The Gonchar-Stahl $\rho^2$-theorem and associated directions in the theory of rational approximations of analytic functions

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Abstract. The Gonchar-Stahl $\rho^2$-theorem characterizes the rate of convergence of best uniform (Chebyshev) rational approximations (with free poles) for one basic class of analytic functions. The theorem itself, modifications and generalizations of it, methods involved in its proof and other related details constitute an important subfield in the theory of rational approximations of analytic functions and complex analysis.

This paper briefly outlines the essentials of the subfield. The fundamental contributions of A. A. Gonchar and H. Stahl are at the heart of the exposition.

Bibliography: 70 titles.

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§ 1. Introduction. Statement of the theorem

The Gonchar-Stahl theorem characterizes the rate of convergence of best uniform rational approximations (with free poles) for one basic class of analytic functions. Its proof combines constructions and methods from different branches of classical analysis and approximation theory. The variety of significant connections explains the fundamental role of the theorem in approximation theory and complex analysis. Some of these facts and connections are discussed briefly below.

1.1. Walsh’s theorem. One of the main predecessors of the Gonchar-Stahl theorem is Walsh’s well-known theorem [68] from the 1930s on the approximation of an arbitrary element $f$ of an analytic function on a continuum $E$ in the (extended) complex plane (see also the book [69]).

Consider the distance from $f$ to the class of rational functions of order $n$ in the uniform metric on $E$,

$$
\rho_n(f) = \rho_n(f, E) = \min_{r \in \mathcal{R}_n} \max_{z \in E} |f(z) - r(z)|,
$$

where $\mathcal{R}_n$ is the set of all rational functions $r_n = P_n/Q_n$ of order $\leq n$ ($P_n, Q_n \in \mathbb{P}_n$ are polynomials of degree $\leq n$). It was a well known fact that $\rho_n(f) \to 0$ as $n \to \infty$. 

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The problem was to determine the rate of convergence. Further, it was known that this rate is at least geometric, and so the problem was to estimate (asymptotically) the order of the associated progression.

The assumption that $f$ is analytic on $E$ means that there exists a domain $\Omega$ containing $E$ such that $f \in H(\Omega)$; that is, $f$ is holomorphic (analytic and single-valued) in $\Omega$.

Walsh’s theorem asserts that $f \in H(\Omega)$ implies that
\[
\lim_{n \to \infty} \rho_n(f)^{1/n} \leq \rho = e^{-1/C(E,F)},
\]
where $F = \partial \Omega$ is the boundary of $\Omega$ and $C(E,F)$ is the capacity of the condenser $(E,F)$. We will assume further that the continuum $E$ has a connected complement. If, in addition, $\Omega$ is simply connected then the quantity $1/\rho = 1/\rho(E,F)$ is also known as the modulus of the ring domain $\Omega \setminus E$ (there is a conformal mapping of $\Omega$ on $\{\rho < |z| < 1\}$).

In 1959 Erokhin [19] presented some examples proving that this estimate is sharp; that is, it cannot be improved without further restrictions on $f$. In particular, he constructed a function $f \in H(\Omega)$ where $\Omega = \{z : |z| < R\}$ is a disk such that for approximations of $f$ on a smaller disk $E = \overline{D_r} = \{z : |z| \leq r\}$ equality in (1) holds (it follows that for this function $f$ rational approximations are not essentially better than polynomial approximations of the same degree). This construction can be modified to prove that for any domain $\Omega$ and any continuum $E \subset \Omega$ there exists $f \in H(\Omega)$ such that equality in (1) holds (for further details see Walsh’s book [69] and a review by Mergelyan [44] included in the Russian translation of this book and concerning the progress in this direction made roughly between 1930 and 1960).

We note that in all these examples there is only a small subsequence of natural numbers along which equality in (2) is reached. It was determined later that ‘on the average’ the rate of convergence is essentially better than in (2) for any function $f \in H(\Omega)$; see §1.2 below.

This means that the whole class $H(\Omega)$ always contains functions for which the sequence of rational approximations has highly irregular behaviour. In this context the Gonchar-Stahl theorem essentially asserts that the behaviour of rational approximations is necessarily regular for functions which have unlimited analytic continuation outside a set of singularities of zero capacity. To state the theorem formally we need the following definitions.

1.2. The Gonchar-Stahl $\rho^2$-theorem. Let $f|_E$ be an element of an analytic function which we want to approximate. If the analytic continuation of this element outside $E$ has branch points, then there exist different (maximal) domains $\Omega$ where $f|_E$ has a holomorphic extension. Any of those domains may be used in combination with Walsh’s inequality (2) and, therefore, $\rho$ in this inequality may be replaced by the following constant $\rho(f)$, called the extremal modulus associated with the holomorphic function (analytic element) $f \in H(E)$:
\[
\rho(f) = \rho(f,E) = \inf\{e^{-1/C(E,F)} : F = \partial \Omega, f \in H(\Omega), E \subset \Omega\}.
\]
There also exists a unique extremal domain $\Omega$ which satisfies the condition $\rho(f, E) = \inf\{\exp\{-1/C(E, \partial \Omega)\}\}$ and is maximal among all such domains with this condition.
Let $\mathcal{A}(\mathcal{D})$ be the class of all analytic elements in a domain $\mathcal{D} \subset \mathbb{C}$ which admit analytic continuation along any path in this domain. We note that this is one of the basic classes of analytic functions. For instance, solutions of differential equations with coefficients in $H(\mathcal{D})$ belong to $\mathcal{A}(\mathcal{D})$ (things are similar for all natural classes of equations).

We now state the Gonchar-Stahl $\rho^2$-theorem.

**Theorem 1.** Let $\mathcal{D}$ be a domain in the extended plane such that $\text{cap}(\mathbb{C} \setminus \mathcal{D}) = 0$. Let $E \subset \mathcal{D}$ be a continuum with connected complement and $f \in H(E) \cap \mathcal{A}(\mathcal{D})$. Then

$$
\lim_{n \to \infty} \rho_n(f, E)^{1/n} = \rho(f, E)^2.
$$

(4)

In particular, the limit in the left-hand side of (4) exists.

This is not the most general form of the theorem; the conditions on $E$ can be relaxed significantly, but for the purposes of our discussion this version is representative enough. In the sequel we mostly consider cases when $E$ is an interval or a disk (including the degenerate or local case when $E$ is a point).

There is also an earlier version of the theorem due to Gonchar [25], related to Markov-type functions. The Markov case is simpler but it is not a particular case of Theorem 1 above (see also [27]). This case is discussed in some detail separately in §2.

The proof of the general version of Theorem 1 was presented in [60] by Stahl and in [34] by Gonchar and the author. In [60] the upper estimate in (4) was proved and in [34] the corresponding lower estimate was obtained. Actually, a general method of rational approximation was developed in these two papers and solutions of several longstanding problems in approximation theory were obtained in them as a direct application of the method. The $\rho^2$-theorem was one of the classical applications of the method. Stahl’s paper [60] contains another classical result; namely, Stahl’s celebrated theorem on the convergence of diagonal Padé approximants for functions with branch points (corresponding to the degenerate case $E = \{\infty\}$ in the $\rho^2$-theorem). The so-called ‘1/9 problem’ on the best rational approximation of the exponential function on a semiaxis was solved in [34] (it was obtained as a corollary of a general theorem on Chebyshev approximation of a sequence of analytic functions). Many other problems were investigated later using the same method or modifications of it (see, for example, [18] and [36]). Further applications and generalizations are on the way (some of them are discussed below in this paper). For details and other references see the recent reviews [36], [4] and [55], and also the memorial articles [37] and [5].

Our purpose is to present the essentials of the method mentioned above. It is not possible to cover all the significant details, so we have selected some of them for discussion. To this end we use the $\rho^2$-theorem and the version of it for Markov-type functions as the starting point of this discussion. We also use it to create a general context in §1 and §2 of the paper.

1.3. Contents of the paper. In §2, which follows, we outline the proof of Gonchar’s $\rho^2$-theorem for Markov-type functions (Theorem 2 below). The proof of this theorem [25] is essentially simpler than the proof of Theorem 1. The simplicity of this situation makes it possible to discuss the main components of the proof in
some detail. In this connection we also mention Stahl’s theorem on the rate of rational approximation of \(|x|\) on \([-1, 1]\) and results and conjectures by Gonchar on the problem of characterizing classes of analytic functions by the rate of their rational approximation.

In §3 we discuss some details related to Stahl’s theorem on the convergence of Padé approximants for functions with branch points and, in particular, the characterization of minimal compact sets for functions with a finite number of branch points. This is a good basis for illustrating the geometric component of the method. We then present some new results on the stability of convergence in Stahl’s theorem. Finally, some new conjectures are presented on the asymptotics of complex orthogonal polynomials (related to problems in the convergence of Hermite-Padé approximants).

In the remainder of this introduction we comment on the main components of the method of proof of the \(\rho^2\)-theorem. Hopefully, this brief review will give a key to the essence of the method. In conclusion we also make a few more comments on its connections and applications.

1.4. Brief description of the method. Now, we give a brief description of the main components of the method including, in particular, a construction of ‘near best’ approximations.

1.4.1. Interpolation by rational function with free poles. We begin by selecting a triangular table of interpolation nodes on \(E\) whose \(n\)th row contains \(2n+1\) points and find the corresponding rational function of order \(n\) interpolating \(f\) at the selected nodes. Now we need to select nodes to obtain a ‘near best’ approximation.

We note that the proof of Walsh’s theorem was based on interpolation with fixed poles. In our ‘free pole approximation’ method, the poles are not known in advance and have to be determined from the interpolating table. To decide which nodes are ‘near optimal’ (see [36]) we need precise information on the location of the poles of the interpolating functions.

The fundamental fact is that the denominators of the approximations (polynomials whose roots are poles of the approximations) satisfy certain complex (non-Hermitian) orthogonality relations. Information on the poles has to be derived from those orthogonality relations, and we arrive at the problem of the asymptotics (in a typical case, of the weak-* zero distribution) for a sequence of complex orthogonal polynomials.

1.4.2. Asymptotics of complex orthogonal polynomials. The method for studying the distribution of the zeros of complex orthogonal polynomials is based on ingenious potential theoretic arguments and was created by Stahl in [59] and [60] and then generalized substantially in [34]. The generalized Stahl method (the so-called GRS-method) reduces the problem of the asymptotics of orthogonal polynomials to an equilibrium problem for the logarithmic potential.

This equilibrium problem is essentially different from ‘standard’ equilibrium problems in complex analysis, related to minimizing an energy functional in the class of measures on a compact ‘conductor’. Robin’s measure of a compact set in the plane and the equilibrium (signed) measure (distribution) of a condenser (a pair of disjoint compact sets) are classical examples. The proof of Walsh’s theorem was based on a condenser equilibrium distribution.
In the case of complex orthogonal polynomials we encounter a different kind of equilibrium, which may be defined as equilibrium in a conducting domain.

