THE HERMITE-HADAMARD’S INEQUALITY FOR SOME CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS AND RELATED RESULTS

ERHAN SET♠, MEHMAT ZEKI SARIKAYA♣, M.EMIN ÖZDEMIR■, AND HÜSEYIN YILDIRIM▼

Abstract. In this paper, firstly we have established Hermite-Hadamard’s inequalities for \( s \)−convex functions in the second sense and \( m \)−convex functions via fractional integrals. Secondly, a Hadamard type integral inequality for the fractional integrals are obtained and these result have some relationships with [11, Theorem 1, page 28-29].

1. Introduction

Let real function \( f \) be defined on some nonempty interval \( I \) of real line \( \mathbb{R} \). The function \( f \) is said to be convex on \( I \) if inequality

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \).

In [10], Hudzik and Maligranda considered, among others, the class of functions which are \( s \)−convex in the second sense. This class of functions is defined as the following:

**Definition 1.** A function \( f : [0, \infty) \to \mathbb{R} \) is said to be \( s \)−convex in the second sense if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)
\]

for all \( x, y \in [0, \infty) \), \( \lambda \in [0, 1] \) and for some fixed \( s \in (0, 1] \). This class of \( s \)-convex functions is usually denoted by \( K^s_2 \).

It can be easily seen that for \( s = 1 \), \( s \)−convexity reduces to ordinary convexity of functions defined on \( [0, \infty) \).

In [12], G. Toader considered the class of \( m \)−convex functions: another intermediate between the usual convexity and starshaped convexity.

**Definition 2.** The function \( f : [0, b] \to \mathbb{R}, b > 0 \), is said to be \( m \)−convex, where \( m \in [0, 1] \), if we have

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

for all \( x, y \in [0, b] \) and \( t \in [0, 1] \). We say that \( f \) is \( m \)−concave if \((-f)\) is \( m \)−convex.
Obviously, for $m = 1$ Definition 2 recaptures the concept of standard convex functions on $[a, b]$, and for $m = 0$ the concept starshaped functions.

One of the most famous inequalities for convex functions is Hadamard’s inequality. This double inequality is stated as follows (see for example [14] and [5]): Let $f$ be a convex function on some nonempty interval $[a, b]$ of real line $\mathbb{R}$, where $a \neq b$. Then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$  \hspace{1cm} (1.1)

Both inequalities hold in the reversed direction if $f$ is concave. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [11]-[16]).

In [8], Hadamard’s inequality which for $s$-convex functions in the second sense is proved by S.S. Dragomir and S. Fitzpatrick.

**Theorem 1.** Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an $s$-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:

$$2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s + 1}.$$  \hspace{1cm} (1.2)

The constant $k = \frac{1}{2^s}$ is the best possible in the second inequality in (1.2).

In [11], Kirmaci et. al. established a new Hadamard-type inequality which holds for $s$-convex functions in the second sense. It is given in the next theorem.

**Theorem 2.** Let $f : I \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on $I^0$ such that $f' \in L^1([a, b])$, where $a, b \in I$, $a < b$. If $|f'|^q$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1)$ and $q \geq 1$, then:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{b - a}{2} \left( \frac{1}{2} \right)^{\frac{s + 1}{s}} \left[ \frac{s + \left( \frac{1}{s} \right)^s}{(s + 1)(s + 2)} \right]^\frac{1}{q} \left[ |f'(a)|^q + |f'(b)|^q \right]^\frac{1}{q}. \hspace{1cm} (1.3)$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 3.** Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+} f$ and $J^\alpha_{b-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \hspace{0.5cm} x > a$$

and

$$J^\alpha_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \hspace{0.5cm} x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$. Here is $J^\alpha_{a+} f(x) = J^\alpha_{b-} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found ([24, 26] and [25]).
For some recent results connected with fractional integral inequalities see ([17]-[27]).

In [27] Sarıkaya et. al. proved a variant of the identity is established by Dragomir and Agarwal in [6, Lemma 2.1] for fractional integrals as the following:

**Lemma 1.** Let $f : [a, b] \rightarrow \mathbb{R}$, be a differentiable mapping on $(a, b)$ with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} = \frac{b - a}{2} \int_{0}^{1} [(1 - t)^\alpha - t^\alpha] f'(ta + (1 - t)b) dt.
\]

The aim of this paper is to establish Hadamard’s inequality and Hadamard type inequalities for $s$–convex functions in the second sense and $m$–conex functions via Riemann-Liouville fractional integral.

