Iterative Reed–Muller Decoding

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Abstract—Reed–Muller (RM) codes are known for their good maximum likelihood (ML) performance in the short block-length regime. Despite being one of the oldest classes of channel codes, finding a low complexity soft-input decoding scheme is still an open problem. In this work, we present a belief propagation (BP) decoding architecture for RM codes based on their rich automorphism group. The decoding algorithm can be seen as a generalization of multiple-bases belief propagation (MBBP) using polar BP as constituent decoders. We provide extensive error-rate performance simulations and compare our results to existing decoding schemes. We report a near-ML performance for the RM(3,7)-code (e.g., 0.05 dB away from the ML bound at BLER of $10^{-4}$) at a competitive computational cost. To the best of our knowledge, our proposed decoder achieves the best performance of all iterative RM decoders presented thus far.

Fig. 1: Block diagram of automorphism ensemble decoding. We focus on the case where $M$ constituent belief propagation (BP) decoders are used. The usage of successive cancellation (SC) and successive cancellation list (SCL) as constituent decoders is presented in the extended version of this paper [1].

I. INTRODUCTION

The current trend of ultra-reliable low-latency communications (URLLC) applications has urged the need for efficient short length coding schemes in combination with the availability of efficient decoders. Besides many other coding schemes, this has lead to the revival of one of the oldest error-correcting codes, namely Reed–Muller (RM) codes [2], [3] – potentially also due to some existent similarities between RM codes and the newly developed family of polar codes [4], [5]. On the one hand, RM codes, as an example of algebraic codes, are known to be capacity-achieving over the Binary Erasure Channel (BEC) for a given rate [6], [7]. Moreover, and practically even more relevant, they enjoy an impressive error-rate performance under maximum likelihood (ML) decoding, which also holds in the short length regime. To this extent, several decoding algorithms have been developed in the course of RM decoding. On the other hand, to the best of our knowledge, there is still a lack of practical decoders that are characterized by near-ML performance and feasible decoding complexity/latency.

RM decoders can be grouped into two main categories, iterative and non-iterative decoders which we will shortly revisit in the following. In the literature, the best known decoder for RM codes over an additive white Gaussian noise (AWGN) channel is Dumer’s recursive list decoding algorithm [8], which is now known under the name SCL decoding [9], and a variant using permutations. Recently, a recursive projection-aggregation (RPA) decoding algorithm for RM codes was proposed in [10], which can be viewed as a weighted BP decoder over a redundant factor graph [11], making use of the symmetry of the RM codes (i.e., its large automorphism group). RM codes under RPA decoding were shown to outperform the error-rate performance of CRC-aided polar codes under SCL decoding. A more general usage of the rich automorphism group of RM codes to aid the decoding process is reported in [5], along with the decoding of RM codes using a redundant parity-check matrix proposed earlier in [12]. In this work, for the sake of comparison, we consider the following iterative decoders: multiple-bases belief propagation (MBBP) decoding [13], minimum weight parity-check belief propagation (MWPC-BP) decoding [14], neural belief propagation (NBP) decoding [15], pruned neural belief propagation (pruned-NBP) decoding [16]. The inherent parallel nature of iterative decoders allows high throughput implementations. Besides, their soft-in/soft-out (SISO) nature makes them suitable for iterative detection and decoding.

In this paper, we propose a new iterative decoding scheme, extending and generalizing some of the previously mentioned decoding algorithms. To the best of our knowledge, our proposed decoding algorithm achieves the best practical iterative decoding performance of the RM(3,7)-code presented thus far (0.05 dB away from the ML bound at BLER of $10^{-4}$). Fig. 1 shows an abstract view of our proposed decoding algorithm.

II. PRELIMINARIES

A. Reed–Muller Codes

We interpret each message of an RM$(r,m)$ code as a multi-linear polynomial $u(z)$ in $m$ binary variables $z_j$ (with
$j \in \{0, \ldots, m-1\}$) and maximum degree $r$, over the finite field $\mathbb{F}_2$. To obtain the codeword $\mathbf{x}$, the message polynomial is evaluated at all points in the space $\mathbb{F}_2^n$, resulting in $N = 2^m$ codeword bits [2], [3].

