KNOTS IN HOMOLOGY SPHERES WHICH HAVE SIMPLE KNOT FLOER HOMOLOGY ARE TRIVIAL

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ABSTRACT. We show that if $K$ is a non-trivial knot inside the homology sphere $X$, then the rank of $\widehat{HF}(X,K)$ is strictly bigger than the rank of $\widehat{HF}(X)$.

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1. INTRODUCTION

The knots inside rational homology spheres with simple knot Floer homology have attracted increasing attention in the past couple of years. Understanding such knots appears in the study of Berge conjecture on the knots admitting a lens space surgery.

By definition, a knot $K$ inside a three-manifold $X$ has simple knot Floer homology if the ranks of $\widehat{HF}(X,\mathbb{Z}/2\mathbb{Z})$ and $\widehat{HF}(X,K;\mathbb{Z}/2\mathbb{Z})$ are equal (c.f. [Hed1, Ras]). If a knot $K \subset S^3$ admits a Lens space surgery $L(p,q) = S^3_p(K)$, we may consider the induced knot $K_p \subset S^3_p(K) = L(p,q)$ generating the

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first homology of this rational homology sphere. If this happens, the knot $K_p$ will either have simple knot Floer homology or $p = 2g(K) - 1$, where $g(K)$ denotes the genus of the knot $K$. In this later case, we will have $\text{rk}(\hat{\text{HF}}(K_p)) = \text{rk}(\hat{\text{HF}}(L(p,q))) + 2 = p + 2$ (see [Hed, Ras]). Note that by Cyclic Surgery Theorem of [CGLS] any Dehn surgery on $K$ yielding a lens space must be integral unless $K$ is a torus knot, and we may thus restrict our attention to integral surgeries. Moreover, the surgery coefficient of an integral surgery resulting in $L(p,q)$ is clearly equal to $p$.

Berge conjecture is then almost reduced to showing that if a knot $K' \subset L(p,q)$ has simple knot Floer homology it is simple, in the sense that it may be represented by a genus 1 doubly pointed Heegaard diagram [Hed, Ras]. This is an example of a situation where we need to understand the topological implications of the assumption that a knot $K$ in a three-manifold $X$ has simple knot Floer homology.

In this paper, we consider the case where $X$ is a homology sphere, and give a complete answer in this situation:

**Theorem 1.1.** If $K$ is a non-trivial knot inside the homology sphere $X$, then $K$ does not have simple knot Floer homology, i.e.

\[ \text{rk}(\hat{\text{HF}}(X,K;\mathbb{Z}/2\mathbb{Z})) > \text{rk}(\hat{\text{HF}}(X;\mathbb{Z}/2\mathbb{Z})). \]

This means that the trivial knot is the only knot inside a homology sphere which has simple knot Floer homology.

The techniques used in this paper are unfortunately limited to integral homology spheres and may not be extended to other (non-homology sphere) three-manifolds. In fact, the theorem, as stated above, is no longer true once we allow rational homology spheres. Namely, we will see in section 5 that surgery on torus knots gives examples of knots with non-zero genus in certain $L$-spaces which have simple knot Floer homology. It is worth mentioning that in fact, the following theorem which will appear in a future paper [Ef4] shows that every example of a knot with simple knot Floer homology which is obtained by surgery on a knot in $S^3$ lives in a $L$-space.

**Theorem 1.2.** If $K \subset X$ is a knot in a homology L-space $X$ (in particular $X$ can be the standard sphere $S^3$) so that the rationally-null homologous knot $L = K_m \subset S^3_m(K) = Y$ obtained by $m$-surgery on $K$ has simple Floer homology, $Y$ is a $L$-space.

We will also discuss the example of the Borromean knot inside $\#^2(S^1 \times S^2)$, which is pretty interesting from the viewpoint of our constructions in this paper. The knot Floer theory of this null-homologous knot behaves very similar, in many aspects, to the knot Floer homology of knots inside integer homology spheres. However, we will describe the more tricky reason why the argument presented in this paper does not work for the Borromean knot.
This paper is organized as follows. In section 2 we will review some background material on knot Floer homology which will be used in this paper. In section 3 an exact sequence for knot Floer homology will be constructed, and in section 4 some surgery formulas for knot Floer homology will be obtained from this exact sequence. This is the central ingredient of the proof of our main result, and the lack of an extension in the case of arbitrary manifolds is probably the main obstruction of generalizing our theorem. In section 5 we will gather all this data to prove our main theorem. Finally in section 6 we will discuss examples of knots inside three-manifolds which are not homology spheres and violate the above conclusion, showing that the above theorem is limited to integer homology spheres.

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2. Knot Floer homology background

2.1. Relative Spin⁶ structures and rationally null-homologous knots. We borrow most part of this subsection from [Ni]. Let $X$ be a rational homology sphere and $K$ be a knot inside $X$, which is not necessarily null-homologous. Consider a tubular neighborhood $\text{nd}(K)$ of $K$ and let $T$ be the torus boundary of this neighborhood. Let $\mu = \mu_K \subset T$ be a meridian of $K$, i.e. $\mu$ bounds a disk in $\text{nd}(K)$, and let $\lambda \subset T$ be a longitude for $K$, which is a curve that is isotopic to $K$ in $\text{nd}(K)$. If $K$ is null-homologous in $X$, we will assume that $\lambda$ is in the kernel of the map

$$\iota_* : H_1(T, \mathbb{Z}) \to H_1(X - \text{nd}(K), \mathbb{Z})$$

induced by the inclusion map $\iota : T = \partial(X - \text{nd}(K)) \to X - \text{nd}(K)$. When $K$ is null-homologous such a curve exists. We may assume that $\lambda$ and $\mu$ intersect each other in a single transverse point. Having fixed these two curves, by $(p,q)$-surgery on $K$ we mean removing $\text{nd}(K)$ and replacing for it a solid torus so that the simple closed curve $p\mu + q\lambda$ bounds a disk in the new solid torus. Denote the resulting three-manifold by $X_{p/q}(K)$. The core of the new solid torus would be an image of $S^1$ in $X_{p/q}(K)$ which will be denoted by $\widetilde{K}_{p/q} \subset X_{p/q}(K)$. Let $\mathbb{H}_{p/q}(K)$ be the Heegaard Floer homology group $\widehat{\text{HFK}}(X_{p/q}(K), \widetilde{K}_{p/q}; \mathbb{Z}/2\mathbb{Z})$ associated with the rationally null homologous knot $K_{p/q}$. Relative Spin⁶ structures on $X - \text{nd}(K)$ which reduce to the translation invariant vector field on the boundary form an affine space $\text{Spin}^c(X, K')$ over $H^2(X, K'; \mathbb{Z})$, and clearly we will have $\text{Spin}^c(X, K') = \text{Spin}^c(X_{p/q}(K), \widetilde{K}_{p/q})$ in a natural way. The group $\mathbb{H}_{p/q}(K)$ is decomposed into subgroups associated with relative Spin⁶
structures:

\[ \mathbb{H}_{p/q}(K) = \bigoplus_{s \in \text{Spin}^c(X,K)} \mathbb{H}_{p/q}(K,s). \]

There is a natural involution

\[ J : \text{Spin}^c(X,K) \rightarrow \text{Spin}^c(X,K) \]

which takes a Spin\(^c\) class \(s\) represented by a nowhere vanishing vector field \(V\) on \(X - \text{nd}(K)\), to the Spin\(^c\) class \(J(s)\) represented by \(-V\). The difference \(s - J(s) \in \mathbb{H}^2(X,K;\mathbb{Z})\) is usually denoted by \(c_1(s)\). There is a symmetry in knot Floer homology of knots inside rational homology spheres which may be described by the following formula

\[ \hat{\text{HF}}(X,K,s) \simeq \hat{\text{HF}}(X,K,J(s) + p\text{PD}[\mu]). \]

Thus we would also have

\[ \mathbb{H}_{p/q}(K,s) \simeq \mathbb{H}_{p/q}(K,J(s) + p\text{PD}[\mu] + q\text{PD}[\lambda]). \]

Suppose that \((\Sigma, \alpha, \beta; u, v)\) is a Heegaard diagram for the knot \(K\), such that \(\beta = \beta_0 \cup \{\mu = \beta_g\}\), \((\Sigma', \alpha, \beta_0)\) is a Heegaard diagram for \(X - \text{nd}(K)\), while \(\mu = \beta_g\) represents the meridian of \(K\) and the two marked points \(u\) and \(v\) are placed on the two sides of \(\beta_g\). Think of the vector space \(\mathbb{B} = \hat{\text{HF}}(X,K;\mathbb{Z}/2\mathbb{Z}) = \mathbb{H}_\infty(K)\) as a vector space computed as \(\mathbb{B} = \hat{\text{HF}}(\Sigma, \alpha, \beta; u, v)\). Letting holomorphic disks pass through the marked point \(v\) in the Heegaard diagram gives a map \(d_\mathbb{B} : \mathbb{B} \rightarrow \mathbb{B}\), which is a filtered differential on the filtered vector space \(\mathbb{B}\), with the filtration induced by Spin\(^c\) structures. To define this filtration we should consider a map

\[ s_{u,v} : T_\alpha \cap T_\beta \rightarrow \text{Spin}^c(X,K) \]

defined in \([N]\).

For \(s, t \in \text{Spin}^c(X,K)\) we will write \(t \geq s\) if

\[ t - s = n\text{PD}[\mu] \in \mathbb{H}^2(X,K;\mathbb{Z}), \quad n \in \mathbb{Z}^\geq 0. \]

Let

\[ \mathbb{B}\{\geq s\} = \bigoplus_{t \in \text{Spin}^c(X,K)} \mathbb{B}(t). \]

Then the subspace \(\mathbb{B}\{\geq s\}\) of \(\mathbb{B}\) is mapped to itself by the differential \(d_\mathbb{B}\). The homology of the complex \((\mathbb{B}, d_\mathbb{B})\) gives \(\hat{\text{HF}}(X;\mathbb{Z}/2\mathbb{Z}) = H_*(\mathbb{B}, d_\mathbb{B})\). In fact, define the relative Spin\(^c\) structures \(t, s \in \text{Spin}^c(X,K)\) equivalent if

\[ t - s = n\text{PD}[\mu] \in \mathbb{H}^2(X,K;\mathbb{Z}), \quad \text{for some } n \in \mathbb{Z}. \]

Denote the equivalence class of \(s\) by \([s]\), and note that the equivalence classes of relative Spin\(^c\) classes in \(\text{Spin}^c(X,K)\) are in correspondence with \(\text{Spin}^c(X)\) if \(K\) is null-homologous. In general, we have the maps

\[ \mathcal{G}_{p/q} : \text{Spin}^c(X,K) \rightarrow \text{Spin}^c(X_{p/q}(K)), \]
obtained by extending the translation invariant vector field on \( \partial(\text{nd}(K_{p/q})) \) to the whole solid torus. The map \( G_{p/q} \) is surjective, and \( G_{1/q} \) is constant on any equivalence class \([s] \subset \text{Spin}^c(X, K)\) for \( q \in \mathbb{Z} \cup \{\infty\}\). In particular, \( G_{\infty}[s] \) is a Spin\(^c\) class in \( \text{Spin}^c(X) \), which will be denoted (abusing the notation) by \([s]\), if there is no confusion.

Let \( \mathbb{B}\{[s]\} \) be the subspace of \( \mathbb{B} \) generated by the generators in relative Spin\(^c\) classes equivalent to \( s \). By the above considerations \( d_{\mathbb{B}} \) reduced to a differential on \( \mathbb{B}\{[s]\} \) and \( \widehat{HF}(X, [s]) = H_*(\mathbb{B}\{[s]\}, d_{\mathbb{B}}) \). In particular, we will focus on the equivalence class of relative Spin\(^c\) structures \( s \in \text{Spin}^c(X, K) \) such that \([c_1(s)] = [s] - [J(s)] \) is equal to \([\text{PD}(\lambda)] \) in \( H^2(X; \mathbb{Z}) \) in section 3.

