INPUT-TO-STATE STABILITY AND NO-INPUTS STABILIZATION OF DELAYED FEEDBACK CHAOTIC FINANCIAL SYSTEM INVOLVED IN OPEN AND CLOSED ECONOMY

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ABSTRACT. The stochastic inflow or withdrawal of funds in the international financial market are both positive or negative external inputs to the domestic financial system. So, in this paper, input-to-state stability criterion of delayed feedback chaotic financial system is investigated, and derived by counterevidence method, Lyapunov functional method, variational method and regional control technique, which was involved to equilibrium solution with the positive interest rate. On the other hand, if these inputs are too small to be ignored, impulse control can be applied to stability analysis of the delayed feedback system, in which the delayed impulse allows the pulse effect to lag for a period of time. The obtained stability criteria show that no matter how complex and chaos the financial system is, high-frequency effective macro-control is conducive to the global asymptotical stability of the economic system, including the open economic model with foreign investment fund inputs. Finally, numerical examples illustrate the effectiveness of all the proposed methods.

1. Introduction. It is well known that control systems are often disturbed, such as some changes in control or some errors in observation, which will affect the final results, thus requiring not only stability of the network system, but also input-to-state stability. Especially, when these inputs become a little larger, maybe the original stable system becomes unstable or even chaotic. For this purpose, we will study the input-to-state stabilization of chaotic financial system in this paper. In fact, some emergencies in economic activities, such as the influx of large amounts of foreign funds, the withdrawal of funds from the international financial market due to economic crisis, etc., may cause positive and negative inputs to the domestic financial system. The input-to-state stability (ISS) property implies that no matter
what the size of the initial state is, the state will eventually approach a neighborhood of the origin whose size is proportional to the magnitude of the input. ISS property has attracted wide attention of scholars. In 2012, the authors of [20] proposed a novel method for ISS property of nonlinear systems, presenting a novel comparison theorem for estimating the upper-bound on the state of the system, in which indefinite derivative of Lyapunov function may be feasible while past literature required the derivative of Lyapunov function to be negative definite. This means a big step forward in this area. In [11], Peng Li, Xiaodi Li and Jinde Cao applied the ADT method to switched systems, improving the method proposed in [20]. Inspired by these documents, we shall study the input-to-state stability of the equilibrium solution with the positive interest rate for a delayed feedback chaotic financial system.

On the other hand, input-to-state stability may not be appropriate for the ordinary circumstances in which such external interferences are too small to be ignored. Usually, the inflows and withdrawals of foreign capital are too small to be neglected as compared with the total funds in the domestic market. Therefore, no-inputs stabilization of delayed feedback chaotic financial system is considered in this paper, too. Specifically, we are interested in the following financial systems that is composed of the production sub-block, currency sub-block, securities sub-block, and labor sub-block (see, e.g., [3,5,7,18,19,23,35,36]):

\[
\begin{align*}
\dot{x} &= z+(y-a)x \\
\dot{y} &= 1-by-x^2 \\
\dot{z} &= -x-cz,
\end{align*}
\]

where \(x\) represents the interest rate, \(y\) represents the investment demand, \(z\) represents the price index, \(a\) represents savings, \(b\) represents the unit investment cost, and \(c\) represents the elasticity of commodity demand. It is well known ([18,19,23]) that the financial system (1) has the unique equilibrium point \(Q_0(0, 1/b, 0)\) if \(c - b - abc \leq 0\), and owns three equilibrium point \(Q_0(0, 1/b, 0), Q_1(\sqrt{\theta}, \frac{1+ac}{c}, \frac{-\sqrt{\theta}}{c})\), \(Q_2(-\sqrt{\theta}, \frac{1+ac}{c}, \frac{-\sqrt{\theta}}{c})\) if \(c - b - abc \geq 0\), where \(\theta = \frac{c-b-abc}{c}\). Under some suitable data, the equilibrium point \(Q_0(0, 1/b, 0)\) may be stable. However, both \(Q_1\) and \(Q_2\) must be unstable equilibrium points. Remark that both the interest rates of \(Q_0\) and \(Q_2\) are non-positive. Moreover, according to the principles of Macroeconomics, the interest rate is usually a positive decimal whenever the product market and the money market are in common equilibrium ([23]). So, in this paper, only the stability of the equilibrium point \(Q_1\) will be studied. On this account, the financial system (1) is translated into the following system:

\[
\begin{align*}
\dot{X}_1 &= -X_1 + \theta X_2 + X_3 + X_1 X_2 \\
\dot{X}_2 &= -2\theta X_1 - bX_2 - X_1^2 \\
\dot{X}_3 &= -X_1 - cX_3,
\end{align*}
\]

where the equilibrium point \(Q_1(\sqrt{\theta}, \frac{1+ac}{c}, \frac{-\sqrt{\theta}}{c})\) of the system (1) becomes the null solution of the system (2). Furthermore, one can rewrite the system (2) in matrix-vector form:

\[
\begin{align*}
\dot{X} &= -AX + f(X), \quad t \geq t_0 = 0, \\
X(t_0) &= X_0,
\end{align*}
\]
where \( X^T = (X_1, X_2, X_3), X_0 = (X_{01}, X_{02}, X_{03})^T \in \mathbb{R}^3 \), and:

\[
A = \begin{pmatrix}
-\frac{1}{2} & -\theta & -1 \\
2\theta & b & 0 \\
1 & 0 & c
\end{pmatrix}, 
\quad f(X) = \begin{pmatrix}
X_1X_2 \\
-X_1^2 \\
0
\end{pmatrix}.
\] (4)

In [23], the author employed impulse control to research the globally asymptotic stability of the equilibrium point \( Q_1(\sqrt{\theta}, \frac{1+\psi c}{c}, -\frac{\sqrt{\theta}}{c}) \) of the following financial system in [23, Theorem 2]:

\[
\begin{align*}
\dot{X} &= -AX + f(X), \quad t \geq t_0 = 0, t \neq t_k, k \in \mathbb{Z}^+ \\
X(t_k^+) &= B_kX(t_k^-), \quad k \in \mathbb{Z}^+ = \{1, 2, \cdots \} \\
X(t_0) &= X_0.
\end{align*}
\] (5)

Since there is a time lag between making economic decisions and the effectiveness of decisions ([3,15,33]), the following delayed feedback financial system was introduced:

\[
\begin{align*}
\dot{x} &= z + (y - a)x + k_1(x - x(t - \tau_1)) \\
\dot{y} &= 1 - by - x^2 + k_2(y - y(t - \tau_2)) \\
\dot{z} &= -x - cz + k_3(z - z(t - \tau_3)),
\end{align*}
\] (6)

where \( k_i (i = 1, 2, 3) \) is the feedback coefficient.

