ONE BASE POINT FREE THEOREM FOR WEAK LOG FANO THREEFOLDS

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Abstract. Let \((X, D)\) be log canonical pair such \(\dim X = 3\) and the divisor \(- (K_X + D)\) is nef and big. For a special class of such \((X, D)\)'s we prove that the linear system \(|- n(K_X + D)|\) is free for \(n \gg 0\).

1. Introduction

Let \(X\) be algebraic variety\(^1\) of dimension \(\geq 2\) with a \(\mathbb{Q}\)-boundary \(D\) such that the pair \((X, D)\) is log canonical and the divisor \(- (K_X + D)\) is nef and big. Then one may consider the following

Conjecture 1.1 (M. Reid (see \[5\], \[11\])). The linear system \(|- n(K_X + D)|\) is free for \(n \gg 0\).

According to \[11\] Proposition 11.1 (see also \[11\]), Conjecture 1.1 is true when \(\dim X = 2\). Unfortunately, it is false when \(\dim X \geq 3\):

Example 1.2 (see \[3\]). Let \(Z\) be a smooth elliptic curve and \(E\) indecomposable rank 2 vector bundle over \(Z\) with \(\deg(E) = 0\) (see \[11\]). Put \(S = \mathbb{P}_Z(E)\) and let \(C\) be the tautological section on \(S\). Then we have \(C^2 = 0\) and \(K_S = -2C\). Let \(F\) be a fibre of the \(\mathbb{P}^1\)-bundle \(S \to \mathbb{P}^1\). Then the cone \(NE(S)\) is generated by two rays \(R_1 = \mathbb{R}_{\geq 0}[C]\), \(R_2 = \mathbb{R}_{\geq 0}[F]\), and there is no curve \(C' \neq C\) with \([C'] \in R_1\) (see \[11\] Example 1.1)). In particular, the linear system \(|- nK_S|\) is not free for any \(n\). Consider the cone \(X\) over \(S \subset \mathbb{P}^N\) with respect to some projective embedding and the blow up \(\sigma : Y \to X\) of the vertex on \(X\) with exceptional divisor \(E\). We have \(- (K_X + E) = \sigma^*(O_X(1)) + \pi^*(-K_S)\), where \(\pi : Y \to S\) is the natural projection, which implies that \(- (K_X + E)\) is nef and big. On the other hand, \((X, E)\) is purely log terminal and \(|- n(K_X + E)|_E = |- nK_E|\) is not free for any \(n\) because \(E \cong S\). Moreover, \((X, E + \pi^*(C))\) is log canonical, \(- (K_X + E + \pi^*(C)) = \sigma^*(O_X(1)) + \pi^*(C)\) is nef and big, but again \(|- n(K_X + E + \pi^*(C))|_E = |(- n/2)K_E|\) is not free. The latter shows that Conjecture 1.1 is not true also for strictly log canonical pairs (the special case of \(\downarrow D_J = 0\) and \(\dim X \leq 4\) was treated in \[3\]).

It follows from Example 1.2 that the case when \(D\) contains a reduced part is far from being trivial. The present paper aims to correct the main result of \[4\]. We shall consider in some sense the simplest case when Conjecture 1.1 is true for \(\dim X \geq 3\) and \(\downarrow D_J \neq 0\):

Theorem 1.3. Let \((X, D)\) be as above. Suppose that

- \(X\) is a smooth 3-fold and \(D = S\) is a smooth surface;
- \(S \cdot Z > 0\) for every curve \(Z\) on \(S\) with \(K_S \cdot Z = 0\).

Then the linear system \(|- n(K_X + D)|\) is free for \(n \gg 0\).

It follows from Example 1.2 that the assertion of Theorem 1.3 is false without the additional assumption on \(K_S\)-trivial curves. This suggests the following

Conjecture 1.4. Let \((X, D)\) be as above. Suppose that \(X\) is \(\mathbb{Q}\)-factorial and \((K_X + D) \cdot S^{\dim X - 1} \geq 0\) for every irreducible component \(S \subseteq \downarrow D_J\). Then the linear system \(|- n(K_X + D)|\) is free for \(n \gg 0\).

