A Kato type criterion for the zero viscosity limit of the incompressible Navier-Stokes flows with vortex sheets data

Franck Sueur

Abstract. There are a few examples of solutions to the incompressible Euler equations which are piecewise smooth with a discontinuity of the tangential velocity across a hypersurface evolving in time: the so-called vortex sheets. An important open problem is to determine whether or not these solutions can be obtained as zero viscosity limits of the incompressible Navier-Stokes solutions in the energy space. In this paper we establish a couple of sufficient conditions similar to the one obtained by Kato in [T. Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. Seminar on nonlinear partial differential equations, 85-98, Math. Sci. Res. Inst. Publ., 2, 1984] for the convergence of Leray solutions to the Navier-Stokes equations in a bounded domain with no-slip condition toward smooth solutions to the Euler equation.

2010 Mathematics Subject Classification: 35B25, 35B30, 76D05, 76D10.

Key words and phrases: Vanishing viscosity, vortex sheets, boundary layer theory.

1. Introduction

In fluid mechanics a vortex sheet is a hypersurface across which the tangential component of the flow velocity is discontinuous while the normal component is continuous. Because of the discontinuity in the tangential velocity the vorticity is infinite on the hypersurface hence the terminology. Examples of solutions to the incompressible Euler equations for which an initial vortex sheet evolves in a smooth hypersurface for positive times are rare because of the Kelvin-Helmholtz instability. Let us mention the examples provided by [14 22] for analytic data and by [3 13] for plane-parallel flows. Let us also refer here to the survey [1] for more on vortex sheets.

Institut de Mathématiques de Bordeaux, UMR CNRS 5251, Université de Bordeaux, Franck.Sueur@math.u-bordeaux.fr
An important open problem is to determine whether or not these solutions can be obtained as zero viscosity limits of solutions to the incompressible Navier-Stokes equations. In particular, Bardos, Titi and Wiedemann have shown in [5] that the zero-viscosity limit can serve as a selection principle for the Euler equations in the case of initial data for which there exist non-unique weak solutions to the incompressible Euler equations satisfying the weak energy inequality, including a vortex sheet solution with plane-parallel symmetry. They prove that the latter is the zero-viscosity limit, in a weak sense, of any sequence of Leray-Hopf solutions to the Navier-Stokes solutions.

In general the difficulty to establish the zero-viscosity limit towards vortex sheets is that the fluid tangential velocity has $O(1)$ variation in a layer containing the hypersurface, similarly to the boundary layer associated with the no-slip condition on a fixed wall.

For the latter a result by Kato [15] establishes the convergence of Leray solutions to the Navier-Stokes equations to a smooth Euler solution in the energy space provided that the energy dissipation rate of the viscous flow in a boundary layer of width proportional to the viscosity vanishes, with an appropriate condition for the initial data. Since then, this result was extended in various ways, see [4, 8, 9, 16, 17, 18, 23]. Let us also mention here the recent survey [19] for more about the zero viscosity limit of the incompressible Navier-Stokes flows.

In this paper we extend Kato’s result to the case of a vortex sheet by establishing a couple of sufficient conditions for the convergence of Leray solutions to the Navier-Stokes in the full space to a vortex sheet. These two conditions: Condition (8) and Condition (9) below, both involve $L^2$ norms of derivatives of the fluid velocity on a boundary layer of width proportional to the viscosity. Indeed Condition (8) involves the energy dissipation rate of the viscous flow and is therefore very similar to Kato’s condition. On the other hand Condition (9) involves the difference of derivatives of the fluid velocity between one side of the hypersurface and the other.

2. Setting

Let $d = 2$ or $3$. We consider $T > 0$ and $\Sigma$ a smooth compact connected hypersurface of $[0, T] \times \mathbb{R}^d$, given as the zero level set of a signed smooth function $\varphi(t, x)$, such that, in a small neighborhood of $\Sigma$,

$$|\varphi(t, x)| = \text{dist}(x, \Sigma_t),$$

where, for every time $t$ in $[0, T]$, we denote by $\Sigma_t \subset \mathbb{R}^d$ the projection of $\Sigma$ on $\{t\} \times \mathbb{R}^d$. We assume that the two connected components $\Omega_{t, \pm}$ of $\mathbb{R}^d \setminus \Sigma_t$ are given by

$$\Omega_{t, \pm} := \{x \in \mathbb{R}^d / \pm \varphi(t, x) > 0\}.$$

We denote by $L^2_\sigma(\mathbb{R}^d)$ the closure in $L^2(\mathbb{R}^d)$ of the space $C^\infty([0, T]; \mathbb{R}^d)$ of smooth divergence free vector fields and we will use the notation $C_{w}([0, T]; L^2_\sigma(\mathbb{R}^d))$ for vector fields depending on time continuously on $[0, T]$ with respect to the weak topology of $L^2(\mathbb{R}^d)$. 
Let \( u^E \) in \( C_w([0, T]; L^2_\sigma(\mathbb{R}^d)) \) such that for every time \( t \in [0, T] \), the restrictions of \( u^E_{\pm}(t, \cdot) \) to \( \Omega_{t,\pm} \) admit some smooth extensions to \( \Omega_{t,\pm} \), with traces \( u^E_{\pm}(t, \cdot) \) on \( \Sigma_t \) satisfying
\[
\partial_t \varphi + u^E_+ \cdot \nabla \varphi = \partial_t \varphi + u^E_- \cdot \nabla \varphi = 0. \tag{2}
\]
Let us precise that in (2) the notation \( \nabla \) refers to the gradient with respect to the space variables only, and it is the same in the sequel.

