ASYMPTOTIC PROFILES OF THE ENDEMIC EQUILIBRIUM
OF A REACTION-DIFFUSION-ADVECTION SIS EPIDEMIC
MODEL WITH SATURATED INCIDENCE RATE

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ABSTRACT. In this paper, we consider a reaction-diffusion SIS epidemic model
with saturated incidence rate in advective heterogeneous environments. The
existence of the endemic equilibrium (EE) is established when the basic re-
production number is greater than one. We further investigate the effects
of diffusion, advection and saturation on asymptotic profiles of the endemic
equilibrium. The individuals concentrate at the downstream end when the
advection rate tends to infinity. As the the diffusion rate of the susceptible
individuals tends to zero, a certain portion of the susceptible population con-
centrates at the downstream end, and the remaining portion of the susceptible
population distributes in the habitat in a non-homogeneous way; on the other
hand, the density of infected population is positive on the entire habitat. The
density of the infected vanishes on the habitat for small diffusion rate of infected
individuals or the large saturation. The results may provide some implications
on disease control and prediction.

1 Introduction

1.1 SIS epidemic reaction-diffusion model It has been widely recognized
that spatial heterogeneity and individual diffusion are significant factors that should
be taken into account in studying disease dynamics. In recent years, for the
susceptible-infected-susceptible (SIS) epidemic reaction-diffusion models, more and
more works have been devoted to study the effect of the spatial heterogeneity of
environment and movement of individuals on dynamics of diseases [1, 9, 10, 11, 12,
13, 14, 15, 17, 20, 21, 27, 28, 29, 30, 38, 39, 42, 44, 52, 53, 54, 55].

The classic SIS reaction-diffusion epidemic model is as follows:

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= d_S \Delta S(x, t) - \beta(x) F(S, I) + \gamma(x) I, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial I(x, t)}{\partial t} &= d_I \Delta I(x, t) + \beta(x) F(S, I) - \gamma(x) I, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

(1)

where $S(x, t)$ and $I(x, t)$ are the densities of susceptible and infected individuals
at position $x$ and time $t$ respectively; $\Omega$ is a bounded domain in $\mathbb{R}^m$ ($m \geq 1$); the
positive constants \( d_S \) and \( d_I \) are diffusion rates for the susceptible and infected individuals; the positive functions \( \beta(x) \) and \( \gamma(x) \) represent the rates of disease transmission and recovery at location \( x \), respectively; \( F(\bar{S}, \bar{I}) \) denotes the infection mechanism.

In [1], Allen et al. studied the model (1) with Neumann boundary condition when the infection mechanism is the standard incidence rate, i.e., \( F(\bar{S}, \bar{I}) = \bar{S} \bar{I}/(\bar{S} + \bar{I}) \). Clearly, the model (1) satisfies the conservation law \( \int_\Omega (\bar{S} + \bar{I}) \, dx = N \), where the positive constant \( N \) is the total population number. The main results of [1] concerned the existence, uniqueness and asymptotic behaviors of the endemic equilibrium as the diffusion rate of the susceptible individuals approaches to zero. For this model, Peng and his collaborators discussed the global stability of the endemic equilibrium in [39] and analyzed the effects of large and small diffusion rates of the susceptible and infected individuals on the persistence and extinction of the disease in [38, 42]. In [44], Peng and Zhao considered the same SIS reaction-diffusion model, but the rates of disease transmission and recovery were assumed to be spatial heterogeneous and temporally periodic. For the SIS model (1) with the mass action infection mechanism (or called density-dependent infection mechanism), i.e, transmission function \( F(\bar{S}, \bar{I}) = \bar{S} \bar{I} \), Deng and Wu in [13] studied the global dynamics and existence of endemic equilibrium, while [54, 55] investigated the asymptotic profiles of endemic equilibrium as the diffusion rate of susceptible or infected populations is small or large. In consideration the infection force of the disease, Capasso and Serio [5] described the saturated incidence rate mechanism \( F(\bar{S}, \bar{I}) = \bar{S} \bar{I}/(1 + m\bar{I}) \), where the positive constant \( m \) is the saturation coefficient. In recent work [53], the authors consider the SIS epidemic model with saturated rate. The main results concern the existence of endemic equilibrium and analyze the effect of diffusion rates and the saturated coefficient on asymptotic profiles of the endemic equilibrium.

It should be noticed that one of the main features of above models is that the total population number is conserved. Thus, it natural to consider the scenario that the susceptible individuals are allowed to have birth and death. There have been many studies on the effect of vary total population on disease persistence, see [12, 20, 27, 28, 29, 30, 31, 41, 46, 48, 56]. These research analyse how the factors such as infection mechanism, the mobility of population migration and source term affect the persistence or extinction of infectious diseases. Such interesting results will help people to predict the pattern of disease occurrence and conduct effective/optimal control strategies of disease eradication.

1.2 SIS epidemic reaction-diffusion-advection model In some circumstances populations sometimes are forced to move in certain direction, for example, population flows with water in streams, rivers, or along water column, or population moves following wind direction or towards more favorable climate zone. More recently, there is growing interest in investigating spatial population dynamics in advective environments (see e.g. [2, 6, 22, 23, 24, 25, 26, 34, 35, 36, 47, 49, 50, 51, 59, 60, 61, 62]). The important question then arise: how could the diffusion and advection jointly affect the dynamics of the infectious diseases?

In [9, 10, 11, 21], the SIS epidemic reaction-diffusion-advection models in one dimensional domain were considered:

\[
\begin{align*}
\bar{S}_t &= d_S \bar{S}_{xx} - q \bar{S}_x - \beta(x) F(\bar{S}, \bar{I}) + \gamma(x) \bar{I}, \quad 0 < x < L, \; t > 0, \\
\bar{I}_t &= d_I \bar{I}_{xx} - q \bar{I}_x + \beta(x) F(\bar{S}, \bar{I}) - \gamma(x) \bar{I}, \quad 0 < x < L, \; t > 0, \\
d_S \bar{S}_x - q \bar{S} &= d_I \bar{I}_x - q \bar{I} = 0, \quad x = 0, L, \; t > 0,
\end{align*}
\]
where the nonnegative constant $q$ is the advection speed, the no-flux boundary conditions are imposed at the boundary $x = 0$ and $x = L$, respectively.

When the infection mechanism $F(S, I)$ is the standard incidence rate $\bar{S}I/(\bar{S} + \bar{I})$, [11] derives the basic reproduction number as follows:

$$\mathcal{R}_0(d_I, q) = \sup_{\varphi \in H^1((0, L))} \left\{ \frac{\int_0^L \beta(x) e^{\frac{\varphi(x)}{d_I}} \varphi^2 \, dx}{d_I \int_0^L e^{\frac{\varphi(x)}{d_I}} \varphi^2 \, dx + \int_0^L \gamma(x) e^{\frac{\varphi(x)}{d_I}} \varphi^2 \, dx} \right\},$$

and the basic reproduction number serves as a threshold parameter. If $\mathcal{R}_0 < 1$, the DFE is globally asymptotically stable. The existence of EE was established when $\mathcal{R}_0 > 1$ in [9], and the asymptotic behavior of EE (provided it exists) with respect to $d_S, d_I$, and $q$ was investigated in [9, 21]. In consideration of the infection mechanism is the mass action mechanism $F(S, I) = \bar{S}I$, [10] shows that the existence of EE whenever $\mathcal{R}_0 > 1$ and analyzes the asymptotic behavior of EE in three cases: large advection; small diffusion of the susceptible individual; small diffusion of the infected individual. These results suggest that advection can help speed up the elimination of disease.

In this paper we lead to investigate a reaction-diffusion-advection SIS epidemic system with saturated incidence rate. Our model is described by the following system of equations:

$$\begin{cases}
\bar{S}_t = d_S \bar{S}_{xx} - q \bar{S}_x - \beta(x) \frac{\bar{S} \bar{I}}{1 + m \bar{I}} + \gamma(x) \bar{I}, & 0 < x < L, \ t > 0, \\
\bar{I}_t = d_I \bar{I}_{xx} - q \bar{I}_x + \beta(x) \frac{\bar{S} \bar{I}}{1 + m \bar{I}} - \gamma(x) \bar{I}, & 0 < x < L, \ t > 0, \\
d_S \bar{S}_x - q \bar{S} = d_I \bar{I}_x - q \bar{I} = 0, & x = 0, L, \ t > 0, \\
\bar{S}(x, 0) = \bar{S}_0(x) \geq 0, \ \bar{I}(x, 0) = \bar{I}_0(x) \geq 0, & 0 < x < L,
\end{cases} \tag{3}$$

where $\bar{S}(x, t)$ and $\bar{I}(x, t)$, respectively, represent the density of susceptible and infected individuals at location $x \in [0, L]$ and time $t > 0$; $L$ is the size of the habitat, and we call $x = 0$ the upstream end and $x = L$ the downstream end; the positive constants $d_S$ and $d_I$ are diffusion coefficients for the susceptible and infected individuals; the positive constant $m$ is the saturated incidence rate; the nonnegative constant $q$ denotes the advection speed/rate which carries the susceptible and infected individuals from the upstream $x = 0$ to the downstream $x = L$; the positive functions $\beta(x)$ and $\gamma(x)$ are Hölder continuous on $[0, L]$ and represent the rates of disease transmission and disease recovery at $x$, respectively. Here we impose no-flux boundary conditions at the upstream and downstream ends, respectively. It means that there is no individual net flux across the boundary $x = 0$ and $x = L$. As mentioned in [11], since the term $\bar{S} \bar{I}/(1 + m \bar{I})$ is a Lipschitz continuous function of $\bar{S}$ and $\bar{I}$ in the open first quadrant, its definition can be extended to the closure of the first quadrant by setting it to be zero when either $\bar{S} = 0$ or $\bar{I} = 0$. We also assume that the number of infected individuals is positive, i.e.,

$$\text{(A)} \quad \int_0^L \bar{I}(x, 0) \, dx > 0, \text{ with } \bar{S}(x, 0) \geq 0 \text{ and } \bar{I}(x, 0) \geq 0 \text{ for } x \in (0, L).$$

As discussed in [11], (3) admits a unique classic solution $(\bar{S}(x, t), \bar{I}(x, t))$ which exists globally in time. Let

$$N \triangleq \int_0^L \left[ \bar{S}(x, 0) + \bar{I}(x, 0) \right] \, dx > 0$$
be the total number of individuals in \((0, L)\) at \(t = 0\). From no-flux boundary conditions, it can be seen that the total population size is constant in time, i.e.,

\[
\int_0^L (\bar{S}(x, t) + \bar{I}(x, t)) \, dx = N, \quad \forall \, t \geq 0.
\] (4)

Unless otherwise stated, we assume that (A) holds and \(N\) is a fixed positive constant throughout this paper.

We are mainly interested in non-negative equilibrium solutions of (3), that is, the non-negative solutions of the following system:

\[
\begin{cases}
  d_S S_{xx} - q S_x - \beta(x) \frac{S I}{1 + m I} + \gamma(x) I = 0, & 0 < x < L, \\
  d_I I_{xx} - q I_x + \beta(x) \frac{S I}{1 + m I} - \gamma(x) I = 0, & 0 < x < L, \\
  d_S S_x - q S = d_I I_x - q I = 0, & x = 0, L.
\end{cases}
\] (5)

Here, \(S(x)\) and \(I(x)\) denote the density of susceptible and infected individuals, respectively, at \(x \in [0, L]\). In view of (4), we impose the total population condition:

\[
\int_0^L [S(x) + I(x)] \, dx = N.
\] (6)

It is clear that only solutions \((S(x), I(x))\) satisfying \(S(x) \geq 0\) and \(I(x) \geq 0\) on \([0, L]\) are of interest. A disease-free equilibrium (DFE) is a solution of (5)-(6) so that \(I(x) = 0\) for every \(x \in (0, L)\); An endemic equilibrium (EE) of (5)-(6) is a solution in which \(I(x) > 0\) for some \(x \in (0, L)\). We denote a DFE by \((\hat{S}(x), 0)\) and an EE by \((S(x), I(x))\). By direct computations and condition (6), it is clear that \(\hat{S}(x) = qN e^{(q/d_S)x} / d_S(e^{qL/d_S} - 1)\). Thus (5)-(6) has a unique disease-free equilibrium, which is spatially inhomogeneous.