I would like to thank Y. Gongyo for pointing out the mistake in \[4\].

\(^1\)All algebraic varieties are assumed to be projective and defined over \(\mathbb{C}\).

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2. Preliminaries

We use standard notation, notions and facts from the theory of minimal models and singularities of pairs (see [6], [9], [7], [8]). In what follows, \((X, S)\) is the pair from Theorem 1.3. In order to prove Theorem 1.3 we assume that \(\text{Bs}(|-n(K_X + S)|) \neq \emptyset\) where \(n \geq 0\).

**Proposition 2.1.** We have
\[
\text{Bs}(|-n(K_X + S)|) \cap S = \text{Bs}(|-n(K_X + S)|_S) = \text{Bs}(|-nK_S|) \neq \emptyset.
\]

**Proof.** Consider the exact sequence
\[
0 \to \mathcal{O}_X(-n(K_X + S) - S) \to \mathcal{O}_X(-n(K_X + S)) \to \mathcal{O}_S(-n(K_X + S)|_S) \to 0.
\]
We have
\[
H^1(X, \mathcal{O}_X(-n(K_X + S) - S)) = H^1(X, \mathcal{O}_X(K_X - (n+1)(K_X + S))) = 0
\]
by Kawamata–Viehweg Vanishing Theorem. This gives the exact sequence
\[
H^0(X, \mathcal{O}_X(-n(K_X + S))) \to H^0(S, \mathcal{O}_S(-n(K_X + S)|_S)) \to 0,
\]
which implies that
\[
\text{Bs}(|-n(K_X + S)|) \cap S = \text{Bs}(|-n(K_X + S)|_S) = \text{Bs}(|-nK_S|).
\]
Finally, if \(\text{Bs}(|-n(K_X + S)|) \cap S = \emptyset\), then \(\text{Bs}(|-n(K_X + S)|) = \emptyset\) (see the proof of the Basepoint-free Theorem in [11]), a contradiction. □

From Proposition 2.1 we get the following

**Corollary 2.2.** Equality \(K_S^2 = 0\) holds.

**Proof.** Since \(- (K_X + S)\) is nef, we have
\[
K_S^2 = (K_X + S)^2 \cdot S \geq 0
\]
(see [9] Theorem 1.38]). Now, if \(K_S^2 > 0\), then \(-K_S\) is nef and big, and the Basepoint-free Theorem implies that \(\text{Bs}(|-nK_S|) = \emptyset\), a contradiction. □

3. Reduction to the non-complementary case

We use notation and conventions of Section 2. Let us show that the surface \(S\) does not have \(\mathbb{Q}\)-complements. Assume the contrary. Then we have the following

**Lemma 3.1.** \(S\) is a rational surface.

**Proof.** Suppose that \(S\) is non-rational. Then it follows from the proof of [2] Theorem 1.3, [11] Corollary 2.2 and [11] Example 2.1 that \(\text{Bs}(|-nK_S|) = \emptyset\), which contradicts Proposition 2.1. □

Thus, there exists a birational contraction \(\chi : S \to \tilde{S}\), where either \(\chi\) is the blow up of \(\tilde{S} = \mathbb{P}^2\) at some points \(p_1, \ldots, p_9\), or \(\chi\) is the blow up of \(\tilde{S} = \mathbb{P}_m, m \in \mathbb{N}\), at some points \(q_1, \ldots, q_8\) (see Corollary 2.2). To simplify the notation, in what follows we assume that all \(p_i\) (respectively, all \(q_j\)) are distinct. Further, by our assumption the equivalence \(K_S + \Delta \sim 0\) holds for some effective \(\mathbb{Q}\)-divisor \(\Delta\) such that the pair \((S, \Delta)\) is log canonical. Then we have
\[
-K_S \sim \sum_{i=1}^{N} \Delta_i,
\]
\(N \in \mathbb{N}\), where \(\Delta_i\) are reduced and irreducible curves such that \(\Delta_i \cap \Delta_j \neq \emptyset\) for all \(i \neq j\) and the intersection is transversal.

