Uniformly Loaded Rectangular Thin Plates with Symmetrical Boundary Conditions

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Abstract

In the article the Fourier series analytical solutions of uniformly loaded rectangular thin plates with symmetrical boundary conditions are considered. For all the cases the numerical values are tabulated.

1 Introduction

This article is motivated by the recent work of Lim et al. ([8]) and Shuang ([16]). The authors consider the deflection of a rectangular plate under a uniform load with symmetrical homogeneous boundary conditions along each plate edge. In general each edge may be simply supported (S), clamped (C) or free (F), so there are 21 different possible combinations of boundary conditions, which are listed in Table 1, where the Reddy notation of the boundary condition is adopted in which the consecutive pair of letters indicates a boundary condition on opposite edges ([15], p. 266). Note that the cases below the diagonal in Table 1 are obtained by rotation of a plate by 90°. As such the problem is one of the oldest in elasticity and it is discussed in more or less detail in any textbook on plate theory ([1], [3], [5], [10], [11], [15], [17], [18], [19], [20], [21]) and even in some books on the theory of elasticity ([2],[9],[22]). Now, for solving the problem the mentioned authors introduce the symplectic method, which reduces a two dimensional plate problem to an eigenvalue problem. Lim et al. ([8]) consider the plate with two opposite edges simply supported and the other two edges arbitrarily supported,
while Shuang ([16]) solves and discusses all the 21 boundary condition cases. Here it must be noted that the method is not quite new in plate theory since it was used, for example, by Morley ([10],[13]) to solve the clamped plate problem. As these authors indicate in their works, one of the advantages of the symplectic method in comparison to other methods is that the symplectic analysis is completely rational while the traditional approach is usually a unitization of the semi-inverse method where one should guess the trial functions which exist only for very special cases of boundary conditions. This may probably be true in general, but not for the case of a rectangular symmetrically supported plate and for all the cases of a plate simply supported on opposite edges, wherein one may obtain all the solutions from the solution of a biharmonic equation obtained by the Fourier method of separation of variables ([21]).

**Table 1.** Possible combinations of boundary conditions.

| Boundary condition | SS   | SC   | SF   | CC   | CF   | FF   |
|--------------------|------|------|------|------|------|------|
| SS                 | SSSS | SSSC | SSSF | SSSC | SSCF | SSSF |
| SC                 | SCSC | SCSF | SCCC | SCCF | SCFF | SCFF |
| SF                 | SFSF | SFCC | SFCF | SFFF |
| CC                 | CCCC | CCCC | CFFF |
| CF                 | CFCF | CFFF |
| FF                 | FFFF |

In this article the consideration is restricted to symmetrical cases of boundary conditions: SSSS, SSCC, SSFF, CCCC, CCFF and FFFF. According to the historical notes of Love ([9]), Timoshenko et al. ([18]) and Melelsho ([12]) the first SSSS plate problem was solved by Navier (1823) by using a double trigonometric series. Later Lévy (1899) provided a single trigonometric series solution of a plate which has two opposite edges simply supported. His solution may be found in all the quoted references. The clamped plate problem was solved by Koialovich (1902), Boobnoff (1914) and Hencky (1913), essentially reducing the problem to the solution of infinite systems of algebraic equations ([9],[12]). The FFFF plate was solved by Galerkin (1915) as a limit case of a plate with elasticity supported edges ([3],[4],[18]) and Nadai ([16]). A recent solution using the symplectic method was given by Lim et al. ([10]) and using the Fourier method by the present author ([1]). For the analytical solution of the
CCFF plate the present author found no historical reference so apparently the only
known analytical solution is given by Shuang ([16]) using the symplectic method.

There are two aims of this paper. The first is to demonstrate that by using the solution of
a biharmonic equation by the Fourier method one could obtain the solutions of a stated
boundary value problem in a unified rational way. The second is to provide a set of
reference values of deflection, moments and shear forces in selected referenced points
of the plate. There are several reasons for yet another set of reference values. First, in
the literature the reference values are scattered and are sometimes given only for a very
specific example. Second, probably the most quoted referenced values are those from
Timoshenko ([18]), however these reference values for displacements are given up to
five decimal places, for the moment up to four and for shear forces up to three. For
practical design purposes this is more than needed but for comportment with the values
obtained by different methods it is sometimes not enough. Also the reference values for
shear forces at the edge of the plate are given only for a simply supported plate; the
reference values for the FFFF plate are given only for a square plate while the reference
values for the CCFF plate are not given. It must be noted that the Shuang work ([16])
covers all the cases, yet the plate aspect ratio for all the symmetrical cases is restricted
from 1 to 2, and for the FFFF plate only a square plate is considered. Also in this work
the accuracy of numerical calculation is not explicitly controlled; rather the author gives
consecutive results with increasing numbers of series terms.

