PERMUTATION GROUP AND DEGREE OF W-OPERATOR

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ABSTRACT. The cut-and-join operator is introduced by Goulden and Jackson. Mironov, Morosov and Natanzon gives a more general construction and call it W-operator $W([n])$. The cut-and-join operator is $W([2])$. In this paper, we show that $W([n])$ can be written as the sum of $n!$ terms and each term corresponds uniquely to a permutation in $S_n$. We also define the degree of each term. The number of terms of $W([n])$ with highest degree is the Catalan number $C_n$.

1. Introduction

The cut-and-join operator $\Delta$

$$\Delta = \sum_{i,j \geq 1} \left( (i + j)p_ip_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right)$$

is introduced by Goulden and Jackson [4]. It is an infinite sum of differential operators in variables $p_i$, $i \geq 1$. This operator plays an important role in calculating the simple Hurwitz number [5].

Mironov, Morosov and Natanzon [11] [12] construct the $W$-operators $W([n])$, where $n$ is a positive integer. The $W$-operators are differential operators acting on the formal power series $\mathbb{C}[[X_{ij}]]_{i,j \geq 1}$, where $X_{ij}$ are coordinate functions on the positive-half-infinite matrix. A subring of $\mathbb{C}[[X_{ij}]]_{i,j \geq 1}$ is $\mathbb{C}[p_1, p_2, ...]$, where $p_k = Tr(X^k)$ and $X = (X_{ij})_{i,j \geq 1}$. A direct calculation shows that $W([2])$ is the cut-and-join operator $\Delta$ on the ring $\mathbb{C}[p_1, p_2, ...]$. In this paper, we study the $W$-operators $W([n])$, $n \geq 1$, as operators on the ring $\mathbb{C}[p_1, p_2, ...]$.

In Section 2, we review the definition of $W$-operator and do some basic calculations of $W$-operators.

In Section 3, we review a basic theorem about the $W$-operator [13].

Theorem 1.1 (3.12). $W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, ...]$ and it can be written as the sum of $n!$ terms, each of which corresponds to a unique element $\beta \in S_n$, i.e.

$$W([n]) = \sum_{\beta \in S_n} FS_\beta,$$

where $FS_\beta$ (free summation) is the term corresponding to the permutation $\beta \in S_n$.

Each $FS_\beta$ itself is the sum of infinitely many differential operators. We will refer to $FS_\beta$ as a summation in $W([n])$. The second part of the theorem relates the permutation group to the study of $W$-operators.

In Section 4, we study some combinatorics properties of the summations. We define the degree of summations in $W([n])$. Roughly speaking, the degree of the
summation $FS_{\beta}$ is the sum of its polynomial part’s degree and the order of its differential part. For example, consider the following summation

$$FS_{(1)} = \sum_{i=1}^{n} ip_i \frac{\partial}{\partial p_i}$$

where (1) is the unique permutation in $S_1$. The degree of this summation is 2. The degree of different summations in $W([n])$ can be different. Given a positive integer $n$, the highest degree of summations in $W([n])$ is $n + 1$. There could also be lower degree summation in $W([n])$. An ordinary summation (OS) is a summation with degree $n + 1$ in $W([n])$ and an $(r, s)$-type OS is an OS summation such that its polynomial part degree is $r$ and its order of differential part is $s$, which also means $r + s = n + 1$. The $W$-operator $W([n])$ and the its ordinary summations part play an important role in calculating the generating function of the Hurwitz numbers. The reader can find more details in [6] [11] [15].

Also, in Section 4, we define the non-crossing sequence. The non-crossing sequence relates to the non-crossing partition [14] and non-crossing permutation [10]. Given a positive integer $n$, we fix a sequence of $n$ integers as following (see Construction 4.15)

$$n \quad n-1 \quad \ldots \quad 2 \quad 1.$$ 

We insert $r$ pairs of brackets into this sequence preserving the order of the brackets () satisfying the following conditions

- any integer is contained in at least one pair of brackets and any pair of brackets contains at least one integer,
- there can be at most one left bracket and at most one right bracket between two successive integers. In another words, only the following four cases are allowed

  $$k + 1 \quad k,$$
  $$k + 1 \quad \quad k,$$
  $$k + 1 \quad ( \quad k,$$
  $$k + 1 \quad ) ( \quad k.$$ 

The main theorem we prove in this section is that each $(r, s)$-type OS corresponds uniquely to such a sequence with $r$ pairs of brackets (see Theorem 4.18).

In Section 5, we construct the dual non-crossing sequence. Given an $(r, s)$-type OS $FS_{\alpha}$, $FS_{\alpha}$ corresponds to a unique non-crossing sequence. The dual non-crossing sequence corresponds to an $(s, r)$-type OS.

In Section 6, we use some properties of Catalan number and Narayana number together with Theorem 4.18 in Section 4 to calculate the number of $(r, n-r+1)$-type ordinary summations.

**Theorem 1.2** [6.1]. The number of $(r, n-r+1)$-type OS in : $tr(D^n)$ : is the Narayana number:

$$|OS(n, r)| = \frac{1}{n+1} \binom{n+1}{r} \binom{n-1}{r-1}.$$
The number of all summations with degree \( n+1 \) in : \( \text{tr}(D^n) \) : is the Catalan number

\[
\sum_{r \geq 1} \frac{1}{n+1} \binom{n+1}{r} \binom{n-1}{r-1} = \frac{1}{n+1} \binom{2n}{n}.
\]

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2. Matrix Variable and W-Operator

**Definition 2.1.** A variable matrix \( X \) is an infinite matrix with variable \( X_{ab} \) in the \((a, b)\)-entry, i.e. \( X := (X_{ab})_{a \geq 1, b \geq 1} \).

**Definition 2.2.** Given \( k \geq 1 \), \( p_k \) is the trace of \( X^k \), i.e. \( p_k = \text{tr}(X^k) \). \( p_k \) is a power series in \( \mathbb{C}[[X_{ab}]]_{a, b \geq 1} \). \( \mathbb{C}[p_1, p_2, ...] \) is a polynomial ring with infinitely many variables \( p_k \).

**Remark 2.3.** If \( X \) is a special variable matrix with \( X_{ab} = 0 \), when \( a \neq b \), then \( p_k \) is exactly the power sum symmetric function \( \sum_{i=1}^{\infty} X_{ii}^k \).

**Definition 2.4.** The operator matrix \( D \) is the infinite matrix with \( D_{ab} \) in the \((a, b)\)-entry, where \( D_{ab} = \sum_{c=1}^{\infty} X_{ac} \frac{\partial}{\partial X_{bc}} \).

In the rest of the paper, we prefer to write \( D_{ab} = X_{ac} \frac{\partial}{\partial X_{bc}} \) with the summation over \( c \) implied.

**Lemma 2.5.**

\[
D_{ab} F(p) = \sum_{k=1}^{\infty} k(X^k)_{ab} \frac{\partial F(p)}{\partial p_k},
\]

for any \( F(p) \in \mathbb{C}[p_1, p_2, ...] \).

\[
D_{cd}(X^k)_{ab} = \sum_{j=0}^{k-1} (X^j)_{ad} (X^{k-j})_{cb}.
\]

In particular, we have

\[
\sum_{k_j=1}^{\infty} D_{a_{n+1}a_n} (X^{k_j})_{a_1 a_j} = \sum_{k_j=1}^{\infty} \sum_{k_n=0}^{k_j} (X^{k_n})_{a_1 a_n} (X^{k_j - k_n})_{a_{n+1} a_j} = \sum_{k_j=1}^{\infty} \sum_{k_n=1}^{\infty} (X^{k_n})_{a_1 a_n} (X^{k_j})_{a_{n+1} a_j}.
\]

**Proof.** See [11], [15].

**Definition 2.6.** The normal ordered product of \( D_{ab} \) and \( D_{cd} \) is

\[
: D_{ab} D_{cd} := X_{ae_1} X_{ce_2} \frac{\partial}{\partial X_{be_1}} \frac{\partial}{\partial X_{de_2}}
\]

(agent with the summation over \( e_1, e_2 \) implied).
Remark 2.8. The formula of normal ordered product \( D_{cd}D_{ab} : \) in Lemma 2.7 comes from the calculation of \( D_{cd}D_{ab} : \). By calculation, we have

\[
D_{an+2\alpha n+1}D_{\alpha n+1\alpha n-1} = \sum_{k,j \geq 1} ((k+j)(X^j)_{\alpha n+1\alpha n+1}(X^k)_{\alpha n+2\alpha n-1} \frac{\partial}{\partial \partial \partial}) \\
+ \sum_{k,j \geq 1} (kj(X^k)_{\alpha n+1\alpha n-1}(X^j)_{\alpha n+2\alpha n+1} \frac{\partial^2}{\partial \partial \partial j}).
\]

The subscript \( j \) in the first summation \( \sum_{k,j \geq 1} ((k+j)(X^j)_{\alpha n+1\alpha n+1}(X^k)_{\alpha n+2\alpha n-1} \frac{\partial}{\partial \partial \partial}) \) goes from 0 to infinity. If we calculate the normal ordered product \( D_{cd}D_{ab} : \), the "zero" term does not appear, which gives the formula in Lemma 2.7. In fact, the zero term comes from \( [\frac{\partial}{\partial X_{\alpha n+1\alpha n}}, X_{\alpha n+1\alpha n+2}] \), since

\[
D_{an+2\alpha n+1}D_{\alpha n+1\alpha n-1} = \sum_{k,j \geq 1} ((k+j)(X^j)_{\alpha n+1\alpha n+1}(X^k)_{\alpha n+2\alpha n-1} \frac{\partial}{\partial \partial \partial}) \\
+ \sum_{k,j \geq 1} (kj(X^k)_{\alpha n+1\alpha n-1}(X^j)_{\alpha n+2\alpha n+1} \frac{\partial^2}{\partial \partial \partial j}).
\]

The reader can use the same method to calculate the normal product \( D_{an+2\alpha n+1} \cdots D_{a2a1} : \) from the product \( D_{an+2\alpha n+1} \cdots D_{a2a1} : \). Compared with the product \( D_{an+2\alpha n+1} \cdots D_{a2a1} : \), the normal product \( D_{an+2\alpha n+1} \cdots D_{a2a1} : \) has no "zero term". More precisely, all subscripts go from one to infinity.

