INTEGRABLE HIERARCHIES IN DONALDSON-WITTEN AND SEIBERG-WITTEN THEORIES *

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Abstract. We review various aspects of integrable hierarchies appearing in \( \mathcal{N} = 2 \) supersymmetric gauge theories. In particular, we show that the blowup function in Donaldson–Witten theory, up to a redefinition of the fast times, is a \( \tau \)–function for a \( g \)-gap solution of the KdV hierarchy. In the case of four-manifolds of simple type, instead, the blowup function becomes a \( \tau \)–function corresponding to a multisoliton solution. We obtain a new expression for the contact terms that links these results to the Whitham hierarchy formulation of Seiberg–Witten theories.

1. Introduction

The Seiberg–Witten ansatz for the low-energy effective action of \( \mathcal{N} = 2 \) super Yang–Mills theories \[1,2\] stands out as the only exact solution that is known at present in four-dimensional Quantum Field Theory. The quantum moduli space of vacua of the theory \( \mathcal{M} \) is identified with the moduli space of an auxiliary hyperelliptic complex curve \( \Sigma \), in such a way that there is a selected meromorphic differential \( dS \)–that induces a special geometry on \( \Sigma \)–, whose periods give the spectrum of BPS states. Interestingly enough, this solution has been shown to display remarkable nonperturbative phenomena such as quark confinement by monopole condensation, when a mass

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term that breaks supersymmetry down to $\mathcal{N} = 1$ is included. In the sake of definiteness and clarity, we will restrict our discussion to the case of $\mathcal{N} = 2$ supersymmetric theories with gauge group $SU(N)$ and without matter content \cite{3,4}, though many of the results reviewed here can be appropriately generalized in different directions.

It was presently realized that the Seiberg–Witten solution could be reformulated, in an elegant and useful way, in terms of classical finite-gap integrable systems, $dS$ being a solution of their averaged (Whitham) dynamics \cite{5}. The spectral curve of the integrable system $\Gamma$ is identified with the auxiliary hyperelliptic curve $\Sigma$, and the effective prepotential of the supersymmetric gauge theory $F(a^i, \Lambda)$ turns out to be given by the logarithm of the quasiclassical $\tau$–function. In the case of pure gauge theory, for example, the corresponding integrable system is the periodic Toda chain \cite{6,7}. The quasiclassical Whitham hierarchy associated to adiabatic deformations of the integrable system naturally endows moduli which, instead of being local invariants, evolve with respect to the slow times $T_n$ \cite{8}. The upshot of this formalism is a prepotential also depending on these new variables, $F(a^i, T_n)$ \cite{9,10}. This prepotential is a deformation of the former in the sense that, roughly speaking, if we put $T_1 = \Lambda$ and $T_{n>1} = 0$, the Seiberg–Witten prepotential is recovered.

The Whitham dynamics can be thought of as a sort of generalization of the Renormalization Group flow \cite{11}. Aside from its intrinsic formal interest, it governs a relevant family of deformations of the Seiberg–Witten solution. In fact, it turns out that the slow times are dual to homogeneous combinations of higher Casimir operators, this revealing that they are the appropriate variables to be promoted to spurion superfields if one is interested in softly broken $\mathcal{N} = 2$ supersymmetry by means of higher than quadratic $\mathcal{N} = 0$ perturbations \cite{12}. The lowest slow time can be identified with the quantum dynamical scale $\Lambda$ of the supersymmetric gauge theory, and its uses as a spurion has been extensively studied in \cite{13,14}. Moreover, this formalism is also very fruitful when restricted to the original Seiberg–Witten variables; new equations arise that provide a powerful technique allowing to compute interesting quantities in the infrared such as instanton corrections up to arbitrary order \cite{12,15}, and the strength of the coupling among different magnetic photons at the monopole singularities \cite{16}. The Whitham hierarchy, and the details of its connection to four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, have been already the focus of many comprehensive reviews \cite{17,18,19,20} and books \cite{21,22}.

There is another context, definitely less explored, where four-dimensional $\mathcal{N} = 2$ supersymmetric theories get involved with integrability. It is the case of topological field theories built out of twisted versions of $\mathcal{N} = 2$ supersymmetric gauge theories \cite{23}. Soon after the appearance of Ref. \cite{1,2,3},
it was realized that the Seiberg–Witten solution may be used to compute Donaldson invariants of a four-manifold $X$ by counting solutions of Abelian monopole equations [24]. Furthermore, for manifolds with $b_2^+ (X) = 1$, a nontrivial contribution comes from each point of the Coulomb branch so that the path integral used as a generating function for Donaldson invariants acquires the form of a sort of integral over $\mathcal{M}$ (the so-called $u$-plane integral [25, 26]). The Whitham hierarchy formalism has been shown to provide an adequate conceptual framework to study several aspects of the behavior of the $u$-plane integral under blowup of a point $p \in X$. The contact terms corresponding to pairs of observables, for example, can be written as derivatives of the prepotential with respect to $T_n$ variables [27, 28, 29]. On the other hand, these terms can be derived from the blowup function [30, 31], which is nothing but a factor appearing in the so-called blowup formula [32, 33] that relates the path integral in $X$ with the one performed in the blowup manifold $\hat{X}$. Furthermore, the blowup formula itself involves in a rather direct way the underlying integrable structure of the low-energy effective theory.

