Research Article
Note on Precise Asymptotics in the Law of the Iterated Logarithm under Sublinear Expectations

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By using Lebesgue bounded convergence theorem, we prove precise asymptotics in the law of the iterated logarithm for independent and identically distributed random variables under sublinear expectation.

1. Introduction

Inspired by the phenomena of uncertainty in risk measure, super-hedging problems, Peng et al. [1, 2] initiated the sublinear expectations space nontrivially. Motivated by the works of Peng, people try to investigate whether or not the corresponding limit theorems in classic probability space could hold under sublinear expectations space. For the corresponding works under sublinear expectations, the interested readers could refer to Gao and Xu [3], Zhang et al. [4–10], Wu [11], Xu and Cheng [12], Zhong and Wu [13], Xu et al. [14, 15], Yu and Wu [16], Wu and Jiang [17], Ma and Wu [18], Chen [19], Fang et al. [20], Hu et al. [21], Hu and Yang [22], Kuczmaszewska [23], Wang and Wu [24], and references therein. Since Gut and Spătaru [25] investigated precise asymptotics in the law of the iterated logarithm, people obtained lots of corresponding results related to precise asymptotics in the law of the iterated logarithm, for which the interested readers could refer to Zhang [26], Huang et al. [27], Jiang and Yang [28], Wu and Wen [29], Xiao et al. [30], and Xu et al. [31–33]. Recently, by using Lebesgue bounded convergence theorem together with the results of Zhang et al. [8, 9], Zhang [10] obtained Heyde’s theorem under sublinear expectations. The methods of Zhang [10] are different from that of Xiao et al. [30] under sublinear expectations. It is interesting to wonder whether or not precise asymptotics in the law of the iterated logarithm under sublinear expectations hold under the same conditions as in Zhang [10]. In this note, we try to use the methods in Zhang [10] to investigate precise asymptotics in the law of the iterated logarithm for independent and identically distributed random variables under sublinear expectations under conditions different from that of Xu and Cheng [12], which complement that obtained by Xu and Cheng [12]. Our results may have potential applications in engineering or finance fields (cf. Peng et al. [2, 34, 35]).

The rest of this paper is organized as follows. In the next section, we summarize necessary basic notions, concepts, and relevant properties and give necessary lemmas under sublinear expectations. In Section 3, we give our main results, Theorems 1 and 2, and we present the proof of Theorem 1 in Section 4. The proof of Theorem 2 is similar to that of Theorem 1, so it is omitted.

2. Preliminary

We use notations of Peng [2]. Let \((\Omega, \mathcal{F})\) be a given measurable space. Assume that \(\mathcal{H}\) is a subset of all random variables on \((\Omega, \mathcal{F})\) satisfying \(I_A \in \mathcal{H}\), where \(A \in \mathcal{F}\) and if \(X_1, \ldots, X_n \in \mathcal{H}\), then \(\varphi(X_1, \ldots, X_n) \in \mathcal{H}\) for each
\( \varphi \in C_{\text{Lip}}(\mathbb{R}^n) \), where \( C_{\text{Lip}}(\mathbb{R}^n) \) stands for the linear space of (local Lipschitz) function \( \varphi \), satisfying
\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)(|x - y|), \forall x, y \in \mathbb{R}^n,
\]
for some \( C > 0, m \in \mathbb{N} \) depending on \( \varphi \). We regard \( \mathcal{H} \) as the space of random variables.

**Definition 1.** A sublinear expectation \( \mathbb{E} \) on \( \mathcal{H} \) is a functional \( \mathbb{E}: \mathcal{H} \rightarrow [\mathbb{R}, \infty] \) satisfying the following properties:
- For all \( X, Y \in \mathcal{H} \), \( \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \).
- For all \( \lambda \geq 0 \), \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \).
- For all \( \lambda > 0 \), \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \).

A set function \( V: \mathcal{H} \rightarrow [0, 1] \) is named a capacity if it obeys
\[
(a) \ V(\emptyset) = 0, \ V(\Omega) = 1.
(b) \ V(X) \leq V(Y), \ X \subseteq Y, A \subseteq B, A, B \in \mathcal{H}.
\]

A capacity \( V \) is called to be subadditive if \( V(A + B) \leq V(A) + V(B), A, B \in \mathcal{H} \).

