New Generalization of Perturbed Ostrowski Type Inequalities and Applications

WEN-JUN LIU, QIAO-LING XUE, JIAN-WEI DONG

Abstract: Generalizations of Ostrowski type inequality for functions of Lipschitzian type are established. Applications in numerical integration and cumulative distribution functions are also given.

1. Introduction

In [10] N. Ujević obtained the following perturbed Ostrowski type inequality.

Theorem 1. Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a differentiable function such that $\gamma \leq f'(t) \leq \Gamma$, $\forall t \in [a, b]$, for some constants $\gamma, \Gamma \in \mathbb{R}$. Then we have

$$\left| (b - a) \left\{ \left[ f(x) - \frac{\Gamma + \gamma}{2} \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{f(a) + f(b)}{2} h \right\} - \int_a^b f(t) dt \right|$$

$$\leq \frac{1}{2} \left[ \frac{(b - a)^2}{4} [h^2 + (h - 1)^2] + \left( x - \frac{a + b}{2} \right)^2 \right] (\Gamma - \gamma),$$

where $a + h((b - a)/2) \leq x \leq b - h((b - a)/2)$ and $h \in [0, 1]$.

In [11], the same author proved the next result.

Theorem 2. Let the assumptions of Theorem 1 hold. Then for all $a + h((b - a)/2) \leq x \leq b - h((b - a)/2)$ and $h \in [0, 1]$, we have

$$\left| (b - a) \left\{ \left[ f(x) - \gamma \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{f(a) + f(b)}{2} h \right\} - \int_a^b f(t) dt \right|$$

$$\leq (b - a) \max \left\{ \frac{h(b - a)}{2}, x - a - h \frac{b - a}{2}, b - x - h \frac{b - a}{2} \right\} (S - \gamma),$$

and

$$\left| (b - a) \left\{ \left[ f(x) - \Gamma \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{f(a) + f(b)}{2} h \right\} - \int_a^b f(t) dt \right|$$

$$\leq (b - a) \max \left\{ \frac{h(b - a)}{2}, x - a - h \frac{b - a}{2}, b - x - h \frac{b - a}{2} \right\} (\Gamma - S),$$

where $S = (f(b) - f(a))/(b - a)$.

2000 Mathematics Subject Classification: 26D10, 41A55, 65D30. Keywords and Phrases: perturbed Ostrowski type inequality, functions of Lipschitzian type, numerical integration, cumulative distribution functions.
All (1.1)–(1.3) have been used to get the tighter error bounds for the midpoint, trapezoid, and Simpson quadrature formulas in numerical integration, respectively.

In this paper, we shall generalize Theorems 1 and 2 to functions of some larger classes. Applications in numerical integration and cumulative distribution functions are also given. For convenience, we define functions of Lipschitzian type as follows:

**Definition 3.** The function $f : [a, b] \rightarrow \mathbb{R}$ is said to be $L$-Lipschitzian on $[a, b]$ if for some $L > 0$ and all $x, y \in [a, b],$

$$|f(x) - f(y)| \leq L|x - y|.$$  

**Definition 4.** The function $f : [a, b] \rightarrow \mathbb{R}$ is said to be $(l, L)$-Lipschitzian on $[a, b]$ if $l(x_2 - x_1) \leq f(x_2) - f(x_1) \leq L(x_2 - x_1)$ for $a \leq x_1 \leq x_2 \leq b,$

where $l, L \in \mathbb{R}$ with $l < L.$

We will need the following well-known results.

**Lemma 5.** (All) Let $h, g : [a, b] \rightarrow \mathbb{R}$ be such that $h$ is Riemann integrable on $[a, b]$ and $g$ is $L$-Lipschitzian on $[a, b].$ Then

$$\left| \int_a^b h(t)dg(t) \right| \leq L \int_a^b |h(t)|dt.$$  

**Lemma 6.** (All) Let $h, g : [a, b] \rightarrow \mathbb{R}$ be such that $h$ is continuous on $[a, b]$ and $g$ is of bounded variation on $[a, b].$ Then

$$\left| \int_a^b h(t)dg(t) \right| \leq \max_{t \in [a, b]} |h(t)| \sup_{a \leq i < b} \int_a^b |g(t)|dt.$$  

2. **Main results**

Our main results are as follows.

