On the neighbour sum distinguishing index of graphs with bounded maximum average degree

H. Hocquard\textsuperscript{a,1}, J. Przybyło\textsuperscript{b,2,3,}\ast

\textsuperscript{a}LaBRI (Université de Bordeaux), 351 cours de la Libération, 33405 Talence Cedex, France
\textsuperscript{b}AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

Abstract

A proper edge $k$-colouring of a graph $G = (V, E)$ is an assignment $c : E \rightarrow \{1, 2, \ldots, k\}$ of colours to the edges of the graph such that no two adjacent edges are associated with the same colour. A neighbour sum distinguishing edge $k$-colouring, or nsd $k$-colouring for short, is a proper edge $k$-colouring such that $\sum_{e \ni u} c(e) \neq \sum_{e \ni v} c(e)$ for every edge $uv$ of $G$. We denote by $\chi'_\Sigma(G)$ the neighbour sum distinguishing index of $G$, which is the least integer $k$ such that an nsd $k$-colouring of $G$ exists. By definition at least maximum degree, $\Delta(G)$, colours are needed for this goal. In this paper we prove that $\chi'_\Sigma(G) \leq \Delta(G) + 1$ for any graph $G$ without isolated edges, and with $\text{mad}(G) < 3$, $\Delta(G) \geq 6$.

Keywords: Neighbour sum distinguishing index, maximum average degree, discharging method.

1. Introduction

A proper edge $k$-colouring of a graph $G = (V, E)$ is an assignment of colours to the edges of the graph such that no two adjacent edges do not host the same colour. We use the standard notation, $\chi'(G)$, to denote the chromatic index of $G$. A neighbour sum distinguishing edge $k$-colouring, or nsd $k$-colouring for short, is a proper edge colouring $c : E \rightarrow \{1, 2, \ldots, k\}$ such that for every edge $uv \in E$, there is no conflict between $u$ and $v$, i.e., $s(u) \neq s(v)$, where $s(u)$ is the sum of colours taken on the edges incident with $u$. In other words, for every vertex $u \in V$, $s(u) = \sum_{e \in E_u} c(e)$, where $E_u$ is the set of edges incident with $u$ in $G$. We denote by $\chi'_\Sigma(G)$ the neighbour sum distinguishing index of $G$, which is the least integer $k$ such that an nsd (edge) $k$-colouring of $G$ exists. This graph invariant binds its two famous archetypes - the parameter associated with so called Zhang’s Conjecture \cite{32}, where the required distinction is weaker and concerns sets of colours rather than their sums, cf. \cite{4, 5, 14, 15, 16, 28, 32} for representative results concerning it, and the problem commonly referred to as 1–2–3 Conjecture \cite{21}, whose objective were not necessarily proper edge colourings in turn, see \cite{1, 2, 20, 30} for a few breakthroughs concerning this. The roots of this branch of graph theory date back to the 1980s, and the papers \cite{7, 8} on degree irregularities in graphs (and multigraphs) and the parameter irregularity strength of a graph. There first integer edge weights (colours) became of use to represent the multiplicities of respective edges in an investigated multigraph with a given underlying simple graph. The sum $s(u)$ defined above corresponds then to the degree of a given vertex $u$ in the multigraph with underlying graph $G$, see \cite{3, 10, 13, 19, 22, 23, 24} for more details and a few crucial results on the irregularity strength.

\ast Corresponding author

Email addresses: herve.hocquard@labri.fr (H. Hocquard), jakubprz@agh.edu.pl (J. Przybyło)

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Note that as for other graph invariants of this type, the value of $\chi^\Sigma(G)$ is well defined for all graphs without isolated edges. By definition, the neighbour sum distinguishing index of every such graph $G$ is not smaller than $\chi(G)$, while by Vizing’s theorem, $\chi(G)$ equals the maximum degree of $G$, $\Delta(G)$, or $\Delta(G) + 1$. The following conjecture was proposed by Flandrin et al. in [12], where it was also verified for a few classical graph families, including, e.g., paths, cycles, complete graphs, complete bipartite graphs and trees.

**Conjecture 1.** If $G$ is a connected graph of order at least three different from the cycle $C_5$, then $\chi^\Sigma(G) \leq \Delta(G) + 2$.

In general it is known that this conjecture is asymptotically correct, as confirmed by the following probabilistic result of Przybyło from [22].

**Theorem 2.** If $G$ is a connected graph of maximum degree $\Delta \geq 2$, then $\chi^\Sigma(G) \leq (1 + o(1))\Delta$.

Other upper bounds can be found in [12, 26, 27, 29]. Recently, Bonamy and Przybyło [6] also confirmed Conjecture 1 for planar graphs with sufficiently large maximum degree proving that:

**Theorem 3.** Any planar graph $G$ with $\Delta(G) \geq 28$ and with no isolated edges satisfies $\chi^\Sigma(G) \leq \Delta(G) + 1$.

