A model of quantum-like decision-making with applications to psychology and cognitive science

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Abstract

We consider the following model of decision-making by cognitive systems. We present an algorithm – quantum-like representation algorithm (QLRA) – which provides a possibility to represent probabilistic data of any origin by complex probability amplitudes. Our conjecture is that cognitive systems developed the ability to use QLRA. They operate with complex probability amplitudes, mental wave functions. Since the mathematical formalism of QM describes as well (under some generalization) processing of such quantum-like (QL) mental states, the conventional quantum decision-making scheme can be used by the brain. We consider a modification of this scheme to describe decision-making in the presence of two “incompatible” mental variables. Such a QL decision-making can be used in situations like Prisoners Dilemma (PD) as well as others corresponding to so called disjunction effect in psychology and cognitive science.

1 Introduction

Recently a new wave of interest to applications of the mathematical formalism of QM (especially its QI part) was generated via interactions of the quantum community with various research groups working in artificial intelligence [1], cognitive science and psychology [2]–[12], finances [13]–[21] and economy [22], [23] (cf. with [24]–[37]). In particular, an important project was started in [6]–[8], namely, creation of quantum-like (QL) models for decision-making by cognitive systems, see also [11], [12]. Since QL-modelling of cognition has always been one of my favorable domains of research [2]–[5], I was happy to contribute to this project on decision-making by QL cognitive systems, see [18]. In this paper I shall combine the QL cognitive model [18] with Bayesian statistical inference in the general framework of quantum decision-making, cf. e. g. , see e.g. [39].
(and references in these works). So, we shall proceed in the same direction as Busemeyer [6–8], La Mura [9, 10] and Franco [11, 12].

We consider the following model of decision-making by cognitive systems. We present an algorithm – quantum-like representation algorithm (QLRA) – which provides a possibility to represent probabilistic data of any origin by complex probability amplitudes. Our conjecture is that cognitive systems developed the ability to use QLRA. Thus they operate with complex probability amplitudes, mental wave functions. Since the mathematical formalism of QM describes as well (under some generalization, see appendix – section 9) processing of such QL mental states, the conventional quantum decision-making scheme can be used by the brain. We consider a modification of this scheme to describe decision-making in the presence of two “incompatible” mental variables. Such a QL decision-making can be used in situations like Prisoner’s Dilemma (PD), see appendix (section 9), as well as others corresponding to so called disjunction effect in psychology and cognitive science, see e.g. [45–50].

We start this paper with a short recollection of the QL representation of contexts which is based on QLRA, see [51], [52] for detailed presentation.

2 Contexts, observables, QL-representation

2.1 Växjö contextual model

Classical as well as quantum probabilistic models can be obtained as particular cases of our general contextual model, the Växjö model, see [51].

A physical, biological, social, mental, genetic, economic, or financial context $C$ is a complex of corresponding conditions. Contexts are fundamental elements of any contextual statistical model $M$.

Thus construction of any probabilistic model $M$ should be started with fixing the collection of contexts of this model. Denote the collection of contexts by the symbol $C$ (so the family of contexts $C$ is determined by the model $M$ under consideration). Another fundamental element of any contextual statistical model $M$ is a set of observables $O$: each observable $a \in O$ can be measured under each complex of conditions $C \in C$. For an observable $a \in O$, we denote the set of its possible values (“spectrum”) by the symbol $X_a$. We do not assume that all these observables can be measured simultaneously. To simplify considerations, we shall consider only discrete observables and, moreover, all concrete investigations will be performed for dichotomous observables.

Axiom 1: For any observable $a \in O$ and its value $y \in X_a$, there is defined a
context, say \( C_y \), corresponding to the \( y \)-selection\(^2\): if we perform a measurement of the observable \( a \) under the complex of physical conditions \( C_y \), then we obtain the value \( a = y \) with probability 1. We assume that the set of contexts \( C \) contains \( C_y \)-selection contexts for all observables \( a \in O \) and \( y \in X_a \).

**Axiom 2:** Contextual (conditional) probabilities \( p^a_y(y) \equiv P(a = y|C) \) are defined for any context \( C \in C \) and any observable \( a \in O \).

Thus, for any context \( C \in C \) and any observable \( a \in O \), there is defined the probability to observe the fixed value \( a = y \) under the complex of conditions \( C \). Especially important role will be played by the “transition probabilities”: \( p^b|a(x|y) \equiv P(b = x|C_y) \), \( a, b \in O \), \( y \in X_a \), \( x \in X_b \), where \( C_y \) is the \([ a = y] \)-selection context. By axiom 2, for any context \( C \in C \), the set of probabilities: \( \{ P(a = y|C) : a \in O \} \) is well defined. We complete this probabilistic data for the context \( C \) by transition probabilities. The corresponding collection of data \( D(O,C) \) consists of contextual probabilities: \( P(a = y|C), P(b = x|C), P(b = x|C_y), P(a = y|C_x), \ldots \), where \( a, b, \ldots \in O \). Finally, we denote the family of probabilistic data \( D(O,C) \) for all contexts \( C \in C \) by the symbol \( D(O,C) \).

**Definition 1.** (Växjö Model) An observational contextual statistical model of reality is a triple \( M = (C,O,D(O,C)) \), where \( C \) is a set of contexts and \( O \) is a set of observables which satisfy to axioms 1, 2, and \( D(O,C) \) is probabilistic data about contexts \( C \) obtained with the aid of observables belonging \( O \).

We call observables belonging the set \( O \equiv O(M) \) reference of observables. Inside of a model \( M \) observables belonging to the set \( O \) give the only possible references about a context \( C \in C \).