In previous remarks we have discussed a connection between integrability and four-dimensional supersymmetric gauge theories that heavily relies in the identification of the relevant geometrical data, and the uses of Whitham theory to study adiabatic deformations thereof. A rough pattern of these interrelations is given in Fig. 1. In all known cases—such as, for instance—
any gauge group and matter content—, the relevant integrable system has a finite number of degrees of freedom. There is, however, a different way in which integrability enters into the game of twisted $N = 2$ supersymmetric gauge theories. It emerges from a detailed analysis of the blowup formula under the light of the theory of hyperelliptic Kleinian functions. The main consequence of such investigation is that the blowup function in Donaldson–Witten theory, up to a redefinition of the fast times, is a $\tau$–function for a $g$-gap solution of the KdV hierarchy \[34\]. For manifolds of the so-called simple type, the blowup function becomes the $\tau$–function of a multisoliton solution of the hierarchy. As a corollary, the correlation functions involving the exceptional divisor on the blowup manifold are governed by the KdV hierarchy, this giving an intriguing connection with two-dimensional topological gravity \[35\]. The uses of the theory of hyperelliptic Kleinian functions also provides powerful techniques that further enhance our knowledge of contact terms and blowup functions. It is our purpose in this talk to present a short survey comprising these latest results.

2. Brief overview of Donaldson-Witten theory

Let $X$ be a smooth, compact, oriented (for simplicity, we also assume that it is simply connected) four-manifold with Riemannian metric $g$, and let $E \to X$ be an $SU(2)$ bundle. The degree $s$ Donaldson polynomial, defined on the homology of $X$ with rational coefficients, is given by \[36\]

$$D_E(p, S) = \sum_{2n+4t = s} d_{n,t} S^n p^t,$$  \hspace{1cm} (2.1)

where $p \in H_0(X, Q), S \in H_2(X, Q)$, and $s$ is the dimension of the moduli space of instantons on $E$. The numbers $d_{n,t}$ are precisely given in terms of intersection theory on this moduli space \[37\].

It is useful to organize these polynomials within a generating function, by summing over all topological types of the bundle $E$ with fixed second Stiefel–Whitney class, $w_2(E)$,

$$\Phi_X(p, S) = \sum_{n,t \geq 0} d_{n,t} S^n \frac{p^t}{n! t!}.$$ \hspace{1cm} (2.2)

The remarkable result obtained a decade ago by Witten \[23\] is that $\Phi_X(p, S)$ is the generating function for the correlators of observables in a twisted version of $\mathcal{N} = 2$ super Yang–Mills theory,

$$\Phi_X(p, S) = Z_X(p, S) = \left\langle \exp \left[ \frac{L}{2} \text{Tr} \phi^2 + \frac{1}{2} \int_S \text{Tr}(\phi F) + \cdots \right] \right\rangle_X.$$ \hspace{1cm} (2.3)
Here, $\phi$ is the scalar field that belongs to the $N = 2$ vector multiplet and $F$ is the Yang–Mills field strength. If $b_2^+(X) > 1$, $\Phi_X$ is independent of $g$, thus defining topological invariants of the four-manifold $X$.

For gauge group $SU(N)$, there is a family of $N - 1$ fundamental observables (that is, local BRST-invariant operators), $\mathcal{O}_k$,

$$\mathcal{O}_k = \frac{1}{k} \text{Tr} \phi^k + \cdots , \quad (2.4)$$

whose vacuum expectation values, $u_k = \langle \mathcal{O}_k \rangle$, are gauge invariant coordinates in $\mathcal{M}$ classifying the various vacua of the theory—, as well as $b_2(X)(N - 1)$ topological descendants thereof, $I_k(S_i)$,

$$I_k(S_i) = \frac{1}{k} \int_{S_i} \text{Tr}(\phi^{k-1}F) + \cdots , \quad (2.5)$$

where the dots stand for lower powers of $\phi$ in (2.4), and superpartner contributions in (2.5). So, the basic problem in Donaldson–Witten theory is to compute the generating function

$$Z_X(p_k, S_i) = \left\{ \exp \left[ \sum_{k=2}^{N} \left( p_k \mathcal{O}_k + \sum_{i=1}^{b_2(X)} f_{k,i} I_k(S_i) \right) \right] \right\}_X . \quad (2.6)$$

Being metric independent, one can consider a uniparametric family of metrics $g_t = t^2 g_0$, with fixed $g_0$, and focus on the limiting cases $t \to 0$ and $t \to \infty$. In the former case, the topological quantum field theory is in the ultraviolet, and thus weakly coupled, so perturbation theory is reliable. Conversely, when $t \to \infty$ the correlation functions result from the infrared behavior of the theory, picking up contributions from every point in the moduli space of vacua. This amounts to an integral over the $u$-plane $\mathbb{C}$. For simple type four-manifolds, the only non-vanishing contributions to this integral come from the maximal singularities of $\mathcal{M}$. The infrared dynamics at those points is that of a weakly coupled theory of Abelian gauge fields and monopoles $\mathbb{1}, \mathbb{2}, \mathbb{38}$. Hence, roughly speaking, Donaldson invariants can be computed in such cases just by counting monopole solutions $\mathbb{24}$.

The $u$-plane integral is performed by means of the Seiberg-Witten solution $\mathbb{25}, \mathbb{26}, \mathbb{27}$. For each descent observable of the microscopic theory, $I_k(S_i)$, there is a corresponding low energy operator $\tilde{I}_k(S_i)$. However, it is not generically true that the product of two or more of these operators map to the analog product in the infrared description. Contact terms appear whenever any pair of the supporting two-cycles intersect $\mathbb{39}, \mathbb{25}$,

$$I_k(S_i) I_l(S_j) \longrightarrow \tilde{I}_k(S_i) \tilde{I}_l(S_j) + \mathcal{T}_{k,l}(S_i \cap S_j) . \quad (2.7)$$
These terms are not deducible from the Seiberg-Witten solution. Their explicit form can be derived from the so-called blowup function \([30, 31]\) by requiring both their duality invariance and semiclassical vanishment \([25]\),

\[
\mathcal{T}_{k,l} = -\frac{1}{2\pi i} \partial_{\tau_i} \left( \log \Theta(\vec{\Delta}, \vec{0})(0|\tau) \frac{\partial u_k}{\partial a^i} \frac{\partial u_l}{\partial a^j} \right),
\]

where \(\vec{\Delta} = (1/2, \ldots, 1/2)\). We will analyze in what follows the case of four-manifolds with \(b_2^+ (X) = 1\).

