INDEX ONE MINIMAL SURFACES IN SPHERICAL SPACE FORMS

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ABSTRACT. We prove that orientable index one minimal surfaces in spherical space forms with large fundamental group have genus at most two. This confirms a conjecture of R. Schoen for an infinite class of 3-manifolds.

1. Introduction

The Morse index is an important analytic quantity in the study of minimal surfaces. Roughly speaking, it counts the maximal number of directions a minimal surface can be deformed in order to decrease its area. Under this analytical point of view, the simplest minimal surfaces are those with small index, namely zero or one. Index zero minimal surfaces, also known as stable, are well studied topic in Differential Geometry. Among classical results we mention the Bernstein problem on the classification of complete minimal graphs in $\mathbb{R}^n$ and those connecting stable minimal surfaces and the topology of manifolds admitting positive scalar curvature metrics due to Schoen-Yau. The existence of stable minimal surfaces depends on the geometry and topology of the ambient space and is in general obtained via a minimization procedure. Such surfaces do not exist in manifolds with positive Ricci curvature. Index one minimal surfaces, on the other hand, do exist in manifolds with positive Ricci curvature. A guiding principle in the theory asserts that in positively curved manifolds, the index of a minimal hypersurface controls its topology and geometry. For instance, it is proved in [3] that the set of minimal surfaces with bounded index in a closed 3-manifold with positive scalar curvature cannot contain sequences of surfaces with unbounded genus or area. More generally, it is conjectured in [16, 22] that if $\Sigma$ is a minimal hypersurface in a closed manifold with positive Ricci curvature $M$, then $\text{index}(\Sigma) \geq C b_1(\Sigma)$, where $b_1(\Sigma)$ is the first Betti number of $\Sigma$ and $C$ is a constant which depends only on $M$. Estimates of this type
have been studied by many authors, see [1, 2, 14, 28] and references therein for further discussion. These estimates are, however, far from being optimal when the index is small in general. A related problem is to describe the geometry and topology of the minimal surfaces with the smallest index. In this direction, we mention the classical result that flat planes and the catenoid are, respectively, the only embedded minimal surfaces with index zero and one in $\mathbb{R}^3$, see [7, 9, 24, 28, 15]. Similar classification has also been proved in other non-compact flat space forms, see [25]. In higher dimensions, we mention the works [6, 33] on the classification of compact minimal hypersurfaces with index one in $\mathbb{R}^n$ and $S^n$, respectively.

Using test functions coming from meromorphic maps and harmonic forms, Ros [28] proved that two sided index one minimal surfaces in 3-manifolds with non-negative Ricci curvature have genus $\leq 3$. This result is sharp as the P Schwarz’s minimal surface in $\mathbb{R}^3$ projects to a closed minimal surface with genus three and index one in the cubic 3-torus [29]. On the other hand, when the ambient space has positive Ricci curvature, the right estimate is given by the following conjecture:

**Conjecture 1.1** (Schoen [22]). Let $M^3$ be a closed three manifold with positive Ricci curvature. If $\Sigma$ is an orientable minimal surface with index one in $M^3$, then genus($\Sigma$) $\leq 2$.

The interest in this conjecture is in part motivated by its implications for the classification of closed 3-manifolds. Namely, it is proved in [13] that every closed 3-manifold with positive Ricci curvature contains an index one minimal surface realizing its Heegaard genus. If Conjecture 1.1 is true, then this Heegaard genus is at most two. Combining this result with the classification of genus two 3-manifolds, one recovers the following classical result of Hamilton:

**Theorem 1.2** (Hamilton [11]). If $(M^3, g)$ is a three manifold with positive Ricci curvature, then $M \cong S^3/G$, where $G$ is a finite group of isometries acting freely on $(S^3, g_0)$.

**Remark 1.3.** With the exception of lens spaces, which has Heegaard genus one, any other spherical space form has Heegaard genus two [21].

The list of 3-manifolds where the Conjecture 1.1 is verified is small. In the case of spherical space forms, the only examples are the sphere $S^3$, the projective space $\mathbb{R}P^3$, and the lens spaces $L(3,1)$ and $L(3,2)$ [27, 35]. The conjecture has also been proved on sufficiently pinched convex hypersurfaces in $\mathbb{R}^4$, see [1, Section 5].

Our main result confirms Schoen’s Conjecture in the class of spherical space forms with large fundamental group.
Theorem 1.4. There exists an integer $p_0$ so that if $\Sigma$ is an orientable index one minimal surface embedded in a spherical space form $M^3$ with $|\pi_1(M^3)| \geq p_0$, then genus$(\Sigma) \leq 2$.

Remark 1.5. The orientability assumption seems to be necessary in Theorem 1.4. It is pointed out in [28, 30] that for every integer $n$, there are lens spaces containing nonorientable area minimizing surfaces with genus greater than $n$.

Let $M$ be a spherical space form and $O_M$ the set of closed orientable minimal surfaces embedded $M$. Define $A_M = \{|\Sigma| : \Sigma \in O_M\}$. By standard compactness theorems [4, 34], we have that $A_M = |\Sigma|$, where $\Sigma \in O_M$. Moreover, the work of Mazet-Rosenberg [18] and Ketover-Marques-Neves [13] imply that $\Sigma$ has index one. From the proof of the Willmore conjecture [16] we know that any orientable minimal surface in the lens space $L(p, q)$ has area bigger than the Clifford torus. In general, the least area minimal surface might have genus bigger than the Heegaard genus of the 3-manifold. In the Berger spheres with small Hopf fibers, the least area minimal surface has genus one and is congruent to the Clifford torus. For the class of spherical space forms in Theorem 1.4, we have:

Corollary 1.6. Let $M$ be a spherical space form with large non-abelian fundamental group. If $\Sigma \in O_M$ satisfies $A_M = |\Sigma|$, then $g(\Sigma) = 2$.

The proof of Theorem 1.4 is inspired by Ritoré and Ros’ work on the compactness of the space of index one minimal surfaces in flat three torus [26]. Among other results, they proved that the flat three torus with small injective radius and unit volume do not contain orientable index one minimal surfaces. Following similar ideas, we show that any rescaled sequence of index one minimal surfaces with genus three in spherical space forms with large fundamental group converges to a totally geodesic surface in a flat 3-manifold. We contradict this statement by showing that the curvature of such surfaces is large somewhere by an application of the Rolling Theorem.

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2. Preliminaries

2.1. Morse index. A surface $\Sigma \subset (M^3, g)$ is called minimal when the trace of its second fundamental form is identically zero. Equivalently, the first variation of its area is zero for all variations generated by flows of compact supported vector fields $X \in \mathcal{X}_0(M)$. If $\Sigma$ is two sided, then its second variation formula is given by:

\begin{equation}
I(f) := \left. \frac{d^2}{dt^2} \right|_{t=0} \text{area}(\Sigma_t) = \int_{\Sigma} |\nabla f|^2 - (\text{Ric}(N, N) + |A|^2)f^2 \, d\Sigma,
\end{equation}

where $f = \langle X, N \rangle$ is the normal component of $X$, Ric($\cdot$, $\cdot$) is the Ricci curvature of $M$, and $A$ is the second fundamental form of $\Sigma$. The quantity $I(f)$ is called the Morse index form of $\Sigma$ and is the quadratic form associated to the Jacobi operator

$$L = \Delta + (\text{Ric}(N, N) + |A|^2).$$

The Morse index of $\Sigma$ is defined as the number of negative eigenvalues of $L$.

