Algebra/Group theory

On the $T^\circ$-slices of a finite group

Sur les $T^\circ$-tranches d’un groupe fini

Ibrahima Tounkara

Laboratoire d’algèbre, de cryptologie, de géométrie algébrique et applications (LACGAA), Département de mathématiques et informatique, Faculté des sciences et techniques, Université Cheikh-Anta-Diop, BP 5005, Dakar, Senegal

Abstract

A slice $(G, S)$ of finite groups is a pair consisting of a finite group $G$ and a subgroup $S$ of $G$. In this paper, we show that some properties of finite groups extend to slices of finite groups. In particular, by analogy with $B$-groups, we introduce the notion of $T^\circ$-slice, and show that any slice of finite groups admits a largest quotient $T^\circ$-slice.

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Résumé

Une tranche $(G, S)$ de groupes finis est un couple formé d’un groupe fini $G$ et d’un sous-groupe $S$ de $G$. Dans cet article, nous démontrons que certaines propriétés des groupes finis s’étendent aux tranches de groupes finis. En particulier, par analogie avec les $B$-groupes, nous introduisons la notion de $T^\circ$-tranche, et nous montrons que toute tranche de groupes finis admet un plus grand quotient qui soit une $T^\circ$-tranche.

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1. Introduction

In this note, we extend to slices of finite groups some notions relative to finite groups. Throughout the paper, $G$ will denote a finite group.

In order to study the lattice of biset subfunctors of the Burnside functor $\mathbb{Q} \otimes B$, Serge Bouc (see [2]) studies the effect of the elementary biset operations on the primitive idempotents of the Burnside algebra $\mathbb{Q} \otimes B(G)$: these idempotents $e_H^G$ are indexed by the subgroups $H$ of $G$, up to conjugation and given (see [6], [8]) by

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H)[G/K]$$

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E-mail address: ibrahima1.tounkara@ucad.edu.sn.

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where \([G/K]\) denotes the isomorphism class of the \(G\)-set \(G/K\), and \(\mu\) is the Möbius function of the poset of subgroups of \(G\).

Serge Bouc shows in particular that, for each normal subgroup \(N\) of \(G\), there is a rational number \(m_{G,N}\) such that

\[
\text{Def}_{G/N}^G e_G^G = m_{G,N} e_{G/N}^G.
\]

This constant \(m_{G,N}\) is given by

\[
m_{G,N} = \frac{1}{|G|} \sum_{\mathclap{X \leq G \atop X \not\leq N}} |X| \mu(X, G).
\]

This leads to the introduction of a special class of finite groups, called \(B\)-groups: a group \(G\) is a \(B\)-group if \(m_{G,N} = 0\) for any non-trivial normal subgroup \(N\) of \(G\).

When \(N\) and \(M\) are maximal subgroups of \(G\) such that \(m_{G,N} \neq 0\) and \(m_{G,M} \neq 0\), then the quotients \(G/N\) and \(G/M\) are isomorphic. Such a quotient \(G/N\) does not depend on \(N\), up to isomorphism. It is denoted by \(\beta(G)\). This quotient \(\beta(G)\) of \(G\) is a \(B\)-group, and moreover, any \(B\)-group \(H\) that is a quotient of \(G\) is actually a quotient of \(\beta(G)\). Further results on this invariant \(\beta(G)\) and \(B\)-groups can be found in [1] and [4].

Recall that a slice of finite groups is a pair \((T, S)\) consisting of a finite group \(T\) and a subgroup \(S\) of \(T\). We say that the slice \((V, U)\) is a quotient of the slice \((T, S)\) if there exists a surjective group homomorphism \(\varphi: T \to V\) such that \(\varphi(S) = U\). When \(\varphi\) is an isomorphism, we say that \((T, S)\) and \((V, U)\) are isomorphic.