Consider now the four-manifold \(\hat{X}\), obtained from \(X\) by blowing up a point \(p\), \(\hat{X} = \text{Bl}_p(X)\). This means that there is a map \(\pi : \hat{X} \to X\) that is the identity everywhere except at \(B = \pi^{-1}(p)\), where \(B \in H_2(\hat{X})\) such that \(B^2 = -1\). \(B\) is called the class of the exceptional divisor. The homology of the blown up manifold is the direct sum \(H_2(\hat{X}) = H_2(X) \oplus \mathbb{Z} \cdot B\). Thus, the twisted theory in \(\hat{X}\) has additional descent observables \(I_k(B)\) that must be included in the generating function. It is also possible to have a non-Abelian magnetic flux \(\vec{\beta}\) through \(B\) of the form \(\beta^i_j = (C^{-1})^i_j n^j\), where the \(n^j\) are arbitrary integers, and \((C^{-1})^i_j\) is the inverse of the Cartan matrix \([27]\). The generating function of the twisted theory in the blown up manifold then reads

\[
Z_{\hat{X}, \vec{\beta}}(p_k, S_i, B) = \left\langle \exp \left[ \sum_{k=2}^{N} (p_k O_k + \sum_{i=1}^{b_2(X)} f_{k,i} I_k(S_i)) \right] \right\rangle_{\hat{X}, \vec{\beta}},
\]

The Donaldson invariants of \(\hat{X}\) are related to those of \(X\), at least in the case in which the exceptional divisor has a small area. This is described by the so-called blowup formula \([32, 33]\). It is reflected in the \(u\)-plane integral, through the following relation between both generating functions:

\[
Z_{\hat{X}, \vec{\beta}}(p_k, S_i) = \left\langle \exp \left[ \sum_{k=2}^{N} (p_k O_k + \sum_{i=1}^{b_2(X)} f_{k,i} I_k(S_i)) \right] \tau_{\vec{\beta}}(t_k|O_k) \right\rangle_X,
\]

where \(\tau_{\vec{\beta}}(t_k|O_k)\) is the above mentioned blowup function. From the point of view of the original manifold, it should be a punctual defect. Thus, it is natural to expect it to be an infinite series of local operators \([27]\),

\[
\tau_{\vec{\beta}}(t_k|O_k) = \sum_{\vec{n} \in \mathbb{Z}^{N-1}_+} t^{\vec{n}} B_{\vec{n}, \vec{\beta}}(O_2, \ldots, O_N),
\]

where \(\vec{n} = (n_2, \ldots, n_N)\), \(t^{\vec{n}} = t_2^{n_2} \cdots t_N^{n_N}\), and the \(n\)-th order term comes from those \(\vec{n}\) with \(|\vec{n}| = \sum_i n_i = n\). By means of the \(u\)-plane integral, the blowup function can be written as \([25, 26, 27]\)

\[
\tau_{\vec{\beta}}(t_k|u_k) = e^{-\sum_{k,l} t_k t_l \mathcal{T}_{k,l}} \frac{\Theta(\vec{\Delta}, \vec{\beta})(\vec{\xi}|\tau)}{\Theta(\vec{\Delta}, \vec{0})(0|\tau)},
\]
where
\[ \xi_i = \sum_{k=2}^{N} \frac{t_k}{2\pi i} \frac{\partial u_k}{\partial a^i}. \] (2.13)

We shall consider in this talk the case in which there is no non-Abelian magnetic flux through the exceptional divisor, \( \vec{\beta} = \vec{0} \). In such a case, the quadratic contribution to the blowup function vanishes \([29]\),
\[ B_{\vec{n},\vec{0}}(O_2, \ldots, O_N) = 0, \quad \text{for } |\vec{n}| = 2. \] (2.14)

In \( SU(2) \), for example, the blowup function is nothing but an elliptic \( \sigma \)-function \([32]\). For higher rank theories, the answer should be some hyperelliptic generalization of the \( \sigma \)-function, as far as the blowup function is defined as a quotient of \( \Theta \)-functions with a prefactor that renders it to be duality invariant; precisely the same features that characterize a \( \sigma \)-function.

### 3. Geometrical detour: Kleinian functions

The theory of hyperelliptic Kleinian functions was developed one century ago by Baker \([40, 41, 42]\) and Bolza \([43, 44, 45, 46]\), among others. A comprehensive modern survey is presented in Ref.\([47]\). We shall introduce in this section some of the algebro-geometric ingredients that will be relevant in the remainder of the talk. Let \( \Sigma \) be a hyperelliptic curve of genus \( g \), which we write in the even form,
\[ y^2 = f(x) = \sum_{i=0}^{2g+2} \lambda_i x^i = Q(x)R(x). \] (3.15)

The last expression above provides a factorization of \( f(x) \) in two polynomials of degree \( g + 1 \). We will eventually be interested in the Seiberg-Witten setup, where
\[
Q_{SW}(x) = P_N(x) - 2\Lambda^N, \quad R_{SW}(x) = P_N(x) + 2\Lambda^N, \quad (3.16a) \\
P_N(x) = x^N - \sum_{k=2}^{N} u_k x^{N-k}. \] (3.16b)

Consider a symplectic basis of homology cycles, \( A^i, B_i \in H_1(\Sigma, \mathbb{Z}) \), and a canonical basis of Abelian differentials of the first kind, \( dv_k = x^{g-k} dx/y \), whose periods are
\[
A^i_k = \frac{1}{2\pi i} \oint_{A^i} dv_k, \quad B_{ik} = \frac{1}{2\pi i} \oint_{B_i} dv_k. \] (3.17)

