SYMBOLIC DYNAMICS OF PLANAR PIECEWISE SMOOTH VECTOR FIELDS

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Abstract. Recently, the theory concerning piecewise smooth vector fields (PSVFs for short) have been undergoing important improvements. In fact, many results obtained do not have an analogous for smooth vector fields. For example, the chaoticity of planar PSVFs, which is impossible for the smooth ones. These differences are generated by the non-uniqueness of trajectory passing through a point. Inspired by the classical fact that one-dimensional discrete dynamic systems can produce chaotic behavior, we construct a conjugation between the shift map and PSVFs. By means of the results obtained and the techniques employed, a new perspective on the study of PSVFs is brought to light and, through already established results for discrete dynamic systems, we will be able to obtain results regarding PSVFs.

1. Introduction

In this paper we investigate some correlations between dynamical systems continuous and discrete in time. In fact, several times, when working with continuous flows, one creates a single transformation of the space (or a subset of it) on itself creating a discrete dynamical system. For example, when studying a physical phenomenon and the measures of such can only be made in a discrete set of time; or when studying a Poincaré Map (first return map) that gives valuable information of a continuous flow concerning stability and cyclicity. The inverse procedure also is useful. For example, a biological phenomenon, where the number of individuals (cells, preys, etc) are natural numbers, can be modeled using an ordinary differential equation continuous in time (instead of a discrete function defined in the set \( \mathbb{N} \) of natural number). In other cases, it is possible to create a transformation of the flow domain (or a subset of it) on a set of discrete objects (for example, numbers either in \( \mathbb{Z} \) or \( \mathbb{N} \)) and using the so called symbolic dynamics to infer some important properties of that flow.

In parallel, there are a lot of applied problems that can be modeled using flows continuous in time but in such a way that the vector field involved is

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not smooth. This happens when the system is suddenly submitted to a on-off change. For example, when we are studying the evolution of cancer cells (resp., HIV infected cells) in a patient submitted to an intermittent protocol of treatment, where chemotherapy (resp., antiretroviral) is administered in periodic periods (see [9, 10, 11]). We will use the term Piecewise Smooth Vector Fields (PSVFs, for short) to refer to these non-smooth vector fields.

One of the most relevant characteristic about PSVFs is the non uniqueness of trajectory passing through a point, which means that if one takes a point, it is possible to ask where its trajectory is going after an amount of time $t$, and we may have several different answers for that. But, if one takes a whole trajectory instead of a point, there is no ambiguity in that question, which indicates that we may have some more properties of the PSVFs when studying the space of orbits of that system.

The literature about PSVFs predicts the existence of chaotic behavior even in the planar (two-dimensional) case (see [3, 2]). The proofs establish in the previous papers take into in count the classical definition of chaos (sensibility to the initial conditions + topological transitivity). However, at the present paper, we introduce a fully new way of addressing this issue. In fact, we induce a discrete dynamics in the space of orbits of the PSVFs and use symbolic dynamics to describe this system. Moreover, the results obtained can be useful, in the future, to answer another open questions, like:

- What is the topological entropy of PSVFs?
- Is it possible the obtainment of Horse Shoes for (planar) PSVFs?
- Is it possible to determine a fractal (box) dimension for PSVFs?

In fact, the deepness of the results and techniques presented in this work is expected to be revealed since a wide range of problems concerning PSVFs can be addressed by following classical scripts in symbolic dynamics, and vice versa.

2. Main results

Given a flow $\varphi(t, x)$ of a vector field $W$ defined in an open set $U$, the time-one map is the function $T_1 : U \to U$, $T_1(x) = \varphi(1, x)$. The next result relates time-one maps of PSVFs (see Definition 14) and (sub)shifts.

Theorem A. (i) There exists a conjugacy between the time-one map of the PSVF

$$Z_2(x, y) = \begin{cases} X_2(x, y) = (1, \frac{y}{2} - 4x^3), & \text{for } y \geq 0 \\ Y_2(x, y) = (-1, \frac{y}{2} - 4x^3), & \text{for } y \leq 0 \end{cases}, \quad (1)$$

restricted to an invariant compact set $\Lambda_2 \subset \mathbb{R}^2$ and the full shift of two symbols $\sigma : \{0, 1\}^2 \to \{0, 1\}^2$

(ii) For each $k \geq 3$, there exists a planar PSVF

$$Z_k(x, y) = \begin{cases} X_k(x, y) = (1, P_k'(x)), & \text{for } y \geq 0 \\ Y_k(x, y) = (-1, P_k'(x)), & \text{for } y \leq 0 \end{cases}, \quad (2)$$
where

\[ P_k(x) = -\left( x + \frac{k-1}{2} \right) \left( x - \frac{k-1}{2} \right)^{k-1} \prod_{i=1}^{k-1} \left( x - \left( \frac{i}{2} \right) \right)^2 \]  

(3)

such that, restricted to an invariant compact set \( \Lambda_k \), the time-one map is conjugated to a subshift of \( 2(k-1) \) symbols.

(iii) There exists a conjugacy between the time-one map of the PSVF

\[ Z_\infty(x,y) = \begin{cases} X_\infty(x,y) = (1, 2\sin(2\pi x)), & \text{for } y \geq 0 \\ Y_\infty(x,y) = (-1, 2\sin(2\pi x)), & \text{for } y \leq 0 \end{cases} \]  

(4)

restricted to an invariant set \( \Lambda_\infty \subset \mathbb{R}^2 \) and a subshift over an infinite alphabet \( \sigma : \Theta_\infty \rightarrow \Theta_\infty \).

(iv) The return map of the PSVF

\[ Z(x,y) = \begin{cases} X(x,y) = (1, -2x), & \text{for } y \geq 0 \\ Y(x,y) = (-2, -4x^3 + 2x), & \text{for } y \leq 0 \end{cases} \]  

(5)

restricted to an invariant compact set \( \Lambda \), is conjugated to \( \sigma : (0,1]^\mathbb{Z} \rightarrow (0,1]^\mathbb{Z} \).

In Section 6 is exhibited an example of PSVF whose time one map is conjugated to sub-shift of 4-symbols.

In the next result we state that the canonical forms presented in items (i) to (iv) of Theorem A represent a bigger class of PSVFs conjugated to the (sub)shifts mentioned. Before announcing it, we indicate Definitions 1 and 5 where the concept of fold points and homoclinic loops are defined.

**Theorem B.**

(i) The PSVF \( Z_2 \) of Theorem A, restrict to \( \Lambda_2 \), is \( \Sigma \)-equivalent to any PSVF presenting a 1-homoclinic loop.

(ii) The PSVF \( Z_k \) of Theorem A, restrict to \( \Lambda_k \), is \( \Sigma \)-equivalent to any PSVF presenting a \((k-1)\)-homoclinic loop.

(iii) The PSVF \( Z_\infty \) of Theorem A, restrict to \( \Lambda_\infty \), is \( \Sigma \)-equivalent to any PSVF presenting a \( \infty \)-homoclinic loop.

(iv) The PSVF \( Z \) of Theorem A, restrict to \( \Lambda \), is \( \Sigma \)-equivalent to any PSVF \( \tilde{Z} \) presenting a compact region \( \tilde{\Lambda} \) bounded by a trajectory of \( \tilde{Z} \) passing through a invisible-visible two-fold \( \tilde{p} \). Moreover, except for \( \tilde{p} \), the PSVF \( \tilde{Z} \) has just more two invisible tangential singularities.

We stress that the restriction to the pictured sets of both PSVFs in Figure 1, according to Theorem B item (i), are \( \Sigma \)-equivalents to \( Z_2 \) restricted to \( \Lambda_2 \). That means that our result is not restrict to general PSVFs with invariant sets with the shape of a “figure eight lying”. The same holds for \( Z_k \), with \( k \in \{3,4,\ldots\} \cup \{\infty\} \).

As an immediate consequence of the previous results we have that:

**Corollary 1.** The time-one maps of PSVFs presented in Theorems A and B have periodic points of any period and are topologically mixing (see Definition 12).
Figure 1. Distinct kinds of PSVF$s whose restriction to an invariant set is $\Sigma$-equivalent to a restriction of $Z_2$ to $\Lambda_2$. Also, it is important to highlight that these PSVF$s could have non-polynomial expressions. An analogous remark could be done for the other cases in Theorem A.

Corollary 2. All PSVF$s presented in items (i) to (iv) of Theorems A and B are chaotic.

3. General theory concerning PSVF$s, Shifts and Diffeomorphisms

3.1. Piecewise Smooth Vector Fields.

Consider a codimension one manifold $\Sigma$ of $\mathbb{R}^n$ given by $\Sigma = f^{-1}(0)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). Assume that $\Sigma$ is an embedded submanifold of $\mathbb{R}^n$. We call $\Sigma$ the switching manifold that is the separating boundary of the regions $\Sigma^+ = \{q \in \mathbb{R}^n \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in \mathbb{R}^n \mid f(q) \leq 0\}$.

Designate by $X^r$ the space of $C^r$-vector fields on $V \subset \mathbb{R}^n$ endowed with the $C^r$-topology, with $r \geq 1$ large enough for our purposes. Call $\Omega^r$ the space of piecewise smooth vector fields (PSVF$s for short) $Z : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$Z(q) = \begin{cases} X(q), & \text{for } q \in \Sigma^+, \\ Y(q), & \text{for } q \in \Sigma^- \end{cases}$$

(6)

where $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n) \in X^r$. We endow $\Omega^r$ with the product topology.

For practical purposes, the contact between the smooth vector field $X$ and the switching manifold $\Sigma = f^{-1}(0)$ is characterized by the expression $Xf(p) = \langle \nabla f(p), X(p) \rangle$ where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^n$.

