Abstract
Let $S$ be a closed, oriented surface of genus $g \geq 2$, and consider the extension $1 \to \pi_1 S \to \text{MCG}(S, p) \to \text{MCG}(S) \to 1$, where $\text{MCG}(S)$ is the mapping class group of $S$, and $\text{MCG}(S, p)$ is the mapping class group of $S$ punctured at $p$. We prove that any quasi-isometry of $\text{MCG}(S, p)$ which coarsely respects the cosets of the normal subgroup $\pi_1 S$ is a bounded distance from the left action of some element of $\text{MCG}(S, p)$. Combined with recent work of Kevin Whyte this implies that if $K$ is a finitely generated group quasi-isometric to $\text{MCG}(S, p)$ then there is a homomorphism $K \to \text{MCG}(S, p)$ with finite kernel and finite index image. Our work applies as well to extensions of the form $1 \to \pi_1 S \to \Gamma_H \to H \to 1$, where $H$ is an irreducible subgroup of $\text{MCG}(S)$—we give an algebraic characterization of quasi-isometries of $\Gamma_H$ that coarsely respect cosets of $\pi_1 S$.

1 Introduction
A surface group extension $1 \to \pi_1(S) \to \Gamma \to H \to 1$ is a short exact sequence where $S$ is a closed, oriented surface of genus $\geq 2$. When $S$ is fixed the universal such extension has two isomorphic descriptions, according to the Dehn-Nielsen-Epstein-Baer theorem:

$$
\begin{array}{cccccc}
1 & \to & \pi_1(S) & \to & \text{MCG}(S, p) & \to & \text{MCG}(S) & \to & 1 \\
& & \parallel & & \parallel & & \parallel & & \\
1 & \to & \pi_1(S) & \to & \text{Aut}(\pi_1(S)) & \to & \text{Out}(\pi_1(S)) & \to & 1 \\
\end{array}
$$

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The universality property says that every surface group extension arises from a homomorphism of short exact sequences

$$
1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \longrightarrow H \longrightarrow 1
$$

$$
1 \longrightarrow \pi_1(S) \longrightarrow \text{MCG}(S,p) \longrightarrow \text{MCG}(S) \longrightarrow 1
$$

and the extension is uniquely determined, up to the appropriate equivalence relation, by the homomorphism $H \to \text{MCG}(S)$. In the special case of an extension determined by inclusion of a subgroup $H < \text{MCG}(S)$, the extension group will be denoted $\Gamma_H$.

In studying the large scale geometry of a finitely generated group $\Gamma$, it is natural to consider the quasi-isometry group $\text{QI}(\Gamma)$, which is the group of self-quasi-isometries of $\Gamma$ with its word metric, modulo the relation that two quasi-isometries from $\Gamma$ to itself are equivalent if their sup distance is finite. When $\Gamma$ is a surface group extension, in many situations it is also natural to focus on the subgroup $\text{QI}_f(\Gamma) < \text{QI}(\Gamma)$ consisting of classes of self quasi-isometries that coarsely respect the decomposition of $\Gamma$ into cosets of $\pi_1(S)$; we call these fiber respecting quasi-isometries. For example, when $H$ is a free group and so $\Gamma_H$ is surface-by-free, then every quasi-isometry of $\Gamma$ is fiber respecting, that is, $\text{QI}_f(\Gamma) = \text{QI}(\Gamma)$ [EM00]. For another example, the left action of $\Gamma$ on itself is fiber respecting, by normality of $\pi_1(S)$, and so the induced homomorphism $\Gamma \to \text{QI}(\Gamma)$ factors through a homomorphism $\Gamma \to \text{QI}_f(\Gamma)$.

Our goal in this paper is to study the group $\text{QI}_f(\Gamma)$ in various cases. For example, when $\Gamma$ is the universal extension group $\text{MCG}(S,p)$ itself we obtain:

**Theorem 1.** The injection $\text{MCG}(S,p) \to \text{QI}_f(\text{MCG}(S,p))$ is an isomorphism.

Kevin Whyte has developed new methods, using uniformly finite homology, for showing that every self quasi-isometry is fiber respecting. Whyte’s methods apply, for example, to a surface-by-free group, but they also apply in more general situations where the quotient group $H$ has dimension greater than 1. In particular, Whyte proves:

**Theorem 2 (Whyte).** Every self quasi-isometry of $\text{MCG}(S,p)$ is fiber respecting, that is, $\text{QI}_f(\text{MCG}(S,p)) = \text{QI}(\text{MCG}(S,p))$.

As mentioned, Whyte’s techniques apply to far more general situations, but for a proof of just this theorem see Mos03.

Combining Theorems 1 and 2 we obtain a strong quasi-isometric rigidity theorem for once-punctured mapping class groups:
Corollary 3 (Mosher–Whyte). The injection $\text{MCG}(S, p) \rightarrow \text{QI}(\text{MCG}(S, p))$ is an isomorphism.

Each of these three results has a more quantitative version—in the last corollary, the conclusion is that for each $K \geq 1$, $C \geq 0$ there exists $A \geq 0$ such that if $\phi : \text{MCG}(S, p) \rightarrow \text{MCG}(S, p)$ is a $K, C$ quasi-isometry, then there exists $g \in \text{MCG}(S, p)$ such that $d(\phi(f), gf) \leq A$ for all $f \in \text{MCG}(S, p)$. From this quantitative version, it follows by a standard argument that:

Corollary 4 (Mosher–Whyte). If $G$ is any finitely generated group quasi-isometric to $\text{MCG}(S, p)$ then there is a homomorphism $G \rightarrow \text{MCG}(S, p)$ with finite kernel and finite index image.

The kind of quasi-isometric rigidity given in Corollaries 3 and 4 is generally conjectured to occur for mapping class groups of nonexceptional finite type surfaces, and this is the first case where the conjecture is verified.

Theorem 1 will follow from a more general result which applies to surface group extensions $1 \rightarrow \pi_1(S) \rightarrow \Gamma_H \rightarrow H \rightarrow 1$ where $H$ is a subgroup of $\text{MCG}(S)$ that contains at least one pseudo-Anosov mapping class. For the reader who wants a direct proof of Theorem 1 without the generalization to groups $\Gamma_H$, and who wants to see details about Theorem 2, we refer to [Mos03a].

By Ivanov’s theorem, a subgroup $H \subset \text{MCG}(S)$ contains a pseudo-Anosov mapping class if and only if $H$ is infinite and irreducible, meaning that there does not exist an essential curve system on $S$ that is preserved by every element of $H$.

Consider the extension $\Gamma_H$ of $\pi_1(S)$ by an infinite, irreducible subgroup $H \subset \text{MCG}(S)$. The homomorphism $\Gamma_H \rightarrow \text{QI}_f(\Gamma_H)$ is an injection, but in general there may be additional elements of $\text{QI}_f(\Gamma_H)$ which can be thought of as “hidden” symmetries of $\Gamma_H$. To describe these, let $S \rightarrow \mathcal{O}_H$ be the orbifold covering map of largest degree such that $H$ descends to a subgroup of $\text{MCG}(\mathcal{O}_H)$, denoted $H'$. Let $C$ be the relative commensurator of $H'$ in $\text{MCG}(\mathcal{O}_H)$, that is, the group of all $g \in \text{MCG}(\mathcal{O}_H)$ for which $g^{-1}H'g \cap H'$ has finite index in both $g^{-1}H'g$ and $H'$. We obtain an extension group

$$1 \rightarrow \pi_1(\mathcal{O}_H) \rightarrow \Gamma_C \rightarrow C \rightarrow 1$$

From this description we obtain an obvious injection $\Gamma_C \rightarrow \text{QI}_f(\Gamma_H)$. The question arises whether there is anything else in $\text{QI}_f(\Gamma_H)$, and the answer is no:

Theorem 5. If $1 \rightarrow \pi_1(S) \rightarrow \Gamma_H \rightarrow H \rightarrow 1$ is a surface group extension where $H$ is an infinite, irreducible subgroup of $\text{MCG}(S)$, then the injection $\Gamma_C \rightarrow \text{QI}_f(\Gamma_H)$ is an isomorphism.
To derive Theorem 1 from Theorem 5, we simply note that $S$ itself is the maximal subcover of $S$ to which $\text{MCG}(S)$ descends, and $\text{MCG}(S)$ is its own relative commensurator in $\text{MCG}(S)$. We will also see how to prove the more quantitative versions of Theorem 1 and Corollary 3.

