Abstract

In this paper, we construct a Spectrum Generating Algebra (SGA) for a quantum system with purely continuous spectrum: the quantum free particle in a Lobachevski space with constant negative curvature. The SGA contains the geometrical symmetry algebra of the system plus a subalgebra of operators that give the spectrum of the system and connects the eigenfunctions of the Hamiltonian among themselves. In our case, the geometrical symmetry algebra is \( \mathfrak{so}(3, 1) \) and the SGA is \( \mathfrak{so}(4, 2) \). We start with a representation of \( \mathfrak{so}(4, 2) \) by functions on a realization of the Lobachevski space given by a two sheeted hyperboloid, where the Lie algebra commutators are the usual Poisson-Dirac brackets. Then, introduce a quantized version of the representation in which functions are replaced by operators on a Hilbert space and Poisson-Dirac brackets by commutators. Eigenfunctions of the Hamiltonian are given and “naive” ladder operators are identified. The previously defined “naive” ladder operators shift the eigenvalues by a complex number so that an alternative approach is necessary. This is obtained by a non self-adjoint function of a linear combination of the ladder operators which gives the correct relation among the eigenfunctions of the Hamiltonian. We give an eigenfunction expansion of
functions over the upper sheet of two sheeted hyperboloid in terms of
the eigenfunctions of the Hamiltonian.

1 Introduction

The notion of the Spectrum Generating Algebra (SGA) was introduced many
years ago by Barut and Bohm [1] and independently by Dothan, Gell-Mann
and Neeman [2] for the construction of multiplets in elementary particle
theory. The notion of SGA in quantum mechanics is suitable for the construction
of the Hilbert space of states for a given system using representation theory.
The point of departure is the geometrical symmetry group for a given system.
The representations of this algebra give the subspace of the whole Hilbert
space of eigenstates corresponding to a fixed energy. Then, we need to add
some generators to the algebra so that the new elements, the ladder opera-
tors, connect states of different energies. This new generators and hence the
ladder operators cannot commute with the Hamiltonian of the system. Thus,
the SGA will generate the whole Hilbert space of eigenfunctions starting from
just one eigenfunction and following some prescriptions on the application of
the operators of the algebra.

In a former publication [5], we have discussed the construction of the
SGA for the free particle in the three dimensional sphere, $S^3$, where the
Hamiltonian has a pure discrete spectrum. In that case, the initial space
isometry algebra or geometrical algebra was $\mathfrak{so}(4)$, while the SGA that we
constructed was isomorphic to $\mathfrak{so}(4, 2)$.

The objective of this paper is to explore the possibility of extending the
notion of SGA for systems with purely continuous spectrum. A typically
non-trivial example in which this situation arises is in the three dimensional
Lobachevskii space. Then, our aim was constructing a SGA for the free par-
ticle on a space of negative constant curvature, which can be realized as the
upper sheet of a two sheeted hyperboloid $\mathcal{H}^3$ embedded in the Minkowskian
space $\mathbb{R}^{3+1}$.

In this situation, the geometrical algebra is $\mathfrak{so}(3, 1)$ and we shall show
that the SGA is again $\mathfrak{so}(4, 2)$. However, the situation is quite different
than in the previous study case concerning the free motion in $S^3$ where the
operator that parameterizes the Hamiltonian is a generator of a compact
subgroup of $SO(4, 2)$, while the analogous operator for $\mathcal{H}^3$ does not have
this property. In the situation under our study, we do not use a maximal
compact subalgebra in order to construct the basis, but instead a subalgebra including generators of noncompact subgroups. Then, ladder operators can be expected to be functions of generators of the algebra not in $\mathfrak{so}(3,1)$.

Once we have constructed ladder operators for the free particle in the two-sheeted hyperboloid, a somehow unexpected situation emerges: the naive choice for ladder operators that should have served to construct the Hilbert space supporting the representation change the energy by a complex number. This result means that the ladder operators take any vector out of the Hilbert space. This illness has a remedy, which is the construction of a complex power of certain linear combinations of ladder operators. This action will solve the problem at the same time that it creates a bridge between quantum theory in Lobachevski space and the Gelfand-Graev transformation [6].

We have organized this paper as follows: In Section 2, we construct the generators of the Lie algebra $\mathfrak{so}(4,2)$ corresponding to either a free classical particle in a one or two sheeted hyperboloid and give their relations in terms of Dirac brackets. In Section 3, we construct the quantized version of the material introduced in Section 2 including the restrictive relations necessary for the determination of an irreducible representation of the algebra. We define the ladder operators and give relations between ladder operators and other generators of the algebra. Finally, in Section 4, we restrict our study to the three dimensional Lobachevski space realized by one sheet of the two sheeted hyperboloid $\mathcal{H}^3$. Here, we obtain a generalization of plane waves for the free particle on the hyperboloid. We find that the previously defined ladder operators shift the energy of these plane waves by a complex number so that a new concept of ladder operators are defined to correct this anomaly. The construction of the SGA for the free particle in $\mathcal{H}^3$ is then complete. This paper closes in Section 5, showing an eigenfunction expansion of functions on the two sheeted hyperboloid in terms of generalized plane waves.

\section{A classical particle in a three dimensional hyperboloid.}

We consider the two sheeted three dimensional hyperboloid $\mathcal{H}^3$ immersed into an ambient Minkowskian space $\mathbb{R}^{3+1}$ with equation $x^i x^j g_{ij} = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = -1$, which can be written in the usual shorthand form as $x^2 = x_i x^i = -1$. Note that here Latin indices will run from 1 to 4, that
the metric is $g_{ij} = \text{diag}(1,1,1,-1)$ and that we have to sum over repeated indices from 1 to 4. Henceforth, we shall use the standard convention relative to the operations of lowering and raising indices using the metric $g_{ij}$.

Now, let us consider the Lagrangian of the free particle with mass $m = 1$ defined in the ambient space $\mathbb{R}^{3+1}$, $L := \frac{1}{2} \dot{x}^i \dot{x}^j g_{ij}$.

Then, its restriction to the hyperboloid $H^3$ is given by

$$L = -\frac{1}{4} (\dot{x}^i \dot{x}^j - \dot{x}^j \dot{x}^i) (\dot{x}_i x_j - \dot{x}_j x_i), \quad (1)$$

where the dot means derivative with respect time. The global minus sign comes from the condition $x^i x_i = -1$ defining $H^3$.

