Periodic orbits and non-integrability of Armbruster-Guckenheimer-Kim potential

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Abstract In this paper we study the periodic orbits of the Hamiltonian system with the Armbruster-Guckenheimer-Kim potential and its $C^1$ non-integrability in the sense of Liouville-Arnold.

Keywords Periodic solution · Averaging method · Non-integrability

1 Introduction and statement of the main results

The main goal of this work is to study the periodic orbits and the non-integrability of the Hamiltonian system with the potential energy given by the Armbruster-Guckenheimer-Kim potential, see Armbruster et al. (1989), which has often been used in the study of the dynamics of galaxies. We investigate the periodic orbits using the averaging theory and the non-integrability is studied through the existence of periodic orbits that do not have all their multipliers equal to 1.

This Hamiltonian consists of a two dimensional harmonic potential plus the following quartic terms

$$\mathcal{H} = \frac{1}{2} (p_x^2 + p_y^2 + x^2 + y^2) - \frac{a}{4} (x^2 + y^2)^2 - \frac{b}{2} x^2 y^2. \quad (1)$$

The Hamiltonian system is given by

$$\dot{x} = p_x,$$
$$\dot{p}_x = -x + ax (x^2 + y^2) + bxy^2,$$
$$\dot{y} = p_y,$$
$$\dot{p}_y = -y + ay (x^2 + y^2) + bx^2 y. \quad (2)$$

As usual the dot denotes derivative with respect to the independent variable $t$, the time. We name (2) the Armbruster-Guckenheimer-Kim Hamiltonian systems, or simply the AGK systems.

In this work we use the averaging method of first order to compute periodic orbits, see Sect. 2. This method allows to find analytically periodic orbits of the AGK systems (2) at any positive values of the energy as a function of the parameters $a$ and $b$. Roughly speaking this method reduces the problem of finding periodic solutions of some differential system to the one of finding zeros of some convenient finite dimensional function. This method was also used by Jiménez-Lara and Llibre (2011a, 2011b).

We divide the plane of parameters $(a, b)$ in the following four parts: the two straight lines

$$L_1 = \{ (a, b) \in \mathbb{R}^2 : b + a = 0 \},$$
$$L_2 = \{ (a, b) \in \mathbb{R}^2 : b = 0 \},$$
and the two regions

$$R_1 = \{ (a, b) \in \mathbb{R}^2 : b - 2a \leq 0, b > 0 \} \cup \{ (a, b) \in \mathbb{R}^2 : b - 2a \geq 0, b < 0 \}.$$
$R_2 = \mathbb{R}^2 \setminus \{ \text{Cl}(R_1) \cup L_1 \}$.

Here the closure of a subset $R$ of $\mathbb{R}^2$ is denoted by $\text{Cl}(R)$.

Our main result on the periodic orbits of the AGK system (2) is summarized as follows.

**Theorem 1** At every positive energy level the Armbruster–Guckenheimer–Kim Hamiltonian system (2) has at least 2 periodic orbits if $(a, b) \in L_2$, 4 periodic orbits if $(a, b) \in R_2$, and 6 periodic orbits if $(a, b) \in R_1$.

Theorem 1 is proved in Sect. 2.

In particular, Theorem 1 states that if $(a, b) \not\in L_1$ then the Hamiltonian system at any positive energy level has periodic orbits and we can use these particular periodic orbits to prove our second main result about the non-integrability in the sense of Liouville-Arnold of the AGK system (2).

**Theorem 2** The Armbruster-Guckenheimer-Kim Hamiltonian system (2) with Hamiltonian $\mathcal{H}$ given by (1) cannot have a $C^1$ second first integral $G$ such that the gradients of $\mathcal{H}$ and $G$ are linearly independent at each point of the periodic orbits found in Theorem 1.

Theorem 2 is proved in Sect. 3.

The proof of Theorem 1 is based on the averaging theory for computing periodic orbits, see the Sect. 2. And the proof of Theorem 2 is based on the Poincaré’s Method that allows to prove the non Liouville-Arnold integrability independently of the class of differentiability of the second first integral, see Sect. 1 for more details. The main difficulty for applying Poincaré’s non-integrability method to a given Hamiltonian system is to find for such system periodic orbits having multipliers different from 1. For applying the Poincaré non-integrability theory to AGK-system, we need to study some of the periodic orbits of these system and to compute their multipliers.