## 2. Hermite-Hadamard Type Inequalities for Some Convex Functions via Fractional Integrals

### 2.1. For $s$–convex functions

Hadamard’s inequality can be represented for $s$–convex functions in fractional integral forms as follows:

**Theorem 3.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If $f$ is a $s$–convex mapping in the second sense on $[a, b]$, then the following inequalities for fractional integrals with $\alpha > 0$ and $s \in (0, 1)$ hold:

\[
2s^{-1} f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J_{\alpha}^a f(b) + J_{\alpha}^b f(a) \right] \leq \frac{1}{(\alpha + s)} + \beta(\alpha, s + 1) \frac{f(a) + f(b)}{2}
\]

where $\beta$ is Euler Beta function.

**Proof.** Since $f$ is a $s$–convex mapping in the second sense on $[a, b]$, we have for $x, y \in [a, b]$ with $\lambda = \frac{1}{2}$

\[
f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2^s}.
\]

Now, let $x = ta + (1 - t)b$ and $y = (1 - t)a + tb$ with $t \in [0, 1]$. Then, we get by (2.2) that:

\[
2s f \left( \frac{a + b}{2} \right) \leq f (ta + (1 - t)b) + f ((1 - t)a + tb)
\]

for all $t \in [0, 1]$. 


Multiplying both sides of (2.3) by \( t^{\alpha-1} \), then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\frac{2^s}{\alpha} f \left( \frac{a + b}{2} \right) \leq \int_0^1 t^{\alpha-1} f (ta + (1-t)b) \, dt + \int_0^1 t^{\alpha-1} f ((1-t)a + tb) \, dt
\]

\[
= \frac{1}{(b-a)^\alpha} \int_a^b (b-u)^{\alpha-1} f(u) \, du - \frac{1}{(a-b)^\alpha} \int_a^b (a-v)^{\alpha-1} f(v) \, dv
\]

\[
= \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right]
\]

i.e.

\[
2^s f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha} \left[ \frac{J_a^\alpha f(b) + J_b^\alpha f(a)}{2} \right]
\]

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if \( f \) is a \( s \)-convex mapping in the second sense, then, for \( t \in [0, 1] \), it yields

\[
f (ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)
\]

and

\[
f ((1-t)a + tb) \leq (1-t)^s f(a) + t^s f(b).
\]

By adding these inequalities we have

\[
(2.4) \quad f (ta + (1-t)b) + f ((1-t)a + tb) \leq [t^s + (1-t)^s] (f(a) + f(b)).
\]

Thus, multiplying both sides of (2.4) by \( t^{\alpha-1} \) and integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\int_0^1 t^{\alpha-1} f (ta + (1-t)b) \, dt + \int_0^1 t^{\alpha-1} f ((1-t)a + tb) \, dt
\]

\[
\leq \int_0^1 t^{\alpha-1} [t^s + (1-t)^s] \, dt
\]

i.e.

\[
\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right] \leq [1 + (\alpha + s) \beta(\alpha, s + 1)] \frac{f(a) + f(b)}{\alpha + s}
\]

where the proof is completed. \( \square \)

**Remark 1.** If we choose \( \alpha = 1 \) in Theorem 3, then the inequalities (2.4) become the inequalities (1.2) of Theorem 1.

Using Lemma 1, we can obtain the following fractional integral inequality for \( s \)-convex in the second sense:

**Theorem 4.** Let \( f : [a, b] \subset [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \) such that \( f' \in L[a, b] \). If \( |f'|^q \) is \( s \)-convex in the second sense on \([a, b]\) for
some fixed $s \in (0, 1)$ and $q \geq 1$, then the following inequality for fractional integrals holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a) \right] \right|$$

(2.5)

$$\leq \frac{b - a}{2} \left[ \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \right] \beta \left( \frac{1}{2^\alpha}; s + 1, \alpha + 1 \right) - \beta \left( \frac{1}{2}; \alpha + 1, s + 1 \right) + \frac{2^{\alpha + s} - 1}{(\alpha + s + 1) 2^\alpha s} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}.$$

Proof. Suppose that $q = 1$. From Lemma [1] and using the properties of modulus, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a) \right] \right|$$

(2.6)

$$\leq \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha| |f'(ta + (1 - t)b)| \, dt.$$

Since $|f'|$ is $s$-convex on $[a, b]$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a) \right] \right|$$

\leq \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha| \left[ t^s |f'(a)| + (1 - t)^s |f'(b)| \right] \, dt

= \frac{b - a}{2} \left\{ \int_0^{\frac{1}{2}} [(1 - t)^\alpha - t^\alpha] \left[ t^s |f'(a)| + (1 - t)^s |f'(b)| \right] \, dt 

