REDUCTIONS OF GALOIS REPRESENTATIONS OF SLOPE 1

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Abstract. We compute the reductions of irreducible crystalline two-dimensional representations of $G_{\mathbb{Q}_p}$ of slope 1, for primes $p > 3$. We give a complete answer for all weights $k \geq 2$, except for weights $k \equiv 4 \mod (p - 1)$ where we provide partial results. The proof uses the mod $p$ Local Langlands Correspondence.

1. Introduction

Let $p$ be an odd prime. This paper is concerned with computing the reductions of certain crystalline two-dimensional representations of the local Galois group $G_{\mathbb{Q}_p}$. The first computations of the reductions in positive slope, after [E92], were carried out by Breuil in [Bre03], for small weights. The reductions are also known for large slopes by [BLZ04]; see also [YY] for results using similar techniques. In the other direction, the reductions are known for small fractional slopes, namely, for slopes in $(0,1)$ by [BG09], [BG13], and for slopes in $(1,2)$ by [BG15], under a mild hypothesis. Here we compute the reduction in the important missing case of integral slope 1.

Let $E$ be a finite extension field of $\mathbb{Q}_p$ and let $v$ be the valuation of $\overline{\mathbb{Q}_p}$ normalized so that $v(p) = 1$. Let $a_p \in E$ with slope $v(a_p) > 0$ and let $k \geq 2$. Let $V_{k,a_p}$ be the irreducible crystalline representation of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $(0,k-1)$ such that $D_{\text{cris}}(V_{k,a_p}^*) = D_{k,a_p}$, where $D_{k,a_p} = E e_1 \oplus E e_2$ is the filtered $\varphi$-module as defined in [Ber11, §2.3]. The semisimplification $\overline{V}_{k,a_p}^{ss}$ of the reduction $\overline{V}_{k,a_p}$ with respect to a lattice in $V_{k,a_p}$ is independent of the choice of the lattice.

Let $\omega = \omega_1$ and $\omega_2$ denote the fundamental characters of level 1 and 2 respectively, and let $\text{ind}(\omega_a^{+1})$ denote the representation of $G_{\mathbb{Q}_p}$ obtained by inducing the character $\omega_a^2$, for $a \in \mathbb{Z}$, from $G_{\mathbb{Q}_2}$ to $G_{\mathbb{Q}_p}$. Let $\mu_x$ be the unramified character of $G_{\mathbb{Q}_p}$ taking (geometric) Frobenius at $p$ to $x \in \overline{\mathbb{F}_p}^\times$.

Set $r = k - 2$ and suppose that $r \equiv b \mod (p - 1)$ with $2 \leq b \leq p$. When $b > 2$, we prove:

Theorem 1.1. Let $p > 3$, $k \geq 2p + 2$, and $v(a_p) = 1$. Then $\overline{V}_{k,a_p}^{ss}$ is as follows:

$$3 \leq b \leq p - 1 \implies \begin{cases} \mu_\lambda \cdot \omega^b \oplus \mu_{\lambda-1} \cdot \omega, & \text{if } p \nmid r - b, \text{ with } \lambda = \frac{b}{b-r} \cdot \frac{a_p}{p} \in \overline{\mathbb{F}_p}^\times \\ \text{ind}(\omega_2^{b+1}), & \text{if } p \mid r - b \end{cases}$$

$$b = p \implies \begin{cases} \text{ind}(\omega_2^2), & \text{if } p \nmid r - b \\ \mu_\lambda \cdot \omega \oplus \mu_{\lambda-1} \cdot \omega, & \text{if } p \mid r - b, \text{ with } \lambda = \frac{1}{\lambda} = \frac{a_p}{p} - \frac{r-p}{\lambda} \in \overline{\mathbb{F}_p}^\times \end{cases}$$
Moreover, if \( b = p - 1 \), \( p \nmid r - b \) and \( \lambda = \frac{a_p}{p(r + 1)} = \pm 1 \), then for every lattice in \( V_{k,a_p} \) whose reduction \( \bar{V}_{k,a_p} \) is a non-split extension of \( \mu_{\pm 1} \) by \( \mu_{\pm 1}\omega \), the extension is “peu ramifiée”.

The proof of Theorem 1.1 uses the \( p \)-adic Local Langlands Correspondence and its compatibility with reduction modulo \( p \) [Ber10]. We compute the reduction of the standard lattice on the automorphic side and then applies the inverse of the mod \( p \) Local Langlands Correspondence defined in [Bre03]. When \( b > 2 \), the question whether the reduction is “peu” or “très ramifiée” as in [Ser87] only arises under the hypotheses on \( b \) and \( \lambda \) in the last statement of the theorem. We show that the reduction is “peu ramifiée” by studying instead an explicit, non-standard lattice on the automorphic side, and by invoking results from [Col10].

When \( b = 2 \), we obtain the following partial results:

**Theorem 1.2.** Let \( p > 3 \) and \( b = 2 \). Suppose that \( k \geq 2p + 2 \) and \( v(a_p) = 1 \).

1. If \( p \nmid r - b \), then \( \bar{V}_{k,a_p}^{ss} \cong \mu_{\lambda}.\omega^2 \oplus \mu_{\lambda-1} \cdot \omega \), if \( p \mid (\frac{r}{2}) \), where \( \lambda = \frac{2(a_p^2 - (\frac{r}{2})p^2)}{(2 - r)p^2} \in \mathbb{F}_p^* \)

2. If \( p \mid r - b \) and \( v(a_p^2 - (\frac{r}{2})p^2) = 2 \), then \( \bar{V}_{k,a_p} \cong \text{ind}(\omega_2^{p+2}) \) or \( \text{ind}(\omega_2^{2p+2}) \).

The theorems above match with all known results for weights \( 2 < k \leq 2p + 1 \) summarized in [Ber11], and could have therefore been stated for \( k > 2 \). It is also interesting to compare our results with work of Serre and Swinnerton-Dyer from the seventies which studied the reductions of \( p \)-adic Galois representations attached to modular forms of level 1 with integral Fourier expansion, in the special case that \( p \) exactly divides the \( p \)-th Fourier coefficient \( a_p \in \mathbb{Z} \) of the form. For instance, for Ramanujan’s Delta function \( \Delta \) of weight \( k = 12 \) and the prime \( p = 5 \), one has \( p \mid a_p = \tau(p) = 4,830 \) and we recover their result \((p\Delta|G_{\mathbb{Q}_p})^{ss} \cong \omega^2 \oplus \omega \) from part (1) of Theorem 1.2 with \( \lambda = 1 \).

It is a classical fact due to Deligne that the reduction \( \bar{V}_{k,a_p}^{ss} \cong \mu_{a_p} \cdot \omega^{b+1} \oplus \mu_{a_p}^{-1} \) is reducible in the ordinary case, that is, when the slope \( v(a_p) = 0 \). The results of this paper show that the reduction is generically reducible when the slope is 1, e.g., when \( 3 \leq b < p - 1 \) and \( p \nmid r - b \). This behaviour for integer slopes 0 and 1 contrasts heavily with the case of fractional slopes less than 2, where \( \bar{V}_{k,a_p} \) is known to be generically irreducible [BG09], [BG15]. However, unlike the ordinary case, the reduction can be irreducible in slope 1.

While preparing this paper, we learned about Arsovski’s recent work for slope 1, when \( b \neq 3 \) or \( p \), giving different possibilities for the reduction [Ars15, Thm. 4]. In this paper, we give a complete solution to the problem, for all congruence classes other than \( b = 2 \), as above. The class \( b = 2 \) is special, in the sense that there are additional complications in this case, and we hope to return to it shortly.

2. Basics

In this section, we recall some notation and well-known facts. Further details can be found in [Bre03] and [BG15].
2.1. Hecke operator $T$. Let $G = \text{GL}_2(\mathbb{Q}_p)$, $K = \text{GL}_2(\mathbb{Z}_p)$ be the standard maximal compact subgroup of $G$ and $Z = \mathbb{Q}_p^*$ be the center of $G$. Let $R$ be a $\mathbb{Z}_p$-algebra and let $V = \text{Sym}^r R^2 \otimes D^*$ be the usual symmetric power representation of $KZ$ twisted by a power of the determinant character $D$, modeled on homogeneous polynomials of degree $r$ in the variables $X, Y$ over $R$. For $g \in G$, $v \in V$, let $[g, v] \in \text{ind}_{KZ}^G V$ be the function with support in $KZg^{-1}$ given by

$$g' \mapsto \begin{cases} g'g \cdot v, & \text{if } g' \in KZg^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

Any function in $\text{ind}_{KZ}^G V$ is compactly supported mod $KZ$ and is a finite linear combination of functions of the form $[g, v]$, for $g \in G$ and $v \in V$. The Hecke operator $T$ is defined by its action on these elementary functions via

$$T([g, v]) = \sum_{\lambda \in \mathbb{F}_p} \left[ g\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), v(X, -[\lambda]X + pY) \right] + \left[ g\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), v(pX, Y) \right],$$

where $[\lambda]$ denotes the Teichmüller representative of $\lambda \in \mathbb{F}_p$. For $m = 0$, set $I_0 = \{0\}$, and for $m > 0$, let $I_m = \{[\lambda_0] + [\lambda_1]p + \cdots + [\lambda_{m-1}]p^{m-1} : \lambda_i \in \mathbb{F}_p\} \subset \mathbb{Z}_p$, where the square brackets denote Teichmüller representatives. For $m \geq 1$, there is a truncation map $[\ ]_{m-1} : I_m \to I_{m-1}$ given by taking the first $m - 1$ terms in the $p$-adic expansion above; for $m = 1$, $[\ ]_{m-1}$ is the 0-map. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. For $m \geq 0$ and $\lambda \in I_m$, let

$$g_{m, \lambda}^0 = \left(\begin{smallmatrix} p^n \lambda \\ 0 \\ 1 \end{smallmatrix}\right) \quad \text{and} \quad g_{m, \lambda}^1 = \left(1, p \lambda, p^{m+1}\right),$$

noting that $g_{0,0}^0 = \text{Id}$ is the identity matrix and $g_{0,0}^1 = \alpha$ in $G$. Recall the decomposition

$$G = \bigotimes_{m \geq 0, \lambda \in I_m, i \in \{0, 1\}} KZ(g_{m, \lambda}^i)^{-1}.$$ 

Thus a general element in $\text{ind}_{KZ}^G V$ is a finite sum of functions of the form $[g, v]$, with $g = g_{m, \lambda}^0$ or $g_{m, \lambda}^1$, for some $\lambda \in I_m$ and $v \in V$. For a $\mathbb{Z}_p$-algebra $R$, let $v = \sum_{i=0}^r c_i X^{r-i}Y^i \in V = \text{Sym}^r R^2 \otimes D^*$. Expanding the formula (2.1) for the Hecke operator $T$ one may write $T = T^+ + T^-$, where

$$T^+([g_{a, \mu}, v]) = \sum_{\lambda \in I_1} \left[ g_{a+1, \mu+p^n\lambda}^0 \sum_{j=0}^r \left(p^{r-i} \sum_{i=j}^r c_i X^{r-j}Y^j \right) \right],$$

$$T^+([g_{a, \mu}, v]) = \left[ g_{n-1, [\mu]_{n-1}}^0 \sum_{j=0}^r \left(p^{r-i} \sum_{i=j}^r c_i X^{r-j}Y^j \right) \right] (n > 0),$$

$$T^-([g_{a, \mu}, v]) = \left[ \alpha \sum_{j=0}^r p^{r-j} c_j X^{r-j}Y^j \right] (n = 0).$$

These explicit formulas for $T^+$ and $T^-$ will be used to compute $(T - a_p)f$, for $f \in \text{ind}_{KZ}^G \text{Sym}^r \mathbb{Q}_p^2$. 

