OLD AND NEW EXAMPLES OF SURFACES OF GENERAL TYPE WITH $p_g = 0$

VIK. S. KULIKOV

Abstract. Surfaces of general type with geometric genus $p_g = 0$, which can be given as Galois covering of the projective plane branched over an arrangement of lines with Galois group $G = (\mathbb{Z}/q\mathbb{Z})^k$, where $k \geq 2$ and $q$ is a prime number, are investigated. The classical Godeaux surface, Campedelli surfaces, Burniat surfaces, and a new surface $X$ with $K_X^2 = 6$ and $(\mathbb{Z}/3\mathbb{Z})^3 \subset \text{Tors}(X)$ can be obtained as such coverings. It is proved that the group of automorphisms of a generic surface of the Campedelli type is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. The irreducible components of the moduli space containing the Burniat surfaces are described. It is shown that the Burniat surface $S$ with $K_S^2 = 2$ has the torsion group $\text{Tors}(S) \simeq (\mathbb{Z}/2\mathbb{Z})^3$, (therefore, it belongs to the family of the Campedelli surfaces), i.e., the corresponding statement in the papers of C. Peters "On certain examples of surfaces with $p_g = 0$" in Nagoya Math. J. 66 (1977), and I. Dolgachev "Algebraic surfaces with $q = p_g = 0"$ in Algebraic surfaces, Liguori, Napoli (1977), and in the book of W. Barth, C. Peters, A. Van de Ven "Compact complex surfaces", p. 237, about the torsion group of the Burniat surface $S$ with $K_S^2 = 2$ is not correct.

0. Introduction

As is known, the self-intersection number of the canonical class of the surfaces of general type with geometric genus $p_g = 0$ can take the values $K^2 = 1, \ldots, 9$, and in the past century the existence of such surfaces for all possible values of $K^2$ was proved. Nevertheless, our knowledge about the surfaces of general type with $p_g = 0$ is far from completeness. In particular, the moduli spaces of such surfaces are not described completely up to now. Moreover, the list of all possible abelian groups, which can be realized as the torsion group of such surfaces, is unknown.

In the paper we investigate surfaces of general type with $p_g = 0$, which can be given as Galois covering of the projective plane branched over an arrangement of lines with Galois group $G = (\mathbb{Z}/q\mathbb{Z})^k$, where

The work was partially supported by RFBR (No. 02-01-00786).
$k \geq 2$ and $q$ is a prime number. In particular, the classical Godeaux surface [God], Campedelli surfaces [Cam], [Mi], Burniat surfaces [Bu], and a new surface $X$ with $K_X^2 = 6$ and
\[(\mathbb{Z}/3\mathbb{Z})^3 \subset \text{Tors}(X) = \text{Tors} H_1(X, \mathbb{Z}) = \text{Tors} H^2(X, \mathbb{Z})\]
can be obtained as such coverings. It is proved that the group of automorphisms of a generic surface of Campedelli type is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. It is shown that the Burniat surface $S$ with $K_S^2 = 2$ has the torsion group $\text{Tors}(S) \simeq (\mathbb{Z}/2\mathbb{Z})^3$ (therefore, it belongs to the family of the Campedelli surfaces ([Mi], see also Proposition 4.24), i.e., the corresponding statement in [Pet], [Dol], and in [B-P-V], p. 237, about the torsion group of the Burniat surface $S$ with $K_S^2 = 2$ is not correct.

The irreducible components of the moduli space containing the Burniat surfaces are described. The description is depicted in the following diagram

\[
\begin{align*}
\mathcal{M}_2 &= \mathcal{C} & M_3 & M_4' \subset M_4 = M_4' \cup M_4'' & M_5 & M_6 \\
\cup & \quad \cup & \cup & \cup & \cup & \cup \\
\tilde{B}_2 & \quad \quad \longrightarrow & \tilde{B}_3 & \quad \longrightarrow & \tilde{B}_4'' \subset \tilde{B}_4 = \tilde{B}_4'' \cup \tilde{B}_4' & \quad \longrightarrow & \tilde{B}_5 & \longrightarrow & \tilde{B}_6
\end{align*}
\]

where $\mathcal{M}_k$ is the union of irreducible components of the moduli space of surface of general type with $p_g = 0$ and $K^2 = k$ containing the Burniat surfaces and $\mathcal{C}$ is the moduli space of the Campedelli surfaces. The points in the subvarieties $\tilde{B}_k$ of $M_k$ correspond to the Burniat surfaces. The varieties $\tilde{B}_k$ for $k \neq 4$ are unirational and $\tilde{B}_4$ consists of two rational surfaces (the points of the irreducible component $\tilde{B}_4''$ parametrize the Burniat surfaces with $K^2 = 4$ having "−2"-curves), $\tilde{B}_2$ consists of a single point, $\tilde{B}_3$ is a rational curve, $\dim \tilde{B}_5 = 3$, and $\dim \tilde{B}_6 = 4$. The subvarieties $\tilde{B}_k$ are everywhere dense in $\mathcal{M}_k$ for $k \geq 4$, $\dim \mathcal{M}_3 = 4$, and as is known (see [Mi]), $\mathcal{C}$ is unirational, $\dim \mathcal{C} = 6$.

The arrows in the diagram show the adjacency of the components (for example, $\tilde{B}_3 \rightarrow \tilde{B}_4''$ means that the Burniat surfaces with $K^2 = 3$ are degenerations of Burniat surfaces with $K^2 = 4$ having "−2"-curves).

The paper is organized as follows. In section 1, we recall the basic facts about Galois coverings $g : Y \rightarrow \mathbb{P}^2$ of the plane $\mathbb{P}^2$ with Galois group $G = (\mathbb{Z}/q\mathbb{Z})^k$ branched along a line arrangement $L \subset \mathbb{P}^2$ and show how to obtain a resolution $X$ of the singular points of $Y$ in terms of the singular points of $L$. Then these results are used in section 2 for the calculations of $K_X^2$ and the topological Euler characteristic $e(X)$. In section 3, we recall an algorithm of calculation of the geometric genus...
Old and New Examples

3

of $X$. Section 4 is devoted to the examples and the result mentioned above.

Acknowledgement. I would like to express my gratitude to the University of Padova (Italy) for its hospitality during the early stages of the preparation of this paper.

1. Abelian coverings of the plane branched over an arrangement of lines

By a Galois covering of a smooth algebraic variety $Y$ we mean a finite morphism $h : X \to Y$ of a normal algebraic variety $X$ to $Y$ such that the function fields imbedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ induced by $h$ is a Galois extension. As is well known, a finite morphism $h : X \to Y$ is a Galois covering with Galois group $G$ if and only if $G$ coincides with the group of covering transformations and the latter acts transitively on every fiber of $h$. Besides, a finite branched covering is Galois if and only if the un-ramified part of the covering (i.e., the restriction to the complements of the ramification and branch loci) is Galois. In addition, a branched covering is determined up to isomorphism by its un-ramified part. Moreover, a covering map from the un-ramified part of one branched covering to the unramified part of another one induces a covering morphism between these branched coverings if the extension of the morphism of underlying varieties to the branch loci is given. Let us recall also that an unramified covering is Galois with Galois group $G$ if and only if it is a covering associated with an epimorphism of the fundamental group of the underlying variety to $G$, and, in particular, the Galois coverings with abelian Galois group $G$ are in one-to-one correspondence with epimorphisms to $G$ of the first homology group with integral coefficients. All these results are well known and their most nontrivial part can be deduced, for example, from the Grauert-Remmert existence theorem [G-R].

In what follows we deal only with coverings of the complex projective plane $\mathbb{P}^2$ ramified over an arrangement of lines $\mathcal{L} = L_1 \cup \cdots \cup L_n$. The simple loops $\lambda_i$, $1 \leq i \leq n$, around the lines $L_i$ generate $H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \simeq \mathbb{Z}^n$. They are subject to the relation

$$\lambda_1 + \cdots + \lambda_n = 0.$$

As for general abelian Galois coverings, a Galois covering $g : Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ with abelian Galois group $G$ branched along $\mathcal{L}$ is determined uniquely by an epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to G$, and it exists for any such epimorphism. The covering $g$ is branched along a line $L_i \subset \mathcal{L}$ if and only if $\varphi(\lambda_i) \neq 0$ and, moreover, the ramification index of $g$ along $L_i$ coincides with the order of the element $\varphi(\lambda_i)$ in $G$. 

Since $H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}) \simeq \mathbb{Z}^{n-1}$, there exists, in particular, an universal covering $g_{u(m)} : Y_{u(m)} \to \mathbb{P}^2$ corresponding to the natural epimorphism
\[ \overline{\varphi} : H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}/m\mathbb{Z}) = H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}) \otimes (\mathbb{Z}/m\mathbb{Z}). \]
The simplest example of such coverings is the following one.

**Example.** Let $\overline{T} = L_0 + L_1 + L_2 \subset \mathbb{P}^2$ be given by equation $x_0x_1x_2 = 0$, where $(x_0 : x_1 : x_2)$ are homogeneous coordinates of $\mathbb{P}^2$. It is easy to see that the covering $g_{u(m)} : \mathbb{P}^2 \to \mathbb{P}^2$ given by $y_i = x_i^m$, $i = 0, 1, 2$, is associated with the epimorphism
\[ \overline{\varphi} : H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}) \simeq \mathbb{Z}^2 \to (\mathbb{Z}/m\mathbb{Z})^2. \]

The following statement is an immediate consequence of the general results on branched coverings mentioned above.

**Proposition 1.1.** If $g : Y \to \mathbb{P}^2$ is a Galois covering with Galois group $G \simeq (\mathbb{Z}/m\mathbb{Z})^k$ branched along $\overline{T}$, then $k \leq n - 1$ and for any epimorphism $H_1(\mathbb{P}^2 \setminus \overline{T}) \to G$ there exists a unique Galois covering $h : Y_{u(m)} \to Y$ inducing this epimorphism and such that $g_{u(m)} = g \circ h$. \quad \Box

In what follows we deal with Galois coverings with Galois group $G \simeq (\mathbb{Z}/q\mathbb{Z})^k$, where $q$ is a prime number, and we construct them in a way described in the above proposition.

Put
\[ G_u = \{ \overline{\gamma} = (\gamma_1, \ldots, \gamma_{n-1}) \mid \gamma_i \in \mathbb{Z}/q\mathbb{Z} \} \simeq (\mathbb{Z}/q\mathbb{Z})^{n-1} \]
and let $\tilde{G}_u \simeq (\mathbb{Z}/q\mathbb{Z})^{n-1}$ be the dual (as a vector space over $\mathbb{Z}/q\mathbb{Z}$) group, the pairing $(\overline{\gamma}, \overline{\alpha})$ is given by
\[ (\overline{\gamma}, \overline{\alpha}) = \sum_{j=1}^{n-1} \gamma_j a_j \in \mathbb{Z}/q\mathbb{Z} \]
for $\overline{\gamma} = (\gamma_1, \ldots, \gamma_{n-1}) \in G_u$ and $\overline{\alpha} = (a_1, \ldots, a_{n-1}) \in \tilde{G}_u$.

Without loss of generality we can assume that the universal covering $g_u : Y_u \to \mathbb{P}^2$ is associated with the epimorphism $\overline{\varphi} : H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}) \to G_u$ sending $\lambda_n$ to $(q - 1, \ldots, q - 1)$ and $\lambda_i$ with $1 \leq i \leq n - 1$ to $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $i$-th place. We choose an additional line $L_\infty \subset \mathbb{P}^2$ in general position with respect to $\overline{T}$ and introduce affine coordinates $(x, y)$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$. Let $l_i(x, y) = 0$ be a linear equation of $L_i \cap \mathbb{C}^2$. Put $z_i = (l_i(l_i)^{-1})^{1/q}$, $1 \leq i \leq n - 1$. Then the function field $K_u = \mathbb{C}(Y_u) = \mathbb{C}(x, y, z_1, \ldots, z_{n-1})$ of a normal variety $Y_u$ is the extension of the function field $K = \mathbb{C}(x, y)$ of $\mathbb{P}^2$ of degree $q^{n-1}$. (In other words, the pull-back of $\mathbb{P}^2 \setminus L_\infty$ in $Y_u$ is naturally isomorphic to
the normalization of the affine subvariety of \( \mathbb{C}^{n+1} \) given in coordinates \( x, y, z_1, \ldots, z_{n-1} \) by equations \( z_1^q = l_1 t_1^{q-1}, \ldots, z_{n-1}^q = l_{n-1} t_{n-1}^{q-1} \).

For a multi-index \( \mathbf{a} = (a_1, \ldots, a_{n-1}) \), \( 0 \leq a_i \leq q - 1 \), we put

\[
    z^{\mathbf{a}} = \prod_{i=1}^{n-1} z_i^{a_i}.
\]

The action of \( \gamma = (\gamma_1, \ldots, \gamma_{n-1}) \in G_u \) on \( K_u \) is given by

\[
    \gamma(z^{\mathbf{a}}) = \mu^{(\gamma, \mathbf{a})} z^{\mathbf{a}},
\]

where \( \mu = e^{2\pi \sqrt{-1}/q} \) is the \( q \)-th root of the unity. Therefore, we have \( \text{Gal}(K_u/\mathbb{C}[x, y]) = G_u \) and

\[
    K_u = \bigoplus_{0 \leq a_i \leq q-1} \mathbb{C}(x, y) z^{\mathbf{a}}
\]

is a decomposition of the vector space \( K_u \) over \( \mathbb{C}(x, y) \) into a finite direct sum of degree 1 representations of \( G_u \).

Let \( \varphi : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to (\mathbb{Z}/q\mathbb{Z})^k \) be an epimorphism given by \( \varphi(\lambda_i) = (a_{i,1}, \ldots, a_{i,k}) \), where \( a_{1,j} + \cdots + a_{n,j} \equiv 0 \mod q \) for every \( j = 1, \ldots, k \), and let \( g : Y \to \mathbb{P}^2 \) be the corresponding Galois covering. The epimorphism \( \varphi \) induces the epimorphism \( \psi : G_u \to G \). By Proposition 1.1, there exists a unique Galois covering \( h : Y_u \to Y \). It determines the inclusion \( h^* : \mathbb{C}(Y) \to K_u \) of the function field \( \mathbb{C}(Y) \) of \( Y \) into the function field \( K_u = \mathbb{C}(Y_u) \).

Since \( \text{Gal}(K_u/h^*(\mathbb{C}(Y))) = \ker \psi \), obviously, the field \( h^*(\mathbb{C}(Y)) \) coincides with the subfield \( K_\varphi = \mathbb{C}(x, y, w_1, \ldots, w_k) \) of \( K_u \), where \( w_j = z_1^{a_{1,j}} \cdots z_{n-1}^{a_{n-1,j}} \), and

\[
    \text{Gal}(K_u/K_\varphi) = \{ (\gamma_1, \ldots, \gamma_{n-1}) \in G \mid \sum_{i=1}^{n-1} a_{i,j} \gamma_i \equiv 0 (q), 1 \leq j \leq k \}.
\]

By construction, \( Y \) is a normal surface with isolated singularities. The singular points of \( Y \) can appear only over the \( r \)-fold points of \( \mathcal{T} \) with \( r \geq 2 \), i.e., over intersection points on \( r \) lines \( L_{i_1}, \ldots, L_{i_r} \) of the arrangement.

In what follows we call 2 elements of \( (\mathbb{Z}/q\mathbb{Z})^k \) linear independent over \( \mathbb{Z}/q\mathbb{Z} \) if they generate in \( (\mathbb{Z}/q\mathbb{Z})^k \) a subgroup isomorphic to \( (\mathbb{Z}/q\mathbb{Z})^2 \).

**Lemma 1.2.** Let \( p \) be a 2-fold point of \( \mathcal{T} \) and \( \varphi(\lambda_{i_1}) \) and \( \varphi(\lambda_{i_2}) \) are linear independent over \( \mathbb{Z}/q\mathbb{Z} \) in \( (\mathbb{Z}/q\mathbb{Z})^k \). Then the surface \( Y \) is nonsingular at each point of \( g^{-1}(p) \).
Lemma 1.3. Let \( p = L_{i_1} \cap \cdots \cap L_{i_r} \) be an \( r \)-fold point of \( \overline{\mathcal{T}} \). Then \( \varepsilon_p = \lambda_{i_1} + \cdots + \lambda_{i_r} \).

Proof. To establish the relation given by the Lemma, it is sufficient to consider a generic line pencil passing through \( p \).

Lemma 1.4. If for each \( r \)-fold point \( p = L_{i_1} \cap \cdots \cap L_{i_r} \) of \( \overline{\mathcal{T}} \) with \( r \geq 3 \) either the pairs \( \varphi(\varepsilon_p) \) and \( \varphi(\lambda_{i_j}) \), \( 1 \leq j \leq r \), are linear independent over \( \mathbb{Z}/q\mathbb{Z} \) in \( (\mathbb{Z}/q\mathbb{Z})^k \) or \( \varphi(\varepsilon_p) = 0 \), then \( X \) is nonsingular.

Proof. It follows from Lemmas 1.2 and 1.3.
the arrangement \( \mathcal{T} \). As a consequence of Lemma 1.4, the constructed surface \( X \) is a resolution of singularities of \( Y \) and the covering \( f \) is included in the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\nu} & Y \\
\downarrow f & & \downarrow g \\
\mathbb{P}^2 & \xrightarrow{\sigma} & \mathbb{P}^2
\end{array}
\]

in which \( \nu \) is a regular birational map.

Let \( N_\varphi \) be the set of all non-branch points with respect to \( \varphi \). Consider the subspace of \( (\mathbb{Z}/q\mathbb{Z})^n = \{ (x_1, \ldots, x_n) \mid x_i \in \mathbb{Z}/q\mathbb{Z} \} \) of solutions of the following system of linear equations

\[
\begin{align*}
\sum_{i=1}^{n} x_i &= 0, \\
\sum_{i \in \{i_1, \ldots, i_r\}} x_i &= 0, \\
p_{i_1, \ldots, i_r} \in N_\varphi.
\end{align*}
\]

Let \( n_\varphi \) be the rank of this linear system over \( \mathbb{Z}/q\mathbb{Z} \). We have \( k \leq n_\varphi = n - n_\varphi \), since the rank of the set of vectors \( A_\varphi = \{(a_{1,j}, \ldots, a_{n,j})\}_{j=1, \ldots, k} \) is equal to \( k \) and the vectors from \( A_\varphi \) satisfy equations (1). Let us add \( k_\varphi - k \) vectors to \( A_\varphi \) to obtain a basis \( A_{u,\varphi} \) over \( \mathbb{Z}/q\mathbb{Z} \) of the space of solutions of (1)

\[ A_{u,\varphi} = \{(a_{1,j}, \ldots, a_{n,j})\}_{j=1, \ldots, k_\varphi} \]

and consider the epimorphism

\[ \psi_\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{T}, \mathbb{Z}) \to G_{u,\varphi} = (\mathbb{Z}/q\mathbb{Z})^{k_\varphi} \]

given by \( \psi_\varphi(\lambda_i) = (a_{i,1}, \ldots, a_{i,k_\varphi}) \). Obviously, the epimorphism \( \varphi \) can be decomposed into the composition \( \varphi = \eta \circ \psi_\varphi \), where \( \eta : (\mathbb{Z}/q\mathbb{Z})^{k_\varphi} \to (\mathbb{Z}/q\mathbb{Z})^k \) is the projection to the first \( k \) coordinates. Let \( \mathcal{T} : \overline{X} \to \overline{\mathbb{P}^2} \) and \( h_{u,\varphi} : \overline{X} \to \overline{X} \) be the Galois coverings associated with \( \psi_\varphi \) and \( \eta \), respectively (see Proposition 1.1). Note that the Galois group of the covering \( h_{u,\varphi} \) is isomorphic to \( (\mathbb{Z}/q\mathbb{Z})^{k_\varphi - k} \).

