Markov $L_2$ inequality with the Gegenbauer weight

Dragomir Aleksov, Geno Nikolov

Abstract

For the Gegenbauer weight function $w_\lambda(t) = (1 - t^2)^{\lambda-1/2}$, $\lambda > -1/2$, we denote by $\| \cdot \|_{w_\lambda}$ the associated $L_2$-norm,

$$\| f \|_{w_\lambda} := \left( \int_{-1}^{1} w_\lambda(t) f^2(t) \, dt \right)^{1/2}.$$ 

We study the Markov inequality

$$\| p' \|_{w_\lambda} \leq c_n(\lambda) \| p \|_{w_\lambda}, \quad p \in P_n,$$

where $P_n$ is the class of algebraic polynomials of degree not exceeding $n$. Upper and lower bounds for the best Markov constant $c_n(\lambda)$ are obtained, which are valid for all $n \in \mathbb{N}$ and $\lambda > -\frac{1}{2}$.

1 Introduction and statement of the results

Throughout this paper $P_n$ stands for the class of algebraic polynomials of degree not exceeding $n$.

For the Gegenbauer weight function $w_\lambda(t) = (1 - t^2)^{\lambda-1/2}$, $\lambda > -1/2$, we denote by $\| \cdot \|_{w_\lambda}$ the associated $L_2$-norm,

$$\| f \|_{w_\lambda} := \left( \int_{-1}^{1} w_\lambda(t) f^2(t) \, dt \right)^{1/2}.$$ 

Here we study the Markov inequality in this norm for the first derivative of polynomials from $P_n$, in particular, we are interested in the best Markov constant

$$c_n(\lambda) = \sup_{p \neq 0} \frac{\| p' \|_{w_\lambda}}{\| p \|_{w_\lambda}}.$$

Let us start with a brief account of the known results.

In the case $\lambda = \frac{1}{2}$ (the case of a constant weight function), E. Schmidt proved that

$$c_n(1/2) = \frac{(2n+3)^2}{4\pi} \left( 1 - \frac{\pi^2 - 3}{3(2n+3)^2} + \frac{16R}{(2n+3)^2} \right)^{-1}, \quad -6 < R < 13.$$

Nikolov studied two other particular cases, $\lambda = 0, 1$, and proved the following two-sided estimates for the corresponding Markov constants:

$$0.472135 n^2 \leq c_n(0) \leq 0.478849 (n+2)^2,$$

$$0.248549 n^2 \leq c_n(1) \leq 0.256861 (n+\frac{5}{2})^2.$$ (1.1)

In [1] we obtained an upper bound for $c_n(\lambda)$, which is valid for all $n$ and $\lambda$:

$$c_n(\lambda) \leq \frac{(n+1)(n+2\lambda+1)}{2\sqrt{2\lambda+1}}.$$

This result has been improved in the recent paper [5], where the following theorem was proved:
**Theorem A** For all \( \lambda > -\frac{1}{2} \) and \( n \geq 3 \), the best constant \( c_n(\lambda) \) in the Markov inequality
\[
\|p_n\|_{w_\lambda} \leq c_n(\lambda)\|p_n\|_{w_\lambda}, \quad p_n \in \mathcal{P}_n,
\]
admits the estimates
\[
\frac{n^2(n+\lambda)^2}{4(\lambda+1)(\lambda+2)} \leq [c_n(\lambda)]^2 < \frac{n(n+2\lambda+2)^2}{(\lambda+2)(\lambda+3)}, \quad \lambda \geq 2;
\]
\[
\frac{(n+\lambda)(n+2\lambda)^2}{(2\lambda+1)(2\lambda+5)} \leq [c_n(\lambda)]^2 < \frac{(n+\lambda+\lambda''+2)^4}{2(2\lambda+1)\sqrt{2\lambda+5}}, \quad \lambda > -\frac{1}{2},
\]
where \( \lambda' = \min\{0, \lambda\} \), \( \lambda'' = \max\{0, \lambda\} \).

It has been also proved in [5] that
\[
[c_n(\lambda)]^2 \asymp \frac{1}{\lambda^2} n(n+2\lambda)^3,
\]
which shows that the upper bound in (1.2) has the right order in both \( n \) and \( \lambda \). The lower bound in (1.2) is inferior to the one in (1.3), it appears in (1.2) just to indicate that, roughly, for a fixed \( \lambda \) and large \( n \) the sharp Markov constant is identified within a factor not exceeding two. Although the upper bound in (1.3) is not of the right order with respect to \( \lambda \), for moderate \( \lambda \) (say, \( \lambda \leq 25 \)) it is superior to the one in (1.2).

In the present paper we prove two-sided estimates for \( c_n(\lambda) \), valid for all \( \lambda > -1/2 \), which are of the same nature as (and slightly sharper than) those in (1.3). The approaches for their derivation however are different. In [5], the results are obtained through estimation of appropriate matrix norms. Here, we identify the reciprocal of the squared best Markov constant as the smallest zero of a related orthogonal polynomial, then exploit the associated three-term recurrence relation to evaluate its lower degree coefficients and eventually derive estimates for its smallest zero. Let us mention that a similar relation between the best constant in the \( L_2 \) Markov inequality with the Laguerre weight function and the smallest zero of an orthogonal polynomial is given in [2, p. 85], and in [4] we applied a similar approach to obtain bounds for the best Markov constant in the Laguerre case.

Our main result is the following theorem:

**Theorem 1.1** For all \( n \geq 3 \) and for every \( \lambda > -\frac{1}{2} \), the best constant \( c_n(\lambda) \) in the Markov inequality
\[
\|p\|_{w_\lambda} \leq c_n(\lambda)\|p\|_{w_\lambda}, \quad p \in \mathcal{P}_n,
\]
admits the estimates
\[
\frac{(n+1)(n+\lambda+\frac{1}{2})(n+2\lambda)}{(2\lambda+1)(2\lambda+5)} \leq c_n^2(\lambda) \leq \frac{(n+5\lambda+\frac{9}{2})^4}{2(2\lambda+1)\sqrt{2\lambda+5}}.
\]

By setting \( \lambda = 0.1 \) in (1.5), we obtain an improvement of the upper bounds in (1.1), and combination with the lower bounds in (1.1) yields rather tight estimates.

