A Simple Sublinear Algorithm for Gap Edit Distance

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Abstract

We study the problem of estimating the edit distance between two $n$-character strings. While
exact computation in the worst case is believed to require near-quadratic time, previous work
showed that in certain regimes it is possible to solve the following gap edit distance problem in
sub-linear time: distinguish between inputs of distance $\leq k$ and $> k^2$. Our main result is a very
simple algorithm for this benchmark that runs in time $\tilde{O}(n/\sqrt{k})$, and in particular settles the
open problem of obtaining a truly sublinear time for the entire range of relevant $k$.

Building on the same framework, we also obtain a $k$-vs-$k^2$ algorithm for the one-sided pre-
processing model with $\tilde{O}(n)$ preprocessing time and $\tilde{O}(n/k)$ query time (improving over a recent
$\tilde{O}(n/k + k^2)$-query time algorithm for the same problem [GRS20]).

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1 Introduction

We study the problem of estimating the edit distance between two \( n \)-character strings. There is a classic \( O(n^2) \) dynamic programming algorithm, and fine-grained complexity results from recent years suggest that it is nearly optimal \[BK15, BHS, AHWW16, AB18\]. There have been long lines of works on beating the quadratic time barrier with approximations \[BEK+03, BJKK04, BES06, AO12, AKO10, BEG+18, CDG+18, CGKK18, HSSS19, RSSS19, BR20, KS20, GRS20, RS20, AN20\], or beyond-worst case \[Ukk85, Apo86, Mye86, LMS98, AK12, ABBK17, BK18, Kus19, HRS19, BSS20\]. Motivated by applications where the strings may be extremely long (e.g. bioinformatics), we are interested in algorithms that run even faster, namely in sub-linear time. For exact computation in the worst case, this is unconditionally impossible — even distinguishing between a pair of identical strings and a pair that differs in a single character requires reading the entire input. But in many regimes sublinear algorithms are still possible \[BEK+03, BJKK04, CK06, AN10, AO12, SS17, GKS19, NRRS19, BCLW19, RSSS19\].

Gap Edit Distance: \( k \) vs \( k^2 \)

We give new approximation algorithms for edit distance that run in sublinear time when the input strings are close. To best understand our contribution and how it relates to previous work, we focus on the benchmark advocated by \[GKS19\] of distinguishing input strings whose edit distance is \( \leq k \) from \( \geq k^2 \); we discuss more general parameters later in this section. Notice that we can assume \( \text{wlog} \) that \( k < \sqrt{n} \) (otherwise the algorithm can always accept). Furthermore, for tiny \( k \) there is an unconditional easy lower bound of \( \Omega(n/k^2) \) for distinguishing even identical strings from ones with \( k^2 \) substitutions. So our goal is to design an algorithm that runs in truly sublinear time for \( 1 < k < \sqrt{n} \).

There are two most relevant algorithms in the literature for this setting:

- \[AO12\] (building on \[OR07\]) gave an algorithm that runs in time \( n^{2+o(1)}/k^3 \); in particular, it is sublinear for \( k \gg n^{1/3} \).
- \[GKS19\] gave an algorithm that runs in time \( \tilde{O}(n/k + k^3) \); in particular, it is truly sublinear for \( k \ll n^{1/3} \).

In particular, \[GKS19\] left as an open problem obtaining a sublinear algorithm for \( k \approx n^{1/3} \).

Our main result is a very simple algorithm that runs in time \( \tilde{O}(n/\sqrt{k}) \) and hence is simultaneously sublinear for all relevant values of \( k \).

**Theorem** (Main result (informal); see Theorem 8). We can distinguish between ED\((A, B) \leq k\) and ED\((A, B) = \Omega(k^2) \) in \( \tilde{O}(n/\sqrt{k}) \) time with high probability.

Our algorithm is better than \[AO12, GKS19\] for \( n^{2/7} < k \ll n^{2/5} \) (and is also arguably simpler than both).

**Independent work of Kociumaka and Saha** The open problem of Goldenberg, Krautgamer, and Saha \[GKS19\] was also independently resolved by Kociumaka and Saha \[KS20a\]. They use essentially the same main algorithm (Algorithm below), but use substantially different techniques to implement approximate queries to the subroutine we call MaxAlign\(_k\). Their running time (\( \tilde{O}(n/k + k^2) \)) is faster than ours in the regime where our algorithm is faster than \[AO12\].
Edit distance with preprocessing: results and technical insights

Our starting point for this paper is the recent work of [GRS20] that designed algorithms for edit distance with preprocessing, namely the algorithm consists of two phases:

**Preprocessing** where each string is preprocessed separately; and

**Query** where the algorithm has access to both strings and outputs of the preprocess phase.

A simple and efficient preprocessing procedure proposed by [GRS20] is to compute a hash table for every contiguous substring. In the query phase, this enables an $O(\log(n))$-time implementation of a subroutine that given indices $i_A, i_B$ returns the longest common (contiguous) substring of $A, B$ starting at indices $i_A, i_B$ (respectively). We use a simple modification of this subroutine, that we call $\text{MaxAlign}_k$: given only an index $i_B$ for string $B$, it returns the longest common (contiguous) substring of $A, B$ starting at indices $i_A, i_B$ (respectively) for any $i_A \in [i_B - k, i_B + k]$. (It is not hard to see that for $k$-close strings, we never need to consider other choices of $i_A$ [Ukk85].)

Given access to a $\text{MaxAlign}_k$ oracle, we obtain the following simple greedy algorithm for $k$-vs-$k^2$ edit distance: Starting from pointer $i_B = 1$, at each iteration it advances $i_B$ to the end of the next longest common subsequence returned by $\text{MaxAlign}_k$.

**Algorithm** [GreedyMatch($A, B, k$)]

\[
i_B \leftarrow 1
\]

\[
\text{for } e \text{ from } 1 \text{ to } 2k + 1
\]

\[
i_B \leftarrow i_B + \max(\text{MaxAlign}_k(A, B, i_B), 1)
\]

\[
\text{if } i_B > n \text{ then return } \text{SMALL}
\]

\[
\text{return } \text{LARGE}
\]

Each increase of the pointer $i_B$ costs at most $2k$ in edit distance (corresponding to the freedom to choose $i_A \in [i_B - k, i_B + k]$). Hence if $i_B$ reaches the end of $B$ in $O(k)$ steps, then $\text{ED}(A, B) \leq O(k^2)$ and we can accept; otherwise the edit distance is $> k$ and we can reject. The above ideas suffice to solve $k$-vs-$k^2$ gap edit-distance in $O(k)$ query time after polynomial preprocessing\(^1\).

Without preprocessing, we can’t afford to hash the entire input strings. Instead, we subsample $\approx 1/k$-fraction of the indices from each string and compute hashes for the sampled subsequences. If the sampled indices perfectly align (with a suitable shift in $[\pm k]$), the hashes of identical contiguous substrings will be identical, whereas the hashes of substrings that are $> k$-far (even in Hamming distance) will be different (w.h.p.). This error is acceptable since we already incur a $\Theta(k)$-error for each call of $\text{MaxAlign}_k$. This algorithm would run in $\tilde{O}(n/k)$ time\(^2\), but there is a caveat: when we construct the hash table, it is not yet possible to pick the indices so that they perfectly align (we don’t know the suitable shift). Instead, we try $O(\sqrt{k})$ different shifts for each of $A, B$; by birthday paradox, there exists a pair of shifts that exactly adds up to the right shift in $[\pm k]$. The total run time is given by $\tilde{O}(n/k \cdot \sqrt{k}) = \tilde{O}(n/\sqrt{k})$.

[GRS20] also considered the case where we can only preprocess one of the strings. In this case, we can mimic the strategy from the previous paragraph, but take all $O(k)$ shifts on the preprocessed string, saving the $O(\sqrt{k})$-factor at query time. This gives the following result:

**Theorem** (Informal statement of Theorem [11]). We can distinguish between $\text{ED}(A, B) \leq k$ and $\text{ED}(A, B) = \tilde{\Omega}(k^2)$ with high probability in $\tilde{O}(n)$ preprocessing time of $A$ and $\tilde{O}(n/k)$ query time.

\(^1\)The preprocessing can be made near-linear, but in this setting our algorithm is still dominated by that of [CGK10].

\(^2\)There is also an additive $\tilde{O}(k)$ term like in the preprocessing case, but it is dominated by $\tilde{O}(n/k)$ for $k < \sqrt{n}$.
Our query time improves over a $\tilde{O}(n/k + k^2)$-algorithm in [GRS20] that used similar ideas. (A similar algorithm with low asymmetric query complexity was also introduced in [GKS19].)

Trading off running time for better approximation

By combining our algorithm with the $h$-wave algorithm of [LMS98], we can tradeoff approximation guarantee and running time in our algorithms. The running times we obtain for $k$ vs $k\ell$ edit distance are:

No preprocessing $\tilde{O}(\sqrt{k} + k^{2.5})$ running time for $\ell \in [\sqrt{k}, k]$. (Theorem 16)

One-sided preprocessing $\tilde{O}(\frac{n}{\ell})$ preprocessing time and $\tilde{O}(\frac{n+k}{\ell})$ query time. (Theorem 19)

Two-sided preprocessing $\tilde{O}(\frac{n}{\ell})$ preprocessing time and $\tilde{O}(\frac{k}{\ell})$ query time. (Corollary 20)

Organization

Section 2 gives an overview of the randomized hashing technique we use, as well as a structural lemma theorem for close strings. Section 3 gives a meta-algorithm for distinguishing $k$ versus $k^2$ edit distance. Sections 4, 5, 6 respectively implement this meta-algorithm for two-, zero-, and one-sided preprocessing. Appendix A explains how to trade off running time for improved gap of $k$ versus $k\ell$ edit distance. Appendix B includes the proof of our structural decomposition lemma.

2 Preliminaries

2.1 Rabin-Karp Hashing

A standard preprocessing ingredient is Rabin-Karp-style rolling hashes (e.g., [CLRS09]). We identify the alphabet $\Sigma$ with $1, 2, \ldots, |\Sigma|$. Assume there is also $\$$ $\not\in \Sigma$, which we index by $|\Sigma| + 1$.

