Nonuniqueness of weak solutions to the Navier-Stokes equation

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November 29, 2017
The Navier–Stokes Equations

The Incompressible Navier–Stokes Equations

The pair \((v, p)\) solves the \textit{incompressible Navier–Stokes equations} if

\[
\partial_t v + \text{div}(v \otimes v) + \nabla p - \nu \Delta v = 0 \\
\text{div} v = 0
\]

for kinematic viscosity \(\nu > 0\), velocity \(v : T^3 \times \mathbb{R} \to \mathbb{R}^3\) and pressure \(p : T^3 \times \mathbb{R} \to \mathbb{R}\).
Weak solutions to the Navier–Stokes equations

We say \( v \in C^0_t L^2_x \) is a weak solution of NSE if for any \( t \in \mathbb{R} \) the vector field \( v(\cdot, t) \) is weakly divergence free, has zero mean, and

\[
\int_{\mathbb{R}} \int_{T^3} v \cdot (\partial_t \varphi + (v \cdot \nabla) \varphi + \nu \Delta \varphi) dx dt = 0,
\]

for any divergence free test function \( \varphi \). Fabes-Jones-Riviere '72, implies such a solutions satisfies the integral equation

\[
v(t) = e^{\nu \Delta(t)} v(\cdot, 0) + \int_0^t e^{\nu \Delta(t-s)} \mathbb{P} \text{div}(v(\cdot, s) \otimes v(\cdot, s)) ds.
\]
Based on the natural scaling of the equations $v(x, t) \mapsto v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t)$:

- A number of partial regularity results have been established: Scheffer ’76, Cafarelli-Kohn-Nirenberg ’82, Lin ’98, Ladyzhenskaya-Seregin ’99, Vasseur ’07, Kukavica ’08, ... 
- Local existence for the Cauchy problem has been proven in scaling-invariant spaces Kato ’84, Giga-Miyakawa ’85, Koch-Tataru ’01, Jia-Sřverák ’14, ... 
- Conditional regularity has been established under geometric structure assumptions (Constantin-Fefferman ’93), or assuming a signed pressure (Seregin-Sřverák ’02). 
- The conditional regularity and weak-strong uniqueness results known under the umbrella of Ladyzhenskaya-Prodi-Serrin conditions: Kiselev-Ladyzhenskaya ’57, Prodi ’59, Serrin ’62, Escauriaza-Seregin-Šverák ’03, ... 
- For the class of weak solutions defined above, if $v \in C^0_t L^3_x$ then such a solution is unique: Furioli–Lemarié-Rieusset–Terraneo ’00, Lions-Masmoudi ’01.
Nonuniqueness of weak solutions

Theorem 1 (B-Vicol '17)

There exists $\beta > 0$, such that for any smooth $e(t) : [0, T] \to \mathbb{R}_{\geq 0}$, there exists a weak solution $v \in C^0_t([0, T]; H^\beta_x(\mathbb{T}^3))$ of the Navier-Stokes equations, such that

$$\int_{\mathbb{T}^3} |v(x, t)|^2 \, dx = e(t),$$

for all $t \in [0, T]$.
Dissipative Euler solutions arise in the inviscid limit

Theorem 2 (B-Vicol ’17)

Let \( u \in C_{t,x}^{\beta}(\mathbb{T}^3 \times [−2T, 2T]) \), for \( \beta > 0 \), is a weak solution of the Euler equations:

\[
\partial_t u + (\text{div} \ u \otimes u) + \nabla p = 0 \quad \text{and} \quad \text{div} \ u = 0
\]

Then there exists \( \beta > 0 \), a sequence \( \nu_n \to 0 \), and a uniformly bounded sequence \( v^{(\nu_n)} \in C^0_t([0, T]; H_x^{\beta}(\mathbb{T}^3)) \) of weak solutions to the Navier-Stokes equations:

\[
\partial_t v^{(\nu_n)} + \text{div} \left( v^{(\nu_n)} \otimes v^{(\nu_n)} \right) + \nabla p - \nu_n \Delta v^{(\nu_n)} = 0 \quad \text{and} \quad \text{div} \ v^{(\nu_n)} = 0
\]

with \( v^{(\nu_n)} \to u \) strongly in \( C^0_t([0, T]; L^2_x(\mathbb{T}^3)) \).
Onsager’s Conjecture

