On the strong uniform consistency for relative error of the regression function estimator for censoring times series model

BOUHADJERA Feriel. 1,2

1 Université Badji-Mokhtar, Lab. de Probabilités et Statistique. BP 12, 23000 Annaba, Algérie.
2 Université du Littoral Côte d’Opale. Lab. de Math. Pures et Appliquées. IUT de Calais. 19, rue Louis David. Calais, 62228, France.

Abstract Consider a random vector $(X,T)$, where $X$ is $d$-dimensional and $T$ is one-dimensional. We suppose that the random variable $T$ is subject to random right censoring and satisfies the $\alpha$-mixing property. The aim of this paper is to study the behavior of the kernel estimator of the relative error regression and to establish its uniform almost sure consistency with rate. Furthermore, we have highlighted the covariance term which measures the dependency. The simulation study shows that the proposed estimator performs well for a finite sample size in different cases.

Keywords Censored data · Kernel estimate · Relative error regression · Strong mixing condition · Uniform almost sure consistency.

1 Introduction

Let $T$ be a strictly positive random variable (r.v.) representing the survival time of an individual taking part to an experimental study and let $X$ be a vector of covariate taking values in $\mathbb{R}^d$ that gives us information about the individuals (age, sex, . . . ). This paper is concerned with the nonparametric regression model:

$$T = m(X) + \varepsilon,$$

where $m$ is a regression function and $\varepsilon$ is a r.v. (corresponding to the residual) such that $E[\varepsilon|X] = 0$. Recall that, $m(\cdot)$ is usually modeled by the following minimization problem $E[(T - m(X))^2|X]$. However this loss function is unsuitable when the data contains some outliers, which is a relatively common case in practice. To avoid this drawback, another approach is to build an efficient estimator of $m(\cdot)$ given by the minimization of the mean squared relative error given by

$$E \left[ \left( \frac{T - m(X)}{T} \right)^2 | X \right], \text{ for } T > 0$$

(1.1)

This kind of model is called relative error regression (RER) which has been studied by many authors. We can refer to Narula and Wellington (1977), Makridakis et al. (1984) in the parametric case. Recall that Park and Stefanski (1998) showed that the solution of (1.1) satisfies, for $x \in \mathbb{R}^d$,

$$\frac{E[T^{-1}|X = x]}{E[T^{-2}|X = x]} \leq E[T|X = x].$$

(1.2)

In the nonparametric analysis, there exist some papers dealing with the estimation of the RER. Without pretending to exhaustivity, we quote Jones et al. (2008) considered kernel and local linear approach to estimate the regression function, Attouch et al. (2015) regarded the problem of estimating the regression function for spatial data and Demongeot et al. (2016) considered the case where the explanatory variable are of functional type of data. Shen and Xie (2013) obtained the strong consistency of the
regression estimator under \(\alpha\)-mixing data. Their result has been generalized to dependent case by Li et al. (2016).

In the case of incomplete data, for independent and identically distributed (i.i.d.) random variables under random right censoring, Guessoum and Ould Said (2008) studied the consistency and asymptotic normality of the kernel estimator of the regression function.

Many statistical results have been established by considering independent samples. It is then interesting to consider the more realistic situation when the observation are no longer i.i.d. This is for example the case, in clinical trials studies, not infrequently, patients from the same hospital have correlated survival times, due to unmeasured variables such as the skill or training of the staff or the quality of the hospital equipment (for more details, see: Lipsitz and Ibrahim, 2000).

Few papers deal with the regression function under censoring in the dependent case. We can cite Cai (1998) who studied the asymptotic properties of Kaplan-Meier’s estimator of censored dependent data and Cai (2001) who addressed the estimation of the distribution function for censored time series data. El Ghouch and Van Keilegom (2008, 2009) estimated the regression and conditional quantile functions by applying local linear method. Guessoum and Ould Said (2010, 2012) studied the consistency and the asymptotic normality of the kernel estimator for the regression function for censored dependant data.

Here we derive the uniform consistency result over a compact set with rate of the RER estimator for dependent case and censored data by highlighting the covariance term which does not appear in many papers. This paper is organized as follows. In Section 2, we recall some notations and definitions needed in our model. The hypotheses and main result are given in Section 3. Simulations study are drawn in Section 4. Finally, the proofs are relegated to Section 5 with some auxiliary results.

### 2 Presentation of the model

Consider a randomly right-censored model given by two non-negative stationary sequences, \((T_i)_{1\leq i\leq n}\) which represents the survival time with common unknown absolutely continuous distribution function (d.f.) \(F\) and \((C_i)_{1\leq i\leq n}\) the censoring time with common unknown d.f. \(G\). In this context, we observe the pairs \((Y_i, \delta_i)\), where

\[
Y_i = T_i \land C_i, \quad \delta_i = \mathbb{1}_{(T_i \leq C_i)}, \quad i = 1, \ldots, n
\]

and \(\mathbb{1}_E\) denotes the indicator function of the set \(E\). Let \((X_i)_{1\leq i\leq n}\) be a sequence of copies of the random vector \(X \in \mathbb{R}^d\) and denote by \(X_1, \ldots, X_d\) the components of \(X\). The study we perform below is then on the set of observations \((Y_i, \delta_i, X_i)_{1\leq i\leq n}\). Having in mind this kind of model, we define a pseudo-estimator of the relative error for the regression function, for all \(x \in \mathbb{R}^d\), by

\[
\hat{m}_n(x) = \frac{\sum_{i=1}^{n} \delta_i Y_i^{-1} G(Y_i) K_d(x - X_i)}{\sum_{i=1}^{n} \delta_i Y_i^{-2} G(Y_i) K_d(x - X_i)} =: \frac{\hat{r}_1(x)}{\hat{r}_2(x)},
\]

(2.1)

with \(\hat{r}_1(\cdot) = \frac{\hat{m}_n(\cdot)}{f(\cdot)}\) and \(\hat{m}_n(\cdot) = \int_{\mathbb{R}^d} y^{-d} f(y|\cdot)dy\), where \(K_d(\cdot) = K_d(\cdot/h_n)\) is a density function defined on \(\mathbb{R}^d\) and \(h_n\) is a sequence of positive numbers.

