L_{w_n}^{u_{w_n}} \right) \right) \]
\[ = \sum_{(v_1, \ldots, v_n) \in \Pi_{j=1}^n V(G; d_j)} \left( \Pi_{j=1}^n q_{v_j} \right) \]
if the previous section, we observed that

\[ \text{Proposition 2.2.} \]

Let \( \mathbb{E} \) be given as above. Then the \( n \)-th moment of \( a \) is

\[
E (d_1 a \ldots d_n a) = \sum_{(v_1, \ldots, v_n) \in \Pi_n^G} \left( \sum_{(w_1, \ldots, w_n) \in \Phi^+(G; a)} \prod_{j=1}^n p_{w_j} \right) \left( \prod_{j=1}^n \delta_{(v_j, x_j, y_j, u_{w_j})} \right) \left( \prod_{j=1}^n \hat{E} (v_j, x_j, y_j, u_{w_j}) \right) E (L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}).
\]

Thus to compute the \( n \)-th moment of \( a \), we have to observe \( E (L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) \). In the previous section, we observed that \( E (L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) \) is nonvanishing if and only if \( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \) has the \( * \)-axis-property.

**Proposition 2.2.** Let \( a \in (W^*(G), E) \) be given as above. Then the \( n \)-th moment of \( a \) is

\[
E (d_1 a \ldots d_n a) = \sum_{(v_1, \ldots, v_n) \in \Pi_n^G} \left( \sum_{(w_1, \ldots, w_n) \in \Phi^+(G; a)} \prod_{j=1}^n p_{w_j} \right) \left( \prod_{j=1}^n \delta_{(v_j, x_j, y_j, u_{w_j})} \right) \left( \prod_{j=1}^n \hat{E} (v_j, x_j, y_j, u_{w_j}) \right) E (L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}).
\]

From now, rest of this paper, we will compute the \( D_G \)-valued cumulants of the given \( D_G \)-valued random variable \( a \). Let \( w_1, \ldots, w_n \in FP(G) \) be finite paths and \( u_1, \ldots, u_n \in \{1, \ast\} \). Then, by the Möbius inversion, we have

\[
(2.2.1)
\]

\[
k_n (L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n}) = \sum_{\pi \in NC(n)} \hat{E} (\pi) (L_{w_1}^{u_1} \otimes \ldots \otimes L_{w_n}^{u_n}) \mu (\pi, 1_n),
\]

where \( \hat{E} = (E^{(n)})_{n=1}^{\infty} \) is the moment multiplicative bimodule map induced by the conditional expectation \( E \) (See [16]) and where \( NC(n) \) is the collection of all noncrossing partition over \( \{1, \ldots, n\} \). Notice that if \( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \) does not have the \( \ast \)-axis-property, then

\[
E (L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) = 0_{D_G},
\]

by Section 2.1. Consider the noncrossing partition \( \pi \in NC(n) \) with its blocks \( V_1, \ldots, V_k \). Choose one block \( V_j = (j_1, \ldots, j_k) \in \pi \). Then we have that

\[
(2.2.2)
\]

\[
\hat{E} (\pi \mid V_j) (L_{w_1}^{u_1} \otimes \ldots \otimes L_{w_n}^{u_n}) = E (L_{w_1}^{u_1} d_{j_1} L_{w_{j_2}}^{u_{j_2}} \ldots d_{j_k} L_{w_{j_k}}^{u_{j_k}}),
\]

where
Definition 2.4. Let random variable \( L \) and let \( \pi \) be the set of all noncrossing partitions over \( \{1, \ldots, n\} \) and fix a \( D \). Let \( d_j, = \begin{cases} 1_{D_G} & \text{if there is no inner blocks between } j_{i-1} \text{ and } j_i \text{ in } V_j \\ L_{v_{j,i}} \neq 1_{D_G} & \text{if there are inner blocks between } j_{i-1} \text{ and } j_i \text{ in } V_j, \end{cases} \)

where \( v_{j1}, \ldots, v_{jk} \in V(G) \). So, again by Section 2.1, \( \hat{E}(\pi | V_j) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jk}}^{u_n}) \) is nonvanishing if and only if \( L_{w_{j1}}^{u_j} d_{j2} L_{w_{j2}}^{u_2} \cdots d_{jk} L_{w_{jk}}^{u_k} \) has the \( * \)-axis-property, for all \( j = 1, \ldots, n \).

Assume that
\[
\hat{E}(\pi | V_j) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jk}}^{u_n}) = L_{v_j},
\]
and
\[
\hat{E}(\pi | V_i) (L_{w_{i1}}^{u_i} \otimes \cdots \otimes L_{w_{in}}^{u_n}) = L_{v_i}.
\]

If \( v_j \neq v_i \), then the partition-dependent \( D \)-moment satisfies that
\[
\hat{E}(\pi) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jn}}^{u_n}) = 0_{D_G}.
\]

This says that \( \hat{E}(\pi) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jn}}^{u_n}) \neq 0_{D_G} \) if and only if there exists \( v \in V(G) \) such that
\[
\hat{E}(\pi | V_j) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jn}}^{u_n}) = L_v,
\]
for all \( j = 1, \ldots, k \).

Definition 2.4. Let \( NC(n) \) be the set of all noncrossing partition over \( \{1, \ldots, n\} \) and let \( L_{w_{j1}}^{u_j}, \ldots, L_{w_{jn}}^{u_n} \in (W^*(G), E) \) be \( D \)-valued random variables, where \( u_1, \ldots, u_n \in \{1, *\} \). We say that the \( D \)-valued random variable \( L_{w_{j1}}^{u_j} \cdots L_{w_{jn}}^{u_n} \) is \( \pi \)-connected if the \( \pi \)-dependent \( D \)-moment of it is nonvanishing, for \( \pi \in NC(n) \). In other words, the random variable \( L_{w_{j1}}^{u_j} \cdots L_{w_{jn}}^{u_n} \) is \( \pi \)-connected, for \( \pi \in NC(n) \), if
\[
\hat{E}(\pi) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jn}}^{u_n}) \neq 0_{D_G}.
\]
i.e., there exists a vertex \( v \in V(G) \) such that
\[
\hat{E}(\pi) (L_{w_{j1}}^{u_j} \otimes \cdots \otimes L_{w_{jn}}^{u_n}) = L_v.
\]

For convenience, we will define the following subset of \( NC(n) \):

Definition 2.5. Let \( NC(n) \) be the set of all noncrossing partitions over \( \{1, \ldots, n\} \) and fix a \( D \)-valued random variable \( L_{w_{j1}}^{u_j} \cdots L_{w_{jn}}^{u_n} \) in \( (W^*(G), E) \), where \( u_1, \ldots, u_n \in \{1, *\} \). For the fixed \( D \)-valued random variable \( L_{w_{j1}}^{u_j} \cdots L_{w_{jn}}^{u_n} \), define
Theorem 2.4. Let $w_1, \ldots, w_n$ be variables, where $w_i \in \mathbb{W}$. We can compute that

$$C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n} \overset{def}{=} \{ \pi \in NC(n) : L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \text{ is } \pi\text{-connected} \},$$

in $NC(n)$. Let $\mu$ be the Möbius function in the incidence algebra $I_2$. Define the number $\mu_{w_1, \ldots, w_n}$, for the fixed $D_G$-valued random variable $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$, by

$$\mu_{w_1, \ldots, w_n}^{u_1, \ldots, u_n} \overset{def}{=} \sum_{\pi \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}} \mu(\pi, 1_n).$$

Assume that there exists $\pi \in NC(n)$ such that $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} = L_\pi$ is $\pi$-connected. Then $\pi \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}$ and there exists the maximal partition $\pi_0 \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}$ such that $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} = L_{\pi_0}$ is $\pi_0$-connected. (Recall that $NC(n)$ is a lattice. We can restrict this lattice ordering on $C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}$ and hence it is a POset, again.) Notice that $1_n \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}$. Therefore, the maximal partition in $C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}$ is $1_n$. Hence we have that:

Lemma 2.3. Let $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable having the $\ast$-axis-property. Then

$$E(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) = \hat{E}(\pi)(L_{w_1}^{u_1} \otimes \ldots \otimes L_{w_n}^{u_n}),$$

for all $\pi \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}$.

Proof. By the previous discussion, we can get the result.

By the previous lemmas, we have that:

Theorem 2.4. Let $n \in 2N$ and let $L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \in (W^*(G), E)$ be $D_G$-valued random variables, where $w_1, \ldots, w_n \in FP(G)$ and $u_j \in \{1, \ast\}, j = 1, \ldots, n$. Then

$$k_n(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) = \mu_{w_1, \ldots, w_n}^{u_1, \ldots, u_n} \cdot E(L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n}),$$

where $\mu_{w_1, \ldots, w_n}^{u_1, \ldots, u_n} = \sum_{\pi \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}} \mu(\pi, 1_n)$.

Proof. We can compute that

$$k_n(L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n}) = \sum_{\pi \in NC(n)} \hat{E}(\pi)(L_{w_1}^{u_1} \otimes \ldots \otimes L_{w_n}^{u_n}) \mu(\pi, 1_n)$$

$$= \sum_{\pi \in C_{w_1, \ldots, w_n}^{u_1, \ldots, u_n}} \hat{E}(\pi)(L_{w_1}^{u_1} \otimes \ldots \otimes L_{w_n}^{u_n}) \mu(\pi, 1_n)$$
by the $\pi$-connectedness
\[
= \sum_{\pi \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}} E \left( L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \right) \mu(\pi, 1_n)
\]
by the previous lemma
\[
= \left( \sum_{\pi \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}} \mu(\pi, 1_n) \right) \cdot E \left( L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \right).
\]

Now, we can get the following $D_G$-valued cumulants of the random variable $a$ ;

**Corollary 2.5.** Let $n \in \mathbb{N}$ and let $a = a_d + a_{(n)} + a_{(non-\ast)} \in (W^*(G), E)$ be our $D_G$-valued random variable. Then $k_1 (d_1 a) = d_1 a_d$ and $k_n (d_1 a, \ldots, d_n a) = 0_{D_G}$, for all odd $n$. If $n \in \mathbb{N} \setminus \{1\}$, then
\[
k_n (d_1 a, \ldots, d_n a) = \sum_{(v_1, \ldots, v_n) \in \mathcal{P}(G; d_1) \cap \mathcal{F}^+} \left( \prod_{j=1}^n \delta_{v_j} \right)
\]
\[
\sum_{(w_1, \ldots, w_n) \in \mathcal{F} \ast (G; a)^n} \left( \prod_{j=1}^n \delta_{w_j} \right) \mu_{w_1, \ldots, w_n} \cdot E \left( L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \right),
\]
where $d_1, \ldots, d_n \in D_G$ are arbitrary. $\square$

