Multiuser quantum communication networks

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We study a quantum state transfer between spins interacting with an arbitrary network of spins coupled by uniform XX interactions. It is shown that in such a system under fairly general conditions, we can expect a nearly perfect transfer of states. Then we analyze a generalization of this model to the case of many network users, where the sender can choose which party he wants to communicate with by appropriately tuning his local magnetic field. We also remark that a similar idea can be used to create an entanglement between several spins coupled to the network.

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I. INTRODUCTION

The fact that certain spin systems can be used for transferring quantum states from a ‘source’ to a ‘destination’ spin have been recently observed and studied by a number of authors (see [11–23]). It has been shown that a perfect transfer of an arbitrary qubit is possible in spin chains [3,4] and some other cases [10–17]. As observed by Bose et al. [1], such a system can also be used for creating an entanglement between the source and destination spins. In this note we develop ideas presented in [6] and prove that one can get a high fidelity transfer of an arbitrary qubit. Let us consider two spins coupled to a network

\[ H_G = \sum_{(i,j)} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \],

where we sum over all network edges \((i,j)\). A ‘source’ spin \(s\) is coupled to a spin \(n_s\) from \(G\), and a ‘destination’ spin \(d\) is coupled to a \(n_d\) from \(G\). The strength of these couplings we denote by \(\epsilon_s, \epsilon_d\), respectively, where we assume that \(\xi_s, \xi_d\) are constants independent of \(N\) but \(\epsilon\) decreases with \(N\). Moreover, spins \(s\) and \(d\) are placed in local magnetic fields \(\omega_s\) and \(\omega_d\), respectively. Then, the Hamiltonian of the whole system can be written as

\[ H = H_G + H_{sd}, \]

where \(H_G\) given above while

\[ H_{sd} = \epsilon_s \xi_s (\sigma_s^x \sigma_n^x + \sigma_n^y \sigma_n^y) + \epsilon_d (\sigma_n^x \sigma_d^x + \sigma_n^y \sigma_d^y) + \omega_s \sigma_s^z + \omega_d \sigma_d^z. \]

Note that \(H\) conserves the total number of excitations. We shall be mostly interested in the case when there is only one excitation in the system. Then, the system remains in the Hilbert space spanned by vectors \(|n\rangle\), where \(n\) denotes the position of the excitation and takes values either \(s, d,\) or \(1, 2, \ldots, N\). Observe that the Hamiltonian written in this basis is similar to the network adjacency matrix.

Let \(\{\lambda\}\) and \(|\lambda\rangle\) be the sets of the eigenvalues and the eigenvectors of \(H_G\), respectively. Then \(H = H_0 + V\), where

\[ H_0 = \omega_s |s\rangle \langle s| + \omega_d |d\rangle \langle d| + \sum_\lambda \lambda |\lambda\rangle \langle \lambda|, \]

\[ V = \epsilon \sum_\lambda (\xi_s g_s \lambda |s\rangle \langle \lambda| + \xi_d g_d \lambda |d\rangle \langle \lambda| + h.c.), \]

and

\[ g_s \lambda = \langle s|H_{sd}|\lambda\rangle / (\epsilon \xi_s), \quad g_d \lambda = \langle d|H_{sd}|\lambda\rangle / (\epsilon \xi_d). \]

Now, let us consider two cases. In the ‘resonant’ case we have \(\omega_s = \omega_d = \lambda_s\), where \(\lambda_s\) is one of the non-degenerated eigenvalues of \(G\). Then all terms of the

\[ \sum_\lambda \lambda |\lambda\rangle \langle \lambda| = H_G, \]