Let $\text{mad}(G) = \max \left\{ \frac{\sum E(H)}{|V(H)|}, H \subseteq G \right\}$ be the maximum average degree of the graph $G$, where $V(H)$ and $E(H)$ are the sets of vertices and edges of $H$, respectively. This is a conventional measure of sparseness of an arbitrary graph (not necessary planar). For more details on this invariant see [18], where properties of the maximum average degree are exhibited and where it is proved that maximum average degree may be computed by a polynomial algorithm. Moreover it can be efficiently computed by translating the question into a flow problem on the right graph [22].

Dong et al. first made the link between maximum average degree and neighbour sum distinguishing index [11]. They proved the following result.

**Theorem 4.** Any graph $G$ with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{8}{7}$ satisfies $\chi^\Sigma(G) \leq \Delta(G) + 1$.

This subject was intensively studied afterwards, and the following improvements have been provided.

**Theorem 5.** [12] Any graph $G$ with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{8}{7}$ satisfies $\chi^\Sigma(G) \leq \Delta(G) + 1$.

**Theorem 6.** [17, 22] Any graph $G$ with no isolated edges, $\Delta(G) \geq 5$ and $\text{mad}(G) < 3$ satisfies $\chi^\Sigma(G) \leq \Delta(G) + 2$.

In this paper, we strengthen all three results above by proving the following (Note that in fact Theorem 7 below implies all Theorems 4–6 above).

**Theorem 7.** Any graph $G$ with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < 3$ satisfies $\chi^\Sigma(G) \leq \Delta(G) + 1$.

2. **Proof of Theorem 7**

2.1. Preliminaries

Fix an integer $k \geq 6$. In the following, $n_i(G)$ denotes the number of vertices of degree $i$ in a graph $G$. We say a graph $G$ is smaller than a graph $H$, $G \prec H$ if $(n_k(H), \ldots, n_2(H), n_1(G))$ precedes $(n_k(H), \ldots, n_2(H), n_1(H))$ with respect to the standard lexicographic order. We say a graph is minimal for a property when no smaller graph verifies it. We shall also call any vertex of degree $d$ ($\geq d, \leq d$) in a given graph a $d$-vertex ($d^+$-vertex, $d^-$-vertex, resp.) of this graph. The same nomenclature shall be used for neighbours as well.
2.2. Structural properties of $H$

Suppose $H$ is a minimal graph without isolated edges such that $\Delta(H) \leq k$, and $\text{mad}(H) < 3$ and $\chi''_c(H) > k + 1$. In the remaining part of the paper we argue that in fact $H$ cannot exist, and thus prove Theorem 7.

In this subsection, we exhibit some structural properties of $H$. The following lemma shall be very useful to this end. Its proof is inspired by the research from [6].

**Lemma 8.** For any finite sets $L_1, \ldots, L_t$ of real numbers with $|L_i| \geq t$ for $i = 1, \ldots, t$, the set \{ $x_1 + \ldots + x_t : x_1 \in L_1, \ldots, x_t \in L_t; x_i \neq x_j$ for $i \neq j$ \} contains at least \( \sum_{i=1}^t |L_i| - t^2 + 1 \) distinct elements.

**Proof.** We begin by first dynamically modifying the lists $L_1, \ldots, L_t$. Thus subsequently, for $i = 1, 2, \ldots, t-1$, we take $\min L_i$ (where every $L_p$, $p \in \{1, \ldots, t\}$ shall always refer to the up-to-date remainder of this list on a given stage of our modifying procedure) and remove it from all current lists $L_j$ with $j > i$.

Then, subsequently, for $i = t, t-1, \ldots, 2$, we find $\max L_i$ and remove it from all up-to-date lists $L_j$ with $j < i$, and denote the finally constructed respective lists by $L'_1, \ldots, L'_t$. As a result at most $t-1$ elements were removed from every list $L_i$ and for every $i < j < t$, $L'_i$ contains neither $\min L'_j$ nor $\max L'_j$. Let $L'_i = \{ c_{i,1}, c_{i,2}, \ldots, c_{i,t} \}$, where $c_{i,1} < c_{i,2} < \ldots < c_{i,t}$, for $i = 1, \ldots, t$. Then it is straightforward to see that the following \( \sum_{i=1}^t |L_i| - t + 1 \geq \sum_{i=1}^t (|L_i| - (t - 1)) - t + 1 = \sum_{i=1}^t |L_i| - t^2 + 1 \) sums are distinct and each consists of $t$ pairwise distinct integers:

\[
\begin{align*}
&c_{i,1} + c_{i,2} + \ldots + c_{i,-2} + c_{i,-1} + 1 + c_{i,1} \\
&< c_{i,1} + c_{i,2} + \ldots + c_{i,-2} + c_{i,-1} + 1 + c_{i,2} \\
&\ldots \\
&< c_{i,1} + c_{i,2} + \ldots + c_{i,-2} + c_{i,-1} + 1 + c_{i,t} \\
&\ldots \\
&< c_{i,t} + c_{i,t+1} + \ldots + c_{t-2,t-1} + c_{t-1,t-2} + c_{t-1,t-1} + 1 + c_{t,t}.
\end{align*}
\]

A 2-vertex or a 3-vertex is called bad if it is adjacent to a vertex of degree 2. Otherwise these are called good. A vertex is called deficient if it is a 1-vertex or a bad 2-vertex, while a vertex is referred to as half-deficient if it is a good 2-vertex or a bad 3-vertex.

