A convergence study of SGD-type methods for stochastic optimization

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Abstract

In this paper, we first reinvestigate the convergence of vanilla SGD method in the sense of $L^2$ under more general learning rates conditions and a more general convex assumption, which relieves the conditions on learning rates and do not need the problem to be strongly convex. Then, by taking advantage of the Lyapunov function technique, we present the convergence of the momentum SGD and Nesterov accelerated SGD methods for the convex and non-convex problem under $L$-smooth assumption that extends the bounded gradient limitation to a certain extent. The convergence of time averaged SGD was also analyzed.

Keywords: SGD, Momentum SGD, Nesterov acceleration, Time averaged SGD, Convergence analysis, Nonconvex

1 Introduction

In this article, we study the convergence analysis of stochastic gradient descent (SGD) type methods to the optimization problem

$$\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{S} \sum_{i=1}^{S} f_i(x), \quad (1.1)$$

where $f, f_i : \mathbb{R}^d \to \mathbb{R}$ are continuously differentiable functions and $S$ is the number of samples in machine learning. Recently, stochastic gradient descent (SGD) has played a significant role in training machine learning models when $S$ is very large and $x$ has many components. The SGD is derived from gradient descent by replacing $\nabla f$ with $\nabla f_{s_k}$, where $s_k$ is a random variable uniformly sampled from $\{1, 2, \ldots, S\}$. The iterative format is often read as

$$x_k = x_{k-1} - \alpha_k \nabla f_{s_k}(x_{k-1}) = x_{k-1} - \alpha_k \nabla f(x_{k-1}) + \alpha_k \xi_k, \quad (1.2)$$

where $\alpha_k$ is the learning rate, which satisfies the assumption (Divergence condition):

$$\lim_{k \to \infty} \alpha_k = 0, \quad \sum_{k=1}^{\infty} \alpha_k = \infty. \quad (1.3)$$

In (1.2), the term $\xi_k = \nabla f(x_{k-1}) - \nabla f_{s_k}(x_{k-1})$. Let $\mathcal{F}_k = \sigma(x_0, \xi_1, \xi_2, \ldots, \xi_k)$ be the filtration generated by $(x_0, \xi_1, \ldots, \xi_k)$, thus $\xi_k$ satisfies $E[\xi_k|\mathcal{F}_{k-1}] = 0$. 

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For iterative format (1.2), it has a mini-batch SGD [9] variant, which utilises \( \frac{1}{m} \sum_{i=1}^{m} \nabla f_{s_k}(x_k) \) to estimate gradient, where \( s_k \) are i.i.d random variables uniformly sampled from \{1, 2, \ldots, S\} and the noise term \( \xi_k = \nabla f(x_{k-1}) - \frac{1}{m} \sum_{i=1}^{m} \nabla f_{s_k}(x_k) \). For convenience, we will choose sample count \( m = 1 \) in this paper, and the results of this paper are consistent for cases where \( m > 1 \).

Many elegant works have been done on the forms of generalization and theoretical analysis of SGD-type methods [2, 4, 11, 15]. Here, the general Markovian iteration forms of SGD-type methods are denoted as

- vanilla SGD (vSGD)
  \[
  x_k = x_{k-1} - \alpha_k F(x_{k-1}, \xi_k),
  \tag{1.4}
  \]

- momentum SGD (mSGD) [19]
  \[
  x_k = x_{k-1} + v_k, \quad v_k = \beta_k v_{k-1} - \alpha_k F(x_{k-1}, \xi_k),
  \tag{1.5}
  \]

- Nesterov accelerated form (NaSGD) [14]
  \[
  y_k = x_k + \beta_k(x_k - x_{k-1}), \quad x_k = y_{k-1} - \alpha_k F(y_{k-1}, \xi_k),
  \tag{1.6}
  \]

respectively. Here \( \mathbb{E}[F(x_{k-1}, \xi_k)|F_{k-1}] = \nabla f(x_{k-1}) \) and \( \beta_k \in [0, 1) \) in (1.5) and (1.6). For the above mentioned SGD-type methods, we assume the noise term \( \{\xi_k\} \) satisfy the following conditional mean and covariance conditions

\[
\mathbb{E}[\xi_k|F_{k-1}] = 0, \quad \mathbb{E}[\|\xi_k\|^2|F_{k-1}] \leq M + V \|\nabla f(x_{k-1})\|^2,
\tag{1.7}
\]

which covers the usual incremental SGD [3] and \( \mathbb{E}[f(x_0)] < \infty \) for the initial value \( x_0 \) in this paper.

In recent years, vSGD (1.4) has attracted an increasing number of researchers. Bertsekas and Tsitsiklis [3] proved that \( \{x_k\} \) converges almost surely to a critical point of \( f \) when the stepsize \( \{\alpha_k\} \) satisfies \( \sum \alpha_k = \infty \) and \( \sum \alpha_k^2 < \infty \) even for non-convex problems. Ghadimi and Guanghui [7] analyzed the complexity of \( \{x_k\} \) to approximate stationary point of a nonlinear problem and showed that this method is non-asymptotically convergent with \( \min_{k \leq n} \mathbb{E}[\|\nabla f(x_k)\|^2] = O(1/\sqrt{n}) \) if the problem is non-convex. By using the variance reduction technique, Reddi et al. [21] showed that the convergence rate can be improved to \( O(1/n) \). For strongly convex problems, this convergence rate can be improved to \( \mathbb{E}[\|\nabla f(x_n)\|^2] = O(1/n) \) when the step size \( \alpha_k = O(1/k) \) [5, 11, 13, 17].

Compared with mSGD method, there are relatively few references about the mSGD and NaSGD. Barakat and Bianchi [2] presented a novel first order convergence rate result of a general class mSGD. Liu, Gao, and Yin [12] established the stationary convergence bound of this time averaged mSGD when the step sizes are constant. For Nesterov acceleration gradient, Su, Boyd, and Candès [24] showed that the convergence rate of \( f(x_n) \) towards \( f(x^*) \) is \( O(n^{-2}) \) in the deterministic case when \( f \) is convex. Assran and Rabbat [1] studied the stationary convergence bound of NaSGD with constant step size.