Assume before any preprocessing that we have picked a prime $p$ with $\Theta(\log n + \log |\Sigma|)$ digits as well a uniformly random value $x \in \{0, 1, \ldots, p-1\}$. We also have $S \subset [n]$, a subsample of the indices which allows for sublinear preprocessing of the rolling hashes while still successfully testing string matching (up to a $\tilde{O}(n/|S|)$ Hamming error).

|Algorithm 1| InitRollingHash($A, S$) |
|---|---|
|**Input:**| $A \in \Sigma^n$; $S$ array of indices to be hashed |
|**Output:**| $H$, a list of $|S|+1$ hashes |
| $H \leftarrow [0]$ |
| $c \leftarrow 0$ |
| for $i \in S$ then |
| $c \leftarrow cx + A[i] \mod p$ |
| append $c$ to $H$. |
| return $H$ |

Observe that InitRollingHash runs in $\tilde{O}(|S|)$ time and RetrieveRollingHash runs in $\tilde{O}(1)$ time.

The correctness guarantees follow from the following standard proposition.

3We assume that all indices out of range of $A[1, n]$ are equal to $\$$.
Algorithm 2 RetrieveRollingHash$(A, S, H, i, j)$

**Input:** $A \in \Sigma^n$; $S$ array of hashed indices; $H$ list of hashes; $i \leq j$ indices from 1 to $n$.

**Output:** $h$, hash of string

\[
i' \leftarrow \text{least index such that } S[i'] \geq i.
\]

\[
j' \leftarrow \text{greatest index such that } S[j'] \leq j.
\]

**return** $h \leftarrow H[j'] - H[i' - 1]x^{j' - i' + 1} \mod p$

---

**Proposition 1.** Let $A, B \in \Sigma^n$ and $S := \{1, 2, \ldots, n\}$. Let $H_A = \text{InitRollingHash}(A, S)$ and $H_B = \text{InitRollingHash}(B, S)$. The following holds with probability at least $1 - \frac{1}{n^4}$ over the choice of $x$. For all $i_A \leq j_A$ and $i_B \leq j_B$, we have that

\[
\text{RetrieveRollingHash}(A, S, H_A, i_A, j_A) = \text{RetrieveRollingHash}(A, S, H_B, i_B, j_B)
\]

if and only if $A[i_A, j_A] = B[i_B, j_B]$.

This claim is sufficient for our warm-up two-sided preprocessing algorithm. However, for the other algorithms, we need to have $|S| = o(n)$ for our hashing to be sublinear. This is captured by another claim.

**Claim 2.** Let $A, B \in \Sigma^n$ and $S \subseteq \{1, 2, \ldots, n\}$ be a random subset with each element included independently with probability at least $\alpha := \min(\frac{4\ln n}{k}, 1)$. Let $H_A = \text{InitRollingHash}(A, S)$ and $H_B = \text{InitRollingHash}(B, S)$. For any $i \leq j$ in $\{1, \ldots, n\}$ we have

1. If $A[i, j] = B[i, j]$ then $
\text{RetrieveRollingHash}(A, S, H_A, i, j) = \text{RetrieveRollingHash}(B, S, H_B, i, j)$.

2. If Ham$(A[i, j], B[i, j]) \geq k$ then with probability at least $1 - \frac{1}{n^4}$ over the choice of $x$ and $S$,

\[
\text{RetrieveRollingHash}(A, S, H_A, i, j) \neq \text{RetrieveRollingHash}(B, S, H_B, i, j)
\]

**Proof.** Let $A_S$ and $B_S$ be the subsequences of $A$ and $B$ corresponding to the indices $S$. Note that if $A[i, j] = B[i, j]$ then $A_S[i', j'] = B_S[i', j']$, where $i'$ and $j'$ are chosen as in RetrieveRollingHash. Property (1) then follows by Proposition 1.

If Ham$(A[i, j], B[i, j]) \geq k$, the probability there exists $i_0 \in S \cap [i, j]$ such that $A[i_0] \neq B[i_0]$ and thus $A_S[i', j'] \neq B_S[i', j']$ is 1 if $\alpha = 1$ and otherwise at least

\[
1 - (1 - (4\ln n)/k)^k \geq 1 - 1/e^{4\ln n} = 1 - 1/n^4.
\]

If $A_S[i', j'] \neq B_S[i', j']$ then by Proposition 1

\[
\text{RetrieveRollingHash}(A, S, H_A, i, j) \neq \text{RetrieveRollingHash}(B, S, H_B, i, j)
\]

with probability at least $1 - 1/n^4$. Therefore, for a random $S$, RetrieveRollingHash$(A, S, H_A, i, j) \neq \text{RetrieveRollingHash}(B, S, H_B, i, j)$ is at least $1 - 1/n^4 - 1/n^4 > 1 - 1/n^3$. Thus, property (2) follows. \qed

### 2.2 Structural Decomposition Lemma

**Definition 1** ($k$-alignment and approximate $k$-alignment).

Given strings $A, B$, we say that a substring $B[i_B, i_B + d - 1]$ with $1 \leq i_B, i_B + d - 1 \leq n$ is in $k$-alignment in $A[i_A, i_A + d - 1]$ if $|i_A - i_B| \leq k$ and $A[i_A, i_A + d - 1] = B[i_B, i_B + d - 1]$. If instead we have $|i_A - i_B| \leq 3k$ and $\text{ED}(A[i_A, i_A + d - 1], B[i_B, i_B + d - 1]) \leq 3k$, we say that $B[i_B, i_B + d - 1]$ is in approximate $k$-alignment with $A[i_A, i_A + d - 1]$. We say that $B[i_B, i_B + d - 1]$ has a (approximate) $k$-alignment in $A$ if there is an $i_A$ with $|i_A - i_B| \leq k$ such that $B[i_B, i_B + d - 1]$ is in (approximate) $k$-alignment with $A[i_A, i_A + d - 1]$. 


For all our algorithms we need the following decomposition lemma. The proof is deferred to Appendix B.

**Lemma 3.** Let $A, B \in \Sigma^*$ be strings such that $ED(A, B) \leq k$. Then, $A$ and $B$ can be partitioned into at $2k + 1$ intervals $I_A^1, \ldots, I_{2k+1}^A; I_B^1, \ldots, I_{2k+1}^B$, respectively, and a partial monotone matching $\pi : [2k + 1] \rightarrow [2k + 1] \cup \{\perp\}$ such that

- Unmatched intervals are of length at most 1, and
- For all $i$ in the matching, $B[I_B^\pi(i)]$ is in $k$-alignment with $A[I_A^i]$.

## 3 A meta-algorithm for distinguishing $k$ vs. $k^2$

In this section, we present GreedyMatch (Algorithm 3), a simple algorithm for distinguishing $ED(A, B) \leq O(k)$ from $ED(A, B) \geq \Omega(k^2)$. The algorithm assumes access to data structure MaxAlign* as defined below. In the following sections, we will present different implementations of this data structure for the case of two-sided, one-sided, and no preprocessing.

Define $\text{MaxAlign}_k(A, B, i_B)$ to be a function which returns $d \in [1, n]$. We say that an implementation of $\text{MaxAlign}_k(A, B, i_B)$ is correct if with probability 1 it outputs the maximum $d$ such that $B[i_B, i_B + d - 1]$ has a $k$-alignment in $A$, and if no $k$-alignment exists, it outputs $d = 0$. We say that an implementation is approximately correct if the following are true.

1. Let $d$ be the maximal such that $B[i_B, i_B + d - 1]$ has a $k$-alignment in $A$. With probability 1, $\text{MaxAlign}_k(A, B, i_B) \geq d$.
2. With probability at least $1 - 1/n^2$, $B[i_B, i_B + \text{MaxAlign}_k(A, B, i_B) - 1]$ has an approximate $k$-alignment in $A$.

We say that an implementation is half approximately correct if the following are true.

1. Let $d$ be the maximal such that $B[i_B, i_B + d - 1]$ has a $k$-alignment. With probability 1, $\text{MaxAlign}_k(A, B, i_B) > d/2$ (unless $d = 0$).
2. With probability at least $1 - 1/n^2$, $B[i_B, i_B + \text{MaxAlign}_k(A, B, i_B) - 1]$ has an approximate $k$-alignment in $A$.

**Algorithm 3** GreedyMatch($A, B, k$)

**Input:** $A, B \in \Sigma^n$, $k \leq n$

**Output:** SMALL if $ED(A, B) \leq k$ or LARGE if $ED(A, B) > 40k^2$

```
\begin{algorithm}
\texttt{i}_B \leftarrow 1
\text{for } e \text{ from 1 to } 2k + 1
\phantom{0} \texttt{i}_B \leftarrow \texttt{i}_B + \max(\text{MaxAlign}_k(A, B, \texttt{i}_B), 1)
\phantom{0} \text{if } \texttt{i}_B > n
\phantom{0} \text{return } \text{SMALL}
\text{return } \text{LARGE}
\end{algorithm}
```

We now give the following correctness guarantee.
Lemma 4. If MaxAlign\(_k\) is approximately correct and ED(\(A, B\)) \leq k, then with probability 1, GreedyMatch(\(A, B, k\)) returns SMALL. If MaxAlign\(_k\) is half approximately correct and ED(\(A, B\)) \leq k/(2\log n), then with probability 1, GreedyMatch(\(A, B, k\)) returns SMALL. If MaxAlign\(_k\) is (half) approximately correct and ED(\(A, B\)) > 40k\(^2\), then with probability 1 - \(\frac{1}{n}\), GreedyMatch(\(A, B, k\)) returns LARGE. Further, GreedyMatch(\(A, B, k\)) makes \(O(k)\) calls to MaxAlign\(_k\) and otherwise runs in \(O(k\log n)\) time.

Proof. If MaxAlign\(_k\) is approximately correct and if ED(\(A, B\)) \leq k then by Lemma 3 \(B\) can be decomposed into \(2k + 1\) intervals such that they are each of length at most 1 or they exactly match the corresponding interval \(A\), up to a shift of \(k\). In the algorithm, if \(i_B\) is in one of these intervals, then MaxAlign\(_k\) finds the rest of the interval (and perhaps more). Then, the algorithm will reach the end of \(B\) in \(2k + 1\) steps and output SMALL.

