Model Comparison of ΛCDM vs $R_h = ct$ using $H(z)$ measurements

Guru Sai Haveesh Singirikonda* and Shantanu Desai†
Department of Physics, Indian Institute of Technology, Hyderabad, Telangana-502285, India

In 2012, Bilicki and Seikel [1] showed that $H(z)$ data reconstructed using Gaussian Process Regression from cosmic chronometers and Baryon Acoustic Oscillations (BAO), conclusively rules out the $R_h = ct$ model. These results were disputed by Melia and collaborators in two different works [2, 3], who showed using both an unbinned analysis and Gaussian Process reconstructed $H(z)$ data from chronometers, that $R_h = ct$ is favored over ΛCDM model. To resolve this imbroglio, we carry out model comparison of ΛCDM versus $R_h = ct$ by independently reproducing the above claims using latest data. We consider $H(z)$ measurements from both cosmic chronometers as well as a combination of chronometers and BAO, and perform model selection between these two models using frequentist, Bayesian, and three different information theoretical criteria. We find that with only cosmic chronometers, no one model is decisively favored. When we combine with BAO data, the information theory and Bayesian model comparison tests decisively favor the ΛCDM model.

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I. INTRODUCTION

The standard Hot-Big Bang model of Cosmology is described by a flat ΛCDM universe, with 70% of the energy density comprising of the Cosmological constant (or any dark energy fluid with equation of state $w < -1/3$) and 25% cold (non-baryonic) dark matter and 5% baryons [4]. This model has two episodes of acceleration (one in the Early universe caused by inflation [5], posited to solve the horizon and flatness problems in standard Hot Big Bang Model [6]) and another in the late universe, caused by dark energy [7]. This model has been spectacularly confirmed by Planck 2018 CMB observations [8] along with other large-scale structure probes. There are however a few data driven lingering problems with the Standard ΛCDM paradigm, such as the Hubble constant tension between local and high redshift measurements [9, 10], $\sigma_8$ tension between CMB and galaxy clusters [11, 12], Lithium-7 problem in Big-Bang Nucleosynthesis [13], anomalies in CMB at low $l$ [14], etc. A few works have also occasionally challenged some of the most well-established tenets of the Standard Cosmological model, viz. cosmic acceleration [15] and even cosmic expansion [16].

Independent of the above data driven problems, there are also conceptual problems with the standard model. The best-fit model of scalar-field driven inflation (an essential pillar of Standard Hot Big Bang Model) with flat potentials also causes lots of conceptual problems [17]. Furthermore, we don’t yet have any laboratory evidence for any cold matter candidate, despite searching for over three decades [18]. If the dark energy turns out to be a cosmological constant, a non-zero value would be very problematic from the point of view of Quantum Field theory [19, 20].

Therefore, because of some of the above problems, many alternatives to the standard model have been constructed. One such model is the $R_h = ct$ universe model, proposed by Fulvio Melia [21–23]. In this model, size of the Hubble sphere given by $R_h(t) = ct$ is upheld for all times in contrast to the case of the ΛCDM model, where this coincidence is true only at the current epoch, i.e. $R_h(t_0) = c t_0$. This model has $a(t) \propto t$ and $H(z) = H_0 (1 + z)$. One direct result of this is that the rate of expansion $\dot{a}$ is constant; and pressure and energy density satisfy an equation of state given by $p = -\frac{\rho}{3}$. The $R_h = ct$ model also has several antecedents and generalizations, discussed in Refs. [24, 25], and an up-to-date review of all such models can be found in Ref. [26].

This model has been tested with a whole slew of cosmological observations by Melia and collaborators, such as cosmic chronometers [3], quasar core angular size measurements [27], quasar X-ray and UV fluxes [28], Type Ia SN [29], strong lensing [30], cluster gas mass fraction [31], etc and found to be in better agreement compared to ΛCDM model. However, other researchers have reached opposite conclusions and have argued that this model is inconsistent with observational data [1, 32, 33]. These results in turn have also been contested by Melia and collaborators [34]. Conceptual problems have also been raised against this model [35–37], although some have also been countered [38].

