Sharp bounds for Hardy-type operators on mixed radial-angular spaces

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Abstract. In this paper, by using the rotation method, we calculate that the sharp bound for $n$-dimensional Hardy operator $\mathcal{H}$ on mixed radial-angular spaces. Furthermore, we also obtain the sharp bound for $n$-dimensional fractional Hardy operator $\mathcal{H}_\beta$ from $L^p_{\|x\|}L^p_\vartheta(\mathbb{R}^n)$ to $L^q_{\|x\|}L^q_\vartheta(\mathbb{R}^n)$, where $0 < \beta < n$, $1 < p, q, \bar{p}, \bar{q} < \infty$ and $1/p - 1/q = \beta/n$. By using duality, the corresponding results for the dual operators $\mathcal{H}^*$ and $\mathcal{H}^*_\beta$ are also established. In addition, the sharp weak-type estimate for $\mathcal{H}$ is also considered.

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1. Introduction

Let $f$ be a non-negative integrable function on $\mathbb{R}$. The classical Hardy operator and its dual operator are defined by

$$H(f)(x) := \frac{1}{x} \int_0^x f(t) dt, \quad H^*(f)(x) := \int_x^{\infty} \frac{f(t)}{t} dt,$$

respectively, where $x \neq 0$.

As we know, the classical Hardy operator was initially introduced by Hardy [21], who showed the following Hardy inequalities:

$$\|H(f)\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}, \quad \|H^*(f)\|_{L^p} \leq p \|f\|_{L^p},$$

where the constants $\frac{p}{p-1}$, $p$ are best possible.

Later, Hardy-type operators were extended to higher dimension by Faris [11]. In 1995, Christ and Grafakos [4] gave an equivalent version of $n$-dimensional Hardy

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operator
\[
\mathcal{H}(f)(x) := \frac{1}{\Omega_n|x|^n} \int_{|y|<|x|} f(y) dy, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
where \(f\) is a non-negative measurable function on \(\mathbb{R}^n\) and \(\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}\) is the volume of the unit ball in \(\mathbb{R}^n\). By a direct computation, the dual operator of \(\mathcal{H}\) can be defined by setting, for any locally integrable function \(f\) and \(x \in \mathbb{R}^n\),
\[
\mathcal{H}^*(f)(x) := \int_{|y|>|x|} \frac{f(y)}{\Omega_n|y|^n} dy, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

By using the rotation method, Christ and Grafakos [4] showed that the norms of \(\mathcal{H}\) and \(\mathcal{H}^*\) on \(L^p(\mathbb{R}^n)\) (1 < \(p < \infty\)) are also \(\frac{p}{p-1}\) and \(p\). Moreover, the sharp weak estimate for \(\mathcal{H}\) was obtained by Zhao et al. [48] as follows:

For 1 ≤ \(p \leq \infty\), we have
\[
\|\mathcal{H}(f)\|_{L^{p,\infty}} \leq 1 \cdot \|f\|_{L^p},
\]
where the constant 1 is best possible.

Similarly, for 0 < \(\beta < n\), the \(n\)-dimensional fractional Hardy operator \(\mathcal{H}_\beta\) and its dual operator \(\mathcal{H}^*_{\beta}\) are defined by
\[
\mathcal{H}_\beta(f)(x) := \frac{1}{(\Omega_n^{1/n}|x|)^{n-\beta}} \int_{|y|<|x|} f(y) dy, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
and
\[
\mathcal{H}^*_\beta(f)(x) := \int_{|y|>|x|} \frac{f(y)}{(\Omega_n^{1/n}|y|)^{n-\beta}} dy, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
respectively.

