Faster Non-Convex Federated Learning via Global and Local Momentum

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Abstract

We propose FedGLOMO, a novel federated learning (FL) algorithm with an iteration complexity of $O(\epsilon^{-1.5})$ to converge to an $\epsilon$-stationary point (i.e., $E[\|\nabla f(x)\|^2] \leq \epsilon$) for smooth non-convex functions – under arbitrary client heterogeneity and compressed communication – compared to the $O(\epsilon^{-2})$ complexity of most prior works. Our key algorithmic idea that enables achieving this improved complexity is based on the observation that the convergence in FL is hampered by two sources of high variance: (i) the global server aggregation step with multiple local updates, exacerbated by client heterogeneity, and (ii) the noise of the local client-level stochastic gradients. By modeling the server aggregation step as a generalized gradient-type update, we propose a variance-reducing momentum-based global update at the server, which when applied in conjunction with variance-reduced local updates at the clients, enables FedGLOMO to enjoy an improved convergence rate. Moreover, we derive our results under a novel and more realistic client-heterogeneity assumption which we verify empirically – unlike prior assumptions that are hard to verify. Our experiments illustrate the intrinsic variance reduction effect of FedGLOMO, which implicitly suppresses client-drift in heterogeneous data distribution settings and promotes communication efficiency.

1 Introduction

Federated learning (FL) is a new edge-computing approach that advocates training statistical models directly on remote devices by leveraging enhanced local resources on each device [24]. In a standard FL setting, there are $n$ clients, each having its own training data, and a central server that is trying to train a model, parameterized by $w \in \mathbb{R}^d$, using the clients’ data. Suppose the data distribution of the $i$th client is $D_i$. Then the $i$th client has an objective function $f_i(w)$ which is the expected loss, with respect to some loss function $\ell$, over data drawn from $D_i$, and the goal of the central server is to optimize the average loss $f(w)$, over the $n$ clients, i.e.,

$$f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w) \quad \text{and} \quad f_i(w) = \mathbb{E}_{x \sim D_i}[\ell(x, w)]. \quad (1)$$

The setting where the data distributions of all the clients are identical, i.e. $D_1 = \ldots = D_n$, is typically known as the “homogeneous” setting. Otherwise, the settings where the data distributions are not identical are referred to as the “heterogeneous” settings.

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†In general this may be a weighted average, but here we only consider uniform weights, i.e., each weight is $1/n$. 
The core algorithmic idea of FL – in the form of FedAvg – was introduced in [24]. In FedAvg (summarized in Algorithm 3), a subset of the clients perform multiple steps of gradient descent based updates on their local data and then communicate back their respective updates to the server, which then averages them to update the global model (hence the name FedAvg). This idea of performing multiple local updates before averaging once mitigates the communication cost required for training. Another strategy to cut down the communication cost is to have the clients send compressed/quantized messages to the server in every round – this is of particular significance for training deep learning models where the number of model parameters is in millions or more.

In practice however, performing multiple local updates on clients with heterogeneous data distributions leads to the so-called phenomenon of “client-drift”, wherein the individual client updates do not align well (due to over-fitting on the local client data) inhibiting the convergence of FedAvg to the optimum of the average loss over all the clients. At the heart of this issue is the high variance associated with the simple averaging step of FedAvg for the global update.

Ever since the development of FL, significant attention has been devoted to analyzing FedAvg under different settings, modifying FedAvg using ideas from centralized optimization to accelerate the training or to reduce the communication cost; we discuss these works in Section 2. Compared to centralized optimization, a formidable challenge in the theoretical analysis of FL algorithms is the use of multiple local updates in the clients which is compounded by the heterogeneous nature of data distribution among the clients. To limit the extent of client heterogeneity, a standard assumption in FL theory is the bounded client dissimilarity (BCD) assumption, i.e.,

\[ E_i[\|\nabla f_i(w) - \nabla f(w)\|^2] \leq G^2 \forall w, \text{ or } \|\nabla f_i(w) - \nabla f(w)\|^2 \leq G^2 \forall w \text{ and } i \in [n], \]  

for some large enough constant \( G \) (e.g., see A1 in 16). However, it is hard to verify in practice and does not allow for arbitrary client heterogeneity.

Recently, [3] showed that the stochastic first-order complexity of any algorithm in the centralized setting to reach an \( \epsilon \)-stationary point (i.e., \( E[\|\nabla f(x)\|^2] \leq \epsilon \)) for smooth non-convex functions is \( \Omega(\epsilon^{-1.5}) \). It is well known that vanilla SGD has a suboptimal complexity of \( \mathcal{O}(\epsilon^{-2}) \) as it cannot mitigate the high variance of the stochastic gradient noise. Recognizing this issue, variance-reducing techniques for SGD [9,10,23,41] have been proposed that attain the optimal complexity of \( \mathcal{O}(\epsilon^{-1.5}) \). Coming to the federated setting, in addition to the noise in the local client-level stochastic gradients, one has to also contend with the high variance associated with the global server aggregation step which depends on the client heterogeneity and the number of local update steps. In this case, applying only local client-level variance-reduction is not enough for improving the iteration complexity of vanilla FedAvg.

To that end, we propose a novel FL algorithm with compressed communication called FedGLOMO (Algorithm 1 and 2), which applies Global as well as Local variance-reducing Momentum to the server update and client updates, respectively. We prove that the iteration complexity of FedGLOMO is \( \mathcal{O}(\epsilon^{-1.5}) \) in the smooth non-convex case, which is better than the \( \mathcal{O}(\epsilon^{-2}) \) complexity of related works in the FL setting; see Table 1 and Theorem 1. Further, our theory does not use the BCD assumption, i.e. eq. 2, which is a standard assumption in related works. Instead, we propose and use Assumption 3, which is more realistic and empirically verified, allowing for arbitrary client heterogeneity. It is worth mentioning here that for FL, [16] also propose an algorithm (MimeMVR) which is shown to attain this improved complexity of \( \mathcal{O}(\epsilon^{-1.5}) \) but with the BCD assumption and no compressed communication; we talk about this at the end of Section 2.

We summarize our contributions next:

(a) We propose FedGLOMO (Alg. 1 and 2), in which we apply a novel global momentum term at the server in addition to SVRG-style local momentum at the clients. The design of FedGLOMO is motivated by two critical issues that need to be alleviated to accelerate convergence in FL; these
are the high variances associated with: (i) the global server aggregation step due to heterogeneity of clients when there are multiple local updates, and (ii) the noise of local client-level stochastic gradients. Global and local momentum result in variance reduction for the global server update and the local client updates, allowing us to tackle (i) and (ii), respectively. This enables FedGLOMO to converge to an $\epsilon$-stationary point (i.e., $\mathbb{E}[\|\nabla f(x)\|^2] \leq \epsilon$) for smooth non-convex functions in $O(\epsilon^{-1.5})$ gradient-based updates, which is better than the $O(\epsilon^{-2})$ complexity of most related works in the FL setting; see Table 1 and Theorem 1.

(b) Unlike prior work, our theory does not use the hard to verify bounded client dissimilarity assumption (i.e., eq. (2)). Instead, to tighten our convergence result, we propose and use Assumption 4 – which is a novel and empirically verified assumption, even allowing for arbitrary client heterogeneity. Moreover, this assumption is not specific to FedGLOMO; we empirically verify that it also holds for FedAvg and derive a novel convergence result for FedAvg using this assumption and without eq. (2) (see Appendix C). Refer to the discussion after Assumption 4 and Remark 2 for details.

(c) Further, FedGLOMO is the first FL algorithm achieving $O(\epsilon^{-1.5})$ complexity while allowing compressed client-to-server communication. For theory, applying compression in FedGLOMO is not trivial and the most obvious approach to do so does not work; see Remark 3.

(d) In Section 6, experiments with neural networks on CIFAR-10 and Fashion-MNIST show that in a highly heterogeneous setting of at most two classes per client, FedGLOMO requires only about one-third the number of bits used by FedAvg with compressed communication, while this ratio further improves to one-fifth in the homogeneous setting; see Figure 1. Our experiments also illustrate the variance reduction provided by our scheme which implicitly mitigates client-drift under heterogeneous data distribution and promotes communication-efficiency.

2 Related Work

FedAvg and related methods: [29] propose FedPAQ which is basically FedAvg [24] with quantized client-to-server communication, and establish its convergence for the homogeneous case. [20] establish the convergence of FedAvg for strongly convex functions with heterogeneity (assuming bounded client dissimilarity) but without any compressed communication. [12] propose FedCOMGATE which incorporates gradient tracking [26] and derive results with data heterogeneity and quantized communication. [17] propose SCAFFOLD which uses control-variates to mitigate the client-drift owing to the heterogeneity of clients. [19] present FedProx which adds a proximal term to control the deviation of the client parameters from the global server parameter in the previous round. [28] propose federated versions of commonly used adaptive optimization methods and prove their convergence under heterogeneity. Local SGD [4, 5, 18, 21, 25, 30, 32, 35, 37, 40, 42] is very similar to FL and is essentially based on the same principle as FedAvg. However, in local SGD, there is usually no data heterogeneity and all the clients participate in each round (known as “full device participation”), both of which do not hold in FL and simplify the derivation of convergence results.

Momentum-based methods in FL: [15, 36] present momentum-based updates at the server but without any improvement in the convergence rate as compared to momentum-free updates. [27] present Nesterov accelerated FedAvg for convex objectives. [16] propose Mime(MVR) which applies momentum at the client-level based on globally computed statistics to control client-drift.

Distributed optimization with compression: There are several papers [1, 2, 4, 6, 8, 11, 14, 22, 29, 31, 33, 34, 38] aiming to minimize the communication bottleneck in distributed op-
Table 1: Number of gradient updates, i.e., $T$, required to achieve $\mathbb{E}[\|\nabla f(w)\|^2] \leq \epsilon$ on smooth non-convex functions. “BCD?” asks if the bounded client dissimilarity assumption (i.e., eq. (2)) is used or not. Here, $n$ is the total number of clients and $r$ is the number of clients participating in each round.

*1: Results are under full device participation, i.e., $r = n$.
*2: Here, $\alpha \leq n$ is a problem-dependent quantity; in practice, we expect $\alpha \ll n$ as confirmed in our experiments.

| Ref.          | $T$                               | Compressed Communication? | BCD? |
|---------------|-----------------------------------|---------------------------|------|
| FedCOMGATE [12] | $O\left(\frac{1}{\epsilon^2}\right)^{\ast 1}$ | Yes                       |      |
| Local SGD [18,36] | $O\left(\frac{1}{\epsilon^2}\right)^{\ast 1}$ | X                         | Yes  |
| SCAFFOLD [17]   | $O\left(\frac{1}{\epsilon r}\right)$          | X                         | Yes  |
| MimeMVR [16]    | $O\left(\frac{1}{\sqrt{r \epsilon^{1.5}}}\right)$ | X                         | Yes  |
| **This work (FedGLOMO)** | $O\left(\max\left(\sqrt{\frac{n}{\epsilon}}, \sqrt{\frac{(n-r)}{r(n-1)}}\right) \frac{1}{\epsilon^{1.5}}\right)^{\ast 2}$ | ✓  | No   |

Optimal complexity/rate for smooth non-convex stochastic optimization: [3] show that the optimal stochastic first-order complexity to reach an $\epsilon$-stationary point (i.e., $\mathbb{E}[\|\nabla f(x)\|^2] \leq \epsilon$) is $O\left(\frac{1}{\epsilon^2}\right)$ where $\sigma^2$ is the variance of the stochastic gradients. SVRG-style algorithms such as SPIDER [10] and SNVRG [41] attain this optimal complexity by periodically using giant batch sizes. [9] propose STORM which also attains this optimal complexity with adaptive learning rates, but without using any large batches. The key idea of STORM is momentum-based variance reduction, obtained by using the stochastic gradient at the previous point *computed over the same batch* on which the stochastic gradient at the current point is computed. [23] present a much simpler proof for essentially the same algorithm by employing a constant learning rate and requiring a large batch size only at the first iteration. Our key idea of global and local momentum is STORM-like variance-reducing momentum applied to the aggregation step at the server, that we interpret as a generalized gradient-type update, and the local updates at the clients, respectively; see Section 4.

Table 1 compares the complexities of the most relevant related works in FL/local SGD with ours (on smooth non-convex functions). Note that only FedGLOMO and MimeMVR [16] attain the improved iteration complexity of $O\left(\epsilon^{-1.5}\right)$ with respect to $\epsilon$. However, unlike [16], our work does not rely on the bounded client dissimilarity assumption (eq. (2)) and allows for compressed client-to-server communication, in which case maintaining the improved complexity is not trivial; for details, see Remark 2 and Remark 3 respectively. There are meaningful algorithmic differences between our work and Mime too. The most noteworthy one is that while we explicitly apply momentum in the server aggregation step (global momentum) as well as in the client updates (local momentum), [16] only apply globally computed momentum in the local client updates and no momentum at the server. For a detailed discussion of the differences of our work from [16], see Appendix B. Since Mime is designed to deal with client drift, we also empirically compare it against FedGLOMO without compression in a highly heterogeneous setting in Section 6.
3 Preliminaries

Recall the setting and the optimization problem that the server is trying to solve as defined in eq. (1). We assume that the clients have access to unbiased stochastic gradients of their individual losses. We denote the stochastic gradient of $f_i$ at $w$ computed over a batch of samples $B_i$ by $\nabla f_i(w;B_i)$. Also in this paper, $K$ is the number of communication rounds, $E$ is the number of local updates per round or the period, and $T = KE$ is the total number of local updates or the (order-wise) number of gradient-based updates. Further, $r$ is the number of clients that the server accesses in each communication round, i.e., the global batch size.

4 FedGLOMO: Global and Local MOMentum-Based Variance Reduction

Algorithm 1 FedGLOMO - Server Update

1: Input: Initial point $w_0$, # of rounds of communication $K$, period $E$, learning rates $\{\eta_k\}_{k=0}^{K-1}$ and global batch size $r$. $Q_D$ is the quantization operator. Set $w_{-1} = w_0$.
2: for $k = 0, \ldots, K-1$ do
3: Server sends $w_k, w_{k-1}$ to a set $S_k$ of $r$ clients chosen uniformly at random w/o replacement.
4: for client $i \in S_k$ do
5: Set $w_{k,0}^{(i)} = w_k$ and $\hat{w}_{k-1,0}^{(i)} = w_{k-1}$. Run Algorithm 2 for client $i$.
6: end for
7: if $k = 0$ then
8: Set $u_k = \frac{1}{r} \sum_{i \in S_k} Q_D(w_k - w_{k}^{(i)})$.
9: else
10: Set $u_k = \frac{2}{r} \sum_{i \in S_k} Q_D((w_k - w_{k}^{(i)}) + (1 - \beta_k)u_{k-1} + \frac{1 - \beta_k}{r} \sum_{i \in S_k} Q_D((w_k - w_{k}^{(i)}) - (w_{k-1} - \hat{w}_{k-1,0}^{(i)})))$. // (Global Momentum)
11: end if
12: Update $w_{k+1} = w_k - u_k$.
13: end for

Algorithm 2 FedGLOMO - Client Update

1: for $\tau = 0, \ldots, E - 1$ do
2: if $\tau = 0$ then
3: Set $v_{k,\tau}^{(i)} = \nabla f_i(w_{k,\tau}^{(i)}), \hat{v}_{k-1,\tau}^{(i)} = \nabla f_i(\hat{w}_{k-1,\tau}^{(i)})$.
4: else
5: Pick a random batch of samples in client $i$, say $B_{k,\tau}^{(i)}$. Compute the stochastic gradients of $f_i$ at $w_{k,\tau}^{(i)}, \hat{w}_{k-1,\tau}^{(i)}, w_{k-1,\tau-1}$ and $\hat{w}_{k-1,\tau-1}$ over $B_{k,\tau}^{(i)}$ viz. $\hat{\nabla} f_i(w_{k,\tau}^{(i)};B_{k,\tau}^{(i)}), \hat{\nabla} f_i(\hat{w}_{k-1,\tau}^{(i)};B_{k,\tau}^{(i)}), \hat{\nabla} f_i(w_{k-1,\tau-1}^{(i)};B_{k,\tau}^{(i)})$ and $\hat{\nabla} f_i(\hat{w}_{k-1,\tau-1}^{(i)};B_{k,\tau}^{(i)})$. // (Local Momentum)
6: Update: $v_{k,\tau}^{(i)} = \hat{\nabla} f_i(w_{k,\tau}^{(i)};B_{k,\tau}^{(i)}) + (v_{k,\tau-1}^{(i)} - \hat{\nabla} f_i(w_{k,\tau-1}^{(i)};B_{k,\tau}^{(i)})$ and $\hat{v}_{k-1,\tau}^{(i)} = \hat{\nabla} f_i(\hat{w}_{k-1,\tau}^{(i)};B_{k,\tau}^{(i)}) + (\hat{v}_{k-1,\tau-1}^{(i)} - \hat{\nabla} f_i(\hat{w}_{k-1,\tau-1}^{(i)};B_{k,\tau}^{(i)})$. // (Local Momentum)
7: end if
8: Update $w_{k,\tau+1} = w_{k,\tau}^{(i)} - \eta_k v_{k,\tau}^{(i)}$ and $\hat{w}_{k-1,\tau+1}^{(i)} = \hat{w}_{k-1,\tau}^{(i)} - \eta_k \hat{v}_{k-1,\tau}^{(i)}$.
9: end for
10: Send $Q_D(w_k - w_{k,E}^{(i)})$ and $Q_D((w_k - w_{k,E}^{(i)}) - (w_{k-1} - \hat{w}_{k-1,E}^{(i)}))$ to the server.
With this, one can clearly see that eq. (4) is the analogue of eq. (3) for the global server aggregation

Algorithm 3

To understand it better, let us revisit $g$ parameter. Note the use of the stochastic gradient at $\xi$. In eq. (3), $\text{STORM}$'s update rule is as follows for the $j$th iteration:

$$z_{j+1} = z_j - \eta_j v_j, \quad \text{where } v_j = \begin{cases} \nabla h(z_j; \xi_j) & \text{for } j = 0 \\ \nabla h(z_j; \xi_j) + (1 - \beta_j)(v_{j-1} - \nabla h(z_{j-1}; \xi_j)) & \text{for } j > 0 \end{cases} \tag{3}$$

In eq. (3), $\xi_j$ denotes the source of randomness in the $j$th iteration and $\beta_j \in [0, 1)$ is the momentum parameter. Note the use of the stochastic gradient at $z_{j-1}$ computed on $\xi_j$. Coming back to Algorithm 1, the quantity $u_k$ plays the role of $v_j$ in eq. (3). To see this clearly, let us analyze $E_{QD}[u_k]$ (see lines 8 and 10 in Algorithm 1). Under Assumption 3, $Q_D$ produces an unbiased estimate of the input. Then defining $g(w_k; S_k) \triangleq \frac{1}{r} \sum_{i \in S_k} (w_k - w_{k,i,E})$ and $\tilde{g}(w_{k-1}; S_k) \triangleq \frac{1}{r} \sum_{i \in S_k} (w_{k-1} - \hat{w}_{k-1,E})$, we have:

$$E_{QD}[u_k] = \begin{cases} g(w_k; S_k) & \text{for } k = 0 \\ g(w_k; S_k) + (1 - \beta_k)(u_{k-1} - \tilde{g}(w_{k-1}; S_k)) & \text{for } k > 0 \end{cases} \tag{4}$$

In eq. (4), $g(w_k; S_k)$ and $\tilde{g}(w_{k-1}; S_k)$ play the roles of $\nabla h(z_j; \xi_j)$ and $\tilde{\nabla} h(z_{j-1}; \xi_j)$, respectively. With this, one can clearly see that eq. (4) is the analogue of eq. (3) for the global server aggregation.
in FL. However, this equivalence is not so apparent without looking at the expected value of \( u_k \) with respect to \( Q_D \); in fact, the choice of quantities that are compressed in line 10 of Alg. 2 and used in line 10 of Alg. 1 is crucial for making our theory work (also see Remark 3).

Now that we understand global momentum, let us move on to local momentum. For this see lines 3, 6 and 8 in Algorithm 2, these give us \( (w_k - w_{k,E}) \) and \( (w_{k-1} - \hat{u}_{k-1,E}) \) after running for \( E \) steps. But notice that these lines are the same as eq. (3) with \( \beta_j = 0 \) and the stochastic gradient at the first iteration replaced by the full gradient. It is worth mentioning here that these local updates are also similar to SPIDER which is an SVRG-style update proposed in [10]. However, recognizing that this is also a special case of the STORM update with \( \beta_j = 0 \), we prefer calling it momentum in order to have a unifying terminology for both the global and local updates.

One might wonder what is the role of global momentum as SPIDER can be extended to improve the complexity in distributed optimization without multiple local updates. For this, in Appendix B, we consider FedLOMO (Alg. 4 and 5) which is a simpler version of FedGLOMO with only local momentum and no global momentum (i.e., plain averaging at the server which is equivalent to setting \( \beta_k = 1 \) in Alg. 1), and show that it does not achieve \( O(\varepsilon^{-1.5}) \) complexity (see Theorem 2). The root cause of this is client-heterogeneity which amplifies its effect under multiple local updates; without incorporating some form of variance reduction in the server aggregation step, the complexity cannot be improved.