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2.2. The mod \( p \) Local Langlands Correspondence. For \( 0 \leq r \leq p-1 \), \( \lambda \in \bar{\mathbb{F}}_p \) and \( \eta : \mathbb{Q}_p^* \to \bar{\mathbb{F}}_p^* \) a smooth character, let
\[
\pi(r, \lambda, \eta) := \text{ind}_{G}^G KZ V_r \otimes (\eta \circ \det)
\]
be the smooth admissible representation of \( G \), where \( \text{ind}_{KZ}^G \) is compact induction, and \( T \) is the Hecke operator. With this notation, Breuil’s semisimple mod \( p \) Local Langlands Correspondence [Bre03, Def. 1.1] is given by:
- \( \lambda = 0 \): \( \text{ind}(\omega r + 1) \otimes \eta \xymatrix{{\text{LL}}\ar@{|->}[rr] & (r,0,\eta)\xymatrix& \pi(r,0,\eta) } \)
- \( \lambda \neq 0 \): \( (\mu \lambda \omega r + 1 \oplus \mu \lambda^{-1}) \otimes \eta \xymatrix{{\text{LL}}\ar@{|->}[rr] & (r,\lambda,\eta)\xymatrix& \pi(r,\lambda,\eta) } \)
\[
\llbracket p - 3 - r \equiv p - 3 \mod (p - 1) \rrbracket \]
where \( \{0, 1, \ldots, p - 2\} \ni \llbracket p - 3 - r \rrbracket \equiv p - 3 \mod (p - 1) \).

Consider the locally algebraic representation of \( G \) given by
\[
\Pi_{k,a} = \text{ind}_{G}^G KZ \text{Sym}^r \bar{\mathbb{Q}}_2^2 \]
where \( r = k - 2 \geq 0 \) and \( T \) is the Hecke operator. Consider the standard lattice in \( \Pi_{k,a} \) given by
\[
(2.2) \quad \Theta = \Theta_{k,a} := \text{image} \left( \text{ind}_{G}^G KZ \text{Sym}^r \bar{\mathbb{Q}}_2^2 \to \Pi_{k,a} \right) \simeq \frac{\text{ind}_{G}^G KZ \text{Sym}^r \bar{\mathbb{Q}}_2^2}{(T - a_p)(\text{ind}_{G}^G KZ \text{Sym}^r \bar{\mathbb{Q}}_2^2) \cap \text{ind}_{G}^G KZ \text{Sym}^r \bar{\mathbb{Q}}_2^2}.
\]
It is known that the semisimplification of the reduction of this lattice satisfies \( \Theta_{k,a}^{ss} \simeq LL(V_{k,a}^{ss}) \), where \( LL \) is the (semi-simple) mod \( p \) Local Langlands Correspondence above [Ber10]. Since the map \( LL \) is clearly injective, it is enough to know \( LL(V_{k,a}^{ss}) \) to determine \( V_{k,a}^{ss} \).

2.3. Useful lemmas. Let us recall some combinatorial results without proof from [BG15]. Lemma [BG15] is not stated there but we skip the proof here, since the proof is similar.

**Lemma 2.1.** For \( r \equiv a \mod (p - 1) \) with \( 1 \leq a \leq p - 1 \), we have
\[
S_r := \sum_{j=0}^{r} \binom{r}{j} \equiv 0 \mod p.
\]
Moreover, we have \( \frac{1}{p} S_r \equiv \frac{a - r}{a} \mod p \), for \( p > 2 \).

**Lemma 2.2.** Let \( 2p \leq r \equiv a \mod (p - 1) \) with \( 2 \leq a \leq p - 1 \). Then one can choose integers \( \alpha_j \in \mathbb{Z} \), for all \( j \) with \( 0 < j < r \) and \( j \equiv a \mod (p - 1) \), such that the following properties hold:
\[
\begin{align*}
(1) & \quad \text{For all } j \text{ as above, } \binom{r}{j} \equiv \alpha_j \mod p, \\
(2) & \quad \sum_{j \geq n} \binom{j}{n} \alpha_j \equiv 0 \mod p^{3-n}, \text{ for } n = 0, 1, \\
(3) & \quad \sum_{j \geq 2} \binom{j}{2} \alpha_j \equiv \begin{cases} 0 \mod p, & \text{if } 3 \leq a \leq p - 1 \\ \binom{r}{2} \mod p, & \text{if } a = 2. \end{cases}
\end{align*}
\]
Lemma 2.3.  
(i) If $r \equiv b \mod (p-1)$, with $2 \leq b \leq p$, then
$$T_r := \sum_{0 < j < r-1 \mod (p-1)} j \equiv b - r \mod p.$$ 
(ii) If $r \equiv b \mod p(p-1)$ with $3 \leq b \leq p$, then one can choose integers $\beta_j$, for all $j \equiv b - 1 \mod (p-1)$ with $b - 1 < j < r - 1$, satisfying the following properties:

1. $\beta_j \equiv \binom{r}{j} \mod p$, for all $j$ as above,
2. $\sum_{j \geq n} \binom{r}{j} \beta_j \equiv 0 \mod p^{3-n}$, for $n = 0, 1, 2$.

Lemma 2.4. Let $p \geq 3$, $2p \leq r \equiv 1 \mod (p-1)$ and let $p$ divide $r$. Then one can choose integers $\alpha_j \in \mathbb{Z}$, for all $j$ with $1 < j < r$ and $j \equiv 1 \mod (p-1)$, such that the following properties hold:

1. For all $j$ as above, $\binom{r}{j} \equiv \alpha_j \mod p$,
2. $\sum_{j \geq n} \binom{r}{j} \alpha_j \equiv 0 \mod p^{3-n}$, for $n = 0, 1, 2$.

3. A quotient of $V_r = \text{Sym}^r \mathbb{F}_p^2$

3.1. Definition of $P$. Let $V_r$ denote the $r + 1$-dimensional $\mathbb{F}_p$-vector space of homogeneous polynomials in two variables of degree $r$ over $\mathbb{F}_p$. The group $\Gamma = \text{GL}_2(\mathbb{F}_p)$ acts on $V_r$ by the formula
$$(a \ b) \cdot F(X, Y) = F(aX + cY, bX + dY).$$

Let $X_r \subset V_r$ denote the $\mathbb{F}_p[\Gamma]$-span of the monomial $X^r$. Let $\theta(X, Y) = X^pY - XY^p$. The action of $\Gamma$ on $\theta$ is via the determinant. We define two important submodules of $V_r$ as follows.

$$V_r^* := \{ F \in V_r : \theta | F \} \cong \begin{cases} 0, & \text{if } r < p + 1 \\ V_{r-p-1} \otimes D, & \text{if } r \geq p + 1 \end{cases},$$

$$V_r^{**} := \{ F \in V_r : \theta^2 | F \} \cong \begin{cases} 0, & \text{if } r < 2p + 2 \\ V_{r-2p-2} \otimes D^2, & \text{if } r \geq 2p + 2 \end{cases}.$$ 

We set $X_r^* := X_r \cap V_r^*$ and $X_r^{**} = X_r \cap V_r^{**}$.

The mod $p$ reduction $\overline{\Theta}_{k,a_p}$ of the lattice $\Theta_{k,a_p}$ is a quotient of $\text{ind}^G_{KZ} V_r$, for $r = k - 2$. By [BCh09] Rem. 4.4] we know that if $v(a_p) = 1$ and $r \geq p$, then the map $\text{ind}^G_{KZ} V_r \rightarrow \overline{\Theta}_{k,a_p}$ factors through $\text{ind}^G_{KZ} P$, where
$$P := \frac{V_r}{X_r + V_r^{**}}.$$ 

We now study the $\Gamma$-module structure of the quotient $P$ of $V_r$. In particular, we will show that $P$ has 3 (and occasionally 2) Jordan-Hölder (JH) factors.

3.2. A filtration on $P$. In order to make use of results in Glover [Glo78], let us abuse notation a bit and model $V_r$ on the space of homogeneous polynomials in two variables $X$ and $Y$ of degree $r$ with coefficients in $\mathbb{F}_p$, rather than in $\mathbb{F}_p$. Once we establish the structure of $P$ over $\mathbb{F}_p$, it will immediately imply the corresponding result over $\overline{\mathbb{F}}_p$, by extension of scalars.
We work with the representatives $1 \leq a \leq p - 1$ of the congruence classes of $r \mod (p - 1)$. Note that $a = 1$ corresponds to $b = p$ from the Introduction.

**Proposition 3.1.** Let $p \geq 3$, and $r \equiv a \mod (p - 1)$ with $1 \leq a \leq p - 1$.

(i) For $r \geq p$, the $\Gamma$-module structure of $V_r/V_r^*$ is given by

$$0 \rightarrow V_a \rightarrow \frac{V_r}{V_r^*} \rightarrow V_{p-a-1} \otimes D^a \rightarrow 0,$$

and the sequence splits if and only if $a = p - 1$.

(ii) For $r \geq 2p + 1$, the $\Gamma$-module structure of $V_r/V_r^{**}$ is given by

$$0 \rightarrow V_{p-2} \otimes D \rightarrow \frac{V_r}{V_r^{**}} \rightarrow V_1 \rightarrow 0,$$

$$0 \rightarrow V_{p-1} \otimes D \rightarrow \frac{V_r}{V_r^{**}} \rightarrow V_0 \otimes D \rightarrow 0,$$

$$0 \rightarrow V_{a-2} \otimes D \rightarrow \frac{V_r}{V_r^{**}} \rightarrow V_{p-a+1} \otimes D^{a-1} \rightarrow 0,$$

and the sequences above split if and only if $a = 2$.

**Proof.** See [BG15, Prop. 2.1, Prop. 2.2].

**Lemma 3.2.** Let $p \geq 3$, $r \geq 2p$ and $r \equiv a \mod (p - 1)$ with $1 \leq a \leq p - 1$. Then

$$X_r^*/X_r^{**} = \begin{cases} V_{p-2} \otimes D, & \text{if } a = 1 \text{ and } p \nmid r, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** This follows from [BG15, Lem. 3.1, Lem. 4.7].

There is a filtration on $V_r/V_r^{**}$ given by

$$0 \subset \frac{(\theta X_r^{r-p-1}) + V_r^{**}}{V_r^{**}} \subset \frac{V_r^*}{V_r^{**}} \subset \frac{X_r + V_r^*}{V_r^{**}} \subset \frac{V_r}{V_r^{**}},$$

where the graded parts of this filtration are non-zero irreducible $\Gamma$-modules. Looking at the image of this filtration inside $P$, we get a filtration:

$$0 \subset W_0 := \frac{(\theta X_r^{r-p-1}) + X_r + V_r^{**}}{V_r^{**}} \subset W_1 := \frac{V_r^* + X_r}{V_r^{**} + X_r} \subset W_2 := \frac{V_r}{V_r^{**} + X_r} = P.$$

Each of the graded pieces of this filtration is either irreducible or zero. We set $J_i := W_i/W_{i-1}$, for $i = 0, 1, 2$, with $W_{-1} = 0$. Note that $W_1 \cong \frac{V_r^*/V_r^{**}}{X_r/X_r^{**}}$ is never zero by Proposition 3.1 (ii) and Lemma 3.2. In fact, $W_1$ has two JH factors $J_0$ and $J_1$ unless $a = 1$ and $p \nmid r$, in which case $W_0 = J_0 = 0$ and $W_1 = J_1$ is irreducible. Also, $J_2 \cong \frac{V_r}{V_r^* + X_r} \cong \frac{V_r/V_r^*}{X_r/X_r}$ is a proper quotient of $V_r/V_r^*$. Using Proposition 3.1 and [Glo78, (4.5)], we obtain:

**Proposition 3.3.** Let $p \geq 3$, $r \geq 2p$ and $r \equiv a \mod (p - 1)$ with $1 \leq a \leq p - 1$. Then the structure of $P$ is given by the following short exact sequences of $\Gamma$-modules:
(i) If \( a = 1 \) and \( p \nmid r \), then
\[
0 \to V_1 \to P \to V_{p-2} \otimes D \to 0,
\]
that is, \( J_0 = 0 \), \( J_1 = V_1 \) and \( J_2 = V_{p-2} \otimes D \).