**Theorem 7.** Let $f : [a, b] \rightarrow \mathbb{R}$ be $(l, L)$-Lipschitzian on $[a, b].$ Then we have

$$\left| (b - a) \left\{ f(x) - L + \left( x - \frac{a + b}{2} \right) (1 - h) + \frac{f(a) + f(b)}{2}h \right\} - \int_a^b f(t)dt \right|$$  

(2.1)

$$\leq \frac{1}{2} \left( \frac{(b - a)^2}{4} [h^2 + (h - 1)^2] + \left( x - \frac{a + b}{2} \right)^2 \right) (L - l),$$  

and

$$\left| (b - a) \left\{ f(x) - l \left( x - \frac{a + b}{2} \right) (1 - h) + \frac{f(a) + f(b)}{2}h \right\} - \int_a^b f(t)dt \right|$$  

(2.2)

$$\leq (b - a) \max \left\{ \frac{h}{2}, x - a - \frac{b - a}{2}, b - x - \frac{b - a}{2} \right\} (S - l),$$

where $S$ is

$$\left| (b - a) \left\{ f(x) - L \left( x - \frac{a + b}{2} \right) (1 - h) + \frac{f(a) + f(b)}{2}h \right\} - \int_a^b f(t)dt \right|$$  

(2.3)

$$\leq (b - a) \max \left\{ \frac{h}{2}, x - a - \frac{b - a}{2}, b - x - \frac{b - a}{2} \right\} (L - S).$$
It is easy to find that both 
the inequality (2.9).

Consequently, the inequality (2.1) follows from substituting (2.5) to the left hand side of
(2.10)

Put

(2.5)

\[ g(t) := f(t) - \frac{L + l}{2} t. \]

It is easy to find that the function 
\( g : [a, b] \to R \) is \( M \)-Lipschitzian on \([a, b]\) with 
\( M = \frac{L - l}{2} \).

So, the Riemann-Stieltjes integral \( \int_a^b p(x, t) dg(t) \) exists. Using the integration by parts 
formula for Riemann-Stieltjes integral, we have

\[
\int_a^b p(x, t) dg(t) = \int_a^x \left( t - \left[ a + \frac{b - a}{2} \right] \right) dg(t) + \int_x^b \left( t - \left[ b - \frac{b - a}{2} \right] \right) dg(t) \\
= (b - a) \left[ g(x)(1 - h) + \frac{g(a) + g(b)}{2}h \right] - \int_a^b g(t) dt.
\]

By Lemma [5] we have

\[
(2.7) \quad \left| (b - a) \left[ g(x)(1 - h) + \frac{g(a) + g(b)}{2}h \right] - \int_a^b g(t) dt \right| \leq \frac{L - l}{2} \int_a^b |p(x, t)| dt.
\]

It is not difficult to find that (see [5])

\[
(2.8) \quad \int_a^b |p(x, t)| dt = \frac{(b - a)^2}{4} [h^2 + (h - 1)^2] + \left( x - \frac{a + b}{2} \right)^2,
\]

and so from (2.7) and (2.8) we get

\[
(2.9) \quad \left| (b - a) \left[ g(x)(1 - h) + \frac{g(a) + g(b)}{2}h \right] - \int_a^b g(t) dt \right| \\
\leq \frac{1}{2} \left| \frac{(b - a)^2}{4} [h^2 + (h - 1)^2] + \left( x - \frac{a + b}{2} \right)^2 \right| (L - l).
\]

Consequently, the inequality (2.1) follows from substituting (2.5) to the left hand side of 
the inequality (2.9).

Now we proceed to prove the inequalities (2.2) and (2.3).

Put

\[
(2.10) \quad g_1(t) := f(t) - lt \quad \text{and} \quad g_2(t) := f(t) - Lt.
\]

It is easy to find that both 
\( g_1, g_2 : [a, b] \to R \) are functions of bounded variation on \([a, b]\) with

\[
(2.11) \quad \bigvee_a^b (g_1) = f(b) - f(a) - l(b - a) \quad \text{and} \quad \bigvee_a^b (g_2) = L(b - a) - [f(b) - f(a)].
\]