Let \(k' = k/(2\log n)\). If MaxAlign\(_k\) is half approximately correct and ED(\(A, B\)) \leq k' then by Lemma 3 \(B\) can be decomposed into \(2k' + 1\) intervals such that they are each of length at most 1 or they exactly match the corresponding interval \(A\), up to a shift of \(k\). In the algorithm, if \(i_B\) is in one of these intervals, then MaxAlign\(_k\) finds more than half of the interval. Thus, it takes at most \(\log n\) steps for the algorithm to get past each of the \(2k' + 1\) intervals. Thus, the algorithm will reach the end of \(B\) in \((2k' + 1)(\log n) < 2k + 1\) steps and output SMALL.

For the other direction, it suffices to prove that if the algorithm outputs SMALL then ED(\(A, B\)) \leq 40k\(^2\). If MaxAlign\(_k\) is (half) approximately correct, and the algorithm outputs SMALL, with probability at least 1 - \(1/n\) over all calls to MaxAlign\(_k\), there exists a decomposition of \(B\) into \(2k + 1\) intervals such that each is either of length 1 or has an approximate \(k\)-alignment in \(A\). Thus, there exists a sequence of edit operations from \(B\) to \(A\) by

1. deleting the at most \(2k + 1\) characters of \(B\) which do not match,
2. modifying at most \(3k\) characters within each interval of \(B\), and
3. adding/deleting \(6k\) characters between each consecutive pair of exactly-matching intervals (and before the first and after the last interval), since each match had a shift of up to \(3k\).

This is a total of \(2k + 1 + 3k(2k + 1) + 6k(2k + 2) \leq 40k^2\) operations. Thus, if ED(\(A, B\)) > 40k\(^2\), GreedyMatch(\(A, B, k\)) return LARGE with probability at least 1 - \(\frac{1}{n}\). The runtime analysis follows by inspection. \(\square\)

By Lemma 3 it suffices to implement MaxAlign\(_k\) efficiently and with 1/\(\text{poly}(n)\) error probability in various models.

4 Warm-up: two-sided Preprocessing

As warm-up, we give an implementation of MaxAlign\(_k\) that first preprocesses \(A\) and \(B\) (separately) for \(\text{poly}(n)\) time\(^4\) and then implement MaxAlign\(_k\) queries in \(O(\log(n))\) time.

Algorithm 3 takes as input a string \(A\) and produces \((H_A, T_A)\), the rolling hashes of \(A\) and a collection of hash tables. We let \(H_B, T_B\) denote the corresponding preprocessing output for \(B\). Algorithm 5 gives a correct implementation of MaxAlign\(_k\) with the assistance of this preprocessing.

**Lemma 5.** TwoSidedMaxAlign\(_k\) is a correct implementation of MaxAlign\(_k\).

\(^4\)It is not hard to improve the preprocessing time to \(\tilde{O}(n)\). We omit the details since this algorithm would still not be optimal for the two-sided preprocessing setting.
Algorithm 4 TwoSidedPreprocessingₖ(ᴬ)

Input: ⁰ ∈ Σₙ, k ≤ n
Output: (Hᴬ, Tᴬ), a collection of hashes

Hᴬ ← InitRollingHash(ᴬ, [1, n])
Tᴬ ← n × n matrix of hash tables

for i from 1 to n
    for j from i to n
        for a from −k to k
            if [i + a, j + a] ⊆ [1, n], add RetrieveRollingHash(ᴬ, [1, n], Hᴬ, i + a, j + a) to T[i, j]
return (Hᴬ, Tᴬ)

Algorithm 5 TwoSidedMaxAlignₖ(ᴬ, B, iᴮ)

Input: ⁰ ∈ Σₙ, B ∈ Σₙ, k ≤ n, iᴮ ∈ [1, n]
Output: d ∈ [0, n].

Binary search to find maximal d ∈ [0, n − iᴮ + 1] such that

RetrieveRollingHash(B, [1, n], Hᴮ, iᴮ, iᴮ + d − 1) ∈ Tᴰ[iᴮ, iᴮ + d − 1]

return d

Proof. Observe that TwoSidedMaxAlign is correct if for all a ∈ [−k, k], RetrieveRollingHash(ᴬ, [1, n], Hᴬ, iᴮ + a, iᴮ + d + a) = RetrieveRollingHash(B, [1, n], Hᴮ, iᴮ, iᴮ + d) if and only if A[iᴮ + a, iᴮ + d + a] = B[iᴮ, iᴮ + d]. By Claim 1 and the union bound, this happens with probability at least 1 − 1/ⁿ. □

Theorem 6. When both A and B are preprocessed for poly(n) time, we can distinguish between ED(A, B) ≤ k and ED(A, B) > 40k² in ˜(k) time with probability 1 − 1/n.

Remark. Note that [CGK16]’s algorithm obtains similar guarantees while only spending O(log(n)) query time. Further, sketching algorithms for edit distance often achieve much better approximation factors, but the preprocessing is often not near-linear (e.g., [BZ16]).

Proof of Theorem 6. By Lemma 4, TwoSidedMaxAlign is correct (and thus approximately correct) so by Lemma 4 succeeds with high enough probability that GreedyMatch outputs the correct answer with probability at least 1 − 1/n.

By inspection, the preprocessing runs in poly(n) time. Further, as the binary search, hash computation, and table lookup are all ˜(1) operations, TwoSidedMaxAlign runs in ˜(1) time, so the two-sided preprocessing version of GreedyMatch runs in ˜(k) time. □

5 Main Result: k vs k² with No Preprocessing

As explained in the introduction, for the no preprocessing case, we take advantage of the fact that any c ∈ [−k, k] can be written as a√k + b, there a, b ∈ [−√k, √k]. Thus, if for A we compute a rolling hash tables according to S + a√k := {s + a√k, s ∈ S} ∩ [1, n] for a ∈ √k. Likewise, for B we compute rolling hash tables according to S − b := {s − b, s ∈ S} ∩ [1, n]. Then, if we

5Document exchange (e.g., [BZ16, Hae19]) is similar to the one-sided preprocessing model, but A and B are never brought together (rather a hash of A is sent to B).

6We have √k as shorthand for √k.
seek to compare \( A[i_B + c, i_B + c + d - 1] \) and \( B[i_B, i_B + d - 1] \), it essentially suffices to compare \( A[i_B + a\sqrt{k} + k, i_B + a\sqrt{k} + d - 1 - k] \) and \( B[i_B - b + k, i_B + d - 1 - b - k] \).

Before calling GreedyMatch, we call two methods ProcessA and ProcessB which compute these hash tables. Note that the procedures are asymmetrical. These take \( \tilde{O}(n/\sqrt{k}) \) time each.

Algorithm 6 ProcessA\(_k\)(A)

Input: \( A \in \Sigma^n \)

\[
\text{for } a \text{ from } -\sqrt{k} \text{ to } \sqrt{k} \\
H_{A,a\sqrt{k}} \leftarrow \text{InitRollingHash}(A, S + a\sqrt{k}) \\
\text{return } \{H_{A,a\sqrt{k}} : a \in [-\sqrt{k}, \sqrt{k}]\}
\]

Algorithm 7 ProcessB\(_k\)(B)

\[
\text{for } b \text{ from } -\sqrt{k} \text{ to } \sqrt{k} \\
H_{B,b} \leftarrow \text{InitRollingHash}(B, S - b) \\
\text{return } \{H_{B,b} : b \in [-\sqrt{k}, \sqrt{k}]\}
\]

Algorithm 8 MaxAlign\(_k\)(A, B, i\(_B\))

Input: \( A \in \Sigma^n, B \in \Sigma^n, k \leq n, i_B \in [1, n] \)

\[
d_0 \leftarrow 2k, d_1 \leftarrow n - i_B + 1 \\
\text{while } d_0 \neq d_1 \text{ do} \\
\quad d_{\text{mid}} \leftarrow [(d_0 + d_1)/2] \\
\quad \text{if } d \leq 2k \text{ then return True} \\
\quad L_A, L_B \leftarrow 0 \\
\quad \text{for } a \text{ from } -\sqrt{k} \text{ to } \sqrt{k} \\
\quad \quad h \leftarrow \text{RetrieveRollingHash}(A, S + a\sqrt{k}, H_{A,a\sqrt{k}}, i_B + a\sqrt{k} + k, i_B + d_{\text{mid}} - k - 1 + a\sqrt{k}) \\
\quad \quad \text{append } h \text{ to } L_A \\
\quad \text{for } b \text{ from } -\sqrt{k} \text{ to } \sqrt{k} \\
\quad \quad h \leftarrow \text{RetrieveRollingHash}(B, S - b, H_{B,b}, i_B + k - b, i_B + d_{\text{mid}} - k - 1 - b) \\
\quad \quad \text{append } h \text{ to } L_B \\
\quad \text{sort } L_A \text{ and } L_B \\
\quad \text{if } L_A \cap L_B \neq \emptyset \\
\quad \quad \text{then } d_0 \leftarrow d_{\text{mid}} \\
\quad \text{else } d_1 \leftarrow d_{\text{mid}} - 1. \\
\text{return } d_0
\]

Lemma 7. MaxAlign\(_k\) is approximately correct.

Proof. First, consider any \( d \geq 1 \) such that \( B[i_B, i_B + d - 1] \) has a \( k \)-alignment in \( A \). We seek to show that MaxAlign\(_k\)(A, B, i\(_B\)) \geq d with probability 1. Note that the output of MaxAlign\(_k\) is always at least \( 2k \), so we may assume that \( d > 2k \). By definition of \( k \)-alignment, there exists \( c \in [-k, k] \) such that \( A[i_B + c, i_B + d - 1 + c] = B[i_B, i_B + d - 1] \). Note that there exists \( a, b \in [-\sqrt{k}, \sqrt{k}] \) such that

\[7\]We need to “shave” \( k \) from each end of the substrings as we need to ensure that \([i_B - b + k, i_B + d - 1 - b - k] \subset [i_B, i_B + d - 1], \text{ etc.} \)
\[ a\sqrt{k} + b = c \] and so
\[ A[i_B + k + a\sqrt{k}, i_B + d - k - 1 + a\sqrt{k}] = B[i_B + k - b, i_B + d - k - 1 - b]. \]

By applying Claim 2, we have with probability 1 that
\[ \text{RetrieveRollingHash}(A, S + a\sqrt{k}, H_{A,a\sqrt{k}}, i_B + k + a\sqrt{k}, d - k - 1 + a\sqrt{k}) = \text{RetrieveRollingHash}(B, S - b, H_{B,b}, i_B + k - b, i_B + d - k - 1 - b). \]

Therefore, in the implementation of MaxAlign_k(A, B, i_B), if \( d_{mid} = d \), then \( L_A \) and \( L_B \) will have nontrivial intersection, so the output of the binary search will be at least \( d \), as desired. Thus, MaxAlign_k(A, B, i_B) will output at least the length of the maximal k-alignment.