Let us try to provide some intuition as to how incorporating global momentum helps. Suppose we keep \( \eta_k = \eta \) and \( \beta_k = \beta < 1 \) for all \( k \). Theoretically, we get a lower bound for \( \beta \) which is approximately \( O(\eta^2) \). Then with this momentum-based aggregation strategy, the variance reduces by a factor of \( O(\beta/\eta) = O(\eta) \) as compared to aggregation by plain averaging. (There are some other terms too but these are sufficiently small.) This reduction in the variance by a factor of \( O(\eta) \) is what enables FedGLOMO to enjoy a faster convergence rate.

It is true that FedGLOMO has to communicate twice the amount of information per round as compared to a FedAvg (or FedPAQ [29] which is FedAvg with compressed communication) per round. One can set the precision of the quantizer sufficiently low to account for the extra per-round communication cost of FedGLOMO – we do this in our experiments.

Also, we only assume access to the full client gradient in line 3 of Algorithm 2 for simplicity of analysis, but our main result (i.e., Theorem 1) can be extended to the case of large enough batch sizes.

5 Main Result for FedGLOMO

First, we state our assumptions.

Assumption 1 (Smoothness). \( \ell(x, w) \) is \( L \)-smooth with respect to \( w \), for all \( x \). Thus, each \( f_i(w) \) (\( i \in [n] \)) is \( L \)-smooth, and so is \( f(w) \).

Assumption 2 (Non-negativity). Each \( f_i(w) \) is non-negative and therefore, \( f_i^* \triangleq \min f_i(w) \geq 0 \).

Most of the loss functions used in practice satisfy this anyways and if not, we can just add a constant offset to achieve non-negativity.

Assumption 3 (Quantization operator). The randomized quantization operator \( Q_D \) in Algorithm 1 and 2 is unbiased, i.e., \( \mathbb{E}[Q_D(x)|x] = x \), and its variance satisfies \( \mathbb{E}[\|Q_D(x) - x\|^2|x] \leq q\|\|x\|\|^2 \) for some \( q > 0 \). The “qsgd” operator proposed in Section 3.1 of [7] satisfies these properties.

Assumption 4 (Heterogeneity). Suppose all clients participate, i.e. \( r = n \), in the \((k+1)^{st}\) round of FedGLOMO (Alg. 1 and 2). Let \( u_{k,r}^{(i)} \) be the \( i^\text{th} \) client’s local parameter at the \((\tau + 1)^{st}\)
Thus, \textit{FedGLOMO}, for \( i \in [n] \). Define \( \tilde{e}^{(i)}_{k,\tau} \triangleq \nabla f_i(w^{(i)}_{k,\tau}) - \nabla f_i(w_{k,\tau}) \), where \( w_{k,\tau} \triangleq \frac{1}{n} \sum_{i \in [n]} w^{(i)}_{k,\tau} \). Then for some \( \alpha \ll n \):

\[
\mathbb{E} \left[ \left\| \sum_{i \in [n]} \tilde{e}^{(i)}_{k,\tau} \right\|^2 \right] \leq \alpha \sum_{i \in [n]} \mathbb{E} \left[ \left\| \tilde{e}^{(i)}_{k,\tau} \right\|^2 \right], \quad \forall \ \tau \in [E].
\]

Obviously, the above assumption always holds with \( \alpha = n \); this follows from the Cauchy-Schwarz inequality. However, we empirically observe \( \alpha \ll n \) in practice; see Section 6.1. The value of \( \alpha \) depends on the degree of heterogeneity – as the heterogeneity increases (decreases), we observe \( \alpha \) to also increase (decrease). Thus, Assumption 1 can characterize the degree of heterogeneity in the system.

From the plots in Section 6.1, we see that for the highly heterogeneous setting that we consider for experiments in Section 6, \( \alpha < 0.06n \) for most of the trajectory of \textit{FedGLOMO} on both CIFAR-10 and Fashion-MNIST (abbreviated as FMNIST). In the homogeneous case, \( \alpha < 0.03n \) and \( \alpha < 0.02n \) for most of the trajectory on CIFAR-10 and FMNIST, respectively.

Further, this assumption is not specific to \textit{FedGLOMO} and is potentially applicable for other FL algorithms. In Appendix E, we empirically observe that this assumption also holds for \textit{FedAvg} and we use it to derive a novel convergence result for \textit{FedAvg} without the bounded client dissimilarity assumption (i.e., eq. (2)).

We now present our main result, followed by some important remarks and a proof sketch. The detailed proof can be found in Appendix F.1.

**Theorem 1 (Smooth non-convex case).** Suppose Assumptions 1, 2, 3 and 4 hold. In \textit{FedGLOMO}, set:

\[
\eta_k = \eta = \frac{1}{6LEK^{1/3}(\frac{1}{n}\alpha + \frac{4}{E}) + 800e^2(1 + q)(E + 1)^2(\frac{4}{n} + \frac{(1 + q)(n - r)}{r(n - 1)})^{1/3}} \quad \text{and} \quad \beta_k = \beta = 160e^2(1 + q)\eta^2L^2E^2(E + 1)^2.
\]

Suppose we use full-device participation (i.e., the global batch size is \( n \)) only at \( k = 0 \). Then if

\[
\frac{1}{1200e^2(1 + q)\left( \frac{4}{n} + \frac{(1 + q)(n - r)}{r(n - 1)} \right)} \leq E + 1 \leq \frac{\sqrt{1 + q(n - r)}}{3\sqrt{r(n - 1)}},
\]

we have:

\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla f(w_k)\|^2] \leq \frac{39Lf(w_0)}{K^{2/3}} \left( \frac{1}{n}\alpha + \frac{4}{E} \right) + 800e^2(1 + q)(E + 1)^2 \left( \frac{4}{n} + \frac{(1 + q)(n - r)}{r(n - 1)} \right)^{1/3}.
\]

Thus, \textit{FedGLOMO} can achieve \( \mathbb{E}[\|\nabla f(w_k^*)\|^2] \leq \epsilon \), where \( k^* \sim \text{Unif}[0, K - 1] \), in \( E = O(1) \) local steps and \( K = O\left(\max\left\{\sqrt{\frac{n}{\epsilon}}, (1 + q)\sqrt{\frac{(n - r)}{r(n - 1)}}\right\}^{-1.5}\right) \) rounds of communication.

Note that the above result is independent of the variance of local stochastic gradients (of the clients). In short, this happens because we use local full gradients at \( \tau = 0 \) and because the local stochastic gradients are Lipschitz.

We now make some remarks to discuss the implications of Theorem 1.

**Remark 1 (Better iteration complexity).** According to Theorem 1, for converging to an \( \epsilon \)-stationary point, \textit{FedGLOMO} needs \( T = KE \) to be \( O\left(\max\left\{\sqrt{\frac{n}{\epsilon}}, (1 + q)\sqrt{\frac{(n - r)}{r(n - 1)}}\right\}^{-1.5}\right) \). This iteration complexity is the same as that of \textit{MimeMVR} but without using the bounded client dissimilarity assumption, i.e., eq. (2), (also see the next remark for more details on this) and better than other related works in the federated setting; see Table 1. We underscore the significance of global momentum here by comparing this complexity of \textit{FedGLOMO} to that of \textit{FedLOMO} (recall this is a simpler version of \textit{FedGLOMO} with only local momentum and no global momentum, described in Appendix E) which is \( O\left(\frac{1}{\epsilon^2}\right) \) (see Theorem 2).
Remark 2 (No requirement of bounded client dissimilarity (BCD) assumption). Divergent from related works, Theorem 1 does not use the commonly used BCD assumption, i.e., eq. (2). This is achieved by utilizing the smoothness and non-negativity of the $f_i$’s, specifically $\frac{1}{n} \sum_{i \in [n]} \|\nabla f_i(w)\|^2 \leq \frac{1}{n} \sum_{i \in [n]} 2L(f_i(w) - f_i^*) \leq 2L f(w)$; see the proof outline of Theorem 1 after Remark 4. In lieu of the hard to verify BCD assumption, we use the empirically verified Assumption 4 to tighten our convergence result. Note that Assumption 4 will always hold for some $\alpha \leq n$, regardless of the degree of client heterogeneity. Thus, our theory also allows for arbitrary client heterogeneity.

Remark 3 (Compressed communication). To our knowledge, FedGLOMO is the first algorithm that attains the aforementioned improved iteration complexity for FL on smooth non-convex functions with compressed communication. We emphasize that the choice of quantities that are compressed in line 10 of Algorithm 2 is important. This particular choice enables deriving the improved rate by first deriving a result analogous to smoothness, i.e., $\| (w_k - w^{(i)}_k) - (w_{k-1} - \hat{w}^{(i)}_{k-1,E}) \| \leq \tilde{L} \| w_k - w_{k-1} \|$ (this derivation is done in Lemma 9 in Appendix F.1). The straightforward choice of sending $Q_D(w_k - w^{(i)}_k)$ and $Q_D(w_{k-1} - \hat{w}^{(i)}_{k-1,E})$ prohibits us from deriving the improved rate, unless we also assume $Q_D(.)$ to be a Lipschitz operator.

In Appendix A for $r \ll n$, we show that using the quantization scheme of [1] with $s = \sqrt{d}$, FedGLOMO achieves more than a five-fold saving in the total communication cost as compared to when there is full-precision communication in FedGLOMO.

Remark 4 (A limitation). Even though our iteration complexity of $T = O(\epsilon^{-1.5})$ is better than that of FedCOMGATE [12] (which is $O(\epsilon^{-2})$), our communication complexity of $K = O(\epsilon^{-1.5})$ is higher than that of FedCOMGATE which is $K = O(\epsilon^{-1})$ (albeit under an extra assumption on the quantizer, namely Assumption 5 in their paper). Ideally, we would like to have $E = O(\epsilon^{-p})$ and $K = O(\epsilon^{-(1.5-p)})$ for some $p > 0$, in order to reduce FedGLOMO’s communication complexity. Exploring whether such a result is obtainable with our proposed style of momentum is an interesting future direction.

Proof Sketch of Theorem 1

Before getting to the proof outline, we would like to mention that the key technical challenge in deriving the improved convergence result with global momentum-based variance reduction is obtaining an analogue of the Lipschitzness of stochastic gradients to the change in local parameters over $E$ local steps. More specifically, for pure stochastic optimization, a key step in proving convergence of momentum-based variance reduction methods is using the Lipschitzness of the stochastic gradients (or the update quantities) [9][23], i.e.,

$$\| \nabla \tilde{f}(x_t, \xi_t) - \nabla \tilde{f}(x_{t-1}, \xi_t) \| \leq L \| x_t - x_{t-1} \|.$$

In the FL setting where aggregation is performed at the server, we need an analogue of this at the server, i.e., something like

$$\| (w_k - w^{(i)}_{k,E}) - (w_{k-1} - \hat{w}^{(i)}_{k-1,E}) \| \leq \tilde{L} \| w_k - w_{k-1} \|.$$

Deriving this result is a part of our contribution and is done in Lemma 9 (in Appendix F.1).

Proof. We set $\eta_k = \eta$ and $\beta_k = \beta \ \forall \ k \in \{0, \ldots, K - 1\}$. Then, using Lemma 1 with full global as well as local batch sizes at $k = 0$ (by which $u_0 = \delta_0$ in the statement of Lemma 1), we have...
at any $k' > 0$:

$$\mathbb{E}[f(w_{k'})] \leq f(w_0) - \frac{\eta E}{4} k' \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]$$

$$+ 160\eta\beta \left( \frac{q}{n^2} + \frac{(1 + q)(n - 1)}{r(n - 1)} \right) \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2], \quad (5)$$

for $4\eta L E^2 \leq 1$ and $\beta \geq \frac{80e^2 (1 + q)\eta^2 L^2 E^2 (E + 1)^2}{(1 - 4\eta LE)}$.

Also, since the $f_i$’s are $L$-smooth and non-negative, using Lemma 11, we have that:

$$\sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \leq \sum_{i \in [n]} 2L (\mathbb{E}[f_i(w_k)] - f_i^*) \leq 2nL \mathbb{E}[f(w_k)] - 2L \sum_{i \in [n]} f_i^* \leq 2nL \mathbb{E}[f(w_k)].$$

This step allows us to circumvent the need for the bounded client dissimilarity assumption. Using this in $(5)$, we get:

$$\mathbb{E}[f(w_{k'})] \leq f(w_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} 2 \mathbb{E}[\|\nabla f(w_k)\|^2]$$

$$+ 64nL^2 \mathbb{E}\left( \frac{\eta^2 L^2 E (\alpha E + 4)}{n} \right) + 5\beta \left( \frac{q}{n} + \frac{(1 + q)(n - 1)}{r(n - 1)} \right) \sum_{k=0}^{k'-1} \mathbb{E}[f(w_k)]. \quad (6)$$

Unfolding the above recursion and simplifying a bit, we get:

$$\sum_{k=0}^{k'-1} \mathbb{E}[f(w_k)] \leq k' f(w_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(w_k)\|^2] + \gamma k' \sum_{k=0}^{k'-1} \mathbb{E}[f(w_k)]. \quad (7)$$

Let us now ensure that $\gamma k' \leq \frac{1}{4}$ for all $k' \in \{1, \ldots, K\}$, so that we can simplify $(7)$ to:

$$\sum_{k=0}^{k'-1} \mathbb{E}[f(w_k)] \leq 2k' f(w_0) - \frac{\eta E}{2} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(w_k)\|^2] \leq 2k' f(w_0). \quad (8)$$

Now for $8\eta L E^2 \leq 1$, it can be verified that $\beta = 160e^2 (1 + q)\eta^2 L^2 E^2 (E + 1)^2$ is a valid choice. Using this, we get that:

$$\gamma k' \leq \gamma K = 64n^2 L^3 E^3 k \left( \frac{1}{n} \left( \alpha + \frac{4}{E} \right) + 800e^2 (1 + q)(E + 1)^2 \left( \frac{q}{n} + \frac{(1 + q)(n - 1)}{r(n - 1)} \right) \right). \quad (9)$$

Setting

$$\eta = \frac{1}{6LEK^{1/3} \left( \frac{1}{n} (\alpha + \frac{4}{E}) + 800e^2 (1 + q)(E + 1)^2 (\frac{q}{n} + \frac{(1 + q)(n - 1)}{r(n - 1)}) \right)^{1/3}},$$

we have $(A) < \frac{1}{2}$. We also need to ensure that $8\eta L E^2 \leq 1$ and $\beta = 160e^2 (1 + q)\eta^2 L^2 E^2 (E + 1)^2 < 1$.

The range of $E$ in the theorem statement is obtained by combining the constraints (on $E$) that we get from these two requirements.

Finally, using $(8)$ in $(6)$ with $k' = K$, substituting our choice of $\eta$ and $\beta$, and then simplifying a bit more, we get the final convergence result.
6 Experiments

To show the efficacy of global momentum in FedGLOMO, we compare it against FedLOMO (recall this has only local momentum and no global momentum; see Appendix E) and the default algorithm of choice for FL, i.e., FedAvg [24] with the standard momentum available in PyTorch applied to its local updates – both with and without compression. Note that FedAvg with compression is referred to as FedPAQ [29]. We call the standard momentum versions of FedAvg and FedPAQ as FedAvg-m and FedPAQ-m henceforth. For quantization, we use the “qsgd” operator proposed in Section 3.1 of [1]. In the no-compression heterogeneous case, we also compare against Mime (specifically, “MimeSGDm”) [16] which is also shown to attain the improved complexity of $O(\epsilon^{-1.5})$ but without compressed communication, and is tailored to handle client heterogeneity.

We consider the task of classification on CIFAR-10 and Fashion-MNIST [39] abbreviated as FMNIST henceforth. The model used is a two-layer neural network with ReLU activation in the hidden layers. The size of both the hidden layers is 300/600 for FMNIST/CIFAR-10. We train the models using the categorical cross-entropy loss with $\ell_2$-regularization. The weight decay value in PyTorch (to apply $\ell_2$-regularization) is set to $1e-4$. We consider both homogeneous and heterogeneous data distribution among the clients. Similar to [24], for the heterogeneous case, we distribute the data among the clients such that each client can have data from either one or (at most) two classes – note that this is a high degree of heterogeneity. The exact procedure is described in Appendix D. The number of clients ($n$) in all the experiments is set to 50, with each client having the same number of samples. In this set of experiments, the global batch-size $r$ is 25, and the number of local updates per round (i.e., $E$) is 10. For FedGLOMO, we use a constant value of $\beta_k = 0.2$. For FedAvg-m and FedPAQ-m, the momentum parameter in Pytorch is set to its standard value, i.e., 0.9. As suggested in [16], we search $\beta$ (momentum hyper-parameter in MimeSGDm) over $\{0, 0.9, 0.99\}$. All full gradients in FedGLOMO, FedLOMO and Mime are replaced by stochastic gradients computed on a (per-client) batch size of 256. The learning rates and some other experimental details are in Appendix D.

In Fig. 1 we compare FedPAQ-m and FedLOMO with 4 (resp., 8) bits per-round against FedGLOMO with 2 (resp., 4) bits per-round on FMNIST (resp., CIFAR-10) in the heterogeneous and homogeneous cases. We set the number of per-round bits used by FedPAQ-m and FedLOMO to be twice that of FedGLOMO so that all algorithms have the same per-round communication budget. All plots depict results over 3 independent runs; the shaded regions represent $\pm 1$ standard deviation whereas the solid lines are the respective means. Please see the discussion in the figure caption. Having shown the suboptimality of FedLOMO in Fig. 1 we compare FedAvg-m, FedGLOMO without compression and MimeSGDm in the heterogeneous case in Fig. 2. These experiments illustrate the power of global momentum. Also see Appendix D.1 for additional experiments.

6.1 Verifying Assumption 4 for FedGLOMO

We compute $\alpha = \max_{r \in [E]} \frac{\|\sum_{i \in [n]} \tilde{e}_r(i)\|^2}{\sum_{i \in [n]} \|\tilde{e}_r(i)\|^2}$, where $\tilde{e}_r(i)$ is as defined in Assumption 4 for 4 and 2 bit FedGLOMO on CIFAR-10 and FMNIST, respectively. Note that we remove the expectation (w.r.t. the stochastic gradients) while computing $\alpha$ for empirical verification. In Fig. 3 we plot $(\alpha/n)$ over different rounds for the heterogeneous as well as homogeneous case on both datasets; see the discussion in the figure caption.
Figure 1: Comparison of FedPAQ-m, FedGLOMO and FedLOMO (recall this has only local momentum and no global momentum) with the same per-round communication budget on FMNIST and CIFAR-10 in the heterogeneous (top four figures) and homogeneous (bottom four figures) settings, respectively. The x-axis is the total number of communicated bits divided by the dimension $d$ and the global batch-size $r$. FedLOMO is the slowest while FedGLOMO is the fastest, showing the ineffectiveness of only local momentum and the power of combining both local and global momentum. For both datasets, in the heterogeneous (resp., homogeneous) case, FedGLOMO attains the final test error of FedPAQ-m with only about a third (resp., less than a fifth) of the number of bits used by FedPAQ-m. In the heterogeneous case, FedGLOMO as well as FedLOMO have a smoother trajectory than FedPAQ-m due to the application of variance-reducing momentum.
Figure 2: **Heterogeneous case:** Comparison of FedAvg-m, FedGLOMO (without compression) and MimeSGDm on FMNIST (top) and CIFAR-10 (bottom). On both datasets, FedAvg-m is the slowest while FedGLOMO is somewhat faster than MimeSGDm. While Mime has an explicit client-drift control mechanism, we do not have that in FedGLOMO, but still our proposed global momentum implicitly mitigates client-drift.

Figure 3: Variation of \( \frac{\alpha}{n} \) over different rounds of 4 and 2 bit FedGLOMO for CIFAR-10 (Fig. 3a) and FMNIST (Fig. 3b) in the heterogeneous and homogeneous cases. In both cases, notice that \( \alpha \ll n \) throughout training. Also, as discussed after the statement of Assumption 4, observe that \( \frac{\alpha}{n} \) is higher for the heterogeneous case (except towards the end of training for FMNIST).
7 Conclusion

We presented FedGLOMO, a communication-efficient algorithm for faster federated learning (FL) via the application of variance-reducing momentum, both in the aggregation step at the server as well as local client updates. We showed that FedGLOMO has better iteration complexity than prior work on smooth non-convex functions with compressed communication. Further, unlike prior work, our result is under Assumption 4, which is a novel and verifiable client-heterogeneity assumption, even allowing for arbitrary client heterogeneity. We also demonstrate the efficacy of FedGLOMO via extensive experiments.

Apart from addressing the limitation discussed in Remark 4, there are several avenues of future work possible such as verifying Assumption 4 for other FL algorithms and deriving convergence results based on it, obtaining lower bounds on the iteration complexity of non-convex FL, coming up with an error-compensated version of FedGLOMO for biased compressors, etc.

8 Acknowledgement

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Appendix

A  Reduction in total communication cost when \( r \ll n \)

Here, we derive the claim made in the second paragraph of Remark 3. We consider the practical regime of \( r \ll n \) (as well as \( \alpha \ll n \)).