Besides the aforementioned convergence analysis, the CLT for the SGD-type methods was studied in [11] under more general divergence condition (1.3). It is a natural question to study the convergence of SGD-type methods under condition (1.3) and whether we can generalize this condition. There are few researches on the convergence of SGD-type methods under this condition. This is a motivation for this work, to give a further investigation on the convergence analysis for the SGD-type methods. We will consider the convergence analysis of the vSGD (1.4), NaSGD (1.6), and the mSGD as follows

\[
\begin{align*}
  x_k &= x_{k-1} + \alpha_k v_k, \quad v_k = (1 - \beta_k)v_{k-1} - \alpha_k F(x_{k-1}, \xi_k),
  \tag{1.8}
\end{align*}
\]
where $\beta_k = \mu_k \alpha_k$ and $\mu_k > 0$ is the damping parameter. We consider the form (1.8) instead of (1.5) mainly because it has a better connection to the continuous time limit.

The main predominant contributions of this paper are as follows.

- **Convergence of vSGD.** We investigate the convergent analysis of vSGD with different setups on the assumptions on noise and step size. Compared with the assumptions of the previous article regarding the step size $\{\alpha_k\}$, such as $\sum \alpha_k = \infty$ and $\sum \alpha_k^2 < \infty$ [3] for convex problem, we relieve the conditions on $\{\alpha_k\}$ and do not need $f(x)$ to be strongly convex.

- **Convergence of mSGD and NaSGD.** For mSGD, this part is classified into two cases: the case with constant damping $\mu_k \equiv \bar{\mu}$, or the case with vanishing damping $\mu_k \to 0$. Taking advantage of the Lyapunov function technique, we can show the convergent results for both cases. Unlike the previous convergence analysis [8], which requires the gradient to be bounded, we assume that $f(x)$ in (1.1) is $L$-smooth: there exists $L > 0$, such that

$$\forall x, y \in \mathbb{R}^d, \quad \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|,$$

which extends the bounded gradient limitation to a certain extent. For NaSGD, its stationary convergence bound was studied in [1] when the learning rates are constant, which limits its application in practical problems. Here, we show the convergent results under $L$-smooth (1.9) and more general learning rates conditions.

- **Convergence of the time average $x_k$.** Similar to [11, 22], we consider the time averaged SGD

$$\bar{x}_n = \frac{\sum_{k=1}^{n} \alpha_k x_{k-1}}{\sum_{k=1}^{n} \alpha_k},$$

which is an analogy of the continuous form $\int_0^T x(t) dt / T$, where $T$ is the summation of step size. Compared with the analysis in [8], we extend the bounded gradient limitation of $f(x)$ to $L$-smooth (1.9).

The rest of this paper is organized as follows. We will prove the convergence of the vSGD, mSGD and NaSGD, and the average $\bar{x}_n$ in Sections 2, 3 and 4, respectively. Finally, we make the conclusion. In the remainder of this paper, we will use $C$ as a $O(1)$ positive constant in different estimates, the value of which may vary in different places.

## 2 Convergence for vSGD

In this section, we will give the convergence analysis of vSGD, which will relieve the constraint on step size $\{\alpha_k\}$ in [3] by a more general convex assumption. The SGD convergence of non-convex function $f$ is already known. In [3], it proves that convergence in probability 1 can be achieved in the sense of the following two limits (2.1) and (2.2). The more general conclusion we give here will also be encountered in mSGD in the next section. It is worth noting that proving the weak convergence (2.1) does not require the condition $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. If some restrictions are imposed on $\{\alpha_k\}$, the convergence can be strengthened, for example, variance reduction SGD [10], SARAH [16] etc. These variants can get exponential convergence for strongly convex function as Gradient Descent and $O(1/t)$ for non-convex function [18].

To obtain the convergent theorems, we first give the following lemmas. The results of (a) and (b) in Lemma 1 are similar to those of Lemma 1 in reference [3]. The results of (c) is to prove the convergence of mSGD later.

**Lemma 1** (Sequence convergence). Assume the step size $\{\alpha_k\}$ satisfy (1.3), the positive sequence $Z_k \to 0$ and $X_k$ has a lower bound $X$. Consider the triplet $\{X_k, Y_k, Z_k\}_{k}$ with the relation

$$X_k \leq X_{k-1} - \alpha_k Y_k + \alpha_k Z_k, \quad Y_k \geq 0.$$
We have: (a) \( \lim_{k \to \infty} Y_k = 0 \), and there exists \( K > 0 \) such that \( \sum_{k=1}^{n} \alpha_k Y_k \leq K + \sum_{k=1}^{n} \alpha_k Z_k \).

(b) If \( \sum \alpha_k Z_k < +\infty \), then \( X_n \) is convergent. (c) Let \( \{Y_{n_k}\} \) and \( \{X_{n_k}\} \) be subsequences of \( \{Y_n\} \) and \( \{X_n\} \), respectively. If \( Y_{n_k} \to 0 \) is a sufficient condition for \( X_{n_k} \to X \), then \( \lim_{n \to \infty} X_n = X \).

**Proof.** For the proof of (b), it is a special case of Lemma 1 in [3].

(a) It is easy to know that for \( m < n \), \( X_n \leq X_{n-1} - \sum_{i=m}^{n} \alpha_i Y_i + \sum_{i=m}^{n} \alpha_i Z_i \), so we have

\[
\sum_{i=1}^{n} \alpha_i Y_i \leq \sum_{i=1}^{n} \alpha_i Z_i + X_0 - X_n \leq \sum_{i=1}^{n} \alpha_i Z_i + X_0 - X.
\]

If \( \lim_{k \to \infty} \sup Y_k > 0 \), it means that there exists \( \varepsilon > 0 \), \( K > 0 \) such that for all \( k \geq K, Y_k \geq \varepsilon \), then there exists \( M \geq K \), for \( m > M, Z_m \leq \varepsilon/2 \), and

\[
X_n \leq X_m - \sum_{i=m+1}^{n} \alpha_i Y_i + \sum_{i=m+1}^{n} \alpha_i Z_i \leq X_m - \frac{\varepsilon}{2} \sum_{i=m+1}^{n} \alpha_i, \text{ for all } n < m,
\]

thus when \( n \to +\infty \), \( X_n \to -\infty \), which leads to contradiction as \( X_n \) has a lower bound.