We assume that there exists a scalar function \( p^E \) which is, for every time \( t \) in \( [0, T] \), smooth in \( \Omega_{t,\pm} \) up to the boundary and continuous at \( \Sigma_t \), such that, in \( \Omega_{t,\pm} \),
\[
\partial_t u^E + \text{div}(u^E \otimes u^E) + \nabla p^E = 0.
\]
Then for every time \( t \) in \( [0, T] \),
\[
\|u^E(t)\|_{L^2(\mathbb{R}^d)} = \|u^E_0\|_{L^2(\mathbb{R}^d)}. \tag{3}
\]
We say that \( u^E \) is a vortex sheet associated with \( \Sigma \). Observe in particular that the tangential component of \( u^E \) can be discontinuous across \( \Sigma \) while the normal component is continuous, as a consequence of (1) and (2). Moreover \( u^E \) is a weak solution to the incompressible Euler equations in \( \mathbb{R}^d \).

**Remark 1.** Observe that we do not use any subscript \( E \) for \( \Sigma \) and \( \varphi \); the reason is that there is no counterpart for the Navier-Stokes equation in the sequel so that there is no ambiguity: \( \Sigma \) and \( \varphi \) are always associated with the Euler solution.

A natural question is whether or not a vortex sheet is a limit when \( \varepsilon \to 0^+ \) of solutions \( u^\varepsilon \) to the incompressible Navier-Stokes equations.
\[
\partial_t u^\varepsilon + \text{div}(u^\varepsilon \otimes u^\varepsilon) + \nabla p^\varepsilon = \varepsilon \Delta u^\varepsilon, \quad \text{div} u^\varepsilon = 0. \tag{4}
\]
Here we will consider weak solution to (4) in the sense of Leray. We recall that, for an initial data \( u_0 \) in \( L^2_\sigma(\mathbb{R}^d) \),
\[
u^\varepsilon \in C_w([0, T]; L^2_\sigma(\mathbb{R}^d)) \cap L^2((0, T); H^1(\mathbb{R}^d))
\]
is a weak Leray solution to (1) associated with \( u_0 \) if it satisfies, for any \( \phi \) in \( C_\infty([0, T], \mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} u_0 \cdot \phi(0, \cdot) \, dx \, dt - \int_{\mathbb{R}^d} u^\varepsilon(T, \cdot) \cdot \phi(T, \cdot) \, dx \, dt + \int_0^T \int_{\mathbb{R}^d} u^\varepsilon \cdot \partial_t \phi \, dx \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^d} \left( u^\varepsilon \cdot \nabla \phi \right) \cdot u^\varepsilon \, dx \, dt - \varepsilon \int_0^T \int_{\mathbb{R}^d} \nabla u^\varepsilon : \nabla \phi \, dx \, dt = 0,
\]
and the strong energy inequality: for almost every \( 0 \leq \tau < t \leq T \),
\[
\frac{1}{2} \|u^\varepsilon(t)\|_{L^2(\mathbb{R}^d)}^2 + \varepsilon \int_{(\tau, t) \times \mathbb{R}^d} |\nabla u^\varepsilon|^2 \, dx \, dt \leq \frac{1}{2} \|u^\varepsilon(\tau)\|_{L^2(\mathbb{R}^d)}^2. \tag{6}
\]
3. Main result

In this section we state the main result of this paper: Theorem 1 below. First we introduce some notations which appear in the statement of this result. Let us recall that we consider $d = 2$ or $3$, $T > 0$ and $\Sigma$ a smooth compact connected hypersurface of $(0, T) \times \mathbb{R}^d$. For any $c > 0$, for every time $t$ in $[0, T]$, we set

$$V_{t,c} := \{x \in \mathbb{R}^d : \text{dist}(x, \Sigma_t) < c\}.$$  

For every $t$ in $[0, T]$, for $c > 0$ small enough, the reflection across $\Sigma_t$ is well-defined on the set of the functions $f$ whose restrictions to $\Omega_t, \pm$ admit smooth extensions to $\overline{\Omega}_{t, \pm}$. This map associates with $f$ a function $\tilde{f}$ defined as follows. If for some time $t$, we denote by $s$ the tangential coordinates so that $(s, \phi)$ are local coordinates then the function $\tilde{f}$ is given explicitly by $\tilde{f}(t, s, \phi) = f(t, s, -\phi)$.

With $f$ we associate the function

$$[f] := f - \tilde{f},$$

which is loosely speaking, at $(t, x)$, the jump of $f$ across $\Sigma_t$ at distance $2|\phi(t, x)|$.