The canonical momenta are determined by

$$p_i := \frac{\partial L}{\partial \dot{x}^i} = -(x^k \dot{x}^j - x^j \dot{x}^k) x_j g_{ik}, \quad (2)$$

and satisfies the primary constraint

$$x^i p_i = x^i p^j g_{ij} = 0. \quad (3)$$

The Legendre transformation of the Lagrangian (1) gives the canonical Hamiltonian for the free motion in $H^3$ as:

$$H = -\frac{1}{2} J_{ij} J^{ij}, \quad J^{ij} = x^i p^j - x^j p^i, \quad (4)$$

where each $J^{ij}$ has the structure of an angular momentum. Our strategy to work in $H^3$ will be the following: Instead of dealing in the 8-dimensional phase space with dynamical variables $x^i, p_i$ satisfying the canonical Poisson brackets $\{x^i, p_j\} = \delta^i_j$, we impose the primary constraint (3) and the gauge fixing condition

$$x_i x^i = -1. \quad (5)$$

According to the usual procedure [3, 4], we also introduce the Dirac brackets

$$\{x^i, x^j\}_D = 0, \quad \{p_i, x^j\}_D = \delta^j_i - x^i x_j, \quad \{p_i, p_j\}_D = J_{ij}. \quad (6)$$

In the sequel, we prefer to use the variables $x^i$ and $p_i$ subject to the Dirac brackets (6) instead of defining a set of independent variables in the hyperboloids. Therefore, from now on, as we shall work in the configuration space $H^3$ where only Dirac brackets will be appropriate, we suppress the label $D$, such as it appears in (6).
Using (6), we obtain:

\[
\{J_{ij}, J_{kl}\} = g_{ij} J_{kl} + g_{jk} J_{il} - g_{ik} J_{jl} - g_{jl} J_{ik},
\]

\[
\{J_{ij}, x_l\} = g_{jl} x_i - g_{il} x_j.
\]

(7)

Therefore, the generators $J_{ij}$ span the geometrical symmetry group for the hyperboloid, $SO(3,1)$. Its Casimirs are

\[
C = \frac{1}{2} J_{ij} J^{ij}, \quad \text{and} \quad \tilde{C} = \epsilon^{ijkl} J_{ij} J_{kl}.
\]

(8)

With this realization, we have that $\tilde{C} = 0$ and $C$ coincides up to a sign with the Hamiltonian derived from the above Lagrangian. The fact that we are moving on the hyperboloid is characterized by the constraint condition (3) plus the gauge condition (5).

Using (7), we can obtain the following commutators:

\[
\{C, x_i\} = -2J_{ik} x^k, \quad \{C, J_{ik} x^k\} = 2C x_i.
\]

(9)

Since $H = -C$ and $H$ is given by formula (49), then we conclude that $-C$ is positive, one may denote by $\sqrt{-C}$ its unique positive square root. Note that $H = -C$ shows that the Hamiltonian $H$ is positive.

Then, using the above commutators we obtain the following new ones:

\[
\{\sqrt{-C}, x_i\sqrt{-C}\} = J_{ik} x^k,
\]

(10)

\[
\{\sqrt{-C}, J_{ik}\sqrt{-C}\} = \sqrt{-C} x_i,
\]

(11)

\[
\{\sqrt{-C} x_i, \sqrt{-C} x_j\} = J_{ij},
\]

(12)

\[
\{J_{ik} x^k, J_{jl} x^l\} = -J_{ij},
\]

(13)

\[
\{\sqrt{-C} x_i, J_{jk} x^k\} = -\sqrt{-C} g_{ij}.
\]

(14)

Then, if we use the following notation:

\[
M_{ij} := J_{ij}, \quad M_{5i} := \sqrt{-C} x_i, \quad M_{6i} := J_{ik} x^k, \quad M_{55} := \sqrt{-C},
\]

(15)

we note that the $\{M_{ab}\}$ satisfy the following commutation relations (henceforth, indices $a, b$ will run out from 1 to 6):
\[ \{M_{ab}, M_{cd}\} = g_{ab} M_{cd} + g_{bc} M_{ad} - g_{ac} M_{bd} - g_{bd} M_{ac}, \] (16)

where the metric \( g_{ab} \) is given by

\[ g_{ab} = \text{diag} (1, 1, 1, -1, -1, 1). \] (17)

Observe that the metric \( g_{ab} \) has the signature \((4, 2)\). This fact and the explicit form of the commutation relations (16) shows that the \( \{M_{ab}\} \) are the generators of the algebra \( \mathfrak{so}(4, 2) \).

It is also important to remark that in the realization given by (15) the following relations hold:

\[ T_{ab} := M_{ac} M_{bd} g^{cd} = 0 \quad \text{and} \quad R^{ab} := \epsilon^{abdef} M_{cd} M_{ef} = 0, \] (18)

where \( \epsilon^{abdef} \) is the completely antisymmetric tensor. Relations (18) are called the restrictive relations for this representation of the algebra \( \mathfrak{so}(4, 2) \). These relations do not change under the action of the algebra, since a direct calculation using (16) shows that \( \{M_{ab}\} \) and \( \{R^{cd}\} \) are \( \mathfrak{so}(4, 2) \) two-tensors. The situation is in complete analogy to the similar problem on \( S^3 \) already studied in [5].

\section{Quantum SGA}

Next, we are going to introduce the quantum version of the previous study. If in Section 2, we have given a representation of the algebra \( \mathfrak{so}(4, 2) \) suitable for a description of the free particle on the hyperboloid \( \mathcal{H}^3 \), now we proceed by giving a representation of this algebra such that its elements are operators on a Hilbert space. To implement this objective, we transform Dirac brackets (16) into commutators, which have the following form:

\[ [M_{ab}, M_{cd}] = -i (g_{ab} M_{cd} + g_{bc} M_{ad} - g_{ac} M_{bd} - g_{bd} M_{ac}), \] (19)

where we require the \( M_{ab} \) to be Hermitian operators on a suitable Hilbert space. In the sequel, we shall use the following standard notation:

\[ J_{ij} := M_{ij}, \quad i, j = 1, \ldots, 4, \]

\[ K_i := M_{5i}, \quad L_i := M_{6i}, \quad h := M_{56}. \] (20)
In matrix form, we can write

\[
M_{ab} = \begin{pmatrix}
J_{ij} & K_i & L_i \\
-K_i & 0 & h \\
-L_i & -h & 0
\end{pmatrix}, \quad i, j = 1, \ldots, 4.
\]  