The group \( \text{Tors}(X) = \text{Tors}H_1(X, \mathbb{Z}) \simeq \text{Tors}H^2(X, \mathbb{Z}) \) is called the \textit{torsion group} of \( X \). Denote by \( \text{Tors}_q(X) \) the subgroup of \( \text{Tors}(X) \) consisting of the elements of order \( q \).

The above consideration gives rise to the following
Proposition 1.5. Let $f : X \to \mathbb{P}^2$ be a Galois covering associated with an epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus \overline{\mathcal{L}}, \mathbb{Z}) \to (\mathbb{Z}/q\mathbb{Z})^k$ such that all singular points of the line arrangement $\overline{\mathcal{L}}$ are $\varphi$-good. Assume also that $\varphi(\lambda_i) \neq 0$ for each $L_i \subset \overline{\mathcal{L}}$. Then $h_{u,\varphi} : X \to X$ is unramified covering.

Corollary 1.6. Let $f : X \to \mathbb{P}^2$ be as in Proposition 1.5. If the irregularity $q(X) = \dim H^1(X, \mathcal{O}_X) = 0$ and $k_\varphi - k > 0$, then the $q$-torsion group $\text{Tors}_q(X)$ is non-trivial. In particular, there is an embedding of $\ker \eta \simeq (\mathbb{Z}/q\mathbb{Z})^{k_\varphi-k}$ to $\text{Tors}_q(X)$.

2. Calculation of $K^2$ and the Euler characteristic

As above, let a Galois covering $g : Y \to \mathbb{P}^2$ with Galois group $G \simeq (\mathbb{Z}/q\mathbb{Z})^k$ branched along a line arrangement $\overline{\mathcal{L}} = L_1 + \cdots + L_n$ be determined by an epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus \overline{\mathcal{L}}, \mathbb{Z}) \to G$ such that $\varphi(\lambda_i) \neq 0$ for each $L_i \subset \overline{\mathcal{L}}$. Assume also that all singular points of $\overline{\mathcal{L}}$ are $\varphi$-good. Denote by $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ the composition of blowups with centers at the all $r$-fold points of $\overline{\mathcal{L}}$ with $r \geq 3$ and at the all double points which are non-branch points with respect to $\varphi$, and let $f : X \to \mathbb{P}^2$ be the covering induced by $\varphi$. Since all singular points of $\overline{\mathcal{L}}$ are $\varphi$-good, the surface $X$ is non-singular.

Denote by $E_p = \sigma^{-1}(p)$ the curve blown up over a $r$-fold point $p$, $L'_i = \sigma^{-1}(L_i)$ the strict transform of $L_i$, $C_i = f^{-1}(L'_i)$, and $D_p = f^{-1}(E_p)$ the strict transforms of the curves $L'_i$ and $E_p$, respectively. Let $T_r$ be the set of all $r$-fold points of $\overline{\mathcal{L}}$. Put

$$T'_r = \{ p \in T_r \mid p \text{ is a non-branch point of } \varphi \},$$

$$T''_r = T_r \setminus T'_r, T' = \bigcup_{r \geq 2} T'_r, T'' = \bigcup_{r \geq 3} T''_r, \text{ and } T = T' \cup T''.$$ Denote by $t'_r = \#T'_r$ (respectively, $t''_r = \#T''_r$) the number of the points belonging to $T'_r$ (respectively, $T''_r$) and put $t_r = t'_r + t''_r$. Note that the total transform $f^*(L'_i) = qC_i$ for each line $L_i \subset \overline{\mathcal{L}}$ and $f^*(E_p) = qD_p$ for each $p \in T''$.

Theorem 2.1. The self-intersection number $K^2_X$ of the canonical class $K_X$ of $X$ is equal to

$$K^2_X = q^{k-2}[(qn-n-3q)^2 - \sum_{r \geq 2} (rq-q-r)^2 t'_r - \sum_{r \geq 3} (rq-2q-r+1)^2 t''_r].\quad (2)$$
Proof. The canonical class of \( \tilde{\mathbb{P}}^2 \) is equal to \( K_{\tilde{\mathbb{P}}^2} = -3L + \sum_{p \in T} E_p \), where \( L = \sigma^*(\mathbb{P}^1) \) is the total transform of a line \( \mathbb{P}^1 \subset \mathbb{P}^2 \), and by adjunction formula,

\[
K_X = f^*(K_{\tilde{\mathbb{P}}^2}) + (q - 1)(\sum_{i \in T'} C_i + \sum_{p \in T''} D_p).
\]

We have

\[
q \sum_{i \in T'} C_i = f^*(nL - \sum_{r \geq 3} \sum_{p \in T_r} rE_p - 2 \sum_{p \in T''_2} E_p)
\]

and

\[
q \sum_{p \in T''} D_p = f^*(\sum_{p \in T''} E_p).
\]

Therefore

\[
qK_X = qf^*(K_{\tilde{\mathbb{P}}^2}) + (q - 1)(q \sum_{i \in T'} C_i + q \sum_{p \in T''} D_p) =
\]

\[
qf^*(-3L + \sum_{p \in T} E_p) +
(q - 1)f^*(nL - \sum_{r \geq 3} \sum_{p \in T_r} rE_p - 2 \sum_{p \in T''_2} E_p) +
(q - 1)f^*(\sum_{p \in T''} E_p)
\]

and, finally,

\[
qK_X = f^*((qn - n - 3q)L -
\sum_{r \geq 2} \sum_{p \in T'_r} (rq - q - r)E_p - \sum_{r \geq 3} \sum_{p \in T''_r} (rq - 2q - r + 1)E_p).
\]

For each divisor \( D \in \text{Pic} \tilde{\mathbb{P}}^2 \), we have

\[
(f^*(D), f^*(D))_X = \deg f \cdot (D, D)_{\tilde{\mathbb{P}}^2} = q^k \cdot (D, D)_{\tilde{\mathbb{P}}^2}.
\]

and Theorem 2.1 follows from the equalities: \((L, L)_{\tilde{\mathbb{P}}^2} = 1\), \((L, E_p)_{\tilde{\mathbb{P}}^2} = 0\) and \((E_p, E_p)_{\tilde{\mathbb{P}}^2} = -1\) for each \( E_p \).

In Section 4, we will apply Theorem 2.1 for the line arrangements \( L \) and epimorphisms \( \varphi \) with the following properties: \( t'_2 = 0 \), \( t'_4 = 0 \), and \( t_r = 0 \) for \( r \geq 5 \). In this case formula (2) takes the following form

\[
K^2_X = q^{k-2}[\left(qn - n - 3q\right)^2 - (2q - 3)^2t_3' - (q - 2)^2t_3'' - (2q - 3)^2t_4'']. \tag{3}
\]

Denote by

\[
D_K = (qn - n - 3q)L - \sum_{r \geq 2} \sum_{p \in T'_r} (rq - q - r)E_p - \sum_{r \geq 3} \sum_{p \in T''_r} (rq - 2q - r + 1)E_p.
\]

Since \( f \) is a finite Galois covering, we have the following claim.

**Claim 2.2.** Let the divisor \( D_K \) be big, i.e., \( D_K^2 > 0 \). Then
(i) the surface $X$ is not minimal if and only if there is an irreducible curve $C \subset \tilde{\mathbb{P}}^2$ such that $(D_K, C)_{\tilde{\mathbb{P}}^2} < 0$;

(ii) the canonical class of $X$ is not ample if and only if there is an irreducible curve $C \subset \tilde{\mathbb{P}}^2$ such that $(D_K, C)_{\tilde{\mathbb{P}}^2} \leq 0$.

**Theorem 2.3.** The topological Euler characteristic of the surface $X$ is equal to

$$e(X) = q^{k-2}(3q^2 - 2n(q^2 - q) + q^2 \sum_{r \geq 2} t'_r + (q - 1)^2 t''_2 + \sum_{r \geq 3} t''_r).$$

(4)

**Proof.** Denote by

$$B = \sum_{i=1}^n L'_i + \sum_{p \in T''} E_p$$

the branch locus of $f$. It is easy to see that

$$e(SingB) = \#SingB = t''_2 + \sum_{r \geq 3} r t''_r,$$

(5)

where $SingB$ is the set of double points of the curve $B$.

The topological Euler characteristic of the curve $B$ is equal to

$$e(B) = 2(n + \sum_{r \geq 3} t''_r) - \#SingB = 2n - t''_2 - \sum_{r \geq 3} (r - 2)t''_r,$$

(6)

since $B$ is a divisor with normal crossings and the topological Euler characteristic of each irreducible component of $B$ is equal to 2.

The topological Euler characteristic of $\tilde{\mathbb{P}}^2$ is equal to

$$e(\tilde{\mathbb{P}}^2) = 3 + \sum_{r \geq 2} t'_r + \sum_{r \geq 3} t''_r.$$  

(7)

We have

$$e(X) = q^k e(\tilde{\mathbb{P}}^2 \setminus B) + q^{k-1} e(B \setminus SingB) + q^{k-2} e(SingB) = q^{k-2}(q^2 e(\tilde{\mathbb{P}}^2) - (q^2 - q) e(B) - (q - 1) e(SingB)).$$

(8)

To complete the proof it is sufficient to substitute (5) – (7) in (8). \(\square\)

For the line arrangements $\mathcal{L}$ and epimorphisms $\varphi$ with the following properties: $t'_2 = 0$, $t'_4 = 0$, and $t_r = 0$ for $r \geq 5$, formula (4) takes the following form

$$e(X) = q^{k-2}(3q^2 - 2n(q^2 - q) + q^2 t'_3 + (q - 1)^2 t''_2 + (2(q - 1)^2 + 1)t''_3 + (3(q - 1)^2 + 1)t''_4).$$

(9)
3. Geometric genus calculation

The aim of this section is to explain a general algorithm we will use for calculation of the geometric genus. In fact, if we calculate the geometric genus of a covering, when we can calculate its irregularity, since their difference is a topological invariant equal to \( \frac{K^2_X + c(X)}{12} - 1 \), due to Noether’s formula. In the calculation we use permanently the invariance of the geometric genus under birational transformations, which allows us at each step to use that nonsingular birational model which is more convenient for the calculation.

The algorithm for calculation which we will use is by no means new. It is contained, for example, in [Kh-Ku]. Recall its main steps.

3.1. Reduction to cyclic coverings. Let \( g : Y_G \to \mathbb{P}^2 \), where \( Y_G \) is supposed to be a normal surface, be a Galois covering with abelian Galois group \( G = (\mathbb{Z}/q\mathbb{Z})^k \) branched along curves \( B_1, \ldots, B_n \subset \mathbb{P}^2 \). As above, such a covering is determined by an epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus \cup B_i) \to G \). Write it in a form

\[
\varphi(\gamma_i) = m_{1,i} \alpha_1 + \cdots + m_{k,i} \alpha_k, \quad i = 1, \ldots, n,
\]

where \( \alpha_j \) are standard generators of \( G = \oplus(\mathbb{Z}/q\mathbb{Z}) \), \( \gamma_i \) are standard generators of \( H_1(\mathbb{P}^2 \setminus \cup B_i) \) dual to \( B_i \) and \( m_{i,j} \in \mathbb{Z}/q\mathbb{Z}, 0 \leq m_{i,j} < q \), are coordinates of \( \varphi(\gamma_i) \) with respect to \( \alpha_j \). In this notation, \( Y_G \) is the normalization of the projective closure of the affine surface \( Y_{G,0} \subset \mathbb{C}^{m+2} \) given by

\[
z_j^q = \prod_{i=1}^n h_i^{m_{j,i}}(x, y), \quad j = 1, \ldots, k,
\]

where \( h_i(x, y) \) are equations of \( B_i \) in some chart \( \mathbb{C}^2 \subset \mathbb{P}^2 \).

Let \( X_G \) be the minimal desingularization of \( Y_G \). As is known, it exists, it is unique and the action of \( G \) on \( Y_G \) lifts uniquely to a regular action on \( X_G \).

Consider the action of \( G \) on the space \( H^0(X_G, \Omega^2_{X_G}) \) of regular 2-forms. It provides a decomposition

\[
H^0(X_G, \Omega^2_{X_G}) = \oplus H_{(s_1, \ldots, s_k)}
\]

into the direct sum of eigen-spaces \( H_{(s_1, \ldots, s_k)} \), where \( \omega \in H_{(s_1, \ldots, s_k)} \) if and only if \( \alpha_j(\omega) = e^{2\pi s_j \sqrt{-1}/q} \cdot \omega \) for all \( j = 1, \ldots, k \). Let \( H \subset G \) be a subgroup and \( G_1 = G/H \). We have the following commutative diagram
where \( f_1 : Y_{G_1} \to \mathbb{P}^2 \) is the Galois covering corresponding to \( \varphi_1 = i \circ \varphi \) with \( i : G \to G_1 = G/H \) being the canonical epimorphism. The map \( h \) induces a rational dominant (i.e., whose image is everywhere dense) map \( X_G \to X_{G_1} \), and the latter, as any rational dominant map between nonsingular varieties, transforms holomorphic 2-forms to holomorphic 2-forms. Thus, the subspace \( h^*(H^0(X_{G_1}, \Omega^2_{X_{G_1}})) \subset H^0(X_G, \Omega^2_{X_G}) \) is well defined, and it coincides with the subspace \( H^0(X_G, \Omega^2_{X_G})^H \subset H^0(X_G, \Omega^2_{X_G}) \) of the elements fixed under the action of \( H \). On the other hand, an eigen-space \( H(s_1, \ldots, s_k) \) is fixed under the action of \( x_1 \alpha_1 + \cdots + x_k \alpha_k \) if and only if \( x_1 s_1 + \cdots + x_k s_k = 0 \) (mod \( q \)). Hence, the sum \( \oplus H(\theta s_1, \ldots, \theta s_k) \) taken over \( \theta \in \mathbb{Z}/q\mathbb{Z} \) coincides with \( H^0(\tilde{X}_G, \Omega^2_{X_G})^H \), where

\[
H = \{ x_1 \alpha_1 + \cdots + x_k \alpha_k \mid x_1 s_1 + \cdots + x_k s_k = 0 (q) \}.
\]

So, this sum is isomorphic to \( H^0(X_{G/H}, \Omega^2_{X_{G/H}}) \). These considerations give rise to the following result.

**Proposition 3.1.** The geometric genus \( p_g(X_G) = \dim H^0(X_G, \Omega^2_{X_G}) \) of \( X_G \) is equal to

\[
p_g(X_G) = \sum_H p_g(X_{G/H}),
\]

where the sum is taken over all subgroups \( H \) of \( G \) of \( rk H = rk G - 1 \).

**3.2. Cyclic coverings.** Now, let \( G = \mathbb{Z}/q\mathbb{Z} \) be a cyclic group. To compute \( p_g(X_G) \), let us choose homogeneous coordinates \((x_0 : x_1 : x_2)\) in \( \mathbb{P}^2 \) such that the line \( x_0 = 0 \) does not belong to the branch locus of \( g : Y_G \to \mathbb{P}^2 \). As above, \( Y_G \) is the normalization of the projective closure of the surface in \( \mathbb{C}^3 \) given by equation

\[
z^p = h(x, y),
\]

where \( x = \frac{x_1}{x_0}, \ y = \frac{x_2}{x_0} \),

\[
h(x, y) = \prod_{i=1}^n h_i^{m_i}(x, y),
\]
$h_i(x, y)$ are equations in $\mathbb{C}^2 \subset \mathbb{P}^2$ of the irreducible curves $B_i$ constituting the branch locus, and $0 < m_i < q$. Note that the degree
\[ \deg h(x, y) = \sum m_i \deg h_i(x, y) = mq \]
is divisible by $q$, since the line $x_0 = 0$ does not belong to the branch locus.

It is easy to see that over the chart $x_1 \neq 0$ the variety $Y_G$ coincides with the normalization of the surface in $\mathbb{C}^3$ given by equation
\[ w^q = \tilde{h}(u, v), \]
where $u = \frac{1}{x}$, $v = \frac{y}{x}$, $\tilde{h}(u, v) = u^{mq}h(\frac{1}{u}, \frac{v}{u})$, and $w = z u^m$.

### 3.2.1. Regularity condition over a generic point of the base.

Consider
\[ \omega \in H^0(Y_G \setminus \text{Sing} Y_G, \Omega_{Y_G \setminus \text{Sing} Y_G}^2) \]
and find a criterion of its regularity outside the ramification and singular loci.

Over the chart $x_0 \neq 0$ the form $\omega$ can be written as
\[ \omega = \left( \sum_{j=0}^{q-1} z^j g_j(x, y) \right) \frac{dx \wedge dy}{z^{q-1}}, \tag{10} \]
where $g_j(x, y)$ are rational functions in $x$ and $y$. The form
\[ \frac{dx \wedge dy}{z^{q-1}} \]
has neither poles nor zeros outside of the preimage of the branch locus. Therefore, $\omega$ is regular at a point $(a, b) \not\in \sum B_i$ if and only if all $g_j(x, y)$ are regular at the point.

In fact, if some $g_j(x, y)$ is not regular at $(a, b)$, then the sum
\[ \sum_{j=0}^{q-1} z^j g_j(x, y) \]
can be written as
\[ \sum_{j=0}^{q-1} z^j P_j(x, y) \frac{P_q(x, y)}{P_q(x, y)}, \]
where $P_j(x, y)$, $j = 0, \ldots, q$, are polynomials such that $P_j(a, b) \neq 0$ for some $j < q$ and $P_q(a, b) = 0$. Therefore,
\[ \sum_{j=0}^{q-1} z^j P_j(a, b) = 0 \]
at all \( q \) points belonging to \( f^{-1}(a, b) \), since otherwise \( \omega \) would not be regular over \((a, b)\). On the other hand, it is impossible, since a non-trivial polynomial of degree less than \( q \) can not have \( q \) roots.

