Hubbard operators in multiqubit systems

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To Professor Bogdan Mielnik, the owner of Bar Quantum

Abstract. The properties of the Kronecker product of $2 \times 2$ matrices are reviewed in terms of Hubbard operators. This framework constitutes a shorthand notation to deal with the tensor algebra of operators acting on multiqubit states. As an application we derive some analytical expressions related to the geometric measure of entanglement of a certain class of multiqubit invariant permutation states. Our results can be straightaway extended for systems of larger dimensions.

1. Introduction

A system made up of several components is called multipartite [1, 2]. Of particular interest are the quantum systems integrated by several qubits because of their promising applications in quantum computing, quantum information, quantum cryptography, etc [3]. The Kronecker product (also known as tensor or direct product) plays an important role in the mathematical description of such systems. However, as the number of qubits increases some calculations involving the tensor product become tedious. For instance, consider the following operator acting on a Hilbert space of $k$ qubits

$$\sigma_3^\otimes k = \sigma_3 \otimes \cdots \otimes \sigma_3, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

As a matrix, $\sigma_3^\otimes k$ is clearly diagonal with entries equal to either 1 or $-1$. A problem arises when one wants to determine the sign of a given element if an explicit calculation should be performed. In a recent work [4] we show that the Hubbard operators (also referred as X-operators) can be used as an accurate tool to deal with the Kronecker product of arbitrary square matrices $A, B, C, \ldots$. In fact, this approach allows to find the element $(i, j)$ of the operator $A \otimes B \otimes C \otimes \cdots$ by using simple relations of subscripts. The relevance of such simplification is clear by considering for instance that the multiple tensor product of operators lies on the basis of the quantum entanglement. Indeed, it is required for a given entanglement measure to be invariant under the action of local unitary operators. That is to say, the $n$-partite state $|\psi\rangle$ must have the same entanglement as the state $|\tilde{\psi}\rangle = U_{\text{loc}} |\psi\rangle$, where the operator $U_{\text{loc}}$ is the tensor product of $n$ unitary operators. However, there is not a unique form to quantify
entanglement, even in the simplest case of bipartite systems. For two qubits the concurrence introduced by Hill and Wootters is a widely used option [5,6]. This is the case for example, in the analysis of the entanglement of a pair of qubits in quantized radiation fields [7]. Nevertheless, the characterization of entanglement in multipartite systems is a tough problem in contemporary physics [8,9]. An interesting proposal is the geometric measure of entanglement [10,11]. This function is related to the maximum overlap of the state we are considering with the closest separable state. Being related to an optimization problem, the evaluation of such overlap is a non trivial problem. Nonetheless, it has been evaluated analytically for some specific states like the multiqubit permutation invariant states [12–14]. However, there is not a general way to compute such a quantity.

In this contribution we construct the appropriate representation of the Hubbard operators for multiqubit systems. The aim is to provide a suitable framework to deal with the involved tensor algebra in a simpler manner. Through this work we use some results concerning the Kronecker product in terms of the X-operators reported in Ref. [4]. The paper is structured as follows. In Section 2 Hubbard notation is introduced for one qubit and multiqubit systems. As an application of this formalism some results related to the geometric measure of entanglement for symmetric states are obtained in Section 3. We also compare our results with those reported in the literature. Finally, in Section 4 some concluding remarks are presented.

2. Formalism
The X-operators were introduced by Hubbard [15–17] and have applications in several branches of physics (see, e.g. [4,18] and references quoted therein). These obey the multiplication rule

\[ X_{i,j} X_{k,m} = \delta_{jk} X_{i,m}, \]  

(2)

and satisfy the following properties

\[ (X_{i,j})^\dagger = X_{j,i}, \quad \sum_k X_{k,k} = I, \quad [X_{i,j}, X_{k,m}]_\pm = \delta_{jk} X_{i,m} \pm \delta_{mi} X_{k,j}. \]  

(3)

where \([A,B]_\pm\) stands for either the commutator (−) or the anticommutator (+) of \(A\) and \(B\). This representation allows a shorthand notation for cumbersome calculations involving several factors or large dimension matrix operators [4].