**Claim 1.** The graph $H$ does not contain any of:

(C1) a 1-vertex $v$ adjacent to a $(\frac{k}{2} + 1)^-$-vertex $u$;

(C2) a 2-vertex $v$ adjacent to a $(\frac{k}{2} + 1)^-$-vertex $u$ and to a $(\frac{k}{2})^-$-vertex $w$, $u \neq w$;

(C3) a 3-vertex $v$ adjacent to a $(\frac{k}{2})^-$-vertex $u$ and to a 2-vertex $w$, $u \neq w$;

(C4) a triangle $uvw$ with $d(u) = 2 = d(w)$;

(C5) a vertex $v$ adjacent to a 1-vertex $u$ and to a bad 2-vertex $w$;

(C6) a vertex $v$ adjacent with two bad 2-vertices $u$ and $w$;

(C7) a vertex $v$ adjacent with two 1-vertices $u_1, u_2$ and to a half-deficient vertex $w$;

(C8) a vertex $v$ of degree $d \geq 3$ adjacent to $d-2$ vertices $u_1, \ldots, u_{d-2}$ of degree 1;

(C9) a vertex $v$ of degree $d \leq \frac{k}{2}$ adjacent with a bad 2-vertex $u$ and to a half-deficient vertex $v$;

(C10) a vertex $v$ of degree $d$ adjacent to exactly one bad 2-vertex $u$, at least one half-deficient vertex and to at most $k - d$ vertices which are neither deficient nor half-deficient;

(C11) a vertex $v$ of degree $d$ adjacent to exactly one 1-vertex $u$ and to at most $k - d + 1$ vertices which are neither deficient nor half-deficient;

(C12) a 5-vertex $v$ adjacent to 5 half-deficient vertices $u_1, \ldots, u_5$;
Figure 1: Forbidden configurations in $H$ (where solid vertices have degrees as presented in the figure, hollow vertices may have additional edges and may coincide with other vertices, while the label ‘h’ indicates a half deficient vertex).

(C13) a 4-vertex $v$ adjacent to at least 3 half-deficient vertices $u_1, u_2, u_3$.

**Proof.** We shall argument ‘reducibility’ of each of these 13 configurations separately, following a similar pattern of reasoning. I.e., we shall first suppose by contradiction that a given configuration exists in $H$. Then we shall consider a graph $H'$ smaller than $H$ with $\Delta(H') \leq k$ and $\text{mad}(H') < 3$ (usually guaranteeing these properties by constructing $H'$ simply via deleting some edges or vertices from $H$), and colour it by minimality, what shall mean from now on that we choose any nsd $(k+1)$-colouring for every component of $H'$ of order at least 3 (such colouring exists as this component is obviously smaller than $H$ then; cf. the definition of $H$) and fix arbitrarily a colour in $\{1, 2, \ldots, k+1\}$ for every isolated edge of $H'$. Finally, in each case, we shall obtain a contradiction by extending the colouring chosen to an nsd $(k+1)$-coloring of the entire $H$.

First note that a vertex of degree $d$ shall certainly be sum-distinguished from its 2-neighbour if

$$d > \frac{1}{2}(\sqrt{8k+9} + 1).$$

Indeed, this inequality is equivalent to $1 + 2 + \ldots + d - 1 > k + 1$ (while a colour of an edge joining two vertices is counted in the sums of the both vertices). Note that this holds e.g. for $d \geq \frac{k+3}{2}$ and for $d \geq \frac{2k+1}{3}$, as $k \geq 6$. Obviously, a 1-vertex is always sum-distinguished from its neighbour in $H$. 


1. Suppose there exists a 1-vertex $v$ adjacent to a $(\frac{k}{2} + 1)^{-}$-vertex $u$. Colour $H' = H - v$ by minimality. In order to colour $uv$ then so that the $(k + 1)$-colouring of $H$ obtained is proper we have to avoid at most $\frac{k}{2}$ colours, and possibly at most $\frac{k}{2}$ more colours to ensure the sum-distinction (of $u$ from its neighbours other than $v$). Hence, we have at least one colour left to extend the colouring to an nsd $(k + 1)$-colouring of $H$, a contradiction.

