ON POWER DEFORMATIONS OF UNIVALENT FUNCTIONS

YONG CHAN KIM AND TOSHIYUKI SUGAWA

Abstract. For an analytic function \( f(z) \) on the unit disk \( |z| < 1 \) with \( f(0) = f'(0) - 1 = 0 \) and \( f(z) \neq 0, 0 < |z| < 1 \), we consider the power deformation \( f_c(z) = z(f(z)/z)^c \) for a complex number \( c \). We determine those values \( c \) for which the operator \( f \mapsto f_c \) maps a specified class of univalent functions into the class of univalent functions. A little surprisingly, we will see that the set is described by the variability region of the quantity \( zf'(z)/f(z), \ 0 < |z| < 1 \), for the class in most cases which we consider in the present paper. As an unexpected by-product, we show boundedness of strongly spirallike functions.

1. Introduction

Let \( \mathcal{A} \) denote the set of analytic functions on the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) of the complex plane \( \mathbb{C} \). Set furthermore \( \mathcal{A}_0 = \{ f \in \mathcal{A} : f(0) = 1 \} \) and \( \mathcal{A}_1 = \{ f \in \mathcal{A} : f(0) = 0, f'(0) = 1 \} \). We note that a function \( h(z) \) belongs to \( \mathcal{A}_0 \) if and only if the function \( zh(z) \) belongs to \( \mathcal{A}_1 \). In what follows, \( f(z)/z \) will be regarded as a function in \( \mathcal{A}_0 \) for \( f \in \mathcal{A}_1 \). More concretely, for a function \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) in \( \mathcal{A}_1 \), the function \( f(z)/z \) is regarded as the analytic function \( 1 + a_2 z + a_3 z^2 + \cdots \). Let \( \mathcal{A}_0^\times \) be the set of invertible elements of \( \mathcal{A}_0 \) with respect to the ordinary multiplication; that is, \( \mathcal{A}_0^\times = \{ h \in \mathcal{A}_0 : h(z) \neq 0, z \in \mathbb{D} \} \). In what follows, \( \text{Log} h \) means the (single-valued) analytic branch of \( \log h \) in \( \mathbb{D} \) determined by \( \text{Log} h(0) = 0 \) for \( h \in \mathcal{A}_0^\times \). We also set \( \text{Arg} h = \text{Im} \text{Log} h \) for \( h \in \mathcal{A}_0^\times \). We note that \( \text{Log} \) maps \( \mathcal{A}_0^\times \) bijectively onto the complex vector space \( \mathcal{V} = \{ f \in \mathcal{A} : f(0) = 0 \} \).

The set \( \mathcal{S} \) consisting of all the univalent funtions in \( \mathcal{A}_1 \) has been the central object to study in the theory of univalent functions since early 20th century.

We are interested in classical subclasses of \( \mathcal{S} \) in the present paper. Let us now introduce them. A function \( f \in \mathcal{A}_1 \) is called \textit{convex} if \( f \) maps \( \mathbb{D} \) univalently onto a convex domain in \( \mathbb{C} \). We denote by \( \mathcal{K} \) the class of convex functions. It is well known that \( f \in \mathcal{A}_1 \) is convex if and only if

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}.
\]

Let \( \lambda \) be a number with \(-\pi/2 < \lambda < \pi/2\). For a point \( a \neq 0 \) in \( \mathbb{C} \), the \( \lambda \)-spirallike segment \([0, a]_\lambda\) is defined to be the set \( \{0\} \cup \{a \exp(-te^{i\lambda}) : 0 \leq t < +\infty\} \). A domain \( \Omega \) in \( \mathbb{C} \) is called \( \lambda \)-spirallike (about the origin) if \([0, a]_\lambda \subset \Omega \) for every \( a \in \Omega \). In particular, a \( 0 \)-spirallike domain is also called starlike as usual. A function \( f \in \mathcal{A}_1 \) is called \( \lambda \)-\textit{spirallike} if \( f \) maps \( \mathbb{D} \) univalently onto a \( \lambda \)-spirallike domain. The class of \( \lambda \)-spirallike functions

2000 Mathematics Subject Classification. Primary 30C45; Secondary 30C55.

Key words and phrases. univalent function, variability region, spirallike function.

The first author was supported by Yeungnam University (2010). The second author was supported in part by JSPS Grant-in-Aid for Scientific Research (B) 22340025.
will be denoted by $\mathcal{SP}(\lambda)$. Set $\mathcal{SP} = \bigcup_{-\pi/2 < \lambda < \pi/2} \mathcal{SP}(\lambda)$. The class of starlike functions $\mathcal{SP}(0)$ is also denoted by $\mathcal{S}^*$. It is also known that $f \in \mathcal{A}_1$ is $\lambda$-spirallike if and only if

$$\Re \left( e^{-i\lambda}zf'(z) \right) > 0, \quad 0 < |z| < 1.$$ 

For a real number $\alpha \leq 1$, a function $f \in \mathcal{A}_1$ is called starlike of order $\alpha$ if $\Re (zf'(z)/f(z)) \geq \alpha$, $z \in \mathbb{D}$. Let $\mathcal{S}^*(\alpha)$ denote the set of starlike functions of order $\alpha$. Similarly, for $0 < \alpha < 1$, a function $f \in \mathcal{A}_1$ is called strongly starlike of order $\alpha$ if $|\Arg (zf'(z)/f(z))| < \pi\alpha/2$, $z \in \mathbb{D}$, and the set of those functions will be denoted by $\mathcal{SS}(\alpha)$.

