A NOTE ON THE WEAK REGULARITY THEORY FOR DEGENERATE KOLMOGOROV EQUATIONS

FRANCESCA ANCESCHI AND ANNALaura REBUCCI

Abstract. The aim of this work is to prove a Harnack inequality and the Hölder continuity for weak solutions to the Kolmogorov equation $\mathcal{L}u = f$ with measurable coefficients, integrable lower order terms and nonzero source term. We introduce a function space $W$, suitable for the study of weak solutions to $\mathcal{L}u = f$, that allows us to prove a weak Poincaré inequality. More precisely, our goal is to prove a weak Harnack inequality for non-negative super-solutions by considering their Log-transform and following S. N. Krážkov (1963). Then this functional inequality is combined with a classical covering argument (Ink-Spots Theorem) that we extend for the first time to the case of ultraparabolic equations.

Key words: Kolmogorov equation, weak regularity theory, Moser iterative method, weak Poincaré inequality, Harnack inequality, Hölder regularity, ultraparabolic, ink-spots theorem

AMS subject classifications: 35K70, 35Q84, 35H20, 35B65, 35B09, 35B45

1. Introduction

The aim of this work is to study the De Giorgi-Nash-Moser regularity theory for weak solutions to the second order partial differential equation of Kolmogorov type of the form

$$\mathcal{L}u(x, t) := \sum_{i,j=1}^{m_0} \partial_{z_i} (a_{ij}(x, t) \partial_{z_j} u(x, t)) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x, t) - \partial_t u(x, t) + \sum_{i=1}^{m_0} b_i(x, t) \partial_{i} u(x, t) + c(x, t) u(x, t) = f,$$

where $z = (x, t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$ and $1 \leq m_0 \leq N$. Moreover, the matrices $A_0 = (a_{ij}(x, t))_{i,j=1,\ldots,m_0}$ and $B = (b_{ij})_{i,j=1,\ldots,N}$ satisfy the following structural assumptions.

Date: August 2, 2021.
The matrix $A_0$ is symmetric with real measurable entries. Moreover, $a_{ij}(x,t) = a_{ji}(x,t)$, for every $i, j = 1, \ldots, m_0$, and there exist two positive constants $\lambda$ and $\Lambda$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(x,t) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad (1.2)$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix $B$ has constant entries.

Note that we allow the operator $L$ to be strongly degenerate whenever $m_0 < N$. However, the first order part of $L$ may induce a strong regularizing property. Indeed, it is known that, under suitable assumptions on the matrix $B$, the operator $L$ is hypoelliptic, namely that every distributional solution $u$ to $Lu = f$ defined in some open set $\Omega \subset \mathbb{R}^{N+1}$ belongs to $C^\infty(\Omega)$ and it is a classical solution to $Lu = f$, whenever $f \in C^\infty(\Omega)$. In the sequel, we will therefore rely on the following assumption.

(H2) The principal part operator $K$ of $L$ is hypoelliptic and homogeneous of degree 2 with respect to the family of dilations $(\delta_r)_{r>0}$ introduced in (2.9), where $K$ is defined as

$$K u(x,t) := \sum_{i=1}^{m_0} \partial_{x_i}^2 u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t), \quad (x,t) \in \mathbb{R}^{N+1}. \quad (1.3)$$

It is clear that if $L$ is uniformly parabolic (i.e. $m_0 = N$ and $B \equiv 0$), then assumption (H2) is satisfied. In fact, in this case the principal part operator $K$ is simply the heat operator, which is known to be hypoelliptic. For further information on the hypoellipticity of $K$ and on the equivalent structural condition for the matrix $B$ we refer to Section 2.

In the sequel we will also make use of the following notation in order to introduce a compact formulation for the operator $L$. More precisely, here and in the sequel

$$D = (\partial_{x_1}, \ldots, \partial_{x_N}), \quad \langle \cdot, \cdot \rangle, \quad \text{div}$$

respectively denote the gradient, the inner product and the divergence in $\mathbb{R}^N$. In addition,

$$D_{m_0} = (\partial_{x_1}, \ldots, \partial_{x_{m_0}}), \quad \text{div}_{m_0}$$

denote the partial gradient and the partial divergence in the first $m_0$ components, respectively. Moreover, we introduce the matrix

$$A(x,t) = (a_{ij}(x,t))_{1 \leq i,j \leq N},$$

where $a_{ij}$, for every $i, j = 1, \ldots, m_0$, are the coefficients appearing in (1.1), while $a_{ij} \equiv 0$ whenever $i > m_0$ or $j > m_0$, and we let

$$b(x,t) := (b_1(x,t), \ldots, b_{m_0}(x,t), 0, \ldots, 0), \quad Y := \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t). \quad (1.4)$$

Now, we are in a position to rewrite the operator $L$ in the following compact form

$$Lu = \text{div}(ADu) + Yu + \langle b, Du \rangle + cu$$

and to state our last assumption on the integrability of $b, c$ and of the source term $f$.

(H3) $c, f \in L^q(\Omega)$ and $b \in (L^q(\Omega))^{m_0}$ for some $q > \frac{4}{Q+2}$. Moreover, we assume

$$\text{div}b \geq 0 \quad \text{in} \ \Omega.$$
The physical interpretation of the sign of the divergence of $b$ can be understood by considering the Vlasov-Poisson-Fokker-Planck equation [36], for which the lower order term $b$ represents the electrostatic or gravitational forces. The equations with the term $b$ satisfying the structural assumption $\text{div}\, b \geq 0$ arise also in some other applications, like the ones contained in [34, 31]. Moreover, the sign assumption on the divergence of $b$ is also quite relevant in the case of parabolic equations, since it has several applications to, for instance, incompressible flows and magnetostrophic turbulence models for the Earth’s fluid core, e.g. [27]. In particular, it is nowadays known that the sign (or either the divergence free, i.e. $\text{div}\, b = 0$) assumption can be considered to relax the regularity assumptions on $b$ under which one can prove the Harnack inequality and other results, see [34, 16]. Nevertheless, in our case as in the parabolic setting presented in [34], there is still the need to require that the divergence of $b$ exists in the sense of distributions and that $b$ is at least locally integrable up to a certain power.

1.1. Main results. Our aim is to prove the local boundedness and the local Hölder continuity, alongside with a Harnack inequality, for weak solutions to $\mathcal{L}\, u = f$ under the assumptions (H1)-(H3). In particular, in order to expose our main results we first need to introduce some preliminary notation. From now on, we consider a set $\Omega = \Omega_{m_0} \times \Omega_{N-m_0+1}$ of $\mathbb{R}^{N+1}$, where $\Omega_{m_0}$ is a bounded Lipschitz domain of $\mathbb{R}^{m_0}$ and $\Omega_{N-m_0+1}$ is a bounded Lipschitz domain of $\mathbb{R}^{N-m_0+1}$. This is not restrictive since the cylinders $\Omega$ introduced in (1.11) that we consider in our local analysis satisfy the Lipschitz boundary assumption. Then we split the coordinate $x \in \mathbb{R}^N$ as

$$x = (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}), \quad x^{(0)} \in \mathbb{R}^{m_0}, \quad x^{(j)} \in \mathbb{R}^{m_j}, \quad j \in \{1, \ldots, \kappa\},$$

(1.5)

where we have that in accordance with the scaling of the differential equation (see (2.9) below) every $m_j$ is a positive integer such that

$$\sum_{j=1}^{\kappa} m_j = N \quad \text{and} \quad N \geq m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1.$$

We denote by $\mathcal{D}(\Omega)$ the set of $C^\infty$ functions compactly supported in $\Omega$ and by $\mathcal{D}'(\Omega)$ the set of distributions in $\Omega$. From now on, $H^1_{x^{(0)}}$ denotes the Sobolev space of functions $u \in L^2(\Omega_{m_0})$ with distribution gradient $D_{m_0} u$ lying in $(L^2(\Omega_{m_0}))^{m_0}$, i.e.

$$H^1_{x^{(0)}} := \left\{ u \in L^2(\Omega_{m_0}) : D_{m_0} u \in (L^2(\Omega_{m_0}))^{m_0} \right\},$$

and we set

$$\|u\|_{H^1_{x^{(0)}}} := \|u\|_{L^2(\Omega_{m_0})} + \|D_{m_0} u\|_{L^2(\Omega_{m_0})}.$$

We let $H^1_{c,x^{(0)}}$ denote the closure of $C_c^\infty(\Omega_{m_0})$ in the norm of $H^1_{x^{(0)}}$ and we recall that $C_c^\infty(\Omega_{m_0})$ is dense in $H^1_{x^{(0)}}$ since $\Omega_{m_0}$ is a bounded Lipschitz domain by assumption. Moreover, $H^1_{c,x^{(0)}}$ is a reflexive Hilbert space and thus we may consider its dual space

$$\left( H^1_{c,x^{(0)}} \right)^* = H^{-1}_{x^{(0)}} \quad \text{and} \quad \left( H^{-1}_{x^{(0)}} \right)^* = H^1_{c,x^{(0)}},$$

where the notation we consider is the classical one. Hence, from now on we denote by $H^{-1}_{x^{(0)}}$ the dual of $H^1_{c,x^{(0)}}$ acting on functions in $H^1_{c,x^{(0)}}$ through the duality pairing $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H^1_{c,x^{(0)}}, H^{-1}_{x^{(0)}}}$. In a standard manner, see for instance [4, 26], we let $\mathcal{W}$ denote the closure of $C_c^\infty(\overline{\Omega})$ in the norm

$$\|u\|_{\mathcal{W}}^2 = \|u\|_{L^2(\Omega_{N-m_0+1}; H^1_{x^{(0)}})}^2 + \|Y u\|_{L^2(\Omega_{N-m_0+1}; H^{-1}_{x^{(0)}})}^2,$$

(1.6)
where the previous norm can explicitly computed as follows:

\[ \|u\|_{W}^{2} = \left( \int_{\Omega_{N-m_{0}+1}} \|u(\cdot,y,t)\|_{H^{1}}^{2} \, dy \, dt \right)^{\frac{1}{2}} + \left( \int_{\Omega_{N-m_{0}+1}} \|Yu(\cdot,y,t)\|_{H^{1}}^{2} \, dy \, dt \right)^{\frac{1}{2}}, \]

where \( y = (x^{(1)}, \ldots, x^{(e)}) \). In particular, \( W \) is a Banach space and we remark that the dual of \( L^{2}(\Omega_{N-m_{0}+1}; H^{1}_{c,x(0)}) \) satisfies

\[ \left( L^{2}(\Omega_{N-m_{0}+1}; H^{1}_{c,x(0)}) \right)^{*} = L^{2}(\Omega_{N-m_{0}+1}; H^{-1}_{c,x(0)}) \quad \text{and} \quad \left( L^{2}(\Omega_{N-m_{0}+1}; H^{-1}_{c,x(0)}) \right)^{*} = L^{2}(\Omega_{N-m_{0}+1}; H^{1}_{c,x(0)}). \]

From now on, we consider the shorthand notation \( L^{2}H^{-1} \) to denote \( L^{2}(\Omega_{N-m_{0}+1}; H^{-1}_{c,x(0)}) \).

The space of functions \( W \) is the most natural framework for the study of the weak regularity theory for the operator \( \mathcal{L} \). In particular, it is an extension of the functional setting proposed by Armstrong and Mourrat in [4] for the study of the kinetic Kolmogorov-Fokker-Planck equation, where the authors show that it is sufficient to control \( u \in L^{2} \) and \( Yu \in L^{2}H^{-1} \) in order to derive new Poincaré inequalities, such as Proposition 1.1, on which we base our analysis later on. Moreover, we refer to [26, Section 2] for some properties of the space \( W \). Lastly, we remark that the major issue when dealing with the space \( W \) is that it requires to handle the duality pairing between \( L^{2}H^{1} \) and \( L^{2}H^{-1} \). To this end, we take advantage of the following remark, see [20, Chapter 4].

**Remark 1.1.** For every open subset \( A \subset \mathbb{R}^{n} \) and for every function \( g \in H^{-1}(A) \) there exist two functions \( H_{0}, H_{1} \in L^{2}(A) \) such that

\[ g = \text{div}_{m_{0}}H_{1} + H_{0} \quad \text{and} \quad \|H_{0}\|_{L^{2}(A)} + \|H_{1}\|_{L^{2}(A)} \leq 2\|g\|_{H^{-1}(A)}. \]

Now, we introduce the definition of weak solution we consider in our work.

**Definition 1.2.** A function \( u \in W \) is a weak solution to (1.1) with source term \( f \in L^{2}(\Omega) \) if for every non-negative test function \( \varphi \in \mathcal{D}(\Omega) \), we have

\[ \int_{\Omega} -\langle ADu, D\varphi \rangle + \varphi Yu + \langle b, Du \rangle \varphi + cu \varphi = \int_{\Omega} f \varphi. \quad (1.7) \]

In the sequel, we will also consider weak sub-solutions to (1.1), namely functions \( u \in W \) that satisfy the following inequality

\[ \int_{\Omega} -\langle ADu, D\varphi \rangle + \varphi Yu + \langle b, Du \rangle \varphi + cu \varphi \geq \int_{\Omega} f \varphi, \quad (1.8) \]

for every non-negative test function \( \varphi \in \mathcal{D}(\Omega) \). A function \( u \) is a super-solution to (1.1) if \(-u\) is a sub-solution.

This framework is quite classical for the study of the weak regularity theory for the kinetic Kolmogorov-Fokker-Planck equation [4, 11, 13, 14], that can be recovered from (1.1) by choosing \( N = 2d, \kappa = 1, m_{0} = m_{1} = d \) and \( c \equiv 0 \). Still, even if it is the most natural framework for (1.1), to the best of our knowledge it has never been considered in literature yet. Indeed, the weak regularity theory for the operator \( \mathcal{L} \) has been widely developed during the last decade starting from the paper [33] by Pascucci and Polidoro, where the authors worked under the stronger assumption \( Yu \in L^{2}_{\text{loc}}(\Omega) \). In the same framework (i.e. \( Yu \in L^{2}(\Omega) \)), Wang and Zang [38, 39]
lately proved the results for equation (1.1) with $a = b = f = 0$ on the local Hölder continuity for solutions.

The aim of this paper is to prove the local Hölder continuity and a Harnack inequality for solutions to (1.1) in the sense of Definition 1.2. Our method is based on the combination of three fundamental ingredients - boundedness of weak solutions, weak Poincaré inequality and Log-transformation - in the same spirit of the recent paper [13] for the Fokker-Planck equation. In particular, we give an answer to Remark 4, p. 2 of [13] and we are also able to simplify the proof proposed in [40] to obtain the local Hölder continuity for weak solutions to (1.1). We classically reduce the local study to the case where $Q^0$ is at unit scale and for some reasons we expose below in Section 5, $Q^0$ takes the form $B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ for some large constant $R_0$ only depending on the dimension $Q$ and the ellipticity constants $\lambda, \Lambda$ in (H1).

As we will see in Section 2, the suitable geometry when dealing with operator $\mathcal{L}$ is given by an homogeneous Lie group structure defined on $\mathbb{R}^{N+1}$. Our results naturally reflect this non-Euclidean setting. Let “$\circ$” denote the composition law introduced in (2.1) and $(\delta_r)_{r>0}$ the family of dilations defined in (2.9). We consider the unit past cylinder

$$Q_1 := B_1 \times B_1 \times \ldots \times B_1 \times (-1, 0),$$

defined through the open balls

$$B_1 = \{ x^{(j)} \in \mathbb{R}^{m_j} : |x| \leq 1 \},$$

where $j = 0, \ldots, \kappa$ and $| \cdot |$ denotes the euclidean norm in $\mathbb{R}^{m_j}$. Now, for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$Q_r(z_0) := z_0 \circ (\delta_r (Q_1)) = \{ z \in \mathbb{R}^{N+1} : z = z_0 \circ \delta_r(\zeta), \zeta \in Q_1 \}.$$  \hspace{1cm} (1.11)

We are now in a position to state our main results.

**Theorem 1.3** (Weak Harnack inequality). Let $Q^0 = B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ and let $u$ be a non-negative weak super-solution to $\mathcal{L} u = f$ in $\Omega \supset Q^0$ under the assumptions (H1)-(H3). Then we have

$$\left( \int_{Q^-} u^p \right)^{\frac{1}{p}} \leq C \left( \inf_{Q^+} u + \| f \|_{L^q(Q^0)} \right),$$

where $Q^+ = B_{\omega} \times B_{\omega^2} \times \ldots \times B_{\omega^{2k+1}} \times (-\omega^2, 0]$ and $Q^- = B_{\omega} \times B_{\omega^2} \times \ldots \times B_{\omega^{2k+1}} \times (-1, -1+\omega^2]$. Moreover, the constants $C$, $p$, $\omega$ and $R_0$ only depend on the homogeneous dimension $Q$ defined in (2.12), and on the ellipticity constants $\lambda$ and $\Lambda$ in (1.2).

We remark that as in [33], the radius $\omega$ is small enough so that when “stacking cylinders” over a small initial one contained in $Q^-$, the cylinder $Q^+$ is captured, see Lemma B.1. As far as we are concerned with $R_0$, it is large enough so that it is possible to apply the expansion of positivity lemma (see Lemma 5.1) to every stacked cylinder. By combining the local boundedness of positive weak sub-solutions proved in Theorem 3.1 and Theorem 1.3 we obtain the following Harnack inequality, which is an extension of the analogous result contained in [1, 11].