In this kind of model, it is well known that the empirical distribution is not a consistent estimator for the df \(G\). Therefore, Kaplan-Meier (1958) proposed a consistent estimator for survival function \(\overline{G}(\cdot) = 1 - G(\cdot)\) which is defined as

\[
\overline{G}_n(t) = \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_i}{n - i + 1}\right)^{\mathbb{1}_{(Y_i \leq t)}} \text{ if } t < Y_n
\]

(2.2)

otherwise,
where \( Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)} \) are the order statistics of the \( Y_i \)’s and \( \delta_i \) is the indicator of non-censoring. The properties of the K-M estimator for dependent variables can be found in Cai (1998, 2001). Then a calculable estimator of \( m(\cdot) \) is given by

\[
\hat{m}(x) = \frac{\hat{m}_1(x)}{\hat{m}_2(x)}
\]

where for \( \ell = 1, 2 \)

\[
\hat{m}_\ell(x) = \frac{\hat{r}_\ell(x)}{\hat{f}_X(x)} = \frac{\sum_{i=1}^n \delta_i Y_{(i)}^{\ell}}{\hat{G}_n(Y_{(i)})} K_{\delta}(x - X_i)
\]

\[
\hat{r}_\ell(x) = \sum_{i=1}^n K_{\delta}(x - X_i)
\]

and \( \hat{f}_X(\cdot) \) is the kernel estimator of the marginal density function \( f_X(\cdot) \).

In order to define the \( \alpha \)-mixing notion, we will use the following notations. Denote by \( \mathcal{F}_k^Z(Z) \) the \( \sigma \)-algebra generated by \( \{Z_j, 1 \leq j \leq k\} \).

**Definition** Let \( \{Z_i, i = 1, 2, \ldots\} \) denote a sequence of rv’s. Given a positive integer \( n \), set

\[
\alpha(n) = \sup \left\{ \|P(A \cap B) - P(A)P(B)\| : A \in \mathcal{F}_k^Z(Z) \text{ and } B \in \mathcal{F}_{k+n}^Z(Z), k \in \mathbb{N}^* \right\}.
\]

The sequence is said to be \( \alpha \)-mixing (strong mixing) if the mixing coefficient \( \alpha(n) \to 0 \) as \( n \to \infty \).

Many processes fulfill the strong mixing property. We quote here, the usual ARMA processes which are geometrically strongly mixing, i.e., there exist \( \rho \in (0, 1) \) and \( a > 0 \) such that, for any \( n \geq 1 \), \( \alpha(n) \leq a \rho^n \) (see, e.g., Jones (1978)). The threshold models, the EXPAR models (see, Ozaki (1979)), the simple ARCH models (see Engle (1982)), their GARCH extension (see Bollerslev (1986)) and the bilinear Markovian models are geometrically strongly mixing under some general ergodicity conditions. We suppose that the sequences \( \{T_i, i \geq 1\} \) and \( \{C_i, i \geq 1\} \) are \( \alpha \)-mixing with coefficients \( \alpha_1(n) \) and \( \alpha_2(n) \), respectively. Cai (2001) Lemma 2 showed that \( \{Y_i, i \geq 1\} \) is then strongly mixing, with coefficient

\[
\alpha(n) = 4 \max(\alpha_1(n), \alpha_2(n)).
\]

From now on, we suppose that \( \{(Y_i, \delta_i, X_i) : i = 1, \ldots, n\} \) is strongly mixing with mixing’s coefficient \( \alpha(n) \) such that \( \alpha(n) = O(n^{-\nu}) \) for some \( \nu > 3 \).

**3 Hypotheses and main results**

Let \( \mathcal{C} \) be a compact set in \( \mathbb{R}^d \). We define the endpoint of \( F \) by \( \tau_F = \sup\{x, \bar{F}(x) > 0\} \) and we assume that \( \tau_F < \infty \) and \( G(\tau_F) > 0 \).

All along the paper, we denote by \( r_\ell(x) = \int_{\mathbb{R}^d} t^{-\ell} f_{\tau,X}(x,t)dt \) where \( f_{\tau,X}(\cdot, \cdot) \) is the joint density of \( (T, X) \). For any generic strictly positive constant \( M \), we assume

\[
\forall T > 0, \exists M, \text{ such that } M \geq T^{-\ell}, \text{ for } \ell = 1, 2.
\]

In order to present our result, we have to introduce the following notations and hypotheses.

**H1.** The bandwidth \( h_n \) satisfy \( \lim_{n \to +\infty} \frac{n h_n^d}{\log n} = +\infty \).

**H2.** \( \lim_{n \to +\infty} \frac{\log n}{n h_n^{2d/v}} = 0 \).
\(\exists \psi > 0, \exists c > 0, \text{ such that } cn^\gamma \leq h_n^d, \text{ for all } \nu > 3.\)

**K1.** The kernel \(K_d\) is bounded and
\[\forall (z_1, z_2) \in C^2, |K_d(z_1) - K_d(z_2)| \leq \|z_1 - z_2\|^\gamma \text{ for } \gamma > 0,\]

**K2.** \[\int_{\mathbb{R}^d} \|t\| K_d(t) dt < +\infty, \text{ with } \|t\| = \sum_{i=1}^n |t_i|,\]

**K3.** \[\int_{\mathbb{R}^d} \|t\| K_d^2(t) dt < +\infty \text{ and } \int_{\mathbb{R}^d} K_d^2(t) dt < +\infty.\]

**D1.** The function \(r_\ell(x)\) defined in (2.1) is continuously differentiable and \[\sup_{x \in C} \left| \frac{\partial r_\ell(x)}{\partial x_i} \right| < +\infty \text{ for } i = 1, \ldots, d,\]

**D2.** The function \(\theta_\ell(x) := \int_{\mathbb{R}^d} \frac{t}{G(t)} f_{X,T}(x, t) dt\) is continuously differentiable and \[\sup_{x \in C} \left| \frac{\partial \theta_\ell(x)}{\partial x_i} \right| < +\infty\]
for \(i = 1, \ldots, d, \text{ and } \ell = 2, 3, 4.\)

**D3.** The joint density \(f_{\ell,j}(\cdot, \cdot)\) of \((X_i, X_j)\) exists and satisfies for \(\ell = 1, 2\)
\[\sup_{i,j} \sup_{u,v \in C} |f_{\ell,i,j}(u,v) - f_{\ell,i}(u)f_{\ell,j}(v)| \leq C < \infty,\]
where \(M\) is a positive constant.

