Field Theory reformulated without self-energy parts.
Divergence-free classical electrodynamics

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Abstract
A manifestly gauge-invariant hamiltonian formulation of classical electrodynamics has been shown to be relativistic invariant by the construction of the adequate generators of the Poincare Lie algebra [1]. The original formulation in terms of reduced distribution functions for the particles and the fields is applied here to the case of two charges interacting through a classical electrodynamical field. On the other hand, we have been able in previous work to introduce irreversibility at the fundamental level of description [2] by reformulating field theory without self-energy parts by integrating all processes associated with self-energy in a kinetic operator, while keeping the equivalence with the original description. In this paper, the two approaches are combined to provide a formalism that enables the use of methods of statistical physics [3] to tackle the problem of the divergence of the self-mass. Our approach leads to expressions that are finite even for point-like charged particles: the limit of an infinite cutoff can be taken in an harmless way on self consistent equations. In order to check our theory, we recover the power dissipated by radiation in geometries where the usual mass divergence does not play a role.
1 Introduction

The derivation of an equation of motion of an electron that includes its reaction to the self-field has been initiated by Abraham and Lorentz hundred years ago and is still a controversial matter \cite{4}, \cite{5}. The main problem is the presence of divergences associated with point-like charged particles. A way of removing them has to be devised without entering in trouble with the special theory of relativity (see Ref. \cite{6} for a recent review and a relevant bibliography). The derivation of the self-force based on energy conservation \cite{7} avoids that problem: the power emitted in the radiating field is due to the work of the radiative reaction force.

In the usual derivation, a given motion is prescribed to the charge and the potentials of Liénard-Wiechert associated with it are computed. The self-fields are then derived and their expression used to get the reaction on the motion of the charged particle, leading to an (infinite) term interpreted as a self-mass. But an infinite self-mass prevents the acceleration, hence the paradox. An approach that uses a finite expansion in the charge does not allow to correct the situation. In a theory without divergence, a natural cut-off for the wave numbers, proportionnal to the inverse of the classical radius of the electron, should appear but that property cannot be obtained in a simple expansion in the charge since the cut-off value should be proportionnal to the square of the inverse of the charge. Moreover, the use of an effective frequency dependent mass $M(\omega)$ \cite{7} leads to an equation that is not free of runaway solutions.

In different contexts, methods of statistical mechanics \cite{3} enable, through resummations of formally divergent class of diagrams, to get relevant finite physical results. Here we present a formalism adapting those methods for problems in classical electrodynamics.

A manifestly gauge-invariant hamiltonian formulation has been developed for systems composed by point particles and fields, the state of which is now described by reduced distribution functions \cite{1}. The dynamical variables are the positions and mechanical momentums for the charged point particles and the transverse components of the electric and magnetic fields. Potentials are absent in that formulation. The Coulomb interaction takes into account the longitudinal part of the electric field. A generalized Liouville equation for the reduced distribution functions is derived. It provides a statistical description that takes into account the Lorentz force between the particles and the Maxwell equations for the fields. The formalism formally ressembles a statistical description of charged particles in Coulomb
gauge but with a different interpretation and the guarantee of satisfying the principle of special relativity. The relativistic invariance is proved by the explicit construction of the generators of the Poincaré Lie algebra.

While Balescu and Poulain have developed their formalism for an arbitrary number of particles, described by reduced distribution functions, we can apply it as such in the simplest case of two charged point particles. They thus interact through the Coulomb interaction and the classical transverse electrodynamical field (electric and magnetic). An alternative possibility is to consider a single charge in interaction with a Coulomb potential (due for instance to an infinitely massive particle) at the origin of the coordinates but the translation invariance is then immediately broken. At the final stage, for the sake of interpretation, we will consider the limit of our expressions when one of the two particles becomes very massive. Working with two particles avoids the consideration of an external force to accelerate the particles: the relativistic and gauge invariance is therefore preserved. A prescribed motion for the charged particle is therefore avoided. The consideration of an incident transverse field is also relevant to the problem but is not treated here.

The Balescu-Poulain formulation seems therefore an adequate starting point to deal with classical electrodynamics thanks to its intrinsic properties: namely relativistic invariance, explicit gauge invariance. The formulation is statistical: the particles and the modes of the field are described by distribution functions. The distribution functions associated with the particles can be spatially extended. A particle does not interact with the electric longitudinal field its generates: the Coulomb interaction is considered only between different particles. The present paper starts with the results of the last section of the paper of R. Balescu and M. Poulain.

A theory of subdynamics has been introduced thirty years ago by the Brussels group (see e.g. [8], [3]) for a dynamics provided by the Liouville-von Neuman equation. A setback of that approach is a limitation on the class of possible initial conditions since they have to belong to the subdynamics. To avoid the trap, we have introduced the so-called single subdynamics approach based on the existence of self-energy contributions to the dynamics. In that way, we obtain a reformulation of field theory that excludes self-energy contributions in the dynamics. However, being able to accommodate also initial conditions outside the scope of the original dynamics: our dynamics is larger that the initial formulation. Since the formal properties of the subdynamics do not depend on a particular realisation of the operators, we have picked up all the formal properties without a need to
The adequate way of dealing with the self-field is provided automatically by the single subdynamics approach. The dynamics is first extended to be able to distinguish the self-field contributions from the other. A subdynamics, inspired by the formalism developed at Brussels \cite{3} enables the obtention of dynamical equations of motion in which the self-field does no longer appear. It has been proven \cite{13} that the description exactly contains the original description and that the effects of the self-field are now taken into account in the new generator of motion. The relevant subdynamics incorporates all the features of usual CED. That description therefore includes not only the original dynamics but could also include a more general class of initial conditions, enabling the inclusion of irreversibility at a fundamental level.

Here, we deliberately restrict ourselves to the derivation of the closed irreversible evolution equations for the interacting charged particles, in the absence of incident field from an outside source, and to the obtention of the emitted fields (velocity and acceleration fields) at the lowest non-vanishing order.

Our paper is structured as follows. First, we present the Balescu-Poulain’s formalism and derive the evolution equations for all reduced distribution functions defining the state of the system. The basic idea for constructing the single subdynamics in CED is the use of a distinction between real and virtual fields (the virtual field forms the self-field). We propose an extension of the dynamics suitable for our purpose and the constitutive relations that connects the original and extended dynamics are displayed. The elements of the extended dynamics bear a tilde accent.

The kinetic operator $\tilde{\Theta}$, considered in section 3, describes the closed evolution of the distribution functions that do not involve the self-field. Their elements are evaluated from the corresponding vacuum-vacuum elements of the subdynamics operator $\tilde{\Sigma}(t)$. The first non-vanishing contribution appears at the second order in the interaction with the transverse fields, without considering, in the first step, the influence of the Coulomb interaction between the charged particles. The various steps of the derivation are illustrated and the final expression for $\tilde{\Theta}$ is given in Appendix B. All the elements are known to examine the putative second order mass correction
for the charged particles, that is found to vanish.

A non-vanishing contributions to the kinetic operator, reflecting the presence of the effect of the transverse self-field, requires to consider either a non-vanishing incident transverse field, either a Coulomb interaction between the charged particles or either the mutual influence of the transverse emitted field: the particle has to be accelerated to receive a radiation reaction force.

To get a better insight of the previous result, we take another road in §4. The kinetic operator can indeed be also evaluated from the knowledge of the self-field determined by the so-called creation operator. The value of the self-field at the location of the particle induces its self-interaction. Since the equivalence conditions require the equality of the emitted and self-field, the creation operator provides us moreover with the expression of the emitted field. Correlation-vacuum elements of the resolvent are considered for evaluating the elements of the subdynamics. A simple computation enables to get explicitly the expression of the common value of the Fourier transform of the emitted and self-field.

To obtain a source of acceleration and to prepare an easy comparison with the usual approaches, the first order effect of the Coulomb acceleration will also be computed in §5 and §6 from two different ways: the direct consideration of the kinetic operator and the recourse to the creation operator for the self-field.

The direct computation of the kinetic operator is performed in the next section §5 from the vacuum-vacuum elements of the resolvent acting in absence of field (field vacuum). All relative orders of the vertices have to be considered: the Coulomb interaction can \textit{a priori} take place before, after or between the two interactions with the transverse field. Only the last two circumstances lead to a non-vanishing contribution. Indeed, when the Coulomb interaction takes place after the two interactions with the transverse field, we receive as factor, as expected, the previous vanishing second order contribution to the kinetic operator. The computation, although lengthy, is straightforward.

For a consistency check, in §6 we consider the creation operator at first order in field-particle interaction and first order in the Coulomb interaction. This enables to get the effect of the acceleration, due to the Coulomb interaction, to the self-field, hence to the retroaction of the emitted field on the accelerated particle. From the equivalence conditions, we deduce for all points the field emitted during the acceleration of the particle. If we use that expression in the kinetic equation, we recover the previous result. From
its expression at the localisation of the particle, the power emitted can be computed.

Our expressions are analysed in §7. We consider a situation in which the distribution functions of the charged particles are infinitely sharp in configuration and momentum space, with an absence of free field. One particle is then considered as infinitely heavy and we use the referential in which the heavy particle is at rest. In the geometry where the position and velocity vectors are orthogonal, the power dissipated by the field due to the motion of the light particle can be computed exactly: all integrals can be performed. In other geometries, we do not avoid the usual divergence. This is natural since our approach contains the usual formalism and no resummation has been performed yet. The usual result is explicitly recovered as a particular case in small velocities circumstances. Indeed, under the equivalence conditions, both theories provide the same equations for the motion of the charged particles.

The finiteness of the theory through resummations is considered in §8. We start from the divergent contribution for the self-field at the lowest order in the charge. We regulate it by introducing a cut-off function, with a cut-off value $K_c$ to smoothen the contributions from very high wave numbers. That contribution is the first term of a series that can be formally resummed, in a self consistent way. We then show that the limit $K_c \to \infty$ can be performed in an harmless way. The effective cut-off resulting from the solutions of the non-linear equations is naturally linked to the classical radius of the electron. Our final expressions do not admit a simple expansion in the charge, since the dependence of the effective cut-off in the charge $e$ is in $\frac{1}{e^2}$. This result is obtained through an analysis that uses several steps.

Some conclusions and perspectives are considered in the last section §9.

2 Electrodynamics in the manifestly gauge invariant Balescu-Poulain formalism

In this section, we define the model for the description of the two charges in interaction with the electromagnetic field. We use the approach by R. Balescu and M. Poulain [1]. The only difference is that for the description of matter, we do not deal with a reduced formalism but keep the two-particle distribution function. Although the transposition is straightforward, we will present it in the following subsection, using their notations (and expressions whenever possible).
2.1 The Balescu-Poulain formalism

The state of the system is described by a distribution vector $\mathcal{F}$, i.e. by a collection of functions describing two different particles and the reduced distribution of $m$ field oscillators, describing the transverse field components which are the only ones that appear explicitly:

$$\mathcal{F} = \{ f_{11[m]}(x^{(1)}, x^{(2)}; \chi^{[1]}, \ldots, \chi^{[m]}; k^{[1]}, \ldots, k^{[m]}) \} ; m = 0, 1, 2, \ldots$$

(2.1)

For $m = 0$, the system does not contain fields variables. An obvious convention in the notation is implicit for $m = 0$. Here $x^{(j)}$ denotes the coordinates $(q^{(j)}, p^{(j)})$ of particle $j$, and $\chi^{[j]}$ denotes the variables describing a given field oscillator associated with the wavevector $k^{[j]}$: $(\eta^{[j]}, \xi^{[j]}, \alpha = 1, 2, \ldots)$ that are the action $(\eta^{[j]})$ and angle variables $(\xi^{[j]})$ associated with the oscillator characterized by the wave number $k^{[j]}$.

If two mutually orthogonal unit vectors, or "polarization vectors", $e^\alpha(k)$ associated with a given wavevector $k$ are introduced such that, together with the unit vector $\vec{k}$, they form a right-handed cartesian frame, the electromagnetic fields are expressed as follows in these variables.

$$E^\perp(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\alpha=1,2} \sum_{a=\pm1} \int d^3k \ k^{\frac{5}{2}} e^\alpha(k) \eta^{\frac{1}{2}}(k) \exp\{ia[k.x - 2\pi\xi(\alpha)(k)]\},$$

(2.2)

$$B(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\alpha=1,2} \sum_{a=\pm1} \int d^3k \ k^{\frac{5}{2}} (-1)^{\alpha'} e^{\alpha'}(k) \eta^{\frac{1}{2}}(k) \exp\{ia[k.x - 2\pi\xi(\alpha)(k)]\},$$

(2.3)

where $\alpha' = 2$ for $\alpha = 1$ and $\alpha' = 1$ for $\alpha = 2$.

The dynamical functions of the system are described by a set $\mathcal{B}$:

$$\mathcal{B} = \{ b_{11[m]}(x^{(1)}, x^{(2)}; \chi^{[1]}, \ldots, \chi^{[m]}; k^{[1]}, \ldots, k^{[m]}) \} ; m = 0, 1, 2, \ldots$$

(2.4)

The average value of an element $b_{11[m]}$ of $\mathcal{B}$ is calculated by the following formula:

$$< b_{11[m]} > = \int d^3k^{[1]} \ldots d^3k^{[m]} \int d^4\chi^{[1]} \ldots d^4\chi^{[m]} \int d^6x^{(1)} d^6x^{(2)}$$

\footnote{In contrast with \cite{13}, the reduction is not performed up to the level of each polarized mode, in the same way that reduced distribution functions for the particles are not considered to only one component of the velocity. This procedure ensures more easily the rotational invariance of the treatment.}
It is well known that to each generator $G$ of the Poincaré Lie algebra corresponds an infinite hierarchy of equations describing the transformation properties of the reduced distribution functions \[\{\chi\}\]. These equations can be written compactly as

$$ \partial_g F = \mathcal{L}_G F, $$

where $\mathcal{L}_G$ is a matrix operator. The components of this equation are written as

$$ \partial_g f_{11}[m] = \sum_{m'=0}^{\infty} <11[m]|\mathcal{L}_G|11[m']> f_{11}[m']. $$

The matrix elements entering these equations are obtained as in [17] and listed below, considering separately the three contributions corresponding to the splitting of the liouvillians in three terms, describing respectively free particles $L_G^{0P}$, free field $L_G^{0F}$ and interactions $L_G'$.

For the free particles, we have:

$$ <11[m]|L_G^{0P}|11[m']> = \delta_{mm'} \left( L_G^{0(1)} + L_G^{0(2)} \right). $$

In this work, we consider only the generator corresponding to the time translation ($g = t, G = H$) and get, using Einstein convention for the summation:

$$ L_H^{0(j)} = -v_r^{(j)} \frac{\partial}{\partial q_r^{(j)}}, $$

where the velocity $v_r^{(j)}$ is connected with the mechanical momentum $p_r^{(j)}$ in the usual way (in the units chosen, $c = 1$ and the div (divergence) of the electric field vector is $4\pi$ the charge density):

$$ v_r^{(j)} = \frac{p_r^{(j)}}{(m_j^2 + p_s^{(j)} p_s^{(j)})^{\frac{1}{2}}}. $$

For the free field, we have:

$$ <11[m]|L_G^{0F}|11[m']> = \delta_{mm'} \sum_{i=1}^{m} L_G^{0[i]}, $$
\[ L^0_{H} = -\frac{1}{2\pi} k^{[i]} \sum_{\alpha=1}^{2} \frac{\partial}{\partial \xi^{[i]}_{\alpha}}. \] (2.12)

For the interaction,
\[ < 11[m]L_{G} | 11[m'] > = \delta_{m'm} \sum_{i=1}^{m} \left( L^1_{G} + L^2_{G} \right) + \delta_{m'm} L^{(12)}_{G} \]
\[ + \delta_{m'm+1} \int d^3 k^{[m+1]} \int d\gamma^{[m+1]} \left( L^1_{G} + L^2_{G} \right), \] (2.13)

where \( \int d\gamma^{[m+1]} \) stands for
\[ \int d\gamma^{[m+1]} = \int_{0}^{\infty} d\eta^{[m+1]} \int_{0}^{\infty} d\eta^{[m+1]} \int_{0}^{1} d\xi^{[m+1]} \int_{0}^{1} d\xi^{[m+1]} \ldots \] (2.14)

The prime on the \( k \) integral means that the values \( k^{[m+1]} = k^{[1]}, \ldots, k^{[m]} \) must be excluded through a principal-part procedure. We have for the interaction of particle \( j \), bearing the charge \( e_j \), with the \( i \) labeled mode:
\[ L^{(j)}_{H} = -e_j \frac{1}{(2\pi)^2} \sum_{\alpha=1,2} \sum_{a=\pm1} \left( \frac{\eta^{[i]}_{\alpha}}{k^{[i]}_{\alpha}} \right)^{\frac{1}{2}} \exp\{ia[k^{[i]}_c q^{(j)}] - 2\pi \xi^{[i]}_{\alpha}]\}
\times \left[ k^{[i]}_c (\alpha^{(i)}_{\alpha}] - g^{st} v^{(j)} (\epsilon^{(i)}_{\alpha} k^{[i]}_{\alpha} - e^{(i)}_{\alpha} k^{[i]}_{\alpha})] \frac{\partial}{\partial p^{(j)}} \right.
\[ - (v^{(j)} \epsilon^{(i)}_{\alpha}] \left( 2\pi \frac{\partial}{\partial \eta^{[i]}_{\alpha}} - \frac{ia}{2\eta^{[i]}_{\alpha}} \frac{\partial}{\partial \xi^{[i]}_{\alpha}} \right) \right]. \] (2.15)

The elements of the metric tensor \( g \) have been chosen as \( g_{rs} = g^{sr} = -\delta_{rs}, \)
\( i, r = 1 \rightarrow 3 \). The last matrix element of interest for us describes the Coulomb interaction between the two charged particles:
\[ L^{(12)}_{H} = e_1 e_2 \left( \frac{\partial q^{(1)} - q^{(2)}}{\partial q^{(2)}} \right) \cdot \left( \frac{\partial}{\partial p^{(1)}} - \frac{\partial}{\partial p^{(2)}} \right). \] (2.16)

2.2 Enlargement of dynamics

We proceed now to an enlargement of dynamics as in previous publications [9], [13]: we multiply the number of variables on physical ground in such a way that the original dynamics (2.7) be included as a particular case. The choice of a particular enlargement is determined by opportunity linked to
physical considerations and the properties to be examined \[18\]. Since all
enlargements provide an equivalent alternative description, that degree of
freedom is welcome. In the present paper, focused on the self-force on each
c particle, our choice is to define the self-field with respect to each particle. If
the interest bears on the field far from the two particles, defining the self-
field with respect to both charged particles would be an alternative useful
option.

The elements of the enlarged dynamics will be noted by a supplementary
upper index tilde “˜”, as well for the variables as for the evolution operator.
Our aim is indeed to eliminate explicit self-interaction processes from the
evolution, while taking their effect into account. We distinguish formally
between 5 varieties of oscillators, based on the recognition of self-energy
parts in the evolution. To each oscillator \([i]\), we associate a discrete index
that determines which interactions are possible for the oscillator (the index
\(j\) takes the two values 1 and 2).

\([i(s_j)]\) will be the label of an oscillator which has previously interacted with
the particle \(j\) and will further interact with it in a future, without interaction
with the other particle \((j' \neq j)\), and without playing a role in a measurement:
by definition, such oscillator does not play a role in the computation of the
mean values.

\([i(e_j)]\) will be the label of an oscillator which has previously interacted with
the particle \((j)\) and will no longer interact with it directly: its next inter-
action should involve the other particle \((j')\), or it should contribute in the
computation of mean values.

\([i(f)]\) will be the label of an oscillator mode which has not previously in-
teracted with the particles (1) or (2). Its excitation has its origin outside
the two charges and such an oscillator is free of constraints on its interac-
tions: either with one of the particle or with an external devise. It provide
a contribution in the computation of mean values.

The free evolution of those oscillators is the same as in the original
dynamics and does not involve a change in their nature.

The vertices for the computation of \(<11|m||\tilde{L}_H'11|m\>\) involve \(L_H'[i]\) for
all \(i : 1 \rightarrow m\). the numerical value will be preserved for the non-vanishing
elements. We have to take into account the (possible) change of nature
of the oscillator after the interaction. We introduce indices corresponding
to the transition of nature of the field \((i(e_1)\) means that a free oscilla-
tor \(i(f)\) becomes of the emitted \(e_1\) variety) and we have the non-vanishing
possibilities: \(\tilde{L}_H'[i(s_1f)], \tilde{L}_H'[i(s_2f)], \tilde{L}_H'[i(e_1f)], \tilde{L}_H'[i(e_2f)], \tilde{L}_H'[i(e_1e_2)],\)
vanish by construction. The other oscillators (1 oscillator only, from the first equation of the hierarchy: the extended dynamics to the original one. The simplest case involves one

Other elements, such as $\tilde{L}_{H}^{'1}[i(f_{s})]$, $\tilde{L}_{H}^{'2}[i(f_{s})]$, $\tilde{L}_{H}^{'1}[i(f_{e})]$, $\tilde{L}_{H}^{'2}[i(f_{e})]$, $\tilde{L}_{H}^{'1}[i(s_{2}e_{1})]$, $\tilde{L}_{H}^{'2}[i(s_{2}e_{1})]$, $\tilde{L}_{H}^{'1}[i(s_{2}e_{2})]$, $\tilde{L}_{H}^{'2}[i(s_{2}e_{2})]$, $\tilde{L}_{H}^{'1}[i(s_{1}e_{2})]$, $\tilde{L}_{H}^{'2}[i(s_{1}e_{2})]$, $\tilde{L}_{H}^{'1}[i(s_{2}e_{2})]$, $\tilde{L}_{H}^{'2}[i(s_{2}e_{2})]$ vanish obviously since the final label of the oscillator does not bear the name of the interacting particle.

The vertices for the computation of $<11[m]|\tilde{L}_{H}^{'j}[11[m']]>(m' \neq m)$ involve a $(m + 1)^{th}$ oscillator mode and its dispersion from the explicit description. The value of the vertices involved, corresponding to $\tilde{L}_{H}^{'j}[m+1]$ is the same as the value of $\tilde{L}_{H}^{'j}[m+1]$: we have to consider the non-vanishing possibilities for the nature of the $(m + 1)^{th}$ oscillator. The oscillator on which the integration is performed is considered belonging to the self variety and

The other oscillators (1 → m) are unchanged by the transition vertex.

2.3 Constitutive relations-Equivalence conditions

Matrix elements of the evolution operator for an enlarged dynamics involve now the five varieties of oscillators. We have to connect the elements of the extended dynamics to the original one. The simplest case involves one oscillator only, from the first equation of the hierarchy:

$$\partial_{t}\tilde{f}_{11[m]} = \sum_{m'=0}^{\infty} <11[m]|\tilde{L}_{H}|11[m']>\tilde{f}_{11[m']}. \tag{2.17}$$

For $m = 0$, we take obviously $f_{11[0]} = \tilde{f}_{11[0]}$. That first equation means:

$$\partial_{t}\tilde{f}_{11[0]} = <11[0]|\tilde{L}_{H}|11[0]>\tilde{f}_{11[0]} + <11[0]|\tilde{L}_{H}|11[1(f)]>\tilde{f}_{11[1(f)]} + <11[0]|\tilde{L}_{H}|11[1(s_{1})]>\tilde{f}_{11[1(s_{1})]} + <11[0]|\tilde{L}_{H}|11[1(s_{2})]>\tilde{f}_{11[1(s_{2})]} + <11[0]|\tilde{L}_{H}|11[1(e_{1})]>\tilde{f}_{11[1(e_{1})]} + <11[0]|\tilde{L}_{H}|11[1(e_{2})]>\tilde{f}_{11[1(e_{2})]} \tag{2.18}$$
\( \mathcal{L}_H \) is composed of the parts part \( \mathcal{L}_H^1 \) and \( \mathcal{L}_H^2 \) according to the particles involved in the interaction. \( \mathcal{L}_H^1 \) acts on \( \tilde{f}_{11[1](f)} \), \( \tilde{f}_{11[1](s_1)} \), \( \tilde{f}_{11[1](e_2)} \) while \( \mathcal{L}_H^2 \) acts on \( \tilde{f}_{11[1](f)}, \tilde{f}_{11[1](s_2)}, \tilde{f}_{11[1](e_1)} \). Since we have to recover the equation

\[
\partial_t f_{11[0]} = <11[0]|\mathcal{L}_H|11[m']> f_{11[0]} + <11[0]|\mathcal{L}_H|11[m']> f_{11[1]}, \quad (2.19)
\]

we are led to the constitutive relation \[9], \[13]\]

\[
f_{11[1]} = \tilde{f}_{11[1](f)} + \tilde{f}_{11[1](e_1)} + \tilde{f}_{11[1](e_2)}. \quad (2.20)
\]

Indeed, if the conditions \( \tilde{f}_{11[1](s_1)} = \tilde{f}_{11[1](e_1)} \) and \( \tilde{f}_{11[1](s_2)} = \tilde{f}_{11[1](e_2)} \) are satisfied at the initial time (equivalence conditions), they will remain satisfied for all times and we recover (2.19) as a particular solution of our set of equations.

Let us consider now the next equations of the hierarchy.

\[
\begin{align*}
\partial_t \tilde{f}_{11[1](f)} &= <11[1](f)|\mathcal{L}_H|11[1](f)> \tilde{f}_{11[1](f)} \\
&+ <11[1](f)|\mathcal{L}_H|11[2](f)|f_{11[2](f)} > \tilde{f}_{11[2](f)} \\
&+ <11[1](f)|\mathcal{L}_H|11[2](s_1)|f_{11[2](s_1)} > \tilde{f}_{11[2](s_1)} \\
&+ <11[1](f)|\mathcal{L}_H|11[2](e_2)|f_{11[2](e_2)} > \tilde{f}_{11[2](e_2)}. \quad (2.21)
\end{align*}
\]

\[
\begin{align*}
\partial_t \tilde{f}_{11[1](s_1)} &= <11[1](s_1)|\mathcal{L}_H|11[1](s_1)> \tilde{f}_{11[1](s_1)} \\
&+ <11[1](s_1)|\mathcal{L}_H|11[1](f)|f_{11[1](f)} > \tilde{f}_{11[1](f)} \\
&+ <11[1](s_1)|\mathcal{L}_H|11[1](e_2)|f_{11[1](e_2)} > \tilde{f}_{11[1](e_2)} \\
&+ <11[1](s_1)|\mathcal{L}_H|11[2](s_1)|f_{11[2](s_1)} > \tilde{f}_{11[2](s_1)} \\
&+ <11[1](s_1)|\mathcal{L}_H|11[2](s_1)|f_{11[2](s_1)} > \tilde{f}_{11[2](s_1)} \\
&+ <11[1](s_1)|\mathcal{L}_H|11[2](s_1)|f_{11[2](s_1)} > \tilde{f}_{11[2](s_1)} \\
&+ <11[1](s_1)|\mathcal{L}_H|11[2](s_1)|f_{11[2](s_1)} > \tilde{f}_{11[2](s_1)}. \quad (2.22)
\end{align*}
\]

and a similar expression for \( \partial_t \tilde{f}_{11[1](s_2)} \). From the equality of the matrix elements, we have also \( \partial_t \tilde{f}_{11[1](e_1)} = \partial_t \tilde{f}_{11[1](s_1)} \) and \( \partial_t \tilde{f}_{11[1](e_2)} = \partial_t \tilde{f}_{11[1](s_2)} \). Those relations have to be compatible with:

\[
\partial_t f_{11[1]} = <11[1]|\mathcal{L}_H|11[1] > f_{11[1]} + <11[1]|\mathcal{L}_H|11[2] > f_{11[2]}, \quad (2.23)
\]

We have, for the terms diagonal in the numbers of oscillators:

\[ \partial_t(\tilde{f}_{11}[1(f)] + \tilde{f}_{11}[1(e_1)] + \tilde{f}_{11}[1(e_2)])_{\text{diag}} \]
\[ = <11[1(f)]|\hat{L}_H|11[1(f)]|\tilde{f}_{11}[1(f)]> + <11[1(s_1)]|\hat{L}_H|11[1(s_1)]|\tilde{f}_{11}[1(s_1)]> + <11[1(s_2)]|\hat{L}_H|11[1(s_2)]|\tilde{f}_{11}[1(s_2)]> \]
\[ + <11[1(s_1)]|\hat{L}_H|11[1(s_2)]|\tilde{f}_{11}[1(s_2)]> + <11[1(s_2)]|\hat{L}_H|11[1(e_1)]|\tilde{f}_{11}[1(e_1)]> + <11[1(s_1)]|\hat{L}_H|11[1(e_2)]|\tilde{f}_{11}[1(e_2)]> \]
\[ + <11[1(s_2)]|\hat{L}_H|11[1(e_2)]|\tilde{f}_{11}[1(e_2)]> \]  

(2.24)

For these terms that do not involve a field oscillator, that equation is manifestly compatible with the previous one. Let us consider the other contributions involving \( \hat{L}_1 \). We have its action on \( \tilde{f}_{11}[1(s_1)] \), \( \tilde{f}_{11}[1(s_2)] \), and this is compatible with the original equation, thanks to the constitutive relations and to the numerical identification of \( \tilde{f}_{11}[1(s_1)] \) with \( \tilde{f}_{11}[1(e_1)] \) inside the equivalence relations. The other terms can be treated in a similar way.

