The moment of core collapse in star clusters with a mass function

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ABSTRACT
Star clusters with multi-mass components dynamically evolve faster than those modeled with equal-mass components. Using a series of direct N-body simulations, we investigate the dynamical evolution of star clusters with mass functions, especially their core collapse time. Multi-mass clusters tend to behave like systems with a smaller number of particles, which we call the effective number of particles ($N_{\text{eff}}$) and for which $N_{\text{eff}} = M/m_{\text{max}}$ (here $M$ and $m_{\text{max}}$ are the total cluster mass and the mass of the most massive star in the cluster, respectively). We find that the time of core collapse is inversely proportional to the mass of the most massive star in the cluster and analytically confirm that this is because the core collapse of clusters with a mass function proceeds on the dynamical friction timescale of the most massive stars. As the mass of the most massive star increases, however, the core-collapse time, which is observed as a core bounce of the cluster core from the evolution of the core density or core radius, becomes ambiguous. We find that in that case the total binding energy of the hard binaries gives a good diagnosis for determining the moment of core collapses. Based on the results of our simulations, we argue that the core bounce becomes ambiguous when the mass of the most massive star exceeds 0.1% of the total mass of the cluster.

Key words: galaxies: star clusters: general — methods: numerical

1 INTRODUCTION
Star clusters are collisional systems with a negative heat capacity, and therefore they dynamically evolve to eventually reach core collapse. The process to core collapse is simply described using a semi-analytic treatment of the energy transfer from the cluster core to the outer part of the cluster (Lynden-Bell & Wood 1968), and it was confirmed using various methods such as gaseous models (Larson 1970), Fokker-Planck simulations (Cohn 1973, 1981; Lynden-Bell & Eggleton 1981; Inagaki 1980), and direct N-body simulations (Aarseth et al. 1974). Because of the high stellar density of the core during its collapse, dynamical binaries form in this phase (Spitzer & Hart 1974). Once binaries form in the core, they generate energy which is transported outward by interactions with other cluster members (Heggie 1975; Hut 1983). Due to the binary heating the core finally bounces, and then the core oscillates when the number of stars is sufficiently large (Goodman 1987). Such gravothermal oscillation was first found using gaseous models (Sugimoto & Bettwieser 1981; Bettwieser & Sugimoto 1984), but the behavior was later confirmed also using Fokker-Planck (Cohn et al. 1989) and direct N-body (Makino 1996) simulations.

The time between cluster birth and the moment of core collapse (what we will call the core-collapse time, or $t_{\text{cc}}$) is proportional to the two-body relaxation time at the half-mass radius, $t_h$ (Spitzer 1987). For clusters in which all stars have the same mass, the core collapses at $t_{\text{cc}} \approx 15-20t_h$. This result has a theoretical background and was confirmed with simulations using a wide variety of techniques: Monte-Carlo methods (Spitzer & Shull 1973), Fokker-Planck calculations (Cohn 1981; Takahashi 1995), and direct N-body simulations (Makino 1996). With a spectrum of stellar masses, however, the core-collapse time becomes much shorter than when all stars have the same mass (Inagaki & Wiyanto 1984). From direct N-body simulations, it is obtained that $t_{\text{cc}} \approx 0.2t_h$ with a realistic mass function (Portegies Zwart & McMillan 2002).

Gürkan et al. (2004) performed a large number of Monte Carlo simulations covering a wide range of the concentration parameters and mass functions. They showed that the core-collapse time scales with the central relaxation
time, $t_{rc}$, rather than with $t_{th}$. Here the central relaxation time is defined as
\begin{equation}
    t_{rc} \equiv \frac{0.065\sigma_{3D}^2}{G^2\langle m \rangle \rho_c \ln \Lambda}.
\end{equation}

Here $\sigma_{3D}$ and $\rho_c$ are the central three-dimensional velocity dispersion and core density, respectively (Spitzer 1985; Girkan et al. 2004). The Coulomb logarithm, $\ln \Lambda$, is written as $\ln \gamma N$, where $N$ is the total number of stars. For the central relaxation time of star clusters, it is numerically obtained that $\gamma = 0.015$ (Giersz & Heggie 1996) (the classic theoretical value for $t_{th}$ with single-mass components is $\gamma = 0.11$ (Spitzer 1987)).

In the case with a power-law mass function in which $m_{\text{max}}$ and $\langle m \rangle$ are the maximum mass and the mean mass of the stellar mass distribution, Girkan et al. (2004) found that the core-collapse time $t_{cc} \propto (m_{\text{max}}/\langle m \rangle)^{-1.3}$. They also demonstrated the existence of a minimum to the core-collapse time, which is $t_{cc}/t_{rc} = 0.15$. This value is also seen in Goswami et al. (2012), in which they use the same Monte Carlo code that was adopted by Girkan et al. (2004), but with a wider range of initial conditions including initially mass-segregated models. The arguments for the particular exponent (being $-1.3$) and the minimum to the core-collapse time, however, was not discussed and remains unclear.

In this paper we show the results of core collapse simulations of star clusters with power-law mass functions using direct N-body simulations. We find that the core collapse time scales $t_{cc} \propto (m_{\text{max}}/\langle m \rangle)^{-1.3}$, contrary to the earlier finding of Girkan et al. (2004), but we support our finding with analytic arguments. We further argue that core collapse is driven by the sinking of the most massive stars to the cluster center, by dynamical friction (as was suggested in Portegies Zwart & McMillan 2002). The time to the core-collapse then corresponds to the time required for the most massive star to reach the cluster center. For the most extreme mass functions, the core-collapse time then naturally depends on the crossing time of the system rather than the dynamical friction timescale.

The core bounce becomes less pronounced for larger values of $m_{\text{max}}/\langle m \rangle$, and this is qualitatively understood from the dynamical evolution being driven by the most massive stars in the cluster. For a mass ratio $M/m_{\text{max}} \lesssim 10^3$, core collapse becomes hard to determine, and it even becomes indistinguishable for $M/m_{\text{max}} \lesssim 100$, because in those cases the core will eventually be composed of only a few massive stars, almost irrespective of the total number of stars in the cluster. In such a case, however, the binding energy of the hardest binary gives us a good indication to detect the moment of core collapse.