It is well known that even and non-singular half-integer characteristics, \( [\vec{\alpha}, \vec{\beta}] \), are in one to one correspondence with the different factorizations of
\( f(x) \) as a product of \( Q(x) \) and \( R(x) \). In particular, for the Seiberg–Witten setup, it is precisely \([\Delta, 0]\), the characteristic appearing in the blowup function when there is no non-Abelian magnetic flux through \( B \). Now, in order to construct hyperelliptic \( \sigma \)–functions, it is also necessary to introduce a canonical basis of Abelian differentials of the second kind. This can be done by means of a Weierstrass polynomial, a function of two variables \( F(x_1, x_2) \) which is at most of degree \( g+1 \) both in \( x_1 \) and \( x_2 \), and satisfies the following conditions:

\[
F(x_1, x_2) = F(x_2, x_1), \quad F(x_1, x_1) = 2f(x_1), \quad (3.18a) \\
\left( \frac{\partial F(x_1, x_2)}{\partial x_1} \right)_{x_1=x_2} = f'(x_2). \quad (3.18b)
\]

Indeed, the following identity \([17]\)

\[
d\bar{v}(x_1) \cdot d\bar{r}(x_2) = - \frac{\partial}{\partial x_2} \left( \frac{y_2}{x_1 - x_2} \right) dx_1 + \frac{F(x_1, x_2)}{4(x_1 - x_2)^2} \frac{dx_1}{y_1} \frac{dx_2}{y_2}, \quad (3.19)
\]

implicitly defines such a basis, \( dr^k(x_2) \). An example of a Weierstrass polynomial that uses the factorization of the curve given above is

\[
F(x_1, x_2) = Q(x_1)R(x_2) + Q(x_2)R(x_1). \quad (3.20)
\]

The corresponding basis of Abelian differentials of the second kind, in the Seiberg–Witten setup, reads

\[
dr^j = \frac{1}{2} P_j'(x)P_N(x) \frac{dx}{y}, \quad j = 1, \cdots, N - 1. \quad (3.21)
\]

Two different Weierstrass polynomials are always seen to be related by

\[
F(x_1, x_2) - \hat{F}(x_1, x_2) = 4(x_1 - x_2)^2 \sum_{i,j=1}^{g} d_{ij} x_1^{g-i} x_2^{g-j}, \quad (3.22)
\]

where \( d_{ij} \) is symmetric in \( i, j \). Correspondingly, the relation between the two basis is

\[
dr^j = \hat{dr}^j + \sum_{k=1}^{g} d_{jk} dv_k. \quad (3.23)
\]

Given a basis of Abelian differentials of the second kind, one can define the analog of elliptic \( \eta \)–periods through

\[
\eta^{ki} = -\frac{1}{2\pi i} \oint_{A_i} dr^k, \quad \eta^{ki} = -\frac{1}{2\pi i} \oint_{B_i} dr^k. \quad (3.24)
\]
Thus, the hyperelliptic $\sigma$-function with even and non-singular characteristic can be defined as [43, 44, 45, 46]

$$\sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v}) = \exp\{v_i\kappa^{il}v_l\} \frac{\Theta[\vec{\alpha}, \vec{\beta}][(2\pi i)^{-1}v_l(A^{-1})^i_l|\tau]}{\Theta[\vec{\alpha}, \vec{\beta}](0|\tau)}, \quad (3.25)$$

where the ($F$-dependent) matrix $\kappa$ is given by

$$\kappa^{il} = \frac{1}{2} \eta^{ij}(A^{-1})^l_j. \quad (3.26)$$

For fixed characteristic, $\sigma$–functions corresponding to different Weierstrass polynomials are related by

$$\sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v}) = \exp\left(\frac{1}{2} \sum_{i,j} d_{ij}v_i v_j\right) \sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v}). \quad (3.27)$$

Notice that all the ingredients defined so far enter in Eq. (3.25). As mentioned above, the $\sigma$–function is duality invariant; that is, invariant under the action of the modular group $\text{Sp}(2g, \mathbb{Z})$. The exponential factor in (3.25) is chosen in such a way that it cancels the duality transformation properties of the quotient of $\Theta$–functions.

The hyperelliptic Kleinian functions are nothing but derivatives of the $\sigma$–function:

$$\zeta^F_{ij}[\vec{\alpha}, \vec{\beta}](\vec{v}) = \frac{\partial \ln \sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v})}{\partial v_j}, \quad (3.28a)$$

$$\wp^F_{ij}[\vec{\alpha}, \vec{\beta}](\vec{v}) = -\frac{\partial^2 \ln \sigma^F[\vec{\alpha}, \vec{\beta}](\vec{v})}{\partial v_i \partial v_j}, \quad (3.28b)$$

further indices denoting higher derivatives with respect to $v_k$. These functions satisfy differential equations that generalize Weierstrass’ cubic relation [17]. Furthermore, Kleinian functions that differ in their generating Weierstrass polynomial are related by

$$\wp^F_{ij}(\vec{v}) = \wp^F_{ij}(\vec{v}) - d_{ij}, \quad (3.29)$$

regardless of the characteristic. It was already shown by Bolza [16] that

$$\sum_{i,j} \wp^F_{ij}[\vec{\alpha}, \vec{\beta}](0) x_1^{g-i} x_2^{g-j} = \frac{F(x_1, x_2) - Q(x_1)R(x_2) - Q(x_2)R(x_1)}{4(x_1 - x_2)^2},$$

where $F$ is an arbitrary Weierstrass polynomial, and $[\vec{\alpha}, \vec{\beta}]$ is the characteristic associated to the factorization $y^2 = Q(x)R(x)$. Thus, it is evident that
\( \psi_{ij}^{\alpha,\beta}(0) \) vanishes for the Weierstrass polynomial (3.20). This is precisely the quadratic contribution to the \( \sigma \)-function. Consequently, as an outcome of this approach, we can fully characterize the blowup function of \( SU(N) \) twisted \( \mathcal{N} = 2 \) super Yang–Mills theories [34]:

\[ \tau(t_k|u_j) = \sigma^{\alpha,\beta}(it_{k+1}) . \]  

(3.30)

after identification of the Jacobian coordinates \( v_k \) with the times \( it_{k+1} \) (the reasons behind the latter name will be clear shortly).