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\[ \sum_\lambda \lambda |\lambda\rangle \langle \lambda| = H_G, \]
Hamiltonian $H$ corresponding to $\lambda \neq \lambda'$ are of the lower order, and the evolution of the system takes place essentially in the space $L$ spanned by vectors $|s\rangle$, $|d\rangle$, $|\lambda'\rangle$. The projection of the Hamiltonian onto $L$ in the basis $\{ |s\rangle, |d\rangle, |\lambda'\rangle \}$ can be written as

$$H_{\text{eff}}^{s\lambda'd} \approx \begin{pmatrix} \lambda' & \varepsilon \xi_d s & 0 \\ \varepsilon \xi_s g_{s\lambda'} & \lambda' & \varepsilon \xi_d g_{d\lambda'} \\ 0 & \varepsilon \xi_s g_{s\lambda'} & \lambda' \end{pmatrix}.$$ 

In order to obtain an effective Hamiltonian for the ‘non-resonant’ case, where $\omega_s \neq \omega$ and $\omega_s \neq \lambda$ for all $\lambda$, i.e., when the local magnetic fields of the source and the destination spins are not ‘tuned’ to any ‘natural frequency’ of the network, we follow the approach presented in [24]. Define a hermitian operator $\lambda$ as

$$S = \varepsilon \sum_{\lambda} (\xi_s s_{s\lambda} |s\rangle \langle \lambda| + \xi_d s_{d\lambda} |d\rangle \langle \lambda| + \text{h.c.}),$$

and set $H' = e^{i S} H e^{-i S}$. Note that

$$H' e^{i S}|\lambda\rangle = e^{i S} H e^{-i S} e^{i S}|\lambda\rangle = \lambda e^{i S}|\lambda\rangle.$$ 

Then, expanding $e^{i S}$ and $e^{-i S}$ into power series, we get

$$H' = H_0 + V + i[S, H_0] + i[S, V] + \frac{i^2}{2!} [S, [S, H_0]] + \ldots.$$ 

Let us choose $s_{s\lambda}$ and $s_{d\lambda}$ so that the terms of the first order in $\varepsilon$ vanish, i.e.,

$$V + i[S, H_0] = 0.$$ 

For such a condition the Hamiltonian $H'$ becomes

$$H'' = H_0 + i[S, V] + \frac{i^2}{2!} [S, [S, H_0]] + O(\varepsilon^3).$$

Thus, the projection of $H''$ onto the space generated by $\{ |s\rangle, |d\rangle \}$ can be written

$$H_{\text{eff}}^{sd} \approx \left( \omega_s - \varepsilon^2 |\xi_s|^2 \sum_{\lambda} \frac{|g_{s\lambda}|^2}{\lambda - \omega_s} \right) |s\rangle \langle s|$$

$$- \varepsilon^2 \xi_s \xi_d \left( \sum_{\lambda} \frac{g_{s\lambda} g_{d\lambda}^*}{\lambda - \omega_s} + \sum_{\lambda} \frac{g_{s\lambda} g_{d\lambda}^*}{\lambda - \omega_d} \right) |s\rangle \langle d|$$

$$- \varepsilon^2 \xi_s^* \xi_d \left( \sum_{\lambda} \frac{g_{s\lambda} g_{d\lambda}}{\lambda - \omega_d} + \sum_{\lambda} \frac{g_{s\lambda} g_{d\lambda}^*}{\lambda - \omega_d} \right) |d\rangle \langle s|$$

$$+ \left( \omega_d - \varepsilon^2 |\xi_d|^2 \sum_{\lambda} \frac{|g_{d\lambda}|^2}{\lambda - \omega_d} \right) |d\rangle \langle d|.$$ 

Using the above formulae for $H_{\text{eff}}^{s\lambda'd}$ and $H_{\text{eff}}^{sd}$ it is not hard to find sufficient conditions under which the state transfer from $s$ to $d$ occurs. In the resonant case, one ought to choose all the off-diagonal case to be of the same absolute value $\beta \varepsilon$. Then, the time of the transfer is roughly, $\frac{\beta \varepsilon}{\sqrt{2 \varepsilon}}$. For the non-resonant case, in order to have a nearly perfect transfer between $s$ and $d$, one should make all the diagonal terms of $H_{\text{eff}}^{sd}$ equal, and ensure that the off-diagonal terms do not vanish. Moreover, if the absolute value of the off-diagonal term is $\beta' \varepsilon^2$, then the transfer occurs in time $T \sim \frac{\beta' \varepsilon^2}{\sqrt{\varepsilon}}$.