### B. Automorphism Group

The automorphism group (or permutation group) $\text{Aut}(\mathcal{C})$ of a code $\mathcal{C}$ is the set of permutations $\pi$ of the codeword bit indices that map $\mathcal{C}$ onto itself, i.e.,

$$\pi(\mathbf{x}) \in \mathcal{C} \quad \forall \mathbf{x} \in \mathcal{C} \quad \forall \pi \in \text{Aut}(\mathcal{C}),$$

where $\pi(\mathbf{x})$ results in the vector $\mathbf{x}'$ with $x'_i = x_{\pi(i)}$. In other words, every codeword is mapped to another (not necessarily different) codeword of the same code. The automorphism group of RM codes is well known as the general affine group $\text{GA}(m)$ over the field $\mathbb{F}_2$ [17]. $\text{GA}(m)$ is the group of all affine bijections over $\mathbb{F}_2^n$, i.e., pairs $(\mathbf{A}, \mathbf{b})$ defining the mapping $\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{b}$, with a non-singular matrix $\mathbf{A} \in \mathbb{F}_2^{m \times m}$ and an arbitrary vector $\mathbf{b} \in \mathbb{F}_2^{m \times 1}$. The vectors $\mathbf{z}, \mathbf{z}' \in \mathbb{F}_2^{m \times 1}$ are the binary representations of the code bit positions $i$ and $\pi(i)$, respectively, i.e., $i = \sum_{j=0}^{m-1} z_j \cdot 2^j$. An important subgroup of $\text{GA}(m)$ is the set of stage-shuffle permutations $\Pi(m)$, corresponding to the special case where $\mathbf{A}$ is a permutation matrix and $\mathbf{b} = \mathbf{0}$.

### C. Iterative RM Decoding

In this section, we briefly revise the different iterative decoding techniques which can be used for RM codes.

1) Naïve Belief Propagation Decoding: BP decoding can be performed over the Tanner graph of a code’s parity-check matrix. However, for high-density parity-check codes like RM codes, the performance of this decoder is usually poor due to the numerous short cycles in the graph.

2) Belief Propagation Decoding over Forney-style Factor Graph: Rather than on a Tanner graph, BP decoding can also be performed over a Forney-style factor graph (FFG), constructed from check and variable nodes of degree three [18]. Fig. 2 shows the FFG of the RM(1,3)-code.

As the frozen nodes always contribute the same Log-likelihood ratios (LLRs) (i.e., a priori LLRs of $+\infty$) to the equations, the FFG can be reduced as shown on the right in Fig. 2, by removing edges of constant value. This potentially reduces the number of performed arithmetic operations per iteration while preserving the same performance [18]. In this work, whenever the BP decoding is used over the FFG, we use the reduced version.

3) Minimum Weight Parity-Check Belief Propagation Decoding: Minimum weight parity-check belief propagation (MWPC-BP) decoding introduced in [14] is based on the concept of iterative decoding over an overcomplete parity-check matrix. An online algorithm tailored to the noisy received sequence $\mathbf{y}$ is used to construct the overcomplete parity-check matrix only based on minimum weight parity-checks.

In this work, we only consider the field $\mathbb{F}_2$ and hence, we omit the size of the field in the notation, i.e., we write $\text{GA}(m)$ instead of $\text{GA}(m,2)$.

4) Neural Belief Propagation Decoding: Neural belief propagation (NBP) decoding as introduced in [15] treats the unrolled Tanner graph of the code as a neural network (NN), while assigning trainable weights to all of its edges leading to a soft Tanner graph. These trainable weights are optimized via stochastic gradient descent (SGD) methods.

5) Pruned Neural Belief Propagation Decoding: Pruned neural belief propagation (pruned-NBP) decoding [16] combines the idea of MWPC-BP together with NBP. To get started, a redundant parity-check matrix containing (all or some of) the minimum weight parity-checks is constructed. During the offline training phase, all edges connected to a check node are assigned a single trainable weight and the least contributing check nodes are pruned (i.e., removed) from the graph. The authors of [16] refer to this decoder as $D_1$. An enhanced version, decoder $D_3$, is the result of assigning trainable weights per edge at the expense of larger memory requirements to save all weights per edges. Furthermore, a pruned NBP decoder without any weights is introduced as $D_2$, however with the expense of a significant degradation in error-rate performance.