(b) Since \( X_n \leq X_0 + \sum_{k=1}^{\infty} \alpha_k Z_k \), we get \( \{X_n\} \) is bounded. From (a), we know that for \( \varepsilon > 0 \), there exists \( M > 0 \) such that

\[
X_n \leq X_m - \sum_{i=m+1}^{n} \alpha_i Y_i + \sum_{i=m+1}^{n} \alpha_i Z_i \leq X_m + \varepsilon, \text{ } n \geq m > M.
\]

If \( X_n \) is not convergent, then assume \( \limsup_k X_k = A_1, \liminf_k X_k = A_2 \) and take \( \varepsilon = (A_1 - A_2)/3 \). Now we can find infinite \( m \) fulfilled \( X_m < A_2 + \varepsilon \) with \( m > M \), then there exists \( n > m \) such that \( X_n > A_1 - \varepsilon \). Then \( X_n > X_m + \varepsilon \), which leads to contradiction.

(c) Without loss of generality, we set \( X = 0 \). From (a), we know \( \liminf_k X_k = 0 \). If there exists \( \varepsilon > 0 \) such that \( \limsup_n X_n \geq \varepsilon \), we can find infinite \( k \) such that \( X_k < \varepsilon/2, X_{m_k} \geq \varepsilon, X_i \in [\varepsilon/2, \varepsilon) \), \( k < i < m_k \) and there exists \( \delta > 0, Y_i \geq \delta \). By \( k \to +\infty \), \( Z_k \leq \delta/2 \), we have

\[
\varepsilon \leq X_{m_k} \leq X_k - \sum_{i=k+1}^{m_k} \alpha_i Y_i + \sum_{i=k+1}^{m_k} \alpha_i Z_i \leq X_k - \sum_{i=k+1}^{m_k} \alpha_i \delta/2 < \varepsilon/2,
\]

which leads to contradiction. \( \Box \)

**Lemma 2.** Assume the step size \( \{\alpha_k\} \) satisfy (1.3), and its partial sums is \( S_n = \sum_{k=1}^{n} \alpha_k \), then we have

\[
\sum_{k=1}^{n} \frac{\alpha_k}{S_k} \to +\infty \text{ as } n \to +\infty.
\]

**Proof.** For \( \forall m, n > 0 \), it is easy to get

\[
\sum_{k=m}^{n} \frac{\alpha_k}{S_k} \geq \frac{S_n - S_m}{S_n} = 1 - \frac{S_m}{S_n}.
\]

Due to \( S_n \to +\infty \), for a given \( m \), there exists \( n(m) \) and \( \delta \in (0, 1) \), such that \( S_m/S_n < 1 - \delta \). Letting \( m_k = n (m_{k-1} + 1) \), we obtain

\[
\sum_{k=1}^{m_k} \frac{\alpha_k}{S_k} \geq \sum_{k=1}^{K} \left( 1 - \frac{S_m}{S(n(m_k))} \right) \geq K \delta \to +\infty.
\]

\( \Box \)
2 CONVERGENCE FOR VSGD

Similar to [3], we give the convergence result of vSGD below. The difference is that we consider $L^2$ convergence instead of convergence with probability one. The reason why we consider $L^2$ convergence here is that it is more convenient to estimate the convergence rate as [7].

**Theorem 1.** If the function $f(x)$ is $L$-smooth and has a lower bound, and the assumptions (1.3), (1.7) hold, then for vSGD we have

$$
\lim_{n \to +\infty} \inf \mathbb{E}[\|\nabla f(x_n)\|^2] = 0. \tag{2.1}
$$

Furthermore, if \( \lim_{n \to +\infty} \sum_{k=1}^{n} \alpha_k^2 < \infty \), we have

$$
\lim_{n \to +\infty} \mathbb{E}[\|\nabla f(x_n)\|^2] = 0. \tag{2.2}
$$

**Proof.** If $f$ is $L$-smooth, it is easy to get

$$
\forall x, y \in \mathbb{R}^d, \quad f(x) \leq f(y) + \nabla f(y)^T(x - y) + \frac{L}{2}\|x - y\|^2. \tag{2.3}
$$

From (2.3), (1.7) and $\mathbb{E}[\xi_k|x_{k-1}|] = 0$, it is easy to know that

$$
\mathbb{E}[f(x_k)|F_{k-1}] \leq f(x_{k-1}) - \alpha_k \|\nabla f(x_{k-1})\|^2 + \frac{L\alpha_k^2}{2}(\|\nabla f(x_{k-1})\|^2 + \mathbb{E}[\|\xi_k\|^2|F_{k-1}])
$$

$$
\leq f(x_{k-1}) - C\alpha_k \|\nabla f(x_{k-1})\|^2 + \frac{LM}{2}\alpha_k^2,
$$

then we have

$$
\mathbb{E}[f(x_k) \leq \mathbb{E}[f(x_{k-1}) - C\alpha_k \|\nabla f(x_{k-1})\|^2 + \frac{LM}{2}\alpha_k^2. \tag{2.4}
$$