First, consider the case where the clients communicate at full precision using 32 bits, i.e., \( q = 0 \). The number of rounds of communication \( K_1 \) needed to reach an \( \epsilon \) stationary point is:

\[
K_1 \approx \left( \frac{39L_f(w_0)}{\epsilon} \right)^{1.5} \left( \frac{800e^2(E + 1)^2 (n - r)}{r(n - 1)} \right)^{0.5}.
\]

Since the communication cost per-round is proportional to \( r \times (32d) \) bits (recall \( d \) is the model dimension), the total communication cost \( C_1 \) in this case is:

\[
C_1 = 32dr \times K_1
= 32dr \left( \frac{39L_f(w_0)}{\epsilon} \right)^{1.5} \left( \frac{800e^2(E + 1)^2 (n - r)}{r(n - 1)} \right)^{0.5}.
\]

Now, let us consider the QSGD quantizer of [1] with \( s = \sqrt{d} \). With this choice, \( q = 1 \). Here, the number of rounds of communication \( K_2 \) needed to reach an \( \epsilon \) stationary point is:

\[
K_2 \approx \left( \frac{39L_f(w_0)}{\epsilon} \right)^{1.5} \left( \frac{3200e^2(E + 1)^2 (n - r)}{r(n - 1)} \right)^{0.5}.
\]

Now using Theorem 3.4 of [1], under the special case of \( s = \sqrt{d} \), the communication cost per-round can be reduced to \( r \times (2.8d + 32) \) bits. Hence, the total communication cost \( C_2 \) in this case is:

\[
C_2 \approx (2.8d + 32)r \times K_2
\approx (2.8d + 32)r \left( \frac{39L_f(w_0)}{\epsilon} \right)^{1.5} \left( \frac{3200e^2(E + 1)^2 (n - r)}{r(n - 1)} \right)^{0.5}.
\]

Therefore,

\[
\frac{C_1}{C_2} \approx \frac{32}{2.8 \times 4^{0.5}} \approx 5.7.
\]

B  Algorithmic and Theoretical Comparison with MIME [16]

We now discuss the major algorithmic and theoretical differences of our work from [16].

- Algorithmically, [16] do not explicitly apply any momentum at the server. Instead, they apply globally computed momentum in the local updates of the clients. On the other hand, FedGLOMO has an explicit momentum-based update at the server to enable global variance reduction, apart from local momentum applied in the client updates.

- Unlike FedGLOMO, the algorithms of [16] do not have any quantized/compressed communication. As we discussed in Remark 3, maintaining the improved complexity of \( \mathcal{O}(\epsilon^{-1.5}) \) with compressed communication is not trivial.

\[\text{[1]} \text{ uses } n \text{ to denote the dimension}\]
- Even in the absence of any compressed communication, FedGLOMO is more communication-efficient than Mime requiring three-fourth / half the number of bits that Mime requires per-round for server to clients as well as clients to server communication / only clients to server communication (which is typically the bottleneck in FL). This is because in Mime, the server needs to send \( x \) (sending some other statistics \( s \) would require even more bits) and \( c \) to the clients, and the clients need to send back \( (y_i, \nabla f_i(x)) \) to the server (please see their notation). In FedGLOMO, the server needs to send \( w_k \) and \( w_{k-1} \) to the clients, but the clients can just send back \( \{ (w_k - w^{(i)}_{k,E}) - (1 - \beta_k)(w_{k-1} - \hat{w}^{(i)}_{k-1,E}) \} \) to the server in the absence of any quantization – this can be verified just by removing the quantization operator \( Q_D \) and expanding the update rule of \( u_k \) (line 10 of Algorithm 1) for \( k > 0 \).

- Our theory for FedGLOMO does not use the bounded client dissimilarity (BCD) assumption, i.e., eq. (2). Instead, we propose and use Assumption 3 which allows for arbitrary client heterogeneity; in the worst case, Assumption 3 will hold with \( \alpha = n \). In contrast, the results of MimeMVR use the BCD assumption.

- See the full version of Theorem V (on page 39 of the latest arXiv draft) of [16] for MimeMVR. Their result is in terms of the gradient of \( f \) at the local client parameters and not the actual server parameters, which is not ideal. Our result for FedGLOMO is completely in terms of the gradient of \( f \) at the server parameters.

C Comparison with [11]

As mentioned in Section 2 [11] also propose algorithms with improved complexity in the distributed setting without multiple local update steps. Since our work is under partial-device participation, we compare against their algorithm for the same case, i.e. PP-MARINA. Note that PP-MARINA has a probability \( p \) of using full gradients from all clients (i.e., full device participation) in each iteration. For a fair comparison against FedGLOMO, which uses gradients from all the clients only in the first round (see Theorem 1), \( p \) should be set to \( \frac{1}{K} \) (\( K \) being the number of rounds) – in which case, their complexity is \( O(\epsilon^{-2}) \) which is worse than ours. See Theorem 4.1 in their paper for this.

D Experimental Details and More Experiments

We first describe the procedure we have used to generate heterogeneous data distribution (among the clients). First, the training data (of both CIFAR-10 and FMNIST) was sorted based on labels and then divided into 100 equal data-shards. Splitting the data into 100 equal shards (after sorting) ensures that each shard contains data from only one class for both CIFAR-10 and FMNIST. Since the number of clients in our experiments is fixed to 50, each client is assigned 2 shards chosen uniformly at random without replacement – this ensures that each client can have data belonging to either just one class or two classes at the most. For the homogeneous case, we distribute the training data uniformly at random among the clients.

In all our experiments (including the ones in Appendix D.1), we use the learning rate schedule suggested in [7] where we decimate the client learning rate by 1% after every round, i.e., \( \eta_k = (0.99)^k \eta_0 \), \( \eta_0 \) being the initial learning rate. Note that this learning rate schedule has been used earlier for FL experiments in [12]. We search the initial learning rates over \( \{10^{-3}, 5 \times 10^{-3}, 10^{-2}, 5 \times 10^{-2}, 10^{-1}\} \); the best performance is obtained with an initial learning rate of \( 10^{-2} \) in almost all the cases.

We make a small modification to FedGLOMO in our experiments for the heterogeneous case. Specifically, we modify line 5 (which is the local momentum application step) of Algorithm 2 as follows:

```
Update: \( v^{(i)}_{k,\tau} = \nabla f_i(w^{(i)}_{k,\tau}; B^{(i)}_{k,\tau}) + 0.8 (v^{(i)}_{k,\tau-1} - \nabla f_i(w^{(i)}_{k,\tau-1}; B^{(i)}_{k,\tau})) \)
```
\( \hat{v}_{k-1, \tau}^{(i)} = \tilde{\nabla} f_i(\hat{w}^{(i)}_{k-1, \tau}; B^{(i)}_{k, \tau}) + 0.8(\hat{v}^{(i)}_{k-1, \tau-1} - \tilde{\nabla} f_i(\hat{w}^{(i)}_{k-1, \tau-1}; B^{(i)}_{k, \tau})). \)

Without applying the above damping factor of 0.8, FedGLOMO seems to diverge – this is probably because we have chosen the number of local updates to be too large.

All experiments are run on a single NVIDIA TITAN Xp GPU.

D.1 More experimental results:

Having shown the suboptimality of FedLOMO in the experiments of the main paper (i.e., in Section 6), we do not compare against FedLOMO anymore. For all the experiments, the figure captions discuss the results in detail.

More comparisons against FedPAQ-m: In Figure [4] similar to Figure [1] we compare FedPAQ-m and FedGLOMO in the heterogeneous case with a smaller per-round communication budget.

In all the experiments so far, we have chosen \( E = 10 \) and \( K = 300 \), and so \( T = KE = 3000 \). In Figure [4] we compare 2 bit FedPAQ-m against 1 bit FedGLOMO in the heterogeneous case on FMNIST with two other combinations of \( E \) and \( K \) such that \( T = 3000 \) – these are \( E = 5, K = 600 \) and \( E = 20, K = 150 \). Compared to \( E = 10, K = 300 \), we halve and double the initial learning rate \( \eta_0 \) for \( E = 20, K = 150 \) and \( E = 5, K = 600 \), respectively; we do this because from our theory, we are required to have \( \eta_0 E < c \) for some constant \( c \). Everything else is the same as described in Section 6.

Now, we move onto changing the number of participating clients or global batch size, i.e., \( r \). Recall that so far, we have set \( r = 0.5n \). In Figure [6] we compare 8-bit FedPAQ-m against 4-bit FedGLOMO with smaller values of \( r \) in the heterogeneous case on CIFAR-10 with everything else remaining the same (including \( K = 300 \) and \( E = 10 \)).

Figure 4: Heterogeneous case: 2 bit FedPAQ-m vs. 1 bit FedGLOMO on FMNIST at the top, and 6 bit FedPAQ-m vs. 3 bit FedGLOMO on CIFAR-10 at the bottom. The x-axis is the total number of communicated bits divided by the dimension \( d \) and the global batch-size \( r \). The trend is similar to Figure [1].

Comparison against FedCOMGATE [12]: As discussed in Section 2, FedCOMGATE [12] is another communication-efficient FL algorithm incorporating gradient tracking to improve the convergence
Figure 5: **FMNIST Heterogeneous case**: 1-bit FedGLOMO vs. 2-bit FedPAQ-m for other values of $E$ and $K$, such that $T = KE = 3000$. Just as Figure 4, the x-axis is the total number of communicated bits divided by the dimension $d$ and the global batch-size $r$. For $E = 5, K = 600$, FedGLOMO reaches the final test error of FedPAQ-m with about a third of the number of bits used by FedPAQ-m. For $E = 20, K = 150$, the corresponding number increases to just above two-thirds.

**Comparison against server-level PyTorch-like momentum:** One can even apply server-level momentum by using the kind of momentum provided by PyTorch. This can be done by implementing the server update as a PyTorch optimizer update with momentum. Here, we show the superiority of our proposed scheme as compared to this kind of server-level momentum. Specifically, we compare FedGLOMO against FedPAQ-m augmented with server-level PyTorch momentum; we call this FedPAQ-scm ("scm" stands for server and client momentum). We tried three different values of server-level momentum which are \{0.9, 0.7, 0.5\} and show the results with the best value in Figure 8 and Figure 9 for the homogeneous and heterogeneous cases, respectively.

**E FedGLOMO: A Simpler Version of FedGLOMO**

Now we consider a simpler version of FedGLOMO, which we call FedLOMO, that applies only local momentum in the client updates and does simple averaging at the server (like FedAvg), i.e., there is no global momentum (and hence the name of this variant does not have a “G”). FedLOMO is summarized in Algorithm 4 and 5. Notice that the momentum application occurs in line 6 of Algorithm 5.

As mentioned in the main paper, FedLOMO does not achieve the optimal convergence rate for smooth non-convex functions due to the absence of global momentum; see Theorem 2 and the subsequent remarks.
Figure 6: CIFAR-10 Heterogeneous case: 4-bit FedGLOMO vs. 8-bit FedPAQ-m for smaller values of global batch size, i.e., $r$. In this case, the x-axis is the total number of communicated bits divided by the dimension $d$ and the number of clients $n$. For $r = 0.3n$, FedGLOMO attains the final test error of FedPAQ-m with only about half the number of bits used by FedPAQ-m. For $r = 0.1n$, the corresponding ratio increases to roughly 80% the number of bits used by FedPAQ-m. Overall, FedGLOMO consistently outperforms FedPAQ-m and has a smoother performance due to the application of variance-reducing momentum.

Just like the results of FedGLOMO, we do not use the BCD assumption (i.e., eq. (2)) to derive the results of FedLOMO.

Algorithm 4 FedLOMO - Server Update

1. Input: Initial point $w_0$, # of rounds of communication $K$, period $E$, learning rates $\{\eta_k\}_{k=0}^{K-1}$, per-client batch size $b$, and global batch size $r$. $Q_D$ is the quantization operator.
2. for $k = 0, \ldots, K - 1$ do
3. Server chooses a set $S_k$ of $r$ clients uniformly at random without replacement and sends $w_k$ to them.
4. for client $i \in S_k$ do
5. Set $w_{k,0}^{(i)} = w_k$ and run Algorithm 5 for client $i$.
6. end for
7. Update $w_{k+1} = w_k + \frac{1}{r} \sum_{i \in S_k} Q_D(w_{k}^{(i)} - w_k)$.
8. end for

E.1 Main Result for FedLOMO

Now, we present the convergence result of FedLOMO for the smooth non-convex case in Theorem 2. Its proof is in Appendix 2. Here, we assume that Assumption 4 holds for FedLOMO; we restate it below.
Figure 7: **Heterogeneous case:** Comparison of FedPAQ-m, FedGLOMO and FedCOMGATE with the same per-round communication budget on FMNIST (top) and CIFAR-10 (bottom). The x-axis is the total number of communicated bits divided by the dimension $d$ and the global batch-size $r$. On FMNIST, FedGLOMO significantly outperforms FedCOMGATE both in terms of training loss as well as test error. However, on CIFAR-10, both FedGLOMO and FedCOMGATE have similar test set performance; with respect to the training loss, FedCOMGATE is initially faster but after about 200 rounds, FedGLOMO takes over. From these experiments, we see that our proposed idea of variance-reducing global and local momentum does have some advantage over gradient tracking (which is the main ingredient of FedCOMGATE) when applied for a sufficiently large number of rounds.

**Algorithm 5** FedLOMO - Client Update

1: for $\tau = 0, \ldots, E - 1$ do
2:  if $\tau = 0$ then
3:   $v_{k,\tau}^{(i)} = \nabla f_i(w_{k,\tau}^{(i)})$.
4:  else
5:    Pick a random batch of $b$ samples in client $i$, say $B_{k,\tau}^{(i)}$. Compute the stochastic gradients of $f_i$ at $w_{k,\tau}^{(i)}$ and $w_{k,\tau-1}^{(i)}$ over $B_{k,\tau}^{(i)}$ viz. $\nabla f_i(w_{k,\tau}^{(i)}; B_{k,\tau}^{(i)})$ and $\nabla f_i(w_{k,\tau-1}^{(i)}; B_{k,\tau}^{(i)})$, respectively.
6:    Update $v_{k,\tau}^{(i)} = \nabla f_i(w_{k,\tau}^{(i)}; B_{k,\tau}^{(i)}) + (v_{k,\tau-1}^{(i)} - \nabla f_i(w_{k,\tau-1}^{(i)}; B_{k,\tau}^{(i)}))$. // (Local Momentum)
7:  end if
8:  Update $w_{k,\tau+1}^{(i)} = w_{k,\tau}^{(i)} - \eta v_{k,\tau}^{(i)}$.
9: end for
10: Send $Q_D(w_{k,E}^{(i)} - w_k)$ to the server.

**Assumption 5** (Assumption 4 for FedLOMO). Suppose all clients participate, i.e. $r = n$, in the $(k+1)^{st}$ round of FedLOMO (Alg. 4 and 5). Let $w_{k,\tau}^{(i)}$ be the $i^{th}$ client’s local parameter at the $(\tau+1)^{st}$
Figure 8: **Heterogeneous case:** 4 (resp., 8) bit FedPAQ-scm vs. 2 (resp., 4) bit FedGLOMO on FMNIST (resp., CIFAR-10) at the top (resp., bottom). The x-axis is the total number of communicated bits divided by the dimension $d$ and the global batch-size $r$. FedGLOMO outperforms FedPAQ-scm and has a smoother performance than it due to the application of variance-reducing momentum.

Figure 9: **Homogeneous case:** Same setting as Figure 8 but in the homogeneous case. Once again, FedGLOMO outperforms FedPAQ-scm. Here, FedPAQ-scm has a smoother performance compared to Figure 8 due to the homogeneous data distribution.
local step of the \((k+1)^{st}\) round of \textit{FedLOMO}, for \(i \in [n]\). Define \(\hat{e}_{k,\tau}^{(i)} \triangleq \nabla f_i(w_{k,\tau}^{(i)}) - \nabla f_i(w_{k,\tau})\), where \(w_{k,\tau} \triangleq \frac{1}{n} \sum_{i \in [n]} w_{k,\tau}^{(i)}\). Then for some \(\alpha \ll n\):

\[
E\left[\left\| \sum_{i \in [n]} \hat{e}_{k,\tau}^{(i)} \right\|^2\right] \leq \alpha \sum_{i \in [n]} E\left[\left\| e_{k,\tau}^{(i)} \right\|^2\right], \quad \forall \tau \in [E].
\]

**Theorem 2** (Smooth non-convex case for \textit{FedLOMO}). Suppose Assumptions 1, 2, 3 and 4 (i.e., Assumption 4 for \textit{FedLOMO} instead of \textit{FedGLOMO}) hold. Define a distribution \(\mathbb{P}\) for \(k \in \{0, \ldots, K-1\}\) such that \(\mathbb{P}(k) = \frac{(1+\zeta)^{(K-1-k)}}{\sum_{k=0}^{K-1} (1+\zeta)^k}\) where \(\zeta\) will be defined later. Sample \(k^*\) from \(\mathbb{P}\). In \textit{FedLOMO}, let \(\eta_k = \frac{1}{8L\sqrt{BK}}\) where \(B = \frac{q}{n} + \frac{4(1+q)(n-r)}{r(n-1)}\). Then for \(K > \frac{1}{64}\frac{1}{n}\left(16\left(\frac{\alpha}{\alpha} + \frac{4}{4}\right)\right)\):

\[
E[\|\nabla f(w_{k^*})\|^2] \leq \frac{64\sqrt{BL}f(w_0)}{K^{1/2}} \quad \text{with} \quad \zeta := \frac{1}{4K} + \frac{1}{16(BK)^{1/2}\left(\frac{1}{n}\left(\frac{\alpha}{\alpha} + \frac{4}{4}\right)\right)}.
\]

So \textit{FedLOMO} needs \(K = \mathcal{O}\left(\frac{1}{\epsilon^2}\right)\) rounds of communication to achieve \(E[\|\nabla f(w_{k^*})\|^2] \leq \epsilon\), for \(\epsilon < \mathcal{O}\left(\frac{n\beta^2}{\alpha}\right) = \mathcal{O}\left(n^{\alpha/\alpha}\right)\).

We make some remarks to discuss implications of this result and establish connections to some claims made in the main paper.

**Remark 5** (Worse iteration complexity than \textit{FedGLOMO}). Since we do not have any constraint on \(E\) depending on \(\epsilon\), \(T = KE\) is also \(\mathcal{O}\left(\frac{1}{\epsilon^2}\right)\) as per the above theorem. So the iteration complexity of \textit{FedLOMO} is poorer than that of \textit{FedGLOMO}. However, it is on a par with the results of \([12,17,18,36]\).

**Remark 6** (High variance of simple averaging at the server). At a high level, \textit{FedLOMO} fails to attain the complexity of \textit{FedGLOMO} because of the high variance of the \textit{FedAvg}-like plain averaging step at the server. The high variance is itself due to the amplified effect of client-heterogeneity with multiple local updates. Without the application of some global variance-reduction technique (like the one in \textit{FedGLOMO}), the complexity cannot be improved. More precisely, in Theorem 2, \(B\) is a constant that is not \(\mathcal{O}(nLE)\) in general, due to which \textit{FedLOMO} does not achieve the improved convergence rate of \(\mathcal{O}(K^{-2/3})\) that \textit{FedGLOMO} attains; see the proof of Theorem 2 in Appendix F.2 for more details. However, in the special case of no compression and full-client participation (i.e., \(r = n\)), \(B\) is 0 which allows \textit{FedLOMO} to also achieve \(\mathcal{O}(K^{-2/3})\) convergence by choosing \(\eta = \mathcal{O}\left(\frac{1}{LKEV^2}\right)\).

**F** Detailed Proofs

**F.1 Detailed Proof of the Result of \textit{FedLOMO}**:

Some details used in the proofs:

\[
\delta_{k}^{(i)} \triangleq \mathbb{E}_{\mathcal{B}_1^{(i)}, \ldots, \mathcal{B}_{E-1}^{(i)}}[w_k - w_{k,E}^{(i)}] \quad \text{for any } E-1 \text{ batches } \{\mathcal{B}_1^{(i)}, \ldots, \mathcal{B}_{E-1}^{(i)}\} \text{ in client } i, \text{ and } \delta_k \triangleq \frac{1}{n} \sum_{i \in [n]} \delta_{k}^{(i)}.
\]

\[
g_Q(w_k; S_k) \triangleq \frac{1}{r} \sum_{i \in S_k} Q_M(w_k - w_{k,E}^{(i)})
\]

\[
\Delta g_Q(w_k, w_{k-1}; S_k) \triangleq \frac{1}{r} \sum_{i \in S_k} Q_M((w_k - w_{k,E}^{(i)}) - (w_{k-1} - w_{k-1,E}^{(i)}))
\]

\[
g(w_k; S_k) \triangleq \frac{1}{r} \sum_{i \in S_k} (w_k - w_{k,E}^{(i)}) = \mathbb{E}_{Q_M} [g_Q(w_k; S_k)]
\]
The last step above follows because the need for the bounded client dissimilarity assumption.

