Correction to the Running of Gauge Couplings out of Extra Dimensional Gravity

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We study the gravitational correction to the gauge couplings in the extra dimension model where the gravity propagates in the (4+n)-dimensional bulk. We show by explicit calculation in the background field method that the one-loop correction coming from graviton Kaluza-Klein states is nontrivial and tends to make the theory anti-asymptotically free. For the theory characterized with asymptotic freedom, the correction will induce a nontrivial UV-stable fixed point. This may lead to an interesting possibility that the non-abelian gauge coupling constants $g_2$ and $g_3$ in the SM and that of the gravity $g_\kappa$ unify at the fixed point in the limit of high energy scale.

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General relativity is not renormalizable after quantization [1, 2]. This is one of the reasons that general relativity was considered to be incompatible to quantum mechanics. A modern point of view to the non-renormalizable theory is that it might be sensible and reliable predictions could still be made from it within the framework of effective field theories. From the value of the only dimensional coupling constant $\kappa$, Newton’s constant, in Hilbert-Einstein Lagrangian, one can see that gravitational effects are tiny at energies $\mu \ll M_{pl} \sim 10^{19} GeV/c^2$. It makes sense to treat general relativity as a low energy effective field theory of some unknown fundamental theory and consider its quantum effects [3]. The effects due to non-renormalizable terms are suppressed by inverse powers of $M_{pl}$, the mass scale of new physics.

The most promising candidate for such new physics is the string theory. A consistent string theory predicts a definite number of extra dimensions. To make contact between string theory and our experienced four-dimensional spacetime, the most straightforward possibility is that the extra dimensions are compactified on an internal manifold whose size is sufficiently small to escape being detected at present.

The field theory with compactified extra dimensions is another example of the non-renormalizability different from the gravitation. When a renormalizable theory in four-dimensional spacetime is generalized to (4+n)-dimensions and reduced to the four-dimensional world, the effective lagrangian contains infinite Kaluza-Klein series, with every operator having mass dimension less than five. Considering that the heavy Kaluza-Klein states are decoupled from the low energy physics of phenomenological interest, one can truncate the Kaluza-Klein tower at some energy scale $\mu$. The truncated theory is renormalizable.

An interesting thing concerned with these two types of effective theory is the quantum corrections to the gauge couplings. In [4], Robinson and Wilczek showed that gravitational corrections will cause gauge couplings to run quadratically. Although the effect is tiny in the regions where perturbative calculations are reliable, it’s of theoretical interest on its own. Pietrykowski [5] redid the calculations and found that the result in [4] is gauge dependent. Up to one-loop order the gravitational contributions to the $\beta$ function are zero in the dimensional regularization (DR) scheme. Later on, Pietrykowski’s result was verified by [6, 7, 8]. More recently, it was argued that the gravitational contribution to the $\beta$ function is nontrivial in a new regularization scheme [9].

The power law running exist in the theory of n extra dimensions. The one-loop correction to the gauge couplings from a tower of gauge boson Kaluza-Klein states is of order $\mu^n [10]$. It drives the theories to run fast so that the gauge coupling unification can be significantly lowered to TeV scale in large extra dimension models.

In this work, we will calculate in DR scheme the one-loop correction to the gauge couplings from the extra dimensional gravity where the gravitons propagate in (4+n) dimensional spacetime while the gauge and matter fields live in the normal four dimensional world [11]. We will show, by explicit calculations in the background field method, that the one-loop correction coming from graviton Kaluza-Klein states is nontrivial. It is universal for all gauge couplings and tends to make the theory anti-asymptotically free. Therefore for the theory characterized with asymptotic freedom, the correction will induce a nontrivial UV-stable fixed point. This may lead to an interesting possibility that the non-abelian gauge coupling constants $g_2$ and $g_3$ in the SM and that of the gravity $g_\kappa$ unify at the fixed point in the limit of some high energy scale, if the numbers of the active Kaluza-Klein particles are properly selected.