An immediate corollary of this result is the following:

\[ \mathcal{T}_{k+1,l+1} \]  

are given by

\[ \mathcal{T}_{k+1,l+1} = \kappa^{k,l} = -\frac{1}{8\pi i} \frac{\partial u_{l+1}}{\partial a_i} \int_{\Delta} P_k(x)P_N(x) \frac{dx}{y} , \]  

(3.31)

where \( \kappa \) is the matrix introduced in (3.26), and we have used the explicit expression for \( dr^k \) given in (3.21).

In the remaining sections we will extract interesting consequences from these formulas. We will show, in particular, that this expression for the contact terms turns out to be very useful when considering the case of manifolds of simple type, where the only nonvanishing contributions to the \( u \)-plane integral come from those points of \( \mathcal{M} \) with maximal number of mutually local massless monopoles. The coincidence of (3.31) with the standard result (2.8), can be shown by means of the Whitham equations (the Renormalization Group equations in the formalism of Ref. [11]) that express the derivatives of the moduli with respect to \( T_n \) –closely related to the contact terms–, in terms of \( A \)-periods of a different basis of Abelian differentials of the second kind [34].

4. The blowup function and the KdV hierarchy

In order to show our main result, i.e. that the blowup function satisfies the differential equations of the KdV hierarchy, we shall first analyze the effect of special linear transformations on it. To this end, it is extremely useful to introduce a tricky symbolic notation –widely used by Baker in the
nineteenth century \[11, 12, 13\] for the geometrical data of the Seiberg–Witten solution. The hyperelliptic curve \(\Sigma\) can be written as

\[y^2 = (\alpha_1 + \alpha_2 x)^{2g+2},\quad (4.32)\]

where, of course, \(\alpha_1\) and \(\alpha_2\) are not complex numbers. The only meaningful object is the combination thereof

\[\lambda_p = \left(\frac{2g+2}{p}\right) \alpha_1^{2g+2-p} \alpha_2^p,\quad (4.33)\]

that renders (4.32) equal to (3.15). This notation allows us to introduce a quite interesting Weierstrass polynomial, the so-called \((g+1)\)-polar of the hyperelliptic curve,

\[\tilde{F}(x_1, x_2) = 2(\alpha_1 + \alpha_2 x_1)^{g+1}(\alpha_1 + \alpha_2 x_2)^{g+1} = 2 \sum_{p,q=1}^{2g+2} \frac{(g+1)(g+1)}{(p+q)} \lambda_{p+q} x_1^p x_2^q,\quad (4.34)\]

which is covariant with respect to an \(\text{Sl}(2, \mathbb{R})\) transformation of the \(x\)-coordinates. Indeed,

\[\tilde{F}(x_1, x_2) = (c + dt_1)^{-g-1}(c + dt_2)^{-g-1} \tilde{F}(t_1, t_2),\quad (4.35)\]

under \(x = (a + bt)/(c + dt),\ bc - ad = 1\). The hyperelliptic curve, in turn, becomes

\[Y^2 = (\beta_1 + \beta_2 t)^{2g+2} = \sum_{i=0}^{2g+2} \tilde{\lambda}_i t^i,\quad (4.36)\]

where \(Y = (c + dt)^{g+1} y,\ \beta_1 = c\alpha_1 + a\alpha_2,\ \beta_2 = d\alpha_1 + b\alpha_2\). It is clear that, by these means, one can always drive the hyperelliptic curve into its canonical form, \(\tilde{\lambda}_{2g+2} = \beta_2^{2g+2} = 0,\ \tilde{\lambda}_{2g+1} = \beta_1\beta_2^{2g+1} = 4\). The Abelian differentials of the first kind transform linearly,

\[x^{g-i} \frac{dx}{y} = (a + bt)^{g-i}(c + dt)^{i-1} \frac{dt}{Y} \Rightarrow dv_i(x) = \Lambda_i^m dv_m(t),\quad (4.37)\]

where \(\Lambda_i^m\) is an invertible matrix. Their periods get modified accordingly, \(A_i^j = \tilde{A}_i^m \Lambda_j^m,\ B_{ij} = \tilde{B}_{im} \Lambda_j^m\) so that

\[\tau_{ij} = \tilde{\tau}_{ij}.\quad (4.38)\]

Now, taking into account the covariance of \(\tilde{F}\), the transformation properties of the \(\eta\)-periods can be extracted –even without knowing the corresponding basis of Abelian differentials of the second kind–,

\[\tilde{\eta}^{ij} = \Lambda_i^k \eta^{kj}.\quad (4.39)\]
Thus, a linear transformation of the times
\[ \hat{v}_l = (\Lambda^{-1})^m v_m , \]  
(4.40)
yields
\[ \sigma^{\hat{F}}(\hat{\alpha}, \hat{\beta})(v_l(x,y)) = \sigma^{\hat{F}}(\hat{\alpha}, \hat{\beta})(\hat{v}_l(t,Y)) , \]  
(4.41)
where \( \hat{F} \) denotes here the polar associated to the corresponding curves. After substituting \( v_l = \Lambda^m v_m \), \( \sigma^{F}(\alpha, \beta)(v_l(x,y)) \) satisfies the same differential equations than \( \sigma^{F}(\hat{\alpha}, \hat{\beta})(\hat{v}_l(t,Y)) \) with respect to the hatted times.