In recent years, much attention has been paid to the sharp bounds for Hardy-type operators on different function spaces. For instance, the sharp bounds for Hardy-type operators and their dual operators on weak Lebesgue spaces were considered in [17, 18, 25, 35, 40, 44, 47, 48, 49]. See also [12, 13, 32, 42] for sharp constants for multilinear Hardy operator on Lebesgue spaces and Morrey-type spaces. Moreover, Hardy-type inequalities have been extended to different settings. For example, Wu et al. [44] and Guo et al. [19] considered the sharp constants for Hardy operators and their dual operators on Heisenberg group in the linear and multilinear situations. Fu et al. [16, 45] investigated sharp constants for Hardy-type operators on p-adic field. Maligranda et al. [34], Guo and Zhao [20], Fan and Zhao [9] studied Hardy \(q\)-type integral inequalities. In addition, there are many extensions of the Hardy-type operators, such as weighted Hardy-Littlewood averages [14, 18, 46] and Hausdorff operators [3, 3, 27, 43]. Readers can refer to the book [22] to get some earlier development of Hardy-type inequalities. We also refer the readers to the review papers [26, 31] for some recent progress on Hardy-type operators and their related topics.

Recently the mixed radial-angular spaces have been successfully used in studying Strichartz estimates and partial differential equations to improve the corresponding results (see [2, 6, 10, 33, 41], etc.). After that, many operators in harmonic analysis have been proved to be bounded on these spaces. For instance, the extrapolation theorems on mixed radial-angular spaces were build by Duoandikoetxea and
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Oruetxebarria [8] to study the boundedness of a large class of operators which are weighted bounded. In addition, the boundedness of some operators with rough kernels on mixed radial-angular spaces were also considered by Liu et al. [28, 29, 30].

Inspired by the references mentioned above, it is natural to ask whether we can obtain the boundedness, or furthermore the sharp bounds, for Hardy-type operators on mixed radial-angular spaces. In this paper, we will give an affirmative answer. More precisely, we will study the sharp bounds for $n$-dimensional Hardy operator $\mathcal{H}$ and $n$-dimensional fractional Hardy operator $\mathcal{H}_\beta$ on mixed radial-angular spaces. By using a duality argument, we also obtain the sharp constants for their dual operators $\mathcal{H}^*$ and $\mathcal{H}_\beta^*$ on mixed radial-angular spaces. Moreover, the sharp weak type estimate for $\mathcal{H}$ is also considered.

Now we recall the definition of mixed radial-angular spaces.

**Definition 1.1.** For $n \geq 2$, $1 \leq p, \bar{p} \leq \infty$, the mixed radial-angular space $L^p_{|x|}L^{\bar{p}}_{\theta} (\mathbb{R}^n)$ consists of all functions $f$ in $\mathbb{R}^n$ for which

$$\|f\|_{L^p_{|x|}L^{\bar{p}}_{\theta}} := \left( \int_0^\infty \left( \int_{S^{n-1}} |f(r,\theta)|^{\bar{p}} \, d\theta \right)^{\frac{p}{\bar{p}}} r^{n-1} \, dr \right)^{1/p} < \infty,$$

where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. If $p = \infty$ or $\bar{p} = \infty$, then we have to make appropriate modifications.

Similarly, we can define the weak mixed radial-angular spaces.

**Definition 1.2.** For $n \geq 2$, $1 \leq p, \bar{p} \leq \infty$, the weak mixed radial-angular space $wL^p_{|x|}L^{\bar{p}}_{\theta} (\mathbb{R}^n)$ consists of all functions $f$ in $\mathbb{R}^n$ for which

$$\|f\|_{wL^p_{|x|}L^{\bar{p}}_{\theta}} := \sup_{\lambda > 0} \lambda \|\chi\{|x|: |f(x)| > \lambda\}\|_{L^p_{|x|}L^{\bar{p}}_{\theta}} < \infty,$$

where $\chi\{|x|: |f(x)| > \lambda\}$ denotes the characteristic function of the set $\{x \in \mathbb{R}^n : |f(x)| > \lambda\}$.

In fact, the mixed radial-angular spaces can be seen as particular cases of mixed-norm Lebesgue spaces introduced by Benedek and Panzone [1]. We refer readers to [7, 23, 24, 36, 37, 39] for more studies on mixed-norm Lebesgue spaces and their applications in PDE.

