Simulation of the Processes in Electrical Engineering Systems via the Two-Point Problem for Telegraph Equation

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Abstract. The physical-mathematical model which describes distribution of an electric current in electrotechnical systems is investigated. This model contains a homogeneous partial differential equation of the second order in time and two-point time conditions. The coefficients of equation are the parameters of the system. Two-point conditions describe the behavior of the process at two points in time. The analytical solution of the two-point problem which is obtained by the differential-symbol method determines the behavior of the process at any time and at arbitrary point of the system. The class of functions in which exists a unique solution is specified. The presented results can be used in the design of electrical engineering systems and to predict the results in adjusting parameters.

1. Introduction

One of the main functions of the elements of electrical systems is the transmission and distribution of electricity with minimal losses and a high level of reliability. The development of methods and means of controlling the modes of operation of perspective power systems elements facilitates the progress of technology. This can be achieved via mathematical modeling, which allows engineers to verify the compliance of system elements with the technical requirements of the project, using virtual rather than physical experiments. The virtual design methods significantly shorten the design cycle and reduces the cost of design. Obtaining a negative result of the experiment encourages the search for alternatives to create a more effective final project [1]. Modern electrical system is a complex set of elements. Its operation at any time is characterized by a certain value of voltage at different nodes, current in different elements, frequency and other values [2]. The state of the system is determined by changing these parameters. During emergencies, these parameters change very quickly and reach emergency values. It can damage expensive electrical equipment, which will negatively affect the state of the system [3]. Therefore, mathematical modeling, in particular, of electromagnetic processes in electrical systems is a relevant topic for their study to further establish the mechanism of their changes [4, 5].

Many works [6-8] are devoted to the study of processes in electrical systems by methods of mathematical modeling. Construction of a mathematical model which describes electromagnetic and electromechanical components of transients is carried out in [9]. It allows to obtain the distribution of current and makes possible to design electrical networks of arbitrary configuration. In [10] a model of a power line in single-phase form is developed. Special attention is paid to the calculation of mutual capacitances of conductors which determine the energy leakage between conductors and ground, voltage and potential of electrostatic and magnetic fields generated by currents.
2. Topicality
Let consider a two-wire line [11]. Since the magnetic field occurs in a conductor in which an electric current flow, the capacitance and inductance can be determined for each segment of the line. In addition, conductors have non-zero resistance and the layer between them has non-zero conductivity. The distribution of electric current at any time point \( t \) and at arbitrary point of a conductor is given by function \( i(t, x) \) which is a solution of the telegraph equation, i.e. the partial differential equation

\[
\frac{\partial^2 i}{\partial x^2} = CL \frac{\partial^2 i}{\partial t^2} + (g_0 L + CR) \frac{\partial i}{\partial t} + g_0 Ri,
\]

where \( R \) is conductor resistance, \( L \) is inductance, \( g_0 \) is conductance, \( C \) is capacitance [12]. These parameters determine the performance of a line. In [12], partial cases of the telegraph equation for a finite and infinite conductor is studied for different variants of a line, in particular, for a line without leakage (resistance is almost insignificant and insulation is quality, \( R = g_0 = 0 \)), for a line without distortion ( \( RC = Lg_0 \) ) and others.

The telegraph equation is also used to model the processes of signal propagation [13], electromagnetic wave propagation [14, 15], inverse scattering [16], etc. In the presented work the problem for the telegraph equation with conditions in which current distribution in an infinite conductor is known in two time moments, instead of at one moment as in Cauchy problem, is investigated. For this problem a differential-symbol method which is convenient to study such problems is used [17]. Applying this method the conditions of existence and uniqueness of solution of the homogeneous two-point in time problem [18] two-point problem for homogeneous equation with nonhomogeneous conditions [19] and two-point problem for nonhomogeneous equation with homogeneous conditions [20, 21] are obtained.

