Poincaré gauge theory in 3D: canonical stability of the scalar sector

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Abstract

We outline the results of the canonical analysis of the three-dimensional Poincaré gauge theory, defined by the general parity-invariant Lagrangian with eight free parameters [11]. In the scalar sector, containing scalar or pseudoscalar (A)dS modes, the stability of the canonical structure under linearization is used to identify dynamically acceptable values of the parameters.

1 Introduction

Models of three-dimensional (3D) gravity, pioneered by Staruskievicz [1], were introduced to help us in clarifying highly complex dynamical behavior of the realistic four-dimensional general relativity (GR). In the last three decades, they led to a number of outstanding results [2]. However, in the early 1990s, Mielke and Baekler [3] proposed a new, non-Riemannian approach to 3D gravity, based on the Poincaré gauge theory (PGT) [4]. In PGT, the basic gravitational variables are the triad $b^i$ and the Lorentz connection $A_{ij}$ (1-forms), and their field strengths are the torsion $T^i := db^i + A_{ij} b^j$ and the curvature $R^{ij} := dA^{ij} + A_{im} A^{mj}$ (we omit the exterior product sign for simplicity). In contrast to the traditional GR, with an underlying Riemannian geometry of spacetime, the PGT approach is characterized by a Riemann–Cartan geometry, with both the curvature and the torsion of spacetime as carriers of the gravitational dynamics. Thus, PGT allows exploring the interplay between gravity and geometry in a more general setting.

Three-dimensional GR with or without a cosmological constant, as well as the Mielke–Baekler (MB) model, are topological theories without propagating modes. From the physical point of view, such a degenerate situation is certainly not quite realistic. Including the propagating modes in PGT is achieved quite naturally by using Lagrangians quadratic in the field strengths [5, 6]. Since the general parity-invariant PGT Lagrangian in 3D is defined by eight free parameters [6], it is a theoretical challenge to find out which values of the parameters are allowed in a viable theory. The simplest approach to this problem is based on the weak-field approximation around the Minkowski background [5]. However, one should be very careful with the interpretation of these results, since the weak-field approximation does not always lead to a correct identification of the physical degrees of freedom.

The constrained Hamiltonian method [7, 4] is best suited for analyzing dynamical content of gauge theories of gravity, respecting fully their nonlinear structure. However, as noticed by Yo and Nester [8, 9], it may happen, for some ranges of parameters, that the canonical structure of a theory (the number and/or type of constraints) is changed after linearization in a way that affects its physical content, such as the number of physical degrees of freedom. Such an effect is called...
the phenomenon of *constraint bifurcation*. Based on the *canonical stability under linearization* as a criterion for an acceptable choice of parameters, Shie et al. [10] proposed a PGT cosmological model that offers a convincing explanation of dark energy as an effect induced by torsion.

In this note, we use the constrained Hamiltonian formalism to study (a) the phenomenon of constraint bifurcation and (b) the stability under linearization of the general parity-invariant PGT in 3D, in order to find out the parameter values that define consistent models of 3D gravity with propagating torsion. Because of the complexity of the problem, we restrict our attention to the scalar sector, with $J^P = 0^+$ or $0^-$ modes, defined with respect to the (A)dS background [11].

The following conventions are of particular importance for our canonical analysis. Let $\mathcal{M}$ be a 3D manifold (spacetime) with local coordinates $x^\mu = (x^0, x^\alpha)$, and $h_i^\mu = h_i^\mu \partial_\mu$ a Lorentz frame on it. Then, if $\Sigma$ is a 2D spacelike surface with a unit normal $n_k$, each tangent vector $V_k$ of $\mathcal{M}$ can be decomposed in terms of its normal and parallel component with respect to $\Sigma$:

$$V_k = n_k V_\perp + \bar{V}_k,$$

where $V_\perp := n^m V_m$, $V_\perp = h^\alpha V_\alpha$.

Note that $\bar{V}_k$ does not contain the time component of $V_\mu$.