(ii) If \( a = 1 \) and \( p \mid r \), or if \( 2 \leq a \leq p-1 \), then
\[
0 \to W_1 \cong V_1^* / V_r^* \to P \to V_{p-a-1} \otimes D^a \to 0,
\]
that is,
(a) \( J_0 = V_{p-2} \otimes D \), \( J_1 = V_1 \) and \( J_2 = V_{p-2} \otimes D \) if \( a = 1 \) and \( p \mid r \),
(b) \( J_0 = V_{p-2} \otimes D \), \( J_1 = V_0 \otimes D \) and \( J_2 = V_{p-3} \otimes D^2 \) if \( a = 2 \), \( r > 2p \) \( (J_1 = 0 \text{ if } r = 2p) \),
(c) \( J_0 = V_{a-2} \otimes D \), \( J_1 = V_{p-a+1} \otimes D^{a-1} \) and \( J_2 = V_{p-1-a} \otimes D^a \) if \( 3 \leq a \leq p-1 \).

The next lemma will be used many times below and describes some explicit properties of the maps \( W_i \to J_i \), for \( i = 0, 1, 2 \).

**Lemma 3.4.** Let \( p \geq 3 \), \( r \geq 2p \), \( r \equiv a \mod (p-1) \) with \( 1 \leq a \leq p-1 \).

If \( 3 \leq a \leq p-1 \), then:

(i) The image of \( \theta X^{r-p-1} \) in \( W_0 \) maps to \( X^{a-2} \in J_0 \).

(ii) The image of \( \theta X^{r-p-a+1}Y^{a-2} \) in \( W_1 \) maps to \( X^{p-a+1} \in J_1 \).

(iii) The image of \( X^{r-i}Y^i \) in \( W_2 \) maps to \( 0 \in J_2 \), for \( 0 \leq i \leq a-1 \), whereas the image of \( X^{r-a}Y^a \) in \( W_2 \) maps to \( X^{p-a-1} \) in \( J_2 \).

If \( a = 2 \), then:

(i) The image of \( \theta X^{r-p-1} \) in \( W_0 \) maps to \( X^{p-1} \in J_0 \).

(ii) For \( r > 2p \), the image of \( \theta X^{r-2p}Y^{r-1} \) in \( W_1 \) maps to \( 1 \in J_1 \).

(iii) The image of \( X^{r-i}Y^i \) in \( W_2 \) maps to \( 0 \in J_2 \), for \( i = 0, 1 \), whereas the image of \( X^{r-2}Y^2 \) in \( W_2 \) maps to \( X^{p-3} \in J_2 \).

If \( a = 1 \), then:

(i) If \( p \mid r \), the image of \( \theta X^{r-p-1} \) in \( W_0 \) maps to \( X^{p-2} \in J_0 \).

(ii) The image of \( \theta X^{r-2p+1}Y^{p-2} \) in \( W_1 \) maps to \( X \in J_1 \).

(iii) The image of \( X^{r-1}Y \) in \( W_2 \) maps to \( X^{p-2} \in J_2 \).

**Proof.** The proof is elementary and consists of explicit calculation with the maps given in [Glo78 (4.2)] and [Bre03 Lem. 5.3]. □

Let \( U_i \) denote the image of \( \text{ind}_{KZ}^G W_i \) under the map \( \text{ind}_{KZ}^G P \to \Theta \), and let \( F_i := U_i / U_{i-1} \), for \( i = 0, 1, 2 \), with \( U_{-1} := 0 \). Then we have the following commutative diagrams of \( G \)-modules:

\[
\begin{array}{cccccc}
0 & \to & \text{ind}_{KZ}^G W_1 & \to & \text{ind}_{KZ}^G P & \to & \text{ind}_{KZ}^G J_2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & U_1 & \to & \Theta (= U_2) & \to & F_2 & \to & 0,
\end{array}
\]
Theorem 4.1. Let \( 2p < k - 2 = r \equiv a \mod (p - 1) \), with \( 3 \leq a \leq p - 1 \), and assume that \( v(a_p) = 1 \).

(i) If \( p \nmid r - a \), then \( \tilde{V}_{k,a_p}^{ss} \cong \mu\lambda \cdot \omega^a \oplus \mu_{\lambda-1} \cdot \omega \) is reducible, with \( \lambda = \tilde{u} \in \mathbb{F}_p^* \).

(ii) If \( p \mid r - a \), then \( \tilde{V}_{k,a_p} \cong \ind(\omega_2^{a+1}) \) is irreducible.

Let us begin with the following elementary lemma.

Lemma 4.2. Let \( r \equiv a \mod (p - 1) \) with \( 3 \leq a \leq p - 1 \). Then we have the congruence

\[
X^{r-1}Y \equiv \frac{a-r}{a} \cdot \theta X^{r-p-1} \mod X_r + V_r^{ss}.
\]

Hence, if \( p \mid r - a \), then \( X^{r-1}Y \in X_r + V_r^{ss} \).

Proof. One checks that the following congruence holds:

\[
aX^{r-1}Y + \sum_{k \in \mathbb{F}_p} k^{r-a}(kX + Y)^r \equiv (a-r) \cdot \theta X^{r-p-1} + F(X,Y) \mod p,
\]

with \( F(X,Y) := (a-r) \cdot X^{r-p}Y^p - \sum_{j \equiv 1 \mod (p-1)} (\binom{r}{j}) \cdot X^{r-j}Y^j \in V_r^{ss} \), by [BG15, Lem 2.3].

We use the notation from Section 3.2 and the Diagrams (3.1) and (3.2). We have \( J_0 = V_{a-2} \otimes D \), \( J_1 = V_{p-a+1} \otimes D^{a-1} \), \( J_2 = V_{p-a-1} \otimes D^a \), by Proposition 3.3. Recall that \( F_i \) denotes the factor of \( \tilde{\Theta} \) which is a quotient of \( \ind_{KZ}^G J_i \), for \( i = 0, 1, 2 \).

Proposition 4.3. Let \( 2p < r \equiv a \mod (p - 1) \) with \( 3 \leq a \leq p - 1 \), and let \( v(a_p) = 1 \).

(i) If \( p \nmid r - a \), then \( F_1 = 0 \) and \( \tilde{\Theta} \) fits in the short exact sequence

\[
0 \to F_0 = U_1 \to \tilde{\Theta} \to F_2 \to 0.
\]

(ii) If \( p \mid r - a \), then \( U_1 = 0 \), and \( \tilde{\Theta} \cong F_2 \) is a quotient of \( \ind_{KZ}^G J_2 \).

Proof. Let us consider

\[
f_0 := \left[ \Id, \frac{X^{r-a-p+2}Y_a^p - X^{r-a+1}Y_{a-1}}{a_p} \right] \in \ind_{KZ}^G \Sym^r \mathbb{Q}_p^2.
\]
As \( r > 2p \), \( T^- f_0 \) is integral and dies mod \( p \). Also \( T^+ f_0 \in \text{ind}_K^G \langle X^{r-1}Y \rangle + p \cdot \text{ind}_K^G \text{Sym}^r \mathbb{Q}_p^2 \) is integral.

(i) If \( p \nmid r - a \), then \( T^+ f_0 \equiv -\frac{p}{a_p} \cdot \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}^0, (-[\lambda])^{a-2} X^{r-1}Y \right] \mod p \). By Lemma 4.2, this is the same as \( -\frac{p}{a_p} \cdot \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}^0, (-[\lambda])^{a-2} \frac{(a-r)}{a} \cdot \theta X^{r-p} \right] \) in \( \text{ind}_K^G P \). Hence \((T - a_p)f_0\) maps to

\[
T^+ f_0 - a_p f_0 \equiv -\frac{p}{a_p} \cdot \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}^0, (-[\lambda])^{a-2} \frac{(a-r)}{a} \cdot \theta X^{r-p} \right] + \left[ \text{Id}, \theta X^{r-p-a+1} Y^{a-2} \right] \in \text{ind}_K^G P.
\]

Note that \( \theta X^{r-p-1} \) maps to 0 in \( J_1 = W_1/W_0 \). By Lemma 3.4(ii), the polynomial \( \theta X^{r-p-a+1} Y^{a-2} \) maps to \( X^{p-a+1} \neq 0 \) in \( J_1 \). We conclude that \((T - a_p)f_0\) maps to \([\text{Id}, X^{p-a+1}] \in \text{ind}_K^G J_1 \), which generates it as a \( G \)-module. Hence \( F_1 = 0 \) and the result follows.

(ii) If \( p \mid r - a \), then by Lemma 4.2 we know that \( X^{r-1}Y \) maps to 0 \( \in \mathcal{P} \). Thus \( T^+ f_0 \) dies in \( \text{ind}_K^G P \). Hence \((T - a_p)f_0\) is integral and maps to the image of \(-a_p f_0 = [\text{Id}, \theta X^{r-p-a+1} Y^{a-2}] \) in \( \text{ind}_K^G P \). By Proposition 3.1(ii), \( W_1 \cong V_1^r / V_1^r \) is a non-split extension of the weight \( J_1 \) by \( J_0 \). By Lemma 3.4(ii), the polynomial \( \theta X^{r-p-a+1} Y^{a-2} \) maps to \( X^{p-a+1} \neq 0 \) in \( J_1 \) and hence generates \( W_1 \) as a \( \Gamma \)-module, so \([\text{Id}, \theta X^{r-p-a+1} Y^{a-2}] \) generates \( \text{ind}_K^G W_1 \) as a \( G \)-module. This proves that the surjection \( \text{ind}_K^G W_1 \twoheadrightarrow U_1 \) is the zero map. \( \Box \)

**Proposition 4.4.** Let \( 2p < r \equiv a \mod (p-1) \) with \( 3 \leq a \leq p - 1 \). Assume that \( p \nmid r - a \) and \( v(a_p) = 1 \), so that \( u := \frac{a}{a-r} \cdot \frac{a}{a_p} \) is a \( p \)-adic unit. Then

(i) \( F_0 \) is a quotient of \( \pi(a-2, \bar{u}, \omega) \),

(ii) \( F_2 \) is a quotient of \( \pi(p-a-1, \bar{u}^{-1}, \omega^a) \).

**Proof.** (i) Consider \( f_0 \in \text{ind}_K^G \text{Sym}^r \mathbb{Q}_p^2 \) given by

\[
f_0 = \left[ \text{Id}, \frac{X^{r-1}Y - X^{r-p} Y^p}{p} \right] = \left[ \text{Id}, \frac{\theta X^{r-p-1}}{p} \right].
\]

Since \( r \geq p + 2 \), \( T^- f_0 \equiv 0 \mod p \). By the formula for the Hecke operator,

\[
(T - a_p) f_0 \equiv T^+ f_0 - a_p f_0 \equiv \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}^0, X^{r-1}Y \right] - \frac{(a_p/p)}{[\text{Id}, \theta X^{r-p-1}]} \mod p.
\]

By Lemma 4.2 \((T - a_p)f_0\) maps to \( \frac{(a-r)}{a} \cdot \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}^0, \theta X^{r-p-1} \right] - \frac{a_p}{p} \left[ [\text{Id}, \theta X^{r-p-1}] \right] \), which in fact lies in the submodule \( \text{ind}_K^G W_0 \). This maps to \( \frac{(a-r)}{a} \cdot [T - \bar{u}] \left[ X^{a-2} \right] \in \text{ind}_K^G J_0 \), by Lemma 3.4(i). The element \([\text{Id}, X^{a-2}] \) generates \( \text{ind}_K^G J_0 \) as a \( G \)-module, so the map \( \text{ind}_K^G J_0 \to F_0 \) must factor through \( \pi(a-2, \bar{u}, \omega) \).
We have $\alpha_j$ where the $\alpha_j$ are integers from Lemma 2.2. We compute that $T^+ f_2 \in \text{ind}_{K Z}^G (X^{r-1} Y) + p \cdot \text{ind}_{K Z}^G \text{Sym}^r \mathbb{Z}_p^2$, which maps to 0 in $\text{ind}_{K Z}^G J_2$, by Lemma 3.4 (iii). By Lemma 2.2 and as $p \geq 5$, both $T^+ f_1$ and $T^- f_1$ die mod $p$. We have

$$-a_p f_2 \equiv \sum_{\lambda \in \mathbb{F}_p} \left[ g_{2,p, \lambda}^0 \cdot \frac{a_p}{p} \cdot X^{p-1} Y^{r-1} \right] \mod p,$$

