MULTIPLICITY OF SOLUTIONS FOR A NONHOMOGENEOUS QUASILINEAR ELLIPTIC PROBLEM WITH CRITICAL GROWTH

MARCOS L. M. CARVALHO AND JOSÉ VALDO A. GONÇALVES
IME - Instituto de Matemática e Estatística
Universidade Federal de Goiás, 74001-970-Goiânia-GO, Brazil

CLAUDINEY GOULART
R. Riachuelo, 1530 - Setor Samuel Graham
Universidade Federal de Goiás, 75804-020- Jataí-GO, Brazil

OLÍMPIO H. MIYAGAKI *
Departamento de Matemática, Universidade Federal de Juiz de Fora
36036-330-Juiz de Fora-MG, Brazil

(Communicated by Zhi-Qiang Wang)

Abstract. It is established some existence and multiplicity of solution results for a quasilinear elliptic problem driven by the Φ-Laplacian operator. One of the solutions is built as a ground state solution. In order to prove our main results we apply the Nehari method combined with the concentration compactness theorem in an Orlicz-Sobolev space framework. One of the difficulties in dealing with this kind of operator is the lost of homogeneity properties.

1. Introduction. In this work we will establish some existence and multiplicity results for the following quasilinear elliptic problem

\[ \begin{align*}
\Delta \Phi u &= |u|^{\ell^*-2}u + f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*} \tag{1.1} \]

where \( \Delta \Phi \) denotes the \( \Phi \)-laplacian operator, which is defined by \( \Delta \Phi u = \text{div}(\phi(|\nabla u|)\nabla u) \), \( \ell^* = \ell N/(N - \ell) \) (\( 1 < \ell < N \)), \( \Omega \subset \mathbb{R}^N \) is bounded and smooth domain, \( f \geq 0 \), and in order to simplify the technicalities we assume \( f \in L_{\text{loc}}^{\frac{\ell N}{N - \ell}}(\Omega) = L^{\frac{\ell^*}{\ell^* - 1}}(\Omega) \equiv L^{\ell^*'}(\Omega) \). With respect to the function \( \phi : (0, \infty) \to (0, \infty) \), we assume that it is \( C^2 \) and satisfies the following conditions

\begin{align*}
(\phi_1): & \lim_{t \to 0} t \phi(t) = 0, \quad \lim_{t \to \infty} t \phi(t) = \infty; \\
(\phi_2): & t \mapsto t \phi(t) \text{ is strictly increasing;}
\end{align*}

2000 Mathematics Subject Classification. Primary: 58E05, 35J20, 35J25, 35J60; Secondary: 35J92.

Key words and phrases. Variational methods, quasilinear elliptic problems, Nehari method, sign-changing solutions.

O. H. Miyagaki is corresponding author and he received research grants from CNPq/Brazil and INCTMAT/CNPQ/Brazil.

* Corresponding author.
(φ₃): \[ -1 < \ell - 2 := \inf_{t>0} \frac{(t\phi(t))''t}{(t\phi(t))'} \leq \sup_{t>0} \frac{(t\phi(t))''t}{(t\phi(t))'} =: m - 2 < N - 2. \]

Furthermore, we shall assume the following hypothesis

\[ (H) \quad 1 < \ell (\ell - m) \leq \ell \leq m < \ell^*. \]

**Remark 1.1.** Notice that the above inequalities still hold when:

1. \( \Phi(t) = pt^{p-2} \) with \( 1 < p < \infty \) and \( \ell = m = p \), in this case \( \Delta_\Phi = \Delta_p \), where \( \Delta_p \) denotes the \( p \)-Laplacian operator.
2. \( \Phi(t) = pt^{p-2} + qt^{q-2} \) with \( 1 < p < q < \infty \), \( \ell = p \) and \( m = q \), in this case \( \Delta_\Phi \) turns the \( \Delta_p + \Delta_q \) operator. Here \( \Delta_p + \Delta_q \) denotes the \( (p,q) \)-laplacian operator. See [21] and [24] for this kind of operators.
3. Other examples, for instance involving anisotropic elliptic problems, can be seen in [7] and references therein.

The main difficulty in dealing with this kind of operator is because it is inhomogeneous, which requires some additional effort to overcome the estimates. As is mentioned in [23] the problem has many physical applications, for instance, in nonlinear elasticity, plasticity, generalized Newtonian fluids, etc. We refer the reader to the following related papers [14, 15, 16, 20, 23] and references therein, where there have been handled different types of nonlinearities involving this kind of operator. Problems like above were started in a beautiful work due to Brézis and Nirenberg [2], when \( \Delta_\Phi = \Delta \), where they treated a nonhomogeneous problem with critical growth obtaining existence result, assuming that \( f \geq 0, f \neq 0 \), together with some additional conditions. Then Tarantello [25] treated the same problem getting existence and multiplicity results under a stronger hypothesis that made in [2]. These works were extended in [19], which was obtained four weak solutions, at least one of them is sign changing solution. On the other hand, in [11] is proved some multiplicity results for symmetric domain by using the category theory. There are only few works involving \( p \)-Laplacian, that is, when \( \Delta_\Phi = \Delta_p \), extending results in [25]. We would like to mention [6, 10] and references therein.

Due to the nature of the operator \( \Delta_\Phi \) we shall work in the framework of Orlicz-Sobolev spaces \( W^{1,\Phi}_0(\Omega) \). Throughout this paper we define

\[ \Phi(t) = \int_0^t s\phi(s)ds, t \geq 0, \]

which is extended as even function, \( \Phi(t) = \Phi(-t) \), for all \( t < 0 \).

Recall that hypotheses (φ₁) - (φ₂) allow us to use the Orlicz and Orlicz-Sobolev spaces, while the hypothesis (φ₃) ensures that the Orlicz-Sobolev spaces are Banach reflexive spaces. There are several publications on Orlicz-Sobolev spaces, we would like to recommend the reader to [1, 12, 15, 18, 22, 23]. However, for the sake of completeness, we recall some definitions and properties in the Appendix.

From the continuous embedding \( W^{1,\Phi}_0(\Omega) \hookrightarrow L^{\ell^*}(\Omega) \), (see [1, 12]), we define

\[ S = \inf \left\{ \frac{|u|_{L^{\ell^*}}}{|u|_{L^{\ell^*}_r}}, \quad u \in W^{1,\Phi}_0(\Omega) \setminus \{0\} \right\}. \tag{1.2} \]

Since our approach is variational method, the functional \( J : W^{1,\Phi}_0(\Omega) \to \mathbb{R} \) associated with our problem is given by

\[ J(u) = \int_{\Omega} \Phi(|\nabla u|) - \frac{1}{\ell^*} \int_{\Omega} |u|^{\ell^*} - \int_{\Omega} fu, \quad u \in W^{1,\Phi}_0(\Omega), \]
is well-defined and of class $C^1$. The Euler-Lagrange equations for $J$ are precisely the weak solutions for problem (1.1). Hence finding weak solutions for the problem (1.1) is equivalent to find critical points for the functional $J$. Here we emphasize that $J$ is in $C^1$ class due to the hypotheses built on the function $\phi$. This is the main reason in order to consider the hypothesis $(\phi_3)$ that is crucial in our arguments. The Gateaux derivative for $J$ possesses the following form

$$\langle J'(u), v \rangle = \int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v - \int_{\Omega} |u|^{r^*-2}uv - \int_{\Omega} fv$$

for any $u, v \in W^{1,\phi}_0(\Omega)$. In general, using hypotheses $(\phi_1) - (\phi_3)$, the functional $J$ is not in $C^2$ class.

In order to perform our precise hypotheses for our results, we will consider the functions $g_{\alpha} : [0, \infty) \to \mathbb{R}$, $\alpha \in \{\ell, m\}$ defined by

$$g_{\alpha}(t) := g_{\alpha, u}(t) = t^{\alpha-1} \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 - t^{r^*-1}|u|^{r^*}, \quad t > 0. \quad (1.3)$$

It is easy to see that there exists $\tilde{t}_{\alpha} > 0$ such that

$$g_{\alpha}(\tilde{t}_{\alpha}) = \max_{t > 0} g_{\alpha}(t).$$

Inspired by [26], given $u \in W^{1,\phi}_0(\Omega)$, with $||u||_{r^*} = 1$, we assume the following assumptions on $f$.\n
$(f_1)$: Suppose either $\tilde{t}_{\ell}, \tilde{t}_m \geq 1$ or $\tilde{t}_{\ell}, \tilde{t}_m \leq 1$. Then

$$||f||_{(r^*)'} \leq \lambda_1 := \min \left\{ \frac{\alpha^{r^*-2}}{S^{(r^*-2)/2}} \left[ \frac{\ell(\ell - 1)}{r^* - 1} \right]^{\frac{r^*-2}{2}}, \left[ \frac{\ell(\ell - m)}{r^* - 1} \right]^{\frac{r^*-2}{2}} : \alpha = \ell, m \right\};$$

$(f_2)$: If $\ell < m$ and $\tilde{t}_{\ell} \leq 1 \leq \tilde{t}_m$ hold, we suppose

$$||f||_{(r^*)'} \leq \min \left\{ \lambda_1, \frac{\ell^* - m}{m - 1} \right\}. \quad \text{(See Lemma 2.7)}$$

We have a second solution to the problem (1.1) considering a more restrictive condition given by:

$(f_2)'$: If $\ell < m$ and $\tilde{t}_{\ell} \leq 1 \leq \tilde{t}_m$ hold, we assume

$$||f||_{(r^*)'} \leq \min \left\{ \frac{1}{m} \lambda_1, \frac{\ell^* - m}{m - 1} \right\}. \quad \text{(See Lemma 2.7)}$$

Our first main result can be read as follows

**Theorem 1.1.** In addition to $(\phi_1) - (\phi_3)$ and $(H)$, suppose $f \geq 0$, and $f \in L^{r'}(\Omega)$. Assume either $(f_1)$ or $(f_2)$ holds. Then there exists $\Lambda_1 > 0$ such that problem (1.1) admits at least one nonnegative ground state solution $u$ satisfying $J(u) \leq 0$ for any $f$ such that $0 < ||f||_{(r^*)'} < \Lambda_1$.

Now we shall consider the following result

**Theorem 1.2.** Suppose $(\phi_1) - (\phi_3)$ and $(H)$. Assume $f \geq 0$, and $f \in L^{r'}(\Omega)$, and either $(f_1)$ or $(f_2)'$ holds. Then there exists $\Lambda_2 > 0$ in such way that problem (1.1) admits at least one nonnegative solution $v$ satisfying $J(v) > 0$ for any $f$ verifying $0 < ||f||_{(r^*)'} < \Lambda_2$. 
Putting together the all results established just above and using a regularity result for quasilinear elliptic problems we can state the following multiplicity result.