Let \( X = (X_1, \ldots, X_m), \ X_i \in \mathcal{H} \) and \( Y = (Y_1, \ldots, Y_n), \ Y_i \in \mathcal{H} \) are two random variables on \( (\Omega, \mathcal{F}, \mathbb{P}) \). \( \mathbb{E} \) is called to be independent of \( X \), if for each \( \varphi \in C_{\text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n), \mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\varphi(X, Y)]_{x=X} \).

Let \( \mathcal{F} = \{ \mathcal{B} \} \) be a sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{P})\), the space of right continuous functions \( \alpha \in \mathbb{R} \), \( \lambda \mathbb{E}[X_{|n}] \leq \mathbb{E}[\lambda X_{|n}] \), \( \mathbb{E}[x^2] \mathbb{E}[|X_{|n}] \leq x \mathbb{E}[\mathbb{E}[X_{|n}]] \).

**Lemma 1** (see Lemma 2.1 in [10]). Suppose that \( \{ Z_{n_k}; k = 1, \ldots, k_n \} \) is an array of independent random variables such that \( \mathbb{E}[Z_{n_k}] \leq 0 \) and \( \mathbb{E}[Z_{n_k}^2] < \infty \), \( k = 1, \ldots, k_n \). Then, for all \( x, y \geq 0 \),
\[
\mathbb{V}(\max_{k \leq k_n} Z_{n_k} \geq x) \leq \mathbb{V}(\max_{k \leq k_n} Z_{n_k} \geq y) + \exp\left(\frac{x}{y} \left(\frac{B_n}{xy} + 1\right) \ln\left(1 + \frac{xy}{B_n}\right)\right),
\]
where \( B_n = \sum_{k=1}^{k_n} \mathbb{E}[Z_{n_k}^2] \).

**Lemma 2** (see Lemma 2.4 in [10]). Assume that \( \{ X_{|n}; n \geq 1 \} \) is a sequence of independent, identically distributed random variables in a sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{P})\), write \( S_n = \sum_{k=1}^{n} X_k \). Suppose that (i) \( \lim_{n \rightarrow \infty} \mathbb{E}[X_{|n}] \) is finite; (ii) \( x \mathbb{E}[|X_{|n}] \geq x \mathbb{E}[\mathbb{E}[X_{|n}]] \).

**Corollary 1** (see Corollary 2.1 in [10]). Let \( F(x) = \mathbb{V}(|\xi| > x) \). Under the assumptions in Lemma 2,
\[
\lim_{n \rightarrow \infty} \mathbb{V}(\max_{k \leq n} |S_k| > x \sqrt{n}) = 2G(x/\sqrt{n}), \ x > 0,
\]
and
\[
\lim_{n \rightarrow \infty} \mathbb{V}(\max_{k \leq n} |S_k| > x \sqrt{n}) = 2G(x/\sqrt{n}), \ x > 0,
\]
where \( G(x) = \sum_{i=0}^{\infty} (-1)^i P(|N| \geq (2i + 1)x) \), and \( N \) is a standard random variable.
In the remainder of this paper, let \( \{X_i, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables under sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{E})\) with \( C_2(X^2) < \infty, \mathbb{E}(X) \leq \text{lim}_{\xi \to \infty} \mathbb{E}([-\xi X \vee 0]) = 0, \mathbb{E}(X^2) = \bar{\sigma}^2 < \infty, \mathbb{E}((-X)^2) = \sigma^2, \) and \( r = \pi^2/2^2\). Set \( S_n = \sum_{i=1}^{n} X_i, n \geq 1 \). Let \( \log x = \ln(x) / \ln(e), \log \log x = \ln(\ln(x)) \), and \( \lfloor x \rfloor = \sup \{l, l \leq x, l \in \mathbb{Z}^+\} \); we denote by \( C \) a positive constant which may change from line to line.

### 3. Main Results

The following are our main results.

**Theorem 1.** For \( b, d > 0 \), we have

\[
\lim_{\epsilon \to 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{P}\left[ \max_{k \leq n} |S_k| \geq \epsilon \sqrt{n} (\log \log n)^d \right] = \frac{\bar{\sigma}^{b/d} C_b(|\xi|^{b/d})}{b},
\]

where \( \xi \sim \mathcal{N}(0, [r, 1]) \). Moreover, for \( b, d > 0 \),

\[
\lim_{\epsilon \to 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{P}\left[ \max_{k \leq n} |S_k| \geq \epsilon \sqrt{n} (\log \log n)^d \right] = \sum_{i=0}^{\infty} \frac{\pi (1/4) (1/2 i + 1)^{b/d}}{\sqrt{\pi} (2i + 1)^{b/d}} = 0.9913425 \ldots.
\]