We can extend strong starlikeness to strong spirallikeness in an obvious way. Let $\lambda \in (-\pi/2, \pi/2)$ and $0 < \alpha < 1$. A function $f \in \mathcal{A}_1$ is called strongly $\lambda$-spirallike of order $\alpha$ if

$$\left| \Arg \frac{zf'(z)}{f(z)} - \lambda \right| < \frac{\pi\alpha}{2}, \quad z \in \mathbb{D}.$$ 

We denote by $\mathcal{SP}(\lambda, \alpha)$ the set of these functions. When we do not specify $\lambda$ and $\alpha$, we simply call it strongly spirallike. This sort of classes were first introduced by Bucka and Ciozda [1].

It is an important observation due to Alexander [2] that $f(z)$ is convex if and only if $g(z) = zf'(z)$ is starlike. The mapping $g \mapsto f$ is sometimes called the Alexander transformation and will be denoted by $J_1[f]$ in the sequel. More explicitly,

$$J_1[f](z) = \int_0^z \frac{f(\zeta)}{\zeta} \, d\zeta = \int_0^1 f(tz) \frac{dt}{t}$$

for $f \in \mathcal{A}_1$. Note also that $J_1(\mathcal{A}_1) = \mathcal{A}_1$.

A function $f \in \mathcal{A}_1$ is called close-to-convex if $\Re (e^{-i\lambda}f'/g') > 0$ in $\mathbb{D}$ for some $g \in \mathcal{K}$ and $\lambda \in (-\pi/2, \pi/2)$. The set of close-to-convex functions will be denoted by $\mathcal{C}$.

We have the inclusion relations $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}$ and $\mathcal{S}^* \subset \mathcal{SP} \subset \mathcal{S}$. See [5] for basic information about these subclasses of $\mathcal{S}$.

Several integral operators have been considered by many authors in connection with univalent functions. For instance, for $c \in \mathbb{C}$, we define

$$I_c[f](z) = \int_0^z (f'(\zeta))^c \, d\zeta$$

for $f \in \mathcal{LU} = \{ f \in \mathcal{A}_1 : f' \in \mathcal{A}_0^\infty \}$ (‘locally univalent’), and

$$J_c[f](z) = \int_0^z \left( \frac{f(\zeta)}{\zeta} \right)^c \, d\zeta$$

for $f \in \mathcal{ZF} = \{ f \in \mathcal{A}_1 : f(z)/z \in \mathcal{A}_0^\infty \}$ (‘zero-free’ except for the origin). Here and hereafter, the complex power $h^c$ for $h \in \mathcal{A}_0^\infty$ will be understood as $h^c = \exp(c \Log h)$. In particular, we see that $h^c \in \mathcal{A}_0^\infty$ for $h \in \mathcal{A}_0^\infty$ and $c \in \mathbb{C}$.

Note that $I_c(\mathcal{LU}) \subset \mathcal{LU}$ and $J_c(\mathcal{ZF}) \subset \mathcal{LU}$. For later convenience, we also set $\mathcal{D}_I = \mathcal{LU}$, $\mathcal{R}_I = \mathcal{LU}$, $\mathcal{D}_J = \mathcal{ZF}$ and $\mathcal{R}_J = \mathcal{LU}$.

In order to deal with these operators at once, let $X$ represent one of $I$, $J$ and $K$ which will be introduced below. For instance, $X_c$ and $D_X$ mean $I_c$ and $D = \mathcal{LU}$, respectively, when $X = I$. 
Theorem 1.1.

f for by $K$ (1.1) and Theorem A.

Thus, several typical subclasses of $S$ are introduced. The operators $K$ defined by $K(f, f)$, then we write $\mathcal{K}$ (1.2) and $K$ for $M \subset \mathcal{D}$ later in the authors’ paper [9].) When $\mathcal{M}$ consists of a single function $f$, then we write $[f, \mathcal{N}] = \{z \in \mathbb{C} : |z - a| < r\}$ and by $\mathcal{D}(a, r)$ its closure. We summarize known relations of this kind.

Theorem A.

1. $\mathcal{D}(0, \frac{1}{2}) \cup \{1\} \subset [\mathcal{S}, \mathcal{S}] \subset \overline{\mathcal{D}}(0, \frac{1}{3}) \cup \{1\}$ (Pfaltzgraf [12] and Royster [14]).
2. $\mathcal{D}(0, \frac{1}{2}) \subset [\mathcal{S}, \mathcal{S}] \subset \overline{\mathcal{D}}(0, \frac{1}{2})$ (Y. J. Kim and Merkes [10]).
3. $[\mathcal{K}, \mathcal{S}] = [\mathcal{S}^*, \mathcal{S}] = \overline{\mathcal{D}}(0, \frac{1}{2}) \cup \{1\}$ (Aksent’ev and Nezhmetdinov [1], cf. [8]).
4. $[\mathcal{S}^*, \mathcal{S}] = \overline{\mathcal{D}}(0, \frac{1}{2})$ (Merkes [11] Corollary 2).
5. $[\mathcal{S}^*, \mathcal{S}] = \overline{\mathcal{D}}(0, \frac{1}{2}) \cup \{1\}$ (Aksent’ev and Nezhmetdinov [1], cf. [8]).

