Affine metrics and algebroid structures: Application to general relativity and unification

N. Elyasi, N. boroojerdian

Abstract  Affine metrics and its associated algebroid bundle are developed. Theses structures are applied to the general relativity and provide an structure for unification of gravity and electromagnetism. The final result is a field equation on the associated algebroid bundle that is similar to Einstein field equation but contain Einstein field equation and Maxwell equations simultaneously and contain a new equation that may have new results.

Keywords: affine metrics, algebroid, curvature, gravitation, electromagnetism, unification

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1 Introduction

Different phenomena can be explained by one theory. This is the main stream of thought in physics. After the invention of GR by Einstein in 1914, the most important question was ”Is it possible to combine gravity and electromagnetism into a unique theory?” This question is also important today, because any unification theory can throw light on the nature of the forces.

Einstein hoped that a good unification theory can solve mysteries of quantum effects and elementary particles too, and spent decades of his life on this project. Of course, he was unsuccessful and it does not seem that the mathematical framework of classical physics be suitable for explanation of quantum effects and elementary particles. To comprehend quantum phenomena, we need to change our viewpoints drastically. So, a unification theory of gravity and electromagnetism in the level of classical physics can not be considered as a true theory and at most we can expect that it be a good approximation in the level of classical physics of the true theory.

The main idea in the unification of gravity and electromagnetism or geometrization of electromagnetism is to provide new geometrical structures such that simultaneously contain both gravitation and electromagnetism naturally. It has been done many attempts to produce convenient geometrical structures. Most of them consider GR as a base structure and enrich it with additional structures or additional degrees of freedom.

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Early attempts in unification of fields have been done by Weyl (considering conformal structures and introducing a new gauge transformation, 1918), Kaluza (adding additional dimension to space-time, 1919), Eddington (considering connection as the central concept and decomposing its Ricci tensor to symmetric and anti-symmetric parts, 1921), Schouten (considering connections with nonzero torsion, 1921), Klein (interpreting fifth dimension of Kaluza theory as a relation to quantum concepts, 1926), Infeld (considering asymmetric metric that its symmetric part represent gravity and its anti-symmetric part represent electromagnetism field, 1928), Einstein and Mayer (5-vector formalism and considering vector bundles and connections on vector bundles, 1931)[3].

The main idea of this paper is to introduce affine metrics and its associated algebroid bundle on a space-time. Connections and curvature of these structures naturally contain elements of gravity and electromagnetism and a field equation can relate these concepts properly.

2 Algebraic preliminary

Let $V$ and $W$ be vector spaces, then a function $S : V \rightarrow W$ is called an affine function iff there exists a linear function $T : V \rightarrow W$ such that:

$$\forall u, v \in V \quad S(u + v) = S(u) + T(v)$$

$T$ is unique and is called the linear part of $S$. All affine function $S : V \rightarrow W$ have the form $S(u) = T(u) + a$ in which $T$ is linear and $a \in W$. In this section $V$ is a fixed vector space.

2.1 2-affine functions and affine inner products

**Definition 2.1** A function $S : V \times V \rightarrow \mathbb{R}$ is called 2-affine iff it is affine in each argument. And $S$ is called affine-linear iff it is affine in first variable and linear in second variable. In a similar way $S$ is called linear-affine iff it is linear in first and affine in second variable.

**Definition 2.2** A 2-affine function $S : V \times V \rightarrow \mathbb{R}$ is called symmetric iff for all $a, b \in E : S(a, b) = S(b, a)$

If $S : V \times V \rightarrow \mathbb{R}$ be a symmetric 2-affine function then there exist a unique linear-affine function $T_1$ and a unique symmetric bilinear function $T$ such that for all $a, b, u, v \in V$:

$$S(a + u, b + v) = S(a, b) + T_1(u, b) + T_1(v, a) + T(u, v)$$

$T_1$ and $T$ are called respectively linear-affine, and bilinear parts of $S$.