Using that $X^{p-1} Y^{r-1} \equiv X^{r-a} Y^a \mod V_r$, and Lemma 3.4 (iii), we get that $-a_p f_2$ maps to $\frac{a_p}{p} \cdot \sum_{\lambda \in \mathbb{F}_p} \left[ g_{2,p, \lambda}^0 \cdot X^{p-1} \right] \in \text{ind}_{K Z}^G J_2$. Moreover, $T^+ f_0 - a_p f_1 + T^- f_2$ is integral and congruent to

$$\left[ g_{1,0, \lambda}^0 \cdot \frac{p-1}{p} \left( \binom{r}{j} - \alpha_j \right) \cdot X^{r-1} Y^j + Y^r \right] \mod p,$$

unless $a = p - 1$, in which case we also have the terms $\sum_{\lambda \in \mathbb{F}_p} \left[ g_{1,0, \lambda}^0 \cdot (\lambda)^{-2} \cdot X^{r-1} Y \right]$ from $T^+ f_0$, in addition to the above. In any case, $T^+ f_0 - a_p f_1 + T^- f_2$ maps to $\left[ g_{1,0}^0 \cdot \frac{p-a}{a} \cdot X^{p-a-1} \right] \in \text{ind}_{K Z}^G J_2$, by Lemma 2.2, Lemma 2.1 and Lemma 3.3 (iii). If $a = p - 1$, then $T^- f_0$ dies mod $p$ and $-a_p f_2 \equiv \left[ \text{Id}, \frac{a_p}{p} \cdot X^{r-1} Y^{p-1} \right] \mod X_r$ maps to $\frac{a_p}{p} \cdot \left[ \text{Id}, 1 \right]$ in $\text{ind}_{K Z}^G J_2 = \text{ind}_{K Z}^G V_0$.

So $(T-a_p) f$ is always integral and maps to $\frac{a_p}{p} \left( T - \frac{a_p}{p} \cdot \frac{(a-1)}{a} \right) \left[ g_{1,0}^0, X^{p-a-1} \right]$ in $\text{ind}_{K Z}^G J_2$. Therefore $F_2$ is a quotient of $\pi (p - a - 1, \bar{u}^{-1}, \omega^a)$, as desired.

**Proof of Theorem 4.7**

(i) If $p \not| r - a$, Proposition 4.3 (i) tells us that $\Theta^{ss} \cong (F_0 \oplus F_0)^{ss}$. By Proposition 4.4 we have surjections $\pi (a - 2, \bar{u}, \omega) \twoheadrightarrow F_0$, and $\pi (p-a-1, \bar{u}^{-1}, \omega^a) \twoheadrightarrow F_2$. Using the fact that $\Theta$ lies in the image of the mod $p$ Local Langlands Correspondence, we deduce that $\Theta^{ss} \cong \pi (a-2, \bar{u}, \omega)^{ss} \oplus \pi (p-a-1, \bar{u}^{-1}, \omega^a)^{ss}$.

(ii) If $p | r - a$, then Proposition 4.3 (ii) and [BG09, Prop. 3.3] together imply that $\tilde{V}_{k,a_p} \cong \text{ind}(\omega_2^{a+1})$ is irreducible.
5. The case \( a = 1 \)

In this section we will treat the case \( r \equiv 1 \mod (p-1) \). The result is the following:

**Theorem 5.1.** Let \( p \geq 5 \), \( r \geq 2p \) and \( r \equiv 1 \mod (p-1) \).

(i) If \( p \nmid r \), then \( V_{k,a_p} \cong \text{ind}(\omega_2^2) \).

(ii) If \( p \mid r \), then \( V_{k,a_p}^{ss} \cong \mu_\lambda \cdot \omega \oplus \mu_{\lambda-1} \cdot \omega \), where \( \lambda^2 - c\lambda + 1 = 0 \) with \( c = \frac{2p}{p} - \frac{r - p}{a_p} \in \overline{\mathbb{F}}_p \).

Recall that by Proposition 3.3 we have \( J_2 = V_{p-2} \otimes D \), \( J_1 = V_1 \), and \( J_0 = \begin{cases} V_{p-2} \otimes D, & \text{if } p \mid r, \\ 0, & \text{if } p \nmid r. \end{cases} \)

**Proposition 5.2.** If \( p \geq 3 \), \( r > 2p \) and \( r \equiv 1 \mod (p-1) \), then \( F_2 = 0 \). As a consequence,

(i) If \( p \nmid r \), then \( \Theta \cong F_1 \) is a quotient of \( \text{ind}^G_{KZ}J_1 \).

(ii) If \( p \mid r \), then \( \Theta \cong U_1 \), and the bottom row of Diagram (3.2) reduces to

\[
0 \to F_0 \to \Theta \to F_1 \to 0.
\]

**Proof.** Consider the function \( f = f_0 \in \text{ind}^G_{KZ}\text{Sym}^2 \mathbb{Q}_p \) given by

\[
f_0 = \left[ \text{Id}, \frac{XY^{r-1} - 2XpY^{r-p} + X^{2p-1}Y^{r-2p+1}}{p} \right].
\]

One checks that \( T^+f_0, T^-f_0, a_pf_0 \) are all integral, and that \( T^+f_0 \equiv 0 \mod p \), \( a_pf_0 \in \text{ind}^G_{KZ}V^r \) and \( T^-f_0 \equiv [\alpha, XY^{r-1}] \mod p \). By Lemma 3.4(iii) and by \( \Gamma \)-linearity, \( XY^{r-1} = (0, 1) \cdot X^{r-1}Y \) maps to \(-Y^{p-2}\), under the map \( W_2 \to J_2 \). So \((T - a_p)f \) maps to \([\alpha, -Y^{p-2}] \in \text{ind}^G_{KZ}J_2 \). As \([\alpha, -Y^{p-2}] \) generates \( \text{ind}^G_{KZ}J_2 \) as a \( G \)-module, we have \( F_2 = 0 \). \( \Box \)

**Proposition 5.3.** Let \( p \geq 3 \), \( r > 2p \), \( r \equiv 1 \mod (p-1) \) and \( p \mid r \). Then we have

(i) \( F_1 = 0 \), and

(ii) \( F_0 \) is a quotient of \( \frac{\text{ind}^G_{KZ}(V_{p-2} \otimes D)}{T^2 - cT + 1} \), where \( c = \frac{a_p}{p} - \frac{r - p}{a_p} \in \overline{\mathbb{F}}_p \).

**Proof.** (i) Consider the function \( f = f_2 + f_1 + f_0 \in \text{ind}^G_{KZ}\text{Sym}^2 \mathbb{Q}_p \) given by

\[
f_2 = \sum_{\lambda \in \mathbb{F}_p^*} \left[ g_{2,p[\lambda]}^0 \cdot \frac{[\lambda]^{p-2}}{a_p} \cdot (Y^r - X^{r-p}Y^p) \right] + \left[ g_{2,0}^0 \cdot \frac{1 - p}{a_p} \cdot (XY^{r-1} - X^{r-p+1}Y^{p-1}) \right],
\]

\[
f_1 = \left[ g_{1,0}^0 \cdot \frac{XpY^{r-p} - XY^{r-1}}{p} + \frac{p - 1}{a_p^2} \cdot \sum_{0 < j < r-1 \mod (p-1)} \beta_j X^{r-j}Y^{j} \right],
\]

\[
f_0 = \left[ \text{Id}, \frac{1 - p}{a_p} \cdot (X^r - X^{pY^{r-p}}) \right],
\]

where \( \beta_j \) are the integers from Lemma 2.3(ii). Using the facts that \( p \mid r - p \) and \( p \geq 3 \), we see that \( T^+f_2 \) and \( T^-f_0 \) die \( \mod p \). We compute that

\[
T^-f_1 - a_pf_0 \equiv [\text{Id}, -XY^{r-1} - (X^r - X^{pY^{r-p}})] \mod p \equiv [\text{Id}, \theta Y^{r-p-1}] \mod X_r.
\]
Note that \( T^+f_0 + T^-f_2 - a_pf_1 \) is congruent mod \( p \) to

\[
\left[ g^0_{1,0} \sum_{j \equiv 0 \ (mod \ (p-1))} \frac{p-1}{a_p} \binom{r}{j} - \beta_j \right] \cdot X^{r-j}Y^j + \frac{p-1}{a_p} (r - p) \cdot XY^{r-1} - \frac{a_p}{p} \cdot \theta Y^{r-p-1},
\]

which is integral because \( v(a_p) = 1, p \mid r - p \) and each \( \beta_j \equiv \binom{r}{j} \ (mod \ p) \). Rearranging the terms, we can write

\[
T^+f_0 + T^-f_2 - a_pf_1 = \left[ g^0_{1,0}, (p-1) \left( F(X,Y) + \frac{p-r}{a_p} \cdot \theta Y^{r-p-1} \right) - \frac{a_p}{p} \cdot \theta Y^{r-p-1} \right],
\]

where \( F(X,Y) := \sum_{j \equiv 0 \ (mod \ (p-1))} \frac{1}{a_p} \binom{r}{j} - \beta_j \cdot X^{r-j}Y^j - \frac{p-r}{a_p} \cdot X^pY^{r-p} \in V_r^{**} \), as it satisfies the criteria given in [BG15, Lem. 2.3]. Thus inside \( \text{ind}_{KZ}^G P \), we have

\[
(5.2) \quad T^+f_0 + T^-f_2 - a_pf_1 = \left[ g^0_{1,0}, \left( -\frac{p-r}{a_p} - \frac{a_p}{p} \right) \cdot \theta Y^{r-p-1} \right].
\]

Then we compute that \( T^+f_1 - a_pf_2 \) is congruent to

\[
\sum_{\lambda \in \mathbb{F}_p} \left[ g^0_{2,p[\lambda]}, (-[\lambda])^{r-p-1}(-p+1) \cdot X^{r-1}Y \right] - \sum_{\lambda \in \mathbb{F}_p} \left[ g^0_{2,p[\lambda]}, (-[\lambda])^{p-2}(-p+1) \cdot (Y^r - X^{r-p}Y) \right] - \left[ g^0_{2,0}, XY^{r-1} - X^{r-p+1}Y^{p-1} \right] \text{ mod } p.
\]

Going modulo \( X_r \) and \( V_r^{**} \), we get

\[
(5.3) \quad T^+f_1 - a_pf_2 = \sum_{\lambda \in \mathbb{F}_p} \left[ g^0_{2,p[\lambda]}, (-[\lambda])^{p-2} \theta X^{r-p-1} \right] + \left[ g^0_{2,0}, \theta Y^{r-p-1} + \frac{(r-2p+1)}{p-1} \cdot \theta X^{r-2p+1}Y^{p-2} \right].
\]

We know \( \theta X^{r-p-1}, \theta Y^{r-p-1} \) map to \( 0 \in J_1 = W_1/W_0 \) and \( \theta X^{r-2p+1}Y^{p-2} \) maps to \( X \in J_1 = V_1 \), by Lemma 5.4(ii). By the equations (5.1), (5.2) and (5.3) above, \((T - a_p)f\) is integral and its image in \( \text{ind}_{KZ}^G P \) lies in \( \text{ind}_{KZ}^G W_1 \) and maps to \([g^0_{2,0}, -X] \in \text{ind}_{KZ}^G J_1\), which generates \( \text{ind}_{KZ}^G J_1 \) over \( G \). Hence \( F_1 = 0 \).

