Negative Flows of the KP-hierarchy

Guido Haak*
Department of Mathematics
405 Snow Hall
University of Kansas
Lawrence, KS 66045

Abstract

We construct a Grassmannian-like formulation for the potential KP-hierarchy including additional “negative” flows. Our approach will generalize the notion of a τ-function to include negative flows. We compare the resulting hierarchy with results by Hirota, Satsuma and Bogoyavlenskii.

1 Introduction

It is a well known fact, that the modified KdV-hierarchy allows an extension to a larger set of commuting differential equations. These are in fact the members of the sine-Gordon hierarchy, as stated for example in [3]. The additional commuting flows are still connected with the action of an infinite dimensional abelian group on an infinite dimensional manifold as it is used in the classical works of Adler and van Moerbeke [4] and Segal and Wilson. In the latter work, a part of a commutative group Γ of multiplication operators acts nontrivial on the quotient \( \text{Gl}_{\text{res}} / \text{Gl}_{\text{res}+} \) of two infinite dimensional groups by left multiplication. The resulting quotient can be interpreted as an infinite dimensional Grassmannian manifold. Out of this viewpoint Sato and later Segal and Wilson derived exact analytic results for a certain class of solutions.

*supported by the Kansas Institute for Theoretical and Computational Science at the University of Kansas
In this paper we investigate the flows of the potential KP-equation in this setting, giving a formulation in terms of $\tau$-functions and group actions on an infinite dimensional group. The extension of the hierarchy is obtained by changing the investigated action of $\Gamma$ on $\text{Gl}_{\text{res}}$ to conjugation, which extends the group of flows acting nontrivial on a certain splitting subgroup of $\text{Gl}_{\text{res}}$.

Due to the natural $\mathbb{Z}$-grading in the investigated hierarchy, we call the additional flows “negative” flows.

The outline of this work follows the paper [2], where the case of the potential KP-equation is investigated:

In section 2 we take as configuration space of the extended integrable system the whole group $\text{Gl}_{\text{res}}$ [1], which replaces the Grassmannian. This group is split into three subgroups represented as lower triangular, diagonal and upper triangular block matrices.

Letting the abelian group $\Gamma$ of Segal and Wilson act by conjugation we obtain in section 3 a nontrivial action of the whole group $\Gamma$.

In section 4 we obtain differential equations relating the matrix elements of the block matrices.

With a suitably defined $\tau$-function (section 5) we show in section 6, that out of these actions new scalar equations result, extending the potential KP-hierarchy.

We also give some thought to the set of solutions, which in section 7 is shown to be much more complicated than in the potential KP case, which was investigated in [11].

In the last chapter we compute the $LU(2)$ reduction of the flows, yielding an equation of Hirota and Satsuma, which commutes with the potential KdV-equation. This equation was also investigated by Bogoyavlenskii.

2 The configuration space

We start out by recalling the setting of Segal and Wilson [1].

Let $H$ be an infinite dimensional separable complex Hilbert space together with a splitting

$$H = H_+ \oplus H_-$$

into the orthogonal direct sum of two infinite dimensional closed subspaces $H_+$ and $H_-$. 

2
We look at the group $G_{l_{res}}$ defined also by Segal and Wilson, of invertible matrices which take w.r.t. the splitting (1) the form

$$\begin{pmatrix} a & r \\ s & d \end{pmatrix}$$

where $a$ and $d$ are Fredholm operators, and $r$ and $s$ are compact. The connected components of $G_{l_{res}}$ are labelled by the Fredholm index of the upper (or lower) diagonal block.

We now look at a factorization of elements of $G_{l_{res}}$ into diagonal, upper and lower triangular block matrices in the following way: We split every element $g$ in $G_{l_{res}}$ as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ab \\ ca & cab + d \end{pmatrix}$$

if this is possible. Here the blocks $b$ and $c$ are again compact and the diagonal blocks are invertible Fredholm. Explicitely the splitting reads:

$$a = A$$
$$b = A^{-1}B$$
$$c = CA^{-1}$$
$$d = D - CA^{-1}B$$

We therefore have a splitting into three subgroups of $G_{l_{res}}$, which we denote by $G_{l_{-}}$, $G_{l_{0}}$ and $G_{l_{+}}$.

The set of elements of $G_{l_{res}}$ which are splittable is characterized by the invertibility of the upper diagonal block, thereby forming a dense open subset of $G_{l_{res}}$. This will lateron motivate the definition of the $\tau$-function in our setting without reference to a Grassmannian.

### 3 The flows

The representation of $G_{l_{res}}$ as a space of block matrices inherits a choice of basis for the separable vector space $H$. We number the basis vectors by integer numbers. Let $e_j$, $j \geq 0$, and $e_j$, $j < 0$, be the basis vectors of $H_+$ and $H_-$, respectively.
We let $GL_{\text{res}}$ act on itself by conjugation and choose the following generators of action:

Let $\Lambda$ be the double right shift, i.e. the operator mapping $e_j$ to $e_{j+1}$. It is represented by the matrix

$$\Lambda = \sum_{j \in \mathbb{Z}} e_{j,j+1},$$

where $e_{j,k}$ are the matrix units $(e_{j,k})_{l,m} = \delta_{j,l} \cdot \delta_{k,m}$.

We define the generator of the $m$-th flow to be $\Lambda^m$. Here $m$ can be any nonzero integer. Then the flows on $GL_{\text{res}}$ are defined by

$$g(t) = \exp(\sum_{m \neq 0} t_m \Lambda^m) g(0) \exp(-\sum_{m \neq 0} t_m \Lambda^m).$$

$t$ denotes the vector with coordinates $t_j$, $j \in \mathbb{Z}$ and $g(0)$ is an initial value for the flow.

As long as only finitly many $t_j$ are nonzero, the flow on $GL_{\text{res}}$ is obviously continuous, which results in the definition of continuous local flows on the three factors $GL_-, GL_0$ and $GL_+$ in (3).

$$g_-(t)g_0(t)g_+(t) := g(t).$$

It is easy to see, that none of these flows acts trivially on $GL_{\text{res}}$ or either of the splitting subgroups $GL_+$ and $GL_-$. Our setting generalizes the Grassmannian flows: Calculating the splitting explicitly we will see, that the negative (positive) flows act by simple translation on $GL_-$ ($GL_+$) and therefore negative flows are not of interest on the big cell of the Grassmannian $\cong GL_{\text{res}}/GL_0GL_+$.

For later use we also define the abelian group $\Gamma \subset GL_{\text{res}}$ generated by the exponentials of the matrices $\Lambda^m$, $m \in \mathbb{Z}$, and its subgroups $\Gamma_+$ and $\Gamma_-$ generated by the exponentials of the matrices $\Lambda^m$ for $m \geq 0$ and $m \leq 0$, respectively. Obviously $\Gamma = \Gamma_-\Gamma_+$. These groups of course coincide with the groups defined in [1].