Taking $(X_k, Y_k, Z_k) = (\mathbb{E}[f(x_k), C\mathbb{E}[\|\nabla f(x_{k-1})\|^2, LM\alpha_k/2)$ in Lemma 1 (a), we get

$$
\lim_{k \to +\infty} \mathbb{E}[\|\nabla f(x_k)\|^2] = 0.
$$

For $\sum \alpha_k^2/2 < +\infty$, then by Lemma 1 (b), we get $\{\mathbb{E}[f(x_k)]\}$ is convergent. If there exists $\varepsilon > 0$, such that $\limsup Y_k \geq \varepsilon$, that means we can find infinite $k$ fulfilled $Y_k \leq \varepsilon/4, Y_m \geq \varepsilon, Y_i \in [\varepsilon/4, \varepsilon], i \in (k, m_k)$, we get

$$
\mathbb{E}[f(x_{m_k}) \leq \mathbb{E}[f(x_k) + \sum_{i=k+1}^{m_k} \alpha_k Z_k - K \sum_{i=k+1}^{m_k} \alpha_k Y_k
$$

with $k \to +\infty$, then we have

$$
\sum_{i=k}^{m_k} \alpha_k \varepsilon/4 \leq \sum_{i=k+1}^{m_k} \alpha_k (\mathbb{E}[\|\nabla f(x_{k-1})\|^2]) \to 0.
$$

With $L$-smooth condition, we obtain

$$
\mathbb{E}[(\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2) \leq L\alpha_k^2 \mathbb{E}[\|\nabla f(x_{k-1})\|^2 - \xi_k\|^2 \leq C\alpha_k^2 (Y_k + 1) \leq C\alpha_k^2.
$$

By Minkowski inequality, we have

$$
\sqrt{\varepsilon/2} \leq Y_k^{1/2} - Y_{m_k}^{1/2} \leq C \sum_{i=m_k+1}^{m_k} \alpha_k \to 0,
$$

which leads to contradiction and the proof of the second part is completed. □
The following theorem give the convergence result of vSGD when \( f(x) \) is convex.

**Theorem 2.** Assume \( f(x) \) is convex and \( L \)-smooth, and the assumptions (1.3), (1.7) hold. If \( f(x) \) has a lower bound and \( \alpha_n \sum_{k=1}^n \alpha_k^2 \to 0 \), then we have

\[
\lim_{n \to +\infty} \mathbb{E} f(x_n) = f(x^*) = f^*,
\]

where \( x^* \) is the minima.

**Proof.** The case for \( \sum \alpha_k^2 < +\infty \) has been proved in Theorem 1. Now we consider \( \sum \alpha_k^2 = +\infty \).

Setting \( \nu_k = \|x_k - x^*\| \), by (1.7), we have

\[
\mathbb{E} \nu_k^2 \leq \mathbb{E} \nu_{k-1}^2 - 2\alpha_k \mathbb{E} (x_k - x^*) \nabla f(x_{k-1}) + (1 + V) \alpha_k^2 \mathbb{E} \| \nabla f(x_{k-1}) \|^2 + M \alpha_k^2.
\]

By the convex condition, we get \((x - x^*) \nabla f(x) > 0\), which gives

\[
\mathbb{E} \nu_k^2 \leq \mathbb{E} \nu_{k-1}^2 + (1 + V) \alpha_k^2 \mathbb{E} \| \nabla f(x_{k-1}) \|^2 + M \alpha_k^2.
\]

Let \( c_n = 1 + \sum_{k=1}^n \alpha_k^2 \). According to Theorem 1, we know \( \sum_{k=1}^n \alpha_k \| \nabla f(x_{k-1}) \|^2 \leq C c_n \), which gives \( \mathbb{E} \nu_k^2 \leq C c_n \). Again with the convex condition, we have

\[
(\mathbb{E} f(x_{k-1}))^2 \leq \mathbb{E} \| \nabla f(x_{k-1}) \|^2 \mathbb{E} \nu_{k-1}^2.
\]

So

\[
\mathbb{E} f(x_k) \leq \mathbb{E} f(x_{k-1}) - C \alpha_k \mathbb{E} \| \nabla f(x_{k-1}) \|^2 + \frac{LM}{2} \alpha_k^2
\]

\[
\leq \mathbb{E} f(x_{k-1}) - C \alpha_k \frac{\mathbb{E} f(x_{k-1})}{\mathbb{E} \nu_{k-1}^2} + \frac{LM}{2} \alpha_k^2
\]

\[
\leq \mathbb{E} f(x_{k-1}) - C \alpha_k c_k^{-1} \mathbb{E} f(x_{k-1}) + \frac{LM}{2} \alpha_k^2.
\]

With condition \( \alpha_n c_n \to 0 \), taking

\[
(X_k, Y_k, Z_k) = (\mathbb{E} f(x_n), C \mathbb{E} f(x_{k-1}), LM \alpha_k c_n / 2)
\]

in Lemma 1 and using \( \{\alpha_n c_n^{-1}\} \) instead of \( \{\alpha_n\} \), it is easy to check that \( \sum \alpha_n c_n^{-1} = +\infty \) by Lemma 2, and we have

\[
X_k = \mathbb{E} f(x_n) \to X^* = f^*.
\]

\[\square\]

**Remark 1.** The condition \( \alpha_n \sum_{k=1}^n \alpha_k^2 \to 0 \) on step size in the above theorem is more general than \( \sum \alpha_k^2 < +\infty \) in [2]. For example, \( \alpha_n = C n^a, a \in (1/3, 1] \) can still ensure the convergence of vSGD for convex problems.