### 3.1 Some comments on the hypotheses

Hypotheses **H1** and **H2** are very common in both independent and dependent cases. Furthermore, **H3** permits to estimate the covariance term. The hypotheses concerning the kernel \(K\) are technicals and it is well-known that it does not improve the quality of the estimation. The **D1** intervenes in Lemma 1, however hypotheses **D2** and **D3** intervene in Lemma 4 to deal with the covariance term.

### 3.2 Bandwidth selection

Note that: It is well-known that the choice of the kernel does not affect the quality of the estimation. In contrast, the bandwidth parameter \(h_n\) has a great influence on the quality of the estimator. A parameter that is too small causes the appearance of artificial details in the graph of the estimator, and for a large enough value of the bandwidth \(h_n\), the majority of the features is on the contrary erased. The choice of the bandwidth \(h_n\) is therefore a central question in nonparametric estimation. Recall that in the literature, there are mainly three methods, the "rule of thumb", "plug-in" and "cross-validation". Each method has its merits and drawbacks. We point out that the latter is very popular and its main idea is to minimize the following criterion

\[CV_{h_n} = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \hat{m}_{-i,h_n}(X_i))^2\]

where \(\hat{m}_{-i,h_n}(X_i)\) is the estimator of \(m(\cdot)\) obtained by raising the observation \((X_i, Y_i)\) in the sense of practical point of view. Even if the latter has the drawbacks that it is very variable and can give an underestimation of \(h_{opt}\), it remains the most common used method. In our entire simulation study, we adopt the cross-validation method (see: Sect. 4). The following theorem establishes the almost sure uniform convergence of \(\hat{m}\) towards \(m\).
Theorem 1 Under hypotheses $H1-3$, $D1-3$, we have, for $\ell = 1, 2$,

$$\sup_{x \in C} |\hat{m}(x) - m(x)| = O_{a.s.} \left( \sqrt{\frac{\log n}{nh_n^2}} + \sqrt{\frac{\log n}{nh_n^{2d/v}}} \right) + O(h_n) \quad \text{as} \quad n \to \infty.$$  

Remark 1 We point out that in our result we highlight the covariance term which gives us how the dependency intervenes. This point is rarely given in the dependent case of many papers. In the latter authors made an additional hypotheses to vanish the covariance term to get an analogous result as in the independent case.

4 Simulation study

The aim of this part is to examine the performance of our estimator $\hat{m}(x)$ by considering some fixed size particular cases. We do it by varying the dependency rate and the censoring percentage (C.P.). We compare the efficiency of the implemented method to the classical regression (CR) estimator defined in Guessoum and Ould Saïd (2010).

In the next paragraph, we recall a result of Port (1994) which permits to calculate the theoretical regression function that will be used throughout this section (see formula (4.1) below).

Proposition 1 Let $q_1(X)$ and $q_2(X)$ be two random variables with means: $\mu_1$ and $\mu_2$ and variances: $\nu_1$ and $\nu_2$ respectively, and covariance $\nu_{12}$. Let $(X_i)_{1 \leq i \leq n}$ be an i.i.d. sequence of r.v. and defined by

$$\hat{\Sigma}_1 = \frac{1}{n} \sum_{1}^{n} q_1(X_i) \quad \text{and} \quad \hat{\Sigma}_2 = \frac{1}{n} \sum_{1}^{n} q_2(X_i)$$

and $\hat{R} = \frac{\hat{\Sigma}_1}{\hat{\Sigma}_2}$, then the second order approximation of $E[\hat{R}]$ is

$$E[\hat{R}] \approx \frac{\mu_1}{\mu_2} + \frac{1}{n} \left( \frac{\mu_1 \nu_2}{\mu_2^2} - \frac{\nu_{12}}{\mu_2^2} \right).$$

Algorithm 1

Require: $0 < \rho < 1$, $X_0 \sim N(1, 0.1)$, $\varepsilon \sim N(0, 1)$, $C \sim N(3 + a, 1)$ with $a$ being a parameter that adapts the censorship percentage C.P.

Step 1: We consider the strong mixing two-dimensional process generated by

$$\begin{align*}
X_i &= c + \rho X_{i-1} + \sqrt{1 - \rho^2} \varepsilon_i, \\
T_i &= X_{i+1}, \quad i = 1, \ldots, n
\end{align*}$$

Step 2: Given $X_1 = x$, we have $T_1 = c + \rho x + \sqrt{1 - \rho^2} \varepsilon_1$. Using Port property (see proposition 1) the theoretical function becomes

$$m(x) = E[T^{-1}\{X = x\}] = c + \rho x + \frac{1 - \rho^2}{c + \rho x}. \quad (4.1)$$

Step 3: Determine $Y_i = T_i \land C_i$ and $\delta_i = I_{\{T_i \leq C_i\}}$, which gives the observed sample $\{(X_i, Y_i, \delta_i), 1 \leq i \leq n\}$.

Step 4: The K-M estimator of $G_1(\cdot)$ is calculated from (2.2).

Step 5: The Gaussian kernel $K(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$ is used as kernel function for the estimator and we choose the optimal bandwidth $h_{opt}$ by the cross validation method (see Subsection 3.2) from $[0.01, 2]$ by step of 0.01 and satisfying $H3$.

Output: Calculate the RER estimator given by (2) for $x \in [1, 4]$ and $h_{opt}$.
4.1 Linear case

In this subsection, we observe the finite sample performance of our estimator (RER) for weak and strong dependency when the theoretical function is of linear form.

