Power variations for a class of Brown-Resnick processes

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Abstract

We consider the class of simple Brown-Resnick max-stable processes whose spectral processes are continuous exponential martingales. We develop the asymptotic theory for the realized power variations of these max-stable processes, that is, sums of powers of absolute increments. We consider an infill asymptotic setting, where the sampling frequency converges to zero while the time span remains fixed. More specifically we obtain biased central limit theorems whose bias depend on the local times of the differences between the logarithms of the underlying spectral processes. We also discuss the estimation of the integral of the extreme value index function for such a class of max-stable processes by considering the normalized total variation.

1 Introduction

In the two last decades there has been an increasing interest in limit theory for power variations of stochastic processes because such functionals are very important in analyzing the fine properties of the underlying model and in statistical inference. Asymptotic theory for power variations of various classes of stochastic processes has been intensively investigated in the literature. We refer e.g. to Jacod and Protter (2011) for limit theory for power variations of Itô semimartingales, to Barndorff-Nielsen, Corcuera and Podolskij (2009) for asymptotic results in the framework of fractional Brownian motion and related processes.

In this paper we study the power variations for a class of max-stable stochastic processes: the simple Brown-Resnick max-stable processes whose spectral processes are continuous exponential martingales. The original process of this class was introduced in the seminal paper of Brown and Resnick (1977) and it has been generalized to form a flexible family of stationary max-stable processes based on Gaussian random fields by Kabluchko, Schlather and de Haan (2009). The characterization of general max-stable stochastic processes in $C[0,1]$, the space of continuous functions on $[0,1]$ has been provided by Giné, Hahn and Vatan (1990) at the beginning of the nineties, while a decade later, de Haan and Lin (2001) investigated the domain of attraction conditions.

To the best of our knowledge, our paper studies for the first time power variations of a max-stable process. We provide central limit theorems in an infill asymptotic setting, i.e. where the sampling frequency converges to zero while the time span remains fixed. It should however be underlined that such an asymptotic approach has been considered for the estimation of the integrated variance of a white noise process with a positive and constant extreme value index by Einmahl, de Haan and Zhou (2016). However such a process is not realistic for a large number of applications for which the assumption of independent observations at any high frequencies may appear as too strong.

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With their assumption they can work with the values of the process (this is more or less equivalent to consider triangular arrays of independent random variables), while we have to consider the increments of the processes to mitigate the (strong) local dependence between the observations.

We propose to use our results to discuss the estimation of the integral of the extreme value index (EVI) function for a such a class of max-stable processes when it is assumed to be no more constant. The main motivation for considering the integral of the EVI function rather than the EVI function itself is that there is no way to obtain a functional convergence for estimators of the EVI function with infill asymptotics because the limiting processes would be white noise processes and, for example, their paths would be almost surely not Borel (this is the same thing as for the estimation of the spot volatility in finance as explained e.g. p. 394 in Jacod and Protter (2011)). Because the EVI function is not identifiable in such an infill asymptotic setting, we first assume that the underlying integrand function with respect to the Brownian motion in the exponential martingale is known. We then provide a functional estimator of the integral of the EVI function and obtain a biased central limit theorem, but without hope of suitable bias corrections. The accuracy of the estimator reaches the parametric rate as it is usually the case in the integrated functionals estimation. In a second step we assume that we also observe a path of the exponential martingale at discrete times and estimate the integrand function before plugging it in the estimator of the integral of the EVI function.

In the recent years important progress has been made in the field of infinite-dimensional extreme value statistics for which it is assumed that the data are continuous independent and identically distributed (iid) functions that belong to the domain of attraction of a max-stable process. de Haan and Lin (2003) established weak consistency of estimators of the extreme value index (EVI) function, the centering and standardizing sequences, and the exponent measure as the number of observations tends to infinity. Einmahl and Lin (2006) proved that the estimators of the extreme-value index function, and the normalizing functions have Gaussian processes as limiting distributions. More recently, Drees et al. (2018) have discussed the influence of linear interpolation on the previous estimators when the processes are only observed at discrete points. They provide conditions on the observational scheme which ensure that the interpolated estimators asymptotically behave in the same way as the estimators which use the fully observed continuous processes (in particular the points of observations get increasingly dense as the number of paths of the iid random functions increases).

We finally also consider $m$ iid sample-continuous stochastic processes that belong to the domain of attraction of our max-stable process and assume that these processes are also regularly sampled over $[0,1]$. We show that the average values of the previous estimators of the integral of the EVI function built over the highest paths of these random elements lead to inconsistent estimators.

The paper is organized as follows. Section 2 is devoted to presenting the setting and to providing definitions and assumptions. Section 3 discusses asymptotic results on the normalized power variations of the maximum of two independent Brownian motions, of the original Brown-Resnick process, and of a general max-stable process in our class of Brown-Resnick processes. These results are useful to understand the way we build our estimator of the integral of the EVI function in Section 4. Moreover we prove in this section that the proposed estimators are consistent and we provide central limit theorems. Section 4 also presents the case where we consider $m$ iid sample-continuous stochastic processes that belong to the domain of attraction of our max-stable process. All proofs are gathered in Appendix.
2 Setting, definitions and assumptions

We consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})\). We denote by \(C^+[0,1] := \{f \in C[0,1] : f > 0\}\) the space of continuous and positive functions on \([0,1]\). Note that we equip \(C[0,1]\) and \(C^+[0,1]\) with the supremum norm \(|f|_\infty = \sup_{s \in [0,1]} |f_s|\).

A stochastic process \(\eta\) on \(C^+[0,1]\) with nondegenerate marginals is called simple max-stable if for all positive integers \(k\)

\[
\frac{1}{k} \sum_{i=1}^{k} \xi_{i,\cdot} \overset{\mathcal{L}}{=} \eta
\]

where \(\eta_1, \eta_2, ...\) are independent and identically distributed (iid) copies of the process \(\eta\), \(\mathbb{P}(\eta_t < 1) = e^{-t}\) for all \(t \in [0,1]\) (i.e. it has a standard Frechet distribution) and \(\overset{\mathcal{L}}{=}\) means equality in distribution. By Corollary 9.4.5 in de Haan and Ferreira (2006), all simple max-stable processes in \(C^+[0,1]\) can be generated in the following way. Consider a Poisson point process, \((R_t)_{t \geq 1}\), on \((0, \infty)\) with mean measure \(dr/r^2\). Further consider iid stochastic processes \(V, V_1, V_2, ... \) in \(C^+[0,1]\) with \(\mathbb{E}[V_i] = 1\) for all \(t \in [0,1]\) and \(\mathbb{E}[\sup_{t \in [0,1]} V_i] < \infty\). Let the point process and the sequence \(V_1, V_2, ...\) be independent. Then

\[
\eta = \bigvee_{i=1}^{\infty} R_i V_i
\]

is a simple max-stable process. Conversely, each simple max-stable process has such a representation (which is not unique). The process \(V\) is called a spectral process associated to \(\eta\).

We now consider a sequence of iid random processes \(\xi, \xi_1, \xi_2, ...\) in \(C[0,1]\). This sequence is said to belong to the domain of attraction of the simple max-stable process \(\eta\) if there exists a sequence of non-random positive normalizing functions \((c_{n,t})_{t \in [0,1]}\), \(n = 1, ...,\) such that

\[
\left\{ c_{n,t}^{-1} \sum_{i=1}^{n} \xi_{i,t} \right\}_{t \in [0,1]} \overset{\mathcal{L}}{=} \eta, \quad n \to \infty,
\]

where \(\overset{\mathcal{L}}{=}\) denotes the convergence in law in \(C[0,1]\), see e.g. Theorem 9.2.1 in de Haan and Ferreira (2006). Let \(S_c := \{f \in C^+[0,1] : |f|_\infty \geq c\}\) where \(c > 0\), and define a sequence of measures \(\nu_n, n = 1, 2, ...,\) by

\[
\nu_n(\cdot) := n \Pr \left( n^{-1} \zeta(\cdot) \in \cdot \right)
\]

to \(S_c\), where \(\zeta_t := (1 - F_t(\xi_t))^{-1}\) and \(F_t(x) = \mathbb{P}(\xi_t \leq x)\). The sequence of measures \(\nu_n\) weakly converges, as \(n \to \infty\), to the restriction of the so-called exponent measure \(\nu\) of \(\eta\) to \(S_c\) for each \(c > 0\). It can be shown that the exponent measure coincides with the distribution of the spectral process, i.e. \(\nu(\cdot) = \mathbb{P}(V(\cdot))\) and that

\[
\lim_{n \to \infty} \frac{c_{n,t}^{|nx|}}{c_{n,t}} = x
\]

locally uniformly for \(x \in (0, \infty)\) and uniformly for \(t \in [0,1]\).

In this paper, we will assume that \(V\) is a continuous exponential martingale defined by

\[
V_t = \exp \left\{ \int_{0}^{t} H_s dW_s - \frac{1}{2} \int_{0}^{t} H_s^2 ds \right\}, \quad t \in [0,1],
\]

where \(H\) is a non-random Hölder function in \(C^+[0,1]\) with exponent \(\alpha > 1/2\) and satisfying \(\int_{0}^{1} H_s^2 ds < \infty\) and \(\inf_{s \in [0,1]} H_s > c\) for some positive constant \(c\). \(\{W_t, t \in [0,1]\}\) is a \(\mathcal{F}\)-adapted
standard Brownian motion on $[0,1]$. All the processes $V_i$, $i \geq 1$, in Eq. (2.1) are $\mathcal{F}$-adapted. We call the family of processes $\eta$ associated with $V$ the family of Brown-Resnick processes because when $H_t = \sigma > 0$ for all $t \in [0,1]$, $\eta$ is the stationary max-stable process introduced in Brown and Resnick (1977).

3 Asymptotic behaviors of normalized power variations of Brown-Resnick processes

The increments of a stochastic process $X$ over the equi-spaced grid with mesh $1/n$ of $[0,1]$ are denoted by

$$\Delta_i^n X = X_{i/n} - X_{(i-1)/n} \quad i = 1, \ldots, n.$$ 

The normalized power variation of order $p \geq 1$ of $X$ is defined by

$$B(p, X)_t^n = n^{p/2-1} \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta_i^n X|^p.$$ 

In this section we discuss the asymptotic behavior of $B(p, X)_t^n$ for several stochastic processes. We begin with the maximum of two independent Brownian motions, then with the logarithm of the original Brown Resnick process for which $H_t = \sigma > 0$ for $t \in [0,1]$, and finally with the logarithm of $\eta$.

We denote by $\overset{u.c.p}{\Rightarrow}$ the convergence in probability, uniform over each compact interval in $[0,1]$. We also need to recall the notion of stable convergence in law, which was introduced in Rényi (1963). Let $Z_n$ be a sequence of $E$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z$ be an $E$-valued random variable defined on an extension, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. We then say that $Z_n$ converges $\mathcal{F}$-stably to $Z$ (and write $Z_n \overset{\mathcal{F}}{\Rightarrow} Z$) if

$$\lim_{n \to \infty} \mathbb{E}[U f(Z_n)] = \mathbb{E}[U f(Z)]$$

for all bounded continuous functions $f$ on $E$ and all bounded $\mathcal{F}$-measurable random variables $U$. This notion of convergence is stronger than convergence in law, but weaker than convergence in probability. We refer to Jacod and Protter (2011) for a detailed exposition of this last type of convergence.

3.1 Normalized power variations of two independent Brownian motions

We here consider the case of the maximum of two Brownian motions $W_1 \lor W_2 = \{W_{1,t} \lor W_{2,t}, t \in [0,1]\}$, where $W_1 = \{W_{1,t}, t \in [0,1]\}$ and $W_2 = \{W_{2,t}, t \in [0,1]\}$ are two independent Brownian motions defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$. Let us recall that (see e.g. p. 10 in Jacod and Protter (2011)) that

$$B(p, W_1)_t^n \overset{u.c.p}{\Rightarrow} m_p t$$

and

$$\sqrt{n} (B(p, W_1)_t^n - m_p t) \overset{\mathcal{F}}{\Rightarrow} \tilde{X}_t$$

where $m_p$ is the expectation of the $p$-th moment of the absolute value of a standard Gaussian random variable, and $\tilde{X}$ is a continuous centered Gaussian martingale with variance $(m_{2p} - m_p^2)t$.

Let us denote by $(x)_+$ the positive part of a real $x$ and let $W_{2 \downarrow 1} = W_2 - W_1$. Since $W_1 \lor W_2 = W_1 + (W_{2 \downarrow 1})_+$, we deduce by Tanaka’s formula that

$$W_{1,t} \lor W_{2,t} = W_{1,t} + \int_0^t \mathbb{I}_{\{W_{2 \downarrow 1,t} > 0\}} dW_{2 \downarrow 1,t} + \frac{1}{2} L_{W_{2 \downarrow 1,t}}^0, \quad t \in [0,1],$$
where $L_{W_{2 \setminus 1}}^0$, $t$ is the local time of $W_{2 \setminus 1}$ at time $t$ and level 0. As a consequence $W_1 \lor W_2$ is not an Ito semi-martingale (since its predictable part of finite variation is not absolutely continuous with respect to the Lebesgue measure). The asymptotic results of functionals of normalized increments of a semi-martingale are often obtained under the assumption that the semi-martingale is an Ito semi-martingale (see e.g. Section 3.4.2 and 5.3 in Jacod and Protter (2011)). Therefore the results given in Jacod and Protter (2011) can not be used directly in our case.

Let $f$ be a real measurable function. By partitioning on the positive and negative values of $W_{2 \setminus 1, (i-1)/n}$ and $W_{2 \setminus 1, i/n}$, we have

$$f \left( \sqrt{n} \Delta^n_i (W_1 \lor W_2) \right) = f \left( \sqrt{n} \Delta^n_i W_1 \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} < 0, W_{2 \setminus 1, i/n} < 0\}} + f \left( \sqrt{n} \Delta^n_i W_2 \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} > 0, W_{2 \setminus 1, i/n} > 0\}} \right) + f \left( \sqrt{n} (W_{2, i/n} - W_{1, (i-1)/n}) \right) \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} \leq 0, W_{2 \setminus 1, i/n} \geq 0\}} + f \left( \sqrt{n} (W_{1, i/n} - W_{2, (i-1)/n}) \right) \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} \geq 0, W_{2 \setminus 1, i/n} \leq 0\}}.$$  

It follows that

$$f \left( \sqrt{n} \Delta^n_i (W_1 \lor W_2) \right) = \left( f \left( \sqrt{n} \Delta^n_i W_1 \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} < 0\}} + \sqrt{n} \Delta^n_i W_2 \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} > 0\}} \right) + \Psi_f \left( \sqrt{n} \Delta^n_i W_1, \sqrt{n} \Delta^n_i W_2, \sqrt{n} W_{2 \setminus 1, (i-1)/n} \right) \right)$$

where

$$\Psi_f (x, y, w) = (f(y + w) - f(x)) \mathbb{1}_{\{x - y < w \leq 0\}} + (f(x - w) - f(y)) \mathbb{1}_{\{0 \leq w < x - y\}}.$$  

One can remark that

$$\Delta^n_i W_1 \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} < 0\}} + \Delta^n_i W_2 \mathbb{1}_{\{W_{2 \setminus 1, (i-1)/n} > 0\}}$$

has the same distribution as $\Delta^n_i W_1$ or $\Delta^n_i W_2$ and is independent of $\sigma(W_{1,t}, 0 \leq t \leq (i-1)/n)$ and of $\sigma(W_{2,t}, 0 \leq t \leq (i-1)/n)$.

Let us now define

$$\varphi_p (w) = \int_{\mathbb{R}^2} \Psi_{|x|^p} (x, y, w) \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy.$$  

We can remark that

$$\varphi_p \left( \sqrt{n} W_{2 \setminus 1, (i-1)/n} \right) = \mathbb{E} \left[ \Psi_{|x|^p} \left( \sqrt{n} \Delta^n_i W_1, \sqrt{n} \Delta^n_i W_2, \sqrt{n} W_{2 \setminus 1, (i-1)/n} \right) \right]_{\mathcal{F}(i-1)/n}.$$  

Since $\int |\varphi_p (w) | dw < \infty$ for any $p \geq 1$, we can deduce from Theorem 1.1 in Jacod (1998), that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - 1} \varphi_p \left( \sqrt{n} W_{2 \setminus 1, (i-1)/n} \right) \xrightarrow{u.c.p} \frac{1}{2} \lambda(\varphi_p) L^0_{W_{2 \setminus 1}, t},$$

where $\lambda(\varphi_p) = \int \varphi_p (w) dw$.