So, the Riemann-Stieltjes integrals \( \int_a^b p(x, t)dg_1(t) \) and \( \int_a^b p(x, t)dg_2(t) \) exist. Using the 
integration by parts formula for Riemann-Stieltjes integral, we have

\[
(2.12) \quad \int_a^b p(x, t)dg_1(t) = (b - a) \left[ g_1(x)(1 - h) + \frac{g_1(a) + g_1(b)}{2}h \right] - \int_a^b g_1(t) dt,
\]

for all \( a + h((b-a)/2) \leq x \leq b - h((b-a)/2) \) and \( h \in [0,1] \), where \( S = (f(b) - f(a))/(b - a) \).
and
\[ \int_a^b p(x,t)dg_2(t) = (b-a) \left[ g_2(x)(1-h) + \frac{g_2(a)+g_2(b)}{2} h \right] - \int_a^b g_2(t)dt. \]

Then by Lemma 6 we can deduce that
\[ \left| (b-a) \left[ g_1(x)(1-h) + \frac{g_1(a)+g_1(b)}{2} h \right] - \int_a^b g_1(t)dt \right| \leq \max_{t \in [a,b]} |p(x,t)| \left\| g_1 \right\|
\]
and
\[ \left| (b-a) \left[ g_2(x)(1-h) + \frac{g_2(a)+g_2(b)}{2} h \right] - \int_a^b g_2(t)dt \right| \leq \max_{t \in [a,b]} |p(x,t)| \left\| g_2 \right\|.

Notice that
\[ \max_{t \in [a,b]} |p(x,t)| = \max \left\{ \frac{b-a}{2}, x-a-h\frac{b-a}{2}, b-x-h\frac{b-a}{2} \right\}, \]
and from (2.11), we get
\[ \left| (b-a) \left[ g_1(x)(1-h) + \frac{g_1(a)+g_1(b)}{2} h \right] - \int_a^b g_1(t)dt \right| \leq (b-a) \max \left\{ \frac{b-a}{2}, x-a-h\frac{b-a}{2}, b-x-h\frac{b-a}{2} \right\} (S-l), \]
and
\[ \left| (b-a) \left[ g_2(x)(1-h) + \frac{g_2(a)+g_2(b)}{2} h \right] - \int_a^b g_2(t)dt \right| \leq (b-a) \max \left\{ \frac{b-a}{2}, x-a-h\frac{b-a}{2}, b-x-h\frac{b-a}{2} \right\} (L-S), \]
where \( S = (f(b)-f(a))/(b-a) \).

Consequently, inequalities 2.2 and 2.3 follow from substituting (2.10) to the left hand sides of (2.14) and (2.15), respectively. □

**Corollary 1.** Under the assumptions of Theorem 7, we have
\[ \left| (b-a) \left[ f(x) - \frac{L+l}{2} \left( x - \frac{a+b}{2} \right) \right] - \int_a^b f(t)dt \right| \leq \frac{1}{2} \left[ \frac{(b-a)^2}{4} + \left( x - \frac{a+b}{2} \right)^2 \right] (L-l), \]
and
\[ \left| (b-a) \left[ f(x) - l \left( x - \frac{a+b}{2} \right) \right] - \int_a^b f(t)dt \right| \leq (b-a) \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (S-l), \]
and

\[
\left| (b-a) \left[ f(x) - L \left( x - \frac{a+b}{2} \right) \right] - \int_a^b f(t)dt \right| \\
\leq (b-a) \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (L - S).
\]

\textbf{Proof.} We set \( h = 0 \) in the above theorem and utilize

\[
\max\{x-a, b-x\} = \frac{1}{2} [b-a + |2x-a-b|] = \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right|.
\]

\( \Box \)

\textbf{Remark 1.} If we set \( x = (a+b)/2 \) in Corollary 1 then we get corresponding mid-point inequalities.

\textbf{Corollary 2.} Under the assumptions of Theorem 7, we have

\[
\left| \frac{b-a}{2} [f(a) + f(b)] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^2}{8} (L - l),
\]

\[
\left| \frac{b-a}{2} [f(a) + f(b)] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^2}{2} (S - l),
\]

and

\[
\left| \frac{b-a}{2} [f(a) + f(b)] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^2}{2} (L - S),
\]

\textbf{Proof.} We set \( h = 1 \) in the above theorem and utilize

\[
\max \left\{ \frac{b-a}{2}, x - \frac{a+b}{2}, \frac{a+b}{2} - x \right\} = \frac{b-a}{2}.
\]

\( \Box \)