In the present paper, we would like to propose yet another operator $K_c$ for $c \in \mathbb{C}$ defined by

$$K_c[f](z) = z \left( \frac{f(z)}{z} \right)^c$$

for $f \in \mathcal{D}$. This will be called the power deformation of $f$ with exponent $c$. Let $\mathcal{D}_K = \mathcal{R}_K = \mathcal{D}$. Of course, the present paper is not the first to define it. Indeed, this simple operation was used at many places before (for instance, [13], [14], [11]). It seems, however, that the operators $K_c$ have not been studied systematically in the literature.

Introduction of this operator is motivated by the following facts:

$$(1.1) \quad K_{e^\lambda, \cos \lambda}(\mathcal{S}^*) = \mathcal{S}^\lambda, \quad -\frac{\pi}{2} < \lambda < \frac{\pi}{2}$$

(see [11]) and

$$(1.2) \quad K_{1, -\lambda}(\mathcal{S}^*) = \mathcal{S}^\lambda, \quad 0 \leq \lambda < 1$$

(see [13], [11]). These relations easily follow from the relation

$$(1.3) \quad \frac{zf'(z)}{f(z)} = 1 - c + \frac{zf'(z)}{f(z)}.$$

Thus, several typical subclasses of $S$ can be obtained as power deformations of $\mathcal{S}^*$. We will show the following relations.

Theorem 1.1.

1. $[\mathcal{S}^*, \mathcal{S}] = [\mathcal{S}^*, \mathcal{S}^\lambda] = \overline{\mathcal{D}}(\frac{1}{2}, \frac{1}{2})$.
2. $[\mathcal{S}^\alpha, \mathcal{S}] = [\mathcal{S}^\alpha, \mathcal{S}^\lambda] = \overline{\mathcal{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$ for $0 \leq \alpha < 1$.
3. $[\mathcal{K}, \mathcal{S}] = [\mathcal{K}, \mathcal{S}^\lambda] = \overline{\mathcal{D}}(1, 1)$.
4. $[\mathcal{S}^\lambda, \mathcal{S}] = [\mathcal{S}^\lambda, \mathcal{S}^\lambda] = \overline{\mathcal{D}}\left(\frac{1-i\tan \lambda}{2\cos \lambda}, \frac{1}{2\cos \lambda}\right)$ for $\lambda \in (-\pi/2, \pi/2)$.
5. $[\mathcal{S}^\lambda, \mathcal{S}] = [\mathcal{S}^\lambda, \mathcal{S}^\lambda] = [0, 1]$. 

It is an interesting problem to describe or estimate the set

$$[\mathcal{M}, \mathcal{N}] = \{c \in \mathbb{C} : X_c[f] \in \mathcal{N} \text{ for all } f \in \mathcal{M}\} = \{c : X_c(\mathcal{M}) \subset \mathcal{N}\}$$

for $\mathcal{M} \subset \mathcal{D}$ and $\mathcal{N} \subset \mathcal{R}$ and a family of operators $X_c : \mathcal{D} \to \mathcal{R}$, $c \in \mathbb{C}$. (This kind of sets appeared earlier in the authors’ paper [9].) When $\mathcal{M}$ consists of a single function $f$, then we write $[f, \mathcal{N}] X = \{z \in \mathbb{C} : |z - a| < r\}$ and by $\mathcal{D}(a, r)$ its closure. We summarize known relations of this kind.

Theorem A.

1. $\mathcal{D}(0, \frac{1}{2}) \cup \{1\} \subset [\mathcal{S}, \mathcal{S}] \subset \overline{\mathcal{D}}(0, \frac{1}{3}) \cup \{1\}$ (Pfaltzgraf [12] and Royster [14]).
2. $\mathcal{D}(0, \frac{1}{2}) \subset [\mathcal{S}, \mathcal{S}] \subset \overline{\mathcal{D}}(0, \frac{1}{2})$ (Y. J. Kim and Merkes [10]).
3. $[\mathcal{K}, \mathcal{S}] = [\mathcal{S}^*, \mathcal{S}] = \overline{\mathcal{D}}(0, \frac{1}{2}) \cup \{1\}$ (Aksent’ev and Nezhmetdinov [1], cf. [8]).
4. $[\mathcal{S}^*, \mathcal{S}] = \overline{\mathcal{D}}(0, \frac{1}{2}) \cup \{1\}$ (Aksent’ev and Nezhmetdinov [1], cf. [8]).
5. $[\mathcal{S}^*, \mathcal{S}] = [\mathcal{S}^*, \mathcal{S}] = \overline{\mathcal{D}}(0, \frac{1}{2}) \cup \{1\}$ (Aksent’ev and Nezhmetdinov [1], cf. [8]).
In particular, if \( f \) gives a space \( \mathcal{S} \), then the following hold:

\[
(6) \quad [\mathcal{S}\mathcal{S}(\alpha), \mathcal{S}]_K = [\mathcal{S}\mathcal{S}(\alpha), \mathcal{S}\mathcal{P}]_K = \overline{D}\left(\frac{1-i \cot \frac{\pi}{2}}{2 \sin \frac{\pi}{2}}, \frac{1}{2 \sin \frac{\pi}{2}}\right) \cup \overline{D}\left(\frac{1+i \cot \frac{\pi}{2}}{2 \sin \frac{\pi}{2}}, \frac{1}{2 \sin \frac{\pi}{2}}\right) \quad \text{for} \quad 0 < \alpha < 1.
\]

\[
(7) \quad [\mathcal{S}\mathcal{P}(\lambda, \alpha), \mathcal{S}]_K = [\mathcal{S}\mathcal{P}(\lambda, \alpha), \mathcal{S}\mathcal{P}]_K = \overline{D}\left(\frac{1-i \tan \lambda_+}{2 \cos \lambda_+}, \frac{1}{2 \cos \lambda_+}\right) \cup \overline{D}\left(\frac{1+i \tan \lambda_-}{2 \cos \lambda_-}, \frac{1}{2 \cos \lambda_-}\right)
\quad \text{for} \quad |\lambda| < \pi \alpha / 2 < \pi / 2, \quad \text{where} \quad \lambda_\pm = \lambda \pm \pi (1 - \alpha) / 2.
\]

\[
(8) \quad [\mathcal{S}, \mathcal{S}]_K = [\mathcal{C}, \mathcal{S}]_K = \{0, 1\}.
\]

As an application of our investigation of power deformations, we obtain the following result, which is used in the second author’s paper \[10\].

**Theorem 1.2.** Let \( f \) be a strongly spirallike function. Then \( \log f(z) / z \) is bounded on \( \mathbb{D} \). In particular, \( f(z) \) is bounded on \( \mathbb{D} \).

We note that boundedness of strongly starlike functions is due to Brannan and Kirwan \[3\].

## 2. Fundamental Facts

In this section, we collect fundamental properties of the operators \( I_c, J_c, K_c \) and the sets \([\mathcal{M}, \mathcal{N}]_X\) of exponents for \( X = I, J, K \).

We first observe that the Alexander transformation \( J_1 \) maps the class \( \mathcal{ZF} \) of zero-free functions onto \( \mathcal{LU} \), the class of locally univalent functions, in a one-to-one manner. By definition, we have

\[
J_c = I_c \circ J_1 = J_1 \circ K_c
\]

for \( c \in \mathbb{C} \). In particular, we have \( K_c = J_1^{-1} \circ I_c \circ J_1 \). Furthermore, Alexander’s observation gives \( J_1(S^*) = K \). Therefore, we have \( J_c(S^*) = I_c(K) \) for \( c \in \mathbb{C} \).

Recall now that the set \( \mathcal{V} = \{ f \in \mathcal{A} : f(0) = 0 \} \) is a subspace of the complex vector space \( \mathcal{A} \). We consider the bijective maps \( \Phi : \mathcal{LU} \to \mathcal{V} \) and \( \Psi : \mathcal{ZF} \to \mathcal{V} \) defined by \( \Phi[f] = \log f' \) and \( \Psi[f](z) = \log f(z) / z \). Then the operators \( I_c \) and \( K_c \) can be viewed as scalar multiplication in \( \mathcal{V} \) when we identify \( \mathcal{LU} \) and \( \mathcal{ZF} \) with \( \mathcal{V} \) through the maps \( \Phi \) and \( \Psi \), respectively. In other words, \( I_c[f] = \Phi^{-1}(c \Phi[f]) \) and \( K_c[f] = \Psi^{-1}(c \Psi[f]) \). In particular, we easily have the relations \( I_c \circ I_c = I_{c'} \) and \( K_c \circ K_{c'} = K_{c c'} \) for \( c, c' \in \mathbb{C} \).

Moreover, we can even introduce linear structures to the sets \( \mathcal{LU} \) and \( \mathcal{ZF} \), although we will not go into details in the present paper. Indeed, such a linear structure on \( \mathcal{LU} \) was first considered by Hornich \[7\] (see also \[8\]).

We now collect obvious properties of the sets \([\mathcal{M}, \mathcal{N}]_X\) for \( X = I, J, K \).

**Lemma 2.1.** Let \( X \) represent one of \( I, J, K \) and let \( \mathcal{M}, \mathcal{M}', \mathcal{M}_\lambda \subset \mathcal{D}_X (\lambda \in \Lambda), \mathcal{N}, \mathcal{N}' \subset \mathcal{R}_X \). Then the following hold:

\[
(1) \quad [\mathcal{M}, \mathcal{N}]_X \supseteq [\mathcal{M}', \mathcal{N}]_X \quad \text{if} \quad \mathcal{M} \subset \mathcal{M}'.
\]

\[
(2) \quad [\mathcal{M}, \mathcal{N}]_X \subset [\mathcal{M}, \mathcal{N}']_X \quad \text{if} \quad \mathcal{N} \subset \mathcal{N}'.
\]

