P-Sasakian Manifold with Quarter-Symmetric Non-Metric Connection

Oğuzhan Bahadır

Department of Mathematics, Faculty of Arts and Sciences, K.S.U. Kahramanmaras, Turkey

Abstract The object of the present paper is to study on a P-Sasakian manifold with quarter symmetric non-metric connection. In this paper, we consider some properties of the curvature tensor, projective curvature tensor, concircular curvature tensor, conformal curvature tensor with respect to quarter symmetric non-metric connection in a P-Sasakian manifolds. Finally, we give an example.

Keywords Para-Sasakian Manifold, Quarter-symmetric Connections

AMS Classification: 53C15, 53C25, 53C40.

1 Introduction

In [3], Friedmann and Schouten introduced the idea of semi-symmetric connection on a differentiable manifold. Many Authors have studied the properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection ([5], [1], [2], [4], [6], [13], [15]). In [4], S.Golab introduced the idea of quarter-symmetric linear connections in a differential manifold. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $T$ is of the form

$$T(X, Y) = u(Y)\varphi X - u(X)\varphi Y,$$

for any vector fields $X, Y$ on a manifold, where $u$ is a 1–form and $\varphi$ is a tensor of type $(1, 1)$. If $\varphi = I$, then the quarter-symmetric connection is reduced to a semi-symmetric connection. Hence quarter-symmetric connection can be viewed as a generalization of semi-symmetric connection. The connection $\nabla$ is said to be a metric connection if there is a Riemannian metric $g$ in $M$ such that $\nabla g = 0$, otherwise it is non-metric. In [12], Sharfuddin and Hussain defined a semi-symmetric metric connection in an almost contact manifold, by setting

$$T(X, Y) = \eta(Y)X - \eta(X)Y.$$

Ajit Barman and Gopal Ghosh studied P-Sasakian manifolds admitting a semi-symmetric non-metric connection whose concircular curvature tensor satisfies certain curvature conditions. Moreover, some properties of a quarter-symmetric non-metric connection on P-sasakian manifolds are investigated in [11].
In the present paper, we will study P-Sasakian manifold with quarter symmetric non-metric connection. Section 2 is devoted to preliminaries. In section 3, we introduce quarter symmetric non-metric connection on a Para-Sasakian manifold. We calculate curvature tensor and Ricci tensor and scalar curvature of a P-Sasakian manifold with respect to quarter symmetric non-metric connection, respectively. Moreover we show that if a Para-Sasakian manifold with quarter symmetric non-metric connection is Ricci semi-symmetric, then the manifold is $\eta$–Einstein manifold with respect to quarter symmetric non-metric connection. In section 4, we find some results for concircular curvature tensor with respect to quarter symmetric non-metric connection. In section 5, it is shown that if a Para-Sasakian manifold is $\phi$–projectively flat with respect to quarter symmetric non-metric connection, then the manifold is an $\eta$–Einstein manifold with respect to quarter symmetric non-metric connection. In section 6, we have proved that if a Para-Sasakian manifold is conformally flat with respect to quarter symmetric non-metric connection, then the manifold is an Einstein manifold with respect to quarter symmetric non-metric connection. In section 7, we give an example which verify the results of Section 3, Section 4 and Section 5.

2 Preliminaries

A differentiable manifold of dimension $n$ is called an almost paracontact Riemannian structure [8, 9, 10], if it admit a $(1,1)$ tensor field $\phi$, a contravariant vector field $\xi$, a 1–form $\eta$ and Riemannian metric $g$ which satisfy

\begin{align*}
\eta(\xi) &= 1, \quad (3) \\
\phi\xi &= 0, \quad \eta(\phi X) = 0, \quad (4) \\
\phi^2(X) &= X - \eta(X)\xi, \quad (5) \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad (6) \\
g(X, \xi) &= \eta(X), \quad (7) \\
(\nabla_X \eta)Y &= g(X, \phi Y) \quad (8)
\end{align*}

for any vector field $X$ and $Y$, where $\nabla$ is Levi-Civita connection with respect to the Riemannian metric $g$. If we write $g(X, \phi Y) = \Phi(X, Y)$, then the tensor field $\phi$ is a symmetric $(0,2)$ tensor field. If $(\phi, \xi, \eta, g)$ satisfy the relations

\begin{align*}
\nabla_X \xi &= \phi X, \quad (9) \\
(\nabla_X \phi)(Y) &= -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (10) \\
d\eta &= 0 \quad (11)
\end{align*}

then the manifold is called para-Sasakian manifold (briefly, P-Sasakian).

