MIRROR SYMMETRY FOR PFAFFIAN CALABI-YAU 3-FOLDS VIA CONIFOLD TRANSITIONS

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Abstract. In this note we construct conifold transitions between several Calabi-Yau threefolds given by Pfaffians in weighted projective spaces and Calabi-Yau threefolds appearing as complete intersections in toric varieties. We use the obtained results to predict mirrors following ideas of [BCFKvS98, Bat04]. In particular we consider the family of Calabi-Yau threefolds of degree 25 in $P^9$ obtained as a transverse intersection of two Grassmannians in their Plücker embeddings.

1. Introduction

Calabi–Yau threefolds with Picard number one are central objects of investigation form the point of view of mirror symmetry. The main reason for this is that for such a manifold the mirror family has one dimensional moduli, and hence can be explicitly studied. There are nowadays more than 100 known families of Calabi–Yau threefolds of Picard number 1. The simplest are complete intersection in toric varieties and for them the mirror symmetry conjecture has been proven. Others, and in fact most of them, appear as smoothings of hypersurfaces in some toric Gorenstein terminal Fano fourfolds (see [BK10]). For these a conjectural mirror construction has been developed.

More precisely basing on ideas of [Mor99] in [BCFKvS98, BCFKvS00] a conjectural method for the construction of mirrors for Calabi–Yau threefolds admitting conifold transitions to complete intersection of toric varieties has been stated. Since then toric degenerations of Fano manifolds and degenerations of Calabi–Yau threefolds to complete intersections in toric varieties has been widely investigated aiming at
derstanding mirror symmetry for new classes of examples.

Recently in [Kap11, Kan12] new families of non-complete intersection Calabi-Yau threefolds with Picard number 1 have been explicitly constructed. They will be denoted $X_5$, $X_7$, $X_{10}$ and $X_{25}$. They are described by Pfaffian equations in some weighted projective spaces. According to the classification of [BK10] they do not admit any conifold transition to a hypersurface in a toric variety.

In [Kan12] the families $X_5$, $X_7$, $X_{10}$ have also been studied, together with the classical family $X_{13}$ of Calabi–Yau threefolds of degree 13 in $P^6$, from the point of view of mirror symmetry. The method used relies on the tropicalization approach introduced in [Boh08]. In this way all these examples have been assigned a candidate mirror family and the period of these families has been computed. The singularities of the elements of the mirror families proposed are however very complicated. In particular it is not clear whether a general elements of the mirror families proposed admit resolutions being Calabi–Yau threefolds.

In this note we study mirror symmetry for all examples $X_5$, $X_7$, $X_{10}$, $X_{25}$ and $X_{13}$ using the methods of [BCFKvS98, Bat04]. We start by interpreting the description of these Calabi–Yau threefolds as complete intersection in some singular Fano varieties related to weighted Grassmannians. By constructing toric degenerations of these ambient varieties we describe conifold transitions between the families $X_5$, $X_7$, $X_{10}$, $X_{13}$, $X_{25}$, and some Calabi–Yau complete intersections in toric varieties. We build on the well known toric degeneration of the Grassmannians $G(2,5)$ described in [Stu96] and used in [BCFKvS98]. More precisely we adapt it to the case of any variety described by Pfaffians of a skew-symmetric $5 \times 5$ matrix in weighted projective space. We next use the methods of [BCFKvS98, Bat04] to compute the main period of the conjectured mirror family. In this way we recover the same periods as in [Kan12] for the examples studied there. One of the advantage of taking our approach is that our constructions involving only conifold transitions leads to candidate mirror families consisting of singular Calabi–Yau threefolds which conjecturally (see

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admit only nodes as singularities and hence can be resolved to smooth Calabi–Yau threefolds.

Moreover our method works also for the family $X_{25}$ consisting of Calabi–Yau threefolds obtained as a complete intersection of two Grassmannians in their Plucker embeddings. Our approach to this case can be extended to work for other Calabi-Yau threefold appearing as the intersection of two Fano varieties each admitting a small toric degeneration.

The family $X_{25}$ consisting of codimension 6 smooth Calabi–Yau threefolds of degree 25 in $\mathbb{P}^9$ is especially interesting because its associated Picard–Fuchs equation appears to be self dual in the sense of Kan12. This phenomenon is related to projective self-duality of the Grassmannian $G(2,5)$ and merits further investigation. We hope that the results obtained in this paper will contribute to it.

Throughout the paper we rely on computer calculations. We use mainly the Toric Package from the computer algebra system Magma (see BCP97).

2. Descriptions of studied families

In this section we recall and study descriptions of the families $X_5$, $X_7$, $X_{10}$ and $X_{25}$ of Calabi-Yau threefolds with Picard number one constructed in Kap11, Kan12. For each of them we provide a description in five different ways:

1. in terms of Pfaffian varieties associated to decomposable vector bundles in weighted projective spaces,
2. using Pfaffian equations in weighted projective space,
3. as a result of a bitransition based on Kustin–Miller unprojection,
4. as a complete intersection in some cone over some universal Pfaffian variety,
5. as the result of a conifold transition with a complete intersection in a smooth toric Fano variety.