**Lemma 3.2.** Equality \(h^0(S, \mathcal{O}_S(-nK_S)) = 1\) holds.
Proof. We have \( h^0(S, \mathcal{O}_S(-nK_S)) > 0 \). Suppose that \( h^0(S, \mathcal{O}_S(-nK_S)) \geq 2 \). Then, since the sum \( \sum_{i=1}^{N} \Delta_i \) is connected and \(-K_S\) is nef with \( K_S^2 = 0\), \((-nK_S)\) is a free pencil, which contradicts Proposition 2.1. \( \square \)

**Proposition 3.3.** If \( N = 1 \), then \( \tilde{S} \neq \mathbb{P}^2 \).

**Proof.** Suppose that \( \tilde{S} = \mathbb{P}^2 \). Let us consider two cases:

- **Case (1).** The curve \( C = \Delta_1 \) is smooth. Write
  \[
  S|_S = \chi^*(aL) + \sum_{i=1}^{9} a_i E_i,
  \]
  where \( L \) is a line on \( \mathbb{P}^2 \), \( a \) and \( a_i \in \mathbb{Z} \), \( E_i = \chi^{-1}(p_i) \). Let \( \varphi : Y \rightarrow X \) be the blow up of \( X \) at \( C \) with exceptional divisor \( E \). For \( S_Y = \varphi^{-1}_*(S) \), \( \varphi \) induces an isomorphism \( \varphi_s : S_Y \cong S \) such that \( \varphi_s(S_Y \cap E) = C \) and \( \varphi_s \) is identical out of \( C_Y = S_Y \cdot E \), which implies that \( \varphi_s \) is an automorphism of \( S \), identical on \( \text{Pic}(S) \). In particular, we have
  \[
  E \cdot C_Y = E \cdot E \cdot S_Y = C_Y^2 = 0,
  \]
  which together with equalities
  \[
  K_Y + S_Y = \varphi^*(K_X + S) \quad \text{and} \quad S_Y = \varphi^*(S) - E
  \]
  implies that to prove Proposition 3.3 we may pass from \((X, S)\) to the pair \((Y, S_Y)\). Moreover, we have
  \[
  S_Y|_{S_Y} = \chi^*(a_Y L_Y) + \sum_{i=1}^{9} a_i Y E_i.
  \]
  where \( L_Y = \varphi^{-1}_*(L) \), \( E_i \cdot Y = \varphi^{-1}_*(E_i) \), \( a_Y \) and \( a_i Y \in \mathbb{Z} \), which implies that
  \[
  -a_i Y = S_Y \cdot E_i \cdot Y = S \cdot E_i - E_i Y \leq S \cdot E_i - 1 = -a_i - 1,
  \]
  and hence \( a_i Y > a_i \).

  Thus, applying the above arguments to \((Y, S_Y)\), after a number of blow ups we obtain that to prove Proposition 3.3 we may assume that \( a_i > 0 \) for all \( i \). In particular, we have
  \[
  (K_X + S) \cdot E_1 = K_S \cdot E_1 < 0 \quad \text{and} \quad S \cdot E_1 = -a_1 < 0,
  \]
  and it follows from the Cone Theorem that equality \( E_1 \equiv \sum R_i + \sum C_j \) holds on \( X \), where \( R_i \) are \((K_X + S)\)-negative extremal rays and \((K_X + S) \cdot C_j = 0 \) for all \( j \). Moreover, by assumption on \( K_S \)-trivial curves we have \( S \cdot C_j \geq 0 \) for all \( j \), which implies that there exists a \((K_X + S)\)-negative extremal ray \( R \) on \( X \) such that \( S \cdot R < 0 \). In particular, we have \( R \subset S \) and the extremal contraction \( \text{cont}_R : X \rightarrow W \) is birational.

