Implications of texture 4 zero lepton mass matrices for $U_{e3}$

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Abstract

Lepton mass matrices similar to texture 4 zero quark mass matrices, known to be quite successful in explaining the CKM phenomenology, have been considered for finding the mixing matrix element $U_{e3}$ ($\equiv s_{13}$) respecting the CHOOZ constraint, with $s_{12}$ and $\Delta m^2_{12}$ constrained by SNP and $s_{23}$ and $\Delta m^2_{23}$ constrained by ANP. Taking charged lepton mass matrix $M_l$ to be diagonal, we find that the ranges of $s_{13}$ corresponding to different SNP solutions very well include the corresponding values of $s_{13}$ found by Akhmedov et al. by considering neutrino mass matrix $M_\nu$ with no texture zeros. Considering $M_l$ and $M_\nu$ both to be real and non-diagonal, $s_{13}$ ranges for the four SNP solutions come out to be: $\sim 0 - 0.19$ (LMA), $0.038 - 0.093$ (SMA), $0.042 - 0.095$ (LOW), $0.038 - 0.096$ (VO), which remain of the same order when $M_l$ and $M_\nu$ are considered to be complex and non-diagonal.

The observation of neutrino oscillations by Super-Kamiokande (SK) [1] as well as by Sudbury Neutrino Observatory (SNO) [2] has provided unambiguous signal for physics beyond the Standard Model (SM). This essentially implies that the neutrinos are massive non-degenerate particles and the observed flavor eigenstates are linear combinations of mass eigenstates, in parallel to the quark mixing phenomenon. In this context, an intense amount of activity, both at the experimental as well as at the phenomenological level, is being carried out to fix the parameters of the neutrino oscillations as well as the underlying textures of the neutrino mass matrices [3]-[6].

Several detailed and exhaustive analyses have been carried out in the case of the solar neutrino data as well as in the case of the atmospheric neutrino data [7]. These analyses along with the results from several other experiments have provided valuable information about the masses and the mixing parameters. The constraints on masses and mixings are presented in terms of the mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$), defined analogous to the quark mixing angles, and the mass square differences defined as $\Delta m^2_{sol} \equiv \Delta m^2_{12} = m^2_2 - m^2_1$, 

\[ \Delta m^2_{sol} \]
\(\Delta m_{atm}^2 \equiv \Delta m_{23}^2 = m_3^2 - m_2^2\). The relationship between the mass eigenstates \((\nu_1, \nu_2, \nu_3)\) and the flavor eigenstates \((\nu_e, \nu_\mu, \nu_\tau)\) is expressed through the mixing matrix, for example

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = 
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}.
\]

In the PDG representation [8], the mixing matrix \(U\) is given as

\[
U = 
\begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

where \(c_{ij} = \cos \theta_{ij}\) and \(s_{ij} = \sin \theta_{ij}\) for \(i,j = 1,2,3\) and \(\delta\) represents the CP violating phase.

The best fit oscillation parameters for the Atmospheric Neutrino Problem (ANP) are [8]:

\[
\Delta m_{23}^2 = 3.0 \cdot 10^{-3} \text{eV}^2, \quad \sin \theta_{23} = 0.71.
\]

Similarly the global analysis incorporating the SNO data [6], implies the following “best fit” parameters for the various solutions of Solar Neutrino Problem (SNP):

\[
\begin{align*}
\Delta m_{12}^2 &= 8.0 \cdot 10^{-10} \text{eV}^2, \quad \sin \theta_{12} = 0.61, \quad \text{VO}, \\
\Delta m_{12}^2 &= 4.2 \cdot 10^{-5} \text{eV}^2, \quad \sin \theta_{12} = 0.54, \quad \text{LMA}, \\
\Delta m_{12}^2 &= 5.0 \cdot 10^{-6} \text{eV}^2, \quad \sin \theta_{12} = 0.025, \quad \text{SMA}, \\
\Delta m_{12}^2 &= 9.0 \cdot 10^{-8} \text{eV}^2, \quad \sin \theta_{12} = 0.61, \quad \text{LOW}.
\end{align*}
\]

Although the above mentioned constraints are for two flavor oscillations, however, in view of the CHOOZ constraint [9]

\[
\sin^2 \theta_{13} \equiv |U_{e3}|^2 \leq (0.06 - 0.018) \quad \text{for} \quad \Delta m_{31}^2 = (1.5 - 5) \times 10^{-3} \text{eV}^2,
\]

these remain largely valid in the case of three flavor oscillations also.