Each of the above description has its own advantages. Descriptions (1) and (2) are strictly related and very explicit, they are used for the definition of the families. In fact the description of varieties using Pfaffians is in general one of the simplest after descriptions as complete intersections and by zero loci of sections of ample vector bundles. As such they can be used for the study of the geometry of varieties involved. However from the point of view of mirror symmetry the description via Pfaffians does not help much. In particular it seems very improbable that a version of Quantum Lefschetz theorem could be developed, since even the usual Barth–Lefschetz theorem does not hold for Pfaffian varieties in general. However having explicit equations one can always try to find explicit degenerations suitable for different approaches to mirror symmetry.

Description (3) tells us about the place the Calabi–Yau threefolds in question take in the Web of Calabi–Yau threefolds. It was introduced in Kap11. It relates our varieties with very standard Calabi–Yau threefolds by composition of a conifold transition and a geometric transition involving a type II primitive contraction morphism. Such constructions are conjecturally (see Mor99) compatible with mirror symmetry, so in principle could lead to the construction of a mirror family for our examples. However contrary to conifold transitions the geometric transition involving a primitive contraction of type II has not yet found a proper counterpart in this theory.

Description (4) is already more suited for mirror symmetry in general. Since in some instances the quantum Lefschetz theorem on singular varieties holds one can in principle reduce the study of mirror symmetry of our Calabi–Yau threefold to the mirror symmetry of the ambient variety in question. In our cases the latter has not yet been studied. It seems however probable that since the ambient varieties obtained in our cases are strictly related to weighted Grassmannians one could generalize the theory developed for Grassmannians to study the quantum cohomology ring of these varieties. In this paper we shall just use the analogy with the Grassmannian to construct a terminal Toric degeneration of our ambient space and get description (5).

The last description is the one that we shall use to study mirror symmetry for our examples. It provides a setting in which the methods of BCFKvS98 conjecturally work. The constructions for all considered families are very similar. We shall write in details only the cases $X_5$. For the remaining we present only the main results and point out where some differences to the case $X_5$ occur. Moreover the construction (3) using bitransitions is only performed for the case $X_5$ which is the only case which was not treated in Kap11. For the remaining cases a detailed analysis of the construction can be found there.
2.1. The family $X_5$. A Calabi-Yau threefold $X_5 \in \mathcal{X}_5$ is naturally embedded in $\mathbb{P}(1^4, 2^3)$. It is defined as a Pfaffian variety associated to the vector bundle:

$$\mathcal{E}_5 = 5\mathcal{O}_{\mathbb{P}(1^4, 2^3)}(1).$$

In other terms it is defined by $4 \times 4$ Pfaffians of a $5 \times 5$ antisymmetric matrix with entries of weighted degree 2. We shall use the following picture to illustrate the weights in the skew symmetric matrix defining $X_5$:

$$\begin{pmatrix}
2 & 2 & 2 & 2 \\
2 & 2 & 2 & \\
2 & 2 & 2 & \\
\end{pmatrix}.$$

The threefold $X_5$ can also be described as a smoothing of a variety obtained as the result of a Kustin–Miller unprojection of a quadric surface in a Calabi-Yau threefold complete intersection of two quartics in $\mathbb{P} = \mathbb{P}(1^4, 2^2)$. More precisely let $D$ be a quadric surface embedded in $\mathbb{P}(1^4, 2^2)$ as a complete intersection of type $(2, 2, 2)$ i.e. defined by three polynomials $q_1, q_2, q_3$ of weighted degree 2. Let us consider a general variety $Y$ obtained as the complete intersection of two quartics containing $D$ i.e. $Y$ is defined by two polynomials $p_1 = a_1q_1 + a_2q_2 + a_3q_3$ and $p_2 = b_1q_1 + b_2q_2 + b_3q_3$, where $a_i, b_i$ are general polynomials of weighted degree 2. Let us now consider the space $\mathbb{P}(1^4, 2^3)$ containing $\mathbb{P}$ with the new weight 2 variable denoted by $l$. Then the variety $Z$ defined by the $4 \times 4$ Pfaffians of the matrix

$$\begin{pmatrix}
l & a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 & \\
q_3 & -q_2 & \ \\
q_1 & \\
\end{pmatrix},$$

is a Gorenstein Calabi–Yau threefold whose projection from the point $p$ with $l(p) = 1$ and the remaining coordinates being zero is $Y$. We can easily prove that this projection factors through a small resolution of nodes on $Y$ and a primitive contraction of a quadric surface. We moreover observe that $Y$ has a smoothing to the family of Calabi–Yau threefolds obtained as intersections of general quartics in $\mathbb{P} = \mathbb{P}(1^4, 2^2)$ whereas $Y$ is smoothed by the family $\mathcal{X}_5$.

The description of the threefold $X_5$ using Pfaffian equations enables us to see $X_5$ as a general complete intersection of a variety $G_5$ described by equations in $\mathbb{P}(1^4, 2^{10})$ given by $4 \times 4$ Pfaffians of a matrix $M$ with weight 2 coordinates as entries. The variety $G_5$ can be interpreted as a weighted cone over the Grassmannian $G(2, 5)$.

Observe that $G_5$ is a normal Gorenstein Fano variety. Let us now consider the following degeneration of $G_5$. Let $\mathcal{G}$ be the family defined in $\mathbb{P}(1^4, 2^{10}) \times \mathbb{C}$ with coordinates $x_1, \ldots, x_4, y_1, \ldots, y_{10}$ by the Pfaffians of the matrix

$$\begin{pmatrix}
\lambda y_1 & y_2 & y_3 & y_4 \\
y_5 & y_6 & y_7 & \\
y_8 & y_9 & \lambda y_{10} & \\
\end{pmatrix}.$$

Proposition 2.1. The family $\mathcal{G}$ is flat over $\mathbb{C}$. Moreover the fiber $F_5 = \mathcal{G}_0$ over $\lambda = 0$ is a terminal Gorenstein toric Fano variety of Picard number one.