2.2. Spherical space forms. We regard $\mathbb{S}^3$ as the unit quaternions, i.e., $(z, w) = z_1 + z_2 i + (w_1 + w_2) j$ and $|z|^2 + |w|^2 = 1$. Let $\phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ be the homomorphism of groups which associate for each pair $(u_1, u_2) \in \mathbb{S}^3 \times \mathbb{S}^3$ the isometry $\phi(u_1, u_2) \in SO(4)$ given by $x \mapsto \phi(u_1, u_2)(x) = u_1 xu_2^{-1}$. The map $\phi$ is surjective and $\text{Ker}\phi = C = \{(\pm 1, \pm 1)\}$. Similarly, one can construct the homomorphism $\psi : \mathbb{S}^3 \to SO(3) \subset SO(4)$ defined by $x \in \mathbb{S}^3 \mapsto \psi(u)(x) = uxu^{-1}$. This map is also surjective and its kernel is $\{\pm 1\}$. It follows that there exists an unique homomorphism $\varphi : SO(4) \to SO(3) \times SO(3)$ such that $\varphi \circ \phi = \psi \times \psi$.

For each finite subgroup $G \subset SO(4)$ we associate $H = \varphi(G) \subset SO(3) \times SO(3)$. The projection of $H$ on each factor of $SO(3) \times SO(3)$ is denoted by $H_1$ and $H_2$ respectively. If $G$ acts freely on $\mathbb{S}^3$, then $H_1$ or $H_2$ must be cyclic [32]. The pre-images in $\mathbb{S}^3$ of $H_1$ and $H_2$ via the homomorphism $\psi$ are denoted by $\widehat{H}_1$ and $\widehat{H}_2$ respectively. Since $H_1$ and $H_2$ are finite subgroups of $SO(3)$, they must be isomorphic to either the cyclic group, the dihedral group $D_n$, the tetrahedral group $T$, the octahedral group $O$ or the icosahedral group $I$.

It is showed in [31] that any finite subgroup $G \subset SO(4)$ is conjugated in $SO(4)$ to a finite subgroup of either $\phi(S^1 \times \mathbb{S}^3)$ or $\phi(S^3 \times S^1)$. Two important remarks that we will use are the following: $\phi(S^1 \times \mathbb{S}^3)$ preserves the Hopf fibers in $\mathbb{S}^3$ and left multiplication by unit quaternions leaves the Hopf fibers invariant. Recall that the Hopf map $h : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$.
sends \((z, w)\) to \(z/w\) where we think of \(S^2\) as \(\mathbb{C} \cup \{\infty\}\). In particular, up to conjugation in \(O(4)\), we may assume that \(G\) is a subgroup of \(\phi(S^1 \times S^3)\). The following describes the classical classification of 3-dimensional spherical space forms:

**Theorem 2.1.** Let \(G\) be a finite subgroup of \(\phi(S^1 \times S^3)\) acting freely on \(S^3\). Then one of the following holds:

1. \(G\) is cyclic;
2. \(H_2\) is \(T\), \(O\), \(I\), or \(D_n\) and \(H_1\) is cyclic of order coprime to the order of \(H_2\). Moreover, \(G = \phi(\hat{H}_1 \times \hat{H}_2)\);
3. \(H_2 = T\) and \(H_1\) is cyclic. Moreover, \(G\) is a subgroup of index three in \(\phi(\hat{H}_1 \times \hat{H}_2)\);
4. \(H_2 = D_n\) and \(H_1\) is cyclic. Moreover, \(G\) is a subgroup of index two in \(\phi(\hat{H}_1 \times \hat{H}_2)\).

**Proof.** See page 455 in [31].

The spherical space forms obtained when \(G\) is cyclic are called lens spaces. If \(p\) and \(q\) are relative primes, then we denote by \(L(p, q)\) the lens space defined by the action of \(\mathbb{Z}_p\) on \(S^3\) as follows: given \(m \in \mathbb{Z}_p\), we define \(m \cdot (z, w) = (e^{2\pi m \frac{q}{p}} z, e^{2\pi m \frac{p}{q}} w)\). The Clifford torus \(T_r \subset S^3\), defined as

\[
T_r := S^1(\cos(r)) \times S^1(\sin(r)) \subset S^3
\]

where \(r \in [0, \frac{\pi}{2}]\), is invariant by the group \(\mathbb{Z}_p\) and the projection of this family foliates \(L(p, q)\) by tori of constant mean curvature. One can check that \(T_{\frac{\pi}{4}}\) projects to an index one minimal tori in \(L(p, q)\) for every \(p \geq 2\) and \(q \geq 1\).

**Example 2.2** (Immersed index one minimal tori). Let \(T\) be a Clifford torus in \(S^3\) containing the geodesics \(T_0\) and \(T_{\frac{\pi}{2}}\). For each \(p\), let \(V_p\) be the varifold defined by \(V_p = \cup_{g \in \mathbb{Z}_p} g \cdot T\), where \(\mathbb{Z}_p\) is the group defined above. The projection of \(V_p\) in \(L(p, q)\) is a minimal immersed torus which fails to be embedded at the critical fibers \(T_0\) and \(T_{\frac{\pi}{2}}\) when \(q \neq 1\). If \(p\) is even, then \(\text{Index}(V_p/\mathbb{Z}_p) = 1\). Moreover, if \(p, q\) are chosen so that \(\lim_{p \to \infty} \text{diam}(T_{\frac{\pi}{2}}/\mathbb{Z}_p) = 0\), then the varifold \(V = \lim_{p \to \infty} V_p\) is the foliation of \(S^3\) where the leaves are Clifford torus containing \(T_0\) and \(T_{\frac{\pi}{2}}\).

**Remark 2.3** (Doubling the Clifford torus). If the group \(G\) satisfies item (4) in Theorem 2.1, i.e., \(H_2 = D_n\), then we call \(S^3/G\) a Prism manifold. These spherical space forms are double covered by lens spaces. In particular, one can sweep-out each Prism manifold with surfaces whose area does not exceed twice the volume of \(S^3/G\), see [13]. Applying the min-max theory, one obtains an orientable index one minimal surface.
with area bounded as above. If the order of $G$ is sufficiently large, then Theorem 1.4 implies that the genus of these min-max surfaces is two. We remark that these manifolds do not contain minimal tori by Frankel’s Theorem [10]. One can visualize these surfaces better when the Prism manifold they live have a double cover $L(p, q)$ which satisfies \( \lim_{p \to \infty} \text{diam}(T_{\pi}^4/Z_p) = 0 \). In this case, the orbit of a point $x \in T_r$ with respect to $G_p$ is becoming dense in the Clifford torus $T_r$. For every $p$, let $\hat{\Sigma}_p$ be the pre-image of these index one minimal surfaces in $S^3$. By Frankel’s Theorem $\hat{\Sigma}_p \cap T_{\pi}^4 \neq \emptyset$ for every $p$; hence, $\hat{\Sigma}_p$ converges as varifolds to the Clifford torus $T_{\pi}^4$ with multiplicity two. The surface $\hat{\Sigma}_p$ pictures like a doubling of the minimal Clifford torus.

