SPECTRAHEDRA

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Positive Semidefinite Matrices

For a real symmetric $n \times n$-matrix $A$ the following are equivalent:

- All $n$ eigenvalues of $A$ are positive real numbers.
- All $2^n$ principal minors of $A$ are positive real numbers.
- Every non-zero vector $x \in \mathbb{R}^n$ satisfies $x^T A \cdot x > 0$.

A matrix $A$ is **positive definite** if it satisfies these properties, and it is **positive semidefinite** if the following equivalent properties hold:

- All $n$ eigenvalues of $A$ are non-negative real numbers.
- All $2^n$ principal minors of $A$ are non-negative real numbers.
- Every vector $x \in \mathbb{R}^n$ satisfies $x^T A \cdot x \geq 0$.

The set of all positive semidefinite $n \times n$-matrices is a convex cone of full dimension $\binom{n+1}{2}$. It is closed and semialgebraic. The interior of this cone consists of all positive definite matrices.
Semidefinite Programming

A *spectrahedron* is the intersection of the cone of positive semidefinite matrices with an affine-linear space. Its algebraic representation is a linear combination of symmetric matrices

\[ A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \succeq 0 \]  (*)

Engineers call this is a *linear matrix inequality.*
Semidefinite Programming

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\[ A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \succeq 0 \quad (\ast) \]

Engineers call this a *linear matrix inequality*.

**Semidefinite programming** is the computational problem of maximizing a linear function over a spectrahedron:

\[
\text{Maximize } c_1 x_1 + c_2 x_2 + \cdots + c_m x_m \text{ subject to } (\ast)
\]

**Example:** The smallest eigenvalue of a symmetric matrix \( A \) is the solution of the SDP \[
\text{Maximize } x \text{ subject to } A - x \cdot \text{Id} \succeq 0.
\]
Convex Polyhedra

*Linear programming* is semidefinite programming for diagonal matrices. If $A_0, A_1, \ldots, A_m$ are diagonal $n \times n$-matrices then

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \succeq 0$$

translates into a system of $n$ linear inequalities in the $m$ unknowns.
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translates into a system of $n$ linear inequalities in the $m$ unknowns. A spectrahedron defined in this manner is a *convex polyhedron*:
Pictures in Dimension Two

Here is a picture of a spectrahedron for $m = 2$ and $n = 3$: 

![Diagram of a spectrahedron for $m = 2$ and $n = 3$.]
Pictures in Dimension Two

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Duality is important in both optimization and projective geometry:
Example: Multifocal Ellipses

Given $m$ points $(u_1, v_1), \ldots, (u_m, v_m)$ in the plane $\mathbb{R}^2$, and a radius $d > 0$, their $m$-ellipse is the convex algebraic curve

$$\left\{ (x, y) \in \mathbb{R}^2 : \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} = d \right\}.$$ 

The 1-ellipse and the 2-ellipse are algebraic curves of degree 2.
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The 1-ellipse and the 2-ellipse are algebraic curves of degree 2. The 3-ellipse is an algebraic curve of degree 8:
2, 2, 8, 10, 32, ...

The 4-ellipse is an algebraic curve of degree 10:

The 5-ellipse is an algebraic curve of degree 32:
Concentric Ellipses

What is the algebraic degree of the \( m \)-ellipse? How to write its equation?

What is the smallest radius \( d \) for which the \( m \)-ellipse is non-empty? How to compute the Fermat-Weber point?
\[ C = \left\{ (x, y, d) \in \mathbb{R}^3 : \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \leq d \right\}. \]
Ellipses are Spectrahedra

The 3-ellipse with foci \((0, 0), (1, 0), (0, 1)\) has the representation

\[
\begin{bmatrix}
    d + 3x - 1 & y - 1 & y & 0 & y & 0 & 0 & 0 \\
    y - 1 & d + x - 1 & 0 & y & 0 & y & 0 & 0 \\
    y & 0 & d + x + 1 & y - 1 & 0 & 0 & y & 0 \\
    0 & y & y - 1 & d - x + 1 & 0 & 0 & 0 & y \\
    y & 0 & 0 & 0 & d + x - 1 & y - 1 & y & 0 \\
    0 & y & 0 & 0 & y - 1 & d - x - 1 & 0 & y \\
    0 & 0 & y & 0 & y & 0 & d - x + 1 & y - 1 \\
    0 & 0 & 0 & y & 0 & y & y - 1 & d - 3x + 1
\end{bmatrix}
\]

The ellipse consists of all points \((x, y)\) where this symmetric 8\(\times\)8-matrix is positive semidefinite. Its boundary is a curve of degree eight:
Theorem: The polynomial equation defining the $m$-ellipse has degree $2^m$ if $m$ is odd and degree $2^m - \binom{m}{m/2}$ if $m$ is even. We express this polynomial as the determinant of a symmetric matrix of linear polynomials. Our representation extends to weighted $m$-ellipses and $m$-ellipsoids in arbitrary dimensions.

[J. Nie, P. Parrilo, B.St.: Semidefinite representation of the $k$-ellipse, in Algorithms in Algebraic Geometry, I.M.A. Volumes in Mathematics and its Applications, 146, Springer, New York, 2008, pp. 117-132]

In other words, $m$-ellipses and $m$-ellipsoids are spectrahedra. The problem of finding the Fermat-Weber point is an SDP.
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Let's now look at some spectrahedra in dimension three. Our next picture shows the typical behavior for $m = 3$ and $n = 3$. 
A Spectrahedron and its Dual
Non-Linear Convex Hull Computation

**Input:** \( \{(t, t^2, t^3) \in \mathbb{R}^3 : -1 \leq t \leq 1\} \)
Non-Linear Convex Hull Computation

Input: \( \{(t, t^2, t^3) \in \mathbb{R}^3 : -1 \leq t \leq 1\} \)

The convex hull of the moment curve is a spectrahedron.

