Topography, Convergence, and Reconstruction of Predictive States

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Predictive equivalence in discrete stochastic processes have been applied with great success to identify randomness and structure in statistical physics and chaotic dynamical systems and to inferring hidden Markov models. We examine the conditions under which they can be reliably reconstructed from time-series data, showing that convergence of predictive states can be achieved from empirical samples in the weak topology of measures. Moreover, predictive states may be represented in Hilbert spaces that replicate the weak topology. We mathematically explain how these representations are particularly beneficial when reconstructing high-memory processes and connect them to reproducing kernel Hilbert spaces.

Keywords: stochastic process, symbolic dynamics, dynamical systems, measure theory, weak topology

I. INTRODUCTION

With an accurate model in hand, an observer can leverage their knowledge of a system’s history to predict its future behavior. For stochastic processes—distributions over time-series data—the task of predicting future behavior from past observations and the associated resource constraints this task imposes on an observer have been studied under the physics of computational mechanics [1]. This subfield of statistical mechanics focuses on the intrinsic information-processing embedded in natural systems.

Its chief insight is the concept of the predictive (or causal) state. A process’ predictive states play a dual role. On the one hand, to accurately predict a process’ future behavior they are the key objects that an observer must be capable of reproducing in their model. On the other hand, the predictive states and their dynamics are central to understanding the intrinsic, model-independent properties of the process itself [1].

The concept of predictive states has found use in numerous settings, such as classical and quantum thermodynamics [2–4], quantum information and computing [5–8], condensed matter [9–11], dynamical systems [12], cellular automata [13], and model inference [14–20]. Additionally, in the setting of processes generated by finite-state, discrete-output hidden Markov models (HMMs) and generalized hidden Markov models (GHMMs), a deep mathematical theory of predictive states is now available [1, 21–25].

Despite their broad utility, a mathematically rigorous definition of predictive states is needed that is applicable and useful for even more general stochastic processes. Here, we have in mind large-memory processes whose long-time correlations cannot be finitely represented by HMMs or GHMMs and processes whose outputs may span a continuous domain in time and space.

The following takes the next major step towards a rigorous and mathematically general definition of predictive states, extending the concept to all processes whose observations are temporally discrete but may otherwise be either discrete or continuous.

Somewhat remarkably, for any stationary and ergodic stochastic process of this kind, predictive states are always well-defined and, furthermore, may always be convergently approximated from empirical observations given a sufficiently large sample. Next, we expand on recent work on Hilbert space embeddings of predictive states [20, 26–28], demonstrate that such embeddings always exist, and discuss their implications for predictive-state geometry and topology. Last, we explore the implications of our results for empirically reconstructing predictive states via reproducing kernel Hilbert spaces, particularly through the addition of new terms in the asymptotic convergence bounds.

II. ASSUMPTIONS AND PRELIMINARIES

A. Stochastic processes

We begin by laying out a series of definitions and identifying the assumptions made. We draw from the combined literature of measures, stochastic processes, and symbolic dynamics [29, 30].

A stochastic process is typically defined as a function-valued random variable $X : \Omega \to \mathcal{X}^T$, where $(\Omega, \Sigma, \mu)$ is a measure space, $T$ is a set of temporal indices (perhaps

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the real line, perhaps a discrete set), and $\mathcal{X}$ is a set of possible observations (also potentially real or discrete in nature). We take the sample space $\Omega$ to be the set $\mathcal{X}^T$ and $X$ to be the identity. In this way, a stochastic process is identified solely with the measure $\mu$ over $\Omega = \mathcal{X}^T$.

When $T$ is $\mathbb{Z}$, we say the process is discrete-time; when it is $\mathbb{R}$ we say continuous-time. Unless specified otherwise we assume discrete-time, later treating continuous-time as an extension of the discrete case. In discrete time, it is convenient to write $X(t)$ as an indexed sequence $(x_i)$, where each $x_i$ is an element of $\mathcal{X}$. When $\mathcal{X}$ is a discrete finite set, we say that the process is discrete-observation; by continuous-observation we typically mean the case where $\mathcal{X}$ is an interval in $\mathbb{R}$ or a Cartesian product of intervals in $\mathbb{R}^d$. These are the only cases we consider rigorously. That said, we believe they are sufficient for many practical purposes or, at least, not too cumbersome to extend if necessary.

The temporal shift operator $\tau : \mathcal{X}^T \to \mathcal{X}^T$ simply translates $t \to t + 1$: $\tau X(t) = X(t + 1)$. It also acts on measures of $\mathcal{X}^T$: $(\tau \mu)(A) = \mu(\tau^{-1}A)$. A stochastic process paired with the shift operator $\tau$ becomes a dynamical system and is stationary if $\tau \mu = \mu$. It is further considered ergodic if, for all shift-invariant sets $I \subseteq \mathcal{X}^T$, either $\mu(I) = 1$ or $\mu(I) = 0$. Here, we assume all processes are both stationary and ergodic.

If $\mathcal{X}$ is discrete, then the measurable sets of $\mathcal{X}^Z$ are generated by the cylinder sets:

$$U_{t,w} := \{ X : x_{t+1} \ldots x_{t+\ell} = w \} ,$$

where $w \in \mathcal{X}^\ell$ is a word of length $\ell$. For a stationary process, the word probabilities:

$$\Pr_\mu ( x_1 \ldots x_\ell ) := \mu ( U_{0,x_1 \ldots x_\ell} )$$

are sufficient to uniquely define the measure $\mu$.

In the continuous-observation case, the issue is more subtle. A cylinder set instead takes the form:

$$U_{t,I_1 \ldots I_\ell} := \{ X : x_{t+1} \in I_1, \ldots, x_{t+\ell} \in I_\ell \} ,$$

where each $I_i$ is an interval in $\mathcal{X}$. This does not lend itself well to expressing simple word probabilities. However, we can define the word measures $\mu_{x}$ by restricting $\mu$ to the set $\mathcal{X}^\ell$ describing the first $\ell$ values.

### III. PREDICTIVE STATES

Each element $X \in \mathcal{X}^Z$ can be decomposed from a bidirectional infinite sequence to a pair of unidirectional infinite sequences in $\mathcal{X}^N \times \mathcal{X}^N$, by the transformation $x_{-\infty}x_{-1} \ldots x_0 x_{-1} \ldots \to (x_0 x_{-1} \ldots, x_1 x_2 \ldots)$. The first sequence in this pair we call the past $\overleftarrow{X}$ and the second we call the future $\overrightarrow{X}$. In this perspective, a stochastic process is a bipartite measure over pasts and futures. The intuitive definition of a predictive state is as a measure over future sequences that arises from conditioning on past sequences. Heuristically, $\Pr_\mu ( \overrightarrow{X} \mid \overleftarrow{X} = x_0 x_{-1} \ldots )$ represents the “predictive state” associated with past $x_0 x_{-1} \ldots$.

Conditioning of measures is a nuanced issue, especially when the involved sample spaces are uncountably infinite [31]. Of the many perspectives that define a conditional measure, the most practical and intuitive is that a conditional measure is a ratio of likelihoods—and, in the continuous case, a limit of such ratios. However, determining the manner in which this limit must be taken is rarely trivial. The following considers first the case of discrete observations, where the matter is relatively straightforward. Then we examine the case of continuous observations, reviewing the previous literature on the nuances of this domain and extending its results for our present purposes. As we will see, in either case, the intuition of predictive states can be born out in a rigorous and elegant manner for any stochastic process satisfying the assumptions heretofore mentioned.

### A. Discrete observations

We first establish likelihood-ratio convergence.

**Theorem 1.** For all measures $\mu$ on $\mathcal{X}^\ell$, all $\ell \in \mathbb{N}$, all $w = x_1 \ldots x_\ell \in \mathcal{X}^\ell$, and $\mu$-almost all pasts $\overleftarrow{X}$, where $\mathcal{X}$ is a finite set, the limit:

$$\Pr_\mu ( w \mid \overleftarrow{X} ) := \lim_{k \to \infty} \frac{\Pr_\mu ( x_{-k} \ldots x_0 x_{-1} \ldots x_\ell )}{\Pr_\mu ( x_{-k} \ldots x_0 )}$$

is convergent.

For all $\overleftarrow{X}$ where Eq. (1) converges, we can define a measure $\epsilon (\overleftarrow{X}) \in \mathcal{M}(\mathcal{X}^N)$ over future sequences, uniquely determined by the requirement $\epsilon (\overleftarrow{X})(U_{0,w}) = \Pr_\mu ( w \mid \overleftarrow{X} )$. This $\epsilon (\overleftarrow{X})$ is the predictive state of $\overleftarrow{X}$ and the function $\epsilon : \mathcal{X}^N \to \mathcal{M}(\mathcal{X}^N)$, the prediction mapping.

The proof strategy consists in redefining the problem. The limit Eq. (1) can be recast as what is called a likelihood ratio. The convergence of likelihood ratios is itself closely related to the theory of Radon-Nikodym derivatives between measures. Specifically, the Radon-Nikodym derivative can be computed as a convergence of likelihood ratios. That convergence is taken over a
particular class of neighborhoods, called a differentiation basis, and that basis supports the Vitali property. We define these concepts for the reader below and use them to prove Theorem 1.