We now state our first result.

**Proposition 1** As $n \to \infty$, we have

$$B (p, W_1 \lor W_2)_{t} \xrightarrow{u.c.p} m_p t$$

and

$$\sqrt{n} (B (p, W_1 \lor W_2)_{t} - m_p t) \overset{L^q}{\to} \tilde{X}_{1,t},$$

where $\tilde{X}_1$ is a process defined on an extension $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, which conditionally on $\mathcal{F}$ is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by

$$\mathbb{E}[\tilde{X}_{1,t} | \mathcal{F}] = \frac{1}{2} \lambda(\varphi_p) L^0_{W_{2 \setminus 1}, t}$$

and

$$\mathbb{V}[\tilde{X}_{1,t} | \mathcal{F}] = \left( m_{2p} - m^2_1 \right) t.$$
We observe that, contrary to the Brownian case, the asymptotic convergence of
\[ \sqrt{n} \left( B(p, W_1 \lor W_2)^n_i - m_p \right) \]
needs the stable convergence in law because of the additional term \( \lambda(\varphi_p)L^0_{W_{2,1},t}/2 \) as the (conditional) mean of \( \tilde{X}_{1,t} \).

### 3.2 Normalized power variations of the logarithm of the Brown Resnick processes

We first consider the case of the original Brown Resnick process for which \( H_t = \sigma > 0 \) for \( t \in [0,1] \). We have
\[
\log \eta_t = \sum_{i=1}^{\infty} \left( \log R_i + \sigma W_{i,t} - \frac{1}{2} \sigma^2 t \right) = \sum_{i=1}^{\infty} \left( \log R_i + \sigma W_{i,t} \right) - \frac{1}{2} \sigma^2 t
\]
where \( (R_i)_{i \geq 1} \) is a \( \mathcal{F}_0 \)-Poisson point process, with mean measure \( dr/r^2 \), and \( W_1, W_2, \ldots \) are independent \( (\mathcal{F}_t) \)-Brownian motions. It is well known (see e.g. Kabluchko, Schlather and de Haan (2009)) that \( \log \eta \) is a stationary process.

Let us study the distribution of its normalized increments
\[
U_i^n = \sqrt{n} \sigma^{-1} \Delta^n_i \log \eta, \quad i = 1, \ldots, n.
\]

The following proposition provides the conditional and marginal distributions of these increments and allows to deduce that they have asymptotically a standard Gaussian distribution.

**Proposition 2** Let \( u \in \mathbb{R} \). The conditional distribution of \( U_i^n \) given \( \eta_{(i-1)/n} = \eta \) is characterized by
\[
\Pr(U_i^n \leq u | \eta_{(i-1)/n} = \eta) = \exp \left( -\frac{1}{\eta} \left[ e^{-\sigma u/\sqrt{n}} \Phi \left( -u + \frac{\sigma}{2\sqrt{n}} \right) - \Phi \left( -u - \frac{\sigma}{2\sqrt{n}} \right) \right] \right) \Phi \left( u + \frac{\sigma}{2\sqrt{n}} \right),
\]
and its marginal distribution by
\[
\Pr(U_i^n \leq u) = \frac{\Phi \left( u + \sigma/(2\sqrt{n}) \right)}{\Phi \left( u + \sigma/(2\sqrt{n}) \right) + e^{-\sigma u/\sqrt{n}} (1 - \Phi \left( u - \sigma/(2\sqrt{n}) \right))},
\]
where \( \Phi \) is the cumulative distribution function of the standard Gaussian distribution. Moreover, we have, for any \( p \geq 1 \),
\[
\lim_{n \to \infty} \sqrt{n} (\mathbb{E} [\left| U_i^n \right|^p] - m_p) = 2p \int_0^{\infty} u^{p-1} \varphi(u) \left[ 1/2 - \Phi(u) - u \Phi(u) \Phi(u)/\varphi(u) \right] du,
\]
where \( \varphi \) and \( \Phi \) are respectively the probability density function and the survival distribution function of the standard Gaussian distribution.

One can observe that
\[
\Pr(U_i^n \leq u) + \Pr(U_i^n \leq -u) = 1, \quad u \in \mathbb{R},
\]
and we therefore conclude that \( U_i^n \) has a symmetric distribution. Moreover it is easily derived that the distribution of \( U_i^n \) converges to a standard Gaussian distribution. The rate of convergence of \( \mathbb{E} [\left| U_i^n \right|^p] \) to \( m_p \) is however relatively slow \( (1/\sqrt{n}) \) and this has for consequence that
\[
\lim_{n \to \infty} \sqrt{n} (\mathbb{E} [\left| B(p, \log \eta)^n_i \right|] - m_p \sigma^p t) = 2p \sigma^p \int_0^{\infty} u^{p-1} \varphi(u) \left[ 1/2 - \Phi(u) - u \Phi(u) \Phi(u)/\varphi(u) \right] du.
\]
Therefore an asymptotic bias is expected in the limit of $\sqrt{n}(B(p, \log \eta)/t - m_p\sigma^p t)$.

Let us now introduce some notation. For $i, j, k \geq 1$, let

$$Z_{i,t} = \log R_i + \sigma W_{i,t} \quad \text{and} \quad Z_{k\setminus j,t} = Z_{k,t} - Z_{j,t}.$$  

Let $f$ be a real measurable function. By partitioning on the values of $j$ and $k$ for which $\forall t \geq 1 Z_{i,(i-1)/n} = Z_{j,(i-1)/n}$ and $\forall t \geq 1 Z_{i,i/n} = Z_{k,i/n}$, we have

$$f(\sqrt{n} \Delta_{i}^{n} \log \eta + \frac{1}{2\sqrt{n}}\sigma^2) = \sum_{j \geq 1} f(\sqrt{n} \sigma \Delta_{i}^{n} W_{j}) \mathbb{I}_{\{\forall t \geq 1 Z_{i,(i-1)/n} = Z_{j,(i-1)/n}, \forall t \geq 1 Z_{i,i/n} = Z_{j,i/n}\}}$$

$$+ \sum_{j \geq 1} \sum_{k \geq j} f(\sqrt{n} \Delta_{j}^{n} W_{j}, \sqrt{n} \Delta_{k}^{n} W_{k}, \sqrt{n} Z_{k\setminus j,(i-1)/n}) \mathbb{I}_{\{\forall t \geq 1 Z_{i,(i-1)/n} = Z_{j,(i-1)/n}, \forall t \geq 1 Z_{i,i/n} = Z_{k,i/n}\}} + H_{f}^{n},$$

where

$$H_{f}^{n} = \sum_{j \geq 1} \sum_{k \geq j} \mathbb{E}[f(\sqrt{n} \Delta_{i}^{n} W_{j}, \sqrt{n} \Delta_{j}^{n} W_{k}, \sqrt{n} Z_{k\setminus j,(i-1)/n}) \times$$

$$\prod_{l \neq j,k} \mathbb{I}_{\{Z_{i,(i-1)/n} \leq 0\}} \times \left(\prod_{l \neq k,j} \mathbb{I}_{\{Z_{i,i/n} \leq 0\}} - \prod_{l \neq k,j} \mathbb{I}_{\{Z_{i,j,(i-1)/n} \leq 0\}}\right),$$

with

$$\Psi_{f,\sigma}(x, y, w) = (f(\sigma y + w) - f(\sigma x)) \mathbb{I}_{\{\sigma(x-y) \leq w \leq 0\}}.$$  

One can observe that

$$\sum_{j \geq 1} \Delta_{j}^{n} W_{j} \mathbb{I}_{\{\forall t \geq 1 Z_{i,(i-1)/n} = Z_{j,(i-1)/n}\}}$$

has the same distribution as $\Delta_{j}^{n} W_{j}$, $j \geq 1$, and are independent of the $\sigma(W_{j,t}, 0 \leq t \leq (i-1)/n)$, $j \geq 1$.

Let us define

$$\varphi_{p,\sigma}(w) = \int_{\mathbb{R}^2} \Psi_{|p,\sigma}(x, y, w) \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy.$$  

We can deduce from a simple modification of Theorem 1.1 in Jacod (1998), that, for $j \geq 1$ and $k > j$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{nt-1} \mathbb{E}[\Psi(\sqrt{n} \Delta_{i}^{n} W_{j}, \sqrt{n} \Delta_{j}^{n} W_{k}, \sqrt{n} Z_{k\setminus j,(i-1)/n}) | \mathcal{F}_{(i-1)/n}] \mathbb{I}_{\{\forall t \geq 1 Z_{i,(i-1)/n} = Z_{j,(i-1)/n}, 0 \leq t \leq (i-1)/n\}}$$

$$\Rightarrow \frac{1}{2\sigma^2} \lambda(\varphi_{p,\sigma}) \int_{0}^{t} \mathbb{I}_{\{\forall t \geq 1 Z_{i,(i-1)/n} = Z_{j,(i-1)/n}, 0 \leq t \leq (i-1)/n\}} dL_{k\setminus j,s},$$
where $L_{k\setminus j}^0$ is the local time of $Z_{k\setminus j}$ at time $t$ and level 0. We can now state our result on the convergence of $B(p, \log \eta)_t^n$.

**Proposition 3** Assume that $H_t = \sigma > 0$ for $t \in [0,1]$. As $n \to \infty$, we have for any integer $p \geq 1$

\[ B(p, \log \eta)_t^n \overset{u.c.p.}{\longrightarrow} m_p \sigma^p t \quad \text{and} \quad \sqrt{n} \left( B(p, \log \eta)_t^n - m_p \sigma^p t \right) \overset{L^q}{\longrightarrow} \mathcal{X}_{2,t}, \]

where $\mathcal{X}_2$ is a process defined on an extension $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, which conditionally on $F$ is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by

\[ \mathbb{E}[\mathcal{X}_{2,t} | F] = \frac{1}{2\sigma^2} \lambda(\varphi_{p,\sigma}) \sum_{j \geq 1} \sum_{k > j} \int_0^t I_{\{\lambda_{t,j,k} Z_{i,s} > \gamma_{i,j,k} Z_{i,s}\}} dL_{Z_{k\setminus j},s}^0 \quad \text{and} \quad \mathcal{V}[\mathcal{X}_{2,t} | F] = (m_{2p} - m_1^2) \sigma^{2p} t. \]

**Remark 1** We only consider integers $p$ for technical reasons in the proof of the central limit theorem, although it is expected that the asymptotic convergence still holds for any $p \geq 1$.

We now consider the case where $H_t$ is not necessarily a constant function and study the power variations of

\[ \log \eta_t = \bigvee_{i=1}^\infty \left( \log R_i + \int_0^t H_s dW_{i,s} - \frac{1}{2} \int_0^t H_s^2 ds \right) = \bigvee_{i=1}^\infty \left( \log R_i + \int_0^t H_s dW_{i,s} \right) - \frac{1}{2} \int_0^t H_s^2 ds \]

where $(R_i)_{i \geq 1}$ is a $\mathcal{F}_0$-Poisson point process, with mean measure $dr/r^2$, $W_1, W_2, \ldots$ are independent $(\mathcal{F}_t)$-Brownian motions, $H$ is a H"older function in $C^+[0,1]$ with exponent $\alpha > 1/2$ and satisfying $\int_0^1 H_s^2 ds < \infty$. For $i, j, k \geq 1$, let us also use the following notation

\[ Z_{i,t} = \log R_i + \int_0^t H_s dW_{i,s}, \quad Z_{k\setminus j,t} = Z_{k,t} - Z_{j,t}. \]

**Proposition 4** As $n \to \infty$, we have, for any integer $p \geq 1$,

\[ B(p, \log \eta)_t^n \overset{u.c.p.}{\longrightarrow} m_p \int_0^t H_s^p ds \quad \text{and} \quad \sqrt{n} \left( B(p, \log \eta)_t^n - m_p \int_0^t H_s^p ds \right) \overset{L^q}{\longrightarrow} \mathcal{X}_{3,t}, \]

where $\mathcal{X}_3$ is a process defined on an extension $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, which conditionally on $F$ is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by

\[ \mathbb{E}[\mathcal{X}_{3,t} | F] = \frac{1}{2} \sum_{j \geq 1} \sum_{k > j} \int_0^t \frac{\lambda(\varphi_{p,H_s})}{H_s^2} I_{\{\lambda_{t,j,k} Z_{i,s} > \gamma_{i,j,k} Z_{i,s}\}} dL_{Z_{k\setminus j},s}^0 \quad \text{and} \quad \mathcal{V}[\mathcal{X}_{3,t} | F] = (m_{2p} - m_1^2) \int_0^t H_s^{2p} ds. \]

4 **Estimating the integral of the EVI function**

In this section, we consider the max-stable process $Y = \eta^\gamma$ where $\gamma$ is assumed to be a positive and differentiable function. This function is called the extreme value index (EVI) function of $Y$ and measures the tail heaviness of $Y$. We decide to estimate the integral of the EVI function

\[ I_t = \int_0^t \gamma_s ds \quad \text{rather than the EVI function itself by using the results of the previous section. The increments of log } Y \text{ are given by} \]

\[ \Delta_i^n \log Y = \log \eta_{(i-1)/n} \Delta_i^n \gamma + \gamma_{i/n} \Delta_i^n \log \eta. \]
Since $\gamma$ is assumed to be a differentiable function, $\Delta_n^i \gamma$ is approximately equal to $\gamma'(i-1)/n$ and $\log \eta(i-1)/n \Delta_n^i \gamma$ will not be a dominant component in $\Delta_n^i \log Y$. It is therefore expected that

$$m_p^{-1} B(p, \log Y)_t^n \xrightarrow{u.c.p} \int_0^t \gamma'_s H_s^p ds.$$  

Because of the presence of $H_s$ in the limit integrals with the same power as $\gamma_s$, we deduce that the EVI function is not identifiable by using the power variations of $\log Y$. Therefore $H_s$ has to be eliminated when computing the increments of $\log Y$. We consider two cases: firstly $H$ is assumed to be known, secondly it is estimated from additional data.

### 4.1 The function $H$ is known

We here assume that $H$ is known and propose the following estimator of the EVI function

$$I^n_t = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - 1} \frac{|\Delta_n^i \log Y|}{m_1 H(i-1)/n}.$$  

**Proposition 5** As $n \to \infty$, we have

$$I^n_t \xrightarrow{u.c.p} I_t \quad \text{and} \quad \sqrt{n}(I^n_t - I_t) \xrightarrow{L^2} \tilde{X}_{4,t},$$

where $\tilde{X}_4$ is a process defined on an extension $(\Omega, \tilde{F}, (\tilde{F})_{t \geq 0}, \mathbb{P})$ of $(\Omega, F, (F)_{t \geq 0}, \mathbb{P})$, which conditionally on $F$ is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by

$$\mathbb{E}[\tilde{X}_{4,t}|F] = \frac{1}{2m_1} \sum_{j \geq 1} \sum_{k > j} \int_0^t \frac{\lambda(\varphi_{1,H_s}) \gamma_s}{H_s^2} \mathbb{1}_{\{z_{i,j,k} > \varphi_{j,k} z_{i,s} \}} dL^0_{z_{i,j,k}} \quad \text{and} \quad \mathbb{V}[\tilde{X}_{4,t}|F] = \frac{(1 - m_1^2)}{m_1^2} \int_0^t \gamma^2_s ds.$$  

### 4.2 The function $H$ is unknown and is estimated

Without observations of an associated spectral process $V$, it will not be possible to estimate $H$. Therefore we assume that we also observe $V$ at discrete times $i/n$, for $i = 0, 1, ..., n$. We then consider the following estimator of $H(i-1)/n$, for $i = 1, ..., n - k_n + 1$,

$$\hat{H}_i^n = \frac{1}{k_n} \sum_{j=0}^{k_n-1} \sqrt{n} |\Delta_{i+j}^n \log V|$$

where $k_n$ is an integer such that $1 \leq k_n \leq n$. It is easy to show that $\hat{H}_{[nt]}^n \xrightarrow{u.c.p} H_t$, as soon as $k_n \to \infty$ and $k_n/n \to 0$.