Proof. We start the proof with the observation that $F_5$ is of expected codimension in $\mathbb{P}(1^4, 2^{10})$ it is hence a (weighted) Pfaffian variety. For flatness of $\mathcal{G}$ we then just need to observe that it is clearly an algebraic family and that the Hilbert polynomial are computed from the same Pfaffian sequence (cf. [Kan12]). To get assertions concerning the fiber $F_5 = \mathcal{G}_0$ we first need to prove that it is normal. Since it is Gorenstein by the Pfaffian construction, normality is equivalent to the computation of the codimension of the singular locus. The latter is done easily by computer. Since the Pfaffians of the degenerate matrix provide binomial equations for $F_5$ normality implies that $F_5$ is a toric variety. The rest of the assertion follows from computer calculation.
using the Toric Package in Magma on the fan of $F_5 = \mathcal{G}_0$ determined by the binomial equations. More precisely we compute that the Polytope associated to $F_5 = \mathcal{G}_0$ polarized by the restriction of $\mathcal{O}_{\mathbb{P}(14,2o)}(1)$ is a reflexive polytope generated by:

\[
\begin{align*}
    e_1 &= (-1, 0, -1, 0, 0, 0, 0, 0, -1), \\
    e_2 &= (0, -1, 0, 0, 0, 0, 0, 0, 1), \\
    e_3 &= (0, -1, 1, 0, 0, 0, 0, 0, 0), \\
    e_4 &= (0, 0, 0, 0, 0, 0, 0, 1, 0), \\
    e_5 &= (0, 0, 0, 0, 0, 0, 1, 0, 0), \\
    e_6 &= (0, 0, 0, 0, 0, 1, 0, 0, 0), \\
    e_7 &= (0, 0, 0, 0, 1, 0, 0, 1, 0), \\
    e_8 &= (0, 0, 0, 0, 1, 0, 0, 0, 0), \\
    e_9 &= (0, 0, 0, 1, 0, 0, 0, 0, 0), \\
    e_{10} &= (0, 0, 0, 1, 0, 0, 0, 0, 0), \\
    e_{11} &= (0, 0, 0, 1, 0, 0, 0, 0, 0), \\
    e_{12} &= (0, 1, 0, -1, -1, -1, -1, -1, 0), \\
    e_{13} &= (1, 0, 0, 0, 0, 0, 0, 0, 0)
\end{align*}
\]

It follows by further calculations with Magma that the toric variety $\tilde{T}_5$ of weighted degree 2 is a nodal Calabi–Yau threefold in a Terminal Gorenstein toric Fano variety polarized by $O_X$ degeneration, similar to the one described for the polynomial of weighted degree 3. In this way we obtain a family of varieties however not Gorenstein. To obtain a degeneration which is Gorenstein we consider $G$.

From the proof of Proposition 2.1 we obtain that the intersection of $F_5$ with 7 general hypersurfaces of weighted degree 2 is a nodal Calabi–Yau threefold $T_5$. Observe that $T_5$ admits a small resolution $\tilde{T}_5$ to a complete intersection in a toric variety obtained as a toric resolution $\tilde{F}_5$ of $F_5$ (with Fan given by a triangulation of our Polytope). This means that $X_5$ and $\tilde{T}_5$ are connected by a conifold transition.

2.2. The family $X_7$. The Calabi-Yau threefold $X_7$ is described in $\mathbb{P}(1^5, 2^2)$ by $4 \times 4$ Pfaffians of a $5 \times 5$ antisymmetric matrix with entries of degrees as shown below:

\[
\begin{pmatrix}
    1 & 1 & 2 & 2 \\
    1 & 2 & 2 \\
    2 & 2 \\
    3
\end{pmatrix}
\]

We can consider $X_7$ as complete intersection of hypersurfaces of degrees 2, 2, 2, 2, 3 and the variety $G_7' \subset \mathbb{P}(1^5, 2^2, 3)$ defined by the $4 \times 4$ Pfaffians of the antisymmetric matrix with entries being coordinates of suitable degree. Now similarly to the case $G(2, 5)$ we can find a toric degeneration of $G_7'$. The latter is however not Gorenstein. To obtain a degeneration which is Gorenstein we consider $G_7$ to be the variety defined by Pfaffians of a generic matrix of the form

\[
\begin{pmatrix}
    x_1 & x_2 & y_1 & y_2 \\
    x_3 & y_3 & y_4 & y_5 \\
    y_6 & c
\end{pmatrix}
\]

in $\mathbb{P}(1^5, 2^6)$ where $x_1, x_2, x_3$ are weight one coordinates $y_1 \ldots y_6$ weight 2 coordinates and $c$ a general polynomial of weighted degree 3. In this way we obtain a family of varieties $G_7$. For each of them we have a degeneration, similar to the one described for $G_5$, to the same variety $F_7$ in $\mathbb{P}(1^5, 2^6)$. This time $F_7$ is a Terminal Gorenstein toric Fano variety polarized by $O_{\mathbb{P}(15, 2o)}(1)$ with Polytope:

\[
\begin{align*}
    e_1 &= (-1, -1, 0, -1, 0, 0, 0), \\
    e_2 &= (0, 0, -1, 1, 0, 0, 0), \\
    e_3 &= (0, 0, -1, 1, 0, 0, 0), \\
    e_4 &= (0, 0, 0, 1, 0, 1), \\
    e_5 &= (0, 0, 0, 1, 1, 1), \\
    e_6 &= (0, 0, 1, 0, -1, -1), \\
    e_7 &= (0, 0, 1, 0, -1, 0), \\
    e_8 &= (0, 1, 0, 0, 0, 0), \\
    e_9 &= (1, 0, -1, 0, 0, 0), \\
    e_{10} &= (1, 0, 0, 0, 0, 1)
\end{align*}
\]
The variety obtained as the intersection of 4 general Cartier divisors from the system corresponding to \( \mathcal{O}_{\mathbb{P}^5}(1,2^5) \) in \( F_7 \) is a nodal Calabi–Yau threefold. Its resolution \( \tilde{F}_7 \) is a complete intersection in a toric resolution \( \tilde{F}_7 \) of \( F_7 \) and is connected to \( X_7 \) by a conifold transition.

2.3. The family \( X_{10} \). The Calabi-Yau threefold \( X_{10} \) is described in \( \mathbb{P}(1^6,2) \) by \( 4 \times 4 \) Pfaffians of a \( 5 \times 5 \) antisymmetric matrix with entries of degrees as shown below:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 
\end{pmatrix}
\]

We consider \( X_{10} \) as complete intersection of hypersurfaces of weighted degrees \( 2,2,2,2,2 \) and the variety \( F_{10} \subset \mathbb{P}(1^6,2^5) \) defined by the \( 4 \times 4 \) Pfaffians of the antisymmetric matrix with entries being coordinates of suitable degree. Now similarly to the case \( G(2,5) \) we find a toric degeneration of \( F_{10} \). It is polarized by the restriction of \( \mathcal{O}_{\mathbb{P}(1^6,2^5)}(1) \) and corresponds to the Polytope.

\[
\begin{align*}
e_1 &= ( -1, -1, -1, 0, -1, 0, 0, 0), \\
e_2 &= ( 0, 0, 0, -1, 0, 1, 0, 0), \\
e_3 &= ( 0, 0, 0, -1, 1, 0, 0, 0), \\
e_4 &= ( 0, 0, 0, 0, 0, 0, 0, 1), \\
e_5 &= ( 0, 0, 0, 0, 0, 1, 1, 0), \\
e_6 &= ( 0, 0, 0, 1, 0, -1, -1, -1), \\
e_7 &= ( 0, 0, 1, 0, 0, 0, 0, 0), \\
e_8 &= ( 0, 1, 0, 0, 0, 0, 0, 0), \\
e_9 &= ( 1, 0, 0, 0, 0, 0, 1, 1), \\
e_{10} &= ( 1, 1, 0, -1, 0, 0, 0, 0), \\
e_{11} &= ( 1, 1, 1, 0, 0, 0, 1, 0).
\end{align*}
\]

The variety obtained as the intersection of 5 general Cartier divisors from the system corresponding to \( \mathcal{O}_{\mathbb{P}(1^6,2^5)}(2) \) in \( F_{10} \) is a nodal Calabi–Yau threefold. Its resolution \( \tilde{F}_{10} \) is a complete intersection in a toric resolution \( \tilde{F}_{10} \) of \( F_{10} \) and is connected to \( X_{10} \) by a conifold transition.

2.4. The family \( X_{13} \). By the same method we can also treat the Tonoli examples of degree 13. A Calabi-Yau threefold \( X_{13} \) from the family \( X_{13} \) is described in \( \mathbb{P}^6 \) by \( 4 \times 4 \) Pfaffians of a \( 5 \times 5 \) antisymmetric matrix with entries of degrees as shown below:

\[
\begin{pmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 
\end{pmatrix}
\]

We consider \( X_{13} \) as a complete intersection of \( 4 \) hypersurfaces of degrees \( 2 \) in \( \mathbb{P}(1^7,2^4) \) and the subvariety \( G_{13} \subset \mathbb{P}(1^7,2^4) \) defined by \( 4 \times 4 \) Pfaffians of a \( 5 \times 5 \) antisymmetric matrix with entries being coordinates of suitable degree. Again we find a toric degeneration of \( G_{13} \) by the same method and in consequence a conifold transition from \( X_{13} \) to a Calabi–Yau threefold obtained as a complete intersection in a toric variety.

3. Mirror symmetry via toric degenerations

In this section we recall a construction based on small toric degenerations which is used to conjecturally predict the principal period of the mirror family of a given Calabi-Yau threefold with Picard number 1. It is the original method of Batyrev ([Bat04]) describing the principal period as a specialization of the hyper-geometric series associated with the toric resolution of the degenerate Fano manifold.

In fact using this method one obtains an explicit candidate for the mirror family of \( X_i \). More precisely, the mirror family \( T^*_i \) of \( T_i \) is computed explicitly in terms of the construction of [BB96]. To get an explicit description of the candidate mirror family for \( X_i \) we use a specialization of the family \( T^*_i \) analogous to
It is conjectured that elements of this specialized family admit only nodes as singularities and their small resolutions are mirrors to $X_j$. Since we are unable to prove the conjecture on singularities in this case we omit the details of the construction of the mirror family here and we concentrate on the computation of its main period.