That relation (2.20) can be easily generalized for two or more oscillators:

\[ f_{11}[2] = \tilde{f}_{11}[2(f_1f_1)] + \tilde{f}_{11}[2(e_1e_1)] + \tilde{f}_{11}[2(e_1f_1)] + \tilde{f}_{11}[2(e_1e_2)] + \tilde{f}_{11}[2(e_2e_1)] + \tilde{f}_{11}[2(e_2e_2)] \]

(2.25)

and similar expressions for the set of all elements \( \{\tilde{f}_{11}[i]\} \).

The consideration of initial conditions that do not satisfy the equivalence conditions requires that the generators of the Poincaré Lie algebra be constructed for the extended dynamics and that point is beyond our aim in this paper. However, we believe that the distinction between the self-field and the external field resists a Lorentz transformation and therefore no problem should arise from the extension of dynamics. Moreover, the subdynamics operator \( \Pi \) has been proved (in another realisation but within a similar framework) by R. Balescu and L. Brenig [14] to be relativistically invariant. Nevertheless, as far as the adequate construction of the Lie brackets for the ten generators of the Lorentz group has not been performed, we have to restrict ourself to the equivalence case. When the compatible (or equivalence) conditions are fulfilled, the new dynamics is simply a reformulation of the original one.

### 2.4 Factorisation properties

The free fields variables are by definition not connected with the charged particles variables. Therefore, the initial conditions concerning the free fields
and particles variables can be chosen as independent. The vacuum components \( \{ \tilde{f}_{11}[i] \} \) can be factorised into a part, describing the particles and the oscillators of the emitted \( e_1 \) and \( e_2 \) varieties, and a part describing the free variety of the field, for instance an incident field that may be or not vanishingly small. We shall therefore write for instance:

\[
\begin{align*}
\tilde{f}_{11}[1(f)] &= \tilde{f}_{11}[0] \tilde{f}_{11}(f) \\
\tilde{f}_{11}[2(ff)] &= \tilde{f}_{11}[0] \tilde{f}_{2}(ff) \\
\tilde{f}_{11}[2(e_1f)] &= \tilde{f}_{11}[1(e_1)] \tilde{f}_{1}(f)
\end{align*}
\] (2.26)

The free field \( \tilde{f}_{11}(f) \) distribution function may be in particular the distribution function \( \tilde{f}_{11}(f) \) describing the absence of field (field vacuum) and considered later on for the computation of the effect of the self-field on the motion of the charged particles. In the extended dynamics, the natural choice is to consider for \( \tilde{f}_{11}(f) \) a distribution function corresponding to a field of null amplitude and no phase dependance. In those circumstances, the function \( \tilde{f}_{11}[1(s_j)] \) receive e.g. contributions from \( \{ \tilde{f}_{11}[0], \tilde{f}_{[n(ff...f)]} \} \) directly through the creation operator \( \langle 11[0] | \tilde{L}_H | 11[1(f)] \rangle \) (Other contributions are written in §4).

Even in absence of field, the factorisation (2.26) is not equivalent to a factorisation \( f_{11}[1] = f_{11}[0] \tilde{f}_{11}(f) \) in the original dynamics. In the equivalence conditions, we have numerically that the functions \( \tilde{f}_{11}[1(s_j)] \) and \( \tilde{f}_{11}[1(e_j)] \) coincide. The constitutive relation (2.20) requires that \( f_{11}[1] = \tilde{f}_{11}[1(f)] + \tilde{f}_{11}[1(e_1)] + \tilde{f}_{11}[1(e_2)] \). Therefore, in the original representation, we are not allowed to consider a factorisation \( f_{11}[1] = f_{11}[0] \tilde{f}_{11}(f) \) corresponding to the absence of field at the time considered. If we impose at some time \( f_{11}[1] = \tilde{f}_{11}[0] \tilde{f}_{11}(f) \), at the same time, we have to consider \( \tilde{f}_{11}[1(f)] = f_{11}[0] \tilde{f}_{11}(f) - \tilde{f}_{11}[1(e_1)] - \tilde{f}_{11}[1(e_2)] \). Therefore, we have to admit the presence in \( \tilde{f}_{11}[1(f)] \) of contributions \( -\tilde{f}_{11}[1(e_1)] - \tilde{f}_{11}[1(e_2)] \). When computing the equation of evolution of the charged particles, those terms play a role directly through, for instance, the element \( \langle 11[0] | \tilde{L}_H | 11[1(f)] \rangle \) of the first equation of the hierarchy (2.18). That contribution has to be combined with the contribution arising from the kinetic operator \( \tilde{\Theta} \). Upon that imposition \( f_{11}[1] = \tilde{f}_{11}[0] \tilde{f}_{11}(f) \), it is mandatory to consider also that contribution to have a valid comparison with the usual results.

In conclusion, we have been able to provide a set of evolution equations for the reduced distribution functions that enables the explicit identification
of the self-field.

3 The kinetic operator

3.1 General concept

We now proceed to the construction of a subdynamics associated with the enlarged dynamics. First of all, we have to define the vacuum and correlation states. A correlation state contains at least one self oscillator while the vacuum (of correlation) is defined as the set \( \{ \tilde{f}_{11[i]} \} \) where all oscillators are of the free \( f \) and emitted \( e_1 \) and \( e_2 \) varieties. The construction of the subdynamics rests on that distinction. All the formal results of the Brussels group, concerning its construction rules and its formal properties, are applied directly, with our specific realisation of the operators involved.

Inside the subdynamics, the vacuum components obey close evolution equations, namely kinetic equations. These equations are not time reversal invariant, hence their name. For their determination, we can limit ourselves to the consideration of the vacuum-vacuum elements of the superoperator \( \tilde{\Sigma}(t) \), its \( t = 0 \) value defining the \( \tilde{\Pi} \) operator. We take for granted the usual properties of idempotency, factorised structure and commutation of \( \tilde{\Pi} \) with the evolution operator \( \tilde{L} \). The explicit verification of those properties requires the explicit knowledge of all the elements of \( \tilde{\Sigma}(t) \), and not only of the vacuum-vacuum ones.

In the enlarged dynamics, the evolution equation takes the form:

\[
\partial_t \tilde{f}_{11[m]} = \sum_{m'=0}^{\infty} <11[m]|\tilde{L}_H|11[m'] > \tilde{f}_{11[m']},
\]

(3.1)

which involves all the varieties of the oscillators. The kinetic operator \( \tilde{\Theta} \) associated with the subdynamics provides in an exact way close equations involving the vacuum oscillators only:

\[
\partial_t V \tilde{f}_{11[m]} = \sum_{m'=0}^{\infty} <11[m]|\tilde{\Theta}|11[m'] > V \tilde{f}_{11[m']},
\]

(3.2)

The value of that operator can be reached by the direct computation of the vacuum-vacuum elements of the superoperator \( \tilde{\Sigma}(t) \).

The hierarchical form of the equations (3.1) \((m' \geq m)\) enables the determination of the elements of \( \tilde{\Theta} \) in a successive way. The elements of \( \tilde{\Theta} \) that
do not involve an oscillator are the same as those of $\tilde{L}_H$, and therefore also the same as $L_H$.

We proceed to the computation of the first non diagonal element $<11[0]|\tilde{\Theta}|11[1(f)]>$. It is based on the evaluation of the corresponding element $<11[0]|\tilde{\Sigma}(t)|11[1(f)]>$. Its evaluation is performed in a perturbative way. The elements will be affected by a couple of upper indices which describes the number of Coulomb interaction and the power of interaction with the oscillators. The simplest element is of course $<11[0]|\tilde{\Sigma}(t)|11[1(f)]>^{(0,1)}$, in which the Coulomb interaction is not considered and only one interaction with the (free) oscillator takes place. Such element involves no self oscillator and we have trivially:

$$<11[0]|\tilde{\Theta}|11[1(1(f)]>^{(0,1)}= <11[0]|\exp{\tilde{L}_H t}|11[1(f)]>^{(0,1)}. \quad (3.3)$$

The vacuum-vacuum elements of $\tilde{\Sigma}(0)$ are noted $\tilde{A}$ and from the general relation valid for vacuum-vacuum elements

$$<11[0]|\tilde{\Sigma}(t)|11[1(f)]> = <11[0]|e^{\tilde{\Theta} t}\tilde{A}|11[1(f)]>, \quad (3.4)$$

we have

$$<11[0]|\tilde{\Theta}|11[1(f)]>^{(0,1)}= <11[0]|\tilde{\Sigma}(t)|11[1(f)]>. \quad (3.5)$$

### 3.2 Construction rules of $V\tilde{\Sigma}(t)V$

The general construction rules are illustrated on the first non trivial element $<11[0]|\tilde{\Sigma}|11[1(f)]>^{(0,2)}$, in which the Coulomb interaction is not considered and two interactions with an oscillator take place. Such element involves one self oscillator if the two interactions involve the same particle. If they involve different particles, only physical states are present in the contribution and we have anew the equivalence of the corresponding elements of $\tilde{\Theta}$ and $\tilde{L}_H$. We dispense ourself of a supplementary index and concentrate on the contribution involving a self oscillator. We have:

$$<11[0]|\tilde{\Sigma}|11[1(f)]>^{(0,2)}= \frac{-1}{2\pi i} \int_C dz e^{-izt} \sum_{j=1,2} \frac{1}{z - i\mathcal{L}_H^0} <11[0]|\tilde{L}_H|11[1(s_j)]> <11[1(s_j)]|\tilde{\Sigma}|11[1(f)]> \quad (3.5)$$
The prime on the integral sign means that only poles corresponding to propagators arising to vacuum states (without self oscillators) have to be included in the path $c$. In the present case, the pole due to the intermediate propagator is thus excluded from the path. That selection of poles corresponds to the recipe to construct the subdynamics. The accidental coincidence of poles due to the correlation and vacuum propagators is avoided by adding a positive imaginary infinitesimal $\iota \epsilon$ to the correlation propagators when computing the residues. Another formulation of the recipe is the following: a positive imaginary infinitesimal $\iota \epsilon$ is first added to all propagators corresponding to the correlation states and the path $c$ encloses then the real axis, above $-\iota \epsilon$. When no resonance can occur, the $\iota \epsilon$ can be dropped.

3.3 Fourier representation

The evaluation is more easy in variables such that the free motion operator is diagonal. For the free motion of particles, those variables are well known and correspond to the Fourier transform of the original spatial variables. Therefore, we will replace the unknown $\tilde{f}_{11[0]}$ where the variables $x^{(1)}, x^{(2)}$ are $(q^{(1)}, p^{(1)}), (q^{(2)}, p^{(2)})$ by new functions depending on variables $(k^{(1)}, p^{(1)}), (k^{(2)}, p^{(2)})$. We will not introduce a new symbol: the nature of the argument specifies the function under consideration. The transition between the two functions is provided by (we use Balescu’s choice for the normalisation factor):

$$\tilde{f}_{11[0]}(q^{(1)}, p^{(1)}, q^{(2)}, p^{(2)}) = \frac{1}{(2\pi)^6} \int d^3 k_1 d^3 k_2 e^{i(k^{(1)}.q^{(1)}+k^{(2)}.q^{(2)})} \tilde{f}_{11[0]}(k^{(1)}, p^{(1)}, k^{(2)}, p^{(2)}),$$

$$\tilde{f}_{11[0]}(k^{(1)}, p^{(1)}, k^{(2)}, p^{(2)}) = \int d^3 q_1 d^3 q_2 e^{-i(k^{(1)}.q^{(1)}+k^{(2)}.q^{(2)})} \tilde{f}_{11[0]}(q^{(1)}, p^{(1)}, q^{(2)}, p^{(2)}).$$

(3.6)

All functions $\tilde{f}_{11[m]}$ have to be similarly replaced.

We have to perform a similar change with respect to the variables associated with the oscillators. As the functions are periodic in the variables $\xi^{[m]}$, Fourier series are relevant. The function $\tilde{f}_{11[1]}$ becomes a new function depending for the oscillator on the new variables $(\eta^{[j]}_{\alpha}, m^{[j]}_{\alpha}, \alpha = 1, 2) (m^{[j]}_{\alpha}$ discrete) in place of the continuous variables $(\eta^{[j]}_{\alpha}, \xi^{[j]}_{\alpha}, \alpha = 1, 2)$.

$$\tilde{f}_{11[1]}(k^{(1)}, p^{(1)}, k^{(2)}, p^{(2)}; \eta^{[1]}_{1}, m^{[1]}_{1}, \eta^{[1]}_{2}, m^{[1]}_{2}; k^{[1]}).$$
\[
\int_0^1 d\xi_1^{[1]} \int_0^1 d\xi_2^{[2]} e^{2\pi i (\xi_1^{[1]} + \xi_2^{[2]})} \\
\times \tilde{f}_{11}[k^{(1)}] \vec{p}^{(1)} \vec{k}^{(2)} \vec{p}^{(2)}; \eta^{(1)}_1, \xi^{(1)}_1, \eta^{(1)}_2, \xi^{(1)}_2; \vec{k}^{[1]}), \tag{3.7}
\]

\[
\tilde{f}_{11}[k^{(1)}] \vec{p}^{(1)} \vec{k}^{(2)} \vec{p}^{(2)}; \eta^{(1)}_1, \xi^{(1)}_1, \eta^{(1)}_2, \xi^{(1)}_2; \vec{k}^{[1]}) \\
= \sum_{m_1^{[1]}, m_2^{[2]}} e^{-2\pi i (m_1^{[1]} \xi_1^{[1]} + m_2^{[2]} \xi_2^{[2]})} \\
\times \tilde{f}_{11}[k^{(1)}] \vec{p}^{(1)} \vec{k}^{(2)} \vec{p}^{(2)}; \eta^{(1)}_1, m_1^{[1]}, \eta^{(1)}_2, m_2^{[2]}; \vec{k}^{[1]}), \tag{3.8}
\]

where the summations run on all integers, positive and negative.

In Fourier variables, the one particle and one oscillator free motion operators take a simple diagonal form:

\[
L_{H}^{0(j)} = -ik_r^{(j)} v_r^{(j)} = -i \vec{k}^{(j)} \cdot \vec{v}^{(j)} \\
L_{H}^{0[i]} = ik[i] \sum_{\alpha = 1}^{2} m_\alpha^{[i]} \tag{3.9}
\]

while we have for \(L_H^{(12)}\)

\[
L_H^{(12)} = i \frac{e_1 e_2}{8\pi^2} \int d^3 l l^{-2} e^{\frac{1}{2} \left[ \frac{\langle 0 | \vec{p}^{(1)} \rangle}{\vec{k}^{[1]}}, \frac{\langle 0 | \vec{p}^{(2)} \rangle}{\vec{k}^{[2]}}, \right]} \left( \frac{\partial}{\partial \vec{p}^{(1)}} - \frac{\partial}{\partial \vec{p}^{(2)}} \right) \tag{3.10}
\]

or (alternative more usual form)

\[
L_H^{(12)} = i \frac{e_1 e_2}{2\pi^2} \int d^3 l l^{-2} e^{\frac{1}{2} \left[ \frac{\langle 0 | \vec{p}^{(1)} \rangle}{\vec{k}^{[1]}}, \frac{\langle 0 | \vec{p}^{(2)} \rangle}{\vec{k}^{[2]}}, \right]} \left( \frac{\partial}{\partial \vec{p}^{(1)}} - \frac{\partial}{\partial \vec{p}^{(2)}} \right) \tag{3.11}
\]

We have for \(L_H^{j[i]}\):

\[
L_H^{j[i]} = -e_j \frac{1}{(2\pi)^2} \sum_{\alpha = 1, 2} \sum_{a = \pm 1} \left( \frac{\eta_\alpha^{[i]}}{m_\alpha^{[i]}} \right)^{1/2} \\
\times \left[ [k^{[i]}c_r^{(\alpha)[i]} - g^{st}v_s^{(j)} (e_r^{(\alpha)[i]} k_r^{[i]} - e_r^{(\alpha)[s]} k_s^{[i]})] \frac{\partial}{\partial p_r^{(j)}} \\
-\pi \langle \vec{v}^{(j)}, \vec{e}^{(\alpha)[i]} \rangle \left( 2 \frac{\partial}{\partial \eta_\alpha^{[i]}} - \frac{a}{\eta_\alpha^{[i]}} (m_\alpha^{[i]} - a) \right) \right] \\
\times \exp a \left\{ -k^{[i]} \frac{\partial}{\partial \vec{k}^{(j)}} - \frac{\partial}{\partial m_\alpha^{[i]}} \right\}. \tag{3.12}
\]
The only difference is the replacement of the variable $q_j^{(r)}$ by the partial derivative $-i\frac{\partial}{\partial k_j^{(r)}}$ and a similar transposition for the angle variable of the field. The notation $\exp -a\frac{\partial}{\partial m'^i_\alpha}$ enables to take into account the non-diagonality of $L'_H$ with respect to the index $m'^i_\alpha$: the transition is $\pm 1$ according to the value of $a$. Another possibility is the introduction of the factor $\sum m'^i_\alpha \delta^i_{m'^i_\alpha,m'^i_\alpha + a}$, writing with a prime the corresponding argument of the function on which the matrix element acts.

### 3.4 First order kinetic operator

In those variables, the operator $\langle 11|0|\tilde{\Theta}|11[1(f)] \rangle^{(0,1)} = \langle 11|0|\tilde{L}_H|11[1(f)] \rangle$ takes a simple form, due to the presence of a front factor $\delta^i_{m'^i_\alpha,m'^i_\alpha + a}$, $a^2 = 1$ and the property

$$
\int_0^\infty dy_{\alpha}^{[1]} (y_{\alpha}^{[1]})^{1/2} \left( 2\frac{\partial}{\partial y_{\alpha}^{[1]}} + \frac{1}{y_{\alpha}^{[1]}} \right) \ldots = 2 \int_0^\infty dy_{\alpha}^{[1]} \frac{\partial}{\partial y_{\alpha}^{[1]}} \left( (y_{\alpha}^{[1]})^{1/2} \ldots \right) = 0,
$$

when acting on a regular function.

$$
\langle 11|0|\tilde{\Theta}|11[1(f)] \rangle^{(0,1)} = -\sum_{j=1,2} e_j \frac{1}{(2\pi)^{3/2}} \int d^3k^{[1]} \int_0^\infty dy_{\alpha}^{[1]} \int_0^\infty dy_{\beta}^{[1]} \times \sum_{m'^i_\alpha,m'^i_\beta} \delta^i_{m'^i_\alpha,m'^i_\beta} \sum_{a=1,2} \sum_{a=\pm 1} \left( \frac{y_{\alpha}^{[1]}}{k^{[1]}} \right)^{1/2} \times \left[ [k^{[1]} e_r^{(\alpha)}] - g^{st} e_s^{(j)} (e_r^{(\alpha)} k^{[1]}_{t} - e_r^{(\alpha)} k^{[1]}_{t}) \frac{\partial}{\partial p^{(j)}} \right]
\times \exp \left\{ -k^{[1]} \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m'^i_\alpha} \right\}. \quad (3.13)
$$

### 3.5 Recovering the Lorentz force

For pedagogical reasons, we show in Appendix A that under conditions of independence (factorisation) of the field and one particle distribution function, describing a particle sharply located at $r(t)$, this expression leads to

$$
\partial_t \tilde{f}(k,p,t) \bigg|_1 = -e <E^\perp(r(t)) + p \times B^\perp(r(t)) > . \nabla_p \tilde{f}(k,p,t). \quad (3.14)
$$
\[ e \cdot \mathbf{E}(\mathbf{r}(t)) + \mathbf{v} \times \mathbf{B}(\mathbf{r}(t)) \] is the usual electromagnetic force acting on the particle. The minus sign is easily accounted for. If the distribution function corresponds to a well defined value of the velocity and an uniform acceleration \( \mathbf{a} \), it can be written \( f(k, \mathbf{v}, t) \propto \delta(\mathbf{v} - \mathbf{v}_0 - \mathbf{a}t) \) and for the time dependence due to the acceleration, we have \( \partial_t \tilde{f}(k, \mathbf{v}, t) \bigg|_1 = -\mathbf{a} \cdot \nabla \mathbf{v} \tilde{f}(k, \mathbf{v}, t) \).

That expression clearly shows that the present formalism yields the well known expression for the Lorentz force. That relation (3.14) will be used in § 6.

3.6 Evaluation of the second order \( \langle 11[0]|\tilde{\Sigma}|11[1(f)] \rangle \)

While the evolution of the field will be considered in the next section, we focus here on the reaction of the particles to the presence of the field, namely the radiative corrections to the direct interaction between the particles and the field. To provide their contribution to the evolution of the two-particle distribution function, only the elements of the kinetic operator that acts on the (reduced) distribution function of the two charged particles and one mode of the field are relevant. When acting on the distribution function corresponding to the absence of field (defined in the extended dynamics), they will determine the first radiative correction to the free motion of the particles.

The operator \( \langle 11[0]|\tilde{\Sigma}|11[1(f)] \rangle \) in the new variables is now determined. The two interactions have to involve the same particle. We use the same convention for the index of particles as for the polari\( \zeta \)ation of the oscillators: \( j' \) is 2 when \( j \) is 1 and vice versa. We get:

\[
\begin{align*}
\langle 11[0]|\tilde{\Sigma}|11[1(f)] \rangle & \overset{(0,2)}{=} \\
&= -\frac{1}{2\pi i} \int_{c} \int_{c}^{'} dz e^{-i\pi t} \sum_{j=1,2} \left( \frac{1}{z - k^{(j)}, \mathbf{v}(j) - k^{(j')}, \mathbf{v}(j')} \right) (-i) e_{j} \frac{1}{(2\pi)^{\frac{3}{2}}} \\
&\times \int \frac{d^{3}k^{[1]}_{1}}{k^{[1]}} \int_{0}^{\infty} d\eta_{1}^{[1]} \int_{0}^{\infty} d\eta_{2}^{[1]} \sum_{m_{1}, m_{2}} \delta_{m_{1}, 0} \delta_{m_{2}, 0} \sum_{a=1,2} \sum_{a=\pm 1} \\
&\times \left( \eta_{1}^{[1]} \right) \left[ [k^{[1]}_{1} e_{\alpha}^{(a)[1]} - g_{s}^{e} v_{s}^{(j)} (e_{e}^{(a)[1]} k_{1}^{[1]} - e_{r}^{(a)[1]} k_{1}^{[1]})] \frac{\partial}{\partial p_{e}^{(j)}} \right] \\
&\times \exp \left\{ -k^{[1]}_{1} \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_{a}^{[1]}} \right\}
\end{align*}
\]
\[
\times \left( \frac{1}{z - {\mathbf{k}}^{(j)} \cdot {\mathbf{v}}^{(j)} - {\mathbf{k}}^{(j')} \cdot {\mathbf{v}}^{(j')} + {k}^{(1)}(m_{\alpha}^{[1]} + m_{\alpha'}^{[1]})} \right)
\]
\[
x \times (-i)e_j \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\beta=1,2} \sum_{b=\pm 1} \left( \frac{\eta_{\beta}^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}}
\]
\[
x \times \left[ \left[ k^{[1]} e_r^{(\beta)[1]} - g^{s't'} v_s^{(j)} (e_t^{(\beta)[1]} k_r^{[1]} - e_r^{(\beta)[1]} k_t^{[1]}) \right] \frac{\partial}{\partial p_r^{(j)}} \right]
\]
\[
- \pi \left( \mathbf{v}^{(j)} \cdot e^{(\beta)[1]} \right) \left( 2 \frac{\partial}{\partial \eta_{\beta}^{[1]}} - \frac{b}{\eta_{\beta}^{[1]}} (m_{\beta}^{[1]} - b) \right) \] \exp b \left\{ -k^{[1]} \cdot \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_{\beta}^{[1]}} \right\}
\]
\[
x \times \left( \frac{1}{z - {\mathbf{k}}^{(j)} \cdot {\mathbf{v}}^{(j)} - {\mathbf{k}}^{(j')} \cdot {\mathbf{v}}^{(j')} + {k}^{(1)}(m_{\alpha}^{[1]} + m_{\alpha'}^{[1]})} \right), \quad (3.15)
\]

The displacement operators can be transferred at the right of the expression to provide

\[
<11[0]|\tilde{\mathcal{S}}|11[1(f)]>^{(0,2)}
\]
\[
= \frac{-1}{2\pi i} \int dz e^{-izt} \sum_{j=1,2} \left( z - {\mathbf{k}}^{(j)} \cdot {\mathbf{v}}^{(j)} - {\mathbf{k}}^{(j')} \cdot {\mathbf{v}}^{(j')} \right)^{-1} (-i)e_j \frac{1}{(2\pi)^{\frac{3}{2}}}
\]
\[
x \times \int d^3k^{[1]} \int_0^{\infty} d\eta_1^{[1]} \int_0^{\infty} d\eta_2^{[1]} \sum_{m_1^{[1]},m_2^{[1]}} \sum_{\alpha=1,2} \sum_{a=\pm 1} \delta m_1^{[1]} \cdot m_2^{[1]} \cdot 0 \sum_{a=\pm 1}
\]
\[
x \times \left( \frac{\eta_{\alpha}^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{s't'} v_s^{(j)} (e_t^{(\alpha)[1]} k_r^{[1]} - e_r^{(\alpha)[1]} k_t^{[1]}) \right] \frac{\partial}{\partial p_r^{(j)}}
\]
\[
x \times \left( z - {\mathbf{k}}^{(j)} \cdot {\mathbf{v}}^{(j)} + ak^{[1]} \cdot {\mathbf{v}}^{(j)} - {\mathbf{k}}^{(j')} \cdot {\mathbf{v}}^{(j')} + {k}^{(1)}(m_{\alpha}^{[1]} + m_{\alpha'}^{[1]} - a) \right)^{-1}
\]
\[
x \times (-i)e_j \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\beta=1,2} \sum_{b=\pm 1} \left( \frac{\eta_{\beta}^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}}
\]
\[
x \times \left[ k^{[1]} e_r^{(\beta)[1]} - g^{s't'} v_s^{(j)} (e_t^{(\beta)[1]} k_r^{[1]} - e_r^{(\beta)[1]} k_t^{[1]}) \right] \frac{\partial}{\partial p_r^{(j)}}
\]
\[
- \pi \left( \mathbf{v}^{(j)} \cdot e^{(\beta)[1]} \right) \left( 2 \frac{\partial}{\partial \eta_{\beta}^{[1]}} - \frac{b}{\eta_{\beta}^{[1]}} (m_{\beta}^{[1]} - b - a\delta_{\alpha,\beta}) \right) \right]
\]
\[
x \times \left( z - {\mathbf{k}}^{(j)} \cdot {\mathbf{v}}^{(j)} + bk^{[1]} \cdot {\mathbf{v}}^{(j)} + ak^{[1]} \cdot {\mathbf{v}}^{(j)} \right)
\]

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\[-k^{(j')}, v^{(j')} + k^{[1]}(m^{[1]}_{j'} + m^{[1]}_{j} - b - a)\]^{-1} \\
\times \exp \left\{ -(a + b)k^{[1]} \frac{\partial}{\partial k^{(j)}} - a \frac{\partial}{\partial m^{[1]}_{\alpha}} - b \frac{\partial}{\partial m^{[1]}_{\beta'}} \right\}. \tag{3.16}

The summation over $m^{[1]}_1$, $m^{[1]}_2 = 0$ and the Kronecker delta functions of the variables $m^{[1]}_{\alpha} = 0$ and $m^{[1]}_{\alpha'} = 0$ are also written at the right of the expression. In the explicit computation, a separation has to be performed between the contributions with $\beta = \alpha$ and $\beta \neq \alpha$ on one hand, $a = b$ and $a \neq b$ on the other hand. In our future expressions, the first sign $=$ or $\neq$ will refer to the polarisation index while the second one to the relative value of $a$ and $b$.