Let $M$ be an $n$-dimensional P-Sasakian manifold. Then the following relations hold:

\begin{align*}
g(R(X, Y)Z, \xi) &= \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (12) \\
R(\xi, X)Y &= \eta(Y)X - g(X, Y)\xi, \quad (13) \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \quad (14) \\
S(X, \xi) &= -(n-1)\eta(X), \quad (15) \\
S(\phi X, \phi Y) &= S(X, Y) + (n-1)\eta(X)\eta(Y) \quad (16)
\end{align*}

for any vector fields $X$, $Y$ and $Z$, where $R$ and $S$ are the curvature and Ricci tensors of $M$, respectively [7, 8].

A P-Sasakian manifold $M$ is said to be $\eta$–Einstein if its Ricci tensor $S$ is of the form

\begin{align*}
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \quad (17)
\end{align*}

for any $X, Y \in \Gamma(TM)$, where $a, b$ are scalar functions such that $b \neq 0$. If $b = 0$ then $M$ is called Einstein manifold.
3 Quarter-Symmetric Non-Metric Connection

Let $M$ be an $n$-dimensional P-Sasakian manifold with Levi-Civita connection $\nabla$. If we set

$$\nabla_X Y = \nabla_X Y + \eta(Y)\phi X$$  \hspace{1cm} (18)

for any vector field $X$ and $Y$, then $\nabla$ is a linear connection on $M$. We know that the torsion tensor $T$ with respect to connection $\nabla$ is given

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

From (18) we get

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y.$$  \hspace{1cm} (19)

Furthermore (18) we have

$$(\nabla_X g)(Y, Z) = -\eta(Y)\Phi(X, Z) - \eta(Z)\Phi(X, Y)$$  \hspace{1cm} (20)

for any vector field $X$ and $Y$, which implies that $\nabla$ is a quarter symmetric non-metric connection on $M$. Also by using (4), and (20) we get

$$(\nabla_\xi g)(Y, Z) = 0,$$

which means that the metric $g$ is $\xi$-parallel with respect to quarter symmetric non-metric connection.

From (5),(9),(10), (4) and (18) we have the following proposition:

**Proposition 1.** Let $M$ be a P-Sasakian manifold. Then we have the following equations:

$$\nabla_X \xi = 2\phi X,$$  \hspace{1cm} (21)

$$(\nabla_X \phi)Y = -g(X, Y)\xi - 2\eta(Y)X + 3\eta(X)\eta(Y)\xi.$$  \hspace{1cm} (22)

The curvature tensor $\mathcal{R}$ of the quarter symmetric non-metric connection $\nabla$ on $M$ is defined by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$  \hspace{1cm} (23)

From (8), (18) and (23) we have

$$\mathcal{R}(X, Y)Z = R(X, Y)Z + \Phi(X, Z)\phi Y - \Phi(Y, Z)\phi X + \eta(Z)\{\eta(X)Y - \eta(Y)X\},$$  \hspace{1cm} (24)

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$  \hspace{1cm} (25)

is the curvature tensor with respect to the Levi-Civita connection $\nabla$. Using (24) and the first Bianchi identity we have the following proposition

**Proposition 2.** Let $M$ be an $n$-dimensional $P$-Sasakian manifold with quarter symmetric non-metric connection. Then the first Bianchi identity of the quarter-symmetric nonmetric connection $\nabla$ on $M$ is provided.
From (14) and (24) we have
\[
\mathcal{K}(X,Y,Z,U) = K(X,Y,Z,U) + \Phi(X,Z)\Phi(Y,U) - \Phi(Y,Z)\Phi(X,U) \\
+ \eta(Z)K(X,Y,\xi,U)
\]
(26)
where \(K\) and \(\mathcal{K}\) are given by
\[
K(X,Y,Z,U) = g(R(X,Y)Z,U)
\]
\[
\mathcal{K}(X,Y,Z,U) = g(\mathcal{R}(X,Y)Z,U).
\]