4.1.1 Weak dependency

- **Effect of sample size:** It is easy to see from Figure 1 that the quality of fit is better when \( n \) increases for a fixed C.P. and \( \rho \).

\[ m(x), \hat{m}(x) \text{ with } c = 3, \rho = 0.1 \text{ and C.P. } \approx 40\% \text{ for } n = 100, 300 \text{ and } 500 \text{ respectively.} \]

- **Effect of C.P.:** To visualize the global performance of the RER estimator under censorship, we set \( \rho = 0.1 \) and vary the C.P. In this case, there is more variation in the resulting estimator, but generally remains close to the theoretical curve even for a high C.P. (Figure 2 displays the results). In conclusion, our estimator is still resistant to the effect of censorship when dependency is weak.

\[ m(x), \hat{m}(x) \text{ with } c = 3, \rho = 0.1 \text{ and } n = 300 \text{ for C.P. } \approx 10, 33 \text{ and } 72\% \text{ respectively.} \]

4.1.2 Strong dependency

- **Effect of sample size:** For the case of highly dependent data (\( \rho = 0.7 \)) and for a fixed C.P., we can observe from Figure 3 that the RER estimator is adjusted to the theoretical curve when \( n \) rises.

- **Effect of C.P.:** We see clearly that the quality of fit is better for large sample size and low percentage of censoring (see Figure 4).
4.2 Nonlinear case

We consider now, three nonlinear functions:

\[ T_i = 1 + \cos\left(\frac{\pi}{2} X_i\right), \quad \text{Cosinus model,} \]
\[ T_i = \exp(\rho^2 X_i), \quad \text{Exponential model,} \]
\[ T_i = \frac{1}{X_i}, \quad \text{Inverse model.} \]

Figure 5 shows that the quality of the fit is good as in linear model. Clearly, we see that the adjustment improves when \( n \) increases.
4.3 Effect of outliers

To show the robustness of our approach, we generate the case where the data contains outliers. To create this outlier effect, 20 values of this sample are multiplied by a factor called M.F.. From Figure 6, we can see that our estimator is close to the theoretical curve knowing that we observe only 70% of the true values. Then, it is absolutely clear that our estimator is resistant in the presence of outliers.

![Figure 6](image)

**Fig. 6** $m(x)$, $\hat{m}(x)$, with $c = 1$, $\rho = 0.7$, C.P. $\approx 30\%$ $n = 300$ and M.F. $= 10, 50$ and $100$ respectively.

4.4 Effect of contamination of the random error $\varepsilon$:

We take the same algorithm as before by changing step 1 which becomes:

- Step 1'. $\varepsilon_i \sim (1 - \beta)\eta_1 + \alpha \eta_2$ where $\eta_1 \sim \mathcal{N}(0,1)$ and $\eta_2 \sim \mathcal{N}(0, \lambda)$. We choose the level of contamination $\beta = 0.01, 0.05$ and $0.1$ and the magnitude of contamination $\lambda = 3$ generally.

We observe from (Figure 7) that the quality of the adjustment to the theoretical function deteriorates when the level of contamination $\alpha$ becomes higher.

![Figure 7](image)

**Fig. 7** $m(x)$, $\hat{m}(x)$, $c = 1$, $\rho = 0.7$, $n = 300$, C.P. $50\%$ and $\beta = 0.01, 0.05$ and $0.1$ respectively.
4.5 Comparaison study

To show the efficiency of the RER estimator, we carry out a comparative study in which we consider the classical regression (CR) estimator defined in Guessoum and Ould Saïd (2010) by

\[
\mu_n(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i Y_i}{G_n(Y_i)} K_d(x - X_i)}{\sum_{i=1}^{n} K_d(x - X_i)},
\]

(4.2)

for weak and strong dependency.

4.5.1 Weak dependency

- **Effect of C.P.:** We fix \( n \) and we vary the censoring rate. We can notice clearly (from Figure 8) that the RER estimator is near to the theoretical curve whereas the CR curve is distant from the true curve when C.P. increases.

![Figure 8](image)

**Fig. 8** \( m(x), \hat{m}(x), \mu_n(x) \) with \( c = 3, \rho = 0.1, n = 300 \) and C.P. \( \approx 10, 50 \) and 80% respectively.

- **Effect of outliers:** We fix \( n, \) C.P. and we vary the M.F. It can be reported from Figure 9 that the RER estimator is overlapped on the true curve in contrast with the CR estimator which is significantly affected by the M.F. when the dependency is weak.

![Figure 9](image)

**Fig. 9** \( m(x), \hat{m}(x), \mu_n(x) \) with \( c = 3, \rho = 0.1, n = 300, \) C.P. \( \approx 35 \) and M.F. \( =10, 25 \) and 50% respectively.
4.5.2 Strong dependency

- **Effect of C.P.**: We fix $\rho$, $n$ and we vary the C.P. to examine the effect of censorship on both RER and CR estimators when the dependency is strong. We can observe from Figure 10 that the RER estimator

![Figure 10](image1)

Fig. 10 $m(x), \tilde{m}(x), \mu_n(x)$ with $c = 1$, $\rho = 0.7$, $n = 300$ and C.P. $\approx 10, 50$ and 80% respectively.

- **Effect of outliers**: We fix $\rho$, $n$, C.P. and we vary the M.F. to evaluate the effect of outliers on both estimators when the dependency is high. As expected, our estimator remains resistant to outliers under a high dependency unlike that of classical regression which is more distant when the M.F. becomes large see Figure 11.

![Figure 11](image2)

Fig. 11 $m(x), \tilde{m}(x), \mu_n(x)$ with $c = 1$, $\rho = 0.7$, $n = 300$, C.P. $\approx 35$ and M.F.=10, 25 and 50% respectively.

4.6 Discussion

In this paper, an estimator for the relative error regression function on the multivariate case has been proposed, when the data are dependent and are subject to censoring. After analyzing and comparing with the CR estimator, we have the following remarks. As expected, the asymptotic behavior of the RER estimator is better for a weak dependency (a small value of $\rho$) and a low censorship rate which is confirmed by the numerical study in Sect. 4, where we show how the quality of the estimation is influenced by several parameters (C.P., $\rho$, M.F., $n$).

Now, concerning the behavior of the RER estimator compared to the CR estimator, we can remark that the comportment of the RER remained almost unchanged in all our results in comparison with the CR estimator, which is significantly affected by the presence of outliers and censorship in the sample. Another interesting remark related to dependency is the fact that with small $\rho$ the estimator remains resistant.
5 Technical lemmas and proofs

We split the proof of Theorem 1 into following Lemmas 1-4.