The basic results of differential equations in this context were stated by Filippov (see [4]). We can distinguish on $\Sigma$ the following regions:

- Crossing Region: $\Sigma^c = \{p \in \Sigma \mid Xf(p) \cdot Yf(p) > 0\}$. Moreover, we denote $\Sigma^{c+} = \{p \in \Sigma \mid Xf(p) > 0, Yf(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma \mid Xf(p) < 0, Yf(p) < 0\}$.
- Sliding Region: $\Sigma^s = \{p \in \Sigma \mid Xf(p) < 0, Yf(p) > 0\}$.
- Escaping Region: $\Sigma^e = \{p \in \Sigma \mid Xf(p) > 0, Yf(p) < 0\}$.
Definition 1. In the case $Xf(p) = 0$ we say that $p$ is a tangential singularity of $X$. A tangential singularity $p \in \Sigma$ is a fold point of $X$ if $Xf(p) = 0$ but $X^2f(p) \neq 0$, where $X^if(p) = \langle \nabla X^{i-1}f(p), X(p) \rangle$ for $i \geq 2$. Moreover, $p \in \Sigma$ is a visible (respectively invisible) fold point of $X$ if $Xf(p) = 0$ and $X^2f(p) > 0$ (respectively $X^2f(p) < 0$). A point $p \in \Sigma$ is a two-fold, if it is a fold point for both $X$ and $Y$, and it is invisible-visible if visible for both (respectively invisible-visible and invisible-invisible).

In addition, a tangential singularity $q$ is singular if $q$ is an invisible tangency for both $X$ and $Y$. On the other hand, a tangential singularity $q$ is regular if it is not singular.

Definition 2. Given a point $p \in \Sigma_+ \cup \Sigma_- \subset \Sigma$, we define the sliding vector field at $p$ as the vector field $Z^T(p) = m - p$ with $m$ being the point of the segment joining $p + X(p)$ and $p + Y(p)$ such that $m - p$ is tangent to $\Sigma$.

In the plane, the sliding vector field is given by the expression

$$Z^T(p) = \frac{Yf(p)X(p) - Xf(p)Y(p)}{Yf(p) - Xf(p)}.$$  \hfill (7)

Moreover, the sliding vector field can be extend to $\Sigma^c \cup \Sigma^c$.

Now we establish the classical convention on the trajectories of orbit-solutions of a PSVF.

Definition 3. The local trajectory (orbit) $\phi_Z(t, p)$ of a PSVF given by (6) through $p \in V$ is defined as follows:

(i) For $p \in \Sigma^+ \setminus \Sigma$ and $p \in \Sigma^- \setminus \Sigma$ the trajectory is given by $\phi_Z(t, p) = \phi_X(t, p)$ and $\phi_Z(t, p) = \phi_Y(t, p)$ respectively, where $t \in I$.

(ii) For $p \in \Sigma^c$ and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t, p) = \phi_Y(t, p)$ for $t \in I \cap \{ t \leq 0 \}$ and $\phi_Z(t, p) = \phi_X(t, p)$ for $t \in I \cap \{ t \geq 0 \}$. For the case $p \in \Sigma^c$ the definition is the same reversing time.

(iii) For $p \in \Sigma^c$ and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t, p) = \phi_Z^T(t, p)$ for $t \in I \cap \{ t \leq 0 \}$ and $\phi_Z(t, p)$ is either $\phi_X(t, p)$ or $\phi_Y(t, p)$ or $\phi_Z^T(t, p)$ for $t \in I \cap \{ t \geq 0 \}$. For $p \in \Sigma^c$ the definition is the same reversing time.

(iv) For $p$ a regular tangency point and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t, p) = \phi_1(t, p)$ for $t \in I \cap \{ t \leq 0 \}$ and $\phi_Z(t, p) = \phi_2(t, p)$ for $t \in I \cap \{ t \geq 0 \}$, where each $\phi_1, \phi_2$ is either $\phi_X$ or $\phi_Y$ or $\phi_Z^T$.

(v) For $p$ a singular tangency point, $\phi_Z(t, p) = p$ for all $t \in \mathbb{R}$.

Definition 4. The global trajectory (orbit) $\Gamma_Z(t, p_0)$ of $Z$ passing through $p_0$ is a union

$$\Gamma_Z(t, p_0) = \bigcup_{i \in \mathbb{Z}} \{ \sigma_i(t, p_i) : t_i \leq t \leq t_{i+1} \}$$

of preserving-orientation local trajectories $\sigma_i(t, p_i)$ satisfying $\sigma_i(t_{i+1}, p_i) = \sigma_{i+1}(t_{i+1}, p_{i+1}) = p_{i+1}$ and $t_i \to \pm \infty$ as $i \to \pm \infty$. 
Definition 5. A **k-homoclinic loop** of a planar PSVF is a global trajectory of $Z = (X, Y)$ presenting $k$ distinct visible-visible two-fold singularities $p_1, \ldots, p_k$ in such a way that, after passes through $p_i$, the trajectory reaches $\Sigma$ either in $p_{i-1}$ or $p_{i+1}$ when $i = 2, \ldots, k - 1$. When $i = 1$ (resp., $i = k$), after passes through $p_i$, the trajectory reaches $\Sigma$ either in a sewing point or $p_{i+1}$ (resp., $p_{i-1}$).

Moreover, an $\infty$-**homoclinic loop** of a planar PSVF is a global trajectory of $Z$ presenting $\infty$ visible-visible two-fold singularities $p_1, p_2, \ldots$ in such a way that, after passes through $p_i$, the trajectory reaches $\Sigma$ either in $p_{i-1}$ or $p_{i+1}$.

The homoclinic loops $\Gamma$ defined above are **regular** if $X(p)$ and $Y(p)$ point to opposite direction, for all two-fold singularity $p \in \Gamma$ (remember that $X(p)$ and $Y(p)$ are parallel). Otherwise, the homoclinic loop is **singular**. Figures 1 and 6 illustrate some regular $k$-homoclinic loops.

Another important definition is the concept of equivalence between two PSVFs.

Definition 6. Two PSVFs $Z = (X, Y), \tilde{Z} = (\tilde{X}, \tilde{Y}) \in \Omega^r$, defined in $U, \tilde{U}$ respectively and with switching manifold $\Sigma$ and $\tilde{\Sigma}$ are $\Sigma$-**equivalent** if there exists an orientation preserving homeomorphism $h : U \rightarrow \tilde{U}$ that sends $U \cap \Sigma$ to $\tilde{U} \cap \tilde{\Sigma}$, the orbits of $X$ restricted to $U \cap \Sigma^+$ to the orbits of $\tilde{X}$ restricted to $\tilde{U} \cap \tilde{\Sigma}^+$, the orbits of $Y$ restricted to $U \cap \Sigma^-$ to the orbits of $\tilde{Y}$ restricted to $\tilde{U} \cap \tilde{\Sigma}^-$ and the orbits of $Z^T$ restricted to $\Sigma$ to the orbits of $\tilde{Z}^T$ restricted to $\tilde{\Sigma}$.

3.2. Symbolic Dynamics.

Consider a set $A_k$ with $k$ elements (say $A_k = \{0, 1, \ldots, k - 1\}$) with the discrete topology. Now, consider $A_k^\mathbb{Z}$, i.e., all the sequences $x = (x_j)_{j \in \mathbb{Z}}$, with $x_j \in A_k$, for all $j$ and the product topology of all discrete topologies.

Definition 7. Let $x = (x_j)_{j \in \mathbb{Z}}$ and $y = (y_j)_{j \in \mathbb{Z}}$ two elements of $A_k^\mathbb{Z}$. Define $d : A_k^\mathbb{Z} \times A_k^\mathbb{Z} \rightarrow \mathbb{R}$ by:

$$d(x, y) = \sum_{j \in \mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}}$$

Definition 8. Define $\sigma_k : A_k^\mathbb{Z} \rightarrow A_k^\mathbb{Z}$ given by $\sigma((a_j)) = (b_j)$, where $b_j = a_{j+1}$. The map $\sigma$ is called **two-sided full shift** and the discrete flow $(A_k^\mathbb{Z}, \sigma)$ is called **symbolic flow** or **shift system**.

Proposition 1. The following holds:

(i) The function $d$ defined above is a metric on $A_k^\mathbb{Z}$ and induces the product topology;

(ii) The space $A_k^\mathbb{Z}$ is a compact Hausdorff space;

(iii) $\sigma_k$ is a homeomorphism.
Proof. In [7], p. 48, the authors assert that the results in items (i) and (ii) of this proposition are true, but without a proof. So, for completeness, we proceed the proof below.

(i) The convergence of $d$ follows from comparison test with the geometric series and the properties for it to be a metric follows easily from absolute value in the numerator. To see that the topology is the same, consider $U$ an open basic set and $x \in U$. Wlog, one can assume that $U = \prod_{j < -\alpha} A \times \{x_{-\alpha}\} \times \cdots \times \{x_{\alpha}\} \times \prod_{j > \alpha} A$, for some $\alpha \in \mathbb{N}$. That is, any $y \in U$ coincides with $x$ in all entries between $-\alpha$ and $\alpha$.

Let $0 < r < \frac{1}{2\alpha}$ and consider the open ball $B = B(x,r)$. Let $y \in B$. Suppose there exists $\beta \in \mathbb{Z}, -\alpha \leq \beta \leq \alpha$ such that $y_\beta \neq x_\beta$. Then

$$d(x,y) = \sum_{j \in \mathbb{Z}} \frac{|x_j - y_j|}{2|j|} \geq \frac{1}{2|\beta|} \geq \frac{1}{2\alpha} > r$$

This is a contradiction, since $y \in B$. Then $y \in U$.