4 Matrix equations

From now on we write for the matrix $\exp(\sum_m t_m \Lambda^m)$ and its inverse $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$, respectively. In addition we write $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ for the matrix elements of...
the splitting (3) at $t$ and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ for the matrix elements of the splitting at $t + h, h = (\ldots, h_{-1}, h_1, \ldots)$. Clearly for the beginning we consider only $g(t)$ which are splittable. Then $g(t + h)$ is splittable for $h$ (i.e. all $h_i$) sufficiently small. In fact, as we will see later, $g(t)$ is not splittable only for isolated values of $t$.

Equations (6) and (7) lead to the following block matrix equations:

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  \tilde{c} & 1
\end{pmatrix}
\begin{pmatrix}
  \tilde{a} & 0 \\
  0 & \tilde{d}
\end{pmatrix}
\begin{pmatrix}
  \hat{a} & \hat{b} \\
  \hat{c} & \hat{d}
\end{pmatrix}

= 
\begin{pmatrix}
  a\tilde{a} + b\tilde{c}\tilde{a} + a\tilde{a}\tilde{b} + b(\tilde{c}\tilde{a} + \tilde{d})\tilde{c} \\
  \tilde{c}\tilde{a}(\hat{a} + \tilde{b}) + d(\tilde{c}\tilde{a} + (\tilde{c}\tilde{a} + \tilde{d})\tilde{c})
\end{pmatrix}

\begin{pmatrix}
  \hat{a} & \hat{b} \\
  \hat{c} & \hat{d}
\end{pmatrix}.
\tag{8}
$$

Notice that $\hat{a} = a^{-1}$ and $\hat{d} = d^{-1}$.

Regarding the matrix $\begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$ as a function of the flow parameter $t$ we easily derive a set of differential equations for the blocks. To write them in a compact notation, we define the following abbreviation for the block $s$ of the matrix $\Lambda^m, m > 0$:

$$
\Lambda^m = \begin{pmatrix}
  \Lambda^{m+}_+ & \Lambda^m_{+-} \\
  0 & \Lambda^{m-}_-
\end{pmatrix}
\tag{9}
$$

$$
\Lambda^{-m} = \begin{pmatrix}
  \Lambda^{-m+}_+ & 0 \\
  \Lambda^{-m+}_+ & \Lambda^{-m-}_-
\end{pmatrix}
\tag{10}
$$

Furtheron we will also use the subscripts $++, +-, --, +-$ and $--$ to denote the blocks of a matrix.

The blocks of $\Lambda^m$ are given by

$$
\Lambda^{m+}_+ = \begin{cases}
  \sum_{k \geq 0} e_{k+m,k}, m > 0 \\
  \sum_{k \geq 0} e_{k,k-m}, m < 0
\end{cases},
\tag{11}
$$

$$
\Lambda^{m+}_- = \sum_{k=0}^{m-1} e_{k,k-m}, m > 0,
\tag{12}
$$

$$
\Lambda^{m-}_+ = \begin{cases}
  \sum_{k<0} e_{k,k-m}, m > 0 \\
  \sum_{k<0} e_{k+m,k}, m < 0
\end{cases},
\tag{13}
$$

$$
\Lambda^{m-}_- = \sum_{k=0}^{-m-1} e_{k+m,k}, m < 0.
\tag{14}
$$

5
Denoting with $\partial_m$ the partial derivative w.r.t. the parameter $t_m$, we end up with the equations:

\[ \partial_m \tilde{a} = \Lambda^m_+ \tilde{a} - \tilde{a} \Lambda^m_+ + \begin{cases} \Lambda^m_+ \tilde{c} \tilde{a}, & m > 0 \\ -\tilde{a} \Lambda^m_- & m < 0 \end{cases}, \quad (15) \]

\[ \partial_m (\tilde{a} \tilde{b}) = \Lambda^m_+ \tilde{a} \tilde{b} - \tilde{a} \tilde{b} \Lambda^m_+ + \begin{cases} \Lambda^m_+ (\tilde{c} \tilde{a} \tilde{b} + \tilde{d}) - \tilde{a} \Lambda^m_+, & m > 0 \\ 0, & m < 0 \end{cases}, \quad (16) \]

\[ \partial_m (\tilde{c} \tilde{a}) = \Lambda^m_- \tilde{c} \tilde{a} - \tilde{c} \tilde{a} \Lambda^m_- + \begin{cases} 0, & m > 0 \\ \Lambda^m_- \tilde{a} - (\tilde{c} \tilde{a} \tilde{b} + \tilde{d}) \Lambda^m_-, & m < 0 \end{cases}, \quad (17) \]

\[ \partial_m \tilde{d} = \Lambda^m_- \tilde{d} - \tilde{d} \Lambda^m_- + \begin{cases} \tilde{c} (\tilde{a} \Lambda^m_- - \Lambda^m_- \tilde{d}), & m > 0 \\ (d \Lambda^m_- - \Lambda^m_- \tilde{a}) \tilde{b}, & m < 0 \end{cases}. \quad (18) \]

From this we get the equations for the offdiagonal blocks $\tilde{b}$ and $\tilde{c}$:

\[ \partial_m \tilde{b} = \Lambda^m_+ \tilde{b} - \tilde{b} \Lambda^m_+ - \begin{cases} \Lambda^m_+ - \tilde{a}^{-1} \Lambda^m_- \tilde{d}, & m > 0 \\ -\tilde{b} \Lambda^m_+ & m < 0 \end{cases}, \quad (19) \]

\[ \partial_m \tilde{c} = \Lambda^m_- \tilde{c} - \tilde{c} \Lambda^m_- - \begin{cases} \tilde{c} \Lambda^m_- \tilde{c}, & m > 0 \\ d \Lambda^m_- \tilde{a}^{-1} - \Lambda^m_- & m < 0 \end{cases}. \quad (20) \]

For $m > 0$, (20) is the Riccati type differential equation already encountered in [4], which has a negative counterpart, (13), for $m < 0$. In coordinates they read $(i < 0, j \geq 0, m > 0)$:

\[ \partial_m \tilde{c}_{i,j} = \tilde{c}_{i-m,j} - \tilde{c}_{i,j+m} - \sum_{k=0}^{m-1} \tilde{c}_{i,k} \tilde{c}_{k-m,j}, \quad (21) \]

\[ \partial_m \tilde{b}_{j,i} = \tilde{b}_{j+m,i} - \tilde{b}_{j,i-m} + \sum_{k=0}^{m-1} \tilde{b}_{j,k-m} \tilde{b}_{k,i}. \quad (22) \]

Therefore each of the offdiagonal blocks is given by its first row or column.

