Infinite-dimensional symmetries of a
two-dimensional generalized Burgers equation

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Abstract

The conditions for a generalized Burgers equation which a priori involves nine ar-
bitrary functions of one, or two variables to allow an infinite dimensional symmetry
algebra are determined. Though this algebra can involve up to two arbitrary functions
of time, it does not allow a Virasoro algebra. This result confirms that variable co-
efficient generalizations of a non-integrable equation should be expected to remain as
such.

1 Introduction

In Ref. [1] this author and Winternitz examined the conditions under which the gen-
eralized KP equation

\[
(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y, t)u_{yy} + a(y, t)u_y \\
+ b(y, t)u_{xy} + c(y, t)u_{xx} + e(y, t)u_x + f(y, t)u + h(y, t) = 0.
\]

(1.1)

has an infinite-dimensional symmetry algebra. In particular, it was shown that the
canonical form of (1.1) obtained for

\[
p = q = 1, \quad \sigma = \varepsilon = \pm 1, \quad e = h = 0
\]

(1.2)

allows the Virasoro algebra as a symmetry algebra if and only if the coefficients satisfy

\[
a = f = 0, \quad b = b(t), \quad c = c_0(t) + c_1(t)y.
\]

(1.3)

Under these conditions, the equation can be transformed by a point transformation to
the standard KP one. In the situation where the symmetry algebra becomes Kac-Moody
type, the equation is slightly more general and we have in addition to \(a = f = 0\)

\[
b(y, t) = b_1(t)y + b_0(t), \quad c(y, t) = c_2(t)y^2 + c_1(t)y + c_0(t).
\]

(1.4)

The purpose of this article is similar, namely to study the symmetry properties of
a generalized Burgers equation

\[
(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y, t)u_{yy} + a(y, t)u_y \\
+ b(y, t)u_{xy} + c(y, t)u_{xx} + e(y, t)u_x + f(y, t)u + h(y, t) = 0
\]

(1.5)

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where we assume that in some neighbourhood we have

\[ p(t) \neq 0, \quad q(t) \neq 0, \quad \sigma(y,t) \neq 0. \]  \hfill (1.6)

The other functions in (1.5) are arbitrary. We intend to determine the cases when eq. (1.5) has an infinite-dimensional symmetry group. More important, we would like to look at the possibility of whether it can have a Kac-Moody-Virasoro structure. The presence of a Virasoro algebra as a subalgebra may exhibit a strong indication of the integrability of the equation. Indeed, this was the case for the GKP equation (1.1)-(1.3).

Painleve analysis can be performed for variable coefficient partial differential equations to decide about integrability or partial integrability, but in spite of computer algebra packages developed for this aim, the computations involved for such equations usually turn up to be unmanageably lengthy.

Even though the algebra has no structure of a Virasoro algebra which is a typical property of integrable equations in 2+1 dimensions, the existence of an infinite-dimensional symmetry group makes it possible to use Lie group theory to obtain large classes of solutions.

The author also showed that the Lie symmetry algebra of the two-dimensional generalized Burgers equation [2, 3]

\[ (u_t + uu_x - u_{xx})_x + \sigma(t)u_{yy} = 0 \]  \hfill (1.7)

which is a special case of (1.5) in which

\[ p = 1, \quad q = -1, \quad \sigma(y,t) = \sigma(t), \quad a = b = c = e = f = h = 0 \]

has a non-Abelian Kac-Moody structure and for arbitrary \( \sigma \) is realized by

\[ \hat{V} = X(f) + Y(g), \]  \hfill (1.8)

\[ X(f) = f(t)\partial_x + \dot{f}(t)\partial_u, \]  \hfill (1.9a)

\[ Y(g) = g(t)\partial_y - \frac{\dot{g}(t)}{2\sigma(t)}y\partial_x - \frac{d}{dt}\left(\frac{\dot{g}(t)}{2\sigma(t)}\right)y\partial_u, \]  \hfill (1.9b)

where \( f(t) \) and \( g(t) \) are arbitrary smooth functions and the primes denote time derivatives. The algebra extends for several special forms of \( \sigma(t) \) (see [2, 3] for the details).