**Corollary 1.2** For the Chebyshev weights \( w_0(x) = \frac{1}{\sqrt{1-x^2}} \) and \( w_1(x) = \sqrt{1-x^2} \), we have
\[
0.472135 n^2 \leq c_n(0) \leq 0.472871 \left( n + \frac{9}{8} \right)^2,
\]
\[
0.248549 n^2 \leq c_n(1) \leq 0.250987 \left( n + \frac{19}{8} \right)^2.
\]

For the proof of Theorem 1.1 we obtain separately estimates for \( c_n(\lambda) \) in the cases of even and odd \( n \) (Theorems 4.2 and 4.3). These estimates are slightly sharper than the ones in Theorem 1.1 in particular, they yield the following asymptotic inequalities:
Corollary 1.3 For every $n \geq 3$, there holds
\[
\frac{(n+2)(n-1)n^2}{4} \leq \lim_{\lambda \to +} (2\lambda + 1) c_n^2(\lambda) \leq \frac{n^2(n+1)^2}{4}.
\] (1.6)

The paper is organised as follows. In Sect. 2 we show that the reciprocal of the squared best Markov constant, $1/[c_n(\lambda)]^2$, is equal to the smallest zero of an orthogonal polynomial of degree $m = [\frac{n+1}{2}]$ (different in the cases $n = 2m$ and $n = 2m - 1$), and we derive the three-term recurrence relation satisfied by these orthogonal polynomials. Based on the three-term recurrence relations, in Sect. 3 we evaluate and estimate the lowest degree coefficients of the $m$-th orthogonal polynomial. In Sect. 4 we prove estimates for $c_n(\lambda)$ in the cases of even and odd $n$ (Theorems 4.2 and 4.4), and derive as consequences Theorem 1.1 and Corollary 1.3.

2 $c_n^2(\lambda)$ and the extreme zero of an orthogonal polynomial

In a recent paper [1] we showed that the extreme polynomial in the Markov inequality (1.4) is even or odd if $n$ is even or odd. The following theorem summarizes some of the results obtained in [1]:

Theorem 2.1 The best constant $c_n(\lambda)$ in the Markov inequality (1.4) is given by
\[
c_n(\lambda) = \begin{cases} 2\sqrt{\nu_m}, & n = 2m, \\ 2\sqrt{\nu_m}, & n = 2m - 1, \end{cases}
\] (2.1)

where $\nu_m$ and $\tilde{\nu}_m$ are the largest eigenvalues of the $m \times m$ positive definite matrices $C_m^\top C_m$ and $\tilde{C}_m^\top \tilde{C}_m$, respectively, given by
\[
C_m = \begin{pmatrix} \alpha_1 \beta_1 & \alpha_1 \beta_2 & \cdots & \alpha_1 \beta_m \\ 0 & \alpha_2 \beta_2 & \cdots & \alpha_2 \beta_m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_m \beta_m \end{pmatrix}, \quad \tilde{C}_m = \begin{pmatrix} \tilde{\alpha}_1 \tilde{\beta}_1 & \tilde{\alpha}_1 \tilde{\beta}_2 & \cdots & \tilde{\alpha}_1 \tilde{\beta}_m \\ 0 & \tilde{\alpha}_2 \tilde{\beta}_2 & \cdots & \tilde{\alpha}_2 \tilde{\beta}_m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\alpha}_m \tilde{\beta}_m \end{pmatrix}.
\] (2.2)

Here,
\[
\alpha_k := (2k - 1 + \lambda)h_{2k-1}, \quad \beta_k := \frac{1}{h_{2k}}; \\
\tilde{\alpha}_k := (2k - 2 + \lambda)h_{2k-2}, \quad \tilde{\beta}_k := \frac{1}{h_{2k-1}},
\] (2.3) \hspace{1cm} (2.4)

with
\[
h_i^2 := h_{i,\lambda}^2 := \frac{\Gamma(i + 2\lambda)}{(i + \lambda)\Gamma(i + 1)}.
\] (2.5)

Clearly, matrices $C_m$ and $\tilde{C}_m$ can be represented as
\[
C_m = \text{diag} (\alpha_1, \ldots, \alpha_m) T_m \text{diag} (\beta_1, \ldots, \beta_m), \\
\tilde{C}_m = \text{diag} (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) T_m \text{diag} (\tilde{\beta}_1, \ldots, \tilde{\beta}_m),
\] (2.6)

where $T_m$ is an upper tri-diagonal $m \times m$ matrix with non-zero entries equal to 1,
\[
T_m = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
\]
Since $C_m^T C_m \sim C_m C_m^T$ and $\tilde{C}_m^T \tilde{C}_m \sim \tilde{C}_m \tilde{C}_m^T$, we conclude that
\[
\nu_m \text{ is the largest eigenvalue of the matrix } A_m := C_m C_m^T,
\]
\[
\tilde{\nu}_m \text{ is the largest eigenvalue of the matrix } \tilde{A}_m := \tilde{C}_m \tilde{C}_m^T.
\] (2.7)

It turns out that it is advantageous to work with the inverse matrices $B_m := A_m^{-1}$ and $\tilde{B}_m := \tilde{A}_m^{-1}$, respectively, as $B_m$ and $\tilde{B}_m$ are tri-diagonal matrices. Below we demonstrate this for $B_m$.

The matrix $T_m^{-1}$ is two-diagonal, namely,
\[
T_m^{-1} = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\] (2.8)

For $B_m = A_m^{-1} = (C_m^T)^{-1} C_m^{-1} = (C_m^{-1})^T C_m^{-1}$, using (2.6), we have
\[
B_m = \left( \text{diag} (\beta_1^{-1}, \ldots, \beta_m^{-1}) T_m^{-1} \text{diag} (\alpha_1^{-1}, \ldots, \alpha_m^{-1}) \right)^T \text{diag} (\beta_1^{-1}, \ldots, \beta_m^{-1}) T_m^{-1} \text{diag} (\alpha_1^{-1}, \ldots, \alpha_m^{-1})
\]
\[
= \text{diag} (\alpha_1^{-1}, \ldots, \alpha_m^{-1}) (T_m^{-1})^T \text{diag} (\beta_1^{-2}, \ldots, \beta_m^{-2}) T_m^{-1} \text{diag} (\alpha_1^{-1}, \ldots, \alpha_m^{-1}).
\]

Making use of (2.8), we perform the multiplications to conclude that, indeed, $B_m$ is tri-diagonal. We formulate the result below:

**Proposition 2.2** The matrix $A_m^{-1} := B_m = (b_{i,j})_{m \times m}$ is symmetric and tri-diagonal, with elements
\[
b_{1,1} = \frac{1}{\alpha_1^2 \beta_1},
\]
(2.9)
\[
b_{k,k} = \frac{1}{\alpha_k^2} \left( \frac{1}{\beta_{k-1}} + \frac{1}{\beta_k} \right), \quad k = 2, \ldots, m,
\]
(2.10)
\[
b_{k,k+1} = -\frac{1}{\alpha_k \alpha_{k+1} \beta_k}, \quad k = 1, \ldots, m - 1.
\] (2.11)

The same conclusion applies to the matrix $\tilde{A}_m^{-1} := \tilde{B}_m = (\tilde{b}_{i,j})_{m \times m}$, with the $b$’s, $\alpha$’s and $\beta$’s replaced by the $\tilde{b}$’s, $\tilde{\alpha}$’s and $\tilde{\beta}$’s.

