FLUCTUATION EXPONENTS FOR DIRECTED POLYMERS IN THE INTERMEDIATE DISORDER REGIME

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Abstract. We compute the fluctuation exponents for a solvable model of one-dimensional directed polymers in random environment in the intermediate regime. This regime corresponds to taking the inverse temperature to zero with the size of the system. The exponents satisfy the KPZ scaling relation and coincide with physical predictions. In the critical case, we recover the fluctuation exponents of the Cole-Hopf solution of the KPZ equation in equilibrium and close to equilibrium.

1. Main results

1.1. Introduction. The directed polymer in a random environment is a statistical physics model that assigns Boltzmann-Gibbs weights to random walk paths as a function of the environment encountered by the walk. It was originally introduced in [29] as a model of an interface in two dimensions. Here is the standard lattice formulation in $d + 1$ dimensions ($d$ space dimensions, one time dimension).

The environment is a collection of i.i.d. random weights $\{\omega(i, x) : i \in \mathbb{N}, x \in \mathbb{Z}^d\}$ with probability distribution $\mathbb{P}$. Let $P$ be the law of simple symmetric random walk $(S_t)_{t \in \mathbb{Z}_+}$ on $\mathbb{Z}^d$ with $S_0 = 0$. Denote expectation under $P$ and $\mathbb{P}$ by $E$ and $\mathbb{E}$, respectively. The quenched partition function of the directed polymer in environment $\omega$ and at inverse temperature $\beta > 0$ is

$$Z_{N,x}(\beta) = E \left[ e^{\beta \sum_{i=1}^N \omega(i, S_i)}, S_N = x \right],$$

where $E[X, A]$ is the expectation of $X$ restricted to the event $A$. This is the point-to-point partition function because the endpoint $S_N$ of the walk is constrained to be $x$. The version that allows $S_N$ to fluctuate freely is the point-to-line partition function. In the point-to-point setting the quenched polymer measure on paths ending at $x$ is

$$Q_{N,x}^\beta(S_1 = x_1, \ldots, S_N = x_N)$$

$$= \frac{1}{Z_{N,x}(\beta)} e^{\beta \sum_{i=1}^N \omega(i, x_i)} P \left[ S_1 = x_1, \ldots, S_N = x_N \right] \cdot 1\{x_N = x\}.$$

These quenched quantities are functions of the environment $\omega$ and thereby random. The averaged distribution of the path is $P_{N,x}^\beta(\cdot) = \mathbb{E}Q_{N,x}^\beta(\cdot)$. We refer the reader to reviews [21, 25, 32] for a deeper discussion of the subject.

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We restrict the discussion to the 1+1 dimensional case. Basic objects of study are the fluctuations of the free energy $\log Z_{N,Nx}(\beta)$ and the path $(S_t)_{0 \leq t \leq N}$. On the crudest level the orders of magnitude of these fluctuations are described by two exponents $\chi$ and $\zeta$:

- fluctuations of $\log Z_{N,Nx}(\beta)$ under $\mathbb{P}$ have order of magnitude $N^\chi$
- fluctuations of the path $S_t$ under $P^\beta_{N,Nx}$ have order of magnitude $N^\zeta$

In the 1+1 dimensional case these exponents are expected to take the values $\chi = \frac{1}{3}$ and $\zeta = \frac{2}{3}$ independently of $\beta$, provided the i.i.d. weights $\omega(i,x)$ satisfy a moment bound. Furthermore, there are specific predictions for the limit distributions of the scaled quantities: for example, the GUE Tracy-Widom distribution for $\log Z_{N,Nx}(\beta)$. These properties are features of the Kardar-Parisi-Zhang (KPZ) universality class to which these models are expected to belong. See [23, 44] for recent surveys. The KPZ regime should be contrasted with the diffusive regime where $\chi = 0$, $\zeta = 1/2$, and the path satisfies a central limit theorem. Diffusive behavior is known to happen for $d \geq 3$ and small enough $\beta$ [22].

There are three exactly solvable 1+1 dimensional models for which KPZ predictions have been partially proved:

(a) the semidiscrete polymer in a Brownian environment [38]
(b) the log-gamma polymer [42]
(c) the continuum directed random polymer, in other words, the solution of the Kardar-Parisi-Zhang (KPZ) equation [2, 4, 31]

In recent years a number of results have appeared, first for exponents and then for distributional properties. This is not a place for a thorough review, but let us cite some of the relevant papers: [4, 15, 16, 17, 10, 24, 37, 42, 43]. To do justice to history, we mention also that KPZ results appeared earlier for zero-temperature polymers (the $\beta \to \infty$ limit of (1.1)–(1.2), known as last-passage percolation), beginning with the seminal papers [7, 30].

Getting closer to the topic of the present paper, physics paper [1] introduced the study of the intermediate disorder regime in model (1.1)–(1.2). This means that $\beta$ is scaled to zero as $N \to \infty$ by taking $\beta = \beta_0 N^{-\alpha}$. The window of interest is $0 \leq \alpha \leq 1/4$. At $\alpha = 0$ one sees the KPZ behavior with exponents $\chi = 1/3$ and $\zeta = 2/3$. At $\alpha = 1/4$ one has the critical case where exponents are diffusive ($\chi = 0$ and $\zeta = 1/2$) but fluctuations are different [3]. When $\alpha > 1/4$ the disorder is irrelevant and the polymer behaves like a simple random walk [2].

Article [1] conjectured the exponents for the entire range:

\[(1.3) \quad \chi(\alpha) = \frac{1}{3}(1 - 4\alpha) \quad \text{and} \quad \zeta(\alpha) = \frac{2}{3}(1 - \alpha) \quad \text{for} \quad 0 \leq \alpha \leq 1/4.\]

In this paper we derive these intermediate disorder exponents for the semidiscrete polymer in the Brownian environment (introduced in [38], hence also called the O’Connell-Yor model). Along the way we offer some improvements to the earlier work [43] which treated the $\alpha = 0$ case. This model has two versions: a stationary version with particular boundary conditions that render the process of $\log Z$ increments shift-invariant, and the point-to-point version without boundary conditions represented by (1.1)–(1.2) above. In general we have better results for the stationary version. In case the reader
We list below the precise contributions of our paper:

(i) For the free energy we derive the exponent \( \chi(\alpha) = \frac{1}{3}(1 - 4\alpha) \) for the entire range \( 0 \leq \alpha \leq 1/4 \) for the stationary version and for \( 0 \leq \alpha < 1/4 \) for the point-to-point version. For the fixed temperature case \( \alpha = 0 \) the lower bound \( \chi \geq 1/3 \) for the point-to-point version was not covered in [43], but is done here.

(ii) We have the path exponent \( \zeta(\alpha) = \frac{2}{3}(1 - \alpha) \) for the stationary version, and the upper bound \( \zeta(\alpha) \leq \frac{2}{3}(1 - \alpha) \) for the point-to-point version.

(iii) Our results refine the prediction (1.3) in the following way. We introduce a macroscopic time parameter \( \tau > 0 \) and conclude that the fluctuations of \( \log Z_{\tau N, \tau N x}(\beta_0 N^{-\alpha}) \) are of magnitude \( \tau^{1/3} N^{\chi(\alpha)} \) while the path fluctuations are of magnitude \( \tau^{2/3} N^{\zeta(\alpha)} \). In other words, in the macroscopic variables we see again the exponents \( \frac{1}{3} \) and \( \frac{2}{3} \).

(iv) In the fixed temperature case \( \alpha = 0 \) the lower bound \( \chi \geq 1/3 \) was already proved in [43] for the stationary version. Here we give a considerably simpler proof of the lower bound, including the case \( \alpha = 0 \).

(v) In the critical case \( \alpha = 1/4 \) we can connect with the KPZ equation. The macroscopic variable \( \tau \) becomes the time parameter of the stochastic heat equation (SHE), and we obtain again the exponent of the stationary Hopf-Cole solution of the KPZ equation, first proved in [10]: \( \text{Var}[\log Z(\tau, 0)] \approx \tau^{4/7} \) where \( Z \) is the solution of SHE. Moreover, we prove similar bounds for solutions where the initial condition is a bounded perturbation of the stationary initial condition.

Some further comments about the state of the field and the place of this work are in order. Presently one can identify the following three approaches to fluctuations of polymer models and of models in the KPZ class more broadly.

(a) The **resolvent method.** This is a fairly robust method used to establish superdiffusivity. It is quite general, for it can often be applied as long as a model has a tractable invariant distribution \([13, 33, 39, 40, 41, 46]\). A drawback of the method is that often it cannot determine the exact exponents but provides only bounds on them. However, here are two exceptions. In [46] the scaling exponent of a 2d TASEP model is identified exactly. In [39, 40] the method is used to give a comparison between the solvable 1d TASEP and more general 1d exclusion models to show that the scaling exponents are the same.

(b) The **coupling method**, represented by the present work and references \([8, 9, 10, 11, 19, 42, 43]\). This approach is able to identify exact exponents, but so far has depended on the presence of special structures such as a Burke-type property.

(c) **Exact solvability methods.** When it can be applied, this approach leads to the sharpest results, namely Tracy-Widom limit distributions. But it is the most specialized and technically very heavy. This approach became available for the semidiscrete polymers after determinantal expressions where found for the
The free energy exponent $\chi = 1/3$ in the fixed temperature case ($\alpha = 0$) is also a consequence of the distributional limits for $\log Z$ in [15, 16]. Presently these results cover the point-to-point case of the semidiscrete polymer for the entire fixed temperature range $0 < \beta < \infty$. It is expected that these methods should work also in the intermediate disorder regime (personal communication from the authors). However, these works do not yet give anything on the stationary versions of the models, or on the path fluctuations in either the point-to-point or stationary version. And even in the cases covered by the distributional limits, our results give sharper bounds on moments and deviations that weak limits alone cannot provide.

The open problem that remains in the coupling approach used here is the lower bound for the path in the point-to-point case.

One more expected universal feature of polymer exponents worth highlighting here is the scaling relation $\chi = 2\zeta - 1$. This is expected to hold very generally across models and dimensions. The exponents we derive satisfy this identity. There is important recent work on this identity that goes beyond exactly solvable models: first [20], and then [5] with a simplified proof, derived this relation for first passage percolation under strong assumptions on the existence of the exponents. These results are extended to positive temperature directed polymers in [6].

Finally, we point out that the coupling method applied to directed polymers first appeared in the work [42] in the context of discrete polymers in a log-gamma environment. Most of the results of [43] have discrete analogues in [42]. The intermediate regime can also be investigated for the polymers in log-gamma environment. Although this model is formulated for $\beta = 1$, the parameters of the environment can be tuned to emulate the situation $\beta \to 0$. We have obtained proofs for the fluctuation exponents of the log-gamma model in the intermediate scaling regime. The methods are very similar to the ones used here for the semidiscrete polymer model, but involve considerably heavier asymptotics so we decided not to include them in the present paper.

**Organization of the paper.** We introduce the directed polymer in a Brownian environment in its point-to-point and stationary versions and state our main theorems in Sections 1.2.1 and 1.2.2. Their proofs are in Section 2. In Section 1.3 we state our results for the KPZ equation. The proofs are given in Section 3. Some basic estimates on polygamma functions are provided in Section 4.
of $\Psi_1$. See Section 4 for more on polygamma functions and their relations to the Gamma($\theta, r$) distribution.

The environment distribution $\mathbb{P}$ has expectation symbol $\mathbb{E}$. Generically expectation under a probability measure $Q$ is denoted by $E^Q$. To simplify notation we drop integer parts. A real value $s$ in a position that takes an integer should be interpreted as the integer part $\lfloor s \rfloor$.

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1.2. The semi-discrete polymer in a Brownian environment. We begin with the results for the semi-discrete polymer in a Brownian environment. This is a semi-discrete version of the generic polymer model described in (1.1). As already mentioned, the model has two versions: a point-to-point and a stationary version.

1.2.1. Point-to-point semi-discrete polymer. The environment consists of a family of independent one-dimensional standard Brownian motions $\{B_i(\cdot) : i \geq 1\}$. These are two-sided Brownian motions with $B_i(0) = 0$. Polymer paths are nondecreasing càdlàg paths $x : [0, t] \to \mathbb{N}$ with nearest-neighbor jumps, $x(0) = 1$, and $x(t) = n$. A path can be coded in terms of its jump times $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = t$. At level $k$ the path collects the increment $B_k(s_{k-1}, s_k) = B_k(s_k) - B_k(s_{k-1})$. The partition function in a fixed Brownian environment at inverse temperature $\beta > 0$ is, for $(n, t) \in \mathbb{N} \times [0, \infty)$,

\[(1.4) \quad Z_{n,t}(\beta) = \int_{0 < s_1 < \cdots < s_{n-1} < t} \exp [\beta (B_1(0, s_1) + B_2(s_1, s_2) + \cdots + B_n(s_{n-1}, t))] \, ds_{1,n-1}.
\]

In the integral $ds_{1,n-1}$ is short for $ds_1 \cdots ds_{n-1}$. The limiting free energy density was computed for a fixed $\beta$ in [36];

\[(1.5) \quad F(\beta) = \lim_{n \to +\infty} \frac{1}{n} \log Z_{n,n}(\beta) = \inf_{t > 0} \left\{ t \beta^2 - \Psi_0(t) \right\} - 2 \log \beta
\]

\[= \Psi^{-1}_1(\beta^2) \beta^2 - \Psi_0(\Psi^{-1}_1(\beta^2)) - 2 \log \beta \quad \text{for } \beta > 0.
\]

We consider this model in the intermediate disorder regime where $\beta = \beta_0 n^{-\alpha}$ for fixed $\beta_0 \in (0, \infty)$ and $\alpha \in [0, 1/4]$. If $0 < \alpha \leq 1/4$, $\log Z_{n,n}(\beta)$ concentrates asymptotically around the value $n F(\beta_0 n^{-\alpha}) = n + O(n^{1-2\alpha})$. (See (4.3) and (4.2) for the asymptotics of the functions $\Psi_0$ and $\Psi_1$.)

