The Shapley Value of Digraph Games

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Abstract

In this paper the Shapley value of digraph (directed graph) games are considered. Digraph games are transferable utility (TU) games with limited cooperation among players, where players are represented by nodes. A restrictive relation between two adjacent players is established by a directed line segment. Directed line segments, connecting the initial player with the terminal player, form the coalition among players. Dominance relation is established between players and this relation determines whether or not a player wants to cooperate. To cooperate, we assume, player joins coalition where he/she is not dominated by any other players. The Shapley value [1] is defines as the average of marginal contribution vectors corresponding to all permutations that do not violate the subordination of players. The shapley value for various digraph games is calculated and analyzed. For a given characteristic function, a quick way to calculated Shapley values is formulated .

Keywords: Cooperative game, TU game, Shapley value, digraph, domination

1 Introduction

Game theory is the mathematical theory that study the conflict and cooperation between rational decision makers. Game theory helps to analyze decision making between two or more individuals which influences one another’s welfare [2]. Cooperative game theory deals with coalitions and allocations, and considers group of players willing to allocate the joint benefits derived form their cooperation [3]. According to cooperative game theory, coalition need to be stable for the formation of coalition. In stable coalition there is less incentives to leave the coalition. When the players in the game form the coalition to work together, it is essential to identify correct way to distribute the profit among themselves. If some of the players in the coalition are unsatisfied with the proposed allocation, then they are free to leave the coalition. The Shapley value provides a unique way to divide a pay-off among players in such a way as to satisfy various fairness criteria. Myerson paper [4] considers the cooperation between players in the undirected graph, where each player has equal chance to move away from coalition by breaking the link between them. Such games assume fair and equal gain through cooperation.

As the structure of this paper, section 2 contains preliminaries, section 3 contains results and section 4 includes examples, where Shapley value of various digraph games is calculated.

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2 Preliminaries

A cooperative TU-game is a pair $(N, v)$, where $N = \{1, ..., n\}$ is a finite set of players with $n \geq 2$ and $v : 2^N \to \mathbb{R}$ is the characteristic function of the game. We interpret $v(s)$ as the benefit that the coalition $S \subseteq N$ can generate. The worth $v(S)$ of coalition is the amount of utility that the coalition can divide among its members. By convention, the pay-off of an empty coalition is zero i.e. $V(\emptyset) = 0$. The complement of $S \subseteq N$, denoted $N \setminus S$, is the set of all players in $N$ that are not in $S$. The collection of all coalitions of $N$ is denoted by $2^N$ and $|S|$ is the number of players in $S$. We refer to the set of all TU-games with fixed set of player $N$ as $\mathbb{G}^N$. For simplicity we use $v$ to refer to $(N, v)$. For $(N, v) \in \mathbb{G}^N$ and $T \subseteq S \subseteq N$, the restriction of $(N, v)$ to the coalition $S$ is the TU-game $(S, v_s)$ such that $v_s(T) = v(T)$. A function $\xi : \mathbb{G} \to \mathbb{R}^N$ such that $\mathbb{G} \subseteq \mathbb{G}^N$ assigns to every game $v \in \mathbb{G}$ a vector $\xi(v) \in \mathbb{R}^N$ where $\xi_i(v)$ is the payoff to player $i \in N$ in game $v$. For any player $i \in N$, player $i$’s minimum payoff is $v(i)$ which he can guarantee to himself without joining any coalition.

In grand coalition $N$, players divide $v(S)$ among themselves. The outcome of this division depends on the power structure in the grand coalition. The marginal contribution of player $i \in N$ to the coalition in game $v \in \mathbb{G}^N$ is given by $m_i(S) = v(S \cup i) - v(S)$. For any $i \in N$ and permutation $\pi : N \to N$, $\pi(i)$ is the position of player $i$ in $\pi$. The set of predecessors of $i$ in $\pi$ denoted $P_{\pi}(i) = \{ j \in N | \pi(j) < \pi(i) \}$ and $P_{\pi}(i) = P_{\pi}(i) \cup \{ i \}$. For any $i \in N$ on a TU game, the marginal contribution vector $m^\pi(S) \in \mathbb{R}^N$ is $m^\pi_i(\pi) = m_i(S) - v(P_{\pi}(i)) - v(P_{\pi}(i))$, where $\pi$ is a permutation on $N$. The Shapley value of a TU game is $Sh(v) = \sum_{\pi \in \Pi} \frac{m^\pi(S)}{n!}$, where $\Pi$ is the set of all permutations on $N$.