The organization of this article is as follows. The sharp bounds for $n$-dimensional Hardy operator and its dual operator on mixed radial-angular spaces are obtained in Sect. 2. We calculate the operator norms of $n$-dimensional fractional Hardy operator on mixed radial-angular spaces in Sect. 3. In addition, we also establish a sharp weak type estimate for $n$-dimensional Hardy operator on mixed radial-angular spaces Sect. 4.

### 2. Sharp bounds for $\mathcal{H}$ and $\mathcal{H}^*$ from $L^p_{|x|}L^{\bar{p}1}_{\theta} (\mathbb{R}^n)$ to $L^p_{|x|}L^{\bar{p}2}_{\theta} (\mathbb{R}^n)$

The main result in this section is the following:
Therefore we have that
\[ |H|_{L_p^w \to L_p^w} = \frac{p}{p - 1} w_n^{1/p_2 - 1/p_1}, \]
where \( w_n = 2\pi^{n/2}/\Gamma(n/2) \) is the induced measure of \( S^{n-1} \).

**Proof.** We borrow some ideas from [4,13] and use the method of rotation. Set
\[ g(x) = \frac{1}{w_n} \int_{S^{n-1}} f(|x|\theta) d\theta, \quad x \in \mathbb{R}^n. \] (6)

Obviously, \( g \) is a radial function and it was proved by Fu et al [13, Proof of Theorem 1] that \( H(g) \) is equal to \( H(f) \). Moreover, we have
\[ \|g\|_{L_p^w}^{p/p_1} = \left( \int_0^\infty \left( \int_{S^{n-1}} |g(r, \theta)|^{p_1} d\theta \right)^{p/p_1} r^{n-1} dr \right)^{1/p} \]
\[ = w_n^{1/p_1} \left( \int_0^\infty |g(r)|^p r^{n-1} dr \right)^{1/p}, \] (7)
where \( g(r) \) can be recognized as \( g(r) = g(x) \) for any \( x \in \mathbb{R}^n \) with \( |x| = r \), since \( g \) is a radial function.

Combining (6) and (7), and using Hölder’s inequality, we get
\[ \|g\|_{L_p^w}^{p/p_1} = w_n^{1/p_1} \left( \int_0^\infty \left( \int_{S^{n-1}} f(r\theta) d\theta \right)^p r^{n-1} dr \right)^{1/p} \]
\[ = w_n^{1/p_1 - 1} \left( \int_0^\infty \left( \int_{S^{n-1}} |f(r\theta)|^{p_1} d\theta \right)^{p/p_1} \left( \int_{S^{n-1}} d\theta \right)^{p/p_1} r^{n-1} dr \right)^{1/p} \]
\[ \leq \left( \int_0^\infty \left( \int_{S^{n-1}} |f(r\theta)|^{p_1} d\theta \right)^{p/p_1} r^{n-1} dr \right)^{1/p} = \|f\|_{L_p^w}^{p/p_1}. \]

Therefore we have that
\[ \frac{\|H(f)\|_{L_p^w}^{p/p_2}}{\|f\|_{L_p^w}^{p/p_1}} \leq \frac{\|H(g)\|_{L_p^w}^{p/p_2}}{\|g\|_{L_p^w}^{p/p_1}}. \]
That is to say, the operator \( H \) and its restriction to radial functions have the same operator norm from \( L_p^w \) to \( L_p^w \). Therefore, we can assume that \( f \) is a radial function.

For a radial function \( f \), \( H(f) \) is also a radial function. Consequently,
\[ \|H(f)\|_{L_p^w}^{p/p_2} = \left( \int_0^\infty \left( \int_{S^{n-1}} |H(f)(r, \theta)|^{p_2} d\theta \right)^{p/p_2} r^{n-1} dr \right)^{1/p} \]
\[ = \left( \int_0^\infty \left( \int_{S^{n-1}} |H(f)(r, \theta)|^{p_2} d\theta \right)^{p/p_2} r^{n-1} dr \right)^{1/p} \]
where \( |\mathcal{H}(f)(r)| \) can be recognized as \( \mathcal{H}(f)(r) = \mathcal{H}(f)(x) \) for any \( x \in \mathbb{R}^n \) with \( |x| = r \), since \( \mathcal{H}(f) \) is a radial function.

