Moments of $S(t,f)$ Associated with Holomorphic Hecke Cusp Forms

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Abstract. Let $S(t,f) := \pi^{-1} \text{arg} L(1/2 + it, f)$, where $f$ is a holomorphic Hecke cusp form for $SL_2(\mathbb{Z})$ of weight $k$. We establish an asymptotic formula for the moments of $S(t,f)$.

1. Introduction

It is well known that the function $S(t) = \frac{1}{\pi} \text{arg} \zeta(1/2 + it)$ is related to the number of nontrivial zeros $\rho$ of the Riemann zeta-function $\zeta(s)$ with $0 < \text{Im}(\rho) \leq t$. There are many research papers that have been devoted to studying the behavior of $S(t)$ (see, for example, [1,2,11,13,14,18]). In particular, Selberg [15,17] showed that for $n \in \mathbb{N}$,

$$\frac{1}{T} \int_{T}^{2T} S(t)^{2n} \, dt = \frac{(2n)!}{n!(2\pi)^{2n}} (\log \log T)^n + O((\log \log T)^{n-1/2}).$$

Using the same method one can derive a similar result for $S(t,\chi) = \pi^{-1} \text{arg} L(1/2 + it, \chi)$, where $\chi$ is a primitive Dirichlet character of modulus $q$. In the subsequent paper [16], Selberg proved an analogous result in the conductor aspect. More precisely, for a prime $q$, he showed

$$\frac{1}{q} \sum_{\chi \bmod q}^* S(t,\chi)^{2n} = \frac{(2n)!}{n!(2\pi)^{2n}} (\log \log q)^n + O_{t,n}((\log \log q)^{n-1/2}),$$

where the summation runs over the primitive characters $\chi \pmod{q}$.

In [6], Hejhal and Luo considered a $GL_2$ analog of (1.1). They proved an asymptotic formula for the spectral moments $S(t, f_j) = \pi^{-1} \text{arg} L(1/2 + it, f_j)$ assuming the Generalized Riemann Hypothesis (GRH), i.e., $S(t, f_j)^n$ is averaged over the Hecke–Maass cusp forms $f_j$ for $SL_2(\mathbb{Z})$ with a smooth test function, for each fixed positive $t$. Recently, a $GL_3$ analogous result was obtained in [12] assuming the GRH. In this paper, we establish a $GL_2$ analogous result of Hejhal and Luo in the weight aspect without assuming the GRH.

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To state our main result, let $H_k$ denote the set of holomorphic Hecke cusp forms $f$ of weight $k$ for $\text{SL}_2(\mathbb{Z})$, where $f(z)$ has the Fourier expansion
\[ f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i nz} \]
with $\lambda_f(1) = 1$. For $f \in H_k$, the $L$-function associated to $f$ is given by
\[ L(s, f) := \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \text{Re}(s) > 1 \]
and this has Euler product
\[
L(s, f) = \prod_p \left( 1 - \lambda_f(p)p^{-s} + p^{-2s} \right)^{-1} \\
= \prod_p \left( 1 - \alpha_f(p)p^{-s} \right)^{-1} \left( 1 - \beta_f(p)p^{-s} \right)^{-1}.
\]

The Ramanujan–Petersson conjecture (proved by Deligne [4]) asserts that
\[ |\alpha_f(p)| = |\beta_f(p)| = 1, \quad \text{and thus} \quad |\lambda_f(p)| \leq 2. \]

The complete $L$-function
\[ \Lambda(s, f) := \pi^{-s} \Gamma \left( \frac{s + (k - 1)/2}{2} \right) \Gamma \left( \frac{s + (k + 1)/2}{2} \right) L(s, f) \]
admits an entire continuation to $s \in \mathbb{C}$ and satisfies the functional equation
\[ \Lambda(s, f) = i^k \Lambda(1 - s, f). \]

Next we define the analog of $S(t)$ for $f$ by
\[ S(t, f) := \frac{1}{\pi} \arg L(1/2 + it, f), \]
where the argument $\arg L(1/2 + it, f)$ is obtained by continuous variation along the line $\text{Im}(s) = t$ from $\sigma = +\infty$ to $\sigma = 1/2$. Our main result is the following theorem.

**Theorem 1.1.** Let $t > 0$ and $n \in \mathbb{N}$ be given. For sufficiently large even integer $k$ we have
\[ \frac{2\pi^2}{k - 1} \sum_{f \in H_k} \frac{S(t, f)^n}{L(1, \text{sym}^2 f)} = C_n (\log \log k)^{n/2} + O_{t,n} \left( (\log \log k)^{(n-1)/2} \right), \]
where
\[ C_n = \begin{cases} 
\frac{n!}{(n/2)!((2\pi)^n)} & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd.}
\end{cases} \]

**Remark 1.2.** Theorem [1.1] and Corollary [2.2] indicate that the values of $|S(t, f)|$ on average have order of magnitude $\sqrt{\log \log k}$. 

2. Preliminaries

The following proposition is the well-known Petersson trace formula, which can be found in [8].

**Proposition 2.1** (Petersson trace formula).

\[
\sum_{f \in H_k} \omega_f^{-1} \lambda_f(m) \lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right),
\]

where \( \omega_f = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \| f \|^2 \), \( \delta_{m,n} \) equals 1 if \( m = n \) and 0 otherwise, \( S(m,n;c) \) is the Kloosterman sum, and \( J_{k-1}(x) \) is the \( J \)-Bessel function.

From the integral representation (see [5, 8.111 10])

\[
J_\nu(x) = \frac{1}{\Gamma(\nu + 1/2)\Gamma(1/2)} \left( \frac{x}{2} \right)^\nu \int_{-1}^{1} e^{ixt}(1-t^2)^{\nu-1/2} \, dt
\]

and the Stirling’s formula, we deduce that

\[
J_{k-1}(x) \ll \left( \frac{e^x}{2k} \right)^{k-1}.
\]

Using the bound (2.1) and the relation (see [10])

\[
\omega_f = \frac{k-1}{2\pi^2} L(1, \text{sym}^2 f),
\]

one can deduce the following corollary.