2. Assume there exists a 2-vertex $v$ adjacent to a $(\frac{k}{2} + 1)^{-}$-vertex $u$ and to a $(\frac{k}{2})^{-}$-vertex $w$, $u \neq w$. Colour $H' = H - v$ by minimality. Then we colour $vu$ first so that the (partial) $(k + 1)$-colouring obtained is proper (at most $\frac{k}{2} - 1 = \frac{k-1}{2}$ forbidden colours), $s(v) \neq s(w)$ (1 constraint), and $u$ is sum-distinguished from its (at most $\frac{k-1}{2}$) neighbours with fixed sums, hence we have at least one colour available for this aim. Finally colour $vw$ so that the colouring is proper (at most $(\frac{k}{2} - 1) + 1 = \frac{k}{2}$ constraints) and $v$ and $w$ are sum-distinguished from their neighbours other than $v$ and $w$ (again $\frac{k}{2}$ constraints). Hence, we obtain a contradiction, as we have at least one colour left to extend the colouring.

3. Suppose there exists a 3-vertex $v$ adjacent to a $(\frac{k}{2})^{-}$-vertex $u$ and to a $2^{-}$-vertex $w$, $u \neq w$. Denote by $v_1$ the third neighbour of $v$ distinct from $u$ and $w$. Colour $H' = H - \{w, vw\}$ by minimality. Then colour $vw$ so that the colouring is proper (at most $(\frac{k}{2} - 1) + 1 = \frac{k}{2}$ constraints), $s(v) \neq s(w)$ (1 constraint), and $u$ is sum-distinguished from its (at most $\frac{k}{2} - 1$) neighbours with fixed sums. Finally colour $vw$ so that the colouring is proper and $v$ and $w$ are sum-distinguished from their neighbours other than $v$ and $w$. We can extend the colouring, since we have at least 7 colours available (where $7 \geq 2 \times 3 = 6$), a contradiction.

4. Assume there exists a triangle $uvw$ with $d(u) = 2 = d(w)$. Colour $H' = H - uw$ by minimality. Note that $s(u) \neq s(w)$ then, as $uw$ and $vw$ must be coloured differently. Then colour $uw$ so that the colouring is proper (2 constraints), $s(u) \neq s(v)$ (1 constraint) and $s(w) \neq s(v)$ (1 constraint). We can extend the colouring, since we have more than 4 colours available, a contradiction.

5. Suppose there exists a vertex $v$ adjacent to a 1-vertex $u$ and to a bad 2-vertex $w$. Denote by $w'$ the neighbour of degree 2 of $w$ ($w' \neq v$). Colour $H' = H - uw'$ by minimality. Next switch the colours of $uv$ and $vw'$ if necessary so that $s(w) \neq s(w')$. Then we easily choose a colour for $uw'$ so that the colouring obtained is proper and no sum conflict arises. Hence, the colouring is extended, a contradiction.

6. Assume there exists a vertex $v$ adjacent to two bad 2-vertices $u$ and $w$. Let $u'$ (resp. $w'$) be the neighbour of degree 2 of $u$ (resp. $w$), $u', w' \neq v$. By (C2), $d(v) \geq 4$ and $u' \neq w'$. Denote by $w''$ (resp. $w''$) the second neighbour of $u'$ (resp. $w'$) distinct from $u$ (resp. $w$). By (C4), $w' \neq w$, while by (C2), $u'$ and $w'$ cannot be adjacent in $H$. Consider $H'' = H + u'w' - \{uw, uw''\}$, and note that $H' \prec H$, as we have decreased the number of vertices of degree 2, creating no new vertices of larger degrees at the same time. It also holds that $\text{mad}(H'') < 3$ (as otherwise there would have to exist a subgraph $H''$ of $H$ with $\text{mad}(H'') \geq 3$, e.g., $H'' = H' - \{u', w'\}$, a contradiction). Consequently, we may colour $H''$ by minimality. Hence $vu$ and $vw$ are coloured differently, and the same holds for $u'w''$ and $w'u''$ (if they were coloured the same, there would be a conflict between $u'$ and $w'$ in $H''$). Then we switch the colours of $vu$ and $vw$ if necessary, so that $vu$ (resp. $vw$) and $u'w''$ (resp. $w'u''$) are coloured differently, in order to ensure the sum-distinction between $u$, $u'$ and $w$, $w'$ in $H$ (where the edge $w'w'$ is not taken into account anymore, as it does not appear in $H$). It then suffices to colour the edges $uw'$ and $uw''$ with colours different from those of their respective adjacent edges and such that $s(u), s(w) \neq s(v), s(u') \neq s(w'')$ and $s(w') \neq s(u'')$. This is possible as there are at least 3 available colours left for this aim in both cases, a contradiction.

7. Suppose there is a vertex $v$ adjacent with two 1-vertices $u_1, u_2$ and to a half-deficient vertex $w$. Let
\(w'\) be the neighbour of \(w\) of degree greater than 2 other than \(v\) (cf. (C1) and (C3)). We consider two cases:

- First suppose \(w\) is a good 2-vertex. We create \(H'\) of \(H\) by splitting the vertex \(w\) in two 1-vertices \(v_1\) and \(w_2\) such that \(v_1\) is adjacent to \(v\) and \(w_2\) is adjacent to \(w'\). Obviously, \(H' \prec H\) and \(\text{mad}(H') \leq \text{mad}(H)\). Hence, we may colour \(H'\) by minimality. Then we switch the colour of \(vw_1\) with the colour of \(vu_1\) or \(vw_2\) if necessary so that the colour of \(vw_1\) is distinct from the colour of \(w_2w'\) and \(s(w) \neq s(w')\) after identifying back \(w_1\) with \(w_2\). Since by (C1), \(d(v) \geq \frac{k+1}{2}\), then by (1), \(s(v) \neq s(w)\), hence we obtain an \(n\)-\((k+1)\)-colouring of \(H\), a contradiction.