In Section 2 we introduce ”allowed transformations” that take equations of the form (1.5) into other equations of the same class. That is, they may change the unspecified functions in eq. (1.5), but not introduce other terms, or dependence on other variables. The allowed transformations are used to simplify eq. (1.5) and transform it into eq. (2.6) that we call the ”canonical generalized Burgers equation” (CGB equation). In Section 3 we determine the general form of the symmetry algebra of the CGB equation and obtain the determining equations for the symmetries. In Section 4 we look at the possibility if the CGB equation can be invariant under arbitrary reparametrization of time at all. Section 5 is devoted to the case when the CGB equation is invariant under a Kac-Moody algebra. Some conclusions are presented in Section 6.
2 Allowed transformations and a canonical generalized Burgers equation

We shall use "allowed transformations" (or equivalence transformations) to map \(1.5\) to some simple form. These transformations are defined to be (invertible) point transformations

\[
\tilde{x} = X(x, y, t, u), \quad \tilde{y} = Y(x, y, t, u), \quad \tilde{t} = T(x, y, t, u), \quad \tilde{u} = U(x, y, t, u),
\]

(2.1)
taking equations of the form \(1.5\) into another equations of the same form, but possibly with different coefficient functions. That is, the transformed equation will be the same as eq. \(1.5\), but the arbitrary functions can change. The typ ical features of the equation are that the new functions \(\tilde{p}(\tilde{t})\) and \(\tilde{q}(\tilde{t})\) depend on \(\tilde{t}\) alone, the others on \(\tilde{y}\) and \(\tilde{t}\), but no \(\tilde{x}\) dependence is introduced. The only \(\tilde{t}\)-derivative is \(\tilde{u}\tilde{t}\), the only nonlinear term is \(\tilde{p}(\tilde{t})(\tilde{u}\tilde{u})\tilde{x}\) and the only derivative higher than a second order one is \(\tilde{q}(\tilde{t})\tilde{u}\tilde{x}\tilde{x}\tilde{x}\). These form-preserving conditions restrict \(2.1\) to the form (the so-called local fiber-preserving transformations)

\[
\begin{align*}
& u(x, y, t) = R(t)\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) - \frac{\dot{\alpha}}{\alpha p} x + S(y, t), \\
& \tilde{x} = \alpha(t)x + \beta(y, t), \quad \tilde{y} = Y(y, t), \quad \tilde{t} = T(t), \\
& \alpha \neq 0, \quad R \neq 0 \quad Y_y \neq 0, \quad \dot{T} \neq 0, \quad \dot{\alpha}f(y, t) = 0.
\end{align*}
\]

(2.2)

We now choose the functions \(R(t), T(t)\) and \(Y(y, t)\) in eq. \(2.2\) to satisfy
\[
\dot{T}(t) = q(t)\alpha^2(t), \quad R(t) = \frac{q}{p}\alpha,
\]
\[
Y_y = \alpha^{3/2} \sqrt{\frac{q(t)}{|\sigma(y, t)|}},
\]
and thus normalize
\[
\tilde{\rho} (\tilde{t}) = 1, \quad \tilde{q} (\tilde{t}) = 1, \quad \tilde{\sigma} (\tilde{y}, \tilde{t}) = \varepsilon = \mp 1.
\]
By an appropriate choice of the functions \( \beta(y, t) \) and \( S(y, t) \) we can arrange to have
\[
\tilde{e}(\tilde{y}, \tilde{t}) = \tilde{h}(\tilde{y}, \tilde{t}) = 0.
\]
Finally, equation (1.5) is reduced to its canonical form
\[
(u_t + uu_x + u_{xx})_x + \varepsilon u_{yy} + a(y, t)u_y + b(y, t)u_{xy} + c(y, t)u_{xx} + f(y, t)u = 0, \quad \varepsilon = \pm 1.
\]
With no loss of generality we can restrict our study to symmetries of eq. (2.6). All results obtained for eq. (2.6) can be transformed into results for eq. (1.5), using the transformations (2.2). We shall call eq. (2.6) the “canonical generalized Burgers equation” (CGB).

We mention that Lie point transformations are particular cases of allowed transformations. When the form of the coefficients is preserved, allowed transformations coincide with symmetry transformations of the equation.