The method presented in [Bat04] is the following. Let $X$ be a Calabi-Yau threefold appearing as a complete intersection of Cartier divisors $D_1, \ldots, D_n$ in a Fano variety $G$ of dimension $n+3$. Assume that $G$ admits a small degeneration to a toric variety $\tilde{F}$, i.e., a flat degeneration such that $F$ is a terminal Gorenstein Fano variety and such that there is a canonical isomorphism between Pic($G$) and Pic($\tilde{F}$); denote $D_1, \ldots, D_n$ the Cartier divisors on $\tilde{F}$ corresponding to $D_1, \ldots, D_n$ via this isomorphism. Let $B = \{e_1, \ldots, e_k\} \subset N$ be the generators of one-dimensional cones of the fan $\Sigma$ of $F$ in the dual lattice $N = \mathbb{Z}^n$. It is well known that the vectors $\{e_1, \ldots, e_k\}$ determine a set $\{E_1, \ldots, E_k\}$ of generators of the divisor class group. Assume that we have a subdivision of $B$ into $n$ disjoint sets $J_1, \ldots, J_n$ such that $J_i$ corresponds to the Cartier Divisor $D_i$ for each $i \in \{1, \ldots, n\}$ (i.e. $D_i = \sum_{j \in J_i} E_j$). Let $L(B) := \{(l_1, \ldots, l_k) \in \mathbb{Z}^k : \sum_{j=1}^k l_j e_j = 0, \ l_j \geq 0\}$. We then have a pairing between $L(B)$ and $\text{Cl}(F)$ given by $\langle (l_1, \ldots, l_k), E_j \rangle = l_j$. Finally we call $A(\Sigma)$ the set of vectors in $\mathbb{C}^k$ admissible for the fan $\Sigma$ of i.e. vectors $(a_1, \ldots, a_k) \in \mathbb{C}^k$ such that there exists a function $\varphi$ on $\mathbb{C}^{n+3}$ linear restricted to each cone of $\Sigma$ and such that $\varphi(e_i) = \log |a_i|$ for each $i \in \{1, \ldots, k\}$. Under this notation, the main period of the mirror family to $X$ is conjectured to be given by the formula:

$$\phi_0(z) = \sum_{l \in L(B)} \prod_{i=1}^n \frac{(\sum_{j \in J_i} l_j)!}{l_1! \cdots l_k!} \prod_{j=1}^k l_j^l_j, \tag{3.1}$$

where $z \in A(\Sigma)$.

**Remark 3.1.** Observe that in [Bat04] the variety $G$ is assumed to be a smooth Fano variety. However the method is conjectured to work for any conifold transition between $X$ and a Calabi–Yau threefold obtained as a complete intersection in a terminal Gorenstein toric variety.

**Example 3.1.** The Calabi-Yau threefold $X_5$ admits a degeneration to a complete intersection in a Gorenstein terminal toric Fano variety $F_5$ of dimension 10. The Picard number of $F_5$ is 1 and the generator of the Picard group is very ample. The following decomposition of the set $B_5 = \{e_1 \ldots e_{13}\}$ of rays of the Fan $\Sigma_5$ of $F_5$ corresponds to 7 sections by elements of the system : $J_1 = \{e_1, e_2\}$, $J_2 = \{e_3, e_{13}\}$, $J_3 = \{e_4, e_6\}$, $J_4 = \{e_5, e_9\}$, $J_5 = \{e_7, e_8, e_{10}\}$, $J_6 = \{e_{11}\}$, $J_7 = \{e_{12}\}$. We compute also the cone $L(B_5)$ and find out that it is generated over $\mathbb{Z}_{\geq 0}$ by vectors:

$$f_1 = (1, 1, 1, 2, 2, 0, 0, 0, 0, 0, 2, 2, 2, 1), \quad f_2 = (1, 1, 1, 2, 0, 0, 0, 2, 2, 0, 2, 2, 1),$$

$$f_3 = (1, 1, 1, 0, 2, 2, 0, 0, 2, 2, 2, 1), \quad f_4 = (1, 1, 1, 2, 1, 0, 0, 1, 1, 2, 2, 1),$$

$$f_5 = (1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 2, 2, 1), \quad f_6 = (1, 1, 1, 1, 0, 1, 1, 1, 2, 0, 2, 2, 1).$$

It is hence a simplicial cone of dimension 3 spanned over a triangle with sides having 3 points belonging to the lattice $\mathbb{Z}^k$ (2 vertices and the midpoint). We observe that the monomials corresponding to the six generators of $L(B_5)$ are equal in $A(\Sigma_5) = \{(z_1, \ldots, z_{13}) \in \mathbb{C}^{11} | z_5 z_{10} = z_8 z_9, \ z_4 z_8 = z_6 z_7\}$. We can hence set a new coordinate $t = z^{13}$ which is independent on $i \in \{1 \ldots 6\}$ on $A(\Sigma_5)$. To make explicit the summation over $L(B_5)$ we observe that every element of $P \in L(B_5)$ has a unique presentation as a sum

$$P = kf_1 + lf_2 + mf_3 + nf_4 + of_5 + pf_6, \quad \text{with} \quad k, l, m, n, o, p \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad |n| + |o| + |p| \leq 1. \tag{3.2}$$

We follow the conjectured formula for the main period of the mirror of $X_5$:

$$\phi_0(t) = \sum_{s=0}^{\infty} \binom{2s}{s}^2 \sum_{k+l+m+n+o+p=0} \binom{2s}{2k+n+o} \binom{2s}{2m+o+p} \binom{2s}{2m+o+p} t^s \tag{3.2}$$

We check that the corresponding Picard-Fuchs equation is the no. 302 from [vEvS06]. It makes our result agree with the computation in [Kan12].