### III. AUTOMORPHISM ENSEMBLE DECODING

Ensemble decoding uses multiple constituent decoders (i.e., a decoder ensemble of size $M$) to generate a set of codeword estimates and selects one of these codewords, using a predefined metric, as the decoder output. Typically, a least-squares metric is used, as this corresponds to the ML decision for the AWGN channel. Hence, this method is also called ML-in-the-list. This can be formulated as

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \| \mathbf{x} - \mathbf{y} \|^2 = \arg\max_{\mathbf{x}} \sum_{i=0}^{N-1} \hat{x}_{i,j} \cdot y_i,$$

where $\hat{x}_{i,j} \in \{\pm 1\}$, $\hat{x}_j$ is the estimated codeword from decoder $j$ for the received vector $\mathbf{y}$ and $\hat{\mathbf{x}}$ is the final codeword estimate of the ensemble.

MBBP is a well-known example for ensemble decoding that uses $M$ BP decoders, each based on a different random parity-check matrix [13]. Another instance of ensemble decoding is belief propagation list (BPL) decoding of polar codes, where the stages of the FFG are randomly permuted for each constituent decoder [19].

In this work, we propose automorphism ensemble BP decoding (Aut-M-BP) for RM codes. The main idea is to make use of the already existent polar BP decoder. Furthermore, we use the RM code symmetry in the decoding algorithm itself, as permitting a valid RM codeword with a permutation from
the code’s automorphism group results in another valid RM codeword.

An abstract view of our proposed Aut-M-BP decoding algorithm is shown in Fig. 1. The input to the decoder is the received noisy codeword \( y \). We randomly sample \( M \) different permutations from the RM automorphism group, where each permutation is denoted by \( \pi_j \), with \( j \) being the decoder index and \( j \in \{1, 2, \ldots, M\} \). The \( y \)-vector is interleaved (i.e., permuted) with the \( M \) different permutations \( \pi_j \) leading to \( M \) permuted noisy codewords \( y'_j \), where \( j \in \{1, 2, \ldots, M\} \).

Now we decode every \( y'_j \)-vector independently using BP, whose output is the interleaved estimated codeword \( \hat{x}'_j \). A de-interleaving phase is applied to all \( M \) interleaved estimated codewords \( \hat{x}'_j \) and, thus, we have \( M \) codeword estimates \( \hat{x}_j \). Let \( \text{BP}(\cdot) \) denote the BP decoding function that maps \( y'_j \) to \( \hat{x}'_j \). Then we can write the interleaved decoding as

\[
\hat{x}_j = \pi_j^{-1}(\text{BP}(\pi_j(y))).
\]

Similar to MBBP and BPL decoding, our proposed decoding algorithm uses the ML-in-the-list picking rule according to Eq. (2) to choose the most likely codeword from the list to get the final decoder output \( \hat{x} \).

As the decoders are linear, their decoding behavior is only dependent on the noise induced by the channel, and not the choice of the transmitted codeword. Therefore, decoding using automorphisms according to Eq. (3) corresponds to permuting the noise. It is reasonable to conclude that suboptimal (i.e., not ML) decoders may react differently to noise realizations in different permutations, which is exactly the property that automorphism ensemble decoding seeks to exploit.

Our proposed algorithm can be therefore seen as a natural generalization of the BPL decoder [19]: We still use \( M \) parallel independent BP decoders. However, we use a more general set of permutations. It was shown in [20] that the stage-shuffling of the FFG is equivalent to a bit-interleaving operation while keeping the factor graph unchanged; these permutations correspond to the automorphism subgroup \( \Pi(m) \).

In contrast, we use permutations from the whole RM code automorphism group \( GA(m) \) (rather than only \( \Pi(m) \), which is used in BPL decoding as proposed in [19]).

It is worth mentioning that the usage of a BP decoder as a constituent decoder has some similarities when compared to automorphism group decoding of Bose-Chaudhuri-Hocquenghem (BCH) and Golay codes for the BEC [21] and for the AWGN channel [22]. Automorphism group decoding is based on permuting the received sequence exploiting automorphisms of the code while applying an iterative message passing algorithm.