Our first result identifies the free energy fluctuation exponent $\chi = \frac{1}{3}(1 - 4\alpha)$ for the point-to-point semi-discrete polymer in the intermediate disorder regime. In the fixed temperature case ($\alpha = 0$) the upper bound was proved in [43] but a lower bound proof with coupling methods is new even in this case. (To clarify, the correct exponent in the $\alpha = 0$ case has of course been identified in the weak convergence results [15, 16] with exact solvability methods.) Note that we see the intermediate regime exponent on the scaling parameter $n$, but for the macroscopic variable $\tau$ we see the exponent $\frac{1}{3}$ corresponding to the KPZ scaling.
Theorem 1.1. Fix $\alpha \in [0, 1/4]$ and $0 < \beta_0 < \infty$. Let $\beta = \beta_0 n^{-\alpha}$. There exist finite positive constants $C, n_0, b_0, \tau_0$ that depend on $(\alpha, \beta_0)$ such that the following bounds hold. For $\tau \geq \tau_0, n \geq n_0$ and $b \geq b_0$,

\begin{equation}
\mathbb{P}\{ | \log Z_{\tau n, \tau n}(\beta) - \tau n\mathcal{F}(\beta) | \geq b \tau^4 n^2(1-4\alpha) \} \leq Cb^{-3/2}
\end{equation}

and

\begin{equation}
C^{-1} \tau^4 n^2(1-4\alpha) \leq \mathbb{E} | \log Z_{\tau n, \tau n}(\beta) - \tau n\mathcal{F}(\beta) | \leq C \tau^4 n^2(1-4\alpha).
\end{equation}

We turn to the fluctuations of the polymer path. The quenched polymer measure $Q_{n,t,\beta}$ on paths is defined, in terms of the expectation of a bounded Borel function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, by

\[ E^{Q_{n,t,\beta}} f(\sigma_1, \ldots, \sigma_{n-1}) = \frac{1}{Z_{n,t}(\beta)} \int_{0<s_1<\cdots<s_{n-1}<t} f(s_1, \ldots, s_{n-1}) \times \exp[\beta(B_1(0, s_1) + \cdots + B_n(s_{n-1}, t))] \, ds_{1,n-1}. \]

The jump times as functions of the path are denoted by $\sigma_i$. Averaged (or annealed) probability and expectation are denoted by $P_{n,t,\beta}(\cdot) = \mathbb{E} Q_{n,t,\beta}(\cdot)$ and $E_{n,t,\beta}(\cdot) = \mathbb{E} E^{Q_{n,t,\beta}}(\cdot)$.

In the point-to-point setting the path exponent $\zeta$ describes the order of magnitude of the deviations of the path from the diagonal. A path close to the diagonal in the rectangle $\{1, \ldots, n\} \times [0, t]$ would have $\sigma_i \approx it/n$. The next theorem shows that the path exponent $\zeta$ is bounded above by its conjectured value $\frac{3}{2}(1-\alpha)$.

**Theorem 1.2.** Fix $\alpha \in [0, 1/4]$ and $0 < \beta_0 < \infty$. Let $\beta = \beta_0 n^{-\alpha}$. There exist finite positive constants $C, n_0, b_0, \tau_0$ that depend on $(\alpha, \beta_0)$ such that the following bound holds. For all $0 < \gamma < 1$, $\tau \geq \tau_0$, $b \geq b_0$, and $n \geq n_0$,

\begin{equation}
P_{n,t,\beta} \left\{ | \sigma_{\gamma n}n - \gamma \tau n | \geq b \tau^2 n^2(1-\alpha) \right\} \leq Cb^{-3}.
\end{equation}

### 1.2.2. Stationary semi-discrete polymer

The proofs of the above theorems rely on comparison with a stationary version of the model. Enlarge the environment by adding another Brownian motion $B$ independent of $\{B_i\}_{i \geq 1}$. Introduce a parameter $\theta \in (0, \infty)$ and restrict to $\beta = 1$ for a moment. The stationary partition function is, for $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

\begin{equation}
Z_{n,t}^\theta = \int_{-\infty<s_0<s_1<\cdots<s_{n-1}<t} \exp[-B(s_0) + \theta s_0 + B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t)] \, ds_{0,n-1}.
\end{equation}

This model has a useful stationary structure described by [38]. Let $Y_0(t) = B(t)$ and, for $k \geq 1$, define inductively

\begin{equation}
r_k(t) = \log \int_{-\infty}^t e^{Y_{k-1}(s,t) - \theta(t-s) + B_k(s,t)} \, ds
\end{equation}

\begin{equation}
Y_k(t) = Y_{k-1}(t) + r_k(0) - r_k(t).
\end{equation}
Induction shows that
\begin{equation}
Z_{n,t}^\theta e^{B(t) - \theta t} = \exp\left(\sum_{k=1}^{n} r_k(t)\right).
\end{equation}

For each fixed $t \geq 0$, \{r_k(t)\}_{k \geq 1} are i.i.d. and $e^{-r_k(t)}$ has Gamma($\theta$) distribution \cite{38}. Thus the law of $Z_{n,t}^\theta e^{B(t) - \theta t}$ is independent of $t$. This stationarity is part of a broader Burke-type property (see \cite{43}, Section 3.1 for more details).

Extend definition (1.4) to $1 \leq k \leq n \in \mathbb{N}$ and $s < t \in \mathbb{R}$ by
\begin{equation}
Z_{(k,n),(s,t)}(\beta) = \int_{s < s_k < \cdots < s_{n-1} < t} \exp\left[\beta(B_k(s, s_k) + B_{k+1}(s_k, s_{k+1}) + \cdots + B_n(s_{n-1}, t))\right] ds_{k,n-1},
\end{equation}
and abbreviate the $\beta = 1$ case as $Z_{(k,n),(s,t)} = Z_{(k,n),(s,t)}(1)$. The stationary partition function can be recovered by integrating these point-to-point partition functions against the boundary Brownian motion:
\begin{equation}
Z_{n,t}^\theta = \int_{-\infty}^{t} ds_0 e^{-B(s_0) + \theta s_0} Z_{(1,n),(s_0,t)}. \tag{1.13}
\end{equation}

We include the inverse temperature in the stationary partition function by defining
\begin{equation}
Z_{n,t}^{\theta,\beta} = \int_{-\infty < s_0 < s_1 < \cdots < s_{n-1} < t} \exp\left[-\beta B(s_0) + \beta s_0 + \beta(B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t))\right] ds_{0,n-1}. \tag{1.14}
\end{equation}

The following theorem identifies the fluctuation exponent $\chi$ for the stationary model. A key difference between the point-to-point and stationary versions is that KPZ fluctuations appear in the stationary version only in a particular characteristic direction $(n,t)$ determined by the parameters. In other directions the diffusive fluctuations of the boundaries dominate (see \cite{42}, Corollary 2.2, in the context of discrete polymers in a log-gamma environment). Once we choose $\beta = \beta_0 n^{-\alpha}$, to make the diagonal a characteristic direction we are forced to pick $\theta = \beta \Psi_1^{-1}(\beta^2) \sim \beta^{-1}$. To simplify notation we suppress the $n$-dependence of the parameters $\beta$ and $\theta$.

**Theorem 1.3.** Let $\alpha \in [0, 1/4]$, $\beta = \beta_0 n^{-\alpha}$, and $\theta = \beta \Psi_1^{-1}(\beta^2)$. Then there exist positive constants $C_1$, $C_2$, $\tau_0$ depending only on $\alpha$ and $\beta_0$ such that
\begin{equation}
C_1 \tau^{\frac{\alpha}{3} n^{\frac{2}{3}} (1 - 4\alpha)} \leq \text{Var}(\log Z_{\tau n, \tau n}^{\theta,\beta}) \leq C_2 \tau^{\frac{\alpha}{3} n^{\frac{2}{3}} (1 - 4\alpha)} \tag{1.15}
\end{equation}
for all $\tau \geq \tau_0$ and $n \geq 1$.

The stationary quenched polymer measure $Q_{n,t}^{\theta,\beta}$ lives on nondecreasing cádlág paths $x : (-\infty, t] \rightarrow \{0, 1, \ldots, n\}$ with boundary conditions $x(-\infty) = 0$, $x(t) = n$. We represent paths again in terms of jump times $-\infty < \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1} \leq t$ where
\[ x(\sigma_i) = i < i + 1 = x(\sigma_i). \] The path measure is defined by

\[
E^{\theta,\beta}_{n,t} f (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) = \frac{1}{Z^{\theta,\beta}_{n,t}} \int_{-\infty < s_0 < \cdots < s_{n-1} < t} f(s_0, s_1, \ldots, s_{n-1})
\]

\[
\times \exp \left[ -\beta B(s_0) + \beta \theta s_0 + \beta (B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t)) \right] ds_{0:n-1}.
\]

Averaged probability and expectation are denoted by \( P^\theta_{n,t}(\cdot) = \mathbb{E} Q^\theta_{n,t}(\cdot) \) and \( E^\theta_{n,t}(\cdot) = \mathbb{E} E Q^\theta_{n,t}(\cdot) \). When \( \beta = 1 \), we simply remove it from the notation.

In the stationary case we can identify the exact path exponent \( \zeta = \frac{2}{3} (1 - \alpha) \).

**Theorem 1.4.** Let \( \alpha \in [0, \frac{1}{4}] \), \( \beta = \beta_0 n^{-\alpha} \) and \( \theta = \beta \Psi^{-1}_1(\beta^2) \). The following bounds hold for all \( \gamma \in (0, 1) \) and \( 0 < \tau < \infty \).

(Upper bound on the tail.) There is a constant \( C = C(\alpha, \beta_0) > 0 \) such that, if \( \tau n \geq 1 \) and \( b \geq 1 \)

\[
P^\theta_{\tau n, \tau n} \left\{ |\sigma_{\tau n} - \gamma \tau n| > b \tau^\frac{2}{3} n^\frac{2}{3}(1-\alpha) \right\} \leq C b^{-3}.
\]

(Bounds on the first absolute moment.)

\[
P^\theta_{\tau n, \tau n} \left\{ |\sigma_{\tau n} - \gamma \tau n| \leq \tau^\frac{2}{3} n^\frac{2}{3}(1-\alpha) \right\} \leq C_1 \tau^\frac{2}{3} n^\frac{2}{3}(1-\alpha)
\]

### 1.3. The KPZ equation close to equilibrium.

The Kardar-Parisi-Zhang (KPZ) equation was introduced in [31] as a model of a randomly growing interface in \( 1 + 1 \) dimension: if we let \( h(t, x) \) denote the height of the interface at site \( x \in \mathbb{R} \) and time \( t \geq 0 \), then the evolution of the interface can be represented by the stochastic partial differential equation

\[
\partial_t h = \frac{1}{2} \Delta h + \frac{1}{2} (\nabla h)^2 + \mathcal{W},
\]

where \( \mathcal{W} \) is a space-time white noise. We will be interested in initial conditions of the form \( B + \varphi \) where \( B \) is a double-sided one-dimensional Brownian motion and \( \varphi \) is a bounded function. It is well known that the meaning of a solution to equation (1.19) is a delicate matter. We will always consider the so called Cole-Hopf solution: we take \( h = -\log \mathcal{Z} \), where \( \mathcal{Z} \) solves the stochastic heat equation

\[
\partial_t \mathcal{Z} = \frac{1}{2} \Delta \mathcal{Z} + \mathcal{W} \mathcal{Z}
\]

\[
\mathcal{Z}(0, x) = e^{\varphi(x) + B(x)}.
\]

The relation between (1.19) and (1.20)-(1.21) can be seen by a formal application of Itô’s formula. For a more detailed overview of the KPZ equation, we refer the reader to the review [23] and its references. See also [28] for recent rigorous work on solving (1.19).

It is expected that, for a wide family of initial conditions, the fluctuations of \( \log \mathcal{Z}(t, x) \) are of order \( t^{1/3} \). This was first proved in [10] in the stationary case with \( \varphi = 0 \). The proof was based on the convergence of the rescaled height function of the weakly asymmetric exclusion process to the Cole-Hopf solution of the KPZ equation [14], together
with non-asymptotic fluctuation bounds on the current of the asymmetric simple exclusion process \([11]\). It is not clear that this approach can be extended to the case of non-zero \(\varphi\).