On the other hand, let $x \in A_k^\mathbb{Z}$ and $B = B(x,r)$ an open ball. There exists $\alpha \in \mathbb{N}$, such that $\frac{k - 1}{2\alpha - 1} < r$. Let $U$ be an basic open set as before. We have $x \in U$ and let $y \in U$. Then:

$$d(x,y) = \sum_{j \in \mathbb{Z}} \frac{|x_j - y_j|}{2|j|} = \sum_{j \in \mathbb{Z} \setminus \{j\} > \alpha} \frac{|x_j - y_j|}{2|j|} \leq 2(k - 1) \sum_{j = \alpha + 1}^{\infty} \frac{1}{2j} = \frac{k - 1}{2\alpha - 1} < r$$

(ii) Each copy of $A$ is compact, then, by Tychonoff’s Theorem, $A_k^\mathbb{Z}$ is compact. Moreover, it is a metric space, hence it is Hausdorff.

(iii) It is enough to note that $\sigma_k$ takes basic open set to basic open set.

When it is clear by the context, the subscript $k$ may be suppressed in the sequel.

Definition 9. Let $K \subset A^\mathbb{Z}$. We say $(K,\sigma)$ is a subshift if $K$ is closed and invariant for $\sigma$.

There are some works (see [5] and [8]) considering one and two sided shifts over enumerable alphabets. To the best of our knowledge, there is not a classic way to work with them, and when considering the full shift, it is difficult to define a topology over it in order to make it metrizable.

In this work, we consider a shift over $\mathbb{Z}$ presenting some restrictions over the sequences which are allowed to occur. These features allow us to define a metric over this space.

Definition 10. Let $\Theta_\infty \subset \mathbb{Z}^\mathbb{Z}$ be the set of bi-infinite sequences with integer entries $(x_j)_{j \in \mathbb{Z}}$, such that the difference between two consecutive entries is
at most 2. That is:

\[ \Theta_\infty = \{ (x_j)_{j \in \mathbb{Z}} \mid x_j \in \mathbb{Z} \text{ and } |x_{j+1} - x_j| \leq 2, \forall j \in \mathbb{Z} \} \]

We can define over \( \Theta_\infty \) the same metric given before.

**Definition 11.** Consider \( x = (x_j)_{j \in \mathbb{Z}} \) and \( y = (y_j)_{j \in \mathbb{Z}} \) two elements of \( \Theta_\infty \).

Define \( d : \Theta_\infty \times \Theta_\infty \rightarrow \mathbb{R} \) by:

\[ d(x, y) = \sum_{j \in \mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}} \]

**Proposition 2.** The function \( d \) defined above is a metric over \( \Theta_\infty \).

**Proof.** Once proven the convergence of \( d \), the properties for it to be a metric follows straightforward from the absolute value on numerator of the series.

Let \( x = (x_j)_{j \in \mathbb{Z}}, y = (y_j)_{j \in \mathbb{Z}} \in \Theta_\infty \), then for all \( j \in \mathbb{Z} \):

\[ |x_j - y_j| \leq |x_j - x_{j-1}| + |x_{j-1} - y_{j-1}| + |y_{j-1} - y_j| \leq |x_{j-1} - y_{j-1}| + 4 \]

Analogously, one can show:

\[ |x_j - y_j| \leq |x_{j+1} - y_{j+1}| + 4. \]

And, recursively, we obtain:

\[ \forall j \in \mathbb{Z} : |x_j - y_j| \leq |x_0 - y_0| + 4|j|. \]

Now we are in conditions to prove the convergence of \( d \). Given \( N \in \mathbb{N} \):

\[
\sum_{j=-N}^{N} \frac{|x_j - y_j|}{2^{|j|}} = |x_0 - y_0| + \sum_{j=1}^{N} \frac{|x_j - y_{-j}|}{2^{|j|}} + \sum_{j=1}^{N} \frac{|x_{-j} - y_j|}{2^{|j|}} \leq |x_0 - y_0| + \sum_{j=1}^{N} \frac{|x_0 - y_0| + 4j}{2^{|j|}} + \sum_{j=1}^{N} \frac{|x_0 - y_0| + 4j}{2^{|j|}} \leq |x_0 - y_0| + 2|x_0 - y_0| \sum_{j=1}^{N} \frac{1}{2^j} + 8 \sum_{j=1}^{N} \frac{j}{2^j} \leq 3|x_0 - y_0| + 16 < \infty \]

Making \( N \rightarrow \infty \), we have that \( d \) converges. \( \square \)

Two important properties of a dynamical system \((X, f)\) are defined below:

**Definition 12.** Let \((X, f)\) be a dynamical system. We say that \( f \) is topologically transitive if for every open sets \( \mathcal{U}, \mathcal{V}, \) there exists \( n, \) such that \( f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset. \) And \( f \) is topologically mixing if for every open sets \( \mathcal{U}, \mathcal{V}, \) there exists \( n_0, \) such that \( f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset \) for all \( n \geq n_0. \)

**Proposition 3.** If \( f \) is topologically mixing, then \( f \) is topologically transitive.

**Proposition 4.** The two-sided shift \( \sigma_k \) is topologically mixing.
Let $\text{Lemma 1.}$ The Hausdorff distance between the sets $A$ and $B$ is given by

$$d_H(A, B) = \max \{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \}$$

4. Proof of Theorem 1

For each natural number $k \geq 2$, let:

$$P_k(x) = -\left( x + \frac{k-1}{2} \right) \left( x - \frac{k-1}{2} \right) \prod_{i=1}^{k-1} \left( x - \left( i - \frac{k}{2} \right) \right)^2$$

and

$$P_\infty(x) = 1 - \cos(2\pi x)$$

Lemma 1. Let $P_k$ and $P_\infty$ as defined above. Then:

(i) $P_k$ has $2k$ roots, being $2$ simple roots at $r_0 = \frac{1-k}{2}$ and $r_1 = \frac{k-1}{2}$ and $k-1$ roots of multiplicity two at $p_j = j - \frac{k}{2}$ for $j = 1, \ldots, k-1$. Moreover, $P'_k(r_0) > 0$, $P'_k(r_1) < 0$, $P'_k(p_j) = 0$ and $P''_k(p_j) > 0$ for every $j = 1, \ldots, k-1$.

(ii) $P_\infty$ has infinitely many roots placed at $p_j = j$ for $j \in \mathbb{Z}$. Moreover $P'_\infty(p_j) = 0$ and $P''_\infty(p_j) > 0$. As consequence, each $p_j$ is a root of multiplicity two.

Proof. The placement of those roots follows easily from the factored form of $P_k$ and the expression for $P_\infty$ presented in (3) and (9).

For the derivatives part, if $k < \infty$, put $g(x) = \left( x + \frac{k-1}{2} \right) \left( x - \frac{k-1}{2} \right)$ and

$$h(x) = \prod_{i=1}^{k-1} \left( x - \left( i - \frac{k}{2} \right) \right)^2,$$

then $P_k(x) = -g(x)h(x)$ and

$$P'_k(x) = -g(x)h'(x) - g'(x)h(x) = -g(x)h'(x) - 2xh(x) \Rightarrow$$

$$\Rightarrow P'_k(r_0) = -2r_0h(r_0) > 0 \text{ and } P'_k(r_1) = -2r_1h(r_1) < 0$$

Since $p_j$ are roots of multiplicity two of $h(x)$, then $h(x) = h'(x) = 0$, which implies $P''_k(p_j) = 0$ and:

$$P''_k(p_j) = -g(p_j)h''(p_j)$$

Now, we have

$$h''(p_j) = 2 \prod_{i=1}^{k-1} \left( x - \left( i - \frac{k}{2} \right) \right)^2 > 0$$

And, if $|x| < \frac{k-1}{2}$:

$$g(x) = \left( x - \frac{1-k}{2} \right) \left( x - \frac{k-1}{2} \right) < 0$$

Thus, $P''_k(p_j) > 0$. 

For $P_\infty$, we have:
\[ P'_\infty(x) = 2 \sin(2\pi x) \Rightarrow P'_\infty(j) = 0, \forall j \in \mathbb{Z} \]
and
\[ P''_\infty(x) = 4 \cos(2\pi x) \Rightarrow P''_\infty(j) = 1 > 0, \forall j \in \mathbb{Z} \]

Now, for every $k \geq 2$ or $k = \infty$, define:

\[
Z_k(x, y) = \begin{cases}
  X_k(x, y) = (1, P'_k(x)), & \text{for } y \geq 0 \\
  Y_k(x, y) = (-1, P'_k(x)), & \text{for } y \leq 0,
\end{cases}
\]

(10)

**Lemma 2** (Tangencies of $Z_k$). Let $Z_k$ be as above, then the following holds:

- If $k < \infty$ then $(r_0, 0)$ and $(r_1, 0)$ are crossing points of $Z_k$;
- The points $(p_j, 0)$ are visible-visible two folds of $Z_k$, $j = 1, \ldots, k - 1$
  (or $j \in \mathbb{Z}$, if $k = \infty$);
- $\Sigma^x \cup \Sigma^e = \emptyset$.

**Proof.** The switching manifold is given by $\Sigma = f^{-1}(0)$, where $f(x, y) = y$, then $X_k f(r_0, 0).Y_k f(r_0, 0) = (P'_k(r_0))^2 > 0$. Analogous for $(r_1, 0)$.

From Lemma 2, $X_k f(p_j, 0) = Y_k f(p_j, 0) = P'_k(p_j) = 0$ and $X^2_k f(p_j, 0) = Y^2_k f(p_j, 0) = P''_k(p_j) > 0$.

For each $k < \infty$, let

\[ \gamma^X_k = \{(x, P_k(x)) \mid x \in [r_0, r_1]\} \quad \text{and} \quad \gamma^Y_k = \{(x, -P_k(x)) \mid x \in [r_0, r_1]\} \]

and, for $k = \infty$,

\[ \gamma^X_\infty = \{(x, P_\infty(x)) \mid x \in \mathbb{R}\} \quad \text{and} \quad \gamma^Y_\infty = \{(x, -P_\infty(x)) \mid x \in \mathbb{R}\}. \]

Define $\Lambda_k = \gamma^X_k \cup \gamma^Y_k$ for both $k < \infty$ or $k = \infty$.