Thus, $B_m$ and $\tilde{B}_m$ are Jacobi matrices, which are positive definite as inverse of the positive definite matrices $A_m$ and $\tilde{A}_m$. The characteristic polynomials of $B_m$ and $\tilde{B}_m$,
\[
P_m(\mu) = \det (\mu E_m - B_m), \quad \tilde{P}_m(\mu) = \det (\mu E_m - \tilde{B}_m),
\]
are determined by three-term recurrence relations, and, by Favard’s theorem, $\{P_m\}$ and $\{\tilde{P}_m\}$ constitute two sequences of orthogonal polynomials with respect to measures supported on the positive axis. Let $\mu_1 < \mu_2 < \cdots < \mu_m$ and $\tilde{\mu}_1 < \tilde{\mu}_2 < \cdots < \tilde{\mu}_m$ be the zeros of $P_m$ and $\tilde{P}_m$, respectively, i.e., the eigenvalues of $B_m$ and $\tilde{B}_m$. Since the latter are reciprocal to the eigenvalues of $A_m$ and $\tilde{A}_m$, in particular, $\nu_m = \mu_1^{-1}$ and $\tilde{\nu}_m = \tilde{\mu}_1^{-1}$, Theorem 2.1, (2.7) and Proposition 2.2 yield the following

**Theorem 2.3** The best constant $c_n(\lambda)$ in the Markov inequality (1.3) is given by
\[
c_n(\lambda) = \begin{cases}
\frac{2}{\sqrt{\nu_1}}, & n = 2m, \\
\frac{2}{\sqrt{\nu_1}}, & n = 2m - 1,
\end{cases}
\] (2.12)
where $\mu_1$ and $\tilde{\mu}_1$ are the smallest zeros of monic polynomials $P_m$ and $\tilde{P}_m$, orthogonal with respect to a measure supported on $\mathbb{R}_+$. The polynomials \{P_k\} are defined by the three-term recurrence relation

$$P_k(\mu) = \left[ \mu - \frac{1}{\alpha_k^2} \left( \frac{1}{\beta_{k-1}^2} + \frac{1}{\beta_k^2} \right) \right] P_{k-1}(\mu) - \frac{1}{\alpha_{k-1}^2 \alpha_k^2 \beta_{k-1}^2} P_{k-2}(\mu), \quad k \geq 2,$$

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu - \frac{1}{\alpha_1^2 \beta_1^2}.$$  \hfill (2.13)

The polynomials \{P_k\} satisfy the same recurrence relation, with the $\alpha$'s and $\beta$'s replaced by the $\tilde{\alpha}$'s and $\tilde{\beta}$'s.

We renormalise polynomials $\{P_k\}_0^m$ and $\{\tilde{P}_k\}_0^m$ by setting $Q_0 = P_0 = \tilde{Q}_0 = \tilde{P}_0 = 1$ and

$$Q_k = d_k P_k, \quad \tilde{Q}_k = \tilde{d}_k \tilde{P}_k, \quad k = 1, \ldots, m$$

so that

$$Q_k(0) = \tilde{Q}_k(0) = 1, \quad k = 0, \ldots, m \hfill (2.14)$$

(note that this is possible because all the zeros of $P_k$ and $\tilde{P}_k$ are positive).

For $k = 1, \ldots, m$, we have $P_k(\mu) = \det(\mu E_k - B_k)$ and $\tilde{P}_k(\mu) = \det(\mu E_k - \tilde{B}_k)$, therefore,

$$P_k(0) = \det(-B_k) = (-1)^k \det(B_k) = (-1)^k \det(A_k^{-1}) = (-1)^k \det(A_k)^{-1},$$

$$\tilde{P}_k(0) = \det(-\tilde{B}_k) = (-1)^k \det(\tilde{B}_k) = (-1)^k \det(\tilde{A}_k^{-1}) = (-1)^k \det(\tilde{A}_k)^{-1}.$$

Since $A_k = C_k C_k^\top$ and $\tilde{A}_k = \tilde{C}_k \tilde{C}_k^\top$, we make use of (2.6) (with $m$ replaced by $k$) to obtain

$$\det(A_k) = \det(C_k)^2 = \prod_{i=1}^k \alpha_i^2 \beta_i^2, \quad \det(\tilde{A}_k) = \det(\tilde{C}_k)^2 = \prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2.$$

Consequently,

$$P_k(0) = \frac{(-1)^k}{\prod_{i=1}^k \alpha_i^2 \beta_i^2}, \quad \tilde{P}_k(0) = \frac{(-1)^k}{\prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2} \Rightarrow d_k = (-1)^k \prod_{i=1}^k \alpha_i^2 \beta_i^2, \quad \tilde{d}_k = (-1)^k \prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2.$$

Thus, the renormalised to satisfy (2.13) polynomials \{Q_k\} and \{\tilde{Q}_k\} are given by

$$Q_k(\mu) = (-1)^k \left( \prod_{i=1}^k \alpha_i^2 \beta_i^2 \right) P_k(\mu), \quad \tilde{Q}_k(\mu) = (-1)^k \left( \prod_{i=1}^k \tilde{\alpha}_i^2 \tilde{\beta}_i^2 \right) \tilde{P}_k(\mu), \quad k = 1, \ldots, m. \hfill (2.15)$$

From (2.13) it is easy to deduce the recurrence relations satisfied by \{Q_k\} and \{\tilde{Q}_k\}.