When the initial condition is \(Z(0, x) = \delta_0(x)\), the asymptotic distribution of the fluctuations of \(\log Z\) is identified in \([4]\) as the Tracy-Widom distribution. The proof is based on heavy asymptotic analysis of exact formulas for the weakly asymmetric simple exclusion process.

We will extend the result of \([10]\) to the case of a bounded perturbation \(\varphi\). Our approach is different as we use an approximation of \(Z\) by partition functions of the Brownian semidiscrete directed polymer in the critical case \(\alpha = \frac{1}{4}\) rather than by particle systems.

Building on the techniques of \([1]\), it is shown in \([35]\) that a suitable renormalization of the partition function of the semi-discrete model with \(\alpha = \frac{1}{4}\) converges to \(Z^\varphi\). More precisely, let \(\varphi_n(x) = \varphi(-\frac{x}{\sqrt{n}})\) and let

\[
Z^{0,\beta,\varphi}_{n,t} = \int_{-\infty}^t \exp \left[ \varphi_n(s_0) - \beta B(s_0) + \beta \theta s_0 \right] Z_{(1,n)(s_0,t)}(\beta) \, ds_0.
\] (1.22)

The renormalized partition function is

\[
Z_n^\varphi(\tau) = e^{-\frac{1}{2} \tau \sqrt{n}} Z_{\tau n, \tau n}^{0,\beta,\varphi}.
\]

When \(\varphi = 0\), we simply denote this by \(Z_N(\tau)\).

**Theorem 1.5.** \([35]\) Let \(\beta = \beta_n = n^{-1/4}\) and \(\theta = \Psi_1^{-1}(\beta_n^2)\). Then as \(n \to +\infty\), the process \((Z_n^\varphi(\tau), \tau \geq 0)\) converges in law to \((Z^\varphi(\tau,0), \tau \geq 0)\), where \(Z^\varphi\) solves the stochastic heat equation (1.20)–(1.21).

Combined with Theorem 1.3 this gives

**Theorem 1.6.** Let \(\varphi\) be a bounded function and let \(Z^\varphi\) be the solution of the stochastic heat equation (1.20)–(1.21). Then there exist constants \(C_1, C_2, \tau_0 > 0\) such that

\[
C_1 \tau^{\frac{2}{3}} \leq \text{Var} \log Z^\varphi(\tau,0) \leq C_2 \tau^{\frac{2}{3}},
\]

for all \(\tau > \tau_0\).

We note that our results for the path of the stationary polymer could in principle have a meaning in the context of the SHE. In \([2]\), \(Z\) is identified as the partition function of a continuum directed polymer. Theorem 1.4 strongly suggests that the fluctuations of the path of the continuum polymer are of order \(t^{2/3}\), in agreement with the KPZ scaling.

## 2. Proofs for the semi-discrete polymer model

The proofs of our Theorems 1.1, 1.2, 1.3 and 1.4 are given in this section. We first prove the results for the stationary model. The results for the point-to-point model are then done by comparison.
2.1. Preliminaries. We recall some facts from [43]. Throughout this section we take \( \beta = 1 \) as we can reduce the situation to this by Brownian scaling (see Section 2.2). The stationary model can be written as

\[
Z_{n,t}^\theta = \int_0^t e^{-B(s)+\theta s} Z_{(1,n),(s,t)} ds + \sum_{j=1}^n \left( \prod_{k=1}^j e^{r_k(0)} \right) Z_{(j,n),(0,t)}
\]

where the \( r_k \) processes are defined recursively in (1.10). Recall that the random variables \( r_k(0) \) are i.i.d. and \( e^{-r_k(0)} \) has Gamma(\( \theta \)) distribution.

The processes \( r_k \) and \( Y_k \) give space and time increments of the partition function:

\[
\begin{align*}
    r_k(t) &= \log Z_{k,t}^\theta - \log Z_{k-1,t}^\theta, \\
    Y_k(s,t) &= Y_k(t) - Y_k(s) = \theta(t-s) - \log Z_{k,t} + \log Z_{k,s}.
\end{align*}
\]

The appearance of polygamma functions in our results is natural because of the identities

\[
\mathbb{E}[r_k(t)] = -\Psi_0(\theta) \quad \text{and} \quad \text{Var}[r_k(t)] = -\Psi_1(\theta).
\]

(See Section 4.)

From (2.2), (2.3) and \( Z_{n,t}^\theta = \exp(-B(t)+\theta t) \) one immediately gets

\[
\mathbb{E}(\log Z_{n,t}^\theta) = -n\Psi_0(\theta) + \theta t.
\]

The variance of \( \log Z_{n,t}^\theta \) was computed in Theorem 3.6 in [43]:

\[
\text{Var}(\log Z_{n,t}^\theta) = n\Psi_1(\theta) - t + 2E_{n,t}(\sigma_0^+) = t - n\Psi_1(\theta) + 2E_{n,t}(\sigma_0^-) = E_{n,t}^\theta |\sigma_0|.
\]

We will also need the following lemma from [43]:

**Lemma 2.1.** [43, Lemma 4.3] For \( \theta, \lambda > 0 \),

\[
|\text{Var}(\log Z_{n,t}^\lambda) - \text{Var}(\log Z_{n,t}^\theta)| \leq n|\Psi_1(\lambda) - \Psi_1(\theta)|.
\]

Finally, we note a shift-invariance of the stationary model (Remark 3.1 of [43]):

\[
E_{Q_{n,t}^\theta} f(\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) \overset{d}{=} E_{Q_{n,0}^\theta} f(t + \sigma_0, t + \sigma_1, \ldots, t + \sigma_{n-1}).
\]

This follows from the stationarity of \( Z_{n,t}^\theta \exp(B(t) - \theta t) \) by observing that the density of \( (\sigma_0, \ldots, \sigma_{n-1}) \) under \( Q_{n,t}^\theta \) can also be written as

\[
\frac{1}{Z_{n,t}^\theta} \exp[\hat{B}(s_0, t) + \hat{B}_1(s_0, s_1) + \cdots + \hat{B}_n(s_{n-1}, t)] \times 1\{s_0 < \cdots < s_{n-1} < t\}
\]

where \( \hat{B}(u) = B(u) - \theta u/2 \) (and similarly for \( \hat{B}_k \)) and \( \hat{Z}_{n,t}^\theta = Z_{n,t}^\theta \exp(B(t) - \theta t) \).

Using the same ideas one can also show a shift-invariance property in \( n \) (see the proof of Theorem 6.1 in [43]):

\[
E_{Q_{n,t}^\theta} f(\sigma_k, \sigma_{k+1}, \ldots, \sigma_{n-1}) \overset{d}{=} E_{Q_{n-k,t}^\theta} f(\sigma_0, \sigma_1, \ldots, \sigma_{n-k-1}).
\]
2.2. Rescaled models and characteristic direction. For the proofs we scale $\beta$ away via the following identity in law which is obtained by Brownian scaling:

\begin{equation}
Z_{(1,n),(0,t)}(\beta) \overset{d}{=} \beta^{-2(n-1)}Z_{(1,n),(0,\beta^2t)}(1).
\end{equation}

We drop $\beta = 1$ from the notation and write $Z_{(1,n),(0,t)} = Z_{(1,n),(0,t)}(1)$. The regime $\beta = \beta_0 n^{-\alpha}$ corresponds to studying $Z_{(1,n),(0,\beta_0^2 n^{1-2\alpha})}$. Similarly we scale $\beta$ away from the stationary partition function (1.14):

\begin{equation}
Z_{n,t}^{\theta,\beta} \overset{d}{=} \beta^{-2n}Z_{n,\beta^2 t}^{\beta^{-1}\theta,1}.
\end{equation}

As we take $(n, t)$ to infinity, we have to follow approximately a characteristic direction determined by $\theta$. The characteristic direction is found by minimizing the right-hand side of (2.4) with respect to $\theta$, or equivalently, by arranging the cancellation of the first two terms on the right of (2.5). The following condition on the triples $(n, t, \theta)$ expresses the fact that $(n, t)$ is close to the characteristic direction:

\begin{equation}
|n\Psi_1(\theta) - t| \leq \kappa n^{2/3}\theta^{-4/3} \quad \text{with a fixed constant } \kappa \geq 0.
\end{equation}

By the scaling relation (2.10), we can see that the choice of parameters in Theorem 1.3 corresponds to the characteristic direction.

2.3. Upper bounds for the stationary model. The main tool for our upper bounds is the following lemma. The proof of the upper bound in Theorem 1.3 will follow by a particular choice of the parameters and can be found at the end of this section.

**Lemma 2.2.** Fix $\theta_0 > 0$ and $\kappa \geq 0$. Assume that $\theta > 0$, $n \in \mathbb{N}$ and $t > 0$ satisfy (2.11) and $\theta_0 \leq \theta \leq \theta_0^{-1}\sqrt{n}$. Then there are constants $\delta > 0$ and $c < \infty$ that depend only on $\theta_0$ such that, for all $n\theta^{-1} \geq u \geq 2\kappa n^{2/3}\theta^{-4/3}$, we have

\begin{equation}
P\{Q_{n,t}^\theta(\sigma_0^+ \geq u) \geq e^{-\delta \theta^2 u^2 n^{-1}}\} \leq c(1 + \kappa)\frac{n^{8/3}}{\theta^{16/3} u^4} + c\frac{n^2}{\theta^4 u^3},
\end{equation}

\begin{equation}
E_{n,t}^\theta(\sigma_0^+) \leq c(1 + \kappa)\frac{n^{2/3}}{\theta^{4/3}},
\end{equation}

\begin{equation}
\text{and} \quad P_{n,t}^\theta(\sigma_0^+ \geq b n^{2/3} \theta^{-4/3}) \leq c(1 + \kappa) b^{-3} \quad \text{for } b \geq (2\kappa) \lor 1.
\end{equation}

The same bounds hold for $\sigma_0^-$, and for $u \geq n\theta^{-1}$ we also have

\begin{equation}
P\{Q_{n,t}^\theta(\sigma_0^- \geq u) \geq e^{-\delta \theta u} \} \leq 2e^{-\delta \theta u}.
\end{equation}

Finally, we also have

\begin{equation}
\text{Var}(\log Z_{n,t}^\theta) \leq c(1 + \kappa)n^{2/3}\theta^{-4/3}.
\end{equation}

**Proof.** We introduce $a = \delta \theta^2 u^2 n^{-1}$. We fix the positive parameter $r$ (its value will be determined later), and set $\lambda = \theta + ru\theta^2 n^{-1}$. From the definition of the path measure,
we have

\[ Q_{n,t}^\theta(\sigma_0^+ \geq u) = \frac{1}{Z_{n,t}^\theta} \int_{u<s_0<\ldots<s_{n-1}<t} \exp[-B(s_0) + \theta s_0 + B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t)] \, ds_{0,n-1} \]

\[ \text{(2.17)} \]

\[ \leq \frac{1}{Z_{n,t}^\theta} \int_{u<s_0<\ldots<s_{n-1}<t} e^{(\theta-\lambda)u} \exp[-B(s_0) + \lambda s_0 + B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t)] \, ds_{0,n-1} \]

\[ \leq \frac{Z_{n,t}^\lambda}{Z_{n,t}^\theta} e^{(\theta-\lambda)u}. \]

Consequently,

\[ \mathbb{P}\{Q_{n,t}^\theta(\sigma_0^+ \geq u) \geq e^{-a}\} \leq \mathbb{P}\{\log Z_{n,t}^\lambda - \log Z_{n,t}^\theta \geq (\lambda - \theta)u - a\} \]

\[ \text{(2.18)} \]

\[ = \mathbb{P}\{\log Z_{n,t}^\lambda - \log Z_{n,t}^\theta \geq n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\lambda - \theta)u - a\}, \]

where \( \overline{X} = X - EX \) denotes the centering of the random variable \( X \). Because of \( (2.4) \) we have \( \log Z_{n,t}^\lambda = \log Z_{n,t}^\theta + n\Psi_0(\theta) - \theta t \).