### 2.2. Knot Floer homology and Seifert genus.

For any Spin\(^c\) structure \( s \) define

\[
\mathfrak{h} : \text{Spin}^c(X, K) \to H^2(X, K; \mathbb{Q}), \quad \mathfrak{h}(s) := \frac{c_1(s) - \text{PD}[\mu]}{2}
\]

\[
\mathfrak{h}_{u,v} : T_\alpha \cap T_\beta \to H^2(X, K; \mathbb{Q}), \quad \mathfrak{h}(x) := \mathfrak{h}(s_{u,v}(x)).
\]

Let \( x, y \in T_\alpha \cap T_\beta \) be a pair of generators. The path \( \epsilon(x, y) \), which is a union of two arcs connecting \( x \) and \( y \) on \( T_\alpha \) and \( T_\beta \) respectively, represents an element in \( H_1(X, \mathbb{Z}) \). Since \( X \) is a rational homology sphere, there is some integer \( k \) such that for a 2-chain \( D \) on \( \Sigma \), \( k\epsilon(x, y) \) is homologous to \( \partial D \), plus a linear combination of \( \alpha \) and \( \beta \) curves. With these assumptions, we will have

\[
\mathfrak{h}_{u,v}(x) - \mathfrak{h}_{u,v}(y) = \frac{n_u(D) - n_v(D)}{k}\text{PD}[\mu] \in H^2(X, K; \mathbb{Q}) = \mathbb{Q}.
\]

This defines a \( \mathbb{Q} \)-grading on \( \mathbb{B} \), which is called the Alexander grading. More precisely, for any class \( j \in H^2(X, K; \mathbb{Q}) = \mathbb{Q} \) we may define

\[
\widehat{CFK}(X, K; j) := \bigoplus_{s \in \text{Spin}^c(X, K), \mathfrak{h}(s) = j} \widehat{CFK}(X, K; s).
\]

The following theorem of Ni [Ni] allows us compute the genus of a knot \( K \subset X \) as above, using Heegaard Floer homology:

**Theorem 2.1.** Suppose that \( h \in H_2(X, K; \mathbb{Q}) \) is an integral class. Then \( h \) may be represented by a properly embedded surface without sphere components. Let \( \chi(h) \) be the maximal possible value of the Euler characteristic of such a surface. Then

\[
-\chi(h) - |h.[\mu]| = \max_{j \in H^2(X, K; \mathbb{Q}) = \mathbb{Q}} \frac{2\langle j, h \rangle}{\widehat{HF}(X, K; j) \neq 0}.
\]

When \( X \) is a homology sphere, the classes \( j \in H^2(X, K; \mathbb{Q}) = \mathbb{Q} \) such that \( \widehat{HF}(X, K; j) \neq 0 \), are all integral classes in \( H^2(X, K; \mathbb{Z}) = \mathbb{Z} \), where this later group is generated by the Poincaré dual of the meridian \( \mu_K \) of
$K$. Furthermore, the correspondence taking $s \in \text{Spin}^c(X, K)$ to $j(s) = (c_1(s) - \text{PD}[\mu])/2 \in H^2(X, K; \mathbb{Z}) = \mathbb{Z}$ gives an identification of Spin$^c$ structures with $\mathbb{Z}$. In this case, the number of direct sum components on the right hand side of equation 7 is either 1 or zero, depending on whether the class $j$ is integral or not.

2.3. Two surgery theorems for Heegaard Floer homology. Suppose that $K$ is a knot inside the homology sphere $X$. Denote the knot Floer complex associated with $K$ by $C = C(K) = CFK^\infty(X, K; \mathbb{Z}/2\mathbb{Z})$. If $C(K)$ is generated by generators $[x, i, j] \in (T_\alpha \cap T_\beta) \times \mathbb{Z} \times \mathbb{Z}$, where $T_\alpha$ and $T_\beta$ are the tori associated with a Heegaard diagram used for the construction of $C(K)$, for any $t \in H^2(X, K; \mathbb{Z}) = \mathbb{Z} = \langle \mu_K \rangle$, let $A_t$ be the free abelian group generated by those generators $[x, i, j]$ such that $\max\{i, j - t\} = 0$ and $j(x) - i + j = 0$. Here $j(x) \in \mathbb{Z}$ is the integer in $\mathbb{Z} = H^2(X, K)$ associated with the relative Spin$^c$ structure $s = s(x)$ via $s \mapsto (c_1(s) - \text{PD}[\mu])/2$. Note that the association $s \mapsto (c_1(s) - \text{PD}[\mu])/2$ gives an identification of Spin$^c(X, K)$ with $\mathbb{Z}$, which will sometimes be implicit in our notation in this paper. Denote the homology of the chain complex $A_t$, with the differential induced from $C$, by $A_t$.

Let $(i, \mathbb{B})$ be a copy of the homology group $\mathbb{B}$ of the complex $B$ generated by the triples $[x, 0, j]$ such that $j(x) + j = 0$ (with the differential induced from $C(K)$). Let

$$\hat{A} = \bigoplus_{s \in \mathbb{Z}} A_s, \quad \hat{\mathbb{B}} = \bigoplus_{s \in \mathbb{Z}} (s, \mathbb{B}),$$

and define $h_n, \nu : \hat{A} \to \hat{\mathbb{B}}$ as the sum of the respective maps

$$h_n^s : A_s \to (s - n, \mathbb{B}), \quad \nu^s : A[s] \to (s, \mathbb{B}).$$

The map $\nu^s$ is defined as the map in homology induced by projecting $A_t = C\{\max\{i, j - t\} = 0\}$ on $C\{i = 0\}$, while $h_n^s$ is defined by first projecting $A_t$ on $C\{j - t = 0\}$, and then using the chain homotopy equivalence of this last complex with $C\{i = 0\}$. Note that $\mathbb{B}$ is the homology of the chain complex $C\{i = 0\}$. Ozsváth and Szabó proved the following theorem in [OS5]:

**Theorem 2.2.** The homology of the mapping cone $\mathcal{M}(d_n)$ of $d_n = h_n + \nu : \hat{A} \to \hat{\mathbb{B}}$ is isomorphic to $\overline{\text{HF}}(X_n(K); \mathbb{Z}/2\mathbb{Z})$ for any integer $n \in \mathbb{Z}$.

The second surgery formula is the combinatorial rational surgery formula from [IZ2]. Let $\mathbb{H}_\bullet(K)$ denote the group $\overline{\text{HF}}(X_\bullet, K_\bullet)$ for $\bullet \in \mathbb{Q} \cup \{\infty\}$, as before. Choose a Heegaard diagram

$$H = (\Sigma, \alpha = \{\alpha_1, ..., \alpha_g\}, \beta_0 = \{\beta_1, ..., \beta_{g-1}\})$$

for $X - K$, and set

$$\beta_\bullet = \{\beta^*_1, ..., \beta^*_{g-1}, \lambda_\bullet\}, \quad \bullet \in \{0, 1, \infty\}$$

where $\beta^*_i$ is an isotopic copy of the curve $\beta_i$. Moreover, $\lambda_\bullet$ denotes an oriented longitude which has framing coefficient $\bullet \in \{0, 1, \infty\}$ (with $\lambda_{\infty} = \mu$
the meridian for \( K \)). One can choose the curves \( \lambda \) so that the pairs \((\lambda_\infty, \lambda_1)\) and \((\lambda_1, \lambda_0)\) have a single intersection point in the Heegaard diagram. Let \((\bullet, \star) \in \{(\infty, 1), (1, 0)\}\) correspond to either of these pairs. There are four quadrants around the intersection point of \( \lambda_\bullet \) and \( \lambda_\star \). If we puncture three of these quadrants and consider the corresponding holomorphic triangle map associated with this triply punctured Heegaard triple, we obtain an induced map \( \mathbb{H}_\bullet \to \mathbb{H}_\star \). More precisely, The triangle map, is constructed from the pointed Heegaard triple \((\Sigma, \alpha, \beta_\bullet, \beta_\star; \text{three punctures})\).

If the punctures are chosen as in figure 1, the result would be two maps \( \phi(K), \phi'(K) : \mathbb{H}_\infty(K) \to \mathbb{H}_1(K) \) and two other maps \( \psi(K), \psi'(K) : \mathbb{H}_1(K) \to \mathbb{H}_0(K) \), such that \( \phi(K) \) and \( \psi(K) \) correspond to the marked points on the right-hand-side of figure 1 and \( \phi'(K) \) and \( \psi'(K) \) correspond to the marked points on the left-hand-side of figure 1. These 4 maps are part of the long exact sequences in homology:

\[
\begin{array}{ccccccccc}
\mathbb{H}_\infty(K) & \phi(K) & \mathbb{H}_1(K) & \psi(K) & \mathbb{H}_0(K) & \mathbb{H}_\infty(K) & \phi(K) & \& \\
\mathbb{H}_\infty(K) & \phi'(K) & \mathbb{H}_1(K) & \psi'(K) & \mathbb{H}_0(K) & \mathbb{H}_\infty(K) & \phi'(K) & .
\end{array}
\]

The homology of the mapping cones of \( \phi(K) \) (or \( \phi'(K) \)) and \( \psi(K) \) (or \( \psi'(K) \)) are \( \mathbb{H}_0(K) \) and \( \mathbb{H}_\infty(K) \), respectively. Let \( \eta(K) = \psi(K) \circ \phi(K) \) and \( \eta(K) = \psi'(K) \circ \phi'(K) \). With the above notation fixed, the following surgery formula is proved in [Ef2]:

**Theorem 2.3.** Let \( K \) be a knot in a homology sphere \( X \) and let the complexes \( \mathbb{H}_\bullet = \mathbb{H}_\bullet(K), \bullet \in \{\infty, 1, 0\} \) and the maps \( \phi(K), \phi'(K), \psi(K) \) and \( \psi'(K) \) between them be as above. The Heegaard Floer homology group of \( X_{p/q}(K) \), the manifold obtained by \( \frac{p}{q} \)-surgery on \( K \) (for a pair of positive integers \( p, q \) with \( (p, q) = 1 \)), may be obtained as the homology of the complex \( \mathbb{E} \) with

\[
\mathbb{E} = \bigoplus_{i=1}^q \mathbb{H}_\infty(i) \oplus \bigoplus_{i=1}^{p-q} \mathbb{H}_1(i) \oplus \bigoplus_{i=1}^p \mathbb{H}_0(i),
\]
where each $\mathbb{H}_i(i)$ is a copy of $\mathbb{H}_i$. Moreover, when $p \geq q$, the differential $d$ is the sum of the following maps

$$
\eta^i : \mathbb{H}_\infty(i) \to \mathbb{H}_0(i+p-q), \quad \psi^i : \mathbb{H}_\infty(i) \to \mathbb{H}_0(i), \quad i = 1, 2, \ldots, q \\
\psi^j : \mathbb{H}_1(j) \to \mathbb{H}_0(j), \quad \psi^j : \mathbb{H}_1(j) \to \mathbb{H}_0(j+q), \quad j = 1, 2, \ldots, p-q,
$$

where $\psi^j$ is the map $\psi(K)$ corresponding to the copy $\mathbb{H}_1(i)$ of $\mathbb{H}_1$, etc.. Whenever $q > p$ the differential $d$ of the complex would be the sum of the following maps

$$
\eta^i : \mathbb{H}_\infty(i) \to \mathbb{H}_0(i+q-p), \quad \psi^i : \mathbb{H}_\infty(i) \to \mathbb{H}_0(i), \quad i = 1, \ldots, p \\
\phi^j : \mathbb{H}_\infty(j) \to \mathbb{H}_1(j), \quad \phi^j : \mathbb{H}_\infty(j+q) \to \mathbb{H}_1(j), \quad j = 1, \ldots, q-p.
$$

**Remark 2.4.** Our notation in [Ef3] and [Ef2] for holomorphic disks connecting two generators $x$ and $y$ corresponding to a Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is different from that of Ozsváth and Szabó in [OS1]. This cooks up Floer cohomology groups rather than Floer homology groups. In order to use surgery formulas of Ozsváth and Szabó, some minor modification in the statement of the results from [OS5] is thus needed. This should justify the difference between the statement of theorem 2.2 and theorem 1.1 from [OS5] (otherwise, we should have defined the map $h^s_n$ from $A_n$ to $(s + n, \mathbb{B})$ rather than $(s - n, \mathbb{B})$).

3. AN EXACT SEQUENCE FOR KNOT FLOER HOMOLOGY

As mentioned in the introduction, formulating a similar theorem for knots (not necessarily null-homologous) inside rational homology spheres is very interesting and is related to Berge conjecture. Yet, the techniques used in this paper are limited to homology spheres, to some extent. In order to make these limitations more clear, we choose to state the central construction of this paper in the context of rationally null-homologous knots.

3.1. The short exact sequence. Consider a Heegaard diagram for the pair $(X, K)$. Suppose that the curve $\mu = \beta_g$ in the Heegaard diagram

$$
H = (\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_g\}, \beta = \{\beta_1, \ldots, \beta_g\}, p)
$$

corresponds to the meridian of $K$ and that the marked point $p$ is placed on $\beta_g$. One may assume that the curve $\beta_g$ cuts $\alpha_g$ once and that $\alpha_g$ is the only element of $\alpha$ that has an intersection point with $\beta_g$. If we put a pair of marked points on the two sides of the meridian $\mu$, we obtain a doubly pointed Heegaard diagram for $K$. Counting disks which miss these two marked points (punctures) gives the Heegaard Floer homology group

$$
\mathbb{B} = \mathbb{H}_\infty(K) = \widehat{HF}(X, K; \mathbb{Z}/2\mathbb{Z}).
$$

As before, if we allow the disks to pass through one of the marked points, we obtain an induced map $d_2 : \mathbb{B} \to \mathbb{B}$, with the property that $d_2 \circ d_2 = 0$. Moreover, the homology group $H_*(\mathbb{B}, d_2)$ is the same as $\mathbb{H}(X) = \widehat{HF}(X; \mathbb{Z}/2\mathbb{Z})$. The induced differential $d_2 : \mathbb{B} \to \mathbb{B}$ respects the Spin$^c$ filtration of $\mathbb{B}$, and we thus have sub-complexes $\mathbb{B}\{\geq s\}$
and quotient complexes $B\{\leq s\}$ of $(B, d_B)$ for any relative Spin$^c$ structure $s \in \text{Spin}^c(X, K)$. The homology of these chain complexes will be denoted by $\mathbb{H}\{\geq s\}$ and $\mathbb{H}\{\leq s\}$ respectively.

