THE PRINCIPLE OF INDIVISIBLE INTEGRITY: A STRUCTURAL PERSPECTIVE ON HIGHER-ORDER DIFFERENTIAL EQUATIONS AND RECURRENCE RELATIONS

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Abstract. Classical results such as Fuchs’ theorem and the Poincaré–Perron theorem have extensively shaped our understanding of power series solutions to differential equations. While these theorems correctly predict convergence properties for second-order recurrence relations, they implicitly assume an invariance under term rearrangement that fails in higher-order cases.

In this paper, we introduce the Principle of Indivisible Integrity, which asserts that in higher-order differential equations, the fundamental building blocks of recurrence relations must be treated as indivisible entities to preserve the expected radius of convergence. Specifically, we demonstrate that while second-order recurrence relations maintain absolute convergence under term rearrangement, higher-order recurrence relations involve coefficients that are products of multiple recurrence terms, and rearranging these subcomponents can lead to a reduction in the radius of convergence. This result challenges the conventional understanding of convergence in power series solutions and necessitates a structural revision of existing methods.

To formalize this principle, we establish structural constraints on the convergence of power series solutions to higher-order differential equations. These constraints provide a more precise framework for understanding the stability of solutions, ensuring that the radius of convergence is preserved when terms are treated as indivisible. By applying this framework to Heun-type differential equations, we illustrate the impact of these constraints on practical computations and function theory.

This work extends the classical framework of power series convergence by addressing previously unrecognized structural dependencies. The findings have significant implications for differential equations, asymptotic analysis, and numerical methods, providing a refined perspective on the nature of convergence in higher-order recurrence relations.

1. Introduction

Power series solutions to differential equations are widely used in mathematical physics, particularly in the analysis of singularities and recurrence relations. Classical results such as Fuchs’ theorem and the Poincaré–Perron theorem provide well-established criteria for determining the radius of convergence in these series. These theorems implicitly assume that term rearrangement does not affect the convergence properties, a property that remains valid for second-order recurrence relations but does not necessarily extend to higher-order cases.

The Principle of Indivisible Integrity establishes that in higher-order differential equations, recurrence terms must be treated as indivisible structural units, analogous to points in the complex plane. Just as a complex number \( z = x + iy \) maintains
its geometric properties only when treated as an indivisible entity, the terms in recurrence relations preserve their convergence properties only when their structural integrity is maintained.

This principle fundamentally differs from classical approaches in several ways:

1. While Fuchs’ theorem traditionally focuses on the relationship between singular points and series convergence, our principle reveals that the structural nature of recurrence terms themselves plays a crucial role in determining convergence properties.

2. The Poincaré-Perron theorem assumes implicitly that term rearrangement does not affect convergence behavior. Our principle demonstrates that this assumption, while valid for second-order recurrence relations, fails for higher-order cases due to the intrinsic structural dependencies between terms.

3. Classical analysis treats recurrence terms primarily as computational elements. The Principle of Indivisible Integrity recognizes them as fundamental structural units whose integrity must be preserved to maintain the theoretical predictions of classical theorems.

This new perspective not only resolves longstanding inconsistencies in the application of classical theorems but also provides a unified framework for understanding how structural properties influence convergence behavior in differential equations. By recognizing the indivisible nature of recurrence terms, we establish a more precise and theoretically sound basis for analyzing power series solutions.

In this paper, we introduce the Principle of Indivisible Integrity, which asserts that in higher-order differential equations, the fundamental recurrence terms must be treated as indivisible entities to ensure the preservation of the expected radius of convergence. Specifically, we demonstrate that while second-order recurrence relations exhibit absolute convergence under arbitrary term rearrangements, higher-order recurrence relations involve coefficients formed from multiple recurrence terms. When these subcomponents are rearranged, the radius of convergence may be reduced, revealing a structural limitation in the classical framework of power series solutions.

To address this issue, we establish structural constraints governing the convergence properties of power series solutions in higher-order differential equations. These constraints clarify the conditions under which convergence remains stable and provide a refined perspective on the interplay between recurrence structure and power series behavior. We apply this framework to Heun-type differential equations, illustrating the practical consequences of these constraints in both analytical and numerical settings.

This paper explores the following key questions:

(1) How does the Principle of Indivisible Integrity redefine the structural constraints necessary for ensuring convergence in higher-order differential equations?

(2) What mathematical conditions must be imposed to preserve the radius of convergence when dealing with multi-term recurrence relations?

By extending the classical framework of power series convergence, this work offers a novel structural interpretation of recurrence relations in differential equations, refining existing methods and revealing previously unrecognized dependencies.
2. Motivation

The classical understanding of power series convergence has been built upon the assumption that term rearrangement does not alter the radius of convergence. While this assumption holds for second-order recurrence relations, it does not necessarily apply to higher-order cases, where recurrence coefficients are composed of multiple interacting terms.

This paper challenges this assumption by introducing the Principle of Indivisible Integrity, which states that in higher-order differential equations, the radius of convergence can be affected if the fundamental recurrence terms are decomposed and rearranged. Unlike second-order cases, where each term remains independent, higher-order recurrence relations inherently introduce structural dependencies between coefficients. By establishing structural constraints on absolute convergence, we clarify how implicit term rearrangement affects power series stability.

This principle extends classical power series analysis by identifying previously unrecognized constraints that govern the convergence behavior of solutions to higher-order differential equations. Our findings highlight the necessity of treating recurrence terms as indivisible units to preserve the expected analytical properties of power series expansions.

Below, we mathematically prove how term rearrangement affects the radius of convergence and demonstrate the necessity of the no-resummation constraint.

3. Proof of the Breakdown of Absolute Convergence in Higher-Order Recurrence Relations

In second-order recurrence relations, the absolute convergence of power series ensures that the radius of convergence remains invariant under term rearrangement. However, in higher-order recurrence relations, this property no longer holds due to the internal structure of the recurrence coefficients.

To formalize this distinction, we introduce the notion of an Indivisible Term, denoted as $d_i$, which is not a simple term but rather a composite structure formed by multiple recurrence coefficients $A_n, B_m, \ldots$. Specifically,

$$d_i = \sum_{i=0}^{j} \prod_{l=0}^{k} A_i B_j \cdots.$$

While absolute convergence guarantees stability under term rearrangement, we demonstrate that rearranging the internal components of $d_i$—a process we refer to as Internal Component Rearrangement—results in a reduction of the radius of convergence.

Definition: (Breakdown of Absolute Convergence in Higher-Order Recurrence Relations) A power series solution is said to exhibit a breakdown of absolute convergence if the rearrangement of its internal components leads to a contraction in the radius of convergence, despite satisfying the recurrence relation. This necessitates treating each $d_i$ as an Indivisible Term, ensuring that structural integrity is preserved.

By analyzing the minimal domain under internal component rearrangement, we establish that structural constraints must be imposed to maintain the original convergence behavior predicted by classical theorems such as Fuchs’ theorem. These
findings highlight the fundamental distinction between second-order and higher-order recurrence relations in terms of convergence stability.

**Note 3.1.** For a fixed $k \in \mathbb{N}$, there is the $(k + 1)$-term recurrence relation with constant coefficients such as

\begin{equation}
\tag{2}
c_{k+1,n+1} = \alpha_1 c_{k+1,n} + \alpha_2 c_{k+1,n-1} + \alpha_3 c_{k+1,n-2} + \cdots + \alpha_k c_{k+1,n-k+1}
\end{equation}

with seed values $c_{k+1,j} = \sum_{i=1}^{j} \alpha_i c_{k+1,j-i}$ where $j = 1, 2, 3, \cdots, k-1$ & $k \in \mathbb{N} - \{1\}$. And $c_{k+1,0} = 1$ is chosen for simplicity from now on. The generating function of the sequence of (2) is given by

\begin{equation}
\tag{3}
y(x) = \sum_{n=0}^{\infty} c_{k+1,n} x^n = \sum_{r=0}^{k} \left( \sum_{j=1}^{r} \alpha_j x^j \right)
\end{equation}

The domain of absolute convergence of (3) is written by

\begin{equation}
\tag{4}
D := \left\{ x \in \mathbb{C} \mid \sum_{j=1}^{k} |\alpha_j x^j| < 1 \right\}
\end{equation}

**Proof.** See Section 3.3 in Flajolet and Sedgewick [17]. A summation series expansion of (3) is

\begin{equation}
\tag{5}
\sum_{n=0}^{\infty} c_{k+1,n} x^n = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{(i_1 + i_2 + \cdots + i_k)!}{i_1! i_2! \cdots i_k!} (\alpha_1 x)^{i_1} (\alpha_2 x^2)^{i_2} \cdots (\alpha_k x^k)^{i_k}
\end{equation}

A real (or complex) series $\sum_{n=0}^{\infty} u_n$ is referred to converge absolutely if the series of moduli $\sum_{n=0}^{\infty} |u_n|$ converge. And the series of absolute values (5) is

\begin{equation}
\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{(i_1 + i_2 + \cdots + i_k)!}{i_1! i_2! \cdots i_k!} |\alpha_1 x|^{i_1} |\alpha_2 x^2|^{i_2} \cdots |\alpha_k x^k|^{i_k} = \sum_{r=0}^{k} \left( \sum_{j=1}^{r} |\alpha_j x^j| \right)^r
\end{equation}

This $k$-tuple series is absolutely convergent for $\sum_{j=1}^{k} |\alpha_j x^j| < 1$. \hfill \Box

A Fuchsian differential equation of order $j$ with variable coefficients is of the form

\begin{equation}
\tag{6}
a_j(x) y^{(j)} + a_{j-1}(x) y^{(j-1)} + \cdots + a_1(x) y' + a_0(x) y = 0
\end{equation}

Assuming its solution as a Frobenius series in the form

\begin{equation}
\tag{7}
y(x) = x^{\lambda} \sum_{n=0}^{\infty} d_n x^n
\end{equation}

where $d_0 = 1$ chosen for simplicity from now on. $\lambda$ is an indicial root. For a fixed $k \in \mathbb{N}$, we suggest that we obtain the $(k + 1)$-term recurrence relation putting (7) in (6)

\begin{equation}
\tag{8}
d_{n+1} = \alpha_1 d_n + \alpha_2 d_{n-1} + \alpha_3 d_{n-2} + \cdots + \alpha_k d_{n-k+1}
\end{equation}

with seed values $d_j = \sum_{i=1}^{j} \alpha_{i,j-1} d_{j-i}$ where $j = 1, 2, 3, \cdots, k-1$ & $k \in \mathbb{N} - \{1\}$. \hfill \Box
Theorem 3.2. To ensure the coefficients stabilize asymptotically, we assume \( \lim_{n \to \infty} \alpha_{l,n} = \alpha_l < \infty \) in \( \mathbb{R} \), the minimal domain under internal component rearrangement for (7) is given by:

\[
D := \left\{ x \in \mathbb{C} \left| \sum_{m=1}^{k} |\alpha_m x^m| < 1 \right. \right\}
\]

where \( D \) represents the minimal domain under internal component rearrangement.

Theorem 3.3. To ensure the coefficients stabilize asymptotically, we assume \( \lim_{n \to \infty} \alpha_{l,n} = \alpha_l < \infty \) in \( \mathbb{R} \), the minimal domain under internal component rearrangement for (7) is given by:

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\]

where \( D \) represents the minimal domain under internal component rearrangement.

Proof. As shown in the introduction, the concept of absolute Convergence plays a crucial role in determining the minimal domain under internal component rearrangement. This theorem formalizes the relationship between term rearrangement and the radius of convergence. Here, absolute convergence refers to the altered convergence domain after term rearrangement. To prove this for all \( k \in \mathbb{N} \), consider the \((k+1)\)-term recurrence relation where \( k = 1, 2, 3, \ldots \). Assuming \( \lim_{n \to \infty} \alpha_{l,n} = \alpha_l < \infty \), let \( \alpha_{l,n} = \alpha_l \overline{\alpha_{l,n}} \).

Here, \( \overline{\alpha_{l,n}} \) represents a normalized coefficient satisfying \( |\overline{\alpha_{l,n}}| < 1 + \epsilon \) for \( n \geq N \) and any positive error bound \( \epsilon \). Let \( \tilde{\alpha}_l = (1 + \epsilon)\alpha_l \) for simplicity. Following the cases for 2-term, 3-term, and 4-term recurrence relations, we generalize the minimal domain under internal component rearrangement condition for a \((k+1)\)-term recurrence relation as follows.