**Remark 2.4** (Desingularizing stationary varifolds). Another family of spherical space forms is given by the quotients $S^3/(I^* \times Z_m)$, where $m$ satisfies $(m, 30) = 1$. By Frankel’s Theorem, there are no minimal spheres or minimal tori in $S^3/(I^* \times Z_m)$. With the help of the Hopf fibration $h : S^3 \to \mathbb{S}^2$, it is possible to construct a sweep-out of $S^3$ which is invariant by $I^* \times Z_m$ and that projects to a sweep-out in $S^3/(I^* \times Z_m)$ by surfaces with genus two and area bounded from above by $C_m$, see [12, Section 6]. Applying the min-max theory, one obtains an index one minimal surface $\Sigma_m$ with genus two in $S^3/(I^* \times Z_m)$ and area satisfying $|\Sigma_m| \leq \frac{C}{m}$. Its pre-image $\hat{\Sigma}_m \subset S^3$ has uniform bounded area and converges, as $m \to \infty$, to a stationary varifold $\mathcal{V}$ which is invariant by the Hopf fibration. In particular, $\mathcal{V} = h^{-1}(\mathcal{T})$, where $\mathcal{T}$ is a $I^*$ invariant geodesic net in $\mathbb{S}^2$. By Allard’s Regularity Theorem, the genus of $\hat{\Sigma}_m$ is concentrated near $h^{-1}(\mathcal{V})$, where $\mathcal{V}$ is the set of vertices of $\mathcal{T}$. The surface $\hat{\Sigma}_m$ pictures like a desingularization of $h^{-1}(\mathcal{T})$ near $h^{-1}(\mathcal{V})$ through Scherk towers.

**2.3. Non compact flat space forms.** Every non-compact orientable flat space form is the quotient of $\mathbb{R}^3$ by a discrete subgroup $G$ of the group Iso($\mathbb{R}^3$) of affine orientation preserving isometries acting properly and discontinuously in $\mathbb{R}^3$. For every subgroup $G$ we denote by $\Gamma(G)$ the subgroup of translations in $G$. The following describe all the possible types of affine diffeomorphic complete non compact orientable flat three manifolds (see [36] for a comprehensive discussion):

If $\text{rank}(\Gamma(G)) = 0$ or 1, then either $G = \{\text{Id}\}$ or $G = S_\theta$, with $0 \leq \theta \leq \pi$, where $S_\theta$ is the subgroup generated by a screw motion given by a rotation of angle $\theta$ followed by a non trivial translation in the direction of the rotation axis.

If $\text{rank}(\Gamma(G)) = 2$, then either $G$ is generated by two linearly independent translations and $\mathbb{R}^3/G$ is the Riemannian product $T^2 \times \mathbb{R}$,
where $T^2$ is a flat torus, or $G$ is generated by a screw motion with angle $\pi$ and a translation orthogonal to the axis of the screw motion.

**Theorem 2.5** (Ritoré [25]). If $\Sigma$ is a complete orientable index one minimal surface properly embedded in a non-compact orientable flat 3-manifold $\mathbb{R}^3/G$, then

$$-8\pi < \int_{\Sigma} K_{\Sigma} d\Sigma \leq -2\pi.$$  

**Remark 2.6.** By [19, 20], the total curvature of a properly embedded minimal surface in $\mathbb{R}^3/G$ is a multiple of $2\pi$ if finite.

**Example 2.7** (Index one Helicoids with total curvature $-2\pi$). Let $\Sigma$ the helicoid in $\mathbb{R}^3$ parametrized by $X(u,v) = (u \cos(v), u \sin(v), v)$. One can check that

$$\int_{\Sigma \cap \{0 \leq v \leq 4\pi\}} K_{\Sigma} d\Sigma = -4\pi. \quad (2.3)$$

Now consider $\Sigma/\mathbb{Z}_{4\pi}$ in $\mathbb{R}^3/\mathbb{Z}_{4\pi}$, where $\mathbb{Z}_{4\pi}$ is the group of vertical translations by multiples of $4\pi$. Recall that the Gauss map $N: \Sigma/\mathbb{Z}_{4\pi} \to S^2$ is conformal and of degree one by (2.3). A standard argument implies that $\text{ind}(L_{\Sigma/\mathbb{Z}_{4\pi}}) = \text{ind}(L_0)$, where $L_{\Sigma} = \Delta + |\nabla N|^2$ is the Jacobi operator of $\Sigma$ and $L_0$ is the operator $L_0 = \Delta + 2$ on $S^2$. Hence, $\text{ind}(\Sigma/\mathbb{Z}_{4\pi}) = 1$. Let $S_{\pi}$ be the subgroup of isometries generated by the screw motion $R(x,y,z) = (-x,-y,z+2\pi)$. Using that $S_{\pi}$ is a subgroup of order two in $\Sigma/\mathbb{Z}_{4\pi}$, we conclude that $\Sigma/S_{\pi}$ is a minimal surface with index one and total curvature $-2\pi$ in $\mathbb{R}^3/S_{\pi}$.

**Remark 2.8.** It is an open question weather there exists an index one minimal surface $\Sigma$ in $\mathbb{R}^3/G$ such that $\int_{\Sigma} K_{\Sigma} d\Sigma = -6\pi$. If $\Sigma$ is an index one minimal surface in a non-compact flat 3-manifold $\mathbb{R}^3/G$ where $G$ contains only translations, then $\int_{\Sigma} K_{\Sigma} d\Sigma = -4\pi$ [26].

**Example 2.9.** Let us show that $\mathbb{R}^3/S_{2\pi}$ can be obtained as a limit of Lens spaces under the Cheeger-Gromov convergence. To see this, consider the sequence of Lens spaces $(L(p_k,k), p_k^2 g_0, x_k)$, where $x_k$ lies on the critical fiber $T_{\frac{\pi}{2}}$ and $p_k = l(k-1)$. This sequence has curvature close to zero and injective radius at $x_k$ bounded from below by $\pi$. We claim that

$$(L(p_k,k), p_k^2 g_0, x_k) \xrightarrow{C-G} (\mathbb{R}^3/S_{2\pi}, \delta, x_\infty).$$

The observation is that the critical fiber $T_{\frac{\pi}{2}}$ has length $2\pi$ whereas the nearby Hopf fibers are equidistant and have length $2\pi l$. 

3. Proof of Theorem 1.4

**Proposition 3.1.** Let $\Sigma$ be a closed minimal surface in $S^3$ and $N : \Sigma_p \to S^3$ be the unit normal vector field of $\Sigma$. If we denote by

$$c = c(\Sigma) = \min \left\{ \arctan \left( \frac{1}{\max \{\lambda_2(x) : x \in \Sigma\}} \right) : \pi \right\},$$

where $\lambda_2(x)$ is the non-negative principal curvature at $x$, and by $F : \Sigma \times [0, c) \to S^3$ the exponential map on $\Sigma$, which is given by

$$(x, t) \mapsto F(x, t) = \cos(t)x + \sin(t)N(x),$$

then $F$ is a diffeomorphism onto its image.

**Proof.** We may assume that $g(\Sigma) \geq 1$ since a minimal sphere in $S^3$ is an equator and the Proposition trivially holds.

