ASYMPTOTIC DISTRIBUTIONS AND BERRY–ESSEEN INEQUALITIES FOR LOTKA–NAGAEV ESTIMATOR OF A POISSON RANDOMLY Indexed BRANCHING PROCESS

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Abstract. Consider a Galton–Watson process \{Z_n\}, the Lotka–Nagaev estimator for offspring mean \(m\) is \(R_n = Z_{n+1}/Z_n\). Let \(N_t\) be a Poisson process independent of \{Z_n\}, the continuous time process \{Z_{N_t}\} is a Poisson randomly indexed branching process. We show the asymptotic distributions for \{R_t := R_{N_t}\}.

1. Introduction

Consider a Galton–Watson process (GW) \{Z_n\} with offspring distribution \{p_i\}. A basic task in statistical inference of GW is the estimation of the offspring mean \(m := \sum_i ip_i\). We assume that \(Z_0 = 1, p_0 = 0, 0 \leq p_i < 1, \forall i\) and GW is supercritical, that is \(m > 1\). One of the most important estimator is Lotka-Nagaev estimator defined as \(R_n = Z_{n+1}/Z_n\), see [12]. Large deviations for \(R_n\) attracted the attention of several researchers in recent years, see [1, 2, 6, 10, 13], etc. [11], [14] and [9] extended these results to the Lotka-Nagaev estimator of a branching process with immigration or random environment.

The model of Poisson randomly indexed branching process (PB) \{Y_t := Z_{N_t}\} was introduced by Epps [5] to study the evolution of stock prices, where \{N_t\} is a Poisson process which is independent of \{Z_n\}. The statistical investigation on various estimates and some parameters of the process were done in [3]. Particularly, \(T_t := \log(Y_t)/\lambda t\) is used to estimate \(\log m\), where \(\lambda\) is the density of underlying Poisson process. The asymptotic normality and Berry–Esseen type inequalities were given in [7].

In this paper, we concentrate on the Lotka–Nagaev estimator \{R_t := R_{N_t}\} for offspring mean \(m\). Wu [15] obtained the large deviations for \(R_t\). These results have been extended to the case that the random index is a renewal process, see [8] for details. In this manuscript, we focus on the asymptotic distribution of \(R_t\).

Note that \(p_0 = 0\), there exists a positive random variable \(W\) such that \(Y_t/C_t \rightarrow W\) a.s., where \(C_t = \exp(\lambda t(m - 1))\), see [17]. Firstly, we consider the asymptotic distribution of normalized process \(\sqrt{C_t}(R_t - m)\) as \(t \rightarrow \infty\).

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Theorem 1. Assume that \( \mathbb{E}(Z_1^2) < \infty \), one has
\[
\lim_{t \to \infty} \mathbb{P}(\sigma^{-1} \sqrt{C_t} (\mathbb{R}_t - m) \leq x) = \int_0^\infty \Phi(xy^{1/2})dG(y),
\]
where \( \sigma^2 = \text{Var}(Z_1) \), \( G(y) \) and \( \Phi(x) \) are the cumulative distribution functions of \( W \) and the standard normal random variable respectively.

Theorem 1 shows that the typical asymptotic distribution of \( \sqrt{C_t} (\mathbb{R}_t - m) \) as \( t \to \infty \) is not a normal distribution. One naturally wonder whether the asymptotic distribution of \( (\mathbb{R}_t - m)/\text{Var}(\mathbb{R}_t) \) is normal distribution, so we consider the rate of \( \text{Var}(\mathbb{R}_t) \) as \( t \to \infty \).

For a PB, we distinguish between the Schröder case and the Böttcher case depending on whether \( p_0 + p_1 > 0 \) or \( p_0 + p_1 = 0 \). Note that \( p_0 = 0 \) in this paper, the Schröder index is defined as \( \alpha = -\log_m p_1 \in (0, +\infty] \). If \( \alpha \in (0, +\infty) \), PB belongs to the Schröder case, else if \( \alpha = -\infty \), PB belongs to the Böttcher case.

