On the Stability of Gravitating Nonabelian Monopoles

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Abstract

The behaviour of magnetic monopole solutions of the Einstein-Yang-
Mills-Higgs equations subject to linear spherically symmetric pertur-
bations is studied. Using Jacobi’s criterion some of the monopoles are
shown to be unstable. Furthermore the numerical results and analytical
considerations indicate the existence of a set of stable solutions.

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1 Introduction

Stimulated by ‘t Hooft’s [1] and Polyakov’s [2] research van Nieuwenhuizen et al. [3] studied the influence of gravity on magnetic monopoles, but their results do neither guarantee the existence nor the stability of the solutions as shown in [4], [5], [6]. Later Breitenlohner et al. [4] and others (Lee et al. [5], Ortiz [6]) investigated ‘t Hooft’s and Polyakov’s monopoles ([1], [2]) in curved space using numerical methods. Introducing the dimensionless parameters

\[ \alpha = \frac{M_W}{g M_{Pl}} \]

(with the Planck mass \( M_{Pl} = 1/\sqrt{G} \)) and

\[ \beta = \frac{M_H}{M_W}, \]

where \( M_W \) and \( M_H \) denote the mass of the corresponding Yang-Mills field and Higgs field respectively, Breitenlohner et al. [4] presented the following results: For \( 0 \leq \beta < \infty \) the monopole mass varies only over a finite domain. Choosing the parameter \( \beta = 0 \) solutions seem to exist for \( 0 \leq \alpha \leq \alpha_{\text{max}} \). For \( \alpha_c \approx 1.386 \leq \alpha \leq 1.403 \approx \alpha_{\text{max}} \) there are two solutions with different masses. Considering the mass as a function of \( \alpha \), one finds that it attains a maximum \( M_{\text{max}} \) for \( \alpha = \alpha_{\text{max}} \), and that it decreases towards \( M_{Pl}/g \) asymptotically as \( \alpha \) tends to \( \alpha_c \). This situation is shown in Figure 1. Each point represents a solution calculated numerically. The right plot of Figure 1 is a magnification of the left plot near the point \( B \). The series of points marked by \( BC \) will be denoted by “upper branch”, the part \( AB \) by “lower branch”.

In this paper stability of the nonabelian gravitating magnetic monopoles is investigated using the technique of spherically symmetric perturbations. Stability considerations of similar nature have been conducted by Zhou and Straumann [7]. The stability problem requires the numerical behaviour of the monopole solutions to be known. The construction of the monopole solutions in question ([4]) is briefly reviewed. In order to investigate their stability the monopoles are subjected to small perturbations in time, which have a Fourier expansion due to its linearity. Eliminating the gravitational degrees of freedom, the remaining terms in the action are written as a quadratic form. Then the application of Jacobi’s criterion ([8], [13]) suggests instability on the branch \( BC \) and stability under spherically symmetric perturbations on the lower branch. Note however that this does not imply stability in general. In the final section the results are summarized and discussed.

2 Solutions of the EYMH Equations

The ‘t Hooft Polyakov monopole in curved space is a static finite energy solution of an SO(3) gauge theory with a triplet of scalar fields. Using a coordinate free notation the action reads

\[
S = \int \left( -\frac{1}{16\pi G} *R - \frac{1}{4\pi} \left( -\frac{1}{4g^2} Tr(F \wedge *F) + \frac{1}{2} Tr(D\Phi \wedge *D\Phi) \right) \right) + U(\Phi),
\]

(1)
where \( R, F \) and \( \Phi \) denote the curvature scalar, the Yang-Mills field and the scalar field respectively. \( G \) and \( g \) are the gravitational constant and the gauge coupling. \( * \) is the Hodge star operator and \( U(\Phi) \) is a potential term which will be specified later. The gravitational field is described by a static spherically symmetric metric tensor given by the line element \( ds^2 \) in Schwarzschild coordinates:

\[
ds^2 = A^2(r)\mu(r) \, dt^2 - \frac{1}{\mu(r)} \, dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