### 2 N-BODY SIMULATIONS AND THE INITIAL CONDITIONS

We performed a series of N-body simulations using King models (King 1966) with a non-dimensional concentration parameter, $W_0$, of 3 and 6 as initial density profiles. Hereafter, we adopt N-body units in which, $G = M = -4E = 1$, where $G$, $M$, and $E$ are the gravitational constant, the total mass, and the total energy of the cluster, respectively.

| $W_0$ | $\rho_c$ | $\sigma_{1D}$ | $r_c$ | $r_h$ | $M_c$ |
|-------|----------|---------------|------|------|------|
| 3     | 0.652    | 0.518         | 0.543| 0.839| 0.238|
| 6     | 2.11     | 0.503         | 0.293| 0.804| 0.117|

Definitions of quantities listed in this table: $W_0$ is a non-dimensional concentration parameter for King models; $\rho_c$ is the core density; $\sigma_{1D}$ is the velocity dispersion in the core; $r_c$ is the core radius; $r_h$ is the half-mass radius; $M_c$ is the mass within the core.

We construct the initial particle distributions using NEMO (Teuben 1995). In table I we summarize the initial conditions for the runs. For each simulation we adopted $N$ particles from $N = 2048$ (2k), 8192 (8k), 32768 (32k) to $N = 131072$ (128k), with a power-law mass function with exponent $-\alpha$ and an upper-mass limit of $m_{\text{max}}$. The value of $m_{\text{max}}/\langle m \rangle$ is varied from 1.0 (equal mass) to 517, but for models with large-$N$ models we adopted a large value of $m_{\text{max}}$ because of the calculation time. For the mass function exponent $\alpha$, we adopted $\alpha = 2.35$ (Salpeter 1955), 1.7, and 1.2. In Tables 2 and 3 we summarize the simulation results.

All simulations are performed using the sixth-order predictor-corrector Hermite scheme (Nitadori & Makino 2008) running on GPU using the Sapporo2 library (Belleman et al. 2008; Bédorf & Portegies Zwart 2012) and also on CPU clusters using the two-dimensional parallelization scheme by Nitadori et al. (2008). We used a time step criterion with accuracy parameter, $\eta = 0.1-0.3$. The energy error is $\lesssim 10^{-4}$ for equal-mass models and $\lesssim 10^{-5}$ for all simulations over the entire duration of the simulation. For the models with $f_{\text{max}} \lesssim 2$, the energy error tends to become larger compared to the models with $f_{\text{max}} > 2$, especially after the formation of a binary of $\sim 10kT$. (We express binding energies in terms of $kT \equiv \langle m \rangle \sigma_{1D}^2$, where $\sigma_{1D}$ is the one-dimensional velocity dispersion of the cluster, $3NkT/2$ is the initial kinetic energy of the entire system.) If we try to adopt small timesteps in order to maintain less energy error, the time step of the calculations dropped below $\sim 10^{-12}$. To prevent such small timesteps, which have unpleasant consequences for the performance, we also performed simulations adopting a small softening $\epsilon$, for the simulations with $W_0 = 3$ we adopted $\epsilon = 1/(200N)$ and $\epsilon = 1/(130N)$ for models with $W_0 = 6$. With this softening we are able to resolve binaries with a semi-major axis of a 1200$kT$ and 780$kT$ for the simulations with $W_0 = 3$ and $W_0 = 6$, respectively. With softening length, energy error is $\lesssim 10^{-5}$ throughout the simulations.

### 3 CORE COLLAPSE IN MULTI-MASS CLUSTERS

The core-collapse time is usually determined by the moment of core bounce, which is seen in the time evolution of the core radius or core density. For some models, it is difficult to distinguish the core collapse, because there does not seem to

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1 See also [http://en.wikipedia.org/wiki/N-body_units](http://en.wikipedia.org/wiki/N-body_units)
Table 2. Models with $W_0 = 3$