Denote \( B(0, R) \) by the ball centered at the origin with radius \( R \). By changing variables, we further have

\[
\mathcal{H}(f)(r) = \frac{1}{\Omega_n} \int_{B(0,1)} f(ry)dy.
\]  

(9)

Combining (8) with (9), and using Minkowski’s inequality, we have

\[
\|\mathcal{H}(f)\|_{L_p^{\beta_2} L_n^{\beta_1}} = \frac{w_n^{1/\beta_2}}{\Omega_n} \left( \int_0^\infty \left( \int_{B(0,1)} |f(ry)dy| r^{n-1}dr \right)^{1/p} \right)
\]

\[
\leq \frac{w_n^{1/\beta_2}}{\Omega_n} \int_{B(0,1)} \left( \int_0^\infty |f(ry)|^p r^{n-1}dr \right)^{1/p} dy
\]

\[
= \frac{w_n^{1/\beta_2 - 1/\beta_1}}{\Omega_n} \int_{B(0,1)} |y|^{-n/p} dy \|f\|_{L_p^{\beta_1}}
\]

\[
= \frac{w_n^{1/\beta_2 - 1/\beta_1}}{\Omega_n} \int_{B(0,1)} |y|^{-n/p} dy \|f\|_{L_p^{\beta_1}}
\]

\[
= \frac{p}{p-1} w_n^{1/\beta_2 - 1/\beta_1} \|f\|_{L_p^{\beta_1}},
\]

where we have used the identity \( w_n = n\Omega_n \).

To prove the constant \( \frac{p}{p-1} w_n^{1/\beta_2 - 1/\beta_1} \) is best possible, we take

\[
f_\varepsilon(x) = \begin{cases} 0, & |x| \leq 1, \\ |x|^{-(\frac{1}{p} + \varepsilon)}, & |x| > 1, \end{cases}
\]

where \( 0 < \varepsilon < 1 \). A direct computation yields

\[
\|f_\varepsilon\|_{L_p^{\beta_1}} = \frac{w_n^{1/\beta_1}}{(p\varepsilon)^{1/p}}.
\]

On the other hand,

\[
\mathcal{H}(f_\varepsilon)(x) = \begin{cases} 0, & |x| \leq 1, \\ \frac{1}{\Omega_n} |x|^{-(\frac{1}{p} + \varepsilon)} \int_{B(0,1)} |y|^{-(\frac{1}{p} + \varepsilon)} dy, & |x| > 1. \end{cases}
\]

So we have

\[
\|\mathcal{H}(f_\varepsilon)\|_{L_p^{\beta_2} L_n^{\beta_1}} = \frac{w_n^{1/\beta_2}}{\Omega_n} \left( \int_1^\infty \left| r^{-(\frac{1}{p} + \varepsilon)} \int_{B(0,1)} |y|^{-(\frac{1}{p} + \varepsilon)} dy \right|^pr^{n-1}dr \right)^{1/p}.
\]
This observation enables us to obtain the sharp bound for $H$ and furthermore, for any $T$.

To state the main result in this section, we need the following Lemma.

Remark 2.1. When $p = \bar{p}_1 = \bar{p}_2 \in (1, \infty)$ in Theorem 2.1, we recover the results in [4, 13].

