Analysis of Subgrid Stabilization Method for Stokes-Darcy Problems

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1. Introduction

The transport of substances between surface water and groundwater has attracted a lot of interest into the coupling of viscous flows and porous media flows [1–5]. In this work we consider coupled problems in fluid dynamics where the fluid in one part of the domain is described by the Stokes equations and in the other, porous media part, by the Darcy equation and mass conservation. Velocity and pressure on these two parts are mutually coupled by interface conditions derived in [6]. Such systems can be discretized by heterogeneous finite elements as analyzed by Layton et al. [1]. In more recent works, unified approaches become more popular. For instance, discontinuous Galerkin methods were analyzed by Girault and Rivière [3], mixed methods by Karper et al. [4], and local pressure gradient stabilized methods by Braack and Nafa [7].

In this work, we consider the $L^2$-formulation of the coupled Stokes-Darcy problem as in [4], but we discretize by equal-order finite elements and use subgrid method and grad-div term to stabilize the pressure and control the natural $H^1(\text{div})$ velocity norm on the Darcy subdomain.

2. Formulations of the Stokes-Darcy Coupled Equations

2.1. Model Equations. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $3$, be a bounded region with Lipschitz boundary $\partial \Omega$. $\Omega_S$ and $\Omega_D$ are, respectively, the fluid and porous media subdomains of $\Omega$ such that $\Omega_S \cap \Omega_D = \emptyset$. The subdomains have a common interface $\Gamma = \overline{\Omega}_S \cap \overline{\Omega}_D$. We denote by $\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D)$ the fluid velocity and by $p = (p_S, p_D)$ the fluid pressure, where $\mathbf{v}_i = \mathbf{v}|_{\Omega_i}$, $p_i = p|_{\Omega_i}$, $i = S, D$. The flow in the domain $\Omega_S$ is assumed to be of Stokes type and governed by the equations

\begin{equation}
-2\nu \text{div}(D(\mathbf{v}_S)) + \nabla p_S = \mathbf{f}, \quad \text{in } \Omega_S
\end{equation}

\begin{equation}
\text{div} \mathbf{v}_S = 0, \quad \text{in } \Omega_S
\end{equation}

with symmetric strain tensor $D(\mathbf{v}_S) = (1/2)(\nabla \mathbf{v}_S + \nabla \mathbf{v}_S^T)$, external force $\mathbf{f}$, and constant viscosity $\nu > 0$. In the porous region $\Omega_D$ the filtration of an incompressible flow through porous media is described by Darcy equations

\begin{equation}
K^{-1} \mathbf{v}_D + \nabla p_D = \mathbf{f}, \quad \text{in } \Omega_D
\end{equation}

\begin{equation}
\text{div} \mathbf{v}_D = g, \quad \text{in } \Omega_D
\end{equation}
where the permeability $K = K(x)$ is a positive definite symmetric tensor and $g$ denotes an external Darcy force.

2.2. Boundary Conditions. On $\Gamma_S = \partial \Omega_S \setminus \Gamma$, we prescribe homogeneous Dirichlet conditions for the velocity $v_S$.

$$v_S = 0, \quad \text{on } \Gamma_S.$$ \hspace{2cm} (3)

The boundary of $\Omega_S$ is split into three parts $\partial \Omega_S = \Gamma \cup \Gamma_{D,1} \cup \Gamma_{D,2}$. We prescribe zero flux on $\Gamma_{D,1}$ and a homogeneous Dirichlet condition for the pressure on $\Gamma_{D,2}$.

$$v_D \cdot n_D = 0, \quad \text{on } \Gamma_{D,1}$$ \hspace{2cm} (4)

$$p_D = 0, \quad \text{on } \Gamma_{D,2}$$

where $n_D$ denotes the outer normal vector on the boundary pointing from $\Omega_D$ into $\Omega_S$. This boundary condition ensures a zero mass flux.

2.3. The Beavers-Joseph-Saffman Condition. The flows in $\Omega_S$ and $\Omega_D$ are coupled across the interface $\Gamma$. Conditions describing the interaction of the flows are as follows [6, 8]:

(i) The continuity of the normal velocity:

$$v_S \cdot n_S = -v_D \cdot n_D, \quad \text{on } \Gamma$$ \hspace{2cm} (5)

(ii) The balance of normal forces:

$$-(p_S l + 2 \nu D(v_S)) n_S \cdot n_S = p_D n_D, \quad \text{on } \Gamma$$ \hspace{2cm} (6)

(iii) The Beavers-Joseph-Saffman condition written in terms of the strain tensor:

$$v_S \cdot \tau = -\frac{2\sqrt{\nu}}{k} (D(v_S) \cdot n_S) \cdot \tau,$$ \hspace{2cm} (7)

where $\nu = \nu K \cdot \tau$ and $k$ is a dimensionless parameter to be determined experimentally, this condition relating the tangential slip velocity $v_S \cdot \tau$ to the normal derivative of the tangential velocity component in the Stokes region.