By the monotonicity of \( \Psi_2(z) = \Psi_0^n(z) \) (see \( (4.1) \)) for any \( \lambda > \theta > 0 \)

\[ 0 \geq \Psi_0(\lambda) - \Psi_0(\theta) - \Psi_1(\theta)(\lambda - \theta) = \int_{\theta}^\lambda \int_{\theta}^y \Psi_2(z) \, dz \, dy \geq -\frac{1}{2}|\Psi_2(\theta)|(\lambda - \theta)^2. \]

Assumptions \( (2.11) \) and \( u \geq 2\kappa n^{2/3}\theta^{-4/3} \) imply

\[ \text{(2.19)} \]

\[ |n\Psi_1(\theta) - t| \leq u/2 \]

and so the right-hand side inside the probability \( (2.18) \) develops as follows:

\[ n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\lambda - \theta)u - a \]

\[ = n(\Psi_0(\lambda) - \Psi_0(\theta) - \Psi_1(\theta)(\lambda - \theta)) + n(\Psi_1(\theta) - t)(\lambda - \theta) + u(\lambda - \theta) - a \]

\[ \geq -\frac{n}{2}|\Psi_2(\theta)|(\lambda - \theta)^2 + \frac{u}{2}(\lambda - \theta) - a \geq \left( -\frac{r^2c_0}{2} + \frac{r}{2} - \delta \right) \theta^2 u^2 n^{-1} \]

\[ \geq \delta \theta^2 u^2 n^{-1}. \]

Above we introduced

\[ \text{(2.20)} \]

\[ c_0 = \sup_{x \geq \theta_0} |\Psi_2(x)| x^2 < \infty \]

(which is finite by \( (4.1) \) and \( (4.2) \)) and then chose \( r = (2c_0)^{-1} \) and \( \delta = c_0^{-1}/16 \).
In the following $c$ denotes a constant that depends only on $\theta_0$, but may change from line to line. From line (2.18), using Lemma 2.1 we get

\[
P \{Q^\theta_{n,t}(\sigma_0^+ \geq u) \geq e^{-a} \} \leq P \left\{ \log Z^\lambda_{n,t} - \log Z^\theta_{n,t} \geq \delta \theta^2 u^2 n^{-1} \right\} \leq c \frac{n^2}{\theta^4 u^4} \text{Var} [\log Z^\lambda_{n,t} - \log Z^\theta_{n,t}] 
\]

(2.21)

Above we used (2.5), (2.19), and the following estimate:

\[
|\Psi_1(\lambda) - \Psi_1(\theta)| \leq |\Psi_2(\theta)(\lambda - \theta)| \leq c_0 \theta^{-2}(\lambda - \theta) = c_0 ru/n = u/(2n).
\]

Let $u_0 \geq 2\kappa n^{2/3} \theta^{-4/3}$.

\[
E^\theta_{n,t}(\sigma_0^+) \leq u_0 + \int_{u_0}^{t} du P^\theta_{n,t} [\sigma_0^+ \geq u] 
\]

\[
\leq u_0 + \int_{u_0}^{t} du \left\{ \int_{e^{-a}}^1 dr P^\theta_{n,t}(\sigma_0^+ \geq u) \geq r \right\} + e^{-a} \]

\[
\leq u_0 + \frac{c n^2}{\theta^4} \int_{u_0}^{t} du \left( E^\theta_{n,t}(\sigma_0^+) \frac{1}{u^4} + \frac{1}{u^3} \right) + \int_{u_0}^{t} e^{-\delta \theta^2 u^2 n} du 
\]

\[
\leq u_0 + \frac{c n^2}{\theta^4 u_0^3} E^\theta_{n,t}(\sigma_0^+) + \frac{c n^2}{\theta^4 u_0^2} + \frac{\delta^{-1} n}{2\theta^2 u_0} e^{-\delta \theta^2 u_0^2 n}. 
\]

The last term comes from $\int_{m}^\infty e^{-x^2} dx \leq (2m)^{-1} e^{-m^2}$ for $m > 0$. Now choose $u_0 = 2(1 + c + \kappa) n^{2/3} \theta^{-4/3}$. The inequality above can be rearranged to give

\[
E^\theta_{n,t}(\sigma_0^+) \leq (c + 4\kappa) \frac{n^{2/3}}{\theta^{4/3}} + \frac{c n^{1/3}}{\theta^{2/3}} \exp(-\delta n^{1/3} \theta^{-2/3})
\]

(2.22)

\[
\leq c(1 + \kappa) \frac{n^{2/3}}{\theta^{4/3}}.
\]

Above, $c$ has been redefined but still depends only on $\theta_0$. This proves (2.13) for $\sigma_0^+$. Substitute this back up in (2.21) to get

(2.23)

\[
P \{Q^\theta_{n,t}(\sigma_0^+ \geq u) \geq e^{-\delta \theta^2 u^2 n^{-1}} \} \leq c(1 + \kappa) \frac{n^{8/3}}{\theta^{10/3} u^4} + c(1 + u) \frac{n^2}{\theta^4 u^4}
\]

which proves (2.12) as $\theta \leq \theta_0 \sqrt{n}$ . To prove (2.14) apply (2.12) with $u = bn^{2/3} \theta^{-4/3}$, and use $b \geq (2\kappa) \vee 1$:

\[
P^\theta_{n,t} \{\sigma_0^+ \geq bn^{2/3} \theta^{-4/3}\} \leq e^{-\delta \theta^2 u^2 n^{-1}} + P \{Q^\theta_{n,t}(\sigma_0^+ \geq bn^{2/3} \theta^{-4/3}) \geq e^{-\delta \theta^2 u^2 n^{-1}} \} 
\]

(2.24)

\[
\leq e^{-\delta \theta^2 u^2 n^{-1} b^{-2/3}} + c(1 + \kappa) b^{-4} + c b^{-3} 
\]

\[
\leq c(1 + \kappa) b^{-3}.
\]
The proof for $\sigma_0^-$ is similar, but we need some modifications. We take $\lambda = \theta - ru\theta^2n^{-1}$. Note that by choosing $r < 1/2$ and using $n\theta^{-1} \geq u$ we have $\lambda > \theta/2$. Now we use the inequality
\[
Q_{n,t}(\sigma_0^- \geq u) \leq \frac{Z_{n,t}^\lambda}{Z_{n,t}^0} e^{-(\theta-\lambda)u},
\]
instead of (2.17), its proof being similar. Bound (2.12) now follows exactly the same way as for $\sigma_0^+$.

In order to get the bound (2.15) for $u \geq n\theta^{-1}$ we set $\lambda = r\theta$ with $r$ to be specified later and proceed with the proof exactly the same way as in the previous case. We have
\[
\mathbb{P}\{Q_{n,t}(\sigma_0^- \geq u) \geq e^{-\theta u}\} \leq \mathbb{P}\{\log Z_{n,t}^\lambda - \log Z_{n,t}^0 \geq n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\theta - \lambda)u - \delta u\}. 
\]
and
\[
n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\lambda - \theta)u - \delta u 
= n(\Psi_0(\lambda) - \Psi_0(\theta) - \Psi_1(\theta)(\lambda - \theta)) + (n\Psi_1(\theta) - t)(\lambda - \theta) + u(\theta - \lambda) - \delta u 
\geq -\frac{n}{2}|\Psi_2(\lambda)|(\lambda - \theta)^2 + \frac{u}{2}(\theta - \lambda) - \delta u \geq -nC(\theta_0)r^2 + \frac{1}{4}ru\theta - \delta u \geq c_0\theta u
\]
with a fixed positive $c_0$. In order to get the last bound we need to choose $r$ and $\delta$ small enough in terms of $c(\theta_0)$. This gives
\[
\mathbb{P}\{Q_{n,t}(\sigma_0^- \geq u) \geq e^{-\theta u}\} \leq \mathbb{P}\{\log Z_{n,t}^\lambda - \log Z_{n,t}^0 \geq c_0\theta u\}
\leq \mathbb{P}\{\log Z_{n,t}^\lambda + B(t) \geq c_0\theta u/2\} + \mathbb{P}\{\log Z_{n,t}^0 + B(t) \geq c_0\theta u/2\}
\]
By (1.12) $\log Z_{n,t}^\theta + B(t) - \theta t$ is the sum of $n$ i.i.d. Gamma($\theta$) variables. Using the exponential Markov inequality (optimizing in the extra parameter) we get for any $y > 0$ that
\[
\mathbb{P}(\log Z_{n,t}^\theta + B(t) \geq n\theta y) \leq e^{-n\theta(y - \log(1+y))}.
\]
We have $y - \log(1 + y) > K(C)y$ if $y > C$. Since $u > Cn\theta^{-1} > C'n$ we have the bound $c_0'un^{-1}/2 > C_1(c_0, C')$ and from this we get
\[
\mathbb{P}(\log Z_{n,t}^\theta + B(t) \geq \theta c_0 u/2) \leq e^{-c_1\theta u}.
\]
and a similar bound for the $\lambda$ term. This completes the proof of (2.15) for $\sigma_0^-$.

To prove (2.13) for $\sigma_0^-$ we use $\mathbb{E}_{\theta}^{0}\sigma_0^{+} - \sigma_0^{-} = t - n\Psi_1(\theta)$ and the fact that we already have (2.13) for $\sigma_0^+$. To prove (2.14) for $\sigma_0^-$ we can follow (2.24) in the case $bn\frac{\theta}{2} - \frac{n}{4} \leq n\theta^{-1}$ and use the bound (2.15) with a similar argument if $bn\frac{\theta}{2} - \frac{n}{4} \geq n\theta^{-1}$.

Finally, bound (2.16) follows from (2.5), (2.11) and (2.13). \hfill \Box

Proof of the upper bound in Theorem 1.3. Introduce variables
\[
(\tilde{n}, \tilde{t}, \tilde{\theta}) = (n, t, \theta) \Gamma_{\alpha}(\theta, \alpha) \tilde{\theta} = \Psi_1^{-1}(\beta_0^2 n^{-2\alpha}).
\]
By the scaling identity (2.10), $\text{Var}(\log Z_{\tilde{n},\tilde{t},\tilde{\theta}}) = \text{Var}(\log Z_{\tilde{n},\tilde{t}}^\theta)$. Condition (2.11) is satisfied by $(\tilde{n}, \tilde{t}, \tilde{\theta})$ with $\kappa = 0$, $\tilde{\theta} \geq \Psi_1^{-1}(\beta_0^2) > 0$ and $\tilde{\theta} \leq C_2\beta_0^{-2} n^{2\alpha} \leq C'\sqrt{n}$, as long
as \( \tau \geq \tau_0 \) for a constant \( \tau_0 = \tau_0(\beta_0) \). This means that we may apply Lemma 2.2 with \((\bar{n},t,\bar{\theta})\). The bound (2.16) gives

\[
\text{Var} \left[ \log Z_{n,t}^\theta \right] \leq C \frac{\bar{n}^{2/3}}{\theta^{4/3}} \leq C \tau^2 n^2 (1 - 4\alpha)
\]

where \( C \) depends only on \( \beta_0 \).

2.4. Lower bound for the stationary model. In this section we prove the lower bound in Theorem 1.3. Again, the proof will follow by a particular choice of the parameters in the next proposition and can be found at the end of the section.

Proposition 2.3. Let \( \theta_0 > 0 \) and \( \kappa \geq 0 \). There are positive constants \( \delta_1, \delta_2, n_0 \) that depend on \((\kappa, \theta_0)\) such that

\[
\mathbb{P}\{ \log Z_{n,t}^\theta - \mathbb{E}(\log Z_{n,t}^\theta) \geq \delta_1 n^{1/3} \theta^{-2} \} \geq \delta_2
\]

whenever \( n \geq n_0 \), \((n,t,\theta)\) satisfies (2.11), and \( \theta_0 \leq \theta \leq \theta_0^{-1} \sqrt{n} \).

Moreover, under the previous assumptions we also have

\[
\text{Var} [\log Z_{n,t}^\theta] \geq cn^{2/3} \theta^{-2/3}
\]

Proof. It is sufficient to prove estimate (2.26) since the lower bound (2.27) follows from this easily. Fix a constant \( 0 < b < \theta^{1/3}n^{1/3} \) and set \( \lambda = \theta + b\theta^{2/3}n^{-1/3} < 2\theta \) and \( \bar{t} = t + n\Psi_1(\lambda) - n\Psi_1(\theta) \). Then we have

\[
v = t - \bar{t} = n(\Psi_1(\theta) - \Psi_1(\lambda)) \geq n|\Psi_2(\lambda)|(\theta - \lambda) \geq 4^{-1}bn^{2/3}\theta^{-4/3}
\]

where we used \(|\Psi_2(\lambda)| \geq \lambda^{-2} \geq \theta^{-2}/4\) from (4.2). We shall take \( b \in (0,\infty) \) large enough in the course of the argument, which is not problematic as \( \theta^{1/3}n^{1/3} \) will be large for large enough \( n \) by our assumption \( \theta \geq \theta_0 \).