While most of the material in this work is well known it also contains a new Fourier
series analytical solution of the CCFF plate and also, in contrast to Timoshenko ([18]),
Lim et al. ([8]) and Shuang ([16]), in all the discussed cases the values of shear forces
at the middle of a plate clamped or simply supported edges are provided and in most
cases the present values cover a wider aspect ratio of the plate – mostly from 1 to 5.
Before proceeding, a note on the numerical calculation strategy for summing a series is
given. In all the discussed cases the calculation was performed in quad precision and the
condition for termination of series summation was \( |s_{n+1} - s_n| < tol \times (1 + |s_n|) \) where \( s_n \) is
the value after summing \( n \) terms and where \( tol \) was taken \( 10^{-9}, 10^{-8} \) and \( 10^{-7} \) for
calculation of deflections, moments and shear forces, respectively, in most cases. In this
way the values are accurate to eight, seven and six decimal places. In all the Tables $N$ indicates the number of the series term used for calculation.

## 2 General considerations

Consider a homogeneous isotropic elastic rectangular thick plate of sides $a' = 2a$ and $b' = 2b$ subject to a uniformly distributed load $q$. The Cartesian coordinate system $Oxy$ is originated at the center of the plate and the plate is orientated in the way that it occupies the region $-a \leq x \leq a$ and $-b \leq y \leq b$. The governing equation of the plate is ([18], p. 82)

$$\Delta^2 w = \frac{q}{D}$$

(1)

where \( \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is a two-dimensional Laplace operator, \( D \equiv \frac{h^3 E}{12(1-\nu^2)} \) is bending stiffness, $E$ is the elastic module, $h$ is plate thickness, $\nu$ is the Poisson ratio, and $w$ is the transverse deflection of the middle plane of the plate. Once $w$ is known one can calculate the bending moments $M_x$, $M_y$ and twisting moment $M_{xy}$ from ([18], pp 81)

$$M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)$$

$$M_{xy} = - (1-\nu) D \frac{\partial^2 w}{\partial x \partial y}$$

(2)

and shear forces $Q_x$, $Q_y$ ([18],pp 81) and effective shear forces $V_x$, $V_y$ from ([18], p. 84)

$$Q_x = -D \frac{\partial \Delta w}{\partial x} \quad Q_y = -D \frac{\partial \Delta w}{\partial y}$$

$$V_x = -D \left[ \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] \quad V_y = -D \left[ \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right]$$

(3)

The equation (1) should be solved in such a way that the boundary conditions at the edge of the plate are satisfied.
For a symmetrical boundary condition the solution of governing equation (1) should be symmetrical in \(x\) and \(y\). The symmetrical solution of (1) obtained by the Fourier method of separation of variables ([21]) may be written in the form

\[
w = w_0 + \frac{q}{D} \sum_{n=0}^{\infty} (-1)^n \left( A_n \frac{\cosh \alpha_n y}{\cosh \alpha_n b} + B_n \frac{y \sinh \alpha_n y}{b \cosh \alpha_n b} \right) \cos \alpha_n x \\
+ \frac{q}{D} \sum_{n=0}^{\infty} (-1)^n \left( C_n \frac{\cosh \beta_n x}{\cosh \beta_n a} + D_n \frac{x \sinh \beta_n x}{a \cosh \beta_n a} \right) \cos \beta_n y
\]

(5)

where \( w_0 \) is a particular solution satisfying \( \Delta^2 w_0 = \frac{q}{D} \) and where

\[
\alpha_n = \frac{(2n+1)\pi}{2a} \quad \beta_n = \frac{(2n+1)\pi}{2b} \quad (n = 0, 1, 2, \ldots)
\]

(6)

The particular solution \( w_0 \) is taken in the form of a symmetrical polynomial of the fourth order in \(x\) and \(y\)

\[
w_0 = c_0 + c_1 x^2 + c_2 y^2 + c_3 x^4 + c_4 x^2 y^2 + c_5 y^4
\]

(7)

This solution must satisfy the plate equation, so

\[
3c_3 + c_4 + 3c_5 = \frac{q}{8D}
\]

(8)

The task is now to determine the unknown coefficients in (5) by boundary conditions.

Before proceeding, a few words on the determination of the particular solution are in order. It is evident that the ease of solution of a specific plate boundary value problem depends on the particular solution \( w_0 \). A strategy for constructing \( w_0 \) which simplifies the solution is the following. Observe that any of the discussed boundary conditions involve either a prescribed zero vertical displacement and/or a zero bending moment. Since according to (5) the displacement and the bending moments are functions of \( \{ \cosh \alpha_n y \} \cos \alpha_n x \) and \( \{ \cosh \beta_n x \} \cos \beta_n y \), one of these function will be zero on a particular edge and only the others will remain without hyperbolic dependence on \(x\) or \(y\). Now if \( w_0 \) also satisfies one or both displacement/moment boundary condition then
this boundary condition becomes homogeneous and one set of unknown coefficients may be directly expressed by another and this may simplify the future treatment of the problem. In the next section the specific problems will be discussed in some detail.

