A KOBAYASHI PSEUDO-DISTANCE FOR HOLOMORPHIC BRACKET GENERATING DISTRIBUTIONS

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Abstract. In this paper, we generalize the Kobayashi pseudo-distance to complex manifolds which admit holomorphic bracket generating distributions. The generalization is based on Chow’s theorem in sub-Riemannian geometry. Let $G$ be a linear semisimple Lie group. For a complex $G$-homogeneous manifold $M$ with a $G$-invariant holomorphic bracket generating distribution $D$, we prove that $(M, D)$ is Kobayashi hyperbolic if and only if the universal covering of $M$ is a canonical flag domain and the induced distribution is the superhorizontal distribution.

1. Introduction

Let $M$ be a complex manifold. We say that $D$ is a holomorphic distribution on $M$ when $D$ is a holomorphic subbundle of the holomorphic tangent bundle $\mathbb{T}_M$ of $M$. We say that it is bracket generating if any local holomorphic frame $\{X_1, \ldots, X_d\}$ for $D$, together with all of its iterated Lie brackets $\{[X_i, X_j], [X_i, [X_j, X_\ell]], \ldots\}$, spans $\mathbb{T}_M$. In this paper, we generalize the Kobayashi pseudo-distance to the complex manifolds that admit holomorphic bracket generating distributions (HBGD for short) on $M$.

Let us first recall the definition of the Kobayashi pseudo-distance. Let $M$ be a complex manifold and $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ a unit disc. For $x, y \in M$ define

$$\delta_M(x, y) = \inf \{d_\Delta(a, b) : h : \Delta \to M \text{ holomorphic, } h(a) = x, h(b) = y\},$$

where $d_\Delta$ is the Poincaré distance on $\Delta$. The Kobayashi pseudo-distance of $M$ is defined by

$$(1.1) \quad d_M(x, y) = \inf \sum_{j=1}^{N} \delta_M(x_{j-1}, x_j)$$

where the infimum is taken over the sets of points $\{x_0, \ldots, x_N\}$ of $M$ such that $x = x_0$ and $y = x_N$. When $d_M$ is a distance, we say that $M$ is Kobayashi hyperbolic.

In the context of complex manifolds, Kobayashi first introduced it as the largest pseudo-distance on a complex manifold satisfying the distance decreasing property with respect to the holomorphic mappings between complex manifolds (13). If a complex manifold has a complete Hermitian metric and its holomorphic sectional curvature is bounded from above by a negative constant, then the manifold is complete Kobayashi hyperbolic. Because of this property, complete Kobayashi hyperbolic manifolds are considered to be a generalization of the Riemann surfaces of genus greater than or equal to 2 equipped with the Poincaré distance.

A holomorphic mapping $f : N \to M$ between complex manifolds $M, N$ is tangential to $D$ if $df(T_N) \subset D$. The generalization of the Kobayashi pseudo-distance to complex manifolds with HBGDs in this paper that we propose is the following.

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Definition 1.1. Let $M$ be a complex manifold with a holomorphic distribution $D$. For $x, y \in M$, define
\[
\delta_{M,D}(x,y) = \inf \left\{ d_{\Delta}(a,b) : h: \Delta \rightarrow M \text{ holomorphic} \right\}
\]
where $d_{\Delta}$ is the Poincaré distance on $\Delta$. If there is no holomorphic disc tangential to $D$ connecting $x$ and $y$, then set $\delta_{M,D}(x,y) = \infty$. For $x, y \in M$, define the Kobayashi pseudo-distance of $(M, D)$ by
\[
d_{M,D}(x,y) = \inf \sum_{j=1}^{N} \delta_{M,D}(x_{j-1},x_{j})
\]
where the infimum is taken over finite sets of points $\{x_0, \ldots, x_N\}$ of $M$ such that $x = x_0$ and $y = x_N$. When $d_{M,D}$ is a distance, we say that $(M, D)$ is Kobayashi hyperbolic. Moreover, if every Cauchy sequence with respect to $d_{M,D}$ has a convergent subsequence, then we say that $(M, D)$ is complete Kobayashi hyperbolic.

In order to make sense of Definition 1.1 we need $d_{M,D}$ to be bounded on all pairs of points belonging to the same connected component.

Theorem 1.2. Let $M$ be a connected complex manifold with a holomorphic bracket generating distribution $D$. Then for any two points $x, y$ in $M$, one has $d_{M,D}(x,y) < \infty$.

The proof of this theorem is based on the one given by Chow in [6] in the context of sub-Riemannian geometry: let $M$ be a connected manifold and $D$ a bracket generating subbundle in the tangent bundle of $M$. Then any two points in $M$ can be joined by a horizontal piecewise curve.

In [8] the Kobayashi-Royden infinitesimal pseudo-metric was generalized by Demailly to complex manifolds with holomorphic distributions.

Definition 1.3 (Demailly [8]). The Kobayashi-Royden infinitesimal pseudo-metric of $(M, D)$ is the Finsler metric on $D$ defined for any $x \in M$ and $v \in D_p$ by
\[
k_{M,D}(v) = \inf \left\{ \lambda > 0 : f: \Delta \rightarrow M \text{ holomorphic} \right\}
\]
where the infimum is taken over finite sets of points $\{x_0, \ldots, x_N\}$ of $M$ such that $x = x_0$ and $y = x_N$. When $k_{M,D}$ is positive definite on every fiber $D_x$ and satisfies a uniform lower bound $k_{M,D}(v) \geq \epsilon|v|_g$ for $\epsilon$ sufficiently small and depending on any smooth Hermitian metric $g$ on $D$, when $x$ describes a compact subset of $M$.

If $D$ is the holomorphic tangent bundle of $M$ itself, then $d_{M,D}$ and $k_{M,D}$ become the usual Kobayashi pseudo-distance $d_M$ and Kobayashi-Royden infinitesimal pseudo-metric $k_M$. Notice that, by definition,
\[
d_M(x,y) \leq d_{M,D}(x,y), \quad k_M(v) \leq k_{M,D}(v)
\]
for any $x, y \in M$ and $v \in D_x$. Hence, if a complex manifold $M$ is (infinitesimally) Kobayashi hyperbolic, then $(M, D)$ is also (infinitesimally) Kobayashi hyperbolic. However, it should be stressed that the converse is not true.

Let $G^C$ be a complex semisimple Lie group. Let $P$ be a parabolic subgroup in $G^C$ and $G$ a non-compact real form of $G^C$. Then an open $G$-orbit in the flag manifold $G^C/P$ is called a flag domain (cf. [10] [17]). Without loss of generality, assume that the $G$-orbit of $eP$ is open in $G^C/P$. Then we call a flag domain $F = G \cdot z_0 = G/V$ with $z_0 = eP \in G^C/P$ a canonical flag domain.
if $V := G \cap P$ is a compact Lie subgroup containing a maximal torus of $G$. Two of the most important examples of canonical flag domains are the period domains and the Mumford–Tate domains. In [4], it is explained that the canonical flag domain $F$ admits a HBGD, denoted by $H_F$, which is referred to as the superhorizontal distribution. We are able to show that the canonical flag domain $F$ with $H_F$ is complete Kobayashi hyperbolic by exploiting that it has holomorphic sectional curvature bounded from above by a negative constant. For more details, see Section 3 and 4. Note that flag domains that are not Hermitian symmetric spaces of noncompact type, always contain compact complex homogeneous submanifolds and hence those are not Kobayashi hyperbolic.

In [15], Nakajima proved that if a complex manifold $M$ is Kobayashi hyperbolic, then it is biholomorphic to a Siegel domain of second type, i.e., to a bounded homogeneous domain in a Euclidean space. As an analogue of Nakajima’s theorem, we can characterize canonical flag domains through Kobayashi hyperbolicity with HBGDs. We say that $(M, D)$ is $G$-homogeneous complex manifold with an invariant distribution if $G$ acts transitively, holomorphically and almost effectively on $M$ and $D$ is invariant with respect to the action of $G$.

