On the Plaque Expansivity Conjecture.

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Abstract

It is one of the main properties of uniformly hyperbolic dynamics that points of two distinct trajectories cannot be uniformly close one to another. This characteristics of hyperbolic dynamics is called expansivity. Hirsch, Pugh and Shub, 1977, formulated the so-called Plaque Expansivity Conjecture, assuming that two invariant sequences of leaves of central manifolds, corresponding to a partially hyperbolic diffeomorphism, cannot be locally close. There are many important statements in the theory of partial hyperbolicity that can be proved provided Plaque Expansivity Conjecture holds true. Here we are proving this conjecture in its general form.

Key words: partial hyperbolicity, central foliation, plaque expansivity, dynamical coherence, shadowing, homoclinic points.

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1 Introduction

The expansivity property plays an important role in the modern theory of Dynamical Systems. Recall the definition of expansivity following [1].

Definition 1. Let $f : M \to M$ be a homeomorphism of a compact metric space $M$ endowed with the metric $d$. We say that $f$ is expansive if there exists $a > 0$ such that given $x \neq y \in M$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq a$.  

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The largest possible constant $a$ is called the expansivity constant of $f$ for the metric $d$.

Quoting J. Lewowicz [2], one may say that expansivity means, from the topological point of view, that all points of the space $M$ have distinctive dynamical behaviors. Therefore, a stronger interaction between the topology of $M$ and the dynamics could be expected.

It is well-known [3] that Anosov, quasi-Anosov and pseudo-Anosov [4] diffeomorphisms of smooth compact manifolds satisfy the expansivity property (see papers [15] and references therein for more examples and properties). J. Lewowicz [12] has demonstrated that expansivity is closely related to topological stability of dynamical systems. To prove this he has introduced some non-classical Lyapunov functions. This approach has been developed in later papers [2,6] and many others.

Of course, the expansivity property does not take place for general non-hyperbolic diffeomorphisms any more. The most evident example for this is the identical mapping of a smooth manifold. Sometimes, the class of expansive diffeomorphisms is not ”much wider” than Anosov ones. For example, it follows from results by R. Mañe [3] that the $C^1$ interior of expansive diffeomorphisms of the two-torus consists of Anosov diffeomorphisms.

Anyway, it seems to be interesting to try to generalize such a useful tool as expansivity to non-hyperbolic maps and see what happens. The most trivial generalization of Anosov diffeomorphisms is the so-called partially hyperbolic homeomorphisms where the stable and the unstable bundles coexist with the central one corresponding to a ”slow” dynamics of linearization (all precise definitions are given at the next section). However, the central bundle is not always integrable [14]. If it is integrable, there exists a foliation of central manifolds usually denoted as $W^c$. It seems to be a natural generalization of expansivity for Anosov mappings, that these central manifolds are expansive in a certain sense.

However, in general these invariant manifolds are non-compact and the distance between them can hardly be defined. In order to define a kind of expansivity for this case one should deal with the so-called central pseudotrajectories (ones where all errors on all steps correspond to shifts along central manifolds). Roughly speaking, the Plaque Expansivity Property implies that any two sufficiently exact and uniformly close central pseudotrajectories belong to the same invariant family of central manifolds. It has been conjectured by Hirsch, Pugh and Shub [15] that if for a fixed partially hyperbolic diffeomorphism of a compact manifold the corresponding central foliation is uniquely integrable, then the Plaque Expansivity property is satisfied.

The main troubles arising during the proof are the following.
Generally, the central bundle is not smooth. The best one can say is that this bundle is Hölder [16]. This makes many methods of smooth dynamics invalid.

Unlike stable and unstable manifolds of an Anosov diffeomorphism, central manifolds of a partially hyperbolic mapping are not uniquely defined by their local parts. For example, in order to construct a central manifold of a fixed point, one must know the global structure of the diffeomorphism. This is a serious obstacle to construct a prolongation of a local central manifold.

We cannot expect any shadowing for a central pseudotrajectory of a partially hyperbolic diffeomorphism. For example, it was proved [17] that diffeomorphisms with $C^1$-robust shadowing property are structurally stable. In [18] Abdenur and Diaz conjectured that $C^1$-generically shadowing is equivalent to structural stability, and proved this statement for so-called tame diffeomorphisms.