According to the similar arguments in [1, 11, 10], we define the basic reproduction number for model (3) as follows:

\[
\mathcal{R}_0(d_S, d_I, q) = \sup_{\varphi \in H^1(0, L), \varphi \neq 0} \left\{ \frac{qN}{d_S(e^{\varphi L} - 1)} \int_0^L \left[ d_S \int_0^L \frac{S(\bar{S} + \varphi)}{\bar{S}} (x) \beta(x) \varphi^2 \, dx \right] + \int_0^L \frac{I_1^2}{d_I} \varphi^2 + \gamma(x) \varphi^2 \, dx \right\}.
\]

From the definition of the basic reproduction number of (3), it can be seen that \(\mathcal{R}_0\) is a smooth function of \(d_S, d_I\) and \(q\). For the sake of convenience, we shall denote the basic reproduction number by \(\mathcal{R}_0\). In [1, 11, 10, 13], it is shown that the basic reproduction number a threshold value for the stability of DFE and the existence of EE: if the basic reproduction number is smaller than unity then DFE is stable, and if the basic reproduction number is greater than unity then the DFE is unstable and the systems admits at least one EE. We can extend these conclusions to the model (3) in the next subsection.

### 1.3 Main results

Our first theorem is to give the existence of EE for system (3).

**Theorem 1.1.** Assume that \(\mathcal{R}_0 > 1\), then there exists a constant \(\delta > 0\) independent of the initial data, such that any solution \((\bar{S}, \bar{I})\) of (3) satisfies

\[
\liminf_{t \to \infty} \bar{S}(x, t) \geq \delta \quad \text{and} \quad \liminf_{t \to \infty} \bar{I}(x, t) \geq \delta \quad \text{uniformly for} \quad x \in [0, L],
\]

and hence, the disease persist uniformly. Furthermore, (3) admits at least one EE when \(\mathcal{R}_0 > 1\).
Theorem 1.1 established the existence of EE of (3) under condition $R_0 > 1$. We next explore the asymptotic profiles of the EE in the four cases: (i) $q \to \infty$; (ii) $d_S \to 0$; (iii) $d_I \to 0$; (iv) $m \to \infty$. To these purposes, it should ensure $R_0 > 1$ with respect to large $q$, or small $d_S$ or small $d_I$ (note that $R_0$ does not depend on $m$). In Subsection 2.1 below, we give some results about the asymptotic behaviors of $R_0$ in above cases. The following result describes the asymptotic profile of the EE when $q \to \infty$.

**Theorem 1.2.** For each fixed $d_S, d_I, m > 0$, let $(S(x; q), I(x; q)) ≜ (S(x), I(x))$ be any EE of (3) (positive solution of (5)-(6)). Then, as $q \to \infty$, we have

(i) $(S(x; q), I(x; q)) \to (0, 0)$ locally uniformly in $[0, L]$;
(ii) at downstream,

(i-1) either

$$\lim_{q \to \infty} \frac{1}{q} S(L - \frac{y}{q}; q) = \frac{m\gamma(L)N}{d_S\beta(L) + m\gamma(L)} e^{-\frac{\lambda}{d_S}},$$

$$\lim_{q \to \infty} \frac{1}{q} I(L - \frac{y}{q}; q) = \frac{\beta(L)N}{d_I\beta(L) + m\gamma(L)} e^{-\frac{\lambda}{d_I}},$$

uniformly for $y$ in any compact subset of $[0, \infty)$,

(ii-2) or

$$\lim_{q \to \infty} \frac{1}{q} S(L - \frac{y}{q}; q) = \frac{N}{d_S} e^{-\frac{\lambda}{d_S}}, \quad \lim_{q \to \infty} \frac{1}{q} I(L - \frac{y}{q}; q) = 0,$$

uniformly for $y$ in any compact subset of $[0, \infty)$.

Theorem 1.2 shows that if the advection rate is sufficiently large, then either both susceptible and infected individuals will leave the habitat $[0, L]$ (Theorem 1.2 (i)) and concentrate at the downstream end $x = L$ (Theorem 1.2 (ii-1)), or the density of the infected vanishes on the entire habitat and susceptible individuals concentrate only at the downstream end $x = L$ (Theorem 1.2 (ii-2)).

In order to present the asymptotic profile of EE as $d_S \to 0$, we need consider the following eigenvalue problem:

$$\begin{cases}
-d_I \psi_{xx} + q \psi_x + \gamma(x) \psi = \lambda \psi, & 0 < x < L, \\
d_I \psi_x(0) - q \psi(0) = 0, \quad d_I \psi_x(L) - (q + \beta(L)N) \psi(L) = 0.
\end{cases}$$ (7)

Let $\hat{\lambda}$ be the principal eigenvalue, it follows the below Lemmas 2.1 and 2.2 that $1 - R_0$ and $\hat{\lambda}$ have the same sign for all sufficiently small $d_S$. Thus, (3) admits at least one EE for all small $d_S$ provided that the condition $\hat{\lambda} < 0$. We describe the asymptotic profile of EE when $d_S \to 0$.

**Theorem 1.3.** Let $d_I, q, m > 0$ fixed. Assume that $\hat{\lambda} < 0$. Let $(S(x; d_S), I(x; d_S)) ≜ (S(x), I(x))$ be any EE of (3). Then, as $d_S \to 0$, we have

$$(S(\cdot; d_S), I(\cdot; d_S)) \to (\hat{S}(\cdot) + \hat{N}\delta_L, \hat{I}(\cdot)) \text{ weakly in } L^1((0, L)),$$

where $\delta_L$ denotes the Dirac measure at $L$, $\hat{N}$ is a constant satisfies $0 < \hat{N} < N$, and $(\hat{S}, \hat{I})$ is a positive solution of the following system:
\begin{equation}
\begin{aligned}
- q \dot{S}_x - \beta(x) \frac{\dot{S} \dot{I}}{1 + m I} + \gamma(x) \dot{I} &= 0, & x \in (0, L), \\
\delta \dot{I}_x - q \dot{I}_x + \beta(x) \frac{\dot{S}}{1 + m I} - \gamma(x) \dot{I} &= 0, & x \in (0, L), \\
\dot{S}(0) &= 0, \\
\delta \dot{I}_x(0) &= q \dot{I}(0) - \beta(L) \dot{N} I(L)/(1 + m I(L)), \\
\int_0^L [\dot{S}(x) + \dot{I}(x)] \, dx + \dot{N} &= N.
\end{aligned}
\end{equation}

Theorem 1.3 shows that when the mobility of the susceptible individuals is sufficiently small, a certain portion of the susceptible population concentrates at the downstream end, and the remaining portion of the susceptible population distributes in \([0, L]\) in a non-homogeneous way; on the other hand, the density of infected population is positive on the entire habitat \([0, L]\).

Our result on the asymptotic profile of EE as \(d_I \to 0\) can be stated as follows.

**Theorem 1.4.** Let \(d_S, q, m > 0\) fixed. Assume that \(\gamma(L) < \beta(L) \frac{q N e^{\frac{q L}{\delta S}}}{d_S (e^{\frac{q L}{\delta S}} - 1)}\). Let \((S(x; d_I), I(x; d_I)) \equiv (S(x), I(x))\) be any EE of (3). Then, as \(d_I \to 0\), we have

(i) \(S(x; d_I) \to \frac{q N}{d_S (1 - e^{-\frac{q L}{\delta S}})} e^{-\frac{q x}{\delta S} (L-x)}\) uniformly on \([0, L]\);

(ii) \(I(x; d_I) \to 0\) locally uniformly in \([0, L]\) and \(\int_0^L I(x; d_I) \, dx \to 0\).

From the below Lemmas 2.1 and 2.2, we know that \(R_0 > 1\) for all small \(d_I > 0\). It follows from Theorem 1.1 that (3) has at least one EE. According to Theorem 1.4, when the diffusion rate of infected individuals approaches zero, the density of infected individuals will vanish and susceptible individuals distribute inhomogeneous everywhere on the whole habitat \([0, L]\).

Finally, we are concerned with the asymptotic profile of EE as \(m \to \infty\). Indeed, we are able to state the following result.

**Theorem 1.5.** Let \(d_S, d_I, q > 0\) fixed. Assume that \(R_0 > 1\). Let \((S(x; m), I(x; m)) \equiv (S(x), I(x))\) be any EE of (5)-(6). Then, as \(m \to \infty\), we have

\((S(x; m), I(x; m)) \to \left(\frac{q N}{d_S (1 - e^{-\frac{q L}{\delta S}})} e^{-\frac{q x}{\delta S} (L-x)}, 0\right)\) uniformly on \([0, L]\).

Moreover, as \(m \to \infty\), \(m I \to \theta\) uniformly on \([0, L]\), where \(\theta(x)\) is the unique positive solution of the following problem:

\begin{equation}
\begin{aligned}
\delta \theta_{xx} - q \theta_x + \left[\beta(x) \frac{q N e^{\frac{q x}{\delta S}}}{d_S (e^{\frac{q x}{\delta S}} - 1)(1 + \theta)} - \gamma(x)\right] \theta &= 0, & 0 < x < L, \\
\delta \theta_x - q \theta &= 0, & x = 0, L.
\end{aligned}
\end{equation}

Theorem 1.5 implies that the EE tends to DFE as the saturated rate approaches to infinity. That means that large saturation can eradicate the disease. Biologically, the disease will be eliminated due to the strong inhibition effect from the crowding of infected individuals.

The rest of the paper is organized as follows. In section 2, we will prove the existence of EE by uniformly persistence theory. In section 3, we analyze the limiting
behavior of EE in four cases: large advection; small diffusion of the susceptible and infected individual; large saturation. Section 4 is devoted to a brief discussion about our results.

2 The existence of EE

2.1 The basic reproduction number and the eigenvalue problem

Since the basic reproduction number plays a critical role in determining the dynamic of DFE and existence of EE, we give some arguments about the basic reproduction number in this subsection.

By the similar argument as the proof of Lemma 2.2 in [11], we can give the following argument that the basic reproduction number is closely related to the principal eigenvalue of an eigenvalue problem.

Lemma 2.1. For any $d_S, d_I > 0$ and $q > 0$, we have $R_0 > 1$ when $\lambda_1(d_S, d_I, q) < 0$, $R_0 = 1$ when $\lambda_1(d_S, d_I, q) = 0$, and $R_0 < 1$ when $\lambda_1(d_S, d_I, q) > 0$, where $\lambda_1(d_S, d_I, q)$ is the principal eigenvalue of the following eigenvalue problem:

$$
\begin{cases}
-d_I \phi_{xx} + q \phi_x + \left[ \gamma(x) - \beta(x) \frac{qN e^{\frac{d_I}{d_S}}}{d_S(e^{\frac{d_I}{d_S}} - 1)} \right] \phi = \lambda \phi, & 0 < x < L, \\
d_I \phi_x(x) - \phi(x) = 0, & x = 0, L.
\end{cases}
$$

2.2 Uniform persistence: proof of Theorem 1.1

In this subsection, we aim to establish the uniform persistence property of solutions to (3) when the basic reproduction number is close to the saturation. Section 4 is devoted to a brief discussion about our results.

By the similar argument as the proof of Lemma 2.2 in [11], we can give the following argument that the basic reproduction number is closely related to the principal eigenvalue of an eigenvalue problem.

Lemma 2.2. The following statements about $\lambda_1(d_S, d_I, q)$ hold.

(i) Given $d_S, d_I > 0$, we have $\lim_{q \to \infty} \lambda_1(d_S, d_I, q) = -\infty$.

