The analysis of some special results of a Lasota–Wazewska model with mixed variable delays

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Abstract

This study is about getting some conditions that guarantee the existence and uniqueness of the weighted pseudo almost periodic (WPAP) solutions of a Lasota–Wazewska model with time-varying delays. Some adequate conditions have been obtained for the existence and uniqueness of the WPAP solutions of the Lasota–Wazewska model, which we dealt with using some differential inequalities, the WPAP theory, and the Banach fixed point theorem. Besides, an application is given to demonstrate the accuracy of the conditions of our main results.

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1 Introduction

In 1976, Wazewska and Lasota [1] presented the delayed logistic differential model

\[ z'(t) = -\varphi(t)z(t) + \sum_{k=1}^{p} \kappa_k(t) e^{-\eta_k(t)(t-t_{\rho_k(t)})} \]  

(1.1)

to define the survival of red cells in an animal [2]. In (1.1) \( p \) is a positive integer, \( z(t) \) stands for the number of red blood cells at time \( t \), \( \varphi(t) \) stands for the death rate of the red blood cell, \( \kappa_k(t) \) and \( \eta_k(t) \) are related to the production of red blood cells per unit time, and \( t_{\rho_k(t)} \) represents the time to produce a red blood cell. For details, see [1, 3], also [4, 5] for logistic-type models from biological models as (1.1), but involving also diffusion and drift contributions.

Zhou [6] considered the following model:

\[ z'(t) = -\delta(t)z(t) + \sum_{j=1}^{m} c_j(t) e^{-\omega_j(t)\int_{t}^{\infty} C_j(s)z(t+s) \, ds} \]  

(1.2)
The author obtained some conditions on the almost periodic solution of this model using the fixed point theorem in cones. In [7], the researchers established some qualitative behaviors of PAP solutions of the following equation with constant delays:

\[ z'(t) = -\alpha(t)z(t) + \sum_{j=1}^{m} A_j(t)e^{-\omega_j(t)} \int_{t-\omega_j(t)}^{0} C_j(t-s)z(s)ds + \sum_{i=1}^{n} B_i(t)e^{-(t-t_i)} \beta_i(t), \quad t \in \mathbb{R}. \]  

(1.3)

The study of almost periodic (AP) and pseudo almost periodic (PAP) differential equations is one of the most interesting issues for the study of almost periodic of many mathematicians: indeed, they are of great importance even in probability for investigating stochastic processes in stability problems tied to oscillatory phenomena [1, 3, 6–24], and [25]. In [26], Diagana familiarized the concept of (WPAP) functions, which is a natural generalization of the concept of (PAP) functions. Since then, some interesting and remarkable results concerning composition theorem, translation invariance, and the ergodicity of (WPAP) have been obtained [26–29]. It is clear that under some limitations of weight function, many of the properties of almost periodic (AP) and pseudo almost periodic (PAP) are valid in this type of class. Thanks to the invariant property under translation, it is quite simple to investigate such solutions in delayed differential equations. For some works on the pseudo almost periodic solutions, oscillation of solutions, and so forth of various differential equations, see [4, 5, 24, 25, 30–34].

Our main purpose is to obtain some sufficient conditions for the existence, uniqueness, and global exponential stability of (WPAP) solutions of the following Lasota–Wazewska model with mixed variable delays:

\[ z'(t) = -\delta(t)z(t) + \sum_{j=1}^{m} A_j(t)e^{-\omega_j(t)} \int_{t-\omega_j(t)}^{0} C_j(t-s)z(s)ds + \sum_{i=1}^{n} B_i(t)e^{-\beta_i(t)}z(t-t_i), \quad t \in \mathbb{R}. \]  

(1.4)

where \( t \in \mathbb{R} \).

As far as we know, there are no studies related to the (WPAP) solutions of (1.4) with variable delays. Therefore, the results attained here are new and complementary to previous studies.