Let $\overset{\leftarrow}{\mu}$ denote the restriction of $\mu$ to pasts, and let $\overset{\leftarrow}{\mu}_{x_1...x_1}$ be the measure on pasts that precede the word $w := x_1 \ldots x_\ell$. These are given by:

$$\text{Pr}_\mu (x_0 \ldots x_{-k}) := \text{Pr}_\mu (x_{-k} \ldots x_0)$$

$$\text{Pr}_\mu (x_1 \ldots x_\ell | \overset{\leftarrow}{X}) = \lim_{k \to \infty} \frac{\overset{\leftarrow}{\mu}_{x_1...x_1}(U_k)}{\mu(U_k)}. \tag{2}$$

This reformulation, though somewhat conceptually cumbersome, is useful due to theorems that relate the convergence of likelihood ratios, taken over a sequence of cylinder sets $U_k := U_{0,x_0...x_{-k}}$ converging on $\overset{\leftarrow}{X}$:

$$\text{Pr}_\mu (x_1 \ldots x_\ell | \overset{\leftarrow}{X}) = \lim_{k \to \infty} \frac{\overset{\leftarrow}{\mu}_{x_i...x_1}(U_k)}{\mu(U_k)}. \tag{2}$$

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The Vitali theorem states that whenever the differentiation basis $D$ possesses the Vitali property with respect to two measures $\mu$ and $\nu$, then for $\mu$-almost all $\overset{\leftarrow}{X}$, the limit of likelihood ratios exists for every sequence of neighborhoods $(V_k) \subset D$ converging on $\overset{\leftarrow}{X}$ and the limit is equal to the Radon-Nikodym derivative $\frac{d\mu}{d\nu}(\overset{\leftarrow}{X})$ at that point [31]. This kind of very flexible limit is denoted by:

$$\lim_{\overset{V \in D}{V \ni \overset{\leftarrow}{X}}} \mu(V) = \frac{d\mu}{d\nu}(\overset{\leftarrow}{X}).$$

The Vitali property has strong and weak forms, but we establish only the strong form here. The differentiation basis $D$ has the strong Vitali property with respect to $\mu$ if for every measurable set $A$ and for every a subdifferentiation basis $D' \subseteq D$ covering $A$, there is an at most countable subset $\{ D_j \} \subseteq D'$ such that $D_j \cap D_{j'}$ is empty for all $j \neq j'$ and:

$$\mu \left( A - \bigcup_j D_j \right) = 0.$$ 

In other words, we must be able to cover “almost all” of $A$ with a countable number of nonoverlapping sets from the differentiation basis [31]. We now demonstrate that the differentiation basis $D$ generated by cylinder sets on $\mathcal{X}^\mathbb{N}$ has the Vitali property for any measure $\mu$.

**Proposition 1** (Vitali property for stochastic processes). For any stochastic process $(\mathcal{X}^\mathbb{N}, \Sigma, \mu)$, let $D$ be the differentiation basis of allowed cylinder sets. Then $D$ has the strong Vitali property.

**Proof.** Let $D' \subseteq D$ be any subdifferentiation basis covering $\mathcal{X}^\mathbb{N}$. (Our proof trivially generalizes to any $A \subseteq \mathcal{X}^\mathbb{N}$.) Since $D'$ is a differentiation basis, for all $\overset{\leftarrow}{X} \in \mathcal{X}^\mathbb{N}$ there must be a sequence $(D_j(\overset{\leftarrow}{X}))$ of cylinder sets converging on $\overset{\leftarrow}{X}$. Without loss of generality, suppose $D_j(\overset{\leftarrow}{X}) = U_{-\ell_j,x_{-\ell_j+1}...x_0}$ with $\ell_j$ monotonically increasing. (If this is not the case, we take a subsequence of $D_j(\overset{\leftarrow}{X})$ for which it is the case.)

Now consider the combination of all such sequences:

$$D'' := \bigcup_{\overset{\overset{\leftarrow}{X} \in \mathcal{X}^\mathbb{N}}{j \in \mathbb{N}}} \{ D_j(\overset{\leftarrow}{X}) | j \in \mathbb{N} \}.$$ 

We note that $D''$, though a union of an uncountable number of sets, itself cannot be larger than a countable set, as the elements of the sets from which it is composed are characterized by finite words, and finite words themselves form a countable set. That is, there is significant redundancy in $D''$ that keeps it countable. Furthermore, $D''$ has a lattice structure given by the set inclusion relation $\subseteq$ with the particular property that for $U,V \in D''$, $U \cap V$ is nonempty only if $U \subseteq V$ or vice versa.

We then choose the set $C$ of all maximal elements of this lattice: that is, those $U \in D''$ such that there is no $V \in D''$ containing $U$. These maximal elements must exist since for each $U \in D''$ there is only a finite number of sets in $D''$ that can contain it.

It must be the case that all sets in $C$ are nonoverlapping. Furthermore, for any $V \in D''$, not in $C$, there can only be a finite number of such sets containing $V$. One of them must be maximal and therefore in $C$. In particular, for every $\overset{\leftarrow}{X} \in \mathcal{X}^\mathbb{N}$, each of its neighborhoods in $D''$ is contained by the union of $C$.

This implies $C$ is a complete covering of $\mathcal{X}^\mathbb{N}$. Since it is also nonoverlapping and countable, the strong Vitali property is proven.

As a consequence, the likelihood ratios in Eq. (2) must converge for $\overset{\leftarrow}{\mu}$-almost every past $\overset{\leftarrow}{X}$ and every finite-length word $w$—proving Theorem 1.

**Note that this result follows as a relatively straightforward application of the Vitali property, which holds for**
FIG. 1: Snapshot of a differentiation basis: A differentiation basis is a collection of neighborhoods in $\mathcal{X}^\mathbb{N}$ that have hierarchical structure. For every point $x \in \mathcal{X}^\mathbb{N}$, there must be a sequence of neighborhoods converging on that point. Pictured above, a line is shown with a partial representation of its differentiation basis above it in the form of a hierarchical collection of rounded rectangles. For two points $x$ and $y$ we show the corresponding sequence of sets $(D_j(x))$, $(D_j(y))$ converging on each.

B. Continuous observations: Overview

Shifting from discrete to real-valued observations, where now $\mathcal{X}$ denotes a compact subset of $\mathbb{R}^d$, multiple subtleties come to the fore.

First, it must be noted that even in $\mathbb{R}$, the existence of a Vitali property is not trivial. For the Lebesgue measure, only a weak Vitali property holds, though this is still sufficient for the equivalence between Radon-Nikodym derivatives and likelihood ratios. The differentiation basis in this setting can be taken to be comprised of all intervals $(a, b)$ on the real line.

Second, to go from $\mathbb{R}$ to $\mathbb{R}^d$, constraints must be placed on the differentiation basis. An “interval” here is really the Cartesian product of intervals, but for a Vitali property to hold we must only consider products of intervals whose edges are held in a fixed ratio to one another, so that the edges converge uniformly to zero. Likelihood ratios for fixed-aspect boxes of this kind can converge to the Radon-Nikodym derivative [31].

This requirement poses a challenge for generalizing the Vitali property to infinite dimensions, as we must to study sequences of real numbers. A fixed-aspect “box” around a sequence of real numbers is not a practical construction. In the empirical setting, we can only observe information about a finite number of past outputs. We therefore cannot obtain any “uniform” knowledge of the entire past. That is, a direct generalization of the case for $\mathbb{R}^d$ does not suffice.

However, integration and differentiation on infinite-dimensional spaces has been considered before, mainly by Jessen [32, 33] and later Enomoto [34]. Their results focused on generalizing Lebesgue measure to $(S^1)^\mathbb{N}$, where $S^1$ is the circle. This section shows that their results can be significantly extended. The primary result we prove is a generalization of Enomoto’s Theorem [34]:

**Theorem 2** (Generalized Enomoto’s Theorem). Let $\mathcal{X}$ be an interval of $\mathbb{R}$, and let $\mu$ be any probability measure over $\mathcal{X}^\mathbb{N}$. Let $f : \mathcal{X}^\mathbb{N} \rightarrow \mathbb{R}^+$ and let $F$ be its indefinite integral under $\mu$. Let $V$ denote the differentiation basis consisting of sets of the form:

$$V_{n,\delta}(\mathcal{X}) = \left\{ Y \mid |y_j - x_j| < \delta, j = 1, \ldots, n \right\}.$$

Then:

$$\lim_{\nu \uparrow \mu} \frac{F(V)}{\mu(V)} = f(\mathcal{X}),$$

for $\nu$-almost all $\mathcal{X}$.

Note that the resulting differentiation basis is a weaker form of that considered above. Each $V_{n,\delta}$ is evidently a
cylinder set, but of a very particular kind. As we take $\delta \to 0$ and $n \to \infty$, we extend the “window” of the cylinder set to the entire past while simultaneously narrowing its width uniformly. This turns out to be sufficient to replicate the same effect as the fixed-aspect boxes in the finite-dimensional case.

As a corollary of Theorem 2, we have the following result for predictive states:

**Corollary 1.** For all measures $\mu$ on $\mathcal{X}^\mathbb{R}$, where $\mathcal{X} \subset \mathbb{R}^d$ is a compact set, all neighborhoods $U \subset \mathcal{X}^\mathbb{R}$, and all $\ell \in \mathbb{N}$, and for $\mu$-almost all pasts $\vec{X} = x_0x_{-1} \ldots$, the limit:

$$ \Pr_{\mu} \left( U \mid \vec{X} \right) \equiv \lim_{n \to \infty} \frac{\mu(V_n,\delta(n) \times U)}{\mu(V)}$$

converges as long as $\delta(n) > 0$ for all $n$ and $\delta(n) \to 0$.

Note here that we allowed $\mathcal{X} \subset \mathbb{R}^d$. This can be obtained from Enomoto’s theorem by simply reorganizing a sequence of $d$-dimensional coordinates from $(x_1, x_2, \ldots)$ to $(x_{11}, \ldots, x_{d1}, x_{12}, \ldots, x_{d2}, \ldots)$. Enomoto’s theorem then requires uniformity of the intervals across past instances as well as within each copy of $\mathbb{R}^d$.