In the special case when $H$ is a Schottky subgroup of $\text{MCG}(S)$, Theorem 5 was proved in [FM02b]. The general proof of Theorem 5 follows the same outline, but with several changes in detail, the most important of which we highlight here. First, whereas the nontrivial elements of a Schottky subgroup are entirely pseudo-Anosov, the most we can say for an irreducible subgroup is that it is generated by its pseudo-Anosov elements (with a simple exception; see Lemma 8); this distinction permeates the proof. Second, the definition of “the orbifold subcover of largest degree to which a subgroup descends” was handled more easily in [FM02b] because we were working only with free subgroups, but it turns out that descent is sufficiently well behaved even for arbitrary subgroups (see Lemma 16). Third, in [FM02b] the key step of proving that $\Gamma_C \to \text{QI}_f(\Gamma_H)$ is surjective depended on an explicit computation of the relative commensurator of a Schottky subgroup of a mapping class group; in the present situation the computation is far less explicit, and so we must work with some general properties of relative commensurators.

2 Preliminaries

Coarse language. A map $f : X \to Y$ between two metric spaces is a $K,C$ quasi-isometry, $K \geq 1, C \geq 0$, if

$$\frac{1}{K} d_X(a,b) - C \leq d_Y(fa,fb) \leq K d_X(a,b) + C$$

for all $a,b \in X$, and for all $c \in Y$ there exists $a \in X$ such that $d_Y(fa,c) \leq C$. A map $f : Y \to X$ is a coarse inverse for $f$ if $d_X(f \circ fx,x)$ and $d_Y(f \circ fy,y)$ are uniformly bounded for $x \in X, y \in Y$.

Given a metric space $X$, let $\widehat{\text{QI}}(X)$ denote the collection of all self-quasi-isometries of $X$ with the operation of composition. Two elements $f,g \in \widehat{\text{QI}}(X)$ are considered to be equivalent if $\sup\{d(fx,gy) \mid x \in X\} < \infty$. Composition descends to a group operation on the quotient $\text{QI}(X)$, whose identity element is the class of the identity. The inverse of the class of $f \in \widehat{\text{QI}}(X)$ is the class of any coarse inverse $\overline{f}$.

Given a metric space $X$ and two subset $A, B$, we say that $A$ is coarsely included in $B$ if there exists $r \geq 0$ such that $A \subset N_r(B)$. We say that $A, B$ are coarsely equivalent if each is coarsely contained in the other; equivalently, the Hausdorff distance $d_H(A,B) = \inf\{r \mid A \subset N_r(B), B \subset N_r(A)\}$ is finite.
We shall need some facts about coarse inclusions among subgroups of a finitely generated group, taken from [MSW03]. Recall that two subgroups $A, B < G$ are *commensurable* if $A \cap B$ has finite index in $A$ and in $B$.

**Proposition 6.** If $G$ is a finitely generated group with the word metric and $A, B$ are subgroups of $G$, then $A$ is coarsely contained in $B$ if and only if $A \cap B$ has finite index in $A$, and $A, B$ are coarsely equivalent if and only if $A, B$ are commensurable in $G$.  

Consider a finitely generated group $G$ and a normal subgroup $N < G$. For each $g \in G$, since $gNg^{-1} = N$ it follows that the Hausdorff distance from each coset $gN = Ng$ to the subgroup $N$ is finite, equal to at most the word length of $g$. A quasi-isometry $\Phi: G \to G$ is fiber preserving with respect to $N$ if there is a constant $R$ such that for each coset $gN$ there exists a coset $g'N$ such that $d_H(\Phi(gN), g'H) \leq R$. A composition of fiber preserving quasi-isometries is fiber preserving, and the identity is fiber preserving, so the subset of $\text{QI}(G)$ represented by fiber preserving quasi-isometries (with respect to $N$) is a subgroup denoted $\text{QI}_f(G)$. Each fiber preserving quasi-isometry $\Phi: G \to G$ induces a quasi-isometry $\phi: G/N \to G/N$ which is well-defined up to equivalence: given the constant $R \geq 0$ as above, for each coset $gN$ choose $g'N$ as above and define $\phi(gN) = g'N$.

**The virtual normalizer and virtual centralizer of a pseudo-Anosov mapping class.** Consider a pseudo-Anosov $f \in \text{MCG}(S)$ with stable and unstable foliations $\mathcal{F}^s$, $\mathcal{F}^u$. Recall that the action of $f$ on $\text{PMF}(S)$ has north-south dynamics with repelling fixed point $\mathcal{P}\mathcal{F}^s$ and attracting fixed point $\mathcal{P}\mathcal{F}^u$. The virtual normalizer $\text{VN}(f)$ and the virtual centralizer $\text{VC}(f)$ of the infinite cyclic subgroup $\langle f \rangle$ generated by $f$ are defined to be

$$\text{VN}(f) = \{ g \in \text{MCG}(S) \mid g^{-1}\langle f \rangle g \cap \langle f \rangle \text{ has finite index in } g^{-1}\langle f \rangle g \text{ and in } \langle f \rangle \}$$

$$\text{VC}(f) = \{ g \in \text{MCG}(S) \mid \exists n \geq 0 \text{ such that } g^{-1}f^n g = f^n \}$$

We need the following alternative description of these subgroups, which is due to McCarthy and can be found in his preprint [McC82].

**Proposition 7.** The subgroup $\text{VN}(f)$ is the stabilizer of the set $\text{Fix}(f) = \{ \mathcal{P}\mathcal{F}^s, \mathcal{P}\mathcal{F}^u \}$, and $\text{VC}(f)$ is the kernel of the action of $\text{VN}(f)$ on this set, and so $\text{VC}(f)$ is a subgroup of index at most 2 in $\text{VN}(f)$. Each of these subgroups is virtually infinite cyclic, in fact, $\text{VN}(f)$ is the maximal virtually cyclic subgroup of $\text{MCG}(S)$ containing $f$, and $\text{VC}(f)$ is the maximal subgroup containing $f$ which is virtually cyclic and contains no $D_\infty$ subgroup.
One can easily show that the inclusion $\text{VC}(f) < \text{VN}(f)$ is proper if and only if there is a pseudo-Anosov element of $\text{VN}(f)$ that is conjugate to its own inverse.

*Proof.* For completeness, here is a proof of these facts.

Consider the Teichmüller geodesic $\gamma$ with endpoints $P,F^s,P,F^u$, and clearly $\text{Stab}(\gamma) = \text{Stab}(\text{Fix}(f))$. The geodesic $\gamma$ is the axis of $f$ in Teichmüller space. The group $\text{Stab}(\gamma)$ acts properly and cocompactly on $\gamma$ and so is a virtually cyclic group, containing $\langle f \rangle$ with finite index, and therefore implying that $\text{Stab}(\gamma) \subset VN(f)$. Conversely, if $g \in \text{MCG}(S) - \text{Stab}(\text{Fix}(f))$ then $g^{-1}(\gamma) \neq \gamma$, and $g^{-1}f^ng$ is a pseudo-Anosov mapping class with axis $g^{-1}(\gamma)$ for any $n$. By uniqueness of axes, for any $m,n \neq 0$ we have $g^{-1}f^ng \neq f^m$. This proves that $\text{VN}(f) = \text{Stab}(\text{Fix}(f))$, and the remaining contentions are easily obtained.

Irreducible subgroups of $\text{MCG}(S)$. Ivanov’s theorem gives a dichotomy for infinite subgroups $G < \text{MCG}(S)$: either $G$ is reducible meaning that there exists a collection $C$ of pairwise disjoint, essential simple closed curves in $S$ such that each element of $G$ preserves $C$ up to isotopy; or $G$ contains a pseudo-Anosov element. Briefly, every infinite, irreducible subgroup of $\text{MCG}(S)$ contains a pseudo-Anosov element.

We wish to show that irreducible subgroups are generated by their pseudo-Anosov elements. Actually, this is not quite true: if the pseudo-Anosov mapping class $f$ has the property that the inclusion $\text{VC}(f) < \text{VN}(f)$ is proper, then the subgroup of $\text{VN}(f)$ generated by the pseudo-Anosov elements is precisely $\text{VC}(f)$. But this is essentially the only counterexample, as the following result shows:

**Proposition 8.** Consider a subgroup $H < \text{MCG}(S)$ which contains a pseudo-Anosov element.

The nonelementary case If $H \nsubseteq \text{VN}(f)$ for any pseudo-Anosov $f \in \text{MCG}(S)$ then $H$ is generated by its pseudo-Anosov elements.

The elementary case If $H < \text{VN}(f)$ for some pseudo-Anosov $f \in \text{MCG}(S)$ then the subgroup of $H$ generated by the pseudo-Anosov elements is $H \cap \text{VC}(f)$.