We can now state the main result of the paper.

**Theorem 1.** Let $d = 2$ or $3$, $T > 0$ and $\Sigma$ a smooth compact connected hypersurface of $(0, T) \times \mathbb{R}^d$. Let $u^E$ a vortex sheet associated with $\Sigma$. Let $(u_0^\varepsilon)$ a family, indexed by $\varepsilon \in (0, 1)$, in $L^2_\sigma(\mathbb{R}^d)$ converging to $u_0^E$. For every $\varepsilon$ in $(0, 1)$, we consider $u^\varepsilon$ a weak Leray solution associated with $u_0^\varepsilon$. Assume that there exists $c > 0$ such that

$$\varepsilon \int_{(0,T)} \int_{V_{t,c\varepsilon}} |\nabla u^\varepsilon|^2 \, dx \, dt \to 0, \text{ when } \varepsilon \to 0,$$

and that

$$\int_{(0,T)} \int_{V_{t,c\varepsilon}\cap \Omega_{t,+}} |[\nabla u^\varepsilon]|^2 \, dx \, dt \to 0, \text{ when } \varepsilon \to 0.$$

Then

$$\sup_{(0,T)} \int_{\mathbb{R}^2} |u^\varepsilon - u^E|^2 \, dx \to 0 \text{ when } \varepsilon \to 0,$$

The proof of Theorem 1 is displayed in three parts corresponding respectively to Sections 5, 6 and 7.

4. A few comments

- Condition (8) is similar to Kato’s original condition for the case of boundary layers attached to a fixed rigid wall, cf. [15]. On the other hand Condition (9), at first look, seems quite a strong extra assumption since there is no factor $\varepsilon$ in front of the integral. However such an assumption is not that bad because of the regularizing effect of the Navier-Stokes equations. Indeed if the solution $u^\varepsilon$ is smooth then for every time $t$ the trace of $[\nabla u^\varepsilon]$ on $\Sigma_t$ is well-defined, vanishes
and Hardy’s inequality can be applied, so that (9) follows from the following condition:
\[ \varepsilon^2 \int_{(0,T)} \int_{\mathcal{V}_{1,\varepsilon}} |\Delta u_\varepsilon|^2 \, dxdt \to 0 \quad \text{when} \quad \varepsilon \to 0; \quad (11) \]
a condition which scales with the energy bound deduced from (6) and Condition (8).

- For shear flows, the Navier-Stokes solutions have variations in $\varphi/\sqrt{\varepsilon}$ so that Condition (8) and Condition (9) are of course satisfied. Indeed our proof of Theorem 1, following Kato’s approach in [15], involves a fake layer with variations in $\varphi/\varepsilon$, see (33) and (34). For more general flows, ansatz with variations in $\varphi/\sqrt{\varepsilon}$ lead to Prandtl-type equations, see [6] and [7].

- In [20] we consider the motion of a rigid body in an incompressible fluid occupying the complementary set in the space, with a no-slip condition at the interface, and we prove that a Kato type condition implies the convergence of both fluid and body velocities. In this paper we extend these results to the case of a vortex sheet, that is to a fluid interface with a more evolved dynamics. Indeed boundary layers associated with the no-slip condition on a fixed wall can be also viewed as vortex sheets with fixed support, see for instance [10, 18]. Loosely speaking Theorem 1 seems to indicate that despite that the dynamics of a vortex sheet is more subtle, the scale which is of interest for the inviscid limit is perhaps not worse than in the case of no-slip boundary layers.

- In the case of the convergence of solutions to the incompressible Navier-Stokes solutions in a bounded domain with no-slip condition to smooth solutions to the incompressible Euler solution in the zero viscosity limit, in addition to Kato’s criterion, another criterion is given by Bardos and Titi in [4, Section 4.4], see also [18, Section 8 and 10]. It involves only the behaviour of the Navier-Stokes solutions on the boundary in the zero viscosity limit and the proof relies on Kato’s construction. This criterion can be adapted to the present setting as a condition on the interface $\Sigma$ by substituting Lemma 2 below instead of Kato’s construction. Indeed, by a direct energy estimate it is not difficult to see that the convergence (10) holds if and only
\[ \varepsilon \int_\Sigma \det(\nabla \varphi, \text{curl} \, u_\varepsilon, [u^E]) \, d\sigma \to 0, \quad \text{when} \quad \varepsilon \to 0, \quad (12) \]
where $\sigma$ is the surface measure on $\Sigma$. (In particular, to prove the direct part, we use that
\[ \varepsilon \int_{(0,T) \times \mathbb{R}^d} |\nabla u_\varepsilon|^2 \, dxdt \to 0, \quad \text{when} \quad \varepsilon \to 0, \]
as a consequence of (3), (6), (10) and of the convergence of the initial data). Moreover, under the assumptions of Theorem 1 for any $\Psi$ in $C_0^\infty(\Sigma; \mathbb{R}^d)$ tangent
to $\Sigma$,
\[ \varepsilon \int_\Sigma \text{curl } u^\varepsilon \cdot \Psi \to 0 \text{ when } \varepsilon \to 0. \]  
(13)

Since the proof of $\text{(13)}$ can be easily adapted from [4, Section 4.4] and from the analysis performed in the course of the proof of Theorem $\text{H}$ (in particular by using Lemma $\text{II}$ below with $\Psi$ instead of $[u^E]$ and following the treatment done in Section $\text{VII}$ of the terms denoted by $R_{(iii)}$, $R_{(iv)}$ and $R_{(v)}$ in Lemma $\text{H}$), the details are left to the reader.