In this work, we try to adjudicate between one such conflicting claim between two of the above works: Ref. [1] (BS12, hereafter) and Refs. [2, 3] (MM13 and MY18, hereafter), which have reached diametrically opposite conclusions, when analyzing Hubble parameter ($H(z)$) measurements. BS12 constructed a non-parametric fit for $H(z)$ using Gaussian Process Regression from 18 Cosmic chronometer measurements and 8 BAO measurements spanning the redshift range $0.09 \leq z \leq 0.73$. They argued based on a visual inspection of the reconstructed $H(z)$ and its derivatives that the ΛCDM model is a much better fit than the $R_h = ct$ model. Soon thereafter, MM13 however pointed out that 19 unbinned $H(z)$
measurements using only chronometers support $R_b = ct$ over the ΛCDM. This assertion was based on AIC, BIC, KIC based tests from information theory and $\chi^2$/dof. Most recently, MY18 used 30 $H(z)$ measurements using cosmic chronometers, and similar to BS12, used Gaussian Process regression to reconstruct a non-parametric $H(z)$. Model comparison of ΛCDM vs $R_b = ct$ was done by calculating the normalized area difference between the model and the reconstructed $H(z)$. They argued that with this procedure, $R_b = ct$ model is a better fit than ΛCDM. Here, we try to do independent analysis of the $H(z)$ data, using latest set of measurements from chronometers and BAO. For model selection between the two, we use multiple tests from frequentist, information theory, Bayesian model comparison techniques, which have previously been applied to different areas in Astrophysics [39–42] and cosmology [43–46].

The outline of this paper is as follows. We discuss the Gaussian Process Regression technique and the different model comparison techniques used here, in Sect. II and Sect. III respectively. The key points made in the two conflicting sets of papers BS12 versus MM13, MY18 are discussed in Sect. IV. The description of our datasets and analysis can be found in Sect. V. Our results can be found in Sect. VI. We conclude in Sect. VII.

II. GAUSSIAN PROCESS REGRESSION

Both the groups (BS12 and MY18) have used Gaussian Process regression (GPR, hereafter) for their analysis. Therefore, we provide an abridged introduction to GPR, before discussing the results of their analysis. Gaussian Process is similar to a Gaussian Distribution but it describes the distribution of functions instead of random variables. To describe the distribution of these functions, we need the mean $\mu(x)$ and a covariance function $\text{cov}(f(x), f(\tilde{x})) = k(x, \tilde{x})$ connecting the values of $f$ evaluated at $x$ and $\tilde{x}$. A more detailed explanation of Gaussian Process can be found in Section 2 of Ref. [47]. There are many choices for the covariance function. Both the papers have used a squared exponential covariance function:

$$k(x, \tilde{x}) = \sigma_f^2 \exp\left(-\frac{(x - \tilde{x})^2}{2l^2}\right)$$

Here $\sigma_f$ and $l$ are hyper-parameters which describe the ‘bumpiness’ of the function.

Even a random function $f(x)$ can be generated using the covariance matrix. Let $X$ be the set of points $x_i$ and one can generate a vector $\mathbf{f}^*$ of function values at $X^*$ with $f_i^* = f(x_i^*)$ as

$$\mathbf{f}^* = \mathcal{N}(\mu^*, K(\mathbf{X}^*, \mathbf{X}^*))$$

Similarly, observational data can be written in the same way as

$$y = \mathcal{N}(\mu, K(\mathbf{X}, \mathbf{X}) + C)$$

where $C$ is the covariance matrix of the data. If data is uncorrelated the covariance matrix is simply $\text{diag}(\sigma_y^2)$. Using the values of $y$ at $X$ we can find reconstruct $\mathbf{f}^*$ using

$$\mathbf{f}^* = \mu^* + K(\mathbf{X}^*, \mathbf{X})[K(\mathbf{X}, \mathbf{X}) + C]^{-1}(y - \mu)$$

and

$$\text{cov}(\mathbf{f}^*) = K(\mathbf{X}^*, \mathbf{X}^*) - K(\mathbf{X}^*, \mathbf{X})[K(\mathbf{X}, \mathbf{X}) + C]^{-1}K(\mathbf{X}, \mathbf{X}^*)$$

where $\mathbf{f}^*$ and $\text{cov}(\mathbf{f}^*)$ are mean and covariance of $\mathbf{f}^*$ respectively. The diagonal elements of $\text{cov}(\mathbf{f}^*)$ gives us the variance of $f^*_i$. More details on this can found in Ref. [47]. Both BS12 and MY18 implement Gaussian Process Regression in Python using the package GaPP which was developed by Seikel and collaborators [47].

III. MODEL COMPARISON SUMMARY

Here, we briefly describe the different model comparison techniques used. These can be broadly classified into three distinct categories: frequentist, information-theory, and Bayesian techniques. More details about these techniques can be found in reviews written for astrophysicists [43, 45, 46, 48, 49]. We have previously applied these techniques to a whole slew of problems in Astrophysics and cosmology such as GRB classification [39], search for Lorentz invariance violation [40], annual modulation search in dark matter experiments [41, 42], periodicities in solar neutrino fluxes [50], etc.

In frequentist model comparison (sometimes also referred to as likelihood ratio test [49]), the first step is parameter estimation for both the models, (cf. Eqns. 8, 9), one wants to compare. The first step in parameter estimation is usually done by minimizing the $\chi^2$ between the observed data and a given model. The frequentist method involves comparing the values of the $\chi^2$ likelihood (denoted by $L_\chi^2$), which is obtained by $\chi^2$ p.d.f. The model with the higher likelihood is the preferred model.