**Theorem 3.** Let \(M\) be an \(n\)-dimensional P–Sasakian manifold with quarter symmetric non-metric connection. Then we have the following equations:
\[
\mathcal{K}(X,Y,Z,U) + \mathcal{K}(Y,X,Z,U) = 0,
\]
(27)
\[
\mathcal{K}(X,Y,Z,U) + \mathcal{K}(X,Y,U,Z) = \eta(Z)K(X,Y,\xi,U) \\
+ \eta(U)K(X,Y,\xi,Z),
\]
(28)
\[
\mathcal{K}(X,Y,Z,U) - \mathcal{K}(Z,U,X,Y) = \eta(X)\eta(U)g(Y,Z) \\
- \eta(Y)\eta(Z)g(X,U),
\]
(29)
for any \(X,Y,Z,U \in \Gamma(TM)\).

**Proof.** From (26) we obtain (27). By using (14) and (26) we get (28). Again from (26) we have (29).

**Proposition 4.** Let \(M\) be an \(n\)-dimensional P–Sasakian manifold with quarter symmetric non-metric connection. Then we have the following equations:
\[
g(\mathcal{R}(X,Y)Z,\xi) = \eta(\mathcal{R}(X,Y)Z) \\
= g(\mathcal{R}(X,Y)Z,\xi),
\]
(30)
\[
\mathcal{R}(X,Y)\xi = 2\mathcal{R}(X,Y)\xi,
\]
(31)
\[
\mathcal{R}(\xi,X)Y = -g(\phi X,\phi Y)\xi
\]
(32)
for any \(X,Y,Z \in \Gamma(TM)\).

**Proof.** From (7), (4) and (24) we obtain (30). By using (3), (14) and (24) we get (31). Again from (3), (13) and (24) we have (32).

The Ricci tensor \(\mathcal{S}\) of an P-Sasakian manifold \(M\) with respect to quarter symmetric non-metric connection \(\nabla\) is given by
\[
\mathcal{S}(X,Y) = \sum_{i=1}^{n} g(\mathcal{R}(e_i,X)Y,e_i).
\]
The scalar curvature of \(M\) with respect to quarter symmetric non-metric connection \(\nabla\) is defined by
\[
\mathfrak{r} = \sum_{i=1}^{n} \mathcal{S}(e_i,e_i)
\]
where \(X,Y \in \Gamma(TM)\), \(\{e_1,e_2,...,e_n\}\) is an orthonormal frame.

From (26) we have
\[
\mathcal{S}(X,Y) = \sum_{i=1}^{n} \left[ g(\mathcal{R}(e_i,X)Y,e_i) + g(\phi X,\phi Y) - \Phi(X,Y)g(\phi e_i,e_i) \right] \\
+ \eta(Y)\eta(X) - n\eta(X)\eta(Y).
\]
(33)
Theorem 5. Let $M$ be an $n$– dimensional $P$–Sasakian manifold. Then we have the following equations:

\[
\mathcal{S}(X,Y) = S(X,Y) + g(X,Y) - \beta \Phi(X,Y) - n\eta(X)\eta(Y),
\]

(34)

\[
\tau = r - \beta^2,
\]

(35)

where $r$ is scalar curvature of Levi-Civita connection and $\beta = \text{trace}(\Phi)$.

Proof. Since the Ricci tensor of the Levi-Civita connection $\nabla$ is given by

\[
S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X), e_i),
\]

then by using (6) and (33) we get (34). Moreover from (6) and (34) we have (35). \qed

From (34) we have the following corollary

Corollary 6. Let $M$ be an $n$– dimensional $P$–Sasakian manifold. Then the Ricci tensor $\mathcal{S}$ of quarter symmetric non-metric connection $\nabla$ is symmetric.

