On robust stability of switched homogeneous systems

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Abstract

This paper analyses the robust stability of a class of homogeneous systems where a relation among the high-order term, the decay rate and the domain of attraction of the system is explicitly established. Such a new observation helps to establish several robust stability conditions for switched homogeneous systems with high-order terms. Several domains of attraction related to asymptotical and finite-time stability properties are also provided. The new results are further applied to supervisory control of spacecraft attitude system with high-order uncertainties to illustrate their efficiency and applicability.

1 | INTRODUCTION

Homogeneous systems can be regarded as local approximations of nonlinear systems in the sense that local asymptotical stability of homogeneous approximated systems implies the asymptotic stability of the original nonlinear systems. Moreover, the degree of homogeneity in closed-loop systems determines the convergence rate [1, 2]. Stability analysis and stabilising controller design for homogeneous systems have been widely investigated (see e.g. [3–7]). Homogeneous system approach has also been widely used in many physical engineering, such as spacecraft attitude control. In [8], the attitude and angular velocity stabilisation is solved for flexible spacecraft against actuator complete failure, in which the homogeneous system property is well utilised to remove the effects of high-order coupling terms.

High-order perturbations often exist in a homogeneous system. It is well known that a stable homogeneous system may remain locally stable even with higher order perturbing terms; the domain of attraction depends on the number of degrees and the magnitude of the high-order term, this domain is often required to be small enough [3]. Such an attractive feature of robustness facilitates the analysis and design of homogeneous system since the higher order terms can be neglected as in [9]. However, the explicit and quantitative relation among the high-order terms, the convergence rate and the domain of attraction is still not clear and has not yet been properly investigated.

On the other hand, fruitful results have been obtained on stability problem of switched systems (see e.g. [10–15, 32, 33, 36], and references therein). Recently, switched homogeneous system has attracted much attention due to its academic meaning and practical value in attitude control of the liquid-filled spacecraft whose system dynamics possesses homogeneous feature and fixed sequential switching property. The growth rate, local and global asymptotical stability and convergence property have been studied in [16–20]. In [16], the growth rate of a class of discrete-time homogeneous systems under arbitrary switching is studied, which is the analogue of joint spectral radius for switched linear systems. In [17], sufficient conditions of the existence of the common Lyapunov function is established for switched homogeneous systems to achieve its global asymptotic stability under arbitrary switching; multiple Lyapunov function case is also considered where the local or global asymptotical stability can be guaranteed subject to admissible switching law. Moreover, a switching control scheme is developed for discrete-time, homogenous, and unstable second order systems under binary sensor limitations by utilising finite state input/output approximation technique in [19].
Robustness problem has also been deeply investigated for switched systems with uncertainties in either each mode [21, 22] or the switching scheme [23, 24]. The main idea behind these works follows the same robust control way that aims to overcome the negative effects of the uncertainties by using control design. This however may not be necessary for a switched homogeneous system with high-order uncertainties. Based on these concerns, the following technical issues arise:

- Whether the robustness of a switched homogeneous system can be achieved without robust control design if the initial states are within a small domain?
- How to construct such a domain?
- What conditions does the switching signal has to be satisfied?

To solve the above problems, this paper addresses the robustness issue of switched homogeneous systems against high-order terms that appear in each mode. For a single homogeneous sub-system with high-order terms, the explicit relationship among the high-order term, the decay rate and its domain of attraction is established. Combining these relationships together yields several robust stability conditions to ensure locally robust exponential, asymptotical, and finite-time stability. To the best of our knowledge, no result has been reported along this direction. Also note that this paper only studies a robust stability problem rather than a tracking problem. The main contributions are twofold:

1. The robustness issue of the homogeneous system is studied and the quantitative relation among the high-order term, the decay rate and the domain of attraction of the system is explicitly established.
2. For the switched homogeneous system with high-order uncertainties, several robust stability conditions are established that are easy to check a priori and several domains of attraction are provided.
3. The new theoretical results are applied to spacecraft attitude system with high-order uncertainties where a family control laws and a supervisory control scheme are proposed that lead to the smaller input amplitudes compared with that of the single robust control law.

The rest of the paper is organised as follows: Section 2 provides some preliminaries. Sections 3 analyses the robustness of homogeneous system, based on which the switched system is considered in Section 4. Section 5 applies the developed results in Sections 3 and 4 to the spacecraft attitude control. Section 6 gives the conclusion.

2 | PRELIMINARIES

The concept of homogeneity in the usual sense and some of its properties will be introduced.

**Definition 1** ([1]). A function \( b : \mathbb{R}^n \rightarrow \mathbb{R} \) is homogeneous of degree \( l \) if there exists a scalar \( l \in \mathbb{R} \) such that, \( \forall \varepsilon > 0, b(\varepsilon x) = \varepsilon^l b(x) \). A vector field \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be homogeneous of degree \( k \) if \( f(x) = x^{1+k} f(\varepsilon x) \) where \( f(\varepsilon x) \) denotes the \( i \)th component of \( f \). A system \( \dot{x} = f(x) \) is called homogeneous if its vector field \( f(x) \) is homogeneous.

The following result can be directly obtained from the well known robust stability result in [3].

**Lemma 1.** Let \( f \) be a homogeneous vector field of degree \( k \), and let \( g \) be a continuous vector field, both defined on \( \mathbb{R}^n \) such that for all \( i = 1, 2, \ldots, n \),

\[
\lim_{\varepsilon \to 0} \frac{g(\varepsilon x)}{\varepsilon^{1+k}} = 0,
\]

Then if the trivial solution \( x = 0 \) of \( \dot{x} = f(x) \) is locally asymptotically stable, the same is true for the trivial solution of the perturbed system \( \dot{x} = f(x) + g(x) \).

Consider a switched homogeneous system

\[
\dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t)),
\]

where \( x \in \mathbb{R}^n \) are the states that are continuous everywhere. Define \( M = \{1, 2, \ldots, m\} \) with \( m \) being the number of modes. \( \sigma(t) : [0, \infty) \rightarrow M \) denotes the switching function, which is a piecewise constant function continuous from the right. \( f_i, i \in M \) are continuous, \( f_i(0) = 0 \). The uncertain term \( g_i, i \in M \) are also continuous, \( g_i(0) = 0 \). Denote by \( f_j, j = 0, 1, \ldots \) the \( j \)th switching instant, \( t_0 = 0 \). It follows that mode \( \sigma(t) \) is activated in \([t_i, t_{i+1})\), for \( i = 0, 1, 2, \ldots \). Define the dwell time \( \tau > 0 \), which is said to be available in period \([t_i, t_{i+1})\) for \( 0 \leq j < s \leq \infty \) if \( \inf_{k}(t_{k+1} - t_k) \geq \tau \) for \( k = j, j + 1, \ldots, s - 1 \).

Throughout the paper, we make the following assumptions:

A1: \( f_i(x), i \in M \), is homogeneous of degree \( k \). The origin of the system \( \dot{x} = f_i(x) \) is locally asymptotically stable.

A2: \( g_i(x), i \in M \), is homogeneous of degree \( k + q \), with \( q > 0 \).

