Weak-localization and Rectification Current in Non-diffusive Quantum Wires

Kang-Hun Ahn

Physics Research Division and Center for Theoretical Physics,
School of Physics, Seoul National University, Seoul 151-747, Korea

(March 22, 2022)

Abstract

We show that electron transport in disordered quantum wires can be described by a modified Cooperon equation, which coincides in form with the Dirac equation for the massive fermions in a 1+1 dimensional system. In this new formalism, we calculate the DC electric current induced by electromagnetic fields in quasi-one-dimensional rings. This current changes sign, from diamagnetic to paramagnetic, depending on the amplitude and frequency of the time-dependent external electromagnetic field.

03.65.-w, 72.15.-v, 73.23.-b
Impurity scattering and quantum coherence of the electron wavefunction are the two key concepts in the transport phenomena of mesoscopic conductors. Conduction electrons are weakly localized by the coherent back-scattering due to impurities, resulting in the effect commonly known as “weak localization”. The conventional theory of weak localization [1] assumes deeply diffusive systems—i.e., the electron mean free path $l$ is much shorter than the system size. Recently, experimental study of transport phenomena in non-diffusive systems have also become important, primarily due to the recent progress in fabrication of clean nanostructures. In this paper, we present a formalism of the weak localization phenomena, valid also in non-diffusive regimes. In particular, we consider electromagnetic(EM)-field-induced current in mesoscopic rings, which is currently an important issue concerning the sign of the measured persistent current [2–4].

Nonlinear properties of field-induced current in mesoscopic rings have been studied in great detail for the case of deeply diffusive regime [3,4]. This problem has recently regained attention due to its relevance to the problem of anomalously large persistent current [2–4] and low-temperature saturation of decoherence time [7–9]. We investigate the same physical model without using diffusion approximation which is valid only for $l \ll L$. The particular system considered in this paper is a quantum wire in ring geometry with finite width much larger than the Fermi wavelength but smaller than the phase coherence length. We show that rectified DC currents in mesoscopic rings induced by high-frequency magnetic fields have oscillating sign depending on the frequency. This result sheds some light on the recent puzzle on the measured sign of the induced DC current in mesoscopic quantum rings [3,4,9].

We start with the conventional weak localization theory. Central to quantum transport in disordered conductors is the concept of so-called ”Cooperon”, the particle-particle diffusion propagator [1,10]. The Cooperon is a two-particle Green function averaged over disorder configurations. In the presence of an electromagnetic field $A$, the Cooperon is the retarded classical propagator of a modified diffusion equation:

$$\left[ \frac{\partial}{\partial t} - D \left( \nabla_{\mathbf{r}} - \frac{2ie}{\hbar c} \mathbf{A} \right)^2 + \frac{1}{\tau_\phi} \right] C(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'),$$  (1)
where $D = v_F l/d$ is the diffusion coefficient for the d-dimensional system, $v_F$ is the Fermi velocity, and $\tau_\phi$ is the phase coherence time. This expression in Eq.(1) has proven to be useful in many cases, since one can easily consider geometrical effects through the boundary condition of the equation. However, it is worthwhile to note that Eq.(1) is only valid in the deeply diffusive regime. Even its original derivation hinged on the realization that the Fourier transformed Cooperon $C(Q, \omega)$ could be approximated as $C(Q, \omega) \approx 1/( -i\omega + DQ^2)$ when

$$Ql \ll 1 \text{ and } \omega \tau \ll 1,$$

where $\tau = l/v_F$ is the elastic mean free time.