Throughout this paper, \( \delta(t) \in \text{AP}(\mathbb{R}, \mathbb{R}^+), \quad \tau_i(t), \beta_i(t) \in \text{PAP}(\mathbb{R}, \mathbb{R}^+), \quad \tau = \max_{1 \leq i \leq K} \{ \sup_{t \in \mathbb{R}} \tau_i(t) \}, \quad (i = 1, 2, \ldots, K) \) and given \( F \in BC(\mathbb{R}, \mathbb{R}^+) \), \( F^+ \) and \( F^- \) are defined as \( F^+ = \sup_{t \in \mathbb{R}} F(t) \) and \( F^- = \inf_{t \in \mathbb{R}} F(t) \). If \( z(t) \) is defined on \([0, \xi] \) with \( 0, \xi \in \mathbb{R} \), then we define \( z_i(\phi) \in D, \) where \( z_i(\phi) = z(t + \phi) \) for all \( \phi \in [-\tau, 0] \) and \( D = D([-\tau, 0], \mathbb{R}) \) is the continuous function space supremum norm \( \| \cdot \| \). For all \( j = 1, 2, \ldots, m, C_j \in C(\mathbb{R}^+, \mathbb{R}^+) \) are integrable, \( \int_{0}^{\infty} C_j(s)ds = 1 \) and \( \int_{0}^{\infty} C_j(x)e^{\xi x}dx < \infty \).

Let us consider the following initial condition:

\[ z(s) = \varphi(s), \quad \varphi \in BC([-\tau, 0], \mathbb{R}^+) \text{ and } \varphi(0) > 0. \]  

(1.5)

2 Preliminary results

Definition 2.1 ([8]) A function \( f \in C(\mathbb{R}, \mathbb{R}) \) is called almost periodic if for any \( \varepsilon > 0 \) there exists a trigonometric polynomial \( T_\varepsilon \) such that

\[ |f(x) - T_\varepsilon(x)| < \varepsilon, \quad x \in \mathbb{R}. \]
Definition 2.2 ([27]) A function $\eta \in C(\mathbb{R}, \mathbb{R})$ is called (PAP) if it can be written as

$$\eta = \eta_1 + \eta_2,$$

with $\eta_1 \in \text{AP}(\mathbb{R}, \mathbb{R})$ and $\eta_2 \in \text{PAP}_0(\mathbb{R}, \mathbb{R})$, where space PAP is defined by

$$\text{PAP}(\mathbb{R}) := \left\{ \eta_2 \in C(\mathbb{R}, \mathbb{R}) \mid \lim_{r \to \infty} \frac{1}{2q} \int_{-q}^{q} \| \eta_2(t) \| \, dt = 0 \right\}.$$

Let $\Lambda$ be the set of functions (weight) $\nu : \mathbb{R} \to (0, \infty)$ which are integrable on $(-\infty, \infty)$. If $\nu \in \Lambda$ and $Q := [-q, q]$ for $q > 0$, we then set

$$\nu(Q_q) := \int_{Q_q} \nu(x) \, dx.$$

The space of weights $\Lambda_{\infty}$ is defined by

$$\Lambda_{\infty} := \left\{ \nu \in \Lambda : \inf_{x \in \mathbb{R}} \nu(x) = \nu_0 > 0 \text{ and } \lim_{r \to \infty} \nu(Q_r) = \infty \right\}$$

and

$$\Lambda_{\infty}^* := \left\{ \nu \in \Lambda_{\infty} : \lim_{|x| \to \infty} \sup_{t \in \mathbb{R}} \frac{\nu(ax)}{\nu(x)} < +\infty, \lim_{|x| \to \infty} \sup_{t \in \mathbb{R}} \frac{\nu([-\alpha q, \alpha q])}{\nu([-q, q])} < +\infty \right\}.$$

Fix $\nu \in \Lambda_{\infty}^*$, $(\text{PAP}(\mathbb{R}, \nu), \| \cdot \|_{\infty})$ is a Banach space.

Definition 2.3 ([27]) Fix $\nu \in \Lambda_{\infty}$. A continuous function is called WPAP if it can be written as

$$\eta = \eta_1 + \eta_2,$$

with $\eta_1 \in \text{AP}(\mathbb{R}, \mathbb{R})$ and $\eta_2 \in \text{PAP}_0(\mathbb{R}, \mathbb{R})$, where space PAP is defined by

$$\text{PAP}_0(\mathbb{R}, \mathbb{R}, \nu) := \left\{ \eta_2 \in C(\mathbb{R}, \mathbb{R}) \mid \lim_{r \to \infty} \frac{1}{\nu([-q, q])} \int_{-q}^{q} \| \eta_2(t) \| \nu(t) \, dt = 0 \right\}.$$

Lemma 2.1 ([27]) Fix $\nu \in \Lambda_{\infty}^*$. For any $s \in (-\infty, \infty)$, assume that

$$\lim_{t \to \infty} \sup_{\nu \in \mathbb{R}} \frac{\nu(t + s)}{\nu(t)} < \infty,$$

the space $\text{PAP}(\mathbb{R}, \mathbb{R}, \nu)$ is translation invariant.