We have the following trivial $D_G$-valued moments and cumulants of an arbitrary $D_G$-valued random variable ;

**Corollary 2.6.** Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable and let $n \in \mathbb{N}$. Then

1. The $n$-th trivial $D_G$-valued moment of $a$ is
\[
E(a^n) = \sum_{(w_1, \ldots, w_n) \in \mathcal{F}^+ (G; a)^n, \{1, \ast\}} \mu_{w_1, \ldots, w_n} \cdot E \left( L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \right).
\]
2. The $n$-th trivial $D_G$-valued cumulant of $a$ is
\[
k_1 (a) = E(a) = a_d
\]
and

\[
\kappa_n \left( a, \ldots, a \right) = \sum_{(w_1, \ldots, w_n) \in FP_n(G: a), \; w_{uj} \in \{1, *\}, \; \ell_{wu_i}^1, \ldots, \ell_{wu_n}^n \in LP_n} \left( \prod_{j=1}^n p_{w_j} \right) \left( \mu_{wu_i}^1, \ldots, \mu_{wu_n}^n \cdot E \left( L_{wu_i}^{u_1}, \ldots, L_{wu_n}^{u_n} \right) \right),
\]

where \(d_1, \ldots, d_n \in D_G\) are arbitrary. □

3. \(D_G\)-Freeness on \((W^*(G), E)\)

Like before, throughout this chapter, let \(G\) be a countable directed graph and \((W^*(G), E)\), the graph \(W^*\)-probability space over its diagonal subalgebra \(D_G\). In this chapter, we will consider the \(D_G\)-valued freeness of given two random variables in \((W^*(G), E)\). We will characterize the \(D_G\)-freeness of \(D_G\)-valued random variables \(L_{wu_1}\) and \(L_{wu_2}\), where \(w_1 \neq w_2 \in FP(G)\). And then we will observe the \(D_G\)-freeness of arbitrary two \(D_G\)-valued random variables \(a_1\) and \(a_2\) in terms of their supports.

Let

\[
(3.0) \quad a = \sum_{w \in F^+(G: a), \; u_w \in \{1, *\}} p_w L_{uw}^w \quad \& \quad b = \sum_{w' \in F^+(G: b), \; u_{w'} \in \{1, *\}} p_{w'} L_{uu'}^{w'}
\]

be fixed \(D_G\)-valued random variables in \((W^*(G), E)\).

Now, fix \(n \in \mathbb{N}\) and let \((a_{i_1}^{\varepsilon_{i_1}}, \ldots, a_{i_n}^{\varepsilon_{i_n}}) \in \{a, b, a^*, b^*\}^n\), where \(\varepsilon_{i_j} \in \{1, *\}\). For convenience, put

\[
a_{ij}^{\varepsilon_{ij}} = \sum_{w_{ij} \in F^+(G: a), \; u_j \in \{1, *\}} p_{w_{ij}}^{(j)} L_{uw_{ij}}^{u_{ij}}, \quad \text{for} \quad j = 1, \ldots, n.
\]

Then, by the little modification of Section 3, we have that:

\[
(3.1) \quad E \left( d_{i_1} a_{i_1}^{\varepsilon_{i_1}} \ldots d_{i_n} a_{i_n}^{\varepsilon_{i_n}} \right) = \sum_{(w_{ij}, \ldots, w_{in}) \in \Pi_{k=1}^n \mathcal{V}(G: d_{ik})} \left( \prod_{k=1}^n p_{ku_k} \right)
\]
Proposition 3.1. Let \( \varepsilon_{i_j} \in \{j \} \mathcal{W} \mu_{i_j} \) where \( 1 \leq k \leq n \). Therefore, we have that

\[
\left( \prod_{j=1}^{n} \delta(u_{i_j}, x_{i_j}, y_{i_j}; u_{i_j}) \right) E \left( L_{u_{i_1}}^{u_{i_n}} \ldots L_{w_{i_1}}^{w_{i_n}} \right).
\]

Therefore, we have that

(3.2)

\[
k_n \left( d_{i_1} a_{i_1}^{\varepsilon_{i_1}}, \ldots, d_{i_n} a_{i_n}^{\varepsilon_{i_n}} \right) = \sum_{(u_{i_1}, \ldots, u_{i_n}) \in \prod_{k=1}^{n} V(G; d_k)} \left( \prod_{k=1}^{n} q_{u_k} \right)
\]

\[
\sum_{(w_{i_1}, \ldots, w_{i_n}) \in \prod_{k=1}^{n} FP(G; a_{i_k}) \cup W^* \cup \ldots \cup \mu_{a_{i_1}}^{u_{i_1}} \ldots \mu_{a_{i_n}}^{u_{i_n}}} \left( \prod_{k=1}^{n} p_{u_k}^{(k)} \right)
\]

\[
\left( \prod_{j=1}^{n} \delta(u_{i_j}, x_{i_j}, y_{i_j}; u_{i_j}) \right) \left( \mu_{a_{i_1}}^{u_{i_1}} \ldots \mu_{a_{i_n}}^{u_{i_n}} \cdot \text{Proj} \left( L_{u_{i_1}}^{u_{i_n}} \ldots L_{w_{i_1}}^{w_{i_n}} \right) \right)
\]

where \( \mu_{a_{i_1}}^{u_{i_1}} \ldots \mu_{a_{i_n}}^{u_{i_n}} = \sum_{\pi \in C_{u_{i_1}}^{u_{i_1}} \ldots C_{w_{i_n}}^{w_{i_n}}} \mu(\pi, 1_n) \) and

\[
C_{u_{i_1}}^{u_{i_1}} \ldots C_{w_{i_n}}^{w_{i_n}} = \{ \pi \in NC(\text{even}) (n) : L_{u_{i_1}}^{u_{i_n}} \ldots L_{w_{i_n}}^{w_{i_n}} \text{ is } \pi\text{-connected} \}.
\]

So, we have the following proposition, by the straightforward computation:

Proposition 3.1. Let \( a, b \in (W^*(G), E) \) be \( D_G \)-valued random variables, such that \( a \notin W^* \{b\}, D_G \), and let \( (a_{i_1}^{\varepsilon_{i_1}}, \ldots, a_{i_n}^{\varepsilon_{i_n}}) \in \{a, b, a^*, b^*\}^n \), for \( n \in \mathbb{N} \setminus \{1\} \), where \( \varepsilon_{i_j} \in \{1, *, \} \), \( j = 1, \ldots, n \). Then

(3.3)

\[
k_n \left( d_{i_1} a_{i_1}^{\varepsilon_{i_1}}, \ldots, d_{i_n} a_{i_n}^{\varepsilon_{i_n}} \right)
\]

\[
= \sum_{(v_{i_1}, \ldots, v_{i_n}) = (x, y, \ldots, x, y) \in \prod_{j=1}^{n} V(G; d_j)} \left( \prod_{j=1}^{n} q_{v_j} \right)
\]

\[
\sum_{(w_{i_1}, \ldots, w_{i_n}) \in \prod_{k=1}^{n} FP(G; a_{i_k}) \cup W^* \cup \ldots \cup \mu_{a_{i_1}}^{u_{i_1}} \ldots \mu_{a_{i_n}}^{u_{i_n}}} \left( \prod_{k=1}^{n} p_{u_k}^{(k)} \right)
\]

\[
\left( \prod_{j=1}^{n} \delta(v_{i_j}, x_{i_j}, y_{i_j}; v_{i_j}) \right) \left( \mu_{a_{i_1}}^{u_{i_1}} \ldots \mu_{a_{i_n}}^{u_{i_n}} \cdot \text{Proj} \left( L_{u_{i_1}}^{u_{i_n}} \ldots L_{w_{i_1}}^{w_{i_n}} \right) \right)
\]

where \( \mu_n = \sum_{\pi \in C_{u_{i_1}}^{u_{i_1}} \ldots C_{w_{i_n}}^{w_{i_n}}} \mu(\pi, 1_n) \) and
\[ W_{w_1, \ldots, w_n} = \{ w \in FP^c_x(G : a) \cup FP^c_y(G : b) : \text{both } L_{w_1}^{u_1} \text{ and } L_{w_n}^{u_n} \text{ are in } L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \}. \]

\[ \square \]

**Corollary 3.2.** Let \( x \) and \( y \) be the \( D_G \)-valued random variables in \((W^*(G), E)\). The \( D_G \)-valued random variables \( a \) and \( b \) are free over \( D_G \) in \((W^*(G), E)\) if

\[ FP_x(G : P(x, x^*)) \cap FP_y(G : Q(y, y^*)) = \emptyset \]

and

\[ W_x^{P(x, x^*), Q(y, y^*)} = \emptyset, \]

for all \( P, Q \in C[z_1, z_2] \). \( \square \)

By using (3.2), we can compute the mixed \( D_G \)-valued cumulants of two \( D_G \)-valued random variables. However, the formula is very abstract. So, we will consider the above formula for fixed two generators of \( W^*(G) \).

**Definition 3.1.** Let \( G \) be a countable directed graph and \( F^+(G) \), the free semigroupoid of \( G \) and let \( FP(G) \) be the subset of \( F^+(G) \) consisting of all finite paths. Define a subset \( \text{loop}(G) \) containing all loop finite paths or loops. (Remark that, in general, loop finite paths are different from loop-edges. Clearly, all loop-edges are loops in \( FP(G) \).) i.e.,

\[ \text{loop}(G) \overset{\text{def}}{=} \{ l \in FP(G) : l \text{ is a loop} \} \subset FP(G). \]

Also define the subset \( \text{loop}^c(G) \) of \( FP(G) \) consisting of all non-loop finite path by

\[ \text{loop}^c(G) \overset{\text{def}}{=} FP(G) \setminus \text{loop}(G). \]

Let \( l \in \text{loop}(G) \) be a loop finite path. We say that \( l \) is a **basic loop** if there exists no loop \( w \in \text{loop}(G) \) such that \( l = w^k \), \( k \in \mathbb{N} \setminus \{1\} \). Define

\[ \text{Loop}(G) \overset{\text{def}}{=} \{ l \in \text{loop}(G) : l \text{ is a basic loop} \} \subseteq \text{loop}(G). \]

Let \( l_1 = w_1^{k_1} \) and \( l_2 = w_2^{k_2} \) in \( \text{loop}(G) \), where \( w_1, w_2 \in \text{Loop}(G) \). We will say that the loops \( l_1 \) and \( l_2 \) are **diagram-distinct** if \( w_1 \neq w_2 \) in \( \text{Loop}(G) \). Otherwise, they are not diagram-distinct.