(ii) Given $d_I, q > 0$, we have $\lim_{d_S \to 0} \lambda_1(d_S, d_I, q) = \hat{\lambda}$, where $\hat{\lambda}$ is the unique principal eigenvalue of the following eigenvalue problem (7);

(iii) Given $d_S, q > 0$, we have $\lim_{d_I \to 0} \lambda_1(d_S, d_I, q) = \gamma(L) - \beta(L) \frac{qN e^{\frac{d_I}{d_S}}}{d_S(e^{\frac{d_I}{d_S}} - 1)}$.

2.2 Uniform persistence: proof of Theorem 1.1

In this subsection, we aim to establish the uniform persistence property of solutions to (3) when the basic reproduction number $R_0 > 1$, which will imply the existence of EE. Theorem 1.1 follows from Lemma 2.4 below.

For any positive solution $(\bar{S}, \bar{I})$ of (3), by the transformation $\bar{S}(x, t) = \xi(x, t) e^{\frac{d_I}{d_S} x}$, $\bar{I}(x, t) = \zeta(x, t) e^{\frac{d_I}{d_S} x}$, we can convert the system (3) to

$$
\begin{cases}
\xi_t = d_S \xi_{xx} + q \xi_x - \beta(x) \frac{e^{\frac{d_I}{d_S} x}}{1 + me^{\frac{d_I}{d_S} x}} \zeta + \gamma(x) e^{\frac{d_I}{d_S} x} \zeta, & 0 < x < L, t > 0, \\
\zeta_t = d_I \zeta_{xx} + q \zeta_x + \beta(x) \frac{e^{\frac{d_I}{d_S} x}}{1 + me^{\frac{d_I}{d_S} x}} \xi - \gamma(x) \xi, & 0 < x < L, t > 0, \\
\xi_x = \zeta_x = 0, & x = 0, L, t > 0, \\
\xi(x, 0) = \bar{S}(x, 0) e^{-\frac{d_I}{d_S} x}, \zeta(x, 0) = \bar{I}(x, 0) e^{-\frac{d_I}{d_S} x}, & 0 < x < L, \\
\int_0^L \left[ e^{\frac{d_I}{d_S} x} \xi(x, t) + e^{\frac{d_I}{d_S} x} \zeta(x, t) \right] dx = N, & t \geq 0.
\end{cases}
$$

(11)
Lemma 2.3. There exists a positive constant $C$ independent of the initial data $(\bar{S}_0, \bar{I}_0)$ with $\int_0^L (\bar{S}_0 + \bar{I}_0) \, dx = N$, such that for the solution $(\bar{S}, \bar{I})$ of (3) satisfies

$$\|\bar{S}(\cdot, t)\|_{L^\infty([0,L])} + \|\bar{I}(\cdot, t)\|_{L^\infty([0,L])} \leq C, \quad \forall t \in [0, \infty).$$

Next, we appeal to the theory of abstract dynamical systems developed in [37, 57] to establish the uniform persistence of (3) when $R_0 > 1$. Specifically, one can state that

Lemma 2.4. Assume that $R_0 > 1$. Then system (3) is uniformly persistent, i.e., there exists a real number $\delta > 0$ independent of the initial data $(\bar{S}_0, \bar{I}_0)$ with $\int_0^L (\bar{S}_0 + \bar{I}_0) \, dx = N$, such that any solution $(\bar{S}, \bar{I})$ of (3) satisfies

$$\liminf_{t \to \infty} \bar{S}(x, t) \geq \delta, \quad \liminf_{t \to \infty} \bar{I}(x, t) \geq \delta,$$

uniformly for $x \in [0, L]$, and hence, the disease persists uniformly. Furthermore, (3) admits at least one EE provided that $R_0 > 1$.

Proof. We first set $X = C([0, L], \mathbb{R}_+^2)$ be the Banach space with the supremum norm $\| \cdot \|_X$, then $X$ is a strongly ordered Banach space. Let us set

$$X_0 = \left\{ \varphi = (\varphi_1, \varphi_2) \in X : \int_0^L (\varphi_1 + \varphi_2) \, dx = N \right\}.$$

Notice that for every $(\bar{S}_0, \bar{I}_0) \in X_0$, (3) admits a unique solution $(\bar{S}(x, t), \bar{I}(x, t)) \in X_0$, which exists for any $t \geq 0$. Thus, we define a semiflow $\Phi(t) : X_0 \mapsto X_0$ by

$$\Phi(t)(\bar{S}_0, \bar{I}_0) = (\bar{S}(x, t), \bar{I}(x, t)), \quad (\bar{S}_0, \bar{I}_0) \in X_0, \quad t \geq 0.$$

Let $U_0 \triangleq \{(\bar{S}_0, \bar{I}_0) \in X_0 : \bar{I}_0 \not\equiv 0\}$ and $\partial U_0 \triangleq \{(\bar{S}_0, \bar{I}_0) \in X_0 : \bar{I}_0 \equiv 0\}$. Then, $X_0 = U_0 \cup \partial U_0$, $\bar{U}_0$ and $\partial U_0$ are relatively open and closed subsets of $X_0$, respectively, and $U_0$ is convex. Clearly, $\Phi(t)U_0 \subset U_0$ and $\Phi(t)\partial U_0 \subset \partial U_0$ for all $t \geq 0$. Hence, by means of Lemma 2.3, standard parabolic theory and embedding theorems ensure that $\Phi(t) : X_0 \mapsto X_0$ is continuous and compact for any $t > 0$. Actually, with the help of Lemma 2.3 we can conclude that $\Phi(t)$ is point dissipative in $X_0$. Hence, [19, Theorem 2.4.7] asserts that $\Phi : X_0 \mapsto X_0$ has a global attractor. Let $\omega((\bar{S}_0, \bar{I}_0))$ be the omega-limit set of the orbit $\gamma^+((\bar{S}_0, \bar{I}_0)) \triangleq \{\Phi(t)(\bar{S}_0, \bar{I}_0) : t \geq 0\}$ and denote $M_0 = (\bar{S}(x, 0), \bar{I}(x, 0))$. We prove the following two claims.

Claim 1. $\omega((\bar{S}_0, \bar{I}_0)) = M_0, \forall (\bar{S}_0, \bar{I}_0) \in \partial U_0$.

Since $(\bar{S}_0, \bar{I}_0) \in \partial U_0$, we notice that $\bar{I}(x, t) \equiv 0$, and so $\bar{S}(x, t)$ solves

$$\begin{align*}
\bar{S}_t &= dS\bar{S}_{xx} - q\bar{S}_x, \quad x \in (0, L), \quad t > 0, \\
dS\bar{S}_x - q\bar{S} &= 0, \quad x = 0, L, \quad t > 0, \\
\bar{S}(x, 0) &= \bar{S}_0(x) \geq 0, \quad x \in (0, L), \\
\int_0^L \bar{S}(x, t) \, dx &= N, \quad t \geq 0.
\end{align*}$$
Then, one can use the similar analysis as in the proof of [9, Lemma 2.5] to conclude that $\lim_{t \to \infty} \hat{S}(x,t) = \hat{S}(x)$ uniformly on $[0,L]$, and so claim 1 holds.

Claim 2. There exists a real number $\delta_1 > 0$ such that

$$\limsup_{t \to \infty} \|\Phi(\hat{S}_0, \bar{I}_0) - M_0\|_X \geq \delta_1, \quad \forall (\hat{S}_0, \bar{I}_0) \in U_0.$$ 

From Lemma 2.1 and our assumption $R_0 > 1$, it follows that $\lambda_1 < 0$. Consequently, by the continuity of the principal eigenvalue $\lambda_1$, we can choose a small positive constant $\epsilon_0$ such that $\lambda_1(\epsilon_0) < -\epsilon_0$, where $\lambda_1(\epsilon_0)$ is the principal eigenvalue of the following eigenvalue problem:

$$
\begin{aligned}
-d_t \psi_{xx} + q \psi_x + \left[ \gamma(x) - \frac{\beta(x)q \epsilon_0 e^{\frac{-\epsilon_0}{L}} - \epsilon_0 d_s (e^{-\frac{\epsilon_0}{L}} - 1)}{\epsilon_0 d_s (e^{-\frac{\epsilon_0}{L}} - 1) (1 + \epsilon_0)} \right] \psi &= \lambda \psi, & 0 < x < L, \\
d_t \psi_x - q \psi &= 0, & x = 0, L.
\end{aligned}
$$

(12)

We now fix such chosen $\epsilon_0$. To prove the claim 2, we argue by contradiction. Suppose there exists some $(\bar{S}_0, \bar{I}_0) \in U_0$ such that the unique solution $\Phi(\bar{S}_0, \bar{I}_0) = (\hat{S}^*(x,t), \hat{I}^*(x,t))$ satisfies

$$\limsup_{t \to \infty} \|\Phi(\bar{S}_0, \bar{I}_0) - M_0\|_X \leq \delta_1.$$ 

Thus, for $\delta_1 < \epsilon_0$, there exists $T_0 > 0$ such that for all $x \in [0, L]$,

$$\hat{S}(x) - \epsilon_0 \leq \hat{S}(x) - \delta_1 \leq \hat{S}^*(x,t) \leq \hat{S}(x) + \delta_1, \quad 0 \leq \hat{I}^*(x,t) \leq \delta_1 < \epsilon_0, \quad \forall t \geq T_0.$$ 

(13)

Let $\psi_{\epsilon_0}$ be a positive eigenfunction corresponding to $\lambda_0(\epsilon_0)$ in (12). On the other hand, as $(\bar{S}_0, \bar{I}_0) \in U_0$, the strong maximum principle for parabolic equations shows $(\hat{S}^*(\cdot, t), \hat{I}^*(\cdot, t)) \in int(X_0)$ for any fixed $t > 0$. Thus, $\bar{I}_0^* \geq c^* \psi_{\epsilon_0}$ on $[0,L]$ for some positive constant $c^*$. From (13) and the choice of $\delta_1$, we find that $\hat{I}^*(x,t)$ is a supersolution to

$$
\begin{aligned}
w_t = d_t w_{xx} - qw_x + \beta(x) \frac{\hat{S}(x) - \epsilon_0}{1 + m \epsilon_0} w - \gamma w, & \quad x \in (0, L), \quad t > T_0, \\
d_t w_x - qw &= 0, & \quad x = 0, L, \quad t > T_0, \\
w(x,T_0) = c^* \psi_{\epsilon_0}, & \quad x \in (0, L).
\end{aligned}
$$

(14)

Furthermore, it is easily checked that $c^* e^{-\lambda_1(\epsilon_0)t} \psi_{\epsilon_0}(x)$ is the unique solution to (14). By the parabolic comparison principle, we deduce

$$\hat{I}^*(x,t) \geq c^* e^{-\lambda_1(\epsilon_0)t} \psi_{\epsilon_0}(x) \to \infty \quad \text{uniformly for } x \in [0,L], \quad \text{as } t \to \infty,$$

which is in contradiction with Lemma 2.3. This proves our claim 2.

The above claims imply that $M_0$ is an isolated invariant set for $\Phi$ in $U$, and $W^s(M_0) \cap U_0 = \emptyset$, where $W^s(M_0)$ is the stable set of $M_0$ for $\Phi$. As a result, [57, Theorem 2.2] (see also [58, Theorem 1.3.1 and Remark 1.3.1]) asserts that $\Phi$ is uniformly persistent with respect to $(U_0, \partial U_0)$. Furthermore, from [57, Theorem 2.3] (see also [37, Theorem 4.5]), we see that $\Phi$ admits a compact global attractor $\hat{A}_0$ in $U_0$, which implies that (3) admits at least one EE in $U_0$. Let $(S^*, I^*)$ be any EE, then $(S^*, I^*) \in \hat{A}_0$. Set $\hat{B}_0 = \bigcup_{t \geq 0} \Phi(t) \hat{A}_0$. Then it follows that $\hat{B}_0 \subset U_0$, and $\lim_{t \to \infty} \|\Phi(t)(S^*, I^*) - \hat{B}_0\|_X = 0$. Clearly, $\hat{A}_0 \subset int(U_0)$, and this implies $\hat{B}_0 \subset int(U_0)$. Obviously, $\Phi(t)(S^*, I^*) \in \hat{B}_0$. Moreover, we know that $\hat{B}_0$ is a global attractor of $\Phi$ and a compact subset in $U_0$. Thus, there exists a positive
constant $\delta > 0$ such that $\liminf_{t \to \infty} (\bar{S}, \bar{I}) \geq (\delta, \delta)$ for all $(\bar{S}_0, \bar{I}_0) \in U_0$. This implies the desired uniform persistence property. The proof is completed. 