**Proposition 5.** $\Lambda_k$ is an invariant set for $Z_k$.

**Proof.** Note that $\gamma^X_k$ and $\gamma^Y_k$ are integral curves of $X_k$ and $Y_k$, respectively. Moreover, $\gamma^X_k$ and $\gamma^Y_k$ coincide at $(p_j, 0)$, for all $j$ and at $(r_0, 0), (r_1, 0)$ (in the case $k < \infty$). By Definitions 3, 4 and the description of these points given by Lemma 2 we obtain that $\Lambda_k$ is invariant. \qed

Our main goal is to prove the conjugacy between the time-one map of the fields $Z_k$ restricted to $\Lambda_k$ and a two-sided shift space. But, since a PSVF does not give uniqueness of trajectory through a point, a map $T_1 : \Lambda_k \to \Lambda_k$, $T_1(x) = \varphi(1, x)$, where $\varphi$ is the flow, is not well-defined, for it may have more than one image (depending on the flow chosen). One way of avoiding this is to work with the space of all possible trajectories, so $T_1$ will be well-defined. This strategy is inspired (and somewhat similar) to the one used in discrete dynamics when working with a non-invertible map, and constructing the inverse limit of the map (see [6]).

In that spirit, we will consider the set $\Omega_k = \{\gamma \text{ global trajectory of } Z_k \mid \gamma(0) \in \Lambda_k\}$ throughout this paper and we re-define the time-one map as follows:
Definition 14. We will call time-one map the function $T_1 : \Omega_k \to \Omega_k$, given by $T_1(\gamma)(\cdot) = \gamma(\cdot + 1)$.

Proposition 6. For any $k < \infty$ (or $k = \infty$), let $\gamma \in \Omega_k$ be a global trajectory, then for all $t \in \mathbb{R}$, there exists unique $t^* \in [t, t + 1)$ such that $\gamma(t^*) \in \{(p_j, 0)\}_{j=1}^{k-1}$ (respectively, $\gamma(t^*) \in \{(p_j, 0)\}_{j \in \mathbb{Z}}$).

Proof. Follows immediately from the expression of $Z_k$ and Lemmas 1 e 2.

The region $\Lambda_k$ can be partitioned into arcs that goes from $p_j$ to the adjacent ones ($p_{j+1}$ and $p_{j-1}$) or to itself (only when $k < \infty$). So, consider $k < \infty$ to be fixed and let $I_0 = \{(x, P_k(x)), x \in [r_0, p_1]\} \cup \{(x, -P_k(x)), x \in [r_0, p_1]\}$, i.e., the arc from $p_1$ to itself passing through $r_0$. For any $j = 1, \ldots, k-2$, let $I_{2j-1} = \{(x, P_k(x)), x \in (p_1, p_2)\}$ and $I_{2j} = \{(x, -P_k(x)), x \in (p_1, p_2)\}$, i.e, the arcs from $p_1$ to $p_{j+1}$ and from $p_{j+1}$ to $p_j$, respectively. And, $I_{2k-3} = \{(x, P_k(x)), x \in (p_{k-1}, r_1)\} \cup \{(x, -P_k(x)), x \in (p_{k-1}, r_1)\}$. In short, we enumerate these arcs top to bottom, left to right (see Figure 6).

For the case $k = \infty$ we will separate $\Lambda_\infty$ into arcs, in the same way as we done for the finite case: let $I_{2j} = \{(x, P_\infty(x)) \mid j < x < j + 1\}$ and $I_{2j+1} = \{(x, -P_\infty(x)) \mid j < x < j + 1\}$.

Definition 15. Let $s : \Omega_k \to \Theta_{2k-2}$, given by $s(\gamma) = (s_j(\gamma))_{j \in \mathbb{Z}}$, where:

$$s_j(\gamma) = \begin{cases} n, & \text{if } \gamma(j) \in I_n \\ m, & \text{if } \gamma(j) \in \{(p_1, 0)\} \text{ and } \gamma(j + 1/2) \in I_m \end{cases}$$

The sequence $s(\gamma)$ is called the itinerary of $\gamma$.

For $k < \infty$, it is clear that $s$ is well-defined. For $k = \infty$ we have to analyse if $s(\gamma) \in \Theta_\infty$. From Definitions 3 and 4, of local and global trajectories, we see that if $\gamma$ goes through some compartment $I_{2j}$, then the next one must be the one below it ($I_{2j+1}$) or the one to its right ($I_{2j+2}$) and if it goes through $I_{2j+1}$ then the following one must be above it ($I_{2j}$) or the one to its left ($I_{2j-1}$). In any case, $0 < |s_j(\gamma) - s_{j+1}(\gamma)| \leq 2$, so $s(\gamma) \in \Theta_\infty$. From Proposition 6, $(s_j(\gamma))_{j \in \mathbb{Z}}$ encodes every compartment $I_j$ that the trajectory $\gamma$ visits in positive and negative time.

Note that, given $\gamma \in \Omega_k$, there exists an infinity amount of different trajectories with the same itinerary of $\gamma$, simply by changing the initial condition of it, without modifying the compartment it is located. To avoid such situation, we will consider two trajectories with the same itinerary to be equivalent. That is, we consider the equivalence relation:

Definition 16. Let $\gamma_1, \gamma_2 \in \Omega_k$. We say $\gamma_1 \sim \gamma_2$ if and only if $s(\gamma_1) = s(\gamma_2)$. Denote $\Omega_k = \Omega_k/\sim$.

Remark 1. Observe that, given $\pi \in \Omega_k$, there exists a representative $\gamma^* \in \pi$ such that $\gamma^*(0) \in \{(p_j, 0), l = 1, \ldots, k - 1\}$, because if $\gamma(0) \in \{(p_j, 0), l = 1, \ldots, k - 1\}$, simply take $\gamma^* = \gamma$, if not, by Proposition 6 there exists unique
−1 < t* < 0 such that γ(t*) ∈ \{(p_l, 0), l = 1, \ldots, k - 1\}. Moreover, if s(γ) = (s_j)_{j \in \mathbb{Z}} then γ((t* + j, t* + j + 1)) = I_{s_j}. Which implies that γ*((j, j + 1)) = I_{s_j} and, consequently, γ*(j + \frac{1}{2}) ∈ I_{s_j}. Then s(γ*) = s(γ).

In the following we construct a metric on the space \(\overline{\Omega}_k\), but for this, we use the Hausdorff distance between two closed sets that is defined below:

**Definition 17.** The Hausdorff distance between the sets \(A\) and \(B\) is given by

\[
d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y)\}
\]

**Definition 18.** Now, define \(\rho_k : \overline{\Omega}_k \times \overline{\Omega}_k \to \mathbb{R}\), by

\[
\rho_k(\gamma_1, \gamma_2) = \sum_{i = -\infty}^{\infty} \frac{d_i(\gamma_1, \gamma_2)}{2|i|}
\]

where \(d_i(\gamma_1, \gamma_2) = d_H(\gamma_1^*([i, i+1]), \gamma_2^*([i, i+1]))\), \(d_H\) is the Hausdorff distance and \(\gamma_1^*, \gamma_2^*\) are those representatives given in the previous remark.

In order to simplify notation, in the sequel we only refer to \(\gamma \in \overline{\Omega}_k\), meaning the equivalence class \(\overline{\gamma}\) with the representative \(\gamma^*\).

**Proposition 7.** \((\overline{\Omega}_k, \rho_k)\) is a metric space.

**Proof.** Let us show that \(\rho_k\) is well-defined, i.e., the series in the previous definition converges. For \(k < \infty\), there exists \(M > 0\) such that \(d_i(\gamma_1, \gamma_2) \leq M\), for any \(i \in \mathbb{Z}\), since \(\gamma_1([i, i+1]), \gamma_2([i, i+1])\) are closed subsets of \(\Lambda_k\) which is compact. So \(\rho_k(\gamma_1, \gamma_2) \leq M \left(1 + 2 \sum_{i=1}^{\infty} \frac{1}{2^i}\right) = 3M\), i.e. it converges for any \(\gamma_1, \gamma_2 \in \overline{\Omega}_k\).

For the case \(k = \infty\), note that, \(\Lambda_\infty\) is contained in the horizontal strip \(\mathbb{R} \times [-2, 2]\). Then, for any \(\gamma \in \overline{\Omega}_\infty\), the arcs \(\gamma([i-1, i]), \gamma([i, i+1]) \subseteq [\lambda, \lambda + 2] \times [-2, 2]\) with diameter \(2\sqrt{5}\). Then, for every \(i \in \mathbb{Z}\):

\[
d_H(\gamma([i - 1, i]), \gamma([i, i + 1])) \leq 2\sqrt{5}
\]

Now, given \(\gamma_1, \gamma_2 \in \overline{\Omega}_\infty\) for every \(i \in \mathbb{Z}\):

\[
d_i(\gamma_1, \gamma_2) = d_H(\gamma_1([i, i+1]), \gamma_2([i, i+1]))
\]

\[
\leq d_H(\gamma_1([i, i+1]), \gamma_1([i-1, i])) + d_H(\gamma_1([i-1, i]), \gamma_2([i-1, i])) + d_H(\gamma_2([i-1, i]), \gamma_2([i, i+1]))
\]

\[
\leq d_H(\gamma_1([i-1, i]), \gamma_2([i-1, i])) + 4\sqrt{5}
\]

\[
\leq d_{i-1}(\gamma_1, \gamma_2) + 4\sqrt{5}
\]

In an analogous way, can be shown that

\[
d_{i-1}(\gamma_1, \gamma_2) \leq d_i(\gamma_1, \gamma_2) + 4\sqrt{5}
\]
Recursively, follows that for every $i \in \mathbb{Z}$:

$$d_i(\gamma_1, \gamma_2) \leq d_0(\gamma_1, \gamma_2) + 4\sqrt{5}|i|$$

So we have

$$\rho_\infty(\gamma_1, \gamma_2) = \sum_{i=-\infty}^{\infty} \frac{d_i(\gamma_1, \gamma_2)}{2^{|i|}} \leq \sum_{i=-\infty}^{\infty} \frac{d_0(\gamma_1, \gamma_2) + 4\sqrt{5}|i|}{2^{|i|}}$$

$$\leq 3d_0(\gamma_1, \gamma_2) + 8\sqrt{5}\sum_{i=1}^{\infty} \frac{i}{2^i} < \infty.$$ 

So $\rho_k$ is well defined for every $k \in \mathbb{N} \cup \{\infty\}$. Let us show it is a metric:  

- every summand in the definition of $\rho_k$ is non-negative, then $\rho_k(\gamma_1, \gamma_2) = 0$ if and only if $d_i(\gamma_1, \gamma_2) = 0$ for every $i \in \mathbb{Z}$. Hence $\gamma_1([i, i+1]) = \gamma_2([i, i+1])$, for all $i \in \mathbb{Z}$. Then $\gamma_1 = \gamma_2$.