We are now able to provide an estimator of the EVI function by

$$\hat{I}_i^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \frac{|\Delta_n^i \log Y|}{m_1 H_i^n}.$$  

This estimator will be consistent but will suffer from a dominating bias when considering a central limit theorem. Therefore we rather propose the following estimator

$$\tilde{I}_i^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \frac{|\Delta_n^i \log Y|}{m_1 H_i^n}.$$
where
\[
\frac{1}{H^n_i} = \frac{1}{H^n_i} \left(1 - \frac{(1 - m_i^2)}{k_n}\right).
\]

Proposition 6 Let \(k_n \to \infty\) and \(k_n^{a/(a-1/2)}/n \to 0\), then
\[
\sqrt{n}(I^n_i - I_t) \overset{d}{=} \tilde{X}_{5_t},
\]
where \(\tilde{X}_5\) is a process defined on an extension \((\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t \geq 0}, \mathbb{P})\) of \((\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})\), which conditionally on \(\mathcal{F}\) is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by
\[
\tilde{\mathbb{E}}[\tilde{X}_{5,t}|\mathcal{F}] = \frac{1}{2m_1} \sum \frac{\lambda(\varphi H_s)\gamma_s}{H_s^3} \int_0^t \mathbb{I}_{\{1 \neq j,k\}} z_{t,s}^{\gamma_s} dL_0 z_{s,k,j}^{\gamma_s} ds \quad \text{and} \quad \tilde{\mathbb{V}}[\tilde{X}_{5,t}|\mathcal{F}] = \frac{(1 - m_i^2)}{m_i^2} \int_0^t \gamma_s^2 ds.
\]

4.3 Normalized approximate total variations of stochastic processes belonging to the domain of attraction of \(Y\)

In this subsection, we introduce the sample-continuous stochastic process \(\xi\) defined by
\[
\xi_t = Z^n V^{\gamma_t}, \quad t \in [0,1],
\]
where \(Z\) has a standard Pareto distribution and is independent of \(V\). It is well-known that \(\xi\) belongs to the domain of attraction of the max-stable process \(Y\) (see e.g. Einmahl and Lin (2006)). We moreover assume that \(\gamma\) is a twice differentiable function and that we observe \(m_n\) iid processes \(\xi_1, \ldots, \xi_{m_n}\) on the equi-spaced grid of \([0,1]\) with mesh \(1/n\).

The previously proposed estimators of the EVI function (based on fully observed processes or on discretely observed processes) only use the observations with the highest values (see de Haan and Lin (2003) and Drees et al. (2017)). We are therefore interested in understanding whether the normalized approximate total variations of the highest \(\xi_j\) for which \(\log \xi > u_n\) tend to infinity as \(n\) tends to infinity. We decide to consider the normalized approximate total variations of the processes \(\xi_j\) for which \(\log \xi_j > u_n\). We denote by \(k_n = \sum_{j=1}^{m_n} \mathbb{I}_{\{\log \xi_j > u_n\}}\) the number of processes \(\xi_j\) that are concerned, and we denote by \(\xi(j)\), \(j = 1, \ldots, k_n\), the process \(\xi_l\), \(l = 1, \ldots, m_n\), such that \(\xi(1), \ldots, \xi(k_n)\). As usual for extreme value statistics, \(k_n\) tends to infinity almost surely and since
\[
\mathbb{E}[k_n] = m_n \Pr(\log \xi > u_n) = m_n \Pr(\gamma_0 \log Z > u_n) = m_n e^{-u_n/\gamma_0},
\]
we have to assume that \(m_n e^{-u_n/\gamma_0} \to \infty\) as \(n \to \infty\).

Now note that
\[
\Delta_n^n \log \xi = \log Z \Delta_n^n \gamma + \log V^{(i-1)/n} \Delta_n^n \gamma + \gamma_0 \Delta_n^n \log V.
\]
We will study the normalized approximate total variation of \(\log \xi\) only when \(\xi_0 = \gamma_0 \log Z > u_n\). Since we have assumed that \(\gamma\) is a differentiable function, we have \(\Delta_n^n \gamma \approx \gamma_0 (\gamma - 1)/n\). To simplify the analysis, we will also assume that \(u_n/\sqrt{n} \to \infty\) such that \(\log Z \Delta_n^n \gamma\) becomes the dominant component in \(\Delta_n^n \log \xi\).
We now consider the average values of the normalized approximate total variations built over the highest paths of the \( \xi_j \), \( j = 1, \ldots, m_n \), defined by

\[
J^n_t = \frac{1}{k_n} \sum_{j=1}^{k_n} \sqrt{n} B(1, \log \xi(j))_n = \frac{1}{k_n} \sum_{j=1}^{k_n} \frac{1}{u_n} \sum_{i=1}^{\floor{nt}} |\Delta_i^n \log \xi(j)|.
\]

Note that we have normalized \( B(1, \log \xi(j))_n \) by \( u_n/\sqrt{n} \). Let us finally introduce the function \( \nu \) defined by

\[
\nu_t = \int_0^t |\gamma'_u| \, du - \frac{1}{2} \int_0^t \left[ \text{sign}(\gamma'_u) \gamma_u + \int_u^t |\gamma'_s| \, ds \right] H^2_u \, du, \quad t \in [0,1].
\]

Let us choose \( m_n = e^{u_n/\gamma_0} x_n \) where \( x_n \to \infty \), and let us define \( w_n = u_n/\sqrt{n} \).

**Proposition 7** Let \((w_n)\) and \((x_n)\) be two sequences such that \( w_n \to \infty \), \( x_n \to \infty \), \( x_n w_n^2/n \to 0 \) and \( n\Phi(w_n) \to 0 \) as \( n \to \infty \), then

\[
J^n_t \overset{\text{u.c.p.}}{\Rightarrow} \gamma_0^{-1} \int_0^t |\gamma'_s| \, ds,
\]

and

\[
\sqrt{k_n u_n} \left( J^n_t - \gamma_0^{-1} \int_0^t |\gamma'_s| \, ds - \frac{1}{u_n} \nu_t \right) \overset{\mathcal{L}}{\Rightarrow} \left( \int_0^t |\gamma'_s| \, ds \right) X_0 + X^c_t
\]

where \( X^c_t \) is a sample-continuous centered Gaussian process with variance \( \int_0^t [\text{sign}(\gamma'_u) \gamma_u + \int_u^t |\gamma'_s| \, ds]^2 H^2_u \, du \), \( X_0 \) has a standard Gaussian distribution, and \( X_0 \) and \( (X^c_t) \) are independent.

We deduce from this proposition that the average of the values of the normalized approximate total variations built over the highest paths of the \( \xi_j \), \( j = 1, \ldots, m_n \), does not lead to a consistent estimator of the EVI function even if \((H_s)_{s \in [0,1]} \) was known.

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5 Appendix

C is a constant that does not depend of \( n \) but can vary from line to line.

5.1 Proof of Proposition 1

We only prove the stable convergence in law of \( \sqrt{n} (B(p, W_1 \lor W_2)^n_t - m_p t) \). We have

\[
\sqrt{n} (B(p, W_1 \lor W_2)^n_t - m_p t) = \sum_{i=1}^{[nt]-1} \zeta_{1,i}^n + \sum_{i=1}^{[nt]-1} \zeta_{2,i}^n + m_p n^{1/2} \left( \frac{|nt| - 1}{n} - t \right),
\]

where

\[
\zeta_{1,i}^n = n^{-1/2} (|\sqrt{n} \Delta^n_p W_1|_{\{W_2; 1, (i-1)/n < 0\}} + \sqrt{n} \Delta^n_p W_2|_{\{W_2; 1, (i-1)/n > 0\}} |^p - m_p
\]

\[
\zeta_{2,i}^n = n^{-1/2} \Psi_{|p} (\sqrt{n} \Delta^n_p W_1, \sqrt{n} \Delta^n_p W_2, \sqrt{n} W_2|_{1, (i-1)/n}).
\]

Step 1) First it is clear that

\[
n^{1/2} \left( \frac{|nt| - 1}{n} - t \right) \overset{u.c.p.}{\rightarrow} 0.
\]
Step 2) Second, by using usual arguments (see e.g. Chapter 5.2 in Jacod and Protter (2011)), we have, as $n \to \infty$,
\[
\sum_{i=1}^{[nt]-1} \zeta_{i,i}^n \overset{L}{\to} \tilde{X}_{0,t},
\]
where $\tilde{X}_0$ is a process defined on an extension $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, which conditionally on $\mathcal{F}$ is a continuous centered Gaussian process, with independent increments, and whose variance is given by
\[
\tilde{\mathbb{V}}[\tilde{X}_{0,t}|\mathcal{F}] = (m_{2p} - m_p^2) t.
\]

Step 3) Third, let us prove that
\[
\sum_{i=1}^{[nt]-1} \zeta_{i,i}^n \overset{u.c.p}{\to} \frac{1}{2} \lambda(\varphi_p) L^0_{W_{21},t}.
\]
We have
\[
\mathbb{E}\left[\zeta_{1,i}^n \mid \mathcal{F}_{(i-1)/n}\right] = n^{-1/2} \varphi_p \left(\sqrt{n} W_{21,(i-1)/n}\right)
\]
\[
\mathbb{E}\left[\left(\zeta_{1,i}^n\right)^2 \mid \mathcal{F}_{(i-1)/n}\right] = n^{-1} \varphi_p^{(2)} \left(\sqrt{n} W_{21,(i-1)/n}\right)
\]
where
\[
\varphi_p^{(2)}(w) = \int_{\mathbb{R}^2} \Psi_{\varphi_p}^2(x, y, w) \frac{1}{2\pi} e^{-(x^2+y^2)/2} \, dx dy.
\]
Note that $\int |\varphi_2(w)| \, dw < \infty$. It follows by Theorem 1.1 in Jacod (1998) that
\[
\sum_{i=1}^{[nt]-1} \mathbb{E}\left[\zeta_{i,i}^n \mid \mathcal{F}_{(i-1)/n}\right] \overset{u.c.p}{\to} \frac{1}{2} \lambda(\varphi_p) L^0_{W_{21},t}.
\]
\[
\sqrt{n} \sum_{i=1}^{[nt]-1} \mathbb{E}\left[\left(\zeta_{i,i}^n\right)^2 \mid \mathcal{F}_{(i-1)/n}\right] \overset{u.c.p}{\to} \frac{1}{2} \lambda(\varphi_p^{(2)}) L^0_{W_{21},t}.
\]
Therefore
\[
\sum_{i=1}^{[nt]-1} \mathbb{E}\left[\left(\zeta_{i,i}^n\right)^2 \mid \mathcal{F}_{(i-1)/n}\right] \overset{u.c.p}{\to} 0,
\]
and the result follows by using Lemma 2.2.12 in Jacod and Protter (2011).

Step 4) Use steps 1), 2) and 3), define $\tilde{X}_{1,t} = \lambda(\varphi_p) L^0_{W_{21},t}/2 + \tilde{X}_{0,t}$ and take into account the properties of the stable convergence in law to conclude.

5.2 Proof of Proposition 2

1) We first characterize the conditional distribution of $U^n_i$ given $\eta_{(i-1)/n} = \eta$. By Proposition 4.1 in Dombry and Eyi Menko (2013), the conditional distribution $\eta_{i/n}|\eta_{(i-1)/n} = \eta$ is given by
\[
\Pr\left(\eta_{i/n} \leq z \mid \eta_{(i-1)/n} = \eta\right) = \exp\left(-\mathbb{E}\left[\left(V_{i/n} - \frac{V_{(i-1)/n}}{\eta}\right)_+\right]\right) \mathbb{E}\left[\mathbb{I}\left(\frac{V_{i/n}}{z} < \frac{V_{(i-1)/n}}{\eta}\right) V_{(i-1)/n}\right],
\]

13
and, by stationarity of $\eta$, it can be rewritten in the following way

$$
\Pr \left( \eta_{i/n} \leq z | \eta_{(i-1)/n} = \eta \right) = \exp \left( -E \left[ \frac{V_{i/n}}{z} - \frac{1}{\eta} \right] \right) \Pr \left( \frac{V_{i/n}}{z} < \frac{1}{\eta} \right).
$$

We have

$$
\E \left[ \mathbb{I} \left( V_{i/n} < \frac{z}{\eta} \right) \right] = \Pr \left( V_{i/n} < \frac{z}{\eta} \right)
= \Pr \left( W_{i/n} < \sigma^{-1} \ln \left( \frac{z}{\eta} \right) + \frac{\sigma}{2n} \right)
= \Phi \left( \frac{1}{\sigma n^{-1/2}} \ln \left( \frac{z}{\eta} \right) + \frac{1}{2} \sigma n^{-1/2} \right),
$$

and

$$
\E \left[ \left( \frac{V_{i/n}}{z} - \frac{1}{\eta} \right)^+ \right] = \frac{1}{z} \Phi \left( -\frac{1}{\sigma n^{-1/2}} \ln \left( \frac{z}{\eta} \right) + \frac{\sigma}{2\sqrt{n}} \right) - \frac{1}{\eta} \Phi \left( -\frac{1}{\sigma n^{-1/2}} \ln \left( \frac{z}{\eta} \right) - \frac{\sigma}{2\sqrt{n}} \right).
$$

Since

$$
\Pr(U_{1/n}^n \leq u | \eta_{(i-1)/n} = \eta) = \Pr \left( \eta_{i/n} \leq \eta e^{u \sigma n^{-1/2}} | \eta_{(i-1)/n} = \eta \right),
$$

we deduce that

$$
\Pr(U_{1/n}^n \leq u | \eta_{(i-1)/n} = \eta) = \exp \left( -\frac{1}{\eta} \left[ e^{-\sigma u / \sqrt{n}} \Phi \left( -u + \frac{\sigma}{2\sqrt{n}} \right) - \Phi \left( -u - \frac{\sigma}{2\sqrt{n}} \right) \right] \right) \Phi \left( u + \frac{\sigma}{2\sqrt{n}} \right).
$$

2) We have

$$
\Pr(U_{1/n}^n \leq u) = \E_{\eta_{(i-1)/n}} \left[ \Pr(U_{1/n}^n \leq u | \eta_{(i-1)/n}) \right],
$$

and since $\eta_{(i-1)/n}$ has a standard Exponential distribution, we derive that

$$
\Pr(U_{1/n}^n \leq u) = \frac{\Phi \left( u + \sigma / (2\sqrt{n}) \right)}{\Phi \left( u + \sigma / (2\sqrt{n}) \right) + e^{-\sigma u / \sqrt{n}} \left( 1 - \Phi \left( u - \sigma / (2\sqrt{n}) \right) \right)}.
$$

3) We have for any $p \geq 1$

$$
\E \left[ \|U_{1/n}^n\|^p \right] = p \int_{-\infty}^0 (-u)^{p-1} \Pr(U_{1/n}^n \leq u) \, du + p \int_0^\infty u^{p-1} \Pr(U_{1/n}^n > u) \, du
= 2p \int_0^\infty u^{p-1} \frac{e^{-\sigma u / \sqrt{n}} \left( 1 - \Phi \left( u - \sigma / (2\sqrt{n}) \right) \right)}{\Phi \left( u + \sigma / (2\sqrt{n}) \right) + e^{-\sigma u / \sqrt{n}} \left( 1 - \Phi \left( u - \sigma / (2\sqrt{n}) \right) \right)} \, du.
$$

Now let us define

$$
V(n, u) = \frac{e^{-\sigma n^{-1/2} u} \Phi(u - \sigma n^{-1/2} / 2)}{\Phi(u)} \text{ and } U(n, u) = \frac{\Phi(u + \sigma n^{-1/2} / 2)}{\Phi(u)},
$$

and note that

$$
\E \left[ \|U_{1/n}^n\|^p \right] = 2p \int_0^\infty u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(nt, u)} \, du.
$$

i) There exists $u_c \in [u, u + \sigma n^{-1/2}]$ such that

$$
\Phi(u + \sigma / (2\sqrt{n})) = \Phi(u) + \frac{1}{\sqrt{n}} \varphi(u) / 2 + \frac{1}{8} \sigma^2 \frac{1}{n} \varphi'(u_c).
$$
where

\[
\beta < \frac{1}{2}
\]