(ii) Consider the function \( f = f_2 + f_1 + f_0 \in \text{ind}_{KZ}^G \text{Sym}^r \mathbb{Q}_p^2 \) given by

\[
f_2 = \sum_{\lambda, \mu \in \mathbb{F}_p} \left[ g^0_{2,p[\mu+[\lambda]], Y^r - X^{r-p}Y^p} \right],
\]

\[
f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[ g^1_{1,[\lambda]}, \frac{1}{p} \cdot (X^{r-1}Y - X^{r-p}Y^p) + \frac{p-1}{a_p^2} \cdot \sum_{j \equiv 1 \ (mod \ (p-1))} \alpha_j X^{r-j}Y^j \right],
\]

\[
f_0 = \left[ \text{Id}, \frac{1-p}{a_p} \cdot (X^{r-1}Y - X^{r-p}Y^p) \right],
\]
where the $\alpha_j$ are the integers from Lemma 2.3. We check that $T^+ f_2$ and $T^- f_0$ die mod $p$, since $p \mid r - p$ and $r \geq 2p$. Next we compute that modulo $p$ and $X_r$, $T^+ f_0 - a_p f_1 + T^- f_2$ is congruent to

$$\sum_{\lambda \in \mathbb{F}_p} \left[ g^0_{1,[\lambda]} \left( \frac{(p-1)(r-p)}{a_p} \right) X^{r-1} Y + \sum_{j=1}^{\lfloor \frac{r}{p} \rfloor} \frac{p-1}{a_p} \left( \frac{r}{j} \right) - \alpha_j \right] \cdot X^{r-j} Y^j + \frac{a_p}{p} \cdot \theta X^{r-p-1}.$$ 

Rearranging the terms, we get

$$T^+ f_0 - a_p f_1 + T^- f_2 \equiv \sum_{\lambda \in \mathbb{F}_p} \left[ g^0_{1,[\lambda]} \left( \frac{a_p}{p} - \frac{r-p}{a_p} \right) \theta X^{r-p-1} - F(X,Y) \right],$$

where

$$F(X,Y) = \frac{r-p}{a_p} \cdot X^{r-p} Y^p + \sum_{j=1}^{\lfloor \frac{r}{p} \rfloor} \frac{1}{a_p} \left( \frac{r}{j} \right) - \alpha_j \right] \cdot X^{r-j} Y^j$$

can be checked to be in $V^*_r$, using [BG15 Lem. 2.3]. Thus in $\text{ind}_{KZ}^G P$, we have

$$T^+ f_0 - a_p f_1 + T^- f_2 \equiv \sum_{\lambda \in \mathbb{F}_p} \left[ g^0_{1,[\lambda]} \left( \frac{a_p}{p} - \frac{r-p}{a_p} \right) \theta X^{r-p-1} \right].$$

(5.4)

We check that $T^+ f_1 - a_p f_2 \equiv \sum_{\mu, \lambda \in \mathbb{F}_p} \left[ g^0_{2,\mu'[\lambda]} - X^{r-1} Y - (Y^{r-1} Y^{r-p}) \right] \mod p$, so

$$T^+ f_1 - a_p f_2 \equiv \sum_{\mu, \lambda \in \mathbb{F}_p} \left[ g^0_{2,\mu'[\lambda]} - \theta X^{r-p-1} \right] \mod X_r.$$ 

(5.5)

Finally, we compute

$$T^- f_1 - a_p f_0 \equiv [\text{Id}, -\theta X^{r-p-1}] \mod p.$$ 

(5.6)

By Lemma 3.4, the image of $\theta X^{r-p-1}$ (in $P$) lands in $W_0$ and maps to $X^{p-2} \in J_0$. Using the formula for the $T$ operator, and the equations (5.4), (5.5), (5.6), we obtain that $(T - a_p)f$ is integral, and in fact it is the image of $-(T^2 - cT + 1)[\text{Id}, X^{p-2}] \in \text{ind}_{KZ}^G J_0$, where

$$c = \frac{a_p}{p} - \frac{r-p}{a_p} \in \mathbb{F}_p.$$

Proof of Theorem 5.1 Part (i) follows from Proposition 5.2 (i) and [BG09 Prop. 3.3], at least if $p > 3$. If $p \mid r$, then $\Theta$ is a quotient of $\frac{\text{ind}_{KZ}^G V_{p-2} \otimes D}{T^2 - cT + 1}$, by Propositions 5.2 (ii) and 6.3. Now part (ii) follows by the mod $p$ semisimple Local Langlands Correspondence.
6. The Case \( a = 2 \)

In this section we study the case \( r \equiv 2 \mod (p - 1) \) with \( r \geq 2p \), for \( p > 3 \). In this case our results are not complete. However, we can prove:

**Theorem 6.1.** Let \( p > 3 \), \( r \geq 2p \) and \( r \equiv 2 \mod (p - 1) \), \( v(a_p) = 1 \).

1. Suppose that \( r \not\equiv 2 \mod p \) and \( v \left( a_p^2 - \frac{r}{(r^2)}p^2 \right) > 2 \). Then \( \tilde{V}_{k,a_p} \cong \text{ind}(\omega_2^{p+2}) \) is irreducible.
2. Suppose that \( p \mid r \). Then we have \( \tilde{V}_{k,a_p} \cong \mu_\lambda \cdot \omega^2 \oplus \mu_{\lambda-1} \cdot \omega \), where \( \lambda = a_p/p \in \overline{F}_p^* \).
3. Suppose \( r \equiv 1 \mod p \). Then we have \( \tilde{V}_{k,a_p} \cong \mu_\lambda \cdot \omega^2 \oplus \mu_{\lambda-1} \cdot \omega \), where \( \lambda = 2a_p/p \in \overline{F}_p^* \).
4. Suppose \( r \equiv 2 \mod p \) and \( v \left( a_p^2 - \frac{r}{(r^2)}p^2 \right) = 2 \). Then \( \tilde{V}_{k,a_p} \) is irreducible, and isomorphic either to \( \text{ind}(\omega_2^p) \) or to \( \text{ind}(\omega_2^p) \).

By Lemma 3.3 the JH factors of \( P \) are \( J_0 = V_{p-1} \otimes D, J_1 = V_0 \otimes D \) (only when \( r > 2p \)), and \( J_2 = V_{p-3} \otimes D^2 \). With the notation of Diagram (3.1), we have

**Proposition 6.2.** Let \( p > 3 \), \( r \geq 2p \) and \( r \equiv 2 \mod (p - 1) \), \( v(a_p) = 1 \).

1. If \( v \left( a_p^2 - \frac{r}{(r^2)}p^2 \right) > 2 \) and \( r \not\equiv 2 \mod p \), then \( F_2 = 0 \).
2. If \( v \left( a_p^2 - \frac{r}{(r^2)}p^2 \right) = 2 \), then \( F_2 \) is a quotient of \( \pi(p - 3, \lambda, \omega^2) \), where \( \lambda \in \overline{F}_p \) is the mod \( p \) reduction of \( \frac{(2 - r)pa_p}{2(a_p^2 - \frac{r}{(r^2)}p^2)} \in \mathbb{Z}_p \).

**Proof.** Let us consider \( f = f_1 + f_0 \in \text{ind}_{KZ}^{G} \text{Sym}^\infty \overline{G}_a^2 \) given by

\[
f_1 = \sum_{\lambda \in \mathbb{F}_p} g_{\lambda, [\lambda]}^0 \left[ \frac{Yr - X^{p-1}Yr-p+1}{a_p} \right], \quad f_0 = \text{Id} \cdot \left[ \frac{p-1}{a_p^2} \sum_{0<j<r \mod (p-1)} \alpha_j \cdot X^{r-j}Y^j \right],
\]

where \( \alpha_j \) are the integers from Lemma 2.2. We check that \( T^- f_0 \) dies mod \( p \), \( T^+ f_1 \) is integral and lies in \( \text{ind}_{KZ}^{G} \langle X^{r-1}Y \rangle + p \cdot \text{ind}_{KZ}^{G} \text{Sym}^\infty \overline{G}_a^2 \). Then we have

\[
T^- f_1 - a_p f_0 \equiv \left[ \text{Id} \cdot \sum_{j \equiv 2 \mod (p-1)} \frac{p-1}{a_p} \left( \binom{r}{j} - \alpha_j \right) \cdot X^{r-j}Y^j + \frac{p}{a_p} \cdot Y^r \right] \mod p
\]

\[
\equiv \left[ \text{Id} \cdot \frac{(p-1)}{a_p} \cdot \sum_{j \equiv 2 \mod (p-1)} \frac{1}{p} \left( \binom{r}{j} - \alpha_j \right) \cdot X^{r-2}Y^2 \right] \mod X_r + V_r^*.
\]

and we have

\[
T^+ f_0 - a_p f_1 \equiv \sum_{\lambda \in \mathbb{F}_p} g_{\lambda, [\lambda]}^0 \left( \frac{(p-1)}{a_p^2} \binom{r}{2} X^{r-2}Y^2 \right) + \sum_{\lambda \in \mathbb{F}_p} \left[ g_{\lambda, [\lambda]}^0 \cdot -Y^r + X^{p-1}Yr-p+1 \right]
\]

\[
\equiv \sum_{\lambda \in \mathbb{F}_p} g_{\lambda, [\lambda]}^0 \left( \frac{\binom{r}{2} p^2}{a_p^2} + 1 \right) \cdot X^{r-2}Y^2 \mod X_r + V_r^*.
\]
Thus $(T - a_p)f$ is integral and we use Lemma 3.3 (iii), Lemma 2.1 and Lemma 2.2 to obtain

$$(T - a_p)f \mapsto \sum_{\lambda \in \mathbb{F}_p} \left[ \frac{1}{2} \left( a_p^2 - \left( \frac{r}{2} \right)^3 \right) \cdot X^{p-3} \right] + \left[ \text{Id}, -\frac{p}{a_p} \frac{2 - r}{2} \cdot X^{p-3} \right] \in \text{ind}_{KZ}^G J_2.$$

(i) If $v \left( a_p^2 - \left( \frac{r}{2} \right)^3 \right) > 2$ and $p$ does not divide $r - 2$, then the left sum above dies mod $p$, and so $(T - a_p)f$ maps to $c \cdot [\text{Id}, X^{p-3}] \in \text{ind}_{KZ}^G J_2$, where $c = \frac{p}{a_p} \left( \frac{2 - r}{2} \right) \in \mathbb{F}_p^*$. Since the element $[\text{Id}, X^{p-3}]$ generates $\text{ind}_{KZ}^G J_2$ over $G$, we conclude that $F_2 = 0$.

(ii) If $v \left( a_p^2 - \left( \frac{r}{2} \right)^3 \right) = 2$, then using the formula for the Hecke operator, we get that $(T - a_p)f$ maps to a non-zero scalar multiple of $(T - \lambda) [\text{Id}, X^{p-3}] \in \text{ind}_{KZ}^G J_2$, where $\lambda \in \mathbb{F}_p$ is the mod $p$ reduction of $\frac{(2 - r)p^2}{2 (a_p^2 - \left( \frac{r}{2} \right)^3)}$, hence the result follows.

\[\square\]

**Theorem 6.3.** Let $p > 3$, $r \geq 2p$ and $r \equiv 2 \mod (p - 1)$, $r \not\equiv 2 \mod p$ and $v(a_p) = 1$. If $v \left( a_p^2 - \left( \frac{r}{2} \right)^3 \right) > 2$, then $\bar{V}_{k,a_p} \cong \text{ind}(\omega_2^{p+2})$ is irreducible.

