Limiting temperature of pion gas
with the van der Waals equation of state

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Abstract

The grand canonical ensemble formulation of the van der Waals equation of state that includes the
effects of Bose statistics is applied to the equilibrium system of interacting pions. If the attractive
interaction between pions is large enough, a limiting temperature $T_0$ emerges, i.e., no thermodynamical
equilibrium is possible at $T > T_0$. The system pressure $p$, particle number density $n$, and energy density
$\varepsilon$ remain finite at $T = T_0$, whereas for $T$ near $T_0$ both the specific heat $C = d\varepsilon/dT$ and the scaled
variance of particle number fluctuations $\omega[N]$ are proportional to $(T_0 - T)^{-1/2}$ and, thus, go to infinity
at $T \to T_0$. The limiting temperature also corresponds to the softest point of the equation of state,
i.e., the speed of sound squared $c_s^2 = dp/d\varepsilon$ goes to zero as $(T_0 - T)^{1/2}$. Very similar thermodynamical
behavior takes place in the Hagedorn model for the special choice of a power, namely $m^{-4}$, in the
pre-exponential factor of the mass spectrum $\rho(m)$.

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I. INTRODUCTION

The van der Waals (VDW) equation of state (EoS) is a simple analytical model of the pressure function $p$ for the system of particles with both attractive and repulsive interactions. In the canonical ensemble (CE), where independent variables are temperature $T$, volume $V$, and number of particles $N$, the VDW EoS has the most simple and transparent form, (see, e.g., Refs. [1, 2]),

$$p(T, n) = \frac{N T}{V - b N} - \frac{a N^2}{V^2} \equiv \frac{n T}{1 - b n} - a n^2,$$  \hspace{1cm} (1)

where $a > 0$ and $b > 0$ are the VDW parameters that describe attractive and repulsive interactions, respectively, and $n \equiv N/V$ is the particle number density. The first term in the right-hand-side of Eq. (1) contains the excluded volume correction (for instance, $b = 16 \pi r^3 / 3$ with $r$ being the particle hard-core radius), the second term comes from the mean field description of the attractive interactions.

In order to apply the VDW EoS to systems with variable number of particles the grand canonical ensemble (GCE) formulation is needed. This was done for the VDW equation (1) in our recent paper [3]. The GCE formulation of EoS in the form $p = p(T, n)$, including the VDW equation, can also be conveniently treated within the thermodynamic mean-field approach [4–6]. As the next step, we proposed in Ref. [7] the generalization of the VDW EoS that includes effects of the quantum statistics. The nuclear matter was considered in [7] as the system of interacting nucleons with Fermi statistics and the VDW EoS. With this model the curve of the first order phase transition and its end point (the so called critical point) were obtained. In Ref. [8] the fluctuations in the vicinity of the critical point were investigated.

In the present paper we apply the VDW EoS with quantum statistics to a description of the gas of interacting pions. In this case the GCE formulation is really necessary: the number of pions is not a conserved quantity and cannot be considered as an independent variable of the CE. Note, the EoS of interacting pions was studied during last decades within different theoretical approaches (see, e.g., [9–13] and references therein).

The paper is organized as follows. In Sec. II we present the VDW EoS that includes the effects of quantum statistics. In Sec. III this formalism is applied to the pion gas. The effects
related to the limiting temperature are investigated in Sec. IV and a comparison with the Hagedorn model is conducted in Sec. V. A summary in Sec. VI closes the article.

II. VDW EQUATION OF STATE FOR QUANTUM STATISTICS

The pressure and the particle number density for the quantum VDW EoS in the GCE are defined as the following [7]:

\[ p(T, \mu) = p^{\text{id}}(T, \mu^*) - a n^2(T, \mu), \quad n(T, \mu) = \frac{n^{\text{id}}(T, \mu^*)}{1 + b n^{\text{id}}(T, \mu^*)}, \]  

(2)

where

\[ \mu^* = \mu - b p(T, \mu) - a b n^2(T, \mu) + 2 a n(T, \mu), \]  

(3)

with \( \mu \) being the chemical potential which regulates the particle number density in the GCE. Quantities \( p^{\text{id}} \) and \( n^{\text{id}} \) correspond to the quantum ideal gas pressure and particle density, respectively:

\[ p^{\text{id}}(T, \mu^*) = \frac{g}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}} \left[ \exp \left( \frac{\sqrt{m^2 + k^2} - \mu^*}{T} \right) + \eta \right]^{-1}, \]  

(4)

\[ n^{\text{id}}(T, \mu^*) = g \int \frac{d^3k}{(2\pi)^3} \left[ \exp \left( \frac{\sqrt{m^2 + k^2} - \mu^*}{T} \right) + \eta \right]^{-1}, \]  

(5)

where \( g \) is the degeneracy factor, \( m \) is the particle mass, \( \eta = +1 \) for Fermi statistics, \( \eta = -1 \) for Bose statistics.

