On the $SU(3)$ Parametrization of Qutrits

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Abstract. Parametrization of qutrits on the complex projective plane $CP^2 = SU(3)/U(2)$ is given explicitly. A set of constraints that characterize mixed state density matrices is found.

Many recent ideas of quantum information theory are based on the notion of qubits. A qubit may be represented by a point on the Poincaré sphere $S^2$ that is homeomorphic to the complex projective line $H^{(2)} = CP^1 = SU(2)/U(1)$. A similar parametrization in the case of higher dimensional quantum systems is desirable both from theoretical [1] and technical points of view [2], [3]. A qutrit may be represented by a point on the complex projective plane $H^{(3)} = CP^2 = SU(3)/U(2)$ . Such a representation is given explicitly in terms of Gell-Mann matrices [4]. We determine a set of constraints that characterize mixed states of qutrits below.

A qutrit $|\psi> = \alpha_0 |0> + \alpha_1 |1> + \alpha_2 |2>$ where $\alpha_0, \alpha_1, \alpha_2 \in C$, $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$, is a state vector in the Hilbert space of states $H^{(3)}$ of a 3-level system. It is spanned by an orthonormal basis $\{|0>, |1>, |2>\}$ which in matrix notation reads

$$|0> \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1> \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |2> \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Therefore

$$|\psi> \rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \in C^3 \cong R^6.$$ 

Since $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$ and since $|\psi>$ is determined up to a multiplicative phase factor, $\dim H^{(3)} = 4$.

Any $3 \times 3$ density matrix can be written as

$$\rho = \frac{1}{3}(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$$

where $\vec{n}$ is a real 8-vector, and components of $\vec{\lambda}$ are the (Hermitian, traceless) Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
\[
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

The product of two Gell-Mann matrices is given by
\[
\lambda_j \lambda_k = \frac{2}{3} \delta_{jk} + \sum_l d_{jkl} \lambda_l + i \sum_l f_{jkl} \lambda_l
\]
where \( j, k = 1, 2, \ldots, 8 \). The \( f \)-symbols (structure constants of the Lie algebra \( \text{su}(3) \)) are totally anti-symmetric:
\[
f_{123} = 1, f_{458} = f_{678} = \frac{\sqrt{3}}{2},
\]
and the \( d \)-symbols are totally symmetric:
\[
\begin{align*}
d_{118} &= d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, & d_{448} &= d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \\
d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}.
\end{align*}
\]

Given two real 8-vectors \( \vec{a} \) and \( \vec{b} \), we define their Euclidean inner product
\[
\vec{a} \cdot \vec{b} = \sum_k a_k b_k
\]
skew-symmetric vector \( \wedge \)-product
\[
(\vec{a} \wedge \vec{b})_j = \sqrt{3} \sum_{k,l} f_{jkl} a_k b_l
\]
and symmetric vector \( \star \)-product
\[
(\vec{a} \star \vec{b})_j = \sqrt{3} \sum_{k,l} d_{jkl} a_k b_l
\]
The pure states that satisfy \( \rho^2 = \rho \) are therefore characterized by
\[
|\vec{n}|^2 = 1 \quad \text{and} \quad \vec{n} \star \vec{n} = \vec{n}.
\]
Suppose that \( \rho = \frac{1}{3} (I + \sqrt{3} \vec{n} \cdot \vec{\lambda}) \) is the density matrix of a mixed state. It is Hermitian, positive with trace equal to 1. Therefore all the eigenvalues \( x_1, x_2, x_3 \) are positive and add to one: \( x_1 + x_2 + x_3 = 1 \). The Cayley-Hamilton equation satisfied by \( \rho \) reads
\[
\rho^3 - \rho^2 + (x_1 x_2 + x_2 x_3 + x_1 x_3) \rho - x_1 x_2 x_3 I = 0.
\]
The following inequalities hold:
\[
\frac{1}{3} \geq x_1 x_2 + x_2 x_3 + x_1 x_3 \geq 0, \quad \frac{1}{27} \geq x_1 x_2 x_3 \geq 0.
\]
Figure 1.