**Lemma 1** Under hypotheses $K_2$ and $D_1$, for $\ell = 1, 2$, we have

$$\sup_{x \in \mathcal{C}} |\mathbb{E}[\hat{r}_\ell(x)] - r_\ell(x)| = O_{a.s.}(h_n) \quad \text{as} \quad n \to \infty.$$ 

*Proof* The proof is standard in the sense that it is not affected by the dependency structure. Using the properties of conditional expectation, a change of variable, and Taylor’s expansion $\zeta \in |x - h_n t, x|$, we have under hypotheses $K_2, D_1$ and for $\ell = 1, 2$

$$\mathbb{E}[\hat{r}_\ell(x)] - r_\ell(x) = h_n^{-d} \int_{\mathbb{R}^d} K_d(x - u)m_\ell(u)f(u)du - r_\ell(x)$$

$$= h_n^{-d} \int_{\mathbb{R}^d} K_d(x - u)[r_\ell(u) - r_\ell(x)]du$$

$$= \int_{\mathbb{R}^d} K_d(t)[r_\ell(x - h_n t) - r_\ell(x)]dt$$

then

$$\sup_{x \in \mathcal{C}} |\mathbb{E}[\hat{r}_\ell(x)] - r_\ell(x)| \leq h_n \sup_{x \in \mathcal{C}} \left| \int_{\mathbb{R}^d} K_d(t) \left( t_1 \frac{\partial r_\ell(\zeta)}{\partial x_1} + \cdots + t_d \frac{\partial r_\ell(\zeta)}{\partial x_d} \right) dt \right|.$$ 

**Lemma 2** Under hypotheses $H_1$ and $K_1-K_3$, for $\ell = 1, 2$, we have

$$\sup_{x \in \mathcal{C}} |\hat{r}_\ell(x) - \tilde{r}_\ell(x)| = O_{a.s.} \left( \left( \frac{\log 2n}{n} \right)^{1/2} \right) \quad \text{as} \quad n \to \infty.$$ 

*Proof*

$$|\hat{r}_\ell(x) - \tilde{r}_\ell(x)| = \left| \frac{1}{nh_n^d} \sum_{i=1}^{n} \delta_i Y_i^{-\ell} K_d(x - X_i) \left( \frac{1}{G_n(Y_i)} - \frac{1}{G(Y_i)} \right) \right|$$

$$= \frac{1}{nh_n^d} \left| \sum_{i=1}^{n} T_i^{-\ell} K_d(x - X_i) \left( \frac{1}{G_n(T_i)} - \frac{1}{G(T_i)} \right) \right|$$

$$\leq \sup_{1 \leq \ell \leq \tau_F} \left| \tilde{G}_n(t) - G(t) \right| \frac{1}{nh_n^d} \sum_{i=1}^{n} |T_i|^{-\ell} K_d(x - X_i).$$

From Cai (2001), under Hypotheses $H_1$ and $K_1-K_3$, we have for $\ell = 1, 2$

$$\sup_{x \in \mathcal{C}} |\hat{r}_\ell(x) - \tilde{r}_\ell(x)| \leq \frac{M}{G^2(\tau_F)} \frac{1}{\sqrt{n}} \frac{\log \log n}{\sqrt{n}}.$$ 

To this step, we introduce the following lemma (Ferraty and Vieu (2006) Proposition A.11 ii), p.237).

**Lemma 3** (Fuk-Nagaev) Let $\{U_i, i \geq 1\}$ be a sequence of real rv’s, with strong mixing coefficient $\alpha(n) = O(n^{-v})$, $v > 1$ such that $\forall n \in \mathbb{N}, \forall i \in \mathbb{N}, 1 \leq i \leq n |U_i| < +\infty$. Then for each $\varepsilon > 0$ and for each $r > 1$

$$P \left( \left| \sum_{i=1}^{n} U_i \right| \geq \varepsilon \right) \leq C \left( 1 + \frac{\varepsilon^2}{rS_n^2} \right)^{-r/2} + \frac{nC}{r} \left( \frac{2r}{\varepsilon} \right)^{v+1}$$

where $S_n^2 = \sum_{i,j} Cov(U_i, U_j)$.

In the following lemma we establish the asymptotic expression for the variance and covariance of the estimator $\hat{m}(x)$. 
Lemma 4 Under hypotheses H2, H3, K1-K3 and D1-D3, we have for \( \ell = 1, 2 \)

\[
\sup_{x \in C} |\hat{r}_\ell(x) - \mathbb{E}[\hat{r}_\ell(x)]| = O_{a.s.} \left( \sqrt{\frac{\log n}{nh_n^d}} + \sqrt{\frac{\log n}{nh_n^{2d/n}}} \right) \text{ as } n \to \infty.
\]

Proof Recall that \( C \) is a compact set, then it admits a covering \( S \) by a finite number \( s_n \) of balls \( B_k(x^*_k, a^d_n) \) centred at \( x^*_k = (x^*_{1,k}, \ldots, x^*_{d,k}) \), \( 1 \leq k \leq s_n \). Then for all \( x \in C \) there exists \( k \) such that \( \|x - x^*_k\| \leq a^d_n \) where \( a_n \) verifies \( a^d_n \times h_n^{\gamma(\gamma + 1)} n^{-\frac{1}{2}} \). With \( \gamma \) is the Lipschitz condition in hypothesis K1. Since \( C \) is bounded then there exist a constant \( M > 0 \) such that \( s_n \leq Ma_n^{-d} \).