**Corollary 2.2.** For any \( m, n \geq 1 \) with \( 8\pi \sqrt{mn} \leq k \) we have

\[
\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{\lambda_f(n) \lambda_f(m)}{L(1, \text{sym}^2 f)} = \delta_{m,n} + O(k^{-A})
\]

for any \( A > 0 \).

We will need the following zero-density estimate which was established by Hough [7]. In fact, [7, Theorem 1.1] states a result without the weight \( 1/L(1, \text{sym}^2 f) \). However, it is easy to derive a weighted version as Proposition 2.3 below by using [7, Proposition 5.1].

**Proposition 2.3.** Let

\[
N_f(\sigma, T) := |\{ \rho = \beta + i\gamma \mid L(\rho, f) = 0, \sigma < \beta, |\gamma| < T \}|.
\]

Let \( \frac{1}{2} + \frac{2}{\log k} < \sigma < 1 \). For some sufficiently small \( \delta_1, \theta_1 > 0 \) we have uniformly in \( \frac{10}{\log k} < T < k^{\delta_1} \),

\[
\sum_{f \in H_k} \frac{N_f(\sigma, T)}{L(1, \text{sym}^2 f)} \ll Tk^{1-\theta_1(\sigma-1/2)} \log k.
\]
3. Proof of the main theorem

We will follow the framework of Selberg [16][17] and Hejhal–Luo [6]. The argument here is more complicated than in [6] since we did not assume the GRH.

For a positive parameter $x$ (to be determined later), let

$$M(t, f) := \frac{1}{\pi} \text{Im} \sum_{p \leq x} \frac{C_f(p)}{p^{1/2+it}} \quad \text{and} \quad R(t, f) := S(t, f) - M(t, f),$$

where $C_f(p) := \alpha_f(p) + \beta_f(p) = \lambda_f(p)$.

Proposition 3.1. Let $t > 0$ be given. For even $k \in \mathbb{N}$ sufficiently large and $x = k^{\delta/3}$ with sufficiently small $\delta > 0$, we have

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{M(t, f)^n}{L(1, \text{sym}^2 f)} = C_n(\log \log k)^{n/2} + O_{t,n}\left((\log \log k)^{n/2-1}\right),$$

(3.1)

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{|R(t, f)|^{2n}}{L(1, \text{sym}^2 f)} = O_{t,n}(1).$$

(3.2)

The proof of Proposition 3.1 shall be given in Section 5. Now we deduce the main theorem from this result.

Proof of Theorem 1.1. By the binomial theorem, we have

$$S(t, f)^n = M(t, f)^n + O_n\left(\sum_{\ell=1}^{n} |M(t, f)|^{n-\ell} |R(t, f)|^\ell\right).$$

(3.3)

For $1 \leq \ell < n$, we apply the generalized Hölder’s inequality with exponents

$$p = 2, \quad q = \frac{2n}{n-\ell} \quad \text{and} \quad r = \frac{2n}{\ell}$$

and Proposition 3.1 to deduce that

$$\ll \left(\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} |M(t, f)|^{n-\ell} |R(t, f)|^\ell\right)^{1/p} \left(\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} |M(t, f)|^{2n}\right)^{1/q} \times \left(\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} |R(t, f)|^{2n}\right)^{1/r} \ll_{t,n} (\log \log k)^{(n-\ell)/2} \ll_{t,n} (\log \log k)^{(n-1)/2}.$$

The assertion follows from (3.3), Proposition 3.1 and this bound. \qed
4. An approximation of $S(t, f)$

In this section we will prove several technical lemmas and derive an approximation of $S(t, f)$.

We denote by $\rho = \beta + i\gamma$ a typical zero of $L(s, f)$ inside the critical strip, i.e., $0 < \beta < 1$. For $\text{Re}(s) > 1$, we have

$$- \frac{L'(s, f)}{L(s, f)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)C_f(n)}{n^s},$$

where $\Lambda(n)$ denotes the von Mangoldt function, and

$$C_f(n) = \begin{cases} 
\alpha_f(p)^m + \beta_f(p)^m & \text{if } n = p^m \text{ for a prime } p, \\
0 & \text{otherwise.}
\end{cases}$$

**Lemma 4.1.** Let $x > 1$. For $s \neq \rho$, and $s \neq -2m - \frac{k+1}{2}$ ($m = 0, 1, 2, \ldots$), we have the following identity

$$- \log 2 x \sum_{n=1}^{\infty} \frac{C_f(n)\Lambda_x(n)}{n^s} = \frac{1}{2\pi i} \int_{(\alpha)} \frac{y^s}{s^3} ds = \begin{cases} 
\log^2 y & \text{if } y \geq 1, \\
0 & \text{if } 0 < y \leq 1
\end{cases}$$

for $\alpha > 0$. It follows from (4.1) and (4.2) that

$$- \log^2 x \sum_{n=1}^{\infty} \frac{C_f(n)\Lambda_x(n)}{n^s} = \frac{1}{2\pi i} \int_{(\alpha)} \frac{x^u(1-x^u)^2}{u^3} \frac{L'(s+u, f)}{L(s+u, f)} du,$$

where $\alpha = \max\{2, 1 + \text{Re}(s)\}$. By moving the line of integration all way to the left, we pick up the residues at $u = 0$, $u = \rho - s$ and $u = -2m - \frac{k+1}{2} - s$ ($m = 0, 1, 2, \ldots$) and
deduce that
\[
\frac{1}{2\pi i} \int_{(2)} \frac{x^u(1-x^u)^2}{u^3} \frac{L'(s+u,f)}{L(s+u,f)} \, du = \frac{L'}{L}(s,f) \log^2 x + \sum_{\rho} \frac{x^\rho-s(1-x^\rho-s)^2}{(\rho-s)^3} + \sum_{m=0}^{\infty} \frac{x^{-2m-\frac{k+1}{2}-s}(1-x^{-2m-\frac{k+1}{2}}-s)^2}{(-2m-\frac{k+1}{2} - s)^3}.
\]
Thus the lemma follows immediately.