1.4.3. Equilibrium in a conducting domain. This is a class of problems which can be described as problems on the critical 'points' of an energy functional. Any such critical point is naturally interpreted as an equilibrium position of an 'electric charge' on an open 'conductor'; the associated equilibrium is always unstable. A classical example of these problems is the Chebotarev problem, which looks for a continuum of minimal logarithmic capacity in the class of all continua containing a given set of points. The Robin measure of the Chebotarev continuum is an equilibrium distribution in the conducting plane with a finite number of insulating points.

In particular, the proof of the $\rho^2$-theorem is related to some generalization of the Chebotarev problem (for the Green potential). A local version of the problem (Padé approximants) is related to a generalization of the standard Chebotarev problem. The solution of Chebotarev-type problems can normally be described in terms of trajectories of a certain quadratic differential associated with the problem. In many cases there are equivalent reformulations in terms of moduli of families of curves, so that this component of the method is also a part of geometric function theory.

In [56]–[60] Stahl observed that the potentials of Chebotarev-type systems of curves satisfy a certain symmetry property (now called the $S$-property, that is, that the normal derivatives of the equilibrium potential in two opposite directions are equal) and this property can be used directly to study complex orthogonal polynomials. He generalized the problem to include curves with the $S$-property ($S$-curves) related to extremal cuts for Padé approximants of functions with branch points.

In [34] (see also [33]) $S$-curves with an external field were part of the method. In the general situation, the existence of an $S$-curve in a given class of functions may be the key part of the whole problem (see [8], [9], [55], [39] and [13]).

1.4.4. Lower bounds for approximations. The first three parts of the method produce an ‘optimal’ sequence of rational approximations to $f$. As an immediate corollary it gives the upper bound in (4). The corresponding lower bound was obtained in [34] using special properties of the sequence of optimal approximations that is constructed. The argument used is rather general and may be stated as a separate theorem. The idea of the method was contained in one of Gonchar’s earlier papers [21], [22], [29] and [30], see also the review [36].

1.4.5. Applications and connections. The method outlined above in connection with the $\rho^2$-theorem potentially has a larger circle of applications. The two components in the proof which are especially important for applications are the asymptotics of complex orthogonal polynomials and the related $S$-equilibrium problems.

Orthogonal polynomials are the key to a great variety of applications. Together with the traditional applications in approximation theory, numerical analysis, and spectral theory, many new applications have been found in the last two or three decades, in particular in mathematical physics.

New classes of problems have arisen in the theory related to random matrices and statistics among other fields. New methods have been created in the theory of orthogonal polynomials, in particular, the method of steepest descent
for matrix Riemann-Hilbert problems. New versions of old methods, such as the Liouville-Green-Steklov method (also known as the WKB method) were developed. In all these cases an $S$-equilibrium configuration, reflecting the geometric component of the problem, plays a key role.

The geometric component arising from the existence problem for $S$-equilibrium configurations leads to an environment somewhat similar to the one related to general moduli problems in geometric function theory (the moduli of families of curves, quadratic differentials, critical trajectories). This component is often present in difficult problems. For instance, many important questions on matrix $S$-problems related to the study of Hermite-Padé polynomials are open; see [2], [55], [15] and [64].

Finally, equilibrium problems create certain connections to integrable systems (soliton theory). In many cases such problems related to approximations are similar to those that come from mathematical physics. For instance, some equilibrium problems associated with the $\rho^2$-theorem and its generalizations are surprisingly close to problems originating in the study of KdV or NLS equations by means of the inverse scattering transform method. For some further comments see [55].

§2. Approximation of Markov-type functions

Markov-type functions $f(z)$, which we shall call M-functions below, are Cauchy transforms of positive measures with compact support on the real line

\[ f(z) := \int_F \frac{d\sigma(t)}{z - t}, \quad z \in \Omega = \mathbb{C} \setminus F, \tag{5} \]

where $F$ is a (finite) interval (we could consider that $F$ is the minimal interval containing the support of $\sigma$).

In particular, under certain assumptions about the measure $\sigma$, an M-function $f$ belongs to $\mathcal{A}(\mathbb{C} \setminus e)$, $e \subset \mathbb{R}$, where $e$ is a finite set, and the jumps of $f$ across the branch-cuts on the real line have constant argument and are integrable. Note that the branch-cuts along $\mathbb{R}$ constitute the boundary of the associated extremal domain. Thus, the class of M-functions and the class $\mathcal{A}(\mathbb{C} \setminus e)$ with $\text{cap} e = 0$ intersect but neither contains the other.

Now, we fix an interval $E$ of the real line not intersecting $F$ and consider best rational approximations to $f$ on $E$. Let $\rho_n(f, E)$ be the distance from $f$ to the class $\mathcal{R}_n$ in the uniform norm on $E$ (see (1)).

2.1. Gonchar’s $\rho^2$-theorem for Markov functions. The following is Gonchar’s version of the $\rho^2$-theorem for M-functions.

**Theorem 2.** Let $\sigma'(x) = d\sigma/dx > 0$ almost everywhere on $F$; then

\[ \lim_{n \to \infty} \rho_n(f, E)^{1/n} = \rho(f, E)^2. \tag{6} \]

We give the main components of the proof.
2.1.1. Interpolation. We begin with an arbitrary triangular table of points \( \{\zeta_{k,n}\} \subset E \) where \( n \) is a natural number and \( k = 1, 2, \ldots, 2n \) for a fixed \( n \). We define \( W_n(z) = \prod_{k=1}^{2n} (z - \zeta_{k,n}) \).

Next, we define the \( n\)^th-order multipoint Padé approximant \( r_n \) to \( f \) associated with the interpolation table \( \{\zeta_{k,n}\} \). For technical reasons it is convenient to use exactly \( 2n \) interpolation points and then use the interpolating function with the condition \( r_n(\infty) = 0 \).

For each \( n \) there exists a pair of polynomials \( P_n \in \mathbb{P}_{n-1} \) and \( Q_n \in \mathbb{P}_n \) such that \( Q_n \neq 0 \) and the condition

\[
F_n(z) = \frac{(Q_nf - P_n)(z)}{W_n(z)} \in H(E)
\]

is satisfied (\( F_n \) is analytic on \( E \)). Indeed, the last condition is equivalent to a system of \( 2n \) linear homogeneous equations for the \( 2n + 1 \) coefficients of the polynomials \( P_n \in \mathbb{P}_{n-1} \) and \( Q_n \in \mathbb{P}_n \) (in the case of distinct nodes those equations are \( (Q_nf - P_n)(\zeta_{k,n}) = 0, \ k = 1, 2, \ldots, 2n \)). Such a system always has a nontrivial solution. This proves the existence of the required polynomials \( P_n \) and \( Q_n \neq 0 \).

We set \( r_n = P_n/Q_n \). This function does not necessarily interpolate \( f \) at all the nodes (\( P_n \) and \( Q_n \) can have common zeros, where the interpolation may be lost). However, this cannot happen for Markov-type functions, as some arguments given below will show.

2.1.2. Orthogonality. Hermite interpolation formula. The denominator \( Q_n \) satisfies the following orthogonality conditions:

\[
\int_F Q_n(x)x^j \frac{d\sigma(t)}{W_n(t)} = 0, \quad j = 0, 1, \ldots, n - 1. \tag{7}
\]

The following identity, called the Hermite interpolation formula, is also important:

\[
f(z) - r_n(z) = \frac{W_n(z)}{Q_n^2(z)} \int_F \frac{Q_n^2(t)}{W_n(t)} \frac{d\sigma(t)}{z-t}, \quad z \in \Omega. \tag{8}
\]

The proof of (7) is obtained by integrating \( z^j F_n(z) \) over a contour \( C \) separating \( F \) from \( E \) and \( \infty \). In particular, (7) implies that the zeros of \( Q_n \) are simple and belong to \( F \). Then (8) follows from the Cauchy integral representation for the function \( Q_nF \in H(\text{Ext} \ C) \), where \( \text{Ext} \ C \) denotes the unbounded connected component of the complement of the contour \( C \).

2.1.3. Zero distribution. Balayage. Conditions (7) present a model situation of orthogonality with varying weights on the real line. The assertion on the asymptotics of the associated orthogonal polynomials \( Q_n \) is formulated in terms of the weak-\( \ast \) convergence of the normalized counting measures. The counting measure of a polynomial \( P \) is defined as the sum of unit masses at the zeros of \( P \) (counting multiplicities) and is denoted by \( \mathcal{X}^\ast(P) = \sum_{\zeta \in \mathcal{X}} \delta(\zeta) \) (here \( \delta(\zeta) \) is the unit measure supported at the point \( \zeta \)).

Now we assume that the interpolation table has a limit distribution (limit density) represented by a positive unit measure \( \mu \) on \( E \). More exactly, this assumption means that the sequence \( \mathcal{X}(W_n)/2n \) is weak-\( \ast \) convergent to the measure \( \mu \).
as \( n \to \infty \). We denote this fact by
\[
\frac{1}{2n} \mathcal{D}(W_n) \xrightarrow{\ast} \mu.
\]

The basic fact regarding free-pole real interpolation of M-functions is that if the interpolation table has a limit density \( \mu \), then the denominators \( Q_n \) have the limit distribution \( \lambda \) which is the balayage of \( \mu \) from \( E \) onto \( F \). Formally,
\[
\frac{1}{2n} \mathcal{D}(W_n) \xrightarrow{\ast} \mu \implies \frac{1}{n} \mathcal{D}(Q_n) \xrightarrow{\ast} \lambda,
\]
where \( \lambda = \lambda(\mu) = \lambda_F \) is the unit measure on \( F \) defined by the condition
\[
U^\lambda(x) - U^\mu(x) = C_F = \text{const}, \quad x \in F
\]
(here \( U^\nu(x) = -\int \log |x-t| d\nu(t) \) denotes the logarithmic potential of the measure \( \nu \)).

