Psi-series method in random trees and moments of high orders

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Abstract
An unusual and surprising expansion of the form
\[ p_n = \rho^{-n-1} \left( 6n + \frac{18}{5} \frac{336}{3125} n^{-5} + \frac{1008}{3125} n^{-6} + \text{smaller order terms} \right), \]
as \( n \to \infty \), is derived for the probability \( p_n \) that two randomly chosen binary search trees are identical (in shape and in labels of all corresponding nodes). A quantity arising in the analysis of phylogenetic trees is also proved to have a similar asymptotic expansion. Our method of proof is new in the literature of discrete probability and analysis of algorithms, and based on the psi-series expansions for nonlinear differential equations. Such an approach is very general and applicable to many other problems involving nonlinear differential equations; many examples are discussed and several attractive phenomena are discovered.

Key words. Psi-series method, nonlinear differential equations, random trees, recursive structures, singularity analysis, asymptotic analysis.

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1 Introduction

The motivating problem. This paper was originally motivated by the following problem. Find the asymptotics of the sequence \( p_n \) defined recursively by
\[ p_n = n^{-2} \sum_{0 \leq j < n} p_j p_{n-1-j} \quad (n \geq 1). \]
with the initial condition $p_0 = 1$. The sequence $p_n$ is nothing but the probability that two randomly chosen binary search trees (BSTs) of size $n$ are identical (having exactly the same shape and with the same labels for corresponding nodes), and was first studied by Martínez in [26] as an auxiliary function for understanding the typical performance of the equality test of two random BSTs; see below for more background details. A minor variation of this sequence was encountered in the analysis of maximum agreement subtrees in [7] under the Yule-Harding model.

While shape parameters defined on a single random tree has been extensively studied in the literature for many varieties of trees, properties of statistics defined on a pair or multiple of random trees received comparatively less attention, partly because of the intrinsic complexity of the underlying analytic problems. Yet many practical situations (such as tanglegrams) naturally lead to such a study, typical example being the so-called “hereditary properties” or “recurrent properties”, which in turn cover the equality, root occurrence, simplification rules, reduction rules, “clashes” as special cases; see [26, 31, 14] for more details.

Recently, there has been more study of statistics defined on two random combinatorial objects; see [6] and the references therein.

**Random BSTs.** For completeness, we first describe BSTs. Given a sequence of distinct numbers $\{x_1, \ldots, x_n\}$, we can construct the corresponding BST as follows. If $n = 0$, then the tree is empty. If $n \geq 1$, then we place $x_1$ at the root; the remaining numbers are compared one after another with $x_1$, and are directed to the left subtree of the root if they are smaller, to the right subtree if larger. Numbers directed to each subtree are constructed recursively by the same procedure according to their original order; see Figure 1 for a plot.

![Figure 1](image)

**Definition: [Equality of two ordered, labeled trees]**. Two ordered, labeled trees of the same size (total number of nodes) are said to be equal or identical if either both trees are empty or they have common root label with all corresponding ordered subtrees equal.

The definition extends to the equality of $d$ trees with $d \geq 2$.

Now we take two random BSTs independently, and our $p_n$ gives the probability that the two trees are identical. Equivalently, we take two random permutations of $n$ elements; then $p_n$
denotes the probability that the BSTs constructed from these two permutations are equal. (A simple example: (2, 1, 3) and (2, 3, 1) lead to the same BST of the shape \(\begin{array}{c}2 \\ \_ \\ 3 \end{array}\).)

**A simple upper bound.** The simple-looking recurrence (1) can be quickly estimated by the following inductive argument. If we assume the form 

\[ p_n \leq c(n+1)q^{-(n+1)} \]

for \( n \geq 0 \), then we see by induction that

\[ p_n \leq \frac{c^2}{n^2} q^{-(n+1)} \sum_{0 \leq j < n} (j+1)(n-j) = \frac{c^2(n+1)(n+2)}{6n} q^{-(n+1)}. \]

In order that the rightmost term is less than \( c(n+1)q^{-(n+1)} \), we can take a positive integer \( n_0 \), let \( \rho := \frac{6n_0}{n_0+2} \), and then choose \( q \) as

\[ q := \min_{0 \leq j \leq n_0} \left( \frac{6n_0(j+1)}{p_j(n_0+2)} \right)^{1/(j+1)}. \]

Then we obtain

\[ p_n \leq \frac{6n_0}{n_0+2} (n+1)q^{-(n+1)}, \]

for all \( n \geq 0 \). This gives successively improving bounds for \( q \) for increasing values of \( n_0 \); see Table 1 where we take only the first four digits after the decimal point without rounding. In particular, taking \( n_0 = 6 \) leads to the bound \( p_n \leq \frac{4}{3} (n+1)3^{-n} \). The simple bound (2) obtained by induction and numerical evidence suggest the possibility that 

\[ p_n \sim 6n\rho^{-(n+1)} \]

for some values of \( \rho \approx 3.14 \) (see Figure 2). How to prove this? And is \( \rho = \pi \)?

**The nonlinear differential equation.** As the elementary argument we used above is not strong enough to derive more precise asymptotic approximations to \( p_n \), we consider instead the generating function \( P(z) := \sum_{n \geq 0} p_n z^n \), which satisfies the nonlinear differential equation (abbreviated throughout as DE)

\[ zP''(z) + P'(z) = P(z)^2, \]

with the initial conditions \( P(0) = P'(0) = 1 \). This nonlinear DE is of Emden-Fowler type for which there is no explicit closed form solution; see [29]. In addition to the apparent singularity determined by the equation, the DE (3) also has singularities determined by the initial conditions, which are often referred to as the movable singularities.

| \( n_0 \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| \( q \) | 2 | 2.4494 | 2.6832 | 2.8284 | 2.9277 | 3 | 3.0274 | 3.0488 | 3.0659 |
| \( n_0 \) | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
| \( q \) | 3.0794 | 3.1235 | 3.1328 | 3.1362 | 3.1378 | 3.1387 | 3.1393 | 3.1396 | 3.1399 |

Table 1: Numerical values of \( q \).
Frobenius method. Starting from the DE (3), the next step is often to apply the Frobenius method (see [23]), namely, we assume the solution of $P(z)$ to be of the form

$$P(z) = \sum_{j \geq 0} c_j (1 - z/\rho)^{j-\alpha},$$

for some $\alpha$ and $\rho > 0$, substitute this form into (3), and then determine $\alpha$ and the coefficients $c_j$ inductively one after another. This classical procedure yields $\alpha = 2, c_0 = 6/\rho$,

$$c_1 = -\frac{12}{5\rho}, c_2 = -\frac{7}{25\rho}, c_3 = -\frac{14}{125\rho}, c_4 = -\frac{63}{1250\rho}, c_5 = -\frac{161}{9375\rho}.$$ (5)

But then inconsistency arises since the coefficient of $(1 - z/\rho)^4$ on

$$\text{LHS of (3)} = \rho^2 \left(12c_6 + \frac{483}{3125}\right) \neq \text{RHS of (3)} = \rho^2 \left(12c_6 + \frac{77}{625}\right),$$ (6)

and $c_6$ cannot be determined by simply matching the coefficients of both sides. This trial suggests that the local expansion of $P$ near the singularity $\rho$ will not be of the form (4) and means that the classical Frobenius method fails for the nonlinear DE (3).

Psi-series method. We will introduce a different type of expansion called psi-series expansion (or Painlevé expansion; see [22]) and it will turn out that $P(z)$ admits an asymptotic expansion of the form

$$U(Z) := \sum_{j \geq 0} Z^{j-2} \sum_{0 \leq \ell \leq [j/6]} c_{j,\ell} (\log Z)\ell, \quad Z := 1 - z/\rho,$$ (7)

when $z$ lies near the singularity $\rho$. This form, first conjectured by Martínez in [27] Ch. 9], also explains why the expansion (4) leads to inconsistency. Thus $z = \rho$ is not a pole but instead a pseudo-pole; see [22]. The first few terms of $U(Z)$ are given as follows.

$$\rho U(Z) = 6 Z^{-2} - \frac{12}{5} Z^{-1} - \frac{7}{25} Z - \frac{14}{125} Z^2 - \frac{63}{1250} Z^3 - \frac{161}{9375} Z^4 + \rho c_6 Z^4 + \rho \sum_{j \geq 7} \sum_{0 \leq \ell \leq [j/6]} c_{j,\ell} Z^{j-2} \log^\ell Z,$$ (8)
for $Z$ small, where $c_6 := c_{6,0}$ and the $c_{j,\ell}$'s are polynomials of the parameter $c_6 \rho$ with degree $\lceil(j - 6\ell)/6\rceil$ for $j \geq 7$.

The approach we use in this paper is roughly as follows. After checking the failure of Frobenius method, we construct a suitable psi-series $U(Z)$ (by matching coefficients) so that $U$ satisfies formally the DE (3). The series in (7) is a priori an asymptotic expansion, but we will show that it is indeed absolutely convergent in the cut-disk $|Z| \leq 1 - \varepsilon$, $Z \notin [-1 + \varepsilon, 0]$. Thus the function $U$ is well defined there and satisfies the DE (3) and differs from $P$ only by their initial conditions. Such a procedure still leaves undetermined two important parameters (similar to the initial conditions of the DE (3)), one is obviously $\rho$ and the other implicit one is $c_6 := c_{6,0}$ due to the same reason as the Frobenius method. This means that $U$ is not only a function of $Z$, but also a function of $\rho$ and $c_6$.

Now to fix $U$ in a unique way, we connect $P(z)$ and $U(Z)$ by first choosing a number $z_0 \in [\varepsilon \rho, \rho - \varepsilon]$, and by considering the solution $(\rho, c_6)$ of the two equations

\[
\begin{cases}
U(Z_0) = P(z_0) \\
U'(Z_0) = -\rho P'(z_0),
\end{cases}
\]

where $Z_0 := 1 - z_0/\rho$. We will show below (Proposition 1) that, as a function of $Z$ (or $\rho$) and $c_6$, the series $U$ has a nonzero radius of convergence for each finite $c_6$. Also we can easily derive simple upper and lower bounds for $\rho$ as above. Thus, as a standard initial-value problem, the system of equations (9) has a unique solution pair of $(\rho, c_6)$. This determines uniquely the pair $(\rho, c_6)$. Furthermore, $P$ and $U$ have a common region of analyticity, and we see by analytic continuation that $U$ is the exact and asymptotic solution we have been looking for.

Although no analytic forms for $\rho$ and $c_6$ are available, we can compute the numerical values of $\rho$ and $c_6$ as follows. First, the values of $U(Z_0)$ and $U'(Z_0)$ can be well approximated by their partial sums since the terms of the series converge in an exponential rate; see (17); similarly, the values of $P(z_0)$ and $P'(z_0)$ can be computed by first computing $p_n$ by its defining recurrence and then summing a sufficiently large number of initial terms up, the convergence rate being also exponential. Then we solve successively the corresponding system of equations by using an increasing number of terms in the partial sums; see next section for details.

Asymptotics of $p_n$. From the expansion (7) and suitable analytic continuation to be clarified below, we deduce our main result for $p_n$.

**Theorem 1** The probability $p_n$ that two randomly chosen binary search trees of $n$ nodes are equal satisfies the asymptotic expansion

\[
p_n \sim \rho^{-n-1} \left( 6n + \frac{18}{5} + \sum_{j \geq 6} n^{-j+1} \sum_{0 \leq \ell < [j/6]} C_{j,\ell} (\log n)^\ell \right),
\]

for explicitly computable constants $C_{j,\ell}$, where $\rho = 3.14085\,75672\,02936\,95160 \ldots$
Thus $\rho \neq \pi$. In particular, the first few terms read

$$p_n = \rho^{-n-1} \left( 6n + \frac{18}{5} + \frac{336}{3125 n^5} + \frac{1008}{3125 n^6} + \frac{10416}{15625 n^7} + \frac{91728}{78125 n^8} + \frac{8234352}{4296875 n^9} + \frac{12228048}{4296875 n^{10}} + \frac{1}{n^{11}} \left( \frac{9483264}{5078125} H_n + \frac{5621191632}{726171875} + \frac{677376}{1625 c_6} \right) \right) + O \left( \frac{\log n}{n^{12}} \right),$$

where $H_n := \sum_{1 \leq j \leq n} j^{-1}$, and we see that no terms of the form $cn^{-j}$ with $j = 1, \ldots, 4$ appear in the expansion. Numerically, the parameter $c_6$ can be determined approximately as $c_6 = -0.00150 84982 09405 93425 \ldots$; see the numerical discussions on Page 14 for details.

As far as we were aware, the asymptotic expansion (10) with missing terms is rare in the analysis of algorithms and applied probability literature. The expansion also indicates that the approximation of $p_n \rho^{n+1}$ by the first two terms $6n + 18/5$ is numerically very precise as can be seen in Figure 2.

