THE VACUOLE MODEL: NEW TERMS IN THE SECOND ORDER DEFLECTION OF LIGHT

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Abstract

The present paper is an extension of a recent work (Bhattacharya et al. 2010) to the Einstein- Strauss vacuole model with a cosmological constant, where we work out the light deflection by considering perturbations up to order $M^3$ and confirm the light bending obtained previously in their vacuole model by Ishak et al. (2008). We also obtain another local coupling term $-\frac{5\pi M^2\Lambda}{8}$ related to $\Lambda$, in addition to the one obtained by Séreno (2008, 2009). We argue that the vacuole method for light deflection is exclusively suited to cases where the cosmological constant $\Lambda$ disappears from the path equation. However, the original Rindler-Ishak method (2007) still applies even if a certain parameter $\gamma$ of Weyl gravity does not disappear. Here, using an alternative prescription, we obtain the known term $-\frac{\gamma R}{2}$, as well as another new local term $\frac{3\pi\gamma M}{2}$ between $M$ and $\gamma$. Physical implications are compared, where we argue that the repulsive term $-\frac{\gamma R}{2}$ can be masked by the Schwarzschild term $\frac{2M}{R}$ in the halo regime supporting attractive property of the dark matter.
I. Introduction

Recently, we confirmed the Rindler-Ishak method (Rindler & Ishak, 2007) by calculating light bending in a more general solution, viz., the Mannheim-Kazanas-de Sitter solution of Weyl conformal gravity (Bhattacharya et al. 2010) that contains two parameters Λ and γ, the latter is assumed to play a prominent role in the galactic halo populated by dark matter. The method indeed delivered the effect of γ exactly as it has been known in the literature for long. Subsequently, the Λ—effect has been calculated by Ishak et al. (2008) within the framework of Einstein-Strauss vacuole (its earlier incarnation is the Kottler vacuole). Our broad aim here is to examine how the effects of both Λ and γ appear from suitable considerations in the Weyl conformal gravity.

Galaxies or clusters of galaxies (hereinafter called lenses for brevity) have been modelled as residing in the womb of a de Sitter vacuole much larger than their sizes (Ishak et al. 2008). The model assumes that, for a given lens, the boundary radius \( r_b \) of the vacuole is determined where the spacetime transitions from a Schwarzschild-de Sitter (SdS) spacetime to a cosmological Friedman-Robertson-Walker (FRW) background. Further, all the light-bending occurs in the SdS vacuole and that once the light transitions out of the vacuole and into FRW spacetime, all bending stops. Ishak et al. (2008) showed that the effect of the cosmological constant Λ appears inside the vacuole in the bending of light by different lens systems. They also obtained an upper bound on Λ, using observational uncertainties in the measurement of the bending of light, which turned out to be only two orders of magnitude away from the cosmologically determined value. For a lens of mass \( M \) and radius \( R \), Ishak et al. (2008) obtained light deflection up to second order in \( \frac{M}{R} \) together with a Λ—repulsion term (\( = -\frac{\Lambda R r_b}{6} \)).

The purpose of the present article is to confirm the light deflection in the second order by using perturbations up to third order\(^1\). When we do that we come up with new extra terms, while confirming the most interesting term \( -\frac{\Lambda R r_b}{6} \). Additionally, one might like to have an idea of how the presence of a conformal parameter γ would affect light deflection by the lenses. To this end, we use an alternative prescription for the more general exact Mannheim-Kazanas-de Sitter solution of Weyl conformal gravity that includes the parameter γ. Pure SdS vacuole with only \( M \) and Λ is readily recovered at γ = 0. New coupling terms arising out of the invariant angle have been obtained.

II. The solution and the approximation scheme

One well discussed solution that contains both the conformal γ and dS Λ effects is the Mannheim-Kazanas-de Sitter (MKdS) solution (Mannheim & Kazanas 1989; Mannheim 1997, 2006) of Weyl conformal gravity field equations. The metric is given, in units \( G = c_0 = 1 \), by (see e.g., Edery & Paranjape

\(^1\)We are indebted to an anonymous referee for suggesting that the correct deflection follows from (at least) third order calculation.
\[ dr^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \]  
\[ B(r) = A^{-1}(r) = 1 - \frac{2M}{r} + \gamma r - \frac{\Lambda}{3} r^2, \]  
where \( M \) is the central mass, \( \Lambda \) and \( \gamma \) are constants. The accepted numerical value from current cosmology is \( \Lambda = 1.29 \times 10^{-56} \text{cm}^{-2} \). However, there seems to be some ambiguity about the sign and magnitude of \( \gamma \). Mannheim and Kazanas fix it from flat rotation curve data to be positive and of the order of the inverse Hubble length, while Pireaux (2004) argues for \( |\gamma| \sim 10^{-33} \text{cm}^{-1} \). Edery & Paranjape (1998) obtained a negative value from the gravitational time delay by galactic clusters. We shall keep the value of \( \gamma \) open, but for purely illustrative purposes, take the value \( \gamma = 3.06 \times 10^{-30} \text{cm}^{-1} \) (Mannheim 2006). We emphasize that the methods adopted here do not need to assume any particular value of \( \gamma \)—it is essentially kept free to be fixed by more accurate observations.