**Proof.** By Proposition 6.2 (i), $F_2 = 0$ and Diagram 3.2 reduces to:

\[(6.1) \quad 0 \longrightarrow \text{ind}_{KZ}^G (V_{p-1} \otimes D) \longrightarrow \text{ind}_{KZ}^G (V^*_r/V^*_r) \longrightarrow \text{ind}_{KZ}^G (V_0 \otimes D) \longrightarrow 0 \]

We argue as in the proof of [CG15] Lem. 29. If both $F_0$ and $F_1$ are non-zero, then $\bar{\Theta}$ is reducible. Since $\bar{\Theta}$ lies in the image of the mod $p$ Local Langlands Correspondence, its semisimplification is of the form $\pi(s, \lambda, \eta)^{ss} \otimes \pi([p - 3 - s], \lambda^{-1}, \eta \omega^{s+1})^{ss}$ for some $s, \eta, \lambda$ with the usual notation. By [BG09] Lem. 3.2, every JH factor of $F_0$ is a subquotient of $\pi(p - 1, \mu, \omega)$, for some $\mu \in \mathbb{F}_p$, and every JH factor of $F_1$ is a subquotient of $\pi(0, \nu, \omega)$, for some $\nu \in \mathbb{F}_p$. This is a contradiction since $(p - 1) + 0 \not\equiv p - 3 \mod (p - 1)$.

Therefore either $F_0 = 0$ or $F_1 = 0$, hence $\bar{\Theta}$ is a quotient of either $\text{ind}_{KZ}^G (V_0 \otimes D)$ or $\text{ind}_{KZ}^G (V_{p-1} \otimes D)$. Now the result follows by [BG09] Prop. 3.3.

\[\square\]

If $p \mid \left( \frac{r}{2} \right)$, i.e., if $r \equiv 0$ or $1 \mod p$, then note that $v \left( a_p^2 - \left( \frac{r}{2} \right)^3 \right) = 2$, and Theorem 6.3 cannot be applied. We will show that $\bar{V}_{k,a_p}$ is in fact reducible in these cases.

**Lemma 6.4.** Let $p \geq 3$, and let $p \mid r \equiv 2 \mod (p - 1)$. Then $X^r - 1 Y \equiv \theta X^{r - p - 1} \mod X_r + V^*_r$.

**Proof.** Consider the polynomial $F(X, Y) = \sum_{k \in \mathbb{F}_p} k^{p-2} (kX + Y)^r \in X_r$. Using the fact $p \mid r$, we obtain

$$F(X, Y) \equiv - \sum_{0 \leq j < r \mod (p - 1)} \binom{r}{j} X^{r-j} Y^j \equiv -2X^{r-p} Y^p + F_1(X, Y) \mod p,$$
where $F_1(X,Y) = 2X^{r-p}Y^p - \sum_{1 < j < r-1 \mod (p-1)} X^{-j}Y^j$. Note that $F_1(X,Y) \in V_r^{**}$, by the criteria given in [BC15] Lem 2.3. Thus we have shown that $X^{r-p}Y^p \in X_r + V_r^{**}$ and the result follows.

**Theorem 6.5.** Let $p > 3$, $r \geq 2p$, $r \equiv 2 \mod (p-1)$ and let $v(a_p) = 1$. If $p \mid r$, then we have $\bar{V}_{k,a_p}^{ss} \cong \mu_\lambda \cdot \omega^2 \oplus \mu_{\lambda-1} \cdot \omega$, where $\lambda = \frac{a_p}{p} \in \bar{F}_p^*$.

**Proof.** By Proposition 6.2 if $p \mid r$, then $F_2$ is a quotient of $\pi(p-3,\lambda^{-1},\omega^2)$, where $\lambda = \frac{a_p}{p} \in \bar{F}_p^*$. We will show that $F_1 = 0$ (if $r > 2p$) and $F_0$ is a quotient of $\pi(p-1,1,\lambda,\omega)$ in this case.

To eliminate the JH factor $J_1$, i.e., to show $F_1 = 0$, which is relevant only when $r > 2p$, we consider the function

$$f_0 = \left[\text{Id}, \frac{X^{r-p}Y^p - X^{r-2p+1}Y^{2p-1}}{a_p}\right] \in \text{ind}_{G_{KZ}}^G \text{Sym}^r \bar{\mathbb{Q}}_p^2,$$

and check that $T^-f_0$ dies mod $p$, and $T^+f_0$ maps to $\sum_{\lambda \in \bar{F}_p} \left[g^0_{\lambda}, \frac{\theta X^{r-p-1}}{a_p}\right]$ in $\text{ind}_{G_{KZ}}^G P$, by Lemma 6.3. By Lemma 3.4 (ii), we get that $T^+f_0$ maps to zero in $\text{ind}_{G_{KZ}}^G J_1$. Thus $(T - a_p)f_0$ maps to the image of $-a_pf_0 \equiv [\text{Id}, -\theta X^{r-2p}Y^{p-1}]$ in $\text{ind}_{G_{KZ}}^G J_1$, which is $[\text{Id}, -1]$ by Lemma 3.4 (ii). As $[\text{Id}, -1]$ generates $\text{ind}_{G_{KZ}}^G J_1$ over $G$, we have $F_1 = 0$.

Next we consider the function

$$h_0 = \left[\text{Id}, \frac{X^{r-1}Y - X^{r-p}Y^p}{p}\right] \in \text{ind}_{G_{KZ}}^G \text{Sym}^r \bar{\mathbb{Q}}_p^2,$$

and check that $(T - a_p)h_0$ is integral and

$$(T - a_p)h_0 \equiv T^+h_0 - a_ph_0 \equiv \sum_{\lambda \in \bar{F}_p} \left[g^0_{\lambda}, X^{r-1}Y\right] - \frac{a_p}{p} \cdot [\text{Id}, \theta X^{r-p-1}] \mod p.$$

By Lemma 6.3 and Lemma 3.4 (i), in $\text{ind}_{G_{KZ}}^G P$ this is the image of $(T - \lambda)[\text{Id}, X^{p-1}] \in \text{ind}_{G_{KZ}}^G J_0$ with $\lambda = \frac{a_p}{p} \in \bar{F}_p^*$. Therefore $F_0$ must be a quotient of $\pi(p-1,1,\lambda,\omega)$.

Now the result follows from the short exact sequence of $G$-modules

$$0 \to F_0 \to \bar{\Theta} \to F_2 \to 0,$$

and the fact that $\bar{\Theta}$ lies in the image of the mod $p$ Local Langlands Correspondence.

**Theorem 6.6.** Let $p > 3$, $r \geq 2p$, $r \equiv 2 \mod (p-1)$ and let $v(a_p) = 1$. If $r \equiv 1 \mod p$, then we have $\bar{V}_{k,a_p}^{ss} \cong \mu_\lambda \cdot \omega^2 \oplus \mu_{\lambda-1} \cdot \omega$, where $\lambda = 2\frac{a_p}{p} \in \bar{F}_p^*$.

**Proof.** By Proposition 6.2 (ii), $F_2$ is a quotient of $\pi(p-3,\lambda^{-1},\omega^2)$ with $\lambda = 2\frac{a_p}{p}$. We will show that $F_0$ is a quotient of $\pi(p-1,1,\lambda,\omega)$, and $F_1$ is a quotient of $\pi(0,1,\lambda,\omega)$ with the same $\lambda$. Then the result follows from the semisimple mod $p$ Local Langlands Correspondence.
First consider the function $f = f_0 + f_1 + f_2 \in \text{ind}^G_{KZ} \text{Sym}^r \bar{\mathbb{Q}}_p^2$ given by
\[
\begin{align*}
f_0 &= \left[ \text{Id}, \frac{2}{p} \cdot (X^{r-1}Y - X^{r-p}Y^p) \right], \\
f_1 &= \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}, \frac{r}{a_p} \cdot (X^{2p-1}Y^{r-2p+1} - 2X^pY^{r-p} + XY^{r-1}) \right], \\
f_2 &= \sum_{\lambda, \mu \in F_p} \left[ g_{2, [\mu]+[\lambda]}, [\mu]^{p-2}(Y^r - X^{p-1}Y^{r-p}) \right].
\end{align*}
\]
Then we check that $T^- f_0, T^+ f_1, T^+ f_2, -a_p f_2 \equiv 0 \mod p,
\begin{align*}
T^+ f_0 &= \sum_{\lambda \in F_p} \left[ g_{0, [\lambda]}, 2X^{r-1}Y \right] \mod p, \\
-a_p f_0 &= \frac{-2a_p}{p} \cdot [\text{Id}, \theta X^{r-p-1}] \mod p, \\
T^- f_1 &= \frac{p}{a_p} \cdot [\text{Id}, F(X, Y)] \mod p,
\end{align*}
where $F(X, Y) = \sum_{0 \leq j < r \mod (p-1)} \binom{r-1}{j} \cdot X^{r-j}Y^j$. We check that $F(X, Y) \in V_r^{**}$, using [BG15, Lem 2.3] and the fact that $p \mid r - 1$. So $T^- f_1$ maps to zero in $\text{ind}^G_{KZ} P$. Further we compute that
\[
\begin{align*}
-a_p f_1 &= \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}, r(X^{2p-1}Y^{r-2p+1} - 2X^pY^{r-p} + XY^{r-1}) \right] \mod p, \ 	ext{and} \\
T^- f_2 &= \sum_{\lambda \in F_p} \left[ g_{2, [\lambda]}, -\sum_{j=1}^{r} \binom{r}{j} X^{r-j}Y^j \right] \mod p.
\end{align*}
\]
Now using the fact $r \equiv 1 \mod p$, and the above computations, we obtain
\[
T^+ f_0 + T^- f_2 - a_p f_1 \equiv \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}, X^{r-1}Y - X^{r-p}Y^p + H(X, Y) \right] \mod p,
\]
where $H(X, Y) = X^{r-p}Y^p - \sum_{p \leq j \leq r \mod (p-1)} \binom{r}{j} X^{r-j}Y^j - 2X^pY^{r-p} + rX^{2p-1}Y^{r-2p+1}$, which lies in $V_r^{**}$ by [BG15, Lem 2.3] and as $r \equiv 1 \mod p$. Therefore $(T - a_p)f$ is integral and its image in $\text{ind}^G_{KZ} P$ is
\[
\frac{-2a_p}{p} \cdot [\text{Id}, \theta X^{r-p-1}] + \sum_{\lambda \in F_p} \left[ g_{1, [\lambda]}, \theta X^{r-p-1} \right],
\]
which is the image of $(T - 2a_p/p)\left[ \text{Id}, X^{p-1} \right] \in \text{ind}^G_{KZ} J_0$. This shows that the submodule $F_0$ (of $\Theta$) is a quotient of $\pi(p - 1, 2a_p/p, \omega)$. 
Next consider the function $h_0 \in \text{ind}_{KZ}^G \text{Sym}^r \mathbb{Q}_p^2$ given by

$$h_0 = \left[ \text{Id}, \frac{X^{r-1}Y - XY^{-1}}{a_p} \right].$$

We compute, using $r \equiv 1 \mod p$, that

$$T^{-} h_0 \equiv \left[ \alpha, -\frac{p}{a_p}XY^{-1} \right] \mod p,$$

$$T^{+} h_0 \equiv \sum_{\lambda \in \mathbb{F}_p} \left[ g_0^{(1,|\lambda|)}, \frac{p}{a_p}X^{r-1}Y \right] \mod p,$$

$$-a_p h_0 \equiv \left[ \text{Id}, XY^{r-1} - X^{r-1}Y \right] \mod p.$$

Then we check that both the images of $X^{r-1}Y, XY^{r-1}$ in $P$ land in $W_1(\cong V_1^*/V_2^{**}) \subseteq P$, and they map to $-\frac{1}{2}$ and $\frac{1}{2}$ respectively under the surjection $W_1 \twoheadrightarrow J_1 = V_0 \otimes D$.

Therefore $(T - a_p) h_0$ is integral and its image in $\text{ind}_{KZ}^G P$ lands in $\text{ind}_{KZ}^G W_1$, and projects to $-\frac{p}{2a_p} \cdot (T - 2a_p/p) [\text{Id}, 1] \in \text{ind}_{KZ}^G J_1$. So $F_1$ must factor through $\pi(0, 2a_p/p, \omega)$. \hfill $\square$

The following (weak) theorem comes out in the wash:

**Theorem 6.7.** Let $p > 3$, $r \geq 2p$ with $r \equiv 2 \mod p(p-1)$ and $v(a_p) = 1$. If $v \left( a_p^2 - \binom{r}{2}p^2 \right) = 2$, then $\bar{V}_{k,a}^{ss}$ is irreducible, and isomorphic either to $\text{ind}(\omega_2^{p+2})$ or to $\text{ind}(\omega_2^3)$.

