Cold matter trapping via slowly rotating helical potential.

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We consider the cold bosonic ensemble trapped by a helical interference pattern in the optical loop scheme. This rotating helical potential is produced by the two slightly detuned counter propagating Laguerre-Gaussian laser beams with counter directed orbital angular momenta $\pm \ell h$. The detuning $\delta \omega$ may occur due to rotational Doppler effect. The superfluid hydrodynamics is analysed for the large number of trapped atoms in Thomas-Fermi approximation. For the highly elongated trap the Gross-Pitaevskii equation is solved in a slowly varying envelope approximation. The speed of axial translation and angular momenta of interacting atomic cloud are evaluated. In the $T \to 0$ limit the angular momentum of the helical cloud is expected to be zero while toroidal trapping geometry leads to $2\ell h$ angular momentum per trapped atom.

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I. INTRODUCTION

The hydrodynamics of the sufficiently cold ($T \sim 10^{-6} K$) bosonic ensemble trapped by optical potential $V(\vec{r}, t)$ is described by the Gross-Pitaevskii equation (GPE) for macroscopic wavefunction $\Psi(\vec{r}, t)$:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(\vec{r}, t) \Psi + \frac{4\pi\hbar^2}{m} a_s |\Psi|^2 \Psi,$$

where $m$ is the mass of atom, $a_s$ is the two body s-wave scattering length. The negative $a_s$ reduces the energy of ensemble and causes the mutual attraction of atoms. This results in formation of bright solitons in 1D and collapse in higher dimensions. On the contrary in repulsive BECs atoms repel each other and dark solitons or vortices are formed. In the periodic potential gratings $V(\vec{r}, t) \sim I(z, r, t) \sim \exp[-r^2] \cos[\delta \omega t - (k_f + k_b)z]$, where $r$ is a distance from propagation axis $z$, $\delta \omega = c \cdot (k_f - k_b) = \omega_f - \omega_b$ is a frequency difference of the counter-propagating $z$-paraxial laser beams with the opposite wave vectors $|\vec{k}_{f,b}| = k_{f,b}$, the many-body nonlinearity leads to the nonlinear tunneling, self-trapping and other quantum interference phenomena. For accelerated 1D optical gratings when $\delta \omega = \delta \omega \cdot t$, i.e. in the non inertial reference frames the interacting bosons demonstrate Bloch oscillations and Landau-Zener tunneling. In rotating 1D lattices the repulsive ensemble may have stable ground states and vortex soliton states. For the attractive ensemble in 2D rotating lattices the localized stable solitons exist in the certain range of angular velocities. Inside the parabolic well with trapping frequency $\omega_\perp$ and rotation frequency $\Omega$ the nonlinear localized matter waves appear when $\Omega \sim \omega_\perp$ at lowest Landau level which appears due to the fictionary magnetic field induced by trap rotation.

The goal of the present work is to study bosonic ensemble in rotating reference frame by a virtue of rotating optical dipole trap whose rotation is due to the frequency detuning $\delta \omega$. The proposed trap configuration is composed of the two counter propagating optical vortices. The shape of the interference pattern is defined by the mutual orientation of their orbital angular momenta $\pm \ell h$ (OAM). When OAMs are co-directed the toroidal traps are formed by Laguerre-Gaussian(LG) or Bessel vortices. This geometry had been used for the single atom trapping and detection and for the analysis of persistent condensate flows in LG beam waist. This rotating experimentally observed for $\delta \omega = 0$. Noteworthy the similar interferometric configuration with counter directed spin optical angular momenta (circular polarizations) which is used for the sub-Doppler polarization gradient cooling. The interference pattern forms the static potential gratings due to Zeeman shift between ground-state sublevels. This causes the Sisyphus deceleration of the atomic beam in optical molasses.

We consider two possible regimes of the cold ensemble trapping when kinetic energy is small compared to the interaction and trapping energies. One Thomas-Fermi solution is obtained as a balance of the red-detuned optical attractive potential and self-defocusing due to positive $a_s$. This solution has atomic density $\rho_h$ perfectly collocated with rotating optical helix $I(z, r, \theta, t)$. In this case the rotating potential imposes rotation to superfluid. The other TFA solution has nonrotating density profile $\rho_{fun}$ relevant to thermodynamic limit when number of atoms $N \to \infty$.  