Among the several issues which need to be thoroughly examined to enhance our understanding regarding the phenomenology of neutrino oscillations is the issue of finding the lower limit of \(|U_{e3}|\), which in PDG representation corresponds to \(s_{13}\). Because of the sensitive dependence on \(s_{13}\) of the probabilities of the long baseline (LBL) experiments and sub-dominant \(\nu_e \to \nu_\mu(\tau)\) oscillations of atmospheric neutrinos, a knowledge of its value would be very helpful in furthering our understanding of the neutrino oscillation phenomenology. Besides this, a knowledge of \(s_{13}\) will be very important in distinguishing various models of mass matrices and mixing schemes, as recently emphasized by Barr and Dorsner [10].

The knowledge of mixing angles naturally leads to the question of finding the underlying mass matrices. This motivates one to go into specific schemes of these for leptons as the mixing matrix can be related to the mass matrices. In this context, texture specific
mass matrices have been considered in the literature \[10\]-\[23\] which are able to explain several general features of the neutrino oscillation data with good deal of success. However, the investigations do not go into the details of the implications of the texture structure on $U_{e3}$. In view of the fact that see-saw mechanism respects the texture structure \[22\] as well as that texture specific matrices can be obtained in the context of SO(10) grand unified theories \[23\], it is desirable to carry out a detailed analysis of such structures.

A particular type of texture 4 zero quark mass matrices of the form

$$M_U = \begin{pmatrix} 0 & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix}, \quad (9)$$

have shown a good deal of success in accommodating the Cabibbo-Kobayashi-Maskawa (CKM) phenomenology \[11, 22, 24\]. Further, it has also been shown that such mass matrices could be generated from grand unified theories (GUTs) \[25\] as well as that these are “natural” in the sense of Peccei and Wang \[26\]. Apriori there is no reason why the lepton mass texture and quark mass texture should be of the similar kind, nevertheless, it is interesting to see the consequences of a similar structure which may serve as guiding stone for the theories of neutrino mass matrices.

The purpose of the present paper is to investigate the implications of the texture 4 zero lepton mass matrices, similar to the ones defined in equation (9), for the neutrino oscillation parameters. In particular, we intend to examine the implications of these matrices for $s_{13}$, keeping in mind the constraints posed by the atmospheric neutrino data, various solutions of the SNP as well as the results from CHOOZ. As a special case we have also investigated the implications of the mass matrices wherein charged lepton mass matrix is taken as diagonal, primarily for the sake of comparison of our results with those of Akhmedov et al. \[15\] as well as because of the recent interest in such a case \[14\]-\[16\]. Further, it would be interesting to estimate the CP violating Jarlskog’s rephasing invariant parameter \[27\], even in the absence of observation of CP violation in the leptonic sector.

Therefore, to begin with, we consider lepton mass matrices which are similar to those given by equation (9). Unlike quark mass matrices where the $V_{\text{CKM}}$ matrix requires the elements of any of the mass matrices to satisfy the hierarchy, $|A| \ll |B| \simeq D < C$ \[14\], no such restriction is imposed on the lepton mass matrices. Therefore, for the leptonic sector, mass matrices $M_l$ and $M_\nu$ for the charged leptons and neutrinos respectively are:

$$M_l = \begin{pmatrix} 0 & A_l & 0 \\ A_l^* & D_l & B_l \\ 0 & B_l^* & C_l \end{pmatrix}, \quad M_\nu = \begin{pmatrix} 0 & A_\nu & 0 \\ A_\nu^* & D_\nu & B_\nu \\ 0 & B_\nu^* & C_\nu \end{pmatrix}, \quad (10)$$

where $A_{l(\nu)} = |A_{l(\nu)}|e^{i\alpha_{l(\nu)}}$ and $B_{l(\nu)} = |B_{l(\nu)}|e^{i\beta_{l(\nu)}}$. As $M_l$ and $M_\nu$ are hermitian, these can be exactly diagonalized by using the unitary matrices $V_l$ and $V_\nu$, for example

$$V_l^\dagger M_l V_l = \text{diag}(m_e, -m_\mu, m_\tau), \quad (11)$$
$$V_\nu^\dagger M_\nu V_\nu = \text{diag}(m_1, -m_2, m_3). \quad (12)$$
The corresponding lepton mixing matrix is given as

$$ U = V_l^\dagger V_{\nu}.$$  

(13)