**Features.** In addition to the unusual form of (10) and its theoretical value per se, the interest of such a psi-series expansion is multifold. First, since no analytic form for the movable singularity $\rho$ is available, the psi-series expansion provides an effective means for obtaining an approximate value to $\rho$ by the argument we mentioned above; see (22) below for more numerical details. Second, from a methodological point of view, the method of proof we use to prove (10) is of some generality. Note that the first two terms on the right-hand side of (10) can be easily obtained by the method of matched coefficients once we assume that $p_n$ has the form (10). Third, the precise approximation we derive has direct consequences in the original motivating problem, as well as several others in the examples we discuss below. Fourth, such a consideration leads to several interesting and unexpected phenomena as we will see in the following sections.

**Outline of this paper.** We describe the psi-series method and give the proof of the asymptotic expansion (10) in the next section. Then we extend in Section 3 the consideration of the probability of equality to either more than two random BSTs or to other variants of BSTs. It turns out that the forms of the asymptotic expansion for the probability of equality of $d$ random BSTs differ drastically according to the parity of $d$, a result not intuitively obvious. Section 3.2 considers the case of two random $m$-ary search trees and we will see that the number of missing terms in the asymptotic expansion increases as $m$ grows. Equality of two random fringe-balanced BSTs is considered in Section 3.3 and there, unlike $m$-ary search trees, the error term beyond the constant term in the asymptotic expansion does not change with the structural parameter once it exceeds one, another unexpected result. Asymptotics of higher-order moments will then be considered in Section 4 with a few representative examples taken from the cost of partial-match queries in random trees, random partition structures and solutions of Boltzmann equations (from statistical physics). We group the details of some proofs in Appendix.

**Notations.** For each problem studied, $\rho$ always denotes the dominant singularity of the associated nonlinear DE and $Z := 1 - z/\rho$. The symbols $c, c', c_j, c_{ij}, C, C_j, C'_j, C_{ij}, K, K'$ all denote suitably chosen constants, not necessarily the same at each occurrence.
2 Psi-series method

We discuss in details the psi-series solution to our nonlinear DE (3) and the tools needed to justify it, then we prove (10).

Analytic properties of $P(z)$. First, the solution $P(z)$ to the DE (3) has positive radius of convergence and is analytic at the apparent fixed singularity $z = 0$ by definition. By simple induction as we discussed in the introduction (Section 1) and Pringsheim’s theorem (since all coefficients $p_n$ are positive; see [19, p. 240]), we expect that $P(z)$ has a finite movable singularity at, say $z = \rho$, and the asymptotics of $p_n$ will be dictated by the local asymptotic expansion of $P(z)$ as $z \sim \rho$.

Martínez [27, p. 117] proved that the function $P(z)$, originally defined only inside the disk $|z| < \rho$ can be analytically continued to the cut-disk $|z| \leq \rho + \varepsilon \setminus [\rho, \rho + \varepsilon]$ with $\rho$ being the sole singularity there.

From a theoretic point of view, the movable singularity $\rho$ for the DE (3) can be either of the following types:

- poles,
- branch points (algebraic or logarithmic),
- essential singularity.

Simple poles and algebraic points are first excluded because of the above trial via Frobenius method. We then show that $P$ can be analytically continued into a function defined by a series expansion of the form (7) that converges absolutely in the cut-region

$$C_R := \{ z : 0 < |z - \rho| \leq R, z \notin [\rho, \rho + R] \},$$

for some $R > 0$. Thus the possibility that $\rho$ is an essential singularity is further excluded, and $\rho$ is a logarithmic branch point (or called pseudo-pole).

Our first focus in this paper is on the determination of the right form of the solution to (3). More detailed and complete introduction and discussions on the theory related to Painlevé analysis can be found in [9, 11] and the references therein.

The ARS method (Type checking). A widely used procedure to check the singularity type (and the local expansion) of nonlinear differential equations is the following procedure, often called the ARS algorithm due to Ablowitz, Ramani and Segur [11], which bears some resemblance to the Frobenius method.

In this method, we start assuming that the solution to the DE (3) admits the formal Laurent expansion (4) about the cut-disk $C_R$ for some positive number $R$.

1 Leading order analysis: Assume $P(z) \sim c_0(1 - z/\rho)^{-\alpha}$. By balancing the dominant terms $\rho P''(z)$ and $P(z)^2$ in (3), we see, as in Frobenius method, that $\alpha = 2$ and the companion constant $c_0 = 6/\rho$. Thus we can exclude the possibility of an algebraic singularity.
Resonance analysis: Starting from this pair \((\alpha, c_0) = (2, 6/\rho)\), if the solution admits only poles, then by substituting (4) into (3) and by equating coefficients, the coefficients \(c_j\)’s are characterized by the recurrence relation of the form

\[
\Phi(j)c_j = (j - 3)^2c_{j-1} + \rho \sum_{1 \leq n < j} c_j c_{n-j} =: G_j(\rho, c_0, c_1, \ldots, c_{j-1}), \quad j \geq 1, \tag{12}
\]

where \(\Phi(j) = (j + 1)(j - 6)\) and \(c_j = 0\) for all \(j < 0\). The roots of \(\Phi(j)\) are called resonance and \(-1\) is always a root of \(\Phi(j)\), reflecting the arbitrariness of the movable singularity \(\rho\). For most of our purposes, a less involved and very commonly used technique is to substitute the test function

\[
c_0(1 - z/\rho)^{-\alpha} + c_r(1 - z/\rho)^r-\alpha
\]

into the DE (3) instead. By collecting the coefficients corresponding to the term \(c_r(1 - z/\rho)^r-1\), we still get the same \(\alpha, c_0\) and \(\Phi(r)\). In this case, we see that \(\Phi\) has only one positive resonance \(6\) that needs to be further examined.

Compatibility: Once we have the system (12) and identify the resonance, the next step is to consider its solvability. Obviously, (4) is the solution to (3) if and only if all the coefficients \(c_k\)’s can be computed recursively by (12). This fact defines the compatibility of the resonance: for any resonance \(r\) of \(\Phi\), if \(G_r(\rho, c_0, c_1, \ldots, c_{r-1}) = 0\) is satisfied, then the resonance \(r\) is said to be compatible; otherwise, \(r\) is incompatible.

From (5) and (6) it follows that \(r = 6\) is incompatible. The formal series solution by introducing suitable logarithmic terms starting at the index 6 has to be considered instead (see (8)). The movable singularity \(\rho\) to (3) is proved to be a logarithmic branch point since we will show that the associated series solution is absolutely convergent in the region \(C_R\) for some \(R > 0\).

In cases when the compatibility of resonance is consistent, the solution of Laurent expansion is the one we need if it has a positive radius of convergence. The above ARS Algorithm is useful in determining if a nonlinear ODE admits the Painlevé property, namely, the DE has only solutions free from movable branch points. In our case, the DE (3) does not satisfy the Painlevé property.

Our approach vs the ARS algorithm. The method of proof we use does not, however, rely completely on this method for two reasons. First, it requires the \textit{a priori} information that \(\rho\) is not an essential singularity, a property often hard to prove. Second, even we can prove that the singularity is not essential, the incompatibility of a resonance (or several) may in some cases very difficult to establish due to the variation of an additional parameter as in the cases of \(d\) random BSTs (Subsection 3.1) and \(m\)-ary search trees (Subsection 3.2).

On the other hand, the ARS algorithm does provide an effective means of computing the exact form of the psi-series expansion for all the examples we discuss, notably the characterization of the resonance. We will thus use the ARS algorithm for two purposes: first, when the resonance equation has no positive integral resonance or when all resonances are compatible, then the solution is given by a Laurent expansion; second, when Laurent expansion fails, we use the ARS algorithm to guess the possible form of the psi-series expansion we are looking
for, and then the proof will be conducted along the same way we do for \( p_n \). Of course, there are also cases for which the ARS algorithm can be easily justified and the singularity is not essential (say, by the absolute convergence of the psi-series).

**Absolute convergence of the psi-series.** We now prove that \( U(Z) \) converges absolutely in a cut-disk \( \mathcal{C}_R \) for some positive \( R > 0 \).

**Proposition 1** For each fixed \( \delta_0 \), the psi-series expansion \( (\tau) \) converges absolutely for \( z \) in the cut-disk \( \mathcal{C}_{(1-\varepsilon)\rho} \) (defined in (11)), where \( \varepsilon > 0 \) is a small number.

The range \(|z - \rho| \leq (1 - \varepsilon)\rho\) is the best that our approach can achieve although it seems to hold true, by numerical evidence, up to \(|z - \rho| \leq \rho\); in particular, this suggests that the psi-series expansion be convergent even for \( Z = 1 \) or \( z = 0 \) for \( P(z) \).

From this proposition, we see that the solution \( P(z) \) can be analytically continued to at least the region

\[
\{ \{ z : |z| \leq \rho + \varepsilon \} \cup \{ z : |z - \rho| \leq (1 - \varepsilon)\rho \} \} \cup \{ \rho, (2 - \varepsilon)\rho \} \quad (\varepsilon > 0),
\]

from which we deduce (10).

To prove Proposition 1 we adopt an approach due to Hille [22] with some new ingredients; see also [21]. The resulting proof can then be extended to cover all the types of DEs we discuss in this paper, whatever their orders.

**Proof of the absolute convergence of the psi-series. I. Recurrence of \( u_k \).** We first rewrite the DE (3) for \( P \) into that for \( U \), which becomes

\[
((1 - Z)U'(Z))' = \rho U(Z)^2.
\]

For convenience, let \( U_0 = \rho U \). Then

\[
((1 - Z)U_0'(Z))' = U_0(Z)^2.
\]

As in [21], we then convert this DE into a first-order differential system by introducing an additional function \( V_0 := (1 - Z)U_0'(Z) \) as follows.

\[
\begin{cases}
U_0'(Z) = \frac{V_0(Z)}{1 - Z}, \\
V_0'(Z) = U_0(Z)^2.
\end{cases} \tag{13}
\]

Let \( \tau = \log Z, U_0(Z) = \sum_{k \geq 0} u_k(\tau) Z^{k-2} \) and \( V_0(Z) = \sum_{k \geq 0} v_k(\tau) Z^{k-3} \), where \( u_k \) and \( v_k \) are polynomials in \( \tau \) of degree at most \( \lfloor k/6 \rfloor \). Note that \((d\tau)/(dZ) = Z^{-1} \) and \( c_0 = 6/\rho \). From (13), we derive an infinite system of equations in \( k \) (\( \hat{u}_k := u_k'(\tau) \))

\[
\begin{cases}
\hat{u}_k + (k - 2)u_k = v_k + \sum_{0 \leq j < k} v_j, \\
\hat{v}_k + (k - 3)v_k = 12u_k + \sum_{1 \leq j < k} u_j u_{k-j}, \quad (k \geq 7).
\end{cases}
\]
We can further express the above system in terms of matrices as follows. Let
\[
\phi_k := \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad A_k := \begin{pmatrix} k - 2 & -1 \\ -12 & k - 3 \end{pmatrix}, \quad \text{and} \quad g_k := \begin{pmatrix} \sum_{0 \leq j < k} v_j \\ \sum_{1 \leq j < k} u_j u_{k-j} \end{pmatrix}.
\]

Then, for \( k \geq 7 \),
\[
\dot{\phi}_k + A_k \phi_k = g_k,
\]
which can be explicitly solved.

**Lemma 1** For \( k \geq 7 \), \( \phi_k \) admits a unique solution satisfying
\[
\lim_{\tau \to -\infty} \| e^{A_k \tau} \phi_k(\tau) \| = 0
\]
of the form
\[
\phi_k(\tau) = \int_{-\infty}^{\tau} e^{-A_k x} g_k(\tau - x) \, dx
\]
\[
= \int_{-\infty}^{\tau} P e^{-D x} P^{-1} g_k(\tau - x) \, dx,
\]
where \( D := \begin{pmatrix} k + 1 & 0 \\ 0 & k - 6 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix} \) and \( P^{-1} = \begin{pmatrix} 4 & -1 \\ 7 & 1 \end{pmatrix} \).