Definition 2.9. For any positive integer \( n \), we define the \( W \)-operator \( W([n]) \) as

\[
W([n]) := \frac{1}{n} : tr(D^n) : = \frac{1}{n} \sum_{a_1, \ldots, a_n \geq 1} : D_{a_1a_n}D_{a_na_{n-1}} \cdots D_{a2a1} : .
\]

3. Quiver and Permutation Group

In this section, we review some main constructions and theorems in [15]. We begin with the quiver. A quiver \( Q = (V, A, s, t) \) is a quadruple, where \( V \) is the set of vertices, \( A \) is the set of arrows, \( s \) and \( t \) are two maps \( A \to V \). If \( a \in A \), \( s(a) \) is the source of this arrow and \( t(a) \) is the target. We assume \( V \) and \( A \) to be finite sets. If \( B \) is a subset of \( A \), \( V_B = \{ s(a), t(a), a \in B \} \), then we call \( (V_B, B, s', t') \) the subquiver of \( Q \), where \( s' = s|_B \), \( t' = t|_B \). A quiver \( Q = (V, A, s, t) \) is connected if the underlying undirected graph of \( Q \) is connected. A connected quiver \( Q = (V, A, s, t) \) is a loop, if for any vertex \( v \in V \), there is a unique arrow \( a \in A \) such that \( s(a) = v \) and a unique arrow \( b \in A \) such that \( t(b) = v \). A chain is obtained by omitting a single arrow in a loop. \( \mathbb{F} Q \) is the set of all quivers.
Definition 3.1. Let $\Phi_n : S_n \to \mathbb{F}Q$ be the map such that $\Phi_n(\alpha) = Q_\alpha$, where

$$Q_\alpha = \{ V_\alpha = \{1, \ldots, n\}, A_\alpha = \{i \mapsto \alpha(i), 1 \leq i \leq n\}, s_\alpha, t_\alpha \}. $$

$Q_\alpha$ consists of disjoint loops which represent disjoint cycles of $\alpha$.

Since the source map and target map is well-defined for any arrow in any quiver, we will use the same symbols $s, t$ for the source and targets maps in any quiver from now on.

Remark 3.2. Every permutation can be written as the product of disjoint cycles. For example, $(123)(45) \in S_6$. But, in this paper, we prefer to write it as $(123)(45)(6)$, which includes the fixed integer 6 as "1-cycle".

Given $\alpha \in S_n$, $Q_\alpha$ is the corresponding quiver. We define a new vertex set

$$\hat{V}_\alpha = \{1, \ldots, n, n+1\}. $$

There is a unique arrow $a$ in $Q_\alpha$ such that $s(a) = 1$. We substitute this arrow by a new one $\hat{a}$, where $s(\hat{a}) = n+1$ and $t(\hat{a}) = t(a)$. Denote by $\hat{A}_\alpha$ the new set of arrows.

Definition 3.3. Denote by $\hat{Q}_\alpha$ the new quiver,

$$\hat{Q}_\alpha = (\hat{V}_\alpha, \hat{A}_\alpha, s, t). $$

Example 3.4. Take $\alpha = (123) \in S_3$, then $Q_\alpha$ is

$$1 \to 2 \to 3 \to 1. $$

$\hat{Q}_\alpha$ is

$$4 \to 2 \to 3 \to 1. $$

Clearly, $Q_\alpha$ is a loop and $\hat{Q}_\alpha$ is a chain.

In general, $\hat{Q}_\alpha$ consists of a chain and possibly a number of loops. Clearly, we can construct $Q_\alpha$ uniquely from $\hat{Q}_\alpha$.

We will consider how to construct $\hat{Q}_\alpha$ from $\hat{Q}_\beta$, where $\alpha \in S_n$ and $\beta \in S_{n+1}$. Given any permutation $\alpha \in S_n$ and $\beta \in S_{n+1}$, compared with $\hat{Q}_\beta$, $Q_\alpha$ has two properties

- For any $\alpha \in S_n$, there is no arrow $a \in \hat{A}_\alpha$ such that $t(a) = n + 1$.
- $n+2 \notin \hat{V}_\alpha$.

Hence, if we want to construct from $\hat{Q}_\beta$, $\beta \in S_{n+1}$, a quiver $\hat{Q}_\alpha$ for some $\alpha \in S_n$, we have to delete the vertex $n+2$ from $\hat{V}_\beta$ and delete one arrow from $\hat{A}_\beta$. Here is the construction.

Construction 3.5. Given $\beta \in S_{n+1}$, we take the arrows $a, b \in \hat{A}_\beta$ such that

$$s(a) = n+2, \quad t(b) = n+1. $$

We also assume that

$$s(b) = j, \quad t(a) = i. $$

If $a$ and $b$ are the same arrow which means $j = n+2, i = n+1$, then delete this arrow from $\hat{A}_\beta$ and delete $n+2$ from $\hat{V}_\beta$. If $a \neq b$, we delete these two arrows $a, b$ and add a new arrow $c$ such that $s(c) = j, t(c) = i$. Also, we delete the vertex $n+2$ from $\hat{V}_\beta$. Denote by $\hat{Q}'_\beta$ the new quiver we construct from $\hat{Q}_\beta$ in this way.
Example 3.6. Let’s consider the example $\beta = (321)$. The quiver $\hat{Q}_{\beta}$ is

\[ 4 \to 3 \to 2 \to 1 . \]

In this case, the arrow $a, b$ are the same $4 \to 3$. Then, we delete this arrow and the vertex 4. We get the following quiver $\hat{Q}'_{(321)}$

\[ 3 \to 2 \to 1 , \]

which corresponds to the quiver $\hat{Q}_{(21)}$.

The second example is $\beta = (3)(21)$ with $\hat{Q}_{(3)(21)}$

\[ 4 \to 2 \to 1 , \quad 3 \to 3 . \]

Now $a$ is $4 \to 2$ and $b$ is $3 \to 3$. By Construction 3.3, we get the following quiver $\hat{Q}'_{(3)(21)}$

\[ 3 \to 2 \to 1 , \]

which corresponds to the same quiver $\hat{Q}_{(21)}$.

The last example is $\beta = (3)(2)(1)$, the identity permutation in $S_3$. By the same argument, we find $\hat{Q}'_{(3)(2)(1)} = \hat{Q}_{(2)(1)}$.

The fact that $\hat{Q}'_{\beta}$ in this example is always of the form $\hat{Q}_{\alpha}$ is no accident.

Lemma 3.7. Given any permutation $\beta \in S_{n+1}$, there is a permutation $\alpha \in S_n$ such that $\hat{Q}_{\alpha} = \hat{Q}'_{\beta}$.

Proof. See [15].

Next we want to go in the opposite direction. For each $\alpha \in S_n$, we want to find all $\beta \in S_{n+1}$ such that $\hat{Q}'_{\beta} = \hat{Q}_{\alpha}$. Given a fixed permutation $\alpha \in S_n$, there turns out to be $n+1$ choices of $\beta$ in $S_{n+1}$.

Given any quiver $\hat{Q}_{\alpha}$, $\alpha \in S_n$, if we want to construct a new quiver $\hat{Q}_{\beta}$ representing an element $\beta \in S_{n+1}$, we should add the vertex $n + 2$ into $\hat{V}_{\alpha}$ and add arrows $a_1, a_2$ in $\hat{A}_{\alpha}$ such that

\[ s(a_1) = n + 2 , \quad t(a_2) = n + 1 , \]

where $a_1, a_2$ can be the same arrow. Here is the construction.

Construction 3.8. Given any $\alpha \in S_n$, we write $\alpha$ as the product of disjoint cycles $\alpha = \alpha_1 \alpha_2 ... \alpha_k$. We assume $1 \in \alpha_1$. So, the corresponding subquiver for $\alpha_1$ in $\hat{Q}_{\alpha}$ is the chain as following

\[ n + 1 \to ... \to 1 . \]

• Case 0

We extend the quiver for $\alpha_1$ directly

\[ n + 2 \to n + 1 \to ... \to 1 . \]

Clearly, this subquiver represents a well-defined cycle $\beta_1$. In this way, we construct a permutation $\beta \in S_{n+1}$, where $\beta = \beta_1 \alpha_2 ... \alpha_k$. In this case, $a_1, a_2$ are the same arrow

\[ a_1 = a_2 : n + 2 \to n + 1 . \]
Next we consider the general case. Roughly speaking, the idea is cutting an arrow in $\hat{Q}_\alpha$ and reconnect the chain and loops in $\hat{Q}_\alpha$. There are $n$ choices of arrows in $\hat{Q}_\alpha$. We first choose an arbitrary arrow $a : i \to j$ in $\hat{Q}_\alpha$.

- **Case 1**, $a \in \hat{Q}_{\alpha_1}$
  In this case, $\hat{Q}_{\alpha_1} \hat{Q}_{\alpha_2}$ is
  \[ n + 1 \to ... \to i \to j \to ... \to 1. \]
  First, cut the arrow $i \to j$, we get
  \[ n + 1 \to ... \to i , j \to ... \to 1. \]
  Then, we add the following two arrows
  \[ a_1 : n + 2 \to j , \quad a_2 : i \to n + 1. \]
  Finally, we get the following quiver,
  \[ n + 2 \to j \to ... \to 1 , \quad i \to n + 1 \to ... \to i. \]
  They represent two disjoint cycles in $S_{n+1}$ by replacing $n+2$ by 1. Call them $\beta_1$ and $\beta_2$. So, $\beta = \beta_1 \beta_2 \alpha_2 \ldots \alpha_k$ is the permutation in $S_{n+1}$ constructed by cutting the arrow $a$.

- **Case 2**, $a \notin \hat{Q}_{\alpha_1}$
  Without loss of generality, we can assume $a \in \hat{Q}_{\alpha_2}$. The corresponding quiver for $\alpha_1$ and $\alpha_2$ are
  \[ n + 1 \to ... \to 1 , \quad i \to j \to ... \to i. \]
  Similar to **Case 1**, we cut the arrow $i \to j$ and we get
  \[ n + 1 \to ... \to 1 , \quad j \to ... \to i. \]
  Then, we add the following two arrows
  \[ a_1 : n + 2 \to j , \quad a_2 : i \to n + 1. \]
  Finally, we get the chain
  \[ n + 2 \to j \to ... \to i \to n + 1 \to ... \to 1 \]
  It represents a cycle in $S_{n+1}$ by replacing $n+2$ by 1 and denote by $\beta_1$. So, $\beta = \beta_1 \alpha_3 \ldots \alpha_k$ is a permutation in $S_{n+1}$.