3. Two-point problem for telegraph equation
Let consider a mathematical model of describing distribution of voltage or current in an infinite conductor, which is given by the function \( u = u(t, x) \). This function is the solution of the telegraph equation, that is the solution of partial differential equation

\[
\left[ \frac{1}{\alpha^2} \frac{\partial^2}{\partial t^2} + D \frac{\partial}{\partial t} + E - \frac{\partial^2}{\partial x^2} \right] u = 0, \quad x \in \mathbb{R}, \quad t > 0.
\]

The coefficients \( \alpha, D, C \) of equation (2) specify the process parameters. Equation (1) can be obtained from (2) for \( \alpha = \frac{1}{\sqrt{LC}} \), \( D = Lg_0 + RC \), \( E = Rg_0 \).

Telegraph equation (2) makes possible to take into account the existing resistances of the medium and associated with them the nature of the wave attenuation. In addition to the processes in electrical systems, equation (2) also describes an elastic displacements of a rope when winding it with load-lifting mechanisms on the drum [22]. Note that the problem for equation (2) with given initial and boundary conditions is investigated in [23, 24]. The solutions of these problems are represented in the integral form using the Riemann functions.

In presented paper, the set of solutions of equation (2) which satisfy two-point time conditions

\[
u(0, x) = \phi_1(x), \quad u(\tau, x) = \phi_2(x), \quad x \in \mathbb{R},
\]

is studied. In conditions (3), the functions \( \phi_1(x) \) and \( \phi_2(x) \) describe the current distribution at two time points \( t = 0 \) and \( t = \tau \), \( \tau > 0 \). The Cauchy problem for telegraph equation (2) with conditions

\[
u(0, x) = \phi_1(x), \quad \frac{\partial u}{\partial t}(0, x) = \phi_2(x), \quad x \in \mathbb{R},
\]

for infinitely long conductor is studied in [12].
4. On the existence of unique solution of the two-point problem and its construction

To study problem (2), (3) we can use the differential-symbol method which is successfully applied, for example, to study oscillation processes which is described by wave equation with two-point time conditions [25].

Let us write the ordinary differential equation

\[
\left[ \frac{1}{\alpha^2} \frac{d^2}{dt^2} + D \frac{d}{dt} + E - \mu^2 \right] G(t, \mu) = 0. \tag{4}
\]

Equation (4) is obtained from (2) with the change \( \frac{\partial}{\partial t} \) by \( \frac{d}{dt} \) and \( \frac{\partial}{\partial x} \) by \( \mu, \mu \in \mathbb{C} \).

The system of functions

\[
G_1(t, \mu) = e^{-\frac{D \alpha^2}{2}} \left\{ \sinh \left[ \frac{\alpha}{2} \eta(\mu)t \right] \right\}, \quad G_2(t, \mu) = e^{-\frac{D \alpha^2}{2}} \frac{\sinh \left[ \frac{\alpha}{2} \eta(\mu)t \right]}{\eta(\mu)}, \tag{5}
\]

where \( \eta(\mu) = \left(D^2 \alpha^2 + 4(\mu^2 - E)\right)^{1/2} \) is the normal fundamental system of solutions of equation (4).

By

\[
\Delta(\mu) = G_2(\tau, \mu), \tag{6}
\]

we denote the value of the function \( G_2(t, \mu) \) at the moment of time \( t = \tau \).

Let \( L \) be the set of zeroes of the function \( \Delta(\mu) \), that is

\[
L = \{ \mu \in \mathbb{C} : \alpha^2 \eta^2(\mu) \tau^2 = -4\pi^2 k^2, k \in \mathbb{N} \}. \tag{7}
\]

We consider quasipolynomial of the form

\[
\varphi(x) = P(x)e^{\beta x}, \tag{8}
\]

where \( P(x) \) is polynomial, coefficient \( \beta \) in exponent is complex number. By \( K_{\mathbb{C} \setminus L} \), we denote a class of quasipolynomials which can be represented as a finite sum of one-term quasipolynomials (8) with arbitrary polynomials \( P(x) \) and different coefficients \( \beta \), moreover these coefficients belong to the set \( \mathbb{C} \setminus L \), where \( L \) is the set (7).