As before, the quantities $\Pr_{\mu} \left( U \mid \vec{X} \right)$ define a unique measure $\epsilon(\vec{X})$ on $\mathcal{X}^\mathbb{R}$. It is determined by:

$$ \epsilon(\vec{X})(U) = \Pr_{\mu} \left( U \mid \vec{X} \right).$$

Enomoto’s theorem itself is the capstone result in a sequence of theorems initiated by Jessen [32]. To prove Theorem 2, we must start from the beginning, generalizing Jessen’s results. Fortunately, the bulk of the effort comes in generalizing the first of these results—Jessen’s correspondence principle. After this, the generalization follows quite trivially from the subsequent theorems. The next section provides the full proof for a generalized correspondence principle and explains how this result impacts the proofs of the subsequent theorems.

For completeness, we also give the full proof of the generalized Enomoto’s theorem, though it does not differ much from Enomoto’s—published in French—once the preceding theorems are secured.

**C. Jessen’s correspondence principle**

The Jessen and Enomoto theory rests on a profound correspondence between cylinder sets on $\mathcal{X}^\mathbb{N}$ and intervals on $\mathbb{R}$. To state it, we must define the concept of a net.

A net is similar to but formally separate from a differentiation basis, but like the latter allows for a notion of differentiation, called differentiation-by-nets. This is weaker than the Vitali property on a differentiation basis, but following on Jessen’s work, Enomoto showed that differentiation-by-nets can be extended to describe a particular differentiation basis with the Vitali property.

Let $\mathcal{X}$ be a finite interval on $\mathbb{R}$. A dissection $D = (b_1, \ldots, b_N)$ of $\mathcal{X}$ is simply a sequence of cut points, that generate a sequence of adjacent intervals $(b_{k}, b_{k+1})$ spanning $\mathcal{X}$, covering all but a finite set of points—the interval edges. See Fig. 2. Denote the intervals $I(D) = \{ (b_{k}, b_{k+1}) \mid k = 1, \ldots, N - 1 \}$. The length of the largest interval in $I(D)$ is denoted $|D|$. (Not to be confused with $D$’s cardinality, that we have no need to reference.) A net $\mathcal{N} = (D_n)$ is a sequence of dissections so that $D_n \subset D_{n+1}$ (that is, each new dissection only adds further cuts) and $|D_n| \to 0$ (the largest interval length goes to zero). The boundary $\partial \mathcal{N} = \bigcup_n D_n$ denotes all the boundary points from the sequence and is always a countable set.

We can similarly define a dissection $D = (d_1, \ldots, d_\ell)$ on $\mathcal{X}^\mathbb{N}$ as a set of $\ell$ dissections, one for each of the first $\ell$ copies of $\mathcal{X}$. $D$ intervals $I(D) = \{ i_1 \times \ldots i_\ell \times \mathcal{X}^\mathbb{N} \mid i_k \in I(d_k) \}$ are the cylinder sets generated by the intervals of each individual dissection. See Fig. 3. The boundary of a dissection is the set of all points that do not belong to these intervals: $\partial D =$

![FIG. 2: Snapshot of a differentiation net. A differentiation net defined on a line segment. $D_1, D_2, D_3, \ldots$ represents the dissections which comprise the net. Each dissection contains the last; new points are indicated in red and old points in gray. These points define intervals; a sequence of these intervals is shown, $(I_k(x))$, converging on the point $x$.](image-url)
\{ X \in \mathcal{X}^N \mid \exists k : x_k \in d_k \}. The size of the dissection is |D| := \max_k |d_k|.
For a finite measure \( \mu \), there are always dissections with \( \mu(\partial D) = 0 \) of any given \( |D| = \max_k |d_k| \), since \( \mu(\mathcal{X}) \) can only have at most countably many singular points.

A net \( \mathcal{N} = (D_n = (d_{1,n}, \ldots, d_{\ell_n,n})) \) of \( \mathcal{X}^N \) is a sequence of dissections of increasing depth \( \ell_n \) so that each sequence \( (d_{k,n}) \) for fixed \( k \) is a net for the \( k \)th copy of \( \mathcal{X} \). \( \partial \mathcal{N} = \bigcup_n \partial D_n \) denotes all the accumulated boundary points of this sequence. Again, for finite measure \( \mu \), nets always exist that have \( \mu(\partial \mathcal{N}) = 0 \) for all \( n \); nets with this property are called \( \mu \)-continuous nets.

Note that for any net, every sequence of intervals \( (I_n) \), \( I_n \in \mathcal{I}(D_n) \) and \( I_{n+1} \subset I_n \), uniquely determines a point \( X \in \mathcal{X}^N \). If \( X \notin \partial D \), then \( X \) uniquely determines a sequence of intervals.

The following result can be proven (generalized from Ref. [32]):

**Theorem 3** (Generalized correspondence principle). Let \( \mathcal{X} \subset \mathbb{R} \) be an interval and let \( \lambda \) be the Lebesgue measure on \( \mathcal{X} \), normalized so \( \lambda(\mathcal{X}) = 1 \). Let \( \mu \) be a finite measure on \( \mathcal{X}^N \) that has no singular points. Let \( \mathcal{N} = (D_n) \) be any \( \mu \)-continuous net of \( \mathcal{X}^N \). Then there exists a net \( \mathcal{M} = (d_n) \) of \( \mathcal{X} \) so that:

1. There exists a function \( \Phi_n \) that maps each interval in \( \mathcal{I}(D_n) \) of positive measure to one and only one interval in \( \mathcal{I}(d_n) \) and vice versa for \( \Phi_n^{-1} \);
2. \( \lambda(\Phi_n(I)) = \mu(I) \) for all \( I \in \mathcal{I}(D_n) \) with \( \mu(I) > 0 \); and
3. The mapping \( \phi : \mathcal{X}^N - \partial \mathcal{N} \to \mathcal{X} - \partial \mathcal{M} \), generated by \( X \mapsto (I_n) \mapsto (\Phi_n(I_n)) \mapsto x \), is measure-preserving.

To summarize this technical statement: For any method of indefinitely dissecting the set \( \mathcal{X}^N \) into smaller and smaller intervals, there is in fact an “equivalent” such method for dissecting the much simpler set \( \mathcal{X} \). It is equivalent in the sense that all the resulting intervals are in one-to-one correspondence with one another, a correspondence that preserves measure. Since interval sequences uniquely determine points (and vice versa for a set of full measure), this induces a one-to-one correspondence between points that is also measure-preserving.

The proof consists of two parts. The first proves the first two claims about \( \mathcal{M} \). Namely, there is an interval correspondence and it is measure-preserving. The second shows this extends to a correspondence between \( \mathcal{X}^N \) and \( \mathcal{X} \) that is also measure-preserving.

**Proof** (Interval correspondence). The proof proceeds by induction. For a given \( \mu \)-continuous net \( \mathcal{N} = (D_n) \), suppose we already constructed dissections \( d_1, \ldots, d_N \) of \( \mathcal{X} \) so that a function \( \Phi_n \) between positive-measure intervals in \( D_n \) and \( d_n \) exists with the desired properties (1) and (2) above, for all \( n = 1, \ldots, N \). Now, for \( D_{n+1} \), a certain set of the intervals in \( \mathcal{I}(D_n) \) is divided. Suppose \( I \in \mathcal{I}_n \) divides into \( I' \) and \( I'' \). If either of these, say \( I'' \), has measure zero then we discard it and set \( \Phi_{n+1}(I') = \Phi_n(I) \). Otherwise, suppose that \( \Phi_n(I) = (a, b) \). Then divide \( \Phi_n(I) \) into the intervals:

\[
\Phi_{n+1}(I') := \left( a, \frac{\mu(I) + (b - a)\mu(I')}{\mu(I)} \right), \\
\Phi_{n+1}(I'') := \left( \frac{\mu(I) + (b - a)\mu(I')}{\mu(I)}, b \right),
\]

that clearly have Lebesgue measures \( \lambda(\Phi_{n+1}(I')) = \mu(I') \) and \( \lambda(\Phi_{n+1}(I'')) = \mu(I'') \), respectively. Generalizing this to more complicated divisions of \( I \) is straightforward.

Now, we can always suppose for a given net \( \mathcal{N} \) that \( D_0 \) is just the trivial dissection that makes no cuts and only one interval. However, this has a trivial correspondence with \( \mathcal{X} \); namely, \( \Phi_0(\mathcal{X}^N) = \mathcal{X} \).

By induction, then, the desired \( \mathcal{M} \) can always be constructed.

With the existence of the interval correspondence established, we further demonstrate the existence of a point correspondence between \( \mu \)-almost-all of \( \mathcal{X}^N \) and \( \lambda \)-almost-all of \( \mathcal{X} \).

**Proof** (Point correspondence). For every \( X \in \mathcal{X}^N - \partial \mathcal{N} \), there is a unique sequence \( (I_n) \) of concentric intervals, \( I_n \in \mathcal{I}(D_n) \) and \( I_{n+1} \subset I_n \), such that \( \bigcap_n I_n = \{ X \} \). If \( X \) is in the support of \( \mu \), then we define:

\[
\phi(X) := \bigcap_n \Phi_n(I_n)
\]

as the corresponding point in \( \mathcal{X} - \partial \mathcal{M} \). Due to the interval correspondence, this mapping is invertible.

By measure-preserving we mean that for all \( A \subset \mathcal{X}^N - \partial \mathcal{N} \), \( \lambda(\phi(A)) = \mu(A) \) and vice-versa for \( \phi^{-1} \). Both the Lebesgue measure and \( \mu \) must be outer regular, due to being finite measures. Outer regular means that the measure of a set \( A \) is the infimum of the measure of all open sets containing \( A \), a property we use to our advantage.

Consider for each \( n \) the minimal covering \( C_n \) of \( A \) by intervals in \( \mathcal{I}(D_n) \). The measure of this covering is denoted \( m_n := \mu(\bigcup C_n) \). Clearly, \( m_n \geq \mu(A) \) and \( m_n \to \mu(A) \). The corresponding covering \( \Phi_n(C_n) \) in \( \mathcal{I}(d_n) \) is a covering of \( \phi(A) \) and has the same measure \( m_n \). By outer regularity, then, \( m_n \geq \lambda(\phi(A)) \) for all \( n \). And so, \( \mu(A) \geq \lambda(\phi(A)) \).