**Theorem 1.3.** In addition to \((\phi_1) - (\phi_3)\) and \((H)\), suppose \(f \geq 0\), and \(f \in L^{\varepsilon'}(\Omega)\).

Assume either \((f_1)\) or \((f_2)\) holds. Then problem \((1.1)\) admits at least two nonnegative solutions \(u, v\) whenever \(0 < \|f\|_{(\varepsilon')} < \Lambda = \min\{\Lambda_1, \Lambda_2\}\). Furthermore, the function \(u\) is a ground state solution for each \(f\) satisfying \(0 < \|f\|_{(\varepsilon')} < \Lambda\).

**Remark 1.2.** We point out that concerning just existence of solution, \(f\) can change sign, see Lemma 2.6. However in such case the solution could change sign, as well.

2. **Preliminary results.** In this section we give some basic results involving the Nehari manifold method, including the fibering maps associated with the functional \(J\), which will give information on the critical points of Euler-Lagrange functional \(J\). We suggest the reader to the book due to Willem [27], for an overview on the Nehari method. The proofs of our results follow closely the arguments used in [8, 9].

The Nehari manifold associated with the functional \(J\) is given by

\[
\mathcal{N} = \{ u \in W_0^{1, \Phi}(\Omega) \setminus \{0\} : \langle J'(u), u \rangle = 0 \} \tag{2.4}
\]

It will be proved later on that \(\mathcal{N}\) is a \(C^1\)-submanifold of \(W_0^{1, \Phi}(\Omega)\).

Initially, note that if \(u \in \mathcal{N}\), by (2.4), we have that

\[
J(u) = \int_\Omega \Phi(|\nabla u|) - \phi(|\nabla u|)|\nabla u|^2 + \left(1 - \frac{1}{\ell}\right) |u|^{\varepsilon},
\tag{2.5}
\]

or equivalently

\[
J(u) = \int_\Omega \Phi(|\nabla u|) - \frac{1}{\ell} \phi(|\nabla u|)|\nabla u|^2 - \left(1 - \frac{1}{\ell}\right) fu. \tag{2.6}
\]

First of all we shall prove some geometric properties of functional \(J\), which allows us to find a critical point for \(J\).

**Proposition 2.1.** The functional \(J\) is coercive and bounded from below on \(\mathcal{N}\).

**Proof.** In virtue of \((\phi_3)\), we have \(m\Phi(t) \geq t^2\phi(t)\) for each \(t \geq 0\). Using this fact and (2.6), we obtain

\[
J(u) \geq \left(\frac{1}{m} - \frac{1}{\ell}\right) \int_\Omega \phi(|\nabla u|)|\nabla u|^2 + \left(\frac{1}{\ell} - 1\right) \int_\Omega fu. \tag{2.7}
\]

Now by combining

\[
\min\{|u|^{\ell}, |u|^m\} \leq \int_\Omega \Phi(|\nabla u|) \leq \frac{1}{\ell} \int_\Omega \phi(|\nabla u|)|\nabla u|^2
\]

with the Hölder inequality and the continuous embedding \(W_0^{1, \Phi}(\Omega) \hookrightarrow L_{\Phi^*}(\Omega) \hookrightarrow L^{\varepsilon'}(\Omega)\), we obtain

\[
J(u) \geq \ell \left(\frac{1}{m} - \frac{1}{\ell}\right) \min\{|u|^{\ell}, |u|^m\} + S^{-\frac{m}{2}} \left(\frac{1}{\ell} - 1\right) \|f\|_{(\varepsilon')} \|u\|, \tag{2.8}
\]

where \(S\) is given by (1.2). Thus, \(J\) is coercive and bounded from below on \(\mathcal{N}\). The proposition is proved. \(\square\)
Now, define the fibering map \( \gamma_u : (0, +\infty) \to \mathbb{R} \) given by
\[
\gamma_u(t) := J(tu) = \int_{\Omega} \Phi(t|\nabla u|) - \frac{t^{\ell^*}}{\ell^*} |u|^{\ell^*} - t f u.
\]
From \((\phi_1) - (\phi_2)\) it follows that \( \gamma_u \) is of \( C^1 \), and its Gateaux derivative is given by
\[
\gamma_u'(t) = t \int_{\Omega} \phi(t|\nabla u|)|\nabla u|^2 - t^{\ell^* - 1}|u|^{\ell^*} - f u. \tag{2.9}
\]
The main feature of the fibering map is the knowledge of the geometry of \( \gamma_u \), which will give information about the existence and multiplicity of solutions. This method was introduced in [13], then it was also employed, for instance, in [3, 4, 5, 25, 26, 28, 29] and references therein.

**Remark 2.1.** Notice that \( tu \in \mathcal{N} \) if, and only if, \( \gamma_u'(t) = 0 \). Therefore, \( u \in \mathcal{N} \) if, and only if, \( \gamma_u'(1) = 0 \). Thus, the stationary points of fibering map are the critical points of \( J \) on \( \mathcal{N} \).

Define \( \psi(u) = \langle J'(u), u \rangle \), \( u \in W_{0}^{1, \Phi}(\Omega) \). Then, for all \( u \in W_{0}^{1, \Phi}(\Omega) \), we have
\[
\langle \psi'(u), u \rangle = \int_{\Omega} \phi'(|\nabla u|)|\nabla u|^3 + 2\phi(|\nabla u|)|\nabla u|^2 - \ell^* |u|^{\ell^*} - f u. \tag{2.10}
\]
As was made in Tarantello in [25, 26], let us split \( \mathcal{N} \) into three sets, namely,
\[
\begin{align*}
\mathcal{N}^+ & := \{ u \in \mathcal{N} : \langle \psi'(u), u \rangle > 0 \}; \\
\mathcal{N}^- & := \{ u \in \mathcal{N} : \langle \psi'(u), u \rangle < 0 \}; \\
\mathcal{N}^0 & := \{ u \in \mathcal{N} : \langle \psi'(u), u \rangle = 0 \},
\end{align*}
\]
which correspond to the critical points of minimum, maximum and inflexion points, respectively.

**Remark 2.2.** For \( u \in \mathcal{N} \), by (2.5) and (2.6), we have
\[
\langle \psi'(u), u \rangle = \int_{\Omega} \phi'(|\nabla u|)|\nabla u|^3 + \phi(|\nabla u|)|\nabla u|^2 - (\ell^* - 1)|u|^{\ell^*} \\
= \int_{\Omega} \phi'(|\nabla u|)|\nabla u|^3 + (2 - \ell^*)\phi(|\nabla u|)|\nabla u|^2 - (1 - \ell^*) f u. \tag{2.11}
\]
The next result is the crucial step in our argument to prove the main result.

**Lemma 2.1.** Suppose either \((f_1)\) or \((f_2)\), and \((\phi_1)\)-(\phi_3) hold. Then,
1. \( \mathcal{N}^0 = \emptyset \).
2. \( \mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \) is a \( C^1 \)-manifold.

**Proof of item (1).** Assume by contradiction that \( \mathcal{N}^0 \neq \emptyset \). Fix \( u \in \mathcal{N}^0 \). Then, \( \gamma_u'(1) = \langle \psi'(u), u \rangle = 0 \). From (2.4) and (2.11), we obtain,
\[
0 = \langle \psi'(u), u \rangle = \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 + \phi'(|\nabla u|)|\nabla u|^3 + (1 - \ell^*) |u|^{\ell^*}.
\]
By hypothesis \((\phi_3)\) we infer that
\[
(\ell - 1) \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \leq (\ell^* - 1)|u|^{\ell^*} \leq (\ell^* - 1) S^{-\frac{\ell^*}{\ell}} |u|^{\ell^*},
\]
where \( S \) is the best constant of the embedding \( W_{0}^{1, \Phi}(\Omega) \hookrightarrow L^{\ell^*}(\Omega) \). On the other hand,
\[
(\ell - 1) \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \geq \ell(\ell - 1) \int_{\Omega} \Phi(|\nabla u|) \geq \ell(\ell - 1) \min\{|u|^{\ell}, ||u||^{m}\}.
\]
Define the function

\[ ||u|| \geq \left( \frac{\ell(\ell - 1)S^{\frac{\ell}{\ell^*} + 1}}{\ell^* - 1} \right)^{\frac{1}{\ell^* - \alpha}}. \]  

(2.12)

Now, using (2.11), we get

\[ 0 = \langle \psi'(u), u \rangle = \int_{\Omega} (2 - \ell^*)\phi(|\nabla u|)|\nabla u|^2 + \phi'(|\nabla u|)|\nabla u|^3 + (\ell^* - 1)f u. \]

From (\phi_3), we obtain

\[ (\ell^* - m) \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \leq (\ell^* - 1) \int_{\Omega} f u. \]

Arguing as above, we get

\[ \ell(\ell^* - m) \min\{||u||^\ell, ||u||^m\} \leq (\ell^* - 1) \int_{\Omega} f u. \]

Therefore, from the H"{o}lder's inequality, we get

\[ \min\{||u||^\ell, ||u||^m\} \leq \frac{(\ell^* - 1)}{\ell(\ell^* - m)} \int_{\Omega} f u \leq \frac{S^{\frac{1}{2}}(\ell^* - 1)}{\ell(\ell^* - m)} ||f||_{(\ell^*)'} ||u||. \]

(2.13)

Comparing (2.12) and (2.13), we get

\[ ||f||_{(\ell^*)'} \geq S^{\frac{\ell^*}{\ell\(\ell^* - 1\)}} \left[ \frac{\ell(\ell - 1)}{\ell^* - 1} \right]^{\frac{\ell^* - 1}{\ell^* - \alpha}} \left[ \frac{\ell(\ell^* - m)}{\ell^* - 1} \right] \geq \lambda_1, \quad \alpha = \ell, m, \]

(2.14)

which is a contradiction if we assume either (f_1) or (f_2).

**Proof of item (2).** Suppose without loss of generality that, u \in N^+.

Define G(u) := \langle J'(u), u \rangle. We can see that

\[ G'(u) = \langle J''(u) \cdot (u, u) \rangle + \langle J'(u), u \rangle = \langle \psi'(u), u \rangle > 0, \quad \forall u \in N^+. \]

Furthermore, using (2.4), we also have that \langle J'(u), u \rangle = 0. Hence, 0 \in \mathbb{R} is a regular value for G and \(N^+ = G^{-1}(0).\) That is, \(N^+\) is a \(C^1\)-manifold. Similarly, we may show that \(N^-\) is a \(C^1\)-manifold. Hence, since we are supposing (f_1) and (f_2), the proof of item (2) follows in virtue of \(N^0 = \emptyset.\)

Next we are going to prove that any critical point for \(J\) on \(N^\lambda\) is a free critical point, i.e., is a critical point in the whole space \(W^{1, p}_0(\Omega).\) Actually, the proof of the Lemma below is fairly standard and we include it for the sake of completeness.