Lars Onsager, in his famous note on statistical hydrodynamics [Onsager ’49]), conjectured the following dichotomy:

(a) Any weak solution $v$ belonging to Hölder spaces $C^\beta$ for $\beta > \frac{1}{3}$ conserves the kinetic energy.

(b) For any $\beta < \frac{1}{3}$ there exist weak solutions $v \in C^\beta$ which do not conserve the kinetic energy.

Part (a) of this conjecture was proven by [Constantin, E and Titi ’94], (cf. [Eyink ’94], [Duchon-Robert ’00], [Cheskidov-Constantin-Friedlander-Shvydkoy ’08])
Previous work

Part (b): Existence of non-conservative solutions

Part b) was recently resolved: $L^2_{x,t}$ [Scheffer ’93]; $L^\infty_t L^2_x$ [Shnirelman ’00]; $L^\infty_{x,t}$ [De Lellis-Székelyhidi Jr. ’09-’11]; $C^0_{x,t}$ [De Lellis-Székelyhidi Jr. ’12]; $C^{1/10}_{x,t}$ [De Lellis-Székelyhidi Jr. ’12]; $C^{1/5}_{x,t}$ [Isett ’13]; $C^{1/5}_{x,t}$ [B.-De Lellis-Székelyhidi Jr. ’13]; $C^{1/3}_{x}$ a.e. in time; [B. ’15]; $L^1_t C^{1/3}_{x}$ [B.-De Lellis-Székelyhidi Jr. ’16].

Theorem 1 (Isett ’16)

For every $\beta < 1/3$, there exists weak solutions $v \in C^\beta_{x,t}$ to the Euler equations with compact support in time.

Theorem 2 (B-De Lellis-Székelyhidi Jr.-Vicol ’17)

For every smooth strictly positive energy profile $e : [0, T] \rightarrow \mathbb{R}$ and $\beta < 1/3$, there exists weak solutions $v \in C^\beta_{x,t}$ such that $\frac{1}{2} \int_{\mathbb{T}^3} |v(x, t)|^2 \, dx = e(t)$. 
Structure functions

Define the structure functions for homogeneous, isotropic turbulence by

\[ S_p(\ell) := \langle [\delta v(\ell)]^p \rangle , \]

where \( \langle \cdot \rangle \) denotes an ensemble average. Kolmogorov’s famous four-fifths law can be stated as

\[ S_3(\ell) \sim -\frac{4}{5} \varepsilon \ell , \]

More generally, Kolmogorov’s scaling laws can be stated as

\[ S_p(\ell) = C_p \varepsilon^{\zeta_p} \ell^{\zeta_p} , \]

for any positive integer \( p \), for \( \zeta_p = p/3 \).
Intermittency

[Landau ’59]: The rate of energy dissipation is intermittent, i.e., spatially inhomogeneous.
Intermittency Corrections

- lognormal model of [Kolmogorov '62]: $\zeta_2 = 0.694444$.
- $\beta$-model [Frisch-Sulem-Nelkin '78]: $\zeta_2 = 0.733333$.
- log-Poisson model of [She-Leveque '94]: $\zeta_2 = 0.695937$.
- mean-field theory of [Yakhot '01]: $\zeta_2 = 0.700758$. 
Intermittent Euler result

Theorem 3 (B.- Masmoudi - Vicol (in preparation))

Fix any $\alpha < 5/14$. There exist infinitely many weak solutions

$$ u \in C^0_t H_x^\alpha $$

of the 3D Euler equations which have compact support in time.