In all cases, we have $\hat{Q}'_{\beta} = \hat{Q}_\alpha$. In the quiver $\hat{Q}_\alpha$, there are $n$ arrows. Hence, we can construct $n$ quivers or permutations from **Case 1,2**. In conclusion, there are $n + 1$ choices of $\beta \in S_{n+1}$ such that $\hat{Q}'_{\beta} = \hat{Q}_\alpha$. It is easy to see that $\beta$ constructed in this way are distinct.

From Construction 3.5 and Construction 3.8, we can get the following theorem.

**Theorem 3.9.** For any $\alpha \in S_n$, we can construct $n + 1$ distinct permutations $\beta$ in $S_{n+1}$ such that $\hat{Q}'_{\beta} = \hat{Q}_\alpha$. In fact, if we do it for all $\alpha \in S_n$, we will get $(n+1)!$ elements, which are exactly all permutations in the group $S_{n+1}$.

**Remark 3.10.** We can summarize the above construction as following. Given any positive integer $n$, there is a map

$$\Psi_n : S_{n+1} \to S_n$$
such that $\Psi_n(\beta) = \alpha$, when $\hat{Q}_\beta = \hat{Q}_\alpha$. Lemma 3.7 says that $\Psi_n$ is well defined and the theorem, Theorem 3.9, says that the preimage $\Psi_n^{-1}(\alpha)$ consists of $n+1$ distinct elements $\beta$. So, $\Psi_n$ is a $n + 1$ to $1$ map.

We define the following notation $[\alpha, j]$, which will be used in the next section.

**Definition 3.11.** Let $\alpha$ be a permutation in $S_n$. Denote by $[\alpha, j]$ the permutation constructed from $\alpha$, where $j$ is an integer, $0 \leq j \leq n$. $[\alpha, 0]$ corresponds to the Case 0 in Construction 3.8 and, if $j \geq 1$, $[\alpha, j]$ corresponds to Case 1,2 by cutting the arrow $a$ such that $t(a) = j$.

In fact, Construction 3.8 comes from the calculation of $W([n])$ (see Definition 3.7), which also implies the following theorem. We will give the basic idea about the theorem without proof.

**Theorem 3.12.** $W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, \ldots]$ and it can be written as the sum of $n!$ summations, each of which corresponds to a unique quiver $Q_\beta$ or a unique permutation $\beta \in S_n$.

**Proof.** We give some examples and ideas about the proof.

To calculate $W([n])$, we have to figure out the operator

$$D_{a_1 a_2 \cdots a_n} : D_{a_1 a_n} D_{a_2 a_{n-1}} \cdots D_{a_2 a_1} :$$

for any $a_i \geq 1, 1 \leq i \leq n$. By Remark 2.8 we have to calculate the product $D_{a_1 a_n} D_{a_2 a_{n-1}} \cdots D_{a_2 a_1}$. Since we want to use induction to calculate this product, we replace $D_{a_1 a_n}$ by $D_{a_{n-1} a_n}$. Now let’s calculate the base step and we will explain how we construct the summation $FS_\beta$ corresponding to $Q_\beta$, where $\beta \in S_2$. Let $n = 1$, by Lemma 2.5 we have

$$D_{a_2 a_1} = \sum_{k_1=1}^{\infty} k_1 (X^{k_1})_{a_2 a_1} \frac{\partial}{\partial p_{k_1}}.$$  

We associate this summation to the quiver

$$2 \rightarrow 1,$$

which corresponds to the subscript of $(X^{k_1})_{a_2 a_1}$.

Now we calculate $D_{a_3 a_2} D_{a_2 a_1}$.

(1)

$$D_{a_3 a_2} D_{a_2 a_1} = \sum_{k_1=1}^{\infty} (D_{a_3 a_2} (k_1 (X^{k_1})_{a_2 a_1})) \frac{\partial}{\partial p_{k_1}} + \sum_{k_1=1}^{\infty} k_1 (X^{k_1})_{a_2 a_1} \left( D_{a_3 a_2} \circ \frac{\partial}{\partial p_{k_1}} \right),$$

By Lemma 2.5 we have

$$D_{a_3 a_2} D_{a_2 a_1} = \sum_{k_1 \geq 1, k_2 \geq 0} ((k_1 + k_2)(X^{k_2})_{a_2 a_2} (X^{k_1})_{a_3 a_1} \frac{\partial}{\partial p_{k_1 + k_2}}$$

$$+ \sum_{k_1, k_2 \geq 1} (k_1 k_2 (X^{k_2})_{a_3 a_2} (X^{k_1})_{a_2 a_1} \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}}).$$

We associate the first summation to the quiver $Q_{(1)(2)}$

$$2 \rightarrow 1, 3 \rightarrow 1,$$
which comes from the subscripts of the polynomial part \((X^{k_2})_{a_2 a_2} (X^{k_1})_{a_3 a_3}\). Similarly, the second summation corresponds to the quiver \(\tilde{Q}_{(12)}\)

\[ 3 \rightarrow 2 \rightarrow 1. \]

We know that \(D_{a_3 a_2}\) acting on \((X^{k_1})_{a_2 a_2}\) gives the first summation, which corresponds to the case of cutting the arrow \(2 \rightarrow 1\) in \(\tilde{Q}_{(1)}\) in Construction 3.8. The same argument holds for the second summation, where \(D_{a_3 a_2}\) acts on \(\frac{\partial}{\partial p_{a_1}}\) and it corresponds to the Case 0 in Construction 3.8. By Lemma 2.7 and Remark 2.8, we know that \( D_{a_3 a_2} D_{a_2 a_1} :\) and \( D_{a_3 a_2} D_{a_2 a_1} \) are almost the same and the only difference comes from the term with subscript \(j = 0\) in the first summation. Hence, we can use quivers to describe the summations of \(D_{a_3 a_2} D_{a_2 a_1}\) : in the same way as \(D_{a_3 a_2} D_{a_2 a_1}\). In conclusion, we find that \( D_{a_3 a_2} D_{a_2 a_1} \) : can be written as the sum of two summations, which correspond to quivers \(Q_\alpha, \alpha \in S_2\),

\[ D_{a_3 a_2} D_{a_2 a_1} : = \sum_{k_1, k_2 \geq 1} ((k_1 + k_2)(X^{k_2})_{a_2 a_2} (X^{k_1})_{a_3 a_3} \frac{\partial}{\partial p_{k_1 + k_2}}) + \sum_{k_1, k_2 \geq 1} (k_1 k_2 (X^{k_2})_{a_2 a_2} (X^{k_1})_{a_3 a_3} \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}}). \]

We use the notation \(\tilde{F}S_\alpha\) for the summation corresponding to \(\alpha \in S_2\). We have

\[ D_{a_3 a_2} D_{a_2 a_1} : = \sum_{\alpha \in S_2} \tilde{F}S_\alpha. \]

Comparing with formula (1), the ranges of integers \(k_1, k_2\) are the same in \(D_{a_3 a_2} D_{a_2 a_1} :\), i.e. from one to infinity (see Remark 2.8). Finally, let \(a_3 = a_1\) and sum over \(a_1, a_2\),

\[ \sum_{a_1, a_2 \geq 1} D_{a_1 a_2} D_{a_2 a_1} : = \sum_{a_1, a_2 \geq 1} \sum_{k_1 \geq 1, k_2 \geq 1} ((k_1 + k_2)(X^{k_2})_{a_2 a_2} (X^{k_1})_{a_1 a_1} \frac{\partial}{\partial p_{k_1 + k_2}}) + \sum_{a_1, a_2 \geq 1} \sum_{k_1, k_2 \geq 1} (k_1 k_2 (X^{k_2})_{a_2 a_2} (X^{k_1})_{a_1 a_1} \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}}). \]

We find that each summation can be written as some polynomial times a differential operator in variable \(p_i\). We get the following formula

\[ W([2]) = \frac{1}{2} \sum_{a_1, a_2 \geq 1} D_{a_1 a_2} D_{a_2 a_1} : = \frac{1}{2} \sum_{k_1, k_2 \geq 1} ((k_1 + k_2)p_{k_1}p_{k_2} \frac{\partial}{\partial p_{k_1 + k_2}} + k_1 k_2 p_{k_1 + k_2} \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}}). \]

By induction on \(n\), we can assume that \( D_{a_{n+1} a_n ... D_{a_2 a_1} :\) can be written in the following way

\[ D_{a_{n+1} a_n ... D_{a_2 a_1} := \sum_{\alpha \in S_n} \tilde{F}S_\alpha, \]

where \(\tilde{F}S_\alpha\) is defined as

\[ \tilde{F}S_\alpha = \sum_{k_1, ..., k_n \geq 1} \left( \prod_{r \in A_n} (X^{k_{1(r)}})_{a_{1(r)} a_{1(r)}} \right) \tilde{D}FS_\alpha (k_1, ..., k_n), \]
where $\hat{A}_\alpha$ is the set of arrows in $Q_\alpha$, $s$ is the source map, $t$ is the target map (see Definition 3.3) and $\widetilde{Dfs}_\alpha(k_1, \ldots, k_n)$ is the differential part with constant coefficients depending on $k_i$, $1 \leq i \leq n$. The differential part $\overline{Dfs}_\alpha(k_1, \ldots, k_n)$ is uniquely determined by the permutation $\alpha$ and integers $k_i$, $1 \leq i \leq n$. We do not need to know how to write it down precisely. Let’s take $\alpha = (21)$ as an example, which is one of the summations in $D_{a_3a_2}D_{a_2a_1}$:

$$\overline{F}s_{(12)} = \sum_{k_1, k_2 \geq 1} \left( (X^{k_2})_{a_3a_2} (X^{k_1})_{a_2a_1} k_1 k_2 \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}} \right).$$

$(X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1}$ is the product of variables described by the arrows and the differential part is

$$\overline{Dfs}_\alpha(k_1, \ldots, k_n) = k_1 k_2 \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}}.$$

Now we try to calculate the product $D_{a_{n+2a_{n+1}}}F{s}_\alpha$. By the product rule, we have

$$D_{a_{n+2a_{n+1}}}F{s}_\alpha =$$

$$\sum_{r' \in \hat{A}_n} \sum_{k_1, \ldots, k_n \geq 1} (D_{a_{n+2a_{n+1}}} (X^{k_{r'}})_{a_{s(r')}a_{t(r')}}) \left( \prod_{r \in \hat{A}_n, r \neq r'} (X^{k_r})_{a_{s(r')}a_{t(r')}} \right) \overline{Dfs}_\alpha(k_1, \ldots, k_n)$$

$$+ \sum_{k_1, \ldots, k_n \geq 1} \left( \prod_{r \in \hat{A}_n} (X^{k_{r'}})_{a_{s(r')}a_{t(r')}} \right) (D_{a_{n+2a_{n+1}}} \circ \overline{Dfs}_\alpha(k_1, \ldots, k_n)).$$