**Lemma 4.2.** For \( s = \sigma + it, s' = \sigma' + it' \) such that \( 1/2 \leq \sigma, \sigma' \leq 10, s \neq \rho, \) and \( s' \neq \rho, \) we have
\[
\text{Im} \left( \frac{L'}{L}(s,f) - \frac{L'}{L}(s',f) \right) = \text{Im} \left( \sum_{\rho} \left( \frac{1}{s-\rho} - \frac{1}{s'-\rho} \right) \right) + O(1),
\]
and
\[
\text{Re} \left( \frac{L'}{L}(s,f) \right) = \sum_{\rho} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} + O(\log(|t| + k)).
\]

**Proof.** By Hadamard’s factorization of the entire function \( \Lambda(s,f) \) we have
\[
\frac{L'}{L}(s,f) = b_f + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \Gamma' \left( \frac{s+(k-1)/2}{2} \right) - \frac{1}{2} \Gamma' \left( \frac{s+(k+1)/2}{2} \right) + \log \pi
\]
for some \( b_f \in \mathbb{C} \) with \( \text{Re}(b_f) = -\text{Re} \sum_{\rho} \frac{1}{\rho} \) (see [9, Propsoition 5.7]). Now Lemma 4.2 follow from
\[
\frac{\Gamma'}{\Gamma}(s) = \log s + O \left( \frac{1}{|s|} \right).
\]

Let \( x \geq 4. \) We define
\[
\sigma_x = \sigma_{x,f} = \sigma_{x,f,t} := \frac{1}{2} + 2 \max_{\rho} \left\{ \left| \beta - \frac{1}{2} \right|, \frac{5}{\log x} \right\},
\]
where \( \rho = \beta + i\gamma \) runs through the zeros of \( L(s,f) \) for which
\[
|t - \gamma| \leq \frac{x^{3|\beta-1/2|}}{\log x}.
\]
We shall display the dependence of \( \sigma_x \) as needed.

**Lemma 4.3.** Let \( x \geq 4. \) For \( \sigma \geq \sigma_x, \) we have
\[
\frac{L'}{L}(\sigma + it, f) = -\sum_{n \leq x^4} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} + O \left( x^{1/4-\sigma/2} \left| \sum_{n \leq x^4} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} \right| \right) + O(x^{1/4-\sigma/2} \log(|t| + k)),
\]
and
\[
\sum_{\rho} \frac{(\sigma_x - 1/2)^2}{(\sigma_x - \beta)^2 + (t - \gamma)^2} = O \left( \sum_{n \leq x^4} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}} \right) + O(\log(|t| + k)).
\]
Proof. By (4.3), we have

\begin{equation}
\text{Re} \, \frac{L'}{L}(\sigma_x + it, f) = \sum_{\rho} \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + O(\log(|t| + k)).
\end{equation}

On the other hand, if \( \beta + i\gamma \) is a zero of \( L(s, f) \), then \( 1 - \beta + i\gamma \) is also a zero of \( L(s, f) \). Thus we have

\begin{equation}
\sum_{\rho} \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} = \frac{1}{2} \sum_{\rho} \left( \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + \frac{\sigma_x - (1 - \beta)}{(\sigma_x - 1 - \beta)^2 + (t - \gamma)^2} \right)
\end{equation}

\begin{equation}
= \left( \sigma_x - \frac{1}{2} \right) \sum_{\rho} \frac{(\sigma_x - 1/2)^2 - (\beta - 1/2)^2 + (t - \gamma)^2}{((\sigma_x - \beta)^2 + (t - \gamma)^2)((\sigma_x - 1 + \beta)^2 + (t - \gamma)^2)}.
\end{equation}

**Case (i):** If \( |\beta - 1/2| \leq \frac{\sigma_x - 1/2}{2} \), then

\[(\sigma_x - 1/2)^2 - (\beta - 1/2)^2 \geq \frac{1}{2}((\sigma_x - 1/2)^2 + (\beta - 1/2)^2)
= \frac{1}{4}((\sigma_x - \beta)^2 + (\sigma_x - 1 + \beta)^2).
\]

Thus

\[(\sigma_x - 1/2)^2 - (\beta - 1/2)^2 + (t - \gamma)^2 \geq \frac{1}{4}((\sigma_x - 1 + \beta)^2 + (t - \gamma)^2).
\]

**Case (ii):** If \( |\beta - 1/2| > \frac{\sigma_x - 1/2}{2} \), then by (4.4) and (4.5) we have

\[|t - \gamma| > x^{3|\beta - 1/2|}/\log x > 3|\beta - 1/2|.
\]

Thus

\[(\sigma_x - 1/2)^2 - (\beta - 1/2)^2 + (t - \gamma)^2
\]

\[= ((\sigma_x - 1/2)^2 + (\beta - 1/2)^2) + (t - \gamma)^2 - 2(\beta - 1/2)^2
\]

\[\geq \frac{1}{2}((\sigma_x - \beta)^2 + (\sigma_x - 1 + \beta)^2) + \frac{7}{9}(t - \gamma)^2
\]

\[\geq \frac{1}{4}((\sigma_x - 1 + \beta)^2 + (t - \gamma)^2).
\]