- Assume now that \(w\) is a bad 3-vertex. Let \(w''\) be the third neighbour of \(w\) (i.e., \(w'' \neq v, w'' \neq w'\) and \(d(w'') = 2\)). We split \(w\) into a 1-vertex \(v_1\) adjacent to \(v\) and a 2-vertex \(w_2\) adjacent to \(w'\) and \(w''\). One can observe that the obtained new graph \(H'\) is smaller than \(H\) (because one 3-vertex has been removed) and \(\text{mad}(H') \leq \text{mad}(H)\). Hence, we may colour \(H'\) by minimality. Then we switch the colour of \(vw_2\) with the colour of \(vw_1\) or \(vw_2\) if necessary so that the colour of \(vw_2\) is distinct from the colour of \(w_2w'\) and \(s(w) \neq s(w'')\) after identifying back \(w_1\) with \(w_2\). If there are still some colour or sum conflicts in \(H\), we change the colour of \(ww''\) to eliminate all of these. This is feasible as we have more than 6 colours available. Hence, we can extend the colouring, a contradiction.

8. Assume there is a vertex \(v\) of degree \(d \geq 3\) adjacent to \(d - 2\) vertices \(u_1, \ldots, u_{d-2}\) of degree 1. By (C1), \(d \geq 5\). Colour \(H' = H - \{u_1, \ldots, u_{d-2}\}\) by minimality. Then every edge \(vu_i\) for \(i \in \{1, \ldots, d-2\}\) has 2 forbidden colours, i.e., \((k+1) - d = k - 1\) available colours left. By Lemma 8, we may complete the proper colouring of \(H\) in different ways, obtaining at least \((d - 2)(k - 1) - (d - 2)^2 + 1 = (k - d + 1)(d - 2) + 1 \geq (d - 2) + 1 \geq 4\) distinct sums for \(v\). Since \(v\) has at most two neighbours of degree greater than 1, then, at least one of these 4 sums is distinct from the sums of these at most two neighbours. Thus again we can extend the colouring, a contradiction.

9. Suppose there is a vertex \(v\) of degree \(d \leq \frac{k}{2}\) adjacent to a bad 2-vertex \(u\) and to a half-deficient vertex \(w\). By (C2), \(d \geq 4\). Denote by \(u'\) the neighbour of degree 2 of \(u\). Denote by \(w'\) the neighbour of degree greater than 2 of \(w\) distinct from \(v\). Note that by (C2) and (C4), neither \(u\) nor \(u'\) is adjacent with \(w\). If \(w\) is a bad 3-vertex, let \(w''\) be its neighbour of degree 2. Colour the graph \(H' = H - \{vu, uu', vv\}\) by minimality. In the case when \(w\) is a bad 3-vertex we uncolour the edge \(ww''\). Regardless if \(d(w) = 2\) or \(d(w) = 3\), for \(vw\) there are at most \((d - 2) + 1\) forbidden colours of the edges adjacent with it and 1 more constraint to guarantee \(s(w) \neq s(w'')\) (if \(d(w) = 3\)) or \(s(w) \neq s(w')\) (if \(d(w) = 2\)). Analogously, as the colour of \(uw'\) is not yet fixed, there are \(d - 2\) colours of the edges incident with \(w\) and at most two more so that \(s(u) \neq s(u')\) and \(s(v) \neq s(w)\). Therefore we have at least \((k + 1) - d = k + 1\) colours available for both, \(vu\) and \(vw\), thus by Lemma 8 we may extend our proper colouring on these two edges obtaining at least \(2(\frac{k}{2} + 1) = 3 = \frac{k}{2} - 1 \geq (d - 2) + 1\) distinct sums for \(v\), one of which is different from the sums of all neighbours of \(v\) other than \(u\) and \(w\). Then we easily complete the construction of an \(n\)-\((k + 1)\)-colouring of \(H\) choosing a right colour for \(uw'\) and one for \(ww''\) (if \(d(w) = 3\)) as in (C7). Thus we obtain an extension of the colouring to the whole \(H\), a contradiction.