**Definition 2.** Let \( a, b \in O \). The observable \( a \) is said to be supplementary\(^3\) to the observable \( b \) if \( p^b(a|x|y) \neq 0 \), for all \( x \in X_b, y \in X_a \).

### 2.2 Law of total probability and its violations

We recall this law in the simplest case of dichotomous random variables, \( a = y_1, y_2 \) and \( b = x_1, x_2 \), see e.g. [53]:

\[
P(b = x) = P(a = y_1)P(b = x|a = y_1) + P(a = y_2)P(b = x|a = y_2)
\]

(1)

Thus the probability \( P(b = x) \) can be reconstructed on the basis of conditional probabilities \( P(b = x|a = y) \) and known or a priori chosen probabilities \( P(a = y) \). This formula plays the fundamental role in modern science. Its consequences are strongly incorporated in modern scientific reasoning. In [3–51] it was pointed out that the quantum formalism induces a modification of

\(^2\)See appendix – section 11 – for discussion of selection contexts and contextual forms of the von Neumann projection postulate.

\(^3\)It might be better to call such observables complementary, but Bohr’s complementarity was rigidly coupled with mutual exclusivity. Our supplementarity may be considered as a version of complementarity, but without mutual exclusivity, see [51].

\(^4\) “The prior probability to obtain the result e.g. \( b = x_1 \) is equal to the prior expected value of the posterior probability of \( b = x_1 \) under conditions \( a = y_1, y_2 \).”
this formula. An additional term appears in the right hand side of (1), so called interference term.

\[
P(b = x) = P(a = y_1)P(b = x|a = y_1) + P(a = y_2)P(b = x|a = y_2) + 2 \cos \theta \sqrt{P(a = y_1)P(b = x|a = y_1)P(a = y_2)P(b = x|a = y_2)}.
\] (2)

The main mathematical consequence of [3]–[5] is that any violation of the formula of total probability (which need not be coupled to quantum physics) induces its interference generalization. However, not any violation induces the ordinary cos-interference. For some contexts violation of (1) induces so called hyperbolic interference. But we shall not consider this type of interference in the present paper.

3 Quantum-like representation algorithm – QLRA

We consider two dichotomous supplementary reference observables \(a\) and \(b\). In [5] we derived the following formula for interference of contextual probabilities for the general Växjö Model:

\[
p^b_C(x) = \sum_y p^a_C(y)p^{b|a}(x|y) + 2\lambda_x \sqrt{\prod_y p^a_C(y)p^{b|a}(x|y)},
\] (3)

where the coefficient of supplementarity (interference):

\[
\lambda_x = \frac{p^b_C(x) - \sum_y p^a_C(y)p^{b|a}(x|y)}{2 \sqrt{\prod_y p^a_C(y)p^{b|a}(x|y)}}.
\] (4)

Contexts such that the interference coefficients \(\lambda_x, x \in X_b\), are bounded by one are called trigonometric, because in this case we have the conventional formula of trigonometric interference:

\[
p^b_C(x) = \sum_y p^a_C(y)p^{b|a}(x|y) + 2 \cos \theta_x \sqrt{\prod_y p^a_C(y)p^{b|a}(x|y)},
\] (5)

where \(\lambda_x = \cos \theta_x\). Parameters \(\theta_x\) are said to be \(b|a\)-phases with respect to the context \(C\). We defined these phases purely on the basis of probabilities. We have not started with any linear space; in contrast we shall define geometry from probability.

We denote the collection of all trigonometric contexts by the symbol \(C^{tr}\). By using the elementary formula: \(D = A + B + 2\sqrt{AB} \cos \theta = |\sqrt{A} + e^{i\theta} \sqrt{B}|^2\), for real numbers \(A, B > 0, \theta \in [0, 2\pi]\), we can represent the probability \(p^b_C(x)\) as the square of the complex amplitude (Born’s rule):

\[
p^b_C(x) = |\psi_C(x)|^2.
\] (6)
Here

\[ \psi(x) \equiv \psi_C(x) = \sqrt{p^b_C(y_1)}p^{b|a}(x|y_1) + e^{i\theta_x} \sqrt{p^b_C(y_2)}p^{b|a}(x|y_2), \ x \in X_b. \] (7)

The formula (7) gives the QL representation algorithm – QLRA. For any trigonometric context \( C \) by starting with the probabilistic data \( p^b_C(x), p^b_C(y), p^{b|a}(x|y) \) – QLRA produces the complex amplitude \( \psi_C \). This algorithm can be used in any domain of science to create the QL-representation of probabilistic data (for a special class of contexts).

We denote the space of functions: \( \psi : X_b \rightarrow \mathbb{C} \) by the symbol \( \Phi = \Phi(X_b, \mathbb{C}) \). Since \( X = \{x_1, x_2\} \), the \( \Phi \) is the two dimensional complex linear space. By using QLRA we construct the map \( J^{b|a} : \mathcal{C}^{\text{tr}} \rightarrow \Phi(X, \mathbb{C}) \) which maps contexts (complexes of, e.g., physical conditions) into complex amplitudes. The representation (6) of probability is nothing other than the famous Born rule. The complex amplitude \( \psi_C(x) \) can be called a wave function of the complex of physical conditions (context) \( C \) or a (pure) state. We set \( e_x^b (\cdot) = \delta(x - \cdot) \) – Dirac delta-functions concentrated in points \( x = x_1, x_2 \). The Born’s rule for complex amplitudes (6) can be rewritten in the following form: \( p^b_C(x) = |\langle \psi_C, e_x^b \rangle|^2 \), where the scalar product in the space \( \Phi(X_b, \mathbb{C}) \) is defined by the standard formula: \( \langle \phi, \psi \rangle = \sum_{x \in X_b} \phi(x) \psi(x) \). The system of functions \( \{e_x^b\}_{x \in X_b} \) is an orthonormal basis in the Hilbert space \( H = (\Phi, \langle \cdot, \cdot \rangle) \).