Suppose that $\lambda$ represents a preferred longitude for the knot $K$ (i.e. it cuts $\beta_g$ once and stays disjoint from other elements of $\beta$). Let $\lambda_n$ be the curve obtained from $\lambda$ by winding it $n$ time around the meridian $\mu$ without creating any new intersection points with the curves in $\beta_0 = \{\beta_1, ..., \beta_{g-1}\}$. The Heegaard diagram

$$H_n = (\Sigma, \alpha, \beta_n = \{\beta_1, ..., \beta_{g-1}, \lambda_n\}, p_n)$$

would give a diagram associated with the rationally null-homologous knot $(X_n(K), K_n)$, where $p_n$ is a marked point at the intersection of $\lambda_n$ and $\beta_g$. The curve $\lambda_n$ intersects the $\alpha$-curve $\alpha_g$ in $(|n| + 2)$ points which appear in the winding region (there may be other intersections outside the winding region). Denote these points of intersection by

$$..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...,$$

where $x_0$ is the intersection point with the property that three of its four neighboring quadrants belong to the regions that contain $p_n$ as a corner.

Any generator which is supported in the winding region is (by definition) of the form

$$\{x_i\} \cup y_0 = \{y_1, ..., y_{g-1}, x_i\},$$

and is in correspondence with the generator

$$y = \{x\} \cup y_0 = \{y_1, ..., y_{g-1}, x\}$$

for the complex associated with the knot $(X, K)$, where $x$ denotes the unique intersection point of $\alpha_g$ and $\beta_g$. Denote the former generator by $(y)_i$, keeping track of the intersection point $x_i$ among those in the winding region.

Choose an intersection point between the curves $\lambda_n$ and $\lambda_{m+n}$ in the middle of the winding region, denoted by $q$, where $m$ is a negative integer with large absolute value. From the 4 quadrants around the intersection point $q$, two of them are parts of small triangles $\Delta_0$ and $\Delta_1$ between $\alpha, \beta_n$ and $\beta_{m+n}$. We may assume that the intersection points between $\alpha_g$ and $\lambda_{m+n}$ in the winding region are

$$..., y_{-2}, y_{-1}, y_0, y_1, y_2, ...,$$

We may also assume that the domain $\Delta_i$ for $i = 0, 1$ is the triangle with vertices $q, x_i$ and $y_i$, and $\Delta_1$ is one of the connected domains in the complement of curves $\Sigma - \alpha - \beta_n - \beta_{m+n}$. Other that $\Delta_0$ and $\Delta_1$ there are two other domains which have $q$ as a corner. One of them is on the right-hand-side of both $\lambda_n$ and $\lambda_{m+n}$, denoted by $D_1$, and the other one is on the left-hand-side of both of them, denoted by $D_2$. The domains $D_1$ and $D_2$ are assumed to be connected regions in the complement of the curves $\Sigma \setminus C$. We may assume that the meridian $\mu$ passes through the regions $D_1, D_2$ and $\Delta_0$, cutting each one of them into two parts:
\[ \Delta_0 = \Delta_0^R \cup \Delta_0^L, \quad D_1 = D_1^R \cup D_1^L, \quad \text{and} \quad D_2 = D_2^R \cup D_2^L. \]

Here \( \Delta_0^R \subset \Delta_0 \) is the part on the right-hand-side of \( \mu \) and \( \Delta_0^L \) is the part on the left-hand-side. Similarly for the other regions \( \bullet, \bullet^R \) is the part of \( \bullet \) on the right-hand-side of \( \mu \) and \( \bullet^L \) is the part of \( \bullet \) on the left-hand-side of \( \mu \).

Choose the marked points \( u, v, w \) and \( z \) so that \( u \) is in \( D_1^R \), \( v \) is in \( D_1^R \), \( w \) is in \( D_2^L \), and \( z \) is in \( D_1^L \) (see figure 2).

Furthermore, choose the marked points \( p \in \Delta_1 \) and \( p' \in \Delta_0^L \) for later use. We thus obtain a Heegaard diagram with 6 marked points \( R_{m,n} = (\Sigma, \alpha, \beta, \beta_{m+n}; u, v, w, z, p, p') \).

Fix a relative Spin\(^c \) class \( s \in \text{Spin}^c(X_n(K), K_n) = \text{Spin}^c(X, K) \) which satisfies \( [s] = [J(s) + PD[\lambda]] \) in \( \text{Spin}^c(X) \) (or equivalently, \( [c_1(s) - PD[\lambda]] = 0 \) in \( H^2(X; \mathbb{Z}) \)).

The first complex we would like to consider is the complex \( \widehat{C\ell}(\Sigma, \alpha, \beta; u, v) \), obtained by forming the hat Heegaard Floer complex associated with the doubly punctured Heegaard diagram \( R_\infty = (\Sigma, \alpha, \beta; u, v) \).

This complex may clearly be identified as \( \widehat{C\ell}(X, \mathbb{Z}/2\mathbb{Z}) \) since both punctures \( u \) and \( v \) are in the same connected component of \( \Sigma - \alpha - \beta \). If we restrict our attention to the generators associated with the Spin\(^c \) structure \( [s] \in \text{Spin}^c(X) \), we obtain a sub-complex

\[ \widehat{C\ell}(X; [s]) = \widehat{C\ell}(\Sigma, \alpha, \beta; u, v; [s]) \subset \widehat{C\ell}(\Sigma, \alpha, \beta; u, v). \]

The second complex which may be associated with the Heegaard diagram \( R_n = (\Sigma, \alpha, \beta_n; u, v) \) is obtained by looking at the sub-complex

\[ \widehat{CFK}(X_n(K), K_n; s) \subset \widehat{CFK}(X_n(K), K_n) = \widehat{C\ell}(R_n). \]

Finally, the last complex would be the sub-complex

\[ \widehat{CFK}(X_{m+n}(K), K_{m+n}, s) \oplus \widehat{CFK}(X_{m+n}(K), K_{m+n}, s + m.PD[\mu]) \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The Heegaard diagram \( R_{m,n} \). The shaded triangles are \( \Delta_0 \) and \( \Delta_1 \). The marked points \( u, v, w \) and \( z \) are placed in \( D_1^R, D_2^R, D_2^L \) and \( D_1^L \) respectively.}
\end{figure}
of the chain complex \(\widehat{CFK}(X_{m+n}(K), K_{m+n}) = \widehat{CF}(R_{m+n})\), where \(R_{m+n} = (\Sigma, \alpha, \beta_{m+n}; u, v)\). We will describe a triangle of chain maps

\[
\begin{array}{ccc}
\widehat{CF}(X; [s]) & \xrightarrow{h^s} & \widehat{CF}(K_n; s) \\
\downarrow f^s & & \downarrow \delta \\
\widehat{CF}(K_{m+n}; s) & \oplus & \widehat{CF}(K_{m+n}; s + mPD[\mu])
\end{array}
\]

such that the compositions \(g^s \circ f^s, h^s \circ g^s\) and \(f^s \circ h^s\) are chain homotopic to zero and the associated long sequence in homology is an exact triangle.

To define the chain map \(f^s\), consider the holomorphic triangle map

\[
(10) \quad \Phi_f^s : \widehat{CF}(X_{m+n}, K_{m+n}; s) \otimes \widehat{CF}(\Sigma, \beta_{m+n}, \beta; u, v; s_0) \to \widehat{CF}(\Sigma, \alpha, \beta; u, v; [s]) = \widehat{CF}(X; [s]).
\]

We define the map \(\Phi_f^s\) on a generator \(x \otimes y\), with \(y\) a generator of the chain complex \(\widehat{CF}(\Sigma, \beta_{m+n}, \beta; u, v) = \widehat{CF}(\#_{g-1}S^1 \times S^2)\) with associated Spin\(^c\) class \(s(y) = s_0\) satisfying \(c_1(s_0) = 0\), as follows:

\[
(11) \quad \Phi_f^s(x \otimes y) := \sum_{z \in V_{\alpha} \cap \beta} \sum_{\Delta \in \pi_2(x, y, z), \mu(\Delta) = 0} \#(\hat{M}(\Delta)).z.
\]

The top generator \(\Theta_f \in \widehat{CF}(\Sigma, \beta_{m+n}, \beta; u, v; s_0) = \widehat{HF}(\#_{g-1}S^1 \times S^2; s_0)\) gives the map \(f^s\) which is defined by

\[
(13) \quad f^s : \widehat{CF}(K_{m+n}; s) \oplus \widehat{CF}(K_{m+n}; s + mPD[\mu]) \to \widehat{CF}(X; [s])
\]

\[
(14) \quad f^s(x \otimes y) = \Phi_f^s(x \otimes \Theta_f).
\]

Similarly we may define a map

\[
\Psi_f^s : \widehat{CF}(X_{m+n}, K_{m+n}; s + mPD[\mu]) \otimes \widehat{CF}(\Sigma, \beta_{m+n}, \beta; w, z; s_0) \to \widehat{CF}(\Sigma, \alpha, \beta; w, z; [s]) = \widehat{CF}(X; [s])
\]

\[
(15) \quad \Psi_f^s(x \otimes y) := \sum_{z \in V_{\alpha} \cap \beta} \sum_{\Delta \in \pi_2(x, y, z), \mu(\Delta) = 0} \#(\hat{M}(\Delta)).z.
\]

The top generator \(\Theta'_f \in \widehat{CF}(\Sigma, \beta_{m+n}, \beta; w, z; s_0) = \widehat{HF}(\#_{g-1}S^1 \times S^2; s_0)\) gives the map \(f^s\) which is defined by

\[
(13) \quad f^s : \widehat{CF}(K_{m+n}; s) \oplus \widehat{CF}(K_{m+n}; s + mPD[\mu]) \to \widehat{CF}(X; [s])
\]

\[
(15) \quad f^s(x \otimes y) = \Xi(\Psi_f^s(z')(y) \otimes \Theta'_f).
\]

Here the duality map

\[
\Xi' : \widehat{CF}(K_{m+n}; s + mPD[\mu]) \to \widehat{CF}(K_{m+n}; J(s + mPD[\mu]) + PD[\lambda_{m+n}])
\]

\[
= \widehat{CF}(K_{m+n}; J(s) + PD[\lambda_n])
\]
is obtained by changing the role of the punctures \( w \) and \( z \), and the duality map \( \Xi : \widehat{\mathcal{CF}}(X; [J(s) + PD[\lambda_n]] = [g]) \rightarrow \widehat{\mathcal{CF}}(X; [g]) \) is the chain equivalence map obtained by interchanging the roles of the marked points \( v \) and \( w \).

The punctured Heegaard diagram \((\Sigma, \alpha, \beta_n, \beta_{m+n}; u, v, p, p')\) gives a pair of holomorphic triangle maps, which are defined as follows. Let \( s_0 \) denote the canonical Spin\(^c\) structure of \( L(m, n)\#(\#^{g-1}S^1 \times S^2)\), in the sense of definition 3.2 of [OS5], and define a pair of maps

\[
\Phi^g_s : \widehat{\mathcal{CF}}(X_n, K_n; s) \otimes \widehat{\mathcal{CF}}(L(m,n)\#(\#^{g-1}S^1 \times S^2); s_0) \\
\rightarrow \widehat{\mathcal{CF}}(X_{m+n}, K_{m+n}; s), \&
\]

\[
\Psi^g_s : \widehat{\mathcal{CF}}(X_n, K_n; s) \otimes \widehat{\mathcal{CF}}(L(m,n)\#(\#^{g-1}S^1 \times S^2); s_0) \\
\rightarrow \widehat{\mathcal{CF}}(X_{m+n}, K_{m+n}; s + mPD[\mu]).
\]

These two maps are defined by the following equations

\[
\Phi^g_s(x \otimes y) := \sum_{z \in T_n \cap T_{m+n}} \sum_{\Delta \in \pi_2(x,y,z), \mu(\Delta) = 0} \#(\widehat{\mathcal{M}}(\Delta)).z, \&
\]

\[
\Psi^g_s(x \otimes y) := \sum_{z \in T_n \cap T_{m+n}} \sum_{\Delta \in \pi_2(x,y,z), \mu(\Delta) = 0} \#(\widehat{\mathcal{M}}(\Delta)).z.
\]

The conditions \( n_{p'} = 0 \) for \( \Phi^g_s \) and \( n_{p'} = 0 \) for \( \Psi^g_s \) guarantee that \( \Phi^g_s(x \otimes y) \) is in \( \widehat{\mathcal{CF}}(K_{m+n}; s) \), and \( \Psi^g_s(x \otimes y) \) is in \( \widehat{\mathcal{CF}}(K_{m+n}; s + \mu PD[\mu]) \) if \( s(x) = s \).