2.1.4. Convergence. Upper bound for the rate on \( E \). Comparing the boundary values on \( F \) and singularities on \( E \), we can verify directly that
\[
|f(z) - r_n(z)|^{1/n} \to \exp\{-2G^\mu(z)\}
\]
uniformly for \( z \in \Omega \setminus E \) and, moreover,
\[
\max_{x \in E} |f(x) - r_n(x)|^{1/n} \to \exp\{-2 \min_{x \in E} G^\mu(x)\}.
\]
To obtain the best possible estimate from (13), we need to find a measure \( \mu \) which maximizes
\[
w(\mu) = \min_{x \in E} G^\mu(x)
\]
in the class of all unit measures \( \mu \) on \( E \). The problem is well known in classical complex analysis; its solution is \( \mu = \lambda_E \), the Green equilibrium measure on \( E \) relative to \( \Omega = \overline{\mathbb{C}} \setminus F \) (cf. [14], [15] and [64]). It follows from (11) that the identity \( G^\mu(x) \equiv \text{const}, \ x \in E \), expresses a characteristic property of the measure \( \mu \). Hence, by (11),
\[
U^{\lambda(\mu)}(x) - U^\mu(x) = C_E = \text{const}, \quad x \in E.
\]
Relations (10) and (15) together mean that the pair of measures \( \mu = \lambda_E = \lambda \) and \( \lambda = \lambda(\mu) = \lambda_F \) form an equilibrium distribution for the condenser \( (E, F) \). The capacity \( C(E, F) \) of the condenser is defined as \( C(E, F) = 1/w \), where in terms of the equilibrium constants in (10) and (15) we set \( w = C_F - C_E \). From here
\[
\lim_{n \to \infty} \rho_n(f)^{1/n} \leq \rho(f)^2.
\]
We note that the equilibrium problem related to this situation is the standard equilibrium for a plane condenser (a pair of disjoint compact sets), and thus we do not seem to have a nonstandard $S$-equilibrium problem here. The reason is that on the real axis any equilibrium potential is automatically symmetric with respect to the real axis (the normal derivatives of the potential in two opposite directions are equal). This means that the associated $S$-property follows from the symmetry of the situation.

2.1.5. Lower bound and strong asymptotics. It follows directly from (8) that $f - r_n$ is real and has exactly $2n$ zeros on $E$. Thus, it makes $2n + 1$ oscillations on $E$, whose amplitudes are asymptotically estimated by (12). These estimates make it possible to use the classical de la Vallée-Poussin inequality [65] (instead of Gonchar’s general complex argument mentioned above) to obtain the corresponding lower bound.

Moreover, under Szegö’s condition on the measure $\sigma$ we can modify the interpolation nodes (the measure $\mu$) slightly, so that the difference $f - r_n$ is asymptotically equi-oscillating. Then, an application of de la Vallée-Poussin’s estimates will give the strong asymptotics for the error of the best approximations

$$\rho_n(f) = \gamma(\sigma, E)\rho(f)^{2n}(1 + \varepsilon_n),$$

where $\varepsilon_n \to 0$ as $n \to \infty$ and $\gamma(\sigma, E)$ is an explicit constant (the same is true for the error of interpolation $\max_{x \in E} |f(x) - r_n(x)|$).

Details related to this and other similar results can be found in the book [62] by Stahl and Totik on general orthogonal polynomials. One example of a more complex (but still Markov) situation, described by Stahl’s theorem on the rational approximation of $\sqrt{x}$ on [0, 1], is in §2.2 below.

For a rather general class of analytic functions (including functions with complex branch points) Aptekarev [1] proved a theorem on the exact constants of approximation by rational functions of order $\leq n$. In particular, for functions in this class he obtained a relation of the type of (16). His theorem yields the following formula for the strong asymptotics of the error $\rho_n = \rho_n(e^{-x})$ in the best uniform approximation to $e^{-x}$ on the semiaxis $[0, +\infty)$ by rational functions of order $\leq n$:

$$\rho_n = 2v^{n+1/2}(1 + o(1)), \quad n \to \infty,$$

where $v$ is the so-called Halphen constant (see [34] and [40]). Formula (17) proves a conjecture of Magnus’s [40] on the exact constant in the rational approximation of $e^{-x}$ on the semiaxis $[0, +\infty)$.

The method in [1] was based on a study of strong asymptotics for complex orthogonal polynomials using the method of steepest descent for the matrix Riemann-Hilbert representation of such polynomials. In this connection see also [6] and [43], where steepest descent is used for both matrix Riemann-Hilbert and WKB.

2.1.6. Generalization. Equilibrium measure. Formulae (9) and (10) were proved in the original papers [27] and [25] only in the case when $\mu = \lambda_E$, which was enough to complete the proof of Theorem 2.

More general orthogonal polynomials $Q_n$ defined by

$$\int_F Q_n(x)e^{-2n\varphi_n(x)}x^jd\sigma(t) = 0, \quad j = 0, 1, \ldots, n - 1,$$

(18)
were studied by Gonchar and the author \cite{31} with the following result (we give a simplified version).

**Theorem 3.** Let \( \sigma'(x) = \frac{d\sigma}{dx} > 0 \) almost everywhere on \( F \) and suppose that the sequence \( \varphi_n(x) \) converges to \( \varphi(x) \) uniformly on \( F \). Then \( \frac{1}{n} \mathcal{E}(Q_n) \to \lambda \) where \( \lambda = \lambda_\varphi \) is the equilibrium measure of \( F \) in the external field \( \varphi(z) \), which means that it is a unit measure on \( F \) defined by

\[
U^\lambda(x) + \varphi(x) = C = \text{const,} \quad x \in \text{supp} \lambda, \quad U^\lambda(x) + \varphi(x) \geq C, \quad x \in F. \tag{19}
\]

This theorem was perhaps the first general result on the distribution of the zeros of orthogonal polynomials with varying weights.

2.2. Stahl’s theorem on the approximation of \(|x|\) on \([-1, 1]\). The problem of estimating \( \rho_n = \rho_n(|x|, [-1, 1]) \) was introduced by Newman \cite{45} in 1964, who proved that

\[
e^{-c_1 \sqrt{n}} \leq \rho_n \leq e^{-c_2 \sqrt{n}} \quad \text{with some } c_1 \geq c_2 > 0.
\]

It is easy to see that \( \rho_{2n} = \rho_n(\sqrt{x}, [0, 1]) \), so the problem is reduced to approximating \( \sqrt{x} \) on \([0, 1]\).

Representing \( \sqrt{z} \) in the domain \( \{|z| < 2\} \setminus (-2, 0) \) by the Cauchy integral and defining

\[
f(z) := \frac{1}{\pi} \int_{[-1,0]} \frac{\sqrt{-t}}{z - t} dt, \quad z \in \Omega = \mathbb{C} \setminus [-1, 0], \tag{20}
\]

we find that \( g(x) = \sqrt{x} - f(x) \) is analytic on \([0, 1]\) and therefore its rational approximations converge to \( g \) geometrically. This, together with Newman’s estimate, implies that \( \rho_n(\sqrt{x}, [0, 1]) / \rho_n(f(x), [0, 1]) \to 1 \) as \( n \to \infty \).

So the problem is reduced to the study of the best rational approximation of the M-function \( f \) on \( E = [0, 1] \). Basically, we can use the method described in §2 above, but this method has to be modified.

Now, this problem is more difficult than the problems discussed in §2.1. The condenser \( (E, F) \) associated with the current situation, \( E = [0, 1], F = [-1, 0] \), is degenerate since the plates \( E \) and \( F \) have a common point and the equilibrium distribution \( \lambda_E - \lambda_F \) for such a condenser does not exist (a ‘collapsing’ situation).

Stahl used a condenser with a logarithmic external field on the plate \( F \) (which comes from the term \( \sqrt{-t} \) in (20)). The external field prevents the equilibrium distribution from ‘collapsing’ and the weighted equilibrium distribution can be used to define an optimal interpolating table. Stahl was able to obtain strong asymptotics for associated orthogonal polynomials. Then he obtained strong asymptotics for the error of approximation. As a result, in \cite{61} he proved the following remarkable theorem.

**Theorem 4.** For \( \rho_n = \rho_n(|x|, [-1, 1]) \)

\[
\lim_{n \to \infty} \rho_n e^{\pi \sqrt{n}} = 8. \tag{21}
\]

This result had been conjectured by Varga \cite{66} on the basis of numerical experiment. The correct constant \( c_1 = c_2 = \pi \) was found earlier by Vyacheslavov \cite{67}. 
2.3. Problems and conjectures by Gonchar related to the $\rho^2$-theorem. A broader context related to the $\rho^2$-theorem is the general problem of characterizing classes of analytic functions through the rate of convergence of their best rational approximations.

The corresponding problem for polynomial approximations in fact has a general solution and the associated theory is well known. For rational approximations the situation is more complicated. It is usually difficult to find a criterion in terms of best rational approximation since direct and inverse theorems are mostly very different. A typical example is related to characterizing the class of functions with supergeometric rate of convergence of best rational approximations,

$$\lim_{n \to \infty} \rho_n(f, E)^{1/n} = 0.$$  \hspace{1cm} (22)

A direct theorem by Pommerenke [50] asserts that if $f \in H(\overline\mathbb{C} \setminus e)$ and $\text{cap}(e) = 0$ then (22) is valid for any $E \subset \overline\mathbb{C} \setminus e$. The converse is not true. Basically, knowing (22) we cannot say anything about the set of singularities of $f$.

On the other hand Gonchar proved in [22] that (22) implies that $f$ is quasi-analytic (there is a uniqueness theorem for such functions which is similar to the one for analytic functions). He also proved in [22] that the function $f$ is single valued in the whole of its Weierstrass domain if (22) is satisfied. There are more theorems by Gonchar of this kind; see [23] and [24] for details.

Soon after the $\rho^2$-theorem for $M$-functions was proved, Gonchar raised the following general question: for what kind of functions does the limit

$$\lim_{n \to \infty} \rho_n(f, E)^{1/n}$$

exist and is positive? In other words, for which functions does the sequence of best rational approximations have regular behaviour? His basic idea was that all the ‘natural’ functions are regular and for any such function we have

$$\lim_{n \to \infty} \rho_n(f, E)^{1/n} = \rho(f, E)^2$$ \hspace{1cm} (23)

for any continuum $E$ in the domain of the function (for example, see [25]). In other words, if $\lim \rho_n(f, E)^{1/n}$ exists then it is equal to $\rho(f)^2$. All subsequent results seem to confirm the conjecture, but it is not clear how it can be proved.