**Theorem 3.** For convex function \( f(x) \) with lower bound and the minima \( x^* \), if \( f(x) \) is \( L \)-smooth, the assumptions (1.3), (1.7) hold, and the following condition (more general than strongly convex) is fulfilled

\[
\exists K_0, \delta > 0, (f(x) - f^*)^2 \leq K_0 \| \nabla f(x) \|^2, \text{ for } x \in \{\| \nabla f(x) \|^2 \leq \delta\},
\]

then we have

\[
\lim_{n \to +\infty} \mathbb{E} f(x_n) = f(x^*) = f^*.
\]
3 CONVERGENCE OF MSGD AND NASGD

Proof. Without loss of generality, we set \( f^* = f(x^*) = 0 \). From Theorem 1, we know \( \lim_{k \to \infty} \| \nabla f(x_k) \| = 0 \). Since \( \cap_n \{ x \| \nabla f(x) < 1/n \} = \{ x \| \nabla f(x) \| = 0 \} \), we get \( \lim_{k \to \infty} \mathbb{E} \| f(x_k) \|^2 = 0 \). Set

\[
Y_k = \begin{cases} 
CK_0^{-1}E\|\nabla f(x_{k-1})\|^2 & \text{if } \|\nabla f(x_{k-1})\|^2 < \delta, \\
CE\|\nabla f(x_{k-1})\|^2 & \text{otherwise}, 
\end{cases}
\]

and \( X_k = \mathbb{E} f(x_n), Z_k = C\alpha_k \). By (2.4), according to (c) in Lemma 1, we have

\[ X_k \to X^* = f^*. \]

That means we have \( f(x_k) \to f^* \) in the sense of expectation. \( \square \)

From the above theorem, it can be seen that the constraints on the step size become less strict by strengthening the conditions of the function \( f(x) \).

In fact, the condition (2.6) is relatively general. For a convex function \( f(x) \) with second derivative, if there exists \( c > 0 \) such that \( D = \{ x \| \nabla f(x) < c \} \) is bounded, then (2.6) can be fulfilled. This situation is illustrated by the following example.

Let \( u = x - x^* \), and \( h(t) = f(tu) - f(x^*) \). We have \( h'(t) = \nabla f(tu)^T u \geq 0 \) and \( h''(t) = u^T \nabla^2 f(tu) u \geq 0 \),

\[
h(t)^2 = h'(\theta t)^2 t^2 = h'(t)^2 t^2 \geq (h'(t)^2 - h'(\theta t)^2) t^2,
\]

where \( \theta \in (0, 1) \). With \( h'(t)^2 t^2 = 2h''(t)h'(t) \), we get

\[
(h'(t)^2 - h'(\theta t)^2) t^2 = (1 - \theta) h''(\eta t) h'(\eta t) t^2 \geq 0,
\]

where \( \eta \in (\theta, 1) \), and

\[
h(t)^2 \leq h'(t)^2 t^2, \quad (f(x) - f^2)^2 \leq \| \nabla f(x) \|^2 \| x - x^* \|^2,
\]

since the set \( D \) is bounded. Let \( K_0 = \sup_{x \in D} \| x - x^* \|^2 \) then (2.6) is fulfilled. As the following corollary, we can also prove the convergence for strongly convex function.

Corollary 1. If \( f(x) \) satisfies conditions in Theorem 3 and is \( \mu \)-strongly convex, then \( x_k \) converge to \( x^* \) in \( L^2 \).

Proof. Because \( f(x) - f(x^*) \leq \frac{L}{2} \| x - x^* \|^2 \leq \frac{L}{2} \| x - x^* \| \| \nabla f(x) \| \), the set \( D \) exists. The condition (2.6) is fulfilled. So \( x_k \) converge to \( x^* \) in \( L^2 \). \( \square \)

3 Convergence of mSGD and NaSGD

3.1 Convergence of mSGD

Assume \( f \) is twice differentiable, \( L \)-smooth (1.9), and has lower bound \( f^* \). Consider the following mSGD iteration

\[
x_k = x_{k-1} + \alpha_k v_k, \\
v_k = v_{k-1} - \mu_k \gamma v_{k-1} - \alpha_k \nabla f(x_{k-1}) + \alpha_k \xi_k,
\]

where \( \mu_k \) is bounded by \( \bar{\mu} \) and \( \bar{\mu} \), i.e., \( 0 < \bar{\mu} \leq \mu_k \leq \bar{\mu} \). The reason why we consider format (3.1) is that it is implicitly mentioned in [6] and this format also has a continuum limit version like [23].
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Theorem 4. If $f(x)$ has lower bound and the assumptions (1.3), (1.7) hold, we have

$$\lim_{n \to +\infty} \mathbb{E}[\|\nabla f(x_n)\|^2] = 0.$$  \hspace{1cm}  \text{(3.2)}$$

If $\lim_{n \to \infty} \sum_{k=1}^{n} a_k^2 < \infty$, we get

$$\lim_{n \to +\infty} \mathbb{E}[\|\nabla f(x_n)\|^2] = 0.$$  \hspace{1cm}  \text{(3.3)}$$

If $f(x)$ is convex with the minima $x^*$ and satisfies the assumption

$$\exists K_0, \delta > 0, (f(x) - f^*)^2 \leq K_0 \|\nabla f(x)\|^2, \text{ for } x \in \{\|\nabla f(x)\|^2 \leq \delta\},$$

we have

$$\lim_{n \to +\infty} \mathbb{E}[f(x_n)] = f(x^*) = f^*.$$  \hspace{1cm}  \text{(3.4)}$$

Proof. We first perform Lyapunov analysis, which is similar to [11]. Define the Hamiltonian $H_k = f(x_k) - f(x^*) + \|v_k\|^2/2$ and $\dot{H}_k = \|\nabla f(x_k)\|^2 + \|v_k\|^2$. By (3.1), $L$-smoothness of $f$ and $\mathbb{E}[\|\xi_k\|^2 | F_{k-1}] \leq M + K_\xi \|\nabla f(x_{k-1})\|^2$, we have

$$\mathbb{E}[f(x_k) | F_{k-1}] \leq f(x_{k-1}) + \alpha_k \mathbb{E}[\|\nabla f(x_{k-1})\|^2 v_k | F_{k-1}] + \frac{L \alpha_k^2}{2} \mathbb{E}[\|\nabla f(x_{k-1})\|^2 | F_{k-1}]$$

$$\leq f(x_{k-1}) + \alpha_k \|\nabla f(x_{k-1})\|^2 v_k + \alpha_k^2 \left(1 + O(\tilde{H}_{k-1})\right)$$

and

$$\mathbb{E}\left[\frac{\|v_k\|^2}{2} | F_{k-1}\right] \leq \frac{\|v_{k-1}\|^2}{2} + \frac{1}{2} \mathbb{E}[\|v_k - v_{k-1}\|^2 | F_{k-1}] - \alpha_k \left(\mu_k \|v_{k-1}\|^2 + \|\nabla f(x_{k-1})v_{k-1}\right)$$

$$\leq \frac{\|v_{k-1}\|^2}{2} - \alpha_k \left(\mu_k \|v_{k-1}\|^2 + \|\nabla f(x_{k-1})v_{k-1}\right) + \alpha_k^2 \left(1 + O(\tilde{H}_{k-1})\right),$$

thus

$$\mathbb{E}[H_k | F_{k-1}] \leq H_{k-1} - \mu_k \alpha_k \|v_{k-1}\|^2 + \alpha_k^2 \left(1 + O(\tilde{H}_{k-1})\right).$$