The top generator \( \Theta^g_s \in \widehat{HF}(L(m,n)\#(\#^{g-1}S^1 \times S^2); s_0) \) determined by the particular intersection point \( q \) of \( \lambda_n \) and \( \lambda_{m+n} \) and the top intersection points of other pairs of \( \beta \) curves, gives a map

\[
g^s : \widehat{\mathcal{CF}}(X_n(K), K_n; s) \rightarrow \widehat{\mathcal{CF}}(X_{m+n}, K_{m+n}; s) \oplus \widehat{\mathcal{CF}}(X_{m+n}, K_{m+n}; s + mPD[\mu]),
\]

\[
g^s(x) := (\Phi^g_s(x \otimes \Theta^g_s), \Psi^g_s(x \otimes \Theta^g_s)).
\]

Finally, the triple Heegaard diagram \((\Sigma, \alpha, \beta, \beta_n; u, v, w, z)\) determines a holomorphic triangle map

\[
\Phi^g_h : \widehat{\mathcal{CF}}(X; [g]) \otimes \widehat{\mathcal{CF}}(\#^{g-1}S^1 \times S^2; s_0) \rightarrow \widehat{\mathcal{CF}}(X_n(K), K_n; s),
\]

defined by counting the following types of holomorphic triangles:

\[
\Phi^g_h(x \otimes y) := \sum_{z \in T_n \cap T_{m+n}} \sum_{\Delta \in \pi_2(x,y,z), \mu(\Delta) = 0} \#(\widehat{\mathcal{M}}(\Delta)).z.
\]
If \( \Theta_h \) is the top generator of \( \widehat{\text{HF}}(\#^g - S^1 \times S^2; s_0) \), we may define the chain map
\[
h^\# : \widehat{\text{CF}}(X; [s]) \rightarrow \widehat{\text{CFK}}(X_n(K), K_n; s)
\]
by \( h^\#(x) = \Phi^\#_h(x \otimes \Theta_h) \).

Straight forward arguments in Heegaard Floer homology (c.f. section 7 of [OS1]) may be used to show the following proposition:

**Proposition 3.1.** The maps \( \Phi^\#_f, \Psi^\#_f, \Phi^\#_g, \Psi^\#_g \) and \( \Phi^\#_h \) as defined above are all chain maps. Thus \( f^\#, g^\# \) and \( h^\# \) are chain maps as well.

### 3.2. Exactness of the long sequence.

The maps defined in the previous subsection give an exact triangle in homology:

**Theorem 3.2.** Suppose that \( K \subset X \) is a framed knot in a rational homology sphere \( X \) with preferred framing \( \lambda \subset \partial(\text{nd}(K)) \) fixed. The maps in homology induced by the chain maps of the triangle in equation (21) form an exact triangle for each relative Spin\(^c\) class \( s \in \text{Spin}^c(X, K) \) with the property that \([c_1(s) - \text{PD}[\lambda]] = 0 \) in \( H^2(X; \mathbb{Z}) \), which looks like

\[
\begin{array}{ccc}
\widehat{\text{HF}}(X; [s]) & \xrightarrow{h^\#} & \widehat{\text{HFK}}(K_n; s) \\
& \Big\downarrow \Phi^\#_s & \\
\text{HFK}(K_{m+n}; s) \oplus \text{HFK}(K_{m+n}; s + m\text{PD}[\mu]) & \xrightarrow{\Phi^\#_s} & \\
\end{array}
\]

if the negative integer \( m \) is a large in absolute value. Moreover, the compositions \( f^\# \circ g^\# \) and \( h^\# \circ f^\# \) are chain homotopic to zero, and thus the complex \( \widehat{\text{CFK}}(K_n, s) \) is quasi-isomorphic to the mapping cone of \( f^\# \).

**Proof.** We first show that the compositions \( f^\# \circ g^\#, g^\# \circ h^\# \) and \( h^\# \circ f^\# \) are chain homotopic to zero. The most complicated vanishing is the first one which we will do in more details. The other two claims are almost straight forward following the existing techniques in the literature. For the first composition, define the homotopy map \( H^\#_h \) by

\[
H^\#_h(x) = \sum_{y \in \pi_2(x, \Theta^\#_h, \Theta_f, y)} \#(\mathcal{M}(\square), y).
\]

If \( y \) is a generator for \( \widehat{\text{CF}}(X; [s]) \) and \( x \) is a generator for \( \widehat{\text{CFK}}(X_n(K), K_n; s) \), and if \( \square \in \pi_2(x, \Theta^\#_h, \Theta_f, y) \) is a square with \( \mu(\square) = 0 \) such that \( n_u(\square) = n_v(\square) = n_p(\square) = 0 \), we may consider the moduli space \( \mathcal{M}(\square) \), which is a smooth 1-dimensional manifold with boundary. The boundary points of this moduli correspond to different types of degenerations of \( \square \). Four of these
Let us denote the top generator in the canonical Spin$^c$erations thus correspond to the coefficient of $\Psi$ in $d(H^p_h(x)) + H^p_h(d(x))$. Then we have a degeneration of $\square$ as $\Delta \ast \Delta'$ with $\Delta \in \pi_2(x, z, y)$ and $\Delta' \in \pi_2(z, \Theta', \Theta_f)$ for some $z \in T_{\beta_n} \cap T_{\beta}$ satisfying

$$\mu(\Delta) = \mu(\Delta') = 0, \quad \&$$
$$n_u(\Delta) = n_v(\Delta) = n_p(\Delta) = n_p(\Delta') = n_u(\Delta') = n_v(\Delta') = 0.$$ 

Such degenerations correspond to the coefficient of $y$ in the expression $\Psi_{h,1}(x \times \Phi_1(\Theta \otimes \Theta_f))$, where the holomorphic triangle maps $\Psi_{h,1}$ and $\Phi_1$ are defined by

$$\Psi_{h,1}^s(X_n(K, K_n; s) \otimes \CF(\Sigma; \beta_n, \beta; u, v; s_0) \rightarrow \CF(X; [s])$$

$$\Phi_1 : \CF(\Sigma; \beta_n, \beta_{m+n}; u, v) \otimes \CF(\Sigma; \beta_{m+n}, \beta; u, v) \rightarrow \CF(\Sigma; \beta_n, \beta; u, v, p)$$

Let us denote the top generator in the canonical Spin$^c$ class of the homology group $\hat{HF}(\Sigma; \beta_n, \beta; u, v; s_0) = \hat{HF}(\#^g-1s^1 \times S^2; s_0)$ by $\hat{\Theta}_h$. The Heegaard triple $(\Sigma, \beta_n, \beta_{m+n}, \beta; u, v, p)$ is a very standard Heegaard diagram and thus the following lemma may be proved without too much difficulty (compare with the proof of theorem 3.1 from [OSS]):

**Lemma 3.3.** With the above notations fixed, we have $\Phi_1(\Theta \otimes \Theta_f) = \hat{\Theta}_h$.

Such degenerations thus correspond to the coefficient of $y$ in $\Psi_{h,1}^s(x \times \hat{\Theta}_h)$. Finally, we have the degeneration of $\square$ as $\Delta \ast \Delta'$ with $\Delta \in \pi_2(x, \Theta \otimes \Theta_f, z)$ and $\Delta' \in \pi_2(z, \Theta_f, y)$ for some $z \in T_{\beta_n} \cap T_{\beta_{m+n}}$ satisfying

$$\mu(\Delta) = \mu(\Delta') = 0, \quad \&$$
$$n_u(\Delta) = n_v(\Delta) = n_p(\Delta) = n_p(\Delta') = n_u(\Delta') = n_v(\Delta') = 0.$$ 

Automatically, the relative Spin$^c$ class associated with $z$ is $s$. Such degenerations thus correspond to the coefficient of $y$ in $f_1^s(g_1^s(x))$. Gathering all this data we conclude that the following relation is satisfied modulo 2.

$$H^p_h \circ d + d \circ H^p_h(x) + \Psi_{h,1}(x \times \hat{\Theta}_h) = (f_1^s \circ g_1^s)(x).$$  (23)
Similarly, we may define the homotopy map \( K_h^s \) by

\[
K_h^s : \CFK(X_n(K), K_n; s) \longrightarrow \CF(\Sigma, \alpha, \beta; w, z; [s]) = \CF(X; [s])
\]

(24) \( K_h^s(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\square \in \pi_2(x, \Theta_\beta, \Theta_f, y)} \#(\mathcal{M}(\square)) \cdot y. \)

If \( y \) is a generator for \( \CF(X; [s]) \) and \( x \) is a generator for \( \CFK(X_n(K), K_n; s) \), and if \( \square \in \pi_2(x, \Theta_\beta, \Theta_f, y) \) is a square with \( \mu(\square) = 0 \) such that \( n_w(\square) = n_z(\square) = n_{p'}(\square) = 0 \), the moduli space \( \mathcal{M}(\square) \) is again a smooth 1-dimensional manifold with boundary. Since \( \Theta_f \) and \( \Theta_\beta^s \) are closed elements, the number of points in the boundary of this moduli space is equal to the coefficient of \( y \) in \( d(K_h^s(x)) + K_h^s(d(x)) \), plus the number of boundary points corresponding to the degenerations of \( \square \) into two holomorphic triangles. The first type of these later degenerations is a degeneration of \( \square \) as \( \Delta \ast \Delta' \) with \( \Delta \in \pi_2(x, z, y) \) and \( \Delta' \in \pi_2(z, \Theta_\beta^s, \Theta_f) \) for some \( z \in T_\alpha \cap T_\beta \) satisfying

\[
\mu(\Delta) = \mu(\Delta') = 0, \quad \&
\]

\[
n_w(\Delta) = n_z(\Delta) = n_{p'}(\Delta) = n_w(\Delta') = n_z(\Delta') = 0.
\]

Such degenerations correspond to the coefficient of \( y \) in \( \Psi_{h,2}^s(x \otimes \Phi_2(\Theta_\beta^s \otimes \Theta_f)) \), where the maps \( \Psi_{h,2}^s \) and \( \Phi_2 \) are defined by

\[
\Psi_{h,2}^s : \CFK(X_n(K), K_n; s) \otimes \CF(\Sigma, \beta_n, \beta; w, z; s_0) \longrightarrow \CF(X; [s])
\]

\[
\Psi_{h,2}^s(x \otimes y) = \sum_{z \in T_\alpha \cap T_\beta} \sum_{\Delta \in \pi_2(x, z, y), \mu(\Delta) = 0, [z(x)] = [s]} \#(\mathcal{M}(\Delta)) \cdot z, \quad \&
\]

\[
\Phi_2 : \CF(\Sigma, \beta_n, \beta_{m+n}; w, z, p') \otimes \CF(\Sigma, \beta_{m+n}, \beta; w, z, p') \longrightarrow \CF(\Sigma, \beta_n, \beta; w, z, p')
\]

\[
\Phi_2(x \otimes y) = \sum_{z \in T_\beta \cap T_\beta} \sum_{\Delta \in \pi_2(x, y, z), \mu(\Delta) = 0, n_w(\Delta) = n_{z(\Delta)} = n_{p'}(\Delta)} \#(\mathcal{M}(\Delta)) \cdot z.
\]

The other type of such degenerations is a degeneration of \( \square \) as \( \Delta \ast \Delta' \) with \( \Delta \in \pi_2(x, \Theta_\beta^s, z) \) and \( \Delta' \in \pi_2(z, \Theta_f, y) \) for some \( z \in T_\alpha \cap T_\beta \) satisfying

\[
\mu(\Delta) = \mu(\Delta') = 0, \quad \&
\]

\[
n_w(\Delta) = n_z(\Delta) = n_{p'}(\Delta) = n_w(\Delta') = n_z(\Delta') = 0.
\]

Automatically, the relative \( \text{Spin}^c \) class associated with \( z \) is \( s + m \text{PD} [\mu] \). Interchanging the role of the marked points \( z \) and \( w \) changes the relative \( \text{Spin}^c \) class associated with \( z \) to \( J(s) + m \text{PD} [\lambda_m] \). Such degenerations thus correspond to the coefficient of \( y \) in \( \Xi^{-1}(f_2^s(\Psi_2^s(x))) \).

Gathering all this data we conclude that the following relation is satisfied
modulo 2.

\begin{equation}
(\Xi \circ K_h^2) \circ d + d \circ (\Xi \circ K_h^2)(x) + \Xi \circ \Psi_{h,2}(x \otimes \widehat{\Theta}_h) = (f^2 \circ g^2)(x).
\end{equation}

Combining equations 23 and 25 we obtain the following equation modulo 2:

\begin{equation}
(\Xi \circ K_h^2 + H_h^2) \circ d + d \circ (\Xi \circ K_h^2 + H_h^2) + f^s \circ g^s
\end{equation}

\[= (\Psi_{h,1}^s + \Xi \circ \Psi_{h,2}^s)(\cdot \otimes \widehat{\Theta}_h).\]

Note that \(\Xi \circ \Psi_{h,2}^s, \cdot \otimes \widehat{\Theta}_h\) and \(\Psi_{h,1}^s, \cdot \otimes \widehat{\Theta}_h\) are chain homotopic maps from \(\widehat{\text{CFK}}(X_n(K), K_n; s)\) to \(\widehat{\text{CF}}(X; [s])\), which induce the same map in homology. In fact, we have to show that under the chain equivalence map

\begin{equation}
\Xi : \widehat{\text{CF}}(X; [s]) = \widehat{\text{CF}}(\Sigma, \alpha, \beta; w, z; [s])
\end{equation}

\[\rightarrow \widehat{\text{CF}}(X; [s]) = \widehat{\text{CF}}(\Sigma, \alpha, \beta; u, v; [s]),\]

the chain maps \(\Psi_{h,1}^s, \cdot \otimes \widehat{\Theta}_h\) and \(\Psi_{h,2}^s, \cdot \otimes \widehat{\Theta}_h\) induce chain equivalent maps from \(\widehat{\text{CFK}}(X_n(K), K_n; s)\) to \(\widehat{\text{CF}}(X; [s])\). In order to see this, we should note that instead of moving \(u\) and \(v\) to the other side of \(\mu\), we may handle-slide \(\mu\) over \(\beta_1, \ldots, \beta_{g-1}\) one by one, and obtain a sequence of Heegaard triple \(H_i = (\Sigma, \alpha, \beta_n, \beta^i; u, v)\), where \(\beta^i = (\beta - \mu) \cup \mu^i\) and \(\mu^i\) is obtained from \(\mu^{i-1}\) by a pair of handle-slides over \(\beta_{i-1}\) (with opposite orientations) as suggested by figure 8 (we set \(\mu^1 = \mu\)). Let \(\Psi_{h}^s(\cdot \otimes \widehat{\Theta}_h)\) be the chain map constructed using the Heegaard diagram \(H_i\). It is then clear that under the natural identification of \(\widehat{\text{CF}}(\Sigma, \alpha, \beta^i; u, v)\) with \(\widehat{\text{CF}}(X)\) by chain equivalence maps, the maps \(\Psi_{h}^s(\cdot \otimes \widehat{\Theta}_h)\) are chain equivalent, since the triples are obtained from each-other by handle-slides which are supported away from the marked points. Our claim is thus proved once we note that \(H_g\) may be identified, after an isotopy of curves supported away from the marked points, with \((\Sigma, \alpha, \beta; w, z)\) (with \(w\) and \(z\) in the same connected domain).