A2 can be relaxed to be \( |g_i(x)| \leq \phi_i(x) \) for a known (not necessary homogeneous) function \( \phi_i(x) \). In this case, the proposed results are still available, but would become conservative.

**Definition 2.** The origin of the switched homogeneous system (1) under \( \sigma(t) \) is

- stable: if for any \( \varepsilon > 0 \), there exists \( \xi > 0 \), such that

\[
|x(0)| \leq \xi \Rightarrow |x(t)| \leq \varepsilon, \quad \forall t \geq 0.
\]

- locally asymptotically stable: if it is stable and there exists a set \( A \subseteq \mathbb{R}^n \), called the domain of attraction, such that

\[
x(0) \in A \Rightarrow \lim_{t \to \infty} |x(t)| = 0.
\]
locally finite-time stable: if it is stable and there exists \( T \in (0, \infty) \), called the settling time, and a set \( A \subseteq \mathbb{R}^n \), such that
\[
x(0) \in A \Rightarrow \lim_{t \to T} x(t) = 0, \text{ and } x(t) \equiv 0, \quad \forall t > T
\]

Definition 2 is an extension of local stability notions of non-linear system in [2] to the switching case with high-order uncertainties. Such a definition implies that for a perturbed switched homogeneous system, the stabilising switching law may depend on the domain that the initial states are within. This fact will be further verified in Section 4. Note that both the globally asymptotical and globally finite-time stability imply that the domain of attraction of the system (1) is \( \mathbb{R}^n \). While in our study, the explicit and quantitative relation among the high-order terms, the convergence rate, and the domain of attraction is established. Therefore, only the locally asymptotical and locally finite-time stability are considered.

The problem to be solved in this paper is to establish local stability conditions and provide the quantitative approximation of the domain of attraction for switched homogeneous system (1) such that its local asymptotical stability or local finite-time stability is achieved at the origin.

## 3 ROBUSTNESS OF HOMOGENEOUS SYSTEMS

This section just focuses on one mode of the switched system. For the sake of clearness, the subscript \( i \) that denotes mode \( i \) is removed temporarily. The dynamics of mode \( i, i \in \mathcal{M} \), takes the simplified form
\[
\dot{x} = f(x) + g(x).
\]

Since the system \( \dot{x} = f(x) \) is asymptotically stable at the origin, it follows from [3] that there exists a Lyapunov function \( V \) which is homogeneous of degree \( m \) and of class \( C^p \), where \( m \) is a real number larger than \( p \) with \( p \) being a positive integer.

As a consequence, \( \frac{\partial V}{\partial x} f \) is homogeneous of degree \( m + k \), and
\[
\frac{\partial V}{\partial x} f < 0, \quad \forall x \neq 0.
\]

Define \( S_2 = \{ z \in \mathbb{R}^n | \| z \| = 1 \} \). The first main result is given:

**Theorem 1.** Consider mode \( i, i \in \mathcal{M} \), of switched system (1) satisfying \( A1 \) and \( A2 \), it holds that
\[
\dot{V}(x) \leq -\bar{\lambda} V(x)^{\frac{m+k}{m}}, \quad \text{if } a + \varepsilon^T b < 0, \quad (3)
\]
\[
\dot{V}(x) \leq \bar{\lambda} V(x)^{\frac{m+k}{m}}, \quad \text{if } a + \varepsilon^T b \geq 0, \quad (4)
\]

where \( \varepsilon > 0 \) is such that \( \frac{\varepsilon}{\varepsilon} \in S_2 \), and
\[
a \triangleq \max_{z \in S_2} \frac{\partial V}{\partial z} f, \quad b \triangleq \max_{z \in S_2} \frac{\partial V}{\partial z} g,
\]
\[
\bar{\lambda} \triangleq -\frac{a + \varepsilon^T b}{\varepsilon}, \quad \bar{\lambda} \triangleq a + \varepsilon^T b.
\]

Moreover, if there exists a constant \( \psi \) such that \( a + \varepsilon^T b < 0, \forall \varepsilon \in (0, \psi) \), then a domain of attraction\(^1\) is
\[
\Omega \triangleq \left\{ x \in \mathbb{R}^n \left| \left( \frac{V}{\varepsilon} \right)^{\frac{1}{m}} < \psi \right. \right\}.
\]

**Proof.** Based on the definition of \( \varepsilon \) and \( \bar{\lambda} \), one has
\[
\frac{\varepsilon}{\varepsilon} \leq V(z) \leq \bar{\lambda} \varepsilon, \quad \forall \varepsilon \in S_2.
\]

Note that \( \forall \varepsilon \in \mathbb{R}^n, x = \varepsilon^T z \) with \( \varepsilon \in S_2 \) for the choice \( \varepsilon = |x| \), thus
\[
\varepsilon |x|^m \leq V(x) \leq \bar{\lambda} |x|^m, \quad \forall \varepsilon \in \mathbb{R}^n.
\]

Since \( \varepsilon = \frac{\varepsilon}{\varepsilon} \in S_2 \), the time derivative of \( V(\varepsilon z) \) along the system (2) takes the form
\[
\dot{V}(\varepsilon z) = \frac{\partial V}{\partial z} (f(\varepsilon z) + g(\varepsilon z))
\]
\[
= \varepsilon^{m+k} \frac{\partial V}{\partial z} f(\varepsilon z) + \varepsilon^{m+k+q} \frac{\partial V}{\partial z} g(\varepsilon z)
\]
\[
\leq (a + \varepsilon^T b) \varepsilon^{m+k}.
\]

Inequalities (3) and (4) follow consequently.

Condition (3) implies that if \( a + \varepsilon^T b < 0 \), then the origin of mode \( i \) is asymptotically (w.r.t. finite-time) stable if \( k \geq 0 \) (w.r.t. \( k < 0 \)).

Suppose there exists a constant \( \psi \) such that \( a + \varepsilon^T b < 0, \forall \varepsilon \in (0, \psi) \), this implies that if \( |x| < \psi \), then condition (3) holds true. Let \( [t_1, t_f] \) be the period in which mode \( i \) is activated. Due to the decreasing property of \( V' \) under (3), it follows that if \( \varepsilon |x(t)|^m \leq V(t) < \bar{\lambda} |x|^m \), then \( |x(t)|^m \leq V(t) < \bar{\lambda} |x|^m \), \( \forall \varepsilon \in [t_1, t_f] \), thus the domain of attraction is
\[
\Omega \triangleq \left\{ x \in \mathbb{R}^n \left| \left( \frac{V}{\varepsilon} \right)^{\frac{1}{m}} < \psi \right. \right\}.
\]

This completes the proof. \( \square \)

Theorem 1 establishes the explicit relations among the homogeneous degrees, the convergence rate, and the domain of attraction. We make some important discussions by following remarks:

**Remark 1** (relations among \( a, b, \) and \( \varepsilon \), with \( q \) fixed).

\(^1\) such a domain may be related to asymptotical or finite-time stability that depends on the value of \( k \).
- For $0 < b \leq -a, \lambda > 0$ cannot be arbitrarily large, the domain of attraction is limited, but it is allowed that $\varepsilon \geq 1$. In this case, $\varepsilon \uparrow \Rightarrow \lambda \downarrow, \varepsilon \downarrow \Rightarrow \lambda \uparrow$, that is, a larger (smaller) domain that the initial states are within allows for a slower (faster) convergence rate.