Let $\{e_1, e_2\}$ be an orthonormal basis with eigenvectors of the second fundamental form $A_{\Sigma}$ and $\{\lambda_1, \lambda_2\}$ the respective eigenvalues. It follows that $dF(e_i) = (\cos(t) - \sin(t)\lambda_i)e_i$ and $dF(\partial t) = -\sin(t)x + \cos(t)N(x)$. Since $\tan(t) \leq \frac{1}{\max \{\lambda_2\}}$ for every $t \in (0, c)$, we conclude that $F$ is a local diffeomorphism. The unit normal vector field along $\Sigma_t = F(\Sigma, t)$ is $N_t = -\sin(t)x + \cos(t)N(x)$. Moreover, if we denote the mean curvature of $\Sigma_t$ by $H_t$, then

$$H_t = \frac{1}{2} \frac{(1 + \lambda_2^2) \sin(2t)}{(\cos^2(t) - \sin^2(t)\lambda_2^2)} > 0,$$

for every $t \in (0, c)$. Let $t_0 = \sup \{t > 0 : F : \Sigma \times [0, t] \to S^3 \text{ is injective} \}$.

If $t_0 < c$, then there exist $(x_1, t_1)$ and $(x_2, t_2)$ in $\Sigma \times [0, t_0]$ with the same image under $F$. Since $\Sigma$ separates $S^3$, these points must lie on $\Sigma \times \{t_0\}$. Hence, we may assume that $F(x_1, t_0) = F(x_2, t_0)$ and that $x_1 \neq x_2$. Since $\Sigma_{t_0}$ has a tangential self intersection at $F(x_1, t_0)$, we conclude that $N_{t_0}(x_1) = \pm N_{t_0}(x_2)$. If $N_{t_0}(x_1) = N_{t_0}(x_2)$, then $t_0 = \frac{\pi}{4}$, contradiction. Consequently, $N_{t_0}(x_1) = -N_{t_0}(x_2)$ since $x_1 \neq x_2$. Hence, $\Sigma_{t_0}$ is locally at $F(x_1, t_0)$, an union of two tangential surfaces $\Gamma_1$ and $\Gamma_2$ with $\Gamma_1 \not\subset \Gamma_2$.

Moreover, the mean curvatures in the $N_{t_0}(x_1)$ direction say satisfies $H_{\Gamma_1} \leq 0 \leq H_{\Gamma_1}$. Applying the Maximum Principle [31, Lemma 1], we conclude the existence of neighborhoods of $x_1$ and $x_2$ in $\Sigma$ with the same image under $F_{t_0}$ and $H_{t_0} = 0$ there. This is a contradiction and the result follows. \hfill \Box

**Lemma 3.2.** Let $\Sigma$ be a orientable minimal surface embedded in $S^3$. If $R < c(\Sigma)$, then there exists $C > 0$ independent of $\Sigma$ such that $\text{vol}(B_{2R}(x)) \geq C R \cdot \text{area}(\Sigma \cap B_R(x))$ for every $x \in \Sigma$. 
Proof. By Lemma 3.1, the following map is a diffeomorphism onto its image:
\[ F : \Sigma \cap B_R(x) \times [0, \frac{R}{2}] \to \mathbb{S}^3. \]
Let us denote the image by \( \Omega \). By the change of variables formula,
\[ \text{Vol}(\Omega) = \int_0^R \int_{\Sigma \cap B_R(x)} \left( \cos^2(s) - \lambda^2_2 \sin^2(s) \right) d\Sigma ds. \]
Hence, we can choose \( 0 < C < \min\{ \cos^2(s)(1 - \tan^2(s/2)) : s \in [0, \frac{\pi}{2}] \} \) such that \( \text{Vol}(\Omega) \geq CR\text{Area}(\Sigma \cap B_R(x)) \) for \( \Omega \subset B_{2R}(x) \), the lemma is proved. \( \square \)

Let \( \{ \Sigma_n \} \) be a sequence of minimal hypersurfaces in a Riemannian manifold \((M, g)\). We say that \( \{ \Sigma_n \} \) converges, in the \( C^\infty \) topology, to a surface \( \Sigma \) if for every \( x \in \Sigma \) and for \( n \) large, the hypersurface \( \Sigma_n \) can be written locally as graphs over an open set of \( T_p \Sigma \), and these graphs converge smoothly to the graph of \( \Sigma \).

We say that \( \{ \Sigma_n \} \) satisfy local area bounds if there exist \( r > 0 \) and \( C > 0 \) such that \( |\Sigma_n \cap B_r(x)| \leq C \) for every \( x \in M \).

**Proposition 3.3.** Let \( \{ \Sigma_n \} \subset (M, g_n) \) be a sequence of properly embedded minimal surfaces such that \( \sup \Sigma_n |A_n| \leq C \) and with local area bounds. Assume that \( g_n \) converges to \( g \), in the \( C^\infty \) topology.

If \( \{ \Sigma_n \}_{n=1}^\infty \) has an accumulation point, then we can extract a subsequence which converges to a minimal surface \( \Sigma \) properly embedded in \((M, g)\).

**Lemma 3.4.** Let \( M_p \) be a 3-manifold with positive Ricci curvature and \( \Sigma_p \subset M_p \) a closed orientable minimal surface with index one and genus \( h \). Assume that \((M_p, g_p, x_p)\) converges, in the Cheeger-Gromov sense, to a flat manifold \((M, \delta, x_\infty)\) and that \((\Sigma_p, x_p)\) converges graphically with multiplicity one to a properly embedded minimal surface \((\Sigma_\infty, x_\infty)\) in \((M, \delta, x_\infty)\).

(1) If \( h = 2 \), then \( \int_{\Sigma_\infty} K_\infty d\Sigma_\infty = -6\pi, -4\pi, \) or \( 0 \).

(2) If \( h = 3 \), then \( \int_{\Sigma_\infty} K_\infty d\Sigma_\infty = 0 \).

Proof. Since the Cheeger-Gromov convergence preserves topology in the compact setting, we conclude that \( M \) is non compact. It follows that \( \Sigma_\infty \) is a complete non compact minimal surface in \( M \) since \( h \geq 2 \). The multiplicity one convergence implies that \( \Sigma_\infty \) is two sided. Moreover, the index of \( \Sigma_\infty \) is at most one by the lower semi continuity of the index. If \( \text{Ind}(\Sigma_\infty) = 0 \), then \( \Sigma_\infty \) is flat and we are done. Hence,
we may assume that Ind(Σ∞) = 1. By classical arguments in [8], Σ∞ is conformally equivalent to Σ − {q1, ..., ql}, where Σ is a closed Riemann surface. Let $D_i(q_i)$ be conformal disks on Σ centered at $q_i$. Given $ε > 0$ we define $U_ε$ to be $Σ − \bigcup_{i=1}^{l} \{ z \in D_i(q_i); |z_i| \leq ε \}$. On the set $U_{ε2}$ we define the function $u_ε$ by:

$$ u_ε = 0 \quad \text{on} \quad U_ε \quad \text{and} \quad u_ε = \frac{\ln(|z_ε|)}{\ln(ε)} \quad \text{for} \quad z \in U_{ε2} − U_ε. $$