Thus the right-hand-side of equation 26 may be written as \(H^s \circ d + d \circ H^s\) for a homotopy map

\[H^s : \widehat{\text{CFK}}(X_n(K), K_n; s) \rightarrow \widehat{\text{CF}}(X; [s]).\]

We have thus proved that \(g^s \circ f^s\) is chain homotopic to zero, where the homotopy is given by the map \(Y_h^s := \Xi \circ K_h^2 + H_h^2 + H^s\). Note that the above argument uses the fact that in the process of going from one Heegaard diagram to another the Heegaard triples remain admissible. This part is thus limited to rational homology spheres.

Similarly, showing that \(f_i^s \circ h^s\) and \(h^s \circ g_i^s\) are chain homotopic to the trivial map for \(i = 1, 2\) is reduced to showing that the following expressions are zero for any fixed \(z\) in \(T_\beta \cap T_{\beta m+n}, T_\beta \cap T_{\beta m+n}, T_{\beta m+n} \cap T_\beta, \) and \(T_{\beta m+n} \cap T_{\beta n}\)
Figure 3. Moving the marked points to the other side of the meridian \( \mu = \mu^1 \) in the Heegaard triple \( H_1 = (\Sigma, \alpha, \beta_n, \beta = \beta^1; u, v) \) may be replaced by a series of handle slides of \( \mu \in \beta = \{ \beta_1, \ldots, \beta_{g-1}, \beta_g = \mu \} \) over \( \beta_1, \ldots, \beta_{g-1} \), to obtain the Heegaard triples \( H_i = (\Sigma, \alpha, \beta_n, \beta^i; u, v) \), where \( \beta^i = \{ \beta_1, \ldots, \beta_{g-1}, \mu^i \} \). The curve \( \mu^i \) is obtained by handle-sliding \( \mu^{i-1} \) over \( \beta_{i-1} \) twice, as indicated in the picture.

respectively:

\[
\begin{align*}
\sum_{\Delta \in \pi_2(\Theta_f, \Theta_h, x)} & \#(M(\Delta)), \quad \sum_{\Delta \in \pi_2(\Theta_f, \Theta_h, x)} \#(M(\Delta)), \\
\sum_{\Delta \in \pi_2(\Theta_f, \Theta_h, x)} & \#(M(\Delta)), \quad \sum_{\Delta \in \pi_2(\Theta_f, \Theta_h, x)} \#(M(\Delta)).
\end{align*}
\]

Since the triangles contributing to the above sums come in canceling pairs, and in fact the corresponding Heegaard triples are pretty standard, the above claims follow (cf. theorem 3.1 from [OS5]). Thus we conclude that
$h^s \circ f^s$ is chain homotopic to zero via a chain homotopy map $\Upsilon^s_g$, and $g^s \circ h^s$ is chain homotopic to zero via a chain homotopy map $\Upsilon^f_h$.

The rest of the argument is almost standard. To show the exactness of the sequence in homology, we may change the curve $\lambda_{m+n}$ by an isotopy, without changing the homology groups and the maps between them. Let us change $\lambda_{m+n}$ so that it gets very close to the juxtaposition of the curves $\lambda_n \ast m \mu$. Choose the curve $\lambda_{m+n}$ so that it cuts $\lambda_n$ near the unique intersection of $\lambda_n$ with $\mu$, winds $m$ times around $\mu$ and then cuts it in a single point, then travels parallel to $\lambda_n$ and very close to it. Any intersection point on $\lambda_{m+n}$ with the curves in $\alpha$ is thus in correspondence either with an intersection point of $\lambda_n$ with the curves in $\alpha$, or an intersection point of $\mu$ with $\alpha$. In fact, associated with any intersection point of $\mu$ with $\alpha$, there are $m$ points of intersection on $\lambda_{m+n}$. For the fixed Spin$^c$ structure $s \in \text{Spin}^c(X, K)$, precisely one of these intersection points generates a generator in Spin$^c$ class $s + m \text{PD}[\mu] \in \text{Spin}^c(X, K)$, if $m$ is large enough in absolute value. There is a small area triangle connecting this generator to the corresponding generator of $\widehat{\text{CF}}(\Sigma, \alpha, \beta; u, v; [s]) = \widehat{\text{CF}}(X; [s])$. Similarly, associated with any generator of $\widehat{\text{CF}}(\Sigma, \alpha, \beta; u, v)$ there is a generator of $\widehat{\text{CF}}(\Sigma, \alpha, \beta_{m+n}; u, v)$.

If the first generator is in relative Spin$^c$ class $s$, so is the second one. The small triangles mentioned earlier give two nearest point maps, which are approximations of $f^s_1$ and $g^s_1$ respectively. In particular, for this choice of the Heegaard diagram, the map $f^s$ is surjective and the map $g^s$ is injective. Combined with the vanishing of the compositions $h^s \circ f^s = 0$, $f^s \circ g^s = 0$, and $g^s \circ h^s = 0$, this surjectivity and injectivity imply the exactness of the triangle in equation 21 (c.f. section 3 of [OS5]).

We may now define the quasi-isomorphism between $\widehat{\text{CFK}}(K_n; s)$ and the mapping cone of $f^s$ as follows:

$$I^s : \widehat{\text{CFK}}(K_n; s) \to \left( \widehat{\text{CFK}}(K_{m+n}; s) \oplus \widehat{\text{CFK}}(K_{m+n}; s + m \text{PD}[\mu]) \right) \oplus \widehat{\text{CF}}(X; [s])$$

$$I^s(x) := (g^s_1(x), g^s_2(x), \Upsilon^s_h(x)).$$

The inverse of this quasi isomorphism is given by the map

$$J^s : \left( \widehat{\text{CFK}}(K_{m+n}; s) \oplus \widehat{\text{CFK}}(K_{m+n}; s + m \text{PD}[\mu]) \right) \oplus \widehat{\text{CF}}(X; [s]) \to \widehat{\text{CFK}}(K_n, s)$$

$$J^s(x, y, z) = \Upsilon^s_g(x, y) + h^s(z).$$

It is straightforward to check that these maps are quasi-isomorphisms, and the proof is thus complete. \qed
4. Surgery formulas and the maps $\phi(K)$ and $\overline{\phi}(K)$

4.1. Surgery formulas. The exact sequence of the previous section may be used to prove the following theorem which gives an explicit formula for the groups $\mathbb{H}_n(K, s)$, if $[s] = [J(s) + PD[\lambda]]$ is satisfied.

**Theorem 4.1.** Suppose that $K \subset X$ is a framed knot inside a rational homology sphere $X$ with the framing $\lambda \subset \partial(\text{nd}(K))$ fixed, and the complex $(\mathbb{B} = \mathbb{B}(K), d_{\mathbb{B}})$, with

$$\mathbb{B}(K) = \widehat{HFK}(X, K) = \bigoplus_{[s] \in \text{Spin}^c(X)} \widehat{HFK}(X; [s]) = \bigoplus_{[s] \in \text{Spin}^c(X)} \mathbb{B}\{[s]\}$$

and the filtration of $\mathbb{B}\{[s]\}$ by sub-complexes $\mathbb{B}\{\geq s\}$ for relative Spin$^c$ classes $s \in \text{Spin}^c(X, K)$ representing $[s] \in \text{Spin}^c(X)$ given as before. Furthermore, let $i_s : \mathbb{B}\{\geq s\} \to \mathbb{B}\{[s]\}$ be the inclusion map, and denote the homology of $(\mathbb{B}\{[s]\}, d_{\mathbb{B}})$ by $\mathbb{H}\{[s]\}$ and the homology of $\mathbb{B}\{\geq s\}$ by $\mathbb{H}\{\geq s\}$. If the relative Spin$^c$ class $s \in \text{Spin}^c(X, K)$ satisfies $[s] = [J(s) + PD[\lambda]]$ in Spin$^c(X)$, the knot Floer homology group $\widehat{HFK}(X_n(K), K_n; s)$ may be computed as the homology of the complex

$$C_n(s) = \mathbb{H}\{\geq s\} \oplus \mathbb{H}\{\geq J(s) + PD[\lambda_n + 2]\} \oplus \mathbb{H}\{[s]\},$$

which is equipped with a differential $d_n : C_n(s) \to C_n(s)$ defined by

$$d_n(x \oplus y \oplus z) = (0, 0, (i_s)_*(x) + (t_{J(s) + PD[\lambda_n + 2]})_*(y)).$$

**Proof.** In order to complete the above computation, consider the Heegaard diagram $H_{n+m} = (\Sigma, \alpha, \beta_{n+m}; p_{n+m})$ and let $u$ and $v$ be two marked points on the two sides of $p_{n+m}$ (separated from each-other by the curve $\lambda_{n+m}$). We may assume that $p_{n+m}$ is almost in the middle of the winding region, so that there are at most $2|n + m|/3$ of the intersection points $x_i$ on each side of $p_{n+m}$. Furthermore, suppose $s \in \text{Spin}^c(X, K)$ is such $[c_1(s)]$ is equal to $[PD[\lambda]]$ in $H^2(X, \mathbb{Z})$. Thus, $s - J(s) - PD[\lambda]$ is a multiple of $PD[\mu]$. Let $|s|$ denote the absolute value of the coefficient of $PD[\mu]$ in $s - J(s) - PD[\lambda]$. We may then assume that $|s|$ is small in comparison with $|n + m|$ (say less than $|n + m|/3$). It is then clear that the only generators of $\widehat{CF}(R_{n+m})$ in the Spin$^c$ class $s \in \text{Spin}^c(X_{n+m}(K), K_{n+m})$ are generators of the form $(x)_i$ with $x \in T_{\alpha} \cap T_{\beta}$ such that $s(x) + iP[\mu] = s$ if $i \leq 0$ and $s(x) + iP[\mu] + PD[\lambda_{n+m}] = s$ if $i > 0$. Note that $|i + (n + m)|$ is greater than $|n + m|/3$ and thus from $s(x) + (i + n + m)PD[\mu] + PD[\lambda] = s$ we may conclude $|J(s(x))| \geq |n + m|/3$. If the absolute value of $m$ is large enough, this can not happen, and we will just have generators of the form $(x)_i, i \leq 0$ in Spin$^c$ class $s$ with $s(x) + iP[\mu] = s$. This observation may be used to identify $\widehat{CF}(X_{n+m}(K), K_{n+m}, s)$ with the sub-complex of $\widehat{CF}(X)$ generated by the generators $x \in T_{\alpha} \cap T_{\beta}$ satisfying $s(x) \geq s$. The homology group $\widehat{HFK}(X_{n+m}(K), K_{n+m}; s)$ may thus be identified with $\mathbb{H}\{\geq s\}$. Note that the map $f_*^x$ takes the generator $(x)_i \in T_{\alpha} \cap T_{\beta_{n+m}}$ to $x + \text{lower energy terms.}.$
Thus, after a change of basis for the filtered chain complex $\mathbb{B}\{\geq s\}$, we may assume that under the above identification of $\widehat{HF}(X_{n+m}(K), K_{n+m}; s)$ with $\mathbb{H}\{\geq s\}$, the map $f_2^s$ is induced by the inclusion map $\iota_s : \mathbb{B}\{\geq s\} \to \mathbb{B}\{[s]\}$ in the level of chain complexes.