**Theorem 1.4.** Let $G$ be a linear semisimple Lie group. Then $(M, D)$ is a Kobayashi hyperbolic $G$-homogeneous complex manifold with an invariant holomorphic bracket generating distribution if and only if the universal covering of $M$ is a canonical flag domain and the induced distribution is the superhorizontal one.

The proof of Theorem 1.4 will be given in Section 5.

The relation between Kobayashi hyperbolicity and negative curvature is induced by a generalization of Schwarz’s lemma: let $f : \Delta \to M$ be a holomorphic mapping, where $M$ is a complex manifold equipped with a Hermitian metric $g$. Suppose that the holomorphic sectional curvature of $M$ is bounded from above by a negative constant. Then $f^*g \leq cg_\Delta$, for some positive constant $c$ and the Poincaré metric $g_\Delta$ on $\Delta$. A generalization of this Schwarz’s lemma for holomorphic discs tangential to a holomorphic distribution with holomorphic sectional curvature bounded from above by a negative constant is already well-known (for instance see [5]). In this paper we will consider generalized Schwarz’s lemma of Yau given in [18] to obtain the version given in [5]. Even though we only need Schwarz’s lemma given in Corollary 3.2, generalized Schwarz’s lemma of Yau for complex manifolds with holomorphic distributions will be described for future reference.

Here is the outline of the paper. In section 2, we recall notions related to holomorphic vector fields and prove Theorem 1.2. The properties are given without proof since they can be obtained through the same proofs used in [13] and in the papers cited. In section 3, we explain the relation between the Kobayashi hyperbolicity of $(M, D)$ and the holomorphic sectional curvature bounded from above by a negative constant. In section 4, we describe two examples of Kobayashi hyperbolic complex manifolds with HBGDs. In section 5, we prove Theorem 1.4.

Throughout this paper, the Roman letters $a, b, c, \ldots$ run from 1 to $m$, the Roman letters $i, j, k, \ldots$ run from 1 to $d$, the Greek letters $\alpha, \beta, \gamma, \ldots$ run from $d + 1$ to $m$ and the Greek letters $\sigma, \mu, \nu, \ldots$ run from 1 to $n$. Let $g, v, \xi, \ldots$ denote Lie algebras and $G, V, K, \ldots$ the Lie groups corresponding to $g, v, \xi, \ldots$. Given a subspace $a \subset g$, let $a^G$ denote its complexification.

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2. A Kobayashi pseudo-distance for complex manifolds with holomorphic bracket generating distributions

2.1. Holomorphic vector fields. Let $M$ be a complex manifold and $T_M$ its holomorphic tangent bundle. A holomorphic vector field $X$ on $M$ is a holomorphic section of $T_M$. Since $T_M$ is naturally isomorphic to the real tangent bundle $TM$, we can identify $X$ with a real vector field that we continue to denote by $X$. A flow $\phi_X$ associated to $X$, defined on an open subset $U$ of $\mathbb{R} \times M$ containing $\{0\} \times M$, is given in the following way: for $(t, p) \in U$, we set $\phi_X(t, p) := c_p(t)$, where $c_p$: $(a(p), b(p)) \to M$ is the unique maximal solution to the initial value problem

$$\begin{align*}
\frac{d}{dt}c(t) &= X \circ c(t), \\
c_p(0) &= p.
\end{align*}$$

Let $\phi_X(t)p$ denote $\phi_X(t, p)$. Fix $p_0 \in M$ and a sufficiently small $\epsilon > 0$. Then, there is an open neighborhood $\tilde{U}$ of $p_0$ such that $p \mapsto \phi_X(t)p$ is holomorphic on $\tilde{U}$ and for each $t \in (-\epsilon, \epsilon)$, the mapping $\phi_X(t)$ is a holomorphic diffeomorphism from $\tilde{U}$ onto its image. Define the complex flow of $X$ to be

$$\phi_X(s + it) := \phi_X(s) \circ \phi_{itX}(t).$$

Then, for sufficiently small $\epsilon' > 0$, the complex flow $\phi_X$ is a well-defined holomorphic mapping on $\{z \in \mathbb{C} : |z| < \epsilon'\}$ and satisfies $\frac{d}{dz} \phi_X(z)p = X \circ \phi_X(z)p$.

2.2. Piecewise connected horizontal discs. Let $M$ be a complex manifold of dimension $m$ and $D$ a HBGD of rank $d$ on $M$. Fix $x \in M$. Let $X_1, \ldots, X_d$ be holomorphic vector fields on a small neighborhood $U$ of $x$ such that $\{X_1, \ldots, X_d\}$ is a basis of $D$ for all $y \in U$. Choose suitable $X_{i,j,k}$ such that

$$X_{d+1} = [X_{i_1,1}, X_{i_1,2}], \quad X_{d+2} = [X_{i_2,1}, X_{i_2,2}], \ldots, \quad X_m = [X_{i_m-d,1}, \ldots, X_{i_m-d,k}].$$

form a basis of $T_yM$ together with $X_1, \ldots, X_d$ at all points $y \in U$. Then, there is a small open neighborhood $\tilde{U} \subset U$ of $x$ and $W \subset \mathbb{C}^m$ such that the holomorphic mapping $F: W \to \tilde{U}$ defined by

$$F(t_1, \ldots, t_m) = \phi_1(t_1) \circ \cdots \circ \phi_m(t_m)x$$

is a holomorphic diffeomorphism. Here $\phi_1, \ldots, \phi_m$ are the complex flows associated to the vector fields $X_1, \ldots, X_m$ described in Section 2.1. In particular, for each $j$ with $d + 1 \leq j \leq m$, the complex flow $\phi_j$ is a finite composition of $\phi_1$ and $\phi_j^{-1}$ with $1 \leq i \leq d$. Hence, for fixed $t_1, \ldots, t_m$, there are holomorphic discs connecting $x$ and $\phi_1(t_1) \circ \cdots \circ \phi_m(t_m)x$. Note that

$$\frac{dF}{dt_i}(t_1, \ldots, t_m) = d\phi_1(t_1) \circ \cdots \circ d\phi_{i-1}(t_{i-1})X_i|_{\phi_i(t_i)}(t_{i+1}) \circ \cdots \circ d\phi_m(t_m)x$$

and in particular $dF|_0 = id$. This implies that there is an open neighborhood of $x$ such that any two points can be connected by finite number of holomorphic discs tangential to $D$. Using standard methods, it is possible to show that local connectivity implies global connectivity. So, if $M$ is connected, every two points in $M$ can be connected by a chain of holomorphic discs that are tangential to $D$. That is for any $x, y \in M$, we can take a finite set $\{x_0, \ldots, x_N\} \subset M$ and a set $\{a_1, \ldots, a_N\} \subset \Delta$ with $x = x_0$ and $y = x_N$ such that there are holomorphic discs $f_j: \Delta \to M$ tangential to $D$ for each $j = 1, \ldots, N$ with $f_j(0) = x_{j-1}$ and $f_j(a_j) = x_j$. Hence this completes the proof of Theorem 1.2.
2.3. Properties of the Kobayashi pseudo-distance and the Kobayashi pseudo-metrics for \((M, D)\). In this section, selected properties of the Kobayashi pseudo-distance given in Definition \[\text{1.1}\] will be presented without proof. Here \(M\), \(M_1\), and \(M_2\) are complex manifolds equipped with HBGDs \(D\), \(D_1\) and \(D_2\), respectively.

**Proposition 2.1.**

1. The Kobayashi pseudo-distance satisfies the triangle inequality.
2. Let \(f\) be a holomorphic mapping from \((M_1, D_1)\) to \((M_2, D_2)\) such that \(df(D_1) \subset D_2\). Then for any \(x, y \in M_1\) and \(v \in D_1\) one has

\[
d_{M_2, D_2}(f(x), f(y)) \leq d_{M_1, D_1}(x, y),
\]

\[
k_{M_2, D_2}(df(v)) \leq k_{M_1, D_1}(v).
\]

**Proposition 2.2.** (Demainily \[\S\] Proposition 1.5.) \(k_{M,D}\) is upper semicontinuous on the total space of \(D\). If \(M\) is compact, \((M, D)\) is infinitesimally Kobayashi hyperbolic if and only if there are no non-constant entire curves \(f: \mathbb{C} \to M\) tangential to \(D\). In that case, \(k_{M,D}\) is a continuous (and positive definite) Finsler metric on \(D\).