Now, we will introduce the more general diagram-distinctness of general finite paths ;

**Definition 3.2.** (Diagram-Distinctness) We will say that the finite paths \( w_1 \) and \( w_2 \) are **diagram-distinct** if \( w_1 \) and \( w_2 \) have different diagrams in the graph.
Suppose that Lemma 3.3.

Let \( H \) be a directed graph with \( V(H) = \{v_1, v_2\} \) and \( E(H) = \{e_1 = v_1e_1v_2, e_2 = v_2e_2v_1\} \). Then \( l = e_1e_2 \) is a loop in \( FP(H) \) (i.e., \( l \in \text{loop}(H) \)). Moreover, it is a basic loop (i.e., \( l \in \text{Loop}(H) \)). However, if we have a loop \( w = e_1e_2e_1e_2 = l^2 \), then it is not a basic loop. i.e.,

\[
l^2 \in \text{loop}(H) \setminus \text{Loop}(H).
\]

If the graph \( G \) contains at least one basic loop \( l \in FP(G) \), then we have

\[
\{l^n : n \in \mathbb{N}\} \subseteq \text{loop}(G) \quad \text{and} \quad \{l\} \subseteq \text{Loop}(G).
\]

Suppose that \( l_1 \) and \( l_2 \) are not diagram-distinct. Then, by definition, there exists \( w \in \text{Loop}(G) \) such that \( l_1 = w^{k_1} \) and \( l_2 = w^{k_2} \), for some \( k_1, k_2 \in \mathbb{N} \). On the graph \( G \), indeed, \( l_1 \) and \( l_2 \) make the same diagram. On the other hands, we can see that if \( w_1 \neq w_2 \in \text{loop}^c(G) \), then they are automatically diagram-distinct.

**Lemma 3.3.** Suppose that \( w_1 \neq w_2 \in \text{loop}^c(G) \) with \( w_1 = v_{11}w_1v_{12} \) and \( w_2 = v_{21}w_2v_{22} \). Then \( L_{w_1} \) and \( L_{w_2} \) are free over \( D_G \) in \( (W^*(G), E) \).

**Proof.** By definition, \( L_{w_1} \) and \( L_{w_2} \) are free over \( D_G \) if and only if all mixed \( D_G \)-valued cumulants of \( W^*\{L_{w_1}\}, D_G \) and \( W^*\{L_{w_2}\}, D_G \) vanish. Equivalently, all \( D_G \)-valued cumulants of \( P(L_{w_1}, L_{w_1}^*) \) and \( Q(L_{w_2}, L_{w_2}^*) \) vanish, for all \( P, Q \in \mathbb{C}[z_1, z_2] \). Since \( w_1 \neq w_2 \) are non-loop edges, we can easily verify that \( w_1^{k_1} \) and \( w_2^{k_2} \) are not admissible (i.e., \( w_1^{k_1} \notin \mathbb{F}^+(G) \) and \( w_2^{k_2} \notin \mathbb{F}^+(G) \)), for all \( k_1, k_2 \in \mathbb{N} \setminus \{1\} \). This shows that

\[
L_{w_j}^k = 0_{D_G} = \left(L_{w_j}^1\right)^{k-1}, \quad \text{for} \ j = 1, 2.
\]

Thus, to show that \( L_{w_1} \) and \( L_{w_2} \) are free over \( D_G \), it suffices to show that all mixed \( D_G \)-valued cumulants of \( P(L_{w_1}, L_{w_1}^*) \) and \( Q(L_{w_2}, L_{w_2}^*) \) vanish, for all \( P, Q \in \mathbb{C}[z_1, z_2] \) such that

\[
P(z_1, z_2) = \alpha_1z_1 + \alpha_2z_1z_2 + \alpha_3z_2z_1 + \alpha_4z_2
\]

and

\[
Q(z_1, z_2) = \beta_1z_1 + \beta_2z_1z_2 + \beta_3z_2z_1 + \beta_4z_2,
\]

where \( \alpha, \beta \in \mathbb{C} \). So, for such \( P \) and \( Q \), we have that

\[
P(L_{w_1}, L_{w_1}^*) = \alpha_1L_{w_1} + \alpha_2L_{w_{11}} + \alpha_3L_{w_{12}} + \alpha_4L_{w_1}^*
\]

and

\[
Q(L_{w_2}, L_{w_2}^*) = \beta_1L_{w_2} + \beta_2L_{w_{21}} + \beta_3L_{w_{22}} + \beta_4L_{w_2}^*.
\]
Thus, we have that
\[ FP_\ast (G : P(L_{w_1}, L_{w_1}^*) ) \supseteq \{ w_1 \}, \quad FP_\ast (G : Q(L_{w_2}, L_{w_2}^*) ) \supseteq \{ w_2 \} \]

and
\[ FP_\ast^c (G : P(L_{w_1}, L_{w_1}^*) ) \supseteq \{ w_1 \}, \quad FP_\ast^c (G : Q(L_{w_2}, L_{w_2}^*) ) \supseteq \{ w_2 \}. \]

Remark that if \( FP_\ast (G : P(L_{w_1}, L_{w_1}^*) ) = \{ w_1 \} \), then \( FP_\ast^c (G : P(L_{w_1}, L_{w_1}^*) ) = \emptyset \), and if \( FP_\ast (G : P(L_{w_1}, L_{w_1}^*) ) = \{ w_1 \} \), then \( FP_\ast (G : P(L_{w_1}, L_{w_1}^*) ) = \emptyset \). The similar relation holds for \( Q(L_{w_2}, L_{w_2}^*) \). So, we have that
\[ FP_\ast (G : P(L_{w_1}, L_{w_1}^*) ) \cap FP_\ast (G : Q(L_{w_2}, L_{w_2}^*) ) = \emptyset \]
and
\[ W_\ast (p(L_{w_1}, L_{w_1}^*), q(L_{w_2}, L_{w_2}^*)) = \emptyset. \]

Therefore, by the formula (3.4.3), we have the vanishing mixed \( D_G \)-valued cumulants of \( P(L_{w_1}, L_{w_1}^*) \) and \( Q(L_{w_2}, L_{w_2}^*) \), for all \( n \in \mathbb{N} \) and for all such \( P, Q \in \mathbb{C}[z_1, z_2] \). So, we can conclude that \( L_{w_1} \) and \( L_{w_2} \) are free over \( D_G \) in \( (W^*(G), E) \).

Now, we will consider the loop case.

**Lemma 3.4.** Let \( l_1 \neq l_2 \in Loop(G) \) be basic loops such that \( l_1 = v_1 l_1 v_1 \) and \( l_2 = v_2 l_2 v_2 \), for \( v_1, v_2 \in V(G) \) (possibly \( v_1 = v_2 \)). i.e., two basic loops \( l_1 \) and \( l_2 \) are diagram-distinct. Then the \( D_G \)-valued random variables \( L_{l_1} \) and \( L_{l_2} \) are free over \( D_G \) in \( (W^*(G), E) \).

**Proof.** Different from the non-loop case, if \( l_1 \) and \( l_2 \) are loops, then \( l_1^{k_1} \) and \( l_2^{k_2} \) exist in \( FP(G) \), for all \( k_1, k_2 \in \mathbb{N} \). To show that \( L_{l_1} \) and \( L_{l_2} \) are free over \( D_G \), it suffices to show that all mixed \( D_G \)-valued cumulants of \( P(L_{w_1}, L_{w_1}^*) \) and \( Q(L_{w_2}, L_{w_2}^*) \) vanish, for all \( P, Q \in \mathbb{C}[z_1, z_2] \), such that
\[ P(z_1, z_2) = f_1(z_1) + f_2(z_2) + P_0(z_1, z_2) \]
and
\[ Q(z_1, z_2) = g_1(z_1) + g_2(z_2) + Q_0(z_1, z_2), \]
where \( f_1, f_2, g_1, g_2 \in \mathbb{C}[z] \) and \( P_0, Q_0 \in \mathbb{C}[z_1, z_2] \) such that \( P_0 \) and \( Q_0 \) does not contain polynomials only in \( z_1 \) and \( z_2 \). So, for such \( P \) and \( Q \), we have that
\[ P(L_{l_1}, L_{l_1}^*) = f_1(L_{l_1}) + f_2(L_{l_1}^*) + P_0(L_{l_1}, L_{l_1}^*) \]
and
\[ Q(L_{l_2}, L_{l_2}^*) = g_1(L_{l_2}) + g_2(L_{l_2}^*) + Q_0(L_{l_2}, L_{l_2}^*). \]
Notice that \( L_{l_j}^k = L_{l_j}^{k_j} \), for all \( k \in \mathbb{N}, \ j = 1, 2 \). Also, notice that
\[ P_0(L_{l_1}, L^*_1) = f^0_1(L_{l_1}) + f^0_2(L^*_1) + \alpha L_{v_1} \]
and
\[ Q_0(L_{l_2}, L^*_2) = g^0_1(L_{l_2}) + g^0_2(L^*_2) + \beta L_{v_2}, \]
where \( f^0_1, f^0_2, g^0_1, g^0_2 \in \mathbb{C}[z] \) and \( \alpha, \beta \in \mathbb{C} \), by the fact that
\[ L^*_j L_{l_j} = L_{v_j} = L_{l_j} L^*_j, \]
under the weak-topology. So, finally, we have that
\[ P(L_{l_1}, L^*_1) = f_1(L_{l_1}) + f_2(L^*_1) + \alpha L_{v_1} \]
and
\[ Q(L_{l_2}, L^*_2) = g_1(L_{l_2}) + g_2(L^*_2) + \beta L_{v_2}, \]
where \( f_1, f_2, g_1, g_2 \in \mathbb{C}[z] \) and \( \alpha, \beta \in \mathbb{C} \). Thus, we have that
\[ FP_*(G : P(L_{l_1}, L^*_1)) \subseteq \{l^k_1\}_{k=1}^\infty, \quad FP_*(G : Q(L_{l_2}, L^*_2)) \subseteq \{l^k_2\}_{k=1}^\infty \]
and
\[ FP^c_*(G : P(L_{l_1}, L^*_1)) \subseteq \{l^k_1\}_{k=1}^\infty, \quad FP^c_*(G : Q(L_{l_2}, L^*_2)) \subseteq \{l^k_2\}_{k=1}^\infty. \]
So, we have that
\[ FP_*(G : P(L_{w_1}, L^*_1)) \cap FP_*(G : Q(L_{w_2}, L^*_2)) = \emptyset, \]
because \( l_1 \) and \( l_2 \) are in Loop(G) (and hence if \( l_1 \neq l_2 \), then they are diagram-distinct.) And we have that
\[ W^*_*(P(L_{w_1}, L^*_1), Q(L_{w_2}, L^*_2)) = \emptyset. \]

Therefore, by the formula (3.4.3), we have the vanishing mixed \( D_G \)-valued cumulants of \( P(L_{l_1}, L^*_1) \) and \( Q(L_{l_2}, L^*_2) \), for all \( n \in \mathbb{N} \) and for all \( P, Q \in \mathbb{C}[z_1, z_2] \). Since \( P \) and \( Q \) are arbitrary, we can conclude that \( L_{l_1} \) and \( L_{l_2} \) are free over \( D_G \) in \( (W^*(G), E) \).