- For $b > -a, \varepsilon$ must be less than 1 such that $\lambda > 0$, the domain of attraction is within the unit sphere. In this case, $\varepsilon \uparrow \Rightarrow \lambda \downarrow, \varepsilon \downarrow \Rightarrow \lambda \uparrow$.

- For $b < 0$: the origin of the system $\dot{x} = g(x)$ is also asymptotically stable, one finds that $\lambda > 0$ always holds for any $\varepsilon > 0$. This implies that the domain of attraction is $\mathbb{R}^n$, and $\varepsilon \uparrow \Rightarrow \lambda$, $\varepsilon \downarrow \Rightarrow \lambda$.

Remark 2 (relations among $a$, $b$, and $g$, with $\varepsilon$ fixed).

- For $0 < b \leq -a, \varepsilon \geq 1$: one has $g \uparrow \Rightarrow \lambda \downarrow, g \downarrow \Rightarrow \lambda \uparrow$, that is, a larger (smaller) homogeneous degree of $g(x)$ allows for a slower (faster) converging speed.

- For $0 < b \leq -a, \varepsilon < 1$: one has $g \uparrow \Rightarrow \lambda \downarrow, g \downarrow \Rightarrow \lambda \uparrow$.

- For $b > -a$: in this case $\varepsilon < 1$, one has $g \uparrow \Rightarrow \lambda \downarrow, g \downarrow \Rightarrow \lambda$.

- For $b < 0, \varepsilon \geq 1$, one has $g \uparrow \Rightarrow \lambda \downarrow, g \downarrow \Rightarrow \lambda$.

- For $b < 0, \varepsilon < 1$, one has $g \uparrow \Rightarrow \lambda \downarrow, g \downarrow \Rightarrow \lambda$.

Remark 3 (relations among $g$ and $\varepsilon$).

- For $b > 0$: two cases can be considered: 1) $\varepsilon < 1$, one has $q \uparrow \Rightarrow \varepsilon \downarrow, q \downarrow \Rightarrow \varepsilon$, that is, the larger (smaller) is the homogeneous degree of $g(x)$, the larger (smaller) is the domain of attraction. 2) one has $q \uparrow \Rightarrow \varepsilon \downarrow, q \downarrow \Rightarrow \varepsilon \uparrow$.

- For $b < 0$: no mutual constraint between $q$ and $\varepsilon$.

Remark 4. Robust local stability analysis and the estimation of robust domain of attraction have been investigated for linear and nonlinear systems with uncertainties that are described by unknown constant or slow-varying parameters (see e.g. [25, 26]). Most of existing methods rely on the uncertainty-dependent Lyapunov function. However, with the aid of homogeneity, the construction of Lyapunov function is completely uncertainty-independent.

Remark 5. As a topic closely related to this work, stability of positive nonlinear systems has also been well studied where the homogeneity is often required for the vector field as in [27–29]. A representative result reported in [27] shows that if the external uncertainty is non-negative, then for the uncertain system, the non-negative domain is still a forward invariant set, and the asymptotical equilibrium point belongs to the positive domain. Another related topic to our study is the robust stability of triangular system under perturbing non-linearities (e.g. [34, 35]). It follows from [35] that if the perturbing non-linearities are high-order homogeneous, then the global robust asymptotic stability is realised. Theorem 1 can be extended to homogeneous positive systems and homogeneous triangular systems with high-order homogeneous uncertainties to achieve their local asymptotic stability at the origin.

![FIGURE 1 State trajectories of systems 1 and 2](image)

Example 1. Consider four homogeneous systems

$\dot{x} = f^{(i)}(x) + g^{(i)}(x), \ i \in \{1, 2, 3, 4\}$

where

$f^{(i)}(x) = \begin{bmatrix} -0.5x_1 + x_2 - x_1 - 0.5x_2 \\ 0.2x_1 + x_2 - x_1 - 0.2x_2 \\ -0.5x_1 + x_2 - x_1 - 0.5x_2 \\ -x_1 + x_2 - x_1 - x_2 \end{bmatrix}$

$g^{(i)}(x) = \begin{bmatrix} 3x_2^2 \\
0.8x_2 \sqrt{x_1} - 0.2x_1x_2 \\
x_2^2 \\
-0.5x_1x_2 + 0.77x_1^2x_2 \end{bmatrix}^T$.

It is clear that $f^{(0)}(x)$ is homogeneous of degree zero, $g^{(1)}(x), g^{(2)}(x)$ and $g^{(3)}(x)$ are homogeneous of degree one and $g^{(4)}(x)$ is homogeneous of degree three. Select a common Lyapunov function $V(x) = x_1^2 + x_2^2$, which is homogeneous of degree two. It follows from Theorem 1 that $a^{(1)} = -1, a^{(2)} = -0.4, a^{(3)} = -1, a^{(4)} = -2, a^{(i)} = 1.5396, a^{(3)} = 0.4619, a^{(3)} = 0.3849, a^{(3)} = 0.1140, a^{(3)} = 0.3849, a^{(3)} = 0.076$. It further has that $\psi^{(1)} = 0.65, \psi^{(2)} = 0.87, \psi^{(3)} = 0.4619, \psi^{(4)} = 2.60$. In the simulation, the initial states $x(0) = [-0.4, 0.4]^T$ for systems 1 and 2, and $x(0) = [-1.5, 1.5]^T$ for systems 3 and 4.

Figure 1 shows the state trajectories of systems 1 and 2. Since the initial states are within the domains of attraction, both two systems are locally asymptotically stable at the origin. Although $g^{(1)} = g^{(2)}$, the converging speed of system 1 is larger than that of system 2 since $\lambda^{(1)} > \lambda^{(2)}$. Figure 2 compares the state trajectories of systems 3 and 4. The local asymptotic stability can also be guaranteed for two systems. Note that two systems have the same domain of attraction, while the converging speed of system 3 is slower than that of system 4 since $\lambda^{(3)} < \lambda^{(4)}$. 
4  ROBUSTNESS OF SWITCHED HOMOGENEOUS SYSTEMS

Based on analysis in Section 3, this section analyses the robustness of the whole switched system; our goal is to provide the stability conditions on the switching signals and to construct the domains of attraction. We shall first analyse the case \( k = 0 \) where the main idea is given, and then consider the other two more complicated cases \( k > 0 \) and \( k < 0 \) consequently. The symbols that have appeared in Section 3 are equipped with the subscript \( i \) in this section, which are related to mode \( i \).

Under assumption A1, the vector field of each mode is homogeneous of degree \( k \), thus the Lyapunov function of each mode is homogeneous of degree \( m \). Define \( \mu \) such that \( V_j \leq \mu V_k, \forall j, k \in \mathcal{M}, j \neq k \). This is a standard formula describing a relation between different \( V_j \) [10, 29]. In the presence of multiple functions \( V_j \), it is clear that \( \mu = \max_{j_{i}, j \in \mathcal{M}, j \neq j_{i}} \frac{\zeta_{j}}{\zeta_{j_{i}}} \). If there is only a common \( V \), then one has \( \mu = 1 \). Also denote \( \lambda_{0} \triangleq \min_{i \in \mathcal{M}} \Lambda_{i} \).