Anyway, Theorem 2 means that Markov-type functions are regular ((23) is satisfied under mild restrictions on the measure), which is also an important argument in favour of Gonchar’s $\rho^2$-conjecture that each $f \in \mathcal{A}(\overline\mathbb{C} \setminus e)$ with a finite set $e$ is regular (a stronger version is contained in the Gonchar-Stahl theorem). In particular, he also conjectured that all algebraic functions are regular and for any $E$ which is free of singularities we have the stronger estimates

$$0 < C_1(f, E) \leq \frac{\rho_n(f, E)}{\rho(f, E)^{2n}} \leq C_2(f, E) < \infty$$ \hspace{1cm} (24)

(this conjecture by Gonchar (see [26]) remains open despite the wide range of existing modern methods).
Another general conjecture by Gonchar was that if a function has worse than \( \rho^2 \)-rate of best rational approximation and
\[
\lim_{n \to \infty} \rho_n(f, E)^{1/n} > \rho(f, E)^2,
\]
then it is ‘not regular’ and there is another subsequence where the rate is better than ‘normal’; that is,
\[
\lim_{n \to \infty} \rho_n(f, E)^{1/n} < \rho(f, E)^2.
\]
In particular, his conjecture stated that
\[
\lim_{n \to \infty} \rho_n(f, E)^{1/n} \leq \rho(f, E)^2 \quad \text{for any } f \in H(E). \tag{25}
\]
Later this conjecture was proved by Parfenov [48] and by Prokhorov [51]–[53], who also obtained the stronger inequality
\[
\lim_{n \to \infty} \prod_{k=1}^{n} \rho_k(f, E)^{1/k} \leq \rho(f, E)^2. \tag{26}
\]
The proofs of the theorems by Parfenov and Prokhorov were based on a combination of fixed poles interpolation and theorems of singular values of Hankel operators. This is essentially another important direction in approximation theory, which is in many ways different from the one under consideration and we shall not go into further details.

§ 3. Stahl’s theorem on Padé approximants

So far we have discussed the rate of convergence of best rational approximations. Now we go over to the convergence properties of the approximating functions; see (12) and (13) as examples. The construction of a near-best rational approximation \( r_n(z) \) to \( f \) in the context of Theorem 1 may also be arranged in such a way that these functions converge to \( f \) uniformly on compact subsets of the whole extremal domain of analyticity of \( f \).

It is more convenient to discuss the convergence problem for rational approximations in the case of (diagonal) Padé approximants, the best local rational approximants to a power series. It is also convenient to select an interpolation point at infinity, so that the singular points of the function are finite.

3.1. Padé approximants for functions with branch points. Let
\[
f(z) = \sum_{n=0}^{\infty} \frac{f_k}{z^k}
\]
be a function analytic at infinity. The Padé approximants to \( f \), \( \pi_n(z) = (P_n/Q_n)(z) \), are defined by the condition
\[
R_n(z) := (Q_n f - P_n)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty, \tag{28}
\]
where \( P_n, Q_n \in \mathbb{P}_n \) and \( Q_n \neq 0 \) (see [7] for details). In connection with best rational approximations we note that the \( \pi_n(z) \) are the limits of the best approximations on \( D_R = \{ z : |z| \geq R \} \) as \( R \to \infty \) [70]. In this sense Padé approximants are the local version of best rational approximants. The function \( R_n \) is called the remainder.
3.1.1. Markov theorem. An old classical convergence theorem proved by Markov [41] in 1895 asserts that if \( f(z) \) is an M-function (5) then the associated sequence \( \pi_n(z) \) converges to \( f \) uniformly on compact subsets of the complement of \( F \) (the minimal interval containing the support \( \text{supp} \sigma \) of the measure \( \sigma \)). Note that \( f \) is holomorphic in the domain \( \Omega = \mathbb{C} \setminus \text{supp} \sigma \), which may be larger than \( \mathbb{C} \setminus F \). The functions \( \pi_n \) may have poles in this larger domain, but they still converge in capacity there.

The fact that the Padé denominators \( Q_n \) are orthogonal polynomials with respect to \( \sigma \) was discovered earlier, in 1855 by Chebyshev [17].

3.1.2. Nuttall’s minimal capacity conjecture. One of the main problems in the theory of Padé approximants in the period 1960–1970 was the convergence problem for functions with branch points. If an element at infinity (27) represents a function \( f \in \mathcal{A}(\mathbb{C} \setminus e) \) where \( e \) is, say, a finite set of branch points, then the Padé approximants to \( f \) can converge to \( f \) only in a domain where \( f \) is single-valued. What is the actual domain of convergence?

The first results on the convergence of Padé approximants for functions with some special type of branch points were obtained by Nuttall, who also made the following conjecture (see [46] and [47]). Let \( f \in \mathcal{A}(\mathbb{C} \setminus e) \) where \( e \) is a finite set and

\[
\mathcal{F} = \{ F \subset \mathbb{C} : f \in H(\mathbb{C} \setminus F) \}
\]

be the set of compact cuts \( F \) which makes \( f \) single-valued. Let, further, \( F_f \in \mathcal{F} \) be the cut of minimal capacity

\[
\text{cap}(F_f) = \min_{F \in \mathcal{F}} \text{cap}(F).
\]

Nuttall’s main conjecture was that the sequence \( \{ \pi_n \} \) converges to \( f \) in capacity in the complement to \( F_f \):

\[
\pi_n \xrightarrow{\text{cap}} f, \quad z \in \mathbb{C} \setminus F_f.
\]

He also formulated a conjecture on the strong asymptotics of the Padé denominators, which he proved in some particular situations [46], [47].

3.2. Stahl’s theorem. A general theorem on the convergence of Padé approximants including, in particular, Nuttall’s conjecture, was proved by Stahl [56]–[59]. Here is the original statement of the theorem where the compact set \( F_f \) of minimal capacity is characterized equivalently in terms of the \( S \)-property.

**Theorem 5.** Let \( e \) be a compact set of zero (logarithmic) capacity \( \text{cap}e = 0 \) and assume that \( f \in \mathcal{A}_e = \mathcal{A}(\mathbb{C} \setminus e) \) is not single-valued in \( \mathbb{C} \setminus e \). Then the following assertions hold.

(A) There exists a unique compact set \( F = F_f \) in the plane which is the union of analytic arcs (up to subsets of capacity zero) with the following properties. The complement of \( F \) is connected, \( f \) is single-valued in \( \mathbb{C} \setminus F \) (so that \( F \in \mathcal{F}(f) \)), the jump of \( f \) across any arc in \( F \) is not identically equal to zero and, finally, the equality

\[
\frac{\partial g}{\partial n_1}(z) = \frac{\partial g}{\partial n_2}(z), \quad z \in F^o
\]
(called the S-property) holds for the Green’s function \(g = g(z, \infty)\) of \(\mathbb{C} \setminus F\) with pole at infinity, where \(F^c\) is the union of the open parts of the arcs constituting \(F\) \((n_1 \text{ and } n_2 \text{ are the two opposite directed normals to } F^c)\).

(B) For the Padé denominators \(Q_n\) associated with \(f\)

\[
\frac{\mathcal{X}(Q_n)}{n} \to \lambda,
\]

where \(\lambda = \lambda_F\) is the Robin measure of the compact set \(F\).

(C) The sequence of Padé approximants \(\pi_n = P_n/Q_n\) associated with \(f\) converges in capacity to the function \(f\) in the interior (that is, on compact subsets) of the domain \(D := \mathbb{C} \setminus F\).

The exact rate of convergence in capacity was also included in Stahl’s statement, but our further discussion is related to assertions (A) and (B). Part (C) is essentially a corollary to (B).

The most important part of the theorem is part (B). Rather sophisticated and entirely original potential theoretic methods were used in this part of the proof. The starting point was the following orthogonality condition for the Padé denominators \(Q_n\)

\[
\int_F Q_n(z) z^k f(z) \, dz = 0, \quad k = 0, 1, \ldots, n - 1, \quad (32)
\]

where integration is taken over any system of contours separating \(F\) from infinity. It is very important to note that Stahl’s proof was the first instance of the effective use of complex orthogonality and this was a significant breakthrough in the theory.

Another interesting fact about Stahl’s proof is that additional assumptions do not lead to any simplifications. The proof of the theorem for a set \(e\) with three (noncollinear) branch points is identical to the original proof for sets \(e\) of capacity zero. Additional assumptions on the character of the branch points do not bring any simplifications either (one exception is the square root of a rational function). It seems that this part of the theorem does not have any simple complex particular cases for noncollinear branch points. The case when the branch points are on a line may essentially be viewed as part of the Markov theorem (some additional assumptions are formally needed).

It is not possible to go into any further details related to this part of the proof here. We have in mind that the Robin measure \(\lambda = \lambda_F\) of the extremal compact set \(F = F_f\) represents the limit zero distribution of the Padé denominators and we will concentrate on the further characterization of \(F\) (the geometric component of the problem). Part (A) of the theorem essentially defines \(F\) in terms of the S-property.

Recall that a similar part of the proof of the \(\rho^2\)-theorem was represented by a Green equilibrium problem. In more general situations, the geometric component can be represented by a more general kind of S-equilibrium problem. Then even the existence of a solution becomes a problem (see [11], [12] and [16], for example). Constructive solutions are another problem. To some extent, further progress in the theory depends on the development of the geometric component of the method.
3.3. ‘Geometry’ of Stahl’s theorem. The geometric part in Stahl’s theorem is a particular case of a general $S$-equilibrium problem; it is the case of a single logarithmic potential and the zero external field. The extremal compact set $F_f$ in such settings always exists and has a comparatively simple constructive characterization. A similar characterization (in the case of existence) can also be obtained in more general cases, with a single, logarithmic or Green, potential and an external field which is harmonic outside a set of capacity zero. Other cases are essentially open; see [55]. Next we discuss the case when the set $e$ of singularities of $f$ is finite (and there is no external field). The formulae related to this case are explicit. At the same time the case is still representative and may shed light on the nature of the problem as a whole.

3.3.1. A quadratic differential. Let $e = \{a_1, a_2, \ldots, a_p\}$ be a finite set of distinct points; we set $A(z) = (z - a_1)(z - a_2) \cdots (z - a_p)$. Let $f \in \mathcal{A}_e = \mathcal{A}(\mathbb{C} \setminus e)$ and let $F_f$ be the associated extremal compact set. In the sequel we call it an $S$-compact set or Stahl’s compact set. The following characterization of $F_f$ is valid.

There exists a polynomial $V_f$ of degree $p - 2$,

$$V_f(z) = (z - v_1)(z - v_2) \cdots (z - v_{p-2}), \quad \text{where } v_j = v_j(f),$$

which depends on $f$ and $e$ and such that the $S$-compact set $F_f$ is the union of some critical trajectories of the quadratic differential $-(V/A)\,(dz)^2$ where $V = V_f$.

The assertion follows from Stahl’s results [56]–[58]. However, the statement is close to some traditional theorems in geometric function theory [63]. An alternative proof based on a max-min energy problem was presented in [49]; see the review [55] for details.

In addition, the following condition is satisfied: $-(V/A)\,(dz)^2$ is a quadratic differential with closed trajectories (in the terminology of [63]). In our particular case it means that all its trajectories, defined by $-(V(z)/A(z))\,(dz)^2 > 0$ are either closed contours or critical trajectories which are analytic arcs connecting some pair of zeros of $AV$.

Moreover, the function $\sqrt{V(z)/A(z)}$ has a holomorphic branch in $\Omega = \mathbb{C} \setminus F_f$ and the Green’s function $g$ for $\Omega$ with pole at infinity can be written as

$$g(z) = \text{Re} \, G(z), \quad G(z) = \int_{a}^{z} \sqrt{\frac{V(t)}{A(t)}} \, dt, \quad a \in e \quad (33)$$

(the branch of the root is selected so that $g(z) = \log |z| + o(1)$ at infinity). The $S$-property (31) of the Green’s function follows directly from this representation.