3 Determining equations for the symmetries

We restrict ourselves to Lie point symmetries. The Lie algebra of the symmetry group is realized by vector fields of the form
\[
\tilde{V} = \xi \partial_x + \eta \partial_y + \tau \partial_t + \phi \partial_u,
\]
where \( \xi, \eta, \tau \) and \( \phi \) are functions of \( x, y, t \) and \( u \). To determine the form of \( \tilde{V} \) we apply the standard infinitesimal algorithm (see, for instance, Olver’s book [4]) which basically consists of requiring that the third prolongation \( \text{pr}^{(3)} \tilde{V} \) of the vector field on the third jet space should annihilate the equation on its solution manifold. This requirement provides an overdetermined set of linear partial differential equations for the coefficients \( \xi, \eta, \tau \) and \( \phi \) in (3.1).

For eq. (2.6) these equations which do not involve the functions \( a, b, c \) and \( f \) can be solved and find that the general element of the symmetry algebra has the form
\[
\tilde{V} = \tau(t)\partial_t + \left( \frac{3}{2}\tau x + \xi_0(y, t) \right) \partial_x + \left( \frac{3}{4}\tau y + \eta_0(t) \right) \partial_y + \left( -\frac{1}{2}\tau u + \frac{1}{2}\tau x + S(y, t) \right) \partial_u,
\]
where
\[
S(y, t) = -\tau c_t - \left( \frac{3}{4}\tau y + \eta_0 \right) c_y + \xi_0, t + b \xi_0, y - \frac{1}{2} c_x. \quad (3.3)
\]
The remaining determining equations for $\tau(t)$, $\eta(t)$ and $\xi_0(y,t)$ are

\begin{align*}
4\tau a_t + (3\dot{\tau}y + 4\eta_0)a_y + 3a\dot{\tau} &= 0, \\
-4\dot{\eta}_0 - 3y\dot{\tau} + 4\tau b_t + (3\dot{\tau}y + 4\eta_0)b_y + b\dot{\tau} - 8\varepsilon\xi_{0,y} &= 0, \\
(a\xi_{0,y} + \varepsilon\xi_{0,yy}) &= 0, \\
f\dddot{\tau} &= 0, \\
6f\ddot{\tau} + 4f_t\tau + f_y(3\dot{\tau}y + 4\eta_0) &= 0, \\
2\dddot{\tau} + 4fS + 4aS_y + 4\varepsilon S_{yy} &= 0.
\end{align*}

At this juncture, there are different directions to go for dealing with determining equations. One is to perform a complete symmetry analysis of eqs. (3.3),..., (3.9) for arbitrary (given) functions $a$, $b$, $c$ and $f$. Of course, one can well proceed to determine the coefficients given that the equation is invariant under low-dimensional Lie algebras. Works in this direction for equations having dependence on several arbitrary functions of both independent and dependent variables and their derivatives exist in the literature (see for example [5, 6, 7]). This approach requires the knowledge of structural results on the classical Lie algebras. Here we shall take another approach and determine the conditions on these functions that permit the symmetry algebra to be infinite-dimensional. This will happen when at least one of the functions $\tau(t)$, $\eta_0(t)$ and $\xi_0(y,t)$ remains an arbitrary function of at least one variable.

\section{Search for the Virasoro symmetries of the CGB equation}

We are looking for conditions on the coefficients $a, b, c$ and $f$ that allow equations (3.4),..., (3.9) to be solved without imposing any conditions on $\tau(t)$. Below we shall see that this can not be realized for any possible choice of the coefficients.

From eq. (3.7) we see that $\tau$ is linear in $t$, unless we have $f(y,t) \equiv 0$. Once this condition is imposed, equations (3.7) and (3.8) are solved identically. Eq. (3.4) leaves $\tau(t)$ free if either we have $a = 0$, or $a = a_0(y + \lambda(t))^{-1}$ where $a_0 \neq 0$ is a constant and $\lambda(t)$ is some function of $t$. We investigate the two cases separately. First let us assume

\begin{equation}
a = \frac{a_0}{y + \lambda(t)}, \quad a_0 \neq 0.
\end{equation}

Then we view eq. (3.4) as an equation for $\eta_0(t)$ and obtain

\begin{equation}
\eta_0(t) = \frac{1}{3}(2\lambda\ddot{\tau} - 3\dot{\lambda}\tau).
\end{equation}

