Accurate approximations of the nonlinear vibration of couple-mass-spring systems with linear and nonlinear stiffnesses

Md. Alal Hosen\(^1\) and MSH Chowdhury\(^2\)

Abstract
An analytical technique has been developed based on the harmonic balance method to obtain approximate angular frequencies. This technique also offers the periodic solutions to the nonlinear free vibration of a conservative, couple-mass-spring system having linear and nonlinear stiffnesses with cubic nonlinearity. Two real-world cases of these systems are analysed and introduced. After applying the harmonic balance method, a set of complicated higher-order nonlinear algebraic equations are obtained. Analytical investigation of the complicated higher-order nonlinear algebraic equations is cumbersome, especially in the case when the vibration amplitude of the oscillation is large. The proposed technique overcomes this limitation to utilize the iterative method based on the homotopy perturbation method. This produces desired results for small as well as large values of vibration amplitude of the oscillation. In addition, a suitable truncation principle has been used in which the solution achieves better results than existing solutions. Comparing with published results and the exact ones, the approximated angular frequencies and corresponding periodic solutions show excellent agreement. This proposed technique provides results of high accuracy and a simple solution procedure. It could be widely applicable to other nonlinear oscillatory problems arising in science and engineering.

Keywords
Nonlinear stiffnesses, harmonic balance method, iterative method, homotopy perturbation method, couple-mass-spring systems, two-degree-of-freedom oscillation systems, Duffing equation

Introduction
In past century, the motion of an oscillating system with a multidegree of freedom was widely considered by researchers. An approximate method was proposed by Moochhala and Raynor\(^1\) for the motions of unequal masses connected by \((n + 1)\) nonlinear springs and anchored to rigid end walls. Huang\(^2\) studied the harmonic oscillations of nonlinear two-degree-of-freedom (TDOF) systems. The free oscillations of conservative quasilinear systems with TDOF have been analysed by Gilchrist.\(^3\) Efstathiades\(^4\) investigated the existence and characteristic behaviour of combination tones in nonlinear systems with TDOF. Alexander and Richard\(^5\) considered the resonant dynamics of a TDOF system which was composed of a linear oscillator which was weakly coupled to a strongly nonlinear one, with an essential (nonlinearizable) cubic stiffness nonlinearity. A generalized Galerkin’s method has been used by Chen\(^6\) on the nonlinear oscillations of TDOF systems. Ladygina and Manevich\(^7\) applied...
the multiscale method for the free oscillations of a conservative system with TDOF having cubic nonlinearities (of symmetric nature) and close natural frequencies. A combination of a Jacobi elliptic function and a trigonometric function has been used by Cveticanin\(^8,9\) to obtain an analytical solution for the motion of a two-mass system with TDOF in which the masses were connected with three springs.

Currently, TDOF systems are very important in physical and engineering disciplines. Many practical engineering vibration systems such as elastic beams supported by two springs and the vibration of a milling machine\(^10\) can be studied by considering them as TDOF systems. The TDOF oscillation systems consist of two second-order differential equations with cubic nonlinearities. The equations of motion for a mechanical system with associated linear and nonlinear springs were solved by the transformation into a set of differential algebraic equations using intermediate variables. The equations of motion for a TDOF system were transformed into the Duffing equation\(^11\) as a result.

In general, finding an exact approximation for nonlinear equations is extremely difficult. This perception of difficulty has led to intensive research over many decades. Many analytical and numerical approaches are currently being investigated. Traditional perturbation methods are the most widely used analytical methods for approximating nonlinear equations. They are not effective for strongly nonlinear equations, however, and there have limitations. In the recent past, many new analytical techniques have been investigated to overcome these limitations. These include the Newton harmonic balance method (HBM),\(^12\) He’s variational approach,\(^13\) the energy balance method,\(^14\) the max-min approach\(^15\) and He’s improved amplitude–frequency formulation method.\(^16–18\) All have been used to derive approximate angular frequencies and corresponding periodic solutions to the TDOF system. In fact, to the best of our knowledge, only one of these methods, the first-order approximation has ever been considered. This does not result in sufficient accuracy.

The HBM\(^19–22\) provides a general technique for determining approximate periodic solutions to strongly nonlinear systems. Usually, a set of complicated higher-order nonlinear algebraic equations appear when the HBM is formulated. It is tremendously difficult to solve these equations analytically, especially in the case of large values of vibration amplitude of the oscillation. In this paper, a analytical technique has been developed to eradicate this limitation. In the proposed technique, an iterative method based on the homotopy perturbation method\(^23,24\) has been used to solve the set of complicated high-order nonlinear algebraic equations that give desired results for small as well as large values of vibration amplitude of the oscillation. In addition, a suitable truncation principle has also been used to these nonlinear higher-order algebraic equations which make the solutions better than the existing ones saving many calculations. The higher-order approximations (mainly third-order approximations) have been applied to the nonlinear free vibration of a conservative, couple-mass-spring system having linear and nonlinear stiffnesses with cubic nonlinearity. Comparison of obtained results with those published and the corresponding exact solutions show that the obtained results are highly accurate. The advantage of the proposed method is that the solution procedure is very easy, direct, concise, and simple to implement compared to other existing methods.

This paper is organized as follows: In the next section, we provide the outline of the solution approach based on the HBM. In the subsequent section, we offer a detailed description of a two-mass system connected with linear and nonlinear stiffnesses and a two-mass system also connected with linear and nonlinear stiffnesses but fixed to a rigid body both geometrically and mathematically. Then, we apply the solution approach to the nonlinear free vibration of a conservative, couple-mass-spring system having linear and nonlinear stiffnesses with cubic nonlinearity. In the penultimate section, results are discussed in detail. Finally, concluding remarks are given in the last section.

**The solution approach based on the HBM**

Consider a second-order nonlinear differential equation as follows

\[
j \dot{y} + \omega_0^2 y = -ef(y) \quad \text{and the initial conditions} \quad y(0) = A_0, \quad \dot{y}(0) = 0
\]  

(1)

where \(f(y)\) is a nonlinear function such that \(f(-y) = -f(y)\), \(\omega_0 \geq 0\) and \(e\) is a constant.

An N-th order periodic solution of equation (1) can be assumed to be

\[
y = A_0 (\rho \cos(\omega t) + u \cos(3\omega t) + v \cos(5\omega t) + w \cos(7\omega t) + z \cos(9\omega t) + \cdots)
\]

(2)
where $A_0$, $\rho$ and $\omega$ are constants. If $\rho = 1 - u - v - \cdots$, then the solution of equation (2) readily satisfies the initial condition given in equation (1).