5 The $\tau$-function

In order to motivate the definition of the $\tau$-function in our setting, we consider for every $k \in \mathbb{N}$ the subgroup $G_{\text{res}}(k)$ of $G_{\text{res}}$ of invertible block matrices w.r.t. the splitting (1), which are of the form

\[ \begin{pmatrix} \mathbb{1} + p & r \\ s & \mathbb{1} + q \end{pmatrix} \]
where \( p \) and \( q \) are operators of trace class, \( r \) is in the \( k \)-th Schatten ideal in \( B(H_-, H_+) \) and \( s \) is in the \( \frac{k}{k-1} \)-th Schatten ideal in \( B(H_+, H_-) \). These groups are proper connected subgroups of \( \text{Gl}_{\text{res}} \) for every parameter \( k \).

As the elements of \( \text{Gl}_{\text{res}}(k) \) have the form

\[
\mathbb{I} + \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where the blocks \( A, B, C, D \) take values in different ideals of operators, the action (1) induces an action of \( \text{Gl}_{\text{res}}(k) \) on itself for every \( k \). The splittable elements of \( \text{Gl}_{\text{res}}(k) \) are those for which \( p \) has no eigenvalue \(-1\). Also in this case all factors of (3) are in \( \text{Gl}_{\text{res}}(k) \).

The definition of \( \text{Gl}_{\text{res}}(k) \) allows us to define the following complex function on \( \text{Gl}_{\text{res}}(k) \): For a fixed splittable element \( g_0 \) in \( \text{Gl}_{\text{res}}(k) \), the function \( \tau_{g_0} : \text{Gl}_{\text{res}}(k) \to \mathbb{C} \) is defined as the quotient of the determinant of the upper diagonal blocks of \( gg_0g^{-1} \) and \( g_0 \), i.e.

\[
\tau_{g_0}(g) := \frac{\det((gg_0g^{-1})++)}{\det(g_0++)g_0++(g^{-1})++}.
\]

By the remark at the end of section 2, this function is nonzero precisely when \( g \) is splittable.

In terms of blocks the \( \tau \)-function is given by

\[
\tau_{g_0}(g) = \det((a + b\hat{c})\hat{a}(\hat{a} + b\hat{c}) + b\hat{d}\hat{c})\det(a\hat{a}\hat{a})^{-1}
\]

\[
= \det(\mathbb{I} + (a^{-1}b\hat{c}\hat{a} + a\hat{b}\hat{c}\hat{a}^{-1} + a^{-1}b(\hat{c}\hat{a}\hat{b} + \hat{d}\hat{c}\hat{a}^{-1}))\hat{a}^{-1})
\]

\[
= \det(\mathbb{I} + \hat{a}^{-1}(a^{-1}b\hat{c}\hat{a} + a\hat{b}\hat{c}\hat{a}^{-1} + a^{-1}b(\hat{c}\hat{a}\hat{b} + \hat{d}\hat{c}\hat{a}^{-1}))),
\]  

which reduces to the known expression \([1,2]\) if we look only at elements \( g \) describing positive flows and to a similar expression for purely negative flows.

The offdiagonal blocks \( b \) and \( \hat{c} \) of the action matrices and therefore all product terms in the last two lines of (24) are of trace class. Thus the last two lines in (24) are well defined and equal for all splittable elements \( g \) of \( \text{Gl}_{\text{res}} \). We use them to define the \( \tau \)-function on the whole of \( \text{Gl}_{\text{res}} \). If \( g \) is not splittable it is natural to set \( \tau_{g_0}(g) = 0 \).

If \( g = g(t) = \exp(\sum m t_m \Lambda^m) \) we also write

\[
\tau(t, g_0) := \tau_{g_0}(g).
\]
We will see later on, that this function is analytic as a function from $\Gamma$ to $\mathbb{C}^\times$.

From its definition $\tau$ inherits an equivariance property w.r.t. the used group action. If $t$ and $s$ are both negative or both positive flows, then as usual

$$\tau(t + s, g_0) = \tau(s, g(t)) \cdot \tau(t, g_0). \quad (25)$$

Otherwise we easily get by explicit calculation using (24) and (8):

$$\tau(t_+ + s_-, g_0) = \tau(t_+, g(s_-)) \cdot \tau(s_-, g_0) \cdot c(g_-, g_+) \quad (26)$$

Here

$$g_+ = g_+(t_+) = \exp\left(\sum_{m > 0} (t_+)_m \Lambda^m\right)$$

describes a positive,

$$g_- = g_-(s_-) = \exp\left(\sum_{m < 0} (s_-)_m \Lambda^m\right)$$

a negative flow and $c(g_-, g_+)$ is the “projective multiplier” introduced by Segal and Wilson [4, Prop. 3.6]:

$$c(g_-, g_+) = \det(a_+a_-^{-1}a_+^{-1}) \quad (28)$$

$a_+$ and $a_-$ being the upper diagonal blocks of $g_+$ and $g_-$, respectively.

Using the fact that $\Gamma_-$ and $\Gamma_+$ act in a noncommuting way on the determinant line bundle over the Grassmannian, they showed that $c(g_-, g_+)$ is a homomorphism in every argument. Therefore

$$c(g_- = e^{f_-}, g_+ = e^{f_+}) = e^{S(f_-, f_+)} \quad (29)$$

with

$$S(f_-, f_+) = -\sum_m a_m b_m \quad (30)$$

if

$$f_- = \sum_{m > 0} a_m \Lambda^{-m}$$

and

$$f_+ = \sum_{m > 0} b_m \Lambda^m.$$
If we complexify the flow variables, it follows from the definition and the
known theorems on the $\tau$-function in the Grassmannian setting, that it is
holomorphic as a function from $\Gamma_+$ to $\mathbb{C}$ and as a function of $\Gamma_-$ of $\mathbb{C}$. Obviously, if we work in the setting of Segal and Wilson, $\tau(t, g_0)$ is also
locally bounded and thus holomorphic as a map from $\Gamma$, being the product
of the abelian groups $\Gamma_-$ and $\Gamma_+$, endowed with the operator norm on the
diagonal and the trace norm on the offdiagonal blocks. The zeros of $\tau$ are
therefore complex hypersurfaces of (real) codimension 2 in $\mathbb{C}^2$.