By using (3),(6), (4), (15), (16), (34), (36) and (37) we obtain

\[
\mathcal{S}(Y,U) = (1 - n)\{g(Y,U) + \eta(Y)\eta(U)\},
\]

(41)

This equation tell us $M$ is an $\eta$ Einstein manifold with respect to quarter symmetric non-metric connection. \qed

4 Concircular Curvature Tensor

Let $M$ be an $n$– dimensional $P$–Sasakian manifold. The concircular curvature tensor of $M$ with respect to quarter symmetric non-metric connection $\nabla$ is defined by

\[
C^*(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}
\]

(42)
By using (24), (35) and (42) we get

\[ C^*(X,Y)Z = R(X,Y)Z + \Phi(X, Z)\phi Y - \Phi(Y, Z)\phi X + \eta(Z)\{\eta(X)Y - \eta(Y)X\} \]

\[ + \frac{\beta^2 - r}{n(n - 1)}(g(Y, Z)X - g(X, Z)Y) \]

(43)

and

\[ C^*(X,Y)Z = C^*(X,Y)Z + \Phi(X, Z)\phi Y - \Phi(Y, Z)\phi X + \eta(Z)\{\eta(X)Y - \eta(Y)X\} \]

\[ + \frac{\beta^2}{n(n - 1)}(g(Y, Z)X - g(X, Z)Y), \]

(44)

where \( C^* \) is concircular curvature tensor with respect to Levi-Civita connection. If we consider \( C^* = C^* \), substituting \( Y \) by \( \xi \) in (44), from (3) and (7) we have

\[ g(X, Z) = \eta(X)\eta(Z). \]

(45)

Using equations (5), (6) and (45) we obtain

\[ \Phi(X, Z) = 0. \]

(46)

From (34), (45) and (46) we have the following theorem

**Theorem 8.** In a P-Sasakian manifold, if concircular curvature tensor is invariant under quarter symmetric non-metric connection then we have

\[ \mathcal{S}(X,Y) = S(X,Y) + (1 - n)\eta(X)\eta(Y) \]

for any \( X, Y \in \Gamma(TM) \).

**Definition 9.** Let \( M \) be an \( n \)-dimensional P-Sasakian manifold. Then \( M \) is \( \xi \)-concirculary flat with respect to quarter symmetric non-metric connection if \( \mathcal{C}^*(X,Y)\xi = 0 \).

Taking \( Z = \xi \) in (43) and using (3), (5), (7), (4) and (15) we have

\[ \mathcal{C}^*(X,Y)\xi = [2 + \frac{r - \beta^2}{n(n - 1)}\}\{\eta(X)Y - \eta(Y)X\} \]

(47)

From (47) we have the following theorem

**Theorem 10.** Let \( M \) be an \( n \)-dimensional P-Sasakian manifold. Then \( M \) is \( \xi \)-concirculary flat with respect to quarter symmetric non-metric connection if and only if the scalar curvature tensor with respect to Levi-Civita connection \( M \) is equal to \( \beta^2 - 2n(n - 1) \).

**Definition 11.** Let \( M \) be an \( n \)-dimensional P-Sasakian manifold. Then \( M \) is \( \phi \)-concirculary flat with respect to quarter symmetric non-metric connection if and only if \( g(\mathcal{C}^*(\phi X, \phi Y)\phi Z, \phi W) = 0 \).

Taking the inner product of (42) with \( W \), we get

\[ \mathcal{C}'(X,Y,Z,W) = \mathcal{K}(X,Y,Z,W) \]

\[ - \frac{r}{n(n - 1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \]

(48)

where \( \mathcal{C}' \) is the concircular curvature tensor of type \((0, 4)\) with respect to quarter symmetric non-metric connection and \( \mathcal{C}'(X,Y,Z,W) = g(\mathcal{C}^*(X,Y)Z, W) \).
From (48) we have
\[
\nabla' \phi X, \phi Y, \phi Z, \phi W) = \nabla(\phi X, \phi Y, \phi Z, \phi W) \\
+ \frac{r}{n(n-1)} \left\{ \phi(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W) \right\}.
\]
(49)

Using (49), \( \phi \) - concircular flatness implies
\[
\nabla(\phi X, \phi Y, \phi Z, \phi W) = \frac{r}{n(n-1)} \left\{ \phi(\phi Y, \phi Z)g(\phi X, \phi W) \\
- g(\phi X, \phi Z)g(\phi Y, \phi W) \right\}.
\]
(50)