3 Asymptotic profiles of EE

In order to investigate the asymptotic profiles of EE, we establish the priori estimates for any EE in subsection 3.1. Theorems 1.2, 1.3, 1.4 and 1.5 are proved in subsections 3.2, 3.3, 3.4 and 3.5, respectively. For notational convenience, in what follows we denote

$$\beta^* = \max_{x \in [0, L]} \beta(x), \quad \beta_* = \min_{x \in [0, L]} \beta(x),$$

$$\gamma^* = \max_{x \in [0, L]} \gamma(x), \quad \gamma_* = \min_{x \in [0, L]} \gamma(x).$$

3.1 A priori estimates

**Lemma 3.1.** Let $(S,I)$ be any EE of (3) (positive solution of (5)-(6)). Then the following assertions hold.

(i) For any $d_S, d_I, q, m > 0$, we have $S(L) e^{-\frac{q}{\beta^*} \left(1 + \frac{\beta^* d_S}{mq^2}\right)(L-x)} \leq S(x)$;

(ii) For any $d_S, d_I, q, m > 0$, we have

$$S(x) \leq \frac{qN \left(1 + \frac{\beta^* d_S}{mq^2}\right)}{d_S \left[1 - e^{-\frac{q}{\beta^*} \left(1 + \frac{\beta^* d_S}{mq^2}\right)L} \right]} e^{-\frac{\beta^* d_S}{mq^2} (L-x)} + \frac{\gamma_* N}{q} \left[1 - e^{-\frac{\beta^* d_S}{mq^2} (L-x)} \right].$$

**Proof.** (i) By using the technique in [11, Lemma 2.5], we employ the change of variable $S(x) = e^{\frac{\beta^*}{m} \left(1 + \frac{\beta^* d_S}{mq^2}\right)x} w(x)$. Thus, $w(x)$ satisfies the following equations:

$$\left\{ \begin{array}{l}
d_S w_{xx} + q \left(1 + \frac{2 \beta^* d_S}{mq^2}\right) w_x + \left[\frac{\beta^*}{m} \left(1 + \frac{\beta^* d_S}{mq^2}\right) - \beta(x) \frac{I}{1 + mI} \right] w \\
\quad + e^{-\frac{\beta^* d_S}{mq^2} (L-x)} \gamma(x) I = 0, & 0 < x < L, \\
\quad w_x(x) = -\frac{\beta^*}{mq} w(x), & x = 0, L.
\end{array} \right. \quad (15)$$

Let $x_1 \in [0, L]$ such that $w(x_1) = \min_{x \in [0, L]} w(x)$. Since $w_x(0) < 0$, then $x_1 \neq 0$. If $x_1 \in (0, L)$, then $w_x(x_1) = 0$ and $w_{xx}(x_1) \geq 0$. By (15) we have

$$\frac{\beta^*}{m} \left(1 + \frac{\beta^* d_S}{mq^2}\right) - \beta(x_1) \frac{I}{1 + mI} \leq 0,$$

which is impossible for any positive $d_S, q, m$. Therefore, $x_1 = L$; i.e., $w(x) \geq w(L)$ for any $x \in [0, L]$. Thus, it follows from the transformation $S(x) = e^{\frac{\beta^*}{m} \left(1 + \frac{\beta^* d_S}{mq^2}\right)x} w(x)$ that

$$S(L) e^{-\frac{q}{\beta^*} \left(1 + \frac{\beta^* d_S}{mq^2}\right)(L-x)} \leq S(x), \quad \forall x \in [0, L]. \quad (16)$$

(ii) First, we claim that

$$S(L) \leq \frac{qN \left(1 + \frac{\beta^* d_S}{mq^2}\right)}{d_S \left[1 - e^{-\frac{q}{\beta^*} \left(1 + \frac{\beta^* d_S}{mq^2}\right)L} \right]}.$$ \quad (17)

Integrating (16) over $(0, L)$ and using (6), we obtain

$$S(L) \int_0^L e^{-\frac{q}{\beta^*} \left(1 + \frac{\beta^* d_S}{mq^2}\right)(L-x)} dx \leq \int_0^L S(x) dx \leq N.$$
Thus, (17) holds by direct calculation.

Integrating the first equation of (5) from 0 to \(x\) with boundary conditions, we have

\[-[d_S S_x(x) - q S(x)] = \int_0^x \gamma(x) I \, dx - \int_0^x \beta(x) \frac{SI}{1 + mI} \, dx \leq \gamma^* \int_0^L I \, dx.\]

Multiplying the above inequality by \(e^{-\frac{\beta x}{q}}\) and integrating over \((x, L)\), we obtain

\[e^{-\frac{\beta x}{q}} S(x) - e^{-\frac{\beta x}{q}} S(L) \leq \frac{\gamma^*}{q} \int_0^L I \, dx \left( e^{-\frac{\beta x}{q}} - e^{-\frac{\beta x}{q} L} \right),\]

that is

\[S(x) \leq S(L) e^{-\frac{\beta x}{q} (L-x)} + \frac{\gamma^*}{q} \int_0^L I \, dx \frac{1}{q} \left[ 1 - e^{-\frac{\beta x}{q} (L-x)} \right]. \tag{18}\]

Substituting (17) into the inequality (18) and combing with (6), we obtain the desired result.

Similar to the proof of Lemma 3.1, we can give the following estimates about \(I\).

**Lemma 3.2.** Let \((S, I)\) be any EE of (3). Then the following assertions hold.

(i) For any \(d_S, d_I, q, m > 0\), we have \(I(L) e^{-\frac{\beta x}{q} \left(1 + \frac{\gamma^* d_I}{d_I q} \right) (L-x)} \leq I(x)\);

(ii) For any \(d_S, d_I, q, m > 0\), we have

\[I(x) \leq \frac{qN (1 + \frac{\gamma^* d_I}{q})}{d_I [1 - e^{-\frac{\beta x}{q} (1 + \frac{\gamma^* d_I}{d_I q}) L}]} e^{-\frac{\beta x}{q} (L-x)} + \frac{\beta^* N}{mq} \frac{1 - e^{-\frac{\beta x}{q} (L-x)}}{1 - e^{-\frac{\beta x}{q} (L-x)}}.\]

In the next Lemma we can give a lower bound of the integral of \(S\) for any EE \((S, I)\).

**Lemma 3.3.** Let \((S, I)\) be any EE of (3). Then we have

\[\int_0^L S(x) \, dx \geq \frac{m \gamma_s N}{m \gamma_s + \beta^*} > 0.\]

**Proof.** Integrating the first equation of (5) in \((0, L)\) and using the boundary condition, we have

\[\frac{\beta^*}{m} \int_0^L S \, dx \geq \int_0^L \beta(x) \frac{SI}{1 + mI} \, dx = \int_0^L \gamma(x) I \, dx \geq \gamma^* \int_0^L I \, dx.\]

By the total population condition (6) and a direct computation, we know that

\[\left( \frac{\beta^*}{m} + \gamma^* \right) \int_0^L S \, dx \geq \gamma^* \int_0^L (I + S) \, dx = \gamma_s N,\]

which completes the proof.

In the following subsections, we are concerned with the asymptotic behavior of EE of (1.3) as \(q\) tends to infinity, or one of the diffusion rates \(d_S, d_I\) tends to zero, or \(m\) tends to infinity.
3.2 The case of $q \to \infty$: Proof of Theorem 1.2

Proof of Theorem 1.2. As discussed before, $\lambda_1(d_S, d_I, q) < 0$ for all large $q > 0$ from Lemma 2.2. Hence, $R_q > 1$ and (3) has at least one EE, denoted by $(S(x; q), I(x; q))$, for all large $q > 0$ by Theorem 1.1. By Lemma 3.2, for any EE $(S(x; q), I(x; q))$ of (3), we know that $(S(\cdot; q), I(\cdot; q)) \to (0, 0)$ locally uniformly in $[0, L]$ as $q \to \infty$. We next will show the concentration profile at the downstream end $x = L$.

**Step 1: Convergence of $(S(x; q), I(x; q))$ at $x = L$.** To achieve the aim, for any EE $(S(x; q), I(x; q))$, we introduce the following transformations:

$$
u(y; q) = \frac{1}{q} S \left( L - \frac{y}{q} \right), \quad \nu(y; q) = \frac{1}{q} I \left( L - \frac{y}{q} \right), \quad 0 \leq y \leq qL.$$

Then $(u, v)$ satisfies the following problem:

$$
\begin{cases}
  d_S u_{yy} + u_y = - \frac{1}{q} \beta \left( L - \frac{y}{q} \right)^{1 + mqv} + \gamma \left( L - \frac{y}{q} \right) \frac{v}{q^2} = 0, & 0 < y < qL, \\
  d_S v_{yy} + v_y = \frac{1}{q} \beta \left( L - \frac{y}{q} \right)^{1 + mqv} - \gamma \left( L - \frac{y}{q} \right) \frac{v}{q^2} = 0, & 0 < y < qL, \\
  d_S u_y + u = d_I v_y + v = 0, & y = 0, qL,
\end{cases} \tag{19}
$$

with the total population condition

$$
\int_0^{qL} [u(y; q) + \nu(y; q)] \, dy = N, \ \forall q > 0. \tag{20}
$$

From Lemmas 3.1 and 3.2, we know that

$$
u(y; q) \leq \frac{N \alpha}{d_S \left[ 1 - e^{-\frac{\alpha}{d_S (1 + \frac{\gamma v}{mqv^2})}} \right]} e^{-\frac{\alpha y}{d_S}} + \frac{\gamma N \alpha}{q^2 \left[ 1 - e^{-\frac{\alpha y}{d_S}} \right]}, \quad \forall y \in [0, qL],$$

and

$$
u(y; q) \leq \frac{N \alpha}{d_I \left[ 1 - e^{-\frac{\alpha y}{d_I (1 + \frac{\gamma v}{mqv^2})}} \right]} e^{-\frac{\alpha y}{d_I}} + \frac{\gamma N \alpha}{q^2 \left[ 1 - e^{-\frac{\alpha y}{d_I}} \right]}, \quad \forall y \in [0, qL].$$

Thus, for any fixed constant $K > 0$, $\{u(\cdot; q)\}$ and $\{\nu(\cdot; q)\}$ are uniformly bounded in $[0, K]$ with respect to $q$ satisfying $q > K/L$. Applying the standard elliptic estimate and the Sobolev imbedding theorem to (19), it follows that

$$
\|u(\cdot; q)\|_{C^{1,\alpha}([0, K])} \leq C, \quad \|\nu(\cdot; q)\|_{C^{1,\alpha}([0, K])} \leq C, \quad \text{for some } \alpha \in (0, 1), \tag{21}
$$

where the positive constant $C$ is independent of large $q$.