The first row of the lower triangular block $\hat{\xi}$ and the first column of
the upper triangular block $\hat{\xi}$ are given by the $\tau$ function. This is clear for
$\hat{\xi}$, as all calculations reduce to those in $[2]$ for positive flows. Introducing
the special element $m_\xi = 1 - \Lambda/\zeta = \exp(\sum_{m>0}(n_\xi)_m \Lambda^m)$, $|\zeta| > 1$, in $\Gamma_+$,
$(n_\xi)_k = -k^{-1}\zeta^{-k}$, $k > 0$, we get

$$
\tau(n_\xi, g(t)) = 1 - \sum_{k=1}^{\infty} \zeta^{-k}\hat{\xi}_{k-1,k-1}(t). \quad (31)
$$

For negative flows everything translates analogously. We choose the special
element $m_\xi = (1 - \xi/\Lambda)^{-1} = \exp(\sum_{m<0}(m_\xi)_m \Lambda^m)$, $|\xi| < 1$, in $\Gamma_-$. Then,
with $(m_\xi)_k = k^{-1}\xi^k$, $k < 0$,

$$
\tau(m_\xi, g(t)) = \det(1 + \hat{\alpha}^{-1}(m_\xi)\hat{b}(t)\hat{c}(m_\xi))
= \det(1 - (1 - \zeta^{-1})^{-1}\hat{b}(t)\Lambda^{-1}_- \xi)
= 1 - \sum_{k=1}^{\infty} \xi^k\hat{b}_{k-1,k-1}(t). \quad (32)
$$

Let $t = t_+ + t_-$ be such that $(t_+)_k = 0$ for $k \leq 0$ and $(t_-)_k = 0$ for $k \geq 0$.
The following calculation is analogous to the one in $[2]$, section 5.8 ($m > 0$):

$$
\partial_{-m}\tau(t, g_0)
= \partial_{t_{-m}}|_{t_{-m}=0}\tau(t + t_{-m}, g_0)
= \partial_{t_{-m}}|_{t_{-m}=0}(\tau(t_+ + (t_- + t_{-m}), g_0))
= \partial_{t_{-m}}|_{t_{-m}=0}(\tau(t_+ + t_{-m}, g(t_+))c(g_-(t_+ + t_{-m}), g_+(t_+))\tau(t_+, g_0))
= \partial_{t_{-m}}|_{t_{-m}=0}(\tau(t_{-m}, g(t))\tau(t_+, g(t_+))\tau(t_+, g_0)c(g_-(t_+ + t_{-m}), g_+(t_+))
= \partial_{t_{-m}}|_{t_{-m}=0}(\tau(t_{-m}, g(t))\tau(t, g_0)c(g_-(t_{-m}), g_+(t_+))). \quad (33)
$$
It follows by the same reasoning as before,
\[
\partial_{-m} \ln(\tau(t, g_0)) = \partial_{t_{-m}} \bigg|_{t_{-m}=0} (\det(\mathbb{1} + \hat{a}^{-1}(t_{-m})\hat{b}(t)\hat{c}(t_{-m})) - c_-(t_{-m}, g_+(t_+)))
\]
\[
= \partial_{t_{-m}} \bigg|_{t_{-m}=0} (\text{tr}(\hat{b}(t)\hat{c}(t_{-m})) - c_-(t_{-m}, g_+(t_+))).
\]
(34)

Explicitly this reads:
\[
\partial_{-m} \ln(\tau(t, g_0)) = -\text{tr}(\hat{b}(t)\Lambda_{-m}^-) - mt_m
\]
\[
= - \sum_{k=1}^m \hat{b}_{m-k,-k} - mt_m, \quad m > 0,
\]
(35)

analogous to
\[
\partial_m \ln(\tau(t, g_0)) = \text{tr}(\Lambda_{-m}^+ \hat{c}(t)) - mt_m
\]
\[
= \sum_{k=1}^m \hat{c}_{-k,m-k} - mt_m, \quad m > 0.
\]
(36)

Especially, together with equation [2, (5.8.1)–(5.8.2)] we have
\[
\partial_{-1} \hat{c}_{-1,0} = \partial_{-1} \partial_1 \ln \tau + 1 = -\partial_1 \hat{b}_{0,-1}.
\]
(37)

This connects the two offdiagonal blocks and at the same time proves, that
the functions \(\hat{c}_{-1,0}\) and \(\hat{b}_{0,-1}\) are meromorphic w.r.t. the variables \(t_1\) and \(t_{-1}\),
being quotients of holomorphic functions. We will see later on how far \(\tau\) is
determined by \(\hat{c}_{-1,0}\) and \(\hat{b}_{0,-1}\).

6 Extension of the potential KP-hierarchy

From [1, 2] we expect \(\hat{c}_{-1,0}\) to be the function satisfying the (potential) KP-
equation. We first look at the equations for \(\hat{c}(t + n_\zeta)\) and \(\hat{b}(t + m_\xi)\).
\[
\hat{c}(t + n_\zeta) (a(n_\zeta) + b(n_\zeta)\hat{c}(t)) = d(n_\zeta)\hat{c}(t)
\]
(38)
\[
(\hat{a}(m_\xi) + \hat{b}(t)\hat{c}(m_\xi))\hat{b}(t + m_\xi) = \hat{b}(t)\hat{d}(m_\xi).
\]
(39)

To get differential equations, we develop \(\hat{b}(t + n_\zeta)\) and \(\hat{c}(t + m_\xi)\) w.r.t. the
parameters \(\zeta\) and \(\xi\), respectively.
\[
\hat{c}(t + n_\zeta) = \sum_{k=0}^\infty \zeta^{-k} P_k \hat{c}(t),
\]
(40)
\[ \hat{b}(t + m_{\xi}) = \sum_{k=0}^{\infty} \xi^k P_{-k} \hat{b}(t). \] (41)

\( P_k, k \in \mathbb{Z} \) are differential operators determined by the special structure of the elements \( n_{\xi} \) and \( m_{\xi} \). For example

\[ P_{-3} = \frac{1}{3} \partial_{-3} + \frac{1}{2} \partial_{-1} \partial_{-2} + \frac{1}{3!} \partial_{-1}^3, \] (42)
\[ P_{-2} = \frac{1}{2} (\partial_{-1}^2 + \partial_{-2}), \] (43)
\[ P_{-1} = \partial_{-1}, \] (44)
\[ P_0 = 1, \] (45)
\[ P_1 = -\partial_1, \] (46)
\[ P_2 = \frac{1}{2} (\partial_1^2 - \partial_2), \] (47)
\[ P_3 = -\frac{1}{3} \partial_3 + \frac{1}{2} \partial_1 \partial_2 - \frac{1}{3!} \partial_1^3, \] (48)

and in general for \( k > 0 \):

\[ P_{-k}(\partial_{-1}, \ldots, \partial_{-k}) = -P_k(\partial_1, \ldots, \partial_k). \]