\textbf{Corollary 3.} Under the assumptions of Theorem 7 we have

\[
\left| (b-a) \left[ \frac{1}{2} f(x) - \frac{L+l}{4} \left( x - \frac{a+b}{2} \right) + \frac{f(a) + f(b)}{4} \right] - \int_a^b f(t)dt \right| \\
\leq \frac{1}{2} \left[ \frac{(b-a)^2}{8} + \left( x - \frac{a+b}{2} \right)^2 \right] (L - l),
\]

\[
\left| (b-a) \left[ \frac{1}{2} f(x) - \frac{l}{2} \left( x - \frac{a+b}{2} \right) + \frac{f(a) + f(b)}{4} \right] - \int_a^b f(t)dt \right| \\
\leq (b-a) \left[ \frac{b-a}{4} + \left| x - \frac{a+b}{2} \right| \right] (S - l),
\]
and
\[
(b-a) \left[ \frac{1}{2} f(x) - \frac{L}{2} \left( x - \frac{a+b}{2} \right) + \frac{f(a) + f(b)}{4} \right] - \int_a^b f(t) dt \leq (b-a) \left[ \frac{b-a}{4} + \left| x - \frac{a+b}{2} \right| \right] (L-S).
\]

**Proof.** We set \( h = 1/2 \) in the above theorem and utilize
\[
\max \left\{ \frac{b-a}{4}, x - \frac{3a+b}{4}, \frac{a+3b}{4} - x \right\} = \frac{b-a}{4} + \left| x - \frac{a+b}{2} \right|.
\]

\[\Box\]

**Remark 2.** If we set \( x = (a+b)/2 \) in Corollary 3, then we get corresponding three point inequalities (i.e. the average of a mid-point and trapezoid type rules).

**Corollary 4.** Under the assumptions of Theorem 7, we have
\[
\left| \frac{b-a}{6} [f(a) + 4f(x) + f(b)] - \frac{L+l}{3} \left( x - \frac{a+b}{2} \right) - \int_a^b f(t) dt \right| \leq \frac{1}{2} \left[ \frac{5}{36} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right] (L-l),
\]

\[
\left| \frac{b-a}{6} [f(a) + 4f(x) + f(b)] - \frac{2L}{3} \left( x - \frac{a+b}{2} \right) - \int_a^b f(t) dt \right| \leq (b-a) \left[ \frac{b-a}{3} + \left| x - \frac{a+b}{2} \right| \right] (S-l),
\]

and
\[
\left| \frac{b-a}{6} [f(a) + 4f(x) + f(b)] - \frac{2L}{3} \left( x - \frac{a+b}{2} \right) - \int_a^b f(t) dt \right| \leq (b-a) \left[ \frac{b-a}{3} + \left| x - \frac{a+b}{2} \right| \right] (L-S).
\]

**Proof.** We set \( h = 1/3 \) in the above theorem and utilize
\[
\max \left\{ \frac{b-a}{6}, x - \frac{5a+b}{6}, \frac{a+5b}{6} - x \right\} = \frac{b-a}{3} + \left| x - \frac{a+b}{2} \right|.
\]

\[\Box\]

**Remark 3.** If we set \( x = (a+b)/2 \) in Corollary 4, then we get corresponding Simpson’s inequalities.

**Remark 4.** It is interesting to note that the smallest bound for (2.1)-(2.3) is obtained at \( h = 1/2 \) for fixed \( x \). Thus the quadrature rules (2.24)-(2.26) comprised of the linear combination of the perturbed mid-point and trapezoidal rules are optimal and has a lower bound than the perturbed Simpson’s rules (2.28)-(2.30).

**Remark 5.** It is clear that Theorem 7 can be regarded as generalization of Theorems 1 and 2.
Theorem 8. Let $f : [a, b] \to \mathbb{R}$ be $L$-Lipschitzian on $[a, b]$. Then we have

\[
(b-a) \left[ f(x)(1-h) + \frac{f(a) + f(b)}{2}h \right] - \int_a^b f(t)dt \leq \left[ \frac{(b-a)^2}{4} h^2 + (h-1)^2 \right] \left[ \left( x - \frac{a+b}{2} \right)^2 \right] L,
\]

(2.32)

and

\[
(b-a) \left\{ f(x) + L \left( x - \frac{a+b}{2} \right) \right\} (1-h) + \frac{f(a) + f(b)}{2}h - \int_a^b f(t)dt \leq (b-a) \max \left\{ h \frac{b-a}{2}, x - a - h \frac{b-a}{2}, b - x - h \frac{b-a}{2} \right\} (S+L),
\]

(2.33)

for all $a + h((b-a)/2) \leq x \leq b - h((b-a)/2)$ and $h \in [0, 1]$, where $S = (f(b) - f(a))/(b-a)$.