There is a third Weierstrass polynomial that plays an important rôle in our proof. Let us call it \( \hat{F} \). It was introduced by Baker \[40\] a long time ago and revisited recently in Ref.\[47\]. It reads:
\[ \hat{F}(x_1, x_2) = 2\lambda_{2g+2}x_1^{g+1}x_2^{g+2} + \sum_{i=0}^{g} x_1^i x_2^i (2\lambda_{2i} + \lambda_{2i+1}(x_1 + x_2)) , \]  
(4.42)
and the corresponding basis of Abelian differentials of the second kind is
\[ dr^j = \sum_{k=g+1-j}^{g+j} (k + j - g) \lambda_{k-j+g+2} x^k dx \frac{dx}{4y} . \]  
(4.43)
As already mentioned above, hyperelliptic Kleinian functions satisfy differential equations which generalize those of the elliptic case like, for example, Weierstrass’ relation \( (\psi'(u))^3 = 4\psi(u)^3 - g_2\psi(u) - g_3 \). They were first studied by Baker in the case of genus two \[42\], and a generalization of his construction has been recently worked out \[47\]. The relevant differential equations are rather implicit. For the derivatives of \( \psi^{\hat{F}}_{11} \) one can, however, write an explicit equation for arbitrary genus:
\[ \psi^{\hat{F}}_{1111} = (6\psi^{\hat{F}}_{11} + \lambda_{2g})\psi^{\hat{F}}_{11} + \frac{1}{4} \lambda_{2g+1}(6\psi^{\hat{F}}_{i+1,1,1} - 2\psi^{\hat{F}}_{i+2} + \frac{1}{2} \delta_{i1} \lambda_{2g-1}) \\
+ \frac{1}{2} \lambda_{2g+2}(6\psi^{\hat{F}}_{i+2,1} - 6\psi^{\hat{F}}_{i+1,1,2} + 2\psi^{\hat{F}}_{i+3} - \delta_{i1} \lambda_{2g-2} - \frac{1}{2} \delta_{i2} \lambda_{2g-3}) . \]  
(4.44)
This equation, being of second order, is independent of the characteristic. It only shows up in the choice of initial conditions. A change in the Weierstrass polynomial amounts, after (3.29), to a \( v \)-independent shift. Let us now drive the curve to its canonical form by means of an \( \text{Sl}(2, \mathbb{R}) \) transformation. The equation above becomes
\[ \psi^{\hat{F}}_{1111} = (6\psi^{\hat{F}}_{11} + \lambda_{2g})\psi^{\hat{F}}_{11} + 6\psi^{\hat{F}}_{i+1,1,1} - 2\psi^{\hat{F}}_{i+2} + \frac{1}{2} \delta_{i1} \lambda_{2g-1} , \]  
(4.45)
where $\hat{F}$ is the polar associated to the canonical curve. It is now easy to show that eq. (4.43) implies that the hyperelliptic Kleinian functions satisfy the equations of the KdV hierarchy [47]. Indeed, take $\mathcal{U} = 2\varphi_{11} + \frac{1}{6}\hat{\lambda}_{2g}$, put $x \equiv v_1$, and let $t_i = v_i$ be the higher evolution times, so that
\[
\frac{\partial \mathcal{U}}{\partial t_2} = \frac{1}{4} \mathcal{U}''' - \frac{3}{2} \mathcal{U}''',
\]
where $'$ denotes derivatives with respect to $x$. This is precisely the KdV equation. In fact, $\mathcal{U}$ is a $g$-gap solution of the KdV hierarchy. To see this, recall that the higher evolution equations of the hierarchy are (for a review, see Appendix A of Ref. [48]),
\[
\frac{\partial \mathcal{U}}{\partial t_i} = R'_i(\mathcal{U}, \mathcal{U}', \cdots), \quad i \geq 3,
\]
where the functions in the right hand side are defined recursively,
\[
R'_{i+1} = \frac{1}{4} R''_i - (\mathcal{U} + \frac{\hat{\lambda}_{2g}}{12}) R'_i - \frac{1}{2} \mathcal{U}' R_i.
\]
The blowup function (3.30) can be finally written as
\[
\tau(v_m = \Lambda_m l \hat{v}_i | \mathcal{O}_i) = e^{\sum c_{ij} \hat{v}_i \hat{v}_j} \sigma \hat{F}[\hat{\Delta}, \hat{0}](\hat{v}_i(t,Y)),
\]
where $\Lambda$ is the appropriate transformation yielding the hyperelliptic curve canonical, and the $c_{ij}$ are constants depending on the parameters of the $\text{SL}(2,\mathbb{R})$ transformation and the moduli of the curve, that can be computed explicitly by comparison of $\sigma$–functions defined for different Weierstrass polynomials. We have thus arrived to the following result:
\[
\mathcal{U} = -2 \frac{\partial^2 \log \tau}{\partial \hat{v}_1^2} + 4c_{11} + \frac{1}{6} \hat{\lambda}_{2g},
\]
is a $g$-gap solution of the KdV hierarchy. In other words, the blowup function is –up to a redefinition of the evolution times and a constant shift–, a $\tau$–function of the KdV hierarchy.

The blowup function appears in the generating function of the correlators involving the exceptional divisor. Thus, a corollary of the above result is that these correlation functions on the manifold $\hat{X}$ are governed by the KdV hierarchy, and they have as initial conditions the generating function of the original manifold $X$. It is intriguing that the differential equations turn out to be essentially the same than those governing the correlation functions of two-dimensional topological gravity [33], though the blowup function –being a $g$-gap solution– lies far apart of these correlation functions in...
the space of solutions of the KdV hierarchy. Finally, it is now clear that the differential equation originally used in the elliptic case \cite{32}, can be understood, under the light of these results, as the reduction of the KdV equation.