With \( b(n_{\xi}) = -\zeta^{-1} \Lambda_{++}, a(n_{\xi}) = \mathbb{I} - \zeta^{-1} \Lambda_{++}, \hat{c}(m_{\xi}) = -\xi \Lambda_{++}^{-1} \) and \( \hat{a}(m_{\xi}) = \mathbb{I} - \xi \Lambda_{++}^{-1} \), we get

\[ \sum_{k \geq 0} \zeta^{-k} P_k \hat{c}(\mathbb{I} - \zeta^{-1} \Lambda_{++} - \zeta^{-1} \Lambda_{+-} \hat{c}) = \hat{c} - \zeta^{-1} \Lambda_{+-} \hat{c}, \] (49)
\[ (\mathbb{I} - \xi \Lambda_{++}^{-1} - \xi \hat{b} \Lambda_{++}^{-1}) \sum_{k \geq 0} \xi^k P_{-k} \hat{b} = \hat{b} - \xi \hat{b} \Lambda_{++}^{-1}, \] (50)

or in coordinates:

\[ P_{k+1} \hat{c}_{-1,j} - P_k \hat{c}_{-1,j+1} - (P_k \hat{c}_{-1,0}) \hat{c}_{-1,j} = 0, \] (51)
\[ P_{-k-1} \hat{b}_{i-1} - P_{-k} \hat{b}_{i+1,-1} - \hat{b}_{i-1} P_{-k} \hat{b}_{0,-1} = 0, \] (52)

for \( k \geq 1, j \geq 0 \). Multiplying the first equation with \( \zeta^{-j-1} \), the second with \( \xi^{i+1} \), we end up after summation with the following equations \((k \geq 1)\):

\[ (P_{k+1} - \zeta P_k - (P_k \hat{c}_{-1,0})) \tau(n_{\xi}, g(t)) = 0, \] (53)
\[ (P_{-k-1} - \xi^{-1} P_{-k} - (P_{-k} \hat{b}_{0,-1})) \tau(m_{\xi}, g(t)) = 0. \] (54)
Using the equations
\begin{align*}
\tilde{a}(t + m_\zeta) &= a(m_\zeta)\tilde{a}(t)(\tilde{a}(m_\zeta) + \tilde{b}(t)\tilde{c}(m_\zeta)), \\
\tilde{a}(t + n_\zeta) &= (a(n_\zeta) + b(n_\zeta)\tilde{c}(t))\tilde{a}(n_\zeta),
\end{align*}
which follow from (8), we get as a byproduct in the same way explicit expressions for \(P_k\tilde{a}\) \((k \geq 1)\):
\begin{align*}
P_k\tilde{a} &= (\tilde{a}\Lambda_{++} - \Lambda_{++}\tilde{a} - \Lambda_{+-}\tilde{c}\tilde{a})\Lambda_{++}^{k-1} \\
&= -\partial_1\tilde{a}\Lambda_{++}^{k-1}, \quad (57)
\end{align*}
and
\begin{align*}
P_{-k}\tilde{a} &= \Lambda_{++}^{1-k}(-\tilde{a}\Lambda_{++}^{-1} - \tilde{a}\Lambda_{+-}^{-1} - \tilde{b}\Lambda_{-+}^{-1}) \\
&= \Lambda_{++}^{1-k}\partial_{-1}\tilde{a}, \quad k \geq 1.
\end{align*}
(58)

It follows that the evolution of the upper diagonal block w.r.t. an arbitrary flow is determined by its evolution w.r.t. the first negative and the first positive flow.

Equations (51) and (52) make it possible to express \(\partial_k\tilde{c}_{-1,j}\) and \(\partial_{-k}\tilde{b}_{j,-1}\) for \(j, k \geq 1\) as differential polynomials in \(\tilde{c}_{-1,0}\) and \(\tilde{b}_{0,-1}\), respectively. The question arises, if this is also possible for the other partial derivatives \(\partial_{-k}\tilde{c}_{-1,j}\) and \(\partial_k\tilde{b}_{j,-1}\).

Using the commutativity of the flows, we can evaluate \(\tau(m_\xi + n_\zeta, g(t))\) in two different ways. We get by (27) and Taylor expansion the following set of equations \((i, j \geq 1)\):
\begin{align*}
P_j\tilde{b}_{i-1,-1}(t) - \sum_{k=1}^{j-1} \tilde{c}_{-1,k-1}(t)P_{j-k}\tilde{b}_{i-1,-1}(t) \\
= P_{-i}\tilde{c}_{-1,j-1}(t) - \sum_{k=1}^{i-1} \tilde{b}_{k-1,-1}(t)P_{k-i}\tilde{c}_{-1,j-1}(t).
\end{align*}
(59)

This doesn’t allow us to express the negative derivatives of \(\tilde{c}_{-1,j+1}\) by derivatives of \(\tilde{c}_{-1,0}\), however, it gives us new relations expressing for example \(\partial_{-k}\tilde{c}_{-1,j}\) as a polynomial in \(\tilde{c}_{-1,i}\), \(i < j\) and positive derivatives of \(\tilde{b}_{0,-1}\).

Let us abbreviate \(u := \tilde{c}_{-1,0}, v := \tilde{b}_{0,-1}\). If we assign the following degrees to the matrix entries and derivation operators:
\begin{align*}
\deg(\tilde{b}_{j,k}) &= \deg(\tilde{c}_{j,k}) = k - j, \\
\deg(\partial_m) &= m.
\end{align*}
(60) (61)
then the equations (21), (22), (52) and (51) respect this gradation.

The picture that develops here is the following: We have in principle two copies of the KP-hierarchy, given by the component \( u = \hat{c}_{-1,0} \) together with the positive flows and \( v = \hat{b}_{0,-1} \) and the negative flows, respectively. They are not completely independent, but are coupled by the equation (37). Therefore the PKP-hierarchy for \( v \) can be pulled back to yield additional differential equations in \( u \) and \( v \), commuting with the PKP-hierarchy for \( u \). These equations are belonging to the \( t_k \) flows for negative \( k \).

For example the \( t_{-3} \)-flow gives the PKP-equation for \( v \):

\[
\frac{3}{4} \partial_{-2}^2 v = -\partial_{-2}^2 \partial_{-1} \ln \tau = \partial_{-2} \partial_{-1} (\hat{b}_{1,-1} + \hat{b}_{0,-2}) = \partial_{-1} (\partial_{-3} v - \frac{1}{4} \partial_{-1}^3 v + \frac{3}{2} (\partial_{-1} v)^2),
\]

by equation (52). For \( u \) this yields:

\[
\partial_{-1} \partial_{-2}^2 u = \partial_{-2}^2 \partial_{1} v = \partial_{-1} \partial_{1} \left( \frac{4}{3} \partial_{-3} v - \frac{1}{3} \partial_{-1}^3 v + (\partial_{-1} v)^2 \right).
\]

Neglecting an integration constant we get the equation in \( u \) and \( v \):

\[
\frac{3}{4} \partial_{-2}^2 u = -\partial_{-1} \partial_{-3} u + \frac{1}{4} \partial_{-1}^4 u - \frac{3}{2} (\partial_{-1}^2 u \partial_{-1} v).
\]

7 The set of solutions

Up to now it may not be clear, why we did not start with a group like \( \text{Gl}_{\text{res}}(k) \) in the first place, thereby avoiding the analytic difficulties we encounter with the infinite determinant and the definition of the \( \tau \)-function.