- Theorem $\text{H}$ proves that the conditions $\text{(8)}$ and $\text{(9)}$ are sufficient for the convergence $\text{(10)}$. The converse statement is an open question. Another open question is whether or not the convergence $\text{(10)}$ implies the interface condition $\text{(13)}$. To contrast with the classical setting, let us recall that Kato’s condition and Bardos and Titi’s condition are proved to be sufficient and necessary, respectively in [15] and [4, Section 4.4].

- As mentioned in Section $\text{I}$, in the case of the convergence of solutions to the incompressible Navier-Stokes solutions in a bounded domain with no-slip condition to smooth solutions to the incompressible Euler solution in the zero viscosity limit, some other variants of Kato’s criterion have been found, see [8, 9, 16, 17, 18, 23]. Similar extensions of Theorem $\text{H}$ can be obtained with minor modifications.

- We hope to extend our analysis to the case of compressible flows. Let us recall that Coulombel and Secchi prove, in [11] and [12], the existence and uniqueness of supersonic compressible vortex sheets in two space dimensions. Therefore it would be interesting to obtain some sufficient conditions for the convergence of solutions to the compressible Navier-Stokes equations to these solutions in the zero-viscosity limit. In this direction let us mention that the case of the inviscid limit of the compressible Navier-Stokes equations in a bounded domain, with the no-slip condition (and also in the case of the Navier slip-with-friction conditions), was tackled in [21].

5. Energy estimate with an abstract corrector

Following Kato’s approach, see [15], we first observe that a corrector may help to deduce a $L^2$ stability estimate. For sake of notations, we write temporarily $u_0$ rather than $u^\varepsilon_0$ and similarly, for the initial data, $u$ rather than $u^\varepsilon$. Moreover the estimate involves an abstract corrector $v$ which will be chosen dependent on $\varepsilon$ in the next sections.
Lemma 1. If \( v \) is such that \( u^E + v \) can be taken as a test function \( \phi \) in (5) then
\[
\frac{1}{2} \| u(t, \cdot) - u^E(t, \cdot) \|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{2} \| u_0 - u_0^E \|_{L^2(\mathbb{R}^d)}^2 + (u, v)_{L^2(\mathbb{R}^d)} - \left( u_0, v \Big|_{t=0} \right)_{L^2(\mathbb{R}^d)} - \int_0^t R(s) ds,
\]
where
\[
R(t) = R_{(i)}(t) + \ldots + R_{(iv)}(t),
\]
with
\[
R_{(i)} := \sum_{\pm} \int_{\Omega_{t, \pm}} ((u - u^E) \cdot \nabla u^E) \cdot (u - u^E) dx,
\]
\[
R_{(ii)} := -\varepsilon \sum_{\pm} \int_{\Omega_{t, \pm}} \nabla u : \nabla u^E dx,
\]
\[
R_{(iii)} := \sum_{\pm} \int_{\Omega_{t, \pm}} u \cdot (\partial_t v + (u^E \cdot \nabla v)) dx,
\]
\[
R_{(iv)} := -\sum_{\pm} \int_{\Omega_{t, \pm}} v \cdot ((u - u^E) \cdot \nabla u) dx,
\]
and
\[
R_{(v)} := -\varepsilon \sum_{\pm} \int_{\Omega_{t, \pm}} \nabla u : \nabla v dx.
\]

Proof. For any \( t \in [0, T] \), we have, thanks to (3) and (6),
\[
\| u(t, \cdot) - u^E(t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq \| u_0 \|_{L^2(\mathbb{R}^d)}^2 + \| u_0^E \|_{L^2(\mathbb{R}^d)}^2 - 2(u, u^E)_{L^2(\mathbb{R}^d)}(t).
\]
By assumption, we can take \( \phi = u^E + v \) as a test function in (5) so that
\[
(u, u^E + v)_{L^2(\mathbb{R}^d)}(t) = \left( u_0, u_0^E + v \Big|_{t=0} \right)_{L^2(\mathbb{R}^d)} + \int_0^t R(s) ds.
\]
where
\[
R(t) := \int_{\mathbb{R}^d} u \cdot \partial_t (u^E + v) dx + \int_{\mathbb{R}^d} (u \cdot \nabla (u^E + v) \cdot u dx
\]
\[
-\varepsilon \int_{\mathbb{R}^d} \nabla u : \nabla (u^E + v) dx.
\]