A slice of a finite group \(G\) is a slice \((T, S)\) consisting of subgroups of \(G\). The group \(G\) acts by conjugation on the set \(\Pi(G)\) of its slices.

In [3], the slice Burnside ring \(\Xi(G)\) is introduced: it is a commutative ring, which has a \(\mathbb{Z}\)-basis \((T, S)_G\) indexed by the conjugacy classes of slices \((T, S)\) of \(G\). It is shown that \(\mathbb{Q} \otimes \Xi(G)\) is a split semisimple \(\mathbb{Q}\)-algebra, whose primitive idempotents are also indexed by conjugacy classes of slices of \(G\), and given by

\[
\xi_{T, S}^G = \frac{1}{|N_G(T, S)|} \sum_{U \leq S < V \leq T} |U| \mu(U, S) \mu(V, T) \mu(V, U)_G.
\]

It is also shown in [3] that the assignments \(G \to \Xi(G)\) and \(G \to \mathbb{Q} \otimes \Xi(G)\) are Green biset functors. On can try to describe the lattice of subfunctors of \(\mathbb{Q} \otimes \Xi\), or its lattice of ideals.

In [7], it is shown that the idempotents \(\xi_{T, S}^G\) play a similar role for \(\mathbb{Q} \otimes \Xi\) as the idempotents \(e_G^G\) do for \(\mathbb{Q} \otimes B\): namely, for any \(N \trianglelefteq G\), there exists a rational number \(m_{G,S,N}\) such that

\[
\text{Def}_{G/N}^G \xi_{G,N}^G = m_{G,S,N} \xi_{G/N,S}^G = m_{G,N} \xi_{G/N}^G.
\]

This constant \(m_{G,S,N}\) is given by

\[
m_{G,S,N} = \frac{|N_G(SN) : SN|}{|N_G(S)|} \sum_{\mathclap{U \leq S < V \leq G \atop SN = S}} |U| \mu(U, S) \mu(V, G).
\]

By Proposition 5.5 in [7] we have

\[
m_{G,S,N} = \frac{|N_G(SN) : SN|}{|N_G(S)|} \sum_{\mathclap{S < X \leq G \atop XN = SN}} m_{S,X,N} \mu(X, G).
\]

where \(m_{G,S,N} = \sum_{\mathclap{S < X \leq G \atop XN = SN}} \mu(X, G)\). We observe that \(m_{G,S,1} = m_{G,S,1} = 1\) for any slice \((G, S)\). Slices \((G, S)\) such that \(m_{G,S,N} = 0\) for any non-trivial normal subgroup \(N\) of \(G\) have been called \(T\)-slices in [7]. By analogy, we say that a slice \((G, S)\) is a \(T^0\)-slice if \(m_{G,S,N} = 0\) for any non-trivial normal subgroup \(N\) of \(G\).

As a first step towards a description of subfunctors of \(\mathbb{Q} \otimes \Xi\), we state the following property of \(T^0\)-slices.

**Theorem 1.1.** Let \((G, S)\) be a slice. Then there exists a slice \((H, U)\) such that

- \((H, U)\) is a \(T^0\)-slice and \((G, S) \twoheadrightarrow (H, U)\).
- \((H, U)\) is a \(T^0\)-slice such that \((G, S) \twoheadrightarrow (K, V)\), then \((H, U) \twoheadrightarrow (K, V)\).

Moreover, any two slices \((H, U)\) with these properties are isomorphic. We set \(\tau^0(G, S) = (H, U)\).
2. Proof of Theorem 1.1

We start with a proposition:

**Proposition 2.1.**

1. Let \( X \) be a subgroup of \( G \), and \( M \) be a normal subgroup of \( G \). Then
   \[
   \mu(X, G) = \sum_{YM=G, Y \not\leq X} \mu(X, Y) \mu(Y, G).
   \]

2. Let \( S \) be a subgroup of \( G \), and \( M, N \) be normal subgroups of \( G \). Then
   \[
   m_{G,S,N} = \sum_{YM=G, Y \geq S} \mu(Y, G) m_{G/M,SM/M,(YM)/M}^G.
   \]
   In particular, if \( N \geq M \), then \( m_{G,S,N} = m_{G,S,M} m_{G/M,SM/M,N/M}^G \).