- From $d_i(\gamma_1, \gamma_2) = d_i(\gamma_2, \gamma_1)$, we obtain $\rho_k(\gamma_1, \gamma_2) = \rho_k(\gamma_2, \gamma_1)$.

- Now, for every $i \in \mathbb{Z}$, we get:

$$d_i(\gamma_1, \gamma_2) \leq d_i(\gamma_1, \gamma_3) + d_i(\gamma_3, \gamma_2)$$

$$\sum_{i=-N}^{N} \frac{d_i(\gamma_1, \gamma_2)}{2^{|i|}} \leq \sum_{i=-N}^{N} \frac{d_i(\gamma_1, \gamma_3)}{2^{|i|}} + \sum_{i=-N}^{N} \frac{d_i(\gamma_3, \gamma_2)}{2^{|i|}}, \forall N \Rightarrow$$

$$\rho_k(\gamma_1, \gamma_2) \leq \rho_k(\gamma_1, \gamma_3) + \rho_k(\gamma_3, \gamma_2)$$

Therefore, $(\Omega_k, \rho_k)$ is a metric space. $\square$

Let $T_\gamma : \Omega_k \rightarrow \Omega_k$ the function induced by $T_1$, that is, $T_\gamma(\tau) = T_1(\gamma)$. It does not depend on the representative, for if $s(\gamma_1) = s(\gamma_2) = (s_j)_{j \in \mathbb{Z}}$, then, for all $j \in \mathbb{Z}$:

$$\gamma_1(j), \gamma_2(j) \in I_{s_j} \Rightarrow \gamma_1(j+1), \gamma_2(j+1) \in I_{s_{j+1}}$$

$$\Rightarrow T_1(\gamma_1)(j), T_1(\gamma_2)(j) \in I_{s_{j+1}} \Rightarrow s(T_1(\gamma_1)) = s(T_1(\gamma_2)).$$

**Proposition 8.** The function $T_\gamma$ given above is a homeomorphism.

**Proof.** The function $T_1$ clearly is invertible, with inverse $(T_1)^{-1}(\gamma)(.) = \gamma(. - 1)$, and it is straightforward to see that $T_\gamma^{-1} = T_1^{-1}$.

Now,

$$d_i(T_1(\gamma_1), T_1(\gamma_2)) = d_H(T_1(\gamma_1)([i, i+1]), T_1(\gamma_2)([i, i+1])) =$$

$$= d_H(\gamma_1([i+1, i+2]), \gamma_2([i+1, i+2])) = d_{i+1}(\gamma_1, \gamma_2).$$

Then:

$$\rho(T_1(\gamma_1), T_1(\gamma_2)) = \sum_{i=-\infty}^{\infty} \frac{d_i(T_1(\gamma_1), T_1(\gamma_2))}{2^{|i|}} = \sum_{i=-\infty}^{\infty} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{|i|}} =$$

$$= \lim_{k \rightarrow \infty} \left( \sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^i} + \sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{-i}} \right) =$$
\[ = \lim_{k \to \infty} \left( \sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{i+1}} + \frac{1}{2} \sum_{i=-k}^{1} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{-i-1}} \right) \leq \]

\[ \leq 2 \lim_{k \to \infty} \left( \sum_{j=1}^{k+1} \frac{d_{j}(\gamma_1, \gamma_2)}{2^{j}} + \sum_{j=-k+1}^{0} \frac{d_{j}(\gamma_1, \gamma_2)}{2^{-j}} \right) = \]

\[ \leq 2 \sum_{j=-\infty}^{\infty} \frac{d_{j}(\gamma_1, \gamma_2)}{2^{j}} = 2 \rho(\gamma_1, \gamma_2). \]

Hence \( T_1 \) is continuous. The proof of continuity of the inverse is analogous. \( \square \)

Now let \( \overline{s} : \overline{\Omega}_k \to \{0, 1, \ldots, 2k - 3\}^\mathbb{Z} \) the function induced by \( s \), that is, \( \overline{s}(\gamma) = s(\gamma) \). It does not depend on the representative chosen, because of the equivalence relation and it is one-to-one.

**Proposition 9.** The map \( \overline{s} \) is a homeomorphism onto its image.

**Proof.** **Case** \( k < \infty \): Put \( \mu = \min\{d_H(I_l, I_j) \mid l \neq j, \text{ and } l, j = 1, \ldots, 2k - 3\} \) and \( \gamma_1, \gamma_2 \in \overline{\Omega}_k \). Suppose \( \rho(\gamma_1, \gamma_2) < \frac{\mu}{2^N} \). Then, for any \( -N \leq i \leq N \):

\[ d_i(\gamma_1, \gamma_2) < \mu \]

because, on the contrary, we have \( \rho(\gamma_1, \gamma_2) = \sum i \frac{d_i(\gamma_1, \gamma_2)}{2^{i}} \geq \frac{\mu}{2^N} \).

Now \( d_i(\gamma_1, \gamma_2) < \mu \Rightarrow d_i(\gamma_1, \gamma_2) = 0 \) and therefore, \( \gamma_1((i, i + 1)) = \gamma_2((i, i + 1)) \) implying that \( s_i(\gamma_1) = s_i(\gamma_2) \), for all \( -N \leq i \leq N \). So

\[ d(s(\gamma_1), s(\gamma_2)) = \sum_{i \in \mathbb{Z}} \frac{|s_i(\gamma_1) - s_i(\gamma_2)|}{2^{|i|}} \]

\[ = \sum_{i \in \mathbb{Z}} \frac{|s_i(\gamma_1) - s_i(\gamma_2)|}{2^{|i|}} \leq \frac{2k - 3}{2^{N-1}}. \]

This proves that \( \overline{s} \) is continuous.

The same argument reverses itself in order to show \( \overline{s} \) is open: let \( \gamma_1, \gamma_2 \in \overline{\Omega}_k \), and \( N \in \mathbb{N} \)

\[ d(\overline{s}(\gamma_1), \overline{s}(\gamma_2)) < \frac{1}{2^N} \Rightarrow s_i(\gamma_1) = s_i(\gamma_2), \forall -N \leq i \leq N \]

Then \( \gamma_1([i, i + 1]) = \gamma_2([i, i + 1]) \), for all \( -N \leq i \leq N \). Hence

\[ \rho(\gamma_1, \gamma_2) = \sum_{i \in \mathbb{Z}} \frac{d_i(\gamma_1, \gamma_2)}{2^{|i|}} \leq \frac{M}{2^{N-1}}. \]

where \( M > 0 \), such that \( \text{diam}(\Lambda_k) < M \).
Case $k = \infty$: Again, let $\mu = \min \{d_H(I_l, I_j) \mid l \neq j, \text{ and } l, j \in \mathbb{Z}\}$ and
$\gamma_1, \gamma_2 \in \overline{\Omega}_\infty$, such that $\rho(\gamma_1, \gamma_2) < \frac{\mu}{2\pi}$. Then, for any $-N \leq i \leq N$ we get:

$$d_i(\gamma_1, \gamma_2) < \mu \Rightarrow d_i(\gamma_1, \gamma_2) = 0,$$
which implies $s_i(\gamma_1) = s_i(\gamma_2)$, for all $-N \leq i \leq N$. So

$$d(s(\gamma_1), s(\gamma_2)) = \sum_{i \in \mathbb{Z}} \frac{|s_i(\gamma_1) - s_i(\gamma_2)|}{2|i|} \leq \sum_{i \in \mathbb{Z}} \frac{|s_0(\gamma_1) - s_0(\gamma_2)| + 4|i|}{2|i|} = 2 \sum_{i = N+1}^{\infty} \frac{4i}{2^i} = \frac{1}{2^{N-4}}.$$  

If $\gamma_1, \gamma_2 \in \overline{\Omega}_\infty$ and $N \in \mathbb{N}$ are such that $d(\overline{s}(\gamma_1), \overline{s}(\gamma_2)) < \frac{1}{2\pi}$, then for all $-N \leq i \leq N$:

$$s_i(\gamma_1) = s_i(\gamma_2) \Rightarrow \gamma_1([i, i+1]) = \gamma_2([i, i+1]).$$

Hence

$$\rho(\gamma_1, \gamma_2) = \sum_{i \in \mathbb{Z}} \frac{d_i(\gamma_1, \gamma_2)}{2|i|} \leq \sum_{i \in \mathbb{Z}} \frac{d_0(\gamma_1, \gamma_2) + 4\sqrt{5}|i|}{2|i|} \leq 8\sqrt{5} \sum_{i = 1}^{\infty} \frac{i}{2^i} = \frac{\sqrt{5}}{2^{N-4}}.$$  

Therefore $\overline{s}$ in a homeomorphism over its image. \hfill \Box

Now we are in conditions to prove items (i) to (iii) of Theorem A.