One can check that $\lim_{ε→0} \int_Σ |∇u_ε|^2 dΣ = 0$. The set $U_ε$ is seen as a subset of $Σ∞$ and, by choosing $ε$ small, we may assume that Ind($U_ε$) = 1. It follows that for $p$ large depending on $ε$, there exist $U_p \subset U_ε' \subset Σ_p$ for which Ind($U_p$) = 1 and such that $U_p$ and $U_ε'$ converge graphically to $U_ε$ and $U_{ε2}$, respectively. Moreover, by means of $u_ε$ we can construct, for each $p$ large enough, an function $u_p$ on $Σ_p$ satisfying $u_p = 0$ on $U_p$, $u_p = 1$ at $Σ_p − U_ε'$, and such that $\lim_{p→∞} \int_Σ |∇u_p|^2 dΣ_p = 0$, i.e., $\int_Σ |∇u_p|^2 dΣ_p = O_1(ε)$. As $Σ_p$ and $U_p$ both have Index one, we concluded that Ind($Σ_p − U_p$) = 0. As supp($u_p$) ⊂ $Σ_p − U_p$, we obtain

$$ 0 \leq \int_Σ |∇u_p|^2 − (\text{Ric}_{g_p}(N_p, N_p) + |A_p|^2) u_p^2 \quad dΣ_p. $$

By the Gauss equation, $2K_p = 2K_p + |A_p|^2$, where $K_p$ is the sectional curvature of $M$ in the direction of $TΣ_p$. Therefore,

$$ 0 \leq \int_Σ (|∇u_p|^2 \quad \text{−} \quad (\text{Ric}_{g_p}(N_p) + 2K_p) u_p^2 + 2K_p u_p^2) \quad dΣ_p $$

$$ = \int_Σ (|∇u_p|^2 dΣ_p + 2 \int_{\{K_p\leq0\}} K_p u_p^2 dΣ_p + 2 \int_{\{K_p>0\}} |K_p| u_p^2 dΣ_p $$

$$ - \int_{K_p\leq0} (\text{Ric}_{g_p}(N_p) + 2K_p) u_p^2 \quad - \int_{K_p>0} (\text{Ric}_{g_p}(N_p) + 2K_p) u_p^2. $$

If $\{e_1, e_2\}$ is an orthonormal base for $TΣ_p$, then $\text{Ric}_{g_p}(e_1) = K_p + K(e_1, N)$, $\text{Ric}_{g_p}(e_2) = K_p + K_p(e_2, N)$, and $\text{Ric}_{g_p}(N_p) = K(e_1, N) + K(e_2, N)$. This immediately implies that $2K + \text{Ric}_{g_p}(N_p) = \text{Ric}_{g_p}(e_1) + \text{Ric}_{g_p}(e_2)$.
\[ \text{Ric}_{p}(e_2). \text{ Hence,} \]
\[ 0 \leq \int_{\Sigma_p} |\nabla u_p|^2 d_{\Sigma_p} + 2 \int_{\{K_p \leq 0\}} K_p u_p^2 d_{\Sigma_p} \]
\[ + \int_{\{K_p > 0\}} \left( 2|K_p| - \text{Ric}_{p}(N_p) - 2K_p \right) u_p^2 \]
\[ = \int_{\Sigma_p} |\nabla u_p|^2 d_{\Sigma_p} + 2 \int_{\{K_p \leq 0\}} K_p u_p^2 d_{\Sigma_p} \]
\[ - \int_{\{K_p > 0\}} \left( |A_p|^2 + \text{Ric}_{p}(N_p) \right) u_p^2. \]
\[ \leq \int_{\Sigma_p} |\nabla u_p|^2 d_{\Sigma_p} + \int_{\{u_p \equiv 1\} \cap \{K_p \leq 0\}} 2K_p d_{\Sigma_p} \]
\[ = O_1(\varepsilon) + \int_{\Sigma_p \cap \{K_p \leq 0\}} 2K_p d_{\Sigma_p} - \int_{U_p \cap \{K_p \leq 0\}} 2K_p d_{\Sigma_p}. \]

On the other hand, we have that \( \int_{U_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p} = \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} + O_2(\varepsilon) \), for the total curvature of \( \Sigma_\infty \) is uniformly close to that of \( U_\varepsilon \) which is uniformly close to that of \( \int_{U_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p} \). Hence,
\[ \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} \leq O_1(\varepsilon) + O_2(\varepsilon) + \int_{\Sigma_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p}. \]

This implies that \( \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} \leq \int_{\Sigma_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p} \) since \( \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} \) and \( \int_{\Sigma_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p} \) are independent of \( \varepsilon \). On the other hand,
\[ \int_{\Sigma_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p} \leq \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} \]
by the upper semi continuity of the limit of non-positive functions. Therefore,
\[ (3.1) \quad -8\pi < \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} = \lim_{p \to \infty} \int_{\Sigma_p \cap \{K_p \leq 0\}} K_p d_{\Sigma_p} \leq 4\pi(1 - h). \]

The first strictly inequality is from Theorem 2.5 and the second inequality if from the Gauss-Bonnet Theorem. If \( h = 2 \), then \( \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} = -4\pi \) or \(-6\pi \) by Remark 2.6. If \( h = 3 \), then (3.1) becomes a contradiction and the lemma is proved.

\[ \textbf{Corollary 3.5.} \text{ If } M_p \text{ is a spherical space form and genus}(\Sigma_p) = 2, \text{ then} \]
\[ \int_{\Sigma_\infty} K_\infty d_{\Sigma_\infty} = -4\pi \quad \text{or} \quad 0. \]
Proof. Since there is no loss of negative Gaussian curvature, then
\[
\lim_{p \to \infty} \int_{\{K_p > 0\}} (2K_p - \text{Ric}_{g_p}(N_p) - 2\overline{K}_p) \, d\Sigma_p = 0.
\]
By the scale invariance of this quantity, we can assume that \(\overline{K}_p = 1\). The Gauss equation then implies that \(\lim_{p \to \infty} |\Sigma_p \cap \{K_p > 0\}| = 0\). Hence, \(\lim_{p \to \infty} \int_{\{K_p > 0\}} K_p \, d\Sigma_p = 0\). The corollary now follows from the
Gauss-Bonnet Theorem. \(\square\)

**Theorem 3.6.** There exists an integer \(p_0\) such that if \(\Sigma\) is an orientable index one minimal surface in an spherical space form \(M^3\) with \(|\pi_1(M)| \geq p_0\), then genus(\(\Sigma\)) \(\leq 2\).

**Proof.** In what follows \(M_{p}\) denotes an spherical space form such that \(|\pi_1(M_{p})| = p\), i.e., \(M_{p} = \mathbb{S}^3/G_{p}\) and \(|G_{p}| = p\). Arguing by contradiction, let us assume the existence of a sequence of spherical space forms \(\{M_{p_i}\}_{i=1}^{\infty}\) such that each \(M_{p_i}\) contains an index one minimal surface \(\Sigma_{p_i}\) of genus three and that \(\lim_{i \to \infty} p_i = \infty\).