Now, let us look at the generators in the relative Spin$^c$ class $s + m PD[\mu] \in \widehat{Spin^c}(X_{n+m}(K), K_{n+m})$. Again, the above assumptions imply that we may assume all relevant generators are of the form $(x)_i$. If $i \leq 0$ and $s(x) + iPD[\mu] = s + m PD[\mu]$, we would have

$$|s(x)| = |(m - i)PD[\mu] + s| \geq \left| |s| - 2|m - i| \right| \geq \frac{|m| - 5|m|}{3}.$$  

Again, if the negative integer $m$ is large in absolute value, this can not happen. The only relevant generators are thus the generators of the form $(x)_i$ for $i > 0$, satisfying

$$s(x) + PD[\mu] + PD[\lambda_{m+n}] = s + m PD[\mu].$$

This is equivalent to $s(x) = s - (n+i)PD[\mu] - PD[\lambda] < s + PD[\lambda_n]$, and we may identify $CF(X_{n+m}(K), K_{n+m}; s + m PD[\mu])$ with the sub-complex of $CF(X)$ (with dual structure) generated by the generators $x \in T_n \cap T_\beta$ satisfying $s(x) < s - PD[\lambda_n]$. The homology group $\widehat{HF}(X_{n+m}(K), K_{n+m}; s + m PD[\mu])$ may thus be identified with $\mathbb{H}^*\{ < s - PD[\lambda_n] \}$, where $\mathbb{H}^*\{ < s - PD[\lambda_n] \}$ is the homology of the sub-complex $\mathbb{B}^*\{ < s - PD[\lambda_n] \}$ of the dual complex $\mathbb{B}^*\{[s]\}$. Under the duality map, this homology group may be identified with

$$\mathbb{H}\{ > J(s - PD[\lambda_n]) + PD[\mu] \} = \mathbb{H}\{ J(s) + PD[\lambda_n + 2\mu] \}$$

$$= \mathbb{H}\{ J(s) + PD[\lambda_{n+2}] \}.$$  

A similar argument shows that under the above identification of the Heegaard Floer homology group $\widehat{HF}(X_{n+m}(K), K_{n+m}; s + m PD[\mu])$ with the group $\mathbb{H}\{ \geq J(s) + PD[\lambda_{n+2}] \}$, the map $f_2^s$ is induced (in homology) by the inclusion map

$$\iota_{J(s) + PD[\lambda_{n+2}]} : \mathbb{B}\{ \geq J(s) + PD[\lambda_{n+2}] \} \to \mathbb{B}\{[s]\}$$

in the level of chain complexes. Together with theorem 3.2 this completes the proof. \hfill \Box

4.2. Understanding the maps. Let $K \subset X$ be a knot inside the rational homology sphere $X$, with a fixed framing $\lambda$, i.e. $\lambda$ is a longitude on the boundary of $\text{nd}(K)$. Let $L = K_{-1} \subset X_{-1}(K) = Y$ be the knot obtained from $K$ by $-1$-surgery. Denote by $\mathbb{B}$ the knot Floer homology group $\widehat{HF}(X, K; \mathbb{Z}/2\mathbb{Z})$ and let $d_\mathbb{B}$ denote the map induced by the differential of $\widehat{CF}(X)$ on $\mathbb{B}$. This map is thus a differential $d_\mathbb{B} : \mathbb{B} \to \mathbb{B}$ and the homology group $\mathbb{H} = H_*(\mathbb{B}, d_\mathbb{B})$ is the Heegaard Floer homology group $\widehat{HF}(X)$. The differential $d_\mathbb{B}$ respects the filtration by relative Spin$^c$ structures induced by
Similarly, map which has a differential of the form \( G \) on homology induced by \( (\) another projection map \( \pi \) two maps explicitly:

In this sub-section we prove the following theorem, which describes the above notation fixed, identify \( H_\infty(L, s) \) with the homology of the complex

\[
C_{-1}(s) = H\{ \geq s \} \oplus H\{ \geq J(s) + PD[\mu + \lambda] \} \oplus H\{ [s] \}
\]

which has a differential of the form \( d_{-1}(x, y, z) = (0, 0, |x| + |y|) \). Then the map \( \phi(K)^s : H_1(K, s) \to H_\infty(K, s) \) may be identified with the map on homology induced by \( G_\phi : C_{-1}(s) \to \mathbb{B}\{ s \} \) defined on an element \( c = (x, y, z) \in C_{-1}(s) \) as follows. First take \( c \) to \( x \in H\{ \geq s \} \) by the quotient map \( C_{-1}(s) \xrightarrow{\pi_1} H\{ \geq s \} \), and then project \( H\{ s \} \to H\{ s \} = H_\infty(K, s) \) by another projection map \( \pi_s \). The composition would be the map \( G_\phi^s = \pi_s \circ \pi_1^s \).

Similarly, \( \overline{\phi}(K)^s \) is induced by \( G_\overline{\phi}^s = \Xi \circ \pi_{J(s)+PD[\lambda]} \circ \pi_2^{J(s)+PD[\lambda]} \) obtained by a composition of quotient maps followed by the duality isomorphism as follows:

\[
(34) \quad C_{-1}(s) \xrightarrow{\pi_2^{J(s)+PD[\mu+\lambda]}} H\{ \geq J(s) + PD[\mu + \lambda] \} \xrightarrow{\pi_{J(s)+PD[\mu+\lambda]}} H_\infty(K, \lambda(s) + PD[\mu + \lambda]) \xrightarrow{\Xi(z)} H_\infty(K, s + PD[\lambda]).
\]

**Proof.** Consider the knot \( L = K_{-1} \) inside the three-manifold \( Y = X_{-1}(K) \), and suppose that \( R_{m-1} = (\Sigma, \alpha, \beta, \beta_{-1}, \beta_{m-1}; u, v, w, z) \) is the Heegaard diagram considered in the previous sub-section for \( K \). In order to compute the map \( \phi = \phi(L) : H_\infty(L) \to H_\infty(K) \), we may use the Heegaard triple \( (\Sigma, \alpha, \beta_{-1}; u, v, w) \). If we do so, for a generator \( x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) of the chain complex \( \mathcal{CF}(\Sigma, \alpha, \beta; u, w) \), the only generators \( x \) in \( \mathbb{T}_\alpha \cap \mathbb{T}_\beta_{-1} \) such that there is a holomorphic triangle between the generators \( x, \Theta_h \) and \( z \) are the
generators of the form \( x = (y)_0 \) for some \( y \in T_\alpha \cap T_\beta \). This implies that
\[
(35) \quad \phi(L)(x) = \begin{cases} 
\text{y + lower energy terms} & \text{if } x = (y)_0 \\
0 & \text{Otherwise}
\end{cases}
\]

It thus suffices to identify the elements of \( C_{-1}(s) \) which are in correspondence with the generators \( x \in T_\alpha \cap T_{\beta-1} \) of the form \( (y)_0 \) for some \( y \in T_\alpha \cap T_\beta \). No such generator can be in the image of the map \( h^s \), since three of the quadrants around the intersection point \( y_0 \in \lambda_{-1} \cap \alpha_g \) are punctured in the Heegaard diagram \( (\Sigma, \alpha, \beta, \beta_{-1}; u, v) \), used for defining \( h^s \). In fact, for orientation reasons, if we have a triangle \( \Delta \in \pi_2(z, \Theta_h, (y)_0) \) with \( n_u(\Delta) = n_v(\Delta) = 0 \), the coefficient of the fourth quadrant around \( y_0 \) should be \(-1\), and \( \mathcal{M}(\Delta) \) should be empty. Thus, we may replace the splitting homomorphism \( \Upsilon^s \) with a new splitting map (see [OS5], sections 2 and 3)
\[
R : \widehat{\text{CFK}}(Y = X_{-1}(K), L = K_{-1}, s) \to \widehat{\text{CF}}(X, [s])
\]
which is trivial on generators of the form \( (y)_0 \) with \( y \in T_\alpha \cap T_\beta \). The modified quasi-isomorphism from \( \widehat{\text{HFK}}(L = K_{-1}, s) \) to the mapping cone of \( f^s \) thus takes \( (y)_0 \) to
\[
g^s((y)_0) \oplus 0 \oplus 0 \in \mathbb{H}\{\geq s\} \oplus \mathbb{H}\{\geq J(s) + PD[\mu + \lambda]\} \oplus \mathbb{H}\{[s]\}.
\]
Again, by looking at the local coefficients of the domains in the winding region of the diagram \( R_{-1,m} \), one finds out that \( g^s((y)_0) = (y)_0 + \) lower energy terms, where the right-hand-side is a sum of generators in \( T_\alpha \cap T_{\beta_{m-1}} \). Under the identification (for large values of \(|m|\))
\[
\widehat{\text{CFK}}(X_{m-1}(K), K_{m-1}, s) \simeq \mathbb{B}\{\geq s\},
\]
the generator \((y)_0\) (with \( s(y) = s \)) corresponds to the class \([y]\) of \( y \) in the sub-complex \( \mathbb{B}\{\geq s\} \). Thus, after a change of basis for \( \mathbb{B}\{\geq s\} \subset C_{-1}(s) \) as a filtered chain complex, we may assume that the map \( \phi(K)^s \) is induced by the chain map obtained by first projecting \( C_{-1}(s) \) over \( \mathbb{B}\{\geq s\} \), and then taking this later complex to the quotient complex \( \mathbb{B}\{s\} = \text{CFK}(K, s) \). This completes the proof for \( \phi(L)^s \). The proof for \( \phi(L)^s \) is completely similar and requires a possible change of basis for \( \mathbb{B}\{\geq J(s) + PD[\mu + \lambda]\} \) as a filtered chain complex.

**Remark 4.3.** In the construction of this section and the exact sequence of the previous section, we have been limited to the relative \( \text{Spin}^c \) classes \( s \in \text{Spin}^c(X, K) \) such that the relative cohomology class \( c_1(s) - PD[\lambda] \) reduces to the trivial class in \( H^2(X; \mathbb{Z}) \). Without this assumption, the exactness of the triangle in equation (34) may not be achieved and the analogue of the above theorem, which plays a central role in the proof of our main theorem in the upcoming section, will no longer be true. These limitations force us to stay in the realm of integral homology spheres, for which any relation is satisfied in \( H^2(X; \mathbb{Z}) \). However, if for a rational homology sphere \( X \) and a framed
knot $K \subset X$ the equation $[c_1(s)] = [\text{PD}[\lambda]]$ is satisfied in $\text{H}^2(X; \mathbb{Z})$ for any relative Spin$^c$ structure $s \in \text{Spin}^c(X,K)$ which carries a non-trivial knot Floer group, the above theorem provides us with all we need for the proof of a similar theorem.

5. Proof of the main result

**Theorem 5.1.** Let $X$ be a homology sphere and $K \subset X$ be a non-trivial knot. Then $K$ does not have simple knot Floer homology, i.e.

$$\text{rk}(\hat{\text{HFK}}(X,K;\mathbb{Z}/2\mathbb{Z})) > \text{rk}(\hat{\text{HF}}(X;\mathbb{Z}/2\mathbb{Z})).$$

**Remark 5.2.** In fact, the same is true if $K \subset X$ is a knot inside a rational homology sphere $X$ and if $\hat{\text{HFK}}(X,K;\mathbb{Z}/2\mathbb{Z}) \neq 0$ for a relative Spin$^c$ class $s \in \text{Spin}^c(X,K)$ we have $[c_1(s)] = [\text{PD}[K]]$ in $\text{H}^2(X;\mathbb{Z})$. Non-trivial examples of this situation are null-homologous knots $K \subset X$ such that $\text{H}^2(X;\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^n$ for some $n$.

**Proof.** The framing $\lambda$ may be set to be the zero framed longitude of $K$. Let $B = \hat{\text{HFK}}(X,K)$ and the induced differential $d_B : B \to B$ from $\hat{\text{CF}}(X)$ be as before (so $H_*(B,d_B) = \hat{\text{HF}}(X)$). If the ranks of $\hat{\text{HFK}}(X,K)$ and $\hat{\text{HF}}(X)$ are equal, $d_B$ will be trivial. We may then use theorem 4.1 for computing Heegaard Floer groups $\mathbb{H}(K,s)$. More precisely, the homology of the complex $C_n(s) = \mathbb{B}\{\geq s\} \oplus \mathbb{B}\{\geq J(s) + (n+2)[\text{PD}][\mu]\} \oplus \mathbb{B}$ is generated by the elements of one of the following two forms: *type one* are the generators of the form $(x,x,0) \in C_n(s)$ where $x \in \mathbb{B}\{t\}$ is a generator and such that the relative Spin$^c$ structure $t$ satisfies $t \geq \max\{s, J(s) + (n+2)[\text{PD}][\mu]\}$, and *type two* are generators of the form $(0,0,x)$, where $x \in \mathbb{B}\{t\}$ and $t < \min\{s, J(s) + (n+2)[\text{PD}][\mu]\}$.

In particular, for $n = -1$, we may use the above description to compute the maps $\phi(L)^s$ and $\overline{\phi}(L)^s$ explicitly for the knot $L = K_{-1} \subset X_{-1}(K) = Y$, from theorem 4.2. A generator $(x,x,0)$ of type one (as above) is in the kernel of $\phi(L)^s$, unless $t = s \geq J(s) + [\text{PD}][\mu]$. If this later assumption is satisfied, $b(s) \geq 0$ and we may identify a subspace of $\mathbb{H}_1(L,s)$ which is isomorphic to $\mathbb{H}_\infty(K,s)$ under $\phi(L)^s$. Similarly, a generator of the above form is in the kernel of $\overline{\phi}(L)^s$ unless $t = J(s) + [\text{PD}][\mu] \geq s$, and in this case we may identify a subspace of $\mathbb{H}_1(L,s)$ which is isomorphic to $\mathbb{H}_\infty(K,J(s) + [\text{PD}][\mu]) \simeq \mathbb{H}_\infty(K,s)$ under $\overline{\phi}(L)^s$. Furthermore, generators of type two are all in the kernels of both $\phi(L)$ and $\overline{\phi}(L)$.