In [1], A-Ying A Zi, Ruofeng Rao, Feng Zhao and Hongyan Huang investigated the equilibrium point \( Q_0(\frac{1}{2}, 0, 0) \) for the following impulsive financial system with probabilistic delays:

\[
\begin{align*}
\dot{X} &= AX + f(X) + c_0K(X - X(t - \tau_1(t))) + (1 - c_0)K(X - X(t - \tau_2(t))) \\
&\quad + (c - c_0)K(X(t - \tau_2(t)) - X(t - \tau_1(t))), \quad t \geq 0, t \neq t_k, \\
X(t_k^+) &= B_kX(t_k^-) + N\xi(X(t_k^- - \tau(t_k))), \quad t = t_k, k = 1, 2, \cdots \\
X(s) &= \xi(s), \quad s \in [-\tau, 0].
\end{align*}
\] (7)

And the globally exponential stability criterion was derived. However, the real economic significance is not great, for the interest rate is zero when the system reaches stable. In fact, the principles of macroeconomics tells us that the interest rate is usually a positive decimal whenever the product market and the money market are in common equilibrium ([23]). Below, we only discuss the equilibrium point \( Q_1 \) with the positive interest rate \( \sqrt{\theta} > 0 \). How to stabilize the unstable equilibrium point \( Q_1 \)? This is the main purpose of this paper. But the stabilization of \( Q_1 \) is much more difficult than that of \( Q_0 \). In fact, The financial system is chaotic as its dynamic approaches the equilibrium point \( Q_1 \) (see e.g., [3,5,7,18,19,23,35,36]).

As pointed out in [23], time delays produce the essential difficulty in deriving the stability of \( Q_1 \) under impulse control. In [23, Theorem 1], the author employed simultaneously regional control and impulse control to derive the globally exponential
stability of the equilibrium point $Q_1$ of the following system:

$$ \begin{align*}
\frac{\partial Y_1}{\partial t} &= d_1 \Delta Y_1 + \frac{1}{c}(Y_1 - \sqrt{\theta}) + \theta(Y_2 - \frac{1 + ac}{c}) + (Y_3 - \frac{-\sqrt{\theta}}{c}) + (Y_1 - \sqrt{\theta})(Y_2 - \frac{1 + ac}{c}) + k_1(r(t))(Y_1 - Y_1(t - \tau(t), x)), \quad t \geq 0, \ t \neq t_k, \\
\frac{\partial Y_2}{\partial t} &= d_2 \Delta Y_2 - 2\theta(Y_1 - \sqrt{\theta}) - b(Y_2 - \frac{1 + ac}{c}) - (Y_1 - \sqrt{\theta})^2 + k_2(r(t))(Y_2 - Y_2(t - \tau(t), x)), \quad t \geq 0, \ t \neq t_k, \\
\frac{\partial Y_3}{\partial t} &= d_3 \Delta Y_3 - (Y_1 - \sqrt{\theta}) - c(Y_3 - \frac{-\sqrt{\theta}}{c}) + k_3(r(t))(Y_3 - Y_3(t - \tau(t), x)), \quad t \geq 0, \ t \neq t_k, \\
Y(t^+_k, x) &= B_k[Y(t^-_k, x) - Q_1], \quad t = t_k, \ k = 1, 2, \ldots \\
Y(s, x) &= \xi(s, x) + Q_1, \quad (s, x) \in [-\tau, 0] \times \Omega, \\
\frac{\partial Y_i}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t \geq 0, \ i = 1, 2, 3. 
\end{align*} $$

In addition, the equilibrium point $Q_1$ is corresponding to the null solution of the following system

$$ \begin{align*}
\frac{\partial Y}{\partial t} &= D \Delta Y - AY(t, x) + f(Y(t, x)) + K(r(t))(Y - Y(t - \tau(t), x)), \\
\quad t \geq t_0 = 0, \ t \neq t_k, \ k \in \mathbb{Z}^+, \\
Y(t^+_k, x) &= B_k[Y(t^-_k, x), \quad t = t_k, \ k \in \mathbb{Z}^+ = \{1, 2, \ldots \} \\
Y(s, x) &= \xi(s, x), \quad (s, x) \in [-\tau, 0] \times \Omega, \\
\frac{\partial Y_i}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t \geq 0, \ i = 1, 2, 3. 
\end{align*} $$

**Remark 1.** Existing stability criteria of the equilibrium point $Q_1(\sqrt{\theta}, \frac{1 + ac}{c}, -\frac{\sqrt{\theta}}{c})$ derived solely by impulse control in some literature are probably erroneous. For example, such literature are always involved in citing the erroneous conclusions of [8]. In fact, Yinping Zhang and Qing-Guo Wang in [34] pointed out the errors of [8, Lemma 3] and [8, Theorem 1]. In the last chapter of [23], the author questioned whether the delayed feedback chaotic financial system could be stabilized by impulse control alone, and the interest rate is positive percentage when the system reaches stable. Now, in this paper, we will give a positive answer to this problem (see Corollary 2). Besides, we admits the delayed impulse (see Theorem 4.1-4.2), for the effect of each macro-control (impulse) by the government often lags behind for a period of time in the real economic market.

As pointed out in [23], the delayed feedback coefficient has the Markov property, we may consider the Markovian jumping model for the delayed financial system. Motivated by some ideas and methods in related literature ([3,5,7,9-19,21-25,27-33,35-37]), we may consider employing regional control to derive the input-to-state stability criterion for the delayed feedback financial system with Markovian jumping. On the other hand, we consider using solely the impulse control to stabilize the delayed feedback financial system. It is worth mentioning that the global asymptotical stability of the equilibrium point with positive interest rate eliminates actually
chaos of the chaotic financial systems, for chaos exists near the equilibrium point with positive interest rate ([2,4,6]).

2. Preparation. In [23, Theorem 1], the author investigated regional control on the feedback Markovian jumping impulsive financial system model involving time-delays partial differential equations. Below, we consider the following system with external inputs:

\[
\begin{cases}
\frac{\partial Y(t, x)}{\partial t} = B\Delta Y(t, x) - AY(t, x) + f(Y(t, x)) + K(r(t))Y(t, x) \\
- Y(t - \tau(t), x) + u(t, x), \quad t \geq 0,
\end{cases}
\]

\[\frac{\partial Y}{\partial \nu} = 0, \quad x \in \partial \Omega,
\]

\[Y(s, x) = \phi(s, x), \quad (s, x) \in [-\tau, 0] \times \Omega,
\]

where \((\Omega, \mathcal{F}, \mathbb{P})\) is the complete probability space with a natural filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). Let \(S = \{1, 2, \cdots, n_0\}\) and the random form process \(\{r(t) : [0, +\infty) \rightarrow S\}\) be a homogeneous, finite-state Markovian process with right continuous trajectories \(r(t)\), and diffusion coefficient matrix \(\gamma\) and transition probability from mode \(i\) at time \(t\) to mode \(j\) at time \(t + \delta\), \(i, j \in S\),

\[\mathbb{P}(r(t + \delta) = j \mid r(t) = i) = \begin{cases} 
\gamma_{ij}\delta + o(\delta), & j \neq i \\
1 + \gamma_{ij}\delta + o(\delta), & j = i,
\end{cases}
\]

where \(\gamma_{ij} \geq 0\) is transition probability rate from \(i\) to \(j\) \((j \neq i)\) and \(\gamma_{ii} = -\sum_{j=1, j \neq i}^{n_0} \gamma_{ij}\), \(\delta > 0\) and \(\lim_{\delta \rightarrow 0} o(\delta)/\delta = 0\). \(x = (x_1, x_2)^T \in \Omega\) with \(\Omega\) being a bounded subset of \(R^2\). \(u\) represents the external input with \(u = (u_1, u_2, u_3)^T\).