From Cases (i), (ii) and (4.7), we have

\[\sum_{\rho} \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \geq \frac{1}{4}(\sigma_x - 1/2) \sum_{\rho} \frac{1}{(\sigma_x - \beta)^2 + (t - \gamma)^2}.
\]

Using this bound and (4.6), we obtain

\begin{equation}
\sum_{\rho} \frac{1}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \leq \frac{4}{\sigma_x - 1/2} \left| \frac{L'}{L}(\sigma_x + it, f) \right| + O\left( \frac{\log(|t| + k)}{\sigma_x - 1/2} \right).
\end{equation}
On the other hand, we have (by Lemma 4.1)
\[
\frac{L'}{L}(\sigma + it, f) = - \sum_{n \leq x^3} \frac{\Lambda_x(n)C_f(n)}{n^{\sigma + it}} + \frac{w(x, \sigma, t)}{\log^2 x} \sum_{\rho} \frac{x^{\beta - \sigma}(1 + x^{\beta - \sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}}
\]
(4.9)
\[\quad + O\left(\frac{x^{-\sigma}}{\log^2 x}\right)\]
with $|w(x, \sigma, t)| \leq 1$.

Next we claim that
\[
\frac{x^{\beta - \sigma}(1 + x^{\beta - \sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} \leq 2 \log x \frac{x^{1/4 - \sigma/2}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}.
\]
(4.10)

If $\beta \leq \frac{\sigma_x + 1/2}{2}$, then
\[
\frac{x^{\beta - \sigma}(1 + x^{\beta - \sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} \leq \frac{4x^{1/4 - \sigma/2}}{(\sigma_x - \beta)((\sigma_x - \beta)^2 + (t - \gamma)^2)}
\]
\[\quad \leq \frac{8}{x^{1/4 - \sigma/2}} \frac{1}{\sigma_x - 1/2((\sigma_x - \beta)^2 + (t - \gamma)^2)}
\]
\[\quad \leq \frac{4}{5} \log x \frac{x^{1/4 - \sigma/2}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}.
\]

If $\beta > \frac{\sigma_x + 1/2}{2}$, then by the definition of $\sigma_x$ in (4.4) and (4.5),
\[
|t - \gamma| > \frac{x^{3|\beta - 1/2|}}{\log x} > 3|\beta - 1/2| \geq 3|\beta - \sigma_x|.
\]

Thus $(t - \gamma)^2 > \frac{8}{5}((\beta - \sigma_x)^2 + (t - \gamma)^2)$. Hence
\[
\frac{x^{\beta - \sigma}(1 + x^{\beta - \sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} \leq \frac{x^{\beta - \sigma}(1 + x^{\beta - 1/2})^2}{(t - \gamma)(t - \gamma)^2} \leq \frac{\log x}{x^{3|\beta - 1/2|}} \frac{9}{8} \frac{x^{\beta - \sigma}(1 + x^{\beta - 1/2})^2}{(\beta - \sigma_x)^2 + (t - \gamma)^2}
\]
\[\quad = \frac{9}{8}(\log x)(1 + x^{-(\beta - 1/2)})^2 \frac{x^{1/2 - \sigma}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}
\]
\[\quad < \frac{9}{8}(1 + e^{-5/2})(\log x) \frac{x^{1/2 - \sigma}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}
\]
\[\quad < 2(\log x) \frac{x^{1/2 - \sigma}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}.
\]

So in both cases, we have (4.10). Using (4.8) and (4.10), we get
\[
\sum_{\rho} \frac{x^{\beta - \sigma}(1 + x^{\beta - \sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}}
\]
\[\leq 8 \log x \frac{x^{1/4 - \sigma/2}}{\sigma_x - 1/2} \left| \frac{L'}{L}(\sigma_x + it, f) \right| + O\left(\frac{\log x x^{1/4 - \sigma/2} \log(|t| + k)}{\sigma_x - 1/2}\right)
\]
\[\leq \frac{4}{5}(\log x)^2 x^{1/4 - \sigma/2} \left| \frac{L'}{L}(\sigma_x + it, f) \right| + O\left((\log x)^2 x^{1/4 - \sigma/2} \log(|t| + k)\right).
\]
Inserting this bound into (4.9), we have

\[
\frac{L'}{L}(\sigma + it, f) = - \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma + it}} + \frac{4}{5} w'(x, \sigma, t) x^{1/4-\sigma/2} \frac{L'}{L}(\sigma x + it, f) + O(x^{1/4-\sigma/2} \log(|t| + k))
\]

(4.11)

with \(|w'(x, \sigma, t)| \leq 1\). By taking \(\sigma = \sigma_x\),

\[
\left(1 - \frac{4}{5} w'(x, \sigma_x, t) x^{1/4-\sigma_x/2}\right) \frac{L'}{L}(\sigma_x + it, f) = O \left( \left| \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + it}} \right| \right) + O(x^{1/4-\sigma/2} \log(|t| + k)).
\]

(4.12)

Since \(1 - \frac{4}{5} w'(x, \sigma_x, t) \geq 1 - \frac{4}{5} = \frac{1}{5}\),

\[
\frac{L'}{L}(\sigma_x + it, f) = O \left( \left| \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + it}} \right| \right) + O(x^{1/4-\sigma/2} \log(|t| + k)).
\]

The results follow from substituting (4.12) into (4.11) and (4.8).

The following theorem provides an approximation of \(S(t, f)\).

**Theorem 4.4.** For \(t \neq 0\), and \(x \geq 4\), we have

\[
S(t, f) = \frac{1}{\pi} \text{Im} \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + it} \log n} + O \left( \left| \frac{\sigma_x - 1/2}{\sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + it}}} \right| \right) + O((\sigma_x - 1/2) \log(|t| + k)),
\]

where \(\sigma_x\) is defined in (4.4).