Let \( f_n(s) \) be the generating function of \( Z_n \), if \( 1 > p_1 > 0 \) (Schröder case), there exists a unique \( Q(s) \) such that \( f_n(s)/p_1^m \to Q(s) \) and (see [1])
\[
Q(f(s)) = p_1 Q(s), \quad Q(0) = 0, \quad Q(s) > 0
\]
for all \( s \in (0, 1) \), where \( f(s) = f_1(s) \).

Theorem 2. Let \( \phi(v) \) be the Laplace transformation of \( V := \lim_{n \to \infty} Z_n/m^n \). Assume that \( 1 > p_1 > 0 \) and \( \mathbb{E}(Z_1^2) < \infty \). One has \( \text{Var}(\mathbb{R}_t) \sim C(\alpha, t) \), where \( f(t) \sim g(t) \) stands for \( f(t)/g(t) \to 1 \) as \( t \to \infty \), \( \alpha \) is the Schröder index and
\[
C(\alpha, t) = \begin{cases} 
\sigma^2 \exp(\lambda t(p_1 - 1)) \int_0^\infty Q(\exp(-v))dv, & \alpha < 1; \\
\sigma^2 p_1 \lambda t \exp(\lambda t(p_1 - 1)) \int_0^m Q(\phi(v))dv, & \alpha = 1; \\
\sigma^2 \exp(\lambda t(m^{-1} - 1)) \int_0^\infty \phi(v)dv, & \alpha > 1.
\end{cases}
\]

In Example 1, we choose \( \lambda = 1, \alpha = 0.5, 1, 2 \). The decay rates of \( C(\alpha, t) \) are illustrated in the Figure 1 and Figure 2. From these figures, we know that the smaller the \( \alpha \), the faster the decay rate of \( \text{Var}(\mathbb{R}_t) \).

Note that \( m > 1 \), then \( m + m^{-1} > 2 \). When \( \alpha \leq 1, p_1 \geq m^{-1} \), thus \( C_t\text{Var}(\mathbb{R}_t) \to \infty \) as \( t \to \infty \). According to Slutsky’s theorem,
\[
\frac{\mathbb{R}_t - m}{\sqrt{\text{Var}(\mathbb{R}_t)}} = \frac{1}{\sqrt{C_t\text{Var}(\mathbb{R}_t)}} \xrightarrow{d} 0,
\]
where \( \xrightarrow{d} \) stands for convergence in distribution. That is \( (\mathbb{R}_t - m)/\sqrt{\text{Var}(\mathbb{R}_t)} \) has no proper asymptotic distribution. In order to balance the fluctuation, we consider the randomly normalized process \( \sqrt{Y_t}(\mathbb{R}_t - m) \), where \( Y_t = Z_{N_t} \).

Theorem 3. Assume that \( \mathbb{E}(Z_1^2) < \infty \). We have
\[
\lim_{t \to \infty} \mathbb{P}(\sigma^{-1} \sqrt{Y_t}(\mathbb{R}_t - m) \leq x) = \Phi(x).
\]
Assume that \( \lambda = 1 \) and the offspring distribution satisfies the following four cases respectively.

(a) \( Z_1 - 1 \sim \text{Geom}(0.5) \): \( p_k = 0.5^k \),
(b) \( Z_1 - 1 \sim \text{Pois}(1) \): \( p_k = 1/(e(k-1)!) \),
(c) \( Z_1 - 1 \sim \text{Binom}(2,0.5) \): \( p_1 = p_3 = 1/4 \), \( p_2 = 1/2 \),
(d) \( Z_1 \sim \text{Unif}(1,2,3) \): \( p_1 = p_2 = p_3 = 1/3 \).

We conduct 10000 simulations for each case. Note that \( \mathbb{E}(Z_1) = 2 \) and \( \mathbb{E}(Z_n) = 2^n \), we choose \( t = 8(2^8 = 128) \) for relatively small sample and \( t = 10(2^{10} = 1024) \) for relatively large sample. Compares for densities of \( t = 8 \), \( t = 10 \) and that of the standard normal distribution are given in Figure 3–6. From these figures, we know that for \( t \) large enough, Theorem 3 is efficient.