The idea of the proof is the following. First of all (Section 5), we demonstrate that two uniformly close pseudotrajectories may be selected in such a way that one belongs to the (un)stable manifold of another one. In Section 6 we study possible structure of homoclinic points for Anosov diffeomorphisms. We demonstrate that this structure can be described in terms of the fundamental group of the manifold. For partially hyperbolic diffeomorphism of simply connected (even non-compact) manifolds we prove that any central unstable leaf cannot intersect a stable leaf in more than one point. In Section 8 we study a ”partial shadowing”. We try to approximate a subsequence of a central pseudotrajectory by a subsequence of a trajectory with the same set of indices. In Section 9 we ”project” some points of two distinct central pseudotrajectories to stable manifolds for points of ”partially” approximating trajectory where the approximation takes place. We prove that these projections give two uniformly close subsequences of trajectories corresponding to the same invariant family of stable manifolds. If those sequences are distinct, we have a contradiction, otherwise initial pseudotrajectories correspond to same center manifolds.

2 Definitions.

Let $M$ be a compact $n$ – dimensional $C^1$ smooth manifold, dist$(\cdot, \cdot)$ be a Riemannian metrics on $M$ and $\exp : TM \to M$ be the exponential mapping. Consider the space $\text{Diff}^1(M)$ of $C^1$ smooth diffeomorphisms $f : M \to M$. Let $|\cdot|$ be the Euclidean norm at $\mathbb{R}^n$ the related and the induced norm on the leaves of the tangent bundle $TM$. Suppose that the metric at the space $\text{Diff}^1(M)$ is
given by the formula
\[
d(f, g) = \sup_{x \in M} \text{dist}(f(x), g(x)) + \sup_{x \in M} |Df(x) - Dg(x)|.
\]
For any \( x \in M \), \( \varepsilon > 0 \) we introduce the \( \varepsilon \) – ball, defined by the formula
\[
B_\varepsilon(x) = \{ y \in \mathbb{R}^n : \text{dist}(x, y) \leq \varepsilon \}.
\]
We shall use the same notation for balls in other Euclidean spaces and ones in Riemannian manifolds.

Consider the following definition of partial hyperbolicity, see also [19].

**Definition 2.** A diffeomorphism \( f \in \text{Diff}^1(M) \) is called partially hyperbolic if there exists \( l \in \mathbb{N} \) such that the mapping \( f^l \) satisfies the following property.

There exists a continuous bundle
\[
T_pM = E^s(p) \oplus E^u(p) \oplus E^c(p), \quad p \in M
\]
and continuous positive functions \( \nu, \hat{\nu}, \gamma, \hat{\gamma} \) such that
\[
\nu, \hat{\nu} < 1, \quad \nu < \gamma < \hat{\gamma} < \hat{\nu}^{-1}
\]
and for all \( p \in M \), \( v \in \mathbb{R}^n \), \( |v| = 1 \)
\[
|Df^l(p)v| < \nu(p) \quad \text{if} \quad v \in E^s(p);
\gamma(p) < |Df^l(p)v| < \hat{\gamma}(p) \quad \text{if} \quad v \in E^c(p);
|Df^l(p)v| > \hat{\nu}^{-1}(p) \quad \text{if} \quad v \in E^u(p).
\]

Without loss of generality, we may assume that \( l = 1 \) in this definition.

Let \( \dim E^s(p) = n^s \), \( \dim E^c(p) = n^c \), \( \dim E^u(p) = n^u \). These dimensions do not depend on the choice of the point \( p \). Denote
\[
E^{cs}(p) = E^c(p) \oplus E^s(p), \quad E^{cu}(p) = E^c(p) \oplus E^u(p).
\]
There exists \( \alpha_0 > 0 \) such that for all \( p \in M \)
\[
\langle E^{sc}(p), E^{uc}(p) \rangle \geq 2\alpha_0.
\]

**Definition 3.** We say that a \( k \) – dimensional distribution \( E \) over \( TM \) is uniquely integrable if there exists a \( k \) – dimensional foliation \( W \) of the manifold \( M \), whose leaves are tangent to \( E \) at every point. Also, any \( C^1 \) – smooth path, tangent to \( E \), is embedded to a unique leaf of \( W \).
Fig. 1. A central pseudotrajectory.

**Definition 4.** [20]. A partially hyperbolic diffeomorphism $f$ is dynamically coherent if both distributions $E^{cs}$ and $E^{cu}$ are uniquely integrable.

Then, as it was proved in [21], both foliations $W^{cs}$ and $W^{cu}$, tangent to $E^{cs}$ and $E^{cu}$ respectively, contain a subfoliation $W^{c}$, that is tangent to $E^{c}$.