Representation (33) establishes a one-to-one correspondence between $S$-compact sets $F_f$ and polynomials $V_f$. Zeros of $V_f$ may serve as the coordinates of $F_f$.

3.3.2. The family of polynomials $V_f$, $f \in \mathcal{A}_e$. The problem of a constructive determination of the compact set $F_f$ for a given $f$ has two components. This compact set depends, first of all, on the branch set $e$ of the function $f \in \mathcal{A}_e$. It also depends on the branch type of the function (determined by indicating the loops along which analytic continuation leaves the original element at infinity unchanged). It is convenient to separate the two dependencies by introducing the family of compact sets $F_f$ associated with all functions $f \in \mathcal{A}_e$ having the fixed set $e$ of branch points.
It is not difficult to prove that this family is finite. How many elements it has
depends on the number of points in the set $e$ and their configuration (we do not
discuss how to calculate this number). Now we concentrate on characterizing this
family.

Since each compact set $F_f$, is uniquely defined by the associated polynomial $V_f$,
the whole family $F_f, f \in \mathcal{A}_e$, can be described in terms of the associated family of
polynomials $V_f$, which we denote by

$$\tilde{V}(e) = \{V_f : f \in \mathcal{A}_e\}.$$ 

A polynomial $V \in \tilde{V}(e)$ is determined by its roots, that is, by $p - 2$ complex
numbers $v_j$ playing the role of coordinates. We can therefore ask if some kind of
equations can be written in terms of coordinates $v_j$. Some equations may, indeed,
be derived from the characterization of $S$-compact sets as critical trajectories of
quadratic differentials (see §3.3.1 above). These equations (written in terms of the
periods of quadratic differentials) belong to a well-known class of equations and
usually they are not easy to deal with. In particular, there is a difficult combi-
natorial element in their structure and a detailed analysis of the situation cannot
be presented here. Below we briefly outline two possible ways the problem can be
approached without going into all the details.

In §3.3.3 we will introduce a family of hyperelliptic Riemann surfaces associated
with the family $F_f, f \in \mathcal{A}_e$, of Stahl compact sets. In terms of this family of Rie-
mann surfaces we define a mapping in the set of monic polynomials of degree $p-2$.
Then the polynomials $V \in \tilde{V}(e)$ are defined as fixed points of this mapping. It seems
that until now nothing has been published related to this approach.

It is possible that a natural way of generalizing Stahl’s theorem to Hermite-Padé
approximation would use a proper generalization of this approach. Anyway, the
associated generalization of the family of Riemann surfaces is already known, at
least in simple situations.

In §3.4 we discuss an approach to the problem of a constructive description
of $S$-compact sets which is based on the embedding of the set of Robin measures
associated with compact sets $F_f, f \in \mathcal{A}_e$, into a larger space of probability measures
in plane which we call $e$-critical measures. These measures constitute a connected
finite dimensional variety and its structure may help to give a better understanding
of the structure of the discrete set of Robin measures for $F_f, f \in \mathcal{A}_e$, (see [42]).

Later, in §4 we also use critical measures to study the problem of the stability
of convergence in Stahl’s theorem under variations of the function $f$ which preserve
the set of branch points.

3.3.3. The family of Riemann surfaces $\mathcal{R}_f, f \in \mathcal{A}_e$. The $S$-property (31) is essen-
tially equivalent to the fact that the real Green’s function $g(z)$ (see (33)) of the
domain $\Omega = \overline{\mathbb{C}} \setminus F_f$ has a harmonic extension to a hyperelliptic Riemann surface
$\mathcal{R} = \mathcal{R}_f$, which may be defined as the Riemann surface of the function $\sqrt{V/A}$
with $V = V_f$. We interpret $\mathcal{R}_f$ in a standard way as a two-sheeted branched
covering over $\overline{\mathbb{C}}$. Formula (33) provides a constructive form of this extension.

Recall that on any hyperelliptic Riemann surface $\mathcal{R}$ there exists a unique function
$g = g_\mathcal{R} : \mathcal{R} \to \mathbb{R}$ defined as a harmonic function on the finite part of $\mathcal{R}$ with
asymptotics $g(z) = \log |z| + o(1)$ as $z \to \infty^{(1)}$ and $g(z) = - \log |z| + o(1)$ as $z \to \infty^{(2)}$
and with normalization $g(z^{(1)}) + g(z^{(2)}) \equiv 0$. We call this function $g = g_\mathcal{R}$ the 
\textit{g-function for the Riemann surface $\mathcal{R}$}. The extension of the Green’s function $g(z)$
from the domain $\mathcal{C} \setminus F_f$ with $f \in \mathcal{A}$ to the Riemann surface $\mathcal{R}_f$ is precisely the 
g-function for $\mathcal{R}_f$.

Consequently, the complex Green’s function $G$ in (33) has a (multivalued) analytic extension to $\mathcal{R}$, which is a standard third-kind Abelian integral on $\mathcal{R}$ with (logarithmic) poles at $\infty^{(1)}$ and $\infty^{(2)}$ and divisor $1, -1$ (we call it the $G$-function
for $\mathcal{R}$).

The representation $G'(z) = \sqrt{V(z)/A(z)}$ where $V = V_f$, asserted in (33) for $z \in \mathcal{C} \setminus F_f$, is valid for $z \in \mathcal{R}$. The extremal compact set $F_f$ is the projection of
the zero level $\{\zeta : g(\zeta) = 0\} \subset \mathcal{R}$ of the $g$-function onto the (extended) plane $\mathcal{C}$.

Now, together with the collection of $S$-compact sets and the associated family
of polynomials $\tilde{V}(e)$, we also have the family of Riemann surfaces $\tilde{\mathcal{R}}(e) = \{\tilde{\mathcal{R}}_f : f \in \mathcal{A}\}$. Next, we will obtain a representation of $\tilde{V}(e)$ in terms of this family.

Recall that we began our constructions with a fixed polynomial $A(z) = z^p + \cdots$ with simple roots. Next, consider a variable polynomial $V(z) = z^{p-2} + \cdots$ (at the
moment we do not relate $V$ to the above constructions). Assume for now, however,
that the zeros of $V$ are simple and distinct from zeros of $A$. Then the Riemann
surface of the function $\sqrt{V/A}$ is a generic hyperelliptic Riemann surface of genus $p - 1$
with $2p - 2$ quadratic branch points at the zeros of $AV$. It is well known that the
$G$-function for such a surface can be written in the form

$$G(z) = G(z; V) = \int_a^z \frac{W(t) \, dt}{\sqrt{A(t)V(t)}}, \quad \text{where } W(z) = z^{p-2} + \cdots \quad (34)$$

($a$ is a root of $A$). The polynomial $W$ is uniquely determined by the polynomial $AV$. Since $A$ is fixed, this defines a mapping $\Phi : V \rightarrow W$. So far, it is defined
under the assumption that the zeros of $V$ are simple and distinct from zeros of $A$
but this mapping has a continuous extension to the whole space $\mathbb{P}_p^{(1)}$ of monic
polynomials of degree $p - 2$ (actually, we only need the restriction of the mapping
$\Phi : \mathbb{P}_p^{(1)} \rightarrow \tilde{\mathbb{P}}_p^{(1)}$ to the space of polynomials with zeros in the convex hull of the
roots of $A$).

It follows from (34) that $G' = W/\sqrt{AV}$. On the other hand (we return to
the original setting) if $V \in \tilde{V}(e)$ then $G'(z) = \sqrt{V(z)/A(z)}$ according to (33).
Combining the two representations we obtain $W = V$. In other words, $V \in \tilde{V}(e)$
implies that $V$ is a fixed point of $\Phi$. Conversely, each fixed point of $\Phi$ lies in $\tilde{V}(e)$, and therefore $\tilde{V}(e)$ is equivalently defined as the set of fixed points of $\Phi$.

It is generally possible for the polynomial $V$ to have common zeros with $A$. Then those common zeros are cancelled in the ratio $V(z)/A(z)$ and the problem is reduced to a similar one with a smaller set $e$ (of zeros of $A$). Such a reduction
is not significant. For instance, let $p = 3$ and let the zeros of $A$ be collinear. Then
the zero of $V$ will cancel the middle zero of $A$ and the problem reduces to the
one with $p = 2$. Cancellations of the other two zeros of $A$ are prohibited by the
assumption that all zeros of $A$ are branch points of functions $f \in \mathcal{A}$.

Reduction of the genus of the surface $\mathcal{R}_f$ may also be the result of the presence
of multiple zeros of $V$, and this is a common occurrence, which is significant. All
the polynomials $V \in \tilde{V}(e)$, apart, possibly, from one, have multiple roots. Loosely
speaking, this fact is a reflection of the possible variety of branch types of the functions \( f \in A_e \). Anyway, the combinatorics of the set \( V(e) \) is in part determined by the multiple roots of \( V \). We add the following observation.

Suppose that \( f \) has a generic branch type, that is, continuation along any non-trivial loop leads to a different branch. Then the associated \( S \)-compact set \( F_f \) is a continuum; it is therefore the Chebotarev continuum for \( e \). In the situation of the configuration of points in the set \( e \) being in ‘common position’ the polynomial \( V^0 \) associated with the Chebotarev continuum will have simple zeros (this may be viewed as a definition of ‘common position’). In this situation, \( V^0 \) is the only fixed point of the mapping \( \Phi \) with simple zeros. All other polynomials \( V \in \tilde{V}(e) \) will necessarily have multiple zeros and so a reduced genus of the associated Riemann surfaces. We shall not go into more detail. We continue our discussion of the structure of the set \( \tilde{V}(e) \) from a different point of view in the next section.

To conclude this section we make the following remark. In the case of a finite set \( e \), Stahl’s theorem can be equivalently formulated in terms of the convergence of the remainder \( R_n \) in (28) on a Riemann surface (in particular, this gives an alternative approach to the way of introducing the Riemann surface \( R = R_f \)).

The theorem can be stated as follows. Given \( f \in A_e \) there exists a hyperelliptic Riemann surface \( R \) such that (with proper normalization) the sequence \( \frac{1}{n} \log |R_n| \) converges in capacity on \( R \) to the \( g \)-function of \( R \). In the equivalent form: the sequence of normalized logarithmic derivatives \( R'_n/(nR_n) \) converge to \( G'(z) \) in the Lebesgue plane measure on \( R \). The surface \( R \) is then uniquely defined by the additional condition that the projection \( F \) of the zero level of \( g \) onto the plane makes \( f \) single-valued and the jump of \( f \) across any arc from \( F \) is not identically equal to zero.