Let \( X_b \subset \mathbb{R} \). By using the Hilbert space representation of the Born’s rule we obtain the Hilbert space representation of the expectation of the observable \( b: E(b|C) = \sum_{x \in X_b} |\psi_C(x)|^2 = \sum_{x \in X_b} x \langle \psi_C, e_x^b \rangle \langle e_x^b, \psi_C \rangle = \langle \hat{b}\psi_C, \psi_C \rangle \), where the (self-adjoint) operator \( \hat{b} : H \rightarrow H \) is determined by its eigenvectors: \( \hat{b}e_x^b = xe_x^b, x \in X_b \). This is the multiplication operator in the space of complex functions \( \Phi(X_b, \mathbb{C}) : \hat{b}\psi(x) = x\psi(x) \). It is natural to represent the \( b \)-observable (in the Hilbert space model) by the operator \( \hat{b} \).

We would like to have Born’s rule not only for the \( b \)-variable, but also for the \( a \)-variable: \( p^a_C(y) = |\langle \psi, e_y^a \rangle|^2, y \in X_a \).

How can we define the basis \( \{e_y^a\} \) corresponding to the \( a \)-observable? Such a basis can be found starting with interference of probabilities. We set \( u_j^a = \sqrt{p^a_C(y_j)}p^{b|a}(x_j|y_i), u_{ij} = \sqrt{p^{b|a}(x_j|y_i)}, \theta = \theta_C(x_j) \). We have:

\[ \psi = u_{11}^ae_{y_1} + u_{12}^ae_{y_2}, \] (8)

where

\[ e_{y_1}^a = (u_{11}, \ u_{12}), \ e_{y_2}^a = (e^{i\theta_1}u_{21}, \ e^{i\theta_2}u_{22}) \] (9)

Suppose now that the matrix of transition probabilities \( P^{b|a} \) is doubly stochastic.\(^3\) Under this condition the system \( \{e_y^a\} \) is an orthonormal basis iff the probabilistic phases satisfy the constraint: \( \theta_2 - \theta_1 = \pi \mod 2\pi \). In this case the \( a \)-observable is also represented by a self-adjoint operator \( \hat{a} \) which is diagonal with eigenvalues \( y_1, y_2 \) in the basis \( \{e_y^a\} \). The conditional average of

\(^3\)It is a square matrix of nonnegative real numbers, each of whose rows and columns sums to 1. Thus, a doubly stochastic matrix is both left stochastic and right stochastic.
the observable $a$ coincides with the quantum Hilbert space average: $E(a|C) = \sum_{y \in X_a} y p_C^a(y) = \langle \hat{a} \psi_C, \psi_C \rangle$.

In the general case (when $P^{b|a}$ need not be doubly stochastic) the $a$-observable is represented as a generalized quantum observable (non self-adjoint operator), see appendix (section 10). We remark that statistical data obtained in cognitive psychology in experimental tests of disjunction effect produce non doubly stochastic matrices of transition probabilities [49, 50].

4 QL Decision-making scheme

As we have seen, if for some context $C$, probability distributions for supplementary observables $a$ and $b$ are known, then the complex probability amplitude $\psi_C$ representing $C$ can be reconstructed by using QLRA. This was the problem of representation of probabilistic data by complex probability amplitude, see section 3. My conjecture is that the brain developed the ability for such a QL representation of probabilistic data, see [38] for details. In such a QL-model the brain uses complex probability amplitudes for decision-making.

We consider the following situation. A (mental) context $C$ is given. The brain must take decision about the $b$-attribute, given by e.g. $b = x_1, x_2, \ldots$ so to choose between $b = x_1$ and $b = x_2$. The crucial point is that it is assumed that another attribute, say $a(=y_1, y_2)$, which is supplementary to $b$, is involved in the process of decision-making. Since variables $a$ and $b$ are supplementary (under the context $C$), interference angles $\theta = (\theta_{x_1}, \theta_{x_2})$ should be considered, see (7).

In the PD, see appendix (section 10), this $a$-attribute is related to actions of another prisoner. In the gambling experiment it is simply the (classical) random generator producing wins and losses. The latter example shows that “quantumness” (qualitatively encoded by the interference angles) is not a feature of $a$ (in fact neither of $b$), but it appears via interrelation of $a$, $b$ and the context $C$. Our scheme of QL decision-making is based on the assumptions that there are given (created by the brain of the basis of previous experience):

a) transition probabilities $p^{b|a}(x|y)$;

b) the probability distribution of the $a$ : $p_C^a(y)$;

c) the probability distribution of the phase angles $\theta = (\theta_{x_1}, \theta_{x_2}) : p_C(\theta)$.

Thus all these distributions are given a priori. One should not always identify prior probabilities with “subjective probabilities.” The previous frequency experience plays an important role in determination of these probability distributions, cf. [54].