Using Bayesian Statistics, we also compute the probability that the data was generated by each model, also called the Bayesian Evidence ($Z$) [48]:

$$P(\Theta|D, M) = \frac{P(D|\Theta, M)P(\Theta, M)}{P(D|M)}$$

where $P(\Theta|D, M)$ is the posterior, $P(D|\Theta, M)$ is the likelihood, $P(\Theta, M)$ is the prior, and $P(D|M)$ is the Evidence, also sometimes referred to as Marginal Likelihood. Note that unlike the other model comparison test, The Bayesian evidence does not use the best-fit value of a given model. It considers the entire range. Again, the model with a higher evidence, i.e., higher probability that the data was generated from that model will be the better model to describe the data. The value for the evidence was computed using the dynesty [51] package. From the
Bayesian evidence of the two models, we can calculate the value of the Bayes’ factor, which is simply the ratio of the evidence for the two models and given by:

\[ B = \frac{Z_1}{Z_2}. \]  

The last set of techniques, for model comparison based on information theory are the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC) and Deviance Information Criterion (DIC).

Akaike Information Criterion (AIC) is defined as

\[ AIC = 2k + \chi^2_{\text{min}} \]  

Bayesian Information Criterion (BIC) is defined as

\[ BIC = k \log N + \chi^2_{\text{min}} \]  

In Eq. 3 and 4, \( k \) is the number of free parameters, and \( N \) is the total number of data points. The model which has a lower value of AIC and BIC will be the suitable model for the given data.

The Deviance Information Criterion combines ideas from both AIC and BIC and based on the concept of Bayesian complexity, as discussed in Ref. [46]. It is computed from the posterior samples created during an Markov Chain Monte Carlo (MCMC) run, used to sample the posterior distribution. The DIC is defined as

\[ DIC = D(\theta) + 2p_D = \bar{D}(\theta) + p_D = 2\bar{D}(\theta) - D(\bar{\theta}), \]

where \( \theta \) is the parameter vector, \( D(\theta) = \chi^2(\theta) + C \) (\( C \) is a constant), and \( p_D = \bar{D}(\theta) - D(\bar{\theta}) \), which is the mean of the deviance minus the deviance of the mean. More details on these information theoretical metrics can be found in Ref. [46].

Here, when we are calculating the difference in the values of the information criteria, we subtract the value corresponding to the ΛCDM model from the value corresponding to the \( R_h = ct \) model. To assess the significance of the difference, we use qualitative strength of evidence rules described in Ref. [43]. For Bayes’ factor, we evaluate the ratio of the evidence of the ΛCDM to the evidence for the \( R_h = ct \) model. The significance can be evaluated using the Jeffreys scale [48]. In frequentist model comparison, if the models are nested, one can use Wilk’s theorem to assess the significance [40–42]. In this example, the models we are testing are not nested. Although, evaluating the significance of non-nested models using frequentist model comparison has been discussed in literature [49, 52], we shall not implement them here, as evaluating significance in these cases is not straightforward or easy to interpret.

IV. SUMMARY OF BS12, MM13, AND MY18

As mentioned in the introduction, there is a large amount of literature comparing the \( R_h = ct \) model with the ΛCDM model. We focus on the particular case of these two sets of papers (BS12 versus MM13/MY18), where they have arrived at conflicting results despite similar analysis.

BS12 reconstructed the value of \( q(z) \) from Union2.1 Type 1a Supernova dataset using Gaussian Process Regression and showed from a visual inspection that the reconstructed \( q(z) \) better fits the ΛCDM model. They also used Hubble rate data from 18 cosmic chronometer measurements and 8 BAO measurements, and reconstructed \( H(z) \) with GPR, and plotted it against the predicted values of \( H(z) \) from the ΛCDM model and the \( R_h = ct \) model. They compared the reconstructed \( H(z) \), its first and second derivative, as well as the \( Om(z) \) diagnostic [53] against the theoretical predictions of the two models. They again used visual inspection from these plots to conclude that the ΛCDM model is a better fit to the data compared to \( R_h = ct \). Very soon after BS12, MM13 considered 19 unbinned \( H(z) \) measurements from cosmic chronometers and fit this data to both the models. They found that the \( \chi^2/\text{DOF} \) (or reduced \( \chi^2 \)) is 0.745 and 0.777 for \( R_h = ct \) and ΛCDM (with parameters given by: \( \Omega_M = 0.32, H_0 = 68.9 \pm 2.4 \) km/sec/Mpc) respectively. Therefore, the reduced \( \chi^2 \) was smaller for ΛCDM. However, when ΛCDM model is fit to the cosmic chronometer data, the estimated values of \( \Omega_M \) and \( H_0 \) (0.27 and 73.8 ± 2.4 km/sec/Mpc respectively) yield a \( \chi^2/\text{DOF} \) of 0.9567, which is greater than that for \( R_h = ct \) universe. However, no comparison of goodness of fit based on \( \chi^2 \) p.d.f. was made. They also found smaller values of AIC, BIC, and KIC for \( R_h = ct \) universe compared to ΛCDM. However, the difference in information criterion between the two models does not cross the threshold of 10, needed for any one model to be decisively favored over the other [43]. They further criticized the SN data analysis in BS12, arguing that the data used was optimized for ΛCDM cosmology. The also point out that BAO data analyzed by BS12 include nonlinear evolution of the matter density and velocity fields, and hence is not model-independent. Therefore, their analysis was done using only chronometers.