Combining (16) with (17) we obtain (14) with \( R(t) \) given by (18). Thus it only remains to prove that (18) can be translated into (15). To do so let us split \( R(t) \) into
\[
R(t) = R_{E, +}(t) + R_{E, -}(t) + R_{F, +}(t) + R_{F, -}(t),
\]
where
\[ R_{E,+} := \int_{\Omega_{t,+}} u \cdot \partial_t u^E dx + \int_{\Omega_{t,+}} (u \cdot \nabla u^E) \cdot u dx - \varepsilon \int_{\Omega_{t,+}} \nabla u : \nabla u^E dx, \]
\[ R_{E,-} := \int_{\Omega_{t,-}} u \cdot \partial_t u^E dx + \int_{\Omega_{t,-}} (u \cdot \nabla u^E) \cdot u dx - \varepsilon \int_{\Omega_{t,-}} \nabla u : \nabla u^E dx. \]

We first use that
\[ \int_{\Omega_{t,\pm}} (u \cdot \nabla u^E) \cdot u dx = \int_{\Omega_{t,\pm}} (u^E \cdot \nabla u^E) \cdot u dx + \int_{\Omega_{t,\pm}} ((u - u^E) \cdot \nabla u^E) \cdot u dx \]

and the continuity of the normal component of \( u - u^E \) at the interface to deduce that
\[ \sum_{\pm} \int_{\Omega_{t,\pm}} ((u - u^E) \cdot \nabla u^E) \cdot u^E dx = 0. \]

Therefore
\[ \sum_{\pm} \int_{\Omega_{t,\pm}} (u \cdot \nabla u^E) \cdot u dx = \sum_{\pm} \int_{\Omega_{t,\pm}} (u^E \cdot \nabla u^E) \cdot u dx \tag{20} \]
\[ + \sum_{\pm} \int_{\Omega_{t,\pm}} ((u - u^E) \cdot \nabla u^E) \cdot (u - u^E) dx \]

Moreover, since \( u^E \) satisfies the Euler equation in a strong sense in both \( \Omega_{t,\pm} \) and \( u \) is divergence free and continuous at the interface, we obtain, upon an integration by parts, the following identity:
\[ \sum_{\pm} \int_{\Omega_{t,\pm}} u \cdot \partial_t u^E dx = - \sum_{\pm} \int_{\Omega_{t,\pm}} (u^E \cdot \nabla u^E) \cdot u dx. \tag{21} \]

Adding \( R_{E,+} \) and \( R_{E,-} \) and using \( (20) \) and \( (21) \) we arrive at
\[ \sum_{\pm} R_{E,\pm} = \sum_{\pm} \int_{\Omega_{t,\pm}} ((u - u^E) \cdot \nabla u^E) \cdot (u - u^E) dx - \varepsilon \int_{\Omega_{t,\pm}} \nabla u : \nabla u^E dx. \tag{22} \]

On the other hand,
\[ R_{F,\pm} = \int_{\Omega_{t,\pm}} u \cdot (\partial_t v + (u^E \cdot \nabla v)) dx + \int_{\Omega_{t,\pm}} u \cdot ((u - u^E) \cdot \nabla v) dx \]
\[ - \varepsilon \int_{\Omega_{t,\pm}} \nabla u : \nabla v dx. \tag{23} \]

Moreover using once again the continuity of the normal component of \( u - u^E \) at the interface we arrive at
\[ \sum_{\pm} \int_{\Omega_{t,\pm}} u \cdot ((u - u^E) \cdot \nabla v) dx = - \sum_{\pm} \int_{\Omega_{t,\pm}} v \cdot ((u - u^E) \cdot \nabla u) dx. \tag{24} \]

Thus combining \( (19) \), \( (22) \), \( (23) \) and \( (24) \) we arrive at \( (15) \) and the proof of Lemma \( \ref{lemma1} \) is completed.
6. Construction of an almost odd transition layer

In the following result, we make use of the Landau notations $o(1)$ and $O(1)$ for quantities respectively converging to 0 and bounded with respect to the limit $\varepsilon \to 0^+$. Let $c > 0$ such that \[8\] and \[9\] are satisfied. Recall, for a function $f$, the notation $\tilde{f}$ in the beginning of Section 3.

Lemma 2. There exists a family $(v^\varepsilon)$, indexed by $\varepsilon$ in $(0, 1)$, in $C([0, T]; L^2_\sigma(\mathbb{R}^d))$ with the following properties: for every $\varepsilon$ in $(0, 1)$,

\[
\text{for every } t \in [0, T], \quad \text{supp } v^\varepsilon(t, \cdot) \subset \mathcal{V}_{t, c\varepsilon},
\]

such that

\[
u^\varepsilon = O(1) \text{ in } L^\infty([0, T] \times \mathbb{R}^3),
\]

\[
v^\varepsilon = O(\varepsilon^{\frac{1}{2}}) \text{ in } C([0, T]; L^2(\mathbb{R}^d)),
\]

\[
\varphi v^\varepsilon = O(\varepsilon) \text{ in } L^\infty([0, T] \times \mathbb{R}^3),
\]

\[
\sup_{t \in (0, T)} \|\nabla v^\varepsilon\|_{L^2(\mathcal{V}_{t, c\varepsilon} \cap \Omega_{t, +})} = O(\varepsilon^{-\frac{1}{2}}),
\]

and

\[
u^\varepsilon + \tilde{v}^\varepsilon = O(\varepsilon) \text{ in } L^\infty([0, T] \times \mathbb{R}^3),
\]

\[
\partial_t v^\varepsilon + u^E \cdot \nabla v^\varepsilon = O(\varepsilon^{\frac{1}{2}}) \text{ in } C([0, T]; L^2(\mathbb{R}^d)),
\]

