Orbital entanglement in an exactly solvable two-electron quantum dot model

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Abstract. We consider an exactly solvable two-dimensional model of a two-electron parabolic quantum dot (QD) under the perpendicular magnetic field. We perform a detailed study of the degree of the orbital entanglement of the ground and first excited singlet state depending on the shape of QD, the strength of the effective interaction and the magnetic field. It was found that a degree of entanglement increases with the growth of interaction between electrons. The magnetic field destroys the entanglement of electrons. However, the entanglement in deformed QDs is more stable to the effect of the magnetic field.

1. Introduction

Few-electron quantum dots (QDs) attract a considerable experimental and theoretical attention in past years [1, 2, 3]. A comprehensive analysis of such systems is due to desire to use these systems in various technological applications starting from quantum transistors to information storage units. Another motivation is to get a deep insight into the nature of quantum correlations, taking advantage of controlled conditions. Altering external fields, number of electrons or parameters of QDs, so-called “artificial atoms”, one can elucidate tiny details of fundamental aspects of a quantum world. In particular, the interaction of two electrons creates an entangled state which lies in the heart of quantum information processes. It is widely believed that the entangled states of these electrons confined in a QD may give a natural realization of a quantum bit or “qubit” [4]. However, an entanglement being one of the most subtle and intriguing phenomena in nature is not yet well understood [5]. The questions how to efficiently produce and control it, for example, in QDs are among fundamental as well as technological problems.

In general, in finite fermionic systems the entanglement is associated with a correlated dynamics of electrons. The correlated motion of electrons in QDs and, therefore, the entanglement could be controlled by externally applied electromagnetic fields or by varying the shape and the depth of the confining potential in QDs. The simplest QD with the essential features of more complex systems contains two electrons. When two electron move in the external field created by the confining potential and by a homogeneous constant magnetic field, the spin part of the total wave function evolves from the singlet to the triplet state [6, 7, 8]. At the same time the spatial part changes in accordance with the antisymmetric nature of the total wave function. The spin singlet wave function is always maximally entangled. In this case one can study how the external field and parameters of the system would influence the spatial
function and, therefore, the spatial entanglement which may have important effects in various applications.

Recently we developed an analytical approach to study the degree of entanglement in a model of two-coupled harmonic oscillators as a function of time [9]. Note, that in condensed matter physics this model is used as a starting point for analysis of electronic properties of QDs in a perpendicular magnetic field [3]. In this communication we report the results of effects produced by: i) the magnetic field; ii) shape parameters, and iii) an effective electron-electron interaction, – on the degree of the orbital entanglement of the singlet state in a solvable model of two-electron QD.

2. Model

We consider a system of two electrons whose motion is restricted to the xy plane by a parabolic potential in an external magnetic field applied in the vertical direction (z). The use of the parabolic confining potential enables one to reproduce quite well experimental ground state transitions under the perpendicular magnetic field, observed in recent experiments with few-electron QDs (see details in [3, 10, 11]). The full Hamiltonian thus reads

\[ H = \sum_{i=1}^{2} \hat{h}_i + V(\mathbf{r}_1, \mathbf{r}_2) + H_{\text{spin}} = \sum_{i=1}^{2} \left[ \frac{1}{2m^*} (\mathbf{p} + eA) \right]^2 + \frac{1}{2} m^* (\omega_x^2 x^2 + \omega_y^2 y^2) \right] + V(\mathbf{r}_1, \mathbf{r}_2) + H_{\text{spin}}, \]