**Lemma 3.4.** \( \text{cont}_R \) is not a divisorial morphism.

**Proof.** Assume the contrary. Then the image of \( S \) is either a point or a curve. But the first case is impossible because \((K_X + S) \cdot C = 0 \). Thus, \( \text{cont}_R(S) \) is a curve. Then there exists a birational contraction \( \chi' : S \rightarrow \mathbb{P}^2 \), which is the blow up at some points \( p'_1, \ldots, p'_9 \) on \( \mathbb{P}^2 \), with exceptional curves \( E'_1, \ldots, E'_9 \) such that

- \( E'_1 \cdot R = 1 \) and \( E'_i \cdot Z = 0 \) for some curve \( Z \) on \( S \) such that \( R = \mathbb{R}_{\geq 0}[Z] \);
- \( R = \mathbb{R}_{\geq 0}[E'_i] \) for all \( i \geq 2 \).

Let \( \varphi : Y \rightarrow X \) be the blow up of \( X \) at \( E'_1 \) with exceptional divisor \( E \). Then \( Y \) possesses a \((K_Y + S_Y)\)-negative extremal ray \( R_Y = \varphi^{-1}_*(R) + a_{\mathbb{Q}} x \), where \( \alpha \in \mathbb{Q} \), \( S_Y = \varphi^{-1}_*(S) \) and \( e \) is the numerical class of a fibre of \( \varphi \). Note that \( \alpha \leq 0 \), which implies that
  \[
  S_Y \cdot R_Y = S \cdot R + \alpha < 0
  \]
  and hence \( R_Y \subset S_Y \). In particular, \( R_Y = \varphi^{-1}_*(R) \). Moreover, for the curves \( E'_{i,Y} = S_Y \cdot E \), \( Z_Y = \varphi^{-1}_*(Z) \) and \( E'_{i,Y} = \varphi^{-1}_*(E'_i) \), \( i \geq 2 \), we have
• $E_{i,Y}' \cdot R_Y = 1$ and $E_{i,Y}' \cdot Z_Y = 0$;
• $R_Y = \mathbb{R}_{\geq 0}[E_{i,Y}'] = \mathbb{R}_{\geq 0}[Z_Y]$ for all $i \geq 2$.

On the other hand, we get

$$0 = E_{i,Y}' \cdot Z_Y = E \cdot Z_Y = E \cdot R_Y = E_{i,Y}' \cdot R_Y = 1,$$

a contradiction.

Thus, $\text{cont}_R$ is a small contraction. Then $R$ is generated by a $(-1)$-curve on $S$. Consider the $(K_X + S)$-flip:

So that the map $\tau$ is an isomorphism in codimension 1, for every curve $R^+ \subset X^+$, which is contracted by $\text{cont}_R^+$, we have $(K_X^+ + S^+) \cdot R^+ > 0$, where $S^+ = \tau_*(S)$, 3-fold $X^+$ is $\mathbb{Q}$-factorial and the pair $(X^+, S^+)$ is purely log terminal (see [3] and [9, Proposition 3.36, Lemma 3.38]). Let

$$X \xrightarrow{\tau} X^+ \xrightarrow{\text{cont}_R^+} W,$$

be resolution of indeterminacies of $\tau$ over $W$. Then $f$ is a sequence of blow ups at smooth centers over $R$ with exceptional divisors $G_1, \ldots, G_s \subset T$ such that $G_i$ constitute the $f^+-$exceptional locus and $Z = f^+(\sum_{i=1}^s G_i)$ is a union of all $\text{cont}_R^-$exceptional curves.