Notice that we assumed that the loops \( l_1 \) and \( l_2 \) are basic loops in the previous lemma. Since they are distinct basic loops, they are automatically diagram-distinct. Now, assume that \( l_1 \) and \( l_2 \) are not diagram-distinct. i.e., there exists a basic loop \( w \in \text{Loop}(G) \) such that \( l_1 = w^{k_1} \) and \( l_2 = w^{k_2} \), for some \( k_1, k_2 \in \mathbb{N} \). In other words, the loops \( l_1 \) and \( l_2 \) have the same diagram in the graph \( G \). Then the \( D_G \)-valued random variables \( L_{l_1} \) and \( L_{l_2} \) are not free over \( D_G \) in \( (W^*(G), E) \). See the next example;
**Example 3.1.** Let $G_1$ be a directed graph with $V(G_1) = \{v\}$ and $E(G_1) = \{l = vlv\}$. So, in this case,

\[ E(G_1) = \text{Loop}(G_1), \quad FP(G_1) = \text{loop}(G_1), \]

and

\[ \text{loop}(G_1) = \{l^k : k \in \mathbb{N}\}. \]

Thus, even if $w_1 \neq w_2 \in \text{loop}(G_1)$, $w_1$ and $w_2$ are not diagram-distinct. Take $l^2$ and $l^3$ in $FP(G_1)$. Then the $D_{G_1}$-valued random variable $L_{i^2}$ and $L_{i^3}$ are not free over $D_{G_1}$ in $(W^*(G_1), E)$. Indeed, let’s take $P, Q \in \mathbb{C}[z_1, z_2]$ as

\[ P(z_1, z_2) = z_1^3 + z_2^3 \quad \text{and} \quad Q(z_1, z_2) = z_1^2 + z_2^2. \]

Then

\[ P \left( L_{i^2}, L_{i^3} \right) = L_{i^2}^3 + L_{i^3}^3 = L_{i^6} + L_{i^6}^* \]

and

\[ Q \left( L_{i^2}, L_{i^3}^* \right) = L_{i^3}^2 + L_{i^3}^{*2} = L_{i^6} + L_{i^6}^*. \]

Then

\[
k_2 \left( P(L_{i^2}, L_{i^3}^*), Q(L_{i^3}, L_{i^3}) \right) = k_2 \left( L_{i^6} + L_{i^6}^*, L_{i^6} + L_{i^6}^* \right) \]

\[ = \mu_{i^6, i^6}^{1,*, 1} \text{Pr o}j \left( L_{i^6}, L_{i^6}^* \right) + \mu_{i^6, i^6}^{*, 1, 1} \text{Pr o}j \left( L_{i^6}^*, L_{i^6} \right) \]

\[ = \mu_{i^6, i^6}^{1,*, 1}L_v + \mu_{i^6, i^6}^{*, 1, 1}L_v = \left( \mu_{i^6, i^6}^{1,*, 1} + \mu_{i^6, i^6}^{*, 1, 1} \right) L_v \]

\[ = 2L_v \neq 0_{D_{G}}, \]

since $\mu_{i^6, i^6}^{1,*, 1} = \mu(1_2, 1_2) = 1 = \mu_{i^6, i^6}^{*, 1, 1}$. This says that there exists at least one nonvanishing mixed $D_{G_1}$-valued cumulant of $W^* \left( \{L_{i^3} \}, D_{G_1} \right)$ and $W^* \left( \{L_{i^2} \}, D_{G_1} \right)$. Therefore, $L_{i^5}$ and $L_{i^2}$ are not free over $D_{G_1}$ in $(W^*(G_1), E)$. \hfill \Box

As we have seen before, if two loops $l_1$ and $l_2$ are not diagram-distinct, then $D_{G}$-valued random variables $L_{l_1}$ and $L_{l_2}$ are not free over $D_{G}$. However, if $l_1$ and $l_2$ are diagram-distinct, we can have the following lemma, by the previous lemma;

**Lemma 3.5.** Let $l_1 \neq l_2 \in \text{loop}(G)$ be loops and assume that $l_1 = w_1^{k_1}$ and $l_2 = w_2^{k_2}$, where $w_1, w_2 \in \text{Loop}(G)$ are basic loops and $k_1, k_2 \in \mathbb{N}$. If $w_1 \neq w_2 \in \text{Loop}(G)$, then the $D_{G}$-valued random variables $L_{l_1}$ and $L_{l_2}$ are free over $D_{G}$ in $(W^*(G), E)$. \hfill \Box

Finally, we will observe the following case when we have a loop and a non-loop finite path.

**Lemma 3.6.** Let $l \in \text{loop}(G)$ and $w \in \text{loop}^c(G)$. Then the $D_{G}$-valued random variables $L_l$ and $L_w$ are free over $D_{G}$ in $(W^*(G), E)$. 

Proof. Let \( l \in \text{loop}(G) \) and \( w \in \text{loop}^{c}(G) \) and let \( L_{l} \) and \( L_{w} \) be the corresponding \( D_{G} \)-valued random variables in \((W^{*}(G), E)\). Then, for all \( P, Q \in \mathbb{C}[z_{1}, z_{2}] \), we have that

\[
FP_{\ast}(G : P(L_{l}, L_{l}^{*})) \cap FP_{\ast}(G : Q(L_{w}, L_{w}^{*})) = \emptyset,
\]

since

\[
FP_{\ast}(G : P(L_{l}, L_{l}^{*})) \subseteq \{ t^{k} : k \in \mathbb{N} \} \subset \text{loop}(G)
\]

and

\[
FP_{\ast}(G : Q(L_{w}, L_{w}^{*})) = \{ w \} \subset \text{loop}^{c}(G).
\]

Also, since \( \text{loop}(G) \cap \text{loop}^{c}(G) = \emptyset \), we can get that

\[
W_{\ast}^{\{P(L_{l}, L_{l}^{*}), Q(L_{w}, L_{w}^{*})\}} = \emptyset,
\]

for all \( P, Q \in \mathbb{C}[z_{1}, z_{2}] \). Therefore, the \( D_{G} \)-valued random variables \( L_{l} \) and \( L_{w} \) are free over \( D_{G} \) in \((W^{*}(G), E)\).}

Now, we can summarize the above lemmas in this section as follows and this theorem is one of the main result of this paper. The theorem is the characterization of \( D_{G} \)-freeness of generators of \( W^{*}(G) \) over \( D_{G} \).

**Theorem 3.7.** Let \( w_{1}, w_{2} \in FP(G) \) be finite paths. The \( D_{G} \)-valued random variables \( L_{w_{1}} \) and \( L_{w_{2}} \) in \((W^{*}(G), E)\) are free over \( D_{G} \) if and only if \( w_{1} \) and \( w_{2} \) are diagram-distinct.

Proof. \( \Leftarrow \) Suppose that finite paths \( w_{1} \) and \( w_{2} \) are diagram-distinct. Then the \( D_{G} \)-valued random variables \( L_{w_{1}} \) and \( L_{w_{2}} \) are free over \( D_{G} \), by the previous lemmas.

\( \Rightarrow \) Let \( L_{w_{1}} \) and \( L_{w_{2}} \) be free over \( D_{G} \) in \((W^{*}(G), E)\). Now, assume that \( w_{1} \) and \( w_{2} \) are not diagram-distinct. We will observe the following cases:

(Case I) The finite paths \( w_{1}, w_{2} \in \text{loop}(G) \). Since they are not diagram-distinct, there exists a basic loop \( l = vlv \in \text{Loop}(G) \), with \( v \in V(G) \), such that \( w_{1} = l^{k_{1}} \) and \( w_{2} = l^{k_{2}} \), for some \( k_{1}, k_{2} \in \mathbb{N} \). As we have seen before, \( L_{w_{1}} \) and \( L_{w_{2}} \) are not free over \( D_{G} \) in \((W^{*}(G), E)\). Indeed, if we let \( k \in \mathbb{N} \) such that \( k_{1} \mid k \) and \( k_{2} \mid k \) with \( k = k_{1}n_{1} = k_{2}n_{2} \), for \( n_{1}, n_{2} \in \mathbb{N} \), then we can take \( P, Q \in \mathbb{C}[z_{1}, z_{2}] \) defined by

\[
P(z_{1}, z_{2}) = z_{1}^{n_{1}} + z_{2}^{n_{1}} \quad \text{and} \quad Q(z_{1}, z_{2}) = z_{1}^{n_{2}} + z_{2}^{n_{2}}.
\]

And then

\[
P(L_{w_{1}}, L_{w_{1}}^{*}) = L_{w_{1}}^{n_{1}} + L_{w_{1}}^{* n_{1}} = L_{l^{k_{1}}}^{n_{1}} + L_{l^{k_{1}}}^{* n_{1}} = L_{l^{k}} + L_{l}^{*}
\]
Recall that we say that the two subsets $X$ are diagram-distinct if the necessary condition for the distinctness of finite paths determine the $D$ and $L$-freeness of two partial isometries. Then by the previous freeness characterization, finite paths $w_1, w_2$ are not free over $D_G$. This shows that $W^*\{L_{w_1}\}, D_G$ and $W^*\{L_{w_2}\}, D_G$ are not free over $D_G$ in $(W^*(G), E)$ and hence $L_{w_1}$ and $L_{w_2}$ are not free over $D_G$. This contradicts our assumption.