**Theorem 1.4** (Harnack inequality). Let $Q^0 = B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ and let $u$ be a non-negative weak solution to $\mathcal{L} u = f$ in $\Omega \supset Q^0$ under the assumptions (H1)-(H3). Then we have

$$\sup_{Q^-} u \leq C \left( \inf_{Q^+} u + \| f \|_{L^q(Q^0)} \right),$$

where
where \( \Omega_+ = B_{\omega} \times B_{\omega}^3 \times \ldots \times B_{\omega}^{2n+1} \times (-\omega^2, 0] \) and \( \Omega_- = B_{\omega} \times B_{\omega}^3 \times \ldots \times B_{\omega}^{2n+1} \times (-1, -1+\omega^2] \). Moreover, the constants \( C, \omega \) and \( R_0 \) only depend on the homogeneous dimension \( Q \) defined in (2.12), \( q \) and on the ellipticity constants \( \lambda \) and \( \Lambda \) in (1.2).

Since our proof of Theorem 1.3 is constructive, as it is based on the combination of an expansion of positivity argument with the weak Poincaré inequality of Proposition 4.1, also the proof of Theorem 1.4 is constructive. Moreover, the weak Harnack inequality also implies the Hölder regularity of weak solutions. In order to state this result, we first need to give the following definition.

**Definition 1.5.** Let \( \alpha \) be a positive constant, \( \alpha \leq 1 \), and let \( \Omega \) be an open subset of \( \mathbb{R}^{N+1} \). We say that a function \( f : \Omega \to \mathbb{R} \) is Hölder continuous with exponent \( \alpha \) in \( \Omega \) with respect to the group \( K = (\mathbb{R}^{N+1}, o) \), defined in (2.1), (in short: Hölder continuous with exponent \( \alpha \), \( f \in C^\alpha_K(\Omega) \)) if there exists a positive constant \( C > 0 \) such that

\[
|f(z) - f(\zeta)| \leq C \, d(z, \zeta)^\alpha \quad \text{for every } z, \zeta \in \Omega,
\]

where \( d \) is the distance defined in (2.11).

To every bounded function \( f \in C^\alpha_K(\Omega) \) we associate the semi-norm

\[
[f]_{C^\alpha_K(\Omega)} = \sup_{z, \zeta \in \Omega \atop z \neq \zeta} \frac{|f(z) - f(\zeta)|}{d(z, \zeta)^\alpha}.
\]

Moreover, we say a function \( f \) is locally Hölder continuous, and we write \( f \in C^\alpha_{K, \text{loc}}(\Omega) \), if \( f \in C^\alpha_K(\Omega') \) for every compact subset \( \Omega' \) of \( \Omega \).

We are now in a position to state the following result.

**Theorem 1.6 (Hölder regularity).** There exists \( \alpha \in (0, 1) \) only depending on dimension \( Q, \lambda, \Lambda \) such that all weak solutions \( u \) to (1.1) under assumption (H1) - (H3) in \( \Omega \supset \Omega_1 \) satisfy

\[
[u]_{C^{\alpha}(Q_2)} \leq C \left( \|u\|_{L^2(\Omega_1)} + \|f\|_{L^q(\Omega_1)} \right),
\]

where the constant \( C \) only depends on the homogeneous dimension \( Q \) defined in (2.12), \( q \) and the ellipticity constants \( \lambda, \Lambda \).

**Remark 1.7.** The estimates presented in Theorem 1.4 and Theorem 1.6 can be scaled and stated in arbitrary cylinders thanks to the invariance of the operator \( \mathcal{L} \) with respect to the group operations (dilations (2.9) and translations (2.1)) introduced in Section 2. Moreover, all of our main results still hold true if we replace assumption (H3) with the following one:

\[
c, f \in L^q(\Omega) \text{ for some } q > \frac{Q+2}{2} \text{ and } b \in (L^q(\Omega))^m \text{ for some } q > Q+2 \text{ in } \Omega.
\]

This is exactly the integrability requirement assumed by Wang and Zhang in [40], thus our work is an improvement of this article for the case homogeneous ultraparabolic operators. Moreover, as we work in the appropriate functional setting, our proofs of the main results are simpler than the ones proposed in [40].

### 1.2. Comparison with existing results and organization of the paper.

The purpose of this paper is to provide a complete characterization of the De Giorgi-Nash-Moser weak regularity theory for the Kolmogorov equation in divergence form with measurable coefficients in the more natural space \( \mathcal{W} \) for weak solutions to \( \mathcal{L}u = f \). First of all, we improve the existing result proved by Wang and Zhang in [38, 39, 40] by providing a simpler and more elegant proof of the weak Harnack inequality and by explicitly showing the derivation of an invariant Harnack inequality, the first one available for weak solutions to \( \mathcal{L}u = f \). In particular, our results hold
true also under the assumptions of [40] as it is pointed out in Remark 1.7 and Remark 3.2, and extend the ones obtained in the particular case of the kinetic Fokker-Planck equation in [13, 11, 26].

Our proof is based on the combination of a weak Poincaré inequality for the space $W$ with the Ink-spots theorem in the same spirit of [19, 13]. We remark that, up to this day, the Ink-spot theorem has only been available just for the Fokker-Planck equation case in $\mathbb{R}^{2n+1}$, and thus also the technical result that we prove in Appendix A is a complete novelty in this more general framework. Moreover, we manage to lower the integrability assumption on the lower order coefficients and on the source term $f$ and to extend the Moser’s iterative method proposed in [33] to this more general case.

The structure of the paper is the following. In Section 2 we recall some known facts about operators $L$ and we state some preliminary results. The proofs of some intermediate theorems (a Sobolev type and a Caccioppoli type inequality), together with the Moser’s iterative method, are presented in Section 3. Section 4 is devoted to the proof of a weak Poincarè inequality. In Section 5 we derive the weak Harnack inequality by combining the expansion of positivity and a covering argument known as Ink-spots theorem, whose proof is contained in Appendix A. Finally, in Appendix B we state a technical lemma regarding stacked cylinders.

Acknowledgements. The authors are grateful to Prof. Sergio Polidoro for suggesting the question and the fruitful discussions. The first author is funded by the research grant PRIN2017 2017AYM8XW “Nonlinear Differential Problems via Variational, Topological and Set-valued Methods”.

2. Preliminaries

In this Section we recall notation and results we need in order to deal with the non-Euclidean geometry underlying the operators $L$ and $K$. We refer to the articles [33, 24] and to the survey [2] for a comprehensive treatment of this subject.

As first observed by Lanconelli and Polidoro in [24], the operator $K$ is invariant with respect to left translation in the group $\mathcal{K} = (\mathbb{R}^{N+1}, \circ)$, where the group law is defined by

$$\begin{align*}
(x,t) \circ (\xi,\tau) &= (\xi + E(\tau)x, t + \tau), \quad (x,t), (\xi,\tau) \in \mathbb{R}^{N+1}, \\
E(s) &= \exp(-sB), \quad s \in \mathbb{R}.
\end{align*}$$

Then $\mathcal{K}$ is a non-commutative group with zero element $(0,0)$ and inverse

$$(x,t)^{-1} = (-E(-t)x, -t).$$

For a given $\zeta \in \mathbb{R}^{N+1}$ we denote by $\ell_\zeta$ the left translation on $\mathcal{K} = (\mathbb{R}^{N+1}, \circ)$ defined as follows

$$\ell_\zeta : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}, \quad \ell_\zeta(z) = \zeta \circ z.$$  

Then the operator $\mathcal{K}$ is left invariant with respect to the Lie product $\circ$, that is

$$\mathcal{K} \circ \ell_\zeta = \ell_\zeta \circ \mathcal{K} \quad \text{or, equivalently,} \quad \mathcal{K} (u(\zeta \circ z)) = (\mathcal{K} u) (\zeta \circ z),$$

for every $u$ sufficiently smooth.
We recall that, by [24] (Propositions 2.1 and 2.2), assumption (H2) is equivalent to assume that, for some basis on \( \mathbb{R}^N \), the matrix \( B \) takes the following form
\[
B = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
B_1 & 0 & \cdots & 0 & 0 \\
0 & B_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B_\kappa & 0 \\
\end{pmatrix}
\]
(2.3)
where every \( B_k \) is a \( m_k \times m_k - 1 \) matrix of rank \( m_j \), \( j = 1, 2, \ldots, \kappa \) with
\[
m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1 \quad \text{and} \quad \sum_{j=0}^\kappa m_j = N.
\]
In the sequel we will assume that \( B \) has the canonical form (2.3). We remark that assumption (H2) is implied by the condition introduced by Hörmander in [15]:
\[
\text{rank Lie}\left( \partial_{x_1}, \ldots, \partial_{x_{m_0}}, Y \right)(x,t) = N + 1, \quad \forall (x,t) \in \mathbb{R}^{N+1},
\]
(2.4)
where \( \text{Lie}\left( \partial_{x_1}, \ldots, \partial_{x_{m_0}}, Y \right) \) denotes the Lie algebra generated by the first order differential operators \( \left( \partial_{x_1}, \ldots, \partial_{x_{m_0}}, Y \right) \) computed at \( (x,t) \). Yet another condition equivalent to (H2), (see [24], Proposition A.1), is that
\[
C(t) > 0, \quad \text{for every } t > 0,
\]
(2.5)
where
\[
C(t) = \int_0^t E(s) A_0 E^T(s) \, ds,
\]
and \( E(\cdot) \) is the matrix defined in (2.2).

We now recall that, under the hypothesis of hypoellipticity, Hörmander in [15] constructed the fundamental solution of \( K \) as
\[
\Gamma((x,t),(0,0)) = \begin{cases}
\frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp \left( -\frac{1}{4} \langle C^{-1}(t)x, x \rangle - t \text{tr}(B) \right), & \text{if } t > 0, \\
0, & \text{if } t < 0.
\end{cases}
\]
(2.7)
Note that condition (2.5) implies that \( \Gamma \) in (2.7) is well-defined.

Let us now consider the second part of assumption (H2). We say that \( \mathcal{K} \) is invariant with respect to \( (\delta_r)_{r>0} \) if
\[
\mathcal{K}(u \circ \delta_r) = r^2 \delta_r(\mathcal{K} u), \quad \text{for every } \quad r > 0,
\]
(2.8)
for every function \( u \) sufficiently smooth. It is known (see Proposition 2.2 of [24]) that this dilation invariance property can be read in the expression of the matrix \( B \) in (2.3). More precisely, \( \mathcal{K} \) satisfies (2.8) if and only if the matrix \( B \) takes the form (2.3). In this case, we have
\[
\delta_r = \text{diag}(r^{2m_0}, r^{2m_1}, \ldots, r^{2\kappa+1}m_\kappa, r^2), \quad r > 0.
\]
(2.9)
We next introduce a homogeneous norm of degree 1 with respect to the dilations \( (\delta_r)_{r>0} \) and a corresponding quasi-distance which is invariant with respect to the group operation (2.1).
Definition 2.1. Let $\alpha_1, \ldots, \alpha_N$ be positive integers such that
\[
\text{diag} (r^{\alpha_1}, \ldots, r^{\alpha_N}, r^2) = \delta_r.
\] (2.10)
If $\|z\| = 0$ we set $z = 0$ while, if $z \in \mathbb{R}^{N+1} \setminus \{0\}$ we define $\|z\| = r$ where $r$ is the unique positive solution to the equation
\[
\frac{x_1^2}{r^{2\alpha_1}} + \frac{x_2^2}{r^{2\alpha_2}} + \ldots + \frac{x_N^2}{r^{2\alpha_N}} + \frac{t^2}{r^4} = 1.
\]
Accordingly, we define the quasi-distance $d$ by
\[
d(z, w) = \|z^{-1} \circ w\|, \quad z, w \in \mathbb{R}^{N+1}.
\] (2.11)

Remark 2.2. As $\det E(t) = e^{t \text{trace } B} = 1$, the Lebesgue measure is invariant with respect to the translation group associated to $\mathcal{K}$. Moreover, since $\det \delta_r = r^{Q+2}$, we also have
\[
\text{meas} (\Omega_r(z_0)) = r^{Q+2} \text{meas} (\Omega_1(z_0)), \quad \forall \ r > 0, \ z_0 \in \mathbb{R}^{N+1},
\]
where
\[
Q = m_0 + 3m_1 + \ldots + (2\kappa + 1)m_\kappa.
\] (2.12)
The natural number $Q+2$ is called the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(\delta_r)_{r>0}$ and the spatial homogeneous dimension $Q$ is the hypoelliptic counterpart of the space dimension $N$ usually considered in the parabolic setting, see [32]. This denomination is proper since the Jacobian determinant of $\delta_r$ equals to $r^{Q+2}$.

Remark 2.3. The semi-norm $\| \cdot \|$ is homogeneous of degree 1 with respect to $(\delta_r)_{r>0}$, i.e.
\[
\|\delta_r(x,t)\| = r \|x,t\| \quad \forall r > 0 \text{ and } (x,t) \in \mathbb{R}^{N+1}.
\]
Since in $\mathbb{R}^{N+1}$ all the norms which are 1-homogeneous with respect to $(\delta_r)_{r>0}$ are equivalent, the norm introduced in Definition 2.1 is equivalent to the following one
\[
\|(x,t)\|_1 = |t|^\frac{1}{2} + |x|, \quad |x| = \sum_{j=1}^N |x_j|^\frac{1}{2j}
\]
where the exponents $\alpha_j$, for $j = 1, \ldots, N$ were introduced in (2.10). We prefer the norm of Definition 2.1 to $\| \cdot \|_1$ because its level sets (spheres) are smooth surfaces.

Remark 2.4. We recall that for every cylinder $\Omega_r(z_0)$ defined in (1.11) there exists a positive constant $\tau$ [39, equation (21)] such that
\[
B_r(x_0^{(0)}) \times B_{r^2}(x_0^{(1)}) \times \ldots \times B_{r^{2\kappa+1}}(x_0^{(\kappa)}) \times (-r_1^2, 0) \subset \Omega_r(z_0) \subset B_{r^2}(x_0^{(0)}) \times B_{r^2}(x_0^{(1)}) \times \ldots \times B_{r^{2\kappa+1}}(x_0^{(\kappa)}) \times (-r_2^2, 0),
\]
where $r_1 = r/\tau$ and $r_2 = \tau r$. From now on, by abuse of notation we will sometimes consider the newly introduced ball representation instead of the definition of cylinder in (1.11).

Since $\mathcal{K}$ is dilation invariant with respect to $(\delta_r)_{r>0}$, also its fundamental solution $\Gamma$ is a homogeneous function of degree $-Q$, namely
\[
\Gamma(\delta_r(z), 0) = r^{-Q} \Gamma(z, 0), \quad \forall z \in \mathbb{R}^{N+1} \setminus \{0\}, \ r > 0.
\]
This property implies a $L^p$ estimate for Newtonian potential (c. f. for instance [9]).
Theorem 2.5. Let $\alpha \in [0, Q + 2]$ and let $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a $\delta_\lambda$-homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(\mathbb{R}^{N+1})$ for some $p \in [1, +\infty]$, then the function
\[
G_f(z) := \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z)f(\zeta) d\zeta,
\]
is defined almost everywhere and there exists a constant $c = c(Q, p)$ such that
\[
\|G_f\|_{L^q(\mathbb{R}^{N+1})} \leq c \max_{\|z\|=1} |G(z)|\|f\|_{L^p(\mathbb{R}^{N+1})},
\]
where $q$ is defined by
\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q + 2}.
\]

Now, we are able to define the $\Gamma$-potential of the function $f \in L^1(\mathbb{R}^{N+1})$ as follows
\[
\Gamma(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)f(\zeta) d\zeta, \quad z \in \mathbb{R}^{N+1}.
\]
(2.13)

We also remark that the potential $\Gamma(D_m f) : \mathbb{R}^{N+1} \to \mathbb{R}^{m_0}$ is well-defined for any $f \in L^p(\mathbb{R}^{N+1})$, at least in the distributional sense, that is
\[
\Gamma(D_m f)(z) := -\int_{\mathbb{R}^{N+1}} D_{m_0}^{(z, \xi)} \Gamma(z, \xi) f(\xi) d\xi,
\]
where $D_{m_0}^{(z, \xi)}$ is the gradient with respect to $\xi_1, \ldots, \xi_{m_0}$. Based on Theorem 2.5, we derive the following explicit potential estimates by substituting $\alpha = 1$ and $\alpha = 2$ when considering the $\Gamma$-potential for $f$ and $D_0 f$, respectively. For the proof of this corollary we refer to [33, Corollary 2.2].

Corollary 2.6. Let $f \in L^p(\Omega_r)$. There exists a positive constant $c = c(T, B)$ such that
\[
\|\Gamma(f)\|_{L^{p^*}(\Omega_r)} \leq c\|f\|_{L^p(\Omega_r)},
\]
(2.14)
\[
\|\Gamma(D_m f)\|_{L^{p^*}(\Omega_r)} \leq c\|f\|_{L^p(\Omega_r)},
\]
(2.15)
where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q+2}$ and $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{Q+2}$.

Lastly, we show that it is possible to use the fundamental solution $\Gamma$ as a test function in the definition of sub and super-solution. The following result extends [33, Lemma 2.5], [6, Lemma 3] and [3, Lemma 2.6] to the functional setting $\Psi$.

