FREUDENTHAL TRIPLE SYSTEMS BY ROOT SYSTEM METHODS

FRED W. HELENUS

Abstract. For certain Lie algebras $g$, we can use a grading $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and define a quartic form and a skew-symmetric bilinear form on $g_1$, thereby constructing a Freudenthal triple system. The structure of the Freudenthal triple system is examined using root system methods available in the Lie algebra context. In the cases $g = E_8$ (where $g_1$ is the minuscule representation of $E_7$) and $g = D_4$, we determine the groups stabilizing the quartic form and both the quartic and bilinear forms.

1. Introduction

Attempts to understand the 56-dimensional minuscule representation of $E_7$ have been based upon an axiomatization of the properties of a bilinear form and a quartic form defined on it, resulting in a so-called Freudenthal triple system. While the minuscule representation of $E_7$ is the prototype Freudenthal triple system, another interesting example can be found in Bhargava’s work on higher composition laws ([Bha04]); the 8-dimensional space with a quartic form defined at the outset also forms a Freudenthal triple system, as observed by Markus Rost.

Freudenthal triple systems have been studied previously using such tools as Jordan algebras ([Spr06], [Sel62]), tensor algebra ([Fre53]), or by an axiomatic development ([Bro69], [Fer72]), but in this paper we exhibit Freudenthal triple systems that are subspaces of Lie algebras with operations defined in terms of the Lie bracket; this allows the Freudenthal triple system to be examined using little more than root system computations. Our approach applies to the exceptional Lie algebras other than $G_2$ as well as to those of types $B$ and $D$; in particular, we obtain both the 56-dimensional prototype and the Bhargava/Rost example. As an application, in each of these cases we determine the group stabilizing the quartic form and the group stabilizing both forms.

We begin (Section 2) by using a $\mathbb{Z}/5\mathbb{Z}$-grading on the Lie algebras in question to define a quartic form and a bilinear form on the grade 1 elements. After establishing some basic properties of these forms (Section 3) and characterizing the so-called strictly regular elements (Section 4), we are able to verify that we have a Freudenthal triple system (Section 5). We next show how to explicitly compute the quartic form in the simply-laced case (Section 6). An eigenspace decomposition (Section 7) into four
subspaces mirrors the construction of Freudenthal triple systems from Jordan algebras. We characterize the orbits of the Freudenthal triple system under a group action in Section 8.

The next three sections examine the groups whose actions stabilize the Freudenthal triple system operations (either exactly or up to scalar multiples). In Section 9, we show that a linear transformation that stabilizes the quartic form up to a scalar factor likewise stabilizes the bilinear form. In the case \( g = E_8 \) (Section 10), we find that \( E_7 \) is the group that stabilizes both the forms on the prototypical Freudenthal triple system, the minuscule representation of \( E_7 \), which was known; we also find the group which stabilizes just the quartic form, which is new. In Section 11, we find the groups stabilizing one or both forms in the case \( g = D_4 \); these are new results.

All the results mentioned are proved under the assumption that \( g \) is the Lie algebra of an algebraic group \( G \) that is split over a field \( F \). In the final section, we show that our results apply equally well to non-split groups.

In [Fer72], Ferrar uses the axiomatic definition of a Freudenthal triple system to study its structure. Our approach moves in the opposite direction: we begin with a structure defined within a Lie algebra, study its properties and eventually show that it satisfies the Freudenthal triple system axioms. Although our hypotheses are totally different, our choice of results to prove was often guided by the content of Ferrar’s article. The table below indicates results here that are parallel to those of Ferrar as well as results in articles by Clerc ([Cle03]) and Krutelevich ([Kru07]).

| Lemma/Prop. | Ref. |
|-------------|------|
| 14          | [Fer72], Cor. 2.5 |
| 20          | [Fer72], Cor. 6.2 |
| 21          | [Fer72], (5) |
| 24          | [Cle03], Lemma 8.5(b); [Fer72], Lemma 3.6 |
| 36          | [Fer72], §4 |
| 38          | [Cle03], §§8.9; [Kru07], Def. 22 |
| 40          | [Fer72], Lemma 7.3 |

Table 1. Parallel results in other papers

2. Preliminaries

Here we establish notation and conventions that will be used throughout and summarize the key results from other papers that are used.

Let \( F \) be an arbitrary field of characteristic \( \neq 2,3 \), and let \( G \) be a simple, connected linear algebraic group that is split over \( F \), and let \( g \) be its Lie algebra. Let \( \Psi \) be the root system of \( g \) with respect to a fixed maximal torus \( h \); thus \( \Psi \subset h^\vee \). Let \( \rho \) be the highest root with respect to a fixed base of \( \Psi \). As is usual (see, for example, [Hum78], §9.1), we define \( \langle \beta, \gamma \rangle = 2(\beta, \gamma)/(\gamma, \gamma) \) for any nonzero \( \beta, \gamma \in h^\vee \). We assume \( g \) is not of type \( A \) or \( C \), so there is a unique simple root \( \alpha \) such that \( \langle \alpha, \rho \rangle = 1 \) and \( \alpha \) is a long root.
We will also assume that the rank of $g$ is at least 4. In the later sections, we will assume that $g$ is simply-laced and thus of type $D$ or $E$.

For each $\beta \in \Psi$, the $\alpha$-height of $\beta$ is given by $\langle \beta, \rho \rangle$; in other words, $\alpha$-height is the coefficient of $\beta$. Thus the $\alpha$-height is one of $-2, -1, 0, 1, 2$. This gives a grading $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$, where, for each $k \neq 0$, $g_k$ is the direct sum of the root subspaces for roots of $\alpha$-height $k$; $g_0$ is the direct sum of the root subspaces for roots of $\alpha$-height 0 and of $h$. Equivalently, each $g_k$ contains all $x \in g$ for which $[h_\rho, x] = kx$. Since $\langle \beta, \rho \rangle = -2$ (resp. 2) only when $\beta = -\rho$ (resp. $\rho$), we see that $g_{-2}$ and $g_2$ are one-dimensional, consisting of the root subspaces corresponding to $-\rho$ and $\rho$, respectively.

We write $x_{\beta}$ for a representative of the root subspace corresponding to $\beta \in \Psi$, and always assume that such representatives have been chosen to lie in a Chevalley basis (see [Hum78], §25).

The grading on $g$ allows us to define several operations on $g_1$ in a natural way. First, we define a quartic form $q(x)$ for $x \in g_1$ by $(\text{ad} x)^4(x, x_\rho) = q(x)x_\rho$. We also define a 4-linear form $q(x, y, z, w)$ by linearization. To specify the scalar factor, we set $q(x, x, x, x) = q(x)$ for all $x \in g_1$.

We also define a skew-symmetric bilinear form $\langle x, y \rangle$ on $g_1$ by $\langle x, y \rangle = \langle x, y \rangle x_\rho$. This form turns out to be nondegenerate (Lemma 2); thus we may also define a symmetric triple product $xyz$ on $g_1$ by requiring $\langle w, xyz \rangle = q(w, x, y, z)$ for all $w, x, y, z \in g_1$. We will show (Theorem 27) that $g_1$ equipped with these operations is a Freudenthal triple system. These operations depend upon the choice of the Chevalley basis as follows: if instead of $x_\rho$ we choose $cx_\rho$ as the basis element in the root subspace corresponding to $\rho$, then the bilinear form is scaled by $c^{-1}$ and the quartic form is scaled by $c^{-2}$.

By Theorem 2 of [ABS90], if $F$ is algebraically closed then the Levi complement of a parabolic subgroup of the linear algebraic group $G$ acts on the unipotent radical of the parabolic subgroup with finitely many orbits. Let $G_0$ be the subgroup of $G$ that corresponds to $g_0$, more precisely, the centralizer of $h_\rho$ in $G$. In terms of the Lie algebra, $G_0$ acts on $g_1$ and actually partitions $g_1$ into finitely many orbits.

In Theorem 2.6 of [Roh93], Röhrle gives the number of $G_0$-orbits in $g_1$ for each Lie algebra $g$ satisfying our common hypotheses. For the Lie algebras $E_6, E_7, E_8$, there are five orbits. Each orbit is represented by an element of the form $\sum_{i=1}^k x_{\beta_i}$ for $k = 0, \ldots, 4$ where $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of mutually orthogonal roots of $\alpha$-height 1 ([Roh93], Theorem 4.8). We refer to these as orbit 0 through orbit 4. We may, and frequently do, take $\beta_1 = \alpha$.

For Lie algebras of type $D_n$, each orbit has a representative as above, but there are either two ($n > 4$) or three ($n = 4$) distinct orbits generated by sums with two terms; that is, “orbit 2” is split into two or three orbits in this case; we refer to each of them as a level 2 orbit. Similarly, for Lie algebras of type $B_n, n \geq 4$, or $F_4$ there are two level 2 orbits.

For all of the types, orbit 4 is also represented by $x_\alpha + x_{-\alpha}$ ([Roh93], Corollary 4.4).
The semisimple part of $G_0$, which we denote by $(G_0)^{ss}$, also acts on $\mathfrak{g}_1$; here there are finitely many orbits in the projective space $\mathbb{P}(\mathfrak{g}_1)$. These projective orbits correspond to the nonzero orbits under the action of $G_0$. The action of $(G_0)^{ss}$ is of interest because of the following fact.

Lemma 1. The quartic form and skew-symmetric bilinear form on $\mathfrak{g}_1$ are preserved by the action of $(G_0)^{ss}$.

Proof. The elements of $(G_0)^{ss}$ act on $\mathfrak{g}$ by Lie algebra homomorphisms. For any basis element of the Lie subalgebra of $\mathfrak{g}$ corresponding to $(G_0)^{ss}$, i.e., any $x_\beta$ where $\beta$ is a root of $\alpha$-height 0 or any $h_\gamma$ where $\gamma$ is a simple root other than $\alpha$, we have $[x_\beta, x_\rho] = 0$ and $[h_\gamma, x_\rho] = 0$ because $\rho$ is orthogonal to every root of $\alpha$-height 0. Similarly, we also have $[x_\beta, x_{-\rho}] = 0$ and $[h_\gamma, x_{-\rho}] = 0$. Thus elements of $(G_0)^{ss}$ fix $x_\rho$ and $x_{-\rho}$. The quartic form and bilinear form we have defined on $\mathfrak{g}_1$ depend only on the Lie bracket, $x_\rho$ and $x_{-\rho}$, so both are preserved by the action of $(G_0)^{ss}$. □

By Théorème 3.13 in Borel & Tits ([BT72]), the closure of any of the $G_0$-orbits is its union with all smaller (i.e., lower level) orbits. In particular, the largest orbit, orbit 4, is dense in $\mathfrak{g}_1$.

The statements about orbits above assume that $F$ is algebraically closed. For a general $F$, geometric statements about orbits will at least be true over the algebraic closure of $F$. The algebraic consequences, such as Lemma 1 above, remain true for any $F$, since they involve polynomial relations defined over $F$. To avoid repetition, we make this convention: all statements about orbits are understood to refer to the orbits over the algebraic closure.

As mentioned earlier, we have assumed for convenience that $G$ is split over $F$. Our results apply more generally, as explained in Section 12.

3. The bilinear and quartic forms

Given any $x, y \in \mathfrak{g}_1$, the Lie algebra product lies in $\mathfrak{g}_2 = Fx_\rho$; thus we may define a bilinear form $\langle x, y \rangle$ on $\mathfrak{g}_1$ by $[x, y] = \langle x, y \rangle x_\rho$. This form is clearly skew-symmetric.

Lemma 2. The bilinear form $\langle - , - \rangle$ on $\mathfrak{g}_1$ is nondegenerate.

Proof. The elements $x_\beta$ with $\beta$ a root of $\alpha$-height 1 form a basis for $\mathfrak{g}_1$. Consider the matrix of the form with respect to this basis; the entries are of the form $\langle x_\beta, x_\gamma \rangle$ with $\beta, \gamma$ roots of $\alpha$-height 1. Such an entry is zero unless $[x_\beta, x_\gamma]$ is a nonzero element of $Fx_\rho$; that is, unless $\beta + \gamma = \rho$. Since $\langle \beta, \rho \rangle = 1$, $\rho - \beta$ is a root (of $\alpha$-height 1); hence each row and each column of the matrix contains exactly one nonzero entry. Such a matrix (sometimes called a monomial matrix) is the product of a diagonal matrix with nonzero entries on the diagonal and a permutation matrix, hence it is invertible. Thus the form is nondegenerate. □

Since $x_{-\rho}$ is in $\mathfrak{g}_{-2}$, for any $x \in \mathfrak{g}_1$ the value $[x, [x, [x, [x, x_{-\rho}]]]]$, or, more briefly, $(\text{ad} x)^4(x_{-\rho})$, is in $\mathfrak{g}_2$. Thus we may define a quartic form $q(x)$
for $x \in \mathfrak{g}_1$ by $(\text{ad} \, x)^4(x_{-\rho}) = q(x)x_\rho$. This in turn gives rise to a fully symmetric 4-linear form $q(x, y, z, w)$ defined by setting $q(x, x, x, x) = q(x)$ and extending by linearization.