Then using Lemma 1, we have that:

\[ \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \leq 2L \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|] - 2L \sum_{i \in [n]} f_i^* \leq 2n \mathbb{E}[f(\mathbf{w}_k)]. \]

for \( 4\eta L^2 \leq 1 \) and \( \beta \geq \frac{8(\alpha + \eta)\eta^2 L^2 E^2(E+1)^2}{(1-\theta L E^2)}. \)

Suppose we use full batch sizes for the local updates as well as the server update at \( k = 0 \) (the latter means \( r = n \) only for \( k = 0 \)). Then, \( \mathbf{u}_0 = \mathbf{0}_0 \) above. Also, since the \( f_i^* \)'s are \( L \)-smooth, using Lemma 1, we have that:

\[ \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \leq \sum_{i \in [n]} 2L [\mathbb{E}[f_i(\mathbf{w}_k)] - f_i^*] \leq 2n \mathbb{E}[f(\mathbf{w}_k)] - 2L \sum_{i \in [n]} f_i^* \leq 2n \mathbb{E}[f(\mathbf{w}_k)]. \]

The last step above follows because the \( f_i^* \)'s are non-negative. This trick allows us to circumvent the need for the bounded client dissimilarity assumption.

Using these in (10), we get:

\[ \mathbb{E}[f(\mathbf{w}_k)] \leq f(\mathbf{w}_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] + \frac{16\eta^2 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \]

\[ + \frac{5}{4\eta E \beta} \mathbb{E}[\|\mathbf{u}_0 - \mathbf{0}\|^2] + 160 \eta E \beta \left( \frac{q}{n^2} + \frac{(1+q)(n-r)}{r(n-1)} \right) \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2]. \]  

Using (11) recursively, we get:

\[ \sum_{k=0}^{k'-1} \mathbb{E}[f(\mathbf{w}_k)] \leq k' f(\mathbf{w}_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-2} (k' - 1 - k) \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] + \gamma \sum_{k=0}^{k'-2} (k' - 1 - k) \mathbb{E}[f(\mathbf{w}_k)] \]

\[ \leq k' f(\mathbf{w}_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] + \gamma k' \sum_{k=0}^{k'-1} \mathbb{E}[f(\mathbf{w}_k)]. \]  

Let us now ensure that \( \gamma k' \leq \frac{1}{2} \) for all \( k' \in \{1, \ldots, K\} \), in which case we can simplify (13) to:

\[ \sum_{k=0}^{k'-1} \mathbb{E}[f(\mathbf{w}_k)] \leq 2k' f(\mathbf{w}_0) - \frac{\eta E}{2} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] \leq 2k' f(\mathbf{w}_0). \]
Now:
\[
\gamma k' \leq \gamma K = 64\eta LE\left(\frac{q^2L^2E(\alpha E + 4)}{n} + 5\beta\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right). \tag{15}
\]

Now if we set \(8\eta LE^2 \leq 1\), then it can be verified that \(\beta = 160e^2(1 + q)\eta^2L^2E^2(E + 1)^2\) is a valid choice. Using this above, we get that:
\[
\gamma k' \leq \gamma K = 64\eta^3L^3E^3K\left(\frac{1}{n}\left(\alpha + \frac{4}{E}\right) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right). \tag{16}
\]

Setting \(\eta = \frac{1}{64L^1/3\left(\frac{1}{n}(\alpha + \frac{4}{E}) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)^{1/3}}\), we have \(A < \frac{1}{2}\). But we must also have
\[
8\eta LE^2 = \frac{4E}{3K^{1/3}\left(\frac{1}{n}(\alpha + \frac{4}{E}) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)^{1/3}} \leq 1. \tag{17}
\]

This holds for \(K^{1/3}(E + 1) \geq \frac{1}{1200e^2(1 + q)}\left(\frac{1}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\).

Further \(\beta\) must be smaller than 1, so
\[
\beta = 160e^2(1 + q)\eta^2L^2E^2(E + 1)^2 = \frac{160e^2(1 + q)(E + 1)^2}{36K^{2/3}\left(\frac{1}{n}(\alpha + \frac{4}{E}) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)^{2/3}} < 1. \tag{18}
\]

This holds for \(E + 1 \leq \frac{\sqrt[n]{(1 + q)(n-r)}}{\sqrt[4]{\eta n}}\).

Now using (14) in (11) with \(k' = K\) and our choice of \(\beta = 160e^2(1 + q)\eta^2L^2E^2(E + 1)^2\) and
\[
\eta = \frac{1}{64L^1/3\left(\frac{1}{n}(\alpha + \frac{4}{E}) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)^{1/3}}
\]
we get:
\[
E[f(w_{K})] \leq f(w_0) - \frac{\eta E}{4} \sum_{k=0}^{K-1} E[\|\nabla f(w_k)\|^2]
\]
\[+ 128\eta^3L^3E^3\left(\frac{1}{n}\left(\alpha + \frac{4}{E}\right) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)Kf(w_0). \tag{19}
\]

Rearranging the above a bit and using the fact that \(f(w_K) \geq 0\), we get:
\[
\frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla f(w_k)\|^2] \leq \frac{4f(w_0)}{\eta E K} + \frac{512\eta^3L^3E^3\left(\frac{1}{n}\left(\alpha + \frac{4}{E}\right) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)}{f(w_0)}. \tag{20}
\]

Substituting the value of \(\eta\) above, we get:
\[
\frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla f(w_k)\|^2] \leq \frac{39L\eta f(w_0)}{K^{2/3}}\left(\frac{1}{n}\left(\alpha + \frac{4}{E}\right) + 800e^2(1 + q)(E + 1)^2\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right)\right)^{1/3}. \tag{21}
\]

This concludes the proof.

\[\blacksquare\]

**Lemmas used in the proof of Theorem 1**

**Lemma 1.** Suppose \(4\eta LE^2 \leq 1\) and \(\beta \geq \frac{80e^2(1 + q)\eta^2L^2E^2(E + 1)^2}{(1 - 4\eta LE)}\). Then for any \(k' \in \{1, \ldots, K\}\), we have:
\[
E[f(w_{k'})] \leq f(w_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} E[\|\nabla f(w_k)\|^2] + \frac{16\eta^3L^3E^3\left(\alpha E + 4\right)}{n^2} \sum_{k=0}^{k'-1} \sum_{i \in [n]} E[\|\nabla f_i(w_k)\|^2]
\]
\[+ \frac{5}{4\eta E\beta} E[\|u_0 - \delta_0\|^2] + 160\eta E\beta\left(\frac{q}{n} + \frac{(1 + q)(n-r)}{r(n-1)}\right) \sum_{k=0}^{k'-1} \sum_{i \in [n]} E[\|\nabla f_i(w_k)\|^2].
\]

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Let us analyze (I*) first.

\[ \mathbb{E}[\nabla f(w_k), -u_k] = \mathbb{E}[\nabla f(w_k), -g(w_k; \mathcal{S}_k) - (1 - \beta)(u_{k-1} - \hat{g}(w_{k-1}; \mathcal{S}_k))] \]

\[ = \mathbb{E}[\nabla f(w_k), -g(w_k; \mathcal{S}_k) - (1 - \beta)\mathbb{E}[\nabla f(w_k), u_{k-1} - \hat{g}(w_{k-1}; \mathcal{S}_k)]] \]

\[ = \mathbb{E}[\nabla f(w_k), \frac{1}{n} \sum_{i \in [n]} (w_{i,k,E} - w_k)] + (1 - \beta)\mathbb{E}[\nabla f(w_k), u_{k-1} - \bar{\delta}_{k-1}] \]

(24) follows by taking expectation with respect to \( Q_D \). (III*) is obtained by taking expectation with respect to \( \mathcal{S}_k \) above. (IV*) is obtained by taking expectation with respect to \( \{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\}_{i=1}^{n} \) and \( \mathcal{S}_k \) above.

From Lemma 2, for \( \eta < \frac{1}{L} \) and \( E < \frac{1}{\eta} \min\left(\frac{1}{\eta L^2}, \frac{1}{\eta L^2} - \frac{1}{\eta L}\right) \), we can bound (III*) as:

\[ (\text{III*}) \leq -\frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{n}{2} (1 - \eta^2 L^2 E^2) \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{v}_{k,\tau}\|^2] + \frac{16\eta^3 L^2 E^2(\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]. \]

(25)

Note that for \( \eta L \ll 1 \) (which is going to be the case eventually), we can combine all the above constraints on \( \eta \) and \( E \) into \( 4\eta L E < 1 \).

As for (IV*):

\[ (\text{IV*}) \leq (1 - \beta)\mathbb{E}[\|\nabla f(w_k)\|\|u_{k-1} - \bar{\delta}_{k-1}\|] \]

\[ \leq (1 - \beta) \left( \frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{2(1 - \beta)\mathbb{E}[\|u_{k-1} - \bar{\delta}_{k-1}\|^2]}{\eta E} \right) \]

\[ = \frac{\eta E}{4} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{(1 - \beta)^2}{\eta E} \mathbb{E}[\|u_{k-1} - \bar{\delta}_{k-1}\|^2]. \]

(27)

above follows by the AM-GM inequality.

Adding (25) and (28), we get:

\[ (\text{I*}) \leq -\frac{\eta E}{4} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{n}{2} (1 - \eta^2 L^2 E^2) \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{v}_{k,\tau}\|^2] + \frac{(1 - \beta)^2}{\eta E} \mathbb{E}[\|u_{k-1} - \bar{\delta}_{k-1}\|^2] + \frac{16\eta^3 L^2 E^2(\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]. \]

(29)

Now, let us analyze (II*). We have:

\[ \mathbb{E}[\|u_k\|^2] \leq 2\mathbb{E}[\|\bar{\delta}_k\|^2] + 2\mathbb{E}[\|u_k - \bar{\delta}_k\|^2] \]

(30)
Notice that:

\[
\overline{\delta}_k = \mathbb{E}_{\{s_{k,1}^{(i)}, g_{k,E}^{(i)}\}_{i=1}^{E}} \left[ \frac{1}{n} \sum_{i \in [n]} (w_k - u_{k,E}^{(i)}) \right] = \mathbb{E}_{\{s_{k,1}^{(i)}, g_{k,E}^{(i)}\}_{i=1}^{E}} \left[ \sum_{\tau=0}^{E-1} \eta v_{k,\tau} \right].
\]  

(31)

Thus:

\[
\mathbb{E}[\|\overline{\delta}_k\|^2] \leq \eta^2 \mathbb{E} \left[ \sum_{\tau=0}^{E-1} \|v_{k,\tau}\|^2 \right] \leq E \eta^2 \mathbb{E} \left[ \sum_{\tau=0}^{E-1} \|v_{k,\tau}\|^2 \right].
\]  

(32)

The expectation above is with respect to all the randomness in the algorithm so far. Using (32) and the result of Lemma 8 in (30) with \(2\eta LE^2 \leq 1\), we have that:

\[
\mathbb{E}[\|u_k\|^2] \leq 2E\eta^2 \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}\|^2] + 2 \left\{ (1 - \beta)^2 \mathbb{E}[\|u_{k-1} - \overline{\delta}_{k-1}\|^2] + 2\beta^2 \mathbb{E}[\|g_Q(w_k; S_k) - \overline{\delta}_k\|^2] + 8\epsilon^2 (1 + q)(1 - \beta)^2 \eta^2 L^2 E^2 (E + 1)^2 \mathbb{E}[\|w_k - w_{k-1}\|^2] \right\}.
\]  

(33)

Recalling that \((\text{II}^*) = \frac{1}{8\eta E} \mathbb{E}[\|u_k\|^2]\), we get:

\[
(\text{II}^*) \leq \frac{\eta}{4} \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}\|^2] + \frac{1}{4\eta E} \left\{ (1 - \beta)^2 \mathbb{E}[\|u_{k-1} - \overline{\delta}_{k-1}\|^2] + 2\beta^2 \mathbb{E}[\|g_Q(w_k; S_k) - \overline{\delta}_k\|^2] + 8\epsilon^2 (1 + q)(1 - \beta)^2 \eta^2 L^2 E^2 (E + 1)^2 \mathbb{E}[\|w_k - w_{k-1}\|^2] \right\}.
\]  

(34)

Adding (29) and (34):

\[
(\text{I}^*) + (\text{II}^*) \leq -\frac{\eta}{4} \mathbb{E}[\|
abla f(w_k)\|^2] - \frac{\eta}{2} \left( 1 - \frac{\eta^2 L^2 E^2}{n^2} \right) \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}\|^2]
\]

\[
+ \frac{16\eta^3 L^2 E^2}{n^2} (\alpha E + 4) \sum_{i \in [n]} \mathbb{E}[\|
abla f_i(w_k)\|^2] + \frac{5(1 - \beta)^2}{4\eta E} \mathbb{E}[\|u_{k-1} - \overline{\delta}_{k-1}\|^2] \quad \text{from Lemma 8}
\]

\[
+ \frac{\beta^2}{2nE} \mathbb{E}[\|g_Q(w_k; S_k) - \overline{\delta}_k\|^2] + 2\epsilon^2 (1 + q)(1 - \beta)^2 \eta^2 L^2 E^2 (E + 1)^2 \mathbb{E}[\|w_k - w_{k-1}\|^2].
\]  

(35)

Therefore, using Lemma 8 recursively, we get:

\[
(\text{I}^*) + (\text{II}^*) \leq -\frac{\eta}{4} \mathbb{E}[\|
abla f(w_k)\|^2] + \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|
abla f_i(w_k)\|^2]
\]

\[
+ \frac{5(1 - \beta)^2}{4\eta E} \mathbb{E}[\|u_0 - \overline{\delta}_0\|^2] + \frac{5\beta^2}{2nE} \sum_{l=1}^{k} (1 - \beta)^{2(k-l)} \mathbb{E}[\|g_Q(w_l; S_l) - \overline{\delta}_l\|^2]
\]

\[
+ 10\epsilon^2 (1 + q) \eta L^2 E (E + 1)^2 \sum_{l=1}^{k} (1 - \beta)^{2(k-l+1)} \mathbb{E}[\|w_l - w_{l-1}\|^2].
\]  

(36)

Using Lemma 10, we get:

\[
(\text{V}^*) \leq 4\eta^2 E \left( \frac{q}{n^2} + \frac{(1 + q)}{r(n - 1)} \left( 1 - \frac{r}{n} \right) \right) \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2].
\]  

(37)
Putting this back in (36), we get:

\[(1^*) + (II^*) \leq -\frac{\eta E}{4} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{16\eta^3 L^2 E^2(\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]
\]
\[+ \frac{5(1 - \beta)^{2k}}{4\eta E} \mathbb{E}[\|u_0 - \delta_0\|^2] + 10\eta^2 \left(\frac{q}{n^2} + \frac{(1 + q)}{r(n - 1)} \left(1 - \frac{r}{n}\right)\right) \sum_{l=1}^{k} (1 - \beta)^{2(k-l)} \sum_{i \in [n]} \tau = 0 E^{-1} \mathbb{E}[\|v_{i,\tau}\|^2]
\]
\[+ 10e^2(1 + q)\eta L^2 E(1 + 2)^{k} \sum_{l=1}^{k} (1 - \beta)^{2(k-l+1)} \mathbb{E}[\|w_l - w_{l-1}\|^2]. \quad (38)
\]

Next, using (38) in (33), we get that:

\[\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta E}{4} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{16\eta^3 L^2 E^2(\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]
\]
\[+ \frac{5(1 - \beta)^{2k}}{4\eta E} \mathbb{E}[\|u_0 - \delta_0\|^2] + 10\eta^2 \left(\frac{q}{n^2} + \frac{(1 + q)}{r(n - 1)} \left(1 - \frac{r}{n}\right)\right) \sum_{l=1}^{k} (1 - \beta)^{2(k-l)} \sum_{i \in [n]} \tau = 0 E^{-1} \mathbb{E}[\|v_{i,\tau}\|^2]
\]
\[+ 10e^2(1 + q)\eta L^2 E(1 + 2)^{k} \sum_{l=1}^{k} (1 - \beta)^{2(k-l+1)} \mathbb{E}[\|w_l - w_{l-1}\|^2] - \left(\frac{1}{8\eta E} - \frac{L}{2}\right) \mathbb{E}[\|w_{k+1} - w_k\|^2]. \quad (39)
\]

Summing the above from \(k = 0\) through \((k' - 1)\) for any \(k' \in \{1, \ldots, K\}\), we get:

\[\mathbb{E}[f(w_{k'})] \leq f(w_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{16\eta^3 L^2 E^2(\alpha E + 4)}{n^2} \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]
\]
\[+ \sum_{l=0}^{\infty} \frac{5(1 - \beta)^{2l}}{4\eta E} \mathbb{E}[\|u_0 - \delta_0\|^2] + 10\eta^2 \left(\frac{q}{n^2} + \frac{(1 + q)}{r(n - 1)} \left(1 - \frac{r}{n}\right)\right) \sum_{l=0}^{\infty} \sum_{k=0}^{k'-1} \tau = 0 \sum_{i \in [n]} \mathbb{E}[\|v_{k,\tau}\|^2] \sum_{l=0}^{\infty} \sum_{l=0}^{\infty} (1 - \beta)^{2l}
\]
\[+ 10e^2(1 + q)\eta L^2 E(1 + 2)^{k-1} \sum_{k=1}^{k'-1} \mathbb{E}[\|w_k - w_{k-1}\|^2] - \frac{1}{8\eta E} \sum_{k=0}^{k'-1} \sum_{k=0}^{k'-1} \mathbb{E}[\|w_{k+1} - w_k\|^2]. \quad (40)
\]

Simplifying the above by noting that \(\sum_{l=0}^{\infty} (1 - \beta)^{2l} \leq \sum_{l=0}^{\infty} (1 - \beta)^{l} = 1/\beta\), we get:

\[\mathbb{E}[f(w_{k'})] \leq f(w_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{16\eta^3 L^2 E^2(\alpha E + 4)}{n^2} \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]
\]
\[+ \frac{5}{4\eta E}\mathbb{E}[\|u_0 - \delta_0\|^2] + 10\eta^2 \left(\frac{q}{n^2} \frac{(1 + q)}{r(n - 1)} \left(1 - \frac{r}{n}\right)\right) \sum_{k=0}^{k'-1} \sum_{i \in [n]} \tau = 0 \sum_{l=0}^{k'-1} \mathbb{E}[\|v_{k,\tau}\|^2]
\]
\[+ 10e^2(1 + q)\eta L^2 E(1 + 2)^{k-1} \sum_{k=1}^{k'-1} \mathbb{E}[\|w_k - w_{k-1}\|^2] - \frac{(1 - 4\eta LE)}{8\eta E} \sum_{k=0}^{k'-1} \sum_{k=0}^{k'-1} \mathbb{E}[\|w_{k+1} - w_k\|^2]. \quad (41)
\]

\[\text{(VI*)} - \text{want this to be } \leq 0 \]

We want (VI*) to be \(\leq 0\). For this, we must have:

\[\beta \geq \frac{80e^2(1 + q)\eta^2 L^2 E^2(1 + 2)^{k}}{(1 - 4\eta LE)}. \quad (42)
\]
Note that the denominator above is positive since we already have a constraint of $4\eta LE \leq 1$.

With $\beta$ satisfying the above constraint, and using the result of Lemma [5] for $\sum_{\tau=0}^{E-1} \mathbb{E}[||\mathbf{v}_{k,\tau}^{(i)}||^2]$, we get:

$$
\mathbb{E}[f(\mathbf{w}_k)] \leq f(\mathbf{w}_0) - \frac{\eta E}{4} \sum_{k=0}^{k'-1} \mathbb{E}[||\nabla f(\mathbf{w}_k)||^2] + \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[||\nabla f_i(\mathbf{w}_k)||^2] 
+ \frac{5}{4n\eta E\beta} \mathbb{E}[||\mathbf{u}_0 - \mathbf{\bar{u}}_0||^2] + 160\eta E\beta \left( \frac{\eta}{n^2} + \frac{(1 + q)}{r(n - 1)} \left( 1 - \frac{r}{n} \right) \right) \sum_{k=0}^{k'-1} \sum_{i \in [n]} \mathbb{E}[||\nabla f_i(\mathbf{w}_k)||^2].
$$

(43)

Finally, note that we have two constraints namely: $4\eta LE \leq 1$ and $2n^2 E^2 \leq 1$. We can merge these constraints into $4\eta LE^2 \leq 1$ for $E \geq 1$ (which is the case).

This gives us the desired result.