### 2 Proof of Theorem 1

To apply Theorem 4 we need a small parameter $\varepsilon$. In system (2) we consider the change of variables $(x, p_x, y, p_y) \mapsto (X, p_X, Y, p_Y) = (\sqrt{x}, p_x/\sqrt{x}, y/\sqrt{x}, p_y/\sqrt{x})$. In the new variables, system (2) becomes

\[
\begin{align*}
\dot{X} &= p_X, \\
\dot{p}_X &= -X + \varepsilon(aX^2 + (a + b)XY^2), \\
\dot{Y} &= p_Y, \\
\dot{p}_Y &= -Y + \varepsilon((a + b)X^2Y + aY^3).
\end{align*}
\]

This system again is Hamiltonian with the Hamiltonian

\[
\frac{1}{2}(p_X^2 + p_Y^2 + X^2 + Y^2) - \varepsilon \frac{a}{4}(X^2 + Y^2)^2 - \varepsilon \frac{b}{2}X^2Y^2.
\]

As the change of variables is only a scale transformation, for all $\varepsilon$ different from zero, the original and the transformed systems (2) and (3) have essentially the same phase portrait, and additionally system (3) for $\varepsilon$ sufficiently small is close to an integrable one.

Notice that system (3) is not in the normal form for applying the averaging theory, see the differential equation (18). We consider the change of variables

\[
\begin{align*}
X &= r \cos \theta, & p_X &= r \sin \theta, \\
Y &= \rho \cos(\theta + \alpha), & p_Y &= \rho \sin(\theta + \alpha).
\end{align*}
\]

Recall that this is a change of variables when $r > 0$ and $\rho > 0$. Moreover doing this change of variables appear in the system the periodic variables $\theta$ and $\alpha$. Later on the variable $\theta$ will be used for obtaining the periodicity necessary for applying the averaging theory.

The energy or Hamiltonian in these new variables becomes

\[
\begin{align*}
H &= \frac{1}{2}(r^2 + \rho^2) - \frac{1}{4}\varepsilon a(r^2 \cos^2 \theta + \rho^2 \cos^2(\theta + \alpha))^2 \\
&\quad - \frac{1}{2}\varepsilon b r^2 \rho^2 \cos^2 \theta \cos^2(\theta + \alpha),
\end{align*}
\]

and the equations of motion are given by

\[
\begin{align*}
\dot{r} &= \varepsilon r \sin \theta \cos \theta (a r^2 \cos^2 \theta + (a + b) \rho^2 \cos^2(\theta + \alpha)), \\
\dot{\theta} &= -1 + \varepsilon(a r^2 \cos^4 \theta + (a + b) \rho^2 \cos^2 \theta \cos^2(\theta + \alpha)), \\
\dot{\rho} &= \varepsilon \rho \sin(\theta + \alpha) \cos(\theta + \alpha)((a + b) r^2 \cos^2 \theta \\
&\quad + a \rho^2 \cos^2(\theta + \alpha)), \\
\dot{\alpha} &= \varepsilon(-a r^2 \cos^4 \theta + (a + b)(r^2 - \rho^2) \cos^2 \theta \cos^2(\theta + \alpha) \\
&\quad + a \rho^2 \cos^4(\theta + \alpha)).
\end{align*}
\]
However the derivatives of the left hand side of these equations are with respect to the time variable $t$, which is not periodic. We change to the $\theta$ variable as the independent one, and we denote by a prime the derivative with respect to $\theta$. Then system (6) goes over to