The rates of convergence in Theorem 3 can be characterized by Berry–Esseen type inequalities. Using the classical Berry–Esseen bound for sums of i.i.d. random sequence and the harmonic moments of \( \{Z_n\} \) one can obtain Theorem 4. Define \( G_t(x) = \mathbb{P}(\sigma^{-1} \sqrt{T_t} (R_t - m) \leq x) \), one has

**THEOREM 4.** Assume that \( 1 > p_1 > 0 \), \( \mathbb{E}(Z_1^2) < \infty \). Then there exists constant \( C \) such that

\[
\sup_{x \in \mathbb{R}} |G_t(x) - \Phi(x)| \leq CH(\alpha,t),
\]

where \( \alpha \) is the Schröder index, \( \mathbb{R} = (-\infty, +\infty) \) and

\[
H(\alpha,t) = \begin{cases} 
\exp(\lambda t (p_1 - 1)), & \alpha < 0.5; \\
p_1 \lambda t \exp(\lambda t (p_1 - 1)), & \alpha = 0.5; \\
\exp(\lambda t (m^{-1/2} - 1)), & \alpha > 0.5.
\end{cases}
\]
Figure 3: $Z_1 - 1 \sim \text{Geom}(0.5)$

Figure 4: $Z_1 - 1 \sim \text{Pois}(1)$

Figure 5: $Z_1 - 1 \sim \text{Binom}(2, 0.5)$

Figure 6: $Z_1 \sim \text{Unif}(1, 2, 3)$
The rest of the paper is organized as follows. In Section 2, we obtain the asymptotic distribution for the normalized process $\sqrt{C_t}(R_t - m)$. Section 3 is devoted to the decay rates of $\text{Var}(R_t)$. Asymptotic normality of the randomly normalized process $\sqrt{Y_t}(R_t - m)$ is given in section 4.

In the rest of the paper, we denote by $C$ an absolute and positive constant which may differ from line to line.

2. Asymptotic distribution of $\sqrt{C_t}(R_t - m)$

Independent of $Y_t$, let $\{X_n\}$ be a sequence of i.i.d random variables with the same distribution as $Z_1$. Define $S_k = X_1 + \cdots + X_k$ for any $k \geq 1$ and

$$L_k(x) = \mathbb{P}\left( \frac{S_k - m}{\sqrt{k}\sigma} \leq x \right), \quad x \in \mathbb{R}, \quad \Delta_k = \sup_{x \in \mathbb{R}} |L_k(x) - \Phi(x)|,$$

where $\sigma^2 = \text{Var}(Z_1) \in (0, \infty)$. Then

$$\Delta_k \to 0 \quad (1)$$
as $k \to \infty$, see [4]P105 for example. The proof of Theorem 1 depends on the convergence $W_t := Y_t/C_t \to W$ a.s. and the independence between $\{N_t\}$ and the underlying $GW \{Z_n\}$.

The proof of Theorem 1.

Conditioning on $Y_t$,

$$\mathbb{P}\left( \frac{\sqrt{C_n}(R_t - m)}{\sigma} \leq x \right) = \sum_{k=1}^{\infty} \mathbb{P}\left( \frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leq x \right) \mathbb{P}(Y_t = k). \quad (2)$$

For any $\varepsilon \in (0, 1)$, we divide the right side of (2) into the following three parts.

$$J_1(\varepsilon, t) = \sum_{k<\varepsilon C_t} \mathbb{P}\left( \frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leq x \right) \mathbb{P}(Y_t = k),$$

$$J_2(\varepsilon, t) = \sum_{\varepsilon C_t \leq k \leq \varepsilon^{-1} C_t} \mathbb{P}\left( \frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leq x \right) \mathbb{P}(Y_t = k),$$

$$J_3(\varepsilon, t) = \sum_{k>\varepsilon^{-1} C_t} \mathbb{P}\left( \frac{\sqrt{C_n}(S_k - m)}{k\sigma} \leq x \right) \mathbb{P}(Y_t = k).$$