Let for \( x \in C \) and \( \ell = 1, 2 \) the given set

\[
A_{\ell,i}(x) = (nh_n^d)^{-1} \left[ \frac{\delta}{G(Y_i)} K_d(x - X_i) - \mathbb{E} \left( \frac{\delta}{G(Y_i)} K_d(x - X_i) \right) \right],
\]

then

\[
\sum_{i=1}^n A_{\ell,i}(x) = \hat{r}_\ell(x) - \mathbb{E}[\hat{r}_\ell(x)],
\]

that we decompose as follows

\[
\sum_{i=1}^n A_{\ell,i}(x) = \sum_{i=1}^n \tilde{A}_{\ell,i}(x) + \sum_{i=1}^n A_{\ell,i}(x^*_k),
\]

from where

\[
\sup_{x \in C} \left| \sum_{i=1}^n A_{\ell,i}(x) \right| \leq \max_{1 \leq k \leq s_n} \sup_{x \in B_k} \left| \sum_{i=1}^n \tilde{A}_{\ell,i}(x) \right| + \max_{1 \leq k \leq s_n} \sum_{i=1}^n A_{\ell,i}(x^*_k)
\]

\[
= B_1 + B_2.
\]

We start by treating the first term \( B_1 \)

\[
\left| \sum_{i=1}^n \tilde{A}_{\ell,i}(x) \right| = \left| [\hat{r}_\ell(x) - \hat{r}_\ell(x^*_k)] - \mathbb{E} [\hat{r}_\ell(x) - \hat{r}_\ell(x^*_k)] \right|
\]

\[
= \frac{1}{nh_n^d} \sum_{i=1}^n \left| \frac{\delta}{G(Y_i)} (K_d(x - X_i) - K_d(x^*_k - X_i)) \right|
\]

\[
+ \frac{1}{h_n^d} \mathbb{E} \left[ \frac{\delta}{G(Y_i)} (K_d(x - X_1) - K_d(x^*_k - X_1)) \right]
\]

\[
\leq \frac{1}{nh_n^d} \sum_{i=1}^n \left| T_{\ell} \right|^{-\ell} |K_d(x - X_i) - K_d(x^*_k - X_i)|
\]

\[
+ \frac{1}{h_n^d} \mathbb{E} \left[ \frac{\left| T_{\ell} \right|^{-\ell}}{G(T_{\ell})} |K_d(x - X_1) - K_d(x^*_k - X_1)| \right]
\]

\[
= D_{1,\ell}(x) + D_{2,\ell}(x),
\]

with

\[
\sup_{x \in B_k} D_{1,\ell}(x) \leq \frac{M}{G(T_{\ell})} \frac{1}{h_n^d} \sup_{x \in C} |K_d(x - X_i) - K_d(x^*_k - X_i)|
\]

\[
\leq \frac{M}{h_n^d G(T_{\ell})} \|x - x^*_k\|^{\gamma}
\]

\[
\leq \frac{Ca_n^{d\gamma}}{h_n^{d\gamma}}.
\]
In the same manner, we have,
\[ \sup_{x \in B_k} D_{2,\ell}(x) \leq \frac{C_{d,\gamma}}{h_n^{d+\gamma}}, \]
then
\[ \sup_{x \in B_k} \left| \sum_{i=1}^{n} \tilde{A}_{\ell,i}(x) \right| = \sup_{x \in B_k} D_{1,\ell}(x) + \sup_{x \in B_k} D_{2,\ell}(x) \leq \frac{2C_{d,\gamma}}{h_n^{d+\gamma}} \leq \frac{C h_n^{d(\gamma+\frac{1}{2})} h^{-\frac{1}{2}}}{h_n^{d+\gamma}} = \frac{C}{\sqrt{h_n^d}} h_n^{d(d-1)}, \]
which allows to
\[ B_1 = \max_{1 \leq k \leq s} \sup_{x \in B_k} \left| \sum_{i=1}^{n} \tilde{A}_{\ell,i}(x) \right| = O \left( \frac{1}{\sqrt{h_n^d}} \right). \] (5.1)

To proceed to the determination of the second term \( B_2 \), we will use Lemma 3. Let
\[ U_i = U_{i,k} = n h_n^d A_{\ell,i}(x_k^*) = \delta_i Y_i \cdot K_d(x_k^* - X_i) - \mathbb{E} \left[ \delta_i Y_i \cdot K_d(x_k^* - X_i) \right]. \]

To apply Lemma 4, we have to calculate first
\[ S_n^2 = \sum_i \sum_j |\text{Cov}(U_i, U_j)| = \sum_{i \neq j} |\text{Cov}(U_i, U_j)| + n \text{Var}(U_1) \]
\[ =: \mathcal{V} + n \text{Var}(U_1). \] (5.2)

On the one hand, we have to start by considering
\[ \text{Var}(U_1) = \text{Var} \left[ \frac{\delta_1 Y_1 - \ell}{G(Y_1)} K_d(x_k^* - X_1) \right] \]
\[ = \mathbb{E} \left[ \frac{\delta_1 Y_1 - 2\ell}{G(Y_1)} K_d^2(x_k^* - X_1) \right] - \mathbb{E}^2 \left[ \frac{\delta_1 Y_1 - \ell}{G(Y_1)} K_d(x_k^* - X_1) \right] \]
\[ =: \mathcal{R}_1 - \mathcal{R}_2. \]

For \( \mathcal{R}_1 \), using the conditional expectation proprieties and a change of variables, we get
\[ \mathcal{R}_1 \leq h_n^d \int_{\mathbb{R}^d} K_d^2(t) \theta_t(x_k^* - h_n t) dt, \]
by a Taylor expansion and from Hypotheses D2 and K3, we obtain
\[ \mathcal{R}_1 = O(h_n^d). \] (5.3)

For \( \mathcal{R}_2 \), under hypothesis D1, we have
\[ \sqrt{\mathcal{R}_2} = \int_{\mathbb{R}^d} K_d(x_k^* - u) r(u) du, \]
using again a change of variable and a Taylor expansion around \( x_k^* \), we have
\[ \mathcal{R}_2 = O(h_n^{2d}). \] (5.4)

Then from (5.3) and (5.4), we get
\[ n \text{Var}(U_1) = n(\mathcal{R}_1 - \mathcal{R}_2) = O(n h_n^d). \] (5.5)
On the other hand, 
\[
|\text{Cov}(U_i, U_j)| = |E[U_i U_j]|
\]
\[
= \mathbb{E} \left[ \frac{\delta_i \delta_j Y_{{i}^{\ell}} Y_{{j}^{\ell}}}{G(Y_i) G(Y_j)} K_d(x^*_k - X_i) K_d(x^*_k - X_j) \right]
\]
\[
- \mathbb{E} \left[ \frac{\delta_i Y_{{i}^{\ell}}}{G(Y_i)} K_d(x^*_k - X_i) \right] \mathbb{E} \left[ \frac{\delta_j Y_{{j}^{\ell}}}{G(Y_j)} K_d(x^*_k - X_j) \right]
\]
\[
\leq h_n^{2d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_d(t) K_d(s) |f_{i,j}(x_k^* - h_n t, x_k^* - h_n s) - f_i(x_k^* - h_n t) f_j(x_k^* - h_n s)| dt ds,
\]
which yields, under Hypothesis D3
\[
|\text{Cov}(U_i, U_j)| = O(h_n^{2d}),
\] (5.6)
uniformly on \( i \) and \( j \).