3.2.2. Regularity condition over the line at infinity. Consider the form \( \omega \) over the chart \( x_1 \neq 0 \),

\[
\omega = -\left( \sum_{j=0}^{q-1} w^j \frac{g_j(u, v)}{u^{j+m+\deg g_j}} \right) \frac{1}{u^{3-m(q-1)}} \frac{du \wedge dv}{w^{q-1}}.
\]

The similar arguments as above show that the regularity criterion is equivalent to the following bound on the degrees of the rational functions \( g_j \)

\[
\deg g_j(x, y) \leq (q - j - 1)m - 3. \tag{11}
\]

3.2.3. Regularity conditions over a nonsingular point of the branch curve. Consider our form

\[
\omega = \left( \sum_{j=0}^{q-1} z^j g_j(x, y) \right) \frac{dx \wedge dy}{z^{q-1}}
\]

over a nonsingular point \((a, b)\) of one of the components, \( B_{io} \), of the branch curve. Let \( r_j \) be the order of zero (or of the pole if \( r_j < 0 \)) of the function \( g_j \) along the curve \( B_{io} \), i.e., \( g_j = \overline{f_j} \cdot h^{r_j} \) with \( \overline{f_j} \) having neither poles nor zeros along \( B_{io} \). Since \((a, b)\) is a nonsingular point of \( B \), we can assume that \( h_{io}(x, y) \) and some function \( g(x, y) \) are local analytic coordinates in some neighborhood \( U \) of \((a, b)\) (denote them by \( u \) and \( v \)). So, over \( U \) the surface \( Y_G \) (after analytic change of variables) is isomorphic to the normalization \( Y_{G,loc} \) of the surface in \( \mathbb{C}^3 \) given by

\[
z^q = u^{k_{io}}.
\]

There is an analytic function \( w \) in \( Y_{G,loc} \) such that \( u = w^q \) and \( z = w^{m_{io}} \), and such that \( w \) and \( y \) are analytic coordinates in \( Y_{G,loc} \). The differential 2-form \( \omega \) considered above has the following form in the new coordinates

\[
\omega = \left( \sum_{j=0}^{q-1} w^{j+m_{io}} \overline{f_j}(x, y)w^{qr_j} \right) \frac{qw^{q-1}dw \wedge dv}{w^{(q-1)m_{io}}}.
\]

It is easy to see that

\[
j_1m_{io} + qr_{j_1} + q - 1 - (q - 1)m_{io} \neq j_2m_{io} + qr_{j_2} + q - 1 - (q - 1)m_{io}
\]

if \( 0 < m_{io} < q, 0 \leq j_1, j_2 \leq q - 1, \) and \( j_1 \neq j_2 \). Therefore, \( \omega \) is a regular form over a nonsingular point \((a, b)\) of \( B_{io} \) if and only if

\[
jm_{io} + qr_j + q - 1 - (q - 1)m_{io} \geq 0
\]
for $0 \leq j \leq q - 1$. Moreover, if $\omega$ is a regular form over $B_{i_0}$, then $r_j$ must be equal or greater than 0, since for $0 < m_{i_0} < q$, $0 \leq j \leq q - 1$, and $r_j \leq -1$, we obtain that

$$jm_{i_0} + qr_j + q - 1 - (q - 1)m_{i_0} < 0.$$ 

It follows that if $\omega$ is a regular form, then all rational functions $g_j(x, y)$ are regular functions everywhere in $\mathbb{C}^2$ outside codimension 2. Thus, $g_j(x, y)$ should be polynomials in $x$ and $y$. Moreover, the polynomials $g_j(x, y)$ must be divisible by $h_{r_j}^{r_j}(x, y)$, where $r_j$ is the smallest non-negative integer satisfying the inequality

$$qr_j \geq (q - j - 1)m_i - q + 1. \quad (12)$$

3.2.4. Regularity conditions over singular points of the branch curve.

Let $\nu : X_G \to Y_G$ be the minimal resolution of singularities of $Y_G$ and $E$ be the exceptional divisor of $\nu$. Pick a composition $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ of $\sigma$-processes with centers at singular points of $B$ (and their preimages) such that $\sigma^{-1} \circ f \circ \nu(E_i)$ is a curve for each irreducible component $E_i$ of $E$. Let $Z$ be the normalization of $\mathbb{P}^2 \times_{\mathbb{P}^2} Y_G$. Denote by $g : X_G \to Z$ the bi-rational map induced by $\nu$ and $\sigma$. It follows from the above choice of $\sigma$ that for any $\omega \in H^0(Z \setminus \text{Sing} Z, \Omega^2_{Z \setminus \text{Sing} Z})$ its pull-back $g^*(\omega)$ is regular at generic points of $E_i$ and, thus, extends to a regular form on the whole $X_G$. Hence, $H^0(X_G, \Omega^2_{X_G})$ is isomorphic to $H^0(Z \setminus \text{Sing} Z, \Omega^2_{Z \setminus \text{Sing} Z})$.

Therefore, it remains to consider a 2-form $\omega$ written as in (10) and to find a criterion of its regularity on $Z \setminus \text{Sing} Z$. It can be done by performing, step by step, the $\sigma$-processes chosen above. Let us accomplish only the first step, since it is sufficient for the calculation in our particular example which follows.

Represent, once more, $Y_G$ as normalization of the surface given by

$$z^q = h(x, y).$$

Denote by $r$ the order of zero of $h(x, y)$ at the point $(0, 0)$, $r = sq + c$, $0 \leq c < q$, and perform the $\sigma$-process with center at this point. In a suitable chart, this $\sigma$-process $\sigma : \mathbb{C}^2_{(u, v)} \to \mathbb{C}^2_{(x, y)}$ is given by $x = u$, $y = uv$. The normalization $Z_1$ of $Y_G \times_{\mathbb{C}^2_{(x, y)}} \mathbb{C}^2_{(u, v)}$ is bi-rational to the normalization of the surface given by

$$w^q = u^c h(u, v),$$
where \( w = z/u^s \) and \( \overline{h}(u, v) = h(u, uv)/u^r \). We have

\[
\omega = \left( \sum_{j=0}^{q-1} z^j g_j(x, y) \right) \frac{dx \wedge dy}{z^{q-1}} = \\
= \left( \sum_{j=0}^{q-1} w^j \overline{g}_j(u, v) u^{s_j+1-s(q-1)} \right) \frac{du \wedge dv}{w^{q-1}},
\]

where \( s_j \) is the order of zero of \( g_j(x, y) \) at \((0, 0)\). Applying (12), we get necessary conditions for the regularity of the pull-back of \( \omega \) at generic points of the exceptional divisor: the order of zero \( s_j \) of each \( g_j(x, y) \) at singular point of the branch locus \( B \) of order \( r \) is the smallest integer satisfying the inequality

\[
qs_j \geq (q - j - 1)r - 2q + 1. \tag{13}
\]

To calculate the geometric genus of each \( X_{G_i} \), we should find explicitly all the regular 2-forms, which are written as in (10) and satisfy criteria (11) – (13).

The above discussion gives rise to the following statements for \( q = 2 \) and 3 in the case of \( q \)-sheeted cyclic covering branched along an arrangement of lines \( \mathcal{L} = L_1 + \cdots + L_n \).

**Claim 3.2.** Let \( X \) be a desingularization of a double covering \( g : Y \to \mathbb{P}^2 \) with branch locus \( \mathcal{L} = L_1 + \cdots + L_n \). Denote by \( T_r \) the set of \( r \)-fold points of \( \mathcal{L} \) and \( T = \bigcup T_r \). Then \( n = 2m \) is an even number and

\[
p_g(X) = \dim_{\mathbb{C}} \{ \overline{s} \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m - 3)) \mid \overline{s} \text{ has zero of order } \geq \left\lceil \frac{r+1}{2} \right\rceil - 2 \text{ at } p \in T, \text{ for } \forall p \in T \},
\]

where \( \left\lceil \frac{a}{b} \right\rceil \) is the smallest integer equal or greater than \( \frac{a}{b} \).

**Claim 3.3.** Let \( X \) be a desingularization of a 3-sheeted Galois covering \( g : Y \to \mathbb{P}^2 \) in non-homogeneous coordinates given by

\[
z^3 = \prod_{i=1}^{n} l_i(x, y)^{m_i},
\]

where \( l_i(x, y) = 0 \) is an equation of \( L_i \), each \( m_i = 1 \) or 2, and \( \sum m_i = 3m \) is divisible by 3. Denote by \( T_r \) the set of \( r \)-fold points of the divisor \( \prod_{i=1}^{n} l_i(x, y)^{m_i} = 0 \), \( T = \bigcup T_r \), and \( \overline{l}(x, y) = \prod_{i=1}^{n} l_i(x, y)^{m_i-1} \). Then

\[
p_g(X) = \dim_{\mathbb{C}} \mathcal{P}_0 + \dim_{\mathbb{C}} \mathcal{P}_1,
\]

where

\[
\mathcal{P}_0 = \{ s \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2m - 3 - \sum (m_i - 1))) \mid s \cdot \overline{l} \text{ has zero of order } \geq 2\left\lceil \frac{r+1}{3} \right\rceil - 2 \text{ at } p \in T, \text{ for } \forall p \in T \}.
\]
and
\[ P_1 = \{ \overline{s} \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m - 3)) \mid \overline{s} \text{ has zero of order } \geq \left\lfloor \frac{r+1}{3} \right\rfloor - 2 \text{ at } p \in T, \forall p \in T \} \]

4. Examples

4.1. Campedelli surfaces. Let \( \mathcal{L} = L_1 + \cdots + L_2 \) be a line arrangement in \( \mathbb{P}^2 \) consisting of seven lines. We numerate them by the non-zero elements \( \alpha_i \in (\mathbb{Z}/2\mathbb{Z})^3 \). We will assume that the arrangement \( \mathcal{L} \) has not \( r \)-fold points with \( r \geq 4 \) and if \( \mathcal{L} \) has a triple point \( p_{\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_1} \cap L_{\alpha_2} \cap L_{\alpha_3} \), then \( \alpha_1 + \alpha_2 + \alpha_3 \neq 0 \). Such arrangement of lines is called a Campedelli arrangement. Consider the covering \( g : Y \to \mathbb{P}^2 \) induced by the epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus \mathcal{T}, \mathbb{Z}) \to G = (\mathbb{Z}/2\mathbb{Z})^3 \) given by \( \varphi(\lambda_{\alpha}) = \alpha_i \).

The surface \( Y \) has the singular points lying only over the triple points \( p_{\alpha_1,\alpha_2,\alpha_3} \). To resolve them, let us blow up the triple points and consider the induced Galois covering \( f : X \to \mathbb{P}^2 \), where \( \sigma : \mathbb{P}^2 \to \mathbb{P}^2 \) is the composition of blowups with centers at the triple points. We call the constructed surface \( X \) a Campedelli surface. Denote by \( E_{\alpha_i,\alpha_j,\alpha_k} = \sigma^{-1}(p_{\alpha_i,\alpha_j,\alpha_k}) \) the exceptional curve lying over \( p_{\alpha_i,\alpha_j,\alpha_k} \). Since \( \alpha_i + \alpha_j + \alpha_k \neq 0 \) for triple points, each curve \( E_{\alpha_i,\alpha_j,\alpha_k} \) is a branch curve of the covering \( f \). It follows from Lemma 1.4 that \( X \) is non-singular, since \( \varphi(\varepsilon_{\alpha_i,\alpha_j,\alpha_k}) = \alpha_i + \alpha_j + \alpha_k \) and \( \alpha_i \) (respectively, \( \alpha_j \alpha_k \)) are linear independent in \( G \). Indeed, \( \alpha_i + \alpha_j + \alpha_k \) and \( \alpha_i \) are linear dependent if and only if \( \alpha_i + \alpha_j + \alpha_k = \alpha_i \), i.e., if and only if \( \alpha_j = \alpha_k \).

Proposition 4.1. The constructed Campedelli surfaces \( X \) are surfaces of general type with \( K^2_X = 2 \), \( p_g = 0 \), and \( \text{Tors}(X) = (\mathbb{Z}/2\mathbb{Z})^3 \).

Proof. Applying Claim 2.2, we have \( 2K_X = | \hat{f}^* (L) | \), where \( L = \sigma^*(\mathbb{P}^1) \) is the total transform of a line \( \mathbb{P}^1 \subset \mathbb{P}^2 \). Therefore \( X \) is a surface of general type. Moreover, it is minimal, since \((L,C)_{\mathbb{P}^2} \geq 0 \) for each curve \( C \subset \mathbb{P}^2 \). Applying (3) and (9), it is easy to see that \( K^2_X = 2 \) and \( e(X) = 10 \). Therefore, by Noether’s formula, \( p_a = 1 - q + p_g = 1 \). As above, to calculate \( p_g \), it is enough to calculate the geometric genera of 7 cyclic coverings corresponding to 7 epimorphisms \( \psi_k, k = 1, \ldots, 7 \), of \( G = (\mathbb{Z}/2\mathbb{Z})^3 \) to the cyclic group \( \mathbb{Z}/2\mathbb{Z} \). It is easy to see that each of these coverings is given in non-homogeneous coordinates by the equation of the form \( w_k^2 = l_{\alpha_{i_1}}l_{\alpha_{i_2}}l_{\alpha_{i_3}}l_{\alpha_{i_4}} \), where \( \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4} \) are the elements of \( G \) such that \( \psi_k(\alpha_{i_j}) = 1 \). Applying Claim 3.2, one can easily check that the geometric genus of each of these coverings is equal to zero. Thus, \( X \) has the geometric genus \( p_g = 0 \).
To show that $\text{Tors}(X) = (\mathbb{Z}/2\mathbb{Z})^3$, consider the universal covering $f_u(2) : X_u(2) \to \tilde{\mathbb{P}}^2$ corresponding to the epimorphism

$$
\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{T}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \mathcal{T}, \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^6,
$$

and the covering $h : X_u(2) \to X$ corresponding to an epimorphism $\psi : (\mathbb{Z}/2\mathbb{Z})^6 \to G = (\mathbb{Z}/2\mathbb{Z})^3$. By Proposition 1.5 and Corollary 1.6, the covering $h$ is unramified and $(\mathbb{Z}/2\mathbb{Z})^3 \subset \text{Tors}(X)$. Therefore, by [Mi], $\text{Tors}(X) = (\mathbb{Z}/2\mathbb{Z})^3$.

The classical Campedelli surface $S$ ([Cam]) is obtained as a resolution of singularities of a double covering $\tilde{g} : \tilde{Y} \to \mathbb{P}^2$ branched along the union of three quadrics $Q_1, Q_2, Q_3$ and a quartic $C_4$ in $\mathbb{P}^2$ such that the curve $B = Q_1 + Q_2 + Q_3 + C_4$ has 6 singular points of the type $[3,3]$ (a singular point of the type $[3,3]$ means that after the blow up with center at the singular point the strict transform of the germ of $B$ consists of 3 irreducible smooth branches each pair of which meets transversally).

Let us show that the classical Campedelli surface $S$ is isomorphic to $X$. Consider a covering $f : X \to \mathbb{P}^2$ branched along a Campedelli arrangement $\mathcal{T} = \sum L_{\alpha_i}$, $\alpha_i \in (\mathbb{Z}/2\mathbb{Z})^3 \setminus \{0\}$, having 3 triple points. The arrangement $\mathcal{T}$ is depicted in Fig. 1.
To see this isomorphism, let us consider the blowup \( \sigma : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) with center at the points \( p_1, p_2, p_3 \) and denote by \( E_i = \sigma^{-1}(p_i) \) the exceptional curve lying over \( p_i \), and the strict transforms \( \sigma^{-1}(L_{\alpha_i}) \subset \tilde{\mathbb{P}}^2 \) we will denote again by \( L_{\alpha_i} \). One can check that

\[
\varphi(\varepsilon_i) = (0, 0, 1) \tag{14}
\]

for \( i = 1, 2, 3 \). The curves \( L_{(1,0,0)}, L_{(1,1,0)}, L_{(0,1,0)} \) in \( \tilde{\mathbb{P}}^2 \) have self-intersection numbers equal \(-1\). Therefore we can blow down them by monoidal transformation \( \tau : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) (the composition \( \tau \circ \sigma^{-1} : \mathbb{P}^2 \to \mathbb{P}^2 \) is the quadratic transformation of the plane with center at the points \( p_1, p_2, p_3 \)). The curves \( L_i = \tau(E_i), i = 1, 2, 3 \), and \( \tau(L_{(1,0,1)}), \tau(L_{(0,1,1)}), \tau(L_{(1,1,1)}) \) are lines and \( \tau(L_{(0,0,1)}) \) is a conic in \( \mathbb{P}^2 \). We have the following commutative diagram

\[
\begin{array}{ccc}
X & \overset{\nu}{\longrightarrow} & \tilde{\mathbb{P}}^2 \\
\downarrow f & & \downarrow \tau \\
\mathbb{P}^2 & \overset{\tilde{g}}{\longrightarrow} & \mathbb{P}^2
\end{array}
\]

where \( \tilde{\mathbb{P}}^2 \) is a normal surface, \( \nu : X \to \tilde{\mathbb{P}}^2 \) is a bi-rational map and \( \tilde{g} : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) is the Galois covering branched along the curves \( L_i, i = 1, 2, 3, \tau(L_{(1,0,1)}), \tau(L_{(0,1,1)}), \tau(L_{(1,1,1)}) \). Since \( \varphi(\lambda_{(0,0,1)}) = (0, 0, 1) \), and taking into account (14) it is easy to see that \( \tilde{g} \) can be decomposed into the composition \( \tilde{g} = g_1 \circ \tilde{g} \), where \( g_1 : \mathbb{P}^2 \to \mathbb{P}^2 \) is the Galois covering with Galois group \( G_1 = (\mathbb{Z}/2\mathbb{Z})^2 \) branched along the lines \( \tau(L_{(1,0,1)}), \tau(L_{(0,1,1)}), \tau(L_{(1,1,1)}) \) (see Example in Section 1) and \( \tilde{g} : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) is the Galois covering with the Galois group \( G_2 = \mathbb{Z}/2\mathbb{Z} \) branched along \( Q_i = g_i^{-1}(L_i), i = 1, 2, 3, \) and \( C_4 = g_4^{-1}(\tau(L_{(0,0,1)})) \), where \( Q_1, Q_2, Q_3 \) are quadrics and \( C_4 \) is a quartic in \( \mathbb{P}^2 \) such that the curve \( B = Q_1 + Q_2 + Q_3 + C_4 \) has 6 singular points of the type \([3, 3] \). \( \square \)

**Theorem 4.2.** For a generic Campedelli surface \( X \), the group \( \text{Aut} \ (X) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^3\) and coincides with the covering transformation group \( G \) of \( f : X \to \tilde{\mathbb{P}}^2 \).

**Proof.** Let \( \mathcal{T} \subset \mathbb{P}^2 \) be a Campedelli arrangement without triple points and assume that if an automorphism \( \tilde{h} \) of \( \mathbb{P}^2 \) leaves fixed \( \mathcal{T} \) (i.e., \( h(\mathcal{T}) = \mathcal{T} \)), then \( h = \text{id} \). Consider the covering \( g : Y \to \mathbb{P}^2 \) associated with \( \varphi : H_1(\mathbb{P}^2 \setminus \mathcal{T}, \mathbb{Z}) \to G = (\mathbb{Z}/2\mathbb{Z})^3 \) given by \( \varphi(\lambda_{\alpha_i}) = \alpha_i \). Since the arrangement \( \mathcal{T} \) has not triple points, \( Y = X \) is a nonsingular surface and \( g = f \). The morphism \( f \) induces an extension of fields \( f^*(\mathbb{C}(\mathbb{P}^2)) \subset \)
Consider an element \( \alpha \in \text{Tors}_2(X) = \text{Tors}(X) \), \( \alpha \neq 0 \). The linear system \( |K_X + \alpha| \) is non-empty and \( D \in |K_X + \alpha| \) for some \( \alpha \in \text{Tors}_2(X) \) if and only if \( 2D = f^*(\tilde{L}) \) for some \( \tilde{L} \in |L| \), where \( L \) is a line in \( \mathbb{P}^2 \). Indeed, the linear system \( |K_X + \alpha| \) is non-empty by Riemann–Roch Theorem, since \( \dim H^2(X, \mathcal{O}_X(K_X + \alpha)) = \dim H^0(X, \mathcal{O}_X(\alpha)) = 0 \). Let \( D_\alpha \in |K_X + \alpha| \). Then \( 2D_\alpha \in |2K_X| \). By Riemann–Roch Theorem, we have \( \dim H^0(X, 2K_X) = K_X^2 + 1 = 3 \). On the other hand, it follows from Claim 2.2 that \( 2K_X = f^*(L) \) and \( \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3 \). Therefore, \( |2K_X| = f^*(|L|) \) and \( D \in |K_X + \alpha| \) for some \( \alpha \in \text{Tors}_2(X) \) if and only if \( 2D = f^*(\tilde{L}) \) for some \( \tilde{L} \in |L| \).

It is easy to see that there are exactly 7 lines \( \tilde{L} \in |L| \) for which the divisors \( f^*(\tilde{L}) \) are divisible by 2, namely, \( L_\alpha \subset \tilde{L} \), \( \alpha \in \text{Tors}_2(X) \), \( \alpha \neq 0 \). So, we have \( \frac{1}{2} f^*(L_\alpha) = D_\alpha \in |K_X + \alpha| \).