For $\tau \in \{s, c, u, cs, cu\}$ we denote by $W^{\tau}(p)$ the connected component of the set $W^{\tau}(p) \cap B_{\epsilon}(p)$, that contains the point $p$. Let $\text{dist}_{\tau}(p, \cdot)$ be the inner distance on $W^{\tau}(p)$ which is a Riemannian manifold itself. Note that

$$\text{dist}(p, q) \leq \text{dist}_{\tau}(p, q), \quad q \in W^{\tau}(p).$$

**Definition 5.** A sequence $\{x_k : k \in \mathbb{Z}\}$ is called $d$-pseudotrajectory ($d > 0$) if $\text{dist}(f(x_k), x_{k+1}) \leq d$ for all $k \in \mathbb{Z}$.

**Definition 6.** An $\epsilon$–pseudotrajectory $\{x_k\}$ is called central if for any $k \in \mathbb{Z}$ we have $f(x_k) \in W^{c}_{\epsilon}(x_{k+1})$ (Fig. 1).

**Definition 7.** [15]. A diffeomorphism $f$ satisfies Plaque Expansivity Property if for any $\epsilon > 0$ there exists a $\delta_0 > 0$ such that for any $\delta \leq \delta_0$, $d \leq \delta_0$ and any two central $d$ pseudotrajectories $p_k$ and $q_k$ the condition

$$\text{dist}(p_k, q_k) \leq \delta \quad \forall k \in \mathbb{Z} \quad (1)$$

implies

$$p_k \in W^{c}(q_k) \quad \forall k \in \mathbb{Z} \quad (2)$$

(Fig. 2).

### 3 Plaque Expansivity Conjecture

The following statement has proved by Hirsch, Pugh and Shub [15] Theorem 7.2.

**Theorem HPS1.** Let $f$ be a partially hyperbolical diffeomorhism. Suppose
that its central bundle $E^c$ is uniquely integrable and the corresponding central foliation $W^c$ is smooth. Then $f$ satisfies the Plaque Expansivity Property.

It was conjectured that this statement is true without assumption on smoothness of $W^c$ (the so-called Plaque Expansivity Conjecture). Note that in general (even if $f$ is $C^\infty$ smooth), the central foliation $f$ is Hölder only.

Sometimes, it is really useful to know that a diffeomorphism satisfies the Plaque Expansivity property. To illustrate this we give a couple of results.

**Theorem HPS2** [15, Theorem 7.1]. Let $f$ be a partially hyperbolical diffeomorphism of a Riemannian manifold $M$, the corresponding central bundle $E^s$ be uniquely integrable and $f$ satisfy the Plaque Expansivity property. Then there exists a neighborhood $U_f$ of $f$ in $\operatorname{Diff}^1(M)$ such that any diffeomorphism $g \in U_f$ is partially hyperbolic, its central foliation is uniquely integrable and $g$ satisfies the Plaque Expansivity property.

Another application concerns the so-called central shadowing problem.

**Definition 8.** We say that the partially hyperbolic diffeomorphism $f$ satisfies the Lipschitz central shadowing property if there exists $L > 0$ such that for any $d > 0$ and any $d$–pseudotrajectory \( \{p_k : k \in \mathbb{Z}\} \) there exists a central $Ld$–pseudotrajectory $q_k$, such that $\operatorname{dist}(p_k, q_k) \leq Ld$ for any $k$.

It is well-known that any Anosov diffeomorphism satisfies the Lipschitz shadowing property. The following result has been proved in the recent author's paper joint with S. Tikhomirov [22] (see also preprints [23,24]).

**Theorem TK.** Let the diffeomorphism $f \in \operatorname{Diff}^1(M)$ be partially hyperbolic and dynamically coherent. Then $f$ satisfies the Lipschitz central shadowing property. If $f$ satisfies Plaque Expansivity property, then there exists $d > 0$ that
for any $d$-pseudotrajectory $p_k$ the central manifolds for all shadowing central pseudotrajectories $q_k$ coincide.

4 The main result.

We are going to prove Plaque Expansivity conjecture in its general form.

**Theorem 1.** Any partially hyperbolic and dynamically coherent diffeomorphism $f \in \text{Diff}^1(M)$ of a smooth manifold $M$ satisfies Plaque Expansivity Property.

5 Projection of a central pseudotrajectory to a family of stable manifolds.

Now we start proving the formulated theorem. Fix a manifold $M$ and a diffeomorphism $f$. First of all, we may assume, without loss of generality that the stable bundle is non-trivial. If both stable and unstable bundles are trivial, the statement, we are going to prove, is evident. If the stable bundle is trivial and the unstable one is not, we may consider the mapping $f^{-1}$ instead of $f$.

In what follows below we will use the following statement, which is consequence of dynamical coherence of $f$.