3. Variational Formulation

As variational formulation we consider the so-called $L^2$-formulation used by Karper et al. [4] and recently by [9, 10]. We denote

$$\langle v, w \rangle_{\Omega_S} = \int_{\Omega_S} vw \, dx, \quad v, w \in L^2(\Omega)^d,$$ \hspace{2cm} (8)

$$\langle v, w \rangle_{\Gamma} = \int_{\Gamma} vw \, ds, \quad v, w \in L^2(\Gamma)^d,$$

where $L^2(\Omega)$ and $H^1(\Omega)$ denote the usual Sobolev spaces.

Next, we define the spaces

$$H^1_{\Gamma_S}(\Omega_S) = \left\{ w \in \left( H^1(\Omega_S) \right)^d \mid w = 0 \text{ on } \Gamma_S \right\},$$

$$H^1(\Gamma_D) = \left\{ w \in L^2(\Omega_D)^d \mid \text{div } w \in L^2(\Omega_D) \right\},$$

$$H^1_{\text{dir}}(\Omega_D) = \left\{ w \in H^1(\Omega_D)^d \mid \text{div } w = 0 \text{ on } \Gamma_D \right\}.$$

Then, multiplying the Stokes equations (1) by the test functions $w_S \in H^1_{\Gamma_S}(\Omega_S), q_D \in L^2(\Omega_D)$, respectively, and integrating by part on the domain $\Omega_S$, we obtain

$$\langle 2\nu D(v_S) \cdot D(w_S) \rangle_{\Omega_S} - \langle p_S \cdot \text{div } w_S \rangle_{\Omega_S} + \langle \rho_S w_S \cdot n_S \rangle_{\Gamma} = \langle f, w_S \rangle_{\Omega_S},$$ \hspace{2cm} (10)

$$\langle \text{div } v_S, q_S \rangle_{\Omega_S} = 0.$$

Using the decomposition $w_S = (w_S \cdot n_S) n_S + (w_S \cdot \tau) \tau$, the fluid normal stress condition (6), and the BJS interface condition (7) in (10), we obtain the weak formulation of the Stokes equations: find $v_S \in H^1_{\Gamma_S}(\Omega_S), p_S \in L^2(\Omega_S)$ such that

$$\langle 2\nu D(v_S) \cdot D(w_S) \rangle_{\Omega_S} + \frac{\alpha_0}{\nu} \langle w_S \cdot \tau, w_S \cdot \tau \rangle_{\Gamma},$$

$$-\langle p_S \cdot \text{div } w_S \rangle_{\Omega_S} + \langle \rho_D w_S \cdot n_S \rangle_{\Gamma} = \langle f, w_S \rangle_{\Omega_S},$$ \hspace{2cm} (11)

$$\langle \text{div } v_S, q_S \rangle_{\Omega_S} = 0,$$

$$\forall w_S \in H^1_{\Gamma_S}(\Omega_S), q_S \in L^2(\Omega_S).$$

Similarly, taking $\nu > 0$ and testing the Darcy equations (2) by $w_D \in H^1_{\Gamma_D}(\Omega_D), q_D \in L^2(\Omega_D)$, respectively, together with the weighted grad-div term we obtain the weak formulation of Darcy equations: find $v_D \in H^1_{\text{dir}}(\Omega_D), p_D \in H^1_{\text{dir}}(\Omega_D)$ such that

$$\langle K^{-1} v_D \cdot w_D \rangle_{\Omega_D} + \langle \nu p_D \cdot w_D \rangle_{\Omega_D},$$

$$\delta \langle \text{div } v_D \cdot \text{div } w_D \rangle_{\Omega_D} = \delta \langle g, \text{div } w_D \rangle_{\Omega_D},$$

$$-\langle v_D, \nu q_D \rangle_{\Omega_D} + \langle v_D \cdot n_D, q_D \rangle_{\Gamma} = \langle g, q_D \rangle_{\Gamma}.$$ \hspace{2cm} (12)

Summing up (11) and (12) the weak form of the coupled problem is given by the following: find $v_S \in H^1_{\Gamma_S}(\Omega_S), p_S \in L^2(\Omega_S), v_D \in H^1_{\text{dir}}(\Omega_D), p_D \in L^2(\Omega_D)$ such that

$$\langle 2\nu D(v_S) \cdot D(w_S) \rangle_{\Omega_S} - \langle p_S \cdot \text{div } w_S \rangle_{\Omega_S} + \langle K^{-1} v_D \cdot w_D \rangle_{\Omega_D} + \langle \nu p_D \cdot w_D \rangle_{\Omega_D},$$

$$\delta \langle \text{div } v_D \cdot \text{div } w_D \rangle_{\Omega_D} + \frac{\alpha_0}{\nu} \langle w_S \cdot \tau, w_S \cdot \tau \rangle_{\Gamma},$$

$$\langle p_D, w_S \cdot n_S \rangle_{\Gamma} = \langle f, w_S \rangle_{\Omega_S} + \delta \langle g, \text{div } w_D \rangle_{\Omega_D},$$