By introducing the term $\tilde{Z}_k = v_k^T \nabla f(x_k)$, with $L$-smoothness of $f$, we get

$$\mathbb{E}[\tilde{Z}_k | F_{k-1}] = \tilde{Z}_{k-1} + \mathbb{E}[v_k^T (\nabla f(x_k) - \nabla f(x_{k-1}) | F_{k-1}) + \mathbb{E}[(v_k - v_{k-1})^T \nabla f(x_{k-1}) | F_{k-1}]$$

$$\leq \tilde{Z}_{k-1} + \alpha_k \left(L \mathbb{E}[\|v_k\|^2 | F_{k-1}] - \|\nabla f(x_{k-1})\|^2 \mu_k v_{k-1}^T \nabla f(x_{k-1})\right)$$

$$= \tilde{Z}_{k-1} + \alpha_k \left(L \|v_{k-1}\|^2 - \|\nabla f(x_{k-1})\|^2 \mu_k v_{k-1}^T \nabla f(x_{k-1})\right) + \alpha_k^2 \left(O(\tilde{H}_{k-1}) + o(1)\right).$$

Now we consider the Lyapunov function $H_k^E = \tilde{E} \tilde{H}_k$ with $\tilde{H}_k = H_k + \zeta \tilde{Z}_k$, where $\zeta > 0$ is small enough. Then

$$\mathbb{E}[\tilde{H}_k | F_{k-1}] \leq \tilde{H}_{k-1} - \alpha_k F_{k-1} + \alpha_k^2 \left(1 + O(\tilde{H}_{k-1})\right),$$

where $F_{k-1} = \zeta \|\nabla f(x_{k-1}) + \mu_k \|v_{k-1}\|^2 + (\tilde{\mu} - \zeta (L + L)) \|v_{k-1}\|^2$. We get $\tilde{F}_k = \Theta(\tilde{H}_k)$, where the notation $a_n = \Theta(b_n)$ means there exists $C_1, C_2 > 0$, such that $C_1 b_n \leq |a_n| \leq C_2 b_n$. So there exists $K > 0$ such that

$$H_k^E \leq H_k^{E_k} - K \alpha_k \tilde{E} \tilde{H}_{k-1} + \alpha_k^2.$$  \hspace{1cm}  \text{(3.5)}$$
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From the $L$-smoothness of $f$ and $f(x)$ with a lower bound $f^*$, we get $\tilde{H}_k \geq \frac{L}{2} \| \nabla f(x_k) \|^2 + C v_k^T \nabla f(x_k) + \| v_k \|^2/2$, which means $\tilde{H}_k, H^F_k$ have lower bound. Let $V_k = E[\| \nabla f(x_k) \|^2]$. Taking $(X_k, Y_k, Z_k) = (H^F_k, E\tilde{H}_{k-1}, C\alpha_k)$ in Lemma 1 (a), we get $\lim \inf \tilde{H}_k = 0$, which means $\lim \inf V_k = 0$.

For $\sum \alpha_k^2 < +\infty$, by in Lemma 1 (b), we get $X_k$ is convergent. If there exists $\varepsilon > 0$, such that $\lim \sup_n V_n \geq \varepsilon$ that means we can find infinite $k$ fulfilled $V_k \leq \varepsilon/4, V_{\alpha_k} \geq \varepsilon, V_k \in [\varepsilon/4, \varepsilon], i \in (k, m_k)$, we get

$$X_{\alpha_k} \leq X_k + \sum_{i=k+1}^{m_k} \alpha_i Z_i - \sum_{i=k+1}^{m_k} \alpha_i Y_i,$$

then we have

$$\sum_{i=k+1}^{m_k} \alpha_i \varepsilon/4 \leq \sum_{i=k+1}^{m_k} \alpha_i V_i \leq C \sum_{i=k+1}^{m_k} \alpha_i Y_i \rightarrow 0.$$

With $L$-smooth condition, we obtain

$$E[\| \nabla f(x_k) - \nabla f(x_{k-1}) \|^2] \leq 2L \alpha_k^2 E\| v_k \|^2 \leq C (E\tilde{H}_k + 1) \alpha_k^2 \leq C \alpha_k^2.$$

By Minkowski inequality, we have

$$\sqrt{\varepsilon}/2 \leq V_k^{1/2} - C \sum_{i=k+1}^{m_k} \alpha_i \rightarrow 0,$$

which leads to contradiction and the proof of the second part is completed.

For the last part with (3.2), similar to SGD, without loss of generality, we set $f^* = f(x^*) = 0$, which means the sharp bound of $\tilde{H}_k$ is 0. We take

$$Y_{k+1} = \begin{cases} 
K (E\| K_0^{-1} \| f(x_k) \|^2 + \| v_k \|^2) & \text{if } \| \nabla f(x_k) \|^2 < \delta, \\
KE\tilde{H}_k & \text{otherwise,} 
\end{cases}$$

(3.4)

$$X_k = E\tilde{H}_k, \text{ and } Z_k = C\alpha_k.$$ 

By (3.3), according to Lemma 1 (c), we have

$$X_k \rightarrow X^* = 0.$$

This means we have $f(x_k) \rightarrow f^*$ in $L^1$.