When $\omega$ is not much small compared to $1/\tau$, one relies on the semiclassical Boltzmann theory \[12\] instead of Eq.(1) for the Cooperon; the semiclassical Boltzmann theory is free from the above constraining approximation in Eq.(2). In this theory \[12\], the electron motion is characterized by a function $F(r, v, t; r', v', t')$ which is a conditional probability density for the particle initially at position $r'$ and time $t'$ with velocity $v'$ to be found at the position $r$ and time $t$ with the velocity $v$. The intrinsic velocity of the particle is fixed to be Fermi velocity, $|v| = |v'| = v_F$. The conditional probability density $F$ is the propagator of the distribution function $f(r, v, t)$ which satisfy the Boltzmann equation

$$\left[ \frac{\partial}{\partial t} + v \cdot \left( \nabla_r - \frac{2ieA}{\hbar c} \right) \right] f = -\frac{f - f_0}{\tau} - \frac{f}{\tau_\phi}$$

$$f_0(r, t) = \frac{1}{N} \sum_v f(r, v, t)$$

where $N$ is the number of available values of $v$.

From this point onwards, we will consider mesoscopic quantum wire with finite width $W$ in two dimension. The boundary condition is that $f(r, v, t)$ is zero if $v_y \neq 0$ at the boundary of the wire $r = x\hat{x} \pm W/2\hat{y}$. This condition is due to the fact that the electron number is conserved in the electron scattering with the boundary. We impose another condition on the width of the wire $W$ is small enough that $W/\tau_\phi << v_F$. In this case, the main contribution to the electron propagator $F$ is essentially zero mode in the transverse direction, i. e.,
there is no $y$-dependence in $F$. To be consistent with the boundary condition, $v_y = 0$ at the boundaries, we have only two values of $v$ in the longitudinal direction, either $v_F \hat{x}$ or $-v_F \hat{x}$. The equation of the propagator for the Boltzmann equation (3) can be rewritten as a differential equation in a 2 by 2 matrix form:

$$\begin{align*}
\frac{\partial}{\partial t} + v_F \sigma_z \left( \frac{\partial}{\partial x} - \frac{2ie}{\hbar c} A \right) + \frac{1}{2\tau}(1 - \sigma_x) + \frac{1}{\tau_\phi} F(x, t; x', t')
&= \delta(x - x')\delta(t - t'),
\end{align*}$$

(5)

where

$$F(x, t; x', t') = \begin{pmatrix} F(x, v_F, t; x', v_F, t') & F(x, v_F, t; x', -v_F, t') \\ F(x, -v_F, t; x', v_F, t') & F(x, -v_F, t; x', -v_F, t') \end{pmatrix}. \quad (6)$$

As it is clear in Eq.(3), each component of the above matrix formalism denotes the relevant chirality of the moving particle. Since the Cooperon $C(x; t; x', t')$ is also a probability density for the particle initially at $(x', t')$ to be found at $x$ after time $t - t'$ [10][12], we get the Cooperon from $F$ by summing all final chiral states and averaging over initial chiral states;

$$C(x, t; x', t') = \frac{1}{2} \sum_{ij} F_{ij}(x, t; x', t').$$

In a matrix form, we get

$$C(x, t; x', t') = \text{Tr} \left[ \frac{1 + \sigma_x}{2} F(x, t; x', t') \right].$$

(7)

Note that the main equation in Eq.(3) coincides in form with a Dirac equation for a massive particles in 1+1 dimension. Interestingly, it is already well known that the conventional Cooperon equation (1) coincides in form with the Schrödinger equation with imaginary time $t \longleftrightarrow -it$ for a particle with mass $m' \leftrightarrow \hbar/2D$. To be more explicit, let us consider the following relativistic propagator $G^{rel}$ for the Dirac equation for the particle with mass $m'$ in 1+1 dimension:

$$\begin{align*}
[i\gamma^0 \left( \frac{\partial}{\partial t} + \frac{im'c^2}{\hbar} \right) - ic\gamma^1 \left( \frac{\partial}{\partial x} - \frac{2ie}{\hbar c} A \right) + \frac{m'c^2}{\hbar}]G^{rel}
&= i\gamma^0 \delta(x - x')\delta(t - t'),
\end{align*}$$