Proof. We get inequality (2.32) and (2.33) immediately by taking $l = -L$ in (2.1) and (2.2).

\[ \Box \]

3. Applications in numerical integration

We restrict further considerations to the perturbed three point rules. We also emphasize that similar considerations can be done for all quadrature rules considered in the previous section.

Theorem 9. Let all the assumptions of Theorem 7 hold. If $I_n = \{ a = x_0 < x_1 < \cdots < x_n = b \}$ is a given subdivision of the interval $[a, b]$ and $h_i = x_{i+1} - x_i, i = 0, 1, 2, \cdots, n-1$, then

\[
\int_a^b f(t)dt = A_{li}(I_n, \xi, f) + R_{li}(I_n, \xi, f),
\]

(3.1)

where

\[
A_{li}(I_n, \xi, f) = \frac{1}{2} \sum_{i=0}^{n-1} f(\xi_i)h_i + \frac{1}{2} \sum_{i=0}^{n-1} f(x_i) + f(x_{i+1})h_i - \frac{L+l}{4} \sum_{i=0}^{n-1} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i,
\]

(3.2)

for $x_i \leq \xi_i \leq x_{i+1}, i = 0, 1, 2, \cdots, n-1$. The remainder term satisfies

\[
|R_{li}(I_n, \xi, f)| \leq \sum_{i=0}^{n-1} \frac{1}{2} \left[ \frac{h_i^2}{8} + \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] (L-l).
\]

(3.3)

Also,

\[
\int_a^b f(t)dt = A_l(I_n, \xi, f) + R_l(I_n, \xi, f),
\]

(3.4)

where

\[
A_l(I_n, \xi, f) = \frac{1}{2} \sum_{i=0}^{n-1} f(\xi_i)h_i + \frac{1}{2} \sum_{i=0}^{n-1} f(x_i) + f(x_{i+1})h_i - \frac{l}{2} \sum_{i=0}^{n-1} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i,
\]

(3.5)
and
\[
|R_t(I_n, \xi, f)| \leq \sum_{i=0}^{n-1} h_i \left[ \frac{h_i}{4} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] (S_i - l),
\]
where 
\[S_i = f(x_{i+1} - f(x_i))/h_i, \ i = 0, 1, 2, \ldots, n - 1.\]

Also,
\[
\int_a^b f(t)dt = A_L(I_n, \xi, f) + R_L(I_n, \xi, f),
\]
where
\[
A_L(I_n, \xi, f) = \frac{1}{2} \sum_{i=0}^{n-1} f(\xi_i)h_i + \frac{1}{2} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h_i - \frac{L}{2} \sum_{i=0}^{n-1} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i,
\]
and
\[
|R_L(I_n, \xi, f)| \leq \sum_{i=0}^{n-1} h_i \left[ \frac{h_i}{4} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] (L - S_i).
\]

**Proof.** Apply Corollary 5 to the interval \([x_i, x_{i+1}], i = 0, 1, 2, \ldots, n - 1\) and sum. Then use the triangle inequality to obtain the desired result. \(\Box\)

**Remark 6.** If we set \(\xi_i = (x_i + x_{i+1})/2\) in Theorem 6 then we get corresponding composite rules which do not depend on \(\xi\).

**Remark 7.** Note that we can apply quadrature rules in 10 and 11 only if \(f \in C^1[a, b]\), while we can apply here if \(f\) is \((l, L)\)-Lipschitzian. Hence, the above obtained result enlarges the applicability of the quadrature rules.

### 4. Applications for cumulative distribution functions

Now we consider some applications for cumulative distribution functions.

Let \(X\) be a random variable having the probability density function \(f: [a, b] \rightarrow R_+\) and the cumulative distribution function \(F(x) = Pr(X \leq x)\), i.e.,
\[
F(x) = \int_a^x f(t)dt, \ x \in [a, b].
\]

\(E(X)\) is the expectation of \(X\). Then we have the following inequality.