From eq. (3.6) we see $\xi_0(y,t)$ may be an arbitrary function of $t$, but never of $y$ (we have $\varepsilon = \pm 1$). Three possibilities for $\xi_0(y,t)$ occur:

1.) $a_0\varepsilon \neq \pm 1$
\[ \xi_0 = \frac{1}{1 - a_0 \varepsilon} \mu_1(t)(y + \lambda)^{-a_0 \varepsilon + 1} + \mu_0(t). \] (4.3)

2.) \( a_0 \varepsilon = 1 \)

\[ \xi_0 = \mu_1(t) \ln(y + \lambda) + \mu_0(t). \] (4.4)

3.) \( a_0 \varepsilon = -1 \)

\[ \xi_0 = \mu_1(t)(y + \lambda)^2 + \mu_0(t). \] (4.5)

We must now put \( \xi_0 \) of (4.3), (4.4) or (4.5) into eq. (3.5) and solve the obtained equation for \( \mu_1(t) \). The expression for \( \mu_1(t) \) must be independent of \( y \) for all values of \( \tau \). Moreover, for \( \tau(t) \) to remain free, there must be no relation between \( b(y,t) \) and \( \tau(t) \). These conditions cannot be satisfied for any value of \( a_0 \varepsilon \). Hence, if \( a(y,t) \) is as in eq. (4.1) the generalized Burgers equation (2.6) does not allow a Virasoro algebra.

The other case to consider is \( a = 0 \) (in addition to \( f = 0 \)). Eq. (3.6) is easily solved in this case and we obtain

\[ \xi_0(y,t) = \mu_1(t)y + \mu_0(t) \] (4.6)

with \( \mu_1(t) \) and \( \mu_0(t) \) arbitrary. We insert \( \xi_0(y,t) \) into eq. (3.5) and try to solve for \( \mu_1(t) \). This is possible if and only if we have \( b = b_1(t)y + b_0(t) \). On the other hand, the \( y \) independent coefficient of (3.5) restricts the form of \( \tau \) which implies that no Virasoro algebra can exist at all. In the following analysis we shall see that in that case there can exist at most two arbitrary functions.

**Theorem 1** The canonical generalized Burgers equation (2.6) can never allow the Virasoro algebra as a symmetry algebra for any choice of the coefficients.

5 Kac-Moody symmetries of the CGB equation

In section 4 we have shown that the symmetry algebra of the canonical generalized Burgers equation cannot contain a Virasoro algebra. In this section we will determine the conditions on the functions \( a(y,t) \), \( b(y,t) \), \( c(y,t) \) and \( f(y,t) \) under which the CGB equation only allows a Kac-Moody algebra. Thus, the function \( \tau(t) \) will not be free, but \( \eta_0(t) \) of eq. (3.2) will be free, or \( \xi_0(y,t) \) will involve at least one free function of \( t \).

5.1 The function \( \eta_0(t) \) free

Eq. (3.4) will relate \( \eta_0 \) and \( a(y,t) \) unless we have \( a_y = 0 \). Hence we put \( a_y = 0 \). For \( a = a(t) \neq 0 \) eq. (3.4) implies \( \tau(t) = \tau_0 a^{-4/3} \). Eq. (3.6) yields

\[ \xi_0(y,t) = \xi_1(t)e^{-a_y} + \xi_0(t). \]

Eq. (3.5) then provides a relation between \( \eta_0(t) \) and \( b(y,t) \). Hence \( \eta_0(t) \) is not free. Thus, if \( \eta_0(t) \) is to be a free function, we must have \( a(y,t) = 0 \). Eq. (3.4) is satisfied identically. From eq. (3.6) we have

\[ \xi_0(y,t) = \rho(t)y + \sigma(t). \] (5.1)
Eq. (3.5) will leave $\eta_0$ free only if we have

$$b(y, t) = b_1(t)y + b_0(t),$$

$$\rho(t) = \frac{\varepsilon}{8}(-4\eta_0 + 4\tau b_0 + 4\eta_0 b_1 + b_0 \dot{\tau}),$$

$$3\ddot{\tau} - 4(\tau b_1) = 0. \tag{5.4}$$

For $f \neq 0$ we have $\dot{\tau} = 0$ and eq. (3.9) will relate $\eta_0(t)$ to $c(y, t)$, $b_1$ and $b_0$. Thus, for $\eta_0(t)$ to be free, we must have $f(y, t) = 0$. Eq. (3.9) reduces to

$$-2\varepsilon \dddot{\tau} + \tau (8c_{yy} + 3y c_{yyy}) + 4(\tau c_{yyt} + \eta_0 c_{yyyy}) = 0.$$  