Lemma 2.7. Let (H1)-(H2) hold. Let $c \in L^q(\Omega)$ and $b \in (L^q(\Omega))^{m_0}$ for some $q > \frac{Q+2}{2}$ and let $f \in L^2(\Omega)$. Moreover, let us assume that div in $\Omega$. Let $v$ be a non-negative weak sub-solution to $\mathcal{L}v = f$ in $\Omega$. For every $\varphi \in D(\Omega)$, $\varphi \geq 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have
\[
\int_{\Omega} -(ADv, D(\Gamma(z, \cdot)\varphi)) + \Gamma(z, \cdot)\varphi Yv + \langle b, Dv \rangle \Gamma(z, \cdot)\varphi + cv\Gamma(z, \cdot)\varphi - \Gamma(z, \cdot)\varphi f \geq 0.
\]
An analogous result holds for weak super-solutions to $\mathcal{L}u = f$.

Proof. For every $\varepsilon > 0$, we set
\[
\psi_\varepsilon(z, \zeta) = 1 - \chi_{\varepsilon, 2\varepsilon}(\|\zeta^{-1} \circ z\|)
\]
(2.16)
where $\chi_{\rho, r} \in C^\infty([0, +\infty))$ is the cut-off function defined by
\[
\chi_{\rho, r}(s) = \begin{cases} 
0, & \text{if } s \geq r, \\
1, & \text{if } 0 \leq s \leq \rho,
\end{cases} \quad |\chi'_{\rho, r}| \leq \frac{2}{r - \rho},
\]
(2.17)
with \( \frac{1}{2} \leq \rho < r \leq 1 \). As \( v \) is a weak-subsolution, for every \( \varepsilon > 0 \) and \( z \in \mathbb{R}^{N+1} \), we have
\[
0 \leq -I_{1,\varepsilon}(z) + I_{2,\varepsilon}(z) - I_{3,\varepsilon}(z) + I_{4,\varepsilon}(z) + I_{5,\varepsilon}(z) + I_{6,\varepsilon}(z)
\]
where
\[
I_{1,\varepsilon}(z) = \int_{\Omega} [(ADv, D\Gamma(z, \cdot))\varphi\psi_{\varepsilon}(z, \cdot)](\zeta) d\zeta \quad I_{5,\varepsilon}(z) = \int_{\Omega} [(ADv, D\psi_{\varepsilon}(z, \cdot))\varphi\Gamma(z, \cdot)](\zeta) d\zeta
\]
\[
I_{2,\varepsilon}(z) = \int_{\Omega} [\Gamma(z, \cdot)\psi_{\varepsilon}(z, \cdot)(-\langle ADv, D\varphi \rangle) + \varphi Yv](\zeta) d\zeta \quad I_{4,\varepsilon}(z) = \int_{\Omega} \langle b, Dv \rangle \Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot)](\zeta) d\zeta
\]
\[
I_{5,\varepsilon}(z) = \int_{\Omega} [c\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot)](\zeta) d\zeta \quad I_{6,\varepsilon}(z) = -\int_{\Omega} [\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot)f](\zeta) d\zeta
\]
Keeping in mind Corollary 2.6, it is clear that the integrals that define \( I_{i,\varepsilon}(z) \), \( i = 1, 2, 3 \) are potentials and therefore convergent for almost every \( z \in \mathbb{R}^{N+1} \). Thus, by a similar argument to the one used in [33] in the proof of Lemma 2.5, we infer that for almost every \( z \in \mathbb{R}^{N+1} \)
\[
\lim_{\varepsilon \to 0^+} I_{1,\varepsilon}(z) = \int_{\Omega} [(ADv, D\Gamma(z, \cdot))\varphi](\zeta) d\zeta \quad \lim_{\varepsilon \to 0^+} I_{3,\varepsilon}(z) = 0
\]
\[
\lim_{\varepsilon \to 0^+} I_{2,\varepsilon}(z) = \int_{\Omega} [\Gamma(z, \cdot)(-\langle ADv, D\varphi \rangle) + \varphi Yv](\zeta) d\zeta
\]
where the passage to the limit for the term \( I_{2,\varepsilon} \) by a density argument. We now take care of the term \( I_{4,\varepsilon} \). Integrating by parts and taking advantage of the assumption \( \text{div} b \geq 0 \), we obtain
\[
I_{4,\varepsilon}(z) = -\int_{\Omega} [\text{div} b \Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot)v](\zeta) d\zeta - \int_{\Omega} [(b, D (\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot))) v](\zeta) d\zeta
\]
\[
\leq -\int_{\Omega} [(b, D (\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot))) v](\zeta) d\zeta
\]
We are left with the estimate of a potential and therefore we exploit once again Corollary 2.6. Since we have \( b \in (L^q(\Omega))^{m_0} \) and \( v \in L^2(\Omega) \), we get
\[
|\Gamma(z, \cdot)||\varphi||Dv| \in L^{2\alpha}(\Omega),
\]
where \( \alpha = \alpha(q) = \frac{q(Q+2)}{q(Q-2)+2(Q+2)} > 1 \) if and only if \( q > \frac{Q+2}{2} \).

Hence, \( |(b, D (\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot))) v| \leq |(b, D (\Gamma(z, \cdot)\varphi)) v| \in L^1(\Omega) \). Thus, the Lebesgue convergence theorem yields
\[
\lim_{\varepsilon \to 0^+} I_{4,\varepsilon}(z) = \lim_{\varepsilon \to 0^+} \int_{\Omega} [(b, D (\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot))) v](\zeta) d\zeta
\]
\[
= \int_{\Omega} [(b, D\Gamma(z, \cdot)\varphi) v](\zeta) d\zeta.
\]
Similarly, we can estimate the term \( I_{5,\varepsilon} \) noting that \( |c||v||\Gamma(z, \cdot)||\varphi| \in L^{2\alpha}(\Omega) \) with \( \alpha \) as above. As a consequence, we have
\[
|c\Gamma(z, \cdot)\varphi\psi_{\varepsilon}(z, \cdot)| \leq |c\Gamma(z, \cdot)\varphi| \in L^1(\Omega), \quad \text{and thus } \lim_{\varepsilon \to 0^+} I_{5,\varepsilon}(z) = \int_{\Omega} |c\Gamma(z, \cdot)\varphi| d\zeta.
\]
Now, we are left with the estimate of term \( I_{6,\varepsilon} \), which is again a \( \Gamma \)-potential such that
\[
|\Gamma(z, \cdot)||\varphi||f| \in L^{2\alpha}(\Omega),
\]
where $\kappa = \frac{Q+2}{Q-2}$. Thus, we infer $|\Gamma(z, \cdot) \varphi \psi_r(z, \cdot)| \leq |\Gamma(z, \cdot) \varphi| \in L^1(\Omega)$. Therefore we conclude the proof by applying the dominated convergence theorem to $I_{6,\varepsilon}(z)$. 

We conclude this Section by recalling the following lemma for whose proof we refer to [6, Lemma 6].

Lemma 2.8. There exists a positive constant $\tau \in (0,1)$ such that

$$z \circ Q_{\tau(r-\rho)} \subseteq Q_r,$$

(2.18)

for every $0 < \rho < r \leq 1$ and $z \in Q_\rho$.

3. Local boundedness for weak solutions to $\mathcal{L}u = f$

Other than the new framework for the study of the weak regularity theory of (1.1), the local Hölder continuity for solutions and the Harnack inequality, the Moser’s iterative scheme for weak solutions to (1.1) with unbounded source term $f$, to whose proof this Section is devoted, is in itself one of the main novelties of our work.

Indeed, this procedure was firstly introduced by Moser in [28, 30] and it is based on the iterative combination of a Caccioppoli and a Sobolev inequality. When considering the classical uniformly elliptic and parabolic settings, the Caccioppoli inequality provides us with a priori estimates for the $L^2$ norm of the complete gradient of the solution in terms of the $L^2$ norm of the solution and we are able to consider the classical Sobolev embedding to obtain a gain of integrability for the solution.

This is not the case when dealing with operator (1.1). In fact, the degeneracy of the diffusion part allows us to estimate only the partial gradient $D_{m_0}u$ of the solution to $\mathcal{L}u = f$ (see Theorem 3.4). In addition, according to our definition of weak solution, $u$ does not lie in a classical Sobolev space. In order to overcome these issues, we adopt a technique based on the representation of a solution $u$ to $\mathcal{L}u = f$ in terms of the fundamental solution $\Gamma$ (see (2.7)) of the principal part operator $\mathcal{K}$. Indeed, following the idea presented for the first time in [33] and later on applied in [3, 6], we have that if $u$ is a solution to $\mathcal{L}u = f$ then

$$\mathcal{K}u = (\mathcal{K} - \mathcal{L})u - f = \text{div}_{m_0} ((I_{m_0} - A_0) D_{m_0}u) - \langle Bx, Du \rangle - \partial u.$$  

(3.1)

Thus, by combining this representation formula with the potential estimates presented in Corollary 2.6, we are able to prove a Sobolev inequality (see Theorem 3.3) and a Caccioppoli inequality (see Theorem 3.4) for weak solutions to $\mathcal{L}u = f$. We remark that in literature there is no similar result available for weak solutions in the sense of Definition 1.2 to the Kolmogorov equation $\mathcal{L}u = f$, with $f$ non zero source term. Indeed, in [11, 13, 14] the authors consider the case of the kinetic Fokker-Planck operator, whereas in [33, 6, 3] the authors consider the Kolmogorov equation $\mathcal{L}u = 0$ under the “strong” notion of weak solution, i.e. $Y u \in L^2$.

Theorem 3.1. Let $u$ be a non-negative weak sub-solution to $\mathcal{L}u = f$ in $\Omega$ under the assumptions \((H1)-(H3)\). Let $z_0 \in \Omega$ and $r, \rho$, with $0 < \rho < r$, be such that $Q_\rho(z_0) \subseteq \Omega$. Then for every $p \in \mathbb{R}$ such that $p > 1/2$ there exist a positive constant $C$ such that

$$\sup_{Q_\rho(z_0)} u^p \leq \frac{C}{(r - \rho)^{4\mu}} \| u^p \|_{L^p(Q_r)}$$

where $C = C(p, \lambda, \Lambda, Q, \| b \|_{L^p(Q_r(z_0))}, \| c \|_{L^p(Q_r(z_0))}, \| f \|_{L^p(Q_r(z_0))})$ and

$$\mu := \frac{\alpha}{\alpha - \beta}, \quad \alpha := \frac{q(Q + 2)}{q(Q - 2) + 2(Q + 2)} \quad \text{and} \quad \beta := \frac{q}{q - 1}. \quad (3.2)$$


Remark 3.2. Theorem 3.1 holds true under the assumptions of Remark 1.7. In particular, in this case the constant $\alpha$ is replaced by $\alpha = 1 + \frac{\gamma}{2}$, that was firstly obtained in [33]. This is due the fact that the Sobolev inequality, Theorem 3.3, holds true with a greater exponent if we assume more integrability for the coefficient $b$. Thus, this allows us to obtain the local boundedness for weak solutions to $L u = f$ with lower integrability for $c$ and $f$, i.e. $c, f \in L^q(\Omega)$, with $q > \frac{Q+2}{Q-2}$.

3.1. Sobolev Inequality. This subsection is devoted to the proof of a Sobolev type inequality for weak solutions to $L u = f$. Our approach is inspired by the paper [33] and allows us to construct an “ad hoc” Sobolev embedding for weak solutions to $L u = f$ by overcoming the difficulties due to the degeneracy of the second order part of the operator $L$ at the cost of lowering the Sobolev exponent, that in our case depends on $q$ and it is defined in (3.2). We remark that the following statement holds true under lower integrability assumption than the one required in (H3).

Theorem 3.3 (Sobolev Type Inequality for sub-solutions). Let $(H1)$-$(H2)$ hold. Let $c \in L^q(\Omega)$ and $b \in (L^q(\Omega))^{m_0}$ for some $q > \frac{Q+2}{Q-2}$ and let $f \in L^2(\Omega)$. Let $v$ be a non-negative weak sub-solution of $L v = f$ in $Q_1$. Then there exists a constant $C = C(\Omega, \lambda, \Lambda) > 0$ such that $v \in L^{2\alpha}(Q_1)$, and the following inequality holds

$$||v||_{L^{2\alpha}(Q_\rho(z_0))} \leq C \cdot \left( \|b\|_{L^q(Q_\rho(z_0))} + \frac{r - \rho + 1}{r - \rho} \|D_{v_0}v\|_{L^2(Q_\rho(z_0))} + C \cdot \left( \|c\|_{L^q(Q_\rho(z_0))} + \frac{\rho + 1}{\rho(r - \rho)} \right) ||v||_{L^2(Q_\rho(z_0))} + C \cdot ||f||_{L^2(Q_\rho(z_0))} \right)$$

for every $\rho, r$ with $\frac{1}{2} \leq \rho < r \leq 1$ and for every $z_0 \in \Omega$, where $\alpha = \alpha(q)$ is defined as (3.2). The same statement holds for non-negative super-solutions.

Proof. Let $v$ be a non-negative weak sub-solution to $L v = f$. We represent $v$ in terms of the fundamental solution $\Gamma$. To this end, we consider the cut-off function $\chi_{\rho,r}$ defined in (2.17) for $\frac{1}{2} \leq \rho < r \leq 1$. Then, if we consider the test function

$$\psi(x,t) = \chi_{\rho,r}(\|x,t\|),$$  

the following estimates hold true

$$|Y \psi| \leq \frac{c_0}{\rho(r - \rho)}, \quad |\partial_{x_j} \psi| \leq \frac{c_1}{r - \rho} \quad \text{for } j = 1, \ldots, m_0$$  

(3.4)

where $c_0, c_1$ are dimensional constants. For every $z \in Q_\rho$, we have

$$v(z) = v \psi(z) = \int_{Q_\rho} \left[ (A_0 D(v \psi), D \Gamma(z, \cdot)) - (\Gamma(z, \cdot) Y(v \psi)) \right] (\zeta) d(\zeta) \quad \text{for } \frac{1}{2} \leq \rho < r \leq 1$$  

(3.5)
where

\[
I_0(z) = \int_{Q_r} [(b, Dv) \Gamma(z, \cdot) \psi] (\zeta) d\zeta + \int_{Q_r} [cv \Gamma(z, \cdot) \psi] (\zeta) d\zeta - \int_{Q_r} [\Gamma(z, \cdot) \psi f] (\zeta) d\zeta \\
I_1(z) = \int_{Q_r} [([A_0 D\psi, D\Gamma(z, \cdot)] \psi] (\zeta) d\zeta - \int_{Q_r} [\Gamma(z, \cdot) v Y \psi] (\zeta) d\zeta = I'_1 + I_1', \\
I_2(z) = \int_{Q_r} [([A_0 - A) Dv, D\Gamma(z, \cdot)] \psi] (\zeta) d\zeta - \int_{Q_r} [\Gamma(z, \cdot) (A Dv, D\psi)] (\zeta) d\zeta \\
I_3(z) = \int_{Q_r} [([A Dv, D(\Gamma(z, \cdot) \psi))] (\zeta) d\zeta - \int_{Q_r} [((\Gamma(z, \cdot) \psi) Yv)] (\zeta) d\zeta \\
- \int_{Q_r} [(b, Dv) \Gamma(z, \cdot) \psi] (\zeta) d\zeta - \int_{Q_r} [cv \Gamma(z, \cdot) \psi] (\zeta) d\zeta + \int_{Q_r} [\Gamma(z, \cdot) \psi f] (\zeta) d\zeta
\]

Since \( v \) is a non-negative weak sub-solution to \( \mathcal{L} v = f \), it follows from Lemma 2.7 that \( I_3 \leq 0 \), then

\[
0 \leq v(z) \leq I_0(z) + I_1(z) + I_2(z) \quad \text{for a.e.} \ z \in Q_r.
\]

To prove our claim is sufficient to estimate \( v \) by a sum of \( \Gamma \)-potentials. We start by estimating \( I_0 \). In order to do so, we recall that

\[
\langle b, Dv \rangle, cv \in L^{\frac{Q_{q+2}}{2}} \quad \text{for} \ b, c \in L^q, \ q > \frac{Q+2}{2} \quad \text{and} \ v \in L^2.
\]

Thus by Corollary 2.6 we get

\[
\Gamma * \langle b, Dv \rangle, \Gamma * (cv) \in L^{2\alpha},
\]

where \( \alpha = \alpha(q) \) is defined in (3.2). In addition, for \( f \in L^2 \), we have

\[
\Gamma * f \in L^{2\kappa}, \quad \kappa = \frac{Q+2}{Q-2}.
\]

Observing that \( \kappa > \alpha \), we obtain that \( \Gamma * f \in L^{2\alpha} \) and therefore

\[
\| I_0(\zeta) \|_{L^{2\alpha}(\Omega_\rho)} \leq \Gamma * \langle (b, D_{m_0} v) \psi \rangle + \Gamma * (cv \psi) \|_{L^{2\alpha}(\Omega_\rho)} + \| \Gamma * f \|_{L^{2\alpha}(\Omega_\rho)} \\
\leq C \cdot \| b \|_{L^r(\Omega_\rho)} \| D_{m_0} v \|_{L^2(\Omega_\rho)} + \| c \|_{L^q(\Omega_\rho)} \| v \|_{L^2(\Omega_\rho)} + \| f \|_{L^2(\Omega_\rho)}.
\]

We now deal with the \( I_1 \). \( I'_1 \) can be estimated by (2.15) of Corollary 2.6 as follows

\[
\| I'_1 \|_{L^{2\alpha}(\Omega_\rho)} \leq C \| I'_1 \|_{L^{2\alpha}(\Omega_\rho)} \leq C \| v D_{m_0} \psi \|_{L^2(\mathbb{R}^{N+1})} \leq \frac{C}{r - \rho} \| v \|_{L^2(\Omega_\rho)}
\]

where the last inequality follows from (3.4). To estimate \( I''_1 \) we use (2.14)

\[
\| I''_1 \|_{L^{2\alpha}(\Omega_\rho)} \leq C \| I''_1 \|_{L^{2\alpha}(\Omega_\rho)} \leq \text{meas}(\Omega_\rho)^{2/Q} \| I''_1 \|_{L^{2\alpha}(\Omega_\rho)} \\
\leq C \| v Y \psi \|_{L^2(\mathbb{R}^{N+1})} \leq \frac{C}{\rho(r - \rho)} \| v \|_{L^2(\Omega_\rho)}.
\]

We can use the same technique to prove that

\[
\| I_2 \|_{L^{2\alpha}(\Omega_\rho)} \leq C \left( 1 + \frac{1}{r - \rho} \right) \| Dv \|_{L^2(\Omega_\rho)},
\]

for some constant \( C = C(Q, \lambda, \Lambda) \). A similar argument proves the thesis when \( v \) is a super-solution to \( \mathcal{L} v = f \). In this case we introduce the following auxiliary operator

\[
\mathcal{K} = \text{div}(A_0 D) + \tilde{Y}, \quad \tilde{Y} \equiv - \langle x, BD \rangle - \partial_t.
\]
Then we proceed analogously as in [33], Section 3, proof of Theorem 3.3.