\(Y = (Y_1, Y_2, Y_3)^T\), and the economic significance \(Y_i(i = 1, 2, 3)\) represents same as that of (8) (see [23]). \(Y = 0\) is corresponding to the equilibrium point \(Q_1\) of financial system (1), \(Y(t - \tau(t), x) = (Y_1(t - \tau_1(t), x), Y_2(t - \tau_2(t), x), Y_3(t - \tau_3(t), x))^T\), and diffusion coefficient matrix \(B\) is positive definite diagonal matrix,

\[A = \begin{pmatrix}
-\frac{1}{\tau} & -\theta & -1 \\
2\theta & b & 0 \\
1 & 0 & c
\end{pmatrix}, \quad f(Y) = \begin{pmatrix}
Y_1Y_2 \\
-\frac{1}{2}Y_1^2 \\
0
\end{pmatrix},
\]

\[K(r(t)) = \begin{pmatrix}
k_1(r(t)) & 0 & 0 \\
0 & k_2(r(t)) & 0 \\
0 & 0 & k_3(r(t))
\end{pmatrix}.
\]

On the other hand, impulse control is the important means of Government Macro-control. But the effect of government macro-control is often shown by lagging time. So we consider the following financial system under delayed impulse:

\[
\begin{cases}
\dot{X} = -AX + f(X) + K(X - X(t - \tau(t))), \quad t \geq 0, \quad t \neq t_k, \\
X(t_k^+) - X(t_k^-) = D_k X(t_k - \rho_k), \quad k \in \mathbb{Z}^+, \quad \mathbb{Z}^+ = \{1, 2, \cdots, n, \cdots\}, \\
X(s) = \xi(s), \quad s \in [-\rho \lor \tau, 0],
\end{cases}
\]

where each \(t_k(k \in \mathbb{Z}^+)\) represents the pulse triggering time, satisfying

\[0 < t_1 < t_2 < \cdots < t_k < \cdots \quad \text{with} \quad \lim_{k \rightarrow \infty} t_k = \infty.
\]

\[
X(t_k^+) = \lim_{\delta \rightarrow 0^+} X(t_k + \delta), \quad X(t_k^-) = \lim_{\delta \rightarrow 0^+} X(t_k - \delta).
\]

Throughout this paper, we assume \(X(t_k) = X(t_k^-)\) for all \(k \in \mathbb{Z}^+\).
It is obvious that the equilibrium point $Q_1$ of the following impulsive financial system
\[
\begin{cases}
\dot{x} = z + (y - a)x + k_1(x - x(t - \tau_1(t))), & t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\dot{y} = 1 - by - x^2 + k_2(y - y(t - \tau_2(t))), & t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\dot{z} = -x - cz + k_3(z - z(t - \tau_3(t))), & t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\},
\end{cases}
\]
\[
\begin{pmatrix}
  x(t_k^+) - \sqrt{\theta}, y(t_k^+) - \frac{1 + ac}{c}, z(t_k^+) + \frac{\sqrt{\theta}}{c} \\
  - \left(x(t_k^-) - \sqrt{\theta}, y(t_k^-) - \frac{1 + ac}{c}, z(t_k^-) + \frac{\sqrt{\theta}}{c}\right)
\end{pmatrix}^T = D_k \begin{pmatrix}
  x(t_k - \rho_k) - \sqrt{\theta}, y(t_k - \rho_k) - \frac{1 + ac}{c}, z(t_k - \rho_k) + \frac{\sqrt{\theta}}{c} \\
  x(s) - \sqrt{\theta}, y(s) - \frac{1 + ac}{c}, z(s) + \frac{\sqrt{\theta}}{c}
\end{pmatrix}^T = \xi(s), \quad s \in [-\rho \lor \tau, 0].
\]
(13)

In addition, we shall investigate Markovian jumping system :
\[
\begin{cases}
\dot{X} = -AX + f(X) + K(r(t))(X - X(t - \tau(t))) + u(t, x), & t \geq 0, t \neq t_k, \\
X(t_k^+) - X(t_k^-) = D_k X(t_k - \rho_k), \\
X(s) = \xi(s), & s \in [-\rho \lor \tau, 0].
\end{cases}
\]
(14)

Similarly, the null solution of the system (15) is corresponding to the equilibrium point $Q_1$ of the following financial system:
\[
\begin{cases}
\dot{x} = z + (y - a)x + k_1(r(t))(x - x(t - \tau_1(t))) + u_1(t, x), & t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\dot{y} = 1 - by - x^2 + k_2(r(t))(y - y(t - \tau_2(t))) + u_2(t, x), & t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\dot{z} = -x - cz + k_3(r(t))(z - z(t - \tau_3(t))) + u_3(t, x), & t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\end{cases}
\]
\[
\begin{pmatrix}
  x(t_k^+) - \sqrt{\theta}, y(t_k^+) - \frac{1 + ac}{c}, z(t_k^+) + \frac{\sqrt{\theta}}{c} \\
  - \left(x(t_k^-) - \sqrt{\theta}, y(t_k^-) - \frac{1 + ac}{c}, z(t_k^-) + \frac{\sqrt{\theta}}{c}\right)
\end{pmatrix}^T = D_k \begin{pmatrix}
  x(t_k - \rho_k) - \sqrt{\theta}, y(t_k - \rho_k) - \frac{1 + ac}{c}, z(t_k - \rho_k) + \frac{\sqrt{\theta}}{c} \\
  x(s) - \sqrt{\theta}, y(s) - \frac{1 + ac}{c}, z(s) + \frac{\sqrt{\theta}}{c}
\end{pmatrix}^T = \xi(s), \quad s \in [-\rho \lor \tau, 0].
\]
(15)

**Definition 2.1.** The system (10) is input-to-state stable in the mean square if for any initial function $\phi$ and the external input $u$, there are two functions $\varphi \in K$ and $\psi \in KL$ such that
\[
E\|Y(t, \phi)\|^2 \leq \psi(t, E\|\phi\|^2) + \varphi(\|u\|^2_\infty).
\]
3. Input-to-state stability. Before presenting the main result of this paper, we may firstly denote

\[ \tau_i(t) \in [0, \tau], \quad i = 1, 2, 3; \quad \|Y(t)\| = \sqrt{\int_{\Omega} Y^T(t, x)Y(t, x)dx}, \]

\[ \|w(t)\|_{\ast} = \sup_{s \in [-\tau, 0]} \|w(t + s)\|. \]