Let \( h : X \to X \) be an isomorphism. Then it induces isomorphisms \( h^* : \text{Tors}(X) \to \text{Tors}(X) \) and

\[
h^* : H^0(X, \mathcal{O}_X(K_X + \alpha)) \to H^0(X, \mathcal{O}_X(K_X + h^*(\alpha)))
\]

for each \( \alpha \in \text{Tors}(X) \). Therefore, \( h^*(D_\alpha) = D_{h^*(\alpha)} \) for \( \alpha \in \text{Tors}_2(X) \), \( \alpha \neq 0 \). The automorphism \( h \) induces the action \( h^* \) on the group \( \text{Div} X \). We have

\[
h^*(f^*(L_{\alpha_1} - L_{\alpha_2})) = h^*(2D_{\alpha_1} - 2D_{\alpha_2}) = 2D_{h^*(\alpha_1)} - 2D_{h^*(\alpha_2)} = f^*(L_{h^*(\alpha_1)} - L_{h^*(\alpha_2)})
\]

for any \( \alpha_1, \alpha_2 \in \text{Tors}(X) \), \( \alpha_1 \neq \alpha_2 \neq 0 \). Therefore,

\[
h^*(f^*(\frac{l_{\alpha_1}(x,y)}{l_{\alpha_2}(x,y)})) = c_{\alpha_1,\alpha_2} f^*(\frac{l_{h^*(\alpha_1)}(x,y)}{l_{h^*(\alpha_2)}(x,y)}), \tag{15}
\]

where \( c_{\alpha_1,\alpha_2} \) is a constant, since any rational function is defined uniquely up to multiplication by a constant by its divisors of zeros and poles. It follows from (15) that \( h^* \) induces an automorphism \( \tilde{h}^* \) of \( \mathbb{C}(x,y) \) such that \( f^* \circ \tilde{h}^* = h^* \circ f^* \), since the functions \( \frac{l_{\alpha_1}(x,y)}{l_{\alpha_2}(x,y)} \) generate the field \( \mathbb{C}(x,y) \). Moreover, the automorphism \( \tilde{h}^* \) induces an automorphism \( \tilde{h} \) of \( \mathbb{P}^2 \) such that \( \tilde{h}(\mathcal{T}) = \mathcal{T} \). Therefore, \( \tilde{h} = \text{id} \) and \( h \in \text{Gal}(X/\mathbb{P}^2) \). \( \square \)

**Theorem 4.3.** (cf. [Mi]) The moduli space \( \mathcal{C} \) of the Campedelli surfaces is an unirational variety, \( \dim \mathcal{C} = 6 \).
Proof. By the same arguments, that were used in the proof of Theorem 4.2, one can show that two Campedelli surfaces $X_1$ and $X_2$, defined by Campedelli line arrangements $\mathcal{L}_1$ and $\mathcal{L}_2$, are isomorphic if and only if there is a linear transformation $h$ of $\mathbb{P}^2$ sending $\mathcal{L}_1$ to $\mathcal{L}_2$.

Applying a suitable linear transformation of $\mathbb{P}^2$ and a suitable automorphism of $(\mathbb{Z}/2\mathbb{Z})^3$, we can assume that for a line arrangement $\mathcal{L} = \sum L_\alpha$, the lines $L_{(1,0,0)}$, $L_{(0,1,0)}$, $L_{(1,1,0)}$, and $L_{(1,1,1)}$ are given respectively by $z_0 = 0$, $z_1 = 0$, $z_2 = 0$, and $z_0 + z_1 + z_2 = 0$. Therefore, a line arrangement $\mathcal{L}$ is defined by a point in an everywhere dense subset $V$ of $(\mathbb{P}^2 \setminus \{\text{four points}\})^3$. Obviously, for any point $v_0 \in V$, the set $A_{v_0} \subset V$ consisting of the points for which the corresponding line arrangements $\mathcal{L}_{v}$, $v \in A_{v_0}$, can be transformed to $\mathcal{L}_{v_0}$ by linear transformations of $\mathbb{P}^2$, is finite. Therefore, the moduli space $\mathcal{C}$ is an unirational variety, $\dim \mathcal{C} = 6$ (see also Corollaries 4.23 and 4.25).

4.2. Burniat surfaces. Let $\mathcal{L}_s = L_1 + \cdots + L_9$ be an arrangement in $\mathbb{P}^2$ of nine lines depicted in Fig. 2. The arrangement $\mathcal{L}_s$ has three 4-fold points $p_1, p_2, p_3$ and $s$ ($s = 0, \ldots, 4$) triple points $p_{3+i}$, $0 < i \leq s$. 

![Fig. 2](image-url)
Such line arrangements we will call *Burinat arrangements*. Consider the covering \( g : Y_s \to \mathbb{P}^2 \) induced by the epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus \mathcal{L}_s, \mathbb{Z}) \to G = (\mathbb{Z}/2\mathbb{Z})^2 \) given by

\[
\varphi(\lambda_1) = \varphi(\lambda_2) = \varphi(\lambda_3) = (1, 0), \\
\varphi(\lambda_4) = \varphi(\lambda_5) = \varphi(\lambda_6) = (0, 1), \\
\varphi(\lambda_7) = \varphi(\lambda_8) = \varphi(\lambda_9) = (1, 1).
\]

The surface \( Y_s \) has \( 3 + s \) singular points lying over the 4-fold points \( p_j, j = 1, 2, 3 \), and the triple points \( p_{3+i}, 1 \leq i \leq s \). Let \( \sigma : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) be the composition of the blowups with centers at these points. Denote by \( E_j = \sigma^{-1}(p_j) \) the exceptional curve lying over \( p_j, 1 \leq j \leq 3 + s \). Consider the induced Galois covering \( f : X_s \to \widetilde{\mathbb{P}}^2 \). We have

\[
\varphi(\varepsilon_1) = (1, 1), \quad \varphi(\varepsilon_2) = (1, 0), \quad \varphi(\varepsilon_3) = (1, 1), \\
\varphi(\varepsilon_{3+i}) = (0, 0) \text{ for } 1 \leq i \leq s.
\]

Therefore, the curves \( E_j \) are the branch curves of \( f \) for \( j = 1, 2, 3 \) and the curves \( E_{3+j} \) are not the branch curves of \( f \) for \( j \geq 1 \). Thus, by Lemma 1.4, \( X_s \) is a smooth surface. Note that the number of triple and 4-fold points of \( \mathcal{L} \) is less than 8 and each 4 of such points do not lie in the same line. Therefore, \( \widetilde{\mathbb{P}}^2 \) is a del Pezzo surface possibly with "\(-2\)"-curves.

**Proposition 4.4.** The constructed above surfaces \( X_s \) (they are called the *Burniat surfaces*) are surfaces of general type with \( K_{X_s}^2 = (6 - s) \) and \( p_g = 0 \).

**Proof.** By Claim 2.2, we have

\[
2K_{X_s} = |f^*(3L - \sum_{i=1}^{3+s} E_i)|,
\]

where \( L = \sigma^*(\mathbb{P}^1) \) is the preimage of a line \( \mathbb{P}^1 \subset \mathbb{P}^2 \). Therefore, \( X_s \) is a surface of general type and it is minimal, since \( \widetilde{\mathbb{P}}^2 \) is a del Pezzo surface possibly with "\(-2\)"-curves. Applying (3) and (9), it is easy to see that \( K_{X_s}^2 = 6 - s \) and \( e(X) = 6 + s \). Therefore, by Noether’s formula, \( p_a = 1 - q + p_g = 1 \). As above, to calculate \( p_g \), it is enough to calculate the geometric genera of the desingularizations \( Z_i \) of 3 cyclic coverings \( g_i : Z_i \to \mathbb{P}^2 \) corresponding to 3 epimorphisms from the group \( G = (\mathbb{Z}/2\mathbb{Z})^2 \) to the cyclic group \( \mathbb{Z}/2\mathbb{Z} \):
where \( g = g_i \circ h_i \) for \( i = 1, 2, 3 \). These coverings are given in non-homogeneous coordinates respectively by the following equations:

\[
\begin{align*}
    w_1^2 &= l_1l_2l_3l_4l_5l_6; \\
    w_2^2 &= l_4l_5l_6l_7l_8l_9; \\
    w_3^2 &= l_1l_2l_3l_7l_8l_9.
\end{align*}
\]  \hspace{1cm} (16)

Applying Claim 3.2, one can easily check that the geometric genus of each of these coverings vanishes, since each of the arrangements given respectively by

\[
\begin{align*}
    l_1l_2l_3l_4l_5l_6 &= 0, \\
    l_4l_5l_6l_7l_8l_9 &= 0, \\
    l_1l_2l_3l_7l_8l_9 &= 0
\end{align*}
\]

has a 4-fold point. Thus, \( X_s \) has the geometric genus \( p_g = 0 \). \hfill\Box

Denote by \( \tilde{L}_j \) the strict transform \( \sigma^{-1}(L_j) \) of the curve \( L_j \subset \mathbb{T}_s \).

Then the divisor \( \sum \tilde{L}_j + \sum_{i=1}^{3} E_i \) is the branch locus of the covering \( f \).

Put

\[
\begin{align*}
    2C_j &= f^*(\tilde{L}_j), \quad j = 1, \ldots, 9, \\
    2D_i &= f^*(E_i), \quad i = 1, 2, 3, \\
    D_k &= f^*(E_k) \quad 3 < k \leq 3 + s,
\end{align*}
\]  \hspace{1cm} (17)

and denote by \( t(L_j) \) the number of singular points of the arrangement \( \mathcal{T}_s \) lying on the line \( L_j \).

**Claim 4.5.** We have

(i) the curves \( C_j, j = 1, \ldots, 9, \) and \( D_i, i = 1, \ldots, 3 + s, \) are non-singular;

(ii) the geometric genus of a curve \( C_j, j = 1, \ldots, 9, \) is equal to
\( g(D_i) = 3 - t(L_j); \)

(iii) the geometric genus of a curve \( D_i \) is equal to \( g(D_i) = 1 \) if
\( i = 1, 2, 3 \) and \( g(D_i) = 0 \) if \( 3 < i \leq 3 + s; \)

(iv) the self-intersection number of a curve \( C_j, j = 1, \ldots, 9, \) is equal to
\( (C_j^2)_{X_s} = 1 - t(L_j); \)

(v) the self-intersection number of a curve \( D_i \) is equal to
\( (D_i^2)_{X_s} = -1 \) if \( i = 1, 2, 3 \) and
\( (D_i^2)_{X_s} = -4 \) if \( 3 < i \leq 3 + s. \)
Proof. Statement (i) is obvious.

(ii) – (iii) The restriction of the covering map $f$ to a curve $C_j$, $j = 1, \ldots, 9$, is a two-sheeted covering of a rational curve branched at $8 - 2t(L_j)$ points. Therefore, $g(C_j) = 3 - t(L_j)$.

The restriction of $f$ to a curve $D_i$, $i = 1, 2, 3$, is a two-sheeted covering of a rational curve branched at 4 points. Therefore $g(D_i) = 1$. Similarly, the restriction of $f$ to a curve $D_i$, $3 < i \leq 3 + s$, is a bi-double covering of a rational curve branched at 3 points. Therefore, the geometric genus $g(D_i) = 0$.

(iv) – (v) Since $(f^*(D), f^*(D))_{X_s} = \deg f \cdot (D, D)_{\tilde{\mathbb{P}}^2} = 4 \cdot (D, D)_{\tilde{\mathbb{P}}^2}$ for any divisor $D$ on $\tilde{\mathbb{P}}^2$, the Claim follows from the equalities: $(L_j^2)_{\tilde{\mathbb{P}}^2} = 1 - t(L_j)$ for $j = 1, \ldots, 9$ and $(\tilde{E}_i^2)_{\tilde{\mathbb{P}}^2} = -1$ for $i = 1, \ldots, 3 + s$. □

Consider the universal Galois covering $\overline{f}_s : \overline{X}_s \to \overline{\mathbb{P}}^2$ and the universal unramified covering $h_{s,\varphi} : \overline{X}_s \to X_s$ with respect to $\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{T}_s, \mathbb{Z}) \to G = (\mathbb{Z}/2\mathbb{Z})^2$ such that $\overline{f}_s = f_s \circ h_{s,\varphi}$. Recall that the covering $\overline{f}_s$ is induced by the epimorphism $\psi_{s,\varphi} : H_1(\mathbb{P}^2 \setminus \mathcal{T}_s, \mathbb{Z}) \to (\mathbb{Z}/q\mathbb{Z})^{k_s}$,
where \((\mathbb{Z}/q\mathbb{Z})^{k_s,\varphi}\) and \(\psi_{s,\varphi}\) are defined by

\[
\begin{cases}
\sum_{j=1}^{9} x_j = 0, \\
x_{j_1(i)} + x_{j_2(i)} + x_{j_3(i)} = 0, & 3 < i \leq 3 + s,
\end{cases}
\tag{18}
\]

where for each \(i\) the triple \((j_1(i), j_2(i), j_3(i))\) is the set of indexes of lines \(L_j\) such that \(p_i = L_{j_1(i)} \cap L_{j_2(i)} \cap L_{j_3(i)}\).

In the case \(s = 0\) (there are not triple points), we have \(k_{0,\varphi} = 8\) and

\[
\deg h_{0,\varphi} = 2^6. \tag{19}
\]

In the case \(s = 1\), we will assume that \(p_4 = L_3 \cap L_6 \cap L_9\) and obtain that \((\mathbb{Z}/q\mathbb{Z})^{k_1,\varphi}\) and \(\psi_{1,\varphi}\) are defined by

\[
\begin{cases}
\sum_{j=1}^{9} x_j = 0 \\
x_3 + x_6 + x_9 = 0.
\end{cases}
\tag{20}
\]

Therefore, \(k_{1,\varphi} = 7\) and

\[
\deg h_{1,\varphi} = 2^5. \tag{21}
\]
In the case $s = 2$, we will assume that $p_4 = L_3 \cap L_6 \cap L_9$ and $p_5$ (see Fig. 3 and 4) is either the intersection $L_2 \cap L_5 \cap L_8$ (a line arrangement $\mathcal{T}_2$) or $L_2 \cap L_5 \cap L_9$ (a line arrangement $\mathcal{T}_2'$).

In the case of a line arrangement $\mathcal{T}_2$, we obtain that $(\mathbb{Z}/q\mathbb{Z})^{k_2,\varphi}$ and $\psi_{2,\varphi}$ are defined by

\[
\begin{align*}
\sum_{j=1}^{9} x_j &= 0 \\
x_3 + x_6 + x_9 &= 0 \\
x_2 + x_5 + x_8 &= 0
\end{align*}
\]

and in the case of $\mathcal{T}_2'$, they are defined by

\[
\begin{align*}
\sum_{j=1}^{9} x_j &= 0 \\
x_3 + x_6 + x_9 &= 0 \\
x_2 + x_5 + x_9 &= 0 \\
x_2 + x_6 + x_8 &= 0
\end{align*}
\]

Therefore in both cases, we have $k_{2,\varphi} = 6$ and

\[
\deg h_{2,\varphi} = 2^4.
\]

In the case $s = 3$, we will assume that $p_4 = L_3 \cap L_6 \cap L_9$, $p_5 = L_2 \cap L_5 \cap L_9$, and $p_6 = L_2 \cap L_6 \cap L_8$. We obtain that $(\mathbb{Z}/q\mathbb{Z})^{k_3,\varphi}$ and $\psi_{3,\varphi}$ are defined by

\[
\begin{align*}
\sum_{j=1}^{9} x_j &= 0 \\
x_3 + x_6 + x_9 &= 0 \\
x_2 + x_5 + x_9 &= 0 \\
x_2 + x_6 + x_8 &= 0
\end{align*}
\]

Therefore, $k_{3,\varphi} = 5$ and

\[
\deg h_{1,\varphi} = 2^3.
\]

In the case $s = 4$, we will assume that $p_4 = L_3 \cap L_6 \cap L_9$, $p_5 = L_2 \cap L_5 \cap L_9$, $p_6 = L_2 \cap L_6 \cap L_8$, and $p_7 = L_3 \cap L_5 \cap L_8$. The line arrangement $\mathcal{T}_4$ is depicted in Fig. 5.
We obtain that \((\mathbb{Z}/q\mathbb{Z})^{k_{4,\varphi}}\) and \(\psi_{4,\varphi}\) are defined by

\[
\begin{align*}
\sum_{j=1}^{9} x_j &= 0 \\
x_3 + x_6 + x_9 &= 0 \\
x_2 + x_5 + x_9 &= 0 \\
x_2 + x_6 + x_8 &= 0 \\
x_3 + x_5 + x_8 &= 0
\end{align*}
\]

(27)

It is easy to see that over \(\mathbb{Z}/2\mathbb{Z}\) the rank of linear system (27) is equal to 4. Therefore \(k_{4,\varphi} = 5\) and

\[
\deg h_{4,\varphi} = 2^3.
\]

(28)

**Claim 4.6.** Let \(X_s\) be a Burniat surface and \(\alpha \in \text{Tors}_2(X_s) = \text{Tors}_2H^2(X_s, \mathbb{Z})\), \(\alpha \neq 0\). Then the linear system \(|K_{X_s} + \alpha|\) is non-empty and \(D \in |K_{X_s} + \alpha|\) for some \(\alpha \in \text{Tors}_2(X_s), \alpha \neq 0\), if and only if \(2D = f^*(D')\) for some \(D' \in |3L - \sum_{i=1}^{3+s} E_i|\).

**Proof.** The linear system \(|K_{X_s} + \alpha|\) is non-empty by Riemann – Roch Theorem, since \(\dim H^2(X_s, \mathcal{O}_{X_s}(K_{X_s} + \alpha)) = \dim H^0(X_s, \mathcal{O}_{X_s}(\alpha)) = 0\).
Let $D_α ∈ |K_{X_s} + α|$. Then $2D_α ∈ |2K_{X_s}|$. By Riemann–Roch Theorem, we have
\[ \dim H^0(X_s, 2K_{X_s}) = K^2_{X_s} + 1 = 7 - s. \]

On the other hand, by Claim 2.2, $2K_{X_s} = f^*(3L - \sum_{i=1}^{3+s} E_i)$ and
\[ \dim H^0(\tilde{D}^2, O_{\tilde{D}}(3L - \sum_{i=1}^{3+s} E_i)) = 7 - s. \]

Therefore
\[ |2K_{X_s}| = f^*(|3L - \sum_{i=1}^{3+s} E_i|) \]

and $D ∈ |K_{X_s} + α|$ for some $α ∈ \text{Tors}_2(X_s)$ if and only if $2D = f^*(\tilde{D})$ for some $\tilde{D} ∈ |3L - \sum_{i=1}^{3+s} E_i|$. \[ \square \]

**Proposition 4.7.** The 2-torsion group of a Burniat surface $X_s$ is isomorphic to $\text{Tors}_2(X_s) \simeq (\mathbb{Z}/2\mathbb{Z})^{6-s}$ if $s ≤ 3$ and $\text{Tors}_2(X_4) \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

**Proof.** It follows from Corollary 1.6 that $(\mathbb{Z}/2\mathbb{Z})^{\text{deg} h_{s,ϕ}} ⊂ \text{Tors}_2(X_s)$. Note that $\text{deg} h_{s,ϕ} = 6 - s$ if $s ≤ 3$ and $\text{deg} h_{4,ϕ} = 3$. Therefore, by Claim 4.6, to prove the Proposition for each case $s = 4, 3, 2, 1, 0$, it is sufficient to show that there are exactly $2^{\text{deg} h_{s,ϕ} - 1}$ complete continuous systems of divisors $\tilde{D}$ belonging to $|3L - \sum_{i=1}^{3+s} E_i|$ and such that the preimage $f^*(\tilde{D})$ of each $\tilde{D}$ is divisible by two (i.e., $f^*(\tilde{D}) = 2D$), and each two divisors $\frac{1}{2}f^*(\tilde{D}_i), \frac{1}{2}f^*(\tilde{D}_j)$ are not linear equivalent if they belong to different systems.