**Definition 2.3** A 2-affine function $S : V \times V \rightarrow \mathbb{R}$ is called an affine inner product on $V$, iff $S$ be symmetric and its bilinear part be an inner product on $V$. 
Every ordinary inner product on $V$, is also an affine inner product.

**Notation:** Suppose $a, b, u, v \in V$. Usually, we show an affine inner product by $(a, b)$ and its linear-affine parts by $< u, b >$ and its bilinear part by $< u, v >$. For simplicity in writing, we also use an affine-linear function denoted by $(b, u >$ that is equal to $< u, b >$.

So,

$$(a + u, b + v) = (a, b) + (a, v > + < u, b )+ < u, v >$$

Let $V$ be a vector space and $(., .)$ be an affine inner product on $V$, then for a unique vector $z \in V$ and a unique scalar $\lambda \in \mathbb{R}$ we have

$$(u, v ) = \lambda + < u - z, v - z >$$

It is sufficient to set $z$ be the vector that for all $v \in V$, $< z, v > = -(0, v >$ and $\lambda = (0, 0) - < z, z >$. Conversely for any ordinary inner product on $V$ and vector $z \in V$ and scalar $\lambda \in \mathbb{R}$, by the above formula we can define an affine inner product on $V$ and all affine inner products on $V$ are obtained in this way.

### 2.2 associated inner product space to affine metrics

Let $(., .)$ be an affine inner product on $V$. Set $\hat{V}$ be the space of real valued affine map on $V$. $\hat{V}$ is a vector space whose dimension is one plus dimension of $V$. For all $x \in V$ set $\hat{x}: V \rightarrow \mathbb{R}$ be the affine map $\hat{x}(y) = (x, y)$. The map $x \mapsto \hat{x}$ is affine and imbed $V$ into $\hat{V}$ as an affine subspace.

For all $x \in V$ set $\hat{x}: V \rightarrow \mathbb{R}$ be the affine map $\hat{x}(y) = < x, y >$. The map $x \mapsto \hat{x}$ is linear and imbed $V$ into $\hat{V}$ as a vector subspace. Denote the set of all $\hat{x}$ by $\hat{V}$. The space of real valued constant function on $V$ is a one dimensional subspace of $\hat{V}$ and is complementary to $\hat{V}$. So, we find a natural projection $\rho: \hat{V} \rightarrow V$ whose kernel is constant functions and its restriction to $\hat{V}$ is $\hat{x} \mapsto x$.

If $(x, y) = \lambda + < x - z, y - z >$ and $\lambda \neq 0$, then there exist a unique inner product on $\hat{V}$ such that for all $x, y \in V$ we have $\langle \hat{x}, \hat{y} \rangle = (x, y)$. By manipulating this property we can find the right definition of this inner product. $\hat{z}$ is the constant function $\hat{z}(x) = \lambda$ and must be orthogonal to $\hat{V}$. Every element of $\hat{V}$ is uniquely written in the form $\hat{x} + \mu \hat{z}$, and we must define

$$< \hat{x} + \mu_1 \hat{z}, \hat{y} + \mu_2 \hat{z} > = < x, y > + \lambda \mu_1 \mu_2$$

Note that for all $x, y \in V$ we have $\hat{x} + \hat{y} = \hat{x} + \hat{y}$, so $\hat{x} = \hat{x} - \hat{z} + \hat{z}$.

### 3 Affine semi-riemannian manifolds and its associated algebroid

In this section, $M$ is a fixed smooth manifold and all functions are smooth.

**Definition 3.1** If for every $p \in M$, we choose on every $T_p M$ an affine inner product smoothly, then we call it an affine metric on $M$ and $M$ is called an affine semi-riemannian manifold.
Every semi-riemannian manifold is also an affine semi-riemannian manifold. The bilinear part of an affine metric on $M$ is a semi-Riemannian metric on $M$ and it is called the associated semi-Riemannian metric.