Case A: The 2-term recurrence relation with non-constant coefficients of a linear ODE. In general, 2-term recurrence relation putting (7) in (6) is given by

\[
d_{n+1} = \alpha_{1,n} d_n
\]

If \( |\overline{\alpha_{1,n}}| < 1 + \epsilon \) when \( n \geq N \), then the series of absolute values, \( 1 + |d_1||x| + |d_2||x|^2 + |d_3||x|^3 + \cdots \), is dominated by the convergent series

\[
1 + |d_1||x| + |d_2||x|^2 + |d_3||x|^3 + \cdots + |d_N||x|^N \left\{ 1 + |\tilde{\alpha}_1||x| + |\tilde{\alpha}_1|^2||x|^2 + |\tilde{\alpha}_1|^3||x|^3 + \cdots \right\}
\]

\[
= \sum_{n=0}^{N-1} |d_n||x|^n + \frac{|d_N||x|^N}{1 - |\tilde{\alpha}_1||x|}
\]

(12) is absolutely convergent for \( |\tilde{\alpha}_1||x| < 1 \). We know \( |\alpha_1||x| < |\tilde{\alpha}_1||x| \). Therefore, a series for the 2-term recurrence relation is convergent for \( |\alpha_1||x| < 1 \).
Case B: The 3-term recurrence relation with non-constant coefficients of a linear ODE. The 3-term recurrence relation putting (7) in (6) is given by

(13) \[ d_{n+1} = \alpha_{1,n} d_n + \alpha_{2,n} d_{n-1} \quad ; n \geq 1 \]

with seed values \( d_1 = \alpha_{1,0} \) and \( d_0 = 1 \). When \( n \geq N \), we have \( |\alpha_{1,n}|, |\alpha_{2,n}| < 1 + \epsilon \).

For \( n = N, N + 1, N + 2, \cdots \) in succession, take the modulus of the general term of \( d_{n+1} \) in (13)

(14) \[ |d_{N+1+j}| = |\alpha_{1,N+j}| |d_{N+j}| + |\alpha_{2,N+j}| |d_{N-1+j}| \leq |\tilde{\alpha}_1| |d_{N+j}| + |\tilde{\alpha}_2| |d_{N-1+j}| \leq c_{3,j+1} |d_N| + c_{3,j} |\alpha_2| |d_{N-1}| \]

Here, \( c_{3,j} \) where \( j \in \mathbb{N}_0 \) is a generated coefficient of the 3-term recurrence relation in (2) with taking \( \alpha_1 \to |\tilde{\alpha}_1| \) and \( \alpha_2 \to |\tilde{\alpha}_2| \).

According to (14), then the series of absolute values, \( 1 + |d_1||x| + |d_2||x|^2 + |d_3||x|^3 + \cdots \), is dominated by the convergent series

(15) \[ \sum_{n=0}^{N-1} |d_n||x|^n + |d_N||x|^N + \sum_{k=0}^\infty (c_{3,k+1} |d_N| + c_{3,k} |\tilde{\alpha}_2| |d_{N-1}|) |x|^{N+1+k} \]

\[ = \sum_{n=0}^{N-1} |d_n||x|^n + (|d_N| + |\tilde{\alpha}_2| |d_{N-1}| |x|) |x|^N \sum_{i=0}^\infty c_{3,i} |x|^i \]

From the generating function for the 3-term recurrence relation in (3), we conclude

(16) \[ \sum_{i=0}^\infty c_{3,i} |x|^i = \sum_{r=0}^\infty \left( |\tilde{\alpha}_1|x| + |\tilde{\alpha}_2|x|^2 \right)^r \]

(16) is absolute convergent for \( |\tilde{\alpha}_1|x| + |\tilde{\alpha}_2|x|^2 | < 1 \). We know \( |\alpha_1 x^l| < |\tilde{\alpha}_1 x^l| \) where \( l = 1, 2 \). Therefore, a series for the 3-term recurrence relation is convergent for \( |\alpha_1 x| + |\alpha_2 x^2| < 1 \).

Case C: The 4-term recurrence relation with non-constant coefficients of a linear ODE. The 4-term recurrence relation putting (7) in (6) is given by

(17) \[ d_{n+1} = \alpha_{1,n} d_n + \alpha_{2,n} d_{n-1} + \alpha_{3,n} d_{n-2} \quad ; n \geq 2 \]

with seed values \( d_1 = \alpha_{1,0} \), \( d_2 = \alpha_{1,0}\alpha_{1,1} + \alpha_{2,1} \) and \( d_0 = 1 \). When \( n \geq N \), we have \( |\alpha_{1,n}|, |\alpha_{2,n}|, |\alpha_{3,n}| < 1 + \epsilon \).

For \( n = N, N + 1, N + 2, \cdots \) in succession, take the modulus of the general term of \( d_{n+1} \) in (17)

(18) \[ |d_{N+1+j}| = |\alpha_{1,N+j}| |d_{N+j}| + |\alpha_{2,N+j}| |d_{N-1+j}| + |\alpha_{3,N+j}| |d_{N-2+j}| \leq |\tilde{\alpha}_1| |d_{N+j}| + |\tilde{\alpha}_2| |d_{N-1+j}| + |\tilde{\alpha}_3| |d_{N-2+j}| \leq c_{4,j+1} |d_N| + (c_{4,j+2} - c_{4,j+1}|\tilde{\alpha}_1|) |d_{N-1}| + c_{4,j} |\alpha_3| |d_{N-2}| \]

Here, \( c_{4,j} \) where \( j \in \mathbb{N}_0 \) is a generated coefficient of the 4-term recurrence relation in (2) with taking \( \alpha_l \to |\tilde{\alpha}_l| \) where \( l = 1, 2, 3 \).
According to (18), then the series of absolute values, \(1 + |d_1||x| + |d_2||x|^2 + |d_3||x|^3 + \cdots\), is dominated by the convergent series

\[
(19) \sum_{n=0}^{N-1} |d_n||x|^n + |d_N||x|^N + \sum_{k=0}^{\infty} \left( c_{4,k+1}|d_N| + (c_{4,k+2} - c_{4,k+1}|\tilde{\alpha}_1|)|d_{N-1}| + c_{4,k}|\tilde{\alpha}_3||d_{N-2}| \right)|x|^{N+1+k}
\]

\[
\leq \sum_{n=0}^{N-1} |d_n||x|^n + |d_N||x|^N + \sum_{k=0}^{\infty} (c_{4,k+1}|d_N| + c_{4,k+2}|d_{N-1}| + c_{4,k}|\tilde{\alpha}_3||d_{N-2}|)|x|^{N+1+k}
\]

\[
\leq \sum_{n=0}^{N-1} |d_n||x|^n + (|\tilde{\alpha}_3||d_{N-2}||x|)|x|^N \sum_{i=0}^{\infty} c_{4,i}|x|^i + |d_{N-1}||x|^{N-1} \sum_{i=2}^{\infty} c_{4,i}|x|^i
\]

From the generating function for the 4-term recurrence relation in (3), we conclude

\[
(20) \sum_{i=0}^{\infty} c_{4,i}|x|^i = \sum_{r=0}^{\infty} \left( |\tilde{\alpha}_1|x| + |\tilde{\alpha}_2|x|^2 + |\tilde{\alpha}_3|x|^3 \right)^r
\]

(20) is absolute convergent for \( |\tilde{\alpha}_1|x| + |\tilde{\alpha}_2|x|^2 + |\tilde{\alpha}_3|x|^3 \) \( < 1 \). We know \( |\alpha_1|x| < |\tilde{\alpha}_1|x| \) where \( l = 1, 2, 3 \). Therefore, a series for the 4-term recurrence relation is convergent for \( |\alpha_1|x| + |\alpha_2|x|^2 + |\alpha_3|x|^3 \) \( < 1 \).

**Case D:** The 5-term recurrence relation with non-constant coefficients of a linear \( ODE \). The 5-term recurrence relation putting (7) in (6) is given by

\[
(21) d_{n+1} = \alpha_{1,n} d_n + \alpha_{2,n} d_{n-1} + \alpha_{3,n} d_{n-2} + \alpha_{4,n} d_{n-3} \quad ; n \geq 3
\]

with seed values \( d_1 = \alpha_{1,0}, d_2 = \alpha_{1,0}\alpha_{1,1} + \alpha_{2,1}, d_3 = \alpha_{1,0}\alpha_{1,1}\alpha_{1,2} + \alpha_{1,0}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,1} + \alpha_{3,2} \) and \( d_0 = 1 \). When \( n \geq N \), we have \( |\alpha_1,n|, |\alpha_{1,2}|, |\alpha_{3,n}|, |\alpha_{4,n}| < 1 + \epsilon \).

For \( n = N, N+1, N+2, \cdots \) in succession, take the modulus of the general term of \( d_{n+1} \) in (21)

\[
(22) |d_{N+1+j}| = |\alpha_{1,N+j}|d_{N+j}| + |\alpha_{2,N+j}|d_{N-1+j}| + |\alpha_{3,N+j}|d_{N-2+j}| + |\alpha_{4,N+j}|d_{N-3+j}|
\]

\[
\leq |\tilde{\alpha}_1||d_{N+j}| + |\tilde{\alpha}_2||d_{N-1+j}| + |\tilde{\alpha}_3||d_{N-2+j}| + |\tilde{\alpha}_4||d_{N-3+j}|
\]

\[
\leq c_{5,j+1}|d_N| + (c_{5,j+2} - c_{5,j+1}|\tilde{\alpha}_1|)|d_{N-1}| + (c_{5,j+3} - (c_{5,j+2}|\tilde{\alpha}_1| + c_{5,j+1}|\tilde{\alpha}_2|))|d_{N-2}| + c_{5,j}|\tilde{\alpha}_4||d_{N-3}|
\]

Here, \( c_{5,j} \) where \( j \in N_0 \) is a generated coefficient of the 5-term recurrence relation in (2) with taking \( \alpha_l \to |\tilde{\alpha}_l| \) where \( l = 1, 2, 3, 4 \).
According to (22), then the series of absolute values, \(1 + |d_1||x| + |d_2||x|^2 + |d_3||x|^3 + \cdots\), is dominated by the convergent series
\[
\sum_{n=0}^{N-1} |d_n||x|^n + |d_N||x|^N + \sum_{k=0}^{\infty} \left( c_{5,k+1}|d_N| + (c_{5,k+2} - c_{5,k+1}|\tilde{\alpha}_1|) |d_{N-1}| + (c_{5,k+3} - (c_{5,k+2}|\tilde{\alpha}_1| + c_{5,k+1}|\tilde{\alpha}_2|)) |d_{N-2}| + c_{5,k}|\tilde{\alpha}_4||d_{N-3}| \right) |x|^{N+1+k} \]
\[
\leq \sum_{n=0}^{N-1} |d_n||x|^n + |d_N||x|^N + \sum_{k=0}^{\infty} \left( c_{5,k+1}|d_N| + c_{5,k+2}|d_{N-1}| + c_{5,k+3}|d_{N-2}| + c_{5,k}|\tilde{\alpha}_4||d_{N-3}| \right) |x|^{N+1+k} \]
\[
\leq \sum_{n=0}^{N-1} |d_n||x|^n + (|d_N| + |\tilde{\alpha}_4||d_{N-3}||x|) |x|^N \sum_{i=0}^{\infty} c_{5,i}|x|^i + |d_{N-1}||x|^{N-1} \sum_{i=2}^{\infty} c_{5,i}|x|^i + |d_{N-2}||x|^{N-2} \sum_{i=3}^{\infty} c_{5,i}|x|^i \]

From the generating function for the 5-term recurrence relation in (3), we conclude
\[
\sum_{i=0}^{\infty} c_{5,i}|x|^i = \sum_{r=0}^{\infty} \left( |\tilde{\alpha}_1x| + |\tilde{\alpha}_2x^2| + |\tilde{\alpha}_3x^3| + |\tilde{\alpha}_4x^4| \right)^r \]

(23) is absolute convergent for \(|\tilde{\alpha}_1x| + |\tilde{\alpha}_2x^2| + |\tilde{\alpha}_3x^3| + |\tilde{\alpha}_4x^4| < 1\). We know \(|\tilde{\alpha}_lx^l| < |\tilde{\alpha}_lx^l|\) where \(l = 1, 2, 3, 4\). Therefore, a series for the 5-term recurrence relation is convergent for \(|\tilde{\alpha}_1x| + |\tilde{\alpha}_2x^2| + |\tilde{\alpha}_3x^3| + |\tilde{\alpha}_4x^4| < 1\).

By the principle of mathematical induction (by repeating similar process for the previous cases such as two, three, four and five term recurrence relations), the series of absolute values for the \((k+1)\)-term recursive relation of a linear ODE where \(k \in \{3, 4, 5, \cdots \}, 1 + |d_1||x| + |d_2||x|^2 + |d_3||x|^3 + \cdots\), is dominated by the convergent series
\[
\sum_{n=0}^{N-1} |d_n||x|^n + \left( |d_N| + |\tilde{\alpha}_4||d_{N-k+1}||x| \right) |x|^N \sum_{j=0}^{k-3} c_{k+1,j}|x|^j + \sum_{i=0}^{k-3} |d_{N-1-i}||x|^{N-1-i} \sum_{j=i+2}^{\infty} c_{k+1,j}|x|^j \]

Here, \(c_{k+1,j}\) where \(j \in \mathbb{N}_0\) is a generated coefficient of the \((k+1)\)-term recurrence relation in (2) with taking \(\alpha_l \rightarrow |\tilde{\alpha}_l|\) where \(l = 1, 2, 3, \cdots, k\). From the generating function for the \((k+1)\)-term recurrence relation in (3), we conclude
\[
\sum_{j=0}^{\infty} c_{k+1,j}|x|^j = \sum_{r=0}^{\infty} \left( \sum_{m=1}^{k} |\tilde{\alpha}_m x^m| \right)^r \]

(25) is minimal absolute convergence domain under internal component rearrangement for \(\left( \sum_{m=1}^{k} |\tilde{\alpha}_m x^m| \right) < 1\). We know \(|\alpha_l x^l| < |\tilde{\alpha}_l x^l|\) where \(l = 1, 2, 3, \cdots, k\).
Therefore, a series for the \((k+1)\)-term recurrence relation of a Fuchsian differential equation is convergent for \(\sum_{m=1}^{k} |\alpha_m x^m| < 1\). □ □

This result provides a rigorous mathematical foundation for understanding the fundamental structural limitations of Fuchs’ theorem in power series analysis. It highlights the necessity of the Indivisible Integrity Principle, which asserts that the intrinsic structure of recurrence terms must be preserved to maintain a stable radius of convergence. By explicitly prohibiting internal component rearrangement, this principle ensures that the radius of convergence remains consistent with classical predictions, even in the presence of complex recurrence relations or perturbative expansions.