The quantum formulation (2)–(5) of the VDW EoS in the GCE has the following basic properties: (i) it transforms to the ideal quantum gas if both \( a = 0 \) and \( b = 0 \); (ii) it becomes equivalent to the classical VDW EoS [1] in a region of thermodynamical parameters where quantum statistics can be neglected; (iii) the entropy found from Eqs. (2)–(5) is a non-negative quantity and it goes to zero at \( T \to 0 \) in accordance with the third law of thermodynamics (see Ref. [7] for details). Note, in the Boltzmann approximation, i.e., \( \eta = 0 \) in Eqs. (4) and (5), EOS obtained in the GCE from (2)–(5) as \( p = p(T, n) \) becomes identical to the VDW equation of state given in Eq. [1].

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III. PION GAS WITH THE VDW EOS

In a case of the pion gas one should use the Bose statistics, \( \eta = -1 \) in Eqs.\( ^4 \) and \( ^5 \). The pion system with zero value of total electric charge is considered, thus, \( \pi^+ \), \( \pi^- \), and \( \pi^0 \) all have zero chemical potentials, i.e., \( \mu = 0 \) in Eq. \( ^3 \). As a result, \( p \) and \( n \) in Eq. \( ^2 \) become the functions of one variable, \( T \), and Eqs. \( ^3 \)–\( ^5 \) take the form:

\[
\mu^* = -b p(T) - a b n^2(T) + 2 a n(T) ,
\]

\[
p^{id}(T, \mu^*) = \frac{1}{2 \pi^2} \int_{0}^{\infty} k^2 dk \frac{k^2}{\sqrt{m^2_{\pi} + k^2}} \left[ \exp \left( \frac{\sqrt{m^2_{\pi} + k^2} - \mu^*}{T} \right) - 1 \right]^{-1} ,
\]

\[
n^{id}(T, \mu^*) = \frac{3}{2 \pi^2} \int_{0}^{\infty} k^2 dk \left[ \exp \left( \frac{\sqrt{m^2_{\pi} + k^2} - \mu^*}{T} \right) - 1 \right]^{-1} ,
\]

where we set \( g = 3 \) for the pion degeneracy factor and \( m_{\pi} \approx 138 \text{ MeV} \) for the pion mass. In what follows we take a fixed value of \( b \approx 0.45 \text{ fm}^3 \) that corresponds to \( r \approx 0.3 \text{ fm}^{-3} \) for the pion hard-core radius \[14 \] \[15 \]. The value of \( a \) is considered as a free model parameter. Note also that the entropy density \( s(T) \) and energy density \( \varepsilon(T) \) can be calculated from the function \( p(T) \) as the following:

\[
s = \frac{dp}{dT} , \quad \varepsilon(T) = T \frac{dp}{dT} - p .
\]

We consider \( T \leq 160 \text{ MeV} \) as a region, where the pion gas may exist. At higher temperatures the effects related to the deconfinement are expected to play a major role. For every \( T \), at \( T \leq 160 \text{ MeV} \), we find \( \mu^* = \mu^*(T) \) by solving numerically the transcendental equation \( ^6 \), the functions \( n(T) \) and \( p(T) \) are then found from Eq. \( ^2 \). If Eq. \( ^6 \) has more then one solution for a given \( T \), only the solution with a larger value of \( p(T) \) should be considered as a stable one according to the Gibbs criteria. For the VDW EoS in the GCE these criteria were considered in Ref. \[7 \].

The results for \( \mu^* \), \( p \), \( n \), and \( \varepsilon \) as functions of \( T \) are shown in Figs.\( ^1 \) (a)-(d) by solid lines for different values of the parameter \( a \). Dotted lines in these figures correspond to the ideal

\[1 \] In the next section we discuss how the model results depend on the numerical value of \( b \), particularly at \( b \to 0 \).
Figure 1: The thermodynamical quantities of the pion gas with the VDW EoS as functions of temperature: shifted chemical potential $\mu^*$ (a), particle number density (b), pressure (c), and energy density (d). The excluded volume parameter is fixed as $b = 0.45 \text{ fm}^3$. The solid lines correspond to the solutions with $a/b = 0, 200, 400, 500, 600 \text{ MeV}$ (from bottom to top). The dotted lines show the behavior of the ideal pion gas, i.e., for both $a = 0$ and $b = 0$. The points at the limiting temperature are depicted by the cross symbols $\times$. In the panel (a) the unstable solutions are depicted by dashed line segments with endpoints at $T = T'$ depicted by the filled circles.