Fix a \( c_1 \in (0,1/2) \). By the shift-invariance (2.6) we have

\[
Q_{n,t}^\lambda(\sigma_0^{+} \leq c_1 v) = Q_{n,t}^\lambda(\sigma_0 \leq c_1 v) \]

\[
\overset{d}{=} Q_{n,t}^\lambda(t - \bar{t} + \sigma_0 \leq c_1 v) = Q_{n,t}^\lambda(\sigma_0 \leq -(1 - c_1)v)
\]

\[
= Q_{n,t}^\lambda(\sigma_0 \geq (1 - c_1)v).
\]

Since \( \lambda \leq 2\theta \), \((n,\bar{t},\lambda)\) satisfies (2.11) with \( \kappa \) replaced by \( 2^{4/3}\kappa \). We can apply the upper bound (2.12) to \( \sigma^- \) with \((n,\bar{t},\lambda)\) and \( u = (1 - c_1)v \) because

\[
(1 - c_1)v \geq \frac{1}{8}bn^{2/3}\theta^{-2/3} \geq 2(2^{4/3}\kappa)\lambda^{-4/3}n^{2/3}
\]

if we choose \( b > 2^{4+4/3}\kappa \). After collecting all the terms on the right of (2.12) we get the upper bound

\[
P(Q_{n,t}^\lambda(\sigma_0^{+} \leq c_1 v) > \varepsilon_0) \leq Cb^{-3}
\]

for any fixed \( \varepsilon_0 > 0 \) if \( b \) is large enough (and hence also \( n \)) relative to \( \varepsilon_0 \) and \( \theta_0 \). This choice of \( b \) is needed to ensure that \( \varepsilon_0 > e^{-\delta\lambda^2u^2n^{-1}} \).

A similar shifting argument gives

\[
Q_{n,t}^\lambda(\sigma_0^{+} \geq (1+c)(t-\bar{t})) \overset{d}{=} Q_{n,t}^\lambda(\sigma_0^{+} \geq c(t-\bar{t})),
\]
and the upper bound (2.12) can be applied with \( \lambda \) and \( \bar{t} \). Hence, we can fix constants \( 0 < c_1 < c_2 < \infty \) such that, for a given \( \varepsilon_0 > 0 \) and large \( n \),

\[
P[Q_{\lambda_1}^{\lambda}\{c_1v \leq \sigma_0 \leq c_2v\} \geq 1 - \varepsilon_0] \geq 1 - Cb^{-3}.
\]

Observe that \( c_1 \) can be taken as close to 0 as we wish.

Introduce temporary notations \( s_1 = c_1v \) and \( s_2 = c_2v \). Assumptions (2.11) and \( \theta \geq \theta_0 > 0 \) guarantee that \( s_2 < t \) for large enough \( n \). Hence

\[
Q_{\lambda_1}^{\lambda}\{s_1 \leq \sigma_0 \leq s_2\} = \frac{1}{Z_{n,t}^{\lambda}n, t} \int_{s_1}^{s_2} e^{-B(s) + \lambda s} Z_{1,n}(s, t) ds.
\]

Using (2.11),

\[
\mathbb{E}(\log Z_{n,t}^{\lambda}) - \mathbb{E}(\log Z_{n,t}^{\theta}) = n(\Psi_0(\theta) - \Psi_0(\lambda)) + (\lambda - \theta)t
\]

\[
\geq n \int_{t}^{\lambda} (\Psi_1(\theta) - \Psi_1(\xi)) d\xi - \kappa n^{2/3}\theta^{-4/3}(\lambda - \theta)
\]

\[
\geq \frac{1}{2}n|\Psi_2(\lambda)||\lambda - \theta|^2 - \kappa b\theta^{-2/3}n^{1/3}
\]

\[
\geq \frac{1}{2}n^{1/3}b^2\theta^{-2/3} - \kappa b\theta^{-2/3}n^{1/3} \geq 2c_s b^2 n^{1/3} \theta^{-\frac{2}{3}},
\]

with the constant \( c_s = \frac{1}{32} \) if we choose \( b \geq 16\kappa \). Then from (2.29)

\[
1 - Cb^{-3} \leq P\left[ Q_{\lambda_1}^{\lambda}\{s_1 \leq \sigma_0 \leq s_2\} \geq 1 - \varepsilon_0 \right]
\]

\[
\leq P\left[ \int_{s_1}^{s_2} e^{-B(s) + \lambda s} Z_{1,n}(s, t) ds \geq (1 - \varepsilon_0)e^{\mathbb{E}(\log Z_{n,t}^{\theta}) + c_s b^2 n^{1/3} \theta^{-\frac{2}{3}}} \right]
\]

\[
+ P\left( Z_{n,t}^{\lambda} \leq e^{\mathbb{E}(\log Z_{n,t}^{\theta}) - c_s b^2 n^{1/3} \theta^{-\frac{2}{3}}} \right).
\]

Bound probability (2.31) with Chebyshev:

\[
P\left( Z_{n,t}^{\lambda} \leq e^{\mathbb{E}(\log Z_{n,t}^{\theta}) - c_s b^2 n^{1/3} \theta^{-\frac{2}{3}}} \right) \leq c_s^{-2} b^{-4} \theta^{\frac{4}{3}} n^{-\frac{2}{3}} \mathbb{V} \mathbb{a} \mathbb{r}(\log Z_{n,t}^{\lambda}).
\]

The variance is estimated by (2.1) and (2.28):

\[
\mathbb{V} \mathbb{a} \mathbb{r}(\log Z_{n,t}^{\lambda}) \leq \mathbb{V} \mathbb{a} \mathbb{r}(\log Z_{n,t}^{\theta}) + n|\Psi_1(\lambda) - \Psi_1(\theta)| \leq C(1 + b)n^{\frac{1}{3}} \theta^{-\frac{2}{3}}.
\]

This implies that (2.31) \( \leq Cb^{-3} \).

Let \( A \) denote the event in probability (2.30):

\[
P(A) = P\left[ e^{-B(s_1) + \lambda s_1} \int_{s_1}^{s_2} e^{-B(s_1, s) + \lambda(s - s_1)} Z_{1,n}(s, t) ds \geq (1 - \varepsilon_0)e^{\mathbb{E}(\log Z_{n,t}^{\theta}) + c_s b^2 n^{1/3} \theta^{-\frac{2}{3}}} \right]
\]

\[
= P\left[ e^{-B(s_1) + \theta s_1} \int_{s_1}^{s_2} e^{-B(s_1, s) + \lambda(s - s_1)} Z_{1,n}(s, t) ds \geq (1 - \varepsilon_0)e^{\mathbb{E}(\log Z_{n,t}^{\theta}) + c_s b^2 n^{1/3} \theta^{-\frac{2}{3}} - (\lambda - \theta)s_1} \right]
\]

We wish to replace \( \lambda \) by \( \theta \) inside the integral to match it with the parameter in \( \mathbb{E}(\log Z_{n,t}^{\theta}) \) on the right-hand side. For this we use the Cameron-Martin-Girsanov formula to add a drift \( \lambda - \theta \) to the Brownian motion \( \{ B(s_1, s) : s_1 \leq s \leq s_2 \} \). Note
that the other random objects in the event \( A \), namely \( \{B(s_1); B_i(\cdot) : 1 \leq i \leq n\} \), are independent of \( B(s_1, \cdot) \). Let

\[
\frac{d\tilde{P}}{dP} = e^{(\lambda-\theta)B(s_1,s_2)-\frac{1}{2}(\theta-\lambda)^2(s_2-s_1)}
\]

so that, under \( \tilde{P} \), \( B(s_1, s) \overset{d}= \tilde{B}(s_1, s) + (\lambda - \theta)(s - s_1) \) where \( \tilde{B} \) is a standard Brownian motion. By Cauchy-Schwarz

\[
\mathbb{P}(A) = \tilde{\mathbb{E}} \left[ \left( \frac{d\tilde{P}}{dP} \right)^2 \right] = \sqrt{\mathbb{E} \left[ \left( \frac{d\tilde{P}}{dP} \right)^2 \right]} \sqrt{\tilde{\mathbb{P}}(A)}.
\]

The first expectation is finite:

\[
\tilde{\mathbb{E}} \left[ \left( \frac{d\tilde{P}}{dP} \right)^2 \right] = \tilde{\mathbb{E}}e^{2(\lambda-\theta)B(s_1,s_2)+\frac{1}{2}(\theta-\lambda)^2(s_2-s_1)} = \tilde{\mathbb{E}}e^{2(\lambda-\theta)(\tilde{B}(s_1,s_2)+(\lambda-\theta)(s-s_1))+\frac{1}{2}(\theta-\lambda)^2(s_2-s_1)}
\]

where \( C \) depends only on \( \theta_0 \). We bound the probability \( \tilde{\mathbb{P}}(A) \) as follows: recall \( c_0 \) from \( (2.20) \),

\[
\tilde{\mathbb{P}}(A) = \tilde{\mathbb{P}} \left[ e^{B(s_1)+c_0 \int_{s_1}^{s_2} e^{-B(s_1,s)+\lambda(s-s_1)} Z_{\tau_1,n}(s,t) \, ds} \geq (1 - \varepsilon_0) e^{\varepsilon \log Z_{\tau_1,n}^{\theta} + c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}} - (\lambda-\theta)s_1} \right]
\]

\[
\leq \mathbb{P} \left( Z_{\tau_1,n}^{\theta} \geq (1 - \varepsilon_0) e^{\varepsilon \log Z_{\tau_1,n}^{\theta} + c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}} - (\lambda-\theta)s_1} \right)
\]

\[
\leq \mathbb{P} \left( \frac{\log Z_{\tau_1,n}^{\theta} \geq \log(1 - \varepsilon_0) + c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}} - c_0 c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}} - c_0 c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}} \right)
\]

where the last line follows after choosing \( c_1 \) small enough.

Put the estimates back on lines \((2.30)-(2.31)\) to conclude that

\[
(1 - C b^{-3}) \exp(-C b^3) \leq \mathbb{P} \left\{ \frac{\log Z_{\tau_1,n}^{\theta} \geq c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}} }{c_* b^2 n \frac{1}{3} \theta^{-\frac{2}{3}}} \right\}
\]

for a constant \( C \) that depends only on \( (\kappa, \theta_0) \). This completes the proof of the proposition. \( \square \)

**Proof of the lower bound in Theorem 1.3.** As in the upper bound proof, condition \((2.11)\) with \( \kappa = 0 \) is satisfied by variables \((\tilde{n}, t, \tilde{\theta})\) from \((2.25)\). By \((4.1)\) and \((4.2)\), for any \( 0 < y_0 < \infty \) there exists a constant \( C(y_0) \) such that \( \Psi^{-1}_1(y) \leq C(y_0)/y \) for \( y \in (0, y_0] \). Hence if \( \alpha > 0, \tilde{\theta} \) satisfies \( 0 < \Psi^{-1}_1(\beta_0) \leq \tilde{\theta} \leq C_1 \beta_0^{-2} n^{2\alpha} \leq C_2 \sqrt{n} \) if \( \tau \geq \tau_0(\beta_0) \), and Proposition 2.3 can be applied. If \( \alpha = 0 \) the hypothesis of Proposition 2.3 is immediately true. By the scaling identity \((2.10)\) this proposition gives, with a constant \( c > 0 \) that depends on \( \beta_0 \),

\[
\text{Var}(\log Z_{\tau_1,n}^{\beta_0}) = \text{Var}(\log Z_{\tilde{n},t}^{\tilde{\theta}}) \geq c n^{2/3} \tilde{\theta}^{-4/3} \geq c n^{2/3} \tilde{\theta}^{2/3} \frac{1}{3} (1 - 4\alpha) \]

\( \square \)
2.5. Bounds on the path for the stationary model.

Proof of Theorem 1.4. We start with the proof of the upper bound (1.17). We introduce the familiar rescaling \( \tilde{\theta} = \Psi^{-1}_1(\beta_0^2 n^{-2\alpha}), \tilde{n} = \tau n \) and \( t = \tau \beta_0^2 n^{1-2\alpha} \). Then

\[
P^{\theta, \beta}_{\tau n, \tau n} \left\{ |\sigma_{\gamma \tau n} - \gamma \tau n| > b r_2 \beta^{-2} n^{2(1-\alpha)} \right\} \leq P^{\tilde{\theta}}_{\tilde{n}, t} \left\{ |\sigma_{\gamma \tilde{n}} - \gamma t| > b \beta_0^{-2} \tilde{n}^{2} \tilde{\theta}^{-\frac{4}{3}} \right\}
\]

(2.32)

\[
P^{\beta}_{(1-\gamma)\tilde{n}, (1-\gamma)\tilde{t}} \left\{ |\sigma_0| > b \beta_0^{-2} \tilde{n}^{2} \tilde{\theta}^{-\frac{4}{3}} \right\} \leq C b^{-3}.
\]

The first inequality is from Brownian scaling (2.10). After the change of variable from the Brownian scaling step, the quantity on the right of the inequality inside the braces develops as follows:

\[
\beta^2 b \tau^{\frac{2}{3}} n^{2(1-\alpha)} = \beta_0^2 b r_2 \beta^{-2} n^{2-\frac{8}{3} \alpha} \geq \beta_0^{-2} \beta \beta_0^2 n^{2} \tilde{\theta}^{-\frac{4}{3}}
\]

using \( \Psi^{-1}_1(x) \geq x^{-1} \) (4.2). The second step (equality) in (2.32) comes from shift invariance in \( t \) and \( n \) (2.6) and (2.8). The last inequality in (2.32) is the upper bound (2.14) for \( \sigma^\pm_0 \). Note that \((1 - \gamma) \tilde{n}, (1 - \gamma) \tilde{t}, \tilde{\theta}\) satisfies (2.11) with \( \kappa = 0 \). The constant \( c \) from (2.14) depends only on the lower bound \( \tilde{\theta}_0 = \Psi^{-1}_1(\beta_0^2) \) and so \( C \) above depends only on \( \beta_0 \). This completes the proof of (1.17).