We introduce another notation. If $j \neq 0$, there is a unique arrow $r' \in \hat{A}_n$ such that $t(r') = j$. We define the operator $D_{a_{n+2a_{n+1},j}}$ acting on $F{s}_\alpha$ as

$$D_{a_{n+2a_{n+1},j}}F{s}_\alpha :=$$

$$\sum_{k_1, \ldots, k_n \geq 1} (D_{a_{n+2a_{n+1}}} (X^{k_{r'}})_{a_{s(r')}a_{t(r')}}) \left( \prod_{r \in \hat{A}_n, r \neq r'} (X^{k_r})_{a_{s(r')}a_{t(r')}} \right) \overline{Dfs}_\alpha(k_1, \ldots, k_n).$$

If $j = 0$, we define $D_{a_{n+2a_{n+1},0}}F{s}_\alpha$ as

$$D_{a_{n+2a_{n+1},0}}F{s}_\alpha :=$$

$$\sum_{k_1, \ldots, k_n \geq 1} \left( \prod_{r \in \hat{A}_n} (X^{k_{r'}})_{a_{s(r')}a_{t(r')}} \right) (D_{a_{n+2a_{n+1}}} \circ \overline{Dfs}_\alpha(k_1, \ldots, k_n)).$$

In terms of the new operators $D_{a_{n+2a_{n+1},j}}$, we have

$$D_{a_{n+2a_{n+1}}}F{s}_\alpha = \sum_{j=0}^n D_{a_{n+2a_{n+1},j}} \overline{F}s_{\alpha},$$

and

$$D_{a_{n+2a_{n+1}}}F{s}_\alpha := \sum_{j=0}^n D_{a_{n+2a_{n+1},j}} \overline{F}s_{\alpha} :.$$

We can define $F{s}_{\beta}$ inductively as

$$F{s}_{\beta} =: D_{a_{n+2a_{n+1},j}} \overline{F}s_{\alpha} :,$$
where \( \beta = [\alpha, j], 0 \leq j \leq n \). Recall the following two formulas in Lemma 2.5

\[
D_{a_{n+2}a_{n+1}} = \sum_{k=1}^{\infty} k(X^k)_{a_{n+2}a_{n+1}} \frac{\partial}{\partial p_k},
\]

\[
\sum_{k_j=1}^{\infty} D_{a_{n+2}a_{n+1}}(X^{k_j})_{a_i a_j} = \sum_{k_j=1}^{\infty} \sum_{k_n=0}^{\infty} (X^{k_n})_{a_i a_{n+1}} (X^{k_j})_{a_{n+2}a_j}.
\]

With the above two formulas, we leave it for the reader to check that \( \tilde{FS}_\beta \) defined by Equation 3 can be written in the same form as \( FS_\alpha \) in Equation 2.

\[
\tilde{FS}_\beta = \sum_{k_1, \ldots, k_{n+1} \geq 1} \left( \prod_{r \in A_{\beta}} (X^{k_{i(r)}})_{a_{i(r)} a_{i(r)}} \right) \tilde{D}FS_\beta(k_1, \ldots, k_n, k_{n+1}).
\]

So, by induction, \( D_{a_{n+2}a_{n+1}} \ldots D_{a_{2}a_1} : \) can be written in the following way

\[
: D_{a_{n+2}a_{n+1}} \ldots D_{a_{2}a_1} := \sum_{\beta \in S_{n+1}} \tilde{FS}_\beta.
\]

Finally, for each \( \tilde{FS}_\beta \), replace \( a_{n+2} \) by \( a_1 \) and take the sum over \( a_i, 1 \leq i \leq n+1 \). Then, we will get a summation in variables \( p_i \) corresponding to \( \tilde{FS}_\beta \). \( W([n+1]) \) can be written as the sum of \( (n+1)! \) summations, each of which corresponds to a unique permutation in \( S_{n+1} \).

**Definition 3.13.** For any permutation \( \beta \in S_{n+1} \), denote by \( FS_\beta \) the summation corresponding to \( \tilde{FS}_\beta \) (or \( \beta \)) in the decomposition of \( W([n+1]) \).

**Remark 3.14.** Recall that \( p_i \) is defined as the trace of \( X^i \). If we define the degree of \( p_i \) to be one, we claim that the degree of the polynomial part of \( FS_\alpha \) is exactly the number of disjoint cycles of \( \alpha \). We will explain it in the rest of this remark.

Given \( \alpha \in S_n \), let \( \alpha = a_1 \ldots a_l \) be the product of disjoint cycles. If we fix integers \( k_i, 1 \leq i \leq n \), the polynomial part of \( FS_\alpha \) with respect to \( k_i \) is

\[
\left( \prod_{r \in A_{a_i}} (X^{k_{i(r)}})_{a_{i(r)} a_{i(r)}} \right).
\]

Now replacing \( a_{n+1} \) by \( a_1 \) and taking the sum over \( a_1, \ldots, a_n \), we have

\[
\sum_{a_1, \ldots, a_n \geq 1} \left( \prod_{r \in A_{a_i}} (X^{k_{i(r)}})_{a_{i(r)} a_{i(r)}} \right) = \prod_{i=1}^{l} \sum_{a_1, \ldots, a_n \geq 1} \left( \prod_{r \in A_{a_i}} (X^{k_{i(r)}})_{a_{i(r)} a_{i(r)}} \right) = \prod_{i=1}^{l} p_{\sum_{r \in A_{a_i}} k_{i(r)}}.
\]

Hence, the degree of the polynomial part of \( FS_\alpha \) is the number of disjoint cycles of \( \alpha \).

Let’s take \( \alpha = (21) \) as an example.

\[
FS_\alpha = \sum_{k_1, k_2 \geq 1} p_{k_1+k_2} \left( k_1 k_2 \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}} \right).
\]
Clearly, the degree of the polynomial part is one which is the number of disjoint cycles of $\alpha$.

4. Degree of Summation and Non-crossing Sequence

Consider the polynomial ring $\mathbb{C}[p_1, p_2, ...]$. Define the degree of each variable $p_i$ to be one. In the last section, we have shown that $W([n])$ can be written as the sum of $n!$ summations. Each summation is a "formal" differential operator. For example, the summation $FS_{(321)}$ in $W([3])$

$$\frac{1}{3} \sum_{i_1, i_2, i_3 \geq 1} (i_1 i_2 i_3 p_{i_1 + i_2 + i_3}) \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}}$$

is an infinite sum of differential operators $i_1 i_2 i_3 p_{i_1 + i_2 + i_3} \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}}$, which has coefficients $i_1 i_2 i_3$, polynomial part $p_{i_1 + i_2 + i_3}$ and differential part $\frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}}$. Now we want to define the summation’s degree, which depends on its polynomial part and differential part.

Definition 4.1. Given any summation $FS_{\alpha}$ of $W([n])$, $dP(FS_{\alpha})$ is the degree of its polynomial part and $dD(FS_{\alpha})$ is the order of its derivative part. The degree of the summation $FS_{\alpha}$ is $d(FS_{\alpha}) = dP(FS_{\alpha}) + dD(FS_{\alpha})$.

Let’s consider the example $W([3])$. There are 6 summations in $W([3])$,

$$W([3]) = \frac{1}{3} \sum_{i_1, i_2, i_3 \geq 1} (i_1 i_2 i_3 p_{i_1 + i_2 + i_3}) \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}} + \text{FS}_{(321)}$$

$$+ i_1 (i_2 + i_3) p_{i_1 + i_2 + i_3} \frac{\partial^2}{\partial p_{i_1} \partial p_{i_2 + i_3}} + \text{FS}_{(13)(2)}$$

$$+ i_2 (i_1 + i_3) p_{i_1 + i_2 + i_3} \frac{\partial^2}{\partial p_{i_1} \partial p_{i_1 + i_3}} + \text{FS}_{(12)(3)}$$

$$+ i_3 (i_1 + i_2) p_{i_1 + i_2 + i_3} \frac{\partial^2}{\partial p_{i_1} \partial p_{i_1 + i_2}} + \text{FS}_{(1)(23)}$$

$$+ (i_1 + i_2 + i_3) p_{i_1 + i_2 + i_3} \frac{\partial}{\partial p_{i_1 + i_2 + i_3}} + \text{FS}_{(1)(2)(3)}$$

$$+ (i_1 + i_2 + i_3) p_{i_1 + i_2 + i_3} \frac{\partial}{\partial p_{i_1 + i_2 + i_3}} + \text{FS}_{(123)}$$

Five of them have degree 4 and the summation $FS_{(123)}$ is of degree 2. If we go back to $W([2])$, all summations are of degree 3. Indeed, the polynomial degree of $FS_{\alpha}$ is the number of the disjoint cycles of $\alpha$, which can be obtained from Theorem 3.12

Question 4.2. Given a positive integer $n$, how many summations in $W([n])$ are of degree $n + 1$? More precisely, what is the number of permutations in $S_n$ such that the degree of the corresponding summation is $n + 1$?

In this section, we prove this problem is equivalent to a special "perfect paring" problem in combinatorics. In Section 5, we figure out this number.

The following lemma describes the relation between the degree of $FS_{\beta}$ and $FS_{\alpha}$, when $\beta = [\alpha, i]$ (see Definition 3.11).

Lemma 4.3. For any $\alpha \in S_n$,
Similarly, in Case 2 for \( \alpha \) in Construction 3.8 tells us that the number of disjoint cycles of \( \beta \) in Construction 3.8 corresponds to Case 1 of \( \alpha \). By the sketch of the proof of Theorem 3.12 and Remark 3.14, we know the number of disjoint cycles is the polynomial degree. Hence, in Case 0, we have
\[
dP(FS_\beta) = dP(FS_\alpha).
\]

**Case 1** in Construction 3.8 corresponds to \( [\beta] = [\alpha, j] \), where \( i \) is a vertex in the chain of \( \hat{Q}_\alpha \). \( \beta \) has one more disjoint cycle than \( \alpha \). So, we have
\[
dP(FS_\beta) = dP(FS_\alpha) + 1.
\]
Similarly, in **Case 2** in Construction 3.8 \( \alpha \) has one more disjoint cycle than \( \beta \). We have
\[
dP(FS_\beta) = dP(FS_\alpha) - 1.
\]

**Remark 4.4.** From the above lemma, the highest degree of summations in \( W([n]) \) is \( n + 1 \) and the other possible degrees are \( n - 1, n - 3, \ldots \).