\quad + \int_{\frac{1}{2}}^1 [t^\alpha - (1 - t)^\alpha] \left[ t^s |f'(a)| + (1 - t)^s |f'(b)| \right] \, dt \right\}

\leq \frac{b - a}{2} \left\{ |f'(a)| \int_0^{\frac{1}{2}} t^s (1 - t)^\alpha \, dt - |f'(a)| \int_0^{\frac{1}{2}} t^{s + \alpha} \, dt 

\quad + |f'(b)| \int_0^{\frac{1}{2}} (1 - t)^{s + \alpha} \, dt - |f'(b)| \int_0^{\frac{1}{2}} (1 - t)^s t^\alpha \, dt 

\quad + |f'(a)| \int_{\frac{1}{2}}^1 t^{s + \alpha} \, dt - |f'(a)| \int_{\frac{1}{2}}^1 t^s (1 - t)^\alpha \, dt 

\quad + |f'(b)| \int_{\frac{1}{2}}^1 (1 - t)^s t^\alpha \, dt - |f'(b)| \int_{\frac{1}{2}}^1 (1 - t)^{s + \alpha} \, dt \right\}.
We obtain
\[\int_0^\frac{1}{2} t^s (1 - t)^\alpha dt = \int_\frac{1}{2}^1 (1 - t)^s t^\alpha dt = \beta \left(\frac{1}{2}; s + 1, \alpha + 1\right),\]
\[\int_0^\frac{1}{2} (1 - t)^s t^\alpha dt = \int_\frac{1}{2}^1 t^s (1 - t)^\alpha dt = \beta \left(\frac{1}{2}; \alpha + 1, s + 1\right),\]
\[\int_0^\frac{1}{2} t^{s+\alpha} dt = \int_\frac{1}{2}^1 (1 - t)^{s+\alpha} dt = \frac{1}{2^{s+\alpha+1} (s + \alpha + 1)}\]
and
\[\int_0^\frac{1}{2} (1 - t)^{s+\alpha} dt = \int_\frac{1}{2}^1 t^{s+\alpha} dt = \frac{1}{(s + \alpha + 1)} - \frac{1}{2^{s+\alpha+1} (s + \alpha + 1)}\]

We obtain
\[\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2 (b - a)^\alpha} \left[ J_{a+} f(b) + J_{b-} f(a) \right] \right|\]
\[\leq \frac{b - a}{2} \left[ |f'(a)| + |f'(b)| \right] \left\{ \beta \left(\frac{1}{2}; s + 1, \alpha + 1\right) - \beta \left(\frac{1}{2}; \alpha + 1, s + 1\right) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1) 2^{\alpha+s}} \right\}\]
which completes the proof for this case. Suppose now that \(q > 1\). Since \(|f'|^q\) is \(s\)-convex on \([a, b]\), we know that for every \(t \in [0,1]\)
\[(2.8)\]
\[|f'(ta + (1 - t)b)|^q \leq t^s |f'(a)|^q + (1 - t)^s |f'(b)|^q,\]
so using well known Hölder’s inequality (see for example [?]) for \(\frac{1}{q} + \frac{1}{q'} = 1, (q > 1)\) and (2.8) in (2.0), we have successively
\[\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2 (b - a)^\alpha} \left[ J_{a+} f(b) + J_{b-} f(a) \right] \right|\]
\[\leq \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha| |f'(ta + (1 - t)b)| dt\]
\[= \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha|^{1 - \frac{s}{q}} |(1 - t)^\alpha - t^\alpha|^{\frac{s}{q}} |f'(ta + (1 - t)b)| dt\]
\[\leq \frac{b - a}{2} \left( \int_0^1 |(1 - t)^\alpha - t^\alpha| dt \right)^{\frac{s}{q}} \left( \int_0^1 |(1 - t)^\alpha - t^\alpha| |f'(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}}\]
\[\leq \frac{b - a}{2} \left[ \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \right]^{\frac{s}{q}} \times \left\{ \beta \left(\frac{1}{2}; s + 1, \alpha + 1\right) - \beta \left(\frac{1}{2}; \alpha + 1, s + 1\right) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1) 2^{\alpha+s}} \right\} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}\]
where we use the fact that
\[\int_0^1 |(1 - t)^\alpha - t^\alpha| dt = \int_0^\frac{1}{2} |(1 - t)^\alpha - t^\alpha| dt + \int_\frac{1}{2}^1 |t^\alpha - (1 - t)^\alpha| dt = \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right)\]
which completes the proof. □
Remark 2. If we take $\alpha = 1$ in Theorem 1, then the inequality (2.3) of Theorem 3 becomes the inequality (1.3) of Theorem 2.