**2 Quadratic PGT and its scalar modes**

Assuming parity invariance, the dynamics of 3D gravity with propagating torsion is determined by the gravitational Lagrangian

$$L_G = -a \varepsilon_i^{jk} b^i R^{jk} - \frac{1}{3} A_0 \varepsilon_i^{jk} b^j b^k + L_{T^2} + L_{R^2}, \quad (1a)$$

where $a = 1/16\pi G$, $A_0$ is a free parameter (bare cosmological constant), the pieces quadratic in the field strengths read

$$L_{T^2} := T^{i*} \left(a_1 (1) T_i + a_2 (2) T_i + a_3 (3) T_i\right),$$

$$L_{R^2} := \frac{1}{2} R^{ij} \left(b_4 (4) R_{ij} + b_5 (5) R_{ij} + b_6 (6) R_{ij}\right), \quad (1b)$$

and $(n)T_i$ and $(n)R_{ij}$ are irreducible components of $T^i$ and $R^{ij}$ [6]. Being interested only in the gravitational degrees of freedom, we disregard the matter contribution.

Particle spectrum of the theory around the Minkowski background $M_3$ is already known [5, 6]. Restricting our attention to the scalar sector, we display here the masses of the spin-$0^+$ and $0^-$ modes:

$$m_{0^+}^2 = \frac{3a(a + a_2)}{a_2(b_4 + 2b_6)}; \quad m_{0^-}^2 = \frac{3a(a + 2a_3)}{(a_1 + 2a_3)b_5}. \quad (2a)$$

These modes have finite masses and propagate if

$$a_2(b_4 + 2b_6) \neq 0; \quad (a_1 + 2a_3)b_5 \neq 0, \quad (2b)$$

respectively.

Transition to the (A)dS background is straightforward; it generalizes the mass formulas (2a) by introducing a dependence on the parameter $q$ that measures the strength of the background curvature [11], but the propagation conditions for the scalar modes remain the same as in (2b).

As we shall see in the next section, the conditions (2b), derived in the weak-field approximation, have a critical role also in the canonical analysis of the *full nonlinear theory*. 

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*Note:* The text above is a simplified version of the provided document, focusing on key aspects and avoiding unnecessary details. The equations and mathematical expressions are presented in a more readable format for better understanding.
3 Primary if-constraints

The canonical momenta corresponding to the basic dynamical variables \( (b^i, A^{ij}) \) are defined by
\[
\pi_i^\mu := \partial \hat{L} / \partial \partial_\mu b^i \quad \text{and} \quad \Pi_{ij}^\mu := \partial \hat{L} / \partial \partial_\mu A^{ij},
\]
respectively. Since the torsion and the curvature do not involve the velocities \( \partial \partial_0 b^i \) and \( \partial \partial_0 A^{ij} \), one obtains the primary constraints
\[
\pi_i^0 \approx 0, \quad \Pi_{ij}^0 \approx 0,
\]
which are always present, independently of the values of coupling constants (“sure” constraints). If the Lagrangian \( \hat{L} \) is singular with respect to some of the remaining velocities \( \partial \partial_\alpha b^i \) and \( \partial \partial_\alpha A^{ij} \), one obtains further primary constraints, known as the primary “if-constraints” (ICs).

The gravitational Lagrangian \( \hat{L} \) depends on the time derivative \( \partial \partial_0 b^i \) only through the torsion tensor, appearing in \( L_{T^2} \). The system of equations defining the parallel gravitational momenta \( \hat{\pi}_i^k = \pi^a b^a_k \) \( (\hat{\pi}_i^k n_k = 0) \) can be decomposed into irreducible parts with respect to the group of two-dimensional spatial rotations in \( \Sigma \):
\[
\phi_{\perp k} := \hat{\pi}_{\perp k} - (a_2 - a_1)T^m_{\perp k} = (a_1 + a_2)T_{\perp \perp k},
\]
\[
S_\phi := \frac{S_{\hat{\pi}}}{J} = -2a_2 T^m_{\perp \perp},
\]
\[
A_{\phi_{ik}} := \frac{A_{\hat{\pi}_{ik}}}{J} - \frac{2}{3} (a_1 - a_3)T_{\perp \perp k} = -\frac{2}{3} (a_1 + 2a_3)T_{[ik]\perp},
\]
\[
T_{\phi_{ik}} := \frac{T_{\hat{\pi}_{ik}}}{J} = -2a_1 T_{ik\perp},
\]
where the terms depending on the velocities \( \partial \partial_0 b^i \) are moved to the right-hand sides. If the critical parameter combinations appearing on the right-hand sides of Eqs. \( \Pi_{ij}^0 \) vanish, the corresponding expressions \( \phi_K \) become additional primary constraints.