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Next the analytical solution of GPE for nonzero kinetic energy $\Psi_{rel}$ is described. It is valid when optical trapping and interaction energies are comparable and subtract each other in GPE. The linear momentum $<p_r>$ and angular momentum $<L_z>$ of this helical atomic cloud are evaluated. The Landau criterion $|\vec{V}| > \epsilon(\vec{p})/|\vec{p}|$ for the appearance of elementary excitations and superfluidity breakup [18] is discussed for helical geometry. 

II. TWISTED WAVETRAINS

It is well known that interference of a two counter propagating waves with a different frequencies $\omega_f, \omega_b$ produces a running sinusoidal roll intensity grating [19]. For the equal wave amplitudes $|\mathbf{E}_f| = |\mathbf{E}_b|$ and the phase difference $\phi$ the distribution of the light intensity $I(z, r, \theta, t)$ has the following form:

$$I(z, r, \theta, t) \sim 2|\mathbf{E}_{f(b)}|^2[1 + \cos(\delta \omega t - (k_f + k_b)z + \phi)]$$

$$\exp[-\frac{r^2}{D_0^2(1+2z^2/z_R^2)}], \quad z_R = k_{f(b)}D_0^2,$$ (4)

provided that a visibility of pattern is good enough $|\mathbf{E}_{f(b)}| \equiv |\mathbf{E}_b| \geq 21$, where $z_R$ is Rayleigh range, $D_0$ is a beam waist radius, the self-similar variable $\chi = (\omega_f - \omega_b)t - (k_f + k_b)z + \phi$ is responsible for the translation of the interference pattern along $z$ axis with the group velocity $V_g = (\omega_f - \omega_b)/(k_f + k_b)$. The transversal (in the plane $(r, \theta)$) confinement of the light amplitudes $\mathbf{E}_f, \mathbf{E}_b$ is typical for the zeroth-order Gaussian beams. The roll interference pattern evolves into the sequence of the equidistantly spaced rotationally invariant in $\theta$ ellipsoids centered at the propagation axis $z$ [11]. For the higher-order propagation modes namely Laguerre-Gaussian beams (LG) with azimuthal quantum number $\ell$ and orbital angular momentum $\ell h$ per photon [21]:

$$\mathbf{E}_{f(b)}(z, r, \theta, t) \sim \mathbf{E}_{f(b)} \exp[i(-\omega_{f(b)}t + k_{f(b)}z) + \ell \theta]$$

$$\frac{1+i z/z_R}{(1+i z/z_R)} \frac{r^2}{(r/D_0)\ell \theta \exp \left[-\frac{r^2}{D_0^2(1+i z/z_R)}\right]},$$ (5)

the interference pattern is different for LG reflected from conventional mirror and phase conjugating mirror. Backward reflection from conventional spherical mirror changes the topological charge of the LG [11], exactly in the same way that circular polarization of light changes from left to right and vice-versa in reflection [19]. The intensity $I_{tor}(z, r, \theta, t)$ vanishes on the beam axis thus interference pattern transforms into the sequence of the equidistant rotationally invariant toroids separated by $\lambda/2$ interval:

$$I_{tor}(z, r, \theta, t) = A_n[1 + \cos(\delta \omega t - (k_f + k_b)z)]\frac{(r/D_0)^2\ell \theta}{2},$$

$$\exp\left[-\frac{2r^2}{D_0^2(1+2z^2/z_R^2)}\right], A_n = \epsilon_b \epsilon \frac{2|\mathbf{E}_{f(b)}|^22^{2\ell(\ell+1)}}{\pi^{\ell}D_0^{2\ell+1}},$$ (6)

where $\epsilon_b$ is the dielectric permittivity of vacuum. The reflection from phase-conjugating mirror (PCM) does not change the topological charge of LG and the interference pattern is twisted [11, 16]:

$$I_{tor}(z, r, \theta, t) = [1 + \cos(\delta \omega t - (k_f + k_b)z + 2\ell \theta)]\times$$

$$\frac{A_n \cdot (r/D_0)^{2\ell \theta}}{2^{(2\ell)^2}} \exp\left[-\frac{2r^2}{D_0^2(1+2z^2/z_R^2)}\right].$$ (7)