A standard argument yields that for $1 < p, \bar{p} < \infty$, the dual space of $L^p_{\|x\|}L^p_{\theta}(\mathbb{R}^n)$ is $L^{p'}_{\|x\|}L^{p'}_{\theta}(\mathbb{R}^n)$, where $p'$ and $\bar{p}'$ satisfy $1/p + 1/p' = 1$ and $1/\bar{p} + 1/\bar{p}' = 1$ (see [1]), and furthermore, for any $f \in L^{p'}_{\|x\|}L^{p'}_{\theta}(\mathbb{R}^n)$,

$$
\|f\|_{L^{p'}_{\|x\|}L^{p'}_{\theta}} = \sup_{\|g\|_{L^{p'}_{\|x\|}L^{p'}_{\theta}}} \int_{\mathbb{R}^n} f(x)g(x)dx.
$$

This observation enables us to obtain the sharp bound for $H^*$ from $L^p_{\|x\|}L^{\bar{p}_1}_{\theta}(\mathbb{R}^n)$ to $L^p_{\|x\|}L^{\bar{p}_2}_{\theta}(\mathbb{R}^n)$ by using duality.

Theorem 2.2. Let $n \geq 2$, $1 < p, \bar{p}_1, \bar{p}_2 < \infty$. Then the operator $H^*$ defined in (2) is bounded from $L^p_{\|x\|}L^{\bar{p}_1}_{\theta}(\mathbb{R}^n)$ to $L^p_{\|x\|}L^{\bar{p}_2}_{\theta}(\mathbb{R}^n)$. Moreover,

$$
\|H^*\|_{L^p_{\|x\|}L^{\bar{p}_1}_{\theta} \rightarrow L^p_{\|x\|}L^{\bar{p}_2}_{\theta}} = p\|w\|_{L^{\bar{p}_2-1/\bar{p}_1}}.
$$

3. Sharp bound for $H_\beta$ from $L^p_{\|x\|}L^{\bar{p}}_{\theta}(\mathbb{R}^n)$ to $L^q_{\|x\|}L^{\bar{q}}_{\theta}(\mathbb{R}^n)$

To state the main result in this section, we need the following Lemma.
Lemma 3.1. Let \( n \geq 2, 0 < \beta < n \) and \( 1 < p < q < \infty \) such that \( 1/p - 1/q = \beta/n \). Then the \( n \)-dimensional fractional Hardy operator \( \mathcal{H}_\beta \) defined in (4) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \). Moreover,

\[
\|\mathcal{H}_\beta\|_{L^p \to L^q} = C_{p,q,n,\beta},
\]

where \( C_{p,q,n,\beta} = \left( \frac{\nu}{q} \right)^{1/q} \left( \frac{\nu}{q} \right) \cdot B \left( \frac{\nu}{q}, \frac{n}{\beta} \right) \right)^{-\beta/n} \) and \( B \) is the Beta function, i.e., \( B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1} \, dt \) for any complex numbers \( z_1 \) and \( z_2 \) with the positive real parts.

Our main result in this section can be stated as follows:

Theorem 3.1. Let \( n \geq 2, 0 < \beta < n, 1 < \bar{p}, \bar{q} < \infty \) and \( 1 < p < q < \infty \) such that \( 1/p - 1/q = \beta/n \). Then the \( n \)-dimensional fractional Hardy operator \( \mathcal{H}_\beta \) defined in (4) is bounded from \( L^\bar{p}_{|x|} L^\bar{q} (\mathbb{R}^n) \) to \( L^q_{|x|} L^\bar{q} (\mathbb{R}^n) \). Moreover,

\[
\|\mathcal{H}_\beta\|_{L^\bar{p}_{|x|} L^\bar{q} \to L^q_{|x|} L^\bar{q}} = C_{p,q,n,\beta} w_n^{1/q-1/p+\beta/n}.
\]

Proof. A similar process as in the proof of Theorem 2.1 yields that the norm of the operator \( \mathcal{H}_\beta \) from \( L^\bar{p}_{|x|} L^\bar{q} (\mathbb{R}^n) \) to \( L^q_{|x|} L^\bar{q} (\mathbb{R}^n) \) is equal to the norm of \( \mathcal{H}_\beta \) acting on radial functions.

