A note on induced Turán numbers

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Abstract

Loh, Tait, Timmons and Zhou introduced the notion of induced Turán numbers, defining
ex(n, {H, F-ind}) to be the greatest number of edges in an n-vertex graph with no copy of H
and no induced copy of F. Their and subsequent work has focussed on F being a complete
bipartite graph. In this short note, we complement this focus by asymptotically determining
the induced Turán number whenever H is not bipartite and F is not an independent set nor
a complete bipartite graph.

1 Introduction

Consider the following induced version of the classical Turán problem. What is the most edges
in an n-vertex graph which does not contain F as an induced subgraph? This question has a
simple answer: if F is a clique, then it is the original subgraph problem (and so answered by
Turán’s theorem [8]), and if F is not complete, then the maximum is plainly \(^n\choose 2\), as witnessed
by the n-vertex complete graph \(K_n\).

In an insightful paper, Loh, Tait, Timmons and Zhou [5] recovered an interesting and natural
induced Turán problem by forbidding both an induced F and a (not necessarily induced) copy
of H. This removes the possibility of \(K_n\) being extremal and so rules out the humdrum answer
of \(^n\choose 2\). To be precise, for a positive integer n and graphs H and F, they defined the induced
Turán number

\[ \text{ex}(n, \{H, F\text{-ind}\}) \]

to be the maximum number of edges in an n-vertex graph with no copy of H and no induced
copy of F.

In [5], Loh, Tait, Timmons and Zhou focussed on complete bipartite F proving general
bounds for \(\text{ex}(n, \{H, K_{s,t}\text{-ind}\})\) with some sharper results when \(s = 2\). Nikiforov, Tait and
Timmons [6] proved spectral improvements of their results. Ergemlidze, Győri and Methuku [3]
asymptotically determined \(\text{ex}(n, \{H, F\text{-ind}\})\) in the important case where H is an odd cycle and
F is \(K_{2,t}\) or \(K_{3,3}\) (except when \(H = C_5\) and F = \(K_{2,2}\) where they strengthened the upper bound).
In [4], the author gave further improvements to the upper bounds for \(\text{ex}(n, \{H, K_{2,t}\text{-ind}\})\).

All the work to date has focussed on complete bipartite F. The following theorem comple-
ments this focus.

Theorem 1. Let H and F be graphs with chromatic numbers r + 1 and s + 1 respectively. Then

\[ \text{ex}(n, \{H, F\text{-ind}\}) = \begin{cases} (1 - \frac{1}{s} + o(1)) \binom{n}{2} & \text{if } s > r \text{ or } F \text{ is not complete multipartite,} \\ (1 - \frac{1}{r} + o(1)) \binom{n}{2} & \text{if } s < r \text{ and } F \text{ is complete multipartite.} \end{cases} \]

This asymptotically determines the induced Turán number except if H is bipartite or F is
an independent set or a complete bipartite graph.

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2 Proof of Theorem 1

We will use $T_r(n)$ to denote the r-partite Turán graph on n vertices. This is complete r-partite with parts as equal in size as possible. We remind the reader that $T_r(n)$ has $(1 - 1/r + o(1))(n^2)$ edges.

Proof of Theorem 1. If $s \geq r$ or $F$ is not complete multipartite, then $F$ is not an induced subgraph of the Turán graph $T_r(n)$. This also does not contain $H$, so

$$\text{ex}(n, \{H, F\text{-ind}\}) \geq e(T_r(n)) = (1 - \frac{1}{r} + o(1))(n^2).$$

On the other hand, the Erdős-Simonovits theorem [1] says that any n-vertex graph not containing $H$ has at most $(1 - 1/r + o(1))(n^2)$ edges. This completes the first part of the theorem.

Now suppose that $s < r$ and $F$ is complete $(s + 1)$-partite. Firstly the Turán graph $T_s(n)$ contains neither $H$ nor $F$ as subgraphs (let alone induced ones) so,

$$\text{ex}(n, \{H, F\text{-ind}\}) \geq e(T_s(n)) = (1 - \frac{1}{s} + o(1))(n^2).$$

As $F$ is complete $(s + 1)$-partite, there is a positive integer $t$ such that $F$ is an induced subgraph of $K_{s+1}(t) = T_{s+1}(t(s+1))$. By Ramsey’s theorem [7], there is a positive integer $m$ such that any $m$-vertex graph contains either an independent $t$-set or a copy of $H$.

Fix $\varepsilon > 0$ and let $G$ be an $n$-vertex graph with at least $(1 - 1/s + \varepsilon)(n^2)$ edges where $n$ is sufficiently large. By the Erdős-Stone theorem [2], $G$ contains a copy of $K_{s+1}(m)$. Let the parts of the $K_{s+1}(m)$ be $V_1, \ldots, V_{s+1}$ so each one has $m$ vertices. If any $V_i$ contains $H$, then $G$ does. Otherwise, by the definition of $m$, each $V_i$ contains an independent set of size $t$. Thus $G$ contains an induced copy of $K_{s+1}(t)$ and so an induced copy of $F$. Thus, for all large $n$,

$$\text{ex}(n, \{H, F\text{-ind}\}) \leq (1 - \frac{1}{s} + \varepsilon)(n^2).$$

This holds for arbitrary $\varepsilon > 0$, as required. □

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