2.2. For $m$-convex functions. We start with the following theorem:

**Theorem 5.** Let $f : [0, \infty] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If $f$ is $m$-convex mapping on $[a, b]$, then the following inequalities for fractional integral with $\alpha > 0$ and $m \in (0, 1]$ hold:

\[
\tag{2.9}
\frac{2}{\Gamma(\alpha + 1)} f \left( \frac{m(a + b)}{2} \right) \leq \frac{1}{(b - a)^{\alpha}} \int_{a}^{b} \frac{f(m(ta + m(1 - t)b + m(1 - t)a + mb))}{(\alpha + 1)} dt
\]

**Proof.** Since $f$ is $m$-convex functions, we have

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

and if we choose $t = \frac{1}{2}$ we get

\[
f \left( \frac{1}{2} (x + my) \right) \leq \frac{f(x) + mf(y)}{2}.
\]

Now, let $x = mta + m(1 - t)b$ and $y = (1 - t)a + tb$ with $t \in [0, 1]$. Then we get

\[
f \left( \frac{1}{2} m(a + b) \right) \leq \frac{f(mta + m(1 - t)b) + mf((1 - t)a + mb)}{2}.
\]

Multiplying both sides of (2.10) by $t^{\alpha - 1}$, then integrating the resulting inequality with respect to $t$ over $[0, 1]$, we obtain

\[
f \left( \frac{1}{2} m(a + b) \right) \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{1} t^{\alpha - 1} dt \leq \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{1} t^{\alpha - 1} f(mta + m(1 - t)b) dt + \frac{m}{2} \int_{0}^{1} t^{\alpha - 1} f((1 - t)a + tb) dt
\]

By the $m$-convexity of $f$, we also have

\[
\frac{1}{\alpha} f \left( \frac{1}{2} m(a + b) \right) \leq \frac{1}{\alpha} \left[ \int_{m(a - b)}^{u - mb} \frac{1}{mb} \int_{ma - mb}^{u - mb} f(u) \frac{du}{m(a - b)} + \frac{m}{2} \int_{a}^{b} \frac{(v - a)}{b - a} f(v) \frac{dv}{b - a} \right]
\]

which the first inequality is proved.
for all $t \in [0, 1]$. Multiplying both sides of above inequality by $t^{\alpha-1}$ and integrating over $t \in [0, 1]$, we get

$$\frac{1}{(mb - ma)^\alpha} \int_{ma}^{mb} (mb - u)^{\alpha-1} f(u) du + \frac{m}{(b-a)^\alpha} \int_{a}^{b} (v - a)^{\alpha-1} f(v) dv \leq \frac{f(ma) + m^2 f\left(\frac{b}{m}\right)}{(\alpha + 1)} + m \frac{f(a) + f(b)}{\alpha(\alpha + 1)}$$

which this gives the second part of (2.9).

**Corollary 1.** Under the conditions in Theorem 5 with $\alpha = 1$, then the following inequality hold:

$$f\left(\frac{m(a + b)}{2}\right) \leq \frac{1}{(b-a)} \int_{a}^{b} \frac{f(mx) + mf(x)}{2} dx \leq \frac{1}{2} \left[ \frac{f(ma) + m^2 f\left(\frac{b}{m}\right)}{2} + m \frac{f(a) + f(b)}{2} \right].$$

**Remark 3.** If we take $m = 1$ in Corollary 1, then the inequalities (2.11) become the inequalities (1.1).

**Theorem 6.** Let $f : [0, \infty] \to \mathbb{R}$ be $m$-convex functions with $m \in (0, 1]$, $0 \leq a < b$ and $f \in L_1[a, b]$. $F(x, y)_{(t)} : [0, 1] \to \mathbb{R}$ are defined as the following:

$$F(x, y)_{(t)} = \frac{1}{2} [f(tx + m(1 - t)y) + f((1-t)x + mt y)].$$

Then, we have

$$\frac{1}{(b-a)^\alpha} \int_{a}^{b} \frac{(b-u)^{\alpha-1} F(u, \frac{a+b}{2})}{(\frac{\Gamma(\alpha)}{2(b-a)^\alpha} J_{a^+}^{\alpha} f(b) + m \frac{f(a + b)}{2})} du \leq \frac{\Gamma(\alpha)}{2(b-a)^\alpha} J_{a^+}^{\alpha} f(b) + m \frac{f(a + b)}{2}$$

for all $t \in [0, 1]$.