A similar analysis using the latest cosmic chronometer data (consisting of 30 measurements) and GPR was carried out in MY18. But here they have used an analytical approach to compare the two models after reconstructing the values of \( H(z) \) using GPR. They argued that the \( R_h = ct \) performs better than the ΛCDM model, contradicting the conclusion of BS12. To quantify this, they constructed a mock data set using Gaussian random variables, and then computed the normalized absolute area difference between this and the real function. For each model they calculated the differential areas by replacing the mock data set with the predictions from the models and then estimate the probability of the model (p-value). Using this analysis they came to the conclusion that the \( R_h = ct \) model is the better model among the two for the chronometer data.
FIG. 1: The ΛCDM model and the $R_h = ct$ model (with best-fit parameters obtained using unbinned data) plotted alongside the reconstructed function of $H(z)$. The reconstruction was done with GPR using the GaPP package, similar to what the authors did in [1]. For this figure, Gaussian Process was used on $H(z)$ data from Cosmic Chronometer data only. The data points used here are tabulated in Tab. I.

FIG. 2: Same as Fig. 1, but using data from both chronometers and BAO. Note that the reconstructed errors are about three times smaller than those in Fig. 1 because the BAO data points at $z > 2$ have very small error bars. The data points used here are tabulated in Tab. II.

V. DATASETS AND ANALYSIS

The $H(z)$ data from cosmic chronometers are obtained by comparing relative ages of galaxies at different redshifts and is given by the following expression, assuming an FRW metric [54]:

$$H(z) = -\frac{1}{1+z} \frac{dz}{dt}$$

(6)

Based on the measurements of the age difference, $\Delta t$, between two passively–evolving galaxies that are separated by a small redshift interval $\Delta z$, we can approximately calculate the value of $dz/dt$ from $\Delta z/\Delta t$. This differential age method is much more reliable than a method based on an absolute age determination for galaxies, as absolute stellar ages are more vulnerable to systematic uncertainties than relative ages.

Even though cosmic chronometers probe only the expansion history of the universe, they have been used for a variety of cosmological inferences, such as determination of $H_0$ [55–57], transition redshift from deceleration to acceleration [58, 59], cosmic distance duality relation [60], $\sigma_8$ [61] determination, dark energy equation of state [62, 63], etc. The complete data set of 31 measurements of $H(z)$ at redshifts $0.07 < z < 1.965$ from Cosmic Chronometers is listed in Table I. This data set taken from the compilation in Table III of [64]. A graphical summary of this data can be found in Fig. 1.

The BAO method for obtaining $H(z)$ is based on clustering of galaxies or quasars, where the acoustic peak observed in radial direction is used as a standard ruler [65]. We used 20 BAO measurements compiled in Table II. The full dataset is shown in Fig. 2 and includes three data points with $z > 2$.

The first step in model comparison is to find the best-fit values of the free parameters in ΛCDM as well as the $R_h = ct$ universe model. This is obtained by minimizing the $\chi^2$ functional given by:

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{H_i(z) - H^{\text{ACDM/R}_h = ct}(z, \theta)}{\sigma_i} \right)^2,$$

(7)

where $H_i(z)$ indicate the various Hubble parameter measurements, $N$ is the total number of datapoints used, $H^{\text{ACDM/R}_h = ct}(z, \theta)$ encapsulates the relation for the Hubble parameter in ΛCDM and $R_h = ct$ cosmology; $\sigma_i$ denotes the error in $H(z)$; and $\theta$ denotes the parameter vector in the two models.

But sometimes the process of minimizing the $\chi^2$ gets stuck in a false minima in the parameter space. Therefore, we use the MCMC technique which can be more robust. MCMC is a stochastic sampling technique used to sample multi-dimensional probability distribution and has been extensively used for parameter estimation in Astrophysics [82–84]. For this work, we also use MCMC in order to calculate DIC. For our work, MCMC was implemented using emcee [85]. For both the models, Gaussian Priors of $H_0 = 68$ and $\sigma = 2.8$ were used for parameter estimation.