One can check that:
- in the case $s = 4$, the elements $\tilde{D} ∈ |3L - \sum_{i=1}^{7} E_i|$, for which $f^*(\tilde{D})$ are divisible by two, are:
  \[ \tilde{L}_3 + \tilde{L}_6 + \tilde{L}_9 + 2E_4, \quad \tilde{L}_2 + \tilde{L}_5 + \tilde{L}_9 + 2E_5, \quad \tilde{L}_2 + \tilde{L}_6 + \tilde{L}_8 + 2E_6, \]
  \[ \tilde{L}_3 + \tilde{L}_5 + \tilde{L}_8 + 2E_7, \quad \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_7 + E_1, \quad \tilde{L}_5 + \tilde{L}_6 + \tilde{L}_1 + E_2, \]
  \[ \tilde{L}_8 + \tilde{L}_9 + \tilde{L}_4 + E_3; \]
- in the case $s = 3$, the elements $\tilde{D} ∈ |3L - \sum_{i=1}^{6} E_i|$, for which $f^*(\tilde{D})$ are divisible by two, are:
  \[ \tilde{L}_3 + \tilde{L}_6 + \tilde{L}_9 + 2E_4, \quad \tilde{L}_2 + \tilde{L}_5 + \tilde{L}_9 + 2E_5, \quad \tilde{L}_2 + \tilde{L}_6 + \tilde{L}_8 + 2E_6, \]
  \[ \tilde{L}_3 + \tilde{L}_5 + \tilde{L}_8, \quad \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_7 + E_1, \quad \tilde{L}_5 + \tilde{L}_6 + \tilde{L}_1 + E_2, \]
  \[ \tilde{L}_8 + \tilde{L}_9 + \tilde{L}_4 + E_3; \]
in the case \( s = 2 \) and \( L_2 = T_2 \), the elements \( \overline{D} \in |3L - \sum_{i=1}^{5} E_i| \), for which \( f^*(\overline{D}) \) are divisible by two, are:

\[
\begin{align*}
&\tilde{L}_2 + \tilde{L}_3 + \tilde{L}_7 + E_1, & \tilde{L}_1 + \tilde{L}_5 + \tilde{L}_6 + E_2, & \tilde{L}_4 + \tilde{L}_8 + \tilde{L}_9 + E_3, \\
&\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_6 + E_1, & \tilde{L}_2 + \tilde{L}_6 + \tilde{L}_7 + E_2, & \tilde{L}_1 + \tilde{L}_3 + \tilde{L}_5 + E_1, \\
&\tilde{L}_3 + \tilde{L}_5 + \tilde{L}_7 + E_2, & \tilde{L}_2 + \tilde{L}_4 + \tilde{L}_9 + E_1, & \tilde{L}_2 + \tilde{L}_7 + \tilde{L}_9 + E_3, \\
&\tilde{L}_3 + \tilde{L}_4 + \tilde{L}_8 + E_1, & \tilde{L}_3 + \tilde{L}_7 + \tilde{L}_8 + E_3, & \tilde{L}_1 + \tilde{L}_5 + \tilde{L}_9 + E_3, \\
&\tilde{L}_4 + \tilde{L}_5 + \tilde{L}_9 + E_2, & \tilde{L}_1 + \tilde{L}_6 + \tilde{L}_8 + E_3, & \tilde{L}_4 + \tilde{L}_6 + \tilde{L}_8 + E_2;
\end{align*}
\]

in the case \( s = 2 \) and \( T_2 = T''_2 \), the elements \( \overline{D} \in |3L - \sum_{i=1}^{5} E_i| \), for which \( f^*(\overline{D}) \) are divisible by two, are:

\[
\begin{align*}
&\tilde{L}_2 + \tilde{L}_3 + \tilde{L}_7 + E_1, & \tilde{L}_1 + \tilde{L}_5 + \tilde{L}_6 + E_2, & \tilde{L}_4 + \tilde{L}_8 + \tilde{L}_9 + E_3, \\
&\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_6 + E_1, & \tilde{L}_2 + \tilde{L}_6 + \tilde{L}_7 + E_2, & \tilde{L}_1 + \tilde{L}_3 + \tilde{L}_5 + E_1, \\
&\tilde{L}_3 + \tilde{L}_5 + \tilde{L}_7 + E_2, & \tilde{L}_1 + \tilde{L}_4 + \tilde{L}_9 + E_1, & \tilde{L}_4 + \tilde{L}_7 + \tilde{L}_9 + E_2, \\
&\tilde{L}_2 + \tilde{L}_5 + \tilde{L}_9 + 2E_5, & \tilde{L}_3 + \tilde{L}_6 + \tilde{L}_9 + 2E_4, & \tilde{L}_2 + \tilde{L}_5 + \tilde{L}_8, \\
&\tilde{L}_1 + \tilde{L}_7 + \tilde{L}_9 + 2E_3, & 2\tilde{L}_4 + \tilde{L}_9 + E_1 + E_2, & \tilde{L}_3 + \tilde{L}_5 + \tilde{L}_8.
\end{align*}
\]

The reader can test that in the case \( s = 1 \) there are exactly the 31 divisors \( \overline{D} \in |3L - \sum_{i=1}^{4} E_i| \), for which \( f^*(\overline{D}) \) are divisible by two, and in the case \( s = 0 \), there are exactly 63 complete continuous systems of divisors \( \overline{D} \) belonging to \( |3L - \sum_{i=1}^{3} E_i| \), for which \( f^*(\overline{D}) \) are divisible by two. Note only that in the case \( s = 0 \), among these systems of divisors, 60 systems consist of single divisors and the last 3 are one-dimensional linear systems. They are

\[
\begin{align*}
&\tilde{L}_1 + 2\tilde{L}_{p_2} + E_2, & \tilde{L}_4 + 2\tilde{L}_{p_3} + E_3, & \tilde{L}_7 + 2\tilde{L}_{p_1} + E_1,
\end{align*}
\]

where \( \tilde{L}_{p_i} = \sigma^{-1}(L_{p_i}) \) is the strict transform of a line belonging to the pencil of lines passing through the point \( p_i \). These three pencils correspond to three elements, say \( \alpha_1, \alpha_2, \alpha_3 \in \text{Tors}_2(X_0) \), for which \( \dim H^1(X_0, O_{X_0}(\alpha_i)) = 1 \) and these elements "come" from three irregular intermediate cyclic coverings of the universal Galois covering \( \overline{f}_0 : \overline{X}_0 \to \overline{\mathbb{P}}^2 \) with respect to \( \varphi : H_1(\mathbb{P}^2 \setminus \overline{T}_0, \mathbb{Z}) \to G = (\mathbb{Z}/2\mathbb{Z})^2 \) (see the end of the proof of Claim 4.8).

**Claim 4.8.** For \( s \geq 1 \) the surfaces \( \overline{X}_s \) are regular, i.e., the irregularities \( q(\overline{X}_s) = 0 \), and \( q(X_0) = 3 \).

**Proof.** The arithmetic genus \( p_a \) of a surface is equal to \( p_a = p_g - q + 1 \). Therefore to calculate \( q \), it is sufficient to calculate \( p_a \) and \( p_g \).

We have \( p_a(X_s) = 1 \). Therefore the arithmetic genus

\[
p_a(\overline{X}_s) = 2^{k_{s,w} - 2}, \tag{29}
\]

since \( h_s \) is unramified and \( \deg h_s = 2^{k_{s,w} - 2} \).
It follows from Claim 3.2 that the contribution to the geometric genus by equation (18) – (27). Thus, for the epimorphisms \( \psi_m \), \( m = 1, \ldots, 2^{k_s, \varphi} - 1 \), from \( G_{u, \varphi} = (\mathbb{Z}/2\mathbb{Z})^{k_s, \varphi} \) to the cyclic group \( \mathbb{Z}/2\mathbb{Z} \), where the group \( G_{u, \varphi} \) is isomorphic to the subgroup of \( (\mathbb{Z}/2\mathbb{Z})^{9} \) given in the coordinates \( (x_1, \ldots, x_9) \) by one of linear equations (18) – (27).

It is easy to see that there is a one-to-one correspondence \( \gamma \) between the epimorphisms \( \psi_m \) and the elements \( (x_1, \ldots, x_9) \in G_{u, \varphi} \) such that for \( \gamma(\psi_m) = (x_1, \ldots, x_9) \) the cyclic covering corresponding to \( \psi_m \) is given by equation

\[
w_m^2 = \prod_{i=1}^{l_i} l_i.
\]

It follows from Claim 3.2 that the contribution to the geometric genus of \( \mathcal{X}_s \) can be given only by the cyclic coverings corresponding to the epimorphisms \( \psi_m \) for which the sum of the coordinates of \( \gamma(\psi_m) \) is equal or greater than 6, and if it is equal to 6 then the corresponding branch locus of the cyclic covering has not 4-fold points.

In the cases \( s = 3 \) or 4 (see linear equations (25) and (27)), we have exactly 7 such coverings given by:

\[
\begin{align*}
z_1^2 &= l_1l_2l_3l_5l_6l_7, & z_2^2 &= l_1l_4l_5l_6l_8l_9, & z_3^2 &= l_2l_3l_4l_7l_8l_9, \\
z_4^2 &= l_1l_2l_4l_5l_7l_8, & z_5^2 &= l_1l_3l_4l_5l_7l_9, & z_6^2 &= l_1l_3l_4l_6l_7l_8, \\
z_7^2 &= l_1l_2l_4l_6l_7l_9.
\end{align*}
\]

Therefore \( p_9(\mathcal{X}_3) = p_9(\mathcal{X}_4) = 7 \) and \( q(\mathcal{X}_3) = q(\mathcal{X}_4) = 0 \).

In the case \( s = 2 \), when a line arrangement \( L_2 = L_2' \) (see linear equations (22)), we have exactly 15 such coverings given by:

\[
\begin{align*}
z_1^2 &= l_1l_2l_3l_5l_6l_7, & z_2^2 &= l_1l_2l_3l_7l_8l_9, & z_3^2 &= l_1l_4l_5l_6l_9, \\
z_4^2 &= l_1l_2l_4l_6l_8l_9, & z_5^2 &= l_1l_3l_4l_5l_8l_9, & z_6^2 &= l_1l_2l_4l_5l_6l_9, \\
z_7^2 &= l_1l_3l_4l_6l_7l_8, & z_8^2 &= l_1l_2l_3l_6l_7l_8, & z_9^2 &= l_1l_2l_3l_5l_7l_9, \\
z_{10}^2 &= l_2l_3l_4l_5l_7l_9, & z_{11}^2 &= l_2l_3l_4l_6l_7l_9, & z_{12}^2 &= l_2l_4l_5l_6l_7l_9, \\
z_{13}^2 &= l_3l_4l_5l_6l_7l_8, & z_{14}^2 &= l_2l_4l_5l_7l_8l_9, & z_{15}^2 &= l_3l_4l_5l_7l_8l_9.
\end{align*}
\]

Therefore, \( p_9(\mathcal{X}_2) = 15 \) and \( q(\mathcal{X}_2) = 0 \).
In the case \( s = 2 \), when a line arrangement \( \mathcal{T}_2 = \mathcal{T}_2'' \) (see linear equations (23)), we have also exactly 15 such coverings given by:

\[
\begin{aligned}
\zeta_1^2 &= l_1l_2l_3l_4l_6l_7, & \zeta_2^2 &= l_1l_2l_5l_6l_8, & \zeta_3^2 &= l_2l_3l_4l_7l_8, \\
\zeta_4^2 &= l_1l_2l_4l_7l_8, & \zeta_5^2 &= l_1l_3l_4l_7l_9, & \zeta_6^2 &= l_1l_3l_6l_7l_8, \\
\zeta_7^2 &= l_1l_2l_4l_7l_9, & \zeta_8^2 &= l_1l_3l_4l_5l_7l_9, & \zeta_9^2 &= l_1l_2l_4l_5l_8, \\
\zeta_{10}^2 &= l_1l_3l_5l_8, & \zeta_{11}^2 &= l_1l_2l_3l_7l_8, & \zeta_{12}^2 &= l_2l_3l_4l_7l_8, \\
\zeta_{13}^2 &= l_2l_4l_6l_7l_8, & \zeta_{14}^2 &= l_3l_4l_5l_7l_8, & \zeta_{15}^2 &= l_1l_2l_3l_4l_5l_6l_7l_8.
\end{aligned}
\]

The branch locus in the 15-th double covering has degree equal to 8 and two 4-fold points. Therefore its geometric genus is equal to 1. Thus, we have again \( p_g(\mathcal{X}_2'') = 15 \) and \( q(\mathcal{X}_2'') = 0 \).

The rest two cases are left for the reader, note only that the non-zero contribution to the irregularity of \( \mathcal{X}_0 \) is given only by the cyclic coverings

\[
\begin{aligned}
\zeta_1^2 &= l_1l_2l_3l_4, & \zeta_2^2 &= l_1l_7l_8, & \zeta_3^2 &= l_4l_5l_6l_7.
\end{aligned}
\]

\( \Box \)

**Corollary 4.9.** The fundamental group of a Burniat surface \( X_0 \) is non-abelian infinite group.

**Proof.** It follows from Claim 4.8. \( \Box \)

It is easy to see that up to a linear transformation of \( \mathbb{P}^2 \) there is the unique Burniat arrangement of lines \( \mathcal{T}_4 \) (depicted in Fig. 5).

Fix homogeneous coordinates \((z_0 : z_1 : z_2)\) in \( \mathbb{P}^2 \). Put \( \mathcal{B}_0-\mathcal{s} = \{ \mathcal{T}_s = L_1 + \cdots + L_9 \} \), \( s \leq 4 \), the family of the ordered Burniat line arrangements such that \( L_1 \), \( L_4 \), and \( L_7 \) are given respectively by \( z_0 = 0 \), \( z_1 = 0 \), \( z_2 = 0 \), and the point \( p_4 = (1 : 1 : 1) \) (in the case \( s = 0 \) the point \( p_4 \) is the intersection \( L_3 \cap L_9 \)). It is easy to see that any Burniat arrangement of lines can be transformed to an arrangement belonging to \( \mathcal{B}_0-\mathcal{s} \) by a linear transformation of \( \mathbb{P}^2 \). Denote by \( F_\mathcal{s} : \mathcal{X}_0-\mathcal{s} \rightarrow \mathcal{B}_0-\mathcal{s} \) the family of Burniat surfaces with fibre \( F_\mathcal{s}^{-1}(\mathcal{T}_s) = \mathcal{X}_s \) over \( \mathcal{T}_s \in \mathcal{B}_0-\mathcal{s} \), where \( \mathcal{X}_s \) is the Burniat surface defined by the line arrangement \( \mathcal{T}_s \).

If \( s = 3 \) then an arrangement \( \mathcal{T}_3 \in \mathcal{B}_3 \) is uniquely determined by the point \( p_5 = L_2 \cap L_5 \cap L_9 \in L_9 \), since the lines \( L_3, L_6, L_9 \) are determined by \( p_4 = (1 : 1 : 1) \), the lines \( L_2 \) and \( L_5 \) are determined by \( p_5 \in L_9 \), and \( L_8 \) is determined by \( p_6 = L_2 \cap L_6 \). Therefore

\[
\mathcal{B}_3 \simeq (\mathbb{P}^1 \setminus \{ \text{three points} \}) \setminus \mathcal{B}_2, \tag{30}
\]

where \( \mathcal{B}_2 = \{ \mathcal{T}_{4,0} \} \) consists of the single arrangement \( \mathcal{T}_{4,0} \) corresponding to the case \( L_3 \cap L_5 \cap L_8 \neq \emptyset \). 
Put \( \mathcal{B}_4 = \mathcal{B}'_4 \cup \mathcal{B}''_4 \), where \( \mathcal{B}'_4 \) (respectively, \( \mathcal{B}''_4 \)) consists of the line arrangements \( \mathcal{L}'_2 \) (respectively, \( \mathcal{L}''_2 \)). It is easy to see that an arrangement \( \mathcal{L}'_2 \in \mathcal{B}'_4 \) is uniquely determined by the point \( p_5 = L_2 \cap L_5 \cap L_8 \). Therefore

\[
\mathcal{B}'_4 \cong \mathbb{P}^2 \setminus \{ \text{six lines} \}. \tag{31}
\]

Similarly, \( \mathcal{L}''_2 \in \mathcal{B}''_4 \) is uniquely determined by the point \( p_5 \in L_9 \) and the line \( L_8 \) belonging to the pencil of lines passing through the point \( p_3 \). Therefore

\[
\mathcal{B}''_4 \cong (\mathbb{P}^1 \setminus \{ \text{three points} \})^2 \setminus (\mathcal{B}_2 \cup \mathcal{B}_3). \tag{32}
\]

As above, it is clear that \( \mathcal{L}_1 \in \mathcal{B}_5 \) is uniquely determined by the lines \( L_2, L_5, \) and \( L_8 \) belonging, respectively, to the pencils of lines passing through the points \( p_1, p_2, \) and \( p_3 \). Therefore

\[
\mathcal{B}_5 \cong (\mathbb{P}^1 \setminus \{ \text{three points} \})^3 \setminus (\mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4). \tag{33}
\]

Similarly,

\[
\mathcal{B}_6 \cong (\mathbb{P}^1 \setminus \{ \text{three points} \})^4 \setminus \mathcal{B}_5, \tag{34}
\]

where the variety \( \mathcal{B}_5 \) is the union of degenerations of the arrangements \( \mathcal{L}_0, \dim \mathcal{B}_5 = 3, \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \subset \mathcal{B}_5 \).

**Claim 4.10.** Any two Burniat surfaces \( X'_2 \) and \( X''_2 \) are not isomorphic to each other.

**Proof.** Assume that there is an isomorphism \( h : X'_2 \to X''_2 \). Then it induces isomorphisms \( h^* : \text{Tors}_2(X''_2) \to \text{Tors}_2(X'_2) \) and

\[
h^* : H^0(X''_2, \mathcal{O}_{X''_2}(K_{X''_2} + \alpha)) \to H^0(X'_2, \mathcal{O}_{X'_2}(K_{X'_2} + h^*(\alpha)))
\]

for each \( \alpha \in \text{Tors}_2(X''_2) \). Note that Claim 4.8 and Riemann – Roch Theorem imply \( \dim H^0(X''_2, \mathcal{O}_{X''_2}(K_{X''_2} + \alpha)) = 1 \) for \( \alpha \neq 0 \). Therefore we should have

\[
K_{h^*(\alpha)}' = h^*(K''_\alpha)
\]

for each \( K''_\alpha \in |K_{X''_2} + \alpha| \). On the other hand, among the irreducible components of the divisors \( K''_\alpha \), there is a rational curve with self-intersection number equal to \(-2\) (the curve \( C_9 \), see Claim 4.5 and Proposition 4.7), but among the irreducible components of the divisors \( K'_\alpha \in |K_{X'_2} + \alpha| \), there is no such a curve. Contradiction. \( \square \)

**Claim 4.11.** For each \( s = 0, \ldots, 4 \) and for each \( \mathcal{L}_{s,0} \in \mathcal{B}_{6-s} \), there are only finitely many arrangements \( \mathcal{T}_s \in \mathcal{B}_{6-s} \) for which the corresponding Burniat surfaces \( X_s = F_{s-1}(\mathcal{T}_s) \) are isomorphic to \( X_{s,0} = F_{s-1}(\mathcal{L}_{s,0}) \).
Proof. Put \( x = \frac{y}{z_0} \) and \( y = \frac{z}{z_0} \), where \( (z_0 : z_1 : z_2) \) are the homogeneous coordinates in \( \mathbb{P}^2 \) chosen above. Then the lines \( L_4, L_3, L_2, L_7, L_9, L_8, L_5, \) and \( L_6 \) are given, respectively, by equations: \( x = 0, x = 1, x = a_1, y = 0, y = 1, y = b_1, x = c_1 y, \) and \( x = c_2 y \) for some \( a_1, b_1, c_1, c_2 \in \mathbb{C} \setminus \{0, 1\} \). Consider the injective map \( r_s : \mathcal{B}_{6-s} \rightarrow \mathbb{C}^{\dim \mathcal{B}_{6-s}} \) given as follows (if \( s = 2 \) we will consider two maps: \( r_2'' \) and \( r_2' \)).