| Model         | $W_0$ | $\alpha$ | $m_{\text{min}}/(m)$ | $m_{\text{max}}/(m)$ | $N$ | $\epsilon$ | $N_{\text{run}}$ |
|---------------|-------|----------|-----------------------|-----------------------|-----|------------|------------------|
| w3-2k-eq     | 3     | -        | 1.0                   | 1.0                   | 2k  | 0          | 5                |
| w3-2k-eq-soft| 3     | -        | 1.0                   | 1.0                   | 2k  | 1/(200N)   | 7                |
| w3-8k-eq-soft| 3     | -        | 1.0                   | 1.0                   | 8k  | 1/(200N)   | 1                |
| w3-2k-m2-Sal | 3     | 2.35     | 0.607                 | 2.02                  | 2k  | 0          | 7                |
| w3-8k-m2-Sal-soft | 3 | 2.35 | 0.607 | 2.02 | 2k | 1/(200N) | 8 |
| w3-32k-m2-Sal | 3     | 2.35     | 0.607                 | 2.02                  | 8k  | 1/(200N)   | 1                |
| w3-8k-m2-Sal | 3     | 2.35     | 0.391                 | 8.07                  | 8k  | 0          | 1                |
| w3-32k-m2-Sal | 3     | 2.35     | 0.391                 | 8.07                  | 32k | 0          | 3                |
| w3-2k-m32-Sal | 3     | 2.35     | 0.329                 | 32.3                  | 2k  | 0          | 10               |
| w3-8k-m32-Sal | 3     | 2.35     | 0.329                 | 32.3                  | 8k  | 0          | 1                |
| w3-32k-m32-Sal | 3 | 2.35 | 0.329 | 32.3 | 32k | 0 | 8 |
| w3-2k-m129-Sal | 3 | 2.35 | 0.296 | 129.2 | 2k | 0 | 10 |
| w3-8k-m129-Sal | 3 | 2.35 | 0.296 | 129.2 | 8k | 0 | 1 |
| w3-32k-m129-Sal | 3 | 2.35 | 0.296 | 129.2 | 32k | 0 | 8 |
| w3-128k-m129-Sal | 3 | 2.35 | 0.279 | 516.6 | 128k | 0 | 7 |
| w3-2k-m258-Sal | 3 | 2.35 | 0.286 | 258.3 | 2k | 0 | 2 |
| w3-8k-m258-Sal | 3 | 2.35 | 0.286 | 258.3 | 8k | 0 | 1 |
| w3-32k-m258-Sal | 3 | 2.35 | 0.286 | 258.3 | 32k | 0 | 1 |
| w3-2k-m517-Sal | 3 | 2.35 | 0.279 | 516.6 | 2k | 0 | 7 |
| w3-8k-m517-Sal | 3 | 2.35 | 0.279 | 516.6 | 8k | 0 | 1 |
| w3-32k-m517-Sal | 3 | 2.35 | 0.279 | 516.6 | 32k | 0 | 8 |
| w3-128k-m517-Sal | 3 | 2.35 | 0.279 | 516.6 | 128k | 0 | 8 |
| w3-2k-m2-a1.2-soft | 3 | 1.2 | 0.442 | 2.02 | 2k | 1/(200N) | 1 |
| w3-8k-m2-a1.2-soft | 3 | 1.2 | 0.442 | 2.02 | 8k | 1/(200N) | 1 |
| w3-2k-m8-a1.2 | 3 | 1.2 | 0.0336 | 8.07 | 2k | 0 | 1 |
| w3-8k-m8-a1.2 | 3 | 1.2 | 0.0336 | 8.07 | 8k | 0 | 1 |
| w3-32k-m8-a1.2 | 3 | 1.2 | 0.0336 | 8.07 | 32k | 0 | 4 |
| w3-2k-m32-a1.2 | 3 | 1.2 | 5.28 × 10^{-4} | 32.3 | 2k | 0 | 1 |
| w3-8k-m32-a1.2 | 3 | 1.2 | 5.28 × 10^{-4} | 32.3 | 8k | 0 | 1 |
| w3-32k-m32-a1.2 | 3 | 1.2 | 5.28 × 10^{-4} | 32.3 | 32k | 0 | 4 |
| w3-2k-m129-a1.2 | 3 | 1.2 | 5.16 × 10^{-6} | 129.2 | 2k | 0 | 1 |
| w3-8k-m129-a1.2 | 3 | 1.2 | 5.16 × 10^{-6} | 129.2 | 8k | 0 | 1 |
| w3-32k-m129-a1.2 | 3 | 1.2 | 5.16 × 10^{-6} | 129.2 | 32k | 0 | 4 |
| w3-2k-m258-a1.2 | 3 | 1.2 | 2.13 × 10^{-8} | 258.3 | 2k | 0 | 1 |
| w3-8k-m258-a1.2 | 3 | 1.2 | 2.13 × 10^{-8} | 258.3 | 8k | 0 | 1 |
| w3-32k-m258-a1.2 | 3 | 1.2 | 2.13 × 10^{-8} | 258.3 | 32k | 0 | 4 |
| w3-2k-m517-a1.2 | 3 | 1.2 | 1.23 × 10^{-8} | 516.6 | 8k | 0 | 1 |
| w3-32k-m517-a1.2 | 3 | 1.2 | 1.23 × 10^{-8} | 516.6 | 32k | 0 | 4 |
| w3-2k-m2-a1.7-soft | 3 | 1.7 | 0.525 | 2.02 | 2k | 1/(200N) | 1 |
| w3-2k-m8-a1.7 | 3 | 1.7 | 0.192 | 8.07 | 2k | 0 | 1 |
| w3-2k-m32-a1.7 | 3 | 1.7 | 0.0856 | 32.3 | 2k | 0 | 1 |
| w3-8k-m32-a1.7 | 3 | 1.7 | 0.0856 | 32.3 | 8k | 0 | 1 |
| w3-2k-m129-a1.7 | 3 | 1.7 | 0.0403 | 129.2 | 2k | 0 | 1 |
| w3-8k-m258-a1.7 | 3 | 1.7 | 0.0304 | 258.3 | 8k | 0 | 1 |

$N_{\text{run}}$ is the number of runs with the same initial parameters, but realized with different random seeds.
be a peak in the density evolution or a depression in the core radius. Another indicator for determining the moment of core collapse is by monitoring the evolution of the binding energy of dynamically formed binaries. During the core collapse, hard binaries form in the cluster. They are hardened by three-body encounters in the cluster core and eventually generate the energy for the core bounce. In this section, we present the evolution of the core radius and the density from the simulations, and then we discuss the relation between the core evolution and the dynamically formed binaries in order to provide an objective determination of the moment of core collapse and define the core collapse.

### 3.1 The evolution of core density and radius

In Figure 1 we present the evolution of the core density and the core radius for an equal-mass model with $W_0 = 3$, $N = 2k$ (left) and the same model but with a Salpeter mass function with $f_{\text{max}} = 8$ (right). We calculate the core radius and density using a method of Casertano & Hut (1985), but we took into account the mass of the particles. Compared to the equal mass case, the core collapse in the models with $f_{\text{max}} = 8$ is less clear, although the core noticeably expands after a slight depression.

If we increase $f_{\text{max}}$, the core collapse becomes more ambiguous. In Figure 2 we present the results of the same simulation as we presented in Figure 1 but with $f_{\text{max}} = 32$ (left in Fig. 2) and 129 (right). After the core shrinks, it keeps the small core radius and slowly expands (see right panel of Figure 2). For models with equal-mass or small $f_{\text{max}}$, we can easily measure the moment of core collapse and confirm the results.

