The Adaptive Dynamic Programming Toolbox

Xiaowei Xing and Dong Eui Chang

Abstract—The paper develops the Adaptive Dynamic Programming Toolbox (ADPT), which solves optimal control problems for continuous-time nonlinear systems. Based on the adaptive dynamic programming technique, the ADPT computes optimal feedback controls from the system dynamics in the model-based working mode, or from measurements of trajectories of the system in the model-free working mode without the requirement of knowledge of the system model. Multiple options are provided such that the ADPT can accommodate various customized circumstances. Compared to other popular software toolboxes for optimal control, the ADPT enjoys its computational precision and speed, which is illustrated with its applications to a satellite attitude control problem.

Index Terms—Optimal control, adaptive dynamic programming, software toolbox.

I. INTRODUCTION

Optimal control is an important branch in control engineering. For continuous-time dynamical systems, finding an optimal feedback control involves solving the so-called Hamilton-Jacobi-Bellman (HJB) equation [1]. For linear systems, the HJB equation becomes the well-known Riccati equation which results in the linear quadratic regulator [2]. For nonlinear systems, solving the HJB equation is generally a formidable task due to its inherently nonlinear nature. As a result, there has been a great deal of research devoted to approximately solving the HJB equation. Al’brekht proposed a power series method for smooth systems to solve the HJB equation [3]. With the assumption that the optimal control and the optimal cost function can be represented in Taylor series, by plugging the series expansions of the dynamics, the cost integrand function, the optimal control and the optimal cost function into the HJB equation and collecting terms degree by degree, the Taylor expansions of the optimal control and the optimal cost function can be recursively obtained. Similar ideas can be found in [4] and [5]. An approach to approximately solve the HJB based on the grid-based discretization of the state and time space is studied in [6]. A recursive algorithm is developed to sequentially improve the control law which converges to the optimal one by starting with an admissible control in [7]. This recursive algorithm is commonly referred to as policy iteration (PI) and can be also found in [8]–[10]. The common limitation of these methods is that the complete knowledge of the system is necessary.

In the past few decades, reinforcement learning (RL) [11] has provided a means to design optimal controllers in an adaptive manner from the point of learning. Adaptive/approximate dynamic programming (ADP), which is an iterative RL-based adaptive optimal control design method, has been proposed in [12]–[14]. An ADP strategy is presented for nonlinear systems with partially unknown dynamics in [12], and the necessity of the knowledge of the system model is fully relaxed in [13] and [14].

Together with the growth of optimal control theory and methods, several software tools for optimal control have been developed. Notable examples are Nonlinear Systems Toolbox [15], Control Toolbox [16], ACADO [17], its successor ACADOS [18], and GPSOP-II [19]. A common feature of these packages is that system equations are used in them. Besides, optimal controls generated by [16]–[19] are open-loop such that an optimal control is computed for each initial state. So, if the initial state changes, optimal controls should be computed again. In contrast, the Nonlinear Systems Toolbox [15] produces an optimal feedback control by solving the HJB equation.

The primary objective of this paper is to develop a MATLAB-based toolbox that solves optimal feedback control problems computationally for nonlinear systems in the continuous-time domain. More specifically, employing the adaptive dynamic programming technique, we derive a computational methodology to compute approximate optimal feedback controls, based on which we develop the Adaptive Dynamic Programming Toolbox (ADPT). The ADPT supports two working modes: the model-based mode and the model-free mode. The knowledge of system equations is required in the model-based working mode. In the model-free working mode, the ADPT produces the approximate optimal feedback control from measurements of system trajectories, removing the requirement of the knowledge of system equations. Moreover, multiple options are provided such that the user can use the toolbox with much flexibility.

The remainder of the paper is organized as follows. Section II reviews the standard optimal control problem for a class of continuous-time nonlinear systems and the model-free adaptive dynamic programming technique. Section III provides implementation details and software features of the ADPT. In Section IV the ADPT is applied to a satellite attitude control problem in both the model-based working mode and the model-free working mode. Conclusions and potential future extensions are given in Section V. The codes of the ADPT are available at https://github.com/Everglow0214/Adaptive_DynamicProgramming_Toolbox.

II. REVIEW OF ADAPTIVE DYNAMIC PROGRAMMING

We review the adaptive dynamic programming (ADP) technique to solve optimal control problems [20]. Consider a continuous-time nonlinear system given by

\[ \dot{x} = f(x) + g(x)u, \]  
\[ (1) \]
where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control, \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \) are locally Lipschitz continuous mappings with \( f(0) = 0 \). It is assumed that the system (1) is stabilizable at \( x = 0 \) in the sense that the system can be locally asymptotically stabilized by a continuous feedback control. To quantify the performance of a control, an integral cost associated with the system (1) is given by

\[
J(x_0, u) = \int_0^\infty (g(x(t)) + u(t)^T R u(t)) \, dt,
\]

where \( x_0 = x(0) \) is the initial state, \( q : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is a positive definite function and \( R \in \mathbb{R}^{n \times m} \) is a symmetric, positive definite matrix. A feedback control \( u : \mathbb{R}^n \to \mathbb{R}^m \) is said to be admissible if it stabilizes the system (1) at the origin, and makes the cost \( J(x_0, u) \) finite for all \( x_0 \) in a neighborhood of \( x = 0 \).

The objective is to find a control policy \( u \) that minimizes \( J(x_0, u) \) given \( x_0 \). Define the optimal cost function \( V^* : \mathbb{R}^n \to \mathbb{R} \) by

\[
V^*(x) = \min_u J(x, u)
\]

for \( x \in \mathbb{R}^n \). Then, \( V^* \) satisfies the HJB equation

\[
0 = \min_u \{ \nabla V^*(x)^T (f(x) + g(x)u) + q(x) + u^T Ru \},
\]

and the minimizer in the HJB equation is the optimal control which is expressed in terms of \( V^* \) as

\[
u^*(x, t) = -\frac{1}{2} R^{-1} g(x)^T \nabla V^*(x).
\]

Moreover, the state feedback \( u^* \) locally asymptotically stabilizes \( x \) at the origin and minimizes (2) over all admissible controls (2). Solving the HJB equation analytically is extremely difficult in general except for linear cases. Hence, approximate or iterative methods are needed to solve the HJB, and a well-known policy iteration algorithm, which is called policy iteration (7), is reviewed in Algorithm 1. Let \( \{V_i(x)\}_{i \geq 0} \) and \( \{u_{i+1}(x)\}_{i \geq 0} \) be the sequences of functions generated by the policy iteration algorithm in Algorithm 1. It is shown in (7) that \( V_{i+1}(x) \leq V_i(x) \) for \( i \geq 0 \), and the limit functions \( V(x) = \lim_{i \to \infty} V_i(x) \) and \( u(x) = \lim_{i \to \infty} u_i(x) \) are equal to the optimal cost function \( V^* \) and the optimal control \( u^* \).