For details of the diagonalizing transformations we refer the reader to reference [12]. Using the hierarchy of charged lepton masses, the diagonalizing transformation $V_l$ for $M_l$ can be simplified as

$$ V_l \equiv \begin{pmatrix} e^{i\alpha_l} & -\sqrt{\frac{m_e}{m_\mu}} e^{i\alpha_l} & \sqrt{\frac{m_e m_\mu (m_\tau + D_l)}{m_\tau^2 (m_\tau - D_l)}} e^{i\alpha_l} \\ \sqrt{\frac{m_\mu (m_\tau - D_l)}{m_\mu m_\tau}} & \sqrt{\frac{m_\tau - D_l}{m_\tau}} & \sqrt{\frac{m_\mu + D_l}{m_\tau}} \\ -\sqrt{\frac{m_e (m_\mu + D_l)}{m_\mu m_\tau}} e^{-i\beta_l} & -\sqrt{\frac{m_\mu + D_l}{m_\tau}} e^{-i\beta_l} & \frac{m_\mu + D_l}{m_\tau} e^{-i\beta_l} \end{pmatrix}. $$  

(14)

In the absence of knowledge about absolute neutrino masses, $V_{\nu}$ can not be simplified in the same manner. However, some simplification can be achieved by noting that texture 4 zero mass matrices considered here are not able to reproduce inverted mass hierarchy, primarily because of (1,1) elements in $M_l$ and $M_\nu$ being zero. Therefore, considering natural hierarchy, we can consider $m_3 \gg m_2, m_1$. Hence the diagonalizing matrix $V_{\nu}$ for $M_\nu$ can be simplified as

$$ V_{\nu} \equiv \begin{pmatrix} \sqrt{\frac{m_\mu}{m_1 + m_2}} e^{i\alpha_{\nu}} & -\sqrt{\frac{m_\mu}{m_1 + m_2}} & \sqrt{\frac{m_1 m_2 (m_2 - m_1 + D_\nu)}{m_3 (m_3 - D_\nu)}} \sqrt{\frac{m_1 m_2 (m_2 - m_1 + D_\nu)}{m_3 (m_3 - D_\nu)}} e^{i\alpha_{\nu}} \\ \sqrt{\frac{m_1 (m_3 - D_\nu)}{m_3 (m_1 + m_2)}} & \frac{m_1 (m_3 - D_\nu)}{m_3 (m_1 + m_2)} & \frac{m_3 - m_3 + D_\nu}{m_3} \sqrt{\frac{m_3 - m_3 + D_\nu}{m_3}} e^{-i\beta_{\nu}} \\ -\sqrt{\frac{m_1 (m_2 - m_1 + D_\nu)}{m_3 (m_2 + m_1)}} e^{-i\beta_{\nu}} & -\sqrt{\frac{m_2 (m_2 - m_1 + D_\nu)}{m_3 (m_2 + m_1)}} e^{-i\beta_{\nu}} & \frac{m_3 - m_3 + D_\nu}{m_3} e^{-i\beta_{\nu}} \end{pmatrix}. $$  

(15)

Using equation (13) and the exact diagonalizing transformations given in reference [12], one can easily construct the corresponding lepton mixing matrix in terms of the charged lepton masses, neutrino masses, $D_l$, $D_\nu$ and phases $\phi_1 (= \alpha_l - \alpha_{\nu})$ and $\phi_2 (= \beta_l - \beta_{\nu})$. For the purpose of calculations we have used the exact expressions, however to facilitate the understanding of the dependence of the elements of the mixing matrix on various parameters, the simplified expressions for $U_{e3}$, $U_{e2}$ and $U_{\mu 3}$, the elements of the mixing matrix directly related to the mixing angles $\theta_{13}$, $\theta_{12}$ and $\theta_{23}$, are as follows:

$$ U_{e3} = \sqrt{\frac{m_1 m_2 (m_2 - m_1 + D_\nu)}{m_3 (m_3 - D_\nu)}} e^{-i\phi_1} + \frac{m_e (m_\tau - D_l)(m_2 - m_1 + D_\nu)}{m_\mu m_\tau m_3}, $$

$$ U_{e2} = -\sqrt{\frac{m_1}{m_1 + m_2}} e^{-i\phi_1} + \frac{m_e m_2 (m_\tau - D_l)(m_3 - D_\nu)}{m_\mu m_\tau m_3 (m_1 + m_2)} e^{i\phi_2}, $$