For $J_1(\varepsilon, t)$, when $\varepsilon$ is a continuous point of $G(x) = \mathbb{P}(W \leq x)$, we have

$$J_1(\varepsilon, t) \leq \sum_{k<\varepsilon C_t} \mathbb{P}(Y_t = k) = \sum_{k<\varepsilon C_t} \mathbb{P}(W_t = k/C_t) = \mathbb{P}(W_t < \varepsilon) \to G(\varepsilon), \quad (3)$$
as $t \to \infty$. Similarly, when $\varepsilon^{-1}$ is a continuous point of $G$, we obtain

$$J_3(\varepsilon, t) \leq \mathbb{P}(W_t > \varepsilon^{-1}) \to 1 - G(\varepsilon^{-1}). \quad (4)$$
Finally, for $J_2(\varepsilon, t)$, one has

$$J_2(\varepsilon, t) = \sum_{\varepsilon C_t \leq k \leq \varepsilon^{-1} C_t} \mathbb{P}\left( \frac{S_k - m}{\sqrt{k\sigma}} \leq x \frac{\sqrt{k}}{C_t} \right) \mathbb{P}(Y_t = k)$$

$$= \sum_{\varepsilon C_t \leq k \leq \varepsilon^{-1} C_t} \mathbb{P}\left( \frac{S_k - m}{\sqrt{k\sigma}} \leq x \frac{\sqrt{k}}{C_t} \right) \mathbb{P}(W_t = k/C_t)$$

$$= \int_{\varepsilon}^{\varepsilon^{-1}} L_{\gamma C_t}(x\sqrt{y})d\mathbb{P}(W_t \leq y)$$

$$= \int_{\varepsilon}^{\varepsilon^{-1}} \Phi(x\sqrt{y})d\mathbb{P}(W_t \leq y) + o(1),$$

as $t \to \infty$, where the last equality follows from formula (1) and $\gamma C_t \geq \varepsilon C_t \to \infty$. Note that $\Phi(x\sqrt{y})$ is a bounded continuous function with respect to $y$ and $W_t \to W$ a.s., we obtain

$$J_2(\varepsilon, t) \to \int_{\varepsilon}^{\varepsilon^{-1}} \Phi(x\sqrt{y})dG(y),$$

as $t \to \infty$. Since $\varepsilon$ is arbitrary, we complete the proof of Theorem 1 by (2)–(6).

### 3. Decay rates of $\operatorname{Var}(R_t)$

Convergence rates for generating function $f_n(s)$ and harmonic moment $\mathbb{E}(Y_t^{-1})$ play important role in estimating the decay rates of $\operatorname{Var}(R_t)$, so we need the following lemmas. Lemma 1 comes from [1].

**Lemma 1.** Assume that $1 > p_1 > 0$, then there exist constants $0 \leq q_k < \infty$ such that

$$\lim_{n \to \infty} \frac{f_n(s)}{p_1^n} = \sum_{k=1}^{\infty} q_k s^k =: Q(s) < \infty, \forall 0 \leq s < 1.$$

Furthermore, $Q(s)$ is the unique solution of the functional equation

$$Q(f(s)) = p_1 Q(s), \quad Q(0) = 0, \quad Q(s) > 0$$

for all $s \in (0, 1)$.

The harmonic moments were given in [13]. We use the following special case.

**Lemma 2.** Assume that $1 > p_1 > 0$, $\mathbb{E}(Z_1^2) < \infty$. Then,

$$\lim_{n \to \infty} \Gamma(r)A_n(r)\mathbb{E}(Z_n^{-r}) = \begin{cases} 
\int_0^{\infty} Q(\exp(-v))v^{r-1}dv, & \alpha < r; \\
\int_{1}^{\infty} Q(\phi(v))v^{r-1}dv, & \alpha = r; \\
\int_{0}^{\infty} \phi(v)v^{r-1}dv, & \alpha > r,
\end{cases}$$

where $Q(x) = \mathbb{E}(Z_1^{-x})$. 

where \( \alpha \) is the Schröder index, \( \phi \) is the Laplace transformation of \( V = \lim_{n \to \infty} Z_n / m^n \), \( \Gamma \) is the \( \Gamma \) function defined as \( \Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx \) and

\[
A_n(r) = \begin{cases} 
p_1^{-n}, & \alpha < r; \\ \frac{1}{n} p_1^{-n}, & \alpha = r; \\ m^n, & \alpha > r.
\end{cases}
\]