4.1 Robust stability in the case \( k = 0 \)

The second main result is given:

**Theorem 2.** Consider a switched homogeneous system (1) with each mode satisfying A1 and A2 and (3), and \( k = 0 \). The origin is locally asymptotically stable with the domain of attraction \( \bigcap_{i \in \mathcal{M}} \Omega_{i} \), if

\[
\tau > \frac{1}{\lambda_{0}} \cdot \max_{i_{j}, j \in \mathcal{M}, j \neq j_{i}} \left[ \ln \left( \frac{\mu \zeta_{i} \psi_{m}}{\zeta_{i_{j}} \psi_{m}} \right) \right],
\]

which is available in period \([0, \infty)\).

**Proof.** We shall prove that condition (10) guarantees that \( x(t) \in \Omega_{i_{j}}(k), k = 1, 2, ..., \) and the local asymptotical stability at the origin.

Note that \( \bigcap_{i \in \mathcal{M}} \Omega_{i} \subseteq \Omega_{i}, \forall i \in \mathcal{M} \). For any initial mode \( \sigma(0) \) and initial states \( x(0) \in \bigcap_{i \in \mathcal{M}} \Omega_{i} \), one has \( x(0) \in \Omega_{i_{j}}(0) \).

Firstly, consider \( \tau \in [0,t_{1}) \) in which mode \( \sigma(0) \) is activated. It follows from (3) that

\[
V_{\sigma(0)}(t) \leq e^{-\frac{\lambda_{0}}{2} t} V_{\sigma(0)}(0).
\]

According to condition (10), it holds that

\[
\tau > \frac{1}{\lambda_{0}} \ln \left( \frac{\mu \zeta_{i} \psi_{m}}{\zeta_{i_{j}} \psi_{m}} \right).
\]

This together with the fact that \( t_{1} \geq \tau \) and \( x(0) \in \Omega_{i_{j}(0)} \) yields

\[
V_{\sigma(0)}(t_{1}) \leq e^{-\frac{\lambda_{0}}{2} t_{1}} V_{\sigma(0)}(0) \leq \mu \zeta_{i} \psi_{m}, \forall i, j \in \mathcal{M}, i \neq j. \]

Thus, \( x(t_{1}) \in \Omega_{i_{j}(0)} \).

By induction, condition (10) guarantees that at each switching instant \( t_{i} \), \( i = 1, 2, ..., \) \( x(t_{i}) \in \Omega_{i_{j}(0)} \). Thus in each interval \([t_{i}, t_{i+1})\), mode \( \sigma(t_{i}) \) satisfies (3).

On the other hand, one can find from (10) that the maximal value of \( \ln(\frac{\mu \zeta_{i} \psi_{m}}{\zeta_{i_{j}} \psi_{m}}) \) can be obtained when \( \frac{\zeta_{i} \psi_{m}}{\zeta_{i_{j}} \psi_{m}} \) reaches its maximal value, while \( \frac{\zeta_{i} \psi_{m}}{\zeta_{i_{j}} \psi_{m}} \) must be no less than 1, and equals 1 when \( \zeta_{i_{j}} \psi_{m} = \zeta_{i_{j}} \psi_{m} \), \( \forall i, j \in \mathcal{M}, i \neq j \). Therefore, one concludes that

\[
\tau > \frac{\ln \mu}{\lambda_{0}}.
\]

Condition (13) is a well-known dwell time condition that ensures \( V_{\sigma}(t_{i+1}) < V_{\sigma}(t_{i}) \), for \( t_{i}, i = 1, 2, ... \). Note that all modes satisfy (3), thus \( V_{\sigma}(t) \) always decreases in \([t_{i}, t_{i+1})\). It further leads to \( \lim_{t \rightarrow \infty} V_{\sigma}(t) = 0, \forall i \in \mathcal{M} \). This together with (7) leads to the results. This completes the proof. \( \square \)

**Remark 6.** Condition (10) is more restrictive than the existing stabilising dwell time condition. This is because the individual domain of attraction of each mode may be different from each other; a dwell time has to be taken for each mode such that the states enter the domain of attraction of the mode that will be activated in at the next switching instant. If all modes share a common Lyapunov function and a common domain of attraction, that is, \( V_{j} = V \) and \( \Omega_{i} = \Omega, \forall i \in \mathcal{M} \), condition (10) holds automatically. In such a case, the local asymptotical stability at the origin can be guaranteed under arbitrary switching and the domain of attraction is \( \Omega \).
Theorem 2 does not impose any constraint on the switching sequence, this indeed makes the domain of attraction conservative. Such a domain can be enlarged provided that the switching sequence is fixed. In this case, the initial mode is fixed, a natural idea is to let the domain of attraction be \( \Omega_{\sigma(0)} \). This however can be further relaxed as shown below.

Define a set \( \mathcal{M}_{\text{seq}} \subseteq \mathcal{M} \) which contains all modes that will be activated along the sequence. Pick any \( s^* \in \mathcal{M}_{\text{seq}} \) and let \( t_{s^*} \) be the time instant that mode \( s^* \) is switched into the plant for the first time. Let \( s_r \) be the number of switches that happen in \([0,t_{s^*}]\). It follows that \( t_{s^*} = t_{s^*} + s_r \) which denotes the \( s_r \)th switching instant of the switched system as defined in Section 2.

Define a set

\[
\Theta_{s^*} \triangleq \left\{ x \in \mathbb{R}^n \left| \left( \frac{\mu^{s^*} V_{\sigma(0)}}{\zeta_{s^*}^{m}} \right)^{\frac{1}{\alpha}} \right| < \psi_{s^*} \right\}. \tag{14}
\]

For a fixed sequence, \( \Theta_{\sigma(0)} = \Omega_{\sigma(0)} \), and \( \Theta_{s^*} \subseteq \Omega_{s^*} \), for \( s^* \in \mathcal{M}_{\text{seq}}. \) Also denote \( \lambda_1 \triangleq \max_{s \in \mathcal{M}_{\text{seq}}} \lambda_i. \)

**Corollary 1.** Consider a switched homogeneous system (1) with each mode satisfying A1 and A2 and (3) and (4), and \( k = 0. \) The switching sequence is fixed. The origin is locally asymptotically stable with the domain of attraction \( \Theta_{s^*} \), \( s^* \in \mathcal{M}_{\text{seq}} \) if

\[
0 < t_{s^*} < \frac{1}{\lambda_1} \ln \left( \frac{\zeta_{s^*} \psi_{s^*}^{m}}{\mu^{s^*} V_{\sigma(0)(0)}} \right), \text{ for } s^* > 0 \tag{15}
\]

and condition (10) holds in \([t_{s^*}, \infty)\).

**Proof.** Two cases are considered:

**Case 1:** \( x(0) \in \Theta_{\sigma(0)} \). In this case, \( s_{r^*} = 0, t_{s^*} = 0, \) condition (15) is not used, condition (10) holds in \([0, \infty)\). The result follows directly from Theorem 2.