The minimal ansatz for the Yang-Mills potential \( A \) which is spherically symmetric is denoted in polar coordinates and abelian gauge as follows (9,10,11):

\[
A = (W(r) \, T_2 + \cot \theta \, T_3) \sin \theta \, d\phi + W(r) \, T_1 \, d\theta.
\]

The \( T_i, i = 1, 2, 3, \) are the generators of the fundamental representation of \( \text{SO}(3) \). Using the potential \( A \) the field \( F \) is expressed by

\[
F = \frac{1}{2} [A, A].
\]

The Higgs field is chosen to be

\[
\Phi = H(r) (\cos \phi \sin \theta \, T_1 + \sin \phi \sin \theta \, T_2 + \cos \theta \, T_3).
\]

In order to reduce the four dimensional action to a one dimensional integral the curvature scalar, the Yang-Mills, and the Higgs terms are substituted into the action (1):

\[
S_{\text{red}} = -\int_0^\infty dr \, A \left( \frac{1}{2\alpha^2} (\mu + r \mu' - 1) - \mu V_1 - V_2 \right),
\]

with

\[
V_1 = W^2 + \frac{r^2}{2} H^2,
\]

\[
V_2 = \frac{(W^2 - 1)^2}{2r^2} + W^2 H^2 + \frac{1}{8} \beta^2 r^2 (H^2 - 1)^2.
\]

The term \( \frac{1}{8} \beta^2 r^2 (H^2 - 1)^2 \) represents the potential \( U(\Phi) \). The equations of motion (EYMH equations) deduced from the action (2) contain the parameters \( \alpha \) and \( \beta \) as well. Their values determine the stability properties of the monopoles.

The “background solution” \( F(r) \) corresponding to global regular monopoles has been obtained solving the equations of motion with a multiple shooting method [12].
3 Stability Analysis

The static background solution \( \mathbf{F}(r) \) is subjected to a spherically symmetric perturbation \( \mathbf{f}(r)e^{i\omega t} \), i.e. the perturbed and the background solutions are related as follows:

\[
\tilde{\mathbf{F}}(r, t) = \mathbf{F}(r) + \mathbf{f}(r)e^{i\omega t},
\]

where \( \mathbf{F}(r) \) is the vector of the functions \( H(r), W(r), Y(r) \) and \( M(r) \) and \( \mathbf{f}(r) \) denotes the vector \( (h, v, y, m) \) of the radial parts of the perturbations.

It is convenient to introduce \( M(r) \) and \( Y(r) \):

\[
\mu(r) = 1 - 2\frac{M(r)}{r}, \quad \text{and} \quad A(r) = e^{Y(r)}.
\]

Instability in this context means the growth of the exponential term in time. The analysis is thus reduced to the question whether admissible perturbations, i.e. perturbations fulfilling the boundary conditions, with an imaginary frequency exist. Substituting \( \tilde{\mathbf{F}}(r, t) \) into the action (1) one obtains a two-dimensional reduced action differing from (2) only by the time-dependent additional term

\[
V_0 = \frac{1}{A\mu} \left( \dot{W}^2 + \frac{r^2}{2} \dot{H}^2 \right).
\]

The part of the reduced action \( \tilde{S}_{\text{red}} \) quadratic in the fields and their derivatives turns out to be

\[
\tilde{S}_{\text{red},2} = \int_0^\infty dr \ e^Y \left\{ y^2 \left( \frac{M'}{\alpha^2} - \mu V_1 - V_2 \right) - 2y\mu \left( 2W'\nu' + r^2 H'h' \right) \\
+ 2y \left( \frac{m'}{\alpha^2} + \frac{2m}{r} V_1 - \frac{2W(W^2 - 1)}{2}\nu - 2WH^2\nu - 2W^2Hh - \frac{\beta^2}{2}r^2H(H^2 - 1)h \right) \\
+ \frac{4m}{r}(2W'\nu' + r^2 H'h') - \mu \left( 2\nu'^2 + r^2 h'^2 \right) - \frac{2(3W^2 - 1)}{r^2}\nu^2 - 2H^2\nu^2 \\
- 8WH\nu h - 2W^2 h^2 - \frac{\beta^2}{2}r^2(3H^2 - 1)h^2 - \omega^2 \frac{e^{-2Y}}{\mu}(2\nu^2 + r^2 h^2) \right\}.
\]