**Lemma 2.2.** Let \(u_0\) be a local minimum (or local maximum) of \(J.\) If \(u_0 \notin N^0,\) then \(u_0\) is a critical point of \(J.\)

**Proof.** Suppose without any loss of generality that \(u_0\) is a local minimum of \(J.\) Define the function

\[ \theta(u) = \langle J'(u), u \rangle = \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 - |u|^{\ell^*} - f u. \]

Then \(u_0\) is a solution for the minimization problem

\[ \min \{J(u), \quad \theta(u) = 0\}. \]

(2.15)

Proceeding as in Carvalho et al. [9], we have

\[ \langle \theta'(u), v \rangle = \int_{\Omega} \phi'(|\nabla u|)|\nabla u|\nabla u \cdot \nabla v + 2\phi(|\nabla u|)|\nabla u|\nabla v - f v - \ell^*|u|^{\ell^*-2}uv \]
Therefore, since \( m < \ell \) above we obtain
\[
\langle \theta'(u_0), u_0 \rangle = \int_{\Omega} \phi'(|\nabla u_0|)|\nabla u_0|^3 + \phi(|\nabla u_0|)|\nabla u_0|^2 - (\ell^\ast - 1) \int_{\Omega} |u_0|^\ell^\ast = \langle \psi'(u_0), u_0 \rangle.
\]

From Lemma 2.1, the problem (2.15) has a solution verifying
\[
\langle J'(u_0), u_0 \rangle = \mu \langle \theta'(u_0), u_0 \rangle = 0,
\]
where \( \mu \in \mathbb{R} \) which is given by Lagrange multipliers Theorem. Notice that \( \langle \theta'(u_0), u_0 \rangle \neq 0 \), then \( \mu = 0 \), i.e, \( u_0 \) is a critical point for \( J \) on \( W^{1,\Phi}_0(\Omega) \). The proof of lemma is complete.

Now we give a complete description on the geometry for the fibering map associated with problem (1.1), where we will focus on the sign of \( \int f u \).

Consider the auxiliary function
\[
m_u(t) = \int_{\Omega} t \phi(t|\nabla u|)|\nabla u|^2 - t^{\ell^\ast - 1} |u|^\ell^\ast,
\]
where the points \( tu \in \mathcal{N} \) will be compared with the function \( m_u \).

**Lemma 2.3.** Let \( t > 0 \) be fixed. Then \( tu \in \mathcal{N} \) if, and only if, \( t \) is a solution of \( m_u(t) = \int f u \).

**Proof.** Fix \( t > 0 \) in such may that \( tu \in \mathcal{N} \). Then
\[
t \int_{\Omega} \phi(|\nabla (tu)|)|\nabla u|^2 - t^{\ell^\ast - 1} \int_{\Omega} |b(x)|u|^{\ell^\ast} = \int_{\Omega} f u.
\]
From the definition of \( m_u \), the proof of the result follows.

The next lemma will give a precise information on the function \( m_u \) and the fibering map.

**Lemma 2.4.** There exists an unique critical point for \( m_u \), i.e, there is an unique point \( t > 0 \) in such way that \( m_u'(t) = 0 \). Furthermore, we know that \( t > 0 \) is a global maximum point for \( m_u \) and \( m_u(\infty) = -\infty \).

**Proof.** Notice that
\[
m_u'(t) = \int_{\Omega} \phi(t|\nabla u|)|\nabla u|^2 + t \int_{\Omega} \phi'(|\nabla (tu)|)|\nabla u|^3 - (\ell^\ast - 1)t^{\ell^\ast - 2}|u|^{\ell^\ast}.
\]
Taking into account \( \phi_3 \) it is easy to verify that
\[
\ell - 2 \leq \frac{\phi'(t)}{\phi(t)} \leq \ell - 2, \quad \text{for any } t \geq 0. \tag{2.16}
\]

Firstly, we prove that \( m_u \) is increasing for \( t > 0 \) small enough and \( \lim_{t \to \infty} m_u(t) = -\infty \). For \( 0 < t < 1 \), using (2.16) we get
\[
m_u'(t) \geq (\ell - 1)t^{m-2} \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 - (\ell^\ast - 1)t^{\ell^\ast - 2}|u|^{\ell^\ast}
\]
Since \( m < \ell^\ast \) we mention that \( m_u'(t) > 0 \) for any \( t > 0 \) small enough. Arguing as above we obtain
\[
m_u(t) \leq \frac{t}{\ell^\ast} \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 - t^{\ell^\ast - 1}|u|^{\ell^\ast}.
\]
Therefore, since \( m < \ell^\ast \), we infer that \( \lim_{t \to \infty} m_u(t) = -\infty \).
Next, we will show that $m_u$ has an unique critical point $\tilde{t} > 0$. Observe that $m_u'(t) = 0$ if, and only if,
\[
t^{2-\ell^*} \int_\Omega \phi(t|\nabla u||\nabla u|^2 + t^{3-\ell^*} \int_\Omega \phi'(t|\nabla (tu)||\nabla u|^3) = (\ell^* - 1) \int_\Omega |u|^\ell^*.
\]
Define the auxiliary function $\eta_u : \mathbb{R} \to \mathbb{R}$ by
\[
\eta_u(t) = t^{2-\ell^*} \int_\Omega \phi(t|\nabla u||\nabla u|^2 + t^{3-\ell^*} \int_\Omega \phi'(t|\nabla (tu)||\nabla u|^3).
\]
Using the inequality below (1.3). As in the proof of the previous Lemma, there exists $\eta_u(t) = 0$ if, and only if,
\[
\int_\Omega \eta(t|\nabla (u)||\nabla u)|}\nabla u|^2 = (\ell^* - 1) \int_\Omega |u|^\ell^*.
\]
Using hypothesis $(\phi_3)$ we have
\[
\left\{
\begin{array}{l}
t^2 \phi''(t) \leq (m - 4)t\phi'(t) + (m - 2)\phi(t), \\
t^2 \phi''(t) \geq (\ell - 4)t\phi'(t) + (\ell - 2)\phi(t),
\end{array}
\right.
\]
which imply that
\[
\eta_u(t) \leq \int_\Omega (m - \ell^*)t^{1-\ell^*} \phi(t|\nabla u||\nabla u|^2 + \int_\Omega (m - \ell^*)t^{2-\ell^*} \phi'(t|\nabla u||\nabla u|^3
\leq (m - \ell^*)(\ell - 1)t^{1-\ell^*} \int_\Omega \phi(t|\nabla u||\nabla u|^2 < 0.
\]
The proof of this lemma is now complete.

Next we will estimate $\max_{t>0} m_u(t)$. To do this, consider $g_{\alpha}, \alpha = \ell, m$, defined in (1.3). As in the proof of the previous Lemma, there exists $\ell_{\alpha} > 0$, given by
\[
\ell_{\alpha} = \left[\frac{(\alpha - 1) \int_\Omega \phi(|\nabla u||\nabla u|^2)}{(\ell^* - 1) \int_\Omega |u|^\ell^*}\right]^{\frac{1}{\alpha - \ell^*}} > 0
\]
such that $g_{\alpha}(\ell_{\alpha}) = \max_{t>0} g_{\alpha}(t)$. 

Remark 2.3. Notice that $g_{m}(t) = g_{t}(t) = m_{u}(t)$ if, only if, $t = 1$.

Lemma 2.5. Suppose either $(f_{1})$ or $(f_{2})$. Then

$$\max_{t>0} m_{u}(t) \geq \int_{\Omega} f_{u}, \ u \in W^{1,\Phi}_{0}(\Omega). \quad (2.22)$$

Proof. If $\int_{\Omega} f_{u}dx \leq 0$, since $\max_{t>0} m_{u}(t) > 0$, the inequality $(2.22)$ is trivially satisfied. Thus, we treat the case $\int_{\Omega} f_{u}dx > 0$. Without loss of generality, take $||u||_{\ell} = 1$ and denote by $A = \int_{\Omega} \phi(\nabla u)|\nabla u|^{2}dx$.

We will consider three possibilities, namely:

(i) $\ell \geq m \geq 1$. Since $\ell \geq 1$, we obtain

$$\ell^{*} - 1 \leq (\ell - 1)A. \quad (2.23)$$

So that,

$$1 \leq \frac{\ell - 1}{\ell^{*} - 1} A \leq m \frac{\ell - 1}{\ell^{*} - 1} \max\{||u||_{\ell}, ||u||_{m}\}$$

$$= \max \left\{ \left\| \left( m \frac{\ell - 1}{\ell^{*} - 1} \right)^{\frac{1}{\alpha}} u \right\|^{\alpha}, \left\| \left( m \frac{\ell - 1}{\ell^{*} - 1} \right)^{\frac{1}{\alpha}} u \right\|^{m} \right\} = m \frac{\ell - 1}{\ell^{*} - 1} ||u||^{m}.$$.

On the other hand, using Proposition 5.1 and inequality

$$\left( m \frac{\ell - 1}{\ell^{*} - 1} \right)^{\frac{1}{\alpha}} ||u|| \geq 1, \ \alpha = \ell, m,$$

we get

$$A \geq \ell \min\{||u||_{\ell}, ||u||_{m}\}$$

$$= \frac{\ell(\ell^{*} - 1)}{m(\ell - 1)} \min \left\{ \left\| \left( m \frac{\ell - 1}{\ell^{*} - 1} \right)^{\frac{1}{\alpha}} u \right\|^{\alpha}, \left\| \left( m \frac{\ell - 1}{\ell^{*} - 1} \right)^{\frac{1}{\alpha}} u \right\|^{m} \right\} = \ell||u||_{\ell}$$

Moreover,

$$\max_{t>0} m_{u}(t) \geq \max g_{t}(t) = g_{t}(\bar{t}_{\ell})$$

$$\geq ||u|| ||u||^{\frac{\ell^{*} - \ell}{\ell^{*} - 1}} \left( \frac{\ell(\ell - 1)}{\ell^{*} - 1} \right)^{\frac{\ell - 1}{\ell^{*} - 1}} \left( \frac{\ell(\ell^{*} - \ell)}{\ell^{*} - 1} \right)$$

$$\geq S^{\frac{\ell^{*} - 1}{\ell^{*} - \ell}} \left( \frac{\ell(\ell - 1)}{\ell^{*} - 1} \right)^{\frac{\ell - 1}{\ell^{*} - 1}} \left( \frac{\ell(\ell^{*} - m)}{\ell^{*} - 1} \right) \frac{1}{||f||_{(\ell^{*})'}} \int_{\Omega} f_{u}$$

$$(f_{1}) \geq \int_{\Omega} f_{u}.$$  

(ii) If $\ell < m$ and $\bar{t}_{\ell} \leq 1 \leq \bar{t}_{m}$, then

$$\frac{\ell^{*} - 1}{\ell - 1} \geq A \geq \frac{\ell^{*} - 1}{m - 1} > 1. \quad (2.24)$$

Therefore, it follows from $(2.24)$ that

$$\max_{t>0} m_{u}(t) \geq m_{u}(1) \geq \frac{\ell^{*} - m}{m - 1} \geq \frac{\ell^{*} - m}{m - 1} ||u||_{\ell^{*}}$$

$$\geq \frac{\ell^{*} - m}{m - 1} \frac{1}{||f||_{(\ell^{*})'}} \int_{\Omega} f_{u} \geq \int_{\Omega} f_{u}. \quad (2.25)$$
Lemma 2.7. Suppose either (f₁) or (f₂). Let the following assertions:

(i) : If $t_1, \bar{t}_m \leq 1$, then
\[(m-1)A \leq \ell^* - 1.\]
As in item (ii) we get
\[
\max_{t>0} m_u(t) \geq \max g_m(t) = g_m(t_1) \geq \frac{\mathcal{S}^m((\ell^* - 1))}{m-1} \frac{1}{\|f\|_{(\ell^*,')}} \int_\Omega f u \geq \langle f\rangle \left(\int_\Omega f u. \right) \geq \int_\Omega f u.
\]
This finishes the proof of lemma.