Therefore, if moreover

\[
\sup_{u \in [u, u + \sigma_n^{-1/2} b]} u \varphi(u)
\]

and we can deduce that

\[
U(n, u) = 1 + \frac{1}{\sqrt{n}} \varphi(u) + \bar{U}(n, u)
\]

where

\[
\sup_{u \in [0, \infty)} \left| \bar{U}(n, u) \right| \leq \frac{C}{n}.
\]

ii) If \( u \in [0, n^\beta) \) with \( \beta < 1/2 \), then

\[
e^{-\sigma_n^{-1/2} u} = 1 - \sigma_n^{-1/2} u + o_{u \in [0, n^\beta)} \left( \frac{1}{n} \right).
\]

There exists \( u_c \in [u - \sigma_n^{-1/2} b, u] \) such that

\[
\Phi(u - \sigma_n^{-1/2} b) = \Phi(u) + \sigma_n^{-1/2} \varphi(u)/2 - \frac{1}{8} \sigma_n^{-1/2} \varphi(u_c).
\]

Therefore, if moreover \( \beta < 1/4 \), we have

\[
\left| \frac{\Phi(u - \sigma_n^{-1/2} b)}{\Phi(u)} - 1 - \sigma_n^{-1/2} \varphi(u) \right| \leq \frac{1}{8} \sigma_n^{-1/2} \varphi(u_c) + o_{u \in [0, n^\beta)} \left( \frac{1}{n} \right)
\]

If \( u \in [0, n^\beta) \) with \( \beta < 1/4 \), we deduce that

\[
V(n, u) = 1 + \frac{1}{\sqrt{n}} \left( \frac{\varphi(u) - \sigma_n^{-1/2} b}{\Phi(u)} - u \right) + \tilde{V}(n, u),
\]

where

\[
\sup_{u \in [0, n^\beta)} \left| \tilde{V}(n, u) \right| = o(n^{-1/2}).
\]

Moreover, as \( u \to \infty \) and \( n \) is fixed, we have

\[
V(n, u) = \frac{e^{-\sigma_n^{-1/2} b} \Phi(u - \sigma_n^{-1/2} b)}{\Phi(u)} = \exp \left( -\frac{\sigma_u}{2\sqrt{n}} - \frac{1}{8} \sigma_n^{-1/2} b + o(1) \right).
\]
iii) Now we have
\[
\int_0^\infty u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du = \int_0^n u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du + \int_n^\infty u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du.
\]
Then, for \( \beta < 1/4, \)
\[
\int_0^n u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du = \int_0^n u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du + \int_n^\infty u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du.
\]

where \( W_n = o(n^{-1/2}). \) Moreover
\[
\int_{n^\\beta}^\infty u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du \leq \int_{n^\\beta}^\infty u^{p-1} \frac{\Phi(u) V(n, u)}{\Phi(u) U(n, u) + \Phi(u) V(n, u)} du \leq C \int_{n^\\beta}^\infty u^{p-1} \Phi(u) du.
\]
Finally, we deduce that
\[
\lim_{n \to \infty} \sqrt{n} \left( \mathbb{E}[|U_i^n|^p] - 2p \int_0^\infty u^{p-1} \Phi(u) du \right) = 2p \int_0^\infty u^{p-1} \Phi(u)[1/2 - \Phi(u) - u\Phi(u)/\varphi(u)] du.
\]

5.3 Proof of Proposition

We only prove the stable convergence in law of \( \sqrt{n} (B(p, \log \eta_t^n) - m_p \sigma ^p t) \). Recall that
\[
\log \eta_t = \sqrt{\sum_{i=1}^\infty \left( \log R_i + \sigma W_{i,t} - \frac{1}{2} \sigma ^2 t \right)} = \sqrt{\sum_{i=1}^\infty Z_{i,t} - \frac{1}{2} \sigma ^2 t},
\]
and that
\[
\left( \sqrt{n} \Delta_i^n \log \eta + \frac{1}{2 \sqrt{n}} \sigma^2 \right)^p = \left( \sum_{j \geq 1}^{n} \sqrt{n} \Delta_i^n W_j \right)^p \# \{ j \geq 1 | z_{i,(i-1)/n} = z_{j,(i-1)/n} \} + \sum_{j \geq 1}^{n} \sum_{k > j}^{n} \Psi_{j,p,q} (\sqrt{n} \Delta_i^n W_j, \sqrt{n} \Delta_i^n W_k, \sqrt{n} Z_{(i-1)/n} \} + H_{1,i}^n.
\]
where
\[
H_{1,i}^n = \sum_{j \geq 1}^{n} \sum_{k \neq j}^{n} \Psi_{j,p,q} (\sqrt{n} \Delta_i^n W_j, \sqrt{n} \Delta_i^n W_k, \sqrt{n} Z_{(i-1)/n} \} \times \prod_{l \neq j,k} \mathbb{I}_{Z_{l,(i-1)/n} \leq 0} \times \left( \prod_{l \neq j,k} \mathbb{I}_{Z_{l,(i-1)/n} \leq 0} - \prod_{l \neq j,k} \mathbb{I}_{Z_{l,(i-1)/n} \leq 0} \right).
\]
Note that, for \(a, u \in \mathbb{R}\),
\[
|a + u| = |a| + \text{sign}(a)u - 2(|a| + \text{sign}(a)u)\mathbb{1}_{\{(a+u)a<0\}},
\]
and, for any integer \(p \geq 1\),
\[
|a + u|^p = |a|^p + p\left[\text{sign}(a)u - 2(|a| + \text{sign}(a)u)\mathbb{1}_{\{(a+u)a<0\}}\right]|a|^{p-1}
+ \sum_{k=2}^{p} \binom{k}{p} \left[\text{sign}(a)u - 2(|a| + \text{sign}(a)u)\mathbb{1}_{\{(a+u)a<0\}}\right]^{k-1} |a|^{p-k}
\]
where \(\binom{k}{p} = p!/(k!(p-k)!\) is the binomial coefficient of order \(k\) and \(p\).

Therefore we have
\[
B(p, \log \eta)^n_t = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor - 1} n\Delta_i^n \log \eta |\sqrt{n\Delta_i^n \log \eta}|^{p-1}
\]
\[
B^n_t = \frac{2}{n} \sum_{i=1}^{\lfloor nt \rfloor - 1} \left(\sqrt{n\Delta_i^n \log \eta} + \text{sign}(\Delta_i^n \log \eta)\frac{1}{2\sqrt{n}}\sigma^2\right) \left|\sqrt{n\Delta_i^n \log \eta}\right|^{p-1} \mathbb{1}_{\{\sqrt{n\Delta_i^n \log \eta + n^{-1/2}\sigma^2/2} \sqrt{n\Delta_i^n \log \eta} < 0\}}
\]
\[
C^n_t = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor - 1} \sum_{k=2}^{p} \binom{k}{p} n\Delta_i^n \log \eta |\sqrt{n\Delta_i^n \log \eta}|^{p-k} \times
\left[\text{sign}(\Delta_i^n \log \eta)\frac{1}{2\sqrt{n}}\sigma^2 - 2(\sqrt{n\Delta_i^n \log \eta} + \text{sign}(\Delta_i^n \log \eta)\frac{1}{2\sqrt{n}}\sigma^2)\mathbb{1}_{\{\sqrt{n\Delta_i^n \log \eta + n^{-1/2}\sigma^2/2} \sqrt{n\Delta_i^n \log \eta} < 0\}}\right]^{k}
\]

It follows that
\[
\sqrt{n}(B(p, \log \eta)^n_t - m_p \sigma^pt)
= \sum_{i=1}^{\lfloor nt \rfloor - 1} \zeta_{p,i} + \sum_{i=1}^{\lfloor nt \rfloor - 1} \zeta_{2,i} \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor - 1} H^p_{|p,i}\]
\[+ \sqrt{n}A^n_t + \sqrt{n}B^n_t + \sqrt{n}C^n_t + m_p \sigma^p n^{1/2} \left(\left\lfloor \frac{nt}{n} \right\rfloor - t\right)
\]

where
\[
\zeta_{p,i} = n^{-1/2}\sigma^p (\left\lfloor \sum_{j \geq 1} \sqrt{n\Delta_i^n W_j} \mathbb{1}_{\sqrt{\Delta_i^n W_j} \leq Z_{i,(i-1)/n} = Z,i,(i-1)/n} \right\rfloor - m_p)
\]
\[
\zeta_{2,i} = n^{-1/2} \sum_{j \geq 1} \sum_{k > j} \Psi|p,\sigma(\sqrt{n\Delta_i^n W_j}, \sqrt{n\Delta_i^n W_k}, \sqrt{n}Z_{k,(i-1)/n} \mathbb{1}_{\{\sqrt{\Delta_i^n W_j} \leq Z_{i,(i-1)/n} = Z,(i-1)/n\}})
\]

Step 1) First it is clear that
\[
n^{1/2} \left(\left\lfloor \frac{nt}{n} \right\rfloor - t\right)^{u.c.p} \rightarrow 0.
\]
Step 2) We have
\[ \sqrt{n}A_i^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} \xi_i^n \]

where
\[ \xi_i^n = -\frac{\sigma^2}{2n} \times \text{sign} \left( \Delta_i^n \log \eta \right) \sqrt{n} \Delta_i^n \log \eta |^{p-1} \cdot
\]

Note that, by Proposition 2, we have for \( p = 1 \)
\[ \mathbb{E} \left[ \xi_i^n \mid \eta_{(i-1)/n} \right] = -\frac{\sigma^2}{2n} \left[ \Pr(U_i^n > 0 \mid \eta_{(i-1)/n}) - \Pr(U_i^n \leq 0 \mid \eta_{(i-1)/n}) \right] 
= -\frac{\sigma^2}{4n} \left[ \frac{1}{2} - \Pr(U_i^n \leq 0 \mid \eta_{(i-1)/n}) \right] 
= -\frac{\sigma^2}{4n} \left[ \frac{1}{2} - \exp \left( -\frac{1}{\eta_{(i-1)/n}} \left[ \Phi \left( \frac{\sigma}{2\sqrt{n}} \right) - \Phi \left( -\frac{\sigma}{2\sqrt{n}} \right) \right] \right) \right] 
= -\frac{\sigma^2}{8n} \left[ \frac{1}{\eta_{(i-1)/n}} - 1 \right] \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi}} (1 + o(1)), \]

and, for any integer larger than 1, the same type of calculations leads to
\[ \mathbb{E} \left[ \xi_i^n \mid \eta_{(i-1)/n} \right] = \frac{C}{n^{3/2}} \left[ \frac{1}{\eta_{(i-1)/n}} - 1 \right] (1 + o(1)). \]

Therefore we have
\[ \sum_{i=1}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[ \xi_i^n \mid \eta_{(i-1)/n} \right] \overset{u.c.p}{\longrightarrow} 0. \]

Moreover we have
\[ \mathbb{E} \left[ (\xi_i^n)^2 \mid \eta_{(i-1)/n} \right] = \frac{C^4}{4n^2} m_{2p-2} (1 + o(1)) \]

and therefore
\[ \sum_{i=1}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[ (\xi_i^n)^2 \mid \mathcal{F}_{(i-1)/n} \right] \overset{u.c.p}{\longrightarrow} 0. \]

By Lemma 2.2.10 in Jacod and Protter (2011), we deduce that
\[ \sqrt{n}A_i^n \overset{u.c.p}{\longrightarrow} 0. \]

Step 3) We have
\[ \sqrt{n}B_i^n = \sum_{i=1}^{\lfloor nt \rfloor - 1} \zeta_i^n \]

with
\[ \zeta_i^n = \frac{2}{\sqrt{n}} \left( \sqrt{n} \Delta_i^n \log \eta + \text{sign}(\Delta_i^n \log \eta) \frac{1}{2\sqrt{n}} \sigma^2 \right) \sqrt{n} \Delta_i^n \log \eta |^{p-1} \mathbb{I}_{\{\sqrt{n} \Delta_i^n \log \eta + n^{-1/2} \sigma^2 / \sqrt{n} \Delta_i^n \log \eta < 0\}}. \]

Note that, for \( a, u \in \mathbb{R} \),
\[ ||a| + \text{sign}(a)u||_{\{(a+u)a<0\}} \leq 2|a||\{a<|u|\}| \leq 2|u||\{|a|<|u|\}|. \]
Therefore we have
\[
\mathbb{E} \left[ |\zeta_i^n| \eta(i-1)/n \right] \leq \frac{4}{\sqrt{n}} \left( \frac{\sigma^2}{2\sqrt{n}} \right)^p \Pr \left( \left| \sqrt{n} \Delta_A^n \log \eta \right| < \frac{\sigma^2}{2\sqrt{n}} \right) \left[ \eta(i-1)/n \right] \\
= \frac{2^{2-p} \sigma^{2p}}{n(p+1/2)} \Pr \left( \left| U^n_i \right| < \frac{\sigma}{2\sqrt{n}} \right) \mathcal{F}_{(i-1)/n}
\]
where \( U^n_i = \sigma^{-1} \sqrt{n} \Delta_A^n \log \eta \). By Proposition 2, we derive that
\[
\mathbb{E} \left[ |\zeta_i^n| \eta(i-1)/n \right] \leq \frac{2^{2-p} \sigma^{2p}}{n(p+1/2)} \exp \left( \frac{1}{\eta(i-1)/n} \left[ \frac{e^{-\sigma^2/2n}}{2} - \Phi \left( \frac{\sigma^2}{n} \right) \right] \Phi \left( \frac{\sigma^2}{n} \right) - \frac{1}{2} \right) \frac{1}{2}
\]
for large \( n \). By Lemma 2.2.10 in Jacod and Protter (2011), it follows that
\[
\sqrt{n} B_i^n = \sum_{i=1}^{[nt]-1} \zeta_i^n \overset{u.c.p}{\longrightarrow} 0.
\]

Step 4) By using the same type of arguments as in Steps 2) and 3), it is easily seen that
\[
\sqrt{n} C_i^n \overset{u.c.p}{\longrightarrow} 0.
\]

Step 5) Let us prove that
\[
\sum_{i=1}^{[nt]-1} \zeta_{2,i}^n \overset{u.c.p}{\longrightarrow} \frac{1}{2\sigma^2} \lambda(\varphi_{p,\sigma}) \sum_{j \geq 1} \sum_{k > j} \int_0^t \mathbb{I}_\{\Lambda_{i=1,j,k}Z_{i,s} > \vee_{1 \leq j,k}Z_{i,s} \} dL_{k,j}^0.
\]
Since
\[
\sum_{j \geq 1} \sum_{k > j} \mathbb{I}_\{\vee_{1 \leq j,k}Z_{i,(i-1)/n} \leq \Lambda_{i=1,j,k}Z_{i,(i-1)/n} \} = 1,
\]
it is enough to prove that for some \((j, k)\) such that \( j \geq 1 \) and \( k > j \)
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} \Psi_{i,p,\sigma}(\sqrt{n} \Delta_A^n W_j, \sqrt{n} \Delta_A^n W_k, \sqrt{n} Z_{k,j(i-1)/n}) \mathbb{I}_\{\vee_{1 \leq j,k}Z_{i,(i-1)/n} \leq \Lambda_{i=1,j,k}Z_{i,(i-1)/n} \} \overset{u.c.p}{\longrightarrow} \frac{1}{2\sigma^2} \lambda(\varphi_{p,\sigma}) \int_0^t \mathbb{I}_\{\Lambda_{i=1,j,k}Z_{i,s} > \vee_{1 \leq j,k}Z_{i,s} \} dL_{k,j}^0.
\]
First note that
\[
\mathbb{E} \left[ \Psi_{i,p,\sigma}(\sqrt{n} \Delta_A^n W_j, \sqrt{n} \Delta_A^n W_k, \sqrt{n} Z_{k,j(i-1)/n}) \mathcal{F}_{(i-1)/n} \right] = \varphi_{p,\sigma} \left( \sqrt{n} Z_{k,j(i-1)/n} \right).
\]
We can deduce from a simple modification of Theorem 1.1 in Jacod (1998), that
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} \mathbb{E} \left[ \Psi_{i,p,\sigma}(\sqrt{n} \Delta_A^n W_j, \sqrt{n} \Delta_A^n W_k, \sqrt{n} Z_{k,j(i-1)/n}) \mathcal{F}_{(i-1)/n} \right] \
\mathbb{I}_\{\vee_{1 \leq j,k}Z_{i,(i-1)/n} \leq \Lambda_{i=1,j,k}Z_{i,(i-1)/n} \} \overset{u.c.p}{\longrightarrow} \frac{1}{2\sigma^2} \lambda(\varphi_{p,\sigma}) \int_0^t \mathbb{I}_\{\Lambda_{i=1,j,k}Z_{i,s} > \vee_{1 \leq j,k}Z_{i,s} \} dL_{k,j}^0.
\]
Moreover

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} \mathbb{E} \left[ \Psi_{|\cdot|^p, \sigma}^2(\sqrt{n} \Delta^n_t W_j, \sqrt{n} \Delta^n_t W_k, \sqrt{n} Z_{k\setminus j, (i-1)/n}) \bigg| \mathcal{F}_{(i-1)/n} \right] \mathbb{I}_{\{\forall i \neq j, k, Z_{k, (i-1)/n} \leq \wedge_{i \neq j, k} Z_{i, (i-1)/n}\}}
\]

\[
\overset{u.c.p}{\longrightarrow} \frac{1}{2\sigma^2} \lambda(\varphi^{(2)}_{p, \sigma}) \int_0^t \mathbb{I}_{\{\forall i \neq j, k, Z_{i, (i-1)/n} > \wedge_{i \neq j, k} Z_{i, (i-1)/n}\}} dL^9_{Z_{k\setminus j, \cdot}}
\]

where

\[
\varphi^{(2)}_{p, \sigma}(w) = \int_{\mathbb{R}^2} \Psi_{|\cdot|^p, \sigma}(x, y, w) \frac{1}{2\pi} e^{-(x^2 + y^2)/2} \, dx \, dy.
\]

The conclusion follows by using Lemma 2.2.12 in Jacod and Protter (2011).