(8)
where $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $g^{\mu\nu} = \text{diag}(1,-1)$ for $\mu, \nu = 0,1$. The term $i\frac{m'c^2}{\hbar}$ within the first parenthesis is included to compensate the time-evolution factor $\exp(-i\frac{m'c^2}{\hbar}t)$ due to the rest-mass energy which is not included in the Schrödinger equation. Let us perform the following transformations, while keeping the electron coupling constant $\frac{2ie}{\hbar c}$ to the EM field,

$$
t \to -it, \quad c \to iv_F, \quad m' \to \hbar/2D,
$$

then we get Eq.(3) using Pauli spin matrices $\gamma^0 = \sigma_x$ and $\gamma^1 = i\sigma_y$.

The physical origin of the transformation $c \to iv_F$ in Eq.(3) is rendered clear by noting that the mean speed of a particle in a disordered conductor is limited by the Fermi velocity while the speed of a particle in the relativistic theory can not exceed the speed of light. Instead of $\sigma_x$ chosen as $\gamma^0$ in the above, a general form of Eq.(7) can be used using $\gamma$ matrices;

$$
C(x, t; x', t') = \text{Tr}\left[\left(1 + \frac{1+\gamma^0}{2}\right)F(x, t; x', t')\right].
$$

The appearance of $(1 + \gamma^0)/2$ in front of $F$ seems to be natural, because $(1 + \gamma^0)/2$ in relativistic mechanics is a projection operator, which projects out negative energy states in the rest reference frame [13].

Pursuing further similarity between the present theory and relativistic theory, let us note how the chiral symmetry is broken in each theory. In relativistic quantum mechanics in 1+1 dimension, the particle’s mass breaks the chiral symmetry. (In other words, since the massive particle moves slower than $c$, there exist reference frames where a right-moving particle can be seen as a left-moving particle.) In mesoscopic quantum wires, the chirality is broken due to the presence of the electron-impurity scattering. These two different mechanisms, which break the chiral symmetry in each theory, are connected to each other as clearly manifested by the correspondence shown in Eq.(3); $m'c^2 \to -h v_F^2/2D = -\hbar/2\tau$.

In the absence of external fields, $F$ is a translationally invariant quantity, which allows us to solve Eq.(5) using Fourier transform:

$$
F(x, t; x', t') = (1/2\pi)^2 \int dQ \int d\omega F(Q, \omega) e^{iQ(x-x')-i\omega(t-t')}.
$$

For $\tau_\phi >> \tau$, $F(Q, \omega)$ is given by

$$
F(Q, \omega) = \frac{1}{-i\omega + DQ^2 - \omega^2\tau}.
$$
\[ \times \begin{pmatrix} 1/2 - i\omega \tau - ilQ & 1/2 \\ 1/2 & 1/2 - i\omega \tau + ilQ \end{pmatrix} \] (10)

The Cooperon in momentum space is written as

\[ C(Q, \omega) = \text{Tr} \frac{1 + \sigma_x}{2} \mathbf{F}(Q, \omega) = \frac{1 - i\omega \tau}{-i\omega + DQ^2 - \omega^2 \tau}. \] (11)

This result coincides with the Cooperon obtained as the total sum of the Dyson series in the Green function approach without the approximations in Eq.(2) [14].

In the presence of time-dependent EM field, \( A = A(t)\hat{x} \), the “Cooperon matrix” \( \mathbf{F} \) explicitly depends on time \( t \) \( (\mathbf{F}_t = \mathbf{F}_t(x, \eta; x', \eta')) \), which is obtained by solving the following equation with time-dependent field \( A_t(\eta) = A(t - \eta/2) + A(t + \eta/2) \) [1];

\[ \left[ \frac{\partial}{\partial \eta} + v_F \sigma_z \left( \frac{\partial}{\partial x} - \frac{ie}{\hbar c} A_t(\eta) \right) \right] \mathbf{F}_t + \frac{1 - \sigma_x}{2\tau} \mathbf{F}_t^* = \delta(x - x')\delta(\eta - \eta'), \] (12)