\[
(3) \quad [\bigcup_{\lambda \in \Lambda} \mathcal{M}_\lambda, \mathcal{N}]_X = \bigcap_{\lambda \in \Lambda} [\mathcal{M}_\lambda, \mathcal{N}]_X.
\]

\[
(4) \quad [\bigcap_{\lambda \in \Lambda} \mathcal{M}_\lambda, \mathcal{N}]_X \supseteq [\bigcup_{\lambda \in \Lambda} \mathcal{M}_\lambda, \mathcal{N}]_X.
\]

\[
(5) \quad [X_c(\mathcal{M}), \mathcal{N}]_X = \frac{1}{c} [\mathcal{M}, \mathcal{N}]_X \quad \text{for} \quad c \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad X = I, K.
\]
(6) \([\mathcal{M}, \mathcal{N}]_X\) is a closed subset of \(\mathbb{C}\) if \(\mathcal{N}\) is closed in the topology of local uniform convergence on \(\mathbb{D}\).

(7) \([\mathcal{M}, \mathcal{N}]_K = [J_1(\mathcal{M}), J_1(\mathcal{N})]_I\).

Here, we define \(cE = \{cz : z \in E\}\) for \(E \subset \mathbb{C}\) and \(c \in \mathbb{C}\). We remark that \(S, S^*(\alpha), K, C, SS(\alpha), SP(\lambda), SP(\lambda, \alpha), SP, LU, \mathcal{ZF}\) are all closed in the topology of local uniform convergence on \(\mathbb{D}\).

The power deformation effects on boundedness. We summarize a few facts about it.

**Lemma 2.2.** For a function \(f \in \mathcal{ZF}\) and \(c \in \mathbb{C}\), let \(f_c = K_c[f]\).

1. If \(\text{Log } f(z)/z\) is bounded in \(\mathbb{D}\), then so is \(\text{Log } f_c(z)/z\) for every \(c \in \mathbb{C}\).
2. If \(\log |f(z)/z|\) is unbounded in \(\mathbb{D}\), then so is \(\log |f_c(z)/z|\) for every \(c > 0\).
3. Suppose that \(f\) is unbounded and univalent in \(\mathbb{D}\) and that \(\text{Arg } f(z)/z\) is bounded in \(\mathbb{D}\). Then \(f_c\) is never univalent when \(\text{Re } c < 0\) while \(f_c\) is unbounded when \(\text{Re } c > 0\).

**Proof.** Assertions (1) and (2) are clear when we look at the relation \(\text{Log } f_c(z)/z = c \text{Log } f(z)/z\).

We prove assertion (3). Let \(c = a + ib\). By assumption, we have a sequence \(z_n (n = 1, 2, \ldots )\) in \(\mathbb{D}\) such that \(|f(z_n)| \to \infty\) and \(|z_n| \to 1\) as \(n \to \infty\). Since we have the relation

\[
\text{Log } f_c(z)/z = a \text{Log } f(z)/z - b \text{Arg } f(z)/z,
\]

\(|f_c(z_n)/z_n| \to 0\) as \(n \to \infty\) if \(b < 0\). Then, \(f_c\) is never univalent. Also, the above relation tells us that \(f_c\) is unbounded if \(b > 0\). \(\square\)

For a subclass \(\mathcal{M}\) of \(\mathcal{ZF}\), we denote by \(V(\mathcal{M})\) the variability region of the quantity \(zf'(z)/f(z)\) for \(f \in \mathcal{M}\); more concretely,

\[
V(\mathcal{M}) = \{zf'(z)/f(z) : f \in \mathcal{M}, z \in \mathbb{D}\}.
\]

Note that \(V(\mathcal{M})\) is a domain (a connected non-empty open set) unless \(\mathcal{M} \subset \{\text{id}\}\). This has a close connection with \([\mathcal{M}, \mathcal{S}]_K\). Let \(T\) be the Möbius transformation defined by

\[
T(w) = \frac{1}{1 - w}.
\]

Then, we have \([\mathcal{M}, \mathcal{S}]_K \subset \mathbb{C} \setminus T(V(\mathcal{M}))\) by the following result.

**Lemma 2.3.** For a subclass \(\mathcal{M}\) of \(\mathcal{ZF}\), the set \([\mathcal{M}, LU]_K\) and the variability region \(V(\mathcal{M})\) of \(zf'(z)/f(z)\) are related by

\[
[\mathcal{M}, LU]_K = \mathbb{C} \setminus T(V(\mathcal{M})).
\]

**Proof.** Let \(c\) be a finite point in \(T(V(\mathcal{M}))\). Then there are \(f \in \mathcal{M}\) and \(z_0 \in \mathbb{D}\) such that \(c = T(z_0f'(z_0)/f(z_0))\); namely, \(z_0f'(z_0)/f(z_0) = 1 - 1/c\). Then by \([13]\) the function \(f_c = K_c[f]\) satisfies

\[
\frac{z_0f'_c(z_0)}{f_c(z_0)} = 1 - c \frac{z_0f'(z_0)}{f(z_0)} = 0.
\]

In particular, \(K_c[f]\) is not locally univalent at \(z_0\) and therefore \(c \notin [\mathcal{M}, LU]_K\).