Taking into account the boundary conditions at \( r = 0 \), the variation with respect to \( y \) yields

\[
\frac{m'}{\alpha^2} + \frac{m}{\alpha^2} y' + D_W L \nu' + D_W L \nu + D_H L H' + D_H L h = 0,
\]

where \( L \) is the integrand of \( \tilde{S}_{\text{red}} \) and \( D_W, D_W, D_H \), and \( D_H \) denote the derivatives with respect to \( W', W, H' \) and \( H \). (4) is solved by

\[
m = \alpha^2 (2\mu W'\nu + r^2 \mu H'h).
\]
This result is used to eliminate $m$ and $y$ in $\tilde{S}^{\text{red},2}$. Partial integration and introduction of $\hat{h} = rh/\sqrt{2}$ and the variable $\sigma$ by $dr = A\mu\,d\sigma$ leads to the normal form of a 2-channel Schrödinger problem $\tilde{S}^{\text{red},2}$:

$$\tilde{S}^{\text{red},2} = \int_0^\infty d\sigma \left\{ D\Psi^\dagger D\Psi + \Psi^\dagger Q(r)\Psi - \omega^2\Psi^\dagger\Psi \right\}. \quad (5)$$

Here

\[
\Psi^\dagger = (\hat{h}, v), \quad D\Psi^\dagger = \left( \frac{d\hat{h}}{d\sigma}, \frac{dv}{d\sigma} \right)
\]

\[
Q(r) = A\mu \begin{pmatrix} Q_{11}(r) & Q_{12}(r) \\ Q_{21}(r) & Q_{22}(r) \end{pmatrix}
\]

\[
Q_{11}(r) = \frac{2W^2}{r^2} + \frac{\beta^2}{2}(3H^2 - 1) + \frac{2\alpha^2}{A} (A\mu r^3H'^2)' + \frac{1}{A} \frac{(A\mu)'}{r}
\]

\[
Q_{12}(r) = \frac{2\sqrt{2}}{r} \left( HW + \alpha^2 \frac{1}{A} (A\mu rW'H')' \right) = Q_{21}(r)
\]

\[
Q_{22}(r) = H^2 + \frac{3W^2 - 1}{r^2} + \frac{4\alpha^2}{A} \left( \frac{A\mu W'^2}{r} \right)'.
\]

The background solution is unstable, iff the Schrödinger problem admits a bound state, that is a solution with $\omega^2 < 0$. The standard technique for a 2-channel Schrödinger problem is to calculate the equations of motion from (5), which then define an eigenvalue problem with the eigenvalue $\omega^2$. Instead of analyzing the spectrum of this operator the positive definiteness of (5) without the last term is investigated applying Jacobi’s criterion [8], [13]. Although in [8] it is formulated for compact intervals only the proof applies in this case to an infinite interval of integration, too, in view of the asymptotic behaviour of $\Psi$. The discussion of Jacobi’s criterion in [8] also provides a normalizable perturbation with negative energy. This criterion states that the monopole solutions are unstable iff

$$\det V(r) = \begin{vmatrix} \hat{h}^{(1)}(r) & \hat{h}^{(2)}(r) \\ v^{(1)}(r) & v^{(2)}(r) \end{vmatrix} = 0 \quad \text{for} \quad r \in (0, \infty). \quad (6)$$

$\hat{h}^{(1)}$, $v^{(1)}$, $\hat{h}^{(2)}$, and $v^{(2)}$ denote the solutions of two linearly independent initial value problems for the equations of motion in the fields $\hat{h}$ and $v$ (Jacobi equation). The initial values are chosen such that they are compatible with the symmetries of the background solution.