The brain uses the formula of total probability with the interference term to find the $b$-probabilities. Under the assumption that the interference angle is $\theta_x$, it produces the probabilities

$$p_C^b(x|\theta) = \sum_y p_C^a(y) p^{b|a}(x|y) + 2 \cos \theta_x \sqrt{p_C^a(y_1) p^{b|a}(x|y_1) p_C^a(y_2) p^{b|a}(x|y_2)}.$$

The crucial point of the decision-making scheme is their interpretation:
For each $x$, $p_C^b(x|\theta)$ is the probability that under the condition that the $b|a$-interference angle is $\theta_x$ (for the context $C$) the decision $b = x$ is “right”, i.e., it would produce some form of reward.

By the (classical) Bayes’ formula the brain finds the joint probability distribution:

$$p_C(x, \theta) = p_C(\theta) \left( \sum_y p_C^a(y) p^{b|a}(x|y) + 2 \cos \theta_x \sqrt{p_C^a(y_1) p^{b|a}(x|y_1) p_C^a(y_2) p^{b|a}(x|y_2)} \right)$$

and finally the total $b$-probabilities

$$\bar{p}_C^b(x) = \int d\theta \, p_C(\theta) \, p_C^b(x|\theta).$$

As the extension of the interpretation of conditional probabilities, the probability $\bar{p}_C^b(x)$ is considered as the probability that the decision $b = x$ is right.

In the present decision-making scheme the brain makes the $b = x_1$-decision if $\bar{p}_C^b(x_1)$ is larger than $\bar{p}_C^b(x_2)$ an vice versa, cf. [40], p. 54. The qualitative meaning of “larger” is determined depending on the cognitive system and may be the context $C$.

We should also mention another QL decision-making scheme. Comparing of the probabilities $\bar{p}_C^b(x_1)$ and $\bar{p}_C^b(x_2)$ is an additional act of mental processing. It needs special neuronal and time recourses. The processing might be especially complicated when these probabilities do not differ essentially. In such a situation a QL cognitive system might choose the regime of “automatic probabilistic decision-making”, namely, by just using a (classical) random generator producing decisions $x_1$ and $x_2$ with the probabilities $\bar{p}_C^b(x_1)$ and $\bar{p}_C^b(x_2)$.

**Remark 1.** (Comparing with classical probability) We remark that a cognitive system $\tau_{CL}$ which uses the classical probabilistic processing of information can apply the conventional formula of total probability (11) to predict the $b$-probabilities on the basis transition probabilities $p^{b|a}(x|y)$ and $a$-probabilities $p_C^a(y)$. Thus one can consider the proposed QL-scheme as simply introduction of an additional – interference – parameter $\theta$ and modification of the formula of total probability. The main source of such a modification of the conventional statistical considerations is the impossibility to combine the context $C$ with the selection contexts $C_y$, and hence to get the probabilities $P(b = x|CC_y)$, cf. with the resolution of “Simpson’s paradox” in [57]. As we have seen, a QL cognitive system $\tau_{QL}$ cannot proceed in the same way. The formula of total probability with the interference term contains not only the transition probabilities and the $a$-probabilities, but also phases and the latter are unknown. Thus even by choosing e.g. prior probabilities $p_C^a(y)$ (under the condition that the transition probabilities were obtained via the frequency experience), the $\tau_{QL}$ could not predict $b$-probabilities.
By using QLRA the cognitive system $\tau_{QL}$ can construct for each $\theta = (\theta_x, \theta_y)$ the complex probability amplitude $\psi_{C,\theta}(x)$. Then the $b$-probabilities can be represented by using the Born’s rule:

$$\bar{p}_C^b(x) = \int d\theta \ p_C(\theta) |\psi_{C,\theta}(x)|^2. \quad (12)$$

5 Bayesian updating of state distribution

Thus by our model the brain of $\tau_{QL}$ proceeds by using the mixture of classical and quantum of probabilities. The whole Bayesian scheme is purely classical, “quantumness” appears in (12) only via Born’s rule.

However, as always, there arises the problem of the choice of prior probability distributions. Since the transition probabilities and the $a$-probabilities are present even in the classical Bayesian framework, only the phase distribution $p_C(\theta)$ makes a new (QL) contribution. A QL cognitive system $\tau_{QL}$ should learn itself to choose $p_C(\theta)$ on the basis of the previous experience of the $b/a$ decision-making under the context $C$. Such a learning can be performed via the (conventional) Bayesian updating procedure.

By combining Bayes’ and Born’s formulas, we get:

$$p_C(\theta|x) = \frac{p_C(x,\theta)}{\bar{p}_C^b(x)} = \frac{p_C(\theta)|\psi_{C,\theta}(x)|^2}{\int d\theta p_C(\theta)|\psi_{C,\theta}(x)|^2}. \quad (13)$$

By following the Bayesian scheme $\tau_{QL}$ would like to maximize the probability $p_C(\theta|x)$, i.e., to construct a map $\mathbf{m} : X_b \to \Theta, \mathbf{m}(x) = \theta_{\max}(x)$, see [56]. Since the denominator in (13) does not depend on $\theta$, this problem is reduced to maximization of the joint probability density $p_C(x,\theta)$.

Suppose now that under the context $C$ the $\tau_{QL}$ made the decision $b = x$ and this decision was successful (so the $\tau_{QL}$ got some form of reward). Then the $\tau_{QL}$ would update the distribution $p_C(\theta)$ by maximizing $p_C(x,\theta)$. To simplify considerations and to extract the main QL factor, we assume that the transition probabilities as well as the $a$-probabilities are fixed. So, optimization is considered only with respect to the interference angles $\theta$.

In the case of the doubly stochastic matrix of transition probabilities $\theta_x = \theta_y + \pi$ and hence we can consider the one dimensional phase parameter $\theta$.