The above observations imply that in an appropriate basis for $\mathbb{H}_1(L)$ and $\mathbb{H}_1(L) \simeq \mathbb{H}_\infty(K)$ the maps $\phi(L)$ and $\overline{\phi}(L)$ will have the following block presentations

$$\phi(K) = \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{\phi}(K) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Xi & 0 & 0 \\ 0 & 0 & I_r & 0 \end{pmatrix}.$$
The matrices $I_r$ and $I_s$ are $r \times r$ and $s \times s$ identity matrices respectively, for some non-negative integers $r$ and $s$. The first, second and third columns correspond to the part generated by $(x, x, 0) \in C_{-1}(s(x))$ such that $h(s(x)) > 0$, $h(s(x)) = 0$ and $h(s(x)) < 0$ respectively. The matrix $\Xi$ describes the duality map $\Xi: H_{\infty}(K, s) \rightarrow H_{\infty}(K, J(s) + \text{PD}[\mu])$. Thus $\Xi$ is an invertible $s \times s$ matrix.

This implies that the matrices $\psi(K)$ and $\overline{\psi}(K)$ will have the following block presentations, by exactness of the short sequences of equation (37)

$$\psi(K) = \begin{pmatrix} \Psi & 0 & 0 \\ \end{pmatrix}, \quad \& \quad \overline{\psi}(K) = \begin{pmatrix} 0 & 0 & \Upsilon \\ \end{pmatrix},$$

which gives the following presentations of the maps $\eta(K)$ and $\overline{\eta}(K)$:

$$\eta(K) = \begin{pmatrix} \Psi & 0 & 0 & 0 \\ \end{pmatrix}, \quad \& \quad \overline{\eta}(K) = \begin{pmatrix} 0 & 0 & \Upsilon & 0 \\ \end{pmatrix}.$$  

We will now use these block presentations, together with theorem 2.3 for computing the rank of $\hat{H}(X_{p/q}(K))$. Note that $X_{p/q}(K) = X_{p/q}(L_1) = Y_{p/p+q}(L)$. If $q > 0$, we will have $p < p + q$ and the second formula from theorem 2.3 can be used for the computation. The differential of the complex

$$d_{\Xi} = \left( \bigoplus_{i=1}^{p+q} H_{\infty}(i) \bigoplus \bigoplus_{i=1}^{q} H_{1}(i) \bigoplus \bigoplus_{i=1}^{p} H_{0}(i) \right),$$

will have the following form

$$d_{\Xi} = \begin{pmatrix} \Phi_{p,q} & 0 & 0 \\ 0 & \Gamma_{p,q} & 0 \\ 0 & 0 & 0 \\ \end{pmatrix},$$

where the matrices $\Gamma_{p,q}$ and $\Phi_{p,q}$ have the following block presentations respectively:

$$\begin{pmatrix} \overline{\eta} & 0 & \ldots & \eta & 0 & \ldots & 0 \\ 0 & \overline{\eta} & \ldots & 0 & \eta & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \overline{\eta} & 0 & \ldots & \eta \end{pmatrix}, \quad \& \quad \begin{pmatrix} \phi & 0 & \ldots & \phi & 0 & \ldots & 0 \\ 0 & \phi & \ldots & 0 & \phi & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi & 0 & \ldots & \phi \end{pmatrix}. $$

The distance between the maps $\eta$ and $\overline{\eta}$ at each row is $q$ and the distance between the maps $\phi$ and $\overline{\phi}$ at each row is $p$. The number of rows in $\Gamma_{p,q}$ is $p$ and the number of rows in $\Phi_{p,q}$ is $q$. Replacing the matrices in equations (36) and (38), we conclude that the rank of $d_{\Xi}$ is equal to

$$\text{rk}(d_{\Xi}) = 2qr + qs + p. \text{rk} \begin{pmatrix} \Psi & \Upsilon \end{pmatrix} = q(2r + s) + px,$$

for some non-negative integer $x$. If we denote the rank of $H_{\bullet}(K)$ by $h_{\bullet}(K)$, this implies that the rank of $H_{\bullet}(\Xi, d_{\Xi})$ is equal to

$$y_{p/q}(K) : = \text{rk}(\hat{H}(X_{p/q}(K); \mathbb{Z}/2\mathbb{Z})) = (p + q)h_{\infty}(L) + qh_1(L) + ph_0(L) - 2(q(2r + s) + px) = q(h_{\infty}(L) + h_{\infty}(K) - 4r - 2s) + p(h_{\infty}(L) + h_0(L) - 2x).$$
Note that $2r + s$ is the rank of $\mathbb{H}_\infty(K)$, which helps us conclude
\[ y_{p/q}(K) = p(h_\infty(L) + h_0(L) - 2x) + q(h_\infty(L) - h_\infty(K)). \]
On the other hand, $y_{\pm 1}(K)$ is asymptotic to $n y_\infty(K) = nh_\infty(K)$ as $n$ grows large. We thus conclude that $h_\infty(K) = h_\infty(L) + h_0(L) - 2x$, and $y_{p/q}(K) = ph_\infty(K) + q(h_\infty(K) - h_\infty(K))$. In particular, $y_1(K) = h_\infty(K)$. If $s \in \text{Spin}^c(X, K)$ is a relative Spin$^c$ structure so that $(c_1(s) - PD[\mu])/2$ is $j = j(s)$ times the generator of $H^2(X, K)$, denote the rank of $\mathbb{H}_\infty(K, s)$ by $\ell_j$. Thus, $y_\infty(K) = h_\infty(K) = \ell_0 + 2(\ell_1 + \ldots + \ell_g)$, where $g = g(K)$ is the genus of $K$. Note that by Ni’s results (theorem 4.1), we may compute $h_\infty(K)$ in terms of $\ell_j$ as follows. Since the differential $d_0$ of the complex $B$ is trivial, the homology of the complex $C_{-1}(s)$ may be identified with
\[ \mathbb{H}\{\geq \max\{s, J(s) + PD[\mu]\}\} \oplus \mathbb{H}\{< \min\{s, J(s) + PD[\mu]\}\}. \]
If we note that $j = j(s) = j(J(s) + PD[\mu]) \geq 0$, this means that the rank of $\mathbb{H}_{-1}(K, s)$ (as well as the rank of $\mathbb{H}_{-1}(K, J(s) + PD[\mu])$) is equal to
\[ (\ell_j + \ell_{j+1} + \ldots + \ell_g) + (\ell_{j+1} + \ell_{j+2} + \ldots + \ell_g) = \ell_j + 2(\ell_{j+1} + \ell_{j+2} + \ldots + \ell_g). \]
Thus the total rank of $\mathbb{H}_{-1}(K)$ is equal to
\[ h_\infty(K) = \text{rk}(\mathbb{H}_{-1}(K, s_0)) + 2 \sum_{i=1}^{g} \text{rk}(\mathbb{H}_{-1}(K, s_0 + iPD[\mu])) \]
\[ = (\ell_0 + 2(\ell_1 + \ldots + \ell_g)) + 2(\sum_{j=1}^{g} (\ell_j + 2(\ell_{j+1} + \ldots + \ell_g))) \]
\[ = \ell_0 + 4(\sum_{j=1}^{g} j\ell_j). \]
Here $s_0$ denotes the unique relative Spin$^c$ structure satisfying $j(s_0) = 0$. On the other hand, $y_{\pm 1}(K)$ may directly be computed using the result of Ozsváth and Szabó [OS5], given that the differential of the complexes $B$ and $A_s$ may be assumed to be trivial. More precisely, the differential $d$ of the $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex $\text{CFK}_\infty(X, K; \mathbb{Z}/2\mathbb{Z})$ may be written as
\[ d = \sum_{i, j \geq 0} d_{i, j}, \]
where $d_{i, j}$ changes the $\mathbb{Z} \oplus \mathbb{Z}$ grading of a generator by the vector $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$. Since the coefficient ring is $\mathbb{Z}/2\mathbb{Z}$, we may assume that the complex $\text{CFK}_\infty(X, K; \mathbb{Z}/2\mathbb{Z})$ is quasi-isomorphic to a $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex $(C^\infty, d^\infty)$ with $d^\infty = \sum_{i, j \geq 0} d^\infty_{i, j}$ and $d^\infty_{0, 0} = 0$. The assumption on the knot $K$ implies that $d^\infty_{i, j}$ and $d^\infty_{i, 0}$ are both trivial for $i, j \geq 0$. If the complex $C^\infty$ is used for constructing the complexes $A_s$, they will have trivial differential as well.

The complex $A_s$ may then be identified, as a vector space, with $B$. Under this identification, the map $v_s : A_s \to B$ may be identified as the zero extension of the identity map from $\mathbb{H}\{\geq s\} \subset A_s$ to $\mathbb{H}\{\geq s\} \subset B$ and the map $h_s : A_s \to B$ may be identified as the zero extension of the duality map.
from $\mathbb{H}\{s\} \subset A_s$ to $\mathbb{H}\{s\} \subset \mathbb{B}$. Here, abusing the notation, we are denoting $\mathbb{H}\{\geq s\}$ by $\mathbb{H}\{s\}$. The kernel of $d_1 : \hat{A} \to \hat{B}$ is thus generated by the elements of the form $y_s \oplus z_s$ where $y_s \in A_{b(s)} = \mathbb{B}$ is a generator $x$ in $\mathbb{B}\{t\}$, for a relative Spin$^c$ structure $t$ satisfying $t < \min\{s, J(s)\}$, and $z_s \in A_{b(s)+1} = \mathbb{B}$ is its dual $\Xi(y_s) \in \mathbb{B}\{J(s) + \text{PD}[\mu]\}$. Thus the generators of the kernel of $d_1$ are in one to one correspondence with the pairs $(s, x)$ such that $x$ is a generator in $\mathbb{B}\{t\}$ and $h(t) < \min\{s, -(s + 1)\}$, and $s$ runs from $1 - g$ to $g - 2$. Associated with these values of $s$, the value of $\min\{s, -(s + 1)\}$ becomes equal to either of the values $-1, -2, \ldots, 1 - g$ twice. Associated with the value $-i$ for $\min\{s, -(s + 1)\}$, the rank of the identified part of the kernel of $d_1$ is equal to $\ell_{i+1} + \ell_{i+2} + \ldots, + \ell_g$. Thus the size of kernel of $d_1$ is equal to

\begin{equation}
|\text{Ker}(d_1)\subset \hat{A}| = 2 \sum_{i=1}^{g} (\ell_{i+1} + \ell_{i+2} + \ldots, + \ell_g) = 2 \left( \sum_{i=1}^{g} (j-1)\ell_j \right).
\end{equation}

The kernel and cokernel of $d_1 : \hat{A} \to \hat{B}$ may be identified with the kernel of the induced map

\begin{equation}
D_1 : \overline{A} = \bigoplus_{s=-g}^{g} A_s \to \overline{B} = \bigoplus_{s=-g}^{g+1} (s, \mathbb{B}).
\end{equation}

Since the size of the kernel of $D_1$ is equal to the size of the kernel of $d_1$, the rank of $D_1$ is equal to

\begin{equation}
\text{rk}(D_1) = |\overline{A}| - 2 \left( \sum_{i=1}^{g} (j-1)\ell_j \right) = (2g + 1)|\mathbb{B}| - 2 \left( \sum_{i=1}^{g} (j-1)\ell_i \right).
\end{equation}

Thus, the homology of the mapping cone of $D_1$, which is isomorphic to the homology of the mapping cone of $d_1$ will be a vector space of size

\begin{equation}
y_1(K) = \text{rk}(H_* (\hat{A} \oplus \hat{B}, d_1)) = (4g + 3)|\mathbb{B}| - 2\text{rk}(D_1)
\end{equation}

\begin{equation}
= (4 \sum_{i=1}^{g} (i-1)\ell_i) + (\ell_0 + 2 \sum_{i=1}^{g} \ell_i)
\end{equation}

\begin{equation}
= \ell_0 + 2 \sum_{i=1}^{g} (2i-1)\ell_i.
\end{equation}

It is worth mentioning that with a similar argument we may also compute $y_{-1}(K)$ to be equal to the above expression. From the equality $y_1 = h_{-1}(K)$ we thus conclude $\sum_{j=1}^{g} \ell_j = 0$. But this can not happen unless $g = g(K) = 0$. This completes the proof of our main theorem. \qed
6. Some examples

6.1. Knots inside arbitrary rational homology spheres. We have already seen that the argument given in the previous sections is quite limited to homology spheres, and may not be extended to the arbitrary rational homology spheres without major modification. In this subsection, we will give examples of knots inside \( L \)-spaces which are not trivial, but have simple knot Floer homology. In fact such examples are quite well known to the people studying Berge conjecture.