In the null geodesic equation, \( \Lambda \) cancels out giving
\[ \frac{d^2 u}{d\varphi^2} = -u + 3Mu^2 - \frac{\gamma}{2}. \]  
We have recently solved the light ray equation using the Rindler-Ishak method (Rindler & Ishak 2007), and have shown how \( \gamma \) and \( \Lambda \) get mixed up in the deflection at higher order (Bhattacharyya et al. 2010). Usual perturbative expansion up to order \( M^3 \) gives the final solution of Eq.(3) as:
\[ u \equiv \frac{1}{r} = \sin \varphi \frac{R}{R} - \frac{\gamma}{2} \\
+ \frac{M}{4R^2} \left[ 6 + 3R^2 \gamma^2 - 3R\gamma(\pi - 2\varphi) \cos \varphi + 2 \cos 2\varphi - 6R\gamma \sin \varphi \right] \\
- \frac{3M^2}{32R^3} \left[ 96R\gamma + 24R^2 \gamma^3 - 10(2 + 3R^2 \gamma^2)(\pi - 2\varphi) \cos \varphi + 32R\gamma \cos 2\varphi \\
- 20 \sin \varphi - 30R^2 \gamma^2 \sin \varphi + 3\pi^2 R^2 \gamma^2 \sin \varphi - 12\pi R^2 \gamma^2 \varphi \sin \varphi \\
+ 12R^2 \gamma^2 \varphi^2 \sin \varphi - 8\pi R\gamma \sin 2\varphi + 16R\gamma \varphi \sin 2\varphi + 2 \sin 3\varphi \right] \\
+ \frac{M^3}{128R^4} \left[ 9R\gamma(\pi - 2\varphi) \left\{ R^2 \gamma^2 (\pi^2 - 150 - 4\pi \varphi + 4\varphi^2) - 260 \right\} \cos \varphi \\
+ 2 \{ 816 + 2916 R^2 \gamma^2 - 36R^2 \gamma^2 (\pi^2 - 27 - 4\pi \varphi + 4\varphi^2) \} \cos 2\varphi \\
+ 27R\gamma(\pi - 2\varphi) \cos 3\varphi - 4 \cos 4\varphi - 1170R\gamma \sin \varphi + 540R^2 \gamma^2 \sin \varphi + 356 \cos 2\varphi \\
+ 108\pi^2 R^2 \gamma^3 \sin \varphi - 360\pi R\gamma \varphi \sin \varphi - 432\pi R^3 \gamma^3 \varphi \sin \varphi + 360R\gamma \varphi^2 \sin \varphi \\
+ 432R^3 \gamma^3 \varphi^2 \sin \varphi - 120\pi \sin 2\varphi - 396\pi R^2 \gamma^2 \sin 2\varphi + 90\pi^2 \gamma \sin \varphi \\
+ 792R^2 \gamma^2 \varphi \sin 2\varphi - 675R^3 \gamma^2 \varphi \sin \varphi + 240\varphi \sin 2\varphi + 126R\gamma \sin 3\varphi \right] \right], \]
where $R$ is related to the closest distance approach $r_0$ defined by ($\varphi = \pi/2$)

$$\frac{1}{r_0} = \frac{1}{16R^3}[4MR(4 - 6R\gamma + 3R^2\gamma^2) - 8R^2(R\gamma - 2)$$
$$+ M^2(33 - 66R\gamma + 45R^2\gamma^2 - 63R^3\gamma^3)]$$
$$+ \frac{3M^3}{64R^4}[152 - 432R\gamma + 648R^2\gamma^2 - 225R^3\gamma^3 + 180R^4\gamma^4]. \tag{5}$$

Note that Eq.(4) is a more general solution involving $\gamma$ and we can recover the SdS vacuole case putting $\gamma = 0$. The Rindler-Ishak method requires another function $A(r, \varphi) = \frac{dr}{d\varphi}$, which yields for the present solution

$$A(r, \varphi) = (-r^3) \times \left[ \frac{\cos \varphi}{32R^3} (32R^2 - 3M^2 \{20 + 3R^2\gamma^2(10 + (\pi - 2\varphi)^2)\}$$
$$+ 32MR(9M\gamma - 2)\sin \varphi - 6M \{3M \cos 3\varphi - 8MR\gamma(\pi - 2\varphi) \cos 2\varphi$$
$$+ (10M - 4R^2\gamma + 9MR^2\gamma^2)(\pi - 2\varphi) \sin \varphi\}\}$$
$$+ M^2 \{130 + \pi^2(10 + 9R^2\gamma^2) - 4\pi\varphi(10 + 9R^2\gamma^2) + 40\varphi^2$$
$$+ 3R^2\gamma^2(25 + 12\varphi^2) \} \cos \varphi - 48(10 + 27R^2\gamma^2)(\pi - 2\varphi) \cos 2\varphi$$
$$+ 648R\gamma \cos 3\varphi + 1620R\gamma \sin \varphi + 486\pi R^3\gamma^3 \sin \varphi - 9\pi R^3\gamma^3 \sin \varphi$$
$$+ 3240R\gamma \varphi \sin \varphi - 972R^3\gamma^3 \varphi \sin \varphi + 54R^2\gamma^3 \varphi \sin \varphi$$
$$- 108R^3\gamma^3 \varphi^2 \sin \varphi + 72R^3\gamma^3 \varphi^3 \sin \varphi - 944R\sin 2\varphi - 2304R^2\gamma^2 \sin 2\varphi$$
$$+ 144\pi^2 R^2\gamma^2 \sin 2\varphi - 576\pi R^2\gamma^2 \varphi \sin 2\varphi + 576R^2\gamma^2 \varphi^2 \sin 2\varphi$$
$$- 162R\gamma \sin 3\varphi \varphi + 3240R\gamma \varphi \sin 3\varphi + 32 \sin 4\varphi\}]. \tag{6}$$