From that expression, the computation of $<11[0] | \tilde{\Sigma} | 11[1(f)] >^{(0,2)}$ is rather straightforward: we have to proceed formally to the derivatives with respect to the mechanical momentums of the particle. A further index can be introduced to reflect the factor on which they act. The last operation is an integral by residue that takes into account the poles due to the first and last propagators that correspond to the vacuum of correlation. For the sake of illustration, let us consider one of the contributions that will matter when acting in absence of free field:

\[
<11[0] | \tilde{\Sigma} | 11[1(f)] >^{(0,2)}_{=, \neq} = -\frac{1}{2\pi i} \int_c' dz e^{-izt} \sum_{j=1,2} (-1) e_j^2 \left( \frac{1}{2\pi i} \right)^3 \int d^3 k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \\
\times \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_\alpha^{[1]}}{k^{[1]}} \right) \left( z - k^{(j)}, v^{(j)} + a k^{[1]} - k^{(j')}, v^{(j')} - a k^{[1]} \right) \\
\times \left\{ \left( \frac{1}{z - k^{(j)}, v^{(j)} - k^{(j')}, v^{(j')}} \right)^2 \\
\times \left[ k^{[1]} e^{(\alpha)[1]}_{r^{(\alpha)[1]}} - g^{\alpha r} v_s^{(j)} \left( e^{(\alpha)[1]}_{r^{(\alpha)[1]}} k^{[1]}_{r^{(\alpha)[1]}} - e^{(\alpha)[1]}_{r^{(\alpha)[1]}} k^{[1]}_{r^{(\alpha)[1]}} \right) \frac{\partial}{\partial p_r^{(j)}} \right] \\
\times \left[ k^{[1]} e^{(\alpha)[1]}_{r^{(\alpha)[1]}} - g^{\alpha r'} v_s^{(j)} \left( e^{(\alpha)[1]}_{r^{(\alpha)[1]}} k^{[1]}_{r^{(\alpha)[1]}} - e^{(\alpha)[1]}_{r^{(\alpha)[1]}} k^{[1]}_{r^{(\alpha)[1]}} \right) \frac{\partial}{\partial p_{r'}^{(j)}} \right) \\
- \pi (v^{(j)}, e^{(\alpha)[1]}_{r^{(\alpha)[1]}}) \left( 2 \frac{\partial}{\partial \eta_\alpha^{[1]}} \right) \right\}
\]
\[
\left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')}} \right)^3 \times \left[ [k^{(1)} e_r^{(\alpha)}[1] - g^{st} v_s^{(j)} (e_t^{(\alpha)}[1] k_t^{[1]} - e_t^{(\alpha)}[1] k_t^{[1]})] \frac{\partial}{\partial p_r^{(j)}} \right] \\
\times \left[ [k^{(1)} e_r^{(\alpha)}[1] - g^{st'} v_s^{(j)} (e_t^{(\alpha)}[1] k_t^{[1]} - e_t^{(\alpha)}[1] k_t^{[1]})] \frac{\partial}{\partial p_r^{(j')}} \right] \\
\times \sum_{m_1^{[1]}, m_2^{[1]}} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \left( -it \right)^2 \left( \frac{1}{ak^{[1]} \cdot v^{(j)} - ak^{[1]}} \right) \\
\left( -it \right)^2 \left( \frac{1}{ak^{[1]} \cdot v^{(j)} - ak^{[1]}} \right)
\]

(3.17)

In that expression, the partial derivative \( \frac{\partial}{\partial p_r^{(j)}} \) acts on everything at its right: the factors \( v_s^{(j)} \) and \( v^{(j)} \) (in the first term), the factors \( v_s^{(j)} \) and \( \left( \frac{\partial v_s^{(j)}}{\partial p_r^{(j')}} \right) \) in the second term and, for both terms, the momentum dependence of the distribution function on which \( < 11[0] | \Sigma | 11[1(f)] >_{=, \#} \) is applied. The only singularity to be included in the close path \( c \) is the pole of \( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')}} \). No coincidence is possible with the pole of \( \frac{1}{z - k^{(j)} \cdot v^{(j)} + ak^{(1)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} - ak^{[1]} \cdot v^{(j)} - ak^{[1]} \cdot v^{(j')} < 1 \) and no \( i \epsilon \) has to be introduced here. The obtention of \( < 11[0] | \Sigma | 11[1(f)] >_{=, \#} \) is therefore straightforward.
for the vacuum intermediate states, we have
between the subdynamics operator $\tilde{\Sigma}(\cdot)$.
Using
\begin{align*}
-2(-it) \left( \frac{1}{ak^{[1]},v^{(j)} - ak^{[1]}} \right)^2 + 2 \left( \frac{1}{ak^{[1]},v^{(j)} - ak^{[1]}} \right)^3 \\
\times \frac{1}{2} \left[ [k^{[1]} e^{[\alpha][1]} - g^{st} v^{(j)} (e^{[\alpha][1]} k^{[1]} - e^{[\alpha][1]} k^{[1]}_t)] \frac{\partial}{\partial p^{(j)}} \right] \\
\times \left[ [k^{[1]} e^{[\alpha][1]} - g^{st} v^{(j)} (e^{[\alpha][1]} k^{[1]} - e^{[\alpha][1]} k^{[1]}_t)] k^{(j)}_t \left( \frac{\partial v^{(j)}}{\partial p^{(j)}} \right) \right] \\
\times \sum_{m_1, m_2} \delta_{m_1}^{[1],0} \delta_{m_2}^{[1],0}.
\end{align*}

(3.18)

3.7 The second order evolution operator $\tilde{\Theta}$

From the general properties of the subdynamics, we have the links (3.4) between the subdynamics operator $\tilde{\Sigma}(t)$ and the evolution operator $\tilde{\Theta}$ for the vacuum-vacuum elements. Since we have a limited choice of possibilities for the vacuum intermediate states, we have
\begin{align*}
\frac{\partial}{\partial t} < 11[0]|\tilde{\Sigma}|11[1(f)] >^{(0,2)}_{=,3,\neq} |_{t=0} \\
= < 11[0]|\tilde{\Theta}|11[1(f)] >^{(0,2)}_{=,3,\neq} < 11[1(f)]|\tilde{A}|11[1(f)] >^{(0,0)} \\
+ < 11[0]|\tilde{L}^{[1]}|11[0] > < 11[0]|\tilde{A}|11[1(f)] >^{(0,2)}_{=,3,\neq}.
\end{align*}

(3.19)

Using $< 11[1(f)]|\tilde{A}|11[1(f)] >^{(0,0)} = 1$ and $< 11[0]|\tilde{L}^{[1]}|11[0] > = -i[k^{(j)},v^{(j)} + k^{(j')}v^{(j')}]$, we can obtain directly $< 11[0]|\tilde{\Theta}|11[1(f)] >^{(0,2)}_{=,3,\neq} = \sum_{j=1,2} (-1) e_j^2 \left\{ \int d^3k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{a=1,2} \sum_{\alpha=\pm 1} \eta_\alpha^{[1]} (k^{[1]}) \left[ [k^{[1]} e^{[\alpha][1]} - g^{st} v^{(j)} (e^{[\alpha][1]} k^{[1]} - e^{[\alpha][1]} k^{[1]}_t)] \frac{\partial}{\partial p^{(j)}} \right] \\
\times \left[ [k^{[1]} e^{[\alpha][1]} - g^{st} v^{(j)} (e^{[\alpha][1]} k^{[1]} - e^{[\alpha][1]} k^{[1]}_t)] k^{(j)}_t \left( \frac{\partial v^{(j)}}{\partial p^{(j)}} \right) \right] \\
- \pi(v^{(j)} e^{[\alpha][1]} \left( 2 \frac{\partial}{\partial \eta_\alpha^{[1]}} \right) + i \left( \frac{1}{ak^{[1]},v^{(j)} - ak^{[1]}} \right)^2 \\
\times \left[ [k^{[1]} e^{[\alpha][1]} - g^{st} v^{(j)} (e^{[\alpha][1]} k^{[1]} - e^{[\alpha][1]} k^{[1]}_t)] \frac{\partial}{\partial p^{(j)}} \right] \right\}.$
\[\sum_{m_1^{[1]}, m_2^{[1]}} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0}. \]

All contributions are treated in a similar way and recombined such that the conservation of the norm is manifest (due to the front factor \(\frac{\partial}{\partial p_{r'}}\)):

\[<11[0]|\tilde{\Theta}|11[1(f)]>^{(0,2)}_{\neq}\]

\[= \sum_{j=1,2} ie^2 \frac{1}{(2\pi)^3} \frac{\partial}{\partial p_r^{(j)}} \int d^3 k^{[1]} \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \sum_{a=1,2} \sum_{a=\pm 1} \left( \alpha k^{[1]} \right) \left\[ k^{[1]} e_r^{(\alpha) [1]} - g^{sr} v_{sr}^{(j)} (e_t^{(\alpha) [1]} k_r^{[1]} - e_r^{(\alpha) [1]} k_t^{[1]}) \right\]

\[\times \left[ \left( \frac{1}{ak^{[1]} \cdot v^{(j)} - ak^{[1]}} \right) \left\{ k^{[1]} e_r^{(\alpha) [1]} - g^{sr} v_{sr}^{(j)} (e_t^{(\alpha) [1]} k_r^{[1]} - e_r^{(\alpha) [1]} k_t^{[1]}) \right\} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \right] \]

\[\times \sum_{m_1^{[1]}, m_2^{[1]}} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0}. \]

\[<11[0]|\tilde{\Theta}|11[1(f)]>^{(0,2)}_{\neq}\]

\[= \sum_{j=1,2} ie^2 \frac{1}{(2\pi)^3} \frac{\partial}{\partial p_r^{(j)}} \int d^3 k^{[1]} \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \sum_{a=1,2} \sum_{a=\pm 1} \left( \alpha k^{[1]} \right) \left\[ k^{[1]} e_r^{(\alpha) [1]} - g^{sr} v_{sr}^{(j)} (e_t^{(\alpha) [1]} k_r^{[1]} - e_r^{(\alpha) [1]} k_t^{[1]}) \right\]

\[\times \left[ \left( \frac{1}{ak^{[1]} \cdot v^{(j)} - ak^{[1]}} \right)^2 \left\{ k^{[1]} e_r^{(\alpha) [1]} - g^{sr} v_{sr}^{(j)} (e_t^{(\alpha) [1]} k_r^{[1]} - e_r^{(\alpha) [1]} k_t^{[1]}) \right\} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \right] \]

\[\times \sum_{m_1^{[1]}, m_2^{[1]}} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0}. \]

Here, the commutation property of \([k^{[1]} e_r^{(\alpha) [1]} - g^{sr} v_{sr}^{(j)} (e_t^{(\alpha) [1]} k_r^{[1]} - e_r^{(\alpha) [1]} k_t^{[1]})]\) and \(\frac{\partial}{\partial p_{r'}}\) has been used.

For the sake of completeness, similar expressions are provided in appendix B for \(<11[0]|\tilde{\Theta}|11[1(f)]>^{(0,2)}_{\neq}\), \(<11[0]|\tilde{\Theta}|11[1(f)]>^{(0,2)}_{\neq}\) and \(<11[0]|\tilde{\Theta}|11[1(f)]>^{(0,2)}_{\neq}\).
3.8 Evolution in the absence of incident field

The knowledge of the operator $\langle 11|0|\tilde{\Theta}|11|1(f)\rangle >^{(0,2)}$ enables us to look for the behaviour of the particles when evolving into the absence of external field. Other matrix elements could also be considered to provide, for instance, the vertex (charge) renormalisation due to the self-field but are outside our scope in this paper.

We can use the factorisation (2.18):

$$\tilde{f}_{11[1(f)]} = \tilde{f}_{11[0]} f_{1[1(f)]},$$  \hspace{1cm} (3.22)

and use for $\tilde{f}_{1[1(f)]}$ the distribution function $\tilde{f}^V_{1[1(f)]}$ corresponding to the absence of field: namely the limit for $\eta_1 \to 0$ and $\eta_2 \to 0$ of:

$$\tilde{f}_{1[1]}(\eta_1^{[1]}, m_1^{[1]}, \eta_2^{[1]}, m_2^{[1]}; k^{[1]}) = \delta(\eta_1^{[1]} - \eta_1)\delta(\eta_2^{[1]} - \eta_2)\delta^{Kr}_{m_1^{[1]}0}\delta^{Kr}_{m_2^{[1]}0}. \hspace{1cm} (3.23)$$

Since the variables $\eta_1^{[1]}$ and $\eta_2^{[1]}$ are integrated from 0 to $\infty$, the limit $\eta_1 \to 0$ and $\eta_2 \to 0$ has to be taken after that we have performed that integration. We identify in $\langle 11|0|\tilde{\Theta}|11|1(f)\rangle >^{(0,2)}$ all the terms which could provide a non vanishing contribution. The summations over $m_1^{[1]}$ and $m_2^{[1]}$ provide a vanishing result if a displacement operator on $m_1^{[1]}$ and $m_2^{[1]}$ is involved: we would then meet a product of Kronecker’s delta functions with incompatible arguments. Therefore, the only possible non-vanishing ones would arise from the contribution $\langle 11|0|\tilde{\Theta}|11|1(f)\rangle >^{(0,2)}$. Since that contribution involves a front factor $\eta_0^{[1]}$, the presence of $\delta(\eta_1^{[1]} - \eta_1)\delta(\eta_2^{[1]} - \eta_2)$ for $\eta_1 \to 0$ and $\eta_2 \to 0$ provides a vanishing result except for the contributions in which the derivative of the Dirac’s delta function appears. The second term in (3.21) provides therefore a vanishing result and we are left with

$$\langle 11|0|\tilde{\Theta}|11|1(f)\rangle >^{(0,2)} \tilde{f}^V_{1[1(f)]}$$

$$= \sum_{j=1,2} i e_j^2 \frac{1}{2(2\pi)^3} \frac{\partial}{\partial p_r^{(2)}} \int d^3 k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{\pm=\pm1} \delta(\eta_\alpha^{[1]} - \eta_1)\delta(\eta_\alpha^{[1]} - \eta_2)\delta^{Kr}_{m_1^{[1]}0}\delta^{Kr}_{m_2^{[1]}0}$$

$$\times \left( \frac{1}{ak^{[1]} \cdot v^{(j)} - ak^{[1]}} \right) \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{s_r (\alpha)} (e_t^{(\alpha)} k_r^{[1]} - e_r^{(\alpha)[1]} k_r^{[1]}) \right]$$

$$\times \left[ -\pi (v^{(j)} \cdot e^{(\alpha)[1]} ) \left( 2 \frac{\partial}{\partial \eta_\alpha^{[1]}} \right) \right]$$

$$\times \sum_{m_1^{[1]},m_2^{[1]}} \delta_{m_1^{[1]}0}\delta_{m_2^{[1]}0} \delta(\eta_1^{[1]} - \eta_1)\delta(\eta_2^{[1]} - \eta_2)\delta^{Kr}_{m_1^{[1]}0}\delta^{Kr}_{m_2^{[1]}0}. \hspace{1cm} (3.24)$$

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The summations and integrations over the fields variables can be performed in a straightforward way using the Kronecker’s and Dirac’s delta functions (after an integration by parts and performing the limits \( \eta_1 \to 0, \eta_2 \to 0 \)) and we have:

\[
<11[0]|\tilde{\Theta}|11[1(f)] >^{(0,2)}_{=\neq} \tilde{f}^V_{[1(f)]} = \frac{1}{(2\pi)^3} \int d^3 k^{[1]} \sum_{a=1,2} \sum_{a=\pm 1} \left( \frac{\eta_a^{[1]}}{k^{[1]}} \right) \left( \frac{1}{ak^{[1]}, v^{(j)} - ak^{[1]}} \right) \times \left[ k^{[1]} c_r^{(\alpha)[1]} \right. \\
- \left. g^{st} v_s^{(j)} (e_t^{(\alpha)[1]} k_r^{[1]} - e_r^{(\alpha)[1]} k_t^{[1]}), \right] \frac{\partial}{\partial p_r^{(j)}} \left( 2\pi (v^{(j)}. e^{(\alpha)[1]}) \right). \quad (3.25)
\]

Nevertheless, this last expression vanishes also by parity for the summation over \( a \).

### 3.9 A useful expression :

For future use, the non vanishing element of \(<11[0]|\tilde{\Theta}|11[1(f)] >^{(0,2)}_{=\neq} \tilde{f}^V_{[1(f)]}\) is required:

\[
<11[0]|\tilde{\Theta}|11[1(f)] >^{(0,2)}_{=\neq} \tilde{f}^V_{[1(f)]} = \frac{1}{(2\pi)^3} \int d^3 k^{[1]} \sum_{a=1,2} \sum_{a=\pm 1} \left( \frac{1}{k^{[1]}} \right) \\
\times \left[ k^{[1]} c_r^{(\alpha)[1]} - g^{st} v_s^{(j)} (e_t^{(\alpha)[1]} k_r^{[1]} - e_r^{(\alpha)[1]} k_t^{[1]}), \right] \frac{\partial}{\partial p_r^{(j)}} \left( 2\pi (v^{(j)}. e^{(\alpha)[1]}) \right). \quad (3.26)
\]

The derivative with respect to \( p_r^{(j)} \) acts of course on all possible dependences at its right.

### 3.10 Physical interpretation

The concept of renormalised mass has now to be extracted from the kinetic equation, by combining radiative corrections with the free motion operator. Our expression of the second order element \( <11[0]|\tilde{\Theta}|11[1(f)] >^{(0,2)}_{=\neq} \tilde{f}^V_{[1(f)]} \) of the kinetic operator shows that it vanishes for a free particle that is not accelerated by external fields nor a Coulomb interaction No mass correction is provided. Indeed, if a relativistic expression is used for the energy of the
particle, the momentum and energy conservations cannot be simultaneously satisfied by an emission act of a non-accelerated charged particle. No resonant process is possible and their absence implies that no $i\epsilon$ is required and the propagator is odd in $a$.

This result could be expected from general considerations from our knowledge of general properties of the subdynamics [8] when the propagator involved cannot be resonant. In Brussels terminology [8], the second order kinetic operator vanishes for parity reasons: it is well known that, at that second order, the contribution to the kinetic operator (called $\psi_2$) arises from a Dirac delta “function” and not from the principal part of the (usually regularised by a $i\epsilon$) propagator. No regularisation has been required here when acting in absence of field and the kinetic operator provides a vanishing contribution.

The effect of the coupling with the field vacuum is therefore to be searched in other terms. Indeed, radiation emission is present when the particles are accelerated. We consider as a first step in §5 $<11|0|\Theta|11|1(f)>^{(1,2)}$. The acceleration provided by the Coulomb interaction will induce a back reaction on the motion of the particles.

The next section §4 is devoted to the computation of the field generated by the particle in their free motion.

## 4 Emitted field in a free motion

In order to get a better understanding of our previous result (3.25), we intend to analyse the field emitted by the particles and their back reaction at the same order $(0,2)$ in the interactions. The contribution is the same as for two particles moving independently. We will use the equivalence conditions that enable to get the emitted field from the self-field. The subdynamics theory determines the last one through the so-called creation operator $\tilde{C}$. Since we know that $\tilde{\Sigma}(t) = \tilde{C}e^{i\Theta}A$ for the correlation-vacuum elements, we focus on the elements of $\tilde{\Sigma}(t)^{(0,1)}$ that provides a contribution to

$$\tilde{f}_{11[1(is_j)]} = <11[1(s_j)][\tilde{C}]11[1(f)] > \tilde{f}_{11[1(f)]}$$
$$+ <11[1(s_j)][\tilde{C}]11[1(e_{j'})] > \tilde{f}_{11[1(e_{j'})]}$$
$$+ <11[1(s_j)][\tilde{C}]11[2(ff)] > \tilde{f}_{11[2(ff)]} + \ldots \quad (4.1)$$

Since numerically, in the equivalence conditions, we have the equality $\tilde{f}_{11[1(ie_{j})]} = \tilde{f}_{11[1(is_j)]}$, the lowest order contribution to $\tilde{f}_{11[1(is_j)]}$ requires $<11[1(s_j)][\tilde{\Sigma}(t)]11[1(f)] >^{(0,1)}$.
that determines the lowest order contribution to the creation operator. Only the terms that provide a contribution when acting in absence of field are considered.

4.1 Correlation-vacuum element of $\tilde{\Sigma}(t)$

In place of (3.15), we start from

$$< 11[1(s_j)]|\tilde{\Sigma}(t)|11[1(f)] > ^{(0,1)} \tilde{f}^{V}_{11(f)}$$

$$= -\frac{1}{2\pi i} \int c dz e^{-izt} \left( \frac{1}{z-k(j),\cdot v(j) - k(j'),\cdot v(j') + k[1](m_\alpha[1] + m_\alpha')} \right)$$

$$\times (-i)e_j \frac{1}{(2\pi)^2} \sum_{\beta=1,2} \sum_{a=\pm 1} \left( \frac{\eta_\beta[1]}{k[1]} \right)^{1/2}$$

$$\times \left[ [k[1] e_{\nu'}(\beta)[1] - g^{s'\nu'} v_s(\nu') (e_{\nu'}(\beta)[1] - c_{\nu'}(\beta)[1] k_{\nu'}[1])] \frac{\partial}{\partial p_{\nu'}[1]} \right.$$

$$\times -\pi (v^{(1)} , e^{(\beta)[1]} (2 \frac{\partial}{\partial \eta_\beta[1]} - \frac{a}{\eta_\beta[1]} (m_\beta[1] - a) ) \exp \left\{ -k[1], \frac{\partial}{\partial k(j)} - \frac{\partial}{\partial m_\beta[1]} \right\}$$

$$\times \left( \frac{1}{z-k(j),\cdot v(j) - k(j'),\cdot v(j') + k[1](m_\beta[1] + m_\beta')} \right)$$

$$\times \delta(\eta_\beta[1] - \eta_\beta) \delta(\eta_\beta[1] - \eta_\beta') \delta_{kr}^{K_\beta} \delta_{m_\beta[1]0} \delta_{m_\beta[1]0}$$

(4.2)

where we have taken into account that the final state ($< 11[1(s_j)]$) is a field correlated state, hence the presence of $k[1](m_\alpha[1] + m_\alpha')$ in the first propagator. Only the pole of the second propagator (due to a vacuum state) is enclosed by the path $c$. Moving the displacement operators to the right and using afterwards $m_1[1] + m_2[1] = m_\beta[1] + m_\beta' = m_\alpha[1] + m_\alpha' = a$, we get

$$< 11[1(s_j)]|\tilde{\Sigma}(t)|11[1(f)] > ^{(0,1)} \tilde{f}^{V}_{11(f)}$$

$$= -\frac{1}{2\pi i} \int c dz e^{-izt} \sum_{\beta=1,2} \sum_{a=\pm 1} \left( \frac{1}{z-k(j),\cdot v(j) - k(j'),\cdot v(j') + ak[1]} \right) (-i)e_j$$

$$\times \frac{1}{(2\pi)^2} \left( \frac{\eta_\beta[1]}{k[1]} \right)^{1/2} \left[ [k[1] e_{\nu'}(\beta)[1] - g^{s'\nu'} v_s(\nu') (e_{\nu'}(\beta)[1] - c_{\nu'}(\beta)[1] k_{\nu'}[1])] \frac{\partial}{\partial p_{\nu'}[1]} \right.$$
\(-\pi (v^{(j)}, e^{(\beta)[1]} \left( 2 \frac{\partial}{\partial \eta^{[1]}} \right) \right) \left( \frac{1}{z - (k^{(j)} - ak^{[1]}, v^{(j)} - k^{(j')}, v^{(j')})} \right) \exp \left( ak^{[1]} \frac{\partial}{\partial k^{(j)}} \right) \delta(\eta^{[1]} - \eta_{\beta}) \delta(\eta^{[1]}_{\beta'} - \eta_{\beta'}) \delta_{m^{[1]}_{\beta}, a} \delta_{m^{[1]}_{\beta'}, 0} \right)

\times \exp \left( - ak^{[1]} \frac{\partial}{\partial k^{(j)}} \right) \delta(\eta^{[1]}_{\beta} - \eta_{\beta}) \delta(\eta^{[1]}_{\beta'} - \eta_{\beta'}) \delta_{m^{[1]}_{\beta}, a} \delta_{m^{[1]}_{\beta'}, 0}. \quad (4.3)

This expression can be computed easily and is identified with the same order of \(< 11[1(s_j)]|\tilde{C} e^{\delta_{\beta'}} A|11[1(f)] > f^{[1]}_{(1(f))} \).

4.2 First order creation operator

Therefore,

\(< 11[1(s_j)]|\tilde{C} |11[1(f)] >^{(0,1)} f^{[1]}_{(1(f))} \)

\[= \sum_{\beta=1, 2} \sum_{a=\pm 1} \left( \frac{1}{-ak^{[1], v^{(j)} + ak^{[1]}} \right) (-i) e_j \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{\eta^{[1]}_{\beta}}{k^{[1]}} \right)^{\frac{1}{2}} \]

\[\times \left[ [k^{[1]} e^{(\beta)[1]} - g^{s'} v^{(j')} (e^{(\beta)[1]} k^{[1]} - e^{(\beta)[1]} k^{[1]})] \frac{\partial}{\partial p^{(j')}_{s'}} \right] \]

\[-\pi (v^{(j)}, e^{(\beta)[1]} \left( 2 \frac{\partial}{\partial \eta^{[1]}} \right) \right) \left( \frac{1}{z - (k^{(j)} - ak^{[1]}, v^{(j)} - k^{(j')}, v^{(j')})} \right) \exp \left( ak^{[1]} \frac{\partial}{\partial k^{(j)}} \right) \delta(\eta^{[1]} - \eta_{\beta}) \delta(\eta^{[1]}_{\beta'} - \eta_{\beta'}) \delta_{m^{[1]}_{\beta}, a} \delta_{m^{[1]}_{\beta'}, 0} \]

\[+ \sum_{\beta=1, 2} \sum_{a=\pm 1} \left( \frac{1}{-ak^{[1], v^{(j)} + ak^{[1]}} \right) (-i) e_j \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{\eta^{[1]}_{\beta}}{k^{[1]}} \right)^{\frac{1}{2}} \]

\[\times \left[ [k^{[1]} e^{(\beta)[1]} - g^{s'} v^{(j')} (e^{(\beta)[1]} k^{[1]} - e^{(\beta)[1]} k^{[1]})] \frac{\partial}{\partial p^{(j')}_{s'}} \right] \]

\[\times \exp \left( - ak^{[1]} \frac{\partial}{\partial k^{(j)}} \right) \delta(\eta^{[1]}_{\beta} - \eta_{\beta}) \delta(\eta^{[1]}_{\beta'} - \eta_{\beta'}) \delta_{m^{[1]}_{\beta}, a} \delta_{m^{[1]}_{\beta'}, 0}. \quad (4.4)\]

The limits \(\eta_{\beta} \rightarrow 0, \eta_{\beta'} \rightarrow 0\) have to be performed after the integration over \(\eta^{[1]}_{\beta}\) and \(\eta^{[1]}_{\beta'}\).

Using (3.6), the original variables \(\xi^{[1]}\) are reintroduced in place of \(m^{[1]}\).
4.3 Field associated with free particles

Since \( \tilde{f}_{11[1(\eps_j)]} = \tilde{f}_{11[1(\eps_j)]} \), we have at first order in the field interaction and zero \(^{th}\) order in the Coulomb interaction:

\[
\tilde{f}_{11[1(\eps_j)]}^{(0,1)} = \sum_{\beta=1,2} \sum_{a=\pm 1} e^{-2\pi i (a\xi^{[1]}_{\beta})} \left( \frac{1}{-ak^{[1]}_s \cdot \nu^{(j)} + ak^{[1]}_s} \right) (-i)\xi^{[1]}_j \frac{1}{(2\pi)^{\frac{1}{2}}} \times \left( \frac{\eta^{[1]}_\beta}{k^{[1]}_s} \right)^\frac{1}{2} \left( \frac{\nu^{(j)} - g^{s'\nu'} (e^{(\beta)[1]}_\nu x^{(j)}_a - e^{(\beta)[1]}_\nu x^{(j)}_a) \partial}{\partial p^{(j)}_a} \right) - \pi (\nu^{(j)} \cdot e^{(\beta)[1]}_\nu) \left( 2 \frac{\partial}{\partial \eta^{[1]}_\beta} \right) \exp - \left( ak^{[1]}_s \frac{\partial}{\partial k^{(j)}_s} \right) \times \delta(\eta^{[1]}_\beta - \eta^{[1]}_\beta) \delta(\eta^{[1]}_\beta - \eta^{[1]}_S) \tilde{f}_{11[0]}.
\]

Combining the form (2.2) for the observables associated with the transverse electric field and the form (2.5), we get the average transverse component of the electric field and we can write

\[
\langle E^\perp_t (x) \rangle^{(0,1)} = \int d^3k^{[1]}_s \int_0^\infty d\eta^{[1]}_\beta \int_0^\infty d\nu^{(j)}_a \int_0^1 d\xi^{[1]}_j \int_0^1 d\xi^{[1]}_s \int d^6x^{(1)} \int d^6x^{(2)} \times \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\alpha=1,2} \sum_{a'=\pm 1} k^{[1]}_s \frac{1}{2} e^{\alpha}_a (k^{[1]}_s) \eta^{[1]}_\beta (k^{[1]}_s) \times \exp \left\{ i a'(k^{[1]}_s \cdot x - 2\pi \xi^a (k^{[1]}_s)) \right\} \tilde{f}_{11[1(\eps_j)]}^{(0,1)} (x^{(1)}_a, x^{(2)}_a; \chi^{[1]}_a; k^{[1]}_s). \tag{4.6}
\]

We now take for \( \tilde{f}_{11[0]} \) in (4.5) a distribution function corresponding to sharp values of the positions and momenta and (3.6) enables to get the expression into the variables \( k^{(j)}_s, p^{(j)}_a; \)

\[
\tilde{f}_{11[0]} (q^{(1)}_a, p^{(1)}_a, q^{(2)}_a, p^{(2)}_a) = \delta(q^{(1)}_a - q^{(1)}_b) \delta(q^{(2)}_a - q^{(2)}_b) \delta(p^{(1)}_a - p^{(1)}_b) \delta(p^{(2)}_a - p^{(2)}_b),
\]

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\[
\tilde{f}_{11|[q]}(k^{(1)}, p^{(1)}, k^{(2)}, p^{(2)}) = \int d^3q^{(1)} \int d^3q^{(2)} e^{-i(k^{(1)} \cdot q^{(1)} + k^{(2)} \cdot q^{(2)})}
\times \delta(q^{(1)} - q_1) \delta(q^{(2)} - q_2) \delta(p^{(1)} - p_1) \delta(p^{(2)} - p_2)
\]
\[
= e^{-i(k^{(1)} \cdot q_1 + k^{(2)} \cdot q_2)} \delta(p^{(1)} - p_1) \delta(p^{(2)} - p_2).
\] (4.7)

Combining the previous terms and performing all trivial integrations, we get:

\[
< E^\bot (x) >^{e_j(0,1)} = (-i)(2\pi)e_j \frac{1}{(2\pi)^3} \int d^3k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} e_\alpha^a(k^{[1]})
\times \exp\{ia[k^{[1]} \cdot (x - q_j)]\}\left( \frac{1}{ak^{[1]},v_j - ak^{[1]}} \right) (v_j.e^{(\alpha)[1]}).\] (4.8)

The value \( < E^\bot (x) >^{e_j(0,1)} \) is now determined by an expression that involves the values of the position \( q_j \) and momentum \( p_j \) of the charged particle \( j \) at the same time.