If the right-hand sides of conditions (3), i.e. the functions \( \phi_1(x) \) and \( \phi_2(x) \) are quasipolynomials of the class \( K_{\mathbb{C} \setminus L} \) [26], then problem (2), (3) has unique solution in the class of quasipolynomials of variables \( t \) and \( x \) which for each fixed \( t > 0 \) belong to \( K_{\mathbb{C} \setminus L} \). This solution can be found by the formula [27]

\[
\varphi(t, x) = \phi_1 \left( \frac{\partial}{\partial \mu} \right) \left[ G_1(t, \mu)e^{\alpha \mu} \right] \bigg|_{\mu=0} + \phi_2 \left( \frac{\partial}{\partial \mu} \right) \left[ G_2(t, \mu)e^{\alpha \mu} \right] \bigg|_{\mu=0}, \tag{9}
\]

where

\[
G_1(t, \mu) = \frac{2e^{-\frac{D \alpha^2}{2}t}}{\Delta(\mu)} \sinh \left[ \frac{1}{2} \frac{\alpha(\tau-x)\eta(\mu)}{\alpha \eta(\mu)} \right], \quad G_2(t, \mu) = \frac{2e^{-\frac{D \alpha^2}{2}t}}{\Delta(\mu)} \left[ \frac{\sinh \left( \frac{1}{2} \frac{\alpha \eta(\mu)}{\alpha \eta(\mu)} \right)}{\alpha \eta(\mu)} \right]. \tag{10}
\]

Note that functions (10) satisfy equation (4) and following conditions

\[
G_1(0, \mu) = 1, \quad G_1(\tau, \mu) = 0, \quad G_2(0, \mu) = 0, \quad G_2(\tau, \mu) = 1.
\]
In formula (9) the expressions \( \phi_1 \left( \frac{\partial}{\partial \mu} \right) \), \( \phi_2 \left( \frac{\partial}{\partial \mu} \right) \) is obtained from \( \phi_1 (x) \), \( \phi_2 (x) \) replacing \( x \) by differential symbol \( \frac{\partial}{\partial \mu} \). For each one-term quasipolynomial (8), the corresponding differential quasipolynomial \( P \left( \frac{\partial}{\partial \mu} \right) e^{i \omega} \) acts on the functions in curly braces of formula (9) by the formula

\[
P \left( \frac{\partial}{\partial \mu} \right) e^{i \omega} \left\{ G(t, \mu) e^{i \omega} \right\}_{\mu=0} = P \left( \frac{\partial}{\partial \mu} \right) \left\{ G(t, \mu) e^{i \omega} \right\}_{\mu=0}.
\]

5. Examples of solving problems for specific parameters of telegraph equation with known behavior of process at two moments of time

Let's consider the examples of the process, which is described by problem (2), (3) and is determined by the specific parameters \( \alpha, D, C \) with given behavior of the process at two points in time.

**Example 1.** We study problem (2), (3) for the case \( \phi_1 (x) = c_1 \), \( \phi_2 (x) = c_2 \), \( t = 1 \), where \( c_1, c_2 \) are positive constants. Since the functions \( \phi_1 (x) = c_1 e^{\alpha x} \), \( \phi_2 (x) = c_2 e^{\alpha x} \) as one-term quasipolynomials of form (8) have index 0 in exponential functions, then \( \eta(0) = \left( D^2 \alpha^2 - 4E \right)^{1/2} \).

We consider three cases.