(1)

where \( H_{\text{spin}} = -g^* \mu_B (\mathbf{s}_1 + \mathbf{s}_2) \cdot \mathbf{B} \) describes the Zeeman energy and \( \mu_B = e\hbar / 2m_e c \) is the Bohr magneton. Here \( m^* \) and \( g^* \) are the effective electron mass and g-factor, respectively, and we use the planar polar coordinates \( (r^2 = x^2 + y^2) \). The confining potential is approximated with a two-dimensional harmonic oscillator with two different confining frequencies \( \omega_x \) and \( \omega_y \), which are not equal in a general case. Although the three-dimensional nature of QDs is important for analysis of experimental data [3], one is able to reproduce the experimental results with the effective two-dimensional electron-electron interaction [10, 12]. We do not take into account the effect of finite temperature; this is appropriate for experiments which are performed at temperatures \( k_B T \ll h\omega_0 \) with \( h\omega_0 = h\sqrt{\omega_x \omega_y} \simeq 3 \) meV being the mean level spacing. In the following, for the sake of illustration, the effective electron mass is chosen as \( m^* = 0.067m_e \) for GaAs. The effective mass determines the orbital magnetic moment \( \mu_B^{\text{eff}} \) for electrons through the relation \( h\omega_L = \mu_B B m_e/m^* = \mu_B^{\text{eff}} B \) and leads to \( \mu_B^{\text{eff}} \approx 15\mu_B \). The magnetic orbital effect is much enhanced in comparison with the magnetic spin effect produced by the Zeeman term with the effective Lande factor \( |g^*| = 0.44 \). In this contribution we focus, however, on singlet states, with the Zeeman term contribution being zero.

The interaction potential between two electrons \( V(\mathbf{r}_1, \mathbf{r}_2) \) is, of course, varies as \( |\mathbf{r}_1 - \mathbf{r}_2| \). To mimic this potential in a simple, yet physically reasonable model, we approximate the interaction between electrons by the Johnson-Payne potential [13]

\[ \hat{V}(\mathbf{r}_1 - \mathbf{r}_2) = V_0 - \lambda^2 \frac{m^*}{2} (\mathbf{r}_1 - \mathbf{r}_2)^2, \]

(2)

where \( V_0 \) and \( \lambda \) are positive parameter within the model.

Introducing new variables

\[ \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \]

(3)

one separates the full Hamiltonian on the the center-of-mass and relative motion terms

\[ \hat{H} = \hat{H}_{\text{CM}} + \hat{H}_{\text{rel}} + \hat{V}_0, \]

(4)
Here, the effective coupling constants are

\[
\hat{H}_{\text{CM}} = \frac{\hat{P}^2}{2M} + \frac{M}{2} \left[ \Omega_1^2 \hat{X}^2 + \Omega_2^2 \hat{Y}^2 \right] + \omega_L \hat{L}_z,
\]
\[
\hat{H}_{\text{rel}} = \frac{\hat{P}^2}{2\mu} + \frac{\mu}{2} \left[ \omega_1^2 \hat{x}^2 + \omega_2^2 \hat{y}^2 \right] + \omega_L \hat{l}_z,
\]
where

\[
\Omega_{1,2}^2 = \omega_{1,2}^2 + \omega_L^2, \quad \omega_{1,2}^2 = \omega_{1,2}^2 + \omega_2^2 - 2\lambda^2, \quad M = 2m^*, \quad \mu = \frac{m^*}{2}, \quad \omega_L = \frac{eB}{2m^*c}.
\]

The quantization of the Hamiltonian (4) can be done in terms of quanta of oscillations for the relative motion of electrons

\[
\hat{x} = \sqrt{\frac{\hbar}{2\mu\omega_1}} (\hat{c}_1^\dagger + \hat{c}_1), \quad \hat{p}_x = i\sqrt{\frac{\hbar\mu\omega_1}{2}} (\hat{c}_1^\dagger - \hat{c}_1),
\]
\[
\hat{y} = \sqrt{\frac{\hbar}{2\mu\omega_2}} (\hat{c}_2^\dagger + \hat{c}_2), \quad \hat{p}_y = i\sqrt{\frac{\hbar\mu\omega_2}{2}} (\hat{c}_2^\dagger - \hat{c}_2),
\]
and for their centre-of-mass motion