**Proof.** Since $r \equiv 2 \mod p$, we see $\lambda = 0$ in Proposition 6.2 (ii). If $F_2 \neq 0$, we obtain the latter possibility. If $F_2 = 0$, then arguing as in the proof of Theorem 6.3, we obtain the former possibility. \hfill $\square$

## 7. Reduction without semi-simplification

In this section we investigate more subtle properties of the reduction $\bar{V}_{k,a_p}$. We shall assume throughout that $p \geq 5$.

### 7.1. Colmez’s Montreal functor

Let $E$ be any finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_E$ and residue field $k_E$. In [Col10], Colmez has defined exact functors $V$ that go from certain categories of representations of $\text{GL}_2(\mathbb{Q}_p)$ on $\mathcal{O}_E$-modules (or $E$-vector spaces, or $k_E$-vector spaces) to categories of representations of $G_{\mathbb{Q}_p}$ with coefficients in $\mathcal{O}_E$ (or $E$, or $k_E$). See [Col10 Intro., 4.] for the precise description of the categories involved, and Théorème IV.2.14 of the same article for a summary of the properties of these functors $V$.

One important property is that these functors are compatible with taking lattices and with reduction modulo $p$. In particular, consider the representation $\Pi_{k,a_p}$ from Section 2 defined say over the finite extension $E$ of $\mathbb{Q}_p$. The functor $V$ attaches to the $E$-representation $\Pi_{k,a_p}$ the Galois representation $V_{k,a_p}$, and if $\Lambda$ is a lattice in $\Pi_{k,a_p}$ stable under the action of $\text{GL}_2(\mathbb{Q}_p)$, hence an $\mathcal{O}_E$-representation of this group, then $V(\Lambda)$ is a lattice in $V_{k,a_p}$. Moreover, the image $V(\bar{\Lambda})$ of the $k_E$-representation $\bar{\Lambda}$ is the reduction of $V(\Lambda)$. 

Hence given a lattice $\Theta$ in $\Pi_{k,a_p}$ we can compute $V(\Theta)$, which gives us the reduction of the lattice $V(\Theta)$ inside $V_{k,a_p}$. This is interesting in the case where we have seen that $V_{k,a_p}$ is (up to a twist) isomorphic to $\omega \oplus 1$. Indeed, in this case there exists inside $V_{k,a_p}$ a lattice that reduces (up to a twist) to a non-split extension of 1 by $\omega$ and one can ask whether this extension is “peu ramifiée” or “très ramifiée” in the sense of [Ser87]. The answer does not depend on the choice of the lattice.

In the context of the results proved so far in this paper, this question arises in three cases:

1. $a = p - 1$ and $p$ does not divide $r + 1$ and $\frac{a_p}{p(r + 1)} = \pm 1$,
2. $a = 2$ and $p$ divides $r$ and $\frac{a_p}{p} = \pm 1$,
3. $a = 2$ and $p$ divides $r - 1$ and $2a_p/p = \pm 1$.

In these cases, the lattice $\Theta = \Theta_{k,a_p}$ we have already studied in Sections I and II does not seem to allow us to answer the question. For instance, in the first case we get a twist of an extension of $\omega$ by 1, and in the second case we get a split extension. However, we can prove:

**Theorem 7.1.** Let $p \geq 5$ and $v(a_p) = 1$. Suppose $p - 1$ divides $r$, $p$ does not divide $r + 1$ and $a_p/p(r + 1) \equiv \pm 1 \mod p$. Then there exists a $GL_2(Q_p)$-stable lattice $\Theta'$ in $\Pi_{k,a_p}$ such that $V(\Theta')$ is a non-split, “peu ramifiée” extension of $\mu_{a_p/p(r+1)}$ by $\omega \mu_{a_p/p(r+1)}$.

### 7.2. Criterion for the extension to be “peu ramifiée”

We give a summary of the results of [Co10] Section VII]. Let $k$ be a finite field of characteristic $p$. Let $H = \text{Hom}_{cont}(Q_p, k)$. In [Co10] Para. VII.4.4, Colmez attaches to any $\tau$ in $H$ a representation $E_\tau$ of $GL_2(Q_p)$ with coefficients in $k$ which is an extension of 1 by the Steinberg representation $St$ and which is non-split if and only if $\tau$ is non-zero. Any non-split extension of 1 by $St$ with coefficients in $k$ (which is smooth, admissible, with a central character) is isomorphic to some $E_\tau$ (Théorème VII.4.18 in [Co10]).

Colmez shows that if $\tau \neq 0$, then there exists a unique non-split extension class of $\pi(p - 3, 1, \omega)$ by $E_\tau$ which we denote by $\Pi_\tau$ (Proposition VII.4.25). Then Colmez’s functor attaches to $\Pi_\tau$ a Galois representation $V(\Pi_\tau)$ which is a non-split extension of 1 by $\omega$ (Proposition VII.4.24) which we denote by $V(\omega, 1, \tau^\perp)$. The description of $V(\omega, 1, \tau^\perp)$ in [Co10] Para. VII] gives:

**Proposition 7.2.** $V(\Pi_\tau)$ is a “peu ramifiée” extension if and only if $\tau$ is zero on $Z_p^\ast$.

### 7.3. A special extension

The representation $\pi(0, 1, 1)$ is a non-split extension of 1 by $St$. We want to know to which $\tau \in H$ it corresponds. We begin by giving a description of this representation in the spirit of [BL95 §3.2].

Let $T$ be the Bruhat-Tits building of $SL_2(Q_p)$. It is a tree with vertices corresponding to homothety classes of lattices in $Q_p^2$. Two lattices $\Lambda$ and $\Lambda'$ are neighbours if and only if, up to homothety, $\Lambda \subset \Lambda'$ and $\Lambda'/\Lambda = Z/pZ$. Let us fix a basis $(e_1, e_2)$ of $Q_p^2$. Let $v_0$ be the class of the lattice $\Lambda_0 = Z_pe_1 \oplus Z_pe_2$ and let $v_n$ be the class of the lattice $\Lambda_n = p^nZ_pe_1 \oplus Z_pe_2$, for $n \in Z$. The group $G$ naturally acts on $T$ by its action on lattices. The vertices are indexed by $KZ \setminus G$ where the class $KZg$ corresponds to the lattice $g^{-1}v_0$.

Let $\gamma$ be the geodesic path in $T$ that links the $(v_n)_{n \in Z}$. Let $\Delta$ be the set of vertices of $T$. We define a function $\delta$ on $\Delta$ as follows. Let $v \in \Delta$ and let $\gamma_v$ be the unique geodesic path from $v_0$ to $v$
in $T$. At first $\gamma_v$ is contained in $\gamma$, then eventually leaves it if $v \not\in \gamma$. We set $\delta(v) = n$ where $n$ is the last integer such that $v_n$ is on $\gamma_v$.

Let $A \subset G$ be the subgroup $\{(0, 0, 1), a \in \mathbb{Q}_p^*\}$. Let $i : \mathbb{Q}_p^* \to A$, $a \mapsto (0, 0, 1)$.

**Lemma 7.3.** For all $v \in \Delta$ and for all $a \in \mathbb{Z}_p^*$, $\delta(i(a)v) = \delta(v)$ and $\delta(i(p)v) = \delta(v) + 1$.

**Proof.** If $a \in \mathbb{Z}_p^*$, then $i(a)v_n = v_n$ for all $n$, and $i(p)v_n = v_{n+1}$. So this is clear when $v$ is on $\gamma$.

Suppose $v \in \Delta$ is not on $\gamma$, and let $\gamma_v$ be the path from $v_0$ to $v$. The vertices on this path are $v_0, \ldots, v_n, v' \ldots, v$ where $n = \delta(v)$ and $v'$ is not on $\gamma$. For $g \in G$, $g\gamma_v$ is the path from $gv_0$ to $gv$.

For $a \in \mathbb{Z}_p^*$, the vertices on $i(a)\gamma_v$ are $v_0, \ldots, v_n, i(a)v', \ldots, i(a)v$ with $i(a)v'$ not on $\gamma$, so $\delta(i(a)v) = \delta(v)$. The vertices on $i(p)\gamma_v$ are $v_1, \ldots, v_{n+1}, i(p)v', \ldots, i(p)v$ with $i(p)v'$ not on $\gamma$. If $n \geq 0$ then the path from $v_0$ to $i(p)v_n$ is the path going from $v_0$ to $v_1$, followed by $i(p)\gamma_v$. If $n < 0$ then the path from $v_0$ to $i(p)v$ is the path obtained by removing the first step of $i(p)\gamma_v$. In both cases, we see that $\delta(i(p)v) = \delta(v) + 1$.

Let $I = \text{ind}^G_{KZ} V_0$. Then $I$ can be identified with the $G$-representation $k(\Delta)$ of functions with values in $k$ and finite support in $\Delta$. The Hecke operator acts on $f \in I$ via: $(Tf)(v) = \sum_{d(v, w) = 1} f(w)$.

The algebraic dual representation $I^\vee$ of $I$ is the representation $k^\Delta$, the set of functions on $\Delta$ with values in $k$. Let $Q = I/(T - 1)$, so that $Q$ is the representation $\pi(0, 1, 1)$. Then $Q^\vee \subset I^\vee$ is the set of harmonic functions on $\Delta$: $Q^\vee = \{(\lambda_v)_{v \in \Delta} \in k^\Delta, \lambda_v = \sum_{d(w, v) = 1} \lambda_w\}$.

There is a special element in $I^\vee$ which is the degree function deg given by $\lambda_v = 1$ for all $v$. This function is in fact in $Q^\vee$ as each vertex of $\Delta$ has $p + 1$ neighbours. It follows from [BL05] Lem. 30 that the kernel of deg inside $Q$ is the representation $St$. Let $e \in I^\vee$ be defined by $e_v = \delta(v) \in k$. Then $e$ is in fact in $Q^\vee$, and the image of $e$ modulo the line generated by deg is an element of $St^\vee$ that is invariant under the action of $A$. For all $a \in \mathbb{Q}_p^*$, $i(a)e - e$ defines a linear form on $Q/\text{St} = 1$ hence is a well-defined scalar. We define $\tau \in H$ by $\tau(a) = i(a)e - e$. We see that $\tau(\mathbb{Z}_p^*) = 0, \tau(p) = 1$.

Following the description of [Co110] VII.4.4, we see that:

**Proposition 7.4.** The representation $Q = \pi(0, 1, 1)$ is isomorphic to the representation $E_\tau$ of Colmez for $\tau \in H$ defined by $\tau(\mathbb{Z}_p^*) = 0$ and $\tau(p) = 1$.

**Corollary 7.5.** Let $\Pi$ be the non-split extension of $\pi(p - 3, 1, \omega)$ by $\pi(0, 1, 1)$. Then $V(\Pi)$ is a non-split, “peu ramifiée” extension of $1$ by $\omega$.

7.4. The case where $a = p - 1$, $p$ does not divide $r + 1$ and $a_p/p(r + 1) \equiv \pm 1$ (mod $p$). Let $\varepsilon = a_p/p(r + 1)$ modulo $p$, so that $\varepsilon = \pm 1$. Let $r \geq 2p$ be divisible by $p - 1$.

7.4.1. Another lattice. Consider the lattice $V_r = \text{Sym}^r \mathbb{Z}_p^2 + \eta \text{Sym}^{r-(p+1)} \mathbb{Z}_p^2$ in $\text{Sym}^r \mathbb{Q}_p^2$, where $\eta = 0/p$. This lattice is stable under the action of $K$, as for all $\gamma \in K$, we have $\gamma \eta \in \mathbb{Z}_p\eta + \text{Sym}^{r+1} \mathbb{Z}_p^2$.