1) If \( D^2 \alpha^2 - 4E > 0 \), then \( \alpha^2 \left( D^2 \alpha^2 - 4E \right)^{1/2} \neq -4\pi^2 k^2 \) for each \( k \in \mathbb{N} \). It means that \( 0 \not\in L \). Let's write functions (10) for \( \mu = 0 \):

\[
G_1(t,0) = e^{\frac{1}{2} \alpha^{2}} \left[ \frac{\sinh \left[ \frac{1}{2} \alpha (1-t) \eta(0) \right]}{\sinh \left( \frac{\alpha}{2} \eta(0) \right)} \right], \quad G_2(t,0) = e^{\frac{1}{2} \alpha^{2} (t-1)} \left[ \frac{\sin \left[ \frac{1}{2} \alpha t \eta(0) \right]}{\sin \left( \frac{\alpha}{2} \eta(0) \right)} \right].
\]

The solution of problem (2), (3) by formula (9) has following form

\[
u(t,x) = c_1 G_1(t,0) + c_2 G_2(t,0)
= \frac{e^{\frac{1}{2} \alpha^{2} t \eta(0)}}{\sinh \left( \frac{\alpha}{2} \eta(0) \right)} \left[ c_1 \sinh \left[ \frac{1}{2} \alpha (1-t) \eta(0) \right] + c_2 e^{\alpha t} \sin \left[ \frac{1}{2} \alpha t \eta(0) \right] \right].
\]

For example, for \( \alpha = 1 \), \( D = 3 \), \( E = 2 \) we get solution of problem (2), (3) in analytical form

\[
U(t,x) = \frac{e^{-2t}}{e-1} \left( c_1 \left( e-e' \right) + c_2 e^{2} \left( e'-1 \right) \right).
\]

Note that the function \( u(t,x) \) at each point \( x \) is the same, i.e. the behavior of process does not depend on the coordinate \( x \) and it goes exponentially to zero, if \( t \to \infty \).

Graphic representation of the solution is shown in Figure 1a), 1b) and 1c).
1a) $u(t,x)$ for $c_1 = 1, c_2 = 2$

1b) $u(t,x)$ for $c_1 = 2, c_2 = 1$

1c) $u(t,x)$ for $c_1 = c_2 = 1$

Figure 1. The graph of the function $u(t,x)$

2) If $D^2 \alpha^2 - 4E = 0$, then $\eta(0) = 0$ and the condition $0 \not\in L$ is fulfilled. This expression is satisfied, for example, for $E = 0$ and $D = 0$, i.e. the conductor is well insulated and the resistance is insignificant. Functions (10) for $\mu = 0$ have the such form

$$G_1(t,0) = (1-t)e^{-\frac{1}{2}x^2D}, \quad G_2(t,0) = e^{-\frac{1}{2}x^2D(1-t)}.$$

We find solution of problem (2), (3) by formula (9):

$$U(t,x) = \left(c_1(1-t) + c_2 e^{\frac{1}{2}\alpha^2D} t\right) e^{-\frac{1}{2}x^2D}.$$ 

In particular, for $\alpha = 1, D = 2, E = 1$ we get solution of problem in analytical form

$$U(t,x) = c_1(1-t)e^{-x^2} + c_2 te^{x^2}$$

and graphic form (Figure 2).

Figure 2. Graph of the function $u(t,x)$ for $c_1 = 1, c_2 = 2$. 

3) Let $D^2 \alpha^2 - 4E < 0$. Then provided that value $\frac{\alpha^2 (4E - D^2 \alpha^2)}{\pi^2}$ is not a multiple of 4, we get $0 \not\in L$. Functions (10) for $\mu = 0$ have following form

$$G_1(t,0) = e^{\frac{1}{2} \alpha D t} \left[ \frac{\sin \left( \frac{1}{2} \alpha (1-t) \sqrt{4E - D^2 \alpha^2} \right)}{\sin \left( \frac{\alpha}{2} \sqrt{4E - D^2 \alpha^2} \right)} \right], \quad G_2(t,0) = e^{\frac{1}{2} \alpha D (t-1)} \left[ \frac{\sin \left( \frac{1}{2} \alpha t \sqrt{4E - D^2 \alpha^2} \right)}{\sin \left( \frac{\alpha}{2} \sqrt{4E - D^2 \alpha^2} \right)} \right].$$