Substituting equation (2) into equation (1), the algebraic identity is obtained as

$$A_0\left(\omega_0^2 - \omega^2\right) \cos(\omega t) + u\left(\omega_0^2 - 9\omega^2\right) \cos(3\omega t) + v\left(\omega_0^2 - 25\omega^2\right) \cos(5\omega t) + \cdots$$

$= -\varepsilon F_1(A_0, u, v, \ldots) \cos(\omega t) + F_3(A_0, u, v, \ldots) \cos(3\omega t) + F_5(A_0, u, v, \ldots) \cos(5\omega t) + \cdots$ (3)

Comparing the coefficients of equal harmonic terms of equation (3), we obtain the following nonlinear algebraic equations

$$A_0\omega_0^2 - \omega^2 = -\varepsilon F_1(A_0, u, v, \ldots), \quad A_0u\omega_0^2 - 9\omega^2 = -\varepsilon F_3(A_0, u, v, \ldots)$$

$$A_0v\omega_0^2 - 25\omega^2 = -\varepsilon F_5(A_0, u, v, \ldots)$$ (4)

With the help of the first equation, $\omega^2$ is eliminated from the rest of equation (4). Subsequently, using $\rho = 1 - u - v - \cdots$, and then simplifying, the second and third equations of equation (4) can be reduced to

$$u = G_1(v, A_0, u, v, \ldots), \quad v = G_2(u, A_0, u, v, \ldots), \ldots$$ (5)

where $G_1, G_2, \ldots$ respectively exclude the linear terms of $u, v, \ldots$.

Now applying the iterative method based on the homotopy perturbation method (see Appendix 1 for details), the values of $u$ and $v$ can be obtained from equation (5) as

$$u = u_0 + u_1 + u_2 + u_3 + \cdots$$ (6)

$$v = v_0 + v_1 + v_2 + v_3 + \cdots$$ (7)

where $u_0$ and $v_0$ are the initial approximations and the unknowns $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are

$$u_1 = -\frac{f(u_0)}{f'(u_0)}; \quad v_1 = -\frac{f(v_0)}{f'(v_0)}$$

$$u_2 = -\frac{f''(u_0)}{f'(u_0)} \left(\frac{f(u_0)}{f'(u_0)}\right)^2; \quad v_2 = -\frac{f''(v_0)}{f'(v_0)} \left(\frac{f(v_0)}{f'(v_0)}\right)^2$$

$$u_3 = \frac{1}{f'(u_0)} \left(\frac{1}{6} \left(\frac{f(u_0)}{f'(u_0)}\right)^3 f''(u_0) + \frac{f(u_0)}{f'(u_0)} \left(\frac{-f''(u_0)}{f'(u_0)} \left(\frac{f(u_0)}{f'(u_0)}\right)^2\right)\right)$$

$$v_3 = \frac{1}{f'(v_0)} \left(\frac{1}{6} \left(\frac{f(v_0)}{f'(v_0)}\right)^3 f''(v_0) + \frac{f(v_0)}{f'(v_0)} \left(\frac{-f''(v_0)}{f'(v_0)} \left(\frac{f(v_0)}{f'(v_0)}\right)^2\right)\right)$$

... 

Finally, substituting the values of $u, v, \ldots$ from equations (6) and (7) into the first equation of equation (4), the angular frequency $\omega$ is determined. This completes the determination of all related unknowns for the proposed N-th order periodic solution as given in equation (2).

**Formulation and mathematical modelling of the problems**

A *two-mass system connected with linear and nonlinear stiffnesses*

The model of a two-mass system connected with linear and nonlinear stiffnesses is considered as shown in Figure 1."
The equations of motion are defined as in Cveticanin

\[ m\ddot{u} + k_1(u - v) + k_2(u - v)^3 = 0 \]  
(8a)

\[ m\ddot{v} + k_1(v - u) + k_2(v - u)^3 = 0 \]  
(8b)

with the initial conditions

\[ u(0) = u_0, \quad \dot{u}(0) = 0 \]  
(9a)

\[ v(0) = v_0, \quad \dot{v}(0) = 0 \]  
(9b)

where the double dots in equations (8a) and (8b) represent double differentiation with respect to time \( t \), \( k_1 \) and \( k_2 \) are linear and nonlinear coefficients of the spring stiffness respectively. Dividing equations (8a) and (8b) by mass \( m \), they can be written as

\[ \ddot{u} + \frac{k_1(u - v)}{m} + \frac{k_2(u - v)^3}{m} = 0 \]  
(10a)

\[ \ddot{v} + \frac{k_1(v - u)}{m} + \frac{k_2(v - u)^3}{m} = 0 \]  
(10b)

Introducing the intermediate variables \( x \) and \( y \) as follows

\[ u := x \]  
(11a)

\[ v - u := y \]  
(11b)

and transforming equations (10a) and (10b), becomes

\[ \ddot{x} - \beta y - \epsilon y^3 = 0 \]  
(12a)

\[ \ddot{y} + \ddot{x} + \beta y + \epsilon y^3 = 0 \]  
(12b)

where \( \beta = \frac{k_1}{m} \) and \( \epsilon = \frac{k_2}{m} \). Rearranging equation (12a) as follows

\[ \ddot{x} = \beta y + \epsilon y^3 \]  
(13)

Substituting equation (13) into equation (12b), it becomes

\[ \ddot{y} + 2\beta y + 2\epsilon y^3 = 0 \]  
(14)
with initial conditions

\[ y(0) = v(0) - u(0) = v_0 - u_0 = A_0, \quad y'(0) = 0 \]  

(15)

Equation (14) is obviously similar to the well-known Duffing equation \( \ddot{y} + \delta \dot{y} + \sigma y^3 = 0 \) with \( \delta = 2\beta \) and \( \sigma = 2e \). In order to solve equation (14) using the proposed method, the approximate solutions of \( y(t) \) can be substituted back into equation (13) as

\[ \ddot{x} = \beta y + ey^3 \]

with the initial conditions

\[ x(0) = u(0) = u_0, \quad \dot{x}(0) = 0 \]

and the intermediate variable \( x(t) \) is obtained by double integration.