**Lemma 1.** For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $p, q \in M$ satisfying $\text{dist}(p, q) < \delta$ there exists a unique point $r: \{r\} = W^u_\varepsilon(x) \cap W^cs_\varepsilon(y)$. Moreover, $\max(\text{dist}(x, z), \text{dist}(y, z)) \leq C \text{dist}(x, y)$.

Of course, a similar statement is true if we replace $cs$ with $cu$ and $u$ with $s$ in superscripts.

This is just the classical local product structure (see Fig. 3 and [25, Definition 1.3]).

Consider standard exponential mappings $\exp_p : T_p M \to M$ and

$$\exp^\tau_p : E^\tau(p) = T_p W^\tau(p) \to W^\tau(p)$$

for $\tau \in \{s, c, u, cs, cu\}$. Note that $D \exp_p(0) = \text{Id}$, $D \exp^s_p(0) = \text{Id}$. Consequently, exponential mappings are locally invertible. Then for inner Riemannian metrics $\text{dist}_\tau$ in manifolds $W^\tau$ ($\tau \in \{s, u, cs, cu, c\}$) the following statement is true.
Lemma 2. For any $\mu > 0$ there exists $\varepsilon > 0$ such that for any point $p \in M$, the following holds:

**A1** For any $q, r \in B_\varepsilon(p)$ and $v_1, v_2 \in T_pM$ such that $|v_1|, |v_2| < \varepsilon$

$$
\frac{1}{1 + \mu} \operatorname{dist}(q, r) \leq |\exp_{p}^{-1}(q) - \exp_{p}^{-1}(r)| \leq (1 + \mu) \operatorname{dist}(q, r),
$$

$$
\frac{1}{1 + \mu} |v_1 - v_2| \leq \operatorname{dist}(\exp_{p}(v_1), \exp_{p}(v_2)) \leq (1 + \mu)|v_1 - v_2|.
$$

**A2** Conditions similar to **A1** hold for $\exp_{\tau}^{\tau}$ and $\operatorname{dist}_{\tau}$, $\tau \in \{s, c, u, cs, cu\}$.

**A3** Angle between tangent space to $\exp_{q}^{-1}W_{\varepsilon}^{\tau}(q)$ and $E_{\tau}^{\circ}(p)$ is less than $\mu$ for $\tau \in \{s, c, u, cs, cu\}$ and $q \in B_\varepsilon(p)$.

**A4** For $q \in W_{\varepsilon}^{\tau}(p)$, $\tau \in \{s, c, u, cs, cu\}$ holds inequality

$$
\operatorname{dist}_{\tau}(p, q) < (1 + \mu) \operatorname{dist}(p, q).
$$

Then we can use the following statement.

Lemma 3. There is a $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, $p \in M$

$$
W_{\varepsilon}^{c}(p) \subset W_{\varepsilon}^{cu}(p) \cap W_{\varepsilon}^{cs}(p) \subset W_{2\varepsilon}^{c}(p).
$$

(3)

**Proof.** The first inclusion of (3) is evident, the second one follows from estimates of interior metrics on manifolds $W_{\tau}^{\tau}(p)$ by metrics $\operatorname{dist}$. □

Later on for any point $p \in M$ we shall consider local stable (unstable, central, etc.) manifolds $W_{\sigma}^{\circ}(p) = W_{\sigma}^{\tau}(p)$. Here $\sigma$ is a positive value chosen so that $\sigma \leq \min(\varepsilon_0, \delta(\varepsilon_0))$ (see Lemmas 1 and 3).

Consider two central pseudotrajectories $p_k$ and $q_k$ of $M$ such that

$$
\operatorname{dist}(p_k, q_k) \leq \sigma
$$
for all $k$.

If $W^c(p_k) \neq W^c(q_k)$, then either $W^{cs}_{loc}(p_k) \cap W^{cs}_{loc}(q_k) = \emptyset$ or

$$W^{cu}_{loc}(p_k) \cap W^{cu}_{loc}(q_k) = \emptyset.$$  \hfill (4)

Later on we always suppose that it is (4) that takes place. Otherwise we may replace $f$ with $f^{-1}$ and $s$ with $u$, $u$ with $s$ everywhere in superscripts. The following statement demonstrates that pseudotrajectories $\{p_k\}$ and $\{q_k\}$ may be chosen so that the local stable manifolds of corresponding points coincide.

Actually, this statement has been already proved by Bohnet and Bonatti [23]. However, formally speaking, the assumptions of the quoted paper are more rigorous, so we have to repeat the proof here.