Remark 2. Above we have written separately the estimates \[27\]-\[30\] which are similar to the ones in Kato’s original paper \[15\] and the estimates \[31\] and \[32\] which are two new requirements useful in the case of vortex sheets.

Proof of Lemma 2. Let $\xi : [0, +\infty) \to [0, +\infty)$ be a smooth cut-off function such that $\xi(0) = 0$, $\xi'(0) = 1$ and $\xi(r) = 0$ for $r \geq c$. Recall the notation $[\cdot]$ in \[7\]. Set, for $t \in [0, T]$, $x \in \Omega_{t, \pm}$ and $\varepsilon$ in $(0, 1),$

\[
v^\varepsilon := -\nabla \frac{\sqrt{d}}{2}\xi(\pm \frac{\varphi}{\varepsilon})[u^E \cdot \nabla \varphi]_{|\varphi=0} \quad \text{ if } d = 2,
\]

\[
v^\varepsilon := \text{curl} \left(\frac{\varepsilon}{2}\xi(\pm \frac{\varphi}{\varepsilon})[u^E]_{|\varphi=0} \times \nabla \varphi\right) \quad \text{ if } d = 3.
\]

Then we easily check that the family $(v^\varepsilon)_{\varepsilon \in (0, 1)}$ is in $C([0, T]; L^2_\sigma(\mathbb{R}^d))$ and satisfies \[25\]-\[32\]. (Observe in particular that we use \[2\] to obtain \[32\].)

Let us display here a remark which could be useful to get an insight of the whole strategy, with some anticipation on the rest of the proof of Theorem 1.

Remark 3. Observe that above the $v^\varepsilon$ are constructed such that $[u^E + v^\varepsilon] = 0$ on $\Sigma$ but without any condition on $[\nabla (u^E + v^\varepsilon)]$ despite that it is expected that a nice physical approximation $u^\varepsilon_\Sigma$ of $v^\varepsilon$ should satisfy $[\nabla u^\varepsilon_\Sigma] = 0$ on $\Sigma$. However in the next section we will combine Lemma 7 and Lemma 2 and we will estimate the right hand side of \[14\] without any further integration by parts of the diffusive terms $R_{(ii)}$ and
so that the lack of information regarding $\nabla (u^E + v^\varepsilon)$ at the interface will not be a problem. We therefore spare a degree of freedom which is used in Lemma 2 to insure the almost oddness of the transition layer stated in (31). Such a condition has no reason to be physical but will be crucial in the treatment of the convective term $R_{(iv)}$ in Section 7.

7. End of the proof of Theorem 1

We now go back to the proof of Theorem 1. We apply Lemma 1 with $v^\varepsilon$ instead of $v$ where $(v^\varepsilon)$ is a family as in Lemma 2. Indeed a density argument, (26) and the piecewise smoothness of $v^\varepsilon$ allows us to take $u^\varepsilon + v^\varepsilon$ as a test function $\varphi$ in (5). (We now stop dropping the index $\varepsilon$ of $u^\varepsilon$ but we will keep the notations $R_{(i)}$, . . . , $R_{(v)}$, without any extra index, being understood that these terms depend on $\varepsilon$). We are now going to bound the various terms in the right hand side of (14).

Since the Navier-Stokes initial data $u_0^\varepsilon$ converges to the Euler one in $L^2(\mathbb{R}^d)$ as the viscosity $\varepsilon$ goes to 0, it is bounded, and so is the corresponding Navier-Stokes solution $u^\varepsilon$ for almost every time, according to the energy estimate (6). Therefore, by Cauchy-Schwarz’ inequality and (28), for almost every time,

$$|(u^\varepsilon, v^\varepsilon)_{L^2(\mathbb{R}^d)}| \leq C\varepsilon^\frac{1}{2} \text{ and } |(u_0^\varepsilon, v^\varepsilon)_{t=0})_{L^2(\mathbb{R}^d)}| \leq C\varepsilon^\frac{1}{2}. \quad (35)$$

Let us warn the reader that we will use the same notation $C$ for various constants which may change from line to line, but always independent of $\varepsilon$.

