The $O(3,2)$ Symmetry derivable from the
Poincaré Sphere

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Abstract

Henri Poincaré formulated the mathematics of the Lorentz transformations, known as the Poincaré group. He also formulated the
Poincaré sphere for polarization optics. It is noted that his sphere contains the symmetry of the Lorentz group applicable to the momentum-energy four-vector of a particle in the Lorentz-covariant world. Since the particle mass is a Lorentz-invariant quantity, the Lorentz group does not allow its variations. However, the Poincaré sphere contains the symmetry corresponding to the mass variation, leading to the $O(3,2)$ symmetry. An illustrative calculation is given.

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1 Introduction

The Poincaré sphere is a mathematical instrument for studying polarization of light waves [1, 2]. This sphere contains the symmetry of the Lorentz group [3]. In addition, the sphere allows us to extend the $O(3, 1)$ symmetry of the Lorentz group to the $O(3, 2)$ symmetry of the de Sitter group [4, 5, 6].

In the Lorentz-covariant world, the energy and momentum are combined into a four-vector, and the particle mass remains invariant under Lorentz transformations. Thus, it is not possible to change the particle mass in the Lorentzian world. However, in the de Sitter space of $O(3, 2)$, there are two energy variables allowing two mass variables $m_1$ and $m_2$, which can be written as [4]

\[ m_1 = m \cos \chi, \quad \text{and} \quad m_2 = m \sin \chi, \]  

(1)

respectively, with

\[ m^2 = m_1^2 + m_2^2. \]  

(2)

For a given momentum whose magnitude is $p$, the energy variables are

\[ E_1 = \sqrt{m^2 \cos^2 \chi + p^2}, \quad \text{and} \quad E_2 = \sqrt{m^2 \sin^2 \chi + p^2}. \]  

(3)

While the Lorentz group is originally formulated in terms of the four-by-four matrices applicable to one time and three space coordinates, it is possible to use two-by-two matrices to perform the same Lorentz transformation [3, 7]. In this representation, the four-vector takes the form of a two-by-two Hermitian matrix with four elements. The determinant of this momentum-energy matrix is the $(mass)^2$ of this determinant. Indeed, the Lorentz-transformation in this representation consists of determinant-preserving transformations.

When Einstein was formulating his special relativity, he did not consider internal space-time structures or symmetries of the particles. It was not until 1939 when Wigner considered the space-time symmetries applicable to the internal space-time symmetries. For this purpose, Wigner in 1939 considered the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant [8, 9]. These subgroups are called Wigner’s little groups, and they define the internal space-time symmetries of the particles.

For a massive particle, the internal space-time symmetry is like the three-dimensional rotation group leading to the particle spin. For a massless particle, the little group has a cylindrical symmetry with one rotational and one translational degrees of freedom, corresponding to the helicity and
gauge transformation respectively [10]. These Wigner’s symmetry problems can also be framed into the two-by-two formulation of the Lorentz group [6].

It is known that the four Stokes parameters are needed for the complete description of the Poincaré sphere [3, 4]. These parameters can also be placed into a two-by-two matrix. It is noted that phase shifts, rotations, and amplitude change lead to determinant-preserving transformations, just like in the case of Lorentz transformations.

However, the determinant of the Stokes parameters becomes smaller as the two transverse components loses their coherence. Since the determinant of the two-by-two four-momentum matrix is the \((\text{mass})^2\), this decoherence could play as an analogy for variations in the mass. The purpose of this paper is precisely to study this decoherence mechanism in detail.

Sec. 2 it is shown possible to study Wigner’s little groups using the two-by-two representation. Wigner’s little groups dictate the internal space-time symmetries of elementary particles. They are the subgroups of the Lorentz group whose transformations leave the four-momentum of the given particle invariant [8, 9].

In Sec. 3 we first note that the same two-by-two matrices are applicable to two-component Jones vectors and Stokes parameters, which allow us to construct the Poincaré spheres. It is shown that the radius of the sphere depends on the degree of coherence between two transverse electric components. The radius is maximum when the system is fully coherent and is minimum when the system is totally incoherent.

In Sec. 4 it is noted that the variation of the determinant of the Stokes parameters can be formulated in terms of the symmetry of the \(O(3, 2)\) group [4]. This allows us to study the extra-Lorentz symmetry which allows variations of the particle mass. We study in detail the mass variation while the momentum is kept constant. The energy takes different values when the mass changes.