Assume a small angle $\varphi_b$ at the vacuole boundary radius $r = r_b$ such that $\sin \varphi_b \simeq \varphi_b$, and $\cos \varphi_b \simeq 1$. Then the above gives

$$\frac{1}{r_b} = \frac{\gamma}{2} + \frac{\varphi_b}{R} + M \left[ \frac{2}{R^2} - \frac{3\pi \gamma}{4R} + \frac{3\gamma^2}{4} \right]$$
$$+ M^2 \left[ \frac{15\pi \gamma}{8R^3} - \frac{12\gamma}{R^2} + \frac{9\pi \gamma^2}{16R} - \frac{9\gamma^2}{16R^3} \right]$$
$$+ M^3 \left[ \frac{73}{4R^4} - \frac{1143\pi \gamma}{64R^3} + \frac{243\gamma^2}{16R^2} - \frac{9\pi \gamma^2}{16R^2} - \frac{9\gamma^2}{8R^3} \right]$$
$$+ \frac{9\pi \gamma^3}{128R} - \frac{135\gamma^4}{16} + \frac{15\pi \varphi_b}{4R^3} + \frac{747\gamma \varphi_b}{32R^3} + \frac{45\gamma \varphi_b}{32R^3}$$
$$+ \frac{5\varphi_b^2}{8R^2} - \frac{81\pi \gamma \varphi_b}{64R} + \frac{187\gamma \varphi_b}{64R} - \frac{81\gamma \varphi_b}{32R} - \frac{45\gamma \varphi_b}{8R^3}$$
$$+ \frac{45\gamma \varphi_b}{16R} - \frac{189\gamma \varphi_b}{32R} + \frac{45\gamma \varphi_b}{8R^3} - \frac{99\gamma \varphi_b}{16R} \right]. \tag{7}$$
and Eq. (6) gives

\[ A_b = A(r_b, \varphi_b) = \frac{r_b^2}{R} - \frac{2M\varphi_b}{R^2} + \frac{M^2\varphi_b}{32R^2} \left[ 48\pi R\gamma - 78 - 90R^2\gamma^2 - 9\pi^2 R^2\gamma^2 - 192R\gamma\varphi_b - 108R^2\gamma^2\varphi_b^2 \right] + O(M^3), \]

where the terms \( O(M^3) \) are straightforward but rather lengthy and hence not displayed here.

Note that observations give us values of \( M \) and \( R \) for a lens, but we have only one equation (7) for two unknowns \( \varphi_b \) and \( r_b \). Hence we need to specify any one of them from independent considerations. Along with Ishak et al. (2008), we shall employ Einstein-Strauss prescription (Einstein & Strauss 1945; Schucking 1954) to determine \( r_b \) assuming that the vacuole has been matched to an expanding FRW universe via the Sen-Lanczos-Darmois-Israel junction conditions (Sen 1924; Lanczos 1924; Darmois 1927; Israel 1966). In general, the vacuole radius \( r_b \) would also change due to cosmic expansion, but we shall consider \( r_b \) at that particular instant \( t_0 \) of cosmic epoch when the light ray happens to pass the point of closest approach to the lens. Thus \( r_b \) is determined by exploiting the Einstein-Strauss prescription [see Ishak et al. (2008)]

\[ r_b \text{ in SdS} = a(t) r_b \text{ in FRW}, \quad M_{\text{SdS}} = \frac{4\pi}{3} r_b^3 \text{ in SdS} \times \rho_{\text{in FRW}}. \]

To achieve exact matching with the exterior FRW universe, the energy density \( \rho \) within the vacuole should have a contribution from \( \Lambda \) besides that of ordinary matter \( \rho_m \), that is, \( \rho = \rho_m + \rho_\Lambda = \frac{3H_0^2}{8\pi} (\Omega_m + \Omega_\Lambda) \), where \( \Omega_m = 8\pi \rho_m/3H_0^2 \), \( \Omega_\Lambda = 8\pi \rho_\Lambda/3H_0^2 \) are the matter and dark energy densities in dimensionless form. Current observations suggest that the universe is spatially flat so that \( \rho = \rho_{\text{critical}} = \frac{3H_0^2}{8\pi} \), which in turn imply that \( \Omega = \Omega_m + \Omega_\Lambda = 1 \). Type Ia supernova observations yield \( \Omega_m = 0.27 \) so that \( \Omega_\Lambda = 0.73 \) (Riess et al. 1998; Perlmutter et al. 1999; Carroll 2001; Page et al. 2003; Peebles & Ratra 2003; Spergel et al. 2007). For computational purposes, we shall take the density to be \( \rho_{\text{in FRW}} = \rho_{\text{critical}} = \frac{3H_0^2}{8\pi} = 1.1 \times 10^{-29} \left( \frac{H_0}{75\text{ km/sec/Mpc}} \right)^2 \text{ gm/cm}^3 \) (Weinberg 1972) inside the vacuole. A slight deviation from this value would not drastically alter our conclusions. Normalizing the scale factor to \( a(t_0) = 1 \) and dropping suffixes, the above prescription translates to

\[ r_b = \left( \frac{3M}{4\pi \rho} \right)^{1/3}, \]

where \( M \) is the lens mass often expressed in units of sun’s mass \( M_\odot = 1.989 \times 10^{33} \text{ gm} = 1.475 \times 10^{15}\text{cm} \).