(16)

$$ U_{e2} = -\sqrt{\frac{m_1}{m_1 + m_2}} e^{-i\phi_1} + \frac{m_e m_2 (m_\tau - D_l)(m_3 - D_\nu)}{m_\mu m_\tau m_3 (m_1 + m_2)} e^{i\phi_2}. $$

(17)
\[
U_{\mu 3} = -\sqrt{\frac{m_\tau m_3}{m_\mu m_3}} e^{-i\phi_1} + \sqrt{\frac{(m_\tau - D_\nu)(m_2 - m_1 + D_\nu)}{m_\tau m_3}} - \sqrt{\frac{(m_\mu + D_\nu)(m_3 - D_\nu)}{m_\tau m_3}} e^{i\phi_2}.
\]

(18)

The calculations have been carried out for the following cases: (i) both \(M_L\) and \(M_\nu\) real, \(M_L\) being diagonal, (ii) both \(M_L\) and \(M_\nu\) real and non-diagonal, and (iii) Both \(M_L\) and \(M_\nu\) being complex and non-diagonal. To begin with, we consider the simplest case, wherein \(M_L\) is diagonal and \(M_\nu\) is real, and calculate \(s_{13}\) corresponding to the different SNP solutions and the constraints implied by ANP data. For this case, the mixing matrix \(U\) corresponds to the diagonalizing matrix \(V_\nu\), given in equation (15) with \(\alpha_\nu = \beta_\nu = 0\).

Before discussing the results, it is perhaps desirable to discuss some of the specific details pertaining to various inputs. While considering the ANP and SNP constraints, we have taken the best fit values corresponding to \(\Delta m^2_{12}\) and \(\Delta m^2_{23}\) given in equations (3)−(7), however in the case of mixing angles we have considered the ranges:

\[
\sin \theta_{23} = 0.55 - 0.84, \quad (19)
\]
\[
\sin \theta_{12} = 0.42 - 0.71 \quad \text{VO}, \quad (20)
\]
\[
\sin \theta_{12} = 0.42 - 0.65 \quad \text{LMA}, \quad (21)
\]
\[
\sin \theta_{12} = 0.015 - 0.05 \quad \text{SMA}, \quad (22)
\]
\[
\sin \theta_{12} = 0.42 - 0.61 \quad \text{LOW}, \quad (23)
\]

which include the the best fit values given in equations (3)-(4). In this case, we have \(D_\nu\) and one of the neutrino masses as the unknown parameters, the other two masses can be deduced from \(\Delta m^2_{23}\) and \(\Delta m^2_{12}\). In principle, the parameter \(D_\nu\) can take any value, however, in accordance with the similar analysis in the quark sector [11], we have restricted its variation within \(0 < D_\nu < m_3\). We choose \(m_1\) to be the free parameter and find a range of \(s_{13}\) for different values of \(m_1\) and \(D_\nu\), such that \(s_{12}\) and \(s_{23}\) are reproduced within the SNP and ANP constraints. In the Table (1), we have presented our results regarding \(s_{13}\) for different values of \(m_1\) and \(D_\nu\) for the LMA solution of SNP. For the sake of completeness, we have also presented in the table the corresponding values of \(s_{12}\) and \(s_{23}\). A general survey of the table reveals the following range of \(s_{13}\):

\[
s_{13} = 0.04 - 0.13, \quad (24)
\]

which is in agreement with the range of \(s_{13}\) found by Akhmedov et al. [15], for example

\[
s_{13} = 0.05 - 0.15. \quad (25)
\]

It is interesting to mention the ranges of the neutrino masses corresponding to the equation (24), for example

\[
m_1 = (1 - 5) \times 10^{-3} \text{eV}, \quad (26)
\]
\[
m_2 = (6 - 8) \times 10^{-3} \text{eV}, \quad (27)
\]
\[
m_3 \simeq 5 \times 10^{-2} \text{eV}, \quad (28)
\]
suggesting the mass hierarchy \( m_1 \approx m_2 < m_3 \) for neutrinos. In the same vein, it is interesting to examine the hierarchy pattern of the elements of the neutrino mass matrix \( M_\nu \). In this context, it may be noted that while carrying out the analysis, we have varied \( D_\nu \) between 0 and \( m_3 \), however the allowed range of \( D_\nu \) is given as \( 0.24 < \frac{D_\nu}{m_3} < 0.6 \). This range of \( D_\nu \) helps determining the hierarchy pattern of mass matrix elements. As an example, when the solution corresponding to the row I of the Table ([1]) is used to construct \( A_\nu, B_\nu, C_\nu \) and \( D_\nu \), we obtain the following mass matrix

\[
M_\nu = \begin{pmatrix}
0 & 0.0041 & 0 \\
0.0041 & 0.022 & 0.028 \\
0 & 0.028 & 0.029 \\
\end{pmatrix}.
\]  