Lemma 2.6. Let $u \in W_0^{1,\Phi}(\Omega)/\{0\}$ be a fixed function. Then we shall consider the following assertions:

1. there exists an unique $t_1 = t_1(u) > \bar{t}$ such that $\gamma_u(t_1) = 0$ and $t_1u \in \mathcal{N}^-$ whenever $\int_\Omega f u \leq 0$.
2. suppose either (f₁) or (f₂). Then, if $\int_\Omega f u > 0$, there exists unique $0 < t_1 = t_1(u) < \tilde{t} < t_2 = t_2(u)$ such that $\gamma_u(t_1) = \gamma_u(t_2) = 0$, $t_1u \in \mathcal{N}^+$ and $t_2u \in \mathcal{N}^-.

Proof. First of all, notice that arguing as in [4], it is easy to see that if $tu \in \mathcal{N}$, then
\[
\langle \psi(tu) , tu \rangle = t^2 m_u'(t). \tag{2.26}
\]

The case $\int_\Omega f u \leq 0$. Notice that the function $m_u$ admits an unique turning point $\tilde{t} > 0$, i.e, we get $m_u'(\tilde{t}) = 0$, $\tilde{t} > 0$ if, only if, $\tilde{t} = \bar{t}$, see Lemma 2.4. Moreover, $\tilde{t}$ is a global maximum point for $m_u$ such that $m_u(\tilde{t}) > 0, m_u(\infty) = -\infty$. As a byproduct there exits an unique $t_1 > \tilde{t}$ such that
\[
m_u(t_1) = \int_\Omega f u.
\]
We emphasize that $m_u'(t_1) < 0$, because $m_u$ is a decreasing function in $(\tilde{t}, \infty)$. Therefore, using Lemma 2.3, we have $t_1u \in \mathcal{N}$, proving that $\gamma_u(t_1) = 0$. Additionally, by the identity (2.26)
\[
m_u(t) = m_u'(t) + \int_\Omega f u,
\]
we get $0 > t^2 m_u'(t_1) = \langle \psi(t_1u) , t_1u \rangle$, proving that $t_1u \in \mathcal{N}^-.

The case $\int_\Omega f u > 0$. We can consider Lemma 2.5 and we get
\[
m_u(\tilde{t}) > \int_\Omega f u,
\]
which $m_u$ is increasing in $(0, \tilde{t})$ and decreasing in $(\tilde{t}, \infty)$. It is not hard to verify that there exist exactly two points $0 < t_1 = t_1(u) < \tilde{t} < t_2 = t_2(u)$ such that
\[
m_u(t_i) = \int_\Omega f u, \; i = 1, 2,
\]
satisfying $m_u'(t_1) > 0$ and $m_u'(t_2) < 0$. As in the previous step we infer that $t_1u \in \mathcal{N}^+$ and $t_2u \in \mathcal{N}^-$. This completes the proof.

Lemma 2.7. Suppose either (f₁) or (f₂). There exist $\delta_1, \lambda_2 > 0$ in such way that $J(u) \geq \delta_1$ for any $u \in \mathcal{N}^-$ where $0 < \|f\|_{(\ell^*,')} < \lambda_2$. \[\square\]
Proof. Since \( u \in N^- \), we have that \( \langle \psi'(u), u \rangle < 0 \). Arguing as in the proof of Lemma 2.1, we obtain
\[
\|u\| > \left[ \frac{\ell (\ell - 1) S^{\frac{\ell - 1}{\ell}}} { (\ell^* - 1)} \right]^{\frac{1}{\ell - 1}}.
\]
Moreover, in view of (2.8) and the Sobolev imbedding, we have that
\[
J(u) \geq \ell \left( \frac{1}{m} - \frac{1}{\ell^*} \right) \min \{|\|u\|^{\ell_*}, |\|u\|^{m_*}\} - \left( 1 - \frac{1}{\ell} \right) \int_{\Omega} f u
\]
\[
\geq \|u\| \left[ \ell \left( \frac{1}{m} - \frac{1}{\ell^*} \right) \min \{|\|u\|^{\ell_*-1}, |\|u\|^{m_*-1}\} - \left( 1 - \frac{1}{\ell} \right) S^{-\frac{1}{2}} |\|f\|_{(\ell^*)'} \right].
\]
By the above inequality, we get
\[
J(u) > \left[ \frac{\ell (\ell - 1) S^{\frac{\ell - 1}{\ell}}} { (\ell^* - 1)} \right]^{\frac{1}{\ell - 1}} \left[ \ell \left( \frac{1}{m} - \frac{1}{\ell^*} \right) \frac{\ell (\ell - 1) S^{\frac{\ell - 1}{\ell}}} { (\ell^* - 1)} \right]^{\frac{\ell - 1}{\ell}} - \left( 1 - \frac{1}{\ell} \right) S^{-\frac{1}{2}} |\|f\|_{(\ell^*)'} |\] \[\left[ A - |\|f\|_{(\ell^*)'} B \right].
\]
Notice that \( A - |\|f\|_{(\ell^*)'} B > 0 \) if, only if, \( |\|f\|_{(\ell^*)'} < \frac{A}{B} = \frac{1}{m} \lambda_1 =: \lambda_2 \), where \( \lambda_1 \) is given by \((f_1)\). On the other hand, if \((f_2)\) holds, we have
\[
|\|f\|_{(\ell^*)'} \leq \min \left\{ \frac{1}{m} \lambda_1, \frac{\ell^* - m}{m - 1} \right\} \leq \frac{1}{m} \lambda_1.
\]
Hence, in either case \((f_1)\) or \((f_2)'\), we conclude that \( J(u) \geq \delta_1 \), for all \( u \in N^- \). \( \square \)

Lemma 2.8. Suppose \((H)\) and either \((f_1)\) or \((f_2)'\). Then, \( \alpha := \inf_{u \in N^-} J(u) = \alpha^+ = \inf_{u \in N^+} J(u) < 0 \).

Proof. Since \( u \in N^+ \) we have that \( \langle \psi'(u), u \rangle > 0 \), i.e.
\[
\int_{\Omega} \phi'(|\nabla u|)|\nabla u|^{\ell_*} + \phi(|\nabla u|)|\nabla u|^{\ell^*} - (\ell^* - 1)|u|^{\ell^*} > 0.
\]
Thus,
\[
(\ell^* - 1) \int_{\Omega} |u|^{\ell^*} < \int_{\Omega} \phi'(|\nabla u|)|\nabla u|^{\ell_*} + \phi(|\nabla u|)|\nabla u|^{\ell^*} < (m - 1) \int_{\Omega} \phi(|\nabla u|)|\nabla u|^{\ell^*}.
\]
Consequently,
\[
\int_{\Omega} |u|^{\ell^*} < \frac{m - 1}{\ell^* - 1} \int_{\Omega} \phi(|\nabla u|)|\nabla u|^{\ell^*}
\]
On the other hand, if \( u \in N \), using the above inequality and \((\phi_3)\), we get
\[
J(u) \leq \left( \frac{1}{\ell} - 1 \right) \int_{\Omega} \phi(|\nabla u|)|\nabla u|^{\ell^*} \left( 1 - \frac{1}{\ell} \right) |u|^{\ell^*}
\]
\[
< \left[ \frac{1 - \ell}{\ell} + \frac{m - 1}{\ell^*} \right] \int_{\Omega} \phi(|\nabla u|)|\nabla u|^{\ell^*} < 0,
\]
because \( \left[ \frac{1 - \ell}{\ell} + \frac{m - 1}{\ell^*} \right] < 0 \), since \((H)\) holds. Consequently, \( \alpha^+ < 0 \).

Since \( N^- = N^- \cup N^+ \) and \( \alpha^- > 0 \), we have that \( \alpha^+ = \alpha \), and the Lemma is proved. \( \square \)
Lemma 3.1. Suppose $(\phi_1) - (\phi_3)$ and $(H)$. Let $u \in \mathcal{N}^+$ be fixed. Then there exist $\epsilon > 0$ and a differentiable function

$$\xi : B(0, \epsilon) \subset W_0^{1, \Phi}(\Omega) \to (0, \infty), \quad \xi(0) = 1, \quad \xi(v)(u - v) \in \mathcal{N}^+, \forall v \in B(0, \epsilon).$$

Furthermore, we have that

$$\langle \xi'(v), v \rangle = \frac{1}{\psi'((u), u)} \int_{\Omega} \left\{\phi'(|\nabla u|)|\nabla u| + 2\phi(|\nabla u|)|\nabla v - \ell^*|u|^{\ell^* - 2}uv - f v \right\}.$$

(3.27)

Proof. Initially, we define $\psi : W_0^{1, \Phi}(\Omega) \setminus \{0\} \to \mathbb{R}$ given by $\psi(u) = \langle J'(u), u \rangle$ for $u \in W_0^{1, \Phi}(\Omega) \setminus \{0\}$. Recall that $\langle \psi'(u), u \rangle$ is given by (2.10), and for any $u \in \mathcal{N}$, $\langle \psi'(u), u \rangle$ was defined in Remark 2.2.