**Proof.** The fundamental matrix solution associated with the homogeneous part of (14) is \( e^{A_k \tau} \), so we can solve (14) by multiplying it by \( e^{x A_k} \) and then by using the fact that \( u_k(\tau) \) and \( v_k(\tau) \) are polynomials in \( \tau \), which gives
\[
e^{A_k x} \dot{\phi}_k(x) + A_k e^{A_k x} \phi_k(x) = \frac{d}{dx} \left( e^{A_k x} \phi_k(x) \right) = e^{A_k x} g_k(x).
\]
Integrating both sides from \(-\infty\) to \( \tau \), we get
\[
e^{A_k \tau} \phi_k \bigg|_{-\infty}^{\tau} = e^{A_k \tau} \phi_k(\tau) = \int_{-\infty}^{\tau} e^{A_k x} g_k(x) \, dx,
\]
or
\[
\phi_k(\tau) = \int_{-\infty}^{\tau} e^{(x-\tau)A_k} g_k(x) \, dx.
\]
The lemma then follows by a change of variables. \( \blacksquare \)

**Proof of the absolute convergence of the psi-series. II. An estimate for \( u_k \).** To estimate the growth order of \( u_k \) and \( v_k \), we now introduce the following norm: for any \( x \in \mathbb{C}^n \) and any matrix \( (a_{ij})_{n \times n} \),
\[
\| x \| = \max_{1 \leq j \leq n} \{|x_j|\}, \quad \| (a_{ij})_{n \times n} \| = \max_{1 \leq j \leq n} \left\{ \sum_{i} |a_{ij}| \right\},
\]
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With this norm, we then have the inequality
\[
\max \{|u_k(\tau)|, |v_k(\tau)|\} \leq \|\phi_k\|
\leq 5 \int_0^\infty e^{-x(k-6)} \max \left\{ \sum_{0 \leq j < k} |v_j|, \sum_{1 \leq j < k} |u_j||u_{k-j}| \right\} \, dx. \tag{16}
\]
Now write \( z = \rho - r e^{i\theta} \), so that \( \tau = \log(r/\rho) + i\theta = \xi + i\theta \), where \( r \leq e^{-\varepsilon}\rho \) and
\[
\mathcal{T} := \{ \xi + i\theta : \xi \in (-\infty, -\varepsilon] \text{ and } |\theta| \leq \pi \},
\]
with \(|1 - \tau| \geq 1 + \varepsilon\). We prove by induction that
\[
\begin{cases}
|u_k(\tau)| \leq \frac{K|1 - \tau|^{6}}{\sqrt{k + 1}}, \\
|v_k(\tau)| \leq \frac{K|1 - \tau|^{6}}{\sqrt{k + 1}},
\end{cases}
\tag{17}
\]
for \( k \geq 0 \) and \( \tau \in \mathcal{T} \), where the constant \( K > 0 \) is easily tuned according to the initial conditions.

Then, by induction hypothesis,
\[
\left| \sum_{0 \leq j < k} v_j(\tau) \right| \leq K \sum_{0 \leq j < k} \frac{|1 - \tau|^{j-6}}{\sqrt{j + 1}} \\
\leq \frac{K}{|1 - \tau| - 1} |1 - \tau|^{k-6} \\
\leq \frac{K}{\varepsilon} |1 - \tau|^{k-6},
\]
and
\[
\left| \sum_{1 \leq j < k} u_j(\tau)u_{k-j}(\tau) \right| \leq K^2|1 - \tau|^{12} \sum_{1 \leq j < k} \frac{1}{\sqrt{(j+1)(k-j+1)}} \\
\leq K^2|1 - \tau|^{12} \int_0^k \frac{1}{\sqrt{x(k-x)}} \, dx \\
= \pi K^2|1 - \tau|^{12}.
\]
Now
\[
\max\{\|u_k(\tau)\|, \|v_k(\tau)\|\} \leq \|\phi_k(\tau)\|
\]
\[
= \left\| \int_{-\infty}^{\tau} \mathcal{P}e^{(x-\tau)}\mathcal{P}^{-1}g_k(x)dx \right\|
\]
\[
= \left\| \int_{0}^{\infty} \mathcal{P}e^{-x}\mathcal{P}^{-1}g_k(\tau - x)dx \right\|
\]
\[
\leq \|\mathcal{P}\|\|\mathcal{P}^{-1}\| \int_{0}^{\infty} e^{-x(k-6)}\|g_k(\tau - x)\|dx
\]
\[
\leq 5 \int_{0}^{\infty} e^{-x(k-6)} \max\left\{ \sum_{0 \leq j < k} |v_j(\tau - x)|, \sum_{1 \leq j < k} |u_j(\tau - x)u_{k-j}(\tau - x)| \right\} dx.
\]

By choosing \( \varepsilon \leq 1/(\pi K) \), so that \( K/\varepsilon \geq \pi K^2 \). We have
\[
|u_{k+6}(\tau)|, |v_{k+6}(\tau)| \leq \frac{5K}{\varepsilon} \int_{0}^{\infty} e^{-xk}|1 - \tau + x|^k dx
\]
\[
\leq \frac{5K}{k\varepsilon} |1 - \tau|^k \int_{0}^{\infty} e^{-x} \left| 1 + \frac{x}{k(1 - \tau)} \right|^k dx.
\]

Since \( |1 - \tau| \geq 1 + \varepsilon \) for \( \tau \in \mathcal{S} \), we see that
\[
\int_{0}^{\infty} e^{-x} \left| 1 + \frac{x}{k(1 - \tau)} \right|^k dx \leq \int_{0}^{\infty} e^{-x} \left( 1 + \frac{x}{k|1 - \tau|} \right)^k dx
\]
\[
\leq \int_{0}^{\infty} e^{-x(1-1/|1-\tau|)} dx
\]
\[
= \frac{|1 - \tau|}{|1 - \tau| - 1}
\]
\[
\leq \frac{1 + \varepsilon}{\varepsilon}.
\] (18)

It follows that
\[
\frac{5K}{k\varepsilon} |1 - \tau|^k \int_{0}^{\infty} e^{-x} \left| 1 + \frac{x}{k(1 - \tau)} \right|^k dx
\]
\[
\leq \frac{5K(1 + \varepsilon)}{k\varepsilon^2} |1 - \tau|^k
\]
\[
\leq \frac{K|1 - \tau|^k}{\sqrt{k + 7}},
\]
for \( k \geq k_0 \geq -7 + (1 + \varepsilon)^2/\varepsilon^4 \). This proves the required estimate.
Proof of the absolute convergence of the psi-series: an estimate for $U(Z)$. From (17), we obtain

$$
\rho|U(Z)| = \left| \sum_{k \geq 0} u_k(\tau) e^{(k-2)\tau} \right|
\leq Ke^{-2R(\tau)} \sum_{k \geq 0} \frac{|1 - \tau|^k e^{kR(\tau)}}{\sqrt{k + 1}}
= O\left(e^{-2R(\tau)} (1 - |1 - \tau| e^{R(\tau)})^{-1/2}\right)
= O(1),
$$

provided that

$$|1 - \tau| e^{R(\tau)} < 1.$$

But this implies that ($R(\tau) = r/\rho$)

$$r < \frac{\rho}{|1 - \tau|} \leq \frac{\rho}{1 + \varepsilon} \leq (1 - \varepsilon')\rho.$$

This proves that the series (8) is absolutely convergent for $z \in \mathcal{G}_{(1-\varepsilon')\rho}$. 

**Numerical approximations to $\rho$ and $c_6$.** As mentioned in Introduction, $P$ is connected to $U$ by choosing a point in $[\varepsilon\rho, \rho - \varepsilon]$; then the values of $(\rho, c_6)$ are determined by solving numerically the two equations $P(z_0) = U(Z_0)$ and $P'(z_0) = -\rho U'(Z_0)$, where $Z_0 := 1 - z_0$.

For numerical purposes, we can compute the approximate values of $P(z_0)$ or $P'(z_0)$ by their corresponding truncated series expansions using, say the first $N$ terms; for example, $P(z_0) \approx \sum_{j < N} p_j z_0^j$. The number of terms used depends on the degree of numerical precision we require, and the remainder $\sum_{j \geq N} p_j z_0^j$ can be well estimated by using the asymptotic expansion (10). More precisely, for large $N$,

$$
\sum_{j \geq N} p_j z_0^j = \frac{6(z_0/\rho)^N}{\rho - z_0} \left( N + \frac{3\rho + 2z_0}{5(\rho - z_0)} + O\left(N^{-4}\right) \right).
$$

Since $z_0 < \rho$, the right-hand side can be made arbitrarily small by choosing $N$ sufficiently large so that the error introduced is under control.

Similarly, $U(Z) \approx U_M(Z) := \rho^{-1} \sum_{k \leq M} u_k(\log Z) Z^{k-2}$ for a sufficiently large $M$ whose choice can be determined by the desired degree of precision and the upper bound (17).

$$
\sum_{k \geq M} u_k(\tau_0) e^{(k-2)\tau_0} = O\left(M^{-1/2} |1 - \tau_0| M e^{MR(\tau_0)}\right),
$$

where $\tau_0 = \log(Z_0)$.

Note that if $z_0$ is too close to zero, then the remainder (19) for $P$ decreases much faster than that (20) for $U$, and if $z_0$ is too close to $\rho$, then the converse is true. So the best choice for $z_0$ will be the one that both remainders are asymptotically of the same order. For practical use, since $p_n$ is easier to compute than $u_k$, we take $M = \beta N$ for some $\beta \in (0, 1)$. Then we solve the equation

$$
\left(\frac{z_0}{\rho}\right)^{1/\beta} = \left|1 - \log\left(1 - \frac{z_0}{\rho}\right)\right| \left(1 - \frac{z_0}{\rho}\right),
$$

(21)
(which obviously has a unique real solution for \(z_0/\rho \in (1/2, 1)\) to find the best \(z_0\). On the other hand, to compute \(u_k\), we take the first entry of \(\phi_k\) in (15) and obtain the recurrence

\[
u_k(\tau) = \frac{1}{7} \int_0^\infty \left(3e^{-(k-6)x} + 4e^{-(k+1)x}\right) \left((k-3)u_{k-1}(\tau - x) + u_{k-1}(\tau - x)\right) \, dx
\]

\[
+ \frac{1}{7} \int_0^\infty \left(e^{-(k-6)x} - e^{-(k+1)x}\right) \sum_{1 \leq j < k} u_j(\tau - x)u_{k-j}(\tau - x) \, dx
\]

\[
-u_{k-1}(\tau) + \frac{1}{7} \int_0^\infty \left(9e^{-(k-6)x} - 16e^{-(k+1)x}\right) u_{k-1}(\tau - x) \, dx
\]

\[
+ \frac{1}{7} \int_0^\infty \left(e^{-(k-6)x} - e^{-(k+1)x}\right) \sum_{1 \leq j < k} u_j(\tau - x)u_{k-j}(\tau - x) \, dx
\]

for \(k \geq 7\). All these polynomials \(u_k\)’s are solvable recursively starting from the initial values

\[
u_0 = 6, \ u_1 = -\frac{12}{5}, \ u_2 = -\frac{7}{25}, \ u_3 = -\frac{14}{125}, \ u_4 = -\frac{63}{625}, \ u_5 = -\frac{161}{3125}, \ u_6 = c_6 - \frac{14\tau}{3125},
\]

with the two free parameters \(\rho\) and \(c_6\). More explicitly, let \(u_k(\tau) := \sum_{0 \leq s \leq \lfloor k/6 \rfloor} u_{k,s} \tau^s\). Then

\[
u_{k,s} = u_{k-1,s} + \frac{1}{s!} \sum_{s \leq \ell \leq \lfloor (k-1)/6 \rfloor} u_{k-1,\ell} (-1)^{\ell-s} \left(\frac{9\ell!}{7(k-6)\ell-s+1} - \frac{16\ell!}{7(k+1)\ell-s+1}\right)
\]

\[
+ \frac{1}{s!} \sum_{\substack{1 \leq j,k,\ell \leq k \\ell_1 + \ell_2 \leq s \\ell_1 \leq j/6 \\ell_2 \leq (k-j)/6}} u_{j,\ell_1} u_{k-j,\ell_2} (-1)^{\ell_1 + \ell_2 - s} \left(\frac{(\ell_1 + \ell_2 - s)!}{7(k-6)\ell_1 + \ell_2 - s+1} - \frac{(\ell_1 + \ell_2 - s)!}{7(k+1)\ell_1 + \ell_2 - s+1}\right),
\]

for \(0 \leq s \leq \lfloor k/6 \rfloor\).

We finally solve numerically the pair \((\rho, c_6)\) from the two equations with \(\rho \in (3, 4)\)

\[
\begin{align*}
P_N(z_0) &= U_M(Z_0) & \text{and} & & P'_N(z_0) &= -\rho U'_M(Z_0).
\end{align*}
\]

(22)

Numerical evidence suggests that the series definition for \(U(Z)\) and \(U'(Z)\) are both convergent for \(Z = 1\), which means that one might even use the two equations

\[
U(1) = 1, \quad U'(1) = -\rho,
\]

to solve for the pair \((\rho, c_6)\). But the convergence is much slower than taking \(z_0\) according to (21).

A quantity arising in phylogenetic trees. Very similar to the original motivations of studying \(p_n\), the following recurrence

\[
q_n = \frac{2}{(n-1)^2} \sum_{1 \leq j < n} q_j q_{n-j} \quad (n \geq 2)
\]

(23)

with \(q_1 = 1\) was introduced in Bryant et al. [7] in the course of analyzing the size of a maximum agreement subtree in two randomly chosen trees according to the Yule-Harding model. The
quantity serves as an effective bound for the probability that the size of a common maximum agreement subtree exceeds a certain given value.

Let \( p_n := 2q_{n+1} \). Then the recurrence (23) becomes

\[
p_n = n^{-2} \sum_{0 \leq j < n} p_j p_{n-1-j} \quad (n \geq 1),
\]

of exactly the same form as (1) but with \( p_0 = 2 \). This means that the DE satisfied by the generating function \( P(z) = \sum_n p_n z^n \) remains the same as (3) but the initial condition differs.