Here, we include the phenomenological dephasing rate \( 1/\tau_\phi^* \), which has its origin from sources other than the external EM field. The weak localization current \( I_{WL}(t) \) [1] (quantum correction to classical ohmic current) is given by

\[ \langle I_{WL}(t) \rangle = \frac{C_\beta e^2 D}{\hbar} \int_0^\infty d\eta \text{Tr} \left[ \frac{1 + \sigma_x}{2} \mathbf{F}_{t-\eta/2}(x, \eta; x, -\eta) \right] \times E(t - \eta), \] (13)

where \( \langle \cdots \rangle \) represents disorder average and \( E(t) = -\frac{1}{e} \frac{\partial A(t)}{\partial t} \) is the applied electric field. \( C_\beta \) is dictated by the Dyson symmetry class ; \( C_\beta = -4/\pi (2/\pi) \) when the spin-orbit scattering is negligible (important) with the characteristic length \( L_{so} >> L \) \( (L_{so} << L) \) [1,4].

Now, let us apply Eq.(12) and Eq.(13) to calculate electric currents induced by the EM field in mesoscopic rings. While the usual equilibrium persistent current is induced by a static magnetic flux \( \phi = \bar{A}L \) only, the rectified direct current is a dynamical phenomenon originating from the time-dependent conductivity of the ring [3,4]. Suppose the EM field, given by \( A(t) = \bar{A} + a(t) \), is applied to the quantum ring with perimeter \( L \), where \( a(t) = \)
\( \frac{1}{2}(a_\omega e^{-i\omega t}+c.c.) \) and \( \bar{A} \) is time-independent. An electric field \( \mathcal{E}(t) = \frac{1}{2}(\mathcal{E}_\omega e^{-i\omega t}+c.c.) \) is induced along the ring, where \( \mathcal{E}_\omega = i\omega a_\omega/c \). The DC component of electric current \( I_0 = \langle I_{WL}(t) \rangle \) is of interest, and it is obtained by averaging the disorder-averaged current \( \langle I_{WL}(t) \rangle \) over time \( t \).

Let us first investigate the case of weakly time-dependent field so that the associated magnetic flux \( \phi_\omega \) is much smaller than the unit flux quantum \( \phi_0 = h/|e|c \):

\[
\phi_\omega = |\mathcal{E}_\omega| L e/\omega << \phi_0.
\]

We calculate up to the first order perturbation term of \( a_{t-\eta/2}(\eta') = a(t-\eta/2-\eta'/2) + a(t-\eta/2+\eta'/2) \) in \( \mathbf{F}_{t-\eta/2} \):

\[
\mathbf{F}_{t-\eta/2}(x, \eta; x, -\eta) = \mathbf{F}^{(0)}_{t-\eta/2}(x, \eta; x, -\eta) + \int dx' \int_{-\eta}^{\eta} d\eta' \mathbf{F}^{(0)}_{t-\eta/2}(x, \eta; x', \eta') \\
\times \left( v_F \frac{ie}{\hbar c} \sigma_z a_{t-\eta/2}(\eta') \right) \mathbf{F}^{(0)}_{t-\eta/2}(x', \eta'; x, -\eta) + \cdots,
\]

where \( \mathbf{F}^{(0)}_{t-\eta/2}(x', \eta'; x, -\eta) \) denotes the \( \mathbf{F} \) matrix in the absence of time-dependent field, \( a_\omega = 0 \). After a long but straightforward calculation, we get the expression for the DC current:

\[
I_0 = C_\beta \left| e \right| \left( \frac{\phi_\omega}{\phi_0} \right)^2 \times \\
\sum_{m=-\infty}^{\infty} 4\pi^2 (\omega\tau_D)^2 k_m \\
\times \left( 1 - \frac{\tau}{\tau_D} \left( k_m^2 - (\omega\tau_f)^2 + \tau_D/\tau_\phi^2 \right)^2 \right),
\]