### Table 3. Models with $W_0 = 6$

| Model          | $W_0$ | $\alpha$ | $m_{\text{min}}/(m)$ | $m_{\text{max}}/(m)$ | $N$ | $\epsilon$ | $N_{\text{run}}$ |
|----------------|-------|----------|-----------------------|-----------------------|-----|-------------|------------------|
| w6-2k-eq       | 6     | -        | 1.0                   | 1.0                   | 2k  | 0           | 5                |
| w6-2k-eq-soft  | 6     | -        | 1.0                   | 1.0                   | 2k  | 1/(130N)    | 1                |
| w6-8k-eq-soft  | 6     | -        | 1.0                   | 1.0                   | 8k  | 1/(130N)    | 1                |
| w6-32k-eq      | 6     | -        | 1.0                   | 1.0                   | 32k | 0           | 1                |
| w6-2k-m2-Sal-soft | 6  | 2.35     | 0.607                 | 2.02                  | 2k  | 1/(130N)    | 1                |
| w6-8k-m2-Sal-soft | 6  | 2.35     | 0.607                 | 2.02                  | 8k  | 1/(130N)    | 1                |
| w6-32k-m2-Sal  | 6     | 2.35     | 0.391                 | 32.3                  | 2k  | 0           | 4                |
| w6-8k-m2-Sal  | 6     | 2.35     | 0.391                 | 32.3                  | 8k  | 0           | 1                |
| w6-32k-m2-Sal  | 6     | 2.35     | 0.391                 | 32.3                  | 32k | 0           | 4                |
| w6-2k-m129-Sal | 6     | 2.35     | 0.296                 | 129.2                 | 2k  | 0           | 1                |
| w6-8k-m129-Sal | 6     | 2.35     | 0.296                 | 129.2                 | 8k  | 0           | 1                |
| w6-32k-m129-Sal | 6  | 2.35     | 0.296                 | 129.2                 | 32k | 0           | 4                |
| w6-2k-m258-Sal | 6     | 2.35     | 0.286                 | 258.3                 | 2k  | 0           | 4                |
| w6-2k-m517-Sal | 6     | 2.35     | 0.279                 | 516.6                 | 2k  | 0           | 1                |
| w6-8k-m517-Sal | 6     | 2.35     | 0.279                 | 516.6                 | 8k  | 0           | 1                |
| w6-32k-m517-Sal | 6  | 2.35     | 0.279                 | 516.6                 | 32k | 0           | 4                |
| w6-2k-m2-a1.2  | 6     | 1.2      | 0.442                 | 2.02                  | 2k  | 0           | 1                |
| w6-8k-m2-a1.2  | 6     | 1.2      | 0.442                 | 2.02                  | 8k  | 0           | 1                |
| w6-2k-m8-a1.2  | 6     | 1.2      | 0.0336                | 8.07                  | 2k  | 0           | 1                |
| w6-8k-m8-a1.2  | 6     | 1.2      | 0.0336                | 8.07                  | 8k  | 0           | 1                |
| w6-2k-m32-a1.2 | 6     | 1.2      | 5.28 × 10^{-4}        | 32.3                  | 2k  | 0           | 1                |
| w6-8k-m32-a1.2 | 6     | 1.2      | 5.28 × 10^{-4}        | 32.3                  | 8k  | 0           | 1                |
| w6-2k-m129-a1.2| 6     | 1.2      | 5.16 × 10^{-6}        | 129.2                 | 2k  | 0           | 1                |
| w6-8k-m129-a1.2| 6     | 1.2      | 5.16 × 10^{-6}        | 129.2                 | 8k  | 0           | 1                |
| w6-32k-m129-a1.2 | 6  | 1.2      | 5.16 × 10^{-6}        | 129.2                 | 32k | 0           | 1                |
| w6-8k-m517-a1.2| 6     | 1.2      | 1.23 × 10^{-8}        | 516.6                 | 8k  | 0           | 1                |
| w6-32k-m517-a1.2 | 6  | 1.2      | 1.23 × 10^{-8}        | 516.6                 | 32k | 0           | 1                |
Figures 1 and 2 reached its minimum (Heggie et al. 2006) for models in which the core evolution is visible ($f_{\text{max}} \lesssim 8$ for $N = 2k$, $f_{\text{max}} \lesssim 32$ for $N = 8k$, $f_{\text{max}} \lesssim 129$ for $N = 32k$, and $f_{\text{max}} \lesssim 517$ for $N = 128k$). We confirmed that there is no large discrepancy and no bias between them.

Based on our simulations we conclude that a clear core bounce occurs if $M/m_{\text{max}} \lesssim 10^3$ ($f_{\text{max}} \lesssim 2$ for $N = 2k$), and the rapid core expansion after a shrink of the core becomes apparent only if $M/m_{\text{max}} \lesssim 100$ ($f_{\text{max}} \lesssim 20$ for $N = 2k$). We illustrate these in the left panels of Figure 3 where we present the same models as in the right panels in Figure 2 but with $N = 128k$. This model satisfies the first criterion ($M/m_{\text{max}} \lesssim 10^3$) and as expected the core bounce is clearly visible. With $f_{\text{max}} = 517$ (the right panels of Figure 3), however, which does not satisfy either of the criteria and as a consequence the core bounce becomes indistinguishable. These criteria are similar to the Spitzer instability (Spitzer 1987). In the case of multi-mass components, the criterion for the Spitzer instability is $M/m_{\text{max}} \sim 10^4$ (Breen & Heggie 2012a). We discuss these criteria further in section 4.

3.2 The determination of core-collapse time using hard binaries

In the previous subsection we discussed the lack of a core bounce for the case where $M/m_{\text{max}} \lesssim 10^3$. In particular if $M/m_{\text{max}} \lesssim 100$, the core in these cases however still expands quite dramatically after some time. In these cases it becomes very hard to use the core size or density peak to determine the moment of core collapse, but the expansion of the core indicates that something like a core collapse must have happened. In order to quantify this we focus on the evolution of hard (dynamically formed) binaries, which are suspected to generate the energy that causes the core to bounce.

3.2.1 Measured binary hardness at core collapse

In the bottom panels in Figures 1, 2, and 3 we present the binding energy of the hardest binaries in the various simulations. By comparison of the evolution of the binding energy with the core density (or core radius), we observe that the moment of the core bounce is consistently occurring at the same moment that the binding energy of the hardest binary reaches $\sim f_{\text{max}} kT$.

In Figure 4 we present the binding energy of the hardest binary at the moment of core collapse measured from the highest core density and the smallest core radius ($E_{\text{bin,cc}}$). In this figure we scale the binding energy by a factor $f_{\text{max}}$. We measure the moment of core collapse using two different methods, both of which give consistent results. The scatter in the binding energy is larger, probably because of the measurement timing. We can only measure the binding energy at the moment of an output time. In particular for equal-mass models and those with $f_{\text{max}} = 2$ the binding energy rapidly increases towards the moment of core collapse (see Figure 1). The softening in the gravitational potential does not appear to affect our measurements of the moment of core collapse, but $E_{\text{bin,cc}}$ is systematically larger is the softened models. We are therefore prone to overestimating the binding energy in these models. We conclude that the average binding energy at the moment of core collapse is $\sim 10 f_{\text{max}} kT$ and this effect appears to be independent of $N$. Hereafter we specify the critical binding energy, $E_{\text{cr}}$, as a minimum binding energy required for core collapse.