*Proof.* The proof follows quickly from Exercise 3a on page 102 of [Iva92], which says that if $f \in \text{MCG}(S)$ is pseudo-Anosov and if $g \in \text{MCG}(S)$ has the property that $g(F^u(f)) \neq F^s(f)$, then $f^ng$ is pseudo-Anosov for all sufficiently large $n \geq 0$. Here are the details, including the solution of the exercise.

Consider first the elementary case, $H < \text{VN}(f)$. Note that each pseudo-Anosov element of $\text{VN}(f)$ fixes $P,F^s$ and $P,F^u$, and so the subgroup they generate is
contained in $VC(f)$. Conversely, given $g \in VC(f) \cap H$ and a pseudo-Anosov element $f \in VN(f) \cap H$, clearly $h = f^ng$ is pseudo-Anosov for sufficiently large $n$, and $g = hf^{-n}$.

Consider next the nonelementary case. For any nonidentity element $g \in MCG(S)$ we may choose a pseudo-Anosov $f \in MCG(S)$ so that $g \notin VN(f)$. It follows that $g$ does not stabilize the set $\{PF^u, PF^s\}$ and so, replacing $f$ by $f^{-1}$ if necessary, we may assume that $g(F^u(f)) \neq F^u(f)$.

We shall prove that $f^ng$ is pseudo-Anosov for sufficiently large $n$ by showing that $f^ng$ has no fixed points in $MF$.

Choose neighborhoods $V^s$ of $PF^s$ and $V^u$ of $PF^u$ so that $V^s \cap V^u = \emptyset$, and so that $g^{-1}V^s \cap V^u = \emptyset$; this is possible because $g(PF^u) \neq PF^s$. Let $W = g^{-1}V^s$. If $n$ is sufficiently large it follows that $f^ng(PMF - W) \subset V^u$, and so the only possible fixed points of $f^ng$ in $PMF$ are in $W$ or in $V^u$.

For a subset $A \subset PMF$ let $R_+A$ denote the preimage of $A$ under the projection $MF \to PMF$. The only possible fixed points of $f^ng$ in $MF$ are in $R_+W$ or in $R_+V^u$.

Consider the action of $f^ng$ on $MF$. Choose a continuous norm $\| \cdot \|$ on $MF$, meaning that $\|r \cdot F\| = r \|F\|$ for each $F \in MF$, $r > 0$. Since $f(F^u) = \lambda F^u$ with $\lambda > 1$, it follows that if $n$ is sufficiently large then there is a constant $m > 1$ such that $\|f^n(g(F))\| \geq m \|F\|$ for all $F \in MF - R_+W$; it follows that $f^ng$ has no fixed points in $R_+V^u$. Similarly, if $n$ is sufficiently large then $\|g^{-1}f^{-n}(F)\| \geq m \|F\|$ for all $F \in R_+W$. It follows that $f$ has no fixed points in $R_+W$. \)

### Teichmüller space and its canonical bundle.

Let $\mathcal{T} = T(S)$ denote the Teichmüller space of $S$. Let $X_T \to \mathcal{T}$ denote the canonical hyperbolic plane bundle over $\mathcal{T}$ on which $\pi_1 S$ acts: the fiber $D_x$ over a point $x \in \mathcal{T}$ is the universal cover of a hyperbolic surface representing the point $x$. The action of $MCG(S)$ on $T(S)$ lifts to an action of $MCG(S, p)$ on $X_T$, indeed $X_T$ can be identified naturally as the Teichmüller space of the once punctured surface $S - p$ \cite{Ber73}.

We fix once and for all an $MCG(S, p)$ equivariant Riemannian metric on $X_T$ whose restriction to each fiber is the hyperbolic metric on that fiber.

A path in $\mathcal{T}$ will always mean a piecewise geodesic map of a subinterval of $R$ into $\mathcal{T}$, and a bi-infinite path has domain $R$. By pulling back the fiber bundle $X_T \to \mathcal{T}$ to the domain of any path $\gamma$ we obtain the canonical $H^2$ bundle $X_\gamma$ over $\gamma$.

Consider two bi-infinite paths $\gamma, \gamma' : R \to \mathcal{T}$. We say that $\gamma, \gamma'$ are fellow travellers if there exists a quasi-isometric homeomorphism $h : R \to R$ and a constant $A$ such that $d_T(\gamma(h(t)), \gamma'(t)) \leq A$ for all $t \in R$. We may assume that $h$ is bilipschitz. It follows that the relation $\gamma(h(t)) \leftrightarrow \gamma'(t)$ lifts to a $\pi_1(S)$-equivariant map
$X_\gamma \to X_\gamma'$, well defined up to moving points a uniformly bounded distance, and this map is a fiber respecting quasi-isometry; see Proposition 4.2 of [FM02a].

The following theorem was proved independently by Mosher and by Bowditch:

**Theorem 9 ([Mos03b]; [Bow02]).** Given a coarse Lipschitz, cobounded, bi-infinite path $\gamma: \mathbb{R} \to T$, the bundle $X_\gamma$ is Gromov hyperbolic if and only if there exists a cobounded Teichmüller geodesic $\ell$ which fellow travels $\gamma$.

We need a uniform version of Theorem 9, proved in [Mos03b]:

**Proposition 10.** For every cocompact, $\text{MCG}(S)$-equivariant subset $B$ of $T$, and for every $\delta > 0$, there exist $K \geq 1$, $C \geq 0$ such that if $\gamma: \mathbb{R} \to T$ is a bi-infinite path with image in $B$, and if $X_\gamma$ is $\delta$ hyperbolic, then there exists a bi-infinite geodesic $\ell$ in $T$, and a $K$ bilipschitz homeomorphism $h: \mathbb{R} \to \mathbb{R}$ such that $d(\gamma(h(t)), \gamma'(t)) \leq C$ for all $t \in \mathbb{R}$.

**Singular solv spaces.** Given a cobounded Teichmüller geodesic $\ell$, there is a natural singular SOLV metric on $X_\ell$ which we denote $X_\ell^\text{SOLV}$. To define the metric, choose a quadratic differential representing a tangent vector on $\ell$, with horizontal and vertical measured foliations $(F_v, d\mu_v)$, $(F_h, d\mu_h)$. On $S \times \mathbb{R}$ we put the singular solv metric $e^{2t}d\mu_v^2 + e^{-2t}d\mu_h^2 + dt^2$. Lifting to the universal cover we obtain the singular SOLV space $X_\ell^\text{SOLV}$, on which $\pi_1 S$ acts by isometries. Note that the assignment $\ell \to X_\ell^\text{SOLV}$ is natural, and so each $f \in \text{MCG}$ induces an isometry $X_\ell^\text{SOLV} \to X_{f(\ell)}^\text{SOLV}$. In particular, if $\ell$ is the axis of a pseudo-Anosov $g \in \text{MCG}(S)$ then the extension group $\pi_1(S) \rtimes _f \mathbb{Z}$ acts on $X_\ell^\text{SOLV}$ by isometries, properly discontinuously and cocompactly; in this context we shall also use $X_g^\text{SOLV}$ to denote $X_\ell^\text{SOLV}$.

Note also that the identity map $X_\ell \to X_\ell^\text{SOLV}$ is a quasi-isometry; see Proposition 4.2 of [FM02a]. When $\ell$ is an axis this is evident from the fact that the identity map is equivariant with respect to the action of $\pi_1(S) \rtimes _f \mathbb{Z}$, and the latter group acts cocompactly by isometries on both $X_\ell$ and $X_\ell^\text{SOLV}$.

**Orbifolds.** All of our orbifolds will be closed, hyperbolic 2-orbifolds with only cone points: no mirror edges and hence no dihedral points. The reason for this restriction is that an orbifold with mirror edges cannot support a pseudo-Anosov homeomorphism.

An orbifold $\mathcal{O}$ has a Teichmüller space $\mathcal{T}(\mathcal{O})$ on which the mapping class group $\text{MCG}(\mathcal{O}) = \text{Homeo}(\mathcal{O})/\text{Homeo}_0(\mathcal{O})$ acts, where $\text{Homeo}(\mathcal{O})$ is the topological group of orbifold homeomorphisms, and $\text{Homeo}_0(\mathcal{O})$ is the normal subgroup $\text{Homeo}_0(\mathcal{O})$ which is the component of the identity map.

8
As with surface mapping class groups, there is a universal extension of $\pi_1\mathcal{O}$ which can be described by two equivalent short exact sequences, generalizing the Dehn-Nielsen-Epstein-Baer theorem; we take the description of this extension from [FM02a]. Unlike surface mapping class groups, when $\mathcal{O}$ has at least one cone point then the universal extension group will not be isomorphic to the once punctured mapping class group $\text{MCG}(\mathcal{O}, o)$, where $o \in \mathcal{O}$ is a generic base point. Instead, the universal extension is the group of lifts of $\text{MCG}(\mathcal{O})$ to the universal cover $\tilde{\mathcal{O}}$.