**Lemma 3.5.** We have $Z \subseteq \text{Bs}([-n(K_X^+ + D^+)])$ and $R^+ \not\subset S^+$ for every $R^+ \subseteq Z$.

**Proof.** The statement follows from conditions $K_X^+ + S^+ = \tau_*(K_X + S)$, $R \not\subset \text{Supp}(-K_S)$, $(K_X^+ + S^+) \cdot R^+ > 0$ for every $R^+ \subseteq Z$ and the fact that $f^{-1}_*(S) \cdot f^{-1}_*(-n(K_X + S)) = f^{-1}_*(S) \cdot (-n(K_X + S))$ (the latter holds because $f$ is a sequence of blow ups at smooth centers).

It follows from Lemma 3.5 that $S^+ \simeq \text{cont}_R(S)$ and $\tau$ induces contraction $\tau_S : S \to S^+$ of the $(-1)$-curve in $R$. Then, since $K_X^+ + S^+ = \tau_*(K_X + S)$, the divisor

$$-(K_X^+ + S^+)\big|_{S^+} \equiv -K_{S^+} = \tau_*(C)$$

is nef and big. Moreover, the surface $S^+$ has only log terminal singularities by the Inversion of adjunction, which implies that $\text{Bs}([-nK_{S^+}]) = \emptyset$ by the Basepoint-free Theorem.

**Lemma 3.6.** Inequality $h^0(S, \mathcal{O}_S(-nK_S)) \geq 2$ holds.

**Proof.** We have $R^1(\text{cont}_R)_*(-n(K_X + S) - S) = 0$ by the relative Kawamata–Viehweg Vanishing Theorem. This and the isomorphism $S^+ \simeq \text{cont}_R(S)$ imply that the push-forwards to $W$ of exact sequences $0 \to \mathcal{O}_X(-n(K_X + S) - S) \to \mathcal{O}_X(-n(K_X + S)) \to \mathcal{O}_S(-n(K_X + S))\big|_S \to 0$ and $0 \to \mathcal{O}_{X^+}(-n(K_X^+ + S^+) - S^+) \to \mathcal{O}_{X^+}(-n(K_X^+ + S^+)) \to \mathcal{O}_{S^+}(-n(K_X^+ + S^+))\big|_{S^+} \to 0$ coincide with exact sequence

$$0 \to \mathcal{O}_W(-n(K_W + S^*) - S^*) \to \mathcal{O}_W(-n(K_W + S^*)) \to \mathcal{O}_{S^*}(-n(K_W + S^*))\big|_{S^*} \to 0.$$

Then it follows from $\text{Bs}([-nK_{S^+}]) = \emptyset$ that $h^0(S, \mathcal{O}_S(-nK_S)) \geq 2$.

From Lemma 3.6 we get contradiction with Lemma 3.2. Thus, **Case (1)** is impossible, and we pass to

**Case (2).** The curve $C = \Delta_1$ is singular. Since $C \sim -K_S$ and the pair $(S, C)$ is log canonical, we have $p_a(C) = 1$ and the only singular point on $C$ is an ordinary double point $O$. 

Let $\varphi : Y \to X$ be the blow up of $X$ at $C$ with exceptional divisor $E$. Locally near $O$ there is an analytic isomorphism

$$(X, S, \Delta) \simeq (\mathbb{C}^3_{x,y,z}, \{x = 0\}, \{yz = 0\}).$$

Then locally over $O$ we have the following representation for $Y$:

$$Y = \{yzt_0 = xt_1\} \subset \mathbb{C}^3_{x,y,z} \times \mathbb{P}^1_{t_0,t_1},$$

which implies that the only singular point on $Y$ is a non-$\mathbb{Q}$-factorial quadratic singularity. Then, since

$$K_Y + \varphi^{-1}_*(S) = \varphi^*(K_X + S),$$

after a small resolution $\psi : \tilde{Y} \to Y$ we may pass from $(X, S)$ to the pair $(\tilde{Y}, \psi^{-1}(\varphi^{-1}(S)))$ as above and apply the arguments from Case (1). \hfill $\Box$

Applying the same arguments as in the proof of Proposition 3.2 to $\tilde{S} = \mathbb{P}_m$, we see that the case $N = 1$ is impossible. Finally, in the case when $N \geq 2$ we apply the same arguments as in the proof of Proposition 3.2 replacing the curve $\Delta_i$ with the cycle $\sum_{i=1}^N \Delta_i$.