**Proposition 2.3** (cf. \[16, 1\]). For \(x, y \in M\) we have

\[
d_{M, D}(x, y) = \inf_{\gamma} \int k_{M, D}(\gamma'(t)) dt,
\]

where the infimum is taken over all piecewise differentiable curve \(\gamma: [0, 1] \to M\) such that \(\gamma'(t) \in D\) for all \(t \in [0, 1]\) and \(\gamma(0) = x, \gamma(1) = y\).

**Corollary 2.4.**

1. \((M, D)\) is Kobayashi hyperbolic if and only if \((M, D)\) is infinitesimally Kobayashi hyperbolic.
2. The Kobayashi distance \(d_{M,D}\) is continuous.

Suppose that \((M, D)\) is Kobayashi hyperbolic. Then the following theorem of van Dantzig and van der Waerden in \[7\] applies: let \(X\) be a connected locally compact metric space. Then the isometry group of \(X\) is locally compact in the compact-open topology and the isotropy subgroup at any point of \(X\) is compact. Furthermore if \(X\) is compact, then the isometry group of \(X\) is also compact. Hence the isometry group of \(M\) with respect to \(d_{M,D}\) is locally compact. By a theorem of Bochner and Montgomery in \[3\] Theorem 1 we obtain the following:

**Proposition 2.5.** Suppose that \((M, D)\) is Kobayashi hyperbolic. Then the set of holomorphic diffeomorphisms of \(M\) preserving \(D\), say \(\text{Aut}(M, D)\), is a Lie group and at any point in \(M\), the isotropy subgroup of \(\text{Aut}(M, D)\) is compact.

**Corollary 2.6.** Suppose that \((M, D)\) is a Kobayashi hyperbolic \(G\)-homogeneous complex manifold with an invariant distribution \(D\). Then \(G\) is not a complex Lie group.

**Proposition 2.7.** Let \(\tilde{M}\) be a covering manifold of \(M\) with the covering projection \(\pi: \tilde{M} \to M\). Define an induced HBGD, denoted by \(\tilde{D}\), on \(\tilde{M}\) by \(d\pi^{-1}(D)\).

1. Let \(x, y \in M\) and choose \(\tilde{x} \in \tilde{M}\) such that \(\pi(\tilde{x}) = x\). Then

\[
d_{M, D}(x, y) = \inf_{\tilde{y}} d_{\tilde{M}, \tilde{D}}(\tilde{x}, \tilde{y})
\]

where the infimum is taken over all \(\tilde{y} \in \tilde{M}\) such that \(y = \pi(\tilde{y})\).

2. \((M, D)\) is (complete) Kobayashi hyperbolic if and only if \((\tilde{M}, \tilde{D})\) is (complete) Kobayashi hyperbolic.
Proposition 2.8. Let $\pi: P \to M$ be a holomorphic fiber bundle over $M$. Suppose that $(M, D)$ is (complete) Kobayashi hyperbolic. Let $T_P = H \oplus V$ be a decomposition where $V$ is the vertical distribution and $H$ is any horizontal distribution (i.e., $d\pi|_H: H\to TM$ is an isomorphism). Set $\tilde{D} = (d\pi|_H)^{-1}(D) \subset H$.

1. If $\tilde{D}$ is bracket generating in $T_P$, then $(P, \tilde{D})$ is (complete) Kobayashi hyperbolic.
2. If the fiber of $P$ is (complete) Kobayashi hyperbolic, then $(P, D \oplus V)$ is also (complete) Kobayashi hyperbolic.

Proposition 2.9. Suppose that there is an embedding $\iota: M_1 \to M_2$ such that $d\iota(D_1) \subset D_2$. If $(M_2, D_2)$ is (complete) hyperbolic, then $(M_1, D_1)$ is also (complete) hyperbolic.

Proposition 2.10. Suppose that $(M, D)$ is complete Kobayashi hyperbolic. Then the set of holomorphic mappings from $M_1$ to $(M, D)$ tangential to $D$ is a normal family: every sequence of holomorphic maps from $M_1$ to $M$ tangential to $D$ either has a subsequence that converges uniformly on compact subsets or has a compactly divergent subsequence.

3. Kobayashi hyperbolicity and negative curvature

3.1. Generalized Schwarz’s lemma for $(M, D)$ and Kobayashi hyperbolicity for $(M, D)$ with negative holomorphic sectional curvatures. Let $M$ be a complex manifold of complex dimension $m$ and $D$ a holomorphic distribution on $M$ of rank $d$ with a Hermitian metric $g$. Let $A^k(D)$ denote the set of $D$-valued $k$-forms on $M$. Given a Hermitian metric $g$, there is a unique connection, called the Chern connection, $\nabla: A^0(D) \to A^1(D)$ which is compatible with both the complex structure of $M$ and the metric. Choose a local orthonormal frame $e_1, \ldots, e_d$ of $D$ and denote its dual frame by $\theta_1, \ldots, \theta_d$, i.e., $g = \sum \theta_i \overline{\theta}_i$. Denote $\theta_{ij}$ a connection 1-form, i.e.,

$$\nabla e_i = \sum \theta_{ij} \otimes e_j.$$ 

If one extends $g$ to a Hermitian metric of $T_M$ and decompose it as $T_M = D \oplus D^\perp$ where $D^\perp$ is the orthogonal complement of $D$ in $T_M$, then $\nabla = \pi_D \circ \nabla_M$ where $\nabla_M$ is the Chern connection with respect to the extended Hermitian metric on $T_M$ and $\pi_D$ is the orthogonal projection onto $D$. By adding a local frame of $D^\perp$ we can complete the given frame to $e_1, \ldots, e_m$ and the dual frame to $\theta_1, \ldots, \theta_m$. Then $d\theta_i - \theta_{ij} \wedge \theta_j$ can be expressed by

$$(3.1) d\theta_i = \theta_{ij} \wedge \theta_j + \Theta_i + N_i,$$

where $N_i$ contains $\theta_\alpha$ or $\overline{\theta}_\alpha$ for $d + 1 \leq \alpha \leq m$ and $\Theta_i$ consists of $\theta_j$ and $\overline{\theta}_j$ for $1 \leq j \leq d$. We can write

$$(3.2) \Theta_i = \Theta_i^{0,0} + \Theta_i^{0,1} + \Theta_i^{1,0}$$

with an $(a, b)$-form $\Theta_i^{a,b}$. Note that $\Theta_i^{0,2} = 0$ and $\Theta_i^{1,1} = 0$ since $M$ is a complex manifold, and $\nabla$ is the Chern connection. The curvature tensor is given by

$$(3.3) \Theta_{ij} = d\theta_{ij} - \theta_{ik} \wedge \theta_{kj} = R_{ijab} \theta_a \wedge \overline{\theta}_b.$$ 

For $X = \sum X_i e_i$ and $Y = \sum Y_i e_i$ in $D_p$, define the holomorphic bisectional curvature of $(M, D, g)$ in the directions $X$ and $Y$ by

$$(3.4) \text{Bisec}_{(D,g)}(X, Y) = \frac{\sum R_{ijk} X_i Y_j \overline{X}_i \overline{Y}_j}{\sum |X_i|^2 \sum |Y_i|^2},$$

and the holomorphic sectional curvature of $(M, D, g)$ in the direction $X$ by

$$(3.5) H_{(D,g)}(X) = \text{Bisec}_{(D,g)}(X, X).$$
For a complex manifold \(N\) equipped with a Hermitian metric \(h\), let \(\{\omega_\sigma\}\) be an orthonormal frame of \((N, h)\). For a holomorphic mapping \(f: N \to M\) tangential to \(D\), let \(u = \sum a_\sigma \overline{a}_\sigma\), where \(f^*(\theta_\sigma) = \sum a_\sigma \omega_\sigma\). Then by the Omori-Yau maximum principle and the same calculations of the Laplacian of \(u\) given in [18], we have the following theorem.