**Proof.** We begin with

\[
\pi S(t, f) = - \int_{1/2}^\infty \text{Im} \frac{L'}{L}(\sigma + it, f) d\sigma
\]

\[
= - \int_{\sigma_x}^\infty \text{Im} \frac{L'}{L}(\sigma + it, f) d\sigma - (\sigma_x - 1/2) \text{Im} \frac{L'}{L}(\sigma_x + it, f)
\]

\[
+ \int_{1/2}^{\sigma_x} \text{Im} \left( \frac{L'}{L}(\sigma_x + it, f) - \frac{L'}{L}(\sigma + it, f) \right) d\sigma
\]

\[=: J_1 + J_2 + J_3.\]
By Lemma 4.3, we have

\[ J_1 = - \int_{\sigma_x}^{\infty} \text{Im} \left( \frac{L'}{L} (\sigma + it, f) \right) d\sigma \]

\[ = \int_{\sigma_x}^{\infty} \text{Im} \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} d\sigma + O \left( \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} \right) \int_{\sigma_x}^{\infty} x^{1/4-\sigma/2} d\sigma \]

\[ + O \left( \log(|t| + k) \int_{\sigma_x}^{\infty} x^{1/4-\sigma/2} d\sigma \right) \]

\[ = \text{Im} \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it} \log n} + O \left( \frac{1}{\log x} \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} \right) + O \left( \frac{\log(|t| + k)}{\log x} \right). \]

Taking \( \sigma = \sigma_x \) in Lemma 4.3, we see

\[ |J_2| \leq (\sigma_x - 1/2) \left| \frac{L'}{L}(\sigma_x + it, f) \right| \]

\[ \ll (\sigma_x - 1/2) \sum_{n \leq x^3} \frac{|C_f(n)\Lambda_x(n)|}{n^{\sigma+it}} + (\sigma_x - 1/2) \log(|t| + k). \]

To bound \( J_3 \), we first apply Lemma 4.2 to see that

\[ \text{Im} \left( \frac{L'}{L}(\sigma + it, f) - \frac{L'}{L}(\sigma + it, f) \right) \]

\[ = \sum_{\rho} \frac{(\sigma_x - \sigma)(\sigma + \sigma_x - 2\beta)(t - \gamma)}{((\sigma - \beta)^2 + (t - \gamma)^2)((\sigma - \beta)^2 + (t - \gamma)^2)} + O(1). \]

Hence

\[ |J_3| \leq \sum_{\rho} \int_{1/2}^{\sigma_x} \frac{(\sigma_x - \sigma)|\sigma + \sigma_x - 2\beta||t - \gamma|}{((\sigma - \beta)^2 + (t - \gamma)^2)((\sigma - \beta)^2 + (t - \gamma)^2)} d\sigma + O(\sigma_x - 1/2) \]

\[ \leq \sum_{\rho} \frac{\sigma_x - 1/2}{(\sigma - \beta)^2 + (t - \gamma)^2} \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma + O(\sigma_x - 1/2). \]

Case (i): If \( |\beta - 1/2| \leq \frac{1}{2}(\sigma_x - 1/2) \), then for \( 1/2 \leq \sigma \leq \sigma_x \),

\[ |\sigma + \sigma_x - 2\beta| = |(\sigma - 1/2) + (\sigma_x - 1/2) - 2(\beta - 1/2)| \]

\[ \leq |\sigma - 1/2| + |\sigma_x - 1/2| + 2|\beta - 1/2| \leq 3(\sigma_x - 1/2). \]

Thus

\[ \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma \leq 3(\sigma_x - 1/2) \int_{-\infty}^{\infty} \frac{|t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma \]

\[ < 10(\sigma_x - 1/2). \]
Case (ii): If $|\beta - 1/2| > \frac{1}{2}(\sigma_x - 1/2)$, then by (4.4) and (4.5),

$$|t - \gamma| > \frac{x^{3|\beta - 1/2|}}{\log x} > 3|\beta - 1/2|$$

and for $1/2 \leq \sigma \leq \sigma_x$,

$$|\sigma + \sigma_x - 2\beta| \leq (\sigma - 1/2) + (\sigma_x - 1/2) + 2|\beta - 1/2| \leq 6|\beta - 1/2|.$$

Thus

$$\int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} \, d\sigma < \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta|}{|t - \gamma|} \, d\sigma < \int_{1/2}^{\sigma_x} \frac{6|\beta - 1/2|}{3|\beta - 1/2|} \, d\sigma = 2(\sigma_x - 1/2).$$

It follows from Cases (i) and (ii) that

$$|J_3| \leq 10(\sigma_x - 1/2) \sum_{\rho} \frac{\sigma_x - 1/2}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + O(\sigma_x - 1/2)$$

$$= O \left( (\sigma_x - 1/2) \sum_{n \leq x^3} \frac{\Lambda_x(n)C_f(n)}{n^{\sigma_x + i\gamma}} \right) + O((\sigma_x - 1/2) \log(|t| + k)),$$

where the last step is obtained by Lemma 4.3. The theorem follows from these bounds for $J_1$, $J_2$ and $J_3$. 

We end this section by giving a consequence of Theorem 4.4, which is an analog to those obtained by Littlewood [11], Selberg [16], and Hejhal–Luo [6].

**Theorem 4.5.** Assuming the GRH, we have

$$S(t, f) \ll \frac{\log(|t| + k)}{\log \log(|t| + k)}.$$ 

**Proof.** We have $\sigma_x = \frac{1}{2} + \frac{10}{\log x}$ by virtue of the GRH. By (1.2),

$$|C_f(p^m)\Lambda_x(p^m)| \leq 2 \log p.$$

Thus

$$\left| \text{Im} \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + i\gamma} \log n} \right| \ll \sum_{p \leq x^3} p^{-1/2} \ll \frac{x^{3/2}}{\log x}$$

and

$$\left| (\sigma_x - 1/2) \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + i\gamma}} \right| \ll \frac{1}{\log x} \sum_{p \leq x^3} p^{-1/2} \log p \ll \frac{x^{3/2}}{\log x}.$$ 

The theorem follows by taking $x = \{\log(|t| + k)\}^{2/3}$ in Theorem 4.4 and above estimates. 