Example 1. (Discrete distribution of phases) Some context $C$ is chosen. Suppose that the transition probabilities as well as the $a$-probabilities are equal to 1/2. Here the formula of total probability with the interference term gives:

$$p_C(x_1|\theta) = \cos^2 \theta/2; \ p_C(x_2,\theta) = \sin^2 \theta/2.$$
the simplest nontrivial case of the parametric set consisting of two points, e.g. \( \Theta = \{ \theta_1 = \pi/2, \theta_2 = \pi \} \). So, this cognitive system reduced (on the basis of some information) phases under the context \( C \) to two possible angles. Hence, \( \tilde{p}_C(x_1) = \frac{1}{2}(\cos^2 \pi/4 + \cos^2 \pi/2) = \frac{1}{4}, \tilde{p}_C(x_2) = \frac{1}{2}(\sin^2 \pi/4 + \sin^2 \pi/2) = \frac{3}{4} \). Thus under the assumption that all phases in \( \Theta \) are equally possible, this cognitive system \( \tau_{QL} \) gets that \( \tilde{p}_C(x_2) \) is essentially larger than \( \tilde{p}_C(x_1) \). Hence, \( \tau_{QL} \) makes the decision \( b = x_2 \). If the result of this decision was positive (i.e. some form of reward was obtained), \( \tau_{QL} \) would like to update the state distribution. Since \( p_C(x_2, \pi/2) = \frac{1}{4} \) and \( p_C(x_2, \pi/2) = \frac{1}{2} \), the cognitive system will put (in future decision-making) more weight to \( \theta_2 = \pi \), e.g. the updated distribution could be \( p_C(\pi/2) = \frac{1}{3}, p_C(\pi) = \frac{2}{3} \).

**Example 2.** (Continuous distribution of phases) Suppose that all transition probabilities are equal. Let us consider the uniform distribution of phases on \( \Theta = [0, 2\pi) \) : \( dp_C(\theta) = \frac{1}{2\pi}d\theta \). Here \( p_C(x_1, \theta) = \frac{1}{2\pi} \cos^2 \theta/2; p_C(x_2, \theta) = \frac{1}{2\pi} \sin^2 \theta/2 \). Hence, \( \tilde{p}_C(x_1) = \tilde{p}_C(x_2) = 1/2 \). Thus the definite decision could not be done.

**Example 3.** Suppose that all transition probabilities are equal. Let us consider the uniform distribution of phases on \( \Theta = [0, \pi/2) \) : \( dp_C(\theta) = \frac{2}{\pi}d\theta \). Here \( p_C(x_1, \theta) = \frac{2}{\pi} \cos^2 \theta/2; p_C(x_2, \theta) = \frac{2}{\pi} \sin^2 \theta/2 \). Hence, \( \tilde{p}_C(x_1) = \frac{1}{2} + \frac{1}{2}, \tilde{p}_C(x_2) = \frac{1}{2} - \frac{1}{2} \). Thus the \( b = x_1 \) decision is preferred. For this decision the maximum is approached for \( \theta = 0 \). Therefore this cognitive system would update \( p_C(\theta) \) by concentrating it at the point \( \theta = 0 \).

### 6 Mixed state representation

We remark that the former Bayesian considerations can be mathematically represented by using mixed quantum states. Let us consider the density matrix:

\[
\rho_C \equiv \int_\Theta d\theta \; p(\theta) \; \rho_{C,\theta}.
\]

\[
\rho_{C,\theta} \equiv \psi_{C,\theta} \otimes \psi_{C,\theta}
\]

We obtain the representation:

\[
\tilde{p}_C(x) = \text{Tr} \; \rho_C \; \pi_x^b,
\]

where \( \pi_x^b \) is the orthogonal projector corresponding to the eigenvalue \( b = x \). Thus quantity

\[
\frac{\tilde{p}_C(x_1)}{\tilde{p}_C(x_2)} = \frac{\text{Tr} \; \rho_C \; \pi_{x_1}^b}{\text{Tr} \; \rho_C \; \pi_{x_2}^b}
\]

is used in the QL decision-making.
7 Comparing with standard quantum decision-making theory

In this section we would like to compare our approach with standard quantum decision-making theory, see e.g. [39], [40]-[43], [44] (and references in these works):

a). Interpretation. The crucial difference is that our formalism is not about really quantum physical systems, but about QL systems. Thus we need not quantum sources of randomness, e.g. electrons or photons, to perform our QL decision making. Moreover, the essence of QL behavior is not consideration of a special class of systems, but of a special class of contexts or to be more precise: interrelation between contexts and observables.

b). Scheme of the decision making. We consider a specific scheme (motivated by PD, see appendix (section 9)) involving two supplementary (“incompatible”) observables \( a \) and \( b \). Moreover, in general one of them, namely, \( a \) is a generalized quantum observable, see section 10.

c). Mathematics. We consider a specific parametrization of a prior quantum state, namely, by the interference angle \( \theta \).

d). Application. We apply our model to modelling of brain’s functioning as a macroscopic QL system or to be more precise: a macroscopic system performing specific interconnections between contexts and observables (inducing nontrivial interference).

8 Bayes risk

As usual in quantum decision-making, we consider Bayes risk corresponding to the deviation function \( W_\theta(x) \), see [40], p. 46:

\[
\mathcal{R}^b_C \equiv \int_\Theta dp(\theta) \sum_x W_\theta(x) p^b_C(x|\theta) = \int_\Theta dp(\theta) \sum_x W_\theta(x) |\psi_{C,\theta}(x)|^2 = (16)
\]

\[
\int_\Theta dp(\theta) \sum_x W_\theta(x) \text{ Tr} \rho_{C,\theta} \pi_x^b.
\]

Typically in quantum decision theory the problem of finding of Bayes decision rule is considered, e.g. [40], p. 46–50. However, we are not interested in this problem, since the decision-making operator \( \hat{b} \) is considered as given.