**Theorem 3.1.** Let \((N, h)\) be a complete Hermitian manifold with Ricci curvature bounded from below by a constant \(K_1\). Let \((M, D, g)\) be a complex manifold with a holomorphic distribution \(D\) equipped with a Hermitian metric \(g\). Suppose that the holomorphic sectional curvature of \((M, D, g)\) is bounded from above by a negative constant \(K_2\). Then, for any holomorphic mapping \(f\) from \(N\) into \(M\) tangential to \(D\), we have

\[
f^*(g) \leq \frac{K_1}{K_2} h.
\]

**Corollary 3.2** (cf. [14]). Let \((\Delta, g_\Delta)\) be the unit disc in \(\mathbb{C}\) endowed with the Poincaré metric \(g_\Delta\) with curvature \(-1\). Let \((M, D, g)\) be a complex manifold with a holomorphic distribution \(D\) and \(g\) a Hermitian metric on \(D\). Suppose that the holomorphic sectional curvature of \((M, D, g)\) is bounded from above by a negative constant \(K\). Then, we have

\[
f^*(g) \leq \frac{1}{K} g_\Delta
\]

for any holomorphic mapping \(f\) from \(\Delta\) into \(M\) tangential to \(D\).

For any \(x, y \in M\), define the Carnot-Caratheodory distance with respect to \((M, D, g)\) as

\[
\rho_{M, D, g}(x, y) = \inf_\gamma \int g(\gamma'(t), \gamma'(t))^{1/2} dt
\]

where the infimum is taken over all \(\gamma: [0, 1] \to M\) tangential to \(D\) such that \(\gamma(0) = x\), \(\gamma(1) = y\). If there is no curve satisfying this condition, set \(\rho_{M, D, g}(x, y) = \infty\).

**Theorem 3.3** (cf. [14]). Suppose that \(D\) is bracket generating. Then \(\rho_{M, D, g}\) is finite, continuous on \(M\) and induces the manifold topology.

Let \(\rho_{N, h}\) be a distance function of \(h\) on \(N\). Then Theorem 3.1 implies that for \(a, b \in N\) one has

\[
\rho_{M, D, g}(f(a), f(b)) \leq \sqrt{\frac{K_1}{K_2}} \rho_{N, h}(a, b).
\]

If the holomorphic sectional curvature of \((M, D, g)\) is bounded from above by a negative constant, for any holomorphic mapping \(f: \Delta \to M\) tangential to \(D\), then by Corollary 3.2 one has \(f^* g \leq cg_\Delta\) for some positive constant \(c\). This implies that for any \(a, b \in \Delta\)

\[
\rho_{M, D, g}(f(a), f(b)) \leq \sqrt{c} \rho_\Delta(a, b).
\]

Suppose that \(d_{M, D}(x, y) = 0\) for some \(x, y \in M\). This implies that there are sequences of elements \(a_i, b_i \in \Delta\) and holomorphic mappings \(f_i: \Delta \to M\) such that \(f(a_i) = x\), \(f(b_i) = y\) and \(\rho_\Delta(a_i, b_i) \to 0\) as \(i \to \infty\). Hence \(\rho_{M, D, g}(x, y) = 0\) and as a consequence we have \(x = y\) by Theorem 3.3. Thus we obtain the following:

**Corollary 3.4.** Suppose that the holomorphic sectional curvature of \((M, D, g)\) is bounded from above by a negative constant. Then \((M, D)\) is Kobayashi hyperbolic. If the Carnot-Caratheodory distance of \(g\) is complete, then the Kobayashi distance of \((M, D)\) is also complete.
Remark 3.5. Suppose that $D$ is a subbundle of the holomorphic tangent bundle of $M$. Here $D$ does not need to be bracket generating. Let $g$ be a Hermitian metric on the tangent bundle of $M$ and $\Theta$ denote the Chern curvature tensor of $g$ on $M$. If $\frac{\Theta(\xi,\xi)}{\langle \xi,\xi \rangle}$ is bounded from above by a negative constant for all $\zeta, \xi \in D$, then the same argument in Theorem 3.7 applies. That is, for every holomorphic mapping $f: N \rightarrow M$ tangential to $D$, the inequality $f^*(g) \leq \frac{K}{K^2} h$ holds. Furthermore Corollary 3.3 also holds as [3] 13.4.1 Schwarz’s lemma.

4. Examples of Kobayashi hyperbolic manifold $(M, D)$

4.1. Canonical flag domains with superhorizontal distributions. In this section, we only consider canonical flag domains. Let $T$ denote a maximal torus, $V = G \cap P$ the isotropy group of $G$ at $z_0$ and $K$ the unique maximal compact subgroup of $G$ containing $V$. Let $\mathfrak{t} \subset \mathfrak{v} \subset \mathfrak{g}$ be the Lie algebras of $T \subset V \subset K \subset G$, respectively. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ be the Cartan decomposition. The Killing form $B$ of $\mathfrak{g}$ is negative definite on $\mathfrak{t}$ and positive definite on $\mathfrak{q}$. Let $\mathfrak{h} := \mathfrak{t}^\mathbb{C}$ be a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. Let $\Phi = \Phi(\mathfrak{g}^\mathbb{C}, \mathfrak{h})$ denote the set of roots of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{h}$. Given a root $\alpha \in \Phi$, let $\mathfrak{g}^\alpha \subset \mathfrak{g}^\mathbb{C}$ denote the associated root space. Given a subspace $s \subset \mathfrak{g}^\mathbb{C}$, set

$$\Phi(s) := \{ \alpha \in \Phi : \mathfrak{g}^\alpha \subset s \}.$$

We say that a root $\alpha$ is compact (non-compact) when $\alpha \in \Phi(\mathfrak{t}^\mathbb{C})$ ( $\alpha \in \Phi(\mathfrak{q}^\mathbb{C})$). Fix a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}^\mathbb{C}$. Then

$$\Phi^+ = \Phi(\mathfrak{b})$$

determines a set of positive roots. Let $\Phi^-$ denote the set of corresponding negative roots. Given a subspace $s \subset \mathfrak{g}^\mathbb{C}$, define

$$\Phi^+(s) := \Phi(s) \cap \Phi^+, \quad \Phi^-(s) := \Phi(s) \cap \Phi^-.$$

Let $\{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ denote the corresponding simple roots. For each $\alpha \in \Phi$, one can choose vectors $e_\alpha \in \mathfrak{g}^\alpha$ and $h_\alpha \in \mathfrak{h}$ such that

1. $B(e_{\alpha}, e_{\beta}) = \delta_{\alpha, -\beta}$ and $[e_\alpha, e_{-\alpha}] = h_\alpha$;
2. $B(h_\alpha, x) = \alpha(x)$ for every $x \in \mathfrak{h}$;
3. $[e_\alpha, e_\beta] = 0$, if $\alpha \neq -\beta$ and $\alpha + \beta \notin \Phi$;
4. $\alpha_\beta = N_{\alpha, \beta} e_{\alpha + \beta}$, if $\alpha, \beta, \alpha + \beta \in \Phi$ where $N_{\alpha, \beta}$ are nonzero real constants such that $N_{-\alpha, -\beta} = -N_{\alpha, \beta}, \quad N_{-\alpha, -\beta} = N_{-\beta, \alpha + \beta} = N_{\alpha + \beta, -\alpha}$;
5. for the complex conjugate $\sigma$ of $\mathfrak{g}$ in $\mathfrak{g}^\mathbb{C}$, we have $\sigma(e_\alpha) = e_{\alpha} e_{-\alpha}$ where $\alpha = -1$ if $\alpha$ is compact and $\epsilon_\alpha = 1$ if $\alpha$ is non-compact.