10. Assume there is a vertex \(v\) of degree \(d\) adjacent to exactly one bad 2-vertex \(u\), at least one half-deficient vertex and to at most \(k - d\) vertices which are neither deficient nor half-deficient. Denote by \(u'\) the neighbour of \(u\) of degree 2. Colour \(H' = H - \{vu, uu'\}\) by minimality. Then, first we choose a colour for \(vu\) so that the colouring is proper (\(d - 1\) constraints), \(s(u) \neq s(u')\) (1 constraint) and the sum of \(v\) is distinct from the sum of every its neighbour which is neither deficient nor half-deficient (there are at most \(k - d\) of these). This is feasible, as we have altogether at most \((d - 1) + 1 + (k - d) = d\) constraints. Subsequently, we choose an appropriate colour for \(uu'\) (avoiding at most 4 constraints). Recall now that \(v\) is adjacent to at least one half-deficient vertex. By (C9), \(d \geq \frac{2k+1}{4}\), and thus \(v\) is sum-distinguished from all its 2-neighbours by (1). Hence, \(v\) can only be in conflict with its adjacent half-deficient vertices which are bad 3-vertices. For every such vertex we can however similarly as
11. Suppose there is a vertex \( v \) of degree \( d \) adjacent to exactly one 1-vertex \( u \) and to at most \( k - d + 1 \) vertices which are neither deficient nor half-deficient. Colour the graph \( H' = H - u \) by minimality. Then, we choose a colour for \( vu \) so that the colouring is proper (\( d - 1 \) constraints) and the sum of \( v \) is distinct from the sum of every its neighbour which is neither deficient nor half-deficient (there are at most \( k - d + 1 \) of these). This is feasible, as we have altogether at most \( (d - 1) + (k - d + 1) = k \) constraints. Since by \((C1)\), \( d \geq \frac{k+1}{2} \), then \( v \) is sum-distinguished from all its 2-neighbours by \((H)\), and hence can only be in conflict with its adjacent bad 3-vertices. For every such vertex we can however similarly as above adjust the colour on the edge joining it with the vertex of degree \( 2 \) in order to eliminate this potential conflict. Finally we obtain an extension of the colouring to the whole \( H \), a contradiction.

12. Assume there is a 5-vertex \( v \) adjacent to at most 3 half-deficient vertices \( u_1, \ldots, u_5 \). Colour the graph \( H' = H - \{vu_1, vu_2\} \) by minimality. Without loss of generality we may assume that \( u_1, u_2 \) are not adjacent in \( H \). In the obtained colouring, for every \( u_i \) which is a bad 3-vertex we uncolour an edge joining it with a vertex of degree 2, \( i = 1, \ldots, 5 \). If \( k \leq 8 \), we then first colour \( vu_2 \) properly (4 constraints) so that \( u_2 \) is sum-distinguished (1 constraint) from its neighbour other than \( v \) which, if possible (i.e. in the case when \( d(u_2) = 3 \) is a 2-vertex. Then we colour \( vu_1 \) properly (5 constraints) so that \( u_1 \) is sum-distinguished (1 constraint) from its neighbour other than \( v \) which, if possible (i.e. in the case when \( d(u_1) = 3 \) is a 2-vertex. As \( k \leq 8 \), by \((H)\), \( v \) is sum-distinguished from all its neighbours of degree \( 2 \). In order to distinguish it from bad 3-neighbours, we subsequently choose new colours for the formerly uncoloured edges incident with them, which is possible as \( k + 1 > 6 \). If on the other hand \( k \geq 9 \), we have at most 4 colours blocked for each of \( vu_1 \) and \( vu_2 \) by the colours of their respective adjacent edges and further 2 for each \( vu_i, i = 1, 2 \), to avoid \( s(v) = s(u_3) \), resp., and the same sum at \( u_i \) and its neighbour of the least degree other than \( v \). Thus both edges \( vu_1 \) and \( vu_2 \) have at least \( k + 1 - 4 - 2 \geq 4 \) colours available left, hence by Lemma \(8\) we may properly extend the colouring to \( vu_1 \) and \( vu_2 \) obtaining at least \( 2 \times 4 - 3 = 5 \) different sums at \( v \). We choose one of these extensions so that \( s(v) \neq s(u_3), s(v) \neq s(u_4), s(v) \neq s(u_5) \). At the end, if necessary, we analogously as in the previous case adjust the colours of uncoloured edges incident with bad 3-vertices adjacent with \( v \). Thus we obtain an extension of the colouring to the whole \( H \), a contradiction.

13. Suppose there exists a 4-vertex \( v \) adjacent to at least 3 half-deficient vertices \( u_1, u_2, u_3 \). If \( u_1 \) is adjacent to \( u_2 \), by \((C3)\) and \((C4)\) it means that one of these vertices, say \( u_2 \) is a bad 3-vertex, and the other \( (u_1) \) is a good 2-vertex. Then we colour \( H' = H - \{u_1v, u_1u_2\} \) by minimality. Next we extend this proper colouring to \( vu_1 \) (at most 3 forbidden colours of the adjacent edges) so that \( s(u_1) \neq s(u_2) \) and \( v \) is sum-distinguished from its remaining two neighbours (other than \( u_1 \) and \( u_2 \)). Finally we choose a colour for \( u_1u_2 \) avoiding the colours of its three adjacent edges and creating no sum conflicts (additional at most 3 constraints), a contradiction. By symmetry, we may thus assume that \( u_1, u_2, u_3 \) form an independent set in \( H \), and denote by \( w \) the remaining neighbour of \( v \).