(21)

Commutation relations (19), along with definitions (20), give explicitly:

\[
[J_{ik}, J_{lm}] = -i(g_{im}J_{kl} + g_{kl}J_{im} - g_{il}J_{km} - g_{km}J_{il})
\]

\[
[K_i, K_j] = i\delta_{ij}
\]

\[
[K_i, L_j] = -[L_i, J_{ij}] = i - J_{ij}, \quad [K_i, L_j] = ig_{ij}h
\]

\[
[h, K_i] = -iL_i, \quad [h, L_i] = -iK_i.
\]  

(22)

Note that \(g_{55}\) and \(g_{66}\) have opposite sign. Then, the generator \(M_{56}\) has always hyperbolic character. In consequence, \(h\) is a noncompact generator corresponding to hyperbolic rotations. It can be easily shown that

\[
H = -\frac{1}{2} J_{ij} J^{ij} = 1 + h^2.
\]  

(23)

At this point it is interesting to note that as is well known [7], the spectrum of \(C = \frac{1}{2} J_{ij} J^{ij}\) is given by \(-1 + \zeta^2\), where \(\zeta\) runs either into the real interval \([-1, 1]\) or into the imaginary axis. This shows that the spectrum of \(C\) is non positive and therefore \(C \leq 0\). Correspondingly, \(SO(3,1)\) has two series of unitary irreducible representations: the principal series labeled by values of \(\zeta = i\rho\) with \(\rho \in (-\infty, \infty)\) and the supplementary series labeled by \(\zeta \in [-1, 1]\). The spectrum of \(C\) has the form \(-1 - \rho^2 \leq -1\) in the first case and \(-1 < -1 + \zeta^2 < 0\) in the second.

### 3.1 Restrictive relations

We have shown in [5] the importance of the quantum version of the restrictive relations in order to fix the representation of the algebra \(\mathfrak{so}(4,2)\). These restrictive relations are the symmetrized version of (13) and can be written
in the following form:
\begin{align}
T_{ab} &= (M_{ad}M_{be} + M_{be}M_{ad})g^{de} + cg_{ab} = 0, \quad (24) \\
R^{ab} &= \varepsilon^{abcdef}(M_{cd}M_{ef} + M_{ef}M_{cd}) = 0, \quad (25)
\end{align}
where \( c \) is a constant to be determined later. In terms of the notation proposed in (20) and (25), \( T_{ab} = 0 \) is equivalent to the following set of equations:
\begin{align}
T_{ij} &= J_{ik}J^k_j + J^k_jJ_{ik} - (K_iK_j + K_jK_i) + (L_iL_j + L_jL_i) + cg_{ij} = 0, \quad (26) \\
T_{5i} &= (hL_i + L_ih) - (J_{ij}K^j_i + K^j_iJ_{ij}) = 0, \quad (27) \\
T_{6i} &= hK_i + K_ih - (J_{ij}L^j_i + L^j_iJ_{ij}) = 0, \quad (28) \\
T_{56} &= K_iL^j_i + L^j_iK_i = 0, \quad (29) \\
T_{55} &= 2(K_i^2 + h^2) - c = 0, \quad (30) \\
T_{66} &= 2(L_i^2 - h^2) + c = 0. \quad (31)
\end{align}

Analogously, for \( R^{ab} = 0 \), we have
\begin{align}
R^{ij} &= 0 \implies K_iL_j + L_jK_i - (L_iK_j + K_jL_i) - 2hJ_{ij} = 0, \quad (32) \\
R^{5i} &= 0 \implies \varepsilon^{ijkl}(L_jJ_{kl} + J_{kl}L_j) = 0, \quad (33) \\
R^{6i} &= 0 \implies \varepsilon^{ijkl}(K_jJ_{kl} + J_{kl}K_j) = 0, \quad (34) \\
R^{56} &= 0 \implies \varepsilon^{ijkl}J_{ij}J_{kl} = 0. \quad (35)
\end{align}

The space \( \mathcal{H} \) supporting this representation of the algebra \( \mathfrak{so}(4, 2) \) is given by the vectors \( \psi \) such that \( T_{ab}\psi = 0 \) and \( R^{ab}\psi = 0 \).

The algebra \( \mathfrak{so}(4, 2) \), together with the above restrictive relations, is the SGA for the quantum free motion on \( \mathcal{H}^3 \). In order to justify this terminology, we need to define creation and annihilation operators (although the spectrum for the free particle in our case cannot be expected to be discrete). As we did in the case of the free particle in \( S^3 \) [5], a natural choice seems to be
\begin{equation}
A_i^\pm := K_i \pm L_i, \quad i = 1, 2, 3, 4. \quad (36)
\end{equation}
Since $K_i$ and $L_i$ should be Hermitian so they are $A_i^\pm$. These operators satisfy the following important commutation relations:

$$A_i^+(A_i^-)^* = A_i^-(A_i^+)^* = 0, \quad i = 1, 2, 3, 4, \quad (37)$$

$$[A_i^+, A_j^+] = [A_i^-, A_j^-] = 0, \quad i, j = 1, 2, 3, 4, \quad (38)$$

$$[A_i^+, A_j^-] = -2i(J_{ij} + g_{ij}h), \quad i, j = 1, 2, 3, 4. \quad (39)$$

We write separately the next commutation relations due to their importance:

$$hA_j^+ = A_j^+(h - i), \quad hA_j^- = A_j^-(h + i), \quad j = 1, 2, 3, 4, \quad (40)$$

where $i$ is here the imaginary unit.

In principle, the operators $A_i^\pm$ are the naive equivalent of the ladder operators defined for $S^3$ in [5]. However, this procedure by analogy does not work here. This is due to the appearance of the term $i$ in (40) as we shall see later. Note that $A_i^\pm$ as well as their real linear combinations belong to the algebra of generators of $SO(4, 2)$, so that for any unitary irreducible representation of $SO(4, 2)$, real linear combinations of $A_i^\pm$ are represented by means of self-adjoint operators on the Hilbert space supporting this representation.