The same psi-series method we used above applies and we obtain the asymptotic expansion

\[
q_n = \rho^{-n} \left( 3n - \frac{6}{5} + \frac{168}{3125} n^{-5} + \frac{336}{3125} n^{-6} + O(n^{-7}) \right).
\]

with \( \rho = 1.57042 \ 87836 \ 01468 \ 47580 \ 40837 \ldots \).

### 3 Probability of equality of random trees

The consideration of the equality of two random BSTs can be easily extended either to more random BSTs or to other variants of BSTs.

#### 3.1 Equality of \( d \) random BSTs

We extend in this subsection the same psi-series analysis to \( d \) random BSTs, \( d \geq 2 \). Surprisingly, the resulting forms of the asymptotic expansions depends on the parity of \( d \).

**Recurrence.** The random BST model is as introduced above. Let \( p_n = p_n(d) \) denote the probability that \( d \) random BSTs, each independent of the others, are identical. More precisely, the probability that \( d \) random permutations whose corresponding BSTs are all the same. Then \( p_n \) satisfies the recurrence

\[
p_n = n^{-d} \sum_{0 \leq j < n} p_j p_{n-1-j} \quad (n \geq 1),
\]

with \( p_0 = 1 \). Let \( P(z) := \sum_{n \geq 0} p_n z^n \) be the generating function of \( p_n \). Then \( P(z) \) satisfies the nonlinear DE of order \( d \)

\[
\left( \frac{d}{dz} \right)^d P(z) = zP(z)^2
\]

with \( p_0 = 1 \) and the first \( d - 1 \) values \( p_n \) for \( 1 \leq n < d \) given by the recurrence (24).

**The ARS Algorithm.** As in the case of two random BSTs above, we begin with applying the ARS Algorithm and check first if there are pseudo-poles and incompatibility.

- Leading order analysis: This part is always easy for the problems we study in this paper and we obtain, by assuming \( P(z) \sim c_0 (1 - z/\rho)^{-\alpha} \) and by matching coefficients, \( \alpha = d \) and \( c_0 = \rho(2d)!/(2d!) \).
Resonance analysis: On the other hand, by collecting the coefficient for the term \( c_r (1 - z/\rho)^{-2d} \) in the resulting expansion for (25), we obtain the polynomial characterizing all possible resonances
\[
\Phi_d(r) = \frac{(2d - 1 - r)!}{(d - 1 - r)!} - \frac{(2d)!}{d!} \Theta_d(r),
\]
where \( \Theta_d(r) \) is a polynomial of even order and has no real zeroes. We see that if \( d \) is odd, then there is no additional integer-valued resonance except \(-1\) for this case. Thus, the movable singularity \( \rho \) is a pole of order \( d \). On the other hand, if \( d \) is even, then there exists an additional, unique, positive, integer-valued resonance \( 3d \) for each \( d \).

Incompatibility: We need only consider the case when \( d \) is even. The incompatibility of the resonance at \( r = 3d \) is easily checked for each specific \( d = 2, 3, \ldots \), but a proof that \( r = 3d \) leads to incompatibility for all \( d \) is not obvious.

The case when \( d \) is odd. From the above quick check by ARS algorithm, we see that the solution for the DE (25) admits the Laurent series expansion
\[
\rho P(z) = \frac{(2d)!}{2 \cdot d!} \left( Z^{-d} - \frac{(3d - 2)(d - 1)}{2(3d - 1)} Z^{-d+1} + \sum_{2 \leq j \leq d} c_j Z^{-j} \right) + \Xi(z),
\]
where \( \Xi(z) = \Xi_d(z) \) is analytic at \( \rho \).

The case when \( d \) is even. By the above procedure of ARS algorithm, we anticipate a psi-series expansion for \( P(z) \) of the form
\[
\rho P(z) = \sum_{j \geq 0} Z^{-j} \sum_{0 \leq \ell \leq [j/3d]} c_{j,\ell} (\log Z)^\ell,
\]
where the \( c_{j,\ell} \)'s are chosen so that the psi-series satisfies the DE (25). In particular, the first few terms read
\[
\rho P(z) = \frac{(2d)!}{2 \cdot d!} Z^{-d} - \frac{(3d - 2)(d - 1)(2d)!}{4(3d - 1)d!} Z^{-d+1} + \sum_{2 \leq j \leq 3d} c_{j,0} Z^{-j} + C_{3d,1} Z^{2d} \log Z + \cdots.
\]
The justification of the psi-series on the right-hand side of (27) follows the same pattern as that for two random BSTs; see Appendix A1 for details.

In summary, we conclude the following asymptotic estimates, the drastic change of the error term according to the parity of \( d \) unveiling an additional surprise.

**Theorem 2** The probability that \( d \geq 2 \) randomly chosen BSTs are all equal satisfies
\[
p_n = \rho^{-d-1} \left( \frac{(2d - 1)!}{(d - 1)!} \left( n^{d-1} + \frac{(d - 1)(2d - 1)}{3d - 1} n^{d-2} + \sum_{0 \leq j \leq d-3} C_j n^j \right) \right) + \begin{cases} O(\rho^{-n}(1 - \varepsilon)^n), & \text{if } d \text{ is odd;} \\ Kn^{-2d-1} \rho^{-n-1} + O(\rho^{-n} n^{-2d-2}), & \text{if } d \text{ is even.} \end{cases}
\]
where \( \varepsilon > 0 \), the \( C_j \)'s are constants, \( \rho = \rho_d \) depends on \( d \) and \( K \) is a constant depending only on \( d \).

More precise asymptotic expansions can be derived, but we content ourselves with the current form for simplicity of presentation. Is there any intuitive reason why the asymptotic expansion of \( p_n = p_n(d) \) differs according to the parity of \( d \)?

### 3.2 Equality of two random \( m \)-ary search trees

The \( m \)-ary search trees are one of the natural extensions of BSTs to branching factors \( m \geq 2 \) beyond binary; see [25] for thorough discussions. Briefly, the first \( m - 1 \) keys are stored in the root and sorted in increasing order, each of the remaining \( n - m + 1 \) keys are then directed to one of the \( m \) subtrees, corresponding to the \( m \) intervals specified by the \( m - 1 \) sorted keys, and are constructed recursively by the same procedure.

In the same vein, the probability \( q_n \) that two random \( m \)-ary search trees are identical is characterized by the following recurrence (\( m \geq 2 \))

\[
q_n = \binom{n}{m-1}^{-2} \sum_{j_1, \ldots, j_m = n-m+1} q_{j_1} \cdots q_{j_m} \quad (n \geq m - 1),
\]

with the initial conditions \( q_j = 1, 0 \leq j \leq m - 2 \). The associated generating function \( Q(z) \) then satisfies the following nonlinear DE

\[
( z^{m-1} Q^{(m-1)}(z) )^{(m-1)} = (m-1)^2 Q^m(z),
\]

with the initial conditions \( Q(z) = 1 + z + \cdots + z^{m-2} + q_{m-1} z^{m-1} + \cdots \) where \( q_j, m - 1 \leq j \leq 2m - 3, \) are determined by the above recurrence.

1. Leading order analysis: The simple form \( Q(z) \sim c_0(1 - z/\rho)^{-\alpha} \) leads to \( \alpha = -2 \) and \( \rho c_0 = ((2m - 1)!/(m - 1)!^2)^{1/(m-1)} \).

2. Resonance analysis: Again, assuming that \( Q(z) \sim c_0(1 - z/\rho)^{-2} + c_r (1 - z/\rho)^{-2+r} \), we obtain the following algebraic equation characterizing all possible resonances

\[
\prod_{2 \leq j < 2m} (r - j) - \frac{(2m)!}{2} = (r + 1)(r - (2m + 2)) \phi_m(r) = 0,
\]

where \( \phi_m(r) \) is a polynomial of degree \( 2(m - 2) \) and admits complex-conjugate zeros only. Thus we need to check if the DE (28) is compatible at the resonance \( r = 2m + 2 \).

3. Incompatibility: Similar to the case of \( d \) random BSTs, the resonance \( r = 2m + 2 \) is easily checked to be incompatible for each finite values of \( m = 2, 3, \ldots \), but it is far from being obvious to prove directly the incompatibility for all \( m \geq 2 \).

Let \( \lambda_m := \left( (2m - 1)!/(m - 1)!^2 \right)^{1/(m-1)} \). Instead of proving the incompatibility of \( r = 2m + 2 \) for all \( m \geq 2 \) and that \( \rho \) is not an essential singularity, we prove that the DE (28) has the psi-series solution

\[
U(Z) = \sum_{j \geq 0} Z^{j-2} \sum_{0 \leq \ell \leq \lfloor j/(2m+2) \rfloor} c_{j,\ell} \log^\ell Z,
\]
| $m$ | $p_n \sim$ | $\lambda_m$ |
|-----|-------------|-------------|
| 2   | $\lambda_2 \rho_2^{-n-1} \left( n + \frac{3}{5} + \frac{56}{3125} n^{-5} \right)$ | 6           |
| 3   | $\lambda_3 \rho_3^{-n-1} \left( n + \frac{4}{7} + \frac{657606}{78236585} n^{-7} \right)$ | $\sqrt{30}$ |
| 4   | $\lambda_4 \rho_4^{-n-1} \left( n + \frac{5}{11} + \frac{10412924224}{15508564099} n^{-9} \right)$ | $\sqrt{140}$ |
| 5   | $\lambda_5 \rho_5^{-n-1} \left( n + \frac{6}{17} + \frac{132601150728198400}{1381408768320} n^{-11} \right)$ | $\sqrt{630}$ |
| 6   | $\lambda_6 \rho_6^{-n-1} \left( n + \frac{7}{23} + \frac{15260615072819840000000}{18729360591890984770} n^{-13} \right)$ | $\sqrt{2772}$ |

Table 2: The asymptotic approximation to the probability that two random $m$-ary search trees are equal for $m = 2, \ldots, 6$. All $O$-terms are omitted.

which converges absolutely in some cut-region $C_R$ (defined in (11)); see Appendix A1 for details. Then we connect $Q(z)$ and $U(Z)$ by the same arguments as those used above for two random BSTs. In this way, we obtain

$$
\rho Q(z) = \lambda_m Z^{-2} - \frac{m \lambda_m}{2m + 1} Z^{-1} + \sum_{2 \leq j \leq 2m+2} c_{j,0} Z^{j-2} + c_{2m+2,1} Z^{2m} \log Z + O \left( Z^{2m+1} \log Z \right).
$$

From this expansion, we then derive the following approximation to $q_n$.

**Theorem 3** The probability $q_n = q_n(m)$ that two random $m$-ary search trees are equal satisfies the asymptotic approximation

$$
q_n = \lambda_m \rho^{-n-1} \left( n + \frac{m + 1}{2m + 1} \right) + K \rho^{-n-1} n^{-2m-1} + O \left( \rho^{-n} n^{-2m-2} \right),
$$

where $\rho = \rho_m$ and $K$ both depend on $m$.

As for BSTs, the consideration can be extended to choose $d \geq 2$ random $m$-ary search trees, and the resonance equation is given by

$$
\prod_{0 \leq j < d(m-1) \atop (d-r+j)} \frac{(d-r+j)(d-r+d(m-1)-1)!}{(d-1)!} = \frac{(d-r+d(m-1))}{\Gamma(d-r)} - \frac{m(dm-1)!}{(d-1)!}.
$$

We then deduce that this equation has no positive integral resonance when $m$ is even and $d$ is odd, and has the positive resonance $d(m+1)$ for all other cases with $d, m \geq 2$. Our approach can be applied and we obtain an asymptotic approximation to the probability that $d$ random $m$-ary search trees are equal, the error terms beyond the constant term being either exponentially small when $m$ is even and $d$ is odd or of order $\asymp n^{-dm-1}$ for all the remaining meaningful cases.

### 3.3 Equality of two random fringe-balanced BSTs

Median-of-$(2t + 1)$ (or fringe-balanced) BSTs represent yet another class of extensions of BSTs. The idea is, instead of placing the first element in the given sequence at the root, which may result in a less balanced binary tree, we take a small sample of size $2t + 1$ and use the
median of this sample as the root element, which then partitions the remaining elements as in
the construction of BSTs, where \( t \geq 0 \). This simple balancing scheme has turned out to be
useful for small \( t \), notably for the corresponding quicksort algorithm. Note the the original
BST corresponds to \( t = 0 \).

For the probability model, assume, as in random BSTs, that we are given a random permuta-
tion; then we construct the corresponding median-of-(\( 2t + 1 \)) BST, which is called a random
median-of-(\( 2t + 1 \)) BST.

Let now \( f_n = f_n(t) \) denote the probability that two randomly chosen permutations lead to
an identical median-of-(\( 2t + 1 \)) BST. Then \( f_n \) satisfies the recurrence

\[
f_n = \sum_{t \leq j \leq n-1-t} \binom{j}{t}^2 \binom{n-1-j}{2t+1}^2 f_j f_{n-1-j} \quad (n \geq 2t + 1),
\]

with the initial conditions \( f_n = 1 \) for \( 0 \leq n \leq 2t \).