**Remark 1.** In (12), when \( b = 3/2, d = 1, \bar{\sigma} = 1 \),

\[
\lim_{\epsilon \to 0} \epsilon^{3/2} \sum_{n=3}^{\infty} \frac{(\log \log n)^{1/2}}{n \log n} \mathbb{P}\left[ \max_{k \leq n} |S_k| \geq \epsilon \sqrt{n} (\log n)^d \right] = \sum_{i=0}^{\infty} \frac{\pi (1/4) (1/2 i + 1)^{b/d}}{\sqrt{\pi} (2i + 1)^{b/d}} = 0.9913425 \ldots
\]

In the following two sections, for \( M \geq 3 \) and \( 0 < \epsilon < 1 \), set \( b(\epsilon) = \lfloor \exp[\exp(M \epsilon^{-1/d})] \rfloor \).

When \( b = 2, d = 1, \bar{\sigma} = 1 \),

\[
\lim_{\epsilon \to 0} \epsilon^{2} \sum_{n=3}^{\infty} \frac{\log \log n}{n \log n} \mathbb{P}\left[ \max_{k \leq n} |S_k| \geq \epsilon \sqrt{n} (\log \log n)^d \right] = \sum_{i=0}^{\infty} (-1)^i 2^{i/2d} \Gamma(2) \left( \frac{\sqrt{2} \pi^{5/2}}{24} \right) = 1.03081 \ldots
\]

Set \( \log_{b+1}(x) \equiv \log \log(x), \log_{b+2}(x) \equiv \log \log \log(x), \ldots, \log_{b+k}(x) \equiv \log \log \log \log \log \ldots \log \log(x), k \geq 2 \). Adapting the proofs of Theorem 1, we also have the following theorem, whose proofs are omitted.

**Theorem 2.** For \( b, d > 0 \), for any integer \( k \geq 3 \), we have

\[
\lim_{\epsilon \to 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log_{b+k}(n))^{b-1}}{n \log_{b+k}(n) \log_{b+k-1}(n) \cdots \log_{b+1}(n)} \mathbb{P}\left[ \max_{k \leq n} |S_k| \geq \epsilon \sqrt{n} (\log \log_{b+k}(n)^d \right] = \frac{\bar{\sigma}^{b/d} C_{b+k}(|\xi|^{b/d})}{b},
\]

For \( b, d > 0 \),

\[
\lim_{\epsilon \to 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log_{b+k}(x))^{b-1}}{n \log_{b+k}(n) \log_{b+k-1}(n) \cdots \log_{b+1}(n)} \mathbb{P}\left[ \max_{k \leq n} |S_k| \geq \epsilon \sqrt{n} (\log \log_{b+k}(n)^d \right] = \frac{\bar{\sigma}^{b/d} C_{b+k}(|\xi|^{b/d})}{b}.
\]

### 4. Proof of Theorem 1

Here, we borrow ideas from the proofs in Zhang [10]. Without loss of restrictions, we suppose \( \bar{\sigma} = 1 \). We first establish (11). Notice for \( n \) large enough,

\[
\int_{n}^{n+1} (\log \log x)^{b-1} \frac{(\log/\log x)^d}{x \log x} \mathbb{P}\left[ \frac{|S_k|}{\sqrt{n}} > \epsilon (\log \log x)^d \right] \, dx \leq \int_{n}^{n+1} \frac{(\log \log x)^{b-1}}{(x - 1) \log (x - 1)} \mathbb{P}\left[ \frac{|S_k|}{\sqrt{n}} > \epsilon (\log \log (x - 1))^d \right] \, dx,
\]
and set \( F(x) = \mathbb{V}(\{x \mid x \geq a\}) \), for \( a \in [0, 1] \),

\[
\lim_{e^{-\delta} \to 0} \int_3^\infty \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} - F(\log \log (x-a)) \, dx = \lim_{e^{-\delta} \to 0} \frac{1}{d} \int_0^{(b(d))^{-1}} F(y) \, dy = \frac{C_F}{b}.
\]

(19)

Therefore, it is enough to establish that

\[
\lim_{e^{-\delta} \to 0} \epsilon \int_3^\infty \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \times \left| \mathbb{V}\left( \frac{S_x}{\sqrt{n}} \right) > \epsilon \right| \leq F(\log \log (x-a)) \, dx = 0.
\]

(20)