(29)

The above matrix corresponds to hierarchy \( A_\nu < B_\nu \approx D_\nu \approx C_\nu \), which looks to be somewhat different compared to the hierarchy pattern of quark mass matrix elements. The mixing matrix corresponding to the above neutrino mass matrix is given as

\[
U = \begin{pmatrix}
0.903 & 0.426 & 0.050 \\
0.284 & 0.682 & 0.673 \\
0.321 & 0.594 & 0.737 \\
\end{pmatrix}.
\]  

(30)

The above mixing matrix is well within the mixing matrix derived recently by Fukugita and Tanimoto [28], by including the SNO data also.

In a similar manner, one can carry out the analysis for the SMA, LOW and VO solutions of SNP. Without presenting the details of results, we summarize the \( s_{13} \) ranges evaluated in these cases as

\[
s_{13} = (0.4 - 2.5) \times 10^{-3} \quad \text{SMA,} \\
s_{13} = (1.7 - 6.7) \times 10^{-3} \quad \text{LOW,} \\
s_{13} = (0.2 - 12.0) \times 10^{-3} \quad \text{VO,}
\]  

(31-33)

which very well include the corresponding values given by Akhmedov et al., for example

\[
s_{13} \sim 10^{-3} \quad \text{(SMA)}, \quad s_{13} \sim 10^{-2} \quad \text{(LOW)}, \quad s_{13} \sim 10^{-4} - 10^{-3} \quad \text{(VO)}.
\]  

(34)

The \( m_1, m_2 \) and \( m_3 \) ranges corresponding to \( s_{13} \), given in equations (31)-(33), are

\[
m_1 \sim 10^{-8} - 10^{-6} \text{ eV}, \quad m_2 \sim 10^{-3} \text{ eV}, \quad m_3 \sim 0.055 \text{ eV}, \quad \text{SMA,} \\
m_1 \sim 10^{-5} - 10^{-4} \text{ eV}, \quad m_2 \sim 10^{-4} \text{ eV}, \quad m_3 \sim 0.055 \text{ eV}, \quad \text{LOW,} \\
m_1 \sim 10^{-5} - 10^{-3} \text{ eV}, \quad m_2 \sim 10^{-4} \text{ eV}, \quad m_3 \sim 0.055 \text{ eV}, \quad \text{VO.}
\]  

(35-37)

In case of LOW and VO solutions, our conclusions regarding the mass hierarchy as well as the hierarchy of mass matrix elements remain the same as that for the LMA case discussed above, however for the SMA case, we find that the allowed values of \( m_1 \) are such that the corresponding mass hierarchy is \( m_1 \ll m_2 < m_3 \).
Having discussed the case with diagonal $M_l$, it is interesting to examine the implications of non-diagonal $M_l$ for $s_{13}$. With non-diagonal and real $M_l$ and $M_\nu$, the $U_{e2}$, $U_{e3}$ and $U_{\mu 3}$ are given by equations (16)-(18) with $\phi_1 = \phi_2 = 0$. In this case, apart from $m_1$ and $D_\nu$, we have $D_l$ as another free parameter. Again, we find a range of $s_{13}$ by varying $D_\nu$ within $0 < D_\nu < m_3$ and $D_l$ within $0 < D_l < m_\tau$, as well as by scanning a suitable range of $m_1$ so that $s_{23}$ and $s_{12}$ are within the limits given by ANP and SNP solutions. In the Table (2), we have presented our results regarding $s_{12}$, $s_{23}$ and $s_{13}$ for different values of $m_1$, $D_\nu$ and $D_l$ for the LMA solution. The range of $s_{13}$ for the LMA solution is given as

$$s_{13} = \sim 0 - 0.19.$$  

The corresponding allowed ranges of $m_1$, $m_2$ and $m_3$ are

$$m_1 = (0.7 - 10.0) \times 10^{-3} \text{eV},$$  

$$m_2 = (6.5 - 11.9) \times 10^{-3} \text{eV},$$  

$$m_3 = (5.5 - 5.6) \times 10^{-2} \text{eV},$$

which are within the ranges found recently by Osland and Wu [29] by considering similar texture for neutrino mass matrix.