**Proposition 2.4** The polynomials \{Q_k\} in (2.15) satisfy the recurrence relation

$$Q_k(\mu) - Q_{k-1}(\mu) = \frac{\beta_k^2}{\beta_{k-1}^2} \left[ Q_{k-1}(\mu) - Q_{k-2}(\mu) \right] - \alpha_k^2 \beta_k^2 \mu Q_{k-1}(\mu), \quad k \geq 2,$$

$$Q_0(\mu) = 1, \quad Q_1(\mu) = 1 - \alpha_1^2 \beta_1^2 \mu. \hfill (2.16)$$

The polynomials \{\tilde{Q}_k\} in (2.15) satisfy the same recurrence relation, with the $\alpha$'s and $\beta$'s replaced by the $\tilde{\alpha}$'s and $\tilde{\beta}$'s.
3 The lowest degree coefficients of $Q_m$ and $\tilde{Q}_m$

In view of (2.14), we may write polynomials $Q_k$, $k \geq 1$, in the form

$$Q_k(\mu) = 1 - A_{i,k} \mu + A_{2,k} \mu^2 - \cdots + (-1)^k A_{k,k} \mu^k,$$

$$\tilde{Q}_k(\mu) = 1 - \tilde{A}_{i,k} \mu + \tilde{A}_{2,k} \mu^2 - \cdots + (-1)^k \tilde{A}_{k,k} \mu^k.$$  

(3.1)

Our goal now is to find expressions for $A_{i,m}$, $\tilde{A}_{i,m}$, $i = 1, 2$. First of all, we make use of (2.3)–(2.4) to find the explicit form of the coefficients occurring in recurrence formulae for $Q_k$ and $\tilde{Q}_k$. We have

$$\frac{\beta_k^2}{\beta_{k-1}^2} = \frac{k(2k-1)(2k+\lambda)}{(k-1+\lambda)(2k-2+\lambda)(2k-1+2\lambda)}, \quad \frac{\alpha_k^2}{\beta_k^2} = \frac{2k(2k-1+\lambda)(2k+\lambda)}{2k-1+2\lambda},$$

(3.2)

$$\frac{\beta_k^2}{\beta_{k-1}^2} = \frac{(k-1)(2k-1+\lambda)}{(k-1+\lambda)(2k-3+\lambda)(2k-2+2\lambda)}, \quad \frac{-2\tilde{\alpha}_k^2}{\alpha_k^2} = \frac{(2k-1)(2k-2+\lambda)(2k-1+\lambda)}{2k-1+2\lambda}.$$  

(3.3)

By substituting these quantities in the recurrence formulae in Proposition 2.4 and replacing $k$ by $m$, we obtain

$$Q_m(\mu) - Q_{m-1}(\mu) = \frac{m(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)} [Q_{m-1}(\mu) - Q_{m-2}(\mu)] - \frac{2m(2m-1+\lambda)(2m+\lambda)}{2m-1+2\lambda} \mu Q_{m-1}(\mu),$$

(3.4)

$$\tilde{Q}_m(\mu) - \tilde{Q}_{m-1}(\mu) = \frac{(m-1)(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-3+\lambda)(2m-2+2\lambda)} [\tilde{Q}_{m-1}(\mu) - \tilde{Q}_{m-2}(\mu)] - \frac{(2m-1)(2m-2+\lambda)(2m-1+\lambda)}{2(m-1+\lambda)} \mu \tilde{Q}_{m-1}(\mu).$$  

(3.5)

Lemma 3.1 For every $m \in \mathbb{N}_0$ there holds

(i) $A_{1,m} = \frac{m(m+1)(m+\lambda)(m+\lambda+1)}{2\lambda+1}$; \hspace{1cm} (ii) $\tilde{A}_{1,m} = \frac{m(m+\lambda)(m^2+\lambda m - \frac{1}{2})}{2\lambda+1}$.

Proof. (i) The formula is true for $m = 0$, since $Q_0(\mu) = 1$, and hence $A_{1,0} = 0$. Clearly, (i) holds for $m = 1$, too, since, by (2.16) and (3.2),

$$A_{1,1} = \alpha_1^2 \beta_1^2 = \frac{2(\lambda+1)(\lambda+2)}{2\lambda+1}.$$  

We set $D_{1,k} := A_{1,k} - A_{1,k-1}, k \in \mathbb{N}$, then claim (i) is equivalent to

$$D_{1,m} = \frac{2m(m+\lambda)(2m+\lambda)}{2\lambda+1}, \quad m \in \mathbb{N},$$

(3.6)

and it is true for $m = 1$, since $D_{1,1} = A_{1,1}$. We shall prove (3.6) by induction with respect to $m$. To this end, we differentiate (3.4) in $\mu$ and then set $\mu = 0$, making use of (3.1), to obtain the recurrence formula

$$D_{1,m} = \frac{m(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)} D_{1,m-1} + \frac{2m(2m-1+\lambda)(2m+\lambda)}{2m-1+2\lambda}. $$

Assuming that (3.6) is true for $m - 1$, $m \geq 2$, we substitute the expression for $D_{m-1}$ in the above formula to verify that (3.6) holds for $m$:

$$D_{1,m} = \frac{m(2m-1)(2m+\lambda)(m-1)(m+\lambda)(2m-2+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)(2\lambda+1)} + \frac{2m(2m-1+\lambda)(2m+\lambda)}{2m-1+2\lambda} = \frac{2m(m+\lambda)(2m+\lambda)}{2\lambda+1}. $$
(ii) Clearly, (ii) holds for \( m = 0 \), since \( \bar{A}_{1,0} = 0 \), and it is also true for \( m = 1 \), since, by Propositions 7.4 and 8.3,

\[
\bar{A}_{1,1} = \bar{a}_1^2 = \frac{\lambda + 1}{2}.
\]

Similarly to the proof of (i), we set \( \bar{D}_{1,k} = \bar{A}_{1,k} - \bar{A}_{1,k-1}, k \in \mathbb{N} \), then (ii) is equivalent to

\[
\bar{D}_{1,m} = \frac{(2m-1)(2m-1+\lambda)(2m-1+2\lambda)}{2(2\lambda+1)},
\]

and the latter is true for \( m = 1 \), since \( \bar{D}_{1,1} = \bar{A}_{1,1} \). Similarly to the proof of (i), we obtain a recurrence relation by differentiating (3.5) and then substituting \( \mu = 0 \):

\[
\bar{D}_{1,m} = \frac{(m-1)(2m-1)(2m-1+\lambda)}{(m-1+\lambda)(2m-3+\lambda)(2m-3+2\lambda)} \bar{D}_{1,m-1} + \frac{(2m-1)(2m-2+\lambda)(2m-1+\lambda)}{2(m-1+\lambda)}.
\]

We observe that the right-hand side of (3.7) is obtained from the right-hand side of (3.6) by the change \( m \mapsto m - 1/2 \), and the same change transforms the recurrence relation for \( D_m \) into the recurrence relation for \( \bar{D}_m \). Therefore, (3.7) is a consequence of (3.6).

Remark 3.2 The coefficients \( A_{1,m} \) and \( \bar{A}_m \) are in fact the traces of matrices \( A_m \) and \( \bar{A}_m \), respectively, and they were evaluated in [11] Lemma 2.3. We incorporate an alternative proof first, for the sake of completeness and, second, because the same approach is applied below for the evaluation of coefficients \( A_{2,m} \) and \( \bar{A}_{2,m} \).