We consider the rescaled sequence \((M_{p_i}, \lambda_{p_i}^2 g_{\mathbb{S}^3}, x_{p_i})\), where \(x_{p_i} \in \Sigma_{p_i}\) and \(\lambda_{p_i} > 0\) is such that \(\lim_{i \to \infty} \lambda_{p_i} \text{inj}_{x_{p_i}} M_{p_i} > 0\). Similarly, we consider \((\Sigma_{p_i}, x_{p_i}) \subset (M_{p_i}, \lambda_{p_i}^2 g_{\mathbb{S}^3}, x_{p_i})\). By Cheeger-Gromov’s compactness theorem, there exists a subsequence \(\{M_{p_i}\}_{i \in \mathbb{N}}\) which converges in the Cheeger-Gromov sense to a flat manifold \((M, \delta, x_{\infty})\).

**Lemma 3.7.** Let \((M_{p_i}, \lambda_{p_i}^2 g_{\mathbb{S}^3}, x_{p_i}) \xrightarrow{C-G} (M, \delta, x_{\infty})\) as above and assume that \(\liminf_{i \to \infty} \lambda_{p_i} \text{inj}_{x_{p_i}} M_{p_i} > 0\). Then \(\{(\Sigma_{p_i}, \lambda_{p_i}^2 g_{\mathbb{S}^3}, x_{p_i})\}_{i \in \mathbb{N}}\) satisfies local area bounds in \(B_{R}(x_{\infty})\) for some \(R > 0\).

**Proof.** As \(\Sigma_{p_i}'\) is \(G_{p}\) invariant, then \(F : \Sigma_{p_i}' \times [0, c(\Sigma_{p_i}')] \to \mathbb{S}^3\) is also \(G_{p}\) invariant. Hence, it makes sense to consider \(F : \Sigma_{p} \times [0, c(\Sigma_{p}')] \to M_{p}\) which is a diffeomorphism onto its image by Proposition 3.1. Let \(r < \frac{1}{4} \min\{1, \liminf_{i \to \infty} \lambda_{p_i} \text{inj}_{x_{p_i}} M_{p_i}\}\), then for every \(y_i \in \Sigma_{p_i} \cap B_{R}(x_{\infty})\) we have that \(\text{Vol}(B_{\frac{r}{\lambda_{p_i}}}(y_{p_i})) \geq C_{1} r \text{Area}(\Sigma_{p} \cap B_{\frac{r}{\lambda_{p_i}}}(y_{p_i}))\) by Lemma 3.2. Since this formula is scale invariant, the lemma is proved. \(\square\)

**Lemma 3.8.** Let \(A_{p}\) be the second fundamental form of \(\Sigma_{p}\) in \(M_{p}\). There exist \(C > 0\) such that \(\sup_{\Sigma_{p}} |A_{p}|_{\lambda_{p}^2 g_{\mathbb{S}^3}} = \sup_{\Sigma_{p}} \frac{1}{\lambda_{p}^2} |A_{p}|^2 \leq C\).

**Proof.** Let \(y_{p} \in \Sigma_{p} \subset M_{p}\) be such that \(|A_{p}||_{(y_{p})} = \max_{\Sigma_{p}} |A_{p}|^2\) and define the quantity \(\rho_{p} = \max_{\Sigma_{p}} |A_{p}||_{(y_{p})}\). Arguing by contradiction, we assume that \(\frac{\rho_{p}}{\lambda_{p}} \to \infty\). We consider the surface \(\tilde{\Sigma}_{p} = (\Sigma_{p}, y_{p}) \subset (M_{p}, \rho_{p}^2 g_{\mathbb{S}^3}, y_{p})\). Under this scale, the sequence \((M_{p}, \rho_{p}^2 g_{\mathbb{S}^3}, y_{p})\) converges to \((\mathbb{R}^3, \delta, 0)\) as \(p \to \infty\). Moreover, the surface \(\tilde{\Sigma}_{p}\) satisfies
Lemma 3.10. \( \max_{\Sigma_p} |A_p'(x)|^2 = |A_p'(0)|^2 = 1 \) and enjoys local area bounds by previous lemma. By Proposition 3.3, \( \Sigma_p \) converges to a non-flat properly embedded minimal surface \( \Sigma_\infty \subset \mathbb{R}^3 \) of index one. The convergence is with multiplicity one. Indeed, applying Proposition 3.1 to \( N \) and \( -N \) we obtain that \( F : (\Sigma_p, x_p) \times (-\alpha, \alpha) \to (M_p, \rho_p^2 g_{g^3}, x_p) \), with \( 0 < \alpha < \liminf_{i \to \infty} \rho_p, c(\Sigma_{p_i}) \), is a diffeomorphism onto its image. Hence, there exists a tubular neighbourhood of radius \( \alpha \) around each \( \Sigma_{p_i} \) in \( (M_p, \rho_p^2 g_{g^3}) \) and the convergence is with multiplicity one. Since \( g(\Sigma_{p_i}) = 3 \), Lemma 3.4 implies that \( \int_{\Sigma_\infty} K d_{\infty} = 0 \). This contradicts \( |A_{\Sigma_\infty}|(0) = 1 \).

Combining Lemma 3.7, Lemma 3.8, and Proposition 3.3 we obtain:

**Lemma 3.9.** There exist a properly embedded orientable minimal surface \( \Sigma_\infty \subset (M, \delta, x_\infty) \) such that:

\[
\{\Sigma_{p_i}\}_{i \in \mathbb{N}} \subset (M_p, \lambda^2_\rho g_{g^3}, x_p) \to \Sigma_\infty \text{ in the } C^k \text{ topology.}
\]

The convergence is with multiplicity one and the Morse index of \( \Sigma_\infty \) is at most one.

**Lemma 3.10.** Let \( x_p \in \Sigma_p \) be such that \( \sup_{\Sigma_p} |A_p| = |A_p|(x_p) \). If \( \lim_{p \to \infty} \lambda_p c(\Sigma_p) < \infty \), then

\[
\lim_{p \to \infty} \frac{|A_p|^2(x_p)}{\lambda_p^2} > 0.
\]

**Proof.** As \( \lim_{p \to \infty} \lambda_p c(\Sigma_p) < \infty \), there exists a positive constant \( C \) such that \( c(\Sigma_{p_i}) \leq \frac{C\pi}{\lambda_p^2} \) for every \( i \geq 1 \). Hence,

\[
c(\Sigma_p) \leq \frac{C\pi}{\lambda_p^2} \Leftrightarrow \arctan \left( \frac{1}{\lambda_p^2(\Sigma_p)} \right) \leq \frac{C\pi}{\lambda_p^2} \Leftrightarrow \lambda_2(x_p) \geq \frac{1}{\tan \left( \frac{C\pi}{\lambda_p^2} \right)},
\]

where \( \lambda_2(x) \) is the largest principal curvature of \( \Sigma_p \) at \( x \). Therefore,

\[
\lim_{p \to \infty} \frac{|A_p|^2(x_p)}{\lambda_p^2} = \lim_{p \to \infty} \frac{2\lambda_2^2(x_p)}{\lambda_p^2} \geq \lim_{p \to \infty} \frac{2}{\lambda_p^2 \tan^2 \left( \frac{C\pi}{\lambda_p^2} \right)} = \frac{2C^2}{\pi^2}
\]

and the lemma is proved.