Let \( \{e_1, e_2, \ldots, e_{n-1}, \xi\} \) be a local orthogonal basis of the vector fields in \( M \) and using the fact that \( \{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\} \) is also a local orthogonal basis, putting \( X = W = e_i \) and summing up with respect to \( i = 1, 2, \ldots, n-1 \) we obtain
\[
\sum_{i=1}^{n-1} \nabla(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{r}{n(n-1)} \sum_{i=1}^{n-1} \left\{ \phi(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\
+ g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \right\}.
\]
(51)

The equation (51) turns into
\[
\nabla(\phi Y, \phi Z) = \frac{r(n-2)}{n(n-1)} g(\phi Y, \phi Z).
\]
(52)

From (3), (5), (7), (4), (35) and (38) we obtain
\[
\nabla(\phi Y, Z) = -g(Y, Z) + \beta \Phi(Y, Z) - (n-2)\eta(Y)\eta(Z) \\
+ \frac{(r-\beta^2)(n-2)}{n(n-1)} g(\phi Y, \phi Z).
\]
(53)

Then contracting the equation (53) over \( Y \) and \( Z \) and from (3), (5), (7), (4) we get
\[
r = \beta^2 - n(n-1).
\]
(54)

Then we have the following theorem:

**Theorem 12.** Let \( M \) be an \( n \)-dimensional \( P \)-Sasakian manifold. If \( M \) is \( \phi \) - concircular flat with respect to quarter symmetric non-metric connection, then the scalar curvature tensor with respect to Levi-Civita connection \( M \) is equal to \( \beta^2 - n(n-1) \).

## 5 Projective Curvature Tensor

Let \( M \) be an \( n \)-dimensional \( P \)-Sasakian manifold. Projective curvature tensor \( \nabla' \) of type \((1,3)\) of \( M \) with respect to quarter symmetric non-metric connection \( \nabla \) is defined by
\[
\nabla'(X, Y)Z = \nabla(X, Y)Z - \frac{1}{n-1} \{ \nabla(\nabla Y, Z)X - \nabla(\nabla X, Z)Y \}.
\]
(55)

From (24) and (34), using (55) we get
\[
\nabla'(X, Y)Z = P(X, Y)Z - \frac{1}{n-1} \left\{ g(Y, Z)X - g(X, Z)Y + \text{trace}(\Phi)\Phi(X, Z)Y \\
- \text{trace}(\Phi)\Phi(Y, Z)X - n\eta(Y)X + n\eta(X)\eta(Z)Y \right\} \\
+ \Phi(X, Z)\phi Y \\
- \Phi(Y, Z)\phi X + \eta(Z)\{ \eta(Y)X - \eta(Y)Y \}.
\]
(56)

where
\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \{ S(Y, Z)X - S(X, Z)Y \}
\]
(57)
is the projective curvature tensor of \( M \) with respect to Levi-Civita connection \( \nabla \).
Definition 13. Let $M$ be an $n-$ dimensional $P-$Sasakian manifold. Then $M$ is said to be $\xi-$ projectively flat with respect to quarter symmetric non-metric connection $\nabla$ if $\Gamma(X,Y)\xi = 0$ on $M$.

Using (31), (36) and (55) we have
\[ \Gamma(X,Y)\xi = 0. \] (58)

From (58) we get the following theorem:

Theorem 14. Let $M$ be an $n-$ dimensional $P-$Sasakian manifold. Then $M$ is $\xi-$ projectively flat with respect to quarter symmetric non-metric connection.

Definition 15. Let $M$ be an $n-$ dimensional $P-$Sasakian manifold. Then $M$ is said to be $\phi-$ projectively flat with respect to quarter symmetric non-metric connection if $g(\Gamma(\phi X, \phi Y)\phi Z, \phi W) = 0$ on $M$.