Now to evaluate the asymptotic behaviour of \( \mathcal{V} \) following the decomposition of Marsy (1986), we define the sets:
\[
E_1 = \{(i, j) \text{ such that } 1 \leq |i - j| \leq \beta_n\} \text{ and } E_2 = \{(i, j) \text{ such that } \beta_n + 1 \leq |i - j| \leq n - 1\}
\]
where \( \beta_n \to \infty \) as \( n \to \infty \) at a slow rate, that is \( \beta_n = o(n) \). Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be the sums of covariances over \( E_1 \) and \( E_2 \), respectively.

\[
\mathcal{V} = \sum_{i=1}^{n} \sum_{E_1} \text{Cov}(U_i, U_j) + \sum_{i=1}^{n} \sum_{E_2} \text{Cov}(U_i, U_j) := \mathcal{V}_1 + \mathcal{V}_2.
\]
We then get, from (5.6)
\[
\mathcal{V}_1 = \sum_{i=1}^{n} \sum_{E_1} \text{Cov}(U_i, U_j) = \sum_{i=1}^{n} \sum_{1 \leq |i - j| \leq \beta_n} h_n^{2d} = O(n h_n^{2d} \beta_n).
\]
For \( \mathcal{V}_2 \), we use the modified Davydov inequality for mixing processes (see Rio (2000)). This leads, for all \( i \neq j \), to
\[
|\text{Cov}(U_i, U_j)| \leq C \alpha(|i - j|),
\]
we then get,
\[
\mathcal{V}_2 \leq C \sum_{i=1}^{n} \sum_{\beta_n + 1 \leq |i - j| \leq n - 1} |i - j|^{-v} = O(n \beta_n^{1-v}).
\]
Choosing \( \beta_n = h_n^{-\frac{2d}{v}} \) permits to get,
\[
\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 = O(n h_n^{2d(v-1)/v}).
\] (5.7)
Finally, from (5.2), (5.7) and (5.5) we obtain
\[
S_n^2 = O(n h_n^d) + O(n h_n^{2d(v-1)/v}) = n h_n^d (1 + h_n^{d(v-2)/v}).
\]
Now, that all the calculus are done. It is convenient to apply the inequality in Lemma 4 with \( \varepsilon > 0 \)
\[
\mathbb{P} \left[ \sum_{i=1}^{n} A_{t,i}(x_k^*) > \varepsilon \right] = \mathbb{P} \left[ \sum_{i=1}^{n} U_i > nh_n^d \varepsilon \right]
\]
\[
\leq C \left( 1 + \frac{nh_n^d \varepsilon^2}{r(1 + h_n^{d(v-2)/v})} \right)^{-\frac{1}{2}} + nCr^{-1} \left( \frac{r}{nh_n^d \varepsilon} \right)^{v+1}
\]
\[
=: C(\mathcal{E}_1 + \mathcal{E}_2).
\]
Taking $\varepsilon = \varepsilon_0 \left( \sqrt{\frac{\log n}{nh_n^2}} + \sqrt{\frac{\log n}{nh_n^2 + 1}} \right)$, we get for the first part

$$\mathcal{E}_1 = \left( 1 + C\frac{\varepsilon_0^2 \log n}{r} \right)^{-\frac{\varepsilon}{r}}. \quad (5.8)$$

By choosing $r = (\log n)^{1+b}$ with $b > 0$, (5.8) becomes

$$\mathcal{E}_1 = \left( 1 + C\varepsilon_0^2 \left( \log n \right)^{-b} \right)^{-\frac{\varepsilon}{(\log n)^{1+b}}},$$

by taking logarithm and using a Taylor expansion of $\log(1 + x)$ we get

$$\log \mathcal{E}_1 \simeq \left( \log n \right)^{-\frac{C\varepsilon_0^2}{b}}$$

which gives

$$\mathcal{E}_1 = n^{-\frac{C\varepsilon_0^2}{b}}. \quad (5.9)$$

For the same choice of $\varepsilon$ and $r$, we have

$$\mathcal{E}_2 \simeq n(\log n)^{v(1+b)\varepsilon_0} - (v+1) \left( nh_n^d \log n \right)^{-\frac{v+1}{2}}. \quad (5.10)$$

By taking again the inequality of Fuk-Nagaev and using $\mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i)$, we can write

$$\mathbb{P} \left[ \max_{1 \leq k \leq S_n} \left| \sum_{i=1}^{n} A_{i,k}(x_k^*) \right| > \varepsilon_n \right] \leq M a_n^{-d} C \left( n^{-\frac{C\varepsilon_0^2}{b}} + n(\log n)^{v(1+b)\varepsilon_0} - (v+1) \left( nh_n^d \log n \right)^{-\frac{v+1}{2}} \right)$$

$$\leq M h_n^{-d(1+\frac{1}{b})} n^{\frac{1}{b}} C \left( n^{-\frac{C\varepsilon_0^2}{b}} + n(\log n)^{v(1+b)\varepsilon_0} - (v+1) \left( nh_n^d \log n \right)^{-\frac{v+1}{2}} \right)$$

$$\leq MCn^{-\frac{C\varepsilon_0^2}{b}} h_n^{-d(1+\frac{1}{b})}$$

$$+ MC\varepsilon_0^{-v+1} n^{1+\frac{1}{b}} h_n^{-d(1+\frac{1}{b})} (\log n)^{v(1+b)(v+1)} \left( nh_n^d \log n \right)^{-\frac{v+1}{2}}$$

$$=: MC(\mathcal{R}_1 + \varepsilon_0^{-v+1} \mathcal{R}_2). \quad (5.11)$$

We have from hypothesis H3

$$\mathcal{R}_2 \leq C n^{1+\frac{1}{b}} \left( \frac{v+1}{2} \right)^{-d(1+\frac{1}{b})+\frac{v+1}{2}} (log n)^{v(1+b)-\frac{v+1}{2}}$$

$$\leq C n^{1+\frac{1}{b}} \left( \frac{v+1}{2} \right)^{-d(1+\frac{1}{b})+\frac{v+1}{2} - \psi d \left[ \frac{(v+1)+2\gamma}{2} \right]} (log n)^{v(1+b)-\frac{v+1}{2}}$$

$$\leq C n^{-\frac{d}{2} - \psi d (\frac{v}{2} + 1)} (log n)^{v(1+b)-\frac{v+1}{2}}. \quad (5.12)$$