Following a similar procedure as discussed above for LMA case, the ranges of $s_{13}$ obtained with the SMA, LOW and VO solutions are given as

$$s_{13} = 0.038 - 0.093$$  

SMA,  

$$s_{13} = 0.042 - 0.095$$  

LOW,  

$$s_{13} = 0.038 - 0.096$$  

VO.

The corresponding allowed ranges of $m_1$, $m_2$ and $m_3$ in each case are

$$m_1 \sim 10^{-8} - 10^{-5} \text{eV},$$  

$$m_2 \sim 10^{-3} \text{eV},$$  

$$m_3 \sim 0.055 \text{eV},$$  

SMA,  

$$m_1 \sim 10^{-5} - 10^{-4} \text{eV},$$  

$$m_2 \sim 10^{-4} \text{eV},$$  

$$m_3 \sim 0.055 \text{eV},$$  

LOW,  

$$m_1 \sim 10^{-5} - 10^{-3} \text{eV},$$  

$$m_2 \sim 10^{-4} \text{eV},$$  

$$m_3 \sim 0.055 \text{eV},$$  

VO.

The ranges of $s_{13}$ given in equations (38), (42) look to be vastly different from the ranges given in equations (24), (31) evaluated using diagonal $M_l$. This warrants a more detailed comparison of the two cases. One finds that in the case of LMA solution, the values of $s_{13}$ obtained are in the same range as obtained with diagonal $M_l$, however the spread is larger. While, in the case of solutions pertaining to SMA, LOW and VO solutions of SNP, $s_{13}$ is much higher compared to that of the case with diagonal $M_l$. Further, the three solutions more or less have same range of values. This can be easily understood by analyzing the expression for $s_{13}$, which can be written as

$$s_{13} = s_{13}^d + \frac{m_e(m_\tau - D_l)(m_2 - m_1 + D_\nu)}{m_\mu m_\tau m_3} - \frac{m_e(m_\mu + D_l)(m_3 - D_\nu)}{m_\mu m_\tau m_3},$$

7
where \( s_{13}^d \) is the expression for \( s_{13} \) with diagonal \( M_l \) and is given as

\[
s_{13}^d = \sqrt{\frac{m_1 m_2 (m_2 - m_1 + D_\nu)}{m_3^2 (m_3 - D_\nu)}}.
\]

A closer scrutiny of (18) reveals that for LMA solution all the three terms are of the same order leading to wider spread for \( s_{13} \). For the SMA, LOW and VO solutions, we find that the contribution of \( s_{13}^d \) is very small compared to the other two terms which are two orders of magnitude larger pushing \( s_{13} \) up compared to the case with diagonal \( M_l \).

Important conclusions can be derived from above results regarding \( s_{13} \). For example, in case \( s_{13} \) is found to be outside the range (0.01-0.1), then the solutions SMA, LOW and VO look to be ruled out for texture 4 zero mass matrices. Further, it is interesting to emphasize that several present analyses, excluding as well as including SNO, favor LMA and LOW solution of SNP, LMA being the preferred solution. The present calculations look to be favoring LMA solution as it is able to encompass the entire possible range of \( s_{13} \).

The hierarchy pattern of the neutrino masses and of the mass matrix elements remains more or less similar to the case with diagonal \( M_l \). For example, for LMA solution the mass matrix corresponding to the first row of Table (2) is given as

\[
M_\nu = \begin{pmatrix} 0 & 0.0069 & 0 \\ 0.0069 & 0.006 & 0.021 \\ 0 & 0.021 & 0.047 \end{pmatrix}.
\] (49)

The corresponding mixing matrix is given as

\[
U = \begin{pmatrix} 0.816 & 0.578 & 0.0002 \\ 0.473 & 0.668 & 0.574 \\ 0.332 & 0.468 & 0.818 \end{pmatrix}.
\] (50)

The above mixing matrix is again within the mixing matrix derived by Fukugita and Tanimoto [28].