Now, we look for quantum operators corresponding to the classical coordinates $x^i$. These operators can now be defined as

$$X_i := f(h)(A_i^+ + A_i^-)f(h), \quad i = 1, 2, 3, 4, \quad (41)$$

where $f(h)$ is a function of $h$ that we determine by the hypothesis that $[X_i, X_j] = 0$ and that $X_iX_i$ is a c-number (which we shall choose to be $\pm 1$). These two conditions not only determine $f(h)$, but also the number $c$ in (26) that happens to be $c = 2$. The function $f(h)$ is

$$f(h) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{h}}, \quad (42)$$

which after (36) and (41) implies that

$$X_i = \frac{1}{\sqrt{h}} K_i \frac{1}{\sqrt{h}} \iff K_i = \sqrt{h} X_i \sqrt{h}, \quad i = 1, 2, 3, 4. \quad (43)$$

The conditions we have enforced on $X_i$ correspond to the requirement that they are coordinates of the ambient Lobachevski space $x^i x_i = -1$ and also that they satisfy the proper gauge condition on the hyperboloid.
Now, let us find an interesting relation for \( L_i \) in terms of \( X_i, J_{ik} \) and \( h \). From (36), it readily follows the first identity in (40) that

\[
hL_i + L_i h = \frac{1}{2} \{ h(A_i^+ - A_i^-) + (A_i^+ - A_i^-)h \} = \frac{1}{2} g(h)(A_i^+ - A_i^-)g(h),
\]

while the second comes from the commutation relations (22) involving \( h \). The function \( g(h) \) should be determined from those relations. Equation \( g(h) \) should fulfil the equation

\[
g(h)g(h + i) = 2h + i. \tag{45}
\]

Equation (45) can be solved after some work and gives as solution:

\[
g(h) = 2 \left[ \frac{\Gamma \left(\frac{i}{2} + \frac{3}{4}\right)}{\Gamma \left(\frac{i}{2} + \frac{1}{4}\right)} \frac{\Gamma \left(-\frac{i}{2} + \frac{3}{4}\right)}{\Gamma \left(-\frac{i}{2} + \frac{1}{4}\right)} \right]^{1/2}. \tag{46}
\]

Due to the fact that \( \Gamma(z^*) = \Gamma^*(z) \), where the star denotes complex conjugation, the function \( g(h) \) is always positive (we take the principal branch in the square root).

Now, combining (44), (36) and the restrictive relation (27), we find that

\[
L_i = g^{-1}(h) \sqrt{h} (J_{ik} X^k + X^k J_{ik}) \sqrt{h} g^{-1}(h). \tag{47}
\]

At this point, we have completed the first task: we have constructed all the generators of the SGA as functions of the operators \( h, X_i \) and the geometrical generators \( J_{ij} \). In particular \( L_i \) and \( K_i \) are given by formulas (43) and (47). In the next section, we shall construct the space of states of our system with the help of the SGA.

4 The quantum free Hamiltonian on \( \mathcal{H}^3 \)

Hereafter, we shall use the following notation: \( x_\alpha^2 = x_1^2 + x_2^2 + x_3^2 \), and \( \dot{x}_\alpha x^\alpha = \dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{x}_3 x_3 \). Note that Greek indices \( \alpha \) and \( \beta \) run from 1 to 3 and that for these indices the distinction between upper and lower makes no sense. From the hyperboloid equations, \( x_i x^i = -1 \), we obtain that \( x_4 = \sqrt{x_\alpha^2 + 1} \). Using this chart of coordinates for \( \mathcal{H}_1^3 \), we obtain the following new expression
for the classical Lagrangian (1),
\[
L = \frac{1}{2} \left( \dot{x}_\alpha - \frac{\dot{x}_\alpha x_\alpha}{x_\alpha^2 + 1} \right).
\]
(48)
This Lagrangian is written in terms of the coordinates \( x_\alpha \). Its Legendre transformation (with respect to the independent variables \( x_\alpha, \alpha = 1, 2, 3 \)) gives the Hamiltonian:
\[
H = \frac{1}{2} \left( p_\alpha^2 + (x_\alpha p_\alpha)^2 \right),
\]
(49)
where \( x_\alpha \) and \( p_\beta \) are the classical canonical conjugate coordinates of the position and momentum respectively.

Then, let us proceed with the quantization of this system.

For \( \mathcal{H}^3 \), the Hilbert space of states will be the space of all Lebesgue measurable functions on the hyperboloid with a metric which is the restriction of the Lebesgue measure on \( \mathbb{R}^{3+1} \) to the hyperboloid. This is
\[
\langle \varphi | \psi \rangle := \int d^4x \delta(x_i x^i + 1) \varphi^*(x) \psi(x) = \int_{\mathbb{R}^3} \frac{d^3x_\alpha}{\sqrt{x_\alpha^2 + 1}} \varphi(x) \psi(x),
\]
(50)
where \( x = (x_1, x_2, x_3) \), \( \alpha = 1, 2, 3 \).

Canonical quantization of the classical free Hamiltonian (49) gives us a quantum Hamiltonian having the same expression as (49), by replacing \( x_\alpha \) and \( p_\beta \) by a pair of canonical conjugate operators for the components of the position \( X_\alpha \) and momentum \( P_\beta \) respectively. Note that \( P_\alpha \) is not just partial derivation with respect to \( x_\alpha \) multiplied by \( i \), since this operator is not Hermitian with respect to the scalar product (50). Instead, we should define
\[
P_\alpha := (X^2 + 1)^{1/4} (-i \partial_\alpha) (X^2 + 1)^{-1/4},
\]
(51)
where \( \partial_\alpha \) is the partial derivative with respect to \( x_\alpha \) and \( X = (X_1, X_2, X_3) \). These \( P_\alpha \) together with \( X_\alpha \) (defined as multiplication operators) satisfy the canonical commutation relations \( [X_\alpha, P_\beta] = i \delta_{\alpha\beta} \). Further, we have to remark that the operators \( X_i, i = 1, 2, 3, 4 \) are realizations of the coordinate operators in the ambient space defined in the previous subsection since they satisfy the same commutation relations and the same gauge condition \( x_i x^i = -1 \). Then, we shall use capital letters to denote these operators \( X_i \) in the sequel. From
this expression for the $P_\alpha$, we can obtain the Hermitian version of the classical Hamiltonian (49) as

$$H = -\frac{1}{2} \sqrt{X^2 + 1} \partial_\alpha \left( \frac{\delta_{\alpha\beta} + X_\alpha X_\beta}{\sqrt{X^2 + 1}} \partial_\beta \right),$$

(52)

where we sum over the repeated indices $\alpha$ and $\beta$ running from 1 to 3.