In light of (21), applying the Ascoli–Arzelà theorem and combining with a diagonal argument, we can take a sequence $q_n$ with $\lim_{n \to \infty} q_n = \infty$ and nonnegative function $(\tilde{u}, \tilde{v})$ such that $(u_n(y), v_n(y)) \to (u(y; q_n), v(y; q_n))$ satisfies

$$
\lim_{n \to \infty} (u_n, v_n) = (\tilde{u}, \tilde{v}) \in C^1_{\text{loc}}([0, \infty)) \times C^1_{\text{loc}}([0, \infty)). \tag{22}
$$

As $(u_n, v_n)$ satisfies (19)-(20) with $q = q_n$, we send $n \to \infty$ to know that $(\tilde{u}, \tilde{v})$ satisfies

$$
\begin{cases}
  d_S \tilde{u}_{yy} + \tilde{u}_y = 0, & y \in (0, \infty), \\
  d_I \tilde{v}_{yy} + \tilde{v}_y = 0, & y \in (0, \infty), \\
  d_S \tilde{u}_y(0) + \tilde{u}(0) = d_I \tilde{v}_y(0) + \tilde{v}(0) = 0. \tag{23}
\end{cases}
$$
A direct computation shows
\[ \hat{\nu}(y) = C_S e^{-\frac{y}{q_S}}, \quad \hat{\nu} = C_I e^{-\frac{y}{q_I}}, \]
for some constants \( C_S, C_I \geq 0 \). So we get from (22) that
\[ \lim_{n \to \infty} (u_n, v_n) = (C_S e^{-\frac{y}{q_S}}, C_I e^{-\frac{y}{q_I}}) \text{ in } C^1_{\text{loc}}([0, \infty]) \times C^1_{\text{loc}}([0, \infty]). \]

**Step 2:** \( C_S > 0 \). First we claim
\[ \lim_{n \to \infty} \int_0^{q_n L} u_n(y) \, dy = \int_0^\infty C_S e^{-\frac{y}{q_S}} \, dy. \]  
(24)

From the above estimate of \( u_n \), we know that there exists \( n_0 \in \mathbb{N} \) such that
\[ 0 \leq u_n(y) \leq \frac{2N}{d_S} e^{-\frac{y}{q_S}} + \frac{\gamma^* N}{q_n^2}, \quad \forall y \in [0, q_n L], \]
for all \( n \geq n_0 \). Then, for any \( \epsilon > 0 \), there exists some \( D_1(\epsilon) > 0 \) such that
\[ \frac{2N}{d_S} \int_{D_1} e^{-\frac{y}{q_S}} \, dy = 2Ne^{-\frac{D_1}{q_S}} \leq \frac{\epsilon}{6}, \quad \int_{D_1} C_S e^{-\frac{y}{q_S}} \, dy < \frac{\epsilon}{3}, \]
and there exists \( n_1 \in \mathbb{N} \) with \( n_1 \geq n_0 \) such that for \( n \geq n_1, q_n L > D_1 \) and
\[ \frac{\gamma^* N(q_n L - D_1)}{q_n^2} < \frac{\epsilon}{6}. \]

Therefore, for \( n \geq n_1 \), we have
\[ \int_{D_1} u_n(y) \, dy \leq \int_{D_1} C_S e^{-\frac{y}{q_S}} \, dy + \int_{D_1} \frac{\gamma^* N}{q_n^2} \, dy \]
\[ \leq 2Ne^{-\frac{D_1}{q_S}} + \frac{\gamma^* N(q_n L - D_1)}{q_n^2} < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}. \]

As \( u_n \to C_S e^{-\frac{y}{q_S}} \) in \( C^1_{\text{loc}}([0, \infty)) \), we can choose \( n_2 \in \mathbb{N} \) with \( n_2 \geq n_1 \) such that
\[ \int_0^{D_1} |u_n(y) - C_S e^{-\frac{y}{q_S}}| \, dy < \frac{\epsilon}{3} \text{ for } n \geq n_2. \]

Hence, for \( n \geq n_2 \), we have
\[ \left| \int_0^{q_n L} u_n(y) \, dy - \int_0^\infty C_S e^{-\frac{y}{q_S}} \, dy \right| \]
\[ \leq \left| \int_0^{D_1} \left[ u_n(y) - C_S e^{-\frac{y}{q_S}} \right] \, dy \right| + \int_{D_1}^{q_n L} u_n(y) \, dy - \int_{D_1}^\infty C_S e^{-\frac{y}{q_S}} \, dy \]
\[ \leq \left| \int_0^{D_1} u_n(y) - C_S e^{-\frac{y}{q_S}} \right| \, dy + \int_{D_1}^{q_n L} u_n(y) \, dy + \int_{D_1}^\infty C_S e^{-\frac{y}{q_S}} \, dy \]
\[ \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \]

This proves (24).

From Lemma 3.3 and the change of variable \( x = L - y/q_n \), we know that
\[ \int_0^{q_n L} u_n(y) \, dy \geq \frac{m\gamma_s N}{m\gamma_s + \beta_s} > 0. \]
Letting $n \to \infty$ in the above inequality, we can obtain
\[ d_S C_S = C_S \int_0^\infty e^{-\frac{y}{n}} dy = \int_0^\infty \hat{u}_n(y) dy \geq \frac{m \gamma_* N}{m \gamma_* + \beta^*}, \]
i.e.,
\[ C_S \geq \frac{m \gamma_* N}{d_S (m \gamma_* + \beta^*)} > 0. \]

**Step 3:** $C_I \geq 0$. There are two possibilities: $C_I > 0$ and $C_I = 0$.

**Case 1:** $C_I > 0$. We claim that
\[ \int_0^{q_n L} \left[ \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y)}{1 + mq_n v_n(y)} - \gamma \left( L - \frac{y}{q_n} \right) \right] v_n(y) dy \]
\[ \to \int_0^\infty \left[ \frac{\beta(L)}{m} C_S e^{-\frac{y}{n}} - \gamma(L) C_I e^{-\frac{y}{n}} \right] dy, \quad \text{as } n \to \infty. \] (25)

Indeed, from the estimate of $v_n$, we know that there exists $n_3 \in \mathbb{N}$ such that
\[ v_n(y) < \frac{2N}{d_I} e^{-\frac{y}{\beta}} + \frac{\beta^* N}{mq_n^2}, \quad y \in [0, q_n L], \]
for all $n \geq n_3$. By the similar argument in Step 2, for any $\epsilon > 0$, there exists some $D_2 = D_2(\epsilon) > 0$ such that
\[ \int_{D_2} \frac{2\beta^* N}{mq_n^2} e^{-\frac{y}{\beta}} < \frac{\epsilon}{12}, \quad \int_{D_2} 2\gamma^* Ne^{-\frac{y}{\beta}} < \frac{\epsilon}{12}, \]
(26)
\[ \int_{D_2} \frac{\beta(L)}{m} C_S e^{-\frac{y}{n}} dy < \frac{\epsilon}{6}, \quad \int_{D_2} \gamma(L) C_I e^{-\frac{y}{n}} dy < \frac{\epsilon}{6}, \]
and there exists $n_4 \in \mathbb{N}$ with $n_4 \geq n_3$ such that for $n \geq n_4$, there holds $q_n L > D_2$,
\[ \frac{\beta^* \gamma^* N}{mq_n^2} (q_n L - D_2) < \frac{\epsilon}{12}. \] (27)

Therefore, for $n \geq n_4$, we have
\[ \frac{\beta^*}{m} \int_{D_2}^q u_n(y) dy \leq \int_{D_2} \int_{D_2} \frac{2\beta^* N}{mq_n^2} e^{-\frac{y}{\beta}} dy + \int_{D_2} \beta^* \gamma^* N \frac{q_n L}{mq_n^2} dy \]
\[ \leq \frac{2\beta^* N}{m} e^{-\frac{D_2}{\beta}} + \frac{\beta^* \gamma^* N}{mq_n^2} (q_n L - D_2) < \frac{\epsilon}{12} + \frac{\epsilon}{12} = \frac{\epsilon}{6}, \]
and
\[ \gamma^* \int_{D_2} v_n(y) dy \leq \int_{D_2} \int_{D_2} 2\gamma^* Ne^{-\frac{y}{\beta}} dy + \int_{D_2} \beta^* \gamma^* N \frac{q_n L}{mq_n^2} dy \]
\[ \leq 2\gamma^* Ne^{-\frac{D_2}{\beta}} + \frac{\beta^* \gamma^* N}{mq_n^2} (q_n L - D_2) < \frac{\epsilon}{12} + \frac{\epsilon}{12} = \frac{\epsilon}{6}. \]

By the uniform convergence of $(u_n, v_n) \to (u^*, v^*)$ in $[0, D_2]$ as $n \to \infty$, there exists some $n_5 \in \mathbb{N}$ with $n_5 \geq n_4$ such that for all $n \geq n_5$,
\[ \int_0^{D_2} \left| \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y)}{1 + mq_n v_n(y)} - \frac{\beta(L)}{m} C_S e^{-\frac{y}{n}} \right| dy \]
\[ + \int_0^{D_2} \left| \gamma \left( L - \frac{y}{q_n} \right) v_n(y) - \gamma(L) C_I e^{-\frac{y}{n}} \right| dy < \frac{\epsilon}{3}. \]
From (26), (27) and above inequality, for \( n \geq n_5 \), we have

\[
\left| \int_0^{q_nL} \left[ \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y) v_n(y)}{1 + m q_n v_n(y)} - \gamma \left( L - \frac{y}{q_n} \right) v_n(y) \right] \, dy - \int_0^{q_nL} \left[ \frac{\beta(L)}{m} C S e^{-\frac{y}{L}} - \gamma(L) C T e^{-\frac{y}{T}} \right] \, dy \right| \\
\leq \left| \int_0^{D_2} \left[ \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y) v_n(y)}{1 + m q_n v_n(y)} - \gamma \left( L - \frac{y}{q_n} \right) v_n(y) \right] \, dy \right| \\
- \int_0^{q_nL} \left[ \frac{\beta(L)}{m} C S e^{-\frac{y}{L}} - \gamma(L) C T e^{-\frac{y}{T}} \right] \, dy \\
+ \int_0^{D_2} \left[ \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y) v_n(y)}{1 + m q_n v_n(y)} + \gamma \left( L - \frac{y}{q_n} \right) v_n(y) \right] \, dy \\
+ \int_0^{\infty} \left[ \frac{\beta(L)}{m} C S e^{-\frac{y}{L}} + \gamma(L) C T e^{-\frac{y}{T}} \right] \, dy \\
\leq \int_0^{D_2} \left| \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y) v_n(y)}{1 + m q_n v_n(y)} - \frac{\beta(L)}{m} C S e^{-\frac{y}{L}} \right| \, dy \\
+ \int_0^{D_2} \left| \gamma \left( L - \frac{y}{q_n} \right) v_n(y) - \gamma(L) C T e^{-\frac{y}{T}} \right| \, dy + \frac{\beta^*}{m} \int_{D_2}^{q_nL} u_n(y) \, dy \\
+ \gamma^* \int_{D_2}^{q_nL} v_n(y) \, dy + \int_0^{\infty} \frac{\beta(L)}{m} C S e^{-\frac{y}{L}} \, dy + \int_{D_2}^{\infty} \gamma(L) C T e^{-\frac{y}{T}} \, dy \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\]

Thus, the claim (25) is proved.

Integrating both sides of (19) with boundary conditions, we have

\[
\int_0^{q_nL} \left[ \beta \left( L - \frac{y}{q_n} \right) \frac{q_n u_n(y) v_n(y)}{1 + m q_n v_n(y)} - \gamma \left( L - \frac{y}{q_n} \right) v_n(y) \right] \, dy = 0. \tag{28}
\]

With the help of the above claim and (22), by sending \( n \to \infty \), we can get

\[
\frac{\beta(L)}{m} \int_0^{\infty} C S e^{-\frac{y}{L}} \, dy = \gamma(L) \int_0^{\infty} C T e^{-\frac{y}{T}} \, dy,
\]

i.e.,

\[
C_S \frac{\beta(L)}{m} d_S = C_I \gamma(L) d_I. \tag{29}
\]

By letting \( n \to \infty \) in (20), arguing similarly as to verify claim, we know that

\[
\int_0^{\infty} \left( C S e^{-\frac{y}{L}} + C T e^{-\frac{y}{T}} \right) \, dy = N,
\]

i.e.,

\[
C_S d_S + C_I d_I = N. \tag{30}
\]

From (29) and (30), we obtain

\[
C_S = \frac{m \gamma(L) N}{d_S [\beta(L) + m \gamma(L)]}, \quad C_I = \frac{\beta(L) N}{d_I [\beta(L) + m \gamma(L)]}.
\]

**Case 2:** \( C_I = 0 \). From (30), we can conclude that \( C_S = \frac{N}{d_S} \).

The proof is completed. \( \square \)
3.3 The case of \( d_S \to 0 \): Proof of Theorem 1.3

Proof of Theorem 1.3. Note that \( R_0 > 1 \) for all small \( d_S > 0 \) provided \( \lambda < 0 \) from Lemma 2.1. Thus, (3) has at least one EE, denoted by \((S(x;\hat{d}_S), \hat{I}(x;\hat{d}_S))\) for all small \( d_S > 0 \) under the condition \( \lambda < 0 \). In the following, for clarity we divide our proof into five steps.