2.1. One qubit
To give a concrete realization of the Hubbard operators consider a system with just two levels. The excited \(|e_2\rangle = |0\rangle\) and the ground states \(|e_1\rangle = |1\rangle\) are orthogonal

\[ \langle e_0 | e_0 \rangle = \langle e_1 | e_1 \rangle = 1, \quad \langle e_0 | e_1 \rangle = \langle e_1 | e_0 \rangle = 0, \]  

(4)

the supra-index makes reference to the dimension of the corresponding Hilbert space \(\mathcal{H}_2 = \text{Span}\{|0\rangle, |1\rangle\}\). The basis vectors in their simplest representation are written

\[ |e_1^2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e_2^2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

(5)

Any vector \(|\psi\rangle \in \mathcal{H}_2\) can be written as a normalized linear combination, that is

\[ |\psi\rangle = c_0 |e_0^2\rangle + c_1 |e_1^2\rangle = c_0 |0\rangle + c_1 |1\rangle, \quad |c_0|^2 + |c_1|^2 = 1, \quad c_0, c_1 \in \mathbb{C}. \]  

(6)
The simplest representation of the Hubbard operators can be written as the outer product of the basis vectors. In Dirac’s notation

\[ X^{ij}_2 := |e^2_{i}| \langle e^2_{j}|, \quad i, j = 0, 1, \]  

the subindex 2 indicates the dimension of the Hilbert space where \( X^{ij}_2 \) acts on. Note that the matrix representation of this last operator has 1 in the entry \((i + 1, j + 1)\) and zero elsewhere. We want to stress that since the Hubbard operators are complete any linear operator \( O : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \) can be be written in X-representation

\[ O = \sum_{i,j=0}^{1} a_{i,j} X^{ij}_2, \quad a_{i,j} = \langle e^2_{i}| o |e^2_{j} \rangle. \]  

The representation in terms of Hubbard operators can generalized to the case of any finite dimension Hilbert space. In particular we are interested in the case of several qubits.

### 2.2. Multiqubit systems

The four-dimensional Hilbert space \( \mathcal{H}_4 \) of a two qubit system is given by

\[ \mathcal{H}_4 = \text{Span}\{ |e^2_{11} \rangle \otimes |e^2_{11} \rangle, \quad i_0, i_1 = 0, 1 \}. \]  

The basis elements of \( \mathcal{H}_4 \) can be expressed using either a single index notation or the binary form

\[ |e^4_{1} \rangle = |e^2_{0} \rangle \otimes |e^2_{0} \rangle = |00\rangle, \quad |e^4_{2} \rangle = |e^2_{1} \rangle \otimes |e^2_{1} \rangle = |01\rangle, \quad |e^4_{3} \rangle = |e^2_{0} \rangle \otimes |e^2_{1} \rangle = |10\rangle, \quad |e^4_{4} \rangle = |e^2_{1} \rangle \otimes |e^2_{0} \rangle = |11\rangle. \]  

In a more compact form we write

\[ |e^4_{i} \rangle \equiv |i\rangle \equiv |i_1 i_0\rangle, \quad i = (i_1 i_0)_2, \]  

where \((a_1 a_0)_2\) indicates that the number \( \alpha \) is expressed in its binary decomposition \( \alpha = 2^1 a_1 + 2^0 a_0 \), for \( 0 \leq \alpha \leq 3 \). An arbitrary normalized vector \( |\psi\rangle \in \mathcal{H}_4 \) can be written as a linear combination of the basis elements

\[ |\psi\rangle = \sum_{i=0}^{3} c_i |e^4_{i} \rangle = \sum_{i=0}^{3} c_i |i\rangle = \sum_{i_1, i_0=0}^{1} c_{i_1, i_0} |i_1 i_0\rangle, \quad c_i \equiv c_{i_1, i_0}, \quad i = (i_1 i_0)_2. \]  

We can generalize this notation to the case of \( n \) qubits. The corresponding Hilbert space \( \mathcal{H}_{2^n} \) is given as

\[ \mathcal{H}_{2^n} = \text{Span}\{ |e^{2^n}_{i_{n-1}} \rangle \otimes \cdots \otimes |e^{2^n}_{i_0} \rangle, \quad i_\ell = 0, 1, \quad \ell = 0, \ldots, n-1 \} = \text{Span}\{ |e^{2^n}_{i} \rangle, \quad i = 0, \ldots, 2^n - 1 \} = \text{Span}\{ |i\rangle, \quad i = 0, \ldots, 2^n - 1 \} = \text{Span}\{ |i_{n-1} \cdots i_0\rangle, \quad i_\ell = 0, 1, \quad \ell = 0, \ldots, n-1 \}, \]  

where

\[ |e^{2^n}_{i} \rangle \equiv |i\rangle \equiv |i_{n-1} \cdots i_0\rangle, \quad i = (i_{n-1} \cdots i_0)_2. \]  