We can also trace back the above argument to prove the converse. \(\square\)
For an $f \in \mathcal{ZF}$, set $V(f) = \{zf'(z)/f(z) : z \in \mathbb{D}\}$. Then, in particular, we have the relation
\[ \{f, \mathcal{LU}\}_K = \mathbb{C} \setminus T(V(f)) \].

We can also derive the following corollary.

**Corollary 2.4.** Let $\mathcal{M}$ be a subclass of $\mathcal{ZF}$ which contains a function $f \neq \text{id}$. Then $[\mathcal{M}, \mathcal{LU}]_K$ is a compact subset of $\mathbb{C}$.

**Proof.** Since $V(\mathcal{M})$ is a domain containing 1, the image $T(V(\mathcal{M}))$ under $T$ is a domain in the Riemann sphere containing $\infty$. Therefore, its complement $[\mathcal{M}, \mathcal{LU}]_K$ is compact in $\mathbb{C}$. $\square$

Therefore, $[\mathcal{M}, \mathcal{N}]_K$ is compact when $\mathcal{M}$ and $\mathcal{N}$ are chosen from $\mathcal{S}, \mathcal{S}'(\alpha), \mathcal{K}, \mathcal{C}, \mathcal{SP}(\lambda, \alpha), \mathcal{SP}$. We summarize information about the variability regions of typical subclasses of $\mathcal{S}$.

**Lemma 2.5.** One has the following relations:

1. $V(\mathcal{S}') = \{w : \text{Re } w > 0\}$.
2. $V(\mathcal{S}'(\alpha)) = \{w : \text{Re } w > \alpha\}$.
3. $V(\mathcal{K}) = \{w : \text{Re } w > 1/2\}$.
4. $V(\mathcal{SP}(\lambda)) = \{w : \text{Re } e^{-i\lambda}w > 0\}$.
5. $V(\mathcal{SP}) = \mathbb{C} \setminus (-\infty, 0]$.
6. $V(\mathcal{SS}(\alpha)) = \{w : |\text{arg } w| < \pi\alpha/2\}$.
7. $V(\mathcal{SP}(\lambda, \alpha)) = \{w : |\text{arg } w - \lambda| < \pi\alpha/2\}$.
8. $V(\mathcal{S}) = V(\mathcal{C}) = \mathbb{C} \setminus \{0\}$.

**Proof.** We have to show a relation of the form $V(\mathcal{M}) = B$ for a class $\mathcal{M}$ and a subdomain $B$ of $\mathbb{C}$ in each case. When $V(\mathcal{M}) \subset B$ is trivial by the definition of $\mathcal{M}$, we just give a function $f \in \mathcal{M}$ such that $zf'(z)/f(z)$ covers the domain $B$ in order to show $B \subset V(\mathcal{M})$.

1. Consider the Koebe function $k(z) = z/(1 - z)^2$.
2. Consider the function $K_{1-\alpha}|k(z)| = z/(1 - z)^{2(1-\alpha)}$.
3. E. Strohacker showed the relation $\mathcal{K} \subset \mathcal{S}'(\frac{1}{2})$ (see [5, p. 251] for instance). Therefore, we have $V(\mathcal{K}) \subset \{w : \text{Re } w > 1/2\}$. On the other hand, $l(z) = z/(1 - z)$ is convex and $zl'(z)/l(z) = 1/(1 - z)$ maps $\mathbb{D}$ conformally onto the half-plane $\text{Re } w > 1/2$. Therefore, we have $V(\mathcal{K}) = \{w : \text{Re } w > 1/2\}$.
4. Consider the function $K_{\alpha, \cos \lambda}|k(z)| = z/(1 - z)^{2e^{i\lambda} \cos \lambda}$.
5. This is clear because $V(\mathcal{SP}) = \bigcup_{\lambda} V(\mathcal{SP}(\lambda))$.
6. Consider the function $f \in \mathcal{A}_1$ determined by $zf'(z)/f(z) = (\frac{1+z}{1-z})^\alpha$.
7. Consider the function $f \in \mathcal{A}_1$ determined by $zf'(z)/f(z) = (\frac{1+ze^{2i\lambda/\alpha}}{1-e^{2i\lambda/\alpha}})^\alpha$.
8. The assertion $V(\mathcal{C}) = \mathbb{C} \setminus \{0\}$ can be found in [17]. Since $V(\mathcal{C}) \subset V(\mathcal{S}) \subset \mathbb{C} \setminus \{0, 1\}$, the other assertion follows, too. $\square$

3. **Proof of main results**

**Proof of Theorem [17]**
We need to prove the assertion \([\mathcal{M}, \mathcal{S}]_K = [\mathcal{M}, \mathcal{S}]_K = A\) for \(\mathcal{M} = S^*, S^*(\alpha), \mathcal{K}, \mathcal{S}, L\mathcal{U}, \mathcal{S}\mathcal{P}(\lambda), \mathcal{S}\mathcal{P}, \mathcal{S}\mathcal{S}(\alpha), \mathcal{S}\mathcal{P}(\lambda, \alpha), \mathcal{S}, \mathcal{C}\) and the subset \(A \subset \mathbb{C}\) which appears in the right-hand side of the relation in the corresponding assertion (though we should omit \([\mathcal{M}, \mathcal{S}]_K\) in the case of (8)). First we observe that the set \(A\) is indeed equal to \(\mathbb{C} \setminus T(V(\mathcal{M}))\) in each case by virtue of Lemma 2.3. Therefore, by Lemma 2.3 and Lemma 2.1 (2), we obtain

\([\mathcal{M}, \mathcal{S}]_K \subset [\mathcal{M}, \mathcal{S}]_K \subset [\mathcal{M}, L\mathcal{U}]_K = \mathbb{C} \setminus T(V(\mathcal{M})) = A\).