3 Plate with two opposite edges simply supported

Consider the plate with edges \( x = \pm a \) simply supported. In this case the boundary conditions are

\[
\begin{align*}
w &= 0 \\
\frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{at} \quad x = \pm a
\end{align*}
\]  

(9)

These boundary conditions are satisfied by the particular solution of the form of for a simply supported beam

\[
w_0 = \frac{q}{24D} \left( x^4 - 6a^2x^2 + 5a^4 \right)
\]  

(10)

By (10) the boundary conditions (9) reduce to a homogeneous system for unknown coefficients \( C_n \) and \( D_n \) so the solution is

\[
C_n = D_n = 0
\]  

(11)

From the remaining two boundary conditions one may calculate the coefficients \( A_n \) and \( B_n \).

3.1 Simply supported plate (SSSS)

In this case boundary conditions which should be satisfied are

\[
\begin{align*}
w &= 0 \\
\frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{at} \quad y = \pm b
\end{align*}
\]  

(12)

From the boundary condition \( \frac{\partial^2 w}{\partial y^2}(x, \pm b) = 0 \) one obtains
\[ A_n = -B_n \left( \tanh \alpha_n b + \frac{2}{\alpha_n b} \right) \]  \hspace{1cm} (13)

and further by expanding \( w(x, b) = 0 \) into a Fourier series in \( \cos \alpha_n x \) one finds

\[ B_n = \frac{b}{a \alpha_n^4} \]  \hspace{1cm} (14)

The coefficients of the series converge very rapidly as \( O(n^{-4}) \).

**Table 2a.** Deflection, bending moment and shear force factors for uniformly loaded simply supported rectangular plate with \( \nu = 0.3 \)

| \( b/a \) | \( \omega = aqa_1^4/D \) | \( M_x = \beta qa_1^2 \) | \( M_y = \beta_i qa_1^2 \) |
|-------|-----------------|-----------------|-----------------|
| 1     | 0.00406235      | 3               | 0.0478864       | 4               | 0.0478864       | 4               |
| 3/2   | 0.00772402      | 2               | 0.0811601       | 3               | 0.0498427       | 3               |
| 2     | 0.01012866      | 2               | 0.1016381       | 2               | 0.0463503       | 2               |
| 3     | 0.01223281      | 1               | 0.1188605       | 2               | 0.0406266       | 2               |
| 4     | 0.01281865      | 1               | 0.1234586       | 1               | 0.0384150       | 1               |
| 5     | 0.01297083      | 1               | 0.1246245       | 1               | 0.0377453       | 1               |
| \( \infty \) | 0.01302083 | 1               | 0.1250000       | -               | 0.0375000       | -               |

**Table 2b.** Shear force factors for uniformly loaded simply supported rectangular plate with \( \nu = 0.3 \)

| \( b/a \) | \( \gamma qa' \) | \( V_x = \delta qa' \) | \( Q_x = \gamma qa' \) | \( V_y = \delta_i qa' \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| 1     | -0.337657 4     | -0.420471 4     | -0.337657 870   | -0.420471 981   |
| 3/2   | -0.423781 3     | -0.485646 3     | -0.364010 862   | -0.479617 961   |
| 2     | -0.465030 2     | -0.503354 2     | -0.369716 860   | -0.495800 956   |
| 3     | -0.492719 1     | -0.504726 2     | -0.371162 860   | -0.500852 955   |
| 4     | -0.498486 1     | -0.501815 1     | -0.371224 860   | -0.501140 955   |
| 5     | -0.499685 1     | -0.500550 1     | -0.371227 860   | -0.501155 955   |
| \( \infty \) | -0.500000 - | -0.500000 -   | -0.371227 870   | -0.501156 -     |
Table 2a contains non-dimensional deflection and bending moments at the center of the plate and Table 2b contains non-dimensional shear forces and reduced shear forces at the middle of the plate edges for various ratios of $b/a$. Values given in Table 2a and Table 2b match those in [18] (Table 8, p. 120) for specified decimal places. The exceptions are the value of the factor for $M_y$ for $b/a=5$ where Timoshenko gives the value for $b/a=\infty$ and the values for $Q_y$ and $V_y$ where the values match in two decimal places. Note that the limit when $b \to \infty$ gives the following values for $Q_y$ and $V_y$ at the middle of the plate edge

$$\lim_{b \to \infty} Q_y(0, \pm b) = -\frac{8G}{\pi^2} qa \quad \lim_{b \to \infty} V_y(0, \pm b) = -\frac{4(3-\nu)G}{\pi^2} qa$$

(15)

where $G \approx 0.9159656$ is Catalan’s constant. These values differ slightly from those for strip reported by Timoshenko. The values given in Table 2a also match those of Lim ([8], Table 1) for all decimal places.