**Example 3.2** If $<.,.>$ be a semi-Riemannian metric on $M$, and $A \in \mathfrak{X}M$ and $\phi \in C^\infty(M)$, then the following formula defines an affine metric on $M$.

$$\forall X, Y \in \mathfrak{X}M \quad (X, Y) = \phi + < X - A, Y - A >$$

Every affine metric on $M$ can be written as above. If $0$ be the zero vector field, it is sufficient to set $A$ be the vector field which for all $X \in \mathfrak{X}M$, $< A, X > = - < 0, X >$ in which $<.,.>$ is the bilinear part of the affine metric and set $\phi = (0, 0) - < A, A >$. In this section $M$ is an affine semi-riemannian manifold and $X, Y \in \mathfrak{X}M$ and its affine metric is as follows:

$$(X, Y) = 1 + < X - A, Y - A >$$

Let $\hat{T}M$ be the vector bundle $\bigcup_{p \in M} \hat{T}_pM$. For every $X \in \mathfrak{X}M$ let $\hat{X}$ and $\check{X}$ be sections of $\hat{T}M$ such that $(\hat{X})_p = \hat{X}_p$, $(\check{X})_p = \check{X}_p$. $\hat{A}$ is the constant function $\hat{A}(u) = 1$ and for simplicity we denote it by $\xi$. Let $\check{T}M$ be the vector bundle $\bigcup_{p \in M} \check{T}_pM$ that is a subvector bundle of $\hat{T}M$. $\check{T}M$ is complementary to line subbundle generated by $\xi$. Every section of $\hat{T}M$ uniquely written in the form $\check{X} + f\xi$ for some $X \in \mathfrak{X}M$ and $f \in C^\infty(M)$. $\check{T}M$ is a semi-riemannian vector bundle by the induced inner product:

$$X, Y \in \mathfrak{X}M, f, g \in C^\infty(M) \quad < \check{X} + f\xi, \check{Y} + g\xi > = < X, Y > + fg$$

$\hat{T}M$ has a natural algebroid structure over $TM$. The anchor map is

$$\rho : \hat{T}M \longrightarrow TM \quad \hat{X} + f\xi \longmapsto X$$

In the definition of Lie bracket on $\hat{T}M$ the vector field $A$ make a crucial role. $\nabla A$ is a 1-1 tensor on $M$ and its anti symmetric part is denoted by $F$. In fact:

$$< F(X), Y > = \frac{1}{2} ( < \nabla_X A, Y > - < X, \nabla_Y A > )$$

$2F$ is equivalent to the exterior derivation of the 1-form equivalent to $A$. Lie bracket on $\hat{T}M$ is defined as follows:

$$[\hat{X}, \hat{Y}] = [X, Y] + 2 < F(X), Y > \xi \quad [\hat{X}, \xi] = 0$$

Jacobi identity is hold because $F$ is equivalent to a closed form. Now, $\hat{T}M$ is a semi-riemannian algebroid over $TM$ and has a unique Levi-civita connection $\nabla$ which can be computed by the following relation\[5\]. For all $U, V, W \in \Gamma(\hat{T}M)$:

$$2 < \nabla_U V, W > = \rho(U) < V, W > + \rho(V) < W, U > - \rho(W) < U, V > + < [U, V], W > - < [V, W], U > + < [W, U], V >$$

4
Proposition 3.3  Levi-civita connection of the algebroid \( \hat{T}M \) satisfies the following relations.

\[
\begin{align*}
\hat{\nabla}_\xi \xi &= 0 \\
\hat{\nabla}_X \xi &= \hat{\nabla}_\xi X = -F(X) \\
\hat{\nabla}_X Y &= \hat{\nabla}_Y X + \langle F(X), Y \rangle \xi
\end{align*}
\]

**proof:** Straightforward computations show these results. \( \square \)

**Definition 3.4** A path \( \hat{\alpha} : I \rightarrow \hat{T}M \) is called a geodesic of \( \hat{\nabla} \) iff for some path \( \alpha : I \rightarrow M \) we have \( \rho(\hat{\alpha}) = \alpha' \) and \( \hat{\nabla}_\alpha \hat{\alpha} = 0 \).