Thus, the surface $S$ does not have $\mathbb{Q}$-complements, and we get the following

**Corollary 3.7.** In the notation of Example 3.2 we have $S = \mathbb{P}_z(\mathcal{E})$ and $\text{Supp}(-n(K_X + S)|_S) = C$. In particular, $\text{Bs}(-n(K_X + S)) \cap S = C$.

**Proof.** The statement follows from [2, Theorem 1.3] and Proposition 2.1 \hfill $\Box$

Let $F$ be a fibre of the $\mathbb{P}^1$-bundle $S \to \mathbb{P}^1$. Write

$$S|_S = aC - bF,$$

where $a, b \in \mathbb{Z}$. Note that $b < 0$ because $C$ is $K_S$-trivial and hence $0 < S \cdot C = S|_S \cdot C = -b$ by assumption on $K_S$-trivial curves.

**Lemma 3.8.** Equality $\text{deg}(\mathcal{N}_{C/X}) = -b$ holds.

**Proof.** Since $C$ is a smooth elliptic curve, we have

$$\text{deg}(\mathcal{N}_{C/X}) = -K_X \cdot C = ((2 + a)C - bF) \cdot C = -b.$$

\hfill $\Box$

Put $\mathcal{L}_n = |-n(K_X + S)|$. Then for the general element $L_n \in \mathcal{L}_n$ we have

$$L_n = M + \sum r_{i,S}B_{i,S} + \sum r_iB_i,$$

where $B_i, B_{i,S}$ are the base components of $\mathcal{L}_n$, $r_i, r_{i,S} \geq 0$ the corresponding multiplicities, $B_i \cap S = \emptyset, B_{i,S} \cap S \neq \emptyset$ for all $i$, and the linear system $|M|$ is movable. According to Corollary 3.7 we have $\text{Bs}(-n(K_X + D)) \cap S = C$ and $B_{i,S} \cap S = C$ for all $i$, which implies that $\text{Bs}(|M|) \cap S = C$ or $\emptyset$. In what follows, we assume that $\text{Bs}(|M|) = \text{Bs}(|M|) \cap S$ (see the proof of the Basepoint-free Theorem in [9]). Furthermore, arguing exactly as in the proof of Proposition 3.2 we can replace $X$ by its blow up at the curve $C$. Then, after applying Corollary 3.1 and a number of blow ups, in what follows we assume that the following conditions are satisfied:

- $r_{i,S} = r > 0$ and $B_{i,S} = B$ for all $i$, where $B = \mathbb{P}_C(\mathcal{N}_{C/X})$ with $B^3 = -\deg(\mathcal{N}_{C/X}) = b$ (see Lemma 3.8);
- $S \cdot B = C$;
- the linear system $|M|$ is free and $M \cap B = \emptyset$;
- $B_j \cap B \neq \emptyset$ for exactly one $j$ and the intersection is transversal, $r_j = r, B_j^2 \cdot B = b$. 

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4. Exclusion of the non-complementary case

We use notation and conventions of Section 3. Let \( \varphi : Y \to X \) be the blow up of \( X \) at the curve \( C \) with exceptional divisor \( E \). Put

\[
S_Y = \varphi^{-1}_*(S), \quad B_Y = \varphi^{-1}_*(B), \quad M_Y = \varphi^{-1}_*(M), \quad B_{i,Y} = \varphi^{-1}_*(B_i).
\]

Then for \( m \gg 0, 0 < \delta_1, \delta_2 \ll 1 \) and \( 0 < c \leq 1 \) we write

\[
R = \varphi^* \left( (K_X + S) + mL_n - cL_n \right) + cM_Y + \delta_1 S_Y + \delta_2 E = \varphi^*(mL_n) + (-1 + \delta_1) S_Y + (\delta_2 - cr) E - cr B_Y - cr B_{j,Y} - \sum_{i \neq j} cr_i B_{i,Y} - K_Y.
\]