**Step 1:** Convergence of \( I \). By the transformation \( \hat{I}(x;\hat{d}_S) = I(x;\hat{d}_S) e^{-\frac{q}{m}x} \), then \( \hat{I}(x;\hat{d}_S) \) satisfies

\[
\begin{aligned}
d_{\hat{x}}x + q\hat{I}_x + \left[ \beta(x) \frac{S(x)}{1 + mI} - \gamma(x) \right] \hat{I} = 0, & \quad 0 < x < L, \\
\hat{I}_x = 0, & \quad x = 0, L.
\end{aligned}
\]

Since

\[
\int_0^L \left| \beta(x) \frac{S(x;\hat{d}_S)}{1 + mI} - \gamma(x) \right| dx \leq \beta^* N + \gamma^* L,
\]

independent of all \( d_S > 0 \). Upon an application of a Harnack inequality (see, e.g., [32, lemma 2.2] or [40, Lemma 2.2]), we get

\[
\max_{x \in [0, L]} \hat{I}(x;\hat{d}_S) \leq C \min_{x \in [0, L]} \hat{I}(x;\hat{d}_S), \quad \forall d_S > 0. \quad (31)
\]

Hereafter, the positive constant \( C \) is independent of small \( d_S > 0 \) and may vary from place to place. In view of the transformation \( I(x;\hat{d}_S) = \hat{I}(x;\hat{d}_S) e^{\frac{q}{m}x} \) and (31), it follows that

\[
\max_{x \in [0, L]} I(x;\hat{d}_S) \leq C \min_{x \in [0, L]} I(x;\hat{d}_S), \quad \forall d_S > 0. \quad (32)
\]

Observe that \( I \) satisfies

\[
\begin{aligned}
d_1 I_{xx} - qI_x = -\beta(x) \frac{SI}{1 + mI} + \gamma(x)I, & \quad 0 < x < L, \\
d_1 I_x - qI = 0, & \quad x = 0, L.
\end{aligned}
\]

Since \( I(\cdot;\hat{d}_S) \) is uniformly bounded for all \( d_S > 0 \) by Lemma 3.2, from the elliptic \( L^1 \)-theory in [3] \(, \) it then follows that

\[
\|I(\cdot;\hat{d}_S)\|_{W^{1,p}([0, L])} \leq C, \quad \text{for any given } p > 1.
\]

Taking \( p \) to be sufficiently large, we see from the Sobolev embedding theorem that

\[
\|I(\cdot;\hat{d}_S)\|_{C^\alpha([0, L])} \leq C, \quad \text{for some } \alpha \in (0, 1).
\]

Therefore, after passing to a sequence if necessary, we may assume that

\[
\lim_{d_S \to 0} I(\cdot;\hat{d}_S) = \hat{I}(\cdot) \quad \text{in } C([0, L]) \quad (34)
\]

for some \( 0 \leq \hat{I} \in C([0, L]) \). Moreover, due to (32), either \( \hat{I} \equiv 0 \) or \( \hat{I} > 0 \) on \([0, L]\).

**Step 2:** Convergence of \( S \). It follows from Lemma 3.1 that \( S(\cdot;\hat{d}_S) \) is locally uniformly bounded in \([0, L]\). Then, for any fixed small \( \epsilon > 0 \), after passing to a subsequence of \( d_S \to 0 \), we may assume that

\[
S(\cdot;\hat{d}_S) \to \hat{S}(\cdot), \quad \text{weakly in } L^2((0, L - \epsilon)),
\]
where $\hat{S} \in L^2((0, L - \epsilon))$ and $\hat{S} \geq 0$ a.e. in $(0, L - \epsilon)$. Since $\int_0^L S(x; d_S) \, dx \leq N$ for all $d_S > 0$, we can apply a diagonal argument, to find a further subsequence of $d_S \to 0$ such that
\[
\int_0^L \hat{S}(x) \, dx = \lim_{\epsilon \to 0} \int_0^{L-\epsilon} \hat{S}(x) \, dx = \lim_{\epsilon \to 0} \lim_{d_S \to 0} \int_0^{L-\epsilon} S(x; d_S) \, dx \leq N. \tag{35}
\]
Thus, $\hat{S} \in L^1((0, L))$ and $\hat{S} \geq 0$ a.e. in $(0, L)$.

Fixing any small $\epsilon > 0$, taking any $\phi \in \{\varphi \in C^2([0, L - \epsilon]) : \varphi = 0$ near $0$ and $L - \epsilon\}$, multiplying the $S$-equation by $\phi$ and integrating by parts, we obtain that
\[
d_S \int_0^{L-\epsilon} S\varphi_{xx} \, dx + q \int_0^{L-\epsilon} S\varphi_x \, dx - \int_0^{L-\epsilon} \left[ \beta(x) \frac{SI}{1 + mL} - \gamma(x)I \right] \phi \, dx = 0.
\]
Sending $d_S \to 0$ and in view of the arbitrariness of $\epsilon$, as $I \in C([0, L])$, we know that $\hat{S}$ is in fact a weak and then a classical solution of
\[
-q\hat{S}_x - \beta(x) \frac{\hat{S}I}{1 + mL} + \gamma(x)I = 0, \quad 0 < x < L. \tag{36}
\]

**Step 3:** $\lim_{d_S \to 0} \min_{x \in [0, L]} S(x; d_S) = \hat{S}(0) = 0$. Set $S(x_{d_S}; d_S) = \min_{x \in [0, L]} S(x; d_S)$ for some $x_{d_S} \in [0, L]$. Without loss of generality, we may assume that $x_{d_S} \to x^* \in [0, L]$ as $d_S \to 0$. We claim that
\[
\lim_{d_S \to 0} S(x_{d_S}; d_S) = 0 \text{ and } x^* = 0. \tag{37}
\]
We argue by contradiction. If the first statement of (37) is false, then there exists some $\delta > 0$ independent of small $d_S > 0$ such that $S(x; d_S) \geq \delta$ for all $x \in [0, L]$ and small $d_S > 0$. Setting
\[
w(x; d_S) = e^{-\frac{\epsilon}{q} (1 - \bar{A}^2 \frac{d_S}{q^2})} S(x; d_S), \quad x \in [0, L],
\]
where $\bar{A}$ is some positive constant to be determined, we know that $w$ satisfies
\[
\begin{cases}
d_S w_{xx} + q \left(1 - 2\frac{\bar{A} d_S}{q^2}\right) w_x + \left[\bar{A} \left(\frac{\bar{A} d_S}{q^2} - 1\right) - \beta(x) \frac{I}{1 + mL}\right] w \\
\quad + \gamma(x)I e^{-\frac{\epsilon}{q^2} (1 - \bar{A}^2 \frac{d_S}{q^2})} x = 0,
\end{cases}
\]
\[
w_x = \frac{\bar{A}}{q} w, \quad x = 0, L. \tag{38}
\]
Let $x_0 \in [0, L]$ such that $w(x_0; d_S) = \max_{x \in [0, L]} w(x; d_S)$. Since $w_x(x_0; d_S) > 0$, then $x_0 \neq 0$. If $x_0 \in (0, L)$, then $w_x(x_0; d_S) = 0$ and $w_{xx}(x_0; d_S) \leq 0$. Thus, it follows from (38) that
\[
\left[\bar{A} \left(\frac{\bar{A} d_S}{q^2} - 1\right) - \beta(x_0) \frac{I(x_0; d_S)}{1 + mL(x_0; d_S)}\right] w(x_0; d_S)
\]
\[
\quad + \gamma(x_0)I(x_0; d_S) e^{-\frac{\epsilon}{q^2} (1 - \bar{A}^2 \frac{d_S}{q^2})} x_0 \geq 0,
\]
i.e.,
\[
\left[\bar{A} \left(\frac{\bar{A} d_S}{q^2} - 1\right) - \beta(x_0) \frac{I(x_0; d_S)}{1 + mL(x_0; d_S)} + \gamma(x_0) \frac{I(x_0; d_S)}{S(x_0; d_S)}\right] S(x_0; d_S) \geq 0. \tag{39}
\]
On the other hand, from Lemma 3.2, we have $I(x; d_S) \leq K_I$ for all $x \in [0, L]$, where $K_I$ is a positive constant independent of all small $d_S > 0$. Since $S$ is bounded from
below by a positive constant independent of small $d_S > 0$ by our hypothesis above, by choosing $A = 2\gamma K_I/\delta$ and requiring $0 < d_S < q^2/2A$, we can obtain that
\[
A \left( A\frac{d_S}{q^2} - 1 \right) - \beta(x_0) \frac{I(x_0; d_S)}{1 + mI(x_0; d_S)} + \gamma(x_0) \frac{I(x_0; d_S)}{S(x_0; d_S)} \\
\leq A \left( A\frac{d_S}{q^2} - 1 \right) + \gamma^* \frac{K_I}{\delta} < 0,
\]
which contradicts (39). Therefore, $x_0 = L$, i.e., $w(x; d_S) \leq w(L; d_S)$ for all $x \in [0, L]$. From the transformation $w(x; d_S) = e^{-\frac{x^2}{2\sigma^2}} \left[ 1 - \left( \frac{2x^* + 2\gamma K_I}{m} \right) \frac{d_S}{q^2} \right] x S(x; d_S)$, then
\[
S(x; d_S) \leq S(L; d_S) e^{-\frac{x^2}{2\sigma^2}} \left[ 1 - \left( \frac{2x^* + 2\gamma K_I}{m} \right) \frac{d_S}{q^2} \right] (L-x), \quad \forall x \in [0, L].
\]
Thanks to (17), it is clear that
\[
S(x; d_S) \leq \frac{qN \left( 1 + \frac{\beta^* d_S}{mq^2} \right)}{d_S \left[ 1 - e^{\frac{-x^2}{2\sigma^2}} \left[ 1 - \left( \frac{2x^* + 2\gamma K_I}{m} \right) \frac{d_S}{q^2} \right] (L-x) \right]}, \quad \forall x \in [0, L].
\]
Therefore, (40) shows that $S(x; d_S)$ converges to zero locally uniformly in $[0, L]$ as $d_S \to 0$, contradicting our assumption that $S(x; d_S) \geq \delta > 0$ on $[0, L]$ for small $d_S > 0$. Thus, we have proved $S(x_d^*; d_S) \to \hat{S}(x^*) = 0$ as $d_S \to 0$.

We next prove $x^* = 0$. Clearly, $x_d \neq L$ by $S_x(L; d_S) > 0$. For each fixed small $d_S > 0$, if $x_d \in (0, L)$, then $S_x(x_d; d_S) = 0$ and $S_{xx}(x_d; d_S) \geq 0$. From the $S$-equation, we have
\[
-\beta(x_d) S(x_d; d_S) \geq m \frac{\gamma(x_d)}{\beta(x_d)} \geq m \min_{x \in [0, L]} \frac{\gamma(x)}{\beta(x)} > 0,
\]
contradicting the above conclusion $\lim_{d_S \to 0} S(x_d^*; d_S) = \hat{S}(x^*) = 0$. Thus, it must hold $x_d = 0$ for all small $d_S > 0$. Thus, $\hat{S}(0) = 0$.