The intensity also vanishes at LG axis $z$ as $r^{-2\ell}$, while a self-similar argument:

$$\chi = [\omega_f - \omega_b]t - (k_f + k_b)z + 2\ell \theta,$$ (8)

keeps the maxima of intensity at the $2\ell$ collocated helices separated from each other by $\lambda/2$ interval (fig. 1). The azimuthal term $2\ell \theta$ appears due to phase conjugation $\mathbf{E}_b \sim \mathbf{E}_f^*$. Thus we have the following strict correspondence between the roll interference pattern (2) and the helical interference pattern (7): the frequency difference $\delta \omega = \omega_f - \omega_b$ is the cause of the translation of rolls with group velocity $V_g = (\omega_f - \omega_b)/(k_f + k_b)$ of the wavetrain, produced by a sum of the two counter-propagating beams ($\mathbf{E}_f + \mathbf{E}_b$) 2. The $\delta \omega$ is responsible also for the rotation of helices with angular velocity $\theta = \delta \omega/2\ell$. The rotation is the cause of the pitch of helical interference maxima along $z$-axis. Consequently there exists a perfect mechanical analogy between the solid body rotation of the helix described by formula (7) and an Archimedean screw. Namely the positive $\delta \omega$ corresponds to the counter-clockwise rotation and this provides the pitch in positive $z$ direction for right helices. On the other hand the negative $\delta \omega$ means clockwise rotation. In this case ($\delta \omega < 0$) the positive translation speed
in z direction takes place for the left-handed helices. Evidently the change of the topological charge $\ell$ changes the direction of helix translation $\vec{V}_z$ due to alternation of the helix hand to the opposite one for a given $\Delta \omega$.

This mechanical analogy is useful for the analysis of the cold atoms motion in the helical trap. The velocity vector of the condensate fragment trapped and perfectly collocated with intensity maxima has two components $\vec{V} = \vec{V}_z + \vec{V}_b(fig. \ 1)$. The axial component is a group velocity $|V_z| = (\omega_f - \omega_b)/(k_f + k_b)$ of the wavetrain, while the azimuthal component $|V_b| = (\omega_f - \omega_b) \cdot D_0$ is of kinematic nature.

The electromagnetic orbital angular momentum inside LG beam waist volume $V \equiv \pi D_0^2 z_R$ located near $z = 0$ plane within Rayleigh range $|z| < z_R$ is the expectation value of the angular momentum operator $L_z = -i\hbar [\hat{r} \times \nabla]$.

$$<L_z >_{(f,b)} = <\Psi^f_{(f,b)}| \hat{L}_z |\Psi^f_{(f,b)}> = 2\epsilon_0 \int (E^+_f)^* (-i\hbar [\hat{r} \times \nabla] E^+_f) d^3r \approx 2\epsilon_0 \times$$

$$\int (E^+_b)^* (-i\hbar \frac{\partial}{\partial z} E^+_b) r dr dz dz \approx \pm \hbar k \frac{I(f,b) V}{\hbar \omega(f,b)c}, \tag{9}$$

where $I(f,b) = \epsilon_0 |\Psi^f_{(f,b)}|^2$ is the light intensity, $\Psi^f_{(f,b)} = \sqrt{2\epsilon_0} E^+_f(z,r,\theta,t)$ are the macroscopic wavefunctions of a single photon inside a forward or backward beam with the winding number $\ell$, $E^+_\ell$ means the positive frequency components in $E(f,t)$ spectrum. The square modulus $|\Psi^f_{(f,b)}|^2$ is a probability density of photon detection which is proportional to the energy density of classical wave $[24]$. In this particular paraxial case the spin-orbit coupling $[23]$ is small enough and the angular momentum of the photon is exactly decoupled to the spin and the orbital component: $J = S + L$. The linear momentum expectation values for the forward and backward LG are as follows:

$$<P_z >_{(f,b)} = <\Psi^f_{(f,b)}| \hat{P}_z |\Psi^f_{(f,b)}> = 2\epsilon_0 \times$$

$$\int (E^+_f)^* (-i\hbar \frac{\partial}{\partial z} E^+_f) d^3r \approx \pm \hbar k \frac{I(f,b) V}{\hbar \omega(f,b)c}. \tag{10}$$

The ratio of the angular and linear momenta is $L_z/P_z \sim \ell \epsilon_0/\omega(f,b) [23]$. The angular and linear momenta of the composite wavefronts $[20]$ and $[24]$ (fig. 1) are:

$$<L_z >_{(f+b)} = <\Psi^f_{(f+b)}| \hat{L}_z |\Psi^f_{(f+b)} > = (\ell \hbar + \epsilon \hbar) \frac{I(f+b) V}{\hbar \omega(f+b)c},$$

$$<P_z >_{(f+b)} = <\Psi^f_{(f+b)}| \hat{P}_z |\Psi^f_{(f+b)} > = (\hbar k_f - \hbar k_b) \frac{I(f+b) V}{\hbar \omega(f+b)c}, \tag{11}$$

where the upper sign in $\epsilon$ corresponds to reflection from conventional mirror while the bottom sign stands for the reflection from PC-mirror with alternation of the photon angular momentum.