2 Poincaré Group, Einstein, and Wigner

The Lorentz group starts with a group of four-by-four matrices performing Lorentz transformations on the Minkowskian vector space of \((t, z, x, y)\), leaving the quantity

\[
t^2 - z^2 - x^2 - y^2
\]

invariant. It is possible to perform this transformation using two-by-two representations [3, 9, 7]. This mathematical aspect is known as the \(SL(2, c)\) as the universal covering group for the Lorentz group.
In this two-by-two representation, we write the four-vector as a matrix

\[
X = \begin{pmatrix}
t + z & x - iy \\
x + iy & t - z
\end{pmatrix}.
\]  

Then its determinant is precisely the quantity given in Eq. (4). Thus the Lorentz transformation on this matrix is a determinant-preserving transformation. Let us consider the transformation matrix as

\[
G = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}, \quad \text{and} \quad G^\dagger = \begin{pmatrix}
\alpha^* & \gamma^* \\
\beta^* & \delta^*
\end{pmatrix},
\]

with

\[
\det (G) = 1.
\]

This matrix has six independent parameters. The group of these \(G\) matrices is known to be locally isomorphic to the group of four-by-four matrices performing Lorentz transformations on the four-vector \((t, z, x, y)\) \[3, 9, 7\]. For each matrix of this two-by-two transformation, there is a four-by-four matrix performing the corresponding Lorentz transformation on the four-dimensional Minkowskian vector.

The matrix \(G\) is not a unitary matrix, because its Hermitian conjugate is not always its inverse. The group can have a unitary subgroup called \(SU(2)\) performing rotations on electron spins. This \(G\)-matrix formalism explained in detail by Naimark in 1954 \[7\]. We shall see first that this representation is convenient for studying the internal space-time symmetries of particles. We shall then note that this two-by-two representation is the natural language for the Stokes parameters in polarization optics.

With this point in mind, we can now consider the transformation

\[
X' = GXG^\dagger.
\]

Since \(G\) is not a unitary matrix, it is not a unitary transformation. For this transformation, we have to deal with four complex numbers. However, for all practical purposes, we may work with two Hermitian matrices

\[
Z(\delta) = \begin{pmatrix}
e^{-i\delta/2} & 0 \\
0 & e^{i\delta/2}
\end{pmatrix}, \quad \text{and} \quad R(\phi) = \begin{pmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix},
\]

plus one symmetric matrix

\[
B(\mu) = \begin{pmatrix}
e^{\mu/2} & 0 \\
0 & e^{-\mu/2}
\end{pmatrix}.
\]
The two Hermitian matrices in Eq.(9) lead to rotations around the $z$ and $y$ axes respectively. The symmetric matrix of Eq.(10) performs Lorentz boosts along the $z$ direction. Repeated applications of these three matrices will lead to the most general form of the $G$ matrix of Eq.(6) with six independent parameters.

It was Einstein who defined the energy-momentum four vector, and showed that it also has the same Lorentz-transformation law as the space-time four-vector. We write the energy-momentum four-vector as

$$P = \begin{pmatrix} E + p_z & p_x - ip_y \\ p_x + ip_y & E - p_z \end{pmatrix},$$

with

$$\det (P) = E^2 - p_x^2 - p_y^2 - p_z^2,$$

which means

$$\det p = m^2,$$

where $m$ is the particle mass.

Now Einstein’s transformation law can be written as

$$P' = GMG^\dagger,$$

or explicitly

$$\begin{pmatrix} E' + p'_z \\ p'_x + ip'_y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E + p_z & p_x - ip_y \\ p_x + ip_y & E - p_z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. $$

Later in 1939 [8], Wigner was interested in constructing subgroups of the Lorentz group whose transformations leave a given four-momentum invariant, and called these subsets “little groups.” Thus, Wigner’s little group consists of two-by-two matrices satisfying

$$P = WPW^\dagger.$$

This two-by-two $W$ matrix is not an identity matrix, but tells about internal space-time symmetry of the particle with a given energy-momentum four-vector. This aspect was not known when Einstein formulated his special relativity in 1905.

If its determinant is a positive number, the $P$ matrix can be brought to the form

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
Table 1: Wigner’s Little Groups. The little groups are the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. They thus define the internal space-time symmetries of particles. The four-momentum remains invariant under the rotation around it. In addition, they remain invariant under the following transformations. They are different for massive and massless particles.

| Particle mass | Four-momentum | Transform matrices |
|---------------|---------------|-------------------|
| massive       | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ |
| Massless      | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ |

corresponding to a massive particle at rest.

If the determinant is zero, we may write $P$ as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

(18)
corresponding to a massless particle moving along the $z$ direction.