The Indivisible Integrity Principle bridges a critical gap between classical power series analysis and modern computational frameworks, where implicit structural modifications often lead to unintended variations in convergence behavior. By formalizing this principle, we extend the theoretical applicability of Fuchs’ theorem to a broader class of power series expansions, ensuring structural consistency across analytical, asymptotic, and numerical methodologies. Beyond addressing a previously overlooked limitation in conventional analysis, this work establishes a mathematically rigorous framework for preserving the structural coherence of power series solutions in various domains, including differential equations, function theory, and applied mathematics.

4. APPLICATION TO THE HEUN’S DIFFERENTIAL EQUATION

The Heun equation, the most general Fuchsian equation with four regular singularities, appears in various applications of modern physics such as quantum gravity, general relativity, and molecular spectroscopy. The solutions to this equation often involve three-term recurrence relations, contrasting with the two-term recurrence relations used in hypergeometric functions. This increased complexity highlights the necessity for a more refined approach to analyzing convergence properties.

Historically, the Poincaré–Perron (P–P) theorem has been employed to construct the radius of convergence for power series solutions of Fuchsian equations, including the Heun equation. However, this study demonstrates that the radius derived through the P–P theorem represents only conditional convergence and fails to account for absolute convergence properties. Specifically, numerical results reveal that the radius derived using Thm 3.3 is consistently smaller than that obtained via the P–P theorem. This reduction occurs because the P–P theorem implicitly assumes non-rearranged coefficients, whereas rearranging coefficients affects the convergence properties.

Using Thm 3.3, we construct the radius of absolute convergence for the power series solution of the Heun equation and identify key differences in the domains of convergence. Furthermore, the study elucidates why the P–P theorem does not capture absolute convergence by revisiting its foundational assumptions. As a result, this new approach provides a more accurate framework for understanding the behavior of Fuchsian equations with multi-term recurrence relations, particularly for modern physical applications that require higher-order differential equations.

The Heun’s equation is a second-order linear ODE of the form [7]:

\[
\frac{d^2y}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a} \right) \frac{dy}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)} y = 0
\]

(26)
with the condition $\varepsilon = \alpha + \beta - \gamma - \delta + 1$. The parameters have distinct roles: $a \neq 0$ is the singularity parameter; $\alpha$, $\beta$, $\gamma$, $\delta$, and $\varepsilon$ are the exponent parameters; and $q$ is the accessory parameter. Additionally, $\alpha$ and $\beta$ are symmetric. The total number of free parameters is six. The equation features four regular singular points, located at $0, 1, a$, and $\infty$, with corresponding exponents $\{0, 1 - \gamma\}$, $\{0, 1 - \delta\}$, $\{0, 1 - \varepsilon\}$, and $\{\alpha, \beta\}$.

We have adopted the restriction $|a| > 1$ for the local Heun function about $x = 0$. However, in this study, we consider $a \in \mathbb{R}$ or $a \in \mathbb{C}$ to observe how the radius of convergence for the local Heun function varies. Assume that $y(x)$ has a series expansion of the form:

$$y(x) = x^\lambda \sum_{n=0}^{\infty} d_n x^n$$

Substituting (27) into (26):

$$d_{n+1} = \alpha_{1,n} d_n + \alpha_{2,n} d_{n-1} \quad : n \geq 1$$

with $\alpha_{1,n} = \frac{1}{\alpha_{1,n}}$ and $\alpha_{2,n} = \frac{1}{\alpha_{2,n}}$:

$$\alpha_1 = 1 + a$$

$$\alpha_2 = \frac{1}{a}$$

$$\alpha_{1,n} = \frac{n^2 + \gamma + \varepsilon + 1 + 2\lambda + \alpha(\gamma + \delta - 1 + 2\lambda)}{n^2 + (1 + \gamma + 2\lambda) n + (1 + \lambda)(\gamma + \lambda) + \frac{\lambda(\gamma + \varepsilon + 1 + 2\lambda)}{1 + a}}$$

$$\alpha_{2,n} = \frac{n^2 + \alpha + \beta - \delta + 2\lambda + \alpha(\gamma + \delta - 1 + 2\lambda)}{n^2 + (1 + \gamma + 2\lambda) n + (1 + \lambda)(\gamma + \lambda) + \frac{\lambda(\alpha + \beta - \delta + 2\lambda + \alpha(\gamma + \delta - 1 + 2\lambda))}{1 + a}}$$

$$d_1 = \alpha_{1,0} d_0 = \alpha_{2,0} \alpha_{1,0} d_0$$

We have two indicial roots, which are $\lambda = 0$ and $1 - \gamma$.

4.1. P–P Theorem and Its Applications for Solutions of Power Series. This section reviews the asymptotic behavior of solutions to linear difference equations with constant coefficients. Consider a linear recurrence relation of order $k + 1$ with constant coefficients $a_i$, where $i = 0, 1, 2, \ldots, k$:

$$(30) \quad u(n + 1) + \alpha_1 u(n) + \alpha_2 u(n - 1) + \alpha_3 u(n - 2) + \cdots + \alpha_k u(n - k + 1) = 0$$

where $\alpha_k \neq 0$.

The characteristic polynomial of the recurrence relation (30) is given by:

$$(31) \quad \rho^k + \alpha_1 \rho^{k-1} + \alpha_2 \rho^{k-2} + \cdots + \alpha_k = 0$$

The roots of the characteristic equation (31) are denoted by $\lambda_1, \ldots, \lambda_k$.

In 1885, H. Poincaré stated that

$$\lim_{n \to \infty} \frac{u(n + 1)}{u(n)}$$
is equal to one of the roots of the characteristic equation \[14\]. This result was extended by O. Perron in 1921 \[13\]: The Poincaré–Perron theorem assumes conditional convergence and does not guarantee absolute convergence, which is necessary to preserve structural invariance and uniqueness.

**Theorem 4.1 (P–P Theorem \[11\]).** If the coefficient of \(u(n)\) in a difference equation of order \(k\) is nonzero for \(n = 0, 1, 2, \ldots\), then the equation has \(k\) fundamental solutions \(u_1(n), \ldots, u_k(n)\) such that:

\[
\lim_{n \to \infty} \frac{u_i(n + 1)}{u_i(n)} = \lambda_i
\]

where \(i = 1, 2, \ldots, k\), \(\lambda_i\) are the roots of the characteristic equation, and \(n \to \infty\) in positive integer increments.

A recurrence relation for the coefficients emerges when substituting a series \(y(x) = \sum_{n=0}^{\infty} d_n x^n\) into a Fuchsian equation. In general, a three-term recurrence relation is expressed as:

\[
d_{n+1} = \alpha_{1,n} d_n + \alpha_{2,n} d_{n-1} \quad ; \quad n \geq 1
\]

with initial values \(d_1 = \alpha_{1,0} d_0\). For the asymptotic behavior of \(33\), \(\lim_{n \to \infty} \alpha_{j,n} = \alpha_j < \infty\) \((j = 1, 2)\) exists.

The asymptotic recurrence relation is then:

\[
\tilde{d}_{n+1} = \alpha_1 \tilde{d}_n + \alpha_2 \tilde{d}_{n-1} \quad ; \quad n \geq 1
\]

where \(\tilde{d}_1 = \alpha_1 \tilde{d}_0\) and \(\tilde{d}_0 = 1\). From the P–P theorem, the characteristic polynomial of the recurrence relation is:

\[
\rho^2 - \alpha_1 \rho - \alpha_2 = 0
\]

The roots of the polynomial \(35\) have two moduli:

\[
\rho_1 = \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4 \alpha_2}}{2}, \quad \rho_2 = \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4 \alpha_2}}{2}
\]

Thus, \(\lim_{n \to \infty} |d_{n+1}/d_n| = \lim_{n \to \infty} |\tilde{d}_{n+1}/\tilde{d}_n|\).

In general:

- If \(|\rho_1| < |\rho_2|\), then \(\lim_{n \to \infty} |\tilde{d}_{n+1}/\tilde{d}_n| = |\rho_2|\), and the radius of convergence is \(|\rho_2|^{-1}\).
- If \(|\rho_2| < |\rho_1|\), then \(\lim_{n \to \infty} |\tilde{d}_{n+1}/\tilde{d}_n| = |\rho_1|\), and the radius of convergence is \(|\rho_1|^{-1}\).

In special cases:

- If \(|\rho_1| = |\rho_2|\) and \(\rho_1 \neq \rho_2\), the series diverges.
- If \(\rho_1 = \rho_2\), the series converges.

More details are provided in Appendix B of Part A \[16\], Wimp (1984) \[21\], Kristensson (2010) \[8\], and Erdélyi (1955) \[4\].

Table 1 lists all possible boundary conditions of a local Heun function at \(x = 0\), corresponding to the exponent zero, where \(a, x \in \mathbb{R}\). These are obtained by applying the P–P theorem with \(\alpha_1 = \frac{1 + a}{x}\) and \(\alpha_2 = -\frac{1}{x}\) in \(36\).

Figure 1 represents the convergence domain of the series for the Heun equation at \(x = 0\). The shaded region indicates convergence, excluding the dotted lines.
Finally, the power series solution derived using the P–P theorem is expressed as:

\[ y^P(x) = \sum_{n=0}^{\infty} d_n x^\lambda \]

\[ = x^\lambda \left( 1 + \alpha_{1,0} x + (\alpha_{2,1} + \alpha_{1,0} \alpha_{1,1}) x^2 + \cdots \right) \]

(37)

4.2. Structural Modifications and Radius of Convergence in a Power Series with a Three-Term Recurrence Relation. By applying Thm. 3.3 the condition of absolute convergence for a local Heun function at \( x = 0 \), where \( \alpha_1 = (1 + a)/a \) and \( \alpha_2 = -1/a \), is given by:

\[ \left| \frac{1 + a}{a} x \right| + \left| -\frac{1}{a} x^2 \right| < 1 \]

(38)

The coefficient \( a \) determines the range of the independent variable \( x \) as shown in (38). The precise ranges of \( a \) and \( x \) are given in Table 2 where \( a, x \in \mathbb{R} \). The radius of convergence from Table 2 is depicted as the shaded region in Fig. 2. It does not include the dotted lines, no solution exists at the origin (black point), and the maximum modulus of \( x \) is unity.

The dotted boundary lines for the shaded area where \( a > 0 \) in Fig. 2 are determined by:

\[ \lim_{a \to N} \frac{-1 - a + \sqrt{a^2 + 6a + 1}}{2} \sim 1, \]

(39)
where \( N \) is a sufficiently large positive real number. Hence, it can be argued that \( |x| < 1 \) as \( a \to N \). For example, if \( a = 10 \), then \( |x| < 0.84429 \), and if \( a = 100 \), then \( |x| < 0.98058 \).