**Lemma 2.** For $\eta < \frac{1}{L}$ and $E < \frac{1}{4}\min\left(\frac{1}{nE}, \frac{1}{n^2 L^2} - \frac{1}{nE}\right)$, (III*) in the proof of Lemma 1 can be bounded as:

$$(III^*) = \mathbb{E}[||\nabla f(\mathbf{w}_k), 1/n \sum_{i \in [n]} (\mathbf{w}^{(i)}_{k,E} - \mathbf{w}_k)||] \leq -\frac{\eta E}{2} \mathbb{E}[||\nabla f(\mathbf{w}_k)||^2] - \frac{\eta E}{2} \left( 1 - \frac{\eta^2 L^2 E^2}{n^2} \right) \sum_{\tau=0}^{E-1} \mathbb{E}[||\mathbf{v}_{k,\tau}||^2] 
+ \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{i \in [n]} ||\nabla f_i(\mathbf{w}_k)||^2.
$$

Proof. (III*) $= \mathbb{E}[||\nabla f(\mathbf{w}_k), 1/n \sum_{i \in [n]} (\mathbf{w}^{(i)}_{k,E} - \mathbf{w}_k)||]$. Then:

\begin{align*}
(III^*) &= \mathbb{E}[||\nabla f(\mathbf{w}_k), \frac{1}{n} \sum_{i \in [n]} (\mathbf{w}^{(i)}_{k,E} - \mathbf{w}_k)||] \\
&= -\eta \sum_{\tau=0}^{E-1} \mathbb{E}[||\nabla f(\mathbf{w}_k), \frac{1}{n} \sum_{i \in [n]} \mathbf{v}^{(i)}_{k,\tau}||] \\
&= \sum_{\tau=0}^{E-1} \left\{ -\frac{\eta}{2} \mathbb{E}[||\nabla f(\mathbf{w}_k)||^2] - \frac{\eta}{2} \mathbb{E}[||\mathbf{v}_{k,\tau}||^2] + \frac{\eta}{2} \mathbb{E}[||\nabla f(\mathbf{w}_k) - \mathbf{v}_{k,\tau}||^2] \right\} \\
&\leq \sum_{\tau=0}^{E-1} \left\{ -\frac{\eta}{2} \mathbb{E}[||\nabla f(\mathbf{w}_k)||^2] - \frac{\eta}{2} \mathbb{E}[||\mathbf{v}_{k,\tau}||^2] + \eta \mathbb{E}[||\nabla f(\mathbf{w}_k) - \mathbf{v}_{k,\tau}||^2] + \eta \mathbb{E}[||\mathbf{v}_{k,\tau}||^2] \right\} \\
&\leq \sum_{\tau=0}^{E-1} \left\{ -\frac{\eta}{2} \mathbb{E}[||\nabla f(\mathbf{w}_k)||^2] - \frac{\eta}{2} \mathbb{E}[||\mathbf{v}_{k,\tau}||^2] + \eta L^2 \mathbb{E}[||\mathbf{w}_k - \mathbf{v}_{k,\tau}||^2] + \eta \mathbb{E}[||\nabla f(\mathbf{w}_k) - \mathbf{v}_{k,\tau}||^2] \right\} \\
&\leq \sum_{\tau=0}^{E-1} \left\{ -\frac{\eta}{2} \mathbb{E}[||\nabla f(\mathbf{w}_k)||^2] - \frac{\eta}{2} \mathbb{E}[||\mathbf{v}_{k,\tau}||^2] + \eta L^2 \mathbb{E}[||\mathbf{w}_k - \mathbf{v}_{k,\tau}||^2] + \eta \mathbb{E}[||\nabla f(\mathbf{w}_k) - \mathbf{v}_{k,\tau}||^2] \right\} \\
&(44)

doesn't\text{ strictly follow by using the fact that for any two vectors } \mathbf{a} \text{ and } \mathbf{b}, \langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{2}(||\mathbf{a}||^2 + ||\mathbf{b}||^2 - ||\mathbf{a} - \mathbf{b}||^2). \text{ Also, } (45) \text{ follows from the fact that for any two vectors } \mathbf{a} \text{ and } \mathbf{b}, ||\mathbf{a} + \mathbf{b}||^2 \leq 2 ||\mathbf{a}||^2 + 2 ||\mathbf{b}||^2.

Per definitions, observe that:

$$
\mathbf{w}_{k,\tau+1} = \mathbf{w}_{k,\tau} - \eta \mathbf{v}_{k,\tau}.
$$

(47)

From this, we have that $\mathbf{w}_k - \mathbf{w}_{k,\tau} = \eta \sum_{\tau=0}^{\tau-1} \mathbf{v}_{k,\tau}$. Hence, $||\mathbf{w}_k - \mathbf{w}_{k,\tau}||^2 = \eta^2 \sum_{\tau=0}^{\tau-1} ||\mathbf{v}_{k,\tau}||^2 \leq \eta^2 \sum_{\tau=0}^{\tau-1} ||\mathbf{v}_{k,\tau}||^2$—this follows from the fact that for any $p > 1$ vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}, ||\sum_{i=1}^p \mathbf{u}_i||^2 \leq \sum_{i=1}^p ||\mathbf{u}_i||^2$.
\( p \sum_{i=1}^{p} \|u_i\|^2 \). Using all this in (16), we get:

\[
\text{(III)} \leq -\frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] + \sum_{\tau=0}^{E-1} \left\{ -\frac{\eta}{2} \mathbb{E}[\|\nabla f(w_{k,\tau})\|^2] + \eta^3 L^2 \tau \sum_{\tau=0}^{\tau-1} \mathbb{E}[\|\nabla f(w_{k,\tau}) - \nabla f(w_k)\|^2] \right\} \]

\[
\leq -\frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{\eta}{2} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(w_{k,\tau})\|^2] + \frac{\eta^3 L^2 E^2}{2} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(w_{k,\tau}) - \nabla f(w_k)\|^2] + \eta \sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(w_{k,\tau}) - \nabla f(w_k)\|^2]
\]

from Lemma 3

(48)

Using Lemma 3 to bound the last term above gives us:

\[
\text{(III)} \leq -\frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{\eta}{2} \left( 1 - \eta^2 L^2 E^2 \right) \sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(w_{k,\tau})\|^2] + \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{i \in [n]} \|\nabla f_i(w_k)\|^2.
\]

This gives us the desired result.

Lemma 3. For \( \eta < \frac{1}{L} \) and \( E < \frac{4}{\eta L} \), we have:

\[
\sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(w_{k,\tau})\|^2] \leq \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{i \in [n]} \|\nabla f_i(w_k)\|^2,
\]

where the expectation is with respect to the randomness due to \( \{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\}_{i=1}^{n} \).

Proof. Let \( \bar{e}_{k,\tau} = \nabla f(w_{k,\tau}) - \nabla f(w_k) \). Then:

\[
\|\bar{e}_{k,\tau}\|^2 = \|\nabla f(w_{k,\tau})\|^2 = \|\nabla f(w_k)\|^2 + \frac{1}{n^2} \sum_{i \in [n]} \left( e_{k,\tau}^{(i)} - \nabla f_i(w_k) \right)^2
\]

\[
\leq \frac{2}{n^2} \sum_{i \in [n]} \|e_{k,\tau}^{(i)}\|^2 + \frac{2}{n^2} \sum_{i \in [n]} \tilde{e}_{k,\tau}^{(i)}
\]

(50)

So:

\[
\mathbb{E}[\|\bar{e}_{k,\tau}\|^2] \leq \frac{2}{n^2} \mathbb{E}\left[ \sum_{i \in [n]} \|e_{k,\tau}^{(i)}\|^2 \right] + \frac{2}{n^2} \mathbb{E}\left[ \sum_{i \in [n]} \tilde{e}_{k,\tau}^{(i)} \right]
\]

(51)

But:

\[
\mathbb{E}\left[ \sum_{i \in [n]} \|e_{k,\tau}^{(i)}\|^2 \right] = \sum_{i \in [n]} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] + \sum_{i \neq j \in [n]} \mathbb{E}[e_{k,\tau}^{(i)}, e_{k,\tau}^{(j)}]
\]

In the cross-term above, we can take expectations individually as \( \{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\} \) and \( \{B_{k,1}^{(j)}, \ldots, B_{k,E-1}^{(j)}\} \) are independent for \( i \neq j \). Next, from Lemma 1 \( \mathbb{E}[e_{k,\tau}^{(i)}] = 0 \ \forall \ i, k, \tau \). Hence:

\[
\mathbb{E}\left[ \sum_{i \in [n]} \|e_{k,\tau}^{(i)}\|^2 \right] = \sum_{i \in [n]} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2].
\]

Using the above result and Assumption 1 in (51), we get that:

\[
\mathbb{E}[\|\bar{e}_{k,\tau}\|^2] \leq \frac{2}{n^2} \sum_{i \in [n]} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] + \frac{2\alpha}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\tilde{e}_{k,\tau}^{(i)}\|^2].
\]

(52)
Now:

\[
E\left[\left\|\tilde{e}_{k,\tau}^{(i)}\right\|^2\right] = E[\|\nabla f_i(w_{k,\tau}) - \nabla f_i(w_{k,\tau})\|^2] \\
= L^2E[\|w_{k,\tau}^{(i)} - w_{k,\tau}\|^2] \\
\leq L^2E[\|w_{k,\tau}^{(i)} - \eta \sum_{t=0}^{\tau-1} v_{k,t}^{(i)} - (w_{k,0} - \eta \sum_{t=0}^{\tau-1} \tilde{w}_{k,t})\|^2]
\]

But since \(w_{k,0} = w_k\) \(\forall\ i\), we have \(w_{k,0} = w_k\). Hence:

\[
E\left[\left\|\tilde{e}_{k,\tau}^{(i)}\right\|^2\right] = \eta^2 L^2E[\sum_{t=0}^{\tau-1} v_{k,t} - \sum_{t=0}^{\tau-1} v_{k,t}^{(i)}]^2 \\
\leq \eta^2 L^2 \sum_{t=0}^{\tau-1} E[\|v_{k,t} - v_{k,t}^{(i)}\|^2] \\
= \eta^2 L^2 \sum_{t=0}^{\tau-1} E[\|v_{k,t}\|^2 + \|v_{k,t}^{(i)}\|^2 - 2\langle v_{k,t}, v_{k,t}^{(i)} \rangle]
\]

Substituting the above in (52), we get:

\[
E[\|\tilde{e}_{k,\tau}\|^2] \leq \frac{2}{n^2} \sum_{i \in [n]} E[\|e_{k,\tau}^{(i)}\|^2] + \frac{2\alpha}{n^2} \sum_{i \in [n]} \eta^2 L^2 \sum_{t=0}^{\tau-1} E[\|v_{k,t}\|^2 + \|v_{k,t}^{(i)}\|^2 - 2\langle v_{k,t}, v_{k,t}^{(i)} \rangle] \\
= \frac{2}{n^2} \sum_{i \in [n]} E[\|e_{k,\tau}^{(i)}\|^2] + \frac{2\alpha \eta^2 L^2}{n^2} \sum_{t=0}^{\tau-1} \sum_{i \in [n]} (E[\|v_{k,t}^{(i)}\|^2] - E[\|v_{k,t}\|^2]) \\
\leq \frac{2}{n^2} \sum_{i \in [n]} E[\|e_{k,\tau}^{(i)}\|^2] + \frac{2\alpha \eta^2 L^2}{n^2} \sum_{t=0}^{\tau-1} \sum_{i \in [n]} E[\|v_{k,t}^{(i)}\|^2]. 
\]

To get (54) from (53), we use the fact \(\sum_{i \in [n]} v_{k,t}^{(i)} = n\tilde{v}_{k,t}\). Now summing up (50) from \(\tau = 0\) through to \(\tau = E - 1\), we get:

\[
\sum_{\tau=0}^{E-1} E[\|\tilde{e}_{k,\tau}\|^2] \leq \frac{2}{n^2} \sum_{i \in [n]} E[\|e_{k,\tau}^{(i)}\|^2] + \frac{2\alpha \eta^2 L^2}{n^2} \sum_{i \in [n]} E[\|v_{k,\tau}^{(i)}\|^2]. 
\]

Now using Lemma 7 and Lemma 5 above with \(\eta < \frac{\lambda}{4}\) and \(E < \frac{1}{4}\min\left(\frac{1}{\eta L}, \frac{1}{\eta^2 L^2} - \frac{1}{\eta L}\right)\), we get:

\[
\sum_{\tau=0}^{E-1} E[\|\tilde{e}_{k,\tau}\|^2] \leq \frac{2}{n^2} \sum_{i \in [n]} 32E^2 \eta^2 L^2 \|\nabla f_i(w_k)\|^2 \\
+ \frac{\alpha \eta^2 L^2 E^2}{n^2} \sum_{i \in [n]} 16E \|\nabla f_i(w_k)\|^2.
\]

This gives us the desired result. ■

Lemma 4. \(E[e_{k,\tau}^{(i)}] = 0 \forall k \in \{0, \ldots, K - 1\}, \tau \in \{1, \ldots, E - 1\}\).
Note that in this lemma, the expectation is with respect to the randomness only due to \( \{B_{k,1}, \ldots, B_{k,\tau-1}\} \).

**Proof.** Note that:
\[
e_{k,0}^{(i)} = v_{k,0}^{(i)} - \nabla f_i(w_{k,0}^{(i)}) = 0.
\]

For \( \tau > 0 \):
\[
E_{B_{k,1}, \ldots, B_{k,\tau}}[e_{k,\tau}^{(i)}] = E_{B_{k,1}, \ldots, B_{k,\tau}}[v_{k,\tau}^{(i)} - \nabla f_i(w_{k,\tau}^{(i)})]
= E_{B_{k,1}, \ldots, B_{k,\tau}}[\nabla f_i(w_{k,\tau}^{(i)}; B_{k,\tau}^{(i)}) + (v_{k,\tau-1}^{(i)} - \nabla f_i(w_{k,\tau-1}^{(i)}; B_{k,\tau-1}^{(i)})) - \nabla f_i(w_{k,\tau}^{(i)}; B_{k,\tau}^{(i)})]
= E_{B_{k,1}, \ldots, B_{k,\tau-1}}[\nabla f_i(w_{k,\tau-1}^{(i)}; B_{k,\tau-1}^{(i)}) - \nabla f_i(w_{k,\tau-1}^{(i)}; B_{k,\tau}^{(i)})]
= E_{B_{k,1}, \ldots, B_{k,\tau-1}}[e_{k,\tau-1}^{(i)}].
\]

Doing this recursively, we get:
\[
E_{B_{k,1}, \ldots, B_{k,\tau}}[e_{k,\tau}^{(i)}] = e_{k,0}^{(i)} = 0.
\] \( (57) \)

Note that this result also holds if we use full gradients at \( \tau = 0 \).

**Lemma 5.** For \( \eta < \frac{1}{L} \) and \( E < \frac{1}{4} \min \left( \frac{1}{\eta L}, \frac{1}{\eta^2 E} - \frac{1}{\eta L} \right) \), we have:
\[
\sum_{\tau=0}^{E-1} E[\|v_{k,\tau}^{(i)}\|^2] \leq 16E\|\nabla f_i(w_k)\|^2.
\]

Note that in this lemma, the expectation is with respect to the randomness only due to \( \{B_{k,1}, \ldots, B_{k,E-1}\} \).

**Proof.** First, recall that \( e_{k,\tau}^{(i)} = v_{k,\tau}^{(i)} - \nabla f_i(w_{k,\tau}^{(i)}) \). Note that \( e_{k,0}^{(i)} = 0 \), as we are using clients’ full gradients at \( \tau = 0 \). We have:
\[
E[\|v_{k,\tau}^{(i)}\|^2] \leq 2E[\|e_{k,\tau}^{(i)}\|^2] + 2E[\|\nabla f_i(w_{k,\tau}^{(i)})\|^2].
\] \( (58) \)

Using Lemma 2.1 of [23] with \( \beta = 0 \), we have:
\[
E[\|e_{k,\tau}^{(i)}\|^2] \leq E[\|e_{k,\tau}^{(i)}\|^2] + 2L^2 \sum_{t=0}^{\tau-1} E[\|w_{k,t}^{(i)} - w_{k,t+1}^{(i)}\|^2]
\leq 2L^2 \sum_{t=0}^{\tau-1} E[\|w_{k,t}^{(i)} - w_{k,t+1}^{(i)}\|^2].
\] \( (59) \)

The last step follows because \( e_{k,\tau}^{(i)} = 0 \).

Summing the above from \( \tau = 0 \) through to \( E - 1 \), we get:
\[
\sum_{\tau=0}^{E-1} E[\|e_{k,\tau}^{(i)}\|^2] \leq 2L^2 \sum_{t=0}^{E-1} \sum_{\tau=t}^{E-1} E[\|w_{k,t}^{(i)} - w_{k,t+1}^{(i)}\|^2]
\leq 2E L^2 \sum_{\tau=0}^{E-1} E[\|w_{k,\tau}^{(i)} - w_{k,\tau+1}^{(i)}\|^2].
\] \( (60) \)

Next, re-arranging equation (11) in Lemma 2.2 of [23] (observe that in our case, \( G_\eta(.) \) is simply the gradient), we get:
\[
E[\|\nabla f_i(w_{k,\tau}^{(i)})\|^2] \leq \frac{2}{\eta} E[f_i(w_{k,\tau}^{(i)}) - f_i(w_{k,\tau+1}^{(i)})] - \frac{1}{\eta^2} (1 - \eta L) E[\|w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)}\|^2] + E[\|e_{k,\tau}^{(i)}\|^2].
\] \( (61) \)
Summing (61) from $\tau = 0$ to $E - 1$ and using (60), we get:

$$
\sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f_i(w_{k,\tau}^{(i)})\|^2] \leq \frac{2}{\eta} (f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})]) - \frac{(1-\eta L)}{\eta^2} \sum_{\tau=0}^{E-1} \mathbb{E}[\|w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)}\|^2] + 2EL^2 \sum_{\tau=0}^{E-2} \mathbb{E}[\|w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)}\|^2].
$$

(62)

Next, summing (60) and (62) gives us:

$$
\sum_{\tau=0}^{E-1} \{\mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] + \mathbb{E}[\|\nabla f_i(w_{k,\tau}^{(i)})\|^2]\} \leq \frac{2}{\eta} (f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})]) - \frac{(1-\eta L)}{\eta^2} \sum_{\tau=0}^{E-1} \mathbb{E}[\|w_{k,\tau}^{(i)} - w_{k,E}^{(i)}\|^2] + 2EL^2 \sum_{\tau=0}^{E-2} \mathbb{E}[\|w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)}\|^2].
$$

(63)

So if we have $\eta < \frac{1}{L}$ and $E < \frac{1}{4\eta^2 L^2} - \frac{1}{\eta L}$, we get:

$$
\sum_{\tau=0}^{E-1} \{\mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] + \mathbb{E}[\|\nabla f_i(w_{k,\tau}^{(i)})\|^2]\} \leq \frac{2}{\eta} (f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})]) - \frac{(1-\eta L)}{\eta^2} \sum_{\tau=0}^{E-1} \mathbb{E}[\|w_{k,\tau}^{(i)} - w_{k,E}^{(i)}\|^2] + 2EL^2 \sum_{\tau=0}^{E-2} \mathbb{E}[\|w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)}\|^2].
$$

(64)

Now from Lemma 6 for $E < \frac{1}{4\min\left(\frac{1}{\eta L}, \frac{1}{\eta^2 L^2} - \frac{1}{\eta L}\right)}$, we have that:

$$
f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})] \leq 4\eta E \|\nabla f_i(w_k)\|^2.
$$

(65)

Putting (65) in (64) and then using it (58) gives us the desired result.

**Lemma 6.** For $\eta < \frac{1}{L}$ and $E < \frac{1}{4\min\left(\frac{1}{\eta L}, \frac{1}{\eta^2 L^2} - \frac{1}{\eta L}\right)}$, we have:

$$
f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})] \leq 4\eta E \|\nabla f_i(w_k)\|^2.
$$

The expectation above is with respect to the randomness only due to $\{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\}_{i=1}^n$.

**Proof.** By $L$-smoothness of each $f_i$, we have:

$$
f_i(w_{k,E}^{(i)}) \geq f_i(w_k) + \langle \nabla f_i(w_k), w_{k,E}^{(i)} - w_k \rangle - \frac{L}{2} \|w_{k,E}^{(i)} - w_k\|^2.
$$

$$
\Rightarrow f_i(w_k) - f_i(w_{k,E}^{(i)}) \leq \langle \nabla f_i(w_k), w_k - w_{k,E}^{(i)} \rangle + \frac{L}{2} \|w_{k,E}^{(i)} - w_k\|^2
$$

$$
\leq \alpha' \|\nabla f_i(w_k)\|^2 + \frac{1}{2\alpha'} \|w_{k,E}^{(i)} - w_k\|^2 + \frac{L}{2} \|w_{k,E}^{(i)} - w_k\|^2 \text{ for } \alpha' > 0.
$$

follows by Young’s inequality

Recall that $w_{k,E}^{(i)} - w_k = \eta\sum_{\tau=0}^{E-1} v_{k,\tau}^{(i)}$. Hence taking expectation above with $\alpha' = 2\eta E$, we get that:

$$
f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})] \leq \eta E \|\nabla f_i(w_k)\|^2 + \eta^2 E \left(\frac{1}{4\eta E} + \frac{L}{2}\right) \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2]
$$

(66)

$$
\leq \eta E \|\nabla f_i(w_k)\|^2 + \frac{3\eta}{8} \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2].
$$

(67)
(67) follows from the fact that \( \eta LE < \frac{1}{4} \). Next, from the proof of Lemma 5, for \( E < \frac{(1 - \eta L)}{4\eta^2L^2} \):
\[
\sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2] \leq \frac{2}{\eta}(f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})]).
\]
Putting this in (67), we get:
\[
f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})] \leq \eta E \|\nabla f_i(w_k)\|^2 + \frac{3}{4}\big(f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})]\big).
\]
\[
\implies f_i(w_k) - \mathbb{E}[f_i(w_{k,E}^{(i)})] \leq 4\eta E \|\nabla f_i(w_k)\|^2.
\]

(68)

Lemma 7. For \( \eta < \frac{1}{4} \) and \( E < \frac{1}{4}\min\left(\frac{1}{\eta L}, \frac{1}{4\eta^2 L^2} - \frac{1}{\eta L}\right) \), we have:
\[
\sum_{\tau=0}^{E-1} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] \leq 32\eta^2 L^2 \|\nabla f_i(w_k)\|^2.
\]
The expectation above is with respect to the randomness only due to \( \{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\}_{i=1}^n \).