3. If \( M \) is a normal subgroup of \( G \), maximal such that \( m_{G,S,M} \neq 0 \), then \((G/M, SM/M)\) is a \( T^0 \)-slice.

**Proof.**

1. Let \([X, G]\) be the lattice of subgroups of \( G \) containing \( X \). In this lattice, a complement of \( XM \) is a subgroup \( Y \) of \( G \) that contains \( X \) such that \((Y, XM) = G \) and \( Y \cap XM = X \).
   Since \( X \leq Y \), the first condition is equivalent to \( YM = G \), and the second one is equivalent to \( X(Y \cap M) = X \), i.e. \( Y \cap M = X \cap M \). Hence \(|Y||M| = |G||XM| \), so \( |Y| \) depends only on \( X \) and \( M \). Hence there is no strict inclusion between complements of \( XM \), and Crapo’s formula (see [5]) gives the following result:
   \[
   \mu(X, G) = \sum_{YM=G, Y \not\leq X} \mu(X, Y) \mu(Y, G)
   \]
   for any normal subgroup \( M \) of \( G \).

2. By 1, we have
   \[
   m_{G,S,N} = \sum_{X,Y} \mu(X, Y) \mu(Y, G)
   \]
   where the sum is over all pair \((X, Y)\) with
   \[
   X \leq Y, \ XN = G, \ YM = G, \ Y \cap M = X \cap M, \ S \leq X \leq G
   \]
i.e.
   \[
   YN = YM = G, \ X(Y \cap N) = Y, \ S \leq X \leq Y, \ X \geq (Y \cap M).
   \]
   For a fixed \( Y \), summing over \( X \) is equivalent to summing over a subgroup \( R (\doteq X/(Y \cap M)) \) of \( Y/(Y \cap M) \) such that
   \[
   R(Y \cap N)/(Y \cap M)/(Y \cap N) = Y/(Y \cap N), \ S(Y \cap M)/(Y \cap M) \leq R \leq Y/(Y \cap M) \tag{\ast}
   \]
   Hence, the sum on \( X \) is equal to
   \[
   \sum_R \mu(R, Y/(Y \cap M)) = m_{Y/(Y \cap M),SM/M,(Y \cap N)/M}^Y = m_{G/M,SM/M,(Y \cap N)M/M}^G,
   \]
   where \( R \) runs through subgroups of \( Y/(Y \cap M) \) fulfilling Condition (\ast). The second equality here follows from the fact that since \( YM = G \), there is a group isomorphism
   \[
   Y/(Y \cap M) \doteq G/M,
   \]
   and this isomorphism sends \( S(Y \cap M)/(Y \cap M) \) to \( SM/M \) and \( (Y \cap N)(Y \cap M)/(Y \cap M) \) to \( (Y \cap N)M/M \).
   Hence,
   \[
   m_{G,S,N} = \sum_{YM=G, Y \geq S} \mu(Y, G) m_{G/M,SM/M,(Y \cap N)M/M}^G.
   \]
If $N \geq M$ and if $YM = G$, then $N = (Y \cap N)M$ and

$$m_{G,S,N}^o = \sum_{YM = G \atop Y \leq S} \mu(Y, G) m_{G/M,SM/M,N/M}^o = m_{G,S,M}^o m_{G/M,SM/M,N/M}^o.$$

3. Straightforward.  □

Let us give a proof of Theorem 1.1.