**Example 3.2.** The Calabi-Yau threefold $X_7$ has a degeneration to a nodal threefold appearing as a complete intersection in a Gorenstein toric Fano variety $F_7$ of dimension 7. The decomposition of the set of $B_7$ of
rays of the fan $\Sigma_7$ is $J_1 = \{e_1, e_3\}$, $J_2 = \{e_4, e_6\}$, $J_3 = \{e_2, e_8, e_9\}$, $J_4 = \{e_5, e_7, e_{10}\}$. The cone $L(B_7)$ is generated by vectors:

$$f_1 = (1, 0, 1, 0, 2, 2, 0, 1, 1, 0), f_2 = (1, 1, 1, 0, 1, 2, 0, 1, 0, 1),$$

$$f_3 = (1, 0, 1, 1, 1, 1, 1, 1, 1, 0), f_4 = (1, 1, 1, 1, 0, 1, 1, 1, 0, 1).$$

It is spanned over a parallelogram. We have $A(\Sigma_7) = \{(z_1, \ldots, z_{13}) \in \mathbb{C}^{11} | z_2 z_{10} = z_5 z_9, \ z_4 z_7 = z_5 z_6\}$ and the monomials corresponding to vectors $f_i$ generating $L(B_7)$ are equal on $A(\Sigma_7)$. Finally to make explicit the summation over $L(B_7)$ we observe that if we denote the above generators of $L(B_7)$ by $f_1, \ldots, f_4$ then every point of $P \in L(B_7)$ has a unique presentation $P = kf_1 + lf_2 + mf_3 + nf_4$ with $kn = 0$. We get the formula for the main period

$$(3.3) \quad \phi_0(t) = \sum_{s=0}^{\infty} \binom{2s}{s} \sum_{k+l+m+n=s, \ k,l,m,n \geq 0, \ kn=0} \binom{2s}{m} \binom{2s}{k+l+m, m, n, l+n} t^s$$

In this way we recover again the same result as [Kan12] getting the Picard-Fuchs equation to be no. 109 from [vEvS06].

**Example 3.3.** Consider our Calabi-Yau threefold $X_{10}$. As described in Section 2 it admits a degeneration to a nodal threefold appearing as a complete intersection in a Gorenstein terminal toric Fano $F_{10}$ variety of dimension 8. The decomposition of the set of rays $B_{10}$ of the Fan $\Sigma_{10}$ of $F_{10}$ is $J_1 = \{e_1, e_3\}$, $J_2 = \{e_2, e_3\}$, $J_3 = \{e_4, e_8\}$, $J_4 = \{e_7, e_9, e_{10}, e_{11}\}$, $J_5 = \{e_6\}$. The cone $L(B_{10})$ is generated by vectors:

$$(1, 1, 1, 2, 1, 2, 1, 0, 0, 0, 1), (1, 0, 1, 2, 2, 2, 1, 0, 1, 0, 1), (1, 1, 1, 1, 1, 1, 1, 0, 0).$$

It is hence a simplicial cone. We also compute that $A(\Sigma_{10}) = \{(z_1, \ldots, z_{11}) \in \mathbb{C}^{11} | z_{21} z_{11} = z_8 z_9, \ z_2 z_{11} = z_5 z_{10}\}$. We observe that the monomials corresponding to the three generators of $L(B_{10})$ are equal in $A(\Sigma_{10})$. We can hence set a new coordinate $t := z^{f_i}$ for $z \in A(\Sigma_{10})$ independent of $i \in \{1, 2, 3\}$. Since every point of $P \in L(B_{10})$ has a unique presentation as $P = kf_1 + lf_2 + mf_3$, with $k, l, m \in \mathbb{Z}_{\geq 0}$. The main period is then given by the formula:

$$(3.4) \quad \phi_0(t) = \sum_{s=0}^{\infty} \binom{2s}{s} \sum_{k+l+m=s, \ k,l,m \geq 0} \binom{2s}{k+m} \binom{2s}{2s-m} \binom{2s-k-m}{l} t^s$$

This corresponds to the Picard-Fuchs equation no. 263 from [vEvS06] as stated in [Kan12].