Consider approximating the solutions to (3) and (4) by ADP instead of obtaining them exactly. For this purpose, choose an admissible feedback control \( u_0 : \mathbb{R}^n \to \mathbb{R}^m \) for (1) and let \( \{V_i(x)\}_{i \geq 0} \) and \( \{u_{i+1}(x)\}_{i \geq 0} \) be the sequences of functions generated by the policy iteration algorithm in Algorithm 1 starting with the control \( u_0(x) \). Choose a bounded time-varying exploration signal \( \eta : \mathbb{R} \to \mathbb{R}^m \), and apply the sum \( u_0(x) + \eta(t) \) to (1) as follows:

\[
\dot{x} = f(x) + g(x)u_0(x) + \eta(t).
\]

Assume that solutions to (5) are well defined for all positive time. Let \( T(x, u_0, \eta, [r, s]) = \{(x(t), u_0(x(t)), \eta(t)) \mid r \leq t \leq s \} \) denote the trajectory \( x(t) \) of the system (5) with the input \( u_0 + \eta \) over the time interval \([r, s] \) with \( 0 \leq r < s \). The system (5) can be rewritten as

\[
\dot{x} = f(x) + g(x)u_i(x) + g(x)\nu_i(x, t),
\]

Algorithm 1 Policy Iteration

**Input:** An initial admissible control \( u_0(x) \), and a threshold \( \epsilon > 0 \).

**Output:** The approximate optimal control \( u_{i+1}(x) \) and the approximate optimal cost function \( V_i(x) \).

1. Set \( i \leftarrow 0 \).
2. while \( i \geq 0 \) do
   3. Policy evaluation: solve for the continuously differentiable cost function \( V_i(x) \) with \( V_i(0) = 0 \) from
      \[
      \nabla V_i(x)^T (f(x) + g(x)u_i(x)) + q(x) + u_i(x)^T Ru_i(x) = 0.
      \]
     4. Policy improvement: update the control policy by
      \[
      u_{i+1}(x) = -\frac{1}{2} R^{-1} g(x)^T \nabla V_i(x).
      \]
   5. if \( \|u_{i+1}(x) - u_i(x)\| \leq \epsilon \) for all \( x \) then
      6. break
   else
      7. Set \( i \leftarrow i + 1 \).
   8. continue
 9. end if
10. end while

where

\[
u_i(x, t) = u_0(x) - u_i(x) + \eta(t).
\]

Combined with (6) and (7), the time derivative of \( V_i(x) \) along the trajectory \( x(t) \) of (6) is obtained as

\[
\dot{V}_i(x) = -q(x) - u_i(x)^T Ru_i(x) - 2u_{i+1}(x)^T Ru_i(x, t)
\]

for \( i \geq 0 \). By integrating both sides of (7) over any time interval \([r, s] \) with \( 0 \leq r < s \), one gets

\[
V_i(x(s)) - V_i(x(r)) = -\int_r^s (q(x) + u_i(x)^T Ru_i(x) + 2u_{i+1}(x)^T Ru_i(x, t)) \, dt.
\]

Let \( \phi_j : \mathbb{R}^n \to \mathbb{R} \) and \( \varphi_j : \mathbb{R}^n \to \mathbb{R}^m \), with \( j = 1, 2, \ldots \), be two infinite sequences of continuous basis functions on a compact set in \( \mathbb{R}^n \) containing the origin as an interior point that vanish at the origin (20). Then, \( V_i(x) \) and \( u_{i+1}(x) \) for each \( i \geq 0 \) can be expressed as infinite series of the basis functions.

For each \( i \geq 0 \) let \( \hat{V}_i(x) \) and \( \hat{u}_{i+1}(x) \) be approximations of \( V_i(x) \) and \( u_{i+1}(x) \) given by

\[
\hat{V}_i(x) = \sum_{j=1}^{N_i} c_{i,j} \phi_j(x),
\]

\[
\hat{u}_{i+1}(x) = \sum_{j=1}^{N_i} w_{i,j} \varphi_j(x),
\]
where $N_1 > 0$ and $N_2 > 0$ are integers and $c_{i,j}$, $w_{i,j}$ are coefficients to be found for each $i \geq 0$. Then, equation (5) is approximated by $\hat{V}_i(x)$ and $\hat{u}_{i+1}(x)$ as follows:

$$\sum_{j=1}^{N_1} c_{i,j}(\phi_j(x(s)) - \phi_j(x(r)))$$

$$+ \int_r^x (2 \sum_{j=1}^{N_2} u_{i,j} \varphi_j(x)^T R \hat{u}_i) \, d\tau$$

$$= - \int_r^x (q(x) + \hat{u}_i(x)^T R \hat{u}_i(x)) \, d\tau,$$  \hspace{1cm} (11)

where

$$\hat{u}_0 = u_0, \quad \hat{u}_i = u_0 - \hat{u}_i + \eta.$$ \hspace{1cm} (12)

Suppose that we have available $K$ trajectories $T(x, u_0, \eta, [r_k, s_k]), \ k = 1, \ldots, K,$ where $x(t)$, $u_0(t)$, and $\eta(t)$ satisfy (6) over the $K$ time intervals $[r_k, s_k], \ k = 1, \ldots, K$. Then, we have $K$ equations of the form (11) for each $i \geq 0$, which can be written as

$$e_{i,k} = 0, \quad k = 1, \ldots, K,$$ \hspace{1cm} (13)

where

$$e_{i,k} := \sum_{j=1}^{N_1} c_{i,j}(\phi_j(x(s_k)) - \phi_j(x(r_k)))$$

$$+ \int_{r_k}^{s_k} (2 \sum_{j=1}^{N_2} w_{i,j} \varphi_j(x)^T R \hat{u}_i) \, d\tau$$

$$+ \int_{r_k}^{s_k} (q(x) + \hat{u}_i(x)^T R \hat{u}_i(x)) \, d\tau.$$  

Then, the coefficients $\{c_{i,j}\}_{j=1}^{N_1}$ and $\{w_{i,j}\}_{j=1}^{N_2}$ are obtained by minimizing

$$\sum_{k=1}^{K} \|e_{i,k}\|^2.$$  

In other words, the $K$ equations in (13) are solved in the least squares sense for the coefficients, $\{c_{i,j}\}_{j=1}^{N_1}$ and $\{w_{i,j}\}_{j=1}^{N_2}$. Thus two sequences $\{\hat{V}_i(x)\}_{i=0}^{\infty}$ and $\{\hat{u}_{i+1}(x)\}_{i=0}^{\infty}$ can be generated from (11). According to (20), Cor. 3.2.4, for any arbitrary $\epsilon > 0$, there exist integers $i^* > 0$, $N_1^{**} > 0$ and $N_2^{**} > 0$ such that

$$\left\| \sum_{j=1}^{N_1} c_{i^*,j} \phi_j(x) - V^*(x) \right\| \leq \epsilon,$$

$$\left\| \sum_{j=1}^{N_2} w_{i^*,j} \varphi_j(x) - u^*(x) \right\| \leq \epsilon$$

for all $x$, if $N_1 > N_1^{**}$ and $N_2 > N_2^{**}$.