Next, we proceed with the evaluation of the coefficients \( A_{2,m} \) and \( \bar{A}_{2,m} \). Let us set

\[
D_{2,1} = 0, \quad D_{2,m} := A_{2,m} - A_{2,m-1}, \quad m \geq 2,
\]

(3.8)

\[
\bar{D}_{2,1} = 0, \quad \bar{D}_{2,m} := \bar{A}_{2,m} - \bar{A}_{2,m-1}, \quad m \geq 2.
\]

(3.9)

Lemma 3.3 (i) The sequence \( \{D_{2,m}\}\) defined by (3.8) satisfies the recurrence relation

\[
D_{2,m} = \frac{m(2m-1)(2m+\lambda)}{(m-1+\lambda)(2m-2+\lambda)(2m-1+2\lambda)} D_{2,m-1} + \frac{2(m-1)m^2(m-1+\lambda)(m+\lambda)(m-1+\lambda)(2m+\lambda)}{(2\lambda+1)(2m-1+2\lambda)}.
\]

(3.10)

The solution of (3.10) with the initial condition \( D_{2,1} = 0 \) is given by

\[
D_{2,m} = \frac{2(m-1)m(m+\lambda)(m+\lambda+1)(2m+\lambda)\left(m^2 + \lambda m - \frac{2}{2\lambda+3}\right)}{(2\lambda+1)(2\lambda+5)}.
\]

(3.11)

(ii) The sequence \( \{\bar{D}_{2,m}\}\) defined by (3.9) satisfies the recurrence relation

\[
\bar{D}_{2,m} = \frac{(m-1)(2m-1)(2m-1+\lambda)}{(m-1+\lambda)(2m-3+\lambda)(2m-3+2\lambda)} \bar{D}_{2,m-1} + \frac{(m-1)(2m-1)(2m-2+\lambda)(2m-1+\lambda)\left(m^2 + (\lambda-2)m - \lambda + \frac{1}{2}\right)}{2(2\lambda+1)}.
\]

(3.12)

The solution of (3.12) with the initial condition \( \bar{D}_{2,1} = 0 \) is given by

\[
\bar{D}_{2,m} = \frac{(m-1)(2m-1)(m+\lambda)(2m-1+\lambda)(2m-1+2\lambda)\left(m^2 + (\lambda-1)m - \frac{2\lambda+1}{2} - \frac{2}{2\lambda+3}\right)}{2(2\lambda+1)(2\lambda+5)}.
\]

(3.13)
Proof. The recurrence formula (3.10) is deduced by two-fold differentiation of (3.4) with respect to $\mu$, then setting $\mu = 0$ and using Lemma 3.1(i) to replace $A_{1,m-1}$ in the resulting identity. The recurrence formula (3.12) is obtained in the same manner: we differentiate (3.5) twice, then set $\mu = 0$ and apply Lemma 3.1(ii) to replace $A_{1,m-1}$ in the resulting identity.

Now it is a straightforward (though rather tedious) task to verify that the sequences $\{D_{2,m}\}$ and $\{\tilde{D}_{2,m}\}$ defined by (3.11) and (3.13) are the solutions of the recurrence relations (3.10) and (3.12), respectively, with the initial conditions $D_{2,1} = 0, \tilde{D}_{2,1} = 0$.

Lemma 3.4 The coefficients $A_{2,m}$ and $\tilde{A}_{2,m}$ are given by

$$A_{2,m} = \frac{(m-1)m(m+1)(m+\lambda)(m+\lambda+1)(m+\lambda+2)(m^2+\lambda+1)m+\frac{4\lambda^2+2\lambda-14}{3\lambda+3}}{2(2\lambda+1)(2\lambda+5)} \tag{3.14}$$

and

$$\tilde{A}_{2,m} = \frac{(m-1)m(m+\lambda)(m+\lambda+1)r_\lambda(m)}{24(2\lambda+1)(2\lambda+3)(2\lambda+5)}, \tag{3.15}$$

$$r_\lambda(m) := 12(2\lambda+3)m^4 + 24\lambda(2\lambda+3)m^3 + 4(6\lambda^3 + 7\lambda^2 - 19\lambda - 32)m^2 - 4\lambda(2\lambda^2 + 19\lambda + 32)m - 8\lambda^3 - 20\lambda^2 + 14\lambda + 71. \tag{3.16}$$

Proof. We have

$$A_{2,m} = \sum_{k=2}^{m} D_{2,k}, \quad \tilde{A}_{2,m} = \sum_{k=2}^{m} \tilde{D}_{2,k},$$

hence, knowing formulae (3.14) and (3.15)–(3.16), one may think of proving them by induction with respect to $m$, especially having in mind that the induction base is obvious. However, performing the induction step by hand, though possible, is a hard work, this is why we highly recommend for that purpose the usage of a computer algebra program, for instance, Wolfram’s Mathematica does perfectly that job.

A reasonable question here is: how do we guess formulae (3.14) and (3.15)–(3.16)? Our approach makes use of the observation that $A_{2,m}$ and $\tilde{A}_{2,m}$ are polynomials in $m$. We evaluate these coefficients for several consecutive values of $m$ (nine values suffice!) and then construct the associated interpolating polynomials to deduce the expressions for $A_{2,m}$ and $\tilde{A}_{2,m}$. Needles to say, we have used a computer algebra program for this purpose.

Next, we obtain two-sided estimates for the coefficients $A_{2,m}$ and $\tilde{A}_{2,m}$.

Lemma 3.5 For all $m \in \mathbb{N}$, $m \geq 2$, and for every $\lambda > -\frac{1}{2}$, the coefficient $A_{2,m}$ admits the estimates

$$\frac{(m-1)m^2(m+1)(m+\lambda)^2(m+\lambda+1)}{2(2\lambda+1)(2\lambda+5)} \leq A_{2,m} \leq \frac{(m-1)m(m+1)^2(m+\lambda)^2(m+\lambda+1)(m+\lambda+2)}{2(2\lambda+1)(2\lambda+5)}.$$

Proof. We use formula (3.14). For the lower estimate, we need to show that

$$(m+\lambda+2)\left[m^2 + (\lambda+1)m + \frac{4\lambda^2 + 2\lambda - 14}{3(2\lambda+3)}\right] \geq m(m+\lambda)(m+\lambda+1).$$

The difference of the left-hand and the right-hand sides is equal to

$$g_0(m) := 2m^2 + \frac{4(4\lambda^2 + 8\lambda + 1)}{3(2\lambda+3)} m + \frac{2(\lambda+2)(2\lambda^2 + \lambda - 7)}{3(2\lambda+3)}.$$