**A two-mass system connected with linear and nonlinear stiffnesses fixed to the body**

Consider a two-mass system connected with linear and nonlinear springs and fixed to a rigid body at two ends as shown in Figure 2.

\[ u \]

\[ k_1 \]

\[ k_2, k_3 \]

\[ v \]

\[ k_1 \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]

\[ m \]
Similar to problem 1, transforming the above equations using the intermediate variables in equations (11a) and (11b), they become

\[
\ddot{x} + 2\dot{x} - \beta y - e y^3 = 0 \tag{19a}
\]
\[
\ddot{y} + \ddot{x} + 2\dot{x} + \alpha y + \beta y + e y^3 = 0 \tag{19b}
\]

where \( \alpha = \frac{k_1}{m}, \beta = \frac{k_2}{m} \) and \( e = \frac{k_3}{m} \). Rearranging equation (19a) as follows

\[
\ddot{x} = \beta y + e y^3 - 2\dot{x} \tag{20}
\]

Substituting equation (20) into equation (19b), it becomes

\[
\ddot{y} + (\alpha + 2\beta)\dot{y} + 2\dot{y} + 2ey^3 = 0 \tag{21}
\]

with initial conditions

\[
y(0) = v(0) - u(0) = v_0 - u_0 = A_0, \quad \dot{y}(0) = 0 \tag{22}
\]

Equation (21) is obviously similar to the well-known Duffing equation \( \ddot{y} + \delta \dot{y} + \sigma y^3 = 0 \) with \( \delta = \alpha + 2\beta \) and \( \sigma = 2\alpha \). In order to solve equation (21) using the proposed method, the approximate solutions of \( y(t) \) can be substituted back into equation (20) to obtain

\[
\ddot{x} + 2\dot{x} = \beta y + e y^3 \tag{23}
\]

with the initial conditions

\[
x(0) = u(0) = u_0, \quad \dot{x}(0) = 0 \tag{24}
\]

Equation (23) is a linear non-homogeneous second-order ordinary differential equation and it can be solved readily using a standard method such as the Laplace transformation.

**Applications**

A two-mass system connected with linear and nonlinear stiffnesses

A second-order approximation for equation (14) can be assumed to be

\[
y = A_0(\rho \cos(\omega_2 t) + u \cos(3\omega_2 t)) \tag{25}
\]

Substituting equation (25) along with \( \rho = 1 - u \) into equation (14) and then taking the coefficients of \( \cos(\omega_2 t) \) and \( \cos(3\omega_2 t) \) equal to zero, the nonlinear algebraic equations can be obtained

\[
2\beta + \frac{3eA_0^2}{2} - \omega_2^2 - 2\beta u - 3eA_0^2 u + \omega_2^2 u + \frac{9eA_0^2 u^2}{2} - 3eA_0^2 u^3 = 0 \tag{26}
\]
\[
\frac{eA_0^2}{2} + 2\beta u + \frac{3eA_0^2 u}{2} - 9\omega_2^2 u - \frac{9eA_0^2 u^2}{2} + 4eA_0^2 u^3 = 0 \tag{27}
\]

Considerable calculation is saved and obtains improved results, if we use the truncation principle in equations (26) and (27). The higher-order terms of \( u \) greater than second-order have no effect on the value of the unknowns \( u \) and \( \omega_2 \). So, we may ignore greater than second-order terms of \( u \), but half of the second-order terms are
considered. This is called the truncation principle. After using the truncation principle, equations (26) and (27) can be transformed into

\[
\begin{align*}
2\beta + \frac{3eA_0^2}{2} - \omega_2^2 - 2\beta u - 3eA_0^2u + \omega_2^2u + \frac{9eA_0^2u^2}{4} &= 0 \\
\frac{eA_0^2}{2} + 2\beta u + \frac{3eA_0^2u}{2} - 9\omega_2^2 - \frac{9eA_0^2u^2}{4} &= 0
\end{align*}
\]

Equation (28) can be easily written as

\[
\omega_2^2 = \left(2\beta + \frac{3eA_0^2}{2} - 2\beta u - 3eA_0^2u + \frac{9eA_0^2u^2}{4}\right)/(1 - u)
\]

Substituting \(\omega_2^2\) into equation (29) and then simplifying, the following nonlinear algebraic equation of \(u\) is

\[
f(u) : \frac{eA_0^2}{2} - 16\beta u - \frac{25eA_0^2u}{2} + 16\beta u^2 + \frac{93eA_0^2u^2}{4} - 18eA_0^2u^3 = 0
\]

Now applying the iterative method based on the homotopy perturbation method (see Appendix 1 for details) in equation (31), the value of \(u\) becomes

\[
u = u_0 + u_1 + u_2 + u_3 + \cdots
\]

where \(u_0\) is an initial approximation and the unknowns \(u_1, u_2, u_3, \ldots\) are

\[
\begin{align*}
u_1 &= -\frac{f(u_0)}{f'(u_0)} \\
u_2 &= -\frac{f'(u_0)}{f''(u_0)} \left(\frac{f(u_0)}{f'(u_0)}\right)^2 \\
u_3 &= \frac{1}{f''(u_0)} \left(\frac{f(u_0)}{f'(u_0)}\right)^3 f''(u_0) + \frac{f(u_0)}{f'(u_0)} \left(\frac{f''(u_0)}{f'(u_0)}\right)^2 f'''(u_0)
\end{align*}
\]

By substituting the value of \(u\) into equation (30), the second-order approximate angular frequency \(\omega_2\) can be determined and equation (30) takes the following form

\[
\omega_2 = \sqrt{\left(2\beta + \frac{3eA_0^2}{2} - 2\beta u - 3eA_0^2u + \frac{9eA_0^2u^2}{4}\right)/(1 - u)}
\]

Thus, the second-order approximate solution of equation (14) is \(y = A_0(\rho\cos(\omega_2t) + uc\cos(3\omega_2t))\) where \(u\) and \(\omega_2\) are respectively given by equations (32) and (33).

In the similar way, the higher-order approximations have been obtained by using the proposed method. In this study, a third-order approximate solution can be considered to be

\[
y = A_0(\rho\cos(\omega_3t) + uc\cos(3\omega_3t) + uc\cos(5\omega_3t))
\]

Substituting equation (34) along with \(\rho = 1 - u - v\) into equation (14) and then taking the coefficients of \(\cos(\omega_3t), \cos(3\omega_3t)\) and \(\cos(5\omega_3t)\) equal to zero, the nonlinear algebraic equations can be obtained

\[
\begin{align*}
x + 2\beta + \frac{3eA_0^2}{2} - \omega_3^2 - 2\beta u - 3eA_0^2u + \omega_3^2u + \frac{9eA_0^2u^2}{2} - 3eA_0^2u^3 \\
xv - 2\beta v - \frac{9eA_0^2v}{2} + \omega_3^2v + 9eA_0^2uv - 6eA_0^2uv^2 + \frac{15eA_0^2v^2}{2} - 9eA_0^2v^3 - \frac{9eA_0^2v^3}{2} = 0
\end{align*}
\]
\[
\frac{eA_0^2}{2} + \omega_3^2 - \omega_3^2 - 2\beta u - \frac{3eA_0^2}{2} + \frac{9eA_0^2uv}{2} - \frac{9eA_0^2u^2}{2} + 4eA_0^2u^3 - 3eA_0^2uv
\]
\[
+ \frac{3eA_0^2u^2v}{2} - \frac{9eA_0^2uv^2}{2} + \frac{9eA_0^2u^2v^2}{2} + eA_0^2v^3 = 0
\]

(36)

\[
\frac{3eA_0^2}{2} + \frac{3eA_0^2}{2} + 2\beta v + 3eA_0^2v - 25\omega_3^2v - 9eA_0^2uv
\]
\[
+ \frac{15eA_0^2u^2v}{2} - 6eA_0^2v^2 + \frac{15eA_0^2u^2v^2}{2} + 9eA_0^2v^3 = 0
\]