We now prove the bound (1.18): using Brownian scaling (2.10) and shift invariance (2.6), (2.8) again,

\[
E^{\theta, \beta}_{\tau n, \tau n} |\sigma_{\gamma \tau n} - \gamma \tau n| = \beta^{-2} E^{\tilde{\theta}}_{\tilde{n}, \tilde{t}} |\sigma_{\gamma \tilde{n}} - \gamma \tilde{t}| = \beta^{-2} E^{\tilde{\theta}}_{(1-\gamma)\tilde{n}, (1-\gamma)\tilde{t}} |\sigma_0|.
\]

By (2.5)

\[
\beta^{-2} E^{\tilde{\theta}}_{(1-\gamma)\tilde{n}, (1-\gamma)\tilde{t}} |\sigma_0| = \beta_0^{-2} n^{4 \alpha} \text{Var}[\log Z^{\tilde{\theta}}_{(1-\gamma)\tilde{n}, (1-\gamma)\tilde{t}}].
\]

Now using (2.16) and (2.27) we have

\[
C^{-1}_1 (1 - \gamma)^{2/3} n^{2/3} \tilde{\theta}^{-4/3} \leq \text{Var} \left[ \log Z^{\tilde{\theta}}_{(1-\gamma)\tilde{n}, (1-\gamma)\tilde{t}} \right] \leq C_1 ((1 - \gamma)^{2/3} n^{2/3} \tilde{\theta}^{-4/3} + 1)
\]

Using the asymptotics for \( \tilde{\theta} \) we get

\[
C^{-1}_1 \tau^{2/3} n^{2(1-\alpha)} \leq E^{\theta, \beta}_{\tau n, \tau n} |\sigma_{\gamma \tau n} - \gamma \tau n| \leq C_1 \tau^{2/3} n^{2(1-\alpha)}.
\]

2.6. Bounds for the point-to-point model. This section derives bounds on the path and free energy fluctuations in the point-to-point case without boundaries, with \( \beta = 1 \), uniformly in \((n, t, \theta)\). Theorems 1.1 and 1.2 follow after a Brownian scaling step. For \( n \in \mathbb{N}, t > 0 \) and events \( D \) on the paths write \( Z_{n,t}(D) = Z^\theta_{n,t} Q^\theta_{n,t}(D) \) for the unnormalized quenched measure.

**Theorem 2.4.** Fix \( 0 < \theta_0 < \infty \). Let \( \theta = \Psi^{-1}_1(t/n) \) satisfy \( \theta_0 \leq \theta \leq \theta_0^{-1} \sqrt{n} \). Then there exist constants \( b_0, C \) that depend only on \( \theta_0 \) so that for \( n \geq n_0, b \geq b_0 \) we have

\[
P \left\{ |\log Z_{(1,n),(0,t)} - (\theta t - n \Psi_0(\theta))| \geq b n^{\frac{1}{2} \theta^{-2}} \right\} \leq C b^{-3/2} + \theta^{-b n^{\frac{1}{2} \theta^{-2}}} \tag{2.33}
\]

and

\[
C^{-1}(n^{\frac{1}{2} \theta^{-2}} - \log n) \leq E |\log Z_{(1,n),(0,t)} - (\theta t - n \Psi_0(\theta))| \leq C n^{\frac{1}{2} \theta^{-2}} + \theta e^{-b n^{\frac{1}{2} \theta^{-2}}} \tag{2.34}
\]
Remark 2.5. The basic strategy of our proofs is the following. If we consider $Z_{n,t}^\theta$ with \( \theta \) defined according to the theorem then $E_{n,t}^\theta \sigma_0 = 0$. Since we expect $\sigma_0$ to be fairly close to its mean, this would suggest that $Z_{n,t}^\theta$ is fairly close to $Z_{(1,n),(0,t)}$. The main components of the proofs will rely on comparisons between the partition functions of the two models and on the results proved about the stationary model.

Note that the centering $\theta - n \Psi_0(\theta)$ inside probability (2.33) is right choice, as it can be seen from the exact expression for the free energy (1.5) and Brownian scaling.

Proof of the lower bound in (2.34). Note that $(n, t, \theta)$ satisfy (2.11) with $\kappa = 0$. Let

\[
(2.35) \quad f_n = \theta t - n \Psi_0(\theta) = E(\log Z_{n,t}^\theta).
\]

Define the event $A = \{ \log Z_{n,t}^\theta \geq f_n + \delta_1 n^{\frac{1}{3}} \theta^{-\frac{2}{3}} \}$. By Proposition 2.3 there exist $\delta_1, \delta_2 > 0$ such that

\[
(2.36) \quad P(A) \geq \delta_2.
\]

From (2.1) we get the simple bound

\[
(2.37) \quad Z_{n,t}^\theta \geq e^{r_1(0)} Z_{(1,n),(0,t)}.
\]

Utilizing this,

\[
\begin{align*}
\delta_1 \delta_2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}} & \leq \delta_1 P(A) n^{\frac{1}{3}} \theta^{-\frac{2}{3}} \leq E(\theta - \log Z_{(1,n),(0,t)}) 1_{A^c} - E r_1(0) 1_{A^c} \\
& \leq E|f_n - \log Z_{(1,n),(0,t)}| + E[r_1(0)] \\
& \leq E|f_n - \log Z_{(1,n),(0,t)}| + C(\theta_0) \log n.
\end{align*}
\]

We used $E|r_1(0)| \leq \sqrt{E r_1(0)^2} = \sqrt{\Psi_0(\theta) + \Psi_0(\theta)^2}$ from (2.3). This proves the lower bound in (2.34). \( \square \)

Proof of the upper bounds in (2.33) and (2.34). Inequality (2.37) gives

\[
P(\log Z_{n,t}^\theta - \log Z_{(1,n),(0,t)} \leq -b n^{\frac{1}{3}} \theta^{-\frac{2}{3}} \leq P(e^{-r_1(0)} \geq e^{b n^{\frac{1}{3}} \theta^{-\frac{2}{3}}}) \leq \theta e^{-b n^{\frac{1}{3}} \theta^{-\frac{2}{3}}}
\]

where the last step comes from Markov’s inequality and $e^{-r_1(0)} \sim \text{Gamma}(\theta)$. Estimate (2.16) gives the Chebyshev bound

\[
P\left(\left|\log Z_{n,t}^\theta - f_n\right| \geq b n^{\frac{1}{3}} \theta^{-\frac{2}{3}}\right) \leq b^{-2} n^{-\frac{2}{3}} \theta^4 \text{Var}(\log Z_{n,t}^\theta) \leq C b^{-2} \leq C b^{-3/2}
\]

which gives

\[
(2.38) \quad P\left(\log Z_{(1,n),(0,t)} - f_n \geq b n^{\frac{1}{3}} \theta^{-\frac{2}{3}}\right) \leq C b^{-3/2} + \theta e^{-b n^{\frac{1}{3}} \theta^{-\frac{2}{3}}}.
\]
The bound on the other tail will be proved in two steps. In Lemma 2.9 below we will show that if \( b > c_0 n^{2/3} \theta^2 / 3 \) then
\[
(2.39) \quad \mathbb{P} \left( \log Z_{(1,n),(0,t)} - (\theta t - n \Psi_0(\theta)) \leq b n^{\frac{1}{3}} \theta^{-\frac{2}{3}} \right) \leq C e^{-b n^{1/3} \theta^{1/3}}.
\]
In Lemma 2.8 below we will show that if \( b \leq c_0 n^{2/3} \theta^2 / 3 \) then there are constants \( C \) and \( n_0 \) depending on \( c_0 \) and \( \theta_0 \) so that
\[
(2.40) \quad \mathbb{P} \left( \frac{Z_{n,t}^\theta}{Z_{(1,n),(0,t)}} \geq e^{b n^{\frac{1}{3}} \theta^{-\frac{2}{3}}} \right) \leq C b^{-3/2}
\]
for all \( n > n_0 \). Using the Chebyshev bound again with (2.40) and then combining it with (2.39) we get
\[
(2.41) \quad \mathbb{P} \left( \log Z_{(1,n),(0,t)} - f_n \leq -b n^{\frac{1}{3}} \theta^{-\frac{2}{3}} \right) \leq C b^{-3/2} + C e^{-b n^{1/3} \theta^{1/3}}.
\]
The estimates (2.38) and (2.41) together establish (2.33).

Integrating out \( b \) in (2.33) gives
\[
\mathbb{E} |\log Z_{n,t}^\theta - \log Z_{(1,n),(0,t)}| \leq C n^{\frac{1}{3}} \theta^{-\frac{2}{3}} + C \theta e^{-b_0 n^{\frac{1}{3}} \theta^{-\frac{2}{3}}}.
\]
Combining the above with
\[
(2.42) \quad \mathbb{E} |\log Z_{n,t}^\theta - f_n| \leq \sqrt{\text{Var} \log Z_{n,t}^\theta} \leq C n^{\frac{1}{3}} \theta^{-\frac{2}{3}}
\]
verifies the upper bound of (2.34). \( \Box \)

Except for the technical estimates postponed to Section 2.7, this completes the proof of Theorem 2.4.

**Proof of Theorem 1.1.** Recall that \( \beta = \beta_0 n^{-\alpha} \). Introduce \( \tilde{n} = \tau n, t = \tau \beta_0^2 n^{1-2\alpha} \) and \( \theta = \Psi^{-1}(t/\tilde{n}) = \Psi^{-1}(\beta_0^2 n^{-2\alpha}) \). From Brownian scaling (2.9) and the explicit free energy density \( F(\beta) \) in (1.5),
\[
\log Z_{\tilde{n},\tau n}(\beta_0 n^{-\alpha}) - \tau n F(\beta)
\]
\[
\overset{\text{d}}{=} -2(\tau n - 1) \log(\beta_0 n^{-\alpha}) + \log Z_{(1,\tau n),(0,\beta_0^2 n^{1-2\alpha})} - \tau n F(\beta_0 n^{-\alpha})
\]
\[
= \log Z_{(1,\tilde{n}),\tilde{n}} - 2(\tilde{n} - 1) \log \beta - \tilde{n}(\theta \beta_0^2 n^{-2\alpha} - \Psi_0(\theta)) - 2 \log \beta
\]
\[
= \log Z_{(1,\tilde{n}),\tilde{n}} - (\theta t - \tilde{n} \Psi_0(\theta)) + 2 \log \beta.
\]
The bounds claimed in Theorem 1.1 follow by quoting Theorem 2.4 for \((\tilde{n}, t)\). Note that because \( \alpha \in [0, 1/4) \), the terms \( \log \beta \) and \( \log n \) are lower order than \( n^{\frac{1}{2}(1-4\alpha)} \), and \( e^{-b n^{1/3} \theta^{2-1/3}} \) is lower order than \( b^{-3/2} \). \( \Box \)

We now turn to the path fluctuations for the model without boundaries.

**Theorem 2.6.** Fix \( 0 < \theta_0 < \infty \) and \( 0 < \varepsilon_0, \varepsilon_1 < 1/2 \). Then there exist positive constants \( b_0, C, C_1 \) that depend only on \( \theta_0, \varepsilon_0 \) such that the following holds for \( n \in \mathbb{N} \) and \( t > 0 \). If \( \theta = \Psi^{-1}(t/n) \) satisfies
\[
(2.43) \quad \theta_0 \leq \theta \leq \theta_0^{-1} n^{1/2-\varepsilon_0}
\]
then for \( n \geq n_0, b \geq b_0, \) and \( \varepsilon_1 \leq \gamma \leq 1 - \varepsilon_1 \) we have

\[
P_{(1,n),(0,t)}^{\beta = 1} \left( \left| \sigma_{[n\gamma]} - \gamma t \right| > bn^2 \theta^{-\frac{2}{3}} \right) \leq Cb^{-3}.
\]

**Proof.** Since \( 0 \leq \sigma_k \leq t = n\Psi_1(\theta) \leq cn\theta^{-1} \) we may assume that

\[
b \leq cn^{1/3}\theta^{1/3}
\]

with a constant \( c \) depending only on \( \theta_0 \).