Similar analysis can be applied to the equations defining the parallel gravitational momenta \( \hat{\Pi}_{ij}^k := \Pi_{ij}^a b^a_k \) \( (\hat{\Pi}_{ij}^k n_k = 0) \), leading to an additional set of primary constraints \( \Phi_K \). The complete set of primary ICs, including their spin-parity characteristics \( (J^P) \), is shown in Table 1.

| Critical conditions | Primary constraints | \( J^P \) |
|---------------------|---------------------|---------|
| \( a_2 = 0 \)      | \( S_\phi \approx 0 \) | \( 0^+ \) |
| \( b_1 + 2b_0 = 0 \)| \( S_{\phi_{\perp}} \approx 0 \) | \( 0^+ \) |
| \( a_1 + 2a_3 = 0 \)| \( A_{\phi_{ik}} \approx 0 \) | \( 0^- \) |
| \( b_5 = 0 \)      | \( A_{\phi_{\perp \perp k}} \approx 0 \) | \( 0^- \) |
| \( a_1 + a_2 = 0 \)| \( \phi_{\perp \perp k} \approx 0 \) | \( 1 \) |
| \( b_1 + b_5 = 0 \)| \( V_{\Phi_{\perp \perp k}} \approx 0 \) | \( 1 \) |
| \( a_1 = 0 \)      | \( T_{\phi_{ik}} \approx 0 \) | \( 2 \) |
| \( b_4 = 0 \)      | \( T_{\Phi_{\perp \perp k}} \approx 0 \) | \( 2 \) |

This classification has a remarkable interpretation: whenever a pair of the ICs with specific \( J^P \) is absent, the corresponding dynamical mode is liberated to become a physical degree of freedom (DoF). Thus, for \( a_2(b_1 + 2b_0) \neq 0 \), the spin-\( 0^+ \) ICs are absent, and the spin-\( 0^+ \) mode becomes physical. Similarly, \( (a_1 + 2a_3)b_5 \neq 0 \) implies that the spin-\( 0^- \) mode is physical. These results, referring to the full nonlinear theory, should be compared to \( (2b) \).
Remark. Once we know the complete set of primary ICs, we can apply Dirac’s consistency algorithm to obtain the secondary constraints, and so on.

4 Spin-0$^+$ sector

As one can see from Table 1, the spin-0$^+$ degree of freedom propagates for $a_2(b_4 + 2b_6) \neq 0$. In order to investigate dynamical features of this sector, we adopt somewhat simplified conditions:

\[
a_2, b_6 \neq 0, \quad a_1 = a_3 = b_4 = b_5 = 0. \tag{5a}
\]

While such a “minimal” choice simplifies the calculations, it is not expected to influence any essential aspect of the spin-0$^+$ dynamics \[8, 9\].

Generic case

Now, we turn to the canonical analysis. First, the form of the Hamiltonian implies that the kinetic energy density is positive definite (no “ghosts”) if

\[
a_2 > 0, \quad b_6 > 0. \tag{5b}
\]

Second, in the simple, generic situation, when all of the ICs are second class (their number is $N_2 = 10$), the complete set of constraints is given in Table 2.

| Table 2. Generic constraints in the 0$^+$ sector |
|-----------------------------------------------|
| **Primary** | **Second class** |
| $\pi^0_i, \Pi^0_{ij}$ | $V_{\Phi\bar{\Pi}}, A_{\Phi}, T_{\Phi}, T_{\Phi}$ |

As always, the Hamiltonian constrains $H_{1}', H_{\alpha}', H_{ij}'$ are first class. With $N = 2 \times 9$ field components, $N_1 = 2 \times 6$ first class constraints and $N_2 = 10$ second class constraints, the dimension of the phase space is $N^* = 2N - 2N_1 - N_2 = 2$, and the theory exhibits a single Lagrangian DoF.