### 3.2 Plate with two opposite edges simply supported and the other two clamped (SSCC)

In this case the remaining boundary conditions which should be satisfied are

$$w = 0 \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = \pm b$$

(16)

From the condition $\frac{\partial w}{\partial y}(x, \pm b) = 0$ one finds

$$A_n = -B_n \left( \coth \alpha_n b + \frac{1}{\alpha_n b} \right)$$

(17)

and further by expanding $w(x, \pm b) = 0$ into a Fourier series in $\cos \alpha_n x$ one finds

$$B_n = \frac{2b}{a} \frac{\tanh \alpha_n b}{\alpha_n^4 \left( \tanh \alpha_n b + \alpha_n b \cosh^{-2} \alpha_n b \right)}$$

(18)

The coefficients of the series converge very rapidly as $O(n^{-4})$. 

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Table 3a. Deflection and bending moments in uniformly loaded rectangular plate with edges \( x = \pm a \) simply supported and other clamped. \( \nu = 0.3 \)

| \( b/a \) | \( w = aqa^4/D \) | \( M_x = \beta qa^2 \) | \( M_y = \beta_1 qa^2 \) | \( \alpha \) | \( N \) | \( \beta \) | \( N \) | \( \beta_1 \) | \( N \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00191714 | 3 | 0.02438749 | 4 | 0.03324494 | 4 |
| 1.5 | 0.00532645 | 2 | 0.05848044 | 3 | 0.04594444 | 3 |
| 2.0 | 0.00844580 | 2 | 0.08686812 | 2 | 0.04736222 | 2 |
| 3.0 | 0.01168129 | 2 | 0.11435721 | 2 | 0.04212632 | 2 |
| 4.0 | 0.01266531 | 1 | 0.12225471 | 1 | 0.03899271 | 1 |
| 5.0 | 0.01293098 | 1 | 0.12431911 | 1 | 0.03792051 | 1 |

Table 3b. Shear forces in uniformly loaded rectangular plate with edges \( x = \pm a \) simply supported and others clamped. \( \nu = 0.3 \)

| \( a/b \) | \( w = aqb^4/D \) | \( M_x = \beta qb^2 \) | \( M_y = \beta_1 qb^2 \) | \( \alpha \) | \( N \) | \( \beta \) | \( N \) | \( \beta_1 \) | \( N \) |
|---|---|---|---|---|---|---|---|---|---|
| 1.5 | 0.00247571 | 5 | 0.01780033 | 6 | 0.04062764 | 6 |
| 2.0 | 0.00261080 | 6 | 0.01417178 | 8 | 0.04206298 | 8 |
| 3.0 | 0.00261488 | 9 | 0.01249867 | 12 | 0.04183111 | 12 |
| 4.0 | 0.00260519 | 12 | 0.01247511 | 15 | 0.04167803 | 15 |
| 5.0 | 0.00260412 | 14 | 0.01249741 | 19 | 0.04166548 | 18 |

Table 3a contains non-dimensional deflection and bending moments at the center of the plate and Table 3b contains non-dimensional bending moment and shear forces at the
middle of the plate edges for various ratios of $b/a$. Values for deflection and moments
given in Table 3a and Table 3b match those given by Lim et al. ([9] Table 3) and
Timoshenko ([18], Table 29, p. 187) for three to four decimal places. There are some
minor differences between values. For example for $b/a = 1.5$ Timoshenko reports the
value 0.00531 for the deflection factor while in Table 3a this factor is 0.00533.

3.3 Plate with two opposite edges simply supported and others free (SSFF)

The remaining boundary conditions which should be satisfied for this case are

$$M_y = 0 \quad V_y = 0 \quad \text{at} \quad y = \pm b$$

From the condition $V_y(x, \pm b) = 0$ one obtains

$$A_n = -B_n \left( \coth \alpha_n b - \frac{1 + \nu}{1 - \nu} \frac{1}{\alpha_n b} \right)$$

and further by expanding $M_y(x, \pm b) = 0$ into a Fourier series in $\cos \alpha_n x$ one finds

$$B_n = \frac{2\nu b}{(3 + \nu)a} \frac{\tanh \alpha_n b}{\alpha_n^4 \left( \tanh \alpha_n b \frac{1 - \nu}{3 + \nu} \alpha_n b \cosh^2 \alpha_n b \right)}$$

The coefficients of the series converge very rapidly as $O(n^{-4})$.