(37)

Considerable calculation is saved and improved results are obtained, if we use the truncation principle in
equations (35) to (37). After implementing it in equations (35) to (37), the following equations are obtained

\[
x + 2\beta + \frac{3eA_0^2}{2} - \omega_3^2 - xu - 2\beta u - 3eA_0^2u + \omega_3^2 + \frac{9eA_0^2uv}{2} - \frac{9eA_0^2u^2}{2} - \frac{3eA_0^2u^3}{2}
\]
\[
- \omega_3^2v - 2\beta v - \frac{9eA_0^2v}{2} = 0
\]

(38)

\[
\frac{eA_0^2}{2} + \frac{3eA_0^2}{2} - \frac{9\omega_3^2u}{2} - 9eA_0^2u - \frac{9eA_0^2u^2}{2} + 2eA_0^2u^3 - \frac{3eA_0^2uv}{2} = 0
\]

(39)

\[
\frac{3eA_0^2u}{2} - \frac{3eA_0^2u^2}{2} + xv + 2\beta v + 3eA_0^2v - 25\omega_3^2v - \frac{9eA_0^2uv}{2} = 0
\]

(40)

Equation (38) can be easily written as

\[
\omega_3^2 = \left( x + 2\beta + \frac{3eA_0^2}{2} - xu - \frac{3eA_0^2u^3}{2} - 2\beta v + \frac{9eA_0^2u^2}{2} + \cdots \right) / (1 - u - v)
\]

(41)

Substituting \(\omega_3^2\) into equations (39) and (40) and then simplifying, the following nonlinear algebraic equations
of \(u\) and \(v\) are obtained

\[
f(u) : \frac{eA_0^2}{2} - 8\omega_3u - 16\omega_3u - \frac{25eA_0^2u}{2} + 8\omega_3u^2 + 16\omega_3u^2 + 21eA_0^2u^3 - 34eA_0^2u^3 + \frac{23eA_0^2u^4}{2}
\]
\[
- \frac{eA_0^2}{2} + 8\omega_3uv + 16\omega_3uv + \frac{75eA_0^2u^3v}{2} - \frac{69eA_0^2u^4}{2} - 2eA_0^2u^3v + \frac{3eA_0^2u^4v}{2} = 0
\]

(42)

\[
f(v) : \frac{3eA_0^2u}{2} - 3eA_0^2u^2 + \frac{3eA_0^2u^3}{2} - 24\omega_3v - 48\beta v - \frac{69eA_0^2v}{2}
\]
\[
+ 24\omega_3uv + 48\beta uv + 66eA_0^2uv - \frac{213eA_0^2u^2v}{2} + \frac{75eA_0^2u^3v}{2}
\]
\[
+ 24\omega_3v^2 + 48\beta v^2 + \frac{219eA_0^2v^3}{2} - 108eA_0^2uv^2 = 0
\]

(43)

Now applying the iterative method based on the homotopy perturbation method (see Appendix 1 for details) in
equations (42) and (43), the values of \(u\) and \(v\) are obtained

\[
u = u_0 + u_1 + u_2 + u_3 + \cdots
\]

(44)

\[
v = v_0 + v_1 + v_2 + v_3 + \cdots
\]

(45)
where \( u_0 \) and \( v_0 \) are the initial approximations and the unknowns \( u_1, u_2, u_3, \ldots \) and \( v_1, v_2, v_3, \ldots \) are

\[
\begin{align*}
\quad u_1 &= -\frac{f(u_0)}{f'(u_0)}; \\
\quad v_1 &= -\frac{f(v_0)}{f'(v_0)};
\end{align*}
\]

\[
\begin{align*}
\quad u_2 &= -\frac{f'(u_0)}{f''(u_0)} \left( \frac{f(u_0)}{f'(u_0)} \right)^2; \\
\quad v_2 &= -\frac{f'(v_0)}{f''(v_0)} \left( \frac{f(v_0)}{f'(v_0)} \right)^2;
\end{align*}
\]

\[
\begin{align*}
\quad u_3 &= \frac{1}{f'(u_0)} \left( \frac{1}{6} \left( \frac{f(u_0)}{f'(u_0)} \right)^3 \right) \frac{f''(u_0)}{f'(u_0)} f'''(u_0) + \frac{f(u_0)}{f'(u_0)} \left( \frac{f''(u_0)}{f'(u_0)} \right)^2; \\
\quad v_3 &= \frac{1}{f'(v_0)} \left( \frac{1}{6} \left( \frac{f(v_0)}{f'(v_0)} \right)^3 \right) \frac{f''(v_0)}{f'(v_0)} f'''(v_0) + \frac{f(v_0)}{f'(v_0)} \left( \frac{f''(v_0)}{f'(v_0)} \right)^2;
\end{align*}
\]

\[
\begin{align*}
\quad &\ldots
\end{align*}
\]

Now substituting the values of \( u \) and \( v \) into equation (41), the third-order approximate angular frequency \( \omega_3 \) is the following

\[
\omega_3 = \sqrt{\left( x + 2\beta + \frac{3eA_0^2}{2} - xu - \frac{3eA_0^2u^3}{2} - 2\beta v + \frac{9eA_0^2\mu v}{2} + \ldots \right) / \left( 1 - u - v \right)}
\]

(46)

Thus, the third-order approximate periodic solution of equation (14) is \( y = A_0(\rho \cos(\omega_3 t) + \mu \cos(3\omega_3 t)) \) where \( u, v \) and \( \omega_3 \) are respectively given by equations (44) to (46).

**Two-mass system connected with linear and nonlinear stiffnesses fixed to the body**

A second-order approximation for equation (21) can be assumed to be

\[
y = A_0(\rho \cos(\omega_2 t) + \mu \cos(3\omega_2 t))
\]

(47)

Substituting equation (47) along with \( \rho = 1 - u \) into equation (21) and then taking the coefficients of \( \cos(\omega_2 t) \) and \( \cos(3\omega_2 t) \) to zero, the nonlinear algebraic equations can be obtained to be

\[
x + 2\beta + \frac{3eA_0^2}{2} - \omega_2^2 - xu - 2\beta u - 3eA_0^2u + \omega_2^2 u + \frac{9eA_0^2\mu u^2}{2} - 3eA_0^2u^3 = 0
\]

(48)

\[
\frac{eA_0^2}{2} + xu + 2\beta u + \frac{3eA_0^2u}{2} - 9\omega_2^2u - \frac{9eA_0^2\mu u^2}{2} + 4eA_0^2u^3 = 0
\]

(49)