To be precise, $\tilde{\text{MCG}}(\mathcal{O}) = \tilde{\text{Homeo}}(\mathcal{O})/\tilde{\text{Homeo}}_0(\mathcal{O})$ where $\tilde{\text{Homeo}}(\mathcal{O})$ is the topological group of homeomorphisms of $\tilde{\mathcal{O}}$ that respect orbits of the action of $\pi_1\mathcal{O}$, and $\tilde{\text{Homeo}}_0(\mathcal{O})$ is the component of the identity element of $\tilde{\text{Homeo}}(\mathcal{O})$ is a normal subgroup; equivalently, $\tilde{\text{Homeo}}_0(\mathcal{O})$ consists of those elements of $\tilde{\text{Homeo}}(\mathcal{O})$ acting trivially on the circle at infinity of $\tilde{\mathcal{O}}$. As a consequence, we have an injection $\tilde{\text{MCG}}(\mathcal{O}) \hookrightarrow \text{Homeo}(S^1\tilde{\mathcal{O}})$. The analogue of the Dehn-Nielsen-Epstein-Baer theorem says

$$
\begin{array}{cccccccc}
1 & \longrightarrow & \pi_1(\mathcal{O}) & \longrightarrow & \tilde{\text{MCG}}(\mathcal{O}) & \longrightarrow & \text{MCG}(\mathcal{O}) & \longrightarrow & 1 \\
\Vert & & \Vert & & \Vert & & \Vert & & \\
1 & \longrightarrow & \pi_1(\mathcal{O}) & \longrightarrow & \text{Aut}(\pi_1(\mathcal{O})) & \longrightarrow & \text{Out}(\pi_1(\mathcal{O})) & \longrightarrow & 1
\end{array}
$$

We say that an automorphism $\hat{f}: \text{Aut}(\pi_1(\mathcal{O}, o))$ represents a mapping class $f \in \tilde{\text{MCG}}(\mathcal{O})$ if $\hat{f}$ maps to $f$ under the homomorphism $\text{Aut}(\pi_1(\mathcal{O}, o)) \rightarrow \text{Out}(\pi_1(\mathcal{O}, o)) \approx \text{MCG}(\mathcal{O})$.

Note that there is a surjective homomorphism $\tilde{\text{MCG}}(\mathcal{O}, o) \rightarrow \text{Aut}(\pi_1(\mathcal{O}, o))$ defined in the obvious manner. But if $\mathcal{O}$ has any cone points then this homomorphism has a nontrivial kernel. The kernel is generated by pushing the base point $o$ around any generic closed curve of the form $\gamma * \rho * \gamma^{-1}$, where $\rho$ is a based closed curve encircling an order $n$ cone point and going around that point $n$ times, and $\gamma$ is a path from $o$ to the base point of $\rho$.

There is a canonical hyperbolic surface bundle $X_T$ over the Teichmüller space $T$ of an orbifold $\mathcal{O}$, on which $\tilde{\text{MCG}}(\mathcal{O})$ acts by isometries, and Theorem 9 and Proposition 10 are true in this context.

**The model space $X_H$.** Consider an irreducible subgroup $H < \mathcal{MCG}(S)$. Construct a graph $G_H$ in Teichmüller space on which $H$ acts properly and cocompactly by isometries, as follows. Take an orbit of $H$, and use a piecewise geodesic to connect points which differ by a generator. Since $H$ is finitely generated and acts properly, we can choose these connecting paths in an $H$-equivariant way so that they are disjoint except at their endpoints. It follows that $G_H \subset T$ is an embedded
Cayley graph of $H$, metrized so that each edge is a piecewise geodesic segment in $T$. Each path in $G_H$ can be regarded as a piecewise geodesic in $T$. The orbit map from $H$ to $G_H$ is a quasi-isometry, and the inclusion $G_H \hookrightarrow T$ is uniformly proper.

By restricting the bundle $X_T \to T$ to $G_H$ we obtain the canonical $H^2$ bundle $X_H \to G_H$ and a piecewise Riemannian metric thereon. The group $\Gamma_H$ acts on $X_H$ by isometries, properly discontinuously and cocompactly, and so any orbit map $\Gamma_H \to X_H$ is a quasi-isometry.

Let $QI_f(X_H)$ denote the subgroup of $QI(X_H)$ represented by quasi-isometries $\Phi: X_H \to X_H$ which uniformly coarsely respect the fibers, in the sense that there is a constant $A$ such that for each $x \in G_H$ there exists $x' \in G_H$ such that $d_H(\Phi(D_x), D_{x'}) \leq A$. Choosing $x' = \phi(x)$ for each $x \in G_H$, we obtain an induced quasi-isometry $\phi: G_H \to G_H$. An orbit map $\phi: \Gamma_H \to X_H$ takes any coset of $\pi_1 S$ in $\Gamma_H$ to with a uniformly finite Hausdorff distance of some fiber of $X_H$, and every fiber in $X_H$ is has uniformly finite Hausdorff distance from the image of some coset. It follows that $\phi$ induces an isomorphism $QI_f(\Gamma_H) \approx QI_f(X_H)$, and henceforth we will identify these two groups. Note that we obtain a commutative diagram of homomorphisms

\[
\begin{array}{ccc}
QI_f(\Gamma_H) & \approx & QI_f(X_H) \\
\downarrow & & \downarrow \\
QI(H) & \approx & QI(G_H)
\end{array}
\]

**Hyperbolic lines and coarse axes in $X_H$.** A bi-infinite path $\gamma$ in $T$ is called a hyperbolic line if $X_\gamma$ is Gromov hyperbolic. By applying Theorem 9 this is equivalent to the existence of a cobounded Teichmüller geodesic $\ell$ which fellow travels $\gamma$ in $T$, and in this case there is an induced fiber respecting quasi-isometry $X_\gamma \approx X_\ell^{\text{solv}}$. In this situation, we adopt the notation $X_\gamma^{\text{solv}}$ for the metric space $X_\ell^{\text{solv}}$.

**Lemma 11.** If $\gamma$ is a hyperbolic line with image contained in $G_H$, fellow travelling a cobounded geodesic $\ell \subset T$, the following are equivalent:

1. There exists a pseudo-Anosov element $f \in G_H$ such that $\gamma$ is coarsely equivalent in $G_H$ to any orbit of $f$.

2. $\ell$ is the axis of some pseudo-Anosov element $f \in \mathcal{MCG}(S)$.

If these happen then we say that $\gamma$ is a coarse $T$-axis.
Proof. To prove that (1) implies (2), the axis of $f$ is coarsely equivalent to any orbit of $f$ and so is coarsely equivalent to $\ell$, but this implies that the axis of $f$ equals $\ell$, because two cobounded Teichmüller geodesics which are coarsely equivalent are equal (Lemma 2.4 of [FM02a]).

To prove that (2) implies (1), suppose that $\ell$ is the axis of a pseudo-Anosov element $g \in \text{MCG}(H)$, and so $\ell$ is coarsely contained in $G_H$. In the word metric on $\text{MCG}(H)$, the infinite cyclic subgroup $\langle g \rangle$ is coarsely contained in $H$. Applying Proposition [6] some finite index subgroup $\langle gn \rangle$ of $\langle g \rangle$ is contained in $H$. It follows that any orbit of $f = gn$ in $G_H$ is coarsely equivalent to $\gamma$.

Theorem 12 ([FM02b]). If $\ell, \ell'$ are cobounded geodesics in $T$, if $\ell$ is the axis of a pseudo-Anosov mapping class, and if $\Phi: X_\ell \to X_{\ell'}$ is a fiber respecting quasi-isometry, then $\ell'$ is the axis of a pseudo-Anosov mapping class and $\Phi$ is a bounded distance from an isometry $X_{\ell}^{\text{SOLV}} \to X_{\ell'}^{\text{SOLV}}$.

Corollary 13. If $\gamma$ is a hyperbolic line in $G_H$ and $\Phi \in \text{QI}(X_H)$ then $\gamma' = \Phi \circ \gamma$ is a hyperbolic line in $G_H$. Moreover, $\gamma$ is a coarse axis if and only if $\gamma'$ is a coarse axis, in which case the fiber respecting quasi-isometry $X_\gamma \to X_{\gamma'}$ induced by $\Phi$ is a bounded distance from an isometry $X_{\gamma}^{\text{SOLV}} \to X_{\gamma'}^{\text{SOLV}}$.