3.2. Caccioppoli inequality. The aim of this subsection is to prove a Caccioppoli type inequality for powers \( v = u^p \) of weak sub-solutions to \( \mathcal{L}u = f \).

**Theorem 3.4** (Caccioppoli type inequality for sub-solutions). Let \((H1)-(H3)\) hold. Let \( r, \rho \) be such that \( \frac{1}{2} \leq \rho < r \leq 1 \). Then for every weak sub-solutions to \( \mathcal{L}u = f \) we have that for every \( p > 1/2 \) it holds

\[
\| D_m v \|_{L^2(Q_r)}^2 \lesssim \frac{4p}{\lambda(2p - 1)} \left( \frac{2p}{2p - 1} \right) \left( \frac{c_1^2 \lambda}{\rho(r - \rho)^2} + \frac{c_0}{\rho(r - \rho)} \right) + \| b \|_{L^q(Q_r)}^2 \| c \|_{L^q(Q_r)}^2 + p \| f \|_{L^q(Q_r)}^2 \| u^p \|_{L^{2q}(Q_r)}^2,
\]

where \( \beta = \beta(q) = \frac{q}{q - 1} \), \( \varepsilon_b = \frac{|2p - 1| \lambda}{2|p|} \) and \( c_0, c_1 \) are defined in (3.4).

**Remark 3.5.** If \( f \in L^2(\Omega) \) then the estimate (3.7) holds true for every \( \frac{1}{2} < p \leq 1 \).

For the sake of completeness and in order to simplify the exposition of the proof of Theorem 3.4 we consider the following lemma, which is the analogous to [3, Proposition 3.2] when considering Definition 1.2 of weak solution.

**Lemma 3.6.** Let \((H1)-(H3)\) hold. Let \( p \in \mathbb{R}, p \neq 0, p \neq 1/2 \) and let \( r, \rho \) be such that \( \frac{1}{2} \leq \rho < r \leq 1 \). Then for every weak sub-solution to \( \mathcal{L}u = 0 \) the following estimate holds true

\[
\lambda \left| \frac{2p - 1}{p} \right| \| D_m u \|_{L^2(Q_r)}^2 \leq \left( \frac{2p}{2p - 1} \right) \left( \frac{c_1^2 \lambda}{(r - \rho)^2} + \frac{c_0}{r - \rho} \right) + \frac{c_1}{r - \rho} \| b \|_{L^q(Q_r)}^2 \| f \|_{L^q(Q_r)}^2 \| u^p \|_{L^{2q}(Q_r)}^2,
\]

where \( \beta = \beta(q) = \frac{q}{q - 1} \) and \( c_0, c_1 \) are defined in (3.4).

**Proof.** Let us consider \( p < 1, p \neq 0, p \neq 1/2 \). First of all, we consider a non-negative weak sub-solution \( u \) to \( \mathcal{L}u = 0 \). For every \( \psi \in C_0^\infty(Q_r) \) we consider the function \( \varphi = u^{2p-1}\psi^2 \). Note that \( \varphi, D_m \varphi \in L^2(Q_r) \), then we can use \( \varphi \) as a test function in the weak formulation (1.7):

\[
0 \leq \int_{Q_r} \langle ADu, D(u^{2p-1}\psi^2) \rangle + u^{2p-1}\psi^2 Y u + \langle b, Du \rangle u^{2p-1}\psi^2 + cu^{2p}\psi^2.
\]

Let \( v = u^p \). Since \( u \) is a weak sub-solution to \( \mathcal{L}u = 0 \) and \( u \geq u_0 \), thanks to Proposition 3.3 we have that \( v, D_m v \in L^2(Q_r) \) and \( Yv \in L^2H^{-1}(Q_r) \) and thus:

\[
0 \leq -\frac{2p - 1}{p} \int_{Q_r} \langle ADv, Dv \rangle \psi^2 - 2 \int_{Q_r} \langle ADv, D\psi \rangle \psi + \int_{Q_r} v Y \psi^2 + \int_{Q_r} v \langle b, D_m v \rangle \psi^2 + p \int_{Q_r} cv^2 \psi^2.
\]

First of all, we estimate of the terms involving the matrix \( A \). In particular, by applying Young’s inequality we have that for every \( \varepsilon > 0 \):

\[
2 \int_{Q_r} |\langle ADv, D\psi \rangle \psi| \leq \varepsilon \int_{Q_r} \langle ADv, Dv \rangle \psi^2 + \frac{1}{\varepsilon} \int_{Q_r} \langle AD\psi, D\psi \rangle v^2.
\]
Thus, taking the absolute value on the right-hand side of (3.9) and considering ε = |(2p − 1)/2p| in (3.10), for every p < 1, p ≠ 0, 1/2 we obtain:

$$\left| \frac{2p - 1}{p} \lambda \int_{Q_r} |D_m v|^2 \right|^2 \leq \left| \frac{2p \lambda}{2p - 1} \right| \int_{Q_r} v^2 + \frac{c_0}{\rho(r - \rho)} \int_{Q_r} \psi^2 +$$

$$+ \int_{Q_r} v \langle b, D_m v \rangle \psi_B^2 + |p| \int_{Q_r} |c \psi^2|^2,$$

(3.11)

where we have considered the definition (3.3) of the cut-off function ψ in order to estimate the first integral on the right-hand side. By considering the bounds on (3.4), assumption (H3) and the Hölder’s inequality with exponent β = \(\frac{q}{q - 1}\) we are in position to estimate the term B

$$\frac{1}{r - \rho} \| b \|_{L^q(Q_r)} \| v \|_{L^{2\beta}(Q_r)}^2,$$

and the term C:

$$|p| \int_{Q_r} |c \psi^2|^2 \leq |p| \| c \|_{L^q(Q_r)} \| v \|_{L^{2\beta}(Q_r)}^2.$$

Thus, by choosing \(\varepsilon_a = \lambda/|p|\) and \(\varepsilon_b = (|2p - 1|\lambda)/(2|p|)\) we recover (3.8).

Now, let us consider the case \(p \geq 1\). For any \(n \in \mathbb{N}\), we define the function \(g_{n,p}\) on \([0, +\infty[\) as follows

$$g_{n,p}(s) = \begin{cases} s^p, & \text{if } 0 < s \leq n, \\ n^p + pn^{p-1}(s - n), & \text{if } s > n, \end{cases}$$

then we let \(v_{n,p} = g_{n,p}(u)\). Note that \(g_{n,p} \in C^1(\mathbb{R}^+)\) and \(g'_{n,p} \in L^\infty(\mathbb{R}^+)\). Thus, since \(u\) is a weak sub-solution to \(\mathcal{L}u = 0\), we have

$$v_{n,p} \in L^2, \quad D_m v_{n,p} \in L^2, \quad Yv_{n,p} \in L^2 H^{-1}.$$

We also note that the function

$$g''_{n,p}(s) = \begin{cases} p(p - 1)s^{p-2}, & \text{if } 0 < s < n \\ 0, & \text{if } s \geq n, \end{cases}$$

is the weak derivative of \(g'_{n,p}\), then \(D_m g_{n,p}(u) = g''_{n,p}(u)D_m (u)\) (for the detailed proof of this assertion, we refer to [10], Theorem 7.8). Hence, by considering

$$\varphi = g_{n,p}(u) g'_{n,p}(u) \psi^2, \quad \psi \in C_0^\infty(Q_r)$$

(3.12)
as a test function in the weak formulation (1.7) and recalling that we find $v_{n,p} = g_{n,p}(u)$ we have

$$0 \leq \int_{\Omega} -\psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle - \int_{\Omega} \frac{v''_{n,p}(u) v_{n,p}(u)}{(v'_{n,p}(u))^2} \psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle +$$

$$- \int_{\Omega} 2\psi \langle ADv_{n,p}, D\psi \rangle v_{n,p} + \int_{\Omega} \psi^2 v_{n,p} Y(v_{n,p}) +$$

$$+ \left[ \int_{\Omega} (b, D_{m_0}v_{n,p}) v_{n,p} \psi^2 \right] + \int_{\Omega} cu_{n,p}(u) v_{n,p}(u) \psi^2 .$$

Let us firstly estimate the term $B$ as in the previous case:

$$\frac{1}{2} \int_{\Omega} \langle b, D_{m_0}v_{n,p}^2 \rangle \psi^2 \leq \int_{\Omega} |\langle b, \psi D_{m_0}\psi \rangle| v_{n,p}^2 \leq \frac{C}{r-\rho} \parallel b \parallel_{L^q(\Omega_r)} \parallel v_{n,p} \parallel_{L^{2q}(\Omega_r)}^2 .$$

Then the term $C$ is treated as follows:

$$\int_{\Omega} cu_{n,p}(u) v_{n,p}(u) \psi^2 \leq p \int_{\Omega} |c| u^{2p} \psi^2 \leq p \parallel c \parallel_{L^q(\Omega_r)} \parallel u^p \parallel_{L^{2q}(\Omega_r)}^2 .$$

Thus we rewrite equation (3.13) as follows

$$\int_{\Omega} \psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle \left( 1 + \frac{v''_{n,p}(u) v_{n,p}(u)}{(p u^{p-1})^2} - \varepsilon \right)$$

$$\leq \left( \frac{c_1^2 \Lambda}{\varepsilon (r-\rho)^2} + \frac{c_0}{\rho (r-\rho)} + \parallel b \parallel_{L^q(\Omega_r)} + p \parallel c \parallel_{L^q(\Omega_r)} \right) \parallel u^p \parallel_{L^{2q}(\Omega_r)}^2 .$$

We now observe that

$$\left| \psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle \frac{v''_{n,p}(u) v_{n,p}(u)}{p u^{2p-2}} \right| \leq \frac{p-1}{p} \langle ADv_{n,p}, Dv_{n,p} \rangle \in L^1(\Omega_r)$$

and thus, keeping in mind that $|Dv_{n,p}| \parallel Du^p|$ as $n \to \infty$, we can apply the dominated convergence theorem to (3.14). The proof is complete if we choose $\varepsilon$ as in the previous case. \( \square \)

**Proof of Theorem 3.4.** We are now in a position to consider weak sub-solutions to $Lu = f$, with $f$ a non zero source term under assumption (H3). First of all, let us consider the case where $1/2 < p \leq 1$. Following the lines of the first part of this proof, for every $\psi \in C_0^\infty(\Omega_r)$ we consider the function $\varphi = u^{2p-1} \psi^2$ as a test function in the weak formulation of (1.7) with non zero source term. Indeed, by denoting $v = u^p$ we have that also in this case $v, D_{m_0}v \in L^2(\Omega_r)$ and $Yv \in L^2 H^{-1}$. Thus, we get exactly estimate (3.11) with the following additional term regarding the source term $f$:

$$|p| \int_{\Omega} |f| u^{2p-1} \psi^2 .$$

In order to conclude the proof, we therefore need to properly estimate $u^{2p-1}$. In particular, if
0 \leq u < 1 \text{ we have that } u^{2p-1} < 1, \text{ thus }
\begin{equation}
|p| \int_{Q_r} |f| u^{2p-1} \psi^2 \leq |p| \int_{Q_r} |f| \psi^2 \tag{3.17}
\end{equation}

• \ u \geq 1 \text{ we have that } u^{2p-1} < u^p = v. \text{ By combining Hölder’s inequality and Young’s inequality (3.10) for every } \varepsilon_f > 0 \text{ we get }
\begin{equation}
|p| \int_{Q_r} |f| u^{2p-1} \psi^2 \leq |p| \|f\|_{L^2(Q_r)} \|v\|_{L^2(Q_r)} \tag{3.18}
\end{equation}
Thus, by combining (3.17) with (3.18) and considering \( \varepsilon_f = 1 \) we get (3.7).

We now address the case \( p > 1 \). Reasoning as above, we set \( v_{n,p} = g_{n,p}(u) \) and we choose the test function as in (3.12). Then the weak formulation of (1.7) for \( v_{n,p} \) reads exactly as (3.13) with the additional term
\begin{equation}
-p \int_{Q_r} f v_{n,p}(u) v'_{n,p}(u) \psi^2.
\end{equation}
We deal with the boxed term \( F \) by distinguishing two different cases. In particular, if
• \ 0 \leq u < 1 \text{ we have that } u^{2p-1} < 1, \text{ thus we exactly recover estimate (3.17).}
• \ u \geq 1 \text{ we have that } u^{2p-1} < u^p. \text{ We now observe that } w^p \text{ is a non-negative weak super-solution to } \mathcal{L} u = f \text{ and therefore } w^p \in L^{2^*} \text{ in virtue of Theorem 3.3. By applying Hölder’s inequality, we then infer }
\begin{equation}
-p \int_{Q_r} f v_{n,p}(u) v'_{n,p}(u) \psi^2 \leq p \int_{Q_r} |f| u^{2p-1} \psi^2 \leq p \|f\|_{L^2(Q_r)} \|u^p\|_{L^{2^*}(Q_r)}^2 \tag{3.19}
\end{equation}
By combining (3.17) and (3.19) we conclude the proof (3.7). \( \square \)