**Theorem 3.1.** If there is a sequence of positive definite diagonal matrices \( P_r = \text{diag}(p_{r1}, p_{r2}, p_{r3})(r \in S) \) with \( p_{r1} = p_{r2} \), positive scalars \( c_i > 0 (i = 1, 2, 3, 4) \) and \( M > 0 \) such that for any mode \( r \in S \),

\[ \max_{r \in S} \lambda_{\max} \left( -2\lambda_1 P_r B - A^T P_r - P_r A + 2P_r K_r + c_1 K_r^2 P_r + c_2 P_r^2 + \sum_{j \in S} \gamma_{rj} P_j \right) \]

\[ \geq \frac{c_1^{-1}}{\min_{r \in S} \lambda_{\min} P_r} e^{c_3 \tau} + c_3 \leq 0, \quad t \geq 0, \quad (16) \]

\[ \max_{r \in S} \lambda_{\max} \left( -2\lambda_1 P_r B - A^T P_r - P_r A + 2P_r K_r + c_1 K_r^2 P_r + c_2 P_r^2 + \sum_{j \in S} \gamma_{rj} P_j \right) \]

\[ \geq \frac{c_1^{-1}}{\min_{r \in S} \lambda_{\min} P_r} + c_2^{-1} \leq 0, \quad t \geq 0, \quad (17) \]

\[ \|\phi\|^2 \leq M \|\phi\|^2 e^{-c_3 s} + c_4 \|u\|^2_{\infty}, \quad (s, x) \in [-\tau, 0] \times \Omega, \quad (18) \]

then the system (10) is input-to-state stable in the mean square, where \( \|\phi\|_{\ast} = \sup_{s \in [-\tau, 0]} \|\phi(s)\| \), and \( \lambda_1 \) is the lowest positive eigenvalue of the Neumann boundary problem

\[ \begin{cases} -\Delta \zeta(x) = \lambda \zeta(x), & x \in \Omega, \\ \frac{\partial \zeta(x)}{\partial \gamma} = 0, & x \in \partial \Omega. \end{cases} \]

**Proof.** Let \( Y \) be a solution of the system (10). With the help of the Gauss formula, the zero boundary condition and the orthogonal decomposition of Sobolev space \( H^1(\Omega) \), the authors of [22] derived the following Poincare inequality:

\[ \int_{\Omega} Y^T P_r B \Delta Y dx \leq -\lambda_1 \int_{\Omega} Y^T P_r B Y dx, \]

where \( Y_i(t, x) \in H^1_0(\Omega) \) for all \( i \in \{1, 2, 3\} \).

**Remark 2.** It is well known that \( \lambda_1 \) is only dependent on \( \Omega \). For example, \( \lambda_1 = \frac{\pi}{d_1}^2 \) if \( \Omega = [0, d] \), and \( \lambda_1 = \min\{\frac{\pi}{d_1}^2, \frac{\pi}{d_2}^2\} \) when \( \Omega = [0, d_1] \times [0, d_2] \).

Consider the Lyapunov functional:

\[ V(t, Y(t), r(t)) = V_r(t) = \int_{\Omega} Y^T(t, x)P_r Y(t, x)dx. \]
Let $\mathcal{L}$ be the weak infinitesimal operator (see, e.g., [16,17,24,27,31]) such that

$$
\mathcal{L}V_t = \int_\Omega Y^T \left( -2\lambda_1 P_r B - A^T P_r - P_r A + 2P_r K_r + \sum_{j \in S} \gamma_{rj} P_j \right) Y \, dx
$$

$$
- \int_\Omega \left( Y^T P_r K_r Y(t - \tau(t), x) + Y^T (t - \tau(t), x) K_r P_r \right) \, dx
$$

$$
+ \int_\Omega (Y^T P_r u + u^T P_r Y) \, dx
$$

$$
\leq \int_\Omega Y^T \left( -2\lambda_1 P_r B - A^T P_r - P_r A + 2P_r K_r + c_1 K_r^2 P_r^2 + c_2 P_r^2 + \sum_{j \in S} \gamma_{rj} P_j \right) Y \, dx
$$

$$
+ c_1^{-1} \|Y(t - \tau(t), x)\|^2 + c_2^{-1} \|u\|_\infty^2
$$

Let $E$ be expectation operator, and $W(t) = V(t, Y(t), r(t))$. Then for any given $t \geq 0$ and $\Delta t > 0$,

$$
EW(t + \Delta t) = EW(t) + \int_t^{t+\Delta} E\mathcal{L}V(s, Y(s), r(s)) \, ds,
$$

which deduces

$$
D^+ EW(t) = \limsup_{\Delta t \to 0^+} \frac{EW(t + \Delta t) - EW(t)}{\Delta t} = E\mathcal{L}V(t, Y(t), r(t)).
$$

Define

$$
U_r(t) = V_r(t) - c_4 \|u\|_\infty^2.
$$

It follows by (18) and the initial condition that

$$
\int_\Omega \phi(t, x)^T P_r \phi(t, x) \, dx - c_4 \|u\|_\infty^2 = U_r(t) \leq M \|\phi\|_2^2 e^{-c_4 t}, t \in [-\tau, 0]. \tag{19}
$$

For any given $\varepsilon > 0$, letting $Z_r(t) = M \|\phi\|_2^2 e^{-c_4 t} e^{(-c_3 + \varepsilon)t}$, we claim that

$$
EU_r(t) \leq ME \|\phi\|_2^2 e^{-C_3 t} e^{(-c_3 + \varepsilon)t}, \quad t \geq 0. \tag{20}
$$

Indeed, (20) holds in the case of $t = 0$ because

$$
EU_r(0) \leq ME \|\phi\|_2^2 e^{(-c_3 \times 0)} = ME \|\phi\|_2^2 < ME \|\phi\|_2^2 e^{c_3 t} = EZ_r(0).
$$

Below, we consider employing the Counterevidence method similarly as that of [25, Theorem 1]. If (20) is not true, the continuity of $V_r(t)$ yields that there must be $\hat{t}_r > 0$, satisfying

$$
EU_r(t) \leq EZ_r(t), t \in [0, \hat{t}_r]; \quad EU_r(\hat{t}_r) = EZ_r(\hat{t}_r), \quad EU_r(t) \geq EZ_r(t), t \in (\hat{t}_r, \hat{t}_r + \delta),
$$

where $\delta > 0$ is small enough. And hence

$$
EU_r(\hat{t}_r) = Z_r(\hat{t}_r), \quad D^+ EU_r(\hat{t}_r) \geq D^+ Z_r(\hat{t}_r). \tag{21}
$$

and

$$
EU_r(t) \leq EZ_r(t), \quad t \in [-\tau, \hat{t}_r]. \tag{22}
$$

Denote for convenience

$$
a = \max_{r \in S} \lambda_{\max} \left( -2\lambda_1 P_r B - A^T P_r - P_r A + 2P_r K_r + c_1 K_r^2 P_r^2 + c_2 P_r^2 + \sum_{j \in S} \gamma_{rj} P_j \right). \quad \max_{r \in S} \lambda_{\max} P_r
$$

$$
\max_{r \in S} \lambda_{\max} P_r
$$
and
\[ b = \frac{c_1^{-1}}{\min_{r \in S} \lambda_{\min} P_r}. \]

Next, we assume that \( \varepsilon \) is the positive scalar small enough.