The simplest conceivable configuration of linear PCM composed of plane mirrors, wavefront curvature compensating lenses and beam splitters is described in $[23, 27]$. The goal of proposed setup is to counter-direct the split LG beams. The four reflections are the necessary minimum. The case is that the incidence angles much above 45 degrees will distort both polarization and spatial structure of LG. In addition a small sliding of a beam along the reflecting surface $[26]$ may occur when mirror is tilted with respect to the LG propagation axis. The evident physical restriction on this $loopt$ setup is to keep the path difference $\Delta L$ of the counter directed LG smaller than coherence length of trapping laser field $\Delta L \ll c\tau_{coh}$ $[23]$. The small frequency shift $\delta \omega \approx 2\pi \times 10^{-1}(3)$ rad/sec required to cause the helix rotation $[11]$ might be induced by a frequency ramp $[5]$ or via rotational Doppler shift which appears due to rotation of the half-wavelength plate $[28]$ or Dove prism $[29]$.

### III. Cold Ensemble Density and Velocity Field in the Helical Trap

Consider a bosonic cloud prepared in an elongated trap $[3]$ and suddenly released afterwards. The well elaborated experimental procedure is to impose a periodic optical potential to study the Bloch oscillations, macroscopic Landau-Zener tunneling and Josephson effects $[4, 6]$. In our case the imposed optical potential is a helical one:

$$V_{opt}(z, r, \theta, t) = -\frac{Re[\alpha(\omega)]}{2\epsilon_0 c} I_{\omega}(z, r, \theta, t),$$

$$\alpha(\omega) = 6\pi \epsilon_0 e^3 \Gamma/\omega_0^2 (\omega_0^2 - \omega^2 - i(\omega_0^2/\Gamma)^2), \tag{12}$$

where $\alpha(\omega)$ is the polarizability of atom, which is real, i.e. $\alpha(\omega) \approx Re[\alpha]$ at large detunings from resonance $\omega - \omega_0$, $\Gamma = e^2\omega_0^2/6\pi\epsilon_0 m e c^3$ is classical damping rate via radiative energy loss, $m_e$ is electron mass $[3]$. The GPE for wavefunction $\Psi$ of ensemble confined by $V_{opt}$ is:

$$ih \frac{\partial \Psi(f,b,t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V_{opt} \Psi + \frac{4\pi \hbar^2 a_s(\vec{B})}{m} \left|\Psi\right|^2 \Psi, \tag{13}$$

with $a_s(\vec{B}) = a_{bg}(1 + \Delta B/|\vec{B} - B_F|)$ magnetic field dependent $s$–wave scattering length, where $a_{bg}$ is background value of $a_s$, $B_F$ and $\Delta B$ are the Feshbach magnetic induction and resonance width respectively.

Consider the sufficiently large number of trapped atoms ($N \approx 10^{6-12}$). Then quantum pressure term following from the uncertainty principle is small $(\hbar^2 \Delta \Psi/2m \approx 0)$ compared to the optical trapping and interaction terms $[3]$:

$$\frac{E_{int}}{E_{kin}} \approx \frac{N \cdot a_{bg}}{a_{ho}}, \tag{14}$$
where \( a_{ho} \cong \sqrt{\hbar/m_0 c^2} \) is the "harmonic oscillator width" \[4\]. The ratio \( E_{int}/E_{kin} \) is much more than unity for the most of the near infrared lasers \( \lambda = 0.8 - 1.5 \mu m \), under the standard focusing requirement \( D_0 \sim 10 - 100 \mu m \) and when \( a_0(\vec{B}) \) is tuned in the range \( \approx 1 - 100nm \) via Feshbach resonance. For the mid-infrared trapping at \( CO_2 \) lasing wavelength \( \lambda = 10.6 \mu m \) the kinetic energy term \( E_{kin} = 2\hbar^2/n^2 \) is about 100 times smaller. The optical dipole trapping energy \( E_{dipole} \) is:

\[
E_{dipole} \cong N \cdot V_{opt} \cong \frac{\vec{p} \cdot \vec{E}}{2} \cong -\alpha\frac{E^2 N}{2}
\]

\[
\cong -\alpha I_{tw} N = -\frac{e^2}{m_0 c^2} \Delta \omega, \quad \Delta \omega = \omega_{f,b} - \omega_0
\]

(15)

where \( I_{tw} \) is the optical intensity of trapping beams \[3\]. Hence one might expect that a Thomas-Fermi approximation (TFA) is adequate in our case when interaction is repulsive \( a_0 > 0 \) \[3\].