(Case II) Suppose that the finite paths $w_1, w_2$ are non-loop finite paths in $\text{loop}^c(G)$ and assume that they are not diagram-distinct. Since they are not diagram-distinct, they are identically equal. Therefore, they are not free over $D_G$ in $(W^*(G), E)$.

(Case III) Let $w_1 \in \text{loop}(G)$ and $w_2 \in \text{loop}^c(G)$. They are always diagram-distinct.

Let $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ and assume that $w_1$ and $w_2$ are not diagram-distinct. Then $L_{w_1}$ and $L_{w_2}$ are not free over $D_G$, by the Case I, II and III. So, this contradicts our assumption. ■

The previous theorem characterize the $D_G$-freeness of two partial isometries $L_{w_1}$ and $L_{w_2}$, where $w_1, w_2 \in FP(G)$. This characterization shows us that the diagram-distinctness of finite paths determine the $D_G$-freeness of corresponding creation operators.

Let $a$ and $b$ be the given $D_G$-valued random variables in (3.0). We can get the necessary condition for the $D_G$-freeness of $a$ and $b$, in terms of their supports. Recall that we say that the two subsets $X_1$ and $X_2$ of $FP(G)$ are said to be diagram-distinct if $x_1$ and $x_2$ are diagram-distinct, for all pairs $(x_1, x_2) \in X_1 \times X_2$.

**Theorem 3.8.** Let $a, b \in (W^*(G), E)$ be $D_G$-valued random variables with their supports $F^+(G : a)$ and $F^+(G : b)$. The $D_G$-valued random variables $a$ and $b$ are free over $D_G$ in $(W^*(G), E)$ if $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct.

**Proof.** For convenience, let’s denote $a$ and $b$ by $a_1$ and $a_2$, respectively. Assume that the supports of $a_1$ and $a_2$, $F^+(G : a_1)$ and $F^+(G : a_2)$ are diagram-distinct. Then by the previous $D_G$-freeness characterization,
are free over $D_G$ in $(W^*(G), E)$. Indeed, since $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct, all summands $L_{u_1}$’s of $a_1$ and $L_{u_2}$’s of $a_2$ are free over $D_G$ in $(W^*(G), E)$. Therefore, $a_1$ and $a_2$ are free over $D_G$ in $(W^*(G), E)$. 

4. $D_G$-valued Semicircular Elements

Throughout this chapter, we will consider the $D_G$-valued semicircularity. Let $B$ be a von Neumann algebra and $A$, a $W^*$-algebra over $B$ and let $E : A \to B$ be a conditional expectation. Then $(A, E)$ be an amalgamated $W^*$-probability space over $B$. We say that the $D_G$-valued random variable $x \in (A, E)$ is a $B$-valued semicircular element if it is self-adjoint and the only nonvanishing $B$-valued cumulants is the second one. i.e., $x \in (A, E)$ is $B$-semicircular if $x$ is self-adjoint in $A$ and

$$k_n(x, \ldots, x) = \begin{cases} k_2(x, x) \neq 0_B & \text{if } n = 2 \\ 0_B & \text{otherwise.} \end{cases}$$

Let $G$ be a directed graph with $V(G) = \{v\}$ and $E(G) = \{e = vev\}$. i.e., $G$ is a one-vertex graph with one loop edge. Canonically, we can construct the graph $W^*$-algebra $W^*(G)$ and its diagonal subalgebra $D_G \simeq \Delta_1 = \mathbb{C}$. So, the canonical conditional expectation $E$ is a linear functional. Moreover, it is a trace on $W^*(G)$. Notice that, by Voiculescu,

$$W^*(G) \simeq L(\mathbb{Z}),$$

where $L(\mathbb{Z})$ is the free group factor. Moreover, the random variable $L_e + L_e^*$ is the Voiculescu’s semicircular element. Therefore, the only nonvanishing cumulants of $L_e + L_e^*$ is the second one. First, we will consider the following combinatorial fact. This is crucial to consider the $D_G$-semicircularity on $(W^*(G), E)$.

**Lemma 4.1.** Let $G$ and $e$ be given as above. Then

$$\sum_{L \in LP^2_{\gamma}} \mu^{L(u_1, u_2)}_{e,c} = 2$$

and

$$\sum_{L \in LP^n_{\gamma}} \mu^{L(u_1, \ldots, u_n)}_{e,\ldots, c} = 0, \forall n \in 2\mathbb{N} \setminus \{2\}.$$
Proof. Define \( a = L_e + L_e^* \). Then it is a semicircular element, in the sense of Voiculescu. So, the only nonvanishing \( D_G \)-valued cumulant of \( a \) is the second one. By the previous lemma, we have that

\[
k_n(a, ..., a) = \sum_{L \in LP_n^*} \mu_{e, ..., e}^{L(u_1, ..., u_n)} \cdot L_v.
\]

Suppose that \( n = 2 \). Then

\[
k_2(a, a) = \mu_{e,e}^{1,*} \cdot L_v + \mu_{e,e}^{1,1} \cdot L_v = \mu_{e,e}^{1,*} + \mu_{e,e}^{1,1},
\]

since \( L_v = 1_{D_G} = 1_C = 1 \). Notice that \( l_{e^{-1}}l_e \) and \( l_{e^{-1}}l_e \) have their lattice paths

\[
\ast \rightarrow \ast \rightarrow \ast \rightarrow \ast,
\]

respectively. Therefore, \( C_{e,e}^{1,*} = C_{e,e}^{*,1} = \{1_2\} \). Therefore,

\[
\mu_{e,e}^{1,*} = \mu_{e,e}^{*,1} = \mu(1_2, 1_2) = 1.
\]

Therefore, \( k_2(a, a) = 2 \). Equivalently,

\[
\sum_{L \in LP_2^*} \mu_{e,e}^{L(u_1, u_2)} = 2,
\]

since \( L_v = 1_{D_G} = 1_C = 1 \). Now, let \( 2 < n \in 2\mathbb{N} \). Then

\[
k_n(a, ..., a) = \sum_{L \in LP_n^*} \mu_{e, ..., e}^{L(u_1, ..., u_n)} \cdot L_v = \sum_{L \in LP_n^*} \mu_{e, ..., e}^{L(u_1, ..., u_n)}
\]

since \( L_v = 1 \)

\[
= 0,
\]

by the semicircularity of \( L_e + L_e^* \). Therefore,

\[
\sum_{L \in LP_n^*} \mu_{e, ..., e}^{L(u_1, ..., u_n)} = 0, \forall n \in 2\mathbb{N} \setminus \{2\}.
\]

By the previous lemma, we can determine the \( D_G \)-semicircular elements in \((W^*(G), E)\).

**Theorem 4.2.** Let \( G \) be a countable directed graph and let \((W^*(G), E)\) be the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \). Let \( w = vvw \in \text{loop}(G) \), with \( v \in V(G) \). Then \( L_w + L_w^* \) is a \( D_G \)-valued semicircular elements, with

\[
k_2(L_w + L_w^*, L_w + L_w^*) = 2L_v.
\]
Proof. Let \( w = vvw \in \text{loop}(G) \), with \( v \in V(G) \). Define a \( D_G \)-valued random variable \( a = L_w + L_w^* \). Then, clearly, it is self-adjoint in \( W^*(G) \). It suffices to show that it has only second nonvanishing \( D_G \)-valued cumulants.

\[
\begin{align*}
  k_n (a, \ldots, a) &= k_n (L_w + L_w^*, \ldots, L_w + L_w^*) \\
  &= \sum_{(u_1, \ldots, u_n) \in \{1, \star\}^n} k_n (L_{u_1}^w, \ldots, L_{u_n}^w)
\end{align*}
\]

by the bimodule map property of \( k_n \)

\[
\begin{align*}
  &= \sum_{l_{w_1} \ldots l_{w_n} : \star \text{-axis-property}} k_n (L_{u_1}^w, \ldots, L_{u_n}^w)
\end{align*}
\]

where \( l_{w_1} \ldots l_{w_n} \in \mathcal{A}_G / R, E \) is the corresponding element of \( L_{u_1}^w \ldots L_{u_n}^w \)

\[
\begin{align*}
  &= \sum_{L \in LP^*_n} \mu_{L, \ldots, L} \cdot L.
\end{align*}
\]

But by the previous theorem,

\[
\begin{align*}
  \sum_{L \in LP^*_n} \mu_{L, \ldots, L} &= \begin{cases} 2 & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}
\end{align*}
\]

Therefore,

\[
\begin{align*}
  k_n (a, \ldots, a) &= \begin{cases} 2L_v & \text{if } n = 2 \\ 0_{D_G} & \text{otherwise.} \end{cases}
\end{align*}
\]

and hence \( a = L_w + L_w^* \), \( w \in \text{loop}(G) \), is a \( D_G \)-semicircular element in \( (W^*(G), E) \). Notice that if \( w \) is a loop (as a finite path), then \( C_{L_{u_1}^w \ldots L_{u_n}^w} = C_{\mathcal{C}^L_{u_1} \ldots \mathcal{C}^L_{u_n}} \), where \( e \) is the given at the beginning of this chapter. Therefore, we have the previous formulæ. \( \blacksquare \)

So, we can conclude that all \( D_G \)-valued random variables having the forms of \( pL_w + pL_w^* \) (\( p \in \mathbb{R}, w \in \text{loop}(G) \)) are \( D_G \)-semicircular elements in \( (W^*(G), E) \).