Lemma 2.2 ([28]) Let $\nu \in \Lambda_{\infty}^*$. If $\eta(t) \in \text{PAP}(\mathbb{R}, R, \nu), \sigma(t) \in C^1(\mathbb{R}, \mathbb{R})$ and $\sigma(t) \geq 0$, $\sigma'(t) \leq 1$, then $f(t - \sigma(t)) \in \text{PAP}(X, \nu)$.

3 Main results

Lemma 3.1 Suppose that

$$\sup_{T > 0} \left\{ \int_{-T}^{T} e^{-\delta(T+t)} \nu(t) \, dt \right\} < \infty.$$ (3.1)
Define a nonlinear operator $G$ for each $z \in \text{PAP}(\mathbb{R}, \mathbb{R}, \nu)$

$$(Gz)(t) = z_1(t) + z_2(t)$$

$$= \int_{-\infty}^{t} e^{-\int_{u}^{t} \beta(s) \, ds} \left[ \sum_{j=1}^{m} A_j(u)e^{-\alpha_j(u) \int_{-\infty}^{u} C_j(u-z) \, dz} \right] dt,$$

where

$$z_1(t) = \int_{-\infty}^{t} \sum_{j=1}^{m} A_j(u)e^{-\alpha_j(u) \int_{-\infty}^{u} C_j(u-z) \, dz} \, du,$$

$$z_2(t) = \int_{-\infty}^{t} \left[ \sum_{i=1}^{n} B_i(u)e^{-z(u-t)} \right] e^{-\int_{u}^{t} \beta(u) \, du} \, du.$$

Then $Gz \in \text{PAP}(\mathbb{R}, \mathbb{R}, \nu)$.

Proof Because of $M[a] > 0$ in [8] and Lemma 3.1 in [7], we have that

$$z_2(t) \in \text{PAP}(\mathbb{R}, \mathbb{R}, \nu). \quad (3.2)$$

Now we show that $z_1(t) \in \text{PAP}(\mathbb{R}, \mathbb{R}, \nu)$.

According to Lemma 2.1, Lemma 2.2, we obtain that there are $z_{11}(t) \in \text{AP}(\mathbb{R}, \mathbb{R})$ and $z_{12}(t) \in \text{PAP}_0(\mathbb{R}, \mathbb{R}, \nu)$ such that

$$z_{11}(t) + z_{12}(t) = \sum_{j=1}^{m} A_j(u)e^{-\alpha_j(u) \int_{-\infty}^{u} C_j(u-z) \, dz} \in \text{PAP}(\mathbb{R}, \mathbb{R}, \nu).$$

Also noting that $M[a] > 0$, we have that

$$\int_{-\infty}^{t} e^{-\int_{u}^{t} \beta(s) \, ds} z_{11}(t) \, dt \in \text{AP}(\mathbb{R}, \mathbb{R}) \quad (3.3)$$

is a solution of the following almost periodic differential equation:

$$w(t) = -\delta(t)w(t) + z_{11}(t).$$

Now, let us show that $\int_{-\infty}^{t} e^{-\int_{u}^{t} \beta(s) \, ds} z_{12}(t) \, dt$ belongs to $\text{PAP}(\mathbb{R}, \mathbb{R}, \nu)$. By using a similar manner in the proof of Theorem 3.5 in [7], it can be displayed that $z_{12}(t) \in \text{BC}(\mathbb{R}, \mathbb{R})$. Also,

$$0 \leq \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \left| \int_{-\infty}^{t} e^{-\int_{u}^{t} \beta(s) \, ds} z_{12}(s) \, ds \right| \right) v(t) \, dt$$

$$\leq \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-\infty}^{t} e^{-\int_{u}^{t} \beta(s) \, ds} \left| z_{12}(s) \right| \, ds \right) v(t) \, dt$$

$$\leq \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-\infty}^{t} e^{-\beta(t-s)} \left| z_{12}(s) \right| \, ds \right) v(t) \, dt$$

$$\leq L_1 + L_2,$$
where

\[
L_1 = \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-r}^{r} e^{-\delta(t-s)} |z_{12}(s)| \, ds \right) v(t) \, dt,
\]

\[
L_2 = \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-r}^{r} e^{-\delta(t-s)} |z_{12}(s)| \, ds \right) v(t) \, dt.
\]