In the $R_h = ct$ model, $H(z)$ is given by:

$$H(z) = H_0(1 + z),$$

(8)
whereas for the the $\Lambda$CDM model, $H(z)$ is given as:

$$H(z) = H_0 \sqrt{\Omega_M(1 + z)^3 + \Omega_\Lambda},$$

where $\Omega_M$ and $\Omega_\Lambda$ denote the matter density and density of Cosmology constant respectively. Note that for a flat $\Lambda$CDM model, $\Omega_\Lambda = 1 - \Omega_M$, which reduces the number of free parameters by one. However, here we do a model selection against the more generalized $\Lambda$CDM model without constraining the sum of $\Omega_M$ and $\Omega_\Lambda$.

To calculate the Bayesian evidence needed for Bayesian model comparison, we need priors for our parameter set. For $\Lambda$CDM we used uniform priors for $\Omega_M$ and $\Omega_\Lambda$ from 0 to 1, and uniform prior for $H_0$ from 0 to 100 km/sec/Mpc. The same prior for $H_0$ is used in calculating the evidence for $R_h = ct$ universe model.

As mentioned earlier, we have used Hubble parameter data, obtained from cosmic chronometers and BAO. But there have been a few concerns about using data from these two sources together for parameter estimation within $\Lambda$CDM as well as for testing $R_h = ct$ universe [86]. One problem in using the BAO data for assessing the viability an alternative to $\Lambda$CDM model arises from the fact that measurement of the Hubble parameter from BAO requires the assumption of a particular cosmological model, unlike the model independent measurements of Cosmic Chronometers. One also needs to model the non-linear evolution of density and velocity fields, which are not model-independent [2, 27]. Therefore, in MY18

| $z$ | $H(z)$ (km/sec/Mpc) | $\sigma$ (km/sec/Mpc) | Ref. |
|-----|---------------------|-----------------------|------|
| 0.07| 69                  | 19.6                  | [66] |
| 0.09| 69                  | 12                    | [67] |
| 0.12| 68.6                | 26.2                  | [66] |
| 0.17| 83                  | 8                     | [67] |
| 0.179| 75                  | 4                     | [68] |
| 0.199| 75                  | 5                     | [68] |
| 0.2 | 72.9                | 29.6                  | [66] |
| 0.27| 77                  | 14                    | [67] |
| 0.28| 88.8                | 36.6                  | [66] |
| 0.352| 83                  | 14                    | [68] |
| 0.3802| 83                 | 13.5                  | [63] |
| 0.4 | 95                  | 17                    | [67] |
| 0.4004| 77                 | 10.2                  | [63] |
| 0.4247| 87.1               | 11.2                  | [63] |
| 0.4497| 92.8               | 12.9                  | [63] |
| 0.47| 89                  | 34                    | [69] |
| 0.4783| 80.9               | 9                     | [63] |
| 0.48| 97                  | 62                    | [70] |
| 0.593| 104                 | 13                    | [68] |
| 0.68| 92                  | 8                     | [68] |
| 0.781| 105                 | 12                    | [68] |
| 0.875| 125                 | 17                    | [68] |
| 0.88| 90                  | 40                    | [70] |
| 0.9 | 117                 | 23                    | [67] |
| 1.037| 154                 | 20                    | [68] |
| 1.3 | 168                 | 17                    | [67] |
| 1.363| 160                 | 33.6                  | [71] |
| 1.43| 177                 | 18                    | [67] |
| 1.53| 140                 | 14                    | [67] |
| 1.75| 202                 | 40                    | [67] |
| 1.965| 186.5               | 50.4                  | [71] |

TABLE I: $H(z)$ data from cosmic chronometers along with references to original sources. This data has been obtained from Cosmic Chronometers. This list was compiled from Ref. [64].

| $z$ | $H(z)$ (km/sec/Mpc) | $\sigma$ (km/sec/Mpc) | Ref. |
|-----|---------------------|-----------------------|------|
| 0.24| 79.69               | 6.65                  | [72] |
| 0.3 | 81.7                | 6.22                  | [73] |
| 0.31| 78.17               | 4.74                  | [74] |
| 0.35| 76.3                | 5.6                   | [75] |
| 0.36| 79.93               | 3.39                  | [74] |
| 0.38| 81.5                | 1.9                   | [76] |
| 0.43| 86.45               | 3.68                  | [72] |
| 0.44| 82.6                | 7.8                   | [77] |
| 0.51| 90.4                | 1.9                   | [76] |
| 0.52| 94.35               | 2.65                  | [74] |
| 0.56| 93.33               | 2.32                  | [74] |
| 0.57| 92.9                | 7.855                 | [78] |
| 0.59| 98.48               | 3.19                  | [74] |
| 0.6 | 87.9                | 6.1                   | [77] |
| 0.61| 97.3                | 2.1                   | [76] |
| 0.64| 98.82               | 2.99                  | [74] |
| 0.73| 97.3                | 7                     | [77] |
| 2.3 | 224                 | 8                     | [79] |
| 2.34| 222                 | 7                     | [80] |
| 2.36| 226                 | 8                     | [81] |