The summations over \( a \) and over the polarisation vectors lead to

\[
< E^\bot (x) >^{e_j(0,1)} = (-i)(2\pi)e_j \frac{1}{(2\pi)^3} \int d^3k^{[1]} \left[ \exp\{i[k^{[1]} \cdot (x - q_j)]\}
\right.
\left. - \exp\{-i[k^{[1]} \cdot (x - q_j)]\}\right] \left( \frac{1}{k^{[1]},v_j - k^{[1]}} \right) \left( v_j - \frac{(v_j,k^{[1]})k^{[1]}}{(k^{[1]}^2)^2} \right).\] (4.9)

This expression behaves obviously as \( \frac{1}{|x - q_j|^2} \) and does not describe a propagating field. It presents a discontinuity and vanishes exactly at the location of the particle \( x = q_j \) since the integrand is identically null for that value.

In view of the value provided by (3.14), this explains why the corresponding terms in the kinetic operator \( \Theta \) vanishes. In our approach, the contribution of each mode to the self-interaction of the electron is computed first. When no acceleration mechanism is provided, each contribution vanishes exactly.

In the usual approach, the emitted fields are computed from the Liénard-Wiechert potential and evaluated, via the Abraham-Lorentz model, at the localisation of the electron.

The expression for the electric field can be further analysed. We can write (4.9) as the sum of two contributions by replacing \( (v_j - (v_j,k^{[1]})k^{[1]}) \) by the sum \( (v_j - k^{[1]}k^{[1]}) + (k^{[1]}k^{[1]} - (v_j,k^{[1]})k^{[1]}) \). The contribution of the second
term is:
\[
< E^\perp(x) >^e_j(0,1) = -(4\pi)e_j \frac{1}{(2\pi)^3} \int d^3k[1] \sin[k[1],(x - q_j)] \frac{k[1]}{(k[1])^2} \\
= (4\pi)e_j \frac{1}{(2\pi)^3} \nabla_x \int d^3k[1] \cos[k[1],(x - q_j)] \frac{1}{(k[1])^2}. \quad (4.10)
\]

From the known relations \(\int d^3k[1] \exp(-i[k[1],x]) \frac{1}{k[1]^2} = 4\pi \frac{1}{x}\) and \(\int d^3k[1] \exp(i[k[1],x]) \frac{1}{(k[1])^2} = 2\pi \frac{1}{x}\), the following identification is possible:
\[
< E^\perp(x) >^e_j(0,1) = e_j \nabla_x \frac{1}{|x - q_j|} = - < E^\parallel(x) >. \quad (4.11)
\]

Therefore, the first contribution \(< E^\perp(x) >^e_j(0,1)\) should be identified with the complete electric field \(< E(x) >^e_j(0,1)\):
\[
< E^\perp(x) >^e_j(0,1) = (-i)(2\pi)e_j \frac{1}{(2\pi)^3} \int d^3k[1] \left[ \exp\{i[k[1],(x - q_j)]\} \right. \\
- \exp\{-i[k[1],(x - q_j)]\} \left( k[1],v_j - k[1] \right) \frac{1}{k[1]} \\
= (-i)(2\pi)e_j \frac{1}{(2\pi)^3} \int d^3k[1] \left[ \exp\{i[k[1],(x - q_j)]\} \right. \\
- \exp\{-i[k[1],(x - q_j)]\} \left( k[1] - k[1],v_j \right) \frac{1}{k[1]} \left( k[1] - k[1],v_j \right) \frac{1}{k[1]}. \quad (4.12)
\]

If the \(x\) axis is placed along \((x - q_j)\) and the \(y\) axis along \(v_{\perp,j}\), defined by \(v_{\perp,j} = v_j - \frac{[v_j \cdot (x - q_j)](x - q_j)}{|x - q_j|^2}\), we show in Appendix C that:
\[
< E^\perp(x) >^e_j(0,1) = [1 - v_{jj}^2] \frac{1}{(1 - v_{jj}^2)^2} \frac{1}{|x - q_j|^2} e_x. \quad (4.13)
\]

On the other hand (11.154) of \([\text{Ref.}]\) gives us the field in terms of the position of the charges at the same time: \((r = |x - q_j|, \beta)\) can be identified with \(v_j\), \(q = e_j, \cos\psi = n \cdot v_j, r = rn, \gamma^2 = \frac{1}{(1 - \beta^2)}\)
\[
E = \frac{qr}{r^3(1 - \beta^2 \sin^2\psi)^{1/2}}. \quad (4.14)
\]

Therefore, our expressions (4.9,4.12) reproduce correctly usual results for the complete and transverse electric field outside the location of the charged particle.
4.4 Physical interpretation

The field that has just been computed is the field generated by a particle in uniform motion since the Coulomb interaction between the charged particles or an outside field is not taken into account in its contribution. That field is therefore equivalent to the field that can be deduced from the static Coulomb field through a Lorentz transformation and this corresponds indeed to our result. This point has to be viewed as a confirmation of the correctness of our alternative approach. The usual (relativistic) expression is recovered outside the location of the particle \( x = q_j \).

On physical grounds, a charged particle in free motion does not emit a field and should experiment no self-force. Our expression (4.9) is in accordance with that property since the self-field vanishes exactly at the location of the point charged particle. That property holds as well for the transverse field \( \langle \mathbf{E} \times (\mathbf{x}) \rangle \) as for the complete one. This explains why the corresponding terms in the kinetic operator \( \Theta \) (3.25) vanish.

5 The radiative reaction force due to the Coulomb interaction

In order to get a contribution to the reactive force due to the self-interaction of the particles, a mechanism of acceleration of the particles has to be provided. We have chosen to consider the Coulomb interaction between the charged particles as responsible for the acceleration. Other mechanisms are possible, such as the presence of a non-vanishing free field, or the consideration of the field emitted by the other particles but they are not treated here. We have to evaluate the elements of \( \tilde{\Sigma}(\hbar) \) (1,2) (corresponding to one Coulomb interaction and two interactions with the transverse fields) that provide a contribution to \( \langle 11[0]|\tilde{\Theta}|11[1(f)] \rangle^{(1,2)} \), when that operator acts on the field vacuum. The Coulomb interaction between the two particles can occur as the first, the second or the last interaction. Since we know that \( \langle 11[0]|\tilde{\Theta}|11[1(f)] \rangle^{(0,2)} \) provides a vanishing result when acting on the field vacuum, we expect that the only non vanishing contribution arises when the Coulomb interaction takes place between or after the interaction of the particles with the transverse field.
5.1 The subdynamics operator

Therefore, we first focus on (the lower index $FPF$ describes the order of the interactions):

\[
<11[0]|\hat{\Sigma}[1|1\{f\}]^{(1,2)}_{FPF} = -\frac{1}{2\pi i} \int_c \frac{dz}{z} e^{-itz} \sum_{j=1,2} \left( \frac{1}{z - k^{(j)}, \nu^{(j)} - k^{(j')}, \nu^{(j')}} \right) \\
\times (-i)e_j \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^3 k^{[1]} \int_0^{\infty} dn_1^{[1]} \int_0^{\infty} dn_2^{[1]} \sum_{m_1^{[1]}, m_2^{[1]}} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \sum_{\alpha=1, 2} \sum_{\beta=1, 2} \sum_{b=\pm 1} \left( \frac{n_1^{[1]} \delta_{l_2^{[1]}, \alpha} - n_2^{[1]} \delta_{l_2^{[1]}, \alpha}}{k^{[1]}} \right) \\
\times \left( \frac{k^{[1]} e^{(\alpha [1]}_{r} - g^{\star v_s^{(j)}}(e^{(\alpha [1]}_{r} k^{[1]}_{r} - e^{(\alpha [1]}_{r} k^{[1]}_{r}) \right) \frac{\partial}{\partial p^{(j)}} \right) \\
\times \exp \left\{ -k^{[1]} \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_1^{[1]}} \right\} \\
\times \left( \frac{1}{z - k^{(j)} \cdot \nu^{(j)} - k^{(j')}, \nu^{(j')} + k^{[1]}(m_1^{[1]} + m_1^{[1]}) \right) \\
\times e_j e_j' \frac{1}{8\pi^2} \int d^3 l^{1, 2} \left( \frac{\partial}{\partial p^{(j)}} - \frac{\partial}{\partial p^{(j')}} \right) \frac{1}{(2\pi)^{\frac{d}{2}}} \left( -\frac{\partial}{\partial k^{(j)}} \right) \\
\times \left( \frac{1}{z - k^{(j)} \cdot \nu^{(j)} - k^{(j')}, \nu^{(j')} + k^{[1]}(m_1^{[1]} + m_1^{[1]}) \right) \\
\times (-i)e_j \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{\beta=1, 2} \sum_{b=\pm 1} \left( \frac{n_1^{[1]} \delta_{l_2^{[1]}, \alpha} - n_2^{[1]} \delta_{l_2^{[1]}, \alpha}}{k^{[1]}} \right) \\
\times \left( -\pi(\nu^{(j)} \cdot e^{(\beta [1]}_{r} - g^{\star v_s^{(j)}}(e^{(\beta [1]}_{r} k^{[1]}_{r} - e^{(\beta [1]}_{r} k^{[1]}_{r}) \right) \frac{\partial}{\partial p^{(j)}} \right) \\
\times \exp \left\{ -k^{[1]} \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_1^{[1]}} \right\} \\
\times \left( \frac{1}{z - k^{(j)} \cdot \nu^{(j)} - k^{(j')}, \nu^{(j')} + k^{[1]}(m_1^{[1]} + m_1^{[1]}) \right), \tag{5.1} \right.
\]

This expression is very similar to the expression of $<11[0]|\hat{\Sigma}[1|1\{f\}]^{(0,2)}$ (3.10), but with the supplementary factors due to the Coulomb interaction: the matrix element (3.10) and a propagator. The order of all the
elements has to be strictly respected, on view of the presence of displacement and derivation operators. The contributions due to the different orders of interaction (\textit{FFP} and \textit{PFF}) are evaluated from similar expressions. We can proceed exactly as for the second order contribution \( <11[0]|\mathcal{A}|11[1](f)>^{(0,2)} \) for the extraction of \( \tilde{f}_{1[f]}^{V} \) \[ (3.26) \] for the presence of a denominator that can be resonant. The subdynamics theory has prescribed, from the beginning of its elaboration, that a propagator corresponding to a correlation state has to be treated with an \( i\epsilon \). A second difference is the consideration of \( <11[0]|\hat{\Theta}|11[1](f)>^{(1,2)} \) for the extraction of \( \hat{f}_{1[f]}^{V} \) from \( <11[0]|\hat{\Sigma}|11[1](f)> > \hat{f}_{1[f]}^{V} \) through \[ (3.4) \]. Moreover, we have to take into account, in the final simplifications, that the matrix elements associated with the Coulomb interaction and the field interaction do not commute. The terms corresponding to that case are affected by a lower index \( \Pi \), the other ones by an index \( I \).

5.2 Kinetic operator

Straightforward but very lengthy computations lead to the following expression (as expected, the \textit{PFF} order of interaction has provided a vanishing result)

\[
<11[0]|\hat{\Theta}|11[1](f)>^{(1,2)} \hat{f}_{1[f]}^{V} = \sum_{j=1,2} (-i) e_j^3 e_j' \left( \frac{1}{2\pi} \right)^3 \frac{1}{4\pi} \int d^3 k_{[1]} \int d^3 l_{1,2} \sum_{a=1,2} \sum_{a=\pm 1} \frac{1}{k_{[1]}} \left[ [k_{[1]} e_r^{(a)[1]} - g^{st} v_s^{(j)} (e_t^{(a)[1]} k_{t[1]} - e_r^{(a)[1]} k_{t[1]})] \frac{\partial}{\partial p_r^{(j)}} \right] \times \left( \frac{1}{i\epsilon + a k_{[1]} \cdot \mathbf{v}^{(j)} - a k_{[1]}} \right) \left( \frac{1}{i\epsilon + (1/2 a k_{[1]} \cdot \mathbf{v}^{(j)} - 1/2 l_{1,2} \cdot \mathbf{v}^{(j)} - a k_{[1]})} \right) \times l_{[1]} (-a k_{u}^{[1]} \frac{\partial v_r^{(j)}}{\partial p_r^{(j)}}) 2\pi (\mathbf{v}^{(j)} \cdot e^{(a)[1]} \mathbf{e}^2) \left( \frac{\partial}{\partial k_{[1]} \cdot \mathbf{v}^{(j)}} \right),
\]

\[ (5.2) \]

\[
<11[0]|\hat{\Theta}|11[1](f)>^{(1,2)} \hat{f}_{1[f]}^{V} = \sum_{j=1,2} (-i) e_j^3 e_j' \left( \frac{1}{2\pi} \right)^3 \frac{1}{4\pi} \int d^3 k_{[1]} \int d^3 l_{1,2} \sum_{a=1,2} \sum_{a=\pm 1} \frac{1}{k_{[1]}} \left[ [k_{[1]} e_r^{(a)[1]} - g^{st} v_s^{(j)} (e_t^{(a)[1]} k_{t[1]} - e_r^{(a)[1]} k_{t[1]})] \frac{\partial}{\partial p_r^{(j)}} \right] \times \left( \frac{1}{i\epsilon + a k_{[1]} \cdot \mathbf{v}^{(j)} - a k_{[1]}} \right) \left( \frac{1}{i\epsilon + (1/2 a k_{[1]} \cdot \mathbf{v}^{(j)} - 1/2 l_{1,2} \cdot \mathbf{v}^{(j)} - a k_{[1]})} \right) \times l_{[1]} (-a k_{u}^{[1]} \frac{\partial v_r^{(j)}}{\partial p_r^{(j)}}) 2\pi (\mathbf{v}^{(j)} \cdot e^{(a)[1]} \mathbf{e}^2) \left( \frac{\partial}{\partial k_{[1]} \cdot \mathbf{v}^{(j)}} \right),
\]

\[ (5.2) \]
\begin{equation}
\times \left( \frac{1}{i\epsilon + (\frac{1}{2}l + ak[1]).v(j) - \frac{1}{2}l.v(j') - ak[1]} \right) \left( \frac{1}{i\epsilon + ak[1].v(j) - ak[1]} \right)
\end{equation}

\begin{equation}
\times l_v \partial v_u \epsilon_{(0)1} e^{(\alpha)[1]} \frac{1}{2} \left( \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial k^{(j')}} \right).
\end{equation}

The order of integration respects the ordering of the apparition of the vertices: in all remaining contributions, the integration over the field modes \( k^{[1]} \) has to be performed after the integration over the wave number \( l \) exchanged by the Coulomb interaction. An opposite order of integration would have been required in the contribution involving the order \( PFF \) of the vertices.

For the sake of completion, let us compute \( \left( \frac{\partial v_u}{\partial p_v} \right) \). We have \( \mathbf{p} = \frac{m\mathbf{v}}{1-v^2} \), from which we deduce \( \left( \frac{\partial v_u}{\partial p_v} \right) = \frac{\delta^{K_{uv}}}{m^2 + (p^{(j)} - p^{(j')})^2} - \frac{(p^{(j)} - p^{(j')})}{m^2 + (p^{(j)} - p^{(j')})^2} \).

The partial derivative \( \frac{\partial}{\partial p_v^{(j)}} \) can be placed in front of the matrix element since the contribution when it acts on the factor \( v_s^{(j)} \) provide a vanishing result. The property can be checked explicitly. Taking into account the value of the matrix tensor \( g_{st} = \delta^{K_{sr}} \), the first term of the derivative, with a \( \delta^{K_{sr}} \), can be seen to involve the scalar product of the vectors \( e^{(\alpha)[1]} \) and \( k^{[1]} \) that vanishes by definition of the polarisation vector. The second contribution, with the product \( p_u^{(j)} p_v^{(j)} \), vanishes by symmetry. This result is not unexpected and reflects that the magnetic force is orthogonal to the velocity vector.

This form (5.2) shows clearly that the norm is not affected by that contribution to the equations of motion. Indeed, the partial derivative \( \frac{\partial}{\partial p_v^{(j)}} \) ensures that the whole contribution vanishes when integrated over \( p_v^{(j)} \). We have the same structure that for the contribution (7.15) of 1 or for the operator \( <1[0]|\tilde{\Theta}|1[1(f)] >^{(0,2)} \).

Since the first propagator in (5.2) and the second one in (5.3) cannot be resonant, the \( i\epsilon \) can be dropped from them.

We do not yet analyse the possible divergence of these contributions to \( <1[0]|\tilde{\Theta}|1[1(f)] >^{(1,2)} \).

5.3 Evolution of the distribution function

We consider anew the case in which the two particles \( j \) and \( j' \) are perfectly localised with a well defined momentum (4.7). If we perform the trivial
integrations, due to simplified form of the distribution function, we get:

\[ \partial_t \tilde{f}_{11}[\eta(q^{(1)}, p^{(1)}, q^{(2)}, p^{(2)})]_{\theta T} \]

\[ = \frac{i}{(2\pi)^3} \frac{1}{4\pi} \sum_{j=1,2} e_j^3 e_{j'} \int d^3 k^{[1]} \int d^3 l \frac{1}{l^2} \sum_{\alpha=1,2} \sum_{a=\pm 1} \frac{1}{k^{[1]}} \]

\[ \times \left[ [k^{[1]} e_r^{(\alpha)[1]} - g^{st} v_s^{(j)} (e_t^{(\alpha)[1]} k^{[1]} - e_r^{(\alpha)[1]} k^{[1]})] \frac{\partial}{\partial p_r^{(j)}} \right] \]

\[ \times \left( \frac{1}{k^{[1]} \cdot v^{(j)} - k^{[1]}} \right)^2 \left( \frac{1}{i\epsilon + \left( \frac{1}{2} l + ak^{[1]} \right) \cdot v^{(j)} - \frac{1}{2} l \cdot v^{(j')} - ak^{[1]} } \right) \]

\[ \times a l_u k_u^{[1]} \frac{\partial v_u^{(j)}}{\partial p_v^{(j)}} (v^{(j)} \cdot e^{(\alpha)[1]}) e^{-i\frac{1}{2} l \cdot (q_j - q_j')} \]

\[ \times \delta(q^{(j)} - q_j) \delta(q^{(j')} - q_j') \delta(p^{(j)} - p_j) \delta(p^{(j')} - p_j'), \quad (5.4) \]

\[ \partial_t \tilde{f}_{11}[\eta(q^{(1)}, p^{(1)}, q^{(2)}, p^{(2)})]_{\theta T} \]

\[ = \frac{i}{(2\pi)^3} \frac{1}{4\pi} \sum_{j=1,2} e_j^3 e_{j'} \int d^3 k^{[1]} \int d^3 l \frac{1}{l^2} \sum_{\alpha=1,2} \sum_{a=\pm 1} \sum_{\pm} \]

\[ \times \left\{ [k^{[1]} e_r^{(\alpha)[1]} - g^{st} v_s^{(j)} (e_t^{(\alpha)[1]} k^{[1]} - e_r^{(\alpha)[1]} k^{[1]})] \frac{\partial}{\partial p_r^{(j)}} \right\} \]

\[ \times \left( \frac{1}{i\epsilon + \left( \frac{1}{2} l + ak^{[1]} \right) \cdot v^{(j)} - \frac{1}{2} l \cdot v^{(j')} - ak^{[1]} } \right) \left( \frac{1}{ak^{[1]} \cdot v^{(j)} - ak^{[1]} } \right) \]

\[ \times a l_u k_u^{[1]} e_u^{(\alpha)[1]} e^{-i\frac{1}{2} l \cdot (q_j - q_j')} \delta(q^{(j)} - q_j) \delta(q^{(j')} - q_j') \]

\[ \times \delta(p^{(j)} - p_j) \delta(p^{(j')} - p_j'). \quad (5.5) \]

5.4 Radiative reaction force

The effect of the coupling of the Coulomb interaction with the field is to provide a supplementary force, the radiative reaction force, that changes the mean value of the momentum of one particle. The expression of the \( r \) component \( F_r^{(j)} \) of the radiative reaction force can be obtained by considering the relation \( F_r^{(j)} = \frac{d}{dt} < p_r^{(j)} > \). By a partial derivative, we get that the \( r \) component \( F_r^{(j)} \) is provided by minus the coefficient of the expression \( 5.4 \) when the partial derivative \( \frac{\partial}{\partial p_r^{(j)}} \) is removed and where the variables
\textbf{q}^{(j)}, \textbf{q}^{(j')}, \textbf{p}^{(j)}$ and $\textbf{p}^{(j')}$ are replaced by their values obtained from the Dirac delta functions.

$$< F^{(j)} _r > I = -i \frac{1}{(2 \pi)^3} \frac{1}{4 \pi} \epsilon_3 \epsilon_3' \int d^3 k^{[1]} \int d^3 l \frac{1}{l^2} \sum_{\alpha =1,2} \sum_{a = \pm 1} \frac{1}{k^{[1]}} e^{-i \frac{1}{2} l \cdot (\textbf{q}_j - \textbf{q}_v)}$$

$$\times [k^{[1]} e_\alpha (\alpha)[1] - g^{st} v_{js} (e_\alpha (\alpha)[1] k^{[1]}_r - e_\alpha (\alpha)[1] k^{[1]}_r)] \left( \frac{1}{k^{[1]} \cdot \textbf{v}_j - k^{[1]}} \right)$$

$$\times \left( \frac{1}{ie + (\frac{1}{2} l + a k^{[1]} \cdot \textbf{v}_j - \frac{1}{2} l \cdot \textbf{v}_j' - ak^{[1]}} \right)$$

$$\times \left[ \frac{\delta K^r_{u,v}}{(m^2_j + p^2_j')^2} - \frac{(p_{ju} p_{jv})}{(m^2_j + p^2_j')^2} \right] \left( \textbf{v}_j \cdot \textbf{e}^{(\alpha)}[1] \right),$$

(5.6)

$$< F^{(j)} _r > II = -i \frac{1}{(2 \pi)^3} \frac{1}{4 \pi} \epsilon_3 \epsilon_3' \int d^3 k^{[1]} \int d^3 l \frac{1}{l^2} \sum_{\alpha =1,2} \sum_{a = \pm 1} e^{-i \frac{1}{2} l \cdot (\textbf{q}_j - \textbf{q}_v)}$$

$$\times \frac{1}{k^{[1]}} \left[ [k^{[1]} e_\alpha (\alpha)[1] - g^{st} v_{js} (e_\alpha (\alpha)[1] k^{[1]}_r - e_\alpha (\alpha)[1] k^{[1]}_r)] \right]$$

$$\times \left[ \left( \frac{1}{ie + (\frac{1}{2} l + a k^{[1]} \cdot \textbf{v}_j - \frac{1}{2} l \cdot \textbf{v}_j' - ak^{[1]}} \right) \left( \frac{1}{a k^{[1]} \cdot \textbf{v}_j - ak^{[1]}} \right) \right]$$

$$\times l_{u} \frac{\partial v^{(j)} _u}{\partial p^{(j')}_v} e_\alpha (\alpha)[1].$$

(5.7)

We focus first on the first contribution. The value of that radiative reaction force depends on the relative orientation of the vectors position and momentum. We explicit the summations over $u$ and $v$. We get then, using the value of the metric tensor $g^{st}$ to replace $g^{st} v_{js} e_\alpha (\alpha)[1] k^{[1]}_r$ by $- \textbf{v} \cdot \textbf{e}^{(\alpha)}[1] k^{[1]}_r$ and $g^{st} v_{js} e_\alpha (\alpha)[1] k^{[1]}_r$ by $- \textbf{v} \cdot \textbf{k}^{[1]} e_\alpha (\alpha)[1]$:

$$< F^{(j)} _r > I = -i \frac{1}{(2 \pi)^3} \frac{1}{4 \pi} \epsilon_3 \epsilon_3' \int d^3 k^{[1]} \int d^3 l \frac{1}{l^2} \sum_{\alpha =1,2} \sum_{a = \pm 1} a \frac{1}{k^{[1]}} e^{-i \frac{1}{2} l \cdot (\textbf{q}_j - \textbf{q}_v)}$$

$$\times [(k^{[1]} - \textbf{v}_j \cdot \textbf{k}^{[1]} e_\alpha (\alpha)[1] + \textbf{v}_j \cdot \textbf{e}^{(\alpha)}[1] k^{[1]}_r)]$$

$$\times \sum_{u,v} l_{u} k^{[1]}_u \left[ \frac{\delta K^r_{u,v}}{(m^2_j + p^2_j')^2} - \frac{(p_{ju} p_{jv})}{(m^2_j + p^2_j')^2} \right] \left( \textbf{v}_j \cdot \textbf{e}^{(\alpha)}[1] \right)$$

$$\times \left( \frac{1}{k^{[1]} \cdot \textbf{v}_j - k^{[1]}} \right)^2 \left( \frac{1}{ie + (\frac{1}{2} l + a k^{[1]} \cdot \textbf{v}_j - \frac{1}{2} l \cdot \textbf{v}_j' - ak^{[1]}} \right).$$

(5.8)
The reality of this expression can be checked by considering the symmetry $a \rightarrow -a$, $l \rightarrow -l$.

### 5.5 Emitted power

We distinguish the components of the radiative reaction force in the direction parallel and perpendicular to the velocity $v_j$ of the $j$ particle. The power emitted is given by $\langle \mathbf{F}^{(j)} \cdot v_j \rangle$. As can be seen, the magnetic force, arising from $-g^{st}v_{js}(e_t^{(0)[1]}k^{[1]}_r - e_r^{(0)[1]}k^{[1]}_t)$ does not contribute. The force parallel to $q_j - q_{j'}$ provides a radiative correction to the Coulomb force that is not considered here. We use $\sum_{\alpha=1,2}(v_j.e^{(0)[1]})(v_j.e^{(0)[1]}) = v_j^2 - \frac{(v_j.k^{[1]})^2}{(k^{[1]})^2}$ to obtain

\[
\langle \mathbf{F}^{(j)} \cdot v_j \rangle = -i \frac{1}{(2\pi)^3} \frac{e^3_j e^{(0)}_{j'}}{4\pi} \int d^3k^{[1]} \int d^3l \frac{1}{\epsilon^2} \sum_{a=\pm 1} a \frac{1}{k^{[1]}} e^{-i\frac{1}{2} l \cdot (q_j - q_{j'})} \times k^{[1]} \left[ v_j^2 - \frac{(v_j.k^{[1]})^2}{(k^{[1]})^2} \right] \left[ \frac{1}{(m_j^2 + p_j^2)\frac{1}{2}} - \frac{(1.p_j)(p_j.k^{[1]})}{(m_j^2 + p_j^2)\frac{1}{2}} \right] \\
\times \left( \frac{1}{k^{[1]}.v_j - k^{[1]}} \right)^2 \left( \frac{1}{ie + (\frac{1}{2} l + ak^{[1]}).v_j - \frac{1}{2} l . v_{j'} - ak^{[1]}} \right)
\]  

(5.9)

That expression is further analyzed in Appendix D, particularly in the situation where the particle $j'$ is much more heavy that the $j$ particle. In the referential in which the heavy particle is at rest at the origin of coordinates, we treat the case where the vectors $q_j$ and $v_j$ are orthogonal (the orbital situation). In such a case, all integrals can be performed explicitly and the final result is

\[
\langle \mathbf{F}^{(j)} \cdot v_j \rangle_{\text{orb}} = \frac{4}{3} e^3_j e^{(0)}_{j'} \frac{m_j^2}{(m_j^2 + p_j^2)^{\frac{1}{2}}} \frac{v_j^2}{q_j^2} \frac{1}{(1 - v_j^2)^3}
\]  

(5.10)

\[
\langle \mathbf{F}^{(j)} \cdot v_j \rangle_{\text{IIorb}} = -\frac{1}{2} e^3_j e^{(0)}_{j'} \frac{m_j^2}{(m_j^2 + p_j^2)^{\frac{1}{2}}} \frac{1}{q_j^2} \frac{1}{v_j} \left[ \ln \frac{1 - v_j}{1 + v_j} + \frac{2v_j}{(1 - v_j^2)} \right]
\]  

(5.11)

These expressions enable to determine the component of the self electric field at the localisation of the particle:

\[
\langle \mathbf{E}^\perp (q_j) \cdot v_j \rangle_{\text{orb}} = e^3_j e^{(0)}_{j'} \frac{m_j^2}{(m_j^2 + p_j^2)^{\frac{1}{2}}} \frac{1}{q_j^2}
\]
\[
\times \left[ -3 + 10v_j^2 - 3v_j^4 \right] - \frac{1}{2v_j} \ln \frac{1 - v_j}{1 + v_j} \right]^{(5.12)}
\]

For the geometry chosen, the radiative reaction force is known exactly by an explicit expression.