Proof. Suppose $\gamma$ is a hyperbolic line in $G_H$, that is, $X_\gamma$ is a Gromov hyperbolic metric space. The quasi-isometry $\Phi: X_H \to X_H$ restricts to a quasi-isometry $X_\gamma \to X_{\gamma'}$, and so $X_{\gamma'}$ is Gromov hyperbolic, that is, $\gamma'$ is a hyperbolic line.

By using Theorem [12] we may replace $X_\gamma$ and $X_{\gamma'}$ with $X_\ell^{\text{SOLV}}$ and $X_{\ell'}^{\text{SOLV}}$ for Teichmüller geodesics $\ell, \ell'$ fellow travelling $\gamma, \gamma'$, respectively, and then we apply Theorem [12].

3 The group $\text{QI}_f(\Gamma_H) \approx \text{QI}_f(X_H)$

The quasi-symmetry group $\text{QSym}(S^1)$. The quasi-isometry group of the hyperbolic plane $\mathbb{H}^2$ acts faithfully on the boundary circle $S^1$, and the image of this action is precisely the group of quasi-symmetric homeomorphisms of $S^1$, denoted $\text{QSym}(S^1)$.

Consider $X_H$. For each $x \in G_H$, the fiber over $x$ is $D_x \subset X_H$. Fix a base point $p_0 \in G_H$ and let $D_0 = D_{p_0}$. Fix an isometric identification of $D_0$ with $\mathbb{H}^2$, so the Gromov boundary of $D_0$ is identified with $S^1$ and $\text{QI}(D_0) \approx \text{QSym}(S^1)$.

There is an induced homomorphism $\theta: \text{QI}_f(X_H) \to \text{QSym}(S^1)$ defined as follows. For each $\Phi \in \text{QI}_f(X_H)$, the set $\Phi(D_0)$ is Hausdorff close to $D_0$ and so a closest point map $\Phi(D_0) \to D_0$, precomposed with $\Phi$, induces a quasi-isometry $D_0 \to D_0$ whose class $\theta(\Phi) \in \text{QI}(\mathbb{H}^2) = \text{QSym}(S^1)$ is well-defined independent of the choice of a closest point map.
Claim 14. The homomorphism \( \theta : \text{QI}_f(X_H) \to \text{QSym}(S^1) \) is injective.

Proof. The proof is in principle the same as in [FM02b], although there it was couched in terms of the limit set of a Schottky group. We shall reformulate the proof without referring to Schottky groups, as follows.

Consider a pseudo-Anosov element \( g \in H \) with coarse axis \( \gamma \), and choose a fiber \( D_g \subset X_g^{\text{SOLV}} \). Since \( D_0 \) and \( D_g \) have finite Hausdorff distance, there is a canonical homeomorphism \( \partial D_g \approx \partial D_0 = S^1 \). With respect to this homeomorphism, let \( E_s(g) \subset S^1 \) be the set of endpoints of leaves of the stable foliation on \( D_g \), let \( E_u(g) \subset S^1 \) be the set of endpoints of unstable leaves, and let \( E(g) = E_s(g) \cup E_u(g) \), a disjoint union. Note that if \( g, g' \in H \) are pseudo-Anosov then either \( \text{VN}(g) = \text{VN}(g') \) and \( E(g) = E(g') \), or \( \text{VN}(g) \cap \text{VN}(g') = \{ \text{Id} \} \) and \( E(g) \cap E(g') = \emptyset \).

Fix a conjugacy class \( C \) of pseudo-Anosov elements of \( H \), fix a coarse axis \( \gamma_g \) for one particular element of \( H \), and for any \( h \in H \) we obtain a coarse axis \( h(\gamma_g) \) for \( hgh^{-1} \). There is a fixed \( \delta > 0 \) so that the spaces \( X_g^{\text{SOLV}} \) are all \( \delta \)-hyperbolic for \( g \in C \). Applying Proposition 10 with \( B = G_H \), it follows that the axis \( \ell_g \subset T \) for \( g \) is uniformly Hausdorff close to \( \gamma_g \).

Let \( \Phi : X_H \to X_H \) be a fiber respecting quasi-isometry such that \( \theta(\Phi) \) is the identity. Let \( \phi : G_H \to G_H \) be the quasi-isometry induced by \( \Phi \). Applying Proposition 13 for each \( g \in C \) the image \( \phi(\gamma_g) \) is a coarse axis for some pseudo-Anosov element \( g' \in H \), and the induced map \( X_{\gamma_g} \to X_{\gamma_g'} \) is a quasi-isometry with uniform constants. It follows that \( X_{\gamma_g'} \) is \( \delta' \)-hyperbolic for \( \delta' \) independent of \( g \). Applying Proposition 10 \( \gamma_{g'} \) is a uniformly finite Hausdorff distance from the axis \( \ell_{g'} \) of \( g' \). We have canonical identifications \( \partial D_g \approx \partial D_{g'} \approx \partial D_0 = S^1 \), and the map \( \partial D_g \to \partial D_{g'} \) induced by \( \Phi \) must agree with the identity map on \( \partial D_0 \), since \( \theta(\Phi) \) is the identity. On the other hand, \( \Phi(E(g)) = E(g') \), implying that \( E(g) = E(g') \), and so \( \text{VN}(g) = \text{VN}(g') \). It follows that \( \ell_g = \ell_{g'} \), and so \( \phi(\gamma) \) is coarsely equivalent to \( \gamma \), with a uniformly finite Hausdorff distance.

The collection of axes \( \gamma_g \) for \( g \in C_x \) coarsely separate points in \( G_H \). That is, there exists constants \( r, s \geq 0 \) so that for each \( x \in G_H \) there is a finite subset \( C_x \subset C \) such that \( x \) is within distance \( r \) of each coarse axis \( \gamma_g \) for \( g \in C_x \), and the set of points within distance \( r \) of these coarse axes has diameter at most \( s \). Since \( \phi(\gamma_g) \) is uniformly coarsely equivalent to \( \gamma_g \) for each \( g \in C \), it follows that \( d(\phi(x), x) \) is uniformly finite, in other words the quasi-isometry \( \phi : G_H \to G_H \) is equivalent to the identity.

For each \( x \in G_H \), it follows that \( \Phi(D_x) \) is a uniformly finite distance from \( D_x \), and since the boundary values of the closest point projection \( \Phi(D_x) \to D_x \) must agree with the identity map on \( S^1 \) it follows that the closest point projection is a uniformly finite distance from the identity on \( D_x \). This proves that \( \Phi \) is equivalent to the identity map on \( X_H \). \( \diamond \)
Having proved that \( \theta \) is injective, we will identify \( \text{QI}_f(X_H) \approx \text{QI}_f(\Gamma_H) \) with its image in \( \text{QSym}(S^1) \).

There are several other subgroups of \( \text{QSym}(S^1) \) that are of significance to us. In particular, for each pseudo-Anosov \( f \in H \) with coarse axis \( \gamma \) fellow travelling a Teichmüller axis \( \ell \) for \( f \), we have constructed a fiber preserving quasi-isometry \( X_\gamma \approx X_\gamma^{\text{SOLV}} = X_\ell^{\text{SOLV}} \). The group \( \text{Isom}(X_\gamma^{\text{SOLV}}) \) takes fibers to fibers. If we choose a fiber \( D_\gamma \subset X_\gamma^{\text{SOLV}} \) then each isometry of \( X_\gamma^{\text{SOLV}} \) takes \( D_\gamma \) to another fiber coarsely equivalent to \( D_\gamma \), and composing with the closest point projection go \( D_\gamma \) we obtain a self quasi-isometry of \( D_\gamma \), thereby constructing an injection \( \text{Isom}(X_\gamma^{\text{SOLV}}) \to \text{QI}(D_\gamma) \). However, \( D_0 \) and \( D_\gamma \) are coarsely equivalent in \( X_H \) and so we obtain an injection \( \text{Isom}(X_\gamma^{\text{SOLV}}) \hookrightarrow \text{QSym}(S^1) \); again we identify \( \text{Isom}(X_\gamma^{\text{SOLV}}) \) with the image of this injection.

**Mapping classes which descend to subcovers.** Consider an orbifold covering map \( \pi: Q \to P \), and fix generic base points \( q \in Q, p = \pi q \in P \), inducing an injection of orbifold fundamental groups \( \pi_1(Q, q) \to \pi_1(P, p) \). A mapping class \( f \in \text{MCG}(Q) \) descends to a mapping class \( g \in \text{MCG}(P) \) if there exist orbifold homeomorphisms \( F: Q \to Q \) and \( G: P \to P \) representing \( f \) and \( g \), respectively, such that \( F \) is a lift of \( G \), that is, \( F \circ \pi = \pi \circ G \).