Our next goal is to solve the Schrödinger equation

$$H \psi = \lambda \psi$$

(53)

associated to this Hamiltonian. By the form of $H$ in (52), we see that $H$ cannot have bound states and that the solutions of $H \psi = \lambda \psi$ are not expected to be normalizable. As the ambient space is $\mathbb{R}^{3+1}$, we expect these wave functions to depend on the four dimensional vector $x^i$. We are looking for those special solutions of the Schrödinger equation (53) which depend on the four variables $x_i$ through the single combination $f := k_i x^i = k_\alpha x^\alpha - \sqrt{x^2 + 1} k_4, x_4 = \sqrt{x^2 + 1}$ showing that the $x_i$ coordinates denote points in the hyperboloid. In terms of this variable $f$, (53) has the form:

$$[f^2 + (k^2 - k_4^2)] \psi''(f) + 3 f \psi'(f) = -\lambda \psi(f),$$

(54)

where the primes indicate derivative with respect to $f$ and $k^2 = k_\alpha k^\alpha$. The simplest situation in (54) happens when $k^2 - k_4^2 = 0$. In this case, the general solution for (54) is given by

$$\psi(f) = C_1 f^{-1+\sqrt{1-\lambda}} + C_2 f^{-1-\sqrt{1-\lambda}},$$

(55)

where $C_i, i = 1, 2$ are arbitrary constants. In the general case, $k_i$ lie on a hyperboloid of the form $k^2 - k_4^2 = -m^2$ ($m^2$ may be positive or negative). Now, the general solution has the form

$$\psi(f) = C_1 \frac{f + \sqrt{f^2 - m^2} i^\rho}{\sqrt{f^2 - m^2}} + C_2 \frac{f + \sqrt{f^2 - m^2}^{-i^\rho}}{\sqrt{f^2 - m^2}},$$

(56)

where $\lambda = (1 + \rho^2)$. Note that in the limiting case $m = 0$, this solution is equal to (55) that in terms of $\rho$ is

$$\psi(f) = C_1 f^{-1+i^\rho} + C_2 f^{-1-i^\rho}.$$  

(57)

We shall use this notation in the sequel. It is important to insist that the parameter which appears in the Schrödinger equation is $\lambda$. For each fixed
value of $\lambda$, there exists two linearly independent solutions as shown in (57). The relation between $\lambda$ and $\rho$ is given by $\lambda = (1 + \rho^2)$ as given before. On the other hand, we may look at $\rho$ as the basic parameter. Then, there is a unique solution for each $\rho \in (-\infty, \infty)$ given by (with $k_i$ in the cone $k^2 - k_i^2 = 0$):

$$\psi(f) = \psi(x_\alpha, k_\alpha, \rho) = (x_\alpha k_\alpha - \sqrt{x^2 + 1} k_4)^{-1 + i\rho}.$$  (58)

Then, we have obtained solutions of the Schrödinger equation with Hamiltonian (52) labeled by $k_\alpha$ and $\rho$, which play the same role than the components of the momentum for the standard Euclidean plane waves $\phi(x, p) = e^{i x \cdot p}$.

Next, let us consider the quantized version of the components of the antisymmetric tensor $J_{ij}$, $i, j = 1, 2, 3, 4$. We have

$$J_{\alpha\beta} = X_\alpha P_\beta - X_\beta P_\alpha, \quad J_{4\alpha} = \sqrt{X^2 + 1} P_\alpha, \quad \alpha, \beta = 1, 2, 3.$$  (59)

The components $J_{ij}$ satisfy the commutation relations for the generators of $SO(3,1)$. We also can readily show that the relation between the quantum analog of Hamiltonian (49) and $J_{ij}$ is given by

$$H = -\frac{1}{2} J_{ij} J^{ij} = 1 + h^2, \quad \text{with} \quad h = M_{56}.$$  (60)

Then, it is clear after (53) and (60) that $h \psi(f) = \rho \psi(f)$. In order to obtain the action of $L_i$ and $K_i$, we shall use an operator with four components $T_i$, $i = 1, 2, 3, 4$. If $X_4 := \sqrt{X^2 + 1}$, let us define $T_i$ by

$$T_i := J_{ij} X^j + X^j J_{ij},$$  (61)

then,

$$T_\alpha = i(X_\alpha(2X_\beta \partial_\beta + 3) + 2\partial_\alpha), \quad T_4 = i\sqrt{X^2 + 1} (2X_\beta \partial_\beta + 3),$$  (62)

where $\alpha, \beta = 1, 2, 3$ and we sum over repeated indices. These four operators are Hermitian with respect to the scalar product (50) and so is the operator $T_i k^i$, where we define the components $k^i$ as above. This is

$$T_i k^i = i(X_i k^i (2X_\beta \partial_\beta + 3) + 2k_\alpha \partial_\alpha).$$  (63)

The action (63) on the wave functions (58) is

$$[T_i k^i] \psi(x_\alpha, k_\alpha, \rho) = [T_i k^i] (x_\alpha k_\alpha - \sqrt{x^2 + 1} k_4)^{-1 + i\rho}[T_i k^i] (x_i k^i)^{-1 + i\rho} = -(2\rho - i)(x_i k^i)^{i\rho} = -(2\rho - i)(x_i k^i)^{-1 + i(\rho - i)} = -(2\rho - i) \psi(x_\alpha, k_\alpha, \rho - i).$$  (64)
As we can see from (64), the action of the operator $T_i k^i$ on $\psi(x_\alpha, k_\alpha, \rho)$ shifts $\rho$ by $-i$.