It is easy to see that $g_0(m) > 0$ for $m \geq 2$ and $\lambda > -1/2$, therefore $g_0$ is monotone increasing, and

$$g_0(m) \geq g_0(2) = \frac{2(2\lambda^3 + 21\lambda^2 + 51\lambda + 26)}{3(2\lambda+3)} > 0.$$
For the upper estimate, we need to prove the inequality
\[
m^2 + (\lambda + 1)m + \frac{4\lambda^2 + 2\lambda - 14}{3(2\lambda + 3)} \leq (m + 1)(m + \lambda).
\]
The latter is equivalent to the inequality
\[
\frac{4\lambda^2 + 2\lambda - 14}{3(2\lambda + 3)} < \lambda,
\]
which is readily verified to be true for \(\lambda > -1/2\). \(\square\)

**Lemma 3.6** For all \(m \in \mathbb{N}, m \geq 2\), the coefficient \(\tilde{A}_{2,m}\) admits the lower estimates

(i) \(\tilde{A}_{2,m} \geq \frac{(m - 1)m(m + \lambda)(m + \lambda + 1)(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{1}{3} - \frac{7}{2})}{2(2\lambda + 1)(2\lambda + 5)}, \quad -\frac{1}{2} < \lambda \leq 0,
\]
(ii) \(\tilde{A}_{2,m} \geq \frac{(m - 1)m(m + \lambda)(m + \lambda + 1)(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{1}{3} - \frac{7}{2})}{2(2\lambda + 1)(2\lambda + 5)}, \quad \lambda \geq 0.
\]

For all \(m \in \mathbb{N}, m \geq 2\), and for every \(\lambda > -\frac{1}{2}\), the coefficient \(\tilde{A}_{2,m}\) admits the upper estimate
\[
\tilde{A}_{2,m} \leq \frac{(m - 1)m^2(m + \lambda)^2(m + \lambda + 1)(m^2 + \lambda m - \frac{1}{2})}{2(2\lambda + 1)(2\lambda + 5)}.
\]

**Proof.** The polynomial \(r_{\lambda}\) in (3.15) satisfies
\[
r_{\lambda}(m) = (2\lambda + 3)(12m^4 + 24\lambda m^3 + (12\lambda^2 - 4\lambda - 32)m^2 - (4\lambda^2 + 32\lambda + 16)m - 4\lambda^2 - 4\lambda + 13) - 16(m - 2)(2m + 1),
\]
therefore
\[
r_{\lambda}(m) \leq (2\lambda + 3)(12m^4 + 24\lambda m^3 + (12\lambda^2 - 4\lambda - 32)m^2 - (4\lambda^2 + 32\lambda + 16)m - 4\lambda^2 - 4\lambda + 13) =: (2\lambda + 3)s_{\lambda}(m).
\]

On the other hand,
\[
s_{\lambda}(m) = (12m^2 + 12\lambda m - 4\lambda - 26)(m^2 + \lambda m - \frac{1}{2}) - (16m + 4\lambda^2 + 6\lambda)
\]
\[
< 12m(m + \lambda)(m^2 + \lambda m - \frac{1}{2}),
\]
hence
\[
r_{\lambda}(m) \leq 12(2\lambda + 3)m(m + \lambda)(m^2 + \lambda m - \frac{1}{2}).
\]
The upper estimate for \(\tilde{A}_{2,m}\) now follows by putting this upper bound for \(r_{\lambda}\) in (3.15).

For the proof of the lower estimates for \(\tilde{A}_{2,m}\), we estimate from below the factor \(r_{\lambda}\) in (3.15). Since \(-16 > -8(2\lambda + 3)\), replacement of \(-16\) by \(-8(2\lambda + 3)\) in the second line of (3.17) yields
\[
r_{\lambda}(m) \geq (2\lambda + 3)(12m^4 + 24\lambda m^3 + (12\lambda^2 - 4\lambda - 48)m^2 - (4\lambda^2 + 32\lambda - 8)m - 4\lambda^2 - 4\lambda + 29) =: (2\lambda + 3)s_{\lambda}(m).
\]

Next, we estimate \(s_{\lambda}\) from below, distinguishing between the cases \(-\frac{1}{2} < \lambda \leq 0\) and \(\lambda \geq 0\). If \(-\frac{1}{2} < \lambda \leq 0\), then from
\[
s_{\lambda}(m) = 12(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{\lambda}{3} - \frac{7}{2}) + 8(2\lambda + 1)m - 4\lambda^2 - 6\lambda + 8
\]
\[
> 12(m^2 + \lambda m - \frac{1}{2})(m^2 + \lambda m - \frac{\lambda}{3} - \frac{7}{2}), \quad -\frac{1}{2} < \lambda \leq 0.
\]
Let us recall that $B$ be a polynomial having only real and positive zeros.

Proposition 4.1

In either place, the equality holds if and only if $\Lambda$

Lemma 3.6 is proved.

4 Estimates for the best Markov constant $c_n(\lambda)$

Let us recall that $Q_m$ and $\tilde{Q}_m$ are the characteristic polynomials of the matrices $B_m = A_m^{-1}$ and $\tilde{B}_m = A_m^{-1}$, respectively, normalized by $Q_m(0) = \tilde{Q}_m(0) = 1$. Hence, their reciprocal polynomials,

\[
R_m(x) = x^m Q_m(x^{-1}) = x^m - A_{1,m} x^{m-1} + A_{2,m} x^{m-2} - \cdots + (-1)^m A_{m,m},
\]

are the monic characteristic polynomials of matrices $A_m$ and $\tilde{A}_m$, respectively. In Sect. 2 we showed that $Q_m$ and $\tilde{Q}_m$ are polynomials orthogonal with respect to measures supported on the positive axis, therefore their zeros are single and positive. Then the same observation applies to the zeros of $R_m$ and $\tilde{R}_m$, which we denote by $\{\nu_i\}$ and $\{\tilde{\nu}_i\}$, respectively, so that

\[
R_m(x) = (x - \nu_1)(x - \nu_2) \cdots (x - \nu_m), \quad 0 < \nu_1 < \nu_2 < \cdots < \nu_m,
\]

Our tool for obtaining two-sided estimates for $\nu_m$ and $\tilde{\nu}_m$ is the following simple observation:

Proposition 4.1 Let

\[
f(x) = x^m - a_{1,m} x^{m-1} + a_{2,m} x^{m-2} - \cdots + (-1)^m a_{m,m}
\]

be a polynomial having only real and positive zeros $\{x_i\}$, $0 < x_1 \leq x_2 \leq \cdots \leq x_m$. Then

\[
a_{1,m} - 2 a_{2,m} a_{1,m} \leq x_m \leq \sqrt{a_{1,m}^2 - 2a_{2,m}}.
\]

In either place, the equality holds if and only if $x_1 = x_2 = \cdots = x_m$.