Second, we verify that MaxAlign_k outputs an approximate k-alignment. Let \( d \) be the output of MaxAlign_k, either \( d = 2k \), in which case \( B[i_B, i_B + d - 1] \) trivially is in approximate k-alignment with \( A[i_B, i_B + d - 1] \) or \( d > 2k \). Thus, for that \( d \), the binary search found that \( L_A \cap L_B \neq \emptyset \) and so there exists \( a, b \in [-\sqrt{k}, \sqrt{k}] \) such that
\[ \text{RetrieveRollingHash}(A, S + a\sqrt{k}, H_{A,a\sqrt{k}}, i_B + k + a\sqrt{k}, d - k - 1 + a\sqrt{k}) = \text{RetrieveRollingHash}(B, S - b, H_{B,b}, i_B + k - b, i_B + d - k - 1 - b). \]

Applying Claim 2 over all \( \tilde{O}(\sqrt{k^2}) = \tilde{O}(k) \) comparisons of hashes made during the algorithm, with probability at least \( 1 - 1/n^3 \), we must have that
\[ \text{ED}(A[i_B + k + a\sqrt{k}, d - k - 1 + a\sqrt{k}], B[i_B + k - b, i_B + d - k - 1 - b]) \leq k. \]
Let \( c := a\sqrt{k} + b \) then we have that
\[ \text{ED}(A[i_B + k + c - b, i_B + d - k - 1 + c - b], B[i_B + k - b, i_B + d - k - 1 - b]) \leq k \]
so
\[ \text{ED}(A[i_B + c, i_B + d - 1 + c], B[i_B, i_B + d - 1]) \leq 3k. \]
Since \( c = a\sqrt{k} + b \in [-3k, 3k] \), we have that \( B[i_B, i_B + d - 1] \) has an approximate k-alignment, as desired.

**Theorem 8.** For \( k \leq O(\sqrt{n}) \), with no preprocessing, we can distinguish between ED(A, B) \( \leq k \) and ED(A, B) \( > 40k^2 \) in \( \tilde{O}(n/\sqrt{k}) \) time with probability at least \( 1 - \frac{1}{n} \).

**Proof.** By Lemma 7, MaxAlign_k is approximately correct so by Lemma 4 succeeds with high enough probability that GreedyMatch outputs the correct answer with probability at least \( 1 - \frac{1}{n} \).

By inspection, both ProcessA_k and ProcessB_k run in \( \tilde{O}(n/\sqrt{k}) \) time in expectation. Further, MaxAlign_k runs in \( \tilde{O}(\sqrt{k}) \) time, so GreedyMatch runs in \( \tilde{O}(n/\sqrt{k} + k^{3/2}) = \tilde{O}(n/\sqrt{k}) \) time.

### 6 One-sided Preprocessing

For the one-sided preprocessing, we desire to get near-linear preprocessing time. To do that, MaxAlign_k shall be half approximately correct rather than approximately correct.

Recall as before we preselect \( S \subset [1, n] \) with each element included i.i.d. with probability \( q := \min(\frac{4k}{n}, 1) \). Also assume that every multiple of \( k \) is in \( S \) and that \( n - 1 \) is in \( S \). This only increases the size of \( S \) by \( n/k \), and does not hurt the success probability of Claim 2. To achieve near-linear preprocessing, we only store \( \text{RetrieveRollingHash}(A, S + a_i, H_{A,a_i}, i + a_i, i + 2^{b_i} - 1 + a_i) \), when \((S + a) \cap [i + a, i + 2^{b} + 1 + a]\) changes. This happens when \( i \in (S + 1) \cup (S - 2^{b} + 1) \).
Algorithm 9 OneSidedPreprocess\(A_k(A)\)

\[
\begin{align*}
\text{for } a \text{ from } -k \text{ to } k & \quad H_{A,a} \leftarrow \text{InitRollingHash}(A, S + a) \\
T_A & \leftarrow [\log n] \times \frac{n}{k} \text{ matrix of empty hash tables} \\
\text{for } i_0 \text{ in } [\lfloor \log n \rfloor] & \\
\text{for } a \text{ from } -k \text{ to } k & \\
\text{for } i \text{ in } ((S + 1) \cup (S - 2^{i_0} + 1)) \text{ with } [i + a, i + 2^{i_0} - 1 + a] \subseteq [n] & \\
h & \leftarrow \text{RetrieveRollingHash}(A, S + a, H_{A,a}, i + a, i + 2^{i_0} - 1 + a) \\
\text{add } h \text{ to } T_A[i_0, [i/k] - 1]. & \\
\text{add } h \text{ to } T_A[i_0, [i/k]]. & \\
\text{add } h \text{ to } T_A[i_0, [i/k] + 1]. & \\
\text{return } T_A & 
\end{align*}
\]

Claim 9. OneSidedPreprocess\(A(A)\) runs in \(\tilde{O}(n)\) time in expectation.

\[\text{Proof. Computing } \text{InitRollingHash}(A, S + a) \text{ takes } |S| = \tilde{O}(n/k) \text{ time in expectation. Thus, computing the } H_{A,a} \text{'s takes } \tilde{O}(n) \text{ time. The other loops take (amortized) } O(1) \cdot O(k) \cdot \tilde{O}(n/k) = \tilde{O}(n) \text{ time.}
\]

Before we call GreedyMatch, we need to initialize the hash function for \(B\) using OneSidedProcess\(B(B)\). This takes \(\tilde{O}(n/k)\) time in expectation.

Algorithm 10 OneSidedProcess\(B(B)\)

\[
\begin{align*}
\text{return } H_B & \leftarrow \text{InitRollingHash}(B, S) 
\end{align*}
\]

Algorithm 11 OneSidedMaxAlign\(_k(A, B, i_B)\)

\[
\begin{align*}
\text{Input: } A \in \Sigma^n, B \in \Sigma^n, k \leq n, i_0 \in [1, n] & \\
\text{for } d \in [2^{\lfloor \log n \rfloor}, 2^{\lfloor \log n \rfloor - 1}, \ldots, 1] & \\
\text{if } \text{RetrieveRollingHash}(B, S, H_B, i_B, i_B + d - 1) \in T_A[\log d, [i_B/k]] & \text{then return } d \\
\text{return } 0 & 
\end{align*}
\]

Lemma 10. OneSidedMaxAlign\(_k(A, B)\) is half approximately correct.

\[\text{Proof. First, consider the maximal } d' \geq 1 \text{ a power of two such that } B[i_B, i_B + d' - 1] \text{ has a } k\text{-alignment in } A. \text{ We seek to show that OneSidedMaxAlign}\(_k(A, B, i_B)\) \geq d' \text{ with probability } 1. \text{ By definition of } k\text{-alignment, there exists } a \in [-k, k] \text{ such that } A[i_B+a, i_B+d'-1+a] = B[i_B, i_B+d'-1]. \text{ By applying Claim }[2] \text{ we have with probability } 1 \text{ that}
\]

\[
\text{RetrieveRollingHash}(A, S + a, H_{A,a}, i_B + a, i_B + d' - 1 + a)
\]

\[= \text{RetrieveRollingHash}(B, S, H_B, i_B, i_B + d' - 1).
\]

Let \(i_B'\) be the least integer in \(((S + 1) \cup (S - d' + 1)) \cap [n]\) which is at least \(i_B\). Since \(S\) contains every multiple of \(k\) (and \(n-1\)), we must have that \(|i_B' - i_B| \leq k\). Therefore,

\[
\begin{align*}
\text{RetrieveRollingHash}(A, S + a, H_{A,a}, i_B + a, i_B + d' - 1 + a)
\quad = \text{RetrieveRollingHash}(A, S + a, H_{A,a}, i_B' + a, i_B' + d' - 1 + a) \\
\quad \in T_A[\log d, [i_B'/k] + \{-1, 0, 1\}].
\end{align*}
\]
Since \(|i_B'/k - i_B/k| \in \{-1, 0, 1\}\). We have that if \(d = d'\), \(\text{RetrieveRollingHash}(B, S, H_B, i_B, i_B + d' - 1) \in T_A[\log d, [i_B/k]].\) Thus, \(\text{OneSidedMaxAlign}_k(A, B, i_B)\) will output at least more than half the length of the maximal \(k\)-alignment.

Second, we verify that \(\text{OneSidedMaxAlign}_k\) outputs an approximate \(k\)-alignment. Let \(d\) be the output of \(\text{OneSidedMaxAlign}_k\), either \(d = 0\), in which case \(B[i_B, i_B + d - 1]\) trivially is in approximate \(k\)-alignment with \(A[i_B, i_B + d - 1]\) or \(d \geq 1\). Thus, for that \(d\), the search found that \(\text{RetrieveRollingHash}(B, S, H_B, i_B, i_B + d' - 1) \in T_A[\log d, [i_B/k]].\) Thus, there exists, \(i_B'\) with \(|i_B'/k - i_B/k| \leq 1\) and \(a \in [-k, k]\) such that

\[
\text{RetrieveRollingHash}(A, S + a, H_A, i_B', i_B' + d' - 1 + a) = \text{RetrieveRollingHash}(B, S, H_B, i_B, i_B + d' - 1).
\]

Applying Claim 2 over all \(\tilde{O}(k)\) potential comparisons of hashes made during the algorithm, with probability at least \(1 - 1/n^3\), we must have that

\[
\text{ED}(A[i_B' + a, i_B' + a + d' - 1], B[i_B, i_B + d' - 1]) \leq k.
\]

Note that \(|i_B' + a - i_B| \leq |i_B - i_B'| + |a| \leq 3k\). Thus \(B[i_B, i_B + d' - 1]\) has an approximate \(k\)-alignment, as desired. \(\square\)

**Theorem 11.** For all \(A, B \in \Sigma^*\). When \(A\) is preprocessed for \(\tilde{O}(n)\) time in expectation, we can distinguish between \(\text{ED}(A, B) \leq k/(2\log n)\) and \(\text{ED}(A, B) > 40k^2\) in \(\tilde{O}(n/k)\) time with probability at least \(1 - 1/n\) over the random bits in the preprocessing (oblivious to \(B\)).