**Lemma 3.** Let $\beta_1, \beta_2, \beta_3, \beta_4$ be roots of $\alpha$-height 1. The value of the 4-linear form $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is given by

$$q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})x_\rho = \frac{1}{4!} \sum_{\pi \in S_4} (\text{ad} \, x_{\beta_{\pi(1)}} \circ \text{ad} \, x_{\beta_{\pi(2)}} \circ \text{ad} \, x_{\beta_{\pi(3)}} \circ \text{ad} \, x_{\beta_{\pi(4)}})(x_{-\rho}),$$

where $S_4$ is the symmetric group on $\{1, 2, 3, 4\}$.

**Proof.** Let $\lambda, \mu, \nu$ be indeterminates. By the definition of the quartic form, we have

$$q(x_{\beta_1} + \lambda x_{\beta_2} + \mu x_{\beta_3} + \nu x_{\beta_4})x_\rho = (\text{ad} \, x_{\beta_1} + \lambda x_{\beta_2} + \mu x_{\beta_3} + \nu x_{\beta_4})^4(x_{-\rho}).$$

Replacing the quartic form by the equivalent 4-linear form, the coefficient of $\lambda \mu \nu$ on the left-hand side is $24q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})x_\rho$. On the right-hand side, the coefficient of $\lambda \mu \nu$ is $\sum_{\pi \in S_4} (\text{ad} \, x_{\beta_{\pi(1)}} \circ \text{ad} \, x_{\beta_{\pi(2)}} \circ \text{ad} \, x_{\beta_{\pi(3)}} \circ \text{ad} \, x_{\beta_{\pi(4)}})(x_{-\rho}).$ Equating the coefficients yields the result. \qed

**Corollary 4.** Let $\beta_1, \beta_2, \beta_3, \beta_4$ be roots of $\alpha$-height 1; then the 4-linear form $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = 0$ whenever $\beta_1 + \beta_2 + \beta_3 + \beta_4 \neq 2\rho$.

**Proof.** If the summand $(\text{ad} \, x_{\beta_{\pi(1)}} \circ \text{ad} \, x_{\beta_{\pi(2)}} \circ \text{ad} \, x_{\beta_{\pi(3)}} \circ \text{ad} \, x_{\beta_{\pi(4)}})(x_{-\rho})$ in the previous lemma is nonzero, it must be some multiple of $x_\rho$; that is, we must have $\beta_1 + \beta_2 + \beta_3 + \beta_4 + (-\rho) = \rho$. To establish that the quartic form is nonzero, we will require some information about the structure constants that define the multiplication in $\mathfrak{g}$. Given roots $\beta, \gamma \in \Psi$, we denote the corresponding structure constant by $c_{\beta, \gamma}$; that is, we define $c_{\beta, \gamma}$ so that $[x_\beta, x_\gamma] = c_{\beta, \gamma} x_{\beta + \gamma}$. In particular, $c_{\beta, \gamma} = 0$ if $\beta + \gamma$ is not a root. As always, we are assuming that the elements $x_\beta, x_\gamma$, etc. are in a Chevalley basis. Theorem 4.1.2 in [Car89] provides the following useful facts about these structure constants:

1. If $\beta, \gamma \in \Psi$, then $c_{\beta, \gamma} = -c_{\gamma, \beta}$.
2. If $\beta, \gamma, \delta \in \Psi$ are long roots such that $\beta + \gamma + \delta = 0$, then $c_{\beta, \gamma} = c_{\gamma, \delta} = c_{\delta, \beta}$.
3. If $\beta, \gamma \in \Psi$ are long roots and $\beta + \gamma$ is a root, then $c_{\beta, \gamma} = \pm 1$.
4. If $\beta, \gamma, \delta, \epsilon \in \Psi$ are long roots such that $\beta + \gamma + \delta + \epsilon = 0$ and no two are opposite, then

$$c_{\beta, \gamma}c_{\delta, \epsilon} + c_{\gamma, \delta}c_{\beta, \epsilon} + c_{\delta, \beta}c_{\gamma, \epsilon} = 0.$$

(5)

For 2, 3 and 4 we have simplified the statements in [Car89] by requiring the roots to be long. Since $x_\beta, x_\gamma$ are in a Chevalley basis, we also have $c_{\beta, \gamma} = -c_{-\beta, -\gamma}$ for all $\beta, \gamma \in \Psi$ and $[x_\beta, x_{-\beta}] = h_\beta$ ([Hum78] §25.2). These facts will be used freely, usually without further comment.

We now use the facts above to compute the value of the 4-linear form on some special arguments.
Lemma 6. If $\beta$ is a long root of $\alpha$-height 1, then

$$q(x_\beta, x_\beta, x_{\rho - \beta}, x_{\rho - \beta}) = 1.$$ \hfill (7)

Proof. By hypothesis, $\langle \beta, \rho \rangle = 1$, so $\rho - \beta$ is also a root. We begin by finding $q(x_\beta + \lambda x_{\rho - \beta})$, which is given by $(\text{ad } x_\beta + \lambda x_{\rho - \beta})^4(x_{\rho - \beta}) = q(x_\beta + \lambda x_{\rho - \beta})x_{\rho - \beta}$. The left-hand side can be calculated by repeatedly applying $\text{ad } x_\beta + \lambda x_{\rho - \beta}$. For the first step,

$$[x_\beta + \lambda x_{\rho - \beta}, x_{\rho - \beta}] = c_{\beta,-\rho} x_{\beta - \rho} + \lambda c_{\rho - \beta,-\rho} x_{\beta - \beta}.$$

Writing $a$ for $c_{\beta,-\rho}$ and $b$ for $c_{\rho - \beta,-\rho}$, we continue:

$$[x_\beta + \lambda x_{\rho - \beta}, ax_{\beta - \rho} + \lambda bx_{\beta - \beta}] = \lambda ah_{\rho - \beta} + \lambda bh_{\beta},$$

$$[x_\beta + \lambda x_{\rho - \beta}, \lambda ah_{\rho - \beta} + \lambda bh_{\beta}] = -2\lambda^2 ax_{\rho - \beta} - 2\lambda bx_{\beta} + \lambda ax_{\beta} + \lambda^2 bx_{\rho - \beta},$$

$$[x_\beta + \lambda x_{\rho - \beta}, -2\lambda^2 ax_{\rho - \beta} - 2\lambda bx_{\beta} + \lambda ax_{\beta} + \lambda^2 bx_{\rho - \beta}] = 3\lambda^2 c_{\beta,\rho - \beta}(b - a)x_{\rho}.$$  

Since $\beta$, $-\rho$ and $\rho - \beta$ are long roots that sum to zero, we have $a = c_{\beta,-\rho} = c_{\rho - \beta,\beta} = -c_{\beta,\rho - \beta}$ and $b = c_{\rho - \beta,-\rho} = c_{\beta,\rho - \beta} = -a$. Since the structure constant $c_{\beta,\rho - \beta}$ is ±1, the result is $6\lambda^2 c_{\beta,\rho - \beta} = 6\lambda^2$. The term in $\lambda^2$ resulting from expanding $q(x_\beta + \lambda x_{\rho - \beta})$ is $6\lambda^2 q(x_\beta, x_\beta, x_{\rho - \beta}, x_{\rho - \beta})$, so we have $q(x_\beta, x_\beta, x_{\rho - \beta}, x_{\rho - \beta}) = 1$, as required. \hfill $\Box$

Since there is always a long root of $\alpha$-height 1 (e.g., $\alpha$ itself), we have established that the 4-linear form and thus also the quartic form are not identically zero. In particular, taking $\beta = \alpha$ and $\lambda = 1$, we have $q(x_\alpha + x_{\rho - \alpha}) = 6$.

In the next section we will also need to know that the 4-linear form is nonzero in another special case. We show this after the next two lemmas. The first is an easy but useful observation; the second is a fact about structure constants that will also be used in Section 6.

Lemma 8. If $\beta$ is root of $\alpha$-height 1, then $\rho - \beta$ is also a root, is also of $\alpha$-height 1, and has the same length as $\beta$. If $\beta$ and $\gamma$ are orthogonal roots of $\alpha$-height 1, then $\rho - \beta$ and $\rho - \gamma$ are also orthogonal.

Proof. We have $\langle \beta, \rho \rangle = 1$, so $\rho - \beta$ is a root. The $\alpha$-height of $\rho - \beta$ is $\langle \rho - \beta, \rho \rangle = \langle \rho, \rho \rangle - \langle \beta, \rho \rangle = 2 - 1 = 1$. The highest root $\rho$ is long, so if $\beta$ is long, then so is $\rho - \beta$. If $\beta$ is short, $\rho - \beta$ cannot be long, for we then have that $\rho - (\rho - \beta) = \beta$ is long.
If \( \langle \beta, \gamma \rangle = 0 \), then
\[
\langle \rho - \beta, \rho - \gamma \rangle = \frac{2}{\langle \rho - \gamma, \rho - \gamma \rangle} \langle \rho - \beta, \rho - \gamma \rangle = \frac{2}{\langle \rho - \gamma, \rho - \gamma \rangle} ((\rho, \rho) - (\gamma, \rho) - (\beta, \rho) + (\beta, \gamma)) = \frac{(\rho, \rho)}{\langle \rho - \gamma, \rho - \gamma \rangle} ((\rho, \rho) - (\gamma, \rho) - (\beta, \rho)) = 0.
\]

**Lemma 9.** Let \( \beta \) and \( \gamma \) be two orthogonal long roots of \( \alpha \)-height 1. Each of \( \beta - \rho, \gamma - \rho \) and \( \rho - \beta - \gamma \) is a root; each of the structure constants \( c_{\beta,\gamma-\rho}, c_{\gamma,\beta-\rho}, c_{\gamma,-\rho} \) is \( \pm 1 \) and their product is 1.

**Proof.** By Lemma 8, \( \rho - \beta \) and \( \rho - \gamma \) are long roots, so their negatives are as well. Since \( \beta \) and \( \gamma \) are orthogonal, \( \langle \rho - \beta, \gamma \rangle = (\rho, \gamma) = 1 \), so \( \rho - \beta - \gamma \) is a root. Since these are roots, the specified structure constants are nonzero; since all roots involved are long, they are \( \pm 1 \).

We apply (15), replacing \( \beta, \gamma, \delta, \epsilon \) with \( \rho - \beta - \gamma, \beta, \gamma, \rho \) to yield
\[
c_{\rho-\beta-\gamma,\beta}c_{\gamma,-\rho} + c_{\beta,\gamma}c_{\rho-\beta-\gamma,-\rho} + c_{\gamma,\rho-\beta-\gamma}c_{\beta,-\rho} = 0.
\]
The structure constants in the middle term are zero since \( \beta + \gamma \) is not a root. Thus we find \( c_{\rho-\beta-\gamma,\beta}c_{\gamma,-\rho} = -c_{\gamma,\rho-\beta-\gamma}c_{\beta,-\rho} \). Substituting \( c_{\rho-\beta-\gamma,\beta} = c_{\beta,\gamma-\rho} \) and \( c_{\gamma,\rho-\beta-\gamma} = c_{\beta,\gamma-\rho} \), we have \( c_{\beta,\gamma-\rho}c_{\gamma,-\rho} = c_{\gamma,\beta-\rho}c_{\beta,-\rho} \). Since each side is \( \pm 1 \), the product of all four structure constants is 1.

**Lemma 10.** If \( \beta \) and \( \gamma \) are two orthogonal long roots of \( \alpha \)-height 1, then
\[
q(\beta, \gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{2}c_{\beta,-\rho}c_{\gamma,-\rho} \neq 0.
\]

**Proof.** By Lemma 8, there are 24 terms to consider. We divide them into three classes.

Class 1: These are the terms in which the first two elements applied to \( x_0 \) are \( x_{\beta} \) and \( x_{\rho-\beta} \), in either order, or, likewise, \( x_{\gamma} \) and \( x_{\rho-\gamma} \). The result in \( g_0 \) is thus in \( h \). By Lemma 8 since \( \beta \) and \( \gamma \) are orthogonal, so are \( \rho - \beta \) and \( \rho - \gamma \). As a result, half the terms in this case are zero; e.g., \( [x_{\rho-\beta}, [x_{\beta}, x_{\rho-\gamma}]] \) is a multiple of \( h_{\rho-\beta} \), and \( [x_{\rho-\gamma}, h_{\rho-\beta}] = 0 \). The other terms are
\[
[x_{\rho-\beta}, [x_{\gamma}, [x_{\rho-\beta}, [x_{\beta}, x_{\rho-\gamma}]]]] = -c_{\rho-\gamma,\beta}c_{\beta,-\rho}x_{\rho},
[x_{\gamma}, [x_{\rho-\gamma}, [x_{\beta}, [x_{\rho-\beta}, x_{\rho-\gamma}]]]] = -c_{\gamma,\rho-\beta}c_{\beta,-\rho}x_{\rho},
[x_{\rho-\beta}, [x_{\beta}, [x_{\rho-\gamma}, [x_{\gamma}, x_{\rho-\gamma}]]]] = -c_{\rho-\beta,\gamma}c_{\gamma,-\rho}x_{\rho},
[x_{\beta}, [x_{\rho-\beta}, [x_{\gamma}, [x_{\rho-\gamma}, x_{\rho-\gamma}]]]] = -c_{\beta,-\rho}c_{\rho-\gamma,\gamma}x_{\rho}.
\]

We have \( c_{\gamma,-\rho} = c_{\rho,\gamma-\rho} = -c_{\rho,-\gamma} = -c_{\rho-\gamma,-\rho} \), and likewise with \( \gamma \) replaced by \( \beta \). Thus each of these four terms is equal to \(-c_{\rho,-\beta}c_{\beta,-\rho}x_{\rho}\).