**Proposition 4.2.** The divisor \( R \) is nef and big for \( \delta_1 \geq \delta_2 \).

**Proof.** Since the divisors \( -(K_X + S) \) and \( M_Y \) are nef and big, it suffices to prove that the divisor

\[
R = \varphi^* \left( (K_X + S) + mL_n - cL_n \right) + cM_Y + \delta_1 S_Y + \delta_2 E
\]

intersects every curve on the surfaces \( S_Y \) and \( E \) non-negatively.

**Lemma 4.3.** Inequality \( R \cdot Z \geq 0 \) holds for every curve \( Z \) on \( S_Y \).

**Proof.** Since \( S_Y \simeq S \), the cone \( \overline{NE}(S_Y) \) is generated by the classes \( [C_Y] = [S_Y \cdot E] \) and \( [F_Y] = [\varphi^{-1}_*(F)] \). Thus, it is enough to consider only the cases when \( Z = C \) or \( F \).

We have \( E \cdot C_Y = 0 \) and

\[
S_Y \cdot C_Y = S \cdot C = -b > 0,
\]

which implies that \( R \cdot C_Y > 0 \). Furthermore, we have

\[
R \cdot F_Y \geq \varphi^*(L_n) \cdot F_Y = L_n \cdot F \geq nC \cdot F = n \gg 0,
\]

and the assertion follows. \( \square \)

**Lemma 4.4.** Inequality \( R \cdot Z \geq 0 \) holds for every curve \( Z \) on \( E \) and \( \delta_1 \geq \delta_2 \).

**Proof.** Let \( F_E \) be a fibre of the \( \mathbb{P}^1 \)-bundle \( E = \mathbb{P}_C(\mathcal{N}_{C/X}) \). We have

\[
\left( B_Y \mid_E \right)^2 = (\varphi^*(B) - E)^2 \cdot E = 2B \cdot C + E^3 = 2C^2 + E^3 = -\deg(\mathcal{N}_{C/X}) = b < 0,
\]

which implies that the cone \( \overline{NE}(E) \) is generated by the classes \( [-E]_E = [B_Y]_E \) and \( [F_E] \) (see [9, Lemma 1.22]). Thus, it is enough to consider only the cases when \( Z = -E \mid_E \) or \( F_E \).

We have

\[
S_Y \cdot (-E \mid_E) = -S_Y \cdot E^2 = 0,
\]

which implies that

\[
R \cdot (-E \mid_E) \geq \delta_2 E \cdot (-E \mid_E) = -b \delta_2 > 0.
\]

Furthermore, we have

\[
S_Y \cdot F_E = 1 \quad \text{and} \quad E \cdot F_E = -1,
\]

which implies that

\[
R \cdot F_E = \delta_1 - \delta_2 \geq 0,
\]

and the assertion follows. \( \square \)

Lemmas 4.3 and 4.4 prove Proposition 4.2.
Take \( c = 1/r \) in [4.1]. Then we obtain
\[
\varphi^*(mL_n) - B_Y - B_{j,Y} + \sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y} - K_Y,
\]
and Proposition 4.2 and Kawamata–Viehweg Vanishing Theorem imply that
\[
H^i(Y, \mathcal{O}_Y(\varphi^*(mL_n) - B_Y - B_{j,Y} + \sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y})) = 0
\]
for all \( i > 0 \).

**Lemma 4.6. Inequality**

\[
H^0(B_Y, \mathcal{O}_{B_Y}((\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y})|_{B_Y})) \neq 0
\]
holds.