**Proof.** By Lemma 10, \(\text{OneSidedMaxAlign}_k\) is half approximately correct so by Lemma 11 succeeds with high enough probability that \(\text{GreedyMatch}\) outputs the correct answer with probability at least \(1 - 1/n\).

By Claim 3 the preprocessing runs in \(\tilde{O}(n)\) time. Also \(\text{OneSidedProcessB}\) runs in \(\tilde{O}(n/k)\) time. Further, \(\text{OneSidedMaxAlign}_k\) runs in \(\tilde{O}(1)\) time, as performing the power-of-two search, computing the hash, and doing the table lookups are \(\tilde{O}(1)\) operations), so the one-sided preprocessing version of \(\text{GreedyMatch}\) runs in \(\tilde{O}(n/k + k) = \tilde{O}(n/k)\) time. \(\square\)

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A Trading off running time for better approximation

In this appendix, we show how to extend the results of the main body to distinguishing edit distance $k$ vs. $k\ell$.

A.1 Preliminaries: the $h$-wave algorithm

The works of [LMS98, GRS20] show that if one preprocesses both $A$ and $B$, then the exact edit distance between $A$ and $B$ can be found by the following $O(k^2)$-sized dynamic program (called an $h$-wave). The DP state is represented by a table $h[i, j]$, where $i \in [0, k]$ and $j \in [-k, k]$, initialized with $h[0, 0] = 0$ and $h[0, j] = -\infty$ for all $j \in [-k, k] \setminus \{0\}$. The transitions for $i \geq 1$ are

$$
h[i, j] = \max\begin{cases}
    h[i - 1, j - 1] + 1 \\
    h[i - 1, j] + \max(d, 1) \\
    h[i - 1, j + 1]
\end{cases}
$$

where $d$ is maximal such that $A[h[i - 1, j] + 1, h[i - 1, j] + d] = B[h[i - 1, j] + j + 1, h[i - 1, j] + j + d]$. Intuitively, $h[i, j]$ is the farthest length $n'$ such that $A[1, n']$ and $B[1, n' - j]$ have edit distance at most $i$. Then, $\text{ED}(A, B) \leq k$ if and only if $h[k, 0] = n$.

A.2 Approximate $h$-wave and MaxShiftAlign$_\ell$

We speed-up the original $h$-wave algorithm by considering a sparsified $h$-wave, where we store $h[i, j]$ with $i \in [0, k]$ and $j \in [-k, k]$ such that $j$ is a multiple of $\ell$. We again initialize $h[0, j] = 0$ for all $j$, but now we have the following transitions.

$$
h[i, j] = \max\begin{cases}
    h[i - 1, j - \ell] + \ell & \text{if } j - \ell \geq -k \\
    h[i - 1, j] + \max(d, \ell) \\
    h[i - 1, j + \ell] + \ell & \text{if } j + \ell \leq k
\end{cases}
$$

where $d$ is maximal such that $A[h[i - 1, j] + 1 + a, h[i - 1, j] + j + a] = B[h[i - 1, j] + j + 1, h[i - 1, j] + j + d]$ for some $a \in [-\ell, \ell]$. Note that when $\ell = 1$ this mostly aligns with the $h$-wave algorithm except we have $h[i - 1, j + \ell] + \ell$ instead of $h[i - 1, j + \ell]$ (as we are only seeking an approximation, we do this to make the analysis simpler).

For the approximate $h$-wave, it is approximately true that $h[i, j]$ is the farthest length $n'$ such that $A[1, n']$ and $B[1, n' - j]$ have edit distance at most $\tilde{O}(i\ell)$ (see Lemma 13 and 14 for more details). Then, if $\text{ED}(A, B) \leq \tilde{O}(k)$, we have $h[k, 0] \geq n$; and if $h[k, 0] < n$ then $\text{ED}(A, B) \geq \tilde{\Omega}(k\ell)$.

[RSSS19] Aviad Rubinstein, Saeed Seddighin, Zhao Song, and Xiaorui Sun. Approximation algorithms for LCS and LIS with truly improved running times. In David Zuckerman, editor, 60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019, pages 1121–1145. IEEE Computer Society, 2019.

[SS17] Michael E. Saks and C. Seshadhri. Estimating the longest increasing sequence in polylogarithmic time. SIAM J. Comput., 46(2):774–823, 2017.

[Ukk85] Esko Ukkonen. Algorithms for approximate string matching. Information and Control, 64(1-3):100–118, 1985.
Note that unlike the main body, we are no longer checking for matches where $A$ and $B$ have a common start point. Instead we require a generalization of MaxAlign, which we call MaxShiftAlign$L,k(A, B, i_A, i_B)$. This algorithm finds the greatest positive integer $d$ such that $A[i_A + c, i_A + d - 1 + c] = B[i_B, i_B + d - 1]$ for some $c \in [-\ell, \ell]$, given the promise that $|i_A - i_B| \leq k$.

**Definition 2** (shifted $(i_A, \ell)$-alignment and approximate shifted $(i_A, \ell)$-alignment).

Given strings $A, B$, and $i_A, i_B \in [1, n]$ we say that a $B[i_B, i_B + d - 1]$ has a shifted-$(i_A, \ell)$-alignment with $A$ if there is $i$ with $|i_A - i| \leq \ell$ and $A[i, i + d - 1] = B[i_B, i_B + d - 1]$. If instead we have that $|i_A - i| \leq \ell$, we say that $B[i_B, i_B + d - 1]$ has an approximate shifted-$(i_A, \ell)$-alignment with $A$.

We say that an implementation of MaxShiftAlign$L,k(A, B, i_A, i_B)$ is approximately correct if whenever $|i_A - i_B| \leq k$ the following are true.

1. Let $d'$ be the maximal $d'$ such that $B[i_B, i_B + d' - 1] = A[i_A, i_A + d' - 1]$ for. With probability at least $1 - 1/n^2$, $B[i_B, i_B + MaxShiftAlign$L,k(A, B, i_A, i_B) - 1]$ an approximate shifted-$(i_A, \ell)$-alignment in $A$.

**Algorithm 12 GreedyWave($A, B, k, \ell$)**

**Input:** $A, B \in \Sigma^n$, $\ell \leq k \leq n$

**Output:** SMALL if $ED(A, B) \leq \tilde{O}(k)$ or LARGE if $ED(A, B) \geq \tilde{O}(k\ell)$

$h \leftarrow$ matrix with indices $[0, k] \times (-k, k) \cap \ell Z$

for $i$ from 0 to $k$

for $j$ multiples of $\ell$ from $-k$ to $k$

if $i = 0$ then $h[i, j] \leftarrow -\infty, h[0, 0] \leftarrow 0$.

else

$h[i, j] \leftarrow h[i - 1, j] + \ell$

if $j - \ell \geq -k$ then $h[i, j] \leftarrow \max(h[i, j], h[i - 1, j - \ell] + \ell)$

if $j + \ell \leq k$ then $h[i, j] \leftarrow \max(h[i, j], h[i - 1, j + \ell] + \ell)$

$h[i, j] \leftarrow \max(h[i, j], h[i - 1, j] + MaxShiftAlign$L,k(A, B, h[i - 1, j] + 1, h[i - 1, j] + j + 1))$

if $h[k, 0] \geq n$ return SMALL

return LARGE

A.3 Analysis of GreedyWave

We first prove that if we do not take any of the "shortcuts" given by MaxShiftAlign$L,k(A, B, h[i - 1, j] + 1, h[i - 1, j] + j + 1)$, we still increase $h$ by a quantifiable amount.

**Claim 12.** Consider $(i, j), (i', j') \in [0, k] \times (-k, k) \cap \ell Z$ such that $i' \geq i$ and

$$|j' - j| \leq \ell(i' - i),$$

then,

$$h[i', j'] \geq h[i, j] + \ell(i' - i).$$

\footnote{Note that a shifted-$(i_A, \ell)$-alignment implies an approximate shifted-$(i_A, k)$-alignment, because $ED(A[i_A, i_A + d - 1], [i, i + d - 1]) \leq 2\ell$ if $|i - i_A| \leq \ell$.}
Proof. We prove this by induction on \( i' - i \). The base case of \( i' - i = 0 \) is immediate.

Now assume \( i' - i \geq 1 \) and that \( |j' - j| \leq \ell(i' - i) \). Note then there exists \( e \in \{-1, 0, 1\} \) such that \( j' + e\ell \) is between \( j' \) and \( j \) and

\[
|(j' + e\ell) - j| \leq \ell(i' - 1 - i).
\]

By the induction hypothesis, we know then that

\[
h[i', j' + e\ell] \geq h[i, j] + \ell(i' - i - 1).
\]

Note then from GreedyWave, since \( i' \geq 1 \), we have that

\[
h[i', j'] \geq h[i' - 1, j' + e\ell] + \ell.
\]

Therefore, combining the previous two inequalities.

\[
h[i', j'] \geq h[i, j] + \ell(i' - i).
\]

\[\square\]

Correctness of Algorithm 12 is proved in the following pair of lemmas.

**Lemma 13.** Assume that MaxShiftAlign_{\ell,K} is approximately correct. If \( \text{ED}(A, B) \leq k/(20 \lceil \log n \rceil) \), then GreedyWave(A, B, k, \ell) outputs SMALL with probability 1.

**Lemma 14.** Assume that MaxShiftAlign_{\ell,K} is approximately correct. If \( \text{ED}(A, B) > 10k\ell \), then GreedyWave(A, B, k, \ell) outputs LARGE with probability at least \( 1 - 1/n \).

**Proof of Lemma 13** Notation and Inductive Hypothesis

Let \( k' = \lfloor k/(20 \lceil \log n \rceil) \rfloor \). Assume that \( \text{ED}(A, B) \leq k' \). Let \( I^A_1, \ldots, I^A_{2k'+1} \) and \( I^B_1, \ldots, I^B_{2k'+1} \) and \( \pi : [2k' + 1] \to [2k' + 1] \cup \{\bot\} \) be as in Lemma 3. Let \((a_1, b_1), \ldots, (a_t, b_t) \in \pi \) be the matching, ordered such that \( a_1 \leq \cdots \leq a_t \) and \( b_1 \leq \cdots \leq b_t \). Let \( A_i = \max I^A_{a_i} \) and \( B_i = \max I^B_{b_i} \).