For a radial function \( f \), \( \mathcal{H}_\beta(f) \) is also a radial function. Consequently,

\[
\|\mathcal{H}_\beta(f)\|_{L^q_{|x|} L^\bar{q}} = \left( \int_0^\infty \left( \int_{S^{n-1}} |\mathcal{H}_\beta(f)(r, \theta)|^\bar{q} r^{n-1} \, d\theta \right)^{q/\bar{q}} r^{n-1} \, dr \right)^{1/q} = w_n^{1/q} \left( \int_0^\infty |\mathcal{H}_\beta(f)(r, \theta)|^q r^{n-1} \, dr \right)^{1/q} = w_n^{1/q-1/q} \left( \int_0^\infty \int_{S^{n-1}} |\mathcal{H}_\beta(f)(r, \theta)|^q r^{n-1} \, d\theta \, dr \right)^{1/q} = w_n^{1/q-1/q} \|\mathcal{H}_\beta(f)\|_{L^\bar{q}} \leq w_n^{1/q-1/q} \times C_{p,q,n,\beta} \cdot \|f\|_{L^\bar{p}} = C_{p,q,n,\beta} \cdot w_n^{1/q-1/q} \cdot w_n^{1/p-1/\bar{p}} \cdot \|f\|_{L^\bar{p}_{|x|} L^\bar{q}} = C_{p,q,n,\beta} w_n^{1/q-1/p+\beta/n} \|f\|_{L^\bar{p}_{|x|} L^\bar{q}},
\]

where we have used Lemma 3.1 in the inequality and \( 1/p - 1/q = \beta/n \) in the last equality.

To get the sharp bound, we take

\[
f_0(x) = \frac{1}{(1 + |x|^{\bar{q}})^{1+\bar{q}}/\bar{q}}.
\]

Since \( f_0 \) is a radial function, we have

\[
\|f_0\|_{L^\bar{p}_{|x|} L^\bar{q}} = w_n^{1/\bar{p}-1/p} \|f_0\|_{L^p}.
\]

Similarly, noting that \( \mathcal{H}_\beta(f_0) \) is also a radial function, we have

\[
\|\mathcal{H}_\beta(f_0)\|_{L^q_{|x|} L^\bar{q}} = w_n^{1/q-1/q} \|\mathcal{H}_\beta(f_0)\|_{L^\bar{q}}.
\]
Due to Zhao and Lu [49], there holds
\[
\|\mathcal{H}_\beta(f_0)\|_{L^{q}} = C_{p,q,n,\beta}\|f_0\|_{L^p}.
\] (12)

Combining (10), (11) with (12), we arrive at
\[
\|\mathcal{H}_\beta(f_0)\|_{L^{q}_w} = C_{p,q,n,\beta}w_n^{1/q-1/q+\beta/n}\|f_0\|_{L^{p}_w L^q_w},
\]

since \(1/p - 1/q = \beta/n\).

The proof is finished. \(\square\)

**Remark 3.1.** Similar to Theorem 2.2, one can also calculate the sharp bound for \(\mathcal{H}_\rho^*\) defined in (5) from \(L^p_{|x|} L^{\bar{p}}_\theta(\mathbb{R}^n)\) to \(L^q_{|x|} L^{\bar{q}}_\theta(\mathbb{R}^n)\) by duality. We omit the details here. In particular, if \(p = \bar{p} \in (1, \infty)\) and \(q = \bar{q} \in (1, \infty)\) in Theorem 3.1, we recover the results in [38-49].

### 4. Sharp weak bound for \(\mathcal{H}\) from \(L^p_{|x|} L^{\bar{p}}_\theta(\mathbb{R}^n)\) to \(w L^p_{|x|} L^{\bar{p}}_\theta(\mathbb{R}^n)\)

This section considers the sharp weak type estimate for \(n\)-dimensional Hardy operator on mixed radial-angular spaces. Our main result can be read as follows.