From (55)
\[ \Gamma(\phi X, \phi Y)\phi Z = \Gamma(\phi X, \phi Y)\phi Z - \frac{1}{n-1}\{S(\phi Y, \phi Z)\phi X - S(\phi X, \phi Z)\phi Y\}. \] (59)

Using (38) and (59), $\phi-$ projectively flatness implies
\[ \mathcal{K}(\phi X, \phi Y, \phi Z, \phi W) = \frac{1}{n-1}\{[S(Y, Z) + g(Y, Z) - \beta\Phi(Y, Z) + (n-2)\eta(Y)\eta(Z)]g(\phi X, \phi U) \]
\[ + \ [S(X, Z) + g(X, Z) - \beta\Phi(X, Z) + (n-2)\eta(X)\eta(Z)]g(\phi Y, \phi U)\}. \] (60)

Let \{e_1, e_2, ..., e_{n-1}, \xi\} be a local orthogonal basis of the vector fields in $M$ and using the fact that \{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\} is also a local orthogonal basis, putting $X = W = e_i$ and summing up with respect to $i = 1, 2, ..., n-1$ we obtain
\[ \sum_{i=1}^{n-1} \mathcal{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(Y, Z) + g(Y, Z) - \beta\Phi(Y, Z) + (n-2)\eta(Y)\eta(Z) \]
\[ - \frac{1}{n-1}\{\sum_{i=1}^{n-1} S(e_i, Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z) - \beta\Phi(Y, Z)\}. \] (61)

We know that
\[ \sum_{i=1}^{n-1} S(e_i, Z)g(\phi Y, \phi e_i) = S(\phi Y, \phi Z). \] (62)

Thus from (34), (38), (62), the equation (61) becomes
\[ \sum_{i=1}^{n-1} \mathcal{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{n-2}{n-1}S(\phi Y, \phi Z). \] (63)

Moreover we have
\[ \sum_{i=1}^{n-1} \mathcal{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \mathcal{S}(\phi Y, \phi Z) - g(\mathcal{R}(\xi, Y)\phi Z, \xi). \] (64)

Using (12) and (64) we get
\[ \sum_{i=1}^{n-1} \mathcal{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \mathcal{S}(\phi Y, \phi Z) - g(\phi Y, \phi Z). \] (65)

Therefore by using (37), (63) and (65) we obtain
\[ \mathcal{S}(Y, Z) = (n-1)\{g(Y, Z) - 3\eta(Y)\eta(Z)\}. \] (66)

From (66), we have the following theorem

Theorem 16. If a $P-$Sasakian manifold is $\phi-$ projectively flat with respect to quarter symmetric non-metric connection, then the manifold is an $\eta-$ Einstein manifold with respect to quarter symmetric non-metric connection.
6 Conformal Curvature tensor

Let $M$ be an $n-$ dimensional P-Sasakian manifold. The conformal curvature tensor of $M$ with respect to quarter symmetric non-metric connection $\nabla$ is given by

$$\bar{C}(X, Y, Z, U) = \bar{K}(X, Y, Z, U)$$

\[= \frac{1}{n-2} \{g(Y, Z)S(X, U) - g(X, Z)S(Y, U)\}
+ S(Y, Z)g(X, U) - S(X, Z)g(Y, U)\]

\[= \frac{r}{(n-1)(n-2)} \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}. \tag{67}\]

Suppose that P-Sasakian manifold is conformally flat with respect to quarter symmetric non-metric connection, that is $\bar{C}(X, Y, Z, U) = 0$. By using (67) we get

$$\bar{K}(X, Y, Z, U) = \frac{1}{n-2} \{g(Y, Z)S(X, U) - g(X, Z)S(Y, U)\}$$

\[+ S(Y, Z)g(X, U) - S(X, Z)g(Y, U)\]

\[- \frac{r}{(n-1)(n-2)} \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}. \tag{68}\]

Putting $Y = Z = \xi$ in (68) and using (31), (35) and (36) we obtain

$$S(X, U) = \{\frac{r - \beta^2}{n-1} + 6 - 2n\}g(X, U) + \{\frac{\beta^2 - r}{n-1} - 4\}\eta(X)\eta(U). \tag{69}\]

from (69) we have the following theorem

**Theorem 17.** If a P-Sasakian manifold is conformally flat with respect to quarter symmetric non-metric connection, then the manifold is Einstein manifold with respect to quarter symmetric non-metric connection.