4.1. Proof of item (i) of Theorem A.

**Proposition 10.** The function $\overline{s} : \overline{\Omega}_2 \to \{0, 1\}^\mathbb{Z}$ is a conjugation between $\overline{T_1}$ and $\sigma$, i.e., $\overline{s} \circ \overline{T_1} = \sigma \circ \overline{s}$ (see Figure 2).

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (0) at (0,0) {$\Omega_2$};
\node (1) at (2,0) {$\overline{\Omega}_2$};
\node (2) at (0,-2) {$\{0, 1\}^\mathbb{Z}$};
\node (3) at (2,-2) {$\{0, 1\}^\mathbb{Z}$};
\draw[->] (0) -- (1) node[midway,above] {$\overline{T_1}$};
\draw[->] (0) -- (2) node[midway,left] {$\overline{s}$};
\draw[->] (1) -- (3) node[midway,above] {$\overline{s}$};
\draw[->] (2) -- (3) node[midway,right] {$\sigma$};
\end{tikzpicture}
\caption{Figure 2}
\end{figure}
Proof. Let us show that \( \pi(\Omega_k) = \{0, 1\}^\mathbb{Z} \). Given \((s_j) \in \{0, 1\}^\mathbb{Z}\), construct \( \gamma \) by concatenating the arcs \( I_0 \) and \( I_1 \) according to \((s_j)\), so \( \gamma \) is a trajectory with \( \gamma(0) = p_1 \), and \( \gamma((0, 1)) = I_0 \). Then \( \gamma(1) = p_1 \) and \( \gamma((1, 2)) = I_1 \). In general, for all \( j \in \mathbb{Z} \), \( \gamma(j) = p_1 \) and \( \gamma((j, j + 1)) = I_s \). By Definition 4, \( \gamma \) is a global trajectory of \( Z_2 \) and therefore \( \gamma \in \Omega_2 \). Moreover, \( s(\gamma) = \pi(\gamma) = (s_j)_{j \in \mathbb{Z}} \).

Now, for the commutative part, let \( \gamma \in \Omega_2 \), \((a_j)_{j \in \mathbb{Z}} = \pi(\gamma)\), and \((b_j)_{j \in \mathbb{Z}} = \pi(T_1(\gamma))\), then:

\[
b_j = \begin{cases} 0, & \text{if } T_1(\gamma)(j) \in I_0 \\ 1, & \text{if } T_1(\gamma)(j) \in I_1 \end{cases} = \begin{cases} 0, & \text{if } \gamma(j + 1) \in I_0 \\ 1, & \text{if } \gamma(j + 1) \in I_1 \end{cases} = a_{j+1} \Rightarrow
\]

\[
(\pi(\gamma))_{j \in \mathbb{Z}} = \sigma((a_j)_{j \in \mathbb{Z}}), \text{ i.e., } \pi \circ T_1 = \sigma \circ \pi.
\]

\[\square\]

4.2. Proof of item (ii) of Theorem A.

Proposition 11. The function \( \bar{s} : \bar{\Omega}_k \to \pi(\bar{\Omega}_k) \) is a conjugation between \( \bar{T}_1 \) and \( \sigma \), i.e., \( \bar{s} \circ \bar{T}_1 = \sigma \circ \bar{s} \) (see Figure 3).

\[
\begin{array}{ccc}
\text{\bar{\Omega}_k} & \xrightarrow{T_1} & \text{\bar{\Omega}_k} \\
\downarrow \pi & & \downarrow \pi \\
\pi(\bar{\Omega}_k) & \xrightarrow{\sigma} & \pi(\bar{\Omega}_k)
\end{array}
\]

\text{Figure 3}

Proof. Let \( \gamma \in \bar{\Omega}_k \), \((a_j)_{j \in \mathbb{Z}} = \pi(\gamma)\), and \((b_j)_{j \in \mathbb{Z}} = \pi(T_1(\gamma))\). Then:

\[
b_j = m, \quad \text{if } T_1(\gamma)(j) \in I_m \\
= m, \quad \text{if } \gamma(j + 1) \in I_m \\
= a_{j+1} \Rightarrow
\]

\[
(\pi(\gamma))_{j \in \mathbb{Z}} = \sigma((a_j)_{j \in \mathbb{Z}}).
\]

It remains to show that \( \pi(\bar{\Omega}_k) \) is a subshift. First, \( \pi \) is continuous and \( \bar{\Omega}_k \) is compact. Then \( \pi(\bar{\Omega}_k) \) is closed in \( \mathcal{A}_k^\mathbb{Z} \). And, By Proposition 9, there exists \( \pi^{-1} : \pi(\bar{\Omega}_k) \to \bar{\Omega}_k \) and it is continuous. By the first part of this proof, we have \( \sigma = \pi \circ T_1 \circ \pi^{-1} \), which proves the invariant part.

Hence \( \pi \) is a conjugation between both systems. \[\square\]
4.3. Proof of item (iii) of Theorem A.

**Proposition 12.** The function \( s : \Omega_\infty \rightarrow s(\Omega_\infty) \subset \Theta_\infty \) is a conjugation between \( T_1 \) and \( \sigma \), i.e., \( s \circ T_1 = \sigma \circ s \) (see Figure 4).

\[
\begin{array}{ccc}
\Omega_\infty & \xrightarrow{T_1} & \Omega_\infty \\
\downarrow{s} & & \downarrow{s} \\
\Theta_\infty & \xrightarrow{\sigma} & \Theta_\infty \\
\end{array}
\]

**Figure 4**

**Proof.** Let \( \gamma \in \Omega_\infty \), \((a_j)_{j \in \mathbb{Z}} = s(\gamma)\), and \((b_j)_{j \in \mathbb{Z}} = s(T_1(\gamma))\). Then:
\[
\begin{align*}
b_j &= m, & & \text{if } T_1(\gamma)(j) \in I_m \\
&= m, & & \text{if } \gamma(j+1) \in I_m \\
&= a_{j+1} \Rightarrow \\
\end{align*}
\]

\( \Rightarrow (b_j)_{j \in \mathbb{Z}} = \sigma((a_j)_{j \in \mathbb{Z}}). \)

By Proposition 9, there exists \( s^{-1} : s(\Omega_\infty) \rightarrow \Omega_\infty \) and it is continuous and then \( \sigma = s \circ T_1 \circ s^{-1} \), which proves that \( \sigma(s(\Omega_\infty)) = s(\Omega_\infty) \). \( \Box \)

4.4. Proof of item (iv) of Theorem A. Consider the PSVF

\[
Z(x,y) = \begin{cases} 
X(x,y) = (1,-2x), & \text{for } y \geq 0 \\
Y(x,y) = (-2,-4x^3 + 2x), & \text{for } y \leq 0. 
\end{cases}
\]

(11)

This PSVF has a compact invariant set \( \Lambda = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } x^4/2 + x^2/2 \leq y \leq 1 - x^2\} \). Furthermore, \( \Lambda \) is a chaotic non-trivial minimal set for \( Z \) (see [1]). As before, we will consider \( \Omega = \{ \gamma \text{ global trajectory of } Z|_{\Lambda} \} \) the set of all possible trajectories contained in the set \( \Lambda \).

In cases (i)-(iii) of Theorem A, one could consider the set \( \{p_j\} \), as a "recurrent" set, in the sense that every trajectory visits it and returns to it an infinity amount of times. Moreover, the time between visits is exactly 1 (see Proposition 6), and that was the basic setting that allowed the previous constructions. In this case we consider the set \( K = \{0\} \times (0,1] \), since every local trajectory intersects it transversally, and every point of \( K \) reaches \( p = (0,0) \) in finite time (positive and negative), thus every trajectory visits it an infinity amount of times. Thus, the set \( K \) will play the role of \( \{p_j\} \) in previous cases. This time, we cannot adjust the expression of \( Z \) in order to make the time between visits equal some constant. So we define the function:

\( \eta : \Omega \rightarrow \mathbb{R} \), as \( \eta(\gamma) = \min\{t > 0 \mid \gamma(t) \in K\} \) to be the least positive value of \( t \) for which \( \gamma(t) \in K \). For simplicity we may denote \( \eta_\gamma = \eta(\gamma) \).
Now define the map $\mathcal{T} : \Omega \to \Omega$, as $\mathcal{T}(\gamma)(.) = \gamma(. + \eta)$. This map takes a trajectory to another one that starts one loop ahead.

**Lemma 3.** Given a global trajectory $\gamma \in \Omega$, there exists a bi-infinite increasing sequence $(t_j^\gamma)_{j \in \mathbb{Z}}$, such that $\gamma(t_j^\gamma) \in K$.

**Proof.** We define $t_0 = \max\{t < 0 \mid \gamma(t) \in K\}$, and for $j > 0$, let $t_j^\gamma = \sum_{k=0}^{j-1} \eta_{\mathcal{T}^k(\gamma)}$. For $j < 0$, consider $\tilde{\eta}(\gamma) = \max\{t < 0 \mid \gamma(t) \in K\}$, and the analogous for $\mathcal{T}$, that is, $\mathcal{\tilde{T}} : \Omega \to \Omega$, $\mathcal{\tilde{T}}(\gamma)(.) = \gamma(. + \tilde{\eta})$. Then and $t_j = t_0 + \sum_{k=0}^{-j-1} \tilde{\eta}_{\mathcal{T}^k(\gamma)}$.