For all three of the above cases, the rotation matrix $Z(\phi)$ of Eq.(9) will satisfy the Wigner condition of Eq.(16). This matrix corresponds to rotations around the $z$ axis.

For the massive particle with the four-momentum of Eq.(17), the two-by-two rotation matrix $R(\theta)$ also leaves the $P$ matrix of Eq.(17) invariant. Together with the $Z(\phi)$ matrix, this rotation matrix lead to the subgroup consisting of unitary subset of the $G$ matrices. The unitary subset of $G$ is $SU(2)$ corresponding to the three-dimensional rotation group dictating the spin of the particle [9].

For the massless case, the transformations with the triangular matrix of the form

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

(19)
leaves the momentum matrix of Eq. (18) invariant. The physics of this matrix has a stormy history, and the variable $\gamma$ leads to gauge transformation applicable to massless particles [10, 11].

Table 1 summarizes the transformation matrices for Wigner’s subgroups for massive and massless particles. Of course, it is a challenging problem to have one expression for both cases, and this problem has been addressed in the literature [13].

3 Geometry of the Poincaré Sphere

The geometry of the Poincaré sphere for polarization optics is determined the four Stokes parameters. In order to construct those parameters, we have to start from the two-component Jones vector.

In studying the polarized light propagating along the $z$ direction, the traditional approach is to consider the $x$ and $y$ components of the electric fields. Their amplitude ratio and the phase difference determine the state of polarization. Thus, we can change the polarization either by adjusting the amplitudes, by changing the relative phase, or both. For convenience, we call the optical device which changes amplitudes an “attenuator” and the device which changes the relative phase a “phase shifter.”

The traditional language for this two-component light is the Jones-matrix formalism which is discussed in standard optics textbooks. In this formalism, the above two components are combined into one column matrix with the exponential form for the sinusoidal function

$$
\begin{pmatrix}
\psi_1(z,t) \\
\psi_2(z,t)
\end{pmatrix}
= \begin{pmatrix}
a \exp \{i (kz - \omega t + \phi_1)\} \\
b \exp \{i (kz - \omega t + \phi_2)\}
\end{pmatrix}.
$$

This column matrix is called the Jones vector. To this vector we can apply the following two diagonal matrices.

$$
Z(\phi) = \begin{pmatrix}
-\exp\{i\phi/2\} & 0 \\
0 & \exp\{i\phi/2\}
\end{pmatrix}, \quad B(\mu) = \begin{pmatrix}
\exp\{\mu/2\} & 0 \\
0 & e^{-\mu/2}
\end{pmatrix},
$$

which leads to a phase shift and a change in the amplitudes respectively. The polarization axis can rotate around the $z$ axis, and it can be carried out by the rotation matrix

$$
R(\theta) = \begin{pmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}.
$$

These two-by-two matrices perform transform clearly defined in optics, while they play the same role in Lorentz transformations as noted in Sec. 2. Their role in the two different fields of physics are tabulated in Table 2.
The physical instruments leading to these matrix operations are mentioned in the literature. With these operations, we can obtain the most general form given in Eq.(20) by applying the matrix \( B(\mu) \) of Eq.(21) to
\[
\begin{pmatrix}
\psi_1(z,t) \\
\psi_2(z,t)
\end{pmatrix}
= \begin{pmatrix}
a \exp \{i(kz - \omega t - \phi/2)\} \\
a \exp \{i(kz - \omega t + \phi/2)\}
\end{pmatrix}.
\] (23)
Here both components have the same amplitude.

However, the Jones vector alone cannot tell whether the two components are coherent with each other. In order to address this important degree of freedom, we use the coherency matrix defined as
\[
C = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix},
\] (24)
with
\[
<\psi_i^* \psi_j> = \frac{1}{T} \int_0^T \psi_i^*(t+\tau)\psi_j(t)dt,
\] (25)
where \( T \) is for a sufficiently long time interval, is much larger than \( \tau \). Then, those four elements become
\[
S_{11} = <\psi_1^* \psi_1> = a^2, \quad S_{12} = <\psi_1^* \psi_2> = a^2 e^{-(\sigma+i\phi)},
\]
\[
S_{21} = <\psi_2^* \psi_1> = a^2 e^{-(\sigma-i\phi)}, \quad S_{22} = <\psi_2^* \psi_2> = a^2.
\] (26)
The diagonal elements are the absolute values of \( \psi_1 \) and \( \psi_2 \) respectively. The off-diagonal elements could be smaller than the product of \( \psi_1 \) and \( \psi_2 \), if the two transverse components are not completely coherent. The \( \sigma \) parameter specifies the degree of coherence.