**Lemma 4.** Let $p_k$ and $q_k$ be two central $\delta$– pseudotrajectories, satisfying condition (1) and not satisfying condition (2). Then for sufficiently small values of $\delta$ there exist two central pseudotrajectories $r_k \in W^s_{loc}(p_k)$ and $r'_k \in W^u_{loc}(q_k)$ such that the following conditions are satisfied (Fig. 4).

1. There exists a constant $C > 0$ such that
   $$\max(\text{dist}(p_k, r_k), \text{dist}(p_k, r'_k), \text{dist}(q_k, r_k), \text{dist}(q_k, r'_k)) \leq C\delta.$$  \hfill (5)

2. Either $r_k \notin W^c(p_k)$ or $r'_k \notin W^c(q_k)$.

**Proof.** Due to Lemma 1 for any $k \in \mathbb{Z}$ there is a non-empty intersection of manifolds $W^s_{loc}(p_k)$ and $W^{cu}_{loc}(q_k)$. This is a singleton $\{r_k\}$. Similarly, we can define $\{r'_k\} = W^{cs}_{loc}(p_k) \cap W^{cu}_{loc}(q_k)$. Note that both points $r_k$ and $r'_k$ belong to $W^{cs}_{loc}(p_k) \cap W^{cs}_{loc}(q_k)$, so $W^c(r_k) = W^c(r'_k)$. Then combination of inclusions $r_k \in W^c(p_k)$ and $r'_k \in W^c(q_k)$ would imply $W^c(p_k) = W^c(q_k)$. Later on we always assume that $r_k \notin W^c(p_k)$, otherwise we can replace $f$ with $f^{-1}$, $r_k$ with $r'_k$ and $p_k$ with $q_k$. Inequalities (5) follow from Lemma 1.
It remains to prove that sequences \( r_k \) and \( r'_k \) form central pseudotrajectories.

For small values of \( \delta \) we have

\[
\{ f(r_k) \} = W_{\sigma/2}^s(f(p_k)) \cap W_{\sigma/2}^{cs}(f(q_k)) \subset W_{\sigma}^{cs}(p_{k+1}) \cap W_{\sigma}^{cu}(q_{k+1}) \subset W_2^c(r_k).
\]

Moreover, due to selection of \( \sigma \) there exists a \( C_0 > 0 \) such that

\[
\text{dist}_c(f(r_k), r_{k+1}) \leq C_0(\text{dist}(r_{k+1}, p_{k+1}) + \text{dist}(f(p_k), p_{k+1}) + \text{dist}(f(p_k), f(r_k))) \leq C_1 \delta
\]

where \( C_{0,1} \) are positive constants. This finishes the proof of the lemma. □

The sense of this lemma is the following: we may project central pseudotrajectories to the same stable manifold, or, in other words, suppose that \( q_k \in W_{loc}^s(p_k) \) for all \( k \).

6 Homoclinic points for Anosov and partially hyperbolic diffeomorphisms

Lemma 5, given below, describes some properties of Anosov diffeomorphisms and is out of the mainstream of the proof. For us it is sufficient to take a much weaker statement of Lemma 6 which is correct for partially hyperbolic diffeomorphisms. However, proofs of these two statements look similar and the result of Lemma 5 may be interesting itself, so we give here both statements.

Let \( f \in \text{Diff}^1(M) \) be an Anosov diffeomorphism of a closed smooth manifold \( M \); \( G = \pi_1(M) \) be the fundamental group of \( M \); \( \text{Diff}^0(M) \) be the group of homeomorphisms of \( M \) onto itself. Denote by \( f^i \) the automorphism of \( G \) induced by \( f \). Let \( f^* : \text{Diff}^0(M) \circ \) be defined by formula \( f^*h = f^{-1} \circ h \circ f \).

Definition 9. We say that a point \( q \in M \) is homoclinic to \( p \in M \) or \( q \in H(p) \) if \( q \in W^s(p) \cap W^u(p) \).

Unlike classical definition, here we have \( p \in H(p) \) for all \( p \).