Now, we shall prove that \( L_1 = L_2 = 0 \),

\[
L_1 = \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-r}^{r} e^{-\delta(t-s)} |z_{12}(s)| \, ds \right) v(t) \, dt
\]

\[
\leq \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{0}^{\infty} e^{-\delta(t-\xi)} |z_{12}(t-\xi)| \, d\xi \right) v(t) \, dt
\]

\[
= \lim_{r \to \infty} \int_{0}^{\infty} e^{-\delta\xi} \left( \frac{1}{2r} \int_{-r}^{r} |z_{12}(t-\xi)| \, dt \right) \, d\xi
\]

\[
= \int_{0}^{\infty} e^{-\delta\xi} \left( \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |z_{12}(t-\xi)| \, dt \right) \, d\xi.
\]

From Lemma 2.1 the function \( z_{12}(t-\xi) \in \text{PAP}_0(\mathbb{R}, \mathbb{R}, v) \), we obtain that

\[
\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |z_{12}(t-\xi)| \, v(t) \, dt.
\]

Therefore,

\[
L_1 = \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-r}^{r} e^{-\delta(t-s)} |z_{12}(s)| \, ds \right) v(t) \, dt = 0. \tag{3.4}
\]

Notice that \( |z_{12}|_{\infty} = \sup_{t \in \mathbb{R}} |z_{12}(t)| = M \) and by (3.1), then

\[
L_2 = \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \int_{-r}^{r} e^{-\delta(t-s)} |z_{12}(s)| \, ds \right) v(t) \, dt
\]

\[
\leq \lim_{r \to \infty} \frac{1}{2r} \int_{-\infty}^{r} e^{-\delta s} |z_{12}(s)| \, ds \int_{r}^{\infty} e^{-\delta t} \, v(t) \, dt
\]

\[
= \frac{M}{\delta} \lim_{r \to \infty} \frac{1}{2r} \left[ e^{\delta r} - e^{-\delta r} \right] \int_{r}^{\infty} e^{-\delta t} \, v(t) \, dt \tag{3.5}
\]

\[
= \frac{M}{\delta} \lim_{r \to \infty} \frac{1}{2r} \int_{r}^{\infty} e^{-\delta(t+r)} v(t) \, dt = 0,
\]

combining with (3.2), (3.3), (3.4), and (3.5), leads to \( Gz \in \text{PAP}(\mathbb{R}, \mathbb{R}, v) \). \qed

**Theorem 3.1** Let \( \max_{1 \leq j \leq K} \{ \inf_{t \in \mathbb{R}} 1 - \tau_j(t) \} > 0 \),

\[
(\delta)^{-1} \left( \sum_{j=1}^{m} (A_j \omega_j)^* + \sum_{i=1}^{n} (\bar{\omega}_i B_i)^* \right) < 1, \tag{3.6}
\]
and by Lemma 2.2, then (1.4) has a unique WPAP solution in the region

\[ C^* = \{ \psi \in \text{PAP}(\mathbb{R}, \mathbb{R}^+, \nu), K_1 \leq |\psi(t)| \leq K_2 \}, \]

where \( K_2 = (\delta)^{-1}(\sum_{j=1}^{m} A_j^2 + \sum_{i=1}^{n} B_i^2) \) and \( K_1 = (\delta)^{-1}(\sum_{j=1}^{m} A_j^2 e^{-\alpha_j^g K_2} + \sum_{i=1}^{n} B_i^2 e^{-\beta_i^g K_2}) \).