It is possible that Stahl’s theorem in this form may be generalized directly for Hermite-Padé approximants of the first-kind for systems of functions with branch points.

3.4. The critical measures \( M_e \). For a finite set \( e = \{a_1, \ldots, a_p\} \) we define \( e \)-critical measures as critical points of the energy functional

\[
\mathcal{E}(\mu) = -\iint \log |x-y| \, d\mu(x) \, d\mu(y)
\]

with respect to local variations with fixed set \( e \). More precisely, for a smooth complex function \( h(z) \) in a neighbourhood of \( \text{supp} \mu \) we define point variations \( z \mapsto z^t = z + th(z) \), where \( |t| \in (0, \epsilon) \) and then variations of the measures \( \mu \mapsto \mu^t \) by \( d\mu(z) = d\mu^t(z^t) \).

The associated variation of the energy (the derivative in the direction \( h \)) is defined by

\[
D_h \mathcal{E}(\mu) = \lim_{|t| \to 0^+} \frac{1}{t} (\mathcal{E}(\mu^t) - \mathcal{E}(\mu)).
\]

Finally, we say that \( \mu \) is \( e \)-critical if for each function \( h \) satisfying \( h(a) = 0 \) for any \( a \in e \) we have \( D_h \mathcal{E}(\mu) = 0 \). The set of all such measures is denoted by \( M_e \).

Critical (stationary) measures were first introduced in [34] and then used in [49]. A systematic study of critical measures (with rational external fields) was presented in [42] in connection with the distribution of the zeros of Heine-Stieltjes polynomials;
see also the review [55]. Here we use critical measures as an approach to describe the set of Robin measures of $S$-compact sets $F_f$ associated with a fixed set $e$. Below, in §4 they will also be used to study the stability of convergence in Stahl’s theorem.

It is important to observe, first, that the Robin measures of all $S$-compact sets $F_f$ are $e$-critical measures. Second, the basic properties of the Robin measures of $S$-compact sets are preserved for critical measures. In particular, the potential of any $e$-critical measure $\mu$ has the $S$-property presented by (31) with $U^\mu$ in place of $g$. Next, for any critical measure $\mu$ there exists a polynomial $V(z) = \prod_{j=1}^{p-2} (z - v_j)$ such that with $A(z) = \prod_{k=1}^{p} (z - a_k)$ we have

$$U^\mu(z) = \Re \int_{a_1}^{z} \sqrt{\frac{V(t)}{A(t)}} \, dt, \quad d\mu(z) = \frac{1}{\pi} \sqrt{\frac{V}{A}} \, dz.$$ (36)

Moreover, $\text{supp} \mu$ is a union of critical trajectories of $-(V(z)/A(z)) \, (dz)^2$ and this differential has closed trajectories just as for the Robin measures of $S$-compact sets $F_f$. Finally, both sets of measures may be characterized in terms of the associated polynomials $V$.

Using the zeros $v_j$ of $V$ as parameters we represent the set $\mathcal{M}_e$ of critical measures as a subset of the space of vectors $\{v = (v_1, \ldots, v_{p-2})\}$ in $\mathbb{C}^{p-2}$. In these coordinates $\mathcal{M}_e$ is represented as a union of $3^{p-2}$ bounded bordered domains, which we call cells. Each cell is a bounded bordered manifold of real dimension $p - 2$. Interior points of each cell correspond to measures $\mu$ whose support $\Gamma = \text{supp} \mu$ consists of exactly $p - 2$ simple disjoint analytic arcs $\Gamma_j$ with endpoints in the set $\{a_k, v_j\}$. Finally, the $v$-coordinates of measures $\mu \in \mathcal{M}_e$ are defined by the system of equations

$$\Re \int_{\Gamma_j} \sqrt{\frac{V(t)}{A(t)}} \, dt = 0, \quad j = 1, \ldots, p - 2, \quad V(t) = (t - v_1) \cdots (t - v_{p-2}).$$ (37)

A particular cell is identified by the homotopy type of the arcs $\Gamma_j$.

The Robin measures of $S$-compact sets are among $e$-critical measures and their representations in terms of the $v$-coordinates lie on the boundaries of cells.

The space $\mathcal{M}_e$ is connected and each critical measure can be connected in a standard way with the Chebotarev continuum associated with $e$, which can be defined as the only continuum (closed connected set) in the set of $S$-compact sets for functions $f \in \mathcal{M}_e$. The roots of the polynomial $V_0 \in \tilde{V}(e)$ associated with the Chebotarev continuum effectively play the role of the origin in the $v$-coordinate system and the corresponding ‘deformation theory’ was in part described in [42].

Further, the equilibrium measures of $S$-compact sets satisfy (37) and also $p - 2$ additional equations which distinguish them among all critical measures. The potential of an ‘interior’ critical measure $\mu$ supported on the arcs $\Gamma_j$ has constant value on these arcs, that is,

$$U^\mu(z) = C_j, \quad z \in \Gamma_j, \quad \text{supp} \mu = \bigcup_{j=1}^{p-1} \Gamma_j.$$

The tuple of constants $C = (C_1, \ldots, C_{p-2})$ may be used to parametrize points in a particular cell in $\mathcal{M}_e$ (note that only $p - 2$ of the constants are independent).
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The differences between these constants correspond to the parametrization of a cell by the ‘heights of cylinders’ in the terminology of the general moduli problems which can be associated with critical measures. There exists a dual parametrization by the ‘lengths of circles’, which corresponds to the masses $\mu(\Gamma_j)$ (see [63]).

Now the additional equations which determine the equilibrium measures of $S$-compact sets in terms of the $C$-coordinates are

$$C_1 = \cdots = C_{p-1}.$$

We have a total of $2p - 2$ real equations for the same number of real parameters in $V$. For some further details see [63], [42] and [55].

§ 4. Some generalizations and conjectures

The method outlined above can be developed in several directions. In this section we make a few remarks about the possible generalizations.

4.1. The dependence of Padé denominators on the function. Let $f \in \mathcal{A}_e$ be a function with a finite set $e$ of branch points which is defined by an element at infinity (27). Let $\pi_n(z) = (P_n/Q_n)(z)$ be the associated Padé approximants at infinity.

Suppose that we make a small variation in the function $f$ in the class $\mathcal{A}_e$. In other words, consider a function $\tilde{f} \in \mathcal{A}_e$ which is close to $f$ in the following sense: its branch points have the same positions but the corresponding exponents (see (38)) may change.

We want to figure out how much the denominator $Q_n = Q_n(\tilde{f})$ will change, say, for a large enough fixed $n$. To be more exact, here we have in mind a significant change and, as a first step towards investigating the problem, we will discuss a possible change in the limit zero distribution. Since the rate of convergence in Stahl’s theorem is determined by the limit zero distribution, the problem is essentially about the (rough) stability of convergence in that theorem.

It turns out that the answer depends on what exactly the $S$-compact set $F_f$ for the function $f$ was. As usual, assume for simplicity that all the points $a_k \in e$, $k = 1, \ldots, p$, are actual branch points for $f$.

If $F_f$ was the Chebotarev continuum $F_e$ for $e$ (in other words, the function $f$ has a ‘generic branch type’) then small variations of $f$ produce no dramatic effect. It is not difficult to see that any such small enough variation of $f$ will still be of ‘generic branch type’ and therefore will have the same $S$-compact set and the same limit distribution (the Robin measure of this set). Thus, the dependence of $Q_n$ on $f$ is essentially continuous (and asymptotically continuous in a certain sense).

If $F_f$ was an $S$-compact set different from a Chebotarev continuum, then this dependence is not continuous since the dependence of $F_f$ on $f$ is not continuous if $F_f$ is not a Chebotarev compact set for $e$. Which branch types can be obtained by small variations of $f$ depends on $F_f$. Anyway, it is clear that the generic branch type can be obtained from any other using arbitrarily small variations and it is enough to prove the discontinuity of $F_f$ as a function of $f$ at any $f$ whose $S$-compact set is not Chebotarev. All the above facts are consequences of Stahl’s theorem.

The situation changes if we consider a sequence of variations depending on $n$ and converging to zero as $n \to \infty$. What exactly happens to the distribution of
the zeros depends on the characteristics of the function, the characteristics of the variation and the relation between $n$ and the magnitude of the variation (It is also possible to consider variations of the positions of the branch points but the effect will be similar).

Formally, let $f, f_n \in A_e$ for $n \in \mathbb{N}$ and suppose the sequence $f_n$ converges to $f$ as $n \to \infty$. Let $Q_n = Q_n(f_n)$. It follows from what was said above that if $F_f$ is the Chebotarev continuum for $e$, then the sequence $\frac{1}{n} \mathcal{E}(Q_n)$ converges weakly to the Robin measure of this continuum.

If $F_f$ is different from the Chebotarev continuum for $e$, the sequence $\frac{1}{n} \mathcal{E}(Q_n)$ is not weakly convergent in general. We can only claim that the weak-$\ast$ limit of any convergent subsequence belongs to the set $\mathcal{M}_e$ of critical measures for $e$. Which measures are included in the limit set for a given $f$ depends on $F_f$ and the character of convergence. However, using different functions $f \in A_e$ we can obtain any $\mu \in \mathcal{M}_e$ as a limit along the whole sequence. In other words, any $\mu \in \mathcal{M}_e$ is a weak limit of the whole sequence $\frac{1}{n} \mathcal{E}(Q_n)$ for some selection of functions $f, f_n \in A_e$.

To state a theorem which presents the above assertions formally, we have to define the convergence $f_n \rightarrow f$. We give a simple example of a theorem of this type, with a particularly simple kind of convergence. Consider the following model class of functions

$$\mathcal{L}_e = \left\{ f : f(z) = \prod_{k=1}^{p} (z - a_k)^{\alpha_k} \right\}, \quad \sum_{k=1}^{p} \alpha_k = 0. \quad \text{(38)}$$

We assume that $e = \{a_k\}$ is fixed and the $\alpha_k$ are parameters; as usual we assume that each $a_k$ is an actual branch point of the function ($\alpha_k$ is not an integer). We have $\mathcal{L}_e \subset A_e$ and the class $\mathcal{L}_e$ is sufficiently representative, in the sense that all the possible branch types are presented by functions in $\mathcal{L}_e$. Convergence $f_n \rightarrow f$ for functions from $\mathcal{L}_e$ is understood to mean that the $\alpha$-parameters of $f_n$ converge to those of $f$, that is, $\alpha_{k,n} \rightarrow \alpha_k$.

Now we can state a version of the theorem related to the class $\mathcal{L}_e$.