TABLE II: $H(z)$ Data along with references to original sources. This data has been obtained from Baryon Acoustic Oscillations. This was compiled from Ref. [64].
and MM12, no BAO data was used, whereas both BAO and chronometer data was used in BS12. To alleviate some of the concerns raised as well to compare our results with BS12, we present our results for model comparison with and without the BAO data.

VI. RESULTS

We now present our results of model comparison using both the chronometer-only dataset as well as a combination of BAO and chronometer dataset. For each of these datasets, we carried out two different analyses. The first analysis involves an unbinned analysis of the raw datasets. The second analysis reconstructing $H(z)$ using the non-parametric GPR method. For this purpose, we repeat the analysis done in BS12, wherein $H(z)$ is reconstructed at many values using GPR. The GPR reconstructed $H(z)$ for chronometer and chronometer+BAO data is shown in Figs. 1 and 2 respectively. We note that errors in GPR reconstructed $H(z)$ using only chronometers are about three times smaller than those obtained using BAO data. This is because the BAO data points at $z > 2$ have negligible error bars, whereas the errors in the reconstructed $H(z)$ using only chronometers are dominated by the errors in the last few data points at $z \approx 2$ of about 40-50 km/sec/Mpc. For doing model comparison with GPR, we use 100 reconstructed measurements uniformly distributed between the lowest and highest available redshift.

We plot the $H(z)$ data along with the predictions made from the $\Lambda$CDM model and the $R_h = ct$ model for a visual inspection. The best-fit values of $\Omega_M$, $\Omega_{\Lambda}$, $H_0$ along with 1σ errors for both the unbinned as well as the GPR data are shown in Tab. III and Tab. IV respectively. The $\Lambda$CDM values are broadly consistent with the Planck 2018 cosmological parameters [8], despite the fact that the chronometer alone data is not that precise enough.

A. Model comparison using unbinned data

Our model comparison results using unbinned analysis for chronometer-only data are summarized in Table V, whereas those for a combination of chronometer and BAO measurements can be found in Table VI. The summary of these results is as follows:

- **Chronometers**: The difference between both the models is marginal using the frequentist comparison technique. The AIC and BIC tests marginally favor $R_h = ct$, although the difference between the two models is less than five. DIC very marginally favors the $\Lambda$CDM model. The Bayes factor (defined as ratio of Bayesian evidence for $\Lambda$CDM model to $R_h = ct$) is close to one, and hence does not prefer any one model over the other. Therefore, we disagree with MM12 that $R_h = ct$ is favored, if you consider only chronometer data since based on our tests no one model is strongly favored.

- **Chronometers and BAO**: If we include BAO data, we find that all the information theory-based tests (AIC, BIC, DIC) and Bayes factor decisively favor $\Lambda$CDM. The frequentist test favors $R_h = ct$, based on comparison of the $\chi^2$ probability. However, because of the large error bars the fractional residual in most of the data points are quite large for $\Lambda$CDM, resulting in values of reduced $\chi^2$ approximately equal to 0.5. Therefore, except the frequentist test, all other tests concur with the conclusions of BS12.

B. Model Comparison using GPR data

Our results for model comparison using data reconstructed with GPR using only chronometers as well as a combination of chronometers and BAO can be found in Tables VII and VIII respectively. We summarize our results here:

- **Chronometers**: We note that the GPR error bars are about three times larger than the corresponding errors constructed after combining with BAO data. Here, AIC and DIC show strong evidence for $\Lambda$CDM. There is no strong preference for either model using BIC and Bayesian model comparison. The frequentist favors $R_h = ct$ universe model. However, the $\chi^2$/dof is less than one in both the cases. Therefore, in summary we disagree with MS18 that $R_h = ct$ provides a better fit than the $\Lambda$CDM model, since no test provides a decisive evidence for either model and most tests strongly favor the $\Lambda$CDM model.

- **Chronometers and BAO**: Here, all tests decisively favor $\Lambda$CDM over $R_h = ct$. We note however that $\chi^2$/dof is greater than 1 for both $\Lambda$CDM as well as $R_h = ct$ model. So neither of these models provide a pristine fit to the $H(z)$ data. Therefore, all the model comparison tests agree with the conclusions of BS12.