3.3. Proof of Theorem 3.1. Keeping in mind Lemma 2.8, it suffices to give a proof in the case \( \varepsilon_0 = 0 \) and to consider \( \frac{1}{2} \leq \rho < r \leq 1 \), since the general statement is obtained by applying to (3.23) the group dilations (2.9) and translations (2.1). Combining Theorems 3.3 and 3.6 for a non-negative sub-solution \( u \), we obtain the following estimate. If \( s > 1/2, \delta > 0 \) verify the condition
\[ |s - 1/2| \geq \delta, \]
then for every \( \rho, r \) such that \( \frac{1}{2} \leq \rho < r \leq 1 \) we have
\begin{equation}
\|u^s\|_{L^{2^*}(Q_r)} \leq \tilde{C}(s, \lambda, \Lambda, Q, \|b\|_{L^\beta(Q_r)}, \|c\|_{L^\gamma(Q_r)}, \|f\|_{L^\gamma(Q_r)}) \|u^s\|_{L^{2^*}(Q_r)} \tag{3.20}
\end{equation}
for a positive constant \( \tilde{C} \) that we be estimated as follows
\begin{equation}
\tilde{C}(s, \lambda, \Lambda, Q, \|b\|_{L^\gamma(Q_r)}, \|c\|_{L^\gamma(Q_r)}, \|f\|_{L^\gamma(Q_r)}) \leq K(s, \lambda, \Lambda, Q, \|b\|_{L^\gamma(Q_r)}, \|c\|_{L^\gamma(Q_r)}, \|f\|_{L^\gamma(Q_r)}) \tag{3.21}
\end{equation}
for every \( \rho, r \) of our choice the following estimates hold
\[
\frac{1}{(\rho(r - \rho))^2} \leq \frac{1}{(r - \rho)^2}, \quad \frac{1}{\rho(r - \rho)} \leq \frac{1}{(r - \rho)^2}, \quad \frac{1}{(\rho(r - \rho))^2} \leq \frac{1}{(r - \rho)^2}.
\]
Thus, by combining (3.20) and (3.21), for every \( n \in \mathbb{N} \cup \{0\} \) the following holds
\[
\| v^{(\alpha)}_{\beta} \|_{L^{2\alpha}(Q_{p+1})} \leq \frac{K(p, \lambda, \Lambda, Q, \| b \|_{L^q(Q_r)}, \| c \|_{L^q(Q_r)}, \| f \|_{L^q(Q_r)}}{(\rho_n - \rho_{n+1})^{2n}} \| v^{(\beta)} \|_{L^{2\beta}(Q_{p+1})},
\]
From now on we denote \( K = K(p, \lambda, \Lambda, Q, \| b \|_{L^q(Q_r)}, \| c \|_{L^q(Q_r)}, \| f \|_{L^q(Q_r)}) \). Since
\[
\| v^{(\alpha)}_{\beta} \|_{L^{2\alpha}} = \left( \| v \|_{L^{2\alpha}(Q_{p+1})} \right)^\alpha \quad \text{and} \quad \| v^{(\beta)} \|_{L^{2\beta}} = \left( \| v \|_{L^{2\beta}} \right)^\beta
\]
we are able to rewrite the previous estimate in the following form for every \( n \in \mathbb{N} \cup \{0\} \)
\[
\| v \|_{L^{2\alpha}(Q_{p+1})} \leq \left( \frac{K}{(\rho_n - \rho_{n+1})^{2n}} \right)^\alpha \| v \|_{L^{2\beta}(Q_{p+1})}.
\]
Iterating this inequality and letting \( n \) go to infinity, we get
\[
\sup_\Omega v \leq \frac{\tilde{K}}{(r - \rho)^{2n}} \| v \|_{L^{2\beta}(Q_r)}, \quad \text{where} \quad \mu = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \tilde{K} = \prod_{j=0}^{+\infty} \left[ K 2^{2(j+2)} \right]^{(\beta/\alpha)^j}.
\]
We remark that \( \tilde{K} \) is a finite constant depending on \( \delta \), since the product over \( j \) corresponds to a converging series. Thus, for every \( p \geq 1 \) which verifies condition (3.22) we have proved that
\[
\sup_\Omega u^p \leq \frac{\tilde{K}^2}{(r - \rho)^{4\mu}} \| u^p \|_{L^{4\mu}(Q_r)}, \quad (3.23)
\]
and thus the statement is proved. We now make a suitable choice of \( \delta > 0 \), only dependent on the homogeneous dimension \( Q \), and on \( q \) in order to show that (3.22) holds for every \( p \geq 1 \). We notice that, if \( p \) is a number of the form
\[
p_m = \frac{1}{2} \left( \frac{\alpha}{\beta} \right)^m \left( \frac{\alpha}{\beta} + 1 \right), \quad m \in \mathbb{Z},
\]
then (3.22) is satisfied with the following choice of \( \delta \) for every \( m \in \mathbb{Z} \)
\[
\delta = \frac{\alpha - \beta}{8\beta}.
\]
Therefore (3.23) holds for such a choice of \( p \), with \( \tilde{K} \) only dependent on \( Q, q, \lambda, \Lambda \) and \( \| a \|_{L^q(Q_r)}, \| b \|_{L^q(Q_r)}, \| c \|_{L^q(Q_r)} \). On the other hand, if \( p \) is an arbitrary positive number, we consider \( m \in \mathbb{Z} \) such that
\[
p_m \leq p < p_{m+1}.
\]
Hence, the proof is complete. \( \square \)
4. Weak Poincaré inequality

This section is devoted to the proof of a weak Poincaré inequality (see Theorem 4.1) that holds true for every function \( u \in W \). As one immediately understands, this Poincaré inequality is independent of the equation \( \mathcal{L} u = f \) and only relies on the structure of the space \( W \). Its importance lies in the fact that it is a necessary tool in the proof of a Harnack inequality (see Theorem 1.4) and of the local Hölder continuity (see Theorem 1.6) of a solution \( u \) to (1.1).

In order to state our result, we first need to introduce the following sets

\[
\begin{align*}
\mathcal{Q}_{\text{zero}} &= \{ (x, t) : |x_j| \leq \eta^{\alpha_j}, j = 1, \ldots, N, -1 - \eta^2 < t \leq -1 \}, \\
\mathcal{Q}_{\text{ext}} &= \{ (x, t) : |x_j| \leq 2^\alpha j R, j = 1, \ldots, N, -1 - \eta^2 < t \leq 0 \},
\end{align*}
\]

where \( R > 1, \eta \in (0, 1] \) and the exponents \( \alpha_j \), for \( j = 1, \ldots, N \), are defined in (2.10). Indeed, \( \mathcal{Q}_{\text{zero}} \) and \( \mathcal{Q}_{\text{ext}} \) are completely equivalent to (1.11) thanks to Remark 2.4, but they are more convenient for the construction of the cut-off function \( \psi_1 \) introduced in Lemma 4.3. Now, we state our weak Poincaré inequality.

**Theorem 4.1** (Weak Poincaré inequality). Let \( \eta \in (0, 1) \) and let \( \mathcal{Q}_{\text{zero}} \) and \( \mathcal{Q}_{\text{ext}} \) be defined as in (4.1). Then there exist \( R > 1 \) and \( \vartheta_0 \in (0, 1) \) such that for any non-negative function \( u \in W \) such that \( u \leq M \) in \( \mathcal{Q}_1 \) for a positive constant \( M \), we have

\[
\| (u - \vartheta_0 M)_+ \|_{L^2(\mathcal{Q}_1)} \leq C \left( \| D_m u \|_{L^2(\mathcal{Q}_{\text{ext}})} + \| Yu \|_{L^2 H^{-1}(\mathcal{Q}_{\text{ext}})} \right),
\]

where \( C > 0 \) is a constant only depending on \( Q \).

The notation we consider here needs to be intended in the sense of (1.6). In particular, we have that \( L^2 H^{-1}(\mathcal{Q}_{\text{ext}}) \) is short for

\[
L^2( B_{2^j R} \times \ldots \times B_{2^j+1} R \times (-1 - \eta^2, 0], H^{-1}_{x(0)}( B_{2 R})),
\]

where we have split \( x = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}) \) according to (1.5).

We observe that a somewhat similar Poincaré inequality was already introduced by W. Wang and L. Zhang for subsolutions of ultraparabolic equations, see for instance [38, Lemmas 3.3 & 3.4] and the corresponding lemmas in [39], for the “strong” notion of weak solution introduced in [33], i.e. \( Yu \in L^2 \). However, the statement of our Theorem 4.1 differs substantially from the ones in the articles mentioned above. This is mainly due to the fact that in our treatment we extend the functional framework introduced in [4] to study the Kolmogorov-Fokker-Planck operator. As already mentioned, this framework seems to be the appropriate one when dealing with operator \( \mathcal{L} \) and allows us to forget the equation under study. Moreover, the proof we propose here is different from the ones in [38, 39], as we avoid using repeatedly the exact form of the fundamental solution of \( \mathcal{L} \) and exploit arguments closer to the classical theory of parabolic equations (developed, for instance, in [25]). Additionally, the information obtained through the log-transform is here summarized in only one weak Poincaré inequality (this is in contrast with [38, 39], where it is split in several lemmas). Finally, we remark that in this paper the geometric settings of the main lemmas are simpler than the ones in [38, 39].

In order to prove Theorem 4.1, the idea is to firstly derive a local Poincaré inequality in terms of an error function \( h \), that is defined as the solution to the following Cauchy problem

\[
\begin{cases}
\mathcal{K} h = u \mathcal{K} \psi, & \text{in } \mathbb{R}^N \times (-\rho^2, 0) \\
h = 0, & \text{in } \mathbb{R}^N \times \{-\rho^2\}
\end{cases}
\]
where $\mathcal{K}$ is the operator defined in (1.3) and $\psi$ is a given cut-off function, and then to explicitly control the error $h$ through the $L^\infty$ norm of the function $u$ (see Lemma 4.4).

**Lemma 4.2.** Let $\mathcal{Q}_{\text{ext}}$ be as defined in (4.1) and let $\psi : \mathbb{R}^{N+1} \rightarrow [0,1]$ be a $C^\infty$ function, with support in $\mathcal{Q}_{\text{ext}}$ and such that $\psi = 1$ in $\mathcal{Q}_1$. Then for any $u \in \mathcal{W}$, the following holds

$$\|(u - h)_+\|_{L^2(\mathcal{Q}_1)} \leq C \left( \|D_{m_0}u\|_{L^2(\mathcal{Q}_{\text{ext}})} + \|Y u\|_{L^2H^{-1}(\mathcal{Q}_{\text{ext}})} \right)$$

(4.5)

where $h$ is defined in (4.4), $C$ is a constant only depending on $|\rho^2|$ and $\|D_{m_0}\psi\|_{L^\infty(\mathcal{Q}_{\text{ext}})}$, and the notation we consider needs to be intended in the sense of (4.3).

This local weak Poincaré inequality is an extension to operator $\mathcal{L}$ and to the space $\mathcal{W}$ of the one proved in [13] and a simplification of the one proved in [38, 39]. Moreover, the result holds true for any cylinder of the form $\mathcal{Q}_{\text{ext}} = B_{R_0} \times \ldots \times B_{R_k} \times (-\rho^2,0]$, provided that $\mathcal{Q}_1 \subseteq \mathcal{Q}_{\text{ext}}$. The proof of Lemma 4.2 is mainly based on the properties of the principal part operator $\mathcal{K}$ and of $\mathcal{W}$.

**Proof of Theorem 4.2.** In virtue of Remark 1.1, the function $g := u\psi$ satisfies the following equation in the sense of distributions

$$\mathcal{K} g = u \mathcal{K}\psi + \text{div}_{m_0} \tilde{H}_1 + \tilde{H}_0, \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (-\rho^2,0)),$$

where $\tilde{H}_1 = (H_1 + D_{m_0}u)\psi$ and $\tilde{H}_0 = H_0\psi - H_1D_{m_0}\psi + \langle D_{m_0}\psi, D_{m_0}u \rangle$. Thus, owing to (4.4), we obtain

$$\mathcal{K}(g - h)_+ = \text{div}_{m_0} \tilde{H}_1 + \tilde{H}_0 =: \tilde{H}, \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (-\rho^2,0)).$$

(4.6)

Now, choosing $2(g - h)_+\psi^2$ as a test function in (4.6) and integrating on the domain $D = \mathbb{R}^N \times [-\rho^2,0]$, we get

$$2 \int_D |D_{m_0}|^2 \psi^2 + \int_D \psi(g - h)_+\langle D_{m_0}((g - h)_+), D_{m_0}\psi \rangle - 2 \int_D (g - h)_+\psi^2 Y((g - h)_+)$$

$$+ 2 \int_D (g - h)_+\psi^2 D_{m_0} \cdot \tilde{H}_1 + 2 \int_D (g - h)_+\psi^2 \tilde{H}_0 = 0.$$ 

(4.7)

We estimate the boxed term $A$ by applying Young’s inequality (3.10) and choosing $\varepsilon = 1$. As far as we are concerned with the boxed term $B$, we rewrite it as

$$- 2 \int_D (g - h)_+\psi^2 Y((g - h)_+) = - \int_D \psi^2 Y_0((g - h)_+^2) + \int_D \psi^2 \partial_t((g - h)_+^2)$$

$$= - \int_D [\psi^2 Y_0((g - h)_+^2) + \partial_t((g - h)_+^2)\psi^2] - (g - h)_+^2 \partial_t(\psi^2)]$$

$$= \int_D (g - h)_+^2 Y(\psi^2) + \int_D \partial_t((g - h)_+^2)\psi^2,$$

where in the first line we defined $Y_0 = \langle Bx, D \rangle$, in the second line we used the equality $\partial_t((g - h)_+^2)\psi^2 = \partial_t((g - h)_+^2\psi^2) - \partial_t(\psi^2)(g - h)_+^2$ and in the third we integrated by parts the term involving $Y_0$. Finally, we take care of the boxed term $C$ and $D$ using Young’s inequality as
By combining (4.7) as
\[
2 \int_{D} (g - h) + \psi^2 D_{m_0} \cdot \tilde{H}_{1} \leq -2 \int_{D} \langle D_{m_0} ((g - h) + \psi^2), \tilde{H}_{1} \rangle \leq \frac{1}{2} \| \nabla^{\perp} D_{m_0} (g - h) \|_{L^2(D)} + \frac{1}{2} \| (g - h) + D_{m_0} \psi \|_{L^2(D)} + 10 \| \tilde{H}_{1} \|_{L^2(D)}.
\]
we infer
\[
2 \int_{D} (g - h) + \psi^2 \tilde{H}_{0} \leq 2 \varepsilon \| (g - h) \|_{L^2(D)} + \frac{1}{2 \varepsilon} \| \tilde{H}_{0} \|_{L^2(D)}.
\]
Combining the previous estimates, for every \( T \in (-\rho^2, 0) \) and \( \varepsilon > 0 \) to be chosen later, we rewrite (4.7) as
\[
\int_{-\rho^2}^{T} \int_{D} \partial_t ((g - h) + \psi^2) dx dt + 2 \int_{D} |D_{m_0} (g - h) + \psi^2 + \int_{D} (g - h)^2 Y (\psi^2)
\]
\[
\leq \frac{1}{2} \| D_{m_0} (g - h) + \psi \|_{L^2(D)} + 10 \| \tilde{H}_{1} \|_{L^2(D)} + 2 \varepsilon \| (g - h) \|_{L^2(D)} + \frac{1}{2 \varepsilon} \| \tilde{H}_{0} \|_{L^2(D)}.
\]
We now apply the fundamental theorem of calculus to the term involving the time derivative and we infer
\[
\int_{D} ((g - h) + \psi^2 (x, T) dx + 2 \int_{D} |D_{m_0} (g - h) + \psi^2 + \int_{D} (g - h)^2 Y (\psi^2)
\]
\[
\leq \frac{1}{2} \| D_{m_0} (g - h) + \psi \|_{L^2(D)} + 10 \| \tilde{H}_{1} \|_{L^2(D)} + 2 \varepsilon \| (g - h) \|_{L^2(D)} + \frac{1}{2 \varepsilon} \| \tilde{H}_{0} \|_{L^2(D)}.
\]
We then integrate in \( T \) from \(-\rho^2\) to 0 and we obtain
\[
\| (g - h) + \psi \|_{L^2(D)} + \rho^2 \int_{D} (g - h)^2 Y (\psi^2) \leq -\frac{3}{2} \rho^2 \| D_{m_0} (g - h) + \psi \|_{L^2(D)} + 10 \rho^2 \| \tilde{H}_{1} \|_{L^2(D)} + 2 \rho^2 \varepsilon \| (g - h) \|_{L^2(D)} + \frac{\rho^2}{2 \varepsilon} \| \tilde{H}_{0} \|_{L^2(D)}
\]
\[
\leq 10 \rho^2 \| \tilde{H}_{1} \|_{L^2(D)} + 2 \rho^2 \varepsilon \| (g - h) \|_{L^2(D)} + \frac{\rho^2}{2 \varepsilon} \| \tilde{H}_{0} \|_{L^2(D)}.
\]
We later observe that \( (g - h) + \psi \) equals \( (u - h)_{+} \) and \( Y (\psi^2) \) equals 0 in \( Q_1 \). In addition, the following estimates hold
\[
\| \tilde{H}_{0} \|_{L^2(D)} \leq \| H_{0} \|_{L^2(Q_{ext})} + \| D_{m_0} \psi \|_{L^\infty (Q_{ext})} \| \| D_{m_0} u \|_{L^2(Q_{ext})} + \| H_{1} \|_{L^2(Q_{ext})},
\]
\[
\| \tilde{H}_{1} \|_{L^2(D)} \leq \| H_{1} \|_{L^2(Q_{ext})} + \| D_{m_0} u \|_{L^2(Q_{ext})}.
\]
By combining (4.8), (4.9) and (4.10) and choosing \( \varepsilon = \frac{1}{4 \rho^2} \) the claim is proved.