\[
\begin{align*}
\dot{r} &= \frac{\varepsilon r \sin \theta \cos \theta (a r^2 \cos^2 \theta + (a + b) r^2 \cos^2 (\theta + \alpha))}{-1 + a r^2 \varepsilon \cos^4 \theta + (a + b) r^2 \varepsilon \cos^2 \theta \cos^2 (\theta + \alpha)}, \\
\dot{\rho} &= \frac{\varepsilon \rho \sin (\theta + \alpha) \cos (\theta + \alpha) ((a + b) r^2 \cos^2 \theta + a r^2 \cos^2 (\theta + \alpha))}{-1 + a r^2 \varepsilon \cos^4 \theta + (a + b) r^2 \varepsilon \cos^2 \theta \cos^2 (\theta + \alpha)}, \\
\dot{\alpha} &= \frac{\varepsilon (a r^2 \cos^4 \theta + (a + b) (r^2 - \rho^2) \cos^2 \theta \cos^2 (\theta + \alpha) + a r^2 \cos^4 (\theta + \alpha))}{-1 + a r^2 \varepsilon \cos^4 \theta + (a + b) r^2 \varepsilon \cos^2 \theta \cos^2 (\theta + \alpha)}.
\end{align*}
\]

Of course this system has now only three equations because we do not need the $\theta$ equation. If we write the previous system as a Taylor series in powers of $\varepsilon$, we have

\[
\begin{align*}
\dot{r} &= -\varepsilon r \sin \theta \cos \theta (a r^2 \cos^2 \theta + (a + b) r^2 \cos^2 (\theta + \alpha)) + O(\varepsilon^2), \\
\dot{\rho} &= -\varepsilon \rho \sin (\theta + \alpha) \cos (\theta + \alpha) ((a + b) r^2 \cos^2 \theta + a r^2 \cos^2 (\theta + \alpha)) + O(\varepsilon^2), \\
\dot{\alpha} &= \varepsilon (a r^2 \cos^4 \theta - (a + b) (r^2 - \rho^2) \cos^2 \theta \cos^2 (\theta + \alpha) - a r^2 \cos^4 (\theta + \alpha)) + O(\varepsilon^2).
\end{align*}
\]

Now system (7) is $2\pi$-periodic in the variable $\theta$. We shall apply Theorem 4 in the Hamiltonian level $H = h$ for $h > 0$, and by solving equation $H = h$ for $\rho$ we obtain

\[
\rho = \sqrt{-\frac{\sec^4 (\alpha + \theta) A + \varepsilon r^2 \cos^2 \theta (a + b) \sec^2 (\alpha + \theta)}{a\varepsilon}},
\]

where

\[
A = -1 + \left[1 + \varepsilon \cos^4 (\alpha + \theta) (\varepsilon b (2a + b) r^4 \cos^4 \theta - 2a (r^2 - 2h))) - 2\varepsilon r^2 \cos^2 \theta (a + b) \cos^2 (\alpha + \theta)\right]^\frac{1}{2}.
\]

Then substituting $\rho$ in (7) and developing in power series of $\varepsilon$, we obtain the two differential equations

\[
\begin{align*}
\dot{r} &= \varepsilon r \sin \theta \cos \theta ((a + b) (r^2 - 2h) \cos^2 (\theta + \alpha)) \\
&\quad - a r^2 \cos^2 \theta + O(\varepsilon^2), \\
\dot{\alpha} &= \varepsilon (2 (a + b) (h - r^2) \cos^2 \theta \cos^2 (\theta + \alpha) \\
&\quad + a (r^2 - 2h) \cos^4 (\alpha + \theta) + a r^2 \cos^4 \theta) + O(\varepsilon^2).
\end{align*}
\]

Clearly system (9) satisfies the assumptions of Theorem 4 and it has the form (18) with $F_1 = (F_{11}, F_{12})$, given by

\[
\begin{align*}
F_{11} &= r \sin \theta \cos \theta ((a + b) (r^2 - 2h) \\
&\quad \times \cos^2 (\alpha + \theta) - a r^2 \cos^2 \theta), \\
F_{12} &= 2 (a + b) (h - r^2) \cos^2 \theta \cos^2 (\alpha + \theta) \\
&\quad + a (r^2 - 2h) \cos^4 (\alpha + \theta) + a r^2 \cos^4 \theta.
\end{align*}
\]

