Filling of a circular pipe with a viscous fluid accounting for the surface tension forces

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Abstract. In the framework of this paper, an investigation of the vertical axisymmetric pipe filling with a viscous fluid against the gravity force is implemented. The fluid flow is described by Navier-Stokes and continuity equations. On the free surface, the shear stress is equal to zero and the normal stress is equal to the sum of external and capillary pressure. The free surface motion is defined by the kinematic condition. On the solid wall, the velocity is set to zero. At the inlet section, the Poiseuille profile is used. Along the symmetry axis, the symmetry condition is assigned. A numerical solution to the problem is obtained using the control volume method and SIMPLE algorithm. The method of invariants is applied for compliance of the natural boundary conditions on the free surface. The effect of dimensionless criteria on the steady-state shape of the free surface was parametrically studied.

1. Introduction

Filling of pipes with a fluid is implemented in various technological processes of chemical industry, metallurgy, etc. Mathematical models of that sort of flows include the systems of nonlinear differential equations which are complicated by initially undefined free surface. Only effective numerical methods are capable to provide successful solutions of such problems.

One of the distinctive features of the flow during mold filling is a three-phase contact line that moves along the solid wall. Numerous studies have been done on the theoretical and experimental investigation of the liquid medium flow with a free surface and moving three-phase contact line which are reviewed in [1,2]. However, there is still lack of understanding of the interaction between phases at the three-phase contact line, and there is no comprehensive mathematical description of the physical hypotheses proposed.

Analysis of the mathematical statement, which includes the Navier-Stokes equations, continuity equation, natural boundary conditions on the free surface, no-slip conditions at the contact line for the contact angle values that are not equal to 0 and π, reveals a singularity at the three-phase contact line [3,4]. A slip condition is used to eliminate this singularity on the solid wall occurred in the vicinity of the contact line [5].

This work is aimed to investigate the impact of proposed model of three-phase contact line dynamics on the free surface shape and distribution of the kinematic and dynamic characteristics during circular pipe filling against the gravity force with account for capillary effect.
2. Mathematical statement of the problem

The process of vertical pipe filling with a viscous incompressible fluid is studied taking into account surface tension forces. The flow is supposed to be axisymmetric, and the filling is implemented against gravity force. The solution domain is illustrated schematically in figure 1. Mathematical statement of the problem includes the Navier-Stokes equation and continuity equation written in dimensionless form as follows

\[
\text{Re} \frac{du}{dt} = -\nabla p + \Delta u + W, \quad \nabla \cdot u = 0.
\]

Here, \(u=(u_r, u_z)\) is the velocity vector, \(t\) is the time, \(p\) is the pressure, \(\text{Re} = \rho UR / \mu\) is the Reynolds number, \(W = (0, -W), W = \rho g R^2 / (\mu U)\) is the parameter that represents the ratio of the gravity forces to the viscous forces, \(\rho\) is the fluid density, \(\mu\) is the fluid viscosity, \(g\) is the acceleration of gravity, \(R\) is the pipe radius, and \(U\) is the average velocity at the inlet section \(\Gamma_2\).

![Figure 1. Flow region.](image)

On the free surface \(\Gamma_1\), the boundary conditions are zero shear stress and the equality of normal stress to the sum of external and capillary pressure. Without loss of generality, the external pressure is supposed to be equal to zero. At the inlet section, a parabolic profile of the velocity is assigned. On the solid wall \(\Gamma_3\), the no-slip condition is applied. Along the symmetry axis \(\Gamma_4\), the symmetry condition is used. Thus, the boundary conditions are given as

\[
\begin{align*}
\Gamma_1 : & \quad \frac{\partial u_r}{\partial s} + \frac{\partial u_z}{\partial n} = 0, \quad -p + 2\frac{\partial u_z}{\partial n} = -\frac{K}{Ca}, \\
\Gamma_2 : & \quad u_r = 0, \quad u_z = 2(1-r^2), \\
\Gamma_3 : & \quad u_r = 0, \quad \frac{\partial u_z}{\partial r} = 0, \\
\Gamma_4 : & \quad u_r = 0, \quad u_z = 0,
\end{align*}
\]
where \( u_n, u_s \) are the normal and tangential velocity components on the boundary \( \Gamma_1 \), respectively, \( \text{Ca} = \mu U / \sigma \) is the capillary number, \( K \) is the sum of the principal curvatures of \( \Gamma_1 \), and \( \sigma \) is the surface tension coefficient. The free surface motion is defined by the following kinematic condition

\[
\frac{dr}{dt} = u_r, \quad \frac{dz}{dt} = u_z.
\]

The velocity of the three-phase contact line motion is determined in accordance with the Hoffman law, which correlates the values of dynamic contact angle \( \theta_d \), static contact angle \( \theta_s \), and capillary number in an empirical functional relation [6]

\[
\frac{\cos \theta_d - \cos \theta_s}{\cos \theta_s + 1} = \tan 4.96 \left( \text{Ca} u_{cl} \right)^{0.702},
\]

where \( u_{cl} \) is the dimensionless velocity of the contact line. On the solid wall near three-phase contact line, the slip condition is assumed. Here, tangential velocity decreases from the value at the contact line \( u_{cl} \) down to zero

\[
v(1, z) = u_{cl} \left( \frac{\varepsilon}{u_{cl}} \right)^{(z_{cl} - z)/l}, \quad z_{cl} - l \leq z \leq z_{cl},
\]

where \( l \) is the dimensionless slip length and \( \varepsilon \) is the small parameter.