By rearranging the coefficients \( \alpha_{1,n} \) and \( \alpha_{2,n} \) in each sequence \( d_n \) in (28), where \( d_0 = 1 \) for simplicity, a local Heun series solution can be expressed as:

\[
y^A(x) = x^\lambda \left( y^A_0(x) + y^A_1(x) \eta + \sum_{\tau=2}^{\infty} y^A_\tau(x) \eta^\tau \right),
\]

where

\[
y^A_0(x) = \sum_{i_0=0}^{\infty} \prod_{i_1=0}^{i_0-1} \alpha_{1,2i_0} \prod_{i_2=0}^{i_2+1} \alpha_{2,2i_2+1} z^{i_0},
\]

\[
y^A_1(x) = \sum_{i_0=0}^{\infty} \prod_{i_1=0}^{i_0-1} \alpha_{1,2i_0} \prod_{i_2=0}^{i_2+1} \alpha_{2,2i_2+1} z^{i_0},
\]

\[
y^A_\tau(x) = \sum_{\tau=2}^{\infty} \left\{ \sum_{i_\tau=0}^{\infty} \prod_{i_2=0}^{i_\tau-1} \alpha_{1,2i_\tau} \prod_{i_3=0}^{i_3+1} \alpha_{2,2i_3+1} \prod_{k=1}^{\tau-1} \left( \sum_{i_{2k}=i_{2(k-1)}}^{\infty} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k+1}+k} \frac{\alpha_{1,2i_{2k+1}+k}}{\alpha_{2,2i_{2k+1}+k+1}} \right) \right\} \times \prod_{i_{2\tau}=i_{2(\tau-1)}}^{i_{2\tau+1}+\tau+1} \alpha_{2,2i_{2\tau+1}+\tau+1} z^{i_\tau} \right\},
\]

Additionally, the variables \( \eta \) and \( z \) are defined as:

\[
\eta = \frac{1 + a}{a} x, \quad z = -\frac{1}{a} x^2.
\]
The sequence $d_n$ is a combination of $\alpha_{1,n}$ and $\alpha_{2,n}$ terms in (28). The series in (40) is constructed by letting $\alpha_{1,n}$ in $d_n$ be the leading term, observing terms in $d_n$ that include:

- Zero terms of $\alpha_{1,n}$ for the sub-power series $y_0^A(x)$,
- One term of $\alpha_{1,n}$ for $y_1^A(x)$,
- Two terms of $\alpha_{1,n}$ for $y_2^A(x)$,
- Three terms of $\alpha_{1,n}$ for $y_3^A(x)$, and so on.

4.3. Numerical Comparison of Absolute Convergence and Internal Component Rearrangement. When comparing Table 2 with Table 1, the boundary conditions for the radius of convergence are equivalent for $a < 0$. However, for $a \geq 1$, their ranges of $x$ differ significantly:

1. The radius of convergence is unity in Table 1 at $a = 1$, while Table 2 suggests it is approximately 0.414214.
2. In the region $0 < a < 1$, the maximum absolute value of $x$ differs significantly between Tables 1 and 2. For positive $x$, Table 2 describes a square root function of $a$, with a slope ranging between 0.207107 and 1. In contrast, Table 1 provides a linear increase with slope 1. For negative $x$, the square root function in Table 2 has a slope between $-\frac{1}{2}$ and $-\frac{1}{207107}$, whereas Table 1 specifies a slope of $-\frac{1}{2}$.
3. For large $a$, the square root function in Table 2 approaches $\pm 1$, validating the absolute Convergence theorem. However, for $0 < a < 1$, the absolute Convergence theorem no longer applies to constructing the radius of convergence for local Heun functions.

The differences between Tables 1 and 2 can also be analyzed numerically. A sequence $d_n$ is derived by inserting a power series into Heun's equation. The boundary condition of $x$ in Table 1 is obtained using the ratio $\lim_{n \to \infty} |d_{n+1}/d_n|$ in Eq. (34) while the radius of convergence in Table 2 is constructed using Theorem 3.1. For numerical computations, we set $\alpha_{1,n}$ and $\alpha_{2,n}$ to unity. The asymptotic recurrence relation for a local Heun function is given by:

\begin{equation}
 d_{n+1} = \alpha_1 d_n + \alpha_2 d_{n-1} = \frac{1 + a}{a} d_n - \frac{1}{a} d_{n-1}.
\end{equation}

A series solution based on the absolute Convergence theorem is:

\begin{equation}
 y(x) = \lim_{N \to \infty} \sum_{n=0}^{N} d_n x^n.
\end{equation}

If Eq. (43) converges absolutely, it can also be expressed as:

\begin{equation}
 y(x) = \lim_{N \to \infty} \sum_{n=0}^{N} \sum_{m=0}^{N} \frac{(n + m)!}{n!m!} \left(\frac{1 + a}{a} x\right)^n \left(-\frac{1}{a} x^2\right)^m.
\end{equation}

For $a = 0.8$, the boundary condition in Table 2 is approximately $-0.368858 < x < 0.368858$, while the radius of convergence in Table 1 is $-0.8 < x < 0.8$. Substituting $a = 0.8$ and $x = 0.3$ or $x = 0.7$ into Eq. (43) and Eq. (44), the numerical results are presented in Table 3.

For $x = 0.7$, the table shows that $y(x) \approx 26.6667$ as $N \to \infty$, indicating convergence in Eq. (43), whereas Eq. (44) demonstrates divergence. This discrepancy highlights the limitation of the absolute Convergence theorem for larger $x$ values.
In summary, Table 3 illustrates that the absolute Convergence theorem cannot reliably construct the radius of convergence for all cases, and the radius of convergence under internal component rearrangement provides a more accurate description. Figure 3 visualizes the domains of convergence.

**Table 3. Numerical Results for $y(x)$ in Different Cases**

| N  | $y(x)$ in (43) $(a = 0.8, \ x = 0.3)$ | $y(x)$ in (44) $(a = 0.8, \ x = 0.3)$ | $y(x)$ in (43) $(a = 0.8, \ x = 0.7)$ | $y(x)$ in (44) $(a = 0.8, \ x = 0.7)$ |
|----|------------------------------------|------------------------------------|------------------------------------|------------------------------------|
| 10 | 2.28559427400                     | 2.276337892064                    | 17.726665066666                  | 1.00791 × 10^9                   |
| 50 | 2.285714285714                    | 2.285714285695                    | 26.62563574231                   | 1.34009 × 10^{28}               |
| 100| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 1.99922 × 10^{57}               |
| 200| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 6.25120 × 10^{115}              |
| 300| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 2.25372 × 10^{174}              |
| 400| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 8.61497 × 10^{232}              |
| 500| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 3.40062 × 10^{291}              |
| 600| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 1.36992 × 10^{350}              |
| 700| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 5.59670 × 10^{408}              |
| 800| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 2.31012 × 10^{467}              |
| 900| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 9.61056 × 10^{525}              |
|1000| 2.285714285714                   | 2.285714285714                    | 26.666666666666                  | 4.02305 × 10^{584}              |

**Figure 3.** Domains of convergence for absolute and the radius of convergence under internal component rearrangement.

### 4.4. Absolute Convergence and Internal Component Rearrangement

A curious case arises as to why the P–P theorem cannot directly construct the radius of convergence for certain power series solutions. This section explores this issue by analyzing the difference between absolute convergence (based on the P–P theorem) and internal component rearrangement (achieved by rearranging the terms of the series).
absolute convergence: The P–P theorem is fundamentally built by observing the ratio $\frac{d_{n+1}}{d_n}$ in the recurrence relation as $n \to \infty$. This ratio corresponds to one of the roots of the characteristic polynomial of the recurrence. The radius of convergence for a power series $\sum_{n=0}^{\infty} d_n x^n$ is determined by requiring that the modulus of a root of the characteristic equation, multiplied by $|x|$, is less than unity:

$$\lim_{n \to \infty} \frac{|d_{n+1}|}{|d_n|} |x| < 1. \quad (45)$$

For the local Heun function at $x = 0$, the recurrence relation is given by:

$$d_{n+1} = \alpha_1 d_n + \alpha_2 d_{n-1} = \frac{1 + a}{a} d_n - \frac{1}{a} d_{n-1}, \quad (46)$$

where $\alpha_1 = (1 + a)/a$ and $\alpha_2 = -1/a$. The closed-form solution for $d_n$ is given by:

$$d_n = -\left(\frac{1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{2}\right)^{n+1} + \left(\frac{1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{2}\right)^{n+1}. \quad (47)$$

To evaluate the radius of convergence, multiply $\lim_{n \to \infty} |\frac{d_{n+1}}{d_n}| |x|$ by $|x|$ and substitute (47):

$$L = \lim_{n \to \infty} \frac{|d_{n+1}|}{|d_n|} |x| = \lim_{n \to \infty} \left| \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}} \right| \left(1 - \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}\right)^{n+1} |x| < 1. \quad (48)$$

Depending on the ratio of roots in (48), two cases arise:

$$\text{If } \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}} < 1, \text{ then } L = \frac{1}{2} |\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}| |x| < 1, \quad (49)$$

$$\text{If } \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}} > 1, \text{ then } L = \frac{1}{2} |\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}| |x| < 1. \quad (50)$$

For the special case where $\alpha_1 = 2$ and $\alpha_2 = -1$, (48) simplifies to:

$$L = |x| < 1. \quad (51)$$

Discrepancy with internal component rearrangement: By substituting $\alpha_1 = (1 + a)/a$ and $\alpha_2 = -1/a$ into (49)–(51), we confirm that the radius of convergence for a local Heun function at $x = 0$ matches the values in Table 1. However, the radius of convergence obtained when internal component rearrangement is applied, as shown in Table 2, is smaller. This discrepancy arises because internal component rearrangement allows the terms $|d_n|$ to be split and reorganized, violating the structural invariance assumed in the P–P theorem.

Findings: While the P–P theorem provides a radius of convergence under the assumption of absolute convergence, it fails to account for the reduced radius observed when internal component rearrangement is applied. This highlights the necessity of imposing a no-rearrangement constraint to preserve the original radius of convergence, as demonstrated in this study.
4.5. **Analysis of Hypergeometric Series and Radius of Convergence.** A hypergeometric function is defined by the power series:

\[
\sum_{n=0}^{\infty} d_n x^n = 1 + \frac{ab}{c!} x + \frac{a(a+1)b(b+1)}{c(c+1)2!} x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!} x^3 + \cdots.
\]

This series consists of a two-term recurrence relation between successive coefficients. To determine the condition for absolute convergence of (52), we analyze the series of moduli \( \sum_{n=0}^{\infty} |d_n||x|^n \):

\[
\sum_{n=0}^{\infty} |d_n||x|^n = 1 + \frac{ab}{c!} |x| + \frac{|a(a+1)b(b+1)|}{c(c+1)2!} |x|^2 + \frac{|a(a+1)(a+2)b(b+1)(b+2)|}{c(c+1)(c+2)3!} |x|^3 + \cdots.
\]

Applying the ratio test to (53), the radius of convergence is obtained as:

\[
\lim_{n \to \infty} \left| \frac{(n+a)(n+b)}{(n+c)(n+1)} \right| |x| < 1.
\]

Thus, \( |x| < 1 \) is required for the absolute convergence of the hypergeometric series.

Asymptotic Expansion: For the local Heun function, an asymptotic series expansion of (46) is:

\[
\sum_{n=0}^{\infty} d_n x^n = 1 + \alpha_1 x + (\alpha_1^2 + \alpha_2) x^2 + (\alpha_1^3 + 2\alpha_1 \alpha_2) x^3 + \alpha_1^4 + 3\alpha_1^2 \alpha_2 + \alpha_2^2 x^4 + (\alpha_1^5 + 4\alpha_1^3 \alpha_2 + 3\alpha_1 \alpha_2^2) x^5 + \cdots.
\]

Structural Constraints in the P–P Theorem: The Poincaré–Perron (P–P) theorem traditionally provides a method for estimating the radius of convergence of a power series solution governed by a recurrence relation. However, this theorem assumes that the structural composition of recurrence terms remains invariant under any ordering.

Our analysis demonstrates that in higher-order recurrence relations, the internal components of the recurrence terms can undergo rearrangement, leading to a reduction in the radius of convergence. We refer to this phenomenon as **Internal Component Rearrangement**, which violates the conventional assumptions of the P–P theorem.

To address this issue, we impose a **Structural Constraint on the P–P Theorem**, explicitly prohibiting internal component rearrangement. This constraint ensures that the radius of convergence aligns with classical predictions, preserving the analytical validity of power series solutions. The necessity of this constraint highlights a fundamental limitation in the conventional application of the P–P theorem, reinforcing the role of structural coherence in determining convergence properties.

The P–P theorem is typically applied to the ratio \( \frac{d_{n+1}}{d_n} \) in the recurrence relation of a local Heun function at \( x = 0 \). However, this approach is flawed when determining the radius of convergence. Instead, we must explicitly consider
absolute values of all terms in (55):
\[
\sum_{n=0}^{\infty} |a_n|x^n = 1 + |\alpha_1|x + (|\alpha_1|^2 + |\alpha_2|)|x|^2 + (|\alpha_1|^3 + 2|\alpha_1||\alpha_2|)|x|^3 \\
+ (|\alpha_1|^4 + 3|\alpha_1|^2|\alpha_2| + |\alpha_2|^2)|x|^4 \\
+ (|\alpha_1|^5 + 4|\alpha_1|^3|\alpha_2| + 3|\alpha_1||\alpha_2|^2)|x|^5 + \cdots .
\]
(56)

To construct the radius of convergence, we replace \( \alpha \to |\alpha_1| = |(1 + a)/a| \) and \( \alpha_2 \to |\alpha_2| = |1/a| \) in (49)–(50). The radius of convergence for the local Heun function at \( x = 0 \) then aligns with Table 2 except in the case of \( a = -1 \). This discrepancy arises because the P–P theorem assumes conditional convergence, whereas the absolute values of the coefficients in (56) must be considered for absolute convergence.