For the boundary, we let \( a_0, b_0 = 0, A_0 = B_0 = 0 \). Also let \( a_{t+1}, b_{t+1} = 2k' + 2 \) and \( A_{t+1} = B_{t+1} = n + 1 \). Let \( r : \mathbb{Z} \to \ell\mathbb{Z} \) be the function which rounds each integer to the nearest multiple of \( \ell \) (breaking ties by rounding down).

It suffices to prove by induction for all \( i \in \{0, 1, \ldots, t+1\} \), we have that

\[
h[a_i + b_i + (\lceil \log n \rceil + 1)i, r(A_i - B_i)] \geq A_i.
\]

The base case of \( i = 0 \) follows from \( h[0, 0] = 0 \) in the initialization. Assume now that

**Inductive hypothesis.** \( h[a_i + b_i + (\lceil \log n \rceil + 1)i, r(A_i - B_i)] \geq A_i. \)

We seek to show that

\[
h[a_{i+1} + b_{i+1} + (\lceil \log n + 1 \rceil)(i + 1), r(A_{i+1} - B_{i+1})] \geq A_{i+1}.
\]

We complete the induction in two steps.
Step 1, \( h[a_{i+1} + b_{i+1} + ([\log n] + 1)i + 1, r(A_{i+1} - B_{i+1})] \geq A_{i+1} + a_{i+1} - a_i + 1 \).

First note that

\[
|(A_{i+1} - B_{i+1}) - (A_i - B_i)| = |(A_{i+1} - A_i - |I_{a_{i+1}}^A|) - (B_{i+1} - B_i - |I_{b_{i+1}}^B|)| \\
\leq |A_{i+1} - A_i| - |I_{a_{i+1}}^A| + |B_{i+1} - B_i| - |I_{b_{i+1}}^B| \\
\leq (a_{i+1} - a_i) + (b_{i+1} - b_i).
\]

Therefore,

\[
|r(A_{i+1} - B_{i+1}) - r(A_i - B_i)| \leq (a_{i+1} - a_i) + (b_{i+1} - b_i) + \ell \\
\leq \ell[(a_{i+1} - a_i) + (b_{i+1} - b_i) + 1] \\
= \ell[(a_{i+1} + b_{i+1} + ([\log n] + 1)i + 1) - (a_i + b_i + ([\log n] + 1)i)].
\]

Therefore, we may apply Claim 12 to get that.

\[
h[a_{i+1} + b_{i+1} + ([\log n] + 1)i + 1, r(A_{i+1} - B_{i+1})] \\
\geq h[a_i + b_i + ([\log n] + 1)i, r(A_i - B_i)] \\
+ \ell[(a_{i+1} + b_{i+1} + ([\log n] + 1)i + 1) - (a_i + b_i + ([\log n] + 1)i)] \\
= A_i + \ell((a_{i+1} - a_i) + (b_{i+1} - b_i) + 1) \text{ (induction hypothesis)} \\
\geq A_i + a_{i+1} - a_i + 1.
\]

Step 2, \( h[a_{i+1} + b_{i+1} + ([\log n] + 1)(i + 1), r(A_{i+1} - B_{i+1})] \geq A_{i+1} \).

For \( j \in [[\log n + 1]] \), let

\[
\hat{h}_j := h[a_{i+1} + b_{i+1} + ([\log n])i + j, r(A_{i+1} - B_{i+1})].
\]

If \( \hat{h}_j \geq A_{i+1} \) for some \( j \in [[\log n]] \), then we know that

\[
h[a_{i+1} + b_{i+1} + ([\log n] + 1)(i + 1), r(A_{i+1} - B_{i+1})] \geq A_{i+1},
\]

which finishes the inductive step.

Otherwise, we know that \( \hat{h}_j \in [A_i + a_{i+1} - a_i + 1, A_{i+1}] \) for all \( j \in [[\log n]] \). If \( i = t \), then \( A_t + a_{t+1} - a_t + 1 \geq n + 1 = A_{t+1} \), because each interval strictly between \( a_t \) to \( a_{t+1} \) has length at most 1. Therefore, \( \hat{h}_j \geq A_{t+1} \), so we are done in this case.

Now assume \( i < t \) and consider any \( j \in [\log n] \). Since every interval between \( I_{a_{i+1}}^A \) and \( I_{a_{i+1}}^A \) has length at most 1, we have that \( A_{i+1} - |I_{a_{i+1}}^A| + 1 \leq A_i + a_{i+1} - a_i + 1 \). Therefore, \( \hat{h}_j \in [A_{i+1} - |I_{a_{i+1}}^A| + 1, A_i] \). Therefore, \( A[\hat{h}_j, A_{i+1}] = B[\hat{h'} - A_{i+1} + B_{i+1}, B_{i+1}] \). By definition of \( r \), \( |r(A_{i+1} - B_{i+1}) - A_{i+1} - B_{i+1}| \leq \ell \). Since MaxShiftAlign\(_\ell\) is approximately correct,

\[
\text{MaxShiftAlign}_{\ell,k}(A, B, \hat{h}_j, \hat{h}_j + r(A_{i+1} - B_{i+1})) > (A_{i+1} - \hat{h}_j)/2.
\]

Therefore,

\[
\hat{h}_{j+1} \geq \hat{h}_j + d > \frac{A_{i+1} + \hat{h}_j}{2}.
\]

By composing these inequalities, we have that

\[
\hat{h}_{j+1} > \frac{(2j - 1)A_{i+1} + \hat{h}_1}{2^j}.
\]
Plugging in \( j = \lfloor \log n \rfloor \), we get that

\[
  \hat{h}_{\lfloor \log n \rfloor + 1} > \frac{(n-1)}{n} A_{i+1} \geq A_{i+1} - 1,
\]

so

\[
  \hat{h}_{\lfloor \log n \rfloor + 1} = h[a_{i+1} + b_{i+1} + (\lfloor \log n \rfloor + 1)(i + 1), r(A_{i+1} - B_{i+1})] \geq A_{i+1},
\]

as desired.

**Conclusion.**

Therefore, we have that \( h[a_{t+2} + b_{t+2} + (\lfloor \log n \rfloor + 1)(t + 1), 0] \geq n + 1 \). Thus, we report SMALL as long as \( a_{t+2} + b_{t+2} + (\lfloor \log n \rfloor + 1)(t + 1) \leq k \). Observe that

\[
  a_{t+2} + b_{t+2} + (\lfloor \log n \rfloor + 1)(t + 1) \leq 2k' + 2 + 2k' + 2 + (\lfloor \log n \rfloor + 1)(2k' + 2) \\
  \leq 20\lfloor \log n \rfloor k' \leq k.
\]

Therefore, our algorithm always reports SMALL when \( ED(A, B) \leq k/(20\lfloor \log n \rfloor) \).

**Proof of Lemma [14]** Assume that GreedyWave\((A, B, k, \ell)\) output SMALL, and that MaxShiftAlign\(_{t,k}\) never failed at being approximately correct. For succinctness, we let \( h_{i,j} \) be shorthand for \( h[i, j] \).

We prove by induction that for all \( i \) and \( j \) for which \( h_{i,j} \neq -\infty \), \( ED(A[1, h_{i,j}], B[1, h_{i,j} + j\ell]) \leq 10i\ell + |j| \). This holds for the base case \( i = j = 0 \). Now, we break into cases depending on how \( h_{i,j} \) was computed.

**Case 1**, \( h_{i,j} = h_{i-1,j} + \ell \).

If \( h_{i,j} = h_{i-1,j} + \ell \), then observe that

\[
  ED(A[1, h_{i,j}], B[1, h_{i,j} + j]) \leq 2\ell + ED(A[1, h_{i-1,j}], B[1, h_{i-1,j} + j]) \\
  \leq 2\ell + 10(i - 1) \ell + |j| \\
  < 10i\ell + |j|.
\]

**Case 2**, \( h_{i,j} = h_{i-1,j-\ell} + \ell \).

If \( h_{i,j} = h_{i-1,j-\ell} + \ell \), then observe that

\[
  ED(A[1, h_{i,j}], B[1, h_{i,j} + j]) \leq ED(A[1, h_{i-1,j-\ell} + \ell], B[1, h_{i-1,j-\ell} + j + \ell]) \\
  \leq 3\ell + ED(A[1, h_{i-1,j-\ell}], B[1, h_{i-1,j-\ell} + j - \ell]) \\
  \leq 3\ell + 10(i - 1) \ell + |j - \ell| < 10i\ell + |j|.
\]

**Case 3**, \( h_{i,j} = h_{i-1,j+\ell} + \ell \).

If \( h_{i,j} = h_{i-1,j+\ell} + \ell \), then observe that

\[
  ED(A[1, h_{i,j}], B[1, h_{i,j} + j\ell]) \leq ED(A[1, h_{i-1,j+\ell} + \ell], B[1, h_{i-1,j+\ell} + j + \ell]) \\
  \leq \ell + ED(A[1, h_{i-1,j+\ell}], B[1, h_{i-1,j+\ell} + j + \ell]) \\
  \leq \ell + 10(i - 1) \ell + |j + \ell| < 10i\ell + |j|.
\]
Case 4, $h_{i,j} = h_{i-1,j} + d$

Finally, if $h_{i,j} = h_{i-1,j} + d$, then

$$\text{ED}(A[h_{i-1,j} + 1, h_{i-1,j} + d], B[h_{i-1,j} + j + 1, h_{i-1,j} + j + d]) \leq 10\ell$$

since MaxShiftAlign$_{\ell,k}$ is approximately correct. Thus,

$$\text{ED}(A[1, h_{i,j}], B[1, h_{i,j} + j]) \leq \text{ED}(A[1, h_{i-1,j}], B[1, h_{i-1,j} + \ell + j])
+ \text{ED}(A[h_{i-1,j} + 1, h_{i-1,j} + d],
B[h_{i-1,j} + j + 1, h_{i-1,j} + j + d])
\leq 10(i - 1)\ell + |j| + 10\ell < 10i\ell + |j|.$$

This completes the induction. Therefore, since GreedyWave$(A, B, k, \ell)$ output SMALL, we have that $h[k, 0] \geq n$. Thus, $\text{ED}(A, B) \leq 10k\ell$, as desired.

\[\square\]

A.4 Implementing MaxShiftAlign$_{\ell,k}$

A.4.1 MaxShiftAlign$_{\ell,k}$ with No Preprocessing

We use a nearly-identical algorithm to that of Section 5, including looking at $\sqrt{k}$ shifts for both $A$ and $B$. But, we now sample $S \subset [1, n]$ so that each element is included with probability at least $\min(4\ln n/\ell, 1)$ (instead of $\min(4\ln n/k, 1)$).