Let $\omega^\alpha$ be dual covectors of $e_\alpha \in \mathfrak{g}^\alpha$ in $(\mathfrak{g}^\mathbb{C})^*$. Let $\{T^1, \ldots, T^r\}$ be the basis of $\mathfrak{h}$ dual to the simple roots $\{\alpha_1, \ldots, \alpha_r\}$. Set $T := \sum_{\alpha \in \Phi(\mathfrak{t}^\mathbb{C})} T^\alpha$. Then $\mathfrak{g}^\mathbb{C}$ can be decomposed as

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-k},$$

where $\mathfrak{g}_l = \{ \xi \in \mathfrak{g}^\mathbb{C} : [T, \xi] = l\xi \}$ with $l \in \mathbb{Z}$. Notice that $[\mathfrak{g}_l, \mathfrak{g}_j] \subset \mathfrak{g}_{l+j}$ and

$$\mathfrak{g}_0 = \mathfrak{v}^\mathbb{C}, \quad \mathfrak{t}^\mathbb{C} = \sum_{i \text{ even}} \mathfrak{g}_i \quad \text{and} \quad q^\mathbb{C} = \sum_{i \text{ odd}} \mathfrak{g}_i.$$

Denote $\mathfrak{g}_+ = \bigoplus_{j > 0} \mathfrak{g}_j$ and $\mathfrak{g}_- = \bigoplus_{j < 0} \mathfrak{g}_j$. Then a homogeneous complex structure on $F$ is given by specifying $T^C F = TF \otimes \mathbb{C} = T^{1,0} F \oplus T^{0,1} F$ with $T^{1,0} F = G \times_V \mathfrak{g}_-$ and $T^{0,1} F = G \times_V \mathfrak{g}_+$, where $T^{1,0} F$ is the holomorphic tangent bundle of $F$. Let

$$H_F := G \times_V \mathfrak{g}_{-1}$$
denote the superhorizontal distribution given in [4, Chapter 4]. Let $H_F$ be the unique HBGD contained in $G \times_V q$. Note that if $F$ is a Hermitian symmetric space of non-compact type, then $H_F$ is the holomorphic tangent bundle $T_F$. Let $g$ be the $G$-invariant Hermitian metric on $H_F$ defined by

$$g(\zeta, \xi) = B(\zeta, \sigma(\xi)).$$

By the expression of Chern curvature given in [12], we obtain

$$\Theta_F = -\sum_{\alpha, \beta \in \Phi^+ \setminus \Phi^+(t^C)} \text{ad}[e_\alpha, e_{-\beta}]_p \otimes \omega^\alpha \wedge \overline{\omega^\beta}$$

$$+ \sum_{\alpha, \beta \in \Phi^+(t^C) \setminus \Phi^+(t^C)} \text{ad}[e_\alpha, e_{-\beta}]_p \otimes \omega^\alpha \wedge \overline{\omega^\beta},$$

(4.5)

where $[e_\alpha, e_{-\beta}]_p \subset \mathfrak{g}$ denotes the projection of $[e_\alpha, e_{-\beta}]$ onto $\mathfrak{v}^C$. For $\zeta, \xi \in H_F|_{xV}$, the holomorphic bisectional curvature of $(F, H_F, g)$ is given by

$$\text{Bisec}_{F, H_F}(\zeta, \xi) = -\frac{B([\zeta, \sigma(\zeta)], [\xi, \sigma(\xi)])}{g(\zeta, \zeta)g(\xi, \xi)} = -\frac{B([\zeta, \sigma(\zeta)], [\xi, \sigma(\xi)])}{g(\zeta, \zeta)g(\xi, \xi)}.$$ 

(4.6)

Let $\zeta \in g_{-1}$. Then $[\zeta, \sigma(\zeta)] \in \sqrt{-1} \mathfrak{v}$. Note that $B$ is positive definite on $\sqrt{-1} \mathfrak{v}$. Hence, there should be a positive constant $c$ such that

$$H_{F, H_F}(\zeta) = -\frac{B([\zeta, \sigma(\zeta)], [\xi, \sigma(\xi)])}{g(\zeta, \zeta)\xi^2} < -c < 0.$$ 

(4.7)

In view of Corollary 3.3 of the homogeneity of $F$ and of the ball-box theorem given in [13, Theorem 2.4.2], we obtain the following.

**Theorem 4.1.** Canonical flag domains with superhorizontal distributions are complete Kobayashi hyperbolic.

**Corollary 4.2.** Let $\Gamma$ be an uniform lattice of $G$, i.e., $\Gamma \setminus G$ is compact. Then $\Gamma \setminus F$, with the distribution induced by the superhorizontal distribution of $F$, is compact and complete Kobayashi hyperbolic.

Suppose that $F$ is not a Hermitian symmetric space of non-compact type. In [17, Lemma 5.1], Wolf showed that there exists the unique closed $K^C$-orbit in $F$. Since it is a compact complex homogeneous submanifold of $F$, this implies that $F$ is not Kobayashi hyperbolic. This $K^C$-orbit is called the base cycle of $F$.

**Corollary 4.3.** There is no holomorphic embedding from a flag domain which is not Hermitian symmetric into the canonical flag domain tangential to the superhorizontal distribution.

**Proof.** Let $F_1$, $F_2$ be flag domains where $F_1$ is not a Hermitian symmetric space, and $D_2$ is the superhorizontal distribution of $F_2$. Let $i: F_1 \to F_2$ be a holomorphic embedding tangential to $D_2$. Then by Proposition 2.9, one has $d_{F_2, D_2}(i(x), i(y)) \leq d_{F_1}(x, y)$. Let $x \neq y$ in the base cycle of $F_1$. Then $d_{F_1}(x, y) = 0$ and this induces the contradiction to that $(F_2, D_2)$ is Kobayashi hyperbolic.

**Remark 4.4.** In [5], the authors proved that the Chern connection on $T_F$ with respect to the same Hermitian metric as in (4.4) has holomorphic sectional curvature bounded from above by a negative constant in non-compact direction $q^C$. Let $i: F_1 \to F_2$ be a holomorphic embedding tangential to $D_2$. Then, by Remark 2.3, we can obtain that $\rho_{F_2, D_2, q^C}(i(x), i(y)) \leq \rho_{F_2, D_2, q^C}(i(x), i(y)) \leq cd_{F_1}(x, y)$ for a positive constant $c$. Hence, there is no holomorphic embedding from a flag domain which is not Hermitian symmetric into the canonical flag domain tangential to the $q^C$-direction.
4.2. Kobayashi hyperbolicity of the complex Euclidean spaces. In [9], Forstneric constructed a holomorphic contact distribution $D$ on $\mathbb{C}^{2n+1}$ such that $(\mathbb{C}^{2n+1}, D)$ is Kobayashi hyperbolic for any $n \in \mathbb{N}$. We generalize his construction to some HBGD on complex Euclidean spaces.

Let $D$ be a HBGD on $\mathbb{C}^n$ of rank $d$. Since $\mathbb{C}^n$ is a contractible Stein manifold, by the Grauert-Oka principle, both $D$ and $T_{\mathbb{C}^n}/D$ are holomorphically trivial vector bundles. Let $X_1, \ldots, X_d, X_{d+1}, \ldots, X_n$ be a global frame of $\mathbb{C}^n$ such that $X_1, \ldots, X_d$ is a frame of $D$. Let $X_j = \sum_{i=1}^n X_{ji} \overline{z}_i$ for each $j = 1, \ldots, n$ with holomorphic functions $X_{ji}$ on $\mathbb{C}^n$. Denote the matrix $(X_{ji})_{j,i=1,\ldots,n}$ by $X$.

(4.8) 
Suppose that there is a $d \times d$ submatrix in $(X_{ji})_{j=1,\ldots,d;i=1,\ldots,n}$ which is invertible at every point in $\mathbb{C}^n$.