Constraint bifurcation

To clarify the term “generic” used above, we calculate the determinant of the $10 \times 10$ matrix $\Delta^+_{MN} = \{X'_M, X'_N\}$, where $X'_M$ is the set of all ICs shown in Table 2. The result is

\[
\Delta^+ \sim W^{10} (W - a_2)^4 \quad \text{where} \quad W := \frac{S \Pi_1}{4J}. \tag{6}
\]

The generic situation corresponds to $\Delta^+ \neq 0$. However, the determinant $\Delta^+$, being a field-dependent object, may vanish in some regions of spacetime, changing thereby the number and/or type of constraints and the number of physical DoF, as compared to the situation described in Table 2. This phenomenon of constraint bifurcation can be fully understood by analyzing dynamical behavior of the critical factors $W$ and $W - a_2$, appearing in $\Delta^+$.

Assuming that $W$ is an analytic function globally, on the whole spacetime manifold $\mathcal{M}$, the analysis of the field equations

\[
-(W - a_2)V_k + 2\partial_k(W - a_2) \approx 0, \tag{7}
\]

leads to the following conclusion [11]:
If there is a point in \( \mathcal{M} \) at which \( W - a_2 \neq 0 \), then \( W - a_2 \neq 0 \) globally.

Hence, by choosing the initial data so that \( W - a_2 \neq 0 \) at \( x^0 = 0 \), it follows that \( W - a_2 \) stays nonvanishing for any \( x^0 > 0 \). The surface \( W - a_2 = \frac{1}{6} b_6 R - a - a_2 \approx 0 \) (on shell) is a dynamical barrier that the spin-0\(^+\) field cannot cross. Moreover, since \( a_2 \) is positive, see (5b), we have:

- By choosing \( W - a_2 > 0 \) at \( x^0 = 0 \), it follows that \( W \neq 0 \) globally.

Thus, with a suitable choice of the initial data, one can ensure the generic condition \( \Delta^+ \neq 0 \) to hold globally, whereupon the constraint structure is described exactly as in Table 2. Any other situation, with \( W = 0 \) or \( W - a_2 = 0 \), would not be acceptable—it would have a variable constraint structure over the spacetime, the property that could not survive the process of linearization.

**Stability under linearization**

Now, we compare the canonical structure of the full nonlinear theory with its weak-field approximation around maximally symmetric background. With the background values \( \bar{R} = -6q \) and \( \bar{W} = \frac{1}{6} b_6 R - a \), the lowest-order critical factors take the form

\[
\bar{W} = -(a + q b_6), \quad \bar{W} - a_2 = -(a + a_2 + q b_6),
\]

which leads to the results shown in Table 3 [11].

| \( a + q b_6 \) | \( a + a_2 + q b_6 \) | DoF | stability |
|-----------------|----------------------|-----|-----------|
| (a) \( \neq 0 \) | \( \neq 0 \) | 1   | stable    |
| (b) \( = 0 \)   | \( \neq 0 \) | 0   | unstable  |
| (c) \( \neq 0 \) | \( = 0 \)  | 1   | stable*   |

Based on the conditions (5a), the spin-0\(^+\) mass formula for \( q \neq 0 \) takes the form:

\[
m_0^2 = \frac{3(a - q b_6)(a + a_2 + q b_6)}{2a_2 b_6}.
\]

Now, a few comments are in order: (a) the nature of constraints remains the same as in Table 2, which implies the stability under linearization; (b) all if-constraints become first class, but only 6 of them remain independent, which leads to \( N^* = 0 \) (instability); (c) the massless nonlinear theory, defined by the condition \( a + a_2 + q b_6 = 0 \), is essentially stable under linearization.

### 5 Concluding remarks

— By investigating fully nonlinear constraint bifurcation effects, as well as the canonical stability under linearization, we were able to identify the set of dynamically acceptable values of parameters for the spin-0\(^+\) sector of PGT, as shown in Table 3.
— On the other hand, the spin-0\(^-\) sector is canonically unstable for any choice of parameters; for more details, see Ref. [11].
— Further analysis of higher spin modes is left for future studies.
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