Considerable calculation is saved and improved results are obtained, if we use the truncation principle in equations (48) and (49). The higher-order terms of \( u \) greater than second-order have no effect on the value of the unknowns \( u \) and \( \omega_2 \). So, we may ignore greater than second-order terms of \( u \), but half of the second-order terms are considered. This is called the truncation principle. After using the truncation principle, equations (48) and (49) can be transformed into

\[
x + 2\beta + \frac{3eA_0^2}{2} - \omega_2^2 - xu - 2\beta u - 3eA_0^2u + \omega_2^2 u + \frac{9eA_0^2\mu u^2}{4} = 0
\]

(50)

\[
\frac{eA_0^2}{2} + xu + 2\beta u + \frac{3eA_0^2u}{2} - 9\omega_2^2u - \frac{9eA_0^2\mu u^2}{4} = 0
\]

(51)
Equation (50) can be easily written as

\[
\omega_2^2 = \left( x + 2\beta + \frac{3eA_0^2}{2} - xu - 2\beta u - 3eA_0^2u + \frac{9eA_0^2u^2}{4} \right)/(1 - u)
\]  

(52)

Substituting \( \omega_2^2 \) into equation (51) and then simplifying, the following nonlinear algebraic equation of \( u \) is

\[
f(u) : \frac{eA_0^2}{2} - 8xu - 16\beta u - \frac{25eA_0^2u}{2} + 8xu^2 + 16\beta u^2 + \frac{93eA_0^2u^2}{4} - 18eA_0^2u^3 = 0
\]  

(53)

Now applying the iterative method based on the homotopy perturbation method (see Appendix 1 for details) in equation (53), the value of \( u \) becomes

\[
u = u_0 + u_1 + u_2 + u_3 + \cdots
\]  

(54)

where \( u_0 \) is an initial approximation and the unknowns \( u_1, u_2, u_3, \ldots \) are

\[
u_1 = -\frac{f(u_0)}{f'(u_0)},
\]

\[
u_2 = -\frac{f'(u_0)}{f'(u_0)} \left( f(u_0) \right)^2,
\]

\[
u_3 = \frac{1}{f'(u_0)} \left( \frac{1}{6} \left( f(u_0) \right)^3 \right) \left( f''(u_0) f(u_0) - f''(u_0) f(u_0)^2 \right) - \frac{f''(u_0)}{f'(u_0)} \left( f(u_0) \right)^2,
\]

\[\cdots\]

Substituting the value of \( u \) into equation (52), the second-order approximate angular frequency \( \omega_2 \) is determined from the following

\[
\omega_2 = \sqrt{\left( x + 2\beta + \frac{3eA_0^2}{2} - xu - 2\beta u - 3eA_0^2u + \frac{9eA_0^2u^2}{4} \right)/(1 - u)}
\]  

(55)

Thus, the second-order approximate solution of equation (21) is \( y = A_0 \left( \rho \cos(\omega_2t) + \mu \cos(3\omega_2t) \right) \) where \( u \) and \( \omega_2 \) are respectively given by equations (54) and (55).

In the similar way, the higher-order approximations have been obtained by using the proposed method. In this study, a third-order approximate solution can be considered to be

\[
y = A_0 \left( \rho \cos(\omega_3t) + \mu \cos(3\omega_3t) + \nu \cos(5\omega_3t) \right)
\]  

(56)

Substituting equation (56) along with \( \rho = 1 - u - v \) into equation (21) and then taking the coefficients of \( \cos(\omega_3t) \), \( \cos(3\omega_3t) \) and \( \cos(5\omega_3t) \) to be zero, the nonlinear algebraic equations become

\[
\begin{align*}
x + 2\beta + \frac{3eA_0^2}{2} - \omega_3^2 - xu - 2\beta u - 3eA_0^2u + \frac{9eA_0^2u^2}{2} - 3eA_0^2u^3 - xv - 2\beta v & \\
- \frac{9eA_0^2v}{2} + \omega_3^2 v + 9eA_0^2uv - 6eA_0^2u^2v + \frac{15eA_0^2v^2}{2} - 9eA_0^2uv^2 - \frac{9eA_0^2v^3}{2} = 0
\end{align*}
\]  

(57)

\[
\begin{align*}
\frac{eA_0^2}{2} + xu + 2\beta u + \frac{3eA_0^2u}{2} - 9\omega_3^2 u - \frac{9eA_0^2u^2}{2} + 4eA_0^2u^3 - 3eA_0^2uv \\
+ 3eA_0^2u^2v - \frac{3eA_0^2v^2}{2} + \frac{9eA_0^2uv^2}{2} + eA_0^2v^3 = 0
\end{align*}
\]  

(58)
\[
\begin{align*}
\frac{3eA^2_x}{2} - \frac{3eA^2_y}{2} + x^v + 2\beta_v + 3eA^2_0 - 25\omega^2_v - 9eA^2_0uv & \\
+ \frac{15eA^2_0uv}{2} - 6eA^2_0v^2 + 15eA^2_0uv^2 + \frac{9eA^2_0}{2} &= 0
\end{align*}
\]

(59)

Considerable calculation is saved and improved results are obtained, if we use the truncation principle in equations (57) to (59). After implementing this in equations (57) to (59), the following equations are obtained

\[
\begin{align*}
x + 2\beta + \frac{3eA^2_0}{2} - \omega^2_x - xu - 2\beta u - 3eA^2_0u + \omega^2_0u + \frac{9eA^2_0u^2}{2} & \\
- \frac{3eA^2_0}{2} - xu - 2\beta v - \frac{9eA^2_0}{2} + \omega^2_0v + \frac{9eA^2_0}{2} &= 0
\end{align*}
\]

(60)

\[
\begin{align*}
\frac{eA^2_x}{2} + xu + 2\beta u + \frac{3eA^2_0}{2} - 9\omega^2_u - \frac{9eA^2_0}{2} + 2eA^2_0u^3 - \frac{9eA^2_0uv}{2} &= 0
\end{align*}
\]

(61)

\[
\begin{align*}
\frac{3eA^2_0}{2} - \frac{3eA^2_0u^2}{2} + xv + 2\beta v + 3eA^2_0 - 25\omega^2_v - 9eA^2_0uv &= 0
\end{align*}
\]

(62)

Equation (60) can be easily written as

\[
\omega^2_x = \left( x + 2\beta + \frac{3eA^2_0}{2} - xu - \frac{3eA^2_0u^3}{2} - 2\beta v + \frac{9eA^2_0uv}{2} + \cdots \right) / (1 - u - v)
\]

(63)

Substituting \(\omega^2_x\) into equations (61) and (62) and then simplifying, the following nonlinear algebraic equations of \(u\) and \(v\) are obtained

\[
\begin{align*}
f(u) : \frac{eA^2_x}{2} - 8xu - 16\beta u - \frac{25eA^2_0}{2} + 8xu^2 + 16\beta u^2 + 21eA^2_0u^2 & \\
- 34eA^2_0u^3 + \frac{23eA^2_0u^4}{2} - \frac{eA^2_0}{2} + 8uv + 16\beta uv + \frac{75eA^2_0uv}{2} & \\
- \frac{69eA^2_0v^2}{2} - 2eA^2_0v^3 + \frac{3eA^2_0uv^2}{2} &= 0
\end{align*}
\]