where \( k_m = 2\pi(m+2\phi/\phi_0) \) (\( \phi = \bar{A}L \) is the static magnetic flux), \( \tau_D = L^2/D \) is the diffusion time, and \( \tau_f = L/v_F \) is a ballistic time scale. By neglecting ballistic parameters in Eq.(16) i.e, \( \tau/\tau_D \rightarrow 0 \), and \( \omega\tau_f = \omega\tau_D\sqrt{\tau/\tau_D} \rightarrow 0 \), we recover the earlier result of \( I_0 \) by Kravtsov and Yudson [5]. Compared with the results for diffusive limit [5], we basically encounter a new parameters \( \tau/\tau_D \) by considering ballistic effects.
Since $I_0$ is periodic with a period $\phi_0/2$, the Fourier components $I^{(n)}$ of $I_0$ are often the quantities under study:

$$I_0(\phi) = C_\beta \frac{|e|}{\tau_D} \sum_n I^{(n)} \sin(4\pi n \frac{\phi}{\phi_0}).$$

(17)

In Fig. 1, we plot the amplitude of the first harmonic $I^{(1)}$ of $I_0(\phi)$ using Eq.(16). In contrast with the current for diffusive limit $\tau/\tau_D = 0$, i.e. $I^{(1)}$ for finite $\tau/\tau_D$ show oscillation behaviour. A new time scale $\tau_f = L/v_f$ appears associated with the oscillation period $\Delta \omega = 2\pi/\tau_f$. Note that when we take into account ballistic effects, (i.e., $\tau/\tau_D \neq 0$), we can not neglect $\omega \tau_f (= \omega \tau_D \sqrt{\tau/\tau_D})$ in the denominator of Eq.(16), which gives oscillating behaviour in Fig. 1. Intuitively, this oscillation is due to the fact that the time period of periodic orbits along the ring match with that of the applied external field.

Now, let us look into a different regime where the disorder potential is very weak but the applied field is arbitrarily strong:

$$\frac{1}{\tau} \ll \omega, \quad \text{and} \quad \frac{1}{\tau_f}. \quad (18)$$

For this case, we use perturbation of the electron-impurity scattering term $\sigma_x/\tau$ with parameter $1/\omega \tau << 1$. The leading terms are written as

$$I^{(n)} \approx \mathcal{F}_n(\pi \frac{\phi_0}{\phi_0}, \omega \tau_f)e^{-n\frac{\tau_f}{\tau}}$$

$$\times \left[ \sin(n\omega \tau_f/4) + \frac{1}{\omega \tau} \left( \cos(n\omega \tau_f/4) + \frac{2}{\omega \tau} \sin(n\omega \tau_f/4) \right) \right] + \cdots,$$

(19)

where

$$\mathcal{F}_n(x, y) = xy J_1(16x \sin(ny/4)/y). \quad (20)$$

Here $J_1$ is the Bessel function of order 1.

As shown in Fig. 2, the first harmonic $I^{(1)}$ of the current may show sign reversal when the applied field is not too weak. When the magnetic flux $\phi_\omega$ associated with the time dependent field is larger than half flux quantum $\phi_0/2$, $I^{(1)}$ is in a regime of negative sign.
depending on the applied frequency. Interestingly, this is also the condition that the applied field can cause dephasing of electrons efficiently.

Experimental configuration in Ref. [4] seems to be promising for the observation of the ballistic effects we have discussed here. However, instead of metals, GaAs samples will be more promising to show ballistic effects, where the mean-free-path is usually order of \( \mu m \). Furthermore, both of the well-defined amplitude and frequency are necessary for the comparison. In case of the samples with GaAs, the applied field of frequency \( \omega \) in order of THz may clearly show the ballistic effects we discussed.