Finally, we remark that if we apply the same approach to the case when both $\omega_s$ and $\omega_d$ are close to a degenerated eigenvalue, we obtain an effective Hamiltonian whose projection $H_{\text{eff}}$ is greater than $3 \times 3$. Thus, typically, no selection of $\xi_s$ and $\xi_d$ can reduce $H_{\text{eff}}$ to a form which guarantees a perfect transfer. Thus, in general, we cannot hope to get a perfect transfer by tuning to a degenerate eigenvalue of $H_G$.

### III. Spin Networks

In the paper [23] we considered a WCS model for spin chains consisting of $N$ spins, where the source and the destination spins $s$, $d$, were coupled to the ends of the chain, and $\omega_s = \omega_d = 0$. We showed that such a system admit a high-fidelity transfer of quantum states but the precise description of this phenomenon depends heavily on the parity of $N$. The reason for that is clear when we compute the effective Hamiltonian for such a system: the case of even $N$ is non-resonant, while for odd $N$, zero is an eigenvalue of the chain, a resonant transmission occurs.

Now let us consider more general case in which $s$ and $d$ are coupled to arbitrary spins of the chain. The eigenvectors and the eigenvalues of the chain are given by

$$|\lambda_k\rangle = \sqrt{\frac{2}{N+1}} \sum_{n=1}^{N} \sin \left( \frac{\pi kn}{N+1} \right) |n\rangle,$$

and

$$\lambda_k = 2 \cos \left( \frac{\pi k}{N+1} \right),$$

respectively, where $N$ is the chain length and $k = 1, 2, \ldots, N$. If $s$ and $d$ are coupled to the $n_s$th and $n_d$th spin in the chain, respectively, then

$$g_{s\lambda}(n_s) = \sqrt{\frac{2}{N+1}} \sin \left( \frac{\pi kn_s}{N+1} \right),$$

for $\alpha = s, d$.

In the ‘symmetric case’, in which $n_d = N+1-n_s$ and $\xi_s = \xi_d = \xi$, we have $g_{s\lambda}(n_s) = g_{d\lambda}(N+1-n_s)$ and, as long as $\omega_s = \omega_d$, both the resonant and non-resonant cases give perfect transfer with transfer times of the orders $\frac{\xi}{\sqrt{N}}$ and $\varepsilon^2 \xi^2$, respectively.

The asymmetric case, when $n_d \neq N+1-n_s$ needs a bit more attention. If we put $\xi_s = \xi_d$, then $g_{s\lambda}(n_s) \neq g_{d\lambda}(n_d)$ and the transfer occurs with probability bounded from
FIG. 1: The probability of the excitation of the destination spin d as a function of time. In this case \( N = 30, n_s = 2 \), and \( n_d = 13 \). Moreover, we have \( \omega_s = \omega_d = \lambda \), so the transfer is resonant. The dashed line corresponds to the case \( \varepsilon_s = \varepsilon_d = 0.01 \), the solid one to the case \( \varepsilon_s = 0.01, \varepsilon_s g_{\lambda}(2) = \varepsilon_d g_{\lambda}(13) \).

above by a constant smaller than one. However, the transfer can be made perfect by switching to a resonant case. In order to do that one needs to select the coupling constants \( \varepsilon_s \) and \( \varepsilon_d \) in such a way that \( \varepsilon_s g_{\lambda}(n_s) = \varepsilon_d g_{\lambda}(n_d) \), so the condition for a perfect transfer is satisfied (see Fig. 1).