In our model the brain is interested to minimize Bayes risk for the fixed observable \( b \) via variation of the prior distribution of interference phases.

We come back to Example 1. Now we do not fix the distribution of phases on \( \Theta = \{\theta_1 = \pi/2, \theta_2 = \pi\} \). Here \( p = p(\theta_1) \) and \( 1 - p = p(\theta_2) \) are parameters of the model. Suppose that the deviation function \( W_{\theta_i}(x_i) = \delta_{ij} \). Thus Bayes risk is \( \mathcal{R}^b_C = p \psi^b_C(x_1|\theta_1) + (1 - p) \psi^b_C(x_2|\theta_2) = p \cos^2 \theta_1/2 + (1 - p) \sin^2 \theta_2/2 = \)

\[
\int_\Theta dp(\theta) \sum_x W_\theta(x) \text{ Tr} \rho_{C,\theta} \pi_x^b.
\]\n
\[^6\text{Of course, it could also be modified in the process of brain’s functioning, but we do not consider this problem in the present paper.}\]
\[ p/2 + (1 - p) = 1 - p/2. \] Thus Bayes risk is minimal for \( p = 1 \). Hence, the brain would modify the prior (mixed) mental state into the (pure) mental state \( \psi_{C,\pi/2} \).

9 Prisoner’s Dilemma

In game theory, PD is a type of non-zero-sum game in which two players can cooperate with or defect (i.e. betray) the other player. In this game, as in all game theory, the only concern of each individual player (prisoner) is maximizing his/her own payoff, without any concern for the other player’s payoff. In the classic form of this game, cooperating is strictly dominated by defecting, so that the only possible equilibrium for the game is for all players to defect. In simpler terms, no matter what the other player does, one player will always gain a greater payoff by playing defect. Since in any situation playing defect is more beneficial than cooperating, all rational players will play defect.

The classical PD is as follows: Two suspects, A and B, are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal: if one testifies for the prosecution against the other and the other remains silent, the betrayer goes free and the silent accomplice receives the full 10-year sentence. If both stay silent, both prisoners are sentenced to only six months in jail for a minor charge. If each betrays the other, each receives a two-year sentence. Each prisoner must make the choice of whether to betray the other or to remain silent. However, neither prisoner knows for sure what choice the other prisoner will make. So this dilemma poses the question: How should the prisoners act? The dilemma arises when one assumes that both prisoners only care about minimizing their own jail terms. Each prisoner has two options: to cooperate with his accomplice and stay quiet, or to defect from their implied pact and betray his accomplice in return for a lighter sentence. The outcome of each choice depends on the choice of the accomplice, but each prisoner must choose without knowing what his accomplice has chosen to do. In deciding what to do in strategic situations, it is normally important to predict what others will do. This is not the case here. If you knew the other prisoner would stay silent, your best move is to betray as you then walk free instead of receiving the minor sentence. If you knew the other prisoner would betray, your best move is still to betray, as you receive a lesser sentence than by silence. Betraying is a dominant strategy. The other prisoner reasons similarly, and therefore also chooses to betray. Yet by both defecting they get a lower payoff than they would get by staying silent. So rational, self-interested play results in each prisoner being worse off than if they had stayed silent, see e.g. wikipedia – “Prisoner’s dilemma.” The following mental contexts are involved in PD:

Context \( C \) representing the situation such that a player has no idea about planned action of another player. Context \( C^a_+ \) representing the situation such that the B-player supposes that A will cooperate and context \( C^a_- \) – A will compete. We can also consider similar contexts \( C^b_+ \). We define dichotomous
observables \( a \) and \( b \) corresponding to actions of players \( A \) and \( B \): \( a = + \) if \( A \) chooses to cooperate and \( a = - \) if \( A \) chooses to compete, \( b \) is defined in the same way.

A priori the law of total probability might be violated for PD, since the \( B \)-player is not able to combine contexts. If those contexts were represented by subsets of a so called space of “elementary events” as it is done in classical probability theory (based on Kolmogorov (1933) measure-theoretic axiomatics), the \( B \)-player would be able to consider the conjunction of the contexts \( C \) and e.g. \( C_a^+ \) and to operate in the context \( C \land C_a^+ \) (which would be represented by the set \( C \cap C_a^+ \)). But the very situation of PD is such that one could not expect that contexts \( C \) and \( C_a^+ \) might be peacefully combined. If the \( B \)-player obtains information about the planned action of the \( A \)-player (or even if he just decides that \( A \) will play in the definite way, e.g. the context \( C_a^+ \) will be realized), then the context \( C \) is simply destroyed. It could not be combined with \( C_a^+ \).