**Remark 1:** The ADP algorithm relies on only the measurements of states, the initial control policy and the exploration signal, lifting the requirement of knowing the precise system model, while the conventional policy iteration algorithm in Algorithm 1 requires the knowledge of the exact system model. Hence, the ADP algorithm is 100% data-based and model-free.

**Remark 2:** Equation (11) depends on the initial control $u_0$, the exploration signal $\eta$, the time interval $[r, s]$ as well as the index $i$, where the first three $u_0$, $\eta$, and $[r, s]$ are together equivalent to the trajectory $T(x, u_0, \eta, [r, s])$ if the initial state $x(r)$ at $t = r$ is given. Hence, we can generate more diverse trajectories by changing $u_0$ and $\eta$ as well as $[r, s]$, and enrich the ADP algorithm accordingly, as follows. Suppose that we have available $K$ trajectories $T(x^k, u_0^k, \eta^k, [r_k, s_k]), \ 1 \leq k \leq K$, where $x^k$, $u_0^k$, and $\eta^k$ satisfy (6), i.e.,

$$x^k(t) = f(x^k(t)) + g(x^k(t)) (u_0^k(x^k(t)) + \eta^k(t))$$

for $r_k \leq t \leq s_k$. Then, we have $K$ equations of the form (11) for each $i \geq 0$, which can be written as $e_{i,k} = 0, \ k = 1, \ldots, K,$ where

$$e_{i,k} := \sum_{j=1}^{N_1} c_{i,j}(\phi_j(x^k(s_k)) - \phi_j(x^k(r_k)))$$

$$+ \int_{r_k}^{s_k} (2 \sum_{j=1}^{N_2} w_{i,j} \varphi_j(x^k)^T R \hat{u}_i^k) \, d\tau$$

$$+ \int_{r_k}^{s_k} (q(x^k) + \hat{u}_i^k(x^k)^T R \hat{u}_i^k(x^k)) \, d\tau$$

with $\hat{u}_0 = u_0^k$ and $\hat{u}^k = u_0^k + \eta^k - \hat{u}_i$. Then, the coefficients $\{c_{i,j}\}_{j=1}^{N_1}$ and $\{w_{i,j}\}_{j=1}^{N_2}$ are obtained by minimizing

$$\sum_{k=1}^{K} \|e_{i,k}\|^2.$$  For the sake of simplicity of presentation, however, in this paper we will fix $u_0$ and $\eta$ and vary only the time intervals to generate trajectory data.

**III. IMPLEMENTATION DETAILS AND SOFTWARE FEATURES**

We now discuss implementation details and features of the Adaptive Dynamic Programming Toolbox (ADPT). We provide two modes to generate approximate optimal feedback controls; one mode requires the knowledge of system model, but the other eliminates this requirement, giving rise to the ADP’s unique capability of handling model-free cases.

**A. IMPLEMENTATION OF COMPUTATIONAL ADAPTIVE DYNAMIC PROGRAMMING**

To approximate $\hat{V}_i(x)$ and $u_{i+1}(x)$ in (13) and (14), monomials composed of state variables are selected as basis functions. For a pre-fixed number $d \geq 1$, define a column vector $\Phi_d(x)$ by ordering monomials in graded reverse lexicographic order as

$$\Phi_d(x) = (x_1, \ldots, x_n; x_1^2, x_1 x_2, \ldots, x_{n^2}; x_1^3, \ldots, x_n^d) \in \mathbb{R}^{N \times 1},$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is the state, $d \geq 1$ is the highest degree of the monomials, and $N$ is given by

$$N = \sum_{i=1}^{d} \binom{i+n-1}{n-1}.$$  

For example, if $n = 3$ and $d = 3$, the corresponding ordered monomials are

$$x_1, x_2, x_3;$$

$x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2;$$

$x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2^2, x_1 x_2 x_3, x_1 x_3^2, x_2^3, x_2^2 x_3, x_2 x_3^2, x_3^3.$
According to (2) and (10), the cost function \( V_i(x) \) and the control \( u_{i+1}(x) \) are approximated by \( \hat{V}_i(x) \) and \( \hat{u}_{i+1}(x) \) which are defined as
\[
\hat{V}_i(x) = c_i \Phi_{d+1}(x), \quad \hat{u}_{i+1}(x) = W_i \Phi_d(x),
\]
where \( d \geq 1 \) is the approximation degree, and \( c_i \in \mathbb{R}^{1 \times N_1} \) and \( W_i \in \mathbb{R}^{m \times N_2} \) are composed of coefficients corresponding to the monomials in \( \Phi_{d+1}(x) \) and \( \Phi_d(x) \) with
\[
N_1 = \sum_{i=1}^{d+1} \left( i + n - 1 \right), \quad N_2 = \sum_{i=1}^{d} \left( i + n - 1 \right).
\]
We take the highest degree of monomials to approximate \( V_i \) greater by one than the approximation degree since \( u_{i+1} \) is obtained by taking the gradient of \( V_i \) in (4) and \( g(x) \) is constant in most cases.