Given the local Poincaré inequality proved in Lemma 4.2, we just need to estimate the function \( h \) defined in (4.4) in order to complete the proof of Theorem 4.1. In particular, our aim is to show that the error function \( h \) is bounded from above by \( \vartheta_{0} M \), where \( \vartheta_{0} \in (0, 1) \) is a constant only depending on \( Q, \lambda \) and \( \Lambda \). In order to prove such result, we first need to explicitly construct an appropriate cut-off function, that differs from the one considered in [13, Lemma 3.3] due to the more involved structure of our drift term \( Y \). We note that our construction of the suitable cut-off function is constructive, in contrast with the one proposed in [13].
Lemma 4.3. Given $\eta \in (0, 1]$ and $T \in (0, \eta^2)$, there exists a smooth function $\psi_1 : \mathbb{R}^N \times [-1 - \eta^2, 0]$, supported in $\{(x, t) : |x_j| \leq 2^{\alpha_j}, j = 1, \ldots, N, t \in [-1 - \eta^2, 0]\}$, equal to 1 in $Q_1$, and such that the following conditions hold

\[
Y \psi_1 \leq 0 \quad \text{everywhere}
\]
\[
Y \psi_1 \leq -1 \quad \text{if } t \in (-1 - \eta^2, -1 - T].
\]

Proof. Let us consider the cut-off function $\chi \in C^\infty([0, +\infty))$ defined by

\[
\chi(s) = \begin{cases} 
0, & \text{if } s > \frac{2}{\sqrt{C}}, \\
1, & \text{if } 0 \leq s \leq C + 1,
\end{cases}
\]

where $C > 1$ is a constant we shall specify later on. In addition, we consider a smooth function $\chi_t : [-1 - \eta^2, 0] \to [0, 1]$ equal to 1 in $[-1, 0]$, with $\chi_t(-1 - \eta^2) = 0$, $\chi'_t \geq 0$ in $[-1 - \eta^2, 0]$ and $\chi'_t = 1$ in $[-1 - \eta^2, -1 - T]$. Now, setting

\[
\chi_0(x, t) = \chi\left(\sum_{j=m_0}^{N} \frac{2x_j^2}{2^{2\alpha_j} \sqrt{2}} - Ct\right),
\]

we define the cut-off function $\psi_1$ as follows

\[
\psi_1(x, t) = \chi(\|(x_1, \ldots, x_{m_0})\|)\chi_0(x, t)\chi_t(t).
\]

We only have to check that conditions (4.11) hold, as the other desired properties immediately follow from the definition of $\psi_1$. To this end, we compute the following derivative

\[
Y \chi_0 = \chi'((\ldots)) \left[ \sum_{i=1}^{N} \sum_{j>m_0} 2x_ib_{ij}x_j2^{-2\alpha_j+1/2} + Ct \right],
\]

where $(\ldots)$ denotes $\left(\sum_{j=m_0}^{N} \frac{2x_j^2}{2^{2\alpha_j} \sqrt{2}} - Ct\right)$. It can be shown (see [39]) that it is well defined a certain constant $C > 1$ such that

\[
C \geq \sum_{i=1}^{N} \sum_{j>m_0} 2x_ib_{ij}x_j2^{-2\alpha_j-1/2}.
\]

Thus, with such a choice of $C$ and keeping in mind that $\chi'_t \geq 0$ in $[-1 - \eta^2, 0]$ and $\chi'_t = 1$ in $[-1 - \eta^2, -1 - T]$, we have

\[
Y \psi_1 = \chi \chi_t Y \chi_0 - \chi \chi_0 \chi'_t \leq 0 \quad \text{everywhere},
\]
\[
Y \psi_1 = \chi \chi_t Y \chi_0 - \chi \chi_0 \chi'_t \leq -1 \quad \text{if } t \in (-1 - \eta^2, -1 - T].
\]

Thus, we are now in a position to state and prove the following result concerning the control of the localization term $h$ defined in (4.4).

Lemma 4.4. Let $\eta \in (0, 1]$ and let $\mathcal{Q}_{ext}$ be as defined in (4.1). Then there exist $R = R(Q, \eta) > 1$, $\vartheta_0 = \vartheta_0(Q, \eta) \in (0, 1)$ and a $C^\infty$ cut-off function $\psi : \mathbb{R}^{N+1} \to [0, 1]$, with support in $\mathcal{Q}_{ext}$ and equal to 1 in $\mathcal{Q}_1$, such that for all $u \in \mathcal{W}$ non-negative bounded functions defined on $\mathcal{Q}_{ext}$, the function $h$ solution to the Cauchy problem (4.4) with $\rho^2 = 1 + \eta^2$ satisfies

\[
h \leq \vartheta_0 \|u\|_{L^\infty(\mathcal{Q}_{ext})}.
\]
Proof. We assume that \( u \) is not identically vanishing in \( \Omega_{\text{ext}} \). Indeed, if \( u = 0 \) in \( \Omega_{\text{ext}} \), then \( h = 0 \) and inequality (4.13) is trivially satisfied. Moreover, we can reduce to the case of a function \( u \) with \( L^\infty \)-norm equal to 1 by taking \( u/\|u\|_{L^\infty(\Omega_{\text{ext}})} \).

We now fix \( T = \eta^2/2 \); then,

\[ |\Omega_{\text{zero}} \cap \{ t \leq -1 - T \}| = \frac{1}{2}|\Omega_{\text{zero}}|. \]

We now consider the cut-off function

\[ \psi(x, t) = \psi_1(x/R, t), \]

where \( R > 1 \) is a constant we will specify later and \( \psi_1 \) is given by Lemma 4.3. We observe that, by definition of \( \psi_1 \), \( \psi \) is supported in \( \Omega_{\text{ext}} \) and equal to 1 in \( \{(x, t) : |x_j| \leq R, j = 1, \ldots, N, t \in (-1, 0]\} \). In addition, it satisfies

\[ K\psi(x, t) = R^2\Delta m_0 \psi_1(x/R, t) + Y \psi_1(x/R, t). \]

Thus, in virtue of (4.4), we have

\[ K(h - \psi) = \frac{u - 1}{R^2} \Delta m_0 \psi_1(x/R, t) + (u - 1)Y \psi_1(x/R, t) \]

and we can write the difference \( h - \psi \) as

\[ h - \psi = E_R + N_R, \quad (4.14) \]

where \( E_R \) and \( N_R \) are solutions in \( \mathbb{R}^N \times (-1 - \eta^2, 0) \) to the following Cauchy problems

\[ K E_R = \frac{u - 1}{R^2} \Delta m_0 \psi_1(x/R, t), \]

\[ K N_R = (u - 1)Y \psi_1(x/R, t), \]

with \( E_R = P_R = 0 \) at the initial time. We first focus on the term involving \( E_R \), and we remark that

\[ K E_R \leq \frac{C'}{R^2}, \quad (4.15) \]

where \( C' = \|\Delta m_0 \psi_1\|_{L^\infty} \) is a constant only depending on \( Q, \lambda, \Lambda \) and \( \eta \). As far as the term involving \( N_R \) is concerned, we observe that, owing to \( Y \psi_1 \leq -1 \) for \( t \in (-1 - \eta^2, -1 - T) \), we have

\[ K N_R \leq -1_{\mathcal{Z}} \quad \text{in} \quad \mathbb{R}^N \times (-1 - \eta^2, 0), \]

where \( \mathcal{Z} := Q_{\text{zero}} \cap \{ t \leq -1 - T \} \). Let \( N \) such that \( \mathcal{K} N = -1_{\mathcal{Z}} \) in \( \mathbb{R}^N \times (-1 - \eta^2, 0) \) and \( N = 0 \) at the initial time \( t = -1 - \eta^2 \). Then, the maximum principle [5] for the principle part operator \( \mathcal{K} \) yields

\[ P \leq N_R \quad \text{in} \quad \Omega_1. \]

We now represent \( N \) in using the fundamental solution \( \Gamma \) of \( \mathcal{K} \) and we infer

\[ N(z) = \int \Gamma(z, \zeta) (-1_{\mathcal{Z}}(\zeta)) d\zeta \leq -\frac{1}{2}m|Q_{\text{zero}}| =: -\delta_0, \]

where \( m = \min_{\Omega_1 \times Q_{\text{zero}} \cap \{ t \leq -1 - T \}} \Gamma(z, \zeta) \). As a consequence,

\[ N_R \leq -\delta_0, \quad (4.16) \]
for a constant $\delta_0$ only depending on $Q$ and $\eta$. Using estimates (4.15) and (4.16) in (4.14), we finally obtain

$$h \leq 1 - \delta_0 + \frac{C'}{R^2}.$$  

We now observe that for $R$ large enough we have $C'/R^2 \leq \delta_0/2$. Thus, setting $\nu_0 = 1 - \delta_0/2 < 1$, we get the desired inequality (4.13). □

5. Main results

This section is devoted to the proof of our main results. The approach we present here is an extension of the method inspired by [21, 22] and then followed by Guerand and Imbert in [13] for the kinetic Kolmogorov-Fokker-Planck equation. In particular, we remark that an analogous approach based on a weak Poincaré type inequality was first introduced by Wang and Zhang in [38, 39] for the Kolmogorov equation $Lu = 0$ under the assumption $Yu \in L^2$, and thus with a stronger notion of weak solution, and with a different log–transform. The main advantage of our approach is that it only relies on the structure of the function space $W$ to which every weak solution belongs to and on the non-Euclidean geometrical structure presented in Section 2 behind the operator $L$. To our knowledge, it is the first time that the study of the weak regularity theory is carried on replacing the classical $L^{Q+2}$ integrability assumptions for the lower order coefficients $b, c$ and the non zero right-hand side $f$ with (H3).

5.1. Weak Harnack inequality. First of all we address the proof of the weak Harnack inequality (Theorem 1.3) that relies on combining the fact that super-solutions to (1.1) expand positivity along times (Lemma 5.1) with the covering argument presented in Appendix B.

The derivation of the weak Harnack inequality in the present paper from the expansion of positivity follows very closely the reasoning in [19] and extends the results presented in [13] for the Kolmogorov-Fokker-Planck case. For reader’s convenience, we here state (and adapt to our more involved case) the results contained in [13, Section 4], sketching their proofs only when they differ from the ones contained in the aforementioned paper.

We observe that, in contrast with parabolic equations, it is not possible to apply a classical Poincaré inequality in the spirit of [29]. Indeed, in our case there is a positive quantity replacing the average in the usual Poincaré inequality (see the statement of Theorem 4.1). Following [13] we circumvent this difficulty by establishing a weak expansion of positivity of super-solutions to (1.1). More precisely, given a small cylinder $Q_{pos}$ lying in the past of $Q_1$ (see Definition (5.1)), we show that the positivity of a non-negative super-solution $u$ lying above 1 in a ”big” part of $Q_{pos}$ is spread to the whole $Q_1$ (see Lemma 5.1). In other words, a positivity in measure in a smaller cylinder $Q_{pos}$ is transformed into a pointwise positivity in a bigger cylinder $Q_1$. We emphasize that such a weak expansion of positivity was already proved in [11] thanks to an intermediate value lemma, following De Giorgi’s original proof, but the argument was not constructive and specific for the Fokker-Planck equation case.

Lastly, we mention that Moser [29] and Trudinger [35] proved a weak Harnack inequality in the spirit of Theorem 1.3 in the setting of parabolic equations. We also recall that Di Benedetto and Trudinger [7] proved that non-negative functions in the elliptic De Giorgi class, which corresponds to super-solutions to elliptic equations, satisfy a weak Harnack inequality. Moreover, let us emphasize that in [37] it is proved a weak Harnack for the corresponding parabolic case, i.e. for functions in the parabolic De Giorgi’s class. We conclude by observing that quantitative interior Hölder regularity estimates for functions in the parabolic De Giorgi class (and for parabolic equations with rough coefficients) can be found in [12].
We first study how equation (1.1) spreads positivity of super-solutions. More precisely, we state the upcoming Lemma 5.1. Given in terms of the cylinders

\[ \mathcal{Q}_{\text{pos}} = B_{\theta} \times B_{\theta^3} \times \ldots \times B_{\theta^{2n+1}} \times (-1 - \theta^2, -1), \]

\[ \tilde{\mathcal{Q}}_{\text{ext}} = B_{3R} \times B_{3\theta R} \times \ldots \times B_{3\theta^{2n+1}R} \times (-1 - \theta^2, 0), \]

where \( R = R(\theta, Q, \lambda, \Lambda) \) is the constant given by Lemma 4.4 and \( \theta \in (0, 1) \) is a parameter we will choose later on. In particular, \( \theta \) will be chosen such that the stacked cylinder \( \mathcal{Q}_{\text{pos}}^m \) (see definition A.1) is contained in \( \Omega_1 \). To stress the dependence of \( R \) on \( \theta \), we will sometimes write \( R_\theta \) instead of \( R \).

**Lemma 5.1.** Let \( \theta \in (0, 1] \) and \( \mathcal{Q}_{\text{pos}} \), \( \tilde{\mathcal{Q}}_{\text{ext}} \) be the cylinders defined in (5.1). Then there exist a small constant \( \eta_0 = \eta_0(\theta, Q, \lambda, \Lambda) \in (0, 1) \) such that for any non-negative super-solution \( u \) of (1.1) in some cylindrical open set \( \Omega \supset \tilde{\mathcal{Q}}_{\text{ext}} \) such that \( |\{u \geq 1\} \cap \mathcal{Q}_{\text{pos}}| \geq \frac{1}{2} |\mathcal{Q}_{\text{pos}}| \), we have \( u \geq \eta_0 \) in \( \Omega_1 \).

**Proof.** For the sake of completeness we hereby state a sketch of the proof, that is an extension of [13, Lemma 4.1]. We consider \( g = G(u + \varepsilon) \), where \( G \) is the convex function defined in [13, Lemma 2.1]. In particular, \( G \) is such that

- \( G'' \geq (G')^2 \) and \( G' \leq 0 \) in \([0, +\infty)\],
- \( G \) is supported in \([0, 1]\),
- \( G(t) \sim -\ln t \) as \( t \to 0^+ \),
- \( -G'(t) \leq \frac{1}{t} \) for \( t \in [0, \frac{1}{4}] \).

Thus, we have that \( |G'(u + \varepsilon)| \leq |G'(\varepsilon)| \leq \varepsilon^{-1} \) as \( u \) is non-negative. Moreover, adapting [13, Lemma 2.2] to our case, we find that \( g \) is a non-negative sub-solution to (1.1) with \( f \) replaced by \( f G'(u + \varepsilon) \). This implies in particular that the drift term \( Yg \) is bounded, i.e. \( Yg \leq \text{div}(ADg) + \langle b, Dg \rangle + cg + \varepsilon^{-1}\|f\|_{L^q(\mathcal{Q}_{\text{ext}})} \). The rest of the proof follows very closely the one of [13, Lemma 4.1], with the only difference that we consider our Theorem 3.1 instead of the classical \( L^2 - L^\infty \) estimate and the weak Poincaré inequality 4.1 instead of [13, Theorem 1.4].

As a straightforward consequence of Lemma 5.1 we have the following result, whose proof is obtained reasoning exactly as in [13, Lemma 4.2].

**Lemma 5.2.** Let \( m \geq 3 \) and let \( R \) be the constant given in Lemma 5.1 for \( \theta \leq m^{-1/2} \). Then there exists a constant \( M = M(m, Q, \lambda, \Lambda) > 1 \) such that for any non-negative super-solution \( u \) to (1.1) with \( f \) equal to \( \theta \) satisfying \(|\{u \geq M\} \cap \Omega_1| \geq \frac{1}{2} |\Omega_1| \), we have \( u \geq 1 \) in \( \mathcal{Q}_{\text{pos}}^m \) (see (A.1)).

Before proving the weak Harnack inequality, we need to show that we can spread positivity along "suitable" cylinders. More precisely, recalling the definition of the open ball in (1.10), we set

\[ \mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2n+1}} \times (-\omega^2, 0), \quad \mathcal{Q}_- = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2n+1}} \times (-1 - 1 + \omega^2), \]

where \( \omega \) is a small positive constant. In particular, we will choose \( \omega \) small enough so that, when expanding positivity from a given cylinder \( \mathcal{Q}_r(z_0) \) in the past, the union of the stacked cylinders where the positivity is spread includes \( \mathcal{Q}_+ \). Moreover, we will choose the radius \( R_0 \) in the statement of Theorem 1.3 so that Lemma 5.1 can be applied to every stacked cylinder. The two previous statements are specified in Lemma B.1. The stacking cylinders Lemma B.1, combined with Lemma 5.1, implies the following result regarding the expansion of positivity for large times.
In the sequel, we will largely use the cylinders $Q_r[k]$, for $k = 1, \ldots, N$ and $Q_{R_{N+1}}[N+1]$, whose definition and properties are presented in Appendix B.

Lemma 5.3. Let $R_{1/2}$ be the constant given by Lemma 5.1 for $\theta = 1/2$ and let $u$ be any non-negative super-solution to (1.1) with $f = 0$ in $\Omega \supset Q$ such that $|\{u \geq M\} \cap Q_r(z_0)| \geq \zeta |Q_r(z_0)|$ for some $M > 0$ and for some cylinder $Q_r(z_0) \subset Q_- \setminus Q_+$. Then there exists a positive constant $p_0$, only depending on $Q$, $\lambda$, $\Lambda$, such that

$$u \geq M \left( \frac{r^2}{4} \right)^{p_0}, \quad \text{in } Q_+.$$  \hfill (5.3)

Proof. We apply Lemma 5.1 for $\theta = \frac{1}{2}$ to the function $u/M$, with $Q_r(z_0)$ and $Q_1$ taking the role of $Q_{pos}$ and $Q_1$ (this is achieved through a rescaling argument) and obtain $u/M \geq \eta_0$ in $Q_r[1]$. We then apply it to $u/(M \eta_0)$ and get $u \geq M \eta_0^2$ in $Q_r[2]$. Reasoning by induction on $k = 1, \ldots, N$ we infer $u \geq M \eta_0^k$ in $Q[k]$.

By exploiting Lemma 5.1 again, we get $u \geq M \eta_0^{N+1}$ in $Q_{R_{N+1}}[N+1]$, which implies that the same inequality holds true in $Q_+$. As $T_N \leq -t_0 < 1$, we have in particular $4N \kappa^2 \leq 1$. Picking $p_0 > 0$ so that $\eta_0 = \left( \frac{1}{4} \right)^{N+1}$, we finally obtain

$$u \geq M \left( \left( \frac{1}{4} \right)^N \right)^{N+1} \geq M \left( \frac{1}{4} \right)^N \geq M \left( \frac{r^2}{4} \right)^{p_0},$$

which concludes the proof. \hfill \Box

From now on we will assume $\omega < 1/\sqrt{2\kappa + 1}$, where $\kappa$ is defined in (1.5). We are in a position to prove the main result of this Section, Theorem 1.3.