Given the sequence constructed above, $\gamma(t_0^\gamma) \in K$, $\gamma(t_1^\gamma) = \gamma(\eta) = \mathcal{T}(\gamma)(0) \in K$

When $j > 1$, we get

$$\gamma(t_j^\gamma) = \gamma(\eta + \eta_{\mathcal{T}(\gamma)} + \ldots + \eta_{\mathcal{T}^{j-1}(\gamma)}) = \mathcal{T}(\gamma)(\eta_{\mathcal{T}(\gamma)} + \ldots + \eta_{\mathcal{T}^{j-1}(\gamma)}) = \mathcal{T}^2(\gamma)(\eta_{\mathcal{T}^2(\gamma)} + \ldots + \eta_{\mathcal{T}^{j-1}(\gamma)}) = \ldots = \mathcal{T}^{j-1}(\gamma)(\eta_{\mathcal{T}^{j-1}(\gamma)}) = \mathcal{T}(\gamma)(0) \in K.$$  

For $j < 0$ it is analogous. □

**Lemma 4.** Given a global trajectory $\gamma \in \Omega$ and $(t_j^\gamma)$ given above, consider the sequence $(t_j^{\mathcal{T}(\gamma)})$. It holds $t_j^{\mathcal{T}(\gamma)} = t_{j+1}^\gamma - \eta$.

**Proof.** In fact,

$$t_0^{\mathcal{T}(\gamma)} = 0 = \eta - \eta = t_1^\gamma - \eta$$

For $j > 0$:

$$t_j^{\mathcal{T}(\gamma)} = \sum_{k=0}^{j-1} \eta_{\mathcal{T}^{k+1}(\gamma)} = \sum_{k=0}^{j} \eta_{\mathcal{T}^k(\gamma)} - \eta = t_{j+1}^\gamma - \eta$$

And if $j < 0$:

$$t_j^{\mathcal{T}(\gamma)} = t_0 + \sum_{k=0}^{-j-1} \tilde{\eta}_{\mathcal{T}^k(\mathcal{T}(\gamma))} = \sum_{k=0}^{j} \eta_{\mathcal{T}^k(\gamma)} - \eta = t_{j+1}^\gamma - \eta$$

□

**Definition 19.** Consider $y : K \to (0, 1]$ the projection on y-coordinate (that is $y(0, \beta) = \beta$), and define the itinerary map $s : \Omega \to (0, 1]^2$, as $s(\gamma) = (s_j(\gamma))_{j \in \mathbb{Z}}$, where $s_j(\gamma) = y(\gamma(t_j^\gamma))$. In this way, every single trajectory of $\Omega$ can be encoded by the y-coordinates of its beats on $K$. 
Clearly $s$ is well-defined, by construction and it is onto, because given a sequence $(x_j)_{j \in \mathbb{Z}} \in (0,1]^{\mathbb{Z}}$ it is possible to construct a trajectory $\gamma \in \Omega$ by concatenating the correct arcs in order to get $s(\gamma) = (x_j)_{j \in \mathbb{Z}}$.

As in previous cases, there are infinitely many trajectories that describes the same curve, simply by changing the initial condition (a shift in time), and the function $s$ takes all of these to the same sequence. So, we consider the set $\overline{\Omega} = \Omega / s$, that is, two trajectories $\gamma_1, \gamma_2$ are equivalent if $s(\gamma_1) = s(\gamma_2)$.

**Remark 2.** Every trajectory $\gamma \in \Omega$ is equivalent to another one $\gamma^*$ such that $\gamma^*(0) \in K$.

**Remark 3.** Another way of avoiding the infinitely many trajectories that looks like the same would be to restrict our function to a subset $\Omega_* \subset \Omega$, where $\Omega_* = \{ \gamma \text{ global trajectory of } \mathbb{Z}_I | \gamma(0) \in K \}$, but we chose to work in a more general setting, without restricting the dynamics as it was done in cases (i)-(iii) of Theorem A.

**Definition 20.** Let $\gamma_1, \gamma_2 \in \overline{\Omega}$, define $\rho : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, by

$$\rho(\gamma_1, \gamma_2) = \sum_{i \in \mathbb{Z}} \frac{d_i(\gamma_1^*, \gamma_2^*)}{2|i|}$$

where $d_i(\gamma_1^*, \gamma_2^*) = d_H(\gamma_1^*(t_i^{j_1}}, t_1^{j_1+1}), \gamma_2^*(t_i^{j_2}}, t_2^{j_2+1}))$. That is, at each step $i$, we take the Hausdorff distance between the $i$-th loops of both trajectories.

**Proposition 13.** $(\overline{\Omega}, \rho)$ is a metric space.

**Proof.** The function $\rho$ defined above is well-defined, since $\Lambda$ is a bounded set, which implies there exists $C > 0$, such that $d_i(\gamma_1^*, \gamma_2^*) < C$, for all $i \in \mathbb{Z}$.

The proof of those properties for it to be a metric is analogous to the one given in Proposition 7. \hfill \Box

Let $\mathcal{T} : \overline{\Omega} \to \overline{\Omega}$ the function induced by $\mathcal{T}$, that is, $\mathcal{T}(\gamma) = \overline{T}(\gamma)$. It does not depend on the representative, for if $s(\gamma_1) = s(\gamma_2) = (s_j)_{j \in \mathbb{Z}}$, then, for all $j \in \mathbb{Z}$, $y(\gamma_1(t_j^{j_1})) = y(\gamma_2(t_j^{j_2})) = s_j$.

$$y(\mathcal{T}(\gamma_1)(t_j^{T(\gamma_1)})) = y(\gamma_1(t_j^{T(\gamma_1)} + \eta_1)) = y(\gamma_1(t_j^{T(\gamma_1)} + \eta_1))$$

$$= y(\gamma_1(t_j^{T(\gamma_1)})) = y(\gamma_2(t_j^{T(\gamma_2)})) = y(\mathcal{T}(\gamma_2))(t_j^{T(\gamma_2)}))$$

$$\Rightarrow s(\mathcal{T}(\gamma_1)) = s(\mathcal{T}(\gamma_2)).$$

**Proposition 14.** The function $\mathcal{T} : \overline{\Omega} \to \overline{\Omega}$ is a homeomorphism.

**Proof.** The function $\mathcal{T}$ clearly is invertible, with inverse $(\mathcal{T})^{-1}(\gamma)(.) = \gamma(.- \eta_1)$, and it is straightforward to see that $\mathcal{T}^{-1} = \overline{T}^{-1}$.

Now,

$$d_i(\mathcal{T}(\gamma_1), \mathcal{T}(\gamma_2)) = d_H(\mathcal{T}(\gamma_1)([t_i^{T(\gamma_1)}}, t_i^{T(\gamma_1)+1}]), \mathcal{T}(\gamma_2)([t_i^{T(\gamma_2)}}, t_i^{T(\gamma_2)+1}])$$

$$= d_H(\gamma_1([t_i^{T(\gamma_1)}}, t_i^{T(\gamma_1)+1}]), \gamma_2([t_i^{T(\gamma_2)}}, t_i^{T(\gamma_2)+1}])) = d_i(\gamma_1, \gamma_2).$$
Then:

\[
\rho(\mathcal{T}(\gamma_1), \mathcal{T}(\gamma_2)) = \sum_{i=-\infty}^{\infty} \frac{d_i(\mathcal{T}(\gamma_1), \mathcal{T}(\gamma_2))}{2^{|i|}} = \sum_{i=-\infty}^{\infty} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{|i|}} = \\
= \lim_{k \to \infty} \left( \sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^i} + \sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{-i}} \right) \\
= \lim_{k \to \infty} \left( 2 \sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{i+1}} + \frac{1}{2} \sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{-i-1}} \right) \leq \\
\leq 2 \lim_{k \to \infty} \left( \sum_{j=1}^{k+1} \frac{d_j(\gamma_1, \gamma_2)}{2^j} + \sum_{j=-k+1}^{0} \frac{d_j(\gamma_1, \gamma_2)}{2^{-j}} \right) = \\
\leq 2 \sum_{j=-\infty}^{\infty} \frac{d_j(\gamma_1, \gamma_2)}{2^{|j|}} = 2\rho(\gamma_1, \gamma_2).
\]

Hence \(\mathcal{T}\) is continuous. The proof of continuity of the inverse is analogous.

On the space \((0, 1]^\mathbb{Z}\) we consider the same metric from those spaces before, that is, given \((x_j)_{j \in \mathbb{Z}}, (y_j)_{j \in \mathbb{Z}} \in (0, 1]^\mathbb{Z}\) the distance between them is the real number

\[
d(x, y) = \sum_{j \in \mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}}.
\]

**Proposition 15.** \(\overline{s} : \overline{\Omega} \to (0, 1]^\mathbb{Z}\) is a homeomorphism.

**Proof.** The function \(s\) is onto, which implies that \(\overline{s}\) is onto and clearly it is injective because it is defined on the quotient \(\Lambda/\overline{s}\). So it remains to show the continuity part.

Note that \(|s_j(\gamma_1) - s_j(\gamma_2)| = ||\gamma_1(t_j^{-1}) - \gamma_2(t_j^{-1})|| \leq d_j(\gamma_1, \gamma_2)\) for all \(j \in \mathbb{Z}\). Then:

\[
d(s(\gamma_1), s(\gamma_2)) = \sum_{j \in \mathbb{Z}} \frac{|s_j(\gamma_1) - s_j(\gamma_2)|}{2^{|j|}} \leq \sum_{j \in \mathbb{Z}} \frac{d_j(\gamma_1, \gamma_2)}{2^{|j|}} = \rho(\gamma_1, \gamma_2).
\]

Thus \(\overline{s}\) is continuous.

The same argument reverses itself in order to show \(\overline{s}\) is open: let \(\gamma_1, \gamma_2 \in \overline{\Omega}\), and \(\varepsilon > 0\) such that

\[
d(\overline{s}(\gamma_1), \overline{s}(\gamma_2)) < \varepsilon.
\]

Then \(d_i(\gamma_1, \gamma_2) < M\varepsilon\), where \(M > 0\) is such that \(\text{diam}(\Lambda) < M\).