**Theorem 6.** Let the sequence of functions $f_n \in \mathcal{L}_e$ converge to $f \in \mathcal{L}_e$. If the extremal compact set $F = F_f$ for $f$ is the Chebotarev continuum $F_e$ for $e$ then $\frac{1}{n} \mathcal{E}(Q_n(f_n)) \rightharpoonup \lambda$ where $\lambda$ is the Robin measure for $F$.

For any $\mu \in \mathcal{M}_e$ there exist a convergent sequence of functions $f_n \in \mathcal{L}_e \rightarrow f \in \mathcal{L}_e$ such that $\frac{1}{n} \mathcal{E}(Q_n(f_n)) \rightharpoonup \mu$ as $n \rightarrow \infty$.

Theorem 6 generalizes Stahl’s theorem in the same way as Theorem 1 in [34] generalizes the $\rho^2$-theorem. The proofs of both theorems can be based on the description of the set of critical measures $\mathcal{M}_e$ outlined above and also on Theorem 1 in [34]. In the next sections we briefly discuss this theorem and some possible generalizations. At the same time Theorem 6 can be proved in a very simple way using a Laguerre-type differential equation for the Padé denominators of the functions from $\mathcal{L}_e$.

4.2. Conjectures on the distribution of the zeros of complex orthogonal polynomials. Here we present some conjectures connected to Hermite-Padé polynomials (their field of application may be larger). Thus, we touch on the general problem of generalizing the theory outlined above in this paper to the case of
Hermite-Padé polynomials. This is one of the central problems in the theory and at the moment the problem is essentially open.

As a starting point we need a version of a general theorem in [34] (the GRS-theorem) which at present is possibly the most advanced known theorem relating to the distribution of the zeros of complex orthogonal polynomials.

4.2.1. The GRS-theorem. To state this theorem we need the following definition.

We say that a compact set $F \subset \mathbb{C}$ has the $S$-property in an external field $\phi$ which is harmonic in a neighbourhood of $F$ if equality in (31) holds for $g = U^\lambda + \phi$, the total potential of the equilibrium measure $\lambda = \lambda_{\phi,F}$ for $F$ in the external field $\phi$. The $S$-property implies that $F$ consists of an at most countable union of disjoint open analytic arcs $F^\circ$ and a set of capacity zero (here we assume from the start that an $S$-compact set associated with the problem exists).

Now we state the assumptions of the theorem.

We assume that we are given a domain $\Omega$ in $\mathbb{C}$, a compact set $F$ in $\Omega$ and a sequence of functions $\Phi_n(z) \in H(\Omega)$ which converges uniformly on compact subsets of $\Omega$ as $n \to \infty$: $\Phi_n(z) \to \Phi(z)$.

Assume that $F$ has the $S$-property in the external field $\phi = \text{Re} \Phi(z)$. Further, let $f \in H(\Omega \setminus F)$ be a function whose jump across any arc in $F^\circ$ is not identically equal to zero and let the polynomials $Q_n(z) \in \mathbb{P}_n$ be defined by orthogonality relations with weights $f_n = f e^{-2n\Phi_n}$:

$$\int_F Q_n(z)P(z)f_n(z) \, dz = 0 \quad \text{for any } P \in \mathbb{P}_{n-1}. \quad (39)$$

Integration in (39) is along a contour (contours) in $\Omega \setminus F$ homotopic to the boundary of $\overline{\Omega} \setminus F$.

Finally, assume that the complement of the support of the equilibrium measure $\lambda = \lambda_{\phi,F}$ for $F$ in the external field $\phi$ is connected.

The following is Theorem 1 in [34].

**Theorem 7.** Under the above assumptions, $\frac{1}{n} \mathcal{Z}(Q_n) \xrightarrow{\ast} \lambda$.

The orthogonality conditions in (39) are fairly general but in a number of situations the theorem cannot be applied directly. This often happens in the study of the distribution of the zeros of Hermite-Padé polynomials. These polynomials are defined by systems of orthogonality relations and reducing such systems to orthogonality with respect to a single weight (if possible) leads to more general forms of orthogonality. Next we give two comparatively simple examples of different types.

4.2.2. A conjecture related to Hermite-Padé polynomials for a Nikishin system. In many cases the study of Nikishin systems can be reduced to the problem of asymptotics for the orthogonal polynomials $Q_n$ defined by relations similar to (39) but with the weight functions $f_n$ in (39) depending not only on $n$ but also on the polynomial $P$. In other words, the polynomials $Q_n$ are orthogonal to some collection of functions which are not polynomials but polynomials $P$ with a multiplier depending on this $P$.

We formulate a conjecture for the case when only the $\Phi_n$ depend on $P$.

Let the polynomials $Q_n$ satisfy orthogonality conditions (39) with $f_n = f e^{-2n\Phi_n}$, where $\Phi_n(z) = \Phi_n(z; P)$. All the assumptions in Theorem 7 above are preserved.
In addition we assume that for any sequence of polynomials $P_n \in \mathbb{P}_{n-1}$ such that \( \frac{1}{n} \mathcal{X}(P_n) \xrightarrow{\ast} \lambda \) we have $\Phi_n(z; P_n) \rightarrow \Phi(z)$.

**Conjecture 1.** Under the above assumptions $\frac{1}{n} \mathcal{X}(Q_n) \xrightarrow{\ast} \lambda$.

Conjecture 1 is partially suggested by the results in [54], where Hermite-Padé polynomials of the first kind were considered for a Nikishin system of two Markov-type functions $f_1, f_2$ on the union $E$ of a finite number of disjoint real closed intervals $E_j$ (see also [38]). We will outline the setting in the paper without going into all the details related to the situation.

It was assumed in [54] that the ratio of the jumps $f(x) := \Delta f_2(x)/\Delta f_1(x)$, $x \in E$, is an analytic complex-valued function on $E$ and $f$ has an analytic continuation from each $E_j$ along any path in $\mathbb{C}$ avoiding the finite set $e_f$ of branch points. It was also assumed that the set $e_f$ is symmetric with respect to the real axis.

It was proved, first, that (under some additional technical assumptions) there exists a unique compact set $F$ such that $f \in H(\mathbb{C} \setminus F)$ and $F$ has the $S$-property with respect to some related equilibrium problem for a mixed Green-logarithmic potential.

Let $Q_{n,0}, Q_{n,1}, Q_{n,2} \in \mathbb{P}_n$, $Q_{n,2} \not\equiv 0$, be the Hermite-Padé polynomials of the first kind for the system $[1, f_1, f_2]$, that is, the following relation holds:

$$
(Q_{n,0} \cdot 1 + Q_{n,1} f_1 + Q_{n,2} f_2)(z) = O \left( \frac{1}{z^{2n+2}} \right), \quad z \to \infty.
$$

The following orthogonality relations of the type of (39) for the polynomial $Q_{n,2}$ were obtained in [54], formula (119):

$$
\int_F Q_{n,2}(z) P_n(z) \left\{ h_{n+m}(z) \frac{\tau_n^2(z)}{q_n(z)} \int_E \frac{q_n^2(\zeta) \tau_n^2(\zeta)}{z - \zeta} \frac{dm_n(\zeta)}{P_n(\zeta)} f(z) \right\} \, dz = 0 \quad (41)
$$

($P_n \in \mathbb{P}_{n-1}$ is an arbitrary polynomial). Finally, these orthogonality relations were used to prove that the sequence $\frac{1}{n} \mathcal{X}(Q_{n,2})$ converges weakly to the equilibrium measure for the problem mentioned above; for more details see [54].

A connection between this result and Conjecture 1 is established by the following fact. The function in curly brackets in (41) which plays the role of a multiplier for $P_n$, satisfies the assumptions of Conjecture 1. Thus, the theorem above supports the conjecture.

4.2.3. A conjecture on incomplete complex orthogonal polynomials. Hermite-Padé polynomials lead also to another type of asymptotic problem for orthogonal polynomials. Before discussing this problem (in §4.2.4) we introduce an auxiliary problem for complex orthogonal polynomials. However, this problem may be of independent interest. We restrict ourselves to the simplest possible version.

Let $f(z) \in \mathcal{A}_e$, where $e = \{a, b\}$. That is, the function $f$ (defined by an element at infinity) has two branch points at $a$ and $b \neq a$. Let $N$ and $n \leq N$ be two natural numbers and let the polynomials $Q_N \in \mathbb{P}_N$ satisfy the relations

$$
\int_F Q_N(z) P(z) f(z) \, dz = 0 \quad \text{for any } P \in \mathbb{P}_{n-1},
$$

(42)
The Gonchar-Stahl $\rho^2$-theorem

where $F$ is a curve connecting $a$ and $b$. Note that here we do not assume that a special curve is given. By the Cauchy integral theorem, any curve $F$ from the class $\mathcal{F}$ of curves connecting $a$ and $b$ can be used in (42). Finding a special curve $\Gamma \in \mathcal{F}$ will be a part of the problem.

Suppose that $n, N \to \infty$ in such a way that $N/n \to k > 1$. What can be said about the distribution of the zeros of $Q_N$?

Clearly, under these assumptions the polynomial $Q_N$ is not uniquely defined and we cannot expect that the sequence of counting measures $\frac{1}{n} \mathcal{X}(Q_N)$ is convergent. Instead, we suggest that any limit point of this sequence satisfies a certain inequality. To state the inequality formally, first we need to select a convergent subsequence

$$\frac{1}{n} \mathcal{X}(Q_N) \rightharpoonup \mu \quad \text{as} \quad n \to \infty, \quad n \in \Lambda$$

($\Lambda$ is a sequence of natural numbers). Since $N/n \to k = 1$, we have $\mu(\mathbb{C}) = k > 1$.

The potential $\varphi = U^\mu$ of $\mu$ will play the role of an external field in the problem we are going to consider. We let

$$E_\mu(\nu) = E(\nu) + 2 \int U^\mu \, d\nu$$

denote the weighted energy of a measure $\nu$ in the external field $\varphi$. Note that here and in what follows we use the abbreviated notation: $E_\mu(\nu)$ stands for $E_\varphi(\nu)$ with $\varphi = U^\mu$ (cf. (19)).

For a fixed $F \in \mathcal{F}$ we denote the minimizing (equilibrium) measure on $F$ in the external field $\varphi = U^\mu$ by $\lambda_{F,\mu} \in \mathcal{M}(F)$:

$$E_\mu(\lambda_{F,\mu}) = \min_{\nu \in \mathcal{M}(F)} E_\mu(\nu),$$

where $\mathcal{M}(F)$ is the set of probability measures on $F$.

Next, we introduce the functional of equilibrium energy $E_\mu[F]$ and assert the existence of a compact set $\Gamma = \Gamma_\mu \in \mathcal{F}$ maximizing this functional (see [55] and cf. [16]):

$$E_\mu[\Gamma] = \max_{F \in \mathcal{F}} E_\mu[F], \quad \text{where} \quad E_\mu[F] = E_\mu(\lambda_{F,\mu}).$$

(44)

Finally, we define a mapping $\mu \mapsto \lambda$ in the space of probability measures in the plane by

$$\lambda(\mu) = \lambda(\mu, \mathcal{F}) = \lambda_{\Gamma,\mu},$$

(45)

where $\Gamma = \Gamma_\mu$ is the extremal compact set in (44). The conjecture is formulated in terms of this function.