**Proof** First, let us prove that \( G \in \text{PAP}(\mathbb{R}, \mathbb{R}^+, \nu) \) into itself. It is clear that

\[
|Gz(t)| = \left| \int_{-\infty}^{t} e^{-\int_{\xi}^{t} (\xi(s)) ds} \left[ \sum_{j=1}^{m} A_j(s) e^{-\alpha_j g (\int_{\xi}^{t} C_j(s) ds)} + \sum_{i=1}^{n} B_i(s) e^{-\beta_i g (\int_{\xi}^{t} C_i(s) ds)} \right] ds \right|
\]

\[
\leq \int_{-\infty}^{t} e^{-\int_{\xi}^{t} (\xi(s)) ds} \left[ \sum_{j=1}^{m} A_j(s) + \sum_{i=1}^{n} B_i \right] ds \leq (\delta)^{-1} \left( \sum_{j=1}^{m} A_j^2 + \sum_{i=1}^{n} B_i^2 \right) = K_2
\]

and

\[
|Gz(t)| = \left| \int_{-\infty}^{t} e^{-\int_{\xi}^{t} (\xi(s)) ds} \left[ \sum_{j=1}^{m} A_j(t) e^{-\alpha_j g (\int_{\xi}^{t} C_j(s) ds)} + \sum_{i=1}^{n} B_i(t) e^{-\beta_i g (\int_{\xi}^{t} C_i(s) ds)} \right] dt \right|
\]

\[
\geq \int_{-\infty}^{t} e^{-\int_{\xi}^{t} (\xi(s)) ds} \left[ \sum_{j=1}^{m} A_j e^{-\alpha_j g K_2} + \sum_{i=1}^{n} B_i e^{-\beta_i g K_2} \right] ds
\]

\[
\geq (\delta)^{-1} \left( \sum_{j=1}^{m} A_j^2 e^{-\alpha_j^g K_2} + \sum_{i=1}^{n} B_i^2 e^{-\beta_i^g K_2} \right) = K_1,
\]

which implies that \( G \in C^* \).

Let \( f_1, f_2 \in C^* \). Then

\[
|(Gf_1)(t) - (Gf_2)(t)|
\]

\[
= \left| \int_{-\infty}^{t} e^{-\int_{\xi}^{t} (\xi(s)) ds} \left[ \sum_{j=1}^{m} A_j(s) \left[ e^{-\alpha_j g (\int_{\xi}^{t} C_j(s) ds)} - e^{-\alpha_j g (\int_{\xi}^{t} C_j(s) ds)} \right] ds \right.ight.
\]

\[
+ \left. \sum_{i=1}^{n} B_i(s) \left[ e^{-\beta_i g (\int_{\xi}^{t} C_i(s) ds)} - e^{-\beta_i g (\int_{\xi}^{t} C_i(s) ds)} \right] \right| ds \right|
\]

\[
\leq \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\int_{\xi}^{t} (\xi(s)) ds} \left[ \sum_{j=1}^{m} A_j(s) \left| e^{-\alpha_j g (\int_{\xi}^{t} C_j(s) ds)} - e^{-\alpha_j g (\int_{\xi}^{t} C_j(s) ds)} \right| ds \right.
\]

\[
+ \left. \sum_{i=1}^{n} B_i(s) \left| e^{-\beta_i g (\int_{\xi}^{t} C_i(s) ds)} - e^{-\beta_i g (\int_{\xi}^{t} C_i(s) ds)} \right| \right| ds.
\]

Obviously, for \( x_1, x_1 \in [0, +\infty], \)

\[
|e^{-x_1} - e^{-x_2}| < |x_1 - x_2|.
\]

Therefore,

\[
|(Gf_1)(t) - (Gf_2)(t)|
\]
ponentially stable.

Proof

Let

\[\Pi(\theta) = \sup_{t \in \mathbb{R}} \left\{ -[\delta(t) - \theta \tau] + e^{\theta \tau} \left( \sum_{j=1}^{m} (A_j \omega_j)^* + \sum_{i=1}^{n} (\beta_i B_i)^* \right) \right\}, \quad \theta \in [0, 1].\]

Then

\[\Pi(0) = \sup_{t \in \mathbb{R}} \left\{ -\delta(t) + \left( \sum_{j=1}^{m} (A_j \omega_j)^* + e^{\theta \tau} \sum_{i=1}^{n} (\beta_i B_i)^* \right) \right\} < 0.\] (3.7)

Since \(\Pi(\theta)\) is continuous, a constant \(\lambda \in (0, \delta^-)\) can be picked out as

\[\Pi(\lambda) = \sup_{t \in \mathbb{R}} \left\{ -[\delta(t) - \lambda \tau] + \left( \sum_{j=1}^{m} (A_j \omega_j)^* + e^{\theta \tau} \sum_{i=1}^{n} (\beta_i B_i)^* \right) \right\} < 0.\] (3.8)