**Case 2:** \( x(0) \in \Theta_{s^*}, s^* \in \mathcal{M}_{\text{seq}}, s^* \neq \sigma(0) \). In this case, \( \ln \left( \frac{\zeta_{s^*} \psi_{s^*}^{m}}{\mu^{s^*} V_{\sigma(0)(0)}} \right) > 0 \), it follows from (15) that mode \( s^* \) will be switched into the plant at \( t = t_{s^*} = t_{s^*} > 0 \). In the interval \([0,t_{s^*}]\), there exists \( s_{r^*} - 1 \) times of switchings, it follows from (4) and condition (15) that

\[
V_{s^*}(t_{s^*}) \leq \mu^{s^*} \sum_{i=1}^{s_{r^*} - 1} \lambda_{s_i} V_{\sigma(i)}(t_{s^*} + 0)V_{\sigma(0)}(0).
\]

\[
< \mu^{s^*} \lambda_{s_{r^*}} V_{\sigma(0)}(0).
\]

\[
\leq \zeta_{s^*} \psi_{s^*}. \tag{16}
\]

Thus, \( x(t_{s^*}) \in \Omega_{s^*} \), the rest of the proof is the same as in that for Theorem 2. \( \square \)

**Remark 7.** The domain of attraction provided by Corollary 1 is not limited to \( \Omega_{\sigma(0)} \). It shows that the stabilising switching law and the domain of attraction are closely related to each other. Different domain of attraction requires different constraint on the switching signal. For the domain of attraction \( \Theta_{s^*} \), \( s^* \in \mathcal{M}_{\text{seq}}, s^* \neq \sigma(0) \), \([x(t)] \) may increase in \([0,t_{s^*}] \) due to the property of dynamics (4). However condition (15) guarantees that \( x(t) \) would never go outside of \( \Omega_{s^*} \) before \( t = t_{s^*}. \)

It should be pointed out that the domain of attraction of single system provided in Theorem 1 is an invariant set. This however is not the case for the switched system. Although the domains of attraction provided in Theorem 2 and Corollary 1 are not invariant sets, both Theorem 2 and Corollary 1 ensure that \( x(t) \in \bigcup_{s^* \in \mathcal{M}} \Omega_s, \forall t \geq 0 \).

**Example 2.** Consider a switched homogeneous system with two modes

\[ x \dot{=} f_i(x) + g_i(x), \; i \in \{1, 2\}, \]

where \( f_1(x) = \begin{bmatrix} -x_1 + 2x_2 \end{bmatrix}^T, \; f_2 = \begin{bmatrix} -x_1 - x_2 \end{bmatrix}^T, \) \( g_1 = \begin{bmatrix} x_2 \end{bmatrix}^T, \; g_2 = \begin{bmatrix} x_2 \end{bmatrix}^T \). One finds that \( f_2(x) \) is homogeneous of degree zero, \( g_2(x) \) is homogeneous of degree one. Select Lyapunov functions as \( V_1(x) \triangleq x_1^2 + 2x_2^2 \) and \( V_2(x) \triangleq 2x_1^2 + x_2^2 \), which are homogeneous of degree one. It follows from Theorem 1 that \( a_1 = a_2 = -2, b_1 = b_2 = 2.3094, \) \( c_1 = c_2 = 2, d_1 = d_2 = 1. \) One can approximate the domains of attraction of two modes as \( \Omega_1 = \{x \in \mathbb{R}^2 | x_1^2 + 2x_2^2 < 0.86\} \) and \( \Omega_2 = \{x \in \mathbb{R}^2 | 2x_1^2 + x_2^2 < 0.86\} \). The switching sequence is supposed to be:

\[
\text{mode 1} \rightarrow \text{mode 2} \rightarrow \text{mode 1} \rightarrow \cdots
\]

We only illustrate Corollary 1 where two cases are considered:

**Case 1:** \( x(0) \in \Omega_1 \). Condition (15) is not used. It follows from condition (10) that the dwell time satisfying \( \tau > 1.39 \). Figure 3 implies the state trajectories w.r.t. time when \( x(0) \in \Omega_1 \). We can easily see that the state trajectories converge no matter...
Consider the switching signal \( \sigma(t) \) modelled by a Markov process in probability, that is,

\[
P\left\{ \sigma(t + \Delta) = j | \sigma(t) = i \right\} = \begin{cases} 
1 - \gamma_{ij} & i \neq j \\
0 & i = j
\end{cases}
\]

where \( \Delta > 0 \) is a given scalar, \( \gamma_{ij} \) denotes the state transition probability from mode \( i \) to mode \( j \), \( \gamma_{ij} = -\Sigma_{i \neq j} \gamma_{ij} \), and \( \lim_{\Delta \to 0} \frac{\gamma_{ij}}{\Delta} = 0 \). The similar robust stability conditions can be established when each mode is exponentially stable in probability.

Suppose that for each mode \( i \), there exists a positive definite function \( V_i(x(t)) \) such that

\[
\frac{d}{dt} V_i(x(t)) \leq \gamma_i x(x(t))
\]

For the initial state satisfying \( x(t_0) \in \Omega \), the following probability inequality

\[
P\left\{ |x(t)| < e^{-\lambda t} |x(t_0)| \right\} \geq 1 - \epsilon
\]

holds for all \( \epsilon > 0 \) with \( \lambda = \min \lambda_i \), then the locally exponential stability of the switched homogeneous system (1) is ensured by using the stochastic switched system theory [36, 37].

For the case \( x(t_0) \in \Omega \setminus \bigcap_{i \in M} \Omega_i \), if for any \( \epsilon > 0 \), there exists a finite positive scalar \( \kappa > 0 \), such that the following probability inequality

\[
P\left\{ x(t_0) + \kappa x \in \Omega \setminus \bigcap_{i \in M} \Omega_i \right\} \geq 1 - \epsilon_0
\]

holds; this together with (17) yields that the switched homogeneous system (1) is locally exponentially stable.
Robust stability in the cases $\gamma > 0$ and $\gamma < 0$}

This section applies the main idea and the method in Section 4.1 to the cases of $\gamma > 0$ and $\gamma < 0$. For the sake of simplification, we only consider a common Lyapunov function $V$ for all modes.

Theorem 3. Consider a switched homogeneous system (1) with each mode satisfying A1 and A2 and (3). If

$$\tau > \frac{m}{\Lambda_0 \delta} \max_{i_j \in M, i_j \neq j} \left[ \left( \xi_j \psi_j^+ \right)^{-\frac{1}{\gamma}} - \left( \xi_j \psi_j^m \right)^{-\frac{1}{\gamma}} \right]$$

(19)

and $\Delta = 0.1$. Let $x(0) = [0.4, 0.4]^T$, which is within the domains of attraction of the all four modes. Figure 7 implies the state trajectories converge no matter which mode is activated. Figure 8 shows the state trajectories, from which one can see that the system is exponentially stable.