### 5.6 Non-relativistic limit of the emitted power

The previous expression can be developed in powers of \( v_j^2 \) to make the connection with the well known result. We have to consider the expression up to order \( v_j^2 \). The result is:

\[
< E_{\perp}(q_j) \cdot v_j >_{NR}^{e_j(1,1)} = \frac{2 e_{j}^{2} e_{j^{'}}}{3 m_j q_j^3} v_j^2
\]

(5.13)

The coulombian acceleration of the charge \( j \) is provided by the dynamical function \( \frac{F_j}{m_j} = \frac{e_j e_j^{'}}{m_j q_j^3} \). The mean value of its time derivative, due to the free motion (2.9), is

\[
\partial_t < F_{j}(q_j) \cdot v_j > = \frac{e_j e_j^{'}}{m_j q_j^3} \frac{d < a_{cj} >}{dt} \times \delta(q_j(q_j) - q_j) \delta(p_j - p_j)
\]

(5.14)

In the geometry where the vectors \( q_j \) and \( v_j \) are perpendicular, we have then \( \frac{d}{dt} < a_{cj} > = \frac{e_j e_j^{'}}{m_j q_j^3} \). Therefore, we get the form

\[
< F^{(j)} \cdot v_j >_{NR orb} = \frac{2 e_j^2 c_j^2}{3 m_j q_j^3} < \frac{d}{dt} a_{cj} > \cdot v_j
\]

(5.15)

If we restore the dimensions, we get

\[
< F^{(j)} \cdot v_j >_{NR orb} = \frac{2 e_j^2 c_j^2}{3 c^3 m_j q_j^3} \frac{d}{dt} a_{cj} > \cdot v_j
\]

(5.16)

The usual result, with a front factor \( \frac{2}{3} \), is recovered directly, without having met any divergence for that contribution. This result is not astonishing. In the usual approach, the divergence appears as an infinite self-mass correction, in front of the acceleration vector. Since we have considered the
geometry where the Coulomb force (hence the acceleration) is perpendicular to the velocity, that divergence has no influence on the emitted power. In the general case, the contribution $< F^{(j)} >_{II}$ would provide the usual divergence.

6 Emitted field due to the Coulomb interaction

6.1 Subdynamics operator

The determination of the emitted field due to the Coulomb interaction requires the determination of the creation operator in the first order in both the Coulomb and the transverse field interactions. The starting expression is:

$$< 11[1(s_j)] | \bar{\Sigma}(t) | 11[1(f)] >^{(1,1)}_{PF}$$

$$= \frac{-1}{2\pi i} \int_c dz e^{-izt} \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} + k^{[1]} (m_{\alpha}^{[1]} + m_{\alpha'}^{[1]})} \right)$$

$$\times (-i) e^j e^{j'} \frac{1}{(2\pi)^{\frac{5}{2}}} \frac{1}{12} \frac{1}{l^2} \left( \frac{\partial}{\partial p^{(j)}} - \frac{\partial}{\partial p^{(j')}} \right) e^{\left( \frac{\partial^2}{\partial k^{(j)} - \partial k^{(j')}} \right)}$$

$$\times \left( z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} + k^{[1]} (m_{\alpha}^{[1]} + m_{\alpha'}^{[1]}) \right)$$

$$\times (-i) e^j \frac{1}{(2\pi)^{\frac{5}{2}}} \sum_{\beta=1, 2} \sum_{b=\pm 1} \left( \eta_\beta^{[1]} \right)^{\frac{1}{2}}$$

$$\times \left[ [k^{[1]} e^{(\beta)}]_{\alpha} - g^{\mu'} v^{(j)} (e^{(\beta)}_{\mu'})_{\alpha'} - (\beta_{\alpha})_{\alpha'} [k^{[1]}]_{\alpha'} \right] \frac{\partial}{\partial p^{(j)}}$$

$$- \pi (v^{(j)} \cdot e^{(\beta)}) \left( \frac{\partial}{\partial \eta^{[1]}_{\beta}} - \frac{b}{\eta^{[1]}_{\beta}} (m_{\beta}^{[1]} - b) \right) \exp b \left\{ -k^{[1]} \cdot \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_{\beta}^{[1]}} \right\}$$

$$\times \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} + k^{[1]} (m_{\beta}^{[1]} + m_{\beta'}^{[1]})} \right) ; \quad (6.1)$$

for the order $PF$ and a similar expression for the order $FP$. For the order $PF$, only the first propagator, at the extreme right, corresponds to a vacuum state while in the other order, $FP$, the first two propagators satisfy that condition and have to be considered inside the path $c$. We can proceed in a straightforward way, as in §5.

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Since the operator $\tilde{A}$ (3.4) can only deviate from unity when two field interactions take place, the expression of $<11|1(s_j)||\Sigma(0)|11|1(f)>^{(1,1)}\tilde{f}_1^{(1)}$ can be identified with the corresponding term $<11|1(s_j)||\tilde{C}|11|1(f)>^{(1,1)}\tilde{f}_1^{(1)}$. As the equivalence conditions imply $\tilde{f}_{11}(\epsilon_j) = \tilde{f}_{11}(s_j)$, the distribution function for the emitted field at first order in the field interaction and first order in the Coulomb interaction is determined in this way.

### 6.2 Transverse emitted field

We have therefore all the elements to deduce the emitted field (for sharp locations and momenta for the particles)

\[
<\mathbf{E}_r^\perp(\mathbf{x})>^{e_j(1,1)} = \int d^3k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm1} k^{[1]}_j \frac{1}{2\pi^2} e_\alpha^0(\mathbf{k}^{[1]}) \exp\{ia[\mathbf{k}^{[1]}] \cdot \mathbf{x}\} e_j^2 e_j^{e_j} \frac{1}{2\pi^2}
\]

\[
\times \frac{1}{(2\pi)^3} \int d^3l \left( \frac{1}{i\epsilon + (1 + a\mathbf{k}^{[1]}), \mathbf{v}_j - 1, \mathbf{v}_j' - a\mathbf{k}^{[1]})} \right) \left( \frac{1}{a\mathbf{k}^{[1]} \cdot \mathbf{v}_j - a\mathbf{k}^{[1]} \cdot \mathbf{v}_j'} \right)
\]

\[
\times \left( \frac{1}{k^{[1]}} \right)^\frac{1}{2} l_v(ak_u^{[1]}) \frac{\partial v_\nu}{\partial p_{j\nu}} 2\pi \mathbf{v}_j(\epsilon^{(\alpha)[1]}_j) e^{-i(1 + a\mathbf{k}^{[1]} \cdot \mathbf{q}_j - 1, \mathbf{q}_j')}
\]

\[
+ \int d^3k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm1} k^{[1]}_j \frac{1}{2\pi^2} e_\alpha^0(\mathbf{k}^{[1]}) \exp\{ia[\mathbf{k}^{[1]}] \cdot \mathbf{x}\} (-) e_j^2 e_j^{e_j} \frac{1}{2\pi^2}
\]

\[
\times \frac{1}{(2\pi)^3} \int d^3l \left( \frac{1}{i\epsilon + (1 + a\mathbf{k}^{[1]}), \mathbf{v}_j - 1, \mathbf{v}_j' - a\mathbf{k}^{[1]})} \right) \left( \frac{1}{a\mathbf{k}^{[1]} \cdot \mathbf{v}_j - a\mathbf{k}^{[1]} \cdot \mathbf{v}_j'} \right)
\]

\[
\times \left( \frac{1}{k^{[1]}} \right)^\frac{1}{2} 2\pi l_v(\epsilon^{(\alpha)[1]}_j) e^{-i(1 + a\mathbf{k}^{[1]} \cdot \mathbf{q}_j - 1, \mathbf{q}_j')}. \tag{6.2}
\]

This new expression is the equivalent of (4.8) in presence of the Coulomb interaction. It determines the field due to the accelerated particles in terms of the actual values of the position $\mathbf{q}_j$ and momentums $\mathbf{p}_j$ of the charged particles. Usually, expressions of the acceleration fields are given in terms of the retarded positions. We look for the comparison only for the radiative force, since we have illustrated in §4 the equivalence of the formalisms outside the locations of the particles. The self-field of the particle due to the Coulomb interaction, is then given at first order by

\[
<\mathbf{E}_r^\perp(\mathbf{q}_j)>^{e_j(1,1)} = -e_j^2 e_j^{e_j} \frac{1}{(2\pi)^2} \int d^3k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm1} e_\alpha^0(\mathbf{k}^{[1]})
\]

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Using (3.14), that expression can be identified with the result obtained from the $\tilde{\Theta}$ operator (in the previous section) that leads to the usual expression for the self-force in the low velocity limit.

7 Finite classical electrodynamics.

7.1 General outline of the approach

We intend to prove that resummations, usual in the context of statistical physics, enables to get rid of the divergences in the kinetic operator, computed from the expression of the self-field at the location of the particle. These resummations involve a renormalisation of the propagators associated with the particles. Our analysis proceeds through several steps.

In the first one, the previous divergent contribution for the self-field is written in an adequate way. A linear dependence in a cut-off wave number $K_c$ is found, as expected. A usual diagrammatic representation, such that it can be found in [3] for instance, is convenient. The dependence of a simple cycle on a cut-off wave number $K_c$ is then analysed and established. The two propagators of the previously divergent contribution are then renormalised using simple cycles. When their dependence in $K_c$ is reported in the expression of the self-field, it induces a supplementary $\frac{1}{K_c^2}$ and the resulting contribution vanishes!

In the second step, we use cycles that are renormalised by themselves: the propagator present in the expression of the cycles is a renormalised one. That first resummation contains only disjoint or inserted cycles in arbitrary numbers, with no overlap: two vertices of a cycle are either disjoint or
completely inserted with respect to two vertices of another one. The contribution of the cycle is thus defined in a self-consistent way. The resulting dependence of the renormalised cycle in $K_c$ is non analytic, in $K_c^{\frac{1}{2}}$. This result of the self consistency can be understood in a simple way. The integral defining the bare cycle behaves as $K_c$. The renormalised cycle involves one propagator that, by hypothesis, provides a factor $K_c^{-\frac{1}{2}}$. When combined, their product reproduces correctly the assumed $K_c^{\frac{1}{2}}$ dependence. When that value is introduced into the expression of the self-field, each of the two propagators provide a factor $K_c^{-\frac{1}{2}}$ that compensates the $K_c$ dependence due to the integration. Therefore, the limit $K_c \to \infty$ is well defined.

That property of the renormalised cycles can be extended to all contributions: For imbricated vertices, each new wave number will be associated with two propagators, that can be renormalised by (renormalised) cycles. The dependence of these new vertices with $K_c$ is then the same as the the simple cycle. Therefore, they can also be used for a global renormalisation while preserving the finiteness of the self-field.

7.2 The previous divergent contribution

The correlated elements describing the self-field are easily obtained from the expression of the creation operator. We have to focus on the divergent contribution to the the self-field. Its computation starts from the expression of $<11|1(s_j)||\Sigma(t)||11|1(f)>^{(1,1)}$ (6.1) and we have to reconsider it in order to be able to add the contributions of higher order that will ensure its finiteness.

The final dependence in the cut-off wave number $K_c$ has to be appreciated after the a last vertex, enabling to get $\tilde{\Theta}$ or the self-field from the creation operator, has been introduced. A displacement operator involving the wave number exchanged through the Coulomb interaction is natural and will remain in our expressions. We keep the dependence in particle operator that has provided, in the previous “$\Pi$” contribution, the ultraviolet divergence, in the computation of the self-interaction. We still restrict ourselves to the effect of the absence of fields: all strenghts of the fields oscillators are close to zero: the distribution function of the action variables of the field oscillator are peaked around zero (the limit is taken afterwards).

We reconsider the contribution, affected by a $d$-index, responsible for the divergence, from the first steps of its computation. In the expression (6.1),
the operator $1. \left( \frac{\partial}{\partial p^\alpha} - \frac{\partial}{\partial p^\beta} \right)$ has to act on $-\pi(v^{(j)} \cdot e^{(\beta)[1]})$. Moreover, in
the factor $\left( 2 \frac{\partial}{\partial \eta^{(j)}_\alpha} - \frac{b}{\eta^{(j)}_\alpha} (m^{[1]}_\alpha - b) \right)$, only the derivative will provide a non-vanishing contribution for the self-interaction.

$$< 11[1(s_j)]|\hat{\Sigma}(t)|11[1(f)] >^{(1,1)}_d$$

$$= \frac{-1}{2\pi i} \int'_c dz \ e^{-izt} \ \frac{1}{(z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} + k^{[1]}(m^{[1]}_\alpha + m^{[1]}_\beta))}$$

$$\times (-i) e_j e_j' \ \frac{-1}{2\pi^2} \int d^3l \ \frac{1}{l^2} e^{i \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial k^{(j')}}}$$

$$\times \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} + k^{[1]}(m^{[1]}_\alpha + m^{[1]}_\beta)} \right)$$

$$\times (-i) e_j \ \frac{1}{(2\pi)^2} \sum_{\beta=1,2} \sum_{b=\pm 1} \left( \frac{\eta^{[1]}_\beta}{k^{[1]}} \right)^{\frac{1}{2}}$$

$$\times \left[ - \left( \frac{\partial}{\partial p^{(j)}} \pi(v^{(j)} \cdot e^{(\beta)[1]}) \right) \ 2 \frac{\partial}{\partial \eta^{[1]}_\beta} \right] \ \exp b \ \left\{ -k^{[1]} \cdot \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m^{[1]}_\beta} \right\}$$

$$\times \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k^{(j')} \cdot v^{(j')} + k^{[1]}(m^{[1]}_\beta + m^{[1]}_\beta)} \right).$$

(7.1)

That expression has to act on the distribution function $\hat{f}^{V}_{[1(f)]}$ describing the absence of field. We change the place of the displacement operators (in variables) to obtain:

$$< 11[1(s_j)]|\hat{\Sigma}(t)|11[1(f)] >^{(1,1)}_d \tilde{f}^{V}_{[1(f)]}$$

$$= (-i) e_j \ \frac{1}{(2\pi)^2} \sum_{\beta=1,2} \sum_{b=\pm 1} \left( \frac{\eta^{[1]}_\beta}{k^{[1]}} \right)^{\frac{1}{2}} \ \exp b \ \left\{ -k^{[1]} \cdot \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m^{[1]}_\beta} \right\}$$

$$\times \frac{-1}{2\pi i} \int'_c dz \ e^{-izt} \ \frac{1}{(z - (k^{(j)} + bk^{[1]}), v^{(j)} - k^{(j')} \cdot v^{(j')} + bk^{[1]}))}$$

$$\times (-i) e_j e_j' \ \frac{-1}{2\pi^2} \int d^3l \ \frac{1}{l^2} \left[ - \left( \frac{\partial}{\partial p^{(j)}} \pi(v^{(j)} \cdot e^{(\beta)[1]}) \right) \ 2 \frac{\partial}{\partial \eta^{[1]}_\beta} \right]$$

$$\times \left( \frac{1}{z - (k^{(j)} + 1 + bk^{[1]}), v^{(j)} - (k^{(j')} - 1) \cdot v^{(j')} + bk^{[1]})} \right).$$

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\[ 1 \times \left( \frac{1}{z - (k^{(j)} + l), v^{(j)} - (k^{(j')} - l), v^{(j')}} \right) \]
\[ \times e \left( \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial k^{(j')}} \right) \delta(\eta[1]_\beta - \eta[1]_\beta') \delta(\eta[2]_{\beta'} - \eta[2]_{\beta'}) \delta^{K_r}{m[1]_{\beta}}_0 \delta^{K_r}{m[1]_{\beta'}}_0. \]

The previous creation operator, enabling the computation of the self-field is obtained from that expression by picking the residue at the pole corresponding to the last propagator.

\[ \left\langle E_{\perp}^* (q_j) > \right\rangle_{(1,1)}^{(0,2)} = \epsilon_j^{(1,1)} \epsilon_j^{(1,1)} \frac{1}{\pi} \frac{1}{(2\pi)^3} \int d^3k^{[1]} \sum_{a=1,2} \sum_{a=\pm 1} e^{(1)}_\alpha (k^{[1]}) \]
\[ \times \int d^3k \frac{1}{E^2} \left( \frac{1}{i\epsilon + (1 + a k^{[1]}), v - 1, v - a k^{[1]}} \right) \]
\[ \times \left( \frac{1}{a k^{[1]}, v_j - a k^{[1]}} \right) \right\rangle_{(1,1)}^{(0,2)} \int d^3l e^{(0)}_\alpha (l) \left( \partial_{p_{j\nu}} \right) e^{(0)}_\alpha (l) \frac{1}{l_{\nu}} \int d^3l e^{(0)}_\alpha (l) \left( \partial_{p_{j\nu}} \right) e^{(0)}_\alpha (l) \]

Each propagator provides a decreasing \( \frac{1}{k^{[1]}} \) factor while the jacobian to get to spherical coordinates provide a \( (k^{[1]})^2 \) factor, hence the linear divergence.

### 7.3 The simple cycle

We intend to renormalise to propagators through simple cycles in a first step. In order to be acquainted with their contribution, we explicit the operator \( \langle 11[0]\left| \tilde{R}(z) \right| 11[1(f)] \rangle_{(0,2)}^{(0,2)} \) that contains only the cycle. The two interactions involve the same particle \( j \).

We consider only contributions that will not contain displacement operators involving the wave number of the interacting field mode acting on the distribution function. Indeed, in their presence, a convergence factor for ultraviolet modes would arise from the dependence of the involved distribution function in that wave number. The selected terms act as \( c \)-number in the particles variables, hence the \( c \)-index. Those contributions are the one responsible for the divergences since the other contributions involve characteristics of the distribution function. In the contribution \( (3.13) \), we have thus \( b = -a \) and \( \frac{\partial}{\partial p_{j\nu}} \) has to act on \( (v^{(j)}, e^{(0)}[1]) \). We act on the field vacuum distribution function \( \tilde{f}^{(1)}_{[1(f)]} \) to get:

\[ \left\langle 11[0]\left| \tilde{R}(z) \right| 11[1(f)] \right\rangle_{(0,2)}^{(0,2)} \tilde{f}^{(1)}_{[1(f)]} \]
\[
\left( \frac{1}{z - \mathbf{k}^{(j)} \cdot \mathbf{v}(j) - \mathbf{k}^{(j')} \cdot \mathbf{v}(j')} \right) (-i) c_j \frac{1}{(2\pi)^{3/2}} \int d^3 k^{[1]} \int_0^\infty d\eta^{[1]}_1 \int_0^\infty d\eta^{[1]}_2 \\
\times \sum_{m^{[1]}_1, m^{[1]}_2} \frac{\delta_{m^{[1]}_1,0} \delta_{m^{[1]}_2,0}}{\alpha = 1, 2} \sum_{a = 1, 2} \sum_{a = \pm 1} \frac{\eta^{[1]}_{\alpha} \eta^{[1]}_1}{k^{[1]}_1} \left[ [k^{[1]}_1 e^{(\alpha)}_r] - \mathbf{g}^a v^{(j)}_s (e^{(\alpha)}_r k^{[1]}_r) \right] \\
\times -e^{(\alpha)}_r k^{[1]}_t \frac{\partial}{\partial p^{(j)}_r} \left[ \exp \left\{ -[k^{[1]}_1 e^{(\alpha)}_r] - \mathbf{g}^a v^{(j)}_s (e^{(\alpha)}_r k^{[1]}_r) \right\} \right] \\
\times \frac{1}{(2\pi)^{3/2}} \left[ \int d^3 k^{[1]} \int_0^\infty d\eta^{[1]}_1 \int_0^\infty d\eta^{[1]}_2 \right]
\]
\[
\times \left[ \delta(\eta^{[1]}_\beta - \eta^{[1]}_\beta) \delta(\eta^{[1]}_\beta - \eta^{[1]}_\beta) \delta^{K_r}_{m^{[1]}_1,0} \delta^{K_r}_{m^{[1]}_2,0} \right]. \quad (7.4)
\]

When acting on the field in its ground state, to obtain a non vanishing contribution, we need an interaction with the same mode ($\alpha = \beta$). The summation-integration over field variables can then be performed:

\[
< 11|0| \mathcal{R}(z)|11|1(f) >_{c}^{(0,2)} F^{V}_{1(f)} = \left( \frac{1}{z - \mathbf{k}^{(j)} \cdot \mathbf{v}(j) - \mathbf{k}^{(j')} \cdot \mathbf{v}(j')} \right) c_j^2 \frac{1}{(2\pi)^{3/2}} \int d^3 k^{[1]} \sum_{a = 1, 2} \sum_{a = \pm 1}
\times \frac{1}{k^{[1]}_1} \left[ [k^{[1]}_1 e^{(\alpha)}_r] - \mathbf{g}^a v^{(j)}_s (e^{(\alpha)}_r k^{[1]}_r) \right] \\
\times \left( \int d^3 k^{[1]} \int_0^\infty d\eta^{[1]}_1 \int_0^\infty d\eta^{[1]}_2 \right)
\times \left[ -2\pi \left( \frac{\partial}{\partial p^{(j)}_r} (\mathbf{v}(j) \cdot e^{(\alpha)} [1]) \right) \left( \frac{1}{z - \mathbf{k}^{(j)} \cdot \mathbf{v}(j) - \mathbf{k}^{(j')} \cdot \mathbf{v}(j')} \right) \right]. \quad (7.5)
\]
We define the cycle contribution $C(z, \nu^{(j)})$ by

$$C(z, \nu^{(j)}) = e^{2j} \frac{1}{(2\pi)^3} \int d^3k^{[1]} \sum_{a^{[1]}=1,2} \sum_{a=\pm 1} \frac{1}{k^{[1]}} \times \left[ [k^{[1]} e^{(\alpha)[1]} - g^{st} e^{(\alpha)[1]} (e^{(\alpha)[1]} k^{[1]} - e^{(\alpha)[1]} k^{[1]})] \right] \times \left( \frac{1}{z + ak^{[1]} \nu^{(j)} + k^{[1]}(-a)} \right) \left[ -2\pi \left( \frac{\partial}{\partial p^{(j)}} (\nu^{(j)} e^{(\alpha)[1]}) \right) \right], \ (7.6)$$

and we recognise $C(z - k^{(j)} \nu^{(j)} - k^{(j')} \nu^{(j')}, \nu^{(j)})$ in (7.5).

Let us examine the dependence of $C$ in the cut off wave number $K_c$. The first matrix element contains a contribution finite for $k^{[1]} \rightarrow \infty$. If we take into account the summation over $a$, the propagator will involve a $(k^{[1]})^{-2}$ dependence. The whole contribution behaves therefore linearly as $K_c$. If $z$ approaches the real axis from above, as it is required from the construction of the subdynamics operator, we can use $z = y + i\epsilon$ and upon replacement in (7.6), the expression of $C(y + i\epsilon, \nu^{(j)})$ contains a Dirac delta function, of argument $y + ak^{[1]} \nu^{(j)} + k^{[1]}(-a)$ and a principal part. The contribution due to the Dirac delta function is independent of the cut off while the contribution due to the principal part provides the $K_c$ dependence.

Moreover, it is easily recognised that $C(z, \nu^{(j)})$ vanishes for $z \rightarrow 0$. Indeed, in that limit, the contribution that could arise from the Dirac delta function $\delta(ak^{[1]} \nu^{(j)} + k^{[1]}(-a))$ vanishes since the vanishing of its argument cannot be satisfied (except for $k^{[1]} = 0$ but the jacobian provides a factor $k^{[1]2}$) and the contribution due to the principal part vanishes by parity in the summation over $a$.

### 7.4 Renormalisation by simple cycles

We first consider the consequences of inserting a simple cycle in the contribution (7.1). For the sake of illustration, we consider the insertion of such a cycle before the vertex corresponding to the Coulomb interaction in (7.1). Details of computation can be found in appendix E. The insertion of the cycle involves of course a supplementary propagator (see the square on the first propagator in (7.1)) and the previously considered cycle contribution (7.6) with the same argument as the new propagator. That property holds for the insertion of an arbitrary number of cycles, enabling a renormalisation for the propagators. The resummation of the class of diagrams where all
the cycles are at the left of the particle vertex (Coulomb interaction) is:

\[
< 11[1(s_j)]|\tilde{\Sigma}(t)|11[1(f)] >_{d,1}^{(1,\infty)} \tilde{f}^V_{[1(f)]}
\]

\[
= (-i)e_j \frac{1}{(2\pi)^2} \sum_{\beta=1,2} \sum_{b=\pm 1} (\frac{\eta_\beta}{k})^{\frac{1}{2}} \exp \left\{ -k \cdot \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_\beta} \right\}
\]

\[
\times \frac{1}{2\pi i} \int_{-}^{t'} dz \ e^{-izt} \left( z - (k^{(j)} + bk).v^{(j)} - k^{(j')} - k^{(j')},v^{(j')} + bk \right)
\]

\[
- C(z - (k^{(j)} + bk).v^{(j)} - k^{(j')} - k^{(j')},v^{(j')} + bk, v^{(j)})^{-1}
\]

\[
\times (-i)e_j e_j \frac{1}{2\pi^2} \int d^2 l \frac{1}{l^2} \left( \frac{1}{z - (k^{(j)} + 1 + bk).v^{(j)} - (k^{(j')} - 1),v^{(j')} + bk} \right)
\]

\[
\times \left[ - \left( 1, \frac{\partial}{\partial p^{(j)},v^{(j)}} \pi(v^{(j)},e^{(j)}) \right) \right] \left[ 2 \frac{\partial}{\partial \eta_\beta} \right]
\]

\[
\times \left( z - (k^{(j)} + l + bk).v^{(j)} - (k^{(j')} - 1),v^{(j')} + bk \right)
\]

\[
- C(z - (k^{(j)} + 1 + bk).v^{(j)} - (k^{(j')} - 1),v^{(j')} + bk, v^{(j)})^{-1}
\]

\[
\times \left[ - \left( 1, \frac{\partial}{\partial p^{(j)},v^{(j)}} \pi(v^{(j)},e^{(j)}) \right) \right] \left[ 2 \frac{\partial}{\partial \eta_\beta} \right]
\]

\[
\times \left( z - (k^{(j)} + 1).v^{(j)} - (k^{(j')} - 1),v^{(j')} \right)
\]

\[
- C(z - (k^{(j)} + 1).v^{(j)} - (k^{(j')} - 1),v^{(j')}, v^{(j)})^{-1}
\]

When an arbitrary number of cycles are also added in the two other propagators, the contribution is:

\[
< 11[1(s_j)]|\tilde{\Sigma}(t)|11[1(f)] >_{d,1}^{(1,\infty)} \tilde{f}^V_{[1(f)]}
\]

\[
= (-i)e_j \frac{1}{(2\pi)^2} \sum_{\beta=1,2} \sum_{b=\pm 1} (\frac{\eta_\beta}{k})^{\frac{1}{2}} \exp \left\{ -k \cdot \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial m_\beta} \right\}
\]

\[
\times \frac{1}{2\pi i} \int_{-}^{t'} dz \ e^{-izt} \left( z - (k^{(j)} + bk).v^{(j)} - k^{(j')} - k^{(j')},v^{(j')} + bk \right)
\]

\[
- C(z - (k^{(j)} + bk).v^{(j)} - k^{(j')} - k^{(j')},v^{(j')} + bk, v^{(j)})^{-1}
\]

\[
\times (-i)e_j e_j \frac{1}{2\pi^2} \int d^2 l \frac{1}{l^2} \left( \frac{1}{z - (k^{(j)} + 1 + bk).v^{(j)} - (k^{(j')} - 1),v^{(j')} + bk} \right)
\]

\[
\times \left[ - \left( 1, \frac{\partial}{\partial p^{(j)},v^{(j)}} \pi(v^{(j)},e^{(j)}) \right) \right] \left[ 2 \frac{\partial}{\partial \eta_\beta} \right]
\]

\[
\times \left( z - (k^{(j)} + l + bk).v^{(j)} - (k^{(j')} - 1),v^{(j')} + bk \right)
\]

\[
- C(z - (k^{(j)} + 1 + bk).v^{(j)} - (k^{(j')} - 1),v^{(j')} + bk, v^{(j)})^{-1}
\]

\[
\times \left[ - \left( 1, \frac{\partial}{\partial p^{(j)},v^{(j)}} \pi(v^{(j)},e^{(j)}) \right) \right] \left[ 2 \frac{\partial}{\partial \eta_\beta} \right]
\]

\[
\times \left( z - (k^{(j)} + 1).v^{(j)} - (k^{(j')} - 1),v^{(j')} \right)
\]

\[
- C(z - (k^{(j)} + 1).v^{(j)} - (k^{(j')} - 1),v^{(j')}, v^{(j)})^{-1}
\]
\[ x e^{\frac{\partial}{\partial k^{(1)}}} \frac{\partial}{\partial k^{(2)}}} \delta(\eta_1^{[1]} - \eta_2^{[1]} - \eta_3^{[1]} - \eta_4^{[1]}) \delta_{(K^r_{m\beta} + \delta_{m\beta}^{[1]})} \delta_{m\beta}^{[1]} \delta_{k^{(1)}}^{[1]} \delta_{k^{(2)}}^{[1]} \theta. \]  

(7.8)

For the evaluation of \( \tilde{\Sigma}(t) \) superoperator, we have to compute the residues to the (multiple) bare poles due to the last propagator. By resummation, the procedure corresponds (cf. Lee model [2]) to the computation of the dressed poles. In view of our previous analysis, \( C(\theta, \nu^{(j)}) = 0 \). A solution \( \theta \) of \( \theta = C(\theta, \nu^{(j)}) \) is thus \( \theta = 0 \).