Let \( F(z) := \sum_{n \geq 0} f_n z^n \) denote the generating function of \( f_n \). Then \( F(z) \) satisfies the DE

\[
(2t+1)^{2t+1} F^{(2t+1)}(z)^{(2t+1)} = \frac{(2t+1)!^2}{t!^4} \left( (z^t F(t)^{(t)}(z)) \right)^2,
\]

with the initial conditions \( F^{(j)}(0) = j! \), \( 0 \leq j \leq 2t \), and \( f_j, 2t + 1 \leq j \leq 4t + 1 \), given by the recurrence \((29)\).

❼ Leading order analysis: With the simple form \( F(z) \sim c_0 (1 - z/\rho)^{-\alpha} \), we obtain \( \alpha = 2 \) and

\[
\rho c_0 = \frac{(4t + 3)! t!^4}{(2t + 1)!^4},
\]

for each \( t \geq 0 \).

❼ Resonance analysis: Again, assuming that \( F(z) \sim c_0 (1 - z/\rho)^{-2} + c_r (1 - z/\rho)^{-2+r} \), we obtain the resonance equation

\[
\Phi_t(r) = \left( \prod_{2 \leq j \leq 2t+1} (r - j) \right) \left( \prod_{2t+2 \leq j \leq 4t+3} (r - j) - 2 \prod_{2t+2 \leq j \leq 4t+3} j \right),
\]

which can be factored into the form

\[
(r + 1)(r - 6t - 6) \phi_t(r) \prod_{2 \leq j \leq 2t+1} (r - j),
\]

where \( \phi_t(r) \) has only complex conjugate zeros since the factor

\[
(r - 2t - 2) \cdots (r - 4t - 3) - 2(2t + 2) \cdots (4t + 3)
\]

\[
= (r - 2t - 2) \cdots (r - 4t - 3) - (2t + 3) \cdots (4t + 4)
\]

never vanishes for \( r \in \mathbb{R} \setminus \{-1, 6t + 6\} \). Thus we get yet another new pattern for the
least positive integer-valued resonance

\[
r = \begin{cases} 
6, & t = 0, \\
2, & t \geq 1.
\end{cases}
\]

(31)
Incompatibility: As \( t = 0 \) has already been addressed in Section 2 we focus on \( t \geq 1 \), which has the constant resonance \( r = 2 \). A direct check of the incompatibility is possible for \( r = 2 \) and \( t \geq 1 \); see Appendix A2.

The same psi-series method applies and we obtain for \( t \geq 1 \)

\[
\rho F(z) = \frac{(4t + 3) t!^4}{(2t + 1) t!^4} \left( Z^{-2} - \frac{2(t + 1)^2}{6t + 5} Z^{-1} + \frac{(22 t^2 + 35 t + 14) (t + 1)^2 t}{(7t + 6) (6t + 5)^2} \log Z \right) + O \left( |Z| |\log Z| \right).
\]

**Theorem 4** The probability \( f_n \) that two random median-of-\((2t + 1)\) BSTs are equal satisfies the asymptotic approximation

\[
f_n = \frac{(4t + 3) t!^4}{(2t + 1) t!^4} \rho^{-n-1} \left( n + \frac{3 + 2t - 2t^2}{6t + 5} - \frac{(22 t^2 + 35 t + 14) (t + 1)^2 t}{(7t + 6) (6t + 5)^2} n^{-1} \right) + O \left( \rho^{-n-2} \right),
\]

for \( t \geq 1 \), where \( \rho = \rho_t \) is an effectively computable constant.

Note that the expansion also holds when \( t = 0 \) but the \( O \)-term becomes \( O(n^{-5}) \); see (10). Also more terms can be computed by the same procedure.

### 4 Moments of high orders

In addition to the equality of random trees, another rich source where nonlinear recurrences and differential equations of the same type as we analyzed above arise is the asymptotics of moments of high orders.

#### 4.1 Partial match queries in random quadtrees

We consider first in this section the cost of partial match queries in random two-dimensional quadtrees. The expected cost was first analyzed in [15] (see also [8]) and the limit law derived in [30] under an idealized model where randomness is preserved throughout the tree.

Let \( v = (\sqrt{17} - 3)/2 \). Then the cost of a random partial match query in a random two-dimensional quadtree of \( n \) nodes tends (under an idealized model where randomness is preserved for all subtrees), after normalized by \( n^v \), to a limit law \( X \) whose moments satisfy (see [30])

\[
\mathbb{E}(X^m) = \frac{a_m}{\Gamma(mv + 1)},
\]

where \( a_1 := \Gamma(2v + 2)/(2\Gamma(v + 1)^2) \) and

\[
a_m = \frac{2}{v(m - 1)((m + 1)v + 3)} \sum_{1 \leq j < m} \binom{m}{j} a_j a_{m-j} \quad (m \geq 2).
\]
Then the generating function

\[ A(z) := 1 + \sum_{m \geq 1} a_m z^m / m! \]

satisfies the differential equation

\[ v^2 z^2 A''(z) + 2z A'(z) + 2A(z) = 2A^2(z), \quad (32) \]

with the initial conditions \( A(0) = 1 \) and \( A'(0) = a_1 \).

The psi-series method we use above can be readily applied with the resonance \( r = 6 \) and we obtain

\[
A(z) = 3v^2Z^{-2} + \frac{6}{5}(9v - 5)Z^{-1} + \sum_{2 \leq j \leq 7} c_jZ^{-j + 2} + \frac{117(39v + 139)}{43750}Z^4 \log Z \\
+ \frac{468(153v + 545)}{109375}Z^5 \log Z + O(|Z|^6 \log |Z|),
\]

(33)

where the \( c_j \)'s are unimportant constants. By singularity analysis ([17]), we conclude the following asymptotic approximation to \( a_n/n! \).

**Theorem 5** The \( m \)-th moment of \( X \) satisfies for large \( m \)

\[
\mathbb{E}(X^m) = \frac{m! \rho^{-m}}{\Gamma(mv + 1)} \left( 3v^2m + \frac{9}{5}v - \frac{1404(39v + 139)}{2185 m^5} + \frac{8424(139v + 495)}{21875 m^6} + O(m^{-7}) \right),
\]

(34)

where \( \rho \approx 1.37649444105715625755 \ldots \)

We omit all details as they are very similar to the case of the equality of two random BSTs.

An interesting implication of our psi-series analysis is that we can derive an asymptotic expansion for the moment generating function of \( X \)

\[
\mathbb{E}(e^{Xz}) = e^{(z/\rho)^{1/v}} \left( 3 \left( \frac{z}{\rho} \right)^{1/v} + \frac{9}{5} - \frac{22464}{21875} \left( \frac{z}{\rho} \right)^{-5/v} + O(|z|^{-6/v}) \right),
\]

as \( |z| \to \infty \) in the sector \( |\arg(z)| \leq (v - \varepsilon)\pi/2 \). This is proved by the integral representation

\[
\mathbb{E}(e^{Xz}) = \frac{1}{2\pi i} \int_{\mathcal{H}} e^{s} s^{-1} A(z/s^v) \, ds,
\]

for a suitable Hankel-type contour, and standard analysis; see Appendix A3. Such an expansion for the moment generating function is unusual in the probability literature and implies in turn that

\[
- \log \mathbb{P}(X > t) \sim (1 - v) t^{v/(1-\varepsilon)} (\rho t)^{1/(1-\varepsilon)},
\]

(36)

for large \( t \), by an application of Tauberian argument; see Section 4.12 of Bingham et al. [5].

Note that the transformations \( z = \xi^{-v} \) and \( A(z) = 2\xi Z(\xi) \) brings the DE (32) to the standard form of the so-called Emden’s equation

\[
\frac{d^2}{d\xi^2} Z(\xi) = \xi^{-1} Z^2(\xi).
\]

But it is not exactly solvable; see [29 § 2.3] or [22 § 12.4].
4.2 Partial match queries in random relaxed $k$-d trees

In a similar setting, the cost of a random partial match query in a random relaxed $k$-d trees (see [12]) tends, after proper normalization, to the limit law $Y$ whose moments satisfy (see [28])

$$E(Y^m) = \frac{b_m}{\Gamma(m\beta + 1)},$$

where $\beta := (-1 + \sqrt{9 - 8s/k})/2$ ($s$ out of the $k$ coordinates in the query pattern is specified, the other $k - s$ being “don’t-cares”), and

$$b_m = \frac{\beta + 1}{(m - 1)((m + 1)\beta + 1)} \sum_{1 \leq j < m} \binom{m}{j} (j\beta + 1)b_j b_{m-j} \quad (m \geq 2),$$

with

$$b_1 = \frac{2\Gamma(2\beta + 2)}{\beta(\beta + 1)^2(2\beta + 1)\Gamma^3(\beta + 1)}.$$

It follows that the generating function $B(z) := 1 + \sum_{m \geq 1} b_m z^m / m!$ satisfies the nonlinear differential equation

$$\beta z^2 B''(z) + (\beta + 1)^2 z B'(z) + (\beta + 1)B(z) = (\beta + 1)B^2(z) + \beta(\beta + 1)zB'(z)B(z),$$

with the initial conditions $B(0) = 1$ and $B'(0) = b_1$.

The psi-series method applies with a resonance at $r = 2$ and we obtain the expansion

$$B(z) = \frac{2}{\beta + 1} Z^{-1} + \frac{\beta - 1}{\beta} c_2 Z + \frac{2(\beta - 1)(\beta + 2)}{3\beta^2(\beta + 1)} Z \log Z + c_3 Z^2 + \frac{(\beta - 1)(\beta + 2)(\beta + 3)}{3\beta^3(\beta + 1)} Z^2 \log Z + c_4 Z^3 + O\left(|Z|^3 \log Z\right),$$

from which we deduce an asymptotic approximation to higher order moments of $Y$.

**Theorem 6** The $m$-th moment of the limit law $Y$ satisfies

$$E(Y^m) = \frac{2m! \rho^{-m}}{(\beta + 1)\Gamma(m\beta + 1)} \left(1 + \frac{(\beta - 1)(\beta + 2)}{3\beta^2 m^2} - \frac{(\beta - 1)(\beta + 2)}{\beta^3 m^3} + O\left(m^{-4} \log m\right)\right),$$

as $m \to \infty$, where $\rho$ depends on $\beta$.

Consequences of this expansion can be derived as those for $X$.

4.3 Recursive partition structures.

In the context of recursive interval splitting, Gnedin and Yakubovich [20] derived the following recurrence relation for the $m$-th moment $h_m$ of certain limit law $W$ (satisfying a fixed-point equation with Dirichlet distribution as prefactors)

$$h_m = \frac{\Gamma(d + \omega)}{\Gamma(\omega)^2 \Gamma(m\lambda + d + \omega)} \sum_{0 \leq j \leq m} \binom{m}{j} \Gamma((m - j)\lambda + \omega) \Gamma(j\lambda + \omega) h_j h_{m-j},$$

for $m \geq 2$ with $h_0 = h_1 = 1$, where $\lambda, \omega > 0$ ($\lambda$ is referred to as the Malthusian exponent) and $d = 2, 3, \ldots$. 

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The case when $d = 2$. Consider first the simplest case when $d = 2$. In this case, the generating function

$$h(z) := \sum_{m \geq 0} \frac{h_m \Gamma(m\lambda + \omega)}{m! \Gamma(\omega)} z^m,$$  \hspace{1cm} (39)$$
satisfies the DE (using the relation $(\lambda + \omega)(\lambda + \omega + 1) = 2\omega(\omega + 1)$)

$$vz^2h''(z) + zh'(z) + h(z) = h^2(z),$$

which is exactly of the type of problems we have been examining in this paper (cf. (32)), where for simplicity

$$v := \frac{\lambda^2}{\omega(\omega + 1)}.$$

For this DE, we can apply the psi-series method and obtain

$$h(z) = 6vZ^{-2} - \frac{6}{5}(6v - 1)Z^{-1} + \sum_{2 \leq j \leq 6} c_j Z^{j-2} + KZ^4 \log Z + O \left(|Z|^5 \log |Z| \right),$$

where

$$K := \frac{(v - 1)^2(v - 6)(6v - 1)(2v + 3)(3v + 2)}{43750v^5}.$$

Consequently, we deduce the asymptotic expansion for the moments of $W$

$$h_m = \frac{6m!\Gamma(\omega)^{-m}}{\Gamma(m\lambda + \omega)} \left(vm - \frac{v - 1}{5} - 4Km^{-5} + O \left(m^{-6} \right) \right),$$

for large $m$.