Then there is a Fatou-Bieberbach map $\Phi: \mathbb{C}^n \to \mathbb{C}^n$ such that $(\mathbb{C}^n, d\Phi^{-1}(D|\Phi(\mathbb{C}^n)))$ is Kobayashi hyperbolic. This can be achieved through Lemma 4.5 and Lemma 4.6. Those proof are analogue to the corresponding lemmas of Forstneric. Note that the condition (1.8) is not true in general.

**Lemma 4.5.** (Generalization of [9] Lemma 2.1) Let $D$ be a holomorphic distribution on $\mathbb{C}^n$ of rank $d$ satisfying (1.8). Then, for each $N \in \mathbb{N}$, there exists $C_N \in \mathbb{R}$ such that for any holomorphic horizontal disc $f: \Delta \to (\mathbb{C}^n/K, D|_{\mathbb{C}^n/K})$ with $f(0) \in 2^{N_0} \Delta^n \subset \mathbb{C}^n$ for some $N_0$ and

(4.9) 
\[ K := \bigcup_{N=1}^{\infty} 2^{N-1} \partial \Delta \times C_N \Delta^{n-d}, \]
we have $|f'(0)| < c_{N_0}$. Here the constant $c_{N_0}$ only depends on $N_0$.

**Proof.** By permutation of coordinates we can assume that $X_1 := (X_{ji})_{j,i=1,\ldots,d}$ satisfies $\det X_1 \neq 0$ for each point of $\mathbb{C}^n$. Furthermore, by the condition (1.8), we can assume that $X$ has the following form

(4.10) 
\[ X = \frac{d}{n-d} \begin{pmatrix} \Pi_1 & \Pi_2 \\ 0 & \Pi_4 \end{pmatrix}. \]

Then the dual frame $\omega_1, \ldots, \omega_n$ of $X_1, \ldots, X_n$ is given by $(X^{-1})^t dz$ where $dz = (dz_1, \ldots, dz_n)^t$.

Note that

\[ (X^{-1})^t := \frac{d}{n-d} \begin{pmatrix} \pi_1 & 0 \\ \pi_3 & \pi_4 \end{pmatrix} = \frac{d}{n-d} \begin{pmatrix} \Pi_1^{-1} & 0 \\ -\Pi_4^{-1} \Pi_3 \Pi_1^{-1} & \Pi_4^{-1} \end{pmatrix}. \]

Let $f = (f_1, \ldots, f_n): \Delta \to (\mathbb{C}^n/K, D|_{\mathbb{C}^n/K})$ be a holomorphic horizontal disc satisfying $f(0) \in 2^{N_0} \Delta^n \subset \mathbb{C}^n$ for some $N_0$. Since $f$ is a horizontal disc, we have $f^* \omega_j = 0$ for each $j = d+1, \ldots, n$. This yields

(4.12) 
\[ (f_{d+1}^t, \ldots, f_n^t)^t = -((\pi_4)^{-1} \pi_3) \circ f (f_1^t, \ldots, f_d^t)^t. \]

Note that each entry of the matrix $((\pi_4)^{-1} \pi_3)$ is a holomorphic function on $\mathbb{C}^n$. Choose a constant $C_N$ such that

(4.13) 
\[ C_N > 2^N + d^2 2^{2N} \sup_{z \in 2^N \Delta^n} \left| \left((\pi_4)^{-1} \pi_3\right)_{ji}(z) \right|. \]

Then, by the same argument given by Forstneric in [9], we obtain the lemma. \qed
of generality we may assume that $G$ is connected. Since a connected semisimple Lie subgroup with finite center in a Lie group is closed (cf. [11, Proposition 6.1.]), $G$ is closed in Aut($G/V$) and hence $V = G \cap \text{Isot}(M, D)$ is compact where \text{Isot}(M, D) denotes the isotropy subgroup of Aut($M, D$) at $x_0$. \hfill \Box

Lemma 5.1. \emph{V is compact.}

Proof. Since $G$ has finite number of components and acts on $M$ almost effectively, without loss of generality we may assume that $G \subset \text{Aut}(M, D)$ and $G$ is connected. Since a connected semisimple Lie subgroup with finite center in a Lie group is closed (cf. [11, Proposition 6.1.]), $G$ is closed in Aut($M, D$) and hence $V = G \cap \text{Isot}(M, D)$ is compact where \text{Isot}(M, D) denotes the isotropy subgroup of Aut($M, D$) at $x_0$. \hfill \Box

Let $g$ denote the Lie algebra of $G$ and $v$ the Lie algebra of $V$ in $g$. Then, there is a vector subspace $m$ of $g$ such that

\begin{equation}
    g = v \oplus m \quad \text{and} \quad [v, m] \subset m.
\end{equation}

The tangent space of $M$ at $x_0$ can be identified with $g/v = m$. Let $g^{\mathbb{R}}_1$ denote the subspace of $m$ corresponding to $D_{x_0}$. Define

\begin{equation}
    g^{\mathbb{R}}_2 = [g^{\mathbb{R}}_1, g^{\mathbb{R}}_1]/(v + g^{\mathbb{R}}_1), \ldots, g^{\mathbb{R}}_k = [g^{\mathbb{R}}_1, g^{\mathbb{R}}_{k-1}]/(v + g^{\mathbb{R}}_1 + \cdots + g^{\mathbb{R}}_{k-1}).
\end{equation}

Then we have

\begin{equation}
    g = v \oplus g^{\mathbb{R}}_1 \oplus \cdots \oplus g^{\mathbb{R}}_k.
\end{equation}

Since $M = G/V$ is a complex manifold and $D$ is $G$-invariant with compact $V$, we may assume that there is an endomorphism $j : g \to g$ induced from the complex structure of $M$, such that

\begin{align}
    jv &= 0, j^2x = -x, \\
    \text{Ad}_v jx &= j \circ \text{Ad}_v x, \\
    [jx, jy] &= [x, y] + j[jx, y] + j[x, jy],
\end{align}

\begin{equation}
    \text{Ad}_v g^{\mathbb{R}}_1 \subset g^{\mathbb{R}}_1,
\end{equation}

where $x, y \in m$ and $v \in V$. Extend $j$ complex linearly to the complexification $g^{\mathbb{C}}$ of $g$, and denote by $g^+, g^-$ the eigenspaces of $j$ with eigenvalues $\sqrt{-1}, -\sqrt{-1}$ respectively. Then $G \times_V g^+$ and $G \times_V g^-$ represent the holomorphic tangent bundle and anti-holomorphic tangent bundle of $M$ respectively. Let $g_1$ denote the subspace in $g^+$ corresponding to $D_{x_0}$. Then $g_1 = \{x - \sqrt{-1}jx \in g^{\mathbb{C}} : x \in g_1^R\}$. Define

\begin{equation}
    g_2 := [g_1, g_1]/(v^C + g_1), \ldots, g_k := [g_1, g_{k-1}]/(v^C + g_1 + \cdots + g_{k-1}).
\end{equation}
Then

(5.9) \( \mathfrak{g}^+ = \mathfrak{g}_{1} \oplus \cdots \oplus \mathfrak{g}_{k} \).

Similarly, let \( \mathfrak{g}_{-1}\{ x + \sqrt{-1}jx \in \mathfrak{g}^C : x \in \mathfrak{g}^R_{1} \} \) and construct \( \mathfrak{g}_{-2}, \ldots, \mathfrak{g}_{-k} \). Set

(5.10) \[
\mathfrak{g}^- = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-k}, \quad \text{and} \\
\mathfrak{g}^C = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \oplus \cdots \oplus \mathfrak{g}_{k}
\]

where \( \mathfrak{g}_{0} := \mathfrak{v}^C \). For the notational convention, denote

(5.11) \[
\mathfrak{g}_{\leq l} := \sum_{j \leq l} \mathfrak{g}_{j} \quad \text{and} \quad \mathfrak{g}_{\geq l} := \sum_{j \geq l} \mathfrak{g}_{j}.
\]

5.2. **Proof of Theorem 1.3** This section is a continuation of Section 5.1. We use the same notation. Since \((M, D)\) is Kobayashi hyperbolic, we obtain

(5.12) \[
[x, jx] \neq 0 \quad \text{for all} \quad x \in \mathfrak{g}^R_{1}.
\]

If not, there is a holomorphic mapping from the complex plane since \( \mathbb{R} x + \mathbb{R} jx \) is a complex Lie subalgebra of \( \mathfrak{g} \). This contradicts Corollary 2.6.