We can introduce the following contextual probabilities:

\[ p_b^{\pm}(\pm) \equiv P(b = \pm|C) \quad \text{-- probabilities for actions of } B \text{ under the complex of mental conditions } C. \]

\[ p_b^{\pm}|a^{\pm}(\pm, +) \equiv P(b = \pm|C_a^+) \quad \text{-- probabilities for actions of } B \text{ under the complex of mental conditions } C_a^+. \]

\[ p_b^{\pm}|a^{\pm}(\pm, -) \equiv P(b = \pm|C_a^-) \quad \text{-- probabilities for actions of } B \text{ under the complex of mental conditions } C_a^-., \]

\[ p_a^{\pm}(\pm) \equiv P(a = \pm|C) \quad \text{-- prior probabilities which } B \text{ assigns for actions of } A \text{ under the complex of mental conditions } C. \]

10 Appendix: Generalization of the QM formalism

Let us consider a finite dimensional Hilbert space \( H \). Let \( \mathcal{E} = \{e_j\}_{j=1}^n \) be an orthonormal basis:

\[ \psi = \sum_j c_j e_j, c_j = c_j(\psi) \in \mathbb{C}. \quad (17) \]

Each \( \mathcal{E} \) generates a class of (conventional) quantum observables, self-adjoint operators, see [59], [58]:

\[ \hat{a}\psi = \sum_j y_j c_j(\psi) e_j, \quad (18) \]

where \( X_a = \{y_1, ..., y_n\}, y_j \in \mathbb{R}, y_j \neq y_i \) is the range of values of \( a \) (so we start with consideration of observables with nondegenerate spectra).

Let now \( \mathcal{E} = \{e_j\}_{j=1}^n \) be an arbitrary basis (thus in general \( \langle e_j, e_i \rangle \neq 0, i \neq j \)) consisting of normalized vectors, i.e., \( \langle e_j, e_j \rangle = 1 \).

We generalize the Dirac-von Neumann formalism by considering observables \( \{15\} \) for an arbitrary \( \mathcal{E} \). We also consider an arbitrary nonzero vector of \( H \) as a

\[ \|e_0\|^2 = 1. \] It is a consequence of stochasticity of an arbitrary matrix of transition probabilities (which was used by QLRA to produce the \( a \)-basis). Thus we consider now a purely linear algebraic version of this situation.
pure quantum state. We postulate (by generalizing Born’s postulate):

\[ P_{\psi}(a = y_j) = \frac{|c_j(\psi)|^2}{\sum_j |c_j(\psi)|^2}, \quad (19) \]

where the coefficients \( c_j(\psi) \) are given by the expansion (17).

If \( \mathcal{E} \) is an orthonormal basis, then \( c_j(\psi) = \langle \psi, e_j \rangle \), \( \sum_j |c_j(\psi)|^2 = \|\psi\|^2 \) and for a normalized vector \( \psi \), we obtain the ordinary Born’s rule.

Our generalization of the Dirac-von Neumann formalism is also very close to another well known (and very popular in QI) generalization of the class of quantum observables, namely, to the formalism of POVM, [54], [40]. To proceed in this way, we introduce projectors on the basis vectors:

\[ \pi_j \psi = c_j(\psi) e_j. \]

We remark that \( \pi_j^2 = \pi_j \), but in general \( \pi_j^* \neq \pi_j \). We have: \( |c_j(\psi)|^2 = \langle \pi_j \psi, \pi_j \psi \rangle = \langle M_j \psi, \psi \rangle \), where \( M_j = \pi_j^* \pi_j \). We remark that each \( M_j \) is self-adjoint and, moreover, positively defined. We also set \( M = \sum_j M_j \). Then our generalization of Born’s rule can be written as:

\[ P_{\psi}(a = y_j) = \frac{\langle M_j \psi, \psi \rangle}{\langle M \psi, \psi \rangle} = \frac{\text{Tr} \rho_{\psi} M_j}{\text{Tr} \rho_{\psi} M}, \quad (20) \]

where \( \rho_{\psi} = \psi \otimes \psi \). We remark that, for an arbitrary nonzero \( \psi \), the operator \( \rho_{\psi} \geq 0 \).

Now we generalize the conventional notion of the density operator, by considering any nonzero \( \rho \geq 0 \) as a generalized density operator (we recall that at the moment we consider a finite-dimensional space). The corresponding generalization of Born’s rule has the following form:

\[ P_{\psi}(a = y_j) = \frac{\text{Tr} \rho M_j}{\text{Tr} \rho M}. \quad (21) \]

The only difference from the POVM formalism is that the operator \( M \neq I \) (the unit operator).

We remark that \( \langle M \psi, \psi \rangle = \sum_j |c_j(\psi)|^2 \neq 0, \psi \neq 0 \). Thus (we are in the finite dimensional case) the inverse operator \( M^{-1} \) is well defined.

We now proceed with our formalization and consider an arbitrary (separable) Hilbert space \( H \).

**Definition 10.1.** A generalized quantum state is represented by an arbitrary trace class nonnegative (nonzero) operator \( \rho : \rho \geq 0, 0 < \text{Tr} \rho < \infty \).

**Definition 10.2.** A generalized quantum observable is represented by an arbitrary (so in general non normalized) positive operator valued measure \( E \) on a measurable space \((X, \mathcal{F})\) such that \( E(X) > 0 \).

Thus, for a generalized quantum observable \( E \), we have:

1. \( E(B) \geq 0 \), for any set \( B \in \mathcal{F} \), and \( E(X) > 0 \);
2. \( E(\bigcup_{j=1}^n B_j) = \sum_{j=1}^n E(B_j) \) for all disjoint sequences \( \{B_j\} \) in \( \mathcal{F} \).
Generalized Born's rule: Let $\rho$ and $E$ be generalized quantum state and observable, respectively. Then the probability to find the result $x$ of the $E$-measurement in a measurable set $B$ (for an ensemble represented by $\rho$) is given by

$$P_{\rho}(x \in B) = \frac{\text{Tr} \rho E(B)}{\text{Tr} \rho E(X)}.$$  \hfill (22)

We remark that $\text{Tr} \rho E(X) > 0$. To prove this, we consider the spectral expansion of the trace class operator $\rho = \sum_j q_j \psi_j \otimes \psi_j$. Here at least one $q_j > 0$. Then $\text{Tr} \rho E(X) = \sum_j q_j \langle E(X)\psi_j, \psi_j \rangle > 0$.