Proof of Theorem 1.3. We start the proof by fixing the parameters $\omega$ and $R_0$ in order to select the appropriate geometric setting. More precisely, we choose $\omega$ so that we capture $Q_+$ when applying Lemma B.1, namely we fix $\omega < \frac{1}{\sqrt{2\kappa + 1}}$. In addition, we choose the radius $R_0$ so that the stacked cylinders do not leak out of $Q_0$, i.e. $R_0 \geq 6 (2\kappa + 1) R_{1/2}$, where $R_{1/2}$ is the constant given by Lemma 5.1 when $\theta = 1/2$. As we want to apply Lemma 5.2 to cylinders contained in $Q_-$, we also assume $R_0 \geq 3(2\kappa + 1) R_{m-1/2} m^{(2\kappa+1)/2} \omega^{2\kappa+1}$, where $R_{m-1/2}$ is the constant given by Lemma 5.1 for $\theta = m^{-1/2}$.

Our aim is to reduce ourselves to the case where

$$\inf_{Q_+} u \leq 1, \quad \text{and} \quad f = 0.$$  \hfill (5.4)

On one hand, if $\inf_{Q_+} u > 1$ we can simply consider $\bar{u} = u / (\inf_{Q_+} u + 1)$ and reduce to the case where $\inf_{Q_+} u \leq 1$. On the other hand, if $f \neq 0$ and $c = 0$ we have that $\bar{u} := u + \vartheta \|f\|_{L^2(Q)}$ is a super-solution to equation (1.1) with source term equal to 0, provided that we choose $\vartheta$ such that

$$\vartheta = R_0^{-\frac{\kappa+2}{\kappa}}.$$  \hfill (5.5)
Indeed, exploiting the fact that \( u \) is non-negative super-solution to \((1.1)\) in \( Q_0 \), we infer
\[
\int_{Q_0} -\langle ADu, D\varphi \rangle + \varphi Yu + (b, Du)\varphi = \int_{Q_0} -\langle ADu, D\varphi \rangle + \varphi Yu - \varphi \|f\|_{L^q(Q_0)} + (b, Du)\varphi
\leq \int_{Q_0} f\varphi - \int_{Q_0} \varphi \|f\|_{L^q(Q_0)}.
\]
We now observe that the last line in \((5.6)\) can be estimated as follows
\[
\int_{Q_0} f\varphi - \int_{Q_0} \varphi \|f\|_{L^q(Q_0)} \leq \left( R_0^{Q+2} \right)^{1-1/q} \|f\|_{L^q(Q_0)} \|\varphi\|_{L^\infty(Q_0)} - \varphi \|f\|_{L^q(Q_0)} R_0^{Q+2} \|\varphi\|_{L^\infty(Q_0)}
= \|f\|_{L^q(Q_0)} R_0^{Q+2} \|\varphi\|_{L^\infty(Q_0)} \left( \left( R_0^{Q+2} \right)^{1-1/q} - \varphi R_0^{Q+2} \right),
\]
which is equal to 0 when \( \varphi = R_0^{-2} \). Thus, \( \tilde{u} \) is a super-solution of equation \((1.1)\) with \( f = 0 \) and the weak Harnack inequality for \( \tilde{u} \) implies the one for \( u \). Lastly, if \( c \neq 0 \) the reasoning follows by replacing the value of \( \varphi \) in \((5.5)\) by
\[
\varphi = \frac{1}{(R_0^{Q+2})^{1/q} - \|c\|_{L^q(Q_0)}}.
\]
We now want to prove that for all \( k \in \mathbb{N} \), the following inequality holds
\[
\left| \{ u > M^k \} \cap Q_1 \right| \leq \tilde{C}(1 - \tilde{\mu})^k,
\]
for some constants \( \tilde{\mu} \in (0, 1) \), \( M > 1 \) and \( \tilde{C} > 0 \) that only depend on \( Q, \lambda \) and \( \Lambda \). The proof of this fact is carried out by induction. For \( k = 1 \) it is sufficient to choose \( \tilde{\mu} \leq \frac{1}{2} \) and \( \tilde{C} \) such that \( |Q_-| \leq \frac{1}{2} \tilde{C} \). We now assume that \((5.7)\) holds true for \( k \geq 1 \) and we prove it for \( k + 1 \). To this end, we consider the sets
\[
E := \{ u > M^{k+1} \} \cap Q_-, \quad F := \{ u > M^k \} \cap Q_1.
\]
We observe that \( E \) and \( F \) satisfy the assumptions of Corollary A.3 with \( Q_1 \) replaced by \( Q_- \) and \( \mu = 1/2 \). Indeed, by definition \( E \) and \( F \) are bounded measurable sets such that \( E \subset Q_- \subset F \). We now consider a cylinder \( Q = Q_r(z) \subset Q_- \) such that \( |Q \cap E| > \frac{1}{2} |Q| \), i.e.
\[
|\{ u > M^{k+1} \} \cap Q | > \frac{1}{2} |Q|.
\]
We show that \( r \) needs to be small, that is to say \( r \) is less than some parameter \( r_0 = r_0(Q, \lambda, \Lambda, k) \). Indeed, applying Lemma 5.3 to \( u \), we obtain \( u \geq M^{k+1}(r^2/4)^{\rho_0} \) in \( Q_+ \). Thus, owing to inf_{Q_+} u \leq 1, we infer \( 1 \geq M^{k+1}(r^2/4)^{\rho_0} \) and therefore it is sufficient to choose \( r_0 \leq 2M^{-k-1/2\rho_0} \). In order to apply Corollary A.3, we are left with proving that \( \Xi^m \subset F \), which holds true if \( \Xi^m \subset \{ u > M^k \} \). To this end, we apply Lemma 5.2 to \( u/M^k \) after rescaling the cylinder \( Q \) in \( Q_1 \).

In virtue of Corollary A.3, there exist \( c_{is} \in (0, 1) \) and \( C_{is} > 0 \) such that
\[
|E| = |\{ u > M^{k+1} \} \cap Q_-| \leq \frac{m+1}{m} \left( 1 - \frac{c_{is}}{2} \right) \left( |\{ u > M^k \} \cap Q_1| + C_{is}mr_0^2 \right) \\
\leq \left( 1 - \frac{c_{is}}{4} \right) \left( |\{ u > M^k \} \cap Q_1| + C_{is}mr_0^2 \right),
\]
provided that we chose \( m \in \mathbb{N} \) so that \( \frac{m+1}{m} \left( 1 - \frac{2}{p} \right) \leq 1 - \frac{2}{q} \). Thanks to the induction assumption and our choice of \( r_0 \) we get

\[
|E| \leq \left( 1 - \frac{C m}{4} \right) \left( \tilde{C}(1 - \tilde{\mu})^k + C \lambda m M^{-1/p_0} \right)
\]

Picking then \( \tilde{\mu} \) small enough so that \( M^{-1/p_0} \leq (1 - \tilde{\mu}) \) and \( \tilde{\mu} \leq \frac{C m}{4} \), we obtain

\[
|E| \leq \tilde{C} \left( 1 - \frac{C m}{4} \right) (1 - \tilde{\mu})^k \left( 1 + \tilde{C}^{-1} m M^{-1/p_0} \right)
\]

Picking \( \tilde{C} \) large enough so that \( \left( 1 + \tilde{C}^{-1} m M^{-1/p_0} \right) \leq 2 \) we conclude the proof of (5.7). By extending estimate (5.7) to the continuous case (i.e. \( k \in \mathbb{R} \) and \( k \geq 1 \)) and applying the layer cake formula to \( \int_{Q_r} f^p \) for some exponent \( p \), we obtain that \( \int_{Q_r} f^p \) is bounded from above by a constant that only depends on \( Q \), \( \lambda \) and \( \Lambda \).

\[ \square \]

5.2. Harnack inequality and local Hölder continuity.

Proof of Theorem 1.4. The full Harnack inequality is a direct consequence of the combination of the local boundedness of weak sub-solutions proved in Theorem 3.1 and the weak Harnack inequality of Theorem 1.3.

\[ \square \]

Proof of Theorem 1.6. The Hölder continuity of weak solutions is classically obtained by proving that the oscillation of the solution decays by a universal factor. This can be achieved in two different ways. Either by applying Lemma 5.1 with \( \theta = 1 \) in the same spirit of [13, Appendix B], or by directly applying the weak Harnack inequality, Theorem 1.3, following a standard argument, for further reference see [10].

\[ \square \]

APPENDIX A. THE INK-SPOTS THEOREM

For the sake of completeness, we provide here the proof of the Ink Spots Theorem for the case of ultraparabolic equations. This theorem involves a covering argument in the spirit of Krylov and Safonov [23] growing ink spots theorem, or the Calderón-Zygmund decomposition, and it is a fundamental ingredient for the proof of the weak Harnack inequality (see Theorem 1.3). In order to give its statement in our setting, we introduce the delayed cylinder

\[
\overline{\Omega}^m_r(z_0) = \left( (0, \ldots, 0, m r^2) \circ \Omega_r(z_0) \right) \cap (\mathbb{R}^{N+1} \times (t_0, +\infty))
\]  

(A.1)

where \( z_0 = (x_0, t_0) = (x_0^{(0)}, \ldots, x_0^{(m)}, t_0) \in \mathbb{R}^{N+1} \). We remark that \( \overline{\Omega}^m_r(z_0) \) starts immediately at the end of \( \Omega_r(z_0) \), with which shares the same values for \( x^{(0)} \), and its structure follows the non Euclidean geometry presented in Section 2 associated to the principal part operator \( \mathcal{K} \). The aim of this section is to prove the following statement.

Theorem A.1. Let \( E \subset \mathcal{F} \) be two bounded measurable sets. We assume there exists a constant \( \mu \in [0, 1] \) such that

- \( E \subset \Omega_1 \) and \( |E| < (1 - \mu)|\Omega_1| \);
- moreover, there exist an integer \( m \) such that for any cylinder \( \Omega \subset \Omega_1 \) such that \( \overline{\Omega}^m \subset \Omega_1 \) and \( |\Omega \cap E| \geq (1 - \mu)|\Omega| \), we have that \( \overline{\Omega}^m \subset \mathcal{F} \).
Then for some universal constant $c_{is} \in (0, 1)$ only depending on $N$, there holds
\[ |E| \leq \frac{m+1}{m} (1 - c_{is} \mu) |F|. \]

**Remark A.2.** Theorem A.1 still holds true if we replace $\Omega_1$ with $\Omega_-$ defined in (5.2).

As it has already been pointed out by Imbert and Silvestre in [18], there is no chance to adapt the Calderón-Zygmund decomposition to this context, because it would require to split a larger piece into smaller ones of the same type and this is impossible due to the non Euclidean nature of our geometry. What we do is a generalization of the procedure proposed in [18], that is in fact an adaptation of the growing ink-spots theorem, whose original construction in the parabolic case dates back to Krylov and Safonov [23, Appendix A].

Moreover, when we need to confine both $E$ and $F$ to stay within a fixed cylinder, the following corollary directly follows.

**Corollary A.3.** Let $E \subset F$ be two bounded measurable sets. We assume
- $E \subset \Omega_1$;
- there exist two constants $\mu, r_0 \in ]0, 1[$ and an integer $m$ such that for any cylinder $\Omega \subset \Omega_1$ of the form $Q_r(z_0)$ such that $|\Omega \cap E| \geq (1 - \mu) |\Omega|$, we have $\overline{\Omega}^m \subset F$ and also $r < r_0$.

Then for some universal constants $c_{is}$ and $C_{is}$ only depending on $N$
\[ |E| \leq \frac{m+1}{m} (1 - c_{is} \mu) (|F \cap \Omega_1| + C_{is} m r_0^2). \]

### A.1. Stacked cylinders

First of all we recall some important properties of the following family of stacked cylinders
\[ k \Omega_r = \left(0, \ldots, 0, \frac{k^2 - 1}{2} r^2 \right) \circ \Omega_{kr} \quad \text{and} \quad k \Omega_r(x_0, t_0) = \left(0, \ldots, 0, \frac{k^2 - 1}{2} r^2 \right) \circ \Omega_{kr}(x_0, t_0), \]
where $(x_0, t_0) \in \mathbb{R}^{N+1}$, that are defined starting from the unit cylinder (1.9) for a certain $k > 0$. By definition, it is clear that $|k \Omega_r(x_0, t_0)| = k^{2+2} |\Omega_r(x_0, t_0)|$, and that the cylinders $\Omega_r(x_0, t_0)$ are not the balls of any metric. Thus, the important properties of the cylinders are explicitly given by the following lemmas.

**Lemma A.4.** Let $\Omega_{r_0}(x_0, t_0)$ and $\Omega_{r_1}(x_1, t_1)$ be two cylinders with non empty intersection, with $(x_0, t_0), (x_1, t_1) \in \mathbb{R}^{N+1}$ and $2r_0 \geq r_1 > 0$. Then
\[ \Omega_{r_1}(x_1, t_1) \subset k \Omega_{r_0}(x_0, t_0) \]
for some universal constant $k$.

**Proof.** Without loss of generality, we may assume $(x_0, t_0) = (0, 0)$. Then we need to choose the constant $k$ in order to satisfy our statement. In particular, if we consider the ball associated to the first $m_0$ variables we get that $B_{r_1}(x_1^{(0)}) \subset B_{kr_0}$ if
\[ kr_0 \geq r_0 + 2 r_1 \quad \Rightarrow \quad k \geq 5. \]
By repeating the same argument for all the $\kappa$ blocks of variables, we get that $k$ must satisfy the following conditions:
\[ k^{2j+1} \geq 1 + 2 \cdot 2^{2j+1} \quad \text{for} \quad j = 0, \ldots, \kappa. \]
As far as we are concerned with the condition regarding the time interval, we need $k$ to be such that
\[ -\frac{k^2 + 1}{2} r_0^2 \leq -r_0^2 - 2 r_1^2 \quad \Rightarrow \quad k^2 \geq 9. \]
All of these inequalities are satisfied when the first one, i.e. the one corresponding to \( j = 0 \), is satisfied. We choose \( k \) to be the smallest parameter satisfying these inequalities.

**Lemma A.5.** Let \( \{ Q_j \}_{j \in J} \) be an arbitrary collection of slanted cylinders with bounded radius. Then there exists a disjoint countable subcollection \( \{ Q_{j_i} \}_{i \in I} \) such that

\[
\bigcup_{j \in J} Q_j = \bigcup_{i=1}^{\infty} k\Omega_{j_i}.
\]

The proof of Lemma A.5 is the same as the classical proof of the Vitali covering lemma, where we employ Lemma A.4 instead of the fact that in any metric space \( B_{r_1}(x_1) \subset 5B_{r_0} \), if \( B_{r_1}(x_1) \cap B_{r_0} \neq \emptyset \) and \( r_1 \leq 2r_0 \).

### A.2. A generalized Lebesgue differentiation theorem.

For the readers convenience, we also recall the definition of maximal function:

\[
Mf(x,t) = \sup_{Q_j \ni (x,t)} \frac{1}{|Q|} \int_{\Omega \cap Q} |f(y,s)| \, dy \, ds,
\]

where the supremum is taken over cylinders of the form \((y,s) + RQ_1\).

**Lemma A.6.** For every \( \lambda > 0 \) and \( f \in L^1(\Omega) \), we have

\[
|\{ Mf < \lambda \} \cap \Omega| \leq \frac{C}{\lambda} \| f \|_{L^1(\Omega)}.
\]

**Proof.** Let us consider \((x,t) \in \{ Mf < \lambda \} \cap \Omega\). Then there exists a cylinder \( \Omega \) such that \((x,t) \in \Omega\) and

\[
\int_{\Omega \cap Q} |f(y,s)| \, dy \, ds \geq \frac{\lambda}{2} |\Omega \cap \Omega|.
\]

Then \( \{ Mf < \lambda \} \cap \Omega \) is covered with cylinders \( \{ Q_j \} \) such that the previous inequality holds. From Lemma A.5, there exists a disjoint countable subcollection \( \{ Q_{j_i} \} \) so that

\[
\{ Mf < \lambda \} \cap \Omega = \bigcup_{j=1}^{\infty} Q_j \subset \bigcup_{i=1}^{\infty} kQ_{j_i},
\]

for some integer \( k \). Thus, we get

\[
\| f \|_{L^1(\Omega)} \geq \int_{\Omega \cap \bigcup_i Q_{j_i}} |f| \geq \frac{\lambda}{2} \sum_{i=1}^{\infty} |Q_{j_i} \cap \Omega| = \frac{\lambda}{2k^{Q+2}} \sum_{i=1}^{\infty} kQ_{j_i} \cap \Omega \geq \frac{\lambda}{2k^{Q+2}} |\{ Mf < \lambda \} \cap \Omega|.
\]

Thus, the claim is proved when \( C = 2k^{Q+2} \). \( \square \)

The following generalized version of the Lebesgue differentiation theorem holds.

**Theorem A.7** (Generalized Lebesgue Differentiation Theorem). Let \( f \in L^1(\Omega, dx \otimes dt) \), where \( \Omega \) is an open subset of \( \mathbb{R}^{N+1} \). Then for a.e. \((x,t) \in \Omega\)

\[
\lim_{r \to 0+} \frac{1}{|Q_r(x,t)|} \int_{Q_r(x,t)} |f(y,s) - f(x,t)| \, dy \, ds = 0.
\]

Theorem A.7 is obtained from the following Lemma A.6 exactly as in [17, Theorem 2.5.1] by considering Lemma A.6.
A.3. Ink-spots theorem without time delay.