Class 2: Here the terms are those in which the first two elements applied to \( x_{\rho-\gamma} \) are \( x_{\beta} \) and \( x_{\rho-\gamma} \), in either order, or, likewise, \( x_{\gamma} \) and \( x_{\rho-\beta} \). Since \( \beta - \gamma \) (resp. \( \gamma - \beta \)) is not a root, these terms are all zero.
Class 3: The remaining terms are those in which the first two elements applied to $x_{-\rho}$ are $x_\beta$ and $x_\gamma$, in either order, or, likewise, $x_{\rho-\gamma}$ and $x_{\rho-\beta}$. Since $\beta + \gamma - \rho$ is a root by Lemma [4], the result in $\mathfrak{g}_0$ is nonzero and not in $\mathfrak{h}$, so we compute each term by simply accumulating the structure constants.

Four of the terms are

$$[x_{\rho-\gamma}, [x_{\rho-\beta}, [x_\gamma, [x_\beta, x_{-\rho}]])] = c_{\rho-\gamma, \gamma} c_{\rho-\beta, \beta} + \gamma - \rho c_{\gamma, \beta, -\rho} c_{\beta, -\rho, \rho}$$

$$= c_{\gamma, -\rho} c_{\gamma, -\rho} c_{\beta, -\rho} c_{\beta, -\rho, \rho}$$

$$= -c_{\gamma, -\rho} c_{\beta, -\rho, \rho}$$,

$$[x_\gamma, [x_\beta, [x_{\rho-\gamma}, [x_{\rho-\beta}, x_{-\rho}]])] = -c_{\gamma, -\rho} c_{\beta, -\rho, \rho}$$,

$$[x_{\rho-\beta}, [x_{\rho-\gamma}, [x_\gamma, [x_\beta, x_{-\rho}]])] = -c_{\beta, -\rho} c_{\gamma, -\rho}$$,

$$[x_\beta, [x_\gamma, [x_{\rho-\gamma}, [x_{\rho-\beta}, x_{-\rho}]])] = -c_{\beta, -\rho} c_{\gamma, -\rho}$$;

the remaining four are obtained by interchanging $\beta$ and $\gamma$. This yields four terms equal to $-c_{\beta, -\rho} c_{\gamma, -\rho} c_{\rho, \beta}$ and four equal to $-c_{\beta, -\rho} c_{\gamma, -\rho} c_{\rho, \beta}$. Combining all the terms, we have

$$q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{3} c_{\beta, -\rho} c_{\gamma, -\rho} - \frac{1}{6} c_{\beta, -\rho} c_{\gamma, -\rho} c_{\rho, \beta}$$;

but it follows from Lemma [4] that the two products of structure constants are equal. Thus we have

$$q(x_\beta, x_\gamma, x_{\rho-\beta}, x_{\rho-\gamma}) = -\frac{1}{2} c_{\beta, -\rho} c_{\gamma, -\rho}$$.

In particular, it is not zero. \qed

4. Strictly regular elements

For any fixed $x, y, z \in \mathfrak{g}_1$, the expression $q(w, x, y, z)$ with $w \in \mathfrak{g}_1$ is a linear function of $w$. Since the skew-symmetric bilinear form $\langle -, - \rangle$ is nondegenerate (Lemma [2]), we may define the triple product of $x, y, z$ to be the unique element $xyz$ of $\mathfrak{g}_1$ such that $q(w, x, y, z) = \langle w, xyz \rangle$ for all $w \in \mathfrak{g}_1$.

Following Ferrar [Per72, §3], we call a nonzero element $x \in \mathfrak{g}_1$ strictly regular if $xxy \in Fx$ for all $y \in \mathfrak{g}_1$. In this section we will give several equivalent characterizations of strictly regular elements.

Lemma 12. The basis element $x_\alpha$ is strictly regular.

Proof. Let $\beta, \gamma$ be roots of $\alpha$-height 1. By Corollary [1] if $\langle x_\gamma, x_\alpha x_\alpha x_\beta \rangle = q(x_\gamma, x_\alpha, x_\alpha, x_\beta)$ is nonzero, then $2\alpha + \beta + \gamma = 2\rho$. Since the simple root $\alpha$ has height 1, this implies $\text{ht}(\beta + \gamma) = 2\text{ht}\rho = 2$. As $\rho$ is the unique highest root, $\beta$ and $\gamma$ have smaller heights than $\rho$, so this can only occur if both have height $\text{ht}\rho - 1$. Since the only simple root not orthogonal to $\rho$ is $\alpha$, the only root of that height is $\rho - \alpha$, and $\langle x_\gamma, x_\alpha x_\alpha x_\beta \rangle$ is therefore zero unless $\beta = \gamma = \rho - \alpha$. The orthogonal complement of any $x_\alpha x_\alpha y$ thus includes the space generated by all the $x_\gamma$, $\gamma \neq \rho - \alpha$. Since this is the orthogonal complement of $x_\alpha$, we have $x_\alpha x_\alpha y \in Fx_\alpha$. \qed
Corollary 13. For any long root $\beta$ of $\alpha$-height 1, $x_\beta$ is strictly regular.

Proof. Since the property of being strictly regular depends only on the triple product, which is in turn defined in terms of the quartic and bilinear forms, it is preserved by the action of $(G_0)^{ss}$ by Lemma 1. It is also preserved by scaling, so it is preserved by the action of $G_0$. By Lemma 2.1 in [Röhl93], all the elements $x_\beta$ with $\beta$ a long root of $\alpha$-height 1 are in the same $G_0$-orbit, so they are are all strictly regular since $x_\alpha$ is.

Lemma 14. Let $x \in g_1$ be such that $xxy = 0$ for all $y \in g_1$; then $x = 0$.

Proof. The set of all $x$ such that $xxy = 0$ is invariant under the action of $G_0$ on $g_1$, so it is a union of $G_0$-orbits; it is also closed (in the Zariski topology). Thus it suffices to show that $xxy \neq 0$ for a representative $x$ of the smallest nonzero orbit (i.e., orbit 1); this follows if there are $y, z \in g_1$ such that $q(x, x, y, z) \neq 0$. A representative of the smallest nonzero orbit is $x = x_\alpha$; we let $y = z = x_{\rho, \alpha}$. By (7), we have $q(x, x, y, z) = 1$. □

Lemma 15. If $\beta_1, \beta_2, \beta_3, \beta_4$ are mutually orthogonal roots of $\alpha$-height 1, then $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$.

This is Corollary 1.4 in [Röhl93].

Proof. Since $\beta_1$ has $\alpha$-height 1, $\rho - \beta_1$ is a root. Since $\beta_2$ is orthogonal to $\beta_1$, $\langle \rho - \beta_1, \beta_2 \rangle = \langle \rho, \beta_2 \rangle - \langle \beta_1, \beta_2 \rangle = 1$, so $\rho - \beta_1 - \beta_2$ is also a root. Continuing in this fashion, we find that $\rho - \beta_1 - \beta_2 - \beta_3 - \beta_4$ is a root; since it has $\alpha$-height $-2$, it must be $-\rho$.

Lemma 16. If four roots of $\alpha$-height 1 are mutually orthogonal, then they are all long roots.

Proof. Call the roots $\beta_1, \beta_2, \beta_3, \beta_4$. By Lemma 15, $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$; since the roots are mutually orthogonal we then have

$$4(\rho, \rho) = (2\rho, 2\rho) = (\beta_1 + \beta_2 + \beta_3 + \beta_4, \beta_1 + \beta_2 + \beta_3 + \beta_4) = (\beta_1, \beta_1) + (\beta_2, \beta_2) + (\beta_3, \beta_3) + (\beta_4, \beta_4).$$

Since $\rho$ is long, $(\beta_i, \beta_i) \leq (\rho, \rho)$ for each $i$, $1 \leq i \leq 4$; thus we must have $(\beta_i, \beta_i) = (\rho, \rho)$ for each $i$.

Lemma 17. Let $\alpha, \beta, \gamma, \delta$ be mutually orthogonal roots of $\alpha$-height 1; then $q(x_\alpha, x_\beta, x_\gamma, x_\delta) \neq 0$.

Proof. Since $x_\alpha + x_\beta + x_\gamma + x_\delta$ is a representative of the dense orbit and $q$ is not identically zero, $q(x_\alpha + x_\beta + x_\gamma + x_\delta) \neq 0$. Expanding the corresponding 4-linear form, we obtain five kinds of terms:

- Those with four equal arguments, e.g., $q(x_\beta, x_\beta, x_\beta, x_\beta)$. Since we cannot have $4\beta = 2\rho$, this expression is zero by Lemma 3.

- Those with three equal arguments and one different argument, e.g., $q(x_\beta, x_\beta, x_\beta, x_\gamma)$. Since we cannot have $3\beta + x_\gamma = 2\rho$, this expression is zero by Lemma 3.

- Those with two equal arguments and two different arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\delta)$. Since we cannot have $2\beta + x_\gamma + x_\delta = 2\rho$, this expression is zero by Lemma 3.

- Those with one equal argument and three different arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\delta)$. Since we cannot have $x_\beta + x_\gamma + x_\delta = 2\rho$, this expression is zero by Lemma 3.

- Those with four different arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\delta)$. Since we cannot have $x_\beta + x_\gamma + x_\delta = 2\rho$, this expression is zero by Lemma 3.

□
• Those with exactly three equal arguments, e.g., $q(x_\beta, x_\beta, x_\beta, x_\gamma)$. The mutually orthogonal roots $\alpha, \beta, \gamma, \delta$ are long by Lemma 16. Thus $x_\beta$ is strictly regular (Corollary 13), so the 4-linear form here is $\langle x_\gamma, x_\beta x_\beta x_\beta \rangle = \lambda \langle x_\gamma, x_\beta \rangle$ for some $\lambda \in F$; but $\langle x_\gamma, x_\beta \rangle = 0$ because $\gamma + \beta$ is not a root. Thus these terms are also zero.

• Those with two pairs of equal arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\gamma)$. Since $\beta + \gamma$ is not a root, it is not $\rho$. Thus $2\beta + 2\gamma \neq 2\rho$, so this expression is zero.

• Those with exactly two equal arguments, e.g., $q(x_\beta, x_\beta, x_\gamma, x_\delta)$. By Lemma 15, $\alpha + \beta + \gamma + \delta = 2\rho$; thus $2\beta + \gamma + \delta \neq 2\rho$, so these terms are zero.

• Those with four unequal arguments, e.g., $q(x_\alpha, x_\beta, x_\gamma, x_\delta)$, which by elimination must be nonzero. □

**Proposition 18.** The strictly regular elements of $g_1$ are those contained in the smallest nonzero orbit.

**Proof.** The set of strictly regular elements is a union of orbits; its union with 0 is a closed set. Since $x_\alpha$ is a representative of the smallest nonzero orbit and is strictly regular by Lemma 12, all elements of the smallest nonzero orbit are also strictly regular. It thus suffices to show that representatives of level 2 orbits are not strictly regular. Let $\alpha, \beta, \gamma, \delta$ be four mutually orthogonal roots of $\alpha$-height 1. We take $x_\alpha + x_\beta$ as a representative of a level 2 orbit.

We compute

$$\langle x_\delta, (x_\alpha + x_\beta)(x_\alpha + x_\beta)x_\gamma \rangle = q(x_\alpha + x_\beta, x_\alpha + x_\beta, x_\gamma, x_\delta)$$

$$= q(x_\alpha, x_\alpha, x_\gamma, x_\delta) + 2q(x_\alpha, x_\beta, x_\gamma, x_\delta)$$

$$+ q(x_\beta, x_\beta, x_\gamma, x_\delta)$$

$$= 2q(x_\alpha, x_\beta, x_\gamma, x_\delta),$$

the other terms being zero since $\alpha + \alpha + \gamma + \delta$ and $\beta + \beta + \gamma + \delta$ cannot equal $2\rho$ since $\alpha + \beta + \gamma + \delta = 2\rho$ by Lemma 15. By Lemma 17, the result is non-zero, so in particular the triple product $(x_\alpha + x_\beta)(x_\alpha + x_\beta)x_\gamma$ is not orthogonal to $x_\delta$. However, $(x_\alpha + x_\beta, x_\delta) = 0$ since neither $\alpha + \delta$ nor $\beta + \delta$ is a root. Hence the triple product $(x_\alpha + x_\beta)(x_\alpha + x_\beta)x_\gamma$ is not a scalar multiple of $x_\alpha + x_\beta$; thus $x_\alpha + x_\beta$ is not strictly regular. □

**Lemma 19.** The strictly regular elements span $g_1$.

**Proof.** By Proposition 18, orbit 1 consists of strictly regular elements. The span of orbit 1 is invariant under the action of $G_0$; thus it is a union of orbits. Both $x_\alpha$ and $x_{\rho - \alpha}$ are in orbit 1, so $x_\alpha + x_{\rho - \alpha}$ is in their span, but is also a representative of the dense orbit. Thus all of the dense orbit is in the span of orbit 1. Since the dense orbit is not contained in a proper subspace, the span of orbit 1 is all of $g_1$. □
An element \( x \in g_1 \) is rank one if \( xxg_1 \) is a one-dimensional vector space over \( F \).