Therefore, any element \( |\psi\rangle \in \mathcal{H}_{2^n} \) is in general a linear combination of the form

\[ |\psi\rangle = \sum_{i=0}^{2^n-1} c_i |e^{2^n}_{i} \rangle = \sum_{i=0}^{2^n-1} c_i |i\rangle = \sum_{i_{n-1}, \ldots, i_0=0}^{1} c_{i_{n-1}, \ldots, i_0} |i_{n-1} \cdots i_0\rangle, \quad c_i \equiv c_{i_{n-1}, \ldots, i_0}, \quad i = (i_{n-1} \cdots i_0)_2. \]
Proposition 1. establishes that the tensor product is closed on the set of X-operators.

As an example consider the Dicke states \([19]\):

\[
|S(n, k)\rangle = \left(\begin{array}{c}
n \\ k \end{array}\right)^{-1/2} \sum_{\pi \in S_n} |\pi(0 \cdots 0 1 \cdots 1)\rangle, \quad \left(\begin{array}{c}n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}.
\]

The sum is performed over all the distinct permutations of the qubits and \(S_n\) denotes the symmetric group of order \(n\). For instance, let us write down the explicit form of a four qubit Dicke state with \(k = 3\):

\[
|S(4, 3)\rangle = \frac{1}{2} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)
\]

where \(1 = (0001)_2, 2 = (0010)_2, 4 = (0100)_2,\) and \(8 = (1000)_2\). On the other hand, any linear operator acting on \(\mathcal{H}_2^n\) can be represented as a \(2^n \times 2^n\) matrix. In this case the Hubbard operators constitute an useful tool to deal with the involved algebra. We start with the case of order 2 and then the general case will be discussed. The following proposition establishes that the tensor product is closed on the set of X-operators.

**Proposition 1.** Let \(X_2^{i_0,j_0}\) and \(X_2^{i_1,j_1}\) be two Hubbard operators of order 2. Then the \(\otimes\)-product

\[
X_2^{i_1,j_1} \otimes X_2^{i_0,j_0} = X_4^{2i_1 + i_0, 2j_1 + j_0}, \tag{18}
\]

is a Hubbard operator of order 4.

The proof can be consulted in \([4]\). This last proposition indicates that to localize the row (column) where the non-vanishing element of the resulting operator is, one has to “count” \(i_1\) (\(j_1\)) blocks of size 2 plus \(i_0\) (\(j_0\)) rows (columns). We can use a single index notation by defining \(i = 2i_1 + i_0\) and \(j = 2j_1 + j_0\). Note that the column (row) indices of the factor operators are the binary coefficients of the column (row) index \(i\) (\(j\)) of the resultant Hubbard operator. This can be expressed as follows

\[
X_2^{i_1,j_1} \otimes X_2^{i_0,j_0} = X_4^{i,j}, \quad i = (i_1i_0)_2, \quad j = (j_1j_0)_2. \tag{19}
\]

Equation (19) is easily generalized by induction. Indeed,

\[
X_2^{i_n,j_n} \otimes \cdots \otimes X_2^{i_0,j_0} = X_2^{i,j}, \quad i = (i_n \cdots i_0)_2, \quad j = (j_n \cdots j_0)_2. \tag{20}
\]

As an example consider the following product

\[
X_2^{1,0} \otimes X_2^{0,0} \otimes X_2^{1,1} = X_8^{5,1}, \quad 5 = (101)_2 \quad 1 = (001)_2.
\]

In the following Theorem we show that the Kronecker product of two arbitrary \(2 \times 2\) matrices of order two can be written in terms of the Hubbard operators.