Let $\Theta'$ be the image of $\text{ind}^G_{KZ} V_r$, inside $\text{ind}^G_{KZ} \text{Sym}^r \mathbb{Q}_p^2/(\langle T - a_p \rangle \text{ind}^G_{KZ} \text{Sym}^r \mathbb{Q}_p^2 \cap \text{ind}^G_{KZ} V_r)$ and let $\overline{\Theta'}$ be its reduction modulo $m_{\mathbb{Q}_p}$, where $\overline{\Theta'}$ is the reduction of $\Theta'$ modulo $m_{\mathbb{Q}_p}$. We denote by $\pi'$ the map $\text{ind}^G_{KZ} V_r \to \overline{\Theta'}$, and by $\pi'$ the map $\text{ind}^G_{KZ} V_r \to \overline{\Theta'}$ where $\overline{\Theta'}$ is the reduction of $\Theta'$ modulo $m_{\mathbb{Q}_p}$.
Theorem 7.6. \( V(\Theta) \) is a non-split, “peu ramifiée” extension of \( \mu_2 \) by \( \omega_2 \).

**Proof.** This follows from Lemma 7.11 and Lemma 7.12 and from Corollary 7.5. \( \square \)

7.4.2. Factors of \( \bar{V}_r \). We define some submodules of \( V_r \).

First, let \( M_1 \) be the submodule \( \text{Sym}^r \mathbb{Z}_p^2 \) of \( \bar{V}_r \). Let \( M_0 \) be the \( \mathbb{Z}_p[K] \)-submodule generated by \( X^r \) inside \( M_1 \). Let \( N \) be the \( \mathbb{Z}_p \)-submodule \( \eta \theta^2 \text{Sym}^{r-3(p+1)} \mathbb{Z}_p^2 \) if \( r \geq 3(p+1) \), and \( N = 0 \) otherwise.

Finally, let \( M_2 = M_1 + N \).

**Lemma 7.7.** The submodules \( M_0, M_1 \) and \( M_2 \) are stable under the action of \( K \).

**Proof.** For \( M_0 \) and \( M_1 \), this is clear from the definition. For \( M_2 \), this follows from the argument used to show that \( V_r \) is stable under the action of \( K \). \( \square \)

Let \( M_0, M_1, N \) and \( M_2 \) be the images in \( \bar{V}_r \) of \( M_0, M_1, N \) and \( M_2 \) respectively. Hence we get a filtration:

\[
0 \subset M_0 \subset M_1 \subset M_2 \subset \bar{V}_r.
\]

**Lemma 7.8.** \( M_1 \) is isomorphic as a \( K \)-representation to \( V_r/V_r^* \). In this isomorphism, \( M_0 \) corresponds to the submodule generated by \( X^r \). In particular, \( M_0 \) is isomorphic to \( V_{p-1} \) and \( M_1/M_0 \) is isomorphic to \( V_0 \). Moreover, a basis of \( M_1 \) is given by the images of the elements \( X^r \) and \( X^i Y^{r-i} \), for \( 0 \leq i \leq p-1 \), and a basis of \( M_0 \) is given by the images of the elements \( X^r \) and \( X^i Y^{r-i} \), for \( 0 \leq i \leq p-2 \).

**Proof.** \( M_1 \) is the image in \( \bar{V}_r \) of \( \hat{M}_1 \), hence is a quotient of \( V_r \). To see which quotient, we must compute \( (m_{\mathbb{Q}_p} \bar{V}_r) \cap \hat{M}_1 \). As a \( \mathbb{Z}_p \)-module, \( \bar{V}_r \) has a decomposition \( \mathbb{Z}_p X^r \oplus (\oplus_{i=0}^{p-1} \mathbb{Z}_p X^i Y^{r-i}) \oplus S \) where \( S = \eta \text{Sym}^{r-3(p+1)} \mathbb{Z}_p^2 \). In this decomposition, \( \hat{M}_1 = \tilde{\mathbb{Z}}_p X^r \oplus (\oplus_{i=0}^{p-1} \mathbb{Z}_p X^i Y^{r-i}) \oplus pS \), and \( pS \) is \( \theta^r \text{Sym}^{r-3(p+1)} \mathbb{Z}_p^2 \). Hence \( (m_{\mathbb{Q}_p} \bar{V}_r) \cap \hat{M}_1 = m_{\mathbb{Q}_p} X^r \oplus (\oplus_{i=0}^{p-1} m_{\mathbb{Q}_p} X^i Y^{r-i}) \oplus pS \). So finally, \( M_1 \) is isomorphic to \( V_r/V_r^* \). The reasoning for \( M_0 \) is similar. \( \square \)

**Lemma 7.9.** The Jordan-Hölder factors of \( \bar{V}_r/M_2 \) are contained in the set \( \{ V_{p-5} \otimes D^2, V_4 \otimes D^{p-3}, V_{p-3} \otimes D, V_2 \otimes D^{p-2} \} \).

**Proof.** In the notation of the previous Lemma, \( \bar{V}_r/M_1 \) is isomorphic to \( S/m_{\mathbb{Q}_p} S \), that is, \( \theta V_{r-(p+1)} \).

Suppose \( r \geq 3(p+1) \). Then the submodule \( \theta^3 V_{r-3(p+1)} \) is sent to the image of \( N \) in this isomorphism, hence \( \bar{V}_r/M_2 \) is a quotient of \( \theta V_{r-(p+1)}/\theta^3 V_{r-3(p+1)} \), whose JH factors comprise the set stated in the lemma.

Suppose \( r < 3(p+1) \). Then either \( r = 3p-3 \) or \( p = 5 \) and \( r = 16 \), as \( p-1 \) divides \( r \). When \( r = 3p-3 \) then \( M_1 = M_2 \) and the JH factors of \( \bar{V}_r/M_2 \) are given by the set \( \{ V_{p-5} \otimes D^2, V_{p-3} \otimes D, V_2 \otimes D^{p-2} \} \). When \( p = 5 \) and \( r = 16 \) then \( M_1 = M_2 \), and the JH factors of \( \bar{V}_{16}/M_2 \) are given by the set \( \{ V_4 \otimes D^2, V_2 \otimes D, V_2 \otimes D^3 \} \). \( \square \)
7.4.3. Study of $\pi'$.

**Proposition 7.10.** We have:

1. For all $F \in \text{Sym}^{r - 3(p + 1)}\mathbb{Z}^2_p$, and all $g \in G$, we have $\pi'( [g, \eta g^2 F]) = 0$.
2. $\text{ind}_{KZ}^G M_0 \subset \ker \pi'$.

**Proof.** For (1): Given $F \in \text{Sym}^{r - 3(p + 1)}\mathbb{Z}^2_p$, we consider $f = (1/p^2)[g, \eta g^2 F] \in \mathcal{V}_r$. Then the computation of $(T - a_p)f$ shows that the image of $[g, \eta g^2 F]$ in $\overline{\Theta}'$ is zero.

For (2): we consider $f = [\text{Id}, (\theta/X)Y^{r-p}]$. \hfill $\Box$

We already know from the computations in Section 1 with the lattice $\Theta$ which is the image of $\text{Sym}^r \mathbb{Z}^2_p$ that the JH factors that appear in $\overline{\Theta}'$ are $\mu_\varepsilon$, $\text{St} \otimes \mu_\varepsilon$ and $\pi(p - 3, \varepsilon, \omega)$.

By Proposition 7.10 (1), the images in $\overline{\Theta}'$ of $\text{ind}_{KZ}^G M_1$ and $\text{ind}_{KZ}^G M_2$ are the same. Hence all the JH factors of $\overline{\Theta}'$ are subquotients of $\text{ind}_{KZ}^G M_1$ or of $\text{ind}_{KZ}^G (\widehat{\mathcal{V}}_r/M_2)$.

Using Lemma 7.8 and $p > 3$ we see that the factor $\pi(p - 3, \varepsilon, \omega)$ appears as a subquotient of $\text{ind}_{KZ}^G (\widehat{\mathcal{V}}_r/M_2)$. The factors $\mu_\varepsilon$ and $\text{St} \otimes \mu_\varepsilon$ can only appear as quotients of $\text{ind}_{KZ}^G \mathcal{V}_0$, or $\text{ind}_{KZ}^G \mathcal{V}_{p-1}$. Hence from Lemma 7.9 we see that these factors can only appear as subquotients of $\text{ind}_{KZ}^G M_1$.

Moreover, by Proposition 7.10 (2), the image of $\text{ind}_{KZ}^G M_0$ is zero. Hence both factors $\mu_\varepsilon$ and $\text{St} \otimes \mu_\varepsilon$ appear as subquotients of $\text{ind}_{KZ}^G (M_1/M_0) \cong \text{ind}_{KZ}^G \mathcal{V}_0$. The only possibility is then that the image of $\text{ind}_{KZ}^G (M_1/M_0)$ in $\overline{\Theta}'$ is isomorphic to $\pi(0, \varepsilon, 1)$, as the only quotient of $\text{ind}_{KZ}^G \mathcal{V}_0$ that has both these JH factors is $\pi(0, \varepsilon, 1)$. Hence $\pi(0, \varepsilon, 1)$ is a subrepresentation of $\overline{\Theta}'$.

We deduce from this:

**Lemma 7.11.** There is an extension: $0 \rightarrow \pi(0, \varepsilon, 1) \rightarrow \overline{\Theta}' \rightarrow \pi(p - 3, \varepsilon, \omega) \rightarrow 0$.

7.4.4. Study of the extension.

**Lemma 7.12.** The extension of Lemma 7.11 is non-split.

**Proof.** Suppose that the extension is split. Then there exists a $G$-equivariant projection $\overline{\Theta}' \rightarrow \pi(0, \varepsilon, 1)$. As $\pi(0, \varepsilon, 1)$ has a quotient $\chi = \mu_\varepsilon \circ \det$ which is of dimension 1, there exists a $G$-equivariant non-zero map $\overline{\Theta}' \rightarrow \chi$. By composing with $\pi'$ we get a non-zero map $\phi : \text{ind}_{KZ}^G \mathcal{V}_r \rightarrow \chi$.

Moreover, the restriction of $\phi$ to $\text{ind}_{KZ}^G M_1$ is still non-zero as $\pi(0, \varepsilon, 1)$ is the image of $\text{ind}_{KZ}^G M_1$ by $\pi'$. But the restriction of $\phi$ to $\text{ind}_{KZ}^G M_0$ is zero, as the image of $\text{ind}_{KZ}^G M_0$ inside $\overline{\Theta}'$ is zero.

Let $s = r - (p + 1)$ and denote by $z$ the image of $\eta Y^s$ in $\mathcal{V}_r$. We have $\phi([\text{Id}, z]) = \phi([\text{Id}, (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}) z])$ as $\det (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}) = 1$. Let us compute $(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}) z$. By computing in $\mathcal{V}_r$ and reducing modulo $m_{\mathcal{V}_r}$, we see that $(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}) z = z + \sum_{i=1}^{p-1} \delta_i X^i Y^{r-i}$ where $\delta_i = (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})/p$, for $1 \leq i \leq p-1$. Hence $\sum_{i=1}^{p-1} \delta_i \phi([\text{Id}, X^i Y^{r-i}]) = 0$.

For $1 \leq i \leq p - 2$, the image of $X^i Y^{r-i}$ inside $\mathcal{V}_r$ is in $M_0$ (see Lemma 7.8). Hence $\phi([\text{Id}, X^i Y^{r-i}]) = 0$, for $1 \leq i \leq p - 2$. The previous equality reduces to $\delta_{p-1} \phi([\text{Id}, X^{p-1} Y^{r-p+1}]) = 0$. Hence $\phi([\text{Id}, X^{p-1} Y^{r-p+1}]) = 0$. But $X^{p-1} Y^{r-p+1}$ generates $M_1/M_0$ as a $K$-representation so we deduce that the restriction of $\phi$ to $\text{ind}_{KZ}^G M_1$ is zero.

We arrive at a contradiction, hence the extension is in fact not split. \hfill $\Box$
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