\[
\hat{X} = \sqrt{\frac{\hbar}{2M\Omega_1}} (\hat{C}_1^\dagger + \hat{C}_1), \quad \hat{P}_X = i\sqrt{\frac{\hbar M\Omega_1}{2}} (\hat{C}_1^\dagger - \hat{C}_1),
\]
\[
\hat{Y} = \sqrt{\frac{\hbar}{2M\Omega_2}} (\hat{C}_2^\dagger + \hat{C}_2), \quad \hat{P}_Y = i\sqrt{\frac{\hbar M\Omega_2}{2}} (\hat{C}_2^\dagger - \hat{C}_2),
\]
in which the Hamiltonians read as

\[
\hat{H}_{\text{rel}} = \hbar\omega_1 (\hat{c}_1^\dagger \hat{c}_1 + 1/2) + \hbar\omega_2 (\hat{c}_2^\dagger \hat{c}_2 + 1/2) - i\hbar g_1 (\hat{c}_1^\dagger \hat{c}_2 - \hat{c}_2^\dagger \hat{c}_1) - i\hbar g_2 (\hat{c}_1^\dagger \hat{c}_2^\dagger - \hat{c}_2 \hat{c}_1^\dagger),
\]
\[
\hat{H}_{\text{CM}} = \hbar\Omega_1 (\hat{C}_1^\dagger \hat{C}_1 + 1/2) + \hbar\Omega_2 (\hat{C}_2^\dagger \hat{C}_2 + 1/2) - i\hbar G_1 (\hat{C}_1^\dagger \hat{C}_2 - \hat{C}_2^\dagger \hat{C}_1) - i\hbar G_2 (\hat{C}_1^\dagger \hat{C}_2^\dagger - \hat{C}_2 \hat{C}_1^\dagger).
\]

Here, the effective coupling constants are

\[
g_1 = \frac{\omega_1}{2} \sqrt{\omega_1\omega_2}, \quad g_2 = \frac{\omega_1}{2} \sqrt{\omega_1\omega_2},
\]
\[
G_1 = \frac{\omega_L}{2} \frac{\Omega_1 + \Omega_2}{\Omega_1\Omega_2}, \quad G_2 = \frac{\omega_L}{2} \frac{\Omega_1 - \Omega_2}{\Omega_1\Omega_2}.
\]

By means of the Bogoliubov transformations

\[
\hat{a}_\pm = \sum_{m=1}^2 \left( A_m^\pm \hat{c}_m + B_m^\pm \hat{c}_m^\dagger \right), \quad \hat{b}_\pm = \sum_{m=1}^2 \left( F_m^\pm \hat{C}_m + D_m^\pm \hat{C}_m^\dagger \right),
\]

one transforms the Hamiltonians to diagonal forms

\[
\hat{H}_{\text{rel}} = \sum_{\pm} \hbar\omega_{\pm} (\hat{a}_{\pm}^\dagger \hat{a}_{\pm} + 1/2), \quad \hat{H}_{\text{CM}} = \sum_{\pm} \hbar\Omega_{\pm} (\hat{b}_{\pm}^\dagger \hat{b}_{\pm} + 1/2),
\]
where the normal frequencies
\[
\omega^2_{\pm} = \frac{1}{2} \left[ \omega_x^2 + \omega_y^2 + 4(\omega_L^2 - \lambda^2) \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 8\omega_L^2(\omega_x^2 + \omega_y^2 + 2\omega_L^2 - 4\lambda^2)} \right],
\]
\[
\Omega^2_{\pm} = \frac{1}{2} \left[ \omega_x^2 + \omega_y^2 + 4\omega_L^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 8\omega_L^2(\omega_x^2 + \omega_y^2 + 2\omega_L^2)} \right].
\]

The coefficients \(A^+_m, B^+_m, F^+_m, D^+_m\) are defined in [14].