For $\eta_0(t)$ to be free, we must have $c(y, t) = c_2(t)y^2 + c_1(t)y + c_0(t),$  

$$-2\varepsilon \dddot{\tau} + (8\tau \dot{c}_2 + 16\dot{\tau} c_2) = 0. \tag{5.6}$$

The only equation that remains to be solved is eq. (5.6). Both functions $\eta_0(t)$ and $\sigma(t)$ remain free. The most general CGB equation allowing $\eta_0(t)$ to be a free function is obtained if eq. (5.6) is solved identically by putting $\tau = 0$. Then $\eta_0(t)$ and $\sigma(t)$ are arbitrary. On the other hand, from (5.6) we see that $\tau(t)$ can not remain free. This again implies that the symmetry algebra can by no means be Virasoro type. Using eq. (3.2) and the above results with the identification $\eta = \eta_0$, $\xi = \sigma$ we obtain the following theorem.

**Theorem 2** The equation

$$(u_t + uu_x + u_{xx})_x + \varepsilon u_{yy} + (b_1(t)y + b_0(t))u_{xy} + (c_2(t)y^2 + c_1(t)y + c_0(t))u_{xx} = 0, \tag{5.7}$$

where $\varepsilon = \pm 1$ and $b_0, b_1, c_0, c_1, c_2$ are arbitrary functions of $t$, is the most general canonical generalized Burgers equation, invariant under an infinite-dimensional Lie point symmetry group depending on two arbitrary functions. Its Lie algebra has a Kac-Moody structure and is realized by vector fields of the form

$$\hat{V} = X(\xi) + Y(\eta), \tag{5.8}$$

where $\xi(t)$ and $\eta(t)$ are arbitrary smooth functions of time and

$$X(\xi) = \xi \partial_x + \dot{\xi} \partial_u, \tag{5.9}$$

$$Y(\eta) = \eta \partial_y + \frac{\varepsilon}{2}y(-\dot{\eta} + b_1 \eta)\partial_x + \{-2c_2 \eta$$

$$+ \frac{\varepsilon}{2}(-\dot{\eta} + b_1 \eta + b_1^2 \eta)\}y - c_1 \eta + \frac{\varepsilon}{2}b_0(-\dot{\eta} + b_1 \eta)\partial_u. \tag{5.10}$$

Several comments are in order:

1. Surprisingly enough, the Kac-Moody symmetry algebras of the canonical generalized KP Eq. (1.1)-(1.2) with $b, c$ given by (1.4) and Burgers (Eq. (5.7)) equations coincide.
2. As in the GKP case, equation (5.7) can be further simplified by allowed transformations. Indeed, let us restrict the transformation (2.2) to

\[ u(x, y, t) = \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) + S_1(t)y + S_0(t), \]
\[ \tilde{x} = x + \beta_1(t)y + \beta_0(t), \quad \tilde{y} = y + \gamma(t), \quad \tilde{t} = t. \]

(5.11)

For any functions \( b_1(t) \) and \( c_2(t) \) we can choose \( S_1, S_0, \beta_0, \beta_1 \) and \( \gamma \) to set \( b_0, c_1 \) and \( c_0 \) equal to zero. Thus, with no loss of generality, we can set

\[ b_0(t) = c_1(t) = c_0(t) = 0 \] (5.12)
in eq. (5.7), (5.9) and (5.10).

3. Let us now consider the cases when eq. (5.7) has an additional symmetry. To do this we should solve equations (5.4)-(5.6).