**Lemma A.8.** Let $E \subset F \subset Q_1$ be two bounded measurable sets. We make the following assumptions for some constant $\mu \in (0, 1)$:

- $E < (1 - \mu)|Q_1|$
- if for any cylinder $Q \subset Q_1$ such that $|Q \cap E| \geq (1 - \mu)|Q|$, then $Q \subset F$.

Then $|E| \leq (1 - c\mu)|F|$ for some universal constant $c$ only depending on $N$.

**Proof.** Thanks to Theorem A.7, for almost all points $z \in E$ there is some cylinder $Q^z$ containing $z$ such that $|Q^z \cap E| \geq (1 - \mu)|Q^z|$. Thus, for all Lebesgue points $z \in E$ we choose a maximal cylinder $Q^z \subset Q_1$ that contains $z$ and such that $|Q^z \cap E| \geq (1 - \mu)|Q^z|$. Here $Q^z = Q_0(z, \bar{r}, \bar{t})$ for some $\bar{r} > 0$ and $(\bar{r}, \bar{t}) \in Q_1$. In particular, we have that $Q^z$ differs from $Q_1$ and $Q^z \subset F$ by our assumption.

First of all we prove that $|Q^z \cap E| = (1 - \mu)|Q^z|$. By contradiction, let us suppose that is not true. Then there exists $\delta > 0$ small enough and $Q$ such that $Q^z \subset Q \subset (1 + \delta)Q^z$, $\overline{Q} \subset Q_1$ and $|\overline{Q} \cap E| > (1 - \mu)|Q^z|$, and this contradicts the maximality of the choice of $Q^z$.

Then we recall that the family of cylinders $\{Q^z\}_{z \in E}$ covers the set $E$. Thanks to Lemma A.5 and considering that $E$ is a bounded set, we can extract a finite subfamily of non overlapping cylinders $Q_j := Q^z_j$ such that $E \subset \bigcup_{j=1}^n kQ_j$. Since $Q_j \subset F$ and $|Q_j \cap E| = (1 - \mu)|Q_j|$, we have that $|Q_j \cap F \setminus E| = \mu|Q^z|$. Therefore,

$$|F \setminus E| \geq \sum_{j=1}^n |Q_j \cap F \setminus E| \geq \sum_{j=1}^n \mu|Q_j| = k^{-\varsigma(Q+2)}\mu \sum_{j=1}^n |kQ_j| \geq k^{-\varsigma(Q+2)}\mu|E|.$$  

Thus, we get that $|F| \geq (1 + \overline{t}\mu)|E|$, with $\overline{t} = k^{-\varsigma(Q+2)}$. Since $\overline{t}\mu \in (0, 1)$, we complete the proof by choosing $c = 1/2$. \qed

A.4. Proof of Theorem A.1 and Corollary A.3. In order to proceed with the proof of the Ink Spots Theorem, we first need to recall two preliminary results.

**Lemma A.9.** Consider a (possibly infinite) sequence of intervals $(a_j - h_k, a_j)$. Then

$$\left| \bigcup_k (a_k, a_k + mh_k) \right| \geq \frac{m}{m+1} \left| \bigcup_k (a_k - h_k, a_k) \right|.$$  

The proof of Lemma A.9 can be found in [19, Lemma 10.8]. Here, we only report the proof of the following lemma, that is an extension of Lemma 10.9 [19].

**Lemma A.10.** Let $\{Q_j\}$ be a collection of slanted cylinders and let $\overline{Q}^m_j$ be the corresponding versions as in (A.1). Then

$$\left| \bigcup_j \overline{Q}^m_j \right| \geq \frac{m}{m+1} \left| \bigcup_j Q_j \right|.$$  

**Proof.** Because of Fubini’s Theorem we know that for any set $\Omega \subset \mathbb{R}^{N+1}$

$$|\Omega| = \int \left| \{(x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \Omega\} \right| dx^{(0)}.$$  

Therefore, in order to prove our statement it is sufficient to show that for every \( x^{(0)} \in \mathbb{R}^{m_0} \)
\[
\left| \left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}^m_j \right\} \right|
\geq \frac{m}{m+1} \left| \left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}_j \right\} \right|
\]
From now on, let us consider a fixed \( \mathcal{Q} \in \mathbb{R}^{m_0} \). Any cylinder \( \mathcal{Q}_j \) is a cylinder with center
\((x^{(0)}_j, x^{(1)}_j, \ldots, x^{(\kappa)}_j, t_j) \in \mathbb{R}^{N+1}\)
and radius \( r_j > 0 \). \( \mathcal{Q}^m_j \) is its delayed version (A.1), that thanks to Remark 2.4 can equivalently be represented as follows
\[
\mathcal{Q}^m_j = (t_0, t_0 + mr_j^2) \times B_r(x^{(0)}_j) \times B_{m+2}(x^{(1)}_j) \times \cdots \times B_{m^2+2}(x^{(\kappa)}_j).
\]
On one hand, when \( |\mathcal{Q} - x^{(0)}_j| \geq r_j \) the set
\[
\left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \mathcal{Q}^m_j \right\}
\]
is empty.
On the other hand, when \( |\mathcal{Q} - x^{(0)}_j| < r_j \) we have that
\[
\left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (\mathcal{Q}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \mathcal{Q}^m_j \right\}
\supset \tilde{Q}_j := (t_j, t_j + mr_j^2) \times B_{2r_j}(x^{(1)}_j) \times \cdots \times B_{2m^2+2}(x^{(\kappa)}_j).
\]
Based on these last observations, we have that
\[
\left| \left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}^m_j \right\} \right| \geq \bigcup_{j : |\mathcal{Q} - x^{(0)}_j| < r_j} \tilde{Q}_j.
\]
Now, thanks to Fubini’s Theorem and Lemma A.9 we have
\[
\left| \left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}^m_j \right\} \right|
\geq \frac{m}{m+1} \bigcup_{j : |\mathcal{Q} - x^{(0)}_j| < r_j} (t_j - r_j^2, 0] \times B_{2r_j}(x^{(1)}_j) \times \cdots \times B_{2m^2+2}(x^{(\kappa)}_j)
\geq \frac{m}{m+1} \bigcup_{j : |\mathcal{Q} - x^{(0)}_j| < r_j} (t_j - r_j^2, 0] \times B_{r_j}(x^{(1)}_j) \times \cdots \times B_{m^2+1}(x^{(\kappa)}_j)
= \frac{m}{m+1} \left| \left\{ (x^{(1)}, \ldots, x^{(\kappa)}, t) : (\mathcal{Q}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}_j \right\} \right|.
\]
Combining all of the above results, the proof is complete.

\textbf{Proof of Theorem A.1.} Let \( Q \) be the collection of all cylinders \( Q \subset \mathcal{Q}_1 \) such that \( |Q \cap E| \geq (1 - \mu)|Q| \). Let \( G := \bigcup_{Q \in Q} Q \). By construction, the sets \( E \) and \( G \) satisfy the assumptions of Lemma A.8. Therefore \( (1 - c_a \mu)|G| \geq |E| \). Combining the assumptions of the theorem with Lemma A.10 we conclude the proof.

\textbf{Proof of Corollary A.3.} The condition \( |E| \leq (1 - \delta)|\Omega_1| \) is implied by the second assumption when \( r_0 < 1 \). Moreover, the result is trivial when \( r_0 \geq 1 \) choosing \( C \) sufficiently large. Let \( Q \) be
the collection of all cylinders $Q \subseteq Q_1$ such that $|Q \cap E| \geq (1 - \mu)|Q|$. Let $G := \cup_{Q \in Q} Q^m$.
From Theorem A.1 we have that $|E| \leq \frac{m}{m+1}(1 - c\mu)|G|$. Moreover, our assumptions tell us $G \subseteq F$. In order to conclude the proof is sufficient to estimate the measure $G \setminus Q_1$ by considering that each cylinder $Q = Q_r(x, t) \subset Q_1$ has radius bounded by $r_0$ (see [19, Corollary 10.2]). □

Appendix B. Stacked cylinders

For the sake of completeness, we here state the stacking cylinders lemma for our operator $\mathcal{L}$.

Such a result is used when applying the Ink-Spots Theorem in the proof of the weak Harnack inequality, Theorem 1.3.

Lemma B.1. Let $\omega < \frac{1}{\sqrt{2r+1}}$ and $\rho = ((3\kappa + 1)\omega)^{\frac{1}{2r+1}}$. We consider any non-empty cylinder $Q_r(z_0) \subset Q_-$ and we set $T_k = \sum_{j=1}^k (2^j r)^2$. Let $N \geq 1$ such that $T_N \leq t_0 < T_{N+1}$ and let

$$Q_r[k] := Q_{2^k r}(z_k), \quad k = 1, \ldots, N$$

$$Q_{R_{N+1}}[N + 1] := Q_{R_{N+1}}(z_{N+1}),$$

where $z_k = z_0 \circ (0, \ldots, 0, T_k)$ and $R = \left| t_0 + T_N \right|^\frac{1}{2r}$, $R_{N+1} = \max(R, \rho)$, and

$$z_{N+1} = \begin{cases} z_N \circ (0, \ldots, 0, R^2), & \text{if } R \geq \rho \\ (0, 0), & \text{if } R < \rho \end{cases}$$

These cylinders satisfy

$$Q_+ \subset Q_{R_{N+1}}[N + 1], \quad \cup_{k=1}^{N+1} Q_r[k] \subset (-1, 0) \times B_2, \quad \bar{Q}[N] \subset Q_r[N],$$

where $\bar{Q}[N] = Q_{R_{N+1}/2}(z_{N+1} \circ (0, \ldots, 0, -R_{N+1}^2))$.

Proof. As our derivation of the previous lemma follows very closely the one contained in [13, Appendix C], we here do not write explicitly the proof. Indeed, the proof of the result is merely geometric and the main difference with [13] lies in the fact that in our case we exploit the more general composition law and dilations defined in (2.1) and (2.9), respectively. This explains why here the constants $\omega$ and $\rho$ differ from the ones in [13]. □

References

[1] F. Anceschi, M. Eleuteri and S. Polidoro A geometric statement of the Harnack inequality for a degenerate Kolmogorov equation with rough coefficients, Comm. Cont. Math. (21): 1–17, 2018.
[2] F. Anceschi and S. Polidoro. A survey on the classical theory for Kolmogorov equation. Le Matematiche, LXXV(Issue I):221–258, 2020.
[3] F. Anceschi, S. Polidoro, and M. A. Ragusa. Moser’s estimates for degenerate Kolmogorov equations with non-negative divergence lower order coefficients. Nonlinear Analysis :1-19, 2019.
[4] S. Armstrong and J.C. Mourrat. Variational methods for the kinetic Fokker-Planck equation. arXiv:1902.04037, preprint, 2019.
[5] J.M. Bony. Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. Ann. Inst. Fourier (Grenoble), 19(fasc. 1):277–304 xii, 1969.
[6] C. Cinti, A. Pascucci, and S. Polidoro. Pointwise estimates for a class of non-homogeneous Kolmogorov equations. Math. Ann., 340(2):237–264, 2008.
[7] E. Di Benedetto and N.S. Trudinger, Harnack inequalities for quasi-minima of variational integrals. Annales de l’I.H.P. Analyse non linéaire, 1(4):295–308, 1984.
[8] B. Fisher and K. Taş The convolution of functions and distributions J. Math. Anal. Appl. 306, 364–374, 2005.
[9] G.B. Folland. Sub-elliptic estimates and function spaces on nilpotent Lie groups. Ark. Mat., 13:161–207, 1975.
A NOTE ON THE WEAK REGULARITY THEORY FOR DEGENERATE KOLMOGOROV EQUATIONS

10. D. Gilbarg and N.S. Trudinger. Elliptic partial differential equations of second order. Springer-Verlag, Berlin-New York, 1977. Grundlehren der Mathematischen Wissenschaften, Vol. 224.

11. F. Golse, C. Imbert and C. Mouhot, and Alexis F. Vasseur. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 19(1):253–295, 2019.

12. J. Guerand. Quantitative regularity for parabolic De Giorgi classes, 2020. arXiv:2103.09646.

13. J. Guerand and C. Imbert. Log-transform and the weak Harnack inequality for kinetic Fokker-Planck equations. arXiv: 2102.04105, preprint, 2021.

14. J. Guerand and C. Mouhot. Quantitative de giorgi methods in kinetic theory. arXiv:2103.09646, preprint, 2021.

15. L. Hörmander. Hypoelliptic second order differential equations. Acta Math., 119:147–171, 1967.

16. M. Ignatova On the continuity of solutions to advection-diffusion equations with slightly super-critical divergence-free drifts. Adv. Nonlinear Anal. 3 (2):81–86 (2014).

17. C. Imbert and L Silvestre. An Introduction to Fully Nonlinear Parabolic Equations. Lecture Notes in Mathematics, Springer, Cham., 2086 in Boucksom S., Eyssidieux P., Guedj V. (eds) An Introduction to the Kähler-Ricci Flow, 2013.

18. C. Imbert and L Silvestre. Global regularity estimates for the boltzmann equation without cut-off. Journal of the American Mathematical Society. Accepted for publication., arXiv:1909.12729v1, 2019.

19. C. Imbert and L Silvestre. The weak Harnack inequality for the Boltzmann equation without cut-off. Journal of the European Mathematical Society, 22(2):507–592, 2020.

20. D. Kinderlehrer and G. Stampacchia. An introduction to variational inequalities and their applications. Academic Press, 1980.

21. S.N. Krúzkov A priori bounds for generalized solutions of second-order elliptic and parabolic equations. Dokl. Akad. Nauk SSSR, 150: 748–751, 1963.

22. S.N. Krúzkov A priori bounds for generalized solutions of second-order elliptic and parabolic equations. Mat. Sb. (N.S.), 65 (107): 522–570, 1964.

23. N.V. Krylov and M.V. Safonov. A certain property of solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR, Ser. Mat., 44:161–175, 1980.

24. E. Lanconelli and S. Polidoro On a class of hypoelliptic evolution operators. Rend. Sem. Mat. Univ. Politec. Torino, 52:29–63, 1994.

25. G.M. Lieberman Second Order Parabolic Differential Equations World Scientific, 1996.

26. M. Litsgard and K. Nyström The Dirichlet problem for Kolmogorov-Fokker-Planck type equations with rough coefficients preprint ArXiV: 2012.11410 (2021).

27. H.K. Moffatt, Magnetostrophic turbulence and the geodynamo. IUTAM Symposium on Computational Physics and New Perspectives in Turbulence, IUTAM Bookser., vol. 4, Springer, Dordrecht, 2008, pp. 339–346.

28. J. Moser. A new technique for the construction of solutions of nonlinear differential equations. Proc. Natl. Acad. Sci. USA, 47(11):1824–1831, 1961.

29. J. Moser. A Harnack inequality for parabolic differential equations. Comm. Pure Appl. Math., 17:101–134, 1964.

30. J. Moser. A rapidly convergent iteration method and non-linear partial differential equations - I. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Ser. 3, 20(2):265–315, 1966.

31. G. Koch, N. Nadirashvili, G.A. Seregin and V. Šverák, Liouville theorems for the Navier–Stokes equations and applications. Acta Mathematica, 203:83 – 105, 2009.

32. Nazarov A.I. , Ural’tseva N.N. The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients Algebra i Analiz, 23 (2011): 136-168

33. A. Pascucci and S. Polidoro. The Moser’s iterative method for a class of ultraparabolic equations. Commun. Contemp. Math., 6(3):395–417, 2004.

34. G. Seregin, L. Silvestre, V. Šverák and A. Zlatos, On divergence-free drifts. Journal of Differential Equations, Volume 252, Issue 1 : 505-540, (2012) https://doi.org/10.1016/j.jde.2011.08.039.

35. N.S. Trudinger. Pointwise estimates and quasilinear parabolic equations. Communications on Pure and Applied Mathematics, 21:205–226, 1968.

36. H.D. Victory, J. On the Existence of Global Weak Solutions for Vlasov-Poisson-Fokker-Planck Systems J. Math. An. App. 160: 525 – 555 (1991)
[37] G. Wang. Harnack inequalities for functions in de giorgi parabolic class. In Shiing-shen Chern, editor, Partial Differential Equations, pages 182–201, Berlin, Heidelberg, 1988. Springer Berlin Heidelberg.

[38] W. Wang and L. Zhang. The $C^\alpha$ regularity of a class of non-homogeneous ultraparabolic equations. Sci. China Ser. A, 52(8):1589–1606, 2009.

[39] W. Wang and L. Zhang. The $C^\alpha$ regularity of weak solutions of ultraparabolic equations. Discrete Contin. Dyn. Syst., 29(3):1261–1275, 2011.

[40] W. Wang and L. Zhang. $C^\alpha$ regularity of weak solutions of non-homogenous ultraparabolic equations with drift terms. arXiv:1704.05323, preprint, 2017.

Dipartimento di Matematica e Applicazioni "Renato Caccioppoli" - Università degli Studi di Napoli "Federico II": Via Cintia, Monte S. Angelo I-80126 Napoli, Italy
Email address: francesca.anceschi@unina.it

Dipartimento di Scienze Matematiche, Fisiche e Informatiche - Università degli Studi di Parma: Parco Area delle Scienze, 7/A 43124 Parma, Italy
Email address: annalaura.rebucci@unipr.it