Now, by the exact reverse argument of the previous paragraph, going from \( \mathcal{X} \) to \( \mathcal{X}^N \) via \( \phi^{-1} \), we can also deduce
that \( \mu(A) \leq \lambda(\phi(A)) \). Therefore \( \mu(A) = \lambda(\phi(A)) \), and the function \( \phi \) is measure-preserving.

### D. Corollaries and Enomoto’s Theorem

Jessen’s correspondence principle is an extremely powerful device. Among its consequences are the following theorems regarding functions on \( \mathcal{X}^\mathbb{N} \). We state their generalized forms here and for the proofs refer to Jessen [32], as each is a direct application of Theorem 3 without making any further assumptions on the measure \( \mu \).

The first offers a much weaker (and on its own, insufficient for our purposes) concept of differentiation of measures that we refer to as **differentiation-by-nets**.

**Corollary 2 (Differentiation-by-nets).** Let \( f : \mathcal{X}^\mathbb{N} \rightarrow \mathbb{R}^+ \) and let \( F \) be the measure defined by its indefinite integral: \( F(A) := \int_A f(X) d\mu(X) \). Further let \( \mathcal{N} = (D_n) \) be a net on \( \mathcal{X}^\mathbb{N} \) and denote by \( f_n \) a piecewise function such that \( f_n(X) = F(I_n)/\mu(I_n) \) for all \( X \in I_n \) and each \( I_n \in D_n \). Then \( f_n(X) \rightarrow f(X) \) as \( n \rightarrow \infty \) for \( \mu \)-almost all \( X \).

Though the full proof is found in Ref. [32], we summarize its key point: Using the correspondence of intervals, we write \( F(I_n)/\mu(I_n) = \hat{F}(\Phi(I_n))/\lambda(\Phi(I_n)) \), where \( \hat{F} \) is the indefinite integral of \( f \circ \phi^{-1} \) with respect to \( \lambda \). The limit then holds due to the Vitali property of \( \lambda \) on \( \mathcal{X} \).

However, we also note that Corollary 2 is *not* an extension of the Vitali property to cylinder sets on \( \mathcal{X}^\mathbb{N} \). Jessen himself offers a counterexample to this effect in a later publication [33].

Jessen’s second corollary is key to demonstrating that \( \mathcal{V} \), the differentiation basis defined in Theorem 2, will have the sought-after Vitali property.

**Corollary 3 (Functions as limits of integrals).** Let \( f : \mathcal{X}^\mathbb{N} \rightarrow \mathbb{R}^+ \), and let \( f_n(X) \) be a sequence of functions given by:

\[
f_n(x_1 x_2 \ldots) := \int_{Y \in \mathcal{X}^\mathbb{N}} f(x_1 \ldots x_n) Y d\mu(Y).
\]

That is, we integrated over all observations after the first \( n \). Thus, \( f_n \) only depends on the first \( n \) observations. Then \( f_n(X) \rightarrow f(X) \) as \( n \rightarrow \infty \) for \( \mu \)-almost all \( X \).

This proof we also skip, again referring the reader to Jessen [32], as no step is directly dependent on the measure \( \mu \) itself and only on properties already proven by the previous theorems.

We now have sufficient knowledge to prove the generalized Enomoto’s theorem; generalized from Ref. [34].

**Proof (Generalized Enomoto’s Theorem).** First, we must demonstrate, for almost every \( X \), that there exists a sequence \( V_j(X) \) converging on \( X \) such that the limit holds. By Corollary 3, there must be, for \( \mu \)-almost all \( X \),
and any $\epsilon > 0$, a $k(X, \epsilon)$ such that $|f_n(X) - f(X)| < \epsilon / 2$ for all $n > k(X, \epsilon)$. Now, from the Vitali property on $\mu_n$ and the fact that $f_n$ only depends on the first $n$ observations, it must be true that for any $\epsilon > 0$ and almost all $X$, there is a $0 < \Delta(X, n, \epsilon) < 1$ so that:

$$|f_n(X) - F(V_n, \Delta(X))| < \epsilon / 2,$$

whenever $\delta < \Delta(X, n, \epsilon)$. For a given $\epsilon$, there is a countable number of conditions (one for each $n$). As such, the set of points $X$ for which all conditions hold is still measure one. Then, taking for each $X$ the integer $K := k(X, \epsilon)$ and subsequently the number $\Delta := \Delta(X, k(X, \epsilon), \epsilon)$, we can choose $V_{K, \Delta}(X)$ and by the triangle inequality we must have:

$$|f(X) - F(V_{K, \Delta}(X))| < \epsilon. \quad (5)$$

This completes the proof’s first part.

However, the second part—that all sequences $V_{n_j, \delta_j}(x)$ of neighborhoods give converging likelihood ratios—further follows from the above statements, as:

$$|f(X) - F(V_{n_j, \delta_j}(X))| < \epsilon$$

must hold for any $n_j > K(X, \epsilon)$ and any $\delta_j < \Delta(X, K(X, \epsilon), \epsilon)$, which must eventually be true for any converging sequence to $X$.

Now, the previous theorem does not directly prove the Vitali property but rather bypasses it. Demonstrating that the differentiation basis $\mathcal{V}$ may be used to recover Radon-Nikodym derivatives. This, then, is sufficient for Corollary 1 to hold, guaranteeing the existence of predictive states $\epsilon[\hat{X}]$ for $\mu$-almost all $\hat{X}$.

**E. Remarks on convergence**

An important task regarding predictive states is to learn a process’s predictive states—that is, the $\epsilon$-mapping from observed pasts to distributions over futures—from a sufficiently large sample of observations. These learned predictive states may then be used to more accurately predict the process’ future behavior based on behaviors already observed.

The previous three sections avoided constraining the measure $\mu$ on $\mathcal{X}^\mathbb{N}$ to be anything other than finite. It was not assumed to be either stationary or ergodic, in particular. In such cases the $\epsilon$-mapping is time-dependent, as it obviously depends on where futures are split from pasts. To adequately reconstruct $\epsilon[\hat{X}]$ from a single, long observation requires that the process be both stationary and ergodic. Stationarity makes $\epsilon[\hat{X}]$ time-independent, and ergodicity ensures that the probabilities in the limits Eqs. (1) and (4) can be approximated by taking the time-averaged frequencies of occurrence.

The next natural question is how rapidly convergence occurs for each past, in a given process. So far, we only guaranteed that convergence exists, but said nothing on its rate. This is process-dependent. Section V gives several examples of processes and process types with their convergence rate. The most useful way to think of the rate is in the form of “probably-almost-correct”-type statements, as exemplified in the following result:

**Proposition 2.** Let $\mu$ be a probability measure on $\mathcal{X}^\ell$. Let $\eta_{n, \delta}[\hat{X}](U) = \mu(V_{n, \delta}(U)) / \mu(V_{n, \delta})$. For every cylinder set $U$ and $\Delta_1, \Delta_2 > 0$, we have for sufficiently large $\ell$ and small $\delta$:

$$\Pr_{\mu} \left( |\eta_{\ell, \delta}[\hat{X}](U) - \epsilon[\hat{X}](U)| > \Delta_1 \right) < \Delta_2.$$

That is, the probability of an error beyond $\Delta_1$ is less than $\Delta_2$.

This is a consequence of the fact that all $\hat{X}$ must eventually converge. The possible relationships between $\Delta_1$, $\Delta_2$, and $\ell$ in particular is explored in our examples.

**IV. PREDICTIVE STATES FORM A HILBERT SPACE**

Thus far, we demonstrated that for discrete and real $\mathcal{X}$, measures over $\mathcal{X}^\mathbb{N}$ possess a well-defined feature called predictive states that relate how past observations constrain future possibilities. These states are defined by convergent limits that can be approximated from empirical time series in the case of stationary, ergodic processes.

We turn our attention now to the topological and geometric structure of these states, the spaces they live in, and how the structure of these spaces may be leveraged in the inference process. The results make contact between predictive states as elements of a Hilbert space and the well-developed arena of reproducing kernel Hilbert spaces. To do this we introduce several new concepts.

Denote the set of real-valued continuous functions on $\mathcal{X}^\mathbb{Z}$ by $C(\mathcal{X}^\mathbb{Z})$. The set of signed measures on $\mathcal{X}^\mathbb{Z}$, that we call $\mathcal{M}(\mathcal{X}^\mathbb{Z})$, may be thought of as dual to $C(\mathcal{X}^\mathbb{Z})$. This allows us to define a notion of convergence of measures on $\mathcal{X}^\mathbb{Z}$ in relation to continuous functions. We say that
a sequence of measures $\mu_n$ converges in distribution if:

$$\lim_{n \to \infty} \int F(X) d\mu_n(X) = \int F(X) d\mu(X)$$

for all $F \in C(\mathcal{X})$. Convergence in distribution is sometimes referred to as weak convergence but we avoid this vocabulary to minimize confusion—as another, distinct kind of weak convergence is needed in the Hilbert space setting.

A kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ generates a reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ if $k(\cdot, \cdot)$ is positive semi-definite and symmetric [35]. $\mathcal{H}$ is typically defined as a space of functions (from $\mathcal{X} \to \mathbb{R}$), but the kernel allows embedding measures on $\mathcal{X}$ into the function space through $f_\mu(x) = \int k(x,y) d\mu(y)$. This elicits an inner product between any two positive measures $\mu$ and $\nu$:

$$\langle f_\mu | f_\nu \rangle_k := \int \int k(x,y) d\mu(x) d\nu(y).$$

The inner-product space on measures generated by this construction is isometric to the RKHS generated by $k(\cdot, \cdot)$. The embedding of measures into this space is unique if the kernel is characteristic. And, convergence in the norm of the Hilbert space is equivalent to convergence in distribution whenever the kernel is universal [36].