**Theorem 10.** With the above assumptions and if there exist constants \(M, m\) such that \(0 \leq m \leq f(t) \leq M\) for all \(t \in [a, b]\), then we have the inequalities
\[
\left| (b - a) \left\{ Pr(X \leq x) - \frac{M + m}{2} \left( x - \frac{a + b}{2} \right) \right\} (1 - h) + \frac{1}{2} h \right| - (b - E(X)) \leq \frac{1}{2} \left[ \frac{(b - a)^2}{4} [h^2 + (h - 1)^2] + \left( x - \frac{a + b}{2} \right)^2 \right] (M - m),
\]
\[
\left| (b - a) \left\{ Pr(X \leq x) - m \left( x - \frac{a + b}{2} \right) \right\} (1 - h) + \frac{1}{2} h \right| - (b - E(X)) \leq (b - a) \max \left\{ \frac{b - a}{2}, x - a - h \frac{b - a}{2}, b - x - h \frac{b - a}{2} \right\} \left( \frac{1}{b - a - m} \right),
\]
where \(a, b, h, M, m, t \in R_+\), \(t \neq 0, 1\), and \(h = (b - a - t)/b - a\).
and
\[
(b - a) \left\{ \left[ P_r(X \leq x) - M \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{1}{2} h \right\} - (b - E(X)) \right]
\] (4.3)
\[
\leq (b - a) \max \left\{ h \frac{b - a}{2}, x - a - h \frac{b - a}{2}, b - x - h \frac{b - a}{2} \right\} \left( M - \frac{1}{b - a} \right),
\]
for all \( a + h((b - a)/2) \leq x \leq b - h((b - a)/2) \) and \( h \in [0, 1] \).

**Proof.** It is easy to show that the function \( F(x) = \int_a^x f(t) \, dt \) is \((m, M)\)-Lipschitzian on \([a, b]\). So, by Theorem 7 we get
\[
(b - a) \left\{ \left[ F(x) - \frac{M + m}{2} \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{F(a) + F(b)}{2} h \right\} - \int_a^b F(t) \, dt \right]
\] (4.4)
\[
\leq \frac{1}{2} \left[ (b - a)^2 \left[ h^2 + (h - 1)^2 \right] + \left( x - a + b \right)^2 \right] (M - m),
\]
\[
(b - a) \left\{ \left[ F(x) - m \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{F(a) + F(b)}{2} h \right\} - \int_a^b F(t) \, dt \right]
\] (4.5)
\[
\leq (b - a) \max \left\{ h \frac{b - a}{2}, x - a - h \frac{b - a}{2}, b - x - h \frac{b - a}{2} \right\} (S - m),
\]
and
\[
(b - a) \left\{ \left[ F(x) - M \left( x - \frac{a + b}{2} \right) \right] (1 - h) + \frac{F(a) + F(b)}{2} h \right\} - \int_a^b F(t) \, dt \right]
\] (4.6)
\[
\leq (b - a) \max \left\{ h \frac{b - a}{2}, x - a - h \frac{b - a}{2}, b - x - h \frac{b - a}{2} \right\} (M - S),
\]
where \( S = (F(b) - F(a))/(b - a) \).

As \( F(a) = 0, F(b) = 1 \), and
\[
\int_a^b F(t) \, dt = b - E(X),
\]
then we can easily deduce inequalities (4.1)-(4.3). \( \square \)

**Corollary 5.** Under the assumptions of Theorem 11, we have
\[
(b - a) \left\{ \frac{1}{2} P_r(X \leq x) - \frac{M + m}{4} \left( x - \frac{a + b}{2} \right) + \frac{1}{4} \right\} - (b - E(X)) \right]
\] (4.7)
\[
\leq \frac{1}{2} \left[ (b - a)^2 \left[ h^2 + \left( x - a + b \right)^2 \right] \right] (M - m),
\]
\[
(b - a) \left\{ \frac{1}{2} P_r(X \leq x) - \frac{m}{2} \left( x - \frac{a + b}{2} \right) + \frac{1}{4} \right\} - (b - E(X)) \right]
\] (4.8)
\[
\leq (b - a) \left\{ \frac{b - a}{4} + \left| x - \frac{a + b}{2} \right| \right\} \left( \frac{1}{b - a} - m \right),
\]
and
\[
(b-a) \left[ \frac{1}{2} Pr(X \leq x) - \frac{M}{2} \left( x - \frac{a+b}{2} \right) + \frac{1}{4} \right] - (b - E(X)) \leq (b-a) \left\{ \frac{b-a}{4} + \left| x - \frac{a+b}{2} \right| \left( M - \frac{1}{b-a} \right) \right\}.
\]