We start with the Einstein-Yang-Mills theory. The extra dimensions are compactified on a torus $T^n$. The compactification scales of the extra n-dimensional spaces $y_i$ are all assumed to be roughly equal to $R$, and they need not to be large extra dimensions in our present consider-
with arbitrary mass parameter renormalized coupling constant, we introduce an arbitrary tensor and 

\[ \bar{g}_{\mu
u} = \hat{g}_{\mu
u} + g f^{abc} \hat{A}^b_{\mu} \hat{A}^c_{\nu}, \]  

(3)

where \( D_\mu \) is the spacetime covariant derivative operator.

In the following we use the background field method \cite{12} and choose the background spacetime to be flat. In DR scheme, we perform all loop momentum integrals in \( d = 4 - \epsilon \) dimensions. The bare coupling constants are not dimensionless. To use a dimensionless renormalized coupling constant, we introduce an arbitrary mass parameter \( \mu \) and replace the Eq. (1) by

\[ g_0 = Z_\mu \mu^\epsilon \hat{g}. \]  

(8)

The renormalized constant \( g \) are functions of \( \mu \), while \( g_0 \) is independent of \( \mu \). By differentiating both sides of Eq. (5), we find the \( \beta \) function

\[ \beta \equiv \mu \frac{\partial \hat{g}}{\partial \mu} = -\epsilon g - g \mu \frac{\partial \ln Z_\mu}{\partial \mu}. \]  

(9)

Using the chain rule

\[ \mu \frac{\partial}{\partial \mu} = \mu \frac{\partial g}{\partial \mu} g + \mu \frac{\partial \ln Z_\mu}{\partial \mu}, \]  

(10)

\( \beta \) function can be written in a simple form

\[ \beta = -\epsilon g - g \mu \frac{\partial \ln Z_\mu}{\partial \mu} - g \beta \frac{\partial \ln Z_\mu}{\partial \mu}. \]  

(11)

The definition of the \( \beta \) function for gravity \( \beta_\kappa \) is the same as that of the gauge sector. It has the form

\[ \beta_\kappa = -\epsilon \kappa - \kappa \mu \frac{\partial \ln Z_\mu}{\partial \mu}. \]  

(12)

From Eqs. (1), (4) and (12) and taking the limit \( \epsilon \to 0 \), one can finally have

\[ \beta = -\frac{1}{2} \epsilon g^2 \frac{\partial \ln Z_A}{\partial g} - \frac{1}{2} \epsilon g^2 \frac{\partial \ln Z_A}{\partial \kappa}, \]  

(13)

where the first term purely comes from the gauge and matter sector, and the second term results from the gravitational correction. It shows that the \( \beta \) function can be determined from the coefficient of the \( 1/\epsilon \) term in the gauge field two-point Green’s function.

Now we continue to calculate the correction to the two-point Green’s function of the gauge field from \( n \)-extra dimensional gravity. Fields \( \hat{g}_{\mu\nu}(x, y) \) and \( \hat{A}^a(x) \) can be written as sums of background fields (\( \hat{n}_{\mu\nu}, \hat{A}^a(x) \)) and quantum fluctuations (\( \hat{h}_{\mu\nu}, a^a(x) \)):

\[ \hat{g}_{\mu\nu}(x, y) = \hat{n}_{\mu\nu} + \hat{h}_{\mu\nu}(x, y), \]  

\[ \hat{A}^a(x) = \hat{A}^a(x) + a^a(x). \]  

(14)

Here \( \hat{n}_{\mu\nu} \) is the Minkowski metric of the bulk. To perform the Kahalu-Klein reduction to four-dimensional spacetime, we parameterize the field \( \hat{h}_{\mu\nu} \) as

\[ \hat{h}_{\mu\nu} = V_n^{-\frac{1}{2}} \left( \hat{h}_{\mu\nu} + \eta_{\mu\nu} \phi, \frac{A_{\mu j}}{2 \phi_{ij}} \right), \]  