pion gas (i.e., both $a = 0$ and $b = 0$). In the region of $T \leq 100 \text{ MeV}$ (not shown in Fig. 1) the interaction effects for the considered $a$ and $b$ values are small, i.e., all lines are very close to those of the ideal gas. This is due to the fact that the VDW interactions depend on the particle number density, and they become negligible at $n \to 0$.

The repulsive interactions suppress the VDW thermodynamic functions $n(T)$, $p(T)$, and
\(\varepsilon(T)\) in comparison to those of the ideal pion gas. This is clearly seen for lines with \(a = 0\) in Figs. 1(b)-(d). The VDW repulsion also introduces the upper limit for the particle number density, namely, \(n \leq 1/b \approx 2.21\ \text{fm}^{-3}\). On the other hand, the presence of the attractive interactions enhances \(n(T), p(T),\) and \(\varepsilon(T)\) in comparison to the ideal pion gas. One observes that the VDW EoS with \(b = 0.45\ \text{fm}^3\) and \(a/b = 200\ \text{MeV}\) appears to be close to that of the ideal gas, i.e., the actions of repulsive and attractive interactions almost cancel each other out in all thermodynamical functions.

We note that the VDW excluded volume correction is consistent with the virial expansion for hard spheres only at low enough densities. At large values of the so-called packing fraction \(\eta = (bn)/4\) (a fraction of the total volume occupied by the particles of finite size), namely \(\eta \gtrsim 0.1\), the VDW equation deviates significantly from the equation of state for hard spheres. The highest value of the pion number density obtained in the present work is \(n_{\text{max}} \approx 0.30\ \text{fm}^{-3}\), as seen in Fig. 1. Therefore, the maximum value of the packing fraction is around \(\eta_{\text{max}} = 0.034\). At such values of \(\eta\) the VDW equation is still fully consistent with the virial expansion for the hard spheres (see, e.g., Fig. 1 in Ref. [6]) and thus can be used.

IV. LIMITING TEMPERATURE

At large values of the parameter \(a\) a new phenomenon takes place: a limiting temperature \(T_0\) emerges, i.e., Eq. (6) has no stable solutions at \(T > T_0\). At the same time in the temperature interval \(T' \leq T \leq T_0\) the unstable solutions for the VDW thermodynamical functions appear. Here the temperature \(T'\) is determined as a starting point of the Bose-Einstein condensation, i.e., \(\mu^*(T') = m_\pi\). The values of the thermodynamical functions at the limiting temperature \(T_0\) are depicted by the cross symbols \(\times\) in Figs. 1(a)-(d), while the values at \(T = T'\) are depicted in Fig. 1(a) by circles, and the unstable solutions are shown by the dashed segments of lines. The stable solutions shown in all figures by solid lines correspond to the Gibbs criteria, i.e., they have pressure larger than that of the unstable solutions at equal temperatures. The unstable solutions are shown only for a shifted chemical potential \(\mu^*(T)\), in the panel (a) of Fig. 1.

\(^2\) Note that only temperatures \(T < 160\ \text{MeV}\) are considered.
To elucidate a physical origin of the limiting temperature $T_0$ for the pion gas with the VDW EoS we calculate the specific heat, $C = d\varepsilon/dT$, the speed of sound squared, $c_s^2 = dp/d\varepsilon$, and the scaled variance of particle number fluctuations,

$$\omega[N] = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = \omega_{\text{id}}(T, \mu^*) \left[ \frac{1}{(1 - bn)^2} - \frac{2an(T, \mu^*)}{T} \omega_{\text{id}}(T, \mu^*) \right]^{-1}, \quad (10)$$
where

\[ \omega_{id}(T, \mu^*) = \frac{1}{n_{id}(T, \mu^*)} \frac{3}{2\pi^2} \int_0^\infty k^2 dk \left[ \exp \left( \frac{\sqrt{m_\pi^2 + k^2} - \mu^*}{T} \right) - 1 \right]^{-2} , \tag{11} \]

is the scaled variance of the particle number fluctuations in the ideal pion gas with the chemical potential \( \mu^* \). Equations (10) and (11) were obtained for the quantum VDW EoS in Ref. [8].