Obviously, we see that for any \( \delta > 0 \),

\[
\lim_{e^{-\delta} \to 0} \epsilon \int_3^\infty \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \times \left| \mathbb{V}\left( \frac{S_x}{\sqrt{n}} \right) > \epsilon \right| \leq \frac{b}{\delta} \int_0^{(b(d))^{-1}} F(z) \, dz.
\]

(21)

For \( M > 1 \geq \delta > \epsilon > 0 \), and \( 0 < a \leq 1 \),

\[
\epsilon \int_0^{(b(d))^{-1}} \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \times \max_{\epsilon \in \{\exp[\delta^{\epsilon - \delta}]\}} \left| \mathbb{V}\left( \frac{S_n^\epsilon}{\sqrt{n}} > y \right) - F(y) \right| \, dx 
\]

\[
\leq \epsilon \int_0^{(b(d))^{-1}} \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \times \max_{\epsilon \in \{\exp[\delta^{\epsilon - \delta}]\}} \left| \mathbb{V}\left( \frac{S_n^\epsilon}{\sqrt{n}} > z \right) - F(z) \right| \, dz.
\]

(22)

As in the proofs of Zhang [10], \( C_F(X^2_1) < \infty \) implies the assumptions of Lemma 2. By Lemma 2 and Corollary 1,

\[
\max_{\epsilon \in \{\exp[\delta^{\epsilon - \delta}]\}} \left| \mathbb{V}\left( \frac{S_n^\epsilon}{\sqrt{n}} > z \right) - F(z) \right| \to 0 \text{ as } \epsilon \to 0.
\]

(23)

If \( z \) is the continuous point of \( F \). Note that \( F(x) \) is a monotone function, whose discontinuous points are countable, \( z \) is one-to-one mapping from \([0, \infty)\) to \([0, \infty)\), and so the above convergence holds except on a set with null Lebesgue measure. From the Lebesgue bounded convergence theorem it follows that

\[
\lim_{e^{-\delta} \to 0} \int_0^{(b(d))^{-1}} \frac{1}{b} \max_{\epsilon \in \{\exp[\delta^{\epsilon - \delta}]\}} \left| \mathbb{V}\left( \frac{S_n^\epsilon}{\sqrt{n}} > z \right) - F(z) \right| \, dz = 0.
\]

(24)

Hence, for \( M \geq 1 \),
\[ \lim_{x \to 0} \int_{b(e)}^{b(c)} \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \]  
\[ \times \mathbb{P} \left( \frac{|S_{\max}|}{\sqrt{|X|}} > \varepsilon (\log \log (x-a))^d \right) \]  
\[ \leq \int_{b(e)}^{b(c)} \frac{1}{x} \mathcal{V}(\varepsilon |x|/b, x) \, dx = 0. \]  

On the other hand, for \( M \geq 2 \), by the study (24) and (25) in [11],

\[ \int_{b(e)}^{b(c)} \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \epsilon (\log \log (x-a))^d \, dx \leq \epsilon (\log \log (x-a))^d \int_{b(e)}^{b(c)} \frac{1}{x} \mathcal{V}(\varepsilon |x|/b, x) \, dx \]

\[ = \epsilon (\log \log (x-a))^d \int_{b(e)}^{b(c)} \frac{1}{x} \mathcal{V}(\varepsilon |x|/b, x) \, dx \rightarrow 0 \text{as} M \rightarrow \infty. \]  

It remains to establish that

\[ \lim_{M \to \infty} \lim_{\varepsilon \to 0} \int_{b(e)}^{b(c)} \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \mathcal{V}(\max_{k \leq x} |S_k|) \]

\[ \geq \frac{\sqrt{b/d}}{2} \varepsilon \sqrt{x - a} (\log \log (x-a))^d \, dx = 0. \]

Let \( b_x = \varepsilon \sqrt{d/b} \sqrt{x - a} (\log \log (x-a))^d / 32 \), \( Y_{ix} = -b_x \sqrt{x}/b_x \). Then,

\[ \mathbb{P} \left( \max_{k \leq x} \sum_{i=1}^{k} (Y_{ix} - \mathbb{E}(Y_{ix})) > \frac{\sqrt{b/d}}{4 \varepsilon \sqrt{x - a} (\log \log (x-a))^d} \right) \leq \exp \left( \frac{\sqrt{b/d}}{4 \varepsilon \sqrt{x - a} (\log \log (x-a))^d} \right) \]

\[ \leq \exp \left( \frac{256}{\varepsilon^4 \left( \frac{\log \log (x-a)^2 d}{4x} \right)^4} \right). \]  