We should now solve the cubic Eq. (7) in \( \varphi_b \) to find three roots designated by \( \varphi_i = \varphi_i(\rho, M, R, \gamma) \) where \( i = 1, 2, 3 \). For the SdS vacuole (\( \gamma = 0 \)), we fortunately get only a single root \( \varphi_b \), which becomes, using the Einstein-Strauss prescription for \( r_b \) from Eq. (10),

\[ \varphi_b = \frac{90\pi M^3 + 96RM^2 - 16(\pi \rho)^{1/3}(6M)^{2/3}R^3}{117M^3 - 48MR^2}. \]
Our interest is to express the deflection angle $\psi$ in terms of $M, R$ and an as yet unspecified $\rho$. We shall use the above value of $\varphi_b$ later. The Rindler-Ishak formula for $\psi$ at $r = r_b$ is

$$\tan \psi = \frac{r_b \sqrt{B(r_b)}}{|A_b|}.$$  \hspace{1cm} (12)

From Eq.(7), one sees that $r_b$ contains the radius $R$ which is a real root of Eq.(5). For large distances, there is little difference between $R$ and $r_0$. So we shall identify $R$ with the Einstein radius where the closest approach distance $r_0$ appears. We are not using the impact parameter here.

Our general algorithm for calculation proceeds along the following analytical steps:

1. Put the expression for $r_b$ from Eq.(7) and $A_b$ from Eq.(8) into Eq.(12) for deflection angle.

2. Expand the right hand side of Eq.(12) in first power of $\gamma$ in order to separate out its contribution from the SdS one. Thus we write formally

$$\tan \psi^{\text{total}} \simeq C(\varphi_b, M, R) + \gamma D(\varphi_b, M, R) \hspace{1cm} (13)$$

where $C, D$ are known functions to be expanded in powers of $M$. For small $\psi^{\text{total}}$, we decompose

$$\tan \psi^{\text{total}} \simeq \psi^{\text{SdS}} + \psi^{\text{MKdS}}. \hspace{1cm} (14)$$

3. Expand both $C$ and $D$ up to the power $M^2$ to see the individual contributions of terms.

III. SdS vacuole: $\Lambda$–effect

This case corresponds to $\gamma = 0$ and we have to be concerned with only $C = \tan \psi^{\text{SdS}} \simeq \psi^{\text{SdS}}$. The boundary radius $r_b$ of the vacuole from Eq.(7) is [step (1)],

$$r_b = \frac{16 R^4}{32 M R^2 + 16 R^3 \varphi_b + M^2 (30 \pi R - 39 R \varphi_b) + M^4 (292 - 60 \pi \varphi_b + 120 \varphi_b^2)}.$$  \hspace{1cm} (15)

and Eq.(8) gives

$$A_b = \frac{r_b^2}{R} - \frac{2 M r_b^2 \varphi_b}{R^2} - \frac{M^2 r_b^2}{R^3} \left\{ \frac{39}{16} + \frac{15 \pi \varphi_b}{8} - \frac{15 \varphi_b^2}{4} \right\} - \frac{M^3 r_b^4}{R^4} \left\{ \frac{15 \pi}{4} + \frac{25 \varphi_b}{4} \right\}.$$  \hspace{1cm} (16)

The half angle of the Einstein ring subtended at the observer is defined as $\theta_E = 2 \pi D_{ls}/D_{os}$, the suffixes $o, l, s$ in $D$ representing angular diameter distances between observer, lens and the source, assuming all to be situated on a "line". We have taken $R = R_E = D_{ol} \tan \theta_E \simeq D_{ol} \theta_E$ for small $\theta_E$. 

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Note that Eqs. (10), (11) and (15) are consistent. Putting the values of \( r_b \) and \( A_b \) in Eq.(12), assuming a small \( \psi^{\text{SdS}} \), and expanding in powers of \( M \) we get [step (2)]

\[
\psi^{\text{SdS}} = \left( 1 - \frac{\Lambda R^2}{6 \varphi_b} \right) \varphi_b + M \left[ \varphi_b \left( \frac{2RA}{3\varphi_b} - \frac{\varphi_b}{R} \right) + \left( 1 - \frac{\Lambda R^2}{6 \varphi_b^2} \right) \left( \frac{2}{R} + \frac{\varphi_b^2}{R} \right) \right]
\]

\[
+ M^2 \left[ \frac{5\pi \Lambda}{8 \varphi_b^3} - \frac{13\Lambda}{16 \varphi_b^3} - \frac{2}{R^2} - \frac{2\Lambda}{\varphi_b^3} \varphi_b + \left( \frac{2RA}{3\varphi_b^3} - \frac{\varphi_b}{R} \right) \left( \frac{2}{R} + \frac{\varphi_b^2}{R} \right) \right]
\]

\[
- \left( 1 - \frac{\Lambda R^2}{6 \varphi_b^2} \right) \left[ - \frac{4\varphi_b}{R^2} - \frac{240\pi R - 312 R \varphi_b}{128 R^3} \right]
\]

\[
+ 128 R^3 \varphi_b \left( - \frac{39}{2048R^5} - \frac{15\pi \varphi_b}{1024 R^2} - \frac{\varphi_b^2}{512 R^5} \right) \bigg] + O(M^3). \quad (17)
\]