The solution of \( z - (k^{(j)} + 1).\nu^{(j)} - (k^{(j')} - 1).\nu^{(j')} - C(z - (k^{(j)} + 1).\nu^{(j)} - (k^{(j')} - 1).\nu^{(j')} = 0 \). can thus be written as \( z = (k^{(j)} + 1).\nu^{(j)} + (k^{(j')} - 1).\nu^{(j')} \); the bare and dressed poles coincide. The residue at the pole disappears when the creation operator \( C \) is computed from the product \( CA \).

Therefore, for our purpose, we do not have to bother about a renormalisation of the vacuum propagator and we can continue to use the simple propagator.

In place of the second term in (6.5), we get the same expression with the cycle contribution in the propagators

\[ < E_r^+ (q_j) >_{IIC} = e_* e_j \frac{1}{\pi} \frac{1}{(2\pi)^3} \int d^3 k^{(1)} \sum_{a=1,2} \sum_{a=\pm 1} e_*^a (k^{(1)}) \]

\[ \times \int d^3 l^2 \left( i \epsilon + (1 + a k^{(1)}).\nu_j - a k^{(1)} \right) \]

\[ - C(i \epsilon + (1 + a k^{(1)}).\nu_j - a k^{(1)}, \nu_j) \]

\[ \times \left( \frac{1}{a k^{(1)}(1) \nu_j - a k^{(1)} + C(i \epsilon + a k^{(1)}(1) \nu_j - a k^{(1)}, \nu_j) \right) \]

\[ \times e^{(\alpha)[1]} e^{(\nu)[1]} e - i \cdot [q_j - q_{j'}]. \]  

(7.9)

In the referenetiels where the velocity \( \nu_{j'} \) vanishes, we get

\[ < E_r^+ (q_j) >_{IIC} = e_* e_j \frac{1}{\pi} \frac{1}{(2\pi)^3} \int d^3 k^{(1)} \sum_{a=1,2} \sum_{a=\pm 1} e_*^a (k^{(1)}) \int d^3 l \]

\[ \times \left( i \epsilon + (1 + a k^{(1)}).\nu_j - a k^{(1)} \right) \]

\[ - C(i \epsilon + (1 + a k^{(1)}).\nu_j - a k^{(1)}, \nu_j) \]

\[ \times \left( \frac{1}{a k^{(1)}(1) \nu_j - a k^{(1)} + C(i \epsilon + a k^{(1)}(1) \nu_j - a k^{(1)}, \nu_j) \right) \]

\[ \times e^{(\alpha)[1]} e^{(\nu)[1]} e - i \cdot [q_j - q_{j'}]. \]  

(7.10)
In the low velocity limit $v_j \to 0$, the previous expression vanishes by symmetry due to the angular integration. (the integral over $l$ and $k$ become independent but the final result could be divergent.)

We introduce a cut-off function in (7.10). The same cut-off will be used in all future expressions.

$$< E_r^\perp (q_j) >_{\perp c}^{e_j (1, 1)} = e_j^2 e_j' \frac{1}{\pi} \frac{1}{(2\pi)^3} \int d^3 k \sum_{\alpha=1,2} \sum_{a=\pm 1} e_\alpha (k[1]) \int d^3 l \frac{1}{l^2}$$

$$\times \left( \frac{1}{i\epsilon + (1 + ak[1]).v_j - ak[1]} - C(i\epsilon + (1 + ak[1]).v_j - ak[1], v_j) \right)$$

$$\times \frac{1}{ak[1].v_j - ak[1]} - C(i\epsilon + ak[1].v_j - ak[1], v_j)$$

$$\times \frac{1}{m_j} l_v e_v (\alpha[1]) e^{-i[l_j - q_j]} \frac{K_c^2}{k[1]^2 + K_c^2}, \quad (7.11)$$

where in the low velocity, neglecting the magnetic component of the force and simplifying the computation of the derivative with respect to $p_j$,

$$C(z, v_j) = - \frac{4\pi}{m_j} e_j^2 \frac{1}{(2\pi)^3} \int d^3 k \left[ \left( \frac{1}{z + k.v_j - k} \right) + \left( \frac{1}{z - k.v_j + k} \right) \right]$$

$$\times \frac{K_c^2}{k^2 + K_c^2}. \quad (7.12)$$

We examine the behaviour for small velocity $v_j$.

$$C(z, 0) = - \frac{16\pi}{m_j} e_j^2 \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 \left[ \left( \frac{1}{z - k} \right) + \left( \frac{1}{z + k} \right) \right] \frac{K_c^2}{k^2 + K_c^2}$$

$$\quad (7.13)$$

For $\Im z > 0$, and $\Re z = y$, we prove in appendix E that

$$C(y + i\epsilon, 0) = \frac{16\pi^3}{m_j} e_j^2 \frac{1}{(2\pi)^3} \pi K_c^3 \frac{y}{y^2 + K_c^2} + i \frac{2e_j^2}{m_j} \frac{K_c^2 y^2}{y^2 + K_c^2}. \quad (7.14)$$

We insert the value of $C(y + i\epsilon, 0)$ in the expression (7.11), neglecting the dependence of $C$ on the velocity $v_j$. That dependence is not seen as capital to get the qualitative behaviour while the remaining dependence in the denominator is required for the coupling between the integrations over
the wave numbers \( l \) and \( k \). We have:

\[
\begin{align*}
\langle \mathbf{E}_r^\perp (\mathbf{q}_j) \rangle_{IL}^{(1,1)} &= \varepsilon_j e_j' \frac{1}{\pi} \frac{1}{(2\pi)^3} \int d^3k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} e_r^\alpha (k^{[1]}) \\
&\times \int d^3l \frac{1}{l^2} \left( \frac{1}{i \varepsilon + (1 + ak^{[1]}).\mathbf{v}_j - ak^{[1]}} \right) \left( \frac{1}{ak^{[1]}.\mathbf{v}_j - ak^{[1]}} \right) \\
&\times \left[ (1 + ak^{[1]}).\mathbf{v}_j - ak^{[1]} \right]^2 + K_c^2 \\
&\times \left[ ak^{[1]}.\mathbf{v}_j - ak^{[1]} \right]^2 + K_c^2 - \frac{\pi}{m_j} e_j^2 K_c^3 - i \frac{2e^2}{m_j} K_c^2 [(1 + ak^{[1]}).\mathbf{v}_j - ak^{[1]}] \\
&\times \frac{1}{m_j} l_v e_v^{(\alpha)[1]} e^{-i l [\mathbf{q}_j - \mathbf{q}_j']} \frac{K_c^2}{k^{[1]}_c^2 + K_c^2}.
\end{align*}
\]

(7.15)

We compare that integral (7.15) with the contribution arising from (6.5):

\[
\begin{align*}
\langle \mathbf{E}_r^\perp (\mathbf{q}_j) \rangle_{II}^{(1,1)} &= \varepsilon_j e_j' \frac{1}{\pi} \frac{1}{(2\pi)^3} \int d^3k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} e_r^\alpha (k^{[1]}) \\
&\times \int d^3l \frac{1}{l^2} \left( \frac{1}{i \varepsilon + (1 + ak^{[1]}).\mathbf{v}_j - ak^{[1]}} \right) \left( \frac{1}{ak^{[1]}.\mathbf{v}_j - ak^{[1]}} \right) \\
&\times \frac{1}{m_j} l_v e_v^{(\alpha)[1]} e^{-i l [\mathbf{q}_j - \mathbf{q}_j']} \frac{K_c^2}{k^{[1]}_c^2 + K_c^2},
\end{align*}
\]

(7.16)

we note the presence of two similar supplementary factors. Except for the domain of values where the wave numbers are of the order of \( K_c^3 \), the denominator in these factors are dominated by \( -\frac{\pi}{m_j} e_j^2 K_c^3 \). In the relevant range of values for the wave numbers, thanks to the cut-off function, the numerator in the factors are of the order of \( K_c^2 \). Therefore, these factors can be replaced by \( \frac{m_j}{e_j^2 K_c} \) to evaluate the dependence of (7.15) with respect to \( K_c \). Since the resulting integral is known to diverge linearly with \( K_c \), the resulting dependence vanishes as \( K_c^{-1} \). The introduction of the (simple) cycle produces a convergence with respect to \( K_c \) that is much more too strong: we go from a divergence in \( K_c \) to a convergence in \( K_c^{-1} \). The consideration of renormalised cycles will produce the correct behaviour.
7.5 The renormalised cycle $\tilde{C}$

The renormalised cycle $\tilde{C}$ is introduced by replacing, in the definition of the cycle, the free propagator by a fully renormalised one. $\tilde{C}$ is thus defined by a self-consistent equation:

$$
\tilde{C}(z, v^{(j)}) = e_j^2 \frac{1}{(2\pi)^3} \int d^3 k^{[1]} \sum_{\alpha[1]=1,2} \sum_{a=\pm 1} \frac{1}{k^{[1]}} \\
\times \left[ [k^{[1]} e^{(\alpha)[1]}_r - g \nu v^{(j)} e^{(\alpha)[1]}_r] k^{[1]}_r - e^{(\alpha)[1]}_r k^{[1]}_r \right] \\
\times \left( z + a k^{[1]} \cdot v^{(j)} + k^{[1]}(-a) - \tilde{C}(z + a k^{[1]}, v^{(j)} + k^{[1]}(-a), v^{(j)}) \right) \\
\times \left[ -2\pi \left( \frac{\partial}{\partial p^{(j)}_r} (v^{(j)} \cdot e^{(\alpha)[1]}) \right) \right] \frac{K^2_c}{k^{[1]2} + K^2_c}.
$$

(7.17)

We analyse the behaviour of $\tilde{C}(z, v^{(j)})$ with respect to $K_c$ and $v^{(j)}$. The self-consistent equation (7.10) is diagonal with respect to the velocity $v^{(j)}$. In a first investigation, we consider the situation in which $v^{(j)}$ is close to zero. In these circumstances, $\left( \frac{\partial}{\partial p^{(j)}_r} (v^{(j)} \cdot e^{(\alpha)[1]}) \right)$ is $\frac{m}{e^{(\alpha)[1]}_r}$ and the summation over $r$ can be performed and we use the unity of the $e^{(\alpha)[1]}$ vector. We get:

$$
\tilde{C}(z, 0) = (-2\pi) \frac{2}{m} e_j^2 \frac{1}{(2\pi)^3} \int d^3 k^{[1]} \frac{K^2_c}{k^{[1]2} + K^2_c} \\
\times \sum_{a=\pm 1} \left( \frac{1}{z + k^{[1]}(-a) - \tilde{C}(z + k^{[1]}(-a), 0)} \right).
$$

(7.18)

We have ($\tilde{C}(z) \equiv \tilde{C}(z, 0)$)

$$
\tilde{C}(z) = \left( -\frac{4\pi}{m} \right) \frac{e_j^2}{(2\pi)^3} \int d^3 k^{[1]} \frac{K^2_c}{k^{[1]2} + K^2_c} \\
\times \sum_{a=\pm 1} \left( \frac{1}{z + k^{[1]}(-a) - \tilde{C}(z + k^{[1]}(-a))} \right) \\
= -\frac{16\pi^2}{m} e_j^2 \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 \frac{K^2_c}{k^{[1]2} + K^2_c} \\
\times \left[ \left( \frac{1}{z - k - \tilde{C}(z - k)} \right) + \left( \frac{1}{z + k - \tilde{C}(z + k)} \right) \right].
$$

(7.19)
In our units in which the velocity of light is unity, the dimension of $\tilde{C}(z)$ is the inverse of a time (as $z$) or the inverse of a wave number as $k$. We can define $K_m$ by $m \pi \beta_j$. We can therefore write $\tilde{C}(z) = z c(\frac{z}{K_m}, \frac{K_c}{K_m})$ and we have.

$$\tilde{C}(\frac{z}{K_m}, \frac{K_c}{K_m}) = -\frac{1}{K_m} \int_0^\infty dk \frac{K_c^2}{k^2 + K_c^2} \left[ \left( 1 \right) \frac{\gamma^2}{z - k - (z - k)c(\frac{z}{K_m}, \frac{K_c}{K_m})} + \left( 1 \right) \frac{\gamma^2}{z + k - (z + k)c(\frac{z}{K_m}, \frac{K_c}{K_m})} \right].$$

(7.20)

We call $\gamma$ the ratio between the cut-off wave number $K_c$ and $K_m$. Introducing $y = \frac{z}{K_m}$ and $u = \frac{k}{K_m}$, we have the equation for $y$ real (from above):

$$c(y, \gamma) = -\frac{1}{y} \int_0^\infty du \frac{u^2 \gamma^2}{\gamma^2 + u^2} \left[ \left( 1 \right) \frac{1}{(i\epsilon + y - u)(1 - c(y - u, \gamma))} + \left( 1 \right) \frac{1}{(i\epsilon + y + u)(1 - c(y + u, \gamma))} \right].$$

(7.21)

What kind of consistency can we deduce from (7.21) about the dependence of $c(y, \gamma)$ with respect to $\gamma$?

Let us be more specific about the behaviour of $c(y, \gamma)$. We assume that

$$c(y, \gamma) = \alpha \gamma^r + \beta \gamma^s$$

(7.22)

in the range $|y| \gg \gamma$ and the self consistency should determine the complex parameters $\alpha$, $\beta$ and the real parameters $r$ and $s$. The parameter $r$ is assumed positive and $s$ has no definite sign a priori. That expression for $c(y, \gamma)$ assumes that $\lim_{y \to \infty} \frac{c(y, \gamma)}{c(-y, \gamma)} = 1$. We first check that property in appendix F by computing the difference $\Delta(y, \gamma) = c(y, \gamma) - c(-y, \gamma)$ for large positive $y$. The asymptotic behaviour of $c(y, \gamma)$ in the domain $y \gg \gamma \gg 1$ is then checked in a self consistent way by the proposal (7.22). In order to make predictions on the behaviour of the self-field when the contributions of the renormalised cycles are included, we need mainly the behaviour of $c(y, \gamma)$ in the other domains, in particularly $\gamma \gg y \gg 1$.

From the expression

$$c(y, \gamma) = -\frac{1}{y} \int_0^\infty du \frac{u^2 \gamma^2}{\gamma^2 + u^2} \left[ \left( 1 \right) \frac{1}{(i\epsilon + y - u)(1 - c(y - u, \gamma))} + \left( 1 \right) \frac{1}{(i\epsilon + y + u)(1 - c(y + u, \gamma))} \right].$$

(7.23)
and our previous results, we have to deduce first the behaviour of \( c(y, \gamma) \) in the other domains: \( y \ll \gamma \) and \( y \) in the range of \( \gamma \).

We assume that
\[
c(y, \gamma) = \gamma^{\frac{3}{2}} g\left(\frac{y}{\gamma}\right)
\]
in the range \( |y| \ll \gamma \) and the intermediary range. Self consistency should hold and determine the complex function \( g\left(\frac{y}{\gamma}\right) \). \( g(0) \) is assumed \( \neq 0 \) while \( g\left(\frac{y}{\gamma}\right) \) is assumed to vanish for \( \frac{y}{\gamma} \to \infty \). That expression is analysed in details in appendix F and self consistency established.

That dependence can be further used to evaluate the dependence of the electric field with respect to the cut-off value \( K_c \) (\( \gamma = \frac{K_c}{K_m} \)).

### 7.6 Behaviour of the self-field

We can now look at the consequences for the self-field. The computation of the residue to get the required creation operator can be pursued in a similar way. The residue at the vacuum pole \( k^{(j)}, v^{(j)} + k^{(j')} v^{(j')} \) does not play a role for the creation operator: only the value of the pole matter. We recover therefore an expression similar to the expression (7.11) in which the simple cycles are replaced by the renormalised one’s:

\[
\begin{align*}
\langle E_\perp (q_j) \rangle_{\text{II}C} &= e_j^2 e_j' \frac{1}{\pi (2\pi)^2} \int d^3 k^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} e_\alpha^a (k^{[1]}) \\
&\times \int d^3 l \frac{1}{l^2} \left( \frac{1}{i\epsilon + (1 + a k^{[1]}).v_j - a k^{[1]} - \tilde{C}(i\epsilon + (1 + a k^{[1]}).v_j - a k^{[1]}, v_j)} \right) \\
&\times \left( \frac{1}{ak^{[1]} . v_j - a k^{[1]} - \tilde{C}(i\epsilon + a k^{[1]} . v_j - a k^{[1]}, v_j)} \right) \\
&\times \frac{1}{m_j} e_v^{(\alpha)[1]} e^{-i[l . (q_j - q_j')] K_c^2}{k^{[1]}}^2 + K_c^2.
\end{align*}
\]

The same manipulations as for the simple cycle lead to a global dependence as the product of a factor \( K_c \) arising from the integral and two factors \( K_c^{-\frac{3}{2}} \) due to the presence in the denominators of the renormalised cycle function \( \tilde{C} \). Therefore, the previous expression (7.25) for the self electric field provides a finite result in the limit \( K_c \to \infty \). We note that the dependence of the self-field on the charge \( e_j \) disappears also in that limit. This property is not at all astonishing and prove that an effective cut-off for wave numbers larger than \( \frac{m c^2}{e_j} \) is present. The final dependence is proportional
to that value and the dependence in $e_j$ cancels out. Therefore, we meet here a typical case where a non-analyticity in the expansion in the charge induces divergence on individual diagrammatic contributions but an adequate resummation enables to obtain finite results.

7.7 Beyond the inserted cycles

If we consider two cycles with an overlap, with respect to the previous case, we get two supplementary propagators. If the propagators are renormalised by the previous considered inserted cycles, the contribution of the imbricated cycles implies a possible $\gamma$ factor arising from the new integration over a wave number and two $K_1^2$ factors arising from the denominators. Therefore, the global behaviour of the imbricated cycles is the same as a single (renormalised) cycle. The imbricated cycles can thus be introduced for renormalising the propagators and the previous analysis is still valid. Therefore, if the contributions are computed in terms of renormalised propagators with all possible vertices (defined also in terms of renormalised propagators), the result of the resummation is a contribution to the self-field that is finite in the limit of an infinite cut-off $K_c$. The replacement of a bare vertex in the expression of the self-field by dressed vertices does not change either the finiteness of the global contribution.

8 Conclusions

Our present work have illustrated the feasibility of a reformulation of classical electrodynamics, that takes explicitly into account the corrections due to the self-fields. Moreover, the procedure avoids the existence of runaway solutions: causality is an ingredient of the construction of the subdynamics operator. Therefore, our expression for the self-force is not in terms of the time derivative of the acceleration but involves the actual position and velocity of the charged particle. We justify in that way the procedure proposed by several author to avoid the runaway solutions: the replacement of the time derivative of the acceleration by the time derivative of the external force. In the traditional approach, the self-force is naturally computed from the characteristics of the trajectory and the replacement has to be added by hand. Here, we have made the opposite step: our expression in terms of the mean field has been shown to be equivalent with the usual expression in terms of the time derivative of the acceleration.
The present approach constitutes a statistical description of interacting charged particles and electromagnetic fields: we are far from the classical view of well defined values for the variables associated to the fields and the particles: all these variables are statistical with a joint distribution function that evolves with time. The use of a reduced formalism enables to treat the distribution functions that are the most relevant for the computation of mean values of all the dynamical functions.

Two distinct ingredients are required. The first one is a relativistic statistical description of interacting fields and charged particles in which no unobservable potential appears as dynamical variables. Balescu-Poulain have developed further the ideas of Bialynicki-Birula [21], [22] and his coworkers to provide such a formalism free from dynamical constraints. The elimination of the Lorentz condition is a key element of the present work that avoids the usual derivation of the self-forces via the Liénard-Wiechert potentials. The second ingredient is the possibility, that we have developed in collaboration with C. George, of getting rid of the self-field by defining an appropriate subdynamics. When both elements are combined, we obtain a finite kinetics for the description of the interacting charges and fields in which no explicit self-energy process is allowed: the kinetic operator takes into account all the effects and its computation, although lengthy, is straightforward.

The formalism developed in this paper offers the basis to tackle in a new way the divergences in classical electrodynamics, through (infinite) re-summations of diagrams. The renormalised propagators are then defined in terms of them-selves (for a similar procedure, see [24]) and the solutions of the resulting non-linear equations do not admit a simple expansion in the charge, enabling the presence of the natural cut-off, linked to the classical radius of the electron.

The present paper illustrates only one of the multiple potentialities of the approach. Many problems can be aborded within the present formalism, such as the charge renormalisation, for instance, of higher order effects. Moreover, we have considered the charged particles outside an external influence: the distribution function corresponding to the field vacuum has been used thoroughly in this paper. The effect of the magnetic field has not been specifically considered: when computing the power dissipated in the motion, its effect disappears. We have not taken advantage of the statistical nature of the formalism: a sharp distribution function has been assumed for the positions and velocities of the particles. A statistical nature for the field has also been ignored.

An irreversible extension of CED, analogous to the treatment of the Lee
model in quantal case, requires the construction of the generators of the Lie
associated with the extended dynamics. The relevance of such an extension
is still to be established.

We dare to state that we have presented new tools for dealing with
problems in classical electrodynamics. The approach may look tedious but is
nevertheless straightforward. It opens new perspectives, not only in classical
electrodynamics but also in quantum electrodynamics [20]. Obviously, a lot
of work remains to be done.

9 Appendix A. The Lorentz force

In this part, we consider only one particle interacting with a free transverse
wave. The particle will be pointlike, with a specific value for the velocity.

From the expression (3.13) for

\[ \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k[1] \int_0^\infty d\eta_1[1] \int_0^\infty d\eta_2[1] \]

\[ \times \sum_{m_1[1],m_2[1]} \delta_{m_1[1],0} \delta_{m_2[1],0} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_\alpha[1]}{k[1]} \right)^{\frac{1}{2}} \]

\[ \times [k[1]e_r^{(\alpha)[1]} - g^{s\alpha}v_s^{(j)} (e_t^{(\alpha)[i]}k_r^{[i]} - e_r^{(\alpha)[i]}k_t^{[i]})] \frac{\partial}{\partial p_r} \]

\[ \times \exp a \left\{ -k[1] \cdot \frac{\partial}{\partial k} - \frac{\partial}{\partial m_\alpha[1]} \right\} \tilde{f}(k, p, t) \tilde{f}[1](\eta_1[1], m_1[1], \eta_2[1], m_2[1], k[1]). \]

(A.1)

If we suppose that \( \tilde{f} \) describes a particle localized at some place \( r(t) \),
\( \tilde{f}(k, v, t) \) is proportional to \( \exp -i k \cdot r(t) \) [3.4]. The action of the displacement
operator \( \exp a \left\{ -k[1] \cdot \frac{\partial}{\partial k} - \frac{\partial}{\partial m_\alpha[1]} \right\} \) can thus be performed and we get easily:

\[ \partial_t \tilde{f}(k, v, t) \bigg|_1 = -\frac{e}{m (2\pi)^{\frac{3}{2}}} \int d^3k[1] \int_0^\infty d\eta_1[1] \int_0^\infty d\eta_2[1] \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_\alpha[1]}{k[1]} \right)^{\frac{1}{2}} \]

\[ \times [k[1]e_r^{(\alpha)[1]} - g^{s\alpha}v_s^{(j)} (e_t^{(\alpha)[i]}k_r^{[i]} - e_r^{(\alpha)[i]}k_t^{[i]})] \frac{\partial}{\partial v_r} \exp a \left\{ -k[1] \cdot r(t) \right\} \]
\( \times \tilde{f}(k, v, t) \tilde{f}_1[1](n_1^i, -a\delta_{a, 1}, n_2^i, -a\delta_{a, 2}; k^1) \). 

(A.2)

The mean values \(< E^\perp_r(x) >\) and \(< B^\perp_r(x) >\) of the fields can be deduced easily from (2.2) and (2.3):

\(< E^\perp_r(x) > = \int d^3k^1 \int_0^\infty dn_1^i \int_0^\infty dn_2^i \int_0^1 d\xi_1^i \int_0^1 d\xi_2^i \\
\times \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{a=1,2} \sum_{a=\pm 1} k^1 \frac{1}{2} e^a_r(k^1) \eta^\frac{1}{2}(k^1) \\
\times \exp\{iak^1[x - 2\pi\xi^a_\alpha(k^1)]\} \tilde{f}_1[1](\chi^1; k^1) \)

\(< B^\perp_r(x) > = \int d^3k^1 \int_0^\infty dn_1^i \int_0^\infty dn_2^i \int_0^1 d\xi_1^i \int_0^1 d\xi_2^i \\
\times \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{a=1,2} \sum_{a=\pm 1} k^1 \frac{1}{2} e^a_r(k^1) \eta^\frac{1}{2}(k^1) \\
\times \exp\{iak^1[x - 2\pi\xi^a_\alpha(k^1)]\} \tilde{f}_1[1](\chi^1; k^1) \)

(A.3)
so that we can proceed to the identification (3.14).