7 Example

We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let $E_1, E_2, E_3$ be a linearly independent global frame on $M$ given by

$$E_1 = -x \frac{\partial}{\partial x}, \quad E_2 = x \frac{\partial}{\partial y}, \quad E_3 = x \frac{\partial}{\partial z}. \tag{70}$$

Let $g$ be the Riemannian metric defined by

$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_1)$, for any $U \in TM$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = 0, \phi E_2 = E_2$ and $\phi E_3 = E_3$. Then, using the linearity of $\phi$ and $g$ we have $\eta(E_1) = 1, \quad \phi^2 U = U - \eta(U)E_1$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_1 = \xi$, $(\phi, \xi, \eta, g)$ defines a paracatonic structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the Riemannian metric $g$. Then, we have

$$[E_1, E_2] = -E_2, \quad [E_1, E_3] = -E_3, \quad [E_2, E_3] = 0, \tag{71}$$

Using Koszul formula for the Riemannian metric $g$, we can easily calculate

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_3 = 0,$$

$$\nabla_{E_2} E_1 = E_2, \quad \nabla_{E_2} E_2 = -E_1, \quad \nabla_{E_2} E_3 = 0,$$

$$\nabla_{E_3} E_1 = E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_1. \tag{72}$$
From the above relations, it can be easily seen that
\[ \nabla_X \xi = \phi X, \quad (\nabla_X \phi) Y = -g(X, Y) \xi - \eta(Y) X + 2\eta(X) \eta(Y) \xi, \]
for all \( E_1 = \xi \) Thus the manifold \( M \) is an \( \xi \)-Sašakian with the structure \( (\phi, \xi, \eta, g) \). Using (18) in the above equations, we get
\[
\begin{align*}
\nabla_{E_1} E_1 &= 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0, \\
\nabla_{E_2} E_1 &= 2E_2, \quad \nabla_{E_2} E_2 = -E_1, \quad \nabla_{E_2} E_3 = 0, \\
\nabla_{E_3} E_1 &= 2E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_1.
\end{align*}
\] (73)

Using the above relations, we can calculate the non-vanishing components of the curvature tensor as follows:
\[
\begin{align*}
R(E_1, E_2) E_1 &= E_2, \quad R(E_1, E_2) E_2 = -E_1, \quad R(E_1, E_3) E_1 = E_3 \\
R(E_1, E_3) E_3 &= -E_1, \quad R(E_2, E_3) E_2 = E_3, \quad R(E_3, E_3) E_3 = -E_2.
\end{align*}
\] (74)

and
\[
\begin{align*}
\overline{R}(E_1, E_2) E_1 &= 2E_2, \quad \overline{R}(E_1, E_2) E_2 = -E_1, \quad \overline{R}(E_1, E_3) E_1 = 2E_3 \\
\overline{R}(E_1, E_3) E_3 &= -E_1, \quad \overline{R}(E_2, E_3) E_2 = 2E_3, \quad \overline{R}(E_2, E_3) E_3 = -2E_2.
\end{align*}
\] (75)

From the equations (74) and (75), we can easily calculate the non-vanishing components of the Ricci tensor as follows:
\[
\begin{align*}
S(E_1, E_1) &= -2, \quad S(E_2, E_2) = -2, \quad S(E_3, E_3) = -2
\end{align*}
\] (76)

and
\[
\begin{align*}
\overline{S}(E_1, E_1) &= -4, \quad \overline{S}(E_2, E_2) = -3, \quad \overline{S}(E_3, E_3) = -3.
\end{align*}
\] (77)

Then the scalar curvature with respect to the Levi-Civita connection and quarter symmetric non-metric connection are \( r = -6 \) and \( \tau = -10 \). This expressions are verifying the some results of the section 3.

Let \( X, Y, Z \) and \( U \) be any four vector fields given by
\[
X = A_1 E_1 + A_2 E_2 + A_3 E_3, \quad Y = B_1 E_1 + B_2 E_2 + B_3 E_3, \quad Z = C_1 E_1 + C_2 E_2 + C_3 E_3 \quad \text{and} \quad U = D_1 E_1 + D_2 E_2 + D_3 E_3,
\]
where \( A_i, B_i, C_i, D_i \), for all \( i = 1, 2, 3 \), are the non-zero real numbers.