Table 4a contains non-dimensional deflection and bending moments at the center of the
plate and Table 4b contains non-dimensional deflection, bending moment and shear
forces at the middle of the plate edges for various ratios of $b/a$. Values for deflection
and moments given in Table 4a and Table 4b match those given by Lim et al. ([9],
Table 2) and those given by Timoshenko ([18], Table 47, p. 187) for three to four
decimal places. There are some minor differences between values. For example for
$b/a = 2$ at the middle of the free plate edge Timoshenko reports the value 0.01521 for
the deflection factor and 0.1329 for the bending moment factor while in Table 4b these
factors are 0.01520 and 0.13280.
### Table 4a. Deflection and bending moments in uniformly loaded rectangular plate with edges $x = \pm a$ simply supported and other two free. $\nu = 0.3$

| $b/a$ | $w = \alpha qa^4/D$ | $M_x = \beta qa^2$ | $M_y = \beta qa^2$ |
|-------|----------------------|---------------------|---------------------|
|       | $\alpha$ | $N$ | $\beta$ | $N$ | $\beta_0$ | $N$ |
| 1     | 0.01309368   | 3   | 0.1225454 | 3   | 0.0270781  | 1   |
| 3/2   | 0.01289772   | 2   | 0.1228061 | 2   | 0.0338624  | 2   |
| 2     | 0.01288730   | 2   | 0.1234678 | 2   | 0.0363888  | 2   |
| 3     | 0.01295984   | 1   | 0.124522  | 1   | 0.0374998  | 1   |
| 4     | 0.01300119   | 1   | 0.1248380 | 1   | 0.0375481  | 1   |
| 5     | 0.01301530   | 1   | 0.1249563 | 1   | 0.0375000  | 1   |
| 10    | 0.01302083   | 1   | 0.1250000 | 1   | 0.0375000  | 1   |

### Table 4b. Deflection, bending moments and shear force in uniformly loaded rectangular plate with edges $x = \pm a$ simply supported and other two free. $\nu = 0.3$

| $b/a$ | $w = \alpha qa^4/D$ | $M_x = \beta qa^2$ | $Q_x = \gamma qa'$ |
|-------|----------------------|---------------------|---------------------|
|       | $\alpha_1$ | $N$ | $\beta_2$ | $N$ | $\gamma$ | $N$ |
| 3/2   | 0.01501126   | 9   | 0.1310877 | 3   | -0.468685 | 3   |
| 2     | 0.01515706   | 9   | 0.1323969 | 45  | -0.485889 | 2   |
| 3     | 0.01520217   | 9   | 0.1328020 | 45  | -0.493610 | 2   |
| 4     | 0.01521806   | 9   | 0.1329447 | 45  | -0.498767 | 1   |
| 5     | 0.01521915   | 9   | 0.1329545 | 45  | -0.499943 | 1   |
| 10    | 0.01521916   | 9   | 0.1329545 | 45  | -0.500000 | 0   |

| $a/b$ | $w = \alpha qb^4/D$ | $M_x = \beta_2qb^2$ | $Q_x = \gamma qb'$ |
|-------|----------------------|---------------------|---------------------|
|       | $\alpha_1$ | $N$ | $\beta_2$ | $N$ | $\gamma$ | $N$ |
| 3/2   | 0.07489906  | 13  | 0.2905851 | 56  | -0.671182 | 5   |
| 2     | 0.23431397  | 15  | 0.5112501 | 65  | -0.866510 | 6   |
| 3     | 1.17352611  | 19  | 1.1378446 | 74  | -1.252217 | 9   |
| 4     | 3.69022839  | 21  | 2.0132905 | 82  | -1.636917 | 11  |
| 5     | 8.98672614  | 21  | 3.1384141 | 85  | -2.021539 | 13  |
4. Plate with all edges clamped (CCCC)

For a plate with clamped edges the boundary conditions are

\[ w = 0 \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = \pm a \] (22)

\[ w = 0 \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = \pm b \] (23)

The particular solution is constructed from the conditions \( w_0(\pm a, y) = 0 \) and \( w_0(x, \pm b) = 0 \). In this way one obtains

\[ w_0 = \frac{q}{8D} \left( x^2 - a^2 \right) \left( y^2 - b^2 \right) \] (24)

By the above particular solution the boundary conditions \( w(\pm a, y) = 0 \) and \( w(x, \pm b) = 0 \) become homogeneous and yield respectively

\[ C_n = -D_n \tanh \beta_n a \quad A_n = -B_n \tanh \alpha_n b \] (25)