**Theorem 1.** The Kronecker product of the \(2\)-square matrices \(A = [a_{i_1,j_1}]\) and \(B = [b_{i_0,j_0}]\) in terms of the X-operators of order 4 reads

\[
A \otimes B = \sum_{i,j=0}^{3} c_{i,j} X_4^{i,j}.
\]
Proof. From the linearity of $\otimes$ and Proposition 1 we have
\[ A \otimes B = \sum_{i_0,j_0=0}^{1} a_{i_1,j_1} b_{i_0,j_0} X_2^{i_1,j_1} \otimes X_2^{i_0,j_0} = \sum_{i_0,j_0=0}^{1} a_{i_1,j_1} b_{i_0,j_0} X_2^{2i_1+i_0,2j_1+j_0}. \]

Now define $i = 2i_1 + i_0$ and $j = 2j_1 + j_0$, so that $i, j = 0, 1, 2, 3$. Then
\[ A \otimes B = \sum_{i,j=0}^{3} a_{i_1,j_1} b_{i_0,j_0} X_2^{i,j}, \quad i = (i_1 i_0)_2, \quad j = (j_1 j_0)_2. \] (21)

Identifying $c_{i,j} := a_{i_1,j_1} b_{i_0,j_0}$ the proof is completed. \qed

The properties of the Kronecker product of two arbitrary matrices in terms of the Hubbard operators are reviewed in [4]. We can use the fact that the $\otimes$-product is associative as well as Theorem 1 to compute the tensor product powers of a $2 \times 2$ matrix with itself
\[ A^{\otimes k} = \sum_{i,j=0}^{2^{k-1}} a_{i_{n-1},j_{n-1}} \cdots a_{i_0,j_0} X_2^{i,j}, \quad i = (i_{n-1} \cdots i_0)_2, \quad j = (j_{n-1} \cdots j_0)_2. \] (22)

The most general case of the tensor product of any size matrices is reviewed in [4]. Equation (22) allows to determine the sign of a given entry of the operator (1). First consider
\[ \sigma_3^{\otimes k} = \sum_{i=0}^{2^{k-1}} (-1)^{i_{k-1}+\cdots+i_0} X_2^{i,i}, \quad i = (i_{k-1} \cdots i_0)_2. \] (23)

For instance, take $k = 4$ and ask for the sign of the entry $(11,11)$. This corresponds to the coefficient of the Hubbard operator $X_2^{10,10}$. Note $10 = (1100)_2$, so the required entry is $(-1)^{1+1} = (-1)^2 = 1$. As another example consider the simplest case of the Hadamard matrices [3, 20]:
\[ H = \frac{1}{\sqrt{2}} \sum_{i_0,j_0=1} (-1)^{i_0,j_0} X_2^{i_0,j_0} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right). \] (24)

This operator is unitary and its action on a vector (6) reads
\[ H|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{i_0,j_0=0} (-1)^{i_0,j_0} c_{j_0}|i_0\rangle = \frac{1}{\sqrt{2}} [(c_0 + c_1)|0\rangle + (c_0 - c_1)|1\rangle]. \] (25)

In quantum computing algorithms it is of interest the calculation of tensor product powers of $H$. Using (22) we immediately get
\[ H^{\otimes k} = \frac{1}{\sqrt{2^k}} \sum_{i,j=0}^{2^k-1} (-1)^{i,j} X_2^{i,j}, \quad i = \sum_{\ell=0}^{k-1} i_\ell j_\ell, \quad i = (i_{k-1} \cdots i_0)_2, \quad j = (j_{k-1} \cdots j_0)_2. \] (26)

The action of the former operator on a vector $|e_z^k\rangle \equiv |x\rangle$ reads
\[ H^{\otimes k}|x\rangle = \frac{1}{\sqrt{2^k}} \sum_{i=0}^{2^k-1} (-1)^x|i\rangle. \] (27)

The following section is devoted to review other applications of the Hubbard operators in quantum entanglement theory.
3. Application in multipartite entanglement

We now consider a multipartite system $S = S_1 + \cdots + S_k$. The $n_k$-dimensional Hilbert space describing the subsystem $S_i$ is given as $\mathcal{H}_{n_k} = \text{Span}\{e_{i}^{(s)} \}$, $s = 0, \ldots, n_k - 1$. A generic multipartite state $|\psi\rangle \in \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_k}$, written

$$|\psi\rangle = \sum_{p_1,\ldots,p_n} c_{p_1,p_2,\ldots,p_n} |e_{p_1}^{(1)}\rangle \otimes |e_{p_2}^{(2)}\rangle \otimes \cdots \otimes |e_{p_n}^{(n)}\rangle,$$  \hspace{1cm} (28)

is said to be completely separable if it can be expressed as a product state of the form

$$|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \cdots \otimes |\varphi_n\rangle,$$

$$|\varphi_\ell\rangle = \sum_{p_\ell} a_{p_\ell} |e_{p_\ell}^{(\ell)}\rangle.$$  \hspace{1cm} (29)