Notice that $F_1$ is $2\pi$-periodic in the variable $\theta$, the independent variable of system (9). Averaging the function $F_1$ with respect to the variable $\theta$ we have

\[
f_1(r, \alpha) = \left(f_{11}(r, \alpha), f_{12}(r, \alpha)\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (F_{11}, F_{12}) \, d\theta,
\]

where

\[
f_{11}(r, \alpha) = -\frac{1}{8} r (a + b) (r^2 - 2h) \sin (2\alpha),
\]

and

\[
f_{12}(r, \alpha) = \frac{1}{4} (h - r^2) ((a + b) \cos (2\alpha) - a + 2b).
\]

We must find the zeros $(r^*, \alpha^*)$ of $f_1(r, \alpha)$, and check that the Jacobian at these points is not zero, i.e.

\[
\det \left( \frac{\partial (f_{11}, f_{12})}{\partial (r, \alpha)} \right)_{(r, \alpha) = (r^*, \alpha^*)} \neq 0.
\]

From $f_{11}(r, \alpha) = 0$ we obtain that either $r = 0$ or $r = \pm \sqrt{2h}$ or $\alpha = 0, \pi/2, -\pi/2, \pi$. The solutions $r = 0$ and $r = -\sqrt{2h}$ are not good, because $r > 0$. So, the good solutions of $f_{11}(r, \alpha) = 0$ are $r = \sqrt{2h}$ and $\alpha = 0, \pi/2, -\pi/2, \pi$.

Now we look for the solutions of $f_{12}(r, \alpha) = 0$. We obtain nine possible solutions $(r^*, \alpha^*)$ with $r^* > 0$:

\[
s_1 = (\sqrt{2h}, 0), \quad s_2 = (\sqrt{2h}, \pi), \quad s_3 = (\sqrt{2h}, \pi/2), \\
s_4 = (\sqrt{2h}, 3\pi/2), \quad s_5 = (\sqrt{2h}, 0), \quad s_6 = (\sqrt{2h}, \pi)
\]
\[ s_7 = (\sqrt{2h}, \pi/2), \]
\[ s_8 = \left( \sqrt{2h}, \frac{1}{2} \arccos \left( \frac{a-2b}{a+b} \right) \right), \]
\[ s_9 = \left( \sqrt{2h}, 2\pi - \frac{1}{2} \arccos \left( \frac{a-2b}{a+b} \right) \right), \]

with corresponding values of \( \rho \) given by (8) tending to \( \sqrt{h} \) for the solutions \( s_1, s_2, s_3, s_4 \) when \( \varepsilon \to 0 \) and tending to 0 for the solutions \( s_5, s_6, s_7, s_8, s_9 \) when \( \varepsilon \to 0 \). Of course in (11) for the solutions \( s_8 \) and \( s_9 \) we assume that \(-1 \leq (a-2b)/(a+b) \leq 1\). These inequalities only occur in the two closed sectors limited by the two straight lines \( b = 0 \) and \( b = 2a \) minus the origin and contained in the quadrants \( \{(a, b) : a > 0, b > 0\} \) and \( \{(a, b) : a < 0, b < 0\} \) of the plane \((a, b)\).

Finally we calculate the Jacobian (10), i.e.

\[
\begin{pmatrix}
\frac{1}{4}(a+b)(2h-3r^2)\sin(2\alpha) & -\frac{1}{2}(a+b)\pi r (r^2-2h)\cos(2\alpha) \\
-\pi r (-a+2b+(a+b)\cos(2\alpha)) & -(a-b)\pi (h-r^2)\sin(2\alpha)
\end{pmatrix}
\]

(12)

at the nine solutions \( s_1, \ldots, s_9 \). Then we obtain the Jacobian

\[
\frac{3}{8}h^2 b(a+b)
\]

(13)
at the solutions \( s_1 \) and \( s_2 \), the Jacobian

\[
\frac{1}{8}h^2 (2a-b)(a+b)
\]

(14)
at the solutions \( s_3 \) and \( s_4 \), the Jacobian 0 at the solutions \( s_5, s_6 \) and \( s_7 \), and the Jacobian

\[
\frac{3}{4}h^2 b(b-2a)
\]

(15)
at the solutions \( s_8 \) and \( s_9 \).