Employing (16)-(18) and (21)-(22) results in
\[ D^+ EU_r(\hat{t}_r) = D^+ EV_r(\hat{t}_r) \]
\[ \max_{r \in S} \lambda_{\max} \left( -2\lambda_1 P_r B - A^T P_r - P_r A + 2P_r K_r + c_1 K_r^2 P_r + c_2 P_r^2 + \sum_{j \in S} \gamma_{rj} P_j \right) \]
\[ \leq \frac{\max_{r \in S} \lambda_{\max} P_r}{\min_{r \in S} \lambda_{\min} P_r} \]
\[ \times EV_r(\hat{t}_r) + c_1^{-1} \min_{r \in S} \left[ EV_r(\hat{t}_r) \right] + c_2^{-1} \| u \|_\infty^2 \]
\[ \leq aME \| \phi \|_2^2 e^{\varepsilon \tau} e^{(-c_3+\varepsilon)\hat{t}_r} + bME \| \phi \|_2^2 e^{\varepsilon \tau} e^{(-c_3+\varepsilon)(\hat{t}_r-\tau)} \]
\[ \leq ME \| \phi \|_2^2 e^{\varepsilon \tau} e^{(-c_3+\varepsilon)\hat{t}_r} (-c_3) \]
\[ \leq ME \| \phi \|_2^2 e^{\varepsilon \tau} e^{(-c_3+\varepsilon)\hat{t}_r} (-c_3 + \varepsilon) = D^+ EZ_r(\hat{t}_r), \]
which is contradictory with (21). Thus (20) holds. Moreover, letting \( \varepsilon \to 0^+ \) in (20), we have
\[ \min_{r \in S} P_r E\|Y\|^2 - c_4 \| u \|_\infty^2 \leq EV_r(t) - c_4 \| u \|_\infty^2 = EU_r(t) \leq ME \| \phi \|_2 e^{-c_3 t}, \quad t \geq 0, \]
which means
\[ E\|Y\|^2 \leq \frac{M}{\min_{r \in S} P_r} E\|\phi\|^2 e^{-c_3 t} + \frac{c_4}{\min_{r \in S} P_r} \| u \|^2, \quad t \geq 0. \]

Therefore, the proof is completed. \( \Box \)

4. **No inputs stabilization by impulse control.** Before giving the main results of this section, we need to present some assumptions and notations.

For simplicity, in this section, we denote the norm \( \| \cdot \| \) as follows,
\[ \| Z \|^2 = Z^T Z, \quad \text{where} \quad Z \in \mathbb{R}^3. \]

Suppose that time delays \( \tau_i(t) \in [0, \tau], \ i = 1, 2, 3. \) Assume that there are two positive scalars \( M_1, M_2 \) such that
\[ 0 < M_1 \leq \| X(s) \|^2 \leq M_2, \quad \forall s \in [-\tau, +\infty). \]

Furthermore, the boundedness assumptions are proposed as follows:

There exists positive number \( c_r \) such that
\[ \left| \| X(t) \|^2 - \| X(t - \tau(t)) \|^2 \right| < c_r. \]

In this section, we assume that time delays \( \tau(t) \in [-\tau, 0], \) and \( \rho_k \in [-\rho, 0] \) for all \( k \in \mathbb{Z}^+. \) In order to obtain the stability of the system, a certain pulse frequency is required, so we assume a smaller pulse interval as follows,
\[ \sup_{k \in \mathbb{Z}^+} (t_k - t_{k-1}) < c_0, \]
where \( c_0 \) is a positive number.
Consider the following Lyapunov function
\[ V(t) = X^T(t)X(t) = \|X(t)\|^2 \]
\[ \left\| X(t) \right\|^2 - \| X(t - \tau(t)) \|^2 < c_r < c_r \frac{\|X(t)\|^2}{M_1}, \]}
which means
\[ \| X(t - \tau(t)) \|^2 \leq \left( 1 + \frac{c_r}{M_1} \right) \| X(t) \|^2, \quad \forall t \geq 0. \]

Then we get
\[ \| X(t) \|^2 \leq \| X(t_{k-1}^+) \|^2 e^{c_\lambda (t - t_{k-1})}, \quad t \in (t_{k-1}, t_k). \]

Using differential mean value theorem, (23) and (31) results in that there exists \( \eta_k(i = 1, 2, 3) \) with \( \eta_k \in (tk - \rho_k, t_k) \subset (t_{k-1}, t_k) \) such that
\[ \| X(t_k^+) \| = \| (D_k + I)X(t_k) + D_k[X(t_k - \rho_k) - X(t_k)] \| \]
\[ \leq \left( \| D_k + I \| + \rho_k \| D_k \| \right) \left( \left\| A \right\| + \sqrt{M_2} + \| K \| \frac{2\sqrt{M_2}}{\sqrt{M_1}} \right) e^{c_\lambda_{k_0} \| X(t_{k-1}^+) \|}. \]

where \( X(\eta_k) = (X_1(\eta_k), X_2(\eta_k), X_3(\eta_k)) \). 

Combining (26) and (32) means that \( \{ \| X(t_k^+) \| \}_{k=1}^\infty \) is a convergent sequence with its limit being zero. For any given \( t \in (t_k, t_{k+1}) \), one can deduce by (31) that
\[ 0 \leq \| X(t) \| \leq \| X(t_{k-1}^+) \| e^{c_\lambda_{k_0} (t - t_{k-1})} \leq e^{c_\lambda_{k_0} \| X(t_{k-1}^+) \|} \to 0, \quad k \to \infty, \]
which completes the proof.

\[ \square \]
Remark 3. The condition $\rho_k < t_k - t_{k-1}$ implies that every macro-control measure (pulse) of the government should be effective enough to see the pulse effect within each pulse interval. Besides, the condition $\sup_{k \in \mathbb{Z}^+} (t_k - t_{k-1}) < c_0$ guarantees pulse (Macro-control) of a certain frequency if $c_0 > 0$ is appropriate small. No matter how complex and chaos the financial system is, high-frequency active macro-control is conducive to the global asymptotical stability of the economic system.

Remark 4. In the real economic market, the interest rate $x$, the investment demand $y$ and the price index $z$ are actually bounded. Since such economic indicators can be considered to be continuous on each time interval $(t_{k-1}, t_k]$ for $k \in \mathbb{Z}^+$. And the impulse are smaller due to impulse control. So the bounded conditions (23)-(25) are usually suitable to the economic indicators.

Theorem 4.2. Suppose that the conditions (23)-(25) hold. And there is a positive scalar $\varepsilon$ and a sequences of positive definite diagonal matrices $P_r = \text{diag}(p_{r1}, p_{r2}, p_{r3})$ $(r \in S)$ with $p_{r1} = p_{r2}$ such that

$$
\|D_k + I\| \leq d_0 < 1, \quad \forall k \in \mathbb{Z}^+, \quad r \in S,
$$

where $d_0$ is a positive scalar,

$$
\bar{c}_\lambda = \frac{\max_{r \in S} \lambda_{\max} \left[ -P_r A - A^T P_r + 2KP_r + \varepsilon K^2 P_r^2 + \varepsilon^{-1} \left( 1 + \frac{\varepsilon}{M_1} \right) I + \sum_{j \in S} \gamma_{rj} P_j \right]}{\min_{r \in S} \lambda_{\min} P_r},
$$

and $I$ represents the identity matrix, $\rho_k < t_k - t_{k-1}$, for all $k \in \mathbb{Z}^+$, then the following two conclusions hold:

(a) the null solution of the system (14) is stochastically globally asymptotically stable;

(b) the equilibrium solution $Q_1$ with the positive interest rate $\sqrt{\theta}$ for the financial system (15) is stochastically globally asymptotically stable.

Proof. Consider the Lyapunov function as follows,

$$
V(t, r) = X^T(t)P_rX(t).
$$

By employing the similar methods used in the proof of Theorem 4.1, we are not difficult to complete the proof of Theorem 4.2.

Remark 5. The effect of the government’s macro-control (impulse control) over the economic market often lags behind for some time. So we consider the time-delays on the impulse control. The systems (12)-(15) are based on such considerations. Particularly, Theorem 4.1 and Theorem 4.2 are also suitable to the case that the time-delays on the impulse control are ignored (see below Corollaries).