We measure the time when the binding energy of the hardest binary for the first time reaches $E_{\text{cr}} = 3, 5, 10,$ and $30 f_{\text{max}} kT$. The results compared to the core collapse time measured from the smoothed core radius are shown in Figure 5. We find that $E_{\text{cr}} = 10 f_{\text{max}} kT$ provides the best comparison. In the following analysis, we associate the first moment when the binding energy of the hardest binary exceeds $10 f_{\text{max}} kT$ as the core collapse time, even in the cases that the core collapse is not obvious upon the inspection of the core radius (see Figure 2 and 3). It turns out that the binding energy of the hardest binary is an excellent indicator for identifying the moment of core collapse. In the following we discuss the argument for $E_{\text{cr}} \sim 10 f_{\text{max}} kT$ from a more theoretical perspective.

3.2.2 Theoretical binary hardness at core collapse

We estimate the critical binding energy for the core bounce from a discussion on the energy emitted by the hard binary via a three-body encounter. In the dynamical evolution of star clusters through core collapse, the cluster responds to a core collapse by a bounce, and the occurrence is associated with the moment when the energy produced by hard binaries exceeds the potential energy of the cluster core $\phi_c$ (Hut 1994; Heggie & Hut 2003). Following the discussion in Hut (1994) assuming that the core is virialized until the moment of the bounce, the potential energy of the core is

$$|\phi_{c}| = N_c \langle m \rangle_c \sigma_{c,3D}^2,$$

where the $N_c$, $\langle m \rangle_c$, and $\sigma_{c,3D}$ are the number, the mean mass, and velocity dispersion of the stars in the core, respectively. The energy released in an encounter between a single star and a binary with binding energy $E_{\text{bin}}$ is estimated by $\Delta E_{\text{bin}} = 0.4 E_{\text{bin}}$ (Heggie 1975) for equal mass cases. The coefficient is ill constrained in multi-mass cases, and we therefore adopt $\Delta E_{\text{bin}} \sim E_{\text{bin}}$ for the first order estimate of the critical binding energy. We then obtain that

$$E_{\text{cr}} \sim N_c \langle m \rangle_{c} \sigma_{c,3D}^2,$$

With a mass function, $\langle m \rangle_c > \langle m \rangle$ due to mass-segregation. Here we assume that $\langle m \rangle_c \sim m_{\text{max}}$. If we rewrite equation 3 with $kT$, we obtain that

$$E_{\text{cr}} \sim N_c m_{\text{max}} \sigma_{c,3D}^2 \langle m \rangle / \sigma_{1D}^2 kT$$

Now we have to estimate $N_c$ and $\sigma_{c,3D}^2$. Initially $\sigma_{c,1D}^2 = 1.5-1.6 \sigma_{1D}^2$ for King models with $W_0 = 3-6$. The core velocity dispersion $\sigma_{c,1D}$ increases towards the core collapse (Giersz & Heggie 1996), but by only a factor of 2 because the core evolves following $\rho_c \propto r^{-\kappa}$ and $\kappa = 2.2-2.3$ (Cohn 1980; Lynden-Bell & Eggleton 1984; Takahashi 1995; Heggie & Hut 2003). We therefore estimate that $\sigma_{c,1D}^2 / \sigma_{1D}^2$ is roughly a factor of 3. For $N_c$ it is theoretically estimated that $N_c \sim 80$ (Hut 1994). Numerically it is obtained that $N_c = 10-30$ for an equal-mass system (Makino 1996). $N_c = 10-100$ for two-component systems (Khalisi et al. 2007), and
Figure 1. Time evolution of the cluster parameters. Top panel: the evolution of the core density. Middle panel: the evolution of the core radius (solid black curve) and the 10%, 30, 50 and 80% (bottom to top) Lagrangian radii (dotted curves). Red curve indicates a smoothed core radius using a method in Heggie et al. (2006). Bottom panel: the binding energy of the hardest binary scaled by $f_{\text{max}}$. Red curves indicate the largest $E_{\text{bin}}$ achieved in this simulations, and black dots indicates the largest $E_{\text{bin}}$ at each time. Blue dashed lines indicate the total energy of the cluster, $E$, and also $0.5E$, and $(M_c/M)E$ from top to bottom. In our model, $E = 1.5NkT$. The left panels give the results for the model with $W_0 = 3$ for equal-mass particles, and the right panels give the results for a mass function with $\alpha = 2.35$ and $f_{\text{max}} = 8$.

Figure 2. Same as Figure 1 but for models with $f_{\text{max}} = 32$ (left) and 129 (right).
The moment of core collapse in star clusters

Figure 3. Same as Figure 1 but for models with \( N = 128k \) and with \( f_{\text{max}} = 129 \) (left) and 517 (right).

Figure 4. The binding energy of the hardest binary at the moment of core collapse measured from time averaged core radius (circles) and from the maximum core density (triangles). Blue and red indicate models with \( \alpha = 2.35 \) without softening. Green and orange are for the same models but with softening. Cyan and magenta indicate models with \( \alpha = 1.2 \) The binding energy is scale by \( f_{\text{max}} \). Here we plot only models with \( \frac{M}{m_{\text{max}}} \gtrsim 100 \) (\( f_{\text{max}} \leq 8 \) for \( N = 2k \), \( f_{\text{max}} \leq 32 \) for \( N = 8k \), \( f_{\text{max}} \leq 129 \) for \( N = 32k \), and \( f_{\text{max}} \leq 517 \) for \( N = 128k \) and with \( N_{\text{run}} > 1 \).
$N_c \sim 25$ for multi-mass systems (Heggie & Hut, 2003). Recent study by Tanikawa et al. (2012) report that there are only $\sim 5$ stars in the core when a hard binary with $\sim 10kT$ forms and that the formation process of such a hard binary is sudden rather than gradual evolution from a softer binary. From our numerical result that $E_{\text{fr}} \approx 9N_c f_{\text{max}} kT$, we roughly estimate that $N_c \sim O(1)$. Hereafter we adopt $E_{\text{bin}} > E_{\text{fr}} = 10 f_{\text{max}} kT$ as the moment of core collapse.