**Theorem 1:** Let a set of trajectories be defined as \( S_T = \{ T(x, u_0, \eta, [r_k, s_k]), k = 1, 2, \ldots, K \} \) with \( K \geq 1 \), and let
\[
\alpha(x) = R \eta \Phi_d(x)^T, \\
\beta(x) = R (u_0(x) + \eta) \Phi_d(x)^T, \\
\gamma(x) = \Phi_{d-1}(x) \Phi_d(x)^T.
\]
Then the coefficients \( c_i \) and \( W_i \) satisfy
\[
A_i \begin{bmatrix} c_i \\ \vec{c}^T \end{bmatrix} = b_i,
\]
where
\[
A_0 = \begin{bmatrix} \Phi_{d+1}^{[r_1, s_1]}(x) & 2 \text{vec}(\int_{r_1}^{s_1} \alpha(x) \, dt)^T \\
\vdots & \vdots \\
\Phi_{d+1}^{[r_K, s_K]}(x) & 2 \text{vec}(\int_{r_K}^{s_K} \alpha(x) \, dt)^T \\
\end{bmatrix} \in \mathbb{R}^{K \times (N_1 + m N_2)},
\]
\[
b_0 = \begin{bmatrix} - \int_{r_1}^{s_1} (q(x) + u_0(x) R u_0(x)) \, dt \\
\vdots \\
- \int_{r_K}^{s_K} (q(x) + u_0(x) R u_0(x)) \, dt \\
\end{bmatrix} \in \mathbb{R}^{K \times 1},
\]
and for \( i = 1, 2, \ldots, \)
\[
A_i = \begin{bmatrix} \Phi_{d+1}^{[r_1, s_1]}(x) & 2 \text{vec}(\int_{r_1}^{s_1} (\beta(x) - R W_{i-1} \gamma(x)) \, dt)^T \\
\vdots & \vdots \\
\Phi_{d+1}^{[r_K, s_K]}(x) & 2 \text{vec}(\int_{r_K}^{s_K} (\beta(x) - R W_{i-1} \gamma(x)) \, dt)^T \\
\end{bmatrix} \in \mathbb{R}^{K \times (N_1 + m N_2)},
\]
\[
b_i = \begin{bmatrix} - \int_{r_1}^{s_1} q(x) \, dt - \langle W_{i-1}^T R W_{i-1}, \int_{r_1}^{s_1} \gamma(x) \, dt \rangle \\
\vdots \\
- \int_{r_K}^{s_K} q(x) \, dt - \langle W_{i-1}^T R W_{i-1}, \int_{r_K}^{s_K} \gamma(x) \, dt \rangle \\
\end{bmatrix} \in \mathbb{R}^{K \times 1},
\]
where
\[
\Phi_{d+1}^{[r, s]}(x) = \Phi_{d+1}(x(s)) \Phi_{d+1}(x(r))
\]
for \( k = 1, 2, \ldots, K \), and the operator \( \text{vec}(\cdot) \) is defined as
\[
\text{vec}(Z) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\
\end{bmatrix} \in \mathbb{R}^{mn \times 1}
\]
with \( z_j \in \mathbb{R}^{m \times 1} \) being the \( j \)-th column of a matrix \( Z \in \mathbb{R}^{m \times n} \) for \( j = 1, \ldots, n \).

**Proof:** See Appendix A

We now give the computational adaptive dynamic programming algorithm in Algorithm 2 for practical implementation. To solve the least squares problem in line 5 in the algorithm, we need to have a sufficiently large number \( K \) of trajectories such that the minimization problem can be well solved numerically. Then the approximate optimal feedback control is generated by the algorithm as \( \hat{u}_{i+1} = W_i \Phi_d(x) \).

**Algorithm 2 Computational Adaptive Dynamic Programming**

**Input:** An approximation degree \( d \geq 1 \), an initial admissible control \( u_0(x) \), an exploration signal \( \eta(t) \), and a threshold \( \epsilon > 0 \).

**Output:** The approximate optimal control \( \hat{u}_{i+1}(x) \) and the approximate optimal cost function \( \hat{V}_i(x) \).

1. Apply \( u = u_0 + \eta \) as the input during a sufficiently long period and collect necessary data.
2. Set \( i \leftarrow 0 \).
3. while \( i \geq 0 \) do
4. Generate \( A_i \) and \( b_i \).
5. Obtain \( c_i \) and \( W_i \) by solving the minimization problem
\[
\min_{c_i, W_i} \left\| A_i \begin{bmatrix} c_i \\ \vec{c}^T \end{bmatrix} - b_i \right\|^2.
\]
6. if \( i \geq 1 \) and \( \| c_i - c_{i-1} \| + \| W_i - W_{i-1} \| \leq \epsilon^2 \) then
7. break
8. else
9. Set \( i \leftarrow i + 1 \).
10. continue
11. end if
12. end while
13. return \( \hat{u}_{i+1}(x) = W_i \Phi_d(x) \) and \( \hat{V}_i(x) = c_i \Phi_{d+1}(x) \)

**Remark 3:** As in the statement of Theorem 1 several integral terms are included in \( A_i \) and \( b_i \) for \( i \geq 0 \). As in (12), \( u_0 \) does not get approximated by the basis functions, so the matrices \( A_0 \) and \( b_0 \) in Theorem 1 are obtained with \( x(r_k) \), \( x(s_k) \), \( \int_{r_k}^{s_k} q(x) \, dt \), \( \int_{r_k}^{s_k} u_0(x) R u_0(x) \, dt \) and \( \int_{r_k}^{s_k} \alpha(x) \, dt \), \( 1 \leq k \leq K \). For \( i \geq 1 \), the matrices \( A_i \) and \( b_i \) in Theorem 1 need, in addition, \( \int_{r_k}^{s_k} \beta(x) \, dt \) and \( \int_{r_k}^{s_k} \gamma(x) \, dt \), \( 1 \leq k \leq K \), as well as \( W_i \).

**Remark 4:** In the situation where the system dynamic equations are known, the ADPT uses the Runge-Kutta method to simultaneously compute the trajectory points \( x(r_k) \) and \( x(s_k) \) and the integral terms that appear in \( A_i \) and \( b_i \). In the case when system equations are not known but trajectory data are available, the ADPT applies the trapezoidal method to evaluate these integrals numerically. In this case, each trajectory \( T(x, u_0, \eta, [r_k, s_k]) \) is represented by a set of its
sample points \( \{x(t_k), u_0(t_k), \eta(t_k)\}_{t_{k=1}}^{L_k} \), where \( \{t_k, L_k\}_{t_{k=1}}^{L_k} \) is a finite sequence that satisfies \( r_k = t_{k,1} < t_{k,2} < \ldots < t_{k,L_k-1} < t_{k,L_k} = s_k \), and then the trapezoidal method is applied on these sample points to numerically evaluate the integrals over the time interval \([r_k, s_k]\). If intermediate points in the interval \([r_k, s_k]\) are not available so that partitioning the interval \([r_k, s_k]\) is impossible, then we just use the two end points \(r_k\) and \(s_k\) to evaluate the integral by the trapezoidal method as
\[
\int_{r_k}^{s_k} h(t)dt \approx \frac{(s_k - r_k)(h(s_k) + h(r_k))}{2}
\]
for a function \( h(t) \).

B. Software Features

The codes of the ADPT are available at [https://github.com/everglow0214/Adaptive_Dynamic_Programming_Toolbox](https://github.com/everglow0214/Adaptive_Dynamic_Programming_Toolbox)

1) Symbolic Expressions: It is of great importance for an optimal control package that the user can describe functions such as system equations, cost functions, etc., in a convenient manner. The idea of the ADPT is to use symbolic expressions. Consider a optimal control problem, where the system model is in the form (1) with
\[
f(x) = \begin{bmatrix} -k_1 x_1 - k_2 x_1^3 - k_3 x_2 \\ k_4 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1/k_4 \end{bmatrix},
\]
where \( x = (x_1, x_2) \in \mathbb{R}^2 \) is the state, \( u \in \mathbb{R} \) is the control, and \( k_1, k_2, k_3, k_4 \in \mathbb{R} \) are system parameters. The cost function is in the form (2) with
\[
q(x) = 5x_1^2 + 3x_2^2, \quad R = 2.
\]
Then in the ADPT the system dynamics and the cost function can be defined in lines 1 – 17 in Listing 1

Listing 1
AN EXAMPLE OF THE MODEL-BASED MODE.