Proof. Note that in Lemma 5, we have already bounded \( \sum_{\tau=0}^{E-1} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] \) (see 60) — but here we expand it more for use in Lemma 3. First, from (60), we have:
\[
\sum_{\tau=0}^{E-1} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] \leq 2EL^2 \sum_{\tau=0}^{E-2} \mathbb{E}[\|w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)}\|^2].
\]
Next, using the fact that \( w_{k,\tau+1}^{(i)} = w_{k,\tau}^{(i)} - \eta v_{k,\tau}^{(i)} \), we get:
\[
\sum_{\tau=0}^{E-1} \mathbb{E}[\|e_{k,\tau}^{(i)}\|^2] \leq 2E\eta^2 L^2 \sum_{\tau=0}^{E-2} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2] \leq 2E\eta^2 L^2 \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2] \leq 2E\eta^2 L^2 (16E \|\nabla f_i(w_k)\|^2).
\]
This gives us the desired result.

Lemma 8. Suppose \( 2\eta LE^2 \leq 1 \). Then:
\[
\mathbb{E}[\|u_k - \bar{\delta}_k\|^2] \leq (1 - \beta)^2\mathbb{E}[\|u_{k-1} - \bar{\delta}_{k-1}\|^2] + 2\beta^2\mathbb{E}[\|\hat{g}(w_k; S_k) - \bar{\delta}_k\|^2]
\]
\[
+ 8\epsilon^2(1 + q)(1 - \beta)^2\eta^2 L^2 E^2 (E + 1)^2 \mathbb{E}[\|w_k - w_{k-1}\|^2].
\]

Proof. First, note that for each \( i \in [n] \), \( \mathbb{E}_{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}}[\hat{g}(w_k; S_k)] = \bar{\delta}_k \). So:
\[
\mathbb{E}_{s_k, \{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\}_{i=1}^n}[g(w_k; S_k)] = \bar{\delta}_k.
\]
(69)

Similarly, for each \( i \in [n] \), \( \mathbb{E}_{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}}[w_{k-1} - w_{k-1,E}^{(i)}] = \delta_{k-1}^{(i)} \). Hence:
\[
\mathbb{E}_{s_k, \{B_{k,1}^{(i)}, \ldots, B_{k,E-1}^{(i)}\}_{i=1}^n}[\hat{g}(w_{k-1}; S_k)] = \bar{\delta}_{k-1}.
\]
(70)
We have:

\[
E[\|\mathbf{u}_k - \mathbf{\overline{f}}_k\|^2] = E[\|\beta g_Q(\mathbf{w}_k; S_k) + (1 - \beta)\mathbf{u}_{k-1} + (1 - \beta)\Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k) - \mathbf{\overline{f}}_k\|^2]
\]

\[
= E[(1 - \beta)(\mathbf{u}_{k-1} - \mathbf{\overline{f}}_{k-1}) + \beta g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k + (1 - \beta)(\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k))]^2
\]

\[
= (1 - \beta)^2 E[\|\mathbf{u}_{k-1} - \mathbf{\overline{f}}_{k-1}\|^2] + E[\|\beta g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k + (1 - \beta)(\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k))\|^2]
\]  

(71)

The cross-term in (71) vanishes by taking expectation with respect to \(Q_D\) and \(S_k\). Next:

\[
E[\|\beta g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k + (1 - \beta)(\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k))\|^2]
\]

\[
= E[\|\beta g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k + (1 - \beta)(\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k)) - \mathbf{\overline{f}}_k\|^2]
\]

\[
\leq 2\beta^2 E[\|g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k\|^2] + 2(1 - \beta)^2 E[\|\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k) - \mathbf{\overline{f}}_k\|^2]
\]  

(72)

Next, note that:

\[
E[\|\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k) - \mathbf{\overline{f}}_k\|^2]
\]

\[
= E[\|\Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k)\|^2] + E[\|\mathbf{\overline{f}}_k - \mathbf{\overline{f}}_{k-1}\|^2] - 2E[\|\Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k), \mathbf{\overline{f}}_k - \mathbf{\overline{f}}_{k-1}\|]
\]

\[
= E[\|\Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k)\|^2] + E[\|\mathbf{\overline{f}}_k - \mathbf{\overline{f}}_{k-1}\|^2] - 2E[\|\mathbf{\overline{f}}_k - \mathbf{\overline{f}}_{k-1}\|]
\]  

(73)

(74)

Follows by first taking expectation with respect to \(Q_D\) and then using (69) and (70). Further:

\[
E[\|\Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k)\|^2] = E\left[\left\|\frac{1}{r} \sum_{i \in S_k} Q_D((\mathbf{w}_k - \mathbf{w}_{k,E}^{(i)})) - (\mathbf{w}_{k-1} - \mathbf{\hat{w}}_{k-1,E}^{(i)})\right\|^2\right]
\]

\[
\leq E_{S_k}\left[\frac{1}{r} \sum_{i \in S_k} E\left[\|Q_D((\mathbf{w}_k - \mathbf{w}_{k,E}^{(i)})) - (\mathbf{w}_{k-1} - \mathbf{\hat{w}}_{k-1,E}^{(i)})\|^2\right]\right]
\]

\[
\leq E_{S_k}\left[\frac{1}{r} \sum_{i \in S_k} (1 + q)E\left[\|\mathbf{w}_k - \mathbf{w}_{k,E}^{(i)}\| - (\mathbf{w}_{k-1} - \mathbf{\hat{w}}_{k-1,E}^{(i)})\|^2\right]\right]
\]  

(75)

(76)

Follows from Assumption B on the variance of \(Q_D\). Further, using Lemma 9, we get

\[
E[\|\mathbf{w}_k - \mathbf{w}_{k,E}^{(i)}\| - (\mathbf{w}_{k-1} - \mathbf{\hat{w}}_{k-1,E}^{(i)})\|^2] \leq 4e^2(1 + q)E[\|\mathbf{w}_k - \mathbf{w}_{k-1}\|^2],
\]  

(77)

for \(2\eta L^E \leq 1\).

Using this in (76):

\[
E[\|\Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k)\|^2] \leq 4e^2(1 + q)E[\|\mathbf{w}_k - \mathbf{w}_{k-1}\|^2].
\]  

(78)

Now using (78) in (74) and then using it in (72), we get:

\[
E[\|\beta g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k + (1 - \beta)(\mathbf{\overline{f}}_{k-1} + \Delta g_Q(\mathbf{w}_k, \mathbf{w}_{k-1}; S_k))\|^2]
\]

\[
\leq 2\beta^2 E[\|g_Q(\mathbf{w}_k; S_k) - \mathbf{\overline{f}}_k\|^2] + 8e^2(1 + q)(1 - \beta)^2\eta^2 L^E + 2E[\|\mathbf{w}_k - \mathbf{w}_{k-1}\|^2].
\]  

(79)

Finally, putting (79) back in (71) gives us the desired result. ■

**Lemma 9.** Suppose \(2\eta L^E \leq 1\). Then \(\forall k \geq 0\) and \(i \in [n]\), we have:

\[
E[\|\mathbf{w}_k - \mathbf{w}_{k,E}^{(i)}\| - (\mathbf{w}_{k-1} - \mathbf{\hat{w}}_{k-1,E}^{(i)})]\| \leq 2e(\eta LE(E + 1))\|\mathbf{w}_k - \mathbf{w}_{k-1}\|.
\]  

36
Proof. We have for any \( i \in [n] \):
\[
\| (w_k - w_{k,E}^{(i)}) - (w_{k-1} - \hat{w}_{k-1,E}^{(i)}) \| = \left\| \sum_{\tau=0}^{E-1} \eta v_{k,\tau}^{(i)} - \sum_{\tau=0}^{E-1} \eta \hat{v}_{k-1,\tau}^{(i)} \right\| 
\leq \sum_{\tau=0}^{E-1} \eta |v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)}|.
\]
(80)
The last step follows by the triangle inequality.

Next, we have:
\[
\| v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)} \| = \| \{ \nabla f_i(w_{k,\tau}; B_k) + (v_{k,\tau-1} - \hat{v}_{k-1,\tau}; B_{k,\tau}) \} 
- \{ \nabla f_i(\tilde{w}_{k-1,\tau}; B_k) + (\hat{v}_{k-1,\tau-1} - \hat{v}_{k-1,\tau}; B_{k,\tau}) \} \|
\]
Note that \( B_{k,\tau}^{(i)} \) can be the full batch too.

Re-arranging the above, using the triangle inequality and the smoothness of the stochastic gradients, we get:
\[
\| v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)} \| \leq L \| w_{k,\tau} - \hat{w}_{k-1,\tau} \| + \| v_{k,\tau-1} - \hat{v}_{k-1,\tau-1} \| + L \| w_{k,\tau-1} - \hat{w}_{k-1,\tau-1} \|. 
\]
(81)
Unfolding the above recursion, we get:
\[
\| v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)} \| \leq 2L \sum_{t=0}^{\tau} \| w_{k,t} - \hat{w}_{k-1,t} \|.
\]
(82)
Just as a sanity check for \([82]\), observe that \( \| v_{k,0}^{(i)} - \hat{v}_{k-1,0}^{(i)} \| = \| \nabla f_i(w_k) - \nabla f_i(w_{k-1}) \| \leq L \| w_k - w_{k-1} \| \). Next:
\[
\| w_{k,\tau+1}^{(i)} - \hat{w}_{k-1,\tau+1}^{(i)} \| = \| w_{k,\tau}^{(i)} - \hat{w}_{k-1,\tau}^{(i)} - \eta(v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)}) \|
\leq \| w_{k,\tau}^{(i)} - \hat{w}_{k-1,\tau}^{(i)} \| + \eta \| v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)} \|
\leq \| w_{k,\tau}^{(i)} - \hat{w}_{k-1,\tau}^{(i)} \| + 2\eta L \sum_{t=0}^{\tau} \| w_{k,t}^{(i)} - \hat{w}_{k-1,t}^{(i)} \|.
\]
The last step follows by using \([82]\). Thus:
\[
\| w_{k,\tau}^{(i)} - \hat{w}_{k-1,\tau}^{(i)} \| \leq \| w_{k,\tau-1}^{(i)} - \hat{w}_{k-1,\tau-1}^{(i)} \| + 2\eta L \sum_{t=0}^{\tau-1} \| w_{k,t}^{(i)} - \hat{w}_{k-1,t}^{(i)} \|. 
\]
(83)
Based on \([83]\), we claim that:
\[
\| w_{k,\tau}^{(i)} - \hat{w}_{k-1,\tau}^{(i)} \| \leq (1 + 2\eta LE)^\tau \| w_k - w_{k-1} \|. 
\]
(84)
We prove this by induction. Let us first examine the base case of \( \tau = 1 \). We have:
\[
\| w_{k,1}^{(i)} - \hat{w}_{k-1,1}^{(i)} \| = \| w_k - w_{k-1} - \eta(v_{k,0}^{(i)} - \hat{v}_{k-1,0}^{(i)}) \|
\leq \| w_k - w_{k-1} \| + \eta L \| w_k - w_{k-1} \| 
\leq (1 + 2\eta LE) \| w_k - w_{k-1} \|.
\]
For ease of notation, let us define $d_k \triangleq \|w_k - w_{k-1}\|$. Now suppose the claim is true for $\tau \leq t$. Then using (83), we have for $\tau = t + 1$:

$$
\|w_{k,t+1}^{(i)} - \hat{w}_{k-1,t+1}^{(i)}\| \leq \left\{ (1 + 2\eta L)^t + 2\eta L \sum_{t_z=0}^t (1 + 2\eta L)^{t_z}\right\} d_k
$$

$$
\leq \left\{ (1 + 2\eta L)^t + 2\eta L(t + 1)(1 + 2\eta L)^t\right\} d_k
$$

$$
\leq (1 + 2\eta L)^t(1 + 2\eta L(t + 1))d_k \leq (1 + 2\eta L)^{t+1}d_k.
$$

(85)

This proves our claim.

Now, using our claim, i.e., (84) in (82), we get:

$$
\|v_{k,\tau}^{(i)} - \hat{v}_{k-1,\tau}^{(i)}\| \leq 2L \sum_{t=0}^\tau (1 + 2\eta L)^t \|w_k - w_{k-1}\| \leq 2L(\tau + 1)(1 + 2\eta L)^\tau \|w_k - w_{k-1}\|
$$

(86)

Note that this bound is independent of $i$.

Finally, using (86) in (80), we get:

$$
\|(w_k - w_{k,E}^{(i)}) - (w_{k-1} - \hat{w}_{k-1,E}^{(i)})\| \leq \sum_{\tau=0}^{E-1} \sum_{\tau=0}^{E-1} 2\eta L(\tau + 1)(1 + 2\eta L)^\tau \|w_k - w_{k-1}\|
$$

$$
\leq 2\eta LE(E + 1)(1 + 2\eta L)^E \|w_k - w_{k-1}\|
$$

$$
\leq 2\eta LE(E + 1)e^{2\eta L^2} \|w_k - w_{k-1}\|.
$$

(87)

The last step follows from the fact that $1 + z \leq e^z \forall z$.

Finally, setting $2\eta L E^2 \leq 1$ gives us the desired result.

**Lemma 10.** $(V^*)$ in the proof of Lemma 4 can be bounded as:

$$(V^*) \leq 4\eta^2 E\left(\frac{q}{n^2} + \frac{(1 + q)}{r(n - 1)} \left(1 - \frac{r}{n}\right)\right) \sum_{i\in[n]} \sum_{\tau=0}^{E-1} E[\|v_{k,\tau}^{(i)}\|^2].$$

**Proof.** We have $(V^*) = \mathbb{E}[\|g_Q(w_l; S_l) - \overline{S}_l\|^2]$. Note that:

$$
\mathbb{E}[\|g_Q(w_l; S_l) - \overline{S}_l\|^2] \leq \eta^2 \mathbb{E}\left[\left\|\frac{1}{r} \sum_{i\in[S_l]} Q_D(w_l - w_{l,E}^{(i)}) \right\|^2 - \frac{1}{n} \sum_{i\in[n]} Q_D(w_l - w_{l,E}^{(i)}) \right]
$$

(A)

$$
+ \eta^2 \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i\in[n]} \left\{ \frac{Q_D(w_l - w_{l,E}^{(i)})}{\eta} - \frac{(w_l - w_{l,E}^{(i)})}{\eta} \right\} \right\|^2
$$

(B)

In (A), we take expectation with respect to $S_k$ and $Q_D(.)$ – for that, we use Lemma 4 of [29]. Note that $x_{k,\tau}^{(i)} - x_k$ in their lemma corresponds to $(w_{k,E}^{(i)} - w_k)$ in our case. Specifically, using
eqn. (59) and (60) in [29] (they also have Assumption [3], we get:

\[
(A) \leq \frac{1}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) \sum_{i \in [n]} \mathbb{E}\left[\|\mathbf{w}_{k,E}^{(i)} - \mathbf{w}_k\|^2\right] \\
= \frac{1}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) \sum_{i \in [n]} \mathbb{E}\left[\|\sum_{\tau=0}^{E-1} \eta \mathbf{v}_{k,r}^{(i)}\|^2\right] \\
\leq \frac{\eta^2}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) E \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}\left[\|\mathbf{v}_{k,r}^{(i)}\|^2\right]
\]

Next, we deal with (B). We have:

\[
(B) = \mathbb{E}\left[\mathbb{E}_{Q_i} \left[\left\| \frac{1}{n} \sum_{i \in [n]} \left\{ Q_i \left(\mathbf{w}_{k,E}^{(i)} - \mathbf{w}_k\right) - \left(\mathbf{w}_{k,E}^{(i)} - \mathbf{w}_k\right)\right\}\right\|^2\right]\right] \\
\leq \frac{q}{n^2} \sum_{i \in [n]} \mathbb{E}\left[\left\| \mathbf{w}_{k,E}^{(i)} - \mathbf{w}_k\right\|^2\right] \\
\leq \frac{qE\eta^2}{n^2} \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{v}_{k,r}^{(i)}\|^2].
\]

Now using (89) and (90) in (88), we get:

\[
(V^*) \leq \frac{\eta^2}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) E \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{v}_{k,r}^{(i)}\|^2] + \frac{r^2qE}{n^2} \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{v}_{k,r}^{(i)}\|^2] \\
\leq 4\eta^2E \left(\frac{q}{n^2} + \frac{(1+q)}{r(n-1)} \left(1 - \frac{r}{n}\right)\right) \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{v}_{k,r}^{(i)}\|^2].
\]

This gives us the desired result.

\[\square\]

**Lemma 11.** For any \(L\)-smooth function \(h(x)\), we have \(\forall x:\)

\[
\|\nabla h(x)\|^2 \leq 2L(h(x) - h^*) \text{ where } h^* = \min_x h(x).
\]

**Proof.** For any \(y\), we have that:

\[
h^* \leq h(y) \leq h(x) + \langle \nabla h(x), y - x \rangle + \frac{L}{2} \|y - x\|^2 \tag{92}
\]

Setting \(\nabla h(x) = 0\), we get that \(\hat{y} = x - \frac{1}{L} \nabla h(x)\) is the minimizer of \(h_2(y)\) (which is a quadratic with respect to \(y\)). Plugging this back in (92) gives us:

\[
h^* \leq h(x) + \left\langle \nabla h(x), -\frac{1}{L} \nabla h(x) \right\rangle + \frac{L}{2} \left\| -\frac{1}{L} \nabla h(x) \right\|^2 = h(x) - \frac{1}{2L} \|\nabla h(x)\|^2.
\]

This gives us the desired result. \[\square\]
Let us denote

\begin{align*}
\overline{w}_{k,r} & \triangleq \frac{1}{m} \sum_{i \in [n]} w^{(i)}_{k,r} \quad \text{and} \quad \overline{v}_{k,r} \triangleq \frac{1}{m} \sum_{i \in [n]} v^{(i)}_{k,r} \\
e^{(i)}_{k,r} & \triangleq u^{(i)}_{k,r} - \nabla f_i(w^{(i)}_{k,r}) \quad \text{and} \quad e^{(i)}_{k,r} \triangleq \nabla f_i(w^{(i)}_{k,r}) - \nabla f_i(\overline{w}_{k,r})
\end{align*}

**Proof of Theorem 2**

Proof. Let us set \( \eta_k = \eta \).

Using Lemma 12 with \( \eta < \frac{1}{4} \) and \( E < \frac{1}{4} \min \left( \frac{1}{\eta L}, \frac{1}{\eta L^2} - \frac{1}{\eta L} \right) \):

\[
\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{\eta}{2} \left( 1 - \eta^2 L^2 E^2 - \eta LE \right) \sum_{r=0}^{E-1} \mathbb{E}[\|\overline{v}_{k,r}\|^2] \\
+ 16\eta LE^2 \left\{ \eta^2 L (\alpha E + 4/n^2) \right\} + \frac{\eta}{2} \left( \frac{q}{n^2} + 4(1 + q)(n - 1)(1 - \frac{r}{n}) \right) \} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \tag{94}
\]

Note here that for \( \eta < \frac{1}{4} \), \( \frac{1}{\eta L} < \frac{1}{\eta L^2} - \frac{1}{\eta L} \) and so \( E < \frac{1}{4} \) or \( \eta LE < \frac{1}{4} \). Since \( E > 1 \), we are just left with \( \eta LE < \frac{1}{4} \).

Next, we circumvent the need for the bounded client dissimilarity assumption by using the fact that each \( f_i \) is \( L \)-smooth and \( \|\nabla f_i(w_k)\|^2 \leq 2L(f_i(w_k) - f_i^*) \) using Lemma 11. Hence:

\[
\sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \leq 2L \sum_{i \in [n]} \mathbb{E}[(f_i(w_k) - f_i^*)] = 2nLE[(f(w_k) - f^* + \Delta^*)], \tag{95}
\]

where \( \Delta^* := f^* - \frac{1}{n} \sum_{i=1}^n f_i^* \). Using all this in \( (94) \), we get:

\[
\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{\eta}{2} \left( 1 - \eta^2 L^2 E^2 - \eta LE \right) \sum_{r=0}^{E-1} \mathbb{E}[\|\overline{v}_{k,r}\|^2] \\
+ 32\eta LE^2 \left\{ \eta^2 L \left( \frac{\alpha E + 4}{n} \right) \right\} + \frac{\eta}{2} \left( \frac{q}{n^2} + 4(1 + q)(n - 1)(1 - \frac{r}{n}) \right) \} \mathbb{E}[(f(w_k) - f^* + \Delta^*)] \tag{96}
\]

Note that \( (1 - \eta^2 L^2 E^2 - \eta LE) > \frac{1}{16} \) for \( \eta LE < \frac{1}{4} \). Further, \( -f^* + \Delta^* = -f^* + f^* - \frac{1}{n} \sum_{i=1}^n f_i^* = -\frac{1}{n} \sum_{i=1}^n f_i^* \); hence, we can ignore the corresponding term when the \( f_i^* \)'s are non-negative.