Another example of communicating through a simple spin network is to attach two spins to \( N \)-cycle. The eigenvectors and eigenvalues for an \( N \)-cycle are

\[
|\lambda_k\rangle = \sqrt{\frac{1}{N}} \sum_{n=1}^{N} e^{2\pi i k n/N} |n\rangle,
\]

and

\[
\lambda_k = 2 \cos \left( \frac{2\pi k}{N} \right),
\]

respectively. For a ‘resonant communication’ through \( N \)-cycle we have only either one or two possible non-degenerated eigenvectors to choose from, namely \(|\lambda_N\rangle\) for an odd \( N \), and \(|\lambda_{N/2}\rangle\) and \(|\lambda_N\rangle\) for an even \( N \). Moreover, one can check that in this case coupling to a degenerated eigenvector cannot lead to a resonant transfer which is nearly perfect. This observation is analogous to a result of Christandl et al. which states that the perfect transfer in the chain with equal couplings is possible only for \( N = 2 \) and \( N = 3 \).

Finally, let us consider a general network \( G \) with the source spin \( s \) and the destination spin \( d \) attached to the nodes \( n_s \) and \( n_d \), respectively. A possible transfer of quantum states between \( s \) and \( d \) depends on the eigenvectors \( \{|\lambda\rangle\} \) of \( G \). Note that each localized state of the network \( |n\rangle \) is a superposition of eigenvectors \( \{|\lambda\rangle\} \). In particular, \( s \) and \( d \) cannot communicate unless there exists at least one eigenvector \(|\lambda'\rangle\) that gives non-zero scalar product with both \(|n_s\rangle\) and \(|n_d\rangle\); such an eigenvector can be viewed as a communication channel. Note however, that for every connected network in which all coupling constants are reals, such a vector exists. Indeed, if \( \lambda' \) is the largest eigenvalue of \( G \) then, by Perron-Frobenius theorem, \(|\lambda'\rangle\) is a vector which corresponds to a stationary distribution of a particle in a classical random walk on \( G \) and so, for connected \( G \), we have \( \langle n|\lambda'\rangle > 0 \), for \( n = 1, 2, \ldots, N \). Observe also that for a connected \( G \) the largest eigenvalue \( \lambda' \) is always non-degenerate, so we can achieve a near perfect transfer setting \( \omega_s = \omega_d = \lambda' \), and choosing \( \xi_s \) and \( \xi_d \) so that \( \xi_s g_{\lambda'}(s) = \xi_d g_{\lambda'}(d) \). We remark however that for some networks it is not possible to achieve a nearly perfect transfer in a non-resonant way, because for any choice of \( \xi \)’s and \( \omega \)’s either the diagonal terms of the \( 2 \times 2 \) projection of the effective Hamiltonian are not equal, or the off-diagonal terms vanish.

IV. MULTIUSER QUANTUM NETWORK

Let us recall that in order to have a nearly perfect transfer between \( s \) and \( d \) we should put \( \omega_s = \omega_d \); it is also not hard to check that the fidelity of transfer drops down rapidly when the difference \( |\omega_s - \omega_d| \) grows. This observation suggests the following multiuser generalization of our communication protocol. Assume that spins \( d_1, d_2, \ldots \) are coupled to some spins of a spin network \( G \) and placed in local magnetic fields \( \omega_{d_1}, \omega_{d_2}, \ldots \), respectively. Then another spin \( s \), coupled to a spin from \( G \) as well, can communicate with any spin \( d_k \) by making its own magnetic field \( \omega_s \) equal to \( \omega_{d_k} \) and calibrating appropriately the coupling strength \( \xi_s \). To guarantee a high fidelity of the state transfer from \( s \) to \( d_k \) we should ensure that the distance between \( \omega_s = \omega_{d_k} \) and the other frequencies \( \omega_{d_i}, i \neq k \), as well as between \( \omega_s \) and the eigenvalues \( \lambda \) of \( G \) (except, perhaps one of them, when in the resonant case we have \( \lambda' = \omega_s \) is large enough. On the other hand, a large magnetic field \( \omega_s \) slows down the transfer from \( s \) to \( d_k \). Thus, choosing frequencies \( \omega_1, \omega_2, \ldots \), we should keep in mind both the fidelity of the transfer (which decreases with the distance between \( \omega_i \) and the closest eigenvalue) and the time of the transmission (which may increase considerably when \( \omega_i \) is large, say, much larger than the largest eigenvalue of the network).