**Step 4:** $(\hat{S}, \hat{I})$ is a classical solution of (8). From the proof of Step 2, we know that $\hat{S}$ is the classical solution of
\[
\begin{cases}
-q\hat{S}_x - \beta(x) \frac{\hat{I}^2}{1 + mI} + \gamma(x) \hat{I} = 0, & 0 < x < L, \\
\hat{S}(0) = 0.
\end{cases}
\]

We take an arbitrary test function $\phi \in \{ \varphi \in C^2([0, L/2]) : \varphi = 0 \text{ near } L/2 \}$. Multiplying the $I$-equation in (5) by $\phi$, integrating by parts over $(0, L/2)$, and then sending $d_S \to 0$, we obtain that $d_I \hat{I}_x(0) = q\hat{I}(0)$. Thus, one can show that $\hat{I}$ is a classical solution of
\[
\begin{cases}
d_I \hat{I}_{xx} - \hat{I}_x + \beta(x) \frac{\hat{I}^2}{1 + mI} - \gamma(x) \hat{I} = 0, & 0 < x < L, \\
d_I \hat{I}_x(0) = q\hat{I}(0).
\end{cases}
\]

We next consider the boundary condition of $\hat{I}$ at $L$. For such a purpose, denote
\[
\hat{N} = N - \int_0^L (\hat{S} + \hat{I}) \, dx.
\]
Clearly, $0 \leq \hat{N} \leq N$. From (35), we notice that

$$\lim_{\epsilon \to 0} \lim_{d_S \to 0} \int_{L-\epsilon}^{L} S(x; d_S) \, dx = N - \lim_{\epsilon \to 0} \lim_{d_S \to 0} \int_{0}^{L-\epsilon} S(x; d_S) \, dx - \lim_{d_S \to 0} \int_{0}^{L} I(x; d_S) \, dx$$

$$= N - \int_{0}^{L} \hat{S}(x) \, dx - \int_{0}^{L} \hat{I}(x) \, dx = \hat{N}. \quad (44)$$

Integrating the $I$-equation in (5) over $(L - \epsilon, L)$, we have

$$d_I I_x(L - \epsilon) - q I(L - \epsilon) - \int_{L-\epsilon}^{L} \left[ \beta(x) \frac{S(x)I(x)}{1 + mI(x)} - \gamma(x)I(x) \right] \, dx = 0. \quad (45)$$

From the mean value theorem, it then follows that

$$\int_{L-\epsilon}^{L} \left[ \beta(x) \frac{S(x)I(x)}{1 + mI(x)} \right] \, dx = \beta(x_\epsilon) \frac{I(x_\epsilon)}{1 + mI(x_\epsilon)} \int_{L-\epsilon}^{L} S(x) \, dx,$$  

for some $x_\epsilon \in (L - \epsilon, L)$. Substituting (46) into (45), we can obtain that

$$d_I \hat{I}_x(L) = q \hat{I}(L) + \beta(L)\hat{N} \frac{\hat{I}(L)}{1 + m\hat{I}(L)}, \quad (47)$$

by sending $d_S \to 0$ first and then sending $\epsilon \to 0$. Therefore, we have verified that $(\hat{S}, \hat{I})$ is a classical solution of (8) with $\int_{0}^{L} [\hat{S}(x) + \hat{I}(x)] \, dx + \hat{N} = N$.

Now, given $\phi \in C([0, L])$, for any small $\epsilon > 0$, it is clear that

$$\int_{0}^{L} S(x; d_S)\phi(x) \, dx = \int_{0}^{L-\epsilon} S(x; d_S)\phi(x) \, dx + \int_{L-\epsilon}^{L} S(x; d_S)\phi(x) \, dx$$

$$= \int_{0}^{L-\epsilon} S(x; d_S)\phi(x) \, dx + \phi(x_\epsilon) \int_{L-\epsilon}^{L} S(x; d_S) \, dx,$$

where $x_\epsilon \in (L - \epsilon, L)$. Thus, by sending $d_S \to 0$ first and then sending $\epsilon \to 0$, and using (44), we conclude

$$\int_{0}^{L} S(x; d_S)\phi(x) \, dx \to \int_{0}^{L} \hat{S}(x)\phi(x) \, dx + \hat{N}\phi(L).$$

Therefore, that means that $S(\cdot; d_S) \to \hat{S}(\cdot) + \hat{N}\delta_L$ weakly in $L^1((0, L))$, where $\delta_L$ denotes the Dirac measure at $L$.

**Step 5:** $\hat{S}(x) > 0$ in $(0, L)$, $\hat{I}(x) > 0$ on $[0, L]$ and $0 < \hat{N} < N$. To this aim, we first claim that

$$q\hat{S}(L) = \hat{N}\beta(L) \frac{\hat{I}(L)}{1 + m\hat{I}(L)}. \quad (48)$$

Actually, adding (41) and (42), integrating over $(0, L)$, and combining with the boundary conditions of $\hat{S}$, $\hat{I}$ and (47), we can get

$$\hat{N} \beta(L) \frac{\hat{I}(L)}{1 + m\hat{I}(L)} - q\hat{S}(L)$$

$$= d_I \hat{I}_x(L) - q\hat{I}(L) - q\hat{S}(L) = d_I \hat{I}_x(0) - q\hat{I}(0) - q\hat{S}(0) = 0,$$

which proves the claim.
To prove $\hat{I}(L) > 0$, we argue by contradiction. Suppose that $\hat{I}(L) = 0$, then (48) implies $\hat{S}(L) = \hat{I}(L) = 0$. From (32) in Step 1, we know that
\[
\max_{x \in [0, L]} \hat{I}(x) \leq C \min_{x \in [0, L]} \hat{I}(x). \tag{49}
\]
It follows from (49) and $\hat{I}(L) = 0$ that $\hat{I} \equiv 0$ on $[0, L]$. Thus, by (41), $\hat{S} \equiv 0$ on $[0, L]$.

Let $\omega(x; d_I) = \frac{I(x; d_I)}{\|I(\cdot; d_I)\|_{L^\infty((0, L))}}$, then $\omega$ satisfies
\[
\begin{align*}
\begin{cases}
d_I \omega_{xx} - q \omega_x + \beta(x) \frac{S}{1 + mI} \omega - \gamma(x) \omega = 0, & 0 < x < L, \\
d_I \omega_x - q \omega = 0, & x = 0, L.
\end{cases}
\end{align*}
\tag{50}
\]
It then follows from $\hat{S} \equiv \hat{I} \equiv 0$ and (43) that $\hat{N} = N$. By passing to a subsequence if necessary, we can show that $\omega \to \hat{\omega}$ in $C([0, L])$ as $d_I \to 0$, where $\hat{\omega} > 0$ and $\|\hat{\omega}\|_\infty = 1$ satisfies
\[
\begin{align*}
\begin{cases}
d_I \hat{\omega}_{xx} - q \hat{\omega}_x - \gamma(x) \hat{\omega} = 0, & 0 < x < L, \\
d_I \hat{\omega}_x(0) = q \hat{\omega}(0), & d_I \hat{\omega}_x(L) = (q + \beta(L) N) \hat{\omega}(L).
\end{cases}
\end{align*}
\]
Here the boundary condition for $\hat{\omega}$ at $x = L$ can be obtained by a similar argument to the proof of (47). From $\hat{\omega} > 0$, then 0 is the principal eigenvalue of the eigenvalue problem (7), which contradicts the assumption of $\hat{\lambda} < 0$. Thus, we must have $\hat{I}(L) > 0$. From (49), it is clear that $\hat{I} > 0$ on $[0, L]$.

It is easily seen that $\hat{S} > 0$ in $(0, L)$ due to (41) is a linear ODE. Clearly, it follows from (48) that $\hat{N} > 0$. Furthermore, one can see from (43) and the assertion of Step 5 that $\hat{N} < N$. This completes the proof.

3.4 The case of $d_I \to 0$: Proof of Theorem 1.4

Proof the Theorem 1.4. By lemma 3.1, we have
\[
S(x; d_I) \leq \frac{q N (1 + \frac{\beta' d_S}{m q})}{d_I \left[1 - e^{-\frac{\beta d_S}{m q} (1 + \frac{\beta' d_S}{m q}) L} \right]} + \frac{\gamma^* N}{q} \leq K_S, \quad \forall x \in [0, L],
\]
where the positive constant $K_S$ is independent of $d_I$.

We apply a similar technique in Lemma 3.1, and employ the change of variable
\[
I(x; d_I) = e^{\frac{d_I}{q}} (1 - C' d_I)^x w(x; d_I),
\]
where $C'$ is a positive constant which will be determined later. Then, $w(x; d_I)$ satisfies
\[
\begin{align*}
\begin{cases}
d_I w_{xx} + q \left(1 - 2 \frac{C' d_I}{q^2} \right) w_x \\
+ [C' \left(\frac{C'}{q^2} - 1\right) - \gamma(x) + \beta(x) \frac{S}{1 + mI}] w = 0, & 0 < x < L, \\
d_I w_x(x) = C' \frac{d_I}{q} w(x), & x = 0, L.
\end{cases}
\end{align*}
\tag{50}
\]
Set \( x^* \in [0, L] \) such that \( w(x^*; d_I) = \max_{x^* \in [0, L]} w(x; d_I) \). Since \( w_x(0; d_I) > 0 \), then \( x^* \neq 0 \). If \( x^* \in (0, L) \), then \( w_x(x^*; d_I) = 0 \) and \( w_{xx}(x^*; d_I) \leq 0 \). By (50) we have

\[
C' \left( \frac{C'd_I}{q^2} - 1 \right) - \gamma(x^*) + \beta(x^*) \frac{S(x^*; d_I)}{1 + mI(x^*; d_I)} \geq 0.
\]

By choosing \( C' = 2\beta^* K_S \) and requiring \( 0 < d_I < q^2/(4\beta^* K_S) \), then

\[
C' \left( \frac{C'd_I}{q^2} - 1 \right) - \gamma(x^*) + \beta(x^*) \frac{S(x^*; d_I)}{1 + mI(x^*; d_I)} < C' \left( \frac{C'd_I}{q^2} - 1 \right) + \beta^* K_S < 0,
\]

it is a contradiction, i.e., \( w(x; d_I) \leq w(L; d_I) \), for any \( x \in [0, L] \). Therefore, for \( 0 < d_I < q^2/(4\beta^* K_S) \),

\[
I(x; d_I) \leq I(L; d_I)e^{-\frac{qN}{d_I} \left( 1 - \frac{q^2 K_S}{\beta^* d_I} \right) (L-x)} \quad \forall x \in [0, L].
\]

Moreover, it follows from Lemma 3.2 that

\[
I(x; d_I) \leq \frac{qN(1 + \frac{x}{d_I})}{d_I \left[ 1 - e^{-\frac{qN}{d_I} \left( 1 + \frac{x}{d_I} \right) L} \right]} e^{-\frac{qN}{d_I} \left( 1 - \frac{2q^2 K_S d_I}{\beta^*} \right) (L-x)} \quad \forall x \in [0, L]. \tag{51}
\]

Thus, one can conclude that \( I(x; d_I) \to 0 \) locally uniformly in \( [0, L] \) as \( d_I \to 0 \).

Next, we claim that \( \int_0^L I(x; d_I) \, dx \to 0 \), as \( d_I \to 0 \). From above arguments, we know that

\[
\left\| \frac{\beta(x)S(x; d_I)}{1 + mI(x; d_I)} \right\|_\infty \leq \beta^* K_S, \quad \forall x \in [0, L], \quad \text{for all small} \quad d_I > 0,
\]

and

\[
0 < \frac{\beta(x)S(x; d_I)I(x; d_I)}{1 + mI(x; d_I)} < \beta^* K_S I(x; d_I) \to 0
\]

for any given \( x \in [0, L] \) as \( d_I \to 0 \). Applying the Lebesgue dominated convergence theorem, we can get

\[
\gamma^* \lim_{d_I \to 0} \int_0^L I(x; d_I) \, dx \leq \lim_{d_I \to 0} \int_0^L \frac{\beta(x)S(x; d_I)I(x; d_I)}{1 + mI(x; d_I)} \, dx = 0,
\]

which confirms the claim.