Starting from these, a straightforward computation shows that the necessary and sufficient conditions for \( \rho = \frac{1}{3} (I + \sqrt{3} n \cdot \lambda) \) to be a density matrix of a mixed state are given by

\[
1 \geq |\vec{n}|^2 \geq 0 \quad \text{and} \quad 1 \geq 3|\vec{n}|^2 - 2\vec{n} \cdot (\vec{n} \times \vec{n}) \geq 0.
\]

An arbitrary diagonal density matrix of a 3-level system will be

\[
\rho = \frac{1}{3} (I + \sqrt{3} (n_3 \lambda_3 + n_8 \lambda_8)).
\]

In this case, the mixed-state density matrix constraints reduce to

\[
0 \leq n_3^2 + n_8^2 \leq 1 \quad \text{and} \quad 0 \leq 2n_3^2 - 6n_3^2n_8 + 3n_3^2 + 3n_8^2 \leq 1.
\]

The region in the \( n_3n_8 \)-plane where both the constraints are satisfied is bound by an equilateral triangle with vertices at the points

\[
(n_3, n_8)_R = (\frac{\sqrt{3}}{2}, \frac{1}{2}) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 
(n_3, n_8)_B = (-\frac{\sqrt{3}}{2}, \frac{1}{2}) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
(n_3, n_8)_G = (0, -1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Vertices of the above triangle correspond to three mutually orthogonal pure-states. We labelled them Red, Blue, Green in analogy with colored quarks [2]. In fact two pure-state vectors \( |\psi> \) and \( |\psi'> > \) are orthogonal if and only if \( <\psi|\psi'> > = 0 \), so that \( Tr(\rho\rho') = 0 \). This implies \( \vec{n} \cdot \vec{n}' = -\frac{1}{2} \). Then \( \arccos(\vec{n} \cdot \vec{n}') = \pm \frac{2\pi}{3} \). This is equal to the geodesic distance between two orthogonal pure-states as measured by the standard Fubini-Study metric on \( CP^2 \).
Points on the edges of the triangle correspond to mixing of two orthogonal pure-states of qutrits. In particular, at mid-points where bi-sectors intersect with the edges we have
\[(n_3, n_8)_C = \left(0, \frac{1}{2}\right) \leftrightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (n_3, n_8)_B = \left(\frac{\sqrt{3}}{4}, -\frac{1}{4}\right) \leftrightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
\[(n_3, n_8)_A = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{4}\right) \leftrightarrow \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\]

Triply mixed-states correspond to points inside the triangle. In particular the origin \[(n_3, n_8)_O = (0, 0) \leftrightarrow \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\]
corresponds to the maximally mixed state.

For 2-level systems, the orbit of any diagonal \(2 \times 2\) density matrix of qubits 
\[\rho = \frac{1}{2} \begin{pmatrix} 1 + n_3 & 0 \\ 0 & 1 - n_3 \end{pmatrix} \leftrightarrow (0, 0, n_3)\]
under the action of the unitary group \(SU(2)\),
\[\rho \to U\rho U^\dagger\]
where \(U \in SU(2)\) (i.e. adjoint representation of \(SU(2)\), which is \(SO(3)\) applied on \(n_3\)) sweeps the whole Poincaré sphere \(S^2\). Adjoint representation \(\text{Ad}\) of a given \(U \in SU(2)\) is explicitly
\[\text{Ad}(U)_{ij} = \frac{1}{2} \text{Tr}(\sigma_i U \sigma_j U^\dagger) \in SO(3),\]
so that \(n_j \to \text{Ad}(U)_{jk} n_k\).

In a similar way, for 3-level systems the orbit of each point \((n_3, n_8)\) of the above triangle under the unitary action of \(SU(3)\) (i.e. adjoint representation of \(SU(3)\)) will provide a generalization of the Poincaré sphere to 3-level systems. Adjoint representation \(\text{Ad}\) of a given \(U \in SU(3)\) is found as follows:
\[\text{Ad}(U)_{ij} = \frac{1}{2} \text{Tr}(\lambda_i U \lambda_j U^\dagger) \in SO(8)\]
so that \(n_j \to \text{Ad}(U)_{jk} n_k\).

In fact \(\text{Ad}(SU(3))\) is an 8-parameter subgroup of the 28-parameter rotation group \(SO(8)\).

We also consider the entropy of mixing of \(\rho\) defined as 
\[E(\rho) = -x_1 \log_3(x_1) - x_2 \log_3(x_2) - x_3 \log_3(x_3)\]
Since in diagonal form
\[\rho = \frac{1}{3} \begin{pmatrix} 1 + \sqrt{3} n_3 + n_8 \\ 1 - \sqrt{3} n_3 + n_8 \\ 1 - 2n_8 \end{pmatrix},\]
the entropy of mixing of \(\rho\) becomes
\[E(\rho) = -(\frac{1 + \sqrt{3} n_3 + n_8}{3}) \log_3(\frac{1 + \sqrt{3} n_3 + n_8}{3}) - (\frac{1 - \sqrt{3} n_3 + n_8}{3}) \log_3(\frac{1 - \sqrt{3} n_3 + n_8}{3}) - (\frac{1 - 2n_8}{3}) \log_3(\frac{1 - 2n_8}{3}) .\]

The equi-mixing curves in the \(n_3n_8\)-plane are shown in the Figure 2.
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