When inhomogeneous ensemble is considered in the thermodynamic limit \( (N \rightarrow \infty) \) the local density approximation for chemical potential is used \( \mu(r) = \mu_{local} = \mu(\vec{r}) + V_{opt}(\vec{r}) \) \[3\]. Let us consider the following TFA wavefunction in the vicinity of the LG-beam waist (i.e. within Rayleigh range \( |z| < z_R \)):

\[
\Psi_{fun} = \Phi(r) \exp \left[ -\frac{i\mu(r) t}{\hbar} + i\Phi(r)^2 \sin(\delta \omega t - 2k_z z + 2\ell \theta) \right],
\]

where the local \( r \)-dependent chemical potential \( \mu(r) \) is:

\[
\mu(r) = 4\pi \hbar^2 a_0 \Phi(r)^2 / m; \quad I_{tw} = 2e_0 c |E_{f,b}|^2;
\]

\[
\Phi(r)^2 = \exp(-2r^2/D_0^2) \cdot (r/D_0)^{2|\ell|} \frac{\alpha(\omega) I_{tw}}{2e_0 c \cdot \hbar \cdot \delta \omega}.
\]

The wavefunction (16) is normalizable and fits the GPE by substitution. This TFA density of atoms \( \rho_{fun}(z, r, \theta, t) = |\Psi_{fun}|^2 \sim \exp(-2(r/D_0)^2) / (r/D_0)^{2|\ell|} \) is \( \delta \)-independent "funnel" collocated with the optical helix \( t \sim \omega_0 \). The phase modulation of \( \Psi_{fun} \) has a maximum near the density maximum \( \rho_{fun}(z, r, \theta, t) \), sinusoidal dependence on azimuthal angle \( \theta \) and decreases down to zero on LG axis and outside the LG waist. Note-worthy \( \Psi_{fun} \) is a multiply valued function of \( \theta \). Hence this solution is of restricted interest. It may be used for evaluations of the thermodynamical parameters of the cold ensemble with \( |\Psi_{fun}|^2 \) density \[3\].

The other solution for dilute Bose gas (e.g. \( N \sim 10^6 \)) is obtained in the TFA for \( r \)-independent chemical potential \( \mu(\vec{r}) = const \). The mean field wavefunction here \( \Psi_{f,b} \) is a sum of the two phase-conjugated vortices with the opposite angular momenta \( \pm \ell \hbar \):

\[
\Psi_{f,b}(z = 0) \equiv \Psi_{+}(z, r, \theta, t) + \Psi_{-}(z, r, \theta, t) \equiv \sqrt{2k_{f,b}} \exp\left[-\frac{r^2}{D_0^2(1+iz/z_R)}\right] \times
\]

\[
\left\{ \exp\left[-\frac{i\mu f/r}{\hbar} + ik_f z + i\ell \theta \right] + \exp\left[-\frac{i\mu b/r - ik_b z - i\ell \theta}{\hbar} \right] \right\},
\]

(18)

where the difference of the partial chemical potentials \( (\mu_f - \mu_b) \), associated with each of "counter-propagating" wavefunctions \( \Psi_{+} \), \( \Psi_{-} \) is adjusted to the frequency difference of counter-propagating optical fields \( (\mu_f - \mu_b)/\hbar = \Delta \omega = \omega_f - \omega_b \). The substitution of this TFA wavefunction into GPE gives the following link for parameters:

\[
\mu = \frac{\alpha(\omega) I_{tw}(0) \cdot [1 + \cos(\delta \omega t + 2k_z z + 2\ell \theta)]}{2e_0 c \cdot g (1 + z^2/z_R^2)} \cdot (r/D_0)^{2|\ell|} \times \exp\left[-\frac{2r^2}{D_0^2(1+iz/z_R)}\right] = \rho_b(z, r, \theta, t), \quad (19)
\]