Output: \( \begin{pmatrix} 1 & x \\ x & y \end{pmatrix} \pm \begin{pmatrix} x & y \\ y & z \end{pmatrix} \succeq 0 \)
Characterization of Spectrahedra

A convex hypersurface of degree $d$ in $\mathbb{R}^n$ is rigid convex if every line passing through its interior meets (the Zariski closure of) that hypersurface in $d$ real points.

**Theorem (Helton–Vinnikov (2006))**

*Every spectrahedron is rigid convex. The converse is true for $n = 2$.***
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Theorem (Helton–Vinnikov (2006))

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Open problem: Is every compact convex basic semialgebraic set $S$ the projection of a spectrahedron in higher dimensions?

Theorem (Helton–Nie (2008))

The answer is yes if the boundary of $S$ is “sufficiently smooth”. 
Questions about 3-Dimensional Spectrahedra

What are the edge graphs of spectrahedra in $\mathbb{R}^3$?
How can one define their *combinatorial types*?
Is there an analogue to Steinitz’ Theorem for polytopes in $\mathbb{R}^3$?

Consider 3-dimensional spectrahedra whose boundary is an irreducible surface of degree $n$. Can such a spectrahedron have $\binom{n+1}{3}$ isolated singularities in its boundary? How about $n = 4$?
Minimizing Polynomial Functions

Let $f(x_1, \ldots, x_m)$ be a polynomial of even degree $2d$. We wish to compute the global minimum $x^*$ of $f(x)$ on $\mathbb{R}^m$.

This optimization problem is equivalent to

Maximize $\lambda$ such that $f(x) - \lambda$ is non-negative on $\mathbb{R}^m$.

This problem is very hard.
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The optimal value of the following relaxation gives a lower bound.

Maximize \( \lambda \) such that \( f(x) - \lambda \) is a sum of squares of polynomials.

The second problem is much easier. It is a semidefinite program.
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Empirically, the optimal value of the SDP almost always agrees with the global minimum. In that case, the optimal matrix of the dual SDP has rank one, and the optimal point $x^*$ can be recovered from this. How to reconcile this with Blekherman’s results?
SOS Programming: A Univariate Example

Let $m = 1$, $d = 2$ and $f(x) = 3x^4 + 4x^3 - 12x^2$. Then

$$f(x) - \lambda = \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Our problem is to find $(\lambda, \mu)$ such that the $3 \times 3$-matrix is positive semidefinite and $\lambda$ is maximal.
SOS Programming: A Univariate Example

Let \( m = 1, \ d = 2 \) and \( f(x) = 3x^4 + 4x^3 - 12x^2 \). Then

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f(x) - \lambda = (x^2 \times 1) \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}
\]

Our problem is to find \((\lambda, \mu)\) such that the \(3 \times 3\)-matrix is positive semidefinite and \(\lambda\) is maximal. The optimal solution of this SDP is

\[
(\lambda^*, \mu^*) = (-32, -2).
\]

Cholesky factorization reveals the SOS representation

\[
f(x) - \lambda^* = ((\sqrt{3} x - \frac{4}{\sqrt{3}}) \cdot (x + 2))^2 + \frac{8}{3} (x + 2)^2.
\]

We see that the global minimum is \( x^* = -2 \).
This approach works for many polynomial optimization problems.
My Favorite Spectrahedron

Consider the intersection of the cone of $6 \times 6$ PSD matrices with the 15-dimensional linear space consisting of all Hankel matrices

$$H = \begin{pmatrix}
\lambda_{400} & \lambda_{220} & \lambda_{202} & \lambda_{310} & \lambda_{301} & \lambda_{211} \\
\lambda_{220} & \lambda_{040} & \lambda_{022} & \lambda_{130} & \lambda_{121} & \lambda_{031} \\
\lambda_{202} & \lambda_{022} & \lambda_{004} & \lambda_{112} & \lambda_{103} & \lambda_{013} \\
\lambda_{310} & \lambda_{130} & \lambda_{112} & \lambda_{220} & \lambda_{211} & \lambda_{121} \\
\lambda_{301} & \lambda_{121} & \lambda_{103} & \lambda_{211} & \lambda_{202} & \lambda_{112} \\
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\end{pmatrix}.$$ 

This is a 15-dimensional spectrahedral cone.
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\end{pmatrix}.
$$

This is a 15-dimensional spectrahedral cone.

Dual to this intersection is the projection

$$
\text{Sym}_2(\text{Sym}_2(\mathbb{R}^3)) \rightarrow \text{Sym}_4(\mathbb{R}^3)
$$

taking a $6 \times 6$-matrix to the ternary quartic it represents. Its image is a cone whose algebraic boundary is a discriminant of degree 27.
Conclusion

Spectrahedra and their geometry deserve to be studied in their own right, independently of their important uses in applications.

A true understanding of these convex bodies will require the integration of three different areas of mathematics:

- Convexity
- Algebraic Geometry
- Optimization Theory

Please join the SIAM Special Interest Group on Algebraic Geometry