If we start with the Jones vector of the form of Eq.(20), the coherency matrix becomes
\[
C = a^4 \begin{pmatrix}
1 & e^{-(\sigma+i\phi)} \\
e^{-(\sigma-i\phi)} & 1
\end{pmatrix}.
\] (27)
This is a Hermitian matrix and can be diagonalized to
\[
D = a^4 \begin{pmatrix}
1 + e^{-\sigma} & 0 \\
0 & 1 - e^{-\sigma}
\end{pmatrix}.
\] (28)

For the purpose of studying the Poincaré sphere, it is more convenient to make the following linear combinations.
\[
S_0 = \frac{S_{11} + S_{22}}{\sqrt{2}}, \quad S_3 = \frac{S_{11} - S_{22}}{\sqrt{2}},
\]
\[
S_1 = \frac{S_{12} + S_{21}}{\sqrt{2}}, \quad S_2 = \frac{S_{12} - S_{21}}{\sqrt{2}i}.
\] (29)
Table 2: Polarization optics and special relativity sharing the same mathematics. Each matrix has its clear role in both optics and relativity. The determinant of the Stokes or the four-momentum matrix remains invariant under Lorentz transformations. It is interesting to note that the decoherency parameter (least fundamental) in optics corresponds to the mass (most fundamental) in particle physics.

| Polarization Optics | Transformation Matrix | Particle Symmetry |
|---------------------|-----------------------|-------------------|
| Phase shift $\phi$  | \[
\begin{pmatrix}
  e^{-i\phi/2} & 0 \\
  0 & e^{i\phi/2}
\end{pmatrix}
\] | Rotation around $z$ |
| Rotation around $z$ | \[
\begin{pmatrix}
  \cos(\theta/2) & -\sin(\theta/2) \\
  \sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}
\] | Rotation around $y$ |
| Squeeze along $x$ and $y$ | \[
\begin{pmatrix}
  e^{\mu/2} & 0 \\
  0 & e^{-\mu/2}
\end{pmatrix}
\] | Boost along $z$ |
| $(a)^4(1 - e^{-\sigma})$ | Determinant | $(\text{mass})^2$ |

These four parameters are called Stokes parameters, and they are like a Minkowskian four-vector which are Lorentz-transformed by the four-by-four matrices constructed from the two-by-two matrices applicable to the coherency matrix [3].

We now have the four-vector $(S_0, S_3, S_1, S_2)$, and the sphere defined in the three-dimensional space of $(S_0, S_3, S_1, S_2)$ is called the Poincaré sphere. If we start from the Jones vector of Eq.(23) with the same amplitude for both components, $S_{11} = S_{22}$, and thus $S_3 = 0$. The Poincaré sphere becomes a two-dimensional circle. The radius of this circle is

$$R = \sqrt{S_1^2 + S_2^2}.$$ (30)

This radius takes its maximum value $S_0$ when the system is completely coherent with $\sigma = 0$, and it vanishes when the system is totally incoherent.
with $\sigma = \infty$. Thus, $R$ can be written as

$$R = S_0 e^{-\sigma}. \quad (31)$$

Let us go back to the four-momentum matrix of Eq. (11). Its determinant is $m^2$ and remains invariant under Lorentz transformations defined by the Hermitian matrices of Eq. (9) and the symmetric matrix of Eq. (10). Likewise, the determinant of the coherency matrix of Eq. (27) should also remain invariant. The determinant in this case is

$$S_0^2 - R^2 = a^4 \left( 1 - e^{-2\sigma} \right). \quad (32)$$

However, this quantity depends on the $\sigma$ variable which measures decoherence of the two transverse components. This aspects is illustrated in Table 2.

While the decoherency parameter is not fundamental and is influenced by environment, it plays the same mathematical role as in the particle mass which remains as the most fundamental quantity since Isaac Newton, and even after Einstein.

### 4 O(3,2) symmetry

The group $O(3,2)$ is the Lorentz group applicable to a five-dimensional space applicable to three space dimensions and two time dimensions. Likewise, there are two energy variables, which lead to a five-component vector

$$(E_1, E_2, p) = (E_1, E_2, p_z, p_x, p_y). \quad (33)$$

In order to study this group, we have to use five-by-five matrices, but we are interested in its subgroups. First of all, there is a three-dimensional Euclidean space consisting of $p_z, p_x,$ and $p_y,$ to which the $O(3)$ rotation group is applicable, as in the case of the $O(3,1)$ Lorentz group.