Proposition 3.5 A path \( \hat{\alpha} : I \rightarrow \hat{T}M \) is a geodesic of \( \hat{\nabla} \) iff for some path \( \alpha : I \rightarrow M \) and scalar \( \lambda \) we have \( \hat{\alpha}(t) = \alpha'(t) + \lambda \xi \) and \( \nabla_{\alpha'} \alpha' = 2\lambda F(\alpha') \).

**Proof:** Since \( \rho(\hat{\alpha}) = \alpha' \), for some function \( f : I \rightarrow \mathbb{R} \) we have \( \hat{\alpha}(t) = \alpha'(t) + f(t) \xi \). To compute easily, assume \( X \) be a local vector field on \( M \) and \( g \) a local function on \( M \) such that \( X_{\alpha(t)} = \alpha'(t) \) and \( g(\alpha(t)) = f(t) \). Consequently, \( \hat{\alpha}(t) = (X + g \xi)_{\alpha(t)} \). Since \( \hat{\nabla}_\alpha \hat{\alpha} = 0 \), we have:

\[
0 = \hat{\nabla}_\alpha \hat{\alpha} = (\hat{\nabla}_X g \xi + g \xi)_{\alpha(t)} = (\hat{\nabla}_X X + g \hat{\nabla}_\xi X + \hat{\nabla}_X g \xi + g \hat{\nabla}_\xi g \xi)_{\alpha(t)}
\]

\[
= \langle \nabla X X - 2g F(X) + X(g) \xi, \alpha' \rangle + \langle \nabla_x F(X), \alpha' \rangle + \langle \nabla_{\alpha'} \alpha', \alpha' \rangle.
\]

second equation means \( g(\alpha(t))' = f'(t) = 0 \), so \( f \) is constant and for some scalar \( \lambda \), \( f(t) = \lambda \). Consequently \( \hat{\alpha}(t) = \alpha'(t) + \lambda \xi \) and \( \nabla_{\alpha'} \alpha' = 2\lambda F(\alpha') \).

**Proposition 3.6** The curvature tensor of \( \hat{\nabla} \) denoted by \( \hat{R} \), and it satisfies the following relations.

\[
\begin{align*}
\hat{R}(\hat{X}, \hat{\xi})(\hat{\xi}) &= -F(\langle F(X), X \rangle) \\
\hat{R}(\hat{X}, \hat{\xi})(\hat{Z}) &= -\langle \nabla X F(X), Z \rangle - \langle F(X), F(Z) \rangle \xi \\
\hat{R}(\hat{X}, \hat{Y})(\hat{\xi}) &= -\langle \nabla_y F(X), X \rangle \\
\hat{R}(\hat{X}, \hat{Y})(\hat{Z}) &= \hat{R}(\hat{X}, \hat{\xi})(\hat{Z}) + \langle Z, F(X) \rangle F(Y) + \langle Z, F(Y) \rangle F(X)
\end{align*}
\]

**proof:** Straightforward computations show these results. \( \square \)

**Proposition 3.7** The Ricci curvature tensor of \( \hat{\nabla} \) denoted by \( \hat{Ric} \), as a 1-1 tensor satisfies the following relations.

\[
\begin{align*}
\hat{Ric}(\hat{\xi}) &= div(F) - tr(F \circ F) \xi \\
\hat{Ric}(\hat{X}) &= \hat{Ric}(X) + 2F(F(X)) + \langle div(F), X \rangle \xi
\end{align*}
\]

**proof:** Let \( X_1, \ldots, X_n \) be an orthonormal local base for \( M \). So, \( \overline{X_1}, \ldots, \overline{X_n}, \xi \) is an orthonormal local base for \( \hat{T}M \). Set \( \epsilon_i = \langle X_i, X_i \rangle = \pm 1 \), note that for any vector \( Y \in \mathfrak{X}M \) we have \( Y = \sum_i \epsilon_i X_i, X_i > X_i \).