The case when $d \geq 2$. From the recurrence (38), the generating function $h(y)$ (defined as in (39)) satisfies the DE

$$y^{1-\omega} \frac{d^d}{dy^d} \left(h(y^\lambda)y^{d+\omega-1} \right) = \omega^\theta h(y^\lambda)^2,$$

where $\omega^\theta = \omega \cdots (\omega + d - 1)$ denotes the rising factorial; see [20]. The DE is however less manageable. We rewrite it as follows. Let $z = y^\lambda$ and $H(z) = z^\kappa h(z)$, where $\kappa := (d + \omega - 1)/\lambda$. Note that the Malthusian exponent $\lambda$ satisfies the relation

$$\frac{\omega^\theta}{(\lambda + \omega)^d} = \frac{1}{2}.$$

Then the function $H(z)$ satisfies the DE

$$\lambda\theta(\lambda\theta - 1) \cdots (\lambda\theta - d + 1)H(z) = z^{-\kappa}\omega^\theta H(z)^2,$$  \hspace{1cm} (40)$$

where the differential operator $\theta$ is defined as $\theta := z(d/dz)$.

The leading order analysis and the resonance analysis give the dominant exponent $-d$ and the resonance equation is exactly the same as (26) for all $d \geq 2$, namely, $(d - r)^\theta - (d + 1)^\theta$. It follows that we have the same asymptotic pattern for $H$ as the case of $d$ random BSTs.
The case when $d$ is odd. The movable singularity $\rho$ is a pole of order $d$ and the solution $H(z)$ admits the Laurent expansion

$$\rho^{-k} H(z) = \frac{(2d)! \lambda^d}{2 \cdot d! \omega^d} \sum_{0 \leq j \leq d} c_j Z^{j-d} + \Xi_1(z),$$

where

$$c_0 = 1, \quad c_1 = -\frac{d}{2} - \frac{(4d - 2) \omega + (d - 1)(5d - 2)}{2(3d - 1) \lambda},$$

and $\Xi_1(z)$ is an analytic function at $z = \rho$.

The case when $d$ is even. In this case, since the resonance equation (26) possesses the unique positive integral resonance $3d$, we see that $z = \rho$ is a pseudo-pole and the psi-series solution to (40) has the form

$$\rho^{-k} H(z) = \sum_{j \geq 0} Z^{j-d} \sum_{\ell \leq [j/3d]} c_{j,\ell} (\log Z)^\ell$$

$$= \frac{(2d)! \lambda^d}{2 \cdot d! \omega^d} \sum_{0 \leq j \leq 3d} c_j Z^{j-d} + K Z^{2d} \log Z + O \left( |Z|^{2d+1} \log Z \right),$$

where, in particular, $c_0$ and $c_1$ are given as in (41), and $K$ is a constant dependent on $\lambda$ and $\omega$.

Expansions for $h$. It is not difficult to verify that $h(z)$ and $H(z)$ have the same dominant singularity $\rho$, dominant exponent $-d$, and the dominant resonance $3d$. Now by the relation between $h(z)$ and $H(Z)$: $h(z) = (1 - Z)^{-k} \rho^{-k} H(z)$, we obtain

$$h(z) = \frac{(2d)! \lambda^d}{2 \cdot d! \omega^d} \times \begin{cases} \sum_{0 \leq j \leq d} c_j' Z^{j-d} + \Xi_2(z), & \text{if } d \text{ is odd;} \\ \sum_{0 \leq j \leq 3d} c_j' Z^{j-d} + K Z^{2d} \log Z + O \left( |Z|^{2d+1} \log Z \right), & \text{if } d \text{ is even}, \end{cases}$$

where $c_0' = 1,$

$$c_1' = \frac{d}{2} \left( \frac{d + 2 \omega - 1}{(3d - 1) \lambda} - 1 \right),$$

and $\Xi_2$ is analytic at $z = \rho$.

Asymptotics of the moments. From the expansions we derived and a similar analysis as for $d$ random BSTs, we can now conclude the following asymptotic approximations to the limit law $W$.

Theorem 7 The $m$-th moment $h_m$ of $W$ satisfies

$$h_m = \frac{(2d)! \Gamma(\omega)^2 \lambda^m \rho^{-m}}{2 \cdot d! (d - 1)! \Gamma(\omega + d) \Gamma(m \lambda + \omega)} \sum_{0 \leq j \leq d} C_j m^{d-1-j}$$

$$+ \begin{cases} O((1 - \varepsilon)^m), & \text{if } d \text{ is odd;} \\ C m^{-2d - 1} + O \left( m^{-2d - 2} \right), & \text{if } d \text{ is even}, \end{cases}$$

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where \( \varepsilon \in (0, 1) \), the \( C_j \) are constants with \( C_0 = 1 \) and

\[
C_1 = \left( \frac{d}{2} \right) \frac{d + 2\omega - 1}{(3d - 1)\lambda},
\]

and \( \rho, C \) are constants depending on \( d, \lambda, \omega \).

### 4.4 An Ansatz solution in Boltzmann equations

The following sequence \( t_n \) arose in the analysis (see [2]) of exact solutions of the Tjon-Wu representation of Boltzmann equations (which represent the major cornerstone of kinetic theory in statistical mechanics). Let \( \nu \) be a positive integer. The sequence \( t_n \) is defined recursively as

\[
\left( \frac{\nu(\nu + 1)}{\nu + 2} n(n - 1) - (n + 1) \right) t_n = - \sum_{0 \leq j \leq n} t_j t_{n-j} \quad (n \geq 2),
\]

(42)

with \( t_0 = t_1 = 1 \). This recurrence translates into the following DE for the generating function \( T(z) := \sum_{n \geq 0} t_n z^n \)

\[
\frac{\nu(\nu + 1)}{\nu + 2} z^2 T''(z) - zT'(z) - T(z) (1 - T(z)) = 0,
\]

(43)

with the initial conditions \( T(0) = T'(0) = 1 \).

Straightforward computations as above give \(-2\) as the dominant exponent for the dominant term of \( T(z) \) and \((r + 1)(r - 6)\) as the resonance equation for each \( \nu = 1, 2, \ldots \). Interestingly, for the resonance \( r = 6 \), the two special cases \( \nu = 1, 2 \) do not lead to incompatible system of equations, in contrast to all higher values of \( \nu \). This is very different from the cases we have been dealing with up to now. According to the ARS method, the cases when \( \nu = 1, 2 \) admit the Painlevé property [9, §1.2, Definition 1.1] and have solutions in terms of Laurent expansion with two free parameters; in other words, they are integrable, and we will derive closed-form solutions for them. The remaining cases when \( \nu \geq 3 \) have psi-series solutions.

**Exactly solvable (integrable) case : \( \nu = 1 \).** We start with the case \( \nu = 1 \). Consider the transformations \( T(z) = 1 - \zeta V(\zeta) \) and \( z = -\zeta \). Note that, by this transform, the coefficients \([\zeta^n]V(\zeta)\) are positive and the transformed DE (after multiplying \( V'(\zeta) \)) becomes

\[
\frac{1}{3} \zeta^{-2} \frac{d}{d\zeta} \left( \zeta \left( \frac{dV}{d\zeta} \right)^2 - V(\zeta)^3 \right) = 0;
\]

or equivalently,

\[
\sqrt{\zeta} \frac{dV}{d\zeta} = \sqrt{V(\zeta)^3 - 1}, \quad V(0) = 1.
\]

(44)

By the relation between \( T(z) \) and \( V(\zeta) \), we deduce that \( V(0) = 1 \) and \( V'(0) = 3 \). Then (44) is solved as

\[
2\sqrt{\zeta} = \int_{1}^{V(\zeta)} \frac{dx}{\sqrt{x^3 - 1}}.
\]

(45)
Let
\[ 2\sqrt{\zeta_\infty} = \int_1^\infty \frac{dx}{\sqrt{x^3 - 1}} \approx 2.42865 06478 87581 61181 \ldots, \]
or \[ \zeta_\infty \approx 1.47458 59923 71192 48035 \ldots. \]
Obviously \( V(\zeta) \to \infty \) as \( \zeta \to \zeta_\infty \). Let \( \Delta := 2(\sqrt{\zeta_\infty} - \sqrt{\zeta}) \). Then (45) can be written as
\[ \Delta = \int_{\cal V(\zeta)}^\infty \frac{dx}{\sqrt{x^3 - 1}}. \]
Since \( V(\zeta) \to \infty \) as \( \zeta \to \zeta_\infty \), we deduce that
\[ \Delta = 2V(\zeta) - \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{1}{2}\right)} + \text{smaller order terms}. \]
Consequently, by inverting the series (justified by analyticity and standard arguments), we obtain
\[ V(\zeta) = 4\Delta - \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{\left(\frac{1}{2}\right)} + \text{smaller order terms}. \]
Finally, let \( \rho := -\zeta_\infty \) and we obtain
\[ t_n = [z^n]T(z) = (-1)^n-1[\zeta^{n-1}]V(\zeta) = \frac{(-1)^n-1}{2\pi i} \oint_{|\zeta| = c,<\zeta_\infty} \zeta^{-n}V(\zeta) d\zeta \]
\[ = \frac{2(-1)^n-1}{2\pi i} \oint_{|y| = c',<\zeta_\infty} y^{-2n+1}V(y^2) dy \]
\[ \sim 8(-1)^n-1[y^{2n-2}] \left(2\sqrt{\zeta_\infty} - 2y\right)^{-2} \]
\[ = (-1)^n-1(4n - 2)\zeta^{-n}_\infty \]
\[ = 2(-1)^n-1(2n - 1)|\rho|^{-n}, \]
the errors omitted being exponentially smaller.

**Exactly solvable (integrable) case**: \( \nu = 2 \). The case when \( \nu = 2 \) is similar. We now adopt the transformations \( T(z) = 1 - \zeta^2 L(\zeta) \) and \( z = -\zeta^3 \). Then the DE (42) becomes
\[ \frac{d^2}{d\zeta^2}L(\zeta) - 6L(\zeta)^2 = 0 \iff \frac{d}{d\zeta} \left( \frac{1}{2} \left( \frac{dL}{d\zeta} \right)^2 - 2L(\zeta)^3 \right) = 0, \]
with the initial values \( L(0) = 0 \) and \( L'(0) = 1 \). Thus, the solution is given by
\[ \zeta = \int_0^{L(\zeta)} \frac{dx}{\sqrt{1 + 4x^3}}. \] (46)
Let \( \zeta_\infty \) denote the dominant singularity of \( L(\zeta) \). Then
\[ \zeta_\infty = \int_0^\infty \frac{dx}{\sqrt{4x^3 + 1}} = 2^{1/3} \frac{\Gamma(1/3)}{6^{1/3}} \text{Beta} \left( \frac{1}{6}, \frac{1}{3} \right) \approx 1.76663 87502 85449 95731 \ldots. \]
Thus the dominant singularity of $T(z)$ when $\nu = 2$ is

$$\rho = -\zeta_\infty^3 = -\frac{1}{108} \Beta \left( \frac{1}{6}, \frac{1}{3} \right)^3 \approx -5.51370 15767 10567 75506 \ldots$$

Furthermore, from (46), we have

$$\Delta := \zeta_\infty - \zeta = \int_{L(\zeta)}^\infty \frac{dx}{\sqrt{4x^3 + 1}},$$

and, by the same procedure as above,

$$\Delta = L(\zeta)^{-1/2} - \frac{1}{56} L(\zeta)^{-7/2} + \frac{3}{1664} L(\zeta)^{-13/2} - \frac{5}{19456} L(\zeta)^{-19/2} + \text{smaller order terms},$$

for $\zeta \sim \zeta_\infty$. By inverting the expansion

$$L(\zeta) = \Delta - \frac{\Delta^4}{28} + \frac{\Delta^{10}}{10192} - \frac{\Delta^{16}}{9868302080} + \frac{3\Delta^{22}}{9868302080} - \text{smaller order terms},$$

Accordingly,

$$t_n = [z^n] T(z) = \frac{1}{2\pi i} \oint_{|z|<\rho} z^{-n-1} T(z) \, dz$$

$$= \frac{3(-1)^n}{2\pi i} \oint_{|\zeta|<\rho} \zeta^{-3n-1} T(-\zeta^3) \, d\zeta$$

$$= \frac{3(-1)^{n-1}}{2\pi i} \oint_{|\zeta|<\zeta_\infty} \zeta^{-3n+1} L(\zeta) \, d\zeta = 3(-1)^{n-1}[\zeta^{3n-2}]L(\zeta)$$

$$\approx 3(-1)^{n-1} [\zeta^{3n-2}] (\zeta_\infty - \zeta)^{-2}$$

$$= 3(-1)^{n-1} (3n-1) \zeta_\infty^{-3n}$$

$$= 3(-1)^{n-1} (3n-1) |\rho|^{-n}.$$

Note that we can use the transforms $z = \zeta^2$ and $T(z) = 1 - V(\zeta)\zeta^2$ to convert the DE for $\nu = 1$ to a DE of same type (differing only by a constant) as the case for $\nu = 2$. Also both solutions can be expressed in terms of Weierstrass $\wp$ functions.