(15)

where \( V_n = R^n \) is the volume of the \( n \)-dimensional compactified torus \( T^n \), \( \phi = \sum_i \phi_i \), the subscript \( \mu, \nu = 0, 1, 2, 3 \) and \( i, j = 4, 5, ..., 3 + n \). The fields \( \hat{h}_{\mu\nu}, A_{\mu i}, \) and \( \phi_{ij} \) are Lorentz tensor, vector and scalar respectively. They have the following mode expansions:

\[ \hat{h}_{\mu\nu}(x, y) = \sum_i \hat{h}_{\mu\nu}^i(x) \exp (i \frac{2\pi \vec{n} \cdot \vec{y}}{R}), \]  

(16)

\[ A_{\mu i}(x, y) = \sum_i A_{\mu i}^i(x) \exp (i \frac{2\pi \vec{n} \cdot \vec{y}}{R}), \]  

(17)

\[ \phi_{ij}(x, y) = \sum_i \phi_{ij}^i(x) \exp (i \frac{2\pi \vec{n} \cdot \vec{y}}{R}), \]  

(18)
with \( \bar{n} = (n_1, n_2, ..., n_n) \).

To continue, one has to fix gauges. Adding a special de Donder gauge fixing term

\[
- \frac{1}{2} \left( \partial_\rho \hat{h}^{\rho \mu \nu} \partial^\phi \hat{h}_{\phi \mu \nu} - \partial_\rho \hat{h}^{\rho \mu \nu} \partial_{\mu} \hat{h} + \frac{1}{4} \partial_\rho \hat{h} \partial^{\rho} \hat{h} \right)
\]

(19)

for the graviton field and the Lorentz gauge fixing term

\[
\frac{1}{2\pi} (\partial_\rho \phi_j^a)^2
\]

for the photon field to the action, the propagators for the massive Kaluza-Klein states \( h_{\bar{\mu} \bar{\nu}}, A_{\bar{\mu} \bar{\nu}}, \phi_j^a \) and the quantum gauge field in momentum space are:

\[
\Delta_{h_{\bar{\mu} \bar{\nu}, \bar{\rho} \bar{\sigma}}}^a(k) = -i \frac{\delta_{\bar{\mu} \bar{\nu}} - \delta_{\bar{\mu} \bar{\rho}} \delta_{\bar{\nu} \bar{\sigma}} \eta_{\bar{\rho} \bar{\sigma}} + \eta_{\bar{\mu} \bar{\sigma}} \eta_{\bar{\nu} \bar{\rho}} - \eta_{\bar{\mu} \bar{\nu}} \eta_{\bar{\rho} \bar{\sigma}}}{k^2 - m_{\bar{n}}^2 + i\epsilon}
\]

\[
\Delta_{A_{\bar{\mu} \bar{\nu}, \bar{j}}}^a(k) = -i \frac{\delta_{\bar{\mu} \bar{\rho}} - \delta_{\bar{\mu} \bar{\sigma}} \delta_{\bar{\rho} \bar{\nu}} \eta_{\bar{\nu} \bar{\sigma}} + \eta_{\bar{\mu} \bar{\sigma}} \eta_{\bar{\nu} \bar{\rho}} - \eta_{\bar{\mu} \bar{\nu}} \eta_{\bar{\rho} \bar{\sigma}}}{k^2 - m_{\bar{n}}^2 + i\epsilon}
\]

\[
\Delta_{\phi_{\bar{j} \bar{k}}^a}(k) = -i \frac{\delta_{\bar{j} \bar{k}}}{k^2 - m_{\bar{n}}^2 + i\epsilon}
\]

\[
\Delta_{a_{\bar{\mu} \bar{\nu}, \bar{b}}}^a(k) = \frac{i \delta_{\bar{a} \bar{b}}}{k^2 + i\epsilon} (\eta_{\bar{\mu} \bar{\nu}} - (1 - \xi) k_{\bar{\mu}} k_{\bar{\nu}} / k^2),
\]

(20)

where \( m_{\bar{n}}^2 \equiv \frac{4\pi^2 \bar{n} \bar{n}}{2\pi} \) is the mass of Kaluza-Klein graviton at \( n \)-th order excitations. Obviously the spin-2 state \( h_{\bar{\mu} \bar{\nu}} \) is not a physical state. This can be seen explicitly from the numerator of its propagator. The physical massive spin-2 state with right polarization tensor is constructed from \( h_{\bar{\mu} \bar{\nu}}, A_{\bar{\mu} \bar{\nu}} \) and \( \phi_j^a \). However, the rearrangement of the physical spin-2, \((n-1)\) spin-1, and \( n(n-1)/2\) spin-0 states into \( h_{\bar{\mu} \bar{\nu}}, A_{\bar{\mu} \bar{\nu}}, \phi_j^a \) states shown above greatly simplifies our calculations.