5. The blowup function for manifolds of simple type

We have already mentioned that the whole nontrivial contribution to the \( u \)-plane integral for manifolds of simple type comes from \( N \) maximal singularities of \( M \). These singularities, where \( N - 1 \) mutually local monopoles get massless, are also known as \( N = 1 \) points because they are the confining vacua after breaking \( N = 2 \) down to \( N = 1 \). The curve \( \Sigma \) can be described in the vicinity of one of these points by Chebyshev polynomials (we set \( \Lambda = 1 \)) \cite{38}

\[
P_N(x) = 2 \cos(N \arccos \frac{x}{2}) ,
\]
and the other \( N = 1 \) points are obtained from the former by means of the \( \mathbb{Z}_N \) symmetry of the theory. From now on we will focus on this \( N = 1 \) point. There, the branch points of the curve become (single) \( e_1 = -e_{2g+2} = 2 \), and (double) \( e_{2k} = e_{2k+1} = \hat{\phi}_k = 2 \cos \hat{\theta}_k \), \( \hat{\theta}_k = (\pi k/N) \). The values of the Casimirs are given by the elementary symmetric polynomials of the eigenvalues \( 2 \cos \theta_i \), \( \theta_i = \pi(i - 1/2)/N \). For example, \( u_2 = N \), \( u_3 = 0 \), \( u_4 = \frac{N}{2}(3 - N) \), etc. The \( B \)-cycles surround the points \( \hat{\phi}_i \) clockwise, while the \( A \)-cycles become curves going from \( \hat{\phi}_i \) to \( 2 \) on the upper sheet and returning to \( \hat{\phi}_i \) on the lower sheet. The hyperelliptic curve becomes

\[
y = \sqrt{x^2 - 4} \prod_{k=1}^{g} (x - \hat{\phi}_k) .
\]

Consider now the normalized \textit{magnetic} holomorphic differentials,

\[
\omega^j = (B^{-1})^{kj} dv_k = -\frac{2i \sin \hat{\theta}_j}{\sqrt{x^2 - 4} (x - \hat{\phi}_j)} ,
\]
that is, the canonical basis of Abelian differentials of the first kind with respect to the \( B \)-cycles,

\[
\frac{1}{2\pi i} \oint_{B_i} \omega^j = -\text{res}_{x=\hat{\phi}_i} \omega^j = \delta_i^j .
\]

We can explicitely compute the derivatives of the moduli with respect to the dual coordinates,

\[
\frac{\partial u_{\ell+1}}{\partial a_{D,m}} = 2i(-1)\ell \sin \hat{\theta}_m E_{\ell-1}(\hat{\phi}_p \neq m) ,
\]
where \( E_0 = 1 \), and \( E_j = \sum_{i_1 < \ldots < i_j} x_{i_1} \cdots x_{i_j} \) are the elementary symmetric polynomials of degree \( j \). Near the \( N = 1 \) points, the diagonal components of the magnetic couplings diverge, but the off-diagonal components are finite. The leading terms of the off-diagonal components have been investigated in Ref. [38], where an implicit expression for them was proposed in terms of an integral involving a scaling trajectory. In the framework of the Whitham hierarchy approach to four-dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theories, some nontrivial constraints arise on those terms, and an explicit expression satisfying the constraints was proposed [16]. We will now derive a very simple expression for the leading terms of the off-diagonal couplings. From the above considerations, it follows that

\[
\tau_{k\ell}^{D} = \frac{1}{\pi i} \int_{\hat{\phi}_k}^{\hat{\phi}_\ell} \omega^\ell = \frac{1}{\pi i} \log \frac{\gamma_\ell - \gamma_k}{\gamma_\ell + \gamma_k}, \quad k < \ell,
\]

where

\[
\gamma_j = -i \sqrt{\frac{\hat{\phi}_j - 2}{\hat{\phi}_j + 2}} = \tan \frac{\pi j}{2N}.
\]

This expression agrees with that conjectured in Ref. [16], as it was proved very recently by Braden and Marshakov [49].

In order to compute the contribution of the \( N = 1 \) points to the blowup function of simple type manifolds, we must first perform a duality transformation to the magnetic coordinates. Being duality invariant, the blowup function remains the same, except for the fact that everything has to be replaced by its dual. In particular, the characteristic dual to \( [\bar{\Delta}, \bar{\Delta}] \) is \( [\bar{\Delta}, \bar{\Theta}] \).

In order to compute the contact terms, it is extremely useful to use the expression we derived before, say (3.31), adapted to the dual frame. We just have to compute the \( B \)-periods of the Abelian differentials of the second kind (3.21) at the \( N = 1 \) point. After the change of variables \( x = 2 \cos \theta \),

\[
dr^\ell = iP'_{\ell}(\theta) \cot N\theta \sin \theta \ d\theta,
\]

and their \( B \)-periods simply read

\[
\eta^\ell_k = \text{res}_{\theta=\hat{\theta}_k} dr^\ell = \frac{i}{N} P'_{\ell}(\hat{\phi}_k) \sin \hat{\theta}_k.
\]

The contact terms are then given by

\[
\tau_{k,\ell} = \frac{i}{2N} P'_{k-1}(\hat{\phi}_m) \sin \hat{\theta}_m \frac{\partial u^\ell}{\partial a_{D,m}}.
\]

We can now write the resulting expression for the blowup function. Notice first that the dual \( \Theta \)-function vanishes at the \( N = 1 \) point. However, after
quotienting by $\Theta[\bar{\partial}, \bar{\Delta}](0|\tau_D)$, we get a finite result. At the end of the day, the blowup function corresponding to simple type manifolds is given by

$$
\tau(t_i) = \left[ \sum_{s_p = \pm 1} \prod_{p < q} (\gamma_q - \gamma_p)^{s_p s_q/2} \right]^{-1} \exp \left\{ - \sum_{k,\ell} t_k t_\ell T_{k,\ell} \right\} \cdot \sum_{s_j = \pm 1} \prod_{p < q} (\gamma_q - \gamma_p)^{s_p s_q/2} \exp \left\{ \sum_{l=2}^N \frac{is_j t_l}{2} \frac{\partial u_l}{\partial a_{D,j}} \right\},
$$