Notice that the above Jacobian at the solutions \( s_5, s_6 \) and \( s_7 \) are zero, then we cannot use Theorem 4 for these solutions. However we have that for \( h \neq 0 \) the Jacobian is non-zero at \( s_1 \) and \( s_2 \) when \( b(a+b) \neq 0 \), the Jacobian is non-zero at \( s_3 \) and \( s_4 \) when \( (2a-b)(a+b) \neq 0 \), and the Jacobian is non-zero at \( s_5, s_6, s_7, s_8, s_9 \) when \( b(b-2a) \neq 0 \).

Summarizing, from Theorem 4, the solutions \( s_1 \) and \( s_2 \) of \( f(r^*, \alpha^*) = 0 \) provide two periodic solutions of system (9) (and consequently of the Hamiltonian system (3) on the level \( h > 0 \)) if \( b \neq 0 \) and \( b = 2a \). Similarly, the solutions \( s_3 \) and \( s_4 \) provide two periodic solutions of system (9) if \( a \neq 0 \) and \( b \neq 0 \). And, if \( b(a+b)(2a-b) \neq 0 \) the solutions \( s_i \) for \( i = 1, 2, 3, 4, 8, 9 \) of \( f_i = 0 \) provide at least six periodic solutions for the Hamiltonian system (3). This completes the proof of Theorem 1.

Note that if \( a + b = 0 \), then we do not have any periodic solution given by \( s_i \) for \( i = 1, 2, 3, 4, 8, 9 \) because either their Jacobian is zero (for \( i = 1, 2, 3, 4 \)) or they are not defined (for \( i = 8, 9 \), see (11)).

### 3 Proof of Theorem 2

We assume that we are under the assumptions of Theorem 1, and that one of the six found periodic solutions corresponding to the solutions \( s_1, s_2, s_3, s_4, s_8, s_9 \) exist, and that their associated Jacobians (13), (14) and (15) are non-zero. So the corresponding multipliers are not all equal to 1. Hence, under the assumptions of Theorem 1, from Theorem 3 it follows Theorem 2.

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### Appendix 1: Periodic orbits and the Liouville-Arnold integrability

We recall that a Hamiltonian system with Hamiltonian \( H \) of two degrees of freedom is **integrable in the sense of Liouville-Arnold** if it has a first integral \( G \) independent with \( H \) (i.e. the gradient vectors of \( H \) and \( G \) are independent in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in **involution** with \( H \) (i.e. the parenthesis of Poisson of \( H \) and \( G \) is zero). The Liouville-Arnold theorem describe the dynamics of the integrable Hamiltonian systems, see for more details see Abraham and Marsden (1978), Arnold et al. (2006) and the Sect. 7.1.2 of Arnold et al. (2006), respectively.

We consider the autonomous differential system

\[ \dot{x} = f(x), \]

(16)

where \( f : U \to \mathbb{R}^n \) is \( C^2 \), \( U \) is an open subset of \( \mathbb{R}^n \) and the dot denotes the derivative respect to the time \( t \). We write its general solution as \( \phi(t, x_0) \) with \( \phi(0, x_0) = x_0 \in U \) and \( t \) belonging to its maximal interval of definition.

We say that \( \phi(t, x_0) \) is **T-periodic** with \( T > 0 \) if and only if \( \phi(T, x_0) = x_0 \) and \( \phi(t, x_0) \neq x_0 \) for \( t \in (0, T) \). The periodic orbit associated to the periodic solution \( \phi(t, x_0) \) is \( \gamma = [\phi(t, x_0), t \in [0, T]] \). The variational equation associated to the \( T \)-periodic solution \( \phi(t, x_0) \) is

\[ \dot{M} = \left( \frac{\partial f(x)}{\partial x} \bigg|_{x=\phi(t,x_0)} \right) M, \]

(17)

where \( M \) is an \( n \times n \) matrix. The **monodromy matrix** associated to the \( T \)-periodic solution \( \phi(t, x_0) \) is the solution \( M(T, x_0) \) of (17) satisfying that \( M(0, x_0) \) is the identity matrix. The eigenvalues \( \lambda \) of the monodromy matrix associated to the periodic solution \( \phi(t, x_0) \) are called the **multipliers** of the periodic orbit.
Theorem 3 If a Hamiltonian system with two degrees of freedom and Hamiltonian $H$ is Liouville-Arnold integrable, and $G$ is a second first integral such that the gradients of $H$ and $G$ are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to 1.