Algorithm 13 ProcessA$_{\ell,k}(A)$

Input: $A \in \Sigma^n$

for $a$ from $-2\sqrt{k}$ to $2\sqrt{k}$

$H_{A,a\sqrt{k}} \leftarrow \text{InitRollingHash}(A, S + a\sqrt{k})$

return \{ $H_{A,a\sqrt{k}} : a \in [-2\sqrt{k}, 2\sqrt{k}]$ \}

Algorithm 14 ProcessB$_{\ell,k}(B)$

for $b$ from $-\sqrt{k}$ to $\sqrt{k}$

$H_{B,b} \leftarrow \text{InitRollingHash}(B, S - b)$

return \{ $H_{B,b} : b \in [-\sqrt{k}, \sqrt{k}]$ \}

Lemma 15. MaxShiftAlign$_{\ell,k}$ is approximately correct.

Proof. First, consider any $d \geq 1$ such that $B[i_B, i_B + d - 1]$ has a $(i_A, \ell)$-alignment with $A$. We seek to show that MaxShiftAlign$_{\ell,k}(A, B, i_A, i_B) \geq d$ with probability 1. Note that the output of MaxShiftAlign$_{\ell,k}$ is always at least $2\ell$, so we may assume that $d > 2k$. By definition of $(i_A, \ell)$-alignment, there exists $c \in [-\ell, \ell]$ such that $A[i_A + c, i_A + c + d - 1] = B[i_B, i_B + d - 1]$. Note that since $|c + i_A - i_B| \leq 2k$, there exists $a \in [-2\sqrt{k}, 2\sqrt{k}]$ and $b \in \sqrt{k}$ such that $a\sqrt{k} + b = c + i_A - i_B$. In fact, we may take $a \in \left[\frac{c + i_A - i_B - \ell}{\sqrt{k}}, \frac{c + i_A - i_B + \ell}{\sqrt{k}}\right]$. Thus, since $b \in [-\sqrt{k}, \sqrt{k}] \subset [-\ell, \ell]$, we have that

$$B[i_B + \ell - b, i_B + d - \ell - 1 - b] = A[i_A + c + \ell - b, i_A + c + d - \ell - 1 - b]$$

$$= A[i_B + \ell + a\sqrt{k}, i_B + d - \ell - 1 + a\sqrt{k}].$$

20
and so there exists \( \ell, k \) such that \( \text{MaxShiftAlign} \) have nontrivial intersection, so the output of the binary search will be at least \( n \).

Applying Claim 2 over all at most \( \tilde{O}(\sqrt{k}) \) comparisons of hashes made during the algorithm, with probability at least 1 that

\[
\text{RetrieveRollingHash}(A, S + a\sqrt{k}, H_{A,a\sqrt{k}}, i_B + \ell + a\sqrt{k}, d - \ell - 1 + a\sqrt{k}) = \text{RetrieveRollingHash}(B, S - b, H_{B,b}, i_B + \ell - b, i_B + d - \ell - 1 - b).
\]

Therefore, in the implementation of \( \text{MaxShiftAlign}_{\ell,k}(A, B, i_A, i_B) \), if \( d_{\text{mid}} = d \), then \( L_A \) and \( L_B \) will have nontrivial intersection, so the output of the binary search will be at least \( d \), as desired. Thus, \( \text{MaxShiftAlign}_{\ell,k}(A, B, i_B) \) will output at least the length of the maximal \( (i_A, \ell) \)-alignment.

Second, we verify that \( \text{MaxShiftAlign}_{\ell,k} \) outputs an approximate \( (i_A, \ell) \)-alignment. Let \( d \) be the output of \( \text{MaxShiftAlign}_{\ell,k} \), either \( d = 2\ell \), in which case \( B[i_B, i_B + d - 1] \) trivially is in approximate \( (i_A, \ell) \)-alignment with \( A \) or \( d > 2\ell \). Thus, for that \( d \), the binary search found that \( L_A \cap L_B \neq \emptyset \) and so there exists \( a \in \left( \left[ \frac{1}{\sqrt{k}} \right], b \right) \) such that

\[
\text{RetrieveRollingHash}(A, S + a\sqrt{k}, H_{A,a\sqrt{k}}, i_B + \ell + a\sqrt{k}, d - \ell - 1 + a\sqrt{k}) = \text{RetrieveRollingHash}(B, S - b, H_{B,b}, i_B + \ell - b, i_B + d - \ell - 1 - b).
\]

Applying Claim 2 over all at most \( \tilde{O}(\sqrt{k}) = \tilde{O}(k) \) comparisons of hashes made during the algorithm, with probability at least 1 that \( 1 - 1/n^3 \), we must have that

\[
\text{ED}(A[i_B + \ell + a\sqrt{k}, d - \ell - 1 + a\sqrt{k}], B[i_B + \ell - b, i_B + d - \ell - 1 - b]) \leq \ell.
\]

Let \( c := a\sqrt{k} + b \), then we have that

\[
\text{ED}(A[i_B + c + \ell - b, i_B + c + d - \ell - 1 - b], B[i_B + \ell - b, i_B + d - \ell - 1 - b]) \leq \ell
\]

Therefore,

\[
\text{ED}(A[i_B + c, i_B + c + d - 1], B[i_B, i_B + d - 1]) \leq 3\ell.
\]
Note that
\[ i_B + c = i_B + a\sqrt{k} + b \]
\[ \geq i_B + [(i_A - i_B - \ell) - \sqrt{k}] - \sqrt{k} \]
\[ \geq i_A - 3\ell. \]
Likewise, \( i_B + c \leq i_A + 3\ell \). Therefore,
\[ \text{ED}(A[i_B + c, i_B + c + d - 1], A[i_A, i_A + d - 1]) \leq 6\ell, \]
since we need at most \( 3\ell \) insertions and \( 3\ell \) deletions to go between the two strings. By the triangle inequality we then have that
\[ \text{ED}(A[i_A, i_A + d - 1], B[i_B, i_B + d - 1]) < 10\ell, \]
as desired.

**Theorem 16.** If \( \ell \geq \sqrt{k} \), with no preprocessing, we can distinguish between \( \text{ED}(A, B) \leq \tilde{O}(k) \) and \( \text{ED}(A, B) \geq \tilde{\Omega}(\ell k) \) in \( \tilde{O}\left(\frac{(n+k^2)\sqrt{k}}{\ell}\right) \) time with probability at least \( 1 - \frac{1}{n} \).

**Proof.** By Lemma 15, MaxShiftAlign_{\ell,k} is approximately correct so by Lemmas 13 and 14 succeeds with high enough probability that GreedyMatch outputs the correct answer with probability at least \( 1 - \frac{1}{n} \).

Both ProcessA_{\ell,k} and ProcessB_{\ell,k} run in \( \tilde{O}\left(\frac{n\sqrt{k}}{\ell}\right) \) time. Further, MaxShiftAlign_{\ell,k} runs in \( \tilde{O}(\sqrt{k}) \) time. Since GreedyWave makes runs in \( \tilde{O}(k^2/\ell) \) and makes that many calls to MaxShiftAlign_{\ell,k}, we have that GreedyWave runs in \( \tilde{O}\left(\frac{(n+k^2)\sqrt{k}}{\ell}\right) \) time. \( \square \)

Setting \( \ell = \tilde{\Theta}(k^{1-\epsilon}) \) with \( \epsilon < 1/2 \), we get that distinguishing \( k \) from \( k^{2-\epsilon} \) can be done in \( \tilde{O}(nk^{-1/2+\epsilon} + k^{3/2+\epsilon}) \) time.

### A.4.2 MaxShiftAlign_{\ell,k} with One-Sided and Two-Sided Preprocessing

For the one-sided preprocessing, we mimic \[6\] We preselect \( S \subseteq [1, n] \) which each element included i.i.d. with probability \( q := \min\left(\frac{\ln n}{\ell}, 1\right) \). Also assume that every multiple of \( \ell \) (and \( n - 1 \)) is in \( S \). This only increases the size of \( S \) by \( n/\ell \), and does not hurt the success probability of Claim \[2\].

**Algorithm 16 OneSidedPreprocess\(_{\ell,k}(A)\)**

```
for a from \(-k\) to \(k\)
    \( H_{A,a} \leftarrow \text{InitRollingHash}(A, S + a) \)
    \( T_A \leftarrow \text{list of } [(\log n) \times \left\lfloor \frac{n}{\ell} \right\rfloor \times \left\lfloor \frac{-k}{\ell} - 1, \frac{k}{\ell} + 1 \right\rfloor \text{ matrix of empty hash tables} \)
for \( e_0 \) in \( (\log n) \)
    for a from \(-k\) to \(k\)
        for i in \((S + 1) \cup (S - 2^{e_0} + 1) \) \( \cap [n] \)
            \( h \leftarrow \text{RetrieveRollingHash}(A, S + a, H_{A,a}, i + a, i + 2^{e_0} - 1 + a) \)
            add h to \( T_A[i_0, [i/\ell] + e_1, [a/\ell] + e_2] \) for \( e_1, e_2 \in \{-1, 0, 1\} \).
return \( T_A \)
```

**Claim 17.** OneSidedPreprocess\(_{\ell,k}(A)\) runs in \( \tilde{O}(nk^{3/2}) \) time in expectation.
Thus, OneSidedMaxShiftAlign\(_{\ell,k}\) takes \(\max\) time in expectation. Therefore, in the implementation of OneSidedMaxShiftAlign\(_{\ell,k}\), this takes \(\tilde{O}(n/\ell)\) time in expectation.