Then, for an appropriate choice of $\psi$, $\mathcal{R}_2$ is the general term of a convergent series. In the same way, we can choose $\varepsilon_0$ such that $\mathcal{R}_1$ is the general term of convergent series. Finally, applying Borel-Cantelli's lemma to (5.11) gives the result.

**Remark 2** The parameter $\psi$ of the hypothesis H3 can be chosen such as:

$$\psi > \frac{1}{\gamma (v+3) + 1}. \quad (5.13)$$

This condition ensures the convergence of the series of Lemma 4.
Proof of Theorem 1. For $x \in \mathbb{R}^d$, we consider the following decomposition:

$$
\tilde{m}(x) - m(x) = \frac{1}{r_2(x)} \left\{ \left[ (\tilde{r}_1(x) - r_1(x)) + (r_1(x) - \mathbb{E}r_1(x)) + (\mathbb{E}r_1(x) - r_1(x)) \right] \\
+ r(x) \left[ (\tilde{r}_2(x) - \tilde{r}_2(x)) + (\mathbb{E}r_2(x) - \tilde{r}_2(x)) + (r_2(x) - \mathbb{E}r_2(x)) \right] \right\}
$$

which by triangle inequality, we have

$$
\sup_{x \in C} |\tilde{m}(x) - m(x)| \\
\leq \frac{1}{\inf_{x \in C} r_2(x)} \left\{ \sup_{x \in C} \left[ |\tilde{r}_1(x) - r_1(x)| + |r_1(x) - \mathbb{E}r_1(x)| + |\mathbb{E}r_1(x) - r_1(x)| \right] \\
+ \sup_{x \in C} |r(x)| \left[ |\tilde{r}_2(x) - \tilde{r}_2(x)| + |\mathbb{E}r_2(x) - \tilde{r}_2(x)| + |r_2(x) - \mathbb{E}r_2(x)| \right] \right\}. \quad (5.14)
$$

Then from the Lemmas 1-4 in conjunction with the inequality (5.14) conclude the proof.

References

1. Attouch M, Laksaci A, Messabihi N (2015), Nonparametric relative error regression for spacial random variables, Stat Papers, 58:987-1008
2. Bollerslev T (1986), General autoregressive conditional heteroscedasticity, J Econom, 31:307-327
3. Bosq D (1998), Nonparametric statistics for stochastics processes Estimation and Prediction, Lecture Notes in Statistics, 110 Springer-Verlag New York
4. Bradley RD (2007), Introduction to strong mixing conditions, Vol I-III Kendrick Pres Utah
5. Cai Z (1998), Asymptotic properties of Kaplan-Meier estimator for censored dependent data, Stat Probab Lett, 37:381-389
6. Cai Z (2001), Estimating a distribution function for censored time series data, J. Multiv. Analysis, 78:299-318
7. Dabrowska M D (1987), Nonparametric regression with censored survival time data, Scand. J. Statist, 14:181-197
8. Demongeot J, Hamie A, Laksaci A, Rachdi M (2016), Relative error prediction in nonparametric functional statistics: Theory and Practice, J. Multivariate Anal., 146:261-268
9. El Ghouch A, Van Keilegom I, (2008), Nonparametric regression with dependent censored data, Scand. J. Elect. J. of statist.
10. Engle R F, (1982), Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation, Econometrica, 50:987-1007
11. Ferraty F, Vieu P, (2006), Nonparametric functionnal data analysis, Theory and Practice, Springer-Verlag, New York
12. Guessoum Z, Ould Said E, (2008), On the nonparametric estimation of the regression function under censorship model, Statist. and Decisions, 26:159-177
13. Guessoum Z, Ould Said E, (2010), Kernel regression uniform rate estimation for censored data under $\alpha$-mixing condition, Elect. J. of statist., 4:117-132.
14. Jones D A, (1978), Nonlinear autoregressive processes, Proc. Roy. Soc. London A, 360, 71-95
15. Kaplan E L, Meier P, (1958), Nonparametric estimation for incomplete observations, J. Amer. Statist. Assoc., 53:457-481
16. Li X, Yang W, Hu S, (2016), Uniform convergence of estimator for nonparametric regression under censorship data, J. of inequal. and Appli. 142
17. Lipsitz S R, Ibrahim J G, (2000), Estimation with correlated censored survival data with missing covariates, Biostatistics, 19:315-327
18. Makridakis S, (1984), The forecasting Accuracy of Major Time Series Methods, Wiley, New York, 1984
19. Masry E, (1986), Recursive probability density estimation for weakly dependent stationary processes, IEEE Trans. Inform. theory, 32:25467
20. Narula S C, Wellington, J F (1977), Prediction, linear regression and the minimum sum of relative errors, Technometrics, 19:185-190
21. Ozaki, T (1979), Nonlinear time series models for nonlinear random vibrations, Technical report. Univ. of Manchester
22. Park H, Stefsanski L A, (1998), Relative error prediction, Statistics and Probability Letters, 40:227-236
23. Park H, Shin K I, Jones M C, Vines S K (2008), Relative error prediction via kernel regression smoothers, Journal of Stat. Plann. and Infer., 138, 2887-2898
24. Rio E, (2000) Theorie asymptotique des processus alatoires faiblement dependants, Math., 42:4347
25. Rosenblatt M, (1956), A central limit theorem and a strong mixing condition, Proc. Nat. Acad. Sci. USA, 42:4347
26. Shen J, Xie Y. (2013), Strong consistency of the internal estimator of the nonparametric regression with dependent data, Statistics and Probability Letters, 83, 1915-1925