What exactly is the set $\mathcal{H}$ of functions? The equivalence of convergence in norm and convergence in distribution tempts identifying $\mathcal{H}$ with the space of continuous functions, but this is overly optimistic. If it were true—that $\mathcal{H} = C(\mathcal{X})$—then the convergence $\langle F | f_\mu_n \rangle \to \langle F | f_\mu \rangle$ for every $F \in \mathcal{H}$ implies $\mu_n \to \mu$ in norm. This, identifies norm convergence on the Hilbert space with weak convergence, which for Hilbert spaces is identified as the convergence of every inner product.

For infinite-dimensional Hilbert spaces, these two types of convergence cannot be identified. This is illustrated by the simple case of any orthogonal basis $e_i$, for which $\langle F | e_i \rangle \to 0$ is necessary for $F$ to have a finite norm, even though $\|e_i\| \to 1$ by definition. So, while convergence in the norm of $\mathcal{H}$ is equivalent to convergence in distribution of measures, we must conclude that $\mathcal{H}$ can only be a proper subspace of the continuous functions; a fact also noted in [37].

The topology of convergence in distribution is closely re-
lated to the definition of continuity on $\mathcal{X}^N$. It behooves us at this juncture to discuss $\mathcal{X}^N$ not only as a topological space but also as a metric space.

Two useful families of distance metrics, equivalent to the product topology on $\mathcal{X}^N$, are the Euclidean metrics, one for the discrete and real case each:

$$D_{E,\gamma}(X, Y)^2 := \begin{cases} \sum_{i=1}^{\infty} (1 - \delta_{x_iy_i}) \gamma^{2t} & \mathcal{X} \text{ discrete} \\ \sum_{i=1}^{\infty} \|x_i - y_i\|^2 \gamma^{2t} & \mathcal{X} \subset \mathbb{R}^d \end{cases},$$

for some $0 < \gamma < 1$. These distance metrics arise from embedding $\mathcal{X}^N$ in a Hilbert space. Given an orthogonal basis $(e_i)$, the components of this embedding for the discrete case are given by:

$$c_i(X) = \begin{cases} \gamma^{i/|\mathcal{X}|} x_{i/|\mathcal{X}|} = i \mod |\mathcal{X}| \\ 0 \text{ otherwise} \end{cases}$$

and in the continuous case ($\mathcal{X} \subset \mathbb{R}^d$) by:

$$c_i(X) = \gamma^t x_{k,t}, \quad i = k \mod d.$$

Using these distance metrics, the following section introduces an inner product structure on $\mathcal{K}(\mu)$ whose norm metrizes the topology of convergence in distribution. This distance metric is equivalent to the total variation norm. This tells us that the norm is less than the $\mathcal{K}(\mu)$'s natural embedding into a Hilbert space of its own is established, we investigate how well this embedding can be approximated by the approach of reproducing kernel Hilbert spaces.

### B. Embedding predictions in a Hilbert space

The space $\mathcal{K}(\mu)$ of predictive states is a subspace $\mathcal{P}(\mathcal{X}^N)$ of the probability measures over $\mathcal{X}^N$. On $\mathcal{P}(\mathcal{X}^N)$, given any symmetric positive-definite kernel $k : \mathcal{X}^N \times \mathcal{X}^N \rightarrow \mathbb{R}$, we can define an inner product over measures:

$$\langle \mu, \nu \rangle_k := \int \int k(X,Y) d\mu(X)d\nu(Y). \quad (6)$$

Positive-definite means that for any finite set $\{X_i\}$ of $X_i \in \mathcal{X}^N$ and any set $\{c_i\}$ of values $c_i \in \mathbb{R}$, both sets have the same cardinality:

$$\sum_{i,j} k(X_i, X_j)c_ic_j \geq 0,$$

with equality only when $c_i = 0$ for all $i$. If this is true, then the inner product Eq. (6) is positive-definite for all measures. That is, $\langle \mu, \mu \rangle_k \geq 0$ with equality only when $\mu = 0$ [36].

Since $\mathcal{X}^N$ is compact, if the kernel $k$ satisfies the property of being universal, then norm convergence under the inner product defined by $k$ is equivalent to convergence in distribution of measures [36]. A simple example of a universal kernel is the Gaussian radial basis function, when paired with an appropriate distance—namely, one defined from embedding $\mathcal{X}^N$ in a Hilbert space, as our $D_{E,\gamma}$ are [38]. These take the form:

$$k_{\beta,\gamma}(X,Y) := \exp \left( -\frac{D_{E,\gamma}(X,Y)^2}{\beta^2} \right).$$

We denote the associated inner products by $\langle \cdot, \cdot \rangle_{\beta,\gamma}$. The Hilbert space $\mathcal{H}_{\beta,\gamma} := (\mathcal{P}(\mathcal{X}^N), \langle \cdot, \cdot \rangle_{\beta,\gamma})$ defines a Hilbert space, since it has the topology of convergence in distribution and $\mathcal{P}(\mathcal{X}^N)$ is complete in this topology.

When referring to a measure $\mu$ as an element of $\mathcal{H}_{\beta,\gamma}$ we denote it $|\mu|_{\beta,\gamma}$. The inner product of measures is given by:

$$\langle \mu, \nu \rangle_{\beta,\gamma} := \int k_{\beta,\gamma}(X,Y) d\mu(Y), \quad (7)$$

so that:

$$\langle \mu \rangle_{\beta,\gamma} := \int f_{\mu}(X) d\nu(Y).$$

Let $\mathcal{F}_{\beta,\gamma}$ denote the space of all $f_{\mu}$ that can be constructed from Eq. (7). This function space, when paired with the inner product $(f_{\mu}, f_{\nu}) := \langle \mu, \nu \rangle_k$, is isomorphic to $\mathcal{H}_{\beta,\gamma}$. $\mathcal{F}_{\beta,\gamma}$ is then a reproducing kernel Hilbert space with kernel $k_{\beta,\gamma}$.

As the start of Section IV discussed, $\mathcal{F}_{\beta,\gamma} \subset C(\mathcal{X})$. Furthermore, the $\mathcal{F}_{\beta,\gamma}$ are not identical to one another, obeying the relationship $\mathcal{F}_{\beta,\gamma} \subset \mathcal{F}_{\beta',\gamma}$ when $\beta > \beta'$ [39]. However, it is also the case that each $\mathcal{F}_{\beta,\gamma}$ is dense in $C(\mathcal{X}^N)$, so their representative capacity is still quite strong [36].

We note an important rule regarding the scaling of our inner products, as constructed. The distances $D_{E,\gamma}(X,Y)$ have finite diameter on our spaces. Let $\Delta$ denote the diameter of $\mathcal{X}$. For discrete $\mathcal{X}$ we simply have $\Delta = 1$; for $\mathcal{X} \subset \mathbb{R}^d$, $\Delta$ is determined by the Euclidean distance. Then $\mathcal{X}^N$'s diameter is given by $\Delta / \sqrt{1 - \gamma^2}$. Since the Gaussian is bounded by below by $1 - D^2/\beta^2$, for arbitrarily large $\beta$:

$$\|\mu - \nu\|^2_{\beta,\gamma} \leq \frac{\|\mu - \nu\|_{TV}^2 \Delta^2}{(1 - \gamma^2)^2} + O(\beta^{-3}), \quad (8)$$

where $\| \cdot \|_{\beta,\gamma}$ is simply the norm of $\mathcal{H}_{\beta,\gamma}$ and $\| \cdot \|_{TV}$ is the total variation norm. This tells us that the norm is less...
discriminating between measures as $\beta \to \infty$. Naturally, this can be remedied by rescaling the kernel with a $\beta^2$ factor. As it happens, Eq. (8) will be useful later.

C. Finite-length embeddings

Our goal is to study how reproducing kernel Hilbert spaces may be used to encode information about predictive states gleaned from empirical observations. Given that such observations are always finite in length, we must determine whether and in what manner the Hilbert space representations of measures over finite-length observations converges to the Hilbert space representation of a measure over infinite sequences.

Let $\mu_\ell$ denote the measure $\mu$ restricted to $\mathcal{X}_\ell$. Define the restricted distance on $\mathcal{X}_\ell$:

$$D_{E,\gamma}^{(\ell)}(X,Y)^2 := \left\{ \begin{array}{ll}
\sum_{t=1}^{\ell} (1 - \delta_{x_t y_t}) \gamma^{2t} & \mathcal{X} \text{ discrete} \\
\sum_{t=1}^{\ell} \|x_t - y_t\|^2 \gamma^{2t} & \mathcal{X} \subset \mathbb{R}^d
\end{array} \right.,$$

for $X, Y \in \mathcal{X}_\ell$. This gives an important Pythagorean theorem for sequences:

$$D_{E,\gamma}(X,Y)^2 = D_{E,\gamma}^{(\ell)}(x_1 \ldots x_\ell, y_1 \ldots y_\ell)^2 + \gamma^{2 \ell} D_{E,\gamma}(x_{\ell+1} \ldots y_{\ell+1} \ldots)^2. \tag{9}$$

Now, using $D_{E,\gamma}^{(\ell)}$ define kernels $k_{\beta,\gamma}^{(\ell)}$ in the same style as for $\mathcal{X}^N$. These generate inner products on $\mathcal{F}(\mathcal{X}_\ell)$, Denote by $\mathcal{H}_{\beta,\gamma}^{(\ell)}$ the resulting Hilbert spaces. These are related to the original $\mathcal{H}_{\beta,\gamma}$ by the following factorization theorem:

**Proposition 3.** The predictive Hilbert space $\mathcal{H}_{\beta,\gamma}$ factors into $\mathcal{H}_{\beta,\gamma}^{(\ell)} \otimes \mathcal{H}_{\beta,\gamma-\ell,\gamma}$.