By formula (9), the solution of problem (2), (3) has the form

$$U(t,x) = c_1 G_1(t,0) + c_2 G_2(t,0)$$

$$= e^{\frac{1}{2} \alpha D t} \left[ \frac{c_1 \sin \left( \frac{1}{2} \alpha (1-t) \sqrt{4E - D^2 \alpha^2} \right) + c_2 e^{\frac{1}{2} \alpha D (t-1)} \sin \left( \frac{1}{2} \alpha t \sqrt{4E - D^2 \alpha^2} \right)}{\sin \left( \frac{\alpha}{2} \sqrt{4E - D^2 \alpha^2} \right)} \right].$$

In particular, for $\alpha = 1$, $D = 2$, $E = 2$ we get the solution of problem (2), (3) in the form

$$U(t,x) = e^{\frac{-t}{\sin 1}} \left( c_1 \sin \left[ 1 - t \right] + c_2 \sin \left[ t \right] \right).$$

A graphical representation of this solution is given in Figure 3.

**Figure 3.** The graph of the function $u(t,x)$ for $c_1 = 2, c_2 = 1$.

**Example 2.** Let’s consider process in electrical engineering system which is modeled by two-point problem

$$\left[ -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 4 \frac{\partial}{\partial t} + 3 \right] u(t,x) = 0,$$  
(11)

$$u(0,x) = \sin x, \quad u(1,x) = 0.$$  
(12)

Equation (11) is telegraph equation (2) for $\alpha = 1$, $D = 4$, $E = 3$. Two-point conditions (12) are conditions (3) for $\tau = 1$, $\phi_1(x) = \sin x$, $\phi_2(x) = 0$.

The function $\phi_1(x) = \sin x = \frac{1}{2i} \left( e^{ix} - e^{-ix} \right)$ is the sum of two one-term quasipolynomials with coefficients $i$ and $-i$ in exponential functions. Since $\eta(\pm i) = 0$ and $\Delta(\pm i) = e^{-2} \neq 0$ then function $\phi_1(x)$ is a quasipolynomial from the class $K_{\mathbb{C} \setminus L}$. According to formula (9) the solution of problem (11), (12) has the form
The graph of this function is represented in Figure 4.

\[ U(t, x) = \frac{1}{2i} \left( \left\{ G_1(t, \mu) e^{\mu x} \right\}_{\mu=i} - \left\{ G_1(t, \mu) e^{\mu x} \right\}_{\mu=-i} \right) \]
\[ = \frac{1}{2i} \left( 2e^{-2(t+1)+\mu t} \sinh \left[ \frac{1}{2} (1-t) \eta(\mu) \right] \right)_{\mu=i} - \left( 2e^{-2(t+1)+\mu t} \sinh \left[ \frac{1}{2} (1-t) \eta(\mu) \right] \right)_{\mu=-i} \]
\[ = \frac{(1-t)}{2i} \left( e^{-2t+\mu t} \right)_{\mu=i} - \left( e^{-2t+\mu t} \right)_{\mu=-i} = (1-t) e^{-2t} \sin x. \]

Note that the obtained solution of problem (11), (12) is \( 2\pi \)-periodical function by variable \( x \) and for \( t \to \infty \) it goes to zero.

Conclusions

The process of electric current distribution in an infinitely long conductor with given system characteristics and process behavior at two moments in time is described by a two-point problem for the telegraph equation. A class of quasipolynomials in which there is a unique solution of the problem is distinguished. According to the differential-symbol method, the formula for finding solution is proposed. The obtained solution of the problem makes possible to determine the process of electric current propagation in the conductor at any time and at any point of the conductor, which greatly facilitates the design of electrical systems. The application of the method is shown in the examples.

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