In the case \( \mathcal{B}_3 \), we have \( c_2 = 1, c_1 = a_1, \) and \( b_1 = a_1, \) and \( a_1 \) is a coordinate in \( \mathcal{B}_3 \). Put \( r_3(\mathcal{L}_3) = a_1 \in \mathbb{C} \). Note also that the arrangement \( \mathcal{A}_3 = \{ \mathcal{L}_{4,0} \} \) has the coordinate \( a_1 = -1 \).

In the case \( \mathcal{B}'_3 \), we have \( c_2 = 1, a_1 = b_1 c_1, \) and \( (b_1, c_1) \) are coordinates in \( \mathcal{B}'_3 \). Put \( r_2'(\mathcal{L}_2) = (b_1, c_1) \in \mathbb{C}^2 \).

Similarly, in the case \( \mathcal{B}'_2 \), we have \( c_2 = 1, c_1 = a_1, \) and \( (b_1, c_1) \) are coordinates in \( \mathcal{B}'_2 \). Put \( r_2'(\mathcal{L}_2) = (b_1, c_1) \in \mathbb{C}^2 \).

In the case \( \mathcal{B}_5 \), we have \( c_2 = 1, \) and \( (a_1, b_1, c_1) \) are coordinates in \( \mathcal{B}_5 \). Put \( r_1(\mathcal{L}_1) = (a_1, b_1, c_1) \in \mathbb{C}^3 \).

In the case \( \mathcal{B}_6 \), \( (a_1, b_1, c_1, c_2) \) are coordinates in \( \mathcal{B}_6 \), and we put \( r_0(\mathcal{L}_0) = (a_1, b_1, c_1, c_2) \in \mathbb{C}^4 \).

Obviously, the image \( r_s(\mathcal{B}_{6-s}) \) is everywhere dense open subset of \( \mathbb{C}^{\dim \mathcal{B}_{6-s}} \).

Let \( \text{Iso}(X_{s,0}) \) be the set of arrangements \( \mathcal{L}_s \) such that the surfaces \( F_{s}^{-1}(\mathcal{L}_s) \) are isomorphic to \( X_{s,0} \). Note that \( \text{Iso}(X_{s,0}) \) is a quasi-projective subvariety of \( \mathcal{B}_{6-s} \). Indeed, each \( X_s \) is a surface with ample canonical class (possibly, modulo \(^{-2}\)-curves). The imbedding of the surfaces \( X_s \) to \( \mathbb{P}^{10K_X^2} \) given by \( |5K_{X_s}| \) defines a morphism \( \mu : \mathcal{B}_{6-s} \rightarrow \text{Hilb}_{P_{X_s}} \) to the Hilbert scheme of the surfaces in \( \mathbb{P}^{10K^2} \) with fixed Hilbert polynomial. The group \( \text{PGL}(10K_X^2 + 1, \mathbb{C}) \) acts on \( \text{Hilb}_{P_{X_s}} \) and

\[
\text{Iso}(X_{s,0}) = \mu^{-1}(\mu(\mathcal{B}_{6-s})) \cap \text{PGL}(10K_X^2 + 1, \mathbb{C})(\mu(\mathcal{L}_{s,0})).
\]

Therefore, to prove Claim 4.11, it is sufficient to show that the image \( r_s(\text{Iso}(X_{s,0})) \) consists of a finite set of points.

For this let us consider two isomorphic Burniat surfaces \( X_{s,0}, X_{s,1} \), and let \( h : X_{s,0} \rightarrow X_{s,1} \) be an isomorphism. As in the proof of Claim 4.10, the isomorphism \( h \) induces isomorphisms \( h^* : \text{Tors}_{2}(X_{s,1}) \rightarrow \text{Tors}_{2}(X_{s,0}) \) and

\[
h^* : H^0(X_{s,1}, \mathcal{O}_{X_{s,1}}(K_{X_{s,1}} + \alpha)) \rightarrow H^0(X_{s,0}, \mathcal{O}_{X_{s,0}}(K_{X_{s,0}} + h^*(\alpha)))
\]

for each \( \alpha \in \text{Tors}_{2}(X_{s,1}) \). Claim 4.8 and Riemann–Roch Theorem imply \( \dim H^0(X_{s,1}, \mathcal{O}_{X_{s,1}}(K_{X_{s,1}} + \alpha)) = 1 \) for each \( \alpha \neq 0 \) if \( s \geq 1 \).
and for almost all \( \alpha \neq 0 \) except three particular values of \( \alpha \) if \( s = 0 \). Therefore we should have

\[
K_{0,h^*(\alpha)} = h^*(K_{1,\alpha})
\]

for each \( K_{1,\alpha} \in |K_{X_{s,1}} + \alpha| \) (in the case \( s = 0 \) we consider only 60 elements \( \alpha \) for which \( \dim H^0(X_{s,1}, \mathcal{O}_{X_{s,1}}(K_{X_{s,1}} + \alpha)) = 1 \)). In notation (17), each divisor \( K_{1,\alpha} \) is a linear combination of the curves \( C_{j,1}, j = 1, \ldots, 9 \), and \( D_{i,1}, i = 1, \ldots, 3 + s \). Therefore \( h^*(R_1) = R_0 \), where \( R_k = \sum_{j=1}^{9} C_{j,k} + \sum_{i=1}^{3+s} D_{i,k} \) for \( k = 0, 1 \), and the invariants of the curves \( D_{i,k} \) and \( C_{j,k} \) are invariants of surfaces \( X_{s,k} \). In particular, the set \( R'(X_{s,k}) \), consisting of the components of \( R_k \) having positive genus, is also an invariant. Note that \( D_{i,k} \in R'(X_{s,k}) \) for \( i = 1, 2, 3 \) and \( C_{j,k} \in R'(X_{s,k}) \) if \( t(L_{j,k}) \leq 2 \), and, in particular, \( C_{1,k}, C_{4,k}, C_{7,k} \in R'(X_{s,k}) \).

Let \( C \) be an elliptic curve. Denote by \( B_C \) the subset of \( \mathbb{C} \setminus \{0, 1\} \) consisting of the complex numbers \( c \) such that the curve \( C \) can be represented as a two-sheeted covering \( f : C \to \mathbb{P}^1 \) branched at four points \( 0, 1, c, \infty \). It is well known that for each elliptic curve \( C \) the set \( B_C \) is finite.

Similarly, for hyperelliptic curve \( C \), \( g(C) = 2 \), denote by \( B_C \) the subset of \( (\mathbb{C} \setminus \{0, 1\})^3 \) consisting of the triples \( (c_1, c_2, c_3) \) of complex numbers, \( c_i \neq c_j \) for \( i \neq j \), such that the curve \( C \) can be represented as a two-sheeted covering \( f : C \to \mathbb{P}^1 \) branched at six points \( 0, 1, c_1, c_2, c_3, \infty \). As in the case of elliptic curves, the set \( B_C \) is finite for each curve \( C \) of genus two.

Consider the restriction of the covering map \( f : X_s \to \tilde{\mathbb{P}}^2 \) to a curve \( C \in R'(X_s) \). It is a two-sheeted covering of \( \mathbb{P}^1 \). Put

\[
B_{X_s} = \bigcup_{C \in R'(X_s)} B_C.
\]

It follows from the above discussion that:

\((*)\) For a Burniat surface \( X_s \), the set \( B_{X_s} \) is finite, and it is an invariant of \( X_s \) up to isomorphism.

Now to complete the proof of Claim 4.11, it is sufficient to notice that:

in the case \( B_3 \), the image \( r_3(T_{3,0}) = a_1 \in B_{X_{s,0}} \), since \( a_1 \in B_{C_{7,0}} \);

in the case \( B_4' \) (and, similarly, in the case \( B_4'' \)), the image \( r_2(T_{2,0}) = (b_1, c_1) \) for some \( b_1, c_1 \in B_{X_{2,0}} \), since \( b_1 \in B_{C_{4,0}} \) and \( c_1 \in B_{D_{2,0}} \);

in the case \( B_5 \), the image \( r_1(T_{1,0}) = (a_1, b_1, c_1) \) for some \( a_1, b_1, c_1 \in B_{X_{1,0}} \), since \( a_1 \in B_{C_{7,0}} \), \( b_1 \in B_{C_{4,0}} \), and \( c_1 \in B_{D_{2,0}} \);

in the case \( B_6 \), the image \( r_0(T_{0,0}) = (a_1, b_1, c_1, c_2) \) for some \( b_1 \) and \( (a_1, c_1, c_2) \in B_{X_{0,0}} \), since \( b_1 \in B_{C_{4,0}} \), and \( (a_1, c_1, c_2) \in B_{D_{9,0}} \).
Denote by $\Theta_{X_s}$ the tangent sheaf and by $\Omega^i_{X_s}$ the sheaf of $i$-differential forms on $X_s$.

**Proposition 4.12.** For $0 \leq s \leq 4$

(i) $\dim H^0(X_s, \Theta_{X_s}) = 0$;
(ii) $\dim H^1(X_s, \Theta_{X_s}) = 2s - 2 + 3 \max(0, 2 - s)$;
(iii) $\dim H^2(X_s, \Theta_{X_s}) = 3 \max(0, 2 - s)$.

*Proof.* It is known that $\dim H^0(X_s, \Theta_{X_s}) = 0$ for surfaces of general type, since the automorphism group of a surface of general type is a discrete group.

By Riemann–Roch Theorem, the Euler characteristic $\chi(\Theta_{X_s})$ of the sheaf $\Theta_{X_s}$ is equal to

$$\chi(\Theta_{X_s}) = \sum_{i=0}^{2} (-1)^i \dim H^i(X_s, \Theta_{X_s}) = 2K^2_{X_s} - 10 = 2s - 2,$$

and by Serre duality, $\dim H^i(X_s, \Theta_{X_s}) = \dim H^{2-i}(X_s, \Omega^1_{X_s} \otimes \Omega^2_{X_s})$. Therefore to prove Proposition 4.12, it is sufficient to prove the following Proposition.

**Proposition 4.13.** For a Burniat surface $X_s$, $s = 0, \ldots, 4$,

$$\dim H^0(X_s, \Omega^1_{X_s} \otimes \Omega^2_{X_s}) = 3 \max(0, 2 - s).$$

*Proof.* Put $X = X_s$. Choose a chart $U = \mathbb{C}^2 \subset \mathbb{P}^2$ such that all singular points of the line arrangement $L$ lie in $\mathbb{C}^2$, and let $x, y$ be non-homogeneous coordinates in $U$, $l_i(x, y) = 0$ an equation of $L_i \subset \mathbb{P}^2$.

The inclusion of the function field $\mathbb{C}(\mathbb{P}^2) \subset \mathbb{C}(X)$ induced by $f$ is the Galois extension with Galois group $G = (\mathbb{Z}/2\mathbb{Z})^2$. Let $\alpha_1 = (1, 0)$, $\alpha_2 = (0, 1)$, and $\alpha_3 = (1, 1)$ be the non-zero elements of $G$. Identifying $\mathbb{C}(\mathbb{P}^2)$ with the field $K_0 = \mathbb{C}(x, y)$ of rational functions in $x, y$ and the field $\mathbb{C}(X)$ with $K = \mathbb{C}(x, y, w_1, w_2, w_3)$, where $x, y, w_1, w_2, w_3$ satisfy the equations (16), without loss of generality, we can assume that $\alpha(x) = x$, $\alpha(y) = y$ for all $\alpha \in G$ and

- $\alpha_1(w_1) = w_1, \alpha_2(w_1) = \alpha_3(w_1) = -w_1$,
- $\alpha_2(w_2) = w_2, \alpha_1(w_2) = \alpha_3(w_2) = -w_2$,
- $\alpha_3(w_3) = w_3, \alpha_1(w_3) = \alpha_2(w_3) = -w_3$.

Put $K_i = \mathbb{C}(x, y, w_i)$ and denote by $g_i : Z_i \to \mathbb{P}^2$ the covering induced by the extension $K_0 \subset K_i$. 
Consider the spaces $M, M_0, M_1, M_2, M_3$ of rational $(1,0) \otimes (2,0)$-forms on $X, \mathbb{P}^2, Z_1, Z_2, Z_3$, respectively. We have $M_0 \subset M_i \subset M$ for $i = 1, 2, 3$ and
\[
M_0 = \mathbb{C}(x, y)dx \otimes (dx \wedge dy) \oplus \mathbb{C}(x, y)dy \otimes (dx \wedge dy),
\]
\[
M = Kdx \otimes (dx \wedge dy) \oplus Kdy \otimes (dx \wedge dy)
\]
and
\[
M_i = K_i dx \otimes (dx \wedge dy) \oplus K_i dy \otimes (dx \wedge dy)
\]
for $i \geq 1$. Moreover, $G$ acts on $M$ and $M^G = M_0$. Besides, $\omega \in M$ belongs to $M_i$, $i = 1, 2, 3$, if and only if $\alpha_i(\omega) = \omega$ and $\alpha_j(\omega) = -\omega$ for $j \neq i$.

The Galois group $G$ acts also on the space $H^0(X, \Omega^1_X \otimes \Omega^2_X)$ and this space is also decomposed in the direct sum of eigen-spaces $H_{(i)}$:
\[
H^0(X, \Omega^1_X \otimes \Omega^2_X) = \bigoplus_{i=0}^{3} H_{(i)},
\]
where $\omega \in H_{(i), i \geq 1}$, if and only if $\alpha_i(\omega) = \omega$ and $\alpha_j(\omega) = -\omega$ for $j \neq i$, and $\omega \in H_{(0)}$ if and only if $\alpha_i(\omega) = \omega$ for all $i$. It is easy to see that
\[
H_{(i)} = H^0(X, \Omega^1_X \otimes \Omega^2_X) \cap M_i
\]
(35)
for $i = 0, 1, 2, 3$.

**Lemma 4.14.** Let $(V, o) \subset \mathbb{C}^2 \times \mathbb{C}^1$ be a germ a surface given in coordinates $(z_1, z_2, w_1)$ by equation $w_1^2 = z_1$. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $H^0(V, \Omega^1_V \otimes \Omega^2_V)$ given by $\alpha(z_1) = z_1, \alpha(z_2) = z_2, \alpha(w_1) = -w_1$, where $\alpha \in \mathbb{Z}/2\mathbb{Z}$ is the non-zero element.

(i) If $\omega \in H^0(V, \Omega^1_V \otimes \Omega^2_V)$ is invariant under the action of $\mathbb{Z}/2\mathbb{Z}$, then
\[
\omega = (P(z_1, z_2)\frac{dz_1}{z_1} + Q(z_1, z_2)dz_2) \otimes (dz_1 \wedge dz_2);
\]
(ii) if $\omega \in H^0(V, \Omega^1_V \otimes \Omega^2_V)$ is anti-invariant under the action of $\mathbb{Z}/2\mathbb{Z}$, then
\[
\omega = (P(z_1, z_2)dz_1 + Q(z_1, z_2)dz_2) \otimes \frac{dz_1 \wedge dz_2}{w_1},
\]
where $P(z_1, z_2)$ and $Q(z_1, z_2)$ are some analytic functions in $z_1$ and $z_2$. 
Proof. Note that \( \frac{dz_1 \wedge dz_2}{w_1} \) is a holomorphic nowhere vanishing two-form on \( V \) and \( \alpha(\frac{dz_1 \wedge dz_2}{w_1}) = -\frac{dz_1 \wedge dz_2}{w_1} \). Therefore, a form \( \omega \in H^0(V, \Omega^1_V \otimes \Omega^2_V) \) can be written in the form

\[
\omega = (h_2 dz_2 + h_3 dw_1) \otimes \frac{dz_1 \wedge dz_2}{w_1},
\]

where \( h_2, h_3 \in H^0(V, \mathcal{O}_V) \). Similarly, the functions \( h_i \) can be written in the form \( h_i = H_i'(z_1, z_2) + w_1 H_i''(z_1, z_2) \), where \( H_i'(z_1, z_2) \) and \( H_i''(z_1, z_2) \) are some analytic functions in \( z_1 \) and \( z_2 \).

It is easy to see that \( \omega \) is invariant if and only if \( H_i'(z_1, z_2) = H_i''(z_1, z_2) = 0 \), and \( \omega \) is anti-invariant if and only if \( H_i''(z_1, z_2) = H_i'(z_1, z_2) = 0 \). To complete the proof of Lemma 4.14, note that \( w_1 dw_1 = \frac{1}{2} dz_1 \).

Claim 4.15. The space \( H(0) = 0 \).

Proof. Let \( \omega \in H(0) \), \( \omega \neq 0 \). It follows from Lemma 4.14 that over the chart \( U = \mathbb{C}^2 \) it can be written in the form \( \omega = \omega_1 \otimes (dx \wedge dy) \), where \( \omega_1 \in H^0(U \setminus \text{Sing } \mathcal{T}, \Omega^1_U \setminus \text{Sing } \mathcal{T}(\log \mathcal{T})) \) is 1-form with logarithmic poles along \( \mathcal{T} \). Assume that it has poles along lines \( L_{i_1}, \ldots, L_{i_k}, 0 \leq k \leq 9 \). Then \( \omega \) can be written in the form

\[
\omega = \frac{P(x, y) dx + Q(x, y) dy}{l_{i_1} \ldots l_{i_k}} \otimes (dx \wedge dy),
\]

where \( P(x, y), Q(x, y) \in \mathbb{C}[x, y] \) are polynomials of degree less or equal \( k - 4 \). Indeed, \( l_{i_1} \ldots l_{i_k} \omega \) is a regular form in \( \mathbb{C}^2 \setminus \text{Sing } \mathcal{T} \). Therefore it can be written in the form

\[
l_{i_1} \ldots l_{i_k} \omega = (P(x, y) dx + Q(x, y) dy) \otimes (dx \wedge dy),
\]

where \( P(x, y), Q(x, y) \in \mathbb{C}[x, y] \). Next, \( \omega \) is regular at the generic point of the line at infinity \( L_{\infty} = \mathbb{P}^1 \setminus C^2 \). Let \( (z_0 : z_1 : z_2) \) be homogeneous coordinates in \( \mathbb{P}^2 \) such that \( x = \frac{z_1}{z_0} \) and \( y = \frac{z_2}{z_0} \). Then in coordinates \( u = \frac{1}{x}, v = \frac{y}{x} \) the form \( \omega \) has the form

\[
\omega = (u^{k-2} \tilde{P}(u, v) + u^{k-2} v \tilde{Q}(u, v)) du - \frac{u^{k-1} \tilde{Q}(u, v)}{u^{deg Q} l_{i_1} \ldots l_{i_k}} dv \otimes \frac{du \wedge dv}{u^3}.
\]

Therefore, \( \deg Q \leq k - 4 \) if \( \omega \) is regular at the generic point of the line at infinity. Similarly, we obtain \( \deg P \leq k - 4 \) if we consider coordinates \( u_1 = \frac{1}{y}, v_1 = \frac{x}{y} \). Therefore, \( k \geq 4 \).
Let \( l_i(x, y) = y + a_i x + b_i \). We have \( a_i \neq a_j \) if \( i \neq j \), since all singular points of \( L \) lie in \( U \), and
\[
dy = dl_i - a_i dx.
\]
Substituting (37) into (36), we obtain
\[
\omega = \frac{(P(x, y) - a_i Q(x, y))dx + Q(x, y)dl_i}{l_i \cdots l_{ik}} \otimes (dx \wedge dy).
\]
Since
\[
\frac{(P - a_i Q)dx + Qdl_i}{l_i \cdots l_{ik}} \in H^0(U \setminus \text{Sing} \mathcal{L}, \Omega^1_{U \setminus \text{Sing} \mathcal{L}}(\log \mathcal{L}_2)),
\]
the polynomials \( P(x, y) - a_i Q(x, y) \) should be divisible by \( l_i(x, y) \) for \( j = 1, \ldots, k \). Therefore \( k \geq 5 \) and the pencil \( P(x, y) - aQ(x, y) = 0 \) of plane curves of degree \( d = k - 4 \) should have \( k \) different fibres containing lines and, by assumption, each of these lines is not a fixed component of the pencil.