$$\langle \text{div } v_S, q_S \rangle_{\Omega_S} - \langle v_D, \nu q_D \rangle_{\Omega_D} + \langle v_D \cdot n_D, q_D \rangle_{\Gamma} = \langle g, q_D \rangle_{\Gamma}.$$ \hspace{2cm} (13)
To analyze the weak formulation of the coupled problem we introduce the following spaces

\[
V = \left\{ v \in H(\text{div}, \Omega) \mid v_S \in \left( H^1(\Omega_S) \right)^d, \ v_S = 0 \text{ on } \Gamma_S, v \cdot \mathbf{n}_S = 0 \text{ on } \Gamma_{D,1} \right\},
\]
(14)

\[
Q = \left\{ q \in L^2(\Omega) \mid p_D \in H^1(\Omega_D), \ p = 0 \text{ on } \Gamma_{D,2} \right\},
\]
\[
X = V \times Q.
\]

The velocity and pressure spaces V and Q are equipped with the natural norms

\[
\|v\|_V = \left( \|\nabla v\|^2_{\Omega_S} + \|\nabla v\|^2_{\Omega_D} + \|\text{div } v\|^2_{\Omega_D} \right)^{1/2},
\]
\[
\|p\|_Q = \left( \|p\|^2_{\Omega_i} + \|\nabla p\|^2_{\Omega_D} \right)^{1/2}.
\]

Further, due to the positive definiteness of K with respect to the \(L^2(\Omega_D)\) norm \(\| \cdot \|_{\Omega_D}\), there exist positive real numbers \(k_1\) and \(k_2\) such that

\[
k_1 \|v\|^2_{\Omega_D} \leq (K^{-1}v, v)_{\Omega_D} \leq k_2 \|v\|^2_{\Omega_D}, \quad \forall v \in V.
\]

Next, we define the bilinear forms for \(v = (v_S, v_D), w = (w_S, w_D)\) in V and \(p = (p_S, p_D), q = (q_S, q_D)\) in Q on the two parts of the domain by

\[
A_S(v, p; w, q) = (2\gamma D(v_S), D(w_S))_{\Omega_S}
+ \frac{\nu k}{\sqrt{k}} \langle v_S \cdot \tau, w_S \cdot \tau \rangle_{\Gamma}
- \langle p_S, \text{div } w_S \rangle_{\Omega_S}
+ \langle \text{div } v_S, q_S \rangle_{\Omega_S},
\]
(17)

\[
A_D(v, p; w, q) = (K^{-1}v_D, w_D)_{\Omega_D}
+ \delta \langle \text{div } v_D, \text{div } w_D \rangle_{\Omega_D}
+ \langle \nabla p_D, w_D \rangle_{\Omega_D} - \langle v_D, \nabla q_D \rangle_{\Omega_D}.
\]

Hence, the bilinear form for the coupled problem is the sum of \(A_S(v, p; w, q), A_D(v, p; w, q)\), and terms to enforce the continuity of the normal part of the velocities across the interface.

\[
A(v, p; w, q) = A_S(v, p; w, q) + A_D(v, p; w, q)
+ \langle p_D, w_S \cdot \mathbf{n}_S \rangle_{\Gamma} - \langle q_D, v_S \cdot \mathbf{n}_S \rangle_{\Gamma}.
\]
(18)

Assuming, for simplicity, that \(f\) and \(g\) are extended by zero to the whole domain, the variational formulation of the coupled Stokes-Darcy system in compact form reads as follows: find \((v, p) \in V \times Q\) solution of

\[
A(v, p; w, q) = F(w, q), \quad \forall (w, q) \in V \times Q,
\]
(19)

with

\[
F(w, q) = (f, w_\Omega) + (g, q_\Omega) + \delta (g, \text{div } w_D)_{\Omega_D}.
\]
(20)

It can easily be shown that a sufficiently regular solution \((v, p) \in V \times Q\) of (19) such that \(v_S \in H^1(\Omega_S)^d, v_D \in H^1(\Omega_D)^d, p \in H^1(\Omega_S \cup \Omega_D)\) is also a classical solution of (1) and (2). We note that there is an alternative variational formulation to the one given here called \(H(\text{div})\)-formulation. The latter uses the term \(-\langle p, \text{div } w \rangle_{\Omega_D} + \langle \text{div } v, q \rangle_{\Omega_D}\) instead of \((w, \nabla p)_{\Omega_D} - \langle w, \nabla q \rangle_{\Omega_D}\).

The existence and uniqueness of the solution of problem (19) follows from Brezzi’s conditions for saddle point problems [11]; namely,

\[
A(v, p; w, p) \geq \gamma \|v\|_V^2, \quad \forall v \in V, \quad \gamma > 0,
\]
(21)

\[
\inf_{q \in L^2(\Omega_S) \cap H^1(\Omega_D)^d} \sup_{v \in V} \frac{(\text{div } v, q)_{\Omega_D}}{\|v\|_V} \geq \beta_S,
\]
(22)

\[
\inf_{q \in H^1(\Omega_S) \cap L^2(\Omega_D)^d} \sup_{v \in V} -\langle v, \nabla q \rangle_{\Omega_D} \geq \beta_D,
\]
(23)

with positive constants \(\beta_S, \beta_D\) [7].

