On topological models of zero entropy loosely Bernoulli systems

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(j.w. with Dominik Kwietniak)
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- There are many connections and parallels between the theory of MPSs and TDSs (ergodic theory and topological dynamics).
Basic relationships

- If we start with a TDS \((X, T)\), there exists a Borel \(T\)-invariant probability measure \(\mu\) (Krylov-Bogoliubov), and hence we obtain a MPS \((X, \mu, T)\).
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- A TDS is **uniquely ergodic** if there is only one Borel \(T\)-invariant probability measure.
- If we start with a MPS \((X, \Sigma, \mu, T)\), there exists a uniquely ergodic TDS \((X', \mu', T')\) isomorphic to \((X, \Sigma, \mu, T)\) (Jewett-Krieger).
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- In this case we say \((X', T')\) is a **topological model for** \((X, \Sigma, \mu, T)\) (uniquely ergodic).
Relationship

- Analogous properties
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  - **Measurable**
    - Entropy
    - Mixing
  - Discrete spectrum on $L^2(X, \mu)$
    - Measure distal
    - $K$-system
  - **Topological**
    - Topological entropy
    - Topologically mixing
  - Discrete spectrum on $C(X)$
    - Distal
    - Completely positive top. entropy
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| MPS                  | Uniquely ergodic TDS                           |
|----------------------|-----------------------------------------------|
| Mixing               | \(\Rightarrow\) Topologically mixing         |
| \(K\)-system        | \(\Rightarrow\) Completely positive top. entropy |
| Measure distal       | \(\Leftarrow\) Distal                         |
| Discrete spectrum on \(L^2(X, \mu)\) | \(\Leftarrow\) Discrete spectrum on \(C(X)\) |
Questions

- Are there any correspondences between isomorphism-invariant properties of ergodic MPSs and conjugacy-invariant properties of their topological models?
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In this talk we will give a topological "version" of zero entropy loosely Bernoulli systems, to give an answer to this question.

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- An ergodic MPS is Kronecker if and only if the (induced) Koopman operator on $L^2(X, \mu)$ has discrete spectrum (Halmos - von Neumann).
- Two Kronecker MPSs are isomorphic if and only if they are spectrally isomorphic (von Neumann).
von Neumann asked if the uniform Bernoulli measure on $\{0, 1\}^\mathbb{Z}$ is isomorphic to the uniform Bernoulli measure on $\{0, 1, 2\}^\mathbb{Z}$. This question was answered by Kolmogorov using entropy ($\log_2 3 = \log_3 2$). Then, Ornstein proved that Bernoulli systems with the same entropy are always isomorphic. In order to understand the isomorphism class of Bernoulli systems, Ornstein introduced the notions of \textit{…initely determined} and \textit{very weak} Bernoulli. At the heart of these definitions lies the Hamming ($d$) metric on words $d(x_1 \ldots x_n, y_1 \ldots y_n) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{x_i \neq y_i}$. This metric is also used in information theory as a way to measure "mistake" noise.
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Now take the periodic measure $\mu = \delta_{(01)^\infty}/2 + \delta_{(10)^\infty}/2$.

Here typical words that start with 0 will be very different to words that begin with 1.
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Ornstein-Rudolph-Weiss approached the isomorphism problem from a different perspective.
The edit ($\bar{f}$) metric is defined as follows

$$\bar{f}(x_1 \ldots x_n, y_1 \ldots y_n) = 1 - \frac{k}{n},$$

where $k$ is the largest integer such that for some

$$1 \leq i(1) < i(2) < \ldots < i(k) \leq n \text{ and }$$
$$1 \leq j(1) < j(2) < \ldots < j(k) \leq n$$

we have that $x_{i(s)} = y_{j(s)}$ for $s = 1, \ldots, k$. 
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- For example $\overline{f}(010101, 001010) = 1 - 5/6 = 1/6$, and $d(010101, 101010) = 0$. 

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- Actually, they proved that if $0 < h_\mu(T), h_\mu(T'), < \infty$ then they are also Kakutani equivalent (and the $A$ can be taking arbitrarily large).
- It is easy to see that (very weak) Bernoulli systems are loosely Bernoulli (because $\overline{f} \leq \overline{d}$)
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A loosely Bernoulli system has zero entropy if and only if it is loosely Kronecker (Feldman, Katok).
Zero entropy loosely Bernoulli

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We will now describe topological models for loosely Kronecker systems.
Dynamical pseudometrics

Let \((X, T)\) be a TDS (with metric \(d\)). We say \(\rho\) is a dynamical pseudo-metric on \(X\), if \(\rho(x, y) = \rho(Tx, Ty)\).
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- We define the **Besicovitch pseudometric** as