**Proof.** Since $f$ and $g$ are $m$-convex functions, we have

$$F(x, y)_{(t)} \leq \frac{1}{2} [tf(x) + m(1-t)f(y) + (1-t)f(x) + mt f(y)]$$

and so,

$$f\left(\frac{a+b}{2}\right)_{(t)} \leq \frac{1}{2} \left[ f(x) + mf\left(\frac{a+b}{2}\right) \right].$$

If we choose $x = ta + (1-t)b$, we have

$$F\left( ta + (1-t)b, \frac{a+b}{2}\right)_{(t)} \leq \frac{1}{2} \left[ f(ta + (1-t)b) + mf\left(\frac{a+b}{2}\right) \right].$$
Thus multiplying both sides of (2.12) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain
\[
\int_0^1 t^{\alpha-1} F \left( (t-1)b, \frac{a+b}{2} \right) dt \leq \frac{1}{2} \left[ \int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt + \int_0^1 t^{\alpha-1} m \left( \frac{a+b}{2} \right) dt \right].
\]
Thus, if we use the change of the variable $u = ta + (1-t)b$, $t \in [0,1]$, then have the conclusion. □

References

[1] M. Alomari and M. Darus, On the Hadamard's inequality for log-convex functions on the coordinates, Journal of Inequalities and Applications, vol. 2009, Article ID 283147, 13 pages, 2009.
[2] A.G. Azpeitia, Convex functions and the Hadamard inequality, Rev. Colombiana Math., 28 (1994), 7-12.
[3] M.K. Bakula, M.E. Özdemir, J. Pečarić, Hadamard type inequalities for $m-$convex and $(\alpha,m)$-convex functions, J. Ineq. Pure and Appl. Math., 9(4) (2008), Art. 96.
[4] M. K. Bakula and J. Pečarić, Note on some Hadamard-type inequalities, Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.
[5] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[6] S. S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. lett., 11(5) (1998), 91-95.
[7] S.S. Dragomir, On some new inequalities of Hermite-Hadamard type for $m-$convex functions, Tamkang J. Math., 3(1) (2002).
[8] S. S. Dragomir and S. Fitzpatrick, The Hadamard’s inequality for $s$-convex functions in the second sense, Demonstratio Math. 32(4), (1999), 687-696.
[9] P. M. Gill, C. E. M. Pearce, and J. Pečarić, Hadamard’s inequality for $r$-convex functions, Journal of Mathematical Analysis and Applications, vol. 215, no. 2, pp. 461–470, 1997.
[10] H. Hudzik and L. Maligranda, Some remarks on $s$–convex functions, Aequationes Math. 48 (1994), 100-111.
[11] U.S. Kirmaci, M.K. Bakula, M.E. Özdemir, J. Pečarić, Hadamard-type inequalities for $s$-convex functions, Appl. Math. and Comp., 193 (2007), 26-35.
[12] G.Toader, Some generalizations of the convexity, Proceedings of The Colloquium On Approximation And Optimization, Univ. Cluj-Napoca, Cluj-Napoca,1985, 329-338.
[13] M. E. Özdemir, M. Avci, and E. Set, On some inequalities of Hermite-Hadamard type via $m$-convexity, Applied Mathematics Letters, vol. 23, no. 9, pp. 1065–1070, 2010.
[14] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992.
[15] E. Set, M. E. Özdemir, and S. S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, Journal of Inequalities and Applications, Article ID 148102, 9 pages, 2010.
[16] E. Set, M. E. Özdemir, and S. S. Dragomir, On Hadamard-Type inequalities involving several kinds of convexity, Journal of Inequalities and Applications, Article ID 286845, 12 pages, 2010.
[17] G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi and F. Moftakharzadeh, Montogomery identities for fractional integrals and related fractional inequalities, J. Ineq. Pure and Appl. Math., 10(4) (2009), Art. 97.
[18] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.
[19] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Sciene, 9(4) (2010), 493-497.
[20] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), 51-58.
[21] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.
[22] Z. Dahmani, L. Tabharit, S. Taf, *New generalizations of Gruss inequality usin Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.

[23] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.

[24] S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, p.2.

[25] I. Podlubni, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[26] M.Z. Sarıkaya and H. Ögünmez, *On new inequalities via Riemann-Liouville fractional integration*, arXiv:1005.1167v1, submitted.

[27] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Boşak, *Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, accepted.

*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY*

E-mail address: erhanset@yahoo.com

E-mail address: sarikayamz@gmail.com

*Ataturk University, K. K. Education Faculty, Department of Mathematics, 25640, Kampus, Erzurum, Turkey*

E-mail address: emos@atauni.edu.tr

*Department of Mathematics, Faculty of Science and Arts, Kahramanmaraş Sütçü İmam University, Kahramanmaraş-TURKEY*

E-mail address: hyildir@ksu.edu.tr