Step 6) Let us prove that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} H_{|\cdot|^p, i} \overset{u.c.p}{\longrightarrow} 0.
\]

In the same way as in Step 3), it is enough to prove that for some \((j, k)\) such that \(j \geq 1, k \geq 1\) and \(k \neq j\),

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} \Psi_{|\cdot|^p, \sigma}^\leq(\sqrt{n} \Delta^n_t W_j, \sqrt{n} \Delta^n_t W_k, \sqrt{n} Z_{k\setminus j, (i-1)/n}) \times I^{(j,k), n}_{i} \overset{u.c.p}{\longrightarrow} 0,
\]

where

\[
I^{(j,k), n}_{i} = \prod_{l \neq j, k} \mathbb{I}_{\{Z_{i \setminus l, (i-1)/n} \leq 0\}} \times \left( \prod_{l \neq k, j} \mathbb{I}_{\{Z_{i \setminus l, (i-1)/n} \leq 0\}} - \prod_{l \neq k, j} \mathbb{I}_{\{Z_{i \setminus l, (i-1)/n} \leq 0\}} \right).
\]

Let

\[
Y_{k, j, t} = R_k + \sigma W_{k, t} - \vee_{l \neq k, j} (R_l + \sigma W_{l, t}).
\]

We have

\[
\prod_{l \neq k, j} \mathbb{I}_{\{Z_{i \setminus l, (i-1)/n} \leq 0\}} - \prod_{l \neq k, j} \mathbb{I}_{\{Z_{i \setminus l, (i-1)/n} \leq 0\}} = \mathbb{I}_{\{Y_{k, j, (i-1)/n} > 0, Y_{k, j, i/n} \leq 0\}} - \mathbb{I}_{\{Y_{k, j, (i-1)/n} \leq 0, Y_{k, j, i/n} > 0\}}.
\]

Note that

\[
\Psi_{|\cdot|^p, \sigma}^\leq(\sqrt{n} \Delta^n_t W_j, \sqrt{n} \Delta^n_t W_k, \sqrt{n} Z_{k\setminus j, (i-1)/n}) = \left( |\sigma \sqrt{n} \Delta^n_t W_k + \sqrt{n} Z_{k\setminus j, (i-1)/n}|^p - |\sigma \sqrt{n} \Delta^n_t W_j|^{p} \right) \mathbb{I}_{\{Z_{k\setminus j, (i-1)/n} \leq 0\}}.
\]

Therefore

\[
\Psi_{|\cdot|^p, \sigma}^\leq(\sqrt{n} \Delta^n_t W_j, \sqrt{n} \Delta^n_t W_k, \sqrt{n} Z_{k\setminus j, (i-1)/n}) \times I^{(j,k), n}_{i}
\]

could be different from 0, if at least \(Z_{k\setminus j, (i-1)/n}\) is close to zero, and \(Y_{k, j, (i-1)/n}\) is also close to zero, or equivalently \(Z_{k, (i-1)/n}\) is close to \(Z_{j, (i-1)/n}\) and to \(\vee_{l \neq k, j} Z_{i, (i-1)/n}\). It is well known that bi-dimensional diffusion processes never revisit a point in the plane, and so they do not in particular have a local time. As a consequence, it is derived that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} \Psi_{|\cdot|^p, \sigma}^\leq(\sqrt{n} \Delta^n_t W_j, \sqrt{n} \Delta^n_t W_k, \sqrt{n} Z_{k\setminus j, (i-1)/n}) \times I^{(j,k), n}_{i} \overset{u.c.p}{\longrightarrow} 0.
\]
Step 7) By using usual arguments (see e.g. Chapter 5.2 in Jacod and Protter (2011)), we have
\[
\sum_{i=1}^{\lfloor nt \rfloor - 1} \zeta_{n,i} \xi_i \Rightarrow \sigma \tilde{X}_{0,t},
\]
where \( \tilde{X}_0 \) is a process defined on an extension \((\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t \geq 0}, \mathbb{P}) \) of \((\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P}) \), which conditionally on \( \mathcal{F} \) is a continuous centered Gaussian martingale with variance
\[
\mathbb{V}[\tilde{X}_{0,t} | \mathcal{F}] = (m_{2p} - m_p^2) t.
\]
Step 8) Use steps from 1) to 7) and define
\[
\tilde{X}_{2,t} = \frac{1}{2\sigma^2} \lambda(p, \sigma) \sum_{j \geq 1} \sum_{k > j} \int_0^t \mathbb{1}_{\{\wedge^{i=j,k} > \vee_{l \neq j,k} Z_{l,s} \}} dL_{Z_{k,j},s} + \sigma \tilde{X}_{0,t}
\]
to conclude.

5.4 Proof of Proposition 4
We only prove the stable convergence in law of \( \sqrt{n} (B(p, \log \eta) t - m_p \int_0^t H_s^p ds) \). Recall that
\[
\log \eta_t = \sqrt{\log R_j + \int_0^t H_s dW_j,s - \frac{1}{2} \int_0^t H_s^2 ds} = \sqrt{\log Z_{j,t} - \frac{1}{2} \int_0^t H_s^2 ds}.
\]
Let
\[
H_s^{(i)} = \begin{cases} H_s & \text{if } s \leq (i-1)/n \\ H_{(i-1)/n} & \text{if } s > (i-1)/n \end{cases}
\]
and define
\[
\log \eta_t^{(i)} = \sqrt{Z_{j,t}^{(i)} - \frac{1}{2} \int_0^t (H_s^{(i)})^2 ds}
\]
with
\[
Z_{j,t}^{(i)} = \log R_j + \int_0^t H_s^{(i)} dW_j,s.
\]
We have
\[
\Delta_i^n \log \eta = \Delta_i^n \log \eta^{(i)} + [\Delta_i^n \log \eta - \Delta_i^n \log \eta^{(i)}]
\]
Now, note that, for \( a, u \in \mathbb{R} \),
\[
|a + u| = |a| + \text{sign}(a)u - 2(|a| + \text{sign}(a)u) \mathbb{1}_{\{(a+u)a < 0\}}.
\]
With
\[
a_i^n = \Delta_i^n \log \eta^{(i)}, \quad u_i^n = [\Delta_i^n \log \eta - \Delta_i^n \log \eta^{(i)}],
\]
and
\[
b_i^n = \text{sign}(a_i^n)u_i^n - 2(|a_i^n| + \text{sign}(a_i^n)u_i^n) \mathbb{1}_{\{(a_i^n + u_i^n)a_i^n < 0\}},
\]
we have
\[
|\Delta_i^n \log \eta|^p = |\Delta_i^n \log \eta| + w_i^n
\]
\[
|\Delta^n \log \eta - \Delta^n \log \eta_i|^q \\
\leq \left| \int_{(i-1)/n}^{i/n} [H_s - H_{(i-1)/n}] dW_{i/n,s} - \frac{1}{2} \int_{(i-1)/n}^{i/n} \left[ H_s^2 - H_{(i-1)/n}^2 \right] ds \right|^q \\
+ \left| \int_{(i-1)/n}^{i/n} [H_s - H_{(i-1)/n}] dW_{j(i-1)/n,s} - \frac{1}{2} \int_{(i-1)/n}^{i/n} \left[ H_s^2 - H_{(i-1)/n}^2 \right] ds \right|^q \\
\leq C \int_{(i-1)/n}^{i/n} \left[ H_s - H_{(i-1)/n} \right] dW_{i/n,s}^q \right| + C \left| \int_{(i-1)/n}^{i/n} \left[ H_s - H_{(i-1)/n} \right] dW_{j(i-1)/n,s} \right|^q \\
+ C \int_{(i-1)/n}^{i/n} \left| H_s^2 - H_{(i-1)/n}^2 \right|^q ds.
\]
Then
\[ \mathbb{E} \left[ |\Delta^n \log \eta - \Delta^n \log \eta_1|^{q} \mid \mathcal{F}_{(i-1)/n} \right] \leq C \frac{1}{n^{aq}} \frac{1}{\eta^{q/2}} \frac{1}{n} + C \frac{1}{n^{aq}} \frac{1}{n} \leq C \frac{1}{n^{aq + (2/\eta)q/2}}. \]

ii) a) Study of \( b^n_i \):

\[ b^n_i = \text{sign}(a^n_i) u^n_i - 2(|a^n_i| + \text{sign}(a^n_i) u^n_i) \mathbb{I}_{\{a^n_i + u^n_i > 0\}} := b^n_{1,i} + b^n_{2,i} \]

with

\[ b^n_{1,i} = \text{sign}(a^n_i) u^n_i \]
\[ b^n_{2,i} = -2(|a^n_i| + \text{sign}(a^n_i) u^n_i) \mathbb{I}_{\{a^n_i + u^n_i > 0\}}. \]

For \( q \geq 1 \),

\[ \mathbb{E} \left[ |b^n_i|^q \mid \mathcal{F}_{(i-1)/n} \right] \leq C \left( \mathbb{E} \left[ |b^n_{1,i}|^q \mid \mathcal{F}_{(i-1)/n} \right] + \mathbb{E} \left[ |b^n_{2,i}|^q \mid \mathcal{F}_{(i-1)/n} \right] \right) \]

Using i), we deduce that

\[ \mathbb{E} \left[ |b^n_{1,i}|^q \mid \mathcal{F}_{(i-1)/n} \right] \leq C \frac{1}{n^{aq + (2/\eta)q/2}}. \]

Now note that

\[ \{(a^n_i + u^n_i) a^n_i < 0\} \subset \{|a^n_i| < |u^n_i|\}. \]

Then

\[ \mathbb{E} \left[ |b^n_{2,i}|^q \mid \mathcal{F}_{(i-1)/n} \right] \leq C \mathbb{E} \left[ (|a^n_i|^q + |u^n_i|^q) \mathbb{I}_{\{(a^n_i + u^n_i) a^n_i < 0\}} \mid \mathcal{F}_{(i-1)/n} \right] \]
\[ \leq C \mathbb{E} \left[ |u^n_i|^q \mathbb{I}_{\{|a^n_i| < |u^n_i|\}} \mid \mathcal{F}_{(i-1)/n} \right]. \]

By H"older's inequality,

\[ \mathbb{E} \left[ |b^n_{2,i}|^q \mid \mathcal{F}_{(i-1)/n} \right] \leq C \left[ \mathbb{E} \left[ |u^n_i|^{2q} \mid \mathcal{F}_{(i-1)/n} \right] \right]^{1/2} \left[ \Pr \left( (a^n_i + u^n_i) a^n_i < 0 \mid \mathcal{F}_{(i-1)/n} \right) \right]^{1/2} \]
\[ \leq C \frac{1}{n^{aq + (2/\eta)q/4}} \left[ \Pr \left( |a^n_i| < |u^n_i| \mid \mathcal{F}_{(i-1)/n} \right) \right]^{1/2} \]
\[ = C \frac{1}{n^{aq + 1/4}} \left[ \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} |a^n_i| < \frac{\sqrt{n}}{H_{(i-1)/n}} |u^n_i| \mid \mathcal{F}_{(i-1)/n} \right) \right]^{1/2}. \]

Now by Proposition 2, note that

\[ \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} a^n_i \leq u \mid \mathcal{F}_{(i-1)/n} \right) = \exp \left( -\frac{1}{\eta_{(i-1)/n}} \left[ e^{-H_{(i-1)/n} n^{-1/2} u \Phi \left( -u + \frac{1}{2} H_{(i-1)/n} n^{-1/2} \right) - \Phi \left( -u - \frac{1}{2} H_{(i-1)/n} n^{-1/2} \right) } \right] \right) \]
\[ \times \Phi \left( u + \frac{1}{2} H_{(i-1)/n} n^{-1/2} \right). \]

Let \( \lambda_n = n^{-1/4} \to 0 \). By Markov's inequality

\[ \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} |u^n_i| > \lambda_n \mid \mathcal{F}_{(i-1)/n} \right) \leq \frac{n^{1/2}}{H_{(i-1)/n}} \frac{\mathbb{E} \left[ |u^n_i| \mid \mathcal{F}_{(i-1)/n} \right]}{\lambda_n} \]
\[ \sim \frac{1}{n^{1/4}} \]
Therefore we have

\[ \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} |a^n_i| < \lambda_n \mid F_{(i-1)/n} \right) \sim C \exp \left( -\frac{1}{n_{(i-1)/n}} \right) \lambda_n \leq C \frac{1}{n^{1/4}} \to 0. \]

Therefore we have

\[ \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} |a^n_i| < \frac{\sqrt{n}}{H_{(i-1)/n}} |u^n_i| \mid F_{(i-1)/n} \right) \leq \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} |a^n_i| < \lambda_n \mid F_{(i-1)/n} \right) + \Pr \left( \frac{\sqrt{n}}{H_{(i-1)/n}} |u^n_i| > \lambda_n \mid F_{(i-1)/n} \right) \leq C \frac{1}{n^{1/4}} \]

for large \( n \). Finally, we have

\[ \mathbb{E} \left[ |b^n_{2,i}|^q \mid F_{(i-1)/n} \right] \leq C \frac{1}{n^{\alpha q + 5/8}} \]

and we deduce that for \( q \geq 1 \)

\[ \mathbb{E} \left[ |b^n_{2,i}|^q \mid F_{(i-1)/n} \right] \leq C \left[ \frac{1}{n^{\alpha q + (2q)/2}} + \frac{1}{n^{\alpha q + 5/8}} \right] \leq C \frac{1}{n^{\alpha q + (q^5/4)/2}}. \]

b) Study of \( w^n_i \):

\[ w^n_i = \sum_{k=1}^{p} C_p^k (b^n_{2,i})^k |a^n_i|^{p-k}. \]

For some \( q > 1 \)

\[ \mathbb{E} \left[ n^{(p-1)/2} |w^n_i| \mid F_{(i-1)/n} \right] \leq \sum_{k=1}^{p} n^{(k-1)/2} C_p^k \mathbb{E} \left[ |b^n_{2,i}|^k |n^{1/2} a^n_i|^{p-k} \mid F_{(i-1)/n} \right] \leq \sum_{k=1}^{p} n^{(k-1)/2} C_p^k \mathbb{E} \left[ |b^n_{2,i}|^k \mid F_{(i-1)/n} \right]^{1/q} \mathbb{E} \left[ n^{1/2} |a^n_i|^{2(p-k)/(1-q)} \mid F_{(i-1)/n} \right]^{1-1/q} \leq C \sum_{k=1}^{p} n^{(k-1)/2} \frac{1}{k^{\alpha k + (kq^5/4)/2q}} \leq C \sum_{k=1}^{p} n^{(k-1)/2} \frac{1}{n^{(\alpha - 1/2)k + 1/2 + (kq^5/4)/2q}}. \]