- If \( k = 6 \), we colour \( H - \{vu_1, vu_2, vu_3\} \) by minimality, and for every \( u_i \) which is a bad 3-vertex we uncolour an edge joining it with a vertex of degree 2, for \( i = 1, 2, 3 \). For every \( vu_i \) we then have at most 2 colours blocked by the colours of its adjacent edges and further at most one to avoid a sum-conflict between \( u_i \ (i = 1, 2, 3) \) and its neighbour of the least degree other than \( v \). We thus have at least \( (k + 1) - 2 - 1 = 4 \) available colours left for every \( vu_i \ (i = 1, 2, 3) \). Therefore, we may first choose a colour for \( vu_1 \) so that \( max\{c(vu_1), c(vw)\} \geq 5 \) (note that this guarantees that if \( u_i \), \( i \in \{2, 3\} \), is of degree 2, then \( s(v) \neq s(u_i) \)). Thus we are left with lists of size at least 3 of available colours for \( vu_2 \) and \( vu_3 \), from which it is sufficient to choose distinct colours so that \( s(v) \neq s(u_1) \) and \( s(v) \neq s(w) \). This is feasible by Lemma \(8\) because we may obtain at least 3 different sums at \( v \). By our construction, in order to eliminate the remaining potential conflicts
it is then sufficient to choose appropriate colours for the formerly uncoloured edges incident with bad 3-vertices.

- We may thus assume that \( k \geq 7 \). Then we colour \( H - \{vu_1, vu_2\} \) by minimality, and for every \( u_i, i = 1, 2 \) which is a bad 3-vertex we uncolour an edge joining it with a vertex of degree 2. Then for every \( vu_i, i = 1, 2 \), we have forbidden at most 3 colours of its adjacent edges and at most 2 more constraints guaranteeing \( s(v) \neq s(u_{3-i}) \) \( (i = 1, 2) \) and \( s(u_i) \neq s(u'_i) \), where \( u'_i \) \( (i = 1, 2) \) is the neighbour of \( u_i \) of minimal degree distinct from \( v \). Altogether we are left with lists of available colours of sizes at least \( k + 1 - 3 - 2 \geq 3 \), and need only choose distinct values from these two lists so that \( s(v) \neq s(w) \) and \( s(v) \neq s(u_3) \). This is feasible by Lemma 8, because we may obtain at least 3 different sums at \( v \). Again by our construction, in order to eliminate the remaining potential conflicts it is then sufficient to choose appropriate colours for the formerly uncoloured edges incident with bad 3-vertices.

In each case, we obtain an extension of the colouring to the whole \( H \), a contradiction. ■

2.3. Discharging procedure

In this subsection we use the discharging technique exploiting the vertices of the graph \( H \). For this aim we first define the weight function \( \omega : V(H) \to \mathbb{R} \) by setting \( \omega(x) = d(x) - 3 \) for every \( x \in V(H) \). Next we shall apply so called Ghost vertices method, introduced earlier by Bonamy, Bousquet and Hocquard [5], and based on the following observation (where given any subsets \( U, U' \subseteq V(H) \) and a vertex \( v \), \( d_U(v) \) denotes the number of neighbours of \( v \) from \( U \), while \( E(U, U') \) is the set of edges joining \( U \) and \( U' \) in the graph \( H \)).

**Observation 9.** Let \( V_1 \cup V_2 \) be a partition of \( V(H) \) where, say \( V_1 \) is the set of vertices of degree at least 2 and \( V_2 \) the set of vertices of degree 1 in \( H \);

- every vertex \( u \) in \( H \) has an initial weight \( w(u) = d(u) - 3 \).
- If we can discharge the weights in \( H \) so that:
  1. every vertex in \( V_1 \) has a non-negative weight;
  2. and every vertex \( u \) in \( V_2 \) has a final weight of at least \( d(u) - 3 + d_{V_1}(u) \), then

  for \( \omega' \) the new weight assignment, we have \( \sum_{v \in V_2} (d(v) - 3 + d_{V_1}(v)) \leq \sum_{v \in V_2} \omega'(v) \), as well as

  \[
  \sum_{v \in V} \omega(v) = \sum_{v \in V} \omega'(v) \text{ and } \sum_{v \in V_1} \omega'(v) \geq 0.
  \]