$\square$
5. Proof of Proposition 3.1

5.1. The main term (3.1)

First recall that \( C_f(p) = \alpha(p) + \beta(p) = \lambda_f(p) \) and

\[
M(t, f) = \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x} \frac{C_f(p)}{p^{1/2+it}} = -\frac{i}{2\pi} \left( \sum_{p \leq x^3} \frac{\lambda_f(p)}{p^{1/2+it}} - \sum_{p \leq x^3} \frac{\lambda_f(p)}{p^{1/2-it}} \right).
\]

Set \( x = k^{\delta/3} \) for a suitably small \( \delta > 0 \) (to be specified later). Thus we have

\[
M(t, f)^n = \frac{(-i)^n n !}{(2\pi)^n} \left( \sum_{p \leq k^\delta} \frac{\lambda_f(p)}{p^{1/2+it}} - \sum_{p \leq k^\delta} \frac{\lambda_f(p)}{p^{1/2-it}} \right)^n.
\]

A general term in the expansion of (5.1) has the form

\[
\frac{\lambda_f(p_1)^{m(p_1)+n(p_1)}}{p_1^{m(p_1)(1/2+it)}(-1)^{n(p_1)p_1^{n(p_1)(1/2-it)}}} \times \cdots \times \frac{\lambda_f(p_r)^{m(p_r)+n(p_r)}}{p_r^{m(p_r)(1/2+it)}(-1)^{n(p_r)p_r^{n(p_r)(1/2-it)}}},
\]

where \( p_1 < p_2 < \cdots < p_r < k^\delta, m(p_j) + n(p_j) \geq 1 \), and \( \sum_{j=1}^r (m(p_j) + n(p_j)) = n \).

To study the general term (5.2), we first recall the following Hecke relation

\[
\lambda_f(m)\lambda_f(n) = \sum_{d|\langle m, n \rangle} \lambda_f\left(\frac{mn}{d^2}\right).
\]

Now we discuss the contribution from the general term (5.2) in the following cases.

**Case (I): In the general term (5.2), \( m(p_{j_0}) \not\equiv n(p_{j_0}) \pmod{2} \) for some \( j_0 \).** By the Hecke relation (5.3) and Corollary 2.2, we have the contribution from these terms are negligible by taking \( 0 < \delta < 2/n \).

Thus for any \( A > 0 \) and odd \( n \geq 1 \), we have

\[
\frac{2\pi^2}{k-1} \sum_{f \in \mathcal{H}_k} \frac{M(t, f)^n}{\pi(1, \text{sym}^2 f)} \ll k^{-A}.
\]

**Case (II): In the general term (5.2), \( m(p_j) \equiv n(p_j) \pmod{2} \) for all \( j \), and \( m(p_{j_0}) + n(p_{j_0}) \geq 4 \) for some \( j_0 \).** In this case, \( n \) is even and \( n > 2r \). By the bound (1.2) and \( \sum_{p \leq x} p^{-1} \ll \log \log x \), we deduce that the contribution from these terms to (3.1) is at most \( O((\log \log k)^{n/2-1}) \).

It remains to discuss the following: In the general term (5.2), \( m(p_j) \equiv n(p_j) \pmod{2} \) and \( m(p_j) + n(p_j) = 2 \) for all \( j \). Clearly \( n \) must be even, say \( n = 2m \) and so \( r = m \).

**Case (III): \( m(p_{j_0}) = 2 \) or \( n(p_{j_0}) = 2 \) for some \( j_0 \).** We deduce that the contribution from these terms to (3.1) is at most \( O((\log \log k)^{n/2-1}) \) by using (1.2) and the convergence of \( \sum_{p \leq x} p^{-1-it} \) for \( t \neq 0 \).
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**Case (IV):** \( m(p_j) = n(p_j) = 1 \) for all \( j \). The contribution from these terms to (3.1) is

\[
\frac{2\pi^2}{k - 1} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \frac{(2m)!}{m!} \left( \frac{(-1)^m (-i)^{2m}}{(2\pi)^{2m}} \right) \sum_{p_1 < \cdots < p_m < k^\delta} \frac{\lambda_f(p_1)^2 \cdots \lambda_f(p_m)^2}{p_1 \cdots p_m}.
\]

(5.4)

By the Hecke relations, we have
\[
\lambda_f(p_1)^2 \cdots \lambda_f(p_m)^2 = \lambda_f(p_1 \cdots p_m)^2 \lambda_f(p_1 \cdots p_m).
\]

Applying Corollary 2.2, we see that (5.4) equals
\[
\frac{(2m)!}{(2\pi)^{2m}} \sum_{p_1 < \cdots < p_m < k^\delta} \frac{1}{p_1 \cdots p_m} \cdot (1 + O(k^{-A}))
\]

\[
= \frac{(2m)!}{m! (2\pi)^{2m}} \sum_{p_1, \ldots, p_m < k^\delta} \frac{1}{p_1 \cdots p_m} \cdot (1 + O(k^{-A}))
\]

\[
= C_{2m}(\log \log k + O(1))^m \cdot (1 + O(k^{-A}))
\]

\[
= C_{2m}(\log \log k)^m + O_m((\log \log k)^m - 1).
\]

Now the asymptotic formula (3.1) follows from Cases (I)–(IV).