To simplify calculations, we shall now expand \( \varphi_b \) of Eq.(11) with \( \rho = \frac{3M}{4\pi r_b} \) obtaining

\[
\varphi_b = \frac{R}{r_b} - \frac{2M}{R} + M^2 \left( \frac{39}{16 R r_b} - \frac{15\pi}{8 R^2} \right). \quad (18)
\]

Using it in \( \psi^{\text{SdS}} \), and collecting terms of similar orders in \( M \), we get

\[
\psi^{\text{SdS}} \simeq \varphi_b - \frac{\Lambda R r_b}{6} + \frac{M}{R} \left[ 2 + \frac{R^2}{r_b^2} - \frac{R^2 \Lambda}{3} \right]
\]

\[
+ \frac{M^2}{R^2} \left[ \frac{15\pi}{8} - \frac{7R^3}{4r_b^3} + \frac{15\pi R^2}{8r_b^4} - \frac{4R}{r_b} - \frac{5\pi R^2 \Lambda}{16} - \frac{R^3 \Lambda}{24 r_b} + \frac{25\Lambda R r_b}{96} \right] \quad (19)
\]

As usual, for small angle, \( \tan \varphi_b \simeq \varphi_b \), so that the deflection \( c^{\text{SdS}} \) for nonzero \( \varphi_b \) is, by definition (Rindler & Ishak 2007)

\[
c^{\text{SdS}} = \tan(\psi^{\text{SdS}} - \varphi_b) \simeq \psi^{\text{SdS}} - \varphi_b. \quad (20)
\]

Assuming that \( r_b >> R, R >> M \), and collecting terms of interest, we get the total deflection

\[
2c^{\text{SdS}} = - \frac{\Lambda R r_b}{3} + \frac{4M}{R} + \frac{15\pi M^2}{4 R^2} - \frac{2MAR}{3} - \frac{5\pi M^2 \Lambda}{8} + \frac{2MR}{r_b}. \quad (21)
\]

Clearly, the above yields the Ishak et al term \( -\frac{\Lambda R r_b}{6} \) as well as the well known Schwarzschild terms. Interestingly, identifying the constant \( R \simeq r_0 \) from Eq.(5), the fourth term, viz., \( t^{\text{SdS}}_4 = -\frac{2MAr_0}{3} \) looks numerically like the same as what Sereno (2009) calls the local coupling term between \( M \) and \( \Lambda \), since the term does not depend on the vacuole radius \( r_b \) (see also the Appendix). Nonetheless, it does depend on the particular path via \( R \). Remarkably, we also discover a new local coupling term in the second order, viz., \( t^{\text{SdS}}_5 = -\frac{5\pi M^2 \Lambda}{8} \), coming from the fifth term in the second square bracket in Eq.(19). This seems to be a more genuine local coupling term because it does not involve the parameter \( R \) of the light trajectory. However, both the terms contribute repulsively to bending.
A certain thing is to be noted here. We might start with the first order differential equation already containing $\Lambda$ through $B(r)$:

$$\frac{1}{r^4} \left( \frac{dr}{d\varphi} \right)^2 + \frac{B(r)}{r^2} - \frac{1}{b^2} = 0, \quad (22)$$

where $b$ is the impact parameter defined as $\ell/e$. Then one can define $b$ using the closest approach distance $r = r_0$, where $\frac{dr}{d\varphi} = 0$, which yields from the first order Eq. (22) the value of $b$ as

$$b = r_0 \left[ \frac{1}{B(r_0)} \right]^{1/2}. \quad (23)$$

This gives

$$r_0 \simeq b \left( 1 - \frac{M}{b} - \frac{b^2 \Lambda}{6} \right). \quad (24)$$

$$\frac{1}{r_0} \simeq \frac{1}{b} \left( 1 + \frac{M}{b} + \frac{b^2 \Lambda}{6} + \frac{M Ab}{3} \right). \quad (25)$$

Using the values of $R = r_0$ and $1/R = 1/r_0$ from above into the expression (8), we find the relevant terms to add to

$$2\varepsilon^{SdS} = -\frac{b\Delta r_0}{3} + \frac{4M}{b} + \frac{2M\Delta b}{3} - \frac{2M\Delta b}{3} + \text{terms in } M^2, \quad (26)$$

so that the local coupling term $\frac{2M\Delta b}{3}$ cancels out!

In our opinion, some caution should be exercised in the interpretation that such a local coupling really vanishes due to that cancellation. Note that both the original Rindler-Ishak (2007) or vacuole (2008) method do not at all use the first order path equation in which $\Lambda$ already appears. Their whole package consists of the second order differential equation in which $\Lambda$ does not appear (which led people to believe that it does not hence affect light bending) and the definition of the invariant angle $\psi$ to capture the effect of $\Lambda$, without needing any further ingredients. To use the first order path equation (22) already containing $\Lambda$ at any stage of the present calculation would mean capturing the effect of $\Lambda$ twice. We argue that both the trajectory equation (3) with $R \simeq r_0$ and the first order equation (22) with $b$ should not be simultaneously used. Integration of the first order equation (22) with $b$ and the present vacuole method should be treated as mutually exclusive ways, both separately yielding the bending with expected local coupling terms. We obtained not only the Sereno-like local coupling term $-\frac{2M\Delta b}{3}$ but also a new term $-\frac{5\pi M^2 A}{8}$ including other new terms, the most notable one being $-\frac{\Delta R r_0}{3}$.