The following lemma is needed in the analysis below and is a consequence of the continuous inf-sup conditions (23) [10].

**Lemma 1.** For every \((v, p) \in X\) there is \(w \in V\) such that \(w_S \cdot \mathbf{n}_S = 0\) on \(\Gamma_S\), satisfying

\[
A(v, p; w, 0) \geq c_2 \|p\|_Q^2 - c_1 \|v\|_V^2,
\]
(24)

\[
\|w\|_V \leq c_3 \|p\|_Q,
\]
with positive constants \(c_1, c_2, c_3\).

**Proof.** Let \((v, p) \in X\). Then, due to Stokes inf-sup condition there exists \(w_S \in H^1(\Omega_S)^d\) with \(w_S = 0\) on \(\Gamma_S\) and \(w_S \cdot \mathbf{n} = 0\) on \(\Gamma\) such that

\[
-\langle \text{div } w_S, p \rangle_{\Omega_S} = \|p\|_Q^2,
\]
(25)

\[
\|\nabla w_S\|_{\Omega_S} \leq c_S \|p\|_{\Omega_S}.
\]

For the Darcy equation, due to the condition \(p = 0\) on \(\Gamma_{D,2}\), there exists \(w_D \in H^1(\Omega_D)^d\) with \(w_D \cdot \mathbf{n} = 0\) on \(\Gamma_{D,2}\) and \(\Gamma\), such that

\[
-\langle \text{div } w_D, p \rangle_{\Omega_D} = \|p\|_Q^2,
\]
(26)

\[
\|\nabla w_D\|_{\Omega_D} \leq c_D \|p\|_{\Omega_D}.
\]

Define

\[
w = \begin{cases} w_S \text{ in } \Omega_S \\ w_D \text{ in } \Omega_D \end{cases}
\]
(27)
and then
\[
\mathcal{A}(v, p; w, 0) = (2vD(v), D(w))\Omega_d - (p, \text{div} w)\Omega_d \\
+ (K^{-1}v, w)\Omega_d + (\nabla p, w)\Omega_d \\
+ \delta (\text{div} v, \text{div} w)\Omega_d
\]
\[
\geq -2\nu ||D(v)||_{\Omega_d} ||D(w)||_{\Omega_d} + ||p||^2_{\Omega_d} \\
- k_2 ||\nabla ||_{\Omega_d} ||\nabla*w||_{\Omega_d} + ||\nabla p||^2_{\Omega_d} \\
- \delta ||\text{div} v||_{\Omega_d} ||\text{div} w||_{\Omega_d}
\]
\[
\geq -2\nu||\nabla||_{\Omega_d} ||\nabla w||_{\Omega_d} + ||p||^2_{\Omega_d} \\
- k_2 ||\nabla||_{\Omega_d} ||\nabla D||_{\Omega_d} + ||\nabla p||^2_{\Omega_d} \\
- \delta ||\text{div} v||_{\Omega_d} ||\text{div} w||_{\Omega_d}
\]
\[
\geq -2\nu\kappa ||\nabla||_{\Omega_d} ||\nabla p||_{\Omega_d} + ||p||^2_{\Omega_d} \\
- c_p c_D k_2 ||\nabla||_{\Omega_d} ||\nabla p||_{\Omega_d} + ||p||^2_{\Omega_d} \\
- \delta c_D ||\text{div} v||_{\Omega_d} ||\text{div} w||_{\Omega_d}
\]

In addition, we also have
\[
||w||^2_{V} = ||\nabla w||^2_{\Omega_d} + ||w||^2_{\Omega_d} + ||\text{div} w||^2_{\Omega_d}
\]
\[
\leq c_s^2 ||p||^2_{\Omega_d} + c_p (c_p^2 + 1) ||\nabla p||^2_{\Omega_d} \leq c_s ||p||_{Q},
\]
where \(c_s^2 = \max\{c_s^2, c_D^2 (c_p + 1)\}\).

4. Finite Element Discretization

Let \(\mathcal{S}_h\) be a shape-regular partition of quadrilaterals for \(d = 2\) or hexahedra for \(d = 3\) [12, 13]. The diameter of element \(T \in \mathcal{S}_h\) will be denoted by \(h_T\) and the global mesh size is defined by \(h = \max(h_T : T \in \mathcal{S}_h)\). Let \(\hat{T}\) be the mapping from \(T\) to element \(T\), and \(Q(\hat{T})\) the space of all polynomials on \(\hat{T}\) with maximal degree \(r \geq 0\) in each coordinate. We assume that the mesh \(\mathcal{S}_h\) is obtained from a coarser mesh \(\mathcal{S}_{2h}\) by global refinement. Hence, \(\mathcal{S}_{2h}\) consists of patches of elements of \(\mathcal{S}_h\). We define the finite element space

\[
X_h^T = \{v \in C(\Omega_h) \cup C(\Omega_{2h}) : v|_T \}
\]

\[
= F_T \in Q(\hat{T}), \forall T \in \mathcal{S}_h\}.
\]