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\rho_B = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(T^i x, T^i y).
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- Even though Kronecker systems are very well understood from a measurable point of view, only recently we are understanding the range of topological behaviours that the models can have (Downarowicz, GR, Glasner, Jägger, Li, Thouvenot, Ye). More on this later.
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- The Besicovitch pseudometric is particularly useful for understanding some families of the models for Kronecker MPSs.
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- Given a TDS and two points $x, y \in X$, we say $S, S' \subset \mathbb{N}$ are $\delta$-matched if there exists a bijective order preserving function $\pi : S \rightarrow S'$ such that $d(T^i x, T^{\pi(i)} y) \leq \delta$. 
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- Given a TDS and two points $x, y \in X$, we say $S, S' \subset \mathbb{N}$ are $\delta$-matched if there exists a bijective order preserving function $\pi : S \rightarrow S'$ such that $d(T^i x, T^{\pi(i)} y) \leq \delta$.
- For $S \subset \mathbb{N}$, $\overline{D}(S) = \limsup_{n \to \infty} \frac{|\{S \cap \{1, \ldots, n\}\}|}{n}$ is the upper density.
Dynamical pseudometrics

- So, if one can consider the Besicovitch pseudometric as an infinite topological Hamming distance.
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- For $S \subseteq \mathbb{N}$, $\overline{D}(S) = \limsup_{n \to \infty} \left| \{ S \cap \{1, \ldots, n\} \} \right| / n$ is the upper density.
- We define the **Feldman-Katok** pseudometric as $\rho_{FK}(x, y) = \inf \{ \delta > 0 : \exists \ \delta$-matched $S, S'$ with $\overline{D}(S'), \overline{D}(S) \geq 1 - \delta \}$.
Theorem 1 (GR-Kwietniak) Let \((X, T)\) be a TDS and \(\mu\) be an ergodic \(T\)-invariant Borel probability measure. Then \((X, \mu, T)\) is loosely Kronecker if and only if there exists a Borel set \(M \subset X\) with \(\mu(M) = 1\) such that \(\rho_{FK}(x, y) = 0\) for every \(x, y \in M\).
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- MPS
  - Zero entropy loosely Bernoulli \(\Leftrightarrow\) topologically loosely Kronecker

F. García-Ramos ()
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Every measure distal MPS has a distal topological realization \textit{(isomorphic for some measure)}, but in some cases these realizations are never uniquely ergodic (Lindenstrauss).

**LK:** There is only one topological analogue of loosely Kronecker; in this case we have that every model is topologically loosely Kronecker.
Models for Kronecker

- We will now describe some topological results for Kronecker systems.
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A TDS is **equicontinuous** if \( \{ T^i \}_{i \in \mathbb{N}} \) is equicontinuous, i.e. for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( d(x, y) \leq \delta \) then \( d(T^i x, T^i y) \leq \varepsilon \) for all \( i \geq 0 \).
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This is essentially the notion of mean-L-stability, introduced by Fomin and studied by Auslander and Oxtoby.
**Theorem** (GR) Let \((X, T)\) be a TDS and \(\mu\) be an ergodic \(T\)-invariant Borel probability measure. \((X, \mu, T)\) is Kronecker if and only if for every \(\tau > 0\) there exists a Borel set \(M \subset X\) with \(\mu(M) \geq 1 - \tau\) such that \(T|_M\) is mean equicontinuous (GR).
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Not every model for Kronecker is mean equicontinuous. It is not known if unique ergodicity gives extra information for the support of Borel Kronecker measures.
We will now briefly explain the hierarchy of topological models for Kronecker systems.
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**Theorem** (GR-Jäger-Ye). A minimal TDS is diam-mean equicontinuous if and only if the maximal equicontinuous factor is almost surely 1-1 (i.e. $\nu(\{x \in X_{eq} : |\pi_{eq}^{-1}(x)| = 1\}) = 1$).

**Theorem** (Glasner, Fuhrmann-Glasner-Jäger-Oertel) Every minimal tame system is diam-mean equicontinuous.

**Theorem** (Kerr-Li) Every null TDS (zero top. sequence entropy) is tame.

We have the following (strict) hierarchy for minimal TDS.

- **null**
- **tame**
- **diam-mean equicontinuous**
- **mean equicontinuous**
- **Kronecker**.
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F. García-Ramos () CONACyT, UASLP (j.w. with Dominik Kwietniak) 20/27
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- equicontinuity $\Rightarrow$ null $\Rightarrow$ tame $\Rightarrow$ diam-mean equicontinuous $\Rightarrow$ mean equicontinuous $\Rightarrow$ Kronecker.
In the second part we will give a sketch of the proof

$(X, \mu, T)$ is ergodic loosely Kronecker $\Rightarrow$ there exists a Borel set

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Thank you
Sketch

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Claim 1: If $(X, \mu, T)$ is ergodic and loosely Kronecker, then for every $\epsilon > 0$, there exists a Borel set $B$ with positive measure such that $\rho_{FK}(x, y) \leq \epsilon$. 