As seen from Figs. 2 (a)-(c), \( c_s^2 \to 0 \), while both \( C \to \infty \) and \( \omega[N] \to \infty \) at \( T \to T_0 \). The straightforward calculations reveal the power-law behavior at \( T \to T_0 \): both \( C \) and \( \omega[N] \) are proportional to \( (T_0 - T)^{-1/2} \), and \( c_s^2 \propto (T_0 - T)^{1/2} \). Fluctuations of the total energy \( E \) behave very similar to fluctuations of the number of pions \( N \). The limiting temperature corresponds, therefore, to the softest point of the EoS (the speed of sound equals zero) and to infinitely large values of the heat capacity and fluctuations. In Fig. 2 (d) a dependence of \( T_0 \) on parameter \( a \) is shown for different values of repulsive parameter \( b \): \( b = 0, b = 0.45 \text{ fm}^3 \) and \( b = 1 \text{ fm}^3 \). It is seen from Fig. 2 (d) that, at each value of VDW repulsion parameter \( b \), there is a minimal value of VDW attraction parameter \( a = a_b \) such that the limiting temperature \( T = T_0 \leq 160 \text{ MeV} \) appears at all \( a \geq a_b \), and \( T_0 \) decreases with increasing \( a \). At \( b = 0.45 \text{ fm}^3 \), one finds that \( T_0 \approx 140 \text{ MeV} \) and \( 155 \text{ MeV} \) at \( a \approx 270 \text{ MeV fm}^3 \) and \( 225 \text{ MeV fm}^3 \), respectively. For the smaller values of \( b \), the same values of the limiting temperature, \( T_0 \approx 140 \text{ MeV} \) and \( 155 \text{ MeV} \), are reached at smaller values of \( a \). Particularly, for \( b = 0 \), the values of \( T_0 \approx 140 \text{ MeV} \) and \( 155 \text{ MeV} \) are obtained at \( a \approx 219 \text{ MeV fm}^3 \) and \( 168 \text{ MeV fm}^3 \), respectively. Note that additional information is needed to have further restrictions on the parameters \( a \) and \( b \). From the present analysis the most interesting physical behavior with \( T_0 = 150 \pm 10 \text{ MeV} \) takes place in the range of \( b \) from 0 to 0.45 \text{ fm}^3 \) and of \( a \) from 150 \text{ MeV fm}^3 \) to 300 \text{ MeV fm}^3 \).

The VDW parameters \( a \) and \( b \) were obtained in Ref. [7] for a system of nucleons. Based on the ground state properties of the nuclear matter it has been estimated that for nucleons \( b \approx 3.42 \text{ fm}^3 \) and \( a \approx 329 \text{ MeV fm}^3 \). Therefore, the numerical values of both \( b \) and \( a \) VDW parameters in the pion gas considered in the present study are smaller than those in the system of interacting nucleons.

In principle, the information about the VDW parameters \( a \) and \( b \) could be inferred from the \( \pi\pi \)-scattering data. Note that second cluster integral for the equilibrium pion gas was calculated in Ref. [10]. The phase shifts in \( \pi\pi \) scattering were described by assuming that both
the hard-core repulsion and resonance attraction are equally important and should be taken into account simultaneously. From the specific calculation in Ref. [16] it could be inferred that parameter $a$ may take values up to few hundred MeV fm$^3$, and could in general be temperature dependent. However, there is no reason to expect that the higher terms in the cluster expansion are negligible. The attractive parts of higher cluster terms can be presented as contributions of multi-pion resonances. The properties of the resonances (masses and widths), and even their existence themselves, are not very well known. Therefore, an accurate estimation of the numerical value of parameter $a$ in the VDW system of pions looks rather problematic, and it is treated as purely phenomenological free parameter in the present study.

V. COMPARISON WITH THE HAGEDORN MODEL

A concept of the limiting temperature for the hadron gas was introduced in physics by Hagedorn [17, 18]. His limiting temperature $T_0 = 160 \pm 10$ MeV emerged due to the exponentially increasing mass spectrum $\rho(m)$ for the hadron excited states at large $m$:

$$\rho(m) \approx c m^{-\alpha} \exp \left( \frac{m}{T_0} \right) \theta(m - M_0),$$

where $c$, $M_0$, $T_0$, and $\alpha$ are the model parameters. These excited states called fireballs were considered as a point-like non-interacting particles. A presentation of particle attractive interactions by the resonance states was first suggested in Ref. [19] in statistical physics and then applied in Ref. [20] to hadron production. This idea was then further extended within the $S$-matrix formulation of statistical mechanics developed in Ref. [21].