Special Case \( a = -1 \): For \( a = -1 \), substituting \( \alpha_1 \to |\alpha_1| = |(1 + a)/a| \) and \( \alpha_2 \to |\alpha_2| = |1/a| \) into (48) yields:
\[
\lim_{n \to \infty} \left| \frac{|A| - \sqrt{|A|^2 + 4|B|}}{2} - \frac{|A| + \sqrt{|A|^2 + 4|B|}}{2} \left( \frac{|A| + \sqrt{|A|^2 + 4|B|}}{|A| - \sqrt{|A|^2 + 4|B|}} \right)^{n+1} \right| = \lim_{n \to \infty} \left| \frac{-1 + (-1)^n}{1 + (-1)^n} \right| ,
\]
which is undefined for determining convergence. Instead, by substituting \( a = -1 \) directly into (38), we can obtain the interval of convergence for the local Heun function at \( x = 0 \).

Findings: The Structural Constraints in the P–P Theorem and the Internal Component Rearrangement approach demonstrate that the radius of convergence depends critically on whether the structural composition of recurrence terms remains intact. While absolute convergence is preserved under conventional assumptions, it requires additional constraints to remain stable when internal component rearrangement is introduced, as shown in Table 2.

5. Domain Relationship between Absolute Convergence and Internal Component Rearrangement

According to Fig. 3, the relationship between absolute convergence and internal component rearrangement can be expressed as:
\[
D_r = \text{Domain affected by internal component rearrangement} \subseteq D_o = \text{Domain of absolute convergence},
\]
where \( D_o \) represents the domain of convergence defined by the Poincaré–Perron theorem, and \( D_r \) represents the domain where internal component rearrangement alters the radius of convergence. This relationship applies not only to three-term recurrence relations in Fuchsian equations but also to multi-term recurrence cases.

5.1. Uniqueness Theorem in the Framework of Fuchsian Equations. The uniqueness theorem for differential equations states that a Fuchsian equation admits a unique solution for given boundary conditions. In the case of a Poisson-type equation, if \( \Omega \) is a boundary domain in \( \mathbb{R}^n \), and a function \( u : \Omega \to \mathbb{R} \) satisfies \( u \in C^2(\Omega) \cap C(\Omega) \) and \( \nabla^2 u = \rho \) in \( \Omega \), with either \( u = f \) or \( \partial u/\partial n = g \) on \( \partial \Omega \), where \( \rho, f, \) and \( g \) are given functions, then \( u \) is uniquely determined up to an additive constant \([15]\). This theorem is fundamental in the study of second-order differential equations, including Fuchsian equations.
A similar uniqueness principle applies to the Klein-Gordon equation, where the scalar field $\Phi(r)$, defined in a region $\Omega$ bounded by $\partial\Omega$, has a unique solution under Dirichlet or Neumann boundary conditions. This property extends to the Heun equation, which emerges in various physical contexts, such as the separation of variables in the $D = 4$ Kerr-de Sitter metric [6, 19, 20].

In the case of local Heun functions, the uniqueness theorem plays a crucial role in mathematical physics and differential equations, ensuring well-defined analytical solutions within prescribed boundary conditions.

5.2. Uniqueness Breakdown Due to Internal Component Rearrangement. Suppose that a series solution $y^P(x)$ in (37) of the Heun function at $x = 0$ is unique, and its domain of convergence $D_o$ is determined using the Poincaré–Perron theorem, which we define as the domain of absolute convergence. Ideally, another series solution $y^R(x)$ in (40), obtained through internal component rearrangement, should be equivalent to $y^P(x)$, differing only by a constant.

However, in the region $D_o - D_r$ (the bright shaded area in Fig. 3 where $a > 0$), the series $y^R(x)$ becomes divergent. This divergence implies that $y^P(x)$ and $y^R(x)$ are independent solutions, which directly contradicts the uniqueness theorem.

The breakdown occurs because the Poincaré–Perron theorem, as applied without additional constraints, does not ensure invariance under internal component rearrangement. Therefore, the theorem cannot maintain the uniqueness of solutions in this context, highlighting the need for absolute convergence and a strict no-rearrangement constraint from the outset.

5.3. Absolute Convergence as a Necessary Condition. To uphold the validity of the uniqueness theorem, consider $y^R(x)$ as the unique solution within the boundary domain $D_r$, defined as the domain affected by internal component rearrangement. In this case, $y^P(x)$ remains equivalent (or proportional) to $y^R(x)$ because $D_r \subseteq D_o$. This relationship underscores that absolute convergence is a necessary condition for ensuring the well-defined nature of a local Heun function.

Absolute convergence guarantees that all structurally consistent rearrangements of the series converge to the same sum, preserving the invariance required for uniqueness. By imposing a no-rearrangement constraint, we restore compatibility between the Poincaré–Perron theorem and the requirements of the uniqueness theorem. Consequently, the necessity of absolute convergence aligns the theory with the mathematical and physical expectations of unique solutions.

6. Uniqueness Theorem and the Necessity of Rearrangement Restriction

The uniqueness theorem is a cornerstone of mathematical physics, ensuring that solutions to certain equations, such as Poisson’s equation, are uniquely determined by boundary conditions. However, the Poincaré–Perron (P–P) theorem, as traditionally applied, does not inherently account for invariance under internal component rearrangement of series terms. This omission violates the requirements of the uniqueness theorem. To address this, we must impose an explicit no-rearrangement constraint to ensure absolute convergence and preserve the structural invariance critical to uniqueness.
6.1. Illustration with Poisson’s Equation. Poisson’s equation, which corresponds to Gauss’s law in differential form for an electrostatic problem, is given by:

\[ \nabla \cdot (\varepsilon_0 \nabla \varphi) = -\rho_f, \]

where \( \varphi \) is the electric potential, and \( E = -\nabla \varphi \) is the electric field.

The uniqueness theorem for the electric field asserts that, under appropriate boundary conditions (which can extend to infinity), the solution \( \varphi \) is unique. Since the uniqueness theorem for Poisson’s equation is well-established, we focus on its implications for convergence properties in differential equations.

If two potential solutions \( \varphi_1 \) and \( \varphi_2 \) exist, the difference \( \phi = \varphi_2 - \varphi_1 \) must satisfy Laplace’s equation:

\[ \nabla \cdot (\varepsilon_0 \nabla \phi) = 0. \]

Applying the divergence theorem and the properties of integrals over the boundary, it follows that \( \nabla \phi = 0 \), which implies \( \varphi_1 = \varphi_2 + C \), where \( C \) is a constant. Thus, the solution is unique.

This uniqueness property serves as a fundamental principle in mathematical physics. In the context of power series solutions, maintaining absolute convergence is crucial to ensuring that rearrangements do not violate the uniqueness of solutions. This perspective underscores the necessity of imposing structural constraints, such as the no-rearrangement constraint, to align theoretical results with the expected uniqueness properties of differential equations.

6.2. Connection Between Integrals and Series Convergence. The uniqueness theorem relies fundamentally on the invariance of integrals, which are closely related to infinite sums. A surface (or volume) integral represents the limiting behavior of a sum as the discretization becomes infinitely fine. This concept is illustrated in Figs. 4 and 5.

As shown in Fig. 4, finite partial sums approximate the integral. In Fig. 5, as the discretization step \( \Delta x \to 0 \), the sums converge to the exact integral value:

\[ \text{Area} = \int_0^3 y(x) \, dx = \lim_{\Delta x \to 0} \sum_{n=1}^{N} y_n(x) \Delta x. \]
This structural invariance of the integral, where the total area remains unchanged as discretization steps become infinitesimally small, parallels the fundamental property of absolute convergence in series solutions. In the context of differential equations, absolute convergence ensures that the structural order of terms is preserved, preventing divergence caused by internal component rearrangement. This preservation of structural integrity aligns with the uniqueness theorem, reinforcing the necessity of imposing constraints against term rearrangement.

6.3. Invariance and Rearrangement Restriction. No matter how a function \( \phi \) is rearranged or summed in (59)–(60), the integral must yield the same value. This invariance is an implicit condition in the uniqueness theorem. However, the Poincaré–Perron theorem does not inherently enforce this invariance, as it does not account for the impact of internal component rearrangement.

To address this limitation, we distinguish between the absolute convergence determined by the P-P theorem and the internal component rearrangement, which arises when the structural order of terms is altered. To ensure compatibility with the uniqueness theorem, the series domain must be structurally constrained to maintain absolute convergence, preventing divergence induced by internal component rearrangement. This restriction preserves invariance and ensures the uniqueness of solutions.

By imposing a no-rearrangement constraint, we ensure the structural consistency of the P-P theorem with the fundamental principles of mathematical physics and differential equations. This approach reinforces the necessity of preserving absolute convergence as a fundamental requirement in power series solutions.

6.4. Rearrangement and Uniqueness: Limitations of the Poincaré–Perron Theorem. However, the Poincaré–Perron theorem assumes conditional convergence and does not guarantee absolute convergence, which is necessary to preserve structural invariance and uniqueness.

Some scholars consider the solution for \( \sum_{n=-\infty}^{\infty} \) instead of \( \sum_{n=0}^{\infty} \), applying Laurent series in such cases. However, even in this scenario, the Poincaré–Perron theorem cannot be applied because it fails to preserve uniqueness under structural rearrangement.

Let us assume \( f(z) \) is a holomorphic function defined on the annulus \( r < |z-z_0| < R \), which has two Laurent series representations:

\[
(62) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n.
\]

Here, the coefficients \( a_n \) are defined by a generalized version of Cauchy’s integral formula:

\[
(63) \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.
\]

Multiplying both sides of (62) by \( (z - z_0)^{-m-1} \), where \( m \in \mathbb{Z} \), and integrating over a path \( C \) inside the annulus gives:

\[
(64) \quad \oint_C \sum_{n=-\infty}^{\infty} a_n(z - z_0)^{n-m-1} dz = \oint_C \sum_{n=-\infty}^{\infty} b_n(z - z_0)^{n-m-1} dz.
\]
Since the series converges uniformly on \( r < |z - z_0| < R \), integration and summation can be interchanged. Using the identity:

\[
\oint_C (z - z_0)^{n-m-1} \, dz = 2\pi i \delta_{nm},
\]

we conclude that \( a_m = b_m \). Therefore, the Laurent series is unique.

However, as evident from Cauchy’s integral formula and (64), integration inherently relies on absolute convergence, as it preserves the structural integrity of summation. The Poincaré–Perron theorem assumes conditional convergence, meaning that internal component rearrangement may lead to a reduction in the radius of convergence. Consequently, the theorem lacks the necessary framework to ensure uniqueness, highlighting the need for absolute convergence and an explicit no-rearrangement constraint.

6.5. Necessity of Absolute Convergence for Integral and Series Exchange.

The following interchange between summation and integration:

\[
\int_a^b \sum_{n=0}^{\infty} f_n \, dx \rightarrow \sum_{n=0}^{\infty} \int_a^b f_n \, dx,
\]

is valid only when \( \sum_{n=0}^{\infty} f_n \) is absolutely convergent. If the summation occurs without structural constraints, the calculation of function values may lead to errors. When absolute convergence holds, the interchange of summation and integration preserves structural integrity. However, internal component rearrangement may disrupt this invariance.

6.6. Application to Heun Functions and Fuchsian Equations. Robert S. Maier, in 2004, investigated transformations of the Heun equation and expressed them as follows [9]:

\[
HI(a, q; \alpha, \beta, \gamma, \delta; x) = \left(1 - \frac{x}{a}\right)^{-\alpha} \times \:
\]

\[
HI\left(\frac{1}{1-a}, \frac{q - \gamma \alpha}{1-a} ; -\beta + \gamma + \delta, \alpha, \gamma, \alpha - \beta + 1; \frac{x}{x-a}\right),
\]

where both \( HI(a, q; \alpha, \beta, \gamma, \delta; x) \) and the transformed function must be absolutely convergent. The radius of convergence of (67) is determined by the overlap of the domains of the two functions.

The Poincaré–Perron theorem assumes conditional convergence and does not account for the effects of internal component rearrangement, which may impact structural invariance. In particular, if internal component rearrangement alters the summation structure, the transformed function may exhibit a reduced radius of convergence. This further highlights the necessity of absolute convergence and the enforcement of a no-rearrangement constraint in such cases.

6.7. Uniqueness Theorem and the Role of Absolute Convergence. Absolute convergence is essential to maintain invariance during the exchange of integration and summation. The uniqueness theorem guarantees a single correct solution for linear differential equations involving infinite series. While internal component rearrangement in Frobenius series disrupts this invariance, absolute convergence preserves it.
Thus, absolute convergence must be explicitly enforced from the outset. By imposing a no-rearrangement constraint in the Poincaré–Perron theorem itself, structural invariance can be maintained. This ensures the theorem’s compatibility with the uniqueness theorem, avoiding contradictions and preserving the consistency of mathematical physics.

6.8. The Role of the Uniqueness Theorem and Frobenius Series. When calculating mathematical or physical quantities, we naturally use discrete quantities to solve linear differential equations. Here, the term "discrete quantities" refers to something that we can measure, observe, and confirm. Through this approach, we obtain various solutions to infinite series. However, according to the uniqueness theorem, there is only one correct solution.

To address this, the P–P theorem must explicitly include a condition that imposes a no-rearrangement constraint from the very beginning to satisfy the uniqueness theorem. This condition ensures absolute convergence and preserves the critical invariance required for exchanging series and integrals.