Proof. The claim is equivalent to

\[
\frac{x_1^2 + x_2^2 + \cdots + x_m^2}{x_1 + x_2 + \cdots + x_m} \leq x_m \leq (x_1^2 + x_2^2 + \cdots + x_m^2)^{\frac{1}{2}},
\]

and both the inequalities and the equality cases are obvious.

We obtain separately estimates for $c_n(\lambda)$ for even and odd $n$. Theorem 1.1 is then obtained as a summary of these results.
4.1 The cases of even and odd $n$

According to Theorem 2.1 for the best Markov constant $c_n(\lambda)$ we have

\begin{align*}
    c_{2m}^2(\lambda) &= 4\nu_m, \tag{4.3} \\
    c_{2m-1}^2(\lambda) &= 4\nu_m. \tag{4.4}
\end{align*}

**Theorem 4.2** For all even $n$, $n \geq 4$, and for every $\lambda > -\frac{1}{2}$ the best Markov constant $c_n(\lambda)$ admits the estimates

\[
\frac{(n + 2)(n + 2\lambda)(n + \lambda + \frac{1}{2})^2}{(2\lambda + 1)(2\lambda + 5)} \leq c_n(\lambda)^2 \leq \frac{n(n + 2\lambda)(n + 2\lambda + 2)}{2(2\lambda + 1)^2\sqrt{(n + 2)(n + 2\lambda + 3)}}. \tag{4.5}
\]

**Proof.** Let us set $n = 2m$. We apply Proposition 4.1 with $f = R_m$, making use of Lemma 3.1 and Lemma 3.5.

1) To derive the lower bound for $c_n(\lambda)^2$, we estimate

\[
\begin{aligned}
    \nu_m &\geq A_{1,m} - 2A_{2,m} \\
    &= \frac{m(m + 1)(m + \lambda)(m + \lambda + 1)}{2\lambda + 1} - \frac{(m - 1)(m + 1)(m + \lambda)(m + \lambda + 2)}{2\lambda + 5} \\
    &= \frac{(m + 1)(m + \lambda)(2m + 2\lambda + \frac{1}{2})^2}{(2\lambda + 1)(2\lambda + 5)}.
\end{aligned}
\]

Hence,

\[
c_n^2(\lambda) = 4\nu_m \geq \frac{4(m + 1)(m + \lambda)(2m + 2\lambda + \frac{1}{2})^2}{(2\lambda + 1)(2\lambda + 5)} = \frac{n(n + 2\lambda)(n + \lambda + \frac{1}{2})^2}{(2\lambda + 1)(2\lambda + 5)}. \tag{4.6}
\]

2) For the upper estimate in Theorem 4.2 we have

\[
\begin{aligned}
    \nu_m^2 &\leq A_{1,m}^2 - 2A_{2,m}^2 \\
    &= \frac{m^2(m + 1)^2(m + \lambda)^2(m + \lambda + 1)^2}{(2\lambda + 1)^2} - \frac{(m - 1)m^2(m + 1)(m + \lambda)^2(m + \lambda + 1)^2}{(2\lambda + 1)(2\lambda + 5)} \\
    &= \frac{4m^2(m + \lambda)^2(m + \lambda + 1)^2(m + 1)(m + \lambda + \frac{1}{2})^2}{(2\lambda + 1)^2(2\lambda + 5)} = \frac{n^2(n + 2\lambda)^2(n + 2\lambda + 2)^2(n + 2\lambda + 3)}{64(2\lambda + 1)^2(2\lambda + 5)}
\end{aligned}
\]

and then (4.3) yields

\[
c_n^2(\lambda) = 4\nu_m \leq \frac{n(n + 2\lambda)(n + 2\lambda + 2)}{2(2\lambda + 1)^2\sqrt{n + 2\lambda + 3}}. \tag{4.7}
\]

The proof of Theorem 4.2 is complete. \hfill \Box

**Remark 4.3** For $\lambda \geq 2$ the upper bound for $c_n^2(\lambda)$ in Theorem 4.2 admits a slight improvement, namely, we have

\[
c_n^2(\lambda) \leq \frac{n(n + 2\lambda)(n + 2\lambda + 2)}{2(2\lambda + 1)^2\sqrt{n + 2\lambda + 3}}, \quad \lambda \geq 2. \tag{4.8}
\]

Indeed, for $\lambda \geq 2$ we can replace the lower bound for $A_{2,m}$ in Lemma 3.5 by the sharper one

\[
A_{2,m} \geq \frac{(m - 1)m^2(m + 1)(m + \lambda)^2(m + \lambda + 1)^2}{2(2\lambda + 1)(2\lambda + 5)},
\]

and then, proceeding in the same way as above, we arrive at the estimate (4.8).
Lemma 3.6 to obtain

Now the lower estimate for Lemma 3.1(ii) and Lemma 3.6.

\[ \nu \tilde{c} m \leq 2 n (2 + \lambda + 1) (2 + \lambda + 5) \]

where \( \lambda' = \max(\lambda, 0) \).

**Proof.** Let us set \( n = 2m - 1, \ m \geq 2 \). We apply Proposition 4.1 with \( f = \tilde{R}_m \), making use of Lemma 3.1(ii) and Lemma 3.6.

1) For the lower bound, we estimate \( \bar{\nu}_m \) from below, using Proposition 4.1, Lemma 3.1(ii) and Lemma 3.6 to obtain

\[
\bar{\nu}_m \geq \bar{A}_{1,m} - 2 \bar{A}_{2,m} \geq \frac{m(m + \lambda)(m^2 + \lambda m - \frac{1}{2})}{2(2 + \lambda + 5)} \geq \frac{(m - 1)m(m + \lambda)(m + \lambda + 1)}{2 + \lambda + 5}.
\]

Now the lower estimate for \( c_n^2(\lambda) \) follows from (4.4):

\[
c_n^2(\lambda) = 4\bar{\nu}_m \geq \frac{2m(2m + 2\lambda)(2m + \lambda + 1)}{(2 + \lambda + 5)^2} = \frac{(n + 1)(n + 2\lambda + 1)}{(2 + \lambda + 5)^2}.
\]

2) Next, we prove the upper estimate for \( c_n^2(\lambda) \).