1 n = 2; \% state dimension
2 m = 1; \% control dimension
3 \%\% Symbolic variables.
4 syms x [n,1] real
5 syms u [m,1] real
6 syms t real
7
8 \%\% Define the system.
9 k1 = 3; k2 = 2; k3 = 2; k4 = 5;
10 f = [x2;-
11 \quad (-k1*x1-k2*x1^3-k3*x2)/k4];
12 g = [0; 1/k4];
13
14 \%\% Define the cost function.
15 q = 5*x1^2+3*x2^2;
16 R = 2;
17
18 \%\% Execute ADP iterations.
19 d = 3; \% approximation degree
20 \\

2) Working Modes: Two working modes are provided in the ADPT: the model-based mode and the model-free mode. The model-based mode deals with the situation where the system model is given while the model-free mode addresses the situation where the system model is not known but only trajectory data are available. An example of the model-based mode is given in Listing 1 where after defining the system model (18), the cost function (19) and the approximation degree \( d \) in lines 1 – 20, the function, adpModelBased, returns the coefficients \( W_i \) and \( c_i \) for the control \( u_{i+1} \) and the cost function \( V_i \), respectively, in lines 21 – 22.

An example of the model-free mode is shown in Listing 2 where the system model (18) is assumed to be unknown. The initial control \( u_0 \) is in the form of \( u_0(x) = -Fx \) with the feedback control gain \( F \) defined in line 18. The exploration signal \( \eta \) is composed of four sinusoidal signals as shown in lines 21 – 22. A list of two initial states \( x(0) = (-3, 2) \) and \( x(0) = (2, 3) \) is given in lines 29 – 30, and a list of the corresponding total time span for simulation is given lines 31 – 32, where the time interval \([0, 6]\) is divided into sub-intervals of size 0.002 so that trajectory data are recorded every 0.002 sec in lines 38 – 44. The time stamps are saved in the column vector \( t_{\text{save}} \) in line 42, and the values of states are saved in the matrix \( x_{\text{save}} \) in line 43, with each row in \( x_{\text{save}} \) corresponding to the same row in \( t_{\text{save}} \). Similarly, the values of the initial control \( u_0 \) and the exploration signal \( \eta \) are saved in vectors \( u_{0\text{save}} \) and \( eta_{\text{save}} \) in lines 46 – 47. These measurements are passed to the function, adpModelFree, in lines 51 – 52 to compute the optimal control and the optimal cost function approximately.

In the both model-based and model-free modes the approximate control is saved in the file, uAdp.p, that is generated automatically and can be applied by calling \( u=u\text{Adp}(x) \) without dependence on other files. Similarly, the user may also check the approximate cost through the file, VAdp.p.

3) Options: Multiple options are provided such that the user may customize optimal control problems in a convenient way. In the model-based mode, the user may set option values through the function, adpSetModelBased, in a name-value manner before calling adpModelBased. That is, the specified values may be assigned to the named options. An example is shown in Listing 3 where two sets of initial states, time intervals and exploration signals are specified in lines 1 – 9. Then, in lines 16 – 17 the output of adpSetModelBased should be passed to adpModelBased for the options to take effect. Otherwise, the default values would be used for the options as in lines 21 – 22 in Listing 1. The options supported by adpSetModelBased are listed in Table II in Appendix C.

Listing 3
A DEMONSTRATION OF CALLING THE FUNCTION adpSetModelBased.

1 \%\% The user may specify settings.
2 xInit = [-3, 2; 2.2, 3];
3 tSpan = [0, 10];
4
% Define the cost function.
q = @(x) 5*x(1)^2+3*x(2)^2;
R = 2;
% Exploration signal.
tau = 1:
eta = 0.8*sin(t) + sin(1.1*t) + sin(1.1*t);
end
\[
\begin{align*}
\eta_{\text{save}} &= \eta(t_{\text{save}}); \\
\text{time}, \text{states} &= \text{t}_{\text{save}}; \\
\text{t}_{\text{save}} &= \text{t}_{\text{save}}; \text{time}; \\
x_{\text{Init}} &= [-3; 2; \ldots; x_{\text{Init}}, \text{tspan}, \text{tspan}, \\
\text{AbsTol} &= 1e-6; \\
x_{\text{save}} &= [x_{\text{save}}, \text{states}]; \\
u_{0, \text{save}} &= -x_{\text{save}}*F; \\
\text{eta}_{\text{save}} &= \text{e}(t_{\text{save}}); \\
end
\]

5 \quad 0, 8];
6 \quad \text{eta} = [0.8*\sin(7*t) + \sin(3*t); \\
9 \quad \sin(1.1*t) + \sin(pi*t)];
10 \quad \text{adpOpt} = \text{adpSetModelBased}(... \\
\quad 'xInit', 'xInit', 'tSpan', 'tSpan', ... \\
\quad 'explSymb', 'eta');
14 \quad \% Execute ADP iterations.
16 \quad [w, c] = \text{adpModelBased}(f, g, x, n, u, m, q, ... \\
17 \quad R, t, d, \text{adpOpt});