\[
\begin{align*}
\hat{Ric}(\hat{\xi}) &= \sum_i \epsilon_i \hat{Ric}(\hat{X}_i, \hat{X}_i) \xi_i + \hat{Ric}(\hat{\xi}, \hat{\xi}) \xi \\
&= \sum_i \epsilon_i \langle \nabla_x F(X_i), X_i \rangle + \sum_i \epsilon_i \langle F(X_i), F(X_i) \rangle \xi_i \\
&= \sum_i \epsilon_i \langle \nabla_x F(X_i), X_i \rangle - \sum_i \epsilon_i \langle F(X_i), X_i \rangle \xi_i \\
&= \hat{div}(F) - tr(F \circ F) \xi
\end{align*}
\]
Denote the scalar curvature of \( \hat{T}M \) by \( \hat{R} \). An easy computation shows that:

\[
\hat{R} = R + tr(F \circ F)
\]

### 4 Application to general relativity

In the rest of the paper, suppose \( M \) is a four manifold and \( X, Y \in \mathfrak{X}M \).

**Definition 4.1** An affine metric \( (, ) \) on \( M \) is called affine-Lorentzian iff its bilinear part be a Lorentzian metric and for some \( A \in \mathfrak{X}M \):

\[
(X, Y) = 1 + < X - A, Y - A >
\]

An affine Lorentzian metric is determined by a Lorentzian linear metric \(<, >\) and a vector field \( A \in \mathfrak{X}M \). It seems plausible to interpret \(<, >\) as a potential for gravity and \( A \) as a potential for electromagnetism.

These interpretations can be rational, if we can interpret projection of the geodesics of \( \hat{T}M \) to \( M \) as world-lines of charged particles. By proposition (3.5) geodesics of \( \hat{T}M \) determine paths \( \alpha \) on \( M \) that satisfy equation

\[
\nabla_{\alpha'} \alpha = 2 \lambda F(\alpha')
\]

where 2\( F \) is the electromagnetism tensor with potential \( A \) and this is exactly the equation of a charged particle that its ration of charge to mass is \( \lambda \) and move under influence of electromagnetism tensor 2\( F \).

#### 4.1 Field equation

We need a field equation that naturally contains Einstein and maxwell equations simultaneously. To have a sound formulation, we use a system of measurement in which \( c = 1 \), \( G = 1 \), and \( \epsilon_0 = \frac{1}{16\pi} \). In this system, Einstein field equation and maxwell equation are written as following[1]:

\[
\begin{align*}
\text{Ric} - \frac{1}{2} Rg &= 8\pi(T^{\text{mas}} + T^{\text{elec}}) \\
\text{div}(2F) &= 16\pi J
\end{align*}
\]

So, \( \text{div}(F) = 8\pi J \). In this system, the momentum-energy tensor of the electromagnetism field is as following:

\[
T^{\text{elec}}_{ij} = \frac{1}{16\pi} (2F_{im}2F_{jm} - \frac{1}{4}g_{ij}2F_{mn}2F^{mn}) = \frac{1}{4\pi} (F_{im}F_{jm} - \frac{1}{4}g_{ij}F_{mn}F^{mn})
\]

In fact \( T^{\text{elec}} = \frac{-1}{4\pi} (F \circ F - \frac{1}{4}tr(F \circ F)g) \).
Einstein field equation can be rewritten as following:

\[ R_{ic} - \frac{1}{2}Rg = 8\pi(T^{\text{mas}} - \frac{1}{4tr(F \circ F)} g) \]

\[ \Rightarrow R_{ic} - \frac{1}{2}Rg + 2(F \circ F - \frac{1}{4}tr(F \circ F)g) = 8\pi T^{\text{mas}} \]

\[ \Rightarrow R_{ic} + 2F \circ F - \frac{1}{2}(R + tr(F \circ F))g = 8\pi T^{\text{mas}} \]