Now we define $F_u : \mathbb{R} \times W_0^{1, \Phi}(\Omega) \setminus \{0\} \to \mathbb{R}$ given by $F_u : \mathbb{R} \times W_0^{1, \Phi}(\Omega) \setminus \{0\} \to \mathbb{R}$ given by

$$F_u(\xi, w) = \langle J'(\xi(w - u)), \xi(u - w) \rangle.$$

Here we observe that $F_u(1, 0) = \psi(u)$. As a consequence, for each $u \in \mathcal{N}$, we have

$$\partial_1 F_u(1, 0) = \int_{\Omega} 2\phi(|\nabla u|)|\nabla u|^2 + \phi(|\nabla u|)|\nabla v - \ell^*|u|^{\ell^* - 2}uv - f v = \psi'(u, u) \neq 0.$$

By using the Inverse Function Theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0, \epsilon) \subset W_0^{1, \Phi}(\Omega) \to (0, \infty)$ satisfying $\xi(0) = 1$ and $F_u(\xi(w), w) = \langle J'(\xi(w)(u - w)), \xi(w)(u - w) \rangle = 0$, i.e. $\xi(w)(u - w) \in \mathcal{N}$, $\forall \xi \in B(0, \epsilon)$. Furthermore, we also have

$$\langle \xi'(w), v \rangle = \frac{\langle \partial_2 F_u(\xi(w), w), v \rangle}{\partial_1 F_u(\xi(w), w)}, \langle \xi'(0), v \rangle = \frac{\langle \partial_2 F_u(\xi(0), w), v \rangle}{\partial_1 F_u(\xi(0), 0)}.$$

Here $\partial_1 F_u$ and $\partial_2 F_u$ denote the partial derivatives on the first and second variable, respectively.

On the other hand, after some manipulations, putting $w = 0$ and $\xi = \xi(0) = 1$, we have

$$- \langle \partial_2 F_u(1, 0), v \rangle = \int_{\Omega} (\phi(|\nabla u|)|\nabla u| + 2\phi(|\nabla u|)|\nabla v - \ell^*|u|^{\ell^* - 2}uv - f v$$

Here was used the fact that $\partial_1 F_u(1, 0) = \langle \psi'(u), u \rangle$ holds for any $u \in \mathcal{N}$. The proof is complete.

Similarly, we have the following

Lemma 3.2. Suppose $(\phi_1) - (\phi_3)$ and $(H)$. Let $u \in \mathcal{N}^-$ be fixed. Then there exist $\epsilon > 0$ and a differentiable function

$$\xi^- : B(0, \epsilon) \subset W_0^{1, \Phi}(\Omega) \to (0, \infty), \quad \xi^-(0) = 1, \quad \xi^-(v)(u - v) \in \mathcal{N}^+, \forall v \in B(0, \epsilon).$$

Furthermore, we obtain

$$\langle (\xi^-)'(0), v \rangle = \frac{1}{\psi'(u, u)} \int_{\Omega} \left\{\phi'(|\nabla u|)|\nabla u| + 2\phi(|\nabla u|)|\nabla v - \ell^*|u|^{\ell^* - 2}uv - f v \right\}.$$

(3.28)

Next, we shall prove that any minimizing sequences on the Nehari manifold in $\mathcal{N}^+$ or $\mathcal{N}^+$ provides us a Palais-Smale sequence.
**Proposition 3.1.** Suppose $(\phi_1) - (\phi_3)$ and $(H)$. Then we have the following assertions

1. there exists a sequence $(u_n) \subset \mathcal{N}$ such that $J(u_n) = \alpha^+ + o_n(1)$ and $J'(u_n) = o_n(1)$ in $W^{-1, \Phi}(\Omega)$.

2. there exists a sequence $(u_n) \subset \mathcal{N}^-$ such that $J(u_n) = \alpha^- + o_n(1)$ and $J'(u_n) = o_n(1)$ in $W^{-1, \Phi}(\Omega)$.

**Proposition 3.2.** Suppose $(\phi_1) - (\phi_3)$ and $(H)$ hold. Let $(u_n)$ be a minimizing sequence for the functional $J$ constrained to the Nehari manifold $\mathcal{N}^+$. Then

\[
\liminf_{n \to \infty} ||u_n|| \geq -\alpha^+ \left[ \frac{\ell^*}{(\ell^* - 1) ||f||(\ell^*) S^{\frac{2}{\ell^*}}} \right] > 0. \tag{3.29}
\]

and

\[
||u_n|| < \left[ \left( \frac{\ell^* - 1}{\ell^* - m} \right) ||f||(\ell^*) S^{\frac{2}{\ell^*}} \right]^{-\frac{1}{\ell^*}}, \tag{3.30}
\]

where $\alpha \in \{\ell, m\}$. The same property can be proved for the Nehari manifold $\mathcal{N}^-$. 

**Proof.** Remember that $(u_n) \subset \mathcal{N}$, $m\Phi(t) \leq \phi(t)t^2$ and arguing as in the proof of Lemma 2.7, we infer that

\[
0 > \alpha^+ + o_n(1) > J(u_n) \geq \int_\Omega \left( 1 - \frac{m}{\ell^*} \right) \Phi(|\nabla u_n|) - \left( 1 - \frac{1}{\ell^*} \right) f u \tag{3.31}
\]

holds for any $n \in \mathbb{N}$ large enough. By using the above inequality and the continuous embedding $W^{1, \Phi}_0(\Omega) \hookrightarrow L^{\ell^*}(\Omega)$, we deduce that

\[
||u_n|| > -\left( \alpha^+ + \frac{1}{n} \right) \frac{\ell^*}{(\ell^* - 1) ||f||(\ell^*) S^{\frac{2}{\ell^*}}},
\]

and (3.29) holds.

Furthermore, using (3.31) and arguing as in (2.13), we obtain that

\[
\min \{ ||u_n|^{\ell^*}, ||u_n||^m \} \leq \int_\Omega \Phi(|\nabla u_n|) < \left( \frac{\ell^* - 1}{\ell^* - m} \right) ||f||(\ell^*) S^{\frac{2}{\ell^*}} ||u_n||.
\]

Hence the last assertions give us

\[
||u_n|| < \left[ \left( \frac{\ell^*}{\ell^* - m} \right) \left( \frac{\ell^* - 1}{\ell^*} \right) ||f||(\ell^*) S^{\frac{2}{\ell^*}} \right]^{-\frac{1}{\ell^*}} = \left[ \left( \frac{\ell^* - 1}{\ell^* - m} \right) ||f||(\ell^*) S^{\frac{2}{\ell^*}} \right]^{-\frac{1}{\ell^*}},
\]

where $\alpha \in \{\ell, m\}$.

Now we will prove two technical results, which will be used to prove that any minimizing sequence for $J$ constrained to the Nehari manifold is a Palais-Smale sequence.

**Proposition 3.3.** Suppose $(\phi_1) - (\phi_3)$ and $(H)$ hold. Then any minimizing sequence $(u_n)$ on the Nehari manifold $\mathcal{N}^-$ or $\mathcal{N}^+$ satisfies

\[
\langle J'(u_n), \frac{u}{||u||} \rangle \leq \frac{C}{n} ||\xi'_n(0)|| + 1. \tag{3.32}
\]

where $\xi_n : B_{\frac{1}{n}}(0) \to \mathbb{R}$ was obtained by Lemmas 3.1 and 3.2.
Proof. Taking $\epsilon_n$ given in Lemma 3.1, put $\rho \in (0, \epsilon_n)$ and $u \in W^{1,\Phi}(\Omega) \setminus \{0\}$. Define the auxiliary function
\[ w_\rho = \frac{\rho u}{||u||} \subseteq B(0, \epsilon_n). \]
Using Lemma 3.1 we infer that
\[ \mu_\rho = \xi(w_\rho)(u_n - w_\rho) \in \mathcal{N}^+ \text{ and } J(\mu_\rho) - J(u_n) \geq -\frac{1}{n}||\mu_\rho - u_n||. \] (3.33)
Notice also that we have the following convergences
\[ w_\rho \to 0, \; \xi_n(w_\rho) \to 1, \; \mu_\rho \to u_n \text{ and } J'(\mu_\rho) \to J'(u_n) \] (3.34)
as $\rho \to 0$, for any $n \in \mathbb{N}$.
Applying Mean Value Theorem, there exists $t \in (0, 1)$ in such way that
\[ J(\mu_\rho) - J(u_n) = \langle J'(\mu_\rho + t(u_n - \mu_\rho)) - J'(u_n), \mu_\rho - u_n \rangle + \langle J'(u_n), \mu_\rho - u_n \rangle. \]
Remind that $||u_n - \mu_\rho|| \to 0$ as $\rho \to 0$. Since $\mu_\rho \in \mathcal{N}^+$ and using (3.33) and (3.34), we obtain
\[ -\frac{1}{n}||\mu_\rho - u_n|| + o_\rho(||\mu_\rho - u_n||) \leq \langle J'(u_n), -w_\rho \rangle + (\xi_n(w_\rho) - 1) \langle J'(u_n), u_n - w_\rho \rangle. \]
where $o_\rho(.)$ denotes a quantity that goes to zero as $\rho$ goes to zero. Using that $\langle J'(\mu_\rho), \mu_\rho \rangle = 0$, we have
\[ -\frac{1}{n}||\mu_\rho - u_n|| \leq \frac{o_\rho(||\mu_\rho - u_n||)}{n} - \rho \left( \langle J'(u_n), \frac{u}{||u||} \rangle + (\xi_n(w_\rho) - 1) \langle J'(u_n) - J'(\mu_\rho), u_n - w_\rho \rangle \right). \]
From the above estimates and (3.34) we obtain
\[ \langle J'(u_n), \frac{u}{||u||} \rangle \leq \frac{||\mu_\rho - u_n||}{n} \frac{o_\rho(||\mu_\rho - u_n||)}{\rho} + \frac{(\xi_n(w_\rho) - 1)}{\rho} \langle J'(u_n) - J'(\mu_\rho), u_n - w_\rho \rangle. \]
Noticing that
\[ \lim_{\rho \to 0} \frac{||\xi_n(w_\rho) - 1||}{\rho} = \langle \xi_n'(0), \frac{u}{||u||} \rangle \leq ||\xi_n'(0)||, \]
from this inequality we have
\[ ||\mu_\rho - u_n|| \leq \rho ||\xi_n(w_\rho)|| + ||\xi_n(w_\rho) - 1|| ||u_n|| \text{ and } \lim_{\rho \to 0} \frac{||\xi_n(w_\rho) - 1||}{\rho} \leq ||\xi_n'(0)||. \] (3.35)
Therefore, using the fact that $(u_n)$ is bounded and (3.35), we infer that
\[ \lim_{\rho \to 0} \frac{||\mu_\rho - u_n||}{n} \leq \lim_{\rho \to 0} \frac{1}{n} \left[ ||\xi_n(w_\rho)|| + \frac{||\xi_n(w_\rho) - 1||}{\rho} ||u_n|| \right] \leq \frac{1}{n} \left[ 1 + ||\xi_n'(0)|| ||u_n|| \right] \leq \frac{C}{n} \left[ 1 + ||\xi_n'(0)|| \right]. \]
On the other hand, since $\frac{\xi_n(w_\rho) - 1}{\rho}$ and $\xi_n(w_\rho)$ are bounded for $\rho > 0$ small enough, we obtain
\[ ||\mu_\rho - u_n|| = \rho \left| \left| \frac{\xi_n(w_\rho) - 1}{\rho} u - \xi_n(w_\rho) \frac{u}{||u||} \right| \right| \leq \rho \left| \left| \frac{\xi_n(w_\rho) - 1}{\rho} \right| ||u_n|| + ||\xi_n(w_\rho)|| \right|. \]
Since \((u_n)\) is bounded there exists a constant \(C > 0\) in such that
\[
\frac{\|u_\rho - u_n\|_\rho}{\rho} \leq C\|\xi'_n(0)\| + 1.
\]