If the momentum is in the $z$ direction, this five-vector becomes

$$(E_1, E_2, p) = (E_1, E_2, p, 0, 0). \quad (34)$$

As for these two energy variables, they take the form

$$E_1 = \sqrt{p^2 + m^2 \cos^2 \chi}, \quad \text{and} \quad E_2 = \sqrt{p^2 + m^2 \sin^2 \chi}, \quad (35)$$

as given in Eq. (3), and they maintain

$$E_1^2 + E_2^2 = m^2 + 2p^2, \quad (36)$$
which remains constant for a fixed value of \( p^2 \). There is thus a rotational symmetry in the two-dimensional space of \( E_1 \) and \( E_2 \). In this section, we are interested in this symmetry for a fixed value of the momentum as described in Fig. 1.

For the present purpose, the most important subgroups are two Lorentz subgroups applicable to the Minkowskian spaces of 

\[
(E_1, p, 0, 0), \quad \text{and} \quad (E_2, p, 0, 0).
\]  

Then, in the two-by-two matrix representation, these four-momenta take the form

\[
\begin{pmatrix}
E_1 + p & 0 \\
0 & E_1 - p
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
E_2 + p & 0 \\
0 & E_2 - p
\end{pmatrix},
\]  

with their determinant are \( m^2 \cos^2 \chi \) and \( m^2 \sin^2 \chi \) respectively. With this understanding, we can now concentrate only on the matrix with \( E_2 \). For \( \chi = 0 \), we are dealing with the massless particle, while the particle mass takes its maximum value of \( m \).

Indeed, this matrix is in the form of the diagonal matrix for the coherency matrix given in Eq. (28). Thus, we can study the property of the four-vector matrix of Eq. (38) in terms of the coherency matrix, whose determinant depends on the decoherency parameter \( \sigma \). Let us now take the ratios of the two diagonal elements for these matrices and write

\[
\frac{1 - e^{-\sigma}}{1 + e^{-\sigma}} = \frac{\sqrt{p^2 + m^2 \sin^2 \chi} - p}{\sqrt{p^2 + m^2 \sin^2 \chi} + p},
\]  

which becomes

\[
\frac{\tanh(\sigma/2)}{1 - \tanh^2(\sigma/2)} = \left( \frac{m}{2p} \right)^2 \sin^2 \chi.
\]  

The right and left sides of this equation consist of the variable of the coherency matrix and that of the four-momentum respectively.

If \( \sigma = 0 \), the optical system is completely coherent, and this leads to the zero particle mass with \( \chi = 0 \). The resulting two-by-two matrices are proportional to the four-momentum matrix for a massless particle given in Eq. (18)

The right side reaches its maximum value of \((m/2p)^2\) when \( \chi = 90^\circ \), while the left side monotonically increases as \( \sigma \) becomes larger. It becomes infinite as \( \sigma \) becomes infinite. For the left side, this is possible only for
Figure 1: Energy-momentum hyperbolas for different values of the mass. The Lorentz group does not allow us to jump from one hyperbola to another, but it is possible within the framework of the $O(3, 2)$ de Sitter symmetry. This figure illustrates the transition while the magnitude of the momentum is kept constant.

vanishing values of momentum. The resulting two-by-two matrices are proportional to the four-momentum matrix for a massive particle at rest given in Eq. (17).

The variable $\sigma$ has a concrete physical interpretation in polarization optics. The variable $\chi$ cannot be explained in the world where the particle mass remains invariant under Lorentz transformations. However, this variable has its place in the $O(3, 2)$-symmetric world.

**Concluding Remarks**

In this report, it was noted first that the group of Lorentz transformations can be formulated in terms of two-by-two matrices. In this formalism, the momentum four-vector can be written in the form of a two-by-two matrix. This two-by-two formalism can also be used for transformations of the coherency matrix in polarization optics. Thus, the set of four Stokes parameters is like a Minkowskian four-vector subject to Lorentz transformations. The geometry of the Poincaré sphere can be extended to accommodate these transformations.

The radius of the Poincaré sphere depends on the degree of coherence between the two transverse components of electric fields of the optical beam.
If the system is completely coherent, the Stokes matrix is like that for the four-momentum of a massless particle. When the system is completely incoherent, the matrix corresponds to that for a massive particle at rest. The variation of the decoherence parameter corresponds to the variation of the mass.

This mass variation is not possible in the $O(3,1)$ Lorentzian world, but is possible if the world is extended to that of the $O(3,2)$ de Sitter symmetry. In this paper, a concrete calculation is presented for the mass variation with a fixed momentum in the de Sitter space.

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