**Corollary 4.3.** Let \( w_j = v_jw_jv_j \in \text{loop}(G) \), \( v_j \in V(G) \), \( j = 1, \ldots, N \), such that they are mutually diagram-distinct (Note that it is possible that \( v_i = v_j \), for any \( i \neq j \) in \( \{1, \ldots, N\} \)). Then the \( D_G \)-valued random variable

\[
a = \sum_{j=1}^N \left(p_j L_{w_j} + p_j L_{w_j}^*\right) \in (W^*(G), E)
\]

is \( D_G \)-semicircular and
\[ \begin{align*}
  k_2(a, a) &= \sum_{j=1}^{N} 2p_j^4 \cdot L_{v_j}.
\end{align*} \]

\[\square\]

5. \textbf{D}_G\text{-Even Elements}

Let \( G \) be a directed graph with \( V(G) = \{v_1, v_2\} \) and \( E(G) = \{e = v_1v_2\} \). We can construct the graph \( W^*\)-algebra \( W^*(G) \) and its diagonal subalgebra \( D_G = \Delta_2 \). Trivially \( L_{v_1} + L_{v_2} = 1_{D_G} \). Define a \( D_G \)-valued random variable \( a = L_e + L^*_e \). Note that

\[ FP(G) = E(G), \]

since \( e^k \notin F^+(G) \), for all \( k \in \mathbb{N} \setminus \{1\} \). We cannot construct the more finite paths other than \( e \), itself. Also, the support of this operator \( a \) is

\[ F^+(G : a) = FP(G : a) = FP_*(G : a) = E(G). \]

Similar to the previous chapter, we will observe the following combinatorial fact:

\textbf{Lemma 5.1.} Let \( G \) and \( e \) be given as above. Then

\[ \sum_{L \in LP^*_n} \mu_e^{L(u_1, \ldots, u_n)} = 2\mu_n \in \mathbb{R}, \]

where \( \mu_n = \mu_{e, e, \ldots, e, e}^{1, \ast, \ldots, 1, \ast} = \mu_{e, e, \ldots, e, e}^{1, \ast, 1, \ldots, \ast, 1} \), for all \( n \in 2\mathbb{N} \).

\textbf{Proof.} Since \( e = v_1v_2 \), with \( v_1 \neq v_2 \in V(G) \), \( e^k \notin F^+(G) \) (i.e., it is not admissible), whenever \( n > 1 \). Therefore, for any even \( n \),

\[ \sum_{L \in LP^*_n} \mu_e^{L(u_1, \ldots, u_n)} = \mu_{e, e, \ldots, e, e}^{1, \ast, \ldots, 1, \ast} + \mu_{e, e, \ldots, e, e}^{1, \ast, \ldots, \ast, 1}. \]

Indeed, if we let \( (u_1, \ldots, u_n) \in \{1, \ast\}^n \) and if it is not alternating, then there exists at least one \( j \in \{1, \ldots, n\} \) such that \( u_j = u_{j+1} \in \{1, \ast\} \). This means that there should be a consecutive increasing or decreasing words of \( e \)'s. But \( e^k \) does not exist in our graph \( G \), whenever \( k > 1 \). Therefore, for such \( L(u_1, \ldots, u_n) \in LP^*_n \),

\[ \mu_e^{L(u_1, \ldots, u_n)} = \sum_{\theta \in C_e^{(u_1, \ldots, u_n)}} \mu(\theta, 1_n) = 0 \in \mathbb{R}. \]
Observe that $C_{e,e,...,e}^1,\ldots,1,\ast$ and $C_{e,e,...,e}^1,\ast,\ldots,1,1$ have the same elements, by the symmetry of the lattice paths of $l_e l_{e-1} \ldots l_{e-i}$ and $l_{e-1} l_{e-2} \ldots l_e$. So,

$$\mu_{e,e,...,e,e}^{1,\ast,\ldots,1,1} = \mu_{e,e,...,e,e}^{1,1,\ast,\ldots,1}.$$ 

Denote this real value by $\mu_n$, for each $n \in 2\mathbb{N}$. Then we can get the above formula. ☐

So, we have that:

**Lemma 5.2.** Let $G$ and $e$ be given as above. Then the $D_G$-valued cumulants of $a = L_e + L_e^*$ is determined by:

1. $k_n(a, \ldots, a) = 0_{D_G}$, whenever $n$ is odd.
2. $k_n(a, \ldots, a) = \mu_n \cdot 1_{D_G}$, for all $n \in 2\mathbb{N}$, where

$$\mu_n = \mu_{e,e,...,e,e}^{1,1,\ast,\ldots,1} = \mu_{e,e,...,e,e}^{1,\ast,\ldots,1,1}.$$

**Proof.** Suppose $n$ is odd. Then there is no lattice path having the $\ast$-axis-property. Therefore, (1) holds true. Now, assume that $n \in 2\mathbb{N}$. Let $L \in LP^*_n$. Since $e$ is a non-loop edge, $e^k \notin FP(G) = E(G)$. Therefore, when we consider $l_{e_1} \ldots l_{e_n} \in (A_G \setminus / R, E)$, the corresponding element of $L_{u_1}^n \ldots L_{u_n}^n$ (where $L_e$ and $L_e^*$ are summands of $a$), the lattice paths of it is

$$\ast / / / \ldots / / \text{ or } \ast / / / \ldots / / 
$$

denoted by $[rfrf...rf]$ and $[frfrfrf]$, respectively (Here $r$ stands for the rising step and $f$ stands for the falling step), because if there are consecutive rising steps or consecutive falling steps, then it represents $l_{e_1} \ldots l_e = 0_{D_G}$ or $l_{e-1} \ldots l_{e-1}$. Therefore, if $n \in 2\mathbb{N}$, then

$$k_n(a, \ldots, a) = \sum_{(u_1, \ldots, u_n) \in \{1,*\}^n} k_n \left(L_{u_1}^{u_1}, \ldots, L_{u_n}^{u_n}\right)$$

$$= \sum_{L \in LP^*_n} \mu_{u_1, \ldots, u_n} E(L_{u_1}^{u_1} \ldots L_{u_n}^{u_n})$$

$$= \mu_{e,e,...,e,e}^{1,\ast,\ldots,1,1} E(L_e L_e^* \ldots L_e L_e^*) + \mu_{e,e,...,e,e}^{1,1,\ast,\ldots,1} E(L_e^* L_e \ldots L_e L_e^*)$$

$$= \mu_n \cdot L_{v_1} + \mu_n L_{v_2} = \mu_n (L_{v_1} + L_{v_2})$$

where $\mu_n = \mu_{e,e,...,e,e}^{1,1,\ast,\ldots,1} = \mu_{e,e,...,e,e}^{1,\ast,\ldots,1,1}$. 


vanish. By the previous lemma, we can easily see that

\[ D \]

be the amalgamated

Definition 5.1. Let \( B \) be a von Neumann algebra and let \( A \) be the \( W^* \)-algebra over

\( \mu \) be a probability space over \( B \).

Theorem 5.3. Let \( w \in FP(G) \). Then the \( D_G \)-valued random variable \( L_w + L_w^* \) is

\[ D_G \]

Indeed, the lattice paths \([rf...rf]\) and \([fr...fr]\) in \( LP_n^* \) induce the same sets

\[ C_{e,e,...,e}^{1,*,...,1,*} \text{ and } C_{e,e,...,e}^{*,1,...,1,*} \] Thus

\[ \mu_{e,e,...,e}^{1,*,...,1,*} = \sum_{\theta \in C_{e,e,...,e}^{1,*,...,1,*}} \mu(\theta, 1_n) = \sum_{\pi \in C_{e,e,...,e}^{*,1,...,1,*}} \mu(\pi, 1_n) = \mu_{e,e,...,e}^{*,1,...,1,*}. \]

Now, we will observe the general case. Define \( \mu_n \overset{def}{=} \mu_{e,e,...,e}^{1,*,...,1,*} = \mu_{e,e,...,e}^{*,1,...,1,*} \)

where \( e \) is given as before. Now, we will observe the general case when \( e \) is a general finite path in an arbitrary countable directed graph \( G \).

Definition 5.1. Let \( B \) be a von Neumann algebra and let \( A \) be the \( W^* \)-algebra over

\( \mu \) be a conditional expectation and

Let \( w \in \{\theta(1_n) \mid \theta \in C_{e,e,...,e}^{1,*,...,1,*} \} \). We say that the \( B \)-valued random variable \( a \in (A, E_B) \) is \( B \)-valued even (in short, \( B \)-even) if it is self-adjoint and if it has all vanishing odd \( B \)-valued moments.

Let \( B \) be a von Neumann algebra and \( A \), a \( W^* \)-algebra over \( B \) and let \( (A, E_B) \)

be the amalgamated \( W^* \)-probability space over \( B \). Recall that \( B \)-valued random variable \( a \in (A, E_B) \) is \( B \)-even if and only if all odd \( B \)-valued cumulants of \( a \) vanish. By the previous lemma, we can easily see that \( D_G \)-valued random variables

\[ L_w + L_w^* \text{ for } w \in FP(G) \], are \( D_G \)-even, because if \( w = v_1 v_2 \), then

\[ k_n (L_w + L_w^*, ..., L_w + L_w^*) = \begin{cases} 0_{D_G} & \text{if } n \text{ is odd} \\ \mu_n \cdot (L_{v_1} + L_{v_2}) & \text{if } n \text{ is even} \end{cases} \]

for all \( n \in \mathbb{N} \), where \( \mu_n = \mu_{w,w,...,w,w}^{1,*,...,1,*} = \mu_{w,w,...,w,w}^{*,1,...,1,*} \). Remark that

\[ C_{w,w,...,w,w}^{1,*,...,1,*} = C_{w,w,...,w,w}^{*,1,...,1,*} = C_{e,e,...,e,e}^{1,*,...,1,*} \]

where \( e \) is the edge given at the beginning of this chapter. Based on it, we can get that

Theorem 5.3. Let \( w \in FP(G) \). Then the \( D_G \)-valued random variable \( L_w + L_w^* \) is

\[ D_G \]

Moreover, if \( w = v_1 v_2 \) with \( v_1, v_2 \in V(G) \) (possibly \( v_1 = v_2 \)), then

\[ k_n (L_w + L_w^*, ..., L_w + L_w^*) = \begin{cases} 0_{D_G} & \text{if } n \text{ is odd} \\ \mu_n \cdot (L_{v_1} + L_{v_2}) & \text{if } n \text{ is even} \end{cases} \]

for all \( n \in \mathbb{N} \), where \( \mu_n = \mu_{w,w,...,w,w}^{1,*,...,1,*} = \mu_{w,w,...,w,w}^{*,1,...,1,*} \). \( \square \)
6. $D_G$-valued R-diagonal Elements

In this chapter, we will consider the $D_G$-valued R-diagonality on the graph $W^*$-probability space $(W^*(G), E)$ over the diagonal subalgebra $D_G$. Recall that

**Definition 6.1.** Let $B$ be a von Neumann algebra and let $A$ be a $W^*$-algebra over $B$. Suppose that we have a conditional expectation $E_B : A \to B$ and hence $(A, E_B)$ is the amalgamated $W^*$-probability space over $B$. We say that the $D_G$-valued random variable $x \in (A, E_B)$ is a $D_G$-valued R-diagonal element if the only nonvanishing mixed cumulants of $x$ and $x^*$ are

$$k_n(b_1x, b_2x^*, ..., b_{n-1}x, b_n x^*)$$

and

$$k_n(b'_1x^*, b_2x, ..., b'_{n-1}x^*, b'_nx),$$

for all $n \in 2\mathbb{N}$, where $b_1, b'_1, ..., b_n, b'_n \in B$ are arbitrary. If $n$ is odd, then automatically the mixed cumulants vanish.