where \( k = k_f \cong k_b, \mu = \mu_f \cong \mu_b \) is a constant \((z, r, \theta, t\)-independent) value of chemical potential, \( g = 4\pi \hbar^2 a_0(\vec{B})/m \) is the interaction parameter. This is the quasiclassical restriction imposed on a homogeneity of chemical potential of the system in an external field \( V_{opt} \). In accordance to this solution the density of the cold atomic ensemble \( \rho_b(z, r, \theta, t) \) is perfectly correlated with the rotating optical helix potential \( I_{tw}(z, r, \theta, t) \) as depicted at fig[3]. The density \( \rho_b \) rotates as a "solid body". The speed of the axial translation is \( V_z = \lambda \cdot \delta \omega / 4\pi \). From TA approximation the truly exact solution with nonzero kinetic energy exists \( \Psi_{ex} = \Psi_{f} + \Psi_{b} \). It is also a superposition of the two counter propagating paraxial matter waves \( \Psi_{f} \) and \( \Psi_{b} \). Let us reduce GPE to the paraxial form relevant to highly elongated geometry, as a helical one in our case:

\[
\frac{i\hbar}{2m} \frac{\partial \Psi_{f,b}(z, r, \theta, t)}{\partial t} = \frac{\hbar^2}{2m} \Delta \Psi_{f,b}(z, r, \theta, t) - \frac{2\hbar^2 k_{f,b}^2}{m} \frac{\partial \Psi_{f,b}(z, r, \theta, t)}{\partial z} + V_{opt} \Psi_{f,b}(z, r, \theta, t) + \frac{\hbar^2}{2m} k_{f,b}^2 \frac{\partial^2 \Psi_{f,b}(z, r, \theta, t)}{\partial z^2} + g(\Psi_f + \Psi_b) |\Psi_{f,b}|^2
\]

(20)

The perfect mutual cancellation of trapping and interaction terms:

\[
- V_{opt}(z, r, \theta, t) \Psi_{f,b}(z, r, \theta, t) = g(\Psi_f + \Psi_b) |\Psi_{f,b}|^2
\]

occurs when nonlinear defocusing due to positive scattering length \( a_0 \) is compensated by attraction to intensity maxima caused by red detuning. Taking again the exact helical solution of (20) as a superposition of the two counter-propagating vortices:

\[
\Psi_{ex}(z, r, \theta, t) = \Psi_{f}(z, r, \theta, t) + \Psi_{b}(z, r, \theta, t) \cong \Psi_{f} \exp\left[-\frac{i\mu_f}{\hbar} + ik_f z + ik_f z \right] + \Psi_{b} \exp\left[-\frac{i\mu_b}{\hbar} - ik_b z \right], \quad (22)
\]

and under the natural assumptions:

\[
k_{f,b} \frac{\partial \Psi_{f,b}(z, r, \theta, t)}{\partial z} >> \frac{\partial^2 \Psi_{f,b}(z, r, \theta, t)}{\partial z^2}
\]

(23)

two following equations for counter propagating and counter rotating matter waves \( \Psi_{f} \) and \( \Psi_{b} \) are valid:

\[
i2k_{f,b} \frac{\partial \Psi_{f,b}(z, r, \theta, t)}{\partial z} + \Delta \Psi_{f,b}(z, r, \theta, t) - (k_{f,b}^2 + \frac{2m(\mu_f + \mu_b)}{\hbar^2}) \Psi_{f,b}(z, r, \theta, t) = 0.
\]

(24)
which have the vortex solutions with charge \( \ell \) for the
\( \Psi_{f,b}(z,r,\theta) \) with initial condition at \( z=0 \) equals \( \Psi_0 \):
\[
\Psi_{(f,b)} \sim \frac{\Psi_0(r/D_0)^{\ell f}. \text{exp} \left[ -\frac{\hbar^2}{2m}(z^2 + (iz + zn))^2 \pm i\ell \theta \right]}{(1 + iz/z_R)}.
\]