### 3.3 The core-collapse time

In Figure 3 we present $t_{\text{cc}}/t_{\text{rc}}$ as a function of $f_{\text{max}} (= m_{\text{max}}/m)$. Here we defined the core-collapse time $t_{\text{cc}}$ as the moment when the binding energy of the hardest binary in the cluster exceeds $E_{\text{fr}} = 10 f_{\text{max}} kT$. The core-collapse time for single-mass component models is $t_{\text{cc}} \approx 20$ for $W_0 = 3$ and 50 for $W_0 = 6$, which are consistent with previous results (Gürkan et al., 2004, and references therein). With a mass function, $t_{\text{cc}}/t_{\text{rc}}$ decreases as we increase $f_{\text{max}}$, and it follows $f_{\text{max}}^{-1}$ (thick dashed line in Figure 3) as far as $f_{\text{max}} \gtrsim 30$.

We analytically derive the core-collapse time for models with stellar mass functions using the dynamical friction timescale of the most massive stars in the cluster, assuming that star clusters collapse when the most massive stars reach the cluster center. The dynamical friction timescale of the most massive star with $m_{\text{max}}$ is estimated from a simple equation. We follow the description in Portegies Zwart & McMillan (2002) (see also section 8.1.1 in Binney & Tremaine, 2008), in which the dynamical friction timescale of a black hole which spirals in to the galactic center is derived. We assume that the massive star has initially a circular orbit with velocity, $v_c$, at a distance $r$ from the cluster center. From equation (8.9) in Binney & Tremaine (2008), the frictional force $F = m_{\text{max}} |d\nu_m/dt|$ on the massive star is

$$F = \frac{4\pi G^2 m_{\text{max}} \rho(r) \ln \Lambda}{v_c^2} \left[ \text{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2) \right],$$

where $X \equiv v_c/(\sqrt{2}\sigma_{\text{1D}}) = 1$, where $\sigma_{\text{1D}}$ is the one-dimensional velocity dispersion and $v_c$ is equivalent to the two-dimensional velocity dispersion. The value of the Coulomb logarithm here is different from that in equation (4), and therefore we write $\ln \Lambda'$. In order to simplify this equation, we assume the density distribution to be a singular isothermal sphere, $\rho(r) = v_c^2/(4\pi G r^2)$, and equation (6) then becomes

$$F = 0.428 \ln \Lambda' \frac{G m_{\text{max}}^2}{r^4}.$$  

The angular momentum change of the massive star due to the friction is

$$\frac{dL}{dt} = -F r$$

$$\simeq 0.428 \ln \Lambda' \frac{G m_{\text{max}}^2}{r}.$$  

In an isothermal sphere the circular velocity is independent of radius, and the angular momentum at radius $r$ is written as $L = m_{\text{max}} r v_c$. With equation (8), we obtain

$$\frac{dr}{dt} = -0.428 \ln \Lambda' \frac{G m_{\text{max}}}{v_c}.$$  

The dynamical friction timescale of the most massive star is finally written as

$$t_{\text{df}} = \frac{1.91}{\ln \Lambda'} \frac{r^2 \sigma_{\text{1D}}}{G m_{\text{max}}}.$$  

Assuming that $t_{\text{df}} = t_{\text{cc}}$, from equations (4) and (11) we
obtain

\[
\frac{t_{cc}}{t_{cc}} = \frac{29.4 \ln \Lambda}{\ln \Lambda} f_{\text{max}} \frac{Gr^2 \rho}{\sigma^3_{\text{3D}}}. \tag{12}
\]

This result is shown in Figure 6 (dashed curves) and is consistent with the simulations. Here, we adopted \( r \) to be the virial radius, \( r_{\text{vir}} = GM/4E = 1 \) in N-body units, and \( \Lambda = 0.1N \) (Giersz & Heggie 1994a; Heggie & Hut 2003). For the Coulomb logarithm for \( t_{cc} \) (equation (1)), we adopted \( \Lambda = 0.1N_c \), where \( N_c \equiv (M_e/M)N \) is the number of particles in the core. The dynamical friction timescale of the most massive star is proportional to \( m_{\text{max}} \), and as a consequence the core-relaxation time is proportional to \( \langle m \rangle^{-1} \) (see Eq. 1), and therefore we obtain that \( t_{cc}/t_{cc} \propto (m_{\text{max}}/\langle m \rangle)^{-1} \).

### 3.4 The minimum core-collapse time

We see in Figure 6 that for models with fewer particles \((N = 2k)\) the value of \( t_{cc}/t_{cc} \) starts to deviate from the analytic result for \( f_{\text{max}} \sim 30 \). This critical value of \( f_{\text{max}} \) however, depends on \( N \); for models with \( N = 128k \) the simulations and theory give consistent results up to \( f_{\text{max}} \sim 100 \). The models between \( N = 2k \) and \( 128k \) show a consistent picture in the sense that the models with a larger \( N \) start to deviate from the theory at a larger value of \( f_{\text{max}} \). We bolster our earlier conclusion that a core bounce requires that \( M/m_{\text{max}} \gtrsim 100 \). The core-collapse time saturates for a smaller value of \( f_{\text{max}} \) in models with fewer particles. We will discuss this critical value of \( M/m_{\text{max}} \) in section 3.5 and now focus on estimating the “minimum” core-collapse time, as indicated in Figure 6.

This minimum in the core-collapse time depends on \( N \) due to the dependency of \( t_{cc} \) on \( N \). We consider that this minimum core-collapse time depends on the crossing time, \( t_{cross} \), of the cluster, because the dynamical friction time cannot be shorter than the crossing time. We adopt \( t_{cross} = r_{\text{vir}}/\sigma_{\text{1D}} \), where \( r_{\text{vir}} \) is the virial radius. The minimum core-collapse time obtained from the simulations is roughly consistent with \( 10t_{cross} \). The dotted curves in Figure 6 give \( 10t_{cross} \), and they depend on \( N \), because \( t_{cross} \) is independent of \( N \) whereas \( t_{cc} \) is.