Lemma 5. For any Anosov diffeomorphism \( f \) of a closed connected manifold \( M \) there exists a homomorphism \( \mathcal{L} : G \rightarrow \text{Diff}^0(M) \) such that

1) for any \( g \in G, p \in M \) we have \( \mathcal{L}g(p) \in H(p) \);
2) for all \( p, q \in M \) such that \( q \in H(p) \) there exists \( g \in G \) such that \( \mathcal{L}g(p) = q \);
3) \( f^* \circ \mathcal{L} = \mathcal{L} \circ f^* \).
Proof. Take two points \( p, p' \in M \), \( q \in H(p) \) and select an oriented arc \( \gamma : [0, 1] \to M \), linking points \( p \) and \( p' \) \( \text{i.e.} \gamma(0) = p, \gamma(1) = p' \). There exists an \( \varepsilon > 0 \) and a uniquely defined continuous function \( h : [0, \varepsilon] \to M \) such that \( h(0) = q, h(t) \in W^u(p) \cap W^s(\gamma(t)) \) for all \( t \in [0, \varepsilon] \). Take the supreme value \( \varepsilon_0 \) such that \( h \) can be uniquely extended to \( [0, \varepsilon_0] \). It is easy to see from continuity of holonomy mappings and transversality of stable and unstable bundles that \( \varepsilon_0 = 1 \). Then we can define \( h_{\gamma}(q) = h(1) \). Roughly speaking, we have moved an intersection of stable and unstable foliations along the arc \( \gamma \).

For a \( g \in G \) and a point \( p \in M \) we take a loop \( \gamma \) from the class \( g \) starting from the point \( p \) and set
\[
\mathcal{L}g(p) = h_{\gamma}(p).
\]

First of all, we need to check that this definition is correct \( \text{i.e.} \) that the right hand side of (6) does not depend on the selection of a loop \( \gamma \) inside the class \( g \). Let \( \gamma_1 \) and \( \gamma_2 \) be two homotopic loops, \( q_{1,2} = h_{\gamma_{1,2}}(p) \). The set \( H(p) \) is at most countable so, all its connected components are singletons. However, \( q_1 \) and \( q_2 \) must be connected by an arc inside \( H(p) \), so \( q_1 = q_2 \).

It is clear that \( \mathcal{L}g \) is a continuous mapping for every fixed \( g \) and that \( \mathcal{L}g_1g_2 = \mathcal{L}g_1 \circ \mathcal{L}g_2 \). Taking \( g_1 = g, g_2 = g^{-1} \), we easily see that all mappings \( \mathcal{L}g \) are homeomorphisms. Statement 3) of the lemma is also trivial. So, it suffices to prove Statement 2).

Take two points \( p, q \in M, q \in H(p) \). There is a loop \( \gamma \), that is concatenation of arcs \( \gamma_s \) and \( \gamma_u \) inside \( W^s(p) \) and \( W^u(p) \) respectively. Both these arcs join \( p \) and \( q \). We write \( \gamma = \gamma_s^{-1}\gamma_u \) that is this is a loop which follows \( \gamma_u \) from \( p \) to \( q \) and then follows \( \gamma_s \) from \( q \) to \( p \). Here we use the power \( -1 \) to underline direction of the arc, nothing more. There is a huge family of such loops \( \gamma \), but all of them are homotopic.

Due to definition \( h_{\gamma_s}(t) = \gamma_u(t) \) for all \( t \in [0, 1] \). As we move a point along the arc \( \gamma_s^{-1} \), the manifold
\[
W^s(\gamma_s^{-1}(t)) = W^s(q) = W^s(p)
\]
for all \( t \). So, \( \mathcal{L}g(p) = q \) where \( g \) is the class of \( \gamma \). \( \Box \)

Remark. We have never used compactness of the manifold \( M \), we just needed to have transversality of invariant bundles.

Lemma 6. Let \( f \) be partially hyperbolic and dynamically coherent diffeomorphism of a simply connected (but eventually non-compact) manifold \( M \). Then for all \( p, q \in M, q \in W^s(p) \cap W^{cs}(p) \) or \( q \in W^u(p) \cap W^{cs}(p) \) imply \( p = q \).
Proof. We use the idea, same to the proof of Lemma 5. Let
\[ q \in H(p) := W^s(p) \cap W^{cu}(p). \]
Take a loop, linking \( p \) and \( q \), similarly to what we have done in Lemma 5. This loop is homotopic to trivial one so, there is an arc, linking \( p \) and \( q \) in \( H(p) \). This implies \( p = q \). \( \square \)

7 Partial shadowing

Here we prove the following result that can be treated as a very weak form of so-called shadowing (see [27] for definitions).

Take two positive sequences \( d_m, \delta_m \to 0 \). If the statement of Theorem 1 is false, we may take two sequences of central pseudotrajectories \( \{ p^m_k \} \) and \( \{ q^m_k \} \) such that
\[
\text{dist}(p^m_k, q^m_k) \leq \delta_m, \quad \text{dist}_c(f(p^m_k), p^m_{k+1}) \leq d_m, \quad k \in \mathbb{Z}, \ m \in \mathbb{N}.
\]

Lemma 7. There exists a number \( m \in \mathbb{N} \), a sequence \( k_j \to -\infty \) and a point \( x_0 \in M \) such that
\[
\text{dist}(p^m_{k_j}, f^{k_j}(x_0)) \leq \sigma
\]
for all \( j \in \mathbb{N} \).