**Proof.** We set \( h = 1/2 \) in the above theorem. \( \square \)

**Corollary 6.** Under the assumptions of Theorem 10 we have
\[
\left| \left[ \frac{1}{2} Pr \left( X \leq \frac{a+b}{2} \right) + \frac{1}{4} \right] - \frac{b - E(X)}{b-a} \right| \leq \frac{b-a}{16} (M - m),
\]
(4.10)
\[
\left| \frac{1}{2} Pr \left( X \leq \frac{a+b}{2} \right) + \frac{1}{4} \right| - \frac{b - E(X)}{b-a} \leq \frac{b-a}{4} \left( \frac{1}{b-a} - m \right),
\]
(4.11)
and
\[
\left| \frac{1}{2} Pr \left( X \leq \frac{a+b}{2} \right) + \frac{1}{4} \right| - \frac{b - E(X)}{b-a} \leq \frac{b-a}{4} \left( M - \frac{1}{b-a} \right).
\]
(4.12)

**Proof.** We set \( x = \frac{a+b}{2} \) in Corollary 5. \( \square \)

**Corollary 7.** Under the assumptions of Theorem 10 we have
\[
\left| E(x) - \frac{a+3b}{4} + \frac{1}{8} (M+m)(b-a)^2 \right| \leq \frac{3}{16} (b-a)^2 (M-m),
\]
(4.13)
\[
\left| E(x) - \frac{a+3b}{4} + \frac{1}{4} m(b-a)^2 \right| \leq \frac{3}{4} (b-a)^2 (M-m),
\]
(4.14)
and
\[
\left| E(x) - \frac{a+3b}{4} + \frac{1}{4} M(b-a)^2 \right| \leq \frac{3}{4} (b-a)^2 (M-m).
\]
(4.15)

**Proof.** We set \( x = a \) or \( x = b \) in Corollary 5. \( \square \)

**Remark 8.** Similar results can be obtained when set \( h = 0, 1 \) or \( 1/3 \) in the above theorem.

**Acknowledgements:** The first author was supported by the Science Research Foundation of NUIST, and the third author was supported by Youth Natural Science Foundation of Zhengzhou Institute of Aeronautical Industry Management under Grant No.Q05K066.

**REFERENCES**

1. N. S. Barnett And S. S. Dragomir, Some inequalities for probability, expectation and variance of random variable defined over a finite interval, Computers and Mathematics with Applications, 43(2002), 1319-1357.
2. P. Cerone and S. S. Dragomir, Trapezoidal-type Rules from an Inequalities Point of View, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, (2000), 65-134.
3. P. Cerone and S. S. Dragomir, Midpoint-type Rules from an Inequalities Point of View, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, (2000), 65-134.

4. S. S. Dragomir, R. P. Agarwal and P. Cerone, *On Simpsons inequality and applications*, J. Inequal. Appl. **5** (2000), 533-579.

5. S. S. Dragomir, P. Cerone and J. Roumeliotis, *A new generalization of Ostrowski’s integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means*, Appl. Math. Lett. **13**(1) (2000), 19-25.

6. W. J. Liu, C. C. Li and J. W. Dong, *Note on Qi’s Inequality and Bougoffa’s Inequality*, J. Inequal. Pure Appl. Math. **7**(4) (2006), Art.129.

7. W. J. Liu, Q. L. Xue and J. W. Dong, *New Generalization of Perturbed Trapezoid and Mid-point Inequalities and Applications*, (submitted).

8. W. J. Liu, J. Luo and J. W. Dong, *Further Generalization of Perturbed Mid-point and Trapezoid Inequalities and Applications*, (submitted).

9. Z. Liu, *Some Ostrowski-Grüss type inequalities and applications*, Computers and Mathematics with Applications, (2007), doi:10.1016/j.camwa.2006.12.021.

10. N. Ujević, *A Generalization of Ostrowski’s Inequality and Applications in Numerical Integration*, Applied Mathematics Letters 17, (2004), 133-137.

11. N. Ujević, *Pertubations of an Ostrowski Type Inequality and Applications*, IJMMS, **32**(8) (2002), 491-500.

12. N. Ujević, *Error inequalities for a generalized trapezoid rule*, Appl. Math. Lett., **19** (2006), 32-37.

College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail: wjliu@nuist.edu.cn

College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail: qlx_1@yahoo.com.cn

Department of Mathematics and Physics, Zhengzhou Institute of Aeronautical Industry Management, Zhengzhou 450015, China.
E-mail: dongjianweiccm@163.com