**Lemma 15.** Consider \( \pi: Q \to P \), and \( q \in Q, \pi q = p \in P \) as above. Given \( f \in \text{MCG}(Q), g \in \text{MCG}(P) \), the following are equivalent:

1. \( f \) descends to \( g \).
2. There exist automorphisms \( \hat{f}: \pi_1(Q, q) \to \pi_1(Q, q) \) and \( \hat{g}: \pi_1(P, p) \to \pi_1(P, p) \) representing \( f, g \), respectively, such that \( \hat{f} \) extends to \( \hat{g} \) (with respect to the injection \( \pi_1(Q, q) \hookrightarrow \pi_1(P, p) \)).
3. For any automorphism \( \hat{f}: \pi_1(Q, q) \to \pi_1(Q, q) \) representing \( f \), there exists an automorphism \( \hat{g}: \pi_1(P, p) \to \pi_1(P, p) \) representing \( g \), such that \( \hat{f} \) extends to \( \hat{g} \).

Moreover, if \( f \in \text{MCG}(Q) \) descends to \( \text{MCG}(P) \), then the element \( g \in \text{MCG}(P) \) to which it descends is unique.

**Proof.** In this proof we will use the fact that if \( O \) is an orbifold with base point \( x \) then we have a surjective homomorphism \( \text{MCG}(O, x) \to \tilde{\text{MCG}}(O, x) \approx \text{Aut}(O, x) \), as described earlier.

Obviously (3) implies (2).

To prove (2) implies (1), suppose that \( \hat{f}, \hat{g} \) exist as in (2). Choose an orbifold homeomorphism \( G: (P, p) \to (P, p) \) that induces \( \hat{g} \). By the lifting lemma in the
category of orbifolds, we may lift $G$ to a homeomorphism $F: (Q, q) \to (Q, q)$ that induces $\hat{f}$. Forgetting base points, clearly $G$ represents $g$ and $F$ represents $f$.

To prove (1) implies (3), suppose there exist orbifold homeomorphisms $F: Q \to Q$ and $G: P \to P$ representing $f$ and $g$, respectively, such that $F$ is a lift of $G$. We may isotop $G$ so that it fixes $p$, and we may lift this to an isotopy of $F$ so that $F$ permutes $\pi^{-1}p$, but it may not be true that $F(q) = q$. If not, pick a path $\gamma$ in $Q$ from $F(q)$ to $q$ whose projection $\pi\gamma$ passes only through generic points of $P$, and postcompose $G$ by the isotopy of $P$ that pushes $p$ around $\pi\gamma$ to obtain a new $G$. This lifts to an isotopy of $Q$ that pushes $F(q)$ along $\gamma$ to $q$, and postcomposing $F$ we obtain a new $F$ so that $F(q) = q$. It follows that $F$ and $G$ induce $\hat{f} \in \text{Aut}(\pi_1(Q, q))$ and $\hat{g} \in \text{Aut}(\pi_1(P, p))$ satisfying the requirements of the lemma.

To prove uniqueness of $g$ given $f$, for each automorphism $\hat{f}: \pi_1(Q, q) \to \pi_1(Q, q)$ representing $f$ it suffices to prove that there is at most one automorphism $\hat{g}: \pi_1(P, p) \to \pi_1(P, p)$ to which $\hat{f}$ extends. We have an isomorphism $\text{Aut}(\pi_1(P, p)) \approx \mathcal{MC}\mathcal{G}(P)$, and an injection $\mathcal{MC}\mathcal{G}(P) \hookrightarrow \text{QSym}(S^1)$; in other words, an automorphism of $\pi_1(P, p)$ is determined by its boundary values. But the finite index injection $\pi_1(Q, q) \hookrightarrow \pi_1(P, p)$ induces a homeomorphism of boundaries, and so the boundary values of an extension of $\hat{f}$ are completely determined by $\hat{f}$. $\diamond$

Given an orbifold covering map $Q \to P$, a subgroup $A \subset \mathcal{MC}\mathcal{G}(Q)$ descends to a subgroup $B \subset \mathcal{MC}\mathcal{G}(P)$ if there is an isomorphism $A \approx B$ such that each element of $A$ descends to the corresponding element of $B$. We also say simply that $A$ descends to $P$.

In [FM02], given a surface $S$ and a free subgroup $H \subset \mathcal{MC}\mathcal{G}(S)$, we studied the problem of finding a smallest orbifold subcovering map $S \to O$ such that $H$ descends to $O$. Since $H$ was free, it was more or less obvious that $H$ descends if and only if the elements of some free basis descend, and this allowed us to ignore some subtleties of the concept of “descent”. In the present situation we are working with arbitrary subgroups, and the question arises whether one can check descent on an element by element basis. The answer turns out to be yes, as we show in Lemma 16 below; the proof is based on the uniqueness of descent for mapping classes proved of Lemma 15. Uniqueness of descent for mapping classes also implies uniqueness of descent for subgroups, as we now see.

**Lemma 16.** Consider $\pi: Q \to P$, $q \in Q$, $p = \pi q \in P$ as above. For any subgroup $A \subset \mathcal{MC}\mathcal{G}(Q)$, there is at most one subgroup of $\mathcal{MC}\mathcal{G}(P)$ to which $A$ descends. Moreover, the following are equivalent:

1. $A$ descends to some subgroup of $\mathcal{MC}\mathcal{G}(P)$.
2. Each element of $A$ descends to some element of $\text{MCG}(\mathcal{P})$.

3. There exists a generating set of $A$ such that each generator descends to some element of $\text{MCG}(\mathcal{P})$.

Proof. Obviously (1) implies (2) implies (3). To prove (3) implies (1), let $G \subset A$ be a generating set such that each $g \in G$ descends to $\text{MCG}(\mathcal{P})$, equivalently, we can choose a representative $\hat{g} \in \text{Aut}(\pi_1(\mathcal{Q}, q))$ that extends to some $\hat{f} \in \text{Aut}(\pi_1(\mathcal{P}, p))$. The composition of each word in the $\hat{g}$ therefore extends to some element of $\text{Aut}(\pi_1(\mathcal{P}, p))$. We must prove that if $w(g)$ is a word in the letters $g \in G$ that represents the identity element of $G$, if $w(\hat{g})$ is the corresponding word in the letters $\hat{g}$, and if $w(\hat{f})$ is the corresponding word in the letters $\hat{f}$, then $w(\hat{f})$ is a representative of the identity element of $\text{MCG}(\mathcal{P})$. But we know that $w(\hat{g})$ is a representative of the identity element of $\mathcal{M}(\mathcal{Q})$, which means that $w(\hat{g})$ agrees with the inner automorphism of $\pi_1(\mathcal{Q}, q)$ defined by some element $\gamma \in \pi_1(\mathcal{Q}, q)$. The word $w(\hat{f})$ represents an extension of $w(\hat{g})$ to an automorphism $\pi_1(\mathcal{P}, p)$, but $\gamma$ acts by conjugation on $\pi_1(\mathcal{P}, p)$ since $\pi_1(\mathcal{Q}, q) < \pi_1(\mathcal{P}, p)$, and the actions of $w(\hat{f})$ and $\gamma$ agree on the circle at infinity because they agree with restricted to $\pi_1(\mathcal{Q}, q)$. It follows that $w(\hat{f})$ and the conjugation action of $\gamma$ are the same automorphism of $\pi_1(\mathcal{P}, p)$. This proves that $w(\hat{f})$ is an inner automorphism of $\pi_1(\mathcal{P}, p)$, and so $w(\hat{f})$ is a representative of the identity element of $\text{MCG}(\mathcal{P})$.  