Let \( T^n = T(2, 2n+1) \) denote the \((2, 2n+1)\) torus knot in the standard sphere and let \( T^m \subset S^3_m(T^n) \) be the knot obtained by \( m \)-surgery on \( T^n \), inside the rational homology sphere \( S^3_m(T^n) \). We may then compute the Heegaard Floer homology groups \( \mathcal{HFK}(T^m, s) \) for different values of \( s \in \mathcal{Spin}^c(S^3_m(T^n), T^m) = \mathcal{Spin}^c(S^3, T^m) \) using theorem 4.1. This group may be identified with the homology of the mapping cone

\[
C_m(s) = \mathbb{H}\{\geq s\} \oplus \mathbb{H}\{\geq J(s) + (m + 2)\text{PD}[\mu]\} \oplus \mathbb{H},
\]

with differential \( d_m \). If we identify \( \mathcal{Spin}^c(S^3, T^n) \) with \( \mathbb{Z} \) using the map \( h : \mathcal{Spin}^c(S^3, T^n) \to \mathbb{Z} = \mathbb{H}^2(S^3, T^n; \mathbb{Z}) \) given by \( s \to c_1(s) - \text{PD}[\mu] \), and if for \( s = h(\bar{s}) \) we denote the complex \( C_m(s) \) by \( C_m(s) \) and the complex \( \mathbb{H}\{\geq s\} \) by \( \mathbb{H}\{\geq s\} \), the complex \( C_m(s) \) may be described as

\[
C_m(s) = \mathbb{H}\{\geq s\} \oplus \mathbb{H}\{\geq m - s\} \oplus \mathbb{H}.
\]

For the torus knot \( T^n \), the complex \( \mathbb{B} \) may easily be described as the complex generated by \( x_s \) with \( s \in \{-n, 1 - n, \ldots, n\} \) so that \( j(x_s) = s \) and \( d_{\mathbb{B}}(x_{2i-n-1}) = x_{2i-n} \) for \( i = 1, ..., n \). This implies that \( \mathbb{H} \) is isomorphic to \( \mathbb{F} = \langle x_{-n} \rangle \), and that

\[
\mathbb{H}\{\geq s\} =
\begin{cases}
0, & \text{if } s > n \\
\mathbb{F} = \langle x_{-n} \rangle, & \text{if } -n \leq s \leq n, \text{ and } s - n = 0 \text{ (mod 2)} \\
\mathbb{F} \oplus \mathbb{F} = \langle x_{-n}, x_s \rangle, & \text{if } -n \leq s \leq n, \text{ and } s - n = 1 \text{ (mod 2)} \\
\mathbb{F} = \langle x_{-n} \rangle, & \text{if } s < -n
\end{cases}
\]

where \( \mathbb{F} \) is the coefficient ring, which is assumed to be \( \mathbb{Z}/2\mathbb{Z} \) in this paper. Let \( \ell(s) \) be 0 if \( s > n \), 2 if \( |s| \leq n \) and \( s - n \) is odd, and 1 otherwise. The above observation implies that the homology group \( \mathbb{H}_m(T^n, s) \) of \( C_m(s) \) may be computed as \( \mathbb{F}^{\ell(s) + \ell(m + 1 - s) - 1} \). In particular, if \( m \geq 2n \), \( \ell(s) = 2 \) implies that \( |s| \leq n \), and thus \( m + 1 - s > n \). Thus, if \( \ell(s) = 2 \), the value of \( |\ell(s) + \ell(m + 1 - s) - 1| \) is equal to 1. On the other hand, if \( \ell(m + 1 - s) = 2 \), we will have \( |m + 1 - s| \leq n \) and thus \( s > n \). Again this implies that \( |\ell(s) + \ell(m + 1 - s) - 1| \) is equal to 1. If both \( \ell(s) \) and \( \ell(m + 1 - s) \) are in \( \{0, 1\} \), the same would be true for \( |\ell(s) + \ell(m + 1 - s) - 1| \). Thus, under the assumption \( m \geq 2n \), the groups \( \mathbb{H}_m(T^n, s) \) are all either isomorphic to \( \mathbb{F} \) or are trivial.

Note that the reduction of the relative \( \mathcal{Spin}^c \) classes associated with \( s, t \in \).
\[ Z = \text{H}^2(S^3_m(T^n), T^n_m) \text{ to Spin}^c \text{ classes in Spin}^c(S^3_m(T^n)) \text{ are the same if and only if } m \text{ divides } s - t. \text{ Suppose that } t = s + km \text{ with } k > 0 \text{ and both } \mathbb{H}_m(T^n, s) \text{ and } \mathbb{H}_m(T^n, t) \text{ are isomorphic to } \mathbb{F}. \text{ This implies that all four of } s, t, m + 1 - s \text{ and } m + 1 - t \text{ are greater than or equal to } -n. \text{ Otherwise, say if } s < -n, \text{ we will have } m + 1 - s > n \text{ and thus } \ell(s) + \ell(m + 1 - s) - 1 = 0. \text{ Also if } m + 1 - s < -n, \text{ we have } s > m + 1 - n > n \text{ and } \ell(s) + \ell(m + 1 - s) - 1 = 0. \text{ We have thus seen that by the non-triviality assumption on } \mathbb{H}_m(T^n, s), s \geq -n \text{ and } m + 1 - s \geq -n. \text{ Similarly } km + s = t \geq -n \text{ and } 1 - s + (1 - k)m = m + 1 - t \geq -n. \text{ These imply that } -n \leq s \leq n + 1 - (k - 1)m. \text{ In particular, the positive integer } k \text{ is forced to be } 1. \text{ Let us assume for a second that } s \neq n + 1. \text{ Thus } |s| \leq n \text{ and } m + 1 - s > n. \text{ Since } \mathbb{H}_m(T^n, s) = \mathbb{F} \text{ we should have that } s - n \text{ is odd. Moreover, } t = s + m > n \text{ and since } \mathbb{H}_m(T^n, t) = \mathbb{F}, \text{ we should have } t - n \text{ is odd, i.e. } m + s - n \text{ is odd. If } m \text{ is odd, this can not happen. Furthermore, if } s = n + 1, \mathbb{H}_m(T^n, s) = \mathbb{F} = \mathbb{F}[(s(n + 1) + \ell(m - n) - 1)] = \mathbb{F}, \text{ while } \mathbb{H}_m(T^n, t) = \mathbb{F}[(s(n + m) + \ell(-m) - 1)] = \mathbb{F}^0 \text{ which contradicts our initial assumption.} \text{ The above assumptions on } m \text{ (that } m \geq 2n \text{ is an odd number) imply that if } s - t \text{ is a multiple of } m, \text{ not both } \mathbb{H}_m(T^n, s) \text{ and } \mathbb{H}_m(T^n, t) \text{ can be isomorphic to } \mathbb{F}. \text{ In other words, } T^n_m \text{ is a knot with simple Floer homology in the } L \text{ space } S^3_m(T^n). \text{ It is of course clear that the genus of } T^n_m \text{ is equal to the genus of } T^n, \text{ i.e. } n, \text{ and } T^n_m \text{ is thus not the trivial knot, although it has simple Floer homology.} \]

6.2. The Borromean knot \( B \subset \#^2(S^1 \times S^2) \). One very interesting example of a knot with simple Heegaard Floer homology is the Borromean knot \( B \subset \#^2(S^1 \times S^2) = X \) which is obtained as follows. If we do 0-surgery on two of the three components of the Borromean link in \( S^3 \), the resulting three-manifold of this surgery will be the connected sum of two copies of \( S^1 \times S^2 \). The third component determines a null-homologous knot \( B \subset \#^2(S^1 \times S^2) = X \) which may be described by the Kirby diagram of figure \[4]. This genus 1 knot is usually called the Borromean knot.

The Borromean knot has simple Floer homology. Note that \( \text{Spin}^c(X, B) = \mathbb{Z}^3 \) and that there is a unique relative \( \text{Spin}^c \) class corresponding to \( 0 \in \mathbb{Z}^3 \), which will be denoted by \( s_0 \in \text{Spin}^c(X, B) \). Then \( \text{HF}^\mathbb{K}(X, B; s_0) = (\mathbb{F}^2)_0 \) (i.e. \( \mathbb{F}^2 \) is supported in homological degree 0) while

\[ \text{HF}^\mathbb{K}(X, B; s_0 \pm \text{PD}([\mu_B])) = (\mathbb{F})_{\mp 1}, \]

where \( \mu_B \) denotes the meridian of \( B \). What makes the Borromean knot interesting for our purposes is that the reduction of the relative \( \text{Spin}^c \) structures on \( X - B \) which support a non-trivial knot Floer homology group to \( X \) is always the canonical \( \text{Spin}^c \) structure of \( X = \#^2(S^1 \times S^2) \). This may suggest that for all practical purposes we may treat \( X \) as a homology sphere, and treat the canonical \( \text{Spin}^c \) structure of \( X \) as the unique \( \text{Spin}^c \) structure.
on homology spheres. In other words, we may hope that theorem 5.1 is still satisfied. The question which then rises is that where exactly our argument in this paper fails when we replace $X$ for the homology spheres considered here?

Below, we will show that the knot surgery formula of theorem 4.1 is not true for the Borromean knot. Namely, we will show that

$$\bigoplus_i \widehat{HFK}(X_{-1}(B), B_{-1}; s_0 + i \text{PD}[\mu]) = \mathbb{F}^4,$$

and thus its rank is equal to the rank of $\widehat{\text{HF}}(X_{-1}(B))$, while the surgery formulas predict that $\widehat{HFK}(X_{-1}(B), B_{-1}) = \mathbb{F}^6$. This later prediction is of course clear from the proof of our main theorem and equation 40, which indicates that $|\widehat{HFK}(X_{-1}(B), B_{-1})| = \ell_0 + 4\ell_1 = 2 + 4 = 6$.

For the actual computation, we will use the explicit Heegaard diagram of figure 5. The inner and outer circles are glued together, and two one-handles are glued to the resulting torus with attaching circles determined by two pairs of small circles to form the Heegaard surface $\Sigma$, which has genus 3. The bold red curve is the meridian of the knot $B_{-1} \subset X_{-1}(B)$. The dotted blue curves are the $\alpha$-curves. The rest of $\beta$-curves (i.e. $\beta_1$ and $\beta_2$) are denoted by regular black curves, and the marked points $u$ and $v$ are placed on the two sides of the curve $\lambda_1$.

Associated with this Heegaard diagram, each generator should contain one element $x_i \in \{x_1, \ldots, x_5\}$, another element $y_j \in \{y_1, \ldots, y_6\}$, and a last element $z_k \in \{z_1, \ldots, z_6\}$. Such a generator will be denoted by $(i, j, k)$. There are 40 generators associated with this Heegaard diagram, which may both be regarded as a Heegaard diagram for $B_{-1} \subset X_{-1}(B)$ and for $X_{-1}(B)$, depending on whether or not we exclude the holomorphic disks passing through the marked point $v$. The generators which are associated with any Spin$^c$ class on $X_{-1}(B)$ other than the canonical Spin$^c$ class cancel each other, even
Figure 5. A Heegaard diagram for the knot obtained by $-1$-surgery on the Borromean knot $B \subset \#^2(S^1 \times S^2) = X$. The bold red curve is the meridian of the knot $B_{-1} \subset X_{-1}(B)$. The dotted blue curves are the $\alpha$-curves. The inner and outer circle are glued together, and two one-handles are glued to the resulting torus with attaching circles determined by two pairs of small circles to form the Heegaard surface $\Sigma$, which has genus 3.

if we only use the disks not passing through $v$. We should thus consider the generators in the canonical Spin$^c$ class. There are 16 such generators which are listed below:

$$I = I_1 \cup I_0 \cup I_{-1},$$

$$I_0 = \left\{ (1, 4, 5), (1, 5, 4), (2, 1, 6), (2, 2, 6), (2, 3, 5), (3, 3, 1), (3, 3, 2), (4, 1, 3), (4, 2, 3), (5, 5, 3), (5, 6, 1), (5, 6, 2) \right\},$$

(46)

$$I_1 = \left\{ (1, 4, 4), (2, 3, 4), (5, 4, 3) \right\},$$

$$I_{-1} = \left\{ (1, 5, 5) \right\}.$$

Here, $I_j$ denotes the set of generators corresponding to the relative Spin$^c$ class $s$ characterized by the property that $\mathfrak{h}(s) = j$. Since $I_{-1}$ consists of a single element, it is clear that $\widehat{\text{HFK}}(X_{-1}(B), B_{-1}; \pm 1) = \mathbb{F}$, and we are down to showing that $\widehat{\text{HFK}}(X_{-1}(B), B_{-1}; 0) = \mathbb{F}^2$. 


Note that there is a small rectangle $R$ with corners $x_1, x_2, y_4$ and $y_3$ and that there is a small bigon $D_1$ connecting $y_1$ to $y_2$ and another small bigon connecting $z_1$ to $z_2$. If we puncture all the domains except these 3 regions, the only differentials in the associated complex between the generators in $I_0$ are differentials from $(i, j, 2)$ to $(i, j, 1)$, from $(i, 2, k)$ to $(i, 1, k)$, and from $(1, 5, k)$ to $(2, 3, k)$. The homology of the complex with this differential is generated by the closed non-exact elements $(1, 5, 4)$ and $(2, 3, 5)$. It is then not hard to see that $\widehat{HFK}(X_{-1}(B), B_{-1}; 0)$ is obtained from the complex $\langle (1, 5, 4), (2, 3, 5) \rangle$ with a differential induced from the original complex $\widehat{CFK}(X_{-1}(B), B_{-1}; 0)$. Thus, $\widehat{HFK}(X_{-1}(B), B_{-1}; 0)$ is either zero or isomorphic to $\mathbb{F}^2$. It is easy to see that it can not be zero, as $\widehat{HFK}(X_{-1}(B)) = \mathbb{F}^4$. We thus have shown that $\widehat{HFK}(X_{-1}(B), B_{-1}; 0) = \mathbb{F}^2$ and so $\widehat{HFK}(X_{-1}(B), B_{-1}) = \mathbb{F}^4$, completing the computation. This indicates that the surgery formula of theorem 4.1 can not be true for this knot (which lives in a three-manifold with positive first betti number).

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