Putting all these estimates together we prove (3.32) holds. \(\square\)

**Proposition 3.4.** Under the hypotheses of Proposition 3.3 there exists \(C > 0\) such that
\[
\|\xi'_n(0)\| \leq C, \forall n \in \mathbb{N}.
\]

**Proof.** Firstly notice that the numerator in (3.27) is bounded from below away from zero by \(b\|v\|\) where \(b > 0\) is a constant. Define the auxiliary function \(\chi_n : W^{1,\Phi}_0(\Omega) \to \mathbb{R}\) given by
\[
\chi_n(v) = \int_\Omega [\phi'(|\nabla u_n|)|\nabla u_n| + 2\phi(|\nabla u_n|)|\nabla u_n|\nabla v - \int_\Omega \ell^*|u_n|^\ell - 2u_nv - f v.
\]

Using that \(\frac{\phi'(t)}{\phi(t)} \leq \max\{\ell - 2, |m - 2|\} =: C_1\) and Holder’s inequality, we obtain
\[
|\chi_n(v)| \leq C_1 \int_\Omega \phi(|\nabla u_n|)|\nabla u_n||\nabla v| + \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|
\]
\[
\leq 2C_1\|\phi(|\nabla u_n|)|\nabla u_n||\nabla v| + \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|
\]
\[
\leq C_2 \max \left\{ \left( \int_\Omega \tilde{\Phi}(\phi(|\nabla u_n|)|\nabla u_n|) \right)^{\frac{\ell - 1}{\ell}}, \left( \int_\Omega \tilde{\Phi}(\phi(|\nabla u_n|)|\nabla u_n|) \right)^{\frac{m - 1}{m}} \right\} |v|
\]
\[
+ \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|.
\]

In virtue of the inequality \(\tilde{\Phi}(t\phi(t)) \leq \Phi(2t) \leq 2^m\Phi(t), t \geq 0\) and (3.30) there exists a constant \(C_3 > 0\) such that
\[
|\chi_n(v)| \leq C_3 \max \left\{ \left( \int_\Omega \Phi(|\nabla u_n|) \right)^{\frac{\ell - 1}{\ell}}, \left( \int_\Omega \Phi(|\nabla u_n|) \right)^{\frac{m - 1}{m}} \right\} |v|
\]
\[
+ \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|
\]
\[
\leq C_4 |v| + \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|
\]
\[
\leq C_5 |v| + \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|
\]
\[
\leq C_6 |v| + \ell^* \int_\Omega |u_n|^{\ell - 1}|v| + \int_\Omega |f||v|.
\]

where \(\beta \in \{\ell - 1, \frac{\ell}{m}(\ell - 1), m - 1, \frac{m}{\ell}(m - 1)\}\).

We shall estimate the terms \(\int |u_n|^{\ell - 1}|v|\) and \(\int |f||v|\). Employing Holder’s inequality and Sobolev imbedding we obtain
\[
\int_\Omega |u_n|^{\ell - 1}|v| \leq \left( \int_\Omega |u_n|^{\ell} \right)^{\frac{\ell - 1}{\ell}} \left( \int_\Omega |v|^{\ell} \right)^{\frac{1}{\ell}} \leq C_6 |u_n|^{\ell - 1}\|v\| \leq C_5 |v|,
\]
and
\[
\int_\Omega |f||v| \leq \left( \int_\Omega |f|^{(\ell')^*} \right)^{\frac{1}{(\ell')^*}} \left( \int_\Omega |v|^{\ell'} \right)^{\frac{1}{\ell'}} = \|f||v\|_{\ell'} \|v\|_{\ell'} \leq S^{\frac{1}{\ell'}} |f||v|_{\ell'} \leq S |f||v|_{\ell'}.
\]

Combining the estimates above there exists a constant \(c > 0\) in such that
\[
|\chi_n(v)| \leq c|v|.
\]
Next, we will show that there exists a constant $d > 0$, independent in $n$, such that $\gamma_{u_n}(1) \geq d$. Indeed, arguing by contradiction that $\gamma_{u_n}(1) = o_n(1)$. It follows from (3.29) that there exists $a_0 > 0$ satisfying

$$\liminf_{n \to \infty} ||u_n|| \geq a > 0$$

(3.36)

Using (2.4) and (2.11), as well as, $\langle \phi'(u_n), u_n \rangle = o_n(1)$, we deduce that

$$o_n(1) = \langle \phi'(u_n), u_n \rangle = \int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^2 + \phi'(|\nabla u_n|)||\nabla u|^3 + (1 - \ell^*)|u_n|^\ell^*.$$

Under hypothesis $(\phi_1)$ and the Sobolev embeddings we infer that

$$(\ell - 1) \int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^2 \leq (\ell^*-1)S^{-\frac{\ell^*}{2}}||u_n||^\ell^* + o_n(1).$$

On the other hand, we observe that

$$\int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^2dx \geq \ell(\ell-1) \int_{\Omega} \Phi(|\nabla u_n|) \geq \ell(\ell-1) \min\{|u_n|, ||u_n||^m\}.$$

Using the above estimates we get

$$\ell(\ell - 1) \min\{|u_n|, ||u_n||^m\} \leq (\ell^*-1)S^{-\frac{\ell^*}{2}}||u_n||^\ell^* + o_n(1).$$

Hence, we have

$$\ell(\ell - 1) \leq (\ell^*-1)S^{-\frac{\ell^*}{2}}||u_n||^\ell^* - \frac{o_n(1)}{||u_n||^\alpha}$$

where $\alpha = \ell$ whenever $||u_n|| \geq 1$ and $\alpha = m$ whenever $||u_n|| \leq 1$. Furthermore, using (3.36), we obtain

$$||u_n|| \geq \left[\frac{\ell(\ell - 1)}{(\ell^*-q)S^{\frac{\ell^*}{2}}}\right]^{\frac{1}{\ell^*-\alpha}} + o_n(1).$$

(3.37)

Using (2.11), $(\phi_3)$ and Holder inequality, we obtain

$$(\ell^*-m) \int_{\Omega} \phi(|\nabla u|)|\nabla u_n|^2 \leq (\ell^*-1)S^{\frac{1}{\ell^*-m}} ||f||(|\ell^*|) ||u_n|| + o_n(1).$$

Combining the above inequalities, we get

$$\frac{\ell(\ell^*-m)}{(\ell^*-1)S^{\frac{1}{\ell^*-m}}||f||(|\ell^*|)} ||u_n||^{\alpha} = \frac{\ell(\ell^*-m)}{(\ell^*-1)S^{\frac{1}{\ell^*-m}}||f||(|\ell^*|)} \min\{|u_n|, ||u_n||^m\} \leq ||u_n|| + o_n(1).$$

To sum up, using the estimate (3.36), we can be shown that

$$||u_n|| \leq \left[\frac{(\ell^*-1)S^{\frac{1}{\ell^*-m}}||f||(|\ell^*|)}{\ell(\ell^*-m)}\right]^{\frac{1}{\ell^*-m}} + o_n(1).$$

Arguing as in the proof of Lemma 2.1, by the above inequality and (3.37) we have a contradiction since either $(f_1)$ or $(f_2)$ hold. This completes the proof.

**Proof of Proposition 3.1.** We shall prove the item (1). The proof of item (2) follows similarly using Lemma 3.2 instead of Lemma 3.1. Applying Ekeland’s variational principle there exists a sequence $(u_n) \subset \mathcal{N}^+$ in such way that

(i): $J(u_n) = \alpha^\ast + o_n(1)$,

(ii): $J(u_n) < J(w) + \frac{1}{n}||w - u||$, $\forall w \in \mathcal{N}^+$.
In what follows we shall prove that \( \lim_{n \to \infty} ||J'(u_n)|| \to 0 \). From Proposition 3.4, there exist \( C > 0 \) independent on \( n \in \mathbb{N} \) such that \( ||\xi_n(0)|| \leq C \). This estimate together with Proposition 3.3

\[
\left\langle J'(u_n), \frac{u}{||u||} \right\rangle \leq \frac{C}{n}, \quad u \in W_0^{1,\Phi}(\Omega)/\{0\}.
\]

This implies that \( ||J'(u_n)|| \to 0 \) as \( n \to \infty \). This finishes the proof. \( \square \)

4. The proof of our main theorems.

4.1. The proof of Theorem 1.1. We are going to apply the following result, whose proof is made by using the concentration compactness principle due to Lions for Orlicz-Sobolev framework, see [27] or else in [8, 15].

Lemma 4.1. (i) \( \phi(|\nabla u_n|)\nabla u_n \to \phi(|\nabla u|)\nabla u \) in \( \prod L_p(\Omega) \);
(ii) \( |u_n|^{\ell^*-2} u_n \to |u|^{\ell^*-2} u \) in \( L^{\ell^*}(\Omega) \).

Let \( ||f||_{(\ell^*)'} < \Lambda_1 = \text{min} \left\{ \lambda_1, \frac{\ell^* - m}{m - 1} \right\} \) where \( \lambda_1 > 0 \) is given by \((f_1)\).

From Lemma 2.8 we infer that

\[
\alpha^+ := \inf_{u \in N^+} J(u) = \inf_{u \in N^+} J(u) < 0.
\]

We will find a function \( u \in N^+ \) such that

\[
J(u) = \min_{u \in N^+} J(u) =: \alpha^+ \quad \text{and} \quad J'(u) = 0.
\]

First of all, using Proposition 3.1, there exists a minimizing sequence denoted by \((u_n) \subset W^{1,\Phi}(\Omega)\) such that

\[
J(u_n) = \alpha^+ + o_n(1) \quad \text{and} \quad J'(u_n) = o_n(1).
\] (4.38)

Since the functional \( J \) is coercive in \( N^+ \), this implies that \((u_n)\) is bounded in \( N^+ \). Therefore, there exists a function \( u \in W_0^{1,\Phi}(\Omega) \) such that

\[
u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\Phi}(\Omega), \quad u_n \to u \quad \text{a.e. in} \quad \Omega, \quad u_n \to u \quad \text{in} \quad L^\Phi(\Omega).
\] (4.39)

We shall prove that \( u \) is a weak solution for the problem elliptic problem \((1.1)\).