A numerical solution to the problem is obtained using the finite-difference method with an application of the control volume method and SIMPLE algorithm for calculating velocity and pressure fields in the internal region [7]. The method of invariants is used for compliance of the boundary conditions on the free surface [8]. Free surface represents a set of marker particles distributed uniformly along this boundary. When implementing computational technology, the sum of principal curvatures of the free surface is smoothed by interpolating spline [9].

3. Results
Evolution of the free surface shape \( \Gamma_1 \) at time step of \( \Delta t=0.1 \) for \( \text{Re}=0.01, \theta_s=120, \text{Ca}=0.01, \) and \( W=0 \) is shown in figure 1. Free surface is flat at the initial time instant and, as time goes by, it acquires a steady-state convex shape that moves along the pipe at average velocity. A set of calculations is implemented on the sequence of grids in order to verify approximating convergence of the computational algorithm. Such parameters as dynamic contact angle \( \theta_d \) and \( \chi \), which indicates position of the point B on the symmetry axis in respect to the point C at the contact line, are considered as characteristics of the free surface steady-state shape. Strain-rate tensor intensity \( I_2 \) at the three-phase contact line is considered as a flow dynamic characteristics, which is calculated by formula

\[
I_2 = \left( 2e_{ij}e_{ij} \right)^{0.5},
\]

where \( e_{ij} \) are the components of strain-rate tensor.

Table 1 demonstrates the results calculated at various grids indicating approximating convergence. The impact of the slip length \( l \) on the free surface characteristics is presented in table 2. Increased value of \( l \) provides a decrease in \( I_2 \) at the three-phase contact line and insignificantly affects the free surface shape.
Table 1. Characteristics of the free surface versus grid step $h$
(Re=0.01, $\theta_s=120$, Ca=0.01, W=50, $l=0.2$, and $\varepsilon=0.01$).

| $h$   | $I_2$ | $\chi$ | $\theta_d$ |
|-------|-------|--------|------------|
| 1/10  | 5.28  | 0.345  | 127.2      |
| 1/20  | 12.83 | 0.331  | 126.8      |
| 1/40  | 20.57 | 0.327  | 126.7      |
| 1/80  | 28.10 | 0.327  | 126.6      |

Table 2. Characteristics of the free surface versus slip length
(Re=0.01, $\theta_s=120$, Ca=0.01, W=50, $\varepsilon=0.01$, and $h=1/40$).

| $l$   | $I_2$ | $\chi$ | $\theta_d$ |
|-------|-------|--------|------------|
| 0.1   | 23.76 | 0.329  | 126.7      |
| 0.2   | 20.57 | 0.327  | 126.7      |
| 0.3   | 18.01 | 0.327  | 126.7      |

The fields of the strain-rate tensor intensity obtained for two different grids are shown in figure 2. Distributions are found to be similar along the flow region except for the small vicinity of the three-phase contact line.

![Figure 2](image-url)

**Figure 2.** Distribution of the intensity of strain-rate tensor $I_2$
(Re=0.01, W=50, $\theta_s=120$, Ca=0.01, $l=0.2$, $\varepsilon=0.01$: (a) $h=1/40$, and (b) $h=1/80$).

The flow kinematic and dynamic characteristics are presented in figure 3. Figure 3d demonstrates distribution of the streamlines in the coordinate system, which moves along the pipe at average velocity. Two zones are distinguished in the flow region: a fountain flow zone appears in the vicinity of the free surface, while one-dimensional steady-state Poiseuille flow is observed in the rest of the region.
Effect of the governing parameters on the free surface shape is shown in figure 4. Enhancement of the capillary effect (decreased $Ca$) leads to a decrease in the surface area, while the shape tends to be spherical, and the dynamic contact angle tends to the value of static contact angle. Evidently, as the static contact angle $\theta_s$ increases, the free surface shape becomes more convex, and the dynamic contact angle increases.

4. Conclusion
The influence of proposed model of the contact line dynamics on the flow pattern during circular pipe filling with a viscous fluid accounting for capillary effects was studied. The calculated dynamic characteristics of the flow in the vicinity of three-phase contact line ensure the approximating
convergence of the computational algorithm. The study of the effect of governing parameters on the steady-state shape of the free surface was carried out.

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