We can show that $L_w$ is $D_G$-valued R-diagonal, whenever $w$ is a finite path in $G$. By the results in the proceeding two chapters, we can get the following theorem:

**Theorem 6.1.** Let $G$ be a countable directed graph and $w \in FP(G)$. Then the $D_G$-valued random variable $L_w$ and $L_w^* \in (W^*(G), E)$ are $D_G$-valued R-diagonal.

**Proof.** It suffices to show that the only nonvanishing mixed cumulants of $L_w$ and $L_w^*$ are alternating ones, i.e., the nonvanishing mixed cumulants are

$$(6.1) \ k_{2n} (L_w^*, L_w, ..., L_w^*, L_w) \ and \ k_{2n} (L_w, L_w^*, ..., L_w^*, L_w^*).$$

Suppose that $w$ is a loop. Then by Lemma 4.1, we have that the only nonvanishing mixed cumulants are

$$k_2 (L_w^*, L_w) \ and \ k_2 (L_w, L_w^*).$$

So, $L_w$ and $L_w^*$ are $D_G$-valued R-diagonal.

Now, assume that $w = v_1 w v_2$ is a non-loop finite path, with $v_1 \neq v_2 \in V(G)$. Then the nonvanishing mixed cumulants of $L_w$ and $L_w^*$ have the forms of (6.1). By Section 2.2, we can easily get that

$$k_{2n} (L_w^*, L_w, ..., L_w^*, L_w) = \mu_{w, w, ..., w, w}^*, L_{v_1}.$$
and

\[ k_{2n}(L_w, L^*_w, \ldots, L_w, L^*_w) = \mu_{w, w, \ldots, w, w} \cdot L_{w^2}. \]

Assume that there exists a nonvanishing mixed \( D_G \)-cumulant of \( L_w \) and \( L^*_w \).

i.e., assume that there exist \( n \in \mathbb{N} \) and an \( n \)-tuple \((u_1, \ldots, u_n)\) of \( \{1, *\} \) such that

\[ k_n(L^{{u_1}}_w, \ldots, L^{{u_n}}_w) \neq 0_{D_G}. \]

By Section 2.2, we have that

\[ k_n(L^{{u_1}}_w, \ldots, L^{{u_n}}_w) = \mu_{w, \ldots, w} E(L^{{u_1}}_w \ldots L^{{u_n}}_w). \]

Notice that since \( w \) is a non-loop finite path, there is no admissible finite path \( w^k \), for \( k \in \mathbb{N} \setminus \{1\} \). So, if \((u_1, \ldots, u_n)\) is not alternating, then there exists at least one \( j \) in \( \{1, \ldots, n - 1\} \) such that \( u_j = u_{j+1} \). Since \( L^{{u_j}}_w L^{{u_{j+1}}} = L^{{u_j}}_w \) or \( L^{{u_{j+1}}} \), the \( D_G \)-valued random variable \( L^{{u_1}}_w \ldots L^{{u_n}}_w \) does not have the *-axis-property and hence

\[ E(L^{{u_1}}_w \ldots L^{{u_n}}_w) = 0_{D_G} = k_n(L^{{u_1}}_w, \ldots, L^{{u_n}}_w). \]

This contradict our assumption. So, \( L_w \) and \( L^*_w \) are \( D_G \)-valued R-diagonal. \( \blacksquare \)

The above theorem shows us that all generators of \( W^*(G) \) generated by finite paths in \( FP(G) \) are \( D_G \)-valued R-diagonal.

7. Generating Operators

In this chapter, as examples, we will compute the trivial \( D_G \)-valued moments and cumulants of the generating operator \( T \) of the graph \( W^* \)-algebra \( W^*(G) \). Let \( G \) be a countable directed graph and let \((W^*(G), E)\) be the graph \( W^* \)-probability space over its diagonal subalgebra. Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable. Recall that the trivial \( D_G \)-valued \( n \)-th moments and cumulants of \( a \) are defined by

\[ E(a^n) \quad \text{and} \quad k_n \left( a, \ldots, a \right) \text{n-times}. \]

In this chapter, we will deal with the following special \( D_G \)-valued random variable

\textbf{Definition 7.1.} Define an operator \( T \) in \( W^*(G) \) by

\[ T = \sum_{e \in E(G)} (L_e + L^*_e). \]
We will call $T$ the generating operator of $W^*(G)$. The self-adjoint operators $L_e + L_e^*$, for $e \in E(G)$, are called the block operators of $T$.

**Example 7.1.** Let $G$ be a one-vertex directed graph with $N$-edges. i.e.,

$$V(G) = \{v\} \text{ and } E(G) = \{e_j = ve_jv : j = 1, ..., N\}.$$  

Then the graph $W^*$-algebra $W^*(G)$ satisfies that

\begin{align*}
W^*(G) = D_G \ast_{D_G} \left( \bigast_{j=1}^N (W^*(\{L_e_j\}, D_G)) \right),
\end{align*}

by Chapter 4. Notice that $D_G = \Delta_1 = \mathbb{C}$. Therefore, the formula (5.2) is rewritten by

\begin{align*}
W^*(G) = \bigast_{j=1}^N (W^*(\{L_e_j\})),
\end{align*}

where $\ast$ means the usual (scalar-valued) free product of Voiculescu. Also notice that $1_{D_G} = L_v = 1 \in \mathbb{C}$ and

\begin{align*}
L_{e_j}L_{e_j} = L_v = 1 = L_{e_j}L_{e_j}^*, \text{ for all } j = 1, ..., N.
\end{align*}

This shows that $L_{e_j}$'s are unitary in $W^*(G)$, for all $j = 1, ..., N$. Now, define the generating operator $T = \sum_{j=1}^N \left( L_{e_j} + L_{e_j}^* \right)$ of $W^*(G)$. It is easy to see that each block operator $L_{e_j} + L_{e_j}^*$ is semicircular, by Voiculescu, for all $j = 1, ..., N$. (Remember the construction of creation operators $L_{e_j}$'s and see [9].) Futhermore, by Chapter 3, we can get that all blocks $L_{e_j} + L_{e_j}^*$'s are free from each other in the graph $W^*$-probability space $(W^*(G), E)$.

By (5.4), the canonical conditional expectation $E : W^*(G) \rightarrow D_G$ is the faithful linear functional. Moreover, by (5.5), this linear functional $E$ is a trace in the sense that $E(ab) = E(ba)$, for all $a, b \in W^*(G)$. From now, to emphasize that $E$ is a trace, we will denote $E$ by $\text{tr}$.

Let's compute the $n$-th cumulant of $T$ ;

\begin{align*}
k_n(T, ..., T) &= k_n \left( \sum_{j=1}^N \left( L_{e_j} + L_{e_j}^* \right), ..., \sum_{j=1}^N \left( L_{e_j} + L_{e_j}^* \right) \right) \\
&= \sum_{j=1}^N k_n \left( (L_{e_j} + L_{e_j}^*), ..., (L_{e_j} + L_{e_j}^*) \right),
\end{align*}

by the mutual freeness of $\{L_{e_j}, L_{e_j}^*\}$'s on $(W^*(G), \text{tr})$, for $j = 1, ..., N$. Observe that
\[ k_n \left( (L_{e_j} + L_{e_j}^*), \ldots, (L_{e_j} + L_{e_j}^*) \right) \]

\[ (5.7) \quad = \begin{cases} 
2 \left( (L_{e_j} + L_{e_j}^*), (L_{e_j} + L_{e_j}^*) \right) & \text{if } n = 2 \\
0 & \text{otherwise},
\end{cases} \]

by the semicircularity of \( L_{e_j} + L_{e_j}^* \), for \( j = 1, \ldots, N \). By (5.7), the formula (5.6) is

\[ k_n (T, \ldots, T) \]

\[ (5.8) \quad = \begin{cases} 
\sum_{j=1}^{N} k_2 \left( L_{e_j} + L_{e_j}^*, L_{e_j} + L_{e_j}^* \right) & \text{if } n = 2 \\
0 & \text{otherwise}
\end{cases} \]

Now, observe \( k_2 \left( L_{e_j} + L_{e_j}^*, L_{e_j} + L_{e_j}^* \right) \):

\[ k_2 \left( L_{e_j} + L_{e_j}^*, L_{e_j} + L_{e_j}^* \right) \]

\[ = k_2 \left( L_{e_j}, L_{e_j} \right) + k_2 \left( L_{e_j}^*, L_{e_j} \right) + k_2 \left( L_{e_j}^*, L_{e_j} \right) + k_2 \left( L_{e_j}^*, L_{e_j} \right) \]

\[ = 0 + k_2 \left( L_{e_j}, L_{e_j}^* \right) + k_2 \left( L_{e_j}^*, L_{e_j} \right) + 0 \]

by Section 2.1

\[ = tr \left( L_{e_j} L_{e_j}^* \right) + tr \left( L_{e_j}^* L_{e_j} \right) = 2 \cdot tr \left( L_{e_j} L_{e_j} \right) \]

since \( tr \) is a trace

\[ = 2 \cdot L_v = 2, \]

for \( j = 1, \ldots, N \), by Section 2.1 and 2.2. So, we can get that

\[ (5.9) \quad k_n (T, \ldots, T) = \begin{cases} 
2N & \text{if } n = 2 \\
0 & \text{otherwise}.
\end{cases} \]

Now, we can compute the trivial moments of \( T \), via the Möbius inversion.

\[ tr (T^n) = \sum_{\pi \in NC(n)} k_\pi (a, \ldots, a) \]
where $k_\pi(a, \ldots, a) = \prod_{V \in \pi} k_{|V|}(a, \ldots, a)$, for each $\pi \in NC(n)$, by Nica and Speicher (See [1] and [17]).

$$
\sum_{\pi \in NC(n/2)} \prod_{V \in \pi} 2N = \sum_{\pi \in NC(n/2)} (2N)^{|\pi|},
$$

where $|\pi| \overset{\text{def}}{=} \text{the number of blocks in } \pi$. Notice that the above formula (5.10) shows us that the $n$ should be even, because $NC_2(n)$ is nonempty when $n$ is even. Therefore,

$$
\begin{cases}
\sum_{\pi \in NC(n/2)} (2N)^{|\pi|} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}.
\end{cases}
$$

Also, notice that if $\pi \in NC_2(n)$, then $|\pi| = \frac{n}{2}$, for all even number $n \in \mathbb{N}$. So,

$$
\begin{cases}
|NC_2(n)| \cdot (2N)^{\frac{n}{2}} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases}
$$

where $c_k = \frac{1}{k+1} \left( \begin{array}{c} 2k \\ k \end{array} \right)$ is the $k$-th Catalan number, for all $k \in \mathbb{N}$. Remember that $|NC(k)| = |NC_2(2k)| = c_k$, for all $k \in \mathbb{N}$.