Corollary 1. Suppose that the conditions (23)-(25) hold. And if there is a positive scalar $\varepsilon$ and a sequences of positive definite diagonal matrices $P_r = \text{diag}(p_{r1}, p_{r2}, p_{r3})$ $(r \in S)$ with $p_{r1} = p_{r2}$ such that

$$
\|D_k + I\| \leq d_0 < 1, \quad \forall k \in \mathbb{Z}^+,
$$
where $d_0$ is a positive scalar,

$$
\bar{c}_\lambda = \frac{\max_{r \in S} \lambda_{\max} \left[- P_r A - A^T P_r + 2 K P_r + \varepsilon K^2 P_r + \varepsilon^{-1} \left(1 + \frac{\varepsilon_r}{M_1}\right) I + \sum_{j \in S} \gamma_{r_j} P_j \right]}{\min_{r \in S} \lambda_{\min} P_r},
$$

and $I$ represents the identity matrix, then the following two conclusions hold:

(a) the null solution of the following system is stochastically globally asymptotically stable:

$$
\begin{aligned}
\dot{X} &= -AX + f(X) + K(r(t)) (X - X(t - \tau(t))), \quad t \geq 0, t \neq t_k, \\
X(t_k^+) - X(t_k^-) &= D_k X(t_k), \\
X(s) &= \xi(s), \quad s \in [-\tau, 0].
\end{aligned}
$$

(b) the equilibrium solution $Q_1$ with the positive interest rate $\sqrt{\theta}$ for the following system is stochastically globally asymptotically stable,

$$
\begin{aligned}
\dot{x} &= z + (y - a)x + k_1(r(t))(x - x(t - \tau_1(t))), \quad t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\dot{y} &= 1 - by - x^2 + k_2(r(t))(y - y(t - \tau_2(t))), \quad t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\dot{z} &= -cx - cz + k_3(r(t))(z - z(t - \tau_3(t))), \quad t \geq 0, t \notin \{t_k, k \in \mathbb{Z}^+\}, \\
\begin{bmatrix}
x(t_k^+) - \sqrt{\theta}, y(t_k^+) - \frac{1 + ac}{c}, z(t_k^+) + \frac{\sqrt{\theta}}{c}
\end{bmatrix}^T
\end{aligned}
$$

$$
\begin{aligned}
= & D_k \begin{bmatrix}
x(t_k) - \sqrt{\theta}, y(t_k) - \frac{1 + ac}{c}, z(t_k) + \frac{\sqrt{\theta}}{c}
\end{bmatrix}^T, \quad k \in \mathbb{Z}^+, \\
\begin{bmatrix}
x(s) - \sqrt{\theta}, y(s) - \frac{1 + ac}{c}, z(s) + \frac{\sqrt{\theta}}{c}
\end{bmatrix}^T
\end{aligned}
$$

$\xi(s), \quad s \in [-\tau, 0].$

**Corollary 2.** Suppose that the conditions (23)-(25) hold. And if there is a positive scalar $\varepsilon$ such that

$$
\| D_k + I \| e^{\frac{c_{\lambda_{\min}}}{2}} \leq d_0 < 1, \quad \forall k \in \mathbb{Z}^+, \quad (33)
$$

where $d_0$ is a positive scalar,

$$
c_\lambda = \lambda_{\max} \left[- A - A^T + 2 K + \varepsilon K^2 + \varepsilon^{-1} \left(1 + \frac{c_r}{M_1}\right) I \right],
$$

and $I$ represents the identity matrix, then the following two conclusions hold:

(a) the null solution of the following system is globally asymptotically stable:

$$
\begin{aligned}
\dot{X} &= -AX + f(X) + K(X - X(t - \tau(t))), \quad t \geq 0, t \neq t_k, \\
X(t_k^+) - X(t_k^-) &= D_k X(t_k), \\
X(s) &= \xi(s), \quad s \in [-\tau, 0].
\end{aligned}
$$
Proof. Consider the following Lyapunov function

\[
V(t) = X^T(t)X(t) = \|X(t)\|^2
\]

This means

\[
\|X(t)\|^2 - \|X(t - \tau(t))\|^2 < c_\tau < c_\tau \frac{\|X(t)\|^2}{M_1},
\]

which requires

\[
\|X(t - \tau(t))\|^2 \leq \left(1 + \frac{c_\tau}{M_1}\right)\|X(t)\|^2, \quad \forall t \geq 0.
\]

Similarly,

\[
D^+ V(t, X) \leq X^T (-A - A^T + 2K)X - (X^T KX(t - \tau(t)) + X(t - \tau(t))^T KX^T) \leq \lambda_{\text{max}} \left[-A - A^T + 2K + \varepsilon K^2 + \varepsilon^{-1} \left(1 + \frac{c_\tau}{M_1}\right) I\right] V(t, X), t \in (t_k - 1, t_k), k \in \mathbb{Z}^+.
\]
Then we get
\[ \|X(t)\|^2 \leq \|X(t_{k-1}^+)^+\|^2 e^{c_\lambda(t-t_{k-1})}, \quad t \in (t_{k-1}, t_k). \]

Besides,
\[ t_k - t_{k-1} \leq \rho_k < t_k - t_{k-2} \]
\[ \Rightarrow t_{k-2} \leq t - \rho_k \leq t_{k-1} \Rightarrow (t - \rho_k) \leq t_{k-1} - t_{k-2} \leq \epsilon_0 \]
and
\[ \rho_k < t_k - t_{k-2} \leq 2\epsilon_0 \]
Using differential mean value theorem yields that there exists \( \gamma_{ki}, \beta_{ki}(i = 1, 2, 3) \) with \( \gamma_{ki} \in (t_{k-2}, t_{k-1}) \) and \( \beta_{ki} \in (t_{k-1}, t_k) \) such that
\[ \|X(t_k^+)\| = \|(D_k + I)X(t_k)\| + \|D_k\| \left[ \|X(\gamma_{ki})\| c_0 + \|X(\beta_{ki})\| c_0 + \|D_{k-1}\| \|X(t_{k-2}^+)\| e^c \right] \]
\[ \leq \|(D_k + I)X(t_k)\| + \|D_k\| \left[ 2c_0 \left( \|A\| + \sqrt{M_2} + \|K\| \frac{2\sqrt{M_2}}{\sqrt{M_1}} \right) + \|D_{k-1}\| \right] \times \|X(t_{k-2}^+)\| e^c \]
\[ = \|(D_k + I) \cdot X(t_{k-1} + D_{k-1}X(t_{k-1} - \rho_{k-1}))\| e^c \]
\[ + \|D_k\| \left[ 2c_0 \left( \|A\| + \sqrt{M_2} + \|K\| \frac{2\sqrt{M_2}}{\sqrt{M_1}} \right) + \|D_{k-1}\| \right] \times \|X(t_{k-2}^+)\| e^c \]
\[ \leq \left\{ \|(D_k + I) \cdot (1 + \|D_{k-1}\|)e^c \|D_k\| \left[ 2c_0 \left( \|A\| + \sqrt{M_2} + \|K\| \frac{2\sqrt{M_2}}{\sqrt{M_1}} \right) \right] \right\} e^c \]
\[ + \|D_{k-1}\| \|X(t_{k-2}^+)\|, \]
which implies that
\[ \|X(t_{k+1}^+)^+\| \leq d_0^{-1} \|X^+\| \to 0 \]
and
\[ \|X(t_{k+1}^+)\| \leq d_0^2 \|X^+\| \to 0. \]
And hence, for any \( t \in (t_{k-1}, t_k) \),
\[ 0 \leq \|X(t)\| \leq \|X(t_{k-1}^+)^+\| e^{c_\lambda(t-t_{k-1})} \leq e^{c_\lambda} \|X(t_{k-1}^+)^+\| \to 0, \quad k \to \infty, \]
which completes the proof. \( \square \)

**Remark 7.** In fact, Theorem 4.3 proposed a novel method, by which and Theorem 4.1 we can actually assume \( \rho_k \in (t_{k-2}, t_{k-1}) \). Of course, the time-delay \( \rho_k \) can be any given thanks to the methods in the proof of Theorem 4.3.