For the command, \text{adpModelFree}, option values can be modified with the function, \text{adpSetModelFree}, in the name-value manner. The options supported by \text{adpSetModelFree} are listed in Table \text{III} in Appendix \text{B}. Among these options, ‘stride’ enables the user to record values of states, initial controls and exploration signals in a high frequency for a long time, while using only a portion of them in the iteration process inside \text{adpModelFree}. To illustrate it, let each trajectory in the set \(S_T\) of trajectories in the statement of Theorem \text{I} be represented by two sample points at time \(r_k\) and \(s_k\), that is, the trapezoidal method evaluates integrals over \([r_k, s_k]\) by taking values at \(r_k\) and \(s_k\) as in (17). Suppose that trajectories in \(S_T\) are consecutive, that is, \(s_k = r_{k+1}\) for \(k = 1, 2, \ldots, K - 1\). By setting ‘stride’ to \(\delta\) with \(\delta = 1, 2, \ldots, K\), the data used to generate \(A_i\) and \(b_i\) in Algorithm \text{I} become \(\{T(x, u_0, \eta, [r_{i+1}, s_{i+1}, \delta])\}, i \in \mathbb{N}, i + 1) \delta \leq K\}. For example, consider 3 consecutive trajectories \(T(x, u_0, \eta, [r_k, r_{k+1}])\) with \(k = 1, 2, 3\). If ‘stride’ is set to 1, one will have three equations from (11) as follows:
\[
\begin{align*}
\sum_{j=1}^{N_1} c_{i,j}(\phi_j(x(r_{k+1})) - \phi_j(x(r_k))) &= \int_{r_k}^{r_{k+1}} (2 \sum_{j=1}^{N_2} w_{i,j} \phi_j(x)^T R u_i) \, d\tau \\
&- \int_{r_k}^{r_{k+1}} (q(x) + \hat{u}_i(x)^T R \hat{u}_i(x)) \, d\tau \\
\end{align*}
\]
for \(k = 1, 2, 3\). These three equations contribute to three rows of \(A_i\) and three rows of \(b_i\) as in Theorem \text{I} if ‘stride’ is set to 3, then one will have only one equation from (11) as follows:
\[
\begin{align*}
\sum_{j=1}^{N_1} c_{i,j}(\phi_j(x(r_4)) - \phi_j(x(r_1))) &= \int_{r_1}^{r_4} (2 \sum_{j=1}^{N_2} w_{i,j} \phi_j(x)^T R u_i) \, d\tau \\
&- \int_{r_1}^{r_4} (q(x) + \hat{u}_i(x)^T R \hat{u}_i(x)) \, d\tau,
\end{align*}
\]
where the integrals over \([r_1, r_4]\) are evaluated by the trapezoidal method with the interval \([r_1, r_4]\) partitioned into the three sub-intervals \([r_1, r_2] \cup [r_2, r_3] \cup [r_3, r_4]\), i.e. with the points at \(r_1, r_2, r_3\) and \(r_4\). Equation (20) will contribute to one row
of $A_i$ and one row of $b_i$ as in Theorem 1. With the assumption that $A_i$ has full rank with ‘stride’ set to 3, by setting ‘stride’ to 3, the number of equations in the minimization problem in Algorithm 1 is two less than that with ‘stride’ set to 1, and as a result, the computation load is reduced in the numerical minimization. It is remarked that with ‘stride’ equal to 3, all the four points at $r_1, \ldots, r_4$ are used by the trapezoidal method to evaluate the integrals over the interval $[r_1, r_4]$ in (20), producing a more precise value of integral than the one that would be obtained with the two end points at $r_1$ and $r_4$ only. An example of calling adpSetModelFree is shown in Listing 4. Similarly, adpModelFree takes the output of adpSetModelFree as an argument to validate the options specified.

Listing 4

A DEMONSTRATION OF CALLING THE FUNCTION adpSetModelFree.

1 \% The user may specify settings.
2 adpOpt = adpSetModelFree('stride',2);
3
4 \% Execute ADP iterations.
5 [w,c] = adpModelFree(t_save,x_save,n,...
6 u0_save,m,eta_save,d,q,R,adpOpt);

IV. APPLICATIONS TO THE SATELLITE ATTITUDE STABILIZING PROBLEM

In this section we apply the ADPT to the satellite attitude stabilizing problem because a stabilization problem can be formulated as an optimal control problem. In the first example, the system model is given and the controller is computed by the function adpModelBased. The same problem is solved again in the second example by the function adpModelFree when the system dynamics is totally unknown. The source codes for these two examples are available at [https://github.com/Everglow0214/The_Adaptive_Dynamic_Programming_Toolbox](https://github.com/Everglow0214/The_Adaptive_Dynamic_Programming_Toolbox)

A. Model-Based Case

Let $\mathbb{H}$ denote the set of quaternions and $S^3 = \{q \in \mathbb{H} \mid \|q\| = 1\}$. The equations of motion of the continuous-time fully-actuated satellite system are given by

$$\dot{q} = \frac{1}{2}q\Omega, \quad \Omega = \mathbb{I}^{-1}((\Omega) \times \Omega) + \mathbb{I}^{-1}u,$$

(21)

(22)

where $q \in S^3$ represents the attitude of the satellite, $\Omega \in \mathbb{R}^3$ is the body angular velocity vector, $\mathbb{I} \in \mathbb{R}^{3 \times 3}$ is the moment of inertia matrix and $u \in \mathbb{R}^3$ is the control input. The quaternion multiplication is carried out for $q\Omega$ on the right-hand side of (21) where $\Omega$ is treated as a pure quaternion. By the stable embedding technique (21), the system (21) and (22) defined on $S^3 \times \mathbb{R}^3$ is extended to the Euclidean space $\mathbb{H} \times \mathbb{R}^3$ (23), (24) as

$$\dot{q} = \frac{1}{2}q\Omega - \alpha(|q|^2 - 1)q, \quad \dot{\Omega} = \mathbb{I}^{-1}((\Omega) \times \Omega) + \mathbb{I}^{-1}u,$$

(23)

(24)

where $q \in \mathbb{H}$, $\Omega \in \mathbb{R}^3$ and $\alpha > 0$.

Consider the problem of stabilizing the system (23) and (24) at the equilibrium point $(q_e, \Omega_e) = ((1,0,0),(0,0,0))$. The error dynamics is given by

$$\dot{e}_q = \frac{1}{2}(e_q + q)e_\Omega - \alpha(|e_q + q|^2 - 1)(e_q + q)e_\Omega,$$

$$\dot{e}_\Omega = \mathbb{I}^{-1}((e_\Omega) \times e_\Omega) + \mathbb{I}^{-1}u,$$

where $e_q = q - q_e$ and $e_\Omega = \Omega - \Omega_e$ are state errors. Since the problem of designing a stabilizing controller can be solved by designing an optimal controller, we pose an optimal control problem with the cost integral $\mathbb{J}$ with $q(x) = x^TQx$, where $x = (e_q, e_\Omega) \in \mathbb{R}^7$ and $Q = 2I_{7 \times 7}$, and $R = I_{3 \times 3}$. The inertia matrix $\mathbb{I}$ is set to $\mathbb{I} = \text{diag}(0.1029, 0.1263, 0.0292)$. The parameter $\alpha$ that appears in the above error dynamics is set to $\alpha = 1$.

We set the option ‘knt’ with three different initial states. For each initial state, the option ‘tSpan’ is set to $[0, 15]$. We use the option ‘explSymb’ to set exploration signals; see Table I in Appendix B for the use of the option ‘explSysb’. For the initial control $u_0$, the default initial control is used, which is an LQR controller computed for the linearization of the error dynamics around the origin with the weight matrices $Q = 2I_{7 \times 7}$ and $R = I_{3 \times 3}$. We then call the function, adpModelBased, to generate controllers of degree $d = 1, 2, 3$. The computation time taken by the function, adpModelBased, to produce the controllers are recorded in Table I. For the purpose of comparison, we also apply Al’brekht’s method with the software package called ACADO [17] to numerically produce the optimal control for each given initial state, which is an open-loop optimal control. Notice however that ACADO produces an open-loop optimal solution, which is on par with ACADO in terms of optimality, speed and usefulness all taken into account.