As argued above, the vacuole method has been exclusively tailored to capture the effect of a parameter (like $\Lambda$) that has disappeared from the path equation. To justify this statement, we might try to assess the influence of the remaining parameter $\gamma$, which has not disappeared from the second order equation (3). If
we still apply the present vacuole method, it can be verified that the known effect $-\frac{\gamma R^2}{2}$ on bending does not appear in $\psi^{MKdS}$ (We omit the details). However, we can still use the initial Rindler-Ishak non-vacuole method (2007) to obtain exactly this effect as already shown in Bhattacharya et al. (2010). Below we shall show that even a reverse prescription in the non-vacuole method yields the same effect $-\frac{\gamma R^2}{2}$.

V. The $\gamma$ effect by alternative prescription

To capture the effect of $\gamma$ by means of the invariant angle, we assume here for simplicity $\Lambda = 0$ and restrict to first order in $M$. In principle, this case corresponds to $C = 0$ and we have to be concerned with only $\gamma_D = \tan \psi^{MKdS} \approx \psi^{MKdS}$.

But as stated above, $\gamma_D$ does not yield the effect $-\frac{\gamma R^2}{2}$. Therefore we proceed as follows.

First, a few words about the original Rindler-Ishak (2007) prescription and the alternative one we are going to follow here. In the asymptotically non-flat metric, the limit $r \to \infty$ makes no sense. Therefore they prescribed that the only intrinsically characterized $r$ value replacing $r \to \infty$ is the one at $\varphi = 0$. The measurable quantities are the various $\psi$ angles that the photon orbit makes with successive coordinate planes $\varphi = \text{const}$. While this assumption is perfectly valid giving the desired results, we shall implement the Rindler-Ishak method following a reverse prescription 3, namely, determining $\varphi \neq 0$ occurring at a point $r = \infty$ on the null orbit and work out how the parameters $\gamma$ and $M$ appear in the light bending.

Next, for the unbound orbits associated with lensing, the distance of closest approach of a light ray to a galaxy will be further from the center of the galaxy than the matter orbiting inside it. Hence our goal here is to calculate the deflection angle $\epsilon = \psi - \varphi$ in the metric $B(r) = 1 - \frac{2M}{r} + \gamma r$ under the approximation $\frac{M}{r} \ll 1$. To this end, we first determine the value of a small nonzero $\varphi$ lying on the null geodesic at $r = \infty$ using the orbit Eq.(4). For small $\varphi$, we take $\sin \varphi \approx \varphi$, $\cos \varphi \approx 1$ and neglect terms of $\varphi^2$ and higher. Then, at $r = \infty$, the orbit Eq.(4) yields

$$0 = -\frac{\gamma}{2} + \frac{\varphi}{R} + \frac{M}{4R^2} [8 + 3R^2\gamma^2 - 3R\gamma(\pi - 2\varphi) - 6R\gamma \varphi]$$

$$-\frac{3M^2}{32R^3} [128R\gamma + 24R^3\gamma^3 - 10(2 + 3R^2\gamma^2)(\pi - 2\varphi)$$

$$- 14\varphi - 16\pi R\gamma \varphi - 30R^2\gamma^2 \varphi + 3\pi^2 R^2\gamma^2 \varphi].$$

3There seems to be no obvious operative way of actually measuring the azimuthal angle on the sky unless one is able to perform the experiment twice, once with the lens in the way, and once without (just as is done with gravitational bending of light by the sun). However, one can define a distance $r$ on the orbit by using a coordinate system in which the lens is at $r = 0$. Then for an observer far enough beyond the lens, the very fact that light reaches the observer entails that the point $r = \infty$ does lie on the geodesic. The form of the geodesic then determines at what value of $\varphi \neq 0$ the source lies. This is what we are calling reverse prescription here.
Solving for \( \varphi \), we obtain

\[
\varphi = \frac{8MR(8 - 3R\gamma + 3R^2\gamma^2) - 16R^3\gamma + 6M^2\{5\pi(2 + 3R^2\gamma^2) - 4R\gamma(16 + 3R^2\gamma^2)\}}{M^2(78 - 48\pi R\gamma + 90R^2\gamma^2 + 9\pi R^2\gamma^2) - 32R^2}.
\]

(29)

Calculating for small \( \psi \), we find

\[
\psi \approx \tan \psi = \frac{rB^{1/2}}{r^2 \frac{du}{d\varphi}} = \left( \frac{du}{d\varphi} \right)^{-1} \sqrt{\frac{1}{r^2} - \frac{2M}{r^3} + \frac{\gamma}{r}},
\]

(30)

which goes to 0 as \( r \to \infty \) since \( \frac{du}{d\varphi} \neq 0 \) at the value of \( \varphi \) derived in Eq.(29). Thus, \( \psi = 0 \) and the one way deflection then is \( \epsilon = 0 - \varphi \), which easily expands to

\[
\epsilon \approx \frac{2M}{R} \left[ 1 + \frac{15\pi M}{16R} \right] - \gamma \left[ \frac{R}{2} + \frac{3\pi M}{4} + \frac{423M^2}{32R} \right].
\]

(31)

We have checked that this result exactly coincides with that obtained by the perturbative Bodenner-Will perturbative method (2003). We find that all terms in the second square bracket are positive, meaning that the effect of \( \gamma > 0 \) is to diminish (and \( \gamma < 0 \) is to enhance) the Schwarzschild bending even up to second order in \( M \). We also find that Eq.(31) nicely reproduces the one way deflection \( \epsilon = \frac{2M}{R} - \frac{\gamma R^2}{2} \) obtained by Edery and Paranjape (1998) using Weinberg’s method. As mentioned, the same result (31) follows also from the unaltered Rindler-Ishak prescription (2007) as well. We shall now discuss some physical implications of Eq.(31).