Therefore, by the proof of Lemma 1 in Zhong and Wu [13],
\[
\varepsilon^{bd} \int_{b(\varepsilon)}^\infty \left( \max_{k \leq x} \frac{\sqrt{b/d}}{2} |\operatorname{log}(x) - \log(x-a)|^d \right) \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \, dx
\]

\[
\leq C \varepsilon^{bd} \int_{b(\varepsilon)}^\infty \left( \max_{k \leq x} \frac{\sqrt{b/d}}{2} |\operatorname{log}(x) - \log(x-a)|^d \right) \frac{(\log \log (x-a))^{b-1}}{(x-a) \log (x-a)} \, dx
\]

\[
+ C \varepsilon^{bd} \int_{b(\varepsilon)}^\infty \left( \max_{k \leq x} \frac{\sqrt{b/d}}{2} |\operatorname{log}(x) - \log(x-a)|^d \right) \frac{256}{(\varepsilon^2 \log (x-a))^{2n}} \, dx
\]

\[
\leq C \int_{\varepsilon M^{-1}}^\infty \mathcal{V} \left( |X| > y \right) dy + C \varepsilon^{-bd} \frac{1}{(M \varepsilon^{-1/d})^b}
\]

\[\leq C \int_{\varepsilon M^{-1}}^\infty \mathcal{V} \left( |X| > y \right) dy + C/M^b \rightarrow 0.
\]

As \( M \rightarrow \infty \), for \((-S_k)\), we have same convergence. Thus, (11) is established.

Next, we prove (12). With (9) in place of (8), we obtain

\[
\lim_{\varepsilon, b \rightarrow 0} \varepsilon^{bd} \int_3^\infty \frac{\left( \log \log (x-a) \right)^{b-1}}{\log \log (x-a)} \times \left| \mathcal{V} \left( \max_{k \leq x} \frac{|S_k|}{\sqrt{x}} > \varepsilon \left( \log \log (x-a) \right)^d \right) \right| \, dx = 0.
\]

Also, for \( 0 \leq a \leq 1 \),

\[
\lim_{\varepsilon, b \rightarrow 0} \varepsilon^{bd} \int_3^\infty \frac{\left( \log \log (x-a) \right)^{b-1}}{\log \log (x-a)} - 2G \left( \varepsilon \left( \log \log (x-a) \right)^d \right) \, dx = \lim_{\varepsilon, b \rightarrow 0} \varepsilon^{bd} \int_3^\infty 2G(y) \frac{1}{b} d\left( \frac{y}{b} \right) = \int_0^\infty \frac{2}{d} G(y) y^{(b-1)} dy.
\]

We see that for any integer \( m \geq 0 \),

\[
\int_0^\infty \frac{2}{d} G(y) y^{(b-1)} dy \leq \int_0^\infty \frac{2m}{d} \sum_{i=0}^m (-1)^i P(|N| \geq (2i + 1)y) y^{(b-1)} dy = \sum_{i=0}^m (-1)^i \int_0^\infty \frac{2}{d} P(|N| \geq (2i + 1)y) y^{(b-1)} dy
\]

\[
= \sum_{i=0}^m (-1)^i \frac{1}{b} \frac{2^{1+b/2} d}{\sqrt{\pi} (2i + 1)^{b/2}}
\]

\[
\int_0^\infty \frac{2}{d} G(y) y^{(b-1)} dy \geq \int_0^\infty \frac{2m+1}{d} \sum_{i=0}^{m+1} (-1)^i P(|N| \geq (2i + 1)y) y^{(b-1)} dy = \sum_{i=0}^{m+1} (-1)^i \int_0^\infty \frac{2}{d} P(|N| \geq (2i + 1)y) y^{(b-1)} dy
\]

\[
= \sum_{i=0}^{m+1} (-1)^i \frac{1}{b} \frac{2^{1+b/2} d}{\sqrt{\pi} (2i + 1)^{b/2}}
\]

Hence, letting \( m \rightarrow \infty \) in the above two inequalities results in

\[
\int_0^\infty \frac{2}{d} G(y) y^{(b-1)} dy = \sum_{i=0}^\infty (-1)^i \frac{1}{b} \frac{2^{1+b/2} d}{\sqrt{\pi} (2i + 1)^{b/2}}
\]

Therefore, (12) is proved.
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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