(64)

\[
\begin{align*}
f(v) : \frac{3eA^2_0}{2} - 3eA^2_0u^2 + \frac{3eA^2_0u^3}{2} - 24xv + 48\beta v - \frac{69eA^2_0}{2} + 24xuv & \\
+ 48\beta uv + 66eA^2_0uv - \frac{213eA^2_0v^2}{2} + \frac{75eA^2_0v^3}{2} + 24xv^2 + 48\beta v^3 & \\
+ \frac{219eA^2_0v^2}{2} - 108eA^2_0uv^2 &= 0
\end{align*}
\]

(65)

Now by applying the iterative method based on the homotopy perturbation method (see Appendix 1 for details) in equations (64) and (65), the values of \(u\) and \(v\) are obtained

\[
u = u_0 + u_1 + u_2 + u_3 + \cdots
\]

(66)

\[
v = v_0 + v_1 + v_2 + v_3 + \cdots
\]

(67)
where \( u_0 \) and \( v_0 \) are the initial approximations and the unknowns \( u_1, u_2, u_3, \ldots \) and \( v_1, v_2, v_3, \ldots \) are

\[
\begin{align*}
  u_1 &= -\frac{f(u_0)}{f'(u_0)}; \quad v_1 = -\frac{f(v_0)}{f'(v_0)} \\
  u_2 &= -\frac{f''(u_0) (f(u_0))^2}{f'(u_0)}; \quad v_2 = -\frac{f''(v_0) (f(v_0))^2}{f'(v_0)} \\
  u_3 &= \frac{1}{f'(u_0)} \left( \frac{1}{6} \left( \frac{f(u_0)}{f'(u_0)} \right)^3 \right) \frac{f'''(u_0)}{f'(u_0)} + \frac{1}{f'(v_0)} \left( \frac{1}{6} \left( \frac{f(v_0)}{f'(v_0)} \right)^3 \right) \frac{f'''(v_0)}{f'(v_0)} \\
  v_3 &= \frac{1}{f'(v_0)} \left( \frac{1}{6} \left( \frac{f(v_0)}{f'(v_0)} \right)^3 \right) \frac{f'''(v_0)}{f'(v_0)} + \frac{1}{f'(u_0)} \left( \frac{1}{6} \left( \frac{f(u_0)}{f'(u_0)} \right)^3 \right) \frac{f'''(u_0)}{f'(u_0)} \\
  \vdots
\end{align*}
\]

Now substituting the values of \( u \) and \( v \) into equation (63), the third-order approximate angular frequency \( \omega_3 \) becomes the following

\[
\omega_3 = \sqrt{\left( \alpha + 2\beta + \frac{3E_A^2}{2} - xu - \frac{3E_A^2 v^2}{2} - 2\beta v + \frac{9E_A^2 \mu v}{2} + \cdots \right)/(1 - u - v)}
\]

(68)

Thus, the third-order approximate periodic solution of equation (21) is \( y = A_0 (\rho \cos(\omega_3 t) + u \cos(3\omega_3 t) + v \cos(5\omega_3 t)) \) where \( u, v \) and \( \omega_3 \) are respectively given by equations (66) to (68).

**Results and discussions**

To demonstrate and verify the accuracy of the proposed method, we present a comparison with already published results and the exact solutions. The second- and third-order approximate solutions are highly accurate, with a significantly improved percentage error for different values of parameters and initial amplitudes. Tables 1 to 4 give the comparison of the approximate results with those previously published\(^{13-16} \) and also the exact solutions for different values of parameters \( m, k_1, k_2, k_3 \) and initial conditions \( u_0 \) and \( v_0 \). For the value of parameters \( m = 10, k_1 = 5, k_2 = 5, u_0 = 10 \) and \( v_0 = 20, \) the approximated maximum relative errors in Problem 1 are 0.0289% and 0.0014% for the second- and third-order analytical approximations, respectively. In Problem 2, almost similar relative errors are found for the value of parameters \( m = 10, k_1 = 5, k_2 = 5, k_3 = 5, u_0 = 10 \) and \( v_0 = 20 \) which are much lower than the errors found using the existing methods including the variational approach,\(^{13} \) the energy balance method,\(^{14} \) the max-min approach\(^{15,16} \) and He’s improved amplitude-formulation.\(^{16} \) Hence, excellent agreement with the exact solutions for the nonlinear Duffing equation is obtained.

**Table 1.** Comparison of the approximate angular frequencies with already published results and exact frequencies for various parameters of the system.

| \( m \) | \( k_1 \) | \( k_2 \) | \( u_0 \) | \( v_0 \) | \( \omega_{16}^{MMA–IAFF} \) | \( \omega_{16}^{MMA–IAFF} \) | \( \omega_{16}^{MMA–IAFF} \) | \( \omega_{16}^{MMA–IAFF} \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.5 | 1 | 5 | 3.605551 | 3.539243 | 3.540101 | 3.539208 |
| 1 | 1 | 1 | 5 | 1 | 5.099020 | 5.005246 | 5.005196 | 5.005098 |
| 10 | 5 | 5 | 10 | 20 | 8.717798 | 8.533856 | 8.536056 | 8.533463 |
| 20 | 40 | 50 | 20 | 10 | 19.46792 | 19.05402 | 19.059843 | 19.054010 |
| 50 | 100 | 50 | -10 | 20 | 36.79674 | 36.00234 | 36.013049 | 36.00179 |
| 100 | 400 | 100 | 50 | -50 | 122.5071 | 119.8489 | 119.884810 | 119.847084 |

(\( \omega_{16}^{MMA–IAFF} \) in this study, \( \omega_{16}^{MMA–IAFF} \) in this study)


Table 2. Comparison of the approximate angular frequencies with already published results and exact frequencies for various parameters of the system.

| m  | k₁  | k₂  | u₀   | ν₀   | \( \Omega_{13}^{(14, EBM)} [\%] \) | \( \Omega_{13}^{(16, IAFF)} [\%] \) | \( \Omega_{20}^{(dO-study)} [\%] \) | \( \Omega_{30}^{(dO-study)} [\%] \) |
|----|-----|-----|------|------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| 1  | 5   | 5   | 5    | 1    | 11.4018                           | 11.1921                           | 11.194784                         | 11.191958                         |
| 1  | 1   | 1   | 10   | -5   | 18.4255                           | 18.0302                           | 18.035502                         | 18.029911                         |
| 5   | 10  | 10  | 20   | 30   | 17.4356                           | 17.0672                           | 17.072113                         | 17.066927                         |
| 10  | 50  | -0.01 | -20 | 40   | 2.1448                            | 2.0795                            | 2.076633                          | 2.079540                          |
| 1   | 10  | 5   | 20   | 25   | 14.4049                           | 14.1514                           | 14.154631                         | 14.151263                         |
| 100 | 200 | 300 | 400  | 200  | 424.2688                          | 415.053                           | 415.17748                         | 415.046582                         |