A graph is a tuple $G = (N, \Gamma)$ where $N$ is the finite set of players or nodes and $\Gamma$ is the set of finite collection of directed links for digraph and $\Gamma$ is the set of finite collection of undirected links for undirected graph. For digraph, a collection of ordered pairs $\Gamma \subseteq \{(i, j) | i, j \in N, i \neq j \}$ and for undirected graph a collection of unordered pairs $\Gamma \subseteq \{(i, j) | i, j \in N, i \neq j \}$. It is obvious that undirected links can be considered as directed links. For a digraph $\Gamma$ on $N$ and coalition $S \subseteq N$, the subgraph of $\Gamma$ on $S$ is $T|_S = \{(i, j) \in \Gamma | i, j \in S \}$. There exists a path from player $i_1$ to player $i_k$ if $\{(i_1, i_{k+1}), (i_{k+1}, i_k)\} \cap \Gamma = \emptyset$ for $k = 1, ..., k - 1$ and directed path in $\Gamma$ from player $i_1$ to $i_k$ if $(i_k, i_{k+1}) \in \Gamma$ for $k = 1, ..., k - 1$. A directed cycle of players is a directed path $(i_1, i_2, ..., i_k) \in \Gamma$ such that $(i_k, i_1) \in \Gamma$. Player $i$ and $j$ in $N$ are connected in $\Gamma$ if there exist a path in $\Gamma$, which results in connected $\Gamma$. A coalition $S \subseteq N$ is connected in $\Gamma$ if the subgraph $\Gamma|_S$ is connected. Player $i$ is a predecessor of player $j$ in $\Gamma$ and a player $j$ is successor of player $i$ if there exists a directed path from $i \in N$ to $j \in N$. For $i \in N$, $S^i(i)$ denotes the set of successors of $i$ in $\Gamma$ and $S^{\Gamma}(i) = S^i(i) \cup \{ i \}$. For digraph $\Gamma$ and $S \subseteq N$, player $j \in S$ is dominated by player $i \in S$ in $\Gamma|_S$ if $j \in S^{\Gamma}(i)$ and $i \notin S^{\Gamma}(j)$. When a player does not have any predecessors, then he or she is undominated. No player is dominated on directed cycle.

Digraph game is a set of players and a pair $(v, \Gamma)$ of a TU-game $v \in \mathbb{G}^N$. A value of digraph game is a function $\xi : \mathbb{G} \to \mathbb{R}^N$ that assigns to every game $(v, \Gamma) \in \mathbb{G}$ a payoff vector $\xi(v, \Gamma)$. A permutation $\pi \in \Pi$ is consistent in $\Gamma$ if it preserves the subordination of players determined by $\Gamma$, i.e., $\pi(j) < \pi(i)$ if player $j$ is not dominated by player $i$ in subgraph for $i, j \in N$ such that $i \neq j$. Finally, $\Pi^\Gamma$ is the set of permutations on $N$ that is consistent with $\Gamma$. 

3 Results

**Lemma 3.1.** In any cyclic digraph \((\Gamma)\),

\[
\text{Sh}(v_k, \Gamma) = \frac{1}{|\Pi^k|} \sum_{\pi \in \Pi^k} \bar{m}^v(\pi) = \sum_{j=1}^{n} \left( (j)^k - (j-1)^k \right)
\]

for any player \(i \in N\) and any \(k \in \mathbb{N}\).

**Proof.** In any cyclic digraph, number of permutation consistent with digraph \((\Gamma)\) is equal to number of player \((n)\). Also,\[
\frac{1}{|\Pi^k|} \sum_{\pi \in \Pi^k} \bar{m}^v(\pi) = \frac{1}{|\Pi^k|} \sum_{\pi \in \Pi^k} (v(\bar{P}_v(i)) - v(P_v(i))) = \frac{1}{|\Pi^k|} \sum_{\pi \in \Pi^k} (v(P_v(i)) \cup v(i) - v(P_v(i))).
\]

We know that \(P_v(i) = \{ j \in N | \pi(j) < \pi(i) \} \). In any cyclic digraph, player \(i \in N\) occurs at most once in every position of \(\pi \in \Pi^F\). Since \(v_k(S) = |S|^k\), the marginal contribution of player \(i \in N\) is \(((n)^k - (n-1)^k) + ((n-1)^k - (n-2)^k) + \ldots + (1^k - 0^k)\). This is equivalent to \(\sum_{j=1}^{n} (j)^k - (j-1)^k\).

**Theorem 3.2.** In any cyclic digraph \((\Gamma)\) with a characteristic function defined by \(v_k(S) \mapsto |S|^k\), Shapley value of digraph game denoted \(\text{Sh}(v_k, \Gamma) = \left(\frac{v^{k-1}, n^{k-1}, n^{k-1}, \ldots, n^{k-1}}{n \text{ times}}\right)\) for any \(k \in \mathbb{N}\).