The issues of dynamical stability of (22) with respect to small perturbations and thermodynamical stability from the point of view of least energy arguments deserve a further careful analysis and will be published elsewhere. For example transformation to the reference frame rotating with angular velocity \( \Omega = \delta \omega / 2 \ell \) synchronously with trapping helix leads to equation [1 3 10]:
\[
\frac{i \hbar}{\ell t} \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Phi + \tilde{V}_{\text{opt}}(z,r,\theta) \Psi + g|\Psi|^2 \Psi - \Omega \hat{L}_z \Psi,
\]
\[
\tilde{V}_{\text{opt}} \sim r^{2|\ell|} \exp\left(-2r^2/D_0^2(1 + z^2/k_D^2D_0^2)\right)[1 + \cos(2kz + 2\ell \theta)],
\]
where stationary solutions for the diluted (\( \mu = \text{const} \)) ensemble \( \Psi = \Phi(z,r,\theta) \exp(-ij\ell t/\hbar) \) are given by:
\[
\mu \Phi = -\frac{\hbar^2}{2m} \Delta \Phi + \tilde{V}_{\text{opt}}(z,r,\theta) \Phi + g|\Phi^2 \Phi + \Omega i\ell \frac{\partial \Phi}{\partial \theta}.
\]

IV. MACROSCOPIC OBSERVABLES

The helical solutions composed of the counter propagating free space LG wavefunctions apparently fit the continuity equation and have realistic field of velocities. In addition to the ensemble density \( \rho(\vec{r},t) \) obtained above in TFA the structure of velocity field \( \vec{V}(\vec{r},t) \) is a consequence of the complex geometry of the helical wavetrain which requires the perfect adjustment of the phase fronts achieved by phase-conjugation of colliding vortices. The one possible application of the (18, 22) might be in using them as variational anzatz for emulation of GPE [30]. Nevertheless the explicit form of solutions \( \Psi_{\mu} \) offers a possibility to evaluate the macroscopic observables of the trapped ensemble. Helical wavetrain has nonzero momentum \( P_{\mu} [10] \):
\[
< P_{\mu} >_h = \langle \Psi_{h} | -i \hbar \frac{\partial}{\partial z} | \Psi_{h} \rangle = Nh(k_f - k_h) \Leftrightarrow \hbar \int_V dV. \exp(-r^2) r^{2|\ell|}[k_f - k_h + k_f \exp(i\chi) - k_h \exp(-i\chi)],
\]
and easily calculated angular momentum \( L_z [9] \):
\[
< L_z >_h = \langle \Psi_{h} | -i \hbar \frac{\partial}{\partial \theta} | \Psi_{h} \rangle = N\ell h(1 \mp 1) \Leftrightarrow \hbar \int_V dV \exp(-r^2) r^{2|\ell|}[1 \mp 1 + \exp(i\chi) \mp \exp(-i\chi)],
\]
due to the apparent identity \( \int_0^{2\pi} \sin(\chi) d\theta = 0 \). The upper \( \mp \) sign in [22] corresponds to counter directed angular momenta and helical interference pattern [17], while bottom \( \mp \) sign corresponds to the toroidal optical interference pattern [10].

The same expectation values \( < P_\mu >_{\text{ex}} = Nh(k_f - k_h) \) and \( < L_z >_{\text{ex}} = N\ell h(1 \mp 1) \) has exact wavefunction \( \Psi_{\mu} \) (22). Quantum mechanically this happens because the wavefunction in both cases is a superposition of the two partial matter waves \( \Psi_f \) and \( \Psi_b \) in (18) (or \( \Psi_f \) and \( \Psi_b \) in [18]) having opposite and quantized (i.e. equal to \( \pm \ell h \)) mutually subtracted angular momenta. This means also that helical optical wavetrain (7) contains OAM of exactly \( 0 \times \ell h \) per photon, while toroidal wavetrain (6) contains \( 2\ell h \) per photon (11) as shown in sec.II.

This feature looks seemingly counter intuitively from the point of view of classical hydrodynamics, but the similar results on vanishing of the moment of inertia for purely superfluid ensemble were summarized in [4]. Namely the density of atomic ensemble \( \rho_h = |\Psi_{h,\text{ex}}|^2 \) rotates as a solid body and one might expect that \( \rho_h \) to have classically the angular momentum \( L_{\text{class}} = I_{zz} \cdot \delta \omega / 2 \ell \), where \( I_{zz} \) is the moment of inertia of the helical wire located in LG beam waist with the density profile \( \rho_h [31] \):
\[
I_{zz} = \int Nm|\Psi_{h,\text{ex}}|^2 r^2 dV = Nm \int |\Psi_{h,\text{ex}}|^2 r^2 dz d\theta d\phi \approx \sim Nm \int (1 + \cos(\delta \omega t + (k_f + k_h)z + 2\ell \theta)) \cdot d\theta d\phi \cdot 
\]
\[
\exp(-r^2/D_0^2)r^{2+2|\ell|} \cdot rdr \sim Nm D_0^2 Z_f. \]