A high order Runge-Kutta-Fehlberg method [12] has been employed for the numerical integration of this system of differential equations.
4 Results and Discussion

First we discuss the results in the Prasad-Sommerfield limit. The numerical analysis shows that the branch $BC$ in Figure 1 is unstable in the sense of linear perturbation theory, since the determinant has a zero in the interior of the integration interval.

In Figure 2 the determinant (4) is shown for some values of the parameter $\alpha$ on the upper branch. For $r_{\mu_{\text{min}}}$, that is the value of $r$ where the function $\mu$ has a minimum (see [4]), $\det V(r)$ attains a local maximum. A second maximum at greater values of $r$ is observed for some values of $\alpha$. For $\alpha$ near $\alpha_{\text{max}}$ the second maximum dominates, whereas, if $\alpha$ is approximately $\alpha_{c}$ the left peak is well developed. For decreasing values of $\alpha$ ($\alpha \to \alpha_{c}$) the zero of the determinant moves to the left.

It is convenient to rescale the determinant by $\mu(r)e^{-\alpha r}$ in order to present the functions for all $\alpha$-values of the upper branch in one plot (see Figure 3). The pronounced minimum in the left part of Figure 3 is due to the rescaling of $\det V(r)$. It suggests that $\lim_{\alpha \to \alpha_{c}} \mu(r_{\mu_{\text{min}}}) \det V(r_{\mu_{\text{min}}}) = 0$, i.e. in the neighbourhood of this point the determinant increases more slowly than $1/\mu$.

For $\alpha_{\text{max}}$ the determinant has a zero at infinity in agreement with a theorem in [14] which states that a change of the stability properties can only happen at a point of bifurcation in the masses. From this theorem it follows that the whole lower branch in Figure 1 consists of stable solutions, because of the stability of the Prasad-Sommerfield solution in flat space ($\alpha = 0$).

The bifurcation point is determined using Jacobi’s criterion. For increasing values of $\beta$ the method becomes more and more sensitive which means that the error becomes smaller and smaller.

The determinant corresponding to solutions on the lower branch is plotted in Figure 4. As expected these functions have no zeros. The curve with the steepest ascent belongs to $\alpha = 0$.

Figure 5 and 6 show the results for some selected values of $\beta \neq 0$. One finds the same qualitative behaviour as for $\beta = 0$. In Figure 5 the determinant is plotted for given $\beta$ and the corresponding maximal $\alpha$-values. The lowest curve belongs to $\beta = 0.05$. Figure 6 uses the same $\beta$-values, but $\alpha$-values on the upper branch. Increasing $\beta$ the zero of the determinant moves to the left.

Some numerical results are collected in the following table. $\alpha_{c}$ is obtained by linear extrapolation:
| $\beta$ | $\alpha_{\text{max}}$ | $\mu_{\alpha_{\text{max}}} (r_{\mu_{\text{min}}})$ | $M_{\text{max}}$ | $\alpha_{c}$ |
|-------|-----------------|-----------------|----------------|--------|
| 0.00  | 1.403034        | 3.611 $\cdot 10^{-2}$ | 1.000219 | 1.3859 |
| 0.05  | 1.397555        | 2.674 $\cdot 10^{-2}$ | 1.000138 | 1.3822 |
| 0.10  | 1.389459        | 2.028 $\cdot 10^{-2}$ | 1.000076 | 1.3806 |
| 0.15  | 1.379974        | 1.402 $\cdot 10^{-2}$ | 1.000038 | 1.3745 |
| 0.20  | 1.369676        | 9.211 $\cdot 10^{-3}$ | 1.000017 | 1.3664 |
| 0.25  | 1.358883        | 5.811 $\cdot 10^{-3}$ | 1.000007 | 1.3571 |

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\[ \text{det}(r) \mu(r) \exp(-\alpha r) \]