Otherwise (28) is referred as entangled. A natural form to quantify entanglement is by considering how “different” is a given state $|\psi\rangle$ from (29). In this sense the maximum overlap of an state $|\psi\rangle$ with the closest separable state is defined as follows [10]:

$$\Lambda_{\text{max}}(\psi) := \max_{|\varphi\rangle} \{ |\langle \psi | \varphi \rangle| : |\varphi\rangle \in \mathcal{H} \text{ is complete separable} \}.$$  \hspace{1cm} (30)

Observe that if $|\psi\rangle$ is completely separable, then $\Lambda_{\text{max}}$ is one. Thus, the geometric measure of entanglement is defined in two different forms [14]:

$$E_{\text{sin}^2}(\psi) := 1 - \Lambda_{\text{max}}^2(\psi), \quad \text{and} \quad E_{\log}(\psi) := -2 \log_2 \Lambda_{\text{max}}(\psi).$$  \hspace{1cm} (31)

The term “geometric” is justified by the fact that $\Lambda_{\text{max}}$ is the angle between two vectors. Besides, notice that the amount of entanglement increases as $\Lambda_{\text{max}}$ decreases. We want to point that evaluating (30) is in general a hard optimization problem, although for specific states analytic expressions have been derived by taking advantage of some symmetries. For instance, the maximum overlap with the closest separable state of the Dicke states (16) is given by [14]:

$$\Lambda_{\text{max}}(n,k) = \sqrt{\frac{n!}{k!(n-k)!}} \left( \frac{k}{n} \right)^{k/2} \left( \frac{n-k}{n} \right)^{(n-k)/2}. \hspace{1cm} (32)$$

This result has been obtained by considering that the product state can be taken as the Kronecker product of the same real single-party state, that is

$$|\varphi\rangle = \bigotimes_{i=1}^n |\varphi_i(\theta)\rangle, \quad |\varphi_i(\theta)\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle, \quad \theta \in [0, \pi/2].$$  \hspace{1cm} (33)

The overlap of $\langle S(n,k)|\phi\rangle$ can be bounded by using the Schwarz and MacLaurin inequalities, so that optimization on $\theta$ is accomplished [14]. However, employing the formalism of Hubbard operators the same result (32) can be obtained in a simpler manner. First, notice that the definition of $\Lambda_{\text{max}}$ is equivalent to the following

$$\Lambda_{\text{max}}(\psi) := \max_{U_{\text{loc}}} \{ |\langle \psi | U_{\text{loc}} | 0 \rangle| : U_{\text{loc}} = U_1 \otimes \cdots \otimes U_N, \ U_\ell \text{ unitary} \}. \hspace{1cm} (34)$$

In the case of Dicke states, the operator $U_{\text{loc}}$ can be taken as

$$U_{\text{loc}}(\theta) = U(\theta)^{\otimes n}, \quad U(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$  \hspace{1cm} (35)
in terms of the X-operators this reads

\[ U(\theta) = \sum_{i,j=0}^{1} u_{i,j}(\theta)X_2^{i,j}, \quad u_{i,j}(\theta) = \cos[\theta + \pi(j - i)/2]. \] (36)

Using (22) we get

\[ U_{\text{loc}}(\theta) = \sum_{i,j=0}^{2^{n-1}} u_{i_{n-1}j_{n-1}}(\theta) \cdots u_{i_0j_0}(\theta)X_2^{i,j}, \quad i = (i_{n-1} \ldots i_0)_2, \quad j = (j_{n-1} \ldots j_0)_2. \] (37)