Let \( \ell = \left\lceil n\gamma \right\rceil, t' = \gamma t \) and \( u = bn^2 \theta^{-\frac{2}{3}} \). By the definitions and (2.37)

\[
Q_{(1,n),(0,t)} (|\sigma_{\ell} - t'| > u) = \frac{1}{Z_{(1,n),(0,t)}} \int_{|\sigma_{\ell} - t'| > u} Z_{(1,t)}(0,s)Z_{(t+1,n)(s,t)} ds
\]

\[
\leq \frac{e^{-r_1(0)}}{Z_{(1,n),(0,t)}} \int \frac{Z^0_{\ell,s} Z_{(t+1,n)(s,t)} ds}{|\sigma_{\ell} - t'| > u} = \frac{e^{-r_1(0)}Z^0_{n,t}}{Z_{(1,n),(0,t)}} Q^0_{n,t} (|\sigma_{\ell} - t'| > u).
\]

Let \( h \in (b^{-3},1) \) (note that if \( b_0 \) is large enough then the interval is non-empty) and set \( r = \delta b^2/(3(1 - \gamma)) \) with \( \delta \) from Lemma 2.2.

\[
\mathbb{P} \left( Q_{(1,n),(0,t)} (|\sigma_{\ell} - t'| > u) > h \right) \leq \mathbb{P}(e^{-r_1(0)} \geq \theta b^3) + \mathbb{P} \left( \frac{Z^0_{n,t}}{Z_{(1,n),(0,t)}} \geq e^{-r_1(0)} \right) + \mathbb{P} \left( Q^0_{n,t} (|\sigma_{\ell} - t'| > u) > e^{-r_1(0)} b^{-3} \right)
\]

On the right hand side, the first term is bounded by \( Cb^{-3} \) by Markov’s inequality, since \( e^{-r_1(0)} \sim \text{Gamma}(\theta) \). The second term is bounded by \( Cr^{-3/2} \leq Cb^{-3} \) by (2.40) above. To see that we can actually apply Lemma 2.8 note that by (2.45) we have

\[
r = \frac{\delta b^2}{3(1 - \gamma)} \leq Cn^{2/3}\theta^{2/3}
\]

with a constant depending on \( \theta_0 \) and \( \varepsilon_1 \) which was the condition needed for the lemma.

Finally, the shift invariance (2.8) and Lemma 2.2 give, for large enough \( n \) and \( b \) and uniformly for \( h \in (b^{-3},1) \),

\[
\mathbb{P} \left[ Q^0_{\ell-\ell,t-t'} (|\sigma_{\ell} - t'| > u) > e^{-r_1(0)} h\theta^{-1}b^{-3} \right] \leq \mathbb{P} \left[ Q^0_{\ell-\ell,t-t'} (|\sigma_{\ell} - t'| > u) > e^{-r_1(0)} h\theta^{-1}b^{-6} \right]
\]

\[
\leq \mathbb{P} \left[ Q^0_{\ell-\ell,t-t'} (|\sigma_{\ell} - t'| > u) > e^{-\delta b^2 u^2/(n-\ell)} \right] \leq Cb^{-3}.
\]

It is above that we need \( \theta \leq \theta_0^{-1}n^{1/2-\varepsilon_0} \) for \( \varepsilon_0 > 0 \), for otherwise the right-hand side \( e^{-r_1(0)} h\theta^{-1}b^{-6} \) cannot be bounded below by \( e^{-\delta b^2 u^2/(n-\ell)} \). Collecting the estimates gives

\[
\mathbb{P} \left[ Q_{(1,n),(0,t)} (|\sigma_{\ell} - t'| > u) > h \right] \leq Cb^{-3}
\]

and from this

\[
P_{(1,n),(0,t)} (\left| \sigma_{[n\gamma]} - \gamma t \right| > bn^2 \theta^{-\frac{2}{3}}) \leq b^{-3} + \int_{b^{-3}}^1 \mathbb{P} \left[ Q_{(1,n),(0,t)} (|\sigma_{\ell} - t'| > u) > h \right] dh \leq Cb^{-3}.
\]
This completes the proof. □

Proof of Theorem 1.2. We again introduce \( \tilde{n} = \tau n, t = \tau \beta_0 n^{1-2\alpha} \) and \( \theta = \Psi_1^{-1}(t/\tilde{n}) = \Psi_1^{-1}(\beta_0 n^{-2\alpha}) \). Assumption (2.43) is satisfied because \( \alpha < 1/4 \). Using (2.9) and Theorem 2.6 the theorem follows. □

2.7. The tail estimates. In this section we prove the missing components of the proofs of Theorem 1.4 and Theorem 2.4. We begin with some definitions.

Augment the family \( Z_{(j,k),(s,t)} = Z_{(j,k),(s,t)}(1) \) defined for \( j \geq 1 \) in (1.13) by introducing, for \( k \in \mathbb{N} \) and \( t \in \mathbb{R}_+ \),

\[
Z_{(0,0),(0,t)} = e^{-B(t)}, 
Z_{(0,k),(0,t)} = \int_{0<s_0<\cdots<s_{k-1}<t} \exp[-B(s_0)+B_1(s_0,s_1)+\cdots+B_k(s_{k-1},t)] \, ds_0,s_1,\cdots,s_{k-1}.
\]

It is also convenient to set, for \( A \subseteq \mathbb{R} \),

\[
Z_{0,t}^\theta(\sigma_0 \in A) = 1_{A \cap \mathbb{R}_+}(t) \exp[-B(t) + \theta t].
\]

The following bounds are proved in Lemma 3.8 of [43].

**Lemma 2.7.** [43] Let \( \theta > 0 \). For \( 0 < s < t \) and \( n \in \mathbb{Z}_+ \)

\[
\frac{Z_{n+1,t}^\theta(\sigma_0 > 0)}{Z_{n,t}^\theta(\sigma_0 > 0)} \leq \frac{Z_{(0,n+1),(0,t)}^\theta}{Z_{(0,n),(0,t)}^\theta} \leq \frac{Z_{n+1,t}^\theta(\sigma_0 < 0)}{Z_{n,t}^\theta(\sigma_0 < 0)}
\]

and

\[
\frac{Z_{n,t}^\theta(\sigma_0 > 0)}{Z_{n,s}^\theta(\sigma_0 > 0)} \geq \frac{Z_{(0,n),(0,t)}^\theta}{Z_{(0,n),(0,s)}^\theta} \geq \frac{Z_{n,t}^\theta(\sigma_0 < 0)}{Z_{n,s}^\theta(\sigma_0 < 0)}.
\]

The second inequality of (2.49) makes sense only for \( n \geq 1 \).

For \( A \subset \mathbb{R} \) note the identity

\[
\frac{Z_{n,t}^\theta(\sigma_0 \in A)}{Z_{1,n}^\theta(0,t)} = \int_A \exp(-B(s) + \theta s) \frac{Z_{1,n}^\theta(s,t)}{Z_{1,n}^\theta(0,t)} \, ds.
\]

We will also define a reversed system: construct a new environment \( \tilde{\omega} \) with

\[
\tilde{B}(s) = -(B_n(t) - B_n(t-s)), \quad \tilde{B}_i(s) = B_{n-i}(t) - B_{n-i}(t-s), \quad 1 \leq i \leq n-1.
\]

Quantities that use environment \( \tilde{\omega} \) are marked with a tilde. From the definitions one checks that

\[
Z_{1,n}(s,t) = \tilde{Z}_{0,n-1}(0,t-s) \quad \text{for any } t > 0 \text{ and } s \in (-\infty, t).
\]

**Lemma 2.8.** Let \( \theta = \Psi_1^{-1}(t/n) \) and assume that \( \theta_0 \leq \theta \leq \theta_0^{-1}\sqrt{n} \) with a fixed \( \theta_0 > 0 \).

Fix a \( c_0 > 0 \). Then there exist finite, positive constants \( C, n_0 \) depending on \( \theta_0, c_0 \) such that if \( n > n_0 \) and \( b \leq c_0 n^{2/3}\theta^{2/3} \), then

\[
\mathbb{P} \left( \frac{Z_{n,t}^\theta}{Z_{(1,n),(0,t)}} \geq e^{bn^{1/3}n^{-2/3}} \right) \leq Cb^{-3/2}.
\]
Proof. Note that once we prove (2.51) for $b > b_0$ with a constant $b_0$ depending on $\theta_0, c_0$ then we can get it for all $b$ by adjusting the constant $C$. Thus we may assume that $b$ is big enough compared to $\theta_0$ and $c_0$.

Let $u = \sqrt{bn^{2/3} \theta^{-4/3}}$ and $\nu = \varepsilon \sqrt{bn^{-1/3} \theta^{2/3}}$ where $\varepsilon > 0$ will be specified later. Then

$$
\mathbb{P} \left( \frac{Z^\theta_{n,t}}{Z_{(1,n),(0,t)}} \geq e^{bn^{2/3} \theta^{-4/3}} \right) = \mathbb{P} \left( \frac{Z^\theta_{n,t}(|\sigma_0| \leq u)}{Z_{(1,n),(0,t)}} \geq e^{bn^{2/3} \theta^{-4/3}} \right)
$$

$$
\leq \mathbb{P} \left( \frac{Z^\theta_{n,t}(|\sigma_0| \leq u)}{Z_{(1,n),(0,t)}} \geq \frac{1}{2} e^{bn^{2/3} \theta^{-4/3}} \right) + \mathbb{P} \left( Q^\theta_{n,t}(|\sigma_0| \leq u) \leq 1/2 \right)
$$

(2.52)

The second probability can be bounded as

$$
\mathbb{P} \left( Q^\theta_{n,t}(|\sigma_0| \leq u) \leq 1/2 \right) = \mathbb{P} \left( Q^\theta_{n,t}(|\sigma_0| > u) \geq 1/2 \right) \leq C b^{-3/2}
$$

by (2.12) of Lemma 2.2 The first probability can be bounded by

$$
\mathbb{P} \left( \frac{Z^\theta_{n,t}(0 \leq \sigma_0 \leq u)}{Z_{(1,n),(0,t)}} \geq \frac{1}{4} e^{bn^{2/3} \theta^{-4/3}} \right) + \mathbb{P} \left( \frac{Z^\theta_{n,t}(-u \leq \sigma_0 < 0)}{Z_{(1,n),(0,t)}} \geq \frac{1}{4} e^{bn^{2/3} \theta^{-4/3}} \right).
$$

We will bound the first term, the second will follow similarly.

Introduce the new parameter $\lambda = \theta - \nu$. Note that by choosing $\varepsilon^2 \leq (4c_0)^{-1}$ we can assume $\lambda > \theta/2$.

Begin with (2.50) and then apply comparison (2.49):

$$
\frac{Z_{(1,n),(s,t)}}{Z_{(1,n),(0,t)}} = \frac{\tilde{Z}_{(0,n-1),(0,t-s)}}{\tilde{Z}_{(0,n-1),(0,t)}} \leq \frac{\tilde{Z}^\lambda_{n-1,t-s}(\sigma_0 < 0)}{Z^\lambda_{n-1,t}(\sigma_0 < 0)} = \frac{\tilde{Z}^\lambda_{n-1,t-s}(\sigma_0 < 0)}{\tilde{Z}^\lambda_{n-1,t}(\sigma_0 < 0)} \cdot \frac{\tilde{Q}^\lambda_{n-1,t-s}(\sigma_0 < 0)}{\tilde{Q}^\lambda_{n-1,t}(\sigma_0 < 0)}
$$

$$
= \exp\left(\tilde{Y}_{n-1}(t-s,t) - \lambda s\right) \cdot \frac{1}{\tilde{Q}^\lambda_{n-1,t}(\sigma_0 < 0)}.
$$

where we used (2.2) for the reversed system. Specializing the above to our context and substituting it in the probability that is to be bounded:

$$
\mathbb{P} \left( \frac{Z^\theta_{n,t}(0 \leq \sigma_0 \leq u)}{Z_{(1,n),(0,t)}} \geq \frac{1}{4} e^{bn^{2/3} \theta^{-4/3} b} \right) = \mathbb{P} \left( \int_0^u \exp(-B(s) + \theta s) Z_{(1,n),(s,t)} S \, ds \geq \frac{1}{4} e^{bn^{2/3} \theta^{-4/3} b} \right)
$$

$$
\leq \mathbb{P} \left( \int_0^u \exp(-B(s) + \tilde{Y}_{n-1}(t-s,t) + (\theta - \lambda)s) \tilde{Q}^\lambda_{n-1,t}(\sigma_0 < 0) \, ds \geq \frac{1}{4} e^{bn^{2/3} \theta^{-4/3} b} \right)
$$

(2.54)

$$
\leq \mathbb{P} \left( \tilde{Q}^\lambda_{n-1,t}(\sigma_0 < 0) \leq 1/2 \right)
$$

(2.55)

$$
+ \mathbb{P} \left( \int_0^u \exp(-B(s) + \tilde{Y}_{n-1}(t-s,t) + \nu s) \, ds \geq \frac{1}{8} e^{bn^{2/3} \theta^{-4/3} b} \right).
$$
To treat probability (2.54) set

\[ \bar{u} = (n - 1)\Psi_1(\lambda) - n\Psi_1(\theta) \geq -\Psi_1(\theta) + (n - 1)\Psi_2(\theta)(\lambda - \theta) \]

\[ \geq -C\theta^{-1} + \frac{1}{2}C\varepsilon \sqrt{bn^2}\theta^{-\frac{3}{4}} \geq C\varepsilon \sqrt{bn^2}\theta^{-\frac{3}{4}}, \]

where we used our assumptions on \( \theta \), the bounds (4.2), and took \( b \) large enough in relation to \( \theta_0 \). Use invariance (2.6) of \( Q \) and upper bound (2.12):

\[ P(\tilde{Q}_{n-1,t}^\lambda(\sigma_0 < 0) < 1/2) = P(\tilde{Q}_{n-1,t}^\lambda(\sigma_0 > 0) \geq 1/2) \]

\[ = P(\tilde{Q}_{n-1,t+\bar{u}}(\sigma_0 > \bar{u}) \geq 1/2) \leq C(\theta_0)b^{-3/2}. \]

To justify our use of the upper bound, note that

\[ (n - 1)\Psi_1(\lambda) - t - \bar{u} = n\Psi_1(\theta) - t = 0 \]

so the upper bound (2.12) is valid for \( \bar{u} \).