By expanding remaining boundary conditions \( \frac{\partial w}{\partial y}(x, \pm b) = 0 \) and \( \frac{\partial w}{\partial x}(\pm a, y) = 0 \) into a trigonometric series in \( \cos \alpha_n x \) and \( \cos \beta_n y \) is yielded the infinite system of algebraic equations

\[ \frac{a}{b} B_n \left( \tanh \alpha_n b + \alpha_n b \cosh^{-2} \alpha_n b \right) = -\frac{4}{a} \sum_{m=0}^{\infty} D_m \frac{\alpha_n \beta_m^2}{(\alpha_n^2 + \beta_m^2)^2} + \frac{qb}{D} \frac{1}{\alpha_n^3} \] (26)

\[ \frac{b}{a} D_n \left( \tanh \beta_n a + \beta_n a \cosh^{-2} \beta_n a \right) = -\frac{4}{b} \sum_{m=0}^{\infty} B_m \frac{\alpha_m \beta_n^2}{(\alpha_m^2 + \beta_n^2)^2} + \frac{qa}{D} \frac{1}{\beta_n^3} \] (27)

The above system is equivalent to the system obtained by Hencky ([9]). The approximate solution of the system may be obtained by successive approximations. It may be shown that with the new unknowns

\[ B_n' = a \alpha_n B_n / b \quad D_n' = -b \beta_n D_n / a \] (28)
the system becomes fully regular so it has a unique bounded solution which can be obtained by the method of reduction ([6],[12]).

Table 5a. Deflection and bending moments in uniformly loaded rectangular plate with clamped edges. \( \nu = 0.3 \)

| \( b/a \) | \( w = \alpha qa^4/D \) | \( M_x = \beta qa^2 \) | \( M_y = \beta qa^2 \) |
|-------|----------------|----------------|----------------|
| \( \alpha \) | \( \beta \) | \( \beta_1 \) | \( \beta_2 \) |
| 1     | 0.00126532   | 3              | 4              |
| 1.5   | 0.00219652   | 5              | 6              |
| 2     | 0.00253296   | 6              | 8              |
| 3     | 0.00261723   | 9              | 11             |
| 4     | 0.00260659   | 12             | 15             |
| 5     | 0.00260423   | 14             | 18             |
| 10    | 0.00260417   | 27             | 34             |

Table 5b. Bending moments and shear forces in uniformly loaded rectangular plate with clamped edges. \( \nu = 0.3 \)

| \( b/a \) | \( M_x = \beta_2 qa^2 \) | \( Q_x = \gamma qa' \) | \( M_y = \beta_3 qa^2 \) | \( Q_y = \gamma qa' \) |
|-------|----------------|----------------|----------------|----------------|
| \( \beta_2 \) | \( \beta_3 \) | \( \gamma_1 \) | \( \gamma_2 \) |
| 1     | -0.0513338    | 183            | -0.441301     | 315            |
| 1.5   | -0.0756586    | 241            | -0.514332     | 427            |
| 2     | -0.0828661    | 298            | -0.516015     | 566            |
| 3     | -0.0837766    | 52             | -0.502594     | 844            |
| 4     | -0.0833867    | 495            | -0.500035     | 1115           |
| 5     | -0.0833324    | 87             | -0.499966     | -1201          |
| 10    | -0.0833333    | 174            | -0.500000     | 175            |

Table 5a contains non-dimensional deflection and bending moments at the center of the plate and Table 5b contains non-dimensional bending moments and also shear forces at the middle of the plate edges for various ratios of \( b/a \). The values of corresponding bending data match those given by Shuang ([16], Table 4.3) and those given by Timoshenko ([18], Table 35, p. 161) from four to five decimal places. There are some minor differences between values. For example for \( b/a = 2 \) Timoshenko reports the
value 0.00254 for the deflection factor in the middle of the plate while in Table 5a this factor is 0.002533.

5. Plate with two opposite edges clamped and the other two free

Consider a plate with edges \( x = \pm a \) clamped and the other edges free. The boundary conditions for this case are

\[
\begin{align*}
 w &= 0 \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = \pm a \\
 M_y &= 0 \quad V_y = 0 \quad \text{at} \quad y = \pm b
\end{align*}
\]

(29) (30)

The particular solution is determined by the conditions \( w_0(\pm a, y) = 0 \) and \( M_{0y}(x, \pm b) = 0 \). In this way one obtains

\[
w_0 = \frac{q}{24(1-2\nu)} D \left( a^2 - x^2 \right) \left( 5a^2 - 6\nu b^2 - x^2 + 6\nu y^2 \right)
\]

(31)

By this the conditions \( M_y(x, \pm b) = 0 \) and \( w(\pm a, y) = 0 \) become homogeneous and yield respectively

\[
A_n = -B_n \left( \tanh \alpha_n b + \frac{2}{1-\nu} \frac{1}{\alpha_n b} \right)
\]

(32)

\[
C_n = -D_n \tanh \beta_n a
\]

(33)