The number $5/14$ is not sharp. Arguments of [C-C-F-S ’08]: for $\alpha > 5/6$ energy is conserved.
The convex integration scheme

The proof proceeds via induction, for each \( q \geq 0 \) we assume we are given a solution \((\nu_q, p_q, \hat{R}_q)\) to the Navier-Stokes-Reynolds system.

\[
\partial_t \nu_q + \text{div}(\nu_q \otimes \nu_q) + \nabla p_q - \nu \Delta \nu_q = \text{div} \hat{R}_q \\
\text{div} \nu_q = 0.
\]

where the stress \( \hat{R}_q \) is assumed to be a trace-free symmetric matrix.
The perturbation

As part of the induction step, the perturbation $w_{q+1} = v_{q+1} - v_q$ is designed such that the new velocity $v_{q+1}$ solves the Navier-Stokes-Reynolds system

$$\begin{aligned}
\partial_t v_{q+1} + \text{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} - \nu \Delta v_{q+1} &= \text{div} \, \hat{R}_{q+1} \\
\text{div} v_{q+1} &= 0.
\end{aligned}$$

with a smaller Reynolds stress $R_{q+1}$. Writing $v_{q+1} = w_{q+1} + v_q$ and using the equation for $v_q$ we may write

$$\begin{aligned}
\text{div} R_{q+1} &= (-\nu \Delta w_{q+1} + \partial_t w_{q+1}) + \text{div}(v_q \otimes w_{q+1} + w_{q+1} \otimes v_q) \\
&\quad + \text{div}(w_{q+1} \otimes w_{q+1} - \mathbb{R}_q) + \nabla(p_{q+1} - p_q) \\
&=: \text{div} \left( \hat{R}_{\text{linear}} + \hat{R}_{\text{quadratic}} + \hat{R}_{\text{oscillation}} \right) + \nabla(p_{q+1} - p_q).
\end{aligned}$$
The perturbation \( w_{q+1} = v_{q+1} - v_q \) is constructed as a superposition of intermittent Beltrami waves at frequency \( \lambda_{q+1} \):

\[
\lambda_q = a(b^q)
\]

for \( a, b \gg 1 \). The perturbation will be of the form

\[
w_{q+1} \sim \sum_{\xi \in \Lambda} a_{\xi} (\dot{R}_q) W_{\xi}
\]

in order to cancel the low frequency (\( \approx \lambda_q \)) error of \( \dot{R}_q \) of size given

\[
\| \dot{R}_q \|_{L^1} \leq \lambda_{q+1}^{-2\beta}
\]

for \( 0 < \beta \ll 1 \). From scaling considerations we expect

\[
\| w_{q+1} \|_{L^2} \leq \lambda_{q+1}^{-\beta}.
\]
Beltrami waves

A stationary divergence free vector field $v$ is called a Beltrami flow if it satisfies the Beltrami condition:

$$\lambda v = \text{curl } v, \quad \lambda > 0 .$$

Given a Beltrami flow $v$, we have the following identity

$$\text{div}(v \otimes v) = v \cdot \nabla v = \nabla \frac{|v|^2}{2} - v \times (\text{curl } v) = \nabla \frac{|v|^2}{2} - \lambda v \times v = \nabla \frac{|v|^2}{2} .$$

Setting $p := \frac{|v|^2}{2}$, then $(v, p)$ is a stationary solution to the Euler equations.
Convex Integration Scheme

Intermittent Beltrami waves

- Gain if can build a version of the Beltrami waves $\mathcal{W}_\xi$ such that
  \[ \|\mathcal{W}_\xi(\lambda q + 1 \cdot \cdot)\|_{L^2} \approx 1, \quad \|\mathcal{W}_\xi(\lambda q + 1 \cdot \cdot)\|_{L^1} \ll \lambda_{q+1} 1 \]

- Recall, in 1D the normalized Dirichlet kernel obeys:
  \[ \left\| \frac{1}{\sqrt{r}} \sum_{-r \leq k \leq r} e^{ikx} \right\|_{L^2} \approx 1, \quad \left\| \frac{1}{\sqrt{r}} \sum_{-r \leq k \leq r} e^{ikx} \right\|_{L^1} \approx \frac{\log r}{\sqrt{r}} \ll 1. \]
Heuristic estimates