5. **Numerical examples. Example 5.1.** Consider the system (10) with the data:

Set the initial condition \( \phi(t, x) = t \cos \frac{5\pi}{6} \cos \frac{5\pi}{6} \) for \( x_1, x_2 \) with \( (x_1, x_2)^T \in \Omega = [0, 1.2] \times [0, 1.2] \in R^2 \), and then \( \lambda_1 = 6.8539 \) (see, Remark 2). \( a = 0.9, b = 0.2, c = 0.2463, \) and

\[
A = \begin{pmatrix}
-4.0601 & -0.0893 & -1.0000 \\
0.1787 & 0.2000 & -0.0000 \\
1.0000 & 0 & 0.2463
\end{pmatrix}, \quad B = \begin{pmatrix}
0.6013 & 0 & 0 \\
0 & 0.0881 & 0 \\
0 & 0 & 0.0933
\end{pmatrix}.
\]

(34)
Let $S = \{1, 2\}$. And the transition rates matrix and Feedback coefficient matrix $\Pi$ and feedback gain coefficient matrix $K_r$ are given as follows:

\[
\Pi = \begin{pmatrix}
-0.013 & 0.013 \\
0.015 & -0.015
\end{pmatrix}, \quad K_1 = \begin{pmatrix}
0.011 & 0 & 0 \\
0 & 0.015 & 0 \\
0 & 0 & 0.012
\end{pmatrix}, \quad (35)
\]

\[
K_2 = \begin{pmatrix}
0.012 & 0 & 0 \\
0 & 0.016 & 0 \\
0 & 0 & 0.013
\end{pmatrix}.
\]

Let $\tau = 0.5$, and the external input $u(t, x) = \sin(100t)\cos\frac{5\pi t}{6}\cos\frac{5\pi x}{6}$. Then we can compute that $\|\phi\|_2^2 = 0.72\sin^21$ and $\|u\|_\infty^2 = 1.44\sin^21$. Moreover, take $M = 8, c_1 = c_2 = 4$, and $c_3 = 0.001, c_4 = 10$. Let

\[
P_1 = \begin{pmatrix}
1.001 & 0 & 0 \\
0 & 1.001 & 0 \\
0 & 0.9899 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0.9987 & 0 & 0 \\
0 & 0.9987 & 0 \\
0 & 0 & 1.013
\end{pmatrix} \quad (36)
\]

then it is easy to verified by Matlab software that the conditions (16)-(18) are satisfied simultaneously. According to Theorem 3.1, the system (10) is input-to-state stable in the mean square.

**Remark 8.** In the numerical examples of this paper and the related literature [23], the same data $a = 0.9, b = 0.2, c = 0.2463$ imply that there is chaos in the financial system (1) (see [23, Remark 10]). But the stability criterion of Theorem 3.1 admits the disturbance of external inputs to a certain extent. Of course, the Theorem 3.1 proposes the more stringent requirement on initial condition than [23, Theorem 1]. In fact, There is the bounded condition (18) on the initial condition in Theorem 3.1, which is particularly depicted in the table below.

| Table 1. Comparisons of Theorem 3.1 and [23, Theorem 1] |
|----------------------------------------------------------|
| stability type | exponential stability | input-to-state stability |
| admitting external inputs | No | Yes |
| Requirements on initial function | No | boundedness restraint |

**Example 5.2.** Consider the following data for the system (12) or (13).

Set $a = 0.9, b = 0.2, c = 0.2463$, and

\[
A = \begin{pmatrix}
-4.0601 & -0.0893 & -1.0000 \\
0.1787 & 0.2000 & 0 \\
1.0000 & 0 & 0.2463
\end{pmatrix}, \quad K = \begin{pmatrix}
0.011 & 0 & 0 \\
0 & 0.015 & 0 \\
0 & 0 & 0.012
\end{pmatrix}.
\]

\[
D_k = \begin{pmatrix}
-0.5 & 0 & 0 \\
0 & -0.4 & 0 \\
0 & 0 & -0.4
\end{pmatrix}, \quad \forall k \in \mathbb{Z}^+.
\]

Let $\varepsilon = 10, c_\varepsilon = 0.8, M_1 = 0.85, M_2 = 2.8, c_0 = 0.055, \rho_k \equiv 0.05, d_0 = 0.95$. By using computer Matlab software to compute, we get $c_\lambda = 8.3433$ and for any $k \in \mathbb{Z}^+$,

\[
\|D_k + I\| + \rho_k\|D_k\| \cdot \left(\|A\| + \sqrt{M_2} + \|K\|\frac{2\sqrt{M_2}}{\sqrt{M_1}}\right)e^{c_\varepsilon d_0} = 0.9446 \leq 0.95 = d_0 < 1.
\]

Now, Theorem 4.1 yields that the equilibrium solution $Q_1$ with the positive interest rate $\sqrt{\theta} = 8.93\%$ for the financial system (13) is globally asymptotically stable.
Remark 9. Replacing $\rho_k \equiv 0.05$ with $\rho_k \equiv 0$ in Example 5.2, we use the other data of Example 5.2 to verify the condition (33) as follows,

$$\|D_k + I\|e^{\frac{c\lambda c}{2}} \leq 0.95 = d_0 < 1, \; \forall \; k \in \mathbb{Z}^+.$$ 

Thus, Corollary 2 yields that the equilibrium solution $Q_1$ with the positive interest rate $\sqrt{\theta} = 8.93\%$ for the following system is globally asymptotically stable.

Table 2. Comparisons of Theorem 4.1, Corollary 2 and [23, Theorem 2]

|                  | [23, Theorem 2] | Corollary 2 | Theorem 4.1 |
|------------------|-----------------|-------------|-------------|
| stability type   | E-S             | A-S         | A-S         |
| interest rate    | 8.93\%         | 8.93\%      | 8.93\%      |
| delayed feedback model | No             | Yes         | Yes         |
| time-delays on impulse | No             | $\rho_k \equiv 0$ | $\rho_k \equiv 0.05$ |

where E-S represents the exponential stability, A-S represents the asymptotical stability.

Remark 10. Table 2 shows that Theorem 4.1 and Corollary 2 improve [23, Theorem 2], which also gives a positive answer to the question proposed in [23] (see Remark 1).