Recently several authors have discussed the possibility of observing CP violation in the leptonic sector [13, 21, 28]. To include CP violation, the corresponding mass matrices \( M_l \) and \( M_\nu \) have to be complex. This necessitates non-zero phases \( \alpha_{l(\nu)} \) and \( \beta_{l(\nu)} \) in the mass matrices given in equation (11) and the corresponding simplified expressions for \( U_{e3} \), \( U_{e2} \) and \( U_{\mu3} \) are given by the equations (16)-(18). Again, following the same procedure as detailed in the previous sections, and by scanning the range of phases \( \phi_1 \) and \( \phi_2 \) from 0 to \( \pi \), we have evaluated the \( s_{13} \) ranges corresponding to the four SNP solutions and satisfying ANP and CHOOZ constraints, for example

\[
s_{13} = \sim 0 - 0.20 \text{ LMA} \quad (51)
\]
\[
s_{13} = 0.038 - 0.094 \text{ SMA} \quad (52)
\]
\[
s_{13} = 0.028 - 0.099 \text{ LOW} \quad (53)
\]
\[
s_{13} = 0.035 - 0.098 \text{ VO} \quad (54)
\]
Comparing with the corresponding results obtained in the last section without CP violation, we find that the range of $s_{13}$ is more or less similar to the case with real and non-diagonal $M_l$ and $M_{\nu}$. For a given values of neutrino masses, $D_{\nu}$ and $D_l$, the $s_{13}$ ranges for SMA, LOW and VO solutions show marginal increase with the introduction of phases compared to the previous case, however for LMA $s_{13}$ goes up significantly, as is evident from the results presented in the Table (3).

As a rough measure of the CP violating phase in the leptonic sector, several authors have estimated the maximum value of $J_{CP}^l$ [21, 27], the Jarlskog’s rephasing invariant parameter in the leptonic sector. In this context, we have also calculated $J_{CP}^l$ from the mass matrices using the following expression:

$$\text{Det} [M_l M_l^\dagger, M_{\nu} M_{\nu}^\dagger] = -2i J_{CP}^l (m_\tau^2 - m_\mu^2)(m_\mu^2 - m_e^2)(m_e^2 - m_\tau^2) \times (m_3^2 - m_2^2)(m_2^2 - m_1^2)(m_1^2 - m_3^2).$$  \hspace{1cm} (55)

Varying phases $\phi_1$ and $\phi_2$ from 0 to $\pi$, with other inputs being the same as used in previous sections, we obtain for LMA, SMA, LOW and VO cases

$$J_{CP}^l < 0.099 \quad \text{LMA}, \hspace{1cm} (56)$$

$$J_{CP}^l < 0.0012 \quad \text{SMA}, \hspace{1cm} (57)$$

$$J_{CP}^l < 0.043 \quad \text{LOW}, \hspace{1cm} (58)$$

$$J_{CP}^l < 0.045 \quad \text{VO}, \hspace{1cm} (59)$$

which are in agreement with most of the contemporary analyses [21, 24, 28].

To summarize the conclusions, we have considered lepton mass matrices similar to the texture 4 zero mass matrices, known to be quite successful in explaining the CKM phenomenology, for finding the mixing matrix element $U_{e3}$ respecting the CHOOZ constraint with $s_{12}$ and $\Delta m_{12}^2$ constrained by SNP and $s_{23}$ and $\Delta m_{23}^2$ constrained by ANP. Interestingly, when the charged lepton mass matrix is taken to be diagonal, the ranges of $s_{13}$ corresponding to different SNP solutions very well include the corresponding ranges found by Akhmedov et al. by considering neutrino mass matrices with no texture zeros. Taking $M_l$ and $M_{\nu}$ both to be real and non-diagonal, $s_{13}$ ranges for the four SNP solutions come out to be: $\sim 0 - 0.19$ (LMA), $0.038 - 0.093$ (SMA), $0.042 - 0.095$ (LOW), $0.038 - 0.096$ (VO). Except for LMA case, the other ranges show marked difference compared to the case with diagonal $M_l$. These conclusions remain largely valid when the effects of phases are included. In fact, in case $s_{13}$ is found to be outside the range (0.01-0.1), then the solutions SMA, LOW and VO look to be ruled out for texture 4 zero mass matrices.