By expanding the remaining boundary conditions \( V_y(x, \pm b) = 0 \) and \( \frac{\partial w}{\partial x}(\pm a, y) = 0 \) into a trigonometric series in \( \cos \alpha_n x \) and \( \cos \beta_n y \) is yielded the infinite system of algebraic equations

\[
\frac{a}{b} B_n \left( \frac{3+\nu}{1-\nu} \tanh \alpha_n b - \alpha_n b \cosh^{-2} \alpha_n b \right) = \frac{4}{(1-\nu)a} \sum_{m=0}^{\infty} D_m \frac{\beta_m^2 \left( (2-\nu)\alpha_n^2 + \beta_m^2 \right)}{\alpha_n \left( \alpha_n^2 + \beta_m^2 \right)^2} + \frac{2(2-\nu)\nu q b}{(1-\nu)(1-2\nu) D} \frac{1}{\alpha_n^2}
\]

(34)
\[
\frac{b}{a} D_n \left( \tanh \beta_n a + \beta_n a \cosh^{-2} \beta_n a \right) = - \frac{4}{(1-v)b} \sum_{n=0}^{\infty} B_n \beta_n \left[ \frac{(2-v)\alpha_n^2 + \beta_n^2}{(\alpha_n^2 + \beta_n^2)^2} \right] + \frac{2qa}{3(1-2v)D} \alpha_n^2 \beta_n^3 - 3v
\]

(35)

The sum of the coefficients on the right side of equation (34) is divergent so the approximate solution of the system may be obtained by solving this system by some direct method. In order to solve the system by the iterative method this equation must be modified and this may be done in the following way. Dividing the first equation by \( \alpha_n \) and the second by \( \beta_n \), then summing each of the results on \( n \) and interchanging the order of summation on \( n \) and \( m \), then multiplying first by \( \frac{1-v}{1+v} \) and subtracting it from the second one finds

\[
S \equiv \sum_{n=0}^{\infty} D_n (1+v \cosh^{-2} \beta_n a) = \frac{qab}{6(1-2v)D} \left[ (1-v+v^2) a^2 - v(1+v)b^2 \right]
- \nu \sum_{n=0}^{\infty} \frac{B_n}{\alpha_n b} \left( \frac{2}{1-v} \tanh \alpha_n b - \alpha_n b \cosh^{-2} \alpha_n b \right)
\]

(36)

If one set \( S \) as new the system (34) may be rewritten as

\[
\frac{a}{b} B_n \left( \frac{3+v}{1-v} \tanh \alpha_n b - \alpha_n b \cosh^{-2} \alpha_n b \right) = \frac{4S}{(1-v)a\alpha_n}
- \frac{4}{(1-v)a} \sum_{n=0}^{\infty} D_n \left[ \frac{\alpha_n^2 (\alpha_n^2 + v\beta_n^2)}{(\alpha_n^2 + \beta_n^2)^2} + v \cosh^{-2} \beta_n a \right]
+ \frac{2(2-v)\nu qb}{(1-v)(1-2v)D} \frac{1}{\alpha_n^3}
\]

(37)

The obtained system of equations (34), (36) and (37) may be solved by the iterative method, however the system, as may be shown, is not regular so the existence and uniqueness of the solution is not guaranteed.

Table 6a contains nondimensional deflection and bending moments at the center of the plate and Table 6b contains nondimensional deflection, bending moment and shear forces at the middle of the plate edges for various ratios of \( b/a \). The values of corresponding bending data match those given by Shuang ([16], Table 4.3 ) from five to six decimal places. There are some minor differences between values. For example
Shuang reports for $b/a = 1$ the value 0.00255911 for the deflection factor and the value 0.0406016 for the bending moment $M_x$ factor. In Table 6a these values are 0.00255977 and 0.0406076.

**Table 6a.** Deflection and bending moments in uniformly loaded rectangular plate with edges $x = \pm a/2$ clamped and other two free. $\nu = 0.3$

| $b/a$ | $w = \alpha qa^4/D$ | $M_x = \beta qa^2$ | $M_y = \beta' qa^2$ |
|-------|---------------------|---------------------|---------------------|
|       | $\alpha'$         | $\beta'$           | $\beta'$           |
|       | $N$                | $N$                | $N$                |
| 1     | 0.00255977         | 4                   | 0.0406076          | 5                   | 0.0109358 | 5 |
| 2/3   | 0.00257164         | 6                   | 0.0411260          | 7                   | 0.0123035 | 8 |
| 2     | 0.00259010         | 8                   | 0.0414649          | 10                  | 0.0125828 | 10 |
| 3     | 0.00260331         | 11                  | 0.0416593          | 14                  | 0.0125334 | 15 |
| 4     | 0.00260428         | 14                  | 0.0416689          | 19                  | 0.0125023 | 19 |
| 5     | 0.00260419         | 18                  | 0.0416670          | 23                  | 0.0124997 | 24 |
| 10    | 0.00260417         | 34                  | 0.0416667          | 44                  | 0.0125000 | 45 |