**Lemma 3.11.** If for each \( p \) there exists \( \lambda_p \) such that \( \text{inj}_x M_p \geq \frac{C}{\lambda_p} \) for every \( x \in \Sigma_p \), then \( \lim_{p \to \infty} \lambda_p c(\Sigma_p) = \infty \).

**Proof.** Let \( x_p \in \Sigma_p \) be such that \( \sup_{\Sigma_p} |A_p| = |A_p|(x_p) \). By Lemma 3.9, \( (M_p, \lambda^2_\rho g_{g^3}, x_p) \to (M, \delta, x_\infty) \) in the Cheeger-Gromov convergence and \( \Sigma_p \to \Sigma_\infty \) in \( (M, \delta, x_\infty) \). If \( \lim_{p \to \infty} \lambda_p c(\Sigma_p) < \infty \), then, by Lemma 3.10, \( \Sigma_\infty \) is not totally geodesic. This contradicts Lemma 3.4.
By Theorem 2.1, we may assume that the subsequence \( \{M_{p_i}\}_{i \in \mathbb{N}} \) satisfies either Case I, II, or III below:

Case I: The sequence \( \{M_{p_i}\}_{i \in \mathbb{N}} \) is such that \( H_2^p = \pi_2(\varphi(G_p)) \) is either \( T, O \), or \( I \).

Lemma 3.12. If \( M_p \) is such that \( H_2^p = T, O, \) or \( I \), then \( \text{inj}_x M_p = O(\frac{1}{p}) \) for every \( x \in M_p \).

Proof. Since the group \( G_p \) preserves the Hopf fibers, the Hopf fibers have size \( O(\frac{1}{p}) \). Let \( h : (M_p, p^2 g_{S^3}) \to (\mathbb{S}^2(\frac{1}{2})/H_2^p, p^2 g_{S^2}) \) be the Hopf fibration. Let \( B_r(x_p) \) be the ball of radius \( r \) in \( (M_p, p^2 g_{S^3}) \). Since \( H_2^p = T, O, \) or \( I \), there exists \( c_0 > 0 \) such that \( \text{vol}(h(B_r(x_p))) \geq c_0 r^2 \).

By the co-area formula,

\[
\text{vol}(B_{2r}(x_p)) \geq \int_{h(B_r(x_p))} \mathcal{H}^1(h^{-1}(y)) \, d\mathcal{H}^2(y) \geq C \, c_0 \, r^2.
\]

Cheeger’s inequality implies that \( \text{inj}_{x_p} (M_p, p^2 g_{S^3}) \geq i_0 \) for \( i_0 > 0 \). \( \square \)

Since \( g(S_p) \geq 3 \), there exist a point \( y_p \in \Sigma_p \) such that the Hopf fiber through \( y_p \) is orthogonal to \( \Sigma_p \). If we parametrize such fiber by \( \gamma : [0, 2\pi] \to M_p \), then the map \( F \) from Proposition 3.1 satisfies \( F(y_p, t) = \gamma(t) \). It follows from Lemma 3.12 that \( c(S_p) \leq \frac{C}{p} \). This contradicts Lemma 3.11.

Case II: The sequence \( \{M_{p_i}\}_{i \in \mathbb{N}} \) is such that \( H_2^p = \pi_2(\varphi(G_p)) \) is \( \mathbb{Z}_m \).

This corresponds to a subsequence of Lens spaces \( L(p_i, q_i) \). The next lemma is useful for the analysis of this case:

Lemma 3.13. If \( M_p = L(p, q) \) and \( \text{diameter}(T_{\mathbb{Z}}/\mathbb{Z}_p) > \varepsilon \) for every \( p \), then \( \text{inj}_x M_p = O(\frac{1}{p}) \) and \( (M_p, p^2 g_{S^3}, x_p) \xrightarrow{C^{-G}} (\mathbb{S}^1 \times \mathbb{R}^2, \delta, x_\infty) \).

Moreover, there exist a unit vector field \( X \in \mathcal{X}(\mathbb{S}^3) \) which is \( \mathbb{Z}_p \) invariant and such that its orbits converge to the standard \( \mathbb{S}^1 \) fibers of \( \mathbb{S}^1 \times \mathbb{R}^2 \).

Proof. See Section 3 in [35]. \( \square \)

For subsequences satisfying Lemma 3.13, we pick \( y_p \in \Sigma_p \) such that \( g_{S^3}(N(y_p), \dot{X}(y_p)) = \pm 1 \). The existence of \( y_p \) is from the Poincaré-Hopf Index Theorem applied to the vector field \( X^T \in \mathcal{X}(\Sigma_p) \). Applying Lemmas 3.9 and 3.4, we conclude that \( \Sigma_\infty \) is an union of planes orthogonal to the fibers of \( \mathbb{S}^1 \times \mathbb{R}^2 \). This implies that \( \lim_{p \to \infty} p \, c(S_p) < \infty \) which contradicts Lemma 3.11.

It remains to study subsequences of Lens spaces \( L(p_i, q_i) \) such that

\[
\lim_{i \to \infty} \text{diameter}(T_{\mathbb{Z}}/\mathbb{Z}_p_i) = 0.
\]
Let us prove that the pre-image of \( \Sigma_p \) in \( S^3 \), denoted by \( \hat{\Sigma}_p \), converges in the Hausdorff sense to \( T_\pi \) as \( p \to \infty \).

**Lemma 3.14.**

\[
\lim_{p \to \infty} d_H(\hat{\Sigma}_p, T_\pi) = 0.
\]

**Proof.** Without loss of generality, we assume that \( \Sigma_p \cap A_p \) is stable, where \( A_p = \{ x \in L(p,q) : r(x) \geq \frac{a}{4} \} \) and \( r(x) = r \) if, and only if, \( x \in T_r \). Let us define the quantities \( a = \liminf_{p \to \infty} \inf \{ r(x) : x \in \Sigma_p \} \) and \( b = \limsup_{p \to \infty} \sup \{ r(x) : x \in \Sigma_p \} \). If \( b < \frac{a}{4} \), then \( T_b \) can be obtained as a limit of \( \hat{\Sigma}_p \) as \( p \to \infty \) since the curvature of \( \Sigma_p \cap \hat{A}_p \) is uniformly bounded and since each orbit of \( \mathbb{Z}_{p_i} \) is becoming dense on the Clifford torus that contains it. Thus, \( b = \frac{a}{4} \) which implies that \( a = \frac{a}{4} \) and the lemma is proved in this case. Indeed, if \( a < \frac{a}{4} \), then \( T_\pi \) would be a stable minimal surface, contradiction. Hence, we may assume that \( b = \frac{a}{4} \).

First we study the case where \( \Sigma_p \cap T_\pi = \emptyset \) for every \( p \). Let \( x_p \in \hat{\Sigma}_p \) be the closest point to \( T_\pi \). By the stability assumption, the connected components of \( \hat{\Sigma}_p \) in \( \hat{A}_p \) converge to leaves of a minimal lamination \( F \) in \( \hat{A}_p \). Since \( T_\pi \) is tangent to every such leaf that it intersects, we conclude that \( T_\pi \) is contained in a leaf \( F_\alpha \). Let \( \Gamma_p \subset S^3 \) be a minimal torus containing the geodesics \( T_0 \) and \( T_\pi \) and perpendicular to \( \hat{\Sigma}_p \) at \( x_p \). The minimal tori \( \Gamma_p \) is a leaf of the singular lamination \( E = \{ E_\beta \} \) by the union of minimal tori containing \( T_0 \) and \( T_\pi \). By compactness, \( \Gamma_p \) converge to a leaf \( \hat{E}_\beta \) perpendicular to \( F_\alpha \) along \( T_\pi \). Consequently, there exist another leaf \( E_{\beta_0} \) which is tangent to \( F_\alpha \) along \( T_\pi \). By the analytical continuation, the lamination \( F \) coincide with the singular lamination \( E \), contradiction.