If \( 1 < q < 5/4 \), we deduce that

\[ \mathbb{E} \left[ n^{(p-1)/2} |w^n_i| \mid F_{(i-1)/n} \right] = o \left( n^{-1} \right). \]

It follows that \( \sum_{i=1}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[ n^{(p-1)/2} |w^n_i| \mid F_{(i-1)/n} \right] \xrightarrow{u.c.p} 0 \) and

\[ \sum_{i=1}^{\lfloor nt \rfloor - 1} n^{(p-1)/2} w^n_i \xrightarrow{u.c.p} 0. \]

- Step 4) We have

\[ \sum_{i=1}^{\lfloor nt \rfloor - 1} \xi^n_i = n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor - 1} \left( n^{1/2} \Delta^n_i \log \eta^{(i)} |p - m_p H_{(i-1)/n}^p \right). \]
Let
\[ \kappa^n_i = \frac{1}{\sqrt{n}} \Psi_{1, P, H_i, n} \left( \sqrt{n} \Delta^n_t W_j, \sqrt{n} \Delta^n_t W_k, \sqrt{n} Z_{k,j}(i-1)/n \right) \mathbb{1}_{\{ \zeta \leq \zeta^n_j \}} \mathbb{1}_{\{ \zeta \leq (i-1)/n \}} \cdot \]

By using similar arguments as in Jacod (1998), it is possible to prove that
\[ \sum_{i=1}^{[nt]} \mathbb{E} \left[ \kappa^n_i | \mathcal{F}_{(i-1)/n} \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \mathbb{1}_{\{ \zeta \leq (i-1)/n \}} \Psi_{1, P, H_{i/n}} \left( \sqrt{n} Z_{k,j}(i-1)/n \right) \frac{1}{2} \int_0^t \frac{\lambda(\varphi_{p,H_s})}{H_s^2} dL_{Z_{k,j},s}^0. \]

We do not give details, but this can be seen from the following intuitive arguments
\[ \sum_{i=1}^{[nt]} \mathbb{E} \left[ \kappa^n_i | \mathcal{F}_{(i-1)/n} \right] \approx \sqrt{n} \int_0^t \mathbb{1}_{\{ \lambda \leq (i-1)/n \}} \Psi_{1, P, H_{i/n}} \left( \sqrt{n} Z_{k,j}(i-1)/n \right) \frac{1}{2} \int_0^t \frac{\lambda(\varphi_{p,H_s})}{H_s^2} dL_{Z_{k,j},s}^0. \]

where we used the occupation time formula for continuous semimartingales (see e.g. Proposition 2.1 p 522 in Revuz and Yor (1999)). Moreover \( \sqrt{n} \sum_{i=1}^{[nt]} \mathbb{E} \left[ \kappa^n_i^2 | \mathcal{F}_{(i-1)/n} \right] \) converges u.c.p. to a non-degenerate process. Therefore
\[ \sum_{i=1}^{[nt]} \kappa^n_i \overset{u.c.p.}{\rightarrow} \frac{1}{2} \int_0^t \frac{\lambda(\varphi_{p,H_s})}{H_s^2} dL_{Z_{k,j},s}^0, \]

Step 5) By using the same arguments as in the proof of Proposition 3 we get
\[ \sum_{i=1}^{[nt]-1} \zeta^n_i \overset{d}{\rightarrow} \tilde{X}_{3,t} \]

where \( \tilde{X}_3 \) is a process defined on an extension \( (\Omega, \tilde{\mathcal{F}}, (\tilde{F}_t)_{t \geq 0}, \mathbb{P}) \) of \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \), which conditionally on \( \mathcal{F} \) is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by
\[ \mathbb{E}[\tilde{X}_{3,t} | \mathcal{F}] = \frac{1}{2} \sum_{j \geq 1} \sum_{k>j} \int_0^t \frac{\lambda(\varphi_{p,H_s})}{H_s^2} dL_{Z_{k,j},s}^0 \quad \text{and} \quad Var[\tilde{X}_{3,t} | \mathcal{F}] = (m_{2p} - m_p^2) \int_0^t H_s^2 ds. \]
5.5 Proof of Proposition 5

We only prove the stable convergence in law of \( \sqrt{n}(I_t^n - \int_0^t \gamma_s ds) \). Recall that

\[
\log \eta_t = \sqrt{\sum_{j=1}^{\infty} \left( \log R_j + \int_0^t H_s dW_{j,s} - \frac{1}{2} \int_0^t H_s^2 ds \right)} = \sqrt{Z_{j,t} - \frac{1}{2} \int_0^t H_s^2 ds}.
\]

Let

\[
H_s^{(i)} = \begin{cases} 
H_s & \text{if } s \leq (i-1)/n \\
H_{(i-1)/n} & \text{if } s > (i-1)/n
\end{cases}
\]

and define

\[
\log \eta_t^{(i)} = \sqrt{Z_{j,t}^{(i)} - \frac{1}{2} \int_0^t (H_s^{(i)})^2 ds}
\]

with

\[
Z_{j,t}^{(i)} = \log R_j + \int_0^t H_s^{(i)} dW_{j,s}.
\]

The increments of \( \log Y \) can be written as

\[
\Delta_i^n \log Y = \log \eta_{(i-1)/n} \Delta_i^n \gamma + \gamma_{i/n} \Delta_i^n \log \eta
\]

\[
= \gamma_{i/n} \Delta_i^n \log \eta^{(i)} + \log \eta_{(i-1)/n} \Delta_i^n \gamma + \gamma_{i/n} [\Delta_i^n \log \eta - \Delta_i^n \log \eta^{(i)}].
\]

Now, note that, for \( a, u \in \mathbb{R} \),

\[
|a + u| = |a| + sign(a)u - 2(|a| + sign(a)u)I_{\{|a+u|<0\}}.
\]

With

\[
a_i^n = \gamma_{i/n} \Delta_i^n \log \eta^{(i)}
\]

\[
u_i^n = \log \eta_{(i-1)/n} \Delta_i^n \gamma + \gamma_{i/n} [\Delta_i^n \log \eta - \Delta_i^n \log \eta^{(i)}],
\]

we have

\[
|\Delta_i^n \log Y| = \gamma_{i/n} |\Delta_i^n \log \eta^{(i)}| + \nu_i^n + w_i^n
\]

where

\[
u_i^n = \gamma_{i/n} \log(\Delta_i^n \log \eta^{(i)}) \log \eta_{(i-1)/n} \Delta_i^n \gamma
\]

and

\[
w_i^n = \gamma_{i/n} \log(\Delta_i^n \log \eta^{(i)}) \Delta_i^n \log \eta - \Delta_i^n \log \eta^{(i)} - 2(|a_i^n| + sign(a_i^n)u_i^n)I_{\{|a_i^n+u_i^n|<0\}}.
\]

Therefore

\[
\sqrt{n}(I_t^n - \int_0^t \gamma_s ds) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{\zeta_i^n}{m_i H_{(i-1)/n}} + \sum_{i=1}^{\lfloor nt \rfloor-1} \frac{\zeta_i^n}{m_i H_{(i-1)/n}} + \sum_{i=1}^{\lfloor nt \rfloor-1} \frac{w_i^n}{m_i H_{(i-1)/n}} + \sum_{i=1}^{\lfloor nt \rfloor-1} \frac{x_i^n}{m_i H_{(i-1)/n}} + \sqrt{n} \int_{\lfloor nt \rfloor/n}^t \gamma_s ds,
\]

where

\[
\zeta_i^n = n^{-1/2} \gamma_{i/n} \left(n^{1/2}|\Delta_i^n \log \eta^{(i)}| - m_i H_{(i-1)/n}\right)
\]

\[
x_i^n = -n^{1/2} m_i \int_{(i-1)/n}^{i/n} (\gamma_s H_s - \gamma_{i/n} H_{(i-1)/n}) ds.
\]

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Step 1) It is clear that
\[ \sqrt{n} \int_{|nt|/n}^{\bar{t}} \gamma_s ds \text{ u.c.p } 0. \]

Step 2) We have
\[ \int_{(i-1)/n}^{i/n} (\gamma_s H_s - \gamma_{i/n} H_{(i-1)/n}) ds = \int_{(i-1)/n}^{i/n} [(\gamma_s H_s - \gamma_{(i-1)/n} H_{(i-1)/n}) + \Delta^n \gamma H_{(i-1)/n}] ds. \]

Since \( s \to \gamma_s H_s \) is Hölder with index \( \alpha \), we derive that
\[ \int_{(i-1)/n}^{i/n} |\gamma_s H_s - \gamma_{i/n} H_{(i-1)/n}| ds \leq C \left( \frac{1}{n^{1+\alpha}} + \frac{1}{n^2} \right) \]
and, as \( n \to \infty \), we have
\[ \sum_{i=1}^{\lfloor nt \rfloor-1} \frac{|x^n_i|}{m_1 H_{(i-1)/n}} \leq C \frac{n^{1/2}}{n^{1/2} \to 0}. \]

Step 3)

i) In the same way as in the previous proof, we have for \( q \geq 1 \),
\[ E \left[ |\Delta^n \log \eta - \Delta^n \log \eta(i)|^q \middle| \mathcal{F}_{(i-1)/n} \right] \leq C \frac{1}{n^{\alpha q + (2\wedge q)/2}}. \]

ii)

a) Study of \( u^n_i \): using i), for \( q \geq 1 \),
\[ E \left[ |u^n_i|^q \middle| \mathcal{F}_{(i-1)/n} \right] \leq C \frac{1}{n^q} + C \frac{1}{n^{\alpha q + (2\wedge q)/2}} \leq C \frac{1}{n^{1+\alpha q + (2\wedge q)/2}}. \]

b) Study of \( v^n_i \): we have (in the same way as in the proof of Proposition 3)
\[ E \left[ |v^n_i|^q \middle| \mathcal{F}_{(i-1)/n} \right] \leq C(\sup_{s \in [0,1]} |\gamma'_s|) n^{-q} |\log n(i-1)/n|^q \leq C n^{-q}. \]

We deduce by Lemma 2.2.10 in Jacod and Protter (2011) that
\[ \sum_{i=1}^{\lfloor nt \rfloor-1} \frac{v^n_i}{m_1 H_{(i-1)/n}} \text{ u.c.p } 0. \]

c) Study of \( w^n_i \): let us define
\[ w^n_i = w^n_{1,i} + w^n_{2,i}, \]
with
\[
\begin{align*}
w^n_{1,i} &= \gamma_{i/n}^2 \text{sign}(\Delta^n \log \eta_i)[\Delta^n \log \eta - \Delta^n \log \eta(i)] \\wedge^n_{1,i} + u^n_{1,i} \mathbb{I}_{\{(a^n_i + u^n_i)\alpha_i < 0\}}. \\
w^n_{2,i} &= -2(|a^n_i| + \text{sign}(a^n_i) u^n_i) \mathbb{I}_{\{(a^n_i + u^n_i)\alpha_i < 0\}}. 
\end{align*}
\]
Using i), we deduce that

\[ \mathbb{E} \left[ |w_{i,i}^n|^q \middle| \mathcal{F}_{(i-1)/n} \right] \leq C \frac{1}{n^{\alpha+q} \wedge q/2}. \]

Now note that

\[ \{(a_i^n + u_i^n)a_i^n < 0\} \subset \{|a_i^n| < |u_i^n|\}. \]

Then

\[
\mathbb{E} \left[ |w_{2,i}^n| \middle| \mathcal{F}_{(i-1)/n} \right] \leq C \mathbb{E} \left[ (|a_i^n| + |u_i^n|) \mathbb{I}_{\{(a_i^n + u_i^n)a_i^n < 0\}} \middle| \mathcal{F}_{(i-1)/n} \right] \\
\leq C \mathbb{E} \left[ |u_i^n| \mathbb{I}_{\{|u_i^n| < |a_i^n|\}} \middle| \mathcal{F}_{(i-1)/n} \right].
\]

By Hölder’s inequality,

\[
\mathbb{E} \left[ |w_{2,i}^n| \middle| \mathcal{F}_{(i-1)/n} \right] \leq C \left[ \mathbb{E} \left[ |a_i^n|^2 \middle| \mathcal{F}_{(i-1)/n} \right] \right]^{1/2} \left[ \mathbb{P} \left( (a_i^n + u_i^n)a_i^n < 0 \middle| \mathcal{F}_{(i-1)/n} \right) \right]^{1/2} \\
\leq C \frac{1}{n} \left[ \mathbb{P} \left( |a_i^n| < |u_i^n| \middle| \mathcal{F}_{(i-1)/n} \right) \right]^{1/2} \\
= C \frac{1}{n} \left[ \mathbb{P} \left( H_{(i-1)/n} |a_i^n| < \frac{\sqrt{n}}{H_{(i-1)/n}} |u_i^n| \middle| \mathcal{F}_{(i-1)/n} \right) \right]^{1/2}.
\]

Now by Proposition \( \mathbb{2} \) note that

\[
\mathbb{P} \left( H_{(i-1)/n} a_i^n \leq u \middle| \mathcal{F}_{(i-1)/n} \right) = \exp \left( -\frac{1}{\eta_{(i-1)/n}} \left[ e^{-H_{(i-1)/n} n^{-1/2} u} \Phi \left( -u + \frac{1}{2} H_{(i-1)/n} n^{-1/2} \right) - \Phi \left( -u - \frac{1}{2} H_{(i-1)/n} n^{-1/2} \right) \right] \right) \\
\times \Phi \left( u + \frac{1}{2} H_{(i-1)/n} n^{-1/2} \right).
\]

Let \( \lambda_n = n^{-1/4} \to 0 \). By Markov’s inequality

\[
\mathbb{P} \left( H_{(i-1)/n} |u_i^n| > \lambda_n \middle| \mathcal{F}_{(i-1)/n} \right) \leq \frac{\mathbb{E} \left[ |u_i^n|^2 \middle| \mathcal{F}_{(i-1)/n} \right]}{\lambda_n} \sim C \frac{1}{n^{1/4}}
\]

and moreover

\[
\mathbb{P} \left( H_{(i-1)/n} |a_i^n| < \lambda_n \middle| \mathcal{F}_{(i-1)/n} \right) \sim C \exp \left( -\frac{1}{\eta_{(i-1)/n}} \right) \lambda_n \leq C \frac{1}{n^{1/4}} \to 0.
\]

Therefore we have

\[
\mathbb{P} \left( H_{(i-1)/n} |a_i^n| < \frac{\sqrt{n}}{H_{(i-1)/n}} |u_i^n| \middle| \mathcal{F}_{(i-1)/n} \right) \\
\leq \mathbb{P} \left( H_{(i-1)/n} |a_i^n| < \lambda_n \middle| \mathcal{F}_{(i-1)/n} \right) + \mathbb{P} \left( H_{(i-1)/n} |u_i^n| > \lambda_n \middle| \mathcal{F}_{(i-1)/n} \right) \\
\leq C \frac{1}{n^{1/4}}
\]

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for large $n$. Finally, we have
\[
\mathbb{E} \left[ \left| w_{2,n}^p \right| \right| \mathcal{F}_{(i-1)/n} \right] \leq C \frac{1}{n^{9/8}}
\]
and we deduce that
\[
\mathbb{E} \left[ \left| w_{i}^n \right| \right| \mathcal{F}_{(i-1)/n} \right] \leq C \left[ \frac{1}{n^{1/2+\alpha}} + \frac{1}{n^{9/8}} \right].
\]
It follows that $\sum_{i=1}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[ \left| w_{i}^n \right| / H_{(i-1)/n} \right| \mathcal{F}_{(i-1)/n} \right] \underset{u.c.p.}{\longrightarrow} 0$ and
\[
\sum_{i=1}^{\lfloor nt \rfloor - 1} \frac{w_{i}^n}{m_1 H_{(i-1)/n}} \underset{u.c.p.}{\longrightarrow} 0.
\]