B. Model-Free Case

Consider solving the same optimal problem as in Section IV-A but the system dynamics in (21) and (22), or equivalently
the error dynamics are not available. Since we do not have real trajectory data available, for the purpose of demonstration we make up some trajectories with four initial states for the error dynamics, where the same initial control \( u_0 \) and exploration signals \( \eta \) are used as the model-based case in Section IV-A. The simulation for data collection is run over the time interval \([0, 20]\) with the recording period being 0.002 sec, producing 10,000 = 20/0.002 sampled points for each run. For the function \( \text{adpModelFree} \), the option of ‘stride’ is set to 4. Then, the function, \( \text{adpModelFree} \), is called to generate controllers of degree \( d = 1, 2, 3 \), the computation time taken for each of which is recorded in Table I. For the purpose of comparison in terms of optimality, we apply the controllers generated by \( \text{adpModelFree} \) to the system with the initial error state \( x_0 = ((\cos(\theta)/2) − 1, \sin(\theta/2), 0, 0), (0, 0, 0)) \) with \( \theta = 1.99999\pi \) and compute the corresponding values of the cost integral; see Table I for the values.

From Table I, we can see that ADPT in the model-free mode takes more computation time than ADPT in the model-based mode, and the cost integrals by ADPT in the model-free working mode is slightly higher than those in the model-based mode, since the integrals in the iteration process are evaluated less accurately. However, ADPT in the model-free mode is superior to NST in terms of optimality and to ACADO in terms of computation time. More importantly, it is noticeable that the result by model-free ADPT is comparable to model-based ADPT, which shows the power of data-based adaptive dynamic programming and the ADP toolbox.

To see how the computed optimal controller works in terms of stabilization, the norm of the state error under the control with \( d = 3 \) generated by ADPT in the model-free mode is plotted in Fig. I together with the norm of state error by the NST controller with degree 3. We can see that the convergence to the origin is faster with the model-free ADP controller than with the controller by NST that is model-based. This comparison result is consistent with the comparison of the two in terms of optimality.

| Table I | Costs at \( x_0 \) and Computation Time by ADPT, NST and ACADO. |
|---------|---------------------------------------------------------------|
|         | \( J(x_0, u) \) | Time [s] |
| ADPT (model-based) |       |         |
| \( d = 1 \) | 37.8259 | 1.6572  |
| \( d = 2 \) | 33.6035 | 2.5878  |
| \( d = 3 \) | 33.4986 | 11.6869 |
| ADPT (model-free) |       |         |
| \( d = 1 \) | 43.8308 | 0.8923  |
| \( d = 2 \) | 36.8319 | 3.5327  |
| \( d = 3 \) | 37.4111 | 90.6225 |
| NST            |       |         |
| \( d = 1 \) | 208.9259 | 0.2702 |
| \( d = 2 \) | 94.6868 | 0.6211  |
| \( d = 3 \) | 64.0721 | 3.6201  |
| ACADO         | -     | 32.6000 | 2359.67 |

\( a \) \( J(x_0, u) \) denotes the integral cost of the corresponding control \( u \).

\( b \) Time [s]’ denotes the computation time taken by the method to obtain the controller.

V. CONCLUSIONS AND FUTURE WORK

The Adaptive Dynamic Programming Toolbox, a MATLAB-based package for optimal control for continuous-time nonlinear systems, has been presented. We propose a computational methodology to approximately produce the optimal control and the optimal cost function by employing the adaptive dynamic programming technique. The ADPT can work in the model-based mode or in the model-free mode. The model-based mode deals with the situation where the system model is given while the model-free mode handles the situation where the system dynamics are unknown but only system trajectory data are available. Multiple options are provided for the both modes such that the ADPT can be easily customized. The optimality, the running speed and the utility of the ADPT are illustrated with a satellite attitude stabilizing problem.

Currently control policies and cost functions are approximated by polynomials in the ADPT. As mathematical principles of neural networks are being revealed, we plan to use deep neural networks in addition to polynomials in the ADPT to approximately represent optimal controls and optimal cost functions to provide users of the ADPT more options.

APPENDIX A

PROOF OF THEOREM I

Proof: Combining (11), (14) and (15), one has

\[
\begin{align*}
&c_0(\Phi_{d+1}(x(s_k)) - \Phi_{d+1}(x(r_k))) + 2 \int_{r_k}^{s_k} \Phi_d(x)^T W_0^T R \eta \, dt \\
&= - \int_{r_k}^{s_k} (q(x) + u_0(x)^T R u_0(x)) \, dt, \quad (25)
\end{align*}
\]

and for \( i = 1, 2, \ldots \),

\[
\begin{align*}
&c_i(\Phi_{d+1}(x(s_k)) - \Phi_{d+1}(x(r_k))) \\
&+ 2 \int_{r_k}^{s_k} \Phi_d(x)^T W_i^T R (u_0(x) + \eta) \, dt \\
&- 2 \int_{r_k}^{s_k} \Phi_d(x)^T W_i^T R W_{i-1} \Phi_d(x) \, dt \\
&= - \int_{r_k}^{s_k} (q(x) + \Phi_d(x)^T W_{i-1} R W_{i-1} \Phi_d(x)) \, dt. \quad (26)
\end{align*}
\]
By applying the property
\[(A, BC) = (AC^T, B) = (B^T A, C)\]
of the Euclidean inner product defined by \(\langle E, F \rangle = \sum_{i,j} E_{ij} F_{ij}\) for matrices \(E = [E_{ij}]\) and \(F = [F_{ij}]\) of equal size, one may rewrite (25) and (26) as
\[
c_0(\Phi_{d+1}(x(s_k)) - \Phi_{d+1}(x(r_k))) + 2\int_{s_k}^{r_k} R \Phi_d(x)^T dt
= -\int_{s_k}^{r_k} (q(x) + u_0(x)^T R u_0(x)) dt, \tag{27}
\]
and for \(i = 1, 2, \ldots\)
\[
c_i(\Phi_{d+1}(x(s_k)) - \Phi_{d+1}(x(r_k))) + 2\int_{s_k}^{r_k} R \Phi_d(x)^T dt
- 2\int_{s_k}^{r_k} R W_{i-1} \Phi_d(x)^T \Phi_d(x)^T dt
= -\int_{s_k}^{r_k} q(x) dt. \tag{28}
\]
Then, the system of linear equations in (16) readily follows from (27) and (28).

**APPENDIX B**

Here we list options supported by `adpSetModelBased` and `adpSetModelFree` in Table I and Table III respectively.

**ACKNOWLEDGMENT**

This work was conducted by Center for Applied Research in Artificial Intelligence (CARAI) grant funded by Defense Acquisition Program Administration (DAPA) and Agency for Defense Development (ADD) (UD190031RD).