Since \( \mathfrak{g} \) is semisimple, there are simple Lie algebras \( s^1, \ldots, s^N \) such that \( \mathfrak{g} = s^1 \oplus \cdots \oplus s^N \). Note that \([s^l, s^m] = 0 \) for any \( l, m \) with \( l \neq m \). Let \( s^1 = s^l \cap \mathfrak{g}^R_{1} \). Then \( s^1 \) is non-empty and bracket generating in \( s^l \). We obtain that \( j(s^1) \subset s^1 \). Otherwise, there exists \( x \in s^1 \) such that \([x, jx] = 0 \) and this contradicts (5.12). By the bracket generating property of \( s^1 \) in \( s^l \) and (5.6), we can get

(5.13) \[
j(s^l) \subset s^l \quad \text{for each} \quad l = 1, \ldots, N.
\]

Suppose that there exists \( l \) such that \( s^l \) is of compact type. Let \( G_l \) be a simple Lie group corresponding to \( s^l \) and \( V_l = G_l \cap V \). Then the \( G_l \)-orbit in \( M \) at \( x_0 (= G_l/V_l) \) with holomorphic distribution \( G_l \times V_l, s^1 \) is a Kobayashi hyperbolic compact homogeneous complex submanifold in \( M \). This implies that \( \text{Aut}(G_l/V_l) \) is a complex Lie group which contradicts Proposition 2.6.

Hence, we have proved that \( \mathfrak{g} \) is a semisimple Lie algebra of non-compact type.

Let \( \mathfrak{k} \) be a maximal compact subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{v} \) and \( \mathfrak{g} = \mathfrak{k} + \mathfrak{a} \) a Cartan decomposition with respect to \( \mathfrak{k} \). Let

(5.14) \[
\mathfrak{g}^C = \mathfrak{g}_{-k} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_{0} + \mathfrak{g}_{1} + \cdots + \mathfrak{g}_{k}
\]

be the decomposition induced by \( D \). Note that \( \mathfrak{g}_{\leq 0} \) is a Lie subalgebra of \( \mathfrak{g}^C \).

Let \( G_{\leq 0} \) denote the connected Lie subgroup corresponding to the Lie subalgebra \( \mathfrak{g}_{\leq 0} \) in \( G^C \). Let \( N_{G^C}(G_{\leq 0}) = \{ g \in G^C : \text{Ad}_g(\mathfrak{g}_{\leq 0}) \subset \mathfrak{g}_{\leq 0} \} \) be the normalizer of \( G_{\leq 0} \). Note that \( N_{G^C}(G_{\leq 0}) \) is a closed Lie subgroup of \( G^C \). Since the Lie algebra of \( N_{G^C}(G_{\leq 0}) \) is given by \( \{ X \in \mathfrak{g}^C : [X, \mathfrak{g}_{\leq 0}] \subset \mathfrak{g}_{\leq 0} \} = \mathfrak{g}_{\leq 0} \), it follows that \( G_{\leq 0} \) is the identity component of \( N_{G^C}(G_{\leq 0}) \).

Hence \( G_{\leq 0} \) is a closed algebraic subgroup of \( G^C \) and \( G^C / G_{\leq 0} \) is a complex homogeneous manifold.

Let \( G_{\mathfrak{u}} \) denote the compact real form of \( G^C \) with respect to \( G \), i.e., the Lie algebra of \( G_{\mathfrak{u}} \) is given by \( \mathfrak{g}_{\mathfrak{u}} = \mathfrak{k} + \sqrt{-1} \mathfrak{q} \). By Lemma 5.2 and the fact that \( \mathfrak{g}_{\leq 0} \cap \mathfrak{g} = \mathfrak{v} \), we obtain that \( \mathfrak{g}_{\mathfrak{u}} \cap \mathfrak{g}_{\leq 0} = \mathfrak{v} \) and hence

(5.15) \[
\dim_{\mathbb{C}}(G_{\mathfrak{u}}\text{-orbit at } eG_{\leq 0}) = \dim_{\mathbb{C}}(G^C / G_{\leq 0}).
\]

It implies that the \( G_{\mathfrak{u}} \)-orbit of \( eG_{\leq 0} \) is open. Besides, since the \( G_{\mathfrak{u}} \)-orbit is compact, it is equal to \( G^C / G_{\leq 0} \). In particular, \( G^C / G_{\leq 0} \) is compact and hence \( G_{\geq 0} \) is a parabolic subgroup (cf. [2] Chapter 3)). By a similar procedure, we may conclude that \( G_{\geq 0} \) is a parabolic subgroup. Since \( \mathfrak{g}_{\leq 0} \cap \mathfrak{g}_{\geq 0} = \mathfrak{g}_{0} = \mathfrak{v}^C \), there exists a Cartan subalgebra of \( \mathfrak{g}^C \) contained in \( \mathfrak{v}^C \) (cf. Corollary 2.1.3
in [10]). This implies that \( M \), the \( G \)-orbit of \( eG_{s0} \) in \( G^C/G_{s0} \), is a canonical flag domain. Since flag domains are simply connected, the isotropy subgroup \( G_{s0} \cap G \) is connected and hence it is the identity component of \( V \). This implies that \( M \) is the universal covering of \( M = G/V \).

Let \( g = t' + q' \) be a Cartan decomposition with corresponding Cartan involution \( \theta' \). It is known that \( \theta' \) is an inner automorphism of \( g \) if and only if \( k' \) contains a Cartan subalgebra of \( g \). Let \( t \) denote \( t^C \cap g \). Note that \( t \subset v \subset t \). Let \( \theta \) be the Cartan involution with respect to \( t \), that is, \( \theta(k + q) = k - q \) for all \( k \in t \) and \( q \in q \). Since \( t \) contains a Cartan subalgebra of \( g \), \( \theta \) is an inner automorphism of \( g \). This implies that \( \theta \) can be expressed as \( \text{Ad}_{exp \xi} \), where \( \xi \) is an element of the center of \( t \). Hence \( \xi \in t \subset v \). By (5.7), we obtain \( \theta(g^R_1) = g^R_1 \) and in particular, \( g^R_1 = g^R_1 \cap \xi + g^R_1 \cap q \).

Suppose that \( t_1 := t \cap g^R_1 \neq \{0\} \), and define a Lie subalgebra

\[
\mathfrak{t}' := v + \mathfrak{t}_1 + \mathfrak{t}_2 + \cdots + \mathfrak{t}_k
\]

where \( \mathfrak{t}_2 = [\mathfrak{t}_1, \mathfrak{t}_1]/v, \ldots, \mathfrak{t}_j = [\mathfrak{t}_1, \mathfrak{t}_{j-1}]/(v + \mathfrak{t}_0 + \cdots + \mathfrak{t}_{j-1}) \). Let \( K' \) denote the connected Lie subgroup of \( G \) with respect to the Lie algebra \( \mathfrak{t}' \). Then \( K' \times V \mathfrak{t}_1 \) is a HBGD and \( (K'/V, K' \times V \mathfrak{t}_1) \) is Kobayashi hyperbolic. However, since \( (K'/V, K' \times V \mathfrak{t}_1) \) is a compact complex manifold such that \( \text{Aut}(K'/V, K' \times V \mathfrak{t}_1) \) acts transitively on \( K'/V \), it cannot be Kobayashi hyperbolic. Hence \( \mathfrak{t}_1 \) should be \( \{0\} \). This implies that \( g^R_1 \subset q \). Since the superhorizontal distribution on the canonical flag domain is the unique invariant HBGD contained in \( G \times V \), we obtain that \( g^R_1 \) is a subalgebra which induces the superhorizontal distribution. \( \square \)

**Lemma 5.2.** \( g_{s0} \) and \( g_{s0} \) are \( \theta \)-invariant where \( \theta : k + q \mapsto k - q \) is a Cartan involution of \( g \) with \( k \in \mathfrak{t} \) and \( q \in q \).