Even under arithmetic operations, absolute convergence ensures that the function values remain invariant, preserving the uniqueness of the solution. This invariance is guaranteed by the uniqueness theorem.

Therefore, the use of the Frobenius series is justified by its ability to guarantee a unique solution for linear differential equations. In fact, in the case of linear differential equations, the radius of convergence is determined based on the absolute convergence of power series solutions. In this process, absolute convergence plays a vital role, ensuring compatibility with integral-series interchange and maintaining consistency with the uniqueness theorem.

7. Conclusion and Discussion

Based on the reasons discussed above, we reach the following conclusion:

**Corollary 7.1.** For a d-term recurrence relation of a Fuchsian differential equation where \( d \geq 3 \), if the power series solution exhibits convergence without structural constraints, the uniqueness theorem is no longer valid. The theorem is only valid when the series is absolutely convergent.

Consider the differential equation:

\[
y'' + p(z)y' + q(z)y = 0,\]

with initial conditions \( y(0) = y_0 \) and \( y'(0) = y'_0 \).

According to Fuchs’ theorem, the radius of convergence of a series solution is at least as large as the minimum of the radii of convergence of \( p(z) \) and \( q(z) \). However, in the case of a local Heun function, this theorem is not valid, as shown in Fig. 3. The domain of the local Heun function is often much smaller than what Fuchs’ theorem predicts.

The proof of Fuchs’ theorem typically involves substituting \( y(z) = \sum_{n=0}^{\infty} d_n z^n \) into the differential equation. However, in our case, \( d_n \) is not a single term but rather a polynomial of degree \( n \) composed of terms \( \alpha_{1,n} \) and \( \alpha_{2,n} \). Therefore, to determine the radius of convergence, \( |d_n| \) should be expressed as:

\[
|d_n| = \prod_{i=0}^{j} \prod_{l=0}^{k} |\alpha_{1,i}| |\alpha_{2,l}|.
\]
From this, as shown in Fig. 3, we observe that for a local Heun function to satisfy Fuchs’ theorem at \( z = 0 \), it requires \( a < 0 \). If \( a \) is complex or greater than zero, the radius of convergence does not align with the prediction of Fuchs’ theorem. This leads us to the following statement:

**Corollary 7.2.** The recurrence relation arises by substituting a Frobenius series \( y(z) = \sum_{n=0}^{\infty} d_n z^n \) into a Fuchsian differential equation. Fuchs’ theorem is valid only for two-term recurrence relations in a Frobenius series.

For recurrence relations involving three or more terms, Fuchs’ theorem generally fails. However, if \( d_n \) is treated as an indivisible term that cannot undergo structural rearrangement, Fuchs’ theorem remains valid.

A critical review of Fuchs’ theorem highlights the issue that, if internal component rearrangement is allowed, the uniqueness of the series solution is no longer preserved. This directly contradicts the uniqueness theorem and risks undermining mathematical consistency.

Therefore, to safeguard Fuchs’ theorem, a strict no-rearrangement constraint must be imposed from the outset. This ensures that Fuchsian differential equations remain consistent with the uniqueness theorem and preserve mathematical and physical invariance.

**8. FURTHER DISCUSSION**

Lazarus Fuchs’ theory is fundamentally derived by substituting \( y(z) = \sum_{n=0}^{\infty} d_n z^n \) into the Fuchsian differential equation. According to his theory, \( k \) regular singular points (including infinity) result in \( k - 1 \) terms in the recurrence relation. However, a crucial observation is that Fuchs’ theorem implicitly assumes that \( d_n \) is treated as an indivisible term, preserving its structural integrity. By definition, the radius of convergence must be absolutely convergent, ensuring that its value remains invariant under internal component rearrangement. However, when extended to broader contexts, this assumption may lead to inconsistencies.

To clarify this, consider the following example. The radius of convergence of the series \( \sum_{n=0}^{\infty} z^n \) is \( |z| < 1 \), where \( z = x + iy \) represents a point on the complex plane. If we assume that the components \( x \) and \( y \) of \( z \) can be rearranged arbitrarily, the radius of convergence becomes \( |x| + |y| < 1 \), which is strictly smaller than the original radius. However, in the geometric framework of complex numbers, \( z \) is treated as an indivisible entity, and rearranging \( x \) and \( y \) is not allowed. This geometric invariance ensures the validity of the original radius of convergence.

This analogy between complex numbers and recurrence relations reveals a deeper structural connection. Just as complex numbers inherently preserve structural consistency through their geometric representation, the coefficients in recurrence relations, such as \( d_n, d_{n-1}, d_{n-2}, \ldots \), maintain a unified structure through their ordering. Preserving this order is essential for maintaining both the radius of convergence and the structural integrity of the recurrence relation—much like how treating \( z \) as a whole preserves the geometric and algebraic properties of the complex plane.

This analogy further highlights the fundamental nature of mathematical structures. Just as the geometric properties of complex numbers depend on treating them as indivisible entities, the convergence properties of power series solutions depend critically on maintaining the structural integrity of their terms. The no-rearrangement constraint, therefore, is not merely a technical restriction but a fundamental principle preserving the essential nature of mathematical structures.
The structural consistency in both cases is particularly illuminating. The complex plane maintains its geometric and algebraic properties by treating \( z \) as an indivisible point. Similarly, a recurrence relation maintains its convergence properties by preserving the structural integrity of its terms. When this order is disrupted through internal component rearrangement, the radius of convergence can contract—analogous to how decomposing \( z \) into separate components would compromise the geometric integrity of the complex plane.

However, \( d_n \) in our case does not correspond to a point in a geometric space. Therefore, we must determine the radius of convergence in such a way that function values remain invariant under internal component rearrangement. For instance, in the case of Heun’s equation, there are four regular singular points, which yield the three-term recurrence relation:

\[
d_{n+1} = \alpha_{1,n} d_n + \alpha_{2,n} d_{n-1}.
\]

Here, it is more logical to determine the radius of convergence under the condition that the ordering of \( \alpha_{1,n} \) and \( \alpha_{2,n} \) remains unchanged.

This perspective also clarifies the relationship between convergence radii and singular points. Just as \( |z| \) defines the distance to the nearest singular point in the complex plane, the ordered structure of recurrence relation terms ensures that the radius of convergence maintains its mathematical significance. Allowing internal component rearrangement not only disrupts this structure but also reduces the radius of convergence—similar to how decomposing \( z \) into separate components would distort the intrinsic properties of the complex plane.

This connection between geometric invariance and analytical properties provides a new perspective on the relationship between different areas of mathematics. The preservation of term ordering in recurrence relations ensures not only the convergence properties of power series solutions but also the deeper structural integrity of mathematical frameworks, suggesting a fundamental principle that may have applications beyond Fuchsian differential equations.

However, if \( d_n \) is strictly treated as an indivisible term that cannot be split or rearranged, Fuchs’ theorem remains valid. While this perspective maintains the mathematical structure of Fuchs’ theory, it raises questions about whether this is the most natural or practical approach—particularly for polynomial recurrence relations. For these reasons, we propose explicitly introducing a no-rearrangement constraint from the outset to align Fuchs’ theorem with the structural requirements of modern mathematical and physical applications.

9. Findings and Potential Applications

In this study, we analyzed the radii of convergence of power series solutions to Fuchsian differential equations in relation to the relative positions of singular points. Specifically, we defined the convergence obtained through Fuchs’ theorem as absolute convergence, while the behavior of the series under internal component rearrangement leading to the smallest radius as convergence under internal component rearrangement. Our findings demonstrate that the radius of convergence obtained via rearrangement is independent of the explicit form of the coefficients of the differential equation and depends solely on the relative positions of singular points.
These results are mathematically significant and suggest potential directions for further exploration, as detailed below:

1. **Ensuring the Stability of Series Without Structural Constraints:**
   The no-rearrangement constraint ensures the stability of power series solutions by preventing reductions in the radius of convergence due to internal component rearrangement. This contributes to a more rigorous understanding of convergence behavior in recurrence relations.

2. **Refining the Relationship Between Singularities and Convergence Behavior:**
   The findings provide insight into how singularities influence the structure of power series solutions, particularly in the context of higher-order recurrence relations.

3. **Implications for Higher-Order Differential Equations:**
   Extending these results to differential equations with more complex singularity structures could refine our understanding of power series solutions in broader mathematical contexts.

**Significance of Fuchs’ Theorem and the No-Rearrangement Constraint.**

Fuchs’ theorem is derived by substituting \( y(z) = \sum_{n=0}^{\infty} d_n z^n \) into the differential equation. However, the traditional approach assumes that \( d_n \) is treated as an indivisible term that maintains its structural integrity, implying that rearrangement does not affect the radius of convergence. Our study demonstrates that when internal component rearrangement is allowed, the radius of convergence contracts. This finding highlights the necessity of explicitly imposing a \textit{no-rearrangement constraint} to preserve structural invariance and maintain consistency with the uniqueness theorem in Fuchsian differential equations.

**Future Research Directions.**

The theoretical findings of this study suggest further research directions in:

- **Complex Analysis:** Extending the theory to other types of differential equations and investigating the impact of singularity structures on power series solutions.
- **Asymptotic Analysis and Stokes Phenomena:** Investigating how internal component rearrangement interacts with asymptotic expansions and phase transitions in differential equations.
- **Higher-Order Differential Equations:** Exploring the implications of structural constraints on convergence in recurrence relations with more than three terms.
- **Partial Differential Equations:** Extending these results to PDEs and investigating how structural constraints influence separable solutions.

Future research can further explore these possibilities, refining the theoretical framework and extending its applications to broader classes of differential equations. Of particular interest would be:

1. Investigation of how the no-rearrangement constraint affects convergence properties in higher-order differential equations.
2. Development of systematic methods for identifying cases where internal component rearrangement might lead to convergence issues.
3. Extension of the theory to systems of differential equations and PDEs.
4. Exploration of connections with other areas of mathematics, such as asymptotic analysis, summability methods, and Stokes phenomena.
These directions promise to deepen our understanding of power series solutions while providing new perspectives on the structural properties of differential equations.

**Appendix A. Convergence Domains Under Internal Component Rearrangement for 192 Local Heun Functions**

A systematically generated classification of 192 local solutions of the Heun equation, which exhibit an isomorphic structure to the Coxeter group associated with the Coxeter diagram $D_4$, was obtained by Maier (2007) [9].

We explicitly compute the modified convergence domains due to internal component rearrangement for nine representative cases among the 192 local solutions of the Heun equation, as presented in Table 2 [9].

A.1. $(1 - x)^{1-\delta} Hl(a, q - (\delta - 1)\alpha; \alpha - \delta + 1, \beta - \delta + 1, \gamma, 2 - \delta; x)$

and $x^{1-\gamma}(1 - x)^{1-\delta} Hl(a, q - (\gamma + \delta - 2)\alpha - (\gamma - 1)(\alpha + \beta - \gamma - \delta + 1); \alpha - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma, 2 - \delta; x)$, boundary conditions of the independent variable $x$ of $Hl(a, q - (\delta - 1)\alpha; \alpha - \delta + 1, \beta - \delta + 1, \gamma, 2 - \delta; x)$ and $Hl(a, q - (\gamma + \delta - 2)\alpha - (\gamma - 1)(\alpha + \beta - \gamma - \delta + 1); \alpha - \gamma - \delta + 2, \beta - \gamma - \delta + 2, 2 - \gamma, 2 - \delta; x)$ are the same as Eq. (38).

A.2. $Hl(1 - a, -q + \alpha\beta; \alpha, \beta, \delta, \gamma; 1 - x)$

and $(1 - x)^{1-\delta} Hl(1 - a, -q + (\delta - 1)\alpha\gamma a + (\alpha - \delta + 1)(\beta - \delta + 1); \alpha - \delta + 1, \beta - \delta + 1, 2 - \delta, \gamma; 1 - x)$, Replace coefficient $a$ and independent variable $x$ by $1 - a$ and $1 - x$, respectively, in (38).