Nevertheless due to the quantization of the angular momentum in free space, the oppositely directed angular momenta cancel each other completely, because they have integer opposite values of \( \pm \ell h \) [22]. In \( T \to 0 \) limit [4] the net angular momentum of the helical wavetrains (18, 22) is zero because the superfluid component remains only. On the contrary, the linear momentum is not quantized in free space and this leads to nonzero net linear momentum \( P_{\mu} \) of the ensembles (18, 22), regardless to the mutual orientation of their OAM’s. The net momentum \( P_{\mu} \) is small because of the smallness of the group velocity of the helical wavetrain \( V_z = \delta \omega / (k_f + k_h) \). For example when frequency splitting \( \delta \omega \) is induced by rotational Doppler effect [27, 28], the speed of the axial component of the helical density profiles (18, 22) \( V_z \approx \) is several \( \mu m \) per second (several rotations of helix per second).

The else interesting physical consequences relevant to experiments with trapped quantum gas traps may be formulated from the point of view of the Landau criterion \( |\vec{V}| > \epsilon (\vec{p})/|\vec{p}| \) for the appearance of elementary excitations (rotons) and superfluidity destruction, where \( \epsilon (\vec{p}) \) is the energy - momentum dispersion relation. Following to [18, 24] consider the flow of quantum gas in a narrow helical channel with velocity \( \vec{V} = \vec{V}_z + \vec{V}_g \). In the rest frame the momentum of excitation \( \vec{p} \) must be opposite to the velocity of superfluid \( \vec{V} \), because of the least energy constraint imposed upon excitation \( \epsilon (\vec{p}) + \vec{p} \cdot \vec{V} < 0 \). Thus \( \epsilon (\vec{p}) - |\vec{p}| \cdot |\vec{V}| < 0 \) and \( |\vec{V}| > \epsilon (\vec{p})/|\vec{p}| \). Because in our case the only significant component of \( \vec{V} \) is \( V_0 = \delta \omega D_0 / 2 \ell \) the
excitations with momentum $\vec{p}$ appear when:
\[ \delta \omega_{\text{crit}} D_0 > 2 \ell \cdot |\epsilon(\vec{p})|/|\vec{p}|. \]  
(31)

The experimentally controllable detuning $\delta \omega$ of counter propagating waves $\omega_f$ and $\omega_b$ by rotational Doppler effect [27] which leads to the change the angular velocity of helix rotation makes possible to determine the critical velocity of superfluid, defined by contact point of roton minimum of $\epsilon(\vec{p})$ with the line $|\vec{p}| V_\theta$. The turbulent excitations (rotons) are assumed to appear due to ejection of superfluid across the trapping potential barrier owing to centrifugal force, rather than because of the roughness or the channel end [18, 22].

V. CONCLUSION

The flow of the degenerate quantum gas in helical trap had been studied analytically in the framework of the Gross-Pitaevskii equation. The necessary conditions were formulated for the appearance of the helical Bose-Einstein condensate flows due to the Thomas-Fermi balance of the self-defocusing of condensate with positive scattering length $a_s$ and "red" detuned optical dipole potential. The minimal achievable ensemble temperature might be approximately evaluated as a recoil one and "red" detuned optical dipole potential. The turbulent excitations (rotons) are assumed to appear due to ejection of superfluid across the trapping potential barrier owing to centrifugal force, rather than because of the roughness or the channel end.

The peculiarities of the cooling mechanisms in this helical configuration were not considered in the current work. But the helical interference pattern (see fig. 1) geometry might reveal the new features of the well elaborated mechanisms as a the Doppler cooling [33], polarization gradient cooling [17] or velocity selective population trapping [34]. The newly found loop and helical features of the optical speckle patterns [35, 36] are also a promising trapping opportunities which may enlighten the features of the Anderson localization of cold atoms in 1D and 3D speckle patterns [37].

Noteworthy the similar helical geometry of the colliding LG optical vortices of picosecond duration with opposite angular momenta proposed recently for the plasma currents excitation via ponderomotive force [23]. As the plasma vortices are the sources of the axial magnetic fields, the superfluid motion in helical trapping environment [39] is to be associated with a so-called artificial magnetic fields [3, 11].

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