Assume \(z(t)\) as an arbitrary solution of (1.4) with (1.5) and \(z^*(t)\) as a WPAP solution of Theorem 3.1. Let us accept \(\rho(t) = z(t) - z^*(t)\), so we obtain

\[
\rho'(t) = -\delta(t) \rho(t) + \sum_{j=1}^{m} A_j(t) \left[ e^{-\delta(t) \tau} \int_{-\infty}^{t} C_j(t-s) \rho(s) \, ds - e^{-\delta(t) \tau} \int_{-\infty}^{t} C_j(t-s) z^*(s) \, ds \right] \\
+ \sum_{i=1}^{n} B_i(t) \left[ e^{-\beta_i(t) \tau} \int_{-(\delta(t)-\tau(t))}^{t} C_j(t-s) z^*(s) \, ds \right].
\] (3.9)

Let

\[
\|\rho(t)\| = \sup_{t \in \mathbb{R}} |\psi(t) - z^*(t)|.
\]

For any \(\varepsilon > 0\), it is trivial to show that

\[
\|\rho(t)\| < M \left( \|\psi - z^*\| + \varepsilon \right) e^{-\delta t} \quad \text{for all } t \in (-\infty, 0],
\]
where $M > 1$ is a constant number. We show that

$$\| \rho(t) \| < M(\| \varphi - z^* \| + \varepsilon) e^{-\lambda t} \quad \text{for all } t > 0. \quad (3.10)$$

Contrarily, there must exist $\theta > 0$

$$\begin{align*}
\| \rho(\theta) \| &= M(\| \varphi - z^* \| + \varepsilon) e^{-\lambda \theta}, \\
\| \rho(t) \| &< M(\| \varphi - z^* \| + \varepsilon) e^{-\lambda t} \quad \forall t \in (-\infty, \theta).
\end{align*} \quad (3.11)$$

Given (3.9) and integrating it on $[0, \theta]$, we have

$$\rho(\theta) = \rho(0)e^{-\int_0^\theta \delta(s) ds}$$

$$+ \int_0^\theta e^{-\int_0^s \delta(s') ds'} \left\{ \sum_{j=1}^m A_j(s) \left[ e^{-\omega_j(t) \int_{-\infty}^s C_j(s-u) z(u) du} - e^{-\omega_j(t) \int_{-\infty}^s C_j(s-u) z^*(u) du} \right] \\
+ \sum_{i=1}^n B_i(s) \left[ e^{-\beta_i(s) z(s-\tau_i(s))} - e^{-\beta_i(s) z^*(s-\tau_i(s))} \right] \right\} ds.$$ 

Hence

$$\begin{align*}
\| \rho(\theta) \| &\leq M(\| \varphi - z^* \| + \varepsilon) e^{-\int_0^\theta \delta(s) ds} \\
&+ \int_0^\theta e^{-\int_0^s \delta(s') ds'} \left\{ \sum_{j=1}^m |A_j(s)| |\omega_j(t)| \int_{-\infty}^s C_j(s-u) |z(u) - z^*(u)| du \\
&+ \sum_{i=1}^n |B_i(s)| |\beta_i(s)| |z(s-\tau_i(s)) - z^*(s-\tau_i(s))| \right\} ds \\
&\leq M(\| \varphi - z^* \| + \varepsilon) e^{-\int_0^\theta \delta(s) ds} \\
&+ \int_0^\theta e^{-\int_0^s \delta(s') ds'} \left\{ \sum_{j=1}^m (A_j \omega_j)^+ \int_{-\infty}^s C_j(s-u) e^{\lambda u} du \right\} M(\| \varphi - z^* \| + \varepsilon) \\
&+ \sum_{i=1}^n (\beta_i B_i)^+ M(\| \varphi - z^* \| + \varepsilon) e^{-\lambda (s-\tau_i(s))} ds \\
&= M(\| \varphi - z^* \| + \varepsilon) e^{-\int_0^\theta \delta(s) ds} \\
&+ \int_0^\theta e^{-\int_0^s \delta(s') ds'} \left\{ \sum_{j=1}^m (A_j \omega_j)^+ \int_{-\infty}^s C_j(s-u) e^{\lambda u} du \right\} M(\| \varphi - z^* \| + \varepsilon) \\
&+ \sum_{i=1}^n (\beta_i B_i)^+ M(\| \varphi - z^* \| + \varepsilon) e^{-\lambda (s-\tau_i(s))} ds \\
&= M(\| \varphi - z^* \| + \varepsilon) e^{-\int_0^\theta \delta(s) ds}.
\end{align*}$$
therefore, it is noticeable that results in [4] are special cases of our results.