Now we are going to calculate the one-loop gravitational correction in the DR scheme. In the present setup, we have the relevant interaction terms \( h_{\bar{\mu} \bar{\nu}, \bar{a} \bar{b}} A_{\bar{a} \bar{b}}, h_{\bar{\mu} \bar{a}}, A_{\bar{\mu} \bar{a}}, \phi_j^a \bar{A}_{\bar{a}} \) at \( \beta \) order, and \( h_{\bar{\mu} \bar{a}}, A_{\bar{\mu} \bar{a}}, \phi_j^a \bar{A}_{\bar{a}} \) at \( \kappa \) order. The Feynman diagrams are shown in Fig. 1 up to \( k^2 \) order. A tedious calculation shows that all of the first three diagrams are zero, though they are not necessary vanished to all appearances because the Kaluza-Klein gravitons are massive.

The last two diagrams have nontrivial contribution to the \( \beta \) function. By a straightforward calculation, it can be shown that the correction obtained in the framework of background field method is independent of the gauge parameter \( \xi \), as it should be. The result of the diagram is

\[
\Pi_{\bar{\mu} \bar{\nu}} = i \frac{\kappa^2}{2\pi^2} \frac{p^2 \eta_{\bar{\mu} \bar{\nu}} - p_{\mu} p_{\nu}}{p^2 + 18m_{\bar{n}}^2}
\]

\[+ \text{finite part}.\]

(21)

The contributions of the extra dimensions are embodied in the summation over the Kaluza-Klein states in a tower. In the limit of four-dimensional spacetime, i.e. \( n=0 \), the second term is vanished because the graviton is massless.

As discussed in Refs. [6, 13], the first term of Eq. (21) corresponds to a higher derivative operator with mass dimension six \(-\frac{1}{4} \partial_\mu F_{\rho\sigma\mu
u} \partial^\rho F_{\rho\sigma\mu
u} \). It can generate a Lee-Wick gauge field. The second term results from the summation of the active Kaluza-Klein states. It contributes to the one-loop \( \beta \) function of the gauge coupling

\[
\beta = -\frac{b}{(4\pi)^2} g^3 + \frac{12g^2}{(4\pi)^2} \sum_{\bar{n}} m_{\bar{n}}^2,
\]

(22)

where \( b \) depends on the gauge and matter contents and is wholly independent of whether gravitation is included in the calculation. It comes from the first term of Eq. (13). The second term in Eq. (22) is contributed by the exchange of graviton Kaluza-Klein states, and is universal for all gauge couplings since gravitons do not carry any gauge charge. It tends to make the theory asymptotically free.

In the theory where \( b \) is positive, such as QCD and the SU(2) gauge sector of the electroweak, there is a nontrivial ultraviolet-stable fixed point at

\[
g_*^2 = \frac{12\kappa^2}{b} \sum_{\bar{n}} m_{\bar{n}}^2.
\]

(23)

In the limit of large momentum, the coupling constant tends to \( g_* \), and therefore runs along with \( \kappa \).

The summation can be written as an integration in term of the mass \( m_{\bar{n}}^2 \) when the Kaluza-Klein states are near degenerate [14]

\[
\sum_{\bar{n}} f(m_{\bar{n}}) = \int_0^{\lambda^2} dm_{\bar{n}}^2 \rho(m_{\bar{n}}) f(m_{\bar{n}}),
\]

(24)

where

\[
\rho(m_{\bar{n}}) = \frac{R^n m_{\bar{n}}^{-2}}{(4\pi)^{n/2} \Gamma(n/2)}
\]

(25)
is the Kaluza-Klein state density. As is well-known, using DR scheme, which is a mass-independent scheme, heavy states do not decouple. Thus we have introduced an explicit cutoff $\Lambda$ to regularize the mass integration. That is, we include only a finite number of low-lying Kaluza-Klein states whose masses are smaller than $\Lambda$ and assume all other states decouple from the low energy physics we are interested. This cutoff does not break any gauge symmetry and our calculation is gauge independent. The near degenerate condition is satisfied when the energy scale $R^{-1}$ characterized the first Kaluza-Klein excitations is much less than the physical scale $\Lambda$.