Using above relations and from (42), (49), (59) we get
\[
\begin{align*}
\nabla'(X, Y) \xi &= \frac{3}{4} (A_1 B_2 - B_1 A_2) E_2 + (A_1 B_3 - B_1 A_3) E_3, \\
\nabla'(\phi X, \phi Y, \phi Z, \phi W) &= -\frac{3}{4} (B_3 A_2 - A_3 B_2)(C_3 D_2 - C_2 D_3),
\end{align*}
\]

and
\[
\begin{align*}
g(\overline{P}(\phi X, \phi Y) \phi Z, \phi W) &= 2(A_3 B_2 - B_3 A_2)(C_3 D_2 - C_2 D_3).
\end{align*}
\]

Then we see that \( P- \) Sašakian manifold \( M \) will be \( \xi- \) concircular flat with respect to quarter symmetric non-metric connection if \( \frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3} \). We also see the that the manifold is \( \phi- \) concircular flat with respect to quarter symmetric non-metric connection if \( \frac{A_1}{B_1} = \frac{A_2}{B_2} \) or \( \frac{C_1}{D_1} = \frac{C_2}{D_2} \). Moreover we see that \( P- \) Sašakian manifold \( M \) is \( \phi- \) projectively flat with respect to quarter symmetric non-metric connection if \( \frac{A_1}{B_1} = \frac{A_2}{B_2} \) or \( \frac{C_1}{D_1} = \frac{C_2}{D_2} \). This is verifying some results of section 4 and section 5.

**Acknowledgements**

The authors are thankful to the referee for his/her valuable comments and suggestions towards the improvement of the paper.
REFERENCES

[1] N. S. Agashe and M. R. Chafle, A semi symmetric non-metric connection in a Riemannian manifold, Indian J. Pure Appl. Math. 23 (1992), 399-409.

[2] U. C. De and D. Kamilya, Hypersurfaces of Riemannian manifold with semi-symmetric non-metric connection, J. Indian Inst. Sci. 75 (1995), 707-710.

[3] A. Friedmann, J. A. Schouten, Über die geometrie der halbsymmetrischen Übertragung, Math. Zeitschr. 21 (1924), 211-223.

[4] S. Golab, On semi-symmetric and quarter-symmetric linear connections, Tensor 29 (1975), 249-254.

[5] H. A. Hayden, Subspaces of a space with torsion, Proc. London Math. Soc. 34 (1932), 27-50.

[6] Y. Liang, On semi-symmetric recurrent-metric connection, Tensor 55 (1994), 107-112.

[7] Adati, T. and Matsumoto, K, On conformally recurrent and conformally symmetric P-Sasakian manifolds, TRU Math., 13, 25-32, 1977.

[8] I. Sato, On a structure similar to the almost contact structure, Tensor new series, vol. 30, no. 3, pp. 219-224, 1976.

[9] T. Adati and T. Miyazawa, On P-Sasakian manifolds satisfying certain conditions, Tensor new series, vol. 33, no. 2, pp. 173-178, 1979.

[10] S. Kaneyuki, M. Konzai, Paracomplex structure and affine symmetric spaces, Tokyo J. Math., 8 (1985), 301-318.

[11] Abdul Kalam Mondal and U.C. De, Quarter-symmetric Nonmetric Connection on P-sasakian manifolds, ISRN Geometry, (2012), 14 pages

[12] A. Sharfuddin, S. I. Husain, Semi-symmetric metric connexions in almost contact manifolds, Tensor. 30 (1976), 133-139.

[13] B. G. Schmidt, Conditions on a connection to be a metric connection, Commun. Math. Phys. 29 (1973), 55-59.

[14] J. A. Schouten, Ricci calculus, Springer, 1954.

[15] M.M Tripathi, A new connection in a Riemannian manifold, International Journal of Geometry. 1 (2006), 15-24.

[16] Gyanvendra Pratap Singh, Sunil Kumar Srivastava. On Kenmotsu Manifold With Quarter Symmetric Non Metric-Connection, International Journal of Pure and Applied Mathematical Sciences. ISSN 0972-9828, Volume 9, Number 1 (2016), pp. 67-74.

[17] Ajit Barman and Gopal Ghosh, Concircular Curvature Tensor of a Semi-symmetric Non-metric Connection on P-Sasakian Manifolds, Analele Universitatii de Vest Timisoara Seria Matematica-informatica, LIV, 2. (2016), 47-58