Hence

\[
\rho(\gamma_1, \gamma_2) = \sum_{i \in \mathbb{Z}} \frac{d_i(\gamma_1, \gamma_2)}{2^{|i|}} \leq 3M\varepsilon.
\]

\(\square\)
Proposition 16. Let $\sigma : (0, 1]^Z \to (0, 1]^Z$ be the shift map. Then $s \circ \pi = \sigma \circ s$, i.e., the following diagram commutes (Figure 5):

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\tau} & \Omega \\
\downarrow s & & \downarrow s \\
(0, 1]^Z & \xrightarrow{\sigma} & (0, 1]^Z
\end{array}
\]

**Figure 5. a**

**Proof.** Let $\gamma \in \Omega$, $(a_j)_{j \in \mathbb{Z}} = s(\gamma)$, and $(b_j)_{j \in \mathbb{Z}} = s(\tau(\gamma))$. Then:

\[
b_j = s_j(\tau(\gamma)) = y(\tau(\gamma)(t_j^{\tau(\gamma)})) = y(\gamma(t_j^{\tau(\gamma)} + \eta_j)) = y(\gamma(t_{j+1}\gamma - \eta_j + \eta_{j+1})) = s_{j+1}(\gamma) = a_{j+1}
\]

\[
\square
\]

5. **Proof of Theorem B**

**Definition 21.** Two arcs $\overline{AB}$ and $\overline{CD}$ are parameterized by an **arc length parameterization** if there exists a homeomorphism $h : \overline{AB} \to \overline{CD}$ such that, for all $x \in \overline{AB}$, then $h(x)$ is the point on $\overline{CD}$ satisfying

\[
\frac{\text{length } (\overline{Ax})}{\text{length } (\overline{AB})} = \frac{\text{length } (\overline{Ch(x)})}{\text{length } (\overline{CD})}.
\]

**Proposition 17.** (i) Let $\tilde{Z}_k$ (respectively $\tilde{Z}_\infty$) be a PSVF presenting a $(k-1)$-homoclinic loop in the set $\tilde{\Lambda}_k$ (respectively $\infty$-homoclinic loop in the set $\tilde{\Lambda}_\infty$). Then $Z_k$ and $\tilde{Z}_k$ restricted to $\Lambda_k$ and $\tilde{\Lambda}_k$, respectively, are $\Sigma$-equivalent (respectively $Z_\infty$ and $\tilde{Z}_\infty$).

(ii) The PSVF $Z$ of Theorem A, restrict to $\Lambda$, is $\Sigma$-equivalent to any PSVF $\tilde{Z}$ presenting a compact region $\tilde{\Lambda}$ bounded by a trajectory of $\tilde{Z}$ passing through a invisible-visible two-fold $\tilde{p}$. Moreover, except for $\tilde{p}$, the PSVF $\tilde{Z}$ has just more two invisible tangential singularities.

**Proof.** (i) Consider $Z_2$ and $\tilde{Z}_2$ restricted to $\Lambda_2$ and $\tilde{\Lambda}_2$. Take $h : \Lambda_2 \rightarrow \tilde{\Lambda}_2$ such that: $h(p_1) = \tilde{p}_1$. Now, consider the positive arc of trajectory $\gamma_{X,p_1}^+$ of $X$ starting at $p_1$ and finishing at $\gamma_1$. Analogously, consider the positive arc of trajectory $\tilde{\gamma}_{X,\tilde{p}_1}^+$ of $\tilde{X}$ starting at $\tilde{p}_1$ and finishing at $\tilde{\gamma}_1$. Using the arc length parameterization we
get \( h(\gamma^{+,-}_{X}) = \tilde{\gamma}^{+,-}_{X} \). Now, it is enough repeat this procedure and extend \( h \) to \( \gamma^{-,-}_{X}, \gamma^{+,-}_{Y} \) and \( \gamma^{-,-}_{Y} \).

- For \( Z_k \) and \( Z_{\infty} \) it is enough repeat the previous construction changing \( p_1 \) by \( p_j \) and \( r_0 \) by \( p_{j+1} \) when it is necessary.

(ii) • Consider \( Z \) and \( \tilde{Z} \) restricted to \( \Lambda \) and \( \tilde{\Lambda} \). Take \( h : \Lambda \rightarrow \tilde{\Lambda} \) such that: \( h(p) = \tilde{p} \). Since \( \Sigma^e = (q_1, p) \times \{0\} \) and \( \tilde{\Sigma}^e \subset \tilde{\Sigma} \) is a continuous curve connecting \( \tilde{q}_1 \) and \( \tilde{p} \), by arc length parameterization we get \( h(\Sigma^e) = \tilde{\Sigma}^e \). Also, we can extend \( h \) in such a way that \( h(q_1) = \tilde{q}_1 \).

For each \( u \in \Sigma^e \cup \{q_1\} \), consider the positive arc of trajectory \( \gamma^{+,-}_{X} \) of \( X \) starting at \( u \) and finishing at \( v \in \Sigma^s \). Analogously, consider the positive arc of trajectory \( \tilde{\gamma}^{+,-}_{X} \) of \( \tilde{X} \) starting at \( \tilde{u} = h(u) \) and finishing at \( \tilde{v} \in \tilde{\Sigma}^s \). Using the arc length parameterization we get \( h(\gamma^{+,-}_{X}) = \tilde{\gamma}^{+,-}_{X} \). Moreover, \( h(\Sigma^e) = \tilde{\Sigma}^e \) and \( h(q_2) = \tilde{q}_2 \).

For each \( u \in \Sigma^e \cup \{p\} \), consider the positive arc of trajectory \( \gamma^{+,-}_{Y} \) of \( Y \) starting at \( u \) and finishing at \( v \in \Sigma^s_+ \). Analogously, consider the positive arc of trajectory \( \tilde{\gamma}^{+,-}_{Y} \) of \( \tilde{Y} \) starting at \( \tilde{u} = h(u) \) and finishing at \( \tilde{v} \in \tilde{\Sigma}^s_+ \). Using the arc length parameterization we get \( h(\gamma^{+,-}_{Y}) = \tilde{\gamma}^{+,-}_{Y} \). Moreover, \( h(\Sigma^e_+) = \tilde{\Sigma}^e_+ \) and \( h(s_1) = \tilde{s}_1 \).

For each \( u \in \Sigma^e_- \), it is possible to repeat the previous argument and conclude that \( h(\gamma^{+,-}_{X}) = \tilde{\gamma}^{+,-}_{X}, h(\Sigma^e_-) = \tilde{\Sigma}^e_- \) (and \( h(s_2) = \tilde{s}_2 \)).

For each \( u \in \Sigma^e_- \), consider the positive arc of trajectory \( \gamma^{+,-}_{Y} \) of \( Y \) starting at \( u \) and finishing at \( v \in \Sigma^s \cup \{p\} \). Analogously, consider the positive arc of trajectory \( \tilde{\gamma}^{+,-}_{Y} \) of \( \tilde{Y} \) starting at \( \tilde{u} = h(u) \) and finishing at \( \tilde{v} \in \tilde{\Sigma}^s \cup \{\tilde{p}\} \). Using the arc length parameterization we get \( h(\gamma^{+,-}_{Y}) = \tilde{\gamma}^{+,-}_{Y} \).

\[ \square \]

6. Examples

Example 1. Consider the case \( k = 3 \). Then \( P_3 = -x^6 + \frac{3x^4}{2} - \frac{9x^2}{16} + \frac{1}{16} \) with roots \( \pm 1 \) and \( \pm \frac{1}{2} \) where the first two are simple and the latter are roots of multiplicity 2. Moreover the PSVF \( Z_3 \) is:

\[
Z_3(x, y) = \begin{cases} 
X_3(x, y) = (1, -6x^5 + 6x^3 - \frac{9x}{8}), & \text{for } y \geq 0 \\
Y_3(x, y) = (-1, -6x^5 + 6x^3 - \frac{9x}{8}), & \text{for } y \leq 0,
\end{cases}
\]

and the points \( p_1 = (-\frac{1}{2}, 0) \) and \( p_2 = (\frac{1}{2}, 0) \) are visible-visible two fold of \( Z_3 \). The invariant region is the set:

\[ \Lambda_3 = \{(x, P(x)) \mid -1 \leq x \leq 1\} \cup \{(x, -P(x)) \mid -1 \leq x \leq 1\}, \]

that is partitioned into the arcs

\[
I_0 = \left\{(x, P(x)) \mid -1 \leq x < -\frac{1}{2}\right\} \cup \left\{(x, -P(x)) \mid -1 \leq x < -\frac{1}{2}\right\}
\]
$I_1 = \left\{ (x, P(x)) \mid -\frac{1}{2} < x < \frac{1}{2} \right\}$

$I_2 = \left\{ (x, -P(x)) \mid -\frac{1}{2} < x < \frac{1}{2} \right\}$

$I_3 = \left\{ (x, P(x)) \mid \frac{1}{2} < x \leq 1 \right\} \cup \left\{ (x, -P(x)) \mid \frac{1}{2} \leq x \leq 1 \right\}$

as shows Figure 6 (the arcs are blue, purple, orange and red, respectively):

---

**Figure 6**

Now, $\Omega_3$ is the set of all trajectories contained in $\Lambda_3$ and $s : \Omega_3 \to \{0, 1, 2, 3\}^\mathbb{Z}$. If we take $\Omega_3 = \Omega_3/\mathcal{S}$ and the functions $\pi$ and $\overline{T}_1$ as before, we have that $\overline{\pi}(\Omega_3)$ is a subshift of $\{0, 1, 2, 3\}^\mathbb{Z}$ and $\pi$ is a conjugation between $\overline{T}_1$ and the shift $\sigma$. In fact, it is easy to see that such subshift is associated to the transition matrix:

$$M = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}$$

by analysing which arc can be reached from the other ones given the orientation of the flow.
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