Before stating the proof, we should explain the above. The factorization $\mathcal{H}_{\beta,\gamma} = \mathcal{H}_{\beta,\gamma}^{(\ell)} \otimes \mathcal{H}_{\beta,\gamma-\ell,\gamma}$ denotes a separation of the infinite-dimensional $\mathcal{H}_{\beta,\gamma}$ into two pieces—one of which is finite-dimensional, but retains the same kernel parameters and another reparametrize infinite-dimensional Hilbert space. The reparametrization is $\beta \rightarrow \beta^{-\ell}$. This constitutes, essentially, a renormalization-group technique, in which the the topology of words starting at depth $\ell$ is equivalent to a reparametrization of the usual topology. This reparametrization works precisely due to the Pythagorean theorem for sequences Eq. (9).

**Proof.** We are demonstrating an isomorphism—a particularly natural one. Let $\delta_X$ be the Dirac delta measure concentrated on $X$. We note that for any measure $\mu$:

$$|\mu\rangle_{\beta,\gamma} = \int \langle \delta_X \rangle_{\beta,\gamma} d\mu(X).$$

Now, consider the linear function from $\mathcal{H}_{\beta,\gamma}$ to $\mathcal{H}_{\beta,\gamma}^{(\ell)} \otimes \mathcal{H}_{\beta,\gamma-\ell,\gamma}$ that maps:

$$|\delta_X\rangle_{\beta,\gamma} \mapsto |\delta_{x_1 \ldots x_\ell}\rangle_{\beta,\gamma}^{(\ell)} \otimes |\delta_{x_{\ell+1} \ldots}\rangle_{\beta,\gamma-\ell,\gamma},$$

for every $X$. Then by Eq. (9) we can see that this preserves the inner product and so is an isomorphism.

Note that for any of these Hilbert spaces there exists an element corresponding to the constant function $1(X) = 1$ for all $X$. This function always exists in $\mathcal{F}_{\beta,\gamma}$. We denote its corresponding element in $\mathcal{H}_{\beta,\gamma}$ as $\langle 1\rangle_{\beta,\gamma}$, so that $\langle 1\rangle_{\beta,\gamma} = 1$ for all $\mu$. Then the operator $\Pi_{\beta,\gamma}^{(\ell)} : \mathcal{H}_{\beta,\gamma} \rightarrow \mathcal{H}_{\beta,\gamma}^{(\ell)}$ is given by:

$$\Pi_{\beta,\gamma}^{(\ell)} := I^{(\ell)} \otimes \langle 1\rangle_{\beta,\gamma},$$

where $I^{(\ell)}$ is the identity on $\mathcal{H}_{\beta,\gamma}$. It provides the canonical mapping from a measure $\mu$ to its restriction $\mu_\ell$: That is, $\Pi_{\beta,\gamma}^{(\ell)} |\mu\rangle_{\beta,\gamma} = |\mu_\ell\rangle_{\beta,\gamma}$.

Consider the “truncation error”—that is, the residual error remaining when representing a measure by its truncated embedding is small:

**Theorem 4.** There exist isometric embeddings $\mathcal{H}_{\beta,\gamma}^{(\ell)} \rightarrow \mathcal{H}_{\beta,\gamma}^{(\ell')}$ and $\mathcal{H}_{\beta,\gamma} \rightarrow \mathcal{H}_{\beta,\gamma}$ for any $\ell \leq \ell'$. Furthermore, let $\mu$ be any measure and $\mu_\ell$ its restriction to the first $\ell$ observations, and let $|\mu_\ell\rangle_{\beta,\gamma}$ be the embedding of $\mu_\ell$ into $\mathcal{H}_{\beta,\gamma}$. Then $|\mu_\ell\rangle_{\beta,\gamma} \rightarrow |\mu\rangle_{\beta,\gamma}$ as $\ell \rightarrow \infty$, with $\|\mu - \mu_\ell\|_{\beta,\gamma} \sim O(\beta^{-1}\gamma^{\ell'})$.

**Proof.** Let $\lambda_{\beta,\gamma} \in \mathcal{P}(\mathcal{X}^N)$ denote the measure such that $\langle \lambda_{\beta,\gamma}\rangle_{\beta,\gamma} = \langle 1\rangle_{\beta,\gamma}$ for a given $\beta, \gamma$. For a measure $\mu$ with restriction $\mu_\ell$ let $\tilde{\mu}_\ell$ denote the measure on $\mathcal{X}_\ell$ with the property:

$$\tilde{\mu}_\ell(A \times B) = \mu_\ell(A) \lambda_{\beta,\gamma-\ell,\gamma}(B),$$

for $A \in \mathcal{X}_\ell$ and $B \in \mathcal{X}_N$. Then the mapping $\mu_\ell \mapsto \tilde{\mu}_\ell$ is
an isomorphism, since:

\[
\langle \hat{\mu}_\ell | \hat{\nu}_\ell \rangle_{\beta, \gamma} = \int \int k_{\beta, \gamma}(X, Y) d\hat{\mu}_\ell(X) d\hat{\nu}_\ell(X) \\
= \int \int k_{\beta, \gamma}^{(f)}(x_1 \ldots x_\ell, y_1 \ldots y_\ell) d\mu d\nu \\
\times \int \int k_{\beta, \gamma-\ell, \gamma}(x_\ell \ldots x_1, y_\ell \ldots y_1) d\lambda_{\beta, \gamma-\ell, \gamma} d\lambda_{\beta, \gamma-\ell, \gamma} \\
= \langle \mu | \nu \rangle_{\beta, \gamma}^{(f)} \int d\lambda_{\beta, \gamma-\ell, \gamma} \\
= \langle \mu | \nu \rangle_{\beta, \gamma}^{(f)} .
\]

Now, as a result of Eq. (9), note that for any two measures \( \mu \) and \( \nu \):

\[
\langle \mu, \nu \rangle = \int d\mu(x_1 \ldots x_\ell) \int d\nu(y_1 \ldots y_\ell) \exp \left( -\beta^2 D_{\gamma}^{(f)}(x_1 \ldots x_\ell, y_1 \ldots y_\ell) \right) \langle \mu(\cdot | x_1 \ldots x_\ell), \nu(\cdot | y_1 \ldots y_\ell) \rangle_{\beta, \gamma, \ell}
\]

If we combine this fact with the bound Eq. (8), we have the result:

\[
\| \mu - \hat{\mu}_\ell \|_{\beta, \gamma}^2 = \int d\mu(x_1 \ldots x_\ell) \int d\mu(y_1 \ldots y_\ell) \exp \left( -\beta^2 D_{\gamma}^{(f)}(x_1 \ldots x_\ell, y_1 \ldots y_\ell) \right) \| \mu(\cdot | x_1 \ldots x_\ell) - \hat{\mu}_\ell(\cdot | y_1 \ldots y_\ell) \|_{\beta, \gamma-\ell}^2 \\
\leq \int d\mu(x_1 \ldots x_\ell) \int d\mu(y_1 \ldots y_\ell) \| \mu - \hat{\mu}_\ell \|_{TV} \Delta_{\gamma}^{2\ell} = \| \mu - \hat{\mu}_\ell \|_{TV} \Delta_{\gamma}^{2\ell} (1 - \gamma^2)^{\ell/2} .
\]

Thus, \( \| \mu - \hat{\mu}_\ell \|_{\beta, \gamma} \sim O(\beta^{-1} \gamma^\ell) \).

In summary, representing measures \( \mu \) over \( X^N \) by their truncated forms \( \mu_\ell \) leads to a Hilbert space representation that admits an approximate isomorphism to the space of full measures. The resulting truncation error is of order \( O(\beta^{-1} \gamma^\ell) \).

We close this part with a minor note about a lower bound on the distance between measures. Given a word \( w \), the function on \( X^\ell \) that equals 1 when \( X = w \) and zero otherwise has a representation \( |w|_{\beta, \gamma} \) in \( H_{\beta, \gamma}^{(f)} \). (This follows since for finite \( X \), all functions on \( X^\ell \) belong to \( F_{\beta, \gamma}^{(f)} \).) The extension of this to \( H_{\beta, \gamma} \) is \( |w|_{\beta, \gamma} := |w|_{\beta, \gamma} \otimes |\lambda_{\beta, \gamma} \rangle_{\beta, \gamma-\ell} \). This has the convenient property that \( \langle w | \mu \rangle_{\beta, \gamma} = \Pr_{\mu}(w) \). Then, by the Cauchy-Schwarz inequality, for any measures \( \mu \) and \( \nu \) and any word \( w \):

\[
\| \mu - \nu \|_{\beta, \gamma} \geq \frac{| \langle w | \mu - \nu \rangle |}{\sqrt{\langle w | w \rangle_{\beta, \gamma}}} = \frac{| \Pr_{\mu}(w) - \Pr_{\nu}(w) |}{\sqrt{\langle w | w \rangle_{\beta, \gamma}}} .
\]

So, word probabilities function as lower bounds on the Hilbert space norm.

D. Predictive states from kernel Bayes’ rule

A prominent use of reproducing kernel Hilbert spaces is to approximate empirical measures [40]. Given a measure \( \mu \) over a space \( X \) and \( N \) samples \( X_k \) drawn from this space, one constructs an approximate representation of \( \mu \) via:

\[
|\hat{\mu}| := \frac{1}{N} \sum_{k=1}^{N} |\delta_{X_k}| .
\]

In other words, \( \mu \) is approximated as a sum of delta functions centered on the observations. Convergence of this approximation to \( |\mu| \) is (almost surely) \( O(N^{-1/2}) \) [40].