IV. RESULTS

Regarding practical applications, both error-rate performance and the computational complexity of the decoding scheme have to be considered. We compare the described decoding schemes for the RM(3,7)-code with \( N = 128 \) and \( k = 64 \). In the following, we specify the parameters of some of the compared decoders for reproducibility:

- MWPC-BP utilizes 5% of the minimum-weight parity-checks, as reported in [14].
- MBBP operates over \( M = 60 \) randomly generated parity-check matrices with 6 iterations each.
- Neural-BP uses all 94488 minimum-weight parity-checks over 6 iterations.
- The pruned neural-BP employs on average 3% of the minimum-weight parity-checks over a total of 6 iterations. We consider the three variants of this decoder as introduced in [16], with tied weights (\( D_1 \)), no weights (\( D_2 \)) and free weights (\( D_3 \)).
- For our proposed Aut-BP, we show results for both \( M = 8 \) and \( M = 32 \) randomly chosen permutations from the full automorphism group. Here, up to \( N_{\text{max}} = 200 \) iterations are performed with, however, an early stopping condition (i.e., when \( \hat{x} = G\hat{G} \)) employed to reduce the average total number of iterations. Furthermore, the FFGs have been reduced from 1792 to 1334 box-plus and addition operations per full iteration by removing operations with constant results, as presented in Section II-C2.

A. ERROR-RATE PERFORMANCE

In Fig. 3 and Fig. 4, we showcase the error-rate performance of the described decoding schemes for the RM(3,7)-code over the AWGN channel using binary phase shift keying (BPSK) mapping. Furthermore, we show the ML performance of the code as provided by [23]. As no data beyond an signal-to-noise ratio (SNR) of 3.5 dB is available, the ML performance is estimated using order-4 ordered statistic decoding (OSD).

Fig. 3 compares the non-SGD-optimized iterative decoders with Aut-BP and ML. One can observe that the naive BP decoding suffers from a very poor performance for RM codes, compared to BP decoding over FFG. Moreover, using multiple H-matrices in MBBP leads to a significant enhancement in performance. All of the previous methods are outperformed
by both Aut-8-BP and MWPC-BP, with similar performance. However, in the high SNR regime, Aut-8-BP beats MWPC-BP by 0.2 dB. Aut-32-BP even closes the gap to ML to less than 0.05 dB at a block error rate (BLER) of $10^{-4}$. We further observe the gains of sampling from $GA(m)$ in Aut-BP compared to $\Pi(m)$, as used in BPL decoding. We can see that for all ensemble sizes $M$, sampling from $GA(m)$ consistently outperforms $\Pi(m)$ by up to 0.3 dB. This confirms the sub-optimality of restricting the automorphisms to a small subgroup. Moreover, Aut-32-BP can even outperform SCL [9] with list size $L = 32$ (i.e., SCL-32) and its permutation variant DS-32 [8] in the high SNR regime.

Fig. 4 compares the SGD-optimized (NN-based) decoders with Aut-BP and ML. Here, the neural-BP decoder is much closer to the ML bound, and the pruned variant with free weights $D_3$ outperforms NBP, which uses all overcomplete parity-checks. The pruned NBP $D_2$ decoder without weights suffers from a significant performance degradation. Over the whole SNR range, $D_1$ and $D_3$ are outperformed by Aut-32-BP. Furthermore, it can be seen that using only $M = 8$ parallel BP decoders (i.e., Aut-8-BP) results in a small performance degradation of less than 0.2 dB over the whole SNR range, offering an attractive trade-off for lower complexity. Simulation results for the RM(4,8)-code are presented in [1].

### B. Iterative Decoding Complexity

For the RM(3,7)-code, we compare the complexity of the iterative decoding algorithms with error-rate performance close to ML by counting the number of computing operations required to decode one RM codeword. The first column of Table I gives the list of the operations we use. However, as non-trivial multiplication is significantly more complex than the other considered operations, we introduce a weighting factor for multiplication $w_{\text{mul}} = 3$ (equivalent to the number of full-adders in a 4+1 bit fixed-point implementation) to make the comparison fair. For all decoders, we assume that the box-plus operation is implemented as

$$L_1 \boxplus L_2 = \sgn(L_1) \cdot \sgn(L_2) \cdot \min(|L_1|, |L_2|) + f_+(|L_1 + L_2|) - f_+(|L_1 - L_2|),$$

with $f_+(|x|) = \log(1 + \exp(-|x|))$ is a correction term that can be well-approximated by a short look-up table (LUT). Furthermore, check nodes (CNs) are assumed to be efficiently implemented using the box-minus operator as

$$L_j \rightarrow i = \bigoplus_{L_{j' \rightarrow j}} \left( \bigoplus_{L_{j' ightarrow j}} L_j - f_+\left(|L_j - L_{j'}|\right) \right) \otimes L_{j' \rightarrow j}.$$  