The proof of Theorem 2.
Conditioning on \( N_t \), using the independence between \( \{N_t\} \) and \( \{Z_n\} \), one has

\[
E(R_t) = \sum_{n=0}^{\infty} E(R_n) \mathbb{P}(N_t = n) = m,
\]
and

\[
\text{Var}(R_t) = \sum_{n=0}^{\infty} \text{Var}(R_n) \mathbb{P}(N_t = n) = \sum_{n=0}^{\infty} E(R_n - m)^2 \mathbb{P}(N_t = n),
\]

where \( R_n = Z_{n+1}/Z_n \).
Note that \( Z_{n+1} = X_1 + \cdots + X_{Z_n} \), where \( \{X_j\} \) are independent and have the same distribution with \( Z_1 \). In addition, \( \{X_j\} \) are independent of \( Z_n \). Conditioning on \( Z_n \), we obtain

\[
\text{Var}(R_t) = \sum_n \sum_k \mathbb{E} \left( \left( k^{-1} \sum_{j=1}^{k} X_j - m \right)^2 \right) \mathbb{P}(Z_n = k) \mathbb{P}(N_t = n)
= \sum_n \sum_k k^{-1} \sigma^2 \mathbb{P}(Z_n = k) \mathbb{P}(N_t = n) = \sigma^2 \sum_n \mathbb{E}(Z_n^{-1}) \mathbb{P}(N_t = n).
\]

Letting \( r = 1 \) in Lemma 2, one has

\[
\lim_{n \to \infty} (A(n, \alpha))^{-1} \mathbb{E}(Z_n^{-1}) = \begin{cases} 
f_0^\infty Q(\exp(-v)) dv, & \alpha < 1; \\ f_1^m Q(\phi(v)) dv, & \alpha = 1; \\ f_0^\infty \phi(v) dv, & \alpha > 1 \end{cases} =: C(\alpha),
\]

where

\[
A(n, \alpha) = \begin{cases} 
p_1^n, & \alpha < 1; \\ np_1^n, & \alpha = 1; \\ m^n, & \alpha > 1.
\end{cases}
\]

For any \( \varepsilon > 0 \), choose \( N \) large enough such that for all \( n \geq N \), we have

\[
\mathbb{E}(Z_n^{-1}) \in ((C(\alpha) - \varepsilon) A(n, \alpha), (C(\alpha) + \varepsilon) A(n, \alpha)).
\]
We can divide $\text{Var}(\mathbb{R}_t)$ into the following three parts.

$$\text{Var}(\mathbb{R}_t) = \sigma^2 \sum_{n=0}^{N} (\mathbb{E}(Z_n^{-1}) - C(\alpha)A(n,\alpha))\mathbb{P}(N_t = n) + \sigma^2 \sum_{n\geq N+1} (\mathbb{E}(Z_n^{-1}) - C(\alpha)A(n,\alpha))\mathbb{P}(N_t = n) + \sigma^2 \sum_{n=0}^{\infty} C(\alpha)A(n,\alpha)\mathbb{P}(N_t = n)$$

$$=: I_1(\alpha,t) + I_2(\alpha,t) + I_3(\alpha,t).$$

According to (7),

$$|I_2(\alpha,t)| \leq \sigma^2 \sum_{n\geq N+1} |\mathbb{E}(Z_n^{-1}) - C(\alpha)A(n,\alpha)| \mathbb{P}(N_t = n) \leq \epsilon I_3(\alpha,t).$$

(8)

For $t$ large enough such that $\lambda t > N$, one has

$$|I_1(\alpha,t)| \leq C\mathbb{P}(N_t \leq N) \leq C(N+1)\frac{(\lambda t)^N}{N!}e^{-\lambda t} \to 0,$$

(9)

as $t \to \infty$.