6. Conclusions and further considerations. In this paper, we employed synthetically Counterevidence method, Lyapunov functional method, variational method and regional control technique to deduce input-to-state stability of the equilibrium point with positive interest rate for delayed feedback financial system. As pointed out in Remark 1, time-delay bring essential difficulties in controlling the chaotic financial system. Instability and unpredictability of economic system sometimes increase the possibility of economic crisis and financial risk. However, the authors overcome such difficulties by proposing bounded hypothesis on the financial indicators, such as the interest rate, the investment demand and the price index, for such financial indicators are usually some percentages.

It is imperative to discuss how to effectively control the stability of the financial system. In this paper, regional control method is employed in deriving the input-to-state stability of the equilibrium point with positive interest rate for delayed feedback financial system. On the other hand, no-inputs stability is also investigated. impulse control technique and differential mean value theorem play roles in stabilizing the chaotic financial system. The obtained criteria improve the previous conclusions in related literature. Finally, numerical examples illustrate the effectiveness of the proposed methods.

As pointed out in [23] and [26], under Lipschitz conditions ensuring the unique existence of the solution of the reaction-diffusion system for any given initial value, Ruofeng Rao, Shouming Zhong, and Zhilin Pu deduced the boundedness conclusion [26, Theorem 3.3] and the stability criterion [26, Theorem 3.4], in which the following formula was derived:

$$\lambda_1(B(0, R_0)) = \begin{cases} \frac{\beta_0^2 \left(\frac{m}{m-n}\right)^\frac{2}{m}}{\left[\text{mes}(B(0, R_0))\right]^\frac{2}{m}} = \frac{\beta_0^2 \left(\frac{\pi}{2}\right)^\frac{2}{n}}{\left[\text{mes}(B(0, R_0))\right]^\frac{2}{n}}, & n = 2m, \\ \frac{\beta_0^2 \left(2(2e)^{-\frac{n}{2}}\right)^{\frac{n}{2}}}{\left(2m+1\right)!} \left[\frac{[\text{mes}(B(0, R_0))]^{\frac{2}{n}}}{\text{mes}(B(0, R_0))}\right]^\frac{2}{n} = \frac{\beta_0^2 \left(2(2e)^{-\frac{n-1}{2}}\right)^{\frac{n-1}{2}}}{\left(2m+1\right)!} \left[\frac{[\text{mes}(B(0, R_0))]^{\frac{2}{n}}}{\text{mes}(B(0, R_0))}\right]^\frac{2}{n}, & n = 2m + 1. \end{cases}$$

(39)
This has actually proven the following conclusion under the assumption (H1) in [26]:

**Theorem 6.1.** ([23, Theorem 3]). If \( f_i(0) = \hat{f}_i(0) = \sigma_{ij}(0) = \hat{\sigma}_{ij}(0) = 0 \), then there must exist a series of spherical regions \( B(0, R_0) \subset \mathbb{R}^n \) with \( R_0 \) moderately small such that the following fuzzy system (40) is globally stochastically exponential stable in the \( p \)th moment, where \( \Upsilon = B(0, R_0) \) in (40).

\[
\begin{align*}
\dot{u}_i(t, x) &= q_i \text{div} \nabla u_i(t, x) dt - \sum_{r=1}^r q_r(\hat{\omega}(t)) \left[ a_{ir}u_i(t, x) - \sum_{j=1}^n b_{ijr}f_j(v_j(t, x)) \right] \\
&\quad - \sum_{j=1}^n c_{ijr}f_j(v_j(t - \tau(t), x)) - \sum_{j=1}^n h_{ijr} \int_{t-\rho(t)}^t f_j(v_j(s, x)) ds dt \\
&\quad + \sum_{j=1}^n \sigma_{ij}(t, u(t, x), v(t - \tau(t), x)) dw_j(t), \quad t \geq 0, x \in \Upsilon,
\end{align*}
\]

\[
\begin{align*}
\dot{v}_i(t, x) &= \hat{q}_i \text{div} \nabla v_i(t, x) dt - \sum_{r=1}^r \hat{q}_r(\hat{\omega}(t)) \left[ \hat{a}_{ir}v_i(t, x) - \sum_{j=1}^n \hat{b}_{ijr}\hat{f}_j(u_j(t, x)) \right] \\
&\quad - \sum_{j=1}^n \hat{c}_{ijr}\hat{f}_j(u_j(t - \hat{\tau}(t), x)) - \sum_{j=1}^n \hat{h}_{ijr} \int_{t-\hat{\rho}(t)}^t \hat{f}_j(u_j(s, x)) ds dt \\
&\quad + \sum_{j=1}^n \hat{\sigma}_{ij}(t, v(t, x), u(t - \hat{\tau}(t), x)) \hat{d}w_j(t), \quad t \geq 0, x \in \Upsilon,
\end{align*}
\]

\[
\begin{align*}
u_i(t, x) &= \zeta_i(t, x), \quad v_i(t, x) = \varpi_i(t, x), \quad \forall (s, x) \in [-\tau, 0] \times \Upsilon
\end{align*}
\]

\[
\begin{align*}
\partial_t u(t, x) &= 0 = \partial_t v(t, x), \quad \forall (t, x) \in [0, +\infty) \times \partial \Upsilon.
\end{align*}
\]

Using coordinate translation, we can actually generalize Theorem 6.1 and [26, Theorem 3.4] from spherical region \( B(0, R_0) \) to more general spherical region \( B(a, R_0) \subset \mathbb{R}^n \) with any point \( a \in \mathbb{R}^n \), where \( B(a, R_0) \) is a spherical region of \( \mathbb{R}^n \) with a radius \( R_0 \). That is,

**Theorem 6.2.** If \( f_i, \hat{f}_i, \sigma_{ij}, \hat{\sigma}_{ij} \) are Lipschitz continuous with \( f_i(0) = \hat{f}_i(0) = \sigma_{ij}(0) = \hat{\sigma}_{ij}(0) = 0 \), then there must exist a series of spherical regions \( B(0, R_0) \subset \mathbb{R}^n \) with \( R_0 \) moderately small such that the following fuzzy system (40) is globally stochastically exponential stable in the \( p \)th moment, where \( \Upsilon = B(a, R_0) \) in (40).

Due to \( \lambda_1(\Omega) \geq \lambda_1(B(0, R_0)) \) when \( \Omega \subset B(0, R_0) \), we can actually generalize Theorem 6.1 and [26, Theorem 3.4] from spherical region \( B(0, R_0) \) to more general region \( \Omega \subset \mathbb{R}^n \), where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with smooth boundary. That is,

**Theorem 6.3.** If \( f_i, \hat{f}_i, \sigma_{ij}, \hat{\sigma}_{ij} \) are Lipschitz continuous with \( f_i(0) = \hat{f}_i(0) = \sigma_{ij}(0) = \hat{\sigma}_{ij}(0) = 0 \), then there must exist a series of domains \( \Omega \subset B(0, R_0) \subset \mathbb{R}^n \) with \( R_0 \) moderately small such that the following fuzzy system (40) is globally stochastically exponential stable in the \( p \)th moment, where \( \Upsilon = \Omega \) in (40).

If the state variables are homogeneous in \( B(a, R_0) \), the system (40) can actually be ordinary differential equations. So, how to improve Theorem 3.1, Theorem 4.1, Theorem 4.2, Corollary 1 and Corollary 2 to such concise conclusion as Theorem 6.2 or Theorem 6.3? This is an interesting problem.
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