Therefore, by (5.9) and (5.12), we can compute the moments and cumulants of the generating operator $T$ of $(W^*(G), tr)$;

$$
\begin{cases}
(2N)^{\frac{n}{2}} \cdot c_{n/2} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd},
\end{cases}
$$
and
\[ k_n (T, ..., T) = \begin{cases} 2N & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases} \]

**Example 7.2.** Let \( N \in \mathbb{N} \) and let \( G \) be the circulant graph with
\[ V(G) = \{v_1, ..., v_N\} \]
and \( E(G) = \{e_1, ..., e_N\} \) with
\[ e_j = v_j e_j v_{j+1}, \text{ for } j = 1, ..., N - 1, \text{ and } e_N = v_N e_N v_1. \]

Define the generating operator \( T = \sum_{j=1}^{N} \left( L e_j + L^* e_j \right) \) of the graph \( W^* \)-algebra \( W^*(G) \). In this case, we can get the diagonal subalgebra \( D_G \) of \( W^*(G) \), as a von Neumann algebra which is isomorphic to \( \Delta_N \), where \( \Delta_N \) is a subalgebra of the matrixial algebra \( M_N(\mathbb{C}) \). Define the canonical conditional expectation \( E : W^*(G) \to D_G \). Then we can compute the trivial \( n \)-th \( D_G \)-valued cumulant of the operator \( T \), by regarding it as a \( D_G \)-valued random variable in the graph \( W^* \)-probability space \(( W^*(G), E )\) over \( D_G = \Delta_N \).

Fix \( n \in \mathbb{N} \). Then
\[
k_n \left( T, \ldots, T \right)_{n\text{-times}} = k_n \left( \sum_{j=1}^{N} \left( L e_j + L^* e_j \right), \ldots, \sum_{j=1}^{N} \left( L e_j + L^* e_j \right) \right)
= \sum_{j=1}^{N} k_n \left( (L e_j + L^* e_j), \ldots, (L e_j + L^* e_j) \right)
\]
by the mutual \( D_G \)-freeness of \( \{L e_j, L^* e_j\} \)'s, for \( j = 1, ..., N \).

(5.13) \[
= \sum_{j=1}^{N} \sum_{(u_1, \ldots, u_n) \in \{1, *\}^n} k_n \left( L_{e_j}^{u_1}, \ldots, L_{e_j}^{u_n} \right).
\]

Recall that, by Section 2.2, we can get that
\[
(5.14) \quad k_n \left( L_{e_j}^{u_1}, \ldots, L_{e_j}^{u_n} \right) = \mu_{e_j, \ldots, e_j}^{u_1, \ldots, u_n} \cdot \text{Pr} \left( L_{e_j}^{u_1} \ldots L_{e_j}^{u_n} \right),
\]
where \( \mu_{e_j, \ldots, e_j}^{u_1, \ldots, u_n} = \sum_{\pi \in C_{e_j}^{u_1, \ldots, u_n}} \mu(\pi, 1_n). \)
Observe that since $e_j$’s are non-loop edges, $e_j^k \notin \mathbb{F}^+(G)$, for all $k \in \mathbb{N} \setminus \{1\}$, for $j = 1, ..., N$. In other words, such $e_j^k$ is not admissible. So, if $(u_1, ..., u_n)$ is not alternating, in the sense that $(u_1, ..., u_n) = (1, *, ..., 1, *)$ or $(*, 1, ..., *, 1)$, then $\text{Pr} \{L^u_{e_j, ..., L^u_{e_j}} = 0_{DG}\}$. For instance, $E \{L^*_{e_j, L^*_{e_j}} = 0_{DG}\}$ or $E \{L^2_{e_j, L^*_{e_j}} = 0_{DG}\}$, by Section 2.1. Therefore, the only nonvanishing case is either

$$k_n \left(L^*_{e_j}, L^*_{e_j}, ..., L^*_{e_j}, L^*_{e_j}\right) \text{ or } k_n \left(L^*_{e_j}, L^*_{e_j}, ..., L^*_{e_j}, L^*_{e_j}\right),$$

where $n$ is even. Notice that

$$\mu_{e_j, e_j, ..., e_j} = \mu_{e_j, e_j, ..., e_j}^{1, *, ..., 1, *}, \tag{5.15}$$

because $C_{e_j, e_j, ..., e_j}^{1, *, ..., 1, *} = C_{e_j, e_j, ..., e_j}^{*, 1, ..., 1, *}$, for all $j = 1, ..., N$. Moreover, since $C_{e_j, e_j, ..., e_j}^{1, *, ..., 1, *} = C_{e_j, e_j, ..., e_j}^{*, 1, ..., 1, *}$, for all $j \neq k$ in $\{1, ..., N\}$,

$$\mu_{e_j, e_j, ..., e_j} = \mu_{e_j, e_j, ..., e_j}^{1, *, ..., 1, *}, \tag{5.16}$$

for all $j, k \in \{1, ..., N\}$. Let’s denote $\mu_{e_j, e_j, ..., e_j}^{1, *, ..., 1, *}$ by $\mu_n$, for all $j = 1, ..., N$. Then, by (5.14), we have that

$$k_n \left(L^u_{e_j}, ..., L^u_{e_j}\right) = \begin{cases} 
\mu_n L^u_{v_j} & \text{if } (u_1, ..., u_n) = (1, *, ..., 1, *) \\
\mu_n L^u_{v_j+1} & \text{if } (u_1, ..., u_n) = (*, 1, ..., *, 1) \\
0_{DG} & \text{otherwise}, 
\end{cases} \tag{5.17}$$

for all $j = 1, ..., N$, where $L^u_{v_{N+1}}$ means $L^u_{v_1}$. So, by (5.13) and (5.17), we can get that

$$k_n (T, ..., T)$$

$$= \sum_{j=1}^{N} \left(k_n \left(L_{e_j}, L^*_{e_j}, ..., L^*_{e_j}, L^*_{e_j}\right) + k_n \left(L^*_{e_j}, L_{e_j}, ..., L^*_{e_j}, L^*_{e_j}\right)\right)$$

$$= \sum_{j=1}^{N} \left(\mu_n L^u_{v_j} + \mu_n L^u_{v_j+1}\right) = \sum_{j=1}^{N} \mu_n \left(L^u_{v_j} + L^u_{v_j+1}\right)$$

where $L^u_{v_{N+1}}$ means $L^u_{v_1}$, for all $n \in 2\mathbb{N}$. Therefore,

$$k_n (T, ..., T) = \begin{cases} 
\sum_{j=1}^{N} \mu_n \left(L^u_{v_j} + L^u_{v_j+1}\right) & \text{if } n \text{ is even} \\
0_{DG} & \text{if } n \text{ is odd.} 
\end{cases}$$
\begin{equation}
(5.18) \quad E(T^n) = \begin{cases} 
2\mu_n \cdot 1_{DG} & \text{if } n \text{ is even} \\
0_{DG} & \text{if } n \text{ is odd.}
\end{cases}
\end{equation}

Unfortunately, it is very hard to compute \( \mu_n \), when \( n \to \infty \). But we have to remark that if we have arbitrary graph \( H \) and its graph \( W^*\)-probability space \( (W^*(H), F) \) over its diagonal subalgebra \( D_H \) and if \( w \in \text{loop}^c(G) \), then

\[
\mu_{1^*, \ldots, 1^*, w} = \mu_n = \mu_{1^*, \ldots, 1^*, w}, \text{ for all } n \in 2N.
\]

Now, let’s compute the trivial \( n \)-th \( DG \)-valued moment of \( T \). Notice that since all odd trivial \( DG \)-valued cumulants of \( T \) vanish (See [11] and [14]). Thus it suffices to compute the even trivial \( DG \)-valued moments of \( T \). Assume that \( n \in 2N \). Then

\begin{equation}
(5.19) \quad E(T^n) = \sum_{\pi \in NC_E(n)} k_\pi (T, \ldots, T),
\end{equation}

where \( k_\pi (T, \ldots, T) \) is the partition-dependent cumulant of \( T \) (See [16]) and

\[
NC_E(n) \overset{\text{def}}{=} \{ \pi \in NC(n) : V \in \pi \iff |V| \in 2N \}.
\]

By (5.18), we can get that \( k_n(T, \ldots, T) \) commutes with all elements in \( W^*(G) \), because \( 1_{DG} \) and \( 0_{DG} \) commutes with \( W^*(G) \) and \( 2\mu_n \in C \), for all \( n \in N \). So, the formula (5.19) can be reformed by

\[
E(T^n) = \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} k_{|V|} (T, \ldots, T) \right)
\]

\[
= \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} 2\mu_{|V|} \cdot 1_{DG} \right)
\]

\begin{equation}
(5.20) \quad = \left( \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} 2\mu_{|V|} \right) \right) \cdot 1_{DG},
\end{equation}

for all \( n \in 2N \). Therefore, by (5.18) and (5.20), we have that if \( T \) is the generating operator of the graph \( W^*\)-algebra of the circulant graph \( G \) with \( N \)-vertices, then

\[
E(T^n) = \begin{cases} 
\left( \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} 2\mu_{|V|} \right) \right) \cdot 1_{DG} & \text{if } n \text{ is even} \\
0_{DG} & \text{if } n \text{ is odd.}
\end{cases}
\]

and

\[
k_n \left( T, \ldots, T \right)_{n\text{-times}} = \begin{cases} 
2\mu_n \cdot 1_{DG} & \text{if } n \text{ is even} \\
0_{DG} & \text{if } n \text{ is odd.}
\end{cases}
\]
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