VII. CONCLUSIONS

In this work we try to independently assess the viability of $\Lambda$CDM vs $R_h = ct$ universe using only $H(z)$ measurements to resolve conflicting claims between two groups of authors. In 2012, Bilicki and Seikel [1] claimed using $H(z)$ measurements from chronometers and BARO, that $R_h = ct$ model is conclusively ruled out. This was contested by Melia and collaborators [2, 3], who showed used $H(z)$ measurements from chronometers that $R_h = ct$ universe is favored over $\Lambda$CDM. We note that
some other works have also used Type 1a SN and BAO to rule out power-law cosmology including $R_h = ct$ universe [32].

In order to settle the conflicting results between the above two groups of authors, we considered measurements from only chronometers (to emulate the analysis in Ref. [2, 3]), as well as using both chronometers and BAO measurements (to emulate the analysis in Ref. [1].) For each of these datasets, we carried out model comparison using both the unbinned data, and also by doing a non-parametric reconstruction using Gaussian Process Regression. To carry out model comparison we did five independent tests, viz., frequentist analysis (or likelihood ratio test), AIC, BIC, DIC, and Bayes factor. A summary of our results can be found in Tables V, VI, VII and VIII. Our conclusions are as follows:

- When we consider chronometer only data, none of the model comparison tests point to decisive evidence for any one model.
- When we include BAO data, all the information theory and Bayesian model comparison tests decisively favor the ΛCDM model.

| Data used | $\Omega_m$ | $\Omega_\Lambda$ | $H_0$ |
|-----------|------------|-----------------|-------|
| CH        | 0.32$^{+0.023}_{-0.036}$ | 0.69$^{+0.104}_{-0.035}$ | 67.58$^{+2.761}_{-2.764}$ |
| CH + BAO  | 0.29$^{+0.026}_{-0.023}$ | 0.79$^{+0.076}_{-0.070}$ | 67.56$^{+2.787}_{-2.761}$ |
| $\mathcal{GP}$ on CH | 0.32$^{+0.022}_{-0.025}$ | 0.72$^{+0.066}_{-0.059}$ | 67.44$^{+2.421}_{-2.821}$ |
| $\mathcal{GP}$ on CH + BAO | 0.3$^{+0.026}_{-0.023}$ | 0.75$^{+0.066}_{-0.059}$ | 67.41$^{+2.755}_{-2.796}$ |

TABLE III: Best fit values of the Cosmological parameters of the ΛCDM model with different data sets (along with 1σ errors) used for parameter estimation. CH represents Hubble parameter measurements from Cosmic Chronometers. CH + BAO represents the data set containing Hubble measurements from both Cosmic Chronometers and BAO. $\mathcal{GP}$ on the data set means that the data obtained after running Gaussian Process on a particular data set.

| Data used | $H_0$ |
|-----------|-------|
| CH        | 62.99$^{+0.955}_{-0.949}$ |
| CH + BAO  | 61.17$^{+0.311}_{-0.318}$ |
| $\mathcal{GP}$ on CH | 62.11$^{+0.181}_{-0.177}$ |
| $\mathcal{GP}$ on CH + BAO | 61.99$^{+0.071}_{-0.072}$ |

TABLE IV: Best fit values of the Cosmological parameters and associated 1σ errors of the $R_h = ct$ universe with different data sets used for parameter estimation. CH represents Hubble parameter measurements from Cosmic Chronometers. CH + BAO represents the data set containing Hubble measurements from both Cosmic Chronometers and BAO. $\mathcal{GP}$ on the data set means that the data obtained after running Gaussian Process on a particular data set.

Therefore, we broadly agree with Ref. [1], that once you include BAO data, ΛCDM is favored. The chronometer only data cannot unambiguously distinguish between the two models.

| Model | $\Lambda$CDM $R_h = ct$ |
|-------|------------------------|
| $\chi^2/\text{dof}$ | 28.5/48 50.8/50 |
| $\mathcal{L}^\chi_2(\chi^2 \text{ pdf})$ | 0.004 0.039 |
| AIC | 34.4 52.8 |
| ΔAIC | 18.4 |
| BIC | 40.2 54.7 |
| ΔBIC | 14.5 |
| DIC | 30.3 51.8 |
| ΔDIC | 21.5 |
| log Z | -186.5 -194.7 |
| Bayes’ Factor | 3640 |

TABLE VI: Model Comparison tests using unbinned measurements of chronometer data listed in Table I. log Z denotes the logarithm of the Bayesian evidence. Bayes factor is defined as the ratio of ΛCDM model to $R_h = ct$ universe model. The frequentist AIC, BIC tests marginally favor $R_h = ct$. The DIC test marginally favors ΛCDM. The Bayesian evidence for the two models are almost identical and no one model is preferred.
Table VII: Model Comparison tests using data generated from Gaussian Process Regression on chronometer data. Almost all the model comparison tests cannot robustly distinguish between the two models. The only exception is DIC, which shows strong (but not decisive) evidence for $\Lambda$CDM model.