**Proposition 20.** An element \( x \in g_1 \) is strictly regular if and only if it is rank one.

**Proof.** Suppose \( x \) is strictly regular. By definition, \( xxg_1 \) is contained in the one-dimensional space \( Fx \). In the case \( x = x_0 \), we know \( x_0x_0g_1 \) is not zero because \( \langle x_{\rho-\alpha}, x_0x_0x_{\rho-\alpha} \rangle = q(x_{\rho-\alpha}, x_0, x_\alpha, x_{\rho-\alpha}) \), which is 1 by (7). The condition that \( xxg_1 \) is not zero is invariant under the action of \( G_0 \), so it holds for all of orbit 1.

As in the proof of the previous proposition, let \( \alpha, \beta, \gamma, \delta \) be four mutually orthogonal roots of \( \alpha \)-height 1, and choose \( x = x_\alpha + x_\beta \) as a representative of a level 2 orbit. Since the set of rank one elements is a closed union of orbits, it will suffice to show that \( x \) is not rank one. We have \( \langle x_{\rho-\beta}, xxx_{\rho-\alpha} \rangle = 2q(x_{\rho-\beta}, x_\alpha, x_\beta, x_{\rho-\alpha}) \neq 0 \), by Corollary 4 and \( \{1\} \). However, \( \langle x_{\rho-\beta}, xxx_\gamma \rangle = q(x_{\rho-\beta}, x_\alpha, x_\gamma) + q(x_{\rho-\beta}, x_\beta, x_\gamma) + 2q(x_{\rho-\beta}, x_\alpha, x_\beta, x_\gamma) = 0 \), where we know the first term is zero because it is \( \langle x_{\rho-\beta}, x_\alpha x_\alpha x_\gamma \rangle \) and the triple product is a scalar multiple of \( x_\alpha \); the other two terms are zero by Corollary 4. On the other hand, we know that \( xxx_\gamma \) is nonzero since \( \langle x_\delta, xxx_\gamma \rangle = 2q(x_\alpha, x_\beta, x_\gamma, x_\delta) \) which is not zero by Lemma 17. Thus \( xxx_{\rho-\alpha} \) and \( xxx_\gamma \) do not lie in the same one-dimensional subspace, so \( x \) is not rank one. \( \square \)

The following result allows us to compute the triple product and the 4-linear form if two of the arguments are the same strictly regular element.

**Lemma 21.** For \( x \) strictly regular and any \( y, z \in g_1 \),

\begin{align*}
\langle xy, x \rangle & = q(y, x)x, \\
q(x, x, y, z) & = \langle y, x \rangle \langle z, x \rangle.
\end{align*}

**Proof.** Since \( x \) is strictly regular, for any \( y \in g_1 \) we have \( xxy \in Fx \). If \( \langle y, x \rangle = 0 \), then for any \( z \in g_1 \) we have \( \langle z, xxy \rangle = q(z, x, x, y) = \langle y, xxx \rangle = 0 \), thus \( xxy = 0 \). Define \( f : g_1 \to F \) by \( xxy = f(y)x \); then \( f \) is a linear form and \( f(y) \) is zero whenever \( \langle y, x \rangle \) is zero. Thus \( f(-) \) is a scalar multiple of \( \langle -, x \rangle \).

By Proposition 18 \( x \) is in orbit 1; by Lemma 12 so is \( x_\alpha \). Hence there is some element \( g \in (G_0)^s \) such that \( g \cdot x = cx_\alpha \) for some \( c \in F^x \). Let \( x' = g^{-1} \cdot x_{\rho-\alpha} \); since the bilinear form is preserved by the action of \( (G_0)^s \) (Lemma 1), we have \( \langle x', x \rangle = \langle x_{\rho-\alpha}, cx_\alpha \rangle = \pm c \). We can now compute \( q(x, x, x', x') \) in two ways. On the one hand, since the 4-linear form is also
On the other hand, it is preserved, we have

\[ q(x, x', x'') = q(cx_\alpha, cx_\alpha, x_{\rho - \alpha}, x_{\rho - \alpha}) \]
\[ = c^2 q(x_\alpha, x_\alpha, x_{\rho - \alpha}, x_{\rho - \alpha}) \]
\[ = c^2 \quad \text{(by (7))} \]
\[ = \langle x', x \rangle^2. \]

On the other hand, it is \( \langle x', xx' \rangle = \langle x', f(x')x \rangle = f(x') \langle x', x \rangle \). Thus, \( f(x') = \langle x', x \rangle \), and therefore \( f(y) = \langle y, x \rangle \) for any \( y \in \mathfrak{g}_1 \).

By the definition of \( f \), we now have \( xx'y = \langle y, x \rangle x \) for all \( y \in \mathfrak{g}_1 \). Further, for any \( z \in \mathfrak{g}_1 \) we have \( q(x, x, y, z) = \langle z, xx'y \rangle = \langle y, x \rangle \langle z, x \rangle \).

**Lemma 24.** Each element in the dense orbit of \( \mathfrak{g}_1 \) can be expressed as the sum of two strictly regular elements in one and only one way.

**Proof.** Since the action of \((G_0)^{ss}\) and scaling by elements of \( F^\times \) both preserve strictly regular elements, it suffices to prove this for any representative of the dense orbit. We choose \( x = x_\alpha + x_{\rho - \alpha} \) as the representative, which establishes the existence of such an expression.

Suppose \( x = u + v \) with \( u, v \) strictly regular. The triple product \( xxx \) is thus

\[ (u + v)(u + v)(u + v) = uuu + 3uvw + 3uvv + vvv \]
\[ = \langle u, u \rangle u + 3\langle v, u \rangle u + 3\langle u, v \rangle v + \langle v, v \rangle v \]
\[ = 3\langle v, u \rangle (u - v); \]

in particular, this is true if \( u = x_\alpha \) and \( v = x_{\rho - \alpha} \), so we have shown that

\[ 3\langle v, u \rangle (u - v) = 3\langle x_{\rho - \alpha}, x_\alpha \rangle (x_\alpha - x_{\rho - \alpha}). \]

The quartic form \( q(x) = \langle x, xxx \rangle \) is thus

\[ \langle u + v, 3\langle v, u \rangle (u - v) \rangle = 3\langle v, u \rangle (-\langle u, v \rangle + \langle v, u \rangle) \]
\[ = 6\langle v, u \rangle^2; \]

again, this must be the same as \( 6\langle x_{\rho - \alpha}, x_\alpha \rangle^2 \). Thus \( \langle v, u \rangle = \pm \langle x_{\rho - \alpha}, x_\alpha \rangle \), so (25) yields \( u - v = \pm (x_\alpha - x_{\rho - \alpha}) \). Combined with \( u + v = x_\alpha + x_{\rho - \alpha} \), one choice of sign yields \( u = x_\alpha, v = x_{\rho - \alpha} \), and the other \( u = x_{\rho - \alpha}, v = x_\alpha \), so the choice of \( u \) and \( v \) is determined up to order. \( \square \)

5. Freudenthal triple systems

A Freudenthal triple system is a finite-dimensional vector space \( V \) over a field \( F \) (with characteristic not 2 or 3) such that

- There is a nonzero quartic form \( q \) defined on \( V \). A corresponding 4-linear form, also called \( q \), is given by linearization, with \( q(x, x, x, x) = q(x) \) for all \( x \in V \).
• There is a nondegenerate skew-symmetric bilinear form \( \langle - , - \rangle \) defined on \( V \). Thus for given \( x, y, z \in V \) we may define the triple product \( xyz \) to be the unique vector in \( V \) such that \( q(w, x, y, z) = \langle w, xyz \rangle \) for all \( w \in V \).

• The triple product satisfies the following identity:

\[
2(xxy)xy = \langle y, x \rangle xx + \langle y, xxx \rangle .
\] (26)

Definitions of Freudenthal triple system in the literature vary. For example, in \cite{Fer72} the 2 on the left-hand side of (26) is omitted; in \cite{Spr06} the 2 becomes a 6 and the triple product is defined so that \( 8q(w, x, y, z) = \langle xyz, w \rangle \). However, these variations are inessential; it is easy to convert one definition to another by rescaling the quartic and bilinear forms as needed.

**Theorem 27.** The vector space \( g_1 \) equipped with the quartic form \( q \) and the bilinear form \( \langle - , - \rangle \) is a Freudenthal triple system.

**Proof.** We established in Section 3 that \( \langle - , - \rangle \) is skew-symmetric and nondegenerate and that \( q \) is nonzero. Hence it remains only to show that the triple product identity (26) is satisfied.

We first set \( x = x_\alpha + x_\rho - \alpha \). As in the proof of Lemma 24 we use (22) to compute \( xxx = 3(x_\rho - \alpha, x_\alpha)(x_\alpha - x_\rho - \alpha) \). Thus the left-hand side of (26) is

\[
2(xxx)xy = 6\langle x_\rho - \alpha, x_\alpha \rangle(x_\alpha - x_\rho - \alpha)y \\
= 6\langle x_\rho - \alpha, x_\alpha \rangle(x_\alpha x_\alpha y - x_\rho - \alpha x_\rho - \alpha y) \\
= 6\langle x_\rho - \alpha, x_\alpha \rangle(\langle y, x_\alpha \rangle x_\alpha - \langle y, x_\rho - \alpha \rangle x_\rho - \alpha).
\]

The right-hand side is

\[
\langle y, x \rangle xxx + \langle y, xxx \rangle x = 3(x_\rho - \alpha, x_\alpha)((\langle y, x_\alpha \rangle + \langle y, x_\rho - \alpha \rangle)(x_\alpha - x_\rho - \alpha) \\
+ 3(x_\rho - \alpha, x_\alpha)((\langle y, x_\alpha \rangle - \langle y, x_\rho - \alpha \rangle)(x_\alpha + x_\rho - \alpha) \\
= 6\langle x_\rho - \alpha, x_\alpha \rangle(\langle y, x_\alpha \rangle x_\alpha - \langle y, x_\rho - \alpha \rangle x_\rho - \alpha);
\]

thus (26) holds for \( x = x_\alpha + x_\rho - \alpha \) and any \( y \in g_1 \).

Since the action of \((G_0)^{ss}\) on \( g_1 \) stabilizes the bilinear form and the triple product, and since (26) is preserved if \( x \) is adjusted by a scalar factor, it holds for the entire orbit of \( x \), which is the dense orbit. Since the identity is a polynomial condition it also holds on the closure, which is all of \( g_1 \). \( \square \)

### 6. Computation of the 4-linear form

In this section we show how to evaluate the expression \( q(x_\beta, x_\gamma, x_\delta, x_\epsilon) \) whenever \( \beta, \gamma, \delta, \epsilon \) are long roots of \( \alpha \)-height 1. Among the Lie algebras we are considering, the roots are always long in types \( D \) and \( E \), so, by linearity, this will suffice to compute \( q \) for any values in \( g_1 \) in these cases.

**Lemma 28.** Suppose \( \beta_1, \beta_2, \beta_3, \beta_4 \) are long roots of \( \alpha \)-height 1 and that their sum is 2\( \rho \). It follows that

\[
(\beta_1, \beta_2) + (\beta_1, \beta_3) + (\beta_1, \beta_4) = 0
\] (29)
and
\[(30) \quad \langle \beta_1, \beta_2 \rangle = \langle \beta_3, \beta_4 \rangle.\]

**Proof.** We may reverse the arguments of \(\langle - , - \rangle\) whenever both are long roots. Thus to show (29) we compute \(\langle \beta_1, \beta_2 \rangle + \langle \beta_1, \beta_3 \rangle + \langle \beta_1, \beta_4 \rangle = \langle \beta_2, \beta_1 \rangle + \langle \beta_3, \beta_1 \rangle + \langle \beta_4, \beta_1 \rangle = 2\rho - \beta_1, \beta_1 \rangle = 2\langle \rho, \beta_1 \rangle - \langle \beta_1, \beta_1 \rangle = 0.

To show (30), we expand the equal expressions \((\beta_1 + \beta_2 + \beta_1 + \beta_2)\) and \((2\rho - \beta_3 - \beta_4, 2\rho - \beta_3 - \beta_4)\). Taking the long roots to have unit length, we have on the one hand \((\beta_1 + \beta_2, \beta_1 + \beta_2) = 2 + 2(\beta_1, \beta_2)\). Keeping in mind that, for example, \(2(\rho, \beta_3) = \langle \rho, \beta_3 \rangle = 1\), we have on the other hand
\[
(2\rho - \beta_3 - \beta_4, 2\rho - \beta_3 - \beta_4) = 6 - 4(\rho, \beta_3) - 4(\rho, \beta_4) + 2(\beta_3, \beta_4) = 2 + 2(\beta_3, \beta_4).
\]

Thus \(2(\beta_1, \beta_2) = 2(\beta_3, \beta_4)\); that is, \(\langle \beta_1, \beta_2 \rangle = \langle \beta_3, \beta_4 \rangle\). \(\square\)

**Proposition 31.** If the sum of four long roots of \(\alpha\)-height 1 is \(2\rho\), then one of the following three cases must hold:

(a) The four roots consist of two equal pairs; that is, they are of the form \(\beta, \beta, \rho - \beta, \rho - \beta\) for some \(\beta\).