Example 3. Consider a switched homogeneous system with four modes

$$\dot{x}(t) = f_i(x(t)) + g_i(x(t)), \quad i \in \{1, 2, 3, 4\},$$

where the four modes are the four systems adopted in Example 1. The state transition probability matrix $\Gamma = (\gamma_{ij})_{4 \times 4}$ can be expressed by

$$\Gamma = \begin{bmatrix}
-2.1 & 0.6 & 0.6 & 0.9 \\
0.6 & -1.6 & 0.4 & 0.6 \\
0.6 & 0.4 & -1.6 & 0.6 \\
0.9 & 0.6 & 0.6 & -2.1
\end{bmatrix}$$

and $\Delta = 0.1$. Let $x(0) = [0.4, 0.4]^T$, which is within the domains of attraction of the all four modes. Figure 7 implies the state trajectories converge no matter which mode is activated. Figure 8 shows the state trajectories, from which one can see that the system is exponentially stable.

4.2 Robust stability in the cases $\gamma > 0$ and $\gamma < 0$

which is available in $[0, \infty)$, then

- For $\gamma > 0$, the origin is locally asymptotically stable;
- For $\gamma < 0$, the origin is locally finite-time stable.

The domain of attraction for both cases is $\bigcap_{i \in M} \Omega_i$.

Proof. We first consider $\gamma > 0$, then consider $\gamma < 0$.

Case 1: $\gamma > 0$. Similar to the proof of Theorem 2, we shall prove that condition (19) guarantees that $x(t) \in \Omega_{\sigma(t)}$, $k = 1, 2, \ldots$, and the local asymptotical stability at the origin. Note that $x(0) \in \Omega_{\sigma(0)}$. Firstly consider $t \in [0, t_1)$. It holds that

$$V(t)^{-\frac{1}{\gamma}} - V(0)^{-\frac{1}{\gamma}} \geq \frac{\Lambda_{\sigma(0)} k t}{m}. \quad (20)$$

Condition (19) leads to

$$V(t_1)^{-\frac{1}{\gamma}} > V(0)^{-\frac{1}{\gamma}} + \left( \xi_{\sigma(t_1)} \psi_{\sigma(t_1)}^m \right)^{-\frac{1}{\gamma}}$$

$$> \left( \xi_{\sigma(0)} \psi_{\sigma(0)}^m \right)^{-\frac{1}{\gamma}}.$$  \quad (21)

It follows that $x(t_1) \in \Omega_{\sigma(t_1)}$. By induction, condition (19) guarantees that at each $t_i$, $i = 1, 2, \ldots$, $x(t_i) \in \Omega_{\sigma(t_i)}$. Thus, in each interval $[t_i, t_{i+1})$, mode $\sigma(t)$ satisfies (3).

Since $t_{i+1} - t_i \geq \tau > 0$, for $i = 0, 1, \ldots$ one has

$$V(t_{i+1})^{-\frac{1}{\gamma}} - V(t_i)^{-\frac{1}{\gamma}} \geq \frac{\Lambda_{\sigma(t_i)} k(t_{i+1} - t_i)}{m}$$

$$> 0. \quad (22)$$
This leads to \( V(t_{i+1}) < V(t_i) \). Note that all modes satisfy (3), thus \( V(t) \) always decreases in \( [t_i, t_{i+1}] \). The result follows.

**Case 2: \( k < 0 \).** Similar to Case 1, we shall prove that condition (19) guarantees that \( x(t_k) \in \Omega_{\sigma(t_k)}, k = 1, 2, \ldots \), and the local finite-time stability at the origin.

Since \( x(0) \in \Omega_{\sigma(0)} \), it holds that for \( t \in [0, t_1) \)

\[
V(t)^{-\frac{k}{m}} - V(0)^{-\frac{k}{m}} \leq \frac{\lambda_{\sigma(t)}}{m}.
\]

Condition (19) leads to

\[
V(t)_{i}^{-\frac{k}{m}} < \left( z_{\sigma(t)}^{m} \psi_{\sigma(t)}^{m} \right)^{-\frac{k}{m}}.
\]

Therefore, \( x(t_i) \in \Omega_{\sigma(t_i)} \). By induction, condition (19) guarantees that at each \( t_i \), \( i = 1, 2, \ldots, x(t_i) \in \Omega_{\sigma(t_i)} \). Thus, in each interval \( [t_i, t_{i+1}] \), mode \( \sigma(t_i) \) satisfies (3).

Since \( t_{i+1} - t_i \geq \tau > 0 \), for \( i = 0, 1, \ldots \) one has

\[
V_{\sigma(t_i)}(t_{i+1})^{-\frac{k}{m}} - V_{\sigma(t_i)}(t_i)^{-\frac{k}{m}} < 0.
\]

Thus, \( V(t_{i+1}) < V(t_i) \). Note that all modes satisfy (3) and thus are individually finite-time stable. \( V(t) \) always decreases until \( x = 0 \), the settling time \( T \) satisfies

\[
T \leq -\frac{m}{\lambda_{\sigma}^k} V(0)^{-\frac{k}{m}}.
\]

This completes the proof.

For the case of fixed switching sequence, the following corollary can be obtained.

**Corollary 2.** Consider a switched homogeneous system (1) with each mode satisfying \( A_1 \) and \( A_2 \) and (3) and (4). The switching sequence is fixed. If

\[
0 < t_{i+1} - t_i = \frac{m}{\lambda_{\sigma}^k} \left( V(0)^{-\frac{k}{m}} - \left( z_{\sigma}^{m} \psi_{\sigma}^{m} \right)^{-\frac{k}{m}} \right), \text{ for } t_{i+1} > 0 \tag{27}
\]

and condition (19) holds in \( [t_{i+1}, \infty) \), then

- For \( k > 0 \), the origin is locally asymptotically stable;
- For \( k < 0 \), the origin is locally finite-time stable.

The domain of attraction for both cases is \( \Omega_{\sigma}, r^* \in M_{\text{opt}} \).

**Proof.** For \( k > 0 \), it follows from (4) that in the interval \( [0, t_{i+1}] \)

\[
V(t_{i+1})^{-\frac{k}{m}} \geq V(0)^{-\frac{k}{m}} - \frac{\lambda_{\sigma}^k t_{i+1}}{m}.
\]

Condition (27) guarantees that \( x(t_{i+1}) \in \Omega_{\sigma} \).

For \( k < 0 \), one has

\[
V(t_{i+1})^{-\frac{k}{m}} \leq V(0)^{-\frac{k}{m}} + \frac{\lambda_{\sigma}^k t_{i+1}}{m}.
\]

Condition (27) also guarantees that \( x(t_{i+1}) \in \Omega_{\sigma} \). The rest of the proof is the same as in that for Theorem 3.

5 | SPACECRAFT ATTITUDE CONTROL

This section applies the results in Sections 3 and 4 to the supervisory control problem of spacecraft attitude system with high-order uncertainties.