**Proof.** If not, there exists an element \( x \in g_{s0} \) such that \( \theta(x) \in g_{s0} \) since \( g_0 = \mathfrak{g}^C \subset \mathfrak{t}^C \) is \( \theta \)-invariant. Denote \( x = x_t - \sqrt{-1}jx_t + x_q - \sqrt{-1}jx_q \) with \( x_t \in t \) and \( x_q \in q \). Let \( \sigma \) be a complex conjugation with respect to \( g \) in \( g^C \). Since \( \sigma(g_{s0}) = g_{s0} \), it follows that \( \sigma \theta(x) = x_t + \sqrt{-1}jx_t - x_q - \sqrt{-1}jx_q \in g_{s0} \). Since \( x + \sigma \theta(x) = 2x_t - 2\sqrt{-1}jx_q \in g_{s0} \), we conclude that \( x_t - \sqrt{-1}jx_q \) is an \( \sqrt{-1} \)-eigenvector with respect to \( j \). This implies that \( x_q - x_t = 0 \) and in particular, \( x_q = x_t = 0 \). Finally, it follows that \( x_q = x_t = 0 \). \( \square \)

### 5.3. Miscellaneous

In this section, we assume that \( G \) is a Lie group without semisimple condition and \( V \) is compact. Under this condition the Lie algebra decompositions given in Section 5.1 hold. Since \( [g_{s0}, g_{s1}] \subset g_{s1} \), we obtain \( [g_{s0}, g_{s\ell}] \subset g_{s\ell} \cap g_{s0} \) for each \( 2 \leq \ell \leq k \). Since \( D \) is a holomorphic distribution, one has

\[
[C^\infty(D), C^\infty(T_M)] \subset C^\infty(D \oplus T_M).
\]

Here \( C^\infty(E) \) denotes the set of smooth sections of the vector bundle \( E \). Using this expression, we obtain

\[
(5.16) \quad [g_{s1}, g_{s\ell}^-] \subset g_{s\ell+1}.
\]

**Lemma 5.3.**

\[
(5.17) \quad \begin{align*}
&[g_{s\ell}, g_{s\ell}] \subset g_{s\ell+1} - g_{s\ell+1} \text{ when } i \geq \ell, \\
&[g_{s\ell}, g_{s\ell}] \subset g_{s\ell+1} - g_{s\ell+1} \text{ when } i \leq \ell, \\
&[g_{s\ell}, g_{s1}] \subset g_{s1} - g_{s0} + g_{s1} \text{ and } [g_{s1}^R, jg_{s1}^R] \subset g_{s0} + g_{s1}, \text{ for all } i > 0.
\end{align*}
\]

**Proof.** It is enough to prove the first expression. Since \( [g_{s1}, g_{s1}] \subset g_{s1} - g_{s0} + g_{s1} \), by taking complex conjugation, we obtain that \( [g_{s1}, g_{s1}] \subset g_{s1} - g_{s0} + g_{s1} \). Suppose that for every \( i < i' \), \( [g_{s1}, g_{s1}] \subset g_{s2} \). Then \( [g_{s1}, g_{s1}] = [g_{s1}, [g_{s1}, g_{s1}]] + [g_{s1}, [g_{s1}, g_{s1}]] \subset g_{s1} \) and hence for every \( i \), \( [g_{s1}, g_{s1}] \subset g_{s2} \).
Fix $i'$ and suppose that for every $i < i'$, one has $[g_i, g_{-i}] \subset g_{\leq i - i' + 1}$ for all $1 \leq i' \leq i$. Furthermore, suppose that for fixed $i'$ with $i' \leq i'$, one has $[g_{i'}, g_{-i'}] \subset g_{\leq i' - i' + 1}$ for all $i < i'$. Then, since $[g_{i'}, g_{-i'}] = [g_{i'}, g_{-i'}^{-1}], g_{-1}] + [g_{i'}, g_{-1}], g_{-i'}^{-1}]$, we obtain that $[g_{i'}, g_{-i'}] \subset g_{\leq i' - i' + 1}$ and the lemma is proved.

Lemma 5.4. Let $\mathfrak{r}$ be an abelian ideal of $g$. Then
\begin{equation}
\mathfrak{r} \cap \mathfrak{v} = 0.
\end{equation}

Proof. Let $w \in \mathfrak{r} \cap \mathfrak{v}$. Then $ad_w^2 = 0$ on $g^C$. Since $\mathfrak{v}$ is compact, we can consider $ad_w$ as a skew-symmetric matrix. Since every skew-symmetric matrix is diagonalizable over $\mathbb{C}$, $ad_w$ is a semisimple element and hence $ad_w = 0$. By almost effectiveness we obtain $w = 0$. □

Lemma 5.5. Let $\mathfrak{r}$ be an abelian ideal in $g$. Then
\begin{equation}
[g, g] \cap \mathfrak{r} = [\mathfrak{r}, g].
\end{equation}

Proof. We will prove it by the induction with respect to the dimension of $g$. If $\dim g = 1$, then the statement is trivial. Suppose the statement to be true for every Lie algebra $g$ of dimension less than or equal to $k$ and every abelian ideal $\mathfrak{r}$ in $g$. Now let $\dim g = k + 1$ and $\mathfrak{r} \subseteq g$ be an abelian ideal. If $[g, \mathfrak{r}] = 0$, then (5.19) is true. If $[g, \mathfrak{r}] = \mathfrak{r}$, then $\mathfrak{r} = [g, \mathfrak{r}] \subset [g, g] \cap \mathfrak{r} \subset \mathfrak{r}$ and again the statement holds. Assume that $\mathfrak{r'} := [g, \mathfrak{r}] \subset \mathfrak{r}$. Then $\mathfrak{r'}$ is an ideal in $g$ and $\mathfrak{r}/\mathfrak{r'}$ is a non-trivial abelian ideal in $g/\mathfrak{r'}$. Since $\dim g/\mathfrak{r'} \leq k$, we obtain $[g/\mathfrak{r'}, g/\mathfrak{r'}] \cap \mathfrak{r}/\mathfrak{r'} = [g/\mathfrak{r'}, \mathfrak{r}/\mathfrak{r'}] = 0$ and this implies (5.19). □

Proposition 5.6. Let $\mathfrak{r}$ be an abelian ideal in $g$.

1. If $\mathfrak{r} \cap g_{l}^R \neq \{0\}$, then there exists an equivariantly embedded bounded homogeneous domain of type $I$ tangential to $D$.

2. If $\mathfrak{r} \cap g_{l}^R = \{0\}$, then $\mathfrak{r} \cap g_{l}^R = \{0\}$ for any $l = 1, \ldots, k$.

Proof. (1) : Let $\mathfrak{r}_1 \supset g_{1}^R$. Then $\mathfrak{r}_1 \cap j\mathfrak{r}_1 = \{0\}$. If not, there are $x, y \in \mathfrak{r}_1$ such that $x = jy$ and hence $[x, jx] = [x, y] = 0$. However, this contradicts (5.12). Furthermore $\mathfrak{r}_1 + j\mathfrak{r}_1$ is a subalgebra in $g$ since for any $x, y \in \mathfrak{r}_1$, $[x, y] = 0$, one has $[x, jy] \in \mathfrak{r} \cap (g_0 + g_1) = \mathfrak{r}_1$ (by (5.18) and (5.17)), and $[jx, jy] \in j\mathfrak{r}_1$ (by (5.3)). Let $R$ be the connected Lie subgroup of $G$ corresponding to $\mathfrak{r}_1 + j\mathfrak{r}_1$. Then the $R$-orbit in $M$ is an equivalently embedded bounded homogeneous domain tangential to $D$.

(2) : It follows from Lemma 5.5. □

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