| Model | $\Lambda$CDM | $R_h = ct$ |
|-------|---------------|------------|
| $\chi^2$/dof | 16.8/97 | 25.7/99 |
| $\mathcal{L}_{\chi^2}$ | $5.2 \times 10^{-21}$ | $9.9 \times 10^{-15}$ |
| AIC | 22.8 | 27.8 |
| $\Delta$AIC | 4.9 |
| BIC | 30.6 | 30.4 |
| $\Delta$BIC | -0.2 |
| DIC | 18.4 | 26.7 |
| $\Delta$DIC | 8.3 |
| $\log Z$ | -276.7 | -277.8 |
| Bayes’ Factor | 2.86 |

Table VIII: Model Comparison tests using data generated from Gaussian Process Regression on Chronometer + BAO data. All tests decisively favor $\Lambda$CDM model.

| Model | $\Lambda$CDM | $R_h = ct$ |
|-------|---------------|------------|
| $\chi^2$/dof | 168.5/97 | 636.3/99 |
| $\mathcal{L}_{\chi^2}(\chi^2 \text{ pdf})$ | $2.1 \times 10^{-6}$ | $9.0 \times 10^{-80}$ |
| AIC | 174.5 | 638.3 |
| $\Delta$AIC | 463.9 |
| BIC | 182.3 | 640.9 |
| $\Delta$BIC | 458.6 |
| DIC | 167.7 | 637.4 |
| $\Delta$DIC | 469.6 |
| $\log Z$ | -279.2 | -509.0 |
| Bayes’ Factor | $1.2 \times 10^{100}$ |

References:
[1] M. Bilicki and M. Seikel, Mon. Not. R. Astron. Soc. 425, 1664 (2012), 1206.5130.
[2] F. Melia and R. S. Maier, Mon. Not. R. Astron. Soc. 432, 2669 (2013), 1304.1802.
[3] F. Melia and M. K. Yennapureddy, JCAP 2018, 034 (2018), 1802.02255.
[4] P. J. Peebles and B. Ratra, Reviews of Modern Physics 75, 559 (2003), astro-ph/0207347.
[5] J. Martin, C. Ringeval, and V. Vennin, Physics of the Dark Universe 5, 75 (2014), 1303.7877.
[6] R. H. Dicke and P. J. E. Peebles, in General Relativity: An Einstein Centenary Survey (1979).
[7] D. Huterer and D. L. Shafer, Reports on Progress in Physics 81, 016901 (2018), 1709.01091.
[8] N. Aghanim et al. (Planck) (2018), 1807.06209.
[9] L. Verde, T. Treu, and A. G. Riess, Nature Astronomy 3, 891 (2019), 1907.10625.
[10] S. Bethapudi and S. Desai, Eur. Phys. J. Plus 132, 78 (2017), 1701.01789.
[11] Planck Collaboration, P. A. R. Ade, N. Aghanim, M. Arnaud, M. Ashdown, J. Aumont, C. Baccigalupi, A. J. Banday, R. B. Barreiro, J. G. Bartlett, et al., Astron. &
[76] S. Alam et al. (BOSS), Mon. Not. Roy. Astron. Soc. \textbf{470}, 2617 (2017), 1607.03155.
[77] C. Blake et al., Mon. Not. Roy. Astron. Soc. \textbf{425}, 405 (2012), 1204.3674.
[78] L. Anderson et al., Mon. Not. Roy. Astron. Soc. \textbf{427}, 3435 (2013), 1203.6594.
[79] N. G. Busca et al., Astron. Astrophys. \textbf{552}, A96 (2013), 1211.2616.
[80] T. Delubac et al. (BOSS), Astron. Astrophys. \textbf{574}, A59 (2015), 1404.1801.
[81] A. Font-Ribera et al. (BOSS), JCAP \textbf{1405}, 027 (2014), 1311.1767.
[82] D. W. Hogg and D. Foreman-Mackey, Astrophys. J. Suppl. Ser. \textbf{236}, 11 (2018), 1710.06068.
[83] J. S. Speagle, arXiv e-prints arXiv:1909.12313 (2019), 1909.12313.
[84] S. Sharma, Ann. Rev. Astron. Astrophys. \textbf{55}, 213 (2017), 1706.01629.
[85] D. Foreman-Mackey, D. W. Hogg, D. Lang, and J. Goodman, Publ. Astron. Soc. Pac. \textbf{125}, 306 (2013), 1202.3665.
[86] X. Zheng, X. Ding, M. Biesiada, S. Cao, and Z. Zhu, Astrophys. J. \textbf{825}, 17 (2016), 1604.07910.