Using that the Euler solution is piecewise smooth, we arrive at

$$|R_{(i)}(t)| \leq C\|u^\varepsilon(t, \cdot) - u^E(t, \cdot)\|^2_{L^2(\mathbb{R}^d)}. \quad (36)$$

By Cauchy-Schwarz’ inequality,

$$|R_{(ii)}| \leq C\varepsilon\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)} \quad (37)$$

Using again Cauchy-Schwarz’ inequality and (32), we arrive at

$$|R_{(iii)}| \leq C\varepsilon^\frac{1}{2}\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}. \quad (38)$$

Let us continue with estimating $R_{(iv)}(t)$, keeping the best for the end. Using again Cauchy-Schwarz’ inequality, (25) and (32), we arrive at

$$|R_{(iv)}(t)| \leq C\varepsilon^\frac{1}{2}\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)}. \quad (39)$$

It remains to deal with $R_{(iv)}(t)$. This is where the treatment is quite different from the one performed in the traditional setting of boundary layers along an impermeable wall. By a change of variable (observe that for $\varepsilon$ small enough, the reflexion across $\Sigma_t$, introduced at the beginning of Section 3 is well-defined on the support of $v^\varepsilon$) and (31),

$$R_{(iv)} = R_{(iv),a} + R_{(iv),b} + R_{(iv),c} + R_{(iv),d},$$

where
with
\[ R_{(iv),a} := - \int_{\Omega_{t,+}} v^\varepsilon \cdot ([u^\varepsilon] \cdot \nabla u^\varepsilon) \, dx, \]
\[ R_{(iv),b} := \int_{\Omega_{t,+}} v^\varepsilon \cdot ([u^E] \cdot \nabla u^\varepsilon) \, dx, \]
\[ R_{(iv),c} := - \int_{\Omega_{t,+}} v^\varepsilon \cdot (\widetilde{u^\varepsilon} - \widetilde{u^E}) \cdot \nabla u^\varepsilon) \, dx, \]
and
\[ R_{(iv),d} := - \int_{\Omega_{t,+}} (v^\varepsilon + \widetilde{v^\varepsilon}) \cdot (\widetilde{u^\varepsilon} - \widetilde{u^E}) \cdot \nabla u^\varepsilon) \, dx. \]

- By (25), (29) and Cauchy-Schwarz’ inequality,
\[ |R_{(iv),a}(t)| \leq \varepsilon \| \varphi^{-1}[u^\varepsilon(t, \cdot)] \|_{L^2(\mathcal{V}_{t,cc} \cap \Omega_{t,+})} \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc} \cap \Omega_{t,+})} \]
Moreover by Hardy’s inequality,
\[ \| \varphi^{-1}[u^\varepsilon(t, \cdot)] \|_{L^2(\mathcal{V}_{t,cc} \cap \Omega_{t,+})} \leq C \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})}, \]
so that
\[ |R_{(iv),a}(t)| \leq C \varepsilon \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})}^2. \] (40)

- By (26), (28) and Cauchy-Schwarz’ inequality,
\[ |R_{(iv),b}(t)| \leq C \varepsilon^{\frac{1}{2}} \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})}. \] (41)

- By (26), (27) and Cauchy-Schwarz’ inequality,
\[ |R_{(iv),(c)}(t)| \leq C \| u^\varepsilon - u^E \|_{L^2(\mathbb{R}^d)} \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc} \cap \Omega_{t,+})}. \] (42)

- Finally, by (25), (31) and Cauchy-Schwarz’ inequality,
\[ |R_{(iv),(d)}(t)| \leq C \varepsilon \| u^\varepsilon - u^E \|_{L^2(\mathbb{R}^d)} \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})}. \] (43)

Gathering (40), (41), (42) and (43) we arrive at
\[ |R_{(iv)}| \leq C \| u^\varepsilon - u^E \|_{L^2(\mathbb{R}^d)}^2 + C \varepsilon \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})}^2 \]
\[ + C \varepsilon^{\frac{1}{2}} \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})} + C \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc} \cap \Omega_{t,+})}. \] (44)

Using (35), (36), (37), (38), (39) and (41) to bound the various terms in the right-hand side of (14) we arrive at
\[ \| u^\varepsilon(t, \cdot) - u^E(t, \cdot) \|_{L^2(\mathbb{R}^d)}^2 \leq \| u_0^\varepsilon - u_0^E \|_{L^2(\mathbb{R}^d)}^2 + C \int_0^t \| u^\varepsilon - u^E \|_{L^2(\mathbb{R}^d)}^2 \, ds \]
\[ + C \varepsilon^{\frac{1}{2}} + C \varepsilon^{\frac{1}{2}} \int_0^t \| u^\varepsilon \|_{L^2(\mathbb{R}^d)} \, ds + C \varepsilon \int_0^t \| \nabla u^\varepsilon \|_{L^2(\mathbb{R}^d)} \, ds \]
\[ + C \int_0^t \left( \varepsilon^{\frac{1}{2}} \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})} + \varepsilon \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc})}^2 + \| \nabla u^\varepsilon \|_{L^2(\mathcal{V}_{t,cc} \cap \Omega_{t,+})} \right) \, ds. \] (45)
Thanks to \((6)\) and Cauchy-Schwarz’ inequality, the terms in the second line above converge to 0 as the viscosity \(\varepsilon\) goes to 0. The terms in the third line above also converge to 0 thanks to Conditions \((8)\) and \((9)\). Therefore, by Gronwall’s lemma, we get the convergence stated in \((10)\) and the proof of Theorem \(1\) is completed.