Proof of claim: Since $(X, \mu, T)$ is LK, there exists a compact abelian group $G$ and $g \in G$, so that $(G, \nu, R)$ is Kakutani equivalent to $(X, \mu, T)$, where $\nu$ is the Haar measure on $G$ and $Rx = g \cdot x$ isometry.
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\((X, \mu, T)\) is ergodic loosely Kronecker \implies there exists a Borel set \(M \subseteq X\) with \(\mu(M) = 1\) such that \(\rho_{FK}(x, y) = 0\) for every \(x, y \in M\).

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where \(\nu_h\) is the Haar measure on \(G\) and \(Rx = g \cdot x\) (isometry).
Proof

Let $\varepsilon > 0$. By ORW, there exists a Borel set $B \subset G$ with $\mu_h(B) \geq 1 - \varepsilon/2$, such that $(X, \mu, T)$ is isomorphic to $(A, \nu_A, R_A)$, using $\phi : X \to A$. 

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By Lusin’s theorem there exists a compact sets $M \subset X$ with $\mu(M)\nu(A) \geq 1 - \varepsilon/3$ such that $\phi|_M : M \rightarrow \phi(M)$ is (uniformly) continuous.
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- Now, let \( Y_M \subseteq \phi(M) \) be the set of points in \( \phi(M) \) which are
  \( \nu \)-generic for \( \phi(M) \) with respect to the map \( S \).
Proof

- Let $\varepsilon > 0$. By ORW, there exists a Borel set $B \subset G$ with $\mu_h(B) \geq 1 - \varepsilon/2$, such that $(X, \mu, T)$ is isomorphic to $(A, \nu_A, R_A)$, using $\phi : X \to A$.

- By Lusin’s theorem there exists a compact sets $M \subset X$ with $\mu(M)\nu(A) \geq 1 - \varepsilon/3$ such that $\phi|_M : M \to \phi(M)$ is (uniformly) continuous.

- There exists $\delta > 0$ is such that if $x, y \in M$ and $d(x, y) \leq \delta$ then $d(\phi^{-1}R^n\phi(x), \phi^{-1}R^n\phi(y)) \leq \varepsilon$ for every $n \in \mathbb{N}$ with $R^n\phi(x), R^n\phi(x) \in \phi(M)$ (1).

- Now, let $Y_M \subset \phi(M)$ be the set of points in $\phi(M)$ which are $\nu$-generic for $\phi(M)$ with respect to the map $S$.

- Thus, we have $\nu(Y_M) = \nu(\phi(M))$, so there is $z \in M$ such that

$$\mu(B_{\delta/2}(z) \cap M \cap \phi^{-1}(Y_M)) > 0.$$
Sketch

- Let $x, y \in B_{\delta/2}(z) \cap M \cap \phi^{-1} Y_M$ (this is our $B$). We will show that $ho_{FK}(x, y) \leq \varepsilon$. 
Let $x, y \in B_{\delta/2}(z) \cap M \cap \phi^{-1}Y_M$ (this is our $B$). We will show that $\rho_{FK}(x, y) \leq \varepsilon$.

Strategy: find a $\delta$-match for the $R_A$ orbits of $\phi(y)$ and $\phi(z)$ that only matches points on $\phi(Y_M)$. 

Use (1) to get $\varepsilon$-match for the $T$ orbits of $x$ and $y$. (see picture)
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One can check that for every $i \in S$ there exists $j$ such that
$$d_G(R^i_B \phi(x), R^\pi(i)_B \phi(y)) = d_G(R^j_B \phi(x), R^j_B \phi(y)) = d(\phi(x), \phi(y)) \leq \delta'.$$

With this we obtain a $\delta_0$-match for the $RA$ orbits of $\phi(y)$ and $\phi(z)$, matches points on $\phi(M)$.
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With this we obtain a $\delta'$-match for the $R_A$ orbits of $\phi(y)$ and $\phi(z)$, matches points on $\phi(M)$.

Since $y, z \in \phi^{-1} Y_M$ then

$$\overline{D}(S), \overline{D}(S') \geq \overline{D}(E_M(y, z)) \geq 1 - \varepsilon.$$ 

Thus $\rho_{FK}(y, z) \leq \varepsilon$ (we finish the claim).
Now we can use the fact that $\rho_{FK}(x, Tx) = 0$, to prove that there exists a Borel set $M_\varepsilon \subset X$ with $\mu(M_\varepsilon) = 1$ such that $\rho_{FK}(x, y) \leq \varepsilon$ for every $x, y \in M_\varepsilon$. 

We conclude the result.

Note that if we assume that the map is uniquely ergodic we do not get directly that $\rho_{FK}(x, y) = 0$ for every $x, y \in M_\varepsilon$. This proof has to be done with a different approach.
Proof

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