(b) The four roots consist of distinct pairs that sum to \(\rho\); that is, they are of the form \(\beta, \rho - \beta, \gamma, \rho - \gamma\) for distinct \(\beta, \gamma\). Moreover, we may take \(\beta\) and \(\gamma\) to be orthogonal.

(c) The four roots are mutually orthogonal.

**Proof.** Let \(\beta_1, \beta_2, \beta_3, \beta_4\) be four such roots. No two can be opposite since all have \(\alpha\)-height 1. If any two are equal, say \(\beta_1 = \beta_2\), then by (30) we have \(2 = \langle \beta_1, \beta_2 \rangle = \langle \beta_3, \beta_4 \rangle\), so \(\beta_3 = \beta_4\) as well. This is case (a).

Suppose some root, say \(\beta_1\), is not orthogonal to all of the others. By (29) we have \(\langle \beta_1, \beta_2 \rangle + \langle \beta_1, \beta_3 \rangle + \langle \beta_1, \beta_4 \rangle = 0\); since each term is \(-1, 0\) or 1 and not all are zero, we must have one of each. Without loss of generality, assume \(\langle \beta_1, \beta_2 \rangle = -1\) and \(\langle \beta_1, \beta_3 \rangle = 0\); then \(\beta_1 + \beta_2\) is a root. Since it has \(\alpha\)-height 2, it must be \(\rho\). By (30), we also have \(\langle \beta_3, \beta_4 \rangle = -1\), thus also \(\beta_3 + \beta_4 = \rho\). Thus we are in case (b). As indicated, we have \(\beta_1\) and \(\beta_3\) orthogonal.

The only remaining possibility is that the four roots are mutually orthogonal, which is case (c). \(\square\)

We now proceed to give the value of \(q(\beta_1, \beta_2, \beta_3, \beta_4)\) in each of the three cases. We remind the reader that we will be making extensive use of the facts about structure constants previously mentioned in Section 8.

The first case was already handled in Lemma 6 where we showed that \(q(x_\beta, x_\beta, x_{\rho - \beta}, x_{\rho - \beta}) = 1\) for any long root \(\beta\) of \(\alpha\)-height 1. The second case was computed in Lemma 10 where we found \(q(x_\beta, x_\gamma, x_{\rho - \beta}, x_{\rho - \gamma}) = -\frac{1}{2}c_{\beta, \rho}c_{\gamma, \rho}\) where \(\beta\) and \(\gamma\) are orthogonal long roots of \(\alpha\)-height 1. The remaining case is covered by the following lemma.
Lemma 32. If $\beta_1, \beta_2, \beta_3, \beta_4$ are mutually orthogonal roots of $\alpha$-height 1, then
\[ q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = c_{\beta_1, \beta_4} - p c_{\beta_2, \beta_1} - r c_{\beta_3, \beta_4} - s c_{\beta_1, \beta_2} \neq 0. \]

Proof. By Lemma 15, the sum of four mutually orthogonal roots of $\alpha$-height 1 is 2$p$, and by Lemma 16 they are all long roots. We will apply (5) with $\beta, \gamma, \delta, \epsilon = \beta_1, \beta_2, \beta_3 - \rho, \beta_4 - \rho$. Observe that $\beta + \gamma + \delta + \epsilon = 0$ and no two of $\beta, \gamma, \delta, \epsilon$ are opposite; for example, $\beta + \delta = 0$ implies $\beta_1 + \beta_3 = \rho$, but $\beta_1$ and $\beta_3$ are orthogonal. With these values, (5) becomes
\[ c_{\beta_1, \beta_2} c_{\beta_3 - \rho, \beta_4 - \rho} + c_{\beta_2, \beta_3 - \rho} c_{\beta_4 - \rho} + c_{\beta_3 - \rho, \beta_1} c_{\beta_2, \beta_4 - \rho} = 0. \]
The structure constants in the first term are zero since $\beta_1 + \beta_2$ is not a root. Since $\beta_2 + \beta_3 - \rho$ and $\beta_1 + \beta_3 - \rho$ are roots the remaining terms are not zero.

We now have $c_{\beta_2, \beta_3 - \rho} c_{\beta_4 - \rho} = -c_{\beta_3 - \rho, \beta_1} c_{\beta_2, \beta_4 - \rho}$. Using $a_{ij}$ as an abbreviation for $c_{\beta_i, \beta_j - \rho}$, we can rewrite this as
\[ a_{23} a_{14} = a_{13} a_{24}. \]
Since the numbering of the indices is arbitrary, we think of this as saying that, in a product of the form $a_{ij} a_{kl}$ that uses four different indices, we may interchange the first subscripts of the two factors.

Since all the $a_{ij}$ are $\pm 1$, we can freely move them across the equals sign; in particular, we also have
\[ a_{13} a_{23} = a_{14} a_{24}; \]
in other words, in a product of the form $a_{ij} a_{kj}$ involving three different indices, the repeated index may be replaced by the unused one.

A typical term in the sum for $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ given by Lemma 3 is
\[ c_{\beta_1, \beta_2} c_{\beta_3, \beta_4} - p c_{\beta_4, \beta_2} c_{\beta_3, \beta_1} - r c_{\beta_3, \beta_4} c_{\beta_2, \beta_1} - s c_{\beta_2, \beta_4} c_{\beta_3, \beta_1} = a_{14} a_{21} a_{34} a_{41}, \]
where we have used Lemma 9 for the second equality. Every other term in the sum is obtained by permuting the indices; we will show that the value is unchanged in each case. Since the two permutations given by $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1$ and by $1 \mapsto 2 \mapsto 1$ generate the symmetric group, it suffices to show that $a_{21} a_{32} a_{41} a_{12}$ and $a_{24} a_{12} a_{34} a_{42}$ are the same as the product above.

We first apply the principle of (34) in the form $a_{14} a_{34} a_{41} = a_{12} a_{21} a_{32} a_{41}$ to find that $a_{14} a_{21} a_{34} a_{41} = a_{12} a_{32} a_{41} a_{12}$, so the first required equality holds. Proceeding from the last expression, we alternately apply (34) and (33) as follows:
\[ a_{21} a_{32} a_{41} a_{12} = a_{23} a_{32} a_{43} a_{12}, \quad (\text{since } a_{21} a_{41} = a_{23} a_{43}) \]
\[ = a_{23} a_{32} a_{13} a_{42}, \quad (\text{since } a_{43} a_{12} = a_{13} a_{42}) \]
\[ = a_{24} a_{32} a_{14} a_{42}, \quad (\text{since } a_{23} a_{13} = a_{24} a_{14}) \]
\[ = a_{24} a_{12} a_{34} a_{42}, \quad (\text{since } a_{32} a_{14} = a_{12} a_{34}) \]
which is the required product.

Thus all 24 summands are equal, so we have

$$q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) = a_{14}a_{21}a_{34}a_{41},$$

which, by substituting for the $a_{ij}$, becomes the desired equation. \qed

To summarize, we have the following result.

**Proposition 35.** If $\beta_1, \beta_2, \beta_3, \beta_4$ are long roots of $\alpha$-height 1, then the value of $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is one of the following:

- 0, if $\beta_1 + \beta_2 + \beta_3 + \beta_4 \neq 2\rho$;
- 1, if $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$ and there are two pairs of equal roots;
- $-\frac{1}{2}c_{\beta,-\rho}c_{\gamma,-\rho}$ if the roots are, in some order, $\beta, \gamma, \rho - \beta, \rho - \gamma$ with $\langle \beta, \gamma \rangle = 0$ for some $\beta, \gamma$; or
- $c_{\beta_1, \beta_3} - \rho c_{\beta_2, \beta_1} - \rho c_{\beta_3, \beta_4} - \rho c_{\beta_4, \beta_1} - \rho$ if the four roots are mutually orthogonal.

In particular, $q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4})$ is nonzero whenever $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\rho$.

7. Eigenspace decomposition of $\mathfrak{g}_1$

In this section we assume that $\mathfrak{g}$ is a Lie algebra of type $D$ or $E$. We show that there is an element $h$ in the torus $\mathfrak{h}$ such that $\mathfrak{g}_1$ is the direct sum of the four eigenspaces under ad $h$ corresponding to the eigenvalues $-3, -1, 1, 3$, and that the eigenspaces corresponding to the eigenvalues $-3$ and $3$ are one-dimensional (cf. [Fer72], §4). This is a consequence of the following proposition about the corresponding root systems.

**Proposition 36.** Let $\Psi$ be a root system of type $D$ or $E$. For any root $\beta \in \Psi$ of $\alpha$-height 1 we have

$$\langle \rho - 2\alpha, \beta \rangle = \begin{cases} 
-3 & \text{if } \beta = \alpha, \\
3 & \text{if } \beta = \rho - \alpha, \\
\pm 1 & \text{otherwise.}
\end{cases}$$

Moreover, the cases $\langle \rho - 2\alpha, \beta \rangle = -1$ and $\langle \rho - 2\alpha, \beta \rangle = 1$ occur equally often.

**Proof.** Let $\beta$ be a root of $\alpha$-height 1. For each such root, $\rho - \beta$ is another root of $\alpha$-height 1, and we have $\langle \alpha, \beta \rangle + \langle \alpha, \rho - \beta \rangle = 1$. Since $\langle \alpha, \beta \rangle = 2$ only if $\beta = \alpha$, it follows that $\langle \alpha, \beta \rangle = -1$ only if $\beta = \rho - \alpha$. Thus for the remaining pairs of roots $\beta, \rho - \beta$ we have $\langle \alpha, \beta \rangle = 0$ or 1 and correspondingly $\langle \alpha, \rho - \beta \rangle = 1$ or 0.

As $\langle \rho, \beta \rangle = 1$, we have $\langle \rho - 2\alpha, \beta \rangle = 1 - 2\langle \alpha, \beta \rangle$. Thus $\langle \rho - 2\alpha, \alpha \rangle = -3$ and $\langle \rho - 2\alpha, \rho - \alpha \rangle = 3$, with the remaining cases split equally between $\langle \rho - 2\alpha, \beta \rangle = 1$ and $\langle \rho - 2\alpha, \beta \rangle = -1$. \qed

The above proposition can be generalized by using $\rho - 2\alpha'$ with $\alpha'$ any root of $\alpha$-height 1 in place of $\rho - 2\alpha$; the proof goes through unchanged. However, we do not make use of this added generality.
At this point, we know that the promised element of \( \mathfrak{h} \) exists because the Chevalley basis gives an isomorphism between \( \mathfrak{h} \) and the coroot lattice with scalars extended to \( F \). To give it explicitly, recall that, for any root \( \beta \), the element \( h_\beta \in \mathfrak{h} \) is defined to be \( [x_\beta, x_{-\beta}] \) and has the property that \( [h_\beta, x_\gamma] = \langle \gamma, \beta \rangle x_\gamma \) for any root \( \gamma \) (see [Hum78, §§8.3, 25.2]). Setting \( h = h_\rho - h_\alpha \in \mathfrak{h} \), we then have \( [h, x_\beta] = ([\beta, \rho - \alpha] - \langle \beta, \alpha \rangle) x_\beta = (\rho - 2\alpha, \beta)x_\beta \), yielding the eigenvalue decomposition described above.

8. Characterization of the orbits

Lemma 37. Let \( \beta, \gamma \) be roots of \( \alpha \)-height 1. The triple product \( x_\beta x_\gamma x_\rho \) is zero unless \( \beta + \gamma = \rho \).

Proof. Since \( x_\beta \) is strictly regular (Corollary [13]), (22) gives \( x_\beta x_\gamma x_\rho = \langle x_\gamma, x_\beta \rangle x_\rho \). As \( \langle x_\gamma, x_\beta \rangle \) is zero unless \( \beta + \gamma = \rho \), the result follows. \( \square \)

Proposition 38. In the cases where there are five \( G_0 \)-orbits in \( \mathfrak{g}_1 \), namely for \( \mathfrak{g} \) of type \( E_6 \), \( E_7 \) or \( E_8 \), the orbits are characterized as follows:

- \( x \) is in orbit 0 iff \( x = 0 \),
- \( x \) is in the closure of orbit 1 iff \( xxy \in Fx \) for all \( y \in \mathfrak{g}_1 \),
- \( x \) is in the closure of orbit 2 iff \( xxx = 0 \),
- \( x \) is in the closure of orbit 3 iff \( q(x) = 0 \), and
- \( x \) is in orbit 4 iff \( q(x) \neq 0 \).

Proof. The statement for orbit 1 is Proposition [18].

The conditions for orbits 2 and 3 are invariant under the action of \( G_0 \) and define closed sets, so it suffices to consider representatives of the orbits. Let \( \beta_1, \beta_2, \beta_3, \beta_4 \) be four mutually orthogonal roots of \( \alpha \)-height 1.

Choose \( x = x_{\beta_1} + x_{\beta_2} \) as a representative of orbit 2. The triple product \( xxx \) contains the terms \( x_{\beta_1} x_{\beta_2} x_{\beta_1}, x_{\beta_2} x_{\beta_3} x_{\beta_2}, x_{\beta_1} x_{\beta_1} x_{\beta_2} \) and \( x_{\beta_1} x_{\beta_2} x_{\beta_2} \). All are zero by Lemma [37] thus \( xxx = 0 \).