For probability (2.55), observe first that \( s \mapsto \tilde{Y}_{n-1}(t - s, t) \) is a standard Brownian motion which is independent of \( B \) by construction. By introducing \( B^\dagger(s) = \frac{1}{\sqrt{2}}(-B(s) + \tilde{Y}_{n-1}(t - s, t)) \) we need to bound

\[ P(\int_0^u \exp(\sqrt{2}B^\dagger((s) + \nu s)ds \geq \frac{1}{8}e^{\nu u}) \leq P(\int_0^u \exp(\sqrt{2}B^\dagger((s) + \nu s)ds \geq e^{3\nu u}). \]

where the upper bound follows by choosing \( \varepsilon \) small enough (for fixed \( b_0, n_0, \theta_0 \)). We will show that

\[ P(\int_0^u \exp(\sqrt{2}B^\dagger((s) + \nu s)ds \geq e^{3\nu u}) \leq Ce^{-\frac{1}{4}\nu^2 u} \]

if \( \nu > 0, u > 0 \). Note that

\[ P(\int_0^u \exp(\sqrt{2}B^\dagger((s) + \nu s)ds \geq e^{3\nu u}) \leq P(\int_{-\infty}^u \exp(\sqrt{2}B^\dagger((s) + \nu s)ds \geq e^{3\nu u}) \]

\[ \leq P(\exp(\sqrt{2}B^\dagger((u) + \nu u) \geq e^{2\nu u}) \]

\[ + P(\int_{-\infty}^u \exp(\sqrt{2}(B^\dagger((s) - B^\dagger((u)) + \nu(s - u))ds \geq e^{\nu u}). \]

The first probability is \( P(\sqrt{2}B^\dagger((u) \geq \nu u) \leq C \exp(-\nu^2 u/4) \). For the second probability we note that by Dufresne’s identity [26] the integral has the same distribution as the reciprocal of a Gamma(\( \nu \)) random variable. Thus the second term is

\[ P(\text{Gamma}(\nu) \leq e^{-\nu u}) \leq \frac{1}{\nu \Gamma(\nu)}e^{-\nu^2 u} \leq Ce^{-\nu^2 u} \]

which proves the estimate (2.57).

Collecting everything we get that

\[ P \left( \frac{Z_{n,t}^\theta(0 < \sigma_0 \leq u)}{Z_{(1,n),0,t}(0)} \geq \frac{1}{4}e^{n/\theta^{-4/3}b} \right) \leq Cb^{-3/2} + Ce^{-\frac{1}{4}\nu^2 \lambda/2} \leq C'b^{-3/2}. \]
The case of \(-u < \sigma < 0\) goes similarly, with small alterations. Now \(\lambda = \theta + \nu \leq 3\theta/2\). Utilizing (2.50) and comparison (2.49) the ratio is developed as follows:

\[
\frac{Z^{\theta}_{n,t}(-u \leq \sigma_0 < 0)}{Z_{(1,n),(0,t)}} = \int_{-u}^{0} \exp(-B(s) + \theta s) \frac{Z_{(n),(1,s,t)}}{Z_{(1,n),(0,t)}} ds \leq \int_{-u}^{0} \exp(-B(s) - \bar{Y}_{n-1}(t, t - s) - (\theta - \lambda)s) ds.
\]

The rest follows along the same lines as above. This proves (2.51). □

**Lemma 2.9.** Fix \(\theta_0 > 0\), suppose that \(\theta_0 < \theta < \theta^{-1}\sqrt{n}\) and let \(t = n\Psi_1(\theta)\). Then there exist constants \(c_0, C\) depending on \(\theta_0\) so that

\[(2.58) \quad P\left(\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) < -x\right) \leq c_1 e^{-c_1^{-1}x\theta}, \quad \text{for } x \geq c_0 n.
\]

The same bound holds for the upper tail.

**Proof.** We first note that

\[
Z_{(1,n),(0,t)} = \int_{0 < s_1 < \ldots < s_{n-1} < t} \exp \left\{ B_1(0, s_1) + \ldots + B(s_{n-1}, 1) \right\} ds_{1,n-1} > \frac{t^{n-1}}{(n-1)!} \exp \left( \min_{0 < s_1 < \ldots < s_{n-1} < t} \left( B_1(0, s_1) + \ldots + B(s_{n-1}, 1) \right) \right)
\]

\[
= \frac{t^{n-1}}{(n-1)!} \exp(-\sqrt{n}t \lambda_{\text{max}}^n)
\]

where \(\lambda_{\text{max}}^n\) is the largest eigenvalue of an \(n \times n\) GUE random matrix where the non-diagonal entries have variance \(1/n\). (This is the normalization where the support of the spectrum converges to \([-2, 2]\).) The fact that

\[
\min_{0 < s_1 < \ldots < s_{n-1} < 1} \left( B_1(0, s_1) + \ldots + B(s_{n-1}, 1) \right) = -\max_{0 < s_1 < \ldots < s_{n-1} < 1} \left( B_1(0, s_1) + \ldots + B(s_{n-1}, 1) \right)
\]

\[
d = -\sqrt{n} \lambda_{\text{max}}^n
\]

was proved independently in [27] and [12].

It is known that the random variable \(\lambda_{\text{max}}^n\) converges to 2, and the following uniform tail bound holds in \(n\) for \(K > K_0 > 2\) (see e.g. [34] and the references within):

\[(2.59) \quad P(\lambda_{\text{max}}^n > K) \leq Ce^{-C^{-1}nK^2}
\]

with a constant \(C\) depending only on \(K_0\). We will use this bound with \(K_0 = 3\).

We have

\[(2.60) \quad P \left( \log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) < -x \right) \leq P \left( e^{-t(\theta t - n\Psi_0(\theta))} \frac{t^{n-1}}{(n-1)!} \exp(-\sqrt{n}t \lambda_{\text{max}}^n) < e^{-x} \right) = P \left( -\theta t + n\Psi_0(\theta) + x + (n-1) \log t - \log(n-1)! < \sqrt{nt}\lambda_{\text{max}}^n \right).
\]
Using Stirling’s formula, the bounds \( \theta_0 \leq \theta \leq \theta_0^{-1} n^{1/2} \) and the bounds (4.2), (4.3) on \( \Psi_0, \Psi_1 \) we get that

\[
| -\theta t + n \Psi_0(\theta) + (n-1) \log t - \log(n-1)! | \leq Cn
\]

where \( C \) depends on \( \theta_0 \). This gives

\[
P \left( \log Z_{(1,n),0,t} - (\theta t - n \Psi_0(\theta)) < -x \right) \leq P \left( -Cn + x < \sqrt{nt\lambda_{\max}^n} \right)
\]

Choosing \( c_0 > 2C \) we get

\[
P \left( -Cn + x < \sqrt{nt\lambda_{\max}^n} \right) \leq P \left( \frac{1}{2} x < \sqrt{nt\lambda_{\max}^n} \right) \leq P \left( \frac{1}{2} x n^{-1} \Psi_1(\theta)^{-1/2} < \lambda_{\max}^n \right).
\]

Now choose \( c_0 \) big enough so that \( \frac{1}{2} c_0 \Psi_1(\theta_0)^{-1/2} > 3 \) (this is possible since \( \theta_0 \leq \theta \)) so that we can use (2.59):

\[
P \left( \log Z_{(1,n),0,t} - (\theta t - n \Psi_0(\theta)) < -x \right) \leq P \left( \frac{1}{2} x n^{-1} \Psi_1(\theta)^{-1/2} < \lambda_{\max}^n \right)
\]

\[
\leq C e^{-C n \left( \frac{1}{2} x n^{-1} \Psi_1(\theta)^{-1/2} \right)^2}
\]

\[
\leq c_1 e^{-c_1^{-1} \theta x}
\]

where in the last step we used the bounds on \( \Psi_1(\theta) \) and \( x \geq c_0 n \). \( \square \)

3. PROOFS FOR THE KPZ EQUATION

**Proof of Theorem 1.6.** We first start with the case \( \varphi = 0 \). From (2.5) one can verify that

\[
\mathbb{E} \log Z_N(\tau) \leq A \tau,
\]

for some constant \( A > 0 \). This, together with the upper bound in Theorem 1.3 yields

\[
\text{Var} \log Z_N(\tau) \leq F(\tau),
\]

for some finite \( F(\tau) \). This implies that \( \{ \log Z_N(\tau) : N \geq 1 \} \) is uniformly integrable, and hence \( \mathbb{E} \log Z_N(\tau) \to \mathbb{E} \log Z(\tau,0) \). Note that this could have been obtained by exact computations for the stochastic heat equation as well. Together with Theorem 1.5, this gives the convergence in law

\[
\log Z_N(\tau) - \mathbb{E} \log Z_N(\tau) \Rightarrow \log Z(\tau) - \mathbb{E} \log Z(\tau).
\]

Fatou’s lemma and Theorem 1.3 then give

\[
\text{Var} \log Z(\tau) \leq \lim \inf \mathbb{E} \left[ (\log Z_N(\tau) - \mathbb{E} \log Z_N(\tau))^2 \right]
\]

\[
\leq C \tau^{\frac{2}{3}},
\]

for some \( C > 0 \). As for the lower bound,

\[
P \left\{ \log Z(\tau) - \mathbb{E} \log Z(\tau) \geq c \tau^{1/3} \right\} = \lim_n P \left\{ \log Z_n(\tau) - \mathbb{E} \log Z_n(\tau) \geq c \tau^{1/3} \right\} \geq \delta,
\]

for some constants \( c, \delta > 0 \), and \( \tau \) large enough, thanks to the lower bound in Proposition 2.3. We have proved that there exist some constant \( C > 0 \) such that

\[
\frac{1}{C} \tau^{\frac{2}{3}} \leq \text{Var} \log Z(\tau) \leq C \tau^{\frac{2}{3}},
\]
for $\tau$ large enough.

We now turn to the case $|\varphi| \leq K$ for some $0 < K < +\infty$. From (1.22), we can verify that

$$e^{-K} Z_N^\varphi(\tau) \leq Z_N(\tau) \leq e^{K} Z_N^\varphi(\tau).$$

This implies that

$$\text{Var} \log Z_N^\varphi(\tau) \leq 8K^2 + 2\text{Var} \log Z_N(\tau),$$

which in turns implies the uniform integrability of $\{\log Z_N^\varphi(\tau) : N \geq 1\}$. Fatou’s lemma and the upper bound on $\text{Var} \log Z_N(\tau)$ show that

$$\text{Var} \log Z^\varphi(\tau) \leq C'' \tau^{\frac{2}{3}},$$

for some $C'' > 0$ and $\tau > 0$ large enough. The lower bound follows from Proposition 2.3

$$P \{\log Z^\varphi(\tau) - E \log Z^\varphi(\tau) \geq c\tau^{1/3}\} = \lim_n P \{\log Z_N^\varphi(\tau) - E \log Z_N^\varphi(\tau) \geq c\tau^{1/3}\} \geq \lim_n P \{\log Z_N(\tau) - E \log Z_N(\tau) \geq c\tau^{1/3} - 2K\} \geq \delta,$$

for suitable $c, \delta > 0$ and all $N$ and $\tau$ large enough. This completes the proof of the theorem. \hfill $\Box$

4. Facts about the gamma function and Gamma distribution

We collect here some basic facts about the gamma function and the Gamma distribution.

Recall that $\Psi_0 = \Gamma'/\Gamma$ and $\Psi_n = \Psi_{n-1}'$ for $n \geq 1$. The lemma below follows from straightforward computations.

**Lemma 4.1.** Let $A \sim \text{Gamma}(\mu, r)$. Then we have

$$E A = \frac{\mu}{r}, \quad \text{Var}(A) = \frac{\mu}{r^2}, \quad E(\log A) = \Psi_0(\mu) - \log r, \quad \text{Var}(\log A) = \Psi_1(\mu).$$

The polygamma functions satisfy

$$\Psi_n(x) = (-1)^{n+1}n! \sum_{k \geq 0} (x + k)^{-n-1}, \quad \text{for } n \geq 1.$$  \hfill (4.1)

From this it follows that for $n \geq 1$ we have

$$\frac{(n-1)!}{x^n} \leq |\Psi_n(x)| \leq \frac{(n-1)!}{x^n} + \frac{n!}{x^{n+1}} \quad \text{for } n \geq 1.$$  \hfill (4.2)

For $\Psi_0$ we have the following asymptotics for large $x > 0$:

$$\Psi_0(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right).$$  \hfill (4.3)
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