Example 3.1 Consider the system

\[
    z'(t) = -(8 + \sin^2 \sqrt{2}t + \sin^2 t)z(t) + (1 + 0.25 \sin^2 \sqrt{2}t + 0.25 \sin^2 \pi t + e^{-t})
    \times
    e^{-(0.25 \cos^2 \sqrt{2}t + 0.25 \cos^2 \pi t + e^{-t}) \int_0^t e^{\delta(s) - \lambda} ds}
    + (1 + 0.25 \sin^2 \sqrt{2}t + 0.25 \sin^2 \pi t + 0.5e^{-5t})
    \times
    e^{-(0.25 \cos^2 \sqrt{2}t + 0.25 \cos^2 \pi t + e^{-t}) \int_0^t (\sin^2 \bar{t})^{-1} (t - \sin^2 t)}
\]

where

\[
    \delta(t) = 8 + \sin^2 \sqrt{2}t + \sin^2 t, \quad A_1(t) = 1 + 0.25 \cos^2 \sqrt{2}t + 0.25 \cos^2 \pi t + e^{-t},
    \\
    \omega_1(t) = 0.25 \cos^2 \sqrt{2}t + 0.25 \cos^2 \pi t + e^{-t},
    \\
    B_1(t) = 1 + 0.25 \cos^2 \sqrt{2}t + 0.25 \cos^2 \pi t + 0.5e^{-5t},
    \\
    \beta_1(t) = 0.25 \cos^2 \sqrt{2}t + 0.25 \cos^2 \pi t + e^{-t}, \quad C_j(t) = e^{-t}, \quad \tau_1(t) = \sin^2 t.
\]

Figure 1 shows weighted pseudo almost periodic solutions of eq. (3.12).
And for $v(t) = e^t$ and $\delta^- = 8, \delta^+ = 10$, $A_1^+ = 2.5, A_1^- = 1, B_1^+ = 2.5, B_1^- = 1, \omega_1^+ = 1.5, \beta_1^- = 1.25, \tau = 1$, we have

$$\left(\delta^-\right)^{-1} \left((A_1 \omega_1)^+ + (\beta_1 B_1)^+\right) = 6.875 \frac{8}{9} = 0.859375 < 1.$$  

$$K_2 = 2.5 + 2.5 = 5 \approx 05555, \quad K_1 = e^{-1.5 \times 0.859375} = 0.027553,$$

$$\sup_{T > 0} \left\{ \int_{-T}^{T} e^{-\delta^- (T + t)} v(t) dt \right\} = \sup_{T > 0} \left\{ \int_{-T}^{T} e^{-8(T + t)} e^t dt \right\} = \sup_{T > 0} \left\{ \int_{-T}^{T} e^{-8T - 7t} dt \right\}$$

$$= -1 \sup_{T > 0} \left\{ \left[ e^{-15T} - e^{-7T} \right] \right\} < \infty.$$

All conditions of Theorems 3.1 and 3.2 are satisfied, then (3.12) has a unique WPAP solution. Therefore, this solution is globally exponentially stable with a convergence rate $\lambda = 0.02$ in the region

$$C^* = \{ \varphi | \varphi BC([\mathbb{R}, \mathbb{R}), 0.027553 \leq |\varphi(t)| \leq 0.5, \text{ for all } t \in \mathbb{R} \}.$$  

**Remark 3.2** According to the results of [4], the globally exponentially stable positive WPAP solution of (3.12) is invalid because

$$A_1(t) = 1 + 0.25 \sin^2 \sqrt{2}t + 0.25 \sin^2 \pi t + e^{-t},$$

$$B_1(t) = 1 + 0.25 \sin^2 \sqrt{2}t + 0.25 \sin^2 \pi t + 0.5e^{-at}$$

are WPAP functions, not almost, and pseudo almost periodic. Consequently, this article is more comprehensive compared to previous studies.
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