### 5.2. The remainder term (3.2)

**Lemma 5.1.** Let \( t > 0 \) be given. For even \( k \in \mathbb{N} \) sufficiently large and \( x = k^{\delta/3} \) with \( 0 < \delta < 3\theta_1/(8n + 3) \), we have

\[
\sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{4n} \lambda_{x}^{4n}(\sigma_{x,f} - 1/2)}{L(1, \text{sym}^2 f)} \ll_{t,n,\delta} \frac{k}{(\log k)^{4n}},
\]

where \( \sigma_{x,f} \) is defined in (4.4) and \( \theta_1 \) is as in Proposition 2.3.

**Proof.** By the definition of \( \sigma_{x,f} \),

\[
(\sigma_{x,f} - 1/2)^{4n} \lambda_{x}^{4n}(\sigma_{x,f} - 1/2)
\]

\[
\leq \left( \frac{10}{\log x} \right)^{4n} x^{4n(\log x) + 2^{4n+1}} \sum_{\beta > \frac{1}{2} + \frac{5}{\log x}, |\beta - \gamma| \leq 2^{3(\beta - 1/2) / \log x}} (\beta - 1/2)^{4n} x^{8n(\beta - 1/2)}.
\]

On the other hand,

\[
\sum_{\beta > \frac{1}{2} + \frac{5}{\log x}, |\beta - \gamma| \leq 2^{3(\beta - 1/2) / \log x}} (\beta - 1/2)^{4n} x^{8n(\beta - 1/2)}
\]
\[
\begin{align*}
\sum_{j=5}^{\lfloor \log x \rfloor} \left( \frac{j+1}{\log x} \right)^{4n} x^{8n(\frac{j+1}{\log x} - \frac{1}{2}) + \beta} & \leq \frac{1}{(\log x)^{4n}} \sum_{j=5}^{\lfloor \log x \rfloor} (j+1)^{4n} e^{8n(j+1)} N_f \left( \frac{1}{2} + \frac{j}{\log x}, |t| + \frac{e^{3(j+1)}}{\log x} \right).
\end{align*}
\]

By Proposition 2.3,

\[
\begin{align*}
\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \sum_{\beta > \frac{1}{2} + \frac{j}{\log x}} \sum_{|t-\gamma| \leq \frac{2^{3\beta-1/2}}{\log x}} (\beta - 1/2)^{4n} x^{8n(\beta-1/2)} & \ll t \frac{1}{(\log x)^{4n}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n(j+1)} e^{3(j+1)} \log x \log k \\
& \ll t \frac{k \log k}{(\log x)^{4n+1}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n(j+1)+3(j+1)-\theta_1 \log \frac{k}{\log x}} \\
& \ll t, n, \delta \frac{k}{(\log k)^{4n}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n+3-\frac{3\theta_1}{\log x}} j \\
& \ll t, n, \delta \frac{k}{(\log k)^{4n}}
\end{align*}
\]

provided that \(0 < \delta < 3\theta_1/(8n+3)\). In addition,

\[
\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left( \frac{10}{\log x} \right)^{4n} x^{4n/\log x} \ll \frac{k}{(\log x)^{4n}} \ll \frac{k}{(\log k)^{4n}}.
\]

The lemma follows from above estimates. \qed

**Lemma 5.2.**

\[
R(t, f) = O \left( \left| \text{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda(p) - \Lambda_x(p))}{p^{1/2+it} \log p} \right| \right) + O \left( \left| \text{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+2it} \log p} \right| \right)
\]

\[
+ O \left( (\sigma_x - 1/2)x^{\sigma_x - 1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right)
\]

\[
+ O((\sigma_x - 1/2) \log(|t| + k)) + O(1).
\]

**Proof.** By Theorem 4.4

\[
R(t, f) = S(t, f) - M(t, f)
\]
Thus, using the bound (1.2), we deduce that

\[
\sum_{m=3}^{\infty} \sum_{p^m \leq x^3} \frac{C_f(p^m)\Lambda_x(p^m)}{p^m(\sigma_x+it)} = O(1) \quad \text{and} \quad \sum_{m=3}^{\infty} \sum_{\sigma_x \leq p^m \leq x^3} \frac{C_f(p^m)\Lambda_x(p^m)}{p^m(\sigma_x+it) \log p} = O(1).
\]

Note that

\[
(\sigma_x - 1/2) \left| \sum_{p \leq x^{1/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{2(\sigma_x+it)}} \right| \ll (\sigma_x - 1/2) \sum_{p \leq x^{1/2}} \frac{\log p}{p} \ll (\sigma_x - 1/2) \log x \ll (\sigma_x - 1/2) \log k.
\]

Thus

\[
R(t, f) = \frac{1}{\pi} \text{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda_x(p) - \Lambda(p))}{p^{1/2+it} \log p} - \frac{1}{\pi} \text{Im} \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)(1 - p^{1/2-\sigma_x})}{p^{1/2+it} \log p} + \frac{1}{\pi} \text{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+it} \log p} + \frac{1}{\pi} \text{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)(p^{-2\sigma_x} - 1)}{p^{1+it} \log p} + O\left((\sigma_x - 1/2) \log(|t| + k)\right) + O(1).
\]

Note that for $1/2 \leq a \leq \sigma_x$,

\[
\left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{a+it}} \right| = x^{a-1/2} \int_a^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \leq x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma.
\]

Thus

\[
\left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)(1 - p^{1/2-\sigma_x})}{p^{1/2+it} \log p} \right| = \int_{1/2}^{\sigma_x} \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{a+it}} da.
\]
\[
\sum_{p \leq x^\lambda} \frac{C_f(p)\Lambda_x(p)}{p^{\sigma+it}} \leq (\sigma_x - 1/2)x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}} \right| \, d\sigma
\]

and

\[
(\sigma_x - 1/2) \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{\sigma+it}} \right| \leq (\sigma_x - 1/2)x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}} \right| \, d\sigma.
\]