The spectrum $\rho(m)$ can be found from the statistical bootstrap equation [22]. This equation requires the spectrum of low-lying hadron states as an input. In a simplest version of the model, this input is reduced to a single lightest hadron – the pion, and the continuous mass spectrum $\rho(m)$ starting from $M_0 > 2m_\pi$. The pressure of the Hagedorn model is then defined as

$$p_H(T) = \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{k^2 + m^2_\pi}} \left[ \exp \left( \frac{\sqrt{k^2 + m^2_\pi}}{T} \right) - 1 \right]^{-1} + \frac{T^2}{2\pi^2} \int_{M_0}^{\infty} dm \rho(m) m^2 K_2(m/T),$$

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where $K_2$ is the modified Bessel function. The first term in the right hand side of Eq. (13) corresponds to the pressure of the ideal Bose gas of pions and the second one to the contribution of all excited states (12) taken in the Boltzmann approximation. Other thermodynamical functions of the Hagedorn model are then given by the thermodynamical relations (9).

From Eqs. (12) and (13) it follows that the thermodynamical functions do not exist at $T > T_0$ because of a divergence of the integral with respect to $m$ in Eq. (13). The singular part of the integral contributes to the pressure as $p_H \propto (T_0 - T)^{\alpha - 5/2}$ and to the energy density as $\varepsilon_H \propto (T_0 - T)^{\alpha - 7/2}$. An explicit solution of the statistical bootstrap equation [22] gives $\alpha = 3$ [23]. Thus, $\varepsilon_H \propto (T_0 - T)^{-1/2}$ and $\varepsilon_H \to \infty$ at $T \to T_0$. On the other hand, for $\alpha > 7/2$ all thermodynamical functions in the Hagedorn model remain finite in a vicinity of the limiting temperature $T = T_0$. Even more, there is a remarkable correspondence between the pion gas with the VDW EoS and the Hagedorn model with $7/2 < \alpha < 9/2$, i.e., the behavior of $p_H$ and $\varepsilon_H$ is very similar to that of $p$ and $\varepsilon$ of the VDW EoS shown in the panels (b) and (c) of Fig. 1 for $a/b = 500$ MeV and 600 MeV. In the vicinity of the limiting temperature $T_0$ the heat capacity and the speed of sound squared behave in Hagedorn model as $C \propto (T_0 - T)^{\alpha - 9/2}$ and $c_s^2 \propto (T_0 - T)^{9/2 - \alpha}$, respectively. Therefore, for $\alpha = 4$ the behavior of $C$ and $c_s^2$ in the Hagedorn model at $T \to T_0$ is the same as in the pion gas with the VDW EoS, if the $T_0$ values in both models are set to be equal. In Fig. 3 these two quantities are compared for the VDW model with $b = 0.45$ fm$^3$, $a/b = 500$ MeV and the Hagedorn model with $T_0 = 155$ MeV and $\alpha = 4$.

VI. SUMMARY

In summary, the interacting pion system has been studied using the quantum version of the VDW EoS within the GCE. The role of repulsive and attractive interactions described by the VDW parameters $b$ and $a$, respectively, was considered. It is found that for $b \cong 0.45$ fm$^3$ and $a/b \cong 200$ MeV the VDW EoS for pions appears to be close to that of the ideal pion gas, i.e., repulsive and attractive interactions are approximately canceled out. At each value of $b$ there is a minimal value of $a = a_b$ such that the limiting temperature $T = T_0$ appears at all $a \geq a_b$, and $T_0$ decreases with increasing $a$. When $T \to T_0$ several remarkable effects happen: the speed of sound approaches zero as $(T_0 - T)^{1/2}$, while the heat capacity as well as the scaled variance of...
fluctuations of the particle multiplicity go to infinity as \((T_0 - T)^{-1/2}\).

The presented VDW model for pions is compared to the Hagedorn model, where the phenomenon of limiting temperature is also present. Even though the realization of the attractive mechanism between particles in the Hagedorn model is different from that in the VDW model, we found that a very similar thermodynamical behavior emerges in both models. It takes place for the special choice of a power, namely \(m^{-4}\), in the pre-exponential factor of the mass spectrum \(\rho(m)\) of the Hagedorn model.

A presence of the limiting temperature \(T_0\) in the VDW pion gas is definitely a signal of the restricted validity of this model. Similar to the Hagedorn model, the limiting temperature of the VDW pion system should be transformed to the temperature of deconfinement transition when the fundamental quark-gluon degrees of freedom are introduced. This consideration is, however, outside of the scope of the present paper.

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