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\begin{tabular}{|c|c|c|c|c|}
\hline
$m_1 \times 10^{-3}$ eV & \frac{D_\nu}{m_1} & $s_{12}$ & $s_{23}$ & $s_{13}$ \\
\hline
1.3 & 0.4 & 0.43 & 0.67 & 0.05 \\
1.4 & 0.5 & 0.45 & 0.74 & 0.06 \\
1.5 & 0.5 & 0.46 & 0.74 & 0.07 \\
1.6 & 0.5 & 0.47 & 0.74 & 0.07 \\
1.6 & 0.6 & 0.49 & 0.80 & 0.08 \\
1.7 & 0.6 & 0.50 & 0.80 & 0.09 \\
2.1 & 0.6 & 0.54 & 0.80 & 0.10 \\
2.5 & 0.6 & 0.57 & 0.80 & 0.11 \\
3.3 & 0.6 & 0.62 & 0.80 & 0.12 \\
3.4 & 0.6 & 0.63 & 0.80 & 0.12 \\
3.6 & 0.6 & 0.64 & 0.80 & 0.13 \\
3.8 & 0.6 & 0.65 & 0.80 & 0.13 \\
3.9 & 0.5 & 0.63 & 0.74 & 0.11 \\
4.0 & 0.5 & 0.63 & 0.74 & 0.11 \\
4.4 & 0.5 & 0.65 & 0.74 & 0.12 \\
\hline
\end{tabular}

Table 1: Calculated values of $s_{12}$, $s_{23}$ and $s_{13}$ for the LMA solution of SNP for real $M_\nu$ and $M_l$, with $M_l$ being diagonal.
| $m_1 \times 10^{-3}$ eV | $D_{\nu} / m_1$ | $D_{l} / m_2$ | $s_{12}$ | $s_{23}$ | $s_{13}$ |
|--------------------------|-----------------|-----------------|---------|---------|---------|
| 5.0                      | 0.1             | 0.7             | 0.58    | 0.57    | 0.0002  |
| 8.2                      | 0.1             | 0.8             | 0.63    | 0.67    | 0.005   |
| 8.6                      | 0.2             | 0.9             | 0.64    | 0.68    | 0.010   |
| 8.8                      | 0.1             | 0.7             | 0.64    | 0.59    | 0.020   |
| 7.6                      | 0.1             | 0.9             | 0.63    | 0.77    | 0.030   |
| 8.4                      | 0.2             | 0.8             | 0.64    | 0.58    | 0.040   |
| 9.1                      | 0.3             | 0.9             | 0.65    | 0.60    | 0.050   |
| 2.1                      | 0.1             | 0.9             | 0.44    | 0.74    | 0.060   |
| 1.3                      | 0.7             | 0.1             | 0.43    | 0.59    | 0.139   |
| 1.7                      | 0.7             | 0.1             | 0.49    | 0.59    | 0.153   |
| 2.1                      | 0.7             | 0.1             | 0.53    | 0.59    | 0.165   |
| 2.2                      | 0.7             | 0.1             | 0.54    | 0.59    | 0.168   |
| 2.3                      | 0.7             | 0.1             | 0.55    | 0.59    | 0.171   |
| 3.0                      | 0.7             | 0.1             | 0.60    | 0.59    | 0.189   |
| 3.4                      | 0.7             | 0.1             | 0.63    | 0.59    | 0.199   |

Table 2: Calculated values of $s_{12}$, $s_{23}$ and $s_{13}$ for the LMA solution of SNP for real and non-diagonal $M_l$ and $M_{\nu}$.

| $s_{13}$                  | Diagonal $M_l$ | Non-diagonal and real $M_l$ and $M_{\nu} | Non-diagonal and complex $M_l$ and $M_{\nu}$ |
|--------------------------|-----------------|---------------------------------------------|-----------------------------------------------|
| Diagonal $M_l$           | Non-diagonal and real $M_l$ and $M_{\nu}$ | Non-diagonal and complex $M_l$ and $M_{\nu}$ |
| LMA                      | 0.0628          | 0.0033                                      | 0.0033 – 0.1422                               |
| SMA                      | 0.0006          | 0.067                                       | 0.067 - 0.071                                 |
| LOW                      | 0.0021          | 0.067                                       | 0.067 - 0.083                                 |
| VO                       | 0.0002          | 0.069                                       | 0.069 - 0.081                                 |

Table 3: The values of $s_{13}$ obtained with fixed $m_1, D_{\nu}$ and $D_{l}$, with $\phi_1$ and $\phi_2$ varying from 0 to $\pi$. 