For the discrete spaces \(V_h\) and \(Q_h\) we use the equal-order finite element functions that are continuous in \(\Omega_h\) and \(\Omega_{2h}\) and piecewise polynomials of degree \(r \geq 1\):

\[
V_h = (X_h^d)^d \cap V,
\]
\[
Q_h = X_h^r \cap Q \cap H^1(\Omega).
\]

We define the Scott-Zhang interpolation operator which preserves the boundary condition [13], as \(j_h^T : H^1(\Omega) \rightarrow X_h^T\) with stability and interpolation properties, respectively as

\[
||\nabla j_h^T \phi||_{\Omega} \leq c_1 ||\phi||_{1,\Omega}, \quad \phi \in H^1(\Omega).
\]
\[
||\phi - j_h^T \phi||_{m,\Omega} \leq c_1 h^{1-m} ||\phi||_{r+1,\Omega}, \quad \phi \in H^{r+1}(\Omega), \quad m = 0 \text{ or } 1,
\]

where \(c_1, c_2\) are positive constants.

We will also use the inverse inequality

\[
\left(\sum_{T \in \mathcal{S}_h} h_T^2 ||\nabla \phi||_{\Omega}^2\right)^{1/2} \leq c_2 ||\phi||_{\Omega}, \quad \forall \phi \in H^1(\Omega).
\]

Similarly, for vector functions we define the interpolation operator

\[
j_h^T : H^1(\Omega)^d \rightarrow (X_h^r)^d,
\]

with interpolation and stability properties as above.

It is known that the standard Galerkin discretizations of the Darcy system are not stable for equal-order elements. This instability stems from the violation of the discrete analogue
on to the inf-sup condition. One possibility to circumvent this condition is to work with a modified bilinear form $\mathcal{A}_h(\cdot; \cdot)$ by adding a stabilization term $\delta_h(\cdot; \cdot)$; that is,
\[
\mathcal{A}_h(v_h, p_h; w, q) = \mathcal{A}(v_h, p_h; w, q) + \delta_h(p_h; q),
\]
such that the stabilized discrete problem reads
\[
\mathcal{A}_h(v_h, p_h; w, q) = \mathcal{F}(w, q) \quad \forall (w, q) \in V_h \times Q_h.
\]

Unlike in [10] where a combination of a generalized mini element and local projection (LPS) is analyzed and in [14] where a method based on two local Gauss integrals for the Stokes equations is used, here we will analyze the problem using a subgrid method [12, 15, 16].

For this method the filter, with respect to the global Lagrange interpolant $I_{2h}$, onto a coarser mesh $\mathcal{S}_{2h}$ is used. Defining $\kappa_{2h} = 1 - I_h$, the subgrid stabilization terms read
\[
\delta_h(p_h; q) = \sum_{M \in \mathcal{S}_{2h}} h_M \left( \gamma V_{2h} p_h, V_{2h} q \right)_M, \quad r \geq 1,
\]
where $\gamma$ is pathwise constant.

A more attractive method from the computational point is obtained using only the fine mesh with smaller stencil. Defining $\kappa_h = 1 - I_h$ the subgrid stabilization term reads
\[
\delta_h(p_h; q) = \sum_{K \in \mathcal{S}_h} h_K \left( \gamma V_{2h} p_h, V_{2h} q \right)_K, \quad r \geq 2. \tag{42}
\]

Next, we prove the stability of the discrete coupled Stokes-Darcy problem with respect to the norm
\[
\| (v, p) \|_{h} = \left( \|v\|_V^2 + \|p\|_Q^2 + \delta_h(p; p) \right)^{1/2}. \tag{44}
\]

### 5. Stability

**Theorem 2.** Let $\mathcal{S}_h$ be a quasi-regular partition [13]. Then, the following discrete inf-sup condition holds for some positive constant $\tilde{\beta}$ independent of the mesh size $h$.

\[
\inf_{(v_h, p_h) \in V_h \times Q_h} \sup_{w_h \in V_h, q_h \in Q_h, \|w_h\|_V = 1, \|q_h\|_Q = 1} \frac{\mathcal{A}(v_h, p_h; w_h, q_h)}{\|w_h\|_V \|q_h\|_Q} \geq \tilde{\beta}. \tag{45}
\]

**Proof.** First, let $(v_h, p_h) \in V_h \times Q_h$, and then the diagonal testing combined with Korn’s inequality and the positivity of $K^{-1}$ give
\[
\mathcal{A}_h(v_h, p_h; v_h, p_h) = \mathcal{A}(v_h, p_h; v_h, p_h) + \delta_h(p_h; p_h) \geq \tilde{\alpha} \|v_h\|_V^2 + \delta_h(p_h; p_h). \tag{46}
\]