Table 3. Comparison of the approximate angular frequencies with already published results and exact frequencies for various parameters of the system.

| m  | k₁  | k₂  | k₃  | u₀   | ν₀   | \( \Omega_{16}^{(MAA-IAFF)} [\%] \) | \( \Omega_{16}^{(IAFF)} [\%] \) | \( \Omega_{20}^{(dO-study)} [\%] \) | \( \Omega_{30}^{(dO-study)} [\%] \) |
|----|-----|-----|-----|------|------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| 1  | 0.5 | 0.5 | 0.5 | 5    | 5    | 3.674235                          | 3.611743                          | 3.612534                          | 3.611713                          |
| 1  | 1   | 1   | 2   | 5    | 1    | 7.141428                          | 7.004694                          | 7.006482                          | 7.004616                          |
| 5   | 2   | 0.5 | 5   | 5    | 10   | 6.17252                           | 6.042804                          | 6.044561                          | 6.042718                          |
| 10  | 5   | 5   | 10  | 10   | 20   | 12.30853                          | 12.04665                         | 12.050172                         | 12.046478                         |
| 20  | 40  | 50  | 50  | 20   | 10   | 19.54482                          | 19.13632                          | 19.141786                         | 19.136055                         |
| 50  | 100 | 50  | 100 | -10  | 20   | 52.00000                          | 50.87391                          | 50.889103                         | 50.873131                         |
| 100 | 400 | 200 | 200 | 50   | -50  | 173.2224                          | 169.4611                          | 169.511909                        | 169.458516                        |
| 200 | 100 | 50  | 400 | 100  | 300  | 346.4174                          | 338.8929                         | 338.988373                        | 338.881543                        |
| 500 | 500 | 1000| 500 | 600  | 400  | 244.951                           | 239.6302                         | 239.710785                        | 239.635257                        |

Table 4. Comparison of the approximate angular frequencies with already published results and exact frequencies for various parameters of the system.

| m  | k₁  | k₂  | k₃  | u₀   | ν₀   | \( \Omega_{13}^{(14, EBM)} [\%] \) | \( \Omega_{13}^{(IAFF)} [\%] \) | \( \Omega_{20}^{(dO-study)} [\%] \) | \( \Omega_{30}^{(dO-study)} [\%] \) |
|----|-----|-----|-----|------|------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| 1  | 1   | 1   | 1   | 5    | 1    | 5.1961                            | 5.1078                            | 5.108895                          | 5.107734                          |
| 1  | 1   | 1   | 5   | 5    | 10   | 13.8022                           | 13.5121                           | 13.516005                         | 13.511928                         |
| 5   | 10  | 20  | 30  | -10  | 10   | 60.8033                           | 58.7856                           | 58.803142                         | 58.784751                         |
| 10  | 50  | 70  | 90  | 20   | -40  | 220.4972                          | 215.7113                          | 215.775905                        | 215.707977                        |
| 10  | 25  | 20  | -0.05 | -10 | 10   | 1.8708                            | 1.8413                            | 1.840531                          | 1.841323                          |
| 100 | 200 | 300 | 400 | -50  | 50   | 244.9653                          | 239.6455                          | 239.717323                        | 239.641804                        |
In Tables 1 to 4, \( m, k_1, k_2, k_3, u_0 \) and \( v_0 \) respectively denote the various parameters of the systems. \( \omega_{[\text{MMA=IAFF}]}^{[16]} \) and \( \omega_{[\text{VA=EBM}]}^{[13,14]} \) represent the already published results which have been obtained using the max-min approach,\(^\text{16}\) He’s improved amplitude–frequency formulation method,\(^\text{16}\) the variational approach\(^\text{13}\) and the energy balance method,\(^\text{14}\) respectively. \( \omega_{[\text{Exact}]}^{[13]} \) and \( \omega_{[\text{Exact}]}^{[16]} \) signify the exact solutions which are obtained in Hashemi Kachapi et al.\(^\text{13}\) and Bayat et al.\(^\text{16}\) \( \omega_{[\text{this study}]}^{[2nd]} \) and \( \omega_{[\text{this study}]}^{[3rd]} \) denote the second- and third-order approximate angular frequencies obtained by using the proposed method. \( E(\%) \) represents the percentage error which is obtained from the relation \( \left| \frac{\omega_{\text{appr}} - \omega_{\text{Ex}}}{\omega_{\text{Ex}}} \right| \times 100 \).

In order further to illustrate and verify the exactness of the approximate analytical solution, a comparison of the time history oscillatory displacement response for the two masses for different values of initial conditions and stiffnesses with fourth-order Runge–Kutta (considered to be exact) solutions are plotted in Figures 3 to 5 for Problem 1 and Figures 6 to 8 for Problem 2. The proposed method is simple, quite easy, and highly efficient and is valid for a wide range of vibration amplitudes of the oscillation. As can be seen in Figures 3 to 8, it is found that the proposed method has excellent agreement with the numerical solution. The proposed method is rapidly convergent and can also be easily generalized to two-degree-of-freedom oscillation systems by combining the transformation technique.

**Figure 3.** Comparison of the approximate solution in equation (14) with the corresponding numerical one for various parameters \( m = 1, k_1 = 0.5, k_2 = 0.5, u_0 = 1, v_0 = 5 \).
Figure 4. Comparison of the approximate solution in equation (14) with the corresponding numerical one for various parameters $m = 100$, $k_1 = 400$, $k_2 = 100$, $u_0 = 50$, $v_0 = -50$.

Figure 5. Comparison of the approximate solution in equation (14) with the corresponding numerical one for various parameters $m = 100$, $k_1 = 200$, $k_2 = 300$, $u_0 = 400$, $v_0 = 200$. 
Figure 6. Comparison of the approximate solution in equation (21) with the corresponding numerical one for various parameters $m = 1$, $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.5$, $u_0 = 1$, $v_0 = 5$.

Figure 7. Comparison of the approximate solution in equation (21) with the corresponding numerical one for various parameters $m = 500$, $k_1 = 500$, $k_2 = 1000$, $k_3 = 500$, $u_0 = 400$, $v_0 = 600$. 
Conclusion

We have developed an analytical technique based on the HBM to a set of second-order coupled differential equations with cubic nonlinearity. These govern the nonlinear free vibration of a conservative, couple-mass-spring system having linear and nonlinear stiffnesses. We obtained the approximate angular frequencies as well as the corresponding periodic solutions using this proposed method. As compared with published results and exact solutions, we have found excellent agreement. The exactness of the results shows that the proposed method can be easily and efficiently used for the analysis of strongly nonlinear vibration problems with high accuracy. Hence, we conclude that the proposed method has great potential and provides an efficient alternative to the previously existing methods for solving strongly nonlinear oscillatory systems.