Proof. In this proof we use ideas of proofs for Krylov-Bogolyubov theorem [28, Theorem 4.1.1] and Poincaré recurrence theorem [28, Theorem 4.1.19].

Consider an integer sequence \( s_j \to -\infty \) such that for any continuous function \( \varphi : M \to \mathbb{R} \) there exists a limit
\[
J_m(\varphi) := \lim_{s_j \to \infty} \frac{1}{|s_j|} \sum_{i=s_j+1}^{0} \varphi(p^m_i).
\]
These functionals \( J_m \) uniquely define Borel probability measures \( \mu_m \) on \( M \) by formula
\[
J_m(\varphi) = \int_M \varphi \, d\mu_m.
\]
Without loss of generality, we may suppose that measures \( \mu_m \) *-weakly converge to a probability measure \( \mu_* \) which is evidently invariant with respect to \( f \) since \( \delta_m \to 0 \). Now we take a point \( \bar{x} \in M \) and a value \( \varepsilon > 0 \) such that \( \mu_*(B) \neq 0 \) where \( B = B_{\varepsilon/2}(\bar{x}) \). This can be done since \( M \) is a compact set. Here the value \( \varepsilon \) can be taken as small as we want.
Note that there exists $m > 0$ such that the ball $B$ contains infinitely many points $p^m_k$, $k < 0$. Otherwise, $J_m(\chi_B) = 0$ for all $m$, where $\chi_B$ is the characteristic function of $B$. So $\mu_m(B) = 0$ for all $m$ and $\mu_*(B) = 0$. Let the sequence $i_j \to -\infty$ be such that $p^m_{i_j} \in B$ for all $j$.

Similarly to Poincaré recurrence theorem we may demonstrate that

$$
\mu_* \left( B \setminus \bigcup_{j=0}^{\infty} f^{-i_j}(B) \right) = 0.
$$

Consequently, there exists a point $x_0 \in B$ and an infinite subsequence $\{k_j\} \subset \{i_j\}$, $k_j \to -\infty$ such that $f^{k_j}(x_0) \in B$. Therefore,

$$
\text{dist}(p^m_{k_j}, f^{k_j}(x_0)) \leq \varepsilon.
$$

To finish the proof, it suffices to take $\varepsilon \leq \sigma$. $\Box$

Select $m$ from the statement of Lemma 7 and take $p_k = p^m_k$, $q_k = q^m_k$, $x_j = f^{k_j}(x_0)$. Let $p'_j$ and $q'_j$ be uniquely defined points of intersections of $W^c_{\text{loc}}(p_k)$ ($W^c_{\text{loc}}(q_k)$) with $W^s_{\text{loc}}(x_j)$. Due to assumptions of previous sections, we have $p'_k \neq q'_k$. Introduce a notation $l_j = k_{j+1} - k_j$.

8 Lifting to the loop-bundle

Let $N$ be the loop-bundle over $M$. This is a smooth (maybe non-compact) manifold that consists of homoclinic classes of arcs linking a point $x \in M$ with points $y \in M$. Manifold $N$ is the universal coverage for $M$; it is always simply connected. One can lift the diffeomorphism $f$ to a diffeomorphism $F : N \to N$. All objects, considered in previous sections e.g. invariant manifolds, corresponding bundles and pseudotrajectories, can be lifted to $N$. Let $P_k, Q_k, P'_j, Q'_j, X_j$ be liftings of points $p_k, q_k, p'_j, q'_j$ and $x_j$ respectively. Select them so that for all $j$

$$
X_j = F^{k_j}(X_0),
$$

$$
\{P'_j\} = W^c_{\text{loc}}(P_k) \cap W^s_{\text{loc}}(X_j),
$$

$$
\{Q'_j\} = W^c_{\text{loc}}(Q_k) \cap W^s_{\text{loc}}(X_j).
$$

**Lemma 8.** For any $j \in \mathbb{N}$

$$
P'_{j+1} = F^{l_j}(P'_j); \quad Q'_{j+1} = F^{l_j}(Q'_j). \quad (7)
$$
Proof. It suffices to prove the first equality of (7). Note that points $P'_{j+1}$ and $F^{l_j}(P'_{j})$ belong to the same central unstable foliation because $P_{k+j+1}$ and $F^{l_j}(P_k)$ do. On the other hand, both these points belong to $W^s(X_j)$. Then statement of the lemma follows from one of Lemma 6. □

Since $P'_{j} \in W^{s}_{loc}(Q'_{j})$, for all $j$ and due to statement of Lemma 8, we obtain $P'_{j} = Q'_{j}$ for all $j$. This finishes the proof of Theorem 1.