**Conjecture 2.** For any subsequential limit $\mu$ of the sequence $\frac{1}{n} \mathcal{X}(Q_N)$, $\mu \geq \lambda(\mu)$.

In a number of situations, Conjecture 2 can be proved. We mention one such situation where the proof can be obtained using the GRS-method. Suppose that the limit distribution is known for a part of the zeros of $Q_N$ containing $N - n$ zeros. Let this limit distribution be represented by a known measure $\sigma$. In other words, we assume that $Q_N = q_n g_n$, with $g_n \in \mathbb{P}_{N-n}$, and that $\frac{1}{n} \mathcal{X}(g_n) \rightharpoonup \sigma$, and $Q_N$ satisfies the orthogonality conditions (42).
Suppose also that the class $\mathcal{F}$ of continua $F$ connecting $a$ and $b$ contains a continuum $\Gamma$ with the $S$-property in the external field $\varphi(z) = \frac{1}{2}U^\sigma(z)$. Then the sequence $\frac{1}{n} \mathcal{E}(q_n)$ is weakly convergent to $\lambda = \lambda_{\varphi, \Gamma}$ according to Theorem 7. It follows that the sequence $\frac{1}{n} \mathcal{E}(Q_N)$ converges to $\mu = \lambda + \sigma$ and finally the sequence $\frac{1}{n} \mathcal{E}(Q_N)$ converges to $\mu = \lambda$. Since the equilibrium measure of an $S$-compact set has the max-min-property (see (44)), we have $\lambda = \lambda(\mu)$ and the assertion of Conjecture 2 follows.

Thus, in this situations the max-min definition of $\lambda(\mu)$ can be equivalently formulated in terms of the $S$-property. In general, we have to define $\lambda(\mu)$ in terms of ‘max-min’ since the external fields associated with the problem need not be harmonic (or even smooth) in a neighbourhood of the extremal compact set.

4.2.4. A conjecture related to the Hermite-Padé polynomials for an Angelesco system. As an example of the possible application of Conjecture 2 we mention the problem of the zero distribution for the denominators of Hermite-Padé approximants of the second-kind in the Angelesco case.

The simplest setting is the following. Two sets, $e_1 = \{a_1, b_1\}$ and $e_2 = \{a_2, b_2\}$, where $a_i \neq b_i$ for $i = 1, 2$, are given. Two functions, $f_1 \in \mathcal{A}_{e_1}$ and $f_2 \in \mathcal{A}_{e_2}$, are defined by their Laurent series at infinity. Assume that $\{a_i, b_i\}$ are actual branch points of $f_i$. Finally, a nontrivial polynomial $Q = Q_{2n} \in \mathbb{P}_{2n}$ is defined by the pair of conditions

$$ (Qf_1 - P_1)(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{and} \quad (Qf_2 - P_2)(z) = O\left(\frac{1}{z^{n+1}}\right) $$

as $z \to \infty$, where $P_i$ is the polynomial part of $Qf_i$ at infinity ($i = 1, 2$).

We assume that the pair of functions $f_1, f_2$ (or rather, the pair of sets $e_1, e_2$) is in the ‘Angelesco case’ which informally speaking means that $e_1$ and $e_2$ are ‘well separated’ (lie far enough apart). The formal definition will be presented below after the related definitions have been introduced. As an example we note that if all the branch points are real then we define the Angelesco case by the condition that the intervals $(a_1, b_1)$ and $(a_2, b_2)$ are disjoint. It is known that in this case the limit zero distribution of the sequence $Q_{2n}$ is defined by a matrix equilibrium problem for the pair of conductors $F_1 = [a_1, b_1]$ and $F_2 = [a_2, b_2]$ (see [28] and [35] for the Markov case). In the complex case we have to use a matrix $S$-equilibrium problem defined next.

For $i = 1, 2$ we denote by $\mathcal{F}_i$ the class of continua in the plane connecting points $a_i$ and $b_i$. We consider the class of ‘vector’ compact sets $\mathcal{F} = (F_1, F_2)$ where $F_i \in \mathcal{F}_i$. For a fixed vector compact set $\bar{F} = (F_1, F_2) \in \mathcal{F}$ we define the class of vector-valued measures

$$ \mathcal{M}(\bar{F}) = \{(\mu_1, \mu_2) : \mu_j \in \mathcal{M}(F_j)\}, $$

where $\mathcal{M}(F_i)$ is the set of probability measures on $F_i$. The energy of the vector measure $\bar{\mu} = (\mu_1, \mu_2)$ is defined by

$$ \mathcal{E}(\bar{\mu}) = [\mu_1, \mu_1] + [\mu_1, \mu_2] + [\mu_2, \mu_2], \quad \bar{\mu} = (\mu_1, \mu_2), $$

where $[\mu, \nu] = \int V^\nu d\mu$ is the mutual energy of $\mu$ and $\nu$. In more general situations the energy of a vector measure is defined by means of a matrix $A$ with constant
elements $a_{ij}, i, j = 1, 2$, so that the matrix-energy is $\mathcal{E}(\bar{\mu}) = \sum a_{ij} [\mu_i, \mu_j]$. In our case the elements of the matrix $A$ are $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 1/2$. This matrix is positive definite and, moreover, $a_{ij} \geq 0$. It follows that for any $\vec{F} = (F_1, F_2) \in \mathcal{F}$ there exists a unique $\vec{\lambda} \in \mathcal{M}$ such that

$$\mathcal{E}[\vec{F}] = \mathcal{E}(\vec{\lambda}) = \min_{\vec{\mu} \in \mathcal{M}(\vec{F})} \mathcal{E}(\vec{\mu}), \quad \vec{\lambda} = (\lambda_1, \lambda_2).$$

The vector-measure $\vec{\lambda}$ is the equilibrium measure for $\vec{F}$ associated with the matrix $A$, and $\mathcal{E}[\vec{F}]$ is the equilibrium energy of $\vec{F}$ (see the original papers [28], [32] and [35] and the recent developments in [10] and [20]).

Further, assume that there exists a vector-compact set $\vec{\Gamma} = (\Gamma_1, \Gamma_2) \in \mathcal{F}$ maximizing the equilibrium energy

$$\mathcal{E}[\vec{\Gamma}] = \max_{\vec{\Gamma} \in \mathcal{F}} \mathcal{E}[\vec{\Gamma}].$$

In some cases the existence of maximizing vector-compact sets $\vec{\Gamma}$ can be proved using the methods presented in [55]. In general, such a set is not unique but the associated equilibrium measure $\vec{\lambda} = (\lambda_1, \lambda_2)$ is.

What can be said about the limit zero distribution of the Hermite-Padé denominators $Q_{2n}$ essentially depends on the structure of $\vec{\Gamma}$ or, to be more precise, the structure of $\vec{\lambda}$. If the supports of $\lambda_1$ and $\lambda_2$ are essentially overlapping then the vector measure $\vec{\lambda}$ does not describe the zero distribution of the polynomials $Q_{2n}$ and the case under consideration is not the Angelesco case. In this situation the equilibrium problem has to be modified; we refer to papers [2] and [3] for further details.

If the supports of $\lambda_1$ and $\lambda_2$ are disjoint, then we have the Angelesco case and we make this assumption in what follows (the case when there is a small—say, finite—intersection can also be included but we will only look at the disjoint situation). Now, the main conjecture on the zero distribution of Angelesco Hermite-Padé polynomials is as follows.

**Conjecture 3.** $\frac{1}{n} \mathcal{Z}^\ast(Q_{2n}) \xrightarrow{\ast} \lambda_1 + \lambda_2$, where $\vec{\lambda} = (\lambda_1, \lambda_2)$ is the equilibrium measure of the extremal compact $\vec{\Gamma} = (\Gamma_1, \Gamma_2)$.

In a number of cases this conjecture was proved under some additional restrictions. First, if the sets $e_1$ and $e_2$ lie far enough, the proof can be given on the basis of the GRS-method. This approach is simple and general. Neither the number of functions nor the number and character of the branch points is actually important if the sets $e_1$ and $e_2$ of branch points are well separated. However, it is difficult to obtain sharp estimates for the critical distance between the sets.

Second, under some additional assumptions on the character of the branch points the strong asymptotics for the $Q_{2n}$ were derived in [3] for two functions, each with algebraic-logarithmic branch points. The proof uses the method of steepest descent for the matrix Riemann-Hilbert representation of $Q_{2n}$. This method is sensitive to the number of functions and how many branch points they have. It is not clear if it may be generalized to arbitrary branch points.
Now, a proof for an arbitrary Angelesco situation can be reduced to Conjecture 2 (or to its proper generalization for more than two functions with any number of branch points). Such a reduction would require some additional potential-theoretic considerations.

We will mention the shortest method of reduction, which is based on yet another conjecture.

Let $\lambda = (\lambda_1, \lambda_2)$ be the equilibrium measure of the extremal compact set $\Gamma = (\Gamma_1, \Gamma_2)$. We define $\mu = \lambda_1 + \lambda_2$ as in Conjecture 3. Assuming that we are in the Angelesco case, the extremal vector-compact set $\Gamma$ has the following important property: both the components $\Gamma_1$ and $\Gamma_2$ have the $S$-property in the external field $\phi(z) = U^{\mu}(z)$. Further, the $S$-property can be rewritten as the ‘energy max-min property’ and, therefore, the three measures above satisfy the following relations:

$$\mu \geq \lambda_1 \geq \lambda(\mu, \mathcal{F}_1) \quad \text{and} \quad \mu \geq \lambda_2 \geq \lambda(\mu, \mathcal{F}_2),$$

(46)

where both the measures $\lambda(\mu, \mathcal{F}_1)$ and $\lambda(\mu, \mathcal{F}_2)$ were defined in (45) and the $\mathcal{F}_i = \mathcal{F}(f_i)$ are the classes of admissible cuts for the functions $f_1, f_2$.

Of course, we actually have the equalities $\lambda_1 = \lambda(\mu, \mathcal{F}_1)$ and $\lambda_2 = \lambda(\mu, \mathcal{F}_2)$ in (46) but we need inequalities to make the converse assertion stronger.

**Conjecture 4.** In the Angelesco case $\mu = \lambda_1 + \lambda_2$ is the only positive Borel measure in the plane satisfying (46) with $\mu(\mathbb{C}) = 2$.

If both Conjectures 2 and 4 are true then Conjecture 3 is also true since it is a direct corollary of the other two.

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