VI. Physical implications

First note that in the halo we have obtained a new coupling term \( \frac{3\pi \gamma M}{2} \) between \( M \) and \( \gamma \), which is independent of the trajectory parameter \( R \). Next, the term \( -\frac{\gamma R}{2} \) shows repulsion for \( \gamma > 0 \), which is consistent with time delay investigations (see e.g., Edery & Paranjape 1998) and attraction if we choose \( \gamma < 0 \). We emphasize that we are not concluding anything about the correct sign of \( \gamma \), which must be decided by independent observations. When \( M = 0 \) and \( \gamma > 0 \), we obtain a negative (repulsive) bending of light or \( \epsilon = -\frac{2\gamma R}{2} \), which coincides with the conclusion by Walker (1994).

On the other hand, in the galactic halo region, where \( R > R_E \) (the Einstein radius) and \( \frac{M}{R} < < 1 \), one would like to obtain a positive (attractive) light bending there. This is possible only if one assumes the condition

\[
\epsilon > 0 \Rightarrow \frac{2M}{R} > \frac{\gamma R}{2}
\]

(32)

to hold. Accurately observed lensing data by galactic clusters are now available. We then find from Table I that the observed values of \( M \) and \( R_E \) do indeed respect the inequality (32). Clearly, even if pure \( \gamma > 0 \) leads to repulsion, in the competition between this repulsion and Schwarzschild attraction, the latter
might win leading to the impression of an overall attractive bending. This can happen in the lensing by galactic clusters, as described in the table below.

Lens data for $M$, $R_E$ and references are taken from Ishak et al. (2008), and converted here to length units using $M_\odot = 1.475 \times 10^5$ cm, 1 kpc = $3.0856 \times 10^{21}$ cm. The references are as follows: 1: Abell 2744 (Smail et al. 1991, Allen 1998), 2: Abell 1689 (Allen 1998, Limousin 2007), 3: SDSS J1004+4112 (Sharon 2006), 4: 3C 295 (Wold et al. 2002), 5: Abell 2219L (Smail et al. 1995a; Allen 1998), 6: AC 114 (Smail et al. 1995b; Allen 1998). We shall take the rotation curve fit value $\gamma = 3.06 \times 10^{-30}$ cm$^{-1}$ purely for illustrative purposes and $\Lambda = 1.29 \times 10^{-56}$ cm$^{-2}$ in both the tables below:

![Table I](image)

It is evident from the above table that the term $\frac{2M}{R_E}$ is smaller than the Schwarzschild term $\frac{2M}{R_E}$, so that the overall bending is always attractive for $\gamma > 0$. One might want to have an idea of the radius $R$ where the leading order Schwarzschild and $\gamma$- bendings balance each other. The value of $R_b$ may be taken to demarcate the boundary of the halo dark matter surrounding each individual cluster. This happens at

$$R_b = 2 \sqrt{\frac{M}{\gamma}} \text{ cm.}$$

The deflection $\epsilon$ below $R < R_b$ is always attractive, as should be the case. Table II shows that the halo boundary $R_b$ can be several times larger than $R_E$. However, the values of $R_b$ tabulated here rely crucially on the value of $\gamma$ and if its value is lowered by one order of magnitude than considered here, $R_b$ will increase by that order. Conversely, if one particular halo boundary is observationally determined, then it would provide us with a determination of $\gamma$. One could then examine if that new value of $\gamma$ explains $R_b$ of other clusters. If it does, then it would support Weyl theory. Observations seem as yet far too inconclusive about the sizes of the halo.

![Table II](image)
Though our interest so far has only been in the galactic clusters, one might still want to compare the magnitudes of the $\gamma$-related effects with the Schwarzschild ones in the solar system although the region around the Sun contains galactic matter. For a light ray grazing the Sun, we have the following numerical values:

$$M_\odot = 1.475 \times 10^5 \text{ cm}, \quad R_\odot = 6.96 \times 10^{10} \text{ cm}, \quad \gamma = 3.06 \times 10^{-30} \text{ cm}^{-1}$$ (34)

so that

$$\frac{2M_\odot}{R_\odot} = 4.24 \times 10^{-6}, \quad \frac{30\pi M_\odot^2}{16R_\odot^2} = 2.65 \times 10^{-11},$$ (35)

$$\frac{\gamma R_\odot}{2} = 1.06 \times 10^{-19}, \quad \frac{3\pi \gamma M_\odot}{4} = 1.06 \times 10^{-24}.$$ (36)

We find that the $\gamma$-correction terms are considerably small compared to $\frac{2M_\odot}{R_\odot}$, therefore the effect of $\gamma$ would be negligible near the Sun. However, as illustrated in Table I, the effect of $\gamma$ near any galactic cluster scale is not as negligible. The fact that $\gamma$ is meaningful only on such large scales has been conjectured in the literature, but here we have found its support from a completely different viewpoint, viz., from the Rindler-Ishak bending.