2.1 In the case \( -\frac{1}{2} < \lambda \leq 0 \), we apply Proposition 4.1, Lemma 3.1(ii) and inequality (i) in Lemma 3.6 to estimate \( \nu^2_m \) from above as follows:

\[
\bar{\nu}_m \leq \bar{A}_{1,m} - 2 \bar{A}_{2,m} \leq \frac{m(m + \lambda)^2(m^2 + \lambda m - \frac{1}{2})}{2(2 + \lambda + 5)^2} - \frac{(m - 1)m(m + \lambda)(m + \lambda + 1)(m^2 + \lambda m - \frac{1}{2})}{2(2 + \lambda + 5)}.
\]

Since \( g_2(\lambda) := \frac{2}{3} \lambda^2 + \frac{7}{3} \lambda + \frac{1}{2} \) is a monotone increasing function in \((-1/2, 0]\), the expression in the last brackets does not exceed \( m^2 + \lambda m + \frac{1}{2} \), hence

\[
\bar{\nu}_m \leq \frac{4m^2(m + \lambda)^2(m^2 + \lambda m - \frac{1}{2})}{(2 + \lambda + 5)^2} = \frac{4m^4(m + \lambda)^4}{(2 + \lambda + 5)^2}.
\]

Now from (4.4) we obtain the desired upper estimate for \( c_n^2(\lambda) \):

\[
c_n^2(\lambda) = 4\bar{\nu}_m \leq \frac{8m^2(m + \lambda)^2}{(2 + \lambda + 5)^2} = \frac{(n + 1)^2(n + 2\lambda + 1)^2}{2(2 + \lambda + 5)^2} = \frac{(n + 1)^2(n + 2\lambda + 1)^2(n + 2\lambda + 1)^2}{(2 + \lambda + 5)^2}.
\]

2.2 In view of (4.4), in the case \( \lambda \geq 0 \) the upper estimate for \( c_n^2(\lambda) \) in Theorem 4.4 is equivalent to

\[
\bar{\nu}_m \leq \frac{4m^4(m + \lambda)^5}{(2 + \lambda + 5)^2}.
\]
We apply Proposition 4.1 Lemma 3.3 ii) and inequality (ii) in Lemma 3.6 to estimate $\nu_m^2$ from above as follows:

$$
\nu_m^2 \leq \frac{m^2(m+\lambda)^2(m^2+\lambda m-\frac{1}{2})}{(2\lambda+1)^2} - \frac{(m-1)m(m+\lambda)(m+\lambda+1)(m^2+\lambda m-\frac{1}{2})}{(2\lambda+1)(2\lambda+5)}\left[(2\lambda+5)(m^2+\lambda m)(m^2+\lambda m-\frac{1}{2})-(2\lambda+1)(m^2+\lambda m-1)(m^2+\lambda m-\frac{\lambda+7}{2})\right]
$$

$$
= \frac{m(m+\lambda)(m^2+\lambda m-\frac{1}{2})}{(2\lambda+1)^2(2\lambda+5)}\left[4m^2(m+\lambda)^2+\frac{1}{2}(6\lambda^2+19\lambda+4)m(m+\lambda)-\frac{1}{8}(2\lambda+1)(\lambda+1)(\lambda+7)\right]
$$

$$
\leq \frac{4m^2(m+\lambda)^2}{(2\lambda+1)^2(2\lambda+5)}\left[m^2+\lambda m-\frac{1}{2}\right]\left[m^2+\lambda m+\frac{1}{8}(6\lambda^2+19\lambda+4)\right].
$$

To prove (4.6), it suffices to show that

$$
(m^2+\lambda m-\frac{1}{2})\left[m^2+\lambda m+\frac{1}{8}(6\lambda^2+19\lambda+4)\right] \leq m(m+\lambda)^3, \quad \lambda \geq 0, \quad m \geq 2.
$$

For $m \geq 3$ the above inequality follows from

$$
m(m+\lambda)^3-(m^2+\lambda m-\frac{1}{2})\left[m^2+\lambda m+\frac{1}{8}(6\lambda^2+19\lambda+4)\right] = \frac{1}{8}\lambda m(m+\lambda)(8m+2\lambda-19)+\frac{1}{16}(6\lambda^2+19\lambda+4),
$$

while for $m = 2$ it is equivalent to the inequality

$$
8\lambda^3 + 10\lambda^2 - 5\lambda + 4 \geq 0, \quad \lambda \geq 0,
$$

which is readily verified to be true.

4.2 Proof of Theorem 1.1 and Corollary 1.3

**Proof of Theorem 1.1.** Clearly, the lower bound for $c_n^2(\lambda)$ in Theorem 1.1 is smaller than the lower bounds in Theorems 4.2 and 4.4 hence it is a lower bound in the cases of both even and odd $n$.

Next, we prove the upper bound for $c_n^2(\lambda)$ in Theorem 1.1. To this end, we apply the geometric mean - arithmetic mean inequality in Theorems 4.2 and 4.4 to obtain

$$
c_n^2(\lambda) \leq \frac{(n+\lambda+\lambda')^4}{2(2\lambda+1)^2}\sqrt{2\lambda+5}, \quad n = 2m, \\
c_n^2(\lambda) \leq \frac{(n+\lambda+\lambda')^4}{2(2\lambda+1)^2}\sqrt{2\lambda+5}, \quad n = 2m-1, \quad \lambda' = \max\{0, \lambda\}
$$

and compare the right-hand sides of these inequalities, observing that the first one is the greater.

**Remark 4.5** Applying the geometric mean – arithmetic mean inequality to the upper bounds for $c_n^2(\lambda)$ in Theorems 4.2 and 4.4 to obtain the upper bound in Theorem 1.1 we certainly lose. For instance, for a fixed $n$, the upper bounds in Theorems 4.2 and 4.4 are $O(\lambda)$ as $\lambda \to \infty$ (notice that the same applies to the lower bounds therein!), while the resulting upper bound in Theorem 1.1 is $O(\lambda^{\lambda/2})$ as $\lambda \to \infty$. However, as was already said, the upper estimates here are good for relatively small $\lambda$, say, $\lambda \leq 25$. For big $\lambda$, we have the better upper estimates (1.1) in Theorem A.

**Proof of Corollary 1.3.** The comparison of the bounds for $c_n^2(\lambda)$ in Theorems 4.2 and 4.4 reveals that for $\lambda < \frac{1}{2}$ the smaller lower bound is the one in Theorem 4.2 while in the limit case $\lambda = -1/2$ the bigger numerator has the upper bound in Theorem 4.4. By taking the limits in the expressions obtained from corresponding bounds we obtain the result.

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Dragomir Aleksov, Geno Nikolov
Department of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mails: dragomira@fmi.uni-sofia.bg, geno@fmi.uni-sofia.bg