3.2 Convergence of NaSGD

In this section, we give the convergence of NaSGD, where $\mu_k$ is bounded by $\tilde{\mu}$ and $\tilde{\mu}$, i.e., $0 < \mu \leq \mu_k \leq \tilde{\mu}$. Assume $f$ is twice differentiable and satisfies $L$-smooth (1.9). Further assume $\lim \sup \alpha_k/\alpha_{k-1} < +\infty$, then $\beta_k = (1 - \mu_k \alpha_k) \frac{\alpha_k}{\alpha_{k-1}}$. Let $\hat{\beta} = \lim \sup \beta_k$. Consider the following NaSGD iteration

$$x_k = x_{k-1} + \alpha_k v_k, \quad v_k = (1 - \mu_k \alpha_k) v_{k-1} - \alpha_k \nabla f(x_{k-1} + \beta_k (x_{k-1} - x_{k-2})) + \alpha_k \xi_k.$$

The following theorem presents the convergent results under $L$-smooth (1.9) and a more general learning rates conditions, which is previously unknown.
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Theorem 5. If \( f(x) \) has lower bound \( f^* \), the assumptions (1.3), (1.7) hold, and \( L\hat{\beta} < \hat{\mu} \) or \( f(x) \) is convex, then we have

\[
\liminf_{n \to +\infty} \mathbb{E}\|\nabla f(x_n)\|^2 = 0.
\]

If \( \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^2 < \infty \), then we get

\[
\lim_{n \to +\infty} \mathbb{E}\|\nabla f(x_n)\|^2 = 0.
\]

If \( f(x) \) is convex and satisfies the assumption

\[
\exists K_0, \delta > 0, (f(x) - f^*)^2 \leq K_0 \|\nabla f(x)\|^2, \text{ for } x \in \{\|\nabla f(x)\|^2 \leq \delta\},
\]

then we obtain

\[
\lim_{n \to +\infty} \mathbb{E}f(x_n) = f(x^*) = f^*.
\]

Proof. For \( L\hat{\beta} < \hat{\mu} \), consider the Hamiltonian function \( H_k = f(x_k) - f^* + \|v_k\|^2/2 \). Let \( \tilde{H}_k = \|\nabla f(x_k)\|^2 + \|v_k\|^2 \). We have

\[
\mathbb{E}[H_k|F_{k-1}] \leq H_{k-1} + \alpha_k [v_{k-1}^T(\nabla f(x_{k-1}) - \nabla f(y_{k-1})) - \mu_k\|v_{k-1}\|^2] + o(\alpha_k)
\]

\[
\leq H_{k-1} - (\hat{\mu} - L\hat{\beta}) \alpha_k\|v_{k-1}\|^2 + C\alpha_k^2 (1 + O(H_{k-1})).
\]

When \( f \) is convex, we have \( v_{k-1}^T(\nabla f(x_{k-1}) - \nabla f(y_{k-1})) < 0 \), and the same result can be obtained except that the coefficient of the \( \|v_{k-1}\|^2 \) term in the above formula is \(-\hat{\mu}\).

By introducing the term \( \tilde{Z}_k = v_{k}^T \nabla f(x_k) \), we get

\[
\mathbb{E}[\tilde{Z}_k|F_{k-1}] \leq Z_{k-1} + \alpha_k (L\|v_{k-1}\|^2 - \nabla f(x_{k-1})^T \nabla f(y_{k-1}) - \hat{\mu} v_{k-1}^T \nabla f(x_{k-1}) + C\alpha_k^2 \Omega(\tilde{H}_{k-1}).
\]

From \( L \)-smoothness of \( f \), we have

\[
|\nabla f(x_{k-1})^T(\nabla f(x_{k-1}) - \nabla f(y_{k-1}))| \leq L\hat{\beta}_k [v_{k-1}^T \nabla f(x_{k-1})] \leq \hat{\beta} L \alpha_k \|\nabla f(x_{k-1})\|^2 + \frac{\hat{\beta} L \alpha_k}{\lambda} v_{k-1}^2.
\]

Set \( \lambda = 1/(2L\hat{\beta}) \), then we obtain

\[
\mathbb{E}[\tilde{Z}_k|F_{k-1}] \leq Z_{k-1} + \alpha_k \left( (L + \frac{L\hat{\beta}}{\lambda})\|v_{k-1}\|^2 - (1 - L\hat{\beta}\lambda) \|\nabla f(x_{k-1})\|^2 - \hat{\mu} v_{k-1}^T \nabla f(x_{k-1}) \right)
\]

\[
+ C\alpha_k^2 (1 + O(H_{k-1})).
\]

Now we consider the Lyapunov function \( H_k^E = \mathbb{E}\tilde{H}_k = \mathbb{E}(H_k + \zeta \tilde{Z}_k) \) with small enough \( \zeta > 0 \). Similar to the analysis of mSGD, there exists \( K > 0 \) such that

\[
H_k^E \leq H_{k-1}^E - K\alpha_k \mathbb{E}H_{k-1} + C\alpha_k^2.
\]

The rest analysis is similar to mSGD, that is, using the lemma 1 to obtain the convergence. We take \( (X_k, Y_k, Z_k) = (H_k^E, \mathbb{E}H_{k-1}, C\alpha_k) \) in Lemma 1 (a), (b) and for the last part we change \( Y_k \) as (3.4) and use Lemma 1 (c), to obtain the convergence with convexity. So we omit it. \( \square \)
3.3 Convergence of mSGD with vanishing damping $\mu_k \to 0$

In this section, we will give the convergence of mSGD with vanishing damping $\mu_k \to 0$. Assume $f$ is twice differentiable. For the vanishing damping case, we need to make some modifications to (1.3). The assumption corresponding to the divergence condition (1.3) of mSGD is

$$\begin{align*}
\lim_{k \to \infty} \alpha_k &= 0, & \lim_{k \to \infty} \mu_k &= 0, & \lim_{k \to \infty} \frac{\alpha_k}{\mu_k} &= 0, & \sum_{k=1}^{\infty} \alpha_k \mu_k &= \infty,
\end{align*}$$