First we will see in the next section, that the reduction to 1+1-dimensional equations like the KdV-equation requires the full group \( \text{Gl}_{\text{res}} \) or at least a dense subset in it.

Second it is hard to see, which solutions we throw away by looking at the restricted subgroups. We will see that, in this setting unexpectedly, we encounter the appearance of another splitting, the analog of the well known Birkhoff factorization of \( \text{Gl}_{\text{res}} \).
The question we are primarily interested in is, what initial conditions give the same $\tau$-function. This amounts to the question how far the diagonal blocks of the initial matrix are determined by the offdiagonal blocks. The latter ones are given by the $\tau$-function. On the other hand for given $\tilde{c}_{-1,0}$ and $\tilde{b}_{0,-1}$ all derivatives $\partial_k \tilde{c}_{-1,j}$ and $\partial_{-k} \tilde{b}_{j,-1}$, $k \geq 1$ and $j \geq 1$ are given by the equations (51) and (52) as differential polynomials in $\tilde{c}_{-1,0}$ and $\tilde{b}_{0,-1}$. The Riccati equations (21) and (22) then give the whole offdiagonal block, and therefore all derivatives of $\ln \tau$ are determined up to integration constants. We can fix these by requiring

$$\partial_k \ln \tau(0, g_0) = 0$$

(65)

for all $k$ with $|k| > 1$, and all splittable $g_0$.

If we restrict ourselves to the subgroup $G_{\text{res}}^{(2)}$ of $G_{\text{res}}$, for which the offdiagonal blocks are Hilbert-Schmidt, then we may actually show, that for all splittable $g_0$ the condition (65) can be enforced by acting with the group $\Gamma_+ \times \Gamma_-$. Denoting with $L_{\gamma_-}$ and $R_{\gamma_+}$ the left multiplication with $\gamma_- = \exp \sum_{k<0} t_k \Lambda^k \in \Gamma_-$ and right multiplication with $\gamma_+ = \exp \sum_{k>0} t_k \Lambda^k \in \Gamma_+$, respectively, we see immediately:

$$\tau(g_+, L_{\gamma_-} g_0) = c(\gamma_-, g_+) \tau(g_+, g_0),$$

(66)

$$\tau(g_-, R_{\gamma_+} g_0) = c(\gamma_-, \gamma_+) \tau(g_-, g_0),$$

(67)

and therefore for $k > 1$:

$$\partial_k \ln \tau(0, L_{\gamma_-} g_0) = \partial_k \ln \tau(0, g_0) - k t_k,$$

(68)

$$\partial_{-k} \ln \tau(0, R_{\gamma_+} g_0) = \partial_k \ln \tau(0, g_0) - k t_{-k}.$$  

(69)

Choosing $t_k$ appropriately for all $k$, $|k| > 1$, we can reach (65) from any initial condition by acting with a uniquely but formally defined element $\gamma_- \times \gamma_+$ in $\Gamma_- \times \Gamma_+$ in the prescribed way. In the case, where the offdiagonal blocks of $g_0$ are Hilbert-Schmidt, the $\gamma_+$ and $\gamma_-$ obtained by this procedure converge in $\Gamma_+$ and $\Gamma_-$. The same holds, as was shown in [11], if one works with certain weighted $\ell_1$ Banach structures instead of $G_{\text{res}}$. For compact offdiagonal blocks, on the other hand, one can employ the example of Pressley and Segal [14, (8.3.4)] to show, that the offdiagonal blocks of $\gamma_- \times \gamma_+$ may not even be bounded operators.
The ambiguity left is an overall multiplicative constant, which we can set to be one, since it doesn’t affect the logarithmic derivatives of $\tau$.

We look at the differential equations (19) and (20) for the offdiagonal blocks $\tilde{b}$ and $\tilde{c}$, respectively. It is obvious, that they, and therefore the solutions, don’t change, if the last terms containing $\tilde{a}$ and $\tilde{d}$ do not. This gives us conditions on the stabilizer of a solution. In the following we will consider only splittable initial conditions. We write out the conditions on the last terms in (19) and (20) as matrix equations:

Let $\bar{a}$, $\bar{d}$ and $\tilde{a}$, $\tilde{d}$ be pairs of matrices, for which

\[
\bar{a}^{-1}\Lambda_{+}^{m} \bar{d} = \tilde{a}^{-1}\Lambda_{+}^{m} \tilde{d}, \tag{70}
\]

\[
\bar{d}\Lambda_{+}^{m} \bar{a}^{-1} = \tilde{d}\Lambda_{+}^{m} \tilde{a}^{-1}. \tag{71}
\]

We may write

\[ D = \bar{d}^{-1}\tilde{d}, \quad A = \bar{a}^{-1}\tilde{a}, \]

and thus obtain from equation (71) the condition

\[ DA^{-m} = \Lambda^{-m} A. \tag{72}\]

In coordinates this is

\[
(D\Lambda_{-+}^{-m})_{ij} = \begin{cases} 
D_{i,-j-m}, & j = 0, \ldots, m-1, \quad i < 0, \\
0, & j \geq m
\end{cases}, \tag{73}
\]

\[
(\Lambda_{-+}^{-m} A)_{ij} = \begin{cases} 
A_{i+m,j}, & i = -m, \ldots, -1, \quad j > 0, \\
0, & i < -m
\end{cases}. \tag{74}
\]

Therefore $A$ and $D$ are lower echelon matrices, which are determined by each other.

The notion “triangular” here means that the entries on the diagonal are all 1, an echelon matrix is a product of an arbitrary diagonal and a triangular matrix.

In the same way we derive from equation (70) the conditions

\[ \tilde{A}\Lambda_{+}^{m} = \Lambda_{+}^{m} \tilde{D}, \tag{75}\]

for the matrices

\[ \tilde{D} = \bar{d}\bar{d}^{-1}, \quad \tilde{A} = \bar{a}\bar{a}^{-1}, \]

15
which, because of

\[
(A_{+}^{m+D})_{ij} = \begin{cases} 
\hat{D}_{i-m,j}, & i = 0, \ldots, m - 1, j < 0, \\
0, & i \geq m, j < 0
\end{cases}, \quad (76)
\]

\[
(\tilde{A}A_{+}^{m+D})_{ij} = \begin{cases} 
\tilde{A}_{i,j+m}, & j = -m, \ldots, -1, i > 0, \\
0, & j < -m
\end{cases}, \quad (77)
\]

amount to \( \tilde{A} \) and \( \hat{D} \) being upper echelon matrices, which are determined by each other.