**Theorem 4.1.** Let \(n \geq 2, 1 \leq p, \bar{p}_1, \bar{p}_2 \leq \infty\). Then the \(n\)-dimensional Hardy operator \(\mathcal{H}\) defined in (7) is bounded from \(L^p_{|x|} L^{\bar{p}}_\theta(\mathbb{R}^n)\) to \(w L^p_{|x|} L^{\bar{p}}_\theta(\mathbb{R}^n)\). Moreover,
\[
\|\mathcal{H}\|_{L^p_{|x|} L^{\bar{p}}_\theta \rightarrow w L^p_{|x|} L^{\bar{p}}_\theta} = w_n^{1/\bar{p}_2-1/\bar{p}_1}.
\]

**Proof.** We only give the proof of the case \(1 < p, \bar{p}_1, \bar{p}_2 < \infty\), with the usual modifications made when \(p = 1, \bar{p}_i = 1\) or \(p = \infty, \bar{p}_i = \infty, i = 1, 2\). For any \(\lambda > 0\), we have

\[
\|\chi_{\{x \in \mathbb{R}^n : |\mathcal{H}(f)(x)| > \lambda\}}\|_{L^p_{|x|} L^{\bar{p}}_\theta} = \left\| \chi_{\{x \in \mathbb{R}^n : \lambda^{1/\bar{p}'_1} \mathcal{H}_\rho f \geq \lambda\}} \right\|_{L^p_{|x|} L^{\bar{p}}_\theta} \\
\leq \left\| \chi_{\{x \in \mathbb{R}^n : \lambda^{1/\bar{p}'_2} \mathcal{H}_\rho f \geq \lambda\}} \right\|_{L^p_{|x|} L^{\bar{p}}_\theta} \\
\leq \left\| \chi_{\{x \in \mathbb{R}^n : \lambda^{1/\bar{p}'_2} \mathcal{H}_\rho f \geq \lambda\}} \right\|_{L^p_{|x|} L^{\bar{p}}_\theta} \\
= w_n^{1/\bar{p}_2} \left( \int_0^{R_n \lambda^{1/\bar{p}'_2} \mathcal{H}_\rho f} \mathcal{H}_\rho f \right)^{1/p} dr \right)^{1/p} \\
= w_n^{1/\bar{p}_2} \left( \int_0^{\lambda} \mathcal{H}_\rho f \right)^{1/p} dr \right)^{1/p}.
\]
where we have used Hölder’s inequality on mixed-norm Lebesgue spaces, see [1].

On the other hand, we need to show that the constant $w_n^{1/p_2-1/p_1}$ is the best possible. Denote by $\chi_r = \chi_{[0,r]}$, $r > 0$. Taking $f_0(x) = \chi_r(|x|)$, $x \in \mathbb{R}^n$, a simple calculation shows that

$$\|f_0\|_{L_n^{p_1},L_n^{p_2}} = \frac{w_n^{1/p_1}}{n^{1/p}}\|f\|_{L_n^{p_1},L_n^{p_2}},$$

Zhao et al. [48, Proof of Theorem 2.1] proved that for

$$\int_{|x| < r; \mathcal{H}(f_0)(x) > \lambda} \|f\|_{L_n^{p_1},L_n^{p_2}},$$

(i) When $|x| < r$, it was showed by Zhao et al. [48, Proof of Theorem 2.1] that $\mathcal{H}(f_0)(x) = 1$. As a consequence, we have

$$\|\chi_{|x| < r; \mathcal{H}(f_0)(x) > \lambda}\|_{L_n^{p_1},L_n^{p_2}} = w_n^{p/p_2} \frac{r^n}{n}.$$

(ii) When $|x| \geq r$, it was also showed by Zhao et al. [48, Proof of Theorem 2.1] that $\mathcal{H}(f_0)(x) = r^n/|x|^p$. Therefore we have

$$\|\chi_{|x| \geq r; \mathcal{H}(f_0)(x) > \lambda}\|_{L_n^{p_1},L_n^{p_2}} = \left\|\chi_{|x| \leq r; |x| < \frac{r^n}{n} \lambda} \right\|_{L_n^{p_1},L_n^{p_2}} = \frac{w_n^{p/p_2} \frac{r^n}{n} \left(\frac{1}{\lambda} - 1\right)}{\lambda}.$$
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