Re-writing the above equation, we get:

\[
\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] + 32\eta LE^2 \left\{ \frac{\eta^2 LE}{n} \left( \alpha + \frac{4}{E} \right) + \eta B \frac{E}{2} \right\} \mathbb{E}[f(w_k)] \\
\leq \mathbb{E}[f(w_k)] \left\{ 1 + \frac{32\eta^3 LE^3}{n} \left( \alpha + \frac{4}{E} \right) + 16B\eta^2 L^2 E^2 \right\} - \frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] \tag{97}
\]

Let us denote \( \frac{32\eta^3 LE^3}{n} \left( \alpha + \frac{4}{E} \right) + 16B\eta^2 L^2 E^2 \) as \( \zeta \) for brevity.

Unfolding the above recursion from \( k = 0 \) through \( K - 1 \), we get:

\[
\mathbb{E}[f(w_K)] \leq f(w_0)(1 + \zeta)^K - \frac{\eta E}{2} \sum_{k=0}^{K-1} (1 + \zeta)^{(K-1-k)} \mathbb{E}[\|\nabla f(w_k)\|^2]. \tag{98}
\]
Re-arranging the above, we get:
\[
\sum_{k=0}^{K-1} p_k E[\|\nabla f(w_k)\|^2] \leq \frac{2 f(w_0)(1 + \zeta)^K}{\eta E} \frac{\sum_{k=0}^{K-1} (1 + \zeta)^k}{\sum_{k=0}^{K-1} (1 + \zeta)^k}, \quad \text{where } p_k = \frac{(1 + \zeta)^{K-1-k}}{\sum_{k=0}^{K-1} (1 + \zeta)^k}.
\]
(99)

Notice that \(p_k\) defines a distribution over \(k\). Hence, the LHS is \(E_{k \sim P(k)}[E[\|\nabla f(w_k)\|^2]]\) with \(P(k) = p_k\). Incorporating this and simplifying further, we get:
\[
E_{k \sim P(k)}[E[\|\nabla f(w_k)\|^2]] \leq 2 \frac{f(w_0)\zeta}{\eta E} \left\{ \frac{1}{1 - (1 + \zeta) - K} \right\}, \quad \text{where } P(k) = \frac{(1 + \zeta)^{K-1-k}}{\sum_{k=0}^{K-1} (1 + \zeta)^k}.
\]
(100)

Also note that: \((1 + \zeta)^{-K} < 1 - \zeta + \zeta^2 K(1 + \zeta)^{K+1}/2 < 1 - \zeta + \zeta^2 K^2\). Hence, \(1 - (1 + \zeta)^{-K} > \zeta K(1 - K)\). Using this in (100), we have for \(\zeta K < 1:\)
\[
E_{k \sim P(k)}[E[\|\nabla f(w_k)\|^2]] \leq 2 \frac{f(w_0)}{\eta E K(1 - K)} \zeta, \quad \text{where } P(k) = \frac{(1 + \zeta)^{K-1-k}}{\sum_{k=0}^{K-1} (1 + \zeta)^k}.
\]
(101)

Plugging in the value of \(\zeta\) in (101), the denominator, \(d(\eta) = \eta E K \left(1 - 16\eta^2 L^2 E^2 \left(\frac{2nLE}{n} \left(\alpha + \frac{4}{E} \right) + B\right) K\right)\).

Before going ahead, we would like to highlight that the reason FedLOMO does not achieve the optimal rate is because \(B\) is a constant that is not \(O(\eta LE)\) in general; if we were to consider the special case of no compression and full-device participation (i.e., \(r = n\)), then \(B\) would be 0 which would allow FedLOMO to achieve the optimal rate.

Let us choose \(\eta = \frac{1}{8LE \sqrt{BK}}\). Note that:
\[
\eta LE \leq \frac{1}{4} \quad \text{for } K \geq \frac{1}{4B}.
\]
(102)

Thus, for sufficiently large \(K\), this choice of \(\eta\) is valid. Also:
\[
\zeta K = \frac{1}{4} + \frac{1}{16B^{1.5}\sqrt{K}} \left(\frac{1}{n} \left(\alpha + \frac{4}{E}\right)\right) < \frac{3}{4} \quad \text{for } K > \frac{1}{64B^{2.5}} \left(\frac{1}{n} \left(\alpha + \frac{4}{E}\right)\right).
\]
(103)

So for \(K > \frac{1}{64B^{2.5}} \left(\frac{1}{n} \left(\alpha + \frac{4}{E}\right)\right)\),
\[
d(\eta) = \eta E K \left(1 - \zeta K\right) \geq \frac{\sqrt{K}}{8L \sqrt{B}} \left(1 - \frac{3}{4}\right) = \frac{\sqrt{K}}{32L \sqrt{B}}.
\]
(104)

Plugging this in (101), we get:
\[
E_{k \sim P(k)}[E[\|\nabla f(w_k)\|^2]] \leq \frac{64E^2LF(w_0)}{K^{1/2}}, \quad \text{where } P(k) = \frac{(1 + \zeta)^{K-1-k}}{\sum_{k=0}^{K-1} (1 + \zeta)^k} \text{ for } k \in \{0, \ldots, K - 1\},
\]
\[
\zeta = \frac{1}{4K} + \frac{1}{16B^{1.5}K^{1.5}} \left(\frac{1}{n} \left(\alpha + \frac{4}{E}\right)\right) \text{ and } B = \frac{q}{n} + \frac{4(1 + q)(n - r)}{r(n - 1)}. \quad \text{(105)}
\]

This concludes the proof.

\[\quad\]

**Key lemma used in the proof of Theorem 2**

**Lemma 12.** For \(\eta_k = \eta\) where \(\eta < \frac{1}{L}\) and \(E < \frac{1}{4} \min \left(\frac{1}{\eta L}, \frac{1}{\eta L^2} - \frac{1}{\eta L}\right)\) in FedLOMO, we have:
\[
E[f(w_{k+1})] \leq E[f(w_k)] - \frac{\eta LE}{2} E[\|\nabla f(w_k)\|^2] - \frac{\eta}{2} \left(1 - \eta^2 L^2 E^2 - \eta LE\right) \sum_{\tau=0}^{E-1} E[\|w_{k+\tau}\|^2] + 16\eta LE^2 \left\{\frac{q}{n} \left(\alpha + \frac{4}{E}\right) + \frac{1}{2} \left(\frac{q}{n^2} + \frac{4(1 + q)(n - r)}{r(n - 1)} \left(1 - \frac{r}{n}\right)\right)\right\} \sum_{i \in [n]} E[\|\nabla f_i(w_k)\|^2].
\]

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Proof. By the $L$-smoothness of $f$, we have:

\[
\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] + \mathbb{E}\left[\left\langle \nabla f(w_k), \frac{1}{r} \sum_{i \in S_k} Q_D(w_{k,E}^{(i)} - w_k) \right\rangle\right] + \frac{L}{2} \mathbb{E}\left[\left\lVert \frac{1}{r} \sum_{i \in S_k} Q_D(w_{k,E}^{(i)} - w_k) \right\rVert^2\right]
\]

Let us analyze (I) first – taking expectation with respect to $S_k$ and $Q_D(.)$ (recall that $Q_D(.)$ is unbiased from Assumption 3), we get:

\[
(I) = \mathbb{E}\left[\langle \nabla f(w_k), \frac{1}{n} \sum_{i \in [n]} (w_{k,E}^{(i)} - w_k) \rangle\right]
\]

But this is the same as (III*) in the proof of Lemma 1 using Lemma 2 and Assumption 5 instead of Assumption 4 we get:

\[
(I) \leq -\frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] - \frac{\eta}{2} (1 - \eta^2 L^2 E^2) \sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f_i(w_k)\|^2] + \frac{16\eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2],
\]

when $\eta < \frac{1}{L}$ and $E < \frac{1}{4} \min\left(\frac{1}{\eta L}, \frac{1}{\eta^2 L^2} - \frac{1}{\eta L}\right)$.

Let us now analyze (II). Recall that:

\[
(II) = \frac{L}{2} \mathbb{E}\left[\left\lVert \frac{1}{r} \sum_{i \in S_k} Q_D(w_{k,E}^{(i)} - w_k) \right\rVert^2\right].
\]

Observe that:

\[
\mathbb{E}_{S_k}\left[\frac{1}{r} \sum_{i \in S_k} Q_D(w_{k,E}^{(i)} - w_k)\right] = \frac{1}{n} \sum_{i \in [n]} Q_D(w_{k,E}^{(i)} - w_k).
\]

Hence:

\[
(II) = \frac{L}{2} \mathbb{E}\left[\left\lVert \frac{1}{n} \sum_{i \in [n]} Q_D(w_{k,E}^{(i)} - w_k) \right\rVert^2\right] + \mathbb{E}\left[\left\lVert \frac{1}{r} \sum_{i \in S_k} Q_D(w_{k,E}^{(i)} - w_k) - \frac{1}{n} \sum_{i \in [n]} Q_D(w_{k,E}^{(i)} - w_k) \right\rVert^2\right].
\]

Note that in (III), the expectation is without $S_k$. In (IV), we take expectation with respect to $S_k$ and $Q_D(.)$ – for that, we use Lemma 4 of [29]. Note that $x_{k,r}^{(i)} - x_k$ in their lemma corresponds to $(w_{k,E}^{(i)} - w_k)$ in our case. Specifically, using eqn. (59) and (60) in [29] (they also have Assumption 5), we get:

\[
(IV) \leq \frac{1}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) \sum_{i \in [n]} \mathbb{E}[\|w_{k,E}^{(i)} - w_k\|^2]
\]

\[
= \frac{1}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) \sum_{i \in [n]} \mathbb{E}[\|\eta v_{k,r}^{(i)}\|^2]
\]

\[
\leq \frac{\eta^2}{r(n-1)} \left(1 - \frac{r}{n}\right) 4(1 + q) E \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,r}^{(i)}\|^2].
\]
Next, we deal with (III). Noting that \( \mathbb{E}_{Q_D} \left[ \frac{1}{n} \sum_{i \in [n]} Q_D(w_{k,E}^{(i)} - w_k) \right] = (w_{k,E} - w_k) \), we get:

\[
(III) = \mathbb{E}[\|w_{k,E} - w_k\|^2] + \mathbb{E}\left[ \mathbb{E}_{Q_D} \left[ \frac{1}{n} \sum_{i \in [n]} \{Q_D(w_{k,E}^{(i)} - w_k) - (w_{k,E}^{(i)} - w_k)\} \right]^2 \right]
\]

\[
\leq \mathbb{E}\left[ \left\| \sum_{\tau=0}^{E-1} \eta \bar{v}_{k,\tau} \right\|^2 \right] + \frac{q}{n^2} \sum_{i \in [n]} \mathbb{E}\left[ \|w_{k,E}^{(i)} - w_k\|^2 \right]
\]

\[
\leq \eta^2 E \sum_{\tau=0}^{E-1} \mathbb{E}[\|\bar{v}_{k,\tau}\|^2] + \frac{q E \eta^2}{n^2} \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2]
\]

Now, using (109) and (110) in (108) gives us:

\[
(II) \leq \frac{LE \eta^2}{2} \left\{ \sum_{\tau=0}^{E-1} \mathbb{E}[\|\bar{v}_{k,\tau}\|^2] + \left( \frac{q}{n^2} + \frac{4(1+q)}{r(n-1)} \left( 1 - \frac{r}{n} \right) \right) \sum_{i \in [n]} \sum_{\tau=0}^{E-1} \mathbb{E}[\|v_{k,\tau}^{(i)}\|^2] \right\}
\]

\[
\leq \frac{LE \eta^2}{2} \left\{ \sum_{\tau=0}^{E-1} \mathbb{E}[\|\bar{v}_{k,\tau}\|^2] + \left( \frac{q}{n^2} + \frac{4(1+q)}{r(n-1)} \left( 1 - \frac{r}{n} \right) \right) 16E \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \right\}
\]

Therefore, using (107) and (111) in (106), we get:

\[
\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta E}{2} \mathbb{E}[\|\nabla f_i(w_k)\|^2] - \mathbb{E}\left[ \frac{L E}{2} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\bar{v}_{k,\tau}\|^2] \right] + \frac{16 \eta^3 L^2 E^2 (\alpha E + 4)}{n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]
\]

\[
+ \frac{LE \eta^2}{2} \left\{ \sum_{\tau=0}^{E-1} \mathbb{E}[\|\bar{v}_{k,\tau}\|^2] + \left( \frac{q}{n^2} + \frac{4(1+q)}{r(n-1)} \left( 1 - \frac{r}{n} \right) \right) 16E \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \right\}
\]

\[
\Rightarrow \mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta E}{2} \mathbb{E}[\|\nabla f_i(w_k)\|^2] - \frac{\eta}{2} \left( 1 - \eta^2 L^2 E^2 - \eta LE \right) \sum_{\tau=0}^{E-1} \mathbb{E}[\|\bar{v}_{k,\tau}\|^2]
\]

\[
+ 16 \eta E^2 \left\{ \frac{\eta^2 L (\alpha E + 4)}{n^2} + \frac{\eta}{2} \left( \frac{q}{n^2} + \frac{4(1+q)}{r(n-1)} \left( 1 - \frac{r}{n} \right) \right) \right\} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2]
\]

This completes the proof.

\section{Convergence of FedAvg under Assumption 4}

Here, we provide a convergence result for FedAvg (Algorithm 3) in the absence of the bounded client dissimilarity assumption (i.e. eq. (2)) and instead assuming that Assumption 4 holds for FedAvg; we restate it below for FedAvg.

**Assumption 6 (Assumption 4 for FedAvg).** Suppose all clients participate, i.e. \( r = n \), in the \((k+1)^{st}\) round of FedAvg (Algorithm 3). Let \( w_{k,E}^{(i)} \) be the \( i \)th client’s local parameter at the \((\tau+1)^{st}\) local step of the \((k+1)^{st}\) round of FedAvg, for \( i \in [n] \). Define \( \tilde{e}_{k,\tau}^{(i)} = \nabla f_i(w_{k,E}^{(i)}) - \nabla f_i(w_{k,\tau}) \), where \( w_{k,\tau} \triangleq \frac{1}{n} \sum_{i \in [n]} w_{k,E}^{(i)} \). Then for some \( \alpha < n \):

\[
\mathbb{E}\left[ \left\| \sum_{i \in [n]} \tilde{e}_{k,\tau}^{(i)} \right\|^2 \right] \leq \alpha \sum_{i \in [n]} \mathbb{E}\left[ \left\| \tilde{e}_{k,\tau}^{(i)} \right\|^2 \right], \forall \tau \in [E].
\]
Again, in the worst case, this assumption will always hold with $\alpha = n$. Also, as discussed after Assumption [4], we expect $\alpha$ to increase as the degree of client heterogeneity increases.

Before presenting the convergence result, we show empirical proof that Assumption [6] holds. For this, we compute and plot $\alpha$ (as we did in Section 6.1) for 8 and 4 bit FedAvg on CIFAR-10 and FMNIST, respectively; the results are in Figure 10.

![Figure 10: Variation of $\alpha$ over different rounds of 8 and 4 bit FedAvg for CIFAR-10 (Fig. 3a) and FMNIST (Fig. 3b) in the heterogeneous and homogeneous cases. In both cases, notice that $\alpha \ll n$ throughout training. Also, as expected, observe that $\frac{\alpha}{n}$ is higher for the heterogeneous case (except towards the end of training for FMNIST).](image)

**Theorem 3 (Smooth non-convex case for FedAvg).** Suppose Assumptions 1, 2 and 6 hold. Let $\sigma^2$ be the maximum variance of the local (client-level) stochastic gradients. In FedAvg (Algorithm 3), set $\eta_k = \frac{1}{LE \sqrt{3 \left( \frac{rE}{6r(6r-1)} + \frac{4\alpha}{9n} \right) K}}$. Define a distribution $\mathbb{P}$ for $k \in \{0, \ldots, K-1\}$ such that $\mathbb{P}(k) = \frac{(1+\zeta)^{(K-1-k)}}{\sum_{k=0}^{K-1} (1+\zeta)^k}$ where $\zeta := \eta^2 L^2 E^2 \left( \frac{(n-r)}{6r(n-1)} + \frac{4\alpha}{9n} \right)$. Sample $k^*$ from $\mathbb{P}$. Then for $K \geq \frac{4}{3 \left( \frac{rE}{6r(6r-1)} + \frac{4\alpha}{9n} \right)}$:

$$
\mathbb{E}[\|\nabla f(w_{k^*})\|^2] \leq \frac{3L^2f(w_0)}{n} \sqrt{\frac{3(n-r)}{6r(n-1)}} + \frac{4\alpha}{9n} + \frac{1}{3 \left( \frac{rE}{6r(6r-1)} + \frac{4\alpha}{9n} \right) K} \left( \frac{1}{E} + \frac{(n-r)}{3r(n-1)} \right) \sigma^2.
$$

So FedAvg needs $K = \mathcal{O}(\frac{1}{rE})$ rounds of communication to achieve $\mathbb{E}[\|\nabla f(w_{k^*})\|^2] \leq \epsilon$ where $\epsilon < \mathcal{O}(\frac{1}{E})$.

Thus, we recover the same complexity for FedAvg/Local SGD (which is basically FedAvg with full-device participation) as [17,18,36] – but without the bounded client dissimilarity assumption.

**Proof.** Using Lemma [13] for $\eta_k LE \leq \frac{1}{2}$, we can bound the per-round progress as:

$$
\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta_k E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] + \eta_k^2 LE^2 \left( \frac{(n-r)}{6r(n-1)} + \frac{8\alpha \eta_k LE}{9n} \right) \left( \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \right) \leq \frac{\eta_k^2 LE}{n} \left( 1 + \frac{8\alpha E}{9} \right) + \frac{1}{r} + \frac{(n-r)E}{3r(n-1)} \sigma^2.
$$

(113)
Now applying our earlier trick of using the $L$-smoothness and non-negativity of the $f_i$'s, we get:

$$\sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \leq \sum_{i \in [n]} 2L(\mathbb{E}[f_i(w_k)] - f_i^*) \leq 2nL \mathbb{E}[f(w_k)] - 2L \sum_{i \in [n]} f_i^* \leq 2nL \mathbb{E}[f(w_k)].$$

Putting this in eq. (113), we get for a constant learning rate of $\eta_k = \eta$:

$$\mathbb{E}[f(w_{k+1})] \leq \left(1 + \eta^2 L^2 E^2 \left(\frac{(n-r)}{6r(n-1)} + \frac{4\alpha}{9n}\right)\right) \mathbb{E}[f(w_k)] - \frac{\eta E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{\eta^2 L E}{2} \left(\frac{\eta L E}{n} \left(1 + \frac{8\alpha E}{9}\right) + \frac{1}{r} + \frac{(n-r)E}{3r(n-1)}\right) \sigma^2. \quad (114)$$

For ease of notation, define $\zeta := \eta^2 L^2 E^2 \left(\frac{(n-r)}{6r(n-1)} + \frac{4\alpha}{9n}\right)$ and $\zeta_2 := \left(\frac{\eta L E}{n} \left(1 + \frac{8\alpha E}{9}\right) + \frac{1}{r} + \frac{(n-r)E}{3r(n-1)}\right)$. Then, unfolding the recursion of eq. (114) from $k = 0$ through to $k = K - 1$, we get:

$$\mathbb{E}[f(w_K)] \leq (1+\zeta)^K f(w_0) - \frac{\eta E}{2} \sum_{k=0}^{K-1} (1+\zeta)^{(K-1-k)} \mathbb{E}[\|\nabla f(w_k)\|^2] + \frac{\eta^2 L^2 E}{2} \zeta_2 \sigma^2 \sum_{k=0}^{K-1} (1+\zeta)^{(K-1-k)}.$$

(115)

Let us define $p_k := \frac{(1+\zeta)^{(K-1-k)}}{\sum_{k'=0}^{K-1}(1+\zeta)^{K-1-k'}}$. Then, re-arranging eq. (115) and using the fact that $\mathbb{E}[f(w_K)] \geq 0$, we get:

$$\sum_{k=0}^{K-1} p_k \mathbb{E}[\|\nabla f(w_k)\|^2] \leq \frac{2(1+\zeta)^K f(w_0)}{\eta E \sum_{k=0}^{K-1} (1+\zeta)^k} + \eta L \zeta_2 \sigma^2 \quad (116)$$

$$= \frac{2\zeta f(w_0)}{\eta E (1 - (1+\zeta)^{-K})} + \eta L \left(\frac{\eta L}{n} \left(1 + \frac{8\alpha E}{9}\right) + \frac{1}{r} + \frac{(n-r)}{3r(n-1)}\right) \sigma^2, \quad (117)$$

where the last step follows by using the fact that $\sum_{k=0}^{K-1} (1+\zeta)^k = \frac{(1+\zeta)^{K-1}}{\zeta}$ and plugging in the value of $\zeta_2$. Now as we did in the proof of Theorem 2, we get:

$$(1+\zeta)^{-K} < 1 - \zeta K + \zeta^2 K \frac{(K+1)}{2} < 1 - \zeta K + \zeta^2 K^2 \implies 1 - (1+\zeta)^{-K} > \zeta K (1 - \zeta K).$$