Case 1. \( b_1 = 0, c_2 \neq 0 \)

We assume (5.12) is already satisfied. From (5.4) we have \( \tau = \tau_1 t + \tau_0 \) and from (5.6)

\[ c_2 = k\tau^{-2} = k(\tau_1 t + \tau_0)^{-2} \]

where \( \tau_1, \tau_0, k \) are constants. The additional symmetry is

\[ T = (\tau_1 t + \tau_0)\partial_t + \frac{1}{2}\tau_1 x\partial_x + \frac{3}{4}\tau_1 y\partial_y - \frac{1}{2}\tau_1 u\partial_u. \]

(5.13)

Under translation of \( t \), it is equivalent to the dilatational symmetry

\[ D = t\partial_t + \frac{1}{2}x\partial_x + \frac{3}{4}y\partial_y - \frac{1}{2}u\partial_u. \]

Case 2. \( b_1 \neq 0, b_0 = c_1 = c_0 = 0 \)

Eq. (5.4) can be integrated to give a first order linear equation for \( \tau \) in terms of \( b_1 \) and eq. (5.6) provides the constraint between \( b_1 \) and \( c_2 \)

\[ \frac{d}{dt}(\tau^2 c_2) = \varepsilon \frac{d^2}{dt^2}(\tau b_1). \]

(5.14)

The additional element of the symmetry algebra in this case is

\[ T = \tau\partial_t + \frac{1}{2}\tau x\partial_x + \frac{3}{4}\tau y\partial_y + \frac{1}{2}\tau x - (\tau c_2 + 2c_2\dot{\tau}) y^2 - \frac{1}{2}\tau u\partial_u \]

(5.15)

with \( \tau \) being a solution of

\[ \ddot{\tau} - \frac{4}{3}b_1\tau = k. \]

5.2 One free function in symmetry algebra

We have established that if \( \tau(t) \) is free in eq. (5.2), then there are three free functions. If \( \tau \) is not free, but \( \eta_0(t) \) is, then there are two free functions. Now let \( \tau(t) \) and \( \eta_0(t) \) be constrained by the determining equations, but let some freedom remain in the function \( \xi_0(y, t) \).
First of all we note that if we put 
\[ \tau = 0, \quad \eta_0 = 0, \quad \xi_0(y, t) = \xi(t) \] (5.16)
in eq. (3.2) then eqs. (3.4), ..., (3.8) are satisfied identically and eq. (3.9) reduces to
\[ f\dot{\xi} = 0. \] (5.17)
Hence
\[ X(\xi) = \xi(t)\partial_x + \dot{\xi}(t)\partial_u, \] (5.18)
with \( \xi(t) \) arbitrary, generates Lie point symmetries of the CGB equation for \( f(y, t) = 0 \) and any functions \( a(y, t), b(y, t), \) and \( c(y, t) \).

For \( f \neq 0 \) we have \( \tau = \tau_1t + \tau_0 \) from eq. (3.7). Eq. (3.6) then determines the \( y \) dependence of \( \xi_0 \).

We skip the details here and just state that the remaining equations (3.5), (3.8) and (3.9) do not allow any solutions with free functions.

We state this result as a theorem.

**Theorem 3** The CGB equation (2.6) is invariant under an infinite-dimensional Abelian group generated by the vector field (5.18) for \( f(y, t) = 0 \) and \( a, b, c \) arbitrary.

Theorems 2 and 3 sum up all cases when the symmetry algebra of the CGB equation is infinite-dimensional.

### 6 Applications and conclusions

We have identified all cases when the generalized Burgers equation has an infinite-dimensional symmetry group. Let us now discuss the implications of this result.

#### 6.1 Equation with nonabelian Kac-Moody symmetry algebra

The symmetry algebra (5.8) of eq. (5.7) is infinite-dimensional and nonabelian. Indeed, we have
\[ [Y(\eta_1), Y(\eta_2)] = X(\xi), \quad \xi = -\frac{\varepsilon}{2}(\eta_1\dot{\eta}_2 - \dot{\eta}_1\eta_2). \] (6.1)

We can apply the method of symmetry reduction to obtain particular solutions. The operator \( X(\xi) \) of eq. (5.9) generates the transformations
\[ \tilde{x} = x + \lambda\xi(t), \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = u(x, y, t) + \lambda\dot{\xi}(t), \] (6.2)
where \( \lambda \) is a group parameter. We see that (6.2) is a transformation to a frame moving with an arbitrary acceleration in the \( x \) direction. For \( \xi \) constant this is a translation, for \( \xi \) linear in \( t \) this is a Galilei transformation. An invariant solution will have the form
\[ u = \frac{\dot{\xi}}{\xi} x + F(y, t). \] (6.3)
Substituting into eq. (5.7) we obtain the family of solutions
\[ u = \frac{\dot{\xi}}{\xi x} - \frac{\varepsilon}{2} \frac{\dot{\xi}}{\xi y^2} + \rho(t)y + \sigma(t) \]  
(6.4)
with \(\rho(t)\) and \(\sigma(t)\) arbitrary.