Notice that, by (4.38), we mention that

\[
o_n(1) = \langle J'(u_n), v \rangle = \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla v - f u_n |^{\ell^* - 2} u_n v
\]

holds for any \( v \in W_0^{1,\Phi}(\Omega) \). In view of (4.39) and Lemma 4.1 we get

\[
\int_\Omega \phi(|\nabla u|)\nabla u \nabla v - f u |^{\ell^* - 2} u v = 0
\]

for any \( v \in W^{1,\Phi}(\Omega) \) proving that \( u \) is a weak solution to the elliptic problem \((1.1)\).

In addition, the weak solution \( u \) is not zero. In fact, using the fact that \( u_n \in N^+ \), we obtain

\[
\int_\Omega f u_n = \int_\Omega \left( \Phi(|\nabla u_n|) - \frac{1}{\ell^*} \phi(|\nabla u_n|) |\nabla u_n|^2 \right) \frac{\ell^*}{\ell^* - 1} - J(u_n) \frac{\ell^*}{\ell^* - 1} \geq \frac{1}{\ell^* - 1} \left( 1 - \frac{m}{\ell^*} \right) \int_\Omega \Phi(|\nabla u_n|) - J(u_n) \frac{\ell^*}{\ell^* - 1} \geq -J(u_n) \frac{\ell^*}{\ell^* - 1}.
\]
From (4.38) and (4.39) we obtain
\[ \int_{\Omega} fu \geq -\alpha^+ \frac{\ell^*}{\ell^* - 1} > 0. \] (4.40)
Hence \( u \neq 0 \).

We shall prove that \( J(u) = \alpha^+ \) and \( u_n \to u \) in \( W_0^{1, \Phi}(\Omega) \). Since \( u \in \mathcal{N} \) we also see that
\[ \alpha^+ \leq J(u) = \int_{\Omega} \Phi(|\nabla u|) - \frac{1}{\ell^*} \phi(|\nabla u|)|\nabla u|^2 \left( - \left( 1 - \frac{1}{\ell^*} \right) fu \right). \]
Notice that
\[ t \mapsto \Phi(t) - \frac{1}{\ell^*} \phi(t) t^2 \]
is a convex function. In fact, by hypothesis \((\phi_3)\) and \( m < \ell^* \), we infer that
\[ \left( \Phi(t) - \frac{1}{\ell^*} \phi(t) t^2 \right)'' = \left[ \left( 1 - \frac{1}{\ell^*} \right) t \phi(t) - \frac{1}{\ell^*} t \phi(t) \right] \phi(t) \]
\[ \geq (t \phi(t))' \left( 1 - \frac{m}{\ell^*} \right) > 0, \quad t > 0. \]
In addition, the last assertion says that
\[ u \mapsto \int_{\Omega} \Phi(|\nabla u|) - \frac{1}{\ell^*} \phi(|\nabla u|)|\nabla u|^2 dx \]
is weakly lower semicontinuous function. Therefore we obtain
\[ \alpha^+ \leq J(u) \leq \liminf \left( \int_{\Omega} \Phi(|\nabla u_n|) - \frac{1}{\ell^*} \phi(|\nabla u_n|)|\nabla u_n|^2 \left( 1 - \frac{1}{\ell^*} \right) fu_n \right) \]
\[ = \liminf J(u_n) = \alpha^+. \]
This implies that \( J(u) = \alpha^+ \). Additionally, using (4.39), we also have
\[ J(u) = \int_{\Omega} \Phi(|\nabla u|) - \frac{1}{\ell^*} \phi(|\nabla u|)|\nabla u|^2 \left( - \left( 1 - \frac{1}{\ell^*} \right) fu \right) \]
\[ = \lim \left( \int_{\Omega} \Phi(|\nabla u_n|) - \frac{1}{\ell^*} \phi(|\nabla u_n|)|\nabla u_n|^2 \right) - \left( 1 - \frac{1}{\ell^*} \right) \int_{\Omega} fu. \]
From the last identity
\[ \lim \left( \int_{\Omega} \Phi(|\nabla u_n|) - \frac{1}{\ell^*} \phi(|\nabla u_n|)|\nabla u_n|^2 \right) = \int_{\Omega} \Phi(|\nabla u|) - \frac{1}{\ell^*} \phi(|\nabla u|)|\nabla u|^2. \]
In view of Brezis-Lieb Lemma, choosing \( v_n = u_n - u \), we infer that
\[ \lim \left( \int_{\Omega} \Phi(|\nabla u_n|) - \frac{1}{\ell^*} \phi(|\nabla u_n|)|\nabla u_n|^2 + \Phi(|\nabla v_n|) - \frac{1}{\ell^*} \phi(|\nabla v_n|)|\nabla v_n|^2 \right) \]
\[ = \int_{\Omega} \Phi(|\nabla u|) - \frac{1}{\ell^*} \phi(|\nabla u|)|\nabla u|^2. \] (4.41)
The previous assertion implies that
\[ 0 = \lim \left( \int_{\Omega} \Phi(|\nabla v_n|) - \frac{1}{\ell^*} \phi(|\nabla v_n|)|\nabla v_n|^2 \right) \geq \lim \left( 1 - \frac{m}{\ell^*} \right) \int_{\Omega} \Phi(|\nabla v_n|) \geq 0. \]
Therefore, we obtain that \( \lim \int_{\Omega} \Phi(|\nabla v_n|) = 0 \) and \( u_n \to u \) in \( W^{1, \Phi}(\Omega) \). Hence we conclude that \( u_n \to u \) in \( W_0^{1, \Phi}(\Omega) \).
We shall prove that $u \in \mathcal{N}^+$. Arguing by contradiction we have that $u \notin \mathcal{N}^+$. Using Lemma 2.6 there are unique $t_0^+, t_0^- > 0$ in such way that $t_0^+ u \in \mathcal{N}^+$ and $t_0^- u \in \mathcal{N}^-$. In particular, we know that $t_0^+ < t_0^- = 1$. Since
\[
\frac{d}{dt} J(t_0^+ u) = 0
\]
and using (4.40) together the Lemma 2.6 we have that
\[
\frac{d}{dt} J(t u) > 0, \quad t \in (t_0^+, t_0^-).
\]
So, there exist $t^- \in (t_0^+, t_0^-)$ such that $J(t_0^- u) < J(t^- u)$.

In addition $J(t_0^+ u) < J(t^- u) \leq J(t_0^- u) = J(u)$ which is a contradiction to the fact that $u$ is a minimizer in $\mathcal{N}^+$. So that $u$ is in $\mathcal{N}^+$.

To conclude the proof of theorem it remains to show that $u \geq 0$ when $f \geq 0$. For this we will argue as in [25]. Since $u \in \mathcal{N}^+$, by Lemma 2.6 there exists a $t_0 \geq 1$ such that $t_0 |u| \in \mathcal{N}^+$ and $t_0 |u| \geq |u|$. Therefore if $f \geq 0$, we get
\[
J(u) = \inf_{w \in \mathcal{N}^+} J(w) \leq J(t_0 |u|) \leq J(|u|) \leq J(u).
\]
So we can assume without loss of generality that $u \geq 0$.

4.2. The proof of Theorem 1.2. Let $\|f\|_{(\ell^*)'} < \Lambda_2 = \min \left\{ \lambda_2, \frac{\ell - m}{m - 1} \right\}$ where $\lambda_2 > 0$ is given by Lemma 2.7.

First of all, from Lemma 2.7, there exists $\delta_1 > 0$ such that $J(v) \geq \delta_1$ for any $v \in \mathcal{N}^-$. So that,
\[
\alpha^- := \inf_{v \in \mathcal{N}^-} J(v) \geq \delta_1 > 0.
\]

Now we shall consider a minimizing sequence $(v_n) \subset \mathcal{N}^-$ given in Proposition 3.1, i.e, $(v_n) \subset \mathcal{N}^-$ is a sequence satisfying
\[
\lim_{n \to \infty} J(v_n) = \alpha^- \quad \text{and} \quad \lim_{n \to \infty} J'(v_n) = 0. \tag{4.42}
\]

Since $J$ is coercive in $\mathcal{N}$ and so on $\mathcal{N}^-$, using Lemma 2.1, we have that $(v_n)$ is bounded sequence in $W_0^{1, \Phi}(\Omega)$. Up to a subsequence we assume that $v_n \rightharpoonup v$ in $W_0^{1, \Phi}(\Omega)$ holds for some $v \in W_0^{1, \Phi}(\Omega)$. Additionally, using the fact that $\ell > 1$, we get $t << \Phi_*(t)$ and $W_0^{1, \Phi}(\Omega) \hookrightarrow L^1(\Omega)$ is also a compact embedding. This fact implies that $v_n \rightharpoonup v$ in $L^1(\Omega)$. In this way, we can obtain
\[
\lim_{n \to \infty} \int_{\Omega} f v_n = \int_{\Omega} f v.
\]

Now we claim that $v \in W_0^{1, \Phi}(\Omega)$ given just above is a weak solution to the elliptic problem (1.1). In fact, using (4.42), we infer that
\[
\langle J'(v_n), w \rangle = \int_{\Omega} \phi(|\nabla v_n|) \nabla v_n \nabla w - f w |v_n|^\ell - 2 v_n w = o_n(1)
\]
holds for any $w \in W_0^{1, \Phi}(\Omega)$. Now using Lemma 4.1 we get
\[
\int_{\Omega} \phi(|\nabla v|) \nabla v \nabla w - f w |v|^\ell - 2 v w = 0, \quad w \in W_0^{1, \Phi}(\Omega).
\]
So that $v$ is a critical point for the functional $J$. Without any loss of generality, changing the sequence $(v_n)$ by $(|v_n|)$, we can assume that $v \geq 0$ in $\Omega$. 