Conversely, for \( x = x_{\beta_1} + x_{\beta_2} + x_{\beta_3} \) in orbit 3, we have \( xxx = 6x_{\beta_1} x_{\beta_2} x_{\beta_3} \) since the other terms vanish by Lemma [37]. Thus we have \( \langle x_{\beta_4}, xxx \rangle = 6q(x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4}) \), which is not zero by Lemma [17]. Hence \( xxx \neq 0 \).

For \( x = x_{\beta_1} + x_{\beta_2} + x_{\beta_3} \) in orbit 3, all the terms arising when \( q(x, x, x, x) \) is expanded are zero: some \( x_{\beta_i} \) must be repeated, so we have terms of the form \( q(x_{\beta_i}, x_{\beta_1}, x_{\beta_j}, x_{\beta_k}) \) with \( i, j, k \) not necessarily distinct; such a term equals \( \langle x_{\beta_1}, x_{\beta_2}, x_{\beta_3}, x_{\beta_4} \rangle \), which is 0 by Lemma [37].

Finally, the fourth orbit is represented by \( x = x_\alpha + x_{\rho - \alpha} \) ([Röh93], Cor. 4.4). By the remark following Lemma [6] we have \( q(x) = 6 \); hence \( q(x) \neq 0 \) for any \( x \) in orbit 4. \( \square \)

A similar result applies for Lie algebras of type \( D_n \), except that the elements \( x \in \mathfrak{g}_1 \) satisfying \( xxx = 0 \) are those that belong to any of the level 2 orbits or their closures. As these orbits are each represented by elements of the form \( x_{\beta_1} + x_{\beta_2} \) but for different choices of \( \beta_1, \beta_2, \beta_3, \beta_4 \), the proof goes through unchanged.
Krutelevich ([Kru07], Definition 22) defines the rank of an element of Freudenthal triple system constructed from a cubic Jordan algebra using characterizations which are nearly the same as those given in the preceding proposition. His definition of rank 1 differs from the characterization of orbit 1; it is equivalent (apart from a different convention on scalars) to (22).

9. Related groups

As in Ferrar ([Fer72], §7), we define two subgroups of the group of linear automorphisms of \( g_1 \). The first, \( Q \), preserves the quartic form on \( g_1 \) up to a nonzero scalar factor, that is,

\[
Q = \{ \eta \in \text{GL}(g_1) : \forall x \in g_1, q(\eta(x)) = rq(x) \text{ for some } r \in F^\times \}.
\]

We call \( r \) the ratio of \( \eta \) in \( Q \).

Similarly, the elements of \( B \) are those that preserve the bilinear form up to a nonzero scalar:

\[
B = \{ \eta \in \text{GL}(g_1) : \forall x, y \in g_1, \langle \eta(x), \eta(y) \rangle = r\langle x, y \rangle \text{ for some } r \in F^\times \}.
\]

In this case, we call \( r \) the ratio of \( \eta \) in \( B \).

**Lemma 39.** The set of strictly regular elements is invariant under any \( \eta \in \text{GL}(g_1) \) that preserves the quartic form.

The following argument is adapted from Ferrar ([Fer72], Cor. 7.2).

**Proof.** Suppose \( x \in g_1 \) is rank one; then \( q(x, x, y, z) = \langle z, xxy \rangle \) is zero for all \( y \in g_1 \) and all \( z \) in a codimension-1 subspace. Conversely, if \( x \neq 0 \) and \( q(x, x, y, z) = \langle z, xxy \rangle \) is zero for all \( y \in g_1 \) and all \( z \) in a codimension-1 subspace, then \( xxg_1 \) lies in a 1-dimensional space. Since \( xxg_1 \) is not zero (Lemma 14), \( x \) is rank one. Thus this condition on the 4-linear form characterizes the rank one elements among the nonzero elements of \( g_1 \).

Since any \( \eta \) in \( \text{GL}(g_1) \) is nonsingular, it preserves the dimension of subspaces. If \( \eta \) preserves the quartic form (and hence the 4-linear form), then the condition on the 4-linear form is true of \( \eta(x) \) if it is for \( x \). Thus \( \eta \) maps rank one elements to rank one elements; by Proposition 20, it thus maps strictly regular elements to strictly regular elements.

**Proposition 40.** \( Q \) is a subgroup of \( B \).

**Proof.** Let \( \eta \) be an element of \( Q \). To show that \( \eta \) preserves \( \langle x, y \rangle \) up to a scalar factor, it suffices to show it for all \( x \) in a spanning set, such as the strictly regular elements (Lemma 19), and all \( y \in g_1 \).

By (23), for \( x \) strictly regular and any \( y \in g_1 \) we have \( q(x, x, y, y) = \langle x, y \rangle^2 \). By Lemma 39, \( \eta(x) \) is also strictly regular, so

\[
\langle \eta(x), \eta(y) \rangle^2 = q(\eta(x), \eta(x), \eta(y), \eta(y)) = r \cdot q(x, x, y, y) = r \langle x, y \rangle^2,
\]

where \( r \) is the ratio of \( \eta \) in \( Q \). Thus \( r \) is a square, say \( r = s^2 \); we then have \( \langle \eta(x), \eta(y) \rangle = \pm s \langle x, y \rangle \). The choice of sign does not depend on \( y \), since for
any $y_1, y_2 \in \mathfrak{g}_1$ we have $\pm s\langle x, y_1 + y_2 \rangle = \langle \eta(x), \eta(y_1 + y_2) \rangle = s\langle x, y_1 \rangle \pm s\langle x, y_2 \rangle$, so the signs must be the same whenever the bilinear forms are nonzero. Let us say that $x$ is associated with $s$ if $\langle \eta(x), \eta(y) \rangle = s\langle x, y \rangle$ for all $y \in \mathfrak{g}_1$, or that $x$ is associated with $-s$ otherwise.

The set of strictly regular elements associated to $s$ (resp., to $-s$) is a relatively closed subset of the set of all strictly regular elements, and the set of strictly regular elements is the disjoint union of these two sets. However, since the set of strictly regular elements is an orbit under the action of the connected set $G_0$ (Proposition 18), it is connected. Thus all strictly regular elements are associated to the same square root of $r$.

**Corollary 41.** Any element $\eta \in \text{GL}(\mathfrak{g}_1)$ that stabilizes the quartic form also preserves orthogonality.

**Proof.** If $\eta$ stabilizes the quartic form, it is in $Q$ (with ratio $1$); thus it is in $B$ (with ratio $\pm 1$). Therefore, for any $x, y \in \mathfrak{g}_1$, we have $\langle x, y \rangle = 0$ if and only if $\langle \eta(x), \eta(y) \rangle = 0$. □

### 10. The Stabilizer of the Quartic Form: $G = E_8$

Suppose that $G$ is of type $E_8$ and $\mathfrak{g}$ is thus the Lie algebra $E_8$, which has dimension 248 ([Bou2], §VI.4.10). In this case the simple root $\alpha$ is, in the labeling of [Bou2], $\alpha_8$. The root subspaces within $\mathfrak{g}_0$ are then generated by the $x_\beta$ where $\beta$ is a root of $\alpha$-height $0$; that is, a root of the Lie algebra $E_7$. There are 126 such roots ([Bou2], §VI.4.11); combined with the 8-dimensional torus of $E_8$, we have $\dim \mathfrak{g}_0 = 134$. Thus $G_0$ is the subgroup $E_7$ plus a one-dimensional torus, so $(G_0)^{\text{ss}}$ is $E_7$.

Since $\dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2 = 1$, we have $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1 = 56$. We see that the action of $(G_0)^{\text{ss}}$ on $\mathfrak{g}_1$ is irreducible since the dense orbit cannot be contained in any proper subspace, so $\mathfrak{g}_1$ is the well-known minuscule representation of $E_7$.

It has been known since Cartan, in the case where $F = \mathbb{C}$, that there is a quartic form on the minuscule representation, $V$, of $E_7$ that is invariant under $E_7$ ([Car52], p. 274). Freudenthal ([Fre53]) later found that the subgroup of $\text{GL}(V)$ stabilizing this quartic form and a skew-symmetric bilinear form is exactly $E_7$ in this case. In this section we use our techniques to establish the subgroup stabilizing the quartic form and the subgroup stabilizing both forms in our more general context.

**Theorem 42.** For $G = E_8$, the subgroup of $\text{GL}(\mathfrak{g}_1)$ stabilizing the quartic form, $\text{Stab}(q)$, is generated by $E_7$ and $\mu_4$, where $\mu_4$ is the group of the fourth roots of unity.

**Proof.** First, $E_7 = (G_0)^{\text{ss}}$ stabilizes the quartic form by Lemma 1. Also, for $k \in \mu_4$, we have $q(k \cdot x) = k^4 q(x) = q(x)$ for any $x \in \mathfrak{g}_1$, so $\mu_4$ also preserves the quartic form. Thus $\text{Stab}(q)$ contains the group generated by $E_7$ and $\mu_4$.

---

1It should be noted that the quartic form is given incorrectly by Cartan; the error seems to have been first observed by Freudenthal ([Fre53]).
To show the reverse inclusion, suppose \( g \in \text{Stab}(q) \). Let \( v = x_\alpha + x_\rho^{-\alpha} \). Since \( v \) is in the dense orbit, we have by Proposition 38 that \( q(v) \neq 0 \) and also, since \( q(g \cdot v) = q(v) \neq 0 \), that \( g \cdot v \) is in the dense orbit. Thus there exists some \( z \in E_7 \) such that \( zg \cdot v = kv \) for some \( k \in F^\times \). Let \( g' = zg \); then \( g' \) is also in \( \text{Stab}(q) \), so \( q(v) = q(g' \cdot v) = k^4q(v) \). Thus \( k \in \mu_4 \). Let \( g'' = k^{-1}g' \), then \( g'' \cdot v = v \), so \( g'' \) both stabilizes \( q \) and fixes \( v \).

Lemma 45 below, which is the key to the proof, shows that any element that stabilizes \( q \) and fixes \( v \) is in the group generated by \( E_7 \) and \( \mu_4 \); thus \( g'' \) is in that group and so is \( g \).

Before completing the proof, we use the preceding theorem to determine the group that stabilizes both \( q \) and the bilinear form \( \langle -, - \rangle \).

**Corollary 43.** For \( G = E_8 \), the subgroup of \( \text{GL}(g_1) \) stabilizing both the quartic form and the skew-symmetric bilinear form, \( \text{Stab}(q, \langle -, - \rangle) \), is \( E_7 \).

**Proof.** The previous proposition and the fact that \( E_7 \) stabilizes both forms yield the following containments:

\[ E_7 \subseteq \text{Stab}(q, \langle -, - \rangle) \subseteq \text{Stab}(q) = \langle E_7, \mu_4 \rangle. \]

Let \( L_0 \) be the root lattice of \( E_7 \) and \( L_1 \) its weight lattice. Then \( L_1/L_0 \) is a group with two elements (see, for example, [Hum78], §13.1 or [Ste68], p. 45). From [Ste68], p. 45, the center of \( E_7 \) is isomorphic to \( \text{Hom}(L_1/L_0, F^\times) \), so the center of \( E_7 \) consists of the elements 1 and \(-1\). Thus the group \( \langle E_7, \mu_4 \rangle \) has two components: \( E_7 \) and \( iE_7 \), where \( i \) is a primitive fourth root of unity. However, \( i \) is not in \( \text{Stab}(q, \langle -, - \rangle) \) since \( \langle ix, iy \rangle = -\langle x, y \rangle \) for any \( x, y \in g_1 \). Therefore \( \text{Stab}(q, \langle -, - \rangle) = E_7 \).

In the remainder of this section we complete the proof of Theorem 42 by showing that we can adjust an element that stabilizes \( q \) and fixes \( x_\alpha + x_\rho^{-\alpha} \) to produce one that preserves even more structure. We will use the same approach in the next section, so we define subspaces of \( g_1 \) and forms on them in a way that is valid when \( g \) is any Lie algebra of type \( D \) or \( E \).

Let \( A \) and \( B \) be the eigenspaces in \( g_1 \) described in Proposition 39 corresponding to the eigenvalues \( +1 \) and \( -1 \), respectively. Thus \( A \) is generated by elements \( x_\beta \) where \( \beta \) has \( \alpha \)-height \( 1 \) and \( \langle \alpha, \beta \rangle = 0 \), whereas \( B \) is generated by elements \( x_\gamma \) where \( \gamma \) has \( \alpha \)-height \( 1 \) and \( \langle \alpha, \gamma \rangle = 1 \).

We define the cubic forms \( f_1 \) on \( A \) and \( f_2 \) on \( B \) as follows:

\[ f_1(a) = \frac{1}{6}q(x_\alpha, a, a, a), \quad f_2(b) = \frac{1}{6}q(x_\rho^{-\alpha}, b, b, b). \]

**Lemma 44.** If \( g \in \text{GL}(g_1) \) is an element that stabilizes the quartic form and fixes \( v = x_\alpha + x_\rho^{-\alpha} \), then there is an element \( g' \) that preserves the spaces \( A \) and \( B \) and stabilizes \( \langle -, - \rangle \) and the cubic forms defined on \( A \) and \( B \) such that \( g'g^{-1} \in \langle (G_0)^s, \mu_4 \rangle \).