Let $S$ be a subgroup of $G$. Let moreover $N$ be a normal subgroup of $G$ such that $m_{G,S,N}^o \neq 0$, and let $(K, V)$ be a $T^o$-slice that is a quotient of $(G, S)$. Then there exists a surjective group homomorphism $\varphi : G \rightarrow K$ such that $\varphi(S) = V$. We denote by $M$ the kernel of $\varphi$, so $(K, V)$ is isomorphic to $(G/M, SM/M)$. Since $(G/M, SM/M)$ is a $T^o$-slice, Proposition 2.1 gives

$$m_{G,S,N}^o = \sum_{YN = YM = G \atop Y \geq S} \mu(Y, G).$$

Since $m_{G,S,N}^o \neq 0$, there exists a subgroup $Y$ of $G$ that satisfies:

$$YN = YM = G, \ (Y \cap N) \leq M, \ S \leq Y.$$  

Then there exists a surjective homomorphism $Y/(Y \cap N) \rightarrow Y/(Y \cap M)$ that sends the group $S(Y \cap N)/(Y \cap N)$ to the group $S(Y \cap M)/(Y \cap M)$.

Since $Y/(Y \cap N) \cong G/N$, $Y/(Y \cap M) \cong G/M$, $S(Y \cap N)/(Y \cap N) \cong SN/N$, $S(Y \cap M)/(Y \cap M) \cong SM/M$, we have a surjective group homomorphism $G/N \rightarrow G/M$ sending $SN/N$ to $SM/M$. In other words,

$$(G/N, SN/N) \rightarrow (G/M, SM/M) \cong (K, V),$$

so $(K, V)$ is a quotient of $(G/N, SN/N)$.

If $M'$ is a normal subgroup of $G$, maximal such that $m_{G,S,M'}^o \neq 0$, then by Proposition 2.1 the slice $(H, U) := (G/M', SM'/M')$ is a $T^o$-slice, so

$$(G/N, SN/N) \rightarrow (H, U).$$

Moreover, since $m_{G,S,M'}^o \neq 0$, the slice $(K, V)$ is a quotient of $(G/M', SM'/M')$ i.e.

$$(H, U) \rightarrow (K, V).$$

Hence $(H, U)$ has the properties required in Theorem 1.1. If $(H', U')$ is another slice with these properties, then $(H, U)$ and $(H', U')$ are quotients of each other, so they are isomorphic. This completes the proof.

**Remark 2.2.** It is natural to ask if the analogue of Theorem 1.1 holds for $T$-slices instead of $T^o$-slices. Namely, for any slice $(G, S)$, does there exist a slice $\tau(G, S) = (H, U)$ such that

- $(H, U)$ is a $T$-slice and $(G, S) \rightarrow (H, U),$
- if $(K, V)$ is a $T$-slice and $(G, S) \rightarrow (K, V)$, then $(H, U) \rightarrow (K, V)$?

The following example shows that the answer is no: consider the direct product $G = C_2 \times D_8$ of a group of order 2 and a dihedral group of order 8. Let $a$ denote the generator of $C_2$, and let $[b, c]$ be a set of generators of $D_8$, where $b$ has order 2 and $c$ has order 4. Set moreover $d = c^2$. Let $S$ be the subgroup of $G$ generated by $a$ and $b$. Thus $S \cong C_2 \times C_2$.

We denote by $N$ the subgroup of $G$ generated by $ad$, and $M$ the subgroup of $G$ generated by $d$. Hence $|N| = |M| = 2$ and the subgroups $N$ and $M$ are central in $G$. We have $G/N \cong D_8$ and $G/M \cong (C_2)^3$.

The slices $(G/N, SN/N)$ and $(G/M, SM/M)$ are both quotients of $(G, S)$. On can check that they are both $T$-slices. If there exists a $T$-slice $(H, U)$ with the required properties, then in particular $H$ is a quotient of $G$, and both $G/N$ and $G/M$ are quotients of $H$. It follows that $H \cong G$, and then $(H, U)$ is isomorphic to $(G, S)$.

This is a contradiction, since $(G, S)$ is not a $T$-slice.

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