Now we deal with $I_3(\alpha,t)$. If $\alpha < 1$, $A(n,\alpha) = p_1^n$, then

$$I_3(\alpha,t) = C(\alpha)\sigma^2 \sum_{n=0}^{\infty} p_1^n \mathbb{P}(N_t = n) = C(\alpha)\sigma^2 \sum_{n=0}^{\infty} \frac{(\lambda tp_1)^n}{n!}e^{-\lambda t}$$

$$= C(\alpha)\sigma^2 \exp(\lambda t(p_1 - 1)).$$

(10)

If $\alpha = 1$, $A(n,\alpha) = np_1^n$, then

$$I_3(\alpha,t) = C(\alpha)\sigma^2 \sum_{n=0}^{\infty} np_1^n \mathbb{P}(N_t = n) = C(\alpha)\sigma^2 \sum_{n=0}^{\infty} \frac{(\lambda tp_1)^n}{n!}e^{-\lambda t}$$

$$= C(\alpha)\sigma^2 \exp(\lambda t(p_1 - 1)) \sum_{n=0}^{\infty} \frac{n}{n!} \frac{(\lambda tp_1)^n}{n!}e^{-\lambda t}p_1$$

$$= C(\alpha)\sigma^2 \lambda tp_1 \exp(\lambda t(p_1 - 1)).$$

(11)

If $\alpha > 1$, $A(n,\alpha) = m^{-n}$, then

$$I_3(\alpha,t) = C(\alpha)\sigma^2 \sum_{n=0}^{\infty} m^{-n} \mathbb{P}(N_t = n) = C(\alpha)\sigma^2 \sum_{n=0}^{\infty} \frac{(\lambda tm^{-1})^n}{n!}e^{-\lambda t}$$

$$= C(\alpha)\sigma^2 \exp(\lambda t(m^{-1} - 1)).$$

(12)

We complete the proof of Theorem 2 by (8)–(13).

We give an example to illustrate Theorem 2. We choose corresponding branching law to satisfy $\alpha < 1, \alpha = 1$ and $\alpha > 1$ respectively.
EXAMPLE 1. Choose three generating functions,

\[ f(s) = \frac{s}{(4 - 3s^2)^{1/2}}, \quad g(s) = \frac{s}{2 - s}, \quad h(s) = \frac{s}{(\sqrt{2} - (\sqrt{2} - 1)s^{1/2})^2}, \]

then corresponding Schröder index \( \alpha_1 = 0.5 < 1, \alpha_2 = 1, \alpha_3 = 2 > 1 \) and

\[
C(\alpha_1, t) = 12\pi \exp(-0.5\lambda t), \\
C(\alpha_2, t) = (\ln 2)\lambda t \exp(-0.5\lambda t), \\
C(\alpha_3, t) = (2 - \sqrt{2})\exp\left(\left(\frac{1}{\sqrt{2}} - 1\right)\lambda t\right).
\]

Proof. For generating function \( f(s) \), we know

\[ f'(s) = \frac{4}{(4 - 3s^2)^{3/2}}, \quad f''(s) = \frac{36s}{(4 - 3s^2)^{5/2}}. \]

Then

\[ p_1 = f'(0) = \frac{4}{8} = 0.5, \quad m_1 = f'(1) = 4, \quad \sigma_1^2 = f''(1) + m_1 - m_1^2 = 24. \]

Thus \( \alpha_1 = -\log_4 0.5 = 0.5 < 1 \). By iteration,

\[ f_n(s) = \frac{s}{(4^n - (4^n - 1)s^2)^{1/2}}. \]

So we have

\[ Q_1(s) = \lim_{n \to \infty} \frac{2^n s}{(4^n - (4^n - 1)s^2)^{1/2}} = \frac{s}{\sqrt{1 - s^2}}. \]

Consequently,

\[ \int_0^\infty Q_1(e^{-v})dv = \int_0^\infty \frac{e^{-v}}{\sqrt{1 - e^{-2v}}}dv = \frac{\pi}{2}. \]

For generating function \( g(s) \), one has

\[ g'(s) = \frac{2}{(2 - s)^2}, \quad g''(s) = \frac{4}{(2 - s)^3}. \]

Then

\[ p_1 = g'(0) = \frac{2}{4} = 0.5, \quad m_2 = g'(1) = 2, \quad \sigma_2^2 = g''(1) + m_2 - m_2^2 = 2. \]

Thus \( \alpha_2 = -\log_2 0.5 = 1 \). By iteration,

\[ g_n(s) = \frac{s}{2^n - (2^n - 1)s}. \]
So we have
\[ Q_2(s) = \lim_{n \to \infty} \frac{2^n s}{2^n - (2^n - 1)s} = \frac{s}{1 - s}. \]