The action of other operators on the wave functions $\psi(x_\alpha, k_\alpha, \rho)$ can also be given. For instance, take expression (47) for the operators $L_i$, which now can be defined with the new $J_{ij}$ and have the same commutation relations than those in the previous section (and therefore they should be identified). We obtain:

$$[L_i k^i] \psi(x_\alpha, k_\alpha, \rho) = [L_i k^i] (x_i k^i)^{-1+ip}$$

$$= \sqrt{\rho (\rho - i)} (g(\rho)g(\rho - i))^{-1} [t_i k^i] (x_i k^i)^{-1+ip}$$

$$= \sqrt{\rho (\rho - i)} (g(\rho)g(\rho - i))^{-1} (2\rho - i)(x_i k^i)^{ip} \quad (65)$$

where $g(\rho)$ has been given in (46) and therefore satisfies relation (45). Then, if we apply (45) into (65), we finally get:

$$[L_i k^i] \psi(x_\alpha, k_\alpha, \rho) = [L_i k^i] (x_i k^i)^{-1+ip}$$

$$= -\sqrt{\rho (\rho - i)} (x_i k^i)^{ip} = -\sqrt{\rho (\rho - i)} \psi(x_\alpha, k_\alpha, \rho - i) . \quad (66)$$

Following similar procedures, we can obtain the action of the operator $K_i k^i$ with $K_i$ as in (43) into $\psi(x_\alpha, k_\alpha, \rho)$. This gives:

$$[K_i k^i] \psi(x_\alpha, k_\alpha, \rho) = \sqrt{\rho (\rho - i)} \psi(x_\alpha, k_\alpha, \rho - i) . \quad (67)$$

From relations (36), we can determine the action of $A_i^\pm k^i$ into $\psi(x_\alpha, k_\alpha, \rho)$. This is

$$A_i^- k^i \psi(x_\alpha, k_\alpha, \rho) = (K_i - L_i) k^i \psi(x_\alpha, k_\alpha, \rho) = 0 \quad (68)$$

$$A_i^+ k^i \psi(x_\alpha, k_\alpha, \rho) = (K_i - L_i) k^i \psi(x_\alpha, k_\alpha, \rho) = 2\sqrt{\rho (\rho - i)} \psi(x_\alpha, k_\alpha, \rho - i) . \quad (69)$$

We discuss the consequences of (69) in the next subsection.
4.1 The ladder operators

Let us go back to equation (40). Assume that $\psi$ is an eigenvector of $h$, $h\psi = \rho \psi$. Then, (40) gives $h(A_i^+ \psi) = A_i^+(h - i)\psi = (\rho - i)(A_i^+ \psi)$. Since $A_i^+$ is Hermitian (and so is $A_i^+ k^i$), this shows that $A_i^+ \psi$ (and also $A_i^+ k^i \psi$) is not in the domain of $h$. Therefore, it is not possible in principle to use $A_i^\pm$ as the ladder operators for the eigenvectors of $h$. This is related to the fact that the spectrum of $h$ is continuous.

Fortunately, this is not the end of the story. In order to find a clue on how to proceed, let us analyze the simplest case of $SO(2,1)$. Its Lie algebra $\mathfrak{so}(2,1)$ has generators $l_0$, $l_1$ and $l_2$ with commutation relations

$$[l_1, l_2] = -i l_0, \quad [l_0, l_1] = i l_2, \quad [l_0, l_2] = -i l_1.$$  

The algebra $\mathfrak{so}(2,1)$ has a generator of a compact subgroup which is $l_0$ (sometimes, generators of compact subgroups are denoted as compact generators, which does not mean that they are compact in the ordinary sense of compact operators on Banach spaces). In order to construct a unitary irreducible representation of $SO(2,1)$, we may use the subspace spanned by the eigenvectors of $l_0$. We may write

$$l_0 |m\rangle = m |m\rangle,$$  

where we have labeled as $m$ the eigenvalues of $l_0$ for convenience. Ladder operators can be defined in this case as $A^\pm = l_1 \pm i l_2$, so that

$$A^\pm |m\rangle = a^\pm (m) |m \pm 1\rangle,$$  

where the coefficients $a^\pm (m)$ should be determined by using commutation relations (70) and unitarity conditions for the elements of $SO(3,1)$.

This is a standard procedure, but it would be interesting to investigate what would happen if instead of $l_0$ we had insisted in building the same construction with a noncompact operator, say $l_1$. If the real number $\lambda$ is in the continuous spectrum of $l_1$ with generalized eigenvector $|\lambda\rangle$ [8], we have

$$l_1 |\lambda\rangle = \lambda |\lambda\rangle.$$  

Commutation relations (70) trivially give

$$[l_1, l_0 \pm l_2] = \mp i (l_0 \pm l_2).$$  

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which implies that the vector given by

\[ |\tilde{\lambda}\rangle := (l_0 + l_2)|\lambda\rangle \] (75)

should be an (generalized) eigenvector of \( l_1 \) with eigenvalue \( \lambda - i \):

\[ l_1|\tilde{\lambda}\rangle = (\lambda - i)|\tilde{\lambda}\rangle . \] (76)

This shows that the subspace spanned by the generalized eigenvectors \(|\lambda\rangle\) of \( l_1 \), with \( \lambda \) in the spectrum of \( l_1 \), cannot support a unitary representation for SO(2,1). This problem was already discussed in the literature [11]. We shall introduce here another point of view.

This new point of view is the essential point in the construction of a SGA for a Hamiltonian with continuous spectrum and we introduce it as follows:

According to (36) operators \( A_i^\pm \) are Hermitian and so are \( A_i^\pm k^i \). Therefore, a measurable function of a self adjoint version of \( A_i^\pm \) is well defined according to the spectral representation theorem. Our goal is to find a workable expression for \((A_i^\pm k^i)^{-iu}\) with \( u \) real and to show that

\[ h(A_i^\pm k^i)^{-iu} = (A_i^\pm k^i)^{-iu}(h - u) . \] (77)

In order to prove this formula, let us note that as a straightforward consequence of (40) is that

\[ h(A_i^\pm k^i)^n = (A_i^\pm k^i)^n(h - in) . \] (78)

Then, for any complex number \( z \) one has

\[ h \exp\{-z(A_i^\pm k^i)\} = \exp\{-z(A_i^\pm k^i)\}(h + iz(A_i^\pm k^i)) , \] (79)

whenever the exponential be correctly defined. As a matter of fact, the exponential in (79) is not always defined and it should be considered as an abbreviate form of writing the formal series

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1Roughly speaking, the vectors \( \psi \) of this space should admit a span in terms of the \(|\lambda\rangle\) in the form \( \psi = \int (|\psi\rangle|\lambda\rangle) d\mu(\lambda) \). Details in [8].