We end up with the following condition: The set of \( g \in \text{Gl}_{\text{res}} \) yielding the same \( \tau \)-function as

\[
\left( \begin{array}{cc} 1 & 1 \\
\hat{c} & 1 \\
\end{array} \right) \left( \begin{array}{cc} \tilde{a} & 0 \\
0 & \tilde{b} \\
\end{array} \right) \left( \begin{array}{cc} 1 & 1 \\
0 & 1 \\
\end{array} \right),
\]

and for which (75) holds, is parametrized by invertible upper and lower triangular Fredholm matrices \( \tilde{A} \) and \( A \), for which

\[
\tilde{a} = \tilde{A}\tilde{a}A^{-1}.
\]

We showed before, that if we look at \( \text{Gl}_{\text{res}}^{(2)} \), \( \tilde{\Gamma} = \Gamma_{+} \times \Gamma_{-} \) acts freely on the matrices \( g_{0} \), which give the same solution. Let’s denote by \( S \) the quotient of the flow action of \( \Gamma \) and the above described action of \( \tilde{\Gamma} \) on \( \text{Gl}_{\text{res}}^{(2)} \). A solution then determines an element of \( S \) up to a transformation of the upper triangular block by (78). I.e. the fibre in \( S \) over a solution, which has an initial condition with upper diagonal block \( \tilde{a} \) is given by the stabilizer of \( \tilde{a} \) w.r.t. the action (78) of invertible upper and lower echelon Fredholm operators.

Notice that the stabilizer of \( \tilde{a} \) may vary in a very complicated way over \( \text{Gl}_{\text{res}} \). We want to investigate the structure of the stabilizer in a special case a little further. We employ the Gauss algorithm in order to split the matrix \( \tilde{a} \) into an upper triangular, a diagonal and a lower triangular matrix.

The Gauss algorithm allows us to write down matrices \( \tilde{a}_{L}, a_{D} \) and \( a_{U} \), s.t.

\[
\tilde{a}a_{L} = a_{U}a_{D},
\]

for \( \tilde{a} \) in a dense subset of \( B(H_{+}) \), to which we restrict ourselves. Even for infinite matrices it is no problem to multiply a lower triangular matrix like
$a_L = \tilde{a}_L^{-1}$ from the right with an arbitrary matrix, as all matrix elements of the product consist of sums of finitely many products of matrix entries. Therefore we have a kind of Birkhoff factorization
\[ \tilde{a} = a_U a_D a_L, \] (79)

where the factors are uniquely determined matrices, but not necessarily bounded operators on $H_\pi$.

Writing down (78) we get
\[ a_U^{-1} \tilde{a} a_U a_D a_L A^{-1} a_L^{-1} = a_D. \]

The only freedom we have, is to choose the diagonal part of $a_U^{-1} \tilde{a} a_U$, because of the uniqueness of the splitting (79). However, as $\tilde{A}$ has to be a bounded operator, we get additional highly nontrivial conditions on this diagonal matrix. To be precise, if
\[ a_U^{-1} \tilde{a} a_U = U \cdot D, \]

$U$ being upper triangular and $D$ being diagonal, then we have to choose $D$ in such a way, that $a_U U D a_U^{-1}$ is a bounded operator.

If $\tilde{a}$ contains a permutation matrix in its Birkhoff decomposition, then the stabilizer gets bigger, including upper triangular matrices, which are transformed by $\tilde{a}$ to lower triangular matrices. It is an open problem, if there is any “solution manifold” as in the Grassmannian case [11].

8 A comment on the connection with the Hirota-Satsuma equation

We want to link up the negative PKP-hierarchy with work done by Bogoyavlenskii and others [6, 7, 8, 9].

To this end we consider the usual reduction from the KP to the KdV-hierarchy [1]:
Let $G_{KdV}$ be the subgroup of $G_{res}$ of all matrices, which commute with the square of the double shift $\Lambda^2$. Taken as initial conditions of the flows (3), these are precisely the ones on which all even numbered flows act trivially.

This subgroup can easily be identified with the loop group $LGL(2, \mathbb{C})$, where the identification in terms of matrix units is
\[ e_{ij} \rightarrow \lambda^{[i]} - [i] e_{i \mod 2, j \mod 2}, \] (80)
[x] denoting the greatest integer less than x. This way the double shift \( \Lambda \) is identified with the matrix
\[
\begin{pmatrix}
0 & 1 \\
\lambda & 0
\end{pmatrix}.
\] (81)
The identification, however, does not respect the splitting (8) as it identifies offdiagonal with diagonal blocks. As we are not really interested in the behaviour of the diagonal blocks we include them into the positive part.

This yields a well known formulation [10] of the potential KdV-equation (PKdV) in terms of a loop group splitting, the PKdV thus being the standard loop group reduction of the potential KP-hierarchy.

Again, we let the flows act by conjugation. The subgroups of the splitting are now
\[
G_+ = \{ g(\lambda) \in \text{LGL}(2, \mathbb{C}) | g(\lambda) \text{ is analytic inside the unit circle} \},
\]
\[
G_- = \{ g(\lambda) \in \text{LGL}(2, \mathbb{C}) | g(\lambda) \text{ is analytic outside the unit circle},
\quad g(0) = 1 1 \}
\] (82)
The product \( G_- G_+ \) is a dense open subset in \( \text{LGL}(2, \mathbb{C}) \) [14]. Since the independent function \( u = \hat{c}_{-1,0} \) which satisfies the extended PKP-hierarchy is mapped to the upper right corner of the \( \lambda^{-1} \) coefficient of \( g_- \), the latter is also the function which satisfies the reduced negative KP-hierarchy.

The first equation plays the same role for the KdV-equation as the sine-Gordon equation does for the modified KdV-equation.

The flow matrices are mapped to the matrix
\[
\Lambda \rightarrow p_1 = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix},
\] (84)
\[
\Lambda^{-1} \rightarrow p_{-1} = \lambda^{-1} p_1.
\] (85)
Therefore the flows read
\[
g(x, t) = \exp(x p_1 + t p_{-1}) g_0 \exp(-x p_1 - t p_{-1}) = g_-(x, t) g_+(x, t)^{-1}.
\] (86)
To reproduce a zero curvature condition for the negative flows we adopt the double group formulation of negative flows [13, 12] This amounts to the following: If we look at (86) we may write the splitting in a slightly different way, \( g_0 = g_0^- g_0^+ \),
\[
g(x, t) = \left( \exp(x p_1 + t p_{-1}), \exp(x p_1 + t p_{-1}) \right) (g_-, g_0^+)\]
\[
= (g_-(x, t), g_+(x, t))(v(x, t), v(x, t)),
\] (87)
where we have identified the set of splittable matrices in $LGL(2, \mathbb{C})$ to the subset $G_- \times G_+$ of the product group $LGL(2, \mathbb{C}) \times LGL(2, \mathbb{C})$ by
\[
g_-g_+^{-1} \rightarrow (g_-, g_+). \tag{88}
\]
The additional factor $(v, v)$ occurs due to the fact, that the flows do not stay in the subgroup $G_- \times G_+$. This amounts to a splitting of the double group $G = LGL(2, \mathbb{C}) \times LGL(2, \mathbb{C})$ into the subgroups $G_- = G_- \times G_+$ and $G_+ = \text{diag}(G) = \{(x, y) \in G | x = y\}$. The flows then act by left multiplication with $(\exp(xp_1 + tp_{-1}), \exp(xp_1 + tp_{-1})) \in G_+$ and reproduce the well known formulation of an integrable system with only positive flows.