In conclusion, we have shown that the mesoscopic electron transport in disordered quantum wires is described by a generalized Cooperon equation which coincides in form with Dirac equation for massive fermions in 1+1 dimensional system. Ballistic effects in a disordered wire are equivalent to the relativistic effects in clean one-dimensional systems. Based on the new Cooperon equation, electric currents in mesoscopic rings induced by oscillating magnetic fields are calculated. It is predicted that, as a ballistic effect, the DC component of the induced electric currents shows oscillating behavior in the domain of external-field frequency. Furthermore, in the high frequency regime, the sign of the induced current can be either diamagnetic or paramagnetic depending on the strength and the frequency of the field.

The author would like to thank C. Kim, P. Mohanty, L.I. Glazman, B.I. Halperin, D. Kim, M. Das, F. Green, Y. D. Park, C. Lee, M. Y. Choi, and T. Yamamoto for stimulating discussions.
REFERENCES

[1] B. L. Altshuler and A. G. Aronov, in *Electron-Electron Interaction in Disordered Systems*, edited by A. L. Efros and M. Pollak (North-Holland, Amsterdam, 1985).

[2] L. P. Levy, et. al., Phys. Rev. Lett. **64**, 2074 (1990); V. Chandrasekhar, et. al., Phys. Rev. Lett. **67**, 3578 (1991); B. Reulet, et. al., Phys. Rev. Lett. **75**, 124 (1995).

[3] E.M.Q. Jariwala, et. al., Phys. Rev. Lett. **86**, 1594 (2001).

[4] R. Deblock, et. al., cond-mat/0109527.

[5] V. E. Kravtsov and V. I. Yudson, Phys. Rev. Lett. **70**, 210 (1993).

[6] A. G. Aronov and V.E. Kravtsov, Phys. Rev. B **47**, 13409 (1993).

[7] P. Mohanty, et. al., Phys. Rev. Lett. **78**, 3366 (1997).

[8] P. Mohanty, Ann. Phys. (Leipzig) **8**, 549 (1999).

[9] V. E. Kravtsov and B. L. Altshuler, Phys. Rev. Lett. **84**, 3394 (2000).

[10] See for a review on various role of Cooperon, e.g., G. Montambaux, in *Quantum Fluctuations*, Proceedings of the Les Houches Summer School, Session LXIII, edited by E. Giacobino *et al.* (Elsevier, Amsterdam, 1996).

[11] See, e.g., G. Bergmann, Phys. Rep. **107**, 1 (1984).

[12] S. Chakravarty and A. Schmid, Phys. Rep. **140**, 193 (1986).

[13] See, e.g., L.H. Ryder, *Quantum Field Theory* (Cambridge University Press, New York, 1985).

[14] The Dyson series solution for the Cooperon including ballistic effects in two dimension was discussed in a recent article, A. Ater and O. Agam, Phy. Rev. B **63**, 205101 (2001) and originally in E. Abrahams, P. W. Anderson, and T. V. Ramakrishnan, Phils. Mag. B **42**, 827 (1980). For a comprehensive study of the ballistic effects in two dimension,
see e.g., A. Altland and Y. Gefen, Phys. Rev. B 51, 10671 (1995).
FIG. 1. The amplitude $I^{(1)}$ of the first harmonic of $I_0(\phi)$ in units of $C_\beta|e|/\tau_D$ for weakly time-dependent field $\phi_\omega \ll \phi_0$ and different “ballisticity” $\tau/\tau_D = 0.5$ (thick solid line), $\tau/\tau_D = 0.1$ (thick dashed line), and $\tau/\tau_D = 0$ (dashed line, Ref.[5]). $\tau^*_\phi$ was chosen to be $10\tau_D$ for all cases.

FIG. 2. The amplitude $I^{(1)}$ of the first harmonic of $I_0(\phi)$ in units of $C_\beta|e|/\tau_D$ using high-frequency approximation $1/\tau \ll \omega, 1/\tau_f$. $\tau_f/\tau = 0.1$ was chosen as a specific case.