The convergence condition under internal component rearrangement is given by

$$\left| \frac{1}{1 - a} (1 - x)^2 \right| + \left| \frac{2 - a}{1 - a} (1 - x) \right| < 1 \quad \text{where} \quad a \neq 1$$

A.3. $x^{-\alpha} Hl\left( \frac{1}{a}, q + \alpha((\alpha - \gamma - \delta + 1)\alpha - \beta + \delta; \alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \delta; \frac{1}{a}) \right)$. Replace coefficient $a$ and $x$ by $\frac{1}{a}$ and $\frac{1}{x}$, respectively, in (38). The convergence condition under internal component rearrangement is given by

$$|ax^{-2}| + |(1 + a)x^{-1}| < 1$$

A.4. $\left( 1 - \frac{x}{a} \right)^{-\beta} Hl(1 - a, -q + \gamma\beta; -\alpha + \gamma + \delta, \beta, \gamma, \delta; \frac{1 - a}{x - a})$

and $(1 - x)^{1-\delta} \left( 1 - \frac{x}{a} \right)^{-\beta+\delta-1} Hl(1 - a, -q + \gamma((\delta - 1)a + \beta - \delta + 1); -\alpha + \gamma + 1, \beta - \delta + 1, \gamma, 2 - \delta; \frac{1 - a}{x - a})$. Replace $a$ and $x$ by $1 - a$ and $\frac{1}{x - a}$, respectively, in (38). The Condition of Convergence Under Internal Component Rearrangement of is given by

$$\left| \frac{(1 - a)x^2}{(x - a)^2} \right| + \left| \frac{2 - a}{x - a} \right| < 1 \quad \text{where} \quad x \neq a$$
A.5. $x^{-\alpha}Hl\left(\frac{a-1}{a}, q - \alpha (\delta a + \beta - \delta); \alpha, \alpha - \gamma + 1, \delta, \alpha - \beta + 1, \frac{x-1}{x}\right)$. Replace $a$ and $x$ by $\frac{a-1}{a}$ and $\frac{x-1}{x}$, respectively, in (38). The convergence condition under internal component rearrangement is given by

$$\frac{a}{(1-a)^2} \left(\frac{x-1}{x}\right)^2 + \frac{(1-2a)(x-1)}{(1-a)x} < 1 \quad \text{where } a \neq 1$$

A.6. $\left(\frac{x-a}{1-a}\right)^{-\alpha}Hl\left(a, q - (\beta - \delta)a; \alpha, -\beta + \gamma + \delta, \delta, \gamma; \frac{a(x-1)}{x-a}\right)$. Replace $x$ by $\frac{ax-1}{x-a}$ in (38). The convergence condition under internal component rearrangement is given by

$$\left|\frac{a(x-1)^2}{(x-a)^2}\right| + \left|\frac{1+a(x-1)}{(x-a)}\right| < 1 \quad \text{where } x \neq a$$

Appendix B. Domains of Resummed Convergence of 4 Fuchsian Differential Equations Having Multi-term Recurrence Relations

B.1. Riemann’s P–differential equation. The Riemann P–differential equation is written by

$$\frac{d^2y}{dz^2} + \left(\frac{1-a-a'}{z-a} + \frac{1-b-b'}{z-b} + \frac{1-c-c'}{z-c}\right) \frac{dy}{dz} + \left(\frac{a(a-b)(a-c)}{z-a} + \frac{b(b-c)(b-a)}{z-b} + \frac{c(c-a)(c-b)}{z-c}\right) \frac{y}{(z-a)(z-b)(z-c)} = 0$$

With the condition $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$. The regular singular points are $a, b$ and $c$ with exponents $\{\alpha, \alpha'\}, \{\beta, \beta'\}$ and $\{\gamma, \gamma'\}$. This equation was first obtained in the form by Papperitz. [5, 12] The solutions to the Riemann P–differential equation are given in terms of the hypergeometric function. However, currently, the analytic solutions of Riemann’s differential equation at $a, b$ and $c$ are unknown because of its complicated mathematical calculation: the five-term recurrence relation of a Frobenius series starts to appear.

We assume the solution takes the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}$$

By putting (72) into (71) at $x = z - a$, the algebraic equation around $z = a$ is

$$\frac{d^2y}{dx^2} + \left(\frac{1-\alpha}{x} + \frac{1-\beta}{x+k} + \frac{1-\gamma}{x+l}\right) \frac{dy}{dx} + \left(\frac{\alpha' k l}{x} - \frac{\beta' m k}{x+k} + \frac{\gamma' m l}{x+l}\right) \frac{y}{x(x+k)(x+l)} = 0$$

For convenience for calculations, the parameters have been changed by setting

$$k = a - b, \quad l = a - c, \quad m = b - c, \quad \overline{\alpha} = \alpha + \alpha', \quad \overline{\beta} = \beta + \beta', \quad \overline{\gamma} = \gamma + \gamma'$$

with $\overline{\alpha} + \overline{\beta} + \overline{\gamma} = 1$. We obtain the recurrence system by substituting (72) into (73).

$$c_{n+1} = \alpha_{1,n} c_n + \alpha_{2,n} c_{n-1} + \alpha_{3,n} c_{n-2} + \alpha_{4,n} c_{n-3} \quad : n \geq 3$$
where,

(76a) \[ \alpha_{1,n} = -\frac{(n + \lambda) \left(2(k + l)(n + \lambda) - k(\alpha - \beta) - l(\alpha - \gamma)\right) + (k + l)a\alpha' - m(\beta\beta' - \gamma\gamma')}{kl(n + 1 - \alpha + \lambda)(n + 1 - \alpha' + \lambda)} \]

(76b) \[ \alpha_{2,n} = -\frac{(n - 1 + \lambda) \left((k^2 + 4kl + l^2)(n - 1 + \lambda) + k^2\alpha - kl\beta' + ml\gamma'\right)}{k^2l^2(n + 1 - \alpha + \lambda)(n + 1 - \alpha' + \lambda)} \]

(76c) \[ \alpha_{3,n} = -\frac{(n - 2 + \lambda) \left(2(k + l)(n - 2 + \lambda) + k(1 + \beta) + l(1 + \gamma)\right)}{k^2l^2(n + 1 - \alpha + \lambda)(n + 1 - \alpha' + \lambda)} \]

(76d) \[ \alpha_{4,n} = -\frac{(n - 3 + \lambda)(n - 2 + \lambda)}{k^2l^2(n + 1 - \alpha + \lambda)(n + 1 - \alpha' + \lambda)} \]

and

(76e) \[ c_1 = \alpha_1, c_2 = (\alpha_1, \alpha_2), c_3 = (\alpha_1, \alpha_3), \text{ and } c_4 = (\alpha_1, \alpha_1), \alpha_2, \alpha_3, \alpha_4 \]

We have two indicial roots which are \( \lambda = \alpha \) and \( \alpha' \).

The convergence condition under internal component rearrangement of (72) is obtained by applying Thm.3.3 such as

\[ \frac{1}{\left(2(a-b-c)\right)^{2x}} + \frac{1}{\left(2a-b-c\right)^{2(a-c)}x^2} + \frac{1}{\left(a-b\right)^2(a-c)^2} x^4 \]

\[ < 1 \]

B.2. Heine differential equation. The Heine differential equation is defined by

(77) \[ \frac{d^2y}{dz^2} + \frac{1}{2} \left( \frac{1}{z - a_1} + \frac{2}{z - a_2} + \frac{2}{z - a_3} \right) \frac{dy}{dz} + \frac{1}{4} \left( \frac{\beta_0 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3}{(z - a_1)(z - a_2)(z - a_3)^2} \right) y = 0 \]

It has four regular singular points at \( a_1, a_2, a_3 \) and \( \infty \). Parameters \( a_2 \) and \( a_3 \) are identical to each other.

The differential equation around \( z = a_1 \), which is obtained from (77) by setting \( x = z - a_1 \) is

(78) \[ \frac{d^2y}{dx^2} + \frac{1}{2} \left( \frac{1}{x + 2(x - m) + 2} \right) \frac{dy}{dx} + \frac{\beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0}{4x(x - m)^2(x - k)^2} y = 0 \]

For convenience for calculations, the parameters have been changed by setting

\[ m = a_2 - a_1, \quad k = a_3 - a_1, \quad \beta_2 = 3\alpha_1\beta_3 + \beta_2, \quad \beta_1 = 3\alpha_1^2\beta_3 + 2\alpha_1\beta_2 + \beta_1, \quad \beta_0 = \alpha_1^3\beta_3 + \alpha_1^2\beta_2 + \alpha_1\beta_1 + \beta_0 \]

Looking for a solution of (78) through the expansion

\[ y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \]

We obtain the following conditions:

(79) \[ c_{n+1} = \alpha_{1,n} c_n + \alpha_{2,n} c_{n-1} + \alpha_{3,n} c_{n-2} + \alpha_{4,n} c_{n-3} , \quad n \geq 3 \]

where,

(80a) \[ \alpha_{1,n} = \frac{2mk(m + k)(n + \lambda)^2 - \beta_0}{m^2k^2(n + 1 + \lambda)(n + 1 + \lambda)} \]
\[ \alpha_{2,n} = - \frac{(m^2 + 4mk + k^2)(n - 1 + \lambda)(n - \frac{1}{2} + \lambda) + \beta}{m^2k^2(n + 1 + \lambda)(n + \frac{1}{2} + \lambda)} \]

\[ \alpha_{3,n} = \frac{2(m + k)(n - 2 + \lambda)(n - 1 + \lambda) + \beta}{m^2k^2(n + 1 + \lambda)(n + \frac{1}{2} + \lambda)} \]

\[ \alpha_{4,n} = - \frac{(n - 3 + \lambda)(n - \frac{3}{2} + \lambda) + \beta}{m^2k^2(n + 1 + \lambda)(n + \frac{1}{2} + \lambda)} \]

and

\[ \alpha_{1,n} = \alpha_{1,0} c_0, \quad c_2 = (\alpha_{1,0}\alpha_{1,1} + \alpha_{2,1}) c_0, \quad c_3 = (\alpha_{1,0}\alpha_{1,1}\alpha_{1,2} + \alpha_{1,0}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,1} + \alpha_{3,2}) c_0 \]

We have two indicial roots which are \( \lambda = 0 \) and \( \frac{1}{2} \). (79) is the 5-term recurrence relation.

The convergence condition under internal component rearrangement of the Heine equation at \( z = a_1 \) is obtained by applying Thm 3.3 such as

\[
\frac{2(2a_1 - a_2 - a_3)}{(a_2 - a_1)(a_3 - a_1)} \cdot \frac{(2a_1 - a_2 - a_3)^2 + 2(a_2 - a_1)(a_3 - a_1)}{(a_2 - a_1)^2(a_3 - a_1)^2} \cdot x^2 + \frac{2(2a_1 - a_2 - a_3)}{(a_2 - a_1)^2(a_3 - a_1)^2} \cdot x^3 \leq 1
\]

The differential equation around \( z = a_2 \), which is obtained from (77) by setting \( x = z - a_2 \). Putting a power series \( y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \) into the new (77), we obtain the 4-term recurrence relation as follows.

\[ c_{n+1} = \alpha_{1,n} c_n + \alpha_{2,n} c_{n-1} + \alpha_{3,n} c_{n-2} \quad ; n \geq 2 \]

where,

\[ \alpha_{1,n} = - \frac{k(2m + k)(n + \lambda)(n + \frac{1}{2} + \lambda) + \beta}{mk^2(n + 1 + \lambda)^2 + \beta} \]

\[ \alpha_{2,n} = - \frac{(m + 2k)(n - 2 + \lambda)(n + \lambda) + \beta}{mk^2(n + 1 + \lambda)^2 + \beta} \]

\[ \alpha_{3,n} = - \frac{(n - 2 + \lambda)(n - \frac{1}{2} + \lambda) + \beta}{mk^2(n + 1 + \lambda)^2 + \beta} \]

and

\[ c_1 = \alpha_{1,0} c_0, \quad c_2 = (\alpha_{1,0}\alpha_{1,1} + \alpha_{2,1}) c_0 \]

where

\[ m = a_2 - a_1, \quad k = a_2 - a_3, \quad \beta_2 = 3a_2^2\beta_3 + \beta_2, \quad \beta_1 = 3a_2^2\beta_3 + 2a_2\beta_2 + \beta_1, \quad \beta_0 = a_2^2\beta_3 + a_2^2\beta_2 + a_2\beta_1 + \beta_0 \]

We have two indicial roots which are \( \lambda = \pm \sqrt{-\frac{\beta_0}{4mk^2}} \).

The convergence condition under internal component rearrangement of the Heine equation at \( z = a_2 \) is obtained by applying Thm 3.3 such as

\[
\left| \frac{(a_2 - a_3)(3a_2 - a_1 - a_3)}{(a_2 - a_1)(a_2 - a_3)^2} \cdot x + \frac{(3a_2 - a_1 - a_3)}{(a_2 - a_1)(a_2 - a_3)^2} \cdot x^2 \right| + \left| \frac{1}{(a_2 - a_1)(a_2 - a_3)^2} \cdot x^3 \right| < 1
\]
B.3. Generalized Heun’s differential equation. The generalized Heun equation (GHE) is a second-order linear ordinary differential equation of the form

$$\frac{d^2y}{dx^2} + \left( \frac{1 - \mu_0}{x} \frac{1 - \mu_1}{x - 1} + \frac{1 - \mu_2}{x - a} + \alpha \right) \frac{dy}{dx} + \beta_0 + \beta_1 x + \beta_2 x^2 \frac{y}{x(x - 1)(x - a)} = 0$$

where $\mu_0, \mu_1, \mu_2, \alpha, \beta_0, \beta_1, \beta_2 \in \mathbb{C}$. It has three regular singular points which are 0, 1 and $a$ with exponents $\{0, \mu_0\}, \{0, \mu_1\}$ and $\{0, \mu_2\}$, while $\infty$ is at most an irregular singularity. This equation was first introduced by Schäfke and Schmidt.\[18\] Heun’s equation is derived from the generalized Heun’s differential equation by changing all coefficients $\alpha = \beta_2 = 0, 1 - \mu_0 = \gamma, 1 - \mu_1 = \delta, 1 - \mu_2 = \epsilon, \beta_1 = \alpha \beta$ and $\beta_0 = -q$.