**Definition 4.5.** Given \( \alpha \in S_n \), \( FS_\alpha \) is an ordinary summation (OS) of type \( (r, s) \), if \( dP(FS_\alpha) = r \), \( dD(FS_\alpha) = s \) and \( r + s = n + 1 \).

**Example 4.6.**
\[
FS_{(1)} = \sum_{k_1 \geq 1} p_{k_1} \frac{\partial}{\partial p_{k_1}}.
\]

\( FS_{(1)} \) is an OS of type \((1, 1)\).

\[
FS_{(1)(2)} = \frac{1}{2} \sum_{k_1, k_2 \geq 1} p_{k_1} p_{k_2} \frac{\partial}{\partial p_{k_1 + k_2}}.
\]

So, \( FS_{(1)(2)} \) is an OS of type \((2, 1)\).

Next we want to find a condition on permutations \( \alpha \in S_n \) such that if \( FS_\alpha \) is an ordinary summation, it satisfies the condition.

**Definition 4.7.** Let \( \alpha \) be a permutation in \( S_n \). Let \( \alpha = \alpha_1 \ldots \alpha_r \) be the decomposition of \( \alpha \) into disjoint cycles. We say \( \alpha \) satisfies the condition \((\ast_1)\), if for each arrow \( a \) in the chain of \( \hat{Q}_\alpha \), we have \( t(a) < s(a) \), and there is only one arrow \( b \) in each loop of \( \hat{Q}_\alpha \) such that \( s(b) < t(b) \).
Remark 4.8. The above condition for $\hat{Q}_\alpha$ is equivalent to the condition for $Q_\alpha$ that there is only one arrow $b$ in each loop of $Q_\alpha$ such that $s(b) < t(b)$. We use the definition in terms of $\hat{Q}_\alpha$ in the proof of Lemma 4.9, 4.12 and Theorem 4.13. We use the definition in terms of $Q_\alpha$ in the proof of Theorem 4.18.

Lemma 4.9. Given $\alpha \in S_n$, if $FS_\alpha$ is an OS, then $\alpha$ satisfies the condition (\(*_1\)).

Proof. We prove this lemma by induction on $n$. For the base step $n = 1$, $\hat{Q}_{(1)}$ is the only quiver and $FS_{(1)}$ is an OS. There is only one arrow $2 \to 1$ in the quiver $\hat{Q}_{(1)}$. Clearly, (1) satisfies the condition (\(*_1\)).

Next, we assume that for all $\alpha \in S_{n-1}$ if $FS_\alpha$ is an OS, then $\alpha$ satisfies (\(*_1\)). Let $\beta \in S_n$ and assume $[\beta] = [\alpha, j]$ in the notation of Definition 4.10. $FS_\beta$ is an OS implies that $FS_\alpha$ is also an OS. Indeed if $FS_\alpha$ is not an OS, then $d(FS_\alpha) < n$. By Lemma 4.9, $d(FS_\beta) < n + 1$, contradicting the fact that $\beta$ is an OS.

Let $\alpha = \alpha_1 \ldots \alpha_r$ be the decomposition of $\alpha$ into disjoint cycles with $1 \in \alpha_1$. By Lemma 4.3, $j$ could be zero or the target of some arrow in the chain of $\hat{Q}_\alpha$. Now we discuss these two cases.

If $j = 0$, then $\beta = \beta_1 \alpha_2 \ldots \alpha_r$, where $\hat{Q}_{\beta_1}$ is constructed from $\hat{Q}_{\alpha_1}$ by adding another arrow $k + 1 \to k$. By induction, the statement is true.

If $j \neq 0$, then $\beta$ is constructed from $\alpha$ by cutting the arrow $a : i \to j$, which is an arrow $a$ in the chain of $\hat{Q}_\alpha$. We use the same notation as Case 1 in Construction 3.8. Let $\beta = \beta_1 \beta_2 \alpha_2 \ldots \alpha_r$. The quiver $\hat{Q}_{\beta_1}$ of the cycle $\beta_1$ is $k + 2 \to j \to \ldots \to 1$, where $j \to \ldots \to 1$ is a subquiver of $\alpha_1$. Hence, all arrows in this chain satisfy that the source is larger than the target. The quiver $\hat{Q}_{\beta_2}$ is $i \to k + 1 \to \ldots \to i$, where $k + 1 \to \ldots \to i$ is a subquiver of $\alpha_1$ by construction. So the only arrow $a$ in the cycle $\hat{Q}_{\beta_2}$ satisfying $s(a) < t(a)$ is $i \to k + 1$. Hence, the statement is true for $n = k$.

The following condition is another condition of permutation $\alpha$ such that $FS_\alpha$ is an ordinary summation. In fact, Theorem 4.14 tells us that $FS_\alpha$ is an OS if and only if $\alpha$ satisfies the following condition and condition (\(*_1\)).

Definition 4.10. $\alpha$ is a permutation in $S_n$. Let $\alpha = \alpha_1 \ldots \alpha_r$ be the decomposition of $\alpha$ into disjoint cycles. We say $\alpha$ satisfies the condition (\(*_2\)), if any two distinct cycles $\alpha_i, \alpha_j$ satisfy at least one of the following conditions,

1. pick an arbitrary element $m$ in $\alpha_i$, then we have $m > n$ for any $n$ in $\alpha_j$ or $m < n$ for any $n$ in $\alpha_j$;
2. pick an arbitrary element $m$ in $\alpha_j$, then we have $m > n$ for any $n$ in $\alpha_i$ or $m < n$ for any $n$ in $\alpha_i$.

Remark 4.11. This remark will give a brief explanation about the condition (\(*_2\)). The two conditions in Definition 4.10 mean that any two cycles are "ordered" or one is "contained" in the other one. If the pair of cycles satisfies both these two conditions, then they are "ordered". If the pair only satisfies one of them, then one is contained in the other one.

For instance, consider the following examples,

$\tau_1 = (123)(45), \quad \tau_2 = (125)(34), \quad \tau_3 = (124)(35)$. 
The two disjoint cycles in $\tau_1$ satisfies both these two conditions. They are "ordered", since any integer in the second cycle is larger than any integer in the first cycle. The disjoint cycle $\alpha_1 = (34)$ in $\tau_2$ is contained in $\alpha_j = (125)$. They satisfy the second condition in Definition 4.10. We prefer to write it as

\begin{align*}
(5 (43) 21).
\end{align*}

We will describe this in Construction 4.13. The last example $\tau_3$ does not satisfy the condition ($*_2$).

**Lemma 4.12.** With the same notation as in Definition 4.10 if $FS_\alpha$ is an OS, then $\alpha$ satisfies the condition ($*_2$).

**Proof.** Similar to the proof of Lemma 4.9, we prove this lemma by induction on $n$. When $n = 1$, it is clear that the unique permutation (1) in $S_1$ satisfies the condition ($*_2$).

Next, we assume that for all $\alpha \in S_{n-1}$ if $FS_\alpha$ is an OS, then $\alpha$ satisfies ($*_2$). Let $\beta \in S_n$ and assume $[\beta] = [\alpha, j]$ in the notation of Definition 4.10. Let $\alpha = \alpha_1...\alpha_r$ be the decomposition of $\alpha$ into disjoint cycles. We will prove that if $FS_\beta$ is an OS, then $\beta$ satisfies the condition ($*_2$). Before we give the proof, recall the property that if $[\beta] = [\alpha, j]$ and $FS_\beta$ is an OS, then $FS_\alpha$ is also an OS by the proof of Lemma 4.9.

If $j = 0$, then $\beta = \beta_1\alpha_2...\alpha_r$, where $\hat{Q}_{\beta_1}$ is constructed from $\hat{Q}_{\alpha_1}$ by adding another arrow $k + 1 \rightarrow k$. In other words, we put another element $k$ into the cycle $\alpha_1$ (see Construction 3.8). By assumption that any two disjoint cycles of $\alpha \in S_{k-1}$ satisfy at least one of the conditions, we only have to check whether the pair $(\beta_1, \alpha_i)$ satisfies the condition ($*_2$), $2 \leq i \leq r$. Since $\alpha_1$ contains the smallest element 1, so if $\alpha_1$ and $\alpha_i$ are "ordered", then any element in $\alpha_1$ is smaller than any element in $\alpha_i$. Since $k$ is the largest element, so the statement is true for $\beta_1$ and $\alpha_i$. Now we consider that $\alpha_1$ and $\alpha_i$ are not "ordered". Since 1 is contained in $\alpha_1$, so $\alpha_i$ is "contained" in $\alpha_1$. Clearly, it still holds for $\beta_1$ and $\alpha_i$. So, $(\beta_1, \alpha_i)$ satisfies the condition ($*_2$).

Now let’s consider the case that $\beta$ is constructed from $\alpha$ by cutting the arrow $a : i \rightarrow j$ lying in the chain of $\alpha$. We use the same notation as Case 1 in Construction 3.8. Let $\beta = \beta_1\beta_2\alpha_2...\alpha_r$. So, we have to check whether the following three types of pairs satisfy the condition:

\begin{align*}
(\beta_1, \beta_2), \quad (\beta_1, \alpha_i), \quad (\beta_2, \alpha_i),
\end{align*}

where $2 \leq i \leq r$.

- $(\beta_1, \beta_2)$

Since $FS_\alpha$ is OS, so all arrows $a$ in $\hat{Q}_{\alpha_1}$ satisfy $t(a) < s(a)$ by Lemma 4.9. Hence, when cutting the arrow $i \rightarrow j$, any elements in $\beta_2$ is larger than any elements in $\beta_1$. It is true in this case.

- $(\beta_1, \alpha_i)$

By induction, we know that the lemma is true for $(\alpha_1, \alpha_i), 2 \leq i \leq r$. Since the elements of $\beta_1$ is a subset of the elements of $\alpha_1$, so it is true for $(\beta_1, \alpha_i), 2 \leq i \leq r$.

- $(\beta_2, \alpha_i)$
If $\beta_2$ is a single disjoint "one cycle" $\mathpzc{(k)}$, the statement is true. If $\beta_2 \neq \mathpzc{(k)}$, assume the largest element in $\beta_2$ except $k$ is $\phi$. If $\phi$ is smaller than the smallest element in $\alpha_i$, then any element $u$ except $k$ in $\beta_2$ $u$ is smaller than any element in $\alpha_i$. Also, $k$ is larger than any element in $\alpha_i$. Hence, the statement is true in this case. Now let’s consider the case that $\phi$ is larger than the smallest element in $\alpha_i$. By construction, $\phi$ is an element in $\alpha_i$, which contains 1. Hence, $\phi$ is larger than any elements in $\alpha_i$ by induction. Similarly, any other elements in $\beta_2$ is larger or smaller to all elements in $\alpha_i$ by induction. So, the statement is true.