Note

\[ U_{\text{loc}}|0\rangle = \sum_{i=0}^{2^{n-1}} u_{i_{n-1}0 \ldots i_0,0}|i\rangle = \sum_{i_{n-1} \ldots i_0} u_{i_{n-1}0 \ldots i_0,0}|i_{n-1}i_{n-1} \ldots i_0\rangle, \] (38)

where we have omitted the \( \theta \) dependence for shortness. The overlap of \( \Lambda_\theta(n,k) = \langle S(n,k)|U_{\text{loc}}|0\rangle \) with the former state can be immediate computed

\[ \Lambda_\theta(n,k) = \frac{1}{\sqrt{C^n_k}} \sum_{i_{n-1} \ldots i_0} \sum_{\pi \in S_n} u_{i_{n-1},0 \ldots u_{i_0,0}}(\pi(0 \ldots 0 1 \ldots 1)|i_{n-1} \ldots i_0\rangle. \]

Remark that the sum is contracted to \( C^n_k \) terms, each one having \( n \) factors \( u_{0,0} \) and \( n-k \) factors \( u_{1,0} \). Explicitly

\[ \Lambda_\theta(n,k) = \sqrt{C^n_k} u_{0,0}(\theta)u_{n-k}(\theta) = \sqrt{C^n_k} \sin^k \theta \cos^{n-k} \theta, \] (39)

so that, maximazing over \( \theta \) the expression (32) is obtained. As an example consider the \( n \)-qubit states

\[ |W_n\rangle = |S(n,1)\rangle, \quad |\tilde{W}_n\rangle = |S(n,n-1)\rangle, \] (40)

for which the maximum overlap with the closest separable state yields

\[ \Lambda_\text{max}(W_n) = \Lambda_\text{max}(\tilde{W}_n) = \left( \frac{n-1}{n} \right)^{(n-1)/2}. \] (41)

For \( n = 3 \) it is known that the maximal state of the geometric measure of entanglement is the \( W \)-state with \( \Lambda_\text{max}(W_3) = 2/3 \) [21]. Using the X-operators we can compute \( \Lambda_\text{max} \) for a more general kind of symmetric states, for instance a linear combination of \( q \) Dicke states

\[ |S(n,\vec{k})\rangle = \sum_{\ell=1}^{q} \alpha_\ell |S(n,k_\ell)\rangle, \quad \alpha_\ell \geq 0, \quad \sum_{\ell} \alpha_\ell^2 = 1, \] (42)

where \( \vec{k} = (k_1, k_2, \ldots, k_q) \) and \( k_\ell \leq n \). For this kind of state we obtain

\[ \Lambda_\text{max}(n,\vec{k}) = \cos^n \theta \sum_{\ell=1}^{q} \alpha_\ell \sqrt{C^n_{k_\ell}} \tan^{k_\ell} \theta, \] (43)

here \( \theta \) is a solution of the following transcendental equation

\[ \sum_{\ell=1}^{q} \frac{\alpha_\ell}{\sqrt{C^n_{k_\ell}}} [k_\ell + (k_\ell - n) \tan^2 \theta] \tan^{k_\ell} \theta = 0. \] (44)
From this last result we can compute $\Lambda_{\text{max}}$ for the following three-qubit state

$$|\tilde{W}(s)\rangle = \sqrt{s}|W\rangle + \sqrt{1-s}|\tilde{W}\rangle, \quad s \in [0, 1],$$  \hfill (45)

with the parameters $q = 2$, $k_1 = 1$, $k_2 = 2$, $\alpha_1 = \sqrt{s}$ and $\alpha_2 = \sqrt{1-s}$ the result reported by Wei and Severini [14] is obtained

$$\Lambda_{\text{max}}(\tilde{W}(s)) = \frac{1}{2}2^{\sqrt{s}}(\sqrt{s} \cos \theta + \sqrt{1-s} \sin \theta) \sin 2\theta,$$  \hfill (46)

where $\theta$ satisfies the equation

$$\sqrt{1-s} \tan^3 \theta + 2\sqrt{s} \tan^2 \theta - 2\sqrt{1-s} \tan \theta - \sqrt{s} = 0.$$  

4. Conclusions

The tensor product algebra in multiqubit systems has been addressed in terms of Hubbard operators. In this notation complicated calculations are turned into simple relations of the binary coefficients of indices. The presented formalism can be easily generalized to qudits, in such a case one should work in base $d$ instead of the binary one. To show the usefulness of our results, we derived an analytic expression for the maximum overlap with respect the closest separable state of a linear superposition of Dicke states. This result includes some cases reported in the literature.

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