Let us show that it is impossible in our case. Indeed, the case \( d = 1 \) (i.e., \( k = 5 \)) is impossible, since only four lines from \( \mathcal{L} \) can lie in the same pencil. The case \( d = 2 \) (i.e., \( k = 6 \)) is impossible, since a pencil of conics can have only three fibers containing lines.

To show that the case \( d \geq 3 \) (i.e., \( k \geq 7 \)) is impossible, note that if the arrangement \( l_i \cdots l_{ik} = 0 \) has a 4-fold point, then the orders of zero of \( P(x, y) \) and \( Q(x, y) \) at the 4-fold point \( p \) should be at least two. Therefore the order of zero of each member of the pencil \( P(x, y) - aQ(x, y) = 0 \) at \( p \) should be also at least two. Indeed, assume that the arrangement \( l_i \cdots l_{ik} = 0 \) has such a point \( p \). Without loss of generality, we can assume that \( p \) has the coordinates \((0, 0)\). Let \( \sigma : \tilde{U} \to U \) be the blow up with center at \( p \). In one of the charts, \( \sigma \) is given by equations \( x = x_1 \) and \( y = x_1 y_1 \). In these new coordinates \( \omega \) has the form
\[
\omega = \frac{(P(x_1, x_1 y_1) + y_1 Q(x_1, x_1 y_1))dx_1 + x_1 Q(x_1, x_1 y_1)dy_1}{x_1^2 l_1 \cdots l_{ik}} \otimes (x_1 dx_1 \wedge dy_1),
\]
and therefore the order of zero of \( Q(x, y) \) at \( p \) should be at least two, since \( \omega \) is a form with logarithmic poles along the exceptional divisor \( x_1 = 0 \) according to Claim 4.14. Similar arguments (consider the map given by \( x = x_2 y_2 \) and \( y = y_2 \)) show that the order of zero of \( P(x, y) \) at \( p \) should be at least two also.

Let us show that the cases \( d = 3, 4, 5 \) (i.e., \( k = 7, 8, 9 \)) are also impossible, since in each of these cases, the pencils \( P(x, y) - aQ(x, y) = 0 \) of degree \( d \) should have a fixed line belonging to the arrangement \( l_i \cdots l_{ik} = 0 \). Indeed, each common point of any two lines (belonging to
different fibers of the pencil \( P(x, y) - aQ(x, y) = 0 \) of the arrangement \( l_1 \ldots l_7 = 0 \) is a base point of the pencil. But, it is easy to check that if we remove any two lines from \( \mathcal{L} \) (the case \( d = 3 \), i.e., \( k = 7 \)), then we obtain a new arrangement consisting of seven lines such that there is a component of the new arrangement passing through four its singular points (counting with multiplicities). Therefore this line should be a fixed component of the pencil of degree 3. Similarly, if we remove any line from \( \mathcal{L} \) (the case \( d = 4 \), i.e., \( k = 8 \)), then we obtain a new arrangement consisting of eight lines such that there is a component of the new arrangement passing through a 4-fold point and three other singular points of the new arrangement. In the case \( d = 5 \) (i.e., \( k = 9 \)), also there is a line (for example, \( L_1 \)) passing through two 4-fold points and two other singular points of the line arrangement \( \mathcal{L} \).

Claim 4.16. Let \( X \) be a Burniat surface \( X_s \). Then
\[
\dim H^{(1)} = \dim H^{(2)} = \dim H^{(3)} = \max(0, 2 - s).
\]

Proof. Consider the space \( H^{(1)} \) (the cases of the spaces \( H^{(2)} \) and \( H^{(3)} \) are similar). Let \( \omega \in H^{(1)}, \omega \neq 0 \). Then, since \( \omega \in M_1 \), we have
\[
\omega = (R_1(x, y)dx + R_2(x, y)dy) \otimes \frac{dx \wedge dy}{w_1},
\]
where \( R_i(x, y) \) are rational functions. Note that the form \( \frac{dx \wedge dy}{w_1} \) has not poles and zeros on \( Y_1 \setminus \text{Sing} Y_1 \), since \( w_1^2 = l_1 \ldots l_6 \) and \( \deg l_1 \ldots l_6 = 6 \) (see Section 3). Therefore, since \( \omega \) is a regular form over the generic point of \( L_\infty \), as in the proof of Claim 4.15, applying Lemma 4.14, one can easily show that \( \omega \) can be written in the form
\[
\omega = \frac{P(x, y)dx + Q(x, y)dy}{l_7l_8l_9} \otimes \frac{dx \wedge dy}{w_1},
\]
where \( P(x, y), Q(x, y) \in \mathbb{C}[x, y] \) are polynomials of degree \( \leq 2 \), and, moreover, the form
\[
\omega_1 = \frac{P(x, y)dx + Q(x, y)dy}{l_7l_8l_9} \in H^0(U \setminus \text{Sing} D_1, \Omega^1_{U \setminus \text{Sing} D_1}(\log D_1)),
\]
where \( D_1 = L_7 + L_8 + L_9 \). Without loss of generality, we can assume that \( p_3 = (0, 0), l_1(x, y) = x - ay, l_7(x, y) = y, l_8(x, y) = x - y, \) and \( l_9(x, y) = x \), where \( a \neq 0, 1 \).

To resolve singularities of \( Y_s \) we should blow up 4-fold and triple points of \( \mathcal{L}_s \). If the form \( \omega \in H^{(1)} \), then it should be regular over the blown up curves \( E_i \).

The following Lemmas 4.17 – 4.18 give necessary and sufficient conditions on the form (39) to be regular over the curve \( E_3 \).
Lemma 4.17. Let \((V, o) \subset (\mathbb{C}^2 \times \mathbb{C}^2, o)\) be a germ of a normal surface given in coordinates \((z_1, z_2, w_1, w_2)\) by equations \(w_1^2 = x - ay\) and \(w_2^2 = xy(x - y)\), where \(a \neq 0, 1,\) and let

\[
\omega = \frac{P(x, y)dx + Q(x, y)dy}{xy(x - y)} \otimes \frac{dx \wedge dy}{w_1} \in H^0(\tilde{V}, \Omega_{\tilde{V}}^1 \otimes \Omega_{\tilde{V}}^2),
\]

where \(P(x, y), Q(x, y) \in \mathbb{C}[x, y]\), \(\deg P(x, y) = \deg Q(x, y) = 2\), and \(\nu : \tilde{V} \to V\) is the minimal resolution of the singular point \(o\) of \(V\). Then

\[
\omega = (e \frac{ydx - xdy}{xy(x - y)} + \frac{P_2(x, y)dx + Q_2(x, y)dy}{xy(x - y)}) \otimes \frac{dx \wedge dy}{w_1},
\]

where \(e\) is a constant and \(P_2(x, y), Q_2(x, y)\) are homogeneous polynomials of degree 2.

Proof. Let \(Z\) be the image \(g(V)\) of the map \(g((x, y, w_1, w_2)) = (x, y)\) and \(\sigma : \tilde{Z} \to Z\) the blow up with center at \(g(o), E = \sigma^{-1}(g(o))\). By Lemma 1.4, the map \(f : \tilde{V} \to \tilde{Z}\) induced by \(g\) is an analytic covering and it can be factorized into the composition \(f = \tilde{f}_1 \circ h_1\), where \(\tilde{f}_1 : \tilde{V}_1 \to \tilde{Z}\) is a \(\mathbb{Z}/2\mathbb{Z}\)-covering and \(\tilde{V}_1\) is bi-meromorphic to the surface given by \(w_1^2 = x - ay\).

The morphism \(\sigma\) is given by \(x = u, y = uv\) in some local coordinates in \(\tilde{Z}\). Then the surface \(\tilde{V}_1\) is given locally by \(w_1^2 = u(1 - av)\). Therefore \(\tilde{f}_1\) is branched along \(E\) and it is easy to see that \(h_1\) is not branched at the generic point of \(\tilde{f}_1^{-1}(E)\). Thus \(h_1\) is a local isomorphism at the generic point of \(\tilde{f}_1^{-1}(E)\).

The form \(\omega\) is a meromorphic form on \(\tilde{V}_1\) and, by assumption, \(h^*(\omega)\) is holomorphic. Therefore, \(\omega\) is holomorphic at the generic point of \(\tilde{f}_1^{-1}(E)\). Moreover, by Lemma 4.14, it can have at most logarithmic poles along the curves given by equations \(y = 0, x - y = 0,\) and \(x - ay = 0\). We have

\[
\omega = \frac{(P(u, uv) + vQ(u, uv))du + uQ(u, uv)dv}{u^3v(1 - v)} \otimes \frac{udu \wedge dv}{w_1} = \frac{(P(u, uv) + vQ(u, uv))du}{u^2v(1 - v)} + \frac{Q(u, uv)dv}{uv(1 - v)} \otimes (dw_1 \wedge dv).
\]

Therefore \(Q(0, 0) = P(0, 0) = 0\) and \(P(u, uv) + vQ(u, uv)\) should be divisible by \(u^2\).

Put \(P = a_1x + a_2y + P_2(x, y)\) and \(Q = b_1x + b_2y + Q_2(x, y)\), where \(P_2\) and \(Q_2\) are homogeneous polynomials of degree two. We have

\[
P(u, uv) + vQ(u, uv) = a_1u + a_2uv + b_1uv + b_2uv^2 + P_2(u, uv) + Q_2(u, uv),
\]
where $P_2(u, uv) + Q_2(u, uv)$ is divisible by $u^2$. Therefore $a_1 = b_2 = 0$ and $a_2 = -b_1$. We have

$$\omega = (a_2 \frac{y dx - x dy}{xy(x-y)} + P_2(x, y)) dx + Q_2(x, y) dy \frac{xy(x-y)}{xy(x-y)} \otimes \frac{dx \wedge dy}{w_1}.$$ 

\[\square\]

**Lemma 4.18.** Let $x, y$ be coordinates in $U = \mathbb{C}^2$, $o = (0, 0)$, $D \subset U$ a divisor given by $xy(x-y) = 0$, and

$$\omega_1 = \frac{P(x, y) dx + Q(x, y) dy}{xy(x-y)} \in H^0(U \setminus \{0\}, \Omega^1_{U \setminus \{0\}}(\log D)),$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of degree two. Then $\omega_1$ is a linear combination of $\frac{dx}{x}$, $\frac{dy}{y}$, and $\frac{d(x-y)}{x-y}$.

**Proof.** Since $\omega_1 \in H^0(U \setminus \{(0, 0)\}, \Omega^1_{U \setminus \{(0, 0)\}}(\log D)$, then $P = yP_1(x, y)$ and $Q = xQ_1(x, y)$. Let $P_1(x, y) = a_1 x + a_2 y$ and $Q_1(x, y) = b_1 x + b_2 y$. Put $l = x - y$. We have $dl = dx - dy$. Therefore

$$\omega_1 = \frac{yP_1(x, y) dx + xQ_1(x, y) dy}{xy(x-y)} =$$

$$\frac{((x-l)P_1(x, x-l) + xQ_1(x, x-l))dx - xQ_1(x, x-l)dl}{x(x-l)l} =$$

$$\frac{((a_1 + a_2 + b_1 + b_2)x^2 + l(\ldots))dx - xQ_1(x, y)dl}{x(x-l)l}$$

and, consequently, we should have

$$a_1 + a_2 + b_1 + b_2 = 0,$$

i.e.,

$$\omega_1 = \frac{(a_1 xy + a_2 y^2) dx + (b_1 x^2 - (a_1 + a_2 + b_1) xy) dy}{xy(x-y)} =$$

$$-a_2 \frac{dx}{x} + b_1 \frac{dy}{y} + (a_1 + a_2) \frac{d(x-y)}{x-y}.$$ 

\[\square\]

It follows from Lemmas 4.17 - 4.18 that if $\omega \in H_{(1)}$, then $\omega$ has the form

$$\omega = (c \frac{y dx - x dy}{y(x-y)(x-ay)} + c_1 \frac{dx}{x} + c_2 \frac{dy}{y} + c_3 \frac{d(x-y)}{x-y}) \otimes \frac{dx \wedge dy}{w_1}. \quad (40)$$

**Lemma 4.19.** A form

$$\omega_1 = c_1 \frac{dx}{x} + c_2 \frac{dy}{y} + c_3 \frac{d(x-y)}{x-y}$$
is regular at the generic point of the line at infinity \( L_{\infty} = \mathbb{P}^2 \setminus \mathbb{C}^2 \) if and only if \( c_1 + c_2 + c_3 = 0 \).

Proof. Let \( x = \frac{1}{u} \) and \( y = \frac{w}{u} \). We have
\[
\omega_1 = c_1 \frac{dx}{x} + c_2 \frac{dy}{y} + c_3 \frac{d(x - y)}{x - y} = (c_1 + c_2 + c_3) \frac{du}{u} + \frac{c_2(1 - v) - c_3v}{v(1 - v)} dv.
\]
\( \square \)

Lemma 4.20. Let \( D \subset \mathbb{P}^2 \) be the projective closure of the curve \( D_0 \subset \mathbb{C}^2 \) given by \( xy(x - y) = 0 \) and \( 0 = (0, 0) \in \mathbb{C}^2 \) the origin. Then the form
\[
y dx - x dy \quad \frac{y dx - x dy}{xy(x - y)} \in H^0(\mathbb{P}^2 \setminus \{0\}, \Omega_{\mathbb{P}^2 \setminus \{0\}}^1(\log D)).
\]
Proof. Straightforward. \( \square \)

It follows from Lemmas 4.19 and 4.20 that if \( \omega \) having the form (40) belongs to \( H(1) \) then
\[
c_1 + c_2 + c_3 = 0. \tag{41}
\]

Let \( p_2 \) has coordinates \((b, 0)\). The following Lemma gives a necessary and sufficient condition on the form (40) to be regular over the curve \( E_2 \).

Lemma 4.21. Let \( \overline{V} \) be the desingularization of a germ of a surface \((V, o) \subset (\mathbb{C}^2 \times \mathbb{C}^2, 0)\) given in coordinates \((x, y, w_1, w_3)\) by equations \( w_1^2 = (x - a_1y)(x - a_2y) \) and \( w_3^2 = y \), and let
\[
\omega = (c \cdot \frac{y dx - (x + b)dy}{(x + b)y(x - y + b)} + c_2 \frac{dy}{y} \otimes \frac{dx \wedge dy}{w_1}) \in H^0(\overline{V}, \Omega_{\overline{V}}^1 \otimes \Omega_{\overline{V}}^2),
\]
where \( a_1, a_2, a_3 \), and \( b \) are constants, \( a_i \neq a_j \) for \( i \neq j \), \( a_i \neq 0 \) for \( i = 1, 2, 3 \), \( b \neq 0 \). Then \( c - bc_1 = 0 \).

Proof. Consider the covering \( g : V \to g(V) = Z \subset \mathbb{C}^2 \) given by \( g((x, y, w_1, w_3)) = (x, y) \) and let \( \sigma : \tilde{Z} \to Z \) be the blow up with center at \( g(o) = (0, 0) \), \( E = \sigma^{-1}(g(o)) \). By Lemma 1.4, the map \( f : \overline{V} \to \tilde{Z} \) induced by \( g \) is a regular covering and it can be factorized into the composition \( f = \overline{f}_1 \circ h_1 \), where \( \overline{f}_1 : \overline{V}_1 \to \tilde{Z} \) is a \( \mathbb{Z}/2\mathbb{Z} \)-covering and \( \overline{V}_1 \) is bimeromorphic to the surface given by \( w_1^2 = (x - a_1y)(x - a_2y)(x - a_3y) \).

Let \( \tau \) be given by \( x = uw, y = v \) in some local coordinates in \( \tilde{Z} \). Then the surface \( \overline{V}_1 \) is given locally by \( \tilde{w}_1^2 = v(u - a_1)(u - a_2)(u - a_3) \), where \( \tilde{w}_1 = \frac{w}{u} \) and the surface \( \overline{V} \) is the normalization of a surface given locally by \( \tilde{w}_1^2 = v(u - a_1)(u - a_2)(u - a_3) \) and \( \tilde{w}_3^2 = v \). Therefore
\( \tilde{f}_1 \) is branched along the exceptional curve \( E \) given by \( v = 0 \) and \( h_1 \) is not branched at the generic point of \( \tilde{f}_1^{-1}(E) \). Thus, \( h_1 \) is a local isomorphism at the generic point of \( f^{-1}(E) \).

The form

\[
\omega = \left( c \frac{ydx - (x + b)dy}{(x + b)y(x - y + b) + c_2 \frac{dy}{y}} \right) \otimes \frac{dx \wedge dy}{w_1} = \frac{cx^2du + (-cb + c_2b^2 + c_2v(...))dv}{v(uv + b)(uv - v + b)} \otimes \frac{du \wedge dv}{\tilde{w}_1}.
\]

Since \( \omega \in H^0(\overline{V}, \Omega^1_{\overline{V}} \otimes \Omega^2_{\overline{V}}) \) and \( b \neq 0 \), it follows from Lemma 4.14 that \( c - c_2b \) should be equal to 0. \( \square \)

It follows from Lemma 4.21 that if \( \omega \) having the form (40) belongs to \( H_{(1)} \) then

\[ c - bc_2 = 0, \]

where \( p_2 = (0, b) \).

One can check that the point \( p_1 \) does not give a restriction on the form (40).

Consider restrictions on the form (40) are given by triple points. Let one of the lines \( L_8 \) or \( L_9 \) (say \( L_9 \)) passes through a triple point \( p_{3+i} \) of \( \tilde{T}_s \). Since the point \( p_{3+i} \in L_9 \), it has coordinates \( (0, b_1) \), \( b_1 \neq 0 \)

**Lemma 4.22.** Let \( \overline{V} \) be the desingularization of a germ of a surface \( (V, o) \subset (\mathbb{C}^2 \times \mathbb{C}^2, o) \) be a given in coordinates \( (x, y, w_1, w_2) \) by equations \( w_1^2 = (x - a_1y)(x - a_2y) \) and \( w_2^2 = x(x - a_1y) \), and let

\[
\omega = \left( c \frac{(y + b_1)dx - xdy}{x(y + b_1)(x - y - b_1)} + c_1 \frac{dx}{x} \right) \otimes \frac{dx \wedge dy}{w_1}
\]

belongs to \( H^0(\overline{V}, \Omega^1_{\overline{V}} \otimes \Omega^2_{\overline{V}}) \), where \( a_1, a_2, \) and \( b_1 \) are constants, \( a_1 \neq a_2, a_i \neq 0, b_1 \neq 0 \). Then \( c - b_1c_1 = 0 \).

**Proof.** As in the proof of Lemma 4.17, consider the map \( g : V \to g(V) = Z \subset \mathbb{C}^2 \) given by \( g((x, y, w_1, w_2)) = (x, y) \) and let \( \sigma : \tilde{Z}^2 \to Z \) be the blow up with center at \( g(o) = (0, 0), E = \sigma^{-1}(g(o)) \). By Lemma 1.4, the map \( f : \overline{V} \to \tilde{Z} \) induced by \( g \) is a regular covering and it can be factorized into the composition \( f = \tilde{f}_1 \circ h_1 \), where \( \tilde{f}_1 : \tilde{V}_1 \to \tilde{Z} \) is a \( \mathbb{Z}/2\mathbb{Z}\)-covering and \( \tilde{V}_1 \) is bimeromorphic to the surface given by \( w_1^2 = (x - a_1y)(x - a_2y) \).

Let \( \sigma \) be given by \( x = u, y = uv \) in some local coordinates in \( \tilde{Z} \).