| $a/b$ | $w = \alpha qb^4/D$ | $M_x = \beta qb^2$ | $M_y = \beta' qb^2$ |
|-------|---------------------|---------------------|---------------------|
|       | $\alpha'$         | $\beta'$           | $\beta'$           |
|       | $N$                | $N$                | $N$                |
| 3/2   | 0.13094422         | 6                   | 0.0906574          | 8                   | 0.0195476 | 8 |
| 2     | 0.04192609         | 8                   | 0.1608362          | 11                  | 0.0278178 | 11 |
| 3     | 0.21632886         | 12                  | 0.3627751          | 16                  | 0.0414663 | 16 |
| 4     | 0.69228326         | 16                  | 0.6479674          | 21                  | 0.0500094 | 22 |
| 5     | 1.70513325         | 20                  | 1.0167348          | 26                  | 0.0544031 | 27 |
| 10    | 27.86972536        | 35                  | 4.1116287          | 51                  | 0.0576706 | 56 |
Table 7b. Deflection and bending moments in uniformly loaded rectangular plate with edges $x = \pm a/2$ clamped and other two free.

| $b/a$ | Clamped: $x = \pm a/2$ $y = 0$ | Free: $x = 0$ $y = \pm b/2$ |
|-------|--------------------------|--------------------------|
| $M_x = \beta_2 qa^2$ | $Q_x = \gamma qa'$ | $w = \alpha_1 qa^4/D$ | $M_y = \beta_3 qa^2$ |
| $\beta_2$ | $N$ | $\gamma$ | $N$ | $\alpha_1$ | $N$ | $\beta_3$ | $N$ |
| 1 | -0.08155 | 118 | -0.3462 | -1201 | 0.00290883 | 77 | 0.04342 | 170 |
| 3/2 | -0.08236 | 172 | -0.4953 | 188 | 0.00291996 | 77 | 0.04362 | 169 |
| 2 | -0.08299 | 224 | -0.4992 | 251 | 0.00291997 | 77 | 0.04362 | 168 |
| 3 | -0.08332 | 321 | -0.5001 | 379 | 0.00291979 | 77 | 0.04362 | 167 |
| 4 | -0.08334 | 410 | -0.5000 | 509 | 0.00291979 | 77 | 0.04361 | 166 |
| 5 | -0.08333 | 493 | -0.5001 | 642 | 0.00291979 | 77 | 0.04361 | 164 |
| 10 | -0.08334 | 832 | -0.5006 | -1201 | 0.00291979 | 77 | 0.04362 | 161 |

| $a/b$ | $M_x = \beta_2 qb^2$ | $Q_x = \gamma qb'$ | $w = \alpha_1 qb^4/D$ | $M_y = \beta_3 qb^2$ |
|-------|--------------------------|--------------------------|--------------------------|--------------------------|
| $\beta_2$ | $\gamma$ | $N$ | $\alpha_1$ | $N$ | $\beta_3$ | $N$ |
| 3/2 | -0.18469 | 81 | -0.2684 | -1201 | 0.01452547 | 110 | 0.09645 | 177 |
| 2 | -0.33449 | 60 | 0.0661 | -1201 | 0.04533656 | 132 | 0.16961 | 175 |
| 3 | -0.78129 | -1201 | 1.8399 | -1201 | 0.22657575 | 152 | 0.37671 | 167 |
| 4 | -1.42875 | -1201 | 5.7571 | -1201 | 0.71335454 | 156 | 0.66520 | 163 |
| 5 | -2.27763 | -1201 | 12.6762 | -1201 | 1.74092355 | 159 | 1.03566 | 164 |
| 10 | -9.53903 | -1201 | 121.5250 | -1201 | 28.03268847 | 191 | 4.13181 | 198 |

6. Plate resting on corner points with all edges free (FFFF)

The detail solution for this case may be found in ([1])

7. Conclusions

It was shown that the solution of the uniformly loaded rectangular plate may be obtained from the general solution of a biharmonic equation by the Fourier method of separation of variables. In the case of a plate with opposite edges simply supported one obtains well known explicate expressions for unknown coefficients of deflection series expansion while for the other cases these coefficients constitute the infinite system of algebraic equations and may be approximately calculated from the truncated system by successive approximations. For the case of the CCCC plate and the FFFF plate the successive approximations converge quickly, while for the CCFF plate the convergence
is slow. Regarding the problem, there is no advantage to using the symplectic method for cases of simply supported edges since both methods lead to the same results and among them the Fourier method results are obtained directly from biharmonic equations. For other cases the symplectic method leads to a solution of the transcendental equation with complex roots and in addition to the solution of infinite system of algebraic equations for unknown eigenvalue expansion coefficients. And this is numerically no simpler than solving the infinite system of equations obtained by the Fourier method.

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