According to Theorem 2, we need to calculate the Laplace transformation of \( W \) which is determined by
\[ \phi_2(v) = \lim_{n \to \infty} g_n(\exp(-v/2^n)) = \lim_{n \to \infty} \frac{\exp(-v/2^n)}{2^n - (2^n - 1)\exp(-v/2^n)} = \frac{1}{1 + v}. \]

Consequently,
\[ \int_1^2 Q_2(\phi_2(v))dv = \int_1^2 \frac{1}{1 - \frac{1}{1 + v}}dv = \ln(2). \]

Finally, for generating function \( h(s) \), one has
\[ h'(s) = \frac{\sqrt{2}}{(\sqrt{2} - (\sqrt{2} - 1)s^{1/2})^3}, \quad h''(s) = \frac{3(2 - \sqrt{2})s^{-1/2}}{2(\sqrt{2} - (\sqrt{2} - 1)s^{1/2})^4}. \]

Then
\[ p_1 = h'(0) = \frac{\sqrt{2}}{2^{3/2}} = 0.5, \quad m_3 = h'(1) = \sqrt{2}, \quad \sigma_3^2 = h''(1) + m_3^2 - m_3^2 = 1 - \frac{\sqrt{2}}{2}. \]

Thus \( \alpha_3 = -\log_{\sqrt{2}}0.5 = 2 > 1 \). By iteration,
\[ h_n(s) = \frac{s}{((\sqrt{2})^n - ((\sqrt{2})^n - 1)s^{1/2})^2}. \]

So we have
\[ \phi_3(v) = \lim_{n \to \infty} h_n\left(\exp\left(-\frac{v}{2^{n/2}}\right)\right) = \lim_{n \to \infty} \frac{\exp(-v/2^{n/2})}{(2^{n/2} - (2^{n/2} - 1)\exp(-v/2^{n/2} + 1))^2} = \frac{4}{(2 + v)^2}. \]

Consequently,
\[ \int_0^\infty \phi_3(v)dv = 2. \]

We complete the proof of Example 1.

4. Asymptotic normality of \( \sqrt{Y_t}(\mathbb{R}_t - m) \)

In this section, we deal with the asymptotic normality of \( \sqrt{Y_t}(\mathbb{R}_t - m) \).
The proof of Theorem 3.
Conditioning on \( Y_t \),
\[
G_t(x) - \Phi(x) = \sum_{k=1}^{\infty} \mathbb{P} \left( \frac{\sqrt{k}(S_k - m)}{k\sigma} \leq x \right) \mathbb{P}(Y_t = k) - \Phi(x)
\]
\[
= \sum_{k=1}^{\infty} (L_k(x) - \Phi(x))\mathbb{P}(Y_t = k), \tag{14}
\]
where \( L_k(x) \) is defined at the beginning of Section 2. According to (1), for any \( \varepsilon \in (0, 1) \), there exist \( N = N(\varepsilon) > 0 \) such that for any \( k \geq N \), we have
\[
L_k(x) \in (\Phi(x) - \varepsilon, \Phi(x) + \varepsilon). \tag{15}
\]
We can divide (14) into the following two parts.
\[
J_1(\varepsilon, t) = \sum_{k<N} (L_k(x) - \Phi(x))\mathbb{P}(Y_t = k),
\]
\[
J_2(\varepsilon, t) = \sum_{k\geq N} (L_k(x) - \Phi(x))\mathbb{P}(Y_t = k).
\]
The rest of the proof is straightforward via (15).

The proof of Theorem 4.
Conditioning on \( Y_t \), according to the Berry–Esseen bound for i.i.d. random variables, we obtain
\[
\sup_{x \in \mathbb{R}} |G_t(x) - \Phi(x)| \leq \sum_{k=1}^{\infty} |L_k(x) - \Phi(x)|\mathbb{P}(Y_t = k) \leq C \sum_{k=1}^{\infty} k^{-1/2}\mathbb{P}(Y_t = k) = C \mathbb{E}(Y_t^{-1/2}).
\]
The rest of the proof is similar to that of Theorem 2.

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