If we construct Einstein tensor \( \hat{G} = \hat{R}_{ic} - \frac{1}{2}\hat{R} \hat{g} \) in the algebroid \( \tilde{T}M \) we can see that the left side of Einstein field equation is exactly the restriction of \( \hat{G} \) to \( \tilde{T}M \). It seems that we can construct a suitable field equation in \( \hat{T}\tilde{M} \) such that contains simultaneously Einstein and maxwell field equations. We need a proper 5-mass momentum-energy tensor \( \hat{T} \) such that the equation:

\[ \hat{R}_{ic} - \frac{1}{2}\hat{R} \hat{g} = 8\pi \hat{T} \]

represent both Einstein and maxwell equations. Left side of this equation is a geometrical object and contains both gravity and electromagnetism as geometrical objects. Right side of this equation must contain information about matter that determine geometry of space. If \( T \) be ordinary mass momentum-energy tensor we must have:

\[ \hat{T}(\tilde{X}, \tilde{Y}) = T(X, Y) \]

Since \( \hat{G}(\tilde{X}, \xi) = < \hat{R}_{ic}(\tilde{X}), \xi > - \frac{1}{2}\hat{R} < \tilde{X}, \xi > = < \text{div}(F), X > \), we must have \( < \text{div}(F), X > = 8\pi \hat{T}(\tilde{X}, \xi) \). But by maxwell equations we have \( \text{div}(F) = 8\pi J \), so we must define:

\[ \hat{T}(\tilde{X}, \xi) = \hat{T}(\xi, \tilde{X}) = < X, J > \]

Definition of \( \hat{T}(\xi, \xi) \) is not straightforward. We need some clue to find right definition. Put \( H = \hat{T}(\xi, \xi) \), so we can consider \((J, H)\) as 5-current. Let \( \eta \) be the charge density and \( \rho \) be the mass density and \( U \) be 4-velocity. If \((U, U_5)\) be 5-velocity, 5-current has the form \((J, H) = \eta(U, U_5)\) and \( \hat{T} \) has the form \( \rho U_i U_j \). So we must have \( \eta U_5 = \rho U_5 U_5 \), and consequently \( U_5 = \frac{\eta}{\rho} \). So, fifth component of 5-velocity must be the ratio of charge to mass. Some other clues support this reason. In computing geodesics of \( \tilde{T}M \) we find that \( \hat{\alpha} \) that is 5-velocity of the geodesic \( \hat{\alpha} \) has the form \( \hat{\alpha}(t) = \hat{\alpha}'(t) + \lambda \xi \). Fifth component of this 5-velocity is \( \lambda \) and it was the ratio of charge to mass.

By this result we can conclude \( \hat{T}(\xi, \xi) = H = \eta U_5 = \frac{\eta^2}{\rho} \). Now, we ready to write field equation in the algebroid \( \tilde{T}M \):

\[ \hat{R}_{ic} - \frac{1}{2}\hat{R} \hat{g} = 8\pi \hat{T} \]

By splitting \( \hat{T}M \) to subbundle \( \tilde{T}M \) and line bundle generated by \( \xi \) the above equation can be written in the following block form.

\[
\begin{pmatrix}
Ric + 2F \circ F \\
-\frac{1}{4}(R + tr(F \circ F))g
\end{pmatrix}
\begin{pmatrix}
div(F) \\
-\frac{1}{4}(R + 3tr(F \circ F))
\end{pmatrix} = 8\pi
\begin{pmatrix}
\frac{T^{\text{mass}}}{J} \\
\frac{\eta^2}{\rho}
\end{pmatrix}
\]

7
This equation produces three other equations, that two of them are Einstein and maxwell field equations and third equation is new:

\[ R + 3 tr(\mathbf{F} \circ \mathbf{F}) = -16\pi \frac{\eta^2}{\rho} \]

5 conclusion

algebroid structures and affine metrics provide a mathematical framework for unification of gravity and electromagnetism. This theory is very similar to Kaluza theory except that it need not an extra dimension in base manifold. Instead, affine metrics naturally produce an extra dimension in tangent space. So, many redundancy of Kaluza theory disappear but one new equation appears that shows an intimate relation between mass and charge densities.

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