Therefore, it is enough to show that \(A \subset [\mathcal{M}, \mathcal{S}]_K\) with the exception of (8). We will take this strategy unless a simpler way is available. We divide the proof into several pieces according to the numbering.

[Proof of (1):] We show the implication \(\overline{\mathbb{D}(\frac{1}{2}, \frac{1}{2})} \subset [S^*, \mathcal{S}]_K\). Let \(f \in S^*\) and set \(f_c = K_c[f]\) for \(c \in \mathbb{C}\). Then, by (1.2), we have \(f_c \in S^*(1-c) \subset S^*\) for \(0 \leq c \leq 1\). Next, by (1.1), we see that \(f_{ce^{-i\lambda} \cos \lambda} = K_{e^{-i\lambda} \cos \lambda}[f_c] \in \mathcal{S}\mathcal{P}(\lambda) \subset \mathcal{S}\mathcal{P}\) for \(\lambda \in (-\pi/2, \pi/2)\). In view of the relation \(e^{i\lambda} \cos \lambda = \left(e^{2i\lambda} + 1ight)/2\), we obtain

\([ce^{-i\lambda} \cos \lambda : 0 \leq c \leq 1, -\pi/2 < \lambda < \pi/2] = \overline{\mathbb{D}(\frac{1}{2}, \frac{1}{2})}\).

Thus we have shown that \(\overline{\mathbb{D}(\frac{1}{2}, \frac{1}{2})} \subset [S^*, \mathcal{S}]_K\).

[Proof of (2) and (4):] We combine Lemma 2.1 (5) with (1.2) and (1.1) to obtain (2) and (4). Here, we note the relation \(1/(e^{i\lambda} \cos \lambda) = 1 - i \tan \lambda\).

[Proof of (3):] By the Strohhäcker theorem: \(\mathcal{K} \subset S^*(\frac{1}{2})\) which is mentioned in the proof of Lemma 2.3 we obtain

\([\mathcal{K}, \mathcal{S}]_K \supset [S^*(\frac{1}{2}), \mathcal{S}]_K = \overline{\mathbb{D}(1, 1)}\).

[Proof of (5):] It is enough to show that \([0, 1] \subset [\mathcal{S}\mathcal{P}, \mathcal{S}]_K\). This follows from the fact that \([0, 1] \subset [\mathcal{S}\mathcal{P}(\lambda), \mathcal{S}]_K\) for every \(\lambda \in (-\pi/2, \pi/2)\) by (1).

[Proof of (6):] Since \(\mathcal{S}\mathcal{S}(\alpha) = \mathcal{S}\mathcal{P}(0, \alpha)\), this follows from (7).

[Proof of (7):] Since \(\mathcal{S}P(\lambda, \alpha) = \mathcal{S}\mathcal{P}(\lambda_+) \cap \mathcal{S}\mathcal{P}(\lambda_-)\), Lemma 2.1 (4) yields the relation

\([\mathcal{S}\mathcal{P}(\lambda_+), \mathcal{S}\mathcal{P}]_K \supset \overline{\mathbb{D}(\frac{1-i \tan \lambda_+}{2}, \frac{1}{2 \cos \lambda_+})} \cup \overline{\mathbb{D}(\frac{1-i \tan \lambda_-}{2}, \frac{1}{2 \cos \lambda_-})}\).

[Proof of (8):] It is enough to see \(\{0, 1\} \subset [\mathcal{S}, \mathcal{S}]_K\). This is trivial. \(\square\)

Remark. As we saw in the proof, we actually showed the relations

\([\mathcal{M}, \mathcal{S}]_K = [\mathcal{M}, L\mathcal{U}]_K = \mathbb{C} \setminus T(V(\mathcal{M}))\)

for \(\mathcal{M} = S^*, S^*(\alpha), \mathcal{K}, \mathcal{S}\mathcal{P}(\lambda), \mathcal{S}\mathcal{P}, \mathcal{S}\mathcal{S}(\alpha), \mathcal{S}\mathcal{P}(\lambda, \alpha), \mathcal{S}, \mathcal{C}\). Under this situation, if a function \(f_0 \in \mathcal{M}\) satisfies \(V(f_0) = V(\mathcal{M})\), then \(\mathcal{C} \setminus T(V(\mathcal{M})) = \mathcal{M}_K \subset [f_0, L\mathcal{U}]_K = \mathbb{C} \setminus T(V(f_0))\) and therefore \([f_0, \mathcal{S}]_K = [\mathcal{M}, \mathcal{S}]_K\). Therefore, the above relations are valid. For instance, the Koebe function \(k\) satisfies \([k, \mathcal{S}]_K = [k, \mathcal{S}\mathcal{P}]_K = \overline{\mathbb{D}(\frac{1}{2}, \frac{1}{2})}\).