We now come back to the model considered at the beginning of this section: a finite-dimensional space. We would like to model in the abstract linear algebra framework the situation considered in section 3. We consider two observables, one is a conventional self-adjoint operator $\hat{a}$ and another is a generalized observable $\hat{b}$. Thus the $b$-basis $\mathcal{E}^b = \{e^b_j\}$ is orthonormal, but the $a$-basis $\mathcal{E}^a = \{e^a_i\}$ need not (but we emphasize that even the latter one is normalized). Any vector $e^b_j$ is a conventional (pure) quantum state. Thus by the rules of the conventional QM we can find “transition probabilities”:

$$p^{b|a}(x_i|y_j) = P_{e^b_i}(b = x_i) = |\langle e^a_i, e^b_j \rangle|^2.$$  \hfill (22)

Since $\mathcal{E}^b$ is orthonormal, we have:

$$\sum_i p^{b|a}(x_i|y_j)) = \sum_i |\langle e^a_i, e^b_j \rangle|^2 = \sum_i |e^b_j|^2 = 1.$$  \hfill (22)

The matrix of $b|a$-transition probabilities $P^{b|a}$ is stochastic (as it should be). However, if $\mathcal{E}^a$ is not orthonormal, then $P^{b|a}$ is not doubly stochastic.

On the other hand, we can expand each $e^b_j$ with respect to $\mathcal{E}^a$: $e^b_j = \sum_i c_j(e^b_j)e^a_i$. By our generalized Born’s rule: $p^{a|b}(y_j|x_i) = P_{e^a_i}(a = y_j) = |c_j(e^b_j)|^2 / \sum_j |c_j(e^b_j)|^2$. We have: $\sum_j p^{a|b}(y_j|x_i) = 1$. Thus even the matrix of transition probabilities $P^{a|b}$ is stochastic.

Finally, we remark that all previous considerations are valid even in the case when both observables are generalized.

11 Appendix: Von Neumann postulate in cognitive science and psychology

In general the transition probabilities can depend on the cognitive context $C$ which was chosen for the first (unconditional) measurement:

$$p^{b|a}(x|y) = p^{b|a}_C(x|y).$$

But in some cases dependence of the transition probabilities $p^{b|a}_C(x|y)$ on $C$ could be reducible. In the experimental situation these probabilities (frequencies) are found in the following way. First cognitive systems interact with a context $C$. In this way an ensemble $S_C$ of cognitive systems representing the context $C$ is created. Then cognitive systems belonging to the ensemble $S_C$ interact with the selection-context $C_y$ which is determined by the value $y$ of the mental observable $a$. For example, students belonging to a group $S_C$ (which was trained
under a complex of mental or social conditions $C$) should answer to the question $a$. If this question is so disturbing for a student $\omega$ that he would totally forget about the previous $C$-training, then the transition probabilities do not depend on $C$:

$$p^{b|a}(x|y).$$

Since we are interested only in probabilities, such an individual blocking can be generalized to “statistical blocking” – dependence on $C$ after sequential $ab$-measurement should be statistically negligible: the number of persons who still use the original $C$-context (e.g. training) to reply to the $b$-question (following the $a$-question) is negligibly small comparing with the total number of persons in a sample $S_C$ representing $C$.

We remark that this is the case in conventional quantum theory. Here for incompatible (noncommutative) observables (with nondegenerate spectra) the transition probabilities $p^{b|a}(x|y) = |(e^b_x, e^a_y)|^2$ do not depend on the original context $C$, i.e., a context preceding the $a = y$ selection (by the QM-terminology: “on the original wave function $\psi$”).

In quantum theory any $a = y$ selection destroys the memory on the preceding physical context $C$. For example, suppose that we prepare electrons with a wave function $\psi$ (which provides symbolic symbolic representation of a context $C$, so $\psi = \psi_C$). We measure spin’s projection on some direction $b$ and then on another direction $a$. The transition probability does not depend on $\psi$ (i.e., on $C$).

This is our contextual interpretation of the von Neumann projection postulate [59].

We do not know the general situation for cognitive systems. Our conjecture is that [6].

**Postulate.** (“von Neumann postulate for mental observable”) For any pair $a, b$ of supplementary mental observables the transition probability $p^{b|a}(x|y)$ is completely determined by the preceding preparation – context $C_y$ corresponding to the $[a = y]$-selection.

We remark that by Axiom 1

$$p^{b|b}(x|x) = 1.$$  

Thus if “a system was prepared in the state $e^b_x$,” then measurement of $a$ would definitely give the value $b = x$.

To proceed in our contextual framework, we could be satisfied even by a weaker form of this postulate – we recall that QLRA works by using only two “reference observables.”

**Postulate.** (“Weak von Neumann postulate for mental observable”) There exist supplementary mental observables $a, b$ such that the transition probability $p^{b|a}(x|y)$ is completely determined by the preceding preparation – context $C_y$ corresponding to the $[a = y]$-selection.

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8 It might be that the von Neumann projection postulate can be violated by cognitive systems. In such a case we would not be able to construct the conventional quantum representation of contexts by complex probability amplitudes.

9 We recall that we consider only observables with nondegenerate spectra.
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