In conclusion, the statement is true when $n = k$. \hfill $\Box$

**Definition 4.13.** Given $\alpha \in S_n$, we say that $\alpha$ satisfies the condition $(\ast)$ if $\alpha$ satisfies the conditions $(\ast_1)$ and $(\ast_2)$.

**Theorem 4.14.** For $\alpha \in S_n$, $FS\alpha$ is OS if and only if $\alpha$ satisfies the condition $(\ast)$.

**Proof.** The "only if" part is exactly Lemma 4.3 and 4.2. So, we only have to prove the "if" part. We will prove it by induction on $n$.

When $n = 1$, it is easy to prove, since (1) is the only permutation. We assume that if $\alpha \in S_{n-1}$ satisfies the condition $(\ast)$, then $FS\alpha$ is an OS. We will prove that if $\beta \in S_n$ satisfies the condition $(\ast)$, then $FS\beta$ is an OS. Assume $[\beta] = [\alpha, j]$ for some $\alpha$ in $S_{n-1}$ and some nonnegative integer $j$. We claim that $j = 0$ or in the chain of $\hat{Q}_\alpha$ ($\text{Claim 1}$). Also, we claim that $\alpha$ also satisfies the condition $(\ast)$ ($\text{Claim 2}$).

Since $\alpha$ satisfies the condition $(\ast)$, $FS\alpha$ is an OS by induction. By Claim 1, $j$ is 0 or in the chain of $\hat{Q}_\alpha$. By Construction 3.8 and Lemma 4.3, we know $FS\beta$ is an OS. Now we begin to prove these two claims.

Proof of Claim 1:

If not, $\beta$ is constructed from $\alpha$ by cutting arrow $a: i \rightarrow j$ which is not the chain of $\hat{Q}_\alpha$. Hence, by Case 2 in Construction 3.8 we will get a long chain

$$k + 1 \rightarrow j \rightarrow \ldots \rightarrow i \rightarrow k \rightarrow \ldots \rightarrow 1.$$ 

In this chain, we have $i < k$, which contradicts with our assumptions that $\beta$ satisfies the condition $(\ast)$. So, $j$ must be in the chain of $\hat{Q}_\alpha$ or $j = 0$.

Proof of Claim 2:

By Claim 1, we know that $j = 0$ or $j$ is in the chain of $\hat{Q}_\alpha$. If $j = 0$, it is easy to prove $\alpha$ satisfies the condition $(\ast)$. We leave it for the reader. Now we assume $j$ is in the chain of $\hat{Q}_\alpha$. With the same notation as in Construction 3.8 let $\beta = \beta_1\beta_2\alpha_2\ldots\alpha_r$ with $1 \in \beta_1$.

First, we have to check $\alpha$ satisfies the condition $(\ast_1)$. By the assumption of $\beta$, there is exactly one arrow $a$ in the quiver of $\alpha_i$ such that $t(a) > s(a)$, where $2 \leq i \leq r$. So, we have to show all arrows $a$ in the chain of $\hat{Q}_\alpha$ satisfying $t(a) < s(a)$. We assume that there is an arrow $a$ in the chain of $\hat{Q}_\alpha$ such that $s(a) < t(a)$. If $t(a) \neq j$, then this arrow will be in either $\beta_1$ or $\beta_2$, which contradicts with the assumption of $\beta$. If $t(a) = j$, then we get $\beta_1$

$$k + 1 \rightarrow j \rightarrow \ldots \rightarrow 1$$

and $\beta_2$

$$i \rightarrow k \rightarrow \ldots \rightarrow i$$
Since $k > j > i$, so $(\beta_1, \beta_2)$ does not satisfy the second condition in the condition (*). Hence, we have $t(a) < s(a)$ for each arrow $a$ in the chain of $\hat{Q}_\alpha$ and there is exactly one arrow $b$ in each loop of $\hat{Q}_\alpha$ such that $s(b) < t(b)$.

Now, we are going to prove that $\alpha$ satisfies the condition (\textsuperscript{2}). The problem pair is $(\alpha_1, \alpha_i)$, $2 \leq i \leq r$. By assumption, $\beta_1$ contains the smallest element 1 and $\beta_2$ contains the element $k$. Hence, by Construction\textsuperscript{4.8} and Lemma\textsuperscript{4.12}, we know that any element in $\beta_1$ is smaller than any element in $\beta_2$. Since $\beta$ satisfies condition (*), so for any cycle $\alpha_i$, $2 \leq i \leq r$, there are three possible cases

- $\alpha_i$ is "contained" in $\beta_1$, i.e. if we pick an arbitrary element $m$ in $\beta_1$, then we have $m > n$ for any $n$ in $\alpha_i$ or $m < n$ for any $n$ in $\alpha_i$;
- $\alpha_i$ is "contained" in $\beta_2$, i.e. if we pick an arbitrary element $m$ in $\beta_2$, then we have $m > n$ for any $n$ in $\alpha_i$ or $m < n$ for any $n$ in $\alpha_i$;
- $\alpha_i$ is between $\beta_1$ and $\beta_2$, i.e. any element in $\alpha_i$ is larger than any element in $\beta_1$ and smaller than any element in $\beta_2$.

In the first case, if $\alpha_i$ is "contained" in $\beta_1$, then any element in $\beta_2$ is larger than any element in $\alpha_i$, because the element in $\beta_2$ is always larger than the element in $\beta_1$. By the construction of $\alpha_1$, the condition is true for $(\alpha_1, \alpha_i)$. The same argument holds for the second case. For the third case, $\beta_1$ and $\beta_2$ are constructed from $\alpha_1$ by cutting the arrow with target $j$ and add another element $k$. Hence, $\alpha_i$ is "contained" in $\alpha_1$. Hence, $\alpha$ satisfies the condition (2) of (*).\hfill \Box

The condition (*) of the permutation corresponds to the non-crossing partition\textsuperscript{[14]} or "non-crossing permutation". In\textsuperscript{[10]}, Mingo and Nica define the non-crossing permutation. The "non-crossing permutation" in this paper is a little different from theirs but with similar idea. The following construction, "non-crossing sequence", and Theorem\textsuperscript{4.18} gives the idea of "non-crossing permutation" in this paper.

\textbf{Construction 4.15.} Given a positive integer $n$, we fix a standard sequence of $n$ integers as follows

\[ n \ n - 1 \ \ldots \ 2 \ 1. \]

We want to insert $r$ pairs of brackets into this sequence satisfying the following condition (**)\textsuperscript{[14]}

- any integer is contained in at least one pair of brackets and any pair of brackets contains at least one integer,
- there can be at most one left bracket and at most one right bracket between two successive integers.

We call the standard sequence with brackets satisfying (**) a non-crossing sequence.

Now we use some examples to explain these conditions.

\textbf{Example 4.16.} We consider the following three examples

\begin{itemize}
  \item (4) 3 2 (1),
  \item (4 3 (2)) (1),
  \item (4 (3) 2) (1).
\end{itemize}

The first one does not satisfy the first condition, since 2 and 3 are not contained in any pair of brackets. The second one does not satisfy the second condition, since there are two right brackets between 2 and 1. The third one satisfies (**).
By the second condition, we can only have at most one left (right) between two successive integers. So, we use the following notation

\[ \Box n \Delta \Box n - 1 \Delta \ldots \Box 1 \Delta, \]

where \( \Box \) is the place for left bracket and \( \Delta \) is for right bracket. Each \( \Box \) or \( \Delta \) contains at most one bracket.

Before we construct the relation between permutations and the non-crossing sequences, we want to give an order to the \( r \) pairs of brackets. We order the \( r \) right brackets as follows: the right most right bracket is \( )_1 \), the next right most right bracket is \( )_2 \), etc. The order of the left brackets is the same as the corresponding right brackets. For example, let’s consider the following non-crossing sequence with three pairs of brackets

\[ ( 4 ( 3 \ ) 2 \ ) ( 1 \ ). \]

We first order the right brackets

\[ ( 4 ( 3 \ )_3 2 \ )_2 ( 1 \ )_1. \]

The order of the left brackets are the same as its corresponding right brackets. We have

\[ ( 2 \ 4 ( 3 \ )_3 2 \ )_2 ( 1 \ 1 \ )_1. \]

Given two positive integers \( n, r \) such that \( n \geq r \), we define three sets \( Pmt(n, r) \), \( Brk(n, r) \) and \( OS(n, r) \).

**Definition 4.17.** \( Pmt(n, r) \) is the set of all permutations in \( S_n \) satisfying the condition \((*)\) (see Definition 4.13) with \( r \) disjoint cycles. \( Brk(n, r) \) is the set of all sequences with \( r \) pairs of brackets satisfying the condition \((**)\) as in Construction 4.15. \( OS(n, r) \) is the set of all ordinary summations of type \((r, n - r + 1)\).

We want to remind the reader that we always insert brackets into the following sequence

\[ n \ n - 1 \ldots \ 2 \ 1. \]

If \( S \) is a finite set, \( |S| \) is its cardinality.

**Theorem 4.18.** Given two positive integers \( n, r \) such that \( n \geq r \), there is a bijective map \( \phi_{n, r} \) between \( Brk(n, r) \) and \( Pmt(n, r) \).

**Proof.** We want to construct a map

\[ \phi_{n, r} : Brk(n, r) \to Pmt(n, r) \]

and show that this map is bijective.

First we will construct a permutation \( \alpha \in S_n \) with \( r \) cycles from a non-crossing sequence in \( Brk(n, r) \). Given a non-crossing sequence in \( Brk(n, r) \), we start with the \( r \)-th pair of brackets

\[ (r \ldots )_r. \]

By construction, the integers in this pair of brackets are not contained in any other pair of brackets, because \( r \) is the largest. Define \( \alpha_r \) as the cycle with integers from this pair of brackets. Then, we delete this pair of brackets and the enclosed integers. We choose the next pair of brackets

\[ (r-1 \ldots )_{r-1} \]
from the remaining sequence and uses it to define another cycle \( \alpha_{r-1} \). Repeating this process, we get a unique permutation \( \alpha \) in \( S_n \) with disjoint cycles \( \alpha_r, ..., \alpha_1 \). Now we have to prove that \( \alpha \) satisfies the condition \((*)\) so that the domain of \( \phi_{n,r} \) is in \( Pmt(n,r) \). In Construction 3.15, we first fix the base sequence

\[
n = n-1 \quad 2 \quad 1.
\]

So, the quiver of any cycle \( \alpha_r \) only contains one arrow \( a \) such that \( s(a) > t(a) \). Hence, \( \alpha \) satisfies the condition \((\ast_1)\). The condition \((\ast_2)\) comes from the property of non-crossing sequence. Consider the following example \((3 \quad 3 \quad 2 \quad 1) \quad (1 \quad 1)\).