In order to prove Theorem 1.2 we recall the following result.
Lemma 3.1 (Goodman [6]). \(|\text{Arg } f(z)/z| \leq 2 \arcsin |z| < \pi, \ |z| < 1, \text{ for } f \in \mathcal{S}^*\).

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( f \) be strongly \( \lambda \)-spirallike of order \( \alpha \) with \( |\lambda| < \pi \alpha/2 < \pi/2 \). Put \( g = K_{e^{-i\lambda}/\cos \lambda}[f] \). By Theorem 1.1 (7) together with Lemma 2.1 (5), we have
\[
[g, \mathcal{S}]_K = [K_{e^{-i\lambda}/\cos \lambda}[f], \mathcal{S}]_K = e^{i\lambda} \cos \lambda [f, \mathcal{S}]_K \\
\subset e^{i\lambda} \cos \lambda \left( \mathbb{D} \left( \frac{1 - i \tan \lambda_+}{2}, \frac{1}{2 \cos \lambda_+} \right) \cup \mathbb{D} \left( \frac{1 - i \tan \lambda_-}{2}, \frac{1}{2 \cos \lambda_-} \right) \right),
\]
where \( \lambda_\pm = \lambda \pm \pi(1 - \alpha)/2 \). Observe that \([g, \mathcal{S}]_K\) is not contained in the closed right half-plane.

On the other hand, by (1.1), \( g \in \mathcal{S}^* \) because \( f \in \mathcal{S}(\lambda) \). Thus \( g \) is univalent and \( \text{Arg } g(z)/z \) is bounded by Lemma 3.1. We now suppose that \( g \) was unbounded in \( \mathbb{D} \). Then Lemma 2.2 implies that \([g, \mathcal{S}]_K\) would be contained in the closed right half-plane \( \text{Re } c \geq 0 \). This is a contradiction. We have shown that \( g \) is bounded, and hence, \( \log g(z)/z \) is bounded. We now have boundedness of \( \log f(z)/z \) by Lemma 2.2 (1). \( \square \)

It is somewhat strange that we obtained a boundedness result for strongly spirallike functions without making any concrete estimate of functions involved. We also note that the above \( g \) satisfies the relation \( zg'(z)/g(z) = czf'(z)/f(z) + 1 - c \), where \( c = e^{-i\lambda}/\cos \lambda = 1 - i \tan \lambda \). Therefore, \( g \) is not necessarily strongly starlike unless \( \lambda = 0 \).

**References**

1. L. A. Aksent’ev and I. R. Nezhmetdinov, *Sufficient conditions for univalence of certain integral transforms* (Russian), Trudy semin. po kraev. zadacham. Kazan 18 (1982), 3–11, English translation in: Amer. Math. Soc. Transl. 136 (2) (1987), 1–9.
2. J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. 17 (1915), 12–22.
3. D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. (2) 1 (1969), 431–443.
4. Cz. Bucka and K. Ciozda, *On a new subclass of the class \( S \)*, Ann. Polon. Math. 28 (1973), 153–161.
5. P. L. Duren, *Univalent Functions*, Springer-Verlag, 1983.
6. A. W. Goodman, *The rotation theorem for starlike univalent functions*, Proc. Amer. Math. Soc. 286 (1953), 278–286.
7. H. Hornich, *Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktionen*, Monatsh. Math. 73 (1969), 36–45.
8. Y. C. Kim, S. Ponnusamy, and T. Sugawa, *Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives*, J. Math. Anal. Appl. 299 (2004), 433–447.
9. Y. C. Kim and T. Sugawa, *The Alexander transform of a spirallike function*, J. Math. Anal. Appl. 325 (2007), 608–611.
10. Y. J. Kim and E. P. Merkes, *On an integral of powers of a spirallike function*, Kyungpook Math. J. 12 (1972), 249–253.
11. E. P. Merkes, *Univalence of an integral transform*, Contemporary Math. 38 (1985), 113–119.
12. J. A. Pfaltzgraaff, *Univalence of the integral of \( f'(z)^k \)*, Bull. London Math. Soc. 7 (1975), 254–256.
13. B. Pinchuk, *Functions of bounded boundary rotation*, Israel J. Math. 10 (1971), 6–16.
14. W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. 12 (1965), 385–387.
15. A. Schild, *On a class of univalent, star shaped mappings*, Proc. Amer. Math. Soc. 9 (1958), 751–757.
16. T. Sugawa, *Quasiconformal extension of strongly spirallike functions*, preprint.
17. L.-M. Wang, in preparation.

DEPARTMENT OF MATHEMATICS EDUCATION, YEUNGNAM UNIVERSITY, 214-1 DAEDONG GYONGSAN 712-749, KOREA
E-mail address: kimyc@ynu.ac.kr

DIVISION OF MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCES, TOHOKU UNIVERSITY, AOBÁ-KU, SENDAI 980-8579, JAPAN
E-mail address: sugawa@math.is.tohoku.ac.jp