Acknowledgement

The authors are grateful for the financial support from the Research Management Centre (RMC), International Islamic University Malaysia.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The authors received financial support from the Research Management Centre (RMC), International Islamic University Malaysia, for publication of this article.

ORCID iD

Md. Alal Hosen https://orcid.org/0000-0002-8801-310X

Figure 8. Comparison of the approximate solution in equation (21) with the corresponding numerical one for various parameters $m = 100, k_1 = 200, k_2 = 300, k_3 = 400, u_0 = -50, v_0 = 50$. 
References

1. Moochhala YE and Raynor S. Free vibration of multi-degree-of-freedom non-linear systems. *Int J Non-Linear Mech* 1972; 7: 651–661.

2. Huang TC. Harmonic oscillations of nonlinear two-degree-of-freedom systems. *J Appl Mech* 1995; 22: 107–110.

3. Gilchrist AO. The free oscillations of conservative quasilinear systems with two degrees of freedom. *Int J Mech Sci* 1961; 3: 286–311.

4. Efstrathiades GJ. Combination tones in single mode motion of a class of non-linear systems with two degrees of freedom. *J Sound Vib* 1974; 34: 379–397.

5. Alexander FV and Richard HR. Non-linear dynamics of a system of coupled oscillators with essential stiffness non-linearities. *Int J Non-Linear Mech* 2004; 39: 1079–1091.

6. Chen G. Applications of a generalized Galerkin's method to non-linear oscillations of two-degree-of-freedom systems. *J Sound Vib* 1987; 119: 225–242.

7. Ladygina YV and Manevich AI. Free oscillations of a non-linear cubic system with two degrees of freedom and close natural frequencies. *J Appl Math Mech* 1993; 57: 257–266.

8. Cveticanin L. Vibrations of a coupled two-degree-of-freedom system. *J Sound Vib* 2001; 247: 279–292.

9. Cveticanin L. The motion of a two-mass system with non-linear connection. *J Sound Vib* 2002; 252: 361–369.

10. Dimarogonas AD and Haddad S. *Vibration for engineers*. Englewood Cliffs, New Jersey: Prentice-Hall, 1992.

11. Telli S and Kopmaz O. Free vibrations of a mass grounded by linear and non-linear springs in series. *J Sound Vib* 2006; 289: 689–710.

12. Lai SK and Lim CW. Nonlinear vibration of a two-mass system with nonlinear stiffnesses. *Nonlinear Dyn* 2007; 49: 233–249.

13. Hashemi Kachapi SHA, Dukkipatic RV, Hashemi SG, et al. Analysis of the nonlinear vibration of a two-mass-spring system with linear and nonlinear stiffness. *Nonlinear Anal* 2010; 11: 1431–1441.

14. Bayata M, Shahidia M, Bararib A, et al. Analytical evaluation of the nonlinear vibration of coupled oscillator systems. *Z Naturforsch* 2011; 66a: 67–74.

15. Ganji SS, Bararib A and Ganji DD. Approximate analysis of two-mass-spring systems and buckling of a column. *Comp Math Appl* 2011; 61: 1088–1095.

16. Bayat M, Pakar I and Shahidi M. Analysis of nonlinear vibration of coupled systems with cubic nonlinearity. *Mechanics* 2011; 17: 620–629.

17. Wang Y and An JY. Amplitude-frequency relationship to a fractional Duffing oscillator arising in microphysics and tsunami motion. *J Low Frequency Noise Vib Active Control* 2018; DOI: 10.1177/1461348418795813.

18. Ren ZF and Hu GF. He's frequency-amplitude formulation with average residuals for nonlinear oscillators. *J Low Frequency Noise Vib Active Control* 2018; DOI:10.1177/1461348418812327.

19. Mickens RE. *Truly nonlinear oscillations*. Singapore: World Scientific Publishing, 2010.

20. Mickens RE. A generalization of the method of harmonic balance. *J Sound Vib* 1986; 111: 515–518.

21. Chowdhury MSH, Hosen MA, Ali MY, et al. An analytical technique to obtain higher-order approximate periods for nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$. *IJUM Eng J* 2018; 19: 182–191.

22. Hosen MA, Rahman MS, Alam MS, et al. An analytical technique for solving a class of strongly nonlinear conservative systems. *Appl Math Comp* 2012; 218: 5474–5486.

23. Javidi M. Iterative methods to nonlinear equations. *Appl Math Comp* 2007; 193: 360–365.

24. Wu Y and He JH. Homotopy perturbation method for nonlinear oscillators with coordinate-dependent mass. *Results Phys* 2018; 10: 270–271.

Appendix 1

A higher-order nonlinear algebraic equation is assumed as the following form

$$f(x) = 0 \quad (69)$$

Considering the nonlinear algebraic equation (69), we construct a homotopy

$$H: R \times [0, 1] \rightarrow R$$

which satisfy

$$H(x, p) = f(x) - f(x_0) + pf(x_0) = 0 \quad (70)$$

$$x \in R, \quad p \in [0, 1]$$
where \( p \) is the embedding parameter and \( x_0 \) is an initial approximation of equation (69). Hence, it is obvious that

\[
H(x, 0) = f(x) - f(x_0) = 0
\]  
(71)

\[
H(x, 1) = f(x) = 0
\]  
(72)

and the process of changing \( p \) from 0 to 1, is equivalent to changing \( H(x, p) \) from \( H(x, 0) \) to \( H(x, 1) \). Applying the perturbation technique\(^{23} \) due to the fact that \( 0 \leq p \leq 1 \) can be considered as a small parameter, we can assume that the solution of equation (70) can be expressed as a series in \( p \)

\[
x = x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \cdots
\]  
(73)

When \( p \to 1 \), equation (70) corresponds to equation (69) and equation (73) becomes the approximate solution of equation (69), that is\(^{23} \)

\[
x = \lim_{p \to 1} x = x_0 + x_1 + x_2 + x_3 + \cdots
\]  
(74)

and in Javidi\(^{23} \) the unknowns are

\[
x_1 = -\frac{f(x_0)}{f'(x_0)}
\]  
(75)

\[
x_2 = -\frac{f''(x_0)}{f'(x_0)} \left( \frac{f(x_0)}{f'(x_0)} \right)^2
\]  
(76)

\[
x_3 = \frac{1}{f'(x_0)} \left( \frac{1}{6} \left( \frac{f(x_0)}{f'(x_0)} \right)^3 \right) f''(x_0) + \frac{f(x_0)}{f'(x_0)} \left( -\frac{f''(x_0)}{f'(x_0)} \right) \left( \frac{f(x_0)}{f'(x_0)} \right)^2
\]  
(77)

\[
\ldots
\]