This fact, combined with our Theorem 4, immediately gives the following result for \( H_{\beta, \gamma} \):

**Proposition 4.** Suppose for some \( \mu \in \mathbb{P}(X^N) \) we take \( N \) samples of length \( \ell \), denoted \( \{X_k \in X^\ell \} (k = 1 \ldots N) \), and construct the state:

\[
|\hat{\mu}_\ell, N \rangle_{\beta, \gamma} = \frac{1}{N} \sum_{k=1}^{N} |\delta_{X_k}|_{\beta, \ell} \otimes |\lambda_{\beta, \gamma-\ell, \gamma} \rangle_{\beta, \gamma-\ell} .
\]

Then \( |\hat{\mu}_\ell, N \rangle_{\beta, \gamma} \rightarrow |\mu\rangle \) converges almost surely as \( N, \ell \to \infty \) with error \( O(N^{-1/2} + \beta^{-1} \gamma^\ell) \).
A more nuanced application of RKHS for measures lies in reconstructing conditional distributions [26–28, 40, 41]. Let μ be a joint measure on some \( \mathcal{X} \times \mathcal{Y} \), and let \( \mu|_\mathcal{X} \) and \( \mu|_\mathcal{Y} \) be its marginalizations. Given \( N \) samples \( (X_k, Y_k) \), construct the covariance operators:

\[
\hat{C}_{XX} := \frac{1}{N} \sum_k |\delta_{X_k}| \langle \delta_{X_k} \rangle \quad \text{and} \\
\hat{C}_{YX} := \frac{1}{N} \sum_k |\delta_{Y_k}| \langle \delta_{X_k} \rangle .
\]

Let \( \mu|_\mathcal{X} \) be the conditional measure for \( X \in \mathcal{X} \). For some \( g \in \mathcal{H}_\mathcal{X} \)—the RKHS constructed on \( \mathcal{Y} \) —let \( F_g(X) := \langle g| \mu|_\mathcal{X} \rangle \) be a function on \( \mathcal{X} \). If \( F_g \in \mathcal{H}_\mathcal{X} \) for all \( g \in \mathcal{H}_\mathcal{Y} \), then \( \hat{C}_{YX} \left( \hat{C}_{XX} - \zeta I \right)^{-1} \) converges to \( |\mu|_\mathcal{X} \) as \( N \to \infty \), \( \zeta \to 0 \), with convergence rate \( O( (N\zeta)^{-1/2} + \zeta^{1/2} ) \).

The result tells us that the structure of the conditional measure is compatible with the structures represented by the RKHS.

This is the kernel Bayes’ Rule [41]. It applies to our \( \mathcal{H}_{\beta,\gamma} \), by combining it with our results on truncated representations:

**Theorem 5.** Let \( \mu \in \mathcal{P}(\mathcal{X}^L) \) be a stationary and ergodic process. Suppose we take a long sample \( X \in \mathcal{X}^L \) and from this sample subwords of length \( 2\ell, w_t = x_{t-\ell+1} \ldots x_{t+\ell} \) for \( t = \ell, \ldots, L - \ell \). (There are \( L - 2\ell + 1 \) such words.) Split each word into a past \( \hat{w}_t = x_{t+1} \ldots x_{t+\ell} \) and a future \( \hat{w}_t = x_{t-\ell+1} \ldots x_t \), each of length \( \ell \). Define the operators:

\[
\hat{C}_{\beta,\gamma}(X) = \frac{1}{L - 2\ell + 1} \sum_t |\hat{\delta}_{\hat{w}_t}|_{\beta,\gamma} \otimes |\hat{\delta}_{\hat{w}_t}|_{\beta,\gamma} \quad \text{and} \\
\hat{C}_{\beta,\gamma}(X) = \frac{1}{L - 2\ell + 1} \sum_t |\hat{\delta}_{\hat{w}_t}|_{\beta,\gamma} \otimes |\hat{\delta}_{\hat{w}_t}|_{\beta,\gamma} .
\]

Now, suppose for every \( g \in \mathcal{F}_{\beta,\gamma} \) that \( \langle \epsilon(X), g \rangle \in \mathcal{F}_{\beta,\gamma} \) and \( \langle \eta(X), g \rangle \to \langle \epsilon(X), g \rangle \) at a rate of \( O(h_X(\ell)) \), see Section III E. Then for all \( X \):

\[
\hat{C}_{\beta,\gamma}(X) \left( \hat{C}_{\beta,\gamma}(X) - \zeta \cdot I_{\beta,\gamma} \right)^{-1} |\delta_{\hat{X}}|_{\beta,\gamma}
\]

almost surely converges to \( |\epsilon(X)|_{\beta,\gamma} \) as \( L \to \infty \), \( \ell \to \infty \), and \( \zeta \to 0 \), at the rate \( O( (L\zeta)^{-1/2} + \zeta^{1/2} + \gamma^{-\ell} + h_X(\ell) ) \).

This integrates all our results thus far with the usual kernel Bayes’ rule. Several observations are in order. First, there will (\( \mu \)-almost) always be an \( h_X(\ell) \) as required by this theorem due to our own Thm. 1 and Cor. 1. Second, since \( \epsilon(X) \) is not generally continuous, the theorem’s strict requirements on \( \epsilon(X) \) are not satisfied. That said, weaker versions hold. If \( \langle \epsilon(X), g \rangle \) as a function of \( X \) does not belong to \( \mathcal{F}_{\beta,\gamma} \) as a function of \( X \), then the representational error scaling depends on the precise form of \( \epsilon(X) \). The latter can be obtained by choosing the \( \zeta \)-parameter through cross-validation analysis [40, 41].

**V. EXAMPLES**

We close with a handful of examples and case studies that give further insight to the convergence of \( ||\eta(X) - \epsilon(X)||_{\beta,\gamma} \) for widely-employed process classes—Markov, hidden Markov, and renewal processes.

**A. Order-\( R \) Markov processes**

A Markov process is a stochastic process where each observation \( x_t \) statistically depends only on the previous observation \( x_{t-1} \). An order-\( R \) Markov process is one where each observation \( x_t \) depends only on the previous \( R \) observations \( x_{t-R} \ldots x_{t-1} \). As such, the predictive states are simply given by:

\[
Pr_{\mu}(x | \hat{X}) = \frac{Pr_{\mu}(x_{t-R+1} \ldots x_0 | x)}{Pr_{\mu}(x_{t-R+1} \ldots x_0)} ,
\]

for each \( \hat{X} = x_0 x_{-1} \ldots \). Since the predictive state is entirely defined after a finite number of observations, and this number is bounded by \( R \), there is no conditioning error when \( R \) is taken as the observation length.

**B. Hidden Markov processes**

A hidden Markov model (HMM) \( (S, \mathcal{X}, \{ T^{(x)} \}) \) is defined here as a finite set \( S \) of states, a set \( \mathcal{X} \) of observations, and a set \( T^{(x)} = (T^{(x)}_{ss'}) \) of transition matrices, labeled by elements \( x \in \mathcal{X} \) and whose components are indexed by \( S \) [21]. The elements are constrained so that \( 0 \leq T^{(x)}_{ss'} \leq 1 \) and \( \sum_{x,s'} T^{(x)}_{ss'} = 1 \) for all \( s, s' \in S \). Let \( T = \sum_x T^{(x)} \) and \( \pi \) be its left-eigenvector such that \( \pi T = \pi \). HMMs generate a stochastic process \( \mu \) defined by the word probabilities:

\[
Pr_{\mu}(x_1 \ldots x_\ell) := \sum_{s'} \pi T^{(x_1)} \ldots T^{(x_\ell)} s' .
\]
An extension of HMMs, called generalized hidden Markov models (GHMMs) [21] (or elsewhere observable operator models [22]), is defined as \((V, X, \{T^{(x)}\})\) where \(V\) is a finite-dimensional vector space. The only constraint on the transition matrices \(T^{(x)}\) is that \(T\) have a simple eigenvector of eigenvalue 1, the left-eigenvector is still denoted \(\pi\), the right-eigenvector denoted \(\phi\), and the word probabilities:

\[
Pr_{\mu}(x_1 \ldots x_\ell) := \pi T^{(x_1)} \ldots T^{(x_\ell)} \phi
\]

are positive [21]. GHMMs generate a strictly broader class of processes than finite hidden Markov models can [21, 22, 42], though their basic structure is very similar.

First off, consider sofic processes. A sofic process is one that is not Markov at any finite order, but that is still expressible in a certain finite way. Namely, a sofic process is any that can be generated by a finite-state hidden Markov model with the unifilar property. An HMM has the unifilar property if \(T^{(x)}_{s'} > 0\) only when \(s' = f(x, s)\) for some deterministic function \(f: S \times \mathcal{X} \to S\). Unifilar HMMs are the stochastic generalization of deterministic finite automata in computation theory [43].

The most useful property of sofic processes is that the states of their minimal unifilar HMM correspond exactly to the predictive states, of which there is always a finite number. Unlike with order-\(R\) Markov processes, there is no upper bound to how many observations it may take to \(\delta\)-synchronize the predictive states. However, closed-form results on the synchronization to predictive states for unifilar HMMs is already known: at \(L\) past observations, with \(L \to \infty\), the conditioning error is exponentially likely (in \(L\)) to be exponentially small (in \(L\)) [24]. In terms of our Hilbert space norm, there are constants \(\alpha\) and \(C\) such that

\[
Pr_{\mu}\left(\left\|\eta_{x_1 \ldots x_\ell} - \epsilon \hat{X}\right\|_{\beta, \gamma} > \alpha^\ell\right) < C\alpha^\ell.
\]

As such, for \(\hat{\mu}\)-almost-all pasts, the corresponding convergence rate for the kernel Bayes’ rule applied to a sofic process is \(O\left((L\delta)^{-1/2} + \delta^{1/2} + \min(\lambda, \gamma)^{-\ell}\right)\).

Not all discrete-observation stochastic processes can be generated with a finite-state unifilar hidden Markov model. Though still encompassing only a small slice of processes, generalized hidden Markov models have a considerably larger scope of representation than finite unifilar models, as noted above.