5.1 | Model setting

Let \( \omega \) be the angular velocity which is within a bounded region, \( J \triangleq \text{diag} [J_1, J_2, J_3] \) be the inertia matrix which is diagonal, \( u \) be the control torque, \( Q_k, k \in \{1, 2, 3, 4\} \) be the quaternion elements [38] which satisfies \( \sum_{i=1}^{3} Q_i^2 + Q_4^2 = 1 \). \( I_{3 \times 3} \) be the identity matrix of dimension \( 3 \times 3 \), \( \tilde{q} \) be a skew-symmetric matrix of a vector \( q \) to represent the vector cross operator which is expressed as

\[
\tilde{q} = \begin{bmatrix}
0 & -q_3 & q_2 \\
q_3 & 0 & -q_1 \\
-q_2 & q_1 & 0
\end{bmatrix}.
\]

\( \lambda_{\text{max}}(\cdot) \) and \( \lambda_{\text{min}}(\cdot) \) represent the maximal and minimal eigenvalues of a matrix, respectively.

Since in the standard spacecraft dynamic model described by the Euler equation and the quaternion kinematic equation has the potential numerical issue in that the quaternion has the different magnitudes near the equilibrium condition, that is, \( Q_k (t) \to 0, k \in \{1, 2, 3\} \) and \( Q_4 (t) \to \pm 1 \) [38], in this work, the original quaternion is replaced by an error quaternion: \( q_k \triangleq Q_k, k \in \{1, 2, 3\} \) and \( q_4 \triangleq \text{sign}(Q_4) - Q_4 \). The resulting attitude model is given as

\[
\dot{\omega} = -J^{-1} \tilde{\omega} \omega + J^{-1} u + \xi(\omega),
\]

\[
\dot{q} = \frac{1}{2} \tilde{q} \omega + \frac{1}{2} (\text{sign}(q_4) - q_4) \omega,
\]

\[
q_4 = \frac{1}{2} q^T \omega,
\]

where \( q \triangleq [q_1, q_2, q_3]^T \). Note that \( \text{sign}(q_4) - q_4 \) never crosses the zero line. Equations (31) and (32) are exactly equivalent to the quaternion equations in [38].

The term \( \xi(\omega, q) \) represents the uncertainty which is supposed to satisfy the homogeneous property expressed by

\[
\xi(\delta_{\xi}(\omega, q)) = \xi^{1+\nu}(\omega, q).
\]
for \( q_0 > 0 \) with \( \delta(\cdot) \) being a given dilation operator. Such kind of uncertainties are often induced by the uncertainties of the inertia matrix that are caused by fuel sloshing and fuel consumption when executing the tasks of attitude adjustment and orbital transfer [8, 39]. It is only assumed that \( |\xi(\omega, q)| \leq \zeta(\omega, q) \), for a known scalar function \( \zeta(\omega, q) \) without imposing any upper bound on \( \zeta(\omega, q) \).

Without loss of generality, only consider \( q_1 > 0 \). Define new variables as \( x \triangleq [\omega, \omega_2, \omega_3, q_1, q_2, q_3, q_4]^T \). The system (30)-(32) can be rewritten as

\[
\dot{x} = \begin{bmatrix} A & 0_{3 \times 1} \\ 0_{1 \times 6} & 0 \end{bmatrix} x + \begin{bmatrix} B \\ 0_{3 \times 3} \end{bmatrix} u + g(x) + \xi(x),
\]

where

\[
A \triangleq \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ A_0 & 0_{3 \times 3} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_0 \\ 0_{3 \times 3} \end{bmatrix}
\]

with \( A_0 \triangleq \text{diag}(1, 1, 1, 1), B_0 \triangleq \text{diag}(1, 1, 1) \), and

\[
g(x) = \begin{bmatrix} a_1 x_2 x_3 \\ a_2 x_1 x_3 \\ a_3 x_1 x_2 \\ -x_1 x_2 - x_3 x_3 \\ x_1 x_1 - x_2 x_1 + x_3 x_3 \\ -x_1 x_3 - x_2 x_2 + x_4 x_3 \\ x_2 x_1 - x_3 x_1 + x_4 x_3 \\ x_3 x_2 - x_4 x_3 \end{bmatrix}, \quad \xi(x) = \begin{bmatrix} \xi_1(x) \\ \xi_2(x) \\ \xi_3(x) \end{bmatrix}
\]

with \( a_1 \triangleq \frac{2(\omega_1 - \omega_2)}{\omega_1}, a_2 \triangleq \frac{2(\omega_2 - \omega_3)}{\omega_1}, a_3 \triangleq \frac{2(\omega_3 - \omega_1)}{\omega_1} \).

One has that \( g(x) \) is homogeneous of degree one since each element of \( g(x) \) satisfies

\[
g_{i}(\varepsilon x) = \varepsilon^{1+1} g_{i}(x), \quad i \in \{1, 2, \ldots, 7\}
\]

Since \((A, B)\) is a controllable pair, there exists a feedback gain \( K \) such that \( A + BK \) is nonsingular and Hurwitz. Obviously that the vector \((A + BK)x\) is homogeneous of degree zero.

### 5.2 Supervisory control strategy

In the presence of uncertainties that satisfy the homogeneity property, an natural idea is to design a single robust controller that makes the domain of attraction cover the bound of initial states. Such a conservative design obviously leads to a large input amplitude, which is often not expected or even not allowed in the practical situation with the input constraints. It will be shown that the input amplitude could be reduced according to different initial states. Moreover, due to the homogeneity property, the robustness can be maintained against any large uncertainties if the states are small enough. This allows for more flexible control strategies.

The control objective to be achieved is to design a family of control laws, denoted as \( u_i, i \in \mathcal{M} \), and a supervisory control scheme among them to stabilize the spacecraft attitude system (34).

Design the control law \( u_i, i \in \mathcal{M} \) as

\[
u_i = \left[ K_i, 0_{3 \times 1} \right] x
\]

where \( K_i \) is such that \( A + BK_i \) is nonsingular and Hurwitz. Also define \( \chi \triangleq [\omega, \omega_2, \omega_3, q_1, q_2, q_3, q_4]^T \), and \( J^T(x) \triangleq [(A + BK)_i x^T, 0]^T \).

Construct a positive definite function which is homogeneous of degree 2 as

\[
V^T(x) \triangleq \chi^T P \chi + z^2_i,
\]

where the symmetric positive definite matrix \( P \) satisfies \( -Q_i \ni (A + BK_i) P + P(A + BK_i) \) with \( Q_i \) being a positive definite matrix. Recall that \( z_i = \varepsilon^{-1} x_i \in S^{n-1} \), the time derivative of \( V(z) \) along the system (34) with \( u_i \) in (35) satisfies

\[
\dot{V}^T(z) \leq (\alpha_i + \beta e + \gamma e^\theta) e^2,
\]

where \( \alpha_i = -\lambda_{\min}(Q_i), \beta = \lambda_{\max}(P)(\frac{\kappa_{\max} + \kappa_{\min}}{3\sqrt{3}} + \frac{1}{2}) + \frac{1}{2}, \text{ and } \gamma = \lambda_{\max}(2P) \max_{\xi \in S^{n-1}} \zeta(\xi) \).