The eigenstates of the Hamiltonian (4) are characterized by four quantum numbers \(n_\pm, N_\pm\):
\[
|n_+n_-N_+N_-\rangle = \frac{1}{\sqrt{n_+!n_-!N_+!N_-!}} (\hat{a}_+^\dagger)^{n_+} (\hat{a}_-^\dagger)^{n_-} (\hat{b}_+^\dagger)^{N_+} (\hat{b}_-^\dagger)^{N_-}|0000\rangle,
\]
which define the ground \(|n_+ = 0, n_- = 0, N_+ = 0, N_- = 0\rangle\) and various excited states of the two-electron QD in a perpendicular magnetic field. The total two-electron QD energy in the singlet state is
\[
E = \hbar\omega_+ (n_+ + 1/2) + \hbar\omega_- (n_- + 1/2) + \hbar\Omega_+ (N_+ + 1/2) + \hbar\Omega_- (N_- + 1/2).
\]

3. Results
We analyse the evolution of the orbital entanglement as a function of the magnetic field \(B\) in the form of the Larmor frequency \(\omega_L\) (6), the deviation from a circular symmetry \(\omega_y/\omega_x\) and the strength of the effective electron-electron interaction \(\lambda\). Evidently, that the effective interaction (2) could be considered as a perturbation in our model. Therefore, in our numerical analysis the strength of interaction is determined from the condition \(\lambda/\omega_x < 1\). Hereafter, we choose \(\omega_x\) as a dimensional parameter of our model.

Let us first consider the QD ground (spin-singlet) state \((N_{sh} = 0; n_\pm = N_\pm = 0)\). As the measure of entanglement we use the logarithmic negativity [15] which is for this state of Gaussian type determined by its covariance matrix
\[
\gamma_{jk} = \frac{1}{\hbar} \text{Tr}(\hat{R}_j \hat{R}_k^\dagger) - \frac{i}{2} \sigma_{jk},
\]
where the operator column \(\hat{R} = (\hat{x}_1, \hat{p}_x, \hat{y}_1, \hat{p}_y, \hat{x}_2, \hat{p}_x, \hat{y}_2, \hat{p}_y)^T\) and the (antisymmetric) symplectic matrix
\[
\sigma = \bigoplus_{j=1}^4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
As a result, the logarithmic negativity
\[
E_N = -\sum_{j=1}^4 \log_2(\min(1, 2\tilde{c}_j))
\]
(21)
is determined by the symplectic spectrum \(\{\tilde{c}_i\}\) of the covariance matrix \(\gamma^{T_1}\) of the partial transpose of our state found in the following expression
\[
-\sigma \gamma^{T_1} \sigma \gamma^{T_1} = \text{diag}(\tilde{c}_1, \tilde{c}_1, \ldots, \tilde{c}_4, \tilde{c}_4).
\]
Here the matrix \(\gamma^{T_1}\) is constructed by multiplying by \(-1\) all matrix elements of the matrix \(\gamma\), which are connected to momentum projections of the first electron.
Figure 1. The dependence of the ground state entanglement on the strength of the effective interaction ($\lambda$), the magnetic field ($\omega_L/\omega_x$), and the QD deformation ($\omega_y/\omega_x$).

At small deformations ($\omega_y/\omega_x = 1.2$) and zero magnetic field (see left panel of Fig. 1), the increase of the interaction strength leads to a visible increase of the entanglement. On the other hand, the increase of the magnetic field weakens the entanglement. In this case the raise of the magnetic field increases the effective confining potential (cf [8, 16]). It results in the decrease of the effective electron-electron interaction at fixed value of the strength $\lambda$, i.e., the decrease of the electron correlations. At fixed value of the strength $\lambda/\omega_x = 0.7$ the QD deformation decreases the entanglement as well (see right panel of Fig. 1). In the absence of the magnetic field our results reproduce qualitatively the evolution of the entanglement properties in the two-electron QD with a plain Coulomb interaction [17].