The transformation corresponding to the general element \(Y(\eta) + X(\xi)\) with \(\eta \neq 0\) is easy to obtain, but more difficult to interpret. An invariant solution will have the form
\[ u = [-c + \frac{\varepsilon}{4}(\dot{b} + b^2 - \frac{\ddot{\eta}}{\eta})]y^2 + \frac{\dot{\xi}}{\eta}y + F(z,t) \]
\[ z = x + \frac{\varepsilon}{4}(-b + \frac{\ddot{\eta}}{\eta})y^2 - \frac{\xi}{\eta}y. \]
(6.5)

We have put \(b_1 = b, c_2 = c, b_0 = c_1 = c_0 = 0\), which can be done with no loss of generality. We now put \(u\) of eq. (6.5) into eq. (5.7) (for \(c_1 = c_0 = b_0 = 0\)) and obtain the reduced equation
\[(F_t + FF_z + F_{zz})z + \frac{\varepsilon}{\eta^2}\dot{\xi}F_{zz} + \frac{1}{2}(\frac{\ddot{\xi}}{\eta} - b)F_z - 2\varepsilon\varepsilon + \frac{1}{2}(\dot{b} + b^2 - \frac{\ddot{\eta}}{\eta}) = 0. \]
(6.6)

Putting
\[ F(z,t) = \tilde{F}(\tilde{z},\tilde{t}), \quad \tilde{z} = z + \beta(t), \quad \tilde{t} = t, \]
\[ \dot{\beta}(t) = -\frac{\varepsilon}{\eta^2} \]
(6.7)
we eliminate the \(F_{zz}\) term. Choosing \(\dot{\eta}/\eta = b(t)\) we obtain the equation
\[(F_t + FF_z + F_{zz})z = 2\varepsilon c(t), \]
(6.8)
an equation that is not integrable (for \(c \neq 0\)). We note that for \(c = 0\) (6.8) reduces to the 1-dimensional Burgers equation.

6.2 Comments

By the results of this paper we have shown that neither 2+1-dimensional Burgers equation nor its generalizations of the form (1.5) can allow a Virasoro type symmetry group. The largest infinite-dimensional symmetry allowed can be Kac-Moody type. In addition to this, for specific choice of the coefficients it has one more symmetry. It should also be worthwhile reiterating the overlapping of the Kac-Moody symmetries allowed by generalized KP and Burgers equations.

The most ubiquitous symmetry of the generalized Burgers equation is the transformation (6.2) to an arbitrary frame moving in the \(x\) direction. Its presence only requires the coefficient \(f(y,t)\) in eq. (1.3) for \(p = 1\) or in (2.6) to be \(f(y,t) \equiv 0\). Invariance of a solution under such a general transformation is very restrictive and leads to solutions that are at most linear in the variable \(x\) and have a prescribed \(y\) dependence (see solutions (6.4)).
The transformations generated by \( Y(\eta) \) leave a more restricted class of generalized Burgers equations invariant, those of eq. (5.7). The invariant solutions have the form (6.5). They are obtained by solving the reduced equation eq. (6.6) with \( F_{zz} \) transformed away or (6.8). For general \( c(t) \), this is difficult, but for \( c(t) = 0 \) this is just the Burgers equation, for arbitrary \( b(t) \), as long as we choose \( \dot{\eta}/\eta = b(t) \). Any solution of the Burgers equation will, via eq. (6.5), provide \( y \) dependent solutions of the corresponding generalized Burgers equation.

One-dimensional additional subalgebras can be imbedded into Kac-Moody subalgebras to form two-dimensional subalgebras. Invariance under them will lead to reductions to ODEs (see [2]).

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