Fig. 5 shows the total number of weighted operations to decode one codeword of the RM(3,7)-code. We can see that out of all methods, neural-BP using the full overcomplete $H$-matrix has the highest complexity. The corresponding pruned decoders $D_1$ and $D_3$ result in approximately 3% of that complexity. MWPC-BP is computationally more expensive, as it uses more parity-check equations and more iterations are required to achieve a good error-rate performance. It has to be noted, however, that we only list the complexity of iterative decoding, not of (adaptively) obtaining the parity-check equations. Hence, the overall complexity of MWPC-BP is higher than the presented number. MBBP has roughly half the complexity of MWPC-BP, while Aut-32-BP without stopping condition has twice the complexity of MWPC-BP. However, when a (G-matrix-based) stopping condition is used,
the average number of iterations of Aut-BP is significantly reduced. To illustrate this, we measure the average required number of iterations until convergence for both $M = 8$ and $M = 32$ with respect to the SNR of the AWGN channel, while $N_{\text{fl,max}} = 200$. At an SNR of 3.65 dB, corresponding to the BLER of $10^{-4}$, each decoder of the Aut-32-BP ensemble requires an average of 4.55 iterations, making Aut-BP the least complex decoder of the compared algorithms (see Fig. 5), without losing any error-rate performance. Aut-8-BP requires an SNR of 3.84 dB to reach this BLER performance, however, reducing the complexity again by a factor of 4, using only 3.96 iterations on average. Note that even though the ML-in-the-list decision can be only made after all constituent decoders are terminated, terminated decoders can already start decoding the next received vector (e.g., in a super-scalar implementation).

It is also worth noting here that the approaches proposed in [16] can also profit from early stopping. However, the effect is less significant since the majority of check node evaluations are performed in the first two iterations. Moreover, the regular structure of the FFG may result in more preferable implementations of Aut-BP compared to the random memory access patterns observed in conventional iterative decoders.

V. CONCLUSION

In this work, we propose an automorphism-based iterative decoding algorithm for RM codes. We present near-MC error-rate performance for the RM(3,7)-code operating only 0.05 dB away from the ML bound at BLER of $10^{-4}$. Furthermore, we report a decoder complexity comparison for the RM(3,7)-code from an operation level perspective. To the best of our knowledge, our proposed iterative Aut-BP decoders using the RM code automorphism group as permutations are the best iterative decoders reported in literature thus far in terms of error-rate performance.

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Table I: Basic operations and their usage in iterative decoding. *For BP with weights (MWPC-BP, NBP, D1, D3).

| Operation     | Weight | 2-input ⋄ | CN (deg. D) | VN (deg. D) | ML out of M | FFG BP Stopping |
|---------------|--------|-----------|-------------|-------------|-------------|----------------|
| sgn(x) ⋄ sgn(y) | 1-1    | 2-1       | 0-0         | 0-0         | m-2N/2+2N-1 |                |
| sgn(x) ⋄ y    | 1-1    | 2D-1      | 0-0         | M-0         |             |                |
| min(|x|, |y|)    | 1-1    | 0-0       | 0-0         | M-1-0       |             |                |
| max(x, y)     | max    | 0-0       | 0-0         | M-1-0       |             |                |
| $f_s(x)$ (LUT) | 1-2    | 4D-2      | 0-0         | 0-0         |             |                |
| $x+y, x-y$    | 1-4    | 8D-4      | 2D-0        | M(N-1)-0    |             |                |
| $x-y$         | 3-0    | 0-0       | [D+1]-0     | 0-0         |             |                |

Weighted total: 16D-9 2D^3-[3D+3] 2MN-1 m-2N/2+2N-1