**REFERENCES**

[1] D. E. Kirk, *Optimal Control Theory: An Introduction*. Englewood Cliffs, NJ, USA: Prentice Hall, 1970.
[2] F. L. Lewis, L. V. Rane, and V. L. Symos, *Optimal Control*. Hoboken, NJ, USA: John Wiley & Sons, Inc., 2012.
[3] E. G. Albrecht, “On the optimal stabilization of nonlinear systems,” *J. Appl. Math. Mech.*, vol. 25, no. 5, pp. 1254–1266, 1961.
[4] W. L. Garrard and J. M. Jordan, “Design of nonlinear automatic flight control system,” *Automatica*, vol. 13, no. 5, pp. 497–505, Sep. 1977.
[5] Y. Nishikawa, N. Sanomiya, and H. Itakura, “A method for suboptimal design of nonlinear feedback systems,” *Automatica*, vol. 7, no. 6, pp. 703–712, Nov. 1971.
[6] J. K. Peterson, “On-line estimation of the optimal value function: HJB-estimators,” in *Adv. Neural Inf. Process. Syst. 5* (NIPS), Denver, CO, USA, Nov. 1992, pp. 519–526.
[7] G. N. Saridis and C.-S. G. Lee, “An approximation theory of optimal control for trainable manipulators,” *IEEE Trans. Syst., Man, Cybern.*, vol. 9, no. 3, pp. 152–159, Mar. 1979.
[8] R. W. Beard, G. N. Saridis, and J. T. Wen, “Galarkin approximations of the generalized Hamilton-Jacobi-Bellman equation,” *Automatica*, vol. 33, no. 12, pp. 2159–2177, Dec. 1997.
[9] ——, “Approximate solutions to the time-invariant Hamilton-Jacobi-Bellman equation,” *J. Optim. Theory Appl.*, vol. 96, pp. 589–626, Mar. 1998.
[10] M. Abu-Khalaf and F. L. Lewis, “Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach,” *Automatica*, vol. 41, no. 5, pp. 779–791, May 2005.
[11] R. S. Sutton and A. G. Barto, *Reinforcement Learning: An Introduction*. Cambridge, MA, USA: MIT Press, 1998.
[12] D. L. Vrabie and F. L. Lewis, “Neural network approach to continuous-time direct adaptive optimal control for partially unknown nonlinear systems,” *Neural Netw.*, vol. 22, no. 3, pp. 237–246, Apr. 2009.
[13] Y. Jiang and Z.-P. Jiang, “Robust adaptive dynamic programming and feedback stabilization of nonlinear systems,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 5, pp. 882–893, May 2014.
[14] J. Y. Lee, J. B. Park, and Y. H. Choi, “Integral reinforcement learning for continuous-time input-affine nonlinear systems with simultaneous invariant explorations,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26,
### TABLE III

OPTIONS SUPPORTED BY THE FUNCTION `adpSetModelFree`.

| Name            | Description |
|-----------------|-------------|
| `basisOpt`      | Supported basis functions (`basisOpt`):
| (`mono`)        | monomials (`mono`). |
| `stride`        | The user may specify data to be used in the iteration process by choosing the stride (`stride`) in each trajectory. The number of rows of `stride` automatically matches with `xInitNum` by default. |
| (`1`)           | |
| `crit`          | The stop criterion (`crit`) can be: |
| (`1`)           | |
| `epsilon`       | The number of rows of `stride` automatically matches with `xInitNum` by default. |
| (`0.001`)       | |
| `maxIter`       | The maximum number of iterations (`maxIter`) can also be specified. |
| (`100`)         | |

---

Name (default) Description

`basisOpt` (`mono`) Supported basis functions (`basisOpt`):
- monomials (`mono`).

`stride` (1) The user may specify data to be used in the iteration process by choosing the stride (`stride`) in each trajectory. The number of rows of `stride` automatically matches with `xInitNum` by default.

`crit` (1) The stop criterion (`crit`) can be:
- 0 : $\| c_i - c_{i-1} \| \leq \epsilon$ (the criterion used in [20]),
- 1 : $\| c_i - c_{i-1} \|^2 + \| W_i - W_{i-1} \|^2 \leq \epsilon^2$,
- 2 : $\| c_i - c_{i-1} \| \leq \epsilon$ · $\| c_{i-1} \|$, $\epsilon = 0.001$,
- 3 : $\| c_i - c_{i-1} \|^2 + \| W_i - W_{i-1} \|^2 \leq \epsilon^2$ · $\| c_{i-1} \|^2 + \| W_{i-1} \|^2$. 123–129.

`maxIter` (100) The maximum number of iterations (`maxIter`) can also be specified.

[15] A. J. Krener, “The Nonlinear Systems Toolbox.” [Online]. Available: https://www.math.ucdavis.edu/~krener/

[16] M. Giffithale, M. Neunet, M. Stauble, and J. Buchli, “The Control Toolbox - an open-source C++ library for robotics, optimal and model predictive control,” in 2018 IEEE Int. Conf. Simul., Model., Program. Auton. Robots (SIMPAR), Brisbane, QLD, Australia, May 2018, pp. 123–129.

[17] B. Houska, H. J. Ferreau, and M. Diehl, “ACADO Toolkit - an open source framework for automatic control and dynamic optimization,” Optim. Contr. Appl. Methods, vol. 32, no. 5, pp. 298–312, May 2011.

[18] R. Verschueren et al., “ACADOS - a modular open-source framework for fast embedded optimal control,” 2019. [Online]. Available: https://arxiv.org/abs/1910.13755

[19] M. A. Patterson and A. V. Rao, “GPOPS-II: a MATLAB software for solving multiple-phase optimal control problems using hp-adaptive Gaussian quadrature collocation methods and sparse nonlinear programming,” ACM Trans. Math. Softw., vol. 41, no. 1, Oct. 2014.

[20] Y. Jiang and Z.-P. Jiang, Robust Adaptive Dynamic Programming. Hoboken, NJ, USA: John Wiley & Sons, Inc., 2017.

[21] D. E. Chang, “On controller design for systems on manifolds in Euclidean space,” Int. J. Robust Nonlinear Contr., vol. 28, no. 16, pp. 4981–4998, Nov. 2018.

[22] W. Ko, “A stable embedding technique for control of satellite attitude represented in unit quaternions,” Master’s thesis, Korea Adv. Inst. Science Techn., Republic of Korea, 2020.

[23] W. Ko, K. S. Phogat, N. Petit, and D. E. Chang, “Tracking controller design for satellite attitude under unknown constant disturbance using stable embedding,” Journal of Electrical Engineering & Technology, In Press. DOI: 10.1007/s42835-020-00622-3

[24] K. Gurney, An Introduction to Neural Networks. UCL, London, UK: UCL Press, 1997.

[25] A. L. Caterini and D. E. Chang, Deep Neural Networks in a Mathematical Framework. New York City, NY, USA: Springer, 2018.