Now we study the case \( \Sigma_p \cap T_\pi \neq \emptyset \). By choosing \( x_p \in \Sigma_p \cap T_\pi \), we have that \( (L(p,q), p^2 g_0, x_p) \to (M, \delta, x_\infty) \), where \( M \) is a quotient of \( \mathbb{R}^3 \) by a screw motion with angle \( \theta \) and \( (\Sigma_p, p^2 g_0, x_p) \to (\Sigma_\infty, \delta, x_\infty) \), where \( \Sigma_\infty \subset M \) is totally geodesic by Lemma 3.4. If \( \Sigma_\infty \) is a plane, then \( \lim_{p \to \infty} p c(\Sigma_p) < \infty \). As this contradicts Lemma 3.11 (note that \( \inf_{x} L(p,q) \geq \frac{a}{p} \) for every \( x \)), we conclude that \( \Sigma_\infty \) is flat cylinder. It is enough to proving that \( \theta \neq 0 \), since there are no totally geodesic cylinders in \( M \) in this case. Let \( T_{r_p} \) be the Clifford torus such that \( \lim_{p \to \infty} p d_{L(p,q)}(T_{r_p}, T_\pi) = c_0 \). It follows that \( (T_{r_p}, p^2 g_0) \) converges to a tube of radius \( c_0 \) around the central fiber in \( (M, \delta) \) through \( x_\infty \). Recall the Hopf fibration \( h : S^3 \to S^2(\frac{1}{2}) \). If \( \gamma_p : [0, 1] \to S^3 \) is the geodesic segment such that \( |\gamma_p| = 2 \inf_{\gamma_p(0)} L(p,q) \) with \( \gamma_p(0) \in T_{r_p} \), then \( h(\gamma_p) \) is a geodesic in \( S^2(\frac{1}{2}) \) whose extremities determine an arc.
Theorem, genus(Σ). Case III: The sequence βₚ converge to an arc β in the geodesic circle of radius c₀ centered at the origin in ℝ² and h(γₚ) converges to a linear segment γ whose extremities are those of β. The angle |β| is independent of the choice of c₀. Hence, |γ| increases as c₀ increases. In particular, the injective radius of M is not constant and, hence, θ ≠ 0.

By Lemma 3.14, there exists λₚ such that \( \frac{1}{\lambda_p} \leq \text{inj}_x L(p, q) = C \frac{1}{\lambda_p} \) for every \( x \in \Sigma_p \). The constant \( C > 0 \) is independent of \( p \) and \( q \). As before, \( (L(p, q), \lambda_p^2 g_0, x_p) \rightarrow (M, \delta, x_\infty) \) and \( \Sigma_p \rightarrow \Sigma_\infty \), where \( \Sigma_\infty \) is a totally geodesic surface in \( (M, \delta) \) by Lemma 3.4. If \( M \) is diffeomorphic to \( T^2 \times \mathbb{R} \), then \( \lim_{p \rightarrow \infty} \lambda_p c(\Sigma_p) < \infty \) and we obtain a contradiction with Lemma 3.11. Similar argument for the case when \( M = \mathbb{R}^3 / T_n \) and \( \Sigma_\infty \) an union of planes. Therefore, we assume, regardless the choices of base points, that \( M \) is diffeomorphic to \( S^1 \times \mathbb{R}^2 \) and that \( \Sigma_\infty \) is a totally geodesic \( S^1 \times \mathbb{R} \). Let us show that this is incompatible with the assumption that genus(\( \Sigma_p \)) > 1.

**Lemma 3.15.** For each \( j \), let \( \Sigma_j \) be a closed minimal surface of genus \( g \) in \( M_j \) and assume that \( (M_j, \lambda_j^2 g_0, x_j) \rightarrow (\mathbb{S}^1 \times \mathbb{R}^2, \delta, x_\infty) \) and that \( \Sigma_j \rightarrow \mathbb{S}^1 \times \mathbb{R} \) for every choice of base points \( x_j \in \Sigma_j \). Then \( g = 1 \).

**Proof.** It follows from the assumptions, that there exist positive constants \( C_1 \) and \( C_2 \) such that \( \frac{C_1}{\lambda_j} \leq \text{inj}_x \Sigma_j \leq \frac{C_2}{\lambda_j} \) and \( \frac{C_1}{\lambda_j} \leq \text{inj}_x M_j \leq \frac{C_2}{\lambda_j} \) for every \( x \in \Sigma_j \) and every \( j \). For each \( j \), let \( \mathcal{F}_j = \{ B_1, \ldots, B_N \} \) be a maximal disjoint collection of balls \( B_i = B_{\frac{C_1}{\lambda_j}}(x_{ij}) \) in \( M_j \) where \( x_{ij} \in \Sigma_j \) and \( R > 4 C_2 \). By the assumption of the lemma, there exists \( j_0 \) such that \( \Sigma_j \cap \bigcup_{i=1}^{N_j} B_i \) is an annulus for every \( j \geq j_0 \). For \( j \) sufficiently large, let \( K_j \) be a connected component of \( \Sigma_j - \bigcup_{i=1}^{N_j} B_i \) and take \( y_j \in K_j \). By assumption, \( (\Sigma_j, \lambda_j^2 g_0, y_j) \rightarrow (\mathbb{S}^1 \times \mathbb{R}, \delta, y_\infty) \). We consider \( \mathcal{F}_\infty \) the disjoint collection of regions in \( \Sigma_\infty \) obtained as the limit of \( \Sigma_j \cap B_{ij} \). Note that each element of \( F_\infty \) is the intersection of geodesic balls in \( \mathbb{S}^1 \times \mathbb{R}^2 \) centered on \( \Sigma_\infty \) and radius \( R \in [C_1, C_2] \), hence, an annulus where each boundary component generates \( \pi_1(\Sigma_\infty) \). Moreover, each connected component of \( \Sigma_\infty - \mathcal{F}_\infty \) is compact by the maximality of \( \mathcal{F}_j \). Since \( K_j \) is connected and \( y_\infty \notin \mathcal{F}_\infty \), we conclude that \( K_\infty \) is also an annulus. Hence, there exists an integer \( j_2 \) such that \( \Sigma_j \) is an union of disjoint annulus for every \( j \geq j_2 \). By the Gauss-Bonnet Theorem, genus(\( \Sigma_j \)) = 1. □

Case III: The sequence \( \{ M_p \}_{i \in \mathbb{N}} \) is such that \( H_2^p = \pi_2(\varphi(G_p)) \) is \( \mathbb{D}_{2n} \).
The spherical space forms in this case are double covered by lens spaces. The arguments in Case II apply \textit{mutatis mutandis}.

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