Moreover,

\[
\left| \sum_{p \leq x^{\lambda/2}} \frac{C_f(p^2)\Lambda_x(p^2)(p^{1-2\sigma_x} - 1)}{p^{1+it}\log p} \right| \ll \sum_{p \leq x^{\lambda/2}} \frac{1}{p}(1 - p^{1-2\sigma_x}) \ll \sum_{p \leq x^{\lambda/2}} \frac{(\sigma_x - 1/2)\log p}{p} \ll (\sigma_x - 1/2)\log k.
\]

The results follows from above estimates. \(\square\)

Now we are ready to prove (3.2). Recall that \(\sigma_x = \sigma_{x,f}\) depending on \(f\). By Lemma 5.2

(5.5)

\[
\sum_{f \in H_k} \frac{|R(t,f)|^{2n}}{L(1, \text{sym}^2 f)} \ll \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left| \text{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda(p) - \Lambda_x(p))}{p^{1/2+it}\log p} \right|^{2n} + \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left| \text{Im} \sum_{p \leq x^{\lambda/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+2it}\log p} \right|^{2n} + \sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{2n}x^{2n(\sigma_{x,f} - 1/2)}}{L(1, \text{sym}^2 f)} \left( \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}} \right| \, d\sigma \right)^{2n} + \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} (\sigma_{x,f} - 1/2)^{2n}(\log(|t| + k))^{2n}.
\]

Since

\[
|\Lambda(p) - \Lambda_x(p)| = O \left( \frac{\log^3 p}{\log^2 x} \right) \quad \text{and} \quad C_f(p^2) = \lambda_f(p^2) - 1,
\]

the first two terms is of \(O(k)\) by the same argument as in Section 5.1. The last term is of \(O(k)\) by Lemma 5.1.
For the third term, it follows from Cauchy’s inequality that

\[
\sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)2n \cdot x^{2n(\sigma_{x,f} - 1/2)}}{L(1, \text{sym}^2 f)} \left( \int_{1/2}^{x^{1/2-\sigma}} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| \, d\sigma \right)^{2n} 
\leq \left( \sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)4n \cdot x^{4n(\sigma_{x,f} - 1/2)}}{L(1, \text{sym}^2 f)} \right)^{1/2} \times \left( \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left( \int_{1/2}^{x^{1/2-\sigma}} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| \, d\sigma \right)^{4n} \right)^{1/2}.
\]

By Hölder’s inequality with the exponents \(4n/(4n - 1)\) and \(4n\),

\[
\left( \int_{1/2}^{x^{1/2-\sigma}} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| \, d\sigma \right)^{4n} \leq \left( \int_{1/2}^{x^{1/2-\sigma}} \left( \sum_{p \leq x^3} C_f(p) \Lambda_x(p) \log(xp) \right)^{4n} \, d\sigma \right)
\leq \frac{1}{(\log x)^{4n-1}} \int_{1/2}^{x^{1/2-\sigma}} \left| \sum_{p \leq x^3} C_f(p) \Lambda_x(p) \log(xp) \right|^{4n} \, d\sigma.
\]

Using the same argument as in Section 5.1, we have

\[
\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right|^{4n} \ll \left( \sum_{p \leq x^3} \frac{|\Lambda_x(p) \log(xp)|^{1/2-\sigma}|^2}{p} \right)^{2n} \ll \left( \sum_{p \leq x^3} \frac{(\log p)^2 (\log x)^2}{p} \right)^{2n} \ll (\log x)^{8n},
\]

where the last inequality is obtained by \(\sum_{p \leq y} \frac{(\log p)^2}{p} \ll (\log y)^2\). Thus

\[
\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left( \int_{1/2}^{x^{1/2-\sigma}} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| \, d\sigma \right)^{4n} \ll \frac{k}{(\log x)^{4n-1}} \int_{1/2}^{x^{1/2-\sigma}} (\log x)^{8n} \, d\sigma \ll k (\log x)^{4n}.
\]

Using this bound and Lemma 5.1, we have the third term in (5.5) is of \(O(k)\).

Finally (3.2) follows from the above discussion.
6. A weighted central limit theorem

As pointed out in Remark 1.2 for a weight \( k \) Hecke cusp forms \( f \), \( S(t, f) \) has order of magnitude \( \sqrt{\log \log k} \). Thus it makes sense to consider the following probability measure \( \mu_k \) on \( \mathbb{R} \), defined by

\[
\mu_k(E) := \frac{\left( \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} 1_E \left( \frac{S(t, f)}{\sqrt{\log \log k}} \right) \right)}{\left( \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \right)}
\]

where \( 1_E \) is the characteristic function on a Borel measurable set \( E \) in \( \mathbb{R} \). As a consequence of Theorem 1.1 we obtain the following weighted central limit theorem, which should be compared with that in [6].

**Theorem 6.1.** As \( k \to \infty \), the probability measure \( \mu_k \) converges to the Gaussian distribution of mean 0 and variance \( (2\pi^2)^{-1} \); that is,

\[
\lim_{k \to \infty} \mu_k([a, b]) = \int_a^b \sqrt{\pi} \exp(-\pi^2 \xi^2) \, d\xi
\]

for any \( a < b \).

**Proof.** The \( n \)th moment of \( \mu_k \) is

\[
\int_{\mathbb{R}} \xi^n \, d\mu_k(\xi) = \frac{\left( \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left( \frac{S(t, f)}{\sqrt{\log \log k}} \right)^n \right)}{\left( \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \right)}.
\]

By Theorem 1.1 and Corollary 2.2 we deduce that for all \( n \),

\[
\lim_{k \to \infty} \int_{\mathbb{R}} \xi^n \, d\mu_k(\xi) = C_n = \int_{\mathbb{R}} \xi^n \sqrt{\pi} \exp(-\pi^2 \xi^2) \, d\xi.
\]

Now the result follows from the theory of moments in probability theory (see, for example, [3, Theorem 30.2]).

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