In addition, let $w$ be as in Lemma 1, corresponding to $(v_h, p_h) \in V_h \times Q_h$, and set $z = j_h^h w - w$. Then,
\[
\mathcal{A}(v_h, p_h; j_h^h w, 0) = \mathcal{A}(v_h, p_h; w, 0) + \mathcal{A}(v_h, p_h; z, 0) \geq c_2 \|p_h\|_Q^2 - c_1 \|v_h\|_V^2
\]
\[
+ \mathcal{A}_S(v_h, p_h; z, 0) + \mathcal{A}_D(v_h, p_h; z, 0). \tag{47}
\]

Next, we estimate $\mathcal{A}_S(v_h, p_h; z, 0)$ and $\mathcal{A}_D(v_h, p_h; z, 0)$ as follows:
\[
\mathcal{A}_S(v_h, p_h; z, 0) = (2\nu D(v_h), D(z))_{\Omega_S} + (v_{hs}, z)_{\Omega_S}
\]
\[
+ \frac{\nu}{\sqrt{K}} (v_{hs}, \tau, z_s, \tau)_{\Gamma_T}, \tag{48}
\]
where the first two terms are bounded using Cauchy inequality together with the interpolation, stability, and inverse inequalities
\[
\left( (v_{hs}, D(z))_{\Omega_S} \leq \nu \|v_h\|_V \|D(z)\|_{\Omega_S} \leq \nu c_3 \|v_h\|_V \|w\|_{\Omega_S} \leq \nu c_3 c_4 \|v_h\|_V \|p_h\|_Q, \right.
\]
\[
\left. \left( \sum_{T \in \mathcal{S}_h \cap \Omega_S} h_T^{-1} \|\nabla z\|_T^2 \right)^{1/2} \leq \left( \sum_{T \in \mathcal{S}_h \cap \Omega_S} h_T^{-2} \|p_h\|_T^2 \right)^{1/2} c_1 \|p_h\|_{\Omega_S} \leq c_2 c_3 \c_4 \|w\|_{\Omega_S} \leq c_2 c_3 \c_4 \|p_h\|_Q^2. \right)
\]

The boundary term is bounded using the trace theorem and the $H^1$ - stability by
\[
\left( \frac{\nu}{\sqrt{K}} (v_{hs}, \tau, z_s, \tau)_{\Gamma_T} \right) \leq \frac{c_1}{\sqrt{\nu c_3}} \|v_h\|_V \|\nabla z\|_{\Omega_S}
\leq \frac{c_1}{\sqrt{\nu c_3}} \c_4 \|v_h\|_V \|p_h\|_Q. \tag{50}
\]

Hence, by Young inequality with
\[
e_1 = \frac{c_2}{8 \nu c_3},
\]
\[
e_2 = \frac{c_2 \sqrt{K}}{8 \nu c_3^2 c_4}, \tag{51}
\]
we obtain
\[
\mathcal{A}_S(v_h, p_h; z, 0) \leq \frac{c_2}{8 \nu c_3} \|p_h\|_Q^2 + c_4 \|v_h\|_V^2, \tag{52}
\]
where $c_4 = (4 (\nu c_3^2)^2 + 0.25 (c_2^2 c_4^2)^2) / c_2$. 


For the Darcy bilinear form we have

$$\mathcal{A}_D (v_h, p_h; z, 0) = (K^{-1} v_h, z)_{\Omega_D} + \delta (\text{div} \, v_h, \text{div} \, z)_{\Omega_D} + (\text{div} \, (p_h - \kappa_{2h} p_h), z)_{\Omega_D} + (\nabla \cdot K_{2h} \, p_h, z)_{\Omega_D}$$

$$\leq \|K^{-1} v_h\|_{\Omega_D} \|z\|_{\Omega_D} + \delta \|\text{div} \, v_h\|_{\Omega_D} \|z\|_{\Omega_D} + \|\nabla (p_h - \kappa_{2h} p_h)\|_{\Omega_D} \|z\|_{\Omega_D} + \|\nabla \cdot K_{2h} \, p_h\|_{\Omega_D} \|z\|_{\Omega_D}$$

$$\leq k_2 \|v_h\|_{\Omega_D} \|z\|_{\Omega_D} + \delta \|\text{div} \, v_h\|_{\Omega_D} \|z\|_{\Omega_D} + \|\nabla (p_h - \kappa_{2h} p_h)\|_{\Omega_D} \|z\|_{\Omega_D} + \|\nabla \cdot K_{2h} \, p_h\|_{\Omega_D} \|z\|_{\Omega_D}$$

$$\leq k_2 \|v_h\|_{\Omega_D} \|z\|_{\Omega_D} + \delta \|\text{div} \, v_h\|_{\Omega_D} \|z\|_{\Omega_D} + \|\nabla (p_h - \kappa_{2h} p_h)\|_{\Omega_D} \|z\|_{\Omega_D} + \|\nabla \cdot K_{2h} \, p_h\|_{\Omega_D} \|z\|_{\Omega_D}$$

(53)