- Step 4) We have
\[
\sum_{i=1}^{\lfloor nt \rfloor - 1} \frac{\zeta_{i}^n}{m_1 H_{(i-1)/n}} = n^{1/2} \sum_{i=1}^{\lfloor nt \rfloor - 1} \frac{\gamma_{i/n}}{m_1 H_{(i-1)/n}} \left( n^{1/2} \frac{\log \eta_{(i)}}{m_1 H_{(i-1)/n}} - 1 \right).
\]
Let
\[
\kappa_{i}^n = \frac{1}{\sqrt{n}} \frac{\gamma_{i/n}}{m_1 H_{(i-1)/n}} \psi_{\cdot, H_{(i-1)/n}} \left( \sqrt{n} \Delta_i^n W_j, \sqrt{n} \Delta_i^n W_k, \sqrt{n} Z_{k,j,(i-1)/n} \right) \mathbb{I}_{\{ \gamma_{(i),k} Z_{i,(i-1)/n} \leq n^{1/2} \log \eta_{(i)} \}}.
\]
By using similar arguments as in Jacod (1998), it is possible to prove that (see also the previous proof)
\[
\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \kappa_{i}^n \right| \mathcal{F}_{(i-1)/n} \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\gamma_{i/n}}{m_1 H_{(i-1)/n}} \mathbb{I}_{\{ \gamma_{(i),k} Z_{i,(i-1)/n} \leq n^{1/2} \log \eta_{(i)} \}} \frac{\lambda(\varphi_{1,H_s})}{m_1 H_s^3} \mathbb{I}_{\{ \gamma_{(i),k} Z_{i,s} \geq n^{1/2} \log \eta_{(i)} \}} dL_{Z_k,j,s}^0.
\]
Moreover $\sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E} \left[ (\kappa_{i}^n)^2 \right| \mathcal{F}_{(i-1)/n} \right]$ converges u.c.p. to a non degenerate process. Therefore
\[
\sum_{i=1}^{\lfloor nt \rfloor} \kappa_{i}^n \underset{u.c.p.}{\longrightarrow} \frac{1}{2} \int_0^t \frac{\lambda(\varphi_{1,H_s})}{m_1 H_s^3} \mathbb{I}_{\{ \gamma_{(i),k} Z_{i,s} \geq n^{1/2} \log \eta_{(i)} \}} dL_{Z_k,j,s}^0.
\]

Step 5) By using the same arguments as in the proof of Proposition 3 we get
\[
\sum_{i=1}^{\lfloor nt \rfloor - 1} \zeta_{i}^n \Rightarrow \tilde{X}_{4,t}
\]
where $\tilde{X}_{4}$ is a process defined on an extension $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, which conditionally on $\mathcal{F}$ is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by
\[
\tilde{\mathbb{E}}[\tilde{X}_{4,t} | \mathcal{F}] = \frac{1}{2} \sum_{j \geq 1} \int_0^t \lambda(\varphi_{H_s}) \gamma_{s/n} \mathbb{I}_{\{ \gamma_{(i),k} Z_{i,s} \geq n^{1/2} \log \eta_{(i)} \}} dL_{Z_k,j,s}^0 \quad \text{and} \quad \tilde{\mathbb{V}}[\tilde{X}_{4,t} | \mathcal{F}] = (m_1^{-2} - 1) \int_0^t \gamma_s^2 ds.
\]
5.6 Proof of Proposition 6

Let $H^n_i = H^n_{(i-1)/n}$ and $\gamma^n_i = \gamma_{(i-1)/n}$. We first consider

\[ R^n_t = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma^n_i \left( \frac{H^n_i}{H^n_i} - 1 \right). \]

**Proposition 8** Let $k_n \to \infty$ and $k_n^{\alpha/(\alpha-1/2)}/n \to 0$, we have

\[ R^n_t \xrightarrow{\mathbb{L}_2} R_t \]

where $R$ is a process defined on an extension $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, which conditionally on $\mathcal{F}$ is a centered continuous Gaussian process, with independent increments, and whose variance is given by $\mathbb{V}[X_{3,t}|\mathcal{F}] = (1 - m^2_t) \int_0^t \gamma^2 ds$.

**Proof:** Note that this proof uses a lot of arguments as developed in the proof of Theorem 3.2 in Jacod and Rosenbaum (2013).

Let us introduce some notation. Let

\[ \alpha^n_i = |\Delta^n_i \log V| - m_1 H^n_i / \sqrt{n} \]
\[ \zeta^n_{i,j} = \alpha^n_i + (H^n_{i+1} - H^n_i) m_1 / \sqrt{n} \]
\[ \beta^n_i = \hat{H}^n_i - H^n_i = \sqrt{n} \sum_{j=0}^{k_n-1} \zeta^n_{i,j}. \]

We have

\[ R^n_t = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma^n_i H^n_i \left( \frac{1}{H^n_i} - 1 \right) = R^{n,1}_t + R^{n,2}_t + R^{n,3}_t, \]

where

\[ R^{n,1}_t = -\frac{1}{\sqrt{n}} m_1 \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \frac{\gamma^n_i}{H^n_i} \frac{1}{k_n} \sum_{j=0}^{k_n-1} (H^n_{i+1} - H^n_i) \]
\[ R^{n,2}_t = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma^n_i H^n_i \left( \frac{1}{H^n_i} - \frac{1}{H^n_i} + \frac{\beta^n_i}{(H^n_i)^2} - \frac{(1 - m^2_t) 1}{k_n H^n_i} \right) \]
\[ R^{n,3}_t = -\frac{1}{k_n} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \frac{\gamma^n_i}{H^n_i} \sum_{j=0}^{k_n-1} \alpha^n_{i+j}. \]

Step 1) We have $R^{n,1}_{nt} \xrightarrow{\mathbb{L}_2} 0$ since

\[ |R^{n,1}_t| \leq C \frac{1}{\sqrt{n}} m_1 \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \frac{\gamma^n_i}{H^n_i} \frac{1}{k_n} \sum_{j=0}^{k_n-1} |H^n_{i+1} - H^n_i| \]
\[ \leq C \frac{1}{\sqrt{n}} m_1 \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \frac{\gamma^n_i}{H^n_i} \frac{1}{k_n} \sum_{j=0}^{k_n-1} \left( \frac{j}{n} \right)^\alpha \]
\[ \leq C \frac{1}{n^{\alpha/2}} \leq C \left( \frac{k_n^{\alpha/(\alpha-1/2)}}{n} \right)^{\alpha-1/2} \xrightarrow{n \to \infty} 0. \]
Step 2) Note that
\[
\frac{1}{H_i^n} - \frac{1}{H_i^n} + \frac{\beta_i^n}{(H_i^n)^2} = \beta_i^n \frac{1}{H_i^n} \left( \frac{1}{H_i^n} - \frac{1}{H_i^n} \right) = (\beta_i^n)^2 \left( \frac{1}{H_i^n} \right)^2 = \frac{1}{(H_i^n)^2} \frac{1}{H_i^n},
\]
and write \( R_t^{n,2} = R_t^{n,2,1} + R_t^{n,2,2} \) where
\[
R_t^{n,2,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \left[ \frac{(\beta_i^n)^2}{(H_i^n)^2} - \frac{(1 - m_i^n)}{k_n} \right],
\]
\[
R_t^{n,2,2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n H_i^n \left[ \frac{(\beta_i^n)^2}{(H_i^n)^2} - \frac{(1 - m_i^n)}{k_n} \right] \left( \frac{1}{H_i^n} - \frac{1}{H_i^n} \right)
\]
and let us prove that
\[
R_t^{n,2,j} \overset{ucp}{\Longrightarrow} 0 \quad j = 1, 2.
\]

i) Note that
\[
(\beta_i^n)^2 = \frac{n}{k_n^2} \sum_{j=0}^{k_n-1} (\zeta_{i,j}^n)^2 + \frac{n}{k_n^2} \sum_{0 \leq j < k_n - 1} \zeta_{i,j}^n \zeta_{i,k}^n + \frac{n}{k_n^2} \sum_{0 \leq k < k_n - 1} \zeta_{i,j}^n \zeta_{i,k}^n.
\]
Recall that
\[
\zeta_{i,j}^n = \alpha_{i+j}^n + (H_{i+j}^n - H_i^n) \frac{m_j}{\sqrt{n}},
\]
with \( \alpha_i^n = |\Delta_i^n \log V| - m_1 n^{-1/2} H_i^n \).

ii) We have
\[
|\Delta_i^n \log V| = \left| \int_{(i-1)/n}^{i/n} H_s dB_s - \frac{1}{2} \int_{(i-1)/n}^{i/n} H_s^2 ds \right|
\]
\[
= \left| H_i^n \Delta_i^n B + \int_{(i-1)/n}^{i/n} (H_s - H_i^n) dB_s - \frac{1}{2} \int_{(i-1)/n}^{i/n} H_s^2 ds \right|
\]
\[
= H_i^n |\Delta_i^n B| + \text{sign}(\Delta_i^n B) \left[ \int_{(i-1)/n}^{i/n} (H_s - H_i^n) dB_s - \frac{1}{2} \int_{(i-1)/n}^{i/n} H_s^2 ds \right]
\]
\[
- 2(|H_i^n| |\Delta_i^n B| + \text{sign}(\Delta_i^n B) \int_{(i-1)/n}^{i/n} (H_s - H_i^n) dB_s - \frac{1}{2} \int_{(i-1)/n}^{i/n} H_s^2 ds) \times \mathbb{I}(H_i^n \Delta_i^n B + \int_{(i-1)/n}^{i/n} (H_s - H_i^n) dB_s - \frac{1}{2} \int_{(i-1)/n}^{i/n} H_s^2 ds) H_i^n \Delta_i^n B < 0) \right|
\]
and therefore, for \( j = 0, ..., k_n - 1 \),
\[
\mathbb{E} \left[ \alpha_{i+j}^n \mathbb{I}((i-1)/n) \right] \leq C \frac{1}{n^{\alpha+1/2}}.
\]
Let \( z_{i,j,n} = \mathbb{E}[\zeta_{i,j}^n | \mathcal{F}_{(i-1)/n}] \), we have, for \( j = 0, ..., k_n - 1 \),
\[
|z_{i,j,n}| \leq C \frac{j^\alpha}{n^{\alpha+1/2}}.
\]
Then we have
\[
\mathbb{E}\left[\frac{n}{k^2_n} \sum_{0 \leq k < j \leq k_n-1} \zeta_{i,j}^{n} \zeta_{i,k}^{n} | \mathcal{F}_{(i-1)/n}\right] = \frac{n}{k^2_n} \sum_{0 \leq k < j \leq k_n-1} z_{j+1,j-k_n,2_{i,j,n}}
\]
and we deduce that
\[
\mathbb{E}\left[\frac{n}{k^2_n} \sum_{0 \leq k < j \leq k_n-1} \zeta_{i,j}^{n} \zeta_{i,k}^{n} | \mathcal{F}_{(i-1)/n}\right] \leq \frac{n}{k^2_n} \sum_{0 \leq k < j \leq k_n-1} \frac{(k-j)^n}{n^{\alpha+1/2}} j^n
\]
and therefore
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \frac{1}{(H_i^n)^2} \mathbb{E}\left[\frac{n}{k^2_n} \sum_{0 \leq k < j \leq k_n-1} \zeta_{i,j}^{n} \zeta_{i,k}^{n} | \mathcal{F}_{(i-1)/n}\right] \leq Cn^{3/2} \frac{1}{n^{2\alpha+1}} k_n^{2n} = C \frac{1}{n^{2\alpha-1/2}} k_n^{2n} = C \left(\frac{k_n^{2\alpha/(2\alpha-1/2)}}{n}\right)^{2\alpha-1/2} \to 0.
\]
iii) It is moreover easy to prove that
\[
\mathbb{E}\left[\left(\zeta_{i,j}^{n}\right)^2 | \mathcal{F}_{(i-1)/n}\right] = \frac{1}{n} (H_i^n)^2 (1 - m_1^2) (1 + o(1)).
\]
iv) We therefore deduce that
\[
\mathbb{E}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \left[\frac{(\beta_i^n)^2}{(H_i^n)^2} - \frac{(1 - m_1^2)}{k_n}\right] | \mathcal{F}_{(i-1)/n}\right] \quad \text{ucp} \quad 0
\]
i.e. \( R_{t}^{n,2,1} \quad \text{ucp} \quad 0. \) The arguments are similar for \( R_{t}^{n,2,2}. \)

Step 3) Let us rewrite \( R_{t}^{n,3} \) as
\[
R_{t}^{n,3} = -\sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor} w_{i,n} \frac{\sigma_i^n}{H_i^n}
\]
where
\[
w_{i,n} = \frac{1}{k_n} \sum_{j=(i-\lfloor nt \rfloor)_+ - 1}^{(i-1) \wedge (k_n-1)} \gamma_{i-j}^n.
\]
In view of definition of \( w_{i,n}, \) we have, for each \( t, \) \( w_{i,(n,t),n} \to \gamma_t \) almost surely if \( |i(n,t)/n - t| \leq k_n/n. \) By using the same arguments as in the proof of Theorem 3.2 in Jacod and Rosenbaum (2013), it follows that
\[
R_{t}^{n} \xrightarrow{\mathcal{L}} R_{t},
\]
where \( R \) is a process defined on an extension \((\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t \geq 0}, \mathbb{P})\) of \((\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P}),\) which conditionally on \( \mathcal{F} \) is a centered continuous Gaussian process, with independent increments, and whose variance is given by \( \mathbb{V}[X_{3,t}|\mathcal{F}] = (1 - m_1^2) \int_0^t \gamma_2^2 ds. \) \( \Box \)
We now consider $\bar{I}_t^n - I_t$. We have

$$\sqrt{n}(\bar{I}_t^n - I_t) = \Delta \bar{I}_t^{n,1} + ... + \Delta \bar{I}_t^{n,5},$$

where

$$\Delta \bar{I}_t^{n,1} = \sqrt{n} \int_{\lfloor nt \rfloor - k_n + 1/n}^t \gamma_i ds,$$

$$\Delta \bar{I}_t^{n,2} = \sqrt{n} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \int_{(i-1)/n}^{i/n} (\gamma_i^n - \gamma_s) ds,$$

$$\Delta \bar{I}_t^{n,3} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \left( \frac{H_i^n}{H_i} - 1 \right) \left( \frac{\sqrt{n} \Delta_i^n \log Y}{m_1 H_i^n \gamma_i^n} - 1 \right),$$

$$\Delta \bar{I}_t^{n,4} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \left( \frac{\sqrt{n} \Delta_i^n \log Y}{m_1 H_i^n \gamma_i^n} - 1 \right),$$

$$\Delta \bar{I}_t^{n,5} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \left( \frac{H_i^n}{H_i} - 1 \right).$$

Step 1) Since $\gamma$ is a differentiable function, we easily derive that $\Delta \bar{I}_t^{n,j} \overset{ucp}{\Rightarrow} 0$ for $j = 1, 2$.