**Proof.** Consider a cyclic digraph \((\Gamma)\) with a characteristic function \(v_k(S) \mapsto |S|^k\), where \(S \subseteq N\) is a coalition of players and \(k \in \mathbb{N}\) is some constant. Also, in cyclic digraph \((\Gamma)\), number of players \((n)\) is equal to number of permutation \((\Pi^F)\) that is consistent with \(\Gamma\). For any player \(i \in N\), \(\text{Sh}(v_k, \Gamma) = \frac{1}{|\Pi^k|} \sum_{\pi \in \Pi^k} \bar{m}^v(\pi)\), where \(\bar{m}^v(\pi) = V(\bar{P}_v(i)) - v(P_v(i))\). Thus, we get

\[
\text{Sh}(V_k, \Gamma)_i = \frac{1}{|\Pi^k|} \sum_{\pi \in \Pi^k} (V(\bar{P}_v(i)) - v(P_v(i))).
\]

We know that in grand coalition, number of players in coalition is equal to number of player in digraph game. So, we have \(S = n\) and this implies \(|S|^k \mapsto |n|^k\). By lemma 3.1, Shapley value of any player \(i \in N\) is

\[
\text{Sh}(v_k, \Gamma)_i = \frac{\sum_{j=1}^{n} ((j)^k - (j-1)^k)}{|\Pi^k|} = \frac{((n)^k - (n-1)^k) + ((n-1)^k - (n-2)^k) + \ldots + (2^k - (1-1)^k)}{|\Pi^k|}.
\]

Since, the number of permutation \(|\Pi^k|\) is equal to number of player in digraph \(\Gamma\),

\[
\text{Sh}(v_k, \Gamma)_i = \frac{((n)^k - (n-1)^k) + ((n-1)^k - (n-2)^k) + \ldots + (2^k - (1-1)^k) + (2^k - (1)^k)}{n}.
\]

In numerator a telescopic series is formed so, by using telescopic sum, each subsequent term cancels each other and leaves only initial and final terms. Thus, \(\text{Sh}(V_k, \Gamma)_i = \frac{n^k - 0^k}{n} = \frac{n^k}{n} = n^{k-1}\). Because our choice for \(i\) is arbitrary so, every player has the Shapley value given
by $\text{Sh}(v_k, \Gamma)_i = n^{k-1}$. Thus, the Shapley value of cyclic digraph $\Gamma$ with $n$ different players and characteristic function defined by $v_k = |S|^k$ is a vector $(n^{k-1}, n^{k-1}, \ldots, n^{k-1}) \in \mathbb{R}^n$

i.e. $\text{Sh}(v_k, \Gamma) = \begin{pmatrix}
n^{k-1} & n^{k-1} & \cdots & n^{k-1} 
\end{pmatrix}$.

\begin{assert}
\textit{Corollary 3.3.} In any cyclic digraph $(\Gamma)$ with a characteristic function defined by $v_k(S) \mapsto |S|^k$, Shapley value is always unit vector, i.e, $\text{Sh}(v_k, \Gamma) = (1, 1, 1, \ldots, 1)$ for $n \geq 3$ and $k = 1$.
\end{assert}

4 Examples

For any coalition $S \subseteq N$ and $k \in \mathbb{N}$, a characteristic function is defined as $v_k(S) \mapsto |S|^k$ on digraph $\Gamma$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph1.png}
\caption{Fig: a) Digraph $\Gamma$}
\end{figure}

Consider a cyclic digraph game $(v, \Gamma)$ with three different players as shown in figure a). The set of all permutations that is consistent with $\Gamma$ is $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. For $k = 0, \text{Sh}(v_0, \Gamma) = \begin{pmatrix} 1 & 1 & 1 \\
\end{pmatrix}$. For $k = 1, \text{Sh}(v_1, \Gamma) = (1, 1, 1)$. For $k = 2, \text{Sh}(v_2, \Gamma) = (3, 3, 3)$.

For $k = 3, \text{Sh}(v_3, \Gamma) = (9, 9, 9)$. For $k = 4, \text{Sh}(v_4, \Gamma) = (81, 81, 81)$, and so on.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph2.png}
\caption{Fig: b) Digraph $\Gamma'$}
\end{figure}

Again, consider a cyclic digraph game $(v, \Gamma')$ with four different players as shown in figure b). The set of all permutations that is consistent with $\Gamma'$ is $\{(1, 2, 3, 4), (2, 3, 4, 1), (3, 4, 1, 2), (4, 1, 2, 3)\}$.
For $k = 0$, $Sh(v_0, \Gamma') = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. For $k = 1$, $Sh(v_1, \Gamma') = (1, 1, 1, 1)$. For $k = 2$, $Sh(v_2, \Gamma') = (4, 4, 4, 4)$. For $k = 3$, $Sh(v_3, \Gamma') = (16, 16, 16, 16)$. For $k = 4$, $Sh(v_4, \Gamma') = (64, 64, 64, 64)$, and so on.

As shown in figure c), consider a cyclic digraph game $(v, \Gamma'')$ with five different players. The set of all permutations that is consistent with $\Gamma''$ is $\{(1, 2, 3, 4, 5), (2, 3, 4, 5, 1), (3, 4, 5, 1, 2), (4, 5, 1, 2, 3), (5, 1, 2, 3, 4)\}$. For $k = 0$, $Sh(v_0, \Gamma'') = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. For $k = 1$, $Sh(v_1, \Gamma'') = (1, 1, 1, 1, 1)$. For $k = 2$, $Sh(v_2, \Gamma'') = (5, 5, 5, 5, 5)$. For $k = 3$, $Sh(v_3, \Gamma'') = (25, 25, 25, 25, 25)$. For $k = 4$, $Sh(v_4, \Gamma'') = (125, 125, 125, 125, 125)$, and so on.

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