We explicitly write down $v(x, t)$:
\[
v(x, t) = g_-(x, t)^{-1} \exp(xp_1 + tp_{-1})g_0- = g_+(x, t)^{-1} \exp(xp_1 + tp_{-1})g_0+. \tag{89}
\]
It follows that the matrices
\[
U = \partial_x vv^{-1}, \tag{90}
V = \partial_t vv^{-1}, \tag{91}
\]
satisfy the zero curvature condition
\[
\partial_t U - \partial_x V + [U, V] = 0. \tag{92}
\]

We further reduce to the loop group $LSU(2)$ and write
\[
g_- = \exp(\lambda^{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \text{lower order terms}). \tag{93}
\]
We therefore get
\[
U(x, t) = (\text{Ad}(g_1^{-1})p_1)_+ = \begin{pmatrix} -b & 1 \\ \lambda + 2a & b \end{pmatrix} \tag{94}
\]
and by
\[
V(x, t)_- = (\text{Ad}(g_1^{-1})p_{-1})_- = \lambda^{-1}V_{-1}, \tag{95}
\]
with $V_{-1}$ independent of $\lambda$, also

$$V(x,t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (-g_{-1}^{-1}\partial_t g_- + g_{-1}^{-1}p_{-1}g_-) - \lambda^{-1}U(x,t) - \begin{pmatrix} a_t & b_t \\ c_t & -a_t \end{pmatrix} \lambda^{-1}. \quad (96)$$

Here the subscripts $+$ and $-$ indicate projection to the Lie algebras of $\mathfrak{gl}_+$ and $\mathfrak{gl}_-$, respectively.

In addition we have by evaluating the upper right corner of the $\lambda^{-1}$-coefficient of

$$g_{-1}^{-1}\partial_x g_- - \text{Ad}(g_{-1}^{-1})p_1 = g_{+1}^{-1}\partial_x g_+ - \text{Ad}(g_{+1}^{-1})p_1 = \frac{1}{2} (b_x + b^2). \quad (97)$$

This is the Riccati equation for the reduced case, where $\dot{c}_{-1,1} + \dot{c}_{-2,0} = \partial_2 \ln(\tau) = 0$.

Evaluating (92) yields the first negative equation in the KdV-hierarchy for the matrix element $b$:

$$b_{xx} - \frac{1}{4}b_{xxx} - 2b_x b_{tx} - b_t b_{xx} = 0, \quad (98)$$

or for $u(x,t) = -4(b(x,t) - t)$:

$$u_{xxx} = u_t u_{xx} + 2u_x u_{xt}, \quad (99)$$

which is Bogoyavlenskii’s version of the Hirota-Satsuma equation for long waves in a medium with nonlinear dispersion. Bogoyavlenskii investigated it as an integrable equation with overturning (breaking) solitons. In the setting of this section it was derived as the first negative flow of the potential KdV-equation by Szmigielski.

In the case of the Grassmannian formulation of the potential KP equation, one recovers the elements of the potential KdV hierarchy as reductions of equations of the PKP-hierarchy, i.e. simply by setting derivatives w.r.t. even numbered variables to zero.

This works even though the splitting in the reduced and the unreduced case is quite different, i.e. in the reduced case we split along the diagonal, in the unreduced case we leave whole block matrices on the diagonal.

In the reduction the $\tau$-function gets lost and is replaced by the zero curvature condition. This yields the fact, that due to the loss of the quations...
and (52) in the reduced case, we can no more express the offdiagonal blocks merely by one corner entry $\tilde{c}_{-1,0}$ or $\tilde{b}_{0,-1}$, respectively.

Instead we end up in the reduction to $LSU(n)$ with $n - 1$ seemingly independent functions, which describe the offdiagonal blocks. The Riccati equations (21) and (22) are still applicable and occur in the form of equation (97).

We plan to investigate the connection between the extended PKdV-hierarchy and the extended PKP-hierarchy in a separate publication.

References

[1] G. Segal, G. Wilson, “Loop Groups and Equations of KdV type”, Publ. IHES 61, 1985.

[2] J. Dorfmeister, E. Neher, J. Szmigielski, “Automorphisms of Banach Manifolds Associated with the KP-equation”, Quart. J. Math. Oxford (2), 40 (1989).

[3] V. G. Drinfeld, V. V. Sokolov, “Equations of Korteweg-de Vries type and simple Lie algebras”, Soviet Math. Dokl. 23 (1981).

[4] M. Adler, P. van Moerbeke, “Completely integrable systems, Euclidean Lie algebras and curves” and “Linearization of Hamiltonian systems, Jacobi varieties and representation theory”, Adv. Math. 38 (1980).

[5] B. Simon, “Trace Ideals and Their Applications”, London Math. Soc. Lecture Notes 35, Cambridge University Press, 1979.

[6] O. I. Bogoyavlenskii, “Overturning Solitons in New Two-Dimensional Integrable Equations”, Math. USSR Izvestiya 34 (1990).

[7] J. Szmigielski, “On Soliton Content of Self Dual Yang-Mills Equations”, Preprint hep-th 9311119, 1993.

[8] R. Hirota, J. Satsuma, “N-soliton of Model Equations for Shallow Water Waves”, J. Phys. Soc. Japan 40 (1976).
[9] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, Stud. Appl. Math. 53, 249 (1974).

[10] J. Dorfmeister, “Banach Manifolds of Solutions to Nonlinear Partial Differential Equations, and Relations with Finite Dimensional Manifolds”.

[11] J. Dorfmeister, “Weighted $\ell_1$-Grassmannians and Banach Manifolds of Solutions to the KP-equation and the KdV-equation”.

[12] G. Haak, “Analytische Untersuchungen an klassischen und quantenintegrablen Systemen”, PhD-thesis, Freie Universität Berlin, FB Physik, 1993.

[13] H.-Y. Wu, “Nonlinear partial differential equations via vector fields on homogenous Banach manifolds”, Ann. Global Anal. Geom. 10 (1992).

[14] A. Pressley, G. B. Segal, “Loop Groups”, Clarendon Press, Oxford, 1986.