(3.5)

Theorem 6. Suppose that function $f(x)$ is $L$-smooth and has minima $x^*$. If (1.3), (1.7) hold and $\mu_k$ satisfy (3.5), then we have

$$\liminf_{n \to +\infty} \mathbb{E}[\|\nabla f(x_n)\|^2] = 0.$$  

If $f(x)$ is convex and satisfies the assumption

$$\exists K_0, \delta > 0, (f(x) - f^*)^2 \leq K_0 \|\nabla f(x)\|^2, \text{ for } x \in \{|\|\nabla f(x)\|^2 \leq \delta\},$$

then we have

$$\lim_{n \to +\infty} \mathbb{E}[f(x_n)] = f(x^*) = f^*.$$  

Proof. For mSGD with $\mu_k \to 0$, consider the Lyapunov function $H_k^F = \mathbb{E}[\tilde{H}_k]$, where $\tilde{H}_k = H_k + \lambda \mu_k Z_k$, with $0 < \lambda < 1/L$. Similar to mSGD, we have

$$\mathbb{E}[\tilde{H}_k | F_{k-1}] \leq \tilde{H}_{k-1} - \alpha_k \tilde{F}_{k-1} - \lambda (\mu_k - \mu_{k-1}) v_{k-1}^T \nabla f(x_{k-1}) + C(1 + O(\tilde{H}_{k-1})) \alpha_k^2,$$

where $\tilde{F}_{k-1} = \lambda \mu_k (\|\nabla f(x_{k-1})\|^2 + v_{k-1}^T \nabla f(x_{k-1})) + (1 - \lambda) \mu_k \|v_{k-1}\|^2$.

Further, let $\lambda < (L + \mu_k^2/4)^{-1}$. By (3.5), we have $\tilde{F}_k + \lambda (\mu_{k+1} - \mu_k) Z_k = \mu_{k+1} \Theta(\tilde{H}_k)$. Similar to the analysis of mSGD, there exists $K > 0$ such that

$$\mathbb{E}[\tilde{H}_k | F_{k-1}] \leq \tilde{H}_{k-1} - K \alpha_k \mu_k \mathbb{E}[\tilde{H}_{k-1}] + C(1 + O(\tilde{H}_{k-1})) \alpha_k^2.$$

Then we have

$$H_k^F \leq \tilde{H}_{k-1} - K \alpha_k \mu_k \mathbb{E}[\tilde{H}_{k-1}] + C \alpha_k^2.$$  

Take $(X_k, Y_k, Z_k) = (H_k^F, \mathbb{E}[\tilde{H}_{k-1}], C \alpha_k / \mu_k)$, and use $\{\alpha_k \mu_k\}$ instead of $\{\alpha_k\}$ in Lemma 1 to get convergence, and for the last part we change $Y_k$ to (3.4) and use Lemma 1 (c), to obtain the convergence with convexity. The rest analysis is similar to mSGD, so we omit it. \qed

Remark 2. The conditions in (3.5) about $\{\alpha_k\}$ and $\{\mu_k\}$ are reasonable. For specific examples, see [11].

4 Convergence of Average SGD

The time average SGD we consider in this section differs from the previous average $\hat{x}_n = \sum_k x_k/n$ in [20]. The form $\hat{x}_n$ has better properties in convex problems, which keeps the convergence $\mathbb{E}[f(\hat{x}_n)] \Rightarrow \mathbb{E}[f(x^*)]$ automatically. The previous researches, such as [22], need $\nabla f(x)$ to be bounded to ensure the convergence. We can use $L$-smooth to replace this.
Theorem 7. If the convex function $f(x)$ is $L$-smooth and has a minima $x^*$, and (1.3) and (1.7) hold, then for Average SGD we have
\[ \lim_{n \to +\infty} \mathbb{E} f(\bar{x}_n) = f(x^*) = f^*, \]
where $x^*$ is the minima.

Proof. For $L$-smooth convex function $f(x)$ with minima $x^*$, we have
\[ f(\bar{x}_n) \leq \frac{\sum_{k=1}^{n} \alpha_k (f(x_k) - f(x^*))}{\sum_{k=1}^{n} \alpha_k}, \]
and
\[ \alpha_k (\mathbb{E} f(x_k) - f(x^*)) \leq \mathbb{E} \|x_{k-1} - x^*\|^2 - \mathbb{E} \|x_k - x^*\|^2 + C\alpha_k^2 (1 + \mathbb{E} \|\nabla f(x_k)\|^2). \] (4.1)

From Theorem 1 and (2.4), we can get $\sum \alpha_k \mathbb{E} \|\nabla f(x_k)\|^2 < +\infty$. With Assumptions (1.3), we obtain $\sum \alpha_k^2 \mathbb{E} \|\nabla f(x_k)\|^2 < +\infty$.

By summing both sides of (4.1) at the same time, we get
\[ \sum_{k=1}^{n} \alpha_k (\mathbb{E} f(x_k) - f(x^*)) \leq C + \sum_{k=1}^{n} \alpha_k^2 - \mathbb{E} \|x_n - x^*\|^2, \]
dividing both sides of the above equation by $\sum_{k=1}^{n} \alpha_k$, we have
\[ \mathbb{E} f(\bar{x}_n) - f(x^*) \leq C \left(1 + \frac{\sum_{k=1}^{n} \alpha_k^2}{\sum_{k=1}^{n} \alpha_k}\right) \to 0. \]

\[ \square \]

5 Conclusion

In this article, we studied the convergence of the vSGD method under more general learning rates conditions and a more general convex assumption. We also investigated the convergence of the mSGD and NaSGD method with usual damping $\mu_k$ and vanishing damping $\mu_k \to 0$ by taking advantage of the Lyapunov function technique, which has been less studied. The convergence of time averaged SGD was also analyzed. The application and further extension of the results obtained in this paper will be the next work.

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