One further has that

\[
\dot{V}(z) \leq -\lambda_{\min} V(z), \quad \text{if } \alpha_i + \beta e + \gamma e^\theta < 0,
\]

\[
\dot{V}(z) \leq -\lambda_{\min} V(z), \quad \text{if } \alpha_i + \beta e + \gamma e^\theta \geq 0,
\]

where \( -\lambda_{\min} \triangleq (\alpha_i + \beta e + \gamma e^\theta) / \zeta \) and \( -\lambda_{\min} \triangleq (\alpha_i + \beta e + \gamma e^\theta) / \zeta \) with \( \zeta \triangleq \max(\lambda_{\max}(P), 1) \) and \( \zeta \triangleq \min(\lambda_{\min}(P), 1) \). Consequently, one obtains the domain of attraction under \( \nu_i(x) \) as

\[
\Omega_i \triangleq \left\{ x \in \mathbb{R}^7 \mid \left| \frac{\sqrt{V_i}}{\zeta} \right| \leq \varepsilon_i \right\}, \quad i \in \mathcal{M},
\]

where \( \varepsilon_i \) is a positive constant satisfying \( \alpha_i + \beta e + \gamma e^\theta < 0 \), \( \forall \varepsilon \in (0, \varepsilon_i) \).

Different \( u_i \) has different \( K_i \) and different \( \alpha_i \), and thus leads to different domain of attraction \( \Omega_i \), from each other. A family of control laws are designed under a common \( P \) such that

\[
\varepsilon_i < \varepsilon_{i+1}, \quad i \in \{1, 2, \ldots, m-1\}.
\]

Condition (40) implies that \( \Omega_i \subset \Omega_{i+1}, \quad i \in \{1, 2, \ldots, m-1\} \) as shown in Figure 9, which means that the control law related to larger domain of attraction can compensate for the uncertainties when the states are within the smaller domain of attraction, while the converse is not true. The design of each \( u_i \) can follow
the following procedure: determine $\varepsilon_i$, calculate $\alpha_i$, obtain $Q_i$, and then choose $K_i$. One can find that for the same states, $u_i$ has the smaller amplitude than that of $u_j$, $j < i$. This motivates the following supervisory control algorithm.

Algorithm 1:

1. Denote $t_0 = 0$, $j = 0$. Given $x(0)$, pick $i$, $i \in \mathcal{M}$ such that
   \begin{align}
   \alpha_i + \beta \varepsilon + \gamma \varepsilon_0 &< 0, \quad (41) \\
   \alpha_{i-1} + \beta \varepsilon + \gamma \varepsilon_0 &> 0, \quad (42)
   \end{align}
   where $\alpha_0 \triangleq 0$. If $i = 1$, then go to 4;
2. Let $t_{j+1} = t_j - \frac{2}{\lambda_i} \ln \frac{\varepsilon_{i+1}}{\varepsilon_i}$, apply $u_i$ until $t = t_{j+1}$;
3. Let $j = j + 1$, let $i = i - 1$. If $i = 1$, then go to 4; else go to 2;
4. Apply $u_1 \forall t \in [t_j, \infty)$.

The main idea behind Algorithm 1 is to pick the control law with minimum subscript that can robustly stabilise the system at the beginning according to the initial states (step 1), then switch into the control law with smaller subscript one by one such that each control law can robustly stabilise the system as the states converge to the origin (steps 2 and 3), the switching is stopped until $u_1$ is applied (step 4). An illustrative trajectory is also shown in Figure 9. Based on the analysis above, the following result is given which can be obtained from Theorem 2.

**Theorem 4.** Consider the spacecraft attitude control system (34) and a family of control laws satisfying (35) and (40), Algorithm 1 guarantees that the origin is exponentially stable.

### 5.3 Simulation results

In the simulation, the inertia matrix of the spacecraft is given in [39]

\[
J = \begin{bmatrix}
350 & 0 & 0 \\
0 & 270 & 0 \\
0 & 0 & 190
\end{bmatrix} \text{kg} \cdot \text{m}^2
\]
The uncertainty $\xi(x)$ is supposed to take the form

$$\xi(x) = [5.1962x_1x_2x_3, 0, 0, 0, 0, 0]^T$$

which is a homogeneous vector of degree two ($q_0 = 2$) since $\xi(x) = \varepsilon^{1+2q} \xi(x)$.

The feedback control laws are designed whose feedback gains are shown as $(44)-(47)$ in the last page.

It follows that $\alpha_1 = -1.0110$, $\alpha_2 = -10.0438$, $\alpha_3 = -36.0876$, $\alpha_4 = -55.1095$, $\beta = 1.0219$, $\gamma = 2$. The domains of attraction for each control law can be calculated whose parameters are $\varepsilon_1 = 0.5$, $\varepsilon_2 = 2$, $\varepsilon_3 = 4$, $\varepsilon_4 = 5$. This satisfies condition (40). The initial states are set as $w(0) = [0.5, -21, 3, 0]^T \text{rad} / \text{sec}$ and $[q^T(0), q_0(0)] = [0.3320, -0.4618, 0.1915, 0.2001]$.

Applying Algorithm 1 yields the switching sequence as $u_3 \to u_2 \to u_1$. Figure 10 shows the trajectory of states norm that converges to the origin, from which we can see that each control law is switched into after the states are within its related domain of attraction. Figure 11 shows the trajectories of the switching function and states, it can be seen that the system is exponentially stable at the origin. This verifies the effectiveness and robustness of the proposed supervisory control scheme in the presence of high-order uncertainties.

6 CONCLUSION

This paper proposes several robust stability conditions for switched homogeneous systems with high-order uncertainties. The obtained results extend the existing stability results of switched systems and are applied to the spacecraft attitude control.

To emphasise the main idea and avoid complex mathematical derivations, we only consider: (1) the homogeneity in the usual sense, the proposed methods are also available for the system satisfying the weighted homogeneity; (2) a common Lyapunov function in the cases $k > 0$ and $k < 0$, the related multiple Lyapunov function technique in [31] can also be combined with the proposed method.

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APPENDIX A

\[ K_1 = \begin{bmatrix} -2.7637 & 0 & 0 & -0.5000 & 0 & 0 \\ 0 & -2.7637 & 0 & 0 & -0.5000 & 0 \\ 0 & 0 & -2.7637 & 0 & 0 & -0.5000 \end{bmatrix} \] (A.1)

\[ K_2 = \begin{bmatrix} -11.5328 & 0 & 0 & -0.5000 & 0 & 0 \\ 0 & -11.5328 & 0 & 0 & -0.5000 & 0 \\ 0 & 0 & -11.5328 & 0 & 0 & -0.5000 \end{bmatrix} \] (A.2)

\[ K_3 = \begin{bmatrix} -22.7993 & 0 & 0 & -0.5000 & 0 & 0 \\ 0 & -22.7993 & 0 & 0 & -0.5000 & 0 \\ 0 & 0 & -22.7993 & 0 & 0 & -0.5000 \end{bmatrix} \] (A.3)

\[ K_4 = \begin{bmatrix} -27.7774 & 0 & 0 & -0.5000 & 0 & 0 \\ 0 & -27.7774 & 0 & 0 & -0.5000 & 0 \\ 0 & 0 & -27.7774 & 0 & 0 & -0.5000 \end{bmatrix} \] (A.4)