For the first excited (spin-singlet) state ($N_{\text{sh}} = 1$: $n_+ = 1$, $n_- = N_\pm = 0$) which is of non-Gaussian type it is convenient to use the linear entropy. It is the correct measure of entanglement for pure bipartite states and has been applied for analysis of various problems related to the entanglement features of two-fermion systems [17, 18]. The linear entropy is determined by the reduced density matrix, say, of subsystem 1 $\hat{\varrho}_1 = \text{Tr}_2(\hat{\varrho})$ (cf [19])

$$E_{LE} = 1 - \text{Tr}(\hat{\varrho}_1^2).$$

(22)

Equivalently, it can be expressed in the phase space

$$E_{LE} = 1 - \int_{\mathcal{R}^4} |\chi(\alpha, \beta)|^2 d^2 \alpha d^2 \beta / \pi^2$$

(23)

with the aid of the symmetric characteristic function of subsystem 1

$$\chi(\alpha, \beta) = \text{Tr}\{\hat{\varrho}\hat{D}(\alpha, \beta)\},$$

(24)

where the displacement operator is defined as

$$\hat{D}(\alpha, \beta) = \exp\{\alpha \hat{d}_x^\dagger - \alpha^* \hat{d}_x + \beta \hat{d}_y^\dagger - \beta^* \hat{d}_y\}$$

(25)

in terms of the quadrature phase operators

$$\hat{d}_x = (m*\omega_x \hat{x}_1 + i \hat{p}_x)/\sqrt{2\hbar m*\omega_x}, \quad \hat{d}_y = (m*\omega_y \hat{y}_1 + i \hat{p}_y)/\sqrt{2\hbar m*\omega_y}.$$

(26)

Similar tendency for the entanglement behavior is found for the first excited state (see Fig. 2). However, the entanglement preserves even at the limit of $\lambda \to 0$, in contrast to the tendency observed for the ground state (see Fig 1).
4. Summary
We have analysed the evolution of the entanglement in the ground and the first excited states in the solvable model for two-electron parabolic QD in the magnetic field. We found that the orbital motion of electrons produces the following specific features of the entanglement in the singlet states:

- in the ground state the entanglement arises due to the interaction between electrons, while in the first excited state it is left even in the weak interaction limit;
- the degree of entanglement increases with the growth of interaction between electrons;
- the magnetic field destroys the entanglement of electrons, suppressing the effective electron-electron interaction;
- the entanglement in deformed QDs is more stable to the effect of the magnetic field.

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References
[1] Kouwenhoven L P, Austing D G and Tarucha S 2001 Rep. Prog. Phys. 64 701
[2] Yannoules C and Landman U 2007 Rep. Prog. Phys. 70 2067
[3] Nazmitdinov R G 2009 Physics of Particles and Nuclei 40 71
[4] Hanson R et al 2007 Rev. Mod. Phys 79 1217
[5] Amico L, Fazio R, Osterloh A and Vedral V 2008 Rev. Mod. Phys 80 517
[6] Wagner M, Merkt U and Chaplik A V 1992 Phys. Rev. B 45 1951
[7] Peeters F M and Schweigert V A 1996 Phys. Rev. B 53 1468
[8] Dineykhan M and Nazmitdinov R G 1999 J.Phys.: Condens. Matter 11 L83
[9] Chizhov A V and Nazmitdinov R G 2008 Phys. Rev. A 78 0641302
[10] Maksym P A et al 2009 Phys. Rev. B 79 115314
[11] Puente A, Pons M and Nazmitdinov R G 2010 J.Phys.: Conf. Series 248 012017
[12] Simonović N S and Nazmitdinov R G 2008 Phys. Rev. A 78 032115
[13] Johnson N F and Payne M C 1991 Phys. Rev. Lett. 67 1157
[14] Chizhov A V and Nazmitdinov R G 1990 Int. J. Mod. Phys. B 4 2335
[15] Vidal G and R. F. Werner R F 2002 Phys. Rev. A 65 032314
[16] Nazmitdinov R G, Simonović N S and Rost J M 2002 Phys. Rev. B 65 155307
[17] Kościk P and Okopińska A 2010 Phys. Lett. A 374 3841
[18] Yañez R J, Plastino A R and Dehesa J S 2010 Eur. Phys. J. D 56 141
[19] Coleman A J I and Yukalov V 2000 Reduced Density Matrices (Berlin: Springer-Verlag)