2If we use a unitary irreducible representation of SO(4,2), the elements of the Lie algebra \( \mathfrak{so}(4,2) \) are represented by self adjoint operators [9] and therefore \( A_i^\pm k^i \) can be represented by self adjoint operators so that \((A_i^\pm k^i)^{-iu}\) can be well defined via spectral theory [10] with a proper choice of a branch for the logarithm.
\[ \sum_{\ell=1}^{\infty} \frac{(A^+_i k^i)^\ell}{\ell!}. \quad (80) \]

In order to prove (77), we shall make use of the following integral representation [12]:
\[ \int_0^\infty dz \frac{z^{-1+iu}}{\Gamma(iu)} e^{-z\mu} = \mu^{-iu}, \quad (81) \]
where \( u \) is a real number. Now replace \( \mu \) by \( A^+_i k^i \) in the above expression and multiply the resulting formal expression by \( h \) to the left. After (81) this gives:
\[ h \int_0^\infty dz \frac{z^{-1+iu}}{\Gamma(iu)} e^{-z\mu} = \int_0^\infty dz \frac{z^{-1+iu}}{\Gamma(iu)} e^{-z\mu}(h + i\mu z) \]
\[ = \int_0^\infty dz \frac{z^{-1+iu}}{\Gamma(iu)} e^{-z\mu} + \int_0^\infty dz \frac{z^{-1+iu}}{\Gamma(iu)} e^{-z\mu} i\mu z. \quad (82) \]

Again, we use (81) in the last term of (82):
\[ \int_0^\infty dz \frac{z^{-1+iu}}{\Gamma(iu)} e^{-z\mu} = i\mu \mu^{-1-iu} \frac{\Gamma(iu + 1)}{\Gamma(iu)} = -u \mu^{-iu}. \quad (83) \]

From there, we readily obtain the commutation relations (77). If we recall that \( h \psi(x_\alpha, k_\alpha, \rho) = \rho \psi(x_\alpha, k_\alpha, \rho) \), it becomes obvious that
\[ h \left[ (A^+_i k^i)^{-iu} \psi(x_\alpha, k_\alpha, \rho) \right] = (\rho - u) (A^+_i k^i)^{-iu} \psi(x_\alpha, k_\alpha, \rho), \quad (84) \]
so that \( (A^+_i k^i)^{-iu} \psi(x_\alpha, k_\alpha, \rho) \) is an eigenfunction of \( h \) with eigenvalue \( \rho - u \). Therefore, it must exist a constant depending on \( u \) and \( \rho \), \( g(u, \rho) \) such that
\[ [(A^+_i k^i)^{-iu} \psi(x_\alpha, k_\alpha, \rho)] = g(u, \rho) \psi(x_\alpha, k_\alpha, \rho - u). \quad (85) \]

The function \( g(u, \rho) \) satisfies the following properties
\[ g(0, \rho) = 0, \quad g(i, \rho) = 2\sqrt{\rho(\rho - i)}, \quad (86) \]
where the first relation in (86) is obvious and the second come from (69). These relations will be useful in order to find the final expression for \( g(u, \rho) \). Another property for \( g(u, \rho) \) can be obtained from
\((A_i^+ k_i)^{-i\nu - i\mu} \psi(x_i, k_i, \rho) = g(u + v) \psi(x_i, k_i, \rho - u - v)\),
\((A_i^+ k_i)^{-i\nu - i\mu} \psi(x_i, k_i, \rho) = (A_i^+ k_i)^{-i\nu} (A_i^+ k_i)^{-i\mu} \psi(x_i, k_i, \rho) = g(u, \rho) (A_i^+ k_i)^{-i\nu} \psi(x_i, k_i, \rho - u) = g(u, \rho) g(v, \rho - u) \psi(x_i, k_i, \rho - u - v)\). (87)

These identities show the following functional identity for \(g(u, \rho)\):
\[ g(u, \rho) g(v, \rho - u) = g(u + v, \rho), \] (88)
which has the following solution:
\[ g(u, \rho) = \left[ 2^{-iu} \frac{\Gamma(-i\rho) \Gamma(1-i\rho) \Gamma(-i(\rho - u)) \Gamma(1 - i(\rho - u))}{\Gamma(-i\rho) \Gamma(1-i\rho) \Gamma(-i(\rho - u)) \Gamma(1 - i(\rho - u))} \right]^\frac{1}{2}. \] (89)

This completes the discussion on the construction of the ladder operators and their action on the hyperboloid plane waves \(\psi(x_i, k_i, \rho)\). In the next section, we shall discuss an interesting formula giving an eigenfunction expansion of functions over \(\mathcal{H}^3_1\).

5 General properties of the eigenfunctions of the Hamiltonian

Here, we start with an infinitely differentiable function \(f(x_i)\) on the hyperboloid \(\mathcal{H}^3_1\), with equation \(x_i x^i = -1\). The function \(f(x_i)\) can be transformed into the function \(h(k_i)\) on the cone \(k_i k^i = 0\) by means of the following integral:
\[ h(k_i) = \int Dx f(x_i) \delta(x_i k^i - 1), \] (90)
where \(Dx\) represents here the invariant measure on the hyperboloid (or equivalently the restriction of the Lebesgue measure on the hyperboloid):
\[ Dx = \frac{d^3x_\alpha}{x_4} = \frac{d^3x_\alpha}{\sqrt{x_\alpha^2 + 1}}, \quad \alpha = 1, 2, 3. \] (91)

The integral (90) for functions \(f(x_i)\) on the hyperboloid gives a function on the cone. This type of transformation has been considered by Gelfand
and Graev [6] and holds their name (Gelfand-Graev transformation). This Gelfand-Graev transform has an inverse which is given by

\[ f(x_i) = -\frac{1}{8\pi^2} \int Dk \, h(k^i) \, \delta''(x_i k^i - 1), \quad (92) \]

where \( Dk \) is the measure on the cone given by

\[ Dk = \frac{d^3 k}{k^4}. \quad (93) \]

Then, let us consider the Mellin transform of the function \( h(k^i) \), which is defined as:

\[ \phi(k^i, \rho) = \int_0^\infty dt \, h(tk^i) \, t^{-i\rho}. \quad (94) \]

This Mellin transform has the following inversion formula:

\[ h(k^i) = \frac{1}{2\pi} \int_{-\infty}^\infty d\rho \, \phi(k^i, \rho). \quad (95) \]

If we replace in (94) the expression (90) for \( h(k^i) \), we find

\[ \phi(k^i, \rho) = \int_0^\infty dt \int Dx \, f(x_i) \, \delta(x_i k^i t - 1) \, t^{-i\rho} \]

\[ = \int Dx \, f(x_i) \int_0^\infty dt \, \delta(x_i k^i t - 1) \, t^{-i\rho} = \int Dx \, f(x_i) (x_i k^i)^{-1+i\rho}. \quad (96) \]