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**References**

[1] R. Patrie, *On the work of Jorge Lewowicz on expansive systems*, arXiv:1208.4394v1.

[2] J. Lewowicz, *Expansive Homomorphisms of Surfaces*, Bol. Soc. Bras. de Mat., 20 (1989), 113–133,

[3] J. Franks, *Anosov Diffeomorphisms*, Proc. Sympos. Pure Math., 14 (1970), 61-93.

[4] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417–431.

[5] Ch. Bonatti, L. J. Diaz, M. Viana, *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*, Springer, Berlin, 2004.

[6] J. Lewowicz, M. Cerminara, *Some open problems concerning expansive systems*, Rend. Istit. Mat. Univ. Trieste 42 (2010), 129–141.

[7] J. Lewowicz, J. Tolosa, *On expansive diffeomorphisms in the $C^0$-border of the set of Anosov diffeomorphisms*, Dynamical systems and partial differential equations (Caracas,1984), 57–64, Univ. Simon Bolivar, Caracas, 1986.
[8] R. Mañe, *Expansive diffeomorphisms*, Lecture Notes in Math. **468** (1975), 162–174.

[9] T. OBrien, W. Reddy, *Each compact orientable surface of positive genus admits an expansive homeomorphism*, Pacific J. Math. **35** (1970), 737–741.

[10] M. Paternain, *Expansive flows and the fundamental group*, Bol. Soc. Brasil. Mat. (N.S.) **24** (1993), no. 2, 179–199.

[11] J. Tolosa, *The Method of Lyapunov Functions of Two Variables*, Contemporary Mathematics **440** (2007), 243–271.

[12] J. Lewowicz, *Lyapunov functions and topological stability*, J. Diff. Equations **38** (1980), 192–209.

[13] K. Moriyasu, K. Sakai, W. Sun, *C^1-stably expansive flows* J. Diff. Eq., **213** (2005), 352–367.

[14] A. Kelley, *The Stable, Center Stable, Center, Center Unstable, Unstable Manifolds*, J. Diff. Equations **3** (1967), 546–570.

[15] M.W. Hirsch, C. C. Pugh, M. Shub, *Invariant Manifolds*, Springer-Verlag, Berlin-Heidelberg, 1977.

[16] M. Brin, *Hölder continuity of invariant distributions*, Smooth ergodic theory and its applications, Eds. A. Katok, R. de la Llave, Ya. Pesin and H. Weiss. Proc. Symp. Pure Math. Providence. RI. AMS, 2001.

[17] K. Sakai, *Pseudo-orbit tracing property and strong transversality of diffeomorphisms of closed manifolds*, Osaka J. Math., **31** (1994), 373–386.

[18] F. Abdenur, L. Diaz, *Pseudo-orbit shadowing in the C^1 topology*, Discrete Contin. Dyn. Syst., **7** (2003), 223–245.

[19] K. Burns, A. Wilkinson, *Dynamical Coherence and Center Bunching*, Discrete and Continuous Dynamical Systems, **22** (2008), 89-100.

[20] C. Pugh and M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. Complexity, **13** (1997), 125-179.

[21] Ya. B. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, Zurich Lectures in Advanced Mathematics (9783037190036), 2006.

[22] S. Kryzhevich, S. Tikhomirov, *Partial hyperbolicity and central shadowing*, Discrete Contin. Dyn. Syst.– A **33** (2013), no. 7, 2901-2909.

[23] D. Bohnet, Ch. Bonatti, *Partially hyperbolic diffeomorphisms with uniformly center foliation: the quotient dynamics*, arXiv:1210.2835.

[24] Huyi Hu, Yunhua Zhou, Yujun Zhu, *Quasi-Shadowing for Partially Hyperbolic Diffeomorphisms*, arXiv:1210.4988.

[25] A. Artigue, J. Brum, R. Potrie, *Local product structure for expansive homeomorphisms*, Topology and its Applications 156 (2009), no. 4, 674 – 685.
[26] M. Paternain, *The principal loop-bundle and dynamical systems*, C. R. Acad. Sci. Paris, *329* (1999), Série I, 1081–1085.

[27] S. Yu. Pilyugin, *Shadowing in Dynamical Systems*, Lect. Notes Math., Vol. 1706, Springer-Verlag, 1999.

[28] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1997.