### Algorithm 17 OneSidedProcess\(_{\ell,k}(B)\)

\[\text{return } H_B \leftarrow \text{InitRollingHash}(B, S)\]

### Algorithm 18 OneSidedMaxShiftAlign\(_{\ell,k}(A, B, i_A, i_B)\)

**Input:** \(A \in \Sigma^n, B \in \Sigma^n, \ell, k \leq n, i_A, i_B \in [1, n] |i_A - i_B| \leq k\)

**for** \(d_0 \in [2^{\log n}, 2^{\log n} - 1, \ldots, 1]\)

**if** RetrieveRollingHash\((B, S, H_B, i_B + d_0 - 1) \in T_A[\log d_0, [i_B/\ell], [(i_A - i_B)/\ell]]\)

**then return** \(d_0\)

**return** \(0\)

### Lemma 18. OneSidedMaxShiftAlign\(_{\ell,k}\) is approximately correct.

**Proof.** First, consider the maximal \(d \geq 1\) a power of two such that \(B[i_B, i_B + d - 1]\) has a \((i_A, \ell)\)-alignment with \(A\). We seek to show that OneSidedMaxShiftAlign\(_{\ell,k}(A, B, i_A, i_B) \geq d\) with probability 1.

Let \(i_B'\) be the least element of \((S + 1) \cup (S - d + 1)\) which is at least \(i_B\). Since every multiple of \(\ell\) is in \(S\) (as well as \(n - 1\)), we have that \(|i_B' - i_B| \leq \ell\) and

\[\text{RetrieveRollingHash}(A, S + a, H_{A,a}, i_B + a, i_B + d - 1 + a) = \text{RetrieveRollingHash}(A, S + a, H_{A,a}, i_B' + a, i_B' + d - 1 + a) \in T_A[\log d, [i_B'/\ell], [a/\ell]].\]

Therefore, in the implementation of OneSidedMaxShiftAlign\(_{\ell,k}(A, B, i_A, i_B)\), if \(d_0 = d\), we have that

\[\text{RetrieveRollingHash}(B, S, H_B, i_B + d - 1) \in T_A[\log d, [i_B'/\ell] + e_1, [a/\ell] + e_2].\]

for all \(e_1, e_2 \in \{-1, 0, 1\}\). Since \(|i_B' - i_B| \leq \ell\) and \(|a - (i_A - i_B)| \leq \ell\), we have that

\[\text{RetrieveRollingHash}(B, S, H_B, i_B + d - 1) \in T_A[\log d, [i_B/\ell], [(i_A - i_B)/\ell]].\]

Thus, OneSidedMaxShiftAlign\(_{\ell,k}(A, B, i_A, i_B)\) will output at least more than half the length of the maximal shift \((i_A, \ell)\)-alignment.

Second, we verify that OneSidedMaxShiftAlign\(_{\ell,k}\) outputs an approximate \((i_A, \ell)\)-alignment. Let \(d\) be the output of OneSidedMaxShiftAlign\(_{\ell,k}\). Either \(d = 0\), in which case \(B[i_B, i_B + d - 1]\)
trivially is in approximate \((i_A, \ell)\)-alignment with \(A\) or \(d \geq 1\). Thus, for that \(d\), the search found that \(\text{RetrieveRollingHash}(B, S, H_B, i_B + d - 1) \in T_A[\log d, [i_B/\ell], [(i_A - i_B)/\ell]]\). Thus, there exists \(i'_A\) with \(|[i'_A/\ell] - [i_B/\ell]| \leq 1\), and \(a \in [-k, k]\) such that

\[
|[a/\ell] - ((i_A - i_B)/\ell)| \leq 1
\]

and

\[
\text{RetrieveRollingHash}(A, S + a, H_A, i_A + a, i'_A + d - 1 + a) = \text{RetrieveRollingHash}(B, S, H_B, i'_A, i'_A + d - 1).
\]

Applying Claim \([2]\) over all \(\tilde{O}(k)\) potential comparisons of hashes made during the algorithm, with probability at least \(1 - 1/n^3\), we must have that

\[
\text{ED}(A[i'_A + a, i'_A + a + d - 1], B[i_B, i_B + d - 1]) \leq \ell.
\]

By \((1)\),

\[
|i'_A + a - i_A| \leq |i'_A - i_B| + |a - (i_A - i_B)| \leq 2\ell + 2\ell = 4\ell.
\]

Therefore, \(\text{ED}(A[i_A, i_A + d - 1], B[i_B, i_B + d - 1]) \leq 9\ell\).

Therefore, \(B[i_B, i_B + d - 1]\) has an approximate \((i_A, \ell)\)-alignment with \(A\), as desired.

**Theorem 19.** For all \(A, B \in \Sigma^n\). When \(A\) is preprocessed for \(\tilde{O}(nk/\ell)\) time in expectation, we can distinguish between \(\text{ED}(A, B) \leq \tilde{O}(k)\) and \(\text{ED}(A, B) \geq \tilde{O}(k\ell)\) in \(\tilde{O}(\frac{n+k^2}{\ell})\) time with probability at least \(1 - \frac{1}{n}\) over the random bits in the preprocessing (oblivious to \(B\)).

**Proof.** By Lemma \([18]\) OneSidedMaxShiftAlign, \(A, k, k\) is approximately correct so by Lemmas \([13]\) and \([14]\) succeeds with high enough probability that GreedyMatch outputs the correct answer with probability at least \(1 - \frac{1}{n}\).

As proved in Claim \([17]\) the preprocessing runs in \(\tilde{O}(nk/\ell)\) time. Also OneSidedProcessB, \(k, k\) runs in \(\tilde{O}(n/\ell)\) time. Further, OneSidedMaxShiftAlign, \(A, k, k\) runs in \(\tilde{O}(1)\) time, as performing the power-of-two search, computing the hash, and doing the table lookups are \(\tilde{O}(1)\) operations. Therefore, GreedyWave runs in \(\tilde{O}(k^2/\ell)\) time. Thus, the whole computation takes \(\tilde{O}(\frac{n+k^2}{\ell})\) time.

Thus, if \(\ell = k^{1-\epsilon}\), the preprocessing is \(\tilde{O}(n \cdot k^\epsilon)\), and the runtime otherwise is \(\tilde{O}(n/k^{1-\epsilon} + k^{1+\epsilon})\).

If we are in the two-sided preprocessing model, we can run both OneSidedPreprocess \(A, k, k\) and OneSidedProcessB \(k, k\) for both \(A\) and \(B\). Then, we can just run GreedyWave which runs in \(\tilde{O}(k^2/\ell)\) time. Thus, we have the following corollary. We have an implication for two-sided preprocessing as a corollary.

**Corollary 20.** For all \(A, B \in \Sigma^n\). When \(A\) and \(B\) are preprocessed for \(\tilde{O}(nk/\ell)\) time in expectation, we can distinguish between \(\text{ED}(A, B) \leq \tilde{O}(k)\) and \(\text{ED}(A, B) \geq \tilde{O}(k\ell)\) in \(\tilde{O}(\frac{n+k^2}{\ell})\) time with probability at least \(1 - \frac{1}{n}\) over the random bits in the pre-processing.
B Omitted Proofs

B.1 Proof of Lemma 3

Proof. We prove this by induction on $k$. If $k = 0$, then we can partition $A$ and $B$ each into a single part which are matched. Assume we have proved the theorem for $k \leq k_0$. Consider $A$ and $B$ with $ED(A, B) = k_0 + 1$. Thus, there exists $B' \in \Sigma^*$ such that $ED(A, B') = k_0$ and $ED(B', B) = 1$. By the induction hypothesis, $B'$ can be partitioned into intervals $I_1, \ldots, I_{2k_0}$ that are each of length at most 1 or are equal to some interval of $A$ up to a shift of $k_0$. We now break up into cases.

Case 1. A character of $B'$ is substituted to make $B$. Let $I_{j_0}$ be the interval this substitution occurs in. We split $I_{j_0}$ into three (some possibly empty) intervals $I^{(1)}_{j_0}, I^{(2)}_{j_0}, I^{(3)}_{j_0}$, the intervals before, at, and after the substitution. We have the partition of $B$ into $2k_0 + 1$ intervals: $I_1, \ldots, I_{j_0-1}, I^{(1)}_{j_0}, I^{(2)}_{j_0}, I^{(3)}_{j_0}, I_{j_0+1}, \ldots, I_{2k_0}$. Every interval is of length at most 1 or corresponds to a equal substring of $A$ up to a shift of $k_0 \leq k_0 + 1$.

Case 2. A character of $B'$ is deleted to make $B$. Let $I_{j_0}$ be the interval this deletion occurs in. We split $I_{j_0}$ into three intervals $I^{(1)}_{j_0}, I^{(2)}_{j_0}, I^{(3)}_{j_0}$, the intervals before, at, and after the deletion (the middle interval is empty). We have the partition of $B$ into $2k_0 + 1$ intervals: $I_1, \ldots, I_{j_0-1}, I^{(1)}_{j_0}, I^{(2)}_{j_0}, I^{(3)}_{j_0}, I_{j_0+1} - 1, \ldots, I_{2k_0} - 1$. Every interval is of length at most 1 or corresponds to a equal substring of $A$ up to a shift of $k_0 + 1$.

Case 3. A character of $B'$ is inserted to make $B$. Let $I_{j_0}$ be the interval this insertion occurs in, or if the insertion is between intervals, take either adjacent interval. We split $I_{j_0}$ into three intervals $I^{(1)}_{j_0}, I^{(2)}_{j_0}, I^{(3)}_{j_0}$, the intervals before, at, and after the intersection. We have the partition of $B$ into $2k_0 + 1$ intervals: $I_1, \ldots, I_{j_0-1}, I^{(1)}_{j_0}, I^{(2)}_{j_0}, I^{(3)}_{j_0}, I_{j_0+1} + 1, \ldots, I_{2k_0} + 1$. Every interval is of length at most 1 or corresponds to a equal substring of $A$ up to a shift of $k_0 + 1$.

In all three cases, if the interval $I_{j_0}$ broken up in $B$ corresponded to an interval in $A$ according to $\pi$, then the interval of $A$ can be broken up in the analogous fashion with $\pi$ suitably modified. If $I_{j_0}$ was an unmatched interval, add two empty intervals to $A$’s partition. \qed