10 Appendix B

This appendix completes the list of the matrix elements of $<11[0]|\tilde{\Theta}|11[1](f)>^{(0,2)}$.

$$<11[0]|\tilde{\Theta}|11[1](f)>^{(0,2)} = \sum_{j=1,2} i e_j^2 \frac{1}{(2\pi)^3} \frac{\partial}{\partial p^{(j)}_r} \int d^3k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_{\alpha}^{[1]}}{k^{[1]}} \right)$$

$$\times \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{st} v^{(j)} (e_t^{(\alpha)[1]} k_r^{[1]} - e_r^{(\alpha)[1]} k_t^{[1]}) \right] \left( \frac{1}{-ak^{[1]} \cdot v^{(j)} + ak^{[1]}} \right)$$

$$\times \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{sr'} v^{(j)} (e_t^{(\alpha')[1]} k_r^{[1]} - e_r^{(\alpha')[1]} k_t^{[1]}) \right] \frac{\partial}{\partial p^{(j')}_{r'}}$$

$$-2\pi (v^{(j)} \cdot e^{(\alpha)[1]}) \left( \frac{\partial}{\partial \eta_{\alpha}^{[1]}} + \frac{1}{\eta_{\alpha}^{[1]}} \right)$$

$$\times \sum_{m_{11}^{[1]}} \sum_{m_{21}^{[1]}} \delta_{m_{11}^{[1]},0} \delta_{m_{21}^{[1]},0} \exp 2a \left\{ -k^{[1]} \frac{\partial}{\partial k^{[j]}}, k^{[1]} \frac{\partial}{\partial m_{i\alpha}^{[1]}} \right\}$$

$$+ \sum_{j=1,2} (-i) e_j^2 \frac{1}{(2\pi)^3} \frac{\partial}{\partial p^{(j)}_r} \int d^3k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_{\alpha}^{[1]}}{k^{[1]}} \right)$$

$$\times \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{st} v^{(j)} (e_t^{(\alpha)[1]} k_r^{[1]} - e_r^{(\alpha)[1]} k_t^{[1]}) \right] \left( \frac{1}{-ak^{[1]} \cdot v^{(j)} + ak^{[1]}} \right)^2$$

$$\times \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{sr'} v^{(j)} (e_t^{(\alpha')[1]} k_r^{[1]} - e_r^{(\alpha')[1]} k_t^{[1]}) \right] \left( k^{(j')}_{sr'} - 2ak^{[1]} \right) \frac{\partial v^{(j')}_{sr'}}{\partial p^{(j')}_{sr'}}$$

$$\times \sum_{m_{11}^{[1]}} \sum_{m_{21}^{[1]}} \delta_{m_{11}^{[1]},0} \delta_{m_{21}^{[1]},0} \exp 2a \left\{ -k^{[1]} \frac{\partial}{\partial k^{[j]}}, k^{[1]} \frac{\partial}{\partial m_{i\alpha}^{[1]}} \right\}, \quad (B.1)$$

$$<11[0]|\tilde{\Theta}|11[1](f)>^{(0,2)} \neq$$

$$= \sum_{j=1,2} i e_j^2 \frac{1}{(2\pi)^3} \frac{\partial}{\partial p^{(j)}_r} \int d^3k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_{\alpha}^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \left( \eta_{\alpha'}^{[1]} \right)^{\frac{1}{2}}$$

$$\times \left( \frac{1}{-ak^{[1]} \cdot v^{(j)} + ak^{[1]}} \right) \left[ k^{[1]} e_r^{(\alpha)[1]} - g^{st} v^{(j)} (e_t^{(\alpha)[1]} k_r^{[1]} - e_r^{(\alpha)[1]} k_t^{[1]}) \right]$$

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\[
\times [k^{[1]} e^{(\alpha')[1]} - g^{\nu' \nu^{'*}} (e^{(\alpha')[1]} k_{\nu^{'*}} - e^{(\alpha')[1]} k_{\nu'})] \frac{\partial}{\partial p_{\nu}^{(j)}}
\]
\[
\times \sum_{m_1^{[1]}, m_2^{[1]}_0} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \exp \left\{ -2a k^{[1]} \frac{\partial}{\partial k^{(j)}} - a \frac{\partial}{\partial m_{\alpha}^{[1]}} - a \frac{\partial}{\partial m_{\alpha'}^{[1]}} \right\}
\]
\[
+ \sum_{j=1,2} (-i) e_j^2 \frac{1}{2(2\pi)^3} \frac{\partial}{\partial p_{\nu}^{(j)}} \int d^3 k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_1^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \left( \frac{\eta_2^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \frac{1}{(ak^{[1]}, v^{(j)} - ak^{[1]})^2}
\]
\[
\times \left[ [k^{[1]} e^{(\alpha)[1]} - g^{\nu' \nu^{'*}} (e^{(\alpha)[1]} k_{\nu^{'*}} - e^{(\alpha)[1]} k_{\nu'})] [k_{\nu'}^{(j)}] - 2ak^{[1]} \right] \frac{\partial v_{\nu'}^{(j)}}{\partial p_{\nu}^{(j)}}
\]
\[
\times \sum_{m_1^{[1]}, m_2^{[1]}_0} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \exp \left\{ -2a k^{[1]} \frac{\partial}{\partial k^{(j)}} - a \frac{\partial}{\partial m_{\alpha}^{[1]}} - a \frac{\partial}{\partial m_{\alpha'}^{[1]}} \right\}, \quad (B.2)
\]

\[
< 11[0] \hat{\Theta} | 11(1, f) > \neq 0,2
\]
\[
= \sum_{j=1,2} i e_j^2 \frac{1}{2(2\pi)^3} \frac{\partial}{\partial p_{\nu}^{(j)}} \int d^3 k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_1^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \left( \frac{\eta_2^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \frac{1}{(ak^{[1]}, v^{(j)} - ak^{[1]})^2}
\]
\[
\times \left[ [k^{[1]} e^{(\alpha)[1]} - g^{\nu' \nu^{'*}} (e^{(\alpha)[1]} k_{\nu^{'*}} - e^{(\alpha)[1]} k_{\nu'})] \frac{\partial}{\partial p_{\nu}^{(j)}} \right]
\]
\[
\times \sum_{m_1^{[1]}, m_2^{[1]}_0} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \exp \left\{ -a \frac{\partial}{\partial m_{\alpha}^{[1]}} + a \frac{\partial}{\partial m_{\alpha'}^{[1]}} \right\}
\]
\[
+ \sum_{j=1,2} (-i) e_j^2 \frac{1}{2(2\pi)^3} \frac{\partial}{\partial p_{\nu}^{(j)}} \int d^3 k^{[1]} \int_0^\infty d\eta_1^{[1]} \int_0^\infty d\eta_2^{[1]} \sum_{\alpha=1,2} \sum_{a=\pm 1} \left( \frac{\eta_1^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \left( \frac{\eta_2^{[1]}}{k^{[1]}} \right)^{\frac{1}{2}} \frac{1}{(ak^{[1]}, v^{(j)} - ak^{[1]})^2}
\]
\[
\times \left[ [k^{[1]} e^{(\alpha)[1]} - g^{\nu' \nu^{'*}} (e^{(\alpha)[1]} k_{\nu^{'*}} - e^{(\alpha)[1]} k_{\nu'})] \frac{\partial v_{\nu'}^{(j)}}{\partial p_{\nu}^{(j)}} \right]
\]
\[
\times \sum_{m_1^{[1]}, m_2^{[1]}_0} \delta_{m_1^{[1]}, 0} \delta_{m_2^{[1]}, 0} \exp \left\{ -2a k^{[1]} \frac{\partial}{\partial k^{(j)}} - a \frac{\partial}{\partial m_{\alpha}^{[1]}} - a \frac{\partial}{\partial m_{\alpha'}^{[1]}} \right\},
\]

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\[ \times \sum_{m_1, m_2} \delta_{m_1, 0} \delta_{m_2, 0} \exp \left\{ -a \frac{\partial}{\partial m_1} + a \frac{\partial}{\partial m_2} \right\}. \]  

(B.3)

11 Appendix C

The expression of the complete electric field \( <E(x) >_{a}^{(0,1)} \) (1.12) is evaluated explicitly, using its identification with \( <E(x) >_{a}^{(1,1)} \). Multiplying numerator and denominator by \((k^{[1]} + \mathbf{k}^{[1]} \cdot \mathbf{v}_j)\), we have

\[ <E^{\perp}(x) >_{a}^{(0,1)} = \frac{e_j (2\pi)^3}{\sin|k^{[1]} |} \int d^3 k^{[1]} |(x - q_j)| \]

\[ \times \left( \frac{1}{|k^{[1]}|^2 - (\mathbf{k}^{[1]} \cdot \mathbf{v}_j)^2} \right) \left( k + (v_{jx} k_x + v_{jy} k_y) \right) \left( (k_x e_x + k_y e_y + k_z e_z) - k \mathbf{v}_j \right) \frac{1}{k}. \]  

(C.1)

Let us place the \( x \) axis along \((x - q_j)\) and the \( y \) axis along \( \mathbf{v}_{\perp j} \), defined by \( \mathbf{v}_{\perp j} = \mathbf{v}_j - \frac{\mathbf{v}_j \cdot (x - q_j)(x - q_j)}{|x - q_j|^2} \).

\[ <E^{\perp}(x) >_{a}^{(0,1)} = \frac{(4\pi)^3}{2\pi} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} \sin k_x |x - q_j| \]

\[ \times \left( \frac{1}{k^2 - (v_{jx} k_x + v_{jy} k_y)^2} \right) \left( k + (v_{jx} k_x + v_{jy} k_y) \right) \left( (k_x e_x + k_y e_y + k_z e_z) - k \mathbf{v}_j \right) \frac{1}{k}. \]  

(C.2)

The integrand has to be even for a simultaneously change of the sign of \( k_x \) and \( k_y \). Therefore,

\[ <E^{\perp}(x) >_{a}^{(0,1)} = e_j \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} \sin k_x |x - q_j| \]

\[ \times \left( \frac{1}{k^2 - (v_{jx} k_x + v_{jy} k_y)^2} \right) \]

\[ \times \left( k(k_x e_x + k_y e_y) - k(v_{jx} k_x + v_{jy} k_y) \mathbf{v}_j \right) \frac{1}{k} \]

\[ = e_j \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} \sin k_x |x - q_j| \]

\[ \times \left( \frac{1}{k^2 - (v_{jx} k_x + v_{jy} k_y)^2} \right) \left( (k_x e_x + k_y e_y) - (v_{jx} k_x + v_{jy} k_y) \mathbf{v}_j \right) \cdot \]  

(C.3)
We use dimensionless variables of integration. We then replace \( \sin k_x \) by \( \frac{1}{2i}(e^{i k_x} - e^{-i k_x}) \) and perform the integration over \( k_x \) by residue at the pole of \( \frac{1}{k^2 - (v_{jx} k_x + v_{jy} k_y)^2} \) in the correct half plane. We have

\[
I_1 = \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z e^{ik_x} \left( \frac{1}{k^2 - (v_{jx} k_x + v_{jy} k_y)^2} \right) \\
\times \left[ (k_x e_x + k_y e_y) - (v_{jx} k_x + v_{jy} k_y) v_j \right].
\]

(C.4)

The pole is obtained by the equation

\[
k^2 - (v_{jx} k_x + v_{jy} k_y)^2 = 0,
\]

\[
k_x^2 (1 - v_{jx}^2) - 2k_x k_y v_{jx} v_{jy} + k_y^2 (1 - v_{jy}^2) + k_z^2 = 0.
\]

(C.5)

Therefore,

\[
k_x = \frac{k_y v_{jx} v_{jy} \pm \sqrt{(k_y v_{jx} v_{jy})^2 - (1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2]}}{(1 - v_{jx}^2)}
\]

\[
= \frac{k_y v_{jx} v_{jy} \pm i \sqrt{(1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2] - (k_y v_{jx} v_{jy})^2}}{(1 - v_{jx}^2)}.
\]

(C.6)

Due to the factor \( e^{ik_x} \), the relevant pole for \( I_1 \) corresponds to the plus sign and we have

\[
I_1 = 2\pi i \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z \frac{\sqrt{(1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2] - (k_y v_{jx} v_{jy})^2}}{(1 - v_{jx}^2)} e^{i k_y v_{jx} v_{jy} - (k_y v_{jx} v_{jy})^2}
\]

\[
\times \frac{1}{2i \sqrt{(1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2] - (k_y v_{jx} v_{jy})^2}} e^{i k_y v_{jx} v_{jy} - (k_y v_{jx} v_{jy})^2}
\]

\[
\times \frac{k_y v_{jx} v_{jy} + i \sqrt{(1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2] - (k_y v_{jx} v_{jy})^2}}{(1 - v_{jx}^2)} [e_x - v_{jx} v_j]
\]

\[
+ k_y [e_y - v_{jy} v_j]
\]

\[
= \pi \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z \frac{\sqrt{(1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2] - (k_y v_{jx} v_{jy})^2}}{(1 - v_{jx}^2)} e^{i k_y v_{jx} v_{jy} - (k_y v_{jx} v_{jy})^2}
\]

\[
\times \frac{1}{\sqrt{(1 - v_{jx}^2)[k_y^2 (1 - v_{jy}^2) + k_z^2] - (k_y v_{jx} v_{jy})^2}} e^{i k_y v_{jx} v_{jy} - (k_y v_{jx} v_{jy})^2}
\]

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\[
\begin{align*}
&\times \left[ k_y v_{jx} v_{jy} + i \sqrt{(1 - v_{jx}^2)(k_y^2(1 - v_{jy}^2) + k_x^2) - (k_y v_{jx} v_{jy})^2} \right] \left[ e_x - v_{jx} v_j \right] \\
&+ k_y \left[ e_y - v_{jy} v_j \right].
\end{align*}
\]

We replace the oscillating factor according to its parity in \( k_y \).

\[
I_{1a} = \pi i \left[ e_x - v_{jx} v_j \right] \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z \left( \frac{k_y^2(1 - v_{jy}^2) + k_x^2 - (k_y v_{jx} v_{jy})^2}{(1 - v_{jx}^2)^2} \right) \cos \left( k_y v_{jx} v_{jy} \right).
\]

Introducing polar coordinates \( r \) and \( \theta \) in the \( k_y, k_z \) plane, we get

\[
I_{1a} = \pi i \left[ e_x - v_{jx} v_j \right] \left( 1 - v_{jx}^2 \right) \int_0^{2\pi} d\theta \int_0^{\infty} dr \left( e^{-r \sqrt{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta}} \cos (r v_{jx} v_{jy} \cos \theta) \right)
\]

\[
= \frac{1}{2} \pi i \left[ e_x - v_{jx} v_j \right] \left( 1 - v_{jx}^2 \right) \int_0^{2\pi} d\theta \int_0^{\infty} dr \left[ e^{-r \sqrt{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta + iv_{jx} v_{jy} \cos \theta}} + e^{-r \sqrt{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta - iv_{jx} v_{jy} \cos \theta}} \right].
\]

The integration over \( r \) is readily performed.

\[
I_{1a} = \frac{1}{2} \pi i \left[ e_x - v_{jx} v_j \right] \left( 1 - v_{jx}^2 \right) \int_0^{2\pi} d\theta
\]

\[
\times \left[ \frac{1}{\sqrt{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta + iv_{jx} v_{jy} \cos \theta}} \right] \left[ \frac{1}{\sqrt{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta - iv_{jx} v_{jy} \cos \theta}} \right]
\]

\[
= \pi i \left[ e_x - v_{jx} v_j \right] \left( 1 - v_{jx}^2 \right) \int_0^{2\pi} d\theta \left[ \frac{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta - v_{jx} v_{jy} \cos^2 \theta}{\left[ 1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta \right]^2} \right]
\]

\[
= \pi i \left[ e_x - v_{jx} v_j \right] \left( 1 - v_{jx}^2 \right) \int_0^{2\pi} d\theta \left[ \frac{1 - v_{jx}^2 - v_{jy}^2 \cos^2 \theta - v_{jx} v_{jy} \cos^2 \theta}{\left[ 1 - v_{jy}^2 \cos^2 \theta \right]^2} \right].
\]

\( \text{(C.9)} \)
Taking $\phi = 2\theta$ as new integration variable, we get

$$I_{1a} = 2\pi i \left[ e_x - v_{jx} v_j \right] \frac{1}{(1 - v_{jx}^2)}$$

$$\times \int_0^\pi d\phi \, \frac{1 - v_{jx}^2 - \frac{1}{2} v_{jy}^2 (1 + v_{jx}^2) - \frac{1}{2} v_{jy}^2 (1 + v_{jx}^2) \cos \phi}{\left[ 1 - \frac{1}{2} v_{jy}^2 - \frac{1}{2} v_{jy}^2 \cos \phi \right]^2}. \quad (C.11)$$

From formulae 2.554.2 and 2.554.2, 148 of [19], we read

$$\int A + B \cos x \left( a + b \cos x \right) dx = \frac{1}{(a + b \cos x)^n} \left[ (aB - Ab) \sin x \right]$$

$$\left[ (Aa - bB)(n - 1) + (n - 2)(aB - Ab) \cos x \right], \quad (C.12)$$

$$\int A + B \cos x \frac{dx}{a + b \cos x} = \frac{B}{b} x + \frac{Ab - aB}{b} \int \frac{1}{a + b \cos x} dx, \quad (C.13)$$

with, formula 2.553.3, for $a^2 > b^2$

$$\int \frac{1}{a + b \cos x} dx = \frac{2}{\sqrt{a^2 - b^2}} \arctan \frac{\sqrt{a^2 - b^2} \tan \frac{x}{2}}{a + b} \quad (C.14)$$

Therefore, the last integration can be performed and we get

$$I_{1a} = 2\pi^2 i \left[ (1 - v_{jx}^2) e_x - v_{jx} v_y e_y \right] \frac{[1 - v_{jy}^2]}{(1 - v_{jx}^2) (1 - v_{jy}^2)^{\frac{3}{2}}} . \quad (C.15)$$

We now turn to the second term of (C.7) that is evaluated in a similar way:

$$I_{1b} = 2\pi^2 i \frac{1}{(1 - v_{jy}^2)^{\frac{3}{2}}} v_{jx} v_{jy} \frac{[1 - v_{jy}^2]}{(1 - v_{jx}^2) (1 - v_{jy}^2)^{\frac{3}{2}}} e_y. \quad (C.16)$$

The sum of the contributions $I_1 = I_{1a} + I_{1b}$ is the contribution along $e_x$ of $I_{1a}$ (C.15) and is given by

$$I_1 = 2\pi^2 i \left[ (1 - v_{jy}^2) e_x \right] \frac{1}{(1 - v_{jy}^2)^{\frac{3}{2}}} . \quad (C.17)$$

The contribution from $I_2$ is obviously its complex conjugate and, from (C.3) and (C.4), we have for $< E^\perp(x) >_a^{e_j(0,1)}$ the expression:

$$< E^\perp(x) >_a^{e_j(0,1)} = e_j \frac{1}{2\pi^2 2i} \frac{1}{|x - q_j|^2} \frac{1}{|x - q_j|^2} I_1$$

$$= \left[ 1 - v_{jy}^2 \right] \frac{1}{(1 - v_{jy}^2)^{\frac{3}{2}}} \frac{1}{|x - q_j|^2} e_x. \quad (C.18)$$
We evaluate first in this section the power dissipated by the radiative force \( \langle \textbf{F}(j) \cdot \textbf{v}_j \rangle_I \) [5.9]. The second contribution is treated afterwards.

We decompose the vector  \( \textbf{k}^{[1]} \) into its component  \( \textbf{k}_{\parallel}^{[1]} \) and perpendicular  \( \textbf{k}_{\perp}^{[1]} \) to the velocity vector  \( \textbf{v}_j \). The scalar product  (\( \textbf{l}.\textbf{k}_{\parallel}^{[1]} \)) becomes the sum  (\( \textbf{l}.\textbf{k}_{\parallel}^{[1]} + \textbf{l}.\textbf{k}_{\perp}^{[1]} \)). By symmetry, the last term will generate a vanishing contribution when integrated over  \( \textbf{k}_{\perp}^{[1]} \). The remaining scalar product  (\( \textbf{l}.\textbf{k}_{\parallel}^{[1]} \)) can be written as  \( p_j^{-2}(\textbf{l}.\textbf{p}_j)(\textbf{k}^{[1]} \cdot \textbf{p}_j) \) and combined with the other contribution.

Since  \( \frac{1}{p_j^2} = \frac{m_j^2}{(m_j^2 + p_j^2)^2} \), we get

\[
\langle \textbf{F}(j) \cdot \textbf{v}_j \rangle_I = -i \frac{1}{(2\pi)^3} \frac{e_j^3 e_{j'}}{4\pi} \int d^3k^{[1]} \int d^3l \frac{1}{l^2} \sum_{a=\pm 1} ae^{-i\frac{1}{2}(\textbf{q}_j - \textbf{q}_{j'})} \\
\times \frac{m_j^2}{p_j^2(m_j^2 + p_j^2)^2} (\textbf{l}.\textbf{p}_j)(\textbf{p}_j, \textbf{k}^{[1]}) \\
\times \left( \frac{1}{\textbf{k}^{[1]}.\textbf{v}_j - k^{[1]}} \right)^2 \left[ \frac{1}{i\varepsilon + (\frac{1}{2}\textbf{l} + ak^{[1]}).\textbf{v}_j - \frac{1}{2}\textbf{l}.\textbf{v}_{j'} - ak^{[1]}} \right], \
\text{(D.1)}
\]

\[
\langle \textbf{F}(j) \cdot \textbf{v}_j \rangle_I = -i \frac{1}{(2\pi)^3} \frac{e_j^3 e_{j'}}{4\pi} \int d^3k^{[1]} \int d^3l \frac{1}{l^2} \\
\times e^{-i\frac{1}{2}(\textbf{q}_j - \textbf{q}_{j'})} (\textbf{l}.\textbf{p}_j)(\textbf{p}_j, \textbf{k}^{[1]}) \\
\times \left( \frac{1}{\textbf{k}^{[1]}.\textbf{v}_j - k^{[1]}} \right)^2 \left[ \left( \frac{1}{i\varepsilon + (\frac{1}{2}\textbf{l} + k^{[1]}).\textbf{v}_j - \frac{1}{2}\textbf{l}.\textbf{v}_{j'} - k^{[1]}} \right) - \left( \frac{1}{i\varepsilon + (\frac{1}{2}\textbf{l} - k^{[1]}).\textbf{v}_j - \frac{1}{2}\textbf{l}.\textbf{v}_{j'} + k^{[1]}} \right) \right], \
\text{(D.2)}
\]

We can consider a situation where the particle  \( j' \) is much more heavy than  \( j \). In the referential in which the heavy particle is at rest at the origin of coordinates, we have:

\[
\langle \textbf{F}(j) \cdot \textbf{v}_j \rangle_I = -i \frac{1}{(2\pi)^3} \frac{e_j^3 e_{j'}}{4\pi} \int d^3k^{[1]} \int d^3l \frac{1}{l^2}
\]

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We consider first the case where the vectors \( q_j \) and \( v_j \) are orthogonal (the orbital situation). We place the \( x \) axis along \( q_j \) and the \( y \) axis along \( v_j \). We have:

\[
\begin{align*}
\langle F^{(j)} \cdot v_j \rangle_{\text{orb}} &= -\frac{i}{4\pi^3} \frac{e^{j} e^{j'}}{p_j^2 (m_j^2 + p_j^2)^2} \times \\
& \times \int d^3 k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dl_x dl_y dl_z \left( \frac{1}{k_j v_j - k_{[1]} v_j} \right)^2 \left[ \left( \frac{1}{i\epsilon + (\frac{1}{2} l_y + k_{[1]} v_j) v_j - k_{[1]} v_j} \right) \right] \\
& \times \left[ \left( \frac{1}{i\epsilon + (\frac{1}{2} l_y - k_{[1]} v_j) v_j + k_{[1]} v_j} \right) \right].
\end{align*}
\]  

(D.3)

The integration over \( l_y \) can be performed by residue, closing the path in the upper plane \( \Im l_y > 0 \). Indeed, the integrand decreases at least as \( l_y^{-3} \). The only pole to be considered is \( l_y = i \sqrt{l_x^2 + l_z^2} \).

\[
\begin{align*}
\langle F^{(j)} \cdot v_j \rangle_{\text{orb}} &= -i \frac{1}{(2\pi)^3} \frac{e^{j} e^{j'}}{4\pi} \frac{m_j^2}{p_j^2 (m_j^2 + p_j^2)^2} \\
& \times \int d^3 k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dl_x dl_y dl_z \frac{2\pi i}{2i \sqrt{l_x^2 + l_z^2}} e^{-i\frac{Q_j}{2}} \\
& \times (i \sqrt{l_x^2 + l_z^2} p_j \cdot k_{[1]}(p_j k_{[1]})) \left( \frac{1}{k_j v_j - k_{[1]} v_j} \right)^2 e^{-i\frac{Q_j}{2}} \\
& \times \left[ \left( \frac{1}{i\epsilon + (\frac{1}{2} i \sqrt{l_x^2 + l_z^2} + k_{[1]} v_j) v_j - k_{[1]} v_j} \right) \right] \\
& \times \left[ \left( \frac{1}{i\epsilon + (\frac{1}{2} i \sqrt{l_x^2 + l_z^2} - k_{[1]} v_j) v_j + k_{[1]} v_j} \right) \right].
\end{align*}
\]  

(D.4)

The \( i\epsilon \) can now be dropped since they have played their role in determining the relative position of the poles in the complex plane. We introduce polar
coordinates in the \( l_x, l_y \) plane to get

\[
\langle \mathbf{F}^{(j)} \cdot \mathbf{v}_j \rangle_{\text{orb}} = -i \frac{1}{(2\pi)^3} \frac{e^j e^{j'}}{4\pi} \frac{m_j^2}{p_j^2 (m_j^2 + p_j^2)^{3/2}} \int d^3k \int_0^\infty dl \int_0^{2\pi} d\theta \\
\times \frac{2\pi i}{2il} e^{-\frac{i}{2} q_j \cos \theta (ilp_j)} (p_j k_y^{[1]}) \left( \frac{1}{k_y^{[1]} v_j - k^{[1]}} \right)^2 \left[ \frac{1}{\frac{1}{2} il + k_y^{[1]} v_j - k^{[1]}} \right] .
\]

Since

\[
\left[ \frac{1}{\frac{1}{2} il + k_y^{[1]} v_j - k^{[1]}} \right] - \left( \frac{1}{\frac{1}{2} il + k_y^{[1]} v_j + k^{[1]}} \right) = \frac{-2\pi ik_y^{[1]} v_j - k^{[1]}}{\left( \frac{1}{2} il + k_y^{[1]} v_j - k^{[1]} \right)^2} ,
\]

we have

\[
\langle \mathbf{F}^{(j)} \cdot \mathbf{v}_j \rangle_{\text{orb}} = -i \frac{1}{(2\pi)^3} \frac{e^j e^{j'}}{4\pi} \frac{m_j^2 (p_j)^2}{p_j^2 (m_j^2 + p_j^2)^{3/2}} \int d^3k \int_0^\infty dl \int_0^{2\pi} d\theta \\
\times e^{-\frac{i}{2} q_j \cos \theta k_y^{[1]} k_y^{[1]} v_j - k^{[1]}} \left( \frac{1}{(k_y^{[1]} v_j - k^{[1]})^2} \right) \left( \frac{2(k_y^{[1]} v_j - k^{[1]})}{\left( \frac{1}{2} il + (k_y^{[1]} v_j - k^{[1]}) \right)^2} \right) .
\]  \tag{D.6}

By definition, \( \int_0^{2\pi} d\theta \cos (y \cos \theta) = 2\pi J_0(y) \), \( J_0 \) being the Bessel function. Therefore,

\[
\langle \mathbf{F}^{(j)} \cdot \mathbf{v}_j \rangle_{\text{orb}} = -i \frac{4\pi}{(2\pi)^3} \frac{e^j e^{j'}}{4\pi} \frac{m_j^2 (p_j)^2}{p_j^2 (m_j^2 + p_j^2)^{3/2}} \int d^3k \int_0^\infty dl \\
\times J_0 \left( \frac{k_y^{[1]} v_j}{k_y^{[1]} v_j - k^{[1]}} \right) \left( \frac{1}{\left( \frac{1}{2} il + (k_y^{[1]} v_j - k^{[1]}) \right)^2} \right) .
\]  \tag{D.7}

From p.686 of [19] we have (formula 6.565.4):

\[
\int_0^\infty J_\nu (bx)x^{\nu + 1} \frac{dx}{(x^2 + a^2)^{\mu + 1}} = \frac{a^{\nu - \mu + 1}}{2^\mu \Gamma(\mu + 1)} K_{\nu - \mu}(ab),
\]  \tag{D.9}

where \( K_\nu(z) \) is a bessel function of imaginary argument \( (-1 < \Re \nu < \Re(2\mu + \frac{3}{2}), a > 0, b > 0) \). We can apply that formula for \( x = l \), with \( \nu = 0, \mu = 0 \), \( b = \frac{1}{2} q_j \), \( a^2 = \frac{4(k_y^{[1]} v_j - k^{[1]})^2}{v_y^{[1]}} \). The function \( K_0(z) \) is represented in 8.432.1 by the integral \( (\nu = 0): K_0(z) = \int_0^\infty e^{-z \cosh \theta} d\theta \). The integral over \( l \) can thus
be performed:

\[
< \mathbf{F}^{(j)} \cdot \mathbf{v}_j >_{Iorb} = - \frac{4\pi}{(2\pi)^3} \frac{e^3 e_j' \cdot m_j^2 (p_j)^2}{4} \frac{4}{p_j^2 (m_j^2 + p_j^2) v_j} \int d^3 k^{[1]}
\]

\[
\times k^{[1]}_y \frac{1}{k_y^{[1]} v_j - k^{[1]}_y} K_0(q_j (k^{[1]}_y - k^{[1]}_y v_j)). \quad (D.10)
\]

We take \( k^{[1]}_y \equiv k^{[1]} \cos \theta \), \( x = \cos \theta \), \( \int d^3 k^{[1]} = \int_0^\infty dk^{[1]} (k^{[1]})^2 \int_{-1}^{+1} dx \int_0^{2\pi} d\phi \ldots \)

\[
< \mathbf{F}^{(j)} \cdot \mathbf{v}_j >_{Iorb} = - \frac{4\pi}{(2\pi)^3} \frac{e^3 e_j' \cdot m_j^2 (p_j)^2}{4} \frac{8\pi}{p_j^2 (m_j^2 + p_j^2) v_j^2} \int_0^\infty \int_{-1}^{+1} dx \int_0 \int_0 \int_0 \int_0 \int \ldots
\]

\[
\times \frac{x}{x v_j - 1} K_0(q_j k^{[1]} (1 - x v_j)). \quad (D.11)
\]

The formula 6.561.16 p. 684 of [19] is:

\[
\int_0^\infty x^\mu K_\nu(a x) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1 + \mu + \nu}{2}\right) \Gamma\left(\frac{1 + \mu - \nu}{2}\right), \quad (D.12)
\]

with \( \Re(\mu + 1 \pm \nu) > 0, \Re a > 0 \). That formula (D.12) can be applied for

\( x = k^{[1]} \), with \( \mu = 2, \nu = 0, a = \frac{q_j (1 - x v_j)}{v_j} \).

\[
< \mathbf{F}^{(j)} \cdot \mathbf{v}_j >_{Iorb} = - \frac{8}{\pi} \left( \Gamma\left(\frac{3}{2}\right) \right)^2 \frac{e^3 e_j' \cdot m_j^2}{4} \frac{1}{(m_j^2 + p_j^2) v_j^2} \int_{-1}^{+1} dx \frac{x}{x v_j - 1} \left( \frac{v_j}{q_j (1 - x v_j)} \right)^3. \quad (D.13)
\]

\[
< \mathbf{F}^{(j)} \cdot \mathbf{v}_j >_{Iorb} = \frac{8}{\pi} \left( \Gamma\left(\frac{3}{2}\right) \right)^2 \frac{e^3 e_j' \cdot m_j^2}{4} \frac{v_j}{(m_j^2 + p_j^2) q_j^3} \int_{-1}^{+1} dx \frac{x}{(1 - x v_j)^4}. \quad (D.14)
\]

The last integral is direct and leads to:

\[
< \mathbf{F}^{(j)} \cdot \mathbf{v}_j >_{Iorb} = \frac{4}{3} e^3 e_j' \frac{m_j^2}{(m_j^2 + p_j^2) q_j^3} \frac{v_j}{(1 - v_j^2)^3}. \quad (D.15)
\]

In the other geometries, some integrals are not known explicitly but can be shown to be more convergent than the orbital case that provides a finite result.
We now turn to the second contribution. In place of (D.4), we have now (by a change of variables, $l$ in this expression is of $\frac{\epsilon}{2}$ in the $l$ contribution):

\[
< \mathbf{F}^\perp(\mathbf{q}_f), \mathbf{v}_j >_{I orb} = e^3 e^{j'} \frac{1}{(2\pi)^3} \int d^3 k^{[1]} \left( \frac{1}{k^{[1]}_y v_j - k^{[1]}} \right) \]
\[
\times \left\{ \frac{1}{(m_j^2 + (p^{(j)}))^\frac{3}{2}} \left[ v_j - \frac{(k^{[1]}_y)(v_j k^{[1]}_y)}{(k^{[1]})^2} \right] \right. 
- \frac{P_j}{(m_j^2 + (p^{(j)}))^\frac{3}{2}} \left[ p_j v_j - \frac{(p_j k^{[1]}_y)(v_j k^{[1]}_y)}{(k^{[1]})^2} \right] \right\} 
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d l_x d l_y d l_z \frac{1}{l_x^2 + l_y^2 + l_z^2} e^{-i l_x q_1 l_y}
\times \left( \frac{1}{i \epsilon + (l_y + k^{[1]}_y) v_j - k^{[1]}} - \frac{1}{i \epsilon + (l_y - k^{[1]}_y) v_j + k^{[1]}} \right). \tag{D.16} \]

The integration over $l_y$ can be performed by residue, closing the path in the upper plane $\Im l_y > 0$. Indeed, the integrand decreases at least as $l_y^{-3}$. The only pole to be considered is $l_y = i \sqrt{l_x^2 + l_z^2}$. Introduce polar coordinates in the $l_x$, $l_y$ plane, we get

\[
< \mathbf{F}^\perp(\mathbf{q}_f), \mathbf{v}_j >_{I orb} = e^3 e^{j'} \frac{1}{(2\pi)^3} \frac{v_j m_j^2}{(m_j^2 + (p^{(j)}))^\frac{3}{2}} \int \int d^3 k^{[1]} \left( \frac{1}{k^{[1]}_y v_j - k^{[1]}} \right) \]
\[
\times \left( 1 - \frac{(k^{[1]}_y)^2}{(k^{[1]})^2} \right) \int_0^\infty \int_0^{2\pi} d l \int_0^{2\pi} d \theta e^{-i \epsilon q_j}
\times \left[ \frac{1}{(i l + k^{[1]}_y) v_j - k^{[1]}} - \frac{1}{(i l - k^{[1]}_y) v_j + k^{[1]}} \right]. \tag{D.17} \]

Since
\[
\left[ \frac{1}{(i l + k^{[1]}_y) v_j - k^{[1]}} - \frac{1}{(i l - k^{[1]}_y) v_j + k^{[1]}} \right] = -\frac{2(k^{[1]}_y v_j - k^{[1]})}{(l v_j)^2 + (k^{[1]}_y v_j - k^{[1]})^2}, \]
and identifying the $J_0$ Bessel function in $\int_0^{2\pi} d \theta \cos (y \cos \theta) = 2\pi J_0(y)$, we get:

\[
< \mathbf{F}^\perp(\mathbf{q}_f), \mathbf{v}_j >_{I orb} = -e^3 e^{j'} \frac{1}{(2\pi)^2} \frac{v_j m_j^2}{(m_j^2 + (p^{(j)}))^\frac{3}{2}} \int d^3 k^{[1]} \]
\[
\times \left( 1 - \frac{(k^{[1]}_y)^2}{(k^{[1]})^2} \right) \int_0^\infty d l J_0(l v_j) \frac{2}{(l v_j)^2 + (k^{[1]}_y v_j - k^{[1]})^2}. \tag{D.18} \]
Using (D.9) and formula 6.561.16 p. 684 of [19] leads then to:

\[
<F_\perp(q_j) \cdot v_j >_{IIorb} = -e_j^3 e_{j'} \frac{2}{\pi} \left( \Gamma \left( \frac{3}{2} \right) \right)^2 \frac{v_j^2 m_j^2}{(m_j^2 + (p(j))^2)^{1/2}} q_j^2
\]

\[
\times \int_{-1}^{+1} dx (1 - x^2) \left( \frac{1}{(1 - xv_j)} \right)^3. \quad (D.19)
\]

The last integral can be performed to provide the result (5.11) of the main text.