### 3.5 The maximum critical binding energy

For models with a large \( f_{\text{max}} \) the critical binding energy \( E_{\text{crit}} \) is comparable to the total energy of the cluster \( (E) \). In those models, for example \( W_0 = 3 \), \( N = 2k \), and \( f_{\text{max}} = 517 \), the cluster dissolves before the binding energy reaches \( 10f_{\text{max}}kT \). We find that \( E_{\text{crit}} = 0.5E \) roughly traces the minimum of the smoothed core radius (see left panels of Figure 7). We therefore adopt \( E_{\text{crit}} = 0.5E \) if \( 10f_{\text{max}}kT > 0.5E \). We are able to detect the moment of core-collapse time even for \( m_{\text{max}} \sim M_e \) if we adopt \( E_{\text{crit}} = 0.5E \) (see right panels of Figure 7).

When \( m_{\text{max}} > M_e \), the evolution of the hardest binary is different from those in models with \( m_{\text{max}} < M_e \). In the former case, a hard sub-systems in which several stars orbits around the most massive star is actually detected. The hardest binary in this sub-system gradually hardens due to repeated scattering encounters with other stars. This evolution is visible in the temporal behavior of the total binding energy of the binaries. In Figure 7 (right panels), we present the total binding energy (green curve), which represents the total energy of the sub-system with the most massive star. The binding energy of the hardest binary is initially much smaller than the total binding energy, but it...
eventually catches up with the total binding energy. When on the other hand \( m_{\text{max}} \gtrsim M_c \), the total binding energy remains comparable to the binding energy of the hardest binary (Figure 7 left panels). In both cases, the binding energy seems to be limited by the total energy of the cluster. In our simulations, the binding energy evolution saturates between \( E \) and \( (M_c/M)E \) (see also Figures 1 and 2). In table 4 we summarize the adopted values of the critical binding energy \( E_{cr} \).

4 DISCUSSION

4.1 \( N \)-dependence and comparison with single-component models

We aim to find an objective criterion for detecting the core collapse in simulated star clusters. We concentrate on those cases where \( M/m_{\text{max}} \gtrsim 10^4 \) and for rapid core expansion in the case that \( M/m_{\text{max}} \gtrsim 100 \). Here we will make an analogy with single-component models. From a wide range of analytic calculations and simulations, it is well established that the dynamical evolution of star clusters such as relaxation, core collapse, core bounce, and gravothermal oscillations only depend the total number of particles in the system (and in the core). For example, gravothermal oscillation occurs only when the number of particles exceed \( \sim 10^4 \) (Goodman 1987; Makino 1996). This criterion comes from the number of particles in the core after core bounce, \( N_c \). (Here we define \( N_c \) as the average number of particles in the core after the actual core collapse, and we adopt \( N_{cb} \) as the number of particles at the moment of core bounce, i.e., at the moment of deepest core collapse.) The gravothermal oscillation occurs only when \( N_c > N_{cb} \). While the value of \( N_{cb} \) is considered a constant in the range of 10 to 80 (Goodman 1987; Makino 1996; Hut 1996), \( N_c \) depends on \( N \). If we adopt \( N_c \simeq N_p^{1/2} \) (Makino 1996), we can confirm that \( N > 10^4 \) satisfies \( N_c > N_{cb} \).

The behavior after core collapse changes when \( N \) decreases. So long as \( N_c \simeq N_{cb} \) (i.e., \( 10^3 \lesssim N \lesssim 10^4 \)) we observe similar evolution but the collapse becomes shallower for smaller \( N_c \) and the gravothermal oscillations damp (see Fig. 1 in Makino 1996). For \( 10^2 \lesssim N \lesssim 10^3 \) the core bounce becomes indiscernible (see Figure 10 in Giersz & Heggie 1994b). This transition is quite similar to those we observe if \( m_{\text{max}} \) is increased. Our results of the multi-mass case are scalable to those of the single mass case if we define an effective number of particles \( (N_{\text{eff}}) \) in the latter. For an equal mass system, \( N_{\text{eff}} \approx N \), but when we introduce a spectrum of masses, \( N_{\text{eff}} = M/m_{\text{max}} \). Interestingly, a similar conclusion is obtained from recent results for two-component and multi-component systems (Breen & Heggie 2012b,a).

If the number of particles drops below \( \sim 100 \), the behavior of the \( N \)-body system changes from being a many-body system to a few-body system, which evolves chaotically (McMillan et al. 1988) rather than deterministic. In few-body systems it is hard to notice the collapse of the core in the evolution of core density and radius. In these cases the binary cannot harden to \( \gtrsim 100kT \) because the total energy budget of the cluster \( \lesssim 100kT \). In such systems the hard binary stops interacting with other cluster members when it becomes too tight and the surrounding density becomes too low. As a result, the “tenured” or sometimes called “frozen binary” remains in the cluster (Casertano 1985). This can be observed in Figures 2 and 4 for the multi-mass models when \( M/m_{\text{max}} \gtrsim 100 \). The dynamical evolution of tenured binaries almost stops in this case, but they remain in the cluster (most likely in the core). In Figure 8 we present the evolution of density profiles of model w3-2k-m129-Sal (we used the same model in the right panel of Figure 2). The tenured binary and its relatively low-density environment are noticeable as the high density peak in the center and a dimple at around \( 0.2r \) in the more extended core. This effect is similar to the core mass-deficiency arguments used in galactic nuclei with binary black holes after a major galaxy merger (Merritt 2011). In the formation process of tenured binaries, massive particles concentrated in the cluster center are ejected from the cluster by strong interactions with the binary. This mechanism leads to the formation of massive runaway stars around dense, young star clusters (Fujii & Portegies Zwart 2011).

4.2 Dynamical evolution driven by the most massive stars

The dynamical evolution of equal-mass models scale with \( t_{cc} \) irrespective of the particle number (Spitzer 1987). Our results are consistent with this hypothesis in the case of multi-mass simulations, so long as \( M/m_{\text{max}} \gtrsim 100 \). For multi-mass models, however, \( t_{cc}/t_{cc} \) decreases for increasing \( m_{\text{max}}/m \) (\( = f_{\text{max}} \)), and we demonstrate that \( t_{cc} \) is determined by the dynamical friction timescale of the most massive stars in the cluster. Here we demonstrate that the dynamical evolution of star clusters with a mass function is driven by the most massive stars in the cluster.