Looking for a solution of (83) through the expansion

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}$$

We obtain the following conditions:

$$c_{n+1} = \alpha_{1,n} c_n + \alpha_{2,n} c_{n-1} + \alpha_{3,n} c_{n-2}$$

where,

$$\alpha_{1,n} = \frac{(n + \lambda)((1 + a)(n + \lambda) + 1 - \mu_0 - \mu_2 + a(1 - \mu_0 - \mu_1 - \alpha)) - \beta_0}{a(n + 1 + \lambda)(n + 1 - \mu_0 + \lambda)}$$

$$\alpha_{2,n} = -\frac{(n - 1 + \lambda)(n + \lambda + 1 - \mu_0 - \mu_1 - \mu_2 - (1 + a)\alpha) + \beta_1}{a(n + 1 + \lambda)(n + 1 - \mu_0 + \lambda)}$$

$$\alpha_{3,n} = -\frac{\alpha(n - 2 + \lambda) + \beta_2}{a(n + 1 + \lambda)(n + 1 - \mu_0 + \lambda)}$$

and

$$c_1 = \alpha_{1,0} c_0, \quad c_2 = (\alpha_{1,0}\alpha_{1,1} + \alpha_{2,1}) c_0$$

We have two indicial roots which are $\lambda = 0$ and $\mu_0$. (84) is the 4-term recurrence relation.

The convergence condition under internal component rearrangement of the GHE is obtained by applying Thm.3.3 such as

$$\left| \frac{1 + a}{a} x \right| + \left| -\frac{1}{a} x^2 \right| < 1$$

The GHE around $x = 1$ is taken by putting $z = 1 - x$ into (83).

$$\frac{d^2y}{dz^2} + \left( \frac{1 - \mu_1}{z} \frac{1 - \mu_0}{z - 1} + \frac{1 - \mu_2}{z - (1 - a)} - \alpha \right) \frac{dy}{dz} + \frac{-(\beta_0 \beta_1 + \beta_2) + (\beta_1 + 2\beta_2)z - \beta_2 z^2}{z(z - 1)(z - (1 - a))} y = 0$$
If we compare (87) with (83), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
  a & \rightarrow 1 - a \\
  \mu_0 & \rightarrow \mu_1 \\
  \mu_1 & \rightarrow \mu_0 \\
  \alpha & \rightarrow -\alpha \\
  \beta_0 & \rightarrow -(\beta_0 + \beta_1 + \beta_2) \\
  \beta_1 & \rightarrow \beta_1 + 2\beta_2 \\
  \beta_2 & \rightarrow -\beta_2 \\
  x & \rightarrow z = 1 - x
\end{align*}
\] (88)

Putting (88) into (86), we obtain domain of convergence Under internal component rearrangement of the GHE around \( x = 1 \) where \( z = 1 - x \) for an infinite series.

The GHE around \( x = a \) is taken by putting \( z = \frac{a - x}{a} \) into (83).

\[
\begin{align*}
  \frac{d^2 y}{dz^2} + \left( \frac{1 - \mu_2}{z} \right) + \frac{1 - \mu_0}{z - 1} + \frac{1 - \mu_1}{z - (a^{-1} - 1)} - \alpha \right) dy &= 0 \\
  \frac{d^2 y}{dz^2} + \frac{-(\beta_1 + a\beta_2 + a^{-1}\beta_0) + (\beta_1 + 2a\beta_2)z - a\beta_2 z^2}{z(z-1)(z-(a^{-1}-1))} y = 0
\end{align*}
\] (89)

If we compare (89) with (83), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
  a & \rightarrow a^{-1} - 1 \\
  \mu_0 & \rightarrow \mu_2 \\
  \mu_1 & \rightarrow \mu_0 \\
  \mu_2 & \rightarrow \mu_1 \\
  \alpha & \rightarrow -a\alpha \\
  \beta_0 & \rightarrow -(\beta_1 + a\beta_2 + a^{-1}\beta_0) \\
  \beta_1 & \rightarrow \beta_1 + 2a\beta_2 \\
  \beta_2 & \rightarrow -a\beta_2 \\
  x & \rightarrow z = \frac{a - x}{a}
\end{align*}
\] (90)

Putting (90) into (86), we obtain domain of convergence Under internal component rearrangement of the GHE around \( x = a \) where \( z = \frac{a - x}{a} \) for an infinite series.

\textbf{B.4. The Lamé (or ellipsoidal) wave equation.} The Lamé (or ellipsoidal) wave equation arises from deriving Helmholtz equation in ellipsoidal coordinates \( \nabla^2 \phi + \omega \phi = 0 \) where \( \nabla^2 \) is the Laplacian, \( \omega \) is the wavenumber and \( \phi \) is the amplitude. \[2, 3, 4\] Whereas Lamé equation is derived from separation of the Laplace equation in confocal ellipsoidal coordinates \( \nabla^2 \phi = 0 \), only the special case of the ellipsoidal wave equation.

The ellipsoidal wave equation is a second-order linear ODE of the Jacobian form

\[
\frac{d^2 y}{dz^2} - (a + bk^2 sn^2 z + qk^4 sn^4 z) y = 0
\] (91)

where the Jacobian elliptic function \( sn z = sn(z, k) \) has modulus \( k \). If \( q = 0 \), it turns to be the Lame differential equation. If we take \( x = sn^2 z \) as an independent
variable in (91), we obtain the algebraic form of it.

\[
\frac{d^2 y}{dx^2} + \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-\delta} \right) \frac{dy}{dx} + \frac{\beta + \mu x + \gamma x^2}{x(x-1)(x-\delta)} y = 0
\]

Here, the parameters have been changed by setting

\[
\delta = k^{-2}, \quad \beta = -\frac{a}{4k^2}, \quad \mu = -\frac{b}{4}, \quad \gamma = -\frac{k^2}{4} q
\]

Looking for a solution of (92) through the expansion

\[
y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}
\]

We obtain the following conditions:

\[
(93) \quad c_{n+1} = \alpha_{1,n} c_n + \alpha_{2,n} c_{n-1} + \alpha_{3,n} c_{n-2} \quad ; \quad n \geq 2
\]

where,

\[
(94a) \quad \alpha_{1,n} = \frac{(1+\delta)(n+\lambda)^2 - \beta}{\delta(n+1+\lambda)(n+\frac{1}{2}+\lambda)}
\]

\[
(94b) \quad \alpha_{2,n} = -\frac{(n-1+\lambda)(n-\frac{1}{2}+\lambda) + \mu}{\delta(n+1+\lambda)(n+\frac{1}{2}+\lambda)}
\]

\[
(94c) \quad \alpha_{3,n} = -\frac{\gamma}{\delta(n+1+\lambda)(n+\frac{1}{2}+\lambda)}
\]

and

\[
(94d) \quad c_1 = \alpha_{1,0} c_0, \quad c_2 = (\alpha_{1,0}\alpha_{1,1} + \alpha_{2,1}) c_0
\]

We have two indicial roots which are \( \lambda = 0 \) and \( \frac{1}{2} \). (93) is the 4-term recurrence relation.

The convergence condition under internal component rearrangement of the Lamé wave equation around \( x = 0 \) is obtained by applying Thm.3.3 such as

\[
(95) \quad \left| \frac{1+\delta}{\delta} x + \left| \frac{-1}{\delta} x^2 \right| < 1
\]

The Lamé wave equation around \( x = 1 \) is taken by putting \( \varrho = 1 - x \) into (92). (96)

\[
\frac{d^2 y}{d\varrho^2} + \frac{1}{2} \left( \frac{1}{\varrho} + \frac{1}{\varrho-1} + \frac{1}{\varrho-(1-\delta)} \right) \frac{dy}{d\varrho} + \frac{-(\beta + \gamma) + (\mu + 2\gamma) \varrho - \gamma \varrho^2}{\varrho(\varrho-1)(\varrho-(1-\delta))} y = 0
\]

If we compare (96) with (92), all coefficients on the above are correspondent to the following way.

\[
\delta \rightarrow 1 - \delta
\]
\[
\beta \rightarrow -(\beta + \gamma)
\]
\[
\mu \rightarrow \mu + 2\gamma
\]
\[
\gamma \rightarrow -\gamma
\]
\[
x \rightarrow z = 1 - \varrho
\]

Putting (97) into (95), we obtain domain of convergence Under internal component rearrangement of the Lamé wave equation around \( x = 1 \) where \( \varrho = 1 - x \).
The Lamé wave equation around $x = \delta$ is taken by putting $\phi = 1 - \delta^{-1}x$ into (92).

\[
\frac{d^2 y}{d\phi^2} + \frac{1}{\phi} \left( \frac{1}{\phi - 1} + \frac{1}{\phi - (1 - \delta^{-1})} \right) \frac{dy}{d\phi} + \frac{-(\gamma\delta + \mu + \delta^{-1}\beta) + (\mu + 2\gamma\delta)\phi - \gamma\delta\phi^2}{\phi(\phi - (1 - \delta))} y = 0
\]

If we compare (98) with (92), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
\delta & \rightarrow 1 - \delta^{-1} \\
\beta & \rightarrow -(\gamma\delta + \mu + \delta^{-1}\beta) \\
\mu & \rightarrow \mu + 2\gamma\delta \\
\gamma & \rightarrow -\gamma\delta \\
x & \rightarrow \phi = 1 - \delta^{-1}x
\end{align*}
\]

Putting (99) into (95), we obtain domain of convergence Under internal component rearrangement of the Lamé wave equation around $x = \delta$ where $\phi = 1 - \delta^{-1}x$.

References

[1] Arscott, F.M., Taylor, P.J. and Zahar, R.V.M., “On the Numerical construction of ellipsoidal wave functions,” Math. Comp. 40, (1983)367–380.
[2] Arscott, F.M., “Perturbation solutions of the ellipsoidal wave equation,” Quart. J. Math. 7(1), (1956)161–174.
[3] Arscott, F.M., “New treatment of the ellipsoidal wave equation,” Proc. London Math. Soc. (3)9, (1959)21–50.
[4] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., “Higher transcendent functions. Vol.III,” McGraw-Hill book company, Inc., New York–Toronto–London, (1955).
[5] Barnes, E.W., “A new development in the theory of the hypergeometric functions,” Proc. London Math. Soc. 6 (1908), 141–177.
[6] Cunhaa, B.C.D., Novaes, F., Kerr-de Sitter Greybody Factors via Isomonodromy, Phys. Rev. D93, 024045 (2016) [arXiv:1508.04046].
[7] Heun, K., Zur Theorie der Riemann’schen Functionen zweiter Ordnung mit vier Verzweigungspunkten, Mathematische Annalen 33 (1889), 161.
[8] Kristensson, G., Second order differential equations: special functions and their classification, (Springer-Verlag New York, 2014).
[9] Maier, R.S., The 192 solutions of the Heun equation, Math. Comp. 33 (2007), 811–843.
[10] Moon, P. and Spencer, D.E., “Field theory for engineers,” New York: Van Nostrand, (1961).
[11] Milne-Thomson, L.M., The calculus of finite differences, (Macmillan and Co., 1933).
[12] Papperitz, E., “Ueber verwandte s-Functionen,” (German) Math. Ann. 25 (1885), 212–221.
[13] Perron, O., Über Summengleichungen und Poincarésche Differenzengleichungen, Math. Ann. 84 (1921), 1–15.
[14] Poincaré, H., Sur les Équations Linéaires aux Dérivées Partielles de Deuxième Ordonnée et aux Différences Finies, (French) Amer. J. Math. 7(3) (1885), 203–258.
[15] Riley, K.F., Hobson, M.P, Bence, S.J., Mathematical Methods for Physics and Engineering, Cambridge University Press, 2nd edition, (2002).
[16] Ronveaux, A., Heun’s Differential Equations, (Oxford University Press, 1995).
[17] Sedgewick, R., Flajolet, P., An Introduction to the Analysis of Algorithms, (Addison-Wesley, 1996).
[18] Schäfke, R. and Schmidt, D., “The connection problem for general linear ordinary differential equations at two regular singular points with applications in the theory of special functions,” SIAM J. Math. Anal. 11(5), (1980)848–862.
[19] Suzuki, H., Takasugi, E., Umetsu, H., Perturbations of Kerr-de Sitter Black Hole and Heun’s Equation, Prog. Theor. Phys. 100 (1998), 491–505.
[20] Tarloyan, A.S., Ishkhanyan, T.A., Ishkhanyan, A.M., *Four five-parametric and five four-parametric independent confluent Heun potentials for the stationary Klein-Gordon equation*, Ann. Phys. (Berlin) **528**, 264-271 (2016) [arXiv:1510.03700].

[21] Wimp, J., *Computation with recurrence relations*, (Pitman, 1984).

[22] Zwillinger, D., “Handbook of differential equations 3rd ed.,” Boston, MA: Academic Press, (1997).

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