(5.61)

with $T_{k,\ell}$ given above. It is, after a linear transformation of the time $s$, a $\tau$ function for an $(N-1)$–soliton solution of the underlying KdV hierarchy [34]. This is a simple consequence of the fact that quasi-periodic solutions of the KdV hierarchy become multisoliton solutions in the limit of maximal degeneracy of the underlying Riemann surface [30] (see also Ref. [49] for recent progress in this direction). Let us finally point out that, regardless of its apparently involved expression, the blowup function for simple type manifolds turns out to be, on general grounds, simplified. For example, in the case of $SU(3)$, eq. (5.61) reads

$$
\tau_{SU(3)}(t_2, t_3) = \frac{1}{3} e^{-\frac{1}{2} t_2^2 - t_3^2} \left\{ \cosh(\sqrt{3}t_2) + 2 \cosh(\sqrt{3}t_3) \right\}.
$$

(5.62)

This fact was already observed in the elliptic case [32], and has to do with the degeneration of hyperelliptic functions into trigonometeric ones.

An important consistency check of our expression for the blowup function can be made by considering the explicit expression of the Donaldson–Witten generating function for manifolds of simple type [27], which can be trivially extended to include more general descent operators [34]:

$$
Z(p_k, f_k, S)^{N=1}_X = \alpha^N \beta^\sigma \sum_{x_j} \prod_{j=1}^{N-1} SW(x_j) \prod_{j < k} (\gamma_j - \gamma_k)^{-(x_j, x_k)/2} \exp \left\{ \sum_{k=2}^N \left( p_k u_k - \frac{i}{2} f_k \frac{\partial u_k}{\partial a_{D,j}} (S, x_j) \right) + S^2 \sum_{k,l} f_k f_l T_{k,l} \right\},
$$

(5.63)

where the values of the $B$–periods and contact terms are those given in (5.55) and (5.60), and $\alpha$ and $\beta$ are universal constants that only depend on $N$. Only the contribution of one of the $N = 1$ points is recorded in the equation above, those of the other points following from $Z_N$ symmetry. For each $i = 1, \ldots, N - 1$, the sum over $x_i$ is over all the Seiberg–Witten basic classes of the manifold $X$ [24], whose Seiberg–Witten invariants are denoted by $SW(x_i)$. $(\ , \ )$ denotes the product in (co)homology. After a blowup, every basic class $x$ of $X$ leads to basic classes $x \pm B$ in $\hat{X}$, where latter $x$
really denotes the pullback to $\tilde{X}$ of the basic class of $X$. The Seiberg–Witten invariants are $SW(x \pm B) = SW(x)$. If we now consider $Z(p, f, S)^{X}_{\tilde{X}} = 1$, we will have to substitute $x_i \to x_i + s_i B$ in (5.63), with $s_i = \pm 1$. The sum over basic classes of $\tilde{X}$ factorizes into a sum over the $x_i$ and a sum over the $s_i$. Taking into account that $(x, B) = 0$ for any cohomology class $x$ pulled back from $X$ to $\tilde{X}$, and that $B^2 = -1$, eq. (5.63) gets an extra factor under blowup that agrees with (5.61) [34]. This is a crucial nontrivial consistency check of the whole set of results presented in this talk, since, when using (5.63), we have to rely on properties of the Seiberg–Witten invariants, while the blowup function (5.64) was derived by means of the $u$-plane integral.

6. Final remarks

An important aspect of blowup functions [25, 26, 27] is that they must admit an expansion of the form (2.11), whose coefficients are polynomials in the local observables of the topological field theory. In the case of $SU(2)$, this is rather explicit from the theory of elliptic functions. As recently shown in Ref. [34], it turns out that an elegant and powerful recursive method to perform this expansion in the case of $SU(N)$ –up to arbitrary order in the $t_i$–, follows from the theory of hyperelliptic Kleinian functions [43, 44, 45, 46]. For example, in the case of $SU(3)$, the outcome result is

$$
\tau_{SU(3)}(t_i|u_i) = 1 - \frac{\Lambda^6}{12} \left[ u_2 t_3^4 + 6t_2^2 t_3^2 \right] - \frac{\Lambda^6}{360} \left[ 3t_2^6 - 15u_2 t_3^4 t_2^2 - 60u_3^3 t_3^3 
- 15u_2^2 t_2^4 t_3^2 - 12u_2 u_3 t_2 t_3^5 - u_2^3 t_3^6 + 3u_2^2 t_3^6 - 12\Lambda^6 t_3^6 \right] + \cdots . \quad (6.64)
$$

A further consistency check comes from duality invariance of the blowup function: this expansion coincides with (5.62) when $u_2 = 3$ and $u_3 = 0$.

The theory of hyperelliptic Kleinian functions has shown to be an appropriate framework to address many aspects of the blowup formulas and the $u$-plane integral, like contact terms and the relation with integrable hierarchies. It would be very interesting to work out the details for theories including massive hypermultiplets and/or other gauge groups. In massive theories, for example, whose Whitham formulation was worked out in [51], the magnetic flux through the class of the exceptional divisor turns out to be fixed by topological constraints, this giving a nonzero value of $\vec{\beta}$ in the blowup function [25, 24].

Another direction to explore is the relation between the hyperelliptic Kleinian functions and the theory of the prepotential. The blowup function gives a natural set of Abelian differentials of the second kind –which differs from the one given in [11]–, and we know from general principles that such
a set is one of the basic ingredients in the construction of a Whitham hierarchy [8]. It would be very interesting to develop this relation in general, at least for hierarchies associated to hyperelliptic curves. This would further clarify the relations between blowup functions in generalizations of Donaldson–Witten theory, and the construction of Whitham hierarchies for supersymmetric $\mathcal{N} = 2$ theories.

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