Acknowledgements

The author thanks the Agence Nationale de la Recherche, Project DYFICOLTI, grant ANR-13-BS01-0003-01, Project IFSMACS, grant ANR-15-CE40-0010 and Project BORDS, grant ANR-16-CE40-0027-01.

References

[1] C. Bardos, D. Lannes. Mathematics for 2d interfaces. Singularities in mechanics: formation, propagation and microscopic description, 37-67, Panor. Synthèses, 38, Soc. Math. France, Paris, 2012.
[2] C. Bardos, E. S. Titi. Euler equations for an ideal incompressible fluid. (Russian) Uspekhi Mat. Nauk 62 (2007), no. 3(375), 5-46; translation in Russian Math. Surveys 62, no. 3, 409-451, 2007.
[3] C. Bardos, E. S. Titi. Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations. DCDS-S. 3(2), 2010.
[4] C. Bardos, E. S. Titi. Mathematics and turbulence: where do we stand?. Journal of Turbulence, 14(3), 42-76, 2013.
[5] C. Bardos, E. S. Titi, Bardos, E. C. Wiedemann. The vanishing viscosity as a selection principle for the Euler equations: the case of 3D shear flow. Comptes Rendus Mathematique, 350(15-16), 757-760. 2012.
[6] D. Benedetto, M. Pulvirenti. From vortex layers to vortex sheets. SIAM J. Appl. Math., 52, 1041-1056, 1992.
[7] R.E. Caflisch, M. Sammartino. Vortex layers in the small viscosity limit. WASCOM 2005–13th Conference on Waves and Stability in Continuous Media, World Sci. Publ., Hackensack, NJ, 59-70, 2006.
[8] P. Constantin, T. Elgindi, M. Ignatova, V. Vicol. Remarks on the Inviscid Limit for the Navier–Stokes Equations for Uniformly Bounded Velocity Fields. SIAM Journal on Mathematical Analysis, 49(3), 1932-1946, 2017.
[9] P. Constantin, I. Kukavica, V. Vicol. On the inviscid limit of the Navier-Stokes equations. Proceedings of the American Mathematical Society, 143(7), 3075-3090, 2015.
[10] G. H. Cottet, P. D. Koumoutsakos. Vortex methods: theory and practice. Cambridge university press, 2000.
[11] J. F. Coulombel, P. Secchi. Nonlinear compressible vortex sheets in two space dimensions. Ann. Sci. cole Norm. Sup.(4), 41(1), 85-139, 2008.
[12] J. F. Coulombel, P. Secchi. Uniqueness of 2-D compressible vortex sheets. Commun. Pure Appl. Anal, 8(4), 1439-1450, 2009.
[13] R. DiPerna and A. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys., 108(4) (1987), 667-689, 1987.
A Kato type criterion for vortex sheets

[14] J. Duchon, J., R. Robert. Global vortex sheet solutions of Euler equations in the plane. Journal of Differential Equations, 73(2), 215-224, 1988.

[15] T. Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. Seminar on nonlinear partial differential equations, 85-98, Math. Sci. Res. Inst. Publ., 2, 1984.

[16] J. P. Kelliher. On Kato’s conditions for vanishing viscosity. Indiana Univ. Math. J. 56, no. 4, 1711-1721, 2007.

[17] J. P. Kelliher. Vanishing viscosity and the accumulation of vorticity on the boundary. Commun. Math. Sci., 6(4):869-880, 2008.

[18] J. P. Kelliher. Observations on the vanishing viscosity limit. Transactions of the American Mathematical Society, 369(3), 2003-2027, 2017.

[19] Y. Maekawa, A. Mazzucato. The Inviscid Limit and Boundary Layers for Navier-Stokes Flows. arXiv preprint arXiv:1610.05372, 2016.

[20] F. Sueur. A Kato type Theorem for the inviscid limit of the Navier-Stokes equations with a moving rigid body. Communications in Mathematical Physics, 316(3):783-808, 2012.

[21] F. Sueur. On the inviscid limit for the compressible Navier-Stokes system in an impermeable bounded domain. Journal of Mathematical Fluid Mechanics, 16(1), 163-178, 2014.

[22] C. Sulem, P.-L. Sulem, C. Bardos, U. Frisch. Finite time analyticity for the two- and three-dimensional Kelvin-Helmholtz instability. Comm. Math. Phys., 80, 485-516, 1981.

[23] X. Wang. A Kato type theorem on zero viscosity limit of Navier-Stokes flows. Indiana Univ. Math. J. 50, Special Issue, 223-241, 2001.

Franck Sueur

7Institut de Mathématiques de Bordeaux, UMR CNRS 5251, Université de Bordeaux, Franck.Sueur@math.u-bordeaux.fr