**Remark 3.2.** A similar computation holds for $X_{13}$ recovering the Picard-Fuchs equation no 99 from [vEvS06].

4. THE FAMILY $X_{25}$

We consider the family of Calabi-Yau threefolds of degree 25 in a separate section because its description is slightly different from the description of earlier studied varieties. It involves two sets of Pfaffian equations. More precisely $X_{25}$ is the family of Calabi-Yau threefold of degree 25 obtained as transversal intersections of two Grassmannians $G(2, 5)$ embedded by Plücker embeddings in $\mathbb{P}^9$. Its equations are given by $4 \times 4$ Pfaffians of two generic $5 \times 5$ matrices of linear forms. We would like to deform both Grassmannians simultaneously and obtain a toric variety as the result of their intersection. This might be impossible to do. However, we can set up a picture in which both deformations do not interfere with each other and the result is really a toric variety. For this we consider $X_{25}$ as a complete intersection of 10 hyperplanes in $\mathbb{P}^{19}$ with the subvariety $G_{25} \subset \mathbb{P}^{19}$ defined by $4 \times 4$ Pfaffians of two $5 \times 5$ skew symmetric matrices with entries being two disjoint sets of coordinates. This means that $G_{25}$ is the intersection of two cones $C_1$ and $C_2$ over Grassmannians $G(2, 5)$ centered in two disjoint $\mathbb{P}^9 \subset \mathbb{P}^{19}$. We find a toric degeneration of $G_{25}$, by degenerating each of the cones $C_1$ and $C_2$ by means of the standard degeneration of varieties given by Pfaffians of a $5 \times 5$ matrix introduced above. More precisely let $C_1, C_2 \subset \mathbb{P}^{19} \times \mathbb{C}$ be the families obtained by multiplying the corner entries of the matrices by $\lambda \in \mathbb{C}$. We then have the following:
Proposition 4.1. The intersection $C_1 \cap C_2 \subset \mathbb{P}^{19} \times \mathbb{C}$ is flat over $\mathbb{C}$. Moreover its fiber over zero denoted by $F_{25}$, is a Gorenstein terminal toric Fano variety $F_{25}$ of Picard number one.

Proof. It is clear that $C_1 \cap C_2$ is an algebraic family. Its fiber $F_{25}$ over 0 is the intersection of two cones $\hat{C}_1$ and $\hat{C}_2$ constructed similarly to $C_1$ and $C_2$ but over small degenerations of the Grassmannians $G(2,5)$. Since $\hat{C}_1$ and $\hat{C}_2$ are both Gorenstein as Pfaffian varieties it follows that $F_{25}$ is a Gorenstein variety of codimension 6. Its Hilbert polynomial is just the product of Hilbert polynomials of $\hat{C}_1$ and $\hat{C}_2$ which are equal to the Hilbert polynomials of $C_1$ and $C_2$. It follows that the family $C_1 \cap C_2$ is flat. We next prove that $F_{25}$ is normal. For this it is enough to compute the codimension of its singular locus. For this observe that by construction, since the equation of $C_1$ and $C_2$ involve disjoint sets of coordinates, a point on $F_{25}$ is singular if and only if it is a singular point of $\hat{C}_1$ or $\hat{C}_2$. Since the centers of the cones are of high codimension this implies that the singular locus of $F_{25}$ is of codimension 3. Hence $F_{25}$ is normal. Moreover we have a set of binomial equations defining $F_{25}$. It follows that $F_{25}$ is a Gorenstein toric variety. The Polytope of $F_{25}$ is generated by the set of rays $B_{25}$ consisting of the following:

$$
eq_1 = (-1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1),$$
$$
eq_2 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_3 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_4 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_5 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_6 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_7 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_8 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_9 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{10} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{11} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{12} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{13} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{14} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{15} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{16} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{17} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$
eq_{18} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

It follows that $F_{25}$ is terminal and of Picard number one.

Remark 4.2. Observe that this set of rays can be thought of as two sets of rays each describing the standard degeneration $P(2,5)$ of the Grassmannian $G(2,5)$.

From the proof of Proposition 4.1 we deduce that $F_{25}$ has 4 codimension 3 singular toric strata obtained by the intersection given by the 2 codimension 3 toric strata in each $\hat{C}_i$ for $i = 1,2$. The variety obtained as the intersection of 10 general Cartier divisors from the system corresponding to $O_{\mathbb{P}^9}(1)$ in $F_{25}$ is hence a nodal Calabi–Yau threefold. It admits a small resolution $\hat{T}_{25}$ being a complete intersection in a toric resolution $\hat{F}_{25}$ of $F_{25}$. It follows that $\hat{T}_{25}$ is connected to $X_{25}$ by a conifold transition.

Remark 4.3. The fact that $T_{25}$ is nodal follows directly form the part of the proof of 4.1 describing the singularities in codimension 3 from which we can easily deduce the local type in a general point of each of these singularities.

The Calabi-Yau threefold $X_{25}$ thus has a degeneration to a nodal threefold appearing as a complete intersection of ten hyperplane sections in a Gorenstein terminal toric Fano variety $F_{25}$ of dimension 13. The decomposition of the set of rays $B_{25}$ of the fan $\Sigma_{25}$ of $F_{25}$ is $J_1 = \{e_1\}$, $J_2 = \{e_9\}$, $J_3 = \{e_{10}\}$, $J_4 = \{e_{18}\}$, $J_5 = \{e_2, e_4\}$, $J_6 = \{e_3, e_7\}$, $J_7 = \{e_{11}, e_{13}\}$, $J_8 = \{e_{12}, e_{16}\}$, $J_9 = \{e_5, e_6, e_8\}$, $J_{10} = \{e_{14}, e_{15}, e_{17}\}$. The Cone $L(B_{25})$ is generated by:

$$f_1 = (1,0,0,1,1,0,0,1,1,0,0,1,1,0,0,1,0,1),$$
$$f_2 = (1,1,0,0,1,1,0,1,0,1,1,0,0,1,1,0,0,1),$$
$$f_3 = (1,1,0,0,1,1,0,1,1,0,0,1,1,0,0,1,0,0),$$
$$f_4 = (1,1,0,0,1,1,0,0,0,0,1,1,0,0,1,0,0,1),$$
$$f_5 = (1,1,0,0,0,1,1,1,0,0,0,0,1,0,0,1,0,0),$$
$$f_6 = (1,1,0,0,0,1,1,1,0,0,0,0,1,0,0,1,0,0),$$
$$f_7 = (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$
$$f_8 = (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$
$$f_9 = (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0).