Step 2) By using the same arguments as for the proof of Proposition 4, we are able to prove that

$$\Delta \bar{I}_t^{n,4} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \gamma_i^n \left( \frac{\sqrt{n} \Delta_i^n \log Y}{m_1 H_i^n \gamma_i^n} - 1 \right) \overset{c- s}{\Rightarrow} \tilde{S}_t,$$

where $\tilde{S}_t$ is a process defined on an extension $(\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}})_{t \geq 0}, \mathbb{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$, which conditionally on $\mathcal{F}$ is a continuous Gaussian process, with independent increments, and whose mean and variance are given respectively by

$$\mathbb{E}[\tilde{S}_t|\mathcal{F}] = \frac{1}{2m_1} \sum_{j \geq 1} \sum_{k > j} \int_0^t \lambda(\varphi_{H_s}) \gamma_s^2 \delta_{\{l_i = j, k, Z_l > \gamma_i, k Z_l > k\}} d\int_0^s dL_z,$$

$$\mathbb{V}[\tilde{S}_t|\mathcal{F}] = \frac{1 - m_1^2}{m_1^2} \int_0^t \gamma_s^2 ds.$$

Step 3) Let

$$\kappa_i^n = \frac{1}{\sqrt{n}} \gamma_i^n \left( \frac{H_i^n}{H_i} - 1 \right) \left( \frac{\sqrt{n} \Delta_i^n \log Y}{m_1 H_i^n \gamma_i^n} - 1 \right).$$

We have by Propositions 8 and 2

$$\mathbb{E} \left[ \frac{\sqrt{n} \Delta_i^n \log Y}{m_1 H_i^n \gamma_i^n} - 1 \bigg| \mathcal{F}_{(i-1)/n} \right] = O_{i=1,..,n} \left( \frac{1}{\sqrt{n}} \right) \quad \text{and} \quad \mathbb{E} \left[ \frac{H_i^n}{H_i} - 1 \bigg| \mathcal{F}_{(i-1)/n} \right] = o_{i=1,..,n}(1).$$

Therefore, by independence between $Y$ and $V$, we deduce that

$$\sum_{i=1}^{\lfloor nt \rfloor - k_n + 1} \mathbb{E} \left[ \kappa_i^n | \mathcal{F}_{(i-1)/n} \right] \overset{ucp}{\Rightarrow} 0.$$
Moreover,
\[
\mathbb{E} \left[ \left( \frac{\sqrt{n} |\Delta_i^n \log Y|}{m_1 H_i^n \gamma_i^n} - 1 \right)^2 \right]_{\mathcal{F}_{(i-1)/n}} = O_{i=1,\ldots,n}(1) \quad \text{and} \quad \mathbb{E} \left[ \left( \frac{H_i^n}{H_i^n - 1} \right)^2 \right]_{\mathcal{F}_{(i-1)/n}} = o_{i=1,\ldots,n}(1),
\]
and then
\[
\sum_{i=1}^{\lfloor nt \rfloor - k_n \gamma_i^n} \mathbb{E} \left[ (\gamma_i^n)^2 \right]_{\mathcal{F}_{(i-1)/n}} \overset{ucp}{\longrightarrow} 0.
\]
It follows by Lemma 2.2.12 in Jacod and Protter that \( \text{DF}^n, \) \( \overset{ucp}{\longrightarrow} 0. \)

- Step 4) To conclude, it suffices to write
\[
\tilde{X}_5 = R + \tilde{S}
\]
and to note that \( R \) and \( \tilde{S} \) are independent conditionally on \( \mathcal{F} \) since \( Y \) and \( V \) are independent.

### 5.7 Proof of Proposition 7

Let us first recall that
\[
\log \xi_t = \gamma_t \log Z + \gamma_t \log V_t,
\]
where \( Z \) has a standard Pareto distribution and
\[
V_t = \exp \left\{ \int_0^t H_s dW_s - \frac{1}{2} \int_0^t H_s^2 ds \right\}, \quad t \in [0, 1].
\]
The increments of \( \log \xi \) are given by
\[
\Delta_i^n \log \xi = \log Z \Delta_i^n \gamma + \log V_{(i-1)/n} \Delta_i^n \gamma + \gamma_i/n \Delta_i^n \log V.
\]

We now study the asymptotic properties of \( B(1, \log \xi)^n_t \) when \( \gamma_0 \log Z > u_n \) where \( (u_n)_{n \geq 1} \) is a sequence that satisfies \( u_n/\sqrt{n} \to \infty \) as \( n \to \infty \). We have that, for \( a, u \in \mathbb{R} \),
\[
|a + u| = |a| + \text{sign}(a)u - 2(|a| + \text{sign}(a)u)\mathbb{I}_{\{a+u<0\}}.
\]
Let us choose
\[
a_i^n = \log Z \Delta_i^n \gamma
\]
\[
u_i^n = \log V_{(i-1)/n} \Delta_i^n \gamma + \gamma_i/n \Delta_i^n \log V.
\]
We then write
\[
|\Delta_i^n \log \xi| = |\log Z \Delta_i^n \gamma| + v_i^n + w_i^n + x_i^n
\]
where
\[
v_i^n = \text{sign}(\Delta_i^n \gamma) \log V_{(i-1)/n} \Delta_i^n \gamma
\]
\[
w_i^n = \text{sign}(\Delta_i^n \gamma) \gamma_i/n \Delta_i^n \log V
\]
\[
x_i^n = -2(|a_i^n| + \text{sign}(a_i^n)w_i^n)\mathbb{I}_{\{a_i^n+w_i^n<0\}}.
\]

Step 1) We begin by studying the asymptotic behaviors of \( \sum_{i=1}^{\lfloor nt \rfloor} v_i^n \), \( \sum_{i=1}^{\lfloor nt \rfloor} w_i^n \) and \( \sum_{i=1}^{\lfloor nt \rfloor} x_i^n \).
First, since $\gamma'$ is a differentiable function, it follows by the definition of the Riemann integral that

$$
\sum_{i=1}^{[nt]} u_i^n = \int_0^t |\gamma'| \log V_s ds + O \left( n^{-1} \right).
$$

Second, let

$$
w_i^n = w_{1,i}^n + w_{2,i}^n,
$$

where

$$
w_{1,i}^n = \text{sign}(\Delta_i^n \gamma) \gamma_i/n \int_{(i-1)/n}^{i/n} H_s dW_s \quad \text{and} \quad w_{2,i}^n = -\frac{1}{2} \text{sign}(\Delta_i^n \gamma) \gamma_i/n \int_{(i-1)/n}^{i/n} H_s^2 ds.
$$

We have

$$
\sum_{i=1}^{[nt]} w_{1,i}^n = \int_0^t \text{sign}(\gamma_i) H_s dW_s + O_{L^2} \left( n^{-1} \right)
$$

and

$$
\sum_{i=1}^{[nt]} w_{2,i}^n = -\frac{1}{2} \int_0^t \text{sign}(\gamma_i) H_s^2 ds + O \left( n^{-1} \right).
$$

Third, we have

$$
\mathbb{E} \left[ |x_i^n| \big| \mathcal{F}_{(i-1)/n} \right] \leq 4 \mathbb{E} \left[ u_i^n \mathbb{I}_{\{|u_i^n| > |a_i^n|\}} | \mathcal{F}_{(i-1)/n} \right]
$$

\leq 4 \log V_{(i-1)/n} ||\Delta_i^n \gamma| | \mathbb{P} \left( |u_i^n| > |a_i^n| | \mathcal{F}_{(i-1)/n} \right)

+ 4 \gamma_i/n \mathbb{E} \left[ |\Delta_i^n \log V| \mathbb{I}_{\{|u_i^n| > |a_i^n|\}} | \mathcal{F}_{(i-1)/n} \right].
$$

Now note that

$$
\{ |u_i^n| > |a_i^n| \} \subset \left\{ \log V_{(i-1)/n} ||\Delta_i^n \gamma| | + \gamma_i/n |\Delta_i^n \log V| > \log Z |\Delta_i^n \gamma| \right\}
$$

$$
= \left\{ |B_i^n| > n^{-1/2} \frac{n^{-1} |\Delta_i^n \gamma|}{\gamma_i/n} \left[ \gamma_0^{-1} u_n + \gamma_0^{-1} E_n - |\log V_{(i-1)/n}| \right] \right\}.
$$

where $B_i^n = n^{1/2} \Delta_i^n \log V$ and $E_n = \gamma_0 \log Z - u_n$. Moreover we have that

$$
B_i^n \overset{d}{=} \mathcal{N} \left( -\frac{1}{2} n^{1/2} \int_{(i-1)/n}^{i/n} H_s^2 ds, n \int_{(i-1)/n}^{i/n} H_s^2 ds \right)
$$

and because $E_n \geq 0$ given that $\gamma_0 \log Z > u_n$, we have

$$
\left\{ |B_i^n| > n^{-1/2} \frac{n^{-1} |\Delta_i^n \gamma|}{\gamma_i/n} \left[ \gamma_0^{-1} u_n + \gamma_0^{-1} E_n - |\log V_{(i-1)/n}| \right] \right\}
$$

$$
\subset \left\{ |B_i^n| > n^{-1/2} \frac{n^{-1} |\Delta_i^n \gamma|}{\gamma_i/n} \left[ \gamma_0^{-1} u_n - |\log V_{(i-1)/n}| \right] \right\}.
$$

It follows that

$$
\mathbb{P} \left( |u_i^n| > |a_i^n| | \mathcal{F}_{(i-1)/n} \right) \leq \mathbb{P} \left( |N| > \theta u_n / \sqrt{n} \right) \sim 2 \Phi \left( \theta u_n / \sqrt{n} \right)
$$

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where $N$ has a standard Gaussian distribution and
\[
\theta = \gamma_0^{-1} \left( \inf_{s \in [0,1]} \frac{|\gamma_s'|}{H_s \gamma_s} \right).
\]

In the same way, we derive that, for large $n$,
\[
\mathbb{E} \left[ \sqrt{n} |\Delta^n \log V| \mathbb{I}_{\{|u^n_t| > \sigma^n_t\}} |F_{(i-1)/n}| \right] \leq K \mathbb{E} \left[ |N|\mathbb{I}_{\{|N| > \theta u_n / \sqrt{n}\}} \right] \leq C \frac{u_n}{\sqrt{n}} \Phi \left( \theta \frac{u_n}{\sqrt{n}} \right).
\]

It follows that
\[
\sum_{i=1}^{[nt]} \mathbb{E} \left[ |x^n_i| |F_{(i-1)/n}| \right] \leq C \frac{u_n}{\sqrt{n}} \Phi \left( \theta \frac{u_n}{\sqrt{n}} \right).
\]

With the assumption that $n \Phi (w_n) \to 0$ and $x_n w_n^2 / n \to 0$, we derive that $\sqrt{n x_n w_n} \Phi (w_n) \to 0$ as $n \to \infty$, and we finally deduce that
\[
\sqrt{k_n} \sum_{i=1}^{[nt]} \mathbb{E} \left[ |x^n_i| |F_{(i-1)/n}| \right] \frac{u_c}{p} \overset{u.c.p}{\rightarrow} 0 \quad \text{and then} \quad \sqrt{k_n} \sum_{i=1}^{[nt]} |x^n_i| \frac{u_c}{p} \overset{u.c.p}{\rightarrow} 0,
\]

by Lemma 2.2.10 in Jacod and Protter (2011).

Step 2) Note that, for large $n$,
\[
\sum_{i=1}^{[nt]} |\Delta_i^n \gamma| = \log Z \sum_{i=1}^{[nt]} |\Delta_i^n \gamma| = \log Z \left( \int_0^t |\gamma'_s| \, ds + O \left( n^{-1} \right) \right).
\]

Step 3) It follows that
\[
\sum_{i=1}^{[nt]} |\Delta_i^n \log \xi| = \gamma_0 \log Z \left( \int_0^t |\gamma'_s| \, ds + O \left( n^{-1} \right) \right) + \int_0^t |\gamma'_s| \log V_s \, ds
+ \int_0^t \text{sign} (\gamma'_s) \gamma_s H_s \, dw_s - \frac{1}{2} \int_0^t \text{sign} (\gamma'_s) \gamma_s H^2_s \, ds + \sum_{i=1}^{[nt]} x^n_i + O_{L^2} \left( n^{-1} \right) + O \left( n^{-1} \right)
= u_n \gamma_0^{-1} \int_0^t |\gamma'_s| \, ds + \gamma_0^{-1} E_n \int_0^t |\gamma'_s| \, ds + X^c_t + \mu_t + \sum_{i=1}^{[nt]} x^n_i + O_{L^2} \left( n^{-1} \right) + O \left( u_n / n \right)
\]

where
\[
X^c_t = \int_0^t \left[ \text{sign} (\gamma'_s) \gamma_u + \int_u^t |\gamma'_s| \, ds \right] \, H_u \, dw_u
\]
\[
\mu_t = -\frac{1}{2} \int_0^t \text{sign} (\gamma'_s) \gamma_s H^2_s \, ds - \frac{1}{2} \int_0^t \left( \int_0^t |\gamma'_s| \, ds \right) \, H^2_s \, du
\]
because
\[ \int_t^1 |\gamma_s'| \log V_s ds = \int_t^1 |\gamma_s'| \left( \int_0^s H_u dW_u - \frac{1}{2} \int_0^s H_u^2 du \right) ds \]

\[ = \int_t^1 \left( \int_u^t |\gamma_s'| ds \right) H_u dW_u - \frac{1}{2} \int_t^1 \left( \int_u^t |\gamma_s'| ds \right) H_u^2 du. \]

Then
\[ \left( \frac{1}{u_n} \sum_{i=1}^{nt} |\Delta_i^n \log \xi_i| - \gamma_0^{-1} \int_0^t |\gamma_s'| ds \right) \]

\[ = \frac{1}{u_n} \left( \gamma_0^{-1} E_n \int_0^t |\gamma_s'| ds + X_t^n + \mu_t + O_{L^2} (n^{-1}) + \sum_{i=1}^{nt} x_i^n \right) + O (n^{-1}) \]

\[ = \frac{1}{u_n} \left( \gamma_0^{-1} E_n - 1 \right) \int_0^t |\gamma_s'| ds + X_t^n + \nu_t + O_{L^2} (n^{-1}) + \sum_{i=1}^{nt} x_i^n \right) + O (n^{-1}), \]

where
\[ \nu_t = \int_0^t |\gamma_u'| du - \frac{1}{2} \int_0^t \left[ \text{sign}(\gamma_u') \gamma_u + \int_u^t |\gamma_s'| ds \right] H_u^2 du. \]

Step 4) Let us now consider the \( m_n \) iid processes \( \xi_{1,\ldots,m_n} \). With our assumptions, \( k_n = \sum_{j=1}^{m_n} 1_{\{\log \xi_{j,n} > u_n\}} \) tends almost surely to infinity. We have (with clear notation)
\[ \left( \frac{1}{k_n} \sum_{j=1}^{k_n} \frac{1}{u_n} \sum_{i=1}^{nt} |\Delta_i^n \log \xi_j| - \gamma_0^{-1} \int_0^t |\gamma_s'| ds \right) \]

\[ = \frac{1}{u_n} \left( \gamma_0^{-1} E_{j,n} - 1 \right) \int_0^t |\gamma_s'| ds + \frac{1}{k_n} \sum_{j=1}^{k_n} X_{j,t}^n + \nu_t + O_{L^2} (n^{-1}) + \frac{1}{k_n} \sum_{j=1}^{k_n} \sum_{i=1}^{nt} x_{i,j,i}^n \right) + O (n^{-1}), \]

and therefore
\[ \sqrt{k_n u_n} \left( \frac{1}{k_n} \sum_{j=1}^{k_n} \frac{1}{u_n} \sum_{i=1}^{nt} |\Delta_i^n \log \xi_j| - \gamma_0^{-1} \int_0^t |\gamma_s'| ds - \frac{1}{u_n} \nu_t \right) \]

\[ = \sqrt{k_n} \left( \frac{1}{k_n} \sum_{j=1}^{k_n} \left( \gamma_0^{-1} E_{j,n} - 1 \right) \int_0^t |\gamma_s'| ds + \frac{1}{k_n} \sum_{j=1}^{k_n} X_{j,t}^n + O_{L^2} (n^{-1}) + \frac{1}{k_n} \sum_{j=1}^{k_n} \sum_{i=1}^{nt} x_{i,j,i}^n + O (u_n n^{-1}) \right). \]

Note that \( E_{j,n} \gamma_0 \log Z > u_n \) weakly converges to an Exponential random variable with parameter \( \gamma_0^{-1} \).

We finally deduce that
\[ \sqrt{k_n u_n} \left( \frac{1}{k_n} \sum_{j=1}^{k_n} \frac{1}{u_n} \sum_{i=1}^{nt} |\Delta_i^n \log \xi_j| - \gamma_0^{-1} \int_0^t |\gamma_s'| ds - \frac{1}{u_n} \nu_t \right) \]

\[ \Rightarrow \left( \int_0^t |\gamma_s'| ds \right) X_0 + X_t^n, \]

where \( X_t^n \) is a sample-continuous Gaussian process with variance \( \int_0^t [\text{sign}(\gamma_u') \gamma_u + \int_0^t |\gamma_s'| ds]^2 H_u^2 du, \]

\( X_0 \) has a standard Gaussian distribution, and \( X_0 \) and \( (X_t^n) \) are independent.