We remark that if in such a protocol another user sets his frequency to the communication frequency, the information will get entangled between him and the intended receiver. An example of such a disturbance in a system of three spins coupled to a 21-cycle is presented on Fig. 2.

V. ENTANGLEMENT GENERATION

The WCS system can be also used for generation of a perfect entanglement. Let us consider first a non-resonant communication between two users. After the time equal to a half of the transfer time, the state of the system is \( \frac{1}{\sqrt{2}} (|s\rangle + e^{i\phi}|d\rangle) \), where \( \phi \) is some angle which
can be easily computed. This state can be also written in the form

\[
\frac{1}{\sqrt{2}} \left( |0\ldots0\rangle_{\text{network}} + e^{i\phi} |01\ldots0\rangle_{\text{network}} \right) = \frac{1}{\sqrt{2}} (|01\rangle + e^{i\phi}|10\rangle)_{sd}|0\ldots0\rangle_{\text{network}},
\]

which clearly corresponds to a maximum entanglement between \(s\) and \(d\).

In order to obtain \(W\) state one has to consider three spins and a non-resonant effective Hamiltonian of the form

\[
H_{\text{eff}}^{\text{multi}} \approx \begin{pmatrix} \gamma & \alpha & e^{i\varphi} \alpha \\ \alpha^* & \gamma & \beta \\ e^{-i\varphi} \alpha^* & \beta^* & \gamma \end{pmatrix}.
\]

Let us set the initial conditions as \(|\Psi(0)\rangle = |s_1\rangle\). After an easily computable time, the state of the network becomes a \(W\) state \(\frac{1}{\sqrt{3}} (|s_1\rangle + e^{i\phi}|s_2\rangle + e^{i\theta}|s_3\rangle)\), for some \(\phi\) and \(\theta\) (cf. Fig. 3).

We can also use a resonant transfer to get an entanglement for \(m \geq 2\) users. To this end we apply the following procedure. First, one user \(s_1\) in an initial state \(|\Psi(0)\rangle = |s_1\rangle\) couples to a non-degenerated eigenvalue \(\lambda'\) of the network and waits until the system evolves into \(|\lambda'\rangle\). This evolution is described by the following effective Hamiltonian

\[
H_{\text{eff}}^{1} = \lambda'|\lambda'\rangle\langle \lambda'| + \lambda'|s_1\rangle\langle s_1| + (\varepsilon_1 g_{s_1,\lambda'} |s_1\rangle\langle \lambda'| + \text{h.c.}).
\]

Then all \(m\) users couple to excited \(|\lambda'\rangle\) state via the effective Hamiltonian

\[
H_{\text{eff}}^{2} = \lambda'|\lambda'\rangle\langle \lambda'| + \sum_{i} \lambda'|s_i\rangle\langle s_i| + \varepsilon \sum_{i} \xi_i g_{s_i,\lambda'} |s_i\rangle\langle \lambda'| + \text{h.c.},
\]

where \(\xi_i g_{s_i,\lambda'} = \xi_j g_{s_j,\lambda'}\) for all pairs \(\{i, j\}\), and wait until the system evolves to the state

\[
\frac{1}{\sqrt{M}} \sum_{j=1}^{M} e^{i\phi_j} |s_j\rangle.
\]
Note, that this method is similar to the dynamics of the spin star network presented in [23]. Observe also that one can get rid of all relative phases by local one qubit operations.

VI. SUMMARY

In the paper we generalized a number of earlier results on quantum information transmission between a source spin and a destination spin using a simple spin network. We show that a near-perfect state transfer between two spins is possible through a large class of networks, provided that we can control local magnetic field in which the source and the destination spins are embedded. We also point out that a source spin can choose the destination spin from a number of ‘users’ of the network by an appropriate ‘tuning’, i.e., by carefully selecting its local magnetic field. The very same mechanism can be used to generate a multispin entanglement state.

VII. ACKNOWLEDGMENTS

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