Observe that if we denote the following vectors in $\mathbb{Z}^9$

$$(1,0,0,1,0,0,0,1,0,1,0,0,0,0,1,0,1,0,1), (1,0,0,0,0,0,1,0,1,0,1,0,0,0,0,1,0,1,0,1), (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1), (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1),$$

by $g_1$, $g_2$, $g_3$ respectively. Then any point of $P \in L(B_{25}) \subset \mathbb{Z}^{18} = \mathbb{Z}^9 \times \mathbb{Z}^9$ can be written in a unique way as $P = (kg_1 + mg_2 + ng_3, ng_1 + og_2 + ge_3)$ with $k + l + m = n + o + p$. We moreover have $A(\Sigma_{25}) = \mathbb{Z}^9 \times \mathbb{Z}^9$. 

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\{(z_1, \ldots, z_{18}) \in \mathbb{C}^{18} | z_2 z_6 = z_4 z_5, \ z_3 z_8 = z_6 z_7, \ , z_{11} z_{15} = z_{13} z_{14}, \ z_{12} z_{17} = z_{15} z_{16}\}. In this way we get the following formula for the period of the mirror of \(X_{25}\).

\[
\phi_0(t) = \sum_{s=0}^{\infty} \left( \sum_{k,l,m=0}^{\infty} \binom{s}{k} \binom{s}{m} \binom{s}{k,l,m} \right) t^s
\]

The latter implies

\[
\phi_0(t) = \left( \sum_{s=0}^{\infty} \sum_{k,l,m=0}^{\infty} \binom{s}{k} \binom{s}{m} \binom{s}{k,l,m} \right)^2 t^s.
\]

The corresponding Picard–Fuchs equation is no 101 in [vEvS06]. Indeed, the invariants of \(X_{25}\) fit with the predicted (in [vEvS06]) invariants of a hypothetical Calabi–Yau threefold of Picard number one with this equation describing the period of its mirror.

**Remark 4.4.** Observe that in the above the vectors \(g_i\) are the generators of the cone \(L(B_{P(2, 5)})\) computed for the standard small toric degeneration \(P(2, 5)\) of the Grassmannian \(G(2, 5)\).

**Remark 4.5.** The approach to the case \(X_{25}\) seems to work for the computation of the main period of the mirror family of any Calabi–Yau threefold (or of a Landau Ginzburg model of any Fano manifold) obtained as a transversal intersection of two Fano varieties admitting Gorenstein terminal toric degenerations. More precisely let \(X\) and \(Y\) be two Fano manifolds in \(\mathbb{P}^N\) intersecting transversely in \(Z\). Assume that \(X\) and \(Y\) admit small toric degenerations \(T_X \subset \mathbb{P}^N\) and \(T_Y \subset \mathbb{P}^N\). Let \(C_X\) and \(C_Y\) be cones in \(\mathbb{P}^{2N+1}\) over \(X\) and \(Y\) respectively with vertices being disjoint \(\mathbb{P}^N\)’s in \(\mathbb{P}^{2N+1}\). Observe that \(Z\) is a complete intersection of \(N\) hyperplane sections of \(C_X \cap C_Y\). Then by analogous proof to Proposition 4.1 we get the intersection of the cones \(C_{T_X} \cap C_{T_Y}\) as if it was smooth. In this way the Cone \(L(B_{T_Z})\) will be interpreted as the intersection of the products of cones \(L(B_{T_X}) \times L(B_{T_Y})\) with a hyperplane. Since the decomposition \(J\) of the set of rays can be done accordingly to the decomposition onto two parts, at the end we obtain the coefficients of the main period of the mirror of \(Z\) to be products of coefficients of two series each obtained by the naive application of the Batyrev formula 3.1 to suitable Calabi–Yau (not necessarily 3 dimensional) complete intersections in \(X\) and \(Y\).

**Remark 4.6.** It is interesting to observe that the above is consistent with another method of computation of the main period of the mirror. The latter method is based on Przyjalkowski constructions of weak Landau–Ginzburg models for smoothings of Gorenstein Terminal toric Fano varieties and the quantum Lefschetz formula in such manifolds. For more details see [Prz07c]. More precisely in this method we consider the Polytope of \(F_i\) and associate to it a Laurent polynomial \(\mathcal{P}\). To the latter we associate its constant term series \(I_P\) and use the formula [Prz07a, Cor 4.2.2.] for the computation of the main period of \(X_i\). In each case we obtain the same result as above even if our variety \(F_i\) has no smoothing. In fact the lack of smoothing may be solved by means of [Prz07b]. More precisely \(F_i\) has a partial smoothing \(G_i\) whose resolution does not affect \(X_i\) we can hence for our purposes work on \(G_i\) as if it was smooth.

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**References**

[Bat04] Victor V. Batyrev. Toric degenerations of Fano varieties and constructing mirror manifolds. In The Fano Conference, pages 109–122. Univ. Torino, Turin, 2004.
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