Next we claim that \( v \neq 0 \). The proof for this claim follows arguing by contradiction assuming that \( v \equiv 0 \). Recall that

\[
J(tv_n) \leq J(v_n)
\]

for any \( t \geq 0 \) and \( n \in \mathbb{N} \). These facts together with Lemma 5.1 imply that

\[
(1 - \frac{m}{\ell^*}) \int_{\Omega} \Phi(|\nabla tv_n|) \leq (t - 1) \left( 1 - \frac{1}{\ell^*} \right) \int_{\Omega} fv_n + \left( 1 - \frac{\ell^*}{\ell} \right) \int_{\Omega} \Phi(|\nabla v_n|).
\]

Using the above estimate, Lemma 5.1 and the fact that \((v_n)\) is bounded, we obtain

\[
\min(t^f, t^m) \left( 1 - \frac{m}{\ell^*} \right) \int_{\Omega} \Phi(|\nabla v_n|) \leq (t - 1) \left( 1 - \frac{1}{\ell^*} \right) \int_{\Omega} fv_n + C
\]

holds for some \( C > 0 \). These inequalities give us

\[
\min(t^f, t^m) \left( 1 - \frac{m}{\ell^*} \right) \int_{\Omega} \Phi(|\nabla v_n|) \leq (t - 1) \left( 1 - \frac{1}{\ell^*} \right) S^{\frac{1}{2}} ||f||_{(\ell^*)'} ||v_n|| + C.
\]

It is no hard to verify that \( ||v_n|| \geq c > 0 \) for any \( n \in \mathbb{N} \). Using Proposition 5.1 we get

\[
\min(t^f, t^m) \leq a_n(1) + C
\]

holds for any \( t \geq 0 \) where \( C = C(l, m, \ell^*, \Omega, a, b) > 0 \) where \( a_n(1) \) denotes a quantity that goes to zero as \( n \to \infty \). Here was used the fact \( v_n \to 0 \) in \( L^1(\Omega) \). This estimate does not make sense for any \( t > 0 \) big enough. Hence \( v \neq 0 \) as claimed. Hence \( v \) is in \( \mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \).

Next, we shall prove that \( v_n \to v \) in \( W^1_{0, \Phi}(\Omega) \). The proof follows arguing by contradiction. Assume that \( \liminf_{n \to \infty} \int_{\Omega} \Phi(|\nabla v_n - \nabla v|) \geq \delta > 0 \). Recall that \( \Psi : \mathbb{R} \to \mathbb{R} \) given by

\[
t \mapsto \Psi(t) := \Phi(t) - \frac{1}{\ell^*} \phi(t) t^2
\]

is a convex function for each \( t \geq 0 \). The Brezis-Lieb Lemma for convex functions says that

\[
\lim_{n \to \infty} \int_{\Omega} \Psi(|\nabla v_n|) - \Psi(|\nabla v_n - v|) = \int_{\Omega} \Psi(|\nabla v|)
\]

In particular, the last estimate give us

\[
\int_{\Omega} \Psi(|\nabla v|) < \liminf_{n \to \infty} \int_{\Omega} \Psi(|\nabla v_n|).
\]

Since \( v \in \mathcal{N} \) there exists unique \( t_0 \) in \((0, \infty)\) such that \( t_0 v \in \mathcal{N}^- \). It is easy to verify that

\[
\int_{\Omega} \Psi(|\nabla t_0 v|) < \liminf_{n \to \infty} \int_{\Omega} \Psi(|\nabla t_0 v_n|).
\]

In view of (4.43), this implies that

\[
\alpha^- \leq J(t_0 v) = \int_{\Omega} \Psi(|\nabla t_0 v|) - \left( 1 - \frac{1}{\ell^*} \right) t_0 v_n \leq \liminf_{n \to \infty} \int_{\Omega} \Psi(|\nabla t_0 v_n|) - \left( 1 - \frac{\ell^*}{\ell} \right) t_0 f v_n = \liminf_{n \to \infty} J(t_0 v_n) \leq \liminf_{n \to \infty} J(v_n) = \alpha^-.
\]

This is a contradiction proving that \( v_n \to v \) in \( W^1_{0, \Phi}(\Omega) \). Therefore \( v \) is in \( \mathcal{N}^- \). This follows from the strong convergence and the fact that \( t = 1 \) is the unique maximum
point for the fibering map $\gamma_v$ for any $v \in \mathcal{N}^\circ$. Hence using the same ideas discussed in the proof of Theorem 1.1 we infer that

$$\alpha^- \leq J(v) \leq \liminf J(v_n) = \alpha^-.$$  

In particular, $\alpha^- = J(v)$ and

$$\lim \int_{\Omega} \Phi(|\nabla v_n|) - \frac{1}{\ell^*} \phi(|\nabla v_n|) |\nabla v_n|^2 = \int_{\Omega} \Phi(|\nabla v|) - \frac{1}{\ell^*} \phi(|\nabla v|) |\nabla v|^2.$$  

Hence, $J(v) \geq \delta_1 > 0$. This finishes the proof of Theorem 1.2. \hfill \square

4.3. **The proof of Theorem 1.3.** In view of Theorems 1.1 and 1.2 there are $u \in \mathcal{N}^+$ and $v \in \mathcal{N}^-$ in such way that $J(u) = \inf_{w \in \mathcal{N}^+} J(w)$ and $J(v) = \inf_{w \in \mathcal{N}^-} J(w)$.

Using that $0 < ||f||_{(c')'} < \Lambda := \min\{\Lambda_1, \Lambda_2\}$ where $\Lambda_1, \Lambda_2 > 0$ are given by Theorem 1.1 and Theorem 1.2 we stress that $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$.

Therefore, $u, v$ are nonnegative solutions to the elliptic problem (1.1), ($u$ being a ground state solution), whenever $0 < ||f||_{(c')} < \Lambda$. This completes the proof. \hfill \square

5. **Appendix.** The reader is referred to [1, 22] regarding Orlicz-Sobolev spaces. The usual norm on $L_{\Phi}(\Omega)$ is (Luxemburg norm),

$$||u||_{\Phi} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \Phi \left( \frac{u(x)}{\lambda} \right) dx \leq 1 \right\}$$

and the Orlicz-Sobolev norm of $W^{1,\Phi}(\Omega)$ is

$$||u||_{1,\Phi} = ||u||_{\Phi} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{\Phi}.$$  

Recall that

$$\bar{\Phi}(t) = \max_{s \geq 0} \left\{ ts - \Phi(s) \right\}, \quad t \geq 0.$$  

It turns out that $\Phi$ and $\bar{\Phi}$ are N-functions satisfying the $\Delta_2$-condition, (cf. [22, p 22]). In addition, $L_{\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive, Banach spaces.

Using the Poincaré inequality for the $\Phi$-Laplacian operator it follows that

$$||u||_{\Phi} \leq C \|\nabla u\|_{\Phi} \text{ for any } u \in W^{1,\Phi}_0(\Omega)$$

holds true for some $C > 0$, see Gossez [17, 18]. As a consequence, $||u|| := ||\nabla u||_{\Phi}$ defines a norm in $W^{1,\Phi}_0(\Omega)$, equivalent to $||.||_{1,\Phi}$. Let $\Phi_*$ be the inverse of the function

$$t \in (0, \infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{1/\alpha}} ds$$

which extends to $\mathbb{R}$ by $\Phi_*(t) = \Phi_*(-t)$ for $t \leq 0$. We say that a N-function $\Psi$ grow essentially more slowly than $\Phi_*$, we write $\Psi << \Phi_*$, if

$$\lim_{t \to \infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} = 0, \text{ for all } \lambda > 0.$$  

The compact embedding below (cf. [1, 12]) will be used in this paper:

$$W^{1,\Phi}_0(\Omega) \overset{cpt}{\hookrightarrow} L_{\Psi}(\Omega), \text{ if } \Psi << \Phi_*.$$
in particular, as $\Phi << \Phi_*$ (cf. [17, Lemma 4.14]),

$$W^{1,\Phi}_0(\Omega) \overset{cpt}{\hookrightarrow} L_\Phi(\Omega).$$

Furthermore, the following continuous embeddings hold (see [1, 12, 17])

$$W^{1,\Phi}_0(\Omega) \overset{cont}{\hookrightarrow} L_{\Phi_*}(\Omega),$$

$$L_{\Phi}(\Omega) \overset{cont}{\hookrightarrow} L^r(\Omega) \text{ and } L_{\Phi_*}(\Omega) \overset{cont}{\hookrightarrow} L^{r^*}(\Omega).$$

**Remark 5.1.** The function $\psi(t) = t^{r-1}, r \in [1, t^*)$ satisfies $\Psi << \Phi_*$ where $\Psi(t) = \int_0^t \psi(s)ds, t \in \mathbb{R}$. In other words, the function $\Psi$ grow essentially more slowly than $\Phi_*$. In fact, we easily see that

$$\lim_{t \to \infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} \leq \frac{\lambda^r}{r\Phi_*(1)} \lim_{t \to \infty} \frac{1}{t^{r-1}} = 0, \text{ for all } \lambda > 0.$$ 

In that case $W^{1,\Phi}_0(\Omega) \overset{cpt}{\hookrightarrow} L_\Phi(\Omega)$.

Now we refer the reader to [15, 23] for some elementary results on Orlicz and Orlicz-Sobolev spaces.

**Proposition 5.1.** Assume that $\phi$ satisfies $(\phi_1) - (\phi_3)$. Set

$$\zeta_0(t) = \min\{t^\ell, t^m\}, \quad \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \geq 0.$$ 

Then $\Phi$ satisfies

$$\zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \quad \rho, t > 0,$$

$$\zeta_0(\|u\|_\Phi) \leq \int_{\Omega} \Phi(u)dx \leq \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega).$$

**Proposition 5.2.** Assume that $(\phi_1) - (\phi_3)$ holds. Define the function

$$\eta_0(t) = \min\{t^{\ell-2}, t^{m-2}\}, \quad \eta_1(t) = \max\{t^{\ell-2}, t^{m-2}\}, \quad t \geq 0.$$ 

Then the function $\phi$ verifies

$$\eta_0(t)\phi(\rho) \leq \phi(\rho t) \leq \eta_1(t)\phi(\rho), \quad \rho, t > 0.$$ 

**Proposition 5.3.** Assume that $\phi$ satisfies $(\phi_1) - (\phi_3)$. Set

$$\zeta_2(t) = \min\{t^{\ell^*}, t^{m^*}\}, \quad \zeta_3(t) = \max\{t^{\ell^*}, t^{m^*}\}, \quad t \geq 0$$

where $1 < \ell, m < N$ and $m^* = \frac{mN}{N-m}, \ell^* = \frac{\ell N}{N-\ell}$. Then

$$\ell^* \leq \frac{t^{2\Phi'_*(t)}}{\Phi_*(t)} \leq m^*, \quad t > 0,$$

$$\zeta_2(t)\Phi_*(\rho) \leq \Phi_*(\rho t) \leq \zeta_3(t)\Phi_*(\rho), \quad \rho, t > 0,$$

$$\zeta_2(\|u\|_{\Phi_*}) \leq \int_{\Omega} \Phi_*(u)dx \leq \zeta_3(\|u\|_{\Phi_*}), \quad u \in L_{\Phi_*}(\Omega).$$

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Received August 2017, revised August 2017.

E-mail address: marcos.leandro.carvalho@ufg.br
E-mail address: goncalves.jva@gmail.com
E-mail address: claudiney@ufg.br
E-mail address: ohmiyagaki@gmail.com