The primary challenge in this setting is to relate the structure of a given HMM to the predictive states of its process. This is achieved through the notion of mixed states. A mixed state \(\rho\) is a distribution over the states of a finite HMM. A given HMM, with the stochastic dynamics between its own states, induces a higher-order dynamic on its mixed states and, critically for analysis, this is an iterated function system (IFS). Under suitable conditions the IFS has a unique invariant measure, and the support of this measure maps surjectively onto the process’ set of predictive states. See Refs. [12] for details on this construction.

If \(\rho = (\rho)\) is a mixed state, then the updated mixed state after observing symbol \(x\) is:

\[
f^{(x)}_s(\rho) := \frac{1}{\sum_{s'} [T^{(x)}\rho]_{s'}} \left[T^{(x)}\rho\right]_s.
\]

Let the matrix \([Df^{(x)}]_{s,s'}(\rho)\) be given by the Jacobian \(\partial f^{(x)} / \partial \rho_s\) at a given value of \(\rho\). There is a statistic, called the Lyapunov characteristic exponent \(\lambda < 0\), such that:

\[
\lambda = \lim_{\ell \to \infty} \frac{1}{\ell} \log \left\| Df^{(x)}(\rho_1) \ldots Df^{(x)}(\rho_1) v \right\|_{\|v\|},
\]

where \(\rho_\ell := f^{(x_{\ell-1})} \ldots f^{(x_1)} (\rho)\), for any vector \(v\) tangent to the simplex, almost any \(\rho\) (in the invariant measure), and almost any \(\hat{X} = x_1, x_2, \ldots\) (in the measure of the prediction induced by \(\rho\)). The exponent \(\lambda\) then determines the rate at which conditioning error for predictive states converges to zero: for all \(\epsilon\) and sufficiently large \(\ell\):

\[
Pr_{\hat{\mu}}\left(\left\|\eta_{x_1 \ldots x_\ell} - \epsilon \hat{X}\right\|_{\beta, \gamma} < C\epsilon^\ell\right) > 1 - \epsilon.
\]

This is somewhat less strict—depending on how rapidly the Lyapunov exponent converges in probability. In any case, for \(\hat{\mu}\)-almost all pasts, the convergence of the kernel Bayes’ rule is \(O\left(\left(L\delta\right)^{-1/2} + \delta^{1/2} + \min(\lambda, \gamma)^{-\ell}\right)\), very similar to the sofic process rate.

We anticipate that these rules still broadly apply to generalized hidden Markov models, though we recommend more detailed analysis on this question.

### C. Renewal processes

A renewal process, usually defined over continuous-time, can be defined for discrete time as follows. A renewal process emits 0s for a randomly selected duration before emitting a single 1 and then randomly selecting a new duration to fill with 0s [44]. Renewal processes can be as simple as Poisson processes, where the probability at any moment of producing another 0 or restarting on 1 is independent of time. Or, they can be far more memoryful, with a unique predictive state for any number of past 0s. While high-memory renewal processes cannot generally be represented by a finite hidden Markov model, they
have only a countable number of predictive states, unlike most hidden Markov models. This follows since every number of past 0s defines a potential predictive state, but the process has no further memory beyond the most recent 1. Said simply, the predictive states are the time since last 1—or some coarse-graining of this indicator in special cases, such as the Poisson process.

A renewal process is specified by the survival probability \( \Phi(n) \) that a contiguous block of 0s has length at least \( n \). The exact probability of a given length is \( F(n) := \Phi(n) - \Phi(n + 1) \). It is always assumed that \( \Phi(1) = 1 \). Further, stationarity requires that \( m := \sum_{n=1}^\infty \Phi(n) \) be finite, as this gives the mean length of a block of 0s. In the most general case the predictive states are given by:

\[
\epsilon_k [\hat{X}] = \left\{ \begin{array}{ll} \epsilon_k & \hat{X} = 0^k 1 \ldots \\ \text{undefined} & \hat{X} = 0^\infty \end{array} \right.,
\]

where the measures \( \epsilon_k \) are recursively defined by the word probabilities:

\[
\Pr_{\epsilon_k} (0^j 1 w) = \frac{F(k + \ell)}{\Phi(k)} \Pr_{\epsilon_0} (w).
\]

Now, it can be easily seen that each past \( \hat{X} \) converges to zero conditioning error at a finite length since (almost) all pasts have the structure \( \ldots 10^k \), and so only the most recent \( k + 1 \) values need be observed to know the predictive state. Therefore the kernel Bayes’ rule has an asymptotic convergence rate for each past \( \hat{X} \) of \( O((L\delta)^{-1/2} + \delta^{1/2} + \gamma^{-\ell}) \). However, this does not tell the entire story, as obviously not all pasts converge uniformly. A probabilistic expression of the conditioning error gives more information:

**Proposition 5.** Suppose \( \mu \) is a renewal process with \( \Phi(n) \propto n^{-\alpha}, \alpha > 1 \). Then there exist constants \( C \) and \( K \) such that:

\[
\Pr_{\mu} \left( \left\| \eta_{x_1 \ldots x_k} - \epsilon(\hat{X}) \right\|_{\beta, \gamma} > C \ell^{-1} \right) > K \ell^{-\alpha}.
\]

That is, the probability the conditioning error decays as \( 1/\ell \) is itself at least power-law decaying in \( \ell \).

**Proof.** Recall from Eq. (10):

\[
\left\| \eta_{x_1 \ldots x_k} - \epsilon(\hat{X}) \right\|_{\beta, \gamma} > \frac{\left| \Pr_{\mu} (w \mid x_1 \ldots x_k) - \Pr_{\mu} (w \mid \hat{X}) \right|}{\sqrt{\langle w | w \rangle_{\beta, \gamma}}},
\]

for every word \( w \), so we can choose any \( w \) and obtain a lower bound on the conditioning error. If our past \( \hat{X} \) has the form \( 0^k 1 \ldots \) for \( k < \ell \), then we are already synchronized to the predictive state and the conditioning error is zero. Thus, we are specifically interested in the case \( k \geq \ell \) and we will further consider the large-\( \ell \) limit.

Now, under our assumptions, \( \Phi(n) = n^{-\alpha} \) for some constant \( \alpha \). For large \( n \), \( F(n) \sim an^{1-\alpha} \). Then for any \( j \):

\[
\Pr_{\mu} (0^j 1 \mid 0^\ell) \sim \frac{\Phi(j + 1)}{\Phi(k)} \sim \alpha^{-1} \frac{(k + j)^{-1} - (k + j + 1)^{-1}}{\ell}.
\]

Meanwhile, so long as \( k \geq \ell \), the truncated prediction has the form:

\[
\Pr_{\mu} (0^j 1 \mid 0^\ell) = \sum_{n=1}^\infty \sum_{p=1}^{n-1} \Phi(p + \ell) F(n + \ell + j) = \frac{\Phi(\ell + j)}{\sum_{p=1}^{\infty} \Phi(p + \ell)} \sim \alpha^{-1} \frac{(1 - \alpha \ell^2)}{\ell} \alpha^{-1} \ell^{-1}.
\]

Now, choose \( 0 < C < \alpha - 1 \) and define:

\[
B = \left( 1 - \frac{C + 1}{\alpha} \right)^{-1}.
\]

Then it can be checked straightforwardly that whenever \( k > B\ell \), we have:

\[
\Pr_{\mu} (1 \mid 0^\ell) - \Pr_{\mu} (1 \mid \hat{X}) \sim \frac{1}{\ell} \left[ \alpha \left( 1 - \frac{\ell}{k} \right) - 1 \right] \geq \frac{C}{\ell}.
\]

The probability that \( k > B\ell \) is given by \( \Phi(B\ell) = B^{-\alpha} \ell^{-\alpha} \). Setting \( K = B^{-\alpha} / \sqrt{(1|1)_{\beta, \gamma}} \) proves the theorem.

Therefore, while every sequence \( \hat{X} \) converges to zero conditioning error at finite length, this convergence is not uniform, to such a degree that the proportion of pasts that retain conditioning error of \( 1/\ell \) has a fat tail in \( \ell \). This is a matter of practical importance that is not cleanly expressed in the big-\( O \) expression of the conditioning error from Thm. 5.

Poisson and renewal processes are only the first and second rungs of the hierarchy of hidden semi-Markov processes [44]. Loosely speaking these are state-dependent renewal processes. The preceding Hilbert space framework extends to this larger process class—a subject recounted elsewhere.
VI. CONCLUDING REMARKS

Taken altogether, the results fill-in important gaps in the foundations of predictive states, while strengthening those foundations for further development, extension, and application. Previously, predictive states were only examined in the context of hidden Markov models, their generalizations, and hidden semi-Markov models. We provided a definition applicable to any stationary and ergodic process with discrete and real-valued observations. Furthermore, we showed that predictive states for these processes are learnable from empirical data, whether through a direct method of partitioning pasts or through indirect methods, such as the reproducing kernel Hilbert space.

One important extension is to continuous-time processes. By exploiting the full generality of Jessen’s and Enomoto’s theorems we believe this extension is quite feasible. As long as the set of possible pasts and futures constitutes a separable space, they should be expressible in the form of a countable basis, to which these theorems may then be applied. The challenge lies in constructing an appropriate and useful basis. We leave this for future work.

We described key properties of the space in which predictive states live. However, predictive states are not merely static objects. They predict the probabilities of future observations. And, once those observations are made, the predictive state may be updated to account for new information. Thus, predictive states provide the stochastic rules for their own transformation into future predictive states. This dynamical process has been explored in great detail in the cases where the process is generated by a finite hidden Markov model—this is found in former work on the \( \epsilon \)-machine and the mixed states of HMMs. Understanding the nature of this dynamic for more general processes, including how it makes contact with other dynamical approaches—such as, stochastic differential equations in the continuous-time setting—also remains for future work.

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