Given a pseudo-Anosov $f \in \mathcal{MCG}(\mathcal{S})$, in [FM02b] we defined the “maximal orbifold subcover” $S \rightarrow \mathcal{O}_f$ to which $f$ descends. The orbifold $\mathcal{O}_f$ is constructed explicitly as follows. Let $\ell \subset \mathcal{T}$ be the axis of $f$. Let $X_{\ell}^{\text{SOLV}}$ be the singular SOLV-manifold associated to $\ell$. Pick a fiber $D_{\ell}$, a singular Euclidean space, with a horizontal and vertical measured foliation, on which $\pi_1S$ acts properly and cocompactly by isometries preserving the two foliations. Consider the full group of isometries $\text{Isom}(X_{\ell}^{\text{SOLV}})$. The fibers of $X_{\ell}^{\text{SOLV}}$ are preserved by $\text{Isom}(X_{\ell}^{\text{SOLV}})$, and so the action of $\text{Isom}(X_{\ell}^{\text{SOLV}})$ descends to an action on $\ell$ which is discrete and cocompact. The quotient group acting on $\ell$ is denoted $C_{\ell}$, and it is isomorphic to either the infinite cyclic group $\mathbb{Z}$ or the infinite dihedral group $D_{\infty}$. The kernel of this action, denoted $\text{Isom}_f(X_{\ell}^{\text{SOLV}})$, the subgroup of $\text{Isom}(X_{\ell}^{\text{SOLV}})$ that preserves each fiber, and the restriction of $\text{Isom}_f(X_{\ell}^{\text{SOLV}})$ to $D_{\ell}$ is the full group of isometries of $D_{\ell}$ that preserve the horizontal and vertical measured foliations. The group $\text{Isom}_f(X_{\ell}^{\text{SOLV}})$ acts properly and cocompactly on $D_{w}$, containing $\pi_1(\mathcal{S})$ as a finite index subgroup. It follows that $\mathcal{O}_f = \mathcal{O}_\ell = D_{\ell}/K_{\ell}$ is an orbifold subcover of $S$; let $\pi: S \rightarrow \mathcal{O}_f$ be the projection. By construction, for any pseudo-Anosov homeomorphism $F$ representing $f$ there is a pseudo-Anosov homeomorphism $G$ of $\mathcal{O}_f$ such that $\pi \circ F = G \circ \pi$, and so $f$ descends to $\mathcal{O}_f$. 

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To explain maximality of $O_f$, suppose that $f$ descends through a subcover $S \to Q$ to some $g \in \mathcal{MCG}(Q)$. Choose $G: Q \to Q$ representing $g$ and a lift $F: S \to S$ representing $f$. Since $f$ is irreducible, $g$ must also be irreducible, and so $g$ has a pseudo-Anosov representative $G'$. The isotopy from $G$ to $G'$ lifts to an isotopy from $F$ to a homeomorphism $F': S \to S$ which is obviously pseudo-Anosov. It follows that $X^\text{solv}_f$ and $X^\text{solv}_g$ are isometric. We obtain inclusions $\pi_1(S) < \pi_1(Q) < \text{Isom}_f(X^\text{solv}_f)$, and so the orbifold covering map $S \to O_f$ factors as orbifold covering maps $S \to Q \to O_f$, which gives maximality of $O_f$.

**Proposition 17.** If $H < \mathcal{MCG}(S)$ is irreducible then there exists an orbifold covering map $\pi: S \to O_H$ to which $H$ descends, so that $O_H$ is maximal: any orbifold covering map $S \to Q$ to which $H$ descends composes with a covering map $Q \to O_f$ to give $\pi$. The group $\pi_1 O_H$ can be characterized in $\text{QSym}(S^1)$ as

$$\pi_1 O_H = \bigcap_g \pi_1 O_g,$$

over all pseudo-Anosov $g \in H$.

**Proof.** In the elementary case, where $H < \text{VN}(f)$ for some pseudo-Anosov $f \in \mathcal{MCG}(S)$, we clearly have $O_H = O_f$.

In the nonelementary case, $H$ is generated by its pseudo-Anosov elements. Each pseudo-Anosov $f \in H$ descends through a maximal subcover $S \to O_f$. We have inclusions $\pi_1 S < \pi_1 O_f < \text{QSym}(S^1)$, the first of which has finite index. Define $\pi_1 O_H$ to be the intersection of $\pi_1 O_f$ over all pseudo-Anosov $f \in H$. Define $O_H$ to be the quotient orbifold of $\pi_1 O_H$. The action of $H$ by conjugation on $\text{QSym}(S^1)$ permutes the subgroups $\pi_1 O_f$, for pseudo-Anosov $f \in H$. In particular the action of any pseudo-Anosov $g \in H$ permutes the $\pi_1 O_f$, and so $g$ preserves $\pi_1 O_H$, that is, $g$ descends to $O_H$. Applying Lemma 16, $H$ descends to $O_H$.

To prove maximality, consider an orbifold covering $S \to Q$ such that $H$ descends to $Q$, and so each pseudo-Anosov $f \in H$ descends to $Q$, which may therefore be composed with a covering map $Q \to O_f$ to obtain the covering map $S \to O_f$. It follows that $\pi_1(S) < \pi_1(Q) < \pi_1 O_f$ in $\text{QSym}(S^1)$, for all pseudo-Anosov $f \in H$, implying that $\pi_1(Q) < \pi_1(O_H)$, which implies that $S \to Q$ composes with a covering map $Q \to O_f$ to yield $S \to O_H$. ◊

**Computation of $\text{QI}_f(X_H)$.** Consider an irreducible subgroup $H < \mathcal{MCG}(S)$. Let $O = O_H$. The subgroup $H < \mathcal{MCG}(S)$ descends to a subgroup of $\mathcal{MCG}(O)$ which will be denoted $H'$. Let $C$ denote the relative commensurator of $H'$ in
\(\mathcal{MCG}(O)\). We have a commutative diagram of extensions

\[
\begin{array}{c}
1 \\
\downarrow \\
1 \\
\end{array}
\begin{array}{c}
\pi_1(O) \\
\downarrow \\
\pi_1(O) \\
\end{array}
\begin{array}{c}
\Gamma_{H'} \\
\downarrow \\
\Gamma_C \\
\end{array}
\begin{array}{c}
H' \\
\downarrow \\
C \\
\end{array}
\begin{array}{c}
1 \\
\uparrow \\
1 \\
\end{array}
\begin{array}{c}
\mathcal{MCG}(O) \\
\downarrow \\
\mathcal{MCG}(O) \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]

in which all vertical arrows are inclusions. We also have a commutative diagram

\[
\begin{array}{c}
1 \\
\downarrow \\
1 \\
\end{array}
\begin{array}{c}
\pi_1(O) \\
\downarrow \\
\pi_1(S) \\
\end{array}
\begin{array}{c}
\Gamma_{H'} \\
\downarrow \\
\Gamma_H \\
\end{array}
\begin{array}{c}
H' \\
\downarrow \\
H \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]

where the vertical arrows are inclusions with finite index image. We may therefore regard \(\Gamma_{H'}\) as a finite index subgroup of \(\Gamma_H\), and so the inclusion induces a quasi-isometry \(\text{QI}_f(\Gamma_{H'}) \approx \text{QI}_f(\Gamma_H)\).

Since \(\mathcal{C}\) is the relative commensurator of \(H'\) in \(\mathcal{MCG}(O)\), it follows that \(\Gamma_C\) is the relative commensurator of \(\Gamma_{H'}\) in \(\mathcal{MCG}(O)\), and so we obtain a homomorphism \(\Gamma_C \to \text{QI}(\Gamma_{H'})\); to be precise, conjugation by \(g \in \Gamma_C\) maps some finite index subgroup of \(\Gamma_{H'}\) isomorphically to another finite index subgroup, inducing a quasi-isometry of \(\Gamma_{H'}\). Moreover, clearly \(\Gamma_C\) preserves cosets of \(\pi_1(O)\), and so we actually obtain a homomorphism \(\Gamma_C \to \text{QI}_f(\Gamma_{H'}) \approx \text{QI}_f(\Gamma_H)\).

To see that this homomorphism is injective, by postcomposing with \(\text{QI}_f(\Gamma_H) \to \text{QSym}(S^1)\) we obtain a homomorphism \(\Gamma_C \to \text{QSym}(S^1)\) which factors through two injections \(\Gamma_C \to \mathcal{MCG}(O) \to \text{QSym}(S^1)\).

Now we prove that the homomorphism \(\Gamma_C \to \text{QI}_f(\Gamma_H)\) is surjective: for each fiber respecting quasi-isometry \(\Phi: X_H \to X_{H'}\) there exists \(F \in \Gamma_C\) such that \(\Phi\) agrees within uniformly bounded distance with the action of \(F\). Let \(\phi: G_H \to G_H\) be the quasi-isometry induced by \(\Phi\).