Theorem 3 is due to Poincaré (1899). It gives us a tool to study the non Liouville-Arnold integrability, independently of the class of differentiability of the second first integral. The main problem for applying this theorem is to find periodic orbits having multipliers different from 1.

Appendix 2: Averaging theory of first order

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst (1991).

Consider the differential equation

\begin{equation}
\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \quad x(0) = x_0 \tag{18}
\end{equation}

with $x \in D \subset \mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F_1(t, x)$ and $F_2(t, x, \varepsilon)$ are $T$-periodic in $t$. Separately we consider in $D$ the averaged differential equation

\begin{equation}
\dot{y} = \varepsilon f_1(y), \quad y(0) = x_0, \tag{19}
\end{equation}

where

\[ f_1(y) = \frac{1}{T} \int_0^T F_1(t, y) \, dt. \]

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T$-periodic solutions of (18).

Theorem 4 Consider the two initial value problems (18) and (19). Suppose:

(i) $F_1$, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, $F_2$ and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of $\varepsilon$ in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.

(ii) $F_1$ and $F_2$ are $T$-periodic in $t$ ($T$ independent of $\varepsilon$).

(iii) $y(t)$ belongs to $\Omega$ on the interval of time $[0, 1/\varepsilon]$.

Then the following statements hold.

(a) For $t \in [1, \varepsilon]$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.

(b) If $p$ is a singular point of the averaged (19) and

\[ \det \left( \frac{\partial f_1}{\partial y} \right)_{y=p} \neq 0, \]

then there exists a $T$-periodic solution $\varphi(t, \varepsilon)$ of (18) which is close to $p$ such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.

(c) The stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the stability or instability of the singular point $p$ of the averaged system (19). In fact, the singular point $p$ has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$.

In the following we use the ideas of the proof of Theorem 4(c). For more details see the Sects. 6.3 and 11.8 of Verhulst (1991). Suppose that $\varphi(t, \varepsilon)$ is a periodic solution of (18) corresponding to $y = p$ an equilibrium point of the averaged system (19). Linearizing (18) in a neighborhood of the periodic solution $\varphi(t, \varepsilon)$ we obtain a linear equation with $T$-periodic coefficients

\begin{equation}
\dot{x} = \varepsilon A(t, \varepsilon)x, \quad A(t, \varepsilon) = \frac{\partial}{\partial x} \left[ F_1(t, x) - F_2(t, x, \varepsilon) \right] \big|_{x=\varphi(t, \varepsilon)}. \tag{20}
\end{equation}

We introduce the $T$-periodic matrices

\[ B(t) = \frac{\partial F_1}{\partial x}(t, p), \quad B_1 = \frac{1}{T} \int_0^T B(t) \, dt, \]

\[ C(t) = \int_0^T (B(s) - B_1) \, ds. \]

From Theorem 4(c) we have

\[ \lim_{\varepsilon \to 0} A(t, \varepsilon) = B(t), \]

and it is clear that $B_1$ is the matrix of the linearized averaged equation. The matrix $C$ has average zero. The near identity transformation

\[ x \mapsto y = \left( I - \varepsilon C(t) \right)x, \tag{21} \]

permits to write (20) as

\[ \dot{y} = \varepsilon B_1 y + \varepsilon \left[ A(t, \varepsilon) - B(t) \right] y + O(\varepsilon^2). \tag{22} \]

Notice that $A(t, \varepsilon) - B(t) \to 0$ as $\varepsilon \to 0$, and also the characteristic exponents of (22) depend continuously on the small parameter $\varepsilon$. It follows that, for $\varepsilon$ sufficiently small, if
the determinant of $B_1$ is not zero, then 0 is not an eigenvalue of the matrix $B_1$ and then it is not a characteristic exponent of (22). By the near-identity transformation we obtain that system (20) has not multipliers equal to 1.

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