**Proof.** Let \( g \) be an element that stabilizes \( q \) and fixes \( v \). By Lemma 39, the action of \( g \) takes strictly regular elements to strictly regular elements, so
$g \cdot x_\alpha$ and $g \cdot x_{\rho - \alpha}$ are strictly regular. Since $g$ fixes $v$, we have $v = g \cdot v = g \cdot x_\alpha + g \cdot x_{\rho - \alpha}$. However, by Lemma 24 the expression of $v$ as a sum of two strictly regular elements is unique, so $g$ must either fix both $x_\alpha$ and $x_{\rho - \alpha}$ or interchange them. By §12.10 in [Gar09], there is an element $z \in i(G_0)^{ss}$ that interchanges $x_\alpha$ and $x_{\rho - \alpha}$; of course, such an element also stabilizes $q$. Hence either $g$ or $zg$ is an element that stabilizes $q$ and fixes $x_\alpha$ and $x_{\rho - \alpha}$; call whichever element does so $g'$. Thus we have $g'g^{-1} \in \langle (G_0)^{ss}, \mu_4 \rangle$.

Let $W$ be the subspace of $\mathfrak{g}_1$ consisting of elements orthogonal to both $x_\alpha$ and $x_{\rho - \alpha}$; by Corollary 41 $W$ is invariant under $g'$. All the basis elements $x_\beta$ with $\beta$ of $\alpha$-height 1 except for $x_\alpha$ and $x_{\rho - \alpha}$ are in $W$, and they form a basis of $W$. Thus $W$ is the direct sum of the $+1$ and $-1$ eigenspaces of Proposition 36, the spaces we have named $A$ and $B$.

Let $A'$ be the subspace of elements $x \in W$ such that $q(x_{\rho - \alpha}, x, y, z) = 0$ for all $y, z \in W$, and define a cubic form on $A'$ by $q'(x_\alpha, x, x, x)$. Clearly $g'$ preserves $A'$ and stabilizes the cubic form. We claim $A'$ is in fact $A$.

On the one hand, if $x_\beta$ is a basis element of the $+1$ eigenspace, then we have $\langle \rho - 2\alpha, \beta \rangle = 1$. Since $\langle \rho, \beta \rangle = 1$, it follows that $\langle \alpha, \beta \rangle = 0$. By writing elements $y, z \in W$ as linear combinations of the basis elements, $q(x_{\rho - \alpha}, x_\beta, y, z)$ expands into a linear combination of terms of the form $q(x_{\rho - \alpha}, x_\beta, x_\gamma, x_\delta)$ with $\gamma, \delta$ such that $\langle \gamma, \alpha \rangle$ and $\langle \delta, \alpha \rangle$ are each either 0 or 1. But then we cannot have $\langle \rho - \alpha \rangle + \beta + \gamma + \delta = 2\rho$, since $\langle (\rho - \alpha) + \beta + \gamma + \delta, \alpha \rangle = -1 + 0 + \langle \gamma, \alpha \rangle + \langle \delta, \alpha \rangle$ is at most 1, but $\langle 2\rho, \alpha \rangle = 2$. Hence all the terms $q(x_{\rho - \alpha}, x_\beta, x_\gamma, x_\delta)$ are zero, so $x_\beta$ is in $A$. Thus $A \subseteq A'$.

Conversely, if $x \in W$ is not in $A$, then it has a nonzero component involving some basis element $x_\beta$ with $\langle \beta, \alpha \rangle = 1$. Thus $\langle \rho - \alpha, \beta \rangle = 0$, so $\rho - \alpha$ and $\beta$ are orthogonal roots of $\alpha$-height 1. It follows from Lemma 2.4 in [Röh93] that any such pair of roots can be extended to a set of four mutually orthogonal roots, say $\rho - \alpha, \beta, \gamma, \delta$. By Lemma 17 $q(x_{\rho - \alpha}, x_\beta, x_\gamma, x_\delta)$ is then nonzero, and thus $q(x_{\rho - \alpha}, x_\beta, x_\gamma, x_\delta)$ is also nonzero, since no other component of $x$ contributes to the value of the form. Thus $x$ is not in $A'$. Therefore $A' \subseteq A$.

Interchanging the roles of $x_\alpha$ and $x_{\rho - \alpha}$, we similarly define $B'$ to be the subspace of elements $x \in W$ such that $q(x_\alpha, x, y, z) = 0$ for all $y, z \in W$, and define a cubic form on $B'$ by $q'(x_\alpha, x, x, x)$. As before, $g'$ preserves $B'$ and stabilizes the cubic form, and the same argument, mutatis mutandis, shows that $B' = B$.

As in the proof of Corollary 41 since $g'$ stabilizes the quartic form, it preserves the bilinear form up to a scalar factor of $\pm 1$. However, since $g'$ fixes $x_\alpha$ and $x_{\rho - \alpha}$ and $\langle x_\alpha, x_{\rho - \alpha} \rangle \neq 0$, the scalar factor is 1; thus $g'$ preserves $\langle -, - \rangle$. 

We now apply the preceding general lemma to the specific case $G = E_8$, thereby completing the proof of Theorem 42.
Lemma 45. When $G = E_8$, the group that stabilizes the quartic form and fixes the element $v = x_\alpha + x_{\rho-\alpha}$ is contained in the group generated by $E_7$ and $\mu_4$.

Proof. We begin by making some observations about the action of the subgroup $E_6$ of $E_7$ on $\mathfrak{g}$. Since $E_7$ fixes $x_\rho$ and $x_{-\rho}$, as shown in the proof of Lemma 1, $E_6$ certainly does as well. Similarly, for any basis element of the Lie algebra $E_6$, i.e., any $x_\beta$ where $\beta$ is a root orthogonal to both $\rho$ and $\alpha$ or any $h_\gamma$ where $\gamma$ is a simple root other than $\alpha = \alpha_8$ or $\alpha_7$, we have $[x_\beta, x_\alpha] = 0$ and $[h_\gamma, x_\alpha] = 0$ and likewise $[x_\beta, x_{\rho-\alpha}] = 0$ and $[h_\gamma, x_{\rho-\alpha}] = 0$. Thus elements of the group $E_6$ fix $x_\alpha$ and $x_{\rho-\alpha}$.

In the proof of Lemma 14 it was shown that $A$ could be characterized in terms of $x_\alpha$, $x_{\rho-\alpha}$, orthogonality and the quartic form; since all these are preserved by elements of $E_6$, $A$ is invariant under $E_6$.

Since $\mathfrak{g}_1$ is 56-dimensional, it follows from Proposition 38 that $A$ is 27-dimensional. It is known ([MP81], p.301) that the minuscule representation of $E_7$ decomposes into the sum of four representations of $E_6$, two 1-dimensional and two 27-dimensional. Thus $A$ is a 27-dimensional minuscule representation of $E_6$.

The cubic form $f_1$ on $A$ is defined in terms of $q$ and $x_\alpha$, so it is stabilized by $E_6$. However, by [SK77], pp. 25–27, we know that the $E_6$-invariant polynomials on $A$ are generated by a cubic, at least in characteristic zero. By [Ses77], this holds in general characteristic. Thus $f_1$ is the unique (up to scalar factor) $E_6$-invariant cubic form on $A$, provided that it is not zero.

To show that $f_1$ is nonzero, take $\alpha$, $\beta$, $\gamma$, $\delta$ to be four mutually orthogonal roots of $\alpha$-height 1. As in the proof of Proposition 38 for $x = x_\beta + x_\gamma + x_\delta$ we have $xxx = 6x_\beta x_\gamma x_\delta$, so $f_1(x) = \frac{1}{2}q(x_\alpha, x, x, x) = q(x_\alpha, x_\beta, x_\gamma, x_\delta)$, which is not zero by Lemma 17.

By Lemma 14, if $g$ is an element that stabilizes $q$ and fixes $v$, there is a $g' \in g\langle E_7, \mu_4 \rangle$ such that $A$ is invariant under $g'$ and the cubic form $f_1$ is stabilized by $g'$. That is, $g'$ is in the stabilizer of the $E_6$-invariant cubic form on the 27-dimensional minuscule representation of $E_6$; by [SV00], Theorem 7.3.2, that stabilizer is $E_6$ itself.

Thus $g' \in E_6$, and therefore $g$ is in $\langle E_7, \mu_4 \rangle$. \hfill \Box

11. The Stabilizer of the Quartic Form: $G = D_4$

In this section, we again consider the group stabilizing the quartic form and the group stabilizing both the quartic and the bilinear forms on $\mathfrak{g}_1$, this time in the case $G = D_4$.

The diagram that results when $\alpha = \alpha_2$ is removed from the Dynkin diagram of $D_4$ consists of three unconnected vertices; that is, it represents the Lie algebra which is the product of three copies of $\mathfrak{sl}_2$. Thus $\mathfrak{g}_0$ is 10-dimensional, generated by the three pairs of roots $x_{\alpha_i}, x_{-\alpha_i}$ for $i = 1, 3, 4$ and the four-dimensional Cartan subalgebra of $D_4$; $(G_0)^{\text{ss}}$ is thus $\text{SL}_3^3$. Since $D_4$ has dimension 28, there are 18 other roots; setting aside $\rho$ and $-\rho$, we
see that \( g_1 \) and \( g_{-1} \) are eight-dimensional. Here is a list of the roots \( \beta \) of \( \alpha \)-height 1, sorted according to the eigenspace decomposition of Proposition 36:

\[
\begin{array}{|c|c|c|}
\hline
\beta & \langle \rho - 2\alpha, \beta \rangle & \langle \alpha, \beta \rangle \\
\hline
\rho - \alpha & 3 & -1 \\
\alpha + \alpha_1 + \alpha_3, \alpha + \alpha_1 + \alpha_4, \alpha + \alpha_3 + \alpha_4 & 1 & 0 \\
\alpha + \alpha_1, \alpha + \alpha_3, \alpha + \alpha_4 & -1 & 1 \\
\alpha & -3 & 2 \\
\hline
\end{array}
\]

As mentioned in the introduction, the quartic form \( q \) on the 8-dimensional space \( g_1 \) is the same as that examined by Bhargava in [Bha04].

To establish the stabilizer of the quartic form, we follow a similar strategy to that employed in the proof of Theorem 42: We define the spaces \( A \) and \( B \) and cubic forms on them as in the previous section. We adjust an element \( g \in \text{GL}(g_1) \) that stabilizes the quartic form to obtain an element that also fixes \( x_{\rho} \), then apply Lemma 44 to obtain a \( g' \) that preserves the spaces \( A \) and \( B \) and stabilizes the cubic forms on them. In this case \( A \) and \( B \) are simple enough so that we can give the cubic forms explicitly and determine a suitable subgroup of \( \text{GL}(g_1) \) that contains \( g' \).

**Theorem 46.** The stabilizer of the quartic form on \( g_1 \) when \( G = D_4 \) is \( \langle \text{SL}_3, \mu_4 \rangle \rtimes S_3 \), where \( S_3 \) is the symmetric group corresponding to the diagram automorphisms of \( D_4 \).

It can be shown that \( S_3 \) acts trivially on \( \mu_4 \) here (see Section 12 of [Gar09]), so we could also write the group as \( \langle \text{SL}_3 \times S_3, \mu_4 \rangle \).

**Proof.** Since \( (G_0)^{ss} = \text{SL}_2 \) and \( \mu_4 \) both stabilize the quartic form, \( \langle \text{SL}_2, \mu_4 \rangle \) is in \( \text{Stab}(q) \). We will now show that the diagram automorphisms also stabilize the quartic form.

It will suffice to show that a diagram automorphism fixes \( x_{\rho} \) and \( x_{-\rho} \).

By Corollaire 5 bis in [SGA3], Exposé 23, an outer automorphism of \( g \) may be taken to act on the Chevalley basis elements \( x_{\alpha_i} \) corresponding to the simple roots by permuting the subscripts, and to act on the elements \( h_i = [x_{\alpha_i}, x_{-\alpha_i}] \) by applying the same permutation to the subscripts; thus the elements \( x_{-\alpha_i} \) are also permuted in the same way. We will write \( x_{\rho} \) in terms of the \( x_{\alpha_i} \), and show that this expression is unaltered by a permutation of the subscripts 1, 3 and 4; the same argument with the negatives of the roots will show that \( x_{-\rho} \) is fixed as well.