There are only two relations between two pairs of brackets: "ordered" or "contained". \((3 \quad 3)\) is contained in \((2 \quad 2)\) and \((2 \quad 2)\), \((1 \quad 1)\) are ordered. This property is exactly the condition \((\ast_2)\). In this way, we see that the image of \( \phi_{n,r} \) is in \( Pmt(n,r) \). Clearly, it is injective.

Now we are going to prove the map \( \phi_{n,r} \) is surjective on \( Pmt(n,r) \). For the base case \( n = 1 \), the only permutation \((1) \in S_1\) corresponds uniquely to the following non-crossing sequence

\[
(1 \quad 1)\).
\]

We use induction on \( n \) and assume that \( \phi_{k-1,r} \) is surjective for any positive integer \( r, r \leq k-1 \). We will show that \( \phi_{k,r} \) is surjective for any \( r \). If \( \beta \in S_k \) satisfies the condition \((*)\), we know that \([\beta] = [\alpha,j]\) where \( j \) is zero or is contained in the chain of \( \hat{Q}_\beta \) and \( \alpha \) satisfies the condition \((*)\) by the proof of Theorem 4.14. By induction, \( \alpha \) corresponds to a unique sequence with brackets as following

\[
( m \quad k-1 \quad \ldots \quad v+1 \quad )_m ( 1 \quad v \quad \ldots \quad 1)\).
\]

where \( m \) is the order of the brackets, i.e. some positive integer smaller than \( r \). \( m \) can be 1. If \( m = 1 \), then \( v = k-1 \).

If \( j = 0 \), then \( \alpha \in Pmt(k-1,r) \). By Construction 3.5, we construct the sequence with brackets corresponds to \( \beta \) as

\[
( 1 \quad k \quad ( m \quad k-1 \quad \ldots \quad v+1 ) \quad )_m \quad \ldots \quad 1 )\).
\]

Here, we add another integer \( k \) into the sequence and move the bracket \((1\) to the left side of \( k \).

If \( j \neq 0 \), then \( \alpha \in Pmt(k-1,r-1) \). By Construction 3.5, we cut the arrow \( a : i \to j \) in the chain of \( \hat{Q}_\alpha \). First, let’s focus on the first pair of brackets \((1 \quad v \quad \ldots \quad 1)\) more precisely,

\[
( 1 \quad v \quad \ldots \quad i \quad ( m_0 \quad \ldots )_{m_0} \quad j \quad \ldots \quad 1 )\).
\]

where \( m_0 \) is the order of the pair of brackets (if exists). We construct the following sequence with bracket, which corresponds to \( \beta \),

\[
( k \quad ( i \quad k-1 \quad \ldots \quad v+1 )_i \quad v \quad \ldots \quad i \quad ) \quad ( m_0 \quad \ldots )_{m_0} \quad ( 1 \quad j \quad \ldots \quad 1 )\).
\]

This non-crossing sequence has one more pair of brackets (the unlabelled pair of brackets above) than that of \( \alpha \), because \( \beta \) has one more disjoint cycle than \( \alpha \) by Construction 3.8. In fact, this non-crossing sequence maps to \( \beta \) under the map \( \phi_{k,r} \). In conclusion, \( \phi_{k,r} \) is surjective.

Combining with the first part of the proof, \( \phi_{k,r} \) is bijective.

The following example will help the reader understanding the proof above.
Example 4.19. Consider the following non-crossing sequence

\[(4)(321)\].

By the construction of \(\phi_{4,2}\) in the proof of Theorem 4.18 it corresponds to the element \(\alpha = (4)(321)\) in \(S_4\) which satisfies the condition (\(*\)). \(\hat{Q}_\alpha\) is

\[5 \rightarrow 3 \rightarrow 2 \rightarrow 1, \quad 4 \rightarrow 4\]

Now, consider the quiver

\[6 \rightarrow 5 \rightarrow 3 \rightarrow 2 \rightarrow 1, \quad 4 \rightarrow 4\]

Clearly, it is \(\hat{Q}_{\beta_1}\), where \([\beta_1] = [\alpha, 0]\), i.e. \(\beta_1 = (4)(5321)\). The corresponding non-crossing sequence of \(\beta_1\) is

\[(5)(4321),\]

which is the case when \(j = 0\). Next, we consider another quiver

\[6 \rightarrow 3 \rightarrow 2 \rightarrow 1, \quad 5 \rightarrow 5, \quad 4 \rightarrow 4\]

which corresponds to \(\beta_2\), where \([\beta_2] = [\alpha, 3]\). By calculation, \(\beta_2 = (5)(4)(321)\). The corresponding non-crossing sequence is

\[(5)(4)(321),\]

which is the case \(j \neq 0\) we discuss above.

Now we want to give some definitions about pairs of brackets.

Definition 4.20. Given any non-crossing sequence in \(\text{Brk}(n, r)\), \((i \ldots )_i\) is of top-level if this pair of brackets is not contained in any other pair of brackets. \((i \ldots )_i\) is embedded if \((i \ldots )_i\) is not top-level. \((i \ldots )_i\) is of bottom-level if there is no embedded pair of brackets in it. Two pairs of brackets are adjacent if there are no positive integers between these two pairs of brackets.

Example 4.21. Let \(\alpha = (531)(2)(4)(6)\), then the corresponding non-crossing sequence is

\[(4 \quad 6 \quad )_4(1 \quad 5 \quad (3 \quad 4 \quad )_3 \quad 3 \quad (2 \quad 2 \quad )_2 \quad 1 \quad ).\]

\[(4 \quad 6 \quad )_4\] is isolated and of top-level, \((3 \quad 4 \quad )_3\) and \((2 \quad 2 \quad )_2\) are embedded and of bottom-level. \((3 \quad 4 \quad )_3\) and \((2 \quad 2 \quad )_2\) are not adjacent, because 3 is between them. Finally, \((1 \ldots )_1\) and \((4 \ldots )_4\) are adjacent.

5. Dual Non-crossing Sequence

In Construction 4.15 we construct the non-crossing sequence with \(r\) pairs of brackets. Now we want to construct the "dual sequence". If we insert \(r\) pairs of brackets into the following sequence satisfying condition (\(**\))

\[n \quad n - 1 \quad \ldots \quad 1,\]

then it represents a permutation \(\alpha \in S_n\) such that \(FS_\alpha\) is a \((r, n + 1 - r)\)-type OS by Theorem 4.14 and 4.18. We want to construct the "dual non-crossing sequence", which corresponds to a permutation \(\alpha'\) such that \(FS_{\alpha'}\) is a \((n + 1 - r, r)\)-type OS.
Construction 5.1. Given any non-crossing permutation $\alpha \in S_n$, the corresponding non-crossing sequence is as following

$$(i_1 \ldots i_s)_{i_1} \ldots (i_s)_{i_s},$$

where all pairs of brackets are separated. There may be some embedded brackets in them. We focus on the first separated pair of brackets and $$(i_1 \ldots i_s)_{i_1} \ldots (i_s \ldots i_s)_{i_s}.$$ For each integer $k$ in this sequence, there are at most four brackets "adjacent" to it,

$$k + 1 \triangle \square k \triangle \square k - 1,$$

the right bracket of $k + 1$, the left bracket of $k - 1$ and the two brackets of $k$. There are 16 possibilities in these four positions. The following construction discuss these possibilities.

| $\triangle \square k \triangle \square$ | $\triangle \square k \triangle \square$ |
|----------------------------------------|----------------------------------------|
| 1                                      | $k$                                    |
| 2                                      | $k$                                    |
| 3                                      | $(k)$                                  |
| 4                                      | $(k)$                                  |
| 5                                      | $)k$                                   |
| 6                                      | $(k)$                                  |
| 7                                      | $)k$                                   |
| 8                                      | $(k)$                                  |
| 9                                      | $)k$                                   |
| 10                                     | $(k)$                                  |
| 11                                     | $)k$                                   |
| 12                                     | $(k)$                                  |
| 13                                     | $(k)$                                  |
| 14                                     | $)k$                                   |
| 15                                     | $(k)$                                  |
| 16                                     | $)k$                                   |

The first column is all of the possible cases in the original non-crossing sequence, the second column is what we will get in the "dual" non-crossing sequence. From the construction we can get this construction are "dual" to each other,

$$1 \leftrightarrow 16, 2 \leftrightarrow 12, 3 \leftrightarrow 13, 4 \leftrightarrow 14, 5 \leftrightarrow 15,$$

$$6 \leftrightarrow 6, 7 \leftrightarrow 7, 8 \leftrightarrow 8, 9 \leftrightarrow 9, 10 \leftrightarrow 11.$$ It is easy to check that given any non-crossing sequence in $Brk(n,r)$, its dual non-crossing sequence is in $Brk(n,n-r+1)$.

Example 5.2. Here is an example of Construction 5.1.
Let \( \alpha = (721)(65)(4)(3) \in Pmt(7, 4) \). The corresponding sequence is
\[
(1 \quad 7 \quad (4 \quad 6 \quad 5)_{4} \quad (3 \quad 4)_{3} \quad (2 \quad 3)_{2} \quad 2 \quad 1)_{1}.
\]
We see that
- 7 is of type 8,
- 6 is of type 3,
- 5 is of type 11,
- 4 is of type 16,
- 3 is of type 12,
- 2 is of type 5,
- 1 is of type 2.

So, the dual sequence is
\[
(2 \quad 7 \quad (4 \quad 6)_{4} \quad (3 \quad 5 \quad 4 \quad 3)_{3} \quad 2)_{2} \quad (1 \quad 1)_{1},
\]
which corresponds to the permutation \((72)(6)(543)(1)\).