  Therefore,

  \[
  \sum_{v \in V_1} (d_{V_1}(v) - 3) \geq \sum_{v \in V_1} (d_{V_1}(v) - 3) + \sum_{v \in V_2} (d(v) - 3 + d_{V_1}(v)) - \sum_{v \in V_2} \omega'(v)
  \]

  \[
  \geq \sum_{v \in V_1} (d_{V_1}(v) - 3) + |E(V_1, V_2)| + \sum_{v \in V_2} (d(v) - 3) - \sum_{v \in V_2} \omega'(v)
  \]

  \[
  \geq \sum_{v \in V_1} (d(v) - 3) + \sum_{v \in V_2} (d(v) - 3) - \sum_{v \in V_2} \omega'(v)
  \]

  \[
  \geq \sum_{v \in V} \omega(v) - \sum_{v \in V_2} \omega'(v)
  \]

  \[
  \geq \sum_{v \in V_1} \omega'(v)
  \]

  \[
  \geq 0.
  \]

  Thus we can conclude that \( \text{mad}(H) \geq \text{mad}(H[V_1]) \geq 3 \).
In other words, the vertices in $V_2$ can be seen but, in a way, do not contribute to the sum analysis.

In order to finish the proof of Theorem 7, it suffices to obtain a contradiction, e.g. with the fact that $\text{mad}(H) < 3$, implying that in fact no counterexample to its thesis may exist. By Observation 9, it is thus enough to redistribute the weight (defined by $\omega$ above) in $H$ so that every vertex of degree at least 2 has a non-negative resulting weight and every vertex of degree one has weight at least $-1$.

The discharging rules we shall use for this aim are defined as follows:

(R1) A vertex of degree $d \geq 5$ gives 1 to every adjacent 1-vertex.
(R2) A vertex of degree $d \geq 4$ gives 1 to every adjacent bad 2-vertex.
(R3) A vertex of degree $d \geq 3$ gives $\frac{1}{2}$ to every adjacent good 2-vertex.
(R4) A vertex of degree $d \geq 4$ gives $\frac{1}{2}$ to every adjacent bad 3-vertex.

Let $v$ be a vertex in $H$. We consider different cases depending on the degree of $v$.

- Assume $d(v) = 1$. By (C1), $v$ is adjacent to a vertex of degree at least 5. Thus, by (R1), $v$ receives 1. So every vertex of degree 1 in $H$ has an initial weight of $-2$, gives nothing according to our rules and receives 1, hence has the final weight of $-1$.

- Assume $d(v) = 2$. First, suppose that $v$ is a bad 2-vertex. Then by (C2), $v$ is adjacent to a vertex of degree at least 4, and thus receives at least 1 by (R2) (and gives away nothing according to the rules above). Suppose now that $v$ is a good 2-vertex. Then by (C1), $v$ is adjacent with two vertices of degree at least 3, and thus receives at least $\frac{1}{2}$ from both by (R3) (and gives away nothing). In both cases $v$ has a non-negative final weight.

- Assume $d(v) = 3$. First, suppose that $v$ is a bad 3-vertex. Then by (C3) it gives away (at most) $\frac{1}{2}$ due to rule (R3), but also receives at least $2 \times \frac{1}{3}$ by (R4), as (C3) implies that $v$ must have two neighbours of degree at least 4. Suppose now that $v$ is a good 3-vertex. Then $v$ gives away nothing and receives nothing. In both cases $v$ has a non-negative final weight.

- Assume $d(v) \geq 4$. Then consider the following subcases:

  - if $v$ has at least 2 deficient neighbours, then by (C5) and (C6), these are both of degree 1 and by (C1), $d(v) \geq 5$. Moreover, in such a case, additionally by (C7), $v$ is adjacent with no other deficient or half-deficient vertices (except 1-vertices), while by (C8), $v$ can be adjacent with at most $d-3$ vertices of degree 1, and thus by (R1), $\omega'(v) \geq 0$;

  - if $v$ has exactly 1 deficient neighbour, then we may assume that it has at least one half-deficient neighbour, as otherwise by (R1) or (R2), $\omega'(v) \geq (d(v) - 3) - 1 \geq 0$. Thus by (C9) and (C1), $d(v) \geq 5$. On the other hand, by (C10) and (C11), at least $k - d(v) + 1$ neighbours of $v$ are neither deficient nor half-deficient, and thus by (R1), (R2), (R3) and (R4), $\omega'(v) \geq (d(v) - 3) - 1 - \frac{1}{3}(d(v) - 1) - (k - d(v) + 1) = \frac{1}{2}k - 3 \geq 0$;

  - assume finally that $v$ has no deficient neighbours. If $d(v) \geq 6$, then by (R3) and (R4), $\omega'(v) \geq d(v) - 3 - \frac{1}{2}d(v) \geq 0$. Consider now the case where $d(v) \leq 5$:

    * if $d(v) = 5$, then by (C12), $v$ has at most 4 half-deficient neighbours, and thus by (R3) and (R4), $\omega'(v) \geq 2 - 4 \times \frac{1}{3} \geq 0$;
    * if $d(v) = 4$, then by (C13), $v$ has at most 2 half-deficient neighbours, and thus by (R3) and (R4), $\omega'(v) \geq 1 - 2 \times \frac{1}{3} \geq 0$.

In both cases $v$ has a non-negative final weight.

This, by Observation 9 completes the proof of Theorem 7. ■

9
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