The relaxation of a multi-mass system is dominated by the dynamics of the most massive star, and the global relaxation time is a factor of \( f_{\text{max}} \equiv \ln(\gamma N/f_{\text{max}})/(f_{\text{max}} \ln(\gamma N)) \)
We present the evolution of the core density scaled with \( \frac{t}{t_{cc}} \) and therefore that \( \rho_{c,\text{max}} \propto t/t_{cc} \), which is consistent with the relation between the maximum density and \( N \) (Giersz & Heggie 1994b). Hud (1996) derived that \( \rho \propto N^{-2} \), in which case we expect that the maximum density decreases \( \propto t^{-2} \). Here we assumed that the core consists of the most massive stars and therefore that \( \rho_{c,\text{max}} \propto N_{\text{eff}}^{-2} \). In Figure 10 however, the power appears to be shallower than \( -2 \), although the trend that the maximum density decreases for increasing \( f_{\text{max}} \) is reproduced. This might be caused by the mean particle mass in the core being smaller than \( m_{\text{max}} \). Here we would like to point out that we measure the core density only in a snapshot, the moment of which is limited by our output frequency. We therefore are likely to miss the highest density peak; it is very difficult to catch the exact moment of core collapse, but shows a core bounce at an earlier time than usual. This discrepancy can be attributed to their Monte-Carlo code, which may not properly be able to follow the cluster all the way to the moment of core collapse, but shows a core bounce at an earlier instance. A hint of the early termination of their calculations is visible in their Fig. 2, where the Lagrangian radius of the core-collapse time is proportional to \( \frac{t}{t_{cc}} \), which is inconsistent with a previous work by Faber et al. (2010) (see also Farr et al. 2011). This discrepancy can be attributed to their Monte-Carlo code, which may not properly be able to follow the cluster all the way to the moment of core collapse, but shows a core bounce at an earlier instance. A hint of the early termination of their calculations is visible in their Fig. 2, where the Lagrangian radius of the simulations with a mass function suddenly truncates. They identify this moment as core collapse, but by inspection of

**Figure 7.** Same as Figure 1 but for models with \( N = 2k \) and \( f_{\text{max}} = 517 \) for \( W_0 = 3 \) (left) and \( W_0 = 6 \) (right). In bottom panels, green curves show the total binding energy of binaries.

**Table 4. Summary of \( E_{\text{cc}} \)**

| \( m_{\text{max}} \) | \( E_{\text{cc}} \) | \( M/m_{\text{max}} \) (\( \equiv N_{\text{eff}} \)) | \( f_{\text{max}} \) in our models \((N = 2k)\) |
|----------------|-------------------|-------------------------------|---------------------|
| \( m_{\text{max}} < M_c \) | \( 10 f_{\text{max}} kT \) | \( \gtrsim 10 \) | \(< 258 \) (Fig. 12) |
| \( m_{\text{max}} \sim M_c \) | \( 0.5 E \) | \( \sim 10 \) | \( 258, 517 \) (Fig. 7) |
| \( m_{\text{max}} > M_c \) | - (no collapse) | \( \gtrsim 10 \) | - |


4.3 Comparison with previous results

We find that \( t_{cc}/t_{cc} \sim f_{\text{max}}^{-1} \), which is consistent with a previous work by G" urkan et al. (2004), who conclude that the core-collapse time is proportional to \( f_{\text{max}}^{-1} \). This discrepancy can be attributed to their Monte-Carlo code, which may not properly be able to follow the cluster all the way to the moment of core collapse, but shows a core bounce at an earlier instance. A hint of the early termination of their calculations is visible in their Fig. 2, where the Lagrangian radius of the simulations with a mass function suddenly truncates. They identify this moment as core collapse, but by inspection of
models with \( N \) present the formation time of the first 3 hard \((\sim)\) our own simulations the sudden break in the inner most Lagrangian radii is generally associated with the formation of the first hard \((\sim 3kT)\) binary. We tested this hypothesis by analyzing the results of our simulation up to the moment of the formation of the first hard \((3kT)\) binary. In figure 11 we present the formation time of the first 3\(kT\)-binary, \( t_{3kT} \), for models with \( W_0 = 3 \) as a function of \( f_{\text{max}} \). If the moment of core collapse is identified by moment of formation of the first 1–3\(kT\) hard binary, we find that \( t_{3kT}/t_{cc} \propto f_{\text{max}}^{-1.3} \), which is consistent with the results of [Güurkan et al. 2004]. In that case the core collapse time saturates at \( t_{cc}/t_{rc} \approx 0.15 \), which they suggested to be associated with the minimum core-collapse time.

5 CONCLUSIONS

We performed a series of \( N\)-body simulations of star clusters with various mass ranges and power laws of the mass function, and found that the core-collapse time follows \( t_{cc}/t_{cc} \propto (m_{\text{max}}/\langle m \rangle)^{-1} \) for clusters with \( M/m_{\text{max}} \gtrsim 100 \). When \( M/m_{\text{max}} \gtrsim 100 \), this relation breaks and \( t_{cc} \) saturates at \( \sim 10 t_{\text{cross}} \). We subsequently argue that star clusters with a mass function reach core collapse on the dynamical-friction timescale of the most massive stars [consistent with the results of Portegies Zwart & McMillan (2002)]. We also showed that the dynamical evolution of star clusters with a mass function are driven by the relaxation timescale of the most massive stars. We define an effective number of stars \( N_{\text{eff}} = M/m_{\text{max}} \) for which a multi-mass cluster shows a similar core collapse behavior as in the equal-mass case.

When the mass of the most massive stars is relatively small \((M/m_{\text{max}} \gtrsim 10^3)\), we notice a pronounced peak in the core density during the evolution. As we increase the mass of the most massive stars, \( m_{\text{max}} \), the peak density at the core bounce becomes lower. We found that the binding energy of the hard binaries are a good indicator for detecting
the moment of the core collapse, even if the density peak is ambiguous. We adopted that the critical binding energy of hard binaries, with which the binary emits sufficient energy to bounce the core, as the moment of the core bounce. We conclude that the binding energy criterion, $E_{\text{cr}} \approx 10 f_{\text{max}} kT$, gives a more robust indicator for the core-collapse time compared to inspection of the core density and radius.

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