Consider a pseudo-Anosov \(g \in H\) with coarse axis \(\gamma \subset G_H\). The image \(\phi(\gamma)\) is a coarse axis for a pseudo-Anosov \(g' \in H\). \(\Phi\) induces a fiber respecting quasi-isometry \(\Phi: X_\gamma \to X_{\phi(\gamma)}\) which, by Corollary 13, is a bounded distance from an isometry \(X_{g'^\text{SOLV}} \to X_{g'^\text{SOLV}}\), and so \(\Phi\) conjugates \(\text{Isom}(X_{g'^\text{SOLV}})\) to \(\text{Isom}(X_{g'^\text{SOLV}})\). Restricting to any fiber, \(\Phi\) conjugates \(\text{Isom}_f(X_{g'^\text{SOLV}}) = \pi_1(O_g)\) to \(\text{Isom}_f(X_{g'^\text{SOLV}}) = \pi_1(O_{g'})\). We may regard this conjugation as taking place in \(\text{QSym}(S^1)\).
We have proved that the conjugation action of \( \Phi \) in \( \text{QSym}(S^1) \) permutes the collection of groups \( \pi_1 \mathcal{O}_g \), over all pseudo-Anosov elements \( g \in H \). It follows that \( \Phi \) preserves the subgroup
\[
\pi_1 \mathcal{O}_H = \bigcap_g \pi_1 \mathcal{O}_g
\]
that is, \( \Phi \in \text{Aut}(\pi_1 \mathcal{O}_H) \approx \hat{\text{MCG}}(\mathcal{O}_H) \). We shall use \( F \) to denote \( \Phi \) regarded as an element of the group \( \hat{\text{MCG}}(\mathcal{O}_H) \), and \( f \) to be its image in the quotient \( \text{MCG}(\mathcal{O}_H) \).

Having identified \( \Phi \) as an element \( F \in \hat{\text{MCG}}(\mathcal{O}_H) \), we must now show that \( F \in \Gamma_C \), that is, \( f \Gamma H f^{-1} \) and \( H' \) are commensurable in \( \hat{\text{MCG}}(\mathcal{O}_H) \). Passing to the quotient group \( \text{MCG}(\mathcal{O}_H) \), it suffices to prove that \( f \Gamma H f^{-1} \) and \( H' \) are commensurable in \( \text{MCG}(\mathcal{O}_H) \). Applying Proposition\textsuperscript{6} we are reduced to showing that \( f \Gamma H f^{-1} \) is coarsely equivalent to \( H' \) in \( \text{MCG}(\mathcal{O}_H) \). Because we have a uniformly proper embedding \( \text{MCG}(\mathcal{O}_H) \to T(\mathcal{O}_H) \) given by any orbit map, and because \( G_{H'} \) is coarsely equivalent to any orbit of the action of \( H' \) on \( T(\mathcal{O}_H) \), it suffices to prove that \( f(G_{H'}) \) is coarsely equivalent to \( G_{H'} \) in \( T(\mathcal{O}_H) \).

We regard \( \Phi \) as a quasi-isometry of \( X_{H'} \). Let \( P \) denote a fixed pseudo-Anosov conjugacy class in \( H' \). For each \( g \in P \) pick a coarse axis \( \gamma_g \subset G_{H'} \) in an \( H' \)-equivariant manner, and let \( L = \{ \gamma_g \mid g \in P \} \). The metric spaces \( X_{\gamma_g} \) are all isometric for \( g \in P \); let \( \delta \) be a hyperbolicity constant for each of them. The action of \( H' \) permutes the collection of coarse axes \( L \), and by cocompactness of the \( H' \) action on \( G_{H'} \) it follows that the union of the coarse axes in \( L \) is coarsely equivalent to \( G_{H'} \).

The map \( \Phi \) induces a fiber respecting quasi-isometry from \( X_{\gamma_g} \) to \( X_{\phi(\gamma_g)} \), with uniform quasi-isometry constants, and taking each fiber to within a uniform Hausdorff distance of some other fiber. It follows that there exists \( \delta' > 0 \) so that for each \( g \in P \), the space \( X_{\phi(\gamma_g)} \) is \( \delta' \)-hyperbolic. Taking \( B = G_{H'} \) and applying Proposition\textsuperscript{10} there is a constant \( D \) such that \( \phi(\gamma_g) \) is within Hausdorff distance \( D \) of the axis \( \ell_{g'} \) of some pseudo-Anosov element \( g' \in \text{MCG}(\mathcal{O}_H) \). Similarly, replacing \( \delta' \) by \( \max\{\delta, \delta'\} \) it follows that \( \gamma_g \) is within Hausdorff distance \( D \) of the axis \( \ell_g \).

For \( g \in P \), the mapping class \( f \in \text{MCG}(\mathcal{O}_H) \) takes \( \ell_g \) to some axis in \( T(\mathcal{O}_H) \), and from the construction of \( f \) the only candidate is the axis \( \ell_{g'} \). It follows that \( f(\gamma_g) \) has a uniformly finite Hausdorff distance from \( \phi(\gamma_g) \). But the union of the coarse axes \( \gamma_g \in L \) is coarsely equivalent to \( G_{H'} \), as is the union of the coarse axes \( \phi(\gamma_g) \). It follows that \( f(G_{H'}) \) is coarsely equivalent to \( G_{H'} \), as required.

**The quantitative version of Corollary\textsuperscript{8}** Recall from the introduction that we need the following quantitative version of Corollary\textsuperscript{9} for all \( K \geq 1, \ C \geq 0 \)
there exists $A \geq 0$ such that if $\phi: \text{Mod}(S, p) \to \text{Mod}(S, p)$ is a $K, C$ quasi-isometry then there exists $F \in \text{Mod}(S, p)$ such that $d(\phi(G), FG) \leq A$ for all $G \in \text{Mod}(S, p)$. The proof of Corollary 1 then follows by a standard argument first found in [Sch96]; see also [Mos03a]. The quantitative version of Corollary 3 follows from quantitative versions of Whyte’s Theorem 2 and Theorem 1.

For Whyte’s Theorem 2 the quantitative version says that for all $K \geq 1$, $C \geq 0$ there exists $R \geq 0$ such that if $\phi: \text{Mod}(S, p) \to \text{Mod}(S, p)$ is a $K, C$ quasi-isometry then each coset of $\pi_1 S$ is taken by $\phi$ to within Hausdorff distance $R$ of another coset of $\pi_1 S$, in this case $\phi$ is said to $R$-coarsely respect the cosets of $\pi_1 S$. For the proof, see [Mos03a].

For Theorem 11 the quantitative version says that for all $K \geq 1$, $C \geq 0$, $R \geq 0$ there exists $A \geq 0$ such that if $\phi: \text{Mod}(S, p) \to \text{Mod}(S, p)$ is a $K, C$ quasi-isometry that $R$-coarsely respects the cosets of $\pi_1 S$ then there exists $F \in \text{Mod}(S, p)$ such that $d(\phi(G), FG) \leq A$ for all $G \in \text{Mod}(S, p)$.

Consider the argument given above for the computation of $\text{QI}_f(\Gamma_H)$, specialized to the case where $H = \text{Mod}(S)$, and so $\mathcal{O}_H = S$. We continue to use the notation $G_H$ for a piecewise geodesic Cayley graph equivariantly embedded in $\mathcal{T}$, and $X_H$ for the canonical $H^2$ bundle over $G_H$. In the coarse of this argument we produced a mapping class $F \in \text{Mod}(S, p)$, with quotient $f \in \text{Mod}(S)$, which is characterized by the following property: for each pseudo-Anosov $g \in \text{Mod}(S)$, with coarse axis $\gamma_g \subset G_H$ and axis $\ell_g \subset \mathcal{T}$, letting $g' = fgf^{-1}$ with axis $\ell_{g'} = f(\ell_g) \subset \mathcal{T}$, we may conclude that $\gamma_g$ and $\phi(\gamma_g)$ are bounded distance in $\mathcal{T}$ from $\ell_g$ and $\ell_{g'}$, respectively, and the quasi-isometry $X_{\gamma_g} \to X_{\phi(\gamma_g)}$ induced by $\Phi$ is a bounded distance from the isometry $X_{\ell_g}^{\text{solv}} \to X_{\ell_{g'}}^{\text{solv}}$ induced by $F$. The bounds in this conclusion depend on the quasi-isometry constants $K, C$ for $\Phi$ as well as on the constant $R$ for coarse preservation of fibers, but the bounds also depend on the pseudo-Anosov $g$ itself, because the proof invoked Proposition 10 which depends on the hyperbolicity constant for $X_{\gamma_g}$.

However, we can get around this dependence by the same trick used already several times: fix a pseudo-Anosov conjugacy class $P$, and choose a coarse axis $\gamma_g$ for each $g \in P$ in an $\text{Mod}(S)$-equivariant manner, and it follows that the metric spaces $X_{\gamma_g}$ are all isometric to each other and hence have the same hyperbolicity constant $\delta$. This constant $\delta$ thus depends only on the genus of the surface $S$, and hence the constants that come in the conclusion of Proposition 10 depend only on $K, C, R$, and the genus of $S$.

Using the additional fact that each point of $G_H$ is a uniformly bounded distance from the axis of some $g \in P$, by a bound which again depends only on the genus of $S$, we obtain the desired conclusion that $d(\phi(G), FG) \leq A$ for all $G \in \text{Mod}(S, p)$, where $A$ depends only on $K, C, R$, and the genus of $S$.  

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