Then, by Young inequality and (52) we obtain

$$\mathcal{A}_h (v_h, p_h; \tilde{j}_h^k, w, 0) \geq \frac{5c_k}{8} \|p_h\|_Q^2 - C \left( \|v_h\|_V^2 + \delta_h (p_h; p_h) \right).$$

(54)

Scaling $\tilde{j}_h^k w$ we obtain

$$\mathcal{A}_h (v_h, p_h; j_h^k, w, 0) \geq \|p_h\|_Q^2 - C_1 \left( \|v_h\|_V^2 + \delta_h (p_h; p_h) \right).$$

(55)

Choosing $(w_h, q_h) = (v_h, p_h) + (1/(1 + C_1)) (\tilde{j}_h^k w, 0)$ we obtain

$$\mathcal{A}_h (v_h, p_h; w_h, q_h) \geq \|v_h\|_V^2 + \frac{1}{1 + C_1} \|p_h\|_Q^2 - C_1 \left( \|v_h\|_V^2 + \|p_h\|_Q^2 \right)$$

$$= \frac{1}{1 + C_1} \|v_h\|_V^2 + \frac{1}{1 + C_1} \|p_h\|_Q^2$$

$$= \frac{1}{1 + C_1} \| (v_h, p_h) \|_{h}^2.$$

(56)

which implies the required result

$$\inf_{(v_h, p_h) \in V_h \times Q_h} \sup_{(w_h, q_h) \in V_h \times Q_h} \mathcal{A}_h (v_h, p_h; w_h, q_h) \| (w_h, q_h) \|_{h} \geq \tilde{\beta},$$

(57)

with $\tilde{\beta} = C_2^{-1}/(1 + C_1)$.

6. Error Analysis

Theorem 3. Assume that the solution $(v, p)$ of the Stokes-Darcy problem (19) is such that $(v_h, p_h) \in V_h \cap H^{1/2} (\Omega_h^d) \times Q \cap \Omega_h^{1/2} (\Omega_h^d), (v_P, p_D) \in V_D \cap \Omega^{1/2} (\Omega_h^d), (v_P, p_D)$ is the solution of the stabilized problem (41). Then, the following error estimate holds with constants $c_1, c_2, \ldots, c_7$ independent of $h$:

$$\|v - v_h, p - p_h\|_{h} \leq \left( c_1 v + c_2 \right) h^{2r} \|v\|_{r+1, \Omega_h^d}$$

$$+ (c_3 + c_4 \delta) h^{2r} \|v\|_{r+1, \Omega_h^d}$$

$$+ (c_5 + c_6 \gamma^{1/2} h^{1/2} + c_7 h) h^{2r} \|p\|_{r+1, \Omega_h^d}$$

$$+ (c_8 + c_9 \gamma^{1/2} h^{1/2} + c_7 h) h^{2r} \|p\|_{r+1, \Omega_h^d}$$

(58)

Proof. Using the stability estimate of Theorem 3, there exists $(w_h, q_h) \in V_h \times Q_h$, with $\|(w_h, q_h)\|_h \leq \tilde{C}$ satisfying

$$\|v - v_h, p - p_h\|_{h} \leq \frac{1}{\tilde{\beta}} \frac{\mathcal{A}_h (v - v_h, p - p_h; w_h, q_h)}{\| (w_h, q_h) \|_{h}}$$

$$\leq \frac{1}{\tilde{\beta}} \frac{\mathcal{A}_h (v - v_h, p - p_h; w_h, q_h)}{\| (w_h, q_h) \|_{h}}$$

$$\leq \frac{1}{\tilde{\beta}} \frac{\mathcal{A}_h (v - v_h, p - p_h; w_h, q_h)}{\| (w_h, q_h) \|_{h}}$$

(59)

Then, by Galerkin orthogonality property, the first term of (59) is bounded by

$$\mathcal{A}_h (v - v_h, p - p_h; w_h, q_h) \leq \delta_h (p; q_h) \| (w_h, q_h) \|_{h}$$

$$\leq \delta_h (p; p)^{1/2} \delta_h (q; q)^{1/2} \leq \delta_h (p; p)^{1/2}.$$
Hence, the approximation properties of $\kappa_{2h}$ and $\kappa_h$ imply
\[
\frac{1}{\beta} A_h(\mathbf{v} - \mathbf{v}_h, p - p_h; \mathbf{w}_h, q_h)
\leq \frac{1}{\beta} \left\| \mathbf{v} \right\|_{\Omega} \left\| \mathbf{w}_h \right\|_{\Omega} + \left\| p - p_h \right\|_{\Omega} + \left\| q_h \right\|_{\Omega}.
\]
(61)

To estimate the second term of (59) we consider separately each individual term of the bilinear form $(1/\beta) A_h(\mathbf{j}^h, \mathbf{v} - \mathbf{v}_h, \mathbf{j}_i^h p - p_h; \mathbf{w}_h, q_h)$.