**Remark 5.3.** In this remark we consider another basic idea and explain why we construct the dual non-crossing sequence as in Construction 5.1. Let's consider the basic idea first and show it does not work. By the first condition of \((**)\), we always have a left bracket of \(n\) and a right bracket of 1. Roughly speaking, we insert another \(r - 1\) pairs of bracket into the following sequence
\[
(n \quad n-1 \quad \ldots \quad 1).
\]
Also, if we want to get the non-crossing sequence of \(\alpha'\), we should insert \(n - r\) pairs of brackets into that sequence. Clearly \((n - 1) - (r - 1) = n - r\), where \((n - 1)\) is the total number of positions for left (right) bracket, \((r - 1)\) is the number of positions occupied. The basic idea is that we can eliminate all of the brackets except the left one of \(n\) and the right one of 1 and insert brackets into all of the other empty positions. In this way, clearly we can get a sequence with \((n - r) + 1\) pairs of bracket. For example, \(\alpha = (54)(321)\). The corresponding non-crossing sequence is
\[
(2 \quad 5 \quad 4)_{2} \quad (1 \quad 3 \quad 2 \quad 1)_{1}.
\]
Then the dual non-crossing sequence is
\[
(4 \quad 5)_{4} \quad (3 \quad 4 \quad 3)_{3} \quad (2 \quad 2)_{2} \quad (1 \quad 1)_{1}.
\]
But, this method is not true for all of cases, which means that the dual non-crossing sequence may not satisfy the condition \((**)\). Let's consider the following example. Let \(\alpha = (5321)(4)\). The corresponding non-crossing sequence is
\[
(1 \quad 5 \quad (2 \quad 4)_{2} \quad 3 \quad 2 \quad 1)_{1}.
\]
By construction, the "dual non-crossing sequence" is
\[
(4 \quad 5)_{4} \quad 4 \quad (3 \quad 3)_{3}(2 \quad 4)_{2}(1 \quad 5)_{1}.
\]
But, this element does not satisfy the condition \((**)\), because 4 is not contained in any pair of brackets.
Construction 5.1 aims at solving this problem. If there are a sequence of adjacent brackets,

\((j_1 \ldots k \ldots j_1)(j_2 \ldots j_2 \ldots)(j_n \ldots j_n)\).

By 3,8,13, \((j_1\) is still exist in the dual sequence and by 2,9,12, the same for \(j_n\). At the same time, by 10,11,12,13,14,15,16, although \(j_1\) will be killed in the dual sequence, \(k\) is still contained in at least one bracket in the dual sequence by our construction. Hence, the dual non-crossing sequence is still a non-crossing sequence satisfying condition \((**)\).

Now, we are ready to prove the number of \((r,s)\)-type OS in \(\text{tr}(D^n)\) is the same as the number of \((s,r)\)-type OS in \(\text{tr}(D^n)\), where \(r + s - 1 = n\).

**Corollary 5.4.** Given two positive integers \(n, r\), we have

\(|\text{OS}(n, r)| = |\text{OS}(n, n - r + 1)|\).

**Proof.** By Theorem 4.14 we have

\(|\text{OS}(n, r)| = |\text{Pmt}(n, r)|, \quad |\text{OS}(n, n - r + 1)| = |\text{Pmt}(n, n - r + 1)|\).

By Proposition 4.18 we know

\(|\text{Pmt}(n, r)| = |\text{Brk}(n, r)|, \quad |\text{Pmt}(n, n - r + 1)| = |\text{Brk}(n, n - r + 1)|\).

By Construction 5.1 we have

\(|\text{Brk}(n, n - r + 1)| = |\text{Brk}(n, r)|\).

Hence,

\(|\text{OS}(n, r)| = |\text{OS}(n, n - r + 1)|\).

\(\square\)

### 6. \(|\text{Brk}(n, r)|\), Catalan Number and Narayana Number

In this section, we will calculate \(|\text{Brk}(n, r)|\), the number of noncrossing sequences with \(r\) pairs of brackets in a sequence of length \(n\), by using properties of the Catalan numbers and Narayana numbers. We first review some properties of the Catalan numbers and Narayana numbers [13]. The Catalan number \(C_n\) is

\[
C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 0.
\]

The generating function of Catalan numbers \(c(x)\) is

\[
c(x) = \sum_{n=0}^{\infty} C_n x^n,
\]

which satisfies the following equation

\[
c(x) = 1 + xc(x)^2.
\]

The Narayana number \(N(n, r)\) is

\[
N(n, r) = \begin{cases} \frac{1}{r+1} \binom{n+1}{r} \binom{n-1}{r-1}, & 0 \leq r \leq n \\ 0, & \text{otherwise}. \end{cases}
\]
The generating function of Narayana numbers is
\[ N(x, y) = \sum_{n, r \geq 0} N(n, r)x^n y^r. \]

The Narayana number \( N(n, r) \) satisfies the following condition
\[ \sum_{r=1}^{n} N(n, r) = C_n, \]
i.e.
\[ N(x, y) = 1 + N(x, y)^2 x. \]

We define a new set \( \overline{Brk}(n, r) \), which contains all sequences in \( Brk(n, r) \) with only one top-level pair of brackets. It means that any non-crossing sequence in \( Brk(n, r) \) can be written in the following form
\[ (1 \ldots )_1, \]
where \( (1 \ldots )_1 \) is the only top-level pair of brackets. Denote by \( \overline{a}_n^r \) the number of non-crossing sequences in \( \overline{Brk}(n, r) \). Also, we introduce the following notation
\[ a_n^r = \begin{cases} |Brk(n, r)|, & 1 \leq r \leq n \\ 0, & \text{otherwise} \end{cases}, \quad \overline{a}_n^r = \begin{cases} |\overline{Brk}(n, r)|, & 1 \leq r \leq n \\ 0, & \text{otherwise} \end{cases}. \]

**Theorem 6.1.** The number of \( (r, n-r+1) \)-type OS in : \( tr(D^n) : \) is the Narayana number:
\[ |OS(n, r)| = \frac{1}{n+1} \binom{n+1}{r} \binom{n-1}{r-1}. \]
The number of all summations with degree \( n+1 \) in : \( tr(D^n) : \) is the Catalan number
\[ \sum_{r \geq 1} \frac{1}{n+1} \binom{n+1}{r} \binom{n-1}{r-1} = \frac{1}{n+1} \binom{2n}{n}. \]

**Proof.** Any non-crossing sequence in \( Brk(n, r) \), it can be written as
\[ (i_1 \ldots )_{i_1} \ldots (i_s \ldots )_{i_s}, \]
where all pairs of brackets \( (i_j \ldots )_{i_j}, 1 \leq j \leq s \) are top-level pairs of brackets. By construction, any integer \( k, 1 \leq k \leq n \), is contained in a unique top-level pair of brackets. \( (i_j \ldots )_{i_j} \) can be considered as a non-crossing sequence with a unique top-level pair of brackets. Let \( n_j \) be the number of integers in \( (i_j \ldots )_{i_j} \) and let \( t_j \) be the number of pairs of brackets. Hence, we have
\[ a_n^r = \sum_{s=1}^{r} \sum_{n_1 + \ldots + n_s = n} \overline{a}_{n_1}^{t_1} \ldots \overline{a}_{n_s}^{t_s}. \]

By the following lemma, we know \( \overline{a}_{n+1}^r = a_n^r \). So, we have
\[ a_n^r = \sum_{s=1}^{r} \sum_{n_1 + \ldots + n_s = n} a_{n_1-1}^{t_1} \ldots a_{n_s-1}^{t_s}. \]
Now we consider the generating function
\[ G(x, y) = \sum_{n, r \geq 0} a_r^n x^n y^r. \]

By (5), we have
\[ \sum_{s=1}^{\infty} G(x, y)^s x^s = G(x, y), \]
which is the generating function of Narayana numbers. If we set \( y = 1 \), we have
\[ G(x, 1) = 1 + G(x, 1)^2 x, \]
which is the generating function for Catalan number \( C_n \).

By the property of Catalan number and Narayana number we state at the beginning of this section, we have
\[ \sum_{r=1}^{n} a_r^n = \frac{1}{n+1} \binom{2n}{n}, \]
\[ a_r^n = \frac{1}{n+1} \binom{n+1}{n+1-r} \binom{n-1}{r-1}. \]

We finish the proof of Theorem 6.1 by showing the following lemma.

**Lemma 6.2.** Given any positive integers \( n, r, n \geq r \geq 1 \), we have \( \overline{a}_{n+1}^r = a_r^n \).

**Proof.** We are going to construct a bijection between \( \overline{Brk}(n+1, r) \) and \( Brk(n, r) \). Take an element in \( \overline{Brk}(n+1, r) \). It has only one top-level pair of brackets. So, the integer \( n+1 \) does not have right bracket and 1 does not have left bracket. We assume the sequence in \( \overline{Brk}(n+1, r) \) as following
\[ (1 \ n+1 \ (j_1 \ n \ ... \ w)_{j_1} \ ... \ (j_k \ n \ ... \ w)_{j_k} \ v \ ... \ 1 \ 1), \]
where the integer \( v \) is the largest integer smaller than \( n+1 \) in the top-level pair of brackets, not contained in any embedded brackets. We construct the sequence in \( Brk(n, r) \) as following
\[ (j_1 \ ... \ j_1)_{j_1} \ ... \ (j_k \ n \ ... \ w)_{j_k} \ (1 \ v \ ... \ 1 \ 1). \]
Indeed, we get rid of the integer \( n+1 \) and move the bracket \( 1 \) to the left side of \( v \). This gives a well defined element in \( Brk(n, r) \).

Now let’s consider how to construct elements in \( \overline{Brk}(n+1, r) \) from elements in \( Brk(n, r) \). In the proof of Theorem 6.1, we already gave the construction. Given an element in \( Brk(n, r) \), we assume it in the following form
\[ (j_1 \ n \ ... \ w)_{j_1} \ ... \ (1 \ v \ ... \ 1 \ 1), \]
where \( (j_1 \ n \ ... \ w)_{j_1} \) is the leftmost top-leveled pair of brackets. Now we give the construction as following
\[ (1 \ n+1 \ (j_1 \ n \ ... \ w)_{j_1} \ v \ ... \ 1 \ 1). \]
Indeed, if we consider the element in $Brk(n, r)$ corresponds to the permutation $\alpha \in S_n$, then the sequence we construct corresponds to the permutation $\beta \in S_{n+1}$, where $\beta = [\alpha, 0]$. It is easy to check that the above construction gives a one-to-one correspondence between $Brk(n, r)$ and $\tilde{Brk}(n + 1, r)$. Hence, $\tilde{a}_{n+1}^r = a_n^r$. □

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