13 Appendix E. The simple cycle

13.1 Insertion of a cycle

The vector \( k^{[1]} \) in (7.1) is replaced by \( k \) with a corresponding change in the other variables associated with the field mode. To avoid ambiguity, \( \eta_\beta \) are replaced by \( \epsilon_\beta \) that has the same use of deferring a vanishing limit. We have to act on the field vacuum to use the previous result:

\[
<11|s_j)|\tilde{\Sigma}(t)|11|f)\rangle \frac{1}{4i} \int_{-1}^{+1} dz e^{-izt} \left( \frac{1}{z - k(j) \cdot v(j) - k'(j) \cdot v'(j) + k(m_\alpha + m_{\alpha'})} \right)
\]

\[
\times e_j^2 \left( \frac{1}{2\pi} \right)^3 \int d^3 k^{[1]} \sum_{\alpha[1]=1,2} \sum_{a=\pm 1}
\]

\[
\times \frac{1}{k^{[1]} \left[ k^{[1]} e^{[a][1]} - g^{st} v_s^{(j)} (e^{[a][1]} k^{[1]} - e^{[a][1]} k^{[1]}) \right]}
\]

\[
\times \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} + ak^{[1]} \cdot v^{(j)} - k'(j) \cdot v'(j) + k^{[1]}(-a) + k(m_\alpha + m_{\alpha'})} \right)
\]

\[
\times \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k'(j) \cdot v'(j) + k(m_\alpha + m_{\alpha'})} \right)
\]

\[
\times (1) \frac{1}{2\pi} \int d^3 l \frac{1}{12} e \left( \frac{\partial}{\partial p(j')} (v^{(j)} \cdot e^{(a)[1]}) \right)
\]

\[
\times (-i)e_j e_{j'} \frac{1}{2\pi} \int d^3 l \frac{1}{12} e \left( \frac{\partial}{\partial k(j')} - \frac{\partial}{\partial k(j')} \right)
\]

\[
\times \left( \frac{1}{z - k^{(j)} \cdot v^{(j)} - k'(j) \cdot v'(j) + k(m_\alpha + m_{\alpha'})} \right) \left( \frac{-i}{(2\pi)^2} \right) \sum_{\beta=1,2} \sum_{b=\pm 1}
\]

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Usual manipulations lead to:

\[<11[1(s_j)][\Sigma(t)|11[1(f)]] >_{11}^{(1,3)} \mathcal{F}_{11[f]} \]

\[= (-i)e_j \frac{1}{(2\pi)^{3}} \sum_{\beta=1,2} \sum_{b=\pm 1} \left( \frac{\eta_{\beta}}{k} \right)^{\frac{3}{2}} \exp b \left\{ -\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{k}^{(j)}} - \frac{\partial}{\partial m_{\beta}} \right\} \]

\[\times \frac{1}{z - (\mathbf{k}^{(j)} + b\mathbf{k}).\mathbf{v}^{(j)} - \mathbf{k}^{(j')} - k(m_{\beta} + m_{\beta'})} \times \delta(\eta_{\beta} - \epsilon_{\beta}) \delta(\eta_{\beta'} - \epsilon_{\beta'}) \delta^{K_{\tau}} \delta^{K_{\tau}}. \quad (E.1)\]

From the definition of the cycle, we have:

\[<11[1(s_j)][\Sigma(t)|11[1(f)]] >_{11}^{(1,3)} \mathcal{F}_{11[f]} \]

\[= (-i)e_j \frac{1}{(2\pi)^{3}} \sum_{\beta=1,2} \sum_{b=\pm 1} \left( \frac{\eta_{\beta}}{k} \right)^{\frac{3}{2}} \exp b \left\{ -\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{k}^{(j)}} - \frac{\partial}{\partial m_{\beta}} \right\} \]

\[\times \frac{1}{z - (\mathbf{k}^{(j)} + b\mathbf{k}).\mathbf{v}^{(j)} - \mathbf{k}^{(j')} - k(m_{\beta} + m_{\beta'})} \times \delta(\eta_{\beta} - \epsilon_{\beta}) \delta(\eta_{\beta'} - \epsilon_{\beta'}) \delta^{K_{\tau}} \delta^{K_{\tau}}. \quad (E.2)\]
\[ \times C(z - (k^{(j)} + bk).v^{(j)} - k^{(j')} .v^{(j')} + bk, v^{(j)}) \]
\[ \times (-i) ejej' \frac{-1}{2\pi^2} \int d^3l \frac{1}{l^2} \left( \frac{1}{z - (k^{(j)} + 1 + bk).v^{(j)} - (k^{(j')} - 1).v^{(j')} + bk} \right) \]
\[ \times \left[ - \left( 1. \frac{\partial}{\partial p^{(j)}(v^{(j)}.e^{(\beta)})} \right) 2 \frac{\partial}{\partial \eta^{\beta}} \right] \]
\[ \times \left( \frac{1}{z - (k^{(j)} + 1).v^{(j)} - (k^{(j')} - 1).v^{(j')}} \right) \]
\[ \times e \left( \frac{\partial}{\partial k^{(j)}} - \frac{\partial}{\partial k^{(j')}} \right) \delta(\eta^{[1]}_{\beta} - \eta_{\beta})\delta(\eta^{[1]}_{\beta'} - \eta_{\beta'}) \delta_{m^{[1]}} \delta_{m^{[1]}} \delta_{\eta^{[1]}} \delta_{\eta^{[1]}} \delta_{K_e} \delta_{K_e} \]  

(E.3)

13.2 Evaluation of the simple cycle

For \( \Im z > 0, \) and \( \Re z = y, \) we have

\[ \Im C(y + i\epsilon, 0) = \frac{16\pi^3}{m_j} \frac{1}{e^2} \frac{1}{(2\pi)^3} \int_0^\infty \frac{dk}{k^2} \left[ \delta(y - k) + \delta(y + k) \right] \frac{K^2_e}{k^2 + K^2_e} \]
\[ = \frac{2e^2}{m_j} K^2_e y^2 / y^2 + K^2_e. \]  

(E.4)

Let us note that the sign of \( \Im C(y + i\epsilon, 0) \) is unusually positive and \( \Im C(y + i\epsilon, 0) \) vanishes at \( y = 0, \) in accordance with the solution \( z = 0 \) of \( z - C(z) = 0. \)

For very large \( y, (y >> K_e), \) it behaves as a constant \( \frac{2e^2 K^2_e}{m_j}. \) Since the imaginary part of \( C(z) \) vanishes for \( z = 0, \) in the denominators in (7.9) and (7.10), that property does not involve a new \( i\epsilon \) rule in the computation of the subdynamics operator. For the real part of \( C(y + i\epsilon, 0), \) involving the principal part, we have

\[ \Re C(y + i\epsilon, 0) \]
\[ = \frac{16\pi^3}{m_j} \frac{1}{e^2} \frac{1}{(2\pi)^3} \int_0^\infty \frac{dk}{k^2} \frac{K^2_e}{k^2 + K^2_e} \left[ \mathcal{P}(\frac{1}{(y - k)}) + \mathcal{P}(\frac{1}{(y + k)}) \right] \]
\[ = \frac{16\pi^3}{m_j} \frac{1}{e^2} \frac{1}{(2\pi)^3} (I_1 + I_2), \]  

(E.5)

\[ I_1 = \lim_{R \to \infty} \int_0^R \frac{dk}{k} \left[ \mathcal{P}(\frac{1}{(y - k)}) + \mathcal{P}(\frac{1}{(y + k)}) \right] \]
\[ = \lim_{R \to \infty} \left[ - \ln(R - y) + \ln(|y|) + \ln(R + y) - \ln(|y|) \right] = 0, \]  

(E.6)
\[ I_2 = \int_0^\infty dk K^2_c \left[ P\left(\frac{1}{y-k}\right) + P\left(\frac{1}{y+k}\right) \right] \frac{K^2_c}{k^2 + K^2_c} \]
\[ = \Re \int_0^\infty dk K^2_c \left[ \frac{1}{(i\epsilon + y - k)} + \frac{1}{(i\epsilon + y + k)} \right] \frac{K^2_c}{k^2 + K^2_c} \]
\[ = \Re \int_{-\infty}^\infty dk \frac{1}{(i\epsilon + y - k) k^2 + K^2_c} = \frac{\pi K^3_c}{y(y^2 + K^2_c)}. \] (E.7)

\( I_2 \) vanishes linearly for \( y \to 0 \) and behaves as \( \pi K^3_c \frac{1}{y} \) for large \( y \).

### 14 Appendix F. Self-consistency analysis

#### 14.1 Consistency check on \( \Delta(y, \gamma) \)

From the expression (7.21), we have:

\[ \Delta(y, \gamma) = c(y, \gamma) - c(-y, \gamma) \]
\[ = -\frac{1}{y} \int_0^\infty du u^2 \frac{\gamma^2}{\gamma^2 + u^2} \times \left[ \frac{1}{(i\epsilon + y - u)[1 - c(y - u, \gamma)]} + \frac{1}{(i\epsilon + y + u)[1 - c(y + u, \gamma)]} \right. \]
\[ + \frac{1}{(i\epsilon - y - u)[1 - c(-y - u, \gamma)]} + \frac{1}{(i\epsilon - y + u)[1 - c(-y + u, \gamma)]} \right]. \] (F.1)

As for (2.30), we separates the contributions arising from the \( \delta \) functions from the contribution arising from the principal parts. That separation does no longer correspond into a separation between a real and an imaginary part. The contribution \( \Delta_\delta(y, \gamma) \) from the \( \delta \) functions is:

\[ \Delta_\delta(y, \gamma) = \frac{1}{y} (-\pi i) \int_0^\infty du u^2 \frac{\gamma^2}{\gamma^2 + u^2} \times \left\{ \delta(y - u) \left[ \frac{1}{[1 - c(y - u, \gamma)]} + \frac{1}{[1 - c(-y + u, \gamma)]} \right] \right. \]
\[ + \delta(y + u) \left[ \frac{1}{[1 - c(y + u, \gamma)]} + \frac{1}{[1 - c(-y - u, \gamma)]} \right] \right\} \]
\[ = \frac{1}{y} (2\pi i) \frac{1}{[1 - c(0, \gamma)]} y^2 \frac{\gamma^2}{\gamma^2 + y^2}. \] (F.2)

\( \Delta_\delta(y, \gamma) \) contributes to the second term of (7.22) since it decreases as \( \frac{1}{y} \).

The parameter \( s \) is thus related to the behaviour in \( \gamma \) of \( \frac{1}{[1 - c(0, \gamma)]} \).
The contribution $\Delta P(y, \gamma)$ from the principal parts is:

$$\Delta P(y, \gamma) = -\frac{1}{y} \int_0^\infty du \frac{u^2 \gamma^2}{\gamma^2 + u^2} \times \left\{ \mathcal{P} \left( \frac{1}{y-u} \right) \left[ \frac{1}{1 - c(y-u, \gamma)} \right] - \frac{1}{1 - c(-y+u, \gamma)} \right\}.$$  \hfill (F.3)

To determine the behaviour for $y >> \gamma$, it is too simple to say that we have two $y$ factors in the denominators in that expression and thus a behaviour as $\frac{1}{y^2}$. We have indeed to make sure that the convergence of the integral is not affected by the limit. Obviously, if we neglect $u$ in front of $y$ in that expression, the remaining integral over $u$ diverges linearly. We analyse therefore the behaviour of the integrand $I(u)$. For the first contribution, the form (7.22) cannot be used since the argument $y - u$ of $c(y - u, \gamma)$ is not large for all values of $u$ inside the domain of integration. For that term $\Delta P_1(y, \gamma)$, we separate the domains $0 < u < 2y$ and $u > 2y$ in the conditions $y >> \gamma >> 1$. In the first domain, the domain of integration over $u$ is finite and we have:

$$\Delta P_{1a}(y, \gamma) = -\frac{1}{y} \int_0^{2y} du \frac{u^2 \gamma^2}{\gamma^2 + u^2} \times \mathcal{P} \left( \frac{1}{y-u} \right) \left[ \frac{1}{1 - c(y-u, \gamma)} \right] - \frac{1}{1 - c(-y+u, \gamma)}.$$  \hfill (F.4)

$$\Delta P_{1a}(y, \gamma) = -\frac{1}{y} \int_0^{2y} du \gamma^2 \mathcal{P} \left( \frac{1}{y-u} \right) \times \left[ \frac{1}{1 - c(y-u, \gamma)} \right] - \frac{1}{1 - c(-y+u, \gamma)} + \frac{1}{y} \int_0^{2y} du \frac{\gamma^4}{\gamma^2 + u^2} \mathcal{P} \left( \frac{1}{y-u} \right) \times \left[ \frac{1}{1 - c(y-u, \gamma)} \right] - \frac{1}{1 - c(-y+u, \gamma)}.$$  \hfill (F.5)

The first term $\Delta P_{1aa}(y, \gamma)$ can be written as:

$$\Delta P_{1aa}(y, \gamma) = \frac{\gamma^2}{y} \int_{-y}^{y} dv \mathcal{P} \left( \frac{1}{v} \right) \times \left[ \frac{1}{1 - c(-v, \gamma)} \right] - \frac{1}{1 - c(v, \gamma)}.$$  \hfill (F.6)
where the principal part symbol can be dropped. For large $y$, we can replace $\pm y$ as limit of integration by $\pm \infty$, providing the convergence of the integral. For large $v$,

$$
\begin{align*}
\left[ \frac{1}{1 - c(-v, \gamma)} - \frac{1}{1 - c(v, \gamma)} \right] &= \left[ \frac{1}{1 - \alpha \gamma^r + \beta \gamma^s} - \frac{1}{1 - \alpha \gamma^r - \beta \gamma^s} \right] \\
&= \left[ \frac{v}{v(1 - \alpha \gamma^r + \beta \gamma^s)} - \frac{v}{v(1 - \alpha \gamma^r - \beta \gamma^s)} \right] \\
&= -2 \beta \gamma^s \left[ \frac{v}{v^2(1 - \alpha \gamma^r)^2 + \beta^2 \gamma^2 s^2} \right].
\end{align*}
$$

That factor behaves as $\frac{1}{v}$ and the integrand in (F.6) behaves as $\frac{1}{v^2}$, ensuring the convergence. $\Delta_{P_{1aa}}(y, \gamma)$ contributes also to the second term of (7.22).

The second term $\Delta_{P_{1ab}}(y, \gamma)$ of (F.5) behaves as least as $\frac{1}{y^2}$ since no convergence problem can arise.

For the second contribution $\Delta_{P_2}(y, \gamma)$ of (F.3), the asymptotic behaviour (7.22) can be used in all the integration domain and we have, keeping the two terms:

$$
\Delta_{P_2}(y, \gamma) = -\frac{1}{y} \int_0^{\infty} du u^2 \left. \frac{\gamma^2}{\gamma^2 + u^2} \right[ \frac{1}{1 - \alpha \gamma^r - \beta \frac{\gamma^s}{y+u}} - \frac{1}{1 - \alpha \gamma^r - \beta \frac{\gamma^s}{y-u}} \right].
$$

Elementary manipulations lead to:

$$
\Delta_{P_2}(y, \gamma) = \frac{1}{y} \int_0^{\infty} du u^2 \frac{\gamma^2}{\gamma^2 + u^2} \left[ \frac{1}{(y+u)^2(1 - \alpha \gamma^r)^2 - \beta^2 \gamma^2 s^2} \right] - \frac{-2 \beta \gamma^s}{\gamma^2 + u^2 \left[ (y+u)^2(1 - \alpha \gamma^r)^2 - \beta^2 \gamma^2 s^2 \right]}.
$$

$\Delta_{P_2}(y, \gamma)$ can be split as (cf. (F.5)):

$$
\Delta_{P_2}(y, \gamma) = \frac{\gamma^2}{y} \int_0^{\infty} du \left[ \frac{-2 \beta \gamma^s}{(y+u)^2(1 - \alpha \gamma^r)^2 - \beta^2 \gamma^2 s^2} \right] - \frac{-\gamma^4}{y} \int_0^{\infty} du \frac{1}{\gamma^2 + u^2} \left[ \frac{-2 \beta \gamma^s}{(y+u)^2(1 - \alpha \gamma^r)^2 - \beta^2 \gamma^2 s^2} \right].
$$

In the first contribution, the integral converges and behaves as $\frac{1}{y}$, providing a global behaviour at least as $\frac{1}{y}$. In the second term, the factor $\frac{1}{\gamma^2 + u^2}$ provides an effective cut to the values of $u$ that contributes in the integral. We can in the integrand neglect $u$ with respect to $y$ and the global behaviour is in $\frac{1}{y^3}$. Therefore, the form (7.22) is compatible with our analysis of $\Delta(y, \gamma)$. 

14.2 First consistency check on $c(y, \gamma)$

We now turn to the analysis of $c(y, \gamma)$ itself. We can choose a positive sign to $y$ to fix the ideas since $\Delta(y, \gamma)$ can then provide the behaviour for negative $y$. We start with (cf. (F.1))

$$c(y, \gamma) = -\frac{1}{y} \int_0^\infty du \frac{u^2 \gamma^2}{\gamma^2 + u^2} \frac{1}{(i\epsilon + y - u)[1 - c(y - u, \gamma)]} + \frac{1}{(i\epsilon + y + u)[1 - c(y + u, \gamma)]}. \quad (F.11)$$

For $y > 0$, we have

$$c_\delta(y, \gamma) = -\frac{1}{y} (-\pi i) \int_0^\infty du u^2 \delta(y - u) \frac{1}{1 - c(y - u, \gamma)} \frac{\gamma^2}{\gamma^2 + u^2} = \frac{1}{y} (\pi i) \left[ \frac{1}{1 - c(0, \gamma)} \right] \frac{\gamma^2}{\gamma^2 + y^2}. \quad (F.12)$$

The contribution $c_P(y, \gamma)$ from the principal parts is:

$$c_P(y, \gamma) = -\frac{1}{y} \int_0^\infty du \frac{u^2 \gamma^2}{\gamma^2 + u^2} \left\{ \mathcal{P} \left( \frac{1}{y - u} \right) \frac{1}{1 - c(y - u, \gamma)} \right\} + \mathcal{P} \left( \frac{1}{y + u} \right) \frac{1}{1 - c(y + u, \gamma)}. \quad (F.13)$$

For the first contribution to $c_P(y, \gamma)$, the form (7.22) cannot be used since the argument $y - u$ of $c(y - u, \gamma)$ is not large for all values of $u$ inside the domain of integration. That integral cannot be written as a sum of integrals involving different principal terms since only the global integrand in (F.13) decreases enough to ensure the convergence at infinity. The following decomposition still holds (F.5):

$$c_P(y, \gamma) = -\frac{\gamma^2}{y} \int_0^\infty du \left[ \mathcal{P} \left( \frac{1}{y - u} \right) \frac{1}{1 - c(y - u, \gamma)} \right] + \mathcal{P} \left( \frac{1}{y + u} \right) \frac{1}{1 - c(y + u, \gamma)} + \frac{\gamma^4}{y} \int_0^\infty du \frac{1}{\gamma^2 + u^2} \left[ \mathcal{P} \left( \frac{1}{y - u} \right) \frac{1}{1 - c(y - u, \gamma)} \right] + \mathcal{P} \left( \frac{1}{y + u} \right) \frac{1}{1 - c(y + u, \gamma)}. \quad (F.14)$$
For the second term, the factor \( \frac{1}{\gamma^2 + u^2} \) ensures the convergence and cut the integral for values of \( u \) limited by \( \gamma \). For that part, the dominant behaviour in \( y \) is obtained by neglecting \( u \) in front of \( y \). For the first term in (F.14), the convergence in \( u \) is ensured by the other factors. For studying the convergence, we take \( u \) much larger than \( y \) and we can replace the \( c \) by their value (7.22) and the integrand \( I_1(u, y) \) is

\[
I_1(u, y) = -\frac{\gamma^2}{y} \left[ \frac{1}{y-u(1-\alpha\gamma^r - \beta\gamma^s)} + \frac{1}{y+u(1-\alpha\gamma^r - \beta\gamma^s)} \right]
\]  

\( \text{(F.15)} \)

\[
I_1(u, y) = -\frac{\gamma^2}{y} \left[ \frac{1}{(y-u)(1-\alpha\gamma^r) - \beta\gamma^s} + \frac{1}{(y+u)(1-\alpha\gamma^r) - \beta\gamma^s} \right]
\]  

\( \text{(F.16)} \)

\[
I_1(u, y) = \frac{\gamma^2}{y} \frac{2[y(1-\alpha\gamma^r) - \beta\gamma^s]}{u^2(1-\alpha\gamma^r)^2 - [y(1-\alpha\gamma^r) - \beta\gamma^s]^2}
\]  

\( \text{(F.17)} \)

Therefore, the factor that ensures the convergence of the integrand in \( u \) contains as a factor \( y \). The integral for \( c(y, \gamma) \) contains therefore a contribution independent of \( y \) for large \( y \). It corresponds to (and determines) the first term of (7.22). The \( \gamma \) dependence factor in the \( y \) independent contribution for very large \( y \) is thus \( \gamma^2 - \gamma^r \), that should be identified with \( \gamma^r \). The self consistency possibility is thus \( r = 1 \).

14.3 Second consistency check on \( c(y, \gamma) \)

In the expression (7.23) for \( c(y, \gamma) \), in the right hand side, three domains for the \( u \) integration variables can be distinguished. In the first one, the argument of \( c(y-u, \gamma) \) and \( c(y+u, \gamma) \) are inside the conditions assumed for (7.24): \( |y| << \gamma \). In another domain, we are in the conditions studied in the preceding section: \( |y| >> \gamma \). We have also the transition domain where \( y \) is of the order of \( \gamma \).

We separate anew in (7.24) the \( c_\delta(y, \gamma) \) and \( c_P(y, \gamma) \) contributions. If we consider \( y > 0 \), we have still the form (F.12):

\[
c_\delta(y, \gamma) = \frac{1}{y} \left( \pi \right) \frac{1}{[1 - c(0, \gamma)]} \frac{1}{y^2 \gamma^2} \frac{\gamma^2}{\gamma^2 + y^2} \sim \frac{\pi i y}{c(0, \gamma)}
\]  

\( \text{(F.18)} \)

Assuming the form (7.24) in \( c_\delta(y, \gamma) \) provides a behaviour as \( y\gamma^{-\frac{1}{2}} \) as dominant contribution, much smaller that the assumed (7.24) behaviour.
For the second term $c_P(y, \gamma)$, we can use (F.13):

$$
c_P(y, \gamma) = -\frac{1}{y} \int_0^\infty du \frac{u^2}{\gamma^2 + u^2} \left\{ \mathcal{P} \left( \frac{1}{y - u} \right) \frac{1}{1 - c(y - u, \gamma)} \right\}.
$$

(F.19)

Combining the behaviour (7.24) and the behaviour (7.22) where $r = 1$, we assume the following behaviour for $c_P(y, \gamma)$ for large $\gamma$.

$$
c_P(y, \gamma) = \gamma \frac{1}{2} g\left(\frac{y}{\gamma}\right) + \gamma g_1\left(\frac{y}{\gamma}\right),
$$

(F.20)

where the function $g_1$ has the qualitative features of $\frac{y^2}{\gamma^2 + (l\gamma)^2}$ ($l$ is a very large number): it is negligible for $y << l\gamma$ and becomes 1 in the other limit $y >> l\gamma$. This expression reproduces qualitatively the behaviour of $c_P(y, \gamma)$ for all values of $y$ with respect to $\gamma$. To check the consistency, we introduce that expression in (4.4) for $y << \gamma$. We have (1 is negligible in front of $\gamma$)

$$
\gamma \frac{1}{2} g\left(\frac{y}{\gamma}\right) = -\frac{1}{y} \int_0^\infty du \frac{u^2}{\gamma^2 + u^2} \left\{ \mathcal{P} \left( \frac{1}{y - u} \right) \frac{1}{\left[ \gamma \frac{1}{2} g\left(\frac{y - u}{\gamma}\right) + \gamma g_1\left(\frac{y - u}{\gamma}\right) \right]} \right\}.
$$

(F.21)

If we introduce variables $t = \frac{y}{\gamma}$, $v = \frac{u}{\gamma}$, we get

$$
\gamma \frac{1}{2} g(t) = -\frac{\gamma}{t} \int_0^\infty du \frac{1}{1 + v^2} \left\{ \mathcal{P} \left( \frac{1}{t - v} \right) \frac{1}{\left[ \gamma \frac{1}{2} g(t - v) + \gamma g_1(t - v) \right]} \right\}
$$

(F.22)

The front factor $\gamma \frac{1}{2}$ can be simplified and we have

$$
g(t) = -\frac{1}{t} \int_0^\infty du \frac{1}{1 + v^2} \left\{ \mathcal{P} \left( \frac{1}{t - v} \right) \frac{1}{\left[ g(t - v) + \gamma g_1(t - v) \right]} \right\} + \mathcal{P} \left( \frac{1}{t + v} \right) \frac{1}{\left[ g(t + v) + \gamma g_1(t + v) \right]}.
$$

(F.23)

The remaining dependence of $\gamma$ plays a role only for very large values of $t - v$ or $t + v$. Its consequence is to diminish the contribution for that part of the
domain of integration. Since that expression has no problem of convergence
due to the factor $\frac{1}{1+\nu}$, that dependence does not play a sensitive role in
the evaluation of $g$. It can therefore qualitatively be dropped and we can
deduce that $c_p(y, \gamma)$ behaves as $\gamma^{2}$ for the the relevant domain of values of
its argument.

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