The rest cases: $\nu \geq 3$. Unlike the preceding two cases, the rest $\nu$’s no longer lead to DEs that are solvable by quadrature. Due to incompatibility, we apply again the psi-series method. Because of the negative sign on the right-hand side of (42), we consider the transform $z = -\zeta$ and $T(z) = 1 - \zeta V(\zeta)$. Then

$$t_n = [z^n] T(z) = (-1)^{n-1} \left[ \zeta^{n-1} \right] V(\zeta),$$

and (43) is translated into

$$\frac{\nu(\nu + 1)}{\nu + 2} \zeta V''(\zeta) + \frac{2\nu^2 + \nu - 2}{\nu + 2} V'(\zeta) - V(\zeta)^2 = 0.$$

A DE is said to be solvable by quadrature if its solution can be expressed in terms of one or more integrations.
Let now $Z = 1 - \zeta/\rho$, where $\rho > 0$ is the dominant singularity of $V$ (having all Taylor coefficients positive). Then we deduce the psi-series expansion for $V$

$$
\rho V(\zeta) = \frac{6\nu(\nu + 1)}{\nu + 2} Z^{-2} - \frac{6(\nu^2 + 2\nu + 2)}{5(\nu + 2)} Z^{-1} + \sum_{0 \leq j \leq 5} c_j Z^j
+ KZ^4 \log Z + O \left( |Z|^5 |\log Z| \right),
$$

where

$$
K := -\frac{(\nu - 1)(\nu - 2)(\nu + 3)(\nu + 4)(2\nu + 1)(2\nu + 3)(3\nu + 2)(3\nu + 4)(\nu^2 + 2\nu + 2)^2}{43750\nu^5(\nu + 1)^5(\nu + 2)}.
$$

This, together with the approximations we derived for $t_n$ in the two cases $\nu = 1, 2$, implies the following asymptotics of $t_n$.

**Theorem 8** The sequence $t_n$ satisfies the asymptotic expansion

$$
(-1)^{n-1} t_n = \rho^n \left( \frac{6\nu(\nu + 1)}{\nu + 2} n - \frac{6(\nu^2 + 2\nu + 2)}{5(\nu + 2)} \right) + \begin{cases} 
O((1 - \varepsilon)^n), & \text{if } \nu = 1, 2; \\
24Kn^{-5} + O(n^{-6}), & \text{if } \nu \geq 3.
\end{cases}
$$

(47)

Note that $K = 0$ when $\nu = 1, 2$.

## 5 Conclusions

Through the examples we studied in this paper, we see that the psi-series method is a powerful approach to handling nonlinear DEs and yields several surprising results, notably asymptotic expansions with the first few terms missing. While psi-series have long been used in many branches of mathematics and physics, little attention has been paid to the corresponding asymptotics of the coefficients. Also the procedure we adapted and improved from Hille for proving the absolute convergence of psi-series is of certain generality and can be applied to other problems of similar nature.

Another feature of the recurrences we studied in this paper is that they are very sensible to small variations, the example of $d$ random BSTs being typical. Note first that the recurrence (24) with $d = 0$ yields the well-known Catalan numbers and the case $d = 1$ gives rise to the trivial sequence $p_n = 1$. The case $d = 1$ in a more general form was studied by Wright [33]; see also Cooper [10] for a study of $p_n$ for real $k \geq 0$.

We now compare the recurrence (24) with the following one by defining $p_1 = 1$ and

$$
p_n = n^{-d} \sum_{1 \leq j \leq n-1} p_j p_{n-j} \quad (n \geq 2).
$$

While the case $d = 0$ still yields the Catalan numbers with their generating function satisfying

$$
P(z) - z = P^2(z),
$$

the case $d = 1$ becomes a nonlinear differential equation of Riccati type

$$
zP'(z) - z = P^2(z), \quad P(0) = 0,
$$

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which can still be explicitly solved \( P(z) = z^{1/2} J_1(2z^{1/2}) / J_0(2z^{1/2}) \), where \( J_\nu(z) \)'s are Bessel functions (see [23]). The case \( d = 2 \) is again of Emden-Fowler type and can be solved asymptotically by psi-series method as well as the remaining cases \( d \geq 3 \).

See [10, 16, 18, 24, 32, 33] and the references therein for some quadratic recurrences of the above “Faltung” type. More examples can be found in the recent papers [3, 4].

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Appendix

A1. Proof of the absolute convergence of psi-series

In this Appendix, we group the details of the proof of the absolute convergence of the psi-series arising in the three cases: \(d\) random BSTs, two random \(m\)-ary search trees, and two random median-of-(\(2t+1\)) BSTs. We first describe briefly the general pattern of the proof and then provide more details for each case.

Our proof begins with rewriting the original DE in \(z\) into a system of linear DEs in \(Z = 1 - z/\rho\) of the form

\[
\frac{d}{dZ} U(Z) = \mathcal{X}(Z, U), \quad U(Z) = \begin{pmatrix} U_1(Z) \\ \vdots \\ U_s(Z) \end{pmatrix},
\]

(A.1)

where \(s \in \{d, 2(m - 1), 4t + 2\}\). Here \(U_j(Z) = \sum_{k \geq 0} u_k[j](\tau) Z^{-\alpha + k - j + 1}\), where \(\alpha\) is the leading order, \(\tau = \log Z\) and \(\mathcal{X} : \mathbb{C}^{s+1} \to \mathbb{C}^s\). Then we derive the infinite system of linear DEs satisfied by the \(u_k[j]\)’s

\[
\dot{\phi}_k + A_k \phi_k = g_k, \quad \phi_k = \begin{pmatrix} u_k[1] \\ \vdots \\ u_k[s] \end{pmatrix},
\]

(A.2)

where \(A_k = kI_{s \times s} - M\) and \(M \in \mathbb{C}^{s \times s}\) are \(s \times s\) matrices.

In terms of such an infinite system, an upper bound for all \(u_k[j]\) (in particular, for \(u_k[1]\)) is of the form

\[
|u_k[1](\tau)| \leq K \psi(k)|1 - \tau|^{k-c(s)},
\]

for \(\tau \in \mathcal{T}\)

\[
\mathcal{T} := \{ \xi + i\theta : \xi \in (-\infty, -\varepsilon] \text{ and } |\theta| \leq \pi \},
\]

(A.3)

with \(|1 - \tau| \geq 1 + \varepsilon\), where \(K\) is a constant and \(\psi(k), c(s)\) depend on the problem in question. Then the absolute convergence can be justified.

An additional common and interesting feature this approach brings is that the resonance equation will be seen to be equal to \(\det(rI_{s \times s} - M)\). We will explain this in more details.

The following relations are useful in converting our DEs in \(z\) into those in \(Z (D = d/dz)\).

\[
z = \rho(1 - Z), \quad zD = -(1 - Z)\frac{d}{dZ}, \quad z^jD^j = (-1)^j(1 - Z)^j\frac{d^j}{dZ^j}.
\]
Equality of \( d \) random BSTs. The corresponding system (A.1) for (25) is

\[
\begin{align*}
U'_j(Z) &= \frac{U_{j+1}(Z)}{1 - Z}, & 1 \leq j < d, \\
U'_d(Z) &= (-1)^d \rho U_1(Z)^2. 
\end{align*}
\]

The associated coefficient matrices \( A_k \) and \( g_k \) in (A.2), \( k \geq 3d + 1 \), are given by

\[
A_k = kI_{d \times d} - M, \quad M = \begin{pmatrix}
  d & 1 & 0 & \cdots & 0 \\
  0 & d+1 & 1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & 2d-2 & 1 \\
  (-1)^d \frac{(2d)!}{d!} & 0 & \cdots & 0 & 2d-1
\end{pmatrix},
\]

and

\[
g_k = \begin{pmatrix}
  \sum_{0 \leq \ell < k} u^{[2]}_k(\tau) \\
  \vdots \\
  \sum_{0 \leq \ell < k} u^{[d]}_k(\tau) \\
  (-1)^d \rho \sum_{1 \leq \ell < k} u^{[1]}_k(\tau) u^{[1]}_{k-\ell}(\tau)
\end{pmatrix}.
\]

Due to the existence of complex-conjugate roots, we can find a \( d \times d \) matrix \( P \) with entries \( P_{ij} \in \mathbb{C} \) such that

\[
PA_kP^{-1} = \begin{pmatrix}
k+1 & 0 & \cdots & \cdots & 0 \\
0 & k-3d & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & k-r_3 \\
0 & \cdots & 0 & 0 & r_d
\end{pmatrix},
\]

for \( k \in \mathbb{N} \). By the same norm and same arguments used for two random BSTs, we derive the inequality (\( C_d := \|P\|\|P^{-1}\| \))

\[
\max_{1 \leq j \leq d} \left\{ \left| u_k^{[j]}(\tau) \right| \right\} \leq \|\phi_k\|
\]

\[
\leq C_d \int_0^\infty e^{-(k-3d)x} \max \left( \sum_{0 \leq \ell < k} \left| u^{[1]}_{\ell} \right|, \ldots, \sum_{0 \leq \ell < k} \left| u^{[d]}_{\ell} \right| \right) dx.
\]

(A.4)

Again, by same the arguments used to prove (17), we have,

\[
\left| u_k^{[j]}(\tau) \right| \leq K(1 + k)^{-1/2} |1 - \tau|^{k-3d} \quad (1 \leq j \leq d, k \geq 0),
\]

for \( \tau \in \mathcal{T} \).
The resonance polynomial equals $\det(r I_{d \times d} - M)$. Direct calculations give the determinant

$$
\det (r I_{d \times d} - M) = \frac{(2d - 1 - r)!}{(d - 1 - r)!} \frac{(2d)!}{d!},
$$

which is nothing but the resonance polynomial (26).

The reason that the two polynomials are equal is as follows. The distinction between Laurent expansion and the psi-series expansion depends crucially either on the existence of positive integer resonance or on whether a relation such as (12) holds for all $k$. This is equivalent to asking whether the linear system $A_k \phi_k = g_k$ is solvable or not for all $k$. If the system (A.2) $A_k \phi_k = g_k$ is solvable under the condition $\det A_k \neq 0$ for all $k$, then by the uniqueness of the solution of (A.2), the solution vectors $\phi_k$’s are constant vectors (independent of $\tau$) and in turn, the series solution $U_1(Z) = \sum_{k \geq 0} u^{[1]}_k Z^{-d+k-j+1}$ is eventually a Laurent’s series. On the other hand, if $\det A_{k_0} \neq 0$ fails to hold for some $k_0$, then we have the following two cases.

— The linear system $A_{k_0} \phi = g_{k_0}$ has a solution depending on the $d - \text{rank} (A_{k_0})$ free parameters, and all the rest constant coefficient vectors $\phi_k$ depend on at least these parameters.

— The linear system is inconsistent. Hence it can no longer provide a solution to (A.2). The real solution should be solved from (A.2) instead and then all the vector functions $\phi_k(\tau)$, $k \geq k_0$, depend on $\tau$. Moreover, the resulting solution $U_1(Z) = Z^{-d} \sum_{k \geq 0} u^{[1]}_k Z^{k-j+1}$ is indeed a psi-series.

In particular, we see that the characteristic polynomial $\det (r I_{d \times d} - M)$ is the same as the polynomial (26) that determines all the possible resonances.

Equality of two random $m$-ary search trees. The transformed first-order differential system in terms of $Z$ for (28) now has the form

$$
\begin{align*}
U'_1(Z) &= U_2(Z) & 1 \leq j \leq m - 2, \\
U'_{m-1}(Z) &= (1 - Z)^{-(m-1)} U'_m(Z), \\
U'_{m-1+j}(Z) &= U_{m+j}(Z), & 1 \leq j \leq m - 2, \\
U'_{2m-2}(Z) &= (m-1)!^2 \sum_{k \geq 0} u^{[1]}_k Z^{k-1} U_1(Z)^m.
\end{align*}
$$

So that the corresponding infinite system (A.2) has the coefficient matrix $A_k = k I_{2(m-1) \times 2(m-1)} - M$, where

$$
M = \begin{pmatrix}
2 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 3 & 1 & 0 & \ddots & \vdots \\
\vdots & 0 & 4 & 1 & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
&m & \cdots & \cdots & 1 & \vdots \\
& & & \ddots & \ddots & \ddots \\
0 & & & & 0 & 2m-2 \\
m(2m-1)! & 0 & \cdots & \cdots & 0 & 2m-1
\end{pmatrix},
$$

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and the vector-valued function

$$g_k = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\sum_{0 \leq j < k} \binom{m-2+k-j}{k-j} u_j^{|m|} \\
\vdots \\
0 \\
\rho^{m-1}(m-1)!^2 \sum_{i_1+i_2+\cdots+i_m=k} u_i^{|1|} u_j^{|1|} \cdots u_l^{|1|} \\
\end{pmatrix}.$$ 

Then similar arguments as those used for (17) leads to the upper bound

$$\left| u_k^{|j|}(\tau) \right| \leq K \left( \frac{k-1+1/m}{k} \right) |1-\tau|^{k-2m-2}, \quad (1 \leq j \leq 2(m-1), k \geq 0),$$

for $\tau \in \mathcal{P}$, where the constant $K$ is easily tuned according to the initial conditions.