Heuristic estimate on dissipation error

Each intermittent Beltrami wave $\tilde{W}_{\xi}$ will be made up of

$$\left(\frac{\lambda_{q+1}}{\lambda_q}\right)^p = \lambda_{q+1}^{p'}$$

distinct frequencies, for some $2 < p' < p < 3$. By setting $\nu = 1$ and writing

$$\Delta w_{q+1} = \text{div}(\nabla w_{q+1})$$

$$= \text{div} \left( \nabla \sum_{\xi} a_{\xi} \tilde{W}_{\xi} \right)$$

The dissipation error's contribution to $\dot{R}_{q+1}$ can be heuristically estimated by

$$\|\nabla w_{q+1}\|_{L^1} \lesssim \sum_{\xi} \|a_{\xi} \tilde{W}_{\xi}\|_{W^{1,1}} \lesssim \lambda_{q+1}^{1-p'/2}.$$
Heuristic estimates

Estimate on the perturbation

A naïve estimate of the perturbation would give

\[ \left\| w_{q+1} \right\|_{L^2} \lesssim \sum_{\xi} \left\| a_\xi W_{\xi} \right\|_{L^2} \lesssim \sum_{\xi} \left\| a_\xi \right\|_{L^\infty} \left\| W_{\xi} \right\|_{L^2} \lesssim \sum_{\xi} \left\| a_\xi \right\|_{L^\infty} \]

However, we have no control on \[ \left\| a_\xi \right\|_{L^\infty} \approx \left\| \hat{R}_q \right\|_{L^\infty}^{1/2} \]!
Lemma 4
Assume $f$ is supported in a ball of radius $\lambda$ in frequency, and that $g$ is a $(\mathbb{T}/\sigma)^3$-periodic function. If $\lambda \ll \sigma$, then

$$\|fg\|_{L^p(\mathbb{T}^3)} \lesssim \|f\|_{L^p(\mathbb{T}^3)} \|g\|_{L^p(\mathbb{T}^3)}.$$ 

Then heuristically we obtain

$$\|v_{q+1} - v_q\|_{L^2} \lesssim \sum_{\xi} \|a_{\xi} \hat{W}_{\xi}\|_{L^2} \lesssim \sum_{\xi} \|a_{\xi}\|_{L^2} \|\hat{W}_{\xi}\|_{L^2} \lesssim \sum_{\xi} \|a_{\xi}\|_{L^2}$$

which gives us the correct estimate since

$$\|a_{\xi}\|_{L^2} \simeq \|\hat{R}_q\|_{L^1}^{1/2} \lesssim \lambda_{q+1}^{-\beta}.$$
Future directions

Given a weak solution $v \in C^0_t L^2_x \cap L^2_t H^1_x$ to the Navier-Stokes equation, we say that $v$ is a Leray-Hopf solution if in addition it satisfies the energy inequality

$$\frac{1}{2} \int_{\mathbb{T}^3} |v(x, t)^2| \, dx + \int_{\mathbb{T}^3 \times [0,t]} |\nabla v(x, s)|^2 \, dx \, ds \leq \frac{1}{2} \int_{\mathbb{T}^3} |v(x, 0)|^2 \, dx.$$ 

In Jia-Šverák ’15 proved that non-uniqueness of Leray-Hopf weak solutions in the regularity class $L^\infty_t L^3_x, \infty$ is implied if a certain spectral assumption holds for a linearized Navier-Stokes operator. Very recently Guillod-Šverák ’17 have provided compelling numerical evidence that the spectral condition is satisfied.

We conjecture that non-uniqueness of Leray-Hopf solutions can be proven via convex integration. This is known in the case where the Laplacian $-\Delta$ is replaced by the fractional laplacian $(-\Delta)^s$ for $s \in (0, 1/5)$, Colombo-De Lellis-De Rosa ’17.
Questions?