Plugging this in eq. (117), we have for $\zeta K < 1$:

$$\sum_{k=0}^{K-1} p_k \mathbb{E}[\|\nabla f(w_k)\|^2] \leq \frac{2f(w_0)}{\eta E K (1 - \zeta K)} + \eta L \left(\frac{\eta L}{n} \left(1 + \frac{8\alpha E}{9}\right) + \frac{1}{r} + \frac{(n-r)}{3r(n-1)}\right) \sigma^2. \quad (118)$$

Let us pick $\eta = \frac{1}{L E \sqrt{3\left(\frac{4}{3r(n-1)} + \frac{4\alpha}{9n}\right)}}$. With this choice, we have $\zeta K = \frac{1}{3} < 1$. Note that we also need to have $\eta L E \leq \frac{1}{2}$; this happens for $K \geq \frac{4}{3\left(\frac{4}{3r(n-1)} + \frac{4\alpha}{9n}\right)}$. Putting $\eta = \frac{1}{L E \sqrt{3\left(\frac{4}{3r(n-1)} + \frac{4\alpha}{9n}\right)}}$ in eq. (118), we get:

$$\sum_{k=0}^{K-1} p_k \mathbb{E}[\|\nabla f(w_k)\|^2] \leq \frac{3L f(w_0)}{\sqrt{K}} \sqrt{3\left(\frac{n-r}{6r(n-1)} + \frac{4\alpha}{9n}\right)} + \frac{1}{L E \sqrt{3\left(\frac{4}{3r(n-1)} + \frac{4\alpha}{9n}\right)}} \left(\frac{1}{r} + \frac{(n-r)}{3r(n-1)}\right) \sigma^2 \quad (119)$$

$$+ \frac{1}{3\left(\frac{4}{3r(n-1)} + \frac{4\alpha}{9n}\right)} \left(\frac{1}{E} + \frac{8\alpha}{9}\right) \sigma^2. \quad (119)$$

This finishes the proof.
Lemma 13. For $\eta_k L E \leq \frac{1}{2}$, we have:

$$\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \frac{\eta_k E}{2} \mathbb{E}[\|\nabla f(w_k)\|^2] + \eta_k^2 L E^2 \left( \frac{(n-r)}{6r(n-1)} + \frac{8\eta_k L E}{9n} \right) \left( \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] \right) + \eta_k^2 L E \left( \frac{\eta_k L E}{n} \left( 1 + \frac{8\alpha E}{9} \right) + \frac{1}{r} + \frac{(n-r)E}{3r(n-1)} \right) \sigma^2.$$

Proof. Define

$$\hat{u}_{k,\tau}^{(i)} := \nabla f_i(w_{k,\tau}; B_{k,\tau}^{(i)}), \quad \hat{u}_{k,\tau} := \frac{1}{n} \sum_{i \in [n]} \hat{u}_{k,\tau}^{(i)}, \quad u_{k,\tau} := \frac{1}{n} \sum_{i \in [n]} \nabla f_i(w_{k,\tau}^{(i)}), \quad w_{k,\tau} := \frac{1}{n} \sum_{i \in [n]} w_{k,\tau}^{(i)} \text{ and } e_{k,\tau}^{(i)} = \nabla f_i(w_{k,\tau}^{(i)}) - \nabla f_i(w_{k,\tau}).$$

Then:

$$w_{k+1} = w_k - \eta_k \sum_{\tau=0}^{E-1} \left( \frac{1}{r} \sum_{i \in S_k} \hat{u}_{k,\tau}^{(i)} \right).$$  \hspace{1cm} (120)

$$w_{k,\tau} = w_k - \eta_k \sum_{t=0}^{\tau-1} \hat{u}_{k,t}.$$

$$\mathbb{E}\{ e_{k,\tau}^{(i)} \} = u_{k,\tau}.$$

$$\mathbb{E}\left[ \| \sum_{t=0}^{\tau-1} \hat{u}_{k,t} \|^2 \right] \leq \tau \sum_{t=0}^{\tau-1} \mathbb{E}\|u_{k,t}\|^2 + \tau \sigma^2.$$  \hspace{1cm} (122)

$$\mathbb{E}\left[ \| \sum_{t=0}^{\tau-1} \hat{u}_{k,t}^{(i)} \|^2 \right] \leq \tau \sum_{t=0}^{\tau-1} \mathbb{E}\|\nabla f_i(w_{k,t}^{(i)})\|^2 + \tau \sigma^2.$$  \hspace{1cm} (123)

Recall that $\sigma^2$ is the maximum variance of the local (client-level) stochastic gradients. In eq. (123), the expectation is w.r.t. $\{e_{k,\tau}^{(i)}\}_{i=1}^{n,\tau-1}$ and it follows due to the independence of the noise in each local update of each client. Similarly, eq. (124), the expectation is w.r.t. $\{B_{k,\tau}^{(i)}\}_{i=1}^{n,\tau-1}$ and it follows due to the independence of the noise in each local update.

Next, using the $L$-smoothness of $f$ and eq. (120), we get

$$\mathbb{E}[f(w_{k+1})] \leq \mathbb{E}[f(w_k)] - \mathbb{E}\left[ \langle \nabla f(w_k), \eta_k \sum_{\tau=0}^{E-1} \left( \frac{1}{r} \sum_{i \in S_k} \hat{u}_{k,\tau}^{(i)} \right) \rangle \right] + \frac{L}{2} \mathbb{E}\left[ \| \sum_{\tau=0}^{E-1} \frac{1}{r} \sum_{i \in S_k} \hat{u}_{k,\tau}^{(i)} \|^2 \right]$$

$$= \mathbb{E}[f(w_k)] - \mathbb{E}[\langle \nabla f(w_k), \eta_k \sum_{\tau=0}^{E-1} \hat{u}_{k,\tau} \rangle] + \frac{\eta_k^2 L}{2} \left( \frac{n(r-1)}{r(n-1)} \mathbb{E}\| \sum_{\tau=0}^{E-1} \hat{u}_{k,\tau} \|^2 \right)$$

$$+ \left( \frac{n-r}{r(n-1)} \right) \left\{ \frac{1}{n} \sum_{i \in [n]} \mathbb{E}\left[ \| \sum_{\tau=0}^{E-1} \hat{u}_{k,\tau}^{(i)} \|^2 \right] \right\}$$

$$\leq \mathbb{E}[f(w_k)] - \eta_k \mathbb{E}[\langle \nabla f(w_k), \sum_{\tau=0}^{E-1} u_{k,\tau} \rangle] + \frac{\eta_k^2 L}{2} \left( \frac{n(r-1)}{r(n-1)} \mathbb{E}\| \sum_{\tau=0}^{E-1} u_{k,\tau} \|^2 + \sigma^2 \right)$$

$$+ \left( \frac{n-r}{r(n-1)} \right) \left\{ \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(w_{k,\tau})\|^2 + \sigma^2] \right\}$$

Note that eq. (126) follows by taking expectation w.r.t. $S_k$ in eq. (125), while eq. (127) follows from eq. (122), eq. (123) and eq. (124).
For any 2 vectors \( \mathbf{a} \) and \( \mathbf{b} \), we have that \( \langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{2}(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2) \). Using this:

\[
\langle \nabla f(\mathbf{w}_k), \sum_{\tau=0}^{E-1} \mathbf{u}_{k,\tau} \rangle = \sum_{\tau=0}^{E-1} \langle \nabla f(\mathbf{w}_k), \mathbf{u}_{k,\tau} \rangle = \frac{1}{2} \sum_{\tau=0}^{E-1} (\|\nabla f(\mathbf{w}_k)\|^2 + \|\mathbf{u}_{k,\tau}\|^2 - \|\nabla f(\mathbf{w}_k) - \mathbf{u}_{k,\tau}\|^2).
\]

Putting this in eq. [127], we get:

\[
\begin{align*}
\mathbb{E}[f(\mathbf{w}_{k+1})] &\leq \mathbb{E}[f(\mathbf{w}_k)] - \frac{\eta_k E}{2} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] - \frac{\eta_k}{2} \left( 1 - \eta_k LE \right) \frac{n(r-1)}{r(n-1)} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{u}_{k,\tau}\|^2] \\
&\quad + \frac{\eta_k^2 LE^2}{2r} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] + \frac{\eta_k^2 LE^2}{2r} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \left( \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \right). \\
\end{align*}
\]

We upper bound (A) and (B) using Lemma [14] and Lemma [15] respectively. Plugging in these bounds, we get:

\[
\begin{align*}
\mathbb{E}[f(\mathbf{w}_{k+1})] &\leq \mathbb{E}[f(\mathbf{w}_k)] - \frac{\eta_k E}{2} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] - \frac{\eta_k}{2} \left( 1 - \eta_k LE \right) \frac{n(r-1)}{r(n-1)} \sum_{\tau=0}^{E-1} \mathbb{E}[\|\mathbf{u}_{k,\tau}\|^2] \\
&\quad + \frac{\eta_k^2 LE^2}{2r} \left( \frac{(n-r)}{6r(n-1)} + \frac{8\alpha \eta_k LE}{9n} \right) \left( \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \right) + \frac{\eta_k^2 LE^2}{2r} \left( \frac{n \eta_k LE}{n} \left( 1 + \frac{8\alpha E}{9} \right) + \frac{1}{r} + \frac{(n-r)E}{3r(n-1)} \right) \sigma^2, \\
\end{align*}
\]

for \( \eta_k LE \leq \frac{1}{2} \). Note that (C) \( \geq 0 \) for \( \eta_k LE \leq \frac{1}{2} \). Thus, for \( \eta_k LE \leq \frac{1}{2} \), we have:

\[
\begin{align*}
\mathbb{E}[f(\mathbf{w}_{k+1})] &\leq \mathbb{E}[f(\mathbf{w}_k)] - \frac{\eta_k E}{2} \mathbb{E}[\|\nabla f(\mathbf{w}_k)\|^2] + \eta_k^2 LE^2 \left( \frac{(n-r)}{6r(n-1)} + \frac{8\alpha \eta_k LE}{9n} \right) \left( \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] \right) + \frac{\eta_k^2 LE^2}{2r} \left( \frac{n \eta_k LE}{n} \left( 1 + \frac{8\alpha E}{9} \right) + \frac{1}{r} + \frac{(n-r)E}{3r(n-1)} \right) \sigma^2. \\
\end{align*}
\]

Lemma 14. For \( \eta_k LE \leq \frac{1}{2} \):

\[
\begin{align*}
\sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(\mathbf{w}_k) - \mathbf{u}_{k,\tau}\|^2] &\leq \eta_k^2 LE^2 \mathbb{E} \sum_{\tau=0}^{E-1} \|\mathbf{u}_{k,\tau}\|^2 + \frac{16\alpha \eta_k^2 LE^2}{9n^2} \sum_{i \in [n]} \mathbb{E}[\|\nabla f_i(\mathbf{w}_k)\|^2] + \frac{\eta_k^2 LE^2}{n} \left( 1 + \frac{8\alpha E}{9} \right) \sigma^2.
\end{align*}
\]
Proof. We have:

\[
\mathbb{E}[\|\nabla f(w_k) - u_{k,\tau}\|^2] = \mathbb{E}[\|\nabla f(w_k) - \nabla f(w_{k,\tau}) + \nabla f(w_{k,\tau}) - u_{k,\tau}\|^2] \\
\leq 2\mathbb{E}[\|\nabla f(w_k) - \nabla f(w_{k,\tau})\|^2] + 2\mathbb{E}[\|\nabla f(w_{k,\tau}) - u_{k,\tau}\|^2] \\
\leq 2L^2\mathbb{E}[\|w_k - w_{k,\tau}\|^2] + 2\mathbb{E}\left[\frac{1}{n} \sum_{i\in[n]} \left(\nabla f_i(w_{k,\tau}) - \nabla f_i(w_{k,\tau}^{(i)})\right)\right]^2 \tag{134}
\]

\[
\leq 2\eta_k^2L^2\mathbb{E}\left[\frac{1}{\tau}\sum_{t=0}^{\tau-1} \|\hat{u}_{k,t}\|^2\right] + \frac{2\alpha}{n^2} \sum_{i\in[n]} \mathbb{E}[\|\epsilon_{i,\tau}\|^2] \tag{135}
\]

\[
\leq 2\eta_k^2L^2\left(\frac{1}{\tau}\sum_{t=0}^{\tau-1} \mathbb{E}[\|u_{k,t}\|^2] + \frac{\tau\sigma^2}{n}\right) + \frac{2\alpha L^2}{n^2} \sum_{i\in[n]} \mathbb{E}[\|u_{k,\tau}^{(i)} - w_{k,\tau}\|^2]. \tag{136}
\]

Equation (134) follows from the L-smoothness of f and the definition of \(u_{k,\tau}\). Equation (135) follows from eq. (121) and Assumption 6. Equation (136) follows from eq. (123) and the L-smoothness of \(f_i\).

But:

\[
\sum_{i\in[n]} \mathbb{E}[\|w_{k,\tau}^{(i)} - w_{k,\tau}\|^2] = \sum_{i\in[n]} \mathbb{E}[\|w_{k,0}^{(i)} - \eta_k \sum_{t=0}^{\tau-1} \hat{u}_{k,t}^{(i)} - (w_{k,0} - \eta_k \sum_{t=0}^{\tau-1} \hat{u}_{k,t})\|^2] \tag{137}
\]

\[
= \eta_k^2 \sum_{i\in[n]} \mathbb{E}\left[\|\sum_{t=0}^{\tau-1} \hat{u}_{k,t} - \hat{u}_{k,t}\|^2\right] \tag{138}
\]

\[
\leq \eta_k^2 \tau \sum_{i\in[n]} \mathbb{E}[\|\hat{u}_{k,t} - \hat{u}_{k,t}\|^2] \tag{139}
\]

\[
= \eta_k^2 \tau \sum_{t=0}^{\tau-1} \sum_{i\in[n]} \mathbb{E}[\|\hat{u}_{k,t}\|^2 + \|\hat{u}_{k,t}\|^2 - 2\langle \hat{u}_{k,t}, \hat{u}_{k,t}\rangle] \tag{140}
\]

Equation (138) follows because \(w_{k,\tau}^{(i)} = w_{k}\ \forall \ i \in [n]\), due to which \(w_{k,0} = w_{k}\). Next, using the fact that \(\hat{u}_{k,\tau} = \frac{1}{n} \sum_{i\in[n]} \hat{u}_{k,t}^{(i)}\), we can simplify eq. (140) to:

\[
\sum_{i\in[n]} \mathbb{E}[\|w_{k,\tau}^{(i)} - w_{k,\tau}\|^2] \leq \eta_k^2 \sum_{t=0}^{\tau-1} \sum_{i\in[n]} (\mathbb{E}[\|\hat{u}_{k,\tau}^{(i)}\|^2] - \mathbb{E}[\|\hat{u}_{k,t}\|^2]) \tag{141}
\]

\[
\leq \eta_k^2 \tau \sum_{i\in[n]} \mathbb{E}[\|\hat{u}_{k,\tau}\|^2] \tag{142}
\]

\[
\leq \eta_k^2 \tau \sum_{i\in[n]} (\mathbb{E}[\|\nabla f_i(w_{k,\tau})\|^2] + \sigma^2). \tag{143}
\]

Next, using Lemma 15 for \(\eta_k L \leq \frac{1}{2}\) in eq. (143), we get:

\[
\sum_{i\in[n]} \mathbb{E}[\|w_{k,\tau}^{(i)} - w_{k,\tau}\|^2] \leq \frac{4\eta_k^2 \tau^2}{3} \sum_{i\in[n]} (2\mathbb{E}[\|\nabla f_i(w_{k})\|^2] + \sigma^2). \tag{144}
\]
Plugging eq. (144) back in eq. (136), we get:

\[
\mathbb{E}[\|\nabla f(w_k) - u_{k,\tau}\|^2] \leq 2\eta_k^2L^2\left(\tau \sum_{t=0}^{\tau-1} \mathbb{E}[\|u_{k,t}\|^2] + \frac{\tau\sigma^2}{n}\right) + \frac{8\alpha\eta_k^2L^2\tau^2}{3n^2} \sum_{i\in[n]} (2\mathbb{E}[\|\nabla f_i(w_k)\|^2] + \sigma^2)
\]

\[
= 2\eta_k^2L^2\tau \sum_{t=0}^{\tau-1} \mathbb{E}[\|u_{k,t}\|^2] + \frac{16\alpha\eta_k^2L^2\tau^2}{9n^2} \sum_{i\in[n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] + \frac{\eta_k^2L^2\tau^2\sigma^2}{n} \left(2 + \frac{8\alpha}{3}\tau\right).
\]

(145)

Summing up eq. (146) for \(\tau \in \{0, \ldots, E-1\}\), we get:

\[
\sum_{\tau=0}^{E-1} \mathbb{E}[\|\nabla f(w_k) - u_{k,\tau}\|^2] \leq \eta_k^2L^2E^2 \sum_{\tau=0}^{E-1} \mathbb{E}[\|u_{k,\tau}\|^2] + \frac{16\alpha\eta_k^2L^2E^3}{9n^2} \sum_{i\in[n]} \mathbb{E}[\|\nabla f_i(w_k)\|^2] + \frac{\eta_k^2L^2E^2}{n} \left(1 + \frac{8\alpha E}{9}\right)\sigma^2.
\]

(146)

**Lemma 15.** For \(\eta_k LE \leq \frac{1}{2}\), we have:

\[
\sum_{t=0}^{\tau-1} \mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] \leq \frac{\tau}{3} \left(8\mathbb{E}[\|\nabla f_i(w_k)\|^2] + \sigma^2\right).
\]

Proof:

\[
\mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] = \mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)}) - \nabla f_i(w_k) + \nabla f_i(w_k)\|^2]
\]

\[
\leq 2\mathbb{E}[\|\nabla f_i(w_k)\|^2] + 2\mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)}) - \nabla f_i(w_k)\|^2]
\]

\[
\leq 2\mathbb{E}[\|\nabla f_i(w_k)\|^2] + 2L^2\mathbb{E}[\|w_{k,t}^{(i)} - w_k\|^2].
\]

(148)

But:

\[
\mathbb{E}[\|w_k - w_{k,t}^{(i)}\|^2] = \mathbb{E}\left[\left\|\eta_k \sum_{t'=0}^{t-1} \nabla f_i(w_{k,t'}^{(i)}; B_{k,t'}^{(i)})\right\|^2\right] \leq \eta_k^2L^2\sum_{t'=0}^{t-1} \mathbb{E}[\|\nabla f_i(w_{k,t'}^{(i)}; B_{k,t'}^{(i)})\|^2] \leq \eta_k^2L^2\sum_{t'=0}^{t-1} \mathbb{E}[\|\nabla f_i(w_{k,t'}^{(i)})\|^2] + \sigma^2.
\]

(149)

Putting this back in eq. (148), we get:

\[
\mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] \leq 2\mathbb{E}[\|\nabla f_i(w_k)\|^2] + 2\eta_k^2L^2t \sum_{t'=0}^{t-1} \mathbb{E}[\|\nabla f_i(w_{k,t'}^{(i)})\|^2] + \sigma^2.
\]

(150)

Now summing up eq. (150) for all \(t \in \{0, \ldots, \tau - 1\}\), we get:

\[
\sum_{t=0}^{\tau-1} \mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] \leq 2\tau(\mathbb{E}[\|\nabla f_i(w_k)\|^2]) + 2\eta_k^2L^2 \sum_{t=0}^{\tau-1} \tau \sum_{t'=0}^{t-1} \mathbb{E}[\|\nabla f_i(w_{k,t'}^{(i)})\|^2] + \sigma^2
\]

\[
\leq 2\tau(\mathbb{E}[\|\nabla f_i(w_k)\|^2]) + \eta_k^2L^2\tau^2 \sum_{t=0}^{\tau-1} \mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] + \sigma^2.
\]

(151)

Let us set \(\eta_k LE \leq 1/2\). Then:

\[
\sum_{t=0}^{\tau-1} \mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] \leq 2\tau(\mathbb{E}[\|\nabla f_i(w_k)\|^2]) + \frac{1}{4} \sum_{t=0}^{\tau-1} \mathbb{E}[\|\nabla f_i(w_{k,t}^{(i)})\|^2] + \frac{\sigma^2\tau}{4}.
\]

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Simplifying, we get:

\[
\sum_{t=0}^{\tau-1} \mathbb{E}[\|\nabla f_i(w_k(t))\|^2] \leq \frac{\tau}{3}(8\mathbb{E}[\|\nabla f_i(w_k)\|^2] + \sigma^2).
\] (152)