**Proof.** Note that \((\sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y})|_{B_Y} = 0\). Let us prove that

\[
H^0(B_Y, \mathcal{O}_{B_Y}((\varphi^*(mL_n) - B_{j,Y})|_{B_Y})) \neq 0.
\]

We have

\[
\varphi^*(mL_n) = mMY + mrBY + mrB_{j,Y} + mr + \sum_{i \neq j} mr_i B_{i,Y},
\]
which implies that

\[
\varphi^*(mL_n)|_{B_Y} = mrBY|_{B_Y} + mrB_{j,Y}|_{B_Y} + mr|_{B_Y}.
\]

Further, since \( B_Y = \varphi^*(B) - E \) and \( \varphi^*(B) \cdot E^2 = -B \cdot C = 0 \), we obtain

\[
(E|_{B_Y})^2 = E^2 \cdot B_Y = -E^3 = -b
\]
and

\[
(B_{j,Y}|_{B_Y})^2 = B^2_j \cdot B = b,
\]
which implies that \( E|_{B_Y} \) is the tautological section of the \( \mathbb{P}^1 \)-bundle \( B_Y = \mathbb{P}_C(N_{C/X}) \) with a fibre \( F_{B_Y} \), and \( B_{j,Y}|_{B_Y} \sim E|_{B_Y} + bF_{B_Y} \). Furthermore, we have

\[
(B_Y|_{B_Y})^2 = B_Y^3 = \varphi^*(B)^3 - E^3 = 0
\]
and

\[
B_Y|_{B_Y} \cdot E|_{B_Y} = B^2_Y \cdot E = E^3 = b,
\]
which implies that \( B_Y|_{B_Y} \sim bF_{B_Y} \). Thus, we get

\[
\varphi^*(mL_n)|_{B_Y} \sim 2mrB_{j,Y}|_{B_Y},
\]
which implies that

\[
\varphi^*(mL_n)|_{B_Y} - B_{j,Y}|_{B_Y} \sim (2mr - 1)B_{j,Y}|_{B_Y}
\]
and hence \( H^0(B_Y, \mathcal{O}_{B_Y}((\varphi^*(mL_n) - B_{j,Y})|_{B_Y})) \neq 0. \)

From [4.5] and the exact sequence

\[
0 \to \mathcal{O}_Y(\varphi^*(mL_n) - B_Y - B_{j,Y} + \sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y}) \to
\]
\[
\to \mathcal{O}_Y(\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y}) \to
\]
\[
\to \mathcal{O}_{B_Y}((\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j} \varphi^* - cr_i \varphi^* B_{i,Y})|_{B_Y}) \to 0
\]
we get exact sequence
\[ 0 \to H^0(Y, \mathcal{O}_Y(\varphi^*(mL_n) - B_Y - B_{j,Y} + \sum_{i \neq j}^{-r}cr_i \gamma B_{i,Y})) \to \]
\[ \to H^0(Y, \mathcal{O}_Y(\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j}^{-r}cr_i \gamma B_{i,Y})) \to \]
\[ \to H^0(B_Y, \mathcal{O}_{B_Y}((\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j}^{-r}cr_i \gamma B_{i,Y})|_{B_Y})) \to 0, \]

which implies, since $-r_i \leq -r - cr_i \leq 0$ and $B_Y, B_{j,Y}, B_{i,Y}$ are the base components of the linear system $[\varphi^*(mL_n)]$, that
\[ H^0(Y, \mathcal{O}_Y(\varphi^*(mL_n) - B_Y - B_{j,Y} + \sum_{i \neq j}^{-r}cr_i \gamma B_{i,Y})) \cong \]
\[ \cong H^0(Y, \mathcal{O}_Y(\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j}^{-r}cr_i \gamma B_{i,Y})) \cong H^0(Y, \mathcal{O}_Y(\varphi^*(mL_n))) \]
and
\[ H^0(B_Y, \mathcal{O}_{B_Y}((\varphi^*(mL_n) - B_{j,Y} + \sum_{i \neq j}^{-r}cr_i \gamma B_{i,Y})|_{B_Y})) = 0, \]
a contradiction with Lemma 4.6. Theorem 1.3 is completely proved.

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