VII. Summary and results

We calculated light deflection in the vacuole model up to third order in $M$ and confirmed that the extension of Rindler-Ishak method to the Einstein-Strauss vacuole, as originally developed by Ishak et al. (2008), reproduces the Schwarzschild $M$-dependent bending terms as well as the $\Lambda$-dependent terms, see Eq.(21). In particular, we have found a local coupling term $-\frac{2M\Lambda r_0}{3}$ similar to that by Sereno. We have also found a more interesting coupling term $-\frac{5\pi M^2 \Lambda}{8}$ including other new terms, the most notable one being $-\frac{\Lambda Rr_0}{3}$. It would be of interest to discuss the recessional impact too (Ishak & Rindler, 2010), but it requires a separate and detailed investigation.

The idea of a cut-off transition region between the halo boundary and the exterior dS cosmology was conjectured, but not implemented, by Edery & Paranjape (1998) over a decade ago. The SdS vacuole model by Ishak et al. (2008) is philosophically the same in idea but different in content. It envisages a transition radius $r_b$ between the SdS vacuole boundary and the exterior FRW cosmology implementing the Einstein-Strauss suggestion. The vacuole surrounding the lens should be devoid of matter, and therefore the model particularly applies to galactic clusters rather than local objects like the Sun, which is surrounded by galactic matter.

We have argued that the vacuole method is exclusive to cases where the cosmological constant $\Lambda$ disappears from the second order differential path equation. To exemplify it, we applied the vacuole model in the calculation of the $\gamma$-dependent effects in Weyl gravity. We note that the parameter $\gamma$ does not disappear from the path equation, and thus the vacuole method does not yield the known Weyl term $-\frac{\gamma R}{3}$. To this end, we point out that the earlier Rindler-Ishak (2007) prescription in their non-vacuole method did nicely yield the otherwise known Weyl term (Bhattacharya et al. 2010). In the present paper, we showed
that an alternative prescription on the azimuthal angle lying on the null orbit also reveal the influence of the Schwarzschild ($M$) and conformal sector ($\gamma$) on light deflection [See Eq. (31)]: It reproduced the correct Schwarzschild bending terms due to $M > 0$ as well as those due to the conformal Weyl parameter $\gamma$. In particular, the known term $-\frac{2\gamma R}{2}$ followed exactly. Also we have found a new local coupling term $\frac{3\pi\gamma M}{2}$ between $M$ and $\gamma$, which is independent of the trajectory parameter $R$. We chose (not mandatorily) the value obtained by Mannheim (2006) from the fit of the galactic flat rotation curve data and applied it to the accurately observed data on several galactic clusters taken from Ishak et al. (2008). We have shown in Table I that, for $R_E \leq R < R_h$, the light bending is attractive since $\epsilon \left( = \frac{2M}{R} - \frac{\gamma R}{2} \right)$ is always positive masking the purely negative Weyl $\gamma-$ term, while Table II gives possible sizes $R_h$ of the halo if the chosen value of $\gamma$ is relied upon. Although galactic halo can be modelled in many ways [see, for instance, the brane world model, Nandi et al (2009)], the interpretations of Weyl gravity in this regime seem as yet conclusive, to our knowledge.

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Appendix

The integration of the first order equation reads

$$\varphi_{\text{Sereno}} = \pm \int \frac{dr}{r} \left[ \frac{1}{b^2} + \frac{\Lambda}{3} - \frac{1}{r^2} + \frac{2M}{r^3} \right]^{-1/2} = \pm \int f(M, \Lambda, b, r)dr \quad (A1)$$

It can't be integrated in a closed form. So expanding the integrand \(f\) in first power of \(M\), we have

\[ f = \frac{1}{r^2} \left( \frac{1}{b^2} - \frac{1}{r^2} - \frac{\Lambda}{3} \right)^{-1/2} - \frac{M}{r^5} \left( \frac{1}{b^2} - \frac{1}{r^2} - \frac{\Lambda}{3} \right)^{-3/2} = f_1 + f_2 \quad (\text{say}). \quad (A2) \]

Then, to first power of \(\Lambda\),

\[ I_1 = \int f_1 dr \]

\[ = \frac{b}{\sqrt{b^2(3 + 2r^2\Lambda) - 3r^4}} \left( \ln r - \ln 2 - \ln \left( \sqrt{b^2(3 + 2r^2\Lambda) - 3r^4} \right) \right) \]

\[ \simeq \frac{b}{r} \frac{b^3}{6r^3} - \frac{3b^5}{40r^5} - \frac{\Lambda b^3}{6r} - \frac{\Lambda b^5}{12r^3} + \text{imaginary terms}. \quad (A3) \]

\[ I_2 = \int f_2 dr \]

\[ = -\frac{M[b^2(3 + 2r^2\Lambda) - 6r^4]}{3b^2 r^2 \sqrt{1 - \frac{b^2}{r^2} - \frac{\Lambda b^2}{3}}} \]

\[ \simeq \frac{2M}{b} - \frac{M\Lambda b}{3} + \frac{Mb^3}{4r^4} + \frac{Mb^5}{4r^6} + \frac{M\Lambda b^5}{8r^4}. \quad (A4) \]
Collecting real terms, we get

\[
\varphi^{\text{Sereno}} = \frac{2M}{b} - \frac{b}{r} - \frac{M\Lambda b}{3} - \frac{b^3}{6r^3} + \frac{Mb^3}{4r^4} - \frac{3b^5}{40r^5} - \frac{\Lambda b^3}{6r} \\
- \frac{M\Lambda^2 b^3}{36} - \frac{\Lambda b^5}{12r^3} + \frac{Mb^5}{4r^6} + \frac{M\Lambda b^5}{8r^4},
\]  

(A5)

which seem to yield that the local coupling term is \(-\frac{2M\Lambda b}{3}\).