The highest root of \( D_4 \) is \( \rho = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \). We write this as \( \rho = \alpha_2 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_2 \); in this expression each partial sum is also a root. Thus we have

\[
x_{\rho} = c[x_{\alpha_2}, [x_{\alpha_4}, [x_{\alpha_3}, [x_{\alpha_1}, x_{\alpha_2}]]]],
\]

where \( c \) is a constant (in fact, \( c = \pm 1 \) since all the roots are long and thus the structure constants are \( \pm 1 \)). Our claim is that this expression is unaltered when the factors \( x_{\alpha_1}, x_{\alpha_3}, x_{\alpha_4} \) are permuted.
To verify the claim for the permutation that interchanges 1 and 3, we must show that \([x_{a_3}, [x_{a_1}, x_{a_2}]] = [x_{a_1}, [x_{a_3}, x_{a_2}]]\); this is equivalent to the following structure constant equation:

\[(48) \quad c_{a_1, a_2} c_{a_3, a_1 + a_2} = c_{a_3, a_2} c_{a_1, a_2 + a_3}.\]

To obtain (48), we apply (5) with \(\beta = \alpha_1 + \alpha_2, \gamma = \alpha_2 + \alpha_3, \delta = -\alpha_2\) and \(\epsilon = -\alpha_1 - \alpha_2 - \alpha_3\); this yields

\[c_{a_1 + a_2, a_2 + a_3} c_{-a_2, -a_1 - a_2 - a_3} + c_{a_2 + a_3, a_2} c_{a_1 + a_2, -a_1 - a_2 - a_3} + c_{-a_2, a_1 + a_2} c_{a_2 + a_3, -a_1 - a_2 - a_3} = 0.\]

The sum \(\alpha_1 + 2\alpha_2 + \alpha_3\) has \(\alpha\)-height 2 but is not equal to \(\rho\), so it is not a root; thus the first term is zero. Applying the rules for structure constants from Section 3, we have \(\langle -\alpha, \alpha \rangle = 1\); this in turn is equivalent to (48).

Let \(g \in G_4\) such that \(g \cdot v = kv\) for all \(v \in A\) and \(B\) and stabilizes \(\langle -, - \rangle\) and the cubic forms on \(A\) and \(B\).

By definition, the subspace \(A\) is generated by the root subspaces corresponding to roots orthogonal to \(\alpha\); examining the list of roots in \(g_1\), these are \(\beta = \alpha + \alpha_1 + \alpha_3, \gamma = \alpha + \alpha_1 + \alpha_4\) and \(\delta = \alpha + \alpha_3 + \alpha_4\). We easily check that \(\alpha, \beta, \gamma\) and \(\delta\) are mutually orthogonal. For an arbitrary element \(x = \lambda_1 x_\beta + \lambda_2 x_\gamma + \lambda_3 x_\delta\) of \(A\), we find that the cubic form is

\[
\frac{1}{6} g(x_\alpha, x, x) = \lambda_1 \lambda_2 \lambda_3 g(x_\alpha, x_\beta, x_\gamma, x_\delta),
\]

since the terms with a repeated argument are zero by Lemma 37. By Proposition 35, this is \(\epsilon \lambda_1 \lambda_2 \lambda_3\), where \(\epsilon = \pm 1\) is a product of structure constants.

Let \(T = (a_{ij})\), \(1 \leq i, j \leq 3\), be the matrix of the linear transformation on \(A\) given by \(x \mapsto g' \cdot x\) with respect to the basis \(x_\beta, x_\gamma, x_\delta\). The value of the cubic form is the same for \(x = \lambda_1 x_\beta + \lambda_2 x_\gamma + \lambda_3 x_\delta\) and \(g' \cdot x\), so we have

\[
\lambda_1 \lambda_2 \lambda_3 = (a_{11} \lambda_1 + a_{12} \lambda_2 + a_{13} \lambda_3)(a_{21} \lambda_1 + a_{22} \lambda_2 + a_{23} \lambda_3)(a_{31} \lambda_1 + a_{32} \lambda_2 + a_{33} \lambda_3)
\]

for all \(\lambda_1, \lambda_2, \lambda_3 \in F\). By unique factorization in \(F[\lambda_1, \lambda_2, \lambda_3]\), the three factors on the right-hand side are (up to units) \(\lambda_1, \lambda_2, \lambda_3\), say \(c_1 \lambda_1, c_2 \lambda_2, c_3 \lambda_3\), with \(c_1 c_2 c_3 = 1\). If the factors occur in that order, then \(T\) is diagonal, with the third entry determined by the first two; each such \(T\) corresponds to an
element \((c_1, c_2, c_3)\) of \(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m\) for which the product of the three components is 1. However, the order of the factors may be different, so in general \(T\) may be an element of \((\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m) \rtimes S_3\).

The subspace \(B\) is generated by the root subspaces corresponding to the roots \(\rho - \beta = \alpha + \alpha_4, \rho - \gamma = \alpha + \alpha_3\) and \(\rho - \delta = \alpha + \alpha_1\). As \(\alpha, \beta, \gamma, \delta\) are mutually orthogonal, so are \(\rho - \alpha, \rho - \beta, \rho - \gamma, \rho - \delta\). The cubic form on \(B\) is given by \(\frac{1}{6}q(x_{\rho-\alpha}, x, x, x)\); for \(x = \lambda_1 x_{\rho-\beta} + \lambda_2 x_{\rho-\gamma} + \lambda_3 x_{\rho-\delta}\) this is, as in the previous case, \(\pm \lambda_1 \lambda_2 \lambda_3\). As before, \(g'\) must map \(x_{\rho-\beta}, x_{\rho-\gamma}\) and \(x_{\rho-\delta}\) to scalar multiples of the same basis elements, possibly permuted.

However, since \(g'\) stabilizes \((-,-)\), the action of \(g'\) on \(B\) can be computed given its action on \(A\). Suppose, for example, that \(g'\) maps \(x_\beta\) to \(cx_\gamma\) in \(A\), then \(\langle x_\beta, x_{\rho-\beta} \rangle = \langle cx_\gamma, x_{\rho-\beta} \rangle\); since this must be \(c_{\beta,\rho-\beta}\), we have that \(g' \cdot x_{\rho-\beta}\) is necessarily \(c_{\beta,\rho-\beta} c_{\gamma,\rho-\gamma} c^{-1} x_{\rho-\gamma}\). In general, \(\beta\) and \(\gamma\) may be replaced by any of \(\beta, \gamma\) or \(\delta\), with a similar result. Hence the action of \(g'\) on \(B\) is determined by its action on \(A\); in particular, if acts diagonally on \(A\), it also does so on \(B\).

It remains only to show that an element \(g'\) that corresponds to element of \(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m\) is an element of \(\text{SL}_3^2\). We will consider the action of an element of \(\text{SL}_3^2\) that corresponds to an element of \(\mathfrak{h}\) of the form \(t_1 h_{\alpha_1} + t_3 h_{\alpha_3} + t_4 h_{\alpha_4}\). By Lemma 19(c) in [Ste68], the action of the element corresponding to \(t_1 h_{\alpha_1}\) takes \(x_\beta\) to \(t_1^{(\beta,\alpha_1)} x_\beta\), which is \(t_1 x_\beta\) since \(\langle \beta, \alpha_1 \rangle = 1\). Similarly, it takes \(x_\gamma\) to \(t_1 x_\gamma\), \(x_\delta\) to \(t_1^{-1} x_\delta\) since \(\langle \delta, \alpha_1 \rangle = -1\); thus its action on \(A\) is that of the element \((t_1, t_1, t_1^{-1})\) in \(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m\). In the same fashion, we find that \(t_3 h_{\alpha_3}\) corresponds to \((t_3, t_3^{-1}, t_3)\) and \(t_4 h_{\alpha_4}\) to \((t_4^{-1}, t_4, t_4)\). Since these classes of elements are multiplicatively independent, they generate \(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m\); the elements with the product of the components equal to 1 come from elements of the form \(t_1 h_{\alpha_1} + t_3 h_{\alpha_3} + t_4 h_{\alpha_4}\) with \(t_1 t_3 t_4 = 1\). Since \(\langle \alpha, \alpha_i \rangle = -1\) for \(i = 1, 3, 4\), this element takes \(x_\alpha\) to \(t_1^{-1} t_3^{-1} t_4^{-1} x_\alpha\), so it fixes \(x_\alpha\) just as \(g'\) does. The action on the remaining basis elements, namely \(x_{\rho-\alpha}\) and those of \(B\), must also correspond to that of \(g'\) because an element of \(\text{SL}_3^2\) stabilizes the bilinear form.

Thus \(g'\) is in \(\text{SL}_3^2 \rtimes S_3\), from which it follows that the original \(g \in \text{GL}(\mathfrak{g}_1)\) stabilizing the quartic form is in \(\langle \text{SL}_3^2, \mu_4 \rangle \rtimes S_3\).

The determination of the group that stabilizes both \(q\) and the bilinear form \((-,-)\) is parallel to Corollary 43.

**Corollary 49.** In the case \(G = D_4\), the subgroup of \(\text{GL}(\mathfrak{g}_1)\) stabilizing both the quartic form and the skew-symmetric bilinear form, \(\text{Stab}(q, (-,-))\), is \(\text{SL}_2^3 \rtimes S_3\).

**Proof.** The previous theorem and the fact that \(\text{SL}_3^2\) and the diagram automorphism stabilize both forms yield the following containments:

\[
\text{SL}_2^3 \rtimes S_3 \subseteq \text{Stab}(q, (-,-)) \subseteq \text{Stab}(q) = \langle \text{SL}_2^3, \mu_4 \rangle \rtimes S_3.
\]
Since $-1 \in \text{SL}_2$, we also have $-1 \in \text{SL}_3^3$. Thus $\text{SL}_3^3 \rtimes S_3$ is an index 2 subgroup of $(\text{SL}_3^3, \mu_4) \rtimes S_3$. However, the coset containing $i$, a primitive fourth root of unity, is not in $\text{Stab}(q, \langle -,- \rangle)$ since $\langle ix, iy \rangle = -\langle x, y \rangle$ for any $x, y \in \mathfrak{g}_1$. Therefore $\text{Stab}(q, \langle -,- \rangle) = \text{SL}_3^3 \rtimes S_3$.

□

12. NON-SPLIT GROUPS

In the preceding sections, we assumed that $G$ was split over $F$. This was only for convenience; in this section we will show that most of our results hold for quite general Freudenthal triple systems.

Suppose $G$ is an absolutely almost simple linear algebraic group, not of type $A$ or $C$, over a field $F$ of characteristic $\neq 2, 3$. Fix a maximal $F$-torus $T$, which we may assume contains a maximal $F$-split torus, and also fix a set $\Delta$ of simple roots for $G$ with respect to $T$ over a separable closure $F_{\text{sep}}$ of $F$.

There is a uniquely determined root $\alpha \in \Delta$ as in Section 2. We require that, in the Tits index of $G$ as defined in [Tit66], the vertex $\alpha$ is circled. This is equivalent to having an $F$-homomorphism $\rho^\vee: \mathbb{G}_m \rightarrow T$ corresponding to the coroot $\rho$ (i.e., such that $\text{Lie}(\text{im} \rho^\vee) \otimes F_{\text{sep}}$ is $F_{\text{sep}}^h \rho$); see Corollaire 6.9 in [BT65]. We grade the Lie algebra $\mathfrak{g}$ of $G$ by setting

$$\mathfrak{g}_i := \{ x \in \mathfrak{g} \mid \rho^\vee(t)x = t^i x \text{ for all } t \in F_{\text{sep}}^x \}.$$

When $G$ is split (e.g., if we extend scalars to $F_{\text{sep}}$), we obtain the same grading as in Section 2. We choose a nonzero vector $x_\rho \in \mathfrak{g}_2$, which gives a skew-symmetric bilinear form $\langle -,- \rangle$ and a quartic form $q$ on $\mathfrak{g}_1$ by the same formulas as in Section 2.

Now Lemmas/Propositions/Theorems/Corollaries [1, 2, 14, 15, 20, 21, 27, 39, 40 and 41] all hold over $F$ without any change in their statements. Indeed, it suffices to verify each over $F_{\text{sep}}$, where $G$ is split.

Theorems/Corollaries [42, 43, 46 and 49] can be viewed as determining the $F_{\text{sep}}$-isomorphism class of their respective stabilizer groups (which are defined over $F$).

For readers interested in Freudenthal triple systems, we now suppose that we are given such a triple system $(V, q, \langle -,- \rangle)$—denoted briefly by $V$—defined over $F$ such that $V \otimes F_{\text{sep}}$ can be identified with one of the triple systems constructed in Sections 2–5. We claim $V$ can be constructed from some group $G$ defined over $F$ by using the construction given earlier in this section and thus the results listed also hold for $V$.

We illustrate the claim in the case where $H = \text{Stab}(q, \langle -,- \rangle)$ is of type $E_7$; equivalently, $V \otimes F_{\text{sep}}$ is obtained from a group of type $E_8$. Since the 56-dimensional representation of $H$ is defined over $F$, $H$ is obtained by twisting the split simply-connected group $E_7^c$ of type $E_7$ by a 1-cocycle $\eta$ (in Galois cohomology) with values in $E_7^c(F_{\text{sep}})$. If the split group of type $E_8$ (which naturally contains $E_7^c$) is also twisted by $\eta$, we find a copy of $H$ inside a group $G$ of type $E_8$. The construction above now yields a Freudenthal triple system $V'$ with automorphism group $H$, which must be similar to $V$.
by [Gar01], Theorem 4.16(2). By scaling $x_\rho$, we can arrange for $V'$ to be isomorphic to $V$, as desired.

In addition to the 56-dimensional representation of a group of type $E_7$, we see in the same way that the results of this paper apply to the half-spin representation of a group of type $D_5$, the natural 20-dimensional representation of a group of type $A_5$ and the natural 8-dimensional representation of a group of type $A_1 \times A_1 \times A_1$, whenever such representations are defined over $F$.

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