Next, Cauchy–schwarz and Poincaré inequality for the boundary terms imply
\[
\frac{1}{\beta} A_h(\mathbf{j}^h, \mathbf{v} - \mathbf{v}_h, \mathbf{j}_i^h p - p_h; \mathbf{w}_h, q_h)
\leq \frac{1}{\beta} \left[ \frac{1}{\left\| \mathbf{v} \right\|_{\Omega}} \left\| \mathbf{v} \right\|_{\Omega} \left\| \mathbf{w}_h \right\|_{\Omega} + \left\| p - p_h \right\|_{\Omega} + \left\| q_h \right\|_{\Omega} \right]
\]
\[
+ \frac{\gamma \Delta x}{\sqrt{k}} \left\| \nabla (\mathbf{j}^h, \mathbf{v} - \mathbf{v}_h) \right\|_{\Omega} \left\| \mathbf{w}_h \right\|_{\Omega}.
\]
(62)

Squaring the norm and applying Young inequality we obtain
\[
\left\| (\mathbf{j}^h, \mathbf{v} - \mathbf{v}_h, \mathbf{j}_i^h p - p_h) \right\|_{l_2}^2
\leq 4 \left( \overline{c}_1 + \overline{c}_2 \right)^2 \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega} + 4 \left( \overline{c}_3 + \overline{c}_4 \delta \right)^2 \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega} + 4 \left( \overline{c}_5 + \overline{c}_6 \right)^2 \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega}.
\]
Next, we estimate the interpolation error by
\[
\left\| (\mathbf{v} - \mathbf{j}^h, \mathbf{v} - \mathbf{j}_i^h p) \right\|_{l_2}^2
\leq \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega} + \left\| \mathbf{v} - \mathbf{j}^h, \mathbf{v} - \mathbf{j}_i^h p \right\|_{l_{2+1}, \Omega}.
\]
(63)

Adding the interpolation error (64) to the projection error (65) we obtain the required result
\[
\left\| (\mathbf{v} - \mathbf{j}^h, \mathbf{v} - \mathbf{j}_i^h p) \right\|_{l_2}^2
\leq \left\{ \left( \overline{c}_1 + \overline{c}_2 \right)^2 \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega} + \left( \overline{c}_3 + \overline{c}_4 \delta \right)^2 \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega} + \left( \overline{c}_5 + \overline{c}_6 \right)^2 \left\| \mathbf{v} \right\|^2_{l_{2+1}, \Omega} \right\}^{1/2}.
\]

Remark 4. We note that the analysis above holds true for the triangular subgrid interpolation $P_t - P_s - P_r$.

Remark 5. Because of the presence of divergence of the velocity and the gradient of the pressure in the discrete norm, the velocity and pressure solutions are $O(h^2)$ and $O(h^2)$, respectively. So, we expect the $L_2$-asymptotic rates to be $O(h^{r+1})$ and $O(h^{r+1})$.

7. Numerical Results

As a test model problem we take $\Omega = (0, 1) \times (0, 1)$ and split it into $\Omega_S = (0, 1/2) \times (0, 1)$ and $\Omega_D = (1/2, 1) \times (0, 1)$. The interface boundary is $\Gamma = \{(0.5, y) \mid 0 < y < 1\}$. We take


| Table 1: Rates of convergence for velocity and pressure solution in the Stokes subdomain. |
|--------------------------------------------------|
| $h = \frac{1}{8}$ | $\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0,\Omega_D}$ | $\left\| \nabla (\mathbf{u} - \mathbf{u}_h) \right\|_{0,\Omega_D}$ | $\left\| p - p_h \right\|_{0,\Omega_D}$ |
| $h = \frac{1}{16}$ | 1.9303 | 1.0284 | 0.8480 |
| $h = \frac{1}{32}$ | 1.9735 | 1.0208 | 0.9149 |
| $h = \frac{1}{64}$ | 1.9890 | 1.0119 | 0.9511 |
| $h = \frac{1}{128}$ | 1.9951 | 1.0055 | 0.9725 |

| Table 2: Rates of convergence for velocity and pressure solution in the Darcy subdomain. |
|--------------------------------------------------|
| $h = \frac{1}{8}$ | $\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0,\Omega_D}$ | $\left\| \text{div} (\mathbf{u} - \mathbf{u}_h) \right\|_{0,\Omega_D}$ | $\left\| p - p_h \right\|_{0,\Omega_D}$ |
| $h = \frac{1}{16}$ | 0.8813 | 0.8412 | 1.0416 |
| $h = \frac{1}{32}$ | 0.9534 | 0.9235 | 1.0318 |
| $h = \frac{1}{64}$ | 0.9642 | 0.9514 | 1.0167 |
| $h = \frac{1}{128}$ | 0.9857 | 0.9657 | 1.0085 |

In addition, In Table 2, we observe that the velocity field and its divergence are of first-order accuracy in the Darcy subdomain, and the pressure is of first-order accuracy with respect to the $L_2$-norm. So, clearly these results are in agreement with the theoretical results of the previous section and are comparable to the ones found in [2, 5].

**Competing Interests**

The author declares that they have no competing interests.

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