Then the surface \( \tilde{V}_1 \) is given locally by \( \tilde{w}_1^2 = (1 - a_1v)(1 - a_2v) \), where \( \tilde{w}_1 = \frac{w_1}{u} \) and the surface \( \overline{V} \) is the normalization of a surface given locally by \( \tilde{w}_1^2 = (1 - a_1v)(1 - a_2v) \) and \( \tilde{w}_2^2 = (1 - a_1v) \), where \( \tilde{w}_2 = \frac{w_2}{u} \).

Therefore \( \tilde{f}_1 \) is not branched along the exceptional curve \( E \) given by
$u = 0$ and $h_1$ is not branched at the generic point of $f^{-1}(E)$. Thus $h_1$ is a local isomorphism at the generic point of $f^{-1}(E)$.

We have

$$
\omega = \left( c \frac{(y + b_1)dx - xdy}{x(y + b_1)(x - y - b_1)} + c_1 \frac{dx}{x} \right) \otimes \frac{dx \wedge dy}{w_1} + \frac{cb_1 - c_1 b_1^2 + u(\ldots)du + u(\ldots)dv}{u(uv + b_1)(u - uv - b_1)} \otimes \frac{du \wedge dv}{\tilde{w}_1}.
$$

Since $\omega \in H^0(\overline{V}, \Omega^1_{\overline{V}} \otimes \Omega^2_{\overline{V}})$, $b_1 \neq 0$, the number $c - c_1 b_1$ should be equal to 0.

It follows from Lemma 4.22 that if $\omega$ having the form (40) belongs to $H_{(1)}$, then

$$
c - c_1 b_1 = 0,
$$

where $p_{3+i} = (0, b_1) \in L_9$.

If the arrangement $L$ has two triple points with coordinates $(0, b_1)$ and $(0, b_2)$ lying in $L_9$, then the equations $c - c_1 b_1 = 0$ and $c - c_1 b_2 = 0$ are linear independent. Similarly, it is easy to see that if a triple point $p_{s+3} \in L_8$, then it also give some linear equation of the form

$$
f(c, c_3) = 0.
$$

As a consequence, we obtain that the space $H_{(1)}$ consists of the forms

$$
\omega = \left( c \frac{ydx - xdy}{xy(x - y)} + c_1 \frac{dx}{x} + c_2 \frac{dy}{y} + c_3 \frac{d(x - y)}{x - y} \right) \otimes \frac{dx \wedge dy}{w_1}
$$

satisfying $2 + s$ linear equations (41) – (44). Note that these equations are linear independent. Therefore $\dim H_{(1)} = \max(0, 2 - s)$.

Denote by $M_{6-s}$, $0 \leq s \leq 4$, the union of irreducible components of the moduli scheme of the surfaces of Burniat type $s$, and denote by $\overline{B}_{6-s}$ the image of $B_{6-s}$ in $M_{6-s}$.

**Corollary 4.23.** The variety $\overline{B}_{6-s}$ is everywhere dense in $M_{6-s}$ if $s \leq 2$ and

(i) the space $M_2$ is non-singular at $X_{4,0} = \overline{B}_2$, $\dim M_2 = 6$;

(ii) the space $M_3$ is non-singular at any point $X_3 \in \overline{B}_3$, $\dim M_3 = 4$, and $\overline{B}_3$ is a rational curve;

(iii) the space $M_4 = M_4' \cup M_4''$ consists of two irreducible rational surfaces $M_4'$ and $M_4''$, $M_4$ is non-singular at each point $X_2 \in \overline{B}_4$;

(iv) the space $M_5$ is unirational 3-fold nonsingular at any point $X_1 \in \overline{B}_5$;
(v) the space $\mathcal{M}_6$ is unirational 4-fold nonsingular at any point $X_0 \in \mathcal{B}_6$.

Proof. It follows from (30) – (34), Claims 4.10, 4.11, and Proposition 4.12.

**Proposition 4.24.** In notation of Section 4.1, let the Campedelli covering $f_C : X_C \to \mathbb{P}^2$ be branched along the Campedelli line arrangement depicted in Fig. 6. Then the Burniat surface $X_4$ is isomorphic to the Campedelli surface $X_C$.

![Fig. 6](image-url)

Proof. First of all, note that $f_C$ can be decomposed into the composition $f_C = f_{C,1} \circ h_{C,1}$, where $f_{C,1} : X_{C,1} \to \mathbb{P}^2$ is the desingularization of the double covering $g_{C,1} : Y_{C,1} \to \mathbb{P}^2$ given in non-homogeneous coordinates by equation

$$w_1^2 = l_{(1,0,0)}l_{(1,1,0)}l_{(1,0,1)}l_{(1,1,1)}$$

(here $l_\alpha(x, y) = 0$ is an equation of the line $L_\alpha$). To resolve the singularities of $Y_{C,1}$, we should blow up the points $p_1 \ldots, p_6$. Let $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ the composition of these blowups, $E_i = \sigma^{-1}(p_i)$, and $\tilde{L}_\alpha = \sigma^{-1}(L_\alpha)$. 
It is easy to see that the curves $E_i$ do not belong to the branch locus of $f_{C,1}$ and for each $(0, a_2, a_3)$ the strict transform $f_{C,1}^{-1}(\tilde{L}_{(0,a_2,a_3)})$ of $\tilde{L}_{(0,a_2,a_3)}$ is the disjoint union of two rational curves $L'_{(0,a_2,a_3)}$ and $L''_{(0,a_2,a_3)}$, since the rational curve $\tilde{L}_{(0,a_2,a_3)}$ does not meet the branch locus of $f_{C,1}$. Therefore the $(\mathbb{Z}/2\mathbb{Z})^2$-covering $h_{C,1} : X_C \to X_{C,1}$ is branched over the union of the curves $E_i$, $i = 1, \ldots, 6$, and the curves $f_{C,1}^{-1}(\tilde{L}_{(0,a_2,a_3)})$, $(a_2, a_3) \in (\mathbb{Z}/2\mathbb{Z})^2$. We have $(L'_{(0,a_2,a_3)}, L''_{(0,a_2,a_3)})x_{C,1} = (L''_{(0,a_2,a_3)}, L''_{(0,a_2,a_3)})x_{C,1} = -1$, since the intersection number
\[
(\tilde{L}_{(0,a_2,a_3)}, \tilde{L}_{(0,a_2,a_3)})_{\mathbb{P}^2} = -1
\]
and $\text{deg} f_{C,1} = 2$. The mutual arrangement of the curves $L'_{(0,a_2,a_3)}$ and $L''_{(0,a_2,a_3)}$, $(a_2, a_3) \in (\mathbb{Z}/2\mathbb{Z})^2$, is depicted in Fig. 7.

![Fig. 7](image)

Similarly, for each $(1, a_2, a_3)$ the intersection number
\[
(\tilde{L}_{(1,a_2,a_3)}, \tilde{L}_{(1,a_2,a_3)})_{\mathbb{P}^2} = -2
\]
and therefore, the strict transform $D_{(1,a_2,a_3)} = f_{C,1}^{-1}(\tilde{L}_{(1,a_2,a_3)})$ has the self-intersection number
\[
(D_{(1,a_2,a_3)}, D_{(1,a_2,a_3)})x_{C,1} = (\frac{1}{2}f_{C,1}^*(\tilde{L}_{(1,a_2,a_3)}), \frac{1}{2}f_{C,1}^*(\tilde{L}_{(1,a_2,a_3)}))x_{C,1} = -1
\]
and
\[
(D_{(1,a_2,a_3)}, L'_{(0,b_2,b_3)})x_{C,1} = (D_{(1,a_2,a_3)}, L''_{(0,b_2,b_3)})x_{C,1} = 0
\]
for all $(a_2, a_3)$ and $(b_2, b_3)$. Note that each $D_{(1,a_2,a_3)}$ is also a rational curve.

It is not hard to see that $X_{C,1}$ is a rational surface and if $\tau : X_{C,1} \to \tilde{X}_{C,1}$ is the blowdown of the curves $L'_{(0,1,0)}$, $L'_{(0,0,1)}$, $L'_{(0,1,1)}$, and four
curves $D_{(a_1,a_2,a_3)}$, then $\tilde{X}_{C,1}$ is isomorphic to the projective plane $\mathbb{P}^2$, the image

$$\tau(\sum_{i=1}^{6} E_i + L'_{(0,1,0)} + L''_{(0,0,1)} + L''_{(0,1,1)})$$

is the Burniat line arrangement $\mathcal{L}_4$, and the covering $h_{C,1}$ coincides with the Burniat covering $f: X_4 \to \mathbb{P}^2$. □

**Corollary 4.25.** The moduli space $\mathcal{M}_2$ coincides with the moduli space $\mathcal{C}$ of the Campedelli surfaces.

4.3 A surface $X$ of general type with $p_g = 0$, $K_X^2 = 6$ and $(\mathbb{Z}/3\mathbb{Z})^3 \subset \text{Tors}(X)$. Let $\mathcal{L} = L_1 + \cdots + L_6$ be an arrangement in $\mathbb{P}^2$ of six lines having 3 triple points $p_1, p_2, p_3$ not lying in the same line. The arrangement $\mathcal{L}$ is depicted in Fig. 8.

![Fig. 8](image)

Consider a covering $g: Y \to \mathbb{P}^2$ associated with the epimorphism $\varphi: H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to G = (\mathbb{Z}/3\mathbb{Z})^2$ given by

$$\varphi(\lambda_1) = \varphi(\lambda_2) = \varphi(\lambda_3) = (1,0),$$

$$\varphi(\lambda_4) = (2,1), \; \varphi(\lambda_5) = (1,1), \; \varphi(\lambda_6) = (0,1).$$

The surface $Y$ has 3 singular points lying over the triple points $p_i$. By Lemma 1.4 to resolve them, it is sufficient to blow up the points $p_i$ and
consider the induced Galois covering \( f : X \to \mathbb{P}^2 \), where \( \sigma : \mathbb{P}^2 \to \mathbb{P}^2 \) is the composition of blowups with centers at the points \( p_i \). Denote by \( E_i = \sigma^{-1}(p_i) \) the exceptional curve lying over \( p_i \).

**Proposition 4.26.** The constructed above surface \( X \) is a surface of general type with \( K_X^2 = 6 \), \( p_g = 0 \), and \( (\mathbb{Z}/3\mathbb{Z})^3 \subset \text{Tors}(X) \).

**Proof.** By Claim 2.2, we have \( 3K_X = |f^*(3L - \sum E_i)| \), where \( L = \sigma^*(\mathbb{P}^1) \) is the total transform of a line \( \mathbb{P}^1 \subset \mathbb{P}^2 \). Therefore, \( X \) is a surface of general type with ample canonical class. Applying (3) and (9), it is easy to see that \( K_X^2 = 6 \) and \( e(X) = 6 \). Therefore, by Noether’s formula, \( p_a = 1 - q + p_g = 1 \). As above, to calculate \( p_g \), it is enough to calculate the geometric genera of 4 cyclic coverings corresponding to 4 epimorphisms from the group \( G = (\mathbb{Z}/3\mathbb{Z})^2 \) to the cyclic group \( \mathbb{Z}/3\mathbb{Z} \). These coverings are given respectively in non-homogeneous coordinates by the following equations:

\[
\begin{align*}
w_2^3 &= l_1l_2l_3l_4l_5; \\
w_3^3 &= l_1l_2l_3l_4l_6; \\
w_4^3 &= l_1l_2l_4l_5l_6.
\end{align*}
\]

Applying Claim 3.3, one can easily check that the geometric genus of each of these coverings is equal to zero. Thus, \( X \) has the geometric genus \( p_g = 0 \).

To see that \( (\mathbb{Z}/3\mathbb{Z})^3 \subset \text{Tors}(X) \), consider the universal covering \( g_{u(3)} : \tilde{Y}_{u(3)} \to \tilde{\mathbb{P}}^2 \) corresponding to the epimorphism

\[
\varphi : H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \overline{T}, \mathbb{Z}/3\mathbb{Z}) \simeq (\mathbb{Z}/3\mathbb{Z})^5
\]

given by

\[
\begin{align*}
\varphi(\lambda_1) &= (1, 0, 0, 0, 0), & \varphi(\lambda_2) &= (1, 0, 1, 0, 0), \\
\varphi(\lambda_3) &= (1, 0, 0, 1, 0), & \varphi(\lambda_4) &= (2, 1, 0, 0, 1), \\
\varphi(\lambda_5) &= (1, 1, 0, 0, 0), & \varphi(\lambda_6) &= (0, 1, 2, 2, 2).
\end{align*}
\]

It is easy to see that the Galois covering \( h_{u,\varphi} : X_u \to X \) induced by the projection \( \psi : (\mathbb{Z}/3\mathbb{Z})^5 \to G = (\mathbb{Z}/3\mathbb{Z})^2 \) to the first two coordinates is unramified. Therefore, by Corollary 1.6, \( (\mathbb{Z}/3\mathbb{Z})^3 \subset \text{Tors}(X) \).

**Claim 4.27.** The surface \( X_u \) has irregularity \( q(X_u) = 3 \).

**Proof.** As in the proof of Claim 4.8, to calculate \( q \), it is sufficient to calculate \( p_a \) and \( p_g \).

We have \( p_a(X) = 1 \). Therefore, the arithmetic genus \( p_a(X_u) = 3^3 \), since \( h_{u,\varphi} \) is unramified and \( \deg h_{u,\varphi} = 3^3 \).

To calculate \( p_g \), it is sufficient to calculate, by Claim 3.3, the geometric genera of \( \frac{3^5 - 1}{2} = 121 \) cyclic coverings corresponding to \( \frac{3^5 - 1}{2} \) epimorphisms \( \psi_m, m = 1, \ldots, 121 \), of \( G_{u,\varphi} = (\mathbb{Z}/3\mathbb{Z})^5 \) to the cyclic
group $\mathbb{Z}/3\mathbb{Z}$. The calculation is left to the reader, note only that the contribution to the irregularity of $X_u$ is given only by the cyclic coverings

$$z_1^3 = l_1l_2l_6, \quad z_2^3 = l_1l_3l_5, \quad z_3^3 = l_2l_3l_4.$$  

\[ \Box \]

**Corollary 4.28.** The fundamental group of the surface $X$, constructed above, is a non-abelian infinite group.

**Proof.** It follows from Claim 4.27. \[ \Box \]

**4.4. The Godeaux surface.** Let $\overline{L} = L_1 + L_2 + L_3 + L_4$ be an arrangement in $\mathbb{P}^2$ of four lines in general position. Consider the following coverings: the universal covering $g_{u(5)} : Y_{u(5)} \to \mathbb{P}^2$ corresponding to the epimorphism

$$\varphi : H_1(\mathbb{P}^2 \setminus \overline{L}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \overline{L}, \mathbb{Z}/5\mathbb{Z}) \simeq (\mathbb{Z}/5\mathbb{Z})^3,$$

a covering $g : Y \to \mathbb{P}^2$ associated with the epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2$ given in some chosen coordinates in $G = (\mathbb{Z}/5\mathbb{Z})^2$ by

$$\varphi(\lambda_1) = (1, 0), \quad \varphi(\lambda_2) = (0, 1), \quad \varphi(\lambda_3) = (1, 2), \quad \varphi(\lambda_4) = (3, 2),$$

and the covering $h : Y_{u(5)} \to Y$ corresponding to an epimorphism $\psi : (\mathbb{Z}/5\mathbb{Z})^3 \to G = (\mathbb{Z}/5\mathbb{Z})^2$ such that $\varphi = \psi \circ \varphi$. By Lemma 1.4, the surface $Y$ is nonsingular and by Proposition 1.5, the covering $h$ is unramified.

**Proposition 4.29.** The constructed above surface $Y$ is a surface of general type with $K_Y^2 = 1$, $p_g = 0$, and $\text{Tors}(Y) = \mathbb{Z}/5\mathbb{Z}$.

**Proof.** By Claim 2.2, we have $5K_Y = |f^*(L)|$, where $L$ is a line in $\mathbb{P}^2$. Therefore $Y$ is a surface of general type with ample canonical class. Applying (3) and (4), it is easy to see that $K_Y^2 = 1$ and $e(Y) = 11$. Therefore, by Noether’s formula, $p_a = 1 - q + p_g = 1$. To calculate $p_g$, it is enough to calculate the geometric genera of 6 cyclic coverings corresponding to 6 cyclic subgroups of $G$ and given respectively in non-homogeneous coordinates by the following equations:

$$w_1^5 = l_1l_3l_4^3; \quad w_2^5 = l_2l_3l_4^2; \quad w_3^5 = l_1l_2l_3^3;$$
$$w_4^5 = l_1^2l_2l_3^2l_4^3; \quad w_5^5 = l_1l_2l_4^2; \quad w_6^5 = l_1l_3^3l_4^2l_5.$$  

Applying calculation made in section 3, one can easily check that the geometric genus of each of these coverings is equal to zero. Thus, $Y$ has the geometric genus $p_g = 0$.

Since the Galois covering $h$ is unramified, we have the following inclusion: $\mathbb{Z}/5\mathbb{Z} \subset \text{Tors}(Y)$. To show that $\text{Tors}(Y) = \mathbb{Z}/5\mathbb{Z}$, it is sufficient to show that $Y_{u(5)}$ is simply connected. Moreover, it is easy to see that
$Y_{u(5)}$ is isomorphic to a smooth surface in $\mathbb{P}^3$. Indeed, let us choose homogeneous coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ such that $x_i = 0$ is an equation of $L_{i+2}$. Let $\sum a_i x_i = 0$ be an equation of $L_1$. Without loss of generality we can assume that the covering $g_{u(5)}$ is associated to the epimorphism $\overline{\varphi} : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^3$ given by

\[
\overline{\varphi}(\lambda_1) = (0, 0, 1), \quad \overline{\varphi}(\lambda_2) = (4, 4, 4), \quad \overline{\varphi}(\lambda_3) = (1, 0, 0), \quad \overline{\varphi}(\lambda_4) = (0, 1, 0).
\]

In this case $Y_{u(5)}$ is given by

\[
z_3^5 = z_1; \quad z_4^5 = z_2; \quad z_5^5 = a_0 + a_1 z_1 + a_2 z_2
\]

in non-homogeneous coordinates $(z_1, z_2, \ldots, z_5)$, where $z_1 = \frac{x_1}{x_0}$ and $z_2 = \frac{x_2}{x_0}$, and therefore $Y_{u(5)}$ is isomorphic to the projective closure of the surface in $\mathbb{C}^3$ given by $z_5^5 = a_0 + a_1 z_3^5 + a_2 z_4^5$ (cf. [God]).

REFERENCES

[B-P-V] Barth W., Peters C., Van de Ven A.: Compact complex surfaces. Springer-Verlag, Berlin - Heidelberg - New York - Tokio (1984).

[Bu] Burniat P.: Sur les surfaces de genre $P_{12} > 0$. Ann. Pura Appl. (4) 71 (1966), 1–24.

[Cam] Campanelli L.: Sopra alcuni piani doppi notevoli con curve di diramazione del decimo ordine. Atti Acad. Naz. Lincei 15 (1932), 536–542.

[Dol] Dolgachev I.: Algebraic surfaces with $q = p_g = 0$. in Algebraic surfaces, Liguori, Napoli (1971).

[God] Godeaux L.: Sur une surface algébrique de genre zero et bigenére deux. Atti. Acad. Naz. Lincei (1971) 14 (1931), 479–481.

[G-R] Grauert H., Remmert R.: Komplexe Räume. Math. Ann. 136 (1958), 245–318.

[Kh-Ku] Kharlamov V., Kulikov Vik.S.: Deformation inequivalent complex conjugated complex structures and applications. Turk. J. Math. 26 (2002), 1–25.

[Mi] Miyaoka Y.: On numerically Campedelli surfaces. in ”Complex analysis and Algebraic geometry”, Iwanami-Shoten, Tokyo (1977), 112–118.

[Pet] Peters C.: On certain examples of surfaces with $p_g = 0$. Nagoya Math. J. 66 (1977), 109–120.

Steklov Mathematical Institute
E-mail address: kulikov@mi.ras.ru