By the transformation \( \tilde{S}(x; d_I) = S(x; d_I)e^{-\frac{x}{\beta^* K_S}} \), then \( \tilde{S}(x; d_I) \) satisfies

\[
\begin{cases}
  \frac{dS}{dx} + qS - \beta(x) \frac{S(x; d_I)I(x; d_I)}{1 + mI(x; d_I)} + \gamma(x)e^{-\frac{x}{\beta^* K_S}}I(x; d_I) = 0, & 0 < x < L, \\
  \tilde{S}_x = 0, & x = 0, L.
\end{cases}
\]

As

\[
\int_0^L \left| - \beta(x) \frac{\tilde{S}I}{1 + mI} + \gamma(x)e^{-\frac{x}{\beta^* K_S}}I \right| \, dx \leq (\beta^* K_S + \gamma^*) \int_0^L I \, dx \leq (\beta^* K_S + \gamma^*)N,
\]

then the elliptic \( L^1 \)-theory in [3] can be applied to (52) to conclude that

\[
|\tilde{S}(x; d_I)|_{W^{1,p}((0, L))} \leq C' \quad \text{for any} \quad p \geq 1,
\]

where the positive constant \( C' \) is independent of all small \( d_I \). Since \( W^{1,p}((0, L)) \) is compactly embedded into \( C([0, L]) \) for any given \( p > 1 \), by passing to a subsequence of \( d_I \), we can get that \( \tilde{S}(.; d_I) \to z \) uniformly on \( [0, L] \) as \( d_I \to 0 \) for some nonnegative \( z \in C([0, L]) \).
Integrating (52) over \((0, x)\), we have
\[
\int_0^x \left[ \beta(\tau) \frac{\hat{S}(\tau; d_I)}{1 + mI(\tau; d_I)} - \gamma(\tau) e^{-\frac{\tau}{\tau_0}} \right] d\tau.
\]
Furthermore, it follows from \(\lim_{d_I \to 0} \int_0^L I(x; d_I) dx = 0\) that
\[
\int_0^L \left[ \beta(\tau) \frac{\hat{S}(\tau; d_I)}{1 + mI(\tau; d_I)} - \gamma(\tau) e^{-\frac{\tau}{\tau_0}} \right] I(\tau; d_I) d\tau \leq (\beta K_S + \gamma^*) \int_0^L I(\tau; d_I) d\tau \to 0,
\]
as \(d_I \to 0\). Thus, by sending \(d_I \to 0\), we see that \(z\) fulfills
\[
\begin{cases}
  d_S z_x + qz = qz(0), & 0 < x < L, \\
  z_x(0) = 0.
\end{cases}
\]
The direct calculation yields that \(z\) is a nonnegative constant. It follows from the transformation \(\hat{S}(x; d_I) = S(x; d_I) e^{-\frac{z}{\tau_0} x}\) that \(S(x; d_I) \to z e^{\frac{z}{\tau_0} x}\) uniformly on \([0, L]\) as \(d_I \to 0\). In view of this fact, \(\lim_{d_I \to 0} \int_0^L I(x; d_I) dx = 0\) and the total population condition (6), one obtain that
\[
z = \frac{qN}{d_S (e^{\frac{z}{\tau_0}} - 1)}.
\]
The proof of Theorem 1.4 is completed. \(\square\)

### 3.5 The case of \(m \to \infty\): Proof of Theorem 1.5

Proof the Theorem 1.5. It follows from Lemmas 3.1 and 3.2 that \((S(x; m), I(x; m))\) is uniformly bounded for fixed \(d_S, d_I, q\) and all sufficiently large \(m\). Then, according to standard elliptic regularity and the Sobolev embedding theorem, we know that
\[
\|S(x; m)\|_{C^{1, \alpha}([0, L])} < C, \quad \|I(x; m)\|_{C^{1, \alpha}([0, L])} < C, \quad \text{for some } \alpha \in (0, 1),
\]
where positive constant \(C\) is independent of large \(m\), which may vary from place to place. Hence there exists a sequence \(\{m_n\}\) with \(m_n \to \infty\) as \(n \to \infty\) and corresponding positive solution \((S_n, I_n) \triangleq (S(x; m_n), I(x; m_n))\) of (5)-(6) with \(m = m_n\), such that
\[
(S_n, I_n) \to (S^\infty, I^\infty) \text{ uniformly on } [0, L], \text{ as } n \to \infty,
\]
where \((S^\infty, I^\infty) \in C([0, L]) \times C([0, L])\) and \(S^\infty, I^\infty \geq 0\) on \([0, L]\). There are two possibilities: \(\|m_n I_n\|_{L^\infty([0, L])} \to \infty\) as \(n \to \infty\) or \(\|m_n I_n\|_{L^\infty([0, L])} \leq C\) for all \(n\).

Let \(\theta_n \triangleq m_n I_n\), we first prove that \(\theta_n\) must be uniformly bounded. It is clear that \(\theta_n\) fulfills
\[
\begin{cases}
  d_I(\theta_n)_{xx} - q(\theta_n) x + \left[ \beta(x) \frac{S_n}{1 + \theta_n e^{\frac{z}{\tau_0} x}} - \gamma(x) \right] \theta_n = 0, & 0 < x < L, \\
  d_I(\theta_n) x - q(\theta_n) = 0, & x = 0, L.
\end{cases}
\]
It follows from the transformation \(\Theta_n(x) = \theta_n(x) e^{-\frac{z}{\tau_0} x}\) that \(\Theta_n\) satisfies
\[
\begin{cases}
  d_I(\Theta_n)_{xx} + q(\Theta_n) x + \left[ \beta(x) \frac{S_n}{1 + \theta_n e^{\frac{z}{\tau_0} x}} - \gamma(x) \right] \Theta_n = 0, & 0 < x < L, \\
  (\Theta_n) x = 0, & x = 0, L.
\end{cases}
\]
Since \( \beta S_n/(1 + \theta_n e^{\frac{q}{\Theta} x}) - \gamma \) is uniformly bounded, from the Harnack inequality (see, e.g., [32, lemma 2.2] or [40, Lemma 2.2]), we can obtain \( \Theta_n \) fulfills
\[
\max_{0 \leq x \leq L} \Theta_n(x) \leq C \min_{0 \leq x \leq L} \Theta_n(x),
\]
for some \( C > 0 \) independent of \( n \). It follows from the transformation and (55) that \( \theta_n \) satisfies
\[
\max_{0 \leq x \leq L} \theta_n(x) \leq C \min_{0 \leq x \leq L} \theta_n(x).
\]
Moreover, integrating (54) gives
\[
\gamma L \min_{0 \leq x \leq L} \theta_n(x) \leq \int_0^L \gamma(x) \theta_n(x) \, dx = \int_0^L \beta(x) \frac{S_n(x) \theta_n(x)}{1 + \theta_n(x)} \, dx \\
\leq \int_0^L \beta(x) S_n(x) \, dx \leq C.
\]
This, in combination with (56), yields that \( \theta_n \) must be uniformly bounded.

Since \( \|m_n I_n\|_{L^\infty((0,L))} \leq C \), then it is easy to see that
\[
I_n \to I^\infty = 0 \text{ uniformly on } [0,L], \text{ as } n \to \infty.
\]
From the total population condition, we can obtain
\[
S_n \to S^\infty = \frac{qN}{d_S(1 - e^{-\frac{q}{\Theta} x})} e^{-\frac{q}{\Theta}(L-x)} \text{ uniformly on } [0,L], \text{ as } n \to \infty.
\]
The standard elliptic regularity and the Sobolev embedding theorem ensure that there exists \( \theta \geq 0 \) in \( C([0,L]) \) such that \( \theta_n \to \theta \) in \( C([0,L]) \).

Next, we will show \( \theta > 0 \) on \([0,L]\). We argue by contradiction and suppose that \( \theta \equiv 0 \), i.e., \( \theta_n \to 0 \) as \( n \to \infty \). Define \( \hat{\theta}_n = \theta_n/\|\theta_n\|_{L^\infty((0,L))} \), then \( \|\hat{\theta}_n\|_{L^\infty((0,L))} = 1 \) for all \( n \geq 1 \), and \( \hat{\theta}_n \) solves
\[
\begin{cases}
d_1(\hat{\theta}_n)_{xx} - q(\hat{\theta}_n)_x + \left[ \beta(x) \frac{S_n}{1 + \theta_n} - \gamma(x) \right] \hat{\theta}_n = 0, & 0 < x < L, \\
\hat{d}_1(\hat{\theta}_n)_x - q\hat{\theta}_n = 0, & x = 0, L.
\end{cases}
\]
As before, by a standard compactness argument for elliptic equations, after passing to a further subsequence if necessary, we may assume that \( \hat{\theta}_n \to \hat{\theta} \) in \( C^1([0,L]) \) as \( n \to \infty \), where \( \hat{\theta} \in C^1([0,L]) \) with \( \hat{\theta} \geq 0 \) on \([0,L]\) and \( \|\hat{\theta}\|_{L^\infty((0,L))} = 1 \). In view of (57), (58) and (59), by sending \( n \to \infty \), it follows that \( \hat{\theta} \) satisfies
\[
\begin{cases}
d_1\hat{\theta}_{xx} - q\hat{\theta}_x + \left[ \beta(x) \frac{qNe^{\frac{q}{\Sigma} x}}{d_S(e^{\frac{q}{\Sigma} x} - 1)} - \gamma(x) \right] \hat{\theta} = 0, & 0 < x < L, \\
\hat{d}_1\hat{\theta}_x - q\hat{\theta} = 0, & x = 0, L.
\end{cases}
\]
Since \( \beta(x)qNe^{\frac{q}{\Sigma} x}/d_S(e^{\frac{q}{\Sigma} x} - 1) - \gamma(x) \) is uniformly bounded, upon a transformation and an application of the Harnack inequality, we can conclude that \( \hat{\theta} > 0 \) on \([0,L]\).

Then 0 is the principal eigenvalue of the eigenvalue problem (10), which contradicts the assumption \( R_0 > 1 \). Thus, \( \theta \geq 0 \) on \([0,L]\) and satisfies (9). Similar the above argument. Again from the transformation and the Harnack inequality, it can be concluded that \( \theta > 0 \) on \([0,L]\). That is, \( \theta_n \to \theta > 0 \) uniformly on \([0,L]\) as \( n \to \infty \).

By the upper-lower solution argument in [4], it can be obtained that \( \theta \) is the unique positive solution of (9). This completes the proof. \(\square\)
4 Discussion In this paper we propose a reaction-diffusion-advection SIS epidemic model with saturated incidence rate in advective heterogeneous environments. We first establish the uniform persistence property of (3), i.e., prove the existence of EE, provided the basic reproduction number greater than one. Moreover, one focus on the asymptotic profiles of the EE in the four cases: (i) large advection; (ii) small diffusion rate of the susceptible individual; (iii) small diffusion rate of the infected individual; (iv) large saturation. Our main results show that the asymptotic profiles of the susceptible and infected individuals obtained here are very different from that of the corresponding system with standard incidence infection mechanism and that of the system without advection.

For the reaction-diffusion-advection SIS model (2) with standard incidence infection mechanism, the asymptotic profiles of the EE were studied in [9, 21]. When the diffusion rate of the susceptible individual tends to zero, it was proved that the density of the infected vanishes on the entire habitat, while the density of the susceptible is positive but inhomogeneous everywhere. However, in this case, our result for (3) shows that a certain portion of the susceptible population concentrates at the downstream end, and the remaining portion of the susceptible population distributes in the habitat in a non-homogeneous way; on the other hand, the density of infected population is positive on the entire habitat. This result is also in sharp contrast with that the model (3) without advection in [53]. Compared with the model with standard incidence infection mechanism, it is clear that the large saturation can help to eradicate the disease. Compared with the one without advection, our results suggest that advection can help speed up the elimination of disease.

Our study shows that the disease will be eliminated if diffusion rate of infected individuals is sufficiently small or saturated incidence rate is sufficiently large, which may provide some implications on disease control and prediction.

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