Equation (96) shows that \( \phi(k^i, \nu) \) is an homogenous function on the cone of degree \(-1 + i\rho\) and therefore it can be defined by its values on any contour which crosses all generatrices of the cone. Equation (96) can be looked as a generalization of the Fourier transform of the function \( f(x_i) \) with respect to the integral kernel \( \{(x_i k^i)^{-1+i\rho}\} \). In fact this integral kernel is formed up to plane waves in the same way that the standard Fourier transform has as integral kernel the Euclidean plane waves \( e^{i(x \cdot p)} \). Then, we can find the inverse transformation of (96) by entering (95) into (92). The result is given under the form of the following integral:

\[ f(x_i) = -\frac{1}{16\pi^3} \int Dk \int_{-\infty}^\infty d\rho \, \phi(k^i, \rho) \, \delta''(x_i k^i - 1). \quad (97) \]
We can rewrite the right hand side of (97) in the following form:

$$f(x_i) = -\frac{1}{16\pi^3} \int_{-\infty}^{\infty} dt \int Dk \int_{-\infty}^{\infty} d\rho \phi(k^i, \rho) \delta''(x_i k^i - t) \delta(t - 1). \quad (98)$$

After integrating by parts twice with respect to the variable $t$, we obtain:

$$f(x_i) = -\frac{1}{16\pi^3} \int_{-\infty}^{\infty} dt \delta''(t - 1) \int Dk \int_{-\infty}^{\infty} d\rho \phi(k^i, \rho) \delta(x_i k^i - t). \quad (99)$$

Now, let us make the change of variables given by $k^i \mapsto t\tilde{k}^i$ in the integrand of (99). The result is

$$\int Dk \int_{-\infty}^{\infty} d\rho \phi(k^i, \rho) \delta(x_i k^i - t) = \int \tilde{k} \int_{-\infty}^{\infty} d\rho \phi(\tilde{k}^i t, \rho) \delta(x_i \tilde{k}^i t - t)$$

$$= \int \tilde{k} \int_{-\infty}^{\infty} d\rho \phi(\tilde{k}^i t, \rho) t^{-1+ip} \delta(x_i \tilde{k}^i - 1)t^{-1}$$

$$= \int \tilde{k} \int_{-\infty}^{\infty} d\rho \phi(\tilde{k}^i t, \rho) t^{-p} \delta(x_i \tilde{k}^i - 1). \quad (100)$$

Then, if we use (100) into (99) and integrate over $t$, we obtain the desired representation of $f(x_i)$ in terms of its generalized Fourier components $\phi(k^i, \rho)$:

$$f(x_i) = \frac{1}{16\pi^3} \int_{-\infty}^{\infty} d\rho \rho^2 \int Dk \phi(k^i i, \rho) \delta(x_i k^i - 1). \quad (101)$$

Thus, we have found an analogue the Fourier transform for functions over an hyperboloid. We may wonder on whether it is also an analogue of the Plancherel formula in this case. Let us consider the following parametrization for the four vector with components $k^i$:

$$k^i = \omega N^i = (\mathbf{n}, 1)\omega, \quad \mathbf{n}^2 = 1, \quad (102)$$

where $\omega = k^4$. We use $\omega$ in order to get rid of the index. In this parametrization, the measure $Dk$ has the following form:

$$Dk = \omega d\omega d\mathbf{n}, \quad (103)$$
where $dn$ is the restriction of the Lebesgue measure in the three dimensional sphere. Using the fact that the function $\phi(k_i, \rho)$ is homogenous of the degree $-1 + i \rho$, we can derive the following expression:

$$f(x_i) = \frac{1}{16\pi^3} \int_{-\infty}^{\infty} d\rho \rho^2 \int \omega d\omega \, dn \, \phi(N_i, \rho) \omega^{-1+i\rho} \delta(x_i N^i \omega - 1).$$  \hspace{1cm} (104)

If we integrate (104) with respect to $\omega$, one obtains

$$f(x_i) = \frac{1}{16\pi^3} \int_{-\infty}^{\infty} d\rho \rho^2 \int dn \, \phi(N_i, \rho)(x_i N^i)^{-1-i\rho}. \hspace{1cm} (105)$$

Relation (105) gives the inverse Fourier transform of $f(x_i)$ in terms of the eigenfunctions $(x_i N^i)^{-1-i\rho}$ of the Hamiltonian, i.e., what we have called the hyperbolic plane waves or Shapiro waves [13, 14]. For real $\rho$, the set of functions $(x_i N^i)^{-1-i\rho}$ realize a unitary representation for the principal series of $SO(3,1)$.

6 Concluding remarks

We have investigated the possibility of constructing Spectrum Generating Algebras (SGA) for quantum systems showing a purely continuous spectrum. In fact, we have obtained a SGA for the free particle in the three dimensional two sheeted hyperboloid $H^3$. We have done this in two steps. First of all, we have obtained a representation of the Lie algebra $so(4,2)$, by functions of coordinates and momenta, suitable for the description of a classical particle on an one or two sheeted hyperboloid. We have obtained the Dirac-Poisson brackets for the generators of the algebra. In the second step, we have obtained another representation of $so(4,2)$ in which functions are replaced by operators and Dirac brackets by commutators. Following a usual procedure, we construct ladder operators as Hermitian members of the algebra $so(4,2)$.

We have found the solutions of the Schrödinger equation in $H^3$ equivalent to the plane waves in the space $\mathbb{R}^3$. As solutions of a time independent Schrödinger equation, these plane waves are eigenvalues of the Hamiltonian, so that they can be labeled by their energies. Instead, we prefer to label them by the eigenvalues of a related operator $h$ as given in (60), which is one of the generators of the algebra. We denote by $\rho$ to the eigenvalues of $h$. 

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We observe that the ladder operators shift the variable $\rho$ in these solutions by a complex number. This may happen because our generalized plane waves are out of the Hilbert space. In order to avoid this inconvenience, we have introduce some operators which are functions of suitable linear combinations of the ladder operators. These operators are not self adjoint but produce real shifts on the label $\rho$ of the generalized plane waves and can be used as a new form of ladder operators for the continuous spectrum.

Finally, we have discussed a generalized Fourier transform between functions on the three dimensional hyperboloid and functions over a three dimensional cone. This is intimately related to the transformation defined by Graev and Gelfand in [6]. A Plancherel type theorem is valid in this context. We have given an eigenfunction expansion of functions over the hyperboloid in terms of the generalized plane waves on the hyperboloid.

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