**Equality of two random median-of-$(2t+1)$ BSTs.** The linear differential system of $4t+2$ equations of (30) is

$$
U_j'(Z) = U_{j+1}(Z), \quad 1 \leq j \leq 2t,
$$

$$
U_{2t+1}'(Z) = (1-Z)^{-(2t+1)} U_{2t+2}(Z),
$$

$$
U_j'(Z) = U_{j+1}(Z), \quad 2t+2 \leq j \leq 4t+1,
$$

$$
U_{4t+2}'(Z) = \frac{(2t+1)!^2}{t!^4} \rho \sum_{0 \leq i_1,i_2 \leq t} \mu(i_1,i_2)(1-Z)^{2t-i_1-i_2} U_{2t+1-i_1}(Z) U_{2t+1-i_2}(Z),
$$

where

$$
\mu(i_1,i_2) := \frac{(-1)^{i_1+i_2} t!^4}{i_1! i_2! (t-i_1)!^2 (t-i_2)!^2}.
$$

Let $U_j(Z) = \sum_{k \geq 0} u_k^{|j|}(\tau) Z^{k-j-1}$ for $1 \leq j \leq 4t+2$, where

$$
u_k^{|j|} = (-1)^{j-1} j! (4t+3)! t!^4 \rho (2t+1)!^2 \quad (1 \leq j \leq 2t+1).
$$

Then coefficient matrix $A_k = k I_{4t+2 \times 4t+2} - M$ in (A.2), $k \geq 4t+2$, is given by

$$
M = \begin{pmatrix}
2 & 1 & 0 & \cdots & 0 \\
0 & 3 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 2t+1 & 1 \\
0 & 2t+2 & 1 & \cdots \\
0 & 2t+3 & 1 & \cdots & 0 \\
0 & 4t+2 & 1 & \cdots \\
0 & 0 & \cdots & 0 & \frac{2(4t+3)!}{(2t+1)!^2} \\
\end{pmatrix},
$$

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and the vector-valued function $g_k$ by

$$g_k = \left( \begin{array}{c}
0 \\
\vdots \\
0 \\
\sum_{0 \leq j < k} \left( \frac{2t + k - j}{k - j} \right)^{[2t+2]} u_j \\
\vdots \\
0 \\
\frac{2t+1}{t!} \rho H \left( u_j^{[t+1]} \left| \begin{array}{c} 0 \leq j < k \\
0 \leq \ell \leq t \end{array} \right. \right) \\
\end{array} \right)$$

where

$$H \left( u_j^{[t+1]} \left| \begin{array}{c} 0 \leq j < k \\
0 \leq \ell \leq t \end{array} \right. \right) = \mu(0,0) \sum_{0 \leq \ell \leq k-j} \sum_{1 \leq j \leq \min(k,t)} (-1)^j \left( \begin{array}{c} 2t \\
j \end{array} \right) u_k^{[2t+1]} u_{k-j-\ell}^{[2t+1]} + \sum_{1 \leq s \leq \min(k,t)} \mu(i,s-i) \sum_{0 \leq \ell \leq k-j} \sum_{1 \leq j \leq \min(s,t)} (-1)^j \left( \begin{array}{c} 2t - s \\
j \end{array} \right) u_k^{[2t+1-i]} u_{k-s-j-\ell}^{[2t+1+i-s]}.$$ 

The the same method of proof used for (17) yields the upper bound ($r_0 = 2$ or $6t + 6$)

$$\left| u_k^{[j]}(\tau) \right| \leq C(1 + k)^{-1/2} |1 - \tau|^{k-r_0}, \quad (1 \leq j \leq 4t + 2, k \geq 0),$$

uniformly for $\tau \in \mathcal{T}$, where the constants $C$ and $K$ are easily tuned according to the initial conditions.

**A2. Proof of the incompatibility of the resonance $r = 2$ for random median-of-$(2t + 1)$ BSTs**

Since the resonance $r = 2$ does not depend on $t$, the incompatibility of the resonance $r = 2$ can be directly checked, which we now do. Let $U(Z) := F(z)$, where $F$ satisfies the DE (30) and $Z = 1 - z/\rho$. Then the DE (30) can be rewritten as

$$((1 - Z)^{2t+1} U^{(2t+1)}(Z))^{(2t+1)} = C_{t,\rho} \left( ((1 - Z)^{t} U^{(t)}(Z))^{(t)} \right)^2,$$  

where all derivatives are with respect to $Z$ and $C_{t,\rho} := (2t + 1)!^2 \rho/t!^4$.

Consider the formal Laurent expansion $f(Z) = \sum_{k \geq 0} u_k Z^{k-\alpha}$. Then for any $s \in \mathbb{N}$, we have

$$((1 - Z)^s f^{(s)}(Z))^{(s)} = \sum_{k \geq 0} (k - \alpha - s)^2 Z^{k-2s-\alpha} \sum_{0 \leq j \leq s} (-1)^j \left( \begin{array}{c} s \\
j \end{array} \right) (k - \alpha - j)^2 u_{k-j},$$  

(A.6)
where \( u_j := 0, j < 0 \). Substituting this into (A.5), we have
\[
\sum_{k \geq 0} (k - \alpha - (2t + 1))^{2t+1} Z^{k-4t-2\alpha} \sum_{0 \leq j \leq 2t+1} (-1)^j \binom{2t+1}{j} (k - \alpha - j)^{2t+1} u_{k-j} = C_{t,\rho} \sum_{k \geq 0} Z^{k-4t-2\alpha} \sum_{0 \leq \ell \leq k} \chi_k \chi_{k-\ell},
\]
where
\[
\chi_k = (k - \alpha - t)^t \sum_{0 \leq j \leq t} (-1)^j \binom{t}{j} (k - \alpha - j)^{2t+1} u_{k-j}.
\]
Equating the dominant term (with \( k = 0 \)) leads to the obvious solution \( \alpha = 2 \). Consider now the relation
\[
(k - \alpha - (2t + 1))^{2t+1} \sum_{0 \leq j \leq 2t+1} (-1)^j \binom{2t+1}{j} (k - \alpha - j)^{2t+1} u_{k-j} = C_{t,\rho} \sum_{0 \leq \ell \leq k} \chi_k \chi_{k-\ell}.
\]
For \( k = 0 \), we get \( \rho u_0 = (4t+3)!/2t+1)!^4 \), and for \( k = 1 \), we get \( u_1 = -2(t+1)^2u_0/(6t+5) \). Now for \( k = 2 \), we have
\[
0 \cdot u_2 = \left( -\frac{(4t+1)!}{(2t)!}, 0^{2t+1} + 2(-1)^{t+1}(2t+1)!/(t-1)! u_0 \cdot 0^t \right. u_2
\]
\[
= C_{t,\rho}(2t)!^2 \left( (2t+1)\binom{t+1}{t/2} + t^2(t+1)^2 \right) u_0^2 + u_1^2 - t(4t+3)u_0u_1
\]
\[
+ (4t+1)!^2u_1 - (4t+1)(2t+1)(2t+2) \binom{2t+1}{2} u_0
\]
\[
= -\frac{(4t+2)!}{4(6t+5)^2} u_0 \left( 216t^4 + 522t^3 + 437t^2 + 141t + 12 \right) \neq 0,
\]
since \( t \geq 1 \). This proves the incompatibility of the resonance \( r = 2 \) for all \( t \geq 1 \).

### A3. Asymptotics of the moment generating function

We prove (35), starting from Hankel’s integral representation of the Gamma function
\[
\frac{1}{\Gamma(w)} = \frac{1}{2\pi i} \int_{\mathcal{H}_0} e^{s w} ds \quad (w \in \mathbb{C}),
\]
where \( \mathcal{H}_0 \) starts at \( -\infty \), encircles the origin once counter-clockwise and returns to its starting point. For definiteness, we may take
\[
\mathcal{H}_0 = \{ s = xe^{\pm i\pi} : R_0 \leq x < \infty \} \cup \{ s = R_0 e^{i\theta} : -\pi \leq \theta \leq \pi \} \quad (R_0 > 0).
\]
This gives
\[
M(z) = \frac{1}{2\pi i} \int_{\mathcal{H}_0} e^{s z} A(z/s^{\alpha-1}) ds,
\]
where \( A(z) \) satisfies the DE (32). Note that \( M \) is an entire function of order \( 1/v > 1 \) and of type \( \rho^{-1/v} \).
Let \( z = |z|e^{i\varphi}, |z| > 0 \) and \( |\varphi| < v\pi/2 \), where \( v = (\sqrt{17} - 3)/2 \). The condition on \( \arg z \) implies that the dominant singularity \( s = (z/\rho)^{1/v} \) of the integrand lies in the half-plane \( \Re(s) > 0 \) (in which \( e^s \to \infty \) with \( z \)). On the other hand, if \( |\arg(-z)| < \pi - v\pi/2 \), then one expects that \( M(z) \to 0 \) with \( z \), but the exact determination of the rate is more delicate. The situation here is similar to the Mittag-Leffler function \( \sum_{j\geq0} z^j/\Gamma(\alpha j + 1) \); see [13] Ch. 18.1.

The change of variables \( z/s^v \to s \) gives

\[
M(z) = \frac{1}{2\pi iv} \int_{\mathcal{H}_1} e^{z/v s^{-1/v}} s^{-1} A(s) \, ds,
\]

where \( \mathcal{H}_1 \) is the cut circle described by

\[
\mathcal{H}_1 = \{ s = xe^{i\varphi \pm i\pi} : 0 \leq x \leq R_1 \} \cup \{ s = R_1 e^{i\varphi i\theta} : -\pi \leq \theta \leq \pi \}.
\]

Here \( 0 < R_1 < \rho \). We then approach in a way similar to the singularity analysis (see [17]) by deforming the contour \( \mathcal{H}_1 \) into \( \mathcal{H}_2 \), where \( \mathcal{H}_2 \) is of the same shape as \( \mathcal{H}_1 \) but with larger radius for the circular part \( |s| = R_2 = \rho + \varepsilon \) and avoiding the cut from \( s = \rho \) to \( \infty \) (in the style of [17]). Symbolically,

\[
\mathcal{H}_2 = \{ s = xe^{i\varphi \pm i\pi} : 0 \leq x \leq R_2 \} \\
\cup \{ s = R_2 e^{i\varphi i\theta} : -\pi \leq \theta \leq \pi \text{ and } |\theta - \varphi| \geq \varepsilon_z \} \\
\cup \Gamma_{\rho},
\]

where \( \varepsilon_z = |z|^{-1/v} \) and \( \Gamma_{\rho} \) is any contour joining the two points \( R_2 e^{-i\varepsilon_z} \) and \( R_2 e^{i\varepsilon_z} \) and lying inside the cut region described by other parts of \( \mathcal{H}_2 \).

The remaining analysis is then easy because the main contribution to \( M(z) \) comes from \( \Gamma_{\rho} \) on which we can apply the local expansion (33) of \( A(z) \), the other parts being negligible

\[
M(z) = \frac{1}{2\pi iv} \int_{\Gamma_{\rho}} e^{z/v s^{-1/v}} s^{-1} A(s) \, ds + O \left( e^{R(z/\rho + \varepsilon)^{1/v}} \right).
\]

By making first the change of variables \( \rho(1 - s) \to s \), using the expansion (33), and then another change of variables \( (z/\rho)^{1/v} s/v \to s \), we deduce that

\[
M(z) = \left. \frac{e^{(z/\rho)^{1/v}}}{2\pi i} \int_{\Gamma_{0}} e^{s} \left( 3 \left( \frac{z}{\rho} \right)^{1/v} s^{-2} + \frac{9}{5} s^{-1} + \sum_{2 \leq j \leq 7} \left( \frac{\bar{c}_j(s) + c_j(s)}{\rho} \right) \left( \frac{z}{\rho} \right)^{-j/v} \right) \right|_{\rho = \rho_{0}} + \frac{936}{21875} \left( \frac{z}{\rho} \right)^{-5/v} s^4 \log s + O \left( |z|^{-6/v} |s|^5 |\log s| \right) ds,
\]

where \( \Gamma_{0} \) denotes the transformed contour of \( \Gamma_{\rho} \) and the \( c_j \)'s are polynomials of \( s \) whose exact values matter less. Extending the contour to infinity and then evaluating the individual terms by Hankel’s integral representation of the Gamma function, we obtain

\[
M(z) = e^{(z/\rho)^{1/v}} \left( 3 \left( \frac{z}{\rho} \right)^{1/v} + \frac{9}{5} - \frac{22464}{21875} \left( \frac{z}{\rho} \right)^{-5/v} + O \left( |z|^{-6/v} \right) \right),
\]

where we also used the formula

\[
\left. \frac{1}{2\pi i} \int_{\mathcal{H}_{0}} e^{s} s^4 \log s \, ds = -\frac{d}{dx} \frac{1}{x} \left. \right|_{x=-4} = -24.
\]

This completes the proof of (35).