In the literatures, there are two different scenarios for calculating the $\beta$ function from the Kaluza-Klein states, depending on how to treat the cutoff scale $\Lambda$ and the evolution scale $\mu$.

One is that the energy scales $\Lambda$ and $\mu$ are different, which implies that, when $\mu < \Lambda$, the particles with mass larger than $\mu$ can contribute to the gauge coupling. This is the case for the QCD where the heavy quark flavors are of significance to the running of gauge coupling at low energy. In general, the cutoff scale $\Lambda$ should be less than $M_{pl(4+n)}$, since the effective theory is only expected to be valid below the fundamental scale $M_{pl(4+n)}$. Unitarity requirement of some explicit extra dimensional models can play some stringent constraints on it \[\text{(13)}.\]

The gauge coupling runs logarithmically

$$ g(\mu)^2 = \frac{g(E)^2}{1 + \frac{2b}{(4\pi)^2} g(E)^2 \ln \frac{\mu}{E} - a \frac{\Lambda^{n+2}}{M_{pl(4+n)}^{n+2}} \ln \frac{\mu}{E}}, \quad \text{(26)} $$

where the coefficient $a$ is purely determined by the number of extra dimensions

$$ a = \frac{48}{(n+2)\pi (4\pi)^2 \Gamma\left(\frac{n}{2}\right)}. \quad \text{(27)} $$

The $\Gamma\left(\frac{n}{2}\right)^{-1}$ factor ensures that the gravitational correction is vanished in the limit of four spacetime dimension, which is in agreement with the result in \[\text{2} \quad \text{6} \quad \text{8} \quad \text{3}.\]

An interesting possibility follows from the UV fixed point. When the Kaluza-Klein cutoff scales $\Lambda_3$ and $\Lambda_1 = \Lambda_2$ in the SM are elaborately set to

$$ \frac{\Lambda_2^{n+2}}{b_2} = \frac{\Lambda_3^{n+2}}{b_3} = \frac{(n+2)(4\pi)^{n/2} \Gamma(n/2) \mu^4 \kappa^2}{24(16\pi)^2 R^n}, \quad \text{(28)} $$

the non-abelian gauge coupling constants $g_2$ and $g_3$ and that of the gravity $g_\kappa$ unify at the UV-stable fixed points $g_{2*} = g_{3*} = g_\kappa$, where $g_\kappa$ is the dimensionless Newton constant, and $g_\kappa \equiv \mu^2 \kappa^2 / 16\pi \text{(11)}$. If $g_1$ can evolve to the fixed point at some high energy scale, the four coupling constants is unified.

The other scenario that the energy scales $\Lambda$ and $\mu$ are the same will lead to the power law running of the gauge coupling

$$ g(\mu)^2 = \frac{g(E)^2}{1 + \frac{2b}{(4\pi)^2} g(E)^2 \ln \frac{\mu}{E} - a \frac{\mu^{n+2}}{M_{pl(4+n)}^{n+2}} \ln \frac{\mu}{E} }, \quad \text{(29)} $$

with

$$ a = \frac{24}{(n+2)\pi (4\pi)^2 \Gamma\left(\frac{n}{2}\right)}. \quad \text{(30)} $$

It implies that only the Kaluza-Klein gravitons with mass less than $\mu$ make contributions to the running of the gauge couplings. Note that the one-loop correction to the gauge couplings from a tower of gauge boson Kaluza-Klein